STOCHASTIC MODELLING AND APPLIED PROBABILITY

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Wendell H. Fleming
H.M. Soner

Controlled Markov Processes and Viscosity Solutions

Second Edition

Springer

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Mathematics Subject Classification 60JXX, 93EXX, 35K55

Library of Congress Control Number: 2005929857

ISBN-10: 0-387-26045-5
ISBN-13: 978-0387-260457

Printed on acid-free paper.

Printed in the United States of America. EB

9 8 7 6 5 4 3 2 1

springeronline.com

We dedicate this edition to
Florence Fleming
Serpil Soner

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Preface to Second Edition

This edition differs from the previous one in several respects. The use of stochastic calculus and control methods to analyze financial market models has expanded at a remarkable rate. A new Chapter X gives an introduction to the role of stochastic optimal control in portfolio optimization and in pricing derivatives in incomplete markets. Risk-sensitive stochastic control has been another active research area since the First Edition of this book appeared. Chapter VI of the First Edition has been completely rewritten, to emphasize the relationships between logarithmic transformations and risk sensitivity. Risk-sensitive control theory provides a link between stochastic control and H-infinity control theory. In the H-infinity approach, disturbances in a control system are modelled deterministically, instead of in terms of stochastic processes. A new Chapter XI gives a concise introduction to two-controller, zero-sum differential games. Included are differential games which arise in nonlinear H-infinity control and as totally risk-averse limits in risk-sensitive stochastic control. Other changes from the First Edition include an updated treatment in Chapter V of viscosity solutions for second-order PDEs. Material has also been added in Section I.11 on existence of optimal controls in deterministic problems. This simplifies the presentation in later sections, and also is of independent interest.

We wish to thank D. Hernandez-Hernandez, W.M. McEneaney and S.-J. Sheu who read various new chapters of this edition and made helpful comments. We are also indebted to Madeline Brewster and Winnie Isom for their able, patient help in typing and revising the text for this edition.

W.H. Fleming
H.M. Soner

May 1, 2005

Preface

This book is intended as an introduction to optimal stochastic control for continuous time Markov processes and to the theory of viscosity solutions. We approach stochastic control problems by the method of dynamic programming. The fundamental equation of dynamic programming is a nonlinear evolution equation for the value function. For controlled Markov diffusion processes on n - dimensional euclidean space, the dynamic programming equation becomes a nonlinear partial differential equation of second order, called a Hamilton – Jacobi – Bellman (HJB) partial differential equation. The theory of viscosity solutions, first introduced by M. G. Crandall and P.-L. Lions, provides a convenient framework in which to study HJB equations. Typically, the value function is not smooth enough to satisfy the HJB equation in a classical sense. However, under quite general assumptions the value function is the unique viscosity solution of the HJB equation with appropriate boundary conditions. In addition, the viscosity solution framework is well suited to proving continuous dependence of solutions on problem data.

The book begins with an introduction to dynamic programming for deterministic optimal control problems in Chapter I, and to the corresponding theory of viscosity solutions in Chapter II. A rather elementary introduction to dynamic programming for controlled Markov processes is provided in Chapter III. This is followed by the more technical Chapters IV and V, which are concerned with controlled Markov diffusions and viscosity solutions of HJB equations. We have tried, through illustrative examples in early chapters and the selection of material in Chapters VI – VII, to connect stochastic control theory with other mathematical areas (e.g. large deviations theory) and with applications to engineering, physics, management, and finance. Chapter VIII is an introduction to singular stochastic control. Dynamic programming leads in that case not to a single partial differential equation, but rather to a system of partial differential inequalities. This is also a feature of other important classes of stochastic control problems not treated in this book, such as impulsive control and problems with costs for switching controls.

Value functions can be found explicitly by solving the HJB equation only in a few cases, including the linear–quadratic regulator problem, and some special problems in finance theory. Otherwise, numerical methods for solving the HJB equation approximately are needed. This is the topic of Chapter IX.

Chapters III, IV and VI rely on probabilistic methods. The only results about partial differential equations used in these chapters concern classical solutions (not viscosity solutions.) These chapters can be read independently of Chapters II and V. On the other hand, readers wishing an introduction to viscosity solutions with little interest in control may wish to focus on Chapter II, Secs. 4–6, 8 and on Chapter V, Secs. 4–8.

We wish to thank M. Day, G. Kossioris, M. Katsoulakis, W. McEneaney, S. Shreve, P. E. Souganidis, Q. Zhang and H. Zhu who read various chapters and made helpful comments. Thanks are also due to Janice D'Amico who typed drafts of several chapters. We are especially indebted to Christy Newton. She not only typed several chapters, but patiently helped us through many revisions to prepare the final version.

W.H. Fleming

June 1, 1992 *H.M. Soner*

Notation

In this book the following system of numbering definitions, theorems, formulas etc. is used. Roman numerals are used to refer to chapters. For example, Theorem II.5.1 refers to Theorem 5.1 in Chapter II. Similarly, IV(3.7) refers to formula (3.7) of Chapter IV; and within Chapter IV we write simply (3.7) for such a reference.

$I\!R^n$ denotes n-dimensional euclidean space, with elements $x = (x_1, \cdots, x_n)$. We write

$$x \cdot y = \sum_{i=1}^{n} x_i y_i$$

and $|x| = (x \cdot x)^{\frac{1}{2}}$ for the euclidean norm. If A is a $m \times n$ matrix, we denote by $|A|$ the operator norm of the corresponding linear transformation from $I\!R^n$ into $I\!R^d$:

$$|A| = \max_{|x| \leq 1} |Ax|.$$

The transpose of A is denoted by A'. If a and A are $n \times n$ matrices,

$$\mathrm{tr}\, aA = \sum_{i,j=1}^{n} a_{ij} A_{ij}.$$

$\mathcal{S}^n$ denotes the set of symmetric $n \times n$ matrices and $\mathcal{S}^n_+$ the set of nonnegative definite $A \in \mathcal{S}^n$. The interior, closure, and boundary of a set B are denoted by $intB, \bar{B}$ and ∂B respectively. If Σ is a metric space,

$\mathcal{B}(\Sigma) = \sigma -$ algebra of Borel sets of Σ

$\mathcal{M}(\Sigma) =$ {all real – valued functions on Σ which are bounded below}

$C(\Sigma) =$ {all real – valued continuous functions on Σ}

$C_b(\Sigma) =$ bounded functions in $C(\Sigma)$}.

If Σ is a Banach space

$C_p(\Sigma) =$ {polynomial growing functions in $C(\Sigma)$}.

A function ϕ is called polynomially growing if there exist constants $K, m \geq 0$ such that

$$|\phi(x)| \leq K(1 + |x|^m), \ \forall x \in \Sigma.$$

For an open set $O \subset I\!R^n$, and a positive integer k,

$C^k(O) =$ {all $k-$ times continuously differentiable functions on O}

$C_b^k(O) = \{\phi \in C^k(O) :$ ϕ and all partial derivatives of ϕ or orders $\leq k$ are bounded}

$C_p^k(O) = \{\phi \in C^k(O) :$ all partial derivatives of ϕ of orders $\leq k$ are polynomially growing}.

For a measurable set $E \subset I\!R^n$, we say that $\phi \in C^k(E)$ if there exist $\tilde{E}$ open with $E \subset \tilde{E}$ and $\tilde{\phi} \in C^k(\tilde{E})$ such that $\phi(x) = \tilde{\phi}(x)$ for all $x \in E$. Spaces $C_b^k(E), C_p^k(E)$ are defined similarly. $C^\infty(E), C_b^\infty(E), C_p^\infty(E)$ denote the intersections over $k = 1, 2, \cdots$ of $C^k(E), C_b^k(E), C_p^k(E)$.

We denote the gradient vector and matrix of second order partial derivatives of ϕ by

$$D\phi = (\phi_{x_1}, \cdots, \phi_{x_n})$$

$$D^2\phi = (\phi_{x_i x_j}), i, j = 1, \cdots, n.$$

Sometimes these are denoted instead by ϕ_x, ϕ_{xx} respectively.

If ϕ is a vector-valued function, with values in $I\!R^m$, then we write $\phi \in C^k(E), \phi \in C_b^k(E)$ etc if each component of ϕ belongs to $C^k(E), C_b^k(E)$ etc. For vector-valued functions, $D\phi$ and $D^2\phi$ are identified with the differentials of ϕ of first and second orders. For vector-valued ϕ, $|D\phi|, |D^2\phi|$ are the operator norms. We denote intervals of $I\!R^1$, respectively closed and half-open to the right, by

$$[a, b], \quad [a, b).$$

Given $t_0 < t_1$

$$Q_0 = [t_0, t_1) \times I\!R^n, \qquad \overline{Q}_0 = [t_0, t_1) \times I\!R^n.$$

Given $O \subset I\!R^n$ open

$$Q = [t_0, t_1) \times O, \qquad \overline{Q} = [t_0, t_1] \times \overline{O}$$

$$\partial^* Q = ([t_0, t_1] \times \partial O) \cup (\{t_1\} \times O).$$

We call $\partial^* Q$ the *parabolic boundary* of the cylindrical region Q. If $\Phi = \phi(t, x), G \subset I\!R^{n+1}$, we say that $\Phi \in C^{\ell,k}(G)$ if there exist $\tilde{G}$ open with $G \subset \tilde{G}$ and $\tilde{\Phi}$ such that $\tilde{\Phi}(t, x) = \Phi(t, x)$ for all $(t, x) \in G$ and all partial derivatives of $\tilde{\Phi}$ or orders $\leq \ell$ in t and of orders $\leq k$ in x are continuous on $\tilde{G}$. For example, we often consider $\Phi \in C^{1,2}(G)$, where either $G = Q$ or $G = \overline{Q}$. The spaces $C_b^{\ell,k}(G), C_p^{\ell,k}(G)$ are defined similarly as above.

The gradient vector and matrix of second-order partial derivatives of $\Phi(t, \cdot)$ are denoted by $D_x \Phi, D_x^2 \Phi$, or sometimes by Φ_x, Φ_{xx}.

If F is a real-valued function on a set U which has a minimum on U, then

$$\arg\min_{v \in U} F(v) = \{v^* \in U : F(v^*) \leq F(v)\ \forall v \in U\}.$$

The supnorm of a bounded function is denoted by $\|\ \|$, and L_p-norms are denoted by $\|\ \|_p$.

I

Deterministic Optimal Control

I.1 Introduction

The concept of control can be described as the process of influencing the behavior of a dynamical system to achieve a desired goal. If the goal is to optimize some payoff function (or cost function) which depends on the control inputs to the system, then the problem is one of *optimal control.*

In this introductory chapter we are concerned with deterministic optimal control models in which the dynamics of the system being controlled are governed by a set of ordinary differential equations. In these models the system operates for times s in some interval I. The state at time $s \in I$ is a vector in n-dimensional euclidean $I\!R^n$. At each time s, a control $u(s)$ is chosen from some given set U (called the *control space.*) If I is a finite interval, namely,

$$I = [t, t_1] = \{s : t \leq s \leq t_1\},$$

then the differential equations describing the time evolution of $x(s)$ are (3.2) below. The cost functional to be optimized takes the form (3.4).

During the 1950's and 1960's aerospace engineering applications greatly stimulated the development of deterministic optimal control theory. Among such applications was the problem of optimal flight trajectories for aircraft and space vehicles. However, deterministic control theory provides methods of much wider applicability to problems from diverse areas of engineering, economics and management science. Some illustrative examples are given in Section 2.

It often happens that a system is being controlled only for $x(s) \in \overline{O}$, where $\overline{O}$ is the closure of some given open set $O \subset I\!R^n$. Two versions of that situation are formulated in Section 3. In one version, control occurs only until the time of exit from a closed cylindrical region $\overline{Q} = [t_0, t_1] \times \overline{O}$. In the other version, only controls which keep $x(s) \in \overline{O}$ for $t \leq s \leq t_1$ are allowed (this is called a *state constrained control problem.*)

The method of dynamic programming is the one which will be followed in this book, to study both deterministic and stochastic optimal control problems. In dynamic programming, a *value function* V is introduced which is the optimum value of the payoff considered as a function of initial data. See Section 4, and also Section 7 for infinite time horizon problems. The value function V for a deterministic optimal control problem satisfies, at least formally, a first order nonlinear partial differential equation. See (5.3) or (7.10) below. In fact, the value function V often does not have the smoothness properties needed to interpret it as a solution to the dynamic programming partial differential equation in the usual ("classical") sense. However, in such cases V can be interpreted as a viscosity solution, as will be explained in Chapter II.

Closely related to dynamic programming is the idea of feedback controls, which will also be called in this book *Markov control policies.* According to a Markov control policy, the control $u(s)$ is chosen based on knowing not only time s but also the state $x(s)$. The Verification Theorems 5.1, 5.2 and 7.1 provide a way to find optimal Markov control policies, in cases when the value function V is indeed a classical solution of the dynamic programming partial differential equation with the appropriate boundary data.

Another approach to optimal deterministic control is via Pontryagin's principle, which provides a general set of necessary conditions for an extremum. In Section 6 we develop, rather briefly, the connection between dynamic programming and Pontryagin's principle. We also give a proof of Pontryagin's principle, for the special case of control on a fixed time interval ($O = \mathbb{R}^n$).

In Section 8 and 9 we consider a special class of control problems, in which the control is the time derivative of the state ($u(s) = \dot{x}(s)$) and there are no control constraints. Such problems belong to the classical calculus of variations. For a calculus of variations problem, the dynamic programming equation is called a Hamilton-Jacobi partial differential equation. Many first-order nonlinear partial differential equations can be interpreted as Hamilton-Jacobi equations, by using duality for convex functions. This duality corresponds to the dual Lagrangian and Hamiltonian formulations in classical mechanics. These matters are treated in Section 10.

Another part of optimal control theory concerns the existence of optimal controls. In Section 11 we prove two special existence theorems which are used elsewhere in this book. The proofs rely on lower semicontinuity of the cost function in the control problem.

The reader should refer to Section 3 for notations and assumptions used in this chapter, for finite-time horizon deterministic optimal control problems. For infinite-time horizon problems, these are summarized in Section 7.

I.2 Examples

We start our discussion by giving some examples. In choosing examples, in this section and later in the book, we have included several highly simplified models chosen from such diverse applications as inventory theory, control of physical devices, financial economics and classical mechanics.

Example 2.1. Consider the production planning of a factory producing n commodities. Let $x_i(s),\ u_i(s)$ denote respectively the inventory level and production rate for commodity $i = 1, \cdots, n$ at time s. In this simple model we assume that the demand rates d_i are fixed constants, known to the planner. Let

$$x(s) = (x_1(s), \cdots, x_n(s)),\ u(s) = (u_1(s), \cdots, u_n(s)),\ d = (d_1, \cdots, d_n).$$

They are, respectively, the inventory and control vectors at time s, and the demand vector. The rate of change of the inventory $x(s) \in I\!\!R^n$ is

$$\frac{d}{ds}x(s) = u(s) - d. \tag{2.1}$$

Let us consider the production planning problem on a given finite time interval $t \le s \le t_1$. Given an initial inventory $x(t) = x$, the problem is to choose the production rate $u(s)$ to minimize

$$\int_t^{t_1} h(x(s))ds + \psi(x(t_1)). \tag{2.2}$$

We call t_1 the *terminal time*, h the *running cost*, and ψ the *terminal cost*. It is often assumed that h and ψ are convex functions, and that $h(x),\ \psi(x)$ have a unique minimum at $x = 0$. A typical example of h is

$$h(x) = \sum_{i=1}^{n} \left[\alpha_i(x_i)^+ + \gamma_i(x_i)^-\right],$$

where α_i, γ_i are positive constants interpreted respectively as a unit holding cost and a unit shortage cost. Here, $a^+ = \max\{a, 0\},\ a^- = \max(-a, 0)$.

The production rate $u(s)$ must satisfy certain constraints related to the physical capabilities of the factory and the workforce. These capacity constraints translate into upper bounds for the production rates. We assume that these take the form $c_1 u_1 + \cdots + c_n u_n \le 1$ for suitable constants $c_i > 0$.

To summarize, this simple production planning problem is to minimize (2.2) subject to (2.1), the initial condition $x(t) = x$, and the control constraint $u(s) \in U$ where

$$U = \{v \in I\!\!R^n :\ v_i \ge 0, i = 1 \cdots, n, \sum_{i=1}^{n} c_i v_i \le 1\}. \tag{2.3}$$

An infinite time horizon, discounted cost version of this problem will be mentioned in Example 7.4, and the solution to it will be outlined there.

Example 2.2. Consider a simple harmonic oscillator, in which a forcing term $u(s)$ is taken as the control. Let $x_1(s), x_2(s)$ denote respectively the position and velocity at time s. Then

$$\frac{d}{ds}x_1(s) = x_2(s) \tag{2.4}$$
$$\frac{d}{ds}x_2(s) = -x_1(s) + u(s).$$

We require that $u(s) \in U$, where U is a closed interval. For instance, if $U = [-a, a]$ with $a < \infty$, then the bound $|u(s)| \le a$ is imposed on the forcing term.

Let us consider the problem of controlling the simple harmonic oscillator on a finite time interval $t \le s \le t_1$. An initial position and velocity $(x_1(t), x_2(t)) = (x_1, x_2)$ are given. We seek to minimize a quadratic criterion of the form

$$\int_t^{t_1} \left[m_1 x_1(s)^2 + m_2 x_2(s)^2 + u(s)^2\right] ds + d_1 x_1(t_1)^2 + d_2 x_2(t_2)^2, \tag{2.5}$$

where m_1, m_2, d_1, d_2 are nonnegative constants. If there is no constraint on the forcing term ($U = I\!R^1$), this is a particular case of the linear quadratic regulator problem considered in Example 2.3. If $U = [-a, a]$ with $a < \infty$, it is an example of a linear regulator problem with a saturation constraint.

One can also consider the problem of controlling the solution $x(s) = (x_1(s), x_2(s))$ to (2.4) on an infinite time horizon, say on the time interval $[0, \infty)$. A suitable modification of the quadratic criterion (2.5) could be used as the quantity to be minimized. Another possible criterion to be minimized is the time for $x(s)$ to reach a given target. If the target is the point $(0, 0)$, then the control function $u(\cdot)$ is to be chosen such that the first time θ when $x(\theta) = (0, 0)$ is minimized.

Example 2.3. We will now describe the *linear quadratic regulator problem* (LQRP). Due to the simplicity of its solution, it has been applied to a large number of engineering problems. Let $x(s) \in R^n,\ u(s) \in R^m$ satisfy

$$\frac{d}{ds}x(s) = A(s)x(s) + B(s)u(s) \tag{2.6}$$

with given matrices $A(s)$ and $B(s)$ of dimensions $n \times n,\ n \times m$ respectively. Suppose we are also given $M(s), N(s)$, and D, such that $M(s)$ and D are nonnegative definite, symmetric $n \times n$ matrices and $N(s)$ is a symmetric, positive definite $m \times m$ matrix. The LQRP is to choose $u(s)$ so that

$$\int_t^{t_1} [x(s) \cdot M(s)x(s) + u(s) \cdot N(s)u(s)]\, ds + x(t_1) \cdot Dx(t_1) \tag{2.7}$$

is minimized. Here $x \cdot y$ denotes the inner product between two vectors. The solution to this problem will be discussed in Example 5.1.

Example 2.4. The simplest kind of problem in classical calculus of variations is to determine a function $x(\cdot)$ which minimizes a functional

$$\int_t^{t_1} L(s, x(s), \dot{x}(s))ds + \psi(x(t_1)), \tag{2.8}$$

subject to given conditions on $x(t)$ and $x(t_1)$). Here, $\dot{} = d/ds$. Let us fix the left endpoint, by requiring $x(t) = x$ where $x \in I\!\!R^n$ is given. For the right endpoint, let us fix t_1 and require that $x(t_1) \in \mathcal{M}$, where $\mathcal{M}$ is given closed subset of $I\!\!R^n$. If $\mathcal{M} = \{x_1\}$ consists of a single point, then the right endpoint (t_1, x_1) is fixed. At the opposite extreme, there is no restriction on $x(t_1)$ if $\mathcal{M} = I\!\!R^n$.

We will discuss calculus of variations problems in some detail in Sections 8 − 10. In the formulation in Section 8, we allow the possibility that the fixed upper limit t_1 in (2.8) is replaced by a time τ which is the smaller of t_1 and the exit time of $x(s)$ from a given closed region $\bar{O} \subset I\!\!R^n$. This is a particular case of the class of control problems to be formulated in Section 3.

I.3 Finite time horizon problems

In this section we formulate some classes of deterministic optimal control problems, which will be studied in the rest of this chapter and in Chapter II. At the end of the section, each of these classes of problems appears as a particular case of a general formulation.

A terminal time t_1 will be fixed throughout. Let $t_0 < t_1$ and consider initial times t in the finite interval $[t_0, t_1)$. (One could equally well take $-\infty < t < t_1$, but then certain assumptions in the problem formulation become slightly more complicated.) The objective is to minimize some payoff functional J, which depends on states $x(s)$ and controls $u(s)$ for $t \le s \le t_1$.

Let us first formulate the state dynamics for the control problem. Let $Q_0 = [t_0, t_1) \times I\!\!R^n$ and $\overline{Q}_0 = [t_0, t_1] \times I\!\!R^n$, the closure of Q_0. Let U be a closed subset of m-dimensional $I\!\!R^m$. We call U the *control space*. The state dynamics are given by a function

$$f : \ \overline{Q}_0 \times U \to I\!\!R^m.$$

It is assumed that $f \in C(\overline{Q}_0 \times U)$. Moreover, for suitable K_ρ:

$$|f(t,x,v) - f(t,y,v)| \le K_\rho |x - y| \qquad (3.1)$$

for all $t \in [t_0, t_1],\ x, y \in I\!\!R^n$ and $v \in U$ such that $|v| \le \rho$. If the control space U is compact, we can replace K_ρ by a constant K, since $U \subset \{v : \ |v| \le \rho\}$ for large enough ρ. If $f(t, \cdot, v)$ has a continuous gradient f_x, (3.1) is equivalent to the condition $|f_x(t,x,v)| \le K_\rho$ whenever $|v| \le \rho$.

A *control* is a bounded, Lebesgue measurable function $u(\cdot)$ on $[t, t_1]$ with values in U. Assumption (3.1) implies that, given any control $u(\cdot)$, the differential equation

$$\frac{d}{ds} x(s) = f(s, x(s), u(s)),\ t \le s \le t_1 \qquad (3.2)$$

with initial condition

$$x(t) = x \tag{3.3}$$

has a unique solution. The solution $x(s)$ of (3.2) and (3.3) is called the *state* of the system at time s. Clearly the state depends on the control $u(\cdot)$ and the initial condition, but this dependence is suppressed in our notation.

Let $\mathcal{U}^0(t)$ denote the set of all controls $u(\cdot)$. In notation which we shall use later (Section 9)

$$\mathcal{U}^0(t) = L^\infty([t, t_1];\ U).$$

This is the space of all bounded, Lebesgue measurable, U - valued functions on $[t, t_0]$. In order to complete the formulation of an optimal control problem, we must specify for each initial data (t, x) a set $\mathcal{U}(t, x) \subset \mathcal{U}^0(t)$ of admissible controls and a payoff functional $J(t, x; u)$ to be minimized. Let us first formulate some particular classes of problems (A through D below). Then we subsume all of these classes in a more general formulation. For classes A and B, all controls $u(\cdot) \in \mathcal{U}^0(t)$ are admitted. However, for classes C and D only controls $u(\cdot)$ in a smaller $\mathcal{U}(t, x)$ are admitted.

A. Fixed finite time horizon. The problem is to find $u(\cdot) \in \mathcal{U}^0(t)$ which minimizes

$$J(t, x; u) = \int_t^{t_1} L(s, x(s), u(s))ds + \psi(x(t_1)), \tag{3.4}$$

where $L \in C(\overline{Q}_0 \times U)$. We call L the *running cost* function and ψ the *terminal cost* function.

B. Control until exit from a closed cylindrical region $\overline{Q}$. Consider the following payoff functional J, which depends on states $x(s)$ and controls $u(s)$ for times $s \in [t, \tau)$, where τ is the smaller of t_1 and the exit time of $x(s)$ from the closure $\overline{O}$ of an open set $O \subset I\!R^n$. We let $Q = [t_0, t_1) \times O$, $\overline{Q} = [t_0, t_1] \times \overline{O}$ the closure of the cylindrical region Q, and

$$\partial^* Q = ([t_0, t_1) \times \partial O) \cup (\{t_1\} \times \overline{O}).$$

We call $[t_0, t_1) \times \partial O$ and $\{t_1\} \times \overline{O}$ the *lateral boundary* and *terminal boundary,* respectively, of Q. Given initial data $(t, x) \in \overline{Q}$, let τ denote the exit time of $(s, x(s))$ from $\overline{Q}$. Thus,

$$\tau = \begin{cases} \inf\{s \in [t, t_1) :\ x(s) \notin \overline{O}\} \text{ or} \\ \\ t_1 \text{ if } x(s) \in \overline{O} \text{ for all } s \in [t, t_1) \end{cases}$$

Note that $(\tau, x(\tau)) \in \partial^* Q$. We let

$$J(t, x; u) = \int_t^{\tau} L(s, x(s), u(s))ds \tag{3.5}$$

$$+g(\tau,x(\tau))\chi_{\tau<t_1}+\psi(x(t_1))\chi_{\tau=t_1}$$

Here χ denotes an indicator function. Thus, for real numbers a,b,

$$\chi_{a<b}=\begin{cases}1 \text{ if } & a<b\\ \\ 0 \text{ if } & a\geq b,\end{cases}$$

and $\chi_{a\leq b}$ is defined similarly. The function g is called a *boundary cost* function, and is assumed continuous.

B′. Control until exit from Q. Let $(t,x)\in Q$, and let τ' be the first time s such that $(s,x(s))\in\partial^*Q$. Thus, τ' is the exit time of $(s,x(s))$ from Q, rather that from $\overline{Q}$ as for class B above. In (3.5) we now replace τ by τ'. We will give conditions under which B and B' are equivalent optimal control problems.

C. Final endpoint constraint. Suppose that in case A, the additional restriction $x(t_1)\in\mathcal{M}$ is imposed, where $\mathcal{M}$ is a given closed subset of $I\!\!R^n$. In particular, if $\mathcal{M}=\{x_1\}$ consists of a single point, then both endpoints (t,x) and (t_1,x_1) of the curve $\gamma=\{(s,x(s)):\ t\leq s\leq t_1\}$ are given. We now admit controls $u(\cdot)\in\mathcal{U}(t,x)$, where

$$\mathcal{U}(t,x)=\{u(\cdot)\in\mathcal{U}^0(t):x(t_1)\in\mathcal{M}\}.$$

The condition that $\mathcal{U}(t,x)$ is nonempty is called a reachability condition. See Sontag [Sg]. If $U=I\!\!R^m$, it is related to the concept of controllability.

In a similar way, one can consider the problem of minimizing J in (3.5) subject to an endpoint constraint $(\tau,x(\tau))\in\mathcal{S}$, where $\mathcal{S}$ is a given closed subset of ∂^*Q.

D. State constraint. This is the problem of minimizing $J(t,x;u)$ in (3.4) subject to the constraint $x(s)\in\overline{O}$. In this case,

$$\mathcal{U}(t,x)=\{u(\cdot)\in\mathcal{U}^0(t):\ x(s)\in\overline{O}\text{ for } t\leq s\leq t_1\}.$$

General problem formulation. Let us now formulate a general class of control problems, which includes each of the classes A through D above. Let $O\subset I\!\!R^n$ be open, with either: (i) $O=I\!\!R^n$, or (ii) ∂O a compact manifold of class C^2. Let $Q=[t_0,t_1)\times O$. In case $O=I\!\!R^n$, we have $Q=Q_0$. Let Ψ be a function, such that

$$\Psi(t,x)=\begin{cases}g(t,x) \text{ if } (t,x)\in[t_0,t_1)\times I\!\!R^n\\ \\ \psi(x) \text{ if } (t,x)\in\{t_1\}\times I\!\!R^n\end{cases}\tag{3.6}$$

We let

$$J(t,x;\ u)=\int_t^\tau L(s,x(s),u(s))ds+\Psi(\tau,x(\tau)),\tag{3.7}$$

where τ is the exit time of $(s, x(s))$ from $\overline{Q}$. This agrees with (3.5), and also with (3.4) in case $O = I\!R^n$. We admit controls $u(\cdot) \in \mathcal{U}(t,x)$, where $\mathcal{U}(t,x)$ is nonempty and satisfies the following "switching" condition (3.9). Roughly speaking, condition (3.9) states that if we replace an admissible control by another admissible one after a certain time, then the resulting control is still admissible. More precisely, let $u(\cdot) \in \mathcal{U}(t,x)$ and $u'(\cdot) \in \mathcal{U}(r, x(r))$ for some $r \in [t, \tau]$. Define a new control by

$$\tilde{u}(s) = \begin{cases} u(s), & t \le s \le r \\ u'(s), & r < s \le t_1. \end{cases} \tag{3.8}$$

Let $\tilde{x}(s)$ be the solution to (3.2) corresponding to control $\tilde{u}(\cdot)$ and initial condition $\tilde{x}(t) = x$. Then we assume that

$$\tilde{u}_s(\cdot) \in \mathcal{U}(s, \tilde{x}(s)), \ t \le s \le \tilde{\tau}, \tag{3.9}$$

where $\tilde{u}_s(\cdot)$ denotes the restriction to $[s, t_1]$ of $\tilde{u}(\cdot)$ and $\tilde{\tau}$ is the exit time from $\overline{Q}$ of $(s, \tilde{x}(s))$. Note that (3.9) implies, in particular, that an admissible control always stays admissible. Indeed, simply take in (3.7) $r = \tau$ and $\tilde{u}_s(\cdot) = u_s(\cdot)$.

The control problem is as follows: given initial data $(t,x) \in \overline{Q}$, find $u^*(\cdot) \in \mathcal{U}(t,x)$ such that

$$J(t,x;\ u^*) \le J(t,x;u) \text{ for all } u(\cdot) \in \mathcal{U}(t,x).$$

Such a $u^*(\cdot)$ is called an *optimal* control.

Relation between classes B and B'. Let us conclude this section by giving some conditions (3.10), (3.11) under which the problem of controlling until the time τ of exit of $(s, x(s))$ from $\overline{Q}$ is equivalent of that of controlling until the time τ' of exit from Q. Let us assume:

$$L \ge 0, \psi \ge 0, \ \psi(x) = 0 \text{ for } x \epsilon \partial O \text{ and } g \equiv 0. \tag{3.10}$$

(3.11) For every $(s, \xi) \in [t_0, t_1] \times \partial O$ there exists $v(s,\xi) \in U$ such that

$$f(s, \xi, v(s,\xi)) \cdot \eta(\xi) > 0,$$

where $\eta(\xi)$ is the exterior unit normal at $\xi \in \partial O$.

We always have $\tau' \le \tau \le t_1$. In particular, $\tau' = t_1$ implies that $\tau' = \tau$. If $\tau' < t_1$, then by (3.5) and the assumption $g \equiv 0$,

$$J(t,x;u) = \int_t^{\tau'} L ds + \left[\int_{\tau'}^{\tau} L ds + \psi(x(t_1)) \chi_{\tau = t_1} \right].$$

Let us denote the first term on the right side by $J'(t,x;u)$. J' is the payoff for the problem of control up to time τ', in case $\tau' < t_1$. Since $L \ge 0$ and $\psi \ge 0$,

$$J(t,x;u) \geq J'(t,x;u) \tag{3.12}$$

for all $u(\cdot) \in \mathcal{U}^0(t)$. On the other hand, given $u(\cdot)$ with $\tau' < t_1$, let

$$\tilde{u}(s) = \begin{cases} u(s), \ t \leq s \leq \tau', \\ \\ v(\tau', x(\tau')), \ \tau' < s \leq t_1, \end{cases}$$

with $v(s,\xi)$ as in (3.11). The corresponding solution $\tilde{x}(s)$ of (3.2) with $\tilde{x}(t) = x$ coincides with $x(s)$ for $t \leq s \leq \tau'$, and exits from $\overline{Q}$ at time τ'. Thus,

$$J'(t,x;u) = J(t,x;\tilde{u}). \tag{3.13}$$

From (3.12) and (3.13), it suffices to minimize J among controls $\tilde{u}(\cdot)$ for which the exit times from Q and $\overline{Q}$ are the same.

I.4 Dynamic programming principle

It is convenient to consider a family of optimization problems with different initial conditions (t,x). Consider the minimum value of the payoff function as a function of this initial point. Thus define a *value function* by

$$V(t,x) = \inf_{u(\cdot)\in\mathcal{U}(t,x)} J(t,x;u), \tag{4.1}$$

for all $(t,x) \in \overline{Q}$. We shall assume that $V(t,x) > -\infty$. This is always true if the control space U is compact, or if U is not compact but the cost functions are bounded below ($L \geq -M, \Psi \geq -M$ for some constant $M \geq 0$.)

The method of dynamic programming uses the value function as a tool in the analysis of the optimal control problem. In this section and the following one we study some basic properties of the value function. Then we illustrate the use of these properties in an example for which the problem can be explicitly solved (the linear quadratic regulator problem) and introduce the idea of feedback control policy.

We start with a simple property of V. Let $r \wedge \tau = \min(r,\tau)$. Recall that g is the boundary cost (see (3.5)).

Lemma 4.1. *For every initial condition $(t,x) \in \overline{Q}$, admissible control $u(\cdot) \in \mathcal{U}(t,x)$ and $t \leq r \leq t_1$, we have*

$$V(t,x) \leq \int_t^{r\wedge\tau} L(s,x(s),u(s))ds + g(\tau,x(\tau))\chi_{\tau<r} + V(r,x(r))\chi_{r\leq\tau}. \tag{4.2}$$

Proof. Suppose that $\tau < r \leq t_1$. Then $\Psi(r \wedge \tau, x(r\wedge\tau)) = g(\tau,x(\tau))$, and (4.2) follows from the definition of the value function. Now suppose that

$r \le \tau$. For any $\delta > 0$, choose an admissible control $u^1(\cdot) \in \mathcal{U}(r, x(r))$ such that

$$\int_r^{\tau^1} L(s, x^1(s), u^1(s))ds + \Psi(\tau^1, x^1(\tau^1)) \le V(r, x(r)) + \delta.$$

Here $x^1(s)$ is the state at time s corresponding to the control $u^1(\cdot)$ and initial condition $(r, x(r))$, and τ^1 is the exit time of $(s, x^1(s))$ from $\overline{Q}$. (Such a control $u^1(\cdot)$ is called δ - *optimal*.) As in (3.8) define an admissible control $\tilde{u}(\cdot) \in \mathcal{U}(t, x)$ by

$$\tilde{u}(s) = \begin{cases} u(s), & s \le r \\ u^1(s), & s > r. \end{cases}$$

Let $\tilde{x}(s)$ be the state corresponding to $\tilde{u}(\cdot)$ with initial condition (t, x), and $\tilde{\tau}$ the exit time of $(s, \tilde{x}(s))$ from $\overline{Q}$. Since $r < \tau, \tau^1 = \tilde{\tau}$ and we have

$$\begin{aligned} V(t,x) &\le J(t,x;\tilde{u}) \\ &= \int_t^{\tilde{\tau}} L(s, \tilde{x}(s), \tilde{u}(s))ds + \Psi(\tilde{\tau}, \tilde{x}(\tilde{\tau})) \\ &= \int_t^r L(s, x(s), u(s))ds + \int_r^{\tau^1} L(s, x^1(s), u^1(s))ds \\ &\qquad + \Psi(\tau^1, x^1(\tau^1)) \\ &\le \int_t^r L(s, x(s), u(s))ds + V(r, x(r)) + \delta. \end{aligned}$$

□

The proof of Lemma 4.1 shows that the right side of (4.2) is a nondecreasing function of r. However, if $u(\cdot)$ is optimal (or nearly optimal), then this function is constant (or nearly constant). Indeed, for a small positive δ, choose a δ-optimal admissible control $u(\cdot) \in \mathcal{U}(t,x)$. Then for any $r \in [t, t_1]$ we have

$$\begin{aligned} \delta + V(t,x) &\ge J(t,x;u) \\ &= \int_t^{\tau} L(s, x(s), u(s))ds + \Psi(\tau, x(\tau)) \\ &= \int_t^{\tau\wedge r} L(s, x(s), u(s))ds + \int_{\tau\wedge r}^{\tau} L(s, x(s), u(s))ds + \Psi(\tau, x(\tau)) \\ &= \int_t^{\tau\wedge r} L(s, x(s), u(s))ds + J(r\wedge\tau, x(r\wedge\tau); u) \\ &\ge \int_t^{\tau\wedge r} L(s, x(s), u(s))ds + g(\tau, x(\tau))\chi_{\tau<r} + V(r, x(r))\chi_{r\le\tau}. \end{aligned}$$

Since δ is arbitrary, we have proved the following.

Lemma 4.2. *For any initial condition* $(t,x) \in \overline{Q}$ *and* $r \in [t, t_1]$,

$$V(t,x) = \inf_{u(\cdot)\in\mathcal{U}(t,x)} \left[\int_t^{r\wedge\tau} L(s,x(s),u(s))ds + g(\tau,x(\tau))\chi_{\tau<r} + V(r,x(r))\chi_{r\le\tau} \right]. \tag{4.3}$$

The above identity is called the *dynamic programming principle*. It is the basis of the solution technique developed by Bellman in the 1950's [Be]. An interesting observation is that an optimal control $u^*(\cdot) \in \mathcal{U}(t,x)$ minimizes (4.3) at every r. Hence to determine the optimal control $u^*(t)$, it suffices to analyze (4.3) with r arbitrarily close to t. Intuitively this yields a simple optimization problem that is minimized by $u^*(t)$. However, as we shall see in later chapters, this approach requires a knowledge of the value function.

Another corollary of the above computations is the following.

Corollary 4.1. *An admissible control* $u(\cdot) \in \mathcal{U}(t,x)$ *is* δ*-optimal at* (t,x) *if any only if it is* δ*-optimal at every* $(r, x(r))$ *with* $r \in [t,\tau]$.

I.5 Dynamic programming equation

In this section, we assume that the value function is continuously differentiable and proceed formally to obtain a nonlinear partial differential equation satisfied by the value function. In general however, the value function is not differentiable. In that case a notion of "weak" solutions to this equation is needed. This will be the subject of Chapter 2. After formally deriving the dynamic programming partial differential equation (5.3), we prove two Verification Theorems (Theorems 5.1 and 5.2) which give sufficient conditions for a solution to the optimal control problem.

Let $0 < h \le t_1 - t$, and take $r = t+h$ in the dynamic programming principle (4.3). Subtract $V(t,x)$ from both sides of (4.3) and then divide by h. This yields

$$\inf_{u(\cdot)\in\mathcal{U}(t,x)} \left\{ \frac{1}{h}\int_t^{(t+h)\wedge\tau} L(s,x(s),u(s))ds + \frac{1}{h} g(\tau,x(\tau))\chi_{\tau<t+h} + \frac{1}{h}[V(t+h,x(t+h))\chi_{t+h\le\tau} - V(t,x)] \right\} = 0. \tag{5.1}$$

Let us assume that:

(5.2) For every $(t,x) \in Q$ and $v \in U$ there exists $u(\cdot) \in \mathcal{U}(t,x)$ such that

$$v = \lim_{s\downarrow t} u(s).$$

If $\mathcal{U}(t,x) = \mathcal{U}^0(t)$, clearly (5.2) holds. For instance, we may take $u(s) \equiv v$. For the state constrained problem (Case D, Section 3), (5.2) holds provided $\mathcal{U}(r,\xi)$ is not empty for every $(r,\xi) \in Q$ (See Theorem II.12.1.). Note that we assume (5.2) for $(t,x) \in Q$, not $(t,x) \in \overline{Q}$. In the state constrained problem, only controls in some subset of U can be used at time t when (t,x) is on the lateral boundary of Q. If $(t,x) \in Q$, then $x \in O$ and $t + h \leq \tau$ if h is sufficiently small. If we formally let $h \downarrow 0$ in (5.2) we obtain, for $(t,x) \in Q$,

$$\frac{\partial}{\partial t}V(t,x) + \inf_{v \in U} \{L(t,x,v) + f(t,x,v) \cdot D_x V(t,x)\} = 0. \tag{5.3}$$

This is a nonlinear partial differential equation of first order, which we refer to as the *dynamic programming equation* or simply DPE. In (5.3), $D_x V$ denotes the gradient of $V(t,\cdot)$. It is notationally convenient to rewrite (5.3) as

$$-\frac{\partial}{\partial t}V(t,x) + H(t,x,D_x V(t,x)) = 0, \tag{5.3$'$}$$

where for $(t,x,p) \in \overline{Q}_0 \times I\!\!R^n$

$$H(t,x,p) = \sup_{v \in U} \{-p \cdot f(t,x,v) - L(t,x,v)\}. \tag{5.4}$$

In analogy with a quantity occurring in classical mechanics, we call this function the *Hamiltonian.* The dynamic programming equation (5.3′) is sometimes also called a *Hamilton–Jacobi–Bellman PDE.*

Equation (5.3) is to be considered in Q, with appropriate terminal or boundary conditions. Let us describe such conditions for problems of the classes A and B in Section 3. Boundary conditions for state constrained problems (class D, Section 3) will be described later in Section II.12. For class A, we have $Q = Q_0$. By (3.4) the terminal (Cauchy) data are

$$V(t_1,x) = \psi(x), \ x \in I\!\!R^n. \tag{5.5}$$

We now state a theorem which connects the dynamic programming equation to the control problem of minimizing (3.4).

Theorem 5.1. $(Q = Q_0)$. *Let* $W \in C^1(\overline{Q}_0)$ *satisfy* (5.3) *and* (5.5). *Then:*

$$W(t,x) \leq V(t,x), \ \forall (t,x) \in \overline{Q}_0. \tag{5.6}$$

Moreover, if there exists $u^*(\cdot) \in \mathcal{U}^0(t)$ *such that*

$$\begin{aligned} L(s,x^*(s),u^*(s)) + f(s,x^*(s),u^*(s)) \cdot D_x W(s,x^*(s)) \\ = -H(s,x^*(s),D_x W(s,x^*(s))) \end{aligned} \tag{5.7}$$

for almost all $s \in [t,t_1]$, *then* $u^*(\cdot)$ *is optimal for initial data* (t,x) *and* $W(t,x) = V(t,x)$.

In Theorem 5.1, $x^*(\cdot)$ denotes the solution to (3.2) with $u(\cdot)=u^*(\cdot)$, $x^*(t)=x$. Theorem 5.1 is called a *Verification Theorem.* Note that, by the definition (5.4) of H, (5.7) is equivalent to

$$(5.7') \qquad u^*(s) \in \arg\min_{v\in U}\{f(s,x^*(s),v)\cdot D_xW(s,x^*(s)) + L(s,x^*(s),v)\}.$$

Proof of Theorem 5.1. Consider any $u(\cdot) \in \mathcal{U}^0(t)$. Using multivariate calculus and the dynamic programming equation (5.3), we obtain

$$(5.8) \quad W(t_1,x(t_1)) = W(t,x) + \int_t^{t_1}[\frac{\partial}{\partial t}W(s,x(s)) + \dot{x}(s)\cdot D_xW(s,x(s)]ds$$

$$= W(t,x) + \int_t^{t_1}[\frac{\partial}{\partial t}W(s,x(s)) + f(s,x(s),u(s))\cdot D_xW(s,x(s))]ds$$

$$\geq W(t,x) - \int_t^{t_1} L(s,x(s),u(s))ds.$$

By (5.5), $W(t_1,x(t_1)) = \psi(x(t_1))$. Hence

$$W(t,x) \leq J(t,x;u).$$

We get (5.6) by taking the infimum over $u(\cdot)$.

To prove the second assertion of the theorem, let $u^*(\cdot) \in \mathcal{U}^0(t)$ satisfy (5.7). We redo the calculation above with $u^*(\cdot)$. This yields (5.8) with an equality. Hence

$$(5.9) \qquad W(t,x) = J(t,x;u^*).$$

By combining this equality with (5.6), we conclude that $u^*(\cdot)$ is optimal at (t,x).

□

Remark 5.1. Condition (5.7) is necessary as well as sufficient. Indeed, from the proof of Theorem 5.1 and the definition (5.4) of H it is immediate that (5.7) holds for almost all s if $u^*(\cdot)$ is optimal.

We illustrate the use of the Verification Theorem 5.1 in an example.

Example 5.1. Consider the linear quadratic regulator problem (LQRP) described in Example 2.3. In this example, $O = I\!R^n, U = I\!R^m, \mathcal{U}(t,x) = \mathcal{U}^0(t)$, and

$$f(t,x,v) = A(t)x + B(t)v$$

$$(5.10) \qquad L(t,x,v) = x\cdot M(t)x + v\cdot N(t)v$$

$$\Psi(t,x) = \psi(x) = x\cdot Dx.$$

The dynamic programming equation (5.3′) becomes

$$(5.11)\qquad -\frac{\partial}{\partial t}V(t,x) + H(t,x,D_xV(t,x)) = 0,\ t_0 \le t < t_1, x \in I\!R^n.$$

The Hamiltonion $H(t,x,p)$ is given by

$$(5.12)\qquad \begin{aligned} H(t,x,p) &= \sup_v \{-f(t,x,v)\cdot p - L(t,x,v)\} \\ &= \tfrac{1}{4}N^{-1}(t)B'(t)p\cdot B'(t)p - A(t)x\cdot p - x\cdot M(t)x, \end{aligned}$$

where $B'(t)$ is the transpose of the matrix $B(t)$ and $N^{-1}(t)$ is the inverse of $N(t)$ which is assumed to be invertible. For later use, we note that the unique maximizer of (5.12) is

$$(5.13)\qquad v^* = -\frac{1}{2}N^{-1}(t)B'(t)p.$$

To use the Verification Theorem 5.1, first we have to solve (5.11) with the terminal condition

$$(5.14)\qquad V(t_1,x) = x\cdot Dx,\ x \in R^n.$$

We guess that the solution of (5.11) and (5.14) is a quadratic function in x. So, let

$$W(t,x) = x\cdot P(t)x$$

for some symmetric matrix $P(t)$. We substitute $W(t,x)$ into (5.11) to obtain

$$\begin{aligned} &-\frac{\partial}{\partial t}W(t,x) + H(t,x,D_xW(t,x)) \\ &\quad = -x\cdot\frac{\partial}{\partial t}P(t)x + N^{-1}(t)B'(t)P(t)x\cdot B'(t)P(t)x \\ &\qquad -2A(t)x\cdot P(t)x - x\cdot M(t)x \\ &\quad = x\cdot[-\frac{\partial}{\partial t}P(t) + P(t)B(t)N^{-1}(t)B'(t)P(t) \\ &\qquad -A(t)P(t) - P(t)A'(t) - M(t)]x. \end{aligned}$$

Hence W satisfies (5.11) provided that

$$(5.15)\qquad \begin{aligned} \frac{d}{dt}P(t) &= P(t)B(t)N^{-1}(t)B'(t)P(t) \\ &\quad -A(t)P(t) - P(t)A'(t) - M(t),\ t\in[0,t_1). \end{aligned}$$

The continuity of W at time t_1 yields that

$$(5.16)\qquad P(t_1) = D.$$

Equation (5.15) is called a *matrix Riccati equation.* It has been studied extensively. If we fix t_1, then (5.15)-(5.16) has a solution $P(t)$ backward in time on some maximal interval $t_{\min} < t \le t_1$, where either $t_{\min} = -\infty$ or $t_{\min} < \infty$. Let us use Theorem 5.1 to show that $V(t,x) = W(t,x)$ for $t_{\min} < t \le t_1$, and to find an explicit formula for the optimal $u^*(s)$. In view of (5.13), (5.7′) holds at any $s \in [t, t_1]$ if and only if

$$\begin{aligned} u^*(s) &= -\frac{1}{2} N^{-1}(s) B'(s) D_x W(s, x^*(s)) \\ &= -N^{-1}(s) B'(s) P(s) x^*(s). \end{aligned} \tag{5.17}$$

Now substitute (5.17) back into the state equation (2.6) to obtain

$$\frac{d}{ds} x^*(s) = [A(s) - B(s) N^{-1}(s) B'(s) P(s)] x^*(s).$$

This equation has a unique solution satisfying the initial condition $x^*(t) = x$. Thus there is a unique control $u^*(\cdot)$ satisfying (5.17). Theorem 5.1 then implies that $u^*(\cdot)$ is optimal at (t, x).

Notice that the optimal control $u^*(s)$ in (5.17) is a linear function of the state $x^*(s)$. The matrix $N^{-1}(s)B'(s)P(s)$ can be precomputed by solving the Riccati differential equation (5.15) with terminal data (5.16), without reference to the initial conditions for $x(s)$. This is one of the important aspects of the LQRP.

In the LQRP as formulated in Example 2.3, the matrices $M(s)$ and D are nonnegative definite and $N(s)$ is positive definite. This implies that $P(t)$ is nonnegative definite and that $t_{\min} = -\infty$. To see this, for $t_{\min} < t \le t_1$, $0 \le V(t,x) \le J(t,x;0)$. Since $V(t,x) = x \cdot P(t)x$, $P(t)$ is nonnegative definite and bounded on any finite interval, which excludes the possibility that $t_{\min} > -\infty$.

In Section VI.8 we will encounter a class of problems in which $M(s)$ is negative definite. Such problems are called LQRP problems with *indefinite sign.* In this case, $P(t)$ may not be nonnegative definite and $t_{\min}$ may be finite. The following example illustrates these possibilities.

Example 5.2. Let $n = 1, f(v) = v$, $L(x,v) = -x^2 + v^2$ and $D = 0$. The solution to (5.15)-(5.16) is $P(t) = -\tan(t_1 - t)$ if $t_1 - t < \frac{\pi}{2}$ and $t_{\min} = t_1 - \frac{\pi}{2}$.

Control until exit from $\overline{Q}$. Let us next consider the problem of control until the time τ of exit from a closed cylindrical region $\overline{Q}$ (class B, Section 3.) We first formulate appropriate boundary conditions for the dynamic programming equation (5.3). Then we outline a proof of a Verification Theorem (Theorem 5.2) similar to Theorem 5.1. When $t = t_1$, we have as in (5.5):

$$V(t_1, x) = \psi(x), \quad x \in \overline{O}. \tag{5.18}$$

Let us assume that (3.11) holds on the lateral boundary $[t_0, t_1) \times \partial O$. This implies that, for $(t,x) \in [t_0, t_1) \times \partial O$, one choice is to exit immediately from $\overline{Q}$ (thus, $\tau = t$). Therefore,

$$(5.19) \qquad V(t,x) \le g(t,x),\ (t,x) \in [t_0,t_1) \times \partial O$$

If it is optimal to exit immediately from $\overline{Q}$, then equality holds in (5.19). However, in many examples, there are points (t,x) of the lateral boundary for which there exists a control $u^0(\cdot)$ such that $J(t,x;u(\cdot)) < g(t,x)$. See Example II.2.3. At such points, strict inequality holds in (5.19). If (3.10) is assumed, in addition to (3.11), then $V(t,x) \ge 0$. Since $g \equiv 0$ when (3.10) holds, (5.19) implies that the lateral boundary condition $V(t,x) = 0$ holds for all $(t,x) \in [t_0,t_1) \times \partial O$, if both (3.10) and (3.11) hold. Note that we have not yet proved that the value function V is continuous on $\overline{Q}$. However, such a result will be proved later (Theorem II.10.2.). Boundary conditions are discussed further in Section II.13.

Theorem 5.2. *Let $W \in C^1(\overline{Q})$ satisfy* (5.3), (5.18) *and* (5.19). *Then*

$$(5.20) \qquad W(t,x) \le V(t,x) \ \text{for all} \ (t,x) \in \overline{Q}.$$

Moreover, suppose that there exists $u^*(\cdot) \in \mathcal{U}^0(t)$ such that (5.7) *holds for almost all $s \in [t,\tau^*]$ and $W(\tau^*,x^*(\tau^*)) = g(\tau^*,x^*(\tau^*))$ in case $\tau^* < t_1$. Then $u^*(\cdot)$ is optimal for initial data (t,x) and $W(t,x) = V(t,x)$.*

Here τ^* is the exit time of $(s,x^*(s))$ from $\overline{Q}$. The proof of Theorem 5.2 is almost the same as for Theorem 5.1. In (5.8) the integral is now from t to the exit time τ, and $W(\tau,x(\tau))$ is on the left side. By (5.18) and (5.19), $W(\tau,x(\tau)) \le \Psi(\tau,x(\tau))$ with Ψ as in (3.6). This gives (5.20). The second half goes exactly as for Theorem 5.1.

Remark 5.2. An entirely similar Verification Theorem is true for the problem of control until the time τ' of exit from Q (rather from $\overline{Q}$.) In fact, since $(s,x(s)) \in Q$ for $t \le s < \tau'$, the proof of Theorem 5.2 shows that it suffices in this case to assume $W \in C^1(Q) \cap C(\overline{Q})$ rather than $W \in C^1(\overline{Q})$. A situation where such a weaker assumption on W is convenient will arise in Example 7.3.

In Example 5.1 we constructed an admissible control by using the value function. To generalize the procedure, let W be as in Theorem 5.2 (or as in Theorem 5.1 in case $Q = Q_0$.) For $(t,x) \in \overline{Q}$ define a set-valued map $F^*(t,x)$ by

$$F^*(t,x) = \{f(t,x,v) : v \in v^*(t,x)\}$$

where $v^*(t,x)$ is another set-valued map

$$(5.21) \qquad v^*(t,x) = \arg\min_{v \in U} \left[f(t,x,v) \cdot D_x W(t,x) + L(t,x,v) \right].$$

We may now restate (5.7′) as $u^*(s) \in v^*(s,x^*(s))$. Substituting this into the state dynamics yields

$$(5.22) \qquad \dot{x}^*(s) \in F^*(s,x^*(s)),\ s \in [t,\tau^*].$$

Thus, we have the following corollary to Theorem 5.2.

Corollary 5.1. *A control* $u^*(\cdot)$ *satisfies the optimality condition* (5.7) *if* $x^*(\cdot)$ *is a solution to the differential inclusion* (5.22).

Feedback controls (Markov control policies). Corollary 5.1 is closely related to the idea of optimal feedback control, according to which the control $u^*(s)$ is chosen based on knowledge not only of time s but also of the state $x^*(s)$. To formulate this idea more precisely, let us call any function $\underline{u} : \overline{Q}_0 \to U$ a *feedback control policy.* Such a function U will also be called in later chapters a *Markov control policy.* Consider the differential equation

$$(5.23) \qquad \frac{d}{ds}x(s) = f(s, x(s), \underline{u}(s, x(s))), \ s \in [t, t_1].$$

If (5.23) with the initial data $x(t) = x$ has a unique solution $x(\cdot)$, and if

$$(5.24) \qquad u(s) = \underline{u}(s, x(s))$$

belongs to $\mathcal{U}^0(t)$, then we call $\underline{u}$ an *admissible* feedback control for initial conditions (t, x).

In particular, suppose that $\underline{u}^*$ is a feedback control policy such that $\underline{u}^*(s, y) \in v^*(s, y)$ for all $(s, y) \in \overline{Q}$. If $\underline{u}^*$ is admissible for initial conditions (t, x), then we let $x^*(s)$ be the corresponding solution of (5.23) with $\underline{u} = \underline{u}^*$, and

$$(5.25) \qquad u^*(s) = \underline{u}^*(s, x^*(s)).$$

For the fixed finite time horizon problem ($Q = Q_0$), Theorem 5.1 and Corollary 5.1 imply that $u^*(\cdot)$ is an optimal control. If $\underline{u}^*$ is admissible for every $(t, x) \in Q_0$, then we call $\underline{u}^*$ an *optimal feedback policy* (or *Markov control policy*).

In the LQRP example above, we take

$$\underline{u}^*(s, y) = -N^{-1}(s)B'(s)P(s)y,$$

which is a linear function of the state variable y. According to (5.17), $\underline{u}^*$ is an optimal feedback control policy.

For the problem of control until exit from a closed cylindrical region $\overline{Q}$, the condition $W(\tau^*, x^*(\tau^*)) = g(\tau^*, x^*(\tau^*))$ in case $\tau^* < t$, needs to be added. Then optimality of $\underline{u}^*$ follows from Theorem 5.2 and Corollary 5.1, provided $\underline{u}^*$ is admissible.

It is a complicated matter to determine, in general, whether an optimal feedback control policy exists. We shall not undertake to deal with it here, but will only indicate some of the difficulties. In order to implement the procedure just outlined above, one needs first a smooth solution W to the dynamic programming equation (5.3) with terminal and boundary conditions. If $L(s, y, v) + f(s, y, v) \cdot D_x W(s, y)$ has a minimum on U at a unique $v^* = \underline{u}^*(s, y)$ this determines a candidate for the optimal feedback policy $\underline{u}^*$. If v^* is not unique, a selection theorem is needed in order to choose $\underline{u}^*(s, y)$. Finally, in many examples $\underline{u}^*(s, y)$ is not even a continuous function of (s, y), and there is no guarantee that (5.23) with the initial conditions has a unique solution.

I.6 Dynamic programming and Pontryagin's principle

In this section we first give a sufficient condition that the value function V satisfies the dynamic programming equation (5.3) at a point (t,x). Then we show how dynamic programming is related to Pontryagin's principle, which gives necessary conditions for $u^*(\cdot)$ to minimize $J(t,x;u)$.

We call a function V *differentiable* at an interior point (t,x) of its domain if there exist a scalar a and a vector b such that

$$\lim_{(h,k)\to(0,0)} (|h|+|k|)^{-1}|V(t+h,x+k) \; - V(t,x) - ah - b\cdot k| = 0.$$

Differentiability of V is equivalent to the existence of a tangent plane to the graph of V at (t,x). If V is differentiable at (t,x), then $a = V_t(t,x)$ and $b = D_xV(t,x)$. A sufficient condition for differentiability of V at (t,x) is that $V\in C^1(N)$ for some neighborhood of N of (t,x). See, for instance, Fleming [F1].

Theorem 6.1. *Let $(t,x)\in Q$ be a point at which the value function V is differentiable. Then:*
(a) $V_t(t,x) + L(t,x,v) + f(t,x,v)\cdot D_xV(t,x) \geq 0,\ \forall v\in U$;

(b) If there exists an optimal control $u^*(\cdot)\in\mathcal{U}(t,x)$ such that $u^*(s)$ tends to a limit v^* as $s\downarrow t$, then

$$V_t(t,x) + L(t,x,v^*) + f(t,x,v^*)\cdot D_xV(t,x) = 0.$$

Hence the dynamic programming equation (5.3) *holds at* (t,x).

Proof. By assumption (5.2), for any $v\in U$ there exists a control $u(\cdot)\in\mathcal{U}(t,x)$ such that $u(s)$ tends to v as $s\downarrow t$. By Lemma 4.1, if $t+h<\tau$, then

$$V(t,x) \leq \int_t^{t+h} L(s,x(s),u(s))ds + V(t+h,x(t+h)), \tag{6.1}$$

where $x(s)$ is the solution of (3.2) with $x(t)=x$. Then

$$\lim_{h\downarrow 0} h^{-1}[x(t+h)-x(t)] = f(t,x,v),$$

$$\lim_{h\downarrow 0} h^{-1}[V(t+h,x(t+h)) - V(t,x)] = V_t(t,x) + f(t,x,v)\cdot D_xV(t,x),$$

since V is differentiable at (t,x). Moreover,

$$\lim_{h\downarrow 0}\int_t^{t+h} L(s,x(s),u(s))ds = L(t,x,v).$$

This proves (a). To prove (b), we use the same argument, observing that equality holds in (6.1) when $u(\cdot)=u^*(\cdot)$ and $x(\cdot)=x^*(\cdot)$ is the corresponding solution of (3-2)-(3.3). □

In particular, assumption (b) holds if a continuous optimal control $u^*(\cdot)$ exists. In Section 9 we will show that there is a continuous optimal control, for a special class of control problems of calculus of variations type. For further results about existence and continuity properties of optimal controls see [FR, Chap III], Cesari [Ce] and Section 11.

If the control space U is compact and $\mathcal{U}(t,x) = \mathcal{U}^0(t)$, assumption (b) is unnecessary:

Theorem 6.1′. *Let U be compact and $\mathcal{U}(t,x) = \mathcal{U}^0(t)$. Then the dynamic programming equation (5.3) holds at every point $(t,x) \in Q$ such that V is differentiable at (t,x).*

Let us merely sketch the proof of Theorem 6.1′. Another proof of this result is also given by using the theory of viscosity solutions (See Corollary II.8.1.). By the dynamic programming principle (4.3), we have (see (5.1)) for small $h > 0$

$$\inf_{u(\cdot)\in\mathcal{U}^0(t)} \left\{ \frac{1}{h}\int_t^{t+h} L(s,x(s),u(s))ds + \frac{1}{h}\left[V(t+h,x(t+h)) - V(t,x)\right] \right\} = 0.$$

Since U is compact, it can be shown that the limits as $h \downarrow 0$ in the proof of Theorem 6.1 are uniform with respect to $u(\cdot)$. Therefore

$$V_t(t,x) + \frac{1}{h}\inf_{u(\cdot)\in\mathcal{U}^0(t)} \left[\int_t^{t+h} L(t,x,u(s))ds \right.$$
$$\left. + \int_t^{t+h} f(t,x,u(s))ds \cdot D_xV(t,x)\right] = O(1),$$

where $O(1) \to 0$ as $h \downarrow 0$. This infimum is attained for $u(s) \equiv v^*$, where v^* is any point of U at which $L(t,x,v) + f(t,x,v) \cdot D_xV(t,x)$ has a minimum. Then

$$V_t(t,x) + L(t,x,v^*) + f(t,x,v^*) \cdot D_xV(t,x) = 0.$$

By combining this with Theorem 6.1 (a), the dynamic programming equation (5.3) holds at (t,x). □

Generalized solutions to dynamic programming equations. Typically the value function V fails to be differentiable at some points (t,x). Thus, V may not satisfy the dynamic programming equation (5.3) everywhere in Q. In such cases, we wish to interpret V as a solution in some extended sense. One such interpretation is as a generalized solution, a concept which we shall now define. We call a function W *locally Lipschitz* on Q if: for every compact set $K \subset Q$ there exists a constant M_K such that

$$|W(t,x) - W(t',x')| \le M_K(|t-t'| + |x-x'|)$$

for all $(t,x),\ (t',x') \in K$. If one can choose $M = M_K$ which does not depend on K, then W is *Lipschitz continuous* on Q. By Rademacher's Theorem ([EG] or [Zi]) every locally Lipschitz function is differentiable at almost all points $(t,x) \in Q$.

Definition. W is a *generalized solution* to the dynamic programming equation in Q if W is locally Lipschitz and satisfies (5.3) for almost all $(t,x) \in Q$.

Corollary 6.1. *Let U be compact and $\mathcal{U}(t,x) = \mathcal{U}^0(t)$. If V is locally Lipschitz on Q, then V is a generalized solution of the dynamic programming equation* (5.3).

Later we will prove two theorems which give sufficient conditions for the value function V to be Lipschitz on Q. See Theorem 9.3, Theorem II.10.2. A local Lipschitz condition for V follows from the estimates in Section IV.8.

Unfortunately, when generalized solutions are considered instead of "classical" solutions of class $C^1(Q)$, a serious lack of uniqueness is encountered. Indeed, elementary examples show that (5.3) with given boundary data can have infinitely many generalized solutions. See Example II.2.2 below. This difficulty is circumvented by choosing the unique generalized solution which is also a viscosity solution, according to the definition to be given in Chapter II.

Pontryagin's Principle. During the 1950's Pontryagin formulated a "maximum principle" which provides a general set of necessary conditions for an extremum in an optimal control problem. A statement and proof of Pontryagin's principle in its full generality is rather lengthy, and will not be given in this book. See for instance [FR, Chap II] [PBGM]. However, the proof is much simpler in special cases. See [FR, Sec. 2.11] and Theorem 6.3 below.

We shall first prove a result (Theorem 6.2) which connects Pontryagin's principle and dynamic programming. For this purpose we assume that the partial derivatives $f_{x_i}, L_{x_i}, g_{x_i}, \psi_{x_i}$ exist and are continuous for $i = 1, \cdots, n$. As above, let $u^*(\cdot)$ denote an optimal control and $x^*(\cdot)$ the corresponding solution to (3.2) with $x^*(t) = x$. Let τ^* be the exit time of $(s, x^*(s))$ from $\overline{Q}$, and let

$$\gamma^* = \{(s, x^*(s)) : \ t \le s \le \tau^*\}.$$

We call γ^* an *optimal trajectory.* A crucial object in Pontryagin's principle is a $I\!R^n$-valued function $P(\cdot) = (P_1(\cdot), \cdots, P_n(\cdot))$ called an *adjoint variable.* It satisfies for $j = 1, \cdots, n, \ t \le s \le \tau^*$ the linear differential equations

$$\frac{d}{ds}P_j(s) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_j} f_i(s, x^*(s), u^*(s)) P_i(s) - \frac{\partial}{\partial x_j} L(s, x^*(s), u^*(s)). \tag{6.2}$$

In addition, for almost all $s \in [t, t^*]$

$$P(s) \cdot f(s, x^*(s), u^*(s)) + L(s, x^*(s), u^*(s)) \tag{6.3}$$

$$= -H(s, x^*(s), P(s)).$$

When Pontryagin's principle is stated in full generality, the term $\partial L/\partial x_i$ in (6.2) and L in (6.3) should be multiplied by some constant $P^0 \geq 0$. In all problems which we shall consider, $P^0 > 0$ and hence one can take $P^0 = 1$.

In addition to (6.2) and (6.3), the adjoint variable satisfies conditions at the final time τ^*, called *transversality conditions*. We will state these transversality conditions below in particular cases of interest to us.

If V is differentiable at each point $(s, x^*(s))$ of the optimal trajectory γ^*, then a candidate for an adjoint variable is

$$P(s) = D_x V(s, x^*(s)). \tag{6.4}$$

We shall give two sets of conditions under which $P(s)$ in (6.4) indeed satisfies (6.2), (6.3) and an appropriate transversality condition.

In the first of these two results, we consider the control problem on a fixed time interval $[t, t_1]$, with no terminal or state constraints on $x(s)$. This is case (A) of Section 3.

Theorem 6.2. *Let $\mathcal{U}(t,x) = \mathcal{U}^0(t)$ and $O = I\!\!R^n$. Let $u^*(\cdot)$ be an optimal control, which is right continuous at each $s \in [t, t_1)$. Assume that the value function V is differentiable at $(s, x^*(s))$ for $t \leq s < t_1$, and let $P(s)$ be as in* (6.4). *Then $P(s)$ satisfies* (6.2), (6.3) *and the transversality condition*

$$P(t_1) = D\psi(x^*(t_1)). \tag{6.5}$$

Proof. The transversality condition (6.5) is immediate since $V(t_1, y) = \psi(y)$ for all $y \in I\!\!R^n$ by (5.5). If in Theorem 6.1 we replace t, x, v^* by $s, x^*(s),\ u^*(s^+)$, then (6.3) follows from Theorem 6.1. It remains to show that $P(s)$ satisfies the linear differential equations (6.2).

Observe that (6.2) is a linear ordinary differential equation. Hence it has a unique solution $\bar{P}(s)$ satisfying (6.5). So the task in front of us is to show that $P(s)$ is indeed equal to $\bar{P}(s)$.

For $t \leq r < t_1$, the restriction $u_r^*(\cdot)$ of $u^*(\cdot)$ to $[r, t_1]$ is admissible. Hence, for any $y \in I\!\!R^n$

$$V(r, y) \leq J(r, y; u_r^*(\cdot)). \tag{6.6}$$

If $y = x^*(r)$, then $u_r^*(\cdot)$ is optimal for the initial data $(r, x^*(r))$ and equality holds in (6.6). Since $J(r, \cdot;\ u_r^*(\cdot)) - V(r, \cdot)$ has a minimum on $I\!\!R^n$ at $x^*(r)$,

$$D_x V(r, x^*(r)) = D_x J(r, x^*(r); u_r^*(\cdot)). \tag{6.7}$$

for all $r \in [t, t_1)$. Hence, it suffices to show that the right side of (6.7) equals $\bar{P}(r)$. In the following calculations it is notationally convenient to make the dependence of the state $x(s)$ on the initial condition explicit. With fixed initial starting time $r \in [t, t_1]$ and control $u_r^*(\cdot)$ let $x(s; y)$ denote the state at time s

with initial conditions $x(r;y) = y$. Clearly $x(s;x^*(r)) = x^*(s)$ for all $s \in [r,t_1]$. We directly calculate that

$$\frac{\partial}{\partial x_i} J(r,x^*(r);u^*) = \sum_{j=1}^{n} \left\{ \int_r^{t_1} L_{x_j}(s,x^*(s),u^*(s)) z_{ij}(s) ds + \psi_{x_j}(x^*(t_1)) z_{ij}(t_1) \right\}, \; r = 1,\cdots,n, \tag{6.8}$$

where $L_{x_j} = \partial L/\partial x_j$, $\psi_{x_j} = \partial\psi/\partial x_j$ and for $i,j = 1,\cdots,n$

$$z_{ij}(s) = \frac{\partial}{\partial y_i} x_j(s;x^*(r)).$$

A straightforward calculation, by standard ordinary differential equations methods, gives

$$\frac{d}{ds} z_{ij}(s) = \sum_{\ell=1}^{n} \frac{\partial}{\partial x_\ell} f_j(s,x^*(s),u^*(s)) z_{i\ell}(s), \; s \in [r,t_1], \tag{6.9}$$

with initial data

$$z_{ij}(r) = \begin{cases} 1, \text{ if } i = j \\ 0, \text{ if } i \neq j. \end{cases} \tag{6.10}$$

We claim that the right-hand side of (6.8) is equal to $\bar{P}_i(r)$. Indeed using (6.2) and (6.9) we calculate that

$$\frac{d}{ds}\left\{ \sum_{j=1}^{n} z_{ij}(s)\bar{P}_j(s) \right\} = -\sum_{j=1}^{n} L_{x_j}(s,x^*(s),u^*(s)) z_{ij}(s),$$
$$i = 1,\cdots,n.$$

We integrate this identity on $[r,t_1]$ and use (6.5), (6.7), (6.8), (6.10) to obtain

$$\begin{aligned}\bar{P}_i(r) &= \sum_{j=1}^{n} z_{ij}(r)\bar{P}_j(r) \\ &= \sum_{j=1}^{n} z_{ij}(t_1)\bar{P}_j(t_1) - \int_r^{t_1} \frac{d}{ds}\left\{ \sum_{j=1}^{n} z_{ij}(s)\bar{P}_j(s) \right\} ds \\ &= \frac{\partial}{\partial x_i} J(r,x^*(r);u_r^*) = \frac{\partial}{\partial x_i} V(r,x^*(r)).\end{aligned}$$

□

When the value function is not differentiable, a version of (6.4) still holds true. However, a weaker notion of gradient is needed. See Section II.15 in this book and [Cle1,2].

Results similar to Theorem 6.2 are known for the problem of control up to the time of exit from the closure $\overline{Q}$ of a cylindrical region $Q = [t_0, t_1) \times O$. See [FR, p.100]. However, in [FR] the assumption that V is differentiable along the optimal trajectory γ^* is replaced by the stronger assumption that V is of class C^2 in some neighborhood of γ^*. We shall not reproduce the proof of this result here. However, we will derive the transversality condition.

Again let $P(s)$ be as as in (6.4). If $\tau^* = t_1$ and $x^*(t_1) \in O$, then the transversality condition is again (6.5). Suppose that $\tau^* < t_1$. Then the transversality condition at the point $(\tau^*, x^*(\tau^*))$ of the lateral boundary is as follows: there exists a scalar λ such that

$$P(\tau^*) = D_x g(\tau^*, x^*(\tau^*)) + \lambda \eta(x^*(\tau^*)) \tag{6.11a}$$

with g as in (3.6) and $\eta(\xi)$ the exterior unit normal at $\xi \in \partial O$. Moreover,

$$g_t(\tau^*, x^*(\tau^*)) = H(\tau^*, x^*(\tau^*), P(\tau^*)). \tag{6.11b}$$

We prove (6.11 a,b) as follows. If $\tau^* < t_1$, then $V(\tau^*, \xi) \leq g(\tau^*, \xi)$ for all $\xi \in \partial O$ by (5.19). Equality holds when $\xi = x^*(\tau^*)$. Hence, the derivative of $V(\tau^*, \cdot) - g(\tau^*, \cdot)$ at $x^*(\tau^*)$ is zero in any direction tangent to ∂O. This implies that $D_x V - D_x g$ at $(\tau^*, x^*(\tau^*))$ is a scalar multiple λ of the exterior unit normal $\eta(x^*(\tau^*))$. Thus,

$$P(\tau^*) - D_x g(\tau^*, x^*(\tau^*)) = D_x(V - g)(\tau^*, x^*(\tau^*)) = \lambda \eta(x^*(\tau^*)),$$

which is (6.11a). Since $V(s, x^*(\tau^*)) \leq g(s, x^*(\tau^*))$, by (5.19), with equality for $s = \tau^*$,

$$\frac{\partial V}{\partial t}(\tau^*, x^*(\tau^*)) = \frac{\partial g}{\partial t}(\tau^*, x^*(\tau^*)).$$

By (5.3′) and (6.4), the left side is $H(\tau^*, x(\tau^*), P(\tau^*))$. Thus, (6.11b) holds.

To conclude this section, we give a proof of Pontryagin's principle on a fixed time interval $t \leq s \leq t_1$ without reference to the value function V or other restrictive assumptions in Theorem 6.2.

Theorem 6.3. *Let $\mathcal{U}(t,x) = \mathcal{U}^0(t)$ and $O = \mathbb{R}^n$. Let $u^*(\cdot)$ be an optimal control, and let $P(s)$ be the solution to* (6.2) *for $t \leq s \leq t_1$ with $P(t_1) = D\psi(x^*(t_1))$. Then* (6.3) *holds for almost all $s \in [t, t_1.]$*

Proof. Let $s \in [t, t_1)$ be any point at which $u^*(\cdot)$ is approximately continuous. (For a definition of approximately continuous function, see [EG] or [McS, p. 224].) Given $v \in U$ and $0 < \delta < t_1 - s$, let

$$u_\delta(r) = \begin{cases} u^*(r) \text{ if } r \notin [s, s+\delta] \\ \\ \quad v \quad \text{if } r \in [s, s+\delta], \end{cases}$$

and let $x_\delta(r)$ be the solution to (3.2) with $u(r) = u_\delta(r)$ and with $x_\delta(t) = x$. Since $J(t, x; u^*) \leq J(t, x; u_\delta)$

$$
\begin{aligned}
0 \leq & \frac{1}{\delta}\int_s^{s+\delta} [L(r, x_\delta(r), v) - L(r, x^*(r), u^*(r)]dr \\
& + \frac{1}{\delta}\int_{s+\delta}^{t_1} [L(r, x_\delta(r), u^*(r)) - L(r, x^*(r), u^*(r))]dr \\
& + \frac{1}{\delta}[\psi(x_\delta(t_1)) - \psi(x^*(t_1))].
\end{aligned}
\tag{6.12}
$$

The first term on the right side tends to $L(s, x^*(s), v) - L(s, x^*(s), u^*(s))$ as $\delta \to 0$. By the mean value theorem, the second term equals

$$
\int_{s+\delta}^{t_1}\int_0^1 L_x(r, x^*(r) + \delta\lambda\zeta_\delta(r), u^*(r)) \cdot \zeta_\delta(r) d\lambda dr
$$

where $L_x = (\partial L/\partial x_1, \ldots, \partial L/\partial x_n)$ and $\zeta_\delta(r) = \delta^{-1}[x_\delta(r) - x^*(r))]$. As $\delta \to 0$, $\zeta_\delta(r)$ tends uniformly on $[t, t_1]$ to $\zeta(r)$ which is the solution to the linear differential equation

$$
\frac{d}{dr}\zeta(r) = f_x(r, x^*(r), u^*(r))\zeta(r), \; s \leq r \leq t_1, \tag{6.13}
$$

with $\zeta(s) = f(s, x^*(s), v) - f(s, x^*(s), u^*(s))$.

In (6.13), f_x is the matrix of partial derivatives $\partial f_l/\partial x_j$. By (6.12)

$$
\begin{aligned}
0 \leq & L(s, x^*(s), v) - L(s, x^*(s), u^*(s)) \\
& + \int_s^{t_1} L_x(r, x^*(r), u^*(r)) \cdot \zeta(r) dr + D\psi(x^*(t_1)) \cdot \zeta(t_1).
\end{aligned}
\tag{6.14}
$$

From (6.2) and (6.13)

$$
\frac{d}{dr}[P(r) \cdot \zeta(r)] = -L_x(r, x^*(r), u^*(r)) \cdot \zeta(r). \tag{6.15}
$$

From (6.14), (6.15) and the values of $\zeta(s)$, $P(t_1)$ we obtain for all $v \in U$

$$
\begin{aligned}
0 \leq & L(s, x^*(s), v) + P(s) \cdot f(s, x^*(s), v) \\
& - L(s, x^*(s), u^*(s)) - P(s) \cdot f(s, x^*(s), u^*(s)).
\end{aligned}
\tag{6.16}
$$

By (5.4), this is equivalent to (6.3). □

I.7 Discounted cost with infinite horizon

In this section we study a class of problems with infinite time horizon ($t_1 = \infty$). With the notation of Section 3, the payoff functional is

$$J(t,x;u) = \int_t^\tau L(s,x(s),u(s))ds + g(\tau,x(\tau))\chi_{\tau<\infty}, \tag{7.1}$$

where τ is the exit time of $(s,x(s))$ from $\overline{Q}$, or $\tau = +\infty$ if $(s,x(s)) \in \overline{Q}$ for all $s \geq t$. In contrast to the finite horizon case, additional assumptions are often needed to ensure the finiteness of the value function.

A standard procedure is to introduce a discount factor $\beta \geq 0$ such that

$$L(t,x,v) = e^{-\beta t}\tilde{L}(t,x,v)$$

$$g(t,x) = e^{-\beta t}\tilde{g}(t,x).$$

If $\beta > 0$ and $\tilde{L}, \tilde{g}$ are bounded, then the value function V is always finite. Unfortunately, these assumptions do not cover many examples of interest. In such cases further assumptions of a technical nature are needed to insure finiteness of V.

To simplify the problem we assume that all the given data are time independent. Thus $\tilde{L}, \tilde{g}$ and f, are independent of t. With an abuse of notation we use $L(x,v)$ and $g(x)$ to denote $\tilde{L}(x,v)$ and $\tilde{g}(x)$ respectively. Then (7.1) becomes

$$J(t,x;u) = \int_t^\tau e^{-\beta s}L(x(s),u(s))ds + e^{-\beta\tau}g(x(\tau))\chi_{\tau<\infty}. \tag{7.1$'$}$$

We will take $\mathcal{U}(t,x) = \mathcal{U}_x$, where $\mathcal{U}_x$ is defined below. If we let $\tilde{u}(s) = u(t+s)$, for $s \geq 0$, then

$$J(t,x;u) = e^{-\beta t}J(0,x;\tilde{u}). \tag{7.2}$$

Hence, it suffices to consider initial time $t = 0$. From now on we shall do so, and will write $J(x;u)$ instead of $J(0,x;u)$. Let us now formulate more precisely the class of infinite horizon control problems which we shall consider. Let $O \subset I\!R^n$ be an open set, such that either $O = I\!R^n$ or ∂O is a compact manifold of class C^2. We assume that $f, L \in C(I\!R^n \times U)$ and $g \in C(I\!R^n)$. Moreover, for suitable K_ρ,

$$|f(x,v) - f(y,v)| \leq K_\rho|x-y| \tag{7.3}$$

for all $x, y \in I\!R^n$ and $v \in U$ such that $|v| \leq \rho$. By a *control* we mean any U-valued, Lebesgue measurable function $u(\cdot)$ on $[0,\infty)$ such that $u(s)$ is bounded on $[0,t]$ for any $t < \infty$. Let $\mathcal{U}^0$ denote the set of all controls.

The dynamics of the state $x(s)$ are now

(7.4) $$\frac{d}{ds}x(s) = f(x(s), u(s)), \quad s \geq 0,$$

with initial condition $x(0) = x$. The payoff functional is

(7.5) $$J(x; u) = \int_0^\tau e^{-\beta s} L(x(s), u(s))ds + e^{-\beta\tau} g(x(\tau))\chi_{\tau<\infty},$$

where τ is the exit time of $x(s)$ for $\overline{O}$ (or $\tau = +\infty$ if $x(s) \in \overline{O}$ for all $s \geq 0$.) If $O = I\!R^n$, then $\tau = +\infty$ and the last term in (7.5) is missing. Let $\mathcal{U}_x \subset \mathcal{U}^0$ denote the set of controls $u(\cdot)$ such that

$$\int_0^\tau e^{-\beta s} |L(x(s), u(s))| \, ds < \infty.$$

Then $J(x; u)$ is defined for all $u(\cdot) \in \mathcal{U}_x$. We assume that

(7.6) $$\mathcal{U}_x \text{ is nonempty for all } \; x \in \overline{O}.$$

We also assume the analogue of (3.11):

(7.7) For every $\xi \in \partial O$ there exists $v(\xi) \in U$ such that

$$f(\xi, v(\xi)) \cdot \eta(\xi) > 0.$$

Here $\eta(\xi)$ is the exterior unit normal at ξ.

The value function V is defined by

(7.8) $$V(x) = \inf_{\mathcal{U}_x} J(x; u), \quad x \in \overline{O}.$$

We will only consider problems in which $V(x) > -\infty$. This is true, in particular, if $L \geq 0$ and $g \geq 0$ since then $V(x) \geq 0$. For simplicity, we consider in this section only the problem of control until exit from $\overline{O}$, rather than the more general formulation in Section 3. In particular, we do not consider the infinite horizon problem with a state constraint $x(s) \in \overline{O}$ for all $s \geq 0$. However such problems can be analyzed exactly as below.

Equation (7.2) suggests that

(7.9) $$V(t, x) = e^{-\beta t} V(x).$$

By substituting this in equation (5.3$'$), we obtain the dynamic programming equation for the infinite horizon, discounted control problem:

(7.10) $$\beta V(x) + H(x, DV(x)) = 0, \; x \in O$$

where for $x, p \in R^n$, $H(x, p) = \sup_{v \in U}\{-p \cdot f(x, v) - L(x, v)\}$.

As in (5.19) we have

(7.11) $$V(x) \leq g(x), \; \forall x \in \partial O.$$

We cannot expect equality in (7.11) at all points of ∂O, although this is true under additional assumptions (for example, if $L \geq 0$ and $g \equiv 0$.) If O is unbounded (in particular, if $O = I\!R^n$) an additional condition will be imposed to exclude solutions to (7.10) which "grow too rapidly" as $|x| \to \infty$. See (7.14).

Following the proofs of Lemma 4.1 and Lemma 4.2 we obtain a similar result for the infinite horizon case.

Lemma 7.1. *For any initial condition $x \in \overline{O}$ and $r \geq 0$*

$$V(x) = \inf_{\mathcal{U}_x} [\int_0^{r\wedge\tau} e^{-\beta s} L(x(s), u(s)) ds + e^{-\beta\tau} g(x(\tau))\chi_{\tau<r} + e^{-\beta r} V(x(r))\chi_{r\leq\tau}]. \tag{7.12}$$

Moreover, an admissible control $u(\cdot) \in \mathcal{U}_x$ is δ - optimal at the initial point x if and only if it is δ - optimal at every $x(s)$ with $s \in [0, \tau]$.

We continue with the proof of a verification theorem. Let $W \in C^1(\overline{O})$ satisfy the stationary dynamic programming equation (7.10) and the boundary conditions (7.11). As in the proof of Theorem 5.1, using the state dynamics and (7.10) we calculate that

$$\begin{aligned} e^{-\beta r} W(x(r)) &= W(x) + \int_0^r e^{-\beta s}[-\beta W(x(s)) + \dot{x}(s) \cdot DW(x(s))] ds \\ &= W(x) + \int_0^r e^{-\beta s}[-\beta W(x(s)) + f(x(s), u(s)) \cdot dW(x(s))] ds \\ &\geq W(x) - \int_0^r e^{-\beta r} L(x(s), u(s)) ds \end{aligned}$$

for any $u(\cdot) \in \mathcal{U}_x$ and $r \in [0, \tau)$. We let $r \uparrow \tau$ to obtain

$$W(x) \leq \int_0^\tau e^{-\beta s} L(x(s), u(s)) ds + \lim_{r\uparrow\tau} e^{-\beta r} W(x(r)). \tag{7.13}$$

If $\tau < \infty$, then (7.11) implies that the above limit is less than or equal to $e^{-\beta\tau} g(x(\tau))$.

However, if $\tau = \infty$ we do not in general know anything about this term. To avoid this difficulty, we assume:

$$\lim_{r\uparrow\infty} e^{-\beta r} W(x(r)) = 0 \tag{7.14}$$

for all $u(\cdot) \in \mathcal{U}_x$ such that $\tau = \infty$.

We have proved the following.

Theorem 7.1. (*Verification Theorem). Let $W \in C^1(\overline{O})$ satisfy* (7.10), (7.11) *and* (7.14). *Then*

(a) $W(x) \leq V(x)$ for all $x \in \overline{O}$.

(b) Suppose that there exists $u^*(\cdot) \in \mathcal{U}_x$ such that

$$(7.15)\qquad L(x^*(s), u^*(s)) + f(x^*(x), u^*(s)) \cdot DW(x^*(s)) = -H(x^*(s), DW(x^*(s)))$$

for almost every $s \in [0, \tau^*)$ *and* $W(x^*(\tau^*)) = g(x^*(\tau^*))$ *provided* $\tau^* < \infty$. *Then* $u^*(\cdot)$ *is optimal for initial data* x *and* $W(x) = V(x)$.

In this theorem, $x^*(\cdot)$ denotes the solution to (7.4) with $u(\cdot) = u^*(\cdot), x^*(0) = x$; and τ^* is the exit time of $x^*(s)$ from $\overline{O}$ (or $+\infty$ if $x^*(s) \in \overline{O}$ for all $s \geq 0$.)

Remark 7.1. A slight generalization of Theorem 7.1 can be proved in which (7.14) is replaced by assumptions like those in Theorem III.9.1 and Theorem IV.5.1. Those results concern stochastic control problems. In the deterministic case, one simply considers control functions $u(\cdot)$ instead of admissible control systems (Section III.9) or progressively measurable control processes (Section IV.5).

The following is one among many conditions sufficient for (7.14).

Lemma 7.l. *Suppose that* $\beta > 0$, *that*

$$(7.16)\qquad \sup\{f(x,v) \cdot x \;:\; x \in \overline{O}, v \in U\} < \infty,$$

and that $|W(x)| \leq C(1 + |x|^m)$ *for some constants* C, m. *Then* (7.14) *holds.*

Proof. We calculate that

$$(7.17)\qquad \frac{d}{ds}(|x(s)|^2) = 2f(x(s), u(s)) \cdot x(s)$$

for any control $u(\cdot)$. By (7.16)

$$|x(s)|^2 \leq |x|^2 + Ks$$

for some K. Then, for suitable C_1

$$|W(x(s))| \leq C_1(1 + |x|^m + s^{\frac{m}{2}}),$$

which implies (7.14). □

Example 7.1. Let $f(x,v) = -x + v,\ x \in I\!R^1, v \in [-1,1]$. Then

$$f(x,v) \cdot x \leq -x^2 + |v||x| \leq \frac{1}{4},$$

and hence (7.16) holds. Consider the problem of minimizing

$$J(x;u) = \int_0^\infty e^{-s}\left[\frac{1}{2}x(s)^2 + \frac{1}{2}u(s)^2\right] ds.$$

The stationary dynamic programming equation (7.10) is

$$(7.18)\qquad V(x) - \min_{|v|\leq 1}\left[\frac{1}{2}v^2 + vV'(x)\right] - \frac{1}{2}x^2 + xV'(x) = 0.$$

Here $' = d/dx$. The value function V turns out to be convex. Moreover, $V(-x) = V(x)$ and $V(\cdot)$ is increasing for $x \geq 0$. Thus, we look for a class C^1 solution W of (7.18) with these properties. Equation (7.18) is equivalent for $x \geq 0$ to

$$(7.19) \qquad V(x) + \frac{1}{2}(V'(x))^2 - \frac{1}{2}x^2 + xV'(x) = 0, \text{ if } V'(x) \leq 1$$

$$(7.20) \qquad V(x) - (\frac{1}{2} + \frac{1}{2}x^2) + (1+x)V'(x) = 0, \text{ if } V'(x) > 1.$$

Equation (7.19) is the same as for the problem without control constraints ($U = I\!R^1$ instead of $U = [-1,1]$.) Motivated by the solution to the linear quadratic regulator problem (Section 5) we try $W(x) = Dx^2$ for $|x| \leq b$, where $W'(b) = 1$. Then (7.19) holds for D the positive root of $2D^2 + 3D - \frac{1}{2} = 0$ and $b = (2D)^{-1}$. Then the linear equation (7.20) is solved for $x \geq b$, with $W(b) = Db^2$; and we set $W(-x) = W(x)$. This solution W to (7.18) is of class $C^1(I\!R^1)$ and grows quadratically as $|x| \to \infty$. We determine $u^*(s) = \underline{u}^*(x^*(s))$ from the control policy

$$\underline{u}^*(x) = \begin{cases} -2Dx, & if\ x \leq b \\ -\operatorname{sgn} x, & if\ |x| \geq b, \end{cases}$$

using the fact that $W'(x) = 2Dx$ if $|x| \leq b$. The Verification Theorem 7.1 implies that $V = W$ and $\underline{u}^*$ is an optimal policy.

When $O = I\!R^n$ there are no boundary conditions. The stationary dynamic programming equation (7.10) may have many solutions W in $I\!R^n$. Condition (7.14) is helpful in finding the one relevant to the control problem. The following simple example illustrates this point.

Example 7.2. Consider the problem of minimizing

$$(7.21) \qquad \frac{1}{2}\int_0^\infty e^{-s}u(s)^2 ds,$$

subject to $\dot{x}(s) = u(s)$ and $u(s) \in I\!R^1$. In this case $O = U = I\!R^1$. The problem is in fact an infinite horizon linear quadratic regulator problem with $n = m = 1$. Since the running cost does not depend on the state, the optimal control is $u^*(s) \equiv 0$. Hence $V(x) \equiv 0$, which is a solution to the stationary dynamic programming equation

$$(7.22) \qquad V(x) + \frac{1}{2}(V'(x))^2 = 0, \ x \in I\!R^1.$$

For any constant a,

$$W(x) = -\frac{1}{2}(x-a)^2$$

is also a solution of (7.22), different from the value function $V(x)$. Formula (7.15) states, in this case, that $\frac{1}{2}v^2 + vW'(x^*(s))$ has a minimum at $v = u^*(s) = \dot{x}^*(s)$. This gives $\dot{x}^*(s) = x^*(s) - a$ with $x^*(0) = x$. If $x \neq a$, then (7.21) is infinite and (7.14) does not hold for this choice of $u^*(s)$.

Remark 7.2. A verification theorem entirely similar to Theorem 7.1 is true for the problem of control until the time τ' when $x(s)$ exits from O, rather than from $\overline{O}$. Since $x(s) \in O$ for $0 \leq s < \tau'$, the proof of Theorem 7.1 shows that in this case it suffices to assume that $W \in C^1(O) \cap C(\overline{O})$ rather than $W \in C^1(\overline{O})$. See Remark 5.2 for the corresponding finite time horizon problem.

Remark 7.2 will be used in the next example, in which O is the interval $(0, \infty)$.

Example 7.3. This example concerns a very simple deterministic model for the consumption and investment behavior of a single agent. In this simple model, the agent has wealth $x(s)$ invested in an asset on which a fixed rate of return r is earned. At each time s, the agent chooses a consumption rate $c(s) \geq 0$, which has the role of a control. The wealth dynamics are

$$\frac{d}{ds}x(s) = rx(s) - c(s), \; s \geq 0. \tag{7.23}$$

Let ℓ be a concave, increasing function on the control set $U = [0, \infty)$. ℓ is called a *utility function*. The problem is to *maximize* the total discounted utility of consumption

$$J(x; c) = \int_0^{\tau'} e^{-\beta s}\ell(c(s))ds, \; x \geq 0, \tag{7.24}$$

where either $\tau' = +\infty$ if $x(s) > 0$ for all $s \geq 0$ or τ' is the first s such that $x(s) = 0$ (bankruptcy). In this model, once bankruptcy occurs the wealth remains at 0, namely, $x(s) = 0$ for all $s \geq \tau'$. Hence, the problem is equivalent to maximizing total discounted utility on the interval $0 \leq s < \infty$, subject to the state constraint $x(s) \geq 0$. Let us take $\ell(c) = \gamma^{-1}c^\gamma$, where $0 < \gamma < 1$. This is a commonly used utility function, of so-called HARA type [KLSS]. In Example IV.5.2 and in Chapter X we will consider stochastic optimal investment-consumption models, in which an investor divides the wealth between a riskless asset with interest rate r and a risky asset with uncertain rate of return.

Let us assume that $r\gamma < \beta$. For $r\gamma \geq \beta$, it can be shown that the value function $V(x)$ is $+\infty$. The stationary dynamic programming equation is

$$\beta V(x) - \frac{1-\gamma}{\gamma}(V'(x))^{\frac{\gamma}{\gamma-1}} - rxV'(x) = 0, \; x > 0, \tag{7.25}$$

with the boundary condition $V(0) = 0$. (To use (7.10), we replace L by $-\ell$, V by $-V$ and maximum by minimum. In (7.25) we implicitly assume that $V'(x) > 0$.) A straightforward calculation shows that (7.25) has a power type solution

$$(7.26) \qquad W(x) = (\frac{1-\gamma}{\beta - r\gamma})^{1-\gamma}\frac{1}{\gamma}x^\gamma.$$

Using (7.26) and the procedure described in Example 5.1, we compute a candidate for the optimal feedback consumption rate

$$(7.27) \qquad \underline{c}^*(x) = (W'(x))^{\frac{1}{\gamma-1}} = (\frac{\beta - r\gamma}{1-\gamma})x.$$

Note that the optimal feedback control is a linear function in this example, as was also true for the linear quadratic regulator problem in Example 5.1.

We must verify that, in fact, W is the value function and $\underline{c}^*(\cdot)$ is optimal. For this purpose, we show that the value function is finite, (7.14) holds and use Theorem 7.1 with Remark 7.2. Since $c(s) \geq 0$, (7.23) implies that

$$(7.28) \qquad x(s) \leq xe^{rs}, \; x = x(0).$$

Then (7.23) and (7.28) imply an upper bound on the total consumption up to time s:

$$\begin{aligned}
\int_0^s c(\rho)d\rho &= \int_0^s [rx(\rho) - \frac{d}{d\rho}x(\rho)]d\rho \\
&= r\int_0^s x(\rho)d\rho - x(s) + x \\
&\leq r\int_0^s xe^{r\rho}d\rho + x = xe^{rs}.
\end{aligned}$$

Using this and Jensen's inequality, we get

$$\begin{aligned}
h(s) &= \int_0^s (c(\rho))^\gamma d\rho \\
&\leq s(\frac{1}{s}\int_0^s c(\rho)d\rho)^\gamma \\
&\leq s^{1-\gamma}x^\gamma e^{r\gamma s}.
\end{aligned}$$

Then integration by parts yields

$$\begin{aligned}
J(x;c) &= \frac{1}{\gamma}\int_0^\infty e^{-\beta s}(c(s))^\gamma ds \\
&= \lim_{T\to\infty}\frac{1}{\gamma}\left[e^{-\beta T}h(T) + \beta\int_0^T e^{-\beta s}h(s)ds\right] \\
&\leq \frac{\beta x^\gamma}{\gamma}\int_0^\infty s^{1-\gamma}e^{-(\beta - r\gamma)s}ds,
\end{aligned}$$

and this is finite since $r\gamma < \beta$. This upper bound for $J(x;c)$ implies the same bound for $V(x)$, and in particular that $V(x) < \infty$. By (7.28)

$$e^{-\beta T} x(T)^\gamma \le x^\gamma e^{-(r\gamma - \beta)T}$$

which tends to 0 as $T \to \infty$ since $r\gamma < \beta$. Thus

$$\lim_{T\to\infty} e^{-\beta T} W(x(T)) = 0$$

as required in (7.14). Consequently, $W(x)$ is the value function and $\underline{c}^*(\cdot)$ is the optimal feedback consumption rate.

Example 7.4. This example is related to the production planning problem described in Example 2.1. In this simple model there are two commodities; $n = 2$. We assume that the running cost is given by

$$h(x) = \alpha|x_1| + |x_2|, \qquad\qquad x = (x_1, x_2) \in I\!R^2$$

with some $\alpha > 0$. Let U be as in (2.3) with $c_1 d_1 + c_2 d_2 < 1$. Then the production planning is to minimize the total discounted holding and shortage cost,

$$J(x,u) = \int_0^\infty e^{-\beta s} h(x(s)) ds.$$

Here $x(s)$ is the solution of (2.1) with initial data $x(0) = x$. and $u(s) \in U$ for all $s \ge 0$.

Clearly if there is no shortage of either commodity then the optimal production rate is zero. Also if only one of the products is in shortage then the optimal strategy is to produce that product with full capacity. Hence the optimal policy $\underline{u}^*$ satisfies

$$\underline{u}^*(x_1, x_2) = \begin{cases} (0,0) \ \text{ if } x_1, x_2 > 0 \\ (\frac{1}{c_1}, 0) \text{ if } x_1 < 0 < x_2 \\ (0, \frac{1}{c_2}) \text{ if } x_1 > 0 > x_2. \end{cases}$$

When both products are in shortage $(x_1, x_2 < 0)$ the optimal production policy is produce one of the products in full capacity until it is no longer in shortage, then produce the other product. The product that has the priority depends on the parameters of the problem. To determine this priority rule we use the dynamic programming equation. An elementary but a tedious computation yields an explicit formula for the value function. See [LSST]. We then obtain the optimal production rate $\underline{u}^*$ by (7.15). For $x_1, x_2 < 0$,

$$\underline{u}^*(x_1, x_2) = \begin{cases} (\frac{1}{c_1}, 0) \text{ if } c_1 \le \alpha c_2 \\ (0, \frac{1}{c_2}) \text{ if } c_1 \ge \alpha c_2. \end{cases}$$

See [S1] for additional examples.

I.8 Calculus of variations I

In the remainder of this chapter we consider a special class of optimal control problems, already mentioned in Example 2.4. The state dynamics (3.2) are now simply

$$\frac{d}{ds}x(s) = u(s), \qquad t \le s \le t_1, \tag{8.1}$$

and there are no control constraints ($U = I\!R^n$). The control problem is then one of calculus of variations, for which there is an extensive literature antedating control theory.

Let us begin by considering calculus of variations problems on a fixed time interval $[t, t_1]$. As endpoint conditions, we require that $x(t) = x$ and $x(t_1) \in \mathcal{M}$ where $\mathcal{M}$ is a given closed subset of $I\!R^n$. See class (C) in Section 3. We recall that $x(\cdot)$ satisfying (8.1) is Lipschitz continuous on $[t, t_1]$ if and only if $u(\cdot)$ is bounded and Lebesgue measurable on $[t, t_1]$. If we write $\dot{} = d/ds$, then the problem is to minimize

$$J = \int_t^{t_1} L(s, x(s), \dot{x}(s))ds + \psi(x(t_1)) \tag{8.2}$$

among all Lipschitz continuous, $I\!R^n$-valued functions $x(\cdot)$ which satisfy the endpoint conditions. We assume that $\psi \in C^3(I\!R^n), L \in C^3(\bar{Q}_0 \times I\!R^n)$ and that for all $(t, x, v) \in \overline{Q}_0 \times I\!R^n$:

$$\begin{aligned} &(a) \quad L_{vv}(t, x, v) > 0, \\ &(b) \quad \frac{L(t, x, v)}{|v|} \to +\infty \text{ as } |v| \to \infty. \end{aligned} \tag{8.3}$$

Here L_{vv} denotes the matrix of second order partial derivatives $L_{v_i v_j}$. Condition (8.3a) implies that $L(t, x, \cdot)$ is a strictly convex function on $I\!R^n$. Condition (8.3b) is called a *coercivity*, or *superlinear growth*, condition. Additional assumptions about L, of a technical nature, will be made in Section 9. See (9.2). It will be shown in Section 9, for the special case $\mathcal{M} = I\!R^n$ (no constraint on $x(t_1)$), that there exists $x^*(\cdot)$ which minimizes J. Moreover, $x^*(\cdot) \in C^2([t, t_1])$. See Theorem 9.2.

In a calculus of variations problem, $f(t, x, v) = v$ and $U = I\!R^n$. The control at time s is $u(s) = \dot{x}(s)$. The Hamiltonian H in (5.4) takes the form

$$H(t, x, p) = \max_{v \in I\!R^n} [-v \cdot p - L(t, x, v)]. \tag{8.4}$$

The following facts are proved in Appendix A. $H \in C^2(\overline{Q}_0 \times I\!R^n)$, and H is strictly convex in the variable p. Moreover, the dual formula

$$L(t, x, v) = \max_{p \in I\!R^n} [-v \cdot p - H(t, x, p)] \tag{8.5}$$

holds. The points v and p where the max occurs in (8.4) and (8.5) are related by the classical Legendre transformation

$$p = -L_v, \quad v = -H_p, \tag{8.6}$$

which for each (t,x) is globally one-one from $\mathbb{R}^n$ onto itself. The duality between L and H is related to the duality between Lagrangian and Hamiltonian formulations in classical mechanics, in which v has the interpretation of velocity and p momentum for a particle of mass 1.

The following formulas relating first and second order partial derivatives of L and H also hold:

$$\begin{aligned} &(a)\ H_t = -L_t, \quad H_x = -L_x; \\ &(b)\ H_{pp} = (L_{vv})^{-1}; \\ &(c)\ H_{tp} = L_{tv}H_{pp}, \quad H_{x_i p} = L_{x_i v}H_{pp}. \end{aligned} \tag{8.7}$$

In (8.7) the partial derivatives of L and H are evaluated at (t,x,v) and (t,x,p), where v and p are related by the Legendre transformation (8.6).

For a calculus of variations problem the adjoint differential equations (6.2) become $\dot{P}(s) = -L_x(s, x^*(s), \dot{x}^*(s))$. Condition (6.3) states that $u^*(s) = \dot{x}^*(s)$ maximizes $-P(s)\cdot v - L(s, x^*(s), v)$ over $\mathbb{R}^n$. From (8.6) and (8.7a) we then obtain the following differential equations for the pair $x^*(s), P(s)$:

$$\begin{aligned} &(a)\ \dot{x}^*(s) = -H_p(s, x^*(s), P(s)) \\ &(b)\ \dot{P}(s) = H_x(s, x^*(s), P(s)). \end{aligned} \tag{8.8}$$

The dynamic programming equation for a calculus of variations problem is

$$-\frac{\partial V}{\partial t} + H(t, x, D_x V) = 0, \tag{8.9}$$

with H as in (8.4). Equation (8.9) is also called a *Hamilton-Jacobi* equation. If there are no constraints on $x(t_1)$, i.e. $\mathcal{M} = \mathbb{R}^n$, then

$$V(t_1, x) = \psi(x), \qquad x \in \mathbb{R}^n. \tag{8.10}$$

Boundary conditions of type (8.10), prescribing V at a fixed time t_1, are called *Cauchy data* for the partial differential equation (8.9). If $\mathcal{M}$ is a proper subset of $\mathbb{R}^n$ (e.g. a single point x_1), then the boundary data for V at time t_1 are incompletely specified by (8.10).

A classical method for studying first-order partial differential equations is the method of characteristics. This method introduces a system of $2n$ differential equations for a pair of functions $x(s), P(s)$. These characteristic differential equations are just (8.8). In general, if $x(s), P(s)$ are any solution to (8.8), then $x(\cdot)$ is called an *extremal* for the calculus of variations problem the sense that it satisfies the Euler equation. Every extremal satisfies the Euler differential equation ((10.1) below). However, if $x^*(\cdot)$ is minimizing then additional conditions must hold. In particular, $(s, x^*(s))$ cannot be a conjugate point for $t < s < t_1$ (the Jacobi necessary condition for a minimum). See [He], [FR, Chap 1], also Section 10.

The duality between H and L offers the possibility of calculus of variations descriptions of solutions to certain classes of first-order nonlinear partial differential equations. If $H(t,x,p)$ satisfies certain conditions, including strict convexity and superlinearity in p, then (8.5) defines an integrand $L(t,x,v)$ for the corresponding calculus of variations problem. We shall return to this point in Section 10.

Example 8.1. Suppose that $L = L(v)$. Then it suffices to consider linear functions $x(\cdot)$ in(8.2). Indeed, given any Lipschitz continuous $x(\cdot)$ let

$$v = \frac{1}{t_1 - t}\int_t^{t_1} \dot{x}(s)ds$$

$$\tilde{x}(s) = x + v(s-t).$$

Then $\tilde{x}(t) = x(t) = x,\ \tilde{x}(t_1) = x(t_1)$ and $\dot{\tilde{x}}(s) \equiv v$. Assumption (8.3a) implies that L is convex. Hence, by Jensen's inequality

$$\int_t^{t_1} L(\dot{\tilde{x}}(s))ds \le \int_t^{t_1} L(\dot{x}(s))ds.$$

(The inequality is, in fact, strict unless $\dot{x}(s)$=constant almost everywhere on $[t,t_1]$.) Assume that ψ_x is bounded. The value function in this example is

$$V(t,x) = \min_{v\in I\!R^n}[(t_1-t)L(v) + \psi(x + v(t_1-t))].$$

Since ψ_x is bounded, $|\psi(x)| \le C(1+|x|)$ for some C. This together with coercivity condition (8.3b) insures that the minimum is attained at some v^*. The corresponding linear function $x^*(s) = x + v^*(s-t)$ is optimal. By (8.4), $H = H(p)$. Any solution $x(s), P(s)$ of the characteristic differential equations (8.8) has $P(s) =$ constant, $\dot{x}(s) =$ constant, which is in accord with what was just shown about minimizing extremals $x^*(\cdot)$.

Control until exit from $\bar{Q}$. As in Section 3, case (B), let us now consider the calculus of variations problem of minimizing

$$J = \int_t^{\tau} L(s, x(s), \dot{x}(s))ds + \Psi(\tau, x(\tau)), \tag{8.11}$$

where τ is the exit time of $(s, x(s))$ from the closed cylindrical region $\bar{Q} = [t_0, t_1] \times \bar{O}$, and Ψ is as in (3.6). Since $U = I\!R^n$ and $f(s,\xi,v) = v$, condition (3.11) is satisfied with $v(s,\xi) = \eta(\xi)$, where $\eta(\xi)$ is the exterior unit normal to ∂O at ξ. The dynamic programming equation is again (8.9), which is to be considered in the cylindrical region Q with the data (5.18) at the final time t_1 and the inequality (5.19).

Let us give sufficient conditions for the lateral boundary condition $V = g$ to be enforced at all $(t,x) \in [t_0,t_1) \times \partial O$. Let us first assume as in (3.10)

$$g = 0, \quad L \geq 0, \quad \psi \geq 0. \tag{8.12}$$

Then, as already noted in the discussion following (5.19), $V(t,x) = 0$ for all $(t,x) \in [t_0,t_1) \times \partial O$.

In the general case, if the function g in (3.5) is sufficiently smooth (for example, $g \in C^4(\overline{Q}_0)$), then

$$g(\tau, x(\tau)) = g(t,x) + \int_t^\tau [g_t(s,x(s)) + D_x g(s,x(s)) \cdot \dot{x}(s)] ds$$

by the Fundamental Theorem of Calculus. We then rewrite J in (8.11) as

$$J = \tilde{J} + g(t,x), \qquad \text{where} \tag{8.11$'$}$$

$$\tilde{J} = \int_t^\tau \tilde{L}(s,x(s),\dot{x}(s)) ds + \tilde{\psi}(x(t_1))\chi_{\tau=t_1},$$

$$\begin{aligned} &(a)\ \tilde{L}(s,y,v) = L(s,y,v) + g_t(s,y) + D_x g(s,y) \cdot v, \\ &(b)\ \tilde{\psi}(y) = \psi(y) - g(t_1,y). \end{aligned} \tag{8.13}$$

We minimize J by minimizing $\tilde{J}$. Thus by replacing L by $\tilde{L}$ and ψ by $\tilde{\psi}$, the problem has been reduced to one with lateral boundary data 0.

Proposition 8.1. *Assume that:*

(a) *For all* $(s,y) \in \bar{Q}, v \in U, g_t(s,y) + D_x g(s,y) \cdot v + L(s,y,v) \geq 0$;

(b) *For* $y \in \bar{O}$, $g(t_1,y) \leq \psi(y)$.

Then

$$V(t,x) = g(t,x) \ \textit{for all} \ (t,x) \in [t_0,t_1) \times \partial O.$$

Proof. We take $\tilde{L}, \tilde{\psi}$ as in (8.13), and $\tilde{g} = 0$. Then $V(t,x) = \tilde{V}(t,x) + g(t,x)$. Since $\tilde{L}, \tilde{\psi}, \tilde{g}$ satisfy (8.12), $\tilde{V}(t,x) = 0$ on the lateral boundary, as noted above.

□

Remark 8.1. When (a) and (b) hold, then g is called a *smooth subsolution* of the Hamilton-Jacobi equation with the boundary conditions. Note that (a) is equivalent to

$$-g_t(s,y) + H(s,y,D_x g(s,y)) \leq 0. \tag{8.14}$$

Example 8.1. (continued) Again let $L = L(v)$, and let $\overline{O}$ be convex. The same argument as above shows that it suffices to minimize J in (8.11) among linear functions $x(s) = x + v(t - s)$. For $(t, x) \in \overline{Q}$, the value function is

$$V(t,x) = \min_{v,\tau}[(\tau - t)L(v) + \Psi(\tau, x + v(\tau - t)].$$

The minimum is taken over all $v \in I\!R^n$ and $\tau \in [t, t_1]$ such that $(\tau, x + v(\tau - t)) \in \partial^* Q$. Suppose that the lateral boundary data do not depend on t, namely, $g = g(x)$ of class $C^4(\overline{O})$. The conditions for g to be a smooth subsolution are $H(Dg(y)) \leq 0$ and $g(y) \leq \psi(y)$ for all $y \in \overline{O}$.

Let us again suppose that (8.12) holds and indicate how the unknown scalar λ in the transversality condition can be determined. Since $g = 0$, the transversality condition (6.11) takes the form

$$(8.15) \qquad P(\tau^*) = \lambda\eta(x^*(\tau^*)), \ \ H(\tau^*, x^*(\tau^*), P(\tau^*)) = 0.$$

Since $V \geq 0$ and $V(s,\xi) = g(s,\xi) = 0$ for (s,ξ) on the lateral boundary, we must have

$$(8.16) \qquad D_x V(s,\xi) \cdot \eta(\xi) \leq 0$$

for each $(s,\xi) \in [t_0, t_1) \times \partial O$ at which V is differentiable. Under perhaps additional smoothness assumptions on V near the lateral boundary, we anticipate that $P(\tau^*) = D_x V(\tau^*, x^*(\tau^*))$. See (6.4). This suggests that we require $\lambda \leq 0$ in (8.15). Let us give two instances in which $P(\tau^*)$ is then determined by (8.15).

Example 8.2. Suppose that $H(s,\xi,0) < 0$ for every $(s,\xi) \in [t_0,t_1) \times \partial O$. Now $h(\lambda) = H(s,\xi,\lambda\eta(\xi))$ is convex, with $h(0) < 0$ and $h(\lambda) \to +\infty$ as $|\lambda| \to \infty$. Hence, there exists a unique $\lambda < 0$ such that $H(s,\xi,\lambda\eta(\xi)) = 0$. By taking $(s,\xi) = (\tau^*, x^*(\tau^*))$ we get $P(\tau^*)$ in (8.15).

Example 8.3. Let $L(x,v) = \frac{1}{2}|b(x) - v|^2$, where $b(\xi) \cdot \eta(\xi) < 0$ for every $\xi \in \partial O$. This arises in a large deviations, exit problem discussed later in Chapter VII. In this example, $H(x,p) = \frac{1}{2}|p|^2 - b(x) \cdot p$, and $H(\xi, \lambda\eta(\xi)) = 0$ is equivalent to

$$\frac{1}{2}|\lambda\eta(\xi)|^2 = b(\xi) \cdot (\lambda\eta(\xi)).$$

Since $|\eta(\xi)| = 1$, this has two solutions: $\lambda = 0$ (which turns out to be irrelevant in the large deviations exit problem) and $\lambda = 2b(\xi) \cdot \eta(\xi) < 0$.

I.9 Calculus of variations II

In this section we consider calculus of variations problems on a fixed time interval $[t, t_1]$, with no constraints on $x(t_1)$, i.e. $\mathcal{M} = I\!R^n$. Our purpose is to prove existence and smoothness of an optimal trajectory, and also to show that the value function V satisfies a Lipschitz condition.

We seek a Lipschitz $\mathbb{R}^n$-valued function $x^*(\cdot)$ which minimizes

$$J = \int_t^{t_1} L(s, x(s), \dot{x}(s))ds + \psi(x(t_1)) \tag{9.1}$$

subject to the left endpoint condition $x(t) = x$.

Let us make the following assumptions about L. We assume that $L \in C^3(\bar{Q}_0 \times \mathbb{R}^n)$, and that for all (t, x, v):

(9.2)

(a) $L_{vv}(t, x, v) > 0$;

(b) $\dfrac{L(t, x, v)}{|v|} \geq \gamma(v)$, where $\gamma(v) \to +\infty$ as $|v| \to \infty$;

(c) For suitable $c_1, L(t, x, v) \geq -c_1$ and $L(t, x, 0) \leq c_1$;

(d) For suitable c_2, c_3, $|L_x(t, x, v)| \leq c_2 L(t, x, v) + c_3$;

(e) For suitable $C(R)$, $|L_v(t, x, v)| \leq C(R)$ whenever $|v| \leq R$;

(f) $\psi \in C_b^2(\mathbb{R}^n) \cap C^4(\mathbb{R}^n)$.

Here c_1, c_2, c_3 are suitable constants, and $C(R)$ a suitable positive function. Condition (9.2a) states that the matrix L_{vv} is positive definite.

Assumption (9.2a) is the same as (8.3a) and (9.2b) is a stronger form of (8.3b). The remaining assumptions (9.2c-f) will be used to obtain various bounds, for optimal controls and for the value function V.

Example 9.1. Let $L(x, v) = \frac{1}{2}|v|^2 - q(x)$. This is the classical action integrand in mechanics, for a particle of mass 1, if $q(x)$ is the potential energy of a particle at position x. Assumptions (9.2) hold if $q \in C^3(\mathbb{R}^n)$ with $q(x)$ and $Dq(x)$ bounded, and if $\psi \in C_b^2(\mathbb{R}^n) \cap C^4(\mathbb{R}^n)$.

Reduction to Lagrange form. In the remainder of this section we will assume that $\psi(x) = 0$. This is the so-called *Lagrange form* of the calculus of variations problem. The more general form (9.1) can be reduced to the Lagrange form as follows. Let

$$\tilde{L}(t, x, v) = L(t, x, v) + D\psi(x) \cdot v.$$

Then $\tilde{L}$ satisfies (9.2)(a)-(e), with different $\tilde{\gamma}(v), \tilde{C}(R)$ and $\tilde{c}_i, i = 1, 2, 3$. As in (8.11′), the Fundamental Theorem of Calculus gives

$$J = \tilde{J} + \psi(x),$$

$$\tilde{J} = \int_t^{t_1} \tilde{L}(s, x(s), \dot{x}(s))ds.$$

As in Section 8, we regard $u(s) = \dot{x}(s)$ as a control. In order to prove existence of a control $u^*(\cdot)$ minimizing $J(t,x;u)$ in (9.1) with $\psi = 0$, let us first impose a bound $|u(s)| \le R < \infty$. Then we show that, for R large enough, the artificially introduced bound on $u(s)$ does not matter. Let

$$\mathcal{U}_R(t) = \{u(\cdot) \in \mathcal{U}^0(t) : |u(s)| \le R \text{ for all } s \in [t,t_1]\}. \tag{9.3}$$

For $K \subset I\!\!R^n$, let $L^\infty([t,t_1];K)$ denote the set of K-valued, bounded measurable functions on $[t,t_1]$. Then $\mathcal{U}^0(t) = L^\infty([t,t_1];I\!\!R^n)$ and $\mathcal{U}_R(t) = L^\infty([t_1,t];U_R)$, where $U_R = \{|v| \le R\}$.

Theorem 9.1. *For each* $(t,x) \in Q_0$, *there exists* $u_R^*(\cdot) \in \mathcal{U}_R(t)$ *such that* $J(t,x;u_R^*) \le J(t,x;u)$ *for all* $u(\cdot) \in \mathcal{U}_R(t)$.

Theorem 9.1 is a special case of Theorem 11.1, which is proved below.

The next lemma is a special case of Pontryagin's principle. See Theorem 6.3. Let $u_R^*(\cdot)$ minimize $J(t,x;u)$ on $\mathcal{U}_R(t)$, and let

$$\text{(9.4)} \quad \begin{aligned} &(a)\ x_R^*(s) = x + \int_t^s u_R^*(r)dr, \\ &(b)\ P_R(s) = \int_s^{t_1} L_x(r, x_R^*(r), u_R^*(r))dr,\ t \le s \le t_1. \end{aligned}$$

Since $f(s,y,v) = v$, and $\psi \equiv 0, P_R(s)$ satisfies the integrated form of the adjoint differential equations (6.2) together with the terminal condition (6.5). By Pontryagin's principle:

Lemma 9.1. *For almost all* $s \in [t,t_1]$,

$$\begin{aligned} &L(s, x_R^*(s), u_R^*(s)) + u_R^*(s) \cdot P_R(s) \\ &\qquad = \min_{|v| \le R} [L(s, x_R^*(s), v) + v \cdot P_R(s)]. \end{aligned} \tag{9.5}$$

Let us next consider the value function for the problem with constraint $|u(s)| \le R$:

$$V_R(t,x) = J(t,x;u_R^*) = \min_{\mathcal{U}_R(t)} J(t,x;u). \tag{9.6}$$

We are going to find bounds for V_R and $u_R^*(s)$ which do not depend on R. Let c_1 be as in (9.2c).

Lemma 9.2. $|V_R(t,x)| \le c_1(t_1 - t)$. *Moreover, there exists* R_1 *such that* $|u_R^*(s)| \le R_1$.

Proof. By (9.2c), $L(s,y,v) \ge -c_1$. Hence, $V_R(t,x) \ge -c_1(t_1 - t)$. On the other hand, by choosing $u(s) \equiv 0, x(s) \equiv x$

$$V_R(t,x) \le \int_t^{t_1} L(s,x,0)ds \le c_1(t_1 - t)$$

by (9.2c). Thus, $|V_R(t,x)| \le c_1(t_1 - t)$. By (9.2d) and (9.4b)

$$|P_R(s)| \leq \int_t^{t_1} (c_2 L(r, x_R^*(r), u_R^*(r))dr + c_3)dr$$

$$= c_2 V_R(t,x) + c_3(t_1 - t),$$

$$|P_R(s)| \leq (c_2 c_1 + c_3)(t_1 - t).$$

Let $B = (c_2 c_1 + c_3)(t_1 - t_0)$. The coercivity condition (9.2b) implies the following. There exists a constant R_1 such that $|p| \leq B$ and $L(s,y,v)+v\cdot p \leq c_1$ imply $|v| \leq R_1$. By (9.2c) and (9.5)

$$L(s, x_R^*(s), u_R^*(s)) + u_R^*(s) \cdot P_R(s) \leq L(s, x_R^*(s), 0) \leq c_1$$

for almost all $s \in [t, t_1]$. By setting $u^*(s) = 0$ for s in the remaining set of measure 0, this holds for all $s \in [t, t_1]$. By taking $y = x_R^*(s), v = u_R^*(s), p = P_R(s)$, we get $|u_R^*(s)| \leq R_1$. □

Since

$$\mathcal{U}^0(t) = \cup_{R>0} \mathcal{U}_R(t),$$

Lemma 9.2 implies that for $R \geq R_1$

$$V(t,x) = V_R(t,x) = J(t,x;u_R^*), \tag{9.7}$$

where V is the value function for the calculus of variations problem formulated in Section 8. From now on we write, with $R \geq R_1$,

$$x^*(s) = x_R^*(s), \quad u^*(s) = u_R^*(s), \quad P(s) = P_R(s).$$

Since $L(s, x^*(s), v) + v \cdot P(s)$ has a minimum on $I\!R^n$ at $v = u^*(s) = \dot{x}^*(s)$, for almost all $s \in [t, t_1]$

$$-L_v(s, x^*(s), \dot{x}^*(s)) = P(s). \tag{9.8}$$

Then (8.6) implies for almost all $s \in [t, t_1]$

$$\dot{x}^*(s) = -H_p(s, x^*(s), P(s)). \tag{9.9}$$

The right side of (9.9) is a continuous function of s. This implies that $x^*(\cdot) \in C^1([t,t_1])$ and that (9.9) holds for all $s \in [t, t_1]$. By differentiating (9.9) with respect to s, one finds that $x^*(\cdot) \in C^2([t, t_1])$; and differentiating (9.8) and (9.4b) with respect to s gives the classical Euler differential equation for $x^*(\cdot)$.

We have proved, in particular, the following:

Theorem 9.2.

(a) *For every $(t,x) \in Q_0$ there exists $x^*(\cdot)$ which minimizes J in* (9.1), *subject to the left end point condition $x(t) = x$.*

(b) *Any such minimizing $x^*(\cdot)$ is of class $C^2([t, t_1])$. Moreover, $|\dot{x}^*(s)| \leq R_1$ for all $s \in [t, t_1]$ with R_1 as in Lemma* 9.2.

Let us next obtain a uniform bound and Lipschitz estimates for V.

Theorem 9.3.

(a) *For every* $(t,x)\in Q_0, |V(t,x)|\le c_1(t_1-t)$ *where* c_1 *is as in* (9.2c).

(b) *There exists* M *such that, for every* $(t,x),(t',x')\in Q_0$,

$$|V(t,x)-V(t',x')|\le M(|t-t'|+|x-x'|). \tag{9.10}$$

Inequality (9.10) states that V satisfies a Lipschitz condition, with Lipschitz constant M.

Proof of Theorem 9.3. Part (a) is immediate from Lemma 9.2, since $V=V_{R_1}$. To prove (b), by (9.2c,e) if $|v|\le R$ then $|L(t,x,v)|\le c_1+C(R)R$. Hence, by (9.2d) there exists N_1 such that $|L_x(s,y,v)|\le N_1$ whenever $|v|\le R_1$. For any control $u(\cdot)\in\mathcal{U}_{R_1}$, let

$$x(s)=x+\int_t^s u(r)dr, \qquad x'(s)=x'-x+x(s).$$

$$J(t,x;u)-J(t,x';u)=\int_t^{t_1}[L(s,x(s),u(s))-L(s,x'(s),u(s))]ds,$$

$$\begin{aligned}|J(t,x;u)-J(t,x';u)| &\le \sup_{|v|\le R_1}|L_x|\cdot\sup_{[t,t_1]}|x(s)-x'(s)|(t_1-t)\\ &\le N_1|x-x'|(t_1-t).\end{aligned}$$

Since this is true for every $u(\cdot)\in\mathcal{U}_{R_1}$, and since $V_{R_1}=V$,

$$|V(t,x)-V(t,x')|\le N_1|x-x'|(t_1-t). \tag{9.11}$$

We may suppose that $t'>t$. Let $x^*(\cdot)$ minimize J for initial data (t,x). Then

$$V(t,x)=\int_t^{t'}L(s,x^*(s),\dot{x}^*(s))ds+V(t',x^*(t')).$$

For $N_2=c_1+C(R_1)R_1$ we have $|L(s,y,v)|\le N_2$ whenever $|v|\le R_1$. Then

$$|V(t,x)-V(t',x)|\le N_2(t'-t)+|V(t',x^*(t'))-V(t',x)|.$$

Moreover, since $|\dot{x}^*(s)|=|u^*(s)|\le R_1$,

$$|x^*(t')-x|\le R_1(t'-t).$$

Therefore, by (9.11)

$$|V(t,x)-V(t',x)|\le (N_2+R_1N_1(t_1-t))|t'-t|. \tag{9.12}$$

Inequalities (9.11) and (9.12) are uniform Lipschitz estimates for $V(t,\cdot)$ and $V(\cdot,x)$. By combining these estimates, we get the Lipschitz estimate (9.10) with $M=N_2+(R_1+1)N_1(t_1-t_0)$. □

I.10 Generalized solutions to Hamilton-Jacobi equations

The Hamilton-Jacobi equation (8.9) is a first-order nonlinear partial differential equation for the value function V. By Theorems 6.1 and 9.2, V indeed satisfies (8.9) at each point $(t,x) \in Q_0$ where V is differentiable. However, we will see that there are generally points where V is not differentiable. In fact, those points (t,x) are exactly the ones for which the calculus of variations problem has more than one solution $x^*(\cdot)$. See Theorem 10.2. As in Section 9 we take $\psi(x) \equiv 0$ (the Lagrange form), and we continue to assume that L satisfies (9.2). At the end of the section, we will consider an application to first order nonlinear PDEs, in which H rather than L is given. The idea in this application is to introduce the calculus of variations problem in which L is obtained from H by the duality formula (8.5).

We recall from Section 6 the definition of generalized solution. By Theorem 9.3(b), the value function V satisfies a Lipschitz condition on Q_0 (not merely a local Lipschitz condition.) From Theorems 6.1 and 9.2 we then have:

Theorem 10.1. *The value function V is a generalized solution to the Hamilton-Jacobi equation in Q_0.*

A difficulty with the concept of generalized solution is that (8.9) together with the boundary data $W(t_1,x) = 0$ typically has many generalized solutions. Among them is one, called a viscosity solution, which is the natural (or physically relevant) generalized solution. As we shall see in Chapter II, this unique viscosity solution turns out to coincide with the value function V.

Let us next characterize those points (t,x) at which the value function V is differentiable.

Definition. A point $(t,x) \in Q_0$, with $t_0 < t < t_1$, is called a *regular point* if there exists a unique $x^*(\cdot)$ which minimizes J in (9.1), subject to the left endpoint condition $x(t) = x$.

The condition $t_0 < t$ is inessential, since the lower bound t_0 was introduced only for convenience. One can always replace t_0 by $\tilde{t}_0 < t_0$, and assume that L satisfies (9.2) on $[\tilde{t}_0, t_1] \times I\!\!R^{2n}$.

Theorem 10.2. *The value function V is differentiable at (t,x) if and only if (t,x) is a regular point.*

In preparation for the proof, let us prove:

Lemma 10.1. *Let (t,x) be a regular point and $x^*(\cdot)$ the unique minimizer of J for left endpoint (t,x). If $(\tau_n, y_n) \to (t,x)$ as $n \to \infty$ and $x_n^*(\cdot)$ minimizes J for left endpoint (τ_n, y_n), then $\dot{x}_n^*(\tau_n) \to \dot{x}^*(t)$ as $n \to \infty$.*

Proof. The Euler equation

$$L_x = \frac{d}{ds} L_v = L_{vt} + L_{vx}\dot{x}_n^* + L_{vv}\ddot{x}_n^* \tag{10.1}$$

holds, where $L_x, L_v, \cdots$ are evaluated at $(s, x_n^*(s), \dot{x}_n^*(s))$. Since $L_{vv} > 0$ this can be rewritten as

$$\ddot{x}_n^*(s) = \Phi(s, x_n^*(s), \dot{x}_n^*(s)), \ \tau_n \le s \le t_1,$$

where $\Phi = L_{vv}^{-1}(L_x - L_{vt} - L_{vx}v)$. By Lemma 9.2, $|\dot{x}_n^*(s)| \le R_1$, where R_1 does not depend on n. Suppose that, for a subsequence of n, $\dot{x}_n^*(\tau_n) \to v^0$, where $v^0 \ne \dot{x}(t)$. If we consider this subsequence instead of the original sequence, then the theory of ordinary differential equations implies that $x_n^*(s) \to x^0(s)$ and $\dot{x}_n^*(s) \to \dot{x}^0(s)$, where

$$\ddot{x}^0(s) = \Phi(s, x^0(s), \dot{x}^0(s)),\ t \le s \le t_1,$$

with $x^0(t) = x, \dot{x}^0(t) = v^0$. Then

$$\int_t^{t_1} L(s, x^0(s), \dot{x}^0(s))ds = \lim_{n\to\infty} \int_{\tau_n}^{t_1} L(s, x_n^*(s), \dot{x}_n^*(s))ds$$

$$= \lim_{n\to\infty} V(\tau_n, y_n).$$

By Theorem 9.3(b), $V(\tau_n, y_n) \to V(t,x)$ as $n \to \infty$. Hence

$$V(t,x) \le \int_t^{t_1} L(s, x^0(s), \dot{x}^0(s))ds = V(t,x).$$

Thus, $x^0(\cdot)$ minimizes J with left endpoint (t,x). Since $v^0 \ne \dot{x}^*(t), x^0(\cdot) \ne x^*(\cdot)$, which contradicts the assumption that (t,x) is a regular point. □

Proof of Theorem 10.2. If V is differentiable at (t,x), then V satisfies (8.9) there by Theorem 6.1. Therefore,

$$0 \le V_t(t,x) + L(t,x,v) + v \cdot D_x V(t,x),$$

with equality when $v = v^* = \dot{x}^*(t)$, where $x^*(\cdot)$ is any minimizer of J with $x^*(t) = x$. (Recall that by Theorem 9.2(b), $u^*(\cdot) = \dot{x}^*(\cdot)$ is continuous, and hence $v^* = \lim_{s\downarrow t} \dot{x}^*(s)$ as required in Theorem 6.1(b).) By (9.2)(a), $L(t,x,v) + v \cdot D_x V(t,x)$ is a strictly convex function of v. Hence v^* is unique. Since $x^*(\cdot)$ is a solution of the Euler equation, it is uniquely determined by its initial data $x^*(t) = x, \dot{x}^*(t) = v^*$. Therefore, (t,x) is a regular point.

Conversely, assume that (t,x) is a regular point. Given $\zeta = (\tau, y, p) \in Q_0 \times \mathbb{R}^n$, let $x_\zeta(\cdot)$ be the solution of the Euler equation (10.1) for $\tau \le s \le t_1$ with initial data (see (8.8)(a))

$$x_\zeta(\tau) = y,\ \dot{x}_\zeta(\tau) = -H_p(\tau, y, p). \tag{10.2}$$

We also let

$$G(\tau, y, p) = \int_\tau^{t_1} L(s, x_\zeta(s), \dot{x}_\zeta(s))ds.$$

Let $\eta = (\tau, y)$ and let $x_\eta^*(\cdot)$ minimize

$$\int_\tau^{t_1} L(s, x(s), \dot{x}(s))ds$$

subject to $x^*_\eta(\tau) = y$. Let $\xi = (t, x)$. Then $x^*_\xi(\cdot) = x^*(\cdot)$ is the unique minimizer of J for left endpoint (t, x). As in (9.4b) let

$$(10.3) \qquad P_\eta(s) = \int_s^{t_1} L_x(r, x^*_\eta(r), \dot{x}^*_\eta(r))dr, \;\; \tau \le s \le t_1.$$

Since $x^*_\eta(\cdot)$ satisfies the Euler equation (10.1), Lemma 10.1 implies that $x^*_\eta(r), \dot{x}^*_\eta(r)$ tends to $x^*_\xi(r), \dot{x}^*_\xi(r)$ as $\eta \to \xi$, for $t < r \le t_1$. Moreover, $x^*_\eta(r)$ and $\dot{x}^*_\eta(r)$ are uniformly bounded. Hence, $P_\eta(\tau) \to P_\xi(t)$ as $\eta \to \xi$. Now $x^*_\eta(\cdot), P_\eta(\cdot)$ satisfy the characteristic differential equations (8.8), and $\dot{x}^*_\eta(s), P_\eta(s)$ are related by the Legendre transformation (8.6). In particular,

$$P_\eta(\tau) = -L_v(\tau, y, \dot{x}^*_\eta(\tau)),$$

$$G(\tau, y, P_\eta(\tau)) = V(\tau, y).$$

Moreover, for every $p \in I\!R^n$

$$G(\tau, y, p) \ge V(\tau, y).$$

By taking $p = P_\xi(t)$ and $p = P_\eta(\tau)$ we get

$$\begin{aligned} G(\tau, y, P_\eta(\tau)) - G(t, x, P_\eta(\tau)) &\le V(\tau, y) - V(t, x) \\ &\le G(\tau, y, P_\xi(t)) - G(t, x, P_\xi(t)). \end{aligned}$$

The function G is of class $C^1(Q_0 \times I\!R^n)$. Let

$$a = G_t(t, x, P_\xi(t)), \;\; b = D_x G(t, x, P_\xi(t)).$$

Then

$$\begin{aligned} &\frac{G(\tau, y, P_\eta(\tau)) - G(t, x, P_\eta(\tau)) - a(\tau - t) - b \cdot (y - x)}{|y - x| + |\tau - t|} \\ &\le \frac{V(\tau, y) - V(t, x) - a(\tau - t) - b \cdot (y - x)}{|y - x| + |\tau - t|} \\ &\le \frac{G(\tau, y, P_\xi(t)) - G(t, x, P_\xi(t)) - a(\tau - t) - b \cdot (y - x)}{|y - x| + |\tau - t|}. \end{aligned}$$

Since $G \in C^1(Q_0 \times I\!R^n)$, the first and last terms tend to 0 as $(\tau, y) \to (t, x)$. Hence V is differentiable at (t, x) and $a = V_t(t, x), b = D_x V(t, x)$. □

Corollary 10.1. *If $x^*(\cdot)$ minimizes J subject to $x(t) = x$, then $(s, x^*(s))$ is a regular point for $t < s < t_1$.*

Proof. When restricted to $[s, t_1]$, $x^*(\cdot)$ minimizes J with left endpoint $(s, x^*(s))$. If $x^{**}(\cdot)$ is another minimizer of J with $x^{**}(s) = x^*(s)$, then $\dot{x}^{**}(s) \ne \dot{x}^*(s)$ since both $x^*(\cdot)$ and $x^{**}(\cdot)$ satisfy the Euler equation. Then

$$\tilde{x}(r) = \begin{cases} x^*(r) & \text{for } t \le r \le s \\ x^{**}(r) & \text{for } s \le r \le t_1 \end{cases}$$

minimizes J subject to $\tilde{x}(t) = x$; and $\tilde{x}(\cdot)$ is not of class $C^1([t, t_1])$. This contradicts Theorem 9.2(b). □

Corollary 10.1 states that only the left endpoint (t, x) of an optimal trajectory $\gamma^* = \{(s, x^*(s)) : t \leq s \leq t_1\}$ can fail to be regular. Let us next impose an additional condition on (t, x), which will imply that the value function is smooth (class C^1) in some neighborhood of γ^*. This condition states that (t, x) is not a conjugate point, a concept which is defined in terms of characteristics as follows. In the proof of Theorem 10.2, we considered a family of solutions parameterized by initial conditions $\zeta = (\tau, y, p)$. Let us now consider another family of solutions parameterized by terminal conditions at time t_1. For $\alpha \in I\!R^n$, let $x(s, \alpha), P(s, \alpha)$ denote the solutions to the characteristic differential equations

$$(10.4) \qquad \begin{aligned} &(a)\ \dot{X}(s, \alpha) = -H_p(s, X(s, \alpha), P(s, \alpha)) \\ &(b)\ \dot{P}(s, \alpha) = H_x(s, X(s, \alpha), P(s, \alpha)) \end{aligned}$$

with the data

$$(10.5) \qquad \begin{aligned} &(a)\ X(t_1, \alpha) = \alpha \\ &(b)\ P(t_1, \alpha) = 0. \end{aligned}$$

[We have taken $\psi(x) \equiv 0$ in (9.1). In general, condition (10.5b) is $P(t_1, \alpha) = D\psi(\alpha)$]. In (10.4), $^{\bullet} = \partial/\partial s$ as usual. From the theory of ordinary differential equations, for each α (10.4) with the data (10.5) have a unique solution $X(\cdot, \alpha), P(\cdot, \alpha)$ on some maximal interval $S(\alpha) < s \leq t_1$. We also observe that if $x^*_\xi(\cdot)$ minimizes J with left endpoint $\xi = (t, x)$, then

$$x^*_\xi(s) = X(s, \alpha_\xi), \ \text{ where } \alpha_\xi = x^*_\xi(t_1),$$

and $S(\alpha_\xi) < t$. In particular, $x = X(t, \alpha_\xi)$.

The function $X(\cdot, \cdot)$ is of class $C^1(\Delta)$ where $\Delta = \{(s, \alpha) : \alpha \in I\!R^n, S(\alpha) < s \leq t_1\}$. For fixed s, consider the Jacobian matrix

$$\frac{\partial X}{\partial \alpha}(s, \alpha) = \Big(\frac{\partial X_i}{\partial \alpha_j}(s, \alpha)\Big), \ \ i, j = 1, \cdots, n.$$

Definition. We call (τ, y) a *conjugate point* of the curve $\gamma_\alpha = \{(s, X(s, \alpha)) : S(\alpha) < s \leq t_1\}$ if $y = X(\tau, \alpha)$ and $\frac{\partial X}{\partial \alpha}(\tau, \alpha)$ is of rank $< n$.

The classical Jacobi necessary condition states that if $x^*_\xi(\cdot)$ minimizes J with left endpoint $\xi = (t, x)$, then $(s, x^*_\xi(s))$ cannot be a conjugate point of γ_{α_ξ} for $t < s < t_1$. See [He], [FR]. However, the left endpoint ξ of a minimizing trajectory may be a conjugate point. If ξ is a regular point, then $x^*_\xi(\cdot)$ is unique. In this case we simply call ξ a conjugate point, without explicitly referring to γ_{α_ξ}.

In the rest of this section, we assume that L satisfies (9.2) on $[t_0, t_1] \times I\!R^{2n}$, for arbitrary $t_0 < t_1$. Thus, we may consider all initial times $t < t_1$. A set N is called *open relative* to $\overline{Q}_0$ if $N = \tilde{N} \cap \overline{Q}_0$ where $\tilde{N} \subset I\!R^{n+1}$ is open.

Theorem 10.3. *Let (t,x) be a regular point and $\gamma^* = \{(s, x^*(s)) : t \leq s \leq t_1\}$ the unique minimizing trajectory with left endpoint (t,x). If (t,x) is not a conjugate point, then there exists a set N open relative to $\overline{Q}_0$ such that $\gamma^* \subset N$ and $V \in C^1(N)$.*

Sketch of proof. Except for one point, Theorem 10.3 is a standard result in the classical theory of first order partial differential equations. See [CH, Chap 2], [He, Chap 3]. Hence, we shall only sketch a proof. The method of characteristics is used to construct a local solution W of the Hamilton-Jacobi equation (8.9) in a neighborhood of γ^*, as follows. Consider the mapping Y from Δ into $I\!R^{n+1}$, such that $Y(s,\alpha) = (s, X(s,\alpha))$. Since $\xi = (t,x)$ is not a conjugate point, there exist positive ρ_0, δ_0 such that Y maps $\{(s,\alpha) : t - \delta_0 < s \leq t_1, |\alpha - \alpha_\xi| < \rho_0\}$ in a one-one way onto N_0 with N_0 open relative to $\overline{Q}_0$. Moreover, for each $s \in (t - \delta_0, t_1]$, the restriction of $Y(s, \cdot)$ to $\{|\alpha - \alpha_\zeta| < \rho_0\}$ has an inverse X_s^{-1} of class C^1. Let

$$w(\tau, \alpha) = \int_\tau^{t_1} L(s, X(s,\alpha), \dot{X}(s,\alpha))ds;$$

and for $(\tau, y) \in N_0$ let

$$W(\tau, y) = w(\tau, X_\tau^{-1}(y)).$$

Then $W \in C^1(N_0)$ and W satisfies (8.9).

It remains to use the fact that (t,x) is regular to show that $W = V$ in a possibly smaller neighborhood N of γ^*. As in the proof of Theorem 10.2, let $x_\eta^*(\cdot)$ be minimizing for left endpoint $\eta = (\tau, y)$, and let $\alpha_\eta = x_\eta^*(t_1)$. Then

$$x_\eta^*(s) = X(s, \alpha_\eta), \;\; \tau \leq s \leq t_1,$$

$$W(\tau, y) = w(\tau, \alpha_\eta).$$

By Lemma 10.1, there exists a relatively open set N containing γ^* such that $|\alpha_\eta - \alpha_\xi| < \rho_0$ for any $\eta \in N$. Then

$$V(\tau, y) = \int_\tau^{t_1} L(s, x_\eta^*(s), \dot{x}_\eta^*(s))ds = w(\tau, \alpha_\eta)$$

and hence $V(\tau, y) = W(\tau, y)$ for $(\tau, y) \in N$. (Note that $V(t_1, y) = W(t_1, y) = 0$ if $(t_1, y) \in N$.) □

Remark 10.1. If L is of class $C^\infty(\overline{Q}_0 \times I\!R^n)$ then the method of characteristics shows that $V \in C^\infty(N)$.

Example 10.1. In dimension $n = 1$, consider the problem of minimizing

$$J = \frac{1}{2} \int_t^{t_1} |\dot{x}(s)|^2 ds + \psi(x(t_1)).$$

From Example 8.1 it is seen that any minimizing $x^*(\cdot)$ is a straight line segment. Thus

$$(10.6), \qquad V(t,x) = \min_{v \in I\!R^1} [\frac{1}{2}(t_1 - t)v^2 + \psi(x + (t_1 - t)v)]$$

where v is the slope of the line $y = x + (s - t)v$. In this example, $L(v) = \frac{1}{2}v^2$ and $H(p) = \frac{1}{2}p^2$. The characteristic equations (10.4) become $\dot{X} = -P, \dot{P} = 0$; and (10.5) becomes $X(t_1, \alpha) = \alpha, P(t_1, \alpha) = \psi'(\alpha)$. Hence

$$X(s, \alpha) = \alpha + (t_1 - s)\psi'(\alpha)$$

$$\frac{\partial X}{\partial \alpha}(s, \alpha) = 1 + (t_1 - s)\psi''(\alpha).$$

A conjugate point appears when $(t_1 - s)\psi''(\alpha) = -1$.

Let us now suppose that $\psi(x) = \psi(-x)$ and $\psi''(0) < 0$ (for example, $\psi(x) = \cos x$). For left endpoint $(t, 0)$, since $-x^*(\cdot)$ is minimizing whenever $x^*(\cdot)$ is minimizing there are two possibilities. Either: (1) $x^*(s) \equiv 0$ is the unique minimizer, in which case $(t, 0)$ is a regular point; or (2) there is a minimizer $x^*(s) \not\equiv 0$, in which case $(t, 0)$ is not a regular point. Let $\lambda = (t_1 - t)v$. From (10.6), $(t, 0)$ is regular if

$$\psi(\lambda) - \psi(0) > -\frac{\lambda^2}{2(t_1 - t)} \text{ for all } \lambda \in I\!R^1, \ \lambda \neq 0.$$

Let us assume that there exists K such that $\psi(\lambda) - \psi(0) \geq -K\lambda^2$ for all λ. Then $(t, 0)$ is regular if $t_1 - t < (2K)^{-1}$. On the other hand, $(t, 0)$ is not regular if $(t_1 - t) > |\psi''(0)|^{-1}$, since there is a conjugate point at $(\bar{t}, 0)$ where $(t_1 - \bar{t})\psi''(0) = -1$. The regular points $(t, 0)$ form a line segment with right endpoint $(t_1, 0)$ and left endpoint $(\tilde{t}, 0)$, with $\bar{t} \leq \tilde{t} < t_1$. Whether $\bar{t} = \tilde{t}$ depends on the global behavior of $\psi(x)$, not just the local behavior near $x = 0$.

Let

$$(10.7) \qquad E = \{(t,x) \in Q_0 : \text{ either } (t,x) \text{ is not regular or } (t,x) \text{ is a conjugate point}\}.$$

Theorem 10.4. *E is a closed subset of Q_0, and $V \in C^1(\overline{Q}_0 \backslash E)$.*

Proof. An easy modification of the proof of Theorem 10.3 shows that any point (t_1, x_1) of the terminal hyperplane $\{t_1\} \times I\!R^n$ has a relative neighborhood N_1 which contains no point of E. For $(t, x) \in N_1, V(t, x) = W(t, x)$ where W is the solution to the Hamilton-Jacobi equation (8.9) by the method of characteristics with $W(t_1, x) = 0$. Thus, $V \in C^1(N_1)$. Let $(t, x) \in Q_0 \backslash E$. By Theorem 10.3, there is a relative neighborhood N of γ^* such that $V \in C^1(N)$. Moreover, the set N constructed in the proof of Theorem 10.3 contains no point of E. □

Remark 10.2. It can be shown that E is a set of Hausdorff dimension $\leq n$. See [F2, p.521]. Thus, in the sense of Hausdorff dimension, the value function V is smooth except at points of a "small" closed set E.

Remark 10.3. In dimension $n = 1$, there is a close connection between Hamilton-Jacobi equations and conservation laws. If we let $Z(t,x) = V_x(t,x)$, then in dimension 1 the Hamilton-Jacobi equation (8.9) is formally equivalent to the conservation law

$$-\frac{\partial Z}{\partial t} + \frac{\partial}{\partial x} H(t,x,Z) = 0.$$

A curve in the (t,x) plane across which V_x is discontinuous corresponds to a shock, in conservation law terminology. See [Daf].

Application to nonlinear PDEs. Consider a Cauchy problem for a first order nonlinear PDE

$$(10.8) \qquad -V_t + H(t,x,D_x V) = 0, \ \ (t,x) \in Q_0,$$

$$(10.9) \qquad V(t_1, x) = 0,$$

where now H rather than L is given. We would like to identify (10.8) as a Hamilton-Jacobi equation, by defining L from the formula (8.5) dual to (8.4).

Let us assume that $H \in C^\infty(\overline{Q}_0 \times I\!\!R^n)$ and that

$$(10.10) \qquad \begin{aligned} &(a)\ H_{pp}(t,x,p) > 0 \\ &(b)\ \lim_{|p|\to\infty} \frac{H(t,x,p)}{|p|} = +\infty. \end{aligned}$$

We define L by (8.5):

$$(10.11) \qquad L(t,x,v) = \max_{p \in I\!\!R^n} [-v \cdot p - H(t,x,p)].$$

The maximum is attained at $p = \Gamma(t,x,v)$, where

$$(10.12) \qquad v = -H_p(t,x,\Gamma(t,x,v)).$$

The implicit function theorem implies that $\Gamma \in C^\infty(\overline{Q}_0 \times I\!\!R^n)$. Since $L = -H(t,x,\Gamma) - v \cdot \Gamma$, $L \in C^\infty(\overline{Q}_0 \times I\!\!R^n)$. First and second order partial derivatives of H and L are related by (8.7).

Since H satisfies the formula (8.4) dual to (10.11), we can apply Theorems 10.1-10.4 to the Cauchy problem (10.8)-(10.9) provided that L satisfies assumptions (9.2). For this purpose we assume:

$$(10.13) \qquad \begin{aligned} &(a)\ \text{Assumptions (10.10)}: \\ &(b)\ p \cdot H_p - H \geq |H_p|\gamma(H_p), \text{ where } \gamma(v) \to \infty \text{ as } |v| \to \infty; \\ &(c)\ \text{For suitable } c_1,\ H(t,x,0) \leq c_1 \text{ and } H(t,x,p) \geq -c_1; \\ &(d)\ \text{For suitable } c_2, c_3,\ |H_x| \leq c_2(p \cdot H_p - H) + c_3; \\ &(e)\ \text{For suitable } C(R), |H_p| \leq R \text{ implies } |p| \leq C(R). \end{aligned}$$

Since H_{pp} is a positive definite symmetric matrix, its inverse L_{vv} is also positive definite as required in (9.2a). Since $L = p \cdot H_p - H$ and v, p are related by (10.12), (9.2b) is just (10.13b). Similarly, from (8.6) and (8.7a), (9.2d, e) are just (10.13d, e). Finally, by (10.11), $H(t,x,0) \le c_1$ implies $L(t,x,v) \ge -c_1$ and $H(t,x,p) \ge -c_1$ implies $L(t,x,0) \le c_1$.

Example 10.2. Let $H(t,x,p) = g(t,x)(1+|p|^2)^k$ where $k > \frac{1}{2}$, g and $D_x g$ are bounded, and $g(t,x) \ge c > 0$. It is elementary to verify (10.13).

Remark 10.4. Assumptions (10.13) imply the following, which will be used later in proving that the value function V is a viscosity solution to (10.8)-(10.9). See Theorem II.10.3. Let

$$H_R(t,x,p) = \max_{|v|\le R} [-v \cdot p - L(t,x,v)]. \tag{10.14}$$

Then there exists $R(M)$ such that

$$H_R(t,x,p) = H(t,x,p) \text{ if } \ |p| \le M, R \ge R(M). \tag{10.15}$$

To see this, let $v^* = -H_p(t,x,p)$. Then

$$-c_1 \le H(t,x,p) = -v^* \cdot p - L(t,x,v^*) \le |v^*|(M - \gamma(v^*)).$$

Since $\gamma(v) \to \infty$ as $|v| \to \infty$, this implies $|v^*| \le R(M)$ for some $R(M)$. If $|p| \le M$, $R \ge R(M)$, then the maximum in (10.14) equals the unconstrained maximum over $v \in I\!R^n$. This implies (10.15).

I.11 Existence theorems

An existence theorem for the control problem formulated in Section 3 asserts that there exists a control $u^*(\cdot)$ which minimizes $J(t,x;u)$ among all admissible controls $u(\cdot)$.

In this section, we prove two such theorems which will be used in later chapters. For a more complete study of existence theorems we refer to [Bk][Ce][FR].

In the first existence result which we prove, control is on the fixed time interval $[t,t_1]$ and hence $Q = Q_0$. Moreover, we assume:

(11.1)

(a) U is compact and convex;

(b) $f(t,x,v) = f_1(t,x) + f_2(t,x)v$, where $f_i \in C^1(\bar{Q}_0 \times U)$ for $i = 1,2$ and f_{1x}, f_2, f_{2x} are bounded;

(c) $L \in C^1(\bar{Q}_0 \times U)$ and $L(t,x,\cdot)$ is a convex function on U for each $(t,x) \in \bar{Q}_0$;

(d) ψ is continuous on $I\!R^n$.

The class of admissible controls $u(\cdot)$ is $\mathcal{U}^0(t) = L^\infty([t,t_1];U)$ and the criterion to be minimized is

$$J(t,x;u) = \int_t^{t_1} L(s,x(s),u(s))ds + \psi(x(t_1)), \tag{11.2}$$

as in (3.4). We say that $u_n(\cdot) \to u(\cdot)$ *weakly* as $n \to \infty$ if, for every $\phi(\cdot) \in L^\infty([t,t_1];\mathbb{R}^n)$

$$\lim_{n\to\infty} \int_t^{t_1} u_n(s)\cdot\phi(s)ds = \int_t^{t_1} u(s)\cdot\phi(s)ds.$$

Since U is compact and convex, $\mathcal{U}^0(t)$ is weakly sequentially compact.

Lemma 11.1. *Let Λ and Λ_v be continuous on $[t,t_1]\times U$, with $\Lambda(s,\cdot)$ convex for each $s\in[t,t_1]$. If $u_n(\cdot)\to u(\cdot)$ weakly, then*

$$\liminf_{n\to\infty} \int_t^{t_1} \Lambda(s,u_n(s))ds \geq \int_t^{t_1} \Lambda(s,u(s))ds.$$

Proof. Since $\Lambda(s,\cdot)$ is convex and of class C^1

$$\Lambda(s,v) \geq \Lambda(s,u(s)) + (v-u(s))\cdot\Lambda_v(s,u(s))$$

for all $v\in U$. In particular, this is true if $v = u_n(s)$. Hence

$$\begin{aligned}\int_t^{t_1} \Lambda(s,u_n(s))ds \geq & \int_t^{t_1} \Lambda(s,u(s))ds \\ & + \int_t^{t_1} [u_n(s)-u(s)]\cdot\Lambda_v(s,u(s))ds.\end{aligned}$$

The last term tends to 0 as $n\to\infty$, since $u_n(\cdot)\to u(\cdot)$ weakly. □

Lemma 11.2. *Let $x_n(s)$ be the solution to (3.2) with $u(s) = u_n(s)$ and $x_n(t) = x$. Suppose that $u_n(s)\in U$, that $u_n(\cdot)$ tends to $u^*(\cdot)$ weakly and that $x_n(s)$ tends to $x^*(s)$ uniformly on $[t_0,t_1]$. Then*

$$\frac{dx^*}{ds} = f_1(s,x^*(s)) + f_2(s,x^*(s))u^*(s), t\leq s\leq t_1 \tag{11.3}$$

and

$$J(t,x;u^*) \leq \liminf_{n\to\infty} J(t,x;u_n). \tag{11.4}$$

Proof. By (11.1)(b) we have

$$\begin{aligned} x_n(s) = x & + \int_t^s [f_1(r,x_n(r)) + f_2(r,x_n(r))u^*(r)]dr \\ & + \int_t^s [f_2(r,x_n(r)) - f_2(r,x^*(r))][u_n(r)-u^*(r)]dr \\ & + \int_t^s f_2(r,x^*(r))[u_n(r)-u^*(r)]dr. \end{aligned}$$

The last two integrals tend to 0 as $n \to \infty$. Hence

$$x^*(s) = x + \int_t^s [f_1(r, x^*(r)) + f_2(r, x^*(r))u^*(r)]dr,$$

which is the integrated form of (11.3).

To obtain (11.4) we write

$$J(t, x; u_n) = \int_t^{t_1} \Lambda(s, u_n(s))ds + \int_t^{t_1} [L(s, x_n(s), u_n(s)) - L(s, x^*(s), u_n(s)]ds \\ + \psi(x_n(t_1)),$$

where $\Lambda(s, v) = L(s, x^*(s), v)$. The second integral tends to 0 and $\psi(x_n(t_1))$ tends to $\psi(x^*(t_1))$ as $n \to \infty$. Then (11.4) follows from Lemma 11.1. □

Theorem 11.1. *Under the assumptions* (11.1) *an optimal control* $u^*(\cdot)$ *exists.*

Proof. Let $u_n(\cdot)$ be a sequence in $\mathcal{U}^0(t)$ such that $J(t, x; u_n)$ tends to $V(t, x)$, where $V(t, x)$ is the infimum of $J(t, x; u)$ among all $u(\cdot) \in \mathcal{U}^0(t)$. Let $x_n(s)$ denote the corresponding solution to (3.2)-(3.3). By assumptions (11.1)(a)(b),

$$|x_n(s)| \le |x| + K(1 + \int_t^s |x_n(r)|dr)$$

for some constant K. Gronwall's inequality implies that $x_n(s)$ is uniformly bounded on $[t, t_1]$. By (3.2), the derivatives $\dot{x}_n(s)$ are also uniformly bounded. We use Ascoli's theorem and weak sequential compactness of $\mathcal{U}^0(t)$ to choose a subsequence of $(u_n(\cdot), x_n(\cdot))$ which satisfies the hypotheses of Lemma 11.2. If we again denote this subsequence by $(u_n(\cdot), x_n(\cdot))$, then

$$V(t, x) \le J(t, x; u^*) \le \liminf_{n\to\infty} J(t, x; u_n) = V(t, x).$$

Thus $J(t, x; u^*) = V(t, x)$. □

In the second existence theorem, we consider a calculus of variations problem in a bounded cylindrical region $Q = [t_0, t_1) \times O$, where ∂O is a manifold of class C^2. Thus

$$J = \int_t^\tau L(s, x(s), \dot{x}(s))ds + \Psi(\tau, x(\tau)) \tag{11.5}$$

where τ is the exit time of $(s, x(s))$ from Q. No constraints are imposed on the control $u(s) = \dot{x}(s)$, and thus $U = \mathbb{R}^n$. We make the following assumptions. $L \in C^3(\bar{Q} \times \mathbb{R}^n)$ and the following hold for all $(t, x, v) \in \bar{Q} \times \mathbb{R}^n$:

(11.6)

(a) $L(t, x, v) \ge 0$ and $L_{vv}(t, x, v) > 0$;

(b) There exist $k > 0, p > 1$ and $R_0 > 0$ such that $k|v|^p \le L(t, x, v)$ whenever $|v| \ge R_0$;

(c) There exists K such that $|L_x(t, x, v)| + |L_v(t, x, v)| \le K(1 + |v|^p)$.

The terminal cost function Ψ satisfies:

$$(11.7)\qquad \begin{aligned}&\text{(a) } \Psi(t,x) = 0 \text{ for } (t,x) \in [t_0, t_1] \times \partial O;\\ &\text{(b) } \Psi(t_1, x) = \psi(x) \text{ for } x \in \overline{O}, \text{ where } \psi\\ &\quad \text{is Lipschitz}, \psi(x) \geq 0 \text{ and } \psi(x) = 0\\ &\quad \text{for } x \in \partial O.\end{aligned}$$

We admit all controls $u(\cdot) \in \mathcal{U}(t)$, where $\mathcal{U}(t) = L^p([t,t_1]; \mathbb{R}^n)$ with $p > 1$ as in (11.6). Without loss of generality, we may assume that $u(s) = 0$ for $\tau < s \leq t_1$, in case $\tau < t_1$.

Theorem 11.2. *An optimal control $u^*(\cdot)$ exists.*

Proof. As in the proof of Theorem 11.1, we choose a sequence $u_n(\cdot)$ such that $J(t,x;u_n)$ tends to $V(t,x)$ as $n \to \infty$, where

$$V(t,x) = \inf_{u(\cdot) \in \mathcal{U}(t)} J(t,x;u).$$

By (11.5)

$$J(t,x;u_n) = \int_t^{\tau_n} L(s, x_n(s), u_n(s)) ds + \Psi(\tau_n, x_n(\tau_n))$$

where $\dot{x}_n(s) = u_n(s), \tau_n$ is the exit time and $u_n(s) = 0$ for $\tau_n < s \leq t_1$. The L^p-norm $\| u_n(\cdot) \|_p$ is bounded by (11.6)(b) and the fact that $L \geq 0, \Psi \geq 0$. By Holder's inequality, for $s_1 < s_2$

$$|x_n(s_1) - x_n(s_2)| = |\int_{s_1}^{s_2} u_n(r) dr| \leq (s_2 - s_1)^{\frac{1}{q}} \| u_n(\cdot) \|_p$$

where $p^{-1} + q^{-1} = 1$. Hence the functions $x_n(\cdot)$ are equicontinuous, and also uniformly bounded on $[t,t_1]$ since $x_n(t) = x$. We take a subsequence (again denoted by $x_n(\cdot)$) such that $x_n(s)$ tends to a limit $x^*(s)$ uniformly on $[t,t_1]$ and $u_n(\cdot)$ tends weakly to $u^*(\cdot)$, where $u^*(s) = \dot{x}^*(s)$. Let τ^* denote the exit time of $(s, x^*(s))$ from Q. Then

$$\tau^* \leq \liminf_{n \to \infty} \tau_n.$$

We consider two cases

Case 1. $x^*(\tau^*) \in \partial O$. Consider any $t_2 < \tau^*$. Then

$$(11.8)\qquad \int_t^{t_2} L(s, x^*(s), u^*(s)) ds \leq \liminf_{n \to \infty} \int_t^{t_2} L(s, x_n(s), u_n(s)) ds.$$

To obtain (11.8), we slightly modify the proof of (11.4), taking into account that $u_n(\cdot)$ is bounded in $\| \, \|_p$ norm but not necessarily pointwise. Let $\Lambda(s,v) = L(s, x^*(s), v)$, and let $\chi_R(s)$ be the indicator function of $\{s \colon |u^*(s)| \leq R\}$. As in Lemma 11.1,

$$\int_t^{t_2} \Lambda(s,u^*(s))\chi_R(s)ds \le \liminf_{n\to\infty} \int_t^{t_2} \Lambda(s,u_n(s))\chi_R(s)ds.$$

Since $\Lambda \ge 0$, $\chi_R(s) \le 1$ and $\chi_R(s) \to 1$ as $R \to \infty$,

$$\int_t^{t_2} \Lambda(s,u^*(s))ds \le \liminf_{n\to\infty} \int_t^{t_2} \Lambda(s,u_n(s))ds.$$

By the mean value theorem

$$|\int_t^{t_2} [L(s,x_n(s),u_n(s)) - L(s,x^*(s),u_n(s)]ds|$$

$$= |\int_t^{t_2}\int_0^1 L_x(s,x_{\lambda n}(s),u_n(s))\cdot(x_n(s)-x^*(s))d\lambda ds|$$
$$\le K \parallel x_n(\cdot) - x^*(\cdot) \parallel \int_t^{t_2} (1+|u_n(s)|^p)ds$$

where K is as in (11.6)(c), $\parallel \ \parallel$ is the supnorm and $x_{n\lambda}(s) = x^*(s) + \lambda(x_n(s) - x^*(s))$. Since $\parallel u_n(\cdot) \parallel_p$ is bounded, the right side tends to 0. Since $L \ge 0$, $\Psi \ge 0, \Psi(\tau^*, x^*(\tau^*)) = 0$ and $t_2 < \tau^*$ is arbitrary, we conclude in the same way as for Lemma 11.2 that

$$J(t,x;u^*) = \int_t^{\tau^*} L(s,x^*(s),u^*(s))ds \le \liminf_{n\to\infty} J(t,x;u_n).$$

Hence, $J(t,x;u^*) = V(t,x)$ as in the proof of Theorem 11.1.

Case 2. $\tau^* = t_1$ and $x^*(\tau^*) \in O$. Since $x_n(s)$ tends to $x^*(s)$ uniformly on $[t,t_1]$, $\tau_n = t_1$ for large n. Then (11.8) holds with t_2 replaced by t_1. Since $\psi(x_n(t_1))$tends to $\psi(x^*(t_1))$ as $n\to\infty$, we again conclude that $J(t,x;u^*) = V(t,x)$. □

We next show that Euler's equation holds in integrated form.

Lemma 11.3. *Let*

(11.9) $$\bar P(s) = \int_t^s L_x(r,x^*(r),u^*(r))dr,\ t \le s \le \tau^*.$$

Then for almost all $s \in [t,\tau^*]$

(11.10) $$\bar P(s) = L_v(s,x^*(s),u^*(s)) + C$$

where C *is some constant.*

Proof. (Sketch) We follow the classical derivation of Euler's equation. Let $t_2 < \tau^*$ and consider any $\xi(\cdot) \in C^1([t,t_2]; I\!R^n)$ such that $\xi(t) = \xi(t_2) = 0$. For $|\delta|$ sufficiently small, $0 \le \lambda \le 1$ and $s \in [t,t_2]$, $x_\lambda(s) \in O$ where $x_\lambda(s) = x^*(s) + \delta\lambda\xi(s)$. Then

$$0 \le \int_t^{t_2} [L(s, x_1(s), \dot{x}_1(s)) - L(s, x^*(s), \dot{x}^*(s)] ds$$
$$= \delta \int_t^{t_2} \int_0^1 [L_x(s, x_\lambda(s), \dot{x}_\lambda(s)) \cdot \xi(s) + L_v(s, x_\lambda(s), \dot{x}_\lambda(s)) \cdot \dot{\xi}(s)] d\lambda ds.$$

We divide by δ and let $\delta \to 0$ to obtain, using (11.6)(b)(c) and the dominated convergence theorem

$$(11.11) \qquad 0 = \int_t^{t_2} [L_x(s, x^*(s), u^*(s)) \cdot \xi(s) + L_v(s, x^*(s), u^*(s)) \cdot \dot{\xi}(s)] ds.$$

Since $\xi(\cdot)$ is arbitrary, (11.10) is obtained by integrating the first term in (11.11) by parts. □

Corollary 11.1. $x^*(\cdot) \in C^2[t, \tau^*]$.

Proof. This is obtained in the same way as Theorem 9.2. By (9.9) we have

$$(11.12) \qquad \dot{x}^*(s) = -H_p(s, x^*(s), P(s))$$

where $P(s) = C - \bar{P}(s)$. □

Theorem 11.3. *There exists R_1, which does not depend on the initial data $(t, x) \in Q$, such that $|u^*(s)| \le R_1$.*

Proof. Let us first show that $|u^*(\tau^*)| \le R_2$, where R_2 does not depend on the initial data (t, x). Corollary 11.1 implies that $u^*(s) = \dot{x}^*(s)$ is continuous on $[t, \tau^*]$. Let $v^* = u^*(\tau^*)$. If $|v^*| < R_0$, there is nothing to prove. Suppose that $|v^*| \ge R_0$.

Case A. $\tau^* = t_1$. For small $\theta > 0$, let $x(s) = x^*(t_1 - \theta)$ for $t_1 - \theta \le s \le t_1$. Then

$$\int_{t_1-\theta}^{t_1} L(s, x^*(s), u^*(s)) ds + \psi(x^*(t_1)) \le \int_{t_1-\theta}^{t_1} L(s, x(s), 0) ds + \psi(x(t_1)).$$

We let $\theta \to 0$ to obtain from (11.6)(b) with $\xi^* = x^*(\tau^*)$,

$$k|v^*|^p \le L(\tau^*, \xi^*, v^*) \le B_1 + B_2|v^*|,$$

where $L(s, x, 0) \le B_1$ for all $(s, x) \in \bar{Q}$ and B_2 is a Lipschitz constant for ψ. Since $p > 1$, this implies that $|v^*| \le R_3$ for some R_3.

Case B. $\tau^* < t_1$. Then $\xi^* \in \partial O$. For small $\theta > 0$ let $\tau_\theta = \tau^* - \theta$, $y = x^*(\tau_\theta)$, $\bar{y}$ the point of ∂O nearest y and $h = |\bar{y} - y|$. Then

$$(11.13) \qquad h \le |y - \xi^*| = \theta |v^*| + o(\theta).$$

Let $x(s) = y + \nu(s - \tau_\theta)$, for $\tau_\theta - h \le s \le t_1$ where $\nu = h^{-1}(\bar{y} - y)$. Then $x(s)$ exits O at time $\tau_\theta + h$. Then

$$\int_{\tau_\theta}^{\tau^*} L(s, x^*(s), u^*(s)) ds \le \int_{\tau_\theta}^{\tau_\theta + h} L(s, x(s), \nu) ds \le B_3 h,$$

where $L(s,x,v) \le B_3$ for all $(s,x) \in \bar{Q}$, $|v| \le R_0$. From (11.6)(b) and (11.13) we have as $\theta \to 0$

$$k|v^*|^p \le L(\tau^*, \xi^*, v^*) \le B_3|v^*|,$$

if $|v^*| \ge R_0$. Hence, $|v^*| \le R_4$ for some R_4. Let $R_2 = \max(R_3, R_4)$.

Since $0 \le J(t,x;u^*) \le J(t,x;0)$ and $J(t,x;0)$ is bounded on $\bar{Q}$, (11.6) gives a bound for $\| u^*(\cdot) \|_p$ which does not depend on the initial data (t,x). From (11.6)(c) and (11.9), $|\bar{P}(s)| \le M_1$ where M_1 does not depend on (t,x). In (11.10) take $s = \tau^*$. Since $|u^*(\tau^*)| \le R_2$, the constant $C = C(t,x)$ is bounded by some C_1. Hence, $|P(s)| \le M_1 + C_1$ where $P(s) = C - \bar{P}(s)$. By (11.12) and $u^*(s) = \dot{x}^*(s)$, $|u^*(s)| \le R_1$ for some R_1. □

I.12 Historical remarks

The deterministic optimal control problems considered in Sections 3 – 7 are of type formulated during the 1950's by Pontryagin and his associates [PBGM]. The method of dynamic programming was developed by Bellman during the same time period [Be]. For the Pontryagin optimal control problem, dynamic programming leads to the Hamilton–Jacobi–Bellman PDE derived in Section 5. In the setting of classical calculus of variations, this connection was known since Hamilton and Jacobi in the 19th century. It played an important role in Caratheodory's approach to calculus of variations [C].

For a good, recent introduction to deterministic control theory, see Sontag [Sg]. There are many other books dealing with mathematical aspects, including [Bk] [Ce] [He] [Y].

Pontryagin - type optimal control theory had roots in calculus of variations, as well as in older engineering work on control of linear and nonlinear systems. For a concise introduction to the special class of calculus of variations problems considered in Section 8, see [FR, Chap. 1]. The classical method of characteristics for the Hamilton–Jacobi PDE, used in Section 10, was called in calculus of variations the method of fields of extremals [He, Chap. 3]. For optimal control problems, the presence of control switching surfaces adds some complications. See [F3, Appendix]. Sections 9 and 10 follow [F2]. Theorem 10.2 is due to Kuznetzov and Šiškin [KSi]. For recent results about representations of generalized solutions to Hamilton-Jacobi PDEs, see [Ca].

Nonsmooth analysis provides another interesting approach to deterministic control and calculus of variations. See Aubin - Frankowska [AF], Clarke [Cle1,2], Rockafellar-Wets [RW].

II

Viscosity Solutions

II.1 Introduction

As we saw in the first chapter, the method of dynamic programing provides a powerful tool for studying deterministic optimal control problems. This method is equally useful in stochastic control, which will be formulated in Chapters III, IV and V. In both cases the *value function* of the control problem is defined to be the infimum of the payoff as a function of the initial data. When the value function is smooth enough, it solves a nonlinear equation which we call the *dynamic programing equation.* For a deterministic optimal control problem, this derivation is given in Section I.5. A similar computation for stochastic problems will be carried out in Section III.7. In general however, the value function is not smooth enough to satisfy the dynamic programming equations in the classical or usual sense. Also there are many functions other than the value function which satisfy the equation almost everywhere, see Section 2 below. Indeed the lack of smoothness of the value function is more of a rule than the exception. Therefore a weak formulation of solutions to these equations is necessary if we are to pursue the method of dynamic programming.

In their celebrated 1984 paper Crandall and Lions [CL1] provided such a weak formulation which they called viscosity solutions. Although the name "viscosity" refers to a certain relaxation scheme, the definition of a viscosity solution is an intrinsic one. In particular as we will see in Section 6, viscosity solutions remain stable under any reasonable relaxation or approximation of the equation. A uniqueness result for first order equations was another very important contribution of that paper. Since then elegant equivalent reformulations of viscosity solutions were obtained by Crandall, Evans and Lions. Jensen proved the uniqueness of viscosity solutions of second order equations in 1986. The recent article of Crandall, Ishii and Lions [CIL1] provides an excellent survey of the development of the theory and contains all the relevant references.

For a large class of optimal control problems the value function is the unique viscosity solution of the related dynamic programming equation. In the case of a deterministic problem this equation is a first order partial differential equation, and for a controlled diffusion process the dynamic programming equation is a second order parabolic partial differential equation. However there are dynamic programming equations which are not differential equations. The optimal control of a Markov chain yields a difference equation, and a piecewise deterministic process gives rise to a system of first order differential equations. To capture this variety in dynamic programing equations we give an abstract discussion of viscosity solutions in Section 4. In this abstract formulation the dynamic programming operator is viewed as the infinitesimal generator of a two parameter nonlinear semigroup satisfying (3.1), (3.2) and (3.11) below. To define this semigroup, we view a given function ψ as the terminal data to our optimal control problem. Then the value function with terminal data ψ is defined to be the evaluation of the semigroup at ψ. In this formalism, the semigroup property (3.2) is equivalent to the dynamic programing principle and (3.11) is nothing but the derivation of the dynamic programing equation for a smooth value function. Then in this abstract setup, the viscosity property of the value function is a very easy consequence of the semigroup property (or equivalently the dynamic programing principle) and (3.11). See Theorem 5.1 below.

We should also note that prior to the formulation of viscosity solutions, Hamilton–Jacobi equations have been used to formulate sufficient and necessary conditions for deterministic optimal control problems. Clarke and his students ([Cle1] Section 3.7) used the theory of nonsmooth analysis to obtain sufficient conditions for optimality. This approach was further developed by Clarke and Vinter in 1983 [CV].

The theory of viscosity solutions is not limited to dynamic programming equations. Indeed this theory applies to any equation with maximum principle as defined in Section 4. In particular second order partial differential equations of (possibly degenerate) parabolic type and first order equations have this property. In the special case of a partial differential equation the definition of a viscosity solution simplifies as we need to consider only local extrema because of the local character of the equations (Definition 4.2.)

In the second part of this chapter we only consider the first order partial differential equations. In Section 7, under standard assumptions we verify that the semigroups related to deterministic optimal control problems have the property (3.11). Also the semigroup property (3.2) follows from the dynamic programing principle which is proved in Section I.4. Then Theorem 5.1 implies that, if the value function is continuous, then it is a viscosity solution of the dynamic programming equation. Sufficient conditions for the continuity of the value function are discussed in Section 10. Although in this chapter we restrict ourselves to continuous viscosity solutions, the theory is not limited to continuous functions. A discussion of discontinuous viscosity solutions is given Section VII.4.

The uniqueness of viscosity solutions is an important property which we discuss in Section 9 and 14. We prove the uniqueness of viscosity solutions of a general first order partial differential equation in Section 9. The generalizations of Theorem 9.1 are then stated in Section 14. In these sections we consider equations which are not necessarily the dynamic programming equation of a control problem. In the special case of a control problem however, the value function is the unique viscosity solution of the dynamic programming equation satisfying appropriate boundary and terminal conditions. This unique characterization of the value function is especially important in the analysis of related singular perturbation problems (Chapter VII) and in the numerical analysis of the control problems (Chapter IX).

To characterize a viscosity solution uniquely we need to specify the solution at the terminal time and at the boundary of the state space. In some cases however, the value of the solution at the boundary is not *a priori* known to us. As an example consider the value function of an optimal control problem with a state constraint. Since there is no need for the boundary cost function, the boundary value of the value function is not a priori known. Also at the boundary of the state space, the value function of an exit time problem may achieve a value smaller than the given boundary cost. Therefore we have to give a weak (viscosity) formulation of the boundary condition satisfied by the value function. This issue of the boundary condition has been given a satisfactory answer through the theory of viscosity solutions. The main idea here is to look for a differential type boundary condition instead of a pure Dirichlet condition. We discuss the state constraint case in Section 12 as a first step towards a general formulation. We then extend our discussion to the general exit time problem in Section 13. Also a uniqueness result with this weak boundary condition is stated in Section 14.

We close the chapter with a brief discussion of the connection between the adjoint variable and generalized gradients as defined in Section 8 below. The main result of Section 15 is a generalization of the classical statement I(6.4). Recall that I(6.4) states that the adjoint variable is equal to the gradient of the value function evaluated at the optimal trajectory. Since the value function is not necessarily differentiable, in general the adjoint variable is a generalized derivative of the value function.

For readers interested in an introduction to viscosity solutions, but not in control theory, Sections 3 – 6 can be read independently of Chapter I by omitting those examples arising from control theory. Similarly, Sections 8 and 9 provide a concise introduction to the theory of viscosity solutions for first order PDEs, independent of any control interpretation.

The following is a partial list of function spaces that will be used in this chapter. A more complete explanation of these notations is given in the general list of Notations preceding Chapter I. For a Banach space Σ, $\mathcal{M}(\Sigma)$ = set of all real-valued functions which are bounded from below, $C_p(\Sigma)$ = set of all continuous, real-valued functions which are polynomially growing. For a measurable subset E of Euclidean $\mathbb{R}^n$, and a positive integer k, the spaces

$C^k(E), C^\infty(E)$ consist of those functions which have an extension to some open set $\bar{E}$ containing E, with continuous partial derivatives of orders $\leq k$ (or of all orders), respectively.

Spaces $C_p^k(E)$, and $C_p^\infty(E)$ are defined similarly by requiring polynomial growth of the corresponding partial derivatives. Set

$$Q_0 = [t_0, t_1) \times I\!R^n, \ \overline{Q}_0 = [t_0, t_1] \times I\!R^n.$$

For a measurable set $G \subset Q_0$,

$C^{1,2}(G) =$ set of all real-valued function on G, which are once continuously differentiable in the t - variable and twice continuously differentiable in the x - variables.

II.2 Examples

In this section we give three one dimensional examples. These examples illustrate that in general the value function is not differentiable, the lateral boundary condition I(5.19) may hold with a strict inequality and that there are many generalized solutions of the dynamic programming equation.

Example 2.1 Consider the calculus of variations problem with $t_0 = 0, t_1 = 1, O = (-1, 1), \Psi \equiv 0$ and $L(t, x, v) = 1 + \frac{1}{4}v^2$, i.e. the problem is to minimize

$$\int_t^\tau \left[1 + \frac{1}{4}(\dot{x}(s))^2\right] ds, \quad t \in [0, 1]$$

where τ is the exit time of $(s, x(s))$ from the closed region $[0, 1] \times [-1, 1]$. Using Example I.8.1, we conclude that any optimal control u^* is constant in time. Indeed for initial condition $(t, x) \in [0, 1] \times [-1, 1]$, an elementary calculation shows that

$$u^*(s) = \begin{cases} 2 \text{ if } x \geq t, \\ 0 \text{ if } |x| < t, \\ -2 \text{ if } x \leq -t, \end{cases} \quad s \in [t, \tau],$$

is an optimal control. Using this control we directly compute that the value function is given by

$$V(t, x) = \begin{cases} 1 - |x| \ , |x| \geq t, \\ 1 - t \ \ , |x| \leq t. \end{cases}$$

Clearly, V is not differentiable if $t = |x|$. However, V satisfies the corresponding dynamic programming equation

$$(2.1) \qquad -\frac{\partial}{\partial t}V(t, x) + \left(\frac{\partial}{\partial x}V(t, x)\right)^2 - 1 = 0,$$

for all $(t, x) \in (0, 1) \times (-1, 1)$ except when $t = |x|$. Finally we note that, in this example the value function satisfies the boundary conditions,

$$V(t,1) = V(t,-1) = 0, \quad t \in [0,1], \tag{2.2}$$

$$V(1,x) = 0, x \in [-1,1]. \tag{2.3}$$
□

Example 2.2. In the previous example V is a generalized solution of (2.1) satisfying (2.2) and (2.3). We now construct a sequence of Lipschitz continuous generalized solutions of (2.1) which also satisfy (2.2) and (2.3).

For a positive integer k define $h_k(x)$ by

$$h_k(x) = \frac{1}{2k+1} - \left| x - \frac{2i}{2k+1} \right|,$$

if $x \in [\frac{2i-1}{2k+1}, \frac{2i+1}{2k+1}]$ for some $i = 0, \pm 1, \ldots, \pm k$.

Then for $(t,x) \in [0,1] \times [-1,1]$, let

$$W_k(t,x) = \min\{h_k(x), 1-t\}.$$

Clearly W_k satisfies (2.2) and (2.3), and $(1-t)$ solves (2.1). Also except at the corners of its graph, h_k is differentiable with $[\frac{d}{dx}h_k(x)]^2 = 1$. Therefore it is easy to verify that W_k satisfies (2.1) at the points of differentiability. Thus there are infinitely many generalized solutions to the dynamic programing equation (2.1) with given boundary conditions (2.2) and (2.3). □

Example 2.3. Consider a simple control problem with $Q = [0,1) \times (-1,1)$, $\dot{x}(s) = u(s), L \equiv 0, \Psi(t,x) = x$ and a control set $U = [-a,a]$ with some constant $a > 0$. Since the boundary data is increasing and the running cost is zero, the optimal control is $u^*(s) \equiv -a$ for all s. Hence the value function is

$$V(t,x) = \begin{cases} -1, & \text{if } x + at \leq a - 1, \\ x + at - a, & \text{if } x + at \geq a - 1, \end{cases} \tag{2.4}$$

for $(t,x) \in \overline{Q}$. V is differentiable except at $x+at = a-1$ and it is a generalized solution of

$$-\frac{\partial}{\partial t}V(t,x) + a\left|\frac{\partial}{\partial x}V(t,x)\right| = 0. \tag{2.5}$$

When $t = t_1 = 1$, the corresponding terminal boundary condition I(5.5),

$$V(1,x) = \Psi(1,x) = x, \ x \in [-1,1], \tag{2.6}$$

is satisfied. However $V(t,1) < 1 = \Psi(t,1)$ for all $t \in [0,1)$. Hence the lateral boundary condition I(5.19),

$$V(t,x) \leq g(t,x), \quad (t,x) \in [0,1) \times \partial O$$

holds with a strict inequality for $(t,x) \in [0,1) \times \{1\}$. □

II.3 An abstract dynamic programming principle

In this section, we discuss a generalization of the dynamic programming principle I(4.3). Then in this abstract set-up, the notion of viscosity solution to dynamic programming equations will be introduced in the next section.

Let Σ be closed subset of a Banach space and $\mathcal{C}$ be a collection of functions on Σ which is closed under addition, i.e.,

$$\phi, \psi \in \mathcal{C} \Rightarrow \phi + \psi \in \mathcal{C}.$$

The main object of our analysis is a two parameter family of operators $\{\mathcal{T}_{tr} : t_0 \le t \le r \le t_1\}$ with the common domain $\mathcal{C}$. In the applications the exact choice of $\mathcal{C}$ is not important. However, when Σ is compact, we will require that $\mathcal{C}$ contains $C(\Sigma)$. For noncompact Σ, additional conditions are often imposed. In most of our examples, $\Sigma \subset I\!R^n$ and in that case we will require that $\mathcal{C}$ contains $\mathcal{M}(\Sigma) \cap C_p(\Sigma)$. (See end of Section 1 for notations.)

We assume that for every $\phi, \psi \in \mathcal{C}$ and $t_0 \le t \le r \le s \le t_1, \mathcal{T}_{tr}\phi$ is a function on Σ satisfying

$$\mathcal{T}_{tt}\phi = \phi, \tag{3.1}$$

$$\mathcal{T}_{tr}\phi \le \mathcal{T}_{ts}\psi \text{ if } \phi \le \mathcal{T}_{rs}\psi, \tag{3.2a}$$

$$\mathcal{T}_{tr}\phi \ge \mathcal{T}_{ts}\psi \text{ if } \phi \ge \mathcal{T}_{rs}\psi. \tag{3.2b}$$

By taking $r = s$ in (3.2) we conclude that the above conditions imply

$$\mathcal{T}_{tr}\phi \le \mathcal{T}_{tr}\psi \text{ if } \phi \le \psi. \tag{3.2$'$)(monotonicity}$$

Moreover if $\mathcal{T}_{rs}\psi \in \mathcal{C}$, by taking $\phi = \mathcal{T}_{rs}\psi$ both in (3.2a) and (3.2b) we obtain

$$\mathcal{T}_{tr}(\mathcal{T}_{rs}\psi) = \mathcal{T}_{ts}\psi \text{ if } \mathcal{T}_{rs}\psi \in \mathcal{C}. \tag{3.3)(semigroup}$$

In general $T_{tr}\psi$ may not be in $\mathcal{C}$ for every $\psi \in \mathcal{C}$. In that case (3.2) is a convenient way of stating the monotonicity and the semigroup conditions. Also (3.2) is a slightly more general condition than (3.2$'$) and (3.3). But when $\mathcal{T}_{tr} : \mathcal{C} \to \mathcal{C}$ for every t and r, (3.2$'$) and (3.3) are equivalent to (3.2). We will use this equivalence in the following examples.

Example 3.1. (Deterministic Optimal Control). Let f, L, g be as in Section I.3, and O be an open subset of $I\!R^n$. Set $\Sigma = \overline{O}$ and $\mathcal{C} = \mathcal{M}(\Sigma)$. We assume that L and g are also bounded from below. Fix $t_0 \le t \le r \le t_1, x \in \Sigma, u(\cdot) \in \mathcal{U}(t,x)$, and $\psi \in \mathcal{C}$. Define

$$T_{t,r;u}\psi(x) = \int_t^{\tau \wedge r} L(s, x(s), u(s))ds + g(\tau, x(\tau))\chi_{\tau < r} + \psi(x(r))\chi_{\tau \ge r},$$

where, as in Section I.3, $\mathcal{U}(t,x)$ is a set of controls satisfying the switching condition I(3.9), $x(\cdot)$ is the solution of I(3.2) and I(3.3), and τ is the exit time of $(s,x(s))$ from $\overline{Q} = [t_0,t_1] \times \overline{O}$. Then the nonlinear semigroup is given by

$$(\mathcal{T}_{t,r}\psi)(x) = \inf_{u(\cdot)\in\mathcal{U}(t,x)} T_{t,r;u}\psi(x). \tag{3.4}$$

Since L, g and ψ are all bounded from below, $\mathcal{T}_{tr}\psi$ is also bounded from below. Note that $\mathcal{U}(t,x)$ is nonempty and therefore for every $\psi \in \mathcal{C}, \mathcal{T}_{tr}\psi$ is well-defined and belongs to $\mathcal{C}$. Clearly $\mathcal{T}_{tr}$ is monotone. Also with the notation of Section I.3, $V(t,x) = (\mathcal{T}_{tt_1}\psi)(x)$. Thus, the dynamic programming principle I(4.3) can be rewritten as

$$(\mathcal{T}_{tt_1}\psi)(x) = (\mathcal{T}_{tr}(\mathcal{T}_{rt_1}\psi))(x),$$

for $(t,x) \in \overline{Q}$, and $\psi \in \mathcal{C}$. Hence the semigroup property, (3.3), holds with $s = t_1$. In fact without any change, the proofs of Lemma I.4.1 and I.4.2 yield (3.3) for any $s \in [t,t_1]$. □

Example 3.2. (Diffusion semigroup). Set $\Sigma = I\!R^n$ and $\mathcal{C} = C_p(I\!R^n)$. For $\psi \in \mathcal{C}, x \in \Sigma$, and $t_0 \le t < r \le t_1$, let

$$(\mathcal{T}_{tr}\psi)(x) = \int_{I\!R^n} \psi(x+z)K(r-t,z)dz, \tag{3.5}$$

where for $h > 0, z \in I\!R^n$,

$$K(h,z) = \frac{1}{(4\pi h)^{n/2}} e^{-|z|^2/4h}.$$

And for $t = r$, let $\mathcal{T}_{tt}$ be the identity operator. Since ψ is polynomially growing, there are constants $K, m \ge 0$ such that,

$$|\psi(x+z)| \le K(1+|x|^m+|z|^m).$$

Also for each $h > 0$ and $k \ge 0, |z|^k K(h,z)$ is an integrable function on $I\!R^n$. Hence $\mathcal{T}_{tr}\psi \in \mathcal{C}$.

The positivity of K implies (3.2′). Also the semigroup property of $\mathcal{T}_{tr}$ is well known. Indeed (3.3) is proved by using the properties of the kernel K and several changes of variables in (3.5).

The linear operator $\mathcal{T}_{tr}$ has a probabilistic representation in terms of the standard Brownian motion. In Chapters III and IV, instead of the Brownian motion we will use controlled diffusions to obtain nonlinear operators, which are generalizations of both (3.4) and (3.5). See Section V.3. □

Example 3.3. (Poisson process). Let Σ be the set of nonnegative integers, $\mathcal{C}$ be the set of all sequences $\{\psi(i) : i \in \Sigma\}$ such that $|\psi(i)|$ grows at most polynomially as i tends to infinity, and $\lambda(\cdot)$ be a continuous, strictly positive function on $[t_0,t_1]$. For $t_0 \le t \le r \le t_1, i \in \Sigma$, and $\psi \in \mathcal{C}$ define

$$(3.6) \qquad (\mathcal{T}_{tr}\psi)(i) = \sum_{k=0}^{\infty} \psi(i+k)\frac{1}{k!}\left[\int_t^r \lambda(u)du\right]^k \exp\left(-\int_t^r \lambda(u)du\right).$$

Then it is straightforward to verify that $\mathcal{T}_{tr}$ satisfies (3.1) and (3.2′). Also, for $t_0 \le t \le r \le s \le t_1$ and $i \in \Sigma$,

$$\begin{aligned}(\mathcal{T}_{tr}(\mathcal{T}_{rs}\psi))(i) &= \textstyle\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \psi(i+k+m)\frac{1}{k!}\frac{1}{m!}[\int_t^r \lambda(u)du]^k \cdot \\ &\cdot [\textstyle\int_r^s \lambda(u)du]^m \exp(-\int_t^s \lambda(u)du) \\ &= \textstyle\sum_{\ell=0}^{\infty} \psi(i+\ell)\frac{1}{\ell!}\exp(-\int_t^s \lambda(u)du)\cdot \\ &\cdot \left\{ \textstyle\sum_{k=0}^{\ell} \frac{\ell!}{k!(\ell-k)!}[\int_t^r \lambda(u)du]^k [\int_r^s \lambda(u)du]^{\ell-k} \right\} \\ &= \textstyle\sum_{\ell=0}^{\infty} \psi(i+\ell)\frac{1}{\ell!}\exp(-\int_t^s \lambda(u)du)[\int_t^s \lambda(u)du]^\ell \\ &= (\mathcal{T}_{ts}\psi)(i).\end{aligned}$$

We used the substitution $\ell = k+m$ in the second step and the Binomial Theorem in the third. Hence $\mathcal{T}_{tr}$ has the semigroup property (3.3).

This linear operator $\mathcal{T}_{tr}$ also has a probabilistic representation. Instead of the Brownian motion, a time inhomogeneous Poisson process appears in the representation of (3.6). Also the nonlinear generalizations of τ_{tr} it are related to the optimal control of jump Markov processes. □

Fix $r = t_1$ and $\psi \in \mathcal{C}$. For $(t,x) \in [t_0, t_1] \times \Sigma$, define

$$(3.7) \qquad V(t,x) = (\mathcal{T}_{tt_1}\psi)(x).$$

If $\mathcal{T}_{tt_1}$ is as in (3.4), $V(t,x)$ is the value function of the deterministic optimal control problem. In analogy with this, $V(t,x)$ is called the *value function.* Using the semigroup property (3.3), we conclude that the value function satisfies

$$(3.8) \qquad V(t,x) = (\mathcal{T}_{tr}V(r,\cdot))(x), \quad \forall x \in \Sigma, t_0 \le t \le r \le t_1,$$

provided that $V(r,\cdot) \in \mathcal{C}$. This identity is just a restatement of the dynamic programming principle I(4.3), if $\mathcal{T}_{tr}$ is as in (3.4). Hence, we refer to (3.8) as the (abstract) *dynamic programming principle.*

Having formulated the dynamic programming principle abstractly, we proceed to derive the corresponding dynamic programming equation. As in Section I.5, let $r = t+h$ in (3.8). Assume that $V(t+h,\cdot) \in \mathcal{C}$. Then

$$(3.9) \qquad -\frac{1}{h}[(\mathcal{T}_{tt+h}V(t+h,\cdot))(x) - V(t,x)] = 0,$$

for all $x \in \Sigma$ and $t_0 \le t < t+h \le t_1$. To continue even formally, we need to assume that the above quantity has a limit as $h \downarrow 0$, when V is “smooth”. So we assume that there exist $\Sigma' \subset \Sigma, \mathcal{D} \subset C([t_0,t_1) \times \Sigma')$ and a one-parameter family of nonlinear operators $\{\mathcal{G}_t\}_{t\in[t_0,t_1]}$ of functions of Σ, satisfying the following conditions with

$$Q = [t_0,t_1) \times \Sigma',$$

(3.10i) For every $w \in \mathcal{D}$, $\frac{\partial}{\partial t}w(t,x)$ and $(\mathcal{G}_t w(t,\cdot))(x)$ are continuous on $(t,x) \in Q$, and $w(t,\cdot) \in \mathcal{C}$ for all $t \in [t_0,t_1]$,

(3.10ii) $w, \tilde{w} \in \mathcal{D}, \lambda \geq 0 \Rightarrow w + \tilde{w} \in \mathcal{D}, \lambda w \in \mathcal{D}$,

$$\lim_{h\downarrow 0} \frac{1}{h}[(\mathcal{T}_{tt+h}w(t+h,\cdot))(x) - w(t,x)] = \frac{\partial}{\partial t}w(t,x) - (\mathcal{G}_t w(t,\cdot))(x), \tag{3.11}$$

for all $w \in \mathcal{D}, (t,x) \in Q$. We refer to the elements of $\mathcal{D}$ as *test functions* and $\mathcal{G}_t$ as the *infinitesimal generator* of the semigroup $\{\mathcal{T}_{tr}\}$. Note that, if w is any test function, then $w(t,x)$ is defined for all $(t,x) \in [t_0,t_1] \times \Sigma$ even though (3.11) is required to hold only for $(t,x) \in Q$.

Like the choice of $\mathcal{C}$, the exact choice of $\mathcal{D}$ is not important. (See Section 6, below.) One should think of $\mathcal{D}$ as the set of "smooth" functions. For example if $\Sigma' = O$ is a bounded subset of $I\!R^n$ and $\Sigma = \overline{O}$, then we require that $\mathcal{D}$ contains $C^\infty(\bar{Q})$, where $\bar{Q} = [t_0,t_1] \times \overline{O}$. Indeed this requirement will be typical when $\mathcal{G}_t$ is a partial differential operator.

If however, $\Sigma' = O$ is an unbounded subset of $I\!R^n$ with $\Sigma = \overline{O}$, then we require that

$$\mathcal{M}(\overline{Q}) \cap C_p^\infty(\overline{Q}) \subset \mathcal{D}.$$

(See end of Section 1 for notations.)

In most applications, Σ' is simply the interior of Σ. In fact, when $\mathcal{G}_t$ is a partial differential operator, this is always the case. However, in the case of a controlled jump Markov process which is stopped after the exit from an open set $O \subset I\!R^n$, we have $\Sigma' = O$, whereas Σ is the closure of the set that can be reached from O. In that case, Σ may be larger than $\overline{O}$, and $[t_0,t_1] \times \Sigma$ is larger than $\overline{Q}$.

Now suppose that $V \in \mathcal{D}$ and let h go to zero in (3.9) to obtain,

$$-\frac{\partial}{\partial t}V(t,x) + (\mathcal{G}_t V(t,\cdot))(x) = 0, \quad (t,x) \in Q. \tag{3.12}$$

Definition 3.1. $V \in \mathcal{D}$ is called a *classical solution* of (3.12) if V satisfies (3.12) for all $(t,x) \in Q$.

In analogy with I(5.3), the above equation is called the (abstract) *dynamic programming equation.*

In general, the value function is not in $\mathcal{D}$ and therefore it is *not* a classical solution of (3.12). In that case the equation (3.12) has to be interpreted in a weaker sense (as a viscosity solution). This will be the subject of the next section.

We continue by verifying the assumption (3.11) for the special cases (3.4), (3.5) and (3.6). For the deterministic optimal control problem in Example 3.1, (3.11) is formally satisfied by

$$
\begin{aligned}
(\mathcal{G}_t\phi)(x) &= H(t,x,D\phi(x)) \\
&= \sup_{v\in U}\{-f(t,x,v)\cdot D\phi(x) - L(t,x,v)\}
\end{aligned}
\tag{3.13}
$$

with $\mathcal{D} = C^1(Q)\cap\mathcal{M}(\overline{Q})$ and $\Sigma' = O$. A rigorous verification of this will be given in Section 7. Note that in this case, (3.12) is the same as I(5.3′).

Now consider Example 3.2, with $\mathcal{T}_{tr}$ is as in (3.5). Let $\Sigma' = O = I\!R^n$ and

$$\mathcal{D} = \{w \in C_p(\overline{Q}_0) \text{ and } w_t, w_{x_i}, w_{x_i x_j} \in C_p(Q_0) \text{ for } i,j = 1,\cdots,n\},$$

where $Q_0 = [t_0,t_1)\times I\!R^n$. Here the subscript denotes the differentiation with respect to that variable. Fix $w\in\mathcal{D}, (t,x)\in[t_0,t_1)\times I\!R^n$. For $\psi\in\mathcal{C}$ and $t < t+h \le t_1$ we have,

$$
\begin{aligned}
(\mathcal{T}_{tt+h}\psi)(x) &= \int_{I\!R^n}\psi(x+z)K(h,z)dz, \\
&= \int_{I\!R^n}\psi(x+\sqrt{h}y)K(1,y)dy,
\end{aligned}
$$

and similarly

$$(\mathcal{T}_{tt+h}\psi)(x) = \int_{I\!R^n}\psi(x-\sqrt{h}y)K(1,y)dy.$$

Since $\int K(1,y)dy = 1$, we obtain

$$
\begin{aligned}
&\frac{1}{h}(\mathcal{T}_{tt+h}w(t+h,\cdot))(x) - w(t,x)] = \\
&= (\mathcal{T}_{t,t+h}[\tfrac{w(t+h,\cdot)-w(t,\cdot)}{h}])(x) + \tfrac{1}{h}[(\mathcal{T}_{t,t+h}w(t,\cdot))(x) - w(t,x)] \\
&= \int_0^1\int_{I\!R^n} w_t(t+\rho h, x+\sqrt{h}y)K(1,y)dyd\rho + \int_{I\!R^n} M(t,x;y)K(1,y)dy,
\end{aligned}
$$

with

$$M(t,x;y) = \frac{1}{2h}[w(t,x+\sqrt{h}y) + w(t,x-\sqrt{h}y) - 2w(t,x)].$$

Observe that M converges to $D_x^2 w(t,x)y\cdot y/2$ as $h\downarrow 0$, where $D_x^2 w(t,x)$ is the Hessian matrix of second order spatial derivatives of w at (t,x). Using the polynomial growth and the continuity of w_t and $D_x^2 w$, and the exponential decay of $K(1,y)$, we obtain

$$
\begin{aligned}
&\lim_{h\downarrow 0}\frac{1}{h}[(\mathcal{T}_{tt+h}w(t+h,\cdot))(x) - w(t,x)] \\
&= \int_0^1\int_{I\!R^n} w_t(t,x)K(1,y)dyd\rho + \frac{1}{2}\int_{I\!R^n} D_x^2 w(t,x)y\cdot yK(1,y)dy \\
&= w_t(t,x) + \Delta w(t,x),
\end{aligned}
$$

where $\Delta w(t,x)$ is Laplacian of w at (t,x). Hence (3.11) is satisfied with

$$(\mathcal{G}_t\phi)(x) = -\Delta\phi(x) \quad , \; (t,x) \in Q_0. \tag{3.14}$$

In Chapters IV and V, on controlled Markov diffusions, $\mathcal{G}_t$ will be a general, fully nonlinear, second order, elliptic partial differential operator. We also remark that the choice of $\mathcal{D}$ is not the largest possible one. A more careful analysis yields that the polynomial growth condition on the derivatives of w is not necessary. However $\mathcal{D}$ contains $C_p^\infty(Q_0)$ and therefore it is large enough for our purposes.

We close this section by analyzing the operator (3.6) in Example 3.3. Observe that for any $k \geq 2$,

$$\lim_{h\downarrow 0} \frac{1}{h}(\int_t^{t+h} \lambda(s)ds)^k \exp{(-\int_t^{t+h} \lambda(s)ds)} = 0,$$

and for $k = 1$ the above limit is $\lambda(t)$. Now using the identity

$$\sum_{k=0}^{\infty} \frac{\Lambda^k}{k!} e^{-\Lambda} = 1, \quad \forall \Lambda \geq 0,$$

together with the dominated convergence theorem, we obtain

$$\lim_{h\downarrow 0}\frac{1}{h}[(\mathcal{T}_{tt+h}\phi)(i) - \phi(i)] =$$

$$\lim_{h\downarrow 0}\sum_{k=1}^{\infty}[\phi(i+k) - \phi(i)]\frac{1}{k!}\frac{1}{h}(\int_t^{t+h} \lambda(s)ds)^k \exp(-\int_t^{t+h} \lambda(s)ds)$$

$$= \lambda(t)[\phi(i+1) - \phi(i)].$$

Hence

$$\mathcal{G}_t\phi(i) = -\lambda(t)[\phi(i+1) - \phi(i)], \; i = 0, 1, 2, \ldots, \tag{3.15}$$

with $\Sigma' = \Sigma$. Note that the above operator $\mathcal{G}_t$ is a nonlocal operator.

II.4 Definition

In this section, we define the notion of *viscosity solutions* of the abstract dynamic programming equation (3.12). This is a straightforward generalization of the original definition given by Crandall and Lions [CL1]. Also see Crandall, Evans and Lions [CEL] and Lions [L4]. Let $Q = [t_0, t_1) \times \Sigma', \mathcal{D}$, and $\mathcal{C}$ be as in Section 3.

Definition 4.1. *Let $W \in C([t_0, t_1] \times \Sigma)$. Then*

(i) W is a *viscosity subsolution* of (3.12) in Q if for each $w \in \mathcal{D}$

$$-\frac{\partial}{\partial t} w(\bar{t}, \bar{x}) + (\mathcal{G}_{\bar{t}} w(\bar{t}, \cdot))(\bar{x}) \leq 0, \tag{4.1}$$

at every $(\bar{t}, \bar{x}) \in Q$ which is a maximizer of $W - w$ on $[t_0, t_1] \times \Sigma$ with $W(\bar{t}, \bar{x}) = w(\bar{t}, \bar{x})$.

(ii) W is a *viscosity supersolution* of (3.12) in Q if for each $w \in \mathcal{D}$

$$-\frac{\partial}{\partial t} w(\bar{t}, \bar{x}) + (\mathcal{G}_{\bar{t}} w(\bar{t}, \cdot))(\bar{x}) \geq 0, \tag{4.2}$$

at every $(\bar{t}, \bar{x}) \in Q$ which is a minimizer of $W - w$ on $[t_0, t_1] \times \Sigma$ with $W(\bar{t}, \bar{x}) = w(\bar{t}, \bar{x})$.

(iii) W is a *viscosity solution* of (3.12) in Q if it is both a viscosity subsolution and a viscosity supersolution of (3.12) in Q.

Any classical solution of (3.12) is also a viscosity solution. See Lemma 5.1.

Although $\mathcal{D}$ is the natural set to use in the above definition, in most applications an equivalent definition is obtained by replacing $\mathcal{D}$ with $C^\infty(Q)$. Indeed this is always the case when $\mathcal{G}_t$ is a partial differential operator. See Theorem 6.1 below. However, if Σ is not compact and $\mathcal{G}_t$ is not a partial differential operator, additional requirements will be needed to obtain a similar equivalent definition.

Remark 4.1. Consider the equation

$$-\frac{\partial}{\partial t} W(t, x) + (\mathcal{G}_t W(t, \cdot))(x) = 0, \quad (t, x) \in Q$$

with a nonlinear operator $\mathcal{G}_t$, which is *not* necessarily the infinitesimal generator of a two-parameter semigroup $\mathcal{T}_{tr}$. Set

$$\mathcal{D} = \{W \in C([t_0, t_1] \times \Sigma) : W_t(t, x), (\mathcal{G}_t W(t, \cdot))(x) \in C(Q)\}.$$

Then we say the W is a *classical solution* of the above equation if $W \in \mathcal{D}$ and if W satisfies the equation at every $(t, x) \in Q$. The viscosity solutions can also be defined as in Definition 4.1. But to have a meaningful theory of viscosity solutions, the notion of a viscosity solution has to be consistent with the notion of classical solutions. That is: every classical solution should also be a viscosity solution. Indeed, if $\mathcal{G}_t$ is the infinitesimal generator of a two-parameter semigroup, there is a stronger connection between classical and viscosity solutions (see Lemma 5.1.) When $\mathcal{G}_t$ is a general operator, an elementary argument shows that every classical solution is also a viscosity solution if $\mathcal{G}_t$ has a property which we call the *maximum principle*: we say that $\mathcal{G}_t$ has the maximum principle if for every $t \in [t_0, t_1]$, and ϕ, ψ in the domain of $\mathcal{G}_t$, we have

$$(\mathcal{G}_t \phi)(\bar{x}) \geq (\mathcal{G}_t \psi)(\bar{x}),$$

whenever $\bar{x} \in \arg\max\{(\phi - \psi)(x) | x \in \Sigma\} \cap \Sigma'$ with $\phi(\bar{x}) = \psi(\bar{x})$. The class of equations that obey the maximum principle is an interesting one. Any infinitesimal generator of a two parameter semigroup $\mathcal{T}_{tr}$ has this property.

Let us now consider the case when $\mathcal{G}_t$ is a partial differential operator. We take $\Sigma' = O, \Sigma = \overline{O}$ where $O \subset I\!\!R^n$ is open. If $\mathcal{G}_t$ is a first order operator, namely

$$(\mathcal{G}_t\phi)(x) = H(t, x, D\phi(x))$$

for some continuous function H, then $\mathcal{G}_t$ has the maximum principle. More generally, if

$$(\mathcal{G}_t\phi)(x) = F(t, x, D\phi(x), D^2\phi(x), \phi(x)) \tag{4.3)(i}$$

for some continuous function F, then $\mathcal{G}_t$ obeys the maximum principle if and only if F is elliptic (possibly degenerate), i.e.,

$$F(t, x, p, A + B, V) \leq F(t, x, p, A, V) \tag{4.3)(ii}$$

for all $(t, x) \in Q, p \in I\!\!R^n, V \in I\!\!R$ and symmetric matrices A, B with $B \geq 0$. Indeed, let $\phi, \psi \in C^2(O)$ and

$$\overline{x} \in \arg\max\{(\phi - \psi)(x) | x \in \overline{O}\} \cap O,$$

with $\phi(\overline{x}) = \psi(\overline{x})$. By calculus $D\phi(\overline{x}) = D\psi(\overline{x})$ and $D^2\phi(\overline{x}) \leq D^2\psi(\overline{x})$. Hence for every $t \in [t_0, t_1)$,

$$F(t, \overline{x}, D\phi(\overline{x}), D^2\phi(\overline{x}), \phi(\overline{x})) \geq F(t, \overline{x}, D\psi(\overline{x}), D^2\psi(\overline{x}), \psi(\overline{x}))$$

if F satisfies (4.3)(ii). Conversely, suppose that F does not satisfy (4.3)(ii) at some $(t, \overline{x}) \in Q$ and $(p, A, V), B > 0$. Define

$$\psi(x) = V + p \cdot (x - \overline{x}) + \frac{1}{2}(A + B)(x - \overline{x}) \cdot (x - \overline{x}), x \in \overline{O},$$

$$\phi(x) = V + p \cdot (x - \overline{x}) + \frac{1}{2}A(x - \overline{x}) \cdot (x - \overline{x}) \quad , x \in \overline{O}.$$

Then $\overline{x} \in \arg\max\{(\phi - \psi)(x) | x \in \overline{O}\} \cap O$ and $\phi, \psi \in C^2(\overline{O})$. So we conclude that if (4.3)(ii) does not hold in $Q, \mathcal{G}_t$ does not have maximum principle. Hence (4.3)(ii) is equivalent to the maximum principle property of F.

We will now give an equivalent definition of a viscosity subsolution and a supersolution of nonlinear partial differential equations.

Definition 4.2. Let O be an open subset of $I\!\!R^n, Q = [t_0, t_1) \times O, W \in C(\overline{O})$ and F be a continuous function satisfying (4.3)(ii). For $(t, x) \in Q$ consider the equation

$$-\frac{\partial}{\partial t}W(t, x) + F(t, x, D_x W(t, x), D_x^2 W(t, x), W(t, x)) = 0. \tag{4.4}$$

(a) W is a *viscosity subsolution* of (4.4) in Q if for each $w \in C^\infty(Q)$,

$$-\frac{\partial}{\partial t}w(\overline{t}, \overline{x}) + F(\overline{t}, \overline{x}, D_x w(\overline{t}, \overline{x}), D_x^2 w(\overline{t}, \overline{x}), w(\overline{t}, \overline{x})) \leq 0 \tag{4.5}$$

at every $(\bar{t}, \bar{x}) \in Q$ which is a local maximum of $W - w$ on $\overline{Q}$, with $W(\bar{t}, \bar{x}) = w(\bar{t}, \bar{x})$.

(b) W is a *viscosity supersolution* of (4.4) in Q if for each $w \in C^\infty(Q)$,

$$-\frac{\partial}{\partial t} w(\bar{t}, \bar{x}) + F(\bar{t}, \bar{x}, D_x w(\bar{t}, \bar{x}), D_x^2 w(\bar{t}, \bar{x}), w(\bar{t}, \bar{x})) \geq 0 \tag{4.6}$$

at every $(\bar{t}, \bar{x}) \in Q$ which is a local minimum of $W - w$ on $\overline{Q}$, with $W(\bar{t}, \bar{x}) = w(\bar{t}, \bar{x})$.

(c) W is a *viscosity solution* of (4.4) in Q if it is both a viscosity subsolution and a viscosity supersolution of (4.4) in Q.

When the infinitesimal generator $\mathcal{G}_t$ is given by (4.3)(i), Definitions 4.1 and 4.2 are equivalent provided that $W \in C_p(\overline{Q}) \cap \mathcal{M}(\overline{Q})$; see Theorem 6.1, below. This equivalence follows from the local character of $\mathcal{G}_t$ and the fact that every test function $w \in \mathcal{D}$ can be approximated by functions from $C^\infty(Q)$. This approximation property also enables us to use test functions $w \in \mathcal{D}$ or $w \in C^\infty(Q)$ or w from any other space which is dense in $\mathcal{D}$; see Remark 6.1 below. Other properties of the second order equations will be the subject of Chapters IV and V on optimal control of diffusion processes.

Remark 4.2. The viscosity property is often stated in the following slightly different way, which does not require $W(\bar{t}, \bar{x}) = w(\bar{t}, \bar{x})$. A viscosity subsolution W of (4.4) in Q satisfies

$$-\frac{\partial}{\partial t} w(\bar{t}, \bar{x}) + F(\bar{t}, \bar{x}, D_x w(\bar{t}, \bar{x}), D_x^2 w(\bar{t}, \bar{x}), W(\bar{t}, \bar{x})) \leq 0$$

at every $(\bar{t}, \bar{x}) \in Q$ which is a local maximum of $W - w$ on $\overline{Q}$. Indeed, define

$$\overline{w}(t, x) = w(t, x) + W(\bar{t}, \bar{x}) - w(\bar{t}, \bar{x}) \quad , \quad (t, x) \in \overline{Q}.$$

Then $(\bar{t}, \bar{x})$ is a local maximum of $W - \overline{w}$ on $\overline{Q}$, and $W(\bar{t}, \bar{x}) = \overline{w}(\bar{t}, \bar{x})$. A similar result also holds for supersolutions.

If the function F in (4.4) does not depend on $W(t, x)$, i.e. $F = F(t, x, p, A)$, then (4.5), (4.6) are unchanged if w is replaced by $w + c$ with c any constant. In this case, this distinction is irrelevant.

Remark 4.3. Definition 4.1 is a weak formulation of sub and supersolutions of (3.12). However, as we have already discussed in Chapter I, the value function of a deterministic control problem satisfies certain boundary conditions on $\{t_1\} \times \overline{O}$ and $[t_0, t_1) \times \partial O$. In Sections 12 and 13, a weak formulation of these boundary conditions will be given for the deterministic control problems. Also a more general definition which does not require the continuity of W will be given in Section VII.4.

Let us return to Example 2.1 and verify that

$$V(t, x) = \min\{1 - t, 1 - |x|\}$$

is a viscosity solution of (2.1) in $Q = [0, 1) \times (-1, 1)$. Let $w \in C^1(Q)$ and choose $(\bar{t}, \bar{x}) \in \arg\max\{(V - w)(t, x) | (t, x) \in \overline{Q}\} \cap Q$. First suppose that V is differentiable at $(\bar{t}, \bar{x})$ or equivalently $|\bar{x}| \neq \bar{t}$. Then

$$(4.7)(i) \qquad \frac{\partial}{\partial x}V(\bar{t},\bar{x}) = \frac{\partial}{\partial x}w(\bar{t},\bar{x}).$$

Note that $\bar{t}$ may equal to $t_0 = 0$ and in that case the time derivatives of V and w may not be equal. Still we have,

$$(4.7)(ii) \qquad \frac{\partial}{\partial t}V(\bar{t},\bar{x}) \le \frac{\partial}{\partial t}w(\bar{t},\bar{x}).$$

Since V satisfies (2.1) at $(\bar{t},\bar{x})$, the above inequality implies that (4.1) is satisfied with $(G_t\phi)(x) = (\phi'(x))^2 - 1$.

Now suppose that $\bar{t} = |\bar{x}| > 0$. Recall that $(\bar{t},\bar{x})$ is a maximizer of the difference $V - w$. Using the directional derivatives of V at $(\bar{t},\bar{x})$, we conclude that

$$\frac{\partial}{\partial t}w(\bar{t},\bar{x}) + \frac{\bar{x}}{|\bar{x}|}\frac{\partial}{\partial x}w(\bar{t},\bar{x}) = -1,$$

$$\frac{\partial}{\partial t}w(\bar{t},\bar{x}) \ge -1, \frac{\bar{x}}{|\bar{x}|}\frac{\partial}{\partial x}w(\bar{t},\bar{x}) \ge -1.$$

Therefore

$$(\frac{\partial}{\partial t}w(\bar{t},\bar{x}), \frac{\partial}{\partial x}w(\bar{t},\bar{x})) = (-\lambda, (\lambda-1)\frac{\bar{x}}{|\bar{x}|})$$

for some $\lambda \in [0,1]$. Also if $\bar{t} = |\bar{x}| = 0$, then a similar analysis yields

$$(\frac{\partial}{\partial t}w(\bar{t},\bar{x}), \frac{\partial}{\partial x}w(\bar{t},\bar{x})) = (-\lambda+\gamma, (\lambda-1)p)$$

for some $\lambda \in [0,1], \gamma \ge 0$ and $|p| \le 1$. Then

$$-\frac{\partial}{\partial t}w(\bar{t},\bar{x}) + (\frac{\partial}{\partial x}w(\bar{t},\bar{x}))^2 - 1 \le \lambda + (\lambda-1)^2 - 1 - \gamma$$

$$\le \lambda^2 - \lambda \le 0.$$

Hence V is a viscosity subsolution of (2.1).

To verify the supersolution property of V, let $(\bar{t},\bar{x})$ be a minimizer of $V - w$. Since V is a concave function of its variables and w is differentiable, V must be differentiable at $(\bar{t},\bar{x})$. Then we have (4.7)(i) and

$$\frac{\partial}{\partial t}V(\bar{t},\bar{x}) \ge \frac{\partial}{\partial t}w(\bar{t},\bar{x}).$$

Therefore (4.2) is satisfied and V is a viscosity supersolution of (2.1).

Now consider W_k defined in Example 2.2. Recall that W_k is a generalized solution of (2.1). However, we claim that for $k > 1$ W_k is *not* a viscosity supersolution of (2.1), and consequently not a viscosity solution of (2.1). Indeed, let $w \equiv 0$. Then $(\bar{t}, \frac{1}{2k+1})$ with $\bar{t} \in [0,1]$, is one of the several minimizers of $W_k - w = W_k$. But

$$-\frac{\partial}{\partial t}w(\bar{t},\frac{1}{k}) + (\frac{\partial}{\partial x}w(\bar{t},\frac{1}{k}))^2 - 1 = -1,$$

and (4.2) is *not* satisfied.

From the above discussion it is clear that if

$$(\mathcal{G}_t\phi)(x) = H(t,x,D\phi(x)), \quad (t,x) \in [t_0,t_1]\times\Sigma,$$

with some continuous function H, then any viscosity solution of (3.12) satisfies the equation at the points of its differentiability. But in addition, a viscosity solution has to satisfy appropriate inequalities when it is not differentiable. These conditions are summarized in an equivalent pointwise definition that is discussed in Section 8.

II.5 Dynamic programming and viscosity property

Recall that the value function is defined by

$$V(t,x) = (\mathcal{T}_{tt_1}\psi)(x).$$

In this section we will show that the abstract dynamic programming principle (3.8) yields that V is a viscosity solution of the dynamic programming equation (3.12), whenever it is a continuous function on $\hat{Q} = [t_0,t_1]\times\Sigma$. For deterministic optimal control, sufficient conditions for the continuity of the value function are obtained in Section 10 and Theorem 13.1.

Theorem 5.1. *Assume* (3.1), (3.2), (3.10), (3.11). *Suppose that* $V \in C(\hat{Q})$. *Then,* V *is a viscosity solution of* (3.12) *in* Q.

Proof. Let $w \in \mathcal{D}$ and $(\bar{t},\bar{x}) \in Q$ be a maximizer of the difference $V - w$ on $\overline{Q}$ satisfying $V(\bar{t},\bar{x}) = w(\bar{t},\bar{x})$. Then, $w \geq V$. Using (3.2)(b) with $\phi = w(r,\cdot)$ and $s = t_1$, we obtain for every $r \in [\bar{t},t_1]$,

$$(\mathcal{T}_{\bar{t}r}w(r,\cdot))(\bar{x}) \geq (\mathcal{T}_{\bar{t}t_1}\psi)(\bar{x}) = V(\bar{t},\bar{x}) = w(\bar{t},\bar{x}).$$

Recall that by (3.10i) $w(r,\cdot)$ is in the domain of $\mathcal{T}_{\bar{t}r}$. Take $r = \bar{t}+h$ and use (3.11) to arrive at

$$-\frac{\partial}{\partial t}w(\bar{t},\bar{x}) + \mathcal{G}_{\bar{t}}w(\bar{t},\cdot))(\bar{x}) = -\lim_{h\downarrow 0}\frac{1}{h}[(\mathcal{T}_{\bar{t}\,\bar{t}+h}w(\bar{t}+h,\cdot))(\bar{x}) - w(\bar{t},\bar{x})] \leq 0.$$

Hence (4.1) is satisfied and consequently V is a viscosity subsolution of (3.12) in Q. The supersolution property of V is proved *exactly* the same way as the subsolution property. □

We close this section by showing that the notion of viscosity solution is consistent with the notion of a classical solution.

Lemma 5.1. *Suppose that* $W \in \mathcal{D}$. *Then* W *is a viscosity solution of* (3.12) *in* Q *if and only if it is a classical solution of* (3.12).

Proof. First suppose that W is a viscosity solution. Since $W \in \mathcal{D}$, $w \equiv W$ is a test function. Moreover every (t,x) is a maximizer and a minimizer of $W-w$. Hence (4.1) and (4.2) hold at every $(t,x) \in Q$ with $w = W$. Therefore W satisfies (3.12), classically, in Q.

To prove the converse, let $w \in \mathcal{D}$ and $(\bar{t},\overline{x}) \in Q$ be a maximizer of $W-w$ satisfying,

$$w(\bar{t},\overline{x}) = W(\bar{t},\overline{x}).$$

Using (3.11) and the monotonicity property (3.2′), together with the inequality $w \geq W$, we obtain

$$\begin{aligned} -\frac{\partial}{\partial t} w(\bar{t},\overline{x}) + (\mathcal{G}_{\bar{t}} w(\bar{t},\cdot))(\overline{x}) = \\ &= -\lim_{h\downarrow 0}\frac{1}{h}[(\mathcal{T}_{\bar{t}\,\bar{t}+h} w(\bar{t}+h,\cdot))(\overline{x}) - w(\bar{t},\overline{x})] \\ &\leq -\lim_{h\downarrow 0}\frac{1}{h}[(\mathcal{T}_{\bar{t}\,\bar{t}+h} W(\bar{t}+h,\cdot))(\overline{x}) - W(\bar{t},\overline{x})] \\ &= -\frac{\partial}{\partial t} W(\bar{t},\overline{x}) + (\mathcal{G}_{\bar{t}} W(\bar{t},\cdot))(\overline{x}) = 0. \end{aligned}$$

Hence W is a viscosity subsolution (3.12). The supersolution property is proved similarly. □

II.6 Properties of viscosity solutions

In this section we obtain equivalent definitions and a stability result (Lemma 6.2), when $\mathcal{G}_t$ is a partial differential operator. A stability result for a general class of equations is also proved at the end of the section. See Lemma 6.3.

Let $\mathcal{G}_t$ be a partial differential operator given by (4.3)(i), $\Sigma' = O$ be an open subset of $I\!R^n$ and $\Sigma = \overline{O}$. We also require that the collections $\mathcal{C}$ and $\mathcal{D}$ of functions are chosen large enough that

$$\text{(6.1)}(i) \qquad C_p(\overline{O}) \cap \mathcal{M}(\overline{O}) \subset \mathcal{C},$$

$$\text{(6.1)}(ii) \qquad C_p^\infty(\overline{Q}) \cap \mathcal{M}(\overline{Q}) \subset \mathcal{D}.$$

As noted earlier, if O is bounded, then (6.1) requires merely that $C(\overline{O}) \subset \mathcal{C}$ and $C^\infty(\overline{Q}) \subset \mathcal{D}$.

Lemma 6.1. *Assume* (6.1). *Let* $\mathcal{G}_t$ *be a partial differential operator as in* (4.3)(i). *Then in Definitions* 4.1 *and* 4.2, *it suffices to consider only the strict extrema of* $W-w$.

Proof. Suppose that (4.2) holds at every strict minimum $(\bar{t},\overline{x}) \in Q$ of $W-w$ with $W(\bar{t},\overline{x}) = w(\bar{t},\overline{x})$. Let $(\bar{t},\overline{x}) \in Q$ be a minimum (not necessarily strict) of $W - w$ satisfying $W(\bar{t}, \bar{x}) = w(\bar{t}, \bar{x})$. For $\varepsilon > 0$, define $w^\varepsilon = w + \varepsilon\xi$, where

$$\xi(t,x) = e^{-(|t-\bar{t}|^2 + |x-\bar{x}|^2)} - 1 \quad , (t,x) \in \overline{Q}.$$

Clearly $\varepsilon\xi \in C_p^\infty(\overline{Q}) \cap \mathcal{M}(\overline{Q})$, and by (6.1)(ii), $\varepsilon\xi \in \mathcal{D}$. Then by (3.10)(ii) $w^\varepsilon \in \mathcal{D}$. Also for every $\varepsilon > 0$, $W - w^\varepsilon$ has a *strict* minimum at $(\bar{t}, \bar{x})$ with $W(\bar{t}, \bar{x}) = w^\varepsilon(\bar{t}, \bar{x})$. Hence

$$-\frac{\partial}{\partial t} w^\varepsilon(\bar{t}, \bar{x}) + F(\bar{t}, \bar{x}, D_x w^\varepsilon(\bar{t}, \bar{x}), D_x^2 w^\varepsilon(\bar{t}, \bar{x}), W(\bar{t}, \bar{x})) \leq 0.$$

We now obtain (4.2) by letting $\varepsilon \downarrow 0$ in the above inequality.

Hence in Definition 4.1(ii) it suffices to consider only strict minima of $W - w$. The remaining assertions are proved similarly. □

Next we will prove that for a partial differential equation, Definitions 4.1 and 4.2 are indeed equivalent. A brief outline of this proof is the following: we first approximate any test function w by a sequence $w^n \in C^\infty(Q)$ satisfying

$$(w_t^n, \mathcal{G}_t w^n) \to (w_t, \mathcal{G}_t w), \text{ as } n \to \infty,$$

locally uniformly. Hence we only need to consider test functions which are in $C^\infty(Q)$. Also, if $\mathcal{G}_t$ is a partial differential operator as in (4.3)(i), then

$$(\mathcal{G}_t \phi)(x_0) = (\mathcal{G}_t \psi)(x_0)$$

if $\phi = \psi$ in a neighborhood of x_0. This local behavior of $\mathcal{G}_t$ enables us to consider only the local extrema in our definition.

Theorem 6.1. *Assume* (6.1). *Let* $W \in C_p(\overline{Q}) \cap \mathcal{M}(\overline{Q}), \mathcal{G}_t$ *be a partial differential operator as in* (4.3)(i) *and* $\mathcal{D} \subset C^{1,2}(Q)$. *Then* W *is a viscosity subsolution (or a supersolution) of* (3.12) *in the sense of Definition* 4.1*, if and only if* W *is a viscosity solution (or a supersolution, respectively) in the sense of Definition* 4.2.

Proof.

(i) (**Necessity**) Suppose that $W \in C_p(\overline{Q}) \cap \mathcal{M}(\overline{Q})$ is a viscosity supersolution of (3.12) in Q. Let $w \in C^\infty(Q)$ and $(\bar{t}, \bar{x}) \in Q$ be a local minimum of $W - w$ with $W(\bar{t}, \bar{x}) = w(\bar{t}, \bar{x})$. Then there exists an open, bounded neighborhood $\mathcal{N}$ of $(\bar{t}, \bar{x})$ such that $W(t,x) \geq w(t,x)$ for all $(t,x) \in \mathcal{N} \cap \overline{Q}$. By Urysohn's Lemma there is $\xi \in C^\infty(\overline{Q})$ satisfying (i) $0 \leq \xi(t,x) \leq 1$, (ii) $\xi = 1$ in a neighborhood $\tilde{\mathcal{N}}$ of $(\bar{t}, \bar{x})$ (iii) $\xi(t,x) = 0, (t,x) \notin \mathcal{N}$. For $(t,x) \in \overline{Q}$, define

$$\tilde{w}(t,x) = \xi(t,x) w(t,x) - (1 - \xi(t,x))K,$$

where $-K$ is a lower bound for W on $\overline{Q}$. Since $W \geq w$ on $\mathcal{N} \cap \overline{Q}$ and $\xi = 0$ outside of $\mathcal{N}$, $\xi W \geq \xi w$ on $\overline{Q}$. Therefore

$$W = \xi W + (1 - \xi) W \geq \xi w - (1 - \xi) K = \tilde{w}.$$

Hence we have

$$(6.2)(i) \qquad w = \tilde{w} \text{ on } \mathcal{N},$$

$$(6.2)(ii) \qquad (\bar{t},\bar{x}) \in \arg\min\{(W-\tilde{w})(t,x) | (t,x) \in \overline{Q}\}.$$

Since $\mathcal{N} \cap Q = \mathcal{N} \cap \overline{Q}$, $w \in C^\infty(\overline{Q})$ and $\xi = 0$ outside $\mathcal{N}$, $\xi w \in C^\infty(\overline{Q})$. Therefore, $\tilde{w} \in C^\infty(\overline{Q})$. Moreover $\tilde{w} \in C^\infty(\overline{Q})$ and equals to a constant outside the compact set $\mathcal{N}$. By (6.1)(ii) $\tilde{w} \in \mathcal{D}$ and

$$-\frac{\partial}{\partial t}\tilde{w}(\bar{t},\bar{x}) + F(\bar{t},\bar{x},D_x\tilde{w}(\bar{t},\bar{x}),D_x^2\tilde{w}(\bar{t},\bar{x}),W(\bar{t},\bar{x})) \geq 0.$$

Due to (6.2)(i), all derivatives of $\tilde{w}$ and w agree at $(\bar{t},\bar{x})$ and the above inequality gives (4.6).

Now let $W \in C_p(\overline{Q}) \cap \mathcal{M}(\overline{Q})$ be a viscosity subsolution of (3.12) in Q, and $(\bar{t},\bar{x})$ be a local maximum of $W - w$ with $W(\bar{t},\bar{x}) = w(\bar{t},\bar{x})$. Then $w(t,x) > W(t,x)$ on $\mathcal{N} \cap \overline{Q}$, where $\mathcal{N}$ is an open, bounded neighborhood of $(\bar{t},\bar{x})$ satisfying $\mathcal{N} \cap Q = \mathcal{N} \cap \overline{Q}$. Since W is polynomially growing, there are a constant $K \geq 0$ and an integer $m \geq 0$ satisfying

$$|W(t,x)| \leq K(1+|x|^{2m}), \quad (t,x) \in \overline{Q}.$$

Let ξ be as before and for $(t,x) \in \overline{Q}$, define

$$\overline{w}(t,x) = \xi(t,x)w(t,x) + (1-\xi(t,x))K(1+|x|^{2m}).$$

Then $\overline{w} \in C_p^\infty(\overline{Q}) \cap \mathcal{M}(\overline{Q})$ and satisfies $w = \bar{w}$ on $\mathcal{N}$, and

$$(\bar{t},\bar{x}) \in \arg\max\{(W-\overline{w})(t,x) | (t,x) \in \overline{Q}\}.$$

Continuing as in the supersolution case, we obtain (4.5).

(ii) (**Sufficiency**) Let W be a viscosity solution according to Definition 4.2. Let $w \in \mathcal{D}$ and $(\bar{t},\bar{x}) \in Q$ be a maximum of $W - w$ with $W(\bar{t},\bar{x}) = w(\bar{t},\bar{x})$. In view of Lemma 6.1, we may assume that $(\bar{t},\bar{x})$ is a strict maximum. Since $\mathcal{D} \subset C^{1,2}(Q)$, $w \in C^{1,2}(Q)$. Therefore there is an open set $Q^* \supset \overline{Q}$ such that $w \in C^{1,2}(Q^*)$. As in Appendix C, extend w to $\mathbb{R}^{n+1}$ by setting it equal to zero on $\mathbb{R}^{n+1} \backslash Q^*$. Let $w^n \in C^\infty(\overline{Q})$ be a mollification of the extension of w. Since $w \in C^{1,2}(Q^*)$ and $Q^* \supset \overline{Q}$, $w^n, w^n_t, w^n_{x_i}, w^n_{x_ix_j}$ converge to $w, w_t, w_{x_i}, w_{x_ix_j}$, uniformly on compact subsets of Q, as $n \to \infty$ (see Appendix C.) The uniform convergence of w^n to w implies that there is $(t_n,x_n) \to (\bar{t},\bar{x})$, as $n \to \infty$, such that (t_n,x_n) is a local maximum of $W - w^n$. Set $\overline{w}^n(t,x) = w^n(t,x) + W(t_n,x_n) - w^n(t_n,x_n)$. Then (t_n,x_n) is a local maximum of $W - \overline{w}^n$ with $W(t_n,x_n) = \overline{w}^n(t_n,x_n)$. Also for sufficiently large $n, (t_n,x_n) \in Q$. So the hypotheses of the lemma imply

$$-\frac{\partial}{\partial t}\overline{w}(t_n,x_n) + F(t_n,x_n,D_x\overline{w}(t_n,x_n),D_x^2\overline{w}(t_n,x_n),W(t_n,x_n)) \leq 0.$$

Now let n go to infinity to arrive at (4.1). Therefore W is a subsolution of (3.12) in Q. The supersolution property is proved similarly. □

Remark 6.1. Take $\mathcal{C} = \mathcal{C}(\overline{\mathcal{O}})$ and $\mathcal{D} = C^{1,2}(Q) \cap \mathcal{M}(\overline{Q})$ or $\mathcal{D} = C^{1,2}(\overline{Q}) \cap \mathcal{M}(\overline{Q})$. Then the above theorem implies that Definition 4.1 with either choice of $\mathcal{D}$, and Definition 4.2 are equivalent for any partial differential equation with maximum principle. From the above proof it is also clear that in Definition 4.2 we could use test functions from any function space between $C^{1,2}(Q)$ and $C^\infty(Q)$. Therefore when applying the viscosity property or when proving such a property, we have the freedom to choose the test function w from $\mathcal{D}$ or $C^\infty(Q)$ or any other dense subset of $C^{1,2}(Q)$ (for example $C^{1,2}(\overline{Q})$.) Indeed this flexibility to choose w from several different classes will be used in the sequel to simplify the presentation. Also in the sequel we will not specify $\mathcal{C}$ and $\mathcal{D}$, since all choices of $\mathcal{C}$ and $\mathcal{D}$ satisfying (6.1) and (3.10)(i) yield equivalent definitions.

Now suppose that $\mathcal{G}_t$ is a first order partial differential operator, namely,

$$(\mathcal{G}_t\phi)(x) = H(t, x, D\phi(x))$$

for $(t,x) \in \overline{Q}$ and $\phi \in C^1(\overline{O})$. Then the proof of Theorem 6.1 implies that the definition obtained by using $\mathcal{C} = \mathcal{M}(\overline{O})$ and $\mathcal{D} = C^1(Q) \cap \mathcal{M}(\overline{Q})$ or $\mathcal{D} = C^1(\overline{Q}) \cap \mathcal{M}(\overline{Q})$ in Definition 4.1 is equivalent to Definition 4.2. Again this equivalence enables us to use test functions w from $\mathcal{D}$ or $C^\infty(Q)$ or any other dense subset of $C^1(Q)$ (eg. $C^1(Q)$ or $C^1(\overline{Q})$). In Sections 8 and 9 we shall use $w \in C^1(Q)$.

Stability Results. Next we will prove a stability property of viscosity solutions. This property states that if the viscosity solutions W^ε of approximate equations depending on a small parameter ε are uniformly convergent as $\varepsilon \to 0$, then the limiting function W is a viscosity solution of the limit equation. Notice that the equation (3.12) is in many cases of interest a nonlinear partial differential equation. Therefore, the importance of this property is in the fact that only the convergence of the solutions, but not their derivatives, is required. See Chapter VII for a more detailed discussion of this subject.

Let W^ε be a viscosity subsolution (or supersolution) of

$$(6.3)^\varepsilon \qquad -\frac{\partial}{\partial t}W^\varepsilon(t,x) + (\mathcal{G}_t^\varepsilon W^\varepsilon(t,\cdot))(x) = 0$$

in Q. One typical example of this situation is

$$(\mathcal{G}_t^\varepsilon\phi)(x) = -\varepsilon\Delta\phi(x) + H(t,x,D\phi(x)),$$

where $H(t,x,p)$ is as in (3.13). But there are other interesting examples such as,

$$(\mathcal{G}_t^\varepsilon\phi)(x) = \sup_{v\in U}\left\{ -\frac{1}{\varepsilon}\int_{R^n}[\phi(x+\varepsilon y) - \phi(x)]G(t,x,y,v)dy - L(t,x,v)\right\},$$

where $G(t,x,y,v) \geq 0$ and $yG(t,x,y,v)$ is an integrable function of $y \in I\!R^n$, for every $(t,x) \in \overline{Q}$ and $v \in U$.

When both $(6.3)^\varepsilon$ and (3.12) are partial differential equations, the proof of the stability result is simpler. So we give two stability results, the second of which applies to all dynamic programming equations.

Lemma 6.2. (**Stability**) *Let W^ε be a viscosity subsolution (or a supersolution) of*

$$-\frac{\partial}{\partial t}W^\varepsilon(t,x) + F^\varepsilon(t,x,D_xW^\varepsilon(t,x),D_x^2W^\varepsilon(t,x),W^\varepsilon(t,x)) = 0$$

in Q, with some continuous function F^ε satisfying the ellipticity condition (4.3)(ii). *Suppose that F^ε converges to F, uniformly on every compact subset of its domain, and W^ε converges to W, uniformly on compact subsets of $\overline{Q}$. Then W is a viscosity subsolution (or a supersolution, respectively) of the limiting equation.*

Proof. Let $w \in C^\infty(Q)$ and $(\bar{t},\bar{x}) \in Q$ be a strict maximizer of $W-w$ with $W(\bar{t},\bar{x}) = w(\bar{t},\bar{x})$. Since W^ε converges to W uniformly on compact subsets of $\overline{Q}$, there exists a sequence $(t_\varepsilon,x_\varepsilon) \to (\bar{t},\bar{x})$, as $\varepsilon \to 0$, such that $(t_\varepsilon,x_\varepsilon)$ is a local maximum of $W^\varepsilon - w$. Set $\tilde{w}(t,x) = w(t,x) + W^\varepsilon(t_\varepsilon,x_\varepsilon) - w(t_\varepsilon,x_\varepsilon)$. Then $(t_\varepsilon,x_\varepsilon)$ is a local maximum of $W^\varepsilon - \tilde{w}$ and $W^\varepsilon(t_\varepsilon,x_\varepsilon) = \tilde{w}(t_\varepsilon,x_\varepsilon)$. Hence the viscosity property W^ε implies that

$$-\frac{\partial}{\partial t}\tilde{w}(t_\varepsilon,x_\varepsilon) + F^\varepsilon(t_\varepsilon,x_\varepsilon,D_x\tilde{w}(t_\varepsilon,x_\varepsilon),D_x^2\tilde{w}(t_\varepsilon,x_\varepsilon),W^\varepsilon(t_\varepsilon,x_\varepsilon)) \leq 0.$$

Let ε go to zero and use the uniform convergence of F^ε to F, to conclude that W is a viscosity subsolution of the limiting equation (4.4). Once again, the supersolution property is proved similarly.

Lemma 6.3. (**Stability**) *Suppose that for every $w, \xi \in \mathcal{D}$, $(t,x) \in \overline{Q}$ and a positive, continuous function h, with $h(0) = 0$,*

$$\lim_{\substack{\varepsilon\downarrow 0\\ (s,y)\to(t,x)\\ (s,y)\in\overline{Q}}} (\mathcal{G}_s^\varepsilon[w(s,\cdot) + h(\varepsilon)\xi(s,\cdot)])(y) = (\mathcal{G}_t w(t,\cdot))(x). \tag{6.4}$$

Let W^ε be a viscosity subsolution (or supersolution) of $(6.3)^\varepsilon$ *in Q. Further assume that there is a nonnegative function $\eta \in \mathcal{D}$ such that $(W^\varepsilon - W)/(1+\eta)$ converges to zero as $\varepsilon \to 0$, uniformly on $\overline{Q}$. Then W is a viscosity subsolution (or supersolution, respectively) of* (3.12) *in Q.*

Before the proof of the lemma, let us briefly discuss its hypotheses. The condition (6.4) is a direct analogue of the local uniform convergence assumption made on F^ε in Lemma 6.2. The uniform convergence assumption on $(W^\varepsilon - W)/(1+\eta)$ however, is in general stronger than the assumption made in Lemma 6.2. Recall that in Lemma 6.2 we only assumed the local uniform convergence of W^ε to W. This minor restriction is caused by the possible non-local character of $\mathcal{G}_t$.

Proof. Let $w \in \mathcal{D}$ and $(\bar{t}, \overline{x}) \in Q$ be a maximizer of $W - w$ on $\overline{Q}$, with $W(\bar{t}, \overline{x}) = w(\bar{t}, \overline{x})$. Set

$$\xi(t,x) = 1 - \exp(-|t-\bar{t}|^2 - |x-\overline{x}|^2), \ \ (t,x) \in \overline{Q},$$

$$h(\varepsilon) = \sup\left\{ \frac{W^\varepsilon(t,x) - W(t,x)}{1+\eta(t,x)} : (t,x) \in \overline{Q} \right\},$$

and for $\delta > 0$,

$$\tilde{w}(t,x) = w(t,x) + h(\varepsilon)[1+\eta(t,x)] + \delta\xi(t,x), \quad (t,x) \in \overline{Q}.$$

By (3.10)(ii) and (6.1)(ii), $\tilde{w} \in \mathcal{D}$. Consider the difference

$$I^\varepsilon(t,x) = W^\varepsilon(t,x) - \tilde{w}(t,x).$$

Since $W \le w$ on $\overline{Q}$, for $(t,x) \in \overline{Q}$,

$$\begin{aligned} I^\varepsilon(t,x) &= W(t,x) - w(t,x) \\ &\quad + (W^\varepsilon(t,x) - W(t,x) - h(\varepsilon)[1+\eta(t,x)]) - \delta\xi(t,x) \le -\delta\xi(t,x) \end{aligned}$$

Recall that $W(\bar{t}, \overline{x}) = w(\bar{t}, \overline{x})$ and $\xi(\bar{t}, \overline{x}) = 0$. Hence the above inequality yields

$$\begin{aligned} I^\varepsilon(t,x) &\le I^\varepsilon(\bar{t}, \overline{x}) - \delta\xi(t,x) - (W^\varepsilon(\bar{t}, \overline{x}) - W(\bar{t}, \overline{x}) - h(\varepsilon)[1+\eta(\bar{t}, \overline{x})]) \\ &\le I^\varepsilon(\bar{t}, \overline{x}) - \delta\xi(t,x) + 2h(\varepsilon)[1+\eta(\bar{t}, \overline{x})]. \end{aligned}$$

Since $h(\varepsilon) \to 0$ as $\varepsilon \to 0$, and $\xi(t,x)$ is bounded away from zero outside any neighborhood of $(\bar{t}, \overline{x})$, the above inequality implies that there are $(t_\varepsilon, x_\varepsilon) \in \arg\max\{I^\varepsilon(t,x) | (t,x) \in \overline{Q}\} \to (\bar{t}, \overline{x})$ as $\varepsilon \to 0$. Therefore the viscosity property of W^ε yields,

$$-\frac{\partial}{\partial t}\tilde{w}(t_\varepsilon, x_\varepsilon) + (\mathcal{G}_{t_\varepsilon}[\tilde{w}(t_\varepsilon, \cdot) + K_\varepsilon]) \le 0,$$

where $K_\varepsilon = W^\varepsilon(t_\varepsilon, x_\varepsilon) - \tilde{w}(t_\varepsilon, x_\varepsilon)$. Now let ε and then δ go to zero and use (6.4), to arrive at (4.1). The supersolution property is proved similarly. □

II.7 Deterministic optimal control and viscosity solutions

In this section, we prove that the value function defined by I(4.1) is a viscosity solution of the dynamic programming equation I(5.3′) under two sets of assumptions. In the first result (Theorem 7.1) the control space U is assumed bounded. The second result (Theorem 7.2) has a simpler proof, and does not require that U is bounded. However, in Theorem 7.2 it is assumed that an optimal control exists.

We start with a brief review of the notation and the assumptions of Section I.3 and Example 3.1. Let O be an open subset of $I\!R^n$, $\Sigma = \overline{O}, \mathcal{C} = \mathcal{M}(\Sigma)$ (see end of Section 1 for the notation) $Q = [t_0, t_1) \times O$ and $Q_0 = [t_0, t_1) \times I\!R^n$. We assume that $f, L \in C(\overline{Q}_0 \times U)$, where U is the control space, and $g \in C(\overline{Q}_0)$. We also assume that f satisfies I(3.1), and L and g are bounded from below. Let $\mathcal{U}(t,x)$ be a set of controls satisfying I(5.2), and the switching condition I(3.9). Then for $\psi \in \mathcal{C}, t_0 \le t \le r \le t_1$ and $x \in \overline{O}$, the semigroup is given by

$$(\mathcal{T}_{tr}\psi)(x) = \inf_{u(\cdot) \in \mathcal{U}(t,x)} \left[\int_t^{r \wedge \tau} L(s, x(s), u(s)) ds + g(\tau, x(\tau))\chi_{\tau < r} \right.$$

$$\left. + \psi(x(r))\chi_{\tau \ge r} \right],$$

where $x(\cdot)$ is the solution of I(3.2), I(3.3) and τ is the exit time of $(s, x(s))$ from the region $\overline{Q}$.

In Example 3.1, it is shown that $\mathcal{T}_{tr}$ satisfies (3.1), (3.2), and the value function is given by

$$V(t,x) = (\mathcal{T}_{tt_1}\psi)(x), \quad (t,x) \in \overline{Q}. \tag{7.1}$$

Then in view of Theorem 5.1 and Remark 6.1, V is a viscosity solution of the dynamic programming equation provided that $V \in C(\overline{Q})$ and the semigroup satisfies (3.11). That is, for every $(t,x) \in Q$ and $w \in C^1(Q) \cap \mathcal{M}(\overline{Q})$,

$$\lim_{h \downarrow 0} \frac{1}{h}[(\mathcal{T}_{tt+h} w(t+h, \cdot))(x) - w(t,x)] = \tag{7.2}$$

$$= \frac{\partial}{\partial t} w(t,x) - H(t, x, D_x w(t,x)),$$

where for $(t,x) \in \overline{Q}$, $p \in I\!R^n$,

$$H(t,x,p) = \sup_{v \in U} \{-f(t,x,v) \cdot p - L(t,x,v)\}. \tag{7.3}$$

We next verify (7.2) under the following assumptions. We assume that for a suitable constant K,

(7.4) (i) U is bounded,

(7.4) (ii) $|f(t,x,v)| \le K(1 + |x|), \qquad \forall (t,x) \in \overline{Q}, v \in U.$

In fact, (7.4)(ii) follows from the boundedness of U and I(3.1).

Theorem 7.1. *Suppose that* (7.4) *is satisfied. Then,* (7.2) *holds for every* $w \in C^1(Q) \cap \mathcal{M}(\overline{Q})$ *and* $(t,x) \in Q$. *In particular, the value function is a viscosity solution of the dynamic programming equation* I(5.3′) *in* Q *provided that* $V \in C(\overline{Q})$.

Proof. First observe that the viscosity property of the value function follows from (7.2), and Theorem 5.1. (See Remark 6.1.) Fix $(t,x) \in Q, v \in U$ and $w \in C^1(Q) \cap \mathcal{M}(\overline{Q})$. In view of I(5.2), there exists an admissible control

$u(\cdot) \in \mathcal{U}(t,x)$ satisfying $\lim_{s\downarrow t} u(s) = v$. Let $x(\cdot)$ be the state corresponding to control $u(\cdot)$ and the initial condition $x(t) = x$. Let τ be the exit time of $(s, x(s))$ from Q. Since $x \in O$ and $t < t_1, \tau > t$. Therefore for sufficiently small $h > 0$, we have $x(s) \in O$ for all $s \in [t, t+h)$ and $t+h < \tau$. Then the definition of $\mathcal{T}_{tt+h}$ yields

$$\frac{1}{h}[(\mathcal{T}_{tt+h}w(t+h,\cdot))(x) - w(t,x)]$$

$$\le \frac{1}{h}\int_t^{t+h} L(s,x(s),u(s))ds + \frac{1}{h}[w(t+h,x(t+h)) - w(t,x)]$$

$$= \frac{1}{h}\int_t^{t+h} \left[L(s,x(s),u(s)) + \frac{\partial}{\partial t}w(s,x(s)) + \frac{d}{ds}x(s)\cdot D_x w(s,x(s))\right] ds$$

$$= \frac{1}{h}\int_t^{t+h} \Big[L(s,x(s),u(s)) + \frac{\partial}{\partial t}w(s,x(s))$$

$$+ f(s,x(s),u(s))\cdot D_x w(s,x(s))\Big]\, ds.$$

Since $(x(s), u(s))$ converges to (x, v) as $s \downarrow t$, the continuity of L and f, and the smoothness of w yield

$$\limsup_{h\downarrow 0} \frac{1}{h}[(\mathcal{T}_{tt+h}w(t+h,\cdot))(x) - w(t,x)]$$

$$\le \frac{\partial}{\partial t}w(t,x) - \{-f(t,x,v)\cdot D_x w(t,x) - L(t,x,v)\}.$$

The above inequality holds for every $v \in U$. Therefore,

$$\text{(7.5)} \qquad \limsup_{h\downarrow 0} \frac{1}{h}[(\mathcal{T}_{tt+h}w(t+h,\cdot))(x) - w(t,x)]$$

$$\le \frac{\partial}{\partial t}w(t,x) - H(t,x,D_x w(t,x)).$$

For later use in the proof of Theorem 7.2, we note that (7.4) has not been used in the derivation of (7.5).

Fix $(t,x) \in Q, u(\cdot) \in \mathcal{U}(t,x)$. Let $x(\cdot)$ be the solution of I(3.2) and I(3.3). For any $r > t$, we integrate I(3.2) over $[t,r]$ and then invoke (7.4)(ii) to obtain,

$$|x(r) - x| \le \int_t^r |f(s,x(s),u(s))|ds$$

$$\le K(r-t) + K\int_t^r |x(s)|ds$$

$$\le K(r-t)(1+|x|) + K\int_t^r |x(s) - x|ds.$$

Then, Gronwall's inequality yields

$$|x(r) - x| \leq (1+|x|)[\exp K(r-t) - 1] \quad , \quad \forall r \geq t. \tag{7.6}$$

Choose $r^* > t$ satisfying,

$$(1+|x|)[\exp K(r^*-t) - 1] = \text{ distance } (x, \partial O).$$

Hence $x(s) \in O$ for all $s \in [t, r^*)$ and consequently

$$\tau \geq r^* \wedge t_1 \quad , \quad \forall u(\cdot) \in \mathcal{U}(t,x).$$

In particular, $\tau \geq t + \frac{1}{n}$ for every $u(\cdot) \in \mathcal{U}(t,x)$ and sufficiently large n. So for n large enough, we may choose an admissible control $u^n(\cdot) \in \mathcal{U}(t,x)$ satisfying

$$\left(\mathcal{T}_{tt+\frac{1}{n}} w\left(t+\frac{1}{n}, \cdot\right)\right)(x) \geq -\frac{1}{n^2} + \int_t^{t+\frac{1}{n}} L(s, x^n(s), u^n(s))ds + w\left(t+\frac{1}{n}, x^n\left(t+\frac{1}{n}\right)\right), \tag{7.7}$$

where $x^n(s)$ denotes the state corresponding to the control $u^n(\cdot)$. We now estimate that

$$\begin{aligned} & n\left[\left(\mathcal{T}_{tt+\frac{1}{n}} w\left(t+\frac{1}{n}, \cdot\right)\right)(x) - w(t,x)\right] \\ & \geq -\frac{1}{n} + n\int_t^{t+\frac{1}{n}} L(s, x^n(s), u^n(s))ds + n\left[w\left(t+\frac{1}{n}, x^n\left(t+\frac{1}{n}\right)\right) - w(t,x)\right] \\ & = -\frac{1}{n} + n\int_t^{t+\frac{1}{n}} \left[L(s, x^n(s), u^n(s)) + \frac{\partial}{\partial t} w(s, x^n(s)) + \right. \\ & \qquad \left. f(s, x^n(s), u^n(s)) \cdot D_x w(s, x^n(s))\right] ds \\ & = \frac{\partial}{\partial t} w(t,x) + n\int_t^{t+\frac{1}{n}} L(t, x, u^n(s))ds \\ & \qquad + (n\int_t^{t+\frac{1}{n}} f(t, x, u^n(s))ds) \cdot D_x w(t,x) + e(n), \end{aligned} \tag{7.8}$$

where

$$\begin{aligned} e(n) = & -\frac{1}{n} + n\int_t^{t+\frac{1}{n}} \left(\frac{\partial}{\partial t} w(s, x^n(s)) - \frac{\partial}{\partial t} w(t,x)\right) ds \\ & + n\int_t^{t+\frac{1}{n}} (L(s, x^n(s), u^n(s)) - L(t, x, u^n(s)))ds \end{aligned} \tag{7.9}$$

$$+n\int_t^{t+\frac{1}{n}}[f(s,x^n(s),u^n(s))\cdot D_xw(s,x^n(s))-f(t,x,u^n(s))\cdot D_xw(t,x)]ds.$$

The estimate (7.6) implies that $x^n(s)\to x$, as $s\to t$, uniformly in n. Therefore the continuity of L and f yields

$$\lim_{n\to\infty} e(n)=0.$$

We now rewrite (7.8) as

$$n\left[\left(\mathcal{T}_{tt+\frac{1}{n}}w\left(t+\frac{1}{n},\cdot\right)\right)(x)-w(t,x)\right]\geq\frac{\partial}{\partial t}w(t,x)-\{-F^n\cdot D_xw(t,x)-L^n\}+e(n),$$

where

$$(F^n,L^n)=n\int_t^{t+\frac{1}{n}}(f(t,x,u^n(s)),L(t,x,u^n(s)))ds.$$

Set

$$FL(t,x)=\{(f,\ell)\in R^{n+1}:(f,\ell)=(f(t,x,v),L(t,x,v))\ \text{ for some }\ v\in U\}.$$

Then (F^n,L^n) belongs to the convex closed hull, $\overline{co}[FL(t,x)]$, of the set $FL(t,x)$. Consequently,

$$\frac{1}{n}\left[\left(\mathcal{T}_{tt+\frac{1}{n}}w\left(t+\frac{1}{n},\cdot\right)\right)(x)-w(t,x)\right]$$

$$\geq\frac{\partial}{\partial t}w(t,x)-\sup\{-f\cdot D_xw(t,x)-\ell:(f,\ell)\in\overline{co}[FL(t,x)]\}+e(n)$$

$$=\frac{\partial}{\partial t}w(t,x)-H(t,x,D_xw(t,x))+e(n).$$

Here we have used the fact that maximization of a linear function over a set yields the same value as the maximization over the convex closed hull of the same set. We now complete the proof of (7.2) after recalling that $\lim_{n\to\infty} e(n)=0$. □

The proof of Theorem 7.1 uses the abstract viscosity solution framework in Section 3, and does not depend on the existence of an optimal control. We next obtain a similar result assuming that optimal controls exist, but that condition (7.4) need not hold. The proof uses the alternate Definition 4.2 of viscosity solutions for PDEs.

Theorem 7.2. *Assume that an optimal control $u^*(\cdot)\in\mathcal{U}(t,x)$ exists, for each $(t,x)\in Q$. Then the value function is a viscosity solution of the dynamic programming equation* I(5.3′) *in Q provided that $V\in C(\overline{Q})$.*

Proof. An argument like the proof of (7.5) above shows that V is a viscosity subsolution of I(5.3′). To show that V is a viscosity supersolution, suppose that $w\in C^\infty(Q)$ and that $V-w$ has a local minimum at $(\bar{t},\bar{x})\in Q$ with $w(\bar{t},\bar{x})=V(\bar{t},\bar{x})$. Let $u^*(\cdot)$ be optimal for initial data $(\bar{t},\bar{x})$, and let $x^*(\cdot)$

be the corresponding state with $x^*(\bar{t}) = \bar{x}$. By the dynamic programming principle, for small enough $h > 0$,

$$V(\bar{t},\bar{x}) = \int_{\bar{t}}^{\bar{t}+h} L(s,x^*(s),u^*(s))ds + V(\bar{t}+h, x^*(\bar{t}+h)),$$

$$0 \geq \int_{\bar{t}}^{\bar{t}+h} L(s,x^*(s),u^*(s))ds + w(\bar{t}+h, x^*(\bar{t}+h)) - w(\bar{t},\bar{x}).$$

Let $F(s,y,v,p) = f(s,y,v)\cdot p + L(s,y,v)$. Then

$$\begin{aligned} 0 &\geq \int_{\bar{t}}^{\bar{t}+h} \left[\frac{\partial}{\partial t} w(s,x^*(s)) + F(s,x^*(s),u^*(s),D_x w(s,x^*(s))\right] ds \\ &\geq \int_{\bar{t}}^{\bar{t}+h} \left[\frac{\partial}{\partial t} w(s,x^*(s)) - H(s,x^*(s),D_x w(s,x^*(s))\right] ds. \end{aligned}$$

We let $h \downarrow 0$ and obtain

$$-\frac{\partial w}{\partial t}(\bar{t},\bar{x}) + H(\bar{t},\bar{x},D_x w(\bar{t},\bar{x})) \geq 0.$$

Thus V is a viscosity supersolution. □

Concerning the existence of optimal controls needed in Theorem 7.2, we refer to Section I.11.1 and to [FR, Chapter 3].

II.8 Viscosity solutions: first order case

In this section, we consider the viscosity sub and supersolutions of a first order partial differential equation

$$(8.1) \qquad -\frac{\partial}{\partial t}V(t,x) + H(t,x,D_x V(t,x)) = 0, \quad (t,x) \in Q,$$

where H is a continuous function of $\overline{Q} \times I\!R^n$. We do *not* assume that H is necessarily related to an optimal control problem. Recall that the notion of viscosity solution applies to equation (8.1); see Definition 4.2.

We start our discussion with a definition.

Definition 8.1. Let $W \in C(\overline{Q})$ and $(t,x) \in Q$.

(a) The set of *superdifferentials* of W at (t,x), denoted by $D^+W(t,x)$, is the collection of all $(q,p) \in I\!R \times I\!R^n$ satisfying

$$(8.2)(i) \qquad (q,p) = \left(\frac{\partial}{\partial t}w(t,x), D_x w(t,x)\right)$$

$$(8.2)(ii) \qquad (t,x) \in \arg\max\{(W-w)(s,y) | (s,y) \in \overline{Q}\},$$

for some $w \in C^1(Q)$.
The set of *subdifferentials* of W at (t,x), denoted by $D^-W(t,x)$, is the collection of all $(q,p) \in I\!R \times I\!R^n$ satisfying

$$(8.3)(i) \qquad (q,p) = \left(\frac{\partial}{\partial t} w(t,x), D_x w(t,x) \right),$$

$$(8.3)(ii) \qquad (t,x) \in \arg\min\{(W-w)(s,y) | (s,y) \in \overline{Q}\},$$

for some $w \in C^1(Q)$.

In view of Remark 6.1, W is a viscosity subsolution of (8.1) in Q if and only if

$$(8.4) \qquad -q + H(t,x,p) \leq 0, \quad \forall (p,q) \in D^+W(t,x), (t,x) \in Q.$$

Similarly W is a viscosity supersolution of (8.1) in Q if and only if

$$(8.5) \qquad -q + H(t,x,p) \geq 0, \quad \forall (q,p) \in D^-w(t,x), (t,x) \in Q.$$

Next, we give different characterizations of the sets $D^+W(t,x)$ and $D^-W(t,x)$.

Lemma 8.1. *Let $W \in C(\overline{Q})$ and $(t,x) \in Q$. Then,*

$$(8.6) \qquad D^+W(t,x) = \Bigg\{ (q,p) \in I\!R^{n+1} \ : \ \limsup_{\substack{(s,y)\to(t,x) \\ (s,y)\in\overline{Q}}} \frac{W(s,y) - W(t,x) - q(s-t) - p\cdot(x-y)}{|s-t| + |x-y|} \leq 0 \Bigg\}$$

$$(8.7) \qquad D^-W(t,x) = \Bigg\{ (q,p) \in I\!R^{n+1} \ : \ \liminf_{\substack{(s,y)\to(t,x) \\ (s,y)\in\overline{Q}}} \frac{W(s,y) - W(t,x) - q(s-t) - p\cdot(x-y)}{|s-t| + |x-y|} \geq 0 \Bigg\}.$$

Proof. Fix $(t,x) \in Q$. Let $(q,p) \in D^+W(t,x)$ and $w \in C^1(Q)$ be a test function satisfying (8.2). Then by (8.2)(ii),

$$W(s,y) - w(s,y) \leq W(t,x) - w(t,x),$$

for every $(s,y) \in \overline{Q}$. Hence

$$\limsup_{\substack{(s,y)\to(t,x) \\ (s,y)\in\overline{Q}}} \frac{W(s,y) - W(t,x) - q(s-t) - p\cdot(y-x)}{|t-s| + |x-y|}$$

$$\leq \limsup_{\substack{(s,y)\to(t,x)\\(s,y)\in\overline{Q}}} \frac{w(s,y)-w(t,x)-q(s-t)-p\cdot(y-x)}{|t-s|+|x-y|} = 0.$$

On the other hand, let (q,p) be an element of the right hand side of (8.6). For $(s,y)\in\overline{Q}$, define

$$w(s,y) = q(s-t) + p\cdot(y-x) + \int_{\alpha(s,y)}^{2\alpha(s,y)} h(\rho)d\rho,$$

where

$$\alpha(s,y) = (|t-s|^2 + |x-y|^2)^{\frac{1}{2}}$$

and for $\rho > 0$,

$$h(\rho) = \sup\{[\frac{W(s,y)-W(t,x)-q(s-t)-p\cdot(y-x)}{|s-t|+|x-y|} \wedge 0] :$$

$$(s,y)\in\overline{Q}, 0<\alpha(s,y)\leq\rho\},$$

with $h(0)=0$. Then $h\in C([0,\infty))$, because (q,p) belongs to the right hand side of (8.6). Consequently $w\in C^1(Q)$. Moreover, it is straightforward to check that w satisfies (8.2). Therefore $(q,p)\in D^+W(t,x)$. See Proposition V.4.1 for a more detailed proof of a similar result. The identity (8.7) is proved similarly. □

Using the above characterization of $D^+W(t,x)$ and $D^-W(t,x)$, we obtain the following result

Corollary 8.1. *Let $W\in C(\overline{Q})$.*

(a) *$D^+W(t,x)$ and $D^-W(t,x)$ are convex for every $(t,x)\in\overline{Q}$.*

(b) *Suppose that $(t,x)\in(t_0,t_1)\times O$. Then W is differentiable at (t,x), in the sense of Section* I.6, *if and only if both $D^+W(t,x)$ and $D^-W(t,x)$ are non-empty. When W is differentiable, we have*

$$(8.8)\qquad D^+W(t,x) = D^-W(t,x) = \left\{\left(\frac{\partial}{\partial t}W(t,x), D_xW(t,x)\right)\right\}.$$

(c) *Let W be a viscosity solution of* (8.1) *in Q. Then W satisfies* (8.1), *classically, at every point of differentiability $(t,x)\in(t_0,t_1)\times O$.*

(d) *Let $N\subset(t_0,t_1)\times O$ be an open, convex set and let W be convex on N. Then for every $(t,x)\in N$, $D^-W(t,x)$ is nonempty and it is equal to the set of subdifferentials in the sense of convex analysis (see Rockafellar* [R1].)

(e) *Let N be as in part* (d), *and W be concave on N. Then $D^+W(t,x)$ is nonempty for every $(t,x)\in N$.*

(f) *Suppose that W is Lipschitz continuous on $\overline{Q}$, i.e.*

$$(8.9)\qquad |W(s,y)-W(t,x)| \leq M_1|x-y| + M_2|t-s|,\ \ (s,y),(t,x)\in\overline{Q},$$

for a suitable constants M_1, M_2. Then,

$$|p| \le M_1, |q| \le M_2 \tag{8.10}$$

for every $(q,p) \in D^+W(t,x) \cup D^-W(t,x)$ *and* $(t,x) \in (t_0,t_1) \times O$.

Proof. The convexity of the sets of sub and superdifferentials follow easily from Lemma 8.1. Also if W is differentiable at $(t,x) \in (t_0,t_1) \times O$, then (8.8) follows immediately. Now suppose that $(q,p) \in D^+W(t,x)$ and $(\overline{q},\overline{p}) \in D^-W(t,x)$, at some $(t,x) \in (t_0,t_1) \times O$. Using (8.6) and (8.7), we obtain

$$\limsup_{(h,z)\to 0} \frac{(\overline{p}-p)\cdot z + (\overline{q}-q)h}{|h|+|z|} \le 0.$$

Choose $(h,z) = \rho(\overline{q}-q, \overline{p}-p)$ with $\rho > 0$ and then let ρ go to zero to obtain $|p-\overline{p}| = |q-\overline{q}| = 0$. Therefore W is differentiable at (t,x) and (8.8) holds.

Suppose W is convex on N. Then, for $(t,x) \in N$ the set of subdifferentials in the sense of convex analysis [R1] is given by

$$\partial W(t,x) = \{(q,p) \in I\!R^{n+1} \;:\; W(s,y) - W(t,x) - q(s-t) - p\cdot(y-x) \ge 0, \quad \forall (s,y) \in N\}.$$

It is well known that this set is non-empty at every $(t,x) \in N$. Also, it is clear that

$$\partial W(t,x) \subset D^-W(t,x),$$

at every $(t,x) \in N$. We claim that the above holds with an equality at every $(t,x) \in N$. Indeed, let $(q,p) \notin \partial W(t,x)$. Then, there exists $(\overline{s},\overline{y}) \in N$ satisfying

$$W(\overline{s},\overline{y}) - W(t,x) - q\overline{h} - p\cdot\overline{z} = -\delta < 0$$

where $\overline{h} = \overline{s} - t$, and $\overline{z} = \overline{y} - x$. For every integer n, set

$$(s_n, y_n) = (t,x) + \frac{1}{2^n}(\overline{h}, \overline{z}).$$

The convexity of W yields,

$$\begin{aligned} W(s_1,y_1) - W(t,x) &\le \frac{1}{2}(W(\overline{s},\overline{y}) - W(t,x)) \\ &= \frac{1}{2}(q\overline{h} + p\cdot\overline{z}) - \frac{1}{2}\delta \\ &= q(s_1 - t) + p\cdot(y_1 - x) - \delta\frac{|s_1-t| + |y_1-x|}{|\overline{h}| + |\overline{z}|}. \end{aligned}$$

By induction, the above argument yields

$$W(s_n,y_n) - W(t,x) \le q(s_n - t) + p\cdot(y_n - x) - \delta\frac{|s_n-t| + |y_n-x|}{|\overline{h}| + |\overline{z}|},$$

for every n. Therefore

$$\liminf_{\substack{(s,y)\to(t,x)\\(s,y)\in\overline{Q}}} \frac{W(s,y)-W(t,x)-q(s-t)-p\cdot(y-x)}{|s-t|+|y-x|}$$

$$\leq \liminf_{n\to\infty} \frac{W(s_n,y_n)-W(t,x)-q(s_n-t)-p\cdot(y_n-x)}{|s_n-t|+|y_n-x|} \leq -\frac{\delta}{|\overline{h}|+|\overline{z}|} < 0.$$

Hence $(q,p)\notin D^-W(t,x)$. Part (e) is proved by using the identity

$$D^+W(t,x) = -D^-[-W](t,x),$$

for every $(t,x)\in Q$. Finally, let W be Lipschitz continuous on $\overline{Q}, (t,x)\in(t_0,t_1)\times O$ and $(q,p)\in D^+W(t,x)$. Let us show that $|p|\leq M_1$. The proof that $|q|\leq M_2$ is similar. Then, (8.6) and (8.9) yield,

$$\begin{aligned}-M_1+|p| &= -M_1+\sup\left\{-\frac{p\cdot z}{|z|} : z\in I\!R^n\right\}\\ &= -M_1+\limsup_{z\to 0}\frac{-p\cdot z}{|z|}\\ &\leq \limsup_{z\to 0}\frac{W(t,x+z)-W(t,x)-p\cdot z}{|z|}\leq 0.\end{aligned}$$

Hence $|p|\leq M_1$. Similarly, $|q|\leq M_2$, which implies (8.10) for every $(q,p)\in D^+W(t,x)$. The case $(q,p)\in D^-W(t,x)$ is proved similarly. □

Remark 8.1. A Lipschitz continuous function W of $\overline{Q}$, is called *semi-concave*, if for every bounded subset B of Q, there exists a constant $K(B)$ such that

$$\overline{W}(t,x) = W(t,x) - K(B)(t^2+|x|^2)$$

is concave on every convex subset of B. W is called *semi-convex* if $-W$ is semi-concave.

When W is semi-concave by part (e) of the above corollary, $D^+\overline{W}(\overline{t},\overline{x})$ is nonempty. Also by part (b), we conclude that $D^-\overline{W}(\overline{t},\overline{x})$ is nonempty if and only if $\overline{W}$ is differentiable at $(\overline{t},\overline{x})$. Observe that for any $(t,x)\in\overline{Q}$

$$(q,p)\in D^{\mp}\overline{W}(t,x) \Longleftrightarrow (q+2K(B)t, p+2K(B)x)\in D^{\mp}W(t,x).$$

Hence, if W is semi-concave then for every $(\overline{t},\overline{x})\in(t_0,t_1)\times O,\ D^+W(\overline{t},\overline{x})$ is non-empty and $D^-W(\overline{t},\overline{x})$ is non-empty if and only if W is differentiable at $(\overline{t},\overline{x})$.

The semi-concavity is a natural property of a large class of value functions. In the special case of controlled diffusions on $I\!R^n$, the semi-concavity of the value function in the spatial variable is proved in Section IV.9; see IV(9.7). Also see [CS]. □

Corollary 8.2. *Let W be a viscosity subsolution (or a supersolution) of* (8.1) *in Q. Suppose W satisfies* (8.9). *Then W is a viscosity subsolution (or a supersolution, respectively) of*

$$-\frac{\partial}{\partial t}W(t,x)+\overline{H}(t,x,D_xW(t,x))=0, \tag{8.11}$$

in Q if

$$\overline{H}(t,x,p)=H(t,x,p)\quad,\quad \forall (t,x)\in\overline{Q} \text{ and } |p|\le M_1, \tag{8.12}$$

where M_1 is the constant appearing in (8.9).

Proof. Let $(q,p)\in D^+W(t,x)$ for some $(t,x)\in Q$. Then (8.4) yields

$$-q+H(t,x,p)\le 0.$$

Also, (8.10) is satisfied. In particular, $|p|\le M_1$ and by (8.12),

$$-q+\overline{H}(t,x,p)=-q+H(t,x,p)\le 0.$$

Hence W is a viscosity subsolution of (8.11). The supersolution case is proved similarly. □

We continue by computing the sets of sub and superdifferentials of several functions. First consider the value function V defined in Example 2.1. Recall that for $(t,x)\in[0,1]\times[-1,1]$,

$$V(t,x)=\min\{1-|x|,1-t\}.$$

If $(t,x)\in(0,1)\times(-1,1)$ and $t\ne|x|$, then V is differentiable and we compute the sub and superdifferentials by (8.8). If $t=|x|\in(0,1)$, then

$$D^+V(t,x)=\{\lambda(-1,0)+(1-\lambda)\left(0,-\frac{x}{|x|}\right)\ :\ \lambda\in[0,1]\}.$$

Since V is concave, $D^-V(t,x)$ is empty if $t=|x|\in(0,1)$.

Now let W_k be as in Example 2.2. For specificity take $k=1$. Then

$$W_1(t,x)=\min\{h_1(x),1-t\}\ ,\ \forall (t,x)\in[0,1]\times[-1,1].$$

Clearly for $(t,x)\in(0,\frac{2}{3})\times(-1,1)$, $W_1(t,x)=h_1(x)$ and

$$D^+W_1(t,x)=\begin{cases}\{(0,1)\}, & x\in(-1,-\frac{2}{3})\cup(-\frac{1}{3},0)\cup(\frac{1}{3},\frac{2}{3}),\\ \{(0,-1)\}, & x\in(-\frac{2}{3},-\frac{1}{3})\cup(0,\frac{1}{3})\cup(\frac{2}{3},1),\\ \{(0,\lambda):\lambda\in[-1,1]\}, & x=-\frac{2}{3},0,\frac{2}{3},\\ \emptyset, & x=-\frac{1}{3},\frac{1}{3},\end{cases}$$

$$D^-W_1(t,x) = \begin{cases} \{(0,\lambda) : \lambda \in [-1,1]\}, & x = -\frac{1}{3}, \frac{1}{3}, \\ \emptyset, & x = -\frac{2}{3}, 0, \frac{2}{3}, \\ D^+W(t,x), & \text{otherwise.} \end{cases}$$

This shows that W_1 is not a supersolution of (2.1), because at $x = -\frac{1}{3}$ or $\frac{1}{3}$ the condition (8.5) is not satisfied. The set of subdifferentials or superdifferentials is not, in general, a line segment. For example

$$D^+W_1\left(\frac{2}{3}, 0\right) = \overline{\text{co}}\{(0,1),(0,-1),(-1,0)\}.$$

Also at a given point both the subdifferential and the superdifferential sets may be empty. To see this, extend $W_1(t,x)$ to $[0,2] \times [-1,1]$ by setting

$$W_1(t,x) = -W_1(2-t,x), \quad \forall t \in [1,2].$$

Then $D^+W_1(1,-\frac{1}{3}) = D^-W_1(1,-\frac{1}{3}) = D^+W_1(1,\frac{1}{3}) = D^-W_1(1,\frac{1}{3}) = \emptyset$.

II.9 Uniqueness: first order case

Let W be a viscosity subsolution of (8.1) in Q and V be a viscosity supersolution of (8.1) in Q, satisfying

$$W(t,x) \le V(t,x), \qquad \forall (t,x) \in \partial^* Q. \tag{9.1}$$

Recall that $\partial^* Q = ([t_0,t_1) \times \partial O) \cup (\{t_1\} \times \overline{Q})$. Under a technical assumption on H (see (9.4)), we will prove that

$$W(t,x) \le V(t,x), \qquad \forall (t,x) \in \overline{Q}. \tag{9.2}$$

See Theorem 9.1. In particular Theorem 9.1 implies that there exists at most one viscosity solution V of (8.1) in Q, satisfying the lateral and terminal boundary conditions

$$V(t,x) = g(t,x), \forall (t,x) \in [t_0,t_1) \times O. \tag{9.3)(a}$$

$$V(t_1,x) = \psi(x), \ \forall x \in \overline{O}. \tag{9.3)(b}$$

Let us now state our assumption on H. We assume that there are a constant K and $h \in C([0,\infty))$ with $h(0) = 0$, such that for all $(t,x),(s,y) \in \overline{Q}$, and $p, \overline{p} \in I\!R^n$ we have,

$$\begin{aligned} &|H(t,x,p) - H(s,y,\overline{p})| \le \\ &\le h(|t-s| + |x-y|) + h(|t-s|)|p| + K|x-y||p| + K|p-\overline{p}|. \end{aligned} \tag{9.4}$$

When H is differentiable, the following condition is sufficient for (9.4):

$$|H_t| + |H_x| \le \hat{K}(1 + |p|) \tag{9.4$'$}$$

and $|H_p| \le \hat{K}$ for a suitable constant $\hat{K}$.

Theorem 9.1. *Let W and V be a viscosity subsolution and a viscosity supersolution of* (8.1) *in Q, respectively. If Q is unbounded, we also assume that W and V are bounded, and uniformly continuous on $\overline{Q}$. Then*

$$\sup_{\overline{Q}}[W - V] = \sup_{\partial^* Q}[W - V]. \tag{9.5}$$

An immediate corollary to this theorem is the following.

Corollary 9.1. *(Uniqueness) There is at most one viscosity solution of* (8.1) *which is bounded and uniformly continuous on $\overline{Q}$, and satisfies the boundary conditions* (9.3).

Remark 9.1. In particular, in (8.1) we may take for V the value function of a deterministic control problem as in I(4.1). Then (8.1) becomes the dynamic programming equation I(5.3). If V is bounded, uniformly continuous on $\overline{Q}$ and satisfies the boundary conditions (9.3), then it is the unique viscosity solution of I(5.3). (See Section 10). Other uniqueness results with a weak (viscosity) formulation of the boundary condition (9.3)(a) are discussed in Sections 13 and 14.

To explain the main idea in the proof of Theorem 9.1, we consider the equation

$$-V_t(t,x) + H(DV(t,x)) = 0, \ (t,x) \in Q.$$

To simplify the discussion we assume that Q is bounded, and

$$W(t,x) \le V(t,x), \quad \forall (t,x) \in \partial^* Q.$$

We continue with a brief outline of the proof of (9.5).

First we define an auxiliary function

$$\psi(t,x;s,y) = W(t,x) - V(s,y) - \frac{1}{2\varepsilon}(|t-s|^2 + |x-y|^2) + \beta(s - t_1)$$

for $(t,x),(s,y) \in \overline{Q}$ and $\beta, \varepsilon > 0$. Let $(\bar{t},\bar{x};\bar{s},\bar{y})$ be a maximizer of ψ on $\overline{Q} \times \overline{Q}$. Then for all $(t,x) \in \overline{Q}$.

$$W(t,x) - V(t,x) \le \psi(\bar{t},\bar{x};\bar{s},\bar{y}) + \beta(t_1 - t_0).$$

Hence it suffices to show that

$$\limsup_{\varepsilon \to 0} \psi(\bar{t},\bar{x};\bar{s},\bar{y}) \le 0,$$

for every $\beta > 0$. First suppose that $(\bar{t},\bar{x}),(\bar{s},\bar{y}) \in Q$. Then the map

$$(t,x) \to \psi(t,x;\bar{s},\bar{y})$$

has a maximum at $(\bar{t},\bar{x})$. Set

$$w(t,x)=\frac{1}{2\varepsilon}[|t-\bar{s}|^2+|x-\bar{y}|^2],\quad (t,x)\in\overline{Q}.$$

Then $W-w$ has a maximum at $(\bar{t},\bar{x})\in Q$ and by Remark 4.2,

$$-w_t(\bar{t},\bar{x})+H(D_xw(\bar{t},\bar{x}))\le 0.$$

We rewrite the above inequality as,

$$-q_\varepsilon+H(p_\varepsilon)\le 0,$$

$$q_\varepsilon=\frac{1}{\varepsilon}(\bar{t}-\bar{s}),\ p_\varepsilon=\frac{1}{\varepsilon}(\bar{x}-\bar{y}).$$

Similarly the map

$$\begin{aligned}(s,y)&\to -\psi(\bar{t},\bar{x};s,y)\\ &=V(s,y)-[-\frac{1}{2\varepsilon}((\bar{t}-s)^2+|\bar{x}-y|^2)+\beta(s-t_1)-W(\bar{t},\bar{x})]\end{aligned}$$

has a minimum at $(\bar{s},\bar{y})$. Therefore $V-\tilde{w}$ with

$$\tilde{w}(s,y)=-\frac{1}{2\varepsilon}(|\bar{t}-s|^2+|\bar{x}-y|^2)+\beta(s-t_1),\ (s,y)\in\overline{Q},$$

has a minimum at $(\bar{s},\bar{y})$. Since V is a viscosity supersolution,

$$-\tilde{w}_s(\bar{s},\bar{y})+H(D_y\tilde{w}(\bar{s},\bar{y}))\ge 0.$$

We directly calculate that

$$\tilde{w}_s(\bar{s},\bar{y})=q_\varepsilon\ +\beta,\ D_y\tilde{w}(\bar{s},\bar{y})=p_\varepsilon.$$

Hence

$$-\beta-q_\varepsilon+H(p_\varepsilon)\ge 0.$$

Subtracting the two inequalities satisfied by p_ε and q_ε, we obtain $\beta\le 0$. So we have proved that if $(\bar{t},\bar{x})\in Q$ and $(\bar{s},\bar{y})\in Q$, then $\beta\le 0$. But $\beta>0$. Therefore we have either $(\bar{t},\bar{x})\in\partial^*Q$ or $(\bar{s},\bar{y})\in\partial^*Q$ or both. Also we formally expect that $|\bar{t}-\bar{s}|,|\bar{x}-\bar{y}|\to 0$ as $\varepsilon\to 0$. Since W and V are continuous and $W\le V$ on ∂^*Q, we have

$$\limsup_{\varepsilon\downarrow 0}\psi(\bar{t},\bar{x};\bar{s},\bar{y})\le 0,$$

for every $\beta>0$. Hence $W\le V$ on $\overline{Q}$.

In the above argument the parameter ε is used to keep $|\bar{t}-\bar{s}|,\ |\bar{x}-\bar{y}|$ small. In the proof of Theorem 9.1 we will use the auxiliary function

$$\Phi(t,x;s,y)=W(t,x)-V(s,y)-\frac{1}{2\varepsilon}|x-y|^2-\frac{1}{2\delta}|t-s|^2+\beta(s-t_1),$$

instead of ψ. The use of two parameters $\varepsilon, \delta > 0$ instead of only ε is technical and allows us to consider Hamiltonians which are less regular in the t - variable (See (9.4)).

Proof of Theorem 9.1. ($\overline{Q}$ bounded case)

For small parameters $\varepsilon, \delta, \beta > 0$, define an auxiliary function for (t, x), $(s, y) \in \overline{Q}$ by,

$$\Phi(t, x; s, y) = W(t, x) - V(s, y) - \frac{1}{2\varepsilon}|x - y|^2 - \frac{1}{2\delta}|t - s|^2 + \beta(s - t_1).$$

Since $\overline{Q}$ is bounded, Φ achieves its maximum on $\overline{Q} \times \overline{Q}$, say at $(\bar{t}, \overline{x}), (\overline{s}, \overline{y})$. We prove (9.5) in several steps.

1. In this step, we estimate the differences $|\bar{t} - \overline{s}|$ and $|\overline{x} - \overline{y}|$. For $\rho \geq 0$, set

$$D_\rho = \{((t, x), (s, y) \in \overline{Q} \times \overline{Q} : |t - s|^2 + |x - y|^2 \leq \rho\},$$

$$m_W(\rho) = 2 \sup\{|W(t, x) - W(s, y)| : ((t, x), (s, y)) \in D_\rho\},$$

$$m_V(\rho) = 2 \sup\{|V(t, x) - V(s, y)| : ((t, x), (s, y)) \in D_\rho\},$$

and

$$K_1 = \sup\{m_W(\rho) : \rho \geq 0\}.$$

Since W and V are continuous, and $\overline{Q}$ is compact, W and V are uniformly continuous on $\overline{Q}$. Thus $m_W, m_V \in C([0, \infty))$ with $m_W(0) = m_V(0) = 0$. We now claim that,

$$|\bar{t} - \overline{s}| \leq \sqrt{K_1 \delta}, \tag{9.6}$$

$$|\overline{x} - \overline{y}| \leq \sqrt{\varepsilon m_W(K_1[\varepsilon + \delta])}. \tag{9.7}$$

Indeed, the inequality $\Phi(\bar{t}, \overline{x}; \overline{s}, \overline{y}) \geq \Phi(\overline{s}, \overline{y}; \overline{s}, \overline{y})$ yields

$$\begin{aligned} \frac{1}{\varepsilon}|\overline{x} - \overline{y}|^2 + \frac{1}{\delta}|\bar{t} - \overline{s}|^2 &\leq 2(W(\bar{t}, \overline{x}) - W(\overline{s}, \overline{y})) \\ &\leq m_W(|\bar{t} - \overline{s}|^2 + |\overline{x} - \overline{y}|^2). \end{aligned} \tag{9.8}$$

Since $m_W(\cdot)$ is bounded by K_1, (9.8) yields (9.6) and $|\overline{x} - \overline{y}|^2 \leq K_1 \varepsilon$. Using these inequalities, (9.8), and the monotonicity of $m_W(\cdot)$, we obtain (9.7).

2. Suppose $(\bar{t}, \overline{x}) \in \partial^* Q$. Then we have,

$$\begin{aligned} \Phi(\bar{t}, \overline{x}; \overline{s}, \overline{y}) &\leq W(\bar{t}, \overline{x}) - V(\overline{s}, \overline{y}) \\ &\leq V(\bar{t}, \overline{x}) - V(\overline{s}, \overline{y}) + \sup_{\partial^* Q}[W - V] \\ &\leq \tfrac{1}{2} m_V(|\bar{t} - \overline{s}|^2 + |\overline{x} - \overline{y}|^2) + \sup_{\partial^* Q}[W - V]. \end{aligned}$$

Using (9.6) and (9.7) in the above inequality, we arrive at

$$\Phi(\bar{t},\bar{x};\bar{s},\bar{y}) \le \frac{1}{2} m_V(K_1(\varepsilon+\delta)) + \sup_{\partial^* Q}[W-V]. \tag{9.9}$$

3. Suppose $(\bar{s},\bar{y}) \in \partial^* Q$. Arguing as in the previous step we obtain,

$$\Phi(\bar{t},\bar{x};\bar{s},\bar{y}) \le \frac{1}{2} m_W(K_1(\varepsilon+\delta)) + \sup_{\partial^* Q}[W-V]. \tag{9.10}$$

4. Suppose that $(\bar{t},\bar{x}),(\bar{s},\bar{y}) \in Q$. Consider the test function

$$w(t,x) = \frac{1}{2\delta}|t-\bar{s}|^2 + \frac{1}{2\varepsilon}|x-\bar{y}|^2, \quad (t,x) \in \overline{Q}.$$

Then, $(\bar{t},\bar{x}) \in \arg\max\{W(t,x) - w(t,x) : (t,x) \in \overline{Q}\}$ and

$$-q_\delta + H(\bar{t},\bar{x},p_\varepsilon) \le 0, \tag{9.11}$$

where

$$q_\delta = \frac{1}{\delta}(\bar{t}-\bar{s}), \quad p_\varepsilon = \frac{1}{\varepsilon}(\bar{x}-\bar{y}).$$

Next, we consider the test function

$$\tilde{w}(s,y) = -\frac{1}{2\delta}|\bar{t}-s|^2 - \frac{1}{2\varepsilon}|\bar{x}-y|^2 + \beta(s-t_1), (s,y) \in \overline{Q}.$$

Then, $(\bar{s},\bar{y}) \in \arg\min\{V(s,y) - \tilde{w}(s,y) : (s,y) \in \overline{Q}\}$ and therefore

$$-\beta - q_\delta + H(\bar{s},\bar{y},p_\varepsilon) \ge 0. \tag{9.12}$$

Subtract (9.12) from (9.11) and then use (9.4) to obtain,

$$\begin{aligned}\beta &\le H(\bar{s},\bar{y},p_\varepsilon) - H(\bar{t},\bar{x},p_\varepsilon)\\ &\le h(|\bar{t}-\bar{s}| + |\bar{x}-\bar{y}|) + h(|\bar{t}-\bar{s}|)|p_\varepsilon| + K|\bar{x}-\bar{y}||p_\varepsilon|.\end{aligned}$$

The estimates (9.6) and (9.7) imply that

$$|p_\varepsilon| = \frac{1}{\varepsilon}|\bar{x}-\bar{y}| \le \frac{1}{\varepsilon}\sqrt{K_1\varepsilon} = \sqrt{\frac{K_1}{\varepsilon}},$$

and

$$|\bar{x}-\bar{y}||p_\varepsilon| = \frac{|\bar{x}-\bar{y}|^2}{\varepsilon} \le m_W(K_1[\varepsilon+\delta]).$$

Combining the above inequalities and (9.6), (9.7) we arrive at

$$\beta \le h(\sqrt{K_1\varepsilon} + \sqrt{K_1\delta}) + h(\sqrt{K_1\delta})\sqrt{\frac{K_1}{\varepsilon}} + K m_W(K_1[\varepsilon+\delta]). \tag{9.13}$$

5. Let $k(\varepsilon,\delta)$ denote the right hand side of (9.13). Then,

$$\lim_{\varepsilon\downarrow 0}[\lim_{\delta\downarrow 0} k(\varepsilon,\delta)] = 0.$$

Since $\beta > 0$, there are $\varepsilon(\beta) > 0$ and $\delta(\varepsilon,\beta) > 0$ such that for all $\varepsilon \le \varepsilon(\beta)$ and $\delta \le \delta(\varepsilon,\beta)$, $k(\varepsilon,\delta) < \beta$. So for $\varepsilon \le \varepsilon(\beta)$, $\delta \le \delta(\varepsilon,\beta)$, (9.13) does not hold and therefore we have either $(\bar{t},\bar{x}) \in \partial^* Q$ or $(\bar{s},\bar{y}) \in \partial^* Q$ or both. Then by (9.9) and (9.10) we obtain,

$$\limsup_{\varepsilon\downarrow 0}\limsup_{\delta\downarrow 0} \Phi(\bar{t},\bar{x};\bar{s},\bar{y}) \le \sup_{\partial^* Q}[W - V], \tag{9.14}$$

for every $\beta > 0$.

6. For any $\varepsilon,\delta,\beta > 0$ and $(t,x) \in \overline{Q}$ we have,

$$W(t,x) - V(t,x) + \beta(t - t_1) = \Phi(t,x;t,x) \le \Phi(\bar{t},\bar{x};\bar{s},\bar{y}).$$

Hence (9.14) yields (9.5), after taking the limit $\beta \downarrow 0$.

($\overline{Q}$ unbounded case).

Since W and V are bounded, for every $\gamma > 0$ there are $(t_\gamma,x_\gamma),(s_\gamma,y_\gamma) \in \overline{Q}$ satisfying

$$\Phi(t_\gamma,x_\gamma;s_\gamma,y_\gamma) \ge \sup_{\overline{Q}\times\overline{Q}} \Phi - \gamma. \tag{9.15}$$

Perturb the auxiliary function Φ by,

$$\Phi_\gamma(t,x;s,y) = \Phi(t,x;s,y) - \frac{\gamma}{2}[|t-t_\gamma|^2 + |s-s_\gamma|^2 + |x-x_\gamma|^2 + |y-y_\gamma|^2],$$

for $(t,x),(s,y) \in \overline{Q}$. We claim that the maximum of Φ_γ on $\overline{Q}\times\overline{Q}$ is achieved, say at $(\bar{t},\bar{x}),(\bar{s},\bar{y}) \in \overline{Q}$. Indeed, for any $|x-x_\gamma|^2+|y-y_\gamma|^2+|t-t_\gamma|^2+|s-s_\gamma|^2 > 2$ we have

$$\begin{aligned} \Phi_\gamma(t,x;s,y) &\le \Phi(t,x;s,y) - \gamma \\ &\le \Phi(t_\gamma,x_\gamma;s_\gamma,y_\gamma) \\ &= \Phi_\gamma(t_\gamma,x_\gamma;s_\gamma,y_\gamma). \end{aligned} \tag{9.16}$$

We now follow the steps of the argument given for the bounded $\overline{Q}$.

1′. Let m_W, m_V and K_1 be as in Step 1. Since W and V are assumed to be bounded and uniformly continuous, K_1 is finite and $m_W, m_V \in C([0,\infty))$ with $m_W(0) = m_V(0) = 0$. Then (9.16) yields,

$$|\bar{t}-t_\gamma|^2 + |\bar{s}-s_\gamma|^2 + |\bar{x}-x_\gamma|^2 + |\bar{y}-\bar{y}_\gamma|^2 \le 2. \tag{9.17}$$

We claim that for $\gamma \le \frac{1}{2}, \delta \le \frac{1}{2}$ and $\varepsilon \le \frac{1}{2}$,

$$(9.18) \qquad |\bar{t} - \bar{s}| \le \sqrt{2(K_1 + 1)\delta},$$

$$(9.19) \qquad |\bar{x} - \bar{y}| \le \sqrt{\varepsilon}\sqrt{2[2\gamma + m_W(2(K_1 + 1)\varepsilon)]}.$$

Indeed, the inequality $\Phi_\gamma(\bar{t}, \bar{x}; \bar{s}, \bar{y}) \ge \Phi_\gamma(\bar{s}, \bar{x}; \bar{s}, \bar{y})$ yields

$$\begin{aligned}
\frac{1}{2\delta}|\bar{t} - \bar{s}|^2 &\le W(\bar{t}, \bar{x}) - W(\bar{s}, \bar{x}) + \frac{\gamma}{2}[|\bar{s} - t_\gamma|^2 - |\bar{t} - t_\gamma|^2] \\
&\le \frac{1}{2}K_1 + \gamma|\bar{t} - \bar{s}|^2 + \frac{\gamma}{2}|\bar{t} - t_\gamma|^2 \\
&\le \frac{1}{2}K_1 + \gamma|\bar{t} - \bar{s}|^2 + \gamma.
\end{aligned}$$

Hence for $\gamma, \delta \le \frac{1}{2}$, (9.18) follows from the above inequality. To prove (9.19), use the inequality $\Phi_\gamma(\bar{t}, \bar{x}; \bar{s}, \bar{y}) \ge \Phi_\gamma(\bar{t}, \bar{y}; \bar{s}, \bar{y})$. Then,

$$\begin{aligned}
\frac{1}{2\varepsilon}|\bar{x} - \bar{y}|^2 &\le W(\bar{t}, \bar{x}) - W(\bar{t}, \bar{y}) + \frac{\gamma}{2}[|\bar{y} - x_\gamma|^2 - |\bar{x} - x_\gamma|^2] \\
&\le \frac{1}{2}m_W(|\bar{x} - \bar{y}|^2) + \gamma|\bar{x} - \bar{y}|^2 + \frac{\gamma}{2}|\bar{x} - x_\gamma|^2 \\
&\le \frac{1}{2}m_W(|\bar{x} - \bar{y}|^2) + \gamma|\bar{x} - \bar{y}|^2 + \gamma.
\end{aligned}$$

For $\gamma, \varepsilon \le \frac{1}{2}$, we obtain

$$\frac{1}{4\varepsilon}|\bar{x} - \bar{y}|^2 \le \frac{1}{2}m_W(|\bar{x} - \bar{y}|^2) + \gamma.$$

Hence $|\bar{x} - \bar{y}|^2 \le 2(K_1 + 1)\varepsilon$ and use this in the above inequality to arrive at (9.19).

2.′ Suppose $(\bar{t}, \bar{x}) \in \partial^* Q$. Then proceed exactly as in Step 2 to obtain

$$(9.20) \qquad \Phi_\gamma(\bar{t}, \bar{x}; \bar{s}, \bar{y}) \le \frac{1}{2}m_V(2(K_1 + 1)(\varepsilon + \delta)) + \sup_{\partial^* Q}[W - V].$$

3′. Suppose that $(\bar{s}, \bar{y}) \in \partial^* Q$ and proceed as in Step 3. Then,

$$(9.21) \qquad \Phi_\gamma(\bar{t}, \bar{x}; \bar{s}, \bar{y}) \le \frac{1}{2}m_W(2(K_1 + 1)(\varepsilon + \delta)) + \sup_{\partial^* Q}[W - V].$$

4′. Suppose that $(\bar{t}, \bar{x}), (\bar{s}, \bar{y}) \in Q$. Consider the test function

$$\overline{w}(t, x) = \frac{1}{2\delta}|t - \bar{s}|^2 + \frac{1}{2\varepsilon}|x - \bar{y}|^2 + \frac{\gamma}{2}|t - t_\gamma|^2 + \frac{\gamma}{2}|x - x_\gamma|^2, (t, x) \in \overline{Q}.$$

Then, $(\bar{t}, \bar{x}) \in \arg\max\{W(t, x) - \overline{w}(t, x) : (t, x) \in \overline{Q}\}$ and

$$(9.22) \qquad -q_\delta - q_\gamma + H(\bar{t}, \overline{x}, p_\varepsilon + p_\gamma) \le 0,$$

where

$$q_\delta = \frac{1}{\delta}(\bar{t} - \overline{s}) \ , \ q_\gamma = \gamma(\bar{t} - t_\gamma),$$

and

$$p_\varepsilon = \frac{1}{\varepsilon}(\overline{x} - \overline{y}) \ , \ p_\gamma = \gamma(\overline{x} - x_\gamma).$$

Next, consider the test function

$$w^*(s,y) = -\frac{1}{2\delta}|\bar{t} - s|^2 - \frac{1}{2\varepsilon}|\overline{x} - y|^2 - \frac{\gamma}{2}|s - s_\gamma|^2 - \frac{\gamma}{2}|y - y_\gamma|^2 + \beta(s - t_1),$$

for $(s,y) \in \overline{Q}$. Then, $(\overline{s}, \overline{y}) \in \arg\min\{V(s,y) - w^*(s,y) : (s,y) \in \overline{Q}\}$ and we have,

$$(9.23) \qquad -\beta - q_\delta - \overline{q}_\gamma + H(\overline{s}, \overline{y}, p_\varepsilon + \overline{p}_\gamma) \ge 0,$$

where

$$\overline{q}_\gamma = \gamma(s_\gamma - \overline{s}) \ , \ \overline{p}_\gamma = \gamma(y_\gamma - \overline{y}).$$

Subtract (9.23) from (9.22) and use (9.4) to obtain,

$$\beta \le H(\overline{s}, \overline{y}, p_\varepsilon + \overline{p}_\gamma) - H(\bar{t}, \overline{x}, p_\varepsilon + p_\gamma) + q_\gamma - \overline{q}_\gamma$$

$$\le h(|\bar{t} - \overline{s}| + |\overline{x} - \overline{y}|) + h(|\bar{t} - \overline{s}|)[|p_\varepsilon| + |\overline{p}_\gamma|] + K|\overline{x} - \overline{y}|[|p_\varepsilon| + |\overline{p}_\gamma|]$$

$$+K|\overline{p}_\gamma - p_\gamma| + q_\gamma - \overline{q}_\gamma.$$

The estimates (9.17), (9.18) and (9.19) yield

$$|p_\varepsilon| = \frac{1}{\varepsilon}|\overline{x} - \overline{y}| \le \sqrt{\frac{2(K_1+1)}{\varepsilon}},$$

$$|p_\varepsilon||\overline{x} - \overline{y}| = \frac{|\overline{x} - \overline{y}|^2}{\varepsilon} = 2[2\gamma + m_W(2(K_1+1)\varepsilon)],$$

and

$$|p_\gamma|, |\overline{p}_\gamma|, |q_\gamma|, |\overline{q}_\gamma| \le 2\gamma.$$

Combining above inequalities, (9.17) and (9.19) we obtain

$$(9.24) \quad \beta \le h(\sqrt{2(K_1+1)}(\sqrt{\varepsilon} + \sqrt{\delta}) + h(\sqrt{2(K_1+1)\delta}) \left[\sqrt{\frac{2(K_1+1)}{\varepsilon}} + 2\gamma\right]$$

$$+2K[2\gamma + m_W(2(K_1+1)\varepsilon) + \sqrt{2(K_1+1)\varepsilon}\gamma] + 2(K+1)\gamma.$$

5′. Using (9.20), (9.21) and (9.24) as in step 5 we arrive at

$$\limsup_{\varepsilon \downarrow 0} \limsup_{\delta \downarrow 0} \limsup_{\gamma \downarrow 0} \varPhi_\gamma(\bar{t}, \overline{x}; \overline{s}, \overline{y}) \le \sup_{\partial^* Q}[W - V],$$

for every $\beta > 0$.

6′. Argue exactly as in step 6 to obtain (9.5). □

Remark 9.2. To analyze the interior maximum case, $(\bar{t}, \bar{x}), (\bar{s}, \bar{y}) \in Q$, we used only the modulus of continuity, m_W, of W. A symmetric argument can also be made by using only m_V. Hence the analysis of the interior maximum requires the continuity of only one of the functions. This observation will be useful in the discussion of semicontinuous viscosity solutions; Chapter VII. Also there is an alternate proof of Theorem 9.1 due to Ishii, which does not use the modulus of continuity of either function [I4], [CIL2].

Extensions of Theorem 9.1 to unbounded solutions and Hamiltonians which do not satisfy (9.4) have been studied by Ishii [I3]. In particular Ishii proved the uniqueness of a viscosity solution V on $Q_0 = [t_0, t_1) \times I\!R^n$ which satisfies the following growth condition,

$$\lim_{|x|\to\infty} \frac{|V(x)|}{1+|x|^m} = 0$$

with an exponent m which is determined by the Hamiltonian H. In his analysis, Ishii modified the auxiliary function Φ as follows:

$$\Phi_\alpha(t, x; s, y) = \Phi(t, x; s, y) - \alpha(1 + |x|^m + |y|^m),$$

where $\alpha > 0$ is a parameter. Then for every $\alpha > 0, \Phi_\alpha$ achieves its maximum, say at $(\bar{t}, \bar{x}), (\bar{s}, \bar{y})$. Using an argument similar to the proof of Theorem 9.1, Ishii first showed that

$$\sup_{\alpha>0} \Phi_\alpha(\bar{t}, \bar{x}; \bar{s}, \bar{y}) < \infty.$$

Then repeating the proof of Theorem 9.1 he completed his proof of uniqueness. (See Theorems 2.1, 2.3 in [I3] and Section 5.D of [CIL1].)

Remark 9.3. For a deterministic control problem the Hamiltonian H takes the form (7.3). When the control set U is bounded, this Hamiltonian H satisfies (9.4) under standard assumptions on the data (see the proof of Theorems 10.1 and 10.2 below.) If however, U is not bounded (9.4) is restrictive. Still Theorem 9.1 and Corollary 8.2 yield comparison and uniqueness results for Lipschitz continuous sub and supersolutions. Indeed in view of Corollary 8.2 any Lipschitz continuous sub or supersolution of the dynamic programming equation of the original problem is also a sub or supersolution of the dynamic programming equation of another control problem with a bounded control set. We then can apply Theorem 9.1, since the Hamiltonian of the second problem satisfies (9.4). Such an argument is carried out in detail in the proofs of Theorems 10.3, 10.4 and Theorem VII.8.2.

Remark 9.4. In some applications the Hamiltonian H in (8.1) may depend on $V(t, x)$, i.e.,

(9.25) $$-\frac{\partial}{\partial t} V(t, x) + H(t, x, D_x V(t, x), V(t, x)) = 0, \text{ in } Q.$$

Let W be a viscosity subsolution of (9.25) in Q and V be a viscosity supersolution of (9.25) in Q. If H *is nondecreasing* in $V(t,x)$ then a minor modification of the proof Theorem 9.1 yields

$$\sup_{\overline{Q}}[W-V] \le \sup_{\partial^* Q}[(W-V)\vee 0]. \tag{9.26}$$

In general, instead of monotonicity of H we assume that there is $\beta \ge 0$ satisfying

$$H(t,x,p,V+v) - H(t,x,p,V) \ge -\beta v, \tag{9.27}$$

for all $(t,x)\in Q,\ p\in I\!\!R^n,\ v\ge 0$ and $V\in I\!\!R$. Set

$$H_\beta(t,x,p,V) = \beta V + e^{\beta t} H(t,x,e^{-\beta t}p, e^{-\beta t}V).$$

Then (9.27) implies that H_β is nondecreasing in V.

Proposition 9.1. *Assume* (9.27) *and that* H_β *satisfies* (9.5). *Let* W *and* V *be a viscosity subsolution and a supersolution of* (9.24) *in* Q, *respectively. Then*

$$\sup_{\overline{Q}}[W-V] \le e^{\beta(t_1-t_0)} \sup_{\partial^* Q}[(W-V)\vee 0].$$

We start the proof with a lemma.

Lemma 9.1. *Let* $\tilde W$ *be a viscosity subsolution (or a supersolution) of* (9.25) *in* Q. *For* $(t,x)\in\overline{Q}$ *and a constant* β, *define*

$$\overline{W}(t,x) = e^{\beta t}\tilde W(t,x).$$

Then $\overline{W}$ *is a viscosity subsolution (or a supersolution, respectively) of*

$$-\frac{\partial}{\partial t}\overline{W}(t,x) + H_\beta(t,x,D_x\overline{W}(t,x),\overline{W}(t,x)) = 0,\ \textit{in}\ Q. \tag{9.28}$$

Proof. Let $\overline{w}\in C^1(Q)$ and $(\bar t,\overline{x})$ be a maximizer of $\overline{W}-\overline{w}$ on $\overline{Q}$, with $\overline{W}(\bar t,\overline{x}) = \overline{w}(\bar t,\overline{x})$. Set

$$w(t,x) = e^{-\beta t}\overline{w}(t,x),\ (t,x)\in\overline{Q}.$$

Then $w\in C^1(Q), (\bar t,\overline{x})$ maximizes $\tilde W - w$ on $\overline{Q}$ and $\tilde W(\bar t,\overline{x}) = w(\bar t,\overline{x})$. Therefore

$$0 \ge -\frac{\partial}{\partial t}w(\bar t,\overline{x}) + H(\bar t,\overline{x},D_x w(\bar t,\overline{x}),w(\bar t,\overline{x}))$$

$$= e^{-\beta t}[-\frac{\partial}{\partial t}\overline{w}(\bar t,\overline{x}) + H_\beta(\bar t,\overline{x},D_x\overline{w}(\bar t,\overline{x}),w(\bar t,\bar x)].$$

The supersolution case is proved exactly the same way. □

Proof of Proposition 9.1. Set

$$\bar W(t,x) = e^{\beta t}W(t,x),\ \bar V(t,x) = e^{\beta t}V(t,x).$$

Then $\bar{W}$ is a viscosity subsolution of (9.28) in Q and $\bar{V}$ is a viscosity supersolution of (9.28) in Q. Also H_β is nondecreasing in V. So by Remark 9.4, we have

$$\sup_{\overline{Q}}[\bar{W} - \bar{V}] \le \sup_{\partial^* Q}[(\bar{W} - \bar{V}) \vee Q].$$

In view of the definitions of $\bar{W}, \bar{V}$, the proof of Proposition 9.1 is now complete. □

II.10 Continuity of the value function

We will prove the continuity of the value function for a deterministic control problem under two different sets of assumptions. We retain the notation and the assumptions of Sections I.3 and II.7. In particular, we assume that f, L, ψ are continuous functions and I(3.1) is satisfied.

Theorem 10.1. *Suppose that U is bounded, $Q = Q_0 = [t_0, t_1) \times I\!R^n$, $\mathcal{U}(t,x) = \mathcal{U}^0(t)$ for every $(t,x) \in \overline{Q}_0$. Moreover assume that f satisfies* I(3.1), *that f, L, ψ are bounded and L, ψ are uniformly continuous on $\overline{Q}_0 \times U$ and $I\!R^n$, respectively. Then the value function is bounded and uniformly continuous on $\overline{Q}_0$. In particular, the value function is the unique viscosity solution of the dynamic programming equation* I(5.3$'$) *in Q_0, satisfying the terminal condition* I(5.5).

Proof. When the control set is bounded, the Hamiltonian H given by (7.3) satisfies the condition (9.4). Indeed for any two functions $g, \tilde{g}$ on U, we have

$$|\sup_{v\in U} g(v) - \sup_{\tilde{v}\in U} \tilde{g}(\tilde{v})| \le \sup_{v\in U} |g(v) - \tilde{g}(v)|.$$

Therefore

$$\begin{aligned}
|H(t,x,p) &- H(s,y,\overline{p})| \\
&\le \sup_{v\in U}\{|L(t,x,v) - L(s,y,v) + f(t,x,v)\cdot p - f(s,y,v)\cdot \overline{p}|\} \\
&\le \sup_{v\in U}|L(t,x,v) - L(s,y,v)| + \sup_{v\in U}|f(s,y,v)||p-\overline{p}| \\
&+ \sup_{v\in U}|f(t,x,v) - f(s,y,v)|\ |p|.
\end{aligned}$$

Then the uniform continuity of L, the boundedness of f and the uniform Lipschitz continuity I(3.1), of f yield (9.4). Hence the last assertion of the theorem follows from Theorem 7.1 and Theorem 9.1, once we establish the boundedness and the uniform continuity of the value function.

The boundedness of the value function follows easily from the boundedness of L and ψ. Fix $(t,x), (t,y) \in \overline{Q}_0$, and $u(\cdot) \in \mathcal{U}^0(t)$. Let $x(\cdot)$ and $y(\cdot)$ be the solutions of I(3.2) with initial conditions $x(t) = x$ and $y(t) = y$, respectively. Set

$$\Lambda(s) = |x(s) - y(s)|, \;\; s \in [t, t_1].$$

Then, I(3.1) and the boundedness of U yield,

$$\begin{aligned}\Lambda(s) &= \left| x + \int_t^s f(r, x(r), u(r)) dr - y - \int_t^s f(r, y(r), u(r)) dr \right| \\ &\le |x - y| + \int_t^s |f(r, x(r), u(r)) - f(r, y(r), u(r))| dr \\ &\le |x - y| + K \int_t^s \Lambda(r) dr, s \in [t, t_1].\end{aligned}$$

Gronwall's inequality implies that

$$\Lambda(s) \le |x - y| e^{K(s-t)}, \;\; s \in [t, t_1]. \tag{10.1}$$

For $\rho \ge 0$, define

$$m_L(\rho) = \sup\{|L(t, x, v) - L(s, y, v)| : |t - s| + |x - y| \le \rho, v \in U\},$$

and

$$m_\psi(\rho) = \sup\{|\psi(x) - \psi(y)| : |x - y| \le \rho\}.$$

The uniform continuity of L and ψ implies that $m_L, m_\psi \in C([0, \infty))$ with $m_L(0) = m_\psi(0) = 0$. Set $T = t_1 - t_0$. By using (10.1) we estimate that,

$$\begin{aligned}|J(t, x; u) - J(t, y; u)| &\le \int_t^{t_1} |L(s, x(s), u(s)) - L(s, y(s), u(s))| ds \\ &\quad + |\psi(x(t_1)) - \psi(y(t_1))| \\ &\le \int_t^{t_1} m_L(\Lambda(s)) ds + m_\psi(\Lambda(t_1)) \\ &\le T m_L(|x - y| e^{KT}) + m_\psi(|x - y| e^{KT}).\end{aligned}$$

Therefore,

$$\begin{aligned}|V(t, x) - V(t, y)| &\le \sup_{u(\cdot) \in \mathcal{U}^0(t)} |J(t, x; u) - J(t, y; u)| \\ &\le T m_L(|x - y| e^{KT}) + m_\psi(|x - y| e^{KT}).\end{aligned} \tag{10.2}$$

Now fix $(t, x) \in \overline{Q}_0, r \in [t, t_1]$ and $u(\cdot) \in \mathcal{U}^0(t)$. The restriction of $u(\cdot)$ on $[r, t_1]$ is an element of $\mathcal{U}^0(r)$, which we call $u(\cdot)$ again. With this notation we have,

$$|J(t,x;u)-J(r,x;u)| \le |J(t,x;u)-J(r,x(r);u)| + |J(r,x(r);u)-J(r,x;u)|$$

$$\le \left|\int_t^r L(s,x(s),u(s))ds\right| + |J(r,x(r);u)-J(r,x;u)|.$$

Set $K_1 = \sup\{|L(t,x,v)| : (t,x) \in \overline{Q}_0, v \in U\}$. Then (10.1), (10.2) yield

$$|J(t,x;u)-J(r,x;u)| \le K_1|r-t| + Tm_L(|x(r)-x|e^{KT}) +$$
$$+ m_\psi(|x(r)-x|e^{KT}).$$

Since f is bounded, $|x(r)-x| \le K_2(r-t)$ for a suitable constant K_2. Therefore,

(10.3) $$|V(t,x)-V(r,x)| \le \sup_{u(\cdot)\in\mathcal{U}^0(t)} |J(t,x;u)-J(r,x;u)|$$

$$\le K_1|t-r| + Tm_L(|t-r|K_2e^{KT}) + m_\psi(|t-r|K_2e^{KT}).$$

Clearly the above argument is symmetric in r and t. Hence (10.3) holds for every $(t,x),(r,x) \in \overline{Q}_0$. □

Remark 10.1. If the hypotheses of Theorem 10.1 are strengthened to require that

(10.4) $$\begin{cases} \text{(a)} & |L(t,x,v)-L(s,y,v)| \le K_L(|t-s|+|x-y|) \\ \text{(b)} & |\psi(x)-\psi(y)| \le K_\psi|x-y|, \end{cases}$$

then the value function V is Lipschitz continuous on $\bar{Q}_0$. This follows by taking $m_L(\rho) = K_L\rho$, $m_\psi(\rho) = K_\psi\rho$ in (10.2) and (10.3).

We next study the bounded Q case. As in Section I.3, $Q = [t_0,t_1] \times O$ where O is bounded and ∂O is a manifold of class C^2. We wish to find conditions under which the value function V is Lipschitz continuous on $\bar{Q}$. The boundary data for V are (9.3). However, examples show that V may not continuously assume the "lateral" boundary data (9.3a) unless further assumptions are made. Let us assume that $g \equiv 0$. The more general case can be reduced to this one by the same methods as in Section I.8 for the calculus of variations problem.

We recall that the value function is

(10.6) $$V(t,x) = \inf_{u(\cdot)\in\mathcal{U}^0(t)} J(t,x;u),$$

(10.7) $$J(t,x;u) = \int_t^\tau L(s,x(s),u(s)ds + \Psi(\tau,x(\tau))$$

where $\mathcal{U}^0(t) = L^\infty([t,t_1];U)$ and $\Psi(\tau,x(\tau)) = 0$ if $\tau < t_1$, $\Psi(\tau,x(\tau)) = \psi(x(t_1))$ if $\tau = t_1$.

Lemma 10.1. *Assume* I(3.1), (10.4) *and that U is bounded. Moreover, assume that there exists M such that*

$$(10.8) \qquad |V(t,x)| \leq M \text{ distance } (x, \partial O), (t,x) \in \bar{Q}.$$

Then V is Lipschitz continuous on $\bar{Q}$.

Proof. Given x, y in $\bar{O}$ and $u(\cdot) \in \mathcal{U}^0(t)$, let $x(s), y(s)$ be the solutions to I(3.2) with $x(t) = x$, $y(t) = y$ respectively. Let τ_x, τ_y denote the exit time of $(s, x(s))$, $(s, y(s))$ from Q, and $\tau' = \min(\tau_x, \tau_y)$. By the dynamic programming principle,

$$V(t,y) \leq \int_t^{\tau'} L(s, y(s), u(s))ds + V(\tau', y(\tau'))$$

and given $\theta > 0$, there exists $u(\cdot)$ such that

$$\int_t^{\tau'} L(s, x(s), u(s))ds + V(\tau', x(\tau')) < V(t,x) + \theta.$$

By (10.1), $|x(s) - y(s)| \leq C_1 |x - y|$, where $C_1 = e^{K(t_1 - t_0)}$. Hence

$$V(t,y) \leq V(t,x) + B_1|x-y| + \theta + |V(\tau', x(\tau')) - V(\tau', y(\tau'))|$$

where $B_1 = C_1 K_L$. We consider three cases.

Case 1. $\tau_x = \tau_y = t_1$. Then

$$|V(\tau', x(\tau')) - V(\tau', y(\tau'))| = |\psi(x(t_1)) - \psi(y(t_1))| \leq B_2|x-y|$$

where $B_2 = C_1 K_\psi$.

Case 2. $\tau' = \tau_x < t_1$. Then $V(\tau', x(\tau')) = 0$ and

$$\begin{aligned} |V(\tau', y(\tau'))| &\leq M \text{ distance } (y(\tau'), \partial O) \\ &\leq M|x(\tau') - y(\tau'))| \leq B_3|x-y| \end{aligned}$$

where $B_3 = C_1 M$.

Case 3. $\tau' = \tau_y < t_1$. As in Case 2, $V(\tau', y(\tau')) = 0$ and $|V(\tau', x(\tau'))| \leq B_3|x-y|$.

Since θ is arbitrary,

$$V(t,y) \leq V(t,x) + M_1|x-y|$$

where $M_1 = B_1 + B_2 + B_3$. Since x and y can be exchanged,

$$(10.9) \qquad |V(t,x) - V(t,y)| \leq M_1|x-y|.$$

We next show that $V(\cdot, x)$ satisfies a uniform Lipschitz condition on $[t_0, t_1]$. Let $r \in [t, t_1]$. By the dynamic programming principle, given $\theta > 0$ there exists $u(\cdot)$ such that

$$(10.10) \quad V(t,x) - V(r,x) \leq \int_t^{\tau_1} L(s, x(s), u(s))ds + V(\tau_1, x(\tau_1)) - V(r,x)$$

$$< V(t,x) - V(r,x) + \theta$$

where $\tau_1 = \min(r,\tau)$. Let K_1, K_2 be as in the proof of Theorem 10.1. If $\tau_1 = r$, then

$$|V(\tau_1, x(\tau_1)) - V(r,x)| \leq M_1|x(r) - x| \leq M_1 K_2(r-t).$$

If $\tau_1 < r$, then $x(\tau_1) \in \partial O$ and $V(\tau_1, x(\tau_1)) = 0$. Then

$$|V(\tau_1, x(\tau_1)) - V(r,x)| \leq M \text{ distance } (x, \partial O) \leq$$
$$\leq M|x(\tau_1) - x| \leq MK_2(r-t).$$

Since θ is arbitrary, from (10.10)

$$|V(t,x) - V(r,x)| \leq M_2(r-t) \tag{10.11}$$

where $M_2 = K_1 + (M + M_1)K_2$. From (10.9) and (10.11), V is Lipschitz continuous. □

Lemma 10.2. *In addition to the hypotheses on f, L, ψ, U in Lemma* 10.1, *assume* I(3.10), I(3.11). *Then* (10.8) *holds for some M.*

Proof. We first show that such an estimate holds in a neighborhood of any given point $(\bar{t}, \bar{x}) \in [t_0, t_1] \times \partial O$. Let $W(x) =$ distance $(x, \partial O)$ and let $N_2(\alpha_2) = \{y \in \bar{O}: |y - \bar{x}| < \alpha_2\}$. Since ∂O is a manifold of class C^2, $W \in C^1(N_2(\alpha_2))$ for small enough $\alpha_2 > 0$. Moreover, $DW(x_0) = -\eta(\bar{x})$ where $\eta(\xi)$ is the exterior unit normal at $\xi \in \partial O$. Let $N_1(\alpha_1) = [t_0, t_1] \cap \{s: |s - \bar{t}| < \alpha_1\}$, and let $N(\alpha_1, \alpha_2) = N_1(\alpha_1) \times N_2(\alpha_2)$. By I(3.11), there exist $v_0 \in U$ and positive c, α_1, α_2 such that

$$f(s, y, v_0) \cdot DW(y) \leq -c, \ (s,y) \in N(\alpha_1, \alpha_2).$$

Choose β_1, β_2 such that $0 < \beta_2 < \alpha_2$ and $0 < c^{-1}\beta_2 + \beta_1 < \alpha_1$. For $(t,x) \in N(\beta_1, \beta_2)$, let $x(s)$ be the solution to I(3.2) with $u(s) = v_0$, $x(t) = x$, and let τ_1 be the exit time of $(s, x(s))$ from $N(\alpha_1, \alpha_2)$. For $t \leq s \leq \tau_1$,

$$\frac{d}{ds} W(x(s)) = f(s, x(s), v_0) \cdot DW(x(s)) \leq -c.$$

Hence

$$0 \leq W(x(\tau_1)) \leq W(x) - c(\tau_1 - t). \tag{10.12}$$

If $x(\tau_1) \in \partial O$, then from (10.12) and $0 \leq L(s, x(s), v_0) \leq K_1$

$$0 \leq V(t,x) \leq K_1 c^{-1} W(x). \tag{10.13}$$

If $x(\tau_1) \notin \partial O$, then from (10.12) and $W(x) < \beta_2$,

$$|\tau_1 - \bar{t}| \leq |\tau_1 - t| + |t - \bar{t}| < c^{-1}\beta_2 + \beta_1 < \alpha_1.$$

Hence $\tau_1 = t_1$ and

$$0 \le V(t,x) \le K_1 c^{-1} W(x) + |\psi(x(t_1))|$$
$$\le K_1 c^{-1} W(x) + L_\psi W(x(t_1))$$

since $\psi(y) = 0$ for all $y \in \partial O$. By (10.12), $W(x(t_1)) \le W(x)$ and hence, for all $(t,x) \in N(\beta_1, \beta_2)$

$$0 \le V(t,x) \le M_0 W(x) \tag{10.14}$$

where $M_0 = K_1 c^{-1} + L_\psi$.

Since ∂O is compact, (10.14) implies that there exist $\bar{M}_1$, $\delta > 0$ such that $0 \le V(t,x) \le \overline{M}_1 W(x)$ for all $(t,x) \in \overline{Q}$ such that $W(x) \le \delta$. Let $M = \max(\overline{M}_1, \delta^{-1}\overline{M}_2)$ where $0 \le V(t,x) \le \overline{M}_2$ for all $(t,x) \in \overline{Q}$. We obtain (10.8).

□

Theorem 10.2. (Q bounded) *Assume that U is bounded and that* I(3.1), I(3.10), I(3.11), (10.4) *hold. Then the value function V is the unique Lipschitz continuous viscosity solution to the dynamic programming equation* I(5.3) *in Q with the boundary data* (9.3).

Proof. Lemmas 10.1 and 10.2 imply that V is Lipschitz continuous. The theorem then follows from Theorem 7.1 and Corollary 9.1. □

We next apply these results to nonlinear PDEs which are Hamilton-Jacobi equations for some calculus of variations problem on a fixed time interval considered in Sections I.9 and I.10. Since $U = I\!R^n$ is not compact, a truncation argument is used in which $I\!R^n$ is replaced by $U_R = \{|v| \le R\}$. Consider the Cauchy problem I(10.8)-I(10.9) where $H(t,x,p)$ satisfies I(10.13). Define $L(t,x,v)$ by the duality formula I(8.5). Let $V(t,x)$ be the value function of the calculus of variations problem in Section I.9 with $\psi(x) = 0$.

Theorem 10.3. ($Q = Q_0$). *Assume that H satisfies* I(10.13). *Then V is the unique bounded, Lipschitz continuous viscosity solution to the Hamilton-Jacobi equation* I(10.8) *with boundary data* I(10.9).

Proof. The function L satisfies I(9.2). By Theorem I.9.3, V is bounded and Lipschitz continuous, with Lipschitz constant M in I(9.10). Moreover, by I(9.7) there exists R_1 such that $V = V_R$ for $R \ge R_1$, where V_R is the value function when the constraint $|u(s)| \le R$ is imposed. By Theorem 7.1, for $R \ge R_1$, V is a viscosity solution to

$$-\frac{\partial V}{\partial t} + H_R(t,x,D_x V) = 0 \tag{10.15}$$

with H_R in I(10.14). Let $R_2 = \max(R_1, R(M))$ where $R(M)$ is as in I(10.15). By Corollary 8.1(f), $(q,p) \in D^+V(t,x) \cup D^-V(t,x)$ implies $|p| \le M$ and hence $H(t,x,p) = H_R(t,x,p)$ for $R \ge R_2$. By (8.4) and (8.5), V is a viscosity solution to I(10.8). Finally, let W be any bounded, Lipschitz continuous viscosity solution to I(10.8)-I(10.9) with Lipschitz constant M_1. The same argument shows that W is a viscosity solution to (10.15) if $R \ge R(M_1)$. By Theorem 10.1, $W = V_R$ for $R \ge R(M_1)$. This implies that $V = W$. □

We conclude this section with a result about the calculus of variations problem in a bounded cylindrical region considered in Section I.11. This result will be applied in Section VII.10 to an exit probability large deviations problem. Note that condition I(3.11), which is needed in Theorem 10.2, is satisfied for $f(v) = v$, $U = U_R = \{|v| \le R\}$.

Theorem 10.4. (Q bounded). *Assume that* I(11.6), I(11.7) *hold. Then the value function V is Lipschitz continuous on $\overline{Q}$. Moreover, V is the unique Lipschitz continuous viscosity solution to the Hamilton-Jacobi equation* I(8.9) *with boundary data $V(t,x) = \Psi(t,x)$ for $(t,x) \in \partial^* Q$.*

Proof. By Theorems I.11.2 and I.11.3, there exists R_1 such that $V(t,x) = V_R(t,x)$ for all $(t,x) \in \overline{Q}$ and $R \ge R_1$, where V_R is the value function with control constraint $|u(s)| \le R$. By Theorem 10.2, V is a Lipschitz continuous viscosity solution to (10.15) and the boundary data, for $R \ge R_1$. The Hamiltonian H, defined by I(8.4), is in $C^2(\overline{Q}_0 \times I\!\!R^n)$. Since Q is bounded, there exists $R(M)$ such that $(t,x) \in \overline{Q}$, $|p| \le M$ imply $|H_p(t,x,p)| \le R(M)$. The proof is completed in the same way as for Theorem 10.3. □

II.11 Discounted cost with infinite horizon

In Section I.7, we studied a class of optimization problems with infinite horizon ($t_1 = \infty$). In that section, we assumed that for $(s,y) \in [0,\infty) \times O, v \in U, f(s,y,v)$ is independent of s, and

$$L(s,y,v) = e^{-\beta s} L(y,v),$$

$$g(s,y) = e^{-\beta s} g(y),$$

for some discount factor $\beta \ge 0$. Then, the value function $V(t,x)$ has a similar form,

$$V(t,x) = e^{-\beta t} V(x) \ , \ (t,x) \in [0,\infty) \times \overline{O}. \tag{11.1}$$

The function $V(x)$, which we call the value function, satisfies the dynamic programming equation

$$\beta V(x) + H(x, D_x V(x)) = 0, \ x \in O, \tag{11.2}$$

where

$$H(x,p) = \sup_{v \in U} \{-f(x,v) \cdot p - L(x,v)\} \ , \ x \in \overline{O}, p \in I\!\!R^n. \tag{11.3}$$

First we give a definition of viscosity sub and supersolutions of (11.2) in O. This definition is obtained by simply substituting the form (11.1) into the Definition 4.2. The following definition does not require H to satisfy (11.3).

Definition 11.1. *Let $W \in C(\overline{O})$.*

(a) W is a *viscosity subsolution* of (11.2) in O if for each $w \in C^\infty(O)$,

$$\beta W(\bar{x}) + H(\bar{x}, Dw(\bar{x})) \le 0, \tag{11.4}$$

at every $\overline{x} \in O$, which is a local maximizer of $W - w$.

(b) W is a *viscosity supersolution* of (11.2) in O if for each $w \in C^\infty(O)$,

$$\beta W(\overline{x}) + H(\overline{x}, Dw(\overline{x})) \ge 0, \tag{11.5}$$

at every $\overline{x} \in O$, which is a local minimizer of $W - w$.

(c) W is a *viscosity solution* of (11.2) in O if it is both a viscosity subsolution and a viscosity supersolution of (11.2) in O.

We should emphasize that here the test function w is only a function of x, *not* (t, x) as it is the case in Definition 4.2. Recall that in Definition 4.2, we require $W = w$ at the extrema. In the above definition however, we do not require this equality but instead, we use $\beta W(\overline{x})$ both in (11.4) and (11.5). This modification is discussed in detail, in Remark 4.2. Also in the above definition we can replace $w \in C^\infty(O)$ by $w \in C^1(O)$ exactly as in the "time dependent" case considered in Sections 4–6, see Remark 6.1.

A straightforward modification of the proof of Theorem 9.1 yields a comparison result for viscosity sub and supersolutions of (11.2), when $\beta > 0$.

Theorem 11.1. *Suppose that $\beta > 0$ and there exists a constant $K \ge 0$ satisfying,*

$$|H(x,p) - H(y,\overline{p})| \le K(1 + |p|)|x - y| + K|p - \overline{p}|,$$

for every $x, y \in \overline{O}$ and $p, \overline{p} \in I\!R^n$. Let W and V be a viscosity subsolution and a viscosity supersolution of (11.2) in O, respectively. If O is unbounded, we also assume that W and V are bounded and uniformly continuous on $\overline{O}$. Then for every $x \in \overline{O}$,

$$W(x) - V(x) \le \sup\{(W(y) - V(y)) \vee 0 \ : \ y \in \partial O\}.$$

The analysis of $\beta = 0$ case is more complicated. If however, $H(x, 0) \le -c_0 < 0$ for all $x \in \overline{O}$, then a result similar to the above theorem holds. See [L4] and [BaS]. This result in particular applies to the dynamic programming equation of time optimal control problems.

When $\beta > 0$, the continuity of the value function can be proved under the assumptions used in Section 10. The proof of the continuity under these assumptions is similar to the arguments of the previous section. We leave these extensions to the reader.

II.12 State constraint

In this section we analyze the optimal control problem with a state constraint. Recall that this is class D of Section I.3 and the class of admissible controls $\mathcal{U}(t, x)$ is given by

$$(12.1) \qquad \mathcal{U}(t,x) = \{u(\cdot) \in \mathcal{U}^0(t) : \quad x(s) \in \overline{O}\,, \ \forall s \in [t,t_1]\}.$$

We assume that for every $(t,x) \in \overline{Q}$

$$(12.2) \qquad \mathcal{U}(t,x) \neq \emptyset.$$

Also as in Section 7, we assume that f satisfies I(3.1), f, L, ψ are continuous functions of $\overline{Q} \times U, \overline{Q} \times U$ and $\overline{O}$, respectively, and L, ψ are bounded from below. Since the process $x(\cdot)$ is not allowed to leave the region $\overline{Q}$ from the lateral boundary $[t_0, t_1) \times \partial O$, the function g is irrelevant in this problem.

The admissible control sets $\mathcal{U}(t,x)$ satisfy the switching condition I(3.9). In the next theorem we verify that $\mathcal{U}(t,x)$ also satisfies I(5.2).

Theorem 12.1. *$\mathcal{U}(t,x)$ satisfies* I(5.2) *at every $(t,x) \in \overline{Q}$. In particular, if f, L satisfy the hypotheses of Theorem* 7.1 *or Theorem* 7.2 *and the value function V is continuous on $\overline{Q}$, then V is a viscosity solution of the dynamic programming equation* I(5.3′) *in Q.*

Proof. Fix $(t,x) \in Q$ and $v \in U$. Let $u(s) = v$ for $s \in [t, t_1]$. Let $x(s)$ be the solution of I(3.2) and I(3.3) and τ be the exit time of $(s, x(s))$ from Q. Then $\tau > t$. Also $(\tau, x(\tau)) \in \overline{Q}$ and by (12.2) there exists $\overline{u}(\cdot) \in \mathcal{U}(\tau, x(\tau))$. Now define a control $\hat{u}(\cdot)$ by

$$\hat{u}(s) = \begin{cases} v, & \text{if } s \in [t,\tau] \\ \overline{u}(s), & \text{if } s \in [\tau, t_1]. \end{cases}$$

Since $\overline{u}(\cdot) \in \mathcal{U}(\tau, x(\tau)), \hat{u}(\cdot) \in \mathcal{U}(t,x)$. Also $\lim_{s \downarrow t} \hat{u}(s) = v$, and therefore I(5.2) is satisfied. Finally we use Theorem 7.1 or Theorem 7.2 to conclude that the value function is a viscosity solution when it is continuous. □

When $t = t_1$, the terminal condition (9.3b) is satisfied. However, there are many viscosity solutions of I(5.3′) in Q satisfying (9.3b). As an example consider the equation (2.1). Then both $V(t,x) = \min\{1 - |x|, 1 - t\}$ and $W(t,x) = 1 - t$ solve (2.1) in $(0,1) \times (-1,1)$ and they both agree at the final time $W(1,x) = V(1,x) = 0$. Indeed in this example W is the value function with a state constraint, while V is the value function with the lateral boundary cost $g(t,x) \equiv 0$. Note that $V = g$ on the lateral boundary $\{-1,1\} \times [0,1]$ and V and W do not agree on the boundary. Also notice that the value of W on the boundary is not a priori given.

In this section, we derive a (differential) boundary condition satisfied by the value function when a state constraint is imposed. This condition plays an essential role in our discussion of boundary conditions when $\mathcal{U}(t,x) = \mathcal{U}^0(t)$ but $O \neq I\!R^n$. A general uniqueness result covering all these cases is stated in Theorem 14.1 and a more general result is proved in Section VII.8.

To derive the boundary condition formally, assume that $u^*(\cdot) \in \mathcal{U}(\overline{t}, \overline{x})$ is optimal at a boundary point $\overline{x} \in \partial O$. Suppose that $u^*(\cdot)$ is continuous at $\overline{t}$. Since $x^*(\cdot)$, the state corresponding to control $u^*(\cdot)$, satisfies the constraint, we have

$$f(\overline{t}, \overline{x}, u^*(\overline{t})) \cdot \eta(\overline{x}) \leq 0,$$

where $\eta(\overline{x})$ is the outward normal vector at $\overline{x} \in \partial O$. The continuity of $u^*(\cdot)$ and Remark I.5.1 imply that

$$-f(\bar{t}, \overline{x}, u^*(\bar{t})) \cdot D_x V(\bar{t}, \overline{x}) - L(\bar{t}, \overline{x}, u^*(\bar{t})) = H(\bar{t}, \overline{x}, D_x V(\bar{t}, \overline{x})),$$

if V is continuously differentiable. Hence for any $\gamma \geq 0$

$$\begin{aligned} (12.4) \qquad & H(\bar{t}, \overline{x}, D_x V(\bar{t}, \overline{x}) + \gamma \eta(\overline{x})) \\ & \quad \geq -f(\bar{t}, \overline{x}, u^*(\bar{t})) \cdot [D_x V(\bar{t}, \overline{x}) + \gamma \eta(\overline{x})] - L(\bar{t}, \overline{x}, u^*(\bar{t})) \\ & \quad = H(\bar{t}, \overline{x}, D_x V(\bar{t}, \overline{x})) - \gamma f(\bar{t}, \overline{x}, u^*(\bar{t})) \cdot \eta(\overline{x}) \\ & \quad \geq H(\bar{t}, \overline{x}, D_x V(\bar{t}, \overline{x})). \end{aligned}$$

This condition is an implicit inequality that has to be satisfied by $D_x V(\bar{t}, \overline{x})$ at the boundary point $(\bar{t}, \overline{x}) \in [t_0, t_1) \times \partial O$. In fact the convexity of H in the gradient variable p yields that the condition

$$H_p(\bar{t}, \overline{x}, D_x V(\bar{t}, \overline{x})) \cdot \eta(\overline{x}) \geq 0$$

is equivalent to (12.4), whenever H_p exists.

To obtain a weak formulation of (12.4), suppose that $V \in C^1(\overline{Q})$ and consider the difference $V - w$ with a smooth function $w \in C^\infty(\overline{Q})$. If the minimum is achieved at a boundary point $(t, x) \in [t_0, t_1) \times \partial O$, then

$$\frac{\partial}{\partial t} w(t, x) \leq \frac{\partial}{\partial t} V(t, x),$$

and

$$D_x w(t, x) = D_x V(t, x) + \gamma \eta(x)$$

for some $\gamma \geq 0$. Since V is a solution of I(5.3′), (12.4) yields

$$\begin{aligned} & -\frac{\partial}{\partial t} w(t, x) + H(t, x, D_x w(t, x)) \\ & \quad \geq -\frac{\partial}{\partial t} V(t, x) + H(t, x, D_x V(t, x) + \gamma \eta(x)) \\ & \quad \geq -\frac{\partial}{\partial t} V(t, x) + H(t, x, D_x V(t, x)) \\ & \quad = 0. \end{aligned}$$

So we make the following definitions, which require neither V nor the boundary ∂O to be differentiable.

Definition 12.1. We say that $W \in C(\overline{Q})$ is a *viscosity supersolution* of I(5.3′) on $[t_0, t_1) \times \overline{O}$ if for each $w \in C^\infty(\overline{Q})$,

$$(12.5) \qquad -\frac{\partial}{\partial t}w(\bar{t},\overline{x}) + H(\bar{t},\overline{x},D_x w(\bar{t},\overline{x})) \geq 0$$

at every $(\bar{t},\overline{x}) \in [t_0,t_1)\times\overline{O}\cap\arg\min\{(W-w)(t,x) : (t,x) \in \overline{Q}\}$ with $W(\bar{t},\overline{x}) = w(\bar{t},\overline{x})$.

In view of Remark 4.2, we do *not* have to require $W(\bar{t},\overline{x}) = w(\bar{t},\overline{x})$. Also, in the above definition we can replace $w \in C^\infty(\overline{Q})$ by $w \in C^1(Q)$ as was done in Section 6. Notice that in the above definition, the minimizer $(\bar{t},\overline{x})$ may be an element of the lateral boundary $[t_0,t_1) \times \partial O$. In that case $(\bar{t},\overline{x})$ is not an interior minimum of $W - w$, but we still require (12.5).

Definition 12.2. W is called a *constrained viscosity solution* of the dynamic programming equation I(5.3′) if W is a viscosity solution of I(5.3′) in Q and a viscosity supersolution of I(5.3′) on $[t_0,t_1) \times \overline{O}$.

We now have the following characterization of the value function. The uniqueness of constrained viscosity solutions will be discussed in Section 14. See Theorem 14.1 and the discussion following Theorem 14.1.

Theorem 12.2. *Suppose that f, L satisfy the hypotheses of Theorem* 7.1 *or Theorem* 7.2. *Assume that V is continuous on $\overline{Q}$. Then the value function V of the optimal control problem with a state constraint is a constrained viscosity solution of* I(5.3′).

Proof. In view of Theorem 12.1, it suffices to prove (12.5) for every smooth function $w \in C^\infty(\overline{Q})$ and a minimizer $(\bar{t},\overline{x}) \in [t_0,t_1) \times \partial O$ of the difference $V - w$ with $V(\bar{t},\overline{x}) = w(\bar{t},\overline{x})$. Since $\tau = t_1$ for every control, the dynamic programming principle I(4.3) states that

$$V(\bar{t},\overline{x}) = \inf_{u(\cdot)\in\mathcal{U}(\bar{t},\overline{x})} \int_{\bar{t}}^{\bar{t}+h} L(s,x(s),u(s))ds + V(\bar{t}+h, x(\bar{t}+h)),$$

for all $h > 0$ such that $\bar{t} + h \leq t_1$. Since $V(t,x) \geq w(t,x)$ for every $(\bar{t},\overline{x}) \in \overline{Q}$ and $V(\bar{t},\bar{x}) = w(\bar{t},\bar{x})$,

$$\begin{aligned} V(\bar{t},\overline{x}) &= w(\bar{t},\overline{x}) \\ &\geq \inf_{u(\cdot)\in\mathcal{U}(\bar{t},\overline{x})} \int_{\bar{t}}^{\bar{t}+h} L(s,x(s),u(s))ds + w(\bar{t}+h, x(\bar{t}+h)). \end{aligned}$$

We now choose $u^n(\cdot) \in \mathcal{U}(\bar{t},\overline{x})$ satisfying (7.7). Starting from (7.7) we follow the arguments of the proof of Theorem 7.1 and use the above inequality to obtain,

$$-\frac{\partial}{\partial t}w(\bar{t},\overline{x}) + H(\bar{t},\overline{x},D_x w(\bar{t},\overline{x})) \geq 0.$$

Hence V is a viscosity supersolution on $[t_0,t_1)\times\overline{O}$. If an optimal control exists, then we argue exactly as in Theorem 7.2. □

Example 12.1. As in Example 2.1, take $Q = [0,1) \times (-1,1), U = I\!R,\ f(t,x,v) = v, L(t,x,v) = 1 + \frac{1}{4}v^2$. This is a calculus of variations problem on the fixed time interval $t \le s \le t_1$, with state constraints $-1 \le x(s) \le 1$. The corresponding dynamic programming equation is,

$$-\frac{\partial}{\partial t}V(t,x) + \left(\frac{\partial}{\partial x}V(t,x)\right)^2 - 1 = 0. \tag{12.6}$$

Suppose that $W \in C^2([0,1]\times[-1,1])$ is a constrained viscosity solution of (12.6). Then (12.6) is satisfied at every interior point $(t,x) \in (0,1)\times(-1,1)$, and by continuity at every $(t,x) \in [0,1]\times[-1,1]$. Now suppose that for $w \in C^\infty([0,1]\times[-1,1])$, $W - w$ achieves its minimum at $(\bar t, 1)$ for some $\bar t \in [0,1)$. The differentiability of W yields that

$$\frac{\partial}{\partial t}W(\bar t,1) \ge \frac{\partial}{\partial t}w(\bar t,1)$$

and

$$\frac{\partial}{\partial x}W(\bar t,1) \le \frac{\partial}{\partial x}w(\bar t,1).$$

In fact for every $p \ge \frac{\partial}{\partial x}W(\bar t,1)$, there is a $w \in C^\infty([0,1]\times[-1,1])$, such that $W - w$ achieves its minimum at $(\bar t,1)$ and $\frac{\partial}{\partial x}w(\bar t,1) = p$. Indeed for $\gamma > 0$, let

$$w^\gamma(t,x) = p(x-1) + \frac{\partial}{\partial t}W(\bar t,1)(t-t_0) + \gamma(x-1)^2 + \gamma(t-\bar t)^2.$$

For γ sufficiently large, $W - w^\gamma$ is a concave function, and consequently $(\bar t,1)$ is the minimizer of it. Set $w = w^\gamma$; then the viscosity supersolution property of W yields

$$\begin{aligned}
0 &\le -\frac{\partial}{\partial t}w(\bar t,1) + (\frac{\partial}{\partial x}w(\bar t,1))^2 - 1 \\
&\le -\frac{\partial}{\partial t}W(\bar t,1) + p^2 - 1 \\
&= -\frac{\partial}{\partial t}W(\bar t,1) + \left(\frac{\partial}{\partial x}W(\bar t,1)\right)^2 - 1 + \left[p^2 - \left(\frac{\partial}{\partial x}W(\bar t,1)\right)^2\right] \\
&= p^2 - \left(\frac{\partial}{\partial x}W(\bar t,1)\right)^2
\end{aligned}$$

for every $p \ge \dfrac{\partial}{\partial x}W(\bar t,1)$. Therefore

$$\frac{\partial}{\partial x}W(t,1) \ge 0, \quad \forall t \in [0,1). \tag{12.7)(i}$$

A similar analysis yields

$$(12.7)(ii) \qquad \frac{\partial}{\partial x} W(t,-1) \le 0, \quad \forall t \in [0,1).$$

In fact, in this example (12.7) is equivalent to the supersolution property at the boundary, when W is continuously differentiable.

Another explanation of the boundary condition (12.7) is obtained by studying the optimal controls. Let $v^*(t,x)$ be as in I(5.21), i.e.,

$$v^*(t,x) = \arg\min_{v \in (-\infty,\infty)} \left\{ v \frac{\partial}{\partial x} W(t,x) + 1 + \frac{1}{4} v^2 \right\}$$

$$= \left\{ -2 \frac{\partial}{\partial x} W(t,x) \right\}, \ (t,x) \in \overline{Q}.$$

Let $W \in C^2([0,1] \times [-1,1])$ be a constrained viscosity solution of (12.6). Substitute $v^*(\cdot,\cdot)$ into I(3.2) to obtain

$$\frac{d}{ds} x^*(s) = -2 \frac{\partial}{\partial x} W(s, x^*(s)) \ , \quad s \ge t$$

with initial data $x^*(t) = x$. Then (12.7) yields that $x^*(s) \in \overline{O}$ for all $s \in [t, t_1]$. Hence $u^*(s) = v^*(s, x^*(s))$ is an admissible control.

As a simple example, take $\psi \equiv 0$. Then the optimal control is $u^*(s) \equiv 0$ and the value function is given by

$$V(t,x) = 1 - t \ , \quad (t,x) \in \overline{Q}.$$

V is the unique solution of (12.6) satisfying (12.7) and the terminal data $V(1,x) = \psi(x) = 0$.

II.13 Discussion of boundary conditions

Consider the deterministic optimal control problem, called class B in Section I.3. Recall that $\mathcal{U}(t,x) = \mathcal{U}^0(t)$ for every $(t,x) \in \overline{Q}$ and the state $x(s)$ is controlled up to its exit time from the closed domain $\overline{Q}$. When f, L, g and ψ satisfy the conditions of Section 10, the value function $V \in C(\overline{Q})$ and the lateral boundary condition (9.3a) is satisfied. However, in Example 2.3 we have shown that (9.3a) is not always satisfied.

In this section we will derive a weak (viscosity) formulation of the boundary condition (9.3). We then verify that the value function satisfies this viscosity formulation. Also a continuity result for the value function is stated at the end of the section.

First let us assume that f satisfies I(3.11). Fix $(\overline{t}, \overline{x}) \in [t_0, t_1) \times \partial O$ and define $u(s) = v(\overline{t}, \overline{x})$, $s \in [\overline{t}, t_1]$, where $v(\cdot,\cdot)$ is as in I(3.11). Then the exit time of the state corresponding to this control and initial data $x(\overline{t}) = \overline{x}$, is equal to $\overline{t}$. Hence,

$$V(\bar{t},\bar{x}) \le g(\bar{t},\bar{x}), \;\; (\bar{t},\bar{x}) \in [t_0,t_1) \times \partial O. \tag{13.1}$$

We continue by a formal derivation of a viscosity formulation of (9.3a). So suppose that

$$g(\bar{t},\bar{x}) > V(\bar{t},\bar{x}), \tag{13.2}$$

at some $(\bar{t},\bar{x}) \in [t_0,t_1) \times \partial O$, and that there exists an optimal control $u^*(\cdot) \in \mathcal{U}^0(\bar{t})$ which is continuous at $\bar{t}$. Let $x^*(\cdot)$ be the solution of I(3.2) with control $u^*(\cdot)$ and initial data $x^*(\bar{t}) = \bar{x}$, and τ^* be the exit time of $(s, x^*(s))$ from $\overline{Q}$. Then (13.2) yields $\tau^* > \bar{t}$. Consequently

$$f(\bar{t},\bar{x},u^*(\bar{t})) \cdot \eta(\bar{x}) \le 0, \tag{13.3}$$

where $\eta(\bar{x})$ is the unit outward normal vector of ∂O at $\bar{x}$. Recall that the above inequality is the starting point in our derivation of (12.4). So formally we conclude that at every $(\bar{t},\bar{x}) \in [t_0,t_1) \times \partial O$, either the boundary condition (9.3a) is satisfied or (12.4) holds.

We now make the following definition, which does not require that the Hamiltonian H is related to a control problem.

Definition 13.1. *Let $W \in C(\overline{Q})$.*

(a) W is a *viscosity subsolution* of (8.1) in Q and the lateral boundary condition (9.3a) if it is a viscosity subsolution of (8.1) in Q and for each $w \in C^\infty(\overline{Q})$,

$$\min\left\{ -\frac{\partial}{\partial t} w(\bar{t},\bar{x}) + H(\bar{t},\bar{x},D_x w(\bar{t},\bar{x}),\; W(\bar{t},\bar{x}) - g(\bar{t},\bar{x}) \right\} \le 0 \tag{13.4}$$

at every $(\bar{t},\bar{x}) \in \arg\max\{(W-w)(t,x) | (t,x) \in \overline{Q}\} \cap [t_0,t_1) \times \partial O$.

(b) W is a *viscosity supersolution* of (8.1) in Q and the lateral boundary condition (9.3a) if it is a viscosity supersolution of (8.1) in Q and for each $w \in C^\infty(\overline{Q})$,

$$\max\left\{ -\frac{\partial}{\partial t} w(\bar{t},\bar{x}) + H(\bar{t},\bar{x},D_x w(\bar{t},\bar{x})), W(\bar{t},\bar{x}) - g(\bar{t},\bar{x}) \right\} \ge 0, \tag{13.5}$$

at every $(\bar{t},\bar{x}) \in \arg\min\{(W-w)(t,x) | (t,x) \in \overline{Q}\} \cap [t_0,t_1) \times \partial O$.

(c) W is a *viscosity solution* of (8.1) in Q and (9.3a) if it is both a viscosity subsolution and a viscosity supersolution of (8.1) in Q and (9.3a).

Notice that (13.1) implies (13.4). Also we have formally argued that if $V(\bar{t},\bar{x}) < g(\bar{t},\bar{x})$ then V is a supersolution at $(\bar{t},\bar{x})$. Hence we have essentially shown that the value function V is a viscosity solution of I(5.3′) in Q and (9.3).

The state constraint problem can also be formulated as an exit control problem with lateral boundary function "$g \equiv +\infty$". In that case (13.4) is automatically satisfied and (13.5) is equivalent to (12.5). Hence the formal limit, "$g \uparrow +\infty$", of the above definition yields the definition of constrained viscosity solutions. In fact the value function $\overline{V}$ of the constrained viscosity is a viscosity solution of I(5.3′) in Q and (9.3) with any $g > \overline{V}$.

Finally observe that the value function V of the exit control problem satisfies (13.1). However, in Definition 13.1 we have used a weaker inequality (13.4). So if we were only interested in dynamic programming equation I(5.3′) we could replace (13.4) by (13.1). But, when the Hamiltonian H is not related to an optimal control problem, the unique viscosity solution of (8.1) in Q and (9.3a) satisfies (13.4) but not necessarily (13.1). So for a general Hamiltonian, Definition 13.1 is the correct formulation.

To obtain the continuity of the value function, let us assume that for every $(t,x) \in [t_0,t_1] \times \partial O$, there exists $\underline{v}(t,x) \in U$ satisfying

$$f(t,x,\underline{v}(t,x)) \cdot \eta(x) < 0. \tag{13.6}$$

The above condition together with I(3.11) yield that the boundary ∂O is "reachable" from nearby points and the converse is also true. Then using the type of argument introduced in Section 10 and the "reachability" of the boundary, whenever the state is near the boundary, one can prove that the value function is continuous. We refer to [S2] and [CGS] for the detailed proof of continuity. Here we only state the result.

Theorem 13.1. (a) *Suppose that f satisfies* I(3.11). *Then the value function V is a viscosity solution of* I(5.3′) *in Q and the lateral boundary condition* (9.3a), *provided that $V \in C(\overline{Q})$.*

(b) *Suppose that L, f, ψ and g satisfy the hypotheses of Theorem 7.1, and f satisfies* I(3.11) *and* (13.6). *Then $V \in C(\overline{Q})$.*

Example 13.1. Consider the equation (2.5) with boundary conditions

$$V(t,1) = 1\ , \qquad t \in [0,1], \tag{13.7)(i}$$

$$V(t,-1) = -1, \qquad t \in [0,1], \tag{13.7)(ii}$$

and the terminal condition (2.6). It is straightforward to check that V given by (2.4), is a viscosity solution of (2.5) in $Q = [0,1) \times (-1,1)$ and (13.7)(ii) is satisfied. Moreover, for every $t \in [0,1)$,

$$V(t,1) = (-1) \vee (1 - a + at) < 1.$$

Also except at $t = [(a-2)/a] \vee 0, V$ is differentiable at $(t,1)$ and

$$\frac{\partial}{\partial x} V(t,1) = \begin{cases} 0 \text{ if } 0 \le t < [(a-2)/a] \vee 0, \\ 1 \text{ if } [(a-2)/a] \vee 0 < t \le 1. \end{cases}$$

Since $\frac{\partial}{\partial x} V(t,1) \ge 0$, an argument similar to the one given in Example 12.1 yields (13.5). Hence V is a viscosity solution of (2.5) in Q and (13.7).

II.14 Uniqueness: first-order case

In this section we state a comparison result between viscosity subsolutions and supersolutions of (8.1) in Q with the lateral boundary condition (9.3a). As in Theorem 9.1 we assume that H satisfies (9.4). We also assume that the boundary of O satisfies a regularity condition: there are $\varepsilon_0, r > 0$ and an $I\!R^n$ valued, bounded, uniformly continuous map $\hat{\eta}$ of $\overline{O}$ satisfying

$$B(x + \varepsilon\hat{\eta}(x), r\varepsilon) \subset O, \;\; \forall x \in O, \varepsilon \in (0, \varepsilon_0]. \tag{14.1}$$

Here $B(x, r)$ denotes the set $\{y \in I\!R^n : |x - y| < r\}$.

Before we state our comparison result, we give an example to clarify which domains satisfy (14.1).

Example 14.1. Let $O = \{(x_1, x_2) \in I\!R^2 : x_1 > 0, x_2 > 0\}$. Then $\hat{\eta}(x) \equiv (1, 1)$ with $r = 1/\sqrt{2}$ and $\varepsilon_0 = 1$ satisfies (14.1).

In general any domain with a uniformly C^1 boundary or a piecewise linear boundary satisfies (14.1).

Theorem 14.1. *Assume* (9.4) *and* (14.1). *Let* W *be a viscosity subsolution and* V *be a viscosity supersolution of (8.1) in* Q *and the lateral boundary condition* (9.3a)*, respectively. If* Q *is unbounded, we also assume that* W *and* V *are bounded and uniformly continuous on* $\overline{Q}$*. Then*

$$\sup_{\overline{Q}} [W - V] \leq \sup_{\overline{O}} (W(t_1, x) - V(t_1, x)) \vee O.$$

A discussion of the above theorem is given in Section VII.8, see Remark VII.8.1. Also in that section a generalization of the above theorem for semi-continuous sub and supersolutions is given. This generalization to semi-continuous functions is a technically powerful tool. It is used elegantly by Barles and Perthame [BP1] in their study of problems with vanishing viscosity. Their approach is described in Chapter VII. We refer to [CGS] for the proof of the above theorem.

We claim that the above result also applies to the constrained viscosity solutions. Indeed let W be a bounded, uniformly continuous constrained viscosity solution of I(5.3′). Let K be an upper bound for W and the value function V of the state constrained problem. Then both W and V are viscosity solutions of I(5.3′) in Q with boundary condition (9.3a) with any g satisfying

$$g(t, x) > K, \;\; (t, x) \in [t_0, t_1) \times \partial O.$$

Then the comparison result Theorem 14.1, yields that $V = W$ provided that the value function V is bounded, uniformly continuous and $V(t_1, x) = W(t_1, x)$. Hence in this case, the value function is the unique constrained viscosity solution of I(5.3′) satisfying the terminal condition (9.3b). Moreover, the value functions of the constrained and unconstrained problems agree if the boundary data g is large enough.

II.15 Pontryagin's maximum principle (continued)

In Section I.6, we have given a proof of the Pontryagin's maximum principle for the fixed finite horizon control problem. The adjoint variable was shown there to be the gradient of the value function when the value function is differentiable. In this section we prove that the adjoint variable in general, belongs to the set of subdifferentials of the value function, as defined in (8.2i).

As in Theorem I.6.2, we assume that $\mathcal{U}(t,x) = \mathcal{U}^0(t)$ and $O = I\!R^n$.

Theorem 15.1. *Let $u^*(\cdot)$ be an optimal control at (t,x) which is right continuous at each $s \in [t,t_1)$, and $P(s)$ be the adjoint variable satisfying* I(6.2), I(6.3) *and* I(6.5). *Then for each $s \in [t,t_1)$,*

$$(H(s,x^*(s),P(s)),P(s)) \in D^+V(s,x^*(s)), \tag{15.1}$$

where $D^+V(t,x)$ is the set of subdifferentials of V at (t,x), as defined in (8.2i).

Proof. Consider the function $J(r,y;u^*(\cdot))$. Since $u^*(\cdot)$ is admissible at every (r,y),

$$V(r,y) \le J(r,y;u^*(\cdot)), \;\; \forall (r,y) \in Q_0.$$

Moreover, equality holds at $(s,x^*(s))$ with $s \in [t,t_1]$. Hence by Definition 8.1, it suffices to show that J is a continuously differentiable function with

$$D_y J(s,x^*(s);u^*(\cdot)) = P(s), \tag{15.2}$$

$$\frac{\partial}{\partial r} J(s,x^*(s);u^*(\cdot)) = H(s,x^*(s),P(s)), \tag{15.3}$$

for all $s \in [t,t_1)$. First note that (15.2) is proved in Section I.6. See I(6.8) and the calculations following it. Using the notation $x(s;r,y)$ to denote the state at time $s \ge r$ with initial condition $x(r;r,y) = y$, we compute as in the proof of I(6.8) that

$$\begin{aligned} \frac{\partial}{\partial r} J(r,x^*(r);u^*(\cdot)) &= -L(r,x^*(r),u^*(r)) + \\ +\sum_{j=1}^n \Bigg\{ \int_r^{t_1} & \frac{\partial}{\partial x_j} L(s,x^*(s),u^*(s)) z_j(s;r,x^*(r)) ds \\ &+ \frac{\partial}{\partial x_j} \psi(t_1,x^*(t_1)) z_j(t_1;r,x^*(r)) \Bigg\}, \end{aligned} \tag{15.4}$$

where

$$z(s;r,y) = \frac{\partial}{\partial r} x(s;r,y).$$

Since

$$x(s;r,y) = y + \int_r^s f(\rho, x(\rho;r,y), u^*(\rho)) d\rho,$$

by differentiation with respect to r, we obtain for $j = 1, \cdots, n$,

$$z_j(s; r, y) = -f_j(r, y, u^*(r)) + \sum_{k=1}^{n} \left\{ \int_r^s \frac{\partial}{\partial x_k} f_j(\rho, x(\rho; r, y), u^*(\rho)) z_k(\rho; r, y) d\rho \right\}.$$

Set $z(s) = z(s; r, x^*(r))$. Then, for each $j = 1, \ldots, n$,

$$\frac{d}{ds} z_j(s) = \sum_{k=1}^{n} \frac{\partial}{\partial x_k} f_j(s, x^*(s), u^*(s)) z_k(s), \ \ \forall s \in (r, t_1), \tag{15.5}$$

and

$$z(r) = -f(r, x^*(r), u^*(r)). \tag{15.6}$$

We claim that

$$\frac{\partial}{\partial r} J(r, x^*(r); u^*(\cdot)) = -L(r, x^*(r), u^*(r)) - f(r, x^*(r), u^*(r)) \cdot P(r).$$

Indeed, (15.5) and I(6.2) yield

$$\frac{d}{ds}[z(s) \cdot P(s)] = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j} L(s, x^*(s), u^*(s)) z_j(s).$$

Hence the initial condition (15.6) and the terminal condition I(6.5) yield

$$\begin{aligned} -f(r, x^*(r), u^*(r)) \cdot P(r) &= z(r) \cdot P(r) \\ &= z(t_1) \cdot P(t_1) - \int_r^{t_1} \frac{d}{ds}[z(s) \cdot P(s)] ds \\ &= \sum_{j=1}^{n} \{ \frac{\partial}{\partial x_j} \psi(t_1, x^*(t_1)) z_j(t_1) + \\ &\quad + \int_r^{t_1} \frac{\partial}{\partial x_j} L(s, x^*(s), u^*(s)) z_j(s)) ds \}. \end{aligned}$$

Therefore using (15.4), we conclude that

$$\frac{\partial}{\partial r} J(r, x^*(r); u^*(\cdot)) = -L(r, x^*(r), u^*(r)) - f(r, x^*(r), u^*(r)) \cdot P(r).$$

Consequently I(6.3) implies (15.3). □

II.16 Historical remarks

The definition of viscosity solutions was first given by Crandall and Lions [CL1] for a general first order partial differential equation. They also proved the uniqueness of viscosity solutions. Then equivalent definitions were provided by Crandall, Evans and Lions [CEL]. Lions' research monograph [L4] provides an account of the earlier theory for dynamic programming equations. The second order equations were then studied by Lions [L1-3] and Jensen [J]. For a complete list of the relevant literature, we refer to the recent survey article of Crandall, Ishii and Lions [CIL1].

In Section 9 we closely followed the proof of Crandall, Evans and Lions. More sophisticated uniqueness proofs are now available. See Ishii [I4] and Crandall, Ishii and Lions [CIL2]. These proofs in some cases provide a modulus of continuity for viscosity solutions. The uniqueness of unbounded solutions was proved by Ishii [I3]. Barron and Jensen provided an alternate proof for dynamic programming equations [BJ1]. The proof of Barron and Jensen also provides more information about the value function.

The analysis of control problems with a state constraint was first given by Soner [S2], and then by Capuzzo-Dolcetta and Lions [CaL]. The general treatment of exit time problems is due to Ishii [I2], and Barles and Perthame [BP2]. Also see Subbotin [Su2] for a different approach using the characteristics.

The connection between the Pontryagin's maximum principle and viscosity solutions was first explored by Barron and Jensen [BJ2]. Our discussion is more closely related to Zhou's proof [Zh]. The connection between the maximum principle and viscosity solutions was extensively studied by Cannarsa and Frankowska [CF1,2] [Fra]. For an elegant and deep treatment of the maximum principle and related topics we refer to Clarke [Cle1,2].

In this book we did not cover several important issues. Existence of viscosity solutions of partial differential equations which are not necessarily dynamic programming equations, differential games [ES1][KSu][LSo1][Su1,2], Perron's method [I6], singularities of viscosity solutions [CS] and the Neumann problem [L7] are some examples of the topics which are not covered in this book. An excellent discussion of these topics with a complete list of references is given by Crandall, Ishii and Lions [CIL1]. Another topic which is not covered is viscosity solutions in infinite dimensions. See [CL3] [CL4] [Ta]. These arise, for instance, in optimal control problems in which the state dynamics are governed by PDEs, rather than ODEs of the type I(3.2).

The sets of sub and superdifferentials defined in Section 8 are closely related to subgradients of nonsmooth analysis. In 1973 Clarke [Cle3] introduced a subgradient $\partial W(t,x)$ which is similar to sub- and superdifferentials. Other subgradients that are more closely related to the sub- and superdifferentials of Section 8 were then introduced by Rockafellar [R2] and Mordukhovich [Mu1,2]. We refer to [Cle1,2] for the connection between these subgradients and their applications to optimal control.

III

Optimal Control of Markov Processes: Classical Solutions

III.1 Introduction

The purpose of this chapter is to give a concise, nontechnical introduction to optimal stochastic control for Markov processes. Just as was done in Chapters I and II for deterministic optimal control problems, the dynamic programming approach will be followed. For a finite time horizon stochastic control problem, the dynamic programming equation is a nonlinear evolution equation, of the form (7.5) below. In the particular case of controlled Markov diffusion processes, the dynamic programming equation becomes a second order nonlinear partial differential equation. Controlled Markov diffusions will be considered in much more detail in Chapters IV and V. When there is a sufficiently regular "classical" solution to the dynamic programming equation, with appropriate terminal data at a final time t_1, then a Verification Theorem provides a solution to the problem. This technique is applied to some illustrative examples in Section 8, and for infinite time horizon models in Section 9. For deterministic control, corresponding Verification Theorems appeared in Sections I.5 and I.7.

Sections 2 - 5 are intended to provide a brief background sketch about continuous time Markov processes, with references and examples of typical kinds of Markov processes encountered in control applications. Experts on Markov processes will recognize that the way in which we define domains for backward evolution operators and generators in Section 2 is not quite standard. However it is convenient for our purposes.

In Sections 6 and 7 we begin the discussion of controlled Markov processes at a formal (heuristic) level. Then in Sections 8 and 9 we give a precise definition of value function, denoted by V_{AS}, using the concept of admissible system and prove the Verification Theorems.

As already seen in case of deterministic control, the value function often lacks smoothness properties needed to be a classical solution to the dynamic programming equation. In succeeding chapters, we study controlled Markov processes without requiring smoothness of value functions. For controlled Markov diffusions, the value function is interpreted as a viscosity solution to the dynamic programming (or Hamilton - Jacobi - Bellman) partial differential equation. The viscosity solution framework also turns out to be quite convenient in studying dependence on small parameters (Chapter VII) and in proving convergence of finite - difference numerical methods for computing value functions approximately (Chapter IX.) In Section 10 we shall consider briefly "general" classes of controlled Markov processes, without assuming that the value function is a classical solution to its dynamic programming equation. The technique, due to M. Nisio, is to construct a corresponding nonlinear semigroup. This approach is closely related to the abstract dynamic programming principle formulated in Section II.3.

III.2 Markov processes and their evolution operators

In this section we summarize some basic concepts and results about continuous time Markov processes. We use the following notations throughout. Σ denotes the state space of a continuous time Markov process $x(s)$. We shall always assume that Σ is a complete separable metric space. In the special case when Σ is discrete, then $x(s)$ is a continuous time Markov chain. Another case of particular interest in this book is when $\Sigma = I\!R^n = n$ - dimensional euclidean space and $x(s)$ is a Markov diffusion process governed by a system of stochastic differential equations (Section 4 and Chapter IV). We let $\mathcal{B}(\Sigma)$ denote the Borel σ - algebra, namely the least σ - algebra containing all open subsets of Σ.

Elements of the state space Σ will be denoted by $x, y, \cdots$. Let $I_0 \subset I\!R^1$ be an interval half open to the right, and let $\overline{I}_0$ be the closure of I_0. The elements of I_0 will be denoted by $s, t, \cdots$. We will consider two cases: (1) $I_0 = [t_0, t_1)$ a finite interval, $\overline{I}_0 = [t_0, t_1]$, or (2) $I_0 = \overline{I}_0 = [0, \infty)$. Let $x(s) = x(s, \omega)$ be a Σ- valued stochastic process, defined for $s \in I$ and $\omega \in \Omega$, where I is some subinterval of $\overline{I}_0$ and $(\Omega, \mathcal{F}, P)$ is some probability space. Roughly speaking, the Markov property asserts that the state $x(t)$ contains all probabilistic information relevant to the evolution of the process for times $s > t$. This can be expressed precisely as follows. Consider any finite set of times $s_1 < s_2 < \cdots < s_m < s$ in I. Then for all $B \in \mathcal{B}(\Sigma)$

$$(2.1) \qquad P(x(s) \in B | x(s_1), \cdots, x(s_m)) = P(x(s) \in B | x(s_m))$$

$P-$ almost surely. The conditional probabilities in (2.1) denote versions of conditional probabilities with respect to the σ - algebras $\mathcal{F}(x(s_1), \cdots, x(s_m))$ and $\mathcal{F}(x(s_m))$, generated respectively by the Σ - valued random variables $x(s_1), \cdots, x(s_m)$ and by $x(s_m)$ respectively.

The transition distribution of a Markov process $x(s)$ is defined for $t < s, s, t \in I_0$ and for $x \in \Sigma$ by

(2.2) $$\hat{P}(t,x,s,B) = P(x(s) \in B | x(t) = x), \ \forall B \in \mathcal{B}(\Sigma).$$

It is required that $\hat{P}(t,\cdot,s,B)$ is $\mathcal{B}(\Sigma)$ - measurable for fixed s,t,B, and that $\hat{P}(t,x,s,\cdot)$ is a probability measure for fixed t,s,x. Moreover, the Chapman-Kolmogorov equation

(2.3) $$\hat{P}(t,x,s,B) = \int_\Sigma \hat{P}(r,y,s,B)\hat{P}(t,x,r,dy)$$

holds for $t < r < s, t, r, s \in I_0$.

For the introductory discussion to follow the reader may find [EK, Chaps. 1, 4], [GS1, Chap. 7], or [Fe, Chaps. IX, X] helpful.

Property (2.1) can be expressed in the following slightly more elegant form. Let $\mathcal{F}_r^x = \mathcal{F}(x(\theta), \theta \le r)$ denote the smallest $\sigma-$ algebra with respect to which the Σ - valued random variables $x(\theta)$ are measurable for all $\theta \le r$ $(\theta, r \in I)$. Then (2.1) is equivalent to

(2.1$'$) $$P[x(s) \in B | \mathcal{F}_r^x] = \hat{P}(r, x(r), s, B)$$

for $r < s$ $(r, s \in I)$.

Let us now define a family of linear operators $T_{t,s}$ associated with the transition distribution of a Markov process. For $t < s$ $(s, t \in I_0)$, let

(2.4) $$T_{t,s}\phi(x) = \int_\Sigma \phi(y)\hat{P}(t,x,s,dy),$$

for all real-valued, $\mathcal{B}(\Sigma)$ - measurable ϕ such that the integral exists. By the Chapman - Kolmogorov equation (2.3),

(2.5) $$T_{tr}[T_{rs}\phi] = T_{ts}\phi, \ t < r < s.$$

In the theory of Markov processes, ϕ is often restricted to belong either to the space $B(\Sigma)$ of all bounded, $\mathcal{B}(\Sigma)$ - measurable functions, or to a subspace of bounded uniformly continuous functions. However, if the state space Σ is not compact, we shall wish to consider unbounded ϕ as well.

We shall denote the right side of (2.4) by $E_{tx}\phi(x(s))$, the subscripts indicating that we have specified the data $x(t) = x$. Thus,

(2.4$'$) $$T_{ts}\phi(x) = E_{tx}\phi(x(s)), \ t < s.$$

Backward evolution operators and equations. Let $I_0 = [t_0, t_1]$ be a finite interval and Φ denote a real - valued function on $\overline{I}_0 \times I\!R^n$. We define a linear operator A by

(2.6) $$A\Phi(t,x) = \lim_{h \to 0^+} h^{-1}[E_{tx}\Phi(t+h, x(t+h)) - \Phi(t,x)]$$

provided the limit exists for each $x \in \Sigma$ and each t which is not the right end point of I. Let $\mathcal{D}(A)$ be a space of functions Φ, such that $A\Phi$ is defined for each $\Phi \in \mathcal{D}(A)$ and moreover the following hold for each $\Phi \in \mathcal{D}(\mathcal{A})$:

(i) $\Phi, \partial\Phi/\partial t$ and $A\Phi$ are continuous on $\bar{I}_0 \times \Sigma$;

(ii) $E_{tx}|\Phi(s,x(s))| < \infty,\ E_{tx}\int_t^s |A\Phi(r,x(r))|dr < \infty$ for $t < s$ $(s,t \in \bar{I}_0)$;

(iii) (Dynkin's formula) For $t < s$,

$$E_{tx}\Phi(s,x(s)) - \Phi(t,x) = E_{tx}\int_t^s A\Phi(r,x(r))dr. \tag{2.7}$$

The Dynkin formula (2.7) is implied by the following property:

$$\Phi(s,x(s)) - \Phi(t,x) - \int_t^s A\Phi(r,x(r))dr \tag{2.7$'$}$$

is a $\{\mathcal{F}_s, P\}$ martingale, where $\{\mathcal{F}_s\}$ is some increasing family of σ - algebras such that $x(s)$ is $\mathcal{F}_s$ - measurable.

We shall need $\mathcal{D}(A)$ to be large enough that the expectations $E_{tx}\Phi(s,x(s))$ for all $\Phi \in \mathcal{D}(A)$ determine the transition distribution $\hat{P}(t,x,s,\cdot)$. This will be true, in particular, if $\mathcal{D}(A)$ contains some subset $\mathcal{D}$ of the space of bounded uniformly continuous functions on $\bar{I}_0 \times \Sigma$, with $\mathcal{D}$ dense in the uniform norm.

We shall call A the *backward evolution operator*, acting on $\mathcal{D}(A)$ satisfying (i), (ii) and (iii) above for all $\Phi \in \mathcal{D}(A)$. If $\Phi(t,x) = \phi(x)$ does not depend on t, then the right side of (2.6) equals to $\lim_{h\to 0^+} h^{-1}[T_{t,t+h}\phi(x) - \phi(x)]$. We denote minus the left side of (2.6) by $G_t\phi(x)$. Thus

$$G_t\phi(x) = -\frac{\partial}{\partial s}T_{ts}\phi(x)|_{s=t^+}. \tag{2.8}$$

A formal calculation suggests that we should have

$$A\Phi = \frac{\partial\Phi}{\partial t} - G_t\Phi(t,\cdot) \tag{2.9}$$

where the notation indicates that the operator G_t acts in the "state variable" x.

In later sections, the choice of $\mathcal{D}(A)$ will depend on the various classes of Markov processes being studied there. For all Markov process which we consider, (2.9) holds.

Backward evolution equation. Given continuous functions $\ell(t,x)$ on $[t_0,t_1] \times \Sigma$ and $\psi(x)$ on Σ, consider the equation

$$0 = A\Phi + \ell(t,x), \quad t_0 \le t \le t_1, \tag{2.10}$$

with the final (Cauchy) data

$$\Phi(t_1,x) = \psi(x). \tag{2.11}$$

The linear inhomogeneous equations (2.10) is called a *backward* evolution equation. If it has a solution $\Phi \in \mathcal{D}(A)$ which also satisfies (2.11), then by the Dynkin formula (2.7) with $s = t_1$

(2.12) $$\Phi(t,x) = E_{tx}\left\{\int_t^{t_1} \ell(s,x(s))ds + \psi(x(t_1))\right\}.$$

This gives a probabilistic formula for solutions to the backward evolution equation. In stochastic control theory, ℓ is often called a *running cost* function and ψ a *terminal cost* function. Formula (2.10) expresses $\Phi(t,x)$ as a total expected cost over the time interval $[t_0, t_1]$.

III.3 Autonomous (time-homogeneous) Markov processes

Let us now take $I_0 = I = [0,\infty)$. A Markov process is called *autonomous* (or *time-homogeneous*) if the transition distribution satisfies

$$\hat{P}(t,x,s,B) = \hat{P}(0,x,s-t,B), \quad 0 \le t < s.$$

In the autonomous case, we will always take initial time $t = 0$. We also write $E_x = E_{0x},\ T_s = T_{0s}$. Thus,

(3.1) $$T_s\phi(x) = E_x\phi(x(s)).$$

Equation (2.5) becomes the semigroup property

(3.2) $$T_{r+s}\phi = T_r(T_s\phi)\ ,\ r,s > 0.$$

In the autonomous case, $G_t = G$ where according to (2.8)

(3.3) $$G\phi(x) = -\lim_{h\to 0^+} h^{-1}[T_h\phi(x) - \phi(x)],\ x \in \Sigma.$$

We suppose that (3.3) holds for $\phi \in D(G)$ where $D(G)$ is a subspace of the space $C(\Sigma)$ of continuous functions on Σ, which has the following property: If $\Phi(t,x) = \eta(t)\phi(x)$ with $\eta \in C^1([0,\infty))$ and $\phi \in D(G)$, then $\Phi \in \mathcal{D}(A) = \mathcal{D}_{[0,t_1]}(A)$ for each $t_1 < \infty$, and

$$A(\eta\phi) = \eta_t\phi - \eta G\phi.$$

We shall call $-G$ the *generator* of the autonomous Markov process $x(s)$. The Hille-Yoshida theorem gives sufficient conditions that $D(G)$ contains "sufficiently many" functions ϕ, and that the transition distributions

$$\hat{P}(s,x,\cdot) = \hat{P}(0,x,s,\cdot)$$

are determined by G. See [EK, Chap. 4].

Discounted infinite horizon expected cost. Let us fix $\beta > 0$, which should be regarded as a discount factor. Let ℓ be continuous on Σ, with

$$E_x \int_0^\infty e^{-\beta s}|\ell(x(s)|ds < \infty$$

for all $x \in \Sigma$. Consider the linear inhomogeneous equation

$$(3.4) \qquad \beta\phi = -G\phi + \ell(x) \quad , \quad x \in \Sigma.$$

Suppose that $\phi \in D(G)$ is a solution to (3.4), with the property that

$$(3.5) \qquad \lim_{t_1 \to \infty} e^{-\beta t_1} E_x \phi(x(t_1)) = 0.$$

Since $A(e^{-\beta t}\phi) = e^{-\beta t}(-G\phi - \beta\phi)$, the Dynkin formula (2.7) with $\Phi = e^{-\beta t}\phi$ and s replaced by t_1 gives

$$e^{-\beta t_1} E_x \phi(x(t_1)) - \phi(x) = -E_x \int_0^{t_1} e^{-\beta s}(G\phi + \beta\phi)(x(s))ds.$$

From (3.4) and (3.5) we get

$$(3.6) \qquad \phi(x) = E_x \int_0^\infty e^{-\beta s}\ell(x(s))ds.$$

The right side of (3.6) is called the *discounted infinite horizon expected cost*, for the running cost function $\ell(x)$.

Remark 3.1. If Σ is compact, then any continuous function ϕ on Σ is bounded. In that case, condition (3.5) holds automatically. Formula (3.6) gives a probabilistic representation of any solution ϕ to (3.4), among solutions satisfying (3.5). However, for noncompact Σ, property (3.5) implicitly prescribes some kind of growth condition on $\phi(x)$ for "large" x. The following simple example illustrates this point.

Example 3.1. Let $\Sigma = I\!R^1$ = real line and $G\phi = -\frac{1}{2}\phi''$. Then $-G$ generates the Markov process $x(s) = x + w(s), s \geq 0$, where $w(s)$ is a standard brownian motion. Let $\ell(x) = x^2, \beta = 1$. Then $\phi(x) = x^2 + 1$ is the desired solution to (3.4), for which (3.5) and therefore (3.6) holds. The general solution to (3.4) is

$$\phi(x) = x^2 + 1 + c_1 \exp(\sqrt{2}x) + c_2 \exp(-\sqrt{2}x).$$

Since $\exp(-s \underline{+} \sqrt{2}x(s))$ is a martingale, (3.5) is satisfied only when $c_1 = c_2 = 0$. The other solutions grow exponentially as $x \to +\infty$ or $x \to -\infty$. These solutions are excluded by requiring that $\phi \in C_p^2(I\!R^1)$ as defined in Section 9 below.

III.4 Classes of Markov processes

Let us now give examples of some kinds of Markov processes which arise in stochastic control applications. In the next section we shall discuss in somewhat greater detail the class of Markov diffusion processes on n dimensional Euclidean $I\!R^n$.

(a) *Finite state, continuous time Markov chains.* In this case, Σ is a finite set, and

$$G_t\phi(x) = -\sum_{y\neq x}\rho(t,x,y)[\phi(y)-\phi(x)], \tag{4.1}$$

where $\rho(s,x,y)$ represents an infinitesimal rate at which $x(s)$ jumps from x to y:

$$\rho(s,x,y) = \lim_{h\to 0} h^{-1}P[x(s+h)=y|x(s)=x].$$

If $\rho(\cdot,x,y)$ is continuous, then given any (t,x) there is a Markov chain $x(s)$ for $s\geq t$ with initial data $x(t)=x$. The Dynkin formula (2.7) holds for any Φ such that $\Phi(\cdot,x)$ and $\Phi_t(\cdot,x)$ are continuous.

For Markov chains with an infinite number of states there are additional technical restrictions which we will not discuss. See [Ch].

(b) *Deterministic evolution in $\mathbb{R}^n$.* Consider an ordinary differential equation in $\mathbb{R}^n$, written in vector-matrix notation as

$$\frac{dx}{ds} = f(s,x(s))\ , \quad t\leq s\leq t_1, \tag{4.2}$$

with initial data $x(t)=x$. Let $\overline{Q}_0 = [t_0,t_1]\times\mathbb{R}^n$. We assume that:

$$\begin{cases} (i) & f \text{ is continuous on } \overline{Q}_0; \\ (ii) & \text{There exists } M \text{ such that} \\ & |f(s,x)|\leq M(1+|x|) \text{ for all } (s,x)\in\overline{Q}_0; \\ (iii) & \text{For each } a, \text{ there exists } K_a \\ & \text{such that } |f(s,x)-f(s,y)|\leq K_a|x-y| \\ & \text{whenever } |x|\leq a,\ |y|\leq a. \end{cases} \tag{4.3}$$

When (4.3) (iii) holds, $f(s,\cdot)$ is said to satisfy a *local Lipschitz condition*, uniformly with respect to s. If $f(s,\cdot)$ is differentiable at each point $x\in\mathbb{R}^n$, then (iii) is equivalent to the condition $|f_x(s,x)|\leq K_a$ whenever $|x|\leq a$, where f_x denotes the gradient of $f(s,\cdot)$. In case $K=K_a$ can be chosen independent of a, then $f(s,\cdot)$ is said to satisfy a *uniform Lipschitz condition.* If f is continuous on $\overline{Q}_0$ and satisfies a uniform Lipschitz condition, then f also satisfies (4.3) (ii) with M the larger of K and $\max_{0\leq s\leq T}|f(s,0)|$. In fact,

$$|f(s,x)|\leq|f(s,0)|+|f(s,x)-f(s,0)|\leq M(1+|x|).$$

The assumptions (4.3) imply that (4.2) with initial data $x(t)=x$ has a unique solution. Although $x(s)$ is nonrandom, we can regard it as a Markov process. The Fundamental Theorem of Calculus gives

$$G_t\phi(x) = -f(t,x)\cdot D\phi(x).$$

For the deterministic evolution on $I\!R^n$ defined by (4.2), the backward evolution operator is a first order linear partial differential operator:

$$A\Phi(t,x) = \Phi_t(t,x) + f(t,x) \cdot D_x\Phi(t,x). \tag{4.4}$$

Here $D\phi$ denotes the gradient of ϕ, and $D_x\Phi$ the gradient of $\Phi(t,\cdot)$. We may take $\mathcal{D}(A) = C^1(\overline{Q}_0)$, the space of all Φ with continuous first order partial derivatives. The Dynkin formula (2.7) is a consequence of the Fundamental Theorem of Calculus.

(c) *Random evolution with Markov chain parameters.* Let $z(s)$ be a finite state Markov chain, with state space a finite set Z. We regard $z(s)$ as a "parameter process". On any time interval where $z(s) = z$ is constant, $x(s)$ satisfies the ordinary differential equation

$$\frac{dx}{ds} = f(s, x(s), z).$$

We assume that $f(\cdot,\cdot,z)$ satisfies the conditions (4.3) for each $z \in Z$. Let $t \le s \le t_1$, and let $\tau_1 < \tau_2 < \cdots < \tau_m$ denote the successive jump times of the parameter process $z(s)$ during $[t, t_1]$. We let $\tau_0 = t, \tau_{m+1} = t_1$, and define $x(s)$ by

$$\frac{dx}{ds} = f(s, x(s), z(\tau_i^+)),\ \ \tau_i \le s < \tau_{i+1},\ \ i = 0,\cdots,m,\ \ x(t) = x, \tag{4.5}$$

with the requirement that $x(\cdot)$ is continuous at each jump time τ_i. The process $x(s)$ is not Markov. However, $(x(s), z(s))$ is a Markov process, with state space $\Sigma = I\!R^n \times Z$. For each $\Phi(t,x,z)$ such that $\Phi(\cdot,\cdot,z) \in C^1(\overline{Q}_0)$ we have

$$\begin{aligned} A\Phi(t,x,z) &= \Phi_t(t,x,z) + f(t,x,z) \cdot D_x\Phi(t,x,z) \\ &\quad + \sum_{\zeta \ne z} \rho(t,z,\zeta)\left[\Phi(t,x,\zeta) - \Phi(t,x,z)\right]. \end{aligned} \tag{4.6}$$

The Dynkin formula (2.7) is a special case of a result which we will derive in Appendix B. The middle term on the right side of (4.6) comes from the deterministic evolution of $x(s)$ during intervals of constancy of $z(s)$, as in Example (b), and the last term comes from the parameter process $z(s)$.

Example (c) is a special case of a class of Markov processes $(x(s), z(s))$ called *random evolutions*, for which the state space is a product $\Sigma = \Sigma_1 \times \Sigma_2$. If for simplicity we consider only the autonomous case, then the generator G takes the form

$$G\phi(x,z) = G_1^z\phi(\cdot,z)(x) + G_2\phi(x,\cdot)(z).$$

For fixed $z \in \Sigma_2$, $-G_1^z$ generates a Markov process with state space Σ_1; and $-G_2$ generates the "parameter process" $z(s)$ with state space Σ_2.

Example (c) can be generalized in a different direction by allowing x-dependent jumping rates $\rho(s,x,z,\zeta)$ for the $z(s)$ process. The resulting $x(s)$

is an example of a *piecewise deterministic* stochastic process. See [Dav 2].
(d) *Jump Markov processes.* In this case, $A = \frac{\partial}{\partial t} - G_t$, with

$$G_t\phi(x) = -\theta(t,x) \int_\Sigma [\phi(y) - \phi(x)]\, \Pi(t,x,dy).$$

Here $\theta(t,x)$ measures the intensity with which jumps occur from state x at time t, and $\Pi(t,x,\cdot)$ is the probability distribution of the post - jump location y. See [EK, p.162].

(e) *Markov diffusion processes on* $\mathbb{R}^n$. This will be discussed in the next section. The backward evolution operator is a second-order partial differential operator, of parabolic type (possibly degenerate parabolic.)

(f) *Processes with generators of Levy form.* This class is sufficiently general to include most "reasonable" Markov processes on $\mathbb{R}^n$. We shall not give the expression for the backward evolution operator A, but refer the reader to [EK, p.379].

III.5 Markov diffusion processes on $\mathbb{R}^n$; stochastic differential equations

In this section we take $\Sigma = \mathbb{R}^n$ and as before let $\overline{Q}_0 = [t_0, t_1] \times \mathbb{R}^n$. A Markov process is called an *n-dimensional diffusion process* if:

(1) For every $\varepsilon > 0$
$$\lim_{h \to 0^+} h^{-1} \int\limits_{|x-y|>\varepsilon} \hat{P}(t,x,t+h,dy) = 0:$$

(2) There exist functions $a_{ij}(t,x), f_i(t,x),\ i,j = 1\cdots,n$,
such that for every $\varepsilon > 0$

$$\lim_{h \to 0^+} h^{-1} \int_{|x-y|\le\varepsilon} (y_i - x_i)\hat{P}(t,x,t+h,dy) = f_i(t,x)$$

$$\lim_{h \to 0^+} h^{-1} \int_{|x-y|\le\varepsilon} (y_i - x_i)(y_j - x_j)\hat{P}(t,x,t+h,dy) = a_{ij}(t,x).$$

These limits hold for each $x \in \mathbb{R}^n$ and $t \in I_0$. The vector function $f = (f_i, \cdots, f_n)$ is called the *local drift coefficient* and the matrix-value function $a = (a_{ij})$ the *local covariance matrix*. The justification for these names is as follows. Suppose that instead of (1) the slightly stronger condition

$$(1^{'}) \qquad \lim_{h \to 0^+} h^{-1} \int_{|x-y|\le\varepsilon} |y-x|^2 \hat{P}(t,x,t+h,dy) = 0$$

holds. Then from $(1^{'})$ and (2), $f(s,x(s))h$ and $a(s,x(s))h$ are good approximations to the mean and covariance matrix of the increment $x(s+h) - x(s)$ conditioned on $x(s)$.

Let $C^{1,2}(\overline{Q}_0)$ denote the space of $\Phi(t,x)$ such that Φ and the partial derivatives $\Phi_t, \Phi_{x_i}, \Phi_{x_i x_j}, i,j = 1,\cdots,n$, are continuous on $\overline{Q}_0$. For $\Phi \in C^{1,2}(\overline{Q}_0)$, let

$$A\Phi = \Phi_t + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(t,x)\Phi_{x_i x_j} + \sum_{i=1}^{n} f_i(t,x)\Phi_{x_i}. \tag{5.1}$$

In later chapters, we will often write (5.1) in the more compact form

$$A\Phi = \Phi_t + \frac{1}{2}\operatorname{tr} a(t,x)D_x^2\Phi + f(t,x)\cdot D_x\Phi. \tag{5.1$'$}$$

The matrices $(a_{ij}(t,x))$ are symmetric and nonnegative definite. If there exists $c > 0$ such that for all $\xi \in I\!\!R^n$,

$$\sum_{i,j=1}^{n} a_{ij}(t,x)\ \xi_i\xi_j \geq c|\xi|^2, \tag{5.2}$$

then A is a *uniformly parabolic* partial differential operator. If (5.2) holds only with $c = 0$, then A is called *degenerate parabolic*. Similarly, in the autonomous case $f = f(x), a = a(x)$, we define for $\phi \in C^2(I\!\!R^n)$

$$-G\phi = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x)\phi_{x_i x_j} + \sum_{i=1}^{n} f_i(x)\phi_{x_i}. \tag{5.3}$$

If (5.2) holds, then $-G$ is a *uniformly elliptic* operator; otherwise $-G$ is *degenerate elliptic.*

We need to state conditions on the local drift f and local covariance a which insure that the corresponding n-dimensional Markov diffusion processes $x(s)$ exists. Moreover, we must choose a suitable subspace $\mathcal{D}(A)$ of $C^{1,2}(\overline{Q}_0)$ such that the Dynkin formula (2.7) holds for all $\Phi \in \mathcal{D}(A)$. This will be done by using results about stochastic differential equations.

Diffusions represented as solutions of stochastic differential equations. Let us suppose that there is a $n \times d$ - matrix valued function $\sigma(t,x)$ such that

$$a_{ij} = \sum_{\ell=1}^{n} \sigma_{i\ell}\sigma_{j\ell}, \quad i,j = 1,\cdots,n.$$

In other words, $\sigma\sigma' = a$. We suppose that the functions f_i, σ_{ij} are continuous on $\overline{Q}_0$ and satisfy the growth and local Lipschitz conditions (4.3) (ii), (iii). Let $t_0 \leq t < t_1$, and let $w(s)$ be a d - dimensional standard brownian motion on the interval $I = [t, t_1]$. The Ito - sense stochastic differential equation

$$dx = f(s,x(s))ds + \sigma(s,x(s))dw(s),\ t \leq s \leq t_1, \tag{5.4}$$

with given initial data $x(t) = x$ $(x \in I\!\!R^n)$ has a pathwise unique solution, which is a Markov diffusion process [EK,Chap. 5], [IW, Chap. 4]. Moreover, for each $m = 1, 2, \cdots$ there exists a constant C_m (depending on m and $t_1 - t$), such that

$$E_{tx}|x(s)|^m \leq C_m(1 + |x|^m),\ t \leq s \leq t_1. \tag{5.5}$$

See Appendix D. The formal derivative dw/ds (which in fact does not exist) is called in engineering literature a *white noise*. Thus, (5.4) describes a dynamical system driven by an additive white noise, with state dependent coefficient $\sigma(s, x(s))$. We say that Φ satisfies a *polynomial growth condition* on $\overline{Q}_0$ if there exist constants K and m such that for all $(t, x) \in \overline{Q}_0$,

$$|\Phi(t, x)| \leq K(1 + |x|^m).$$

We take $\mathcal{D}(A) = C_p^{1,2}(\overline{Q}_0)$, where $C_p^{1,2}(\overline{Q}_0)$ denotes the space of $\Phi \in C^{1,2}(\overline{Q}_0)$ such that $\Phi, \Phi_t, \Phi_{x_i}, \Phi_{x_i x_j}, i, j = 1 \cdots, n$ satisfy a polynomial growth condition. For $x(s)$ satisfying (5.4) with $x(t) = x$, the Dynkin formula (2.7) follows from the Ito differential rule applied to $\Phi(s, x(s))$. By the Ito differential rule,

$$\begin{aligned} \Phi(s, x(s)) - \Phi(t, x) - \int_t^s A\Phi(r, x(r))dr \\ = \int_t^s (D_x\Phi \cdot \sigma)(r, x(r))dw(r). \end{aligned} \tag{5.6}$$

Since $\sigma(s, x)$ satisfies a linear growth condition (4.3)(ii), $\Phi_x \sigma$ also has polynomial growth. From (5.5), it follows that the right side of (5.6) is a martingale, and we get (2.7) by taking expectations.

Remark 5.1. The method of diffusion approximation is used to reduce to problems in partial differential equations more complicated questions about stochastic processes which are not diffusions. The technique has been applied to problems from a wide variety of applications, in engineering, chemical physics and genetics.

One common use of diffusion approximations is to replace a Markov chain with many states and nearest - neighbor transitions by a diffusion obtained after rescaling time and state variables and passing to a limit. Convergence of the rescaled Markov chains is proved by martingale/weak convergence methods [EK, Chap. 7]. For queuing systems, the diffusion limit is called the heavy traffic limit. See Harrison [Har] concerning the use of heavy traffic limits for flow control.

In other applications a diffusion limit is obtained for processes which are not Markov, or which are Markov on a higher dimensional state space. For a treatment of such situations and applications in communications engineering, see [Ku2].

III.6 Controlled Markov processes

We now consider problems in which the time - evolution of $x(s)$ is actively influenced by another stochastic process $u(s)$, called a *control process.* The control process has values $u(s) \in U$, where U is a complete separable metric space. We refer to Σ as the *state space* and U as the *control space.*

Example 6.1. (Controlled Markov chain.) For each constant control $v \in U$, infinitesimal jumping rates $\rho(s,x,y,v)$ for a finite state Markov chain are given, as in Section 4, Example (a). If control $u(s)$ is used at time s, the jumping rates are $\rho(s,x,y,u(s))$.

Example 6.2. (Controlled Markov diffusion.) In this case we suppose that $x(s)$ satisfies a stochastic differential equation of the form.

$$dx = f(s,x(s),u(s))ds + \sigma(s,x(s),u(s))dw(s). \tag{6.1}$$

We will specify later (Chapter IV) conditions of f, σ, and the control process under (6.1), with initial data $x(t) = x$, has a unique solution.

In formulating a general class of control problems with a finite time horizon, we suppose that for each constant control $v \in U$, the state process $x(s)$ is Markov with backward evolution operator A^v. The domain $\mathcal{D}(A^v)$ may depend on v. However, we assume that there is a "large enough" space $\mathcal{D}$ such that $\mathcal{D} \subset \mathcal{D}(A^v)$ for all $v \in U$. For instance, for controlled Markov diffusions on $I\!R^n$, we shall take $\mathcal{D} = C_p^{1,2}(\overline{Q}_0)$ as in Section 5.

One must also specify what kind of information is available to the controller of time s. Without yet being mathematically precise, throughout this book the controller is allowed to know the past history of states $x(r)$ for $r \le s$ when control $u(s)$ is chosen. The Markovian nature of the problem suggests that it should suffice to consider control processes of the form

$$u(s) = \underline{u}(s,x(s)). \tag{6.2}$$

Such a function $\underline{u}$ from Q_0 into U is called a *Markov control policy.* Formally, we expect that when (6.2) holds $x(s)$ should be a Markov process with backward evolution operator satisfying, for $\Phi \in \mathcal{D}$,

$$A^{\underline{u}}\Phi(t,x) = A^{\underline{u}(t,x)}\Phi(t,x). \tag{6.3}$$

For instance, for a controlled Markov diffusion

$$A^{\underline{u}}\Phi = \Phi_t + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(t,x,\underline{u}(t,x))\Phi_{x_i x_j} + \sum_{i=1}^{n} f_i(t,x,\underline{u}(t,x))\Phi_{x_i}. \tag{6.4}$$

Various assumptions, depending on the type of control processes being considered, are needed to insure that $A^{\underline{u}}$ indeed defines a Markov process. Discontinuous Markov policies $\underline{u}$ must often be admitted, in order to obtain a policy $\underline{u}^*$ which minimizes an expected cost criterion J of the type (6.5) below. This introduces additional mathematical complications; in fact, a Markov control policy $\underline{u}^*$ satisfying (7.7) below is a natural candidate for an optimal policy. However, in some cases there is no Markov process corresponding to $\underline{u}^*$. This difficulty is encountered for degenerate controlled Markov diffusions, and in particular in case of deterministic optimal control (with $a_{ij} \equiv 0$ in (6.4)). This point will be discussed further in Section IV.3.

For various technical reasons, it will be convenient to allow control processes $u(s)$ which depend on the past in some more complicated way than through a Markov policy in (6.2). This will be formalized later through the idea of admissible control system. In case of controlled diffusions, we shall consider in Chapter IV a class of admissible control systems in which $u(s)$ is progressively measurable with respect to some increasing family $\{\mathcal{F}_s\}$ of σ - algebras to which the brownian motion $w(s)$ is adapted.

Criteria to be optimized. Roughly speaking, the control problem on a finite time interval $t \le s \le t_1$ is to minimize

$$J = E_{tx}\{\int_t^{t_1} L(s,x(s),u(s))ds + \psi(x(t_1))\}. \tag{6.5}$$

We call $L(s,x,v)$ a *running cost function* and ψ a *terminal cost* function We always assume that L and ψ are continuous, together with further (integrability) assumptions needed to insure that J is well defined. If $\psi(x) \equiv 0$ then the problem is said to be in *Lagrange form.* If $L(t,x,v) \equiv 0$, the problem is in *Mayer form.* These names are used similarly in the calculus of variations. (Sec. I.9.)

III.7 Dynamic programming: formal description

In this section we shall describe in a purely formal way the principle of dynamic programming, the corresponding dynamic programming equation, and a criterion for finding optimal Markov control policies. Our first mathematically rigorous statements about dynamic programming for controlled Markov diffusions will be made in Section 8. These will take the form of "Verification Theorems" which require that the dynamic programming equation have a sufficiently well behaved "classical solution". It often happens that there is no classical solution. In such cases, we must resort to solutions to the dynamic programming equation which hold in some weaker sense (in particular, viscosity solutions.)

Finite time horizon. Let $t \le s \le t_1$. Since the controller is allowed to observe the states $x(s)$ of the process being controlled, we may as well assume that the initial state $x(t) = x$ is known ($x \in \Sigma$). The starting point for dynamic programming is to regard the infimum of the quantity J in (6.5) being minimized as a function $V(t,x)$ of the initial data:

$$V(t,x) = \inf_{C} J(t,x;\ \text{control}), \tag{7.1}$$

the infimum being over the class C of controls admitted. V is called the value function (or optimal cost function). Since we have not yet specified C, formula (7.1) remains only heuristic. The next step (at least conceptually) is to obtain Bellman's *principle of dynamic programming.* This states that for $t \le t+h \le t_1$,

$$V(t,x) = \inf_C E_{tx} \left\{ \int_t^{t+h} L(s,x(s),u(s))ds + V(t+h,x(t+h)) \right\}. \tag{7.2}$$

Speaking intuitively, the expression in brackets represents the sum of the running cost on $[t,t+h]$ and the minimum expected cost obtained by proceeding optimally on $[t+h,t_1]$ with $(t+h,x(t+h))$ as initial data.

The dynamic programming equation is obtained formally from (7.2) by the following heuristic derivation. If we take constant control $u(s) = v$ for $t \le s \le t+h$, then

$$V(t,x) \le E_{tx} \int_t^{t+h} L(s,x(s),v)ds + E_{tx}V(t+h,x(t+h)).$$

We subtract $V(t,x)$ from both sides, divide by h and let $h \to 0$:

$$\lim_{h\to 0^+} h^{-1} E_{tx} \int_t^{t+h} L(s,x(s),v)ds = L(t,x,v),$$

$$\lim_{h\to 0^+} h^{-1} \left[E_{tx}V(t+h,x(t+h) - V(t,x)\right] =$$

$$\lim_{h\to 0^+} h^{-1} E_{tx} \int_t^{t+h} A^v V(s,x(s))ds = A^v V(t,x).$$

Hence, for all $v \in U$

$$0 \le A^v V(t,x) + L(t,x,v). \tag{7.3}$$

Among the various assumptions needed to make this argument rigorous would be an assumption (such as $V \in \mathcal{D}$) which justifies using the Dynkin formula.

On the other hand, if $\underline{u}^*$ is an optimal Markov control policy, we should have

$$V(t,x) = E_{tx} \int_t^{t+h} L(s,x^*(s),\underline{u}(s,x^*(s))ds + E_{tx}V(t+h,x^*(t+h)),$$

where $x^*(s)$ is the Markov process generated by $A^{\underline{u}^*}$. A similar argument gives, under sufficiently strong assumptions (including continuity of $\underline{u}^*$ at (t,x))

$$0 = A^{\underline{u}^*} V(t,x) + L(t,x,\underline{u}^*(t,x)). \tag{7.4}$$

Inequality (7.3) together with (7.4) are equivalent to the *dynamic programming equation*

$$(7.5)\qquad 0 = \min_{v \in U} \left[A^v V(t,x) + L(t,x,v) \right].$$

Equation (7.5) is to be considered in $[t_0, t_1] \times \Sigma$, with the terminal (Cauchy) data

$$(7.6)\qquad V(t_1, x) = \psi(x), \ \ x \in \Sigma.$$

The above formal argument also suggests that an optimal Markov control policy should satisfy

$$(7.7)\qquad \underline{u}^*(t,x) \in \arg\min[A^v V(t,x) + L(t,x,v)],$$

where

$$\arg\min g(v) = \{u^* \in U : \ g(u^*) \leq g(v) \text{ for all } v \in U\}.$$

The dynamic programming equation (7.5) is a kind of *nonlinear evolution equation.* Let $x(t)$ be a controlled Markov chain with finite state space Σ and jump rates $\rho(s,x,y,v)$. In the notation of Section 4, let

$$(7.8)\qquad F(t,x;\ \phi) = \min_{v \in U} \left[\sum_{y \neq x} \rho(t,x,y,v)[\phi(y) - \phi(x)] + L(t,x,v) \right].$$

In this case (7.5) becomes a system of ordinary differential equations

$$(7.9)\qquad \frac{d}{dt} V(t,x) + F(t,x;\ V(t,\cdot)) = 0, \ \ x \in \Sigma.$$

These differential equations evolve backward in time, for $t_0 \leq t \leq t_1$, with the terminal data (7.6).

Next, consider a controlled random evolution with Markov chain parameters of the kind in Section 4 (c). Now $f = f(t,x,z,v)$ depends on a control variable v. As in Section I.5, for $(t,x,z,p) \in \overline{Q}_0 \times Z \times I\!\!R^n$, let

$$H(t,x,z,p) = \max_{v \in U} \left[-f(t,x,z,v)p - L(t,x,z,v) \right].$$

Then $V = V(t,x,z)$ and (7.5) has the form

$$(7.10)\qquad -\frac{\partial V}{\partial t} + H(t,x,z,D_x V) - \sum_{\zeta \neq z} \rho(t,z,\zeta)[V(t,x,\zeta) - V(t,x,z)] = 0.$$

In the variables (t,x), (7.10) describes a system of first-order partial differential equations indexed by $z \in Z$ (recall that Z is a finite set.) These equations are coupled through the zeroth order terms.

For a controlled Markov diffusion, governed by the stochastic differential equation (6.1), the dynamic programming equation (7.5) becomes a partial differential equation IV(3.3) of second order for $V(t,x)$. This case will be studied in detail in Chapter IV.

Infinite horizon discounted cost control problem. Consider the problem of minimizing

$$J = E_x \int_0^\infty e^{-\beta s} L(x(s), u(s)) ds \tag{7.11}$$

and introduce (again formally) the value function

$$V(x) = \inf_{C_1} J(x;\ \text{control}), \tag{7.12}$$

where C_1 is some class of admissible controls and $x = x(0)$ is the initial state. By a formal derivation similar to the finite time horizon case, we get the following dynamic programming equation for the discounted cost control problem on the infinite time interval $[0, \infty)$:

$$\beta V(x) = \min_{v \in U} [-G^v V(x) + L(x, v)]. \tag{7.13}$$

For this problem we may consider *stationary* Markov control policies $\underline{u}(x)$. The formal derivation of (7.13) suggests that an optimal stationary Markov control policy $\underline{u}^*$ should satisfy

$$\underline{u}^*(x) \in \arg\min [-G^v V(x) + L(x, v)]. \tag{7.14}$$

III.8 A Verification Theorem; finite time horizon

We again consider a finite time horizon, with $I_0 = [t_0, t_1)$. Let us call $W(t, x)$ a *classical solution* of the dynamic programming equation (7.5) with terminal data (7.6) if $W \in \mathcal{D}$ and

$$0 = \min_{v \in U} [A^v W(t, x) + L(t, x, v)],\ (t, x) \in I_0 \times \Sigma \tag{8.1}$$

$$W(t_1, x) = \psi(x),\ \text{for all}\ \ x \in \Sigma. \tag{8.2}$$

(The space $\mathcal{D}$ was defined in Section 6.) In this section, we show that if W is a classical solution, then $W(t, x)$ equals the minimum total expected cost among an appropriately defined class of admissible control systems. See Theorem 8.1. The proof is quite simple, but the assumption that W is a classical solution is quite restrictive. For the deterministic control problem, considered in Chapter I, the partial derivatives W_t and $W_{x_i}, i = 1, \cdots, n$, of a classical solution must be continuous. For controlled Markov diffusions, $W \in \mathcal{D}$ requires in addition continuity of the second order partial derivatives $W_{x_i x_j}$ together with polynomial growth of W and $A^v W$. (Later we show that polynomial growth of $A^v W$ is not, in fact, needed; see Theorem IV.3.1).

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $x(s) = x(s, \omega)$ a Σ - valued stochastic process defined on $[t, t_1] \times \Omega$. This process is called *measurable* if

$x(\cdot,\cdot)$ is measurable with respect to the σ - algebras $\mathcal{B}([t,t_1]\times\mathcal{F})$ and $\mathcal{B}(\Sigma)$. It is called *corlol* if the sample paths $x(\cdot,\omega)$ are right continuous and have left hand limits. (Some further background about stochastic processes and stochastic differential equations is summarized in Appendix D.)

Given initial data (t,x), we call

$$\pi = (\Omega, \{\mathcal{F}_s\}, P, x(\cdot), u(\cdot))$$

an *admissible control system* if $(\Omega,\mathcal{F}_{t_1},P)$ is a probability space, $\{\mathcal{F}_s\}$ is an increasing family of σ-algebras $(t\le s\le t_1)$, and $x(\cdot)$, $u(\cdot)$ are stochastic processes on $[t,t_1]$ such that:

(8.3)

$$\begin{aligned}
&(i)\quad x(s)\in\Sigma,\ t\le s\le t_1,\ x(t)=x\\
&\qquad x(\cdot)\ \text{ is corlol and } x(s) \text{ is } \mathcal{F}_s-\text{measurable};\\
&(ii)\ \ u(s)\in U,\ t\le s\le t_1,\\
&\qquad u(s) \text{ is } \mathcal{F}_s-\text{measurable and } u(\cdot,\cdot) \text{ is measurable};\\
&(iii)\ \text{For all } \Phi\in\mathcal{D},\ \text{satisfying } E_{tx}|\Phi(t_1,x(t_1))|<\infty,\ \text{ and}\\
&\qquad E_{tx}\int_t^{t_1}|A^{u(s)}\Phi(s,x(s))|ds<\infty,
\end{aligned}$$

the Dynkin formula holds:

$$E_{tx}\Phi(t_1,x(t_1)) - \Phi(t,x) = E_{tx}\int_t^{t_1} A^{u(s)}\Phi(s,x(s))ds.$$

The total expected cost corresponding to π is

$$J(t,x;\pi) = E_{tx}\left\{\int_t^{t_1} L(s,x(s),u(s))ds + \psi(x(t_1))\right\}.$$

Theorem 8.1. *Let $W\in\mathcal{D}$ be a classical solution to* (8.1) - (8.2). *Then for all $(t,x)\in[t_0,t_1]\times\Sigma$:*

(a) $W(t,x)\le J(t,x;\pi)$ for every admissible control system π.
(b) If there exists an admissible system $\pi^*=(\Omega^*,\{\mathcal{F}_s^*\},P^*,x^*(\cdot),u^*(\cdot))$ such that

$$u^*(s)\in\arg\min[A^vW(s,x^*(s))+L(s,x^*(s),v)]$$

for Lebesgue $\times P^*$ - almost all $(s,\omega)\in[t,t_1]\times\Omega^*$, then $W(t,x)=J(t,x;\pi^*)$.

Proof of (a): Let π be any admissible control system. Since $u(s)\in U$,

$$A^{u(s)}W(s,x(s)) + L(s,x(s),u(s)) \ge 0.$$

From (8.2) and the Dynkin formula

$$W(t,x) = E_{tx}\left\{\int_t^{t_1} -A^{u(s)}W(s,x(s))ds + \psi(x(t_1))\right\}$$
$$\leq E_{tx}\left\{\int_t^{t_1} L(s,x(s),u(s))ds + \psi(x(t_1))\right\}.$$

This proves (a). For part (b), the inequality becomes equality. □

We call Theorem 8.1 a *Verification Theorem.* If we let

$$V_{AS}(t,x) = \inf_C J(t,x;\pi) \tag{8.4}$$

where C is the class of all admissible control systems, then $V_{AS} = W$ provided the assumptions of Theorem 8.1 hold. In addition to the requirement $W \in \mathcal{D}$, it may not be easy to show that π^* exists with the property required in (b). A natural way to proceed is to select a Markov control policy $\underline{u}^*$ such that, for each $(t,x) \in I_0 \times \Sigma$,

$$\underline{u}^*(t,x) \in \arg\min[A^v W(s,x) + L(s,x,v)].$$

If $\underline{u}^*$, together with any initial data (t,x), determine a Markov process $x^*(s)$ with backward evolution operator $A^{\underline{u}^*}$, then we can take

$$u^*(s) = \underline{u}^*(s,x^*(s)). \tag{8.5}$$

Once the corresponding control system π^* is verified to be admissible, π^* is optimal. When this procedure works, we call u^* an *optimal Markov control policy.*

We conclude this section with two examples, both of which involve controlled Markov diffusions. Additional examples will be given in Section 9.

Example 8.1. (Stochastic linear regulator). This is a stochastic perturbation of the linear quadratic regulator problem (Example I.2.3 and I.5.1). The stochastic differential equations for $x(s)$ are linear:

$$dx = [A(s)x(s) + B(s)u(s)]ds + \sigma(s)dw(s). \tag{8.6}$$

There are no control constraints ($U = \mathbb{R}^m$) and the expected total cost is

$$J = E_{tx}\{\int_t^{t_1} [x(s)\cdot M(s)x(s) + u(s)\cdot N(s)u(s)]ds + x(t_1)\cdot Dx(t_1)\}. \tag{8.7}$$

We make the same assumptions as in Example I.2.3, and assume that $\sigma(\cdot)$ is continuous on $[t_0,t_1]$.

To apply the Verification Theorem 8.1, let us seek a solution W of the dynamic programming equation (8.1) which has the special form

$$W(t,x) = x\cdot P(t)x + g(t). \tag{8.8}$$

When $t = t_1$ we must have $W(t_1, x) = x \cdot Dx$. Thus,

$$P(t_1) = D, \ g(t_1) = 0. \tag{8.9}$$

A slight modification of the calculation used in Example I.5.1 for the deterministic case shows that $P(t)$ satisfies the Riccati equation I(5.15), with the same data I(5.16). In addition, $dg/ds = -\mathrm{tr}\, a(s)P(s)$, where $a = \sigma\sigma'$ and tr is the trace. Hence, by (8.9)

$$g(t) = \int_t^{t_1} \mathrm{tr}\, a(s)P(s)ds. \tag{8.10}$$

The unique v which minimizes $A^v W(t,x) + L(t,x,v)$ is $v = \underline{u}^*(t,x)$, where

$$\underline{u}^*(t,x) = -N^{-1}(t)B^{'}(t)P(t)x. \tag{8.11}$$

This feedback control is precisely the same as for the deterministic case. See I(5.17). In order to verify that $\underline{u}^*$ is an optimal Markov control policy, consider any $(\Omega, \{\mathcal{F}_s\}, P, w)$, where $(\Omega, \mathcal{F}_{t_1}, P)$ is a probability space, $\{\mathcal{F}_s\}$ is an increasing family of σ - algebras and $w(\cdot)$ is a $\mathcal{F}_s$ - adapted brownian motion on $[t, t_1]$. (In the terminology of Section IV.2 below, this is called a reference probability system.) The linear stochastic differential equation

$$dx^* = [A(s)x^*(s) + B(s)\underline{u}^*(s, x^*(s))]ds + \sigma(s)dw(s) \tag{8.12}$$

with initial data $x^*(t) = x$ has a unique solution $x^*(\cdot)$. Let

$$u^*(s) = \underline{u}^*(s, x^*(s)).$$

The system $\pi^* = (\Omega, \{\mathcal{F}_s\}, P, x^*(\cdot), u^*(\cdot))$ is admissible, if we take $\mathcal{D} = C_p^{1,2}(\overline{Q}_0)$ as in Section 5 and Example 6.2. By Theorem 8.1, $W(t,x)$ is the minimum of $J(t,x;\pi)$ in the class C of all admissible control systems π, and $J(t,x;\pi^*) = W(t,x)$. Thus, $\underline{u}^*$ is an optimal Markov control policy. A more detailed treatment of the stochastic linear regulator problem is given in [YZ, Chapt. 6].

Example 8.2. Let us take $\Sigma = U = I\!R^n$, and let $x(s)$ satisfy

$$dx = u(s)ds + dw(s). \tag{8.13}$$

In the setting of Nelson's theory of stochastic mechanics [Ne], one can regard $x(s)$ as the position of some "particle" at time s. If (8.13) is the model taken for the motion, then the velocity is undefined since brownian paths are nowhere differentiable with probability 1. However, one can regard $u(s)$ as a kind of local "average velocity". Consider the problem of minimizing

$$J = E_{tx}\left\{\int_t^{t_1}\left[\frac{1}{2}|u(s)|^2 - q(x(s))\right]ds + \psi(x(t_1))\right\}$$

in the class of admissible systems π. In this example

$$L(x,v) = \frac{1}{2}|v|^2 - q(x) \tag{8.14}$$

is the classical action integrand, if one interprets v as the velocity of a particle of mass 1 and $q(x)$ as the potential energy when the particle is at position x. If $\psi = 0$ we can call J the *mean average action.*

In Example 8.2, the dynamic programming equation (8.1) becomes

$$0 = \frac{\partial W}{\partial t} + \frac{1}{2}\Delta_x W - \frac{1}{2}|D_x W|^2 - q(x), \tag{8.15}$$

with the terminal data (8.2). The procedure used to get (8.5) leads to minimizing $\frac{1}{2}|v|^2 + v \cdot D_x W$ over $U = I\!R^n$. We obtain as candidate for an optimal Markov control policy

$$\underline{u}^*(t,x) = -D_x W(t,x). \tag{8.16}$$

The nonlinear term in (8.15) is quadratic in $D_x W$. The dynamic programming equation is linearizable by the following transformation. Let

$$\Phi(t,x) = \exp[-W(t,x)], \quad \phi(x) = \exp[-\psi(x)].$$

Then (8.15), (8.2) become

$$0 = \frac{\partial \Phi}{\partial t} + \frac{1}{2}\Delta_x \Phi + q(x)\Phi \tag{8.17}$$

$$\Phi(t_1, x) = \phi(x). \tag{8.18}$$

Equation (8.17) is just the backward heat equation with a potential term $q(x)\Phi$. This kind of substitution will be considered more systematically in Chapter VI, when we discuss logarithmic transformations.

If $q(x) = x \cdot Cx$ and $\psi(x)$ are quadratic functions, with $q(x) \leq 0$ and $\psi(x) \geq 0$, we have a special case of the stochastic linear regulator in Example 8.1. Then $\Phi(t,x)$ has the gaussian form

$$\Phi(t,x) = \theta(t)\exp[-x \cdot P(t)x],$$

with $\theta = \exp(-g)$, as could have shown directly from (8.17) - (8.18).

Remark 8.1. This formulation is an oversimplification of Nelson's ideas. To allow for time-reversibility in quantum mechanical phenomena, Nelson allows forward and backward local average velocities.

Stochastic control interpretations for solutions of the Schrödinger equation, in a stochastic mechanics setting, have been given. See Nelson [Ne], Guerra - Morato [GM], Zambrini [Za]. There has not been unanimity in deciding which combination of average forward and backward velocities is most suitable to replace the kinetic energy term $\frac{1}{2}|v|^2$ in (8.14).

III.9 Infinite Time Horizon

In this section we consider problems of minimizing an infinite horizon, discounted expected cost:

$$J = E\int_0^\infty e^{-\beta t} L(x(s), u(s))ds, \ \beta > 0. \tag{9.1}$$

In this case, the control problem is specified by a collection $\{G^v\}, v \in U$, such that $-G^v$ is the generator of an autonomous Markov process (Section 3.) We assume that $D(G^v) \supset D$ for each v, where D is a "sufficiently large" class of functions.

Given an initial state x, we call

$$\pi = (\Omega, \{\mathcal{F}_s\}, P, x(\cdot), u(\cdot))$$

an *admissible control system* if $(\Omega, \mathcal{F}, P)$ is a probability space, $\{\mathcal{F}_s\}$ an increasing family of σ - algebras $(s \geq 0)$ with $\mathcal{F}_s \subset \mathcal{F}$, and $x(\cdot), u(\cdot)$ are stochastic processes on $[0, \infty)$ such that:

(9.2)

(a) Assumptions (8.3) (i) (ii) hold for $s \geq 0$;

(b) Assumption (8.3) (iii) holds with

$$\Phi(t, x) = e^{-\beta t}\phi(x), \text{ for all } \phi \in D \text{ and } 0 < t_1 < \infty;$$

(c) $E_x\int_0^\infty e^{-\beta s}|L(x(s), u(s))|ds < \infty.$

We note that, for $\Phi = e^{-\beta t}\phi$, the Dynkin formula becomes

$$\begin{aligned} &e^{-\beta t_1}E_x\phi(x(t_1)) - \phi(x) \\ &\qquad = E_x\int_0^{t_1} e^{-\beta s}[-G^{u(s)}\phi - \beta\phi](x(s))ds. \end{aligned} \tag{9.3}$$

The dynamic programming equation for this infinite time horizon control problem is

$$\beta W(x) = \min_{v \in U}[-G^v W(x) + L(x, v)]. \tag{9.4}$$

We call W a *classical solution* if $W \in D$ and W satisfies (9.4) for all $x \in \Sigma$.

Lemma 9.1. *Let $W \in D$ be a classical solution to* (9.4). *Then:*

(a) *$W(x) \leq J(x; \pi)$ for every admissible π such that*

$$\liminf_{t_1 \to \infty} e^{-\beta t_1}E_x W(x(t_1)) \leq 0. \tag{9.5}$$

(*b*) *If there exists an admissible system* π^* *such that*

$$u^*(s) \in \arg\min[-G^v W(x^*(s)) + L(x^*(s), v)]$$

for Lebesgue $\times P$ *- almost all* $(s, \omega) \in [0, \infty) \times \Omega$, *and*

$$\limsup_{t_1 \to \infty} e^{-\beta t_1} E_x W(x^*(t_1)) \geq 0, \tag{9.6}$$

then $W(x) \geq J(x; \pi^*)$.

Proof of (a). Since $u(s) \in U$,

$$-G^{u(s)} W(x(s)) - \beta W(x(s)) + L(x(s), u(s)) \geq 0.$$

By applying Dynkin's formula to $\Phi = e^{-\beta t} W$, as in the derivation of (9.3), we get

$$W(x) \leq E_x \int_0^{t_1} e^{-\beta s} L(x(s), u(s) ds + e^{-\beta t_1} E_x W(x(t_1)). \tag{9.7}$$

We let $t_1 \to \infty$ through a sequence for which the last term tends to a limit ≤ 0.

Proof of (b). In the proof of (a), equality now replaces inequality in (9.7). We let $t_1 \to \infty$ through a sequence for which the last term tends to a limit ≥ 0.

□

Let C_1 denote the class of admissible π such that (9.5) holds, and let

$$V_{AS}(x) = \inf_{C_1} J(x; \pi). \tag{9.8}$$

Theorem 9.1. *Let* $W \in D$ *be a classical solution to* (9.4). *Then* $W(x) \leq V_{AS}(x)$. *If there exists* $\pi^* \in C_1$ *such that* (9.6) *holds and*

$$u^*(s) \in \arg\min[-G^v W(x^*(s)) + L(x^*(s), v)]$$

for Lebesgue $\times P$ *- almost all* $(s, \omega) \in [0, \infty) \times \Omega$, *then*

$$W(x) = V_{AS}(x) = J(x; \pi^*).$$

Theorem 9.1 is an immediate consequence of Lemma 9.1. The control system π^* is optimal in the class C_1. As in the finite horizon case (Section 8) we may seek to find an optimal stationary Markov control policy $\underline{u}^*$ such that

$$\underline{u}^*(x) \in \arg\min[-G^v W(x) + L(x, u)], \tag{9.9}$$

$$u^*(s) = \underline{u}^*(x^*(s)),$$

and $x^*(s)$ is a Markov process for $s \geq 0$, with generator $-G^{\underline{u}^*}$ and initial state $x^*(0) = x$. However, to prove the existence of $\underline{u}^*$ with these properties is a separate matter from the Verification Theorem 9.1.

If in Example 6.2 we take $f = f(x,v), \sigma = \sigma(x,v)$ we have an infinite time horizon problem for a controlled Markov diffusion processes. The generator $-G^v$ is then

$$(9.10) \qquad -G^v\phi = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x,v)\phi_{x_i x_j} + \sum_{i=1}^{n} f_i(x,v)\phi_{x_i},$$

with $a = \sigma\sigma'$. To apply Theorem 9.1, we can take $D = C_p^2(I\!R^n)$, where $C_p^2(I\!R^n)$ is the set $\phi \in C^2(I\!R^n)$ such that $\phi, \phi_{x_i}, \phi_{x_i x_j}, i,j = 1, \cdot, n$ satisfy a polynomial growth condition. Another verification theorem, with less restrictive growth assumptions on a classical solution W to (9.4), will be proved in Section IV.5.

Example 9.1. Consider an infinite horizon discounted stochastic linear regulator. For simplicity, we take scalar state $x(s)$, control $u(s)$ and brownian motion $w(s)$. Using notations similar to Example 8.1

$$(9.11) \qquad dx = [Ax(s) + Bu(s)]ds + \sigma dw(s),\ B \neq 0,$$

with $x(0) = x$. The problem is to minimize

$$(9.12) \qquad J = E_x \int_0^\infty e^{-\beta s}[Mx(s)^2 + Nu(s)^2]ds,$$

where $M > 0, N > 0$. In this example the dynamic programming equation (9.4) is

$$(9.13) \qquad \beta W = \frac{\sigma^2}{2}W'' + AxW' - \frac{B^2}{4N}(W')^2 + Mx^2.$$

Let us seek a solution of (9.4) of the form

$$(9.14) \qquad W(x) = Kx^2 + g$$

where $K > 0$ and g are constants. A calculation gives K as the positive root of the quadratic equation

$$(9.15) \qquad \beta K = 2AK - \frac{B^2}{N}K^2 + M$$

and $g = \beta^{-1}\sigma^2 K$. The optimal stationary Markov control policy is linear (as in (8.11)):

$$(9.16) \qquad \underline{u}^*(x) = -N^{-1}BKx.$$

Since $M > 0, N > 0$, by (9.12) and (9.14)

$$\liminf_{t_1\to\infty} e^{-\beta t_1} E_x W(x(t_1)) = 0$$

for any control such that $J < \infty$. Hence (9.5) holds for every admissible π, and (9.6) is automatic since $W \geq 0$. To verify that $\underline{u}^*$ is optimal, we use Theorem 9.1 with

$$\pi^* = (\Omega, \{\mathcal{F}_s\}, P, x^*(\cdot), u^*(\cdot))$$

where $x^*(s),\ u^*(s)$ satisfy for $s \geq 0$

$$dx^* = [Ax^*(s) + Bu^*(s)]ds + \sigma dw(s) \tag{9.17}$$

with $x^*(0) = x$ and

$$u^*(s) = -N^{-1}BKx^*(s). \tag{9.18}$$

Thus, $\underline{u}^*$ is an optimal Markov control policy, and $V_{AS}(x) = W(x)$ is the minimum expected discounted cost.

The multidimensional infinite horizon, discounted stochastic linear regulator problem can be solved in a similar way, provided that a detectability condition holds. An expected average cost per unit time criterion can also be considered. See [Dav1, Sec 5.4].

Example 9.2. This is a highly simplified production planning model, in which demand is random. Let $x(s)$ be the inventory level of some good at time s, $u(s)$ the production rate, and $z(s)$ the demand rate. We allow $x(s) < 0$, which corresponds to unfilled orders (a shortage). The demand process $z(s)$ is assumed to be a finite state Markov chain, with state space Z. The production rate is constrained by $u(s) \geq 0$, and the inventory dynamics are

$$\frac{dx}{ds} = u(s) - z(s). \tag{9.19}$$

The pair $(x(s), z(s))$ is the state and $u(s)$ is the control. This is an instance of a controlled random evolution with Markov chain parameters, of the kind in Section 4(c). Let us take

$$L(x,v) = h(x) + g(v), \quad \text{where}$$

$$h(x) = \begin{cases} h_1 x, \ x \geq 0, \quad h_1 > 0 \\ h_2|x|, \ x \leq 0, h_2 > 0 \end{cases} \tag{9.20}$$

$$g(0) = g'(0) = 0,\ g''(v) \geq c > 0 \text{ for } \ v \geq 0.$$

If we assume that the Markov chain $z(s)$ has autonomous jumping rates $\rho(z,\zeta)$,

$$-G^v\phi(x,z) = (v-z)\frac{\partial\phi}{\partial x} + \sum_{\zeta \neq z} \rho(z,\zeta)[\phi(x,\zeta) - \phi(x,z)] \tag{9.21}$$

Then $W(x,z)$ is a classical solution to the dynamic programming equation (9.4), if $W(\cdot,z) \in C^1(I\!R^1)$ for each $z \in Z$, and W satisfies

$$\beta W = \min_{v \geq 0}\{(v-z)\frac{\partial W}{\partial x} + g(v)\} + h(x) + \sum_{\xi \neq z} \rho(z,\xi)[W(x,\xi) - W(x,z)].$$

It can be shown that a classical solution $W(x) \geq 0$ exists. Moreover, $W(\cdot, z)$ is a strictly convex function on $I\!R^1$, $W(x,z) \to +\infty$ as $|x| \to \infty$ and $\partial W/\partial x$ is bounded. See [FSS, Thm. 2.2]. A candidate optimal control policy $\underline{u}^*(x,z)$ is found by minimizing $g(v) + vW_x$ for $v \geq 0$. We get

$$\text{(9.22)} \qquad \underline{u}^*(x,z) = \begin{cases} (g')^{-1}(-W_x(x,z)) \, , & \text{if } W_x(x,z) < 0 \\ \\ 0 \, , & \text{if } W_x(x,z) \geq 0. \end{cases}$$

To use Theorem 9.1, one needs to verify (9.5) for any admissible π. Condition (9.6) is automatic since $W \geq 0$. Since W_x is bounded, $|W(x)| \leq C(1+|x|)$ for suitable C. Then (9.5) holds provided

$$\text{(9.23)} \qquad \liminf_{t_1 \to \infty} e^{-\beta t_1} E_x |x(t_1)| = 0.$$

By (9.20) $L(x,v) \geq k|x|$, where $k = \min(h_1, h_2) > 0$. Therefore, if $J(x;\pi) < \infty$, then

$$E_x \int_0^\infty e^{-\beta s}|x(s)|ds \leq k^{-1}J(x;\pi) < \infty.$$

This implies (9.23) for any admissible system π, and thus $\pi \in C_1$.

If we use $\underline{u}^*$ in (9.22), then (9.19) becomes

$$\text{(9.24)} \qquad \frac{dx^*}{ds} = \underline{u}^*(x^*(s), z(s)) - z(s).$$

Since $(g')^{-1}$ and $W_x(\cdot, z)$ are nondecreasing continuous functions, $\underline{u}^*(\cdot, z)$ is nonincreasing and continuous. This implies that (9.24) with $x^*(0) = x$ has a unique solution $x^*(s)$. See Hartman [Hn, Thm 6.2]. Moreover, $|x^*(s)|$ is bounded by a constant depending on the initial data $x = x(0)$ [FSS, Lemma 4.1]. Let $u^*(s) = \underline{u}^*(x^*(s), z(s))$; and let $(\Omega, \mathcal{F}, P)$ denote the probability space on which the Markov chain $z(s)$ is defined, with $\mathcal{F}_s = \mathcal{F}_s^z$ the σ-algebra generated by $z(r),\ 0 \leq r \leq s$. By a version of the Dynkin formula proved in Appendix B,

$$\pi^* = (\Omega, \{\mathcal{F}_s\}, P, x^*(\cdot), u^*(\cdot))$$

is an admissible system, with $J(x, \pi^*) \leq W(x) < \infty$ by Lemma 9.1(b). By Theorem 9.1, $J(x,\pi^*) = W(x)$. Thus, $\underline{u}^*$ is an optimal Markov control policy and $V_{AS}(x) = W(x)$ is the minimum expected discounted cost.

Example 9.3. This is another simple production planning model, in which demand is fixed but random machine breakdown and repairs are allowed. It was analyzed by Akella–Kumar [AK]. We will formulate the model and summarize their results. There is one machine, which is either working or not. If the machine is working, it produces a good at rate $u(s)$, subject to

$0 \le u(s) \le K$. The constant demand is at rate $d < K$. The inventory $x(s)$ changes according to

$$\frac{dx}{ds} = u(s) - d$$

during intervals when the machine is working, and according to

$$\frac{dx}{ds} = -d$$

when the machine is not working. Let $z(s)$ be a 2-state Markov chain, with state space $Z = \{1, 2\}$, such that the machine is working when $z(s) = 1$ and not working when $z(s) = 2$. We take $L(x, v) = h(x)$ with $h(x)$ as in (9.20). Thus, the running cost function depends on holding costs for $x > 0$ or shortage costs for $x < 0$, but not on the production rate v. In this example, the control space depends on z, namely, $U(1) = [0, K]$ and $U(2)$ is the empty set. This minor change in the problem formulation does not cause any difficulty. The dynamic programming equation (9.4) becomes

$$\beta W(x, z) = \min_{v \in U(z)} [-G^v W(x, z) + h(x)]. \tag{9.25}$$

When $z = 2$, the control v does not enter. From (9.21) we get the following pair of differential equations for $W(x, 1)$ and $W(x, 2)$;

(9.26)

$$\beta W(x, 1) = \min_{0 \le v \le K} [(v - d)W_x(x, 1)] + h(x) + \rho(1, 2)[W(x, 2) - W(x, 1)],$$

$$\beta W(x, 2) = -dW_x(x, 2) + h(x) + \rho(2, 1)[W(x, 1) - W(x, 2)].$$

A candidate optimal Markov control policy is

$$\underline{u}^*(x, 1) = \begin{cases} 0 & \text{if } W_x(x, 1) > 0 \\ d & \text{if } W_x(x, 1) = 0 \\ K & \text{if } W_x(x, 1) < 0. \end{cases}$$

Akella-Kumar [AK] show that (9.25) has a classical solution $W \ge 0$, with $W(x, 1)$, $W(x, 2)$ strictly convex and $W_x(x, 1)$, $W_x(x, 2)$ continuous and bounded. Define $\overline{x}$ by $W_x(\overline{x}, 1) = 0$. Then

$$\underline{u}^*(x, 1) = \begin{cases} 0 & \text{if } x > \overline{x} \\ d & \text{if } x = \overline{x} \\ K & \text{if } x < \overline{x}. \end{cases} \tag{9.27}$$

An explicit formula for $\overline{x}$ is given in [AK]. Arguments like those in Example 9.2 can be used to verify the $W(x, z) = V(x, z)$ and the $\underline{u}^*(x, 1)$ is an optimal Markov control policy. It is interesting to note that for some choice of parameters $\overline{x} = 0$.

Additional examples, of controlled Markov diffusion processes on an infinite time horizon will be given in Section IV.5.

III.10 Viscosity solutions

Let us again consider controlled Markov processes on a finite time horizon. In Section 8 we assumed that the dynamic programming equation (7.5) with terminal data (7.6) had a "classical" solution $W \in \mathcal{D}$. Under the rather restrictive assumptions of Theorem 8.1, it turned out that $W = V_{AS}$, where V_{AS} defined by (8.4) is the value function using the class C of *all* admissible control systems π. In this section we discuss some results which hold without such restrictive assumptions. The idea is to consider, as in (7.1),

$$V(t,x) = \inf_{C} J(t,x;\pi) \tag{10.1}$$

where C is a *suitable* class of admissible control systems π (not necessarily including all admissible π.) Although one cannot generally expect V to be a classical solution of the dynamic programming equation, it is often possible to interpret V as a viscosity solution in the sense of Definition II.4.1. In this section, we present briefly some results of this kind, but do not include any proofs.

Let us rewrite the dynamic programming equation (7.5) in the form II(3.12), proceeding formally as in Section 7. We recall from Section 6 that a controlled Markov process is associated with a family A^v of backward evolution operators, $v \in U$. We rewrite A^v in the form (2.9):

$$A^v \Phi = \frac{\partial \Phi}{\partial t} - G_t^v \Phi(t,\cdot). \tag{10.2}$$

Consider the nonlinear operators $\mathcal{G}_t$, such that

$$\mathcal{G}_t \psi(x) = \sup_{v \in U} \left[G_t^v \psi(x) - L(t,x,v) \right] \tag{10.3}$$

Then the dynamic programming equation (7.5) becomes

$$-\frac{\partial V}{\partial t}(t,x) + (\mathcal{G}_t V(t,\cdot))(x) = 0, \tag{10.4}$$

which is just II(3.12).

The case of controlled Markov diffusions will be considered in detail in Chapters IV and V. In that case, a convenient class C of admissible control systems turns out to be those which arise via progressively measurable control processes. See Section IV.2. The dynamic programming equation (10.4) is a second-order nonlinear partial differential equation, which we call of Hamilton–Jacobi–Bellman (HJB) type. It will be shown that a dynamic programming principle holds, from which it follows that the value function is a viscosity solution of the HJB equation (Section V.2.) From uniqueness results for solutions to HJB equations, the particular choice of class C turns out not to be especially important. Every choice of admissible controls for which a dynamic programming principle holds leads to a value function which is the same as the unique viscosity solution of the HJB equation with given boundary data.

For controlled Markov processes formulated in the general framework of Section 7, the known results are less complete. To show that the function V in (10.1) is a viscosity solution, one can try to use Theorem II.5.l. For this purpose, consider the nonlinear operators $\mathcal{T}_{tr}$ defined (for ψ in a suitable class $\mathcal{C}$ of functions) by

$$\mathcal{T}_{tr}\psi(x) = \inf_{C} E_{tx}\left\{\int_t^r L(s,x(s),u(s))ds + \psi(x(r))\right\}. \tag{10.5}$$

The monotonicity property II(3.2$'$) is immediate. However, properties II(3.1) and II(3.2) are not at all immediate, and indeed depend on further assumptions about the Markov process control problem. As already noted in the deterministic case (Example II.3.1) the semigroup property II(3.3) corresponds to a dynamic programming principle. By taking $r = t_1$, we have by (10.1):

$$V(t,x) = \mathcal{T}_{tt_1}\psi(x). \tag{10.6}$$

Definition II.4.1 of viscosity solutions also requires a set $\mathcal{D}$ of "smooth" test functions such that II(3.10), (3.11) hold. For controlled Markov processes we typically require that $\mathcal{D} \subset \mathcal{D}(A^v)$ for all $v \in U$, that $\Sigma' = \Sigma$ and that $\mathcal{D}$ is dense in the uniform norm in the space of bounded, uniformly continuous functions on $[t_0, t_1] \times \Sigma$.

The best results which are known in this general setting can be obtained from Nisio's construction of a nonlinear semigroup for controlled Markov processes [Ni1]. Let us sketch Nisio's construction, in the case of controlled time homogeneous Markov processes on $\Sigma = I\!R^n$, following [Ni1]. To simplify matters, let us assume that there is no running cost ($L \equiv 0$.) This is no real restriction (see Remark 10.2). Let $\mathcal{C}$ be the space of bounded, uniformly continuous on $I\!R^n$ and assume that U is compact. For each $v \in U$, we are given a semigroup T_r^v of positive linear operators on $\mathcal{C}$ such that $T_r^v 1 = 1$. In the stochastic control interpretation, the semigroup T_r^v corresponds to a time homogeneous Feller Markov process under constant control v (Section 3.) Let $\mathcal{L}$ denote the class of all bounded, Lipschitz continuous functions on $I\!R^n$. For $\psi \in \mathcal{L}$, let

$$\lambda(\psi) = \sup_{x \neq y} \frac{|\psi(x) - \psi(y)|}{|x - y|}.$$

Assume that there exists $\beta \geq 0$ such that

$$\lambda(T_r^v\psi) \leq e^{\beta r}\lambda(\psi), \ \forall \psi \in \mathcal{L} \text{ and } v \in U. \tag{10.7}$$

For $h > 0, r = Mh, \ M = 1, 2, \cdots$, define the nonlinear operator $\mathcal{T}_r^h$ inductively by

$$\mathcal{T}^h_{r+h}\psi(x) = \inf_{v\in U} T^v_h(\mathcal{T}^h_r\psi)(x), \tag{10.8}$$

with $\mathcal{T}^h_0\psi = \psi$. Then $\mathcal{T}^h_r$ is monotone, contracting; and using (10.7) it can be shown that $\mathcal{T}^h_r : \mathcal{C} \to \mathcal{C}$. Now let $h = h_N = 2^{-N}$. If $r = M2^{-N_0}$ is a dyadic rational number, then for $N \geq N_0$

$$\mathcal{T}^{h_N}_r\psi \geq \mathcal{T}^{h_{N+1}}_r\psi.$$

The basic idea of the Nisio construction is to obtain the nonlinear semigroup $\mathcal{T}_r$ on $\mathcal{C}$ by

$$\mathcal{T}_r\psi = \lim_{\substack{N\to\infty \\ r_N\to r}} \mathcal{T}^{h_N}_{r_N}\psi, \tag{10.9}$$

where r_N is dyadic rational. The results of [Ni1, Chap. 2] imply that the operators $\mathcal{T}_{tr} = \mathcal{T}_{r-t}$ satisfy II(3.1)-(3.2). Moreover, the function V defined by (10.6) is continuous on $\overline{Q}_0 = [t_0, t_1] \times I\!R^n$.

For constant control v, let S^v_r denote the corresponding semigroup for the Markov process $(s, x(s))$:

$$S^v_r\Phi(t,x) = T^v_r\Phi(t+r,\cdot)(x). \tag{10.10}$$

Corresponding to (10.7), assume that there exists $\gamma \geq 0$ such that

$$\lambda(S^v_r\Phi) \leq e^{\gamma r}\lambda(\Phi) \tag{10.11}$$

for all Lipschitz continuous Φ on $I\!R^{n+1}$. In (10.2), $G^v_t = G^v$ where $-G^v$ is the generator of the semigroup T^v_t for a time-homogeneous controlled Markov process. Assume that there exists a dense subspace $\mathcal{D}_0$ of the space of uniformly continuous, bounded functions on $I\!R^{n+1}$, such that for each $\Phi \in \mathcal{D}_0$;

$$\begin{gathered}\Phi_t \text{ is continuous on } I\!R^{n+1} \text{ and} \\ G^v\Phi(t,\cdot)(x) \text{ is continuous on } I\!R^{n+1}\times U.\end{gathered} \tag{10.12}$$

$$\sup_{v\in U}\|A^v\Phi\| < \infty \tag{10.13}$$

$$\lim_{h\downarrow 0} h^{-1}\|S^v_h\Phi - \Phi - hA^v\Phi\| = 0. \tag{10.14}$$

(As usual, $\| \;\; \|$ is the sup norm.) Let $\mathcal{D}$ denote the space of Φ restricted to $\overline{Q}_0$, $\Phi \in \mathcal{D}_0$. (In this chapter we denote "smooth" functions of (t,x) by Φ rather than w as in Chapter II.) Then II(3.10) follows from (10.12). By [Ni1, Theorem 2.1]

$$\lim_{h\downarrow 0} h^{-1}[\mathcal{T}_h\Phi - \Phi] = \Phi_t - \mathcal{G}\Phi, \tag{10.15}$$

$$\mathcal{G}\Phi = \sup_{v \in U} G^v \Phi.$$

Thus, II(3.11) holds for all $\Phi \in \mathcal{D}$, and by Theorem II.5.1, V is a viscosity solution of (10.4).

Remark 10.1. Unfortunately, we do not know a uniqueness theorem for viscosity solutions to dynamic programming equations of the general form (10.4). However, uniqueness theorems are known if $\mathcal{G}_t$ is a second order partial differential operator (Section V.8). Uniqueness results for (10.4) are also known for some particular classes of nonlocal operators $\mathcal{G}_t$. See [Sa] [S3] [LeB][AT].

Stochastic control interpretation. Nisio's construction of the nonlinear semigroup $\mathcal{T}_r$, outlined above, is purely analytical. A stochastic control interpretation of $V(t,x)$ can be given in terms of piecewise constant controls. If $t_1 - t = Mh, M = 1, 2 \cdots$, let

$$V^h(t,x) = \mathcal{T}^h_{t_1 - t}\psi(x).$$

Then $V^h(t,x)$ turns out to be the value function obtained by requiring that $u(s)$ is constant on each interval $[t + mh, t + (m+1)h), m = 0, 1, \cdots, M - 1$, and (10.8) can be interpreted as a corresponding discrete time dynamic programming equation. According to (10.9),

$$V(t,x) = \lim_{\substack{N \to \infty \\ t_N \to t}} V^{h_N}(t_N, x),$$

where $t_1 - t_N$ is dyadic rational. In case $t_1 - t = M2^{-N_0}$ is dyadic rational, $V(t,x)$ is the infimum of $E_{tx}\psi(x(t_1))$ among controls which are constant on some dyadic partition of $[t, t_1]$, with $h = h_N$ and $N \geq N_0$. For a more precise formulation of these statements and proofs, we refer to [Ni1, Chap.2].

Remark 10.2. In this section, we assumed that $L \equiv 0$. This restriction can be removed by the following device. Consider an "augmented" state space $\Sigma \times I\!R^1$, with elements denoted by $(x, \tilde{x})$. At time s, the augmented state is $(x(s), \tilde{x}(s))$, where

$$\tilde{x}(s) = \int_t^s L(x(r), u(r))dr.$$

Let $\Psi(x, \tilde{x}) = \psi(x) + \tilde{x}$. Then

$$J(t, x; \pi) = \tilde{J}(t, x, 0; \pi), \text{ where}$$

$$\tilde{J}(t, x, \tilde{x}; \pi) = E_{tx\tilde{x}}\Psi(x(t_1), \tilde{x}(t_1)).$$

III.11 Historical remarks

The study of optimal control for continuous time Markov processes began in the early 1960's with the stochastic linear regulator problem (Example

8.1.). That model has several features which are appealing from the viewpoint of engineering applications. The problem reduces to solving a matrix Riccati equation, and optimal control policies are linear in the state. Moreover, the optimal policy is insensitive to the intensity of noise entering the state dynamics. A good introduction to linear-quadratic models in estimation and stochastic control is Davis [Dav1]. There is also a substantial literature on control of discrete time Markov processes and applications. Bertsekas [Bs] provides a good introduction to this topic.

For continuous time controlled Markov processes, the theory is most extensively developed for Markov diffusion processes. See [BL1] [Bo] [ElK] [FR] [Kr1] [L1-L3]. For optimal control of jump markov processes, see [El, Chap 17]; and for control of piecewise deterministic processes see [Dav2] [Ve].

In Section 6 - 9 we formulated a general class of Markov process control problems. The Verification Theorems 8.1 and 9.1 allow one to solve examples in which the value function is a classical solution to the dynamic programming equation. However, these sections include only rather elementary results. The Nisio nonlinear semigroup in Section 10 provides a further step toward a theory which does not refer to particular types of controlled Markov processes (diffusions, jump processes, etc.).

There are interesting types of control problems for Markov processes which do not fit the model considered in this chapter. Among them we mention singular stochastic control (Chapter VIII), impulsive control [BL2] and problems with switching costs [LeB].

IV

Controlled Markov Diffusions in $I\!\!R^n$

IV.1 Introduction

This chapter is concerned with the control of Markov diffusion processes in n-dimensional $I\!\!R^n$. The dynamics of the process $x(s)$ being controlled are governed by a stochastic differential equation of the form (2.1). Section 2 is concerned with the formulation of finite time horizon control problems for Markov diffusions, where control occurs until exit from a given cylindrical region $Q \subset I\!\!R^{n+1}$. Several candidates for the value function are considered, which are shown later to agree under appropriate assumptions.

For a controlled Markov diffusion, the dynamic programming equation becomes a second order, nonlinear partial differential equation, which we call a Hamilton - Jacobi - Bellman (HJB) equation. Sections 3–5 are concerned with cases when the HJB equation has a sufficiently smooth "classical" solution. Verification theorems in the same spirit as those in Sections I.5, III.8 and III.9 are proved. This is done for finite horizon problems in Section 3, and for infinite horizon discounted problems in Section 5. Section 5 also provides illustrative examples.

We saw in Chapters I and II that the value function V for a deterministic control problem is generally not of class $C^1(Q)$. Thus, V cannot generally be a classical solution to the first - order HJB equation of deterministic optimal control. In contrast, if the second order HJB equation for a controlled Markov diffusion is of uniformly parabolic type, then the corresponding boundary problem indeed has a unique classical solution. Theorems of this type depend on the theory of parabolic partial differential equations, and are quoted without proof in Section 4.

In Sections 6–10 the assumption of uniform parabolicity is abandoned. Hence, the value function need not be a classical solution to the HJB equation. The goal of these sections is to provide a systematic analysis of value functions, for problems on a *fixed* finite time interval. The methods are probabilistic. They depend on standard results about stochastic differential equations, which are reviewed in Appendix D. A strong form of the dynamic programming principle is proved in Section 7, by making approximations which reduce the result to the uniformly parabolic case. In Section 8, we obtain bounds for first-order difference quotients of the value function V. For second - order difference quotients, one sided bounds are obtained in Section 9. If V is a classical solution to the HJB equation, these bounds imply the same bounds for the corresponding first and second order partial derivatives. The bounds on difference quotients depend only on certain constants associated with the functions f, σ describing the state dynamics, and with the cost function. Hence, they can be shown to hold uniformly if the stochastic control problem is approximated in various ways. In Section 10, we use such bounds in studying generalized subsolutions and solutions to HJB equations.

IV.2 Finite time horizon problem

Let us consider a control model in which the state evolves according to an $I\!R^n$-valued process $x(s)$ governed by a system of stochastic differential equations of the form

$$(2.1) \qquad dx = f(s, x(s), u(s))ds + \sigma(s, x(s), u(s))dw(s), \ t \le s \le t_1,$$

where $u(s) \in U$ is the control applied at time s and $w(s)$ is a brownian motion of dimension d. As in previous chapters, we fix $t_0 < t_1$, and let $Q_0 = [t_0, t_1) \times I\!R^n$, $\overline{Q}_0$ the closure of Q_0.

We make the following assumptions: $U \subset I\!R^m$ for some m, and U is closed. The functions f, σ are continuous on $\overline{Q}_0 \times U$, and $f(\cdot,\cdot,v), \sigma(\cdot,\cdot,v)$ are of class $C^1(\overline{Q}_0)$. Moreover, for some constant C

$$(2.2) \qquad \begin{aligned} &(a)\ |f_t| + |f_x| \le C, \ |\sigma_t| + |\sigma_x| \le C; \\ &(b)\ |f(t,x,v)| \le C(1 + |x| + |v|) \\ &\quad\ |\sigma(t,x,v)| \le C(1 + |x| + |v|). \end{aligned}$$

Here $f_t, \sigma_t, f_x, \sigma_x$ denote the t-partial derivatives and gradients with respect to x, respectively. The inequalities (2.2a) are equivalent to the following:

$$(2.2)(a') \qquad \begin{aligned} &|f(s,y,v) - f(t,x,v)| \le \bar{C}[|s-t| + |y-x|], \\ &|\sigma(s,y,v) - \sigma(t,x,v)| \le \bar{C}[|s-t| + |y-x|] \end{aligned}$$

for all $(s,y), (t,x) \in \overline{Q}_0$ and suitable constant $\bar{C}$. The notation $|\sigma|$ denotes the operator norm. We also recall that σ_t, σ_x denote respectively the t partial derivative and the differential $D_x\sigma$ of the matrix valued function $\sigma(t,x,v)$; and $|\sigma_t|, |\sigma_x|$ are the operator norms. Since $\sigma(\cdot,\cdot,v) \in C^1(\overline{Q}_0)$, the equivalence of (2.2)(a) and (2.2)(a′) follows from the integrated form of the Mean Value theorem.

Progressively measurable controls. Let us consider control processes $u(\cdot)$ which are progressively measurable, in a sense which we now define. Fix an initial time $t \in [t_0, t_1)$. For $t \le s \le t_1$, let $\mathcal{B}_s$ denote the Borel σ-algebra on $[t,s]$. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\{\mathcal{F}_s\}$ an increasing family of σ-algebras with $\mathcal{F}_s \subset \mathcal{F}$ for all $s \in [t, t_1]$.

Definition 2.1. A U-valued process $u(\cdot)$, defined on $[t,t_1] \times \Omega$, is $\mathcal{F}_s$-*progressively measurable* if the map $(r,\omega) \to u(r,\omega)$ from $[t,s] \times \Omega$ into U is $\mathcal{B}_s \times \mathcal{F}_s$-measurable for each $s \in [t, t_1]$.

We call a progressively measurable control process *admissible*, if moreover

$$E \int_t^{t_1} |u(s)|^m ds < \infty \quad \text{for} \quad m = 1, 2, \cdots. \tag{2.3}$$

In later sections we shall often assume that the control space U is compact. For U compact, $|u(s)| \le M$ for some $M < \infty$, and (2.3) automatically holds.

Let $w(\cdot)$ be a $\mathcal{F}_s$-adapted brownian motion on $[t,t_1]$, and consider (2.1) with fixed initial data

$$x(t) = x, \quad x \in I\!R^n. \tag{2.4}$$

From standard theory of stochastic differential equations with random coefficients, (2.1) - (2.4) has a pathwise unique solution $x(s)$ which is $\mathcal{F}_s$-progressively measurable and has continuous sample paths. Moreover, for each $m = 1, 2, \cdots$, the m^{th} order absolute moments $E_{tx}|x(s)|^m$ are bounded for $t \le s \le t_1$. These facts are reviewed in Appendix D.

Criterion to be minimized. We consider the problem of minimizing a criterion J similar to the one in III(6.5). However, instead of controlling on the fixed interval $[t,t_1]$ we control up to the smaller of t_1 and the exit time of $x(s)$ from a given open set $O \subset I\!R^n$. We assume that either $O = I\!R^n$ or that ∂O is a compact $(n-1)$-dimensional manifold of class C^3. Let $Q = [t_0, t_1) \times O$. for $(t,x) \in Q$, let

$$\tau = \inf\{s : (s, x(s)) \notin Q\}. \tag{2.5}$$

Note that τ is the exit time of $x(s)$ from O, if exit occurs by time t_1, and $\tau = t_1$ if $x(s) \in O$ for all $s \in [t, t_1]$. We call τ the *exit time* from *the cylinder* Q. One always has $(\tau, x(\tau)) \in \partial^* Q$, where

$$\partial^* Q = ([t_0, t_1] \times \partial O) \cup (\{t_1\} \times O). \tag{2.6}$$

Of course, in the particular case $O = I\!R^n, Q = Q_0$ and $\partial^* Q$ is the hyperplane $\{t_1\} \times I\!R^n$.

Let L, Ψ be continuous functions which satisfy the following polynomial growth conditions:

$$(2.7) \qquad \begin{cases} (a)\ |L(t,x,v)| \le C(1+|x|^k+|v|^k) \\ \\ (b)\ |\Psi(t,x)| \le C(1+|x|^k) \end{cases}$$

for suitable constants C, k. Given initial data $(t,x) \in Q$ and any admissible progressively measurable control process $u(\cdot)$, let

$$(2.8) \qquad J(t,x;u) = E_{tx}\left\{\int_t^\tau L(s,x(s),u(s))ds + \Psi(\tau,x(\tau))\right\}.$$

The problem is to choose $u(\cdot)$ to minimize J.

Let us restate the problem in a somewhat more systematic way. By a *reference probability system* ν let us mean a 4-tuple

$$\nu = (\Omega, \{\mathcal{F}_s\}, P, w)$$

where $(\Omega, \mathcal{F}_{t_1}, P)$ is a probability space, $\{\mathcal{F}_s\}$ is an increasing family of σ-algebras and $w(\cdot)$ is a $\mathcal{F}_s$-adapted brownian motion on $[t,t_1]$. Let $\mathcal{A}_{t\nu}$ denote the collection of all $\mathcal{F}_s$-progressively measurable, U-valued processes $u(\cdot)$ on (t,t_1) which satisfy (2.3). We consider the infimum of (2.8) among all $u(\cdot) \in \mathcal{A}_{t\nu}$:

$$(2.9) \qquad V_\nu(t,x) = \inf_{\mathcal{A}_{t\nu}} J(t,x;u).$$

We also consider the infimum of $V_\nu(t,x)$ among all reference probability systems ν:

$$(2.10) \qquad V_{PM}(t,x) = \inf_\nu V_\nu(t,x).$$

We call $u^*(\cdot) \in \mathcal{A}_{t\nu}$ *ν-optimal* if $V_\nu(t,x) = J(t,x;u^*)$. We call $u^*(\cdot)$ an *optimal admissible progressively measurable control process* if $u^*(\cdot) \in \mathcal{A}_{t\nu^*}$ for some

$$\nu^* = (\Omega^*, \{\mathcal{F}_s^*\}, P^*, w^*)$$

and $V_{PM}(t,x) = J(t,x;u^*)$. We will show later in the chapter that, under suitable assumptions, $V_\nu = V_{PM}$ for each ν.

Consider the case $Q = Q_0$ and hence $\tau = t_1$. We defined the concept of admissible control system in Chapter III.8 and corresponding candidate V_{AS} for the "value function". Given $\nu, u(\cdot) \in \mathcal{A}_{t\nu}$ and initial data (2.4), let $x(\cdot)$ be the corresponding solution of (2.1). Then the system

$$\pi = (\Omega, \{\mathcal{F}_s\}, P, x(\cdot), u(\cdot))$$

is admissible with respect to the class $\mathcal{D} = C_p^{1,2}(\overline{Q}_0)$. To see this, the operators A^v in Section III.6 now have the form

$$A^v\Phi = \Phi_t + \frac{1}{2}\sum_{i,j=1}^n a_{ij}(t,x,v)\Phi_{x_ix_j} + \sum_{i=1}^n f_i(t,x,v)\Phi_{x_i}, \tag{2.11}$$

where $a = \sigma\sigma'$. As in III(5.6) the Ito differential rule applied to $\Phi \in C_p^{1,2}(\overline{Q}_0)$ gives

$$\begin{aligned}\Phi(s,x(s)) - \Phi(t,x) - \int_t^s A^{u(r)}\Phi(r,x(r))dr \\ = \int_t^s D_x\Phi(r,x(r))\cdot\sigma(r,x(r),u(r))dw(r).\end{aligned} \tag{2.12}$$

Assumption (2.3) and the boundedness of $E_{tx}|x(s)|^m$ on $[t,t_1]$ for each m imply

$$E_{tx}|\Phi(s,x(s))| < \infty,\ E_{tx}\int_t^s |A^{u(r)}\Phi(r,x(r)|dr < \infty,$$

$$E_{tx}\int_t^s |D_x\Phi(r,x(r))|^2|\sigma(r,x(r),u(r)|^2dr < \infty.$$

The right side of (2.12) is then a martingale, and we obtain Dynkin's formula

$$E_{tx}\Phi(t_1,x(t_1)) - \Phi(t,x) = E_{tx}\int_t^{t_1} A^{u(r)}\Phi(r,x(r))dr. \tag{2.13}$$

Therefore, π is an admissible system.

Every admissible, progressively measurable $u(\cdot)$ defines an admissible system π, with $J(t,x;u) = J(t,x;\pi)$. Hence, we always have $V_{AS} \le V_{PM}$, where V_{AS} was defined by formula III(8.4). While we do not know in general whether equality holds, this will be proved in many cases. In some instances, the Verification Theorem in Section 3 can be used to see that $V_{AS} = V_{PM}$.

Remark 2.1. Additional restrictions are sometimes made on the class of reference probability systems admitted. One such restriction would be to require that $\mathcal{F}_s$ is the σ - algebra generated by the brownian motion $w(\cdot)$, instead of assuming (as we have done) only that $w(\cdot)$ is $\mathcal{F}_s$ - adapted. In the formulation of singular stochastic control problems in Chapter VIII, we shall assume right continuity of the σ - algebras $\mathcal{F}_s$. Since we will make assumptions under which V_ν turns out to be the same for all reference probability systems ν, such additional restrictions turn out to be unimportant for our purposes.

IV.3 Hamilton-Jacobi-Bellman PDE

For controlled Markov diffusion processes, the dynamic programming equation III(7.5) becomes a nonlinear partial differential equation (PDE) of second order. See (3.3) below. Let $\mathcal{S}_+^n$ denote the set of symmetric, nonnegative definite $n \times n$ matrices $A = (A_{ij}), i,j = 1,\cdots,n$. Let $a = \sigma\sigma'$ and

$$\operatorname{tr} aA = \sum_{i,j=1}^{n} a_{ij}A_{ij}. \tag{3.1}$$

For $(t,x) \in \overline{Q}_0, p \in I\!R^n, A \in \mathcal{S}_+^n$, let

$$\mathcal{H}(t,x,p,A) = \sup_{v\in U}[-f(t,x,v)\cdot p - \frac{1}{2}\operatorname{tr} a(t,x,v)A - L(t,x,v)]. \tag{3.2}$$

If U is compact, the supremum is always attained. Otherwise, some extra conditions are needed to insure this. For each fixed (t,x,v), $f\cdot p+\frac{1}{2}\operatorname{tr} aA$ is a linear function of (p,A). Hence, $\mathcal{H}(t,x,\cdot,\cdot)$ is a convex function on $I\!R^n \times \mathcal{S}_+^n$. As in Chapter III, let $C^{1,2}(B)$ denote the set of functions $\Phi(t,x)$ such that Φ and its partial derivatives $\Phi_t, \Phi_{x_i}, \Phi_{x_ix_j}$, $i,j = 1,\cdots,n$ are continuous on $B \subset I\!R^{n+1}$. Let $D_x\Phi$ denote the gradient of Φ in x and $D_x^2\Phi = (\Phi_{x_ix_j}), i,j = 1,\cdots,n$. The dynamic programming equation III(7.5) can then be written as

$$-\frac{\partial V}{\partial t} + \mathcal{H}(t,x,D_xV,D_x^2V) = 0,\ (t,x)\in Q. \tag{3.3}$$

Equation (3.3) is often called the *Hamilton-Jacobi-Bellman* (HJB) partial differential equation associated with the optimal stochastic control problem formulated in Section 2. It is to be considered in the cylinder $Q = [t_0,t_1)\times O$ with the boundary data

$$V(t,x) = \Psi(t,x) \ \text{ for } \ (t,x) \in \partial^*Q. \tag{3.4}$$

The HJB partial differential equation is called *uniformly parabolic* if there exists $c > 0$ such that, for all $(t,x,v) \in Q_0 \times U$ and $\xi \in I\!R^n$

$$\sum_{i,j=1}^{n} a_{ij}(t,x,v)\xi_i\xi_j \geq c|\xi|^2. \tag{3.5}$$

When (3.5) holds, results from the theory of second order nonlinear PDEs of parabolic type imply existence and uniqueness of a solution to the boundary value problem (3.3)-(3.4). These results will be summarized, without proofs, in Section 4.

When (3.5) does not hold, the HJB equation (3.3) is said to be of *degenerate parabolic type*. In this case a smooth solution $V(t,x)$ cannot be expected. Instead, the value function will be interpreted as a solution in some broader sense, for instance as a generalized solution (Section 10). Another convenient interpretation of the value function is as a viscosity solution to the HJB equation (Chapter V.) In addition, in the degenerate parabolic case the value function may not assume continuously the boundary data (3.4) at points $(t,x) \in [t_0,t_1)\times\partial O$. An additional assumption will be made in Section V.2, to insure that $V_{PM}(t,x)$ is continuous in $\overline{Q}$.

Let us next prove a verification result (Theorem 3.1), which has somewhat different assumptions than Theorem III.8.1. For $B \subset \overline{Q}_0$, let $C_p(B)$ denote the set of all Φ continuous on B and satisfying a polynomial growth condition

$$|\Phi(t,x)| \leq K(1+|x|^m) \tag{3.6}$$

for some constants K and m.

Theorem 3.1. *(Verification Theorem). Let $W \in C^{1,2}(Q) \cap C_p(\overline{Q})$ be a solution to* (3.3)-(3.4). *Then:*

(a) *$W(t,x) \leq J(t,x;u)$ for any admissible progressively measurable control process $u(\cdot)$ and any initial data $(t,x) \in Q$.*
(b) *If there exist $\nu^* = (\Omega^*, \{\mathcal{F}_s^*\}, P^*, w^*)$ and $u^*(\cdot) \in \mathcal{A}_{t\nu^*}$ such that*

$$u^*(s) \in \arg\min[f(s,x^*(s),v) \cdot D_x W(s,x^*(s)) + \frac{1}{2}\operatorname{tr} a(s,x^*(s),v) D_x^2 W(s,x^*(s)) + L(s,x^*(s),v)] \tag{3.7}$$

for Lebesgue$\times P^*$ - almost all $(s,\omega) \in [t,\tau^*] \times \Omega^*$, then $W(t,x) = V_{PM}(t,x) = J(t,x;u^*)$.

Here $x^*(s)$ is the solution of (2.1) corresponding to ν^* and $u^*(\cdot)$, with $x^*(t) = x$, and τ^* is the exit time of $(s,x^*(s))$ from Q.

Theorem 3.1 will be obtained as a special case of the following result, in which the exit time τ is replaced by an $\mathcal{F}_s$ - stopping time θ with $t \leq \theta \leq \tau$. (We recall that θ is an $\mathcal{F}_s$ - stopping time provided that the event $\theta \leq s$ is $\mathcal{F}_s$ - measurable for each $s \in [t,t_1]$.)

Lemma 3.1. *Let W be as in Theorem* 3.1. *Then:*

(i) If $u(\cdot) \in \mathcal{A}_{t\nu}$ and θ is an $\mathcal{F}_s$-stopping time with $t \leq \theta \leq \tau$, then

$$W(t,x) \leq E_{tx}\left\{\int_t^\theta L(s,x(s),u(s))ds + W(\theta,x(\theta))\right\}.$$

(ii) If $\nu^*, u^*(\cdot)$ are as in Theorem 3.1(b), *then for every $\mathcal{F}_s^*$-stopping time θ^* with $t \leq \theta^* \leq \tau^*$*

$$W(t,x) = E_{tx}\left\{\int_t^{\theta^*} L(s,x^*(s),u^*(s))ds + W(\theta^*,x^*(\theta^*))\right\}.$$

Proof of Lemma 3.1 (a). We return to notation A^v of Chapter III and recall that (3.3) is the same as III(7.5) in case of controlled Markov diffusions. Let us first assume that Q is bounded, i.e. that O is bounded, and that $W \in C^{1,2}(\overline{Q})$. For each $(t,x) \in \overline{Q}, v \in U$,

$$0 \leq A^v W(t,x) + L(t,x,v).$$

Let us replace t, x, v by $s, x(s), u(s), t \leq s \leq \tau$:

$$0 \leq A^{u(s)} W(s, x(s)) + L(s, x(s), u(s)). \tag{3.8}$$

We next use the Ito differential rule, integrated from t to θ in III(5.6), to get

$$W(\theta, x(\theta)) - W(t, x) - \int_t^\theta A^{u(s)} W(s, x(s)) ds = M(\theta) \tag{3.9}$$

where

$$M(s) = \int_t^s \chi_\tau(r) D_x \Phi(r, x(r)) \cdot \sigma(r, x(r), u(r)) dw(r)$$

is a $\mathcal{F}_s$-martingale. Here $\chi_\tau(r)$ is the indicator function of the event $r \leq \tau$. Then $E_{tx} M(\theta) = 0$. We get (i) by taking expectations in (3.9) and using (3.8).

In the general case, for $0 < \rho^{-1} < (t_1 - t_0)$ let

$$O_\rho = O \cap \{|x| < \rho, \text{ dist } (x, \partial O) > \frac{1}{\rho}\} \ , \ Q_\rho = [t_0, t_1 - \frac{1}{\rho}) \times O_\rho.$$

Let τ_ρ be the exit time of $(s, x(s))$ from Q_ρ, and $\theta_\rho = \theta \wedge \tau_\rho = \min(\theta, \tau_\rho)$. Since Q_ρ is bounded and $W \in C^{1,2}(\overline{Q}_\rho)$, by part (i)

$$W(t, x) \leq E_{tx} \left\{ \int_t^{\theta_\rho} L(s, x(s), u(s)) ds + W(\theta_\rho, x(\theta_\rho)) \right\}. \tag{3.10}$$

As $\rho \to \infty$, θ_ρ increases to θ with probability 1. Since

$$E_{tx} \int_t^{\theta_\rho} |L(s, x(s), u(s))| ds \leq E_{tx} \int_t^{t_1} |L(s, x(s), u(s))| ds < \infty,$$

$$\lim_{\rho \to \infty} E_{tx} \int_t^{\theta_\rho} L(s, x(s), u(s)) ds = E_{tx} \int_t^\theta L(s, x(s), u(s)) ds.$$

Since $W \in C_p(\overline{Q})$, $W(\theta_\rho, x(\theta_\rho)) \to W(\theta, x(\theta))$ with probability 1 as $\rho \to \infty$, and

$$|W(\theta_\rho, x(\theta_\rho))| \leq K(1 + |x(\theta_\rho)|^k| \leq K(1 + ||x(\cdot)||^k)$$

for suitable k, K, where $||\ ||$ is the sup norm on $[t, t_1]$. By Appendix D, $E_{tx} ||x(\cdot)||^m < \infty$ for each $m = 1, 2, \cdots$. We take $m > k$ and let $\alpha = mk^{-1}$. Then $E_{tx} |W(\theta_\rho, x(\theta_\rho))|^\alpha$ is bounded, which implies uniform integrability of the random variables $W(\theta_\rho, x(\theta_\rho))$. Hence

$$\lim_{\rho \to \infty} E_{tx} W(\theta_\rho, x(\theta_\rho)) = E_{tx} W(\theta, x(\theta)).$$

This proves (i).

Proof of Lemma 3.1(ii). Inequality (3.8) is an equality when $u(s) = u^*(s)$ and $x(s) = x^*(s)$. Then (3.10) also becomes an equality. □

Remark 3.1. From (3.8)

$$\zeta_s = W(s \wedge \tau, x(s \wedge \tau)) - \int_t^{s\wedge\tau} L(r, x(r), u(r))dr \tag{3.11}$$

is a local submartingale, and a local martingale when $u(s) = u^*(s), x(s) = x^*(s)$ and $\tau = \tau^*$. While we do not explicitly use this property, it is essentially the basis for the proof of Lemma 3.1.

Proof of Theorem 3.1. For part (a), take $\theta = \tau$. Since W satisfies (3.4), $W(\tau, x(\tau)) = \Psi(\tau, x(\tau))$. Use Lemma 3.1(i); similarly, use Lemma 3.1(ii) for part (b). □

Markov control policies. As in earlier chapters we call a Borel measurable function $\underline{u}: \overline{Q}_0 \to U$ a Markov control policy. Let $\mathcal{L}$ denote the class of Markov control policies $\underline{u}$ with the following properties:

(i) $\underline{u}$ is continuous on $\overline{Q}_0$;

(3.12) (ii) For each $\rho > 0$ there exists K_ρ such that for $t_0 \le t \le t_1$,

$$|\underline{u}(t,x) - \underline{u}(t,y)| \le K_\rho |x - y|, \ |x| \le \rho, |y| \le \rho;$$

(iii) There exists C such that $|\underline{u}(t,x)| \le C(1 + |x|)$.

Let $\underline{u} \in \mathcal{L}$. Given any $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}, P)$ and $\mathcal{F}_s$-adapted brownian motion $w(s)$, the stochastic differential equation

$$\begin{cases} dx = f(s, x(s), \underline{u}(s, x(s))ds + \sigma(s, x(s), \underline{u}(s, x(s))dw(s), \\ x(t) = x \end{cases} \tag{3.13}$$

has a pathwise unique solution. The process

$$u(s) = \underline{u}(s, x(s)) \tag{3.14}$$

is $\mathcal{F}_s$ progressively measurable and admissible, since $E_{tx}|x(s)|^m$ is bounded on $[t, t_1]$ for each m. We then write $J(t, x; \underline{u})$ instead of $J(t, x; u)$.

Theorem 3.1 then has the following:

Corollary 3.1. *If there exists $\underline{u}^* \in \mathcal{L}$ such that*

$$\begin{aligned} \underline{u}^*(s,y) \in \arg\min[& f(s,y,v) \cdot D_x W(s,y) \\ & + \tfrac{1}{2} \operatorname{tr} a(s,y,v) D_x^2 W(s,y) + L(s,y,v)] \end{aligned} \tag{3.15}$$

for all $(s,y) \in Q$, then $W(t,x) = J(t,x;\underline{u}^*)$ for all $(t,x) \in Q$.

Proof. Given any reference probability system ν, let $x^*(s)$ be the solution to (3.13) with $\underline{u} = \underline{u}^*$, and

$$u^*(s) = \underline{u}^*(s, x^*(s)) \ , \ t \le s \le \tau^*. \tag{3.16}$$

Then we use Theorem 3.1(b). □

Remark 3.2. We call $\underline{u}^*$ *an optimal Markov control policy.* For $\underline{u} \in \mathcal{L}$ the reference probability system ν can be arbitrary, since (3.13) has a pathwise unique solution. Thus, if there is an optimal $\underline{u}^* \in \mathcal{L}$, then

$$W(t,x) = V_\nu(t,x) = V_{PM}(t,x) = J(t,x;\underline{u}^*).$$

If, in addition, $Q = Q_0$ and $W \in C_p^{1,2}(\overline{Q}_0)$, then Theorem III.8.1 gives $W(t,x) = V_{AS}(t,x)$. In that case, all of the candidates V_ν, V_{PM}, V_{AS} for the name "value function" agree.

Unfortunately, there is generally no $\underline{u}^*$ which satisfies both (3.12) and (3.15). If, for instance, the control space U is compact and convex, then order to have a continuous function $\underline{u}^*$ satisfying (3.15) one generally needs to know that $f \cdot D_x W + \frac{1}{2}\operatorname{tr} a D_x^2 W + L$, considered as a function of v, has a minimum on U at a *unique* $v^* = \underline{u}^*(s,y)$. When this uniqueness property fails, one generally must allow discontinuous Markov control policies in order to get an optimal policy $\underline{u}^*$. In Corollary 3.2, $\underline{u}^*$ is merely required to be bounded and Borel measurable. However, in addition the stochastic differential equation (3.13) with initial data is required to have a solution $x^*(s)$ which has a probability density. In Section 4 we will show how Corollary 3.2 can be applied in the uniformly parabolic case.

Corollary 3.2. *Let U be compact. Let $\underline{u}^*$ be Borel measurable and satisfy (3.15) for almost all $(s,y) \in Q$. Given $(t,x) \in Q$, assume that there exist $\nu^* = (\Omega^*, \{\mathcal{F}_s^*\}, P^*, w^*)$ and a solution $x^*(\cdot)$ to*

$$dx^* = f(s,x^*(s),\underline{u}^*(s,x^*(s))ds + \sigma(s,x^*(s),\underline{u}^*(s,x^*(s))dw^*(s) \tag{3.17}$$

with $x^*(t) = x$, such that

$$\int_t^{t_1} P_{tx}\{(s,x^*(s)) \in N\} ds = 0 \tag{3.18}$$

for every $N \in \mathcal{B}(Q)$ with $(n+1)$-dimensional Lebesgue measure 0. *Then*

$$W(t,x) = V_{PM}(t,x) = J(t,x;\underline{u}^*).$$

Proof. By hypothesis, there exists $N \in \mathcal{B}(Q)$ with $(n+1)$-dimensional Lebesgue measure 0 such that (3.15) holds for all $(s,y) \in Q \backslash N$. Let $\chi(s)$ be the indicator function of the event $(s,x^*(s)) \in N$. Then

$$E_{tx}\int_t^{t_1} \chi(s) ds = \int_t^{t_1} P_{tx}\{(s,x^*(s)) \in N\} ds = 0.$$

The conclusion follows from Theorem 3.1(b). □

Remark 3.3. The results of this section extend readily to the case when the criterion J to be minimized has the following more general form. Let

$\ell(t,x,v)$ be a continuous function which is bounded above ($\ell \le M$ for some M). Let

$$(3.19)\qquad J(t,x;u) = E_{tx}\left\{\int_t^\tau \Gamma(s)L(s,x(s),u(s))ds + \Gamma(\tau)\Psi(\tau,x(\tau))\right\},$$

$$(3.20)\qquad \Gamma(s) = \exp\int_t^s \ell(r,x(r),u(r))dr.$$

In case $\ell = 0$, (3.19) becomes (2.8). The Hamilton - Jacobi - Bellman equation now has the form

$$(3.21)\qquad -\frac{\partial V}{\partial t} + \mathcal{H}(t,x,D_xV(t,x),D_x^2V(t,x),V(t,x)) = 0,$$

where for $(t,x)\in\overline{Q}_0, p\in I\!R^n, A\in\mathcal{S}_+^n, V\in I\!R$

$$(3.22)\qquad \mathcal{H}(t,x,p,A,V) = \sup_{v\in U}[-f(t,x,v)\cdot p$$

$$-\frac{1}{2}\mathrm{tr}\, a(t,x,v)A - L(t,x,v) - \ell(t,x,v)V].$$

In contrast to (3.2), the "zeroth order term" $-\ell V$ appears in (3.22). Correspondingly, in the statement of the Verification Theorem 3.1, a term $\ell(s,x^*(s),v)W(s,\ x^*(s))$ must be added on the right side of formula (3.7). To prove this form of the Verification Theorem, in the proof of Lemma 3.1 a Feynman-Kac formula (see Appendix formula (D.15)) is used instead of the Ito differential rule.

In Section VI.8 we will consider again criteria of the form (3.19)-(3.20), in case $L = 0, \Psi = 1$. The problem of minimizing J is called there a risk sensitive control problem.

IV.4 Uniformly parabolic case

When the condition (3.5) of uniform parabolicity holds we expect the boundary value problem (3.3) - (3.4) to have a unique solution $W(t,x)$ with the properties required in the Verification Theorem 3.1. In this section we state, without proof, some results of this kind. These results are proved by methods in the theory of second order nonlinear parabolic partial differential equations. In the first result which we cite the following assumptions are made:

(4.1)
- (a) U is compact;
- (b) O is bounded with ∂O a manifold of class $C^{(3)}$;
- (c) For $g = a, f, L,$ the function g and its partial derivatives $g_t, g_{x_i}, g_{x_i x_j}$ are continuous on $\overline{Q}\times U, i,j = 1,\cdots,n.$
- (d) $\Psi\in C^3(\overline{Q}_0)$.

Theorem 4.1. *If assumptions* (3.5) (4.1) *hold, then* (3.3)-(3.4) *has a unique solution* $W \in C^{1,2}(Q) \cap C(\overline{Q})$.

This theorem is due to Krylov. For a proof see [Kr2, Chap. 6, esp. pp. 279 and 297].

In case $\sigma(t,x)$ does not depend on the control parameter v, the HJB equation (3.3) takes the form

$$-\frac{\partial V}{\partial t} - \frac{1}{2}\operatorname{tr} a(t,x) D_x^2 V + H(t,x,D_x V) = 0, \tag{4.2}$$

where as in the case of deterministic optimal control (see formula I(5.4))

$$H(t,x,p) = \sup_{v \in U} [-f(t,x,v)\cdot p - L(t,x,v)]. \tag{4.3}$$

A partial differential equation of the form (4.2) is called *semilinear*, since $D_x^2 V$ appears linearly. Results for semilinear, uniformly parabolic HJB equations were proved before corresponding results for fully nonlinear uniformly parabolic PDEs were known. See [LSU]. If some standard technical estimates for linear, uniformly parabolic PDEs are taken for granted, then one can also consult [FR, Sec. VI.6 and Appendix E.]

Results like Theorem 4.1 in an unbounded region Q require additional boundedness or growth assumptions on a, f, L, Ψ and certain of their partial derivatives. To simplify matters, in the unbounded case let us consider only $O = \mathbb{R}^n$ (hence $Q = Q_0$). Let

$$\Psi(t_1, x) = \psi(x). \tag{4.4}$$

The next result which we cite is a special case of a theorem of Krylov [Kr 2, p. 301]. We denote by $C_b^k(\mathbb{R}^n)$ the space of ϕ such that ϕ and its partial derivatives of orders $\leq k$ are continuous and bounded. The space $C_b^{m,k}(\overline{Q}_0)$ is defined similarly. See the Notation list before Chapter I. The following assumptions are made:

(4.5)

- (a) U is compact;
- (b) a, f, L are continuous and bounded on $\overline{Q}_0 \times U$;
- (c) For $g = a, f, L,$ the function g and its partial derivatives $g_t, g_{x_i}, g_{x_i x_j}$ are continuous and bounded on $\overline{Q}_0 \times U, i, j = 1, \cdots, n$;
- (d) $\psi \in C_b^3(\mathbb{R}^n)$.

Theorem 4.2. *Let* $Q = Q_0$. *If assumptions* (3.5) (4.5) *hold, then* (3.3) *with the Cauchy data* $W(t_1, x) = \psi(x)$ *has a unique solution* $W \in C_b^{1,2}(\overline{Q}_0)$.

When the strong boundedness restrictions in (4.5) do not hold, one can often find a solution $W \in C^{1,2}(\overline{Q}_0) \cap C_p(\overline{Q}_0)$ by approximating a, f, L, ψ by functions which satisfy (4.5). Let us cite a result of this type, for the semilinear case. We assume that $\sigma(t,x)$ is a nonsingular $n \times n$ matrix. Moreover:

(4.6)
(a) U is compact;
(b) $f(t,x,v) = \tilde{b}(t,x) + \sigma(t,x)\Theta(t,x,v)$;
(c) $\tilde{b}$, σ are in $C^{1,2}(\overline{Q}_0)$; moreover, $\sigma, \sigma^{-1}, \sigma_x, \tilde{b}_x$ are bounded on $\overline{Q}_0$, while $\Theta \in C^1(\overline{Q}_0 \times U)$ with Θ, Θ_x bounded;
(d) $L \in C^1(\overline{Q}_0 \times U)$, and L, L_x satisfy a polynomial growth condition;
(e) $\psi \in C^3(I\!R^n)$, and ψ, ψ_x satisfy a polynomial growth condition.

Theorem 4.3. *Let $Q = Q_0$ and assume* (3.5), (4.6). *Then* (4.2) *with the Cauchy data $W(t_1,x) = \psi(x)$ has a unique solution $W \in C^{1,2}(\overline{Q}_0) \cap C_p(\overline{Q}_0)$.*

This is [FR, Thm 6.2, p. 169]. The proof of existence of W given there uses PDE arguments; see [FR, p. 210]. However, uniqueness is obtained using the stochastic control interpretation of the HJB equation (3.3) and a verification theorem.

Remark 4.1. The existence Theorems 4.1 - 4.3 all extend readily to HJB equations of the more general form (3.21) - (3.22). It is sufficient to add the assumption that (4.1)(c) if Q is bounded, or (4.5)(c) if $Q = Q_0$, also holds for $g = \ell$.

Optimal Markov control policies. Let us next sketch how to apply Corollary 3.2 to show that an optimal Markov control policy $\underline{u}^*$ exists. As already pointed out in Section 3, the requirement that $\underline{u}^* \in \mathcal{L}$ is generally too restrictive. Therefore, we seek $\underline{u}^*$ which may be only bounded and Borel measurable. For $(s,y,v) \in Q \times U$, let

$$F(s,y,v) = f(s,y,v) \cdot D_x W(s,y) + \frac{1}{2}\mathrm{tr}\, a(s,y,v) D_x^2 W(s,y) + L(s,y,v),$$

where $W \in C^{1,2}(Q)$ is a solution to the HJB equation (3.3). By a measurable selection theorem there exist Borel measurable $\underline{u}^* : Q \to U$ and $N \in \mathcal{B}(Q)$ with $(n+1)$-dimensional Lebesgue measure 0, such that

$$\underline{u}^*(s,y) \in \arg\min F(s,y,v)$$

for all $(s,y) \in Q \backslash N$. See [FR, Appendix B]. For $(s,y) \in \overline{Q}_0 \backslash Q$, let $\underline{u}^*(s,y) = v_0$, where $v_0 \in U$ is arbitrary. In order to apply Corollary 3.2, we need ν^* and a solution $x^*(s)$ to (3.17) with $x^*(t) = x$, such that (3.18) holds. Let us cite two results which provided sufficient conditions for this.

Case 1. Let us assume that $\sigma(t,x,v)$ is an $n \times n$ matrix, such that for all $(t,x,v) \in \overline{Q}_0 \times U$,

$$\sum_{i,j=1}^{n} \sigma_{ij}(t,x,v)\xi_i\xi_j \geq \gamma|\xi|^2, \quad \text{where} \quad \gamma > 0. \tag{4.7}$$

Assumption (4.7) implies that $a = \sigma\sigma'$ satisfies the uniform parabolicity assumption (3.5). By a result of Krylov about existence of a solution to a stochastic differential equation with measurable coefficients [Kr 1, p. 87], there exist ν^* and a solution $x^*(s)$ to (3.17) with $x^*(t) = x$. [It is not asserted that this solution is unique in probability law.] By an estimate of Krylov [Kr 1, p. 66], for each $p \geq n$ there exists M_p with the following property. For every g which is Borel measurable on $\overline{Q}_0$,

$$E_{tx} \int_t^{t_1} |g(s, x^*(s))| ds \leq M_p ||g||_{p+1}, \tag{4.8}$$

where $||\ \ ||_{p+1}$ is the norm in $L^{p+1}(\overline{Q}_0; m)$ and m denotes $(n+1)$-dimensional Lebesgue measure. If g is the indicator function of N, with $m(N) = 0$, the right side of (4.8) is 0. Therefore, (3.18) holds.

Case 2. Let us assume that $\sigma(t, x)$ is a nonsingular $n \times n$ matrix (which does not depend on v). Moreover, we assume that there exists C such that, for all $(t, x) \in \overline{Q}_0$:

$$|\sigma^{-1}(t, x)| \leq C. \tag{4.9}$$

Assumption (4.9) implies (3.5). In this case, the existence of ν^* and a solution $x^*(s)$ to (3.17) with $x^*(t) = x$ is immediate from Girsanov's Theorem. [This solution is unique in probability law.] It also follows from Girsanov's Theorem that $x^*(s)$ has a probability density $p^*(s, y)$ for $t < s \leq t_1$. See [FR, p. 143]. If $m(N) = 0$, then

$$\int_t^{t_1} P_{tx}\{(s, x^*(s)) \in N\} ds = \int\int_N p^*(s, y) dy ds = 0.$$

Therefore, (3.18) holds.

We summarize these results in the following theorem.

Theorem 4.4. *Assume* (4.1) *if O is bounded, or* (4.5) *if $O = \mathbb{R}^n$. Moreover, assume either* (4.7) *or* (4.9)*. Then $\underline{u}^*$ is an optimal Markov control policy, and $V_{PM}(t, x) = W(t, x)$ for all $(t, x) \in Q$.*

Theorem 4.4 will not be used in the developments which follow.

IV.5 Infinite time horizon

Let us now suppose that the stochastic differential equations (2.1) which describe the state dynamics are autonomous: $f = f(x, v), \sigma = \sigma(x, v)$. We take $t = 0$ as the initial time. Then (2.1) becomes

$$dx = f(x(s), u(s))ds + \sigma(x(s), u(s))dw(s), \quad s \geq 0, \tag{5.1}$$

with initial data $x(0) = x$. We assume that f, σ are continuous on $\mathbb{R}^n \times U$, with $f(\cdot, v), \sigma(\cdot, v)$ of class $C^1(\mathbb{R}^n)$ and (as in (2.2))

$$|f_x| \leq C, \quad |\sigma_x| \leq C \tag{5.2}$$

$$(5.3)\qquad |f(x,v)| \le C(1+|x|+|v|),\ |\sigma(x,v)| \le C(1+|x|+v|)$$

for some constant C. As in Section 2, let $O \subset I\!R^n$ be open, with either $O = I\!R^n$ or ∂O a compact manifold of class C^3. Let τ denote the exit time of $x(s)$ from O, or $\tau = +\infty$ if $x(s) \in O$ for all $s \ge 0$.

Let L be continuous on $I\!R^n \times U$, and g continuous on $I\!R^n$. We also assume that L satisfies the polynomial growth condition (2.7a). Given initial data $x \in O$, we wish to minimize

$$(5.4)\qquad J(x;u) = E_x\Big\{\int_0^\tau e^{-\beta s} L(x(s),u(s))ds + \chi_{\tau<\infty} e^{-\beta\tau} g(x(\tau))\Big\},$$

where $\beta \ge 0$ is a discount factor. In order to formulate this problem more precisely, we define admissible progressively measurable control processes as follows. Let $\nu = (\Omega, \{\mathcal{F}_s\}, P, w)$ be a reference probability system (Section 2), where now $(\Omega, \mathcal{F}, P)$ is a probability space, $\mathcal{F}_s \subset \mathcal{F}$, and $w(\cdot)$ is a $\mathcal{F}_s$-adapted brownian motion on $[0,\infty)$. Let $\mathcal{A}_\nu$ denote the set of all $\mathcal{F}_s$-progressively measurable, U-valued processes $u(\cdot)$ on $[0,\infty)$ which satisfy (2.3) with $t = 0$ and any $t_1 < \infty$, together with

$$(5.5)\qquad E_x \int_0^\tau e^{-\beta s} |L(x(s),u(s))| ds < \infty.$$

Let

$$(5.6)\qquad V_\nu(x) = \inf_{\mathcal{A}_\nu} J(x;u)$$

$$(5.7)\qquad V_{PM}(x) = \inf_{\nu} V_\nu(x).$$

The dynamic programming equation (HJB equation) is now the second order nonlinear partial differential equation

$$(5.8)\qquad \beta V + \mathcal{H}(x, DV, D^2V) = 0, \quad x \in O,$$

where $\mathcal{H}(x,p,A)$ is defined by (3.2). See III(9.4). It is considered with the boundary data

$$(5.9)\qquad V(x) = g(x), \quad x \in \partial O.$$

If there exists $c > 0$ such that

$$(5.10)\qquad \sum_{i,j=1}^n a_{ij}(x,v)\xi_i\xi_j \ge c|\xi|^2$$

for all $\xi \in I\!R^n$ and $v \in U$, then the HJB equation is uniformly elliptic. In that case, one may expect (5.8)-(5.9) to have a smooth solution which is unique if O is bounded. For O unbounded and $\beta > 0$ there will generally by many solutions to (5.8)-(5.9). Among these solutions the value function turns out to be the one which does not grow too rapidly as $|x| \to \infty$.

When (5.10) does not hold, then the HJB equation is of *degenerate elliptic type*. In that case, one cannot expect the value function V_{PM} to be a smooth solution to (5.8)-(5.9). Indeed, additional assumptions are needed even to ensure that V_{PM} is continuous on $\overline{O}$, and that V_{PM} satisfies (5.8) and (5.9) in some weaker sense (e.g. as a viscosity solution as defined in Chapter V).

At the end of this section we will mention, without proof, some results about existence and regularity of solutions to (5.8)-(5.9). However, we first prove a Verification Theorem similar to Theorems III.9.1 and 3.1, and give some examples.

Theorem 5.1 (Verification Theorem). *Let $W \in C^2(O) \cap C_p(\overline{O})$ be a solution to* (5.8)-(5.9). *Then for every $x \in O$:*

(a) $W(x) \leq J(x;u)$ for any admissible progressively measurable control process $u(\cdot)$ such that

$$\liminf_{t_1 \to \infty} e^{-\beta t_1} E_x \left[\chi_{\tau \geq t_1} W(x(t_1)) \right] \leq 0. \tag{5.11}$$

(b) *Suppose that there exist $\nu^* = (\Omega^*, \{\mathcal{F}_s^*\}, P^*, w^*)$ and $u^*(\cdot) \in \mathcal{A}_{\nu*}$ such that*

$$\begin{aligned} u^*(s) \in \arg\min \Big[& f(x^*(s),v) \cdot DW(x^*(s)) \\ & + \frac{1}{2} \operatorname{tr} a(x^*(s),v) D^2 W(x^*(s)) + L(x^*(s),v) \Big] \end{aligned} \tag{5.12}$$

for Lebesgue $\times P^*$- almost all (s,ω) such that $0 \leq s < \tau^*(\omega)$, and

$$\lim_{t_1 \to \infty} e^{-\beta t_1} E_x \left[\chi_{\tau^* \geq t_1} W(x^*(t_1)) \right] = 0. \tag{5.13}$$

Then $W(x) = J(x;u^*)$.

Sketch of proof. We apply the Ito differential rule to $\Phi(x,t) = W(x) e^{-\beta t}$. As in the proof of Lemma 3.1, for any $t_1 < \infty$,

$$\begin{aligned} W(x) \leq E_x \Big\{ & \int_0^{t_1 \wedge \tau} e^{-\beta s} L(x(s),u(s)) ds \\ & + e^{-\beta t_1} \chi_{\tau \geq t_1} W(x(t_1)) + e^{-\beta \tau} \chi_{\tau < t_1} g(x(\tau)) \Big\}. \end{aligned}$$

For $u(\cdot) = u^*(\cdot)$ equality holds. We then proceed as in the proof of Lemma III.9.1.

□

Theorem 5.1 does not quite allow us to conclude that $W(x) = V_{PM}(x)$. However, if we let $\tilde{V}_{PM}(x)$ be the infimum of $J(x;u)$ among all ν and $u(\cdot) \in \mathcal{A}_\nu$ such that (5.11) holds, then

$$W(x) = \tilde{V}_{PM}(x) = J(x;u^*).$$

This complication can often be avoided. See Corollary 5.1, and also Examples 5.1, 5.2.

Optimal stationary Markov control policies. As in (9.4) we may seek optimal controls of the form $u^*(s) = \underline{u}^*(x^*(s))$, where $\underline{u}^*$ is a stationary Markov control policy such that

$$\underline{u}^*(x) \in \arg\min[-G^v W(x) + L(x,v)] \tag{5.14}$$

$$-G^v W(x) = f(x,v)\cdot DW(x) + \frac{1}{2}\mathrm{tr}\, a(x,v)D^2W(x). \tag{5.15}$$

An analogue of Theorem 4.4, asserting the existence of an such optimal $\underline{u}^*$ can be proved. However, we shall not do so.

If W is bounded on $\overline{O}$ (in particular, if O is bounded and $W \in C(\overline{O})$) conditions (5.11), (5.13) are readily verified:

Corollary 5.1. *Let $W \in C^2(O)\cap C_b(\overline{O})$ be a solution to* (5.8)-(5.9). *Moreover, assume either that $\beta > 0$ or that $\tau < \infty$ with probability 1 for every admissible progressively measurable control process $u(\cdot)$. Then $W(x) = J(x;u^*) = V_{PM}(x)$.*

Minimum mean exit time from O. Let us apply Corollary 5.1 in the special case $L(x,v)\equiv 1, g(x)\equiv 0, \beta = 0$. Let O be bounded and U compact. Moreover, assume the uniform ellipticity condition (5.10), which implies $\tau < \infty$ with probability 1. Let $J(x;u) = E_x\tau$, the mean exit time of $x(s)$ from O. The HJB equation in this case becomes

$$\min_{v\in U}[-G^v V(x) + 1] = 0, \quad x\in O. \tag{5.16}$$

The boundary data are

$$V(x) = 0, \quad x \in \partial O. \tag{5.17}$$

Example 5.1. Let $n = 1, O = (-b,b)$ and let

$$dx = u(s)ds + \alpha dw(s), \quad \alpha > 0,$$

where the control constraint $|u(s)|\le 1$ is imposed. Then (5.16) becomes

$$\frac{\alpha^2}{2}V''(x) + \min_{|v|\le 1} v\cdot V'(x) + 1 = 0, \quad |x| < b,$$

with $V(b) = V(-b) = 0$. We expect the value function $V(x) = V_{PM}(x)$ to be symmetric with $V(x)$ decreasing for $0 < x < b$. Moreover, $\underline{u}^*(x) =$ sgn x should turn out to be a stationary optimal Markov control policy. With this in mind let $W(x)$ be the solution to

$$\frac{\alpha^2}{2}W''(x) + W'(x) + 1 = 0, \quad 0 < x < b,$$

$$W'(0) = 0, \ W(b) = 0.$$

For $-b \leq x \leq 0$, let $W(x) = W(-x)$. The explicit formula for $W(x)$ is

$$W(x) = \frac{\alpha^2}{2}[e^{-\frac{2b}{\alpha^2}} - e^{-\frac{2|x|}{\alpha^2}}] + b - |x|.$$

Then $W \in C^2([-b,b])$ and W satisfies (5.16)-(5.17). As in the discussion preceding Theorem 4.4, Girsanov's Theorem implies that the stochastic differential equation

$$dx^* = \text{ sgn } x^*(s)ds + \alpha dw^*(s), \ s \geq 0,$$

with initial data $x^*(0) = x$ has a solution for some reference probability system $\nu^* = (\Omega^*, \{\mathcal{F}_s^*\}, P^*, w^*)$. We take $u^*(s) = \text{ sgn } x^*(s)$. Corollary 5.1 can then be applied to show that $V_{PM}(x) = W(x)$ and that $\underline{u}^*(x) = \text{ sgn } x$ is an optimal stationary Markov control policy.

The next example comes from financial economics.

Example 5.2 (Merton's portfolio problem). Chapter X of this book is concerned with applications of stochastic control theory to finance. The earliest such application was Merton's classical portfolio optimization problem [Mer1]. This is formulated as follows. An investor has a portfolio consisting of two assets, one "risk free" and the other "risky". The price $p(s)$ per share of the risk-free asset changes according to $dp = prds$ while the price $S(s)$ of the risky asset changes according to $dS = S(\mu ds + \sigma dw(s))$. Here r, μ, σ are constants with $r < \mu$, $\sigma > 0$ and $w(s)$ is a brownian motion. Let $x(s)$ denote the investor's wealth at time s, $\pi(s)$ the fraction of wealth in the risky asset and $c(s)$ the consumption rate. No constraint is imposed on $\pi(s)$, but it is required that $x(s) \geq 0$, $c(s) \geq 0$. Then $x(s)$ changes according to the stochastic differential equation

$$(5.18) \qquad dx = (1 - \pi(s))x(s)rds + \pi(s)x(s)(\mu ds + \sigma dw(s)) - c(s)ds.$$

We stop the process if the wealth $x(s)$ reaches 0 (bankruptcy). Thus $O = (0, \infty)$. If we write $u_1(s) = \pi(s)$, $u_2(s) = c(s)$, then the control is the two-dimensional vector $u(s) = (u_1(s), u_2(s))$, subject to the constraint $u_2(s) \geq 0$. Thus $U = (-\infty, \infty) \times [0, \infty)$ The "running cost" is

$$(5.19) \qquad L(x, v_2) = \ell(v_2)$$

where $\ell(c)$ is the utility of consuming at rate $c > 0$. We assume that $\ell(0) = 0, \ell'(0^+) = +\infty, \ell'(c) > 0, \ell''(c) < 0$ for $c > 0$. The problem is to *maximize* the total expected utility, discounted at rate $\beta > 0$:

$$(5.20) \qquad J(x; u_1, u_2) = E_x \int_0^\tau e^{-\beta s} \ell(c(s))ds$$

where $c(s) = u_2(s)$ is the consumption rate and either $\tau = +\infty$ or τ is the bankruptcy time.

For $v = (v_1, v_2) \in U$,

$$-G^v W(x) = \frac{\sigma^2 v_1^2 x^2}{2} W_{xx} + (\mu - r) v_1 x W_x + r x W_x - v_2 W_x.$$

The dynamic programming equation (5.8) can be put in the form:

$$\text{(5.21)} \qquad \beta W = \max_{v_1} \left[\frac{\sigma^2 v_1^2 x^2}{2} W_{xx} + (\mu - r) v_1 x W_x \right] + r x W_x + \max_{v_2 \geq 0} [\ell(v_2) - v_2 W_x].$$

More precisely, if ℓ is replaced by $-\ell$, then the problem is to minimize $-J$. Thus $-W$ must satisfy the equation corresponding to (5.21), with min instead of max and ℓ replaced by $-\ell$. This has the form (5.8). The value function V should be increasing on $[0, \infty)$. Moreover, this example has a linear–concave structure which implies that V is concave. See Example 10.1 below. Accordingly, we look for a solution to (5.21) for $x > 0$ with $W_x > 0, W_{xx} < 0$ and

$$0 = \lim_{x \to 0^+} W(x).$$

By an elementary calculation, we get the following candidates for optimal investment and consumption policies:

$$\text{(5.22)} \qquad \underline{u}_1^*(x) = -\frac{(\mu - r) W_x(s)}{\sigma^2 x W_{xx}(s)}, \;\; \underline{u}_2^*(x) = (\ell^{'})^{-1}(W_x(x)).$$

Let us now assume that

$$\text{(5.23)} \qquad \ell(c) = \frac{1}{\gamma} c^\gamma, \;\; 0 < \gamma < 1.$$

Such an ℓ is called of hyperbolic absolute risk aversion (HARA) type. As a solution to (5.21), we try

$$\text{(5.24)} \qquad W(x) = K x^\gamma.$$

Then (5.22) becomes

$$\text{(5.25)} \qquad \underline{u}_1^*(x) = \frac{\mu - r}{(1 - \gamma)\sigma^2}, \;\; \underline{u}_2^*(x) = (\gamma K)^{\frac{1}{\gamma - 1}} x.$$

Note that $\underline{u}_1^*$ does not, in fact, depend on x, and $\underline{u}^*(x)$ is linear in x.

It remains to determine K. By substituting (5.24) and (5.25) in (5.21) we get a nonlinear equation for K, which has a positive solution provided

$$\text{(5.26)} \qquad \beta > \frac{(\mu - r)^2 \gamma}{2\sigma^2 (1 - \gamma)} + r\gamma.$$

Let us verify that $\underline{u}_1^*, \underline{u}_2^*$ are optimal policies and that $W(x)$ is the maximum expected discounted utility. Since $W(x) \geq 0$, condition (5.11) is automatically satisfied by $-W$ (recall that signs change for a problem of maximum rather than minimum). Hence by Theorem 5.1(a) by $W(x) \geq J(x; u_1, u_2)$ for all admissible $u_1(\cdot), u_2(\cdot)$. Instead of verifying (5.13), let us check directly that, for all $x > 0$,

$$W(x) = J(x; \underline{u}_1^*, \underline{u}_2^*). \tag{5.27}$$

Since $\underline{u}_1^*$ is constant and $\underline{u}_2^*(x)$ is linear in x, $x^*(s)$ satisfies a linear, constant coefficient stochastic differential equation (5.18) which can be solved explicitly. By doing so, we find that $x^*(s) > 0$ for all $s \geq 0$. (In fact by applying the Ito differential rule to $\log x^*(s)$, we find that $\log x^*(s)$ is a brownian motion with drift). Thus $\tau^* = +\infty$. As in the proof of Theorem 5.1

$$W(x) = E_x \left\{ \frac{1}{\gamma} \int_0^{t_1} e^{-\beta s} \left[\underline{u}_2^*(x^*(s))\right]^\gamma ds + e^{-\beta t_1} W(x^*(t_1)) \right\}$$

for any $t_1 > 0$. Since $W(x^*(t_1)) \geq 0$ this implies by (5.25)

$$E_x \int_0^\infty e^{-\beta s} [x^*(s)]^\gamma ds < \infty. \tag{5.28}$$

Since $W(x^*(t_1)) = K[x^*(t_1)]^\gamma$, we get (5.27) by sending $t_1 \to \infty$ through a sequence such that $e^{-\beta t_1} E_x [x^*(t_1)]^\gamma$ tends to 0.

Remark 5.1. In this model there is no way to create wealth or to consume once bankruptcy occurs. The problem which we have formulated is equivalent to one with a state constraint ($x(s) \geq 0$ and $\tau = +\infty$ in (5.20)).

Remark 5.2. It was shown by Karatzas - Lehoczky - Sethi - Shreve [KLSS] that the Merton problem has a relatively explicit solution even if $\ell(c)$ does not have the special HARA form (5.23). They show that by introducing a new "independent variable" the dynamic programming equation for $W(x)$ is changed into an inhomogeneous second order *linear* differential equation. This will be discussed in Section X.4.

When $u_1(s) > 1$, the term $(1 - u_1(s))x(s)$ in (5.18) is negative. This term represents money borrowed by the investor; and it has been implicitly assumed that money is borrowed at the same rate r as the return rate for the riskless asset. When the borrowing rate is larger than r, the dynamic programming equation no longer has an explicit solution $W(x)$. This case has been studied by Fleming-Zariphopoulou [FZ] and Fitzpatrick-Fleming [FF] using viscosity solution techniques and numerical solutions of the dynamic programming equation. Another simplification in Merton's problem is that transaction costs were ignored. When these are included in the model, one has a stochastic control problem of "singular" type. The portfolio selection problem with transaction cost is solved by viscosity solution methods in Section X.5.

For results like Theorem 4.1 concerning the existence of a classical solution $W \in C^2(O) \cap C(\bar{O})$ when O is bounded and the uniform ellipticity condition

(5.10) holds, we refer to Evans [E1] [E4], Gilbarg - Trudinger [GT, Thm. 17.17], Trudinger [Tu].

IV.6 Fixed finite time horizon problem: Preliminary estimates

We will take $Q = Q_0 = [t_0, t_1) \times I\!R^n$ in the rest of this chapter. Thus, control occurs on a fixed finite interval $[t, t_1]$. Our goal is to study systematically the value function V_{PM}, without the uniform parabolicity condition (3.5). For notational simplicity, let us write

$$V(t, x) = V_{PM}(t, x).$$

We can no longer expect that $V \in C^{1,2}(Q_0)$; hence V cannot be in general a classical solution at the HJB equation (3.3). However, we will find that V satisfies (3.3) in some extended sense (as a generalized solution in Section 10, or a viscosity solution in Chapter V.)

We make the following assumptions:

$$\text{(6.1)} \qquad \begin{aligned} &(a)\ U \text{ is compact;} \\ &(b)\ f, \sigma \text{ are continuous on } \overline{Q}_0 \times U, \text{ and } f(\cdot,\cdot,v), \sigma(\cdot,\cdot,v) \\ &\quad\ \text{are of class } C^1(\overline{Q}_0) \text{ for each } v \in U; \\ &(c)\ \text{For suitable } C_1, C_2 \\ &\quad\ |f_t| + |f_x| \le C_1,\ |\sigma_t| + |\sigma_x| \le C_1, \\ &\quad\ |f(t,0,v)| + |\sigma(t,x,v| \le C_2. \end{aligned}$$

Remark 6.1. The assumptions (6.1) are similar to (2.2). However, compactness of U and boundedness of σ are now assumed. We have not sought utmost generality, under the weakest possible assumptions. For example, for the estimates in this section one could assume instead of class $C^1(\overline{Q}_0)$ that $f(t,\cdot,v), \sigma(t,\cdot,v)$ satisfy a Lipschitz condition uniformly with respect to t, v.

Unless stated otherwise, we shall also suppose that

$$\psi(x) = \Psi(t_1, x) = 0,\ x \in I\!R^n. \tag{6.2}$$

This is scarcely a restriction. Indeed, if $\psi \in C_p^{1,2}(\overline{Q}_0)$ then the Dynkin formula implies that

$$J(t, x; u) = \psi(x) + E_{tx} \int_t^{t_1} \tilde{L}(s, x(s), u(s)) ds,$$

with $\tilde{L} = L - G^v \psi$. The first term $\psi(x)$ on the right side is unaffected by the control $u(\cdot)$. Hence to minimize J is equivalent to minimizing the last term.

We assume that L is continuous on $\overline{Q}_0 \times U$ and that

$$|L(t, x, v)| \le C_3(1 + |x|^k) \tag{6.3}$$

for suitable C_3 and k. We shall use standard results about stochastic differential equations with random (progressively measurable) coefficients. These are summarized in Appendix D. In the notation of the Appendix D, given a progressively measurable $u(\cdot)$, let

$$b(s,y) = f(s,y,u(s)), \quad \gamma(s,y) = \sigma(s,y,u(s)).$$

Then (2.1) can be rewritten (Appendix (D.1)) as

$$(2.1') \qquad dx = b(s,x(s))ds + \gamma(s,x(s))dw(s),$$

with initial data $x(t) = x$. If we let

$$\Lambda(s,y) = L(s,y,u(s))$$

then

$$J(t,x;u) = E_{tx}\int_t^{t_1} \Lambda(s,x(s))ds.$$

From Appendix (D.7)

$$(6.4) \qquad E_{tx}\|x(\cdot)\|^m \le B_m(1+|x|^m),$$

where $\| \ \|$ is the sup norm on $[t,t_1]$. The constant B_m depends on C_1 and C_2 in (6.1c) as well as U and $t_1 - t_0$. See Appendix (D.5). By (6.3) and (6.4) with $m = k$,

$$(6.5) \qquad |J(t,x;u)| \le M(t_1 - t)(1+|x|^k),$$

with $M = C_3(1+B_k)$. From the definition (2.10) of $V = V_{PM}$ and (6.5):

$$(6.6) \qquad |V(t,x)| \le M(t_1-t)(1+|x|^k), \ (t,x) \in Q.$$

The fact the M depends on f, σ, L only through the constants C_1, C_2, C_3 and k will be useful when we make various approximations to f, σ and L.

Let us first obtain bounds for the change in the value function V when f and L are approximated by certain functions f_ρ and L_ρ with compact support. For each $\rho > 0$ choose $\alpha_\rho \in C^\infty(I\!R^n)$ such that

$$0 \le \alpha_\rho(x) \le 1, \ |D\alpha_\rho(x)| \le 2,$$

$$\alpha_\rho(x) = 1 \text{ for } \ |x| \le \rho, \quad \alpha_\rho(x) = 0 \text{ for } \ |x| \ge \rho + 1.$$

Let $L_\rho = \alpha_\rho L, f_\rho = \alpha_\rho f$. Note L_ρ and f_ρ satisfy (6.1) and (6.3) with the same C_2, C_3 and C_1' which does not depend on ρ.

Given a probability reference system ν and progressively measurable $u(\cdot) \in \mathcal{A}_{t\nu}$, let $x_\rho(s)$ denote the solution to

$$(2.1_\rho) \qquad dx_\rho = f_\rho(s,x_\rho(s),u(s))ds + \sigma(s,x_\rho(s),u(s))dw(s)$$

with $x_\rho(t) = x(t) = x$. From the theory of stochastic differential equations, with probability 1

$$(6.7) \qquad x_\rho(s) = x(s) \;\text{ for }\; t \le s \le \tau_\rho,$$

where τ_ρ is the exit time of $(s,x(s))$ from $Q_\rho = [t_0,t_1) \times \{|x| < \rho\}$. See [GS2] or [IW, Chap. 4]. By (6.4) with $m = 1$,

$$(6.8) \qquad P_{tx}(\tau_\rho < t_1) \le P_{tx}(\|x(\cdot)\| \ge \rho) \le B_1\rho^{-1}(1+|x|).$$

Let

$$\Lambda_\rho(s,y) = L_\rho(s,y,u(s)).$$

Lemma 6.1. *There exists C such that*

$$(6.9) \qquad E_{tx}\int_t^{t_1} |\Lambda_\rho(s,x_\rho(s)) - \Lambda(s,x(s))|ds \le C\rho^{-\frac{1}{2}}(1+|x|^{2k+1})^{\frac{1}{2}},$$

where k is as in (6.3).

Proof. Let χ_ρ be the indicator function of the event $\tau_\rho < t_1$. Using the Cauchy–Schwarz inequality

$$E_{tx}\int_{\tau_\rho}^{t_1} |\Lambda_\rho|ds \le E_{tx}\int_t^{t_1} \chi_\rho|\Lambda_\rho|ds$$

$$\le [(t_1 - t)P_{tx}(\tau_\rho < t_1)E_{tx}\int_t^{t_1} |\Lambda_\rho|^2 ds]^{\frac{1}{2}}.$$

By (6.3) and (6.4) with $m = 2k$, for suitable constant Γ

$$E_{tx}\int_t^{t_1} |\Lambda_\rho|^2 ds \le \Gamma(1+|x|^{2k}).$$

Thus, by (6.8) and Cauchy-Schwarz, for suitable C

$$E_{tx}\int_{\tau_\rho}^{t_1} |\Lambda_\rho|ds \le \frac{C}{2}\rho^{-\frac{1}{2}}(1+|x|^{2k+1})^{\frac{1}{2}}.$$

Similarly,

$$E_{tx}\int_{\tau_\rho}^{t_1} |\Lambda|ds \le \frac{C}{2}\rho^{-\frac{1}{2}}(1+|x|^{2k+1})^{\frac{1}{2}}.$$

Finally, by (6.7) and definition of L_ρ, $\Lambda_\rho(s,x_\rho(s)) = \Lambda(s,x(s))$ for $t \le s \le \tau_\rho$. □

Let

$$J_\rho(t,x;u) = E_{tx}\int_t^{t_1} L_\rho(s,x_\rho(s),u(s))ds.$$

Then Lemma 6.1 implies

$$|J_\rho(t,x;u) - J(t,x;u)| \le C\rho^{-\frac{1}{2}}(1+|x|^{2k+1})^{\frac{1}{2}}.$$

Since this is true for all ν and $u(\cdot) \in \mathcal{A}_{t\nu}$,

$$(6.10) \qquad |V_\rho(t,x) - V(t,x)| \le C\rho^{-\frac{1}{2}}(1+|x|^{2k+1})^{\frac{1}{2}},$$

where V_ρ is the value function with f, L replaced by f_ρ, L_ρ. The same inequality (6.10) holds for $|V_{\rho\nu} - V_\nu|$, where ν is any reference probability system.

Let us next obtain estimates in the sup norm $\|\ \ \|$. Let $\tilde{x}(s)$ be the solution of

$$(\tilde{2.1}) \qquad d\tilde{x} = \tilde{f}(s,\tilde{x}(s),u(s))ds + \tilde{\sigma}(s,\tilde{x}(s),u(s))dw(s),$$

with $\tilde{x}(t) = x(t) = x$. By Appendix (D.9), there exist $\bar{B}_m$ such that

$$(6.11) \qquad E_{tx}\|x(\cdot) - \tilde{x}(\cdot)\|^m \le \bar{B}_m[\ \|f-\tilde{f}\|^m + \|\sigma - \tilde{\sigma}\|^m\ ].$$

We also consider an integrand $\tilde{L}$ instead of L, and let

$$\tilde{\Lambda}(s,y) = \tilde{L}(s,y,u(s)).$$

Lemma 6.2. *Assume that $L, \tilde{L}, \tilde{L}_x$ are continuous, and that $L - \tilde{L}, \tilde{L}_x$ are bounded on $\overline{Q}_0 \times U$. Then*

$$(6.12) \qquad E_{tx}\int_t^{t_1} |\Lambda(s,x(s)) - \tilde{\Lambda}(s,\tilde{x}(s))|ds$$

$$\le (t_1 - t)\{\|L-\tilde{L}\| + \bar{B}_1\|\tilde{L}_x\|\ [\ \|f-\tilde{f}\| + \|\sigma-\tilde{\sigma}\|\ ]\ \},$$

with $\bar{B}_1$ as in (6.11) *with $m = 1$.*

Proof. We have

$$|\Lambda(s,x(s)) - \tilde{\Lambda}(s,\tilde{x}(s))| \le |\Lambda(s,x(s)) - \tilde{\Lambda}(s,x(s))| + |\tilde{\Lambda}(s,x(s)) - \tilde{\Lambda}(s,\tilde{x}(s))|$$

$$\le \|L-\tilde{L}\| + \|\tilde{L}_x\|\ \|x(\cdot) - \tilde{x}(\cdot)\|.$$

By (6.11) with $m=1$ and Cauchy-Schwarz

$$E\|x(\cdot) - \tilde{x}(\cdot)\| \le \bar{B}_1[\ \|f-\tilde{f}\| + \|\sigma - \tilde{\sigma}\|\ ].$$

□

If we set

$$\tilde{J}(t,x;u) = E_{tx}\int_t^{t_1} \tilde{L}(s,\tilde{x}(s),u(s))ds,$$

then $|J - \tilde{J}|$ is no more than the left side of (6.12), for every ν and $u(\cdot) \in \mathcal{A}_{t\nu}$. Therefore,

$$(6.13) \quad |V(t,x) - \tilde{V}(t,x)| \le (t_1-t)\{\|L-\tilde{L}\| + \bar{B}_1\|\tilde{L}_x\|\ [\ \|f-\tilde{f}\| + \|\sigma-\tilde{\sigma}\|\ ]\}.$$

The same inequality (6.13) holds for $|V_\nu - \tilde{V}_\nu|$, where ν is any reference probability system.

Let us apply Lemma 6.2 to the case when (2.1) is perturbed by adding a "small" term $\epsilon^{\frac{1}{2}}\sigma_1 dw_1$, where $w_1(\cdot)$ is a d_1-dimensional brownian motion independent of $w(\cdot)$. Let $u(\cdot) \in \mathcal{A}_{t\mu}$, where $\mu = (\Omega, \{\mathcal{F}_s\}, P, w, w_1)$ is a reference probability system. Let $x^\epsilon(s)$ be the solution of

$$dx^\epsilon = f(s, x^\epsilon(s), u(s))ds + \sigma(s, x^\epsilon(s), u(s))dw(s) + \epsilon^{\frac{1}{2}}\sigma_1 dw_1 \tag{6.14}$$

with $x^\epsilon(t) = x(t) = x$. Define

$$J^\epsilon(t, x; u) = E_{tx} \int_t^{t_1} L(s, x^\epsilon(s), u(s))ds,$$

$$V^\epsilon(t, x) = \inf_{\mu} \inf_{\mathcal{A}_{t\mu}} J^\epsilon(t, x; u). \tag{6.15}$$

In Lemma 6.2 we take $\tilde{f} = f, \tilde{L} = L$ and $\tilde{\sigma} = (\sigma, \epsilon^{\frac{1}{2}}\sigma_1)$. The matrices $\tilde{\sigma}(t, x, v)$ are $n \times (d + d_1)$ dimensional, where d, d_1 are the respective dimensions of the brownian motions $w(\cdot), w_1(\cdot)$. Then

$$|J - J^\epsilon| \leq K\epsilon^{\frac{1}{2}}, \tag{6.16}$$

where $K = (t_1 - t_0)\bar{B}_1 \|L_x\| \, |\sigma_1|$. From (6.16) we will obtain:

Lemma 6.3. *If L_x is continuous and bounded on $\overline{Q}_0 \times U$, then*

$$|V(t, x) - V^\epsilon(t, x)| \leq K\epsilon^{\frac{1}{2}}. \tag{6.17}$$

Proof. To obtain (6.17) from (6.16) there is the minor difficulty that V is defined in (2.10) using 4-tuples ν, while V^ϵ is defined in terms of 5-tuples μ. However, given $\mu = (\Omega, \{\mathcal{F}_s\}, P, w, w_1)$, let $\nu = (\Omega, \{\mathcal{F}_s\}, P, w)$. Any $u(\cdot) \in \mathcal{A}_{t\mu}$ is also in $\mathcal{A}_{t\nu}$. By (6.16)

$$J^\epsilon(t, x; u) \geq J(t, x; u) - K\epsilon^{\frac{1}{2}} \geq V(t, x) - K\epsilon^{\frac{1}{2}},$$

and therefore

$$V^\epsilon(t, x) \geq V(t, x) - K\epsilon^{\frac{1}{2}}.$$

On the other hand, given ν consider another reference probability system $(\Omega'$, $\{\mathcal{F}'_s\}$,$P', w')$, with $w'(s)$ an n-dimensional brownian motion adapted to $\{\mathcal{F}'_s\}$. For $(\omega, \omega') \in \Omega \times \Omega'$, let $w_1(s, \omega, \omega') = w'(s, \omega')$. Let

$$\mu = (\Omega \times \Omega', \{\mathcal{F}_s \times \mathcal{F}'_s\}, P \times P', w, w_1).$$

If $u(\cdot) \in \mathcal{A}_{t\nu}$, we can regard $u(\cdot)$ as an element of $\mathcal{A}_{t\mu}$, which does not depend on ω'. Then

$$J(t, x; u) \geq J^\epsilon(t, x; u) - K\epsilon^{\frac{1}{2}} \geq V^\epsilon(t, x) - K\epsilon^{\frac{1}{2}},$$

and therefore

$$V(t, x) \geq V^\epsilon(t, x) - K\epsilon^{\frac{1}{2}}.$$

□

IV.7 Dynamic programming principle

Let us next obtain a result which is a stronger version of the traditional dynamic programming principle. We again write $V_{PM} = V$. When we refer to a stopping time θ in this section, we always assume that $\theta \in [t, t_1]$.

Definition 7.1. The fixed finite time horizon problem has property (DP) if, for every $(t,x) \in Q_0$, the following hold: For every $\nu, u(\cdot) \in \mathcal{A}_{t\nu}$ and $\{\mathcal{F}_s\}$-stopping time θ,

$$V(t,x) \le E_{tx}\left\{\int_t^\theta L(s,x(s),u(s))ds + V(\theta, x(\theta))\right\}. \tag{7.1a}$$

For every $\delta > 0$, there exist ν and $u(\cdot) \in \mathcal{A}_{t\nu}$ such that

$$V(t,x) + \delta \ge E_{tx}\left\{\int_t^\theta L(s,x(s),u(s))ds + V(\theta, x(\theta))\right\} \tag{7.1b}$$

for every $\{\mathcal{F}_s\}$-stopping time θ.

We will establish property (DP) by using Theorem 4.2 together with an approximation argument. A similar dynamic programming principle when Q_0 is replaced by a cylindrical region Q will be obtained in the next chapter. See Section V.2.

We shall not use Theorem 4.4 about the existence of optimal Markov control policies. Instead, under the assumptions of Theorem 4.2 we will construct directly $u(\cdot)$ with the properties needed in (7.1b). For this purpose we define the concept of discrete-time Markov control policy. Let $\underline{u}_\ell : I\!R^n \to U$ be Borel measurable for $\ell = 0, 1, \cdots, M-1$, and let $\underline{\mathbf{u}} = (\underline{u}_0, \underline{u}_1, \cdots, \underline{u}_{M-1})$. We partition $[t, t_1)$ into M subintervals $I_j = [s_j, s_{j+1}), j = 0, 1, \cdots, M-1$, where $t = s_0 < s_1 < \cdots < s_M = t_1$ and $s_{j+1} - s_j = (t_1 - t)M^{-1}$. Then $\underline{\mathbf{u}}$ is called a *discrete-time Markov control policy*, relative to this partition.

Given initial data (t,x) and any reference probability system ν, a discrete-time Markov control policy $\underline{\mathbf{u}}$ defines $u(\cdot) \in \mathcal{A}_{t\nu}$ and solution $x(\cdot)$ to (2.1), such that

$$u(s) = \underline{u}_j(x(s_j)), \text{ if } s \in I_j, j = 0, 1, \cdots, M-1. \tag{7.2}$$

This is done by induction on j. For $s \in \bar{I}_j$, $x(s)$ is the solution to (2.1) with $\mathcal{F}_{s_j}$-measurable initial data $x(s_j)$.

Lemma 7.1. *Assume* (3.5) *and* (4.5). *Let W be as in Theorem* 4.2. *Given any reference probability system ν and $\delta > 0$, there exists $u(\cdot) \in \mathcal{A}_{t\nu}$ such that*

$$W(t,x) + \delta \ge E_{tx}\left\{\int_t^\theta L(s,x(s),u(s))ds + W(\theta, x(\theta))\right\} \tag{7.3}$$

for every $\{\mathcal{F}_s\}$-stopping time θ.

Proof. Let us fix $\rho, \alpha > 0$. Since $W \in C_b^{1,2}(\overline{Q}_0)$, the partial derivatives $W_t, W_{x_i}, W_{x_i x_j}$ are uniformly continuous on $\overline{Q}_\rho = [t, t_1] \times \overline{O}_\rho$, where $O_\rho = \{|x| < \rho\}$. Hence, there exists $\gamma > 0$ such that $|s' - s| < \gamma$ and $|y' - y| < \gamma$ imply

$$(7.4) \qquad |A^v W(s,y) + L(s,y,v) - A^v W(s',y') - L(s',y',v)| < \frac{\alpha}{2}$$

for all $v \in U$ and $(s,y) \in \bar{Q}_\rho$. Here $A^v W$ is as in (2.11), and (3.3) can be written as

$$0 = \min_{v \in U} [A^v W + L].$$

Let $\overline{O}_\rho = B_1 \cup \cdots \cup B_N$, where $B_1, \cdots, B_N$ are disjoint Borel sets of diameter less than $\gamma/2$. We choose M large enough that $M^{-1}(t_1 - t) < \min(\gamma, 1)$. We partition $[t, t_1)$ into M subintervals $I_j = [s_j, s_{j+1})$ of length $M^{-1}(t_1 - t)$, and choose $y_k \in B_k$. Since W satisfies the HJB equation (3.3), there exist $v_{jk} \in U$, such that

$$A^{v_{jk}} W(s_j, y_k) + L(s_j, y_k, v_{jk}) < \frac{\alpha}{2}.$$

By (7.4)

$$(7.5) \qquad A^{v_{jk}} W(s,y) + L(s,y,v_{jk}) < \alpha$$

for all $s \in I_j$ and y such that $|y - y_k| < \gamma$.

We define the discrete time Markov control policy $\underline{\mathbf{u}} = (\underline{u}_0, \underline{u}_1, \cdots, \underline{u}_{M-1})$ by

$$\underline{u}_j(y) = \begin{cases} v_{jk} \ \text{ if } \ y \in B_k, \\ \\ v_0 \ \text{ if } \ (s,y) \in \overline{Q}_0 \backslash \overline{Q}_\rho, \end{cases}$$

where $v_0 \in U$ is arbitrary. Define, by induction on j, $u(\cdot) \in \mathcal{A}_{t\nu}$ and solution $x(s)$ to (2.1) with $x(t) = x$ such that (7.2) holds. Thus

$$(7.6) \qquad u(s) = v_{jk} \ \text{ if } \ s \in I_j, x(s_j) \in B_k.$$

Let us show that (7.3) holds, if ρ, α and M are suitably chosen. By Dynkin's formula

$$(7.7) \qquad W(t,x) = E_{tx} \left\{ -\int_t^\theta A^{u(s)} W(s, x(s)) ds + W(\theta, x(\theta)) \right\}$$

$$= E_{tx} \left\{ \int_t^\theta L ds + W(\theta, x(\theta)) \right\} - E_{tx} \left\{ \int_t^\theta [A^{u(s)} W + L] ds \right\}.$$

In using Dynkin's formula, we recall that $W \in C_b^{1,2}(\overline{Q}_0)$. We must estimate the last term from below. Let

$$\Gamma = \{ \|x(\cdot)\| \le \rho, |x(s) - x(s_j)| < \frac{\gamma}{2}, s \in I_j, j = 0, 1, \cdots, M-1 \}.$$

By (7.5) and (7.6), and the fact that $|x(s_j) - y_k| < \gamma/2$ if $x(s_j) \in B_k$, we have on Γ

$$A^{u(s)}W(s,x(s)) + L(s,x(s),u(s)) < \alpha. \tag{7.8}$$

Note that f and σ are bounded by (4.5b). By a standard estimate for solutions to stochastic differential equations (Appendix (D.12)) there exists C such that

$$P_{tx}(\max_{I_j} |x(s) - x(s_j)| \geq \frac{\gamma}{2}) \leq \frac{C(s_{j+1} - s_j)^2}{\gamma^4}.$$

Since $s_{j+1} - s_j = M^{-1}(t_1 - t)$ and there are M intervals I_j, we have

$$P_{tx}(\max_{j,I_j} |x(s) - x(s_j)| \geq \frac{\gamma}{2}) \leq \frac{C(t_1 - t)^2}{M\gamma^4}. \tag{7.9}$$

We then have by (7.8) and the fact that $\theta \leq t_1$,

$$E_{tx} \int_t^\theta [A^{u(s)}W + L]ds \leq \alpha(t_1 - t) + \|A^v W + L\| P_{tx}(\Gamma^c). \tag{7.10}$$

By (6.8) and (7.9), the last term is less than $\delta/2$, if ρ and M are chosen large enough. If $\alpha(t_1 - t) < \delta/2$, then (7.3) follows from (7.7) and (7.10). □

As a consequence of Lemma 7.1 we have:

Lemma 7.2. *Assume* (3.5) *and* (4.5). *Let W be as in Theorem* 4.2. *Then $W = V = V_\nu$ for every ν, and property* (DP) *holds.*

Proof. By Dynkin's formula

$$W(t,x) \leq E_{tx} \left\{ \int_t^\theta L(s,x(s),u(s))ds + W(\theta, x(\theta)) \right\} \tag{7.11}$$

for every $\nu, u(\cdot) \in \mathcal{A}_{t\nu}$ and $\mathcal{F}_s$-stopping time θ. (See proof of Lemma 3.1.). In particular, if we take $\theta = t_1$ and use Lemma 7.1 we get

$$W(t,x) = \inf_{\mathcal{A}_{t\nu}} J(t,x;u) = V_\nu(t,x).$$

Therefore, $W = V_\nu = V$. Property (DP) is then just (7.3) and (7.11).

□

Theorem 7.1. *Assume* (6.1)-(6.3). *Then V is continuous on $\overline{Q}_0$ and property* (DP) *holds. Moreover, $V = V_\nu$ for every reference probability system ν.*

Proof. *Step 1.* Assume that (3.5) and (4.5) hold. Then the conclusions of Theorem 7.1 follow from Lemma 7.2.

Step 2. Next, assume that there exists ρ such that

$$f(t,x,v) = 0,\ L(t,x,v) = 0 \text{ if } |x| \geq \rho.$$

By a standard smoothing technique (Appendix C), for $n = 1, 2 \cdots$ there exists continuous L_n such that $||L_n - L|| \to 0$ as $n \to \infty$ and $L_n(\cdot,\cdot,v) \in C_b^{1,2}(\overline{Q}_0)$.

Similarly, by smoothing f and σ, there exist f_n, σ_n with the same smoothness property, such that

$$\lim_{n\to\infty} ||L_{nx}|| \, [||f - f_n|| + ||\sigma - \sigma_n||] = 0.$$

Choose ε_n such that $0 < \varepsilon_n < (n||L_{nx}||)^{-1}$, and let

$$\eta_n = ||L_n - L|| + \bar{B}_1 ||L_{nx}|| \left[||f - f_n|| + ||\sigma - \sigma_n|| + \varepsilon_n^{\frac{1}{2}} \right],$$

with $\bar{B}_1$ as in (6.11). Then $\eta_n \to 0$ as $n \to \infty$. In (2.1) let us replace f, σ by f_n, σ_n and add a small $\varepsilon_n^{\frac{1}{2}} dw_1$ term, where $w_1(\cdot)$ is a brownian motion independent of $w(\cdot)$. Consider any reference probability system $\mu = (\Omega, \{\mathcal{F}_s\}, P, w, w_1)$ and $u(\cdot) \in \mathcal{A}_{t\mu}$. Let $x_n(s)$ be the solution to

$$dx_n = f_n(s, x_n(s), u(s))ds + \sigma_n(s, x_n(s), u(s))dw(s) + \varepsilon_n^{\frac{1}{2}} dw_1(s),$$

with $x_n(t) = x(t) = x$. Let

$$\begin{aligned} J_n(t, x; u) &= E_{tx} \int_t^{t_1} \Lambda_n(s, x_n(s))ds, \\ \Lambda_n(s, y) &= L_n(s, y, u(s)), \\ V_{n\mu}(t, x) &= \inf_{\mathcal{A}_{t\mu}} J_n(t, x; u) \\ V_n(t, x) &= \inf_{\mu} V_{n\mu}(t, x). \end{aligned}$$

We apply Lemma 6.2 with $\tilde{f} = f_n, \tilde{L} = L_n, \tilde{\sigma} = (\sigma_n, \varepsilon_n^{\frac{1}{2}} I)$, $I =$ identity matrix, to conclude that

$$\begin{aligned} &|J_n(t, x; u) - J(t, x; u)| \\ &\le E_{tx} \int_t^{t_1} |\Lambda(s, x(s)) - \Lambda_n(s, x_n(s))| ds \le (t_1 - t)\eta_n. \end{aligned} \tag{7.12}$$

Since this is true for arbitrary $u(\cdot)$ and $(t, x) \in Q_0$

$$||V_{n\mu} - V_\mu|| \le (t_1 - t_0)\eta_n. \tag{7.13}$$

By Step 1, for each $n = 1, 2, \cdots$, property (DP) holds and $V_{n\mu} = V_n$. Since each V_n is uniformly continuous on $\overline{Q}_0$ and $\eta_n \to 0$ as $n \to \infty$, we conclude from (7.13) that V_n tends uniformly on $\overline{Q}_0$ to a limit $\tilde{V}$, and $\tilde{V} = V_\mu$ for each μ. Given any reference probability system $\nu = (\Omega, \{\mathcal{F}_s\}, P, w)$ and $u(\cdot) \in \mathcal{A}_{t\nu}$, the product space construction in the proof of Lemma 6.3 provides a brownian motion w_1 independent of w and reference probability system μ such that

$u(\cdot) \in \mathcal{A}_{t\mu}$. Since dw_1 appears with coefficient 0 in (2.1), we have $V_\nu = V_\mu = V$. Since this is true for each $\nu, V = V_\nu = \tilde{V}$.

It remains to verify property (DP). We have

$$|E_{tx}V(\theta, x(\theta) - E_{tx}V_n(\theta, x_n(\theta))|$$
$$\leq \|V - V_n\| + E_{tx}|V(\theta, x(\theta)) - V(\theta, x_n(\theta))|.$$

By (6.11), for each $\alpha > 0$

$$P_{tx}(\|x(\cdot) - x_n(\cdot)\| > \alpha) \leq \alpha^{-2}E_{tx}\|x(\cdot) - x_n(\cdot)\|^2$$
$$\leq \bar{B}_2\alpha^{-2}[\|f - f_n\|^2 + \|\sigma - \sigma_n\|^2 + \varepsilon_n].$$

From this inequality, (7.13b) and uniform continuity of V,

$$\lim_{n\to\infty} \sup_\theta E_{tx}|V(\theta, x(\theta)) - V_n(\theta, x_n(\theta))| = 0.$$

Therefore, from (7.12) and (7.13)

(7.14)
$$\lim_{n\to\infty} \sup_\theta \left| E_{tx}\left\{\int_t^\theta Lds + V(\theta, x(\theta))\right\} - E_{tx}\left\{\int_t^\theta L_n ds + V_n(\theta, x_n(\theta))\right\}\right| = 0,$$

where $L = L(s, x(s), u(s)), L_n = L_n(s, x_n(s), u(s))$. We get (7.1a) from Step 1, (7.13) and passage to the limit as $n \to \infty$. To get (7.1b) we choose μ and $u(\cdot) \in \mathcal{A}_{t\mu}$ such that

$$V_n(t, x) + \frac{\delta}{2} \geq E_{tx}\left\{\int_t^\theta L_n(s, x_n(s), u(s))ds + V_n(\theta, x_n(\theta)\right\}$$

for all $\{\mathcal{F}_s\}$-stopping times θ, where n is chosen large enough that in (7.14)

$$\sup_\theta |E_{tx}\{\cdots\} - E_{tx}\{\cdots\}| < \frac{\delta}{2}.$$

Note that $u(\cdot) \in \mathcal{A}_{t\nu}$, where $\nu = (\Omega, \{\mathcal{F}_s\}, P, w)$ is obtained by omitting the last component w_1 of the 5-tuple μ. Then (7.1b) holds for this choice of ν. Therefore, property (DP) holds.

Step 3. Given f, σ, L satisfying (6.1)(b)(c) and (6.3), let $f_\rho = \alpha_\rho f,\ L_\rho = \alpha_\rho L$ as in Section 6. By Step 2, the corresponding value function V_ρ is continuous and property (DP) holds for the problem determined by f_ρ, σ, L_ρ. By (6.10), V_ρ tends to V uniformly on each compact set as $\rho \to \infty$. Hence V is continuous. Fix ρ_0, and consider $|x| \leq \rho_0$. Let $Q = [t_0, t_1) \times \{|x| < \rho_0\}$, and τ the exit time of $(s, x(s))$ from Q. If $\theta \leq \tau$ and $\rho > \rho_0$, then $x_\rho(\theta) = x(\theta)$ by (6.7). Hence

$$E_{tx}|V_\rho(\theta, x_\rho(\theta)) - V(\theta, x(\theta))| \leq$$

$$\leq \sup_Q |V_\rho - V| + E_{tx}[|V_\rho(\theta, x_\rho(\theta))| + |V(\theta, x(\theta))|; \theta > \tau].$$

Since V and V_ρ satisfy (6.6), by (6.10)

$$E_{tx}|V_\rho(\theta, x_\rho(\theta)) - V(\theta, x(\theta))| \leq C\rho^{-\frac{1}{2}}(1 + \rho_0^{2k+1})^{\frac{1}{2}}$$

$$+M(t_1 - t_0)\{2P_{tx}(\tau < t_1) + E_{tx}[\|x_\rho(\cdot)\|^k + \|x(\cdot)\|^k; \tau < t_1]\}.$$

Here, we have used $\{\tau < \theta\} \subset \{\tau < t_1\}$. Let χ be the indicator function of $\{\tau < t_1\}$. Then, using Cauchy-Schwarz and (6.4), (6.8)

$$E_{tx}[\|x_\rho(\cdot)\|^k; \tau < t_1] = E_{tx}[\|x_\rho(\cdot)\|^k \chi]$$

$$\leq [E_{tx}\|x_\rho(\cdot)\|^{2k} P_{tx}(\tau < t_1)]^{\frac{1}{2}} \leq \frac{K\rho_0^{-\frac{1}{2}}}{2}(1 + |x|^{2k+1})^{\frac{1}{2}}$$

for suitable K. The same estimate holds for $E_{tx}[\|x(\cdot)\|^k; \tau \leq t_1]$. Hence

$$E_{tx}|V_\rho(\theta, x_\rho(\theta)) - V(\theta, x(\theta))| \leq C\rho^{-\frac{1}{2}}(1 + \rho_0^{2k+1})^{\frac{1}{2}} + K\rho_0^{-\frac{1}{2}}(1 + |x|^{2k+1})^{\frac{1}{2}}.$$

Moreover, by (6.9)

$$|E_{tx}\int_t^\theta [L_\rho(s, x_\rho(s), u(s)) - L(s, x(s), u(s))]ds| \leq C\rho^{-\frac{1}{2}}(1 + |x|^{2k+1})^{\frac{1}{2}}.$$

Consequently,

$$\limsup_{\rho\to\infty} \sup_\theta \left| E_{tx}\left\{\int_t^\theta L ds + V(\theta, x(\theta))\right\} - E_{tx}\left\{\int_t^\theta L_\rho ds + V_\rho(\theta, x_\rho(\theta))\right\}\right|,$$

$$\leq K\rho_0^{-\frac{1}{2}}(1 + |x|^{2k+1})^{\frac{1}{2}}.$$

However, given (t, x), the right side can be made arbitrarily small by choosing ρ_0 large enough. As in the last part of Step 2, we conclude that property (DP) holds. Moreover, $V = V_\nu$ since $V_\rho = V_{\rho\nu}$ for each ρ and $V_{\rho\nu} \to V_\nu$ uniformly on compact sets as $\rho \to \infty$. □

In the discussion above we have taken terminal data $\psi \equiv 0$. For nonzero terminal data ψ, the following extension of Theorem 7.1 holds.

Corollary 7.1. *Assume* (6.1), (6.3), *and that* ψ *is bounded and uniformly continuous. Then the conclusions of Theorem* 7.1 *are true.*

Proof. (a) Suppose first that $\psi \in C_b^2(I\!R^n)$. We replace L by $\tilde{L} = L - G^v\psi$ and ψ by terminal data 0 as in the discussion following (6.2).

(b) If ψ is bounded and uniformly continuous, then a standard smoothing technique (Appendix C) gives a sequence $\psi_m \in C_b^2(I\!R^n)$ such that $\|\psi - \psi_m\| \to 0$ as $m \to \infty$. Let $V_{m\nu}, V_m$ be the corresponding value functions. Then

$$\|V_{m\nu} - V_\nu\| \leq \|\psi_m - \psi\|, \ \|V_m - V\| \leq \|\psi_m - \psi\|.$$

We obtain Corollary 7.1 from part (a). □

As a particular case, we may consider nonrandom $\theta = r$. For brevity, let us write

$$\inf_{u(\cdot)} \cdots = \inf_{\nu} \inf_{u(\cdot)\in\mathcal{A}_{t\nu}} \cdots .$$

Thus, (2.10) becomes (recall that $V = V_{PM}$)

$$V(t,x) = \inf_{u(\cdot)} J(t,x;u).$$

Corollary 7.2. *Let $t < r < t_1$. Then*

$$V(t,x) = \inf_{u(\cdot)} E_{tx}\left\{\int_{t_1}^{r} L(s,x(s),u(s))ds + V(r,x(r))\right\}.$$

Corollary 7.2 is a traditional statement of the dynamic programming principle.

Remark 7.1. As in Remark 3.3, let us consider a more general criterion J of the form (3.19), where $\tau = t_1$, since we consider control on the time interval $[t,t_1]$ and $Q = Q_0$. We again define $V = V_{PM}$ by (2.10). On the right side of (7.1a) and (7.1b) we now put

$$E_{tx}\left\{\int_{t}^{\theta} \Gamma(s)L(s,x(s),u(s))ds + \Gamma(\theta)V(\theta,x(\theta))\right\}, \tag{7.15}$$

with $\Gamma(s)$ as in (3.20). Theorem 7.1 again holds. In the proof of Theorem 7.1 and Corollary 7.1, we used the Verification Theorem 3.1 and Theorem 4.2, which guarantees smoothness of the value function under the strong assumptions (4.5). If we add to (4.5b) a corresponding smoothness assumption on ℓ then Theorem 4.2 is still true. See Remark 4.1. Only routine changes are then needed in the proof of Theorem 7.1.

IV.8 Estimates for first order difference quotients

In Section 6, we obtained a bound (6.6) for the value function $V(t,x)$, which depends only on certain constants C_1, C_2, C_3 and k associated with f, σ, and L. In this section we will find bounds for first-order difference quotients of V, and in Section 9 a one sided bound for certain second-order difference quotients. In case that $V \in C^{1,2}(Q_0)$ these estimates provide corresponding estimates for partial derivatives of V. If V does not possess that degree of smoothness, then the estimates for difference quotients are useful in showing that V satisfies the HJB equation in a generalized sense (Section 10).

We continue to assume that (6.1)–(6.3) hold. Let $\xi \in I\!R^n$ be a direction ($|\xi| = 1$). For any function $\Phi(t,x)$ and $h > 0$ we consider the difference quotients

$$\Delta_x \Phi = \frac{1}{h}[\Phi(t, x + h\xi) - \Phi(t, x)]$$

$$\Delta_t \Phi = \frac{1}{h}[\Phi(t + h, x) - \Phi(t, x)]$$

$$\Delta_x^2 \Phi = \frac{1}{h^2}[\Phi(t, x + h\xi) + \Phi(t, x - h\xi) - 2\Phi(t, x)].$$

In order to obtain bounds for $\Delta_x V, \Delta_t V$ we need a stronger assumption than (6.3) about L. Let us assume that L is continuous on $\overline{Q}_0 \times U$, that $L(\cdot, \cdot, v) \in C^1(\overline{Q}_0)$ for each $v \in U$ and that

$$\text{(8.1)} \qquad \begin{aligned} &(a)\ |L(t, x, v)| \leq C_3(1 + |x|^k) \\ &(b)\ |L_t| + |L_x| \leq C_4(1 + |x|^\ell) \end{aligned}$$

for suitable constants $C_3, C_4, k \geq 0, \ell \geq 0$. Let

$$\Delta_x J = \Delta_x J(t, \cdot, u) = \frac{1}{h}[J(t, x + h\xi, u) - J(t, x, u)].$$

Lemma 8.1. *There exists M_1 such that*

$$\text{(8.2)} \qquad |\Delta_x J| \leq M_1(1 + |x|^\ell)$$

for every direction ξ and $0 < h \leq 1$. The constant M_1 depends on C_1, C_2, C_4, ℓ and $t_1 - t_0$.

Proof. Given $(t, x^0) \in Q_0$ let $x(s)$ be the solution of (2.1) with $x(t) = x^0$; and let $x_h(s)$ be the solution of (2.1) with $x_h(t) = x^0 + h\xi$. Also, let $\Delta x(s) = \frac{1}{h}[x_h(s) - x(s)]$; and as in Section 6, let $L(s, y, u(s)) = \Lambda(s, y)$. Then

$$\Delta_x J = E \int_t^{t_1} \frac{1}{h}[\Lambda(s, x_h(s)) - \Lambda(s, x(s))] ds$$

$$= E \int_t^{t_1} \int_0^1 \Lambda_x(s, x^\lambda(s)) \cdot \Delta x(s) d\lambda ds,$$

where $x^\lambda(s) = (1 - \lambda)x(s) + \lambda x_h(s)$. By (8.1b) and $\Lambda_x = L_x$,

$$|\Lambda_x(s, x^\lambda(s))| \leq C_4(1 + |x^\lambda(s)|^\ell)$$

$$\leq C_4(1 + |x(s)|^\ell + |x_h(s)|^\ell).$$

By Cauchy–Schwarz

$$|\Delta_x J| \leq C_4 [E \int_t^{t_1} (1 + |x(s)|^\ell + |x_h(s)|^\ell)^2 ds]^{\frac{1}{2}} [E \int_t^{t_1} |\Delta x(s)|^2 ds]^{\frac{1}{2}}.$$

We bound the first term on the right side using (6.4) with $m = 2\ell$ and $x = x^0, x^0 + h\xi$. By Appendix (D.10), $E|\Delta x(s)|^2 \le B$, where B depends on bounds for $|f_x|$ and $|\sigma_x|$, and hence on the constant C_1 in (6.1)(c). Since $|\xi| = 1, 0 < h \le 1$,

$$1 + |x|^{2\ell} + |x + h\xi|^{2\ell} \le c_\ell(1 + |x|^{2\ell})$$

for suitable c_ℓ. This implies (8.2), for suitable M_1 depending on C_1, C_2, C_4, ℓ and $t_1 - t_0$. □

Since the bound in (8.2) is the same for all $u(\cdot)$, we obtain from Lemma 8.1:

$$(8.3) \qquad |\Delta_x V| \le M_1(1 + |x|^\ell).$$

In order to estimate $\Delta_t V$, let us make a time shift, after which the initial time is 0. Thus, we let

$$\bar{u}(r) = u(t + r),\ \bar{w}(r) = w(t + r),\ 0 \le r \le t_1 - t,$$

and let $\bar{x}(r) = x(t + r)$ be the corresponding solution to

$$(8.4) \qquad d\bar{x} = f(t + r, \bar{x}(r), \bar{u}(r))dr + \sigma(t + r, \bar{x}(r), \bar{u}(r))d\bar{w}(r)$$

with $\bar{x}(0) = x^0$. Moreover,

$$(8.5) \qquad J(t, x^0; u) = E\int_0^{t_1 - t} L(t + r, \bar{x}(r), \bar{u}(r))dr.$$

If we denote the right side by $\bar{J}(t, x^0; \bar{u})$, then

$$(8.6) \qquad V(t, x^0) = \inf \bar{J}(t, x^0; \bar{u}),$$

where the infimum is taken over reference probability systems $\bar{\nu}$ on the time interval $0 \le r \le t_1 - t$ and $\bar{u} \in \mathcal{A}_{0\bar{\nu}}$. If t is replaced by $t + h$, with $0 < h \le t_1 - t$, then

$$(8.7) \qquad V(t + h, x) = \inf \bar{J}(t + h, x^0; \bar{u}), \quad \text{where}$$

$$(8.8) \qquad \bar{J}(t + h, x^0; \bar{u}) = E\int_0^{t_1 - t - h} L(t + h + r, \bar{x}_h(r), \bar{u}(r))dr,$$

$$(8.9) \qquad d\bar{x}_h = f(t + h + r, \bar{x}_h(r), \bar{u}(r))dr + \sigma(t + h + r, \bar{x}_h(r), \bar{u}(r))d\bar{w}(r),$$

with $\bar{x}_h(0) = \bar{x}(0) = x^0$. The inf in (8.7) is taken over the same class of $\bar{\nu}, \bar{u} \in \mathcal{A}_{0\bar{\nu}}$ as in (8.6). The values of $\bar{u}(r)$ for $t_1 - t - h \le r \le t_1 - t$ do not affect $\bar{J}$, but it is easy to verify that this makes no difference. Let $\Delta_t \bar{J} = h^{-1}[\bar{J}(t + h, x^0; \bar{u}) - \bar{J}(t, x^0; \bar{u})]$.

Lemma 8.2. *There exists M_2 such that*

$$|\Delta_t \bar{J}| \leq M_2(1 + |x|^k + |x|^\ell) \tag{8.10}$$

for $0 < h \leq t_1 - t$. *The constant M_2 depends on $C_i, i = 1, \cdots, 4, k, \ell$ and $t_1 - t_0$ where k, ℓ are as in* (8.1).

Proof. Let

$$\tilde{f}(s, y, v) = f(s + h, y, v),\ \tilde{\sigma}(s, y, v) = \sigma(s + h, y, v).$$

Then

$$\|\tilde{f} - f\| \leq \|f_t\| h \leq C_1 h, \quad \|\tilde{\sigma} - \sigma\| \leq \|\sigma_t\| h \leq C_1 h$$

with C_1 as in (6.1c). Fix an initial time t and initial state $x = x^0$. By (6.11)

$$E \sup_{0 \leq r \leq r_1} |\bar{x}(r) - \bar{x}_h(r)|^m \leq 2\bar{B}_m C_1^m h^m,\ m = 1, 2, \cdots, \tag{8.11}$$

where $r_1 = t_1 - t$. Let

$$\bar{\Lambda}(r, y) = L(t + r, y, \bar{u}(r)), \quad \bar{\Lambda}_h(r, y) = L(t + h + r, y, \bar{u}(r)).$$

By (8.1b) and the mean value theorem

$$\begin{aligned} &(a)\ |\bar{\Lambda}_h(r, \bar{x}_h(r)) - \bar{\Lambda}(r, \bar{x}_h(r))| \leq C_4 h(1 + |\bar{x}_h(r)|^\ell, \\ &(b)\ |\bar{\Lambda}(r, \bar{x}_h(r)) - \ \bar{\Lambda}(r, \bar{x}(r))| \leq C_4 |\Delta \bar{x}(r)| h(1 + |\bar{x}(r)|^\ell + |\bar{x}_h(r)|^\ell) \end{aligned} \tag{8.12}$$

where $\Delta \bar{x} = h^{-1}(\bar{x}_h - \bar{x})$. We have

$$\Delta_t \bar{J} = \frac{1}{h} E \int_0^{r_1 - h} [\bar{\Lambda}_h(r, \bar{x}_h(r)) - \bar{\Lambda}(r, \bar{x}_h(r))] dr$$

$$+ \frac{1}{h} E \int_0^{r_1 - h} [\bar{\Lambda}(r, \bar{x}_h(r)) - \bar{\Lambda}(r, \bar{x}(r))] dr$$

$$- \frac{1}{h} E \int_{r_1 - h}^{r_1} \bar{\Lambda}(r, \bar{x}(r)) dr.$$

If (i), (ii), (iii) denote the terms on the right side, then by (8.12a) and (6.4) with $x(\cdot)$ replaced by $\bar{x}_h(\cdot)$,

$$|(\mathrm{i})| \leq K_1(1 + |x|^\ell),$$

for suitable K_1. The same proof as for Lemma 8.1 gives

$$|(\mathrm{ii})| \leq K_2(1 + |x|^\ell).$$

By (8.1a) and (6.4)

$$|(\text{iii})| \le K_3(1+|x|^k).$$

By combining these estimates, we get (8.10). □

From Lemma 8.2, (8.6) and (8.7) we get

$$|\Delta_t V| \le M_2(1+|x|^k+|x|^\ell) \tag{8.13}$$

As an immediate consequence of Lemmas 8.1 and 8.2 we have:

Theorem 8.1. *Assume* (6.1), (6.2) *and* (8.1)*. If V is differentiable at $(t,x) \in Q_0$, then*

$$\begin{aligned} &(a)\ |D_x V(t,x)| \ \le M_1(1+|x|^\ell) \\ &(b)\ |V_t(t,x)| \ \le M_2(1+|x|^k+|x|^\ell). \end{aligned} \tag{8.14}$$

Proof. Use (8.2) and let $h \to 0^+$. Then

$$|V_\xi(t,x)| \le M_1(1+|x|^\ell)$$

where V_ξ denotes the derivative of $V(t,\cdot)$ in direction ξ. Since ξ is arbitrary, this gives (8.14 a). Similarly, (8.14b) follows from (8.10). □

IV.9 Estimates for second-order difference quotients

We will now obtain a bound for second difference quotients $\Delta_x^2 J = \Delta_x^2 J(t,\cdot,u)$. This will imply a one-sided bound (9.7) for $\Delta_x^2 V$. In addition to the previous assumptions let us now assume that $f(\cdot,\cdot,v), \sigma(\cdot,\cdot,v), L(\cdot,\cdot,v)$ are of class $C^{1,2}(\overline{Q}_0)$ for each $v \in U$. Moreover, their second order partial derivatives in x satisfy

$$\begin{aligned} &(a)\ |f_{xx}| + |\sigma_{xx}| \le C_5 \\ &(b)\ |L_{xx}| \le C_5(1+|x|^m) \end{aligned} \tag{9.1}$$

for suitable C_5 and $m \ge 0$.

Lemma 9.1. *There exists M_3 such that*

$$|\Delta_x^2 J| \le M_3(1+|x|^\ell+|x|^m) \tag{9.2}$$

for every direction ξ and $0 < h \le 1$. The constant M_3 depends on $C_1, \cdots, C_5, \ell, m$ and $t_1 - t_0$.

Proof. Given $(t,x^0) \in Q_0$ and $u(\cdot)$, we again let $x(s)$ denote the solution to (2.1) with $x(t) = x^0$. Let $x_h^+(s)$ denote the solution to (2.1) with $x_h^+(t) = x^0 + h\xi$, and $x_h^-(s)$ the solution to (2.1) with $x_h^-(t) = x^0 - h\xi$. Let

$$\Delta^+ x(s) = \frac{1}{h}[x_h^+(s) - x(s)],\ \Delta^- x(s) = \frac{1}{h}[x_h^-(s) - x(s)],$$

$$\Delta_2 x(s) = \frac{1}{h^2}[x_h^+(s) + x_h^-(s) - 2x(s)].$$

Similarly, if $g \in C^2(I\!R^n)$ and $x, x^+, x^- \in I\!R^n$, let

$$\Delta^+ g = \frac{1}{h}[g(x^+) - g(x)],\ \ \Delta^- g = \frac{1}{h}[g(x^-) - g(x)],$$

$$\Delta_2 g = \frac{1}{h^2}[g(x^+) + g(x^-) - 2g(x)] = \frac{1}{h}[\Delta^+ g + \Delta^- g].$$

By Taylor's formula with remainder, applied to $g(x^+)-g(x)$ and $g(x^-)-g(x)$,

$$\text{(9.3)}\quad \Delta_2 g(x) = g_x(x)(\Delta_2 x) + \frac{1}{2}(\Delta^+ x)\cdot g^+_{xx}(\Delta^+ x) + \frac{1}{2}(\Delta^- x)\cdot g^-_{xx}(\Delta^- x),$$

where

$$g^{\pm}_{xx} = 2\int_0^1\int_0^1 g_{xx}(x + \lambda\mu\Delta^{\pm}x)\lambda d\lambda d\mu.$$

Since $x(s), x_h^+(s), x_h^-(s)$ all satisfy (2.1) and

$$\Delta_2 x(t) = h^{-2}[x^0 + h\xi + x^0 - h\xi - 2x^0] = 0,$$

$\Delta_2 x(s)$ satisfies the stochastic integral equation

$$\Delta_2 x(s) = \int_t^s \Delta_2 f dr + \int_t^s \Delta_2 \sigma dw(r).$$

We apply (9.3) with $g = f_i, \sigma_{ij}, i = 1, \cdots, n, j = 1, \cdots, d$ to get

$$\text{(9.4)}\qquad \Delta_2 x(s) = \int_t^s f_x \Delta_2 x(r) dr + \int_t^s \sigma_x \Delta_2 x(r) dw(r) + h_1(s) + h_2(s),$$

where f_x, σ_x are evaluated at $(r, x(r), u(r))$ and

$$h_1(s) = \frac{1}{2}\int_t^s [(\Delta^+ x(r))\cdot f^+_{xx}(\Delta^+ x(r)) + (\Delta^- x(r))\cdot f^-_{xx}(\Delta^- x(r))]dr$$

$$h_2(s) = \frac{1}{2}\int_t^s [(\Delta^+ x(r))\cdot \sigma^+_{xx}(\Delta^+ x(r)) + (\Delta^- x(r))\cdot \sigma^-_{xx}(\Delta^- x(r))]dw(r),$$

$$f^{\pm}_{xx} = 2\int_0^1\int_0^1 f_{xx}(r, x(r) + \lambda\mu\Delta^{\pm}x(r), u(r))\lambda d\lambda d\mu,$$

$$\sigma^{\pm}_{xx} = 2\int_0^1\int_0^1 \sigma_{xx}(r, x(r) + \lambda\mu\Delta^{\pm}x(r), u(r))\lambda d\lambda d\mu.$$

By a standard estimate for stochastic differential equations, there exist constants Γ_j depending on the constant C_1 in (6.1c) and $t_1 - t_0$, such that

$$\text{(9.5)}\qquad E|\Delta^{\pm}x(s)|^{2j} \le \Gamma_j, j = 1, 2, \cdots .$$

By (9.1a), f_{xx} and σ_{xx} are bounded. Hence, by (9.5) with $j = 1$,

$$E[|h_1(s)|^2 + |h_2(s)|^2] \leq K$$

for suitable K. Since $|f_x| \leq C_1, |\sigma_x| \leq C_1$, (9.4) and Gronwall's inequality applied to

$$m(s) = \sup_{t \leq r \leq s} E|\Delta_2 x(r)|^2$$

imply that

$$E|\Delta_2 x(s)|^2 \leq K_1 \quad , \quad t \leq s \leq t_1 \tag{9.6}$$

for suitable K_1.

We again let $\Lambda(s, y) = L(s, y, u(s))$. Then

$$\Delta_2 J = E \int_t^{t_1} \Delta_2 \Lambda(s, x(s)) ds.$$

From (9.3)

$$\Delta_2 \Lambda(s, x(s)) = \Lambda_x(s, x(s)) \cdot \Delta_2 x(s) + \frac{1}{2}(\Delta^+ x(s)) \cdot \Lambda^+_{xx}(\Delta^+ x(s)) + \frac{1}{2}(\Delta^- x(s)) \cdot \Lambda^-_{xx}(\Delta^- x(s)).$$

By Cauchy - Schwarz

$$|\Delta_2 J| \leq (E \int_t^{t_1} |\Lambda|_x^2 ds)^{\frac{1}{2}} (E \int_t^{t_1} |\Delta_2 x(s)|^2 ds)^{\frac{1}{2}}$$

$$+ \frac{1}{2}(E \int_t^{t_1} |\Lambda^+_{xx}|^2 ds)^{\frac{1}{2}} (E \int_t^{t_1} |\Delta^+ x(s)^4| ds)^{\frac{1}{2}}.$$

$$+ \frac{1}{2}(E \int_t^{t_1} |\Lambda^-_{xx}|^2 ds)^{\frac{1}{2}} (E \int_t^{t_1} |\Delta^- x(s)|^4 ds)^{\frac{1}{2}}.$$

As in the proof of Lemma 8.1, assumption (8.1b) implies that

$$E \int_t^{t_1} |\Lambda_x(s, x(s))|^2 ds \leq D_1 (1 + |x^0|^\ell)^2$$

for suitable D_1. Since

$$\Lambda^+_{xx}(s, x(s)) = \int_0^1 \int_0^1 L_{xx}(s, x(s) + \lambda\mu\Delta^+ x(s)) \lambda d\lambda d\mu,$$

(9.1b) implies that

$$|\Lambda^+_{xx}(s, x(s))| \leq C_5 (1 + |x(s)|^m + |x^+(s)|^m).$$

From (6.4) we then get

$$E\int_t^{t_1} |\Lambda_{xx}^+|^2 ds \le D_2(1+|x^0|^m)^2.$$

The same inequality holds with Λ_{xx}^- instead of Λ_{xx}^+. From these inequalities together with (9.5) with $j = 2$ and (9.6), we get (9.2). A careful inspection of the proof shows that M_3 can be chosen to depend only on $C_1, C_2, \cdots$ as stated in Lemma 9.1. □

From Lemma 9.1, let us obtain a one sided bound for $\Delta_x^2 V$:

$$\Delta_x^2 V(t,x) \le M_3(1+|x|^\ell + |x|^m). \tag{9.7}$$

To obtain (9.7), given $\delta > 0$ choose $u(\cdot)$ such that

$$J(t,x;u) < V(t,x) + \frac{\delta h^2}{2}.$$

Then

$$\begin{aligned} &V(t,x+h\xi) + V(t,x-h\xi) - 2V(t,x) \\ &\qquad \le J(t,x+h\xi;u) + J(t,x-h\xi;u) - 2J(t,x;u) + \delta h^2 \\ &\qquad = h^2[\Delta_x^2 J + \delta], \\ &\Delta_x^2 V \le \Delta_x^2 J + \delta \le M_3(1+|x|^\ell + |x|^m) + \delta. \end{aligned}$$

Since δ is arbitrary, we get (9.7). Let

$$D_{\xi\xi}^2 V(t,x) = \frac{d^2}{dh^2} V(t,x+h\xi)|_{\xi=0} \tag{9.8}$$

if this second derivative exists. In particular, if the second order partial derivatives $V_{x_i x_j}$ are continuous, then

$$D_{\xi\xi}^2 V(t,x) = \lim_{h\to 0^+} \Delta_x^2 V(t,x).$$

By (9.7), for every direction ξ

$$D_{\xi\xi}^2 V(t,x) \le M_3(1+|x|^\ell + |x^m|). \tag{9.9}$$

In case V is a smooth solution to the HJB partial differential equation (3.3), the estimates (8.14) and (9.9) provide a bound for $A^v V$. An essential feature of this bound is that it does not depend on a positive lower bound c for the eigenvalues of $a(t,x,v)$ as in (3.5).

Theorem 9.2. *Assume* (6.1), (6.2), (8.1), (9.1). *Let $Q \subset Q_0$ be an open set such that $V \in C^{1,2}(Q)$ and V satisfies* (3.3) *in Q. Then there exists M_4 such that, for all $(t,x) \in Q$ and $v \in U$,*

$$|A^v V(t,x)| \le M_4(1+|x|^p), \tag{9.10}$$

where $p = \max(k, \ell{+}1, m)$. *The constant* M_4 *depends on* $C_i, i = 1, \cdots, 5, k, \ell, m$ *and* $t_1 - t_0$.

Proof. We recall that

$$A^v V = V_t + \frac{1}{2}\operatorname{tr} a D_x^2 V + f \cdot D_x V.$$

By (6.1c) and (8.14)

$$\begin{aligned} |V_t| &\le M_2(1+|x|^k+|x|^\ell), \\ |f||D_x V| &\le M_1 \bar{C}(1+|x|^\ell)(1+|x|), \end{aligned} \tag{9.11}$$

where $\bar{C} = \max(C_1, C_2)$. Given t, x, v choose an orthonormal basis $\xi^1, \cdots, \xi^n$ for $I\!R^n$, such that

$$\operatorname{tr} a(t,x,v) D_x^2 V(t,x) = \sum_{i=1}^n \lambda_i(t,x,v) D_{\xi_i \xi_i} V(t,x),$$

where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of the symmetric, nonnegative definite matrix a. Since $a = \sigma\sigma'$, and $|\sigma| \le C_2$ by (6.1c), (9.9) implies for suitable K,

$$\operatorname{tr} a D_x^2 V \le K(1+|x|^\ell+|x|^m). \tag{9.12}$$

Inequalities (9.11), (9.12) give an upper bound for $A^v V$. On the other hand, since V satisfies the HJB equation (3.3) in Q, $A^v V + L(t,x,v) \ge 0$. Hence, by (8.1a) we have the lower bound

$$A^v V \ge -L \ge -C_3(1+|x|^k).$$

These upper and lower bounds give (9.10). □

Remark 9.1. Inequality (9.7) implies that $V(t,\cdot)$ is a semiconcave function on $I\!R^n$ (see Remark II.8.1 for the definition.)

IV.10 Generalized subsolutions and solutions

Let us now use the estimates for difference quotients in Sections 8 and 9 to describe V as a generalized solution to the HJB equation. This will be done in the semilinear case, when $\sigma(t,x)$ does not depend on the control. However, V is shown to be a generalized subsolution without this restriction.

We make the same assumptions about f, σ, L as in Theorem 9.1. These functions are continuous on $\overline{Q}_0 \times U$, are of class $C^{1,2}(\overline{Q}_0)$ as functions of (t,x), and satisfy (6.1) (8.1) (9.1). Let us introduce the following notations. For $1 \le p \le \infty$, let $L^p_{\ell oc}(Q_0)$ denote the space of Lebesgue measurable functions Ψ such

$\chi_B \Psi \in L^p(Q_0)$ for every compact $B \subset Q_0$, where χ_B is the indicator function of B. Let $C_0^\infty(Q_0)$ denote the space of Φ such that all partial derivatives of Φ of every order are continuous on Q_0, and Φ has compact support in int Q_0 (i.e. $\Phi(t,x) = 0$ for $(t,x) \notin B$, where $B \subset$ int Q_0 is compact).

Generalized partial derivatives of V are defined as follows [Zi]. Suppose that there exists $\Psi_i \in L^1_{\ell oc}(Q_0)$ such that

$$\int_{Q_0} \Psi_i \Phi dx dt = -\int_{Q_0} V \Phi_{x_i} dx dt, \text{ for all } \Phi \in C_0^\infty(Q_0) \tag{10.1}$$

Then Ψ_i is called a *generalized first order partial derivative* of V with respect to x_i. If such Ψ_i exists, it is unique up to Lebesgue $(n+1)$-measure 0. We write $\Psi_i = V_{x_i}$. Similarly, the generalized partial derivatives $V_t, V_{x_i x_j}$ (if they exist) are defined as those functions in $L^1_{\ell oc}(Q_0)$ such that, for all $\Phi \in C_0^\infty(Q_0)$,

$$\int_{Q_0} V_t \Phi dx dt = -\int_{Q_0} V \Phi_t dx dt, \tag{10.2}$$

$$\int_{Q_0} V_{x_i x_j} \Phi dx dt = \int_{Q_0} V \Phi_{x_i x_j} dx dt. \tag{10.3}$$

Remark 10.1. If V is locally Lipschitz in Q_0, then by Rademacher's theorem, V is differentiable almost everywhere in Q_0. Integrations by parts show that the usual partial derivatives V_t, V_{x_i}, which exist at each point of differentiability, are also generalized first order partial derivatives of V.

We define $A^v V$ in the generalized sense, as follows. Consider the formal adjoint of the linear operator A^v:

$$(A^v)^* \Phi = -\Phi_t + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij}(t,x,v)\Phi) - \sum_{i=1}^n \frac{\partial}{\partial x_i} (f_i(t,x,v)\Phi). \tag{10.4}$$

If there is a function $\Psi^v \in L^1_{\ell oc}(Q_0)$ such that

$$\int_{Q_0} \Psi^v \Phi dx dt = \int_{Q_0} V (A^v)^* \Phi dx dt. \tag{10.5}$$

for all $\Phi \in C_0^\infty(Q_0)$, then we set $A^v V = \Psi^v$. The function Ψ^v is unique up to Lebesgue $(n+1)$-measure 0.

Remark 10.2. An equivalent interpretation of generalized partial derivatives is in terms of the Schwartz theory of distributions [Zi]. Any function in $L^1_{\ell oc}(Q_0)$ has (by definition) partial derivatives of all orders in the Schwartz distribution sense. If a distribution theory partial derivative of V can be identified with a function in $L^1_{\ell oc}(Q_0)$, then this function is the corresponding generalized partial derivative of V. Similarly (10.5) is equivalent to the statement that $A^v V$, regarded as a Schwartz distribution, can be identified with the function $\Psi^v \in L^1_{\ell oc}(Q_0)$. Note that in defining $A^v V$, we did not require that the individual generalized second order partial derivatives $V_{x_i x_j}$ in (10.3) exist.

We say that a sequence Ψ_n converges to Ψ *weakly** in $L^\infty_{\ell oc}(Q_0)$ if

$$\lim_{n\to\infty} \int_{Q_0} \Psi_n \Phi dx dt = \int_{Q_0} \Psi \Phi dx dt \tag{10.6}$$

for every $\Phi \in L^1(Q_0)$ with support in some compact set $B \subset Q_0$.

Lemma 10.1. *Let $\Psi_n \in L^1_{\ell oc}(Q_0)$ be such that, for every compact set $B \subset Q_0, |\Psi_n(t,x)| \le K_B$ for all $(t,x) \in B$ and $n = 1, 2, \cdots$.*
Then:

(a) There is a subsequence Ψ_{n_k} which tends to a limit Ψ weakly* in $L^\infty_{\ell oc}(Q_0)$;
(b) If (10.6) *holds for all $\Phi \in C^\infty_0(Q_0)$, then Ψ_n tends to Ψ weakly* in $L^\infty_{\ell oc}(Q_0)$.*
(c) If $\Psi_n \ge 0$, then $\Psi \ge 0$.

Lemma 10.1 follows from [Ro, Thm 17, p. 202].

Here K_B denotes a suitable constant. In (c) the inequalities are understood to hold for almost all (t,x).

Lemma 10.2. *Let $V_n \in C^1(Q_0)$ for $n = 1, 2, \cdots$. Assume that, for every compact set $B \subset$ int Q_0, $V_n \to V$ uniformly on B as $n \to \infty$ and that $|V_{nt}(t,x)| + |D_x V_n(t,x)| \le K_B$ for all $(t,x) \in B$. Then the generalized partial derivatives V_t, V_{x_i} exist for $i = 1, \cdots, n$ and are the respective limits of V_{nt}, V_{nx_i} weakly* in $L^\infty_{\ell oc}(Q_0)$.*

Proof. This is an easy consequence of Lemma 10.1, together with (10.1), (10.2). □

Let us next consider $f_n, \sigma_n, n = 1, 2, \cdots$, and also $a_n = \sigma_n \sigma_n'$. Let $(A^v_n)^*$ denote the formal adjoint as in (10.4). We require that, for every $\Phi \in C^\infty_0(Q_0)$,

$$\lim_{n\to\infty} (A^v_n)^* \Phi = (A^v)^* \Phi, \text{ uniformly on every compact set } B \subset Q_0. \tag{10.7}$$

In particular, (10.7) holds provided f_n, σ_n, their first order partial derivatives in t, x and $f_{nx_i x_j}$, $\sigma_{nx_i x_j}$ are all continuous and converge uniformly on compact sets to f, σ and the corresponding partial derivatives.

Lemma 10.3. *Assume that:*

(a) f_n, σ_n are such that (10.7) *holds;*
(b) $V_n \in C^{1,2}(Q_0)$;
(c) For every compact set $B \subset Q_0$, $V_n \to V$ uniformly on B and $|A^v_n V_n(t,x)| \le K_B$ for all $(t,x) \in B$, $v \in U$.

Then $A^v V$ exists in the generalized sense, and is the weak* limit in $L^\infty_{\ell oc}(Q_0)$ of $A^v_n V_n$.

Proof. For any $\Phi \in C^\infty_0(Q_0)$

$$\int_{Q_0} (A^v_n V_n) \Phi dx dt = \int_{Q_0} V_n (A^v_n)^* \Phi dx dt,$$

$$\lim_{n\to\infty}\int_{Q_0} V_n(A_n^v)^*\Phi dxdt = \int_{Q_0} V(A^v)^*\Phi dxdt.$$

Since $A_n^v V_n$ is uniformly bounded on every compact set $B \subset Q_0$, Lemma 10.1 implies that $A_n^v V_n$ tends weakly in $L^\infty_{\ell oc}(Q_0)$ to a limit Ψ^v. Since Ψ^v satisfies (10.5), $\Psi^v = A^v V$, interpreted in the generalized sense. □

Definition 10.1. We call W a *generalized subsolution* of the HJB equation (3.3) if, for every $v \in U, A^v W$ exists in the generalized sense and

$$A^v W(t,x) + L(t,x,v) \geq 0 \tag{10.8}$$

for Lebesgue almost all $(t,x) \in Q_0$.

Theorem 10.1. *Let V be the value function. Then $A^v V$ exists in the generalized sense. Moreover, V is a generalized subsolution of the HJB equation.*

Proof. The same kind of approximations used in the proof of Theorem 6.1 provide f_n, σ_n, L_n with the following properties: (a) f_n, σ_n, L_n converge uniformly on compact sets to f, σ, L as $n \to \infty$; (b) the partial derivatives of f_n, σ_n with respect to $t, x_i, x_i x_j,\ i,j = 1,\cdots,n$ are continuous on $\overline{Q}_0 \times U$ and converge uniformly on compact sets to the corresponding partial derivatives of f, σ; (c) f_n, σ_n, L_n satisfy (6.1), (8.1), (9.1) with the same $C_1, \cdots, C_5, k, \ell, m$; (d) the HJB equation

$$0 = \min_{v\in U}[A_n^v V_n + L_n] \tag{10.9}$$

has a solution $V_n \in C_b^{1,2}(\overline{Q}_0)$ with $V_n(t_1,x) = 0$. Moreover, $V_n \to V$ as $n \to \infty$ uniformly on compact sets. By (9.10) $A_n^v V_n(t,x)$ is uniformly bounded on each compact set $B \subset \text{int } Q_0$. We conclude from Lemma 10.3 that $A_n^v V_n$ tends to $A^v V$ weakly* in $L^\infty_{\ell oc}(Q_0)$. Since $L_n(t,x,v)$ tends to $L(t,x,v)$ uniformly on compact sets, $A_n^v V_n + L_n$ tends to $A^v V + L$ weakly* in $L^1_{\ell oc}(Q_0)$, for each $v \in U$. Since $A_n^v V_n + L_n \geq 0$, Lemma 10.1(c) implies that $A^v V + L \geq 0$ for Lebesgue almost all $(t,x) \in Q_0$. □

Remark 10.3. It can be shown that the value function V is the maximal generalized subsolution. See [L1][L9] for results of this type. Another approach using convex duality ideas leads to a characterization of the value function as the supremum of smooth subsolutions. See [VL] [FV].

Generalized solutions to HJB. The HJB partial differential equation is

$$0 = \min_{v\in U}[A^v V(t,x) + L(t,x,v)]. \tag{10.10}$$

Since the HJB equation is nonlinear and convergence of $A_n^v V_n + L_n$ to $A^v V + L$ is only weakly* in $L^\infty_{\ell oc}(Q_0)$, we cannot pass directly to the limit from (10.9) to conclude that the value function V satisfies (10.10) in any sense. However let us now show how this can be done in the semilinear case, with $\sigma = \sigma(t,x)$. In this case

$$A^v \Phi(t,x) = A_0\Phi(t,x) + f(t,x,v)\cdot D_x\Phi(t,x),$$

$$A_0\Phi = \Phi_t + \frac{1}{2}\mathrm{tr}\, aD_x^2\Phi, \tag{10.11}$$

$$a(t,x) = \sigma(t,x)\sigma(t,x)'.$$

Equation (10.10) can then be rewritten as

$$0 = -A_0 V(t,x) + H(t,x,D_x V), \tag{10.12}$$

where as in I(5.4)

$$H(t,x,p) = \max_{v\in U}[-f(t,x,v)\cdot p - L(t,x,v)]. \tag{10.13}$$

We will accomplish the passage to the limit from (10.9) to (10.12) by first showing that $D_x V_n$ tends to $D_x V$ Lebesgue almost everywhere in Q_0 as $n \to \infty$.

Lemma 10.4. *If $V(t,\cdot)$ is differentiable at x, then $D_x V_n(t,x) \to D_x V(t,x)$ as $n \to \infty$.*

Proof. Consider any direction $\xi \in I\!R^n$ ($|\xi| = 1$). Then Taylor's formula with remainder gives, since $V_n \in C^{1,2}(Q_0)$,

$$V_n(t,x+h\xi) = V_n(t,x) + hD_x V_n(t,x)\cdot\xi + \frac{h^2}{2}D_{\xi\xi}V_n(t,x+\lambda h\xi).$$

where $0 < \lambda < 1$. Since V_n satisfies (9.9) for some M_3 not depending on n, for $0 < h < 1$ we have

$$\frac{1}{h}[V_n(t,x+h\xi) - V_n(t,x)] - D_x V_n(t,x)\cdot\xi \le Ch \tag{10.14}$$

for suitable C (which may depend on t,x). Since V_n satisfies (8.3) with M_1 not depending on $n, D_x V_n(t,x)$ is a bounded sequence in $I\!R^n$. If $D_x V_n(t,x)$ tends to p as $n \to \infty$ through some subsequence, then by (10.14)

$$\frac{1}{h}[V(t,x+h\xi) - V(t,x)] - p\cdot\xi \le Ch \tag{10.15}$$

for every direction ξ. In particular, (10.15) holds with ξ replaced by $-\xi$. Since $V(t,\cdot)$ is differentiable at x,

$$D_x V(t,x)\cdot\xi = \lim_{h\to 0}\frac{1}{h}[V(t,x+h\xi) - V(t,x)]$$

for every ξ. Therefore, $p = D_x V(t,x)$. Since the limit p is the same for every convergent subsequence, $D_x V_n(t,x) \to D_x V(t,x)$ as $n \to \infty$. □

Lemma 10.5. *$A_{0n}V_n$ tends to $A_0 V$ weakly* in $L^\infty_{\ell oc}(Q_0)$, where $A_0 V$ exists in the generalized sense.*

Proof. $A_{0n}V_n(t,x)$ is uniformly bounded on each compact set $B \subset Q_0$ since $A_{0n} = A_n^v - f_n\cdot D_x$ and $A_n^v V_n, f_n\cdot D_x V_n$ are uniformly bounded on compact sets. The proof is then the same as for Lemma 10.3. □

Definition. Let W be locally Lipschitz on Q_0. We call W a *generalized solution* of the HJB equation if $A_0 W$ exists in the generalized sense and (10.12) holds for Lebesgue almost all $(t,x) \in Q_0$.

In the deterministic case, $\sigma(t,x) \equiv 0$ and $A_0 \Phi = \Phi_t$. Hence, this definition of generalized solution agrees with the one in Section I.6.

Theorem 10.2. *In the semilinear case* $(\sigma = \sigma(t,x))$ *the value function* V *is a generalized solution the* HJB *equation* (10.12).

Proof. We use the same approximations f_n, σ_n, L_n to f, σ, L as in the proof of Theorem 10.1. In the semilinear case, (10.9) becomes

$$0 = -A_{0n} V_n + H_n(t,x,D_x V_n), \text{ where} \tag{10.16}$$

$$A_{0n}\Phi = \Phi_t + \frac{1}{2}\operatorname{tr} a_n D_x^2 \Phi,$$

$$H_n(t,x,p) = \max_{v \in U}[-f_n(t,x,v) \cdot p - L_n(t,x,v)]$$

and $a_n = \sigma_n \sigma_n'$. Since $f_n \to f, L_n \to L$ uniformly on compact sets, we have

$$H(t,x,p) = \lim_{n \to \infty} H_n(t,x,p)$$

uniformly on compact subsets of $\overline{Q}_0 \times I\!R^n$. By Lemma 10.4 we then have

$$H(t,x,D_x V(t,x)) = \lim_{n \to \infty} H_n(t,x,D_x V_n(t,x)), \tag{10.17}$$

for each (t,x) at which $V(t,\cdot)$ is differentiable. However, (8.3) implies that $V(t,\cdot)$ satisfies a Lipschitz condition on each compact subset of $I\!R^n$. By Rademacher's theorem, $V(t,\cdot)$ is differentiable at Lebesgue almost every $x \in I\!R^n$. Therefore, (10.17) holds for Lebesgue almost every $(t,x) \in Q_0$. Moreover $H_n(t,x,D_x V_n(t,x))$ is uniformly bounded on compact sets. For every $\Phi \in C_0^\infty(Q_0)$

$$\int_{Q_0} [-A_0 V + H(t,x,D_x V)]\Phi dx dt$$

$$= \lim_{n \to \infty} \int_{Q_0} [-A_{0n} V_n + H_n(t,x,D_x V_n)]\Phi dx dt = 0.$$

This implies (10.12) for Lebesgue almost all (t,x). □

Remark 10.4. The upper bound (9.7) for $\Delta_x^2 V$ was used together with the inequality $A^v V + L \geq 0$ to obtain the bound (9.10) for $|A^v V|$ in case V is smooth. Then it was shown, by approximating V by smooth value functions V_n, that $A^v V$ exists in the generalized sense. It was not claimed that the generalized second derivatives $V_{x_i x_j}$ exist; and in fact that need not be true. For instance, if $n = 1$, $\sigma \equiv 0$ (the deterministic case) simple examples show that $V_x(t,\cdot)$ may have jump discontinuities. When that occurs, there is no generalized second derivative V_{xx} in the sense of (10.3). By (9.7) $V(t,\cdot)$ is semiconcave. When $n = 1$ the Schwartz distribution theory second derivative of $V(t,\cdot)$ can be identified with a measure on $I\!R^1$. By a theorem of Alexandrov, $V(t,\cdot)$ is twice differentiable at almost all points of $I\!R^1$. See [CIL1, Theorem A2]. This pointwise second derivative can be identified with the generalized second derivative V_{xx} if and only if $V_x(t,\cdot)$ is absolutely continuous on each finite subinterval of $I\!R^1$. Similarly, for $n > 1$ the semiconcavity of $V(t,\cdot)$ implies that the Schwartz distribution second derivatives $V_{x_i x_j}$ are measures. Viscosity solution methods (see [L3]) and Alexandrov's theorem can then be used to show that the HJB equation is satisfied almost everywhere. Similar results were obtained by Lions [L9] for an infinite-horizon discounted problem.

Problems with affine - convex structure. Let us now mention a special class of problems for which $V(t,\cdot)$ is a convex function on $I\!R^n$. Convexity of $V(t,\cdot)$ is equivalent to $\Delta^2 V \geq 0$. One has then a bound for $|\Delta_x^2 V|$, which will imply the existence of generalized second derivatives $V_{x_i x_j}$ which are bounded on every compact set.

Let us now assume that:

(a) U is convex and compact, $U \subset I\!R^m$.

(10.18) (b) $f(t,x,v) = A_1(t)x + A_2(t)v + A_3(t),\ \sigma(t,v) = \gamma_1(t)v + \gamma_2(t)$, where$A_1, A_2, A_3, \gamma_1, \gamma_2$ are of class $C^1([t_0,t_1])$.

(c) L is continuous on $\overline{Q}_0 \times U, L(t,\cdot,\cdot)$ is convex on $I\!R^n \times U$, and (6.3) holds.

Models with this special structure often arise in such applications as finance and inventory theory. An instance is the Merton portfolio problem (Example 10.1 below.)

Let us fix a reference probability system ν. By Theorem 7.1, $V = V_\nu$. Since U is convex, the set $\mathcal{A}_{t\nu}$ of admissible progressively measurable $u(\cdot)$ is also convex.

Lemma 10.5. *$J(t,\cdot,\cdot)$ is convex on $I\!R^n \times \mathcal{A}_{t\nu}$.*

Proof. Let $x^0, x^1 \in I\!R^n$ and $u_0(\cdot), u_1(\cdot) \in \mathcal{A}_{t\nu}$. For $0 \leq \lambda \leq 1$, let

$$x^\lambda = (1-\lambda)x^0 + \lambda x^1,\ u_\lambda(\cdot) = (1-\lambda)u_0(\cdot) + \lambda u_1(\cdot).$$

Let $x_\lambda(s)$ be the solution to (2.1) for $t \leq s \leq t_1$, with $x_\lambda(t) = x^\lambda$. By (10.18)(b),

$$x_\lambda(s) = \lambda x_0(s) + (1-\lambda)x_1(s).$$

By convexity of $L(s,\cdot,\cdot)$

$$L(s, x_\lambda(s), u_\lambda(s)) \leq (1-\lambda)L(s, x_0(s), u_0(s)) + \lambda L(s, x_1(s), u_1(s)),$$

which implies

$$J(t, x^\lambda, u_\lambda) \leq (1-\lambda)J(t, x^0, u_0) + \lambda J(t, x^1, u_1). \qquad \square$$

Lemma 10.6. *$V(t,\cdot)$ is convex on $\mathbb{R}^n$.*

Proof. Given $x^0, x^1 \in \mathbb{R}^n$, we define x^λ as above. Given $\delta > 0$, choose $u_0(\cdot), u_1(\cdot)$ such that

$$J(t, x^i, u_i) < V(t, x^i) + \delta \text{ for } i = 0, 1.$$

By Lemma 10.5,

$$V(t, x^\lambda) \leq J(t, x^\lambda, u_\lambda) \leq (1-\lambda)V(t, x^0) + \lambda V(t, x^1) + \delta.$$

Since δ is arbitrary, $V(t,\cdot)$ is convex. □

Remark 10.5. Lemmas 10.5 and 10.6 remain true if $\sigma = \gamma_0(t)x + \gamma_1(t)v + \gamma_2(t)$. Since we have assumed in Sections 6–10 that $\sigma(t,x,v)$ is bounded (see(6.1c)), we have omitted the term $\gamma_0(t)x$.

Let us denote by $W^{2,\infty}_{\ell oc}(\mathbb{R}^n)$ the space of functions g such that $g \in C^1_b(\mathbb{R}^n)$ and $D_x g$ satisfies a Lipschitz condition on every compact set $\Gamma \subset \mathbb{R}^n$. See [Zi]. For every $g \in W^{2,\infty}_{\ell oc}(\mathbb{R}^n)$ the generalized second derivatives $g_{x_i x_j}, i, j = 1, \cdots, n$ exist and $g_{x_i x_j} \in L^\infty_{\ell oc}(\mathbb{R}^n)$. If g is both convex and semiconcave, then for every bounded set $B \subset \mathbb{R}^n$, $0 \leq D^2 g \leq K(B)$ in the sense of Schwartz distributions. Thus:

Lemma 10.7. *If g is both convex and semiconcave, then $g \in W^{2,\infty}_{\ell oc}(\mathbb{R}^n)$.*

By taking $g = V(t,\cdot)$ we get:

Theorem 10.3. *In addition to* (10.18) *assume that $L(\cdot,\cdot,v)$ is of class $C^{1,2}(\overline{Q}_0)$ and satisfies* (8.1b), (9.1b)*. Then $V(t,\cdot)$ belongs to $W^{2\infty}_{\ell oc}(\mathbb{R}^n)$ for each $t \in [t_0, t_1)$.*

Example 10.1. Let us return to the Merton portfolio problem in Example 5.2. Let

$$\phi(s) = u_1(s)x(s), \; c(s) = u_2(s).$$

Then $\phi(s)$ is the amount of wealth in the risky asset and $c(s)$ is the consumption rate at time s. The wealth dynamics are equation (5.18). Moreover, $L = \ell(c)$ is concave. Let us regard $\phi(s)$ and $c(s)$ as the controls. Given a reference probability system $(\Omega, \{\mathcal{F}_s\}, P, w), s \geq 0$, we admit all $\mathcal{F}_s$ - progressively measurable controls $(\phi(\cdot), c(\cdot))$ such that $c(s) \geq 0$, $x(s) \geq 0$ and $J < \infty$ in (5.20) with $\tau = +\infty$. See Remark 5.1. Just as in Lemma 10.6, the linear-concave structure of this problem implies that the value function is concave.

IV.11 Historical remarks

The theory of optimal control for Markov diffusion processes had an extensive development since the 1960's. This chapter is intended as introduction to some aspects related to dynamic programming and the Hamilton–Jacobi–Bellman PDE. For another introduction to older results, see [FR, Chap. 6]. Other aspects of the theory of controlled Markov diffusions are described in [BL1]

[Bo] [ElK] [El] [Hau][YZ] as well as other books and research papers cited there. We have not considered problems with partial state observations, for which the method of dynamic programming leads to an infinite dimensional HJB equation. See [L5] [Hi] [I5] [Ni2].

In Section I.4 we showed that it is quite easy to obtain a dynamic programming principle for deterministic optimal control problems. At an intuitive level, one should be able to proceed similarly in case of stochastic control. However, various technical difficulties are encountered in doing so. In Section 7 we obtained a dynamic programming principle, for problems on a fixed finite time horizon, by an approximation method which reduces everything to the case when the Hamilton–Jacobi–Bellman PDE has a classical solution. An extension of this result, and a remark about other approaches to dynamic programming, will be given in Sections V.2 and V.10. The book of Yong-Zhou [YZ] gives another treatment of controlled Markov diffusions, by dynamic programming and stochastic maximum principle methods.

The one sided estimate (9.7) for second difference quotients is due to Krylov [Kr1, Sec. 4.2]. It is useful in proving regularity properties of value functions, such as the existence of $A^v V$ in Theorem 10.1. Krylov's estimate was proved for control problems on a fixed time interval $[t, t_1]$. Lions [L3] proved corresponding estimates for control until exit from a bounded cylindrical region $Q = [t_0, t_1) \times O$, or until exit from O in the infinite horizon discounted cost problem (Section 5). In the latter case, the discount factor must be sufficiently large.

Following Krylov [Kr1, p. 209] let us say that the stochastic control problem is nondegenerate if, for each (t, x), there exists $v \in U$ such that the matrix $a(t, x, v)$ is positive definite. This is weaker than condition (3.5). For nondegenerate problems, Krylov [Kr1, Sec. 4.7] proved by probabilistic methods that all generalized derivatives $V_t, V_{x_i}, V_{x_i x_j}, i, j = 1, \cdots, n$ of the value function V exist, and that the Hamilton–Jacobi–Bellman PDE holds almost everywhere in Q_0. Similar results were proved by Lions [L3] for the problem of control until exit from O (or from Q), using probability and PDE methods. See also Lions [L9] and Evans [E5]

As already noted in Remark 10.4, the fact that V satisfies the Hamilton – Jacobi – Bellman PDE almost everywhere can be proved by the theory of viscosity solutions (Chapter V) and the differentiability properties of semiconcave functions. In the semilinear case this result however is slightly weaker than Theorem 10.2 which also proves that $A_0 V \in L^1_{\ell oc}$, with $A_0 V$ interpreted in the generalized sense.

The regularity of solutions to fully nonlinear, elliptic equations of type (5.8) was proved by Evans [E1] [E4]. Also see [GT, Chapter 17].

V

Viscosity Solutions: Second-Order Case

V.1 Introduction

In this chapter we study the exit time control of a Markov diffusion process as formulated in Section IV.2. With the exception of the last section, we assume that the state space is a bounded finite-dimensional set. The main purpose of this chapter is to study viscosity solutions of the dynamic programming equation which is a second-order, nonlinear parabolic partial differential equation, (4.1) below. If the controlled Markov process is uniformly parabolic, then there are classical solutions to this equation and the related results were discussed in Section IV.4. In this chapter we do *not* assume the uniform parabolicity and therefore we only expect the value function to be a viscosity solution. Indeed when the value function is continuous, Theorem II.5.1 implies that the value function is a viscosity solution of (4.1).

Among the hypotheses of Theorem II.5.1, condition II(3.2) is equivalent to the dynamic programming principle. In the literature several proofs of the dynamic programming principle were given by using either measurable selection theorems or compactness type arguments. In Section 2 we give an alternate proof of Lions when appropriate assumptions are satisfied. See Theorem 2.1. Lions' proof is based on an approximation argument and uses the results of Chapter IV. This proof also shows that under these assumptions, the value function is continuous and satisfies the boundary data pointwise.

In Sections 4–8, we prove a uniqueness result, or more generally a comparison result for viscosity solutions of (4.1). Equation (4.1) is a second-order equation and the uniqueness results of Chapter II do not apply to (4.1) (see Section 5, below). Moreover, second-order equations introduce new difficulties which cannot be resolved by simply modifying the techniques discussed in Chapter II. These difficulties were overcome by Jensen in 1986. Jensen first extended the classical Alexandrov maximum principle to semi convex functions. At its maxima a semiconvex function is differentiable but may fail to be twice differentiable. However, Jensen proved that there are points of twice differentiability which are arbitrarily close to its maxima and at these points Hessian is nonnegative and the gradient is small. Jensen used this result to obtain a first comparison result. To use this generalized maximum principle in a general uniqueness proof, we need to approximate the subsolutions by semiconvex subsolutions and supersolutions by semi-concave supersolutions. The so-called inf and sup convolutions provide exactly that. These approximations are also used as powerful approximations of viscosity solutions in general. The definition and the properties of these operations are given in Section 5. The combination of these tools enabled Jensen to prove the first uniqueness result for second-order equations when the coefficients of equation (4.1) are state-independent. Later the uniqueness result was generalized to cover all equations satisfying standard Lipschitz regularity assumptions. The main tool in this extension is a lemma due to Ishii. Later Crandall and Ishii streamlined these results in a powerful analysis result about the maxima of semi-continuous functions. In recent years, Crandall-Ishii lemma has been the cornerstone of every comparison result proved in the literature. In our presentation we chose to state the Crandall-Ishii Lemma, Theorem 6.1 and refer to the survey article "*Users' Guide to Viscosity Solutions for Second Order Partial Differential equations*" by Crandall, Ishii and Lions [CIL1] and to the paper of Crandall and Ishii [CI] for its proof. We then prove a general comparison result in Section 8, Theorem 8.1. This result implies that if a continuous subsolution W of (4.1) and a continuous supersolution V satisfy $W \leq V$ at the boundary, then this inequality holds everywhere.

A discussion of viscosity solutions defined on the whole space is given in Section 9. The techniques of Sections 4–8 can easily be modified to apply to this case. However, to acquaint the reader with approximation arguments, we approximate the given equation by a sequence of equations defined on large balls. We then pass to the limit by letting the radii of the balls go to infinity.

For readers interested in viscosity solutions but not in stochastic control theory, Sections 4–9 can be read independently of Sections 2 and 3.

V.2 Dynamic programming principle

In Section IV.7 we established a dynamic programming principle for Markov diffusion processes on a fixed finite time interval $[t, t_1]$. This principle was obtained as a consequence of a somewhat stronger property (DP). See Corollary IV.7.1 and Corollary IV.7.2. In this section, we will obtain similar results for the problem of control until time τ of exit from a bounded cylindrical region $Q = [t_0, t_1) \times O$. We again write $V = V_{PM}$, where V_{PM} is as in IV(2.10). Thus

$$V(t,x) = \inf_{u(\cdot)} J(t,x;u),$$

where for brevity

$$\inf_{u(\cdot)} \cdots = \inf_{\nu} \inf_{u(\cdot)\in\mathcal{A}_{t\nu}} \cdots .$$

The definition of property (DP) in Section IV.7 is modified by replacing any stopping time θ by $\tau \wedge \theta = \min(\tau, \theta)$ where τ is the exit time of $(s, x(s))$ from Q. Thus, property (DP) holds if:

(a) For every $\nu, u(\cdot) \in \mathcal{A}_{t\nu}$ and $\{\mathcal{F}_s\}$-stopping time θ,

$$\text{(2.1a)} \qquad V(t,x) \le E_{tx}\left\{\int_t^{\tau\wedge\theta} L(s,x(s),u(s))ds + V(\tau\wedge\theta, x(\tau\wedge\theta))\right\}.$$

(b) For every $\delta > 0$ there exist ν and $u(\cdot) \in A_{t\nu}$ such that

$$\text{(2.1b)} \quad V(t,x) + \delta \ge E_{tx}\left\{\int_t^{\tau\wedge\theta} L(s,x(s),u(s))ds + V(\tau\wedge\theta, x(\tau\wedge\theta))\right\}.$$

for every $\{\mathcal{F}_s\}$-stopping time θ.

In particular, by taking $\theta = r$ (nonrandom) with $t < r < t_1$, property (DP) implies

$$\text{(2.2)} \qquad V(t,x) = \inf_{u(\cdot)} E_{tx}\left\{\int_t^{\tau\wedge r} L(s,x(s),u(s))ds + V(\tau\wedge r, x(\tau\wedge r))\right\}.$$

Let us assume that Q and the control set U are bounded, and that IV(6.1), IV(6.3) hold. We will obtain property (DP) by using a device of Lions to reduce it to the corresponding result in Section IV.7. However, additional assumptions will be needed to ensure continuity of the value function V on $\overline{Q}$.

Let us begin by considering an auxiliary control problem, in which the running cost L and terminal cost Ψ in IV(2.8) are replaced by continuous functions $\tilde{L}$, $\tilde{\Psi}$ such that:

$$\text{(2.3)} \qquad \begin{aligned} &\text{(a)} \quad \tilde{L} \ge 0,\ \tilde{\Psi} \ge 0, \\ &\text{(b)} \quad \tilde{\Psi}(t,x) = 0,\ (t,x) \in [t,t_1] \times \partial O. \end{aligned}$$

Let $\tilde{\psi}(x) = \tilde{\Psi}(t_1, x)$ for $x \in \overline{O}$, and $\tilde{\psi}(x) = 0$ for $x \in I\!R^n \backslash \overline{O}$. Then $\tilde{\psi}$ is bounded and uniformly continuous on $I\!R^n$. The criterion to be minimized is

$$\text{(2.4)} \qquad \tilde{J}(t,x;u) = E_{tx}\left\{\int_t^{\tau} \tilde{L}(s,x(s),u(s))ds + \tilde{\psi}(x(t_1))\chi_{\tau=t_1}\right\}.$$

Let $\tilde{V}(t,x)$ be the corresponding value function. By (2.3), $\tilde{V} \ge 0$. If the reference probability system ν is fixed, $\tilde{V}_\nu$ is defined similarly.

Let us next introduce an approximation of $\tilde{J}$. Let $\hat{d}(x)$ be the distance between x and $\overline{O}$. For $\varepsilon > 0, (t,x) \in Q_0$ we define

$$J^\varepsilon(t,x;u) = E_{tx}\left[\int_t^{t_1} \Gamma(s,\varepsilon)\tilde{L}(s,x(s),u(s))ds + \Gamma(t_1,\varepsilon)\tilde{\psi}(x(t_1))\right], \tag{2.5}$$

where

$$\Gamma(s,\varepsilon) = \exp\left(-\frac{1}{\varepsilon}\int_t^s \hat{d}(x(r))dr\right).$$

Clearly Γ depends on (t,x) and $u(\cdot)$, but this dependence is suppressed in our notation. Since $\hat{d} = 0$ on $\overline{O}$, we have

$$\Gamma(s,\varepsilon) = 1, \qquad s \in [t,\tau]. \tag{2.6}$$

Hence, for $(t,x) \in Q$,

$$J^\varepsilon(t,x;u) = \tilde{J}(t,x;u) + E_{tx}\int_\tau^{t_1} \Gamma(s,\varepsilon)\tilde{L}(s,x(s),u(s))ds + \Gamma(t_1,\varepsilon)\tilde{\psi}(x(t_1))\chi_{\tau=t_1}. \tag{2.7}$$

The positivity of $\tilde{L}$ and $\tilde{\psi}$ imply that $J^\varepsilon \geq \tilde{J}$ on Q. Also $\Gamma(s,\varepsilon') \leq \Gamma(s,\varepsilon)$ for $\varepsilon' < \varepsilon$. Hence J^ε is nondecreasing in $\varepsilon > 0$. Let

$$V_\nu^\varepsilon = \inf_{\mathcal{A}_{t\nu}} J^\varepsilon, \qquad V^\varepsilon = V_{PM}^\varepsilon = \inf_\nu V_\nu^\varepsilon.$$

In Chapter IV it is shown that $V^\varepsilon = V_\nu^\varepsilon$ for every ν and V^ε satisfies property IV(7.1). See Remark IV.7.1.

Let $\hat{\rho}(x)$ be the signed distance to the boundary of O, i.e., for $x \notin O$, $\hat{\rho}(x) = \hat{d}(x)$ is the distance between x and $\bar{O}$, and for $x \in O$, $-\hat{\rho}(x)$ is the distance between x and the complement of O. To prove the convergence of V^ε in Q, we also assume a condition analogous to I(3.11), i.e., assume that ∂O is smooth and hence $\hat{\rho}$ is smooth near ∂O and for every $(t,x) \in [t_0,t_1] \times \partial O$, there exists $v(t,x) \in U$ satisfying

$$A^{v(t,x)}\hat{\rho}(x) = \frac{1}{2}\sum_{i,j=1}^n a_{ij}(t,x,v(t,x))\hat{\rho}_{x_i x_j}(x) + f(t,x,v(t,x))\cdot\ \nabla\hat{\rho}(x) > 0, \tag{2.8}$$

where A^u is as in IV(2.11). Further we assume that v has a smooth extension. We set $\tilde{V}(t,x) = \tilde{\Psi}(t,x)$ on $\partial^* Q$. Thus,

$$\begin{aligned} &\tilde{V}(t,x) = 0,\ (t,x) \in [t_0,t_1], \times\partial O, \\ &\tilde{V}(t_1,x) = \tilde{\psi}(x),\ x \in O. \end{aligned} \tag{2.9}$$

We will show that these boundary data are assumed continuously.

Lemma 2.1. *Assume* (2.8)*. Then* $\tilde{V} \in C(\overline{Q})$ *and* $\tilde{V}$ *satisfies* (2.1a), (2.1b) *with* L *replaced by* $\tilde{L}$*. Moreover,* $\tilde{V} = \tilde{V}_\nu$ *for each* ν*.*

Proof. Let $x \in \partial O, t \in [t_0, t_1)$. Set $\bar{u}(s) \equiv v(t,x)$, with $v(t,x)$ as in (2.8). Then

$$0 \le V^\varepsilon(t,x) \le J^\varepsilon(t,x;\bar{u})$$

and we claim that

$$\lim_{\varepsilon \downarrow 0} J^\varepsilon(t,x;\overline{u}) = 0. \tag{2.10}$$

Let $x(s)$ be the state process corresponding the control $\overline{u}$ and the initial condition $x(t) = x$. Then, by Ito's formula and (2.8), $\hat{\rho}(x(s))$ is locally a strict submartingale. Since, $\hat{d}$ is the positive part of $\hat{\rho}$, $\hat{d}(x(s))$ P-almost surely satisfies

$$\int_t^s \hat{d}(x(r))dr > 0, \quad \forall s \in [t, t_1].$$

Hence P-almost surely $\Gamma(\varepsilon, s) \to 0$ as $\varepsilon \downarrow 0$. Then (2.10) follows from the dominated convergence theorem.

Hence V^ε converges to $\tilde{V} \equiv 0$ on $[t_0, t_1) \times \partial O$, as ε tends to zero. Since $V^\varepsilon = \tilde{V} = \tilde{\psi}$ on $\{t_1\} \times \overline{O}, V^\varepsilon \to \tilde{V}$ on $\partial^* Q$. Observe that $\partial^* Q$ is compact and since $\tilde{\psi}(x) = 0$ for $x \in \partial O, \tilde{V}$ is continuous on $\partial^* Q$. Then by Dini's Theorem [Ro, p. 162] this monotone convergence is uniform on $\partial^* Q$. Set

$$h(\varepsilon) = \sup\{V^\varepsilon(t,x) - \tilde{V}(t,x) : (t,x) \in \partial^* Q\}.$$

We have just argued that $h(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$.

Since V^ε satisfies the dynamic programming principle and $\tilde{V}(\tau, x(\tau)) = 0$ when $\tau < t_1$, yields

$$V^\varepsilon(t,x) \le \tilde{J}(t,x;u) + E_{t,x}\Gamma(\tau,\varepsilon)[V^\varepsilon(\tau,x(\tau)) - \tilde{V}(\tau,x(\tau))],$$

for any $(t,x) \in \overline{Q}$ and $u(\cdot) \in \mathcal{A}_{t\nu}$. Since $(\tau, x(\tau)) \in \partial^* Q$ and $\Gamma(\tau,\varepsilon) = 1$, we now have

$$V^\varepsilon(t,x) \le \tilde{J}(t,x;u) + h(\varepsilon), \ \forall (t,x) \in \overline{Q}.$$

Therefore

$$\tilde{V}(t,x) \le V^\varepsilon(t,x) \le \tilde{V}(t,x) + h(\varepsilon), \ \forall (t,x) \in \overline{Q}.$$

Hence V^ε converges to $\tilde{V}$ uniformly on $\overline{Q}$. Consequently $\tilde{V} \in C(\overline{Q})$. Also for a reference probability system ν,

$$\tilde{V} \le \tilde{V}_\nu \le V^\varepsilon_\nu = V^\varepsilon \le \tilde{V} + h(\varepsilon).$$

Therefore $\tilde{V} = \tilde{V}_\nu$ for any ν.

It remains to show that $\tilde{V}$ satisfies (2.1). By Corollary IV.7.1 and Remark IV.7.1, for every $u(\cdot)$ and stopping time θ,

$$V^\varepsilon(t,x) \le E_{tx}\bigg\{\int_t^\theta \Gamma(s,\varepsilon)\tilde{L}(s,x(s),u(s))ds$$

$$+\Gamma(\theta,\varepsilon)V^\varepsilon(\theta,x(\theta))\Big\}.$$

Since $\tilde{V}=0$ on ∂^*Q, $\Gamma(\theta,\varepsilon)\to 0$ when $\theta>\tau$, and $\Gamma(s,\varepsilon)=1$ for $t\le s\le\tau\wedge\theta$, we get (2.1a) by letting $\varepsilon\to 0$. For every $\delta>0$ there exists $u^\varepsilon(\cdot)$ such that

$$V^\varepsilon(t,x)\ge E_{tx}\left[\int_t^\theta \Gamma(s,\varepsilon)\tilde{L}(s,x(s),u(s))ds\right.$$
$$\left.+\Gamma(\theta,\varepsilon)V^\varepsilon(\theta,x(\theta))\right]-\frac{\delta}{3}$$

for every stopping time θ. If we take ε small enough that $\|V^\varepsilon-V\|<\delta/3$, then $u(\cdot)=u^\varepsilon(\cdot)$ satisfies (2.1b). Here we use the positivity of $\tilde{L}$ and the fact that $\Gamma(s,\varepsilon)=1$ for $t\le s\le\tau\wedge\theta$. □

Let us now consider running cost function L and boundary cost function Ψ which do not necessarily satisfy (2.3). As in I(3.6) set

$$g(t,x)=\Psi(t,x),\quad (t,x)\in[t_0,t_1)\times\partial O,$$

$$\psi(x)=\Psi(t_1,x),\quad x\in\overline{O}.$$

We also assume that g can be extended to $\overline{Q}_0$ such that $g\in C_b^3(\overline{Q}_0)$ and

(2.11a) $$-g_t(t,x)+\mathcal{H}(t,x,D_xg(t,x),D_x^2g(t,x))\le 0,\quad \forall (t,x)\in\overline{Q}_0,$$

(2.11b) $$g(t_1,x)\le\psi(x),\quad \forall x\in I\!R^n,$$

where for $(t,x)\in\overline{Q}_0, p\in I\!R^n$ and a symmetric matrix $A, \mathcal{H}(t,x,p,A)$ is as in IV(3.2), i.e,

(2.12)
$$\mathcal{H}(t,x,p,A)=\sup_{v\in U}\left\{-\frac{1}{2}\,\mathrm{tr}(\sigma\sigma')(t,x,v)A]-f(t,x,v)\cdot p-L(t,x,v)\right\}.$$

Assumptions (2.11) state that g is a smooth subsolution of the HJB equation IV(3.3). Observe that (2.11) is satisfied if $L,\psi\ge 0$ and $g\equiv 0$, as in (2.3). Condition (2.11) is very similar to the hypotheses of Proposition I.8.1. See Remark I.8.1. Using I(8.13) as our motivation, we define

$$\tilde{L}(t,x,v)=L(t,x,v)+A^vg(t,x),$$

$$\tilde{\psi}(x)=\psi(x)-g(t_1,x),$$

for $(t,x)\in\overline{Q}_0, v\in U$ (see IV(2.11) for the definition of A^v). Then $\tilde{L}\ge 0,\tilde{\psi}\ge 0$ as required in (2.3). Recall that in Section IV.2, Ψ is assumed continuous. Hence

$$g(t_1,x)=\psi(x),\quad \forall x\in\partial O,$$

and an application of Ito's formula yields

$$g(\tau, x(\tau)) = g(t,x) + E_{tx}\int_t^\tau A^{u(s)} g(s,x(s))ds,$$

and

$$\begin{aligned}\tilde{J}(t,x;u) + g(t,x) &= E_{t,x}\Big\{\int_t^\tau [\tilde{L}(s,x(s),u(s) - A^{u(s)}g(s,x(s))]ds \\ &\qquad + g(\tau,x(\tau)) + \tilde{\psi}(x(t_1))\chi_{\tau=t_1}\Big\} \\ &= J(t,x;u).\end{aligned}$$

Since this is true for all $u(\cdot)$, we get

$$V = \tilde{V} + g,\ V_\nu = \tilde{V}_\nu + g. \tag{2.13}$$

From Lemma 2.1 and (2.13) we obtain the main result of this section.

Theorem 2.1. *Assume* IV(6.1), IV(6.3), (2.8), (2.11) *and that ψ is continuous on $\overline{Q}$. Then V is continuous on $\overline{Q}$ and property (DP) holds. Moreover, $V = V_\nu$ for every reference probability system ν.*

Remark 2.1. Our method is to approximate the exit time control problem by a sequence of problems with state space $\overline{Q}_0$. We then use the results of Chapter IV. For a general proof of dynamic programming we refer to the books of Bensoussan-Lions [BL1], Krylov [Kr1] and Borkar [Bo]. Bensoussan-Lions and Krylov prove dynamic programming by discretization. Their proof requires continuity of the value function. Borkar's proof however [Bo, Sec. III.1] is more general. Also a probabilistic proof based on a deep measurable selection theorem of Brown and Purves [BrP] is possible. In the discrete-time setup such a proof is given in Bertsekas and Shreve [BsS]. Borkar's [Bo, Lemma III.1.1] is an analogue of this measurable selection theorem. Recently this method was used by Soner and Touzi [ST1] to prove a geometric dynamic programming. El Karoui and her collaborators [ENJ] and Kurtz [Kz] avoid using measurable selection by proving the compactness of the set of admissible controls.

V.3 Viscosity property

Except in Section 9, we will take Q to be bounded in the rest of this chapter. Let ν be a reference probability system and define a two parameter family of nonlinear operators on $\mathcal{C} = C(\overline{O})$ by

$$(\mathcal{T}_{tr}\phi)(x) = \inf_{u(\cdot)} E_{tx}\Big\{\int_t^{\tau\wedge r} L(s,x(s),u(s))ds + g(\tau,x(r))\chi_{\tau<r}$$

$$+\phi(x(r))\chi_{\tau\geq r}\Big\},$$

where $t_0 \leq t \leq r \leq t_1, \phi \in \mathcal{C}$, τ is the exit time of $(s, x(s))$ from Q and $g \in C(\overline{Q})$ is a given function, which we call the *lateral boundary data.* Clearly $\mathcal{T}_{tr}$ satisfies II(3.1)-II(3.2$'$). The semigroup property II(3.3) however, is equivalent to the dynamic programming principle (2.2).

To apply the results of Chapter II to this situation, we also have to verify II(3.11). Indeed we shall prove that II(3.11) holds with $\Sigma = \overline{O}$, $\Sigma' = O$, $\mathcal{D} = C^{1,2}(\overline{Q})$ and the infinitesimal generator,

$$(\mathcal{G}_t\phi)(x) = \mathcal{H}(t, x, D\phi(x), D^2\phi(x)), \;\; (t, x) \in Q,$$

where $\mathcal{H}$ is as in (2.12). In view of Theorem II.5.1 and Remark II.6.1, this result will imply that the value function is a viscosity solution of the dynamic programming equation provided that it is continuous, see Corollary 3.1 below.

Theorem 3.1. *Suppose that f, σ satisfy* IV(2.2)*, U is compact, and g and L are continuous. Then for every $w \in \mathcal{D}$ and $(t, x) \in Q$ we have*

$$\lim_{h\downarrow 0} \frac{1}{h}[(\mathcal{T}_{t,t+h}w(t+h,\cdot))(x) - w(t,x)] = \frac{\partial}{\partial t}w(t,x) - (\mathcal{G}_t w(t,\cdot))(x).$$

Proof. We start with a probabilistic estimate. Let $x(\cdot)$ be the solution of IV(2.1) with control $u(\cdot)$ and initial condition $x(t) = x \in O$. Since Q is bounded, f and σ are bounded and for any positive integer m and $h \in (0, 1]$ we have

$$(3.1) \qquad E_{tx} \sup_{t\leq\rho\leq t+h} |x(\rho) - x|^{2m}$$

$$= E_{tx} \sup_{t\leq\rho\leq t+h} \left| \int_t^\rho f(s, x(s), u(s))ds + \int_t^\rho \sigma(s, x(s), u(s))dw(s) \right|^{2m}$$

$$\leq \hat{C}_m E_{tx} \left(\int_t^{t+h} |f(s, x(s), u(s))|ds \right)^{2m}$$

$$+ \hat{C}_m E_{tx} \sup_{t\leq\rho\leq t+h} \left| \int_t^\rho \sigma(s, x(s), u(s))dw(s) \right|^{2m}$$

$$\leq \hat{C}_m \|f\|^{2m} h^{2m} + \tilde{C}_m \|\sigma\|^{2m} h^m$$

$$\leq C_m h^m,$$

where $\|\cdot\|$ denotes the sup-norm on Q and $C_m, \hat{C}_m, \tilde{C}_m$ are suitable constants. (See (D.4) in Appendix D for a more general probabilistic estimate of the above type.) Set $d(x) = \operatorname{dist}(x, \partial O)$ and recall that τ is the exit time from Q. Then for $t + h \le t_1$,

$$
\begin{aligned}
P_{tx}(\tau \le t+h) &\le P_{tx}\Big(\sup_{t\le\rho\le t+h} |x(\rho) - x| \ge d(x)\Big) \\
&\le \Big(E_{tx} \sup_{t\le\rho\le t+h} |x(\rho)-x|^{2m}\Big)(d(x))^{-2m} \qquad (3.2)\\
&\le C_m h^m / d(x)^{2m}.
\end{aligned}
$$

Fix $v \in U$ and let $u(s) \equiv v$. Then the definition of $\mathcal{T}_{t,t+h}$ yields

$$
\begin{aligned}
I(h) &= \frac{1}{h}[(\mathcal{T}_{t,t+h} w(t+h, \cdot))(x) - w(t,x)] \\
&\le \frac{1}{h} E_{tx} \int_t^{(t+h)\wedge\tau} L(s, x(s), v)\,ds \qquad (3.3)\\
&\quad + \frac{1}{h} E_{tx}[w(t+h, x(t+h)) - w(t,x)]\chi_{\tau \ge t+h} \\
&\quad + \frac{1}{h} E_{tx}[g(\tau, x(\tau)) - w(t,x)]\chi_{\tau < t+h}.
\end{aligned}
$$

The estimate (3.2) with $m = 2$ yields

$$\lim_{h\downarrow 0} \frac{1}{h} P_{tx}(\tau \le t+h) = 0$$

for every $(t,x) \in Q$. Hence

$$\lim_{h\downarrow 0} \frac{1}{h} E_{tx} \int_t^{(t+h)\wedge\tau} L(s, x(s), v)\,ds = L(t,x,v)$$

and

$$\lim_{h\downarrow 0} \frac{1}{h} E_{tx}[g(\tau, x(\tau)) - w(t,x)]\chi_{\tau<t+h} = 0.$$

Also, by Ito's formula

$$
\begin{aligned}
&\lim_{h\downarrow 0} \frac{1}{h} E_{tx}[w(t+h, x(t+h)) - w(t,x)]\chi_{\tau\ge t+h} \\
&= \lim_{h\downarrow 0} \frac{1}{h} E_{tx}[w((t+h)\wedge\tau, x((t+h)\wedge\tau)) - w(t,x)] \\
&= \lim_{h\downarrow 0} \frac{1}{h} E_{tx} \int_t^{(t+h)\wedge\tau} A^v w(s, x(s))\,ds \\
&= A^v w(t,x).
\end{aligned}
$$

Substitute the above into (3.3) to obtain

$$\limsup_{h\downarrow 0} I(h) \leq L(t,x,v) + A^v w(t,x),$$

for all $v \in U$. By taking the infimum over v, we get

$$\limsup_{h\downarrow 0} I(h) \leq \frac{\partial}{\partial t} w(t,x) - (\mathcal{G}_t w(t,\cdot))(x).$$

For any sequence $h_n \downarrow 0$, there exists $u_n(\cdot)$ satisfying

$$\begin{aligned}
&(\mathcal{T}_{t,t_n} w(t_n,\cdot))(x) \\
&\quad \geq E_{tx}\left[\int_t^{\tau_n} L(s,x_n(s),u_n(s))ds + g(\tau_n, x_n(\tau_n))\chi_{\tau_n<t_n}\right. \\
&\qquad \left. + w(t_n, x_n(t_n))\chi_{\tau_n=t_n}\right] - (h_n)^2,
\end{aligned}$$

where $t_n = t + h_n, \tau_n = \hat{\tau}_n \wedge t_n$, $x_n(\cdot)$ is the solution of IV(2.1), IV(2.4) with control u_n, and $\hat{\tau}_n$ is the exit time of $(s, x_n(s))$ from Q. Therefore

$$\begin{aligned}
I(h_n) \geq &\frac{1}{h_n} E_{tx} \int_t^{\tau_n} L(s,x_n(s),u_n(s))ds \\
&+\frac{1}{h_n} E_{tx}[w(t_n,x(t_n)) - w(t,x)]\chi_{\tau_n=t_n} \qquad (3.4)\\
&+\frac{1}{h_n} E_{tx}[g(\tau_n,x_n(\tau_n)) - w(t,x))]\chi_{\tau_n<t_n} - h_n.
\end{aligned}$$

The probabilistic estimate (3.2) with $m = 2$ implies that the limit of the third term is zero and

$$\begin{aligned}
&\lim_{n\to\infty}\left|\frac{1}{h_n} E_{tx}\left(\int_t^{t_n} L(t,x,u_n(s))ds - \int_t^{\tau_n} L(s,x_n(s),u_n(s))ds\right)\right| \\
&\quad \leq \lim_{n\to\infty} \|L\| \frac{1}{h_n} E_{tx}(t_n - \tau_n) \qquad (3.5)\\
&\qquad + \lim_{n\to\infty} \frac{1}{h_n} E_{tx} \int_t^{t_n} |L(t,x,u_n(s)) - L(s,x_n(s),u_n(s))|ds.
\end{aligned}$$

Since $\overline{Q} \times U$ is compact, L is uniformly continuous. Also (3.1) implies that for every $\delta > 0$,

$$\lim_{n\to\infty} P(\sup_{t\leq\rho\leq t+h_n} |x_n(\rho) - x| \geq \delta) = 0.$$

Therefore the uniform continuity of L and (3.1) imply that the limits in (3.5) are zero. We now use (3.2) and Dynkin's formula IV(2.13) to obtain

$$\lim_{n\to\infty}\frac{1}{h_n}\left|E_{tx}\left\{[w(t_n,x(t_n))-w(t,x)]\chi_{\tau_n=t_n}-\int_t^{t_n}A^{u_n(s)}w(t,x)ds\right\}\right|$$

$$\leq \lim_{n\to\infty}\frac{1}{h_n}\sup_v|A^v w(t,x)|E_{tx}(t_n-\tau_n)$$

$$+\lim_{n\to\infty}\frac{1}{h_n}E_{tx}\int_t^{\tau_n}|A^{u_n(s)}w(s,x_n(s))-A^{u_n(s)}w(t,x)|ds.$$

Since $w\in C^{1,2}(\overline{Q})$, $A^v w(s,y)$ is a uniformly continuous function of $\overline{Q}\times U$. As in (3.5), the dominated convergence theorem and (3.1) imply that the above limit is zero. Combine this with (3.4) and (3.5) to obtain

$$I(h_n)\geq L^n+G^n-e(n),$$

where

$$L^n=\frac{1}{h_n}E\int_t^{t+h_n}L(t,x,u_n(s))ds,$$

$$G^n=\frac{1}{h_n}E\int_t^{t+h_n}A^{u_n(s)}w(t,x)ds,$$

and the error term $e(n)$ converges to zero as $n\to\infty$. Define a set

$$\hat{U}=\left\{(L,G)\in I\!R^2:L=L(t,x,v),G=A^v w(t,x)\ \text{for some}\ v\in U\right\}.$$

Then $(L^n,G^n)\in\overline{\text{co}}(\hat{U})$, where $\overline{\text{co}}$ denotes the convex, closed hull of $\hat{U}$. Also,

$$L^n+G^n\geq\inf\{L+G:(L,G)\in\overline{\text{co}}(\hat{U})\}$$

$$=\inf\{L+G:(L,G)\in\hat{U}\}$$

$$=\frac{\partial}{\partial t}w(t,x)-(\mathcal{G}_t w(t,\cdot))(x).$$ □

As in Section 2, let V be the value function.

Corollary 3.1. *Suppose that $V\in C(\overline{Q})$ satisfies* (2.1)*. Then V a viscosity solution of the dynamic programming equation*

$$-\frac{\partial}{\partial t}V(t,x)+(\mathcal{G}_t V(t,\cdot))(x)=0\ \text{ in } Q. \tag{3.6}$$

Proof. We have shown that the two parameter family of nonlinear operators $\{\mathcal{T}_{tt_1}\}$ satisfy II(3.1), II(3.2), II(3.11). Hence the viscosity property of V follows from Theorem II.5.1 and Remark II.6.1. □

V.4 An equivalent formulation

Recall that in Section II.8 we obtained an equivalent definition for the viscosity solutions of a first-order equation. In this section, we follow the procedure devised in Section II.8 to obtain an equivalent definition of viscosity solutions of

$$(4.1)\quad -\frac{\partial}{\partial t}W(t,x) + \mathcal{H}(t,x,D_xW(t,x),D_x^2W(t,x),W(t,x)) = 0, \ \ (t,x) \in Q.$$

Here we do not assume that (4.1) is related to a control problem.

We start with the definition of second subdifferentials and superdifferentials of continuous functions. Let $\mathcal{S}^n$ be the set of all $n \times n$ symmetric matrices.

Definition 4.1. Let $W \in C(\overline{Q})$.
(i) The set of *second (parabolic) superdifferentials* of W at $(t,x) \in Q$ is

$$D^{+(1,2)}W(t,x) = \Bigg\{ (q,p,A) \in I\!R \times I\!R^n \times \mathcal{S}^n :$$

$$\limsup_{\substack{(h,y)\to 0\\ (t+h,x+y)\in Q}} \frac{W(t+h,x+y) - W(t,x) - qh - p\cdot y - \frac{1}{2}Ay\cdot y}{|h| + |y|^2} \le 0 \Bigg\}.$$

(ii) The set of *second (parabolic) subdifferentials* of W at $(t,x) \in Q$ is

$$D^{-(1,2)}W(t,x) = -D^{+(1,2)}(-W)(t,x)$$

$$= \Bigg\{ (q,p,A) \in I\!R \times I\!R^n \times \mathcal{S}^n :$$

$$\liminf_{\substack{(h,y)\to 0\\ (t+h,x+y)\in Q}} \frac{W(t+h,x+y) - W(t,h) - qh - p\cdot y - \frac{1}{2}Ay\cdot y}{|h| + |y|^2} \ge 0 \Bigg\}.$$

In the literature, for instance in the User's Guide of Crandall, Ishii and Lions [CIL1, Section 8], the above sets are also parabolic sub or superjets and the notation $\mathcal{P}^{2,\pm}$ is used.

The closure of the set of sub and superdifferentials are also useful in the theory. We define,

Definition 4.2. For $W \in C(\overline{Q})$, $(t,x) \in Q$, $(q,p,A) \in cD^{\pm(1,2)}W(t,x)$ if and only if there are sequences $(t_n,x_n) \in Q \to (t,x)$ and $(q_n,p_n,A_n) \in D^{\pm(1,2)}W(t_n,x_n) \to (q,p,A)$.

It is clear that if $(q, p, A) \in cD^{+(1,2)}W(t,x)$, then $(q, p, A + B) \in cD^{+(1,2)}W(t,x)$ for any positive semidefinite matrix B. Also for $W \in C^{1,2}(Q)$, and $(t,x) \in Q$,

$$cD^{+(1,2)}W(t,x) = \left\{ \left(\frac{\partial}{\partial t}W(t,x), D_x W(t,x), D_x^2 W(t,x) + B \right) : B \geq 0 \right\},$$

$$cD^{-(1,2)}W(t,x) = \left\{ \left(\frac{\partial}{\partial t}W(t,x), D_x W(t,x), D_x^2 W(t,x) - B \right) : B \geq 0 \right\}.$$

Now, also assume that $W \in C^{1,2}(Q)$ is a classical solution of (4.1). Since for every $B \geq 0$

$$\mathrm{tr}[(\sigma\sigma')(t,x,v)B] \geq 0,$$

the above characterization of the second sub- and superdifferentials yields

$$-q + \mathcal{H}(t,x,p,A,W(t,x)) \leq 0, \quad \forall (q,p,A) \in cD^{+(2,1)}W(t,x), \tag{4.2}$$

$$-q + \mathcal{H}(t,x,p,A,W(t,x)) \geq 0, \quad \forall (q,p,A) \in cD^{-(2,1)}W(t,x). \tag{4.3}$$

As in the first order equations, the above inequalities form an equivalent definition for the viscosity solutions. We first prove the following results towards this equivalence.

Lemma 4.1. *Let $(t,x) \in Q$ be given. Then $(q,p,A) \in D^{+(1,2)}W(t,x)$ if and only if there exists $w \in C^{1,2}(\overline{Q})$ satisfying*

$$\left(\frac{\partial}{\partial t} w(\bar{t},\bar{x}), D_x w(\bar{t},\bar{x}), D_x^2 w(\bar{t},\bar{x}) \right) = (q,p,A). \tag{4.4}$$

such that $W - w$ achieves its maximum at $(t,x) \in Q$. Similarly, $(q,p,A) \in D^{-(1,2)}W(t,x)$ if and only if there exists $w \in C^{1,2}(\overline{Q})$ satisfying (4.4) such that $W - w$ achieves its minimum at $(t,x) \in Q$.

Proof. Suppose that $w \in C^{1,2}(\overline{Q})$ and $W - w$ achieves its maximum at (t,x) with $W(t,x) = w(t,x)$. Then it is easy to show that

$$\left(\frac{\partial}{\partial t} w(t,x), D_x w(t,x), D_x^2 w(t,x) \right) \in D^{+(1,2)}W(t,x).$$

To prove the opposite direction: Let $(q,p,A) \in D^{+(1,2)}W(\bar{t},\bar{x})$ for some $(\bar{t},\bar{x}) \in Q$ be given. We continue with constructing a test function $w \in C^{1,2}(\overline{Q}_0)$ satisfying (4.4) and $(t,x) \in \operatorname{argmax}\{(W-w)(t,x)\colon (t,x) \in \overline{Q}\}$ with $W(\bar{t},\bar{x}) = w(\bar{t},\bar{x})$. For $r \geq 0$ define

$$h(r) = \sup \left\{ \frac{\left(W(\bar{t}+h,\bar{x}+y) - W(\bar{t},\bar{x}) - qh - p\cdot y - \frac{1}{2}Ay\cdot y\right)^+}{(|y|^4 + h^2)^{\frac{1}{2}}} : \right.$$

$$(\bar{t}+h, \bar{x}+y) \in \overline{Q}, (|y|^4+h^2)^{\frac{1}{2}} \le r\Big\}.$$

Since

$$\limsup_{(y,h)\to 0} \frac{|y|^2+|h|}{(|y|^4+h^2)^{\frac{1}{2}}} < \infty$$

and $(q,p,A) \in D^{+(1,2)}W(\bar{t},\bar{x}), h$ is continuous and nondecreasing on $[0,\infty)$ with $h(0)=0$. Now let

$$w(t,x) = F(r(t,x)) + W(\bar{t},\bar{x}) + q(t-\bar{t}) + p\cdot(x-\bar{x}) + \frac{1}{2}A(x-\bar{x})\cdot(x-\bar{x}),$$

with

$$r(t,x) = ((t-\bar{t})^2 + |x-\bar{x}|^4)^{\frac{1}{2}}, \;\; (t,x)\in Q_0,$$

$$F(r) = \frac{2}{3r}\int_r^{2r}\int_\xi^{2\xi} h(\rho)d\rho d\xi, \quad r>0,$$

and $F(0)=0$. We claim that $w \in C^{1,2}(\overline{Q}_0)$. We will verify this by a straightforward computation. So to simplify the notation we take $(\bar{t},\bar{x}) = (0,0)$. Since by the monotonicity of $h, h(\rho) \le h(4r)$ for every $\rho \in [0,4r]$, we obtain

$$0 \le F(r) \le rh(4r).$$

Also for $r>0$,

$$F_r(r) = \frac{4}{3r}\int_{2r}^{4r} h(\xi)d\xi - \frac{2}{3r}\int_r^{2r} h(\xi)d\xi - \frac{1}{r}F(r),$$

$$F_{rr}(r) = \frac{2}{3r}[8h(4r) - 6h(2r) + h(r)] - \frac{2}{r}F_r(r).$$

Again the monotonicity and positivity of h yield

$$|F_r(r)| \le \frac{8}{3}h(4r),$$

$$|F_{rr}(r)| \le \frac{28}{3r}h(4r).$$

Hence F and F_r continuous at $r=0$ with $F(0)=F_r(0)=0$. We now calculate that for $(t,x)\neq(0,0)$,

$$\frac{\partial}{\partial t}F(r(t,x)) = \frac{t}{r(t,x)}F_r(r(t,x)),$$

$$\frac{\partial}{\partial x_i}F(r(t,x)) = \frac{2|x|^2 x_i}{r(t,x)}F_r(r(t,x))$$

and since $|t| \le r(t,x)$ and $|x|^2|x_i| \le r(t,x)$,

$$\lim_{(t,x)\to(0,0)} \frac{\partial}{\partial t} F(r(t,x)) = 0,$$

$$\lim_{(t,x)\to(0,0)} \frac{\partial}{\partial x_i} F(r(t,x)) = 0.$$

Also for $(t,x) \neq (0,0)$,

$$\frac{\partial^2}{\partial x_i \partial x_j} F(r(t,x)) = \frac{4|x|^4}{r^2} x_i x_j F_{rr}(r(t,x))$$
$$+ \left(\frac{2|x|^2}{r} \delta_{i,j} + \frac{4x_i x_j t^2}{r^3} \right) F_r(r(t,x)),$$

where $\delta_{i,j} = 0$ for $i \neq j$ and $\delta_{i,i} = 1$. Since $|x|^2, |t| \leq r(t,x)$, the previous estimates of F_r and F_{rr} yield

$$\left| \frac{\partial^2}{\partial x_i \partial x_j} F(r(t,x)) \right| \leq 4r|F_{rr}(r)| + 6|F_r(r)| \leq \frac{160}{3} h(4r(t,x)).$$

Therefore $F(r(t,x)) \in C^{1,2}(\overline{Q}_0)$ with zero time derivative, gradient and Hessian at $(t,x) = (0,0)$. This also implies that w satisfies (4.4).

We claim that $W - w$ achieves its maximum at $(\bar{t}, \overline{x})$. Indeed the monotonicity of h yields

$$\begin{aligned} F(r) &\geq \frac{2}{3r} \int_r^{2r} \int_\xi^{2\xi} h(r) d\rho d\xi \\ &= \frac{2}{3r} \int_r^{2r} \xi h(r) d\xi \\ &= r h(r). \end{aligned} \tag{4.5}$$

Set

$$Q(t,x) = W(\bar{t}, \overline{x}) + q(t - \bar{t}) + p \cdot (x - \overline{x}) + \frac{1}{2} A(x - \overline{x}) \cdot (x - \overline{x}).$$

The definition of h implies that

$$W(t,x) - Q(t,x) \leq r(t,x) h(r(t,x)).$$

In view of (4.5), this gives

$$\begin{aligned} w(t,x) &= F(r(t,x)) + Q(t,x) \\ &\geq r(t,x) h(r(t,x)) + Q(t,x) \\ &\geq W(t,x), \qquad \forall (t,x) \in \overline{Q}. \end{aligned}$$

We complete the proof of this lemma, after observing that $w(\bar{t},\bar{x}) = W(\bar{t},\bar{x})$.

The second statement is proved in exactly the same way □

An immediate corollary of the above result is this.

Proposition 4.1. *Let $\mathcal{H}$ be continuous. Then, $W \in C(\overline{Q})$ is a viscosity subsolution of* (4.1) *in Q if and only if* (4.2) *holds for all $(t,x) \in Q$. Similarly $W \in C(\overline{Q})$ is a viscosity supersolution of* (4.1) *in Q if and only if* (4.3) *holds for all $(t,x) \in Q$.*

V.5 Semiconvex, concave approximations

In this section we introduce an approximation procedure of Lasry and Lions [LL] and Jensen [J]. These approximations are used in an essential way in the comparison proof. In particular, they are used in the proof of Crandall-Ishii lemma stated in the next section. We record their definition and few basic properties as they are powerful tools in the analysis of optimal control problems in addition to their central role in the proof of comparison.

We recall that Q is assumed to be bounded. For $\varepsilon > 0, W \in C(\overline{Q})$, and $(t,x) \in \overline{Q}$ define

$$(5.1) \qquad W^\varepsilon(t,x) = \sup\left\{ W(s,y) - \frac{1}{2\varepsilon^2}(|t-s|^2 + |x-y|^2) : (s,y) \in \overline{Q} \right\},$$

$$(5.2) \qquad W_\varepsilon(t,x) = \inf\left\{ W(s,y) + \frac{1}{2\varepsilon^2}(|t-s|^2 + |x-y|^2) : (s,y) \in \overline{Q} \right\}.$$

Now let $(\bar{s},\bar{y}) \in \overline{Q}$ be a maximizer of (5.1). Then

$$W^\varepsilon(t,x) = W(\bar{s},\bar{y}) - \frac{1}{2\varepsilon^2}((t-\bar{s})^2 + |x-\bar{y}|^2) \geq W(t,x).$$

Therefore,

$$|t-\bar{s}|^2 + |x-\bar{y}|^2 \leq 4\|W\|\varepsilon^2$$

and

$$0 \leq W^\varepsilon(t,x) - W(t,x) \leq W(\bar{s},\bar{y}) - W(t,x).$$

Hence W^ε converges uniformly to W on $\overline{Q}$ as $\varepsilon \to 0$. Similarly we show that W_ε converges uniformly to W on $\overline{Q}$.

Lemma 5.1. *On $\overline{Q}$, W^ε and W_ε are semiconvex, and semiconcave, respectively. Moreover, W^ε and W_ε converge to W uniformly on $\overline{Q}$ as $\varepsilon \to 0$.*

Proof. We claim that

$$w^\varepsilon(t,x) = W^\varepsilon(t,x) + \frac{1}{2\varepsilon^2}(t^2 + |x|^2)$$

is convex on every convex subset of $\overline{Q}$. Indeed let $(t+h, x+z)$, $(t-h,\, x-z)$, $(t,x) \in \overline{Q}$, and $(\overline{s},\overline{y})$ be a maximizer of (5.1). Using the definition of W^ε, we obtain

$$W^\varepsilon(t \pm h, x \pm z) \geq W(\overline{s},\overline{y}) - \frac{1}{2\varepsilon^2}(|t \pm h - \overline{s}|^2 + |x \pm z - \overline{y}|^2).$$

Therefore,

$$\begin{aligned}
&w^\varepsilon(t+h,x+z) + w^\varepsilon(t-h,x-z) - 2w^\varepsilon(t,x)\\
&\geq \frac{1}{2\varepsilon^2}\{[(t+h)^2 + (t-h)^2 - 2t^2]\\
&\quad - [(t+h-\overline{s})^2 + (t-h-\overline{s})^2 - 2(t-\overline{s})^2]\}\\
&\quad + \frac{1}{2\varepsilon^2}\{[|x+z|^2 + |x-z|^2 - 2|x|^2]\\
&\quad - [|x+z-\overline{y}|^2 + |x-z-\overline{y}|^2 - 2|x-\overline{y}|^2]\}\\
&= 0.
\end{aligned}$$

Thus w^ε is convex. The properties of W_ε are proved similarly. □

For $\gamma > 0$, set $k_0 = (1 + 4\|W\|)^{\frac{1}{2}}$, and

$$O_\gamma = \{x \in O: \text{ distance}(x, \partial O) > \gamma\}, \quad Q_\gamma = [t_0+\gamma, t_1-\gamma) \times O_\gamma.$$

Lemma 5.2. *Fix $\varepsilon > 0$. For each $(\overline{t},\overline{x}) \in Q_{\varepsilon k_0}$, there exists $(\overline{s},\overline{y}) \in Q$ such that*

$$D^{+(1,2)}W^\varepsilon(\overline{t},\overline{x}) \subset D^{+(1,2)}W(\overline{s},\overline{y}).$$

Proof. Let $(\overline{s},\overline{y})$ be a maximizer of (5.1) at $(\overline{t},\overline{x})$. Then,

$$W(\overline{s},\overline{y}) - \frac{1}{2\varepsilon^2}(|\overline{t}-\overline{s}|^2 + |\overline{x}-\overline{y}|^2) \geq W(\overline{t},\overline{x}).$$

Therefore

$$|\overline{t}-\overline{s}|^2 + |\overline{x}-\overline{y}|^2 < \varepsilon^2 k_0^2.$$

Since $(\overline{t},\overline{x}) \in Q_{\varepsilon k_0}, (\overline{s},\overline{y}) \in (t_0,t_1) \times O$. Therefore for every (t,x) sufficiently near $(\overline{t},\overline{x})$, the point $(\overline{s}+t-\overline{t}, \overline{y}+x-\overline{x}) \in Q$. In particular,

$$W^\varepsilon(t,x) \geq W(\overline{s}+t-\overline{t}, \overline{y}+x-\overline{x}) - \frac{1}{2\varepsilon^2}[|\overline{s}-\overline{t}|^2 + |\overline{y}-\overline{x}|^2].$$

Suppose that $w \in C^{1,2}(\overline{Q})$ and $W^\varepsilon - w$ attains its maximum at $(\overline{t}, \overline{x})$. Since $(\overline{s}, \overline{y})$ maximizes (5.1), this inequality implies that

$$W(\overline{s} + t - \overline{t}, \overline{y} + x - \overline{x}) - w(t, x)$$

also attains its maximum at $(\overline{t}, \overline{x})$. In view of Lemma 4.1, this completes the proof of the lemma. □

Proceeding exactly as in the above proof, we obtain the following dual result.

Lemma 5.3. *For each $(\overline{t}, \overline{x}) \in Q_{\varepsilon k_0}$, there exists and $(\overline{s}, \overline{y}) \in Q$ such that $D^{-(1,2)} W_\varepsilon(\overline{t}, \overline{x}) \subset D^{-(1,2)} W(\overline{s}, \overline{y})$.*

V.6 Crandall-Ishii Lemma

In this section we state an analysis lemma formulated by Crandall and Ishii. It is the cornerstone of the theory of viscosity solution, and is the key result in the comparison proof that will be given in Section 8. This lemma summarizes the analytic part of the original comparison proofs of Jensen and Ishii and is very useful in streamlining the comparison proof. We state it without proof and refer the reader to Theorem 8.3 in the User's Guide Crandall, Ishii and Lions [CIL1] and to the article by Crandall and Ishii [CI].

Theorem 6.1. (Crandall-Ishii maximum principle) *Suppose $W, V \in C(\bar{Q})$ are two semi-continuous functions such that for every $M > 0$ there exist a constant $C = C(M)$ so that for any $(t, x) \in Q$,*

$$(q, p, A) \in cD^{+(1,2)} W(t, x), \quad \|(t, x, p, A, W(t, x))\| \le M, \quad \Rightarrow \quad q \ge -C(M), \tag{6.1}$$

$$(q, p, A) \in cD^{-(1,2)} V(t, x), \quad \|(t, x, p, A, V(t, x))\| \le M, \quad \Rightarrow \quad q \le C(M), \tag{6.2}$$

where $\|\cdot\|$ is the standard Euclidean norm.

Then, for every $\varphi \in C^{1,2}([t_0, t_1] \times \bar{O} \times \bar{O})$, $\epsilon > 0$ and any local maximizer $(t^*, x^*, y^*) \in (t_0, t_1) \times O \times O)$ of the difference $W(t, x) - V(t, y) - \varphi(t, x, y)$, there exist q, $\hat{q} \in I\!R^1$ and symmetric matrices A, B satisfying,

$$\begin{cases} (i)\ (q, p, A) \in cD^{+(1,2)} W(t^*, x^*), \quad p = D_x \varphi(t^*, x^*, y^*), \\ \\ (ii)\ (\hat{q}, \hat{p}, B) \in cD^{-(1,2)} V(t^*, y^*), \quad \hat{p} = -D_y \varphi(t^*, x^*, y^*), \\ \\ (iii)\ q - \hat{q} = D_t \varphi(t^*, x^*, y^*), \\ \\ (iv)\ -\left(\dfrac{1}{\epsilon} + \|X\|\right) I \le \begin{bmatrix} A & 0 \\ 0 & -B \end{bmatrix} \le X + \epsilon X^2, \end{cases} \tag{6.3}$$

where I is the identity matrix with appropriate dimension, $X = D^2_{x,y}\varphi(t^*, x^*, y^*)$ is a $2n \times 2n$ symmetric matrix,

$$\|X\| = \sup\{\ X\eta \cdot \eta \ :\ |\eta| = 1\ \},$$

and the above inequalities are in the sense of symmetric matrices.

Remark 6.1. Several remarks about the proof and the use of the above important result are in order.

1. Conditions (6.1) and (6.2) are stated with reverse inequalities in the User's Guide. However, we immediately obtain the above result from Theorem 8.3 of the User's Guide by considering the functions $w(t,x) := W(-t,x)$ and $v(t,x) := V(-t,x)$. These conditions are needed only in the parabolic case. Notice that in the above theorem, while we are only doubling up the spatial variables x and y but not the time variable t. This is the reason for the additional conditions (6.1) and (6.2). They are used to obtain bounds in the limiting arguments.

2. An elliptic version of the Crandall-Ishii lemma is available and is very useful, see the User's Guide [CIL1] Theorem 3.2. The elliptic version holds true under minimal assumptions. In particular, conditions like (6.1) and (6.2) are not needed for the elliptic problem.

3. The unusual form of (6.3)(iv) is extremely important and it follows from the following classical observation. Suppose that W and V are smooth functions. Set $\Phi(t,x,y) := W(t,x) - V(t,y)$, and apply the classical maximum principle from calculus to the difference $\Phi - \varphi$. The result is

$$D^2_{x,y}\Phi(t^*, x^*, y^*) = \begin{bmatrix} D^2W(t^*, x^*) & 0 \\ 0 & -D^2V(t^*, y^*) \end{bmatrix} \le D^2_{x,y}\varphi(t^*, x^*, y^*).$$

4. We will use Crandall-Ishii lemma with $\varphi(x,y) = \alpha|x-y|^2/2 + h(t)$, with a smooth function h. Then,

$$X = D^2_{x,y}\varphi(x^*, y^*) = \alpha \begin{bmatrix} I & -I \\ -I & I \end{bmatrix},$$

and $X^2 = 2\alpha^2 X$, $\|X\| = 2\alpha$. Further choosing $\epsilon = 1/\alpha$ in (6.3)(iv) yields a symmetric inequality

$$-3\alpha \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \le \begin{bmatrix} A & 0 \\ 0 & -B \end{bmatrix} \le 3\alpha \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}. \tag{6.4}$$

5. Finally, we note that since the matrix on the right hand side of the above inequality annihilates all vectors of the form (η, η), this inequality implies that $A \le B$. Indeed,

$$A\eta \cdot \eta - B\eta \cdot \eta = \begin{bmatrix} A & 0 \\ 0 & -B \end{bmatrix} \begin{bmatrix} \eta \\ \eta \end{bmatrix} \cdot \begin{bmatrix} \eta \\ \eta \end{bmatrix} \le \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} \begin{bmatrix} \eta \\ \eta \end{bmatrix} \cdot \begin{bmatrix} \eta \\ \eta \end{bmatrix} = 0.$$

□

In all uses of the Crandall-Ishii lemma, W is a subsolution and V is a supersolution of the same equation. In this case, the conditions (6.1) and (6.2) follow from the properties of the equation. We close this section proving this statement for the sub and supersolutions of the dynamic programming equation (4.1).

Lemma 6.1. *Assume $\mathcal{H}$ in (4.1) is continuous. Then, every continuous viscosity subsolution W of (4.1) satisfies (6.1). Also, every continuous viscosity supersolution V of (4.1) satisfies (6.2).*

Proof. This is an immediate consequence of the continuity of $\mathcal{H}$ and (4.2) or (4.3). Indeed, we simply take

$$C(M) = \sup\{\ |\mathcal{H}(t,x,p,A,w)|\ :\ \|(t,x,p,A,w)\| \le M\ \}.$$

□

V.7 Properties of $\mathcal{H}$

In this section, we prove an elementary property of the nonlinear function $\mathcal{H}$ defined in (2.12). Recall that we always assume that Q is bounded.

Lemma 7.1. *Let $\mathcal{H}$ be as in (2.12). Assume IV(2.2) and that $L(\cdot,\cdot,v) \in C^1(\overline{Q})$. Then, there exists a continuous function $\omega : [0,\infty) \to [0,\infty)$ that satisfies $\omega(0) = 0$ such that*

$$\mathcal{H}(t,y,\alpha(x-y),B) - \mathcal{H}(t,x,\alpha(x-y),A) \le \omega(\alpha|x-y|^2 + |x-y|), \tag{7.1}$$

for every $(t,x),(t,y) \in Q$, $\alpha > 0$, and symmetric matrices A, B satisfying (6.4).

Proof. Recall that

$$\mathcal{H}(t,x,p,A) = \sup_{v\in U}\left\{-\frac{1}{2}\mathrm{tr}(\sigma\sigma')(t,x,v)A - f(t,x,v)\cdot p - L(t,x,v)\right\}.$$

Set $p_\alpha = \alpha(x-y)$, $D = \sigma(t,x,v)$, $C = \sigma(t,y,v)$ so that

$$\begin{aligned}\mathcal{H}(t,y,\alpha(x-y),B) - \mathcal{H}(t,x,\alpha(x-y),A) &\le \sup_{v\in U}\left\{\frac{1}{2}\mathrm{tr}(DD'A - CC'B)\right\}\\ &\quad + \sup_{v\in U}\{|f(t,x,v) - f(t,y,v)|\ |p_\alpha| + |L(t,x,v) - L(t,y,v)|\}.\end{aligned}$$

By assumption IV(2.2),

$$\begin{aligned}|f(t,x,v) - f(t,y,v)|\ |p_\alpha| + |L(t,x,v) - L(t,y,v)| &\le C|x-y||p_\alpha| + C|x-y|\\ &= C[\alpha|x-y|^2 + |x-y|].\end{aligned}$$

We now use (6.4) to obtain,

$$\begin{aligned}
\mathrm{tr}(DD'A - CC'B) &= \mathrm{tr}\left(\begin{bmatrix} DD' & DC' \\ CD' & CC' \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & -B \end{bmatrix}\right) \\
&\leq 3\alpha\ \mathrm{tr}\left(\begin{bmatrix} DD' & DC' \\ CD' & CC' \end{bmatrix}\begin{bmatrix} I & -I \\ -I & I \end{bmatrix}\right) \\
&= 3\alpha\ \mathrm{tr}(DD' - DC' - CD' + CC') \\
&= 3\alpha\ \mathrm{tr}([D-C][D'-C']) \\
&= 3\alpha\ \|D-C\|^2 = 3\alpha\ \|\sigma(t,x,v) - \sigma(t,y,v)\|^2 \\
&\leq C\alpha|x-y|^2.
\end{aligned}$$

These inequalities yield (7.1) with $\omega(r) = Cr$. □

V.8 Comparison

In this section we will prove a general comparison result for second order partial differential equations under the structural assumption (7.1). In particular, this comparison result applies to the dynamic programming equation (4.1) provided that the nonlinearity $\mathcal{H}$ for the dynamic programming equation for controlled Markov diffusions defined in (2.12) satisfies (7.1). Therefore, by Lemma 7.1, the equation (4.1) has comparison under the assumption IV(2.2). However, since the assumption IV(2.2) yields (7.1) with a linear ω, comparison for (4.1) can be proved under assumptions that are considerably weaker than IV(2.2).

Recall that $Q = (t_0, t_1) \times O$ and O is open and bounded.

Theorem 8.1. *Suppose that* $\mathcal{H}$ *is continuous and satisfies* (7.1). *Let* $W \in C(\overline{Q})$ *be a viscosity subsolution of* (4.1) *in* Q, *and* $V \in C(\overline{Q})$ *be a viscosity supersolution of* (4.1) *in* Q. *Then*

$$\sup_{\overline{Q}}(W - V) = \sup_{\partial^* Q}(W - V). \tag{8.1}$$

Proof. Suppose to the contrary, i.e.,

$$\sup_{\overline{Q}}(W - V) - \sup_{\partial^* Q}(W - V) > 0.$$

1. For $\rho > 0$ define

$$W^\rho(t,x) = W(t,x) - \frac{\rho}{t - t_0}, \ \ (t,x) \in Q.$$

Since

$$\frac{d}{dt}\left(-\frac{\rho}{t-t_0}\right) = \frac{\rho}{(t-t_0)^2} > 0,$$

W^ρ is a viscosity subsolution of (4.1) in Q.

2. For $\rho, \beta, \alpha > 0$ consider the auxiliary function

$$\Phi(t,x,y) = W^\rho(t,x) - V(t,y) - \alpha|x-y|^2 + \beta(t-t_1), \qquad t \in [t_0,t_1], x,y \in \overline{O}.$$

Note that W, V are continuous on $\overline{Q}$. Now proceed as in Step 2 of Theorem II.9.1 to construct β_0, ρ_0, $\alpha_0 > 0$ such that for all positive $\beta \le \beta_0$, $\rho \le \rho_0$, $\alpha \ge \alpha_0$ we have

$$\sup_{[t_0,t_1]\times\overline{O}\times\overline{O}} \Phi > \sup_{\partial[(t_0,t_1)\times\overline{O}\times\overline{O}]} \Phi.$$

Let $(\bar{t},\overline{x},\overline{y}) \in (t_0,t_1)\times O \times O$ be a local maximum of Φ. Then the hypotheses of Theorem 6.1 are satisfied with W^ρ, V and

$$\varphi(t,x,y) = \alpha|x-y|^2 - \beta(t-t_1).$$

In view of Lemma 6.1, and Remark 6.1(3), we conclude that (6.3) and (6.4) are satisfied with symmetric matrices A and B, i.e.,

$$(q, p_\alpha, A) \in cD^{+(1,2)}W^\rho(\bar{t},\overline{x}), \qquad (\hat{q}, p_\alpha, B) \in cD^{-(1,2)}V(\bar{t},\overline{y}),$$

where

$$p_\alpha = \frac{1}{\alpha}(\overline{x}-\overline{y}), \qquad q - \hat{q} = \varphi_t(\bar{t},\overline{x},\overline{y}) = -\beta.$$

3. Viscosity properties of W^ρ and V yield,

$$-q + \mathcal{H}(\bar{t},\overline{x},p_\alpha,A) \le 0$$

and

$$-\hat{q} + \mathcal{H}(\bar{t},\overline{y},p_\alpha,B) \ge 0.$$

Recall that $\hat{q} - q = \beta$, and A, B satisfy (6.4). Hence, by Lemma 7.1, (7.1) is satisfied. Subtract the above inequalities and then use (7.1). The result is

$$(8.2) \quad \beta = \hat{q} - q \le \mathcal{H}(\bar{t},\overline{y},p_\alpha,B) - \mathcal{H}(\bar{t},\overline{x},p_\alpha,A) \le \omega(\alpha|\overline{x}-\overline{y}|^2 + |\overline{x}-\overline{y}|).$$

4. Set,

$$h(r) = \sup\{|V(t,x) - V(t,y)| : x,y \in \overline{O}, t \in [t_0,t_1],\ |x-y|^2 \le r\},$$

so that for any $x,y \in \overline{O}$, $t \in [t_0,t_1]$,

$$|V(t,x) - V(t,y)| \le h(|x-y|^2).$$

Since $(\bar{t},\overline{x},\overline{y})$ maximizes Φ over $[t_0,t_1]\times\overline{O}\times\overline{O}$,

$$\begin{aligned}\Phi(\bar{t},\overline{x},\overline{y}) &= W^\rho(\bar{t},\overline{x}) - V(\bar{t},\overline{y}) - \alpha|\overline{x}-\overline{y}|^2 + \beta(\bar{t}-t_0)\\ &\ge \Phi(\bar{t},\overline{x},\overline{x})\\ &= W^\rho(\bar{t},\overline{x}) - V(\bar{t},\overline{x}) + \beta(\bar{t}-t_0).\end{aligned}$$

Hence,

$$\alpha|\overline{x}-\overline{y}|^2 \leq V(\bar{t},\overline{x}) - V(\bar{t},\overline{y}) \leq h(|\overline{x}-\overline{y}|^2).$$

Since, $V \in C(\bar{Q})$ and O is bounded, h is bounded by some constant K. This implies that

$$\alpha|\overline{x}-\overline{y}|^2 \leq K.$$

The definition of h yields,

$$\alpha|\overline{x}-\overline{y}|^2 + |\overline{x}-\overline{y}| \leq h(|\overline{x}-\overline{y}|^2) + \sqrt{(K/\alpha)} \leq h(K/\alpha) + \sqrt{(K/\alpha)}.$$

Substitute this into (8.2) to obtain

$$\beta \leq k(\alpha) = \omega(h(K/\alpha) + \sqrt{(K/\alpha)}\ ).$$

It is clear that

$$\lim_{\alpha\uparrow\infty} k(\alpha) = 0.$$

But this contradicts with the fact that $\beta > 0$. □

The following uniqueness result is an immediate consequence of Theorem 8.1.

Corollary 8.1. (Uniqueness). *Assume* (7.1)*. Then there is at most one viscosity solution* $V \in C(\overline{Q})$ *of* (4.1) *in* Q *satisfying the boundary and terminal conditions*

$$V(t,x) = g(t,x), \quad (t,x) \in [t_0,t_1) \times \partial O, \tag{8.8a}$$

$$V(t_1,x) = \psi(x), \quad x \in \overline{O}. \tag{8.8b}$$

In particular, in (4.1) we may take V to be the value function of the stochastic control problem defined in Chapter IV. Then (4.1) becomes the dynamic programming equation IV(3.3). If $V \in C(\overline{Q})$ satisfies (8.8) and the dynamic programming principle (2.2), then it is the unique viscosity solution of IV(3.3). See Section 2 and Corollary 3.1.

Remark 8.1. Discontinuous viscosity sub- and supersolutions of (4.1) will be defined in Chapter VII (see Definition VII.4.2 and Remark VII.4.1). An attendant modification of this chapter also yields a comparison result for discontinuous sub- and supersolutions. More precisely: suppose W and V are a viscosity subsolution and a viscosity supersolution of (4.1) in Q, respectively. Assume (7.1) and that W, V are upper semicontinuous and lower semicontinuous on $\overline{Q}$, respectively. Then (8.1) holds. In particular $W \leq V$ on $\overline{Q}$ if this inequality holds on $\partial^* Q$. The elliptic version of the comparison between semicontinuous sub- and supersolutions was proved by Ishii [I1, Theorem 3.3]. Also the extension of the comparison for semicontinuous sub- and supersolutions on Q_0 can be proved exactly as it will be done for continuous solutions in the next section.

Remark 8.2. As in the first-order case if $\mathcal{H}$ depends on $V(t,x)$, a minor modification of the above proof yields results similar to (8.1) and Corollary 8.1. Indeed, if $\mathcal{H}$ is nondecreasing in $V(t,x)$, then we have II(9.26). Also, a result entirely similar to Proposition II.9.1 holds for second-order equations.

V.9 Viscosity solutions in Q_0

In Section IV.7 we have shown that the value function satisfies the dynamic programming principle, IV(7.1). In addition to IV(2.2), let us now assume that

$$\text{(9.1a)} \qquad U \text{ is compact,}$$

$$\text{(9.1b)} \qquad f, \sigma, L, L_x, \text{ and } L_t \text{ are bounded on } Q_0 \times U.$$

Then a straightforward modification of results of Section 3 yields that the value function is a uniformly continuous viscosity solution of the dynamic programming equation IV(3.3) in Q_0.

In this section, we will obtain an analogue of Theorem 8.1 for viscosity subsolutions and supersolutions of the dynamic programming equation IV(3.3) in Q_0. See Theorem 9.1 below. This result, in particular, implies that the value function is the unique bounded viscosity solution of IV(3.3) in Q_0 satisfying the terminal data

$$V(t_1, x) = \psi(x), \;\; x \in I\!R^n.$$

Indeed an attendant modification of the proof of Theorem 8.1 yields the comparison result on Q_0. In the case of the first-order equations, an analogous modification is carried out in detail. See Theorem II.9.1. However to acquaint the reader with a different technique, we will prove Theorem 9.1 by using Theorem 8.1 on large balls $B_R = \{x : \; |x| < R\}$ and then letting R go to infinity.

To motivate our analysis, let us first suppose that $W \in C^{1,2}(\overline{Q}_0)$ is a classical solution of IV(3.3) in Q_0. Let $\xi^R \in C^2(B_R)$ be a function satisfying $\xi^R(x) > 0$ whenever $|x| \le R$. Set

$$\hat{W}(t,x) = \xi^R(x) W(t,x), \quad (t,x) \in [t_0, t_1] \times \overline{B}_R.$$

We directly calculate that

$$\hat{W}_t = \xi^R W_t, \quad \hat{W}_{x_i} = \xi^R W_{x_i} + \xi^R_{x_i} W,$$

$$\hat{W}_{x_i x_j} = \xi^R W_{x_i x_j} + \xi^R_{x_i} W_{x_j} + \xi^R_{x_j} W_{x_i} + \xi^R_{x_i x_j} W.$$

Multiplying IV(3.3) by $\xi^R(x)$ we obtain

$$\text{(9.2)} \qquad -\xi^R(x) W_t(t,x) + \sup_{v \in U} \left\{ -\xi^R(x) \left[\frac{1}{2} \operatorname{tr}(\sigma\sigma')(t,x,v) D_x^2 W(t,x) \right.\right.$$
$$\left.\left. + f(t,x,v) \cdot D_x W(t,x) + L(t,x,v) \right] \right\} = 0.$$

Observe that

$$\xi^R W_{x_i} = \hat{W}_{x_i} - W\xi^R_{x_i}$$

$$\xi^R W_{x_i x_j} = \hat{W}_{x_i x_j} - W\xi^R_{x_i x_j} - \xi^R_{x_i} W_{x_j} - \xi^R_{x_j} W_{x_i}$$

$$= \hat{W}_{x_i x_j} - W\xi^R_{x_i x_j} - \frac{\xi^R_{x_i}}{\xi^R}\hat{W}_{x_j} - \frac{\xi^R_{x_j}}{\xi^R}\hat{W}_{x_i} + 2W\frac{\xi^R_{x_i}\xi^R_{x_j}}{\xi^R}.$$

Substitute these into (9.2) to obtain

$$(9.3) \qquad -\hat{W}_t(t,x) + H^R_W(t,x,D_x\hat{W}(t,x),D^2_x\hat{W}(t,x)) = 0, \quad (t,x) \in Q_R$$

where $Q_R = [t_0,t_1) \times B_R$ and

$$H^R_W(t,x,p,A) = \sup_{v\in U}\left\{ -\frac{1}{2}\operatorname{tr}(\sigma\sigma')(t,x,v)A - f^R(t,x,v)\cdot p - L^R_W(t,x,v)\right\},$$

with

$$f^R(t,x,v) = f(t,x,v) - \frac{1}{\xi^R(x)}(\sigma\sigma^{'})(t,x,v)D\xi^R(x),$$

$$L^R_W(t,x,v) = \xi^R(x)L(t,x,v) - W(t,x)\Big[f(t,x,v)\cdot D\xi^R(x)$$

$$+\frac{1}{2}\operatorname{tr}(\sigma\sigma')(t,x,v)D^2\xi^R(x) - \frac{1}{\xi^R(x)}(\sigma\sigma')(t,x,v)D\xi^R(x)\cdot D\xi^R(x)\Big].$$

Lemma 9.1. *Let $W \in C(\overline{Q}_0)$ be a viscosity subsolution (or supersolution) of* IV(3.3) *in Q_0. Then $\hat{W}$ is a viscosity subsolution (or supersolution, respectively) of* (9.3) *in Q_R.*

Proof. Suppose that $\hat{w} \in C^\infty(\overline{Q}_R)$ and $\hat{W} - \hat{w}$ has a maximum at $(\bar{t},\bar{x}) \in Q_R$, satisfying $(\hat{W} - \hat{w})(\bar{t},\bar{x}) = 0$. Set $w(t,x) = \hat{w}(t,x)/\xi^R(x)$. Then $w \in C^\infty(Q_R)$ and $W - w$ has a maximum at $(\bar{t},\bar{x}) \in Q_R$. Since W is a viscosity subsolution of (4.1),

$$-w_t(\bar{t},\bar{x}) + H(\bar{t},\bar{x},D_x w(\bar{t},\bar{x}),D^2_x w(\bar{t},\bar{x})) \le 0.$$

Also $\hat{w} = \xi^R w$. So the calculations preceding the lemma yield

$$-\hat{w}_t(\bar{t},\bar{x}) + H^R_w(\bar{t},\bar{x},D_x\hat{w}(\bar{t},\bar{x}),D^2_x\hat{w}(\bar{t},\bar{x})) \le 0.$$

Since $\hat{W}(\bar{t},\bar{x}) = \hat{w}(\bar{t},\bar{x})$, we have $W(\bar{t},\bar{x}) = w(\bar{t},\bar{x})$ and consequently

$$H^R_w(\bar{t},\bar{x},p,A) = H^R_W(\bar{t},\bar{x},p,A).$$

Hence $\hat{W}$ is a viscosity subsolution of (9.3) in Q_R. The supersolution property is proved similarly. □

Theorem 9.1. *Assume* IV(2.2) *and* (9.1)*. Let $W \in C(\overline{Q}_0)$ be a bounded viscosity subsolution of* (4.1) *in Q_0 and $V \in C(\overline{Q}_0)$ be a bounded viscosity supersolution of* (4.1) *in Q_0. Then,* (9.4) $$\sup_{\overline{Q}_0}(W-V) = \sup_{I\!R^n}(W(t_1,y)-V(t_1,y)).$$

Proof. Set
$$\xi^R(x) = \left(\frac{|x|^2}{R^2}-1\right)^2 + \frac{1}{R}, \quad |x|<R.$$
Then $\xi^R \in C^2(B_R)$ and for a suitable constant K_0
$$|D\xi^R(x)| \le \frac{K_0}{R}, \quad |D^2\xi^R(x)| \le \frac{K_0}{R^2}, \quad \frac{|D\xi^R(x)|^2}{\xi^R(x)} \le \frac{K_0}{R^2}.$$
Hence the definition of L_W^R and the boundedness of W, f and σ yield
$$|L_W^R(t,x,v) - \xi^R(x)L(t,x,v)| \le \frac{K_1}{R},$$
with a constant K_1 depending on the sup norm of W, f and σ. For $(t,x) \in Q_R, p \in I\!R^n$ and a symmetric matrix A, set $W^R(t,x) = \xi^R(x)W(t,x) - K_1(t-t_1)/R$ and
$$H^R(t,x,p,A) = \sup_{v\in U}\left\{-f^R(t,x,v)\cdot p - \mathrm{tr}[(\sigma\sigma')(t,x,v)A] - \xi^R(x)L(t,x,v)\right\}.$$
Then $W^R(t,x)$ is a viscosity subsolution of
$$-W_t^R(t,x) + H^R(t,x,D_xW^R(t,x),D_x^2W^R(t,x)) \le 0, \quad (t,x)\in Q_R.$$
Similarly there exists K_2 such that
$$|L_V^R(t,x,v) - \xi^R(x)L(t,x,v)| \le \frac{K_2}{R}$$
and $V^R(t,x) = \xi^R(t,x)V(t,x) + K_2(t-t_1)/R$ is a viscosity supersolution of
$$-V_t^R(t,x) + H^R(t,x,D_xV^R(t,x),D_x^2V^R(t,x)) \ge 0, \quad (t,x)\in Q_R.$$
Notice that $\xi^R \ge 1/R$ and therefore f^R satisfies the hypotheses of Lemma 8.1. Therefore H^R satisfies (8.1) and by Theorem 8.1 we obtain

(9.5) $$\sup_{\overline{Q}_R}(W^R - V^R) = \sup_{\partial^* Q_R}(W^R - V^R).$$

Also as R tends to infinity, W^R, V^R converge to W and V uniformly on bounded subsets of Q_0, and
$$\sup_{[t_0,t_1]\times\partial B_R}(W^R - V^R) \le \frac{1}{R}((K_1+K_2)(t_1-t_0) + \|W\| + \|V\|).$$
Hence we obtain (9.4) by letting R go to infinity in (9.5). □

V.10 Historical remarks

The first treatment of viscosity solutions of second-order dynamic programming equations was given by Lions [L1-2]. Lions proved that any viscosity solution is the value function of the related stochastic optimal control problem. For general second-order equations which are not necessarily dynamic programming equations, this technique is clearly not appropriate. Jensen was first to prove a uniqueness result for a general second-order equation [J]. In [J] semiconvex and concave approximations of a function were given by using the distance to the graph of the function. Afterward it was observed ([JLS]) that the construction given by Jensen is the same as the sup and inf convolutions as defined by Lasry and Lions [LL]. Another important step in the development of the second-order problems is Ishii's lemma [I1]. Since then the proofs and the statements of the results have been greatly improved. In particular, the analysis result of Crandall and Ishii [CI], which we quote as Theorem 6.1, have been used in almost all comparison results. We refer the reader to the survey article of Crandall, Ishii and Lions [CIL1] and Crandall [Cr] for more information.

The proof of the dynamic programming principle given in Section 2 is taken from [L2]. Krylov [Kr1] and Bensoussan-J. Lions [BL1] prove the dynamic programming principle by a space discretization. Their method requires the continuity of the value function. Borkar [Bo] proves the dynamic programming by considering Markov policies. El Karoui et al. [ENJ] and Kurtz [Kz] first prove that the set of all control processes is compact. Then for each initial data they choose an optimal control process. Moreover the compactness of the control processes makes it possible to make this selection in a measurable way. Their approach requires the convexification of the set of control processes. In the discrete-time setup Bertsekas and Shreve [BsS] prove the dynamic programming by using a deep measurable selection theorem of Brown and Purves. A similar approach in the continuous time case is also possible. Indeed this approach was recently used to prove a geometric dynamic programming by Soner and Touzi [ST1].

Several interesting questions arise in defining viscosity solutions for second order equations. We refer to [CCKS] and references therein. For uniformly elliptic equations, an elegant regularity theory has been established by Caffarelli [Caf], also see the book by Cabre and Caffarelli [CC]. Several important research areas related to second order equations are not covered in this book. We refer to Evans & Spruck [ES] and Chen, Giga & Goto [CGG] for level set equations and to Ambrosio & Soner [AS] for level set applications in higher co-dimensions. For homogenization, see Evans [E3], Souganidis [Sou2]. For viscosity solutions of stochastic partial differential equations, we refer to Lions & Souganidis [LSo2,3] and Buckdahn & Ma [BM]. For effective Hamiltionians and weak KAM theory, see Evans & Gomes [EG1,2] Fathi & Siconolfi [FS] and the references therein. Evans, Soner & Souganidis [ESS] and Barles, Soner &

Souganidis [BSS] used viscosity solutions in analyzing an evolution problem related to the Ginzburg-Landau functional.

VI

Logarithmic Transformations and Risk Sensitivity

VI.1 Introduction

In this chapter we are concerned with the idea of risk sensitivity and its connections with stochastic control. If $\mathcal{J}$ is some cost function associated with a random variable, then not all values of $\mathcal{J}$ may be equally significant. For example, large values of $\mathcal{J}$ might be given greater weight. The expectation of a nonlinear function $F(\mathcal{J})$ takes this into account. We take $F(\mathcal{J}) = \exp(\rho\mathcal{J})$ of exponential form, where the parameter ρ is a measure of risk sensitivity. Equivalently, one can consider the certainty-equivalent expectation defined in Section 2. It is shown that the certainty-equivalent expectation can be rewritten as an ordinary expectation, after a change of probability measure.

In Section 3 and 4 we consider Markov diffusion processes $x(s)$ governed by stochastic differential equations, and take $\mathcal{J}$ of the form (3.4) involving a running cost and a terminal cost. To find the certainty-equivalent expectation, we introduce a logarithmic transformation of the following kind. The expectation of $\exp(\rho\mathcal{J})$ is a function $\Phi(t, x)$ of the initial data for $x(s)$. With Φ is associated the linear PDE (3.16) using the Feynman-Kac formula. The certainty-equivalent expectation is $V(t, x) = \rho^{-1} \log \Phi(t, x)$. It satisfies the nonlinear PDE (3.19), which turns out to be the H-J-B equation for a controlled Markov diffusion process described in Sections 4 and 5.

Logarithmic transformations of this kind have proved useful in studying asymptotic problems which arise when $x(s) = x^\varepsilon(s)$ depends on a small parameter ε and $x^\varepsilon(s)$ tends to a deterministic limit as $\varepsilon \to 0$. Included are problems of "large deviations" of so-called Freidlin-Wentzell type for dynamical systems subject to small random perturbations [FW]. Associated with large deviations analyses are deterministic control problems, which are seen in Section 6 to arise in a natural way from the logarithmic transform method. An example is to estimate asymptotically the probability that $x^\varepsilon(s)$ exits from a given region $O \subset I\!R^n$ during a given time interval.

The theory of H-infinity norms and H-infinity control provides an alternative to stochastic dynamical system and stochastic control models. In the

H-infinity approach, stochastic differential equations are replaced by corresponding ordinary differential equations, in which unknown (deterministic) disturbances take the place of white noises (formal time derivatives of Brownian motions). A concise introduction to the H-infinity approach to nonlinear dynamical systems is given in Section 7, using viscosity solution methods and results from Section 6.

In Section 8 we consider controlled Markov diffusions with expected exponential-of-integral criteria to be minimized. These are called problems of risk sensitive stochastic control, on a finite time horizon. The methods of Chapter IV extend readily to this case. An extension of the "small noise limit" analysis in Section 6 leads to a deterministic differential game, which will be described later in Chapter XI. This analysis again depends on viscosity solution methods.

Finally, in Section 9 we consider logarithmic transformations for other classes of Markov processes, besides Markov diffusions. To illustrate the basic ideas without technical complications, the details are presented only in case of finite-state Markov chains.

VI.2 Risk sensitivity

Let $\mathcal{J}$ denote some random variable, for example a "cost" associated with the sample paths $x(\cdot)$ of some Markov process $x(s)$. The expectation $E^0(\mathcal{J})$ with respect to a probability measure P^0 is then an "average cost" associated with $\mathcal{J}$. However, it may happen that not all possible values of $\mathcal{J}$ are equally significant. For example, large values of $\mathcal{J}$ might be given greater weight. For this reason, we consider in this chapter a risk sensitive criterion $E^0[F(\mathcal{J})]$ where F is some nonlinear function. It is assumed that

$$F'(\mathcal{J}) \neq 0, \qquad F''(\mathcal{J}) \neq 0. \tag{2.1}$$

The following function $r_F(\mathcal{J})$ is a measure of risk sensitivity:

$$r_F(\mathcal{J}) = \frac{|F''(\mathcal{J})|}{|F'(\mathcal{J})|}. \tag{2.2}$$

Large $r_F(\mathcal{J})$ indicates great sensitivity to risk. The *certainty-equivalent expectation* $\mathcal{E}^0(\mathcal{J})$ is defined, as in [BJ3], by:

$$\mathcal{E}^0(\mathcal{J}) = F^{-1}(E^0[F(\mathcal{J})]). \tag{2.3}$$

Example 2.1. Let $F(\mathcal{J}) = \exp(\rho\mathcal{J}),\ \rho \neq 0$, where exp denotes the exponential function. Then $r_F(\mathcal{J}) = |\rho|$ is constant. This exponential function F is the one which will be considered in this chapter. The certainty equivalent expectation is then

$$\mathcal{E}^0(\mathcal{J}) = \rho^{-1} \log E^0[\exp(\rho\mathcal{J})]. \tag{2.4}$$

As $\rho \to 0$, the right side is $E^0(\mathcal{J}) + \frac{\rho}{2}\text{var}(\mathcal{J}) + O(\rho^2)$ where $\text{var}(\mathcal{J})$ is the variance under P^0. Thus, for $|\rho|$ small $\mathcal{E}^0(\mathcal{J})$ is approximately a weighted combination of the mean and variance.

Example 2.2. In mathematical finance applications, the following function F is often used. (See Chapter X.) For $\mathcal{J} > 0$, let

$$F(\mathcal{J}) = \begin{cases} \gamma^{-1}\mathcal{J}^\gamma & \text{if} \quad \gamma < 1, \quad \gamma \neq 0 \\ \log \mathcal{J} & \text{if} \quad \gamma = 0. \end{cases} \tag{2.5}$$

F is called a *hyperbolic absolute risk aversion* (HARA) utility function. The parameter γ is interpreted as a measure of the risk which an investor will accept. In this example

$$r_F(\mathcal{J}) = \frac{1-\gamma}{\mathcal{J}}.$$

Thus, an investor who chooses $1-\gamma$ very large is quite risk averse, and an investor with $\gamma = 1$ is "risk neutral".

In the sections which follow, $x(s)$ will satisfy the stochastic differential equation (3.1) and $\mathcal{J}$ will have the form (3.4). Under certain assumptions, the certainty equivalent expectation $\mathcal{E}^0(\mathcal{J})$ is shown to equal the maximum expected cost in a suitably defined "auxiliary" stochastic control problem. The method depends on changes of probability measure. Let us begin by explaining the basic idea at an abstract level.

We consider changes of probability measure, from P^0 to another P, with Radon-Nikodym derivative of the form

$$\frac{dP}{dP^0} = \exp(\rho\zeta), \tag{2.6}$$

where ζ is some random variable and $\rho \neq 0$ is a constant. Let $\Omega, \mathcal{F}$ be the sample space and σ-algebra of subsets of Ω on which P^0 is defined. Then

$$\int_\Omega \Phi dP = \int_\Omega \Phi \exp(\rho\zeta) dP^0$$

for every nonnegative $\mathcal{F}$-measurable P. In particular, by tking Φ of the form $\Phi = \exp(-\rho\zeta)\Psi$, we see that the Radon-Nikodym derivative of P^0 with respect to P is

$$\frac{dP^0}{dP} = \exp(-\rho\zeta). \tag{2.6$'$}$$

Let E^0 and E denote expectations under P^0 and under P, respectively. By Jensen's inequality and (2.6′)

$$E^0[\exp(\rho\mathcal{J})] = E\{\exp[\rho(\mathcal{J} - \zeta)]\} \geq \exp\{E[\rho(\mathcal{J} - \zeta)]\} \tag{2.7}$$

provided that the expectation $E(\mathcal{J}-\zeta)$ exists. Since the exponential function is strictly convex, equality holds in (2.7) if an only if the random variable $\mathcal{J}-\zeta$ is constant P-almost surely.

Case 1. $\rho > 0$. By (2.4) and (2.7)

$$\mathcal{E}^0(\mathcal{J}) \geq E(\mathcal{J}-\zeta). \tag{2.8}$$

We may think of ζ as a "decision variable" in a stochastic optimization problem, with the goal to maximize $E(\mathcal{J}-\zeta)$. Suppose that there exists ζ^* in the class of admissible decision variables such that $\mathcal{J}-\zeta$ is constant P-almost surely when $\zeta=\zeta^*$. Then $E(\mathcal{J}-\zeta)$ achieves its maximum for $\zeta=\zeta^*$ and $\mathcal{E}^0(\mathcal{J}) = E(\mathcal{J}-\zeta^*)$.

Case 2. $\rho < 0$. In this case $\mathcal{E}^0(\mathcal{J}) \leq E(\mathcal{J}-\zeta)$ with equality when $\mathcal{J}-\zeta$ is constant P-almost surely. The "decision variable" ζ is chosen to minimize $E(\mathcal{J}-\zeta)$, rather than to maximize $E(\mathcal{J}-\zeta)$ as in Case 1.

Minimizing a certainty-equivalent expectation. Let us take $\rho > 0$, and suppose that $\mathcal{J}=\mathcal{J}(\alpha)$ depends also on some random decision variable α. Consider the problem of choosing α in some admissible class to minimize the certainty equivalent expectation $\mathcal{E}^0(\mathcal{J})$. The change of probability measure method outlined above introduces an "auxiliary" decision variable ζ which is chosen to maximize $E(\mathcal{J}-\zeta)$. The problem of minimizing $\mathcal{E}^0(\mathcal{J})$ turns into a min-max problem with criterion $\mathcal{J}-\zeta$. This idea will be mentioned again in Section 8 in the context of risk sensitive stochastic control.

If $\rho < 0$ and α is chosen to maximize $\mathcal{E}^0(\mathcal{J})$, then the change of probability measure method leads to a max-min problem with criterion $\mathcal{J}-\zeta$. In this case, the auxiliary decision variable ζ is minimizing. The case $\rho > 0$ if often called "risk averting" and $\rho < 0$ "risk seeking".

VI.3 Logarithmic transformations for Markov diffusions

In Section 2 a change of probability measure technique was outlined in an abstract setting. Let us now use this technique in the context of Markov diffusion processes, governed by stochastic differential equations (SDEs). The changes in probability measure correspond to changes of drift in the SDE and are obtained using Girsanov's theorem.

Let us take risk sensitivity parameter $\rho > 0$ in Example 2.1. In the notation of Section IV.2, let $\nu^0 = (\Omega, \{\mathcal{F}_s\}, P^0, w^0)$ be a reference probability system. Thus $w^0(\cdot)$ is a d-dimensional brownian motion under probability P^0, which is $\mathcal{F}_s$-adapted. Let $x(s)$ be the (pathwise) solution to the SDE

$$dx = b(s,x(s))ds + \rho^{-\frac{1}{2}}\sigma(s,x(s))dw^0(s), t \leq s \leq t_1, \tag{3.1}$$

with initial data $x(t)=x$. Note that (3.1) corresponds to equation III(5.4) with f replaced by b, σ replaced by $\rho^{-\frac{1}{2}}\sigma$ and $w(s)$ by $w^0(s)$. As in previous chapters, let $Q_0 = [t_0, t_1) \times I\!R^n$. We make the following assumptions:

$$\text{(3.2)} \quad \begin{aligned} &\text{(i) } b \in C^1(\bar{Q}_0) \text{ and } b_x \text{ is bounded on } \bar{Q}_0; \\ &\text{(ii) } \sigma \in C^1(\bar{Q}_0) \text{ and } \sigma, \sigma_x \text{ are bounded on } \bar{Q}_0. \end{aligned}$$

The process $x(\cdot)$ is a Markov diffusion on $I\!R^n$ and its backward evolution operator has the form (see III(5.1))

$$\text{(3.3)} \qquad A\Phi = \Phi_t + \frac{1}{2\rho} \operatorname{tr} a(t,x) D_x^2 \Phi + b(t,x) \cdot D_x \Phi,$$

with $a = \sigma\sigma'$. Let

$$\text{(3.4)} \qquad \mathcal{J} = \int_t^{t_1} \ell(s, x(s)) ds + \psi(x(t_1)).$$

Let us assume throughout that the functions ℓ, ψ are continuous on Q_0 with additional assumptions on ℓ, ψ made later as needed.

We consider changes of probability measure, from P^0 to another probability measure P, via a Girsanov transformation. These changes of probability measure correspond to change of drift in (3.1), from $b(s, x(s))$ to $b(s, x(s)) + \sigma(s, x(s))z(s)$ where $z(s)$ is a $\mathcal{F}_s$-progressively measurable $I\!R^d$-valued process called an auxiliary control process. We write (3.1) as

$$\text{(3.5)} \qquad dx = [b(s, x(s)) + \sigma(s, x(s))z(s)]ds + \rho^{-\frac{1}{2}} \sigma(s, x(s)) dw(s)$$

$$\text{(3.6)} \qquad w(s) = w^0(s) - \rho^{\frac{1}{2}} \int_t^s z(r) dr,$$

on the time interval $[t, t_1]$. Let

$$\text{(3.7)} \qquad \zeta(s) = \int_t^s \left[\rho^{-\frac{1}{2}} z(r) \cdot dw^0(r) - \frac{1}{2} |z(r)|^2 \right] dr.$$

It is assumed that this integral exists and moreover that

$$\text{(3.8)} \qquad E^0[\exp(\rho \zeta(t_1))] = 1.$$

By Girsanov's theorem [LSh] the process $w(s)$ is a brownian motion under probability P, where

$$\text{(3.9)} \qquad P(d\omega) = \exp(\rho \zeta(t_1)) P^0(d\omega).$$

We rewrite (3.7) as

$$-\zeta(s) = \int_t^s \left[-\rho^{-\frac{1}{2}} z(r) \cdot dw(r) - \frac{1}{2}|z(r)|^2 dr \right]. \tag{3.10}$$

By (2.6′) assumption (3.8) implies that

$$E\left[\exp\left(-\rho\zeta(t_1)\right)\right] = 1, \tag{3.8′}$$

$$P^0(d\omega) = \exp(-\rho\zeta(t_1))P(d\omega), \tag{3.9′}$$

where E is expectation under probability P. Let us also require that

$$E \int_t^{t_1} |z(s)|^2 ds \quad < \infty. \tag{3.11}$$

Then $E[\zeta(t_1)] = E(Z)$, where

$$Z = \frac{1}{2} \int_t^{t_1} |z(s)|^2 ds. \tag{3.12}$$

Lemma 3.1. *Assume* (3.8),(3.11) *and that the expectation* $E(\mathcal{J})$ *exists. Then*
(3.13)
 (a) $\mathcal{E}^0(\mathcal{J}) \geq E(\mathcal{J} - Z)$

 (b) $\mathcal{E}^0(\mathcal{J}) = E(\mathcal{J} - Z)$ *if* $\mathcal{J} - \zeta(t_1)$ *is constant* P *– almost surely.*

Proof. This is immediate from Jensen's inequality. See (2.8) above.□

Lemma 3.2. *Assume that* $|z(s)| \leq K(1 + |x(s)|)$ *for some constant* K*. Then* (3.8), (3.11) *hold.*

Proof. An estimate for solutions to stochastic differential equations (Appendix (D.13)) implies that there exist positive constants k, C such that

$$E^0[\exp(k|x(s)|^2)] \leq C$$

for $t \leq s \leq t_1$. Therefore, $E^0[\exp(k_1|z(s)|^2)] \leq C_1$ for some positive constants k_1, C_1. This implies (3.8). See [LSh, p. 220]. Property (3.11) follows from (3.2), (3.5) and the estimate Appendix (D.7) with $m = 2$. □

To make use of Lemma 3.1, we consider the certainty equivalent expectation as a function of (t, x), where $x = x(t)$ is the initial data for (3.1) and (3.5). Let

$$\Phi(t,x) = E^0_{tx}[\exp(\rho\mathcal{J})] \tag{3.14}$$

$$V(t,x) = \mathcal{E}^0_{tx}(\mathcal{J}) = \rho^{-1}\log\Phi(t,x). \tag{3.15}$$

Associated with (3.14) is the linear PDE

$$A\Phi + \rho\ell(t,x)\Phi = 0, \quad (t,x)\in Q_0, \tag{3.16}$$

with A as in (3.3). The boundary condition for (3.16) is

$$\Phi(t_1,x) = \exp[\rho\psi(x(t_1))], \quad x\in I\!R^n. \tag{3.17}$$

If $\tilde{\Phi}$ is a smooth solution to (3.16)-(3.17), we would like to verify that $\tilde{\Phi} = \Phi$ (under some additional assumptions). One method is to use the Feynman-Kac formula, as in the following:

Proposition 3.1. *Let $\tilde{\Phi}\in C^{1,2}(Q_0)\cap C(\bar{Q}_0)$ be a positive, bounded solution to* (3.16)-(3.17) *and assume that ℓ is bounded above. Then $\tilde{\Phi}(t,x) = \Phi(t,x)$.*

Proof. For $R>0$, let $Q_R = [t_0, t_1 - R^{-1})\times O_R$, where $O_R = \{x : |x| < R\}$. Let τ_R be the exit time of $(s, x(s))$ from Q_R. By the Feynman-Kac formula (Appendix (D.15)) and (3.16)

$$\tilde{\Phi}(t,x) = E^0_{tx}\left\{\tilde{\Phi}(\tau_R, x(\tau_R))\exp\left[\rho\int_t^{\tau_R}\ell(s,x(s))ds\right]\right\}. \tag{3.18}$$

Since $\tilde{\Phi}$ is bounded, $\rho>0$ and ℓ is bounded above, the right side of (3.18) tends to $\Phi(t,x)$ as $R\to\infty$. □

Theorem 3.1 at the end of this section gives sufficient conditions that a solution $\tilde{\Phi}$ exists with the properties required in Proposition 3.1.

Another method for showing that $\tilde{\Phi} = \Phi$ is based on making a logarithmic transformation. Let $\tilde{\Phi}\in C^{1,2}(\bar{Q}_0)$ be a positive solution to (3.16)-(3.17), with $\tilde{\Phi}$ not necessarily bounded. Let $W = \rho^{-1}\log\tilde{\Phi}$. An elementary calculation gives the following nonlinear PDE for W:

$$AW + \frac{1}{2}a(t,x)D_xW\cdot D_xW + \ell(t,x) = 0 \tag{3.19}$$

with the boundary condition

$$W(t_1,x) = \psi(x). \tag{3.20}$$

We now define a particular auxiliary control process $z(s)$ as follows.

$$z(s) = \sigma'(s,x(s))D_xW(s,x(s)) \tag{3.21}$$

where $x(s)$ is the solution to

$$dx = [b(s,x(s)) + a(s,x(s))D_xW(s,x(s))]ds + \rho^{-\frac{1}{2}}\sigma(s,x(s))dw(s) \tag{3.22}$$

with $x(t) = x$. Note that (3.22) is the same as (3.5) in this case. The solution $x(s)$ is well defined provided there exists a constant M such that

$$|D_xW(s,x)| \le M(1+|x|), \quad (s,x) \in \bar{Q}_0. \tag{3.23}$$

Lemma 3.3. *Assume* (3.23) *and let* $z(s)$ *be as in* (3.21). *Then* (3.8′) *and* (3.11) *hold. Moreover,* $\mathcal{J} - \zeta(t_1) = W(t,x)$ *P-almost surely.*

Proof. By (3.2)(ii) and (3.23), $|z(s)| \le K(1+|x(s)|)$ for some K. By Lemma 3.2, (3.8) and (3.11) hold. Since W satisfies (3.19), the Ito differential rule gives

$$dW(s,x(s)) = -\left[\frac{1}{2}a(s,x(s))D_xW\cdot D_xW \right.$$
$$\left. + \ell(s,x(s))\right] ds + \rho^{-\frac{1}{2}}D_xW\cdot\sigma(s,x(s))dw^0(s)$$

where D_xW is evaluated at $(s,x(s))$. From (3.7) and (3.21) we can rewrite this as

$$dW(s,x(s)) = -\ell(s,x(s))ds + d\zeta(s). \tag{3.24}$$

The lemma follows by integrating from t to t_1 and using (3.20). □

Lemmas 3.1 and 3.3 can be used to characterize the certainty-equivalent expectation $\mathcal{E}^0_{tx}(\mathcal{J}) = V(t,x)$ as the maximum of $E_{tx}(\mathcal{J}-Z)$ in an appropriate class of auxiliary control processes $z(\cdot)$. This will be made precise in Theorem 4.1. In Section 4 we will fix a reference probability system $\nu = (\Omega, \{\mathcal{F}_s\}, P, w)$, and will consider solutions $x(s)$ to (3.5). Girsanov's Theorem gives $x(s)$ as a solution to (3.1) with P^0, w^0 determined from P, w by (3.6) and (3.9′). Note that the expectation $E^0[\exp(\rho\mathcal{J})]$ depends only on the probability law of $x(\cdot)$ and not on the particular reference probability system $\nu^0 = (\Omega, \{\mathcal{F}_s\}, P^0, w^0)$.

We conclude this section with a theorem which gives conditions under which $\Phi(t,x)$ defined by (3.14) is a solution to (3.16) in the classical sense.

Theorem 3.1. *Assume that* (3.2) *and either of the following two assumptions* (3.25) *or* (3.26) *hold. Then* $\Phi \in C^{1,2}(\bar{Q}_0)$ *is a solution to* (3.16)-(3.17) *with* Φ *and* $D_x\Phi$ *bounded.*

(i) *The matrices* $a(t,x) = \sigma(t,x)\sigma(t,x)'$ *are positive definite with bounded inverse* $a(t,x)^{-1}$*;*

(3.25) (ii) $\ell \in C^1(\bar{Q}_0)$, $\psi \in C^3(I\!R^n)$. *Moreover* ℓ, ψ *are bounded above and have bounded first order partial derivatives.*

(i) *For* $\phi = b, \sigma, \ell$, $\phi \in C^{1,2}(\bar{Q}_0)$. *Moreover,* ϕ *together with its first*

(3.26) *order partial derivatives and second order partial derivatives* $\phi_{x_ix_j}$,

$i, j = 1, \ldots, n$ *are bounded.*
(ii) $\psi \in C_b^2(I\!R^n)$.

When (3.25) holds, an existence theorem for linear, uniformly parabolic PDEs implies that (3.16)-(3.17) has a solution $\tilde{\Phi} \in C^{1,2}(\bar{Q}_0)$ with $\tilde{\Phi}$ and $D_x\tilde{\Phi}$ bounded. See [LSU, Chapt. 4, Thm. 9.1]. By Proposition 3.1, $\Phi = \tilde{\Phi}$. When (3.26) holds, probabilistic arguments show that Φ has the properties stated in Theorem 3.1. The method is based on the fact that the solution $x(s)$ to (3.1) depends smoothly on the initial state $x = x(t)$. See [GS2,Sec. 11].

Remark 3.1. Theorem 3.1 implies that $V = \rho^{-1} \log \Phi$ is a solution to (3.19)-(3.20). Since ℓ and ψ are bounded, $\Phi(t,x)^{-1}$ is bounded. Hence, $D_xV = \rho^{-1}\Phi^{-1}D_x\Phi$ is bounded on $\bar{Q}_0$. In particular, (3.23) holds with $W = V$. When D_xV is bounded, it would suffice to consider changes of probability measure with bounded auxiliary control process $z(s)$.

Remark 3.2. It can be shown under less restrictive assumptions that Φ is a viscosity solution to (3.16)-(3.17), by the method used to prove Theorem 8.1 below.

VI.4 Auxiliary stochastic control problem

Let us reformulate the maximization problem introduced in Section 3 as a stochastic control problem of the kind considered in Chapter IV. As in Section 3, we take $\rho > 0$. Let $\nu = (\Omega, \{\mathcal{F}_s\}, P, w)$ be a reference probability system, which is fixed throughout this section. The state is $x(s) \in I\!R^n$ and the control is $z(s) \in I\!R^d$. The state dynamics are

(4.1) $$dx = [b(s, x(s)) + \sigma(s, x(s))z(s)]ds + \rho^{-\frac{1}{2}}\sigma(s, x(s))dw(s)$$

for $t \le s \le t_1$, with $x(t) = x$. Let

(4.2) $$L(t, x, z) = \ell(t, x) - \frac{1}{2}|z|^2,$$

(4.3) $$J(t, x; z) = E_{tx}\left\{\int_t^{t_1} L(s, x(s), z(s))ds + \psi(x(t_1))\right\}.$$

The goal is to find an "auxiliary" control process $z(\cdot) \in \mathcal{A}_{t\nu}$, which maximizes $J(t, x, z)$, where $\mathcal{A}_{t\nu}$ was defined in Section IV.2. In the notation of Chapter IV, the "control space" is $U = I\!R^d$, and $z(s)$ has the role of $u(s)$ there. Also maximizing J is equivalent to minimizing $-J$.

We recall the notations of Section IV.3. In particular, let $\mathcal{L}$ denote the class of Markov control policies $\underline{z}(s, x)$ defined there. Then the solution $x(s)$ to (4.1) is well defined, with

$$z(s) = \underline{z}(s, x(s)). \tag{4.4}$$

See IV(3.13)-(3.14).

The dynamic programming equation for this stochastic control problem is (3.19). The following theorem gives a sufficient condition that the function $V(t,x)$ in (3.15) is the value function.

Theorem 4.1. *Let $W \in C^{1,2}(\bar{Q}_0)$ be a solution to (3.19)-(3.20) such that $|D_x W(t,x)| \le M(1+|x|)$ for some constant M. Let*

$$\underline{z}^*(s,x) = \sigma'(s,x) D_x W(s,x), \tag{4.5}$$

with corresponding $z^*(s)$ as in (4.4). Then $V = W$ where $V = \rho^{-1}\log\Phi$. Moreover, $z^*(\cdot) \in \mathcal{A}_{t\nu}$ and

$$\begin{aligned} &(a) \quad V(t,x) \ge J(t,x;z) \text{ for all } z(\cdot) \in \mathcal{A}_{t\nu}; \\ &(b) \quad V(t,x) = J(t,x;z^*). \end{aligned} \tag{4.6}$$

Proof. The linear growth condition on $D_x W(t,x)$ implies that $W \in C_p(\bar{Q}_0)$ with $p = 2$. By Corollary IV.3.1, $W(t,x) \ge J(t,x;z)$ with equality when $z(s) = z^*(s)$, Lemmas 3.1(b) and 3.3 imply that $W(t,x) = V(t,x)$. □

We note that the dynamic programming equation can be rewritten as

$$-\frac{\partial V}{\partial t} - \frac{1}{2\rho}\operatorname{tr} a(t,x) D_x^2 V + H(t,x,D_x V) = 0, \tag{4.7}$$

$$-H(t,x,p) = b(t,x)\cdot p + \frac{1}{2}a(t,x)p\cdot p + \ell(t,x). \tag{4.8}$$

This corresponds to the form IV(3.3) of the dynamic programming PDE used in Chapters IV and V.

Example 4.1. Using the notation in (3.1), let

$$dx = A(s)x(s)ds + \rho^{-\frac{1}{2}}\sigma(s)dw^0(s)$$

with $x(t) = x$ and

$$\ell(t,x) = -x \cdot M(t)x$$

$$\psi(x) = -x \cdot Dx$$

where $M(t)$ and D are nonnegative definite matrices. Then $x(s)$ is a gaussian process under probability P^0, and

$$\Phi(t,x) = E^0_{tx} \exp\left\{-\rho\left[\int_t^{t_1}[x(s)\cdot M(s)x(s)ds + x(t_1)\cdot Dx(t_1)]\right]\right\}. \tag{4.9}$$

Let $\tilde{J} = -J$. The problem of choosing the control $z(s)$ to minimize $\tilde{J}$ is a stochastic linear regulator problem (see Example III.8.1). In III(8.6) we now have $B(s) = \sigma(s)$, $u(s)$ replaced by $z(s)$ and $\sigma(s)dw(s)$ replaced by $\rho^{-\frac{1}{2}}\sigma(s)dw(s)$. In III(8.7), J is now replaced by $\tilde{J}$ and $2N(s)$ is the identity matrix. The value function $\tilde{V}(t,x)$ for the problem of minimizing $\tilde{J}$ is

$$\tilde{V}(t,x) = x \cdot P(t)x + \rho^{-1}g(t)$$

where the symmetric matrix $P(t)$ and $g(t)$ are the same as in Example III.8.1 with $B(s) = \sigma(s)$ and $2N(s)$ the identity matrix. By Theorem 4.1, $V(t,x) = \rho^{-1}\log\Phi(t,x)$ has the form

$$V(t,x) = -x \cdot P(t)x - \rho^{-1}g(t). \tag{4.10}$$

We recall that $P(t)$ satisfies the Riccati differential equation I(5.15) with terminal data $P(t_1) = D$. From III(8.11), or from (4.5), the optimal auxiliary control policy is $\underline{z}^*(t,x) = -2\sigma'(t)P(t)x$.

Example 4.2. In Example 4.1, suppose that the assumption that $M(t)$ and D are nonnegative definite does not hold. Then the stochastic linear regulator problem is of indefinite sign. The solution in Example 4.1 is valid for $t_{min} < t \le t_1$, where $t_{min} \ge -\infty$ is the same as for the deterministic LQR problem (Example I.5.1).

The case $\rho < 0$. In this case we replace ρ by $|\rho|$ in the SDE (4.1). The logarithmic transformation $V = \rho^{-1}\log\Phi$ transforms the linear PDE (3.16) into

$$-\frac{\partial V}{\partial t} - \frac{1}{2|\rho|}\operatorname{tr} a(t,x)D_x^2 V + \bar{H}(t,x,D_xV) = 0. \tag{4.11}$$

$$-\bar{H}(t,x,p) = b(t,x)\cdot p - \frac{1}{2}a(t,x)p\cdot p + \ell(t,x). \tag{4.12}$$

Equation (4.11) is the HJB PDE for the problem of minimizing

$$\bar{J}(t,x;z) = E_{tx}\left\{\int_t^{t_1}\left[\ell(s,x(s)) + \frac{1}{2}|z(s)|^2\right]ds + \psi(x(t_1))\right. .$$

As in Theorem 4.1, if $W \in C^{1,2}(\bar{Q}_0)$ is a solution to (4.11)-(3.20) such that $|D_xW(t,x)| \le M(1+|x|)$, then $W = V$. Moreover, the control policy

$$z^*(s,x) = -\sigma'(s,x)D_xW(s,x) \tag{4.13}$$

is optimal.

VI.5 Bounded region Q

Let $Q = [t_0, t_1] \times O$ where O is a bounded open set with ∂O a manifold of class C^3. Let us indicate modifications in the previous results when

$$\mathcal{J} = \int_t^\tau \ell(s, x(s))ds + \Psi(\tau, x(\tau)), \tag{5.1}$$

where τ is the exit time of $(s, x(s))$ from Q. In addition to the previous assumptions (3.2), let us assume that $\ell \in C^1(\bar{Q}_0)$. Moreover:

(5.2)
- (a) The matrices $a(t,x) = \sigma(t,x)\sigma(t,x)'$ are nonsingular with bounded inverses $a(t,x)^{-1}$.
- (b) For $x \in \bar{O}$, $\Psi(t_1, x) = \psi(x)$, where $\psi \in C^3(I\!R^n)$.
- (c) For $(t,x) \in [t_0, t_1] \times \partial O$, $\Psi(t,x) = g(x)$ where $g \in C^3(I\!R^n)$.

Theorem 5.1. *Let $\Phi(t,x)$ be as in* (3.14), *with $\mathcal{J}$ as in* (5.1). *Then $\Phi \in C^{1,2}(Q) \cap C(\bar{Q})$ and Φ satisfies* (3.16) *in Q. Moreover, $D_x\Phi$ is bounded on Q and continuous on $\bar{Q}\backslash(\{t_1\} \times \partial O)$.*

An existence theorem for linear, uniformly parabolic PDEs implies that (3.16) with boundary data $\Phi(t,x) = \exp[\rho\Psi(t,x)]$, $(t,x) \in \partial^* Q$ has a solution $\tilde{\Phi}$ with the properties stated in Theorem 5.1 [LSU, Chap. 4]. The same proof as for Proposition 3.1 shows that $\Phi = \tilde{\Phi}$. □

Example 5.1. Let $\ell = 0$, $g = 0$. The PDE (3.16) becomes $A\Phi = 0$, and (3.14) becomes

$$\Phi(t,x) = P^0_{tx}(\tau < t_1) + E^0_{tx}\left\{\exp[\rho\psi(x(t_1))]; \tau = t_1\right\}. \tag{5.3}$$

The first term on the right side is the exit probability. If we formally set $\psi(x) = -\infty$, then the second term is 0. We will return to the exit probability problem in Section 6 and again in Section VII.10.

For the auxiliary stochastic control problem in Section 4, we now have

$$J(t,x;z) = E_{tx}\left\{\int_t^\tau L(s, x(s), z(s))ds + \Psi(\tau, x(\tau))\right\}, \tag{5.4}$$

which is the same as IV(2.8) with the control $u(s)$ replaced by $z(s)$. An optimal Markov control policy $\underline{z}^*(s,x)$ is obtained in a way similar to Theorem 4.1. For $(s,x) \in Q$, define $\underline{z}^*(s,x)$ by (4.5), with $W = \rho^{-1}\log\Phi$. The technique of Appendix C extends $\underline{z}^*(s,x)$ to Q_0, such that $\underline{z}^*$ is Lipschitz on $[t_0, t_2]$ for each $t_2 < t_1$ and $\underline{z}^*(s,x)$ is bounded. Then $\underline{z}^*(s)$ in (4.4) is well defined and is an optimal control process.

VI.6 Small noise limits

Let $\varepsilon > 0$ be a parameter, and let $x^\varepsilon(s)$ denote the solution to

$$dx^\varepsilon = b(s, x^\varepsilon(s))ds + \varepsilon^{\frac{1}{2}}\sigma(s, x^\varepsilon(s))dw^0(s) \tag{6.1^ε}$$

with initial data $x^\varepsilon(t) = x$. For $\varepsilon = 0$, $x^0(s)$ satisfies the ordinary differential equation

$$\frac{dx^0}{ds} = b(s, x^0(s)) \tag{6.1^0}$$

with $x^0(t) = x$. For $\varepsilon > 0$, $x^\varepsilon(s)$ can be regarded as a random perturbation of $x^0(s)$. If $\mathcal{J}^\varepsilon$ is some functional of the sample paths $x^\varepsilon(\cdot)$, then a result which states that $\varepsilon \log E^0[\exp(\varepsilon^{-1}\mathcal{J}^\varepsilon)]$ tends to a limit as $\varepsilon \to 0$ is called a *large deviations theorem.* If $\rho = \varepsilon^{-1}$, then ρ is the risk sensitivity parameter in previous sections ($\rho > 0$).

As in (3.4) let us take

$$\mathcal{J}^\varepsilon = \int_t^{t_1} \ell(s, x^\varepsilon(s))ds + \psi(x^\varepsilon(t_1)), \tag{6.2}$$

and as in (3.15) let

$$V^\varepsilon(t, x) = \varepsilon \log E^0_{tx}[(\exp \varepsilon^{-1}\mathcal{J}^\varepsilon)]. \tag{6.3}$$

By (4.7) the dynamic programming PDE for V^ε is

$$-\frac{\partial V^\varepsilon}{\partial t} - \frac{\varepsilon}{2}\mathrm{tr}\, a(t, x)D_x^2 V^\varepsilon + H(t, x, D_x V^\varepsilon) = 0 \tag{6.4^ε}$$

with H as in (4.8). For $\varepsilon = 0$, (6.4^ε) becomes the first order PDE

$$-\frac{\partial V^0}{\partial t} + H(t, x, D_x V^0) = 0, \qquad (t, x) \in Q_0 \tag{6.4^0}$$

$$V^0(t_1, x) = \psi(x). \tag{6.5}$$

This is the dynamic programming equation for the following deterministic control problem. Let $x(s)$ denote the state and $z(s)$ the control at time s. They are deterministic functions of time, not random variables. The state dynamics are

$$\frac{d}{ds}x(s) = b(s, x(s)) + \sigma(s, x(s))z(s) \tag{6.6}$$

with initial data $x(t) = x$. There are no control constraints. The goal is to maximize

$$J^0(t,x;z) = \int_t^{t_1} \left[\ell(s,x(s)) - \frac{1}{2}|z(s)|^2\right] ds + \psi(x(t_1)). \tag{6.7}$$

Let us assume that b, σ satisfy (3.2) and that:

$$\begin{aligned} &\text{(a)} \quad \ell \in C^1(\bar{Q}_0) \text{ with } \ell \text{ and } \ell_x \text{ bounded.} \\ &\text{(b)} \quad \psi \in C^1(I\!R^n) \text{ with } \psi \text{ and } \psi_x \text{ bounded.} \\ &\text{(c)} \quad b \text{ is bounded.} \end{aligned} \tag{6.8}$$

Let $\mathcal{Z}(t) = L^\infty\big([t,t_1]; I\!R^d\big)$ denote the space of all bounded, Lebesgue measurable control functions, and $\mathcal{Z}_R(t)$ the space of such controls such that $|z(s)| \le R$ almost everywhere in $[t,t_1]$. Let

$$\begin{aligned} &\text{(a)} \quad V^0(t,x) = \sup_{z(\cdot)\in\mathcal{Z}(t)} J^0(t,x;z) \\ &\text{(b)} \quad V_R^0(t,x) = \sup_{z(\cdot)\in\mathcal{Z}_R(t)} J^0(t,x;z) \end{aligned} \tag{6.9}$$

Lemma 6.1. *There exists $z_R^*(\cdot) \in \mathcal{Z}_R(t)$ such that $J(t,x;z_R^*) = V_R^0(t,x)$.*

Lemma 6.1 is a special case of Theorem I.11.1.

Lemma 6.2. *There exists R_1 such that $|z_R^*(s)| \le R_1$ for all R.*

Proof. Let $x^*(s)$ be the solution to (6.6) with $z(s) = z_R^*(s)$ and $x^*(t_1) = x$. Let $P(s)$ be the solution to

$$\frac{dP}{ds} = -P'(s)F_x(s,x^*(s),z_R^*(s)) - \ell_x(s,x^*(s)) \tag{6.10}$$

with $P(t_1) = \psi_x(x^*(t_1))$, where

$$F(s,x,z) = b(s,x) + \sigma(s,x)z.$$

By Pontragin's principle (Theorem I.6.3), $P(s)\cdot\sigma(s,x^*(s))\xi - \frac{1}{2}|\xi|^2$ is maximum on the set $|\xi| \le R$ when $\xi = z_R^*(s)$. Let us show that $|P(s)|$ is bounded by a constant which does not depend on R. We have

$$\log(1+|P(s)|^2) = \log(1+|P(t_1)|^2) - 2\int_s^{t_1} \frac{P(r)\cdot\dot{P}(r)}{1+|P(r)|^2}. \tag{6.11}$$

We use (3.2), (6.8)(a) and (6.10) to get

$$|P(r)\cdot\dot{P}(r)| \le |P(r)|^2(C_1 + C_2|z_R^*(r)|) + C_3|P(r)|$$

for suitable C_1, C_2, C_3. From (6.11) and Cauchy-Schwartz,

$$\log(1+|P(s)|^2) \le \log(1+|P(t_1)|^2) + \int_{t_1}^{s} (C_4 + C_5|z_R^*(r)|^2)dr.$$

Since ℓ, ψ are bounded,

$$-K(t) \le J^0(t,x;0) \le V_R^0(t,x) \le K(t)$$

where $K(t) = \|\ell\|(t_1 - t) + \|\psi\|$ and $\| \ \|$ is the sup norm. Since $J(t,x;z_R^*) = V_R^0(t,x)$, we have

$$\|z_R^*(\cdot)\|_2 \le 4K(t)$$

where $\| \ \|_2$ is the L_2 norm. Since ψ_x is bounded, $|P(t_1)| \le \|\psi_x\|$. Therefore, $|P(s)| \le M$ for $t \le s \le t_1$ where M does not depend on R. Since σ is bounded, $|z_R^*(s)| \le R_1$ for some R_1. □

Theorem 6.1. *Assume* (3.2) *and* (6.8) *and let* R_1 *be as in Lemma* 6.2. *Then:*

(a) $V_R^0(t,x) = V^0(t,x)$ *if* $R \ge R_1$*;*
(b) V^0 *is bounded and Lipschitz continuous on* $\bar{Q}_0$*;*
(c) V^0 *is a viscosity solution to* (6.4^0)-(6.5)*;*
(d) V^0 *is unique in the class of bounded, Lipschitz continuous viscosity solutions to* (6.4^0)-(6.5).

Proof. Part (a) is immediate from Lemma 6.2. From Remark II.10.1 V_R^0 is Lipschitz. Moreover, V_R^0 is bounded since ℓ and ψ are bounded. Hence, (b) follows from (a). By Theorem II.7.1 and part (a), for $R \ge R_1$, V^0 is a viscosity solution of

$$(6.12) \qquad -\frac{\partial V^0}{\partial t} + H_R(t,x,D_x V^0) = 0,$$

$$(6.13) \qquad H_R(t,x,p) = -b(t,x)\cdot p - \max_{|\xi|\le R}\left[p\cdot\sigma(t,x)\xi - \frac{1}{2}|\xi|^2\right].$$

As a function of ξ, $p\cdot\sigma\xi - \frac{1}{2}|\xi|^2$ has a maximum on $I\!R^d$ at $z^* = \sigma' p$. Since σ is bounded, $|\sigma' p| \le k|p|$ for some k. Since $V^0 = V_{R_1}^0$ is Lipschitz,

$$|V^0(t,x) - V^0(t,y)| \le M_1|x-y|$$

for some M_1. For $R \ge \max(R_1, kM_1)$, V^0 is a viscosity solution to (6.4^0)-(6.5) by Corollary II.8.1(f). Finally, let $\tilde{V}$ be any bounded, Lipschitz continuous viscosity solution to (6.4^0)-(6.5). Then V^0 and $\tilde{V}$ are both viscosity solutions to (6.12)-(6.5) for large R. By Corollary II.9.1, $V^0 = \tilde{V}$. □

Remark 6.1. There is another proof of Theorem 6.1 which does not use the existence theorem I.11.1. See Theorem XI.7.1 for differential games, which includes Theorem 6.1 as a special case (with no minimizing control $u(s)$).

Theorem 6.2. *Assume* (3.2) *and that* ℓ, ψ *are bounded and uniformly continuous on* $\bar{Q}_0$, $I\!R^n$ *respectively. Then* $V^\varepsilon(t,x)$ *tends to* $V^0(t,x)$ *uniformly on compact subsets of* $\bar{Q}_0$.

Theorem 6.2 will be proved in Section VII.11 using viscosity solution methods. It can also be obtained by probabilistic methods, based on the Freidlin-Wentzell theory of large deviations. If σ is constant, there is another rather easy proof based on the characterization in Section 4 of $V^\varepsilon(t,x)$ as a value function. Let us sketch this argument. We suppose that for each $\varepsilon > 0$, the verification Theorem 4.1 holds. Let $x^\varepsilon(s)$, $y^\varepsilon(s)$ satisfy (6.1^ε) with initial data $x^\varepsilon(t) = x$, $y^\varepsilon(t) = y$. Since σ is constant,

$$|x^\varepsilon(s) - y^\varepsilon(s)| \le |x-y| \exp(||b_x||(s-t)).$$

If we write $J = J^\varepsilon$ in (4.3), then for any $z(\cdot)$

$$|J^\varepsilon(t,x;z) - J^\varepsilon(t,y;z)| \le M_1|x-y|$$

where the constant M_1 depends on $||b_x||$, $||\ell_x||$ and $||\psi_x||$. Hence the value-function V^ε satisfies

$$|V^\varepsilon(t,x) - V^\varepsilon(t,y)| \le M_1|x-y|. \tag{6.14}$$

An argument similar to that for Theorem 6.1(c) implies that there exists R_1 such that $V^\varepsilon_R(t,x) = V^\varepsilon(t,x)$ for $R \ge R_1$, where V^ε_R is the value function with auxiliary control constraint $|z(s)| \le R$. Lemma IV.6.3 then implies that $|V^\varepsilon(t,x) - V^0(t,x)| \le K\varepsilon^{\frac{1}{2}}$ for some K.

Remark 6.2. We recall from Section 2 that $V^\varepsilon(t,x) = \mathcal{E}^0_{tx}(\mathcal{J}^\varepsilon)$ where $\mathcal{E}^0$ is the certainty equivalent expectation. The limit $V^0(t,x)$ as $\varepsilon \to 0$ is the value function of the control problem described above. While this control problem is deterministic, $V^0(t,x)$ does have a "probabilistic" interpretation in terms of the Maslov idempotent probability calculus. For any bounded functional $\mathcal{J} = \mathcal{J}(z(\cdot))$, define the max-plus expectation as

$$E^+(\mathcal{J}) = \sup_{z(\cdot)} \left[\mathcal{J}(z(\cdot)) - \frac{1}{2}\int_t^{t_1} |z(s)|^2 ds \right].$$

In particular, let $\mathcal{J}$ be as in (3.4) with $x(s)$ the solution to (6.6), $x(t) = x$. By (6.7), (6.9)(a), $V^0(t,x) = E^+_{tx}(\mathcal{J})$. The operator E^+ is linear with respect to max-plus addition and scalar multiplication, in which

$$a \oplus b = \max(a,b), \qquad a \otimes b = a + b.$$

These ideas are developed in [BCOQ][F6][MS] and references cited there.

Negative logarithmic transformation. For some applications it is more natural to let

$$V^\varepsilon(t,x) = -\varepsilon \log E^0_{tx}[\exp(\varepsilon^{-1}\mathcal{J}^\varepsilon)] \tag{6.3$'$}$$

The PDE for V^ε is again (6.4$^\varepsilon$). However, instead of (4.8) we now have

$$H(t,x,p) = -b(t,x)\cdot p + \frac{1}{2}a(t,x)p\cdot p + \ell(t,x). \tag{6.15}$$

For $\varepsilon = 0$, the auxiliary control $z(s)$ is chosen to minimize $-J^0(t,x;z)$ in (6.7). In particular, let us now assume that the matrices $\sigma(t,x)$ are $n\times n$ nonsingular with $\sigma^{-1}(t,x)$ bounded. Then from (6.6), $z(s) = \sigma^{-1}(s,x(s))(\dot{x}(s) - b(s,x(s)))$. Let

$$\bar{L}(t,x,v) = \frac{1}{2}(b(t,x)-v)\cdot a^{-1}(t,x)(b(t,x)-v) - \ell(t,x) \tag{6.16}$$

$$\bar{J}^0 = \int_t^{t_1} \bar{L}(s,x(s),\dot{x}(s))ds - \psi(x(t_1)). \tag{6.17}$$

Then $J^0(t,x;z) = -\bar{J}^0$. Choosing $z(s)$ to maximize J^0 is equivalent to the calculus of variations problem of choosing $x(s)$ to minimize $\bar{J}^0$, subject to the initial condition $x(t) = x$.

Example 6.1. Let $b = 0$, σ the identity matrix, $\ell(t,x) = q(x)$ and $\psi = 0$. Then $\bar{L}(x,v) = \frac{1}{2}|v|^2 - q(x)$ is the classical action integrand corresponding to the potential energy function $q(x)$, already mentioned in Example III.8.2 Assume that $q \in C^1_b(I\!R^n)$ and let

$$\Phi^\varepsilon(t,x) = E^0_{tx}\left[\exp\left(\varepsilon^{-1}\int_t^{t_1} q(x^\varepsilon(s))\right)ds\right].$$

Then Φ^ε satisfies

$$\Phi^\varepsilon_t + \frac{\varepsilon}{2}\Delta_x\Phi^\varepsilon + \frac{q(x)}{\varepsilon}\Phi^\varepsilon = 0 \tag{6.18}$$

with $\Phi^\varepsilon(t_1,x) = 1$. As $\varepsilon \to 0$, $V^\varepsilon = -\varepsilon\log\Phi^\varepsilon$ tends to V^0, where $V^0(t,x)$ is the least action for initial data $x(t) = x$.

If Φ^ε_t were replaced by $-i\Phi^\varepsilon_t (i^2 = -1)$, then (6.18) would become the Schrödinger equation of quantum mechanics. For a particle of mass 1 in a field with the potential function $q(x)$, the parameter ε corresponds to Planck's constant. Equation (6.18) is sometimes called the "imaginary time" analogue of Schrödinger's equation. In this example, Theorem 6.2 becomes an "imaginary time" analogue of so-called semiclassical limit results, which recover classical mechanics from quantum mechanics.

The exit problem. Let $O \subset I\!R^n$ be open and bounded, with ∂O a manifold of class C^3. Let θ^ε denote the exit time from O for the solution $x^\varepsilon(s)$ to (6.1$^\varepsilon$), and let

$$\Phi^\varepsilon(t,x) = P^0_{tx}(\theta^\varepsilon < t_1), \quad (t,x) \in Q, \tag{6.19}$$

where as before $Q = [t_0, t_1) \times O$. We again make the nondegeneracy assumption that $\sigma(t,x)$ is $n \times n$ and nonsingular with $\sigma^{-1}(t,x)$ bounded. The exit probability, as a function of the initial data (t,x), has the following properties:

$$\begin{aligned}
&\text{(a) } \Phi^\varepsilon \in C^{1,2}(\bar{Q}\backslash(\{t_1\} \times \partial O)) \text{ and } A\Phi = 0;\\
&\text{(b) } \Phi^\varepsilon(t,x) > 0 \quad \text{for} \quad (t,x) \in Q;\\
&\text{(c) } \Phi^\varepsilon(t,x) = 1 \quad \text{for} \quad (t,x) \in [t_0,t_1) \times \partial O;\\
&\text{(d) } \Phi^\varepsilon(t_1,x) = 0 \quad \text{for} \quad x \in O.
\end{aligned} \tag{6.20}$$

Properties (6.20) can be proved by making continuous approximations to the discontinuous data for Φ^ε at points $(t_1, x), x \in \partial O$, as in [F4], and using a priori estimates for solutions to linear, uniformly parabolic PDE.

We make the logarithmic transformation $V^\varepsilon = -\varepsilon \log \Phi^\varepsilon$. Then V^ε satisfies the HJB equation

$$-V^\varepsilon_t - \frac{\varepsilon}{2} \operatorname{tr} a(t,x) D^2_x V^\varepsilon - b(t,x) \cdot D_x V^\varepsilon + \frac{1}{2} a(t,x) D_x V^\varepsilon \cdot D_x V^\varepsilon = 0 \tag{6.21}$$

with the boundary data

$$\begin{aligned}
&V^\varepsilon(t,x) = 0 \quad \text{for} \quad (t,x) \in [t_0,t_1) \times \partial O\\
&V^\varepsilon(t_1,x) = +\infty \quad \text{for} \quad x \in O.
\end{aligned} \tag{6.22}$$

The calculus of variations problem which arises when $\varepsilon = 0$ is to minimize

$$\bar{J}^0 = \int_t^\tau \bar{L}(s, x(s), \dot{x}(s)) ds, \tag{6.23}$$

$$\bar{L}(t,x,v) = \frac{1}{2}(b(t,x) - v) \cdot a^{-1}(t,x)(b(t,x) - v)$$

among all $x(\cdot) \in C^1([t,t_1])$ such that $x(t) = x$ and $\tau < t_1$, where τ is the exit time of $(s, x(s))$ from Q. The requirement that $x(s)$ must reach ∂O before time t_1 reflects the data $V^0(t_1, x) = +\infty$ for $x \in O$.

The result that $V^\varepsilon \to V^0$ as $\varepsilon \to 0$, where $V^0(t,x)$ is the value function for this calculus of variations problem, is called a *large deviations* theorem for exit probabilities. This large deviations result has been proved probabilistically; see Freidlin-Wentzell [FW]. Later another proof by stochastic control methods was given [F4]. In Section VII.10 a different proof, based on PDE-viscosity solution methods, will be given.

VI.7 H-infinity norm of a nonlinear system

In this section we consider the following deterministic dynamical system model. Let $x(s)$ denote the state at time $s \geq 0$. It satisfies the ordinary differential equation

$$\frac{d}{ds}x(s) = b(x(s)) + \sigma(x(s))z(s), \quad s \geq 0 \tag{7.1}$$

with initial state $x(0) = x_0$. In (7.1), $z(s)$ represents an unknown "disturbance" at time s. Note that (7.1) is the same as (6.6), except that we now consider "autonomous" state dynamics on the interval $0 \leq s < \infty$. We assume that $z(\cdot) \in L^2([0,T]; I\!R^d)$ for every $T < \infty$. The class of all such functions $z(\cdot)$ is denoted by $\mathcal{Z}$. We assume that $b, \sigma \in C^1(I\!R^n)$. Moreover,

(7.2)
(a) b_x, σ, σ_x are bounded;

(b) $b(0) = 0$ and there exists $c > 0$ such that, for all $x, y \in I\!R^n$

$$(x-y)\cdot(b(x) - b(y)) \leq -c|x-y|^2.$$

Condition (7.2)(b) is equivalent to:

(7.2) (b′) $\quad b(0) = 0$ and $b_x(x)\eta \cdot \eta \leq -c|\eta|^2$ for all $x, \eta \in I\!R^n$.

It implies that the unperturbed system (with $z(s) = 0$) is globally asymptotically stable to 0. We also consider a "running cost" function $\ell(x)$ such that:

(7.3)
(a) ℓ is continuous on $I\!R^n$;

(b) $\ell(0) = 0$, $\ell(x) > 0$ for all $x \neq 0$ and there exists K such that $\ell(x) \leq K|x|^2$ for all $x \in I\!R^n$.

Additional assumptions on ℓ will be imposed later as needed.

Definition. For $\gamma > 0$, the system described by (7.1) is said to have L^2-gain $\leq \gamma$ if there exists $W(x) \geq 0$ with $W(0) = 0$, such that: for every $T, x_0, z(\cdot)$

$$\int_0^T \ell(x(s))ds \leq \gamma^2 \left[\int_0^T |z(s)|^2 ds + 2W(x_0) \right]. \tag{7.4}$$

We will see that (7.4) holds if γ is sufficiently large (Example 7.1). The infimum of those γ for which (7.4) holds is called the *H-infinity norm* [HJ,p. 47]. This term came from linear H-infinity systems theory, which we mention later in this section. In the linear theory, γ^2 can be interpreted as a L^2-space operator norm, which becomes a H-infinity norm in the frequency domain.

It is convenient to rewrite (7.4) as follows. Let $\mu = (2\gamma^2)^{-1}$. Then (7.4) is equivalent to

$$\mu \int_0^T \ell(x(s))ds \leq \frac{1}{2} \int_0^T |z(s)|^2 ds + W(x_0), \tag{7.4$'$}$$

for all $T, x_0, z(\cdot)$.

Definition. A function W is called a *storage function* if $W(x) \geq 0$ for all $x \in I\!\!R^n$, $W(0) = 0$ and: for $0 \leq t < s$ and any $z(\cdot)$

$$W(x(s)) + \mu \int_t^s \ell(x(r))dr \leq \frac{1}{2} \int_t^s |z(r)|^2 dr + W(x(t)). \tag{7.5}$$

Moreover, (7.5) is called a *dissipation inequality.*

Proposition 7.1. *If $W(x)$ is any storage function, then* (7.4′) *holds.*

Proof. Take $t = 0$, $s = T$ in (7.5) and use the fact that $W(x(T)) \geq 0$. □

As in (4.8), let

$$H(x,p) = -b(x) \cdot p - \frac{1}{2} a(x)p \cdot p - \mu\ell(x). \tag{7.6}$$

Proposition 7.2. *Let $W \in C^1(I\!\!R^n)$ satisfy $W(x) \geq 0$, $W(0) = 0$ and $H(x, D_x W(x)) \geq 0$ for all $x \in I\!\!R^n$. Then W is a storage function.*

Proof. We have

$$\frac{d}{dr} W(x(r)) + \mu\ell(x(r)) - \frac{1}{2}|z(r)|^2 \leq -H(x(r), DW(x(r))) \leq 0.$$

We integrate from t to s to obtain (7.5) □

Example 7.1. Let $W(x) = B|x|^2$, $B > 0$. Then

$$H(x, DW(x)) = -2Bb(x) \cdot x - 2B^2 a(x)x \cdot x - \mu\ell(x) \geq (2Bc - 2kB^2 - \mu K)|x|^2,$$

with c, K as in (7.2), (7.3) and $k > 0$ some constant. Choose $B < k^{-1}c$. Then $W(x)$ is a storage function if μ is small enough, by Proposition 7.2.

Let

$$\tilde{J}(T, x; z) = \int_0^T \left[\mu\ell(x(s)) - \frac{1}{2}|z(s)|^2\right] ds \tag{7.7}$$

$$\tilde{V}(T,x) = \sup_{z(\cdot)\in\mathcal{Z}} \tilde{J}(T,x;z) \tag{7.8}$$

where as usual $x = x_0 = x(0)$ is the initial state. Then for $0 < T_1 < T_2$

$$0 \le \tilde{V}(T_1,x) \le \tilde{V}(T_2,x). \tag{7.9}$$

Since $\ell \ge 0$, the left hand inequality in (7.9) is immediate (take $z(s) = 0$). The right hand inequality then follows by taking $z(s) = 0$ for $T_1 \le s \le T_2$. Let

$$\tilde{W}(x) = \lim_{T\to\infty} \tilde{V}(T,x). \tag{7.10}$$

Proposition 7.3. *Assume that $\tilde{W}(x) < \infty$ for all $x \in I\!R^n$. Then $\tilde{W}$ is a storage function. Moreover $\tilde{W} \le W$ for any storage function W.*

Proof. Inequality (7.4′) for all $z(\cdot)$ is equivalent to $\tilde{V}(T,x) \le W(x)$. Hence, $\tilde{W}(x) \le W(x)$ if a storage function W exists. It remains to show that $\tilde{W}(x)$ is a storage function, if $\tilde{W}(x) < \infty$. By dynamic programming, for $0 \le t < s \le T$

$$\tilde{V}(T-t,x(t)) = \sup_{z(\cdot)\in\mathcal{Z}} \left[\int_t^s \left[\mu\ell(x(r)) - \frac{1}{2}|z(r)|^2 \right] dr + \tilde{V}(T-s,x(s)) \right].$$

Since $\tilde{V}(T-t,x(t))$ and $\tilde{V}(T-s,x(s))$ increase to $\tilde{W}(x(t))$ and $\tilde{W}(x(s))$ as $T\to\infty$, $\tilde{W}$ satisfies the dissipation inequality (7.5). Moreover, $\tilde{W}(x) \ge 0$. It remains to show that $\tilde{W}(0) = 0$. Given $\delta > 0$, choose x^δ such that

$$\tilde{W}(x^\delta) < \inf_x \tilde{W}(x) + \delta.$$

The dissipation inequality with $t = 0$, $z(r) = 0$ implies that for all $s > 0$

$$\int_0^s \mu\ell\big(x^0(r)\big)dr \le \delta,$$

where $x^0(s)$ is the solution to (7.1) with $z(s) = 0$ and $x^0(0) = x^\delta$. Since $\ell(x) > 0$ for all $x \ne 0$, $x^\delta \to 0$ as $\delta \to 0$. Thus $\tilde{W}(x)$ has a strict minimum at $x = 0$. If we take $x_0 = 0$, then the dissipation inequality implies

$$0 \le \int_0^T \mu\ell\big(x(s)\big)ds \le \frac{1}{2}\int_0^T |z(s)|^2 ds.$$

Hence $\tilde{V}(T,0) = 0$ for all T, which implies $\tilde{W}(0) = 0$. □

The dynamic programming PDE for the value function $\tilde{V}$ is

$$\text{(7.11)} \qquad \frac{\partial \tilde{V}}{\partial T} + H(x, D_x \tilde{V}) = 0$$

with initial data $\tilde{V}(0, x) = 0$. This is obtained as follows. In Section 6, write $V^0 = V^0(t, x, t_1)$ to indicate dependence on the final time t_1. Then $V^0(t, x, t_1) = \tilde{V}(t_1 - t, x)$ and (6.4^0) becomes (7.11). At least formally, $\tilde{W}$ should satisfy the steady state form of (7.11):

$$\text{(7.12)} \qquad H(x, D_x \tilde{W}) = 0.$$

Later in the section, we will show that (7.12) holds at least in the viscosity sense, under some additional assumptions.

Linear-quadratic case. Let

$$\text{(7.13)} \qquad b(x) = Ax, \quad \sigma(x) = \sigma, \quad \ell(x) = x \cdot \bar{M} x$$

where $Ax \cdot x \leq -c|x|^2$ for all $x \in I\!\!R^n (c > 0)$ and $\bar{M}$ is a positive definite matrix. Let

$$\text{(7.14)} \qquad J = \int_0^T \left[-\mu x(s) \cdot \bar{M} x(s) + \frac{1}{2} |z(s)|^2 \right] ds$$

Then $J = -\tilde{J}$ with $\tilde{J}$ as in (7.7). The problem of minimizing J is equivalent to a $LQRP$ problem with indefinite sign, considered in Example I.5.1. Let $\tilde{P}(T)$ be the $n \times n$ symmetric matrix solution to the Riccati equation

$$\text{(7.15)} \qquad \frac{d\tilde{P}}{dT} = 2\tilde{P}(T) a \tilde{P}(T) + A\tilde{P}(T) + \tilde{P}(T) A' + \mu \bar{M}$$

with $\tilde{P}(0) = 0$, where $a = \sigma\sigma'$. The solution $\tilde{P}(T)$ exists for $0 \leq T < T_1$, where either T_1 is finite or $T_1 = +\infty$. Let

$$\text{(7.16)} \qquad \tilde{W}(T, x) = x \cdot \tilde{P}(T) x, \qquad 0 \leq T < T_1.$$

If we fix t_1 then in the notation of Section I.5, $\tilde{W}(T, x) = -W(t_1 - T, x)$. The ODE (7.15) is equivalent to I(5.15) with $B = \sigma$, $N(t) = \frac{1}{2} I$ and $M(t) = -\mu \bar{M}$ (I is the identity matrix). The Verification Theorem I.5.1 implies that $\tilde{W}$ is the value function $\tilde{V}$ in (7.8). Moreover, by (7.9) $\tilde{P}(T)$ is nonnegative and is a nondecreasing function of T in the ordering of symmetric matrices. If $T_1 = +\infty$ and $\tilde{P}(T)$ tends to a finite limit P_∞ as $T \to \infty$, then

$$\text{(7.17)} \qquad 0 = 2P_\infty a P_\infty + AP_\infty + P_\infty A' + \mu \bar{M}.$$

Moreover, $W_\infty(x) = x \cdot P_\infty x$ is the minimal storage function.

Proposition 7.4. *The linear system in* (7.13) *has H-infinity norm $\leq \gamma$ if and only if* (7.17) *has a symmetric, nonnegative definite solution P, with $\mu = (2\gamma^2)^{-1}$.*

Proof. If P is such a solution to (7.17), let $W(x) = x \cdot Px$. Then $H(x, D_x W(x)) = 0$ and $W(x)$ is a storage function by Proposition 7.2. Conversely, if the H-infinity norm is $\leq \gamma$, then the value function $\tilde{V}(T, x)$ is bounded above by $W(x)$ in (7.4′). This implies that $T_1 = +\infty$ and that $\tilde{V} = \tilde{W}$ in (7.16). The limit P_∞ of $\tilde{P}(T)$ is a solution to (7.17) with the required properties. □

In the remainder of this section, we assume in addition to (7.2), (7.3) that $\ell \in C_b^1(I\!R^n)$. By Theorem 6.1, for any $T_1 < \infty$, $\tilde{V}(T, x)$ is bounded and Lipschitz continuous on $[0, T_1] \times I\!R^n$. Moreover, $\tilde{V}$ is a viscosity solution to

$$\frac{\partial \tilde{V}}{\partial T} + H(x, D_x \tilde{V}) = 0, \qquad T > 0 \tag{7.18}$$

with $\tilde{V}(0, x) = 0$. The Lipschitz constant may depend on T_1. However, if σ is constant, then the following result gives a uniform Lipschitz constant for $\tilde{V}(T, \cdot)$ which does not depend on T.

Theorem 7.1. *If σ is constant, then*

$$|\tilde{V}(T, x) - \tilde{V}(T, y)| \leq \mu c^{-1} \|\ell_x\| \cdot |x - y| \tag{7.19}$$

where the constant c is as in (7.2) *and* $\|\ \ \|$ *is the sup norm.*

Proof. For any $z(\cdot) \in \mathcal{Z}$, x and y, let $x(s)$ be the solution to (7.1) with $x(0) = x$ and $y(s)$ the solution to (7.1) with $y(0) = y$. Let $\zeta(s) = x(s) - y(s)$. Since σ is constant, the mean value theorem gives $\dot{\zeta}(s) = A(s)\zeta(s)$ where

$$A(s) = \int_0^1 b_x(y(s) + \lambda \zeta(s)) d\lambda.$$

with $\zeta(0) = x - y$. By (7.2)(b')

$$\frac{d}{ds} |\zeta(s)|^2 = 2A(s)\zeta(s)\dot{\zeta}(s) \leq -2c|\zeta(s)|^2,$$

which implies $|\zeta(s)| \leq |x - y| e^{-cs}$. Then

$$|\tilde{J}(T, x; z) - \tilde{J}(T, y; z)| \leq \mu \int_0^T |\ell(x(s)) - \ell(y(s))| ds$$

$$\leq \mu \|\ell_x\| \int_0^T |\zeta(s)| ds \leq \mu c^{-1} \|\ell_x\| \cdot |x - y|.$$

Since this is true for all $z(\cdot)$, we obtain (7.19). □

Corollary 7.1. *Assume that σ is constant and that $\tilde{W}(x) < \infty$ for all x. Then $\tilde{W}$ is a Lipschitz continuous viscosity solution to* (7.11)*.*

Remark 7.1. The proof of Theorem 7.1 uses the bound for ℓ_x, but not a bound for ℓ. If ℓ_x is bounded but not ℓ, then a refinement of the arguments above give $\tilde{V}(T,x)$ as a viscosity solution to (7.11) which may grow linearly with $|x|$ as $|x| \to \infty$. Another interesting case is when $\ell(s)$ grows quadratically as $|x| \to \infty$, as for the linear-quadratic problem above. This more difficult case has been studied in [Mc3][DLM]. In addition, a local Lipschitz estimate for $\tilde{V}(\tau,\cdot)$ independent of T is obtained in [Mc3], with $\sigma(x)$ nonconstant and weaker assumptions on ℓ. See also [Mc5].

Remark 7.2. If $\sigma(x)$ is not constant, then it is more difficult to obtain a Lipschitz estimate for $\tilde{V}(T,\cdot)$ independent of T. If $|x||\sigma_x(x)\sigma'(x)|$ is bounded on $I\!R^n$, then such an estimate can be found by the method of [MI].

Remark 7.3. Let $\bar{\gamma}$ denote the H-infinity norm. By (7.4) if $\bar{\gamma} < \gamma$ then $\tilde{V}(T,x)$ tends to the finite limit $W(x)$ which is the smallest storage function by Proposition 7.3. If $\bar{\gamma} > \gamma$, then the behavior of $\tilde{V}(T,x)$ as T increases depends on the behavior of $\ell(x)$ for $|x|$ large. We mention two cases.

Case 1. As in the discussion preceding Theorem 7.1, assume that $\ell \in C_b^1(I\!R^n)$ and that σ is constant. For $\bar{\gamma} > \gamma$, $T^{-1}\tilde{V}(T,x)$ tends to a constant $\lambda > 0$ as $T \to \infty$. See [FM2][FJ]. This result is proved using asymptotic properties as $\varepsilon \to 0$, $T \to \infty$ of solutions to the SDE (see (6.1^ε))

$$dx^\varepsilon = b(x^\varepsilon(s))ds + \varepsilon^{\frac{1}{2}}\sigma dw^0(s), \qquad s \geq 0.$$

Case 2. Assume (as in the linear-quadratic case above) that $\ell(x)$ is unbounded, with at most quadratic growth as $|x| \to \infty$. For $\bar{\gamma} > \gamma$, $\tilde{V}(T,x)$ may tend to infinity as $T \to T_\infty < \infty$. See [BN2][DLM][KN1,2].

VI.8 Risk sensitive control

In this section we again consider controlled Markov diffusion processes. The criterion J to be minimized is now of the exponential form (8.3) below instead of the form IV(2.8) considered in Chapter IV. This is called a finite time horizon, risk-sensitive stochastic control problem. Risk sensitivity is indicated by a parameter $\rho > 0$, which has the same role as in Sections 2–5 above. Later in the section, we mention small noise limits, in a way similar to Section 6. Deterministic, two-controller, zero-sum differential games are obtained as small noise limits of risk-sensitive stochastic control problems. Such games will be considered in Section XI.7.

The state $x(s) \in I\!\!R^n$ for the risk sensitive stochastic control problem evolves according to the stochastic differential equation

$$dx = f(s, x(s), u(s))ds + \rho^{-\frac{1}{2}}\sigma(s, x(s), u(s))dw^0(s), t \le s \le t_1, \tag{8.1}$$

where $u(s) \in U$ is the control at time s and $w^0(s)$ is a d-dimensional brownian motion. Let

$$\mathcal{J} = \int_t^{t_1} \ell(s, x(s), u(s))ds \tag{8.2}$$

$$J(t, x; u) = E^0_{tx}[\exp(\rho\mathcal{J})]. \tag{8.3}$$

In (8.3), either $J < \infty$ or $J = +\infty$. Note that when the control $u(s)$ does not appear in (8.1) or (8.2), then (8.1) is the same as (3.1) with $f = b$ and (8.2) is the same as (3.4) with $\psi = 0$. A terminal cost function ψ could be included, but this slightly complicates the exposition.

We assume throughout this section that $f, \sigma \in C^1(\bar{Q}_0 \times U)$. Moreover

(8.4)

(a) The first order partial derivatives of f are bounded;

(b) σ and its first order partial derivatives are bounded, *i.e.* $\sigma \in C^1_b(\bar{Q}_0 \times U)$.

The "running cost" function ℓ is assumed continuous throughout. Further assumptions about ℓ will be made as needed.

The risk sensitive stochastic control problem is as follows. Given initial data $x(t) = x$, choose the control process $u(\cdot)$ to minimize $J(t, x; u)$. More precisely (see Section IV.2) a reference probability system ν and a progressively measurable control process $u(\cdot)$ are to be chosen to minimize $J(t, x; u)$. Let $\Phi(t, x)$ denote the value function for this risk sensitive control problem. It is well defined provided $J(t, x; u) < \infty$ for some control process $u(\cdot)$. In particular, $\Phi(t, x)$ is a bounded, positive function if ℓ is bounded above. The associated Hamilton-Jacobi-Bellman PDE is (see IV(3.21)):

$$-\frac{\partial \Phi}{\partial t} + \mathcal{H}(t, x, D_x\Phi, D_x^2\Phi, \Phi) = 0, \qquad (t, x) \in Q_0, \tag{8.5}$$

$$\mathcal{H}(t, x, p, A, \Phi) = -\inf_{v \in U}\left[f(t, x, v) \cdot p + \frac{1}{2\rho}\operatorname{tr} a(t, x, v)A + \rho\ell(t, x, v)\Phi\right]. \tag{8.6}$$

Theorem 8.1. *Assume that U is compact, that* (8.4) *holds and that* $\ell \in C^1_b(\bar{Q}_0 \times U)$. *Then the value function Φ is a Lipschitz continuous viscosity solution of* (8.5) *with terminal data* $\Phi(t_1, x) = 1$.

Proof. As noted above, Φ is bounded and $\Phi(t,x) > 0$, since ℓ is bounded. Consider an "augmented" state $\hat{x}(s) = (x(s), x_{n+1}(s))$, where

$$dx_{n+1} = \rho\ell(s,x(s),u(s))x_{n+1}(s)ds\,, \qquad s \in [t,t_1] \tag{8.7}$$

with $x_{n+1}(t) > 0$. Then $|x_{n+1}(s)| \le R|x_{n+1}(t)|$, where $R = \exp(\rho\|\ell\|(t_1 - t_0))$. Choose $h \in C_b^2(I\!R^1)$ such that $h(x_{n+1}) = x_{n+1}$ if $|x_{n+1}| \le 2R$. Let

$$\hat{J}(t,\hat{x};u) = E^0_{t\hat{x}}\left[h(x_{n+1}(t_1))\right],$$

and $\hat{\Phi}(t,\hat{x})$ the infimum of $\hat{J}$ over all ν and $u(\cdot)$. Then $\hat{\Phi}(t,\hat{x}) = x_{n+1}\Phi(t,x)$ for $0 < x_{n+1} < 2$. By Lemmas IV.8.1, IV.8.2 and comments after IV(6.2), $\hat{\Phi}$ is Lipschitz continuous. Hence $\Phi(t,x) = \hat{\Phi}(t,x,1)$ is Lipschitz continuous.

To show that Φ is a viscosity solution to (8.5), suppose that $\phi \in C^{1,2}(Q_0)$ and that $\Phi(t,x) - \phi(t,x)$ has a local maximum at $(\bar{t},\bar{x})$ with $\Phi(\bar{t},\bar{x}) = \phi(\bar{t},\bar{x})$. Let $\hat{\phi}(t,\hat{x}) = x_{n+1}\phi(t,x)$. Then $\hat{\Phi}(t,\hat{x}) - \hat{\phi}(t,\hat{x})$ has a local maximum at $(\bar{t},\bar{x},1)$ with $\hat{\Phi}(\bar{t},\bar{x},1) = \phi(\bar{t},\bar{x},1)$. By Theorem V.3.1, $\hat{\Phi}$ is a viscosity solution of

$$-\frac{\partial\hat{\Phi}}{\partial t} + \hat{\mathcal{H}}(t,x,D_x\hat{\Phi},D_x^2\hat{\Phi},\hat{\Phi}_{x_{n+1}}) = 0, \tag{8.8}$$

$$\hat{\mathcal{H}}(t,x,p,A,p_{n+1}) = \mathcal{H}(t,x,p,A,x_{n+1}p_{n+1}).$$

Since $\hat{\phi}_{x_{n+1}} = \phi$ in a neighborhood of $(\bar{t},\bar{x},1)$, the definition of viscosity solution for PDEs implies

$$-\frac{\partial\phi}{\partial t} + \mathcal{H}(\bar{t},\bar{x},D_x\phi,D_x^2\phi,\phi) \le 0$$

at $(\bar{t},\bar{x})$. Hence Φ is a viscosity subsolution. Similarly, Φ is a viscosity supersolution of (8.5). □

Remark 8.1. Under stronger assumptions, the value function $\Phi \in C_b^{1,2}(\bar{Q}_0)$ and satisfies (8.5) in the classical sense. As in Theorem IV.4.2, it suffices to assume the uniform parabolicity condition IV(3.5) and that $a = \sigma\sigma'$, f and ℓ satisfy IV(4.5)(b)(c).

Remark 8.2. Theorem 8.1 remains true if a terminal cost $\psi(x(t_1))$ is included in (8.2), provided ψ is bounded and Lipschitz continuous. In the proof above, take

$$\hat{J}(t,\hat{x};u) = E^0_{t\hat{x}}\left[h(x_{n+1}(t_1))\exp(\rho\psi(x(t_1))\right].$$

Logarithmic transformation of Φ. We now let

$$V(t,x) = \rho^{-1} \log \Phi(t,x). \tag{8.9}$$

To obtain a PDE for V, we make the following observation. Suppose that $\phi \in C^{1,2}(Q_0)$ with $\phi > 0$ and let $w = \rho^{-1} \log \phi$. A direct calculation gives

$$-\frac{\partial \phi}{\partial t} + \mathcal{H}(t,x,D_x\phi, D_x^2\phi, \phi) = \phi \left[-\frac{\partial w}{\partial t} + \bar{\mathcal{H}}(t,x,D_x w, D_x^2 w) \right] \tag{8.10}$$

$$\bar{\mathcal{H}}(t,x,p,A) = -\min_{v \in U} \left[f \cdot p + \frac{1}{2\rho} \mathrm{tr}\, aA + \frac{1}{2} a\, p \cdot p + \ell \right] \tag{8.11}$$

with f, a, ℓ evaluated at (t,x,v). From (8.10), Theorem 8.1 and the definition II.4.2 of viscosity solution for PDEs:

Corollary 8.1. *V is a bounded, Lipschitz continuous viscosity solution to*

$$-\frac{\partial V}{\partial t} + \bar{\mathcal{H}}(t,x,D_x V, D_x^2 V) = 0 \tag{8.12}$$

with terminal data $V(t_1,x) = 0$.

Note that when f, a and ℓ depend only on (t,x) then (8.12) is the same as (4.7) with $b = f$. In Sections 3 and 4, the logarithmic transformation introduced an auxiliary control process $z(s)$. The same happens for the risk-sensitive stochastic control problem. In (8.11)

$$\frac{1}{2} a(t,x,v) p \cdot p = \max_{\xi \in I\!R^d} \left[p \cdot \sigma(t,x,v)\xi - \frac{1}{2}|\xi|^2 \right]. \tag{8.13}$$

In this way, $-\bar{\mathcal{H}}$ becomes a min-max, taken over $v \in U$, $\xi \in I\!R^d$. There is an associated two-controller, zero-sum stochastic differential game, with controls $u(s) \in U$, $z(s) \in I\!R^d$. The PDE (8.12) is called the Isaacs equation for this stochastic differential game [FS3][Sw]. In this book, we will not discuss stochastic differential games further. However, deterministic differential games which arise as small noise limits of risk sensitive stochastic control problems are considered in Section XI.7. If we write $\varepsilon = \rho^{-1}$ and let $\varepsilon \to 0$, then in the limit $\bar{\mathcal{H}}$ becomes a first-order partial differential operator, which will be denoted by $H_+(t,x,p)$ in Chapter XI. An analogue of the small noise limit Theorem 6.2 will be proved there (Theorem XI.7.2).

Verification Theorem. We conclude this section with a result for the risk sensitive control problem which is similar to the verification Theorem IV.3.1. For simplicity, let us assume that $\sigma = \sigma(t,x)$. Then (8.12) takes the form:

$$-\frac{\partial V}{\partial t} - \frac{1}{2\rho} \mathrm{tr}\, a(t,x) D_x^2 V - \frac{1}{2} a(t,x) D_x V \cdot D_x V + \bar{H}(t,x,D_x V) = 0, \tag{8.14}$$

$$\bar{H}(t,x,p) = -\min_{v\in U}\left[f(t,x,v)\cdot p + \ell(t,x,v)\right]. \tag{8.15}$$

Let $\mathcal{L}$ denote the class of Markov control policies $\underline{u}$ defined by IV(3.12). For $\underline{u}\in\mathcal{L}$, let

$$b^{\underline{u}}(t,x) = f(t,x,\underline{u}(t,x))$$

$$\ell^{\underline{u}}(t,x) = \ell(t,x,\underline{u}(t,x)).$$

The solution $x(t)$ to (8.1) with $u(s)=\underline{u}(s,x(s))$ and $x(t)=x$ is the same as for (3.1) with $b=b^{\underline{u}}$. Also $J(t,x;u)=J(t,x;\underline{u})$ is the same as (3.14) with $\ell=\ell^{\underline{u}}$.

Theorem 8.2. *Assume that* (8.4) *holds, that* $\sigma=\sigma(t,x)$ *and that* $\ell\in C^1(\bar{Q}_0\times U)$. *Also suppose that* $W\in C^{1,2}(\bar{Q}_0)$ *is a solution to* (8.14) *such that* $|D_xW(t,x)|\le M(1+|x|)$ *and* $W(t_1,x)=0$. *Then:*

(a) $W(t,x)\le\rho^{-1}\log J(t,x;\underline{u})$ *for all* $\underline{u}\in\mathcal{L}$*;*
(b) *If there exists* $\underline{u}^*\in\mathcal{L}$ *such that*

$$\bar{H}(t,x,D_xW(t,x)) = -\left[b^{\underline{u}^*}(t,x)\cdot D_xW(t,x)+\ell^{\underline{u}^*}(t,x)\right] \tag{8.16}$$

for all $(t,x)\in\bar{Q}_0$, *then* $W(t,x)=\rho^{-1}\log J(t,x;\underline{u}^*)$.

Proof. We write

$$\bar{H}(t,x,D_xW(t,x)) = -b^{\underline{u}}(t,x)\cdot D_xW(t,x) - \Lambda^{\underline{u}}(t,x) \tag{8.17}$$

where $\Lambda^{\underline{u}}(t,x)\le\ell^{\underline{u}}(t,x)$ by (8.15). By Theorem 4.1(b) with $b=b^{\underline{u}}$, $\ell=\Lambda^{\underline{u}}$,

$$\begin{aligned} W(t,x) &= \rho^{-1}\log E^0_{tx}\left[\exp\rho\int_t^{t_1}\Lambda^{\underline{u}}(s,x^{\underline{u}}(s))ds\right] \\ &\le \rho^{-1}\log J(t,x;\underline{u}). \end{aligned}$$

Here $x^{\underline{u}}(s)$ is the solution to (3.1) with $x(t)=x$ and $b=b^{\underline{u}}$, which is the same as $x(s)$ in (8.1) with $u(t)=\underline{u}(t,x(t))$. This proves (a). Since $\Lambda^{\underline{u}^*}=\ell^{\underline{u}^*}$ we get (b). □

The Markov control policy $\underline{u}^*$ in Theorem 8.2(b) is optimal for the risk sensitive stochastic control problem.

Example 8.1. Linear exponential-of-quadratic regulator (LEQR) problem. In a way similar to the stochastic linear regulator problem (Example III.8.1) let $U=I\!R^m$, $f(t,x,v)=A(t)x+B(t)v$, $\sigma=\sigma(t)$, $\ell(t,x,v)=$

$x \cdot M(t)x + v \cdot N(t)v$, where $M(t), N(t)$ are symmetric matrices and $N(t)$ is positive definite. (We now take $D = 0$ in III(8.7)). The problem of minimizing $J(t, x; u)$ is the called a LEQR problem. For the LEQR problem, $\bar{H} = H$ where H is the Hamiltonian I(5.12) for the linear regulator problem. Then (8.14) becomes

$$(8.18)\quad -\frac{\partial V}{\partial t} - \frac{1}{2\rho}\operatorname{tr} a(t)D_x^2 V + \frac{1}{4}\tilde{a}(t)D_x V \cdot D_x V - A(t)x \cdot D_x V - x \cdot M(t)x = 0$$

$$(8.19)\qquad \tilde{a}(t) = B(t)N^{-1}B'(t) - 2a(t).$$

By the same method as for the stochastic linear regulator problem, there is a solution $W(t, x)$ to (8.18) with $W(t_1, x) = 0$ of the form

$$(8.20)\qquad W(t, x) = x \cdot \tilde{P}(t)x + \tilde{g}(t), \quad t_{\min} < t \leq t_1,$$

with either $t_{\min}$ finite or $t_{\min} = -\infty$. The symmetric matrices $\tilde{P}(t)$ satisfy the matrix Riccati differential equation (see I(5.15))

$$(8.21)\quad \frac{d}{dt}\tilde{P}(t) = \tilde{P}(t)\tilde{a}(t)\tilde{P}(t) - A(t)\tilde{P}(t) - \tilde{P}(t)A'(t) - M(t), t_{\min} < t \leq t_1$$

with $\tilde{P}(t_1) = 0$ and

$$(8.22)\qquad \tilde{g}(t) = \rho^{-1}\int_t^{t_1} \operatorname{tr} \tilde{a}(s)\tilde{P}(s)ds.$$

The optimal control policy for the LEQR problem is

$$(8.23)\qquad \underline{u}^*(t, x) = -N^{-1}(t)B'(t)\tilde{P}(t)x, \quad t_{\min} < t \leq t_1.$$

A sufficient condition that $t_{\min} = -\infty$ is that the matrices $M(t)$ and $\tilde{a}(t)$ are positive definite for all t. See the discussion for the linear regulator problem (Example I.5.1). Similarly, $t_{\min} = -\infty$ if $M(t)$ and $\tilde{a}(t)$ are negative definite.

VI.9 Logarithmic transformations for Markov processes

In this section we shall outline how to extend some of the results described in Sections 3-5 for Markov diffusion processes to other classes of Markov processes. These results are based mainly on Sheu [Sh1]. Following the notation of Section III.2, let A be the backward evolution operator of a Markov process $x(s)$, with state space Σ. Let $\Phi(t, x)$ be a positive solution to

$$(9.1)\qquad A\Phi + \ell(t, x)\Phi = 0, \quad (t, x) \in [t_0, t_1] \times \Sigma.$$

We make the logarithmic transformation

$$V = -\log \Phi. \tag{9.2}$$

Thus, in the notation of Section 4 we take $\rho = -1$. An equation for V is derived formally, as follows. As in III(2.9), write

$$A\Phi = \frac{\partial \Phi}{\partial t} - G_t \Phi.$$

Then V satisfies (at least formally)

$$-\frac{\partial V}{\partial t} + \mathcal{H}(V) = 0, \text{ where} \tag{9.3}$$

$$\mathcal{H}(V) = -e^V G_t(e^{-V}) + \ell. \tag{9.4}$$

If A is the backward evolution operator of a Markov diffusion in $I\!\!R^n$ then (9.3) becomes the HJB partial differential equation (4.11). We wish to interpret (9.3) as the dynamic programming equation for a stochastic control problem, when A is the backward evolution operator for some other kind of Markov process. To avoid all technical difficulties, we will describe the procedure in complete detail for finite state Markov chains, and merely indicate how it extends to other classes of Markov processes.

Logarithmic transformations for Markov chains. We now let Σ be a finite set. To further simplify matters, we consider time-homogeneous Markov chains $x(s)$ on the time interval $[0, \infty)$ with state space Σ. According to formula III(4.1)

$$G\phi(x) = -\sum_{y \neq x} \rho(x,y)[\phi(y) - \phi(x)], \;\; x \in \Sigma, \tag{9.5}$$

where $\rho(x,y)$ is the infinitesimal rate at which $x(s)$ jumps from x to y if $x(s) = x$. Let $v(\cdot)$ by any positive function on Σ ($v(x) > 0$ for all $x \in \Sigma$); and let G^v be the linear operator defined by

$$G^v \phi = \frac{1}{v}[G(\phi v) - \phi G v]. \tag{9.6}$$

From (9.5)

$$G^v \phi(x) = -\sum_{y \neq x} \frac{\rho(x,y)v(y)}{v(x)}[\phi(y) - \phi(x)]. \tag{9.7}$$

Thus $-G^v$ is the generator of a Markov chain with infinitesimal jumping rates $v(x)^{-1}\rho(x,y)v(y)$. It is not difficult to show that the corresponding semigroup T_t^v satisfies

$$T_t^v\phi(x) = \frac{1}{v(x)} E_x\left[\phi(x(s))v(x(s))\exp\left\{\int_0^s \frac{Gv(x(r))}{v(x(r))}dr\right\}\right]. \tag{9.8}$$

(We will not use (9.8).) Let

$$U = \{v(\cdot) : v(x) > 0 \text{ for all } x \in \Sigma\}. \tag{9.9}$$

U will be the control space for the stochastic control problem which we are going to formulate. The key step in the construction of this control problem is the following lemma.

Lemma 9.1. *For each $v(\cdot) \in U$, let*

$$k^v = -G^v(\log v) + \frac{Gv}{v}. \tag{9.10}$$

Then, for each ϕ,

$$\min_U[-G^v\phi + k^v] = e^\phi(Ge^{-\phi}). \tag{9.11}$$

The minimum is attained when $v = \exp(-\phi)$.

Proof. Fix $x \in \Sigma$ and consider the function

$$\begin{aligned} F(y) = \frac{1}{v(x)}\Big[&(v\phi)(y) - \phi(x)v(y) + (v\log v)(y) \\ &-(\log v(x))v(y) - v(y)\Big] + \exp(\phi(x) - \phi(y)) \\ = \frac{v(y)}{v(x)}&(\phi(y) - \phi(x)) + \frac{v(y)}{v(x)}\log\frac{v(y)}{v(x)} \\ &-\frac{v(y)}{v(x)} + \exp(\phi(x) - \phi(y)). \end{aligned}$$

An elementary calculation shows that

$$\min_{z \in \mathbb{R}^1}[e^{-z} + az] = a - a\log a, \quad a > 0. \tag{9.12}$$

By taking $z = \phi(y) - \phi(x), \quad a = v(x)^{-1}v(y)$, we obtain

$$F(y) \geq 0, \text{ all } y \in \Sigma \quad , \quad F(x) = 0.$$

Since F has a minimum at x, $GF(x) \leq 0$. This is equivalent to

$$-G^v\phi(x) + k^v(x) \geq e^{\phi(x)}(Ge^{-\phi})(x), \tag{9.13}$$

for every $v = v(\cdot) \in U$. If we take $v = \exp(-\phi)$, then $F \equiv 0$ and hence $GF \equiv 0$. In that case equality holds in (9.13). □

As running cost, let

$$L(x,v) = k^v(x) - \ell(x). \tag{9.14}$$

As in (4.3), consider the stochastic control problem: minimize

$$J(t,x;u) = E_{tx}\left\{\int_t^{t_1} L(x(s),u(s))ds + \psi(x(t_1))\right\}, \tag{9.15}$$

where $\exp(-\psi) = \Phi(t_1,\cdot)$. The dynamic programming equation for this problem is (see III(7.5))

$$V_t(t,x) + \min_{v\in U}\left[-G^v V(t,\cdot)(x) + L(x,v)\right] = 0. \tag{9.16}$$

By Lemma 9.1, with $\phi = V(t,\cdot)$, this is the same as equation (9.3). The minimum in (9.16) is attained for $v = \underline{u}^*(t,x)$, where

$$\underline{u}^*(t,x) = \exp(-V(t,x)) = \Phi(t,x). \tag{9.17}$$

We summarize these results as follows.

Theorem 9.1. (Σ finite) *Let Φ be a positive solution to* (9.1), *with $G_t = G$, where $-G$ is the generator of a finite state time-homogeneous Markov chain. Then $V = \exp(-\Phi)$ is the value function for a stochastic control problem, with* (9.16) *as dynamic programming equation. Moreover, $\underline{u}^* = \Phi$ is an optimal Markov control policy.*

Extensions of Theorem 9.1. The definitions of G^v and k^v in (9.6) and (9.10), and the proof of Lemma 9.1, do not use at all the particular form (9.5) of the generator. Hence, it is only a technical matter to extend Theorem 9.1 to a much wider class of time-homogeneous Markov processes. For simplicity, let us consider the case when the generator G is a bounded operator on $C_b(\Sigma)$. Then G^v and k^v are exactly as in (9.6), (9.10), provided $v = v(\cdot) \in U$, where

$$U = \{v(\cdot) : v(x) > 0 \text{ for all } x \in \Sigma, v(\cdot), v^{-1}(\cdot) \in C_b(\Sigma)\}.$$

In the extension of Theorem 9.1 one needs to assume that $\Phi \in C_b([t_0,t_1]\times\Sigma)$, with $\Phi(t,\cdot) \in U$ for each $t \in [t_0,t_1]$.

Example 9.1. Let $-G$ be the generator of a jump Markov process, of the form

$$G\phi(x) = -\int_\Sigma \rho(x,y)[\phi(y)-\phi(x)]\lambda(dy)$$

where $\rho \in C_b(\Sigma\times\Sigma)$ and λ is a finite positive measure on $\mathcal{B}(\Sigma)$. Then G^v has the same form, with $\rho(x,y)$ changed to $v(x)^{-1}v(y)\rho(x,y)$.

VI.10 Historical remarks

The use of logarithmic transformations goes back at least to Schrödinger [Sch] in his fundamental work on wave mechanics in 1926. In 1950, E. Hopf [Hf] connected the Burgers equation and the heat equation using a logarithmic transformation. This paper was influential for subsequent developments in the theory of weak solutions to conservation laws. In a stochastic control setting, logarithmic transformations were introduced independently by Fleming [F4] and by Holland [Ho] in the late 1970's. Evans and Ishii [EI] were the first to apply PDE/viscosity solutions methods to the Freidlin Wentzell small-noise asymptotics for exit problems (Section 6). For further work in that direction see [BP1] [FS1], and for more refined results in the form of asymptotic series [FS2] [FSo]. An application of the logarithmic transformation to the exit problem in Section 5 was made in [Day].

The notions of H-infinity norm and H-infinity control theory were initially developed for linear systems, using frequency domain methods. See Green-Limbeer [GL] and references cited there. Glover-Doyle [GD] considered a state space formulation for the linear case, and its relation to risk sensitive control. Since then a substantial nonlinear H-infinity theory has developed. See Ball-Helton [BH], Basar-Bernhard [BB], Helton-James [HJ], Soravia [Sor] and vander Schaft [VdS]. The treatment of H-infinity norms in Section 7 follows [F6].

In 1973 Jacobson [Jac] found the solution of the LEQR problem. A theory of linear risk sensitive control, with complete or partial state information, was developed by Whittle [Wh1] and Bensoussan-van Schuppen [BvS]. For further results about nonlinear risk sensitive control, with complete state information, see Bensoussan-Frehse-Nagai [BFN], Nagai [Na1] and Fleming-McEneaney [FM2]. For the discrete time case, with partial state information, see James-Baras-Elliott [JBE].

VII

Singular Perturbations

VII.1 Introduction

In Section II.6, we proved that any uniform limit of a sequence of viscosity solutions is again a viscosity solution of the limiting equation. This stability of viscosity solutions under any approximation or relaxation is an extremely important property which has been applied to many different problems in large deviations, homogenization, numerical analysis, singular perturbations and large time behavior of reaction-diffusion equations. In this chapter we outline a procedure of Barles and Perthame which is common in all these applications. We then apply this procedure to the problem of vanishing viscosity and to a large deviations problem of probability theory. Analytical applications to homogenization and to reaction-diffusion equations require nontrivial modifications of the procedure outlined there. Several papers of Barles, Evans and Souganidis provide an excellent discussion of these problems [BES], [E3], [ES2]. Applications to the numerical analysis of control problems will be the subject of Chapter IX.

Now consider a situation in which we want to prove the convergence of a sequence of viscosity solutions V^ε to the viscosity solution V of the limiting equation. Examples of this situation are given in Section 2 below. In view of Lemma II.6.2 or II.6.3, it suffices to show the precompactness of the sequence V^ε in the uniform topology and the uniqueness of viscosity solutions to the limiting equation with appropriate boundary and terminal data. Then using the results of Section II.6 we conclude that any uniform limit of the sequence V^ε is a viscosity solution of the limiting equation. We may then obtain the desired convergence result by invoking the uniqueness result for the limiting equation. This procedure has been successfully applied to several problems. However the main difficulty in applying the outlined procedure is to obtain the precompactness of the sequence V^ε in the uniform topology. In some cases approximating equations can be used to prove a uniform Lipschitz (or Hölder) estimate for the sequence V^ε. Then the precompactness and consequently the convergence of V^ε follows from the Ascoli-Arzela Theorem.

Barles and Perthame [BP1] modified the above procedure. In their modified procedure the precompactness condition is replaced by a much weaker uniform boundedness condition. This assumption enabled them to define two possible limit functions: V^* the largest possible limit point, and V_* the smallest possible one, see (3.2) and (3.3) below. Then Barles and Perthame observed that V^* is a discontinuous viscosity subsolution of the limiting equation. Similarly V_* is a discontinuous viscosity supersolution of the same equation. Since for a large class of problems subsolutions are less than supersolutions, we formally expect that $V^* \leq V_*$. Also the reverse inequality is evident from the construction. Therefore the largest limit point V^* is equal to the smallest limit point V_* and consequently the sequence V^ε is uniformly convergent. Moreover the limit point $V = V^* = V_*$ is both a viscosity subsolution and a viscosity supersolution of the limiting equation.

The Barles-Perthame procedure requires a discussion of discontinuous viscosity sub and supersolutions. In Definition 4.1 below, we give the definition of discontinuous viscosity sub and supersolution. The main difference between Definition 4.1 and Definition II.4.2 is the use of the upper semi-continuous envelope in the case of a subsolution and the use of the lower semi-continuous envelope in the case of a supersolution. Implicit in the Barles-Perthame procedure is a comparison result between a discontinuous viscosity subsolution and a discontinuous viscosity supersolution of the limiting equation satisfying the terminal and boundary conditions. We prove several comparison results in Section 8. However to apply these comparison results to V^* and V_*, we need to show that V^* and V_* satisfy the same terminal and boundary conditions. In Section 5 we show that V^* and V_* pointwise satisfy the terminal data (2.4)(b). The discussion of the lateral boundary data is more delicate as V^* and V_* satisfy (2.4)(a) only in the viscosity sense. The viscosity formulation of boundary conditions for continuous solutions was given in Section II.13. In Section 6 below, we first give an extension of Definition II.13.1 and then show that V^* and V_* satisfy (2.4)(a) in the viscosity sense. The combination of all these results yields several convergence results which are stated in Section 7.

We close the chapter with some applications. In the first one we consider the classical problem of vanishing viscosity. In this application the convergence of V^ε follows immediately from the results of Section 7, if V^ε is uniformly bounded in ε. This uniform bound is an easy consequence of the classical maximum principle, Theorem 9.1 below. Our second application is a large deviations problem for exit probabilities. We prove the exponential convergence of the exit probabilities in Section 10. In that section we also identify the rate of their exponential convergence as the value function of a deterministic control problem. A similar large deviations result on a fixed time interval is proved in Section 11. The same method will be used in Section XI.7 for small noise, risk sensitive control limits.

VII.2 Examples

In Sections 2 through 10 of this chapter we assume that $Q = [t_0, t_1) \times O$, where $O \subset I\!R^n$ is open and bounded. V^ε is a viscosity solution of

$$(2.1^\varepsilon) \qquad -\frac{\partial}{\partial t} V^\varepsilon(t,x) + (\mathcal{G}_t^\varepsilon V^\varepsilon(t,\cdot))(x) = 0, \ (t,x) \in Q.$$

satisfying

$$(2.2) \qquad V^\varepsilon(t,x) = \Psi^\varepsilon(t,x) = \begin{cases} g^\varepsilon(t,x), t \in [t_0,t_1), x \in \partial O, \\ \\ \psi^\varepsilon(x), \quad t = t_1, x \in \overline{O}. \end{cases}$$

As remarked in Section II.4, viscosity solutions of $(2.1)^\varepsilon$ are defined if the nonlinear operators $\mathcal{G}_t^\varepsilon$ have the maximum principle property. Since the perturbation theory we will develop extends to such operators with no difficulty, we only assume that $\mathcal{G}_t^\varepsilon$ has the maximum principle. Recall that when $\mathcal{G}_t^\varepsilon$ has the maximum principle property, any classical solution of $(2.1)^\varepsilon$ is also a viscosity solution of $(2.1)^\varepsilon$ in Q.

When $\mathcal{G}_t^\varepsilon$ is a nonlocal infinitesimal generator, the space Σ introduced in Chapter II may not be equal to $\overline{O}$. However to simplify the exposition we assume $\Sigma = \overline{O}$ and that the domain of $\mathcal{G}_t^\varepsilon$ contains $C^\infty(\overline{Q})$.

Let us assume that g^ε and ψ^ε converge uniformly to g and ψ, respectively, and $\mathcal{G}_t^\varepsilon$ converges to a first order partial differential operator H, in the following sense:

$$\lim_{\substack{(s,y)\to(t,x) \\ \varepsilon\downarrow 0}} (\mathcal{G}_s^\varepsilon[w(s,\cdot) + K^\varepsilon])(y) = H(t,x,D_x w(t,x), w(t,x)),$$

for all $(t,x) \in \overline{Q}, w \in C^\infty(\overline{Q})$ and real numbers $K^\varepsilon \to 0$ as $\varepsilon \to 0$. (The theory that will be developed in this chapter also applies to the cases when the limiting operator is not a first-order partial differential operator.) Then under these assumptions we formally expect that V^ε converges to a solution of the limiting equation

$$(2.3) \qquad -\frac{\partial}{\partial t} V(t,x) + H(t,x,D_x V(t,x), V(t,x)) = 0, \ (t,x) \in Q.$$

Indeed we will show that for a general class of problems V^ε converges, as $\varepsilon \downarrow 0$, to the unique viscosity solution V of (2.3) satisfying the boundary data

$$(2.4)(a) \qquad V(t,x) = g(t,x), \ (t,x) \in [t_0,t_1) \times \partial O,$$

$$(2.4)(b) \qquad V(t_1,x) = \psi(x), \quad x \in \overline{O},$$

in the viscosity sense. Recall that a viscosity formulation of the boundary data is given in Section II.13 and a further discussion is included in Section 6.

We give two illustrative examples of nonlinear operators $\mathcal{G}_t^\varepsilon$.

Example 2.1. (Vanishing viscosity) A classical approximation of the equation (2.3) is

$$(2.5^\varepsilon) \qquad -\frac{\partial}{\partial t}V^\varepsilon(t,x) - \frac{\varepsilon}{2}\operatorname{tr} a(t,x)D_x^2 V^\varepsilon(t,x) + H^\varepsilon(t,x,D_xV^\varepsilon(t,x)) = 0,$$

where $a(t,x)$ is an $(n\times n)$ symmetric matrix satisfying

$$\sum_{i,j=1}^n a_{ij}(t,x)\xi_i\xi_j \geq c|\xi|^2, \ \ \forall \xi \in I\!R^n, (t,x)\in \overline{Q},$$

with an appropriate constant $c>0$. We also assume that H^ε approximates the nonlinear function H. A special case of this is obtained when H^ε is as in I(5.4),

$$H^\varepsilon(t,x,p) = \sup_{v\in U}\{-f^\varepsilon(t,x,v)\cdot p - L^\varepsilon(t,x,v)\}, \ (t,x)\in\overline{Q}, p\in I\!R^n,$$

and $f^\varepsilon, L^\varepsilon$ are uniformly convergent to f, L. Then Theorem IV.4.1 implies that there exists a classical solution V^ε of (2.5^ε) and (2.2), provided that $a, f^\varepsilon, L^\varepsilon, g^\varepsilon$ and Ψ^ε satisfy IV(4.1).

Now suppose that there exists a classical solution V^ε of (2.5^ε) and (2.2), and that V^ε converges to V uniformly on $\overline{Q}$ as ε tends zero. Then due to the stability result, Lemma II.6.2, V is a viscosity solution of (2.3). Hence this procedure yields an existence result for viscosity solutions of (2.3), when the equation (2.3) is *not* necessarily a dynamic programming equation.

This formal discussion also explains the terminology "viscosity solution". In fluid mechanics, the coefficient of the second order term is related to the viscosity of the fluid. Then in the context of fluid mechanics, the viscosity solution is the zero viscosity limit of solutions with positive but small viscosity.

Example 2.2. (Deterministic limit of controlled jump Markov processes). For $\phi\in C^1(\overline{Q}), (t,x)\in\overline{Q}$ and $v\in U$ let

$$(\mathcal{G}_t\phi)(x) = \sup_{v\in U}\{-b(t,x,v)D\phi(x) - L(t,x,v)$$

$$-\lambda(t,x,v)\int_{\overline{O}}[\phi(y)-\phi(x)]m(t,x,v,dy)\},$$

where $\lambda(t,x,v)\geq 0$ and $m(t,x,v,\cdot)$ is a probability measure. Then $\mathcal{G}_t$ is related to a controlled jump Markov process $\{X_s\}$. Indeed for fixed $v\in U$, $\lambda(t,x,v)$ is the *jump intensity* (or *jump rate*) and $m(t,x,v,\cdot)$ is the *post-jump location distribution* of $\{X_s\}$. We also assume that the support of $m(t,x,v,\cdot)$ is a subset of $\overline{O}$. Then the state space Σ of $\{X_s\}$ is $\overline{O}$ and the domain of $\mathcal{G}_t$ includes $C^1(\overline{O})$.

Let $\{X_s\}$ be the corresponding Markov process started at $X_t = x$ with a fixed control v. Suppose that O is convex and for $\varepsilon\in(0,1]$, define a rescaled process X_s^ε by

$$X_s^\varepsilon = x + \varepsilon(X_{s/\varepsilon} - x),\ s \geq t.$$

Then the infinitesimal generator of the rescaled semigroup is

$$(2.6)\qquad \begin{aligned}(\mathcal{G}_t^\varepsilon \phi)(x) = \sup_{v \in U}\{&-b(t,x,v)\cdot D\phi(x) - L(t,x,v)\\ &-\frac{\lambda(t,x,v)}{\varepsilon}\int_{\overline{O}}[\phi(x+\varepsilon z) - \phi(x)]\hat{m}(t,x,v,dz)\},\end{aligned}$$

where

$$\hat{m}(t,x,v,A) = m(t,x,v,\{y:\ y = x+z \ \text{for some}\ z \in A\}).$$

The jump intensity of the rescaled process is $\lambda(t,x,v)/\varepsilon$. Hence the average number of jumps in a given time interval is of order ε^{-1}. However the average size of a jump is of order ε. Therefore one expects to have a limit result, analogous to the law of large numbers. In particular, the limit of $(2.1)^\varepsilon$ should be related to a *deterministic* process.

Indeed for $\phi \in C^1(\overline{Q})$ the limit of $(\mathcal{G}_t^\varepsilon \phi)(x)$, as ε tends to zero, is

$$\mathcal{G}_t \phi(x) = H(t,x,D_x\phi(x)),$$

where

$$H(t,x,p) = \sup_{v\in U}\{-b(t,x,v)\cdot p - L(t,x,v) - \lambda(t,x,v)\int_{\overline{O}} p\cdot z\ \ \hat{m}(t,x,v,dz)\}.$$

Notice that the limiting equation is a first order equation corresponding to a *deterministic* optimal control problem with

$$f(t,x,v) = b(t,x,v) + \lambda(t,x,v)\int_{\overline{O}} z\hat{m}(t,x,v,dz).$$

VII.3 Barles and Perthame procedure

Let V^ε be a viscosity solution of (2.1^ε) and (2.2). Suppose that on a subsequence $\varepsilon_n \to 0, V^{\varepsilon_n}$ converges to $\overline{V}$ uniformly on $\overline{Q}$. Then Lemma II.6.2 implies that $\overline{V}$ is a viscosity solution of (2.3) and (2.4). If we also know that there is a unique viscosity solution V of (2.3) satisfying the boundary conditions (2.4), we conclude that the uniform limit $\overline{V}$ is equal to the unique viscosity solution V. We now claim that under the above uniqueness assumption, the convergence of V^ε to V follows from equicontinuity and uniform boundedness of the sequence $\{V^\varepsilon\}$. Indeed, from the Ascoli-Arzela theorem we conclude that any sequence converging to zero has a further subsequence ε_n such that V^{ε_n} is uniformly convergent on $\overline{Q}$. Then we have argued that this limit is equal to V. Therefore V^ε converges to V uniformly.

The above procedure requires the equicontinuity and the uniform boundedness of $\{V^\varepsilon\}$. For certain applications one may obtain a uniform Lipschitz estimate

$$\sup_{0<\varepsilon<\varepsilon_0} \sup_{(t,x)\in\overline{Q}} |\frac{\partial}{\partial t} V^\varepsilon(t,x)| + |D_x V^\varepsilon(t,x)| < \infty. \tag{3.1}$$

In this case, the equicontinuity of $\{V^\varepsilon\}$ is immediate. However, in many applications it is difficult to establish the equicontinuity of $\{V^\varepsilon\}$ by (3.1) or other means.

In 1988, Barles and Perthame [BP1] devised a procedure which assumes only the uniform boundedness of the sequence $\{V^\varepsilon\}$, but a slightly stronger uniqueness (in fact a comparison) result for (2.3) and (2.4). We now outline this procedure.

For $(t,x) \in \overline{Q}$, define

$$V^*(t,x) = \limsup_{\substack{(s,y)\to(t,x)\\ \varepsilon\downarrow 0\\ (s,y)\in\overline{Q}}} V^\varepsilon(s,y), \tag{3.2}$$

$$V_*(t,x) = \liminf_{\substack{(s,y)\to(t,x)\\ \varepsilon\downarrow 0\\ (s,y)\in\overline{Q}}} V^\varepsilon(s,y). \tag{3.3}$$

We assume that both V^* and V_* are finite. These functions however, are not necessarily continuous. In fact from definitions (3.2) and (3.3) we may only infer that they are semi-continuous. Still V^* and V_* are viscosity sub and supersolutions of (2.3) - (2.4), respectively (see Sections 4, 5 and 6). However the notion of viscosity solutions has to be modified so as to include semi-continuous functions. This modified definition is given in the next section. We then use the equation (2.3) and the boundary conditions (2.4), to show that any viscosity subsolution of (2.3)-(2.4) is dominated by any viscosity supersolution of (2.3)-(2.4) (see Section 8). In particular $V^* \le V_*$. But by construction $V_* \le V^*$ and therefore $V^* = V_*$. Hence V^ε is uniformly convergent on Q.

The main steps of the above procedure are;

(a) V^* and V_* are viscosity sub and supersolutions of (2.3), (2.4), respectively.
(b) A general comparison result for semi-continuous viscosity sub and supersolutions of (2.3)- (2.4).

These will be the subjects of following sections.

VII.4 Discontinuous viscosity solutions

Let W be a bounded real-valued function on $\overline{Q}$. We first define the upper and lower semi-continuous envelope of W.

Definition 4.1.

(a) The *upper semi-continuous envelope* of W is

$$(4.1) \qquad (W)^*(t,x) = \limsup_{\substack{(s,y)\to(t,x)\\ (s,y)\in\overline{Q}}} W(s,y), \ \ \forall (t,x)\in\overline{Q}.$$

(b) The *lower semi-continuous envelope* of W is

$$(4.2) \qquad (W)_*(t,x) = \liminf_{\substack{(s,y)\to(t,x)\\ (s,y)\in\overline{Q}}} W(s,y), \ \ \forall (t,x)\in\overline{Q}.$$

Observe that $(W)^*$ is the smallest upper semi-continuous function which is greater than or equal to W and a dual statement holds for $(W)_*$.

We now are ready to give the definition of viscosity sub and supersolutions for discontinuous functions next. This definition is very similar to the one for continuous functions, Definition II.4.2.

Definition 4.2. We say that W is a:

(a) *Viscosity subsolution* of (2.3) in Q if for each $w\in C^\infty(Q)$,

$$(4.3) \qquad -\frac{\partial}{\partial t}w(\bar t,\overline{x}) + H(\bar t,\overline{x},D_x w(\bar t,\overline{x}),w(\bar t,\bar x)) \le 0$$

at every $(\bar t,\overline{x})\in Q$ which is a strict maximizer of $(W)^*-w$ on $\overline{Q}$ with $(W)^*(\bar t,\bar x) = w(\bar t,\bar x)$.

(b) *Viscosity supersolution* of (2.3) if for each $w\in C^\infty(Q)$

$$(4.4) \qquad -\frac{\partial}{\partial t}w(\bar t,\overline{x}) + H(\bar t,\overline{x},D_x w(\bar t,\overline{x}),w(\bar t,\bar x)) \ge 0,$$

at every $(\bar t,\overline{x})\in Q$ which is a strict minimizer of $(W)_*-w$ on $\overline{Q}$ with $(W)_*(\bar t,\bar x) = w(\bar t,\bar x)$.

(c) *Viscosity solution* of (2.3) in Q if it is both a viscosity subsolution and a viscosity supersolution of (2.3) in Q.

Notice that for any $\bar w\in C(\overline{Q})$, the difference $(W)^*-\bar w$ is an upper semi-continuous function on $\overline{Q}$. Hence $(W)^*-\bar w$ attains its maximum on $\overline{Q}$. This is an indication why the upper semi-continuous envelope $(W)^*$ is used in the definition of a viscosity subsolution. Similarly $(W)_*-\bar w$ attains its minimum on $\overline{Q}$, and this explains the use of $(W)_*$ in the definition of a viscosity supersolution.

Several remarks are now in order.

Remark 4.1. The (discontinuous) viscosity subsolutions and supersolutions of a second order equation,

$$-\frac{\partial}{\partial t}W(t,x) + F(t,x,D_xW(t,x),D_x^2W(t,x),W(t,x)) = 0, \ \ (t,x)\in Q,$$

are defined exactly the same way.

Remark 4.2. Following the arguments of Section II.6, we can prove that it is enough to consider $w \in C^1(\overline{Q})$ or $w \in C^1(Q)$ and the local extrema $(\bar{t}, \overline{x}) \in Q$, which need not be strict (see Remark II.6.1).

Our goal in this section is to show that V^* and V_* are viscosity subsolution and supersolutions of (2.3) in Q, respectively. First, observe that V^*, V_* are upper and lower semi-continuous on $\overline{Q}$, respectively. Therefore $(V^*)^* = V^*$ and $(V_*)_* = V_*$.

We are now ready to state a stability result which is a generalization of Lemma II.6.2. We assume that

$$\sup_{(t,x)\in\overline{Q},\varepsilon\in(0,1]} |V^\varepsilon(t,x)| = K_1 < \infty, \tag{4.5}$$

$$\lim_{\substack{(s,y)\to(t,x)\\ \varepsilon\downarrow 0}} (\mathcal{G}_s^\varepsilon[w(s,\cdot) + K^\varepsilon]) = H(t,x,D_x w(t,x), w(t,x)), \tag{4.6}$$

for all $(t,x) \in Q, w \in C^\infty(\overline{Q})$ and real numbers $K^\varepsilon \to 0$, as $\varepsilon \to 0$.

Proposition 4.1 (Stability). *Let H in* (4.6) *be a continuous function. Then V^* is a viscosity subsolution of* (2.3) *in Q and V_* is a viscosity supersolution of* (2.3) *in Q.*

Proof. Let $w \in C^\infty(\overline{Q})$ and $(\bar{t}, \bar{x}) \in Q$ be a strict maximizer of $V^* - w$ on $\overline{Q}$ with $V^*(\bar{t}, \bar{x}) = w(\bar{t}, \bar{x})$. Choose a sequence

$$(t_\varepsilon, x_\varepsilon) \in \arg\max\{V^\varepsilon(t,x) - w(t,x) : (t,x) \in \overline{Q}\}.$$

We claim that there is a sequence $\varepsilon_n \to 0$, such that $(t_n, x_n) = (t_{\varepsilon_n}, x_{\varepsilon_n}) \to (\bar{t}, \overline{x})$. Indeed pick $\varepsilon_n \to 0$ and $(s_n, y_n) \to (\bar{t}, \overline{x})$ satisfying

$$V^*(\bar{t}, \overline{x}) = \lim V^{\varepsilon_n}(s_n, y_n).$$

Since Q is bounded, the sequence (t_n, x_n) has limit points. Let $(\tilde{t}, \tilde{x})$ be one of them. Set $k^n = V^{\varepsilon_n}(t_n, x_n) - w(t_n, x_n)$. Then

$$\begin{aligned} 0 = V^*(\bar{t}, \overline{x}) - w(\bar{t}, \overline{x}) &= \lim[V^{\varepsilon_n}(s_n, y_n) - w(s_n, y_n)] \\ &\le \liminf k^n \le \limsup k^n \qquad (4.7) \\ &\le V^*(\tilde{t}, \tilde{x}) - w(\tilde{t}, \tilde{x}). \end{aligned}$$

Since $(\bar{t}, \overline{x})$ is a strict maximum of $V^* - w$, we conclude that $(\bar{t}, \overline{x}) = (\tilde{t}, \tilde{x})$. Hence $(t_n, x_n) \to (\bar{t}, \overline{x})$ and $k^n \to 0$. In particular $(t_n, x_n) \in Q$ for sufficiently large n. Therefore the equation $(2.1)^\varepsilon$ gives

$$-\frac{\partial}{\partial t} w(t_n, x_n) + (\mathcal{G}_{t_n}^{\varepsilon_n}[w(t_n, \cdot) + k^n])(x_n) \le 0.$$

Using the smoothness of w and (4.6), we conclude that the limit of the above inequality is

$$-\frac{\partial}{\partial t}w(\bar{t},\bar{x})+H(\bar{t},\overline{x},D_x w(\bar{t},\overline{x}),w(\bar{t},\bar{x}))\leq 0.$$

Hence V^* is a viscosity subsolution of (2.3). The supersolution property of V_* is proved similarly. □

VII.5 Terminal condition

In this section, we show that V^* and V_* satisfy (2.4)(b). The case (2.4)(a) will be discussed in the next section.

Proposition 5.1. *Suppose that* $V^\varepsilon \in C(\overline{Q})$ *and* $\Psi^\varepsilon \in C(\overline{Q})$ *converges uniformly to* Ψ *on* $\overline{Q}$, *as* $\varepsilon \to 0$. *Then*

$$V^*(t_1,x)=V_*(t_1,x)=\Psi(t_1,x)=\psi(x),\ \ \forall x\in\overline{O}. \tag{5.1}$$

Proof. Since V^ε satisfies (2.2), the construction of V^* and V_* yields $V^*(t_1,x)\geq\psi(x)\geq V_*(t_1,x)$ for all $x\in\overline{O}$. Now suppose that

$$V^*(t_1,\overline{x})=\psi(\overline{x})+\delta \tag{5.2}$$

for some $\delta>0$ and $\overline{x}\in\overline{O}$. Since Ψ^ε is uniformly convergent, there are $\rho>0$ and $\varepsilon_0>0$ satisfying

$$\Psi^\varepsilon(t,x)<\psi^\varepsilon(\overline{x})+\frac{\delta}{2},\ \text{ whenever }\ |x-\overline{x}|^2+t_1-t\leq\rho,\varepsilon\leq\varepsilon_0.$$

Let

$$w(t,x)=\gamma(t_1-t)+K|x-\overline{x}|^2,\ \ (t,x)\in\overline{Q},$$

where $\gamma\geq K$ is an arbitrary constant, and

$$K=(K_1+1-\psi(\overline{x}))/\rho$$

with K_1 as in (4.5). When $\Psi^\varepsilon(t,x)\geq\psi^\varepsilon(\overline{x})+\frac{\delta}{2}$, and $\varepsilon\leq\varepsilon_0$ we have

$$|x-\overline{x}|^2+t_1-t>\rho.$$

Consequently

$$\begin{aligned} w(t,x) &\geq K(t_1-t+|x-\overline{x}|^2)\\ &\geq K\rho=K_1+1-\psi(\overline{x})\\ &\geq V^\varepsilon(t,x)-\psi^\varepsilon(\overline{x})+[1+\psi^\varepsilon(\overline{x})-\psi(\overline{x})]. \end{aligned}$$

Therefore for sufficiently small $\varepsilon>0$,

$$w(t,x)>V^\varepsilon(t,x)-\psi^\varepsilon(\overline{x}),\ \ \text{whenever }\ \Psi^\varepsilon(t,x)\geq\psi^\varepsilon(\overline{x})+\frac{\delta}{2}. \tag{5.3}$$

For $\varepsilon > 0$, choose

$$(t_\varepsilon, x_\varepsilon) \in \arg\max\{V^\varepsilon(t,x) - w(t,x) : (t,x) \in \overline{Q}\}.$$

Also the definition of V^* implies that there exists $\varepsilon_n \to 0$ and $(s_n, y_n) \to (t_1, \overline{x})$ such that

$$\psi(\overline{x}) + \delta = V^*(t_1, \overline{x}) = \lim V^{\varepsilon_n}(s_n, y_n).$$

Set $(t_n, x_n) = (t_{\varepsilon_n}, x_{\varepsilon_n})$. Then,

$$\liminf_{n\to\infty} V^{\varepsilon_n}(t_n, x_n) - w(t_n, x_n) \geq \lim_{n\to\infty} V^{\varepsilon_n}(s_n, y_n) - w(s_n, y_n) = \psi(\overline{x}) + \delta.$$

We claim that $(t_n, x_n) \notin \partial^* Q$ for sufficiently large n. Indeed if $(t_n, x_n) \in \partial^* Q$, (2.2) yields $V^{\varepsilon_n}(t_n, x_n) = \Psi^{\varepsilon_n}(t_n, x_n)$. Therefore the above inequality yields that there is n_0 such that, for all $n \geq n_0$

$$\Psi^{\varepsilon_n}(t_n, x_n) = V^{\varepsilon_n}(t_n, x_n) \geq \psi^{\varepsilon_n}(\overline{x}) + \frac{\delta}{2} + w(t_n, x_n) \geq \psi^{\varepsilon_n}(\overline{x}) + \frac{\delta}{2}.$$

Then using (5.3) we conclude that $w(t_n, x_n) > V^{\varepsilon_n}(t_n, x_n) - \psi^{\varepsilon_n}(\overline{x})$, which in turn gives

$$\limsup V^{\varepsilon_n}(t_n, x_n) - w(t_n, x_n) \leq \psi(\overline{x}).$$

Hence for all $n \geq n_0$ $(t_n, x_n) \notin \partial^* Q$ and the viscosity property of V^ε gives

$$0 \geq -\frac{\partial}{\partial t} w(t_n, x_n) + (\mathcal{G}^{\varepsilon_n}_{t_n}[w(t_n, \cdot) + K^n])(x_n) \tag{5.4}$$

$$= \gamma + (\mathcal{G}^{\varepsilon_n}_{t_n}[w(t_n, \cdot) + K^n])(x_n),$$

where $K^n = V^{\varepsilon_n}(t_n, x_n) - w(t_n, x_n)$. Set $C^n = V^{\varepsilon_n}(t_n, x_n) - K|x_n - \overline{x}|^2$. Then C^n is independent of γ and $w(t_n, x) + K^n = K|x - \overline{x}|^2 + C^n$. We now rewrite (5.4) as

$$\gamma + (\mathcal{G}^{\varepsilon_n}_{t_n}[K|\cdot - \bar{x}|^2 + C^n])(x_n) \leq 0,$$

for all γ and $n \geq n_0$. Observe that K, C^n and n_0 are independent of γ. Also C^n, (t_n, x_n) are bounded. Let $\overline{C}$, $(\bar{t}, \bar{x})$ be a limit point of C^n, (t_n, x_n) as $n \to \infty$ through a subsequence. Now let $n \to \infty$ on this subsequence in the above inequality. Then (4.6) yields

$$\gamma + H(\bar{t}, \bar{x}, D_x w(\bar{t}, \bar{x}, w(\bar{t}, \bar{x}) + \overline{C}) \leq 0.$$

Therefore

$$\gamma \leq \sup\{|H(t,x,p,\lambda)| : (t,x) \in \overline{Q}, |p| \leq \| Dw \|, |\lambda| \leq \| w \| + \overline{C}\}.$$

Then by taking γ large enough we obtain a contradiction in the above inequality. Therefore $V^*(t_1, x) = \psi(x)$ for every $x \in \overline{O}$. The other equality in (5.1) is proved similarly. □

VII.6 Boundary condition

Recall that the viscosity formulation of the boundary value problem (2.3) and (2.4)(a) is given in Section II.13. In that formulation the lateral boundary condition is imposed only in the viscosity sense. In the context of singular perturbations, we also expect that V^* and V_* satisfy (2.4)(a) not pointwise but in the viscosity sense, which we will define next.

Definition 6.1. We say that

(a) W is a *viscosity subsolution* of (2.3) and (2.4)(a) if it is a viscosity subsolution of (2.3) in Q and for each $w \in C^\infty(\overline{Q})$

(6.1)

$$\min\left\{-\frac{\partial}{\partial t}w(\bar t,\bar x)+H(\bar t,\bar x,D_x w(\bar t,\bar x),w(\bar t,\bar x)),\ (W)^*(\bar t,\bar x)-g(\bar t,\bar x)\right\}\le 0,$$

at every $(\bar t,\bar x) \in [t_0,t_1)\times\partial O$ which maximizes $(W)^*-w$ on $\overline{Q}$ with $(W)^*(\bar t,\bar x)=w(\bar t,\bar x)$.

(b) W is a *viscosity supersolution* of (2.3) and (2.4)(a) if it is a viscosity supersolution of (2.3) in Q and for each $w \in C^\infty(\overline{Q})$

(6.2)

$$\max\left\{-\frac{\partial}{\partial t}w(\bar t,\bar x)+H(\bar t,\bar x,D_x w(\bar t,\bar x),w(\bar t,\bar x)),\ (W)_*(\bar t,\bar x)-g(\bar t,\bar x)\right\}\ge 0$$

at every $(\bar t,\bar x) \in [t_0,t_1)\times\partial O$ which maximizes $(W)_*-w$ on $\overline{Q}$ with $(W)_*(\bar t,\bar x)=w(\bar t,\bar x)$.

(c) W is a *viscosity solution* of (2.3) and (2.4)(a) if it is both a viscosity subsolution and a viscosity supersolution of (2.3) and (2.4)(a).

The above definition is very similar to definition given in Section II.13. In the case of a subsolution, (6.1) states that if $(W)^*$ is not less than or equal to g at a boundary point, then $(W)^*$ is a subsolution of (2.3) at that point. In particular if (2.4)(a) holds pointwise, then (6.1) is automatically satisfied at the boundary points. However when (2.4)(a) fails, we have to interpret (2.4)(a) in the viscosity sense. It should be emphasized that this formulation of (2.4)(a) is connected to the equation (2.3).

Proposition 6.1. *Suppose that $V^\varepsilon \in C(\overline{Q})$ and $\Psi^\varepsilon \in C(\overline{Q})$ converges to Ψ uniformly on $\overline{Q}$, as ε tends to zero. Then V^* and V_* are a viscosity subsolution and a viscosity supersolution of* (2.3)-(2.4)(a)*, respectively.*

Proof. We already know that V^* is a viscosity subsolution of (2.3) in Q. We continue by verifying (2.4)(a) in the viscosity sense, i.e. in the sense of the above definition. So let $w \in C^\infty(\overline{Q})$ and let $(\bar t,\bar x) \in [t_0,t_1)\times\partial O$ be a strict maximizer of V^*-w with $V^*(\bar t,\bar x)=w(\bar t,\bar w)$. In view of Remark 4.2 and the results of Section II.6, we may assume that $(\bar t,\bar x)$ is a strict maximizer. Notice that (6.1) holds trivially if $V^*(\bar t,\bar x)\le g(\bar t,\bar x)$. Therefore we may assume that

$$V^*(\bar t,\bar x) > g(\bar t,\bar x). \tag{6.3}$$

Pick $(t_\varepsilon,x_\varepsilon) \in \arg\max\{V^\varepsilon(t,x)-w(t,x) : (t,x)\in\overline{Q}\}$. Then arguing as in the proof of Proposition 4.1 (see (4.7)), we construct a sequence $\varepsilon_n \to 0$ satisfying

$(t_n, x_n) = (t_{\varepsilon_n}, x_{\varepsilon_n}) \to (\bar{t}, \overline{x})$ and

$$V^{\varepsilon_n}(t_n, x_n) \to V^*(\bar{t}, \overline{x}) > g(\bar{t}, \overline{x}) = \Psi(\bar{t}, \overline{x}). \tag{6.4}$$

Since $V^{\varepsilon_n} = \Psi^{\varepsilon_n}$ on $\partial^* Q$ and Ψ^{ε_n} is uniformly convergent to Ψ, we conclude that $(t_n, x_n) \notin \partial^* Q$. Consequently the viscosity property of V^{ε_n} at (t_n, x_n) gives

$$-\frac{\partial}{\partial t} w(t_n, x_n) + (\mathcal{G}_{t_n}^{\varepsilon_n}[w(t_n, \cdot) + K^n])(x_n) \leq 0,$$

where $K^n = V^{\varepsilon_n}(t_n, x_n) - w(t_n, x_n)$. Since $V^*(\bar{t}, \bar{x}) = w(\bar{t}, \bar{x})$, (6.4) yields that $K^n \to 0$ as $n \to \infty$. Now let n go to infinity to obtain

$$-\frac{\partial}{\partial t} w(\bar{t}, \overline{x}) + H(\bar{t}, \overline{x}, D_x w(\bar{t}, \overline{x}),\ w(\bar{t}, \overline{x})) \leq 0.$$

Hence V^* is a viscosity subsolution of (2.3) and (2.4)(a). Similarly V_* is a viscosity supersolution of (2.3) and (2.4)(a). □

VII.7 Convergence

To complete the procedure of Barles and Perthame, we need a comparison between viscosity sub and supersolutions of (2.3) and (2.4). In this section, we assume that there is such a comparison. Then we prove the convergence of V^ε as $\varepsilon \to 0$. A proof of the comparison principle under some structural conditions on H is given in the next section.

Let W be a viscosity subsolution of (2.3), (2.4)(a), in the sense of Definition 6.1, and a pointwise subsolution of (2.4)(b), i.e. $(W)^*(t_1, x) \leq \psi(x)$ for $x \in \overline{O}$. In the sequel we call any such function a *viscosity subsolution of* (2.3), (2.4). Similarly we define *viscosity supersolutions of* (2.3), (2.4). Let $\overline{W}$ be one of them.

Definition 7.1. We say that the equation (2.3) with boundary condition (2.4) has a *weak comparison principle* if;

$$(W)^*(t, x) \leq (\overline{W})_*(t, x),\ \ \forall (t, x) \in \overline{Q} \backslash [t_0, t_1) \times \partial O, \tag{7.1}$$

for any viscosity subsolution W and supersolution $\overline{W}$ of (2.3), (2.4).

We should emphasize that (7.1) is *not* assumed to hold on the lateral boundary $[t_0, t_1) \times \partial O$. In fact if (2.3), (2.4) does not admit a continuous viscosity solution satisfying (2.4)(a) pointwise, then we do not expect (7.1) to hold on $[t_0, t_1) \times \partial O$. Recall that such an equation is given in Example II.2.3. However, in the next section conditions implying (7.1) on $\overline{Q}$ are also discussed.

We continue by applying the comparison principle to V^* and V_*. Notice that Propositions 5.1 and 6.1 imply that V^* and V_* are viscosity sub and supersolutions of (2.3), (2.4), respectively. Hence the weak comparison principle gives

(7.2) $$V^*(t,x) \le V_*(t,x), \ \forall (t,x) \in \overline{Q}\backslash [t_0,t_1) \times \partial O.$$

Since by construction $V_* \le V_*$, we have

(7.3) $$V^*(t,x) = V_*(t,x), \ \forall (t,x) \in \overline{Q}\backslash [t_0,t_1) \times \partial O.$$

However if (7.1) holds on $\overline{Q}$, we conclude that $V^* = V_*$ on $\overline{Q}$. Therefore $V = V^* = V_*$ is a continuous viscosity solution of (2.3)-(2.4). Moreover by definition $V = V_* \le \Psi \le V^* = V$ on $\partial^* Q$. Hence V satisfies (2.4) at every boundary point. We have proved the following theorem.

Theorem 7.1. *Suppose that $V^\varepsilon \in C(\overline{Q})$ and $\Psi^\varepsilon \in C(\overline{Q})$ converges uniformly to Ψ on $\overline{Q}$ as $\varepsilon \to 0$. If in addition* (2.3), (2.4) *has a weak comparison principle, then* (7.3) *holds. In particular V^ε converges to $V^* = V_*$ uniformly on compact subsets of $\overline{Q}\backslash [t_0,t_1) \times \partial O$. This convergence is uniform on $\overline{Q}$ if (7.1) holds on $\overline{Q}$.*

VII.8 Comparison

The weak comparison principle as formulated in Definition 7.1, is the main focus of this section. In the previous section, we argued that (7.1) fails at a boundary point if the unique viscosity solution of (2.3)-(2.4) does not satisfy (2.4)(a) pointwise. Hence (7.1) is not expected to hold on $\overline{Q}$ for every boundary data Ψ. However, under certain circumstances the value function of the optimal control problem associated with (2.3), satisfies (2.4)(a) pointwise. See Sections II.10 and I.8. We will first prove the weak comparison principle when there exists such a viscosity solution. Recall that by Proposition 5.1 the terminal condition (2.4)(b) is satisfied pointwise. A more general condition under which the weak comparison holds is stated at the end of the section. To simplify our discussion we assume that the Hamiltonian H does not depend on $w(t,x)$, i.e.,

$$H = H(t,x,D_x w(t,x)).$$

For our first result, Theorem 8.1, we also assume that H satisfies II(9.4), i.e. there are a constant K and $h \in C([0,\infty))$ with $h(0) = 0$ such that for all $(t,x),(s,y) \in \overline{Q}$ and $p, \overline{p} \in I\!R^n$ we have,

(8.1) $$|H(t,x,p) - H(s,y,\overline{p})| \le$$

$$\le h(|t-s| + |x-y|) + h(|t-s|)|p| + K|x-y||p| + K|p-\overline{p}|.$$

Although we are not assuming that H is related to a control problem, the condition (8.1) is motivated by Hamiltonian given by I(5.4). In particular, (8.1) holds if U is compact and $f, L \in C^1(\overline{Q} \times U)$. However if U is not compact (8.1) holds only locally in the p-variable. The weak comparison principle for that type of Hamiltonian is obtained in Theorem 8.2.

We also impose a regularity condition II(14.1) on the boundary of O. The statement and a discussion of this condition is given in Section II.14. We just recall its statement here: there are $r, \varepsilon_0 > 0$ and an $I\!R^n$-valued, uniformly continuous function $\hat{\eta}$ on $\overline{O}$ satisfying

$$B(x + \varepsilon\hat{\eta}(x), \varepsilon r) \subset O, \ \forall x \in O, \varepsilon \in (0, \varepsilon_0], \tag{8.2}$$

where $B(z, \rho)$ is the ball with radius ρ and center z.

Theorem 8.1. *Assume* (8.1), (8.2), $\Psi \in C(\overline{Q})$ *and that there exists a viscosity solution* $V \in C(\overline{Q})$ *of* (2.3) *in* Q*, satisfying* (2.4)(a) *pointwise. Then* (2.3)-(2.4) *has a weak comparison principle.*

Proof. Let W and $\overline{W}$ be a viscosity subsolution and a supersolution of (2.3)-(2.4), respectively. Set $W^* = (W)^*$ and $\overline{W}_* = (\overline{W})_*$. We will show that for $(t, x) \in \overline{Q}$,

$$W^*(t, x) \leq V(t, x) \tag{8.3)(i}$$

and

$$V(t, x) \leq \overline{W}_*(t, x). \tag{8.3)(ii}$$

Clearly (7.1) follows from the above inequalities. For $\beta \in (0, 1]$, choose

$$(\bar{t}, \bar{x}) \in \arg\max\{W^*(t, x) - V(t, x) + \beta(t - t_1) : (t, x) \in \overline{Q}\}.$$

Then to prove (8.3)(i), it suffices to show that

$$\liminf_{\beta \to 0} [W^*(\bar{t}, \bar{x}) - V(\bar{t}, \bar{x})] + \beta(\bar{t} - t_1) \leq 0. \tag{8.4}$$

We analyze several cases separately.

Case 1: $\bar{t} = t_1$**.** The inequality (8.4) follows from the fact that (2.4)(b) is satisfied by V and $W^*(t_1, \bar{x}) \leq \psi(\bar{x})$.

Case 2: $\bar{t} < t_1, \bar{x} \in O$**.** For $\varepsilon, \delta > 0$ consider an auxiliary function

$$\phi(t, x; s, y) = W^*(t, x) - V(s, y) - (t - \bar{t})^2 - |x - \bar{x}|^2$$
$$-\frac{1}{2\varepsilon}|y - x|^2 - \frac{1}{2\delta}(t - s)^2 + \beta(t - t_1), (t, x), (s, y) \in \overline{Q}.$$

Let $(t^*, x^*), (s^*, y^*)$ be a maximizer of ϕ. Then $t^*, s^* \to \bar{t}, x^*, y^* \to \bar{x}$. In particular for sufficiently small $\varepsilon, \delta > 0, (t^*, x^*), (s^*, y^*) \in Q$. We now follow the proof of Theorem II.9.1 to obtain (8.4). Here we should recall that, in the proof of Theorem II.9.1 the analysis of the interior maximum case $((t^*, x^*), (s^*, y^*) \in Q)$ requires the continuity of only the subsolution or the supersolution (see Remark II.9.2).

Case 3: $\bar{t} < t_1, \bar{x} \in \partial O$**.** Since V satisfies (2.3a) pointwise, $V(\bar{t}, \bar{x}) = g(\bar{t}, \bar{x})$ and therefore we may assume that $W^*(\bar{t}, \bar{x}) > g(\bar{t}, \bar{x})$. For two positive parameters ε, γ, consider an auxiliary function

$$\phi(t,x;s,y) = W^*(t,x) - V(s,y) - (t-\bar{t})^2 - |x-\overline{x}|^2$$
$$-|\frac{y-x}{\varepsilon} - \hat{\eta}(\overline{x})|^2 - (\frac{t-s}{\gamma})^2 + \beta(t-\bar{t}),\ (t,x),(s,y) \in \overline{Q},$$

where $\hat{\eta}$ is as in (8.2). Let $(t^*,x^*),(s^*,y^*) \in \overline{Q}$ be a maximizer of ϕ on $\overline{Q} \times \overline{Q}$. Notice that $(t^*,x^*),(s^*,y^*)$ depend on ε and γ, but for simplicity we suppress this dependence in the notation.

Before we continue the proof in several steps, one remark is now in order. The main difference between the above auxiliary function and the ones used in Section II.9 is the introduction of $\hat{\eta}$. This modification enables us to show that

$$y^* \in O,$$

(see Step 2, below.) Since V does not satisfy the equation on ∂O, this modification is the essential ingredient of this proof.

1. For $\rho \geq 0$, set

$$m(\rho) = \sup\{|V(t,x) - V(s,y)| : (t,x),(s,y) \in \overline{Q} \text{ and } |t-s| + |x-y| \leq \rho.\}$$

Since $V \in C(\overline{Q}), m$ is continuous on $[0,\infty)$ with $m(0) = 0$. We claim that there is a constant C, *independent of* ε *and* γ, such that

$$|t^* - s^*| \leq C\gamma, \tag{8.5}$$

$$|y^* - x^*| \leq C\varepsilon, \tag{8.6}$$

$$|\frac{y^*-x^*}{\varepsilon} - \hat{\eta}(\overline{x})|^2 + (\bar{t} - t^*)^2 + |\overline{x} - x^*|^2 \leq 2m(C[\varepsilon+\gamma]) + C\varepsilon^2. \tag{8.7}$$

Indeed the inequality $\phi(t^*,x^*;s^*,y^*) \geq \phi(t^*,x^*;t^*,y^*)$ yields

$$(\frac{t^*-s^*}{\gamma})^2 \leq V(t^*,y^*) - V(s^*,y^*).$$

Since Q is bounded and V is continuous, V is bounded. Consequently (8.5) follows from the above inequality. Also for $\varepsilon \leq \varepsilon_0$ (where ε_0 is as in (8.2)), $\overline{x} + \varepsilon\hat{\eta}(\overline{x}) \in O$. Therefore

(8.8)
$$\begin{aligned}
\phi(t^*,x^*;s^*,y^*) &\geq \phi(\bar{t},\overline{x};\bar{t},\overline{x}+\varepsilon\hat{\eta}(\overline{x})) \\
&= W^*(\bar{t},\overline{x}) - V(\bar{t},\overline{x}+\varepsilon\hat{\eta}(\overline{x})) - \varepsilon^2 \|\hat{\eta}\|^2 \\
&\geq W^*(\bar{t},\overline{x}) - V(\bar{t},\overline{x}) - m(\|\hat{\eta}\|\varepsilon) - \varepsilon^2 \|\hat{\eta}\|^2 \\
&\geq W^*(t^*,x^*) - V(t^*,x^*) - m(\|\hat{\eta}\|\varepsilon) + \beta(t^*-\bar{t}) - \varepsilon^2 \|\hat{\eta}\|^2 \\
&\geq W^*(t^*,x^*) - V(s^*,y^*) + \beta(t^*-\bar{t}) \\
&\qquad -m(\|\hat{\eta}\|\varepsilon) - m(|t^*-s^*| + |x^*-y^*|) - \varepsilon^2 \|\hat{\eta}\|^2,
\end{aligned}$$

where $\|\hat{\eta}\|$ is the sup-norm of $\hat{\eta}$. This inequality gives

$$(8.9)\quad |\frac{y^*-x^*}{\varepsilon}-\hat{\eta}(\overline{x})|^2+(\overline{t}-t^*)^2+|\overline{x}-x^*|^2 \le m(\|\hat{\eta}\|\varepsilon)+m(|t^*-s^*|+|x^*-y^*|) + \varepsilon^2 \| \hat{\eta} \|^2 .$$

Since V is bounded, so is m. Consequently (8.9) implies (8.6). We then obtain (8.7), by using (8.5) and (8.6) in (8.9) and redefining C, if necessary.

2. Since at the boundary V does not necessarily satisfy the equation in the viscosity sense, we need to show that

$$(8.10)\qquad y^* \in O.$$

Indeed, $y^* = x^* + \varepsilon\hat{\eta}(x^*) + \varepsilon\nu$, where

$$\nu = (\frac{y^*-x^*}{\varepsilon} - \hat{\eta}(\overline{x})) + (\hat{\eta}(\overline{x}) - \hat{\eta}(x^*)).$$

Since $\hat{\eta}$ is uniformly continuous and $m(0)=0$, (8.7) implies that $|\nu|$ converges to zero as ε, γ tends to zero. Hence for sufficiently small ε and γ, we have $|\nu| < r$, where r is as in (8.2). Therefore

$$y^* \in B(x^* + \varepsilon\hat{\eta}(x^*), \varepsilon r),$$

and (8.10) follows from (8.2) for all sufficiently small ε, γ.

3. In general, x^* may be a boundary point. But the viscosity formulation of the boundary condition (2.4) implies that equation (4.3) still holds at (t^*, x^*), provided that

$$(8.11)\qquad W^*(t^*, x^*) > g(t^*, x^*).$$

Recall that $W^*(\overline{t}, \overline{x}) > g(\overline{t}, \overline{x})$ and that g is continuous. Since for every $(t,x), (s,y) \in \overline{Q}$

$$W^*(t,x) - V(s,y) + \beta(t-\overline{t}) \ge \phi(t,x;\ s,y),$$

(8.8) yields

$$\begin{aligned}
&\liminf_{\varepsilon,\gamma\to 0}\,[W^*(t^*,x^*) - V(s^*,y^*) + \beta(t^*-\overline{t})] \\
&\qquad \ge \liminf_{\varepsilon,\gamma\to 0} \phi(t^*,x^*;s^*,y^*) \\
&\qquad \ge W^*(\overline{t},\overline{x}) - V(\overline{t},\overline{x}) \\
&\qquad \ge \limsup_{\varepsilon,\gamma\to 0} [W^*(t^*,x^*) - V(s^*,y^*) + \beta(t^*-\overline{t})].
\end{aligned}$$

Since V is continuous, and $(s^*,y^*), (t^*,x^*) \to (\bar{t}, \bar{x})$, we have

$$\lim_{\varepsilon,\gamma\to 0} W^*(t^*,x^*) = W^*(\bar{t},\bar{x}) > g(\bar{t},\bar{x}). \tag{8.12}$$

Now for small ε and γ, (8.11) follows from the continuity of g.

4. Consider the map $(s,y) \to \phi(t^*,x^*;s,y)$. It has a maximum at $(s^*,y^*) \in Q$. Since V is a viscosity subsolution of (2.3) in Q,

$$-q_\gamma + H(s^*,y^*,p_\varepsilon) \geq 0, \tag{8.13}$$

where

$$q_\gamma = 2(t^*-s^*)/\gamma^2,$$

$$p_\varepsilon = 2(x^*-y^*+\varepsilon\hat{\eta}(\bar{x}))/\varepsilon^2.$$

Notice that (8.10) is used in the derivation of (8.13).

5. Similarly $W^*(t,x)$ is a viscosity subsolution of (2.3) in Q. Also the map $(t,x) \to \phi(t,x;s^*,y^*)$ has a maximum at $(t^*,x^*) \in [t_0,t_1) \times \overline{O}$. Suppose that $x^* \in O$, then as in Step 4 of the proof of Theorem II.9.1 we have

$$\beta - q_\gamma - 2(t^*-\bar{t}) + H(t^*,x^*,p_\varepsilon + 2(x^*-\bar{x})) \leq 0. \tag{8.14}$$

If $x^* \in \partial O$, the strict inequality (8.11) and the boundary viscosity formulation (6.1) implies that (8.14) is still valid in this case.

We should remark that in contrast with the previous step, x^* is not in general an interior point of O. But when $x^* \in \partial O$, the viscosity formulation of the boundary condition is used to get around this problem.

6. We subtract (8.13) from (8.14), and then use (8.1) with the estimates (8.9), (8.6), (8.7). This gives

$$\beta \leq 2(t^*-\bar{t}) + H(s^*,y^*,p_\varepsilon) - H(t^*,x^*,p_\varepsilon + 2(x^*-\bar{x}))$$

$$\leq 2(t^*-\bar{t}) + h(C[\gamma+\varepsilon]) + h(C\gamma)|p_\varepsilon| + KC\varepsilon|p_\varepsilon| + 2K|x^*-\bar{x}|.$$

In view of (8.7)

$$|t^*-\bar{t}|, \varepsilon|p_\varepsilon|, |\bar{x}-x^*| \leq (2m(C[\varepsilon+\gamma]) + C\varepsilon^2)^{\frac{1}{2}}.$$

Combining these inequalities, we obtain $\beta \leq k(\varepsilon,\gamma)$ where

$$\lim_{\gamma\downarrow 0} k(\varepsilon,\gamma) = h(C\varepsilon) + (2+CK+2K)(2m(C\varepsilon)+C\varepsilon^2)^{\frac{1}{2}}.$$

Therefore

$$\lim_{\varepsilon\downarrow 0}\,[\lim_{\gamma\downarrow 0} k(\varepsilon,\gamma)] = 0.$$

Since $\beta > 0$, this proves (8.3)(i).

We continue with the proof of (8.3)(ii). Again pick $(\bar{t},\bar{x}) \in \arg\max\{(V(t,x) - W_*(t,x)) + \beta(t-t_1) : (t,x) \in \overline{Q}\}$. We will show that $V(\bar{t},\bar{x}) - W_*(\bar{t},\bar{x}) \leq 0$. If $\bar{t} = t_1$ or $\bar{t} < t_1$ and $\bar{x} \in O$, we proceed exactly as in the proof of (8.3)(i). So we may assume that $\bar{t} < t_1$ and $\overline{x} \in \partial O$. Then $V(\bar{t}, \overline{x}) = g(\bar{t}, \overline{x})$ and we need to consider only the case $W_*(\bar{t}, \overline{x}) < g(\bar{t}, \overline{x})$. Then define an auxiliary function by

$$\phi(t, x; s, y) = V(s, y) - W_*(t, x) - (t - \bar{t})^2 - |x - \overline{x}|^2$$
$$-|\frac{y - x}{\varepsilon} - \hat{\eta}(\overline{x})|^2 - (\frac{t - s}{\gamma})^2 + \beta(t - \bar{t}), \ \ (t, x), (s, y) \in \overline{Q}.$$

Now follow the Steps 1 through 6 with minor and obvious modifications, to obtain

$$\lim_{\beta \to 0} [V(\bar{t}, \overline{x}) - W_*(\bar{t}, \overline{x})] \leq 0.$$

□

Remark 8.1. An attendant modification of the above proof can be used to prove Theorem II.14.1. See [CGS, Theorem 3.1]. Recall that in Theorem II.14.1 we assume that sub and supersolutions are continuous on $\overline{Q}$. However the existence of a continuous solution satisfying the boundary condition is *not* assumed. Still the above proof with attendant modifications yields a proof for Theorem II.14.1. So in particular under the hypotheses of Theorems II.13.1 and II.14.1, the value function is the unique continuous viscosity solution of I(5.3$'$) in Q and the boundary condition II(9.3). But the value function may not satisfy II(9.3a) pointwise.

As mentioned earlier the condition (8.1) is restrictive if (2.3) is related to a control problem with an unbounded control set. In fact in the applications to the theory of large deviations,

$$(8.15) \qquad H(t, x, p) = -b(t, x) \cdot p + \frac{1}{2} a(t, x) p \cdot p \ \ , \ \ (t, x) \in \overline{Q}, p \in I\!R^n,$$

where $b(t, x) \in I\!R^n$ and $a(t, x)$ is a symmetric, $n \times n$, positive-definite matrix (Section 10 below). Then using the Legendre transform introduced in Section I.8, we rewrite H as

$$H(t, x, p) = \max_{v \in R^n} \{-v \cdot p - L(t, x, v)\}$$

with

$$L(t, x, v) = \frac{1}{2} a^{-1}(t, x)[v - b(t, x)] \cdot [v - b(t, x)].$$

Suppose that a^{-1}, a, b are of class $C^3(\overline{Q} \times I\!R^n)$ and Ψ is nonnegative and Lipschitz continuous on $\overline{Q}$ with $\Psi(t, x) = 0$ for all $(t, x) \in [t_0, t_1] \times \partial O$. Then Theorem II.10.4 implies that the value function is Lipschitz continuous on $\overline{Q}$.

Our next result is modelled after the above example. We assume the Lipschitz continuity of V, and relax (8.1) to hold only locally. However we do *not* assume that H is related to a calculus of variations problem. We assume that for every $R > 0$ there are a constant K_R and a nondecreasing, continuous function $h(R, \cdot)$ with $h(R, 0) = 0$, such that

$$(8.16) \qquad |H(t, x, \overline{p}) - H(s, y, p)|$$

$$\leq h(R,|t-s|+|x-y|)+h(R,|t-s|)|p|+K_R|x-y||p|+K_R|p-\overline{p}|,$$

for all $(t,x),(s,y)\in\overline{Q}$ and $|p|,|\overline{p}|\leq R$. Inequality (8.16) is satisfied by the "large deviations" example if a, a^{-1} and b are Lipschitz continuous on $\overline{Q}$.

Theorem 8.2. *Assume* (8.2), (8.16) *and that there exists a Lipschitz continuous viscosity solution of* (2.3) *in* Q*, satisfying* (2.4)(a) *pointwise. Then* (2.3)-(2.4) *has a weak comparison principle.*

Proof. A minor modification of the proof of the previous theorem yields this result. Indeed follow that proof without a change until the sixth step of the last case. Replace that step by the following:

6′. Then $(q_\gamma,p_\varepsilon)\in D^+V(s^*,y^*)$. Since V is Lipschitz continuous, by Corollary II.8.1(f),

$$|q_\gamma|+|p_\varepsilon|\leq L(V), \tag{8.17}$$

where $L(V)$ is the Lipschitz constant of V. Then, (8.16) yields

$$|H(t^*,x^*,p_\varepsilon+2(x^*-\overline{x}))-H(s^*,y^*,p_\varepsilon)|$$

$$\leq h(R,|t^*-s^*|+|x^*-y^*|)+h(R,|t^*-s^*|)|p_\varepsilon|+K_R|x^*-y^*||p_\varepsilon|+2K_R|x^*-\overline{x}|,$$

where $R=\sup\{|p_\varepsilon|+2|x^*-x|:\varepsilon\in(0,1]\}$. Due to (8.17) and the boundedness of $\overline{Q}, R$ is uniformly bounded in ε and γ.

Now continue as in the proof of Theorem 8.1, but use the above estimate instead of (8.1). □

Combining Theorem 7.1, and Theorems 8.1 or 8.2 we obtain a convergence result for V^ε. However the existence of a continuous or a Lipschitz continuous viscosity solution satisfying (2.4)(a) is assumed in these theorems. The existence of such functions discussed in Section II.10 when the equation (2.3) is related to a control problem. So we now assume that H is as in I(5.4), i.e.,

$$H(t,x,p)=\sup_{v\in U}\{-f(t,x,v)\cdot p-L(t,x,v)\}. \tag{8.18}$$

Then Theorem II.10.2 or Theorem II.10.4 together with Theorems 7.1 and 8.2 yield the following convergence result.

Theorem 8.3. *Let* $V^\varepsilon\in C(\overline{Q})$ *be a viscosity solution of* $(2.1)^\varepsilon$ *in* Q *satisfying* $(2.2)^\varepsilon$*. Suppose* Ψ^ε *converges to* Ψ *uniformly on* $\overline{Q}$*, as* $\varepsilon\to 0, \partial O$ *satisfies* (8.2)*, and* H *is as in* (8.18) *with* L,f,Ψ *satisfying the hypotheses of Theorem* II.10.2 *or Theorem* II.10.4*. Then as* $\varepsilon\to 0, V^\varepsilon$ *converges to the unique Lipschitz continuous viscosity solution of* (2.3), (2.4) *uniformly on* $\overline{Q}$*.*

Next we state a more general sufficient condition for the weak comparison principle. Let us assume that $\partial O\in C^2$.

Theorem 8.4. *Assume* $\Psi\in C(\overline{Q})$ *and let* H *be as in* (8.18) *with* L,f *satisfying the hypotheses of Theorem* II.13.1(b)*. Then,* (2.3)-(2.4) *has a weak comparison principle.*

Sketch of proof. Using Theorem II.13.1 we obtain a viscosity solution $V\in C(\overline{Q})$ of (2.3), (2.4). Also the assumptions on L and f imply (8.1), and the differentiability of ∂O implies (8.2). Then as in the proof of Theorem 8.1, we compare the viscosity subsolution W and the supersolution $\overline{W}$ to V separately. But V does not necessarily satisfy (2.4)(a) pointwise. So the proof of Theorem 8.1 has to be modified to account for this. This modification is technical and we refer the reader to Barles and Perthame [BP2], or Ishii [I2]. However their results do not directly apply to this case. The conditions I(3.11) and II(13.6) have to be used once again to verify the hypotheses of theorem of Barles and Perthame or Ishii. □

Remark 8.2. Under the hypotheses of Theorem II.13.1(b) the two exit time control problems (classes B and B′ of the Section I.3) are equivalent. When they are no longer equivalent one may create easy examples with no weak comparison principle. So the hypotheses of the above theorem are almost optimal.

VII.9 Vanishing viscosity

Consider a classical solution $V^\varepsilon \in C(\overline{Q}) \cap C^{1,2}(Q)$ of (2.5^ε) and (2.2). Then for $(t,x) \in \overline{Q}, \phi \in C^2(\overline{O})$,

$$(\mathcal{G}_t^\varepsilon \phi)(x) = -\frac{\varepsilon}{2} \operatorname{tr} a(t,x) D^2\phi(x) + H^\varepsilon(t,x,D\phi(x)),$$

and by the classical maximum principle V^ε is a viscosity solution of (2.5^ε) in Q. Also the condition (4.6) is equivalent to the uniform convergence of H^ε to H on compact subsets of $\overline{Q} \times I\!R^n$. We assume this and that H satisfies (8.1) or (8.16). To be able to apply the results of the previous sections, we have to show only the uniform boundedness of V^ε. This bound is an elementary consequence of the maximum principle.

We continue with the simple proof of the uniform bound. Set $\overline{W}(t,x) = Ke^{-t}$ where K is an arbitrary constant. We claim that for sufficiently large $K, \overline{W}$ and $-\overline{W}$ are sub and supersolutions of (2.5^ε), respectively. Indeed, for all $\varepsilon \in (0,1]$,

$$-\frac{\partial}{\partial t}\overline{W}(t,x) + (\mathcal{G}_t^\varepsilon \overline{W}(t,\cdot))(x) = Ke^{-t} + H^\varepsilon(t,x,0) \geq 0,$$

provided that $K \geq K^* = \sup\{e^t H^\varepsilon(t,x,0) : (t,x) \in \overline{Q}, \varepsilon \in (0,1]\}$. Notice that

$$\lim_{\varepsilon \downarrow 0} H^\varepsilon(t,x,0) = H(t,x,0)$$

and by the continuity of H, $H(t,x,0)$ is uniformly bounded in (t,x). Since $H^\varepsilon(t,x,0)$ converges to $H(t,x,0)$ uniformly on $\overline{Q}, K^*$ is finite. Similarly one shows that $-\overline{W}$ is a supersolution of (2.5^ε). Then the classical maximum principle yields that

$$|V^\varepsilon(t,x)| \leq Ke^{-t}, \;\; \forall (t,x) \in \overline{Q}, \varepsilon \in (0,1],$$

provided that $K \geq \max\{K^*, e^t|\Psi(t,x)| : (t,x) \in \overline{Q}\}$. Therefore V^ε satisfies (4.5) and using the results of the previous section, we obtain the following convergence result.

Theorem 9.1. *Assume that the hypotheses of Theorem* 8.1 *or Theorem* 8.2 *or Theorem* 8.3 *or Theorem* 8.4 *are satisfied. Then* V^ε *converges uniformly to the unique solution of* (2.3)-(2.4) *on* $\overline{Q}$ *or on every compact subset of* $\overline{Q}\backslash[t_0, t_1) \times \partial O$, *in the case of Theorem* 8.4.

We give an example to show that uniform convergence on $\overline{Q}$ is not true in general.

Example 9.1. Set

$$G^\varepsilon(x) = e^{-x/\varepsilon} \sinh\,(\frac{\sqrt{1+2\varepsilon}}{\varepsilon}x),\;\; x \in [0,1],$$

and

$$V^\varepsilon(t,x) = e^{-t}G^\varepsilon(x)/G^\varepsilon(1).$$

Then V^ε is the unique solution of

$$-\frac{\partial}{\partial t}V^\varepsilon(t,x) - \frac{\varepsilon}{2}\frac{\partial^2}{\partial x^2}V^\varepsilon(t,x) - \frac{\partial}{\partial x}V^\varepsilon(t,x) = 0, (t,x) \in (0,1) \times (0,1),$$

with the boundary condition

$$(9.1)(i) \qquad V^\varepsilon(t,0) = 0,\;\; V^\varepsilon(t,1) = e^{-t},\;\; \forall t \in [0,1],$$

and the terminal condition

$$(9.1)(ii) \qquad V^\varepsilon(1,x) = G^\varepsilon(x)[eG^\varepsilon(1)]^{-1},\;\; \forall x \in [0,1].$$

A direct calculation gives,

$$\lim_{\varepsilon \downarrow 0} V^\varepsilon(t,x) = \begin{cases} e^{-t+x-1} & \text{if } (t,x) \in [0,1] \times (0,1] \\ 0 & \text{if } (t,x) \in [0,1] \times \{0\}. \end{cases}$$

In fact the function $V(x) = e^{-t+x-1}$ is the unique viscosity solution of

$$-\frac{\partial}{\partial t}V(t,x) - \frac{\partial}{\partial x}V(t,x) = 0,\;\; \forall (t,x) \in (0,1) \times (0,1)$$

with the boundary and the terminal conditions (9.1). Clearly V does not satisfy the boundary condition pointwise (9.1)(i) at $x = 0$, but (9.1)(i) is satisfied in the viscosity sense.

Also,

$$\frac{\partial}{\partial x}V^\varepsilon(0,t) = e^{-t}\frac{\sqrt{1+2\varepsilon}}{\varepsilon G^\varepsilon(1)}.$$

This tends to infinity as $\varepsilon \to 0$. Hence V^ε forms a boundary layer near $x = 0$.

VII.10 Large deviations for exit probabilities

Let us consider the exit probability problem described in Section VI.6. In that problem,

$$\Phi^\varepsilon(t,x) = P_{tx}(\theta^\varepsilon < t_1)$$

satisfies a linear equation

(10.1)
$$-\frac{\partial}{\partial t}\Phi^\varepsilon(t,x) - \frac{1}{2}\mathrm{tr}\, a(t,x)D_x^2\Phi^\varepsilon(t,x) - b(t,x)\cdot D_x\Phi^\varepsilon(t,x) = 0,\ (t,x)\in Q,$$

with boundary data

(10.2a)
$$\Phi^\varepsilon(t,x) = 1,\ (t,x)\in[t_0,t_1)\times\partial O,$$

(10.2b)
$$\Phi^\varepsilon(t_1,x) = 0,\ x\in\overline{O}.$$

The logarithmic transformation

$$V^\varepsilon = -\varepsilon\log\Phi^\varepsilon,$$

transforms (10.1) into (2.5^ε) with

(10.3)
$$H(t,x,p) = -b(t,x)\cdot p + \frac{1}{2}a(t,x)p\cdot p,\ (t,x)\in\overline{Q}, p\in I\!R^n.$$

Also the boundary data (10.2) yields,

(10.4a)
$$V^\varepsilon(t,x) = 0,\ (t,x)\in[t_0,t_1)\times\partial O,$$

(10.4b)
$$\lim_{t\uparrow t_1} V^\varepsilon(t,x) = +\infty \text{ uniformly on compact subsets of } O.$$

Hence with the notation of previous sections, $g\equiv 0$ and $\psi\equiv+\infty$. Because of the nonstandard form of the terminal data ψ, the results of the previous sections are *not* directly applicable to V^ε. However, a careful use of those results yields a convergence result for V^ε (Theorem 10.1, below).

As in Chapter VI, let us assume that a is invertible and that

(10.5)
$$h\in C^3(\overline{Q}) \text{ for } h = a, a^{-1}, b.$$

Then using the Legendre transform, introduced in Section I.8, we may rewrite $H^\varepsilon = H$ as

$$H(t,x,p) = \sup_{v\in R^n}\{-v\cdot p - L(t,x,v)\},\ (t,x)\in\overline{Q}, p\in I\!R^n,$$

where

$$L(t,x,v) = \frac{1}{2}a^{-1}(t,x)(v-b(t,x)) \cdot (v-b(t,x)). \tag{10.6}$$

Notice that, here $U = I\!R^n$ and $f(t,x,v) = v$. Also the boundedness of $a(t,x)$ yields

$$L(t,x,v) \geq c_0|v-b(t,x)|^2, \ \ (t,x) \in \overline{Q} \ \ , v \in I\!R^n, \tag{10.7}$$

with an appropriate constant $c_0 > 0$. This estimate together with (10.5) imply that the conditions I(11.6) with $p = 2$ are satisfied by L. Moreover, there exists C_R such that $|H(t,x,\bar{p}) - H(s,y,p)| \leq C_R(|t-s|+|x-y|+|\bar{p}-p|)$ whenever (t,x), $(s,y) \in \overline{Q}$ and $|p|$, $|\bar{p}| \leq R$. Hence (8.16) is satisfied by H. We continue with two elementary estimates of V^ε. These estimates can also be derived by probabilistic methods.

Lemma 10.1. *Suppose that the boundary of O is the class C^3. Then, there exists a constant K satisfying*

$$V^\varepsilon(t,x) \leq \frac{K \operatorname{dist}(x, \partial O)}{t_1 - t}, \ \ (t,x) \in \overline{Q}, \varepsilon \in (0,1]. \tag{10.8}$$

Proof. Since O is bounded, there is a constant μ satisfying $x_1 + \mu > 0$ for all $x \in \overline{O}$ where x_1 is the first component of the vector x. For $\lambda, \gamma > 0$, define an auxiliary function

$$g^\varepsilon(t,x) = \exp\left(-\frac{\lambda(x_1+\mu)}{\varepsilon(t_\gamma - t)}\right), \ \ (t,x) \in \overline{Q}_\gamma, \tag{10.9}$$

$$t_\gamma = t_1 - \gamma, \ \ Q_\gamma = [t_0, t_\gamma) \times O.$$

Then a simple calculation shows that

$$-\frac{\partial}{\partial t}g^\varepsilon(t,x) - \frac{\varepsilon}{2}\operatorname{tr} a(t,x)D_x^2 g^\varepsilon(t,x) - b(t,x) \cdot D_x g^\varepsilon(t,x)$$

$$= -\frac{g^\varepsilon(t,x)}{\varepsilon(t_\gamma - t)^2}[\frac{1}{2}a_{11}(t,x)\lambda^2 - \lambda(x_1+\mu) - (t_\gamma - t)\lambda b_1(t,x)].$$

Since $a^{-1}(t,x)$ is bounded, $a_{11}(t,x)$ is uniformly bounded away from zero on $\overline{Q}$. Hence the above expression is non-positive for a sufficiently large $\lambda = \lambda^*$. Therefore g^ε is a subsolution of (10.1) in Q and the boundary data (10.2a). Also $g^\varepsilon(t_\gamma, x) \equiv 0 \leq \Phi^\varepsilon(t_\gamma, x)$. Consequently the maximum principle for parabolic equations (or an application of Ito's formula) yields

$$\Phi^\varepsilon(t,x) \geq g^\varepsilon(t,x), \ \ (t,x) \in \overline{Q}.$$

Since ∂O is C^3, there is $\delta > 0$ such that $d(x) = \operatorname{dist}(x, \partial O)$ is twice continuously differentiable on

$$O_\delta = \{x \in O \ : \ d(x) < \delta\}.$$

Set

$$\tilde{g}^\varepsilon(t,x) = \exp\left(- \frac{Kd(x)}{\varepsilon(t_\gamma - t)} \right), \ \ (t,x) \in Q_\gamma,$$

with $K > 0$ satisfying

$$K \geq \frac{1}{\delta} \sup\{\lambda^*(x_1 + \mu) \ \ : \ \ x \in \overline{O}\}.$$

Then $\tilde{g}^\varepsilon(t,x) \leq g^\varepsilon(t,x) \leq \Phi^\varepsilon(t,x)$ whenever $d(x) \geq \delta$. In particular

$$\tilde{g}^\varepsilon(t,x) \leq \Phi^\varepsilon(t,x), \ \ \forall (t,x) \in [t_0, t_\gamma) \times (O \backslash O_\delta). \tag{10.10}$$

Also for $(t,x) \in [t_0, t_\gamma) \times \partial O$, $\tilde{g}^\varepsilon(t,x) = \Phi^\varepsilon(t,x) = 1$. A direct computation gives

$$I^\varepsilon(t,x) = -\frac{\partial}{\partial t}\tilde{g}^\varepsilon(t,x) - \frac{\varepsilon}{2}\operatorname{tr} a(t,x) D_x^2 \tilde{g}^\varepsilon(t,x) - b(t,x) \cdot D_x \tilde{g}^\varepsilon(t,x)$$

$$= -\frac{K\tilde{g}^\varepsilon(t,x)}{\varepsilon(t_\gamma - t)^2}\left[\frac{K}{2} a(t,x) Dd(x) \cdot Dd(x) - \varepsilon \frac{(t_\gamma - t)}{2} \operatorname{tr} a(t,x) D^2 d(x) \right.$$

$$\left. -(t_\gamma - t) b(t,x) \cdot Dd(x) - d(x) \right].$$

Since $a^{-1}(t,x)$ is bounded, there is $\alpha_0 > 0$ such that

$$a(t,x)\xi \cdot \xi \geq \alpha_0 |\xi|^2, \ \ \forall (t,x) \in \overline{Q}, \xi \in I\!R^n.$$

Using this and the fact that $|Dd(x)| = 1$ on O_δ we obtain

$$a(t,x) Dd(x) \cdot Dd(x) \geq \alpha_0 |Dd(x)|^2 = \alpha_0, \ \ \forall (t,x) \in \overline{Q}.$$

Therefore for $(t,x) \in [t_0, t_1) \times O_\delta$,

$$I^\varepsilon(t,x) \leq -\frac{K\tilde{g}^\varepsilon(t,x)}{\varepsilon(t_\gamma - t)}\left[K\alpha_0 - \frac{\varepsilon}{2}(t_\gamma - t)|a(t,x)||D^2 d(x)| + (t_\gamma - t)|b(t,x)| - d(x) \right].$$

Since d is $C^2(\overline{O}_\delta)$, the above expression is negative for sufficiently large K. In other words, $\tilde{g}^\varepsilon$ is a subsolution of (10.1) on $[t_0, t_\gamma) \times O_\delta$. We have already shown that $\tilde{g}^\varepsilon \leq \Phi^\varepsilon$ on $[t_0, t_\gamma] \times \partial O_\delta \cup \{t_\gamma\} \times O_\delta$. Hence the maximum principle yields, $\tilde{g}^\varepsilon \leq \Phi^\varepsilon$ on $[t_0, t_1] \times \overline{O}_\delta$. This combined with (10.10) implies (10.8) after letting $\gamma \to 0$. □

Lemma 10.2. *For any positive constant M and $\tilde{\psi} \in C^2(\overline{O})$ with $\tilde{\psi}(x) = 0$ for all $x \in \partial O$, there exists $K_M > 0$ satisfying*

$$V^\varepsilon(t,x) \geq M\tilde{\psi}(x) - K_M(t_1 - t), \ \ (t,x) \in \overline{Q}, \varepsilon \in (0,1]. \tag{10.11}$$

Proof. Set

$$K_M = \sup_{(t,x)\in\overline{Q}} \left\{ -\frac{M}{2}\operatorname{tr} a(t,x)D^2\tilde{\psi}(x) + H(t,x,MD\tilde{\psi}(x)) \right\}$$

and

$$\overline{g}^\varepsilon(t,x) = \exp\left(-\frac{M\tilde{\psi}(x) - K_M(t_1-t)}{\varepsilon} \right).$$

Then $\overline{g}^\varepsilon$ is a supersolution of (10.1) and (10.2). Moreover, although Φ^ε is not continuous at $\{t_1\}\times\partial O$, $\overline{g}^\varepsilon$ is continuous and $\overline{g}^\varepsilon = 1$ on $\{t_1\}\times\partial O$. We now obtain (10.11) by using the maximum principle. □

Since $\Phi^\varepsilon \le 1, V^\varepsilon \ge 0$. Hence by (10.8), $V^\varepsilon(t,x)$ is uniformly bounded for $\varepsilon\in(0,1], (t,x)\in[t_0,T]\times\overline{O}$ with any $T<t_1$. In particular, we may use (3.2) and (3.3) to define $V^*(t,x)$ and $V_*(t,x)$ at any $(t,x)\in[t_0,t_1)\times\overline{O}$. Although the terminal data (10.4)(b) is infinite, the stability result, Proposition 4.1, still applies to this situation. Thus we conclude that V^* and V_* are respectively, viscosity subsolution and supersolution of (2.3) in $[t_0,T)\times O$ for every $T<t_1$. Moreover (10.11) implies that $V^*(t,x)$ and $V_*(t,x)$ converges to $+\infty$, as $t\uparrow t_1$, uniformly on compact subsets of O. But this convergence is controlled by the estimate (10.8).

The above properties of V_* and V^* will be used to show the convergence of V^ε to $V=V_*=V^*$ which is the unique viscosity solution of (2.3), (2.4) (with $\psi(x)=+\infty$ in (2.4b)); see Theorem 10.1 below. This unique solution is the value function of an optimal control problem which we will define now. Let $U=I\!R^n, f(t,x,v)=v, g\equiv 0, L$ be as in (10.6), and τ the exit time of $(s,x(s))$ from Q. Let

$$\mathcal{U}(t,x) = \{u(\cdot)\in\mathcal{U}^0(t) : \tau<t_1\}.$$

Then $\dot{x}(s)=u(s)$ and

$$V(t,x) = \inf_{x(\cdot)\in\mathcal{A}(t,x)} J(t,x;u) \tag{10.12}$$

where

$$\mathcal{A}(t,x) = \{x(\cdot):[t,t_1]\to I\!R^n:\ \text{Lipschitz}, x(t)=x\ \text{and}\ \tau<t_1\},$$

$$J(t,x;u) = \int_t^\tau L(s,x(s),\dot{x}(s))ds.$$

Lemma 10.3. *For every $T<t_1, V$ is a viscosity solution of* (2.3) *in $[t_0,T)\times O$ with H as in* (10.3) *and it is Lipschitz continuous on $[t_0,T]\times\overline{O}$. Moreover it satisfies the boundary condition* (10.4).

Proof. We give the proof in several steps.

1. There exists M such that

$$0\le V(t,x)\le M dist(x,\partial O), (t,x)\in[t_0,T]\times O. \tag{10.13}$$

To show this, let $\bar{x}$ be a point of ∂O nearest x,

$$x_0(s) = x + v_0(s-t),\ v_0 = c|\bar{x}-x|^{-1}(\bar{x}-x),$$

where $c = (t_1 - T)^{-1} diam\ O$. Let τ_0 be the exit time from O of $x_0(s)$. Then

$$\tau_0 - t = c^{-1}|\bar{x}-x| \le c^{-1} diam\ O = t_1 - T$$

and hence $\tau_0 < t_1$ when $t \le T$. For every $x(\cdot) \in \mathcal{A}(t,x)$,

$$0 \le J(t,x;u) \le \int_t^{\tau_0} L(s,x_0(s),v_0)ds \le C(\tau_0 - t) \le M dist(x, \partial O)$$

where $L(s,y,v) \le C$ for all $(s,y) \in Q, |v| \le c$ and $M = c^{-1}C$.

2. From (10.6), there exist $k > 0, R_0 > 0$ such that $L(s,y,v) \ge k|v|^2$ whenever $|v| \ge R_0$. By (10.13), $V(t,x)$ is bounded on $[t_0,T] \times \overline{O}$. Therefore, there exists C_1 such that

$$\int_t^{\tau} |\dot{x}(s)|^2 ds \le C_1,\ (t,x) \in [t_0,T] \times O \tag{10.14}$$

for all $x(\cdot)$ such that $J(t,x;u) < V(t,x) + 1$.

3. We next show that $V(t,\cdot)$ is Lipschitz continuous on $\overline{O}$. For any $x,y \in O$ and $0 < \theta < 1$, choose $x(\cdot) \in \mathcal{A}(t,x)$ such that $J(t,x;u) < V(t,x) + \theta$. For $0 \le \lambda \le 1$, let $y_\lambda(s) = x(s) + \lambda(y-x)$. Let τ_1 be the exit time of $(s,y_1(s))$ from Q and $\tau_2 = \min(\tau,\tau_1)$. By the dynamic programming principle,

$$V(t,y) \le \int_t^{\tau_2} L(s,y_1(s),\dot{x}(s))ds + V(\tau_2,y_1(\tau_2))$$

$$= \int_t^{\tau_2} L(s,x(s),\dot{x}(s))ds + \int_t^{\tau_2}\int_0^1 L_x(s,y_\lambda(s),\dot{x}(s))\cdot\lambda(y-x)d\lambda ds + V(\tau_2,y_1(\tau_2)).$$

From (10.6) there exists K such that $|L_x(s,y,v)| \le K(1+|v|^2)$ for all $(s,y) \in \overline{Q}, v \in I\!\!R^n$. Hence,

$$V(t,y) \le J(t,x;u) + K|x-y| \int_t^{\tau_2} (1+|\dot{x}(s)|^2)ds + V(\tau_2,y_1(\tau_2)).$$

If $\tau_2 = \tau_1 \le \tau$, then $y_1(\tau_2) \in \partial O$ and $V(\tau_2,y_1(\tau_2)) = 0$. If $\tau_2 = \tau < \tau_1$, then $x(\tau_2) \in \partial O$. By (10.13)

$$V(\tau_2,y_1(\tau_2)) \le M|y_1(\tau_2) - x(\tau_2)| = M|x-y|.$$

Therefore,

$$V(t,y) \le V(t,x) + \theta + M_1|x-y|$$

where $M_1 = K(t_1 - t_0 + C_1) + M$. Since θ is arbitrary and x,y can be exchanged,

$$|V(t,x) - V(t,y)| \le M_1|x-y|, \tag{10.15}$$

for $t_0 \le t \le T, x,y \in \overline{O}$.

4. Let $Q_T = [t_0, T) \times O$. For $(t,x) \in Q_T$, the dynamic programming principle implies that

$$(10.16) \qquad V(t,x) = \inf_{u(\cdot)\in\mathcal{U}^0(t)} \left[\int_t^{\tilde{\tau}} L(s,x(s),u(s))ds + V(\tilde{\tau}, x(\tilde{\tau})) \right]$$

where $\tilde{\tau} = \min(\tau, T)$. The integrand L in (10.6) satisfies I(11.6) with $p = 2$. Let

$$(10.17) \qquad \tilde{\Psi}(t,x) = \begin{cases} 0 \ \text{ if } \ (t,x) \in [t_0,T] \times \partial O \\ \\ V(T,x) \ \text{ if } \ t = T, x \in O. \end{cases}$$

By Step 3, $\tilde{\Psi}$ satisfies I(11.7). By Theorem II.10.4, V is Lipschitz continuous in $\bar{Q}_T$ and is a viscosity solution to the Hamilton-Jacobi equation (2.3) in Q_T.

5. We continue by verifying (10.4). First note that the positivity of L yields $V \geq 0$. Then for $(t,x) \in [t_0, t_1) \times \partial O$, choose $x(\cdot) \in \mathcal{A}(t,x)$ satisfying $\tau = t$. Hence (10.4a) is satisfied by V. To prove (10.4b), let $x \in O$ and $x(\cdot) \in \mathcal{A}(t,x)$. Since $x(\tau) \in \partial O$, we have

$$\operatorname{dist}(x, \partial O) \leq |x(\tau) - x| = \left| \int_t^\tau \dot{x}(s)ds \right|.$$

Also (10.7) yields

$$\begin{aligned} \int_t^\tau L(s,x(s),\dot{x}(s))ds &\geq c_0 \int_t^\tau |\dot{x}(s) - b(s,x(s))|^2 ds \\ &\geq c_0 \int_t^\tau |\dot{x}(s)|^2 ds - K \\ &\geq \frac{c_0}{\tau - t} |\int_t^\tau \dot{x}(s)ds|^2 - K, \end{aligned}$$

where K is a suitable constant. Since the above inequality holds for every $x(\cdot) \in \mathcal{A}(t,x)$, and $\tau - t \leq t_1 - t$, we obtain

$$(10.18) \qquad V(t,x) \geq \frac{c_0 (\operatorname{dist}(x,\partial O))^2}{t_1 - t} - K.$$

□

We are now ready to prove the convergence of V^ε to V.

Theorem 10.1. *Suppose ∂O is of class C^3. Assume* (10.5). *Then V^ε converges to V uniformly on compact subsets of* $[t_0, t_1) \times \overline{O}$, *as $\varepsilon \downarrow 0$.*

Proof. It suffices to show that for all $\delta > 0$,

$$(10.19) \qquad V^*(t,x) \leq V(t+\delta, x), \ \ (t,x) \in [t_0, t_1 - \delta) \times \overline{O},$$

and

$$V(t-\delta,x) \le V_*(t,x), \quad (t,x) \in [t_0+\delta, t_1) \times \overline{O}. \tag{10.20}$$

First we claim that for all $\delta > 0$,

$$\liminf_{T \uparrow t_1} a(T,\delta) \ge 0, \tag{10.21}$$

where

$$a(T,\delta) = \left[\sup_{x \in \overline{O}} \{V(T,x) - V^*(T-\delta,x)\} \right]^+ .$$

Indeed, pick any sequence $(T_m, x_m) \to (t_1, \overline{x})$. Suppose that $\overline{x} \in \partial O$, then the estimate (10.8) and the positivity of V gives (10.21). But if $\overline{x} \in O$, (10.4)(b) or equivalently (10.18) together with (10.8) yield (10.21).

Fix $\delta > 0$ and $T < t_1$. Then $V(t+\delta,x) + a(T,\delta)$ and $V^*(t,x)$ are a viscosity subsubsolution and a viscosity supersolution of (2.3) in $[t_0, T-\delta) \times O$ (with H as in (10.3)) and boundary condition (2.4) with $g(t,x) \equiv a(T,\delta)$ and $\psi(x) = V(T,x) + a(T,\delta)$ respectively. Since V is Lipschitz continuous, Theorem 8.2 applies to this case giving $V(t+\delta,x)+a(T,\delta) \ge V^*(t,x), \forall (t,x) \in [t_0, T-\delta] \times \overline{O}$. We then obtain (10.19) by letting T go to t_1. The inequality (10.20) again follows from Theorem 8.2 and

$$\liminf_{t \uparrow t_1} \inf_{x \in \overline{O}} \{V_*(t,x) - V(t-\delta,x)\} \ge 0.$$

The above inequality is a consequence of (10.4)(a) and (10.11). □

The above convergence result can be restated as

$$\Phi^\varepsilon(t,x) = \exp(-\frac{1}{\varepsilon}[V(t,x) + h^\varepsilon(t,x)]), \tag{10.22}$$

where $|h^\varepsilon|$ converges to zero uniformly on compact subsets of $[t_0,t_1) \times \overline{O}$, as $\varepsilon \to 0$. The leading term in this expansion, $V(t,x)$ is non-negative. When $V(t,x) > 0$, we conclude that $\Phi^\varepsilon(t,x) \to 0$ exponentially fast as $\varepsilon \to 0$. However if $V(t,x) = 0$ the expansion (10.22) does not provide any information. So it is of some interest to discuss the cases when $V > 0$ on Q. By a probabilistic argument we see that the event $\{\theta^\varepsilon < t_1\}$ is rare if the deterministic flow

$$\frac{d}{ds} y(s) = b(s, y(s))$$

is pointing into the region O. So we claim that $V > 0$ on Q if

$$b(t,x) \cdot \eta(x) < 0, \;\; x \in \partial O, t \in [t_0, t_1], \tag{10.23}$$

where $\eta(x)$ is the unit outward normal vector at $x \in \partial O$. Indeed, let $(t,x) \in Q$ and $x(\cdot) \in \mathcal{A}(t,x)$. Then by the definition of $\mathcal{A}(t,x), \tau < t_1$. Set

$$u(s) = \dot{x}(s) - b(s, x(s)), \; s \in [t, t_1],$$

and let $\zeta(s)$ be the solution of

$$\dot{\zeta}(s) = b(s, \zeta(s)), \;\; s \in (t, t_1)$$

with initial data $\zeta(t) = x$. Using these equations we obtain the following estimate for the time derivative of $\xi(s) = \frac{1}{2}|x(s) - \zeta(s)|^2$,

$$\begin{aligned}
\dot{\xi}(s) &= (\dot{x}(s) - \dot{\zeta}(s)) \cdot (x(s) - \zeta(s)) \\
&= (b(s, x(s)) - b(s, \zeta(s))) \cdot (x(s) - \zeta(s)) + u(s) \cdot (x(s) - \zeta(s)) \\
&\leq K\xi(s) + |u(s)|^2,
\end{aligned}$$

with a suitable constant K, depending on the Lipschitz constant of b. Using the Gronwall inequality and the fact that $\xi(t) = 0$, we have

$$\xi(s) \leq e^{K(s-t)} \int_t^s |u(z)|^2 dz \quad , \quad s \in [t, \tau].$$

The condition (10.23), implies that for $x \in O$ there is a constant $\beta(x)$ satisfying dist $(\zeta(s), \partial O) \geq \beta(x) > 0$ for all $s \in [t, t_1]$. Since $x(\tau) \in \partial O$, we have

$$\begin{aligned}
[\beta(x)]^2 &\leq \text{dist}\,(\zeta(\tau), \partial O)^2 \\
&\leq 2\xi(\tau) \\
&\leq 2e^{K(\tau-t)} \int_t^\tau |u(z)|^2 dz \\
&= 2e^{K(\tau-t)} \int_t^\tau |\dot{x}(s) - b(s, x(s))|^2 ds \\
&\leq \frac{2}{c_0} e^{K(t_1-t)} \int_t^\tau L(s, x(s), \dot{x}(s)) ds.
\end{aligned}$$

In the last inequality we used (10.7). Since the above holds for every $x(\cdot) \in \mathcal{A}(t, x)$, we have

$$V(t, x) \geq \frac{c_0}{2} e^{-K(t_1-t)} [\beta(x)]^2.$$

We summarize the above result in the following.

Lemma 10.4. *Suppose that the hypotheses of Theorem* 10.1 *hold and* (10.23) *is satisfied. Then* $V(t, x) > 0$ *for* $(t, x) \in [t_0, t_1] \times O$. *In particular* Φ^ε *converges to zero exponentially fast on* $[t_0, t_1] \times O$.

VII.11 Weak comparison principle in Q_0

In this section we prove a weak comparison principle (Theorem 11.1) similar to Theorem 8.2, which holds when $O = I\!R^n$ and thus $Q = Q_0 = [t_0, t_1] \times I\!R^n$. Since ∂O is empty, the proof is simpler than the arguments used in Section 8 to prove Theorems 8.1 and 8.2. Theorem 11.1 is then used to prove a large deviations result (Theorem VI.6.2) for the small noise limit problem. A similar risk-sensitive control limit result will be proved later (Theorem XI.7.2).

We consider the first order PDE

$$(11.1) \qquad -\frac{\partial V}{\partial t} + H(t, x, D_x V) = 0, \quad (t, x) \in Q_0$$

with the terminal data (see (2.4b))

$$(11.2) \qquad V(t_1, x) = \psi(x).$$

We assume that:

(11.3)

(a) ψ is bounded and Lipschitz continuous on $I\!R^n$;

(b) $H(t, x, p)$ is bounded on $Q_0 \times \{|p| \le R\}$ and and (8.16) holds, for each $R > 0$.

Lemma 11.1. *Assume* (11.3) *and that* (11.1)-(11.2) *has a bounded, Lipschitz continuous viscosity solution* V. *Let* $W^*(t, x)$ *be any bounded, upper semicontinuous viscosity subsolution of* (11.1) *such that* $W^*(t_1, x) \le \psi(x)$ *for all* $x \in I\!R^n$. *Then* $W^* \le V$.

The proof is a modification of arguments used to prove Theorem II.9.1 in the unbounded case and uses arguments similar to those for Theorem 8.2. We merely give a sketch. Introduce the function $\Phi_\gamma(t, x; s, y)$ as in the proof of Theorem II.9.1, with $W(t, x)$ replaced by $W^*(t, x)$. Assume that

$$\sup_{(t,x) \in \overline{Q}_0} [W^*(t, x) - V(t, x)] = \alpha > 0.$$

If the parameters β, γ in the proof of Theorem II.9.1 are small enough, then the sup of Φ_γ on $\overline{Q}_0 \times \overline{Q}_0$ is at least $\alpha/2$, and is achieved at some $(\bar{t}, \bar{x}), (\bar{s}, \bar{y})$. The previous proof shows that $|\bar{t} - \bar{s}| \le C_1 \delta^{\frac{1}{2}}$, $|\bar{x} - \bar{y}| \le C_2 \epsilon^{\frac{1}{2}}$ for some constants C_1, C_2. Morover, for fixed γ there is a compact set K such that $(\bar{t}, \bar{x})$, $(\bar{s}, \bar{y}) \in K$ for ϵ, δ sufficiently small. Let us show that $\bar{s} < t_1$ and $\bar{t} < t_1$ for small enough ϵ, δ. Let

$$\phi(\theta) = \sup\{W^*(t, x) - \psi(x) \colon |t - t_1| < \theta, (t, x) \in K\}.$$

Since W^* is upper semicontinuous, ψ is Lipschitz and $W^*(t, x) \le \psi(x)$ for all (t, x),

$$\limsup_{\theta \to 0} \phi(\theta) \le 0.$$

If $\bar{s} = t_1$, then from the definition of Φ_γ,

$$\frac{\alpha}{2} < \Phi_\gamma(\bar{t}, \bar{x}; t_1, \bar{y}) \leq W^*(\bar{t}, \bar{x}) - \psi(\bar{y})$$
$$\leq \psi(\bar{x}) - \psi(\bar{y}) + \phi(C_1 \delta^{1/2}).$$

This is impossible for small enough ϵ, δ. If $\bar{t} = t_1$, then

$$\frac{\alpha}{2} < \Phi_\gamma(t_1, \bar{x}; \bar{s}, \bar{y}) \leq \psi(\bar{x}) - V(\bar{s}, \bar{y})$$
$$\leq \psi(\bar{x}) - \psi(\bar{y}) + \psi(\bar{y}) - V(\bar{s}, \bar{y}) \leq C_3(\epsilon^{\frac{1}{2}} + \delta^{\frac{1}{2}}).$$

since ψ and V are Lipschitz. Hence $\bar{t} < t_1$, and $(\bar{t}, \bar{x}), (\bar{s}, \bar{y}) \in Q_0$ if ϵ and δ are small enough.

Define the comparison functions $\overline{w}(t, x),\ w^*(s, y)$ as in Step $4'$ of the proof of Theorem II.9.1. Then inequalities II(9.22), II(9.23) hold. Since V is Lipschitz, there exist M_1, M_2 such that

$$|D_y w^*(\bar{s}, \bar{y})| = |p_\epsilon + \bar{p}_\gamma| \leq M_1$$

$$\left| \frac{\partial w^*}{\partial s}(\bar{s}, \bar{y}) \right| = |\beta + q_\delta + \bar{q}_\gamma| \leq M_2,$$

by Corollary II.8.1(f). If we fix $\beta > 0$ and take ϵ, δ, γ small enough, then a contradiction is obtained. □

In the same way, if $W_*(t, x)$ is a bounded lower semicontinuous supersolution of (11.1) with $\psi(x) \leq W_*(t_1, x)$, then $V \leq W_*$. We have therefore:

Theorem 11.1. *Assume* (11.3) *and that* (11.1)-(11.2) *has a bounded, Lipschitz continuous viscosity solution V. Then a weak comparison principle holds.*

An easy consequence of Theorem 11.1 is the following result stated in Section VI.6.

Proof of Theorem VI.6.2. Since VI(6.8) holds, the Hamiltonian $H(t, x, p)$ in VI(4.8) satisfies (11.3)(b). Moreover, VI(6.8)(a)(b) imply (11.3)(a) and that $|V^\epsilon(t, x)| \leq K_1$ for some K_1. By Theorem VI.8.1 and Remark VI.8.2, V^ϵ is a viscosity solution to VI(6.4^ϵ) with $V^\epsilon(t_1, x) = \psi(x)$. Define $V^*(t, x)$, $V_*(t, x)$ by (3.2), (3.3). By Proposition 4.1, V^* is a viscosity subsolution of (11.1) and V_* is a viscosity supersolution. By Lemma 5.1, $V^*(t_1, x) = V_*(t_1, x) = \psi(x)$. By Theorem VI.6.1, V^0 is a bounded Lipschitz continuous viscosity solution of (6.4^0) with $V^0(t_1, x) = \psi(x)$. By Theorem 11.1, $V^* = V_* = V^0$. Hence, V^ϵ tends to V^0 uniformly on compact subsets of $\overline{Q}_0$. □

Remark 11.1. The assumptions VI(3.2) VI(6.8) are not enough to guarantee that $V^\epsilon \in C^{1,2}(\overline{Q}_0)$, although this is true under the stronger assumptions VI(3.25) or VI(3.26).

VII.12 Historical remarks

Fleming was first to study the vanishing viscosity problem by using control theoretic techniques [F1]. The probabilistic treatment of exit time problems and the large deviations theory for Markov diffusion processes is pursued by Freidlin and Wentzell [FrW]. For other probabilistic results we refer the reader to Varadhan [V] and Stroock [St].

The first application of the viscosity solutions to large deviations problem is given by Evans and Ishii [EI]. Since then there have been several such applications. In [FS2] Fleming and Souganidis obtained asymptotic expansions for Markov diffusions, Dupuis, Ishii and Soner [DIS] studied the large deviations problem of a queueing system, and asymptotic expansions for jump Markov processes were derived by Fleming and Soner [FSo].

The application to homogenization is due to Evans [E3]. Evans and Souganidis [ES2] used the viscosity theory to analyze the large time asymptotic of a reaction-diffusion equation of KPP type. The extension of this result to a system of reaction-diffusion equations is given by Barles, Evans and Souganidis [BES]. Along the same direction Evans, Soner and Souganidis [ESS] studied the asymptotics of the zero level set of a solution of a reaction-diffusion equation with a cubic nonlinearity. They proved that in the limit the zero level set moves by motion by mean curvature. Another application is obtained by Soner [S1], who studied a sequence of singularly perturbed optimal control problems arising in manufacturing.

The discontinuous viscosity solutions were defined independently by Barles-Perthame [BP1] and by Ishii [12]. See Barles [Ba] for another account of this theory. The proof of Theorem 8.1 is an extension of [S2, Thm.2.1]. The applications of this definition to singular perturbation problems are included in [BP2] and [IK2]. Perthame [P] combined these techniques with some other analytical tools to study the asymptotics of the exit location distribution. Alternate approaches to discontinuous value functions have been formulated by Frankowska [Fra] and Subbotin [Su2].

VIII

Singular Stochastic Control

VIII.1 Introduction

In contrast to classical control problems, in which the displacement of the state due to control effort is differentiable in time, the singular control models we consider allow this displacement to be discontinuous. Bather-Chernoff were the first to formulate such a problem in their study of a simplified model of spacecraft control. Since then singular control has found many other applications in diverse areas of communications, queueing systems and mathematical finance.

We start our analysis in Section 2 with a formal derivation of the dynamic programming equation. This discussion leads us to a formulation of singular control problems with controls which are processes of bounded variation. The related verification theorem is then proved in Section 4. We discuss the viscosity property of the value function in Section 5. Since we want to emphasize only the new issues arising in singular control, in Sections 2–5 we restrict our attention to a simple infinite horizon problem with no absolutely continuous control component. However the theory is not limited only to these problems. In Section X.5, we demonstrate the potential of this theory in a portfolio selection problem with transaction costs. The portfolio selection problem has "mixed" type controls; consumption rate and transactions between the bond and the stock. Here the consumption rate is an absolutely continuous control and the transactions between the bond and the stock is a singular type control.

The dynamic programming equation (2.7) below is a pair of differential inequalities. Also at every point of the state space either one of the inequalities is satisfied by an equality. So the state space splits into two regions, the "no-action" region and the "push region," corresponding to the active inequality in (2.7). Starting from the push region, the optimal state process moves immediately into the no-action region, where its exit is prevented by reflection at the boundary in an appropriate direction. In Section 4 we exhibit this qualitative character of the optimal control process in several examples.

The portfolio selection problem with transaction costs is an important example of a singular control problem. It will be studied in Chapter X. This problem is a generalization of the Merton's portfolio problem, Example IV.5.2. In the portfolio selection problem the Markov diffusion process is degenerate and there are only two possible push directions. These facts render this problem difficult. However if the utility function is of HARA type (see Example IV.5.2), the value function has an homothety property which reduces the problem to a one-dimensional one. Using this property we prove that the no-action region is a wedge in the plane, Theorem X.5.2 below.

In the spacecraft control example the cumulative control action is constrained by the total amount of the initial fuel. In the literature generalizations of problems with this type of constraint are known as finite fuel problems. In Section 6, we formulate the finite fuel problem and derive the dynamic programming equation. In the case of a single push direction, Chow-Menaldi-Robin [CMR] discovered a striking connection between the optimal solutions of finite fuel and the unconstrained problem. We prove this connection in Proposition 6.1.

VIII.2 Formal discussion

In this section, we consider a special case of the infinite horizon problem described in Section IV.5. We let $O \subset I\!R^n$ and $U \subset I\!R^n$ be a closed cone in $I\!R^n$, i.e.,

$$v \in U, \quad \lambda \geq 0 \Rightarrow \lambda v \in U. \tag{2.1}$$

We also assume that there are $\hat{f}, \hat{\sigma} \in C^1(I\!R^n)$ with bounded first order partial derivatives and $\hat{c}, \hat{L} \in C(I\!R^n)$ satisfying

$$f(x,v) = v + \hat{f}(x), \tag{2.2i}$$

$$\sigma(x,v) = \hat{\sigma}(x), \tag{2.2ii}$$

$$L(x,v) = \hat{L}(x) + \hat{c}(v), \tag{2.2iii}$$

$$\hat{c}(\lambda v) = \lambda \hat{c}(v), \quad \forall \lambda \geq 0, \tag{2.2iv}$$

for all $x \in I\!R^n, v \in U$. For simplicity we take the boundary condition $g \equiv 0$ and $\hat{L}, \hat{c} \geq 0$.

The control set U is not bounded as was often assumed in Chapter IV.

Let us continue by calculating the Hamiltonian $\mathcal{H}(x, p, A)$ defined in IV(3.2). For $x, p \in I\!R^n$ and a symmetric matrix A

$$\mathcal{H}(x, p, A) = -\frac{1}{2} \operatorname{tr} \hat{a}(x) A - \hat{f}(x) \cdot p - \hat{L}(x) + \hat{\mathcal{H}}(p),$$

where $\hat{a}(x) = \hat{\sigma}(x)\hat{\sigma}'(x)$ and

$$\hat{\mathcal{H}}(p) = \sup_{v \in U} \{-p \cdot v - \hat{c}(v)\}. \tag{2.3}$$

Observe that if $-p \cdot v - \hat{c}(v) > 0$ for some $v \in U$, then by (2.1) and (2.2iv) we conclude that $\hat{\mathcal{H}}(p) = +\infty$. Therefore

$$\hat{\mathcal{H}}(p) = \begin{cases} +\infty & \text{if } H(p) > 0 \\ 0 & \text{if } H(p) \leq 0, \end{cases}$$

where

$$H(p) = \sup_{v \in \hat{K}} \{-p \cdot v - \hat{c}(v)\}, \tag{2.4}$$

$$\hat{K} = \{v \in U : |v| = 1\}.$$

One can think of $\hat{K}$ as the set of allowable directions in which control may act.

The above calculation indicates that the dynamic programming equation IV(3.3) has to be interpreted carefully. However, we formally expect that the value function V satisfies

$$H(DV(x)) \leq 0, \quad x \in O, \tag{2.5i}$$

$$\mathcal{L}V(x) = \beta V(x) - \frac{1}{2} \operatorname{tr} \hat{a}(x) D^2 V(x) - \hat{f}(x) \cdot DV(x) \leq \hat{L}(x), \; x \in O. \tag{2.5ii}$$

Now suppose $H(DV(x)) < 0$ for some $x \in O$. Then in a neighborhood of x, the unique maximizer in (2.3) is zero. Hence at least formally, the optimal feedback control should be equal to zero in a neighborhood of x. Since the uncontrolled diffusion processes are related to linear equations, we expect

$$\mathcal{L}V(x) = \hat{L}(x), \; \text{whenever } H(DV(x)) < 0. \tag{2.6}$$

We now rewrite (2.5) and (2.6) in the following more compact form:

$$\max\{\mathcal{L}V(x) - \hat{L}(x), H(DV(x))\} = 0, \quad x \in O. \tag{2.7}$$

Since $g \equiv 0$, the boundary condition is

$$V(x) = 0, \quad x \in \partial O. \tag{2.8}$$

In Section 4 we will prove a Verification Theorem, Theorem 4.1, for the dynamic programming equation (2.7) and the boundary condition (2.8). This provides a rigorous basis to our formal derivation. However due to the linear dependence of L on v, in general there are no optimal controls and nearly optimal controls take arbitrarily large values. For this reason it is convenient to reformulate the above problem by using the integral of $u(s)$ as our control process. This reformulation will be the subject of the next section.

VIII.3 Singular stochastic control

As in Chapters IV and V, let $\nu = (\Omega, \{\mathcal{F}_s\}, P, w)$ be a probability reference system with a right continuous filtration $\mathcal{F}_s$. Let us rewrite the state dynamics IV(5.1) as follows, taking into account the special form (2.2i) of f: Let

$$\hat{u}(s) = \begin{cases} |u(s)|^{-1}u(s) & \text{if } u(s) \neq 0 \\ 0 & \text{if } u(s) = 0, \end{cases}$$

$$\xi(t) = \int_0^t |u(s)|ds.$$

Then IV(5.1) becomes

(3.1) $$dx(s) = \hat{\sigma}(x(s))dw(s) + \hat{f}(x(s))ds + \hat{u}(s)d\xi(s), \ \ s > 0.$$

We now regard

(3.2) $$z(t) = \int_{[0,t)} \hat{u}(s)d\xi(s)$$

as control variable at time t. However in order to obtain optimal controls, we must enlarge the class of controls to admit $z(t)$ which may not be an absolutely continuous function of t. But we assume that each component of $z(t)$ is a function of bounded variation on every finite interval $[0,t]$; namely, each component of $z(t)$ is the difference of two monotone functions of t. Let $\mu(\cdot)$ be the total variation measure of $z(\cdot)$ and

$$\xi(t) = \int_{[0,t)} d\mu(s).$$

Then

(3.3a) $\xi(\cdot)$ nondecreasing, real-valued, left continuous with $\xi(0) = 0$.

Moreover by Radon-Nikodynm theorem, there exists $\hat{u}(s) \in I\!\!R^n$ satisfying (3.2) and $|\hat{u}(s)| \leq 1$. In this chapter we will identify the process $z(\cdot)$ by the pair $(\xi(\cdot),\hat{u}(\cdot))$. Note that for a given $z(\cdot)$, $\xi(t)$ is uniquely determined for all $t\geq 0$, and $\hat{u}(s)$ is uniquely determined for μ-almost every $s\geq 0$. Also if $z(s)$ is $\mathcal{F}_s$-progressively measurable, then $\xi(s)$ is $\mathcal{F}_s$-progressively measurable and there is a version of $\hat{u}(s)$ which is $\mathcal{F}_s$-progressively measurable (See Appendix D). In the sequel we always assume that $z(s)$ is $\mathcal{F}_s$-progressively measurable and that $\hat{u}(s)$ is a $\mathcal{F}_s$ - progressively measurable function satisfying (3.2). Further we assume that

$$\hat{u}(s)\in U,\ \text{for } \mu\text{-almost every } s\geq 0, \tag{3.3b}$$

$$E|z(t)|^m<\infty,\ m=1,2,\cdots. \tag{3.3c}$$

Let $\hat{\mathcal{A}}_\nu$ denote the set of all progressively measurable $z(\cdot)=(\xi(\cdot),\hat{u}(\cdot))$ satisfying (3.3). Then for a given $x\in O$, the usual Picard iteration gives $x_m(t), m=1,2,\cdots$ such that $x_m(t)-z(t)$ converges to $x(t)-z(t)$ with probability 1, uniformly for bounded t. The process $x(\cdot)$ is the unique, left continuous solution to

$$x(t)=x+\int_0^t\hat{\sigma}(x(s))dw(s)+\int_0^t\hat{f}(x(s))ds+z(t),\ t\geq 0, \tag{3.4}$$

with

$$x(t^+)-x(t)=z(t^+)-z(t).$$

Observe that $x(t)$ is not in general continuous. Indeed at every $t\geq 0$,

$$x(t^+)=\lim_{s\downarrow t}x(s)=x(t)+z(t^+)-z(t)=x(t)+\hat{u}(t)(\xi(t^+)-\xi(t)).$$

Let τ be the exit time of $x(s)$ from $\overline{O}$. Since $x(\cdot)$ is left continuous and $\mathcal{F}_t$ is right continuous, τ is a $\mathcal{F}_t$-stopping time. We now wish to minimize

$$J(x;\xi,\hat{u})=E_x\int_{[0,\tau]}e^{-\beta s}[\hat{L}(x(s))ds+\hat{c}(\hat{u}(s))d\xi(s)] \tag{3.5}$$

over all $(\xi(\cdot),\hat{u}(\cdot))\in\hat{\mathcal{A}}_\nu$ and reference probability systems ν. Observe that since $\hat{L},\hat{c}\geq 0$, $J(x;\xi,\hat{u})$ is defined for all $(\xi(\cdot),\hat{u}(\cdot))\in\hat{\mathcal{A}}_\nu$ but it may take the value $+\infty$. This is why we do not need to impose an additional condition like IV(5.5) on the control $(\xi(\cdot),\hat{u}(\cdot))$. Finally, let

$$V_\nu(x)=\inf_{\hat{\mathcal{A}}_\nu}J(x;\xi,\hat{u}),$$

$$V(x)=V_{PM}(x)=\inf_\nu V_\nu(x).$$

We close this section by proving some elementary properties of V.

Lemma 3.1. Let $x\in O,\hat{v}\in U$ *and* $h>0$. *If* $x+h\hat{v}\in O$, *then*

$$V(x)-V(x+h\hat{v})\leq h\hat{c}(\hat{v}). \tag{3.6}$$

In particular if V is differentiable at x, then (2.5i) holds at x.

Proof. In view of (2.2iv), we may assume that $|\hat{v}| \leq 1$. For $(\xi(\cdot), \hat{u}(\cdot)) \in \hat{\mathcal{A}}_\nu$, let

$$\xi_h(s) = \begin{cases} 0, & s = 0 \\ \xi(s) + h, & s > 0, \end{cases} \qquad \hat{u}_h(s) = \begin{cases} \hat{v}, & s = 0 \\ \hat{u}(s), & s > 0. \end{cases}$$

Then $(\xi_h(\cdot), \hat{u}_h(\cdot)) \in \hat{\mathcal{A}}_\nu$. Let $x_h(s)$ be the solution of (3.1) with control $(\xi_h(\cdot), \hat{u}_h(\cdot))$ and initial condition $x_h(0) = x$. Then, $x_h(s) = x(s) + h\hat{v}$ for all $s > 0$. Moreover,

$$V(x) \leq J(x; \xi_h, \hat{u}_h) = J(x + h\hat{v}; \xi, \hat{u}) + h\hat{c}(\hat{v}).$$

Since $\xi(\cdot)$ and $\hat{u}(\cdot)$ are arbitrary, (3.6) follows from the above inequality. □

Lemma 3.2. *Suppose that $O = I\!\!R^n$, U is convex, $\hat{f}$ and $\hat{\sigma}$ are affine functions and $\hat{L}, \hat{c}$ are convex. Then V_ν is also convex for every ν.*

Proof. For $i = 1, 2$, let $x_i \in I\!\!R^n$ and $(\xi_i(\cdot), \hat{u}_i(\cdot)) \in \hat{\mathcal{A}}_\nu$. Set

$$y = \frac{1}{2}(x_1 + x_2),$$

$$\xi(t) = \frac{1}{2}(\xi_1(t) + \xi_2(t)), \quad t \geq 0,$$

$$z(t) = \int_{[0,t)} \frac{1}{2}(\hat{u}_1(s)d\xi_1(s) + \hat{u}_2(s)d\xi_2(s)), \quad t \geq 0.$$

For a Borel subset $B \subset [0, \infty)$ define

$$\mu_i(B) = \int_B d\xi_i(s), \ \mu(B) = \int_B d\xi(s).$$

Then both μ_1 and μ_2 are absolutely continuous with respect to μ. For $i = 1, 2$ let $r_i(s)$ be the Radon-Nikodyn derivative of μ_i with respect to μ. We may choose $r_1(s)$ and $r_2(s)$, so that for every $s \geq 0$,

$$\frac{1}{2}(r_1(s) + r_2(s)) = 1.$$

Let $|\hat{u}(s)| \leq 1$ be a progressively measurable function satisfying

$$z(t) = \int_{[0,t)} \hat{u}(s)d\xi(s), \ t \leq 0.$$

Then for μ-almost every $s \geq 0$,

$$\hat{u}(s) = \frac{1}{2}[\hat{u}_1(s)r_1(s) + \hat{u}_2(s)r_2(s)], \ s \geq 0. \tag{3.7}$$

Also by the convexity of U, $\hat{u}(s) \in U$ for μ-almost every $s \geq 0$. Thus $z(\cdot) = (\xi(\cdot), \hat{u}(\cdot)) \in \hat{\mathcal{A}}_\nu$.

Now the convexity of $\hat{c}$, together with (2.2iv) and (3.7), yields

$$\hat{c}(\hat{u}(s)) \leq \frac{1}{2}[\hat{c}(\hat{u}_1(s))r_1(s) + \hat{c}(\hat{u}_2(s))r_2(s)],$$

for μ-almost every $s \geq 0$. Multiply both sides of the above inequality by $e^{-\beta s}$ and then integrate with respect to $d\xi(s)$ to obtain

$$\text{(3.8)} \qquad \int_0^\infty e^{-\beta s}\frac{1}{2}[\hat{c}(\hat{u}_1(s))d\xi_1(s) + \hat{c}(\hat{u}_2(s))d\xi_2(s)]$$

$$\geq \int_0^\infty e^{-\beta s}\hat{c}(\hat{u}(s))d\xi(s).$$

Finally the convexity of $\hat{L}$ and (3.8) imply that

$$V_\nu(y) \leq J(y;\xi,\hat{u}) \leq \frac{1}{2}[J(x_1;\xi_1,\hat{u}_1) + J(x_2;\xi_2,\hat{u}_2)].$$

After taking the inf over $(\xi_i(\cdot), \hat{u}_i(\cdot)) \in \hat{\mathcal{A}}_\nu$ we obtain

$$V_\nu(y) \leq \frac{1}{2}[V_\nu(x_1) + V_\nu(x_2)].$$

Now the convexity of V_ν follows from elementary considerations. □

The convexity assumption on U and $\hat{c}$ are satisfied by the following examples:

(i) $\hat{K} = S^{n-1} = \{v \in I\!\!R^n : |v| = 1\},\ \hat{c}(v) = |v|,$

(ii) $\hat{K} = \{\nu_0\},$

(iii) $\hat{K} = \{\nu_0, -\nu_0\},$

where $\nu_0 \in S^{n-1}$ and $\hat{K}$ is as in (2.4).

VIII.4 Verification theorem

We start with the definition of classical solutions of (2.7). Let $W^{1,\infty}_{\ell oc}(O; I\!\!R^n)$ be the set of all $I\!\!R^n$-valued functions of O which are Lipschitz continuous on every bounded subset of O.

Definition 4.1. Let $W \in C_p(\bar{O}) \cap C^1(\bar{O})$ with $DW \in W^{1,\infty}_{\text{loc}}(O; I\!\!R^n)$ be given. Define

$$\mathcal{P} = \{x \in I\!\!R^n : H(DW(x)) < 0\}.$$

We say that W is a (*classical*) *solution* of (2.7) if $W \in C^2(\mathcal{P})$,

$$\mathcal{L}W(x) = \hat{L}(x), \ \forall x \in \mathcal{P},$$

$$H(DW(x)) \le 0, \ \forall x \in \bar{O},$$

and

$$\mathcal{L}W(x) \le \hat{L}(x),$$

for almost every $x \in I\!R^n$.

At the end of this section we will construct explicit solutions of (2.7) for specific problems.

The following verification theorem is very similar to Theorem IV.5.1.

Theorem 4.1. (Verification) *Assume O is convex. Let W be a classical solution of* (2.7) *with the boundary condition* (2.8)*. Then for every $x \in \bar{O}$*
(a) $W(x) \le J(x; \xi, \hat{u})$ for any $(\xi(\cdot), \hat{u}(\cdot)) \in \hat{\mathcal{A}}_\nu$ such that

$$\liminf_{t\to\infty} e^{-\beta t} E_x[W(x(t))\chi_{\tau=\infty}] = 0. \tag{4.1}$$

(b) Assume that $W \ge 0$ and that there exists $(\xi^*(\cdot), u^*(\cdot)) \in \hat{\mathcal{A}}_\nu$ such that with probability 1:

$$x^*(t) \in \mathcal{P}, \ \textit{Lebesgue almost every } t \le \tau, \tag{4.2i}$$

$$\int_{[0,t)} [u^*(s) \cdot DW(x^*(s)) + \hat{c}(u^*(s))] d\xi^*(s) = 0, \ \forall t \le \tau, \tag{4.2ii}$$

$$W(x^*(t)) - W(x^*(t^+)) = \hat{c}(u^*(t))[\xi^*(t^+) - \xi(t)], \ \forall t \le \tau, \tag{4.2iii}$$

$$\lim_{t\to\infty} E_x[e^{-\beta(t\wedge\tau)} W(x^*(t\wedge\tau))\chi_{\tau=\infty}] = 0. \tag{4.2iv}$$

Then

$$J(x; \xi^*, u^*) = W(x).$$

Proof. (a) Extend W to $I\!R^n$ by setting $W(x) = 0$ for $x \notin O$. Since $W(x) = 0$ for $x \in \partial O$ and since W is continuous on $\bar{O}$, the extension of W is also continuous on $I\!R^n$. Let $\zeta \in C^\infty(I\!R^n)$ be a nonnegative function satisfying

$$\int_{R^n} \zeta(x) dx = 1,$$

$$\text{supp}\ \zeta \subset B_1(0) = \{x \in I\!R^n = |x| < 1\}.$$

For a positive integer m, set $\zeta_m(x) = m^{-n}\zeta(mx)$ and

$$W_m = W * \zeta_m, \quad \hat{L}_m = \hat{L} * \zeta_m,$$

where $*$ denotes the convolution operator. Since W and $\hat{L}$ are continuous on $I\!R^n$, W_m and $\hat{L}_m$ converge to W and $\hat{L}$ uniformly on compact subsets of $I\!R^n$.

Similarly since $DW \in C(\bar{O})$, DW_m converges to DW uniformly on compact subset of O. See Appendix C.

Let

$$O_m = \left\{ x \in O : \text{ dist } (x, \partial O) > \frac{1}{m} \right\}, \quad O_{m,N} = O_m \cap B_N(0).$$

Since ζ_m is supported in $B_{1/m}(0)$, for any function $h, h * \zeta_m(x_0)$ depends only on the values of h in $B_{1/m}(x_0) = \{x \in I\!R^n : |x - x_0| < 1/m\}$. Hence if $x_0 \in \bar{O}_m, h * \zeta_m(x_0)$ depends on the values $h(x)$ with $x \in O$. Therefore the local boundedness of $D^2 W$ in O and the continuity of the coefficients of the operator $\mathcal{L}$ yield

$$\lim_{m \to 0} \sup_{x \in \bar{O}_{m,N}} |(\mathcal{L} W_m)(x) - (\mathcal{L} W * \zeta_m)(x)| = 0.$$

Set

$$K_{m,N} = \sup_{x \in \bar{O}_{m,N}} [\mathcal{L} W_m(x) - \hat{L}_m(x)].$$

Since $\mathcal{L} W(x) \leq \hat{L}(x)$ for almost every $x \in O$,

$$(\mathcal{L} W * \xi_m)(x) \leq \hat{L}_m(x), \ \forall x \in O_m.$$

Hence

$$\limsup_{m \to \infty} K_{m,N} \leq 0.$$

Also for every $\hat{v} \in U$ and $x \in \bar{O}_m$

$$-\hat{v} \cdot DW_m(x) - \hat{c}(\hat{v}) = -[(\hat{v} \cdot DW + \hat{c}(\hat{v})) * \zeta_m](x) \leq 0.$$

Combining the above inequalities, we obtain

$$\max\{\mathcal{L} W_m(x) - \hat{L}_m(x), H(DW_m(x))\} \leq K_{m,N} \vee 0, \ \forall x \in \bar{O}_{m,N}. \tag{4.3}$$

Let $x \in O_{m,N}$ and $(\xi(\cdot), \hat{u}(\cdot)) \in \hat{\mathcal{A}}_\nu$ be given. Since $W_m \in C^\infty(I\!R^n)$, we may apply the Ito's rule for semimartingales (Meyer [M, pp. 278-301]) to $e^{-\beta t} W_m(x(t))$. Fix N and let τ_m be the exit time of $x(s)$ from $\overline{O}_{m,N}$. Then for $t \geq 0$,

$$\begin{aligned} W_m(x) = {} & E_x e^{-\beta(t \wedge \tau_m)} W_m(x(t \wedge \tau_m)) \\ & + E_x \int_0^{t \wedge \tau_m} e^{-\beta s} \mathcal{L} W_m(x(s)) ds \\ & + E_x \int_0^{t \wedge \tau_m} e^{-\beta s} [-\hat{u}(s) \cdot DW_m(x(s))] d\xi^c(s) \end{aligned} \tag{4.4}$$

$$+E_x \sum_{0 \le s < t \wedge \tau_m} e^{-\beta s}[W_m(x(s)) - W_m(x(s^+))],$$

where for $t \ge 0$

$$\xi^c(t) = \xi(t) - \sum_{0 \le s < t} [\xi(s^+) - \xi(s)],$$

$$\xi^J(t) = \xi(t) - \xi^c(t) = \sum_{0 \le s < t} [\xi(s^+) - \xi(s)].$$

Note that both $\xi^c(\cdot)$ and $\xi^J(\cdot)$ are nondecreasing. (When $\xi(\cdot)$ has finitely many discontinuities with a bounded derivative except at these points, (4.4) can easily be derived by using the classical Ito's rule between the discontinuities of ξ.)

Next we will let m go to infinity in (4.4). Since $x(\cdot)$ is left continuous as m tends to infinity, $\tau_m \to \theta_N$, where θ_N is the exit time of $x(s)$ from $O_N = O \cap B_N(0)$. Also

$$\limsup_{m\to\infty} \mathcal{L}W_m(x(s)) \le \limsup_{m\to\infty}[\hat{L}_m(x(s)) + K_{m,N}] \le \hat{L}(x(s)),$$

$$-\hat{u}(s) \cdot DW_m(x) \le \hat{c}(\hat{u}(s)), \quad \forall s \ge 0, x \in O_m.$$

Now by the Mean Value Theorem and $x(s^+) = x(s) + \hat{u}(s)[\xi(s^+) - \xi(s)]$, we obtain

$$\begin{aligned} &W_m(x(s)) - W_m(x(s^+)) \\ &\quad = -\int_0^1 \hat{u}(s) \cdot DW_m(x(s) + \lambda[x(s^+) - x(s)])(\xi(s^+) - \xi(s))d\lambda \\ &\quad \le \hat{c}(\hat{u}(s))(\xi(s^+) - \xi(s)). \end{aligned}$$

In the above inequality we used the fact that

$$x(s) + \lambda[x(s^+) - x(s)] \in \bar{O}$$

for every $s < \tau, \lambda \in [0,1]$, since O is convex and $x(s), x(s^+) \in \bar{O}$. Therefore by taking the limit $m \to \infty$ in (4.4), we obtain

$$\begin{aligned} W(x) \le\ & E_x e^{-\beta(t\wedge\theta_N)} W(x(t \wedge \theta_N)) \\ & + E_x \int_0^{t\wedge\theta_N} e^{-\beta s} \hat{L}(x(s))ds \\ & + E_x \int_0^{t\wedge\theta_N} e^{-\beta s} \hat{c}(\hat{u}(s))d\xi^c(s) \\ & + E_x \sum_{0 \le s < t\wedge\theta_N} e^{-\beta s} \hat{c}(\hat{u}(s))(\xi(s^+) - \xi(s)). \end{aligned}$$

Now let $N \to \infty$. By (3.3c) we obtain

$$\lim_{N\to\infty} E_x e^{-\beta(t\wedge\theta_N)} W(x(t\wedge\theta_N)) = E_x e^{-\beta(t\wedge\tau)} W(x(t\wedge\tau)).$$

Also

$$E_x \sum_{0\le s<t\wedge\tau} e^{-\beta s}\hat{c}(\hat{u}(s))(\xi(s^+)-\xi(s))$$

$$= E_x \int_{[0,t\wedge\tau)} e^{-\beta s}\hat{c}(\hat{u}(s))d\xi^J(s).$$

So we have

$$W(x) \le E_x e^{-\beta(t\wedge\tau)} W(x(t\wedge\tau))$$

$$+ E_x \int_{[0,t\wedge\tau)} e^{-\beta s}[\hat{L}(x(s))ds + \hat{c}(\hat{u}(s))d\xi(s)].$$

Send t to infinity and use (4.1) to obtain

$$W(x) \le E_x e^{-\beta\tau} W(x(\tau))\chi_{\tau<\infty}$$

$$+ E_x \int_{[0,\tau)} e^{-\beta s}[\hat{L}(x(s))ds + \hat{c}(\hat{u}(s))d\xi(s)].$$

We now claim that when $\tau < \infty$ we have

$$W(x(\tau)) \le \hat{c}(\hat{u}(\tau))[\xi(\tau^+)-\xi(\tau)].$$

Indeed if $x(\tau) \in \partial O$, then the above inequality follows from (2.8). Now suppose that $x(\tau) \in O$. Since $x(\tau^+) \notin O$ and O is convex, the line segment connecting $x(\tau)$ to $x(\tau^+)$ intersects ∂O. Let $x^* \in \partial O$ be an intersection point. Then the line segment connecting $x(\tau)$ to x^* stays entirely in $\bar{O}$. Hence for every $\lambda \in [0,1]$,

$$-\hat{u}(\tau)\cdot DW(x_\lambda) - \hat{c}(\hat{u}(\tau)) \le H(DW(x_\lambda)) \le 0,$$

$$x_\lambda = x(\tau) + \lambda[x^* - x(\tau)].$$

Also there is $\gamma \in (0,1]$ such that

$$x^* - x(\tau) = \gamma[x(\tau^+) - x(\tau)] = \gamma\hat{u}(\tau)[\xi(\tau^+)-\xi(\tau)].$$

Now by the Mean Value Theorem and (2.8),

$$W(x(\tau)) = W(x^*) - \int_0^1 DW(x_\lambda)d\lambda\cdot(x^* - x(\tau))$$

$$= -\gamma\int_0^1 \hat{u}(\tau)\cdot DW(x_\lambda)d\lambda[\xi(\tau^+)-\xi(\tau)]$$

$$\le \gamma\hat{c}(\hat{u}(\tau))[\xi(\tau^+)-\xi(\tau)].$$

Then the desired inequality follows from the inequalities $\hat{c} \geq 0, \gamma \leq 1$. Hence

$$W(x) \leq E_x e^{-\beta\tau} \hat{c}(\hat{u}(\tau))[\xi(\tau^+) - \xi(\tau)]$$

$$+ E_x \int_{[0,\tau)} e^{-\beta s} [\hat{L}(x(s))ds + \hat{c}(\hat{u}(s))d\xi(s)]$$

$$= J(x; \xi, \hat{u}),$$

for all $(\xi(\cdot), \hat{u}(\cdot)) \in \hat{\mathcal{A}}_\nu$, satisfying (4.1).

(b) Suppose that at $s \geq 0,\ x^*(s) \in \mathcal{P}$. Since $\mathcal{P}$ is an open set and $\mathcal{L}W = \hat{L}$ on $\mathcal{P}$, we have

$$\mathcal{L}W_m(x^*(s)) \to \hat{L}(x^*(s)) \ \text{ as } m \to \infty.$$

Moreover for fixed $N, |\mathcal{L}W_m(x)|$ is uniformly bounded in m and $x \in O_{m,N}$. Hence by (4.2i) and the dominated convergence theorem,

$$\lim_{m\to\infty} E_x \int_0^{t\wedge\theta_m} e^{-\beta s} \mathcal{L}W_m(x^*(s))ds = E_x \int_0^{t\wedge\theta_N} e^{-\beta s} \hat{L}(x^*(s))ds.$$

Now use (4.4) with $(\xi^*(\cdot), u^*(\cdot))$ and let $m \to \infty, N \to \infty$ to obtain

$$W(x) = E_x e^{-\beta(t\wedge\tau)} W(x^*(t\wedge\tau)) + E_x \int_0^{t\wedge\tau} e^{-\beta s} \hat{L}(x^*(s))ds$$

$$+ E_x \int_0^{t\wedge\tau} e^{-\beta s} [-u^*(s) \cdot DW(x^*(s))] d\xi^{*,c}(s)$$

$$+ E_x \sum_{0\leq s<t\wedge\tau} e^{-\beta s} [W(x^*(s)) - W(x^*(s^+))].$$

We then let $t \to \infty$ and use (4.2ii), (iii), (iv) to obtain

$$W(x) = J(x; \xi^*, u^*).$$ □

Remark 4.1. The convexity hypotheses on O is made to simplify the presentation. The Verification Theorem holds for any domain O which is not necessarily convex. Indeed let $\bar{\mathcal{A}}_\nu$ be the set of all $(\xi(\cdot), \hat{u}(\cdot)) \in \hat{\mathcal{A}}_\nu$ satisfying

$$x(t) + \lambda[x(t^+) - x(t)] \in O$$

for all $t \leq \tau, \lambda \in [0,1]$. Then it is easy to show that for each $(\xi(\cdot), \hat{u}(\cdot)) \in \hat{\mathcal{A}}_\nu$ there exists $(\bar{\xi}(\cdot), \bar{u}(\cdot)) \in \bar{\mathcal{A}}_\nu$ satisfying $J(x; \bar{\xi}, \bar{u}) \leq J(x; \xi, \hat{u})$. (We will not use this fact in our subsequent analysis.) We then follow the proof of Theorem 4.1 for a control pair $(\xi(\cdot), \hat{u}(\cdot)) \in \bar{\mathcal{A}}_\nu$ to prove the Verification Theorem for nonconvex domains.

Example 4.1. Consider a one dimensional problem with $O = (-\infty, \infty)$, $\hat{f} \equiv 0$, $\hat{\sigma} \equiv \sqrt{2}$, $\hat{c}(v) = |v|$, $\hat{K} = \{-1\}$ and $\hat{L}$ is convex. Then $U = (-\infty, 0]$

and the hypotheses of Lemma 3.2 are satisfied. Hence the value function V is convex and equation (2.7) takes the form

$$\max\left\{\beta V(x) - V_{xx}(x) - \hat{L}(x), V_x(x) - 1\right\} = 0, \ x \in (-\infty, \infty). \tag{4.5}$$

We will first construct a convex, polynomially growing solution W of (4.5) and then using the Verification Theorem we will show that $W = V$. Now suppose that W is indeed a convex solution of (4.5). Set

$$a = \sup\{x : \ W_x(x) < 1\}.$$

Here a may be equal to $+\infty$. The convexity of W yields

$$W_x(x) < 1, \ \forall x < a.$$

Then using (4.5) we conclude that

$$\beta W(x) - W_{xx}(x) = \hat{L}(x), \ \forall x < a, \tag{4.6}$$

and

$$W_x(x) = 1, \ \forall x \geq a. \tag{4.7}$$

The value of a is not a priori given to us. So we will solve (4.6) and (4.7) for every real number a, and then determine the value of a by using (4.5) again. Let $W_a(x)$ be the polynomially growing solution of (4.6) and (4.7). Now we assume that $\beta = 1, \ \hat{L}(x) = \alpha x^2$ for some $\alpha > 0$. Then, $W_a(x)$ is given by

$$W_a(x) = \begin{cases} (1 - 2\alpha a)e^{x-a} + \alpha x^2 + 2\alpha, & x \leq a \\ \\ W_a(a) + x - a, & x > a. \end{cases}$$

In addition to (4.6) and (4.7), $W_{a,x}(x) \leq 1$ for every x. Hence W_a solves (4.5) provided that

$$W_a(x) - W_{a,xx}(x) - \alpha x^2 \leq 0, \ \forall x > a.$$

Since W_a is linear on (a, ∞), the above inequality is equivalent to

$$0 \geq W_a(x) - \alpha x^2 = W_a(a) + x - a - \alpha x^2, \ \forall x \geq a.$$

The above inequality implies

$$0 \geq W_a(a) - \alpha a^2.$$

Also from (4.6) we obtain

$$W_a(a) = \lim_{x \uparrow a} W_{a,xx}(x) + \alpha a^2.$$

Since W_a is convex, the above argument shows that if W_a is a solution of (4.5), then

$$\lim_{x\uparrow a} W_{a,xx}(x) = 0. \tag{4.8}$$

Note that (4.8) implies $W_a \in C^2((-\infty,\infty))$ and for that reason Benes-Shepp-Witsenhausen call (4.8) "*the principle of smooth fit*" [BSW].

We solve (4.8) to obtain

$$(1 - 2\alpha a) + 2\alpha = 0 \Rightarrow \; a = \frac{1}{2\alpha} + 1.$$

Now it is easy to check that

$$W(x) = \begin{cases} -2\alpha e^{x-a} + \alpha x^2 + 2\alpha, & x \le a \\ \\ \alpha a^2 + x - a, & x > a, \end{cases} \tag{4.9}$$

is a convex, quadratically growing solution of (4.5). Let $(\xi(\cdot), \hat{u}(\cdot)) \in \hat{\mathcal{A}}_\nu$ be such that $J(x;\xi,\hat{u}) < \infty$. Then

$$\liminf_{t\to\infty} e^{-t} E_x(x(t))^2 = 0.$$

Since W is growing quadratically, (4.1) is satisfied by $(\xi(\cdot), \hat{u}(\cdot))$. Therefore

$$W(x) \le J(x;\xi,\hat{u})$$

for all $(\xi(\cdot), \hat{u}(\cdot)) \in \hat{\mathcal{A}}_\nu$ with $J(x;\xi;\hat{u}) < \infty$. Hence

$$W(x) \le V_\nu(x),$$

for every reference probability system ν.

To prove that $W(x) = V_\nu(x)$, we need to construct $(\xi^*(\cdot), u^*(\cdot)) \in \hat{\mathcal{A}}_\nu$ satisfying (4.2). Since $\hat{K} = \{-1\}$, we take $u^*(s) \equiv -1$. Then for a given x, we look for a process $\xi^*(\cdot)$ satisfying (3.3) and

$$x^*(t) = x + \sqrt{2}w(t) - \xi^*(t) < a \;\text{ for a. e. } t \ge 0, \tag{4.10}$$

$$\int_{[0,t)} [-DW(x^*(s)) + 1] d\xi^*(s) = 0, \;\forall t \ge 0. \tag{4.11}$$

Since $DW(x) - 1 < 0$ unless $x \ge a$, (4.11) is equivalent to

$$\int_{[0,t)} \chi_{\{x^*(s)\ge a\}} d\xi^*(s) = \xi^*(t), \;\forall t \ge 0. \tag{4.11$'$}$$

The problem of finding $\xi^*(\cdot)$ satisfying (3.3a), (4.10) and (4.11$'$) is known as the Skorokhod problem and its solution is given by

(4.12) $$\xi^*(t^+) = \sup\{(x + \sqrt{2}w(s) - a) \vee 0 : s \le t\},\ t \ge 0.$$

If $x \le a$, the process $x^*(\cdot)$ is a *Brownian motion reflected at* a and $\xi^*(\cdot)$ is *the local time at* a. (See (3.8) in Chapter 6 of [KS4].) If $x > a$, $\xi^*(0^+) = x - a$ and $x^*(0^+) = a$. Since W is linearly growing, (4.2iv) is easily verified. Therefore $W = V = V_\nu$ for every ν and $\xi^*(\cdot)$ is the optimal control process.

It is well known that the local time is not absolutely continuous. This is why we need to admit controls $\xi(s)$ which are not necessarily absolutely continuous functions of s. □

Example 4.2. Consider the same problem as in the previous example but with $\hat{K} = \{1, -1\}$. Then $U = (-\infty, \infty)$ and (2.7) takes the form

(4.13) $$\max\{V(x) - V_{xx}(x) - \alpha x^2, |V_x(x)| - 1\} = 0,\ \forall x \in (-\infty, \infty).$$

Following the procedure devised in Example 4.1, we look for a convex, polynomially growing function W and constants $-\infty \le a < b \le +\infty$ satisfying

$$W(x) - W_{xx}(x) = \alpha x^2, \qquad \forall a < x < b,$$

$$W_x(x) = -1, \qquad \forall x \le a,$$

$$W_x(x) = 1, \qquad \forall x \ge b,$$

$$\lim_{x \downarrow a} W_{xx}(x) = \lim_{x \uparrow b} W_{xx}(x) = 0.$$

The last condition is analogous to (4.8) and it implies that $W \in C^2((-\infty, \infty))$. Also as in Example 4.1, this last condition follows from (4.13) and the convexity of W. Now an elementary computation yields

$$W(x) = \begin{cases} \alpha x^2 + 2\alpha + \dfrac{(1-2\alpha b)}{\sinh(b)}\cosh(x), & |x| < b \\ \alpha b^2 + x - b, & x \ge b \\ \alpha b^2 - x + b, & x \le -b, \end{cases}$$

and $b = -a$ is the unique positive solution of

$$\tanh(b) = b - \frac{1}{2\alpha}.$$

Using the explicit form of W, we reduce (4.2) to the following equivalent form:

$$x^*(t) \in (-b, b) \quad \text{a.e.}, t \ge 0,$$

$$\hat{u}^*(t) = 1 \text{ if } x^*(t) \le 0, \quad \hat{u}^*(t) = -1 \text{ if } x^*(t) > 0,$$

$$\int_{[0,t)} \chi_{\{|x^*(s)| \ge b\}} d\xi^*(s) = \xi^*(t), \quad \forall t \ge 0.$$

Then the solution $x^*(t)$ of the above equations is the Brownian motion reflected at the boundary of $(-b, b)$. Also ξ^* is the sum of the local times of $x^*(\cdot)$ at b and $-b$ (see [KS4] or Definition 7.3 in Chapter 4 of [IW]). □

In the above examples we used a procedure based on the convexity of the value function. Indeed for any one dimensional problem satisfying the hypotheses of Lemma 3.2, the value function can be constructed by this procedure. In particular the value function can be shown to be twice continuously differentiable and the optimal state process to be a diffusion process reflected in an interval. However when the value function is not convex, neither one of these hold. We give an example to illustrate this point.

Example 4.3. Let $O = (-\infty, \infty)$, $\hat{f} \equiv 0$, $\hat{\sigma} \equiv \sqrt{2}$, $\hat{c}(v) = |v|$ $\beta = 1$, $U = (-\infty, \infty)$. Set

$$\alpha = (e^2 + 1)/4, \; x_0 = 9 - \alpha, \tag{4.14}$$

and

$$W(x) = \begin{cases} \alpha x^2 + 2\alpha - e\cosh x, & |x| \le 1 \\ 10 - 2e^{-(|x|-x_0)/2}, & |x| \ge x_0 \\ |x| - 1 + \alpha, & |x| \in [1, x_0]. \end{cases}$$

We claim that there exists a smooth function $\hat{L}$ satisfying

$$\hat{L}(x) = \alpha x^2, \; |x| \le 1, \tag{4.15i}$$

$$\hat{L}(x) = 10 - \frac{3}{2}e^{-(|x|-x_0)/2}, \; |x| \ge x_0, \tag{4.15ii}$$

$$\hat{L}(x) \ge |x| - 1 + \alpha, \quad |x| \in [1, x_0]. \tag{4.15iii}$$

Indeed for $|x| \notin (1, x_0)$, define $\hat{L}(x)$ by (4.15i) and (ii). Set

$$h(x) = \hat{L}(x) - \||x| - 1| - \alpha, \; |x| \notin (1, x_0)$$

Then (4.14) yields

$$h(x_0) = \frac{1}{2}, \; h(1) = 0.$$

Moreover

$$\lim_{x \uparrow 1} \frac{d}{dx} h(x) = 2\alpha - 1 = \frac{e^2 - 1}{2} > 0.$$

Hence $\hat{L}(\cdot)$ has an extension to $(-\infty, \infty)$ which satisfies (4.15iii). We then compute that $W_{xx}(x) \ge 0$ for $|x| \le x_0$ and

$$W(x) - W_{xx}(x) = \hat{L}(x), \qquad \forall |x| \notin [1, x_0],$$

$$|W_x(x)| = 1, \qquad \forall |x| \in [1, x_0],$$

$$|W_x(x)| < 1, \qquad \forall |x| \notin (1, x_0),$$

$$W(x) - W_{xx}(x) = |x| - 1 + \alpha \leq \hat{L}(x), \quad \forall |x| \in [1, x_0].$$

Hence, W is a classical solution of

$$\max\{W(x) - W_{xx}(x) - \hat{L}(x), |W_x(x)| - 1\} = 0, \ \forall x \in (-\infty, \infty).$$

We construct the optimal processes in three separate cases:

(a) $|x| \leq 1$. Let $x^*(\cdot)$ be the Brownian motion reflected in $(-1, 1)$ and $\xi^*(\cdot)$ be the sum of its local times at 1 and -1. Let $\hat{u}^*(t) = -x^*(t)$.

(b) $|x| \in (1, x_0]$. Let $\xi^*(0^+) = |x| - 1$ and $\hat{u}^*(0) = -$ sign x. Then $|x^*(0^+)| = 1$. For $t > 0$ we construct $x^*(t)$, $u^*(t)$, $\xi^*(t)$ as in case (a), but starting from $x(0^+)$.

(c) $|x| > x_0$. Let τ be the first time when

$$|x + \sqrt{2}w(\tau)| = x_0.$$

We then let $\xi^*(t) = 0$ for $t \leq \tau$ and $\xi^*(\tau^+) = x_0 - 1$. Also let $\hat{u}^*(t) = -$ sign x for $t \leq \tau$. Then

$$x^*(t) = x + \sqrt{2}w(t), \qquad t \leq \tau,$$

$$x^*(\tau^+) = \text{sign } x.$$

For $t > \tau$ we construct $x^*(t)$, $\xi^*(t)$, $\hat{u}^*(t)$ as in case (a) but starting from $x^*(\tau^+)$.

It is easy to check that $(\xi^*(\cdot), \hat{u}^*(\cdot)) \in \hat{\mathcal{A}}_\nu$ satisfy (4.2). Hence $W = V = V_\nu$. In this example

$$\mathcal{P} = (-\infty, -x_0) \cup (-1, 1) \cup (x_0, \infty)$$

is not connected. Also W is not convex and not twice differentiable at $x = x_0$ and $-x_0$.

The following is a simple two-dimensional example.

Example 4.4. In this example $O = I\!R^2, \hat{f} \equiv 0$, $\beta = 1$, $\hat{\sigma}$ is equal to $\sqrt{2}$ times a 2×2 identity matrix,

$$U = (-\infty, 0] \times (-\infty, 0]$$

and for $v \in (v_1, v_2) \in U, x = (x_1, x_2) \in O$,

$$\hat{c}(v) = -v_1 - v_2,$$

$$\hat{L}(x) = \alpha(x_1^2 + x_2^2).$$

Then (2.7) reduces to

$$\max\{V(x) - \Delta V(x) - \alpha|x|^2, V_{x_1}(x) - 1, V_{x_2}(x) - 1\} = 0, \ \forall x \in I\!R^2. \tag{4.16}$$

Let W be the function defined by (4.9). Then

$$V(x_1, x_2) = W(x_1) + W(x_2)$$

is a classical solution of (4.16). Moreover

$$\mathcal{P} = (-\infty, a) \times (-\infty, a)$$

and $\partial\mathcal{P}$ is not differentiable.

When $U = I\!R^n, \hat{c}(v) = |v|, \hat{\sigma} \equiv \sqrt{2}$ times the identity, and $\hat{f} \equiv 0$ (2.7) reduces to

$$\max\{\beta V(x) - \Delta V(x) - \hat{L}(x), |DV(x)| - 1\} = 0, \ \forall x \in O. \tag{4.17}$$

Then by analytical arguments Evans [E2] proved the following.

Theorem 4.2. *Suppose that $\hat{L} \in C^2(\bar{O})$ and O is bounded. Then the value function V is the unique classical solution of* (4.17) *and* (2.8).

The $W^{2,\infty}$ regularity proved by Evans is the best possible general result. However for convex problems (i.e., those satisfying the hypotheses of Lemma 3.2) we expect the value function V to be twice continuously differentiable. Indeed C^2 regularity of the value function is proved by Soner and Shreve [SSh1-2] under either one of the following assumptions:

$$O = U = I\!R^2, \hat{f} \equiv 0, \hat{\sigma} = \text{ identity}, \ \hat{c}(v) = |v|, \hat{L} \text{ strictly convex}, \tag{4.18i}$$

$$U = \{\lambda v_0 : \ \lambda \geq 0\}, \text{ and hypotheses of Lemma 3.2}, \tag{4.18ii}$$

where $v_0 \in I\!R^n$ is any nonzero vector. Also the finite horizon problem for the second case is considered in [SSh2].

Now consider the case (4.18i). Then in [SSh1] the following change of variables was useful. Let x^* be the minimum of V and $\delta > 0$ be sufficiently small. The existence of x^* was shown in [SSh1]. For $t \geq 0$ and $\theta \in S^{n-1}$ let $z(t;\theta) \in I\!R^n$ be the unique solution of

$$\frac{d}{dt} z(t;\theta) = DV(z(t;\theta)), \quad t > 0, \theta \in S^{n-1},$$

$$z(0;\theta) = x^* + \delta\theta.$$

In [SSh1] it is shown that the map

$$(t, \theta) \to z(t, \theta)$$

is a diffeomorphism between $[0,\infty)\times S^{n-1}$ onto $I\!R^n - B_\delta(x^*)$. Also for $t \geq 0$.

$$\frac{d}{dt}[|DV(z(t;\theta))|^2] = 2D^2V(z(t;\theta))DV(z(t;\theta))\cdot DV(z(t,\theta)).$$

Since V is convex, $|DV(z(t;\theta))|^2$ is nondecreasing in t. Hence we may characterize the region $\mathcal{P}$ by

$$\mathcal{P} = B_\delta(x^*) \cup \{z(t;\theta) : \theta \in S^{n-1}, t < T(\theta)\},$$

where

$$T(\theta) = \inf\{t \geq 0 : |DV(z(t;\theta))|^2 \geq 1\}.$$

This calculation proves that $\mathcal{P}$ is a connected subset of $I\!R^n$. Also the above characterization of $\mathcal{P}$ is the first step in studying the regularity of $\partial\mathcal{P}$ and V.

For a general problem with a convex value function the appropriate change of variables is

$$\frac{d}{dt}z(t;\theta) \in \partial H(DV(z(t;\theta))),$$

where $\partial H(p)$ is the subdifferential of H in the sense of convex analysis. However the properties of this change of variables have not yet been studied.

We finally remark that the region $\mathcal{P}$ does not have a special geometric property. In particular, $\mathcal{P}$ in general is not a convex set even if the hypotheses of Lemma 3.2 hold. In fact under (4.18ii), an analytical argument based on the connection between singular control and optimal stopping shows that for a large class of problems the complement of $\mathcal{P}$ is convex.

VIII.5 Viscosity solutions

In this section we will prove that the value function V is a viscosity solution of (2.7) in O. We assume that $V \in C_p(\overline{O})$ and it satisfies the dynamic programming principle, i.e., for every $x \in O$ and stopping time $\theta > 0$,

(5.1)
$$V(x) = \inf_{\nu}\inf_{\mathcal{A}_\nu}\{E_x e^{-\beta(\tau\wedge\theta)}V(x(\tau\wedge\theta)) + E_x\int_{[0,\tau\wedge\theta]} e^{-\beta s}[\hat{L}(x(s))ds + \hat{c}(\hat{u}(s))d\xi(s)]\},$$

where τ is the exit time of $x(\cdot)$ from $\overline{O}$. Equation (5.1) is a straightforward modification of V(2.2) and I(7.12). Indeed the continuity of V and (5.1) can be proved as in Section IV.7 when $O = I\!R^n$ and as in Section V.2 when O is bounded. In the latter case we need to assume I(3.11) and V(2.3).

Theorem 5.1. *Assume $V \in C_p(\overline{O})$ and* (5.1). *Then V is a viscosity solution of* (2.7) *in O.*

Recall that in Sections II.7 and V.2, we proved the viscosity property of the value function of other control problems by using Theorem II.5.1. The following proof however does not use Theorem II.5.1. Instead we will give a more direct proof. As we will see, the subsolution property follows from elementary considerations and a counterposition argument is used for the supersolution part.

Proof. (Subsolution). Let $w \in C^2(O) \cap C_p(\overline{O})$ and $\bar{x} \in O$ be a maximizer of $V - w$ on $\overline{O}$ with $V(\bar{x}) = w(\bar{x})$. Then for any reference probability system ν, $(\xi(\cdot), \hat{u}(\cdot)) \in \hat{\mathcal{A}}_\nu$ and $h > 0$, (5.1) with $\theta = h$ implies

$$w(\bar{x}) = V(\bar{x}) \leq E_{\bar{x}} \int_{[0,\tau\wedge h]} e^{-\beta s}[\hat{L}(x(s))ds + \hat{c}(\hat{u}(s))d\xi(s)] + E_{\bar{x}} e^{-\beta(\tau\wedge h)} w(x(\tau \wedge h)). \tag{5.2}$$

1. For $\delta > 0$ and $\hat{v} \in U$, set $\hat{u}(s) \equiv \hat{v}$ and

$$\xi(s) = \begin{cases} 0, & s = 0 \\ \delta, & s > 0. \end{cases}$$

Then by (3.5), $x(0^+) = \bar{x} + \delta\hat{v}$. Let h go to zero in (5.2) to obtain

$$w(\bar{x}) \leq \delta\hat{c}(\hat{v}) + w(\bar{x} + \delta\hat{v}),$$

for every $\delta > 0$, $\hat{v} \in U$. Now divide both sides by δ and let δ go to zero. Then in the notation of Section 2,

$$H(Dw(\bar{x})) \leq 0. \tag{5.3}$$

2. In this step we use (5.2) with $\xi \equiv 0$. Then the standard Ito's rule applied to $e^{-\beta s} w(x(s))$ yields

$$E_{\bar{x}} e^{-\beta(t\wedge h)} w(x(t \wedge h)) = -E_{\bar{x}} \int_0^{t\wedge h} e^{-\beta s} \mathcal{L} w(x(s))ds + w(\bar{x}). \tag{5.4}$$

Use (5.4) in (5.2) and recall that $\xi \equiv 0$. Then a straightforward limiting argument yields

$$\mathcal{L}w(\bar{x}) - \hat{L}(\bar{x}) \leq 0. \tag{5.5}$$

3. Combining (5.3) and (5.5) we conclude that V is a viscosity subsolution of (2.7) in O.

(Supersolution). Let $w \in C^2(O) \cap C_p(\overline{O})$ and $\bar{x} \in O$ be a minimizer of $V - w$ on O with $V(\bar{x}) = w(\bar{x})$. We need to show that

$$\max\{\mathcal{L}w(\bar{x}) - \hat{L}(\bar{x}), H(Dw(\bar{x}))\} \geq 0. \tag{5.6}$$

4. Suppose the contrary. Hence the left-hand side of (5.6) is negative and by the smoothness of w, there are $\delta, \gamma > 0$ satisfying

$$\max\{\mathcal{L}w(x) - \hat{L}(x), H(Dw(x))\} \leq -\gamma, \ \forall x \in B_\delta(\bar{x}), \tag{5.7}$$

where $B_\delta(\bar{x}) = \{x : |x - \bar{x}| \le \delta\}$. Since O is open by changing δ, if necessary, we may assume that $B_\delta(\bar{x}) \subset O$.

In the next four steps we will obtain a contradiction to (5.7). A brief summary of these steps is the following. In Step 5, we will use Ito's rule and (5.7) to derive (5.14). In Step 6, we will show that the sum of the last two terms in (5.14) is uniformly negative. See (5.15) and (5.16). We then use the dynamic programming (5.1) to obtain a contradiction in Step 8.

5. Let $(\xi(\cdot), \hat{u}(\cdot) \in \hat{\mathcal{A}}_\nu$ be given and θ be the exit time of $x(\cdot)$ from $B_\delta(\bar{x})$. Since $B_\delta(\bar{x}) \subset O$, $\theta < \tau$. Moreover, since $x(\cdot)$ is left continuous with right limits, the filtration $\mathcal{F}_t$ is right continuous and $B_\delta(\bar{x})$ is closed, θ is an $\mathcal{F}_t$-stopping time.

As in (4.4) we apply the Ito's rule for semimartingales to $e^{-\beta t} w(x(t))$ for $t \in (0, \theta)$ to obtain

$$(5.8) \qquad \begin{aligned} w(\bar{x}) &= E_{\bar{x}} e^{-\beta\theta} w(x(\theta)) \\ &+ E_{\bar{x}} \int_0^\theta e^{-\beta s} \mathcal{L} w(x(s)) ds \\ &+ E_{\bar{x}} \int_0^\theta e^{-\beta s} [-\hat{u}(s) \cdot Dw(x(s))] d\xi^c(s) \\ &+ E_{\bar{x}} \sum_{0 \le s < \theta} e^{-\beta s} [w(x(s)) - w(x(s^+))], \end{aligned}$$

where, as in (4.4),

$$\xi^c(t) = \xi(t) - \sum_{0 \le s < t} [\xi(s^+) - \xi(s)].$$

For $0 \le s < \theta$, (5.7) implies that

$$(5.9) \qquad \begin{aligned} \mathcal{L} w(x(s)) &\le \hat{L}(x(s)) - \gamma \\ H(Dw(x(s))) &\le -\gamma. \end{aligned}$$

Taking $v = \hat{u}(s)/|\hat{u}(s)|$ in (2.4) and then using (2.2iv) we obtain

$$(5.10) \qquad -\hat{u}(s) \cdot Dw(x(s)) \le \hat{c}(\hat{u}(s)) - \gamma |\hat{u}(s)|.$$

Also

$$w(x(s)) - w(x(s^+)) = \int_0^1 -(x(s^+) - x(s)) \cdot Dw(\rho x(s^+) + (1-\rho) x(s)) d\rho,$$

$$x(s^+) - x(s) = \hat{u}(s)[\xi(s^+) - \xi(s)].$$

If $s < \theta$, then $x_\rho = \rho x(s^+) + (1-\rho) x(s) \in B_\delta(\bar{x})$ for all $\rho \in [0,1]$. Using (5.7) at x_ρ we obtain

$$H(Dw(x_\rho)) \leq -\gamma.$$

Combining the three previous statements, we obtain that for $s < \theta$

$$\begin{aligned} w(x(s)) - w(x(s^+)) &= \int_0^1 -\hat{u}(s)[\xi(s^+) - \xi(s)] \cdot Dw(x_\rho)d\rho \\ &\leq [\hat{c}(\hat{u}(s)) - \gamma|\hat{u}(s)|]\ [\xi(s^+) - \xi(s)]. \end{aligned} \tag{5.11}$$

Substituting (5.9), (5.10), (5.11) into (5.8),

$$\begin{aligned} w(\bar{x}) \leq\ & E_{\bar{x}} e^{-\beta\theta} w(x(\theta)) \\ & + E_{\bar{x}} \int_{[0,\theta)} e^{-\beta s}[\hat{L}(\hat{u}(s))ds + \hat{c}(\hat{u}(s))d\xi(s)] \\ & - \gamma E_{\bar{x}} \int_{[0,\theta)} e^{-\beta s}[ds + |\hat{u}(s)|d\xi(s)]. \end{aligned} \tag{5.12}$$

Since $x(\cdot)$ is left continuous and θ is the exit time from $B_\delta(\bar{x})$, $x(\theta^+) \notin \operatorname{int} B_\delta(\bar{x})$ and $x(\theta) \in B_\delta(\bar{x})$. Also

$$x(\theta^+) = x(\theta) + \hat{u}(\theta)[\xi(\theta^+) - \xi(\theta)].$$

Then there is a random variable $\lambda \in [0,1]$ such that

$$x_\lambda = x(\theta) + \lambda\hat{u}(\theta)[\xi(\theta^+) - \xi(\theta)] \in \partial B_\delta(\bar{x}) = \{x : |x - \bar{x}| = \delta\}.$$

Moreover the argument that led us to (5.11) yields

$$w(x(\theta)) \leq w(x_\lambda) + \lambda[\hat{c}(\hat{u}(\theta)) - \gamma|\hat{u}(\theta)|]\ [\xi(\theta^+) - \xi(\theta)].$$

Since $V - w$ is minimized at $\bar{x}$ with $V(\bar{x}) = w(\bar{x})$, $w \leq V$. Suppose that $x(\theta^+) \in \bar{O}$. Then (3.6) yields

$$\begin{aligned} w(x_\lambda) &\leq V(x_\lambda) \\ &\leq V(x(\theta^+)) + (1-\lambda)\hat{c}(\hat{u}(\theta))[\xi(\theta^+) - \xi(\theta)] \end{aligned}$$

Hence

$$w(x(\theta)) \leq V(x(\theta^+)) + [\hat{c}(\hat{u}(\theta)) - \lambda\gamma|\hat{u}(\theta)|]\ [\xi(\theta^+) - \xi(\theta)], \tag{5.13}$$

if $x(\theta^+) \in \bar{O}$. If $x(\theta^+) \notin \bar{O}$, we set $V(x(\theta^+)) = 0$. Then arguing as in the proof of Theorem 4.1(a) we obtain (5.13). Now substitute (5.13) into (5.12),

$$
\begin{aligned}
w(\bar{x}) \leq\ & E_{\bar{x}} e^{-\beta\theta} V(x(\theta^+)) \\
& + \int_{[0,\theta]} e^{-\beta s}[\hat{L}(x(s))ds + \hat{c}(\hat{u}(s))d\xi(s)] \\
& - \gamma E_{\bar{x}} \int_{[0,\theta)} e^{-\beta s}[ds + |\hat{u}(s)|d\xi(s)] \\
& - \gamma E_{\bar{x}} e^{-\beta\theta} \lambda |\hat{u}(\theta)|[\xi(\theta^+) - \xi(\theta)].
\end{aligned} \tag{5.14}
$$

6. Let

$$
\begin{aligned}
\alpha(\xi, \hat{u}) = E_{\bar{x}} & \int_{[0,\theta)} e^{-\beta\theta}[ds + |\hat{u}(s)|d\xi(s)] \\
& + E_{\bar{x}} e^{-\beta\theta} \lambda |\hat{u}(\theta)|[\xi(\theta^+) - \xi(\theta)].
\end{aligned}
$$

We claim that there is $\alpha_0 > 0$ satisfying

$$
\alpha(\xi, \hat{u}) \geq \alpha_0 > 0, \tag{5.15}
$$

for every $(\xi(\cdot), \hat{u}(\cdot)) \in \hat{\mathcal{A}}_\nu$ and reference probability system ν. Indeed consider the singular stochastic control problem with $\overline{O} = B_\delta(\bar{x})$, state dynamics (3.1) and $\hat{L}(x) \equiv 1$, $\hat{c}(\hat{v}) = |\hat{v}|$, i.e., the payoff functional is given by

$$
\tilde{J}(x; \xi, \hat{u}) = E_x \int_{[0,\theta]} e^{-\beta s}[ds + |\hat{u}(s)|d\xi(s)].
$$

For $(\xi(\cdot), \hat{u}(\cdot)) \in \hat{\mathcal{A}}_\nu$ set

$$
\bar{\xi}(t) = \begin{cases} \xi(t), \quad t \leq \theta \\ \xi(\theta) + \lambda[\xi(\theta^+) - \xi(\theta)], \ t > \theta, \end{cases}
$$

where λ is as in the previous step. Since the random variable λ is an explicit function of $\xi(\theta^+)$ and $\xi(\theta)$, $\bar{\xi}(\cdot)$ is progressively measurable. Therefore, $(\bar{\xi}(\cdot), \hat{u}(\cdot)) \in \hat{\mathcal{A}}_\nu$ and

$$
\alpha(\xi; \hat{u}) = \tilde{J}(\bar{x}; \bar{\xi}, \hat{u}).
$$

So we need to show that

$$
\tilde{V}(\bar{x}) \geq \alpha_0,
$$

where

$$
\tilde{V}(x) = \inf_{\nu} \inf_{\mathcal{A}_\nu} \tilde{J}(x; \xi, \hat{u}), \ |x - \bar{x}| < \delta.
$$

For $C > 0$, set

$$
W(x) = C(\delta^2 - |x - \bar{x}|^2), \ x \in B_\delta(\bar{x}).
$$

Then

$$\mathcal{L}W(x) = C[\beta\delta^2 - \beta|x-\bar{x}|^2 + \operatorname{tr}\hat{a}(x) + 2\hat{f}(x)\cdot(x-\bar{x})] \le 1, \quad x \in B_\delta(\bar{x}),$$

if

$$C \le C_0 = \left\{ \sup_{|x-\bar{x}|\le\delta} [\beta\delta^2 + \operatorname{tr}\hat{a}(x) + 2|\hat{f}(x)\cdot(x-\bar{x})|] \right\}^{-1}.$$

Hence

$$\max\{\mathcal{L}W(x)-1, |DW(x)|-1\} \le 0, \; x \in B_\delta(x),$$
$$W(x) = 0, \qquad x \in \partial B_\delta(x),$$

provided that

$$C = C^* = \min\left\{C_0, \frac{1}{2\delta}\right\}.$$

Now a standard application of Ito's rule for semimartingales imply that

$$\tilde{V}(x) \ge W(x), \; x \in B_\delta(x).$$

Hence

$$\alpha(\xi,\hat{u}) \ge \tilde{V}(\bar{x}) \ge W(\bar{x}) = C^*\delta^2 > 0.$$

7. Since $\alpha(\xi,\hat{u}) \ge \alpha_0 > 0$, (5.14) yields that

$$V(\bar{x}) = w(\bar{x}) \le E_{\bar{x}} e^{-\beta\theta} V(x(\theta^+))$$

(5.16)
$$+ E_{\bar{x}} \int_{[0,\theta]} e^{-\beta s}[\hat{L}(x(s))ds + \hat{c}(\hat{u}(s))d\xi(s)] - \gamma\alpha_0,$$

for every $(\xi(\cdot),\hat{u}(\cdot)) \in \hat{\mathcal{A}}_\nu$.

8. For $h > 0$, $\theta + h$ is a stopping time. Hence (5.1) yields

$$V(\bar{x}) = \inf_{\xi,\hat{u},\nu} \left\{ E_{\bar{x}} \int_{[0,(\theta+h)\wedge\tau)} e^{-\beta s}[\hat{L}(x(s))ds + \hat{c}(\hat{u}(s))d\xi(s)] \right.$$
$$\left. + E_{\bar{x}} e^{-\beta((\theta+h)\wedge\tau)} V(x((\theta+h)\wedge\tau)) \right\}.$$

We let h go to zero. Then the positivity of $\hat{L}$ and $\hat{c}$ and the continuity of V imply that

$$V(\bar{x}) = \inf_{\xi,\hat{u},\nu} \left\{ E_{\bar{x}} \int_{[0,\theta]} e^{-\beta s}[\hat{L}(x(s))ds + \hat{c}(\hat{u}(s))d\xi(s)] \right.$$
$$\left. + E_x e^{-\beta\theta} V(x(\theta^+)) \right\}.$$

Clearly the above equation contradicts (5.16). Hence V is a supersolution of (2.7) in O. □

Comparison principles for classical solutions were obtained by Evans [E2] and Soner and Shreve [SSh1-2]. A combination of Evans' arguments and the theory developed in Chapter V yields comparison principles for viscosity sub- and supersolutions. See [Zhu].

VIII.6 Finite fuel problem

A simplified model of spacecraft control led Bather and Chernoff [BC1-2] to singular problems similar to the kind formulated in Section 3. In the context of spacecraft control, the control variable $\xi(t)$ is the total amount of fuel used by the spacecraft up to time $t \geq 0$ and $\hat{u}(t)$ is the push direction at time $t \geq 0$. With this interpretation it is appropriate to impose a constraint

$$\xi(t) \leq z, \quad \forall t \geq 0, \tag{6.1}$$

with a given constant $z > 0$. In the literature the singular stochastic control problem with the above constraint is called the *finite fuel* problem.

For $x \in O$ and $z \geq 0$, let $\hat{\mathcal{A}}_\nu(z)$ be the set of all controls $(\xi(\cdot), \hat{u}(\cdot)) \in \hat{\mathcal{A}}_\nu$ satisfying (6.1). Then the value function is defined by

$$V_\nu(x,z) = \inf_{\hat{\mathcal{A}}_\nu(z)} J(x;\xi,\hat{u}),$$

$$V(x,z) = \inf_\nu V_\nu(x,z), \quad x \in \overline{O}, \quad z \geq 0.$$

The dynamic programming principle takes the form

$$V(x,z) = \inf_{\nu, \hat{\mathcal{A}}_\nu(z)} \left\{ E \int_{[0,\theta]} e^{-\beta s}[\hat{L}(x(s))ds + \hat{c}(\hat{u}(s))d\xi(s)] \right.$$
$$\left. + e^{-\beta\theta} V(x(\theta), z - \xi(\theta)) \right\}$$

for any stopping time $\theta \leq \tau$. Hence the dynamic programming equation is

$$\max\{(\mathcal{L}V(\cdot,z))(x) - \hat{L}(x), \hat{H}(D_x V(x,z), D_z V(x,z))\} = 0, \tag{6.2}$$

$$\forall x \in O, \quad z > 0,$$

where $\mathcal{L}$ is as in (2.5ii) and for $p \in I\!R^n$, $p_z \in I\!R$,

$$\hat{H}(p,p_z) = \sup_{v \in \hat{K}} \{-p \cdot v - \hat{c}(v)\} + p_z.$$

Note that for the finite fuel problem, in addition to $x(s)$, $z(s) = z - \xi(s)$ is also a state variable. This new state variable gives rise to the extra term p_z in the definition of $\hat{H}$.

Appropriate boundary conditions are (2.8) for $x \in \partial O,\ z > 0$, whereas for $x \in O, z = 0$

$$V(x,0) = h(x) = E_x \int_0^\tau e^{-\beta s} \hat{L}(x(s))ds, \tag{6.3}$$

where $x(s)$ is the uncontrolled process, i.e.,

$$dx(s) = \hat{f}(x(s))ds + \hat{\sigma}(x(s))dw(s), \quad s > 0.$$

When $\hat{L}$ is convex, polynomially growing, $\hat{\sigma}, \hat{f}$ are affine and $\hat{K} = \{\nu_0\}$ an elegant simple solution of the finite fuel problem was found by Chow-Menaldi-Robin [CMR]. A probabilistic explanation was then given by Karatzas [K2]. For simplicity let us assume $O = I\!R$, $\nu_0 = -1$, $\hat{c}(v) = v$, $\hat{\sigma} \equiv \sqrt{2}$, $\hat{f} \equiv 0$. Then

$$\mathcal{L}\phi(x) = \beta\phi(x) - \phi_{xx}(x), \ \ x \in I\!R,$$

$$\hat{H}(p_x, p_z) = p_x + p_z - 1.$$

Proposition 6.1. *Let $\hat{V}(x)$ be the value function of the unconstrained problem (i.e., with no upper bound z for $\xi(t)$), and $h(x)$ be as in (6.3). Then*

$$V(x,z) = \hat{V}(x) - \hat{V}(x-z) + h(x-z), \ \ \forall x \in I\!R, \quad z \geq 0. \tag{6.4}$$

Moreover for $z > 0$ the optimal Markov policy for the finite fuel problem is equal to that of unconstrained problem.

Proof. Let $W(x,z)$ be the right-hand side of (6.4). We will show that W is a solution of (6.2). First recall that (see Example 4.1) $\hat{V} \in C^2(-\infty,\infty)$ and there exists a such that

$$\beta\hat{V}(x) - \hat{V}_{xx}(x) = \hat{L}(x), \quad \forall x \leq a,$$

$$\hat{V}_x(x) = 1, \quad \forall x \geq a.$$

Then $\hat{V}(x) = \hat{V}(a) + x - a$ for $x \geq a$. Since $\hat{V}$ also satisfies

$$\beta\hat{V}(x) - \hat{V}_{xx}(x) \leq \hat{L}(x), \quad \forall x,$$

we conclude that

$$\beta[\hat{V}(a) + x - a] \leq \hat{L}(x), \quad \forall x \geq a,$$

and $\beta\hat{V}(a) = \hat{L}(a)$. Hence $\hat{L}_x(a) \geq \beta$ and by convexity of $\hat{L}$,

$$\hat{L}(x-z) + \beta z \leq \hat{L}(x), \quad \forall x - z \geq a, \quad z \geq 0. \tag{6.5}$$

Also h satisfies the linear equation

$$\beta h(x) - h_{xx}(x) = \hat{L}(x), \quad \forall x. \tag{6.6}$$

Combining above equations we obtain

$$\hat{H}(DW(x,z)) = W_x(x,z) + W_z(x,z) - 1 = \hat{V}_x(x) - 1 \leq 0, \ \ \forall x,$$

$$\hat{H}(DW(x,z)) = 0, \quad \forall x \geq a,$$

$$\beta W(x,z) - W_{xx}(x,z) = \hat{L}(x), \quad \forall x \leq a.$$

Also we need to show that

$$\beta W(x,z) - W_{xx}(x,z) \le \hat{L}(x), \quad \forall x \ge a.$$

The above inequality follows immediately if $x \ge a \ge x-z$. When $x \ge x-z \ge a$, by (6.5) we obtain

$$\begin{aligned}\beta W(x,z) - W_{xx}(x,z) &= \beta[\hat{V}(a) + x - a] - \beta[\hat{V}(a) + x - z - a] + \hat{L}(x-z) \\ &= \beta z + \hat{L}(x-z) \\ &\le \hat{L}(x).\end{aligned}$$

Hence W is a polynomially growing classical solution of (6.2) and (6.3). Since $\xi(t)$ is bounded from above by z, an application of Ito's formula yields (6.4).

Now observe that

$$\beta V(x,z) - V_{xx}(x,z) = \hat{L}(x), \quad \forall x \le a, \quad z > 0,$$

$$V_x(x,z) + V_z(x,z) = 1, \quad \forall x \ge a, \quad z > 0.$$

Therefore if $z > 0$, then the optimal policy can be summarized as follows,

(i) jump immediately to a, if $x > a$ and $z \ge x - a$,
(ii) jump immediately to $x - z$, if $x \ge a + z$,
(iii) remain in the interval $(-\infty, a]$ by reflection, if $x < a$, $z > 0$. □.

If $n > 1$, (6.4) is replaced by

$$V(x,z) = \hat{V}(x) - \hat{V}(x + z\nu_0) + h(x + z\nu_0).$$

VIII.7 Historical remarks

The first examples of stochastic singular control problems were formulated by Bather and Chernoff [BC1-2]. In 1980 Benes, Shepp and Witsenhaussen explicitly solved a one dimensional example by observing that the value function in their example is twice continuously differentiable [BSW]. Since this regularity of the value function reduces to a condition at the interface, Benes et al. called this regularity property the *principle of smooth fit*. In the regularity direction, Evans proved that the gradient of the value function is Lipschitz continuous [E2]. See also Ishii and Koike [IK1]. In the absence of convexity this is the best possible regularity result since there are value functions which are not twice differentiable; see Example 4.3. Recently, Soner and Shreve proved the twice differentiability of the value function under two sets of assumptions, (4.18i) and (4.18ii) [SSh1-2].

One dimensional convex problem received much attention in the early 1980's. Karatzas [K1], Harrison and Taksar [HT], Menaldi and Robin [MR] and Chow, Menaldi and Robin [CMR] provided an almost complete analysis of one dimensional problems. The connection between singular control and stopping time problems was pursued by Karatzas and Shreve [KS1-2]. Finite fuel problems were studied by Karatzas [K2], Karatzas and Shreve [KS3] and Chow, Menaldi and Robin [CMR].

If the value function is smooth, the optimal control can be constructed by using the reflected Brownian motion. In one space dimension, this is always possible. However in higher space dimensions the construction of the reflected Brownian motion requires the smoothness of the interface separating the two phases of the equation. See Lions and Sznitman [LS] and Varadhan and Williams [VW] for the construction of reflected Brownian motions. Soner and Shreve proved the regularity of the interface and then constructed the optimal process as the reflected Brownian motion in [SSh1-2]. Williams-Chow-Menaldi [WCM] also proved partial regularity results for the interface when there are finitely many push directions.

The existence of an optimal control can also be obtained by abstract compactness arguments. In general Krylov introduced a framework in which the singular control problems are also included; see Section 1.2, Exercise 4 in [Kr1]. For a convex singular control problem Menaldi and Taksar proved existence by more direct arguments [MT].

IX

Finite Difference Numerical Approximations

IX.1 Introduction

In many of the examples given in earlier chapters to illustrate the theory, the dynamic programming equation could be solved rather explicitly to obtain the value function and optimal Markov control policies. However, these examples are exceptional. For most optimal stochastic control models arising from applications, the dynamic programming equation can only be solved approximately by numerical computations.

This chapter is intended as a brief introduction to the topic of numerical solution of HJB partial differential equations. We consider a finite difference scheme due to Kushner [Ku1] for computing approximately the value function $V(t,x)$ for a controlled Markov diffusion on a finite time horizon. In this approximation scheme, first order partial derivatives are replaced by corresponding forward or backward finite difference quotients. Similarly, second order partial derivatives are replaced by appropriate second order finite difference quotients (Section 3.) An important feature of Kushner's scheme is that the discretized HJB equation is itself the dynamic programming equation for a suitably defined stochastic control problem for Markov chains. This fact was exploited by Kushner [Ku1], [Ku3] to give a probabilistic proof of convergence of the discrete value function to $V(t,x)$ as step sizes tend to zero. Viscosity solution methods provide another way to prove convergence; and it is that method which we will follow in Sections 4 and 5. The method was introduced, in a more abstract setting, by Barles and Souganidis [BS].

Once the HJB equation has been discretized, there remains the important question of efficient computational methods for solving the discrete dynamic programming equation. We shall merely allude briefly to the considerable body of literature in that direction. The reader should consult the book by Kushner and Dupuis [KuD] which provides a thorough treatment and many references.

IX.2 Controlled discrete time Markov chains

In this section we will describe briefly the method of dynamic programming in discrete time, for finite time horizon and infinite time horizon with discounted cost criterion. The book by Bertsekas [Bs] provides a good detailed account of this topic. We first formulate a finite horizon stochastic control problem. Let Σ be a set, which is either finite or countably infinite. We consider times $\ell = k, k+1, \cdots, M$, where k denotes an initial time and M a terminal time. The state at time ℓ is denoted by x^ℓ and the control chosen at time ℓ by u^ℓ. These are discrete-time stochastic processes, with $x^\ell \in \Sigma,\ u^\ell \in U$. The state dynamics are prescribed by a family of one step transition probabilities $p_\ell^v(x,y)$. In analogy with Section III.8 for the continuous time case, we define discrete time admissible control systems π as follows. We call

$$\pi = (\Omega, \{\mathcal{F}_\ell\}, P, x^\bullet, u^\bullet)$$

admissible if $(\Omega, \mathcal{F}, P)$ is a probability space, $\{\mathcal{F}_\ell\}$ is an increasing family of σ - algebras $(\ell = k, k+1, \cdots, M)$, $\mathcal{F}_\ell \subset \mathcal{F}$ and

$$\text{(2.1)}\qquad \begin{array}{l} (i)\ \ x^k = x,\ x^\ell \text{ is } \mathcal{F}_\ell - \text{measurable.} \\ (ii)\ u^\ell \text{ is } \mathcal{F}_\ell - \text{measurable.} \\ (iii)\ P(x^{\ell+1} = y|\mathcal{F}_\ell) = p_\ell^{u^\ell}(x^\ell, y) \quad P - \text{almost surely for} \\ \qquad \ell = k, \cdots, M-1. \end{array}$$

The problem is to minimize a criterion (or payoff functional) of the form:

$$\text{(2.2)}\qquad J_k(x;\pi) = E_{kx}\left\{\sum_{\ell=k}^{M-1} L_\ell(x^\ell, u^\ell) + \psi(x^M)\right\}.$$

To avoid undue technical complications we make the following rather strong assumptions:

$$\text{(2.3)}\qquad \begin{array}{l} (a)\ \text{There exists } K \text{ such that} |L_\ell(x,v)| \le K, |\psi(x)| \le K \\ \quad\ \text{for all } x \in \Sigma,\ v \in U,\ \ell = k, \cdots, M-1. \\[1ex] (b)\ U \text{ is compact.} \\[1ex] (c)\ p_\ell^v(x,y) \text{ is continuous on } U,\ \text{for all } x, y \in \Sigma. \\[1ex] (d)\ \text{For each } x \in \Sigma \text{ there exists a finite set} \\ \quad\ \Gamma_x \text{ such that } p_\ell^v(x,y) = 0 \ \text{ for } y \notin \Gamma_x. \end{array}$$

In analogy with the continuous time case (Section III.7) let us introduce the value function

$$\text{(2.4)}\qquad V_k(x) = \inf_\pi J_k(x;\pi),\ x \in \Sigma.$$

A straightforward argument (which we shall omit) shows that the value function is the unique bounded solution to the dynamic programming equation

$$V_k(x) = \min_{v \in U} \left[\sum_{y \in \Sigma} p_k^v(x,y) V_{k+1}(y) + L_k(x,v) \right], \ k < M, \tag{2.5}$$

with the terminal data

$$V_M(x) = \psi(x). \tag{2.6}$$

Moreover, an optimal discrete time Markov control policy $\underline{u}_k^*(x)$ is found by taking arg min over U on the right side of (2.5).

Infinite horizon discounted control problem. Let us now consider times $\ell = 0, 1, 2 \cdots$ and autonomous state dynamics for a controlled Markov chain, prescribed by one step transition probabilities $p^v(x,y)$. The concept of admissible control system π is defined as above. Let

$$J(x;\pi) = E_x \sum_{\ell=0}^{\infty} \lambda^\ell L(x^\ell, u^\ell), \tag{2.7}$$

where λ is a discount factor $(0 < \lambda < 1.)$ Let us make the same assumptions about L, U and $p^v(x,y)$ as in (2.3). Let

$$V(x) = \inf_{\pi} J(x;\pi). \tag{2.8}$$

The dynamic programming equation is now [Bs, Chap V]

$$V(x) = \min_{v \in U} [L(x,v) + \lambda \sum_{y \in \Sigma} p^v(x,y) V(y)]. \tag{2.9}$$

An optimal stationary Markov control policy $\underline{u}^*(x)$ is found by taking arg min over U. Let us denote the right side of (2.9) by $F(V)(x)$. Then (2.9) states that $V = F(V)$; i.e. V is a fixed point of F. It is easy to verify that

$$||F(V) - F(W)|| \leq \lambda ||V - W||, \tag{2.10}$$

where $||\ \ ||$ is the sup norm. Since $0 < \lambda < 1$, the contraction property (2.10) implies that there is a unique fixed point V, which is in fact the value function in (2.8). For proofs of these facts, see Bertsekas [Bs, Sec. 5.3].

For infinite horizon discounted problems, two standard methods for computing the value function V are successive approximation (or *value iteration*) and *approximation in policy space*. The method of value iteration gives V as the uniform limit of a sequence $W^m, m = 0, 1, 2 \cdots$, where $W^{m+1} = F(W^m)$. From the definition (2.9), the operator F is monotone:

$$F(\phi_1) \leq F(\phi_2) \text{ if } \phi_1 \leq \phi_2. \tag{2.11}$$

Therefore, if W^0 is chosen such that $W^0 \le F(W^0)$, then the approximating sequence is monotone nondecreasing: $W^m \le W^{m+1}$.

The method of approximation in policy space proceeds as follows. Let $\underline{u}^0$ be an initial choice of stationary Markov control policy. Define $W^m, \underline{u}^{m+1}$ successively for $m = 0, 1, 2, \cdots$ by

$$W^m(x) = L(x, \underline{u}^m(x)) + \lambda \sum_{y \in \Sigma} p^{\underline{u}^m(x)}(x, y) W^m(y), \; x \in \Sigma, \tag{2.12}$$

$$\underline{u}^{m+1}(x) \in \arg\min[L(x, v) + \lambda \sum_{y \in \Sigma} p^v(x, y) W^m(y)]. \tag{2.13}$$

Note that (2.12) is a linear system of equations for $W^m(x)$. The sequence $W^m(x)$ is monotone nonincreasing ($W^m \ge W^{m+1}$) and it converges to $V(x)$ as $m \to \infty$. The method is closely related to Newton's method for solving nonlinear equations [Bs, p. 236].

For refinements of these methods for computing the value function and illustrative examples, we refer to [Bs, Chapter 5].

IX.3 Finite difference approximations to HJB equations

In this section we describe a finite difference scheme for approximate solution of the Hamilton-Jacobi-Bellman PDE IV(3.3) for a controlled Markov diffusion. This approximation scheme is a slight simplification of one introduced by Kushner [Ku1], who used probabilistic methods to prove convergence as step sizes tend to 0. In Sections 4 and 5 we will obtain convergence by another method based on viscosity solution techniques. This method uses a technique introduced by Barles and Souganidis [BS].

In Kushner's scheme, a controlled Markov diffusion in $I\!R^n$ is approximated by a controlled Markov chain on a lattice in $I\!R^n$ with nearest neighbor transitions. The dynamic programming equation for this controlled Markov chain turns out to be related to the HJB equation IV(3.3) by replacing first and second order partial derivatives by appropriate finite difference quotients.

To simplify notations, let us assume autonomous state dynamics and running cost functions $f(x, v), \sigma(x, v), L(x, v)$ in Section IV.2. Let us also assume in addition to IV(2.2) that:

(3.1)
(a) U is compact
(b) f, σ, L, L_x and L_t are bounded on $Q_0 \times U$.

We consider the HJB partial differential equation

$$-V_t + \mathcal{H}(x, D_x V, D_x^2 V) = 0, \tag{3.2}$$

with $\mathcal{H}(x,p,A)$ as in IV(3.2). As in Chapters IV and V, we consider (3.2) either in Q_0 with bounded terminal (Cauchy) data

$$V(t_1,x)=\psi(x),\ x\in I\!R^n, \tag{3.3}$$

or in a cylindrical region Q with the boundary data IV(3.4) on ∂^*Q.

To begin with, let us consider dimension $n=1$ and the case of Cauchy data (3.3). Afterward we include lateral boundary conditions, and outline extensions to dimension $n>1$. According to IV(3.2) we have for $n=1$.

$$\mathcal{H}(x,p,A)=\max_{v\in U}[-f(x,v)p-\frac{1}{2}a(x,v)A-L(x,v)]$$

with $a=\sigma^2$.

Consider a *time step* $h>0$ and a *spatial step* $\delta>0$, which will be related in such a way that inequality (3.7) below holds. The approximating controlled discrete time Markov chain has as state space the 1 dimensional lattice

$$\Sigma_0^h=\{x=j\delta : j=0,\pm1,\pm2,\cdots\}. \tag{3.4}$$

Let

$$\begin{aligned} f^+(x,v)&=\max(f(x,v),0)\\ f^-(x,v)&=\max(-f(x,v),0). \end{aligned} \tag{3.5}$$

We call f^+ and f^- the *positive* and *negative* parts of f. The dynamics of the controlled Markov chain are specified by the one step transition probabilities

$$\begin{aligned} p^v(x,x+\delta)&=\frac{h}{\delta^2}[\frac{a(x,v)}{2}+\delta f^+(x,v)]\\ p^v(x,x-\delta)&=\frac{h}{\delta^2}[\frac{a(x,v)}{2}+\delta f^-(x,v)]\\ p^v(x,x)&=1-p^h(x,x+\delta)-p^h(x,x-\delta). \end{aligned} \tag{3.6}$$

If $y=x+j\delta$ with $j\neq 0,\pm1$, then $p^v(x,y)=0$. Thus, one-step transitions are to nearest neighbor states. By definition, $p^v(x,x\pm\delta)\geq 0$. We also require that $p^v(x,x)\geq 0$, which imposes a restriction on h and δ. A sufficient condition that $p^v(x,x)\geq 0$ is that

$$h[a(x,v)+\delta|f(x,v)|]\leq\delta^2 \tag{3.7}$$

for all $(x,v)\in I\!R^1\times U$. From now on we choose $\delta=\delta(h)$ such that (3.7) holds.

Let us consider the Markov chain control problem of minimizing

$$J_k^h(x;\pi)=E_{kx}\left\{\sum_{\ell=k}^{M-1}hL(x^\ell,u^\ell)+\psi(x^M)\right\} \tag{3.8}$$

Thus, in (2.2) we take $L_\ell = hL$. Let $t_0^h = t_1 - Mh$, where $t_0^h \to t_0$ as $h \downarrow 0$. We write

$$Q_0^h = \left\{(t,x) : t = t_0^h + kh, k = 0, 1, \cdots, M, x \in \Sigma_0^h\right\}.$$

We denote the value function in (2.4) by

$$V_k(x) = V^h(t,x),\ (t,x) \in Q_0^h.$$

Thus, one discrete time step corresponds to a step of length h on the time scale for the controlled Markov diffusion process. The dynamic programming equation (2.5) becomes

$$(3.9)\qquad V^h(t,x) = \min_{v\in U}[p^v(x,x+\delta)V^h(t+h,x+\delta)$$

$$+p^v(x,x-\delta)V^h(t+h,x-\delta) + p^v(x,x)V^h(t+h,x) + hL(x,v)],$$

with the terminal data (3.3) for $t = t_1$. In order to rewrite (3.9) in a form which resembles the HJB equation (3.2) we introduce the following notations. For any function $W(t,x)$, let

$$\Delta_x^+ W = \frac{W(t,x+\delta) - W(t,x)}{\delta}$$

$$\Delta_x^- W = \frac{W(t,x) - W(t,x-\delta)}{\delta}$$

$$\Delta_x^2 W = \frac{W(t,x+\delta) + W(t,x-\delta) - 2W(t,x)}{\delta^2}$$

These are respectively the forward and backward first order difference quotients, and second order difference quotient in x. Similarly, we consider the first order difference quotient backward in time

$$\Delta_t^- W = \frac{W(t,x) - W(t-h,x)}{h}.$$

Let us replace t by $t-h$ and $t+h$ by t in (3.9). By using (3.6), rearranging terms and dividing by h we get

$$(3.9')\qquad -\Delta_t^- V^h + \tilde{\mathcal{H}}(x, \Delta_x^+ V^h, \Delta_x^- V^h, \Delta_x^2 V^h) = 0$$

where

$$(3.10)\qquad \tilde{\mathcal{H}}(x,p^+,p^-,A) = \max_{v\in U}[-f^+(x,u)p^+ + f^-(x,v)p^-$$

$$-\frac{a(x,v)}{2}A - L(x,v)].$$

Observe that

$$\tilde{\mathcal{H}}(x,p,p,A) = \mathcal{H}(x,p,A). \tag{3.11}$$

Equation (3.9$'$) is called an *explicit* finite difference scheme, backward in time. Since

$$V^h(t-h,x) = V^h(t,x) - h\tilde{\mathcal{H}}(x, \Delta_x^+ V^h, \Delta_x^- V^h, \Delta_x^2 V^h) \tag{3.12}$$

and the difference quotients $\Delta_x^{\pm} V^h, \Delta_x^2 V^h$ are evaluated at (t,x), the values of V^h at time $t-h$ are explicitly expressed in terms of the values of V^h at time t. We expect that $V^h \to V$ as $h \to 0$, where V is the value function for the controlled diffusion process. This will be proved in Section 4.

Remark 3.1. If $\psi(x) \le \phi(x)$ for all $x \in \Sigma_0^h$ and $\psi(\bar{x}) = \phi(\bar{x})$, then

$$\Delta^+\psi(\bar{x}) \le \Delta^+\phi(\bar{x}),\ \Delta^-\psi(\bar{x}) \ge \Delta^-\phi(\bar{x}),\ \Delta^2\psi(\bar{x}) \le \Delta^2\phi(\bar{x}).$$

By (3.10) we have for $x = \bar{x}$

$$\tilde{\mathcal{H}}(\bar{x}, \Delta_x^+\phi, \Delta_x^-\phi, \Delta_x^2\phi) \le \tilde{\mathcal{H}}(\bar{x}, \Delta_x^+\psi, \Delta_x^-\psi, \Delta_x^2\psi).$$

This form of monotonicity is a direct analogue of the maximum principle formulated in Section II.4. Since viscosity solution methods depend in an essential way on maximum principles, this observation will be important when proving the convergence of V^h to V by a viscosity solution method.

Implicit finite difference scheme. If the backward time difference Δ^- in (3.9$'$) is replaced by a forward time difference Δ^+, then we obtain instead of (3.12) the following equation:

$$W^h(t,x) = W^h(t+h,x) - h\tilde{\mathcal{H}}(x, \Delta_x^+ W^h, \Delta_x^- W^h, \Delta_x^2 W^h), \tag{3.13}$$

where $\Delta_x^{\pm} W^h, \Delta_x^2 W^h$ are again evaluated at (t,x). This is called a backward *implicit* finite difference scheme, since (nonlinear) equations must be solved to determine $W^h(t,\cdot)$ from $W^h(t+h,\cdot)$. The implicit scheme (3.13) also has a stochastic control interpretation, under restrictions on the step sizes h and δ similar to (3.7). See Kushner [Ku1, Sec. 7.3].

Boundary conditions. In the discussion above we considered $x \in \Sigma_0^h$, where Σ_0^h is the infinite lattice defined by (3.4) with $\delta = \delta(h)$. For actual numerical calculations Σ_0^h must be replaced by some finite subset Σ^h, and then the one step transition probabilities must be changed at boundary points of Σ^h. Let us suppose that

$$\Sigma^h = \left\{ x \in \Sigma_0^h : |x| \le B_h \right\}, \tag{3.14}$$

where B_h represents some finite "cutoff" parameter ($B_h \in \Sigma_0^h$.) We need to redefine the controlled Markov chain so that $x_\ell \in \Sigma^h$. If $x \in \Sigma^h$ and $|x| < B_h$, then the one step transition probabilities are again as in (3.6). Thus (3.9) is again the dynamic programming equation at interior points of Σ^h. We must have

$$p^v(B_h, B_h + \delta) = p^v(-B_h, -B_h - \delta) = 0. \tag{3.15}$$

If only nearest neighbor transitions are allowed from $\pm B_h$, then the values assigned to $p^v(B_h, B_h - \delta)$ and $p^v(-B_h, -B_h + \delta)$ effectively prescribe boundary conditions for V^h at $\pm B_h$. We may, for instance, take these to be the same as in (3.6). This corresponds, roughly speaking, to assigning in continuous variables the Neumann boundary condition $\partial V/\partial x = 0$ at the endpoints. If $B_h \to \infty$ as $h \to 0$, it follows from Theorem 5.3 below that the same limit for V^h is obtained no matter which boundary conditions at $\pm B_h$ are chosen. The limit V is the value function for the controlled Markov diffusion on $I\!R^1$.

The boundary conditions described above were introduced artificially at the "cutoff" endpoints $\pm B_h$. Let us next consider the problem of optimally controlling a diffusion until the time τ when $(s, x(s))$ exits from a finite rectangle $Q = [t_0, t_1) \times (x_0, x_1)$. This problem was formulated in Section IV.2, where now $O = (x_0, x_1)$ is a finite 1-dimensional interval. We now take $\delta = N^{-1}(x_1^h - x_0^h)$, where x_0^h, x_1^h are approximations to x_0, x_1 with N a "large" positive integer and

$$\Sigma^h = \left\{ x_0^h + j\delta : j = 0, 1, \cdots, N \right\}.$$

For the controlled discrete time Markov chain, the one step transition probabilities are defined by (3.6) at interior points $x \in \Sigma^h$, i. e. those points with $1 \le j \le N - 1$. The endpoints of Σ^h are absorbing:

$$p^v(x_0^h, x_0^h) = p^v(x_1^h, x_1^h) = 1. \tag{3.16}$$

The objective is to chose an admissible control system π which minimizes

$$J_k^h(x; \pi) = E_{kx} \left\{ \sum_{\ell=k}^{\mu-1} hL(x^\ell, u^\ell) + \Psi(t^\mu, x^\mu)) \right\} \tag{3.17}$$

where $t^\mu = t_0^h + \mu h$ and either μ is the first step at which x^μ is an endpoint of Σ^h or $\mu = M$ if x^ℓ is interior to Σ^h for $k \le \ell \le M - 1$. The dynamic programming equation $(3.9^{'})$ is satisfied by the value function $V^h(t, x)$ if x is an interior point of Σ^h. In addition, we have the same boundary condition as in IV(3.4):

$$V^h(t, x) = \Psi(t, x), \tag{3.18}$$

for all $(t, x) \in ([t_0, t_1) \times \{x_0^h, x_1^h\} \cup (\{t_1\} \times [x_0^h, x_1^h])$.

Infinite horizon discounted problem. Let us consider the controlled diffusion problem formulated in Section IV.5. For simplicity, we again take $O = I\!R^1$. The dynamics of the approximating discrete time Markov chain are again (3.6), and the criterion to be minimized is

$$J^h(x; \pi) = E_x \sum_{\ell=0}^{\infty} h\lambda^\ell L(x^\ell, u^\ell), \ x \in \Sigma_0^h, \tag{3.19}$$

where $\lambda = \exp(-\beta h)$ and $\beta > 0$. The value function

$$V^h(x) = \inf_{\pi} J^h(x;\pi) \tag{3.20}$$

satisfies (2.9), which becomes after using (3.6) and rearranging terms

$$0 = (\frac{1-e^{-\beta h}}{h})V^h + \tilde{\mathcal{H}}(x, \Delta_x^+ V^h, \Delta_x^- V^h, \Delta_x^2 V^h). \tag{3.21}$$

In view of (3.11) this can be regarded as a discretization of the HJB equation IV(5.8) for the infinite horizon controlled diffusion problem.

Controlled diffusion in $I\!R^n, n > 1$. We merely indicate the changes needed in explicit finite difference scheme $(3.9^{'})$ and refer to [Ku1, Sec. 6.2] for details. We again take autonomous $f(x,v), \sigma(x,v)$ and $L(x,v)$, where now $x \in I\!R^n, f = (f_1, \cdots, f_n)$ is $I\!R^n$ - valued and $a = \sigma\sigma'$ is $n \times n$ - matrix valued. As in the one dimensional case, let f_i^+ and f_i^- denote the positive and negative parts of $f_i, i = 1, \cdots, n$. The matrices $a(x,v) = (a_{ij}(x,v)), i,j = 1, \cdots, n$, are nonnegative definite. Hence $a_{ii} \geq 0$. For $j \neq i$, let a_{ij}^+, a_{ij}^- denote the positive and negative parts of a_{ij}. Let us assume that

$$a_{ii}(x,v) - \sum_{j \neq i} |a_{ij}(x,v)| \geq 0, \tag{3.22}$$

$$h \sum_{i=1}^{n} \left[a_{ii}(x,v) - \frac{1}{2} \sum_{j \neq i} |a_{ij}(x,v)| + \delta |f_i(x,v)| \right] \leq \delta^2. \tag{3.23}$$

Condition (3.22) is less restrictive than might at first seem. For instance, if the noise coefficient matrix σ is constant, then (3.22) can always be achieved by a suitable rotation of coordinates in $I\!R^n$ such that a becomes a diagonal matrix. For $n = 1$, condition (3.23) is the same as (3.7).

Let $e_1, \cdots, e_n$ denote the standard basis for $I\!R^n$. Thus

$$x = (x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i e_i.$$

The approximating controlled Markov chain has a state space the n-dimensional lattice

$$\Sigma_0^h = \left\{ x = \delta \sum_{i=1}^{n} j_i e_i \right\} \tag{3.24}$$

where $j_1, \cdots, j_m$ are any integers. The one step transition probabilities are as follows:

$$p^v(x, x \pm \delta e_i) = \frac{h}{2\delta^2}[a_{ii}(x,v) - \sum_{j \neq i} |a_{ij}(x,v)| + 2\delta f_i^{\pm}(x,v)]$$

(3.25)
$$p^v(x, x + \delta e_i \pm \delta e_j) = \frac{h}{2\delta^2} a_{ij}^{\pm}(x,v), \ i \neq j$$

$$p^v(x, x - \delta e_i \pm \delta e_j) = \frac{h}{2\delta^2} a_{ij}^{\mp}(x,v), \ i \neq j$$

$$p^v(x,x) = 1 - \frac{h}{\delta^2} \sum_{i=1}^{n} [a_{ii}(x,v) - \frac{1}{2} \sum_{j \neq i} |a_{ij}(x,v)| + \delta |f_i(x,v)|].$$

Moreover, $p^v(x,y) = 0$ for all other y. The dynamic programming equation is

$$V^h(t,x) = \min_{v \in U} [\sum_{y \in \Sigma_0^h} p^v(x,y) V^h(t+h,y) + hL(x,v)], \tag{3.26}$$

which is just (3.9) in dimension $n = 1$. By rearranging terms in (3.26) and dividing by h, we obtain a n-dimensional analogue of $(3.9^{'})$. Instead of writing this out explicitly, let us recall the definition IV(3.2) of $\mathcal{H}(x,p,A)$ in the HJB equation and explain which finite difference quotients are used to approximate the corresponding partial derivatives. For $i = 1, \cdots, n$, and any function $W(t,x)$ let

$$\Delta_{x_i}^{\pm} W = \delta^{-1}[W(t, x \pm \delta e_i) - W(t,x)]$$

$$\Delta_{x_i}^{2} W = \delta^{-2}[W(t, x + \delta e_i) + W(t, x - \delta e_i) - 2W(t,x)].$$

The time derivative V_t is replaced by $\Delta_t^- V^h$, just as in $(3.9^{'})$. If $f_i(x,v) \geq 0$, then V_{x_i} is replaced by $\Delta_{x_i}^+ V^h$, and if $f_i(x,v) < 0$ by $\Delta_{x_i}^- V^h$. Similarly, $V_{x_i x_i}$ is replaced by $\Delta_{x_i}^2 V^h$. For the mixed second order partial derivatives, when $i \neq j$ $V_{x_i x_j}$ is replaced by $\Delta_{x_i x_j}^+ V^h$ if $a_{ij}(x,v) \geq 0$ and by $\Delta_{x_i x_j}^- V^h$ if $a_{ij}(x,v) < 0$, where

$$\Delta_{x_i x_j}^+ W = \frac{1}{2}\delta^{-2}[2W(t,x) + W(t, x + \delta e_i + \delta e_j) + W(t, x - \delta e_i - \delta e_j)]$$
$$-\frac{1}{2}\delta^{-2}[W(t, x + \delta e_i) + W(t, x - \delta e_i) + W(t, x + \delta e_j) + W(t, x - \delta e_j)]$$

$$\Delta_{x_i x_j}^- W = -\frac{1}{2}\delta^{-2}[2W(t,x) + W(t, x + \delta e_i - \delta e_j)] + W(t, x - \delta e_i + \delta e_j)]$$
$$+\frac{1}{2}\delta^{-2}[W(t, x + \delta e_i) + W(t, x - \delta e_i) + W(t, x + \delta e_j) + W(t, x - \delta e_j)].$$

In order to rewrite (3.26) as a backward difference equation like (3.12), we introduce the following notation. For each $x, p_i^{\pm}, A_{ii}, A_{ij}^{\pm}, i,j = 1, \cdots, n$, let

$$\tilde{\mathcal{H}}(x, p_i^{\pm}, A_{ii}, A_{ij}^{\pm}) = \max_{v \in U} \Big\{ \sum_{i=1}^{n} [-f_i^+(x,v) p_i^+ + f_i^-(x,v) p_i^- \tag{3.27}$$

$$-\frac{a_{ii}(s,v)}{2}A_{ii} + \sum_{j\neq i}\Big(-\frac{a_{ij}^{+}(x,v)}{2}A_{ij}^{+} + \frac{a_{ij}^{-}(x,v)}{2}A_{ij}^{-}\Big)\Big] - L(x,v)\Big\}.$$

Then, as in (3.12),

$$V^h(t-h,x) = V^h(t,x) - h\tilde{\mathcal{H}}(x, \Delta_{x_i}^{\pm}V^h, \Delta_{x_i}^2 V^h, \Delta_{x_i x_j}^{\pm}V^h). \tag{3.28}$$

For the infinite horizon discounted problem in $I\!R^n$, the generalization of (3.21) is obtained in the same way. The discussion of boundary conditions is essentially the same as for dimension $n = 1$ above. For the problem of control until the time of exit from a cylindrical region $Q = [t_0, t_1) \times O$, we take $\Sigma^h \subset \Sigma_0^h$ which "approximates" O and assign absorbing boundary conditions like (3.16) at "boundary points" of Σ^h. We postpone this discussion to Section 5.

Computational methods. After the value function has been replaced by a finite difference approximation V^h, it remains to compute $V^h(t,x)$. For the finite time horizon problem, the dynamic programming equation (3.26) or (3.9) in dimension $n = 1$ can be solved backward in time, at least in principle. However, up to now these calculations can only be done in practice for quite low dimension n. In (3.26) a minimum over U must in general be computed repeatedly. Fortunately, in many problems of interest there is an explicit formula for $\mathcal{H}(x,p,q)$, and this tedious step can be avoided. For numerical solutions of linear parabolic PDEs, implicit finite difference schemes are often found to be advantageous. See [KuD, Sec. 12.4]. However, there does not seem to be a great deal of experience with implicit schemes in the context of nonlinear PDE's of HJB type.

At the end of Section 2, we mentioned two standard methods for computing $V^h(x)$, namely value iteration and approximation in policy space. The discount factor $\lambda = \exp(-\beta h)$ in (3.19) is nearly 1 for small h, and the contraction in (2.10) is weak. For small h, the successive approximations to V^h through value iteration tend to converge quite slowly. For approximations in policy space, two computations (2.12) and (2.13) are needed at each iteration. Multigrid methods have been used by Akian [Ak] to speed up the solution of the linear system of equations (2.12).

IX.4 Convergence of finite difference approximations I

We wish to show that the value function V^h obtained from the finite difference scheme in Section 3 converges to the value function V for the controlled Markov diffusion as $h \to 0$. This has been proved by Kushner [Ku1] using stochastic control and weak convergence of probability measure methods. Later, another method to show convergence of V^h to V by viscosity solution techniques was introduced by Barles and Souganidis[BS]. This method is the one which we will follow. Both Kushner's stochastic control method and the Barles-Souganidis viscosity solution method apply not only to the explicit scheme described in Section 3, but to other finite difference schemes as well. The stochastic control method specifically requires that the finite difference scheme has a Markov chain interpretation. When such an interpretation is available, one can easily verify the monotonicity and stability properties (4.3), (4.5) below needed for the viscosity solution method.

Let us first describe the Barles - Souganidis method for the HJB equation (3.2) in Q_0, with the terminal (Cauchy) data (3.3). In Section 5, we will consider the HJB equation in a cylindrical region Q, with both terminal and lateral boundary conditions. Let Σ^h be a discrete subset of $I\!R^n$, for $0 < h \le 1$; and let $B(\Sigma^h)$ denote the space of bounded functions on Σ^h. We assume that

$$\lim_{h \downarrow 0} \text{ dist } (x, \Sigma^h) = 0, \text{ for all } x \in I\!R^n.$$

Let F_h be an operator on $B(\Sigma^h)$. We consider the "abstract" finite difference equation, backward in time

$$V^h(t,x) = F_h[V^h(t+h, \cdot)](x), \; x \in \Sigma^h, \tag{4.1}$$

$$t = t_0^h + kh, k = 0, 1, \cdots, M-1 \text{ with}$$

$$V^h(t_1, x) = \psi(x), \; x \in \Sigma^h. \tag{4.2}$$

We make the following assumptions:

$$F_h(\phi_1) \le F_h(\phi_2) \text{ if } \phi_1 \le \phi_2. \qquad \text{(monotonicity)} \tag{4.3}$$

$$F_h(\phi + c) = F_h(\phi) + c, \text{ for all } c \in I\!R. \tag{4.4}$$

(4.5) For $0 < h < 1$, there exists a solution V^h to (4.1), (4.2) and a constant K such that $\|V^h\| \le K$. (stability)

$$\begin{aligned} &\lim_{\substack{(s,y)\to(t,x)\\ h\downarrow 0}} h^{-1}[F_h[w(s+h,\cdot)](y) - w(s,y)] \\ &\qquad = w_t(t,x) - \mathcal{H}(x, D_x w(t,x), D_x^2 w(t,x)) \end{aligned} \tag{4.6}$$

for every "test function" $w \in C^{1,2}(I\!R^{n+1})$. (consistency)

Example 4.1. Consider the explicit finite difference scheme (3.26) in Section 3. Then

$$F_h(\phi)(x) = \min_{v \in U}[\sum_{y \in \Sigma_0^h} p^v(x,y)\phi(y) + hL(x,y)]. \tag{4.7}$$

Then $\Sigma^h = \Sigma_0^h$ and $p^v(x,y)$ is as in (3.25), or (3.9) when $n=1$. (If a finite cutoff $|x_i| \le B_h, i=1,\cdots,n$, is introduced, then $p^v(x,y)$ is suitably redefined at boundary points. See Section 5.) We define $V^h(t,x)$ to be the value function for the discrete time Markov chain control problem. Properties (4.3) and (4.4) are immediate since $p^v(x,y)$ are one step transition probabilities. From (3.8) and $(M-k)h = t_1 - t$,

$$|J_k^h(x;\pi)| \le \|L\|(t_1-t) + \|\psi\|.$$

The value function $V^h(t,x)$ therefore satisfies the same inequality, and in (4.5) we may take $K = (t_1-t_0)\|L\| + \|\psi\|$. We can rewrite (4.7) as

$$(4.7') \qquad F_h\phi(x) = \phi(x) - h\tilde{\mathcal{H}}(x, \Delta_{x_i}^{\pm}\phi, \Delta_{x_i}^2\phi, \Delta_{x_ix_j}^{\pm}\phi).$$

The explicit scheme (3.26) in this notation becomes (4.1), if in (3.28) we replace $t-h$ by t and t by $t+h$.

In addition to the explicit scheme (3.26), one may expect other first-order accurate finite difference schemes (including implicit schemes) to satisfy the monotonicity and stability conditions provided time and spatial step sizes are related by a condition like (3.23). However, monotonicity can not be expected for higher-order accurate finite difference schemes.

Remark 4.1. The condition (4.6) is similar to VII(4.6). When the Hamiltonian $\mathcal{H}$ in (3.2) depends on $V(t,x)$, we need to modify the above conditions. One possibility is to drop (4.4) and replace (4.6) by following conditions,

$$\lim_{\substack{(s,y)\to(t,x)\\ h\downarrow 0}} h^{-1}\{F[w(s+h,\cdot) - K(h)](y) - [w(s,y) - K(h)]\}$$

$$= w_t(t,x) - \mathcal{H}(x, D_xw(t,x), D_x^2w(t,x), w(t,x)),$$

for every test function $w \in C^{1,2}(I\!R^{n+1})$ and every bounded sequence $K(h)$ converging to 0 as $h \downarrow 0$.

As in Section VII.4, for $(t,x) \in Q_0$ let

$$(4.8) \qquad \begin{aligned} V^*(t,x) &= \limsup_{\substack{(s,y)\to(t,x)\\ h\downarrow 0}} V^h(s,y) \\ V_*(t,x) &= \liminf_{\substack{(s,y)\to(t,x)\\ h\downarrow 0}} V^h(s,y), \end{aligned}$$

where V^h satisfies the abstract finite difference equation (4.1) with terminal data (4.2).

Note that V^* is upper semicontinuous on Q_0 and V_* is lower semicontinuous on Q_0. We recall from Remark VII.4.1 the definitions of viscosity sub and super solution.

Lemma 4.1. *V^* is a viscosity subsolution of the HJB equation, and V_* is a viscosity supersolution.*

Proof. To show that V^* is a viscosity subsolution, suppose that w is a test function such that $V^* - w$ has a maximum at $(\bar{t}, \bar{x}) \in Q_0$. As noted earlier (Definition VII.4.2 and Remark VII.4.2) we can assume that the maximum is strict. Then there is a sequence converging to zero denoted by h, such that $V^h - w$ has a maximum on Q_0^h at (s_h, y_h) which tends to $(\bar{t}, \bar{x})$ as $h \downarrow 0$. For all $y \in \Sigma_0^h$,

$$V^h(s_h, y_h) - w(s_h, y_h) \geq V^h(s_h + h, y) - w(s_h + h, y),$$

$$w(s_h + h, y) - w(s_h, y_h) \geq V^h(s_h + h, y) - V^h(s_h, y_h).$$

By (4.3), (4.4)

$$\begin{aligned} F_h\left[w(s_h + h, \cdot)\right](y_h) - w(s_h, y_h) \\ \geq F_h\left[V^h(s_h + h, \cdot)\right](y_h) - V^h(s_h, y_h). \end{aligned} \tag{4.9}$$

By (4.1), the right side is 0. We divide by h and let $h \downarrow 0$. By (4.6)

$$w_t(\bar{r}, \bar{x}) - \mathcal{H}(\bar{x}, D_x w(\bar{t}, \bar{x}), D_x^2 w(\bar{t}, \bar{x}) \geq 0. \tag{4.10}$$

Thus, V^* is a viscosity subsolution. Similarly, V_* is a viscosity supersolution. □

We assume that (3.1) and IV(2.2) hold. Let $V(t,x)$ be the value function for the controlled diffusion process, as in IV(2.10).

We also make the following assumption, which ensures that V^h assumes the terminal data (4.2) in a uniform way:

$$\lim_{\substack{(s,y) \to (t_1, x) \\ h \downarrow 0}} V^h(s, y) = \psi(x) \tag{4.11}$$

uniformly for x in any compact subset of $\mathbb{R}^n$.

Theorem 4.1. *Let V^h be a solution to* (4.1) *and* (4.2). *Assume that* (4.3) - (4.6) *and* (4.11) *hold. Then*

$$\lim_{\substack{(s,y) \to (t, x) \\ h \downarrow 0}} V^h(s, y) = V(t, x) \tag{4.12}$$

uniformly on any compact subset of $\overline{Q}_0$.

Proof. V is a bounded, uniformly continuous viscosity solution of the HJB equation (3.2) with the terminal data (3.3). (See comment following V(9.1).) By Lemma 4.1, V^* is a bounded uppersemicontinuous subsolution of (3.2) and by Lemma 4.2 $V^*(t_1, x) = \psi(x)$ pointwise for $x \in \mathbb{R}^n$. By a comparison result (Remark V.8.1). $V^* \leq V$. Similarly, $V_* \geq V$. Since $V_* \leq V^*$, we obtain Theorem 4.1. □

Sub and supersolutions. In order to apply Theorem 4.1 to the explicit finite difference scheme in Section 3, let us first introduce the idea of sub and supersolutions to difference equations. Sub and supersolutions will also play a key role in Section 5 in enforcing lateral boundary conditions.

We call a bounded function $W(t,x)$ a *supersolution* of (4.1) - (4.2) if

$$W(t,x) \geq F_h\left[W(t+h,\cdot)\right](x) \tag{4.13}$$

for $t = t_0^h + kh, k = 0, 1, \cdots, M-1, x \in \Sigma^h$, and

$$W(t_1, x) \geq \psi(x), \ x \in \Sigma^h. \tag{4.14}$$

Similarly $W(t,x)$ is a *subsolution* of (4.1), (4.2) if the inequalities are reversed in (4.13) and (4.14). By using the monotonicity property (4.3) of F_h and backward induction on k, we have:

Lemma 4.2. *$V^h \leq W$ for any supersolution W of* (4.1) - (4.2)*, and $Z \leq V^h$ for any subsolution Z of* (4.1) - (4.2)*.*

Theorem 4.2. *Let $V^h(t,x)$ be defined by* (3.26)*, with terminal data* (3.3)*. Suppose that ψ is bounded and uniformly continuous in addition to assumptions* (3.1) *and* IV(2.2)*. Then* (4.12) *holds uniformly on $\overline{Q}_0$.*

Proof. Let us first suppose that $\psi, \psi_{x_i}\psi_{x_i x_j}, i,j = 1,\cdots,n$ are bounded and uniformly continuous. By Taylor's formula, the difference quotients $\Delta^{\pm}_{x_i}\psi, \Delta^2_{x_i}\psi, \Delta^{\pm}_{x_i x_j}\psi$ are also bounded. Let

$$W(t,x) = K(t_1 - t) + \psi(x), \tag{4.15}$$

where the constant K is chosen large enough that

$$K > |\tilde{\mathcal{H}}(x, \Delta^{\pm}_{x_i}\psi, \Delta^2_{x_i}\psi, \Delta^{\pm}_{x_i x_j}\psi)|$$

for all $x \in \Sigma_h$. By $(4.7^{'})$

$$\begin{aligned} F_h W(t+h,\cdot)(x) &= W(t+h,x) - h\tilde{\mathcal{H}}(x, \Delta^{\pm}_{x_i}\psi, \Delta^2_{x_i}\psi, \Delta^{\pm}_{x_i x_j}\psi) \\ &= W(t,x) - h\left[K + \tilde{\mathcal{H}}(x, \Delta^{\pm}_{x_i}\psi, \Delta^2_{x_i}\psi, \Delta^{\pm}_{x_i x_j}\psi)\right] \\ &\leq W(t,x). \end{aligned}$$

Thus, W is a supersolution. By Lemma 4.2, $V^h \leq W$. Similarly,

$$Z(t,x) = -K(t_1 - t) + \psi(x)$$

is a subsolution, and hence $Z \leq V^h$. The inequality

$$|V^h(s,y) - \psi(x)| \leq |V^h(s,y) - \psi(y)| + |\psi(y) - \psi(x)|$$

$$\leq K(t_1 - s) + \|D\psi\| \, |y - x|$$

implies that (4.11) holds uniformly for $x \in I\!R^n$. As already noted in Example 4.1, F_h satisfies (4.3) - (4.6). Theorem 4.2 then follows in this case from Theorem 4.1.

If ψ is bounded and uniformly continuous a standard smoothing technique (Appendix C) gives for any $a > 0$ a function $\tilde{\psi}$ such that $\|\tilde{\psi} - \psi\| \leq a$ with $\tilde{\psi}, \tilde{\psi}_{x_i}, \tilde{\psi}_{x_i x_j}$ bounded and uniformly continuous. Let $\tilde{V}, \tilde{V}^h$ denote the corresponding value functions, for terminal cost function $\tilde{\psi}$ instead of ψ. From the definition of value function,

$$\|V - \tilde{V}\| \leq a, \ \|V^h - \tilde{V}^h\| \leq a.$$

We have already shown that

$$\lim_{\substack{(s,y)\to(t,x)\\ h\downarrow 0}} \tilde{V}^h(s,y) = \tilde{V}(t,x)$$

uniformly on compact subsets of $\overline{Q}_0$. Since a is arbitrary, the same is true for V^h and V, as required in (4.12).

□

In Section 5 we will show that the conclusion of Theorem 4.2 remains true uniformly on compact sets, if in (3.26) the infinite lattice Σ_0^h is replaced by the finite lattice Σ^h obtained by imposing a numerical cutoff $|x_i| \leq B_h, i = 1, \cdots, n$, where $B_h \to \infty$ as $h \downarrow 0$. The assumption in Theorem 4.2 that ψ is uniformly continuous will also be removed. See Theorem 5.3.

IX.5 Convergence of finite difference approximations. II

In this section we consider the HJB equation (3.2) in a cylindrical region $Q = [t_0, t_1) \times O$, with boundary data

(5.1) $$V(t,x) = \Psi(t,x), \ (t,x) \in \partial^* Q.$$

Let us assume that Ψ is continuous, and that O is bounded with ∂O a manifold of class C^3. We will impose conditions V(2.3) which insure that the value function V, as defined by IV(2.10), is continuous in $\overline{Q}$ and is the unique viscosity solution to (3.2) - (5.1). See Theorem V.2.1 and Corollary V.8.1.

Let $\Sigma^h \subset \Sigma_0^h$ be a finite lattice. We call $x \in \Sigma^h$ an interior point of Σ^h if all nearest neighbor points $x \pm \delta e_i, \ x \pm e_i \pm e_j, i, j = 1, \cdots, n$, are in Σ^h. Let $\partial\Sigma^h$ denote the set of $x \in \Sigma^h$ which are not interior points of Σ^h. We assume that Σ^h approximates O in the sense that

(5.2) $$\lim_{h\downarrow 0} \operatorname{dist}(\partial O, \partial\Sigma^h) = 0.$$

We define one step transition probabilities $p^v(x,y), x,y \in \Sigma^h$, as follows. If x is an interior point of Σ^h, then $p^v(x,y)$ is as in (3.25), which is (3.6) in dimension $n=1$. As in (3.16), boundary points of Σ^h are absorbing:

$$p^v(x,x) = 1,\ x \in \partial\Sigma^h. \tag{5.3}$$

The objective is to minimize (3.17), where now in (3.17) μ is the smaller of M and the first time step ℓ such that $x^\ell \in \partial\Sigma^h$. Let $V^h(t,x)$ denote the value function, for $(t,x) \in Q^h$, where

$$Q^h = \{(t,x) : t = t_0^h + kh, k = 0,1,\cdots,M, x \in \Sigma^h\}.$$

If x is an interior point of Σ^h, then V^h satisfies the same dynamic programming equation (3.28) as for the case $O = I\!R^n$. Moreover,

$$V^h(t,x) = \Psi(t,x),\ (t,x) \in \partial^* Q^h, \tag{5.4}$$

where $\partial^* Q^h = \{(t,x) \in Q^h : t < t_1, x \in \partial\Sigma^h \text{ or } t = t_1, x \in \Sigma^h\}$.

Lemma 5.1. *Assume that $V \in C(\overline{Q})$ and that*

$$\lim_{\substack{(s,y)\to(t,x)\\ h\downarrow 0}} V^h(s,y) = \Psi(t,x) \tag{5.5}$$

uniformly with respect to $(t,x) \in \partial^* Q$. Then

$$\lim_{\substack{(s,y)\to(t,x)\\ h\downarrow 0}} V^h(s,y) = V(t,x) \tag{5.6}$$

uniformly on $\overline{Q}$.

Sketch of proof. The proof is almost the same as for Theorem 4.1, and we merely sketch it. For $(t,x) \in Q$, define $V^*(t,x)$ and $V_*(t,x)$ by (4.8). As in Lemma 4.1, V^* is a viscosity subsolution and V_* a viscosity supersolution in Q. By Theorem V.8.l, the value function V is the unique continuous viscosity solution of the HJB equation with the boundary conditions (5.1). The lemma then follows from (5.5) and a comparison result, Remark V.8.1.

□

It remains to find conditions under which Lemma 5.1 can be applied. The discrete time dynamic programming equation for V^h is

$$V^h(t,x) = F_h[V^h(t+h,\cdot)](x), \tag{5.7}$$

if $(t,x) \in Q^h - \partial^* Q^h$, with the boundary data (5.4). Here F_h is the same as in (4.7$'$). In analogy with (4.13) - (4.14) we call W a *supersolution* of (5.7) - (5.4) if

$$W(t,x) \geq F_h[W(t+h,\cdot)](x),\ (t,x) \in Q^h - \partial^* Q^h, \tag{5.8}$$

$$W(t,x) \geq \Psi(t,x), (t,x) \in \partial^* Q^h. \tag{5.9}$$

By reversing the inequalities in (5.8) and (5.9) we define subsolutions of (5.7) - (5.4). Just as in Lemma 4.2,

$$Z \leq V^h \leq W \tag{5.10}$$

if W is supersolution and Z a subsolution of (5.7) - (5.4).

Lemma 5.2. *Let $W \in C^{1,2}(\overline{Q})$ be such that*

$$-W_t + \mathcal{H}(x, D_x W, D_x^2 W) \geq 0, \ (t,x) \in \overline{Q}, \tag{5.11}$$

$$W(t,x) \geq \Psi(t,x), \ (t,x) \in \partial^* Q. \tag{5.12}$$

Then for every $a > 0$ there exists $h_0 > 0$, such that $V^h(t,x) \leq W(t,x) + a$ for all $(t,x) \in \Sigma^h, 0 < h < h_0$.

Proof. There is an open set Q_1 such that $\overline{Q} \subset Q_1$ and W has an extension to Q_1 of class $C^{1,2}(Q_1)$. Choose b such that

$$0 < b(t_1 - t_0 + 1) < a,$$

and let

$$\overline{W}(t,x) = W(t,x) + b(t_1 - t + 1).$$

For small $h, Q^h \subset Q_1$, and

$$-\Delta_t^- \bar{W} + \tilde{\mathcal{H}}(x, \Delta_{x_i}^{\pm} \bar{W}, \Delta_{x_i}^2 \bar{W}, \Delta_{x_i x_j}^{\pm} \bar{W}) \geq 0$$

for all $(t,x) \in Q^h$. This implies that $\bar{W}$ satisfies (5.8). By (5.2), $\bar{W}$ satisfies (5.9) for small h. Hence $V^h \leq \bar{W} \leq W + a$ for small h.

□

Inequalities (5.11) - (5.12) state that W is a smooth supersolution of the HJB equation with the boundary data (5.1). By reversing inequalities in Lemma 5.2, we obtain a corresponding statement about subsolutions, and an estimate for V^h from below.

In order to apply Lemma 5.2, let us consider separately the terminal and lateral parts of $\partial^* Q$. As in I(3.6) we write

$$\psi(x) = \Psi(t_1, x).$$

For the terminal part of $\partial^* Q$ we obtain (5.5) from the following lemma.

Lemma 5.3. $\displaystyle\lim_{\substack{(s,y)\to(t_1,x)\\ h\downarrow 0}} V^h(s,y) = \psi(x)$ *uniformly with respect to $x \in \overline{O}$.*

Proof. Choose $\tilde{\psi} \in C^2(\overline{O})$ such that $\psi(x) < \tilde{\psi}(x)$ for all $x \in \overline{O}$. Let

$$W(t,x) = K(t_1 - t) + \tilde{\psi}(x).$$

For K sufficiently large, W satisfies both (5.11) and (5.12). Hence, by Lemma 5.2 given $a > 0$, for small h

$$V^h(s,y) \leq K(t_1 - s) + \tilde{\psi}(y) + a,$$

$$V^h(s,y) \leq \psi(x) + K(t_1 - s) + |\psi(y) - \psi(x)| + \|\tilde{\psi} - \psi\| + a.$$

Since $\tilde{\psi}$ can be chosen such that $\|\tilde{\psi} - \psi\|$ is arbitrarily small and a is arbitrary,

$$\limsup_{\substack{(s,y)\to(t_1,x) \\ h\downarrow 0}} V^h(s,y) \leq \psi(x)$$

uniformly with respect to $x \in \overline{O}$. Similarly, by considering subsolutions of the form

$$Z(t,x) = -K(t_1 - t) + \overset{\approx}{\psi}(x)$$

with $\overset{\approx}{\psi} \in C^2(\bar{O})$ and $\overset{\approx}{\psi} < \psi$, we get

$$\liminf_{\substack{(s,y)\to(t_1,x) \\ h\downarrow 0}} V^h(s,y) \geq \psi(x)$$

uniformly with respect to $x \in \bar{O}$.

□

To arrive at conditions under which (5.5) holds on the lateral boundary, let us first consider the case when

$$(5.13) \qquad L \geq 0,\ \psi \geq 0,\ \Psi(t,x) = 0 \text{ for } (t,x) \in [t_0, t_1] \times \partial O.$$

Then $V^h \geq -\alpha_h$ where $\alpha_h \to 0$ as $h \to 0$. We wish to show that $V^h(s,y)$ is uniformly small for small h and for (s,y) near the lateral boundary of Q. For that purpose we make the following assumption (5.14) and then construct a suitable strict supersolution in a neighborhood of the lateral boundary.

Let

$$O_\delta = \{x \in O : \text{ dist } (x, \partial O) < \delta\}.$$

We assume: There exist $\delta > 0, c > 0$ and $\phi \in C^2(\bar{O}_\delta)$ such that

$$(5.14) \qquad \begin{aligned} &(i) \quad \phi(x) = 0, \quad x \in \partial O, \\ &(ii) \quad \phi(x) > 0, \quad x \in \bar{O}_\delta - \partial O, \\ &(iii) \max_{v \in U} G^v \phi(x) \geq c,\ x \in \bar{O}_\delta, \text{ where} \end{aligned}$$

$$-G^v\phi(x) = f(x,v)\cdot D\phi(x) + \frac{1}{2}tr\ a(x,v)D^2\phi(x).$$

Example 5.1. Assume that the uniform ellipticity condition IV(3.5), or IV(5.10), holds. Choose any $v_1 \in U$. The linear elliptic PDE

$$G^{v_1}\phi(x) = 1,\ x \in O,$$

with the boundary $\phi(x) = 0, x \in \partial O$, has a positive solution $\phi \in C^2(\bar{O})$. In fact, $\phi(x)$ is the mean exit time from O for the solution to the stochastic differential equation IV(5.1) with $u(s) \equiv v_1$ and $x(0) = x$. Then (5.14) holds with δ arbitrary and $c = 1$.

Example 5.2. Consider a deterministic control problem ($a(x,v) \equiv 0$) in which the following slightly stronger form of I(3.11) holds. Assume that for every $\xi \in \partial O$ there exists $v(\xi) \in U$ such that

$$f(\xi, v(\xi))\cdot\eta(\xi) \geq b > 0 \tag{5.15}$$

when $\eta(\xi)$ is the exterior unit normal at ξ and b is some constant. Let $\phi(x) =$ dist $(x, \partial O)$. Since ∂O is a C^3 manifold, $\phi \in C^2(\bar{O}_\delta)$ for δ small enough. Moreover, $D\phi(\xi) = -\eta(\xi)$ for $\xi \in \partial O$. Hence

$$\sup_{v\in U} G^v\phi(\xi) = \sup_{v\in U} f(\xi,v)\cdot\eta(\xi) \geq f(\xi,v(\xi))\cdot\eta(\xi) \geq b.$$

Let $c = b/2$. Then (5.14) holds if δ is small enough. The distance function $\phi(x)$ also can be used in (5.14) if (5.15) holds and $|a(x,v)|$ is sufficiently small, rather than $a(x,v) \equiv 0$. This situation arises when small stochastic perturbations of a deterministic control problem are considered (Chapter VII.)

Lemma 5.4. *Assume* (5.13), (5.14). *Then*

$$\lim_{\substack{(s,y)\to(t,x)\\ h\downarrow 0}} V^h(s,y) = 0$$

uniformly for $(t,x) \in [t_0,t_1]\times\partial O$.

Proof. Let $Q_\delta = [t_0,t_1)\times O_\delta$, and let

$$W(x) = K\phi(x) + a_1$$

where K is to be suitably chosen and $a_1 > 0$ is arbitrary. Then

$$\mathcal{H}(x, DW, D^2W) = \max_{v\in U}[KG^v\phi - L(x,v)] \geq Kc - \|L\|$$

with $c > 0$ as in (5.14)(iii). We have $\|V^h\| \leq M$, where $M = (t_1-t_0)\|L\|+\|\Psi\|$. We choose K large enough that

$$Kc - \|L\| \geq 0,\ W(x) \geq \psi(x),\ x \in \overline{O}_\delta,\ W(x) \geq M, x \in \partial O^\delta \backslash \partial O.$$

For the latter inequality we use (5.14)(ii) and the fact that by (5.13) $\psi(x) = 0$ for $x \in \partial O$. Thus W satisfies (5.11) - (5.12) in $\overline{Q}_\delta$. Let

$$\Sigma_\delta^h = \left\{ x \in \Sigma^h :\ \text{dist}\ (x, \partial O) < \delta \right\}.$$

Consider the controlled Markov chain with state space Σ_δ^h, with the same transition probabilities $p^v(x,y)$ for x interior to Σ_δ^h and $p^v(x,x) = 1$ for $x \in \partial \Sigma_\delta^h$. Instead of (5.4), we take $V^h(t,x)$ as boundary data for $x \in \partial \Sigma_\delta^h, t < t_1$ or $x \in \Sigma_\delta^h, t = t_1$. By the dynamic programming principle $V^h(t,x)$ is the value function for this controlled Markov chain, for $x \in \Sigma_\delta^h$. In Lemma 5.2, we replace Q by Q_δ and Σ^h by Σ_δ^h. For small h,

$$-\alpha_h \leq V^h(t,x) \leq W(x) + a = K\phi(x) + a_1 + a.$$

Since $\phi(x) = 0$ for $x \in \partial O$ and a_1, a are arbitrary we get Lemma 5.4.

□

Assumptions (5.13), (5.14), I(3.11) imply that the value function V for the controlled Markov diffusion process is continuous on $\overline{Q}$, Theorem V.2.1. From Lemmas 5.1, 5.3, 5.4 we then obtain:

Theorem 5.1. *Let* $V^h(t,x)$ *be the solution to* (5.7) *for* $(t,x) \in Q^h \backslash \partial^* Q^h$ *with the boundary data* (5.4) *for* $(t,x) \in \partial^* Q^h$. *Assume that* (5.13), (5.14), I(3.11) *hold. Then*

$$\lim_{\substack{(s,y)\to(t,x) \\ h\downarrow 0}} V^h(s,y) = V(t,x)$$

uniformly on $\overline{Q}$.

Remark 5.1. The continuity of V needed for Theorm 5.1 actually holds if (5.13), (5.14) but not I(3.11) are assumed. However we shall not prove this.

To remove the special assumptions (5.13) on the running and boundary cost functions, let us introduce the idea of smooth subsolution.

We recall from section V.2 that a function $g \in C^{1,2}(\overline{Q})$ is a *smooth subsolution* of the HJB equation with the boundary conditions (5.1) if

$$\text{(5.16)} \qquad \begin{gathered} -g_t + \mathcal{H}(x, D_x g, D_x^2 g) \leq 0,\ (t,x) \in \overline{Q}, \\ g(t,x) \leq \Psi(t,x),\ (t,x) \in \partial^* Q. \end{gathered}$$

A smooth subsolution is of course also a viscosity subsolution according to the definition in Section II.4.

Theorem 5.2. *Assume that* (5.14) *holds and that there exists a smooth subsolution* g *such that* $g(t,x) = \Psi(t,x)$ *for all* $(t,x) \in [t_0, t_1] \times \partial O$. *Then the conclusion of Theorem* 5.1 *is true, provided* $\Sigma^h \subset \overline{O}$ *for small* h.

Proof. To simplify matters slightly let us prove Theorem 5.2 only in case $g = g(x)$ and $\Psi(x) = \psi(x)$. The case when Ψ and g depend on t is entirely similar. Thus

$$\mathcal{H}(x, Dg(x), D^2 g(x)) \le 0,\ x \in \overline{O},$$

$$g(x) \le \psi(x),\ x \in \overline{O};\ g(x) = \psi(x), x \in \partial O.$$

We reduce the result to Theorem 5.1 by the following device, already used in Sections I.8, IV.6 and V.2. Let

$$\tilde{L}(x,v) = L(x,v) - G^v g(x),$$

$$\tilde{\psi}(x) = \psi(x) - g(x).$$

Consider the controlled Markov diffusion problem with running cost $\tilde{L}$, with $\tilde{\psi}$ as terminal cost and with $\tilde{\Psi}(x) = 0$ on the lateral boundary of Q. By using the Dynkin formula, as in Sec. IV. 6, for any admissible progressively measurable control process $u(\cdot)$

$$J(t,x;u) = g(x) + \tilde{J}(t,x;u),$$

$$\tilde{J}(t,x;u) = E_{tx}\{\int_t^\tau \tilde{L}(x(s),u(x))ds + \tilde{\Psi}(x(\tau))\}.$$

Hence, the respective value functions satisfy

$$V(t,x) = \tilde{V}(t,x) + g(x).$$

Since g is a smooth subsolution, $\tilde{L} \ge 0, \tilde{\psi} \ge 0$ as required in (5.13). Let $\tilde{V}^h$ be the value function for the discrete control problem, using these running and terminal cost functions. By Theorem 5.1, $\tilde{V}^h(s,y) \to \tilde{V}(t,x)$ as $(s,y) \to (t,x), h \downarrow 0$, uniformly on $\overline{Q}$.

For every admissible control system π for the discrete control problem, we have from (3.17)

$$J_k^h(x;\pi) = E_{kx}\left\{\sum_{\ell=k}^{\mu-1} hL(x^\ell,u^\ell) + \psi(x^\mu)\right\}$$

$$= \tilde{J}_k^h(x;\pi) + E_{kx}\left\{g(x^\mu) + \sum_{\ell=k}^{\mu-1} hG^{u^\ell} g(x^\ell)\right\},$$

where $\tilde{J}_k^h(x;\pi)$ is defined similarly using $\tilde{L}, \tilde{\psi}$. [Since $\Sigma^h \subset \overline{O}$, these quantities are all well defined without considering $L(x,v), \psi(x), g(x)$ for $x \notin \overline{O}$.] Define G_h^v for $x \in \Sigma^h \backslash \partial\Sigma^h$ by

$$G_h^v \phi(x) = -\frac{1}{h} \sum_{y\in\Sigma^h} p^v(x,y)[\phi(y) - \phi(x)].$$

The discrete version of Dynkin's formula implies that

$$E_{kx} g(x^\mu) = g(x) - E_{kx} \sum_{\ell=k}^{\mu-1} h G_h^{u^\ell} g(x^\ell).$$

Since $g \in C^2(\overline{O})$, for all $(y, v) \in \overline{O} \times U$,

$$|G^v g(y) - G_h^v g(y)| \le c_h,$$

where $c_h \to 0$ as $h \downarrow 0$. By taking $v = u^\ell, y = x^\ell$, we get

$$|J_k^h(x;\pi) - \tilde{J}_k^h(x;\pi) - g(x)| \le c_h(t_1 - t).$$

Since this is true for any π,

$$|V^h(t,x) - \tilde{V}^h(t,x) - g(x)| \le c_h(t_1 - t).$$

We conclude that $V^h(s,y) \to V(t,x)$ as $(s,y) \to (t,x), h \downarrow 0$ uniformly on $\overline{Q}$. □

Numerical Cutoffs. Let us consider as in Section 4 the HJB equation in Q_0, with terminal data ψ. We again assume (3.1) and IV(2.2). In particular, recall that f, σ, L, ψ are assumed to be bounded. We consider the finite lattice

$$\Sigma^h = \{x \in \Sigma_0^h : |x| \le B_h\}, \tag{5.17}$$

where B_h is a "cutoff" parameter such that $B_h \to \infty$ as $h \downarrow 0$. In dimension $n = 1$, this is just (3.14). The one step transition probabilities $p^v(x, y)$ are as in (3.25) if $x \in \Sigma^h \backslash \partial\Sigma^h$. At points of $\partial\Sigma^h$, the one step transition probabilities are assigned arbitrarily. We wish to show that the effect of the cutoff is "small" provided $x \in K$, where K is any compact set.

Define $Q^h, \partial^* Q^h$ as in the discussion preceding Lemma 5.1, with Σ^h as in (5.17). For $(t,x) \in Q^h$, let $\bar{V}^h(t,x)$ denote the value function for this Markov chain control problem. For $(t,x) \in Q^h \backslash \partial^* Q^h$, $\bar{V}^h$ satisfies the dynamic programming equation:

$$\bar{V}^h(t,x) = \bar{V}^h(t+h,x) - h\tilde{\mathcal{H}}(x, \Delta_{x_i}^{\pm}\bar{V}^h, \Delta_{x_i}^2 \bar{V}^h, \Delta_{x_i x_j}^{\pm}\bar{V}^h), \tag{5.18}$$

where $\Delta_{x_i}^{\pm}\bar{V}^h$ etc. are evaluated at $(t+h, x)$. Moreover,

$$\begin{aligned} &(a)\ \bar{V}^h(t_1, x) = \psi(x), \quad x \in \Sigma^h. \\ &(b)\ \|\bar{V}^h\| \le (t_1 - t_0)\|L\| + \|\psi\|. \end{aligned} \tag{5.19}$$

As in Section 4, let $V^h(t,x)$ denote the value function without numerical cutoff. Then V^h also satisfies (5.18) and (5.19). In order to compare $\bar{V}^h$ and V^h, we first prove the following lemmas.

Lemma 5.5. *Let $\eta \in C^2(I\!R^n)$ with $D\eta$ and $D^2\eta$ bounded. Then there exists M such that, for all $\alpha > 0$, $(x, p, A) \in I\!R^{2n} \times \mathcal{S}_+^n$*

$$|\tilde{\mathcal{H}}(x, p_i^{\pm}, A_{ii}, A_{ij}^{\pm}) - \tilde{\mathcal{H}}(x, (p_i + \alpha\eta_{x_i})^{\pm}, A_{ii} + \alpha\eta_{x_ix_i},$$
$$(A_{ij} + \alpha\eta_{x_ix_j})^{\pm})| \le M\alpha.$$

Proof. We recall the definition (3.27) of $\tilde{\mathcal{H}}$, and that f, σ are assumed bounded. Then

$$|f_i^+(p_i + \alpha\eta_{x_i})^+ - f_i^+ p_i^+| \le \alpha \|f_i^+\| \, \|\eta_{x_i}\|,$$

with similar estimates for the other quantities $(p_i + \alpha\eta_{x_i}), \cdots$. Lemma 5.5 follows from these estimates.

□

Lemma 5.6. *Assume that* V^h *and* $\tilde{V}^h$ *satisfy* (5.18) *for* $(t, x) \in Q^h \backslash \partial^* Q^h$ *and* (5.19). *Then given any compact set* $K \subset I\!R^n$, *there exist* C_K, h_K *such that*

$$(5.20) \qquad |V^h(t,x) - \tilde{V}^h(t,x)| \le \frac{C_K}{B_h}(t_1 - t)$$

for all $(t, x) \in Q^h$, $x \in K$, *and* $0 < h < h_K$.

Proof. There exists r_K such that $K \subset \{|x| \le r_K\}$. We choose $\zeta \in C^2(I\!R^1)$ with the following properties:

$$\zeta(r) \ge 0, \ \zeta(r) = 0 \text{ for } r \le r_K,$$
$$\zeta'(r), \zeta''(r) \text{ are bounded},$$
$$\zeta(r) \ge c_1 r - c_2, \ c_1 > 0.$$

In Lemma 5.5 we take $\eta(x) = \zeta(|x|)$ and choose later $\alpha = \alpha_h$ suitably. Let

$$W^h(t,x) = V^h(t,x) + \alpha_h \eta(x) + M\alpha_h(t_1 - t).$$

By (4.7′) and Lemma 5.5, for $(t, x) \in Q^h \backslash \partial^* Q^h$

$$W^h(t,x) - F^h[W(t+h,\cdot)](x)$$
$$= W^h(t,x) - W^h(t+h,x) + h\tilde{\mathcal{H}}(x, \Delta_{x_i}^{\pm} W^h, \Delta_{x_i}^2 W^h, \Delta_{x_ix_j}^{\pm} W^h)$$
$$\ge M\alpha_h h + V^h(t,x) - V^h(t+h,x)$$
$$+ h\tilde{\mathcal{H}}(x, \Delta_{x_i}^{\pm} V^h, \Delta_{x_i}^2 V^h, \Delta_{x_ix_j}^{\pm} V^h) - M\alpha_h h.$$

Since V^h satisfies (5.18) in $Q^h \backslash \partial^* Q^h$, the last term is 0, which implies that W^h satisfies (5.8). Since $\eta(x) \ge 0$ and $\tilde{V}^h(t_1, x) = \psi(x)$, we have $W^h(t_1, x) \ge \tilde{V}^h(t_1, x)$. For $x \in \partial\Sigma^h$, $|x| \ge B_h - \gamma_h$ where $\gamma_h \to 0$ as $h \to 0$. Hence, for $x \in \partial\Sigma^h$

$$W^h(t,x) \geq -\|V^h\| + \alpha_h \zeta(|x|)$$

$$\geq -M_1 + \alpha_h [c_1(B_h - \gamma_h) - c_2],$$

$$M_1 = (t_1 - t_0)\|L\| + \|\psi\|.$$

We choose

$$\alpha_h = \frac{3M_1}{c_1 B_h}. \tag{5.21}$$

For small h,

$$W^h(t,x) \geq M_1 \geq \tilde{V}^h(t,x),\ x \in \partial\Sigma^h$$

Thus, $\tilde{V}^h(t,x) \leq W^h(t,x)$ for all $(t,x) \in \partial^* Q^h$. By (5.10), $\tilde{V}^h \leq W^h$. Since $\eta(x) = 0$ for $x \in K$,

$$\tilde{V}^h(t,x) \leq V^h(t,x) + M\alpha_h(t_1 - t)$$

if $(t,x) \in Q^h, x \in K$ and h is small enough. Since V^h and $\tilde{V}^h$ can be exchanged, we get Lemma 5.6 with $C_K = 3c_1^{-1} M M_1$.

□

Theorem 5.3. *Let $\bar{V}^h$ be the value function for the explicit finite difference scheme with numerical cutoff. Then*

$$\lim_{\substack{(s,y)\to(t,x) \\ h\downarrow 0}} \bar{V}^h(s,y) = V(t,x)$$

uniformly on any compact subset of $\overline{Q}_0$.

Proof. By Theorem 4.1, we need only verify that (4.11) holds uniformly on compact sets. If ψ is bounded and uniformly continuous, this follows from Lemma 5.6 with $\tilde{V}^h = \bar{V}^h$ and Theorem 4.2. Next, suppose that ψ is merely bounded and continuous. Let K be compact. Chose $\tilde{\psi}$ with compact support K_1, where $K \subset int\ K_1,\ \psi(x) = \tilde{\psi}(x)$ for $x \in K$ and $\|\tilde{\psi}\| \leq \|\psi\|$. Let $\tilde{V}^h$ be the value function for the problem with numerical cutoff and with ψ replaced by $\tilde{\psi}$. By applying Lemma 5.6 to $\bar{V}^h$ and $\tilde{V}^h$, we have for $x \in K,\ y \in K_1$ and small h

$$|\bar{V}^h(s,y) - \psi(x)| \leq |\bar{V}^h(s,y) - \tilde{V}^h(s,y)| + |\tilde{V}^h(s,y) - \tilde{\psi}(x)|$$

$$\leq \frac{C_{K_1}}{B_h}(t_1 - s) + |\tilde{V}^h(s,y) - \tilde{\psi}(x)|.$$

Since $\tilde{\psi}$ is uniformly continuous, the last term tends to 0 uniformly as $(s,y) \to (t,x),\quad h \downarrow 0$ by Theorem 4.2.

□

IX.6 Historical remarks

The finite difference numerical method used in this chapter is due to Kushner [Ku1]. We have given only a concise introduction to the topic, with convergence proofs based on viscosity solution methods following Barles and Souganidis [BS]. Kushner–Dupuis [KuD] provides a much more thorough account of the method, together with more refined versions of it and applications. The convergence proofs in [Ku1, Chap. 9] and [KuD, Chap. 10] use weak convergence of probability measure techniques.

For further work on numerical approximations in control problems see [BF] [HW] [FF] [Me], also the survey article [Ku3]. A viscosity solution method was used in [Zhu] to prove convergence of finite difference approximations in singular stochastic control. For related work in a mathematical finance context, see [TZ].

X

Applications to Finance

X.1 Introduction

During the past decade, mathematical finance has matured using optimal control together with stochastic processes, probability theory and partial differential equations. This development has its roots in two seminal papers. Merton's 1971 paper [Mer1] on optimal consumption and investment strategies uses optimal control theory to model the utility of a rational investor. Black & Scholes in their 1973 paper [BSc] on pricing of options uses probability theory and partial differential equations. Since these papers, stochastic optimal control has been widely used in the financial economics literature as an important modelling tool.

In this chapter, we outline several techniques in mathematical finance through several central examples. The financial model used in all of these examples is common and it is described in the next section. The classical Merton problem in a complete financial market is solved in different levels of generality in Sections 3 and 4. Section 5 is devoted to a financial model with proportional transaction costs. Mathematically, this is a singular stochastic control problem. We next consider pricing of derivative securities (options). After the introduction of the classical Black-Scholes theory, we outline different approaches to pricing in Sections 6-10. In the final section, we discuss a Merton-type model with random coefficients.

X.2 Financial market model

In the literature a simple financial market model is often used to study the basic properties. This model, already introduced in Example IV.5.2, consists of two assets; one "risk free" and the other "risky." We call the risk free asset as *bond* and the risky asset as *stock*. We deviate from the notation used in Example IV.5.2 and use the familiar notation from the mathematical finance literature by letting $p(t)$ and $S(t)$, respectively, be the price per share for the risk-free and risky asset at time $t \geq 0$. Following the literature, we make the further simplifying assumptions that these price processes follow simple stochastic differential equations with constant coefficients:

$$
\begin{aligned}
dp &= p(t)r dt, \\
dS &= S(t)[\mu dt + \sigma dw(t)],
\end{aligned}
\tag{2.1}
$$

where r, μ, σ are constants with $0 < r < \mu, \sigma > 0$, and $w(\cdot)$ is a standard one-dimensional Brownian motion. In Section 11, we will discuss a model with random coefficients.

Let us now assume that there are no transaction costs and no constraints on the structure of the portfolio. In particular, any investor in this market may transfer funds from one account to the other instantaneously and with no costs. Moreover, she may hold short positions of any size in both accounts. We will remove these assumptions later in this chapter.

Under these assumptions, the market is complete as defined in the economics literature (c.f. Pliska [Pl2]) and the investor's wealth $x(t)$ at time t changes according to the stochastic differential equation

$$dx = x(t)\left[(1 - \pi(t))\frac{dp(t)}{p(t)} + \pi(t)\frac{dS(t)}{S(t)}\right] - c(t)dt,$$

where $\pi(t)$ is the fraction of wealth invested in the risky asset at time t and $c(t)$ is a consumption rate. Then, with initial wealth x, the state equation has the form

$$
\begin{aligned}
dx &= x(t)\ [(r + \pi(t)[\mu - r])dt + \pi(t)\sigma dw(t)]\ -\ c(t)dt, \\
x(0) &= x.
\end{aligned}
\tag{2.2}
$$

The control is the two-dimensional vector $u(t) = (\pi(t), c(t))$, subject to the constraint $c(t) \geq 0$. Thus, $U = (-\infty, \infty) \times [0, \infty)$. We stop the process if the wealth $x(\cdot)$ reaches zero (bankruptcy). Thus, $O = (0, \infty)$.

X.3 Merton portfolio problem

This is a model problem to study the optimal investment and consumption strategies of a rational investor. This investor is assumed to make decisions so as to maximize a certain utility function. There are several alternate utilities or equivalently objective functionals one may consider, and we will study several of them in this chapter.

In this section, we start with the classical model already studied in Example IV.5.2. We assume that the utility is derived from the consumption through a consumption utility rate function and then an exponentially discounted integral over the whole future is taken as the objective function:

$$J(x;\pi,c) = E_x \int_0^\infty e^{-\beta t}\ell(c(t))dt, \tag{3.1}$$

where $\ell(c)$ is the utility rate of consuming at rate $c \geq 0$. Here we refrained from using the classical notation U for the utility function as it clashes with our already established notation.

The classical *Merton optimal investment consumption problem* is to *maximize* the total discounted expected utility, J, discounted at rate $\beta > 0$. As stated earlier, $U = (-\infty,\infty) \times [0,\infty)$ and $O = (0,\infty)$. In Example IV.5.2 we formulated the problem as an exit time problem. However, due to the absorbing nature of the boundary $x = 0$, it is equivalent to consider it as a state constrained problem. In the present chapter, only control processes $\pi(\cdot)c(\cdot)$, will be admitted for which the wealth constraint $x(t) \geq 0$ holds for all t. The boundary point $x = 0$ is absorbing. If $x(\tau) = 0$, then $x(t) = c(t) = \ell(c(t)) = 0$ for all $t \geq \tau$. Hence the bankruptcy time τ in IV(5.20) is replaced by infinity in (3.1).

The dynamic programming equation and the optimal control policies are derived in Example IV.5.2. Also the HARA utility case is solved explicitly in that example as well.

X.4 General utility and duality

Now we consider the portfolio selection problem with a general utility function ℓ. A general solution is obtained by Karatzas, Lechozky, Sethi and Shreve [KLSS] and by Pliska [Pl1]. Here we present an outline of their method through a differential equations type of arguments. Also to simplify the presentation, we make several technical assumptions on the utility function. However, their method applies to more general utility functions. We refer to the book of Karatzas and Shreve [KS5] for these generalizations and for the general probabilistic approach. For a general utility function $\ell(c)$, it turns out that the dynamic programming equation for the value function $V(x)$ can be linearized, by introducing the variable $y = V_x(x)$. The solution depends on two functions $G(y)$, $W(y)$, which satisfy the linear differential equations (4.10), (4.13) and which have Feynman-Kac type representations (4.6), (4.9). The optimal consumption control $c^*(t)$ will be obtained from a convex duality argument. Then the optimal investment control $\pi^*(t)$ is obtained from the dynamic programming equation.

We assume that the utility function $\ell(c)$ is defined for $c \geq 0$ and

$$\ell(0) = 0, \quad \ell^{'}(0^+) = +\infty, \quad \ell^{'}(\infty) = 0, \quad \ell^{'}(c) > 0, \quad \ell^{''}(c) < 0 \text{ for } c > 0. \tag{4.1}$$

We recall several basic facts from convex analysis (cf. Rockafellar [R1], Section 3.4 [KS5].) By (4.1), the inverse function I of ℓ' exists on $[0,\infty)$ and satisfies

$$I(0) = \infty, \quad I(\infty) = 0, \quad I(p) < \infty, \quad I'(p) < 0, \ \forall\, p > 0.$$

The convex dual ℓ^* is given by

$$\ell^*(p) := \sup_{c \geq 0} \ \{ \ \ell(c) - cp \ \}, \quad \text{for } p \geq 0.$$

Then,

$$(\ell^*)' = -I, \quad \text{and} \quad \ell^*(p) = \ell(I(p)) - pI(p) \geq \ell(c) - pc, \quad \forall\, p, c > 0. \tag{4.2}$$

In this chapter we say that progressively measurable control processes $\pi(\cdot), c(\cdot)$ are *admissible at* x if the following hold: with probability one

$$\int_0^T c(t)dt < \infty, \qquad \int_0^T \pi^2(t)dt < \infty, \qquad \forall\, T > 0,$$

and $x(t) \geq 0$ for all $t \geq 0$. This is consistent with the usage in [KS5, Section 3.9]. Next, we derive an inequality satisfied by all admissible consumption processes. Let P_0 be the absolutely continuous probability measure so that

$$w_0(t) := w(t) + \theta t, \quad \theta = \frac{\mu - r}{\sigma}, \tag{4.3}$$

is a P_0 Brownian motion. By the Girsanov theorem, we know that P_0 exists and for any $\mathcal{F}_t$ measurable random variable B

$$E^{P_0}[B] = E^P[z(t)B]$$

where

$$z(t) = \exp\left(-\frac{\theta^2}{2} t - \theta w(t) \right). \tag{4.4}$$

In terms of w_0, the wealth equation (2.2) has the form

$$dx = x(t)[rdt + \pi(t)\sigma dw_0(t)] - c(t)dt.$$

P_0 is called the "risk neutral" probability measure. We integrate to obtain

$$e^{-rT}x(T) - x = \int_0^T \ e^{-rt}x(t)\pi(t)\sigma dw_0(t) - \int_0^T \ e^{-rt}c(t)dt.$$

It is standard to show that the stochastic integral is a local martingale and therefore, for every T,

$$E_0[\int_0^T \ e^{-rt}c(t)dt] \leq x,$$

where E_0 is integration with respect to P_0. Since $c(t) \geq 0$, by Fatou's lemma we pass to the limit as T tends to infinity. Then we state the result in terms of the original measure P using the density z. The result is

$$E[\int_0^\infty e^{-rt} z(t)c(t)dt] \le x. \tag{4.5}$$

The above constraint is known as the *budget constraint* in the literature. It is clear that we would like to saturate the above inequality in order to maximize J. In fact this is the only relevant constraint in the above problem. The reason for this is the following result.

Theorem 4.1 [Theorem 3.9.4 [KS5]]. *Given $x \ge 0$, and a consumption process $c(\cdot)$ satisfying the budget constraint* (4.5) *with an equality, there exists a portfolio process $\pi(\cdot)$, such that (π, c) is admissible at x. Moreover, the corresponding wealth process is*

$$x(t) = \frac{1}{e^{-rt}z(t)} E\left[\int_t^\infty e^{-r\rho} z(\rho)c(\rho)d\rho \mid \mathcal{F}_t \right].$$

The proof is an application of the martingale representation result. In view of this result we have the following corollary,

Corollary 4.1. *The Merton portfolio selection problem is equivalent to maximizing* (3.1) *under the budget constraint* (4.5).

We solve the equivalent problem using a Lagrange multiplier technique. For $y \ge 0$, consider

$$\begin{aligned} H(x, y, c(\cdot)) &:= E \int_0^\infty e^{-\beta t} \ell(c(t))dt + y \left(x - E[\int_0^\infty e^{-rt} c(t)z(t)dt] \right) \\ &= xy + E \int_0^\infty e^{-\beta t} \left[\ell(c(t))dt - ye^{(\beta - r)t} z(t)c(t) \right] dt. \end{aligned}$$

Given x, we want to maximize H over y and c. In view of (4.2), for any y and $c(\cdot) \ge 0$,

$$H(x, y, c(\cdot)) \le xy + E \int_0^\infty e^{-\beta t} \left[\ell(\hat{c}(t; y))dt - ye^{(\beta - r)t} z(t)\hat{c}(t; y) \right] dt,$$

with equality when $c(t) = \hat{c}(t, y)$, where

$$\hat{c}(t; y) := I(ye^{(\beta - r)t} z(t)),$$

and $z(t)$ is as in (4.4). The above calculation suggests that $\hat{c}$ with a carefully chosen y is the optimal consumption process. We determine y by using the budget constraint (4.5). Set

$$W(y) := E[\int_0^\infty e^{-rt} z(t)\hat{c}(t; y)dt], \tag{4.6}$$

so that for $\hat{c}$ to satisfy (4.5) with equality, y has to be equal to $W^{-1}(x)$, where W^{-1} is the inverse function of W. So for a given initial wealth x, the candidate for the optimal consumption process is

$$c^*(t) := \hat{c}(t; W^{-1}(x)) = I(y(t)) \tag{4.7}$$

$$y(t) = ye^{(\beta-r)t}z(t), \qquad y = W^{-1}(x). \tag{4.8}$$

From (4.7) and (4.8) the value function should equal

$$\begin{aligned} &V(x) := G(W^{-1}(x)), \\ &G(y) := E\left[\int_0^\infty e^{-\beta t}\ell(\hat{c}(t;y))dt\right] = E[\int_0^\infty e^{-\beta t}\ell(I(y(t))dt]. \end{aligned} \tag{4.9}$$

Assumption (4.11) below will imply that $G(y)$ is finite. This conjectured relationship between the value function $V(x)$ and $G(y)$ is true (Theorem 4.2), and the optimal controls $\pi^*(t), c^*(t)$ can be written as functions of $y(t)$ via (4.7) and (4.15).

Let us first state some properties of the functions W and G. By (4.4) and (4.8)

$$dy = y(t)[(\beta - r)dt - \theta dw(t)].$$

A formal application of the Feynman-Kac formula suggests that G satisfies the linear differential equation for $y > 0$

$$\beta G - (\beta - r)yG_y - \frac{\theta^2 y^2}{2}G_{yy} - \ell(I(y)) = 0. \tag{4.10}$$

The associated homogeneous linear differential equation has solutions $g(y) = y^\lambda$ for $\lambda = \lambda_1 < 0$ and $\lambda = \lambda_2 > 1$ which satisfy

$$\frac{1}{2}\theta^2\lambda^2 - \left(r - \beta + \frac{1}{2}\theta^2\right)\lambda - \beta = 0.$$

Assume that for any $a > 0$

$$\int_0^a y^{-\lambda_1} I(y)dy < \infty. \tag{4.11}$$

Then $G(y)$ is a class C^2 solution to (4.10). In fact,

$$G(y) = \frac{2}{\theta^2(\lambda_2 - \lambda_1)}\left[y^{\lambda_1}\psi_1(y) + y^{\lambda_2}\psi_2(y)\right] \tag{4.12}$$

where $\psi_{1y}(y) = y^{-\lambda_1 - 1}\ell(I(y)), \psi_{2y}(y) = -y^{-\lambda_2-1}\ell(I(y))$ with $\psi_1(0) = 0$, $\psi_2(\infty) = 0$. See [KS5,Thm. 3.9.18]. To obtain an equation for W, using the probability measure P_0, we rewrite (4.6) as

$$W(y) = E_0\left[\int_0^\infty e^{-rt}\ I(y(t))dt\right],$$

where $y(t)$ is as above and we write its dynamics using the Brownian motion $w_0(t) = w(t) + \theta t$. The result is

$$dy = y(t)[(\beta - r)dt - \theta dw(t)] = y(t)[(\beta + \theta^2 - r)dt - \theta dw_0(t)].$$

Hence, the associated linear differential equation for $W(y)$ is

$$rW - (\beta + \theta^2 - r)yW_y - \frac{\theta^2 y^2}{2} W_{yy} - I(y) = 0. \tag{4.13}$$

By [KS5, Thm. 3.9.14], $W(y)$ is a class C^2 solution of (4.13) which has a representation similar to (4.12). Moreover,

$$G_y(y) = y\, W_y(y). \tag{4.14}$$

Theorem 4.2. *The value function is $V(x) = G(W^{-1}(x))$ and V_x is the inverse function of W. The controls $\pi^*(t)$, $c^*(t)$ are optimal, where $c^*(t)$ is as in* (4.7) *and*

$$\pi^*(t) = -\frac{\theta}{\sigma} \frac{y(t) W_y(y(t))}{W(y(t))}. \tag{4.15}$$

Proof. We rewrite the dynamic programming equation IV(5.21) as

$$\beta V(x) - rxV_x(x) - \ell^*(V_x(x)) + \frac{\theta^2}{2} \frac{V_x(x)^2}{V_{xx}(x)} = 0, \tag{4.16}$$

where $\theta = (\mu - r)/\sigma$ and ℓ^* are as before. Let $\hat{V}(x) = G(W^{-1}(x))$. We directly calculate that

$$\hat{V}_x(W(y)) = G_y(y)(W^{-1})_x(W(y)) = \frac{G_y(y)}{W_y(y)},$$

where we used the fact that $W^{-1}(W(y)) = y$ implies that $(W^{-1})_x(W(y))W_y(y) = 1$. By (4.14), we conclude that $\hat{V}_x(W(y)) = y$. Hence, W is equal to the inverse of $\hat{V}_x$. Let

$$\mathcal{I}(x) := \beta \hat{V}(x) - rx\hat{V}_x(x) - \ell^*(\hat{V}_x(x)) + \frac{\theta^2}{2} \frac{(\hat{V}_x(x))^2}{\hat{V}_{xx}(x)}.$$

In order to show that $\hat{V}$ solves the dynamic programming equation (4.16), we need to show that $\mathcal{I}$ is identically equal to zero. Equivalently, we will prove that $\mathcal{I}(W(y)) = 0$ for all y.

By the previous step, $\hat{V}(W(y)) = G(y)$, $\hat{V}_x(W(y)) = y$ and by differentiating $\hat{V}_{xx}(W(y))W_y(y) = 1$. Hence,

$$\mathcal{I}(W(y)) = \beta G(y) - ryW(y) - \ell^*(y) + \frac{\theta^2}{2}\, y^2 W_y(y).$$

By (4.2), $\ell^*(y) = \ell(I(y)) - yI(y)$. We use this together with equation (4.10) for G. The result is

$$\mathcal{I}(W(y)) = \frac{\theta^2 y^2}{2} G_{yy}(y) + (\beta - r)yG_y(y) - ryW(y) + yI(y) + \frac{\theta^2}{2}\, y^2 W_y(y).$$

By (4.14), $G_{yy} = W_y + y\,W_{yy}$. Hence

$$\begin{aligned}\mathcal{I}(W(y)) &= \frac{\theta^2 y^2}{2}[W_y(y) + yW_{yy}(y)] + (\beta - r)y^2 W_y(y) - ryW(y)\\ &\quad + yI(y) + \frac{\theta^2}{2}\, y^2 W_y(y)\\ &= y\left[\frac{\theta^2 y^2}{2} W_{yy}(y) + (\beta + \theta^2 - r)yW_y(y) - rW(y) + I(y)\right].\end{aligned}$$

By (4.13), this is zero. Since the range of W is all of $[0, \infty)$, we conclude that $\hat{V}$ is a solution of the dynamic programming equation (4.16).

It remains to verify that $V(x) = \hat{V}(x)$ and that the controls $\pi^*(t)$, $c^*(t)$ are optimal. We use the dynamic programming technique in Section IV.5. Since $\ell \geq 0$ and $\hat{V} \geq 0$, we obtain $J(x; \pi, c) \leq \hat{V}(x)$ for all $\pi(\cdot), c(\cdot)$ as in Theorem IV.5.1(a). The dynamic programming technique in Section IV.5 gives the following candidates for optimal stationary control policies (see formula IV(5.21)):

$$\begin{aligned}&\underline{\pi}^*(x) = -\frac{\theta}{\sigma}\frac{\hat{V}_x(x)}{x\hat{V}_{xx}(x)}\\ &\underline{c}^*(x) = (\ell')^{-1}(\hat{V}_x(x)) = I(\hat{V}_x(x)).\end{aligned} \tag{4.17}$$

Let $x^*(t) = W(y(t))$. The previous calculations show that $\pi^*(t) = \underline{\pi}^*(x^*(t))$, $c^*(t) = \underline{c}^*(x^*(t))$. Moreover, $x^*(t)$ satisfies (2.2) with $\pi(t) = \pi^*(t)$, $c(t) = c^*(t)$, with initial data $x = W^{-1}(y)$ where $x = x(0)$, $y = y(0)$. By (4.9),

$$J(x; \pi^*, c^*) = G(W^{-1}(x)) = \hat{V}(x).$$

□

X.5 Portfolio selection with transaction costs

In this section, we outline a model of a portfolio selection problem with transaction costs and consumption. It was first introduced in the financial literature by Constantinides [Co] and analyzed by Davis & Norman [DN]. We refer to Shreve & Soner [ShS] for the viscosity approach.

As in Merton's portfolio problem the investor may consume from her investment in the bond, but simultaneously she may transfer her stock holdings to the bond. However, this results in a transaction cost. We assume this cost is linearly proportional in the size of the transaction. We remark on the fixed cost case in Remark 5.1 below. Let $\lambda_1 \in (0,1)$ be the proportional cost of transaction from stock to bond. We also allow transactions from bond to stock. Let $\lambda_2 \in (0,1)$ be the cost of these type of transactions.

Let $x_1(t)$ and $x_2(t)$ be the dollars invested at time t in the bond and the stock, respectively. Using the equation (2.1) we see that $x_1(\cdot)$ and $x_2(\cdot)$ change according to

$$\begin{aligned} dx_1(t) &= [rx_1(t) - c(t)]dt + (1-\lambda_1)dM_1(t) - dM_2(t), \\ dx_2(t) &= \mu x_2(t)dt - dM_1(t) + (1-\lambda_2)dM_2(t) + \sigma x_2(t)dw(t), \end{aligned} \tag{5.1}$$

where $c(t) \geq 0$ is the consumption rate and $M_1(t)$ and $M_2(t)$ are the total transactions up to time $t \geq 0$, from stock to bond and bond to stock, respectively.

The agent's goal is to maximize his discounted total utility of consumption

$$J(x_1, x_2; c, M_1, M_2) = E_x \int_0^\infty e^{-\beta t} \ell(c(t))dt,$$

over all progressively measurable $c(\cdot) \geq 0$, and progressively measurable, left continuous, nondecreasing $M_i(\cdot)$'s with $M_i(0) = 0$. Moreover, we impose an additional constraint

$$x_1(t) + (1-\lambda_1)x_2(t) \geq 0, \;\; (1-\lambda_2)x_1(t) + x_2(t) \geq 0, \;\; \forall t \geq 0.$$

This condition ensures that at any time the investor has sufficient funds so that if needed her portfolio can be liquidated to result in a non-negative bond holding and no stock holding. For this reason Davis & Norman [DN] call the cone O defined below the "solvency region."

It is more convenient to state the above condition as a state constraint. Let

$$O = \{x = (x_1, x_2) \in I\!\!R^2 : x_1 + (1-\lambda_1)x_2 > 0, (1-\lambda_2)x_1 + x_2 > 0\}.$$

Then, the state $x = (x_1, x_2)$ is required to satisfy $(x_1(t), x_2(t)) \in \bar{O}$, for all $t \geq 0$. We claim that for any $x \in \bar{O}$ there are $c(\cdot), M_1(\cdot), M_2(\cdot)$ satisfying (5.1). Indeed let $x \in \bar{O}$ with $x_1 + (1-\lambda_1)x_2 = 0$ be given. Then $x_2 \geq 0$. Set

$$c(t) \equiv M_2(t) \equiv 0, \;\; M_1(t) \equiv x_2, \;\; \forall t > 0.$$

Then $x(t) \equiv (0,0)$ for all $t > 0$, and therefore this control satisfies the state constraint. Similarly for $x \in \bar{O}$ with $(1-\lambda_2)x_1 + x_2 = 0$, we use $c(t) \equiv M_2(t) \equiv 0$ and $M_2(t) \equiv x_1$ for all $t > 0$.

Since there are admissible controls for each boundary point $x \in \partial O$, there are admissible controls for every $x \in \bar{O}$. Hence the set of all admissible controls $\mathcal{A}_\nu(x)$ is non-empty. Moreover, at the boundary ∂O the only admissible consumption policy is $c(\cdot) \equiv 0$. Hence

$$V(x) = 0, \quad \forall\, x \in \partial O. \tag{5.2}$$

The optimal portfolio selection problem with transaction costs is similar to the singular stochastic control problems considered in Chapter VIII. However, the criterion $J(x_1, x_2; c, M_1, M_2)$ involves the nonlinear function $\ell(c(t))$ of consumption, and is not of the form VIII(3.5). In the notation of Section VIII.3

$$U = \{m(1-\lambda_1, -1) : m \geq 0\} \cup \{n(-1, 1-\lambda_2) : n \geq 0\},$$

$$\hat{f}(x) = (rx_1, \mu x_2),$$

$$\hat{\sigma}(x) = \begin{bmatrix} 0 & 0 \\ 0 & \sigma x_2 \end{bmatrix}.$$

Although there are differences, the model with consumption is very similar to the one discussed in Section VIII.3. Indeed one can prove an entirely similar Verification Theorem. The corresponding dynamic programming equation for $x \in O$ is

$$\min\{\mathcal{L}V - \ell^*(V_{x_1})\ ,\ -(1-\lambda_1)V_{x_1} + V_{x_2}\ ,\ V_{x_1} - (1-\lambda_2)V_{x_2}\} = 0, \tag{5.3}$$

where

$$\mathcal{L}V = \beta V - \left\{ \frac{1}{2}\sigma^2 x_2^2 V_{x_2 x_2} + rx_1 V_{x_1} + \mu x_2 V_{x_2} \right\}.$$

Let us now assume that $\ell(c)$, the utility of consuming at rate $c \geq 0$, is of HARA type, i.e., for some $\gamma \in (0,1)$,

$$\ell(c) = \frac{1}{\gamma} c^\gamma, \quad c \geq 0.$$

In this book we will not consider the cases $\gamma < 0$ and the logarithmic utility function. However, a similar analysis can be carried out for those cases.

Observe that if $(c(\cdot), M_i(\cdot)) \in \mathcal{A}_\nu(x)$ and $\rho > 0$, then $\rho(c(\cdot), M_i(\cdot)) \in \mathcal{A}_\nu(\rho x)$ with

$$J(\rho x; \rho c, \rho M_i) = \rho^\gamma J(x; c, M_i).$$

Hence

$$V(\rho x) = \rho^\gamma V(x), \quad \forall \rho \geq 0. \tag{5.4}$$

Set

$$\hat{W}(z) = V(z, 1-z), \;\; z \in \left(-\frac{1-\lambda_1}{\lambda_1}, \frac{1}{\lambda_2} \right),$$

so that for $x \in O$

$$V(x) = (x_1 + x_2)^\gamma \hat{W}(z), \quad z = \frac{x_1}{x_1 + x_2}.$$

Now, one can rewrite the dynamic programming equation in terms of the function $\hat{W}$ which only has one independent variable. This transformation makes the problem more accessible to both theoretical and numerical analysis. Note that there are other possible reductions to one variable. However, this particular one has the advantage of having the new state variable z to be confined in a bounded interval.

Studying an equivalent one dimensional problem, Davis & Norman [DN] constructed a solution $\hat{W} \in C^2$ of (5.3) and (5.2), under an implicit assumption on the coefficients. Then using this solution, they also constructed the optimal consumption and transaction strategies. As in the Examples in Section VIII.4, the optimal process $x^*(t)$ is a reflected diffusion process. Further analysis of this problem is carried out in Shreve & Soner [ShS]. They prove that the value function is smooth enough to carry out the stochastic calculus needed in the construction of the optimal strategies. The only assumption they impose is the finiteness of the value function. We formulate their results without proof in the following theorem.

Theorem 5.1[DN,ShS]. *Assume that the value function V is finite. Then, it is concave and a classical solution of the dynamic programming equation* (5.3) *together with the boundary condition* (5.2).

We continue by constructing the optimal strategies. First let

$$\mathcal{P} = \{x \in O : -(1-\lambda_1)V_{x_1}(x) + V_{x_2}(x) > 0, V_{x_1}(x) - (1-\lambda_2)V_{x_2}(x) > 0\}.$$

The concavity of V, homothety (5.4) and the dynamic programming equation yield that $\mathcal{P}$ is a connected cone, and there are

$$z_1, z_2 \in [-\frac{1-\lambda_1}{\lambda_1}, \frac{1}{\lambda_2}]$$

such that

$$\mathcal{P} = \left\{(x_1, x_2) \in O : z_1 < \frac{x_1}{x_1+x_2} < z_2\right\}.$$

We call $\mathcal{P}$ the *no transaction* region and define *sell bond* region SB and *sell stock* region SSt by

$$\text{SB} := \left\{(x_1, x_2) \in O : \frac{x_1}{x_1+x_2} > z_2\right\}, \quad V_{x_1} - (1-\lambda_2)V_{x_2} = 0 \ \Leftrightarrow \ x \in \text{SB},$$

$$\text{SSt} := \left\{(x_1, x_2) \in O : \frac{x_1}{x_1+x_2} < z_1\right\} \quad -(1-\lambda_1)V_{x_1} + V_{x_2} = 0, \ \Leftrightarrow \ x \in \text{SSt}.$$

Recall that for $p > 0$,

$$\ell^*(p) = \sup_{c \geq 0} \left\{\frac{1}{\gamma}c^\gamma - cp\right\},$$

and the minimizer is $c = p^{1/\gamma - 1}$. Therefore the candidate for an optimal Markov consumption policy is

$$c^*(x) = (V_{x_1}(x))^{\frac{1}{\gamma-1}}.$$

Since $\gamma < 1$ and V is concave, we have

$$c^*(x_1, x_2) > c^*(\bar{x_1}, x_2) \text{ if } x_1 > \bar{x_1}.$$

Due to this monotonicity, for every progressively measurable nondecreasing processes $M_i(\cdot)$ there exists a unique solution $x(\cdot)$ of (4.1) with $c(t) = c^*(x(t))$. We refer to [ShS] for details.

We continue by constructing a control $(c^*(\cdot), M_i^*(\cdot))$ for a given reference probability system ν. We assume for simplicity that the sets SB and SSt are non empty and

$$-\frac{1-\lambda_1}{\lambda_1} < z_1 < z_2 < \frac{1}{\lambda_2}.$$

We refer to [ShS] for the other cases.

1. $(x_1, x_2) \in \overline{\mathcal{P}}$**.** Then by a result of Lions and Sznitman [LS] on reflected diffusion processes (also see Davis and Norman [DN, Theorem 4.1]) there are unique processes $x^*(\cdot)$ and nondecreasing processes $M_i^*(\cdot)$ such that for $t < \tau^* = \inf\{\rho : (x^*(\rho)) \in \partial O\}$,

$$M_1^*(t) = \int_0^t \chi_{\{(x^*(\rho)) \in \partial_1 \mathcal{P}\}} dM_1^*(\rho),$$

$$M_2^*(t) = \int_0^t \chi_{\{(x^*(\rho)) \in \partial_2 \mathcal{P}\}} dM_2^*(\rho)$$

$$x^*(\cdot) \text{ solves (5.1) with } c(t) = c^*(x^*(t)),$$

$$x^*(0) = x, \;\; x^*(t) \in \overline{\mathcal{P}}, \;\; \forall t \geq 0,$$

where

$$\partial_i \mathcal{P} = \left\{ x \in O : \frac{x_1}{x_1 + x_2} = z_i \right\} \text{ for } i = 1, 2.$$

Then x^* is a diffusion process reflected at $\partial \mathcal{P}$. The reflection angle is $(1 - \lambda_1, -1)$ on $\partial_1 \mathcal{P}$ and $(-1, 1 - \lambda_2)$ on $\partial_2 \mathcal{P}$. Moreover $M_i^*(t)$'s are the local times of x^* on the lines $\partial_i \mathcal{P}$. Since the region $\mathcal{P}$ has a corner at the origin, the result of Lions and Sznitman is not directly applicable to the above situation. However the process $x^*(t)$ is stopped if it reaches the origin. Hence we may construct $x^*(t)$ up to any stopping time $\tau < \tau^*$ and therefore up to τ^*.

2. Suppose that $x \in$ SSt. Then

$$\frac{x_1}{x_1 + x_2} \in \left(-\frac{1-\lambda_1}{\lambda_1}, z_1 \right),$$

and in particular $z_1 > -(1 - \lambda_1)/\lambda_1$. Set

$$M_1^*(0^+) = \lim_{t\downarrow 0} M_1^*(t) = [z_1x_2 + (z_1-1)x_1]\frac{1}{1-\lambda_1+z_1\lambda_1},$$

and $M_2^*(0^+) = 0$. Since $x \in \text{SSt}$, $M_1^*(0^+) > 0$. Also

$$x_1^*(0^+) = x_1 + (1-\lambda_1)M_1^*(0^+),$$

$$x_2^*(0^+) = x_2 - M_1^*(0^+),$$

and we calculate that

$$\frac{x_1^*(0^+)}{x_1^*(0^+) + x_2^*(0^+)} = z_1.$$

Therefore $x^*(0^+) \in \overline{\mathcal{P}}$. We now construct $x^*(\cdot), c^*(\cdot), M_i^*(\cdot)$ as in Case 1 starting from $x^*(0^+)$ and $M_1^*(0^+)$.

3. Suppose that $x \in \text{SB}$. Then

$$\frac{x_1}{x_1+x_2} \in \big(z_2, \frac{1}{\lambda_2}\big),$$

and in particular $z_2 < 1/\lambda_2$. Set

$$M_2^*(0^+) = \frac{(1-z_2)x_1 - z_2x_2}{1-z_2\lambda_2},$$

and $M_1^*(0^+) = 0$. As in the previous case, $M_2^*(0^+) > 0$ since $x \in \text{SB}$. Also

$$x_1^*(0^+) = x_1 - M_2^*(0^+),$$

$$x_2^*(0^+) = x_2 + (1-\lambda_2)M_2^*(0^+).$$

Then $x^*(0^+) \in \partial_2\mathcal{P}$ and we construct $x^*(\cdot), c^*(\cdot), M_i^*(\cdot)$ starting from $x^*(0^+)$ and $M_2^*(0^+)$.

Theorem 5.2. *For any reference probability system ν, and $x \in O$,*

$$J(x; c^*, M_i^*) = V_\nu(x) = V(x).$$

We refer to [ShS] for the proof.

Remark 5.1. In practice it is typical to have a fixed transaction cost perhaps in addition to a proportional one. However, for a large investor the proportional transaction costs are much larger than the fixed ones and that is the main reason for neglecting the fixed transaction costs. However, the model with of both proportional transaction and fixed costs can be analyzed by dynamic programming. In such a model, the control processes M_i must be restricted to piecewise constant functions,

$$M_i(t) = \sum_{k=1}^{\infty} m_i^k \; H(t-\tau_i^k),$$

where H is the heaviside function ($H(r) = 0$ if $r \leq 0$ and $H(r) = 1$ if $r > 0$), τ_i^k are nondecreasing sequence of stopping times determining when the transactions are made and m_i^k's positive transaction sizes with usual measurability conditions. Then, the state $x(\cdot)$ follows an modification of the equation (5.1):

$$\begin{aligned} dx_1(t) &= [rx_1(t) - c(t)]dt + (1 - \lambda_1)dM_1(t) - dM_2(t) \\ &\qquad -\alpha_1 \sum_k H(s - \tau_1^k), \\ dx_2(t) &= \mu x_2(t)dt - dM_1(t) + (1 - \lambda_2)dM_2(t) + \sigma x_2(t)dw(t) \\ &\qquad -\alpha_2 \sum_k H(s - \tau_1^k), \end{aligned}$$

where $\alpha_i > 0$ are fixed transaction costs. This problem is a mixture of an optimal stopping time and singular control problems. Unfortunately, the value function does not satisfy the property (5.4) and therefore the optimal solution is not as elegant. Still the solution is expected to have a similar structure as in the pure proportional case. Indeed, there must be two concentric continuation sets $\mathcal{P}_1 \subset \mathcal{P}_2 \subset O$ so that it is optimal to consume according to c^* in the continuation set $\mathcal{P}_2$. When the state process hits the boundary of $\mathcal{P}_2$, it is pushed onto the boundary of $\mathcal{P}_1$ along one of the push directions. If however, the initial state is outside $\mathcal{P}_2$ then again it is optimal to push it on to the boundary of $\mathcal{P}_1$. □

Results about the case of both fixed and proportional transactions costs are given in [Ko].

X.6 Derivatives and the Black-Scholes price

A *contingent claim* or equivalently a *derivative* is a financial instrument whose future pay-off is determined by a *deterministic* function of the future (and therefore random) value of a set of financial instruments, namely the *underlying* instruments. Although more than one instrument is also used in practice, for brevity we restrict our attention to the model described in Section 2. Then the risky asset, or the stock is the underlying instrument. Let T be the prescribed future date of settlement, called maturity. The value of the pay-off of the derivative is $\varphi(S(T))$, where φ is a given deterministic function and $S(\cdot)$ is the per share price of the stock.

Example 6.1. Standard examples of derivatives are

$$\varphi_C(S(T)) = (S(T) - K)^+, \quad \text{Call option.}$$

In fact, the holder of this call option has the option to buy one share of the underlying stock for the price of K; the *strike price*. Of course it is only rational to exercise this option when the stock price $S(T)$ at time T is larger than K resulting in a net gain of $S(T) - K$, when $S(T) \geq K$. Other important examples are

$$\varphi_P(S(T)) = (K - S(T))^+, \quad \text{Put option,}$$

which gives the holder the option to sell the stock for a price of K, and

$$\varphi_D(S(T)) = H(S(T) - K), \quad \text{Digital option,}$$

pays one dollar when $S(T) \geq K$. □

Derivatives are essential tools in risk management allowing investors to reduce the risk of their portfolio. We refer to Hull's book [Hu] for this very important point. In this book we concentrate on the mathematical problem of pricing and hedging. The pricing question is to determine the fair value (or equivalently the fair price) of this derivative at any time t prior to maturity T.

Let us introduce the idea of arbitrage pricing in a formal manner. We will rederive the same pricing formula through two different ways in the following sections.

Consider the financial market described in Section 2 and suppose that the fair value is a function $V(t, s)$ of the current stock price s and the time t. If this is the fair value one can buy or sell this derivative in the market for this value. Suppose that we sell it and obtain a cash amount of $V(t, s)$. However, this sale also introduces a future liability of $\varphi(S(T))$ at time T. To protect ourselves against this uncertain liability, we invest our in the financial market consisting of only two instruments. Let $\pi(\cdot)$ be our adapted investment strategy, i.e., we invest $\pi(t)$ times of our wealth in the stock and the rest in the bond. Let $x(\cdot)$ be the resulting wealth process. Then, x solves the equation (2.2) with no consumption:

$$(6.1) \qquad dx(\rho) = x(\rho)[(r + \pi(\rho)\ [\mu - r])d\rho + \pi(\rho)\sigma dw(\rho)].$$

with the initial condition $x(t) = V(t, s)$. Arbitrage arguments imply that the value of the derivative must be equal to the wealth process with an appropriate investment strategy $\pi(\cdot)$. Since the pricing equation will be derived rigorously two different ways, we omit the proof of this statement and refer to the book of Karatzas & Shreve [KS5] for its proof. Then, by the Ito formula

$$dV(\rho, S(\rho)) = [V_t + \mu S(\rho)V_s + \frac{\sigma^2 S^2(\rho)}{2} V_{ss}]d\rho + \sigma S(\rho)V_s dw(\rho).$$

We argued that $V(\rho, S(\rho)) = x(\rho)$ for all $\rho \in [t, T]$. We now equate the drift and the diffusion terms in the above and (6.1). The result is

$$\sigma S(\rho)V_s(\rho, S(\rho)) = \sigma\pi(\rho)x(\rho),$$

$$[V_t + \mu S(\rho)V_s + \frac{\sigma^2 S^2(\rho)}{2} V_{ss}](\rho, S(\rho)) = [r + \pi(\rho)\ (\mu - r)]x(\rho).$$

Since $V(\rho, S(\rho)) = x(\rho)$, above equations imply that

$$\pi(\rho) = \frac{S(\rho) V_s(\rho, S(\rho))}{V(\rho, S(\rho))}, \tag{6.2}$$

$$V_t(t,s) + rsV_s(t,s) + \frac{\sigma^2 s^2}{2} V_{ss}(t,s) = rV(t,s), \tag{6.3}$$

for $s > 0, t < T$ together with the final condition

$$V(T,s) = \varphi(s). \tag{6.4}$$

The equation (6.3) is known as the *Black-Scholes equation* for derivative pricing. It has the following Feynman-Kac representation,

$$V(t,s) = e^{-r(T-t)}\ E[\varphi(\hat{S}(T)) \mid \hat{S}(t) = s], \tag{6.5}$$

where

$$d\hat{S} = \hat{S}(\rho)[rd\rho + \sigma dw(\rho)].$$

See [KS5, Sec. 2.4], also [KS4, Sec. 5.8.B][MR, Sec. 5.1.5]. Note that the mean return rate coefficient μ does not influence the Black-Scholes price. Also in the case of a Call option the above expression can be computed using the standard error function (c.f. Hull [Hu].)

X.7 Utility pricing

The Black-Scholes arbitrage pricing is not available in incomplete markets or in markets with friction. One has to bring in the risk preferences of the investor in the pricing theory. Hodges & Neuberger [HN] proposed a utility based theory to price derivatives. This approach was later developed by Davis, Panas & Zariphopoulou [DPZ] and applied to models with transactions costs. Another application is to markets with both traded and non-traded securities [Z4][MZ][Hen]. We will briefly describe this method and show that in the financial market described in Section 2, it yields the Black-Scholes price.

Suppose that an investor at time $t < T$ has an initial wealth of x and can offer the possibility of buying one share of a derivative for a price of v dollars. So she has two alternatives; one is to sell the derivative. In this case her wealth will increase to $x + v$. She will optimally invest this in the market. However, at time T she has to pay $\varphi(S(T))$ dollars from her portfolio. In the second alternative, she does not sell the derivative and invest x dollars in the market.

Hodges and Neuberger assume that there is no consumption and the investor is trying to maximize the utility she gets from her wealth at time T. Let ℓ be her utility function satisfying the assumptions (4.1). Mathematically let

$$V_f(t,x) := \sup_{\pi(\cdot)}\ E[\ell(x(T)) \mid x(t) = x], \tag{7.1}$$

$$(7.2)\qquad \hat{V}(t,x,s;v) := \sup_{\pi(\cdot)} E[\ell(x(T) - \varphi(S(T)) \mid x(t) = x + v, S(t) = s],$$

where in both cases the wealth process $x(t)$ solves the equation (6.1). Description of admissible controls requires a lower bound on the wealth process which needs to be specified. The exact nature of this condition depends on the financial model. Here we assume that admissible controls are required to satisfy $x(\cdot) \geq 0$. We refer to Davis, Panas & Zariphopoulou [DPZ] for the formulation in a market with proportional transaction costs.

If we assume that the utility function ℓ is unbounded, then for large v, $V_f < \hat{V}$ and she will choose to sell the derivative for a price of v. On the other hand, $\hat{V} < V_f$ when $v = 0$. So given (t,x,s), there exists a solution of the algebraic equation

$$V_f(t,x) = \hat{V}(t,x,s;v).$$

Since the dependence of $\hat{V}$ on v is strictly increasing, the solution is unique. Hodges and Neuberger call this solution the *utility price.*

Theorem 7.1. *In the financial market of Section 2 (without consumption), for any* (t,s,x) *the utility price of a derivative is equal to the Black-Scholes price given* (6.5).

Proof. When consumption $c(t)$ is omitted from the model in Section 2, equation (2.2) becomes (6.1). Let x and π be as in (6.1). For the purposes of this proof, it is more convenient to work with $\alpha := x\ \pi$. With this notation, (6.1) has the form

$$(7.3)\qquad dx = [rx(\rho) + \alpha(\rho)(\mu - r)]d\rho + \alpha(\rho)\sigma dw(\rho).$$

For a given portfolio $\alpha(\cdot)$ and an initial condition $x(t) = \xi$, let $x_\xi^\alpha(\cdot)$ be the solution of (7.3). By the linearity of the state equation (7.3),

$$(7.4)\qquad x_{\xi_1}^{\alpha_1}(\cdot) + x_{\xi_2}^{\alpha_2}(\cdot) = x_{\xi_1+\xi_2}^{\alpha_1+\alpha_2}(\cdot).$$

Let π_{BS} be as in (6.2). Define $\alpha_{BS}(\rho) := \pi_{BS}(\rho)\ V(\rho, S(\rho))$. By the linearity (7.4), α is an admissible portfolio starting from (t,x) if and only if $\alpha + \alpha_{BS}$ is admissible starting from $(t, x + V(t,s))$. Indeed by construction,

$$x_{V(t,s)}^{\alpha_{BS}}(\cdot) = V(\cdot, S(\cdot)).$$

Hence by (7.4) and (6.4),

$$x_{x+V(t,s)}^{\alpha+\alpha_{BS}}(T) - \varphi(S(T) = x_x^\alpha(T) + x_{V(t,s)}^{\alpha_{BS}}(T) - \varphi(S(T)) = x_x^\alpha(T).$$

Hence,

$$V_f(t,x) = \hat{V}(t,x,s;V(t,s)).$$

□

Remark 7.1. In the above proof the chief property of the financial model is the linearity (7.4). Hence the result extends to a more general "linear" cases easily (cf. Hodges & Neuberger [HN]).

A striking feature of the above result is that the resulting price is independent of the utility function. In fact, no particular form of the function ℓ is used in the proof.

In general, the utility price depends on the utility function and also it depends on the current wealth of the investor and that it is not linear, i.e., in general the utility price of n shares of the derivative is not necessarily equal to n times the price of one share. The dependence of the utility function is natural, as in incomplete markets there are many possibilities and the investor chooses the price according to her risk profile. However, linearity is a desired property and several solutions for it were proposed. In a market with proportional transaction costs, Barles & Soner [BaS] overcomes this difficulty through an asymptotic analysis.

Since in markets with friction (markets with portfolio constraints or with transaction costs) arbitrage pricing is not available, utility price provides an interesting method of pricing. One alternate approach to pricing will be considered in the next section. □

X.8 Super-replication with portfolio constraints

In this section, we introduce the concept of super-replication in the particular problem of portfolio constraints. However, the method of super-replication applies to other models as well.

The portfolio constraint is simply a bound on the control parameter π in the equation (6.1). We assume that $\pi(t)$ satisfies

$$-a \le \pi(t) \le b, \quad \forall t \ge 0, \tag{8.1}$$

where a and b are positive constants and we assume that $b > 1$. Since $\pi(t)$ denotes the fraction of the wealth invested in the stock, the constraint $\pi(t) \le b$ implies that the investor is not allowed to borrow (or equivalently, short-sell the bond) more than $(b-1)$ times her wealth. The constraint $-a \le \pi(t)$ puts a restriction on short-selling the stock. The investor is not allowed to shortsell stocks of value more than a times her wealth.

Let $x^{\pi}_{t,x}(\cdot)$ be the solution of (6.1) with initial data $x^{\pi}_{t,x}(t) = x$ and $S_{t,s}(\cdot)$ be the solution of the stock equation with initial condition $S_{t,s}(t) = s$. Let $\mathcal{A}$ be the set of all adapted portfolio processes π satisfying the constraint (8.1).

Given a derivative with pay-off φ and maturity T, the minimal super-replication cost is given by

$$\bar{V}(t,s) := \inf\{x \mid \exists \pi(\cdot) \in \mathcal{A} \;\text{ such that }\; x^{\pi}_{t,x}(T) \ge \varphi(S_{t,s}(T)) \text{ a.s. }\}. \tag{8.2}$$

The above problem is not in the standard form of an optimal control problem. Still, Soner & Touzi [ST1] showed that $\bar V$ satisfies a dynamic programming principle; for any stopping time $\tau \in [t,T]$,

$$\bar V(t,s) := \inf\{x \mid \exists \pi(\cdot) \in \mathcal{A} \ \text{ such that } \ x^{\pi}_{t,x}(\tau) \geq \bar V(\tau, S_{t,s}(\tau)) \text{ a.s. }\}.$$

Hence, the nonlinear semigroup property is also satisfied by this problem. Indeed for any function Ψ, the two-parameter semigroup $\mathcal{T}_{t,r}$ is given by

$$\mathcal{T}_{t,r}\Psi(s) := \inf\{x \mid \exists \pi(\cdot) \in \mathcal{A} \ \text{ such that } \ x^{\pi}_{t,x}(r) \geq \Psi(S_{t,s}(r)) \text{ a.s. }\}.$$

Then, $\bar V(t,s) = \mathcal{T}_{t,T}\varphi(s)$ and the dynamic programming formally implies the semigroup property II(3.3). So the viscosity theory for $\bar V$ can be developed as in Chapter II. We refer to [ST1] for this theory. Here we only state (without proof) that the infinitesimal operator in II(3.11) is

$$\mathcal{G}_t\phi(s) = \begin{cases} \mathcal{L}\Phi(s), & \text{if } \dfrac{s\Phi_s(s)}{\Phi(s)} \in [-a,b], \\ \infty, & \text{otherwise}, \end{cases} \tag{8.3}$$

where

$$\mathcal{L}\Phi(s) = \frac{\sigma^2 s^2}{2}\Phi_{ss}(s) + rs\Phi_s(s) - r\Phi(s). \tag{8.4}$$

We refer to [S4] for an analysis of this problem.

X.9 Buyer's price and the no-arbitrage interval

In the previous subsection, we considered the problem from the perspective of the writer of the option. For a potential buyer, there is a different possibility of arbitrage, if the quoted price x of a certain claim is low. She would take advantage of a low price by buying the option for a price x. She would finance this purchase by using the instruments in the market. Mathematically she tries to optimize her wealth (or minimize her debt)with initial wealth of $-\,x$. If at maturity,

$$x^{\pi}_{t,-x}(T) + \varphi(S_{t,s}(T)) \geq 0, \quad a.s.,$$

then this provides arbitrage. Hence the largest of these initial data provides the lower bound of all prices that do not allow arbitrage. So we define (after observing that $x^{\pi}_{t,-x}(T) = -x^{\pi}_{t,x}(T)$),

$$\underline{V}(t,s) := \sup\{x \mid \exists \pi(\cdot) \in \mathcal{A} \ \text{ such that } \ x^{\pi}_{t,x}(T) \leq \varphi(S_{t,s}(T)) \text{ a.s. }\}.$$

Then, the *no-arbitrage interval* is given by

$$\underline{V}(t,s), \bar V(t,s)]\ .$$

with $\bar V(t,s)$ as in (8.2).

This approach can be applied to other markets. In general, there are many approaches to pricing and the above interval must contain all the prices obtained by any method.

X.10 Portfolio constraints and duality

In general for solving super-replication problems, two approaches are available. In the first, after a clever duality argument, the problem is transformed into a standard optimal control problem and then solved by dynamic programming. We refer to lecture notes of Rogers [Rog] for an introduction. In the second approach, dynamic programming is used directly. Although, when available the first approach provides more insight, it is not always possible to apply the dual method. For instance, the problem with Gamma constraint (c.f. [CST]) is an example for which the dual method is not yet known. The second approach is a direct one and applicable to all super-replication problems.

In this section, we solve the super-replication problem through a probabilistic approach. We refer to Karatzas & Shreve [KS5] for details of the material covered in this section.

Let us recall the problem briefly. We consider a market with one stock and one bond. By multiplying $e^{r(T-t)}$ we may take $r = 0$, (or equivalently taking the bond as the numeriare). Also by a Girsanov transformation, we may take $\mu = 0$. So the resulting simpler equations for the stock price and wealth processes are

$$dS(t) = \sigma S(t)dW(t) \; ,$$

$$dx(t) = \sigma\pi(t)x(t)dw(t) \; .$$

Suppose a derivative with payoff $\varphi : [0,\infty) \to [0,\infty)$ is given. The minimal super-replication cost (from the seller's point of view) is as in (8.2)

$$\bar{V}(t,s) = \inf\{x : \exists\pi(\cdot) \in \mathcal{A} \;\text{ s.t. } x^{\pi}_{t,x}(T) \geq \varphi(S_{t,s}(T)) \text{ a.s. }\},$$

where $\mathcal{A}$ is the set of all essentially bounded, adapted processes $\pi(\cdot)$ with values in a convex set $K = [-a,b]$.

Black-Scholes case. Let us start with the unconstrained case, $K = (-\infty,\infty)$. Since $x^{\pi}_{t,x}(\cdot)$ is a martingale, if there is x and a super-replicating portfolio $\pi(\cdot) \in \mathcal{A}$, then

$$x = x^{\pi}_{t,x}(t) = E[x^{\pi}_{t,x}(T) \mid \mathcal{F}_t] \geq E[\varphi(S_{t,s}(T)) \mid \mathcal{F}_t] := x_0.$$

Our claim is that indeed the above inequality is an equality for $x = x_0$. Set

$$Y(\rho) := E[\varphi(S_{t,s}(T)) \mid \mathcal{F}_\rho] \; .$$

By the martingale representation theorem, $Y(\cdot)$ is a stochastic integral. We choose to write it as

$$Y(\rho) = E[\varphi(S_{t,s}(T))|\, \mathcal{F}_t] + \int_t^\rho \sigma\pi^*(\rho')Y(\rho')dw(\rho') \; ,$$

with an appropriate $\pi^*(\cdot) \in \mathcal{A}$. Then, $Y(\cdot) = x^{\pi^*}_{t,x_0}(\cdot)$. Hence, $\bar{V}(t,s) \geq x_0$. But we have already shown that if an initial capital supports a super-replicating portfolio then, it must be larger than x_0. Hence,

$$\bar{V}(t,s) = x_0 = E[\varphi(S_{t,s}(T)) \mid \mathcal{F}_t] := V(t,s) \ ,$$

which is the Black-Scholes price already computed in Section 6. Formula (6.5) differs from above as we have already discounted the prices. From the above formula we can obtain (6.5) after appropriate discounting. Finally we note that in this case, starting with x_0, there always exists a replicating portfolio.

In this example, it can be shown that the buyer's price is also equal to the Black-Scholes price $v_{BS} = V(t,s)$. Hence, the no-arbitrage interval defined in the previous subsection is the singleton $\{v^{BS}\}$. Thus, that is the only fair price.

General Case. Let us first introduce several elementary facts from convex analysis (c.f. [R1]). Set

$$\delta_K(\nu) := \sup_{\pi \in K} -\pi\nu, \quad \tilde{K} := \{\nu : \quad \delta_K(\nu) < \infty\} \ .$$

In the convex analysis literature, δ_K is the support function of the convex set K. In one dimension, we may directly calculate these functions. However, we use this notation, as it is suggestive of the multidimensional case. Then, it is a classical fact that

$$\pi \in K \Leftrightarrow \pi\nu + \delta_K(\nu) \geq 0 \quad \forall \nu \in \tilde{K} \ .$$

Let $x, \pi(\cdot)$ be an initial capital, and respectively, a super-replicating portfolio. For any $\nu(\cdot)$ with values in $\tilde{K}$, let P^ν be such that

$$w^\nu(u) := w(u) + \int_t^u \frac{\nu(\rho)}{\sigma} d\rho$$

is a P^ν martingale. This measure exists under integrability conditions on $\nu(\cdot)$, by the Girsanov theorem. Here we assume essentially bounded processes, so P^ν exits. Set

$$\tilde{x}(u) := x^\pi_{t,x}(u) \ \exp(-\int_t^u \delta_K(\nu(\rho))d\rho).$$

By calculus,

$$d\tilde{x}(\rho) = \tilde{x}(\rho)[-(\delta_K(\nu(\rho)) + \pi(\rho)\nu(\rho))d\rho + \sigma dw^\nu(\rho)].$$

Since $\pi(\rho) \in K$ and $\nu(\rho) \in \tilde{K}$, $\delta_K(\nu(\rho) + \pi(\rho)\nu(\rho) \geq 0$ for all ρ. Therefore, $\tilde{x}(\rho)$ is a super-martingale and

$$E^\nu[\tilde{x}(T) \mid \mathcal{F}_t] \leq \tilde{x}(t) = x^\pi_{t,x}(t) = x.$$

Also $x^\pi_{t,x}(T) \geq \varphi(S_{t,s}(T))$ $P-$a.s, and therefore, P^ν-a.s. as well. Hence,

$$\begin{aligned}\tilde{x}(T) &= \exp(-\int_t^T \delta_K(\nu(\rho))d\rho) \ x^\pi_{t,x}(T) \\ &\geq \exp(-\int_t^T \delta_K(\nu(\rho))d\rho) \ \varphi(S_{t,s}(T)) \quad P^\nu - \text{a.s.} \ .\end{aligned}$$

All of these together yield,

$$\bar{V}(t,s) \geq x^\nu := E^\nu[\exp(-\int_t^T \delta_K(\nu(\rho))d\rho)\varphi(S_{t,s}(T)) \mid \mathcal{F}_t] \ .$$

Since this holds for any $\nu(\cdot) \in \tilde{K}$,

$$\bar{V}(t,s) \geq \sup_{\nu \in \tilde{K}} x^\nu \ .$$

The equality is obtained through a super-martingale representation for the right hand side, c.f. Karatzas & Shreve [KS5]. The final result is

Theorem 10.1 [CK]. *The minimal super-replicating cost $\bar{V}(t,s)$ is the value function of the standard optimal control problem,*

$$\bar{V}(t,s) \ = \ E\left[\exp(-\int_t^T \delta_K(\nu(\rho))d\rho) \ \varphi(S^\nu_{t,s}(T)) \mid \mathcal{F}_t\right] \ ,$$

where $S^\nu_{t,s}$ solves

$$dS^\nu_{t,s} = S^\nu_{t,s}(\rho) \ [-\nu(\rho)d\rho + \sigma dw(\rho)] \ .$$

Now this problem can be solved by dynamic programming. Indeed, an explicit solution was obtained by Broadie, Cvitanic & Soner [BCS]. Here we state it without proof.

Theorem 10.2 [BCS]. *The minimal super-replicating cost $\bar{V}(t,s)$ is equal to the Black-Scholes price with a modified pay-off $\hat{\varphi}$ given by*

$$\hat{\varphi}(s) := \sup_{\nu \in \tilde{K}} \{ \ e^{-\delta_K(\nu)} \ \varphi(e^\nu s) \ \}.$$

Then,

$$\bar{V}(t,s) = E[\ \hat{\varphi}(S_{t,s}(T) \ ].$$

Recall that we are assuming that the interest rate is zero. One needs to discount the above formula appropriately to obtain the formula in models with non-zero interest rate.

In this problem, buyer's price can be calculated as above.

X.11 Merton problem with random parameters

In the Merton problem formulated in Section 3, the riskless interest rate r, the risky mean rate of return μ and the volatility coefficient σ are assumed to be constant. However, this assumption is unrealistic, particularly over long time intervals. Instead, we now assume that these parameters may depend on some "economic factor" $\eta(t)$ which is a Markov diffusion process. Instead of equation (2.2), the wealth $x(t)$ now satisfies

$$(11.1) \qquad dx = x(t)\left\{(r(t) + \pi(t)[\mu(t) - r(t)])\right\}dt + \pi(t)\sigma(t)dw(t) - c(t)dt,$$

$$(11.2) \qquad r(t) = R(\eta(t)), \quad \mu(t) = M(\eta(t)), \quad \sigma(t) = \Sigma(\eta(t)).$$

It is assumed that $\eta(t)$ satisfies the stochastic differential equation

$$(11.3) \qquad d\eta = g(\eta(t))dt + \tilde{\sigma}\left[\lambda dw(t) + (1-\lambda^2)^{\frac{1}{2}} d\tilde{w}(t)\right]$$

where $\tilde{\sigma}, \lambda$ are constants with $|\lambda| < 1$ and $\tilde{w}(t)$ is a Brownian motion independent of $w(t)$. The constant λ has the role of a correlation coefficient. As in Section 3, the goal is to choose investment and consumption control processes $\pi(t)$, $c(t)$ to maximize (3.1). We assume that $\ell(c) = \gamma^{-1}c^{\gamma}$ is HARA with $0 < \gamma < 1$.

This problem is an infinite time horizon stochastic control problem, with two dimensional state $(x(t), \eta(t))$ governed by equations (11.1), (11.3) and with controls $\pi(t)$, $c(t)$. In the notation of Section IV.5, we have $O = (0,\infty) \times (-\infty,\infty)$ and $U = (-\infty,\infty) \times [0,\infty)$. We seek a solution $W(x,\eta)$ to the dynamic programming equation of the form

$$(11.4) \qquad W(x,\eta) = \gamma^{-1}K(\eta)x^{\gamma}, \quad K(\eta) > 0,$$

where $x = x(0)$, $\eta = \eta(0)$. By substituting in the dynamic programming equation IV(5.8), $K(\eta)$ must satisfy the differential equation

$$(11.5) \qquad \beta K = g(\eta)K_\eta + \frac{\tilde{\sigma}^2}{2}K_{\eta\eta} + \max_c[-c\gamma K + c^\gamma]$$
$$+\gamma \max_\pi \left\{ \left[R(\eta) + (M(\eta) - R(\eta))\pi + \frac{\gamma-1}{2}\pi^2\Sigma^2(\eta)\right]K \right.$$
$$\left. +\tilde{\sigma}\lambda\pi\Sigma(\eta)K_\eta \right\}$$

This can more conveniently be written in the following logarithmic form. Let $Z(\eta) = \log K(\eta)$. A routine calculation shows that $Z(\eta)$ must satisfy

$$(11.6) \qquad \beta = \bar{g}(\eta)Z_\eta + \frac{\tilde{\sigma}^2}{2}Z_{\eta\eta} + \frac{\bar{\sigma}^2}{2}Z_\eta^2 + (1-\gamma)e^{\frac{Z}{\gamma-1}} + \gamma Q(\eta)$$

$$(11.7) \qquad \bar{g}(\eta) = g(\eta) + \frac{\gamma\tilde{\sigma}\lambda(M(\eta) - R(\eta))}{(1-\gamma)\Sigma(\eta)}$$

$$(11.8) \qquad \bar{\sigma} = \tilde{\sigma}\left[\frac{1-\gamma(1-\lambda^2)}{1-\gamma}\right]^{\frac{1}{2}}$$

(11.9) $$Q(\eta) = R(\eta) + \frac{(M(\eta) - R(\eta))^2}{2(1-\gamma)\Sigma^2(\eta)}.$$

By taking argmax in (11.5), one obtains the following candidates for optimal control policies:

(11.10) $$\underline{\pi}^*(\eta) = \frac{M(\eta) - R(\eta)}{(1-\gamma)\Sigma^2(\eta)} + \frac{\tilde{\sigma}\lambda Z_\eta(\eta)}{(1-\gamma)\Sigma(\eta)}$$

(11.11) $$\underline{c}^*(x,\eta) = x\, e^{\frac{Z(\eta)}{\gamma-1}}.$$

We now make the following assumptions, under which the dynamic programming formalism just outlined indeed gives a solution to the Merton portfolio optimization problem with random parameters. We assume that:

(i) $R(\eta)$, $M(\eta)$, $\Sigma(\eta)$ are in $C_b^1(\mathbb{R}^n)$, and $\Sigma(\eta) \geq \underline{\Sigma} > 0$;

(11.12) (ii) $\beta - \gamma Q(\eta) \geq \underline{\beta} > 0$ for all η;

(iii) There exist $\eta_1,\ a_1 > 0$ such that $\bar{g}_\eta(\eta) \leq -a_1$ whenever $|\eta| \geq \eta_1$.

We note in particular that $\bar{g}(\eta) = g(\eta)$ in the "uncorrelated" case $\lambda = 0$. When $\lambda = 0$, formula (11.10) for the optimal investment policy $\underline{\pi}^*(\eta)$ has the same form as IV(5.25) for the classical Merton problem. However, in this formula the constants $r,\ \mu,\ \sigma$ are now replaced by $R(\eta)$, $M(\eta)$, $\Sigma(\eta)$. We also note that (11.12) (iii) is implied by the following:

(11.12) (iii′) $R_\eta(\eta)$, $M_\eta(\eta)$, $\Sigma_\eta(\eta)$ tend to 0 as $|\eta| \to \infty$ and there exist $\zeta_2, a_2 > 0$ such that $g_\eta(\eta) \leq -a_2$ whenever $|\eta| \geq \eta_2$.

Condition (11.12)(iii′) implies that the factor process $\eta(t)$ is ergodic.

Theorem 11.1. *Equation* (11.6) *has a unique solution* $Z \in C^3(\mathbb{R}^1)$ *such that* $Z(\eta)$ *is bounded and* $Z_\eta(\eta)$ *tends to* 0 *as* $|\eta| \to \infty$. *Moreover,* $W(x,\eta) = \gamma^{-1}x^\gamma e^{Z(\eta)}$ *is the value function, and* $\underline{\pi}^*, \underline{c}^*$ *in* (11.10),(11.11) *are optimal control policies.*

Proof. (Sketch) We first find K^+, K^- which are constant super- and subsolutions to (11.6). Let $\underline{\beta}$ be as in (11.12)(ii) and choose $\bar{\beta}$ such that $\beta - \gamma Q(\eta) \leq \bar{\beta}$ for all ζ. We take

$$Z^+ = (\gamma - 1)\log[\underline{\beta}(1-\gamma)^{-1}]$$

$$Z^- = (\gamma - 1)\log[\bar{\beta}(1-\gamma)^{-1}].$$

Standard arguments then imply the existence of a solution Z to (11.6) with $Z^- \leq Z(\eta) \leq Z^+$. See [FP][KSh1]. Let us show that $Z_\eta(\eta)$ tends to zero as

$n \to \infty$. The same argument applies as $\eta \to -\infty$. Suppose that $Z_\eta(\eta)$ has a positive limsup as $\eta \to \infty$. Since $Z(\eta)$ is bounded, the liminf of $|Z_\eta(\eta)|$ as $\eta \to \infty$ is 0. Hence there is a sequence η_n tending to infinity such that $Z_\eta(\eta)$ has a local max at η_n with $Z_\eta(\eta_n) \geq \delta > 0$. We differentiate with respect to η in (11.6) and use $Z_{\eta\eta}(\eta_n) = 0$, $Z_{\eta\eta\eta}(\eta_n) \leq 0$ to obtain for large n

$$\frac{a_1}{2} Z_\eta(\eta_n) \leq \left(-\bar{g}_\eta(\eta_n) + e^{\frac{Z(\eta_n)}{\gamma-1}}\right) Z_\eta(\eta_n) \leq \gamma Q_\eta(\eta_n).$$

Since Q_η is bounded, $Z_\eta(\eta_n)$ is bounded. Then from (11.6), $\bar{g}(\eta_n) Z_\eta(\eta_n)$ is bounded. Since $\bar{g}(\eta_n) \to -\infty$ as $n \to \infty$, $Z_\eta(\eta_n) \to 0$. This is a contradiction. Similarly, $Z_\eta(\eta)$ cannot have a negative liminf as $\eta \to \infty$.

To show that $W(x, \eta)$ is the value function and $\underline{\pi}^*, \underline{c}^*$ are optimal control policies, we use the same verification argument as for the constant parameter case (Example IV.5.2). Note that $\underline{\pi}^*(\eta)$ is bounded and that $x^{-1}\underline{c}^*(x, \eta)$ has positive upper and lower bounds since $Z(\eta)$ is bounded. This verification argument also implies uniqueness of Z. □

Remark 11.1. The sub- and super-solution technique sketched in the proof above can be used under less restrictive assumptions than (11.12). In [FP] the case when μ, σ are constant but $R(\eta) = \eta$. Thus, the interest rate $\eta(t) = r(t)$ varies according to (11.3). Under the assumptions in [FP], an unbounded solution $Z(\zeta)$ to (11.6) is obtained with polynomial growth rate as $|\zeta| \to \infty$. The verification argument in this case is considerably more complicated. In [FHH2] the case when r, μ are constant, but $\sigma(t) = \Sigma(\eta(t))$ varies is considered.

Merton terminal wealth problem with random parameters. Another problem which is often considered is the following. Suppose that consumption is omitted from the model $(c(t)) = 0$ in equation (11.1), and that the factor $\eta(t)$ again satisfies (11.3). The goal is to choose the investment control on a finite time interval $0 \leq t \leq T$ to maximize the expected HARA utility of wealth $\gamma^{-1} E(x(T)^\gamma)$ at the final time T. The value function V is of the form

$$V(T, x, \eta) = \gamma^{-1} x^\gamma \Psi(T, \eta), \tag{11.13}$$

where as in (11.5) $\Psi(T, \eta)$ must satisfy the PDE

$$\frac{\partial \Psi}{\partial T} = g(\eta)\Psi_\eta + \frac{\tilde{\sigma}^2}{2}\Psi_{\eta\eta} + \max_\pi \left[f(\eta, \pi)\Psi_\eta + \gamma \ell(\eta, \pi)\Psi\right] \tag{11.14}$$

$$\begin{cases} f(\eta, \pi) = g(\eta) + \gamma \tilde{\sigma} \lambda \pi \Sigma(\eta) \\ \ell(\eta, \pi) = \gamma \left[R(\eta) + (M(\eta) - R(\eta))\pi + \frac{\gamma-1}{2}\pi^2 \Sigma^2(\eta)\right]. \end{cases} \tag{11.15}$$

$\Psi(T, \eta)$ is the value function for a risk sensitive stochastic control problem of the kind considered in Section VI.6.8. To see this we use a Girsanov transformation, as in [FSh1][FHH2]. Let

$$\zeta(t) = \gamma \int_0^t \pi(s)\Sigma(\eta(s))dw(s) - \frac{1}{2}\gamma^2 \int_0^t \pi^2(s)\Sigma^2(\eta(s))ds$$

$$\hat{w}(t) = w(t) - \gamma \int_0^t \pi(s)\Sigma(\eta(s))ds.$$

Then $\hat{w}(t)$ and $\tilde{w}(t)$ are independent Brownian motions under the probability measure $\hat{P}$ such that $d\hat{P} = \exp(\zeta(T))dP$. Then (11.3) becomes

$$d\eta = f(\eta(t)), \pi(t))dt + \tilde{\sigma}dz(t)) \tag{11.16}$$

with $z(t) = \lambda d\hat{w}(t) + \sqrt{1-\lambda^2}d\tilde{w}(t)$ a Brownian motion. A calculation using (11.1) and the Ito differential rule gives

$$E(x(T)^\gamma) = \hat{E}\left[\exp \int_0^T \ell(\eta(t), \pi(t))dt\right] \tag{11.17}$$

with $\hat{E}$ the expectation under $\hat{P}$.

In the notation of Section VI.8, let $\rho = 1$ and let $\Phi(\bar{t}, \eta; t_1)$ denote the value function of the risk sensitive control problem on an interval $\bar{t} \le t \le t_1$ with initial data $\eta(\bar{t}) = \eta$. Note that the control $\pi(t)$ is maximizing, rather than minimizing as in Section VI.8. Since $f(\eta, \pi), \ell(\eta, \pi)$ do not depend on time, $\Phi(\bar{t}, \eta; \bar{t}_1) = \Psi(t_1 - \bar{t}, \eta)$. The dynamic programming equation VI(8.5) reduces to (11.14).

The Merton terminal wealth problem, and associated risk sensitive control problems, have been considered by several authors, including [BP][FSh3][KSh3][Na2]. In [Z4] it is shown that (11.14) can be transformed into a linear PDE for $\psi(T, \eta)$, where $\Psi = \psi^\delta$ and $\delta = (1-\gamma+\lambda^2\gamma)^{-1}(1-\gamma)$. In [FPS, Chap. 10] an approximate solution is found in case of stochastic volatility (r, μ constant), using a perturbation analysis. In [FPS] volatility is assumed mean-reverting on a "fast" time scale.

If $R(\eta), M(\eta), g(\eta)$ are linear functions and σ is constant, then the risk sensitive control problem is a LEQR problem considered in Example VI.8.1, with min replaced by max. In this case, the problem has an explicit solution [BP][FSh3][KSh3].

X.12 Historical remarks

The classical references to the Merton problem are [Mer1], [Mer2]. For the general utility functions, we refer to [KLSS], [Pl1] and the textbooks [KS5] and [Pl2]. For option pricing we refer to the original paper by Black & Scholes [BSc] and the textbooks [KS5] and [Hu].

The portfolio selection problem with transaction costs was first studied by Constantinides [Co], then by Davis and Norman [DN] who solved the problem under an implicit assumption on the coefficients. Later Shreve and Soner [ShS] studied the problem only assuming that the value function is finite. A similar problem with no consumption was analyzed by Taksar, Klass and Assaf [TKA]. The deterministic model was "explicitly" solved by Shreve et al. [SSX] with a general utility function which is not necessarily a HARA type function. For the utility based approach, we refer to Hodges & Neuberger [HN]. An interesting application of this approach with proportional transaction costs is given by Davis, Panas and Zariphopoulou [DPZ]. An asymptotic analysis of this problem is given in [BaS], [WW] and more recently in [JS]. For other related models we refer to Zariphopoulou [Z1], [Z2] and [FZ]. We refer to [KS5] and [ST2] and the references therein for results on super-replication. References for the model with random coefficients are given in Section 11: [FHH2], [FP], [FPS], [BP], [FSh3], [KSh3], [Na2].

XI

Differential Games

XI.1 Introduction

In this chapter we give a concise introduction to zero-sum differential games. The game dynamics are governed by ordinary differential equations, which are affected by two controllers with opposite goals. One controller chooses at time s a control $u(s)$ with the goal of minimizing the game payoff P. The other controller chooses a control $z(s)$ with the goal of maximizing P. The notion of value function has a key role in the theory of differential games. In his pioneering work, R. Isaacs [Is] introduced corresponding nonlinear first-order PDEs which are now called Isaacs equations. As for the HJB equations of deterministic control (Chapters I,II), solutions to Isaacs PDEs are typically not smooth and should be considered in the viscosity sense.

We will consider only differential games on a fixed time interval $t \le s \le t_1$. Rather strong assumptions will be made about the game dynamics and payoff, to simplify the presentation. See (3.4). We begin in Section 2 with the definition of upper and lower values V_+, V_- of a static game. The upper value V_+ is obtained if the minimizing control is chosen first, and V_- is obtained if the maximizing control is chosen first. In Section 3, a description of differential games on $[t, t_1]$ is given, and PDEs (3.6), (3.9) for the upper and lower game values $V_+(t,x)$, $V_-(t,x)$ are derived formally. For discrete-time dynamic games, upper and lower value functions are easily defined, by specifying which controller chooses first at each step. However, this is not so in continuous time, since control choices can be changed instantaneously. In Section 4 we define upper and lower value functions using the idea of progressive strategy, in a way similar to Elliott-Kalton [ElKa]. The upper and lower value functions V_+, V_- are shown to satisfy dynamic programming principles (5.1), (5.4). Then V_+ and V_- are shown in Section 6 to be the unique bounded, Lipschitz continuous viscosity solutions to the upper and lower Isaacs PDEs with given terminal data. When the Isaacs minimax condition (3.11) holds upper and lower game values are equal ($V_+ = V_-$).

In Section 7, we consider differential games which arise as small noise limits from risk-sensitive stochastic control problems on a finite time interval. The upper game value $V_+(t,x)$ is the limit of the minimum certainty-equivalent expected cost in the risk sensitive control problem.

At an intuitive level, a progressive strategy α allows the minimizing controller to choose $u(s)$ knowing $z(r)$ for $r \leq s$. Such a strategy α can be chosen which guarantees payoff P no more than the lower value $V_-(t,x)$ plus an arbitrarily small constant. In Section 9, the smaller class of strictly progressive strategies is introduced, which corresponds intuitively to excluding knowledge of $z(s)$ when $u(s)$ is chosen. With strictly progressive α, the upper value $V_+(t,x)$ is obtained (Theorem 9.1). We call the difference $V_+(t,x) - V_-(t,x)$ an instantaneous information gap in game value. Our proof of Theorem 9.1 depends on time discretizations. A different proof is given in [KSh2].

This chapter depends on viscosity solution results in Chapter II, and some ideas from Chapter I. Except for Theorem 7.2, it may be read independently of the other chapters.

XI.2 Static games

Let us first review some basic concepts concerning zero sum games. Let $\mathcal{U}$ and $\mathcal{Z}$ be two sets, and $P(u,z)$ a bounded function on $\mathcal{U} \times \mathcal{Z}$ called the game payoff. There are two controllers, one of which chooses $u \in \mathcal{U}$ and wishes to minimize P. The other controller chooses $z \in \mathcal{Z}$ and wishes to maximize P. (In the game theory literature, the term "player" rather than "controller" is frequently used. See for instance [Au][O][PZ].) To complete the description of the game, the information available to each controller must be specified. We distinguish three cases.

Case 1. The minimizing controller chooses u first, and u is known when the maximizing controller chooses z. Let

$$V_+ = \inf_u \sup_z P(u,z). \tag{2.1}$$

Case 2. The maximizing controller chooses z first, and z is known when the minimizing controller chooses u. Let

$$V_- = \sup_z \inf_u P(u,z). \tag{2.2}$$

V_+ and V_- are called the *upper game value* and *lower game value* respectively. We always have $V_- \leq V_+$. If $V_- = V_+$, then the game is said to have a *saddle point.*

Case 3. Both controllers choose simultaneously. In this case, a game value can be defined by introducing "mixed strategies" which are probability measures on $\mathcal{U}$ and $\mathcal{Z}$. This case is not considered in this book.

The upper game value can be rewritten as follows. Let us call any function β from $\mathcal{U}$ into $\mathcal{Z}$ a *strategy* for the maximizing controller. Since u is known before z is chosen in Case 1, the maximizing controller chooses a strategy β. Then (2.1) can be rewritten as

$$V_+ = \inf_u \sup_\beta P(u, \beta(u)) = \sup_\beta \inf_u P(u, \beta(u)). \tag{2.3}$$

The first equality in (2.3) is just (2.1), taking $z = \beta(u)$. The second equality is a consequence of the following remark: for every $\theta > 0$ there exist u_θ, β_θ, such that

$$P(u_\theta, z) < V_+ + \theta, \quad P(u, \beta_\theta(u)) > V_+ - \theta \tag{2.4}$$

for all $u \in \mathcal{U}$, $z \in \mathcal{Z}$.

In the same way, a function α from $\mathcal{Z}$ into $\mathcal{U}$ is called a strategy for the minimizing controller. We have

$$V_- = \sup_z \inf_\alpha P(\alpha(z), z) = \inf_\alpha \sup_z P(\alpha(z), z). \tag{2.5}$$

XI.3 Differential game formulation

We consider two controller, zero sum differential games on a finite time interval $[t, t_1]$. The state of the differential game at time s is $x(s) \in I\!\!R^n$, which satisfies the differential equation

$$\frac{d}{ds}x(s) = G(s, x(s), u(s), z(s)), \quad t \le s \le t_1, \tag{3.1}$$

with initial data

$$x(t) = x. \tag{3.2}$$

At time s, $u(s)$ is chosen by a minimizing controller and $z(s)$ is chosen by a maximizing controller, with $u(s) \in U$, $z(s) \in Z$. The sets $U,\ Z$ are called the control spaces. The game payoff is

$$P(t, x; u, z) = \int_t^{t_1} L(s, x(s), u(s), z(s))ds + \psi(x(t_1)). \tag{3.3}$$

We make the following assumptions:

(a) $U \subset I\!\!R^{m_1},\ Z \subset I\!\!R^{m_2}$ and U, Z are compact;

(b) G, L are bounded and continuous on $\bar{Q}_0 \times U \times Z$;

(c) There exist constants K_G, K_L such that

$$|G(t,x,u,z) - G(t,y,u,z)| \le K_G|x-y|$$

$$(3.4)\qquad |L(t,x,u,z) - L(t,y,u,z)| \le K_L|x-y|$$

$$\text{for all } t \in [t_0,t_1], \quad x,y \in I\!\!R^n, \quad u \in U, \quad z \in Z;$$

(d) ψ is bounded and Lipschitz continuous.

These assumptions could be relaxed in various ways, which would however complicate the exposition. For example, for differential games which arise in nonlinear H-infinity robust control, it is often assumed that $L(t,x,u,z)$ may grow quadratically in x as $|x| \to \infty$. See [Mc3].

In order to complete the description of the differential game, one must take into account what information is available to the controllers at each time s. To begin with, let us try to do this at an intuitive level. At time s, both controllers are allowed complete information about the past before time s. In addition, one controller is given additional information at time s. At an intuitive level, the maximizing controller has this advantage if $u(s)$ is known before $z(s)$ is chosen. Similarly, the minimizing controller has the advantage if $z(s)$ is known before $u(s)$ is chosen. This idea is easy to make precise for multistage discrete time games. See [F7][Fr2][PZ]. However, it is not easy to do so in continuous time, since control choices can be changed instantaneously. This issue will be addressed in Sections 4 and 9, where the concepts of progressive and strictly progressive strategies are used.

Upper and lower Isaacs PDEs. The pioneering work of R. Isaacs on differential games [Is] makes extensive use of first order PDEs, which are now called *Isaacs equations.* They correspond to the Hamilton-Jacobi-Bellman PDEs considered in Chapters I and II when there is no maximizing control $z(s)$. There are two PDEs, (3.6) and (3.9) below, which are called upper and lower Isaacs equations. Speaking intuitively, these result from giving the information advantage to either the maximizing or the minimizing controller.

To derive formally the upper Isaacs PDE, suppose that an "upper value function" $V_+(t,x)$ has been defined which satisfies a dynamic programming principle. If one could consider only constant controls $u(s) = u, \ z(s) = z$ on a small time interval $t \le s \le t+h$, then formally

$$(3.5)\qquad V_+(t,x) \approx \inf_u \sup_z \left[\int_t^{t+h} L(s,x(s),u,z)ds + V_+(t+h,x(t+h)) \right].$$

Moreover, if V_+ were a smooth function, then

$$V_+(t+h,x(t+h)) \approx V_+(t,x) + \left[\frac{\partial V_+}{\partial t} + D_xV_+ \cdot G(t,x,u,z) \right] h$$

where $\partial V_+/\partial t$ and D_xV_+ are evaluated at (t,x). If we substitute into (3.5), divide by h and let $h \to 0$, we get the *upper Isaacs PDE*:

(3.6) $$0 = -\frac{\partial V_+}{\partial t} + H_+(t, x, D_x V_+)$$

(3.7) $$H_+(t, x, p) = -\min_{u \in U} \max_{z \in Z} [G(t, x, u, z) \cdot p + L(t, x, u, z)].$$

The boundary condition for (3.6) is

(3.8) $$V_+(t_1, x) = \psi(x).$$

The above is merely a formal derivation, not mathematically precise. A rigorous definition of upper value function V_+ is given in Section 4, and V_+ is shown in Section 6 to satisfy the upper Isaacs PDE in the viscosity sense.

In the same way, the *lower Isaacs PDE* is derived formally by giving the minimizing controller the information advantage. This PDE is

(3.9) $$0 = -\frac{\partial V_-}{\partial t} + H_-(t, x, D_x V_-)$$

(3.10) $$H_-(t, x, p) = -\max_{z \in Z} \min_{u \in U} \left[G(t, x, u, z) \cdot p + L(t, x, u, z)\right].$$

The lower value function V_-, as rigorously defined in Section 4, will satisfy (3.9) in the viscosity sense.

In many examples, it happens that

(3.11) $$H_+(t, x, p) = H_-(t, x, p)$$

for all $(t, x) \in \bar{Q}_0,\ p \in I\!R^n$. This is called the *Isaacs minimax condition.* When (3.11) holds, then $V_+ = V_-$. See Corollary 6.1.

Control policies. For differential games, control policies can be introduced in a way similar to Markov control policies in Chapter IV. Consider the upper differential game, in which the maximizing controller has the information advantage. A control policy for the minimizing controller is a function $\underline{u}$ from Q_0 into U. For the maximizer, a control policy is a function $\underline{z}$ from $Q_0 \times U$ into Z. If $\underline{u}$ is Lipschitz continuous, then for any choice of $z(s)$ equation (3.1) has a well defined solution $x(s)$ with $u(s) = \underline{u}(s, x(s))$ and $x(t) = x$. Similarly, if $\underline{z}$ is Lipschitz continuous, the solution $x(s)$ is well defined for each $u(s)$ when $z(s) = \underline{z}(s, x(s), u(s))$.

At a purely formal level, equations (3.6)-(3.7) suggest the following candidates for optimal control policies $\underline{u}^*, \underline{z}^*$. Let

$$F(t, x, u, z) = G(t, x, u, z) \cdot D_x V_+(t, x) + L(t, x, u, z).$$

Choose $\underline{u}^*(t, x)$ and $\underline{z}^*(t, x, u)$ such that

(3.12) $$\underline{u}^*(t, x) \in \operatorname*{argmin}_{u \in U} \max_{z \in Z} F(t, x, u, z)$$

$$\underline{z}^*(t,x,u) \in \operatorname*{argmax}_{z\in Z} F(t,x,u,z) \tag{3.13}$$

Unfortunately, there are technical obstacles to obtaining a general theorem which puts this idea on a mathematically rigorous basis. Typically the upper value function is not of class C^1, and thus $\underline{u}^*$, $\underline{z}^*$ are not well defined. Even if $\underline{u}^*(t,x)$ and $\underline{z}^*(t,x,u)$ were suitably defined at points (t,x) where $D_x V_+$ does not exist, but $\underline{u}^*, \underline{z}^*$ are not Lipschitz, there remains the problem of interpreting (3.1) when these control policies are used.

Control policies have had a limited role in developing a general theory of value for differential games, which is based instead on the idea of strategies. On the other hand, if one wishes to solve differential games in specific examples, then strategies lead to unreasonably large computations and are of little use. Instead, solutions in terms of optimal control policies are sought, using some modified version of the procedure outlined above.

We conclude this section with the following result, in which very restrictive assumptions are made. If $\underline{u}$ is a Lipschitz continuous policy, then we write (3.3) as $P(t,x;\underline{u},z)$, where as above $u(s)=\underline{u}(s,x(s))$. Similarly, if z is Lipschitz continuous, then we write (3.3) as $P(t,x;u,\underline{z})$ where $z(s)=\underline{z}(s,x(s),u(s))$.

Theorem 3.1. *Let $W \in C^1(\bar{Q}_0)$ be a solution to the upper Isaacs PDE* (3.6) *and* (3.8). *Moreover, let $\underline{u}^*, \underline{z}^*$ be Lipschitz continuous and satisfy* (3.12,(3.13)). *Then:*

(a) For every $z(\cdot) \in L^\infty([t,t_1];Z)$

$$P(t,x;\underline{u}^*,z) \le W(t,x); \tag{3.14}$$

For every $u(\cdot) \in L^\infty([t,t_1];U)$

$$W(t,x) \le P(t,x;u,\underline{z}^*). \tag{3.15}$$

Proof. Let $F(t,x,u,z)$ be as above, with V_+ replaced by W. Then

$$F(s,x(s),\underline{u}^*(s,x(s)),z(s)) \le -H_+(s,x(s),D_xW(s,x(s))$$

for all $s\in[t,t_1]$. Then (3.14) follows from (3.6),(3.8) and the fundamental theorem of calculus, in the same way as for Theorem I.5.1. The proof of (3.15) is similar. □

When inequalities (3.14), (3.15) hold we say that the policies $\underline{u}^*$, $\underline{z}^*$ are optimal among Lipschitz continuous control policies, for the differential game corresponding to the upper Isaacs PDE (3.6). Moreover, $W(t,x)$ in Theorem 3.1 turns out to be the same as the upper differential game value $V_+(t,x)$, which is defined in Section 4 using progressive strategies rather than control policies. See Remark 6.1.

XI.4 Upper and lower value functions

In this section we give precise definitions of upper and lower value functions. To introduce the upper value function, we first define the concept of progressive strategy for the maximizing controller. Let

$$\mathcal{U}^0(t,t_1) = L^\infty([t,t_1];U), \qquad \mathcal{Z}^0(t,t_1) = L^\infty([t,t_1];Z).$$

Let $\mathcal{U}(t,t_1) \subset \mathcal{U}^0(t,t_1)$ be a class of functions $u(\cdot)$ with the following properties:

(a) $\mathcal{U}(t,t_1)$ contains all constant functions, $u(s) \equiv u \in U$;

(b) If $t < r < t_1,\ u_1(\cdot) \in \mathcal{U}(t,r),\ u_2(\cdot) \in \mathcal{U}(r,t_1)$, then $u(\cdot) \in \mathcal{U}(t,t_1)$, where

$$u(s) = \begin{cases} u_1(s) & \text{for } t \le s < r \\ u_2(s) & \text{for } r \le s \le t_1. \end{cases} \tag{4.1}$$

(c) Conversely, if $u(\cdot) \in \mathcal{U}(t,t_1)$, then the restrictions $u_1(\cdot), u_2(\cdot)$ are in $\mathcal{U}(t,r), \mathcal{U}(r,t_1)$ respectively.

As ususal, any two functions which agree for almost all $s \in [t,t_1]$ define the same element of $\mathcal{U}(t,t_1)$. In particular, the value of $u(\cdot)$ for the particular time $s = r$ is unimportant. From (4.1), $\mathcal{U}(t,t_1)$ contains all piecewise constant functions $u(\cdot)$. Similarly, let $\mathcal{Z}(t,t_1) \subset \mathcal{Z}^0(t,t_1)$ be a class of functions with the same properties as in (4.1), with $u(t),\ u_1(t),\ u_2(t)$ replaced by $z(t),\ z_1(t), z_2(t)$.

A progressive strategy β for the maximizing controller is a function from $\mathcal{U}(t,t_1)$ into $\mathcal{Z}(t,t_1)$ with the following property: for $t < r < t_1$, $u(s) = \tilde{u}(s)$ for almost all $s \in [t,r]$ implies $\beta(u)(s) = \beta(\tilde{u})(s)$ for almost all $s \in [t,r]$. Let $\Delta(t,t_1)$ denote the class of all progressive strategies β. The *upper value* for initial data (t,x) is defined as

$$V_+(t,x) = \sup_{\beta \in \Delta} \inf_{u \in \mathcal{U}} P(t,x;u,\beta(u)), \tag{4.2}$$

where for notational brevity we put in (4.2) $\Delta = \Delta(t,t_1)$, $\mathcal{U} = \mathcal{U}(t,t_1)$. It will be shown later (Theorem 6.1) that $V_+(t,x)$ does not depend on the particular choices of $\mathcal{U}(t,t_1),\ \mathcal{Z}(t,t_1)$ satisfying (4.1). If $\mathcal{U}(t,t_1) = \mathcal{U}^0(t,t_1)$ and $\mathcal{Z}(t,t_1) = \mathcal{Z}^0(t,t_1)$, then $V_+(t,x)$ is called the *Elliott-Kalton upper value* [ElKa].

Lemma 4.1. *There exists a constant M_1 such that*

$$|V_+(t,x) - V_+(t,y)| \le M_1|x-y| \tag{4.3}$$

for all $t \in [t_0, t_1]$, $x, y, \in I\!R^n$.

Lemma 4.1 is proved in the same way as II(10.2), making use of the uniform Lipschitz bounds in (3.4)(c) and (d).

Similarly, a progressive strategy for the minimizing controller is a function α from $\mathcal{Z}(t, t_1)$ into $\mathcal{U}(t, t_1)$ with the property that $z(s) = \tilde{z}(s)$ for almost all $s \in [t, r]$ implies $\alpha(z)(s) = \alpha(\tilde{z})(s)$ for almost $s \in [t, r]$ where $t < r < t_1$. Let $\Gamma(t, t_1)$ denote the class of all progressive strategies α. The lower value for initial data (t, x) is

$$(4.4) \qquad V_-(t, x) = \inf_{\alpha \in \Gamma} \sup_{z \in \mathcal{Z}} P(t, x; \alpha(z), z)$$

where again for brevity $\mathcal{Z} = \mathcal{Z}(t, t_1)$, $\Gamma = \Gamma(t, t_1)$.

XI.5 Dynamic programming principle

Let us now show that the upper value function V_+ satisfies the following dynamic programming principle. For this purpose, we introduce the following notation. Given $r \in (t, t_1)$, let

$$\mathcal{U}_1 = \mathcal{U}(t, r), \qquad \mathcal{U}_2 = \mathcal{U}(r, t_1)$$

$$\Delta_1 = \Delta(t, r), \qquad \Delta_2 = \Delta(r, t_1).$$

Theorem 5.1. *(Dynamic programming principle) For* $t < r < t_1$

(5.1)

$$V_+(t, x) = \sup_{\beta_1 \in \Delta_1} \inf_{u_1 \in \mathcal{U}_1} \left[\int_t^r L(s, x(s), u_1(s), \beta_1(u_1)(s)) ds + V_+(r, x(r)) \right],$$

where $x(s)$ *is the solution of* (3.1) *on* $[t, r]$ *with* $x(t) = x$ *and controls* $u_1(s)$, $z_1(s) = \beta_1(u_1)(s)$.

Proof. Let $W(t, x)$ denote the right side of (5.1). Given $\theta > 0$, choose $\beta_1 \in \Delta_1$ such that

$$W(t, x) - \theta < \int_t^r L\, ds + V_+(r, x(r))$$

for all $u_1(\cdot) \in \mathcal{U}_1$, with $L = L(s, x(s), u_1(s), z_1(s))$ as in (5.1). The definition (4.2) of upper value implies that for any $y \in I\!R^n$, there exists $\tilde{\beta}_y \in \Delta_2$ such that

$$V^+(r, y) - \theta < P(r, y; u_2, \tilde{\beta}_y(u_2))$$

for all $u_2(\cdot) \in \mathcal{U}_2$. Given any $u(\cdot) \in \mathcal{U}(t_1, t_2)$, let $u_1(\cdot), u_2(\cdot)$ be the restrictions of $u(\cdot)$ to $[t, r)$ and $[r, t_1]$ respectively. Define β_0 by

$$\beta_0(u)(s) = \begin{cases} \beta_1(u_1)(s), & t \le s < r \\ \\ \tilde{\beta}_{x(r)}(u_2)(s), & r \le s \le t_1. \end{cases}$$

Then $\beta_0 \in \Delta(t, t_1)$. Moreover,

$$P(t, x; u, \beta_0(u)) = \int_t^r L\, ds + P(r, x(r); u_2, \tilde{\beta}_{x(r)}(u_2))$$

$$> \int_t^r L\, ds + V_+(r, x(r)) - \theta > W(t, x) - 2\theta.$$

Since $u(\cdot)$ is arbitrary,

$$V_+(t, x) \ge \inf_{u \in \mathcal{U}} P(t, x; u, \beta_0(u)) \ge W(t, x) - 2\theta.$$

Since θ is arbitrary, $W \le V_+$.

To prove the opposite inequality, consider any $\beta \in \Delta(t, t_1)$. Define $\beta_1 \in \Delta_1$ by $\beta_1(u_1)(s) = \beta(u)(s)$, where $u_1(\cdot)$ is the restriction of $u(\cdot)$ to $[t, r)$. Given $\theta > 0$ there exists $\hat{u}_1 \in \mathcal{U}_1$ such that

$$W(t, x) \ge \inf_{u \in \mathcal{U}_1} \left[\int_t^r L\, ds + V_+(r, x(r)) \right]$$

$$> \int_t^r L(s, \hat{x}(s), \hat{u}_1(s), \hat{z}_1(s)) ds + V_+(r, \hat{x}(r)) - \theta$$

where $\hat{x}(s)$ is the solution to (3.1)-(3.2) with controls $\hat{u}_1(s)$, $\hat{z}_1(s) = \beta_1(\hat{u}_1)(s)$. Define $\beta_2 \in \Delta_2$ by $\beta_2(u_2)(s) = \beta(\hat{u}_1, u_2)(s)$, for $s \in [r, t_1]$. Then

$$V^+(r, \hat{x}(r)) \ge \inf_{u_2 \in \mathcal{U}_2} P(r, \hat{x}(r); u_2, \beta_2)$$

$$> P(r, \hat{x}(r); \hat{u}_2, \beta_2) - \theta$$

for some $\hat{u}_2(\cdot) \in \mathcal{U}_2$. Let $\hat{u}(s) = \hat{u}_1(s)$ for $s \in [t, r)$ and $= \hat{u}_2(s)$ for $s \in [r, t_1]$. Then

$$\inf_{u \in \mathcal{U}} P(t, x; u, \beta(u)) \le P(t, x; \hat{u}, \beta(\hat{u}))$$

$$= \int_t^r L(s, \hat{x}(s), \hat{u}_1(s), \hat{z}_1(s)) ds + P(r, \hat{x}(r); \hat{u}_2, \beta_2(\hat{u}_2))$$

$$< W(t, x) + 2\theta.$$

Since $\beta \in \Delta(t, t_1)$ and θ are arbitrary, $V_+ \le W$. □

Theorem 5.2. *The upper value function V_+ is bounded and Lipschitz continuous on $\bar{Q}_0$.*

Proof. By the definition (4.2) and (3.4)(b)(d),

$$|V_+(t,x)| \le (t_1 - t)\|L\| + \|\psi\| \tag{5.2}$$

where $\| \ \|$ is the sup norm. Lemma 4.1 gives a uniform Lipschitz bound for $V_+(t,\cdot)$. For $t < r \le t_1$ and any $u_1(\cdot) \in \mathcal{U}_1, \beta_1 \in \Delta_1$,

$$\left| \int_t^r L(s, x(s), u_1(s), \beta_1(u_1)(s))ds + V^+(r, x(r)) - V^+(r,x) \right|$$

$$\le \|L\|(r-t) + M_1|x(r) - x| \le M_2(r-t)$$

by (4.3), where $M_2 = \|L\| + M_1\|G\|$. Since $u_1(\cdot)$ and β_1 are arbitrary, Theorem 5.1 implies that

$$|V^+(t,x) - V^+(r,x)| \le M_2(r-t). \tag{5.3}$$

Hence V_+ is Lipschitz continuous. □

In the same way, the lower value function V_- is Lipschitz continuous on $\bar{Q}_0$ and satisfies a dynamic programming principle

(5.4)

$$V_-(t,x) = \inf_{\alpha_1 \in \Gamma_1} \sup_{z_1 \in \mathcal{U}_1} \left[\int_t^r L(s, x(s), \alpha_1(z_1(s)), z_1(s))ds + V_-(r, x(r)) \right].$$

XI.6 Value functions as viscosity solutions

The upper value function V_+ is not generally of class C^1, although V_+ is Lipschitz continuous according to Theorem 5.2. In this section we show that V_+ satisfies the upper Isaacs PDE in the viscosity sense. Similarly the lower value V_- is a viscosity solution of the lower Isaacs PDE. If the Isaacs minimax condition (3.11) holds, then $V_+ = V_-$. We begin with:

Lemma 6.1. *Let Φ be continuous on $U \times Z$. Then for every $\theta > 0$ there exists a Borel measurable function ζ from U into Z such that*

$$\max_{z \in Z} \Phi(u,z) < \Phi(u, \zeta(u)) + \theta, \qquad \textit{for all} \quad u \in U. \tag{6.1}$$

Proof. Given $\delta > 0$, partition the compact set U into Borel sets $A_1, \cdots, A_M$ of diameter less than δ. Choose $u_i \in A_i$ and z_i which maximizes $\Phi(u_i, \cdot)$ on Z. Let $\zeta(u) = z_i$ for all $u \in A_i$. Then for $i = 1, \cdots, M, u \in A_i$

$$\max_{z\in Z}\Phi(u,z) < \max_{z\in Z}\Phi(u_i,z) + m(\delta)$$

$$= \Phi(u_i,z_i) + m(\delta) < \Phi(u,\zeta(u)) + 2m(\delta),$$

where $m(\delta) \to 0$ as $\delta \to 0$. We choose δ small enough that $2m(\delta) < \theta$. □

Given $w \in C^1(Q_0)$ let

(6.2) $$F(s,y,u,z) = G(s,y,u,z)\cdot D_x w(s,y) + L(s,y,u,z).$$

Lemma 6.2. *For $w \in C^1(Q_0)$, $(t,x) \in Q_0$*

(6.3)
$$\lim_{h\downarrow 0}\frac{1}{h}\left[\sup_{\beta\in\Delta}\inf_{u\in\mathcal{U}}\int_t^{t+h} F(s,x(s),u(s),\beta(u)(s))ds\right] = -H_+(t,x,D_x w(t,x)),$$

where $\Delta = \Delta(t,t+h)$, $\mathcal{U} = \mathcal{U}(t,t+h)$.

Proof. Since $|x(s)-x| \le \|G\|h$ for $s \in [t,t+h]$,

(6.4) $$|F(s,x(s),u(s),\beta(u)(s)) - F(t,x,u(s),\beta(u)(s)))| \le \chi_1(h)$$

where $\chi_1(h)$ does not depend on $u(\cdot)$ or β and $\chi_1(h) \to 0$ as $h \to 0$. Let $\Phi(u,z) = F(t,x,u,z)$ and note that by (3.7)

$$\min_{u\in U}\max_{z\in Z}\Phi(u,z) = -H_+(t,x,D_x w(t,x)).$$

Choose $u_0(s) = u_0$ for $s \in [t,t+h]$ where

$$u_0 \in \arg\min_{u\in U}\max_{z\in Z}\Phi(u,z).$$

Let $\beta_0(u)(s) = \zeta[u(s)]$ with ζ as in Lemma 6.1. Then for every $u(\cdot)$ and β

(6.5) $$\Phi(u_0,\beta(u_0)(s)) \le -H_+(t,x,D_x w(t,x))$$

(6.6) $$\Phi(u(s),\beta_0(u)(s) \ge -H_+(t,x,D_x w(t,x)) - \theta.$$

Denote by $(*)$ the expression in brackets in (6.3). Then by (6.4)-(6.6)

$$\limsup_{h\downarrow 0}\frac{1}{h}(*) \le \limsup_{h\downarrow 0}\sup_{\beta\in\Delta}\frac{1}{h}\int_t^{t+h}\Phi(u_0,\beta(u_0)(s))ds \le -H_+(t,x,D_x w(t,x))$$

$$\liminf_{h\downarrow 0}\frac{1}{h}(*) \ge \liminf_{h\downarrow 0}\inf_{u(\cdot)\in\mathcal{U}}\frac{1}{h}\int_t^{t+h}\Phi(u(s),\beta_0(u)(s))ds \ge -H_+(t,x,D_x w(t,x)) - \theta.$$

Since θ is arbitrary, we obtain (6.3). □

Theorem 6.1. *The upper value function V_+ is the unique bounded, uniformly continuous viscosity solution to the upper Isaacs PDE* (3.6) *with the terminal condition* (3.8).

Proof. By Theorem 5.2, V_+ is bounded and Lipschitz continuous on $\bar{Q}_0$ (hence uniformly continuous on $\bar{Q}_0$). We show that V_+ is a viscosity solution to (3.6). Uniqueness follows from Corollary II.9.1. Following the notation of Section II.3 let $\mathcal{C} = \{$bounded, Lipschitz continuous functions on $\mathbb{R}^n\}$. For $\phi \in \mathcal{C}$, let

$$(6.7)\qquad \mathcal{T}_{tr}\phi(x) = \sup_{\beta\in\Delta}\inf_{u\in\mathcal{U}}\left[\int_t^r L(s,x(s),u(s),\beta(u)(s))ds + \phi(x(r))\right]$$

where $\Delta = \Delta(t,r)$, $\mathcal{U} = \mathcal{U}(t,r)$. By Lemma 4.1, $\mathcal{T}_{tr}$ maps $\mathcal{C}$ into $\mathcal{C}$. It is immediate that conditions II(3.1), II(3.2) hold and the semigroup property II(3.3) is a consequence of Theorem 5.1. By Theorem II.5.1 is suffices (see II(3.11)) to show that, for each $(t,x) \in Q_0$ and $w \in C^1(Q_0)$

(6.8)
$$\lim_{h\downarrow 0}\frac{1}{h}\left[\mathcal{T}_{t\,t+h}w(t+h,(\cdot))(x) - w(t,x)\right] = \frac{\partial}{\partial t}w(t,x) - H_+(t,x,D_x w(t,x)).$$

By the fundamental theorem of calculus, applied to $w(t+h,x(t+h)) - w(t,x)$, this is equivalent to

$$(6.9)\qquad \lim_{h\downarrow 0}\frac{1}{h}\left\{\sup_{\beta\in\Delta}\inf_{u\in\mathcal{U}}\int_t^{t+h}\left[\frac{\partial w}{\partial t}(s,x(s)) + F(s,x(s),u(s),\beta(u)(s))\right]ds\right\}$$
$$= \frac{\partial}{\partial t}w(t,x) - H_+(t,x,D_x w(t,x)).$$

Since $w \in C^1(Q_0)$ and $|x(s) - x| \le \|G\|(s-t)$ for $t \le s \le t+h$

$$\left|\frac{\partial}{\partial t}w(s,x(s)) - \frac{\partial}{\partial t}w(t,x)\right| \le \chi_2(h)$$

where $\chi_2(h)$ does not depend on $u(\cdot)$ or β and $\chi_2(h) \to 0$ as $h \to 0$. Then (6.9) follows from Lemma 6.2. □

Remark 6.1. In particular, if there exists a bounded uniformly continuous classical solution of (3.6)–(3.8) $W \in C^1(\bar{Q}_0)$, then $W = V_+$.

Remark 6.2. By uniqueness of viscosity solutions in Theorem 6.1, the upper value function V_+ does not depend on the particular choices for $\mathcal{U}(t,t_1)$ and $\mathcal{Z}(t,t_1)$ in the definition (4.2). In particular, $V_+(t,x)$ is the same as the Elliott-Kalton upper value.

The same proof as for Theorem 6.1 gives:

Theorem 6.2. *The lower value function V_- is the unique bounded uniformly continuous viscosity solution to the lower Isaacs PDE* (3.9) *with the terminal condition* (3.8).

Corollary 6.1. *If the Isaacs minimax condition* (3.11) *holds, then $V_+ = V_-$*

If (3.11) does not hold, then we may have $V_-(t,x) < V_+(t,x)$. When that happens, we call the $V_+(t,x) - V_-(t,x)$ the *instantaneous information* gap in value for the differential game.

Example 6.1. Let $n = 1$, $U = Z = [-1,1]$ and $\dot{x}(s) = |u(s) - z(s)|$, where $\cdot = d/ds$. The game payoff is $P(t,x;u,z) = \psi(x(t_1))$, where $\psi \in C^1(\mathbb{R}^1)$ and $\psi_x(x) > 0$. Since the payoff is a strictly increasing function of x, it suffices to compute $H_\pm(p)$ for $p \geq 0$. We obtain $H_+(p) = -p$ and $H_-(p) = 0$ if $p \geq 0$. By explicitly solving (3.6) and (3.8), $V_+(t,x) = \psi(x + t_1 - t)$. The optimal control policy for the minimizer is $\underline{u}^*(t,x) = 0$ (see Theorem 3.1). Similarly, $V_-(t,x) = \psi(x) < V_+(t,x)$ since ψ is strictly increasing.

XI.7 Risk sensitive control limit game

In this section we will assume that the differential game dynamics (3.1) have the form

$$\frac{d}{ds}x(s) = f(s,x(s),u(s)) + \sigma(s,x(s),u(s))z(s). \tag{7.1}$$

Moreover, the game payoff is

$$P(t,x;u,z) = \int_t^{t_1} \left[\ell(s,x(s),u(s)) - \frac{1}{2}|z(s)|^2\right] ds. \tag{7.2}$$

We assume that $z(s) \in \mathbb{R}^d$. In the notation of Section 3, $G = f + \sigma z$, $L = \ell - \frac{1}{2}|z|^2$ and $\psi = 0$. We also assume that $f, \sigma, \ell \in C_b^1(\bar{Q}_0 \times U)$.

The differential equation (7.1) is a deterministic analogue of the stochastic differential equation VI(8.1) for the state dynamics of the risk-sensitive stochastic control problem. Theorem 7.2 below justifies the idea of regarding such differential games as small noise limits of risk-sensitive control problems. We also note that if the control $u(s)$ is absent from the model, then (7.1) is the same as VI(6.6) with $f(t,x,u) = b(t,x)$ and with $L(t,x,z)$ as in VI(4.2).

The upper Isaacs PDE for this differential game is (3.6), where

$$H_+(t,x,p) = -\min_{v\in U}\left[f(t,x,v)\cdot p + \frac{1}{2}a(t,x,v)p\cdot p + \ell(t,x,v)\right], \tag{7.3}$$

where $a = \sigma\sigma'$.

In the definition of upper value in Section 4, the control space is assumed compact (see (3.4a)). Since $Z = I\!R^d$ is not compact, we now define $V_+(t,x)$ by (4.2) where

$$\triangle = \bigcup_{R>0} \triangle_R$$

and $\triangle_R = \triangle_R(t,t_1)$ is the set of all progressive strategies which map $\mathcal{U}(t,t_1)$ into $L^\infty([t,t_1];Z_R)$, $Z_R = \{z : |z| \le R\}$. Let $V_{+R}(t,x)$ denote the upper value with the constraint $|z(s)| \le R$. Then $V_{+R}(t,x)$ tends to $V_+(t,x)$ as $R \to \infty$. In fact, we will show that $V_{+R} = V_+$ for large enough R.

Lemma 7.1. *There exists M_1 such that*

$$|V_{+R}(t,x) - V_{+R}(t,y)| \le M_1|x-y| \tag{7.4}$$

for all $t \in [t_0,t_1]$, $x,y \in I\!R^n$.

Proof. Let $\beta_0(u)(s) = 0$ for all $u(\cdot) \in \mathcal{U}$, $s \in [t,t_1]$. Then

$$-(t_1-t)\|\ell\| \le \inf_{u\in\mathcal{U}} P(t,x;u,\beta_0) \le V_{+R}(t,x) \le (t_1-t)\|\ell\|. \tag{7.5}$$

By (7.2) and (7.5), it suffices to consider strategies β such that $\|\beta(u)\|_2 \le C_1$ for all $u(\cdot)$, where $\|\ \|_2$ is the L^2-norm. Given β and $u(\cdot)$, let $x(s), y(s)$ satisfy (7.1) with $x(t) = x$, $y(t) = y$, and $z(s) = \beta(u)(s)$. Let $\zeta(s) = x(s) - y(s)$. Since $f(s,\cdot,u)$ and $\sigma(s,\cdot,u)$ satisfy a uniform Lipschitz condition, there exists K such that

$$\frac{d}{ds}|\zeta(s)|^2 \le 2|\zeta(s)||\dot{\zeta}(s)| \le K(1+|z(s)|)|\zeta(s)|^2,$$

$$|\zeta(s)|^2 \le |x-y|^2 \exp\left[K\int_t^s (1+|z(r)|)dr\right].$$

Since $\|z\|_2 \le C_1$, $|\zeta(s)| \le C_2|x-y|$ for some C_2. Note that C_1, C_2 do not depend on $(t,x) \in Q_0$ or on R. Since $\ell(s,\cdot,u)$ satisfies a uniform Lipschitz condition, this implies

$$|P(t,x;u,\beta(u)) - P(t,y;u,\beta(u))| \le M_1|x-y| \tag{7.6}$$

for all β such that $\|\beta(u)\|_2 \le C_1$ which satisfy (7.5). This implies Lemma 7.1. □

Theorem 7.1. (a) *There exists R_1 such that $V_{+R}(t,x) = V_+(t,x)$ for every $R \ge R_1$.*
(b) *V_+ is the unique bounded, Lipschitz continuous viscosity solution to the upper Isaacs PDE* (3.6) *with terminal data $V_+(t_1,x) = 0$.*

Proof. Let us prove (a). Part (b) follows from (a) by the same proof as the Theorem VI.6.1. Let

$$H_{+R}(t,x,p) = -\min_{v\in U}\max_{|\xi|\le R}\left[p\cdot(f(t,x,v)+\sigma(t,x,v)\xi)+\ell(t,x,v)-\frac{1}{2}|\xi|^2\right].$$

Let M_1 be as in Lemma 7.1. Since σ is bounded, there exists k such that $kM_1 \le R$ and $|p| \le M_1$ imply $H_{+R}(t,x,p) = H_+(t,x,p)$. Let $R_1 = kM_1$. For $R \ge R_1$, Corollary II.8.1(f) implies that $|p| \le M_1$ for all $(q,p) \in D^+(V_{+R}(t,x)) \cup D^-(V_{+R}(t,x))$. Therefore, by Theorem 6.1, V_{+R} is a bounded, Lipschitz continuous viscosity solution of

$$-\frac{\partial V}{\partial t} + H_{+R_1}(t,x,D_xV) = 0$$

for every $R \ge R_1$, with $V_{+R}(t_1,x) = 0$. Since V_{+R_1} is also a viscosity solution, the uniqueness part of Theorem 6.1 implies that $V_{+R} = V_{+R_1}$. Since $V_{+R}(t,x)$ tends to $V_+(t,x)$ as $R \to \infty$, this proves (a). □

Remark 7.1. If $\sigma = \sigma(t,x)$ does not depend on v, then the Isaacs minimax condition (3.11) holds. By Corollary 6.1, $V_{+R} = V_{-R}$. Since $V_{-R} \le V_- \le V_+ = V_{+R}$ for $R \ge R_1$, this implies $V_- = V_+$.

Small noise, risk sensitive control limit. In Section VI.8, let us now take $\rho = \varepsilon^{-1}$ and let $\Phi^\varepsilon(t,x)$ denote the value function for the risk sensitive stochastic control problem. As in VI(8.9), let $V^\varepsilon = \varepsilon \log \Phi^\varepsilon$.

Theorem 7.2. *$V^\varepsilon(t,x)$ tends to $V_+(t,x)$ uniformly on compact subsets of $\bar{Q}_0$.*

Proof. By Corollary VI.8.1, V^ε is a viscosity solution of

$$(7.7)\qquad -\frac{\partial V^\varepsilon}{\partial t} + \bar{\mathcal{H}}_\varepsilon(t,x,D_xV^\varepsilon,D_x^2V^\varepsilon) = 0$$

with $\bar{\mathcal{H}}_\varepsilon$ as in VI(8.11) with $\rho = \varepsilon^{-1}$. Moreover, $|V^\varepsilon(t,x)| \le (t_1 - t)\|\ell\|$. For every $w \in C^\infty(\bar{Q}_0)$,

$$\lim_{\varepsilon\to 0}\bar{\mathcal{H}}_\varepsilon(t,x,D_xw,D_x^2w) = H_+(t,x,D_xw)$$

uniformly for (t,x,p,A) in any compact set. Theorem 7.2 then follows from Theorem 7.1, by using the weak comparison principle for viscosity solutions in the same way used to prove Theorem VI.6.2. See Section VII.11. □

Remark 7.2. In the terminology of Chapter VI, $V^\varepsilon(t,x)$ is the infimum over admissible control processes $u(\cdot)$ of the certainty equivalent expectation $\mathcal{E}^0_{tx}(\mathcal{J}^\varepsilon)$, where

$$\mathcal{J}^\varepsilon = \int_t^{t_1} \ell(s, x^\varepsilon(s), u(s))ds$$

with $x^\varepsilon(s)$ as in VI(8.1) with $\rho = \varepsilon^{-1}$. According to Theorem 7.2, this tends to $V_+(t,x)$ as $\varepsilon \to 0$. The upper game value has a "max-plus stochastic control" interpretation, in terms of the Maslov idempotent probability calculus mentioned in Remark VI.6.2. See [F5]. Let

$$\mathcal{J}^0 = \int_t^{t_1} \ell(s, x(s), u(s))ds$$

with $x(s)$ as in (7.1). Given a strategy α for the minimizing controller,

$$\sup_{z \in \mathcal{Z}} P(t,x;\alpha(z),z) = E^+_{tx}(\mathcal{J}^0)$$

where E^+ denotes the max-plus expectation. The *max-plus stochastic control problem* is to minimize $E^+_{tx}(\mathcal{J}^0)$ in an appropriate class $\bar{\Gamma}$ of strategies α. If $\bar{\Gamma} = \Gamma$ is the class of all progressive strategies, then the infimum of $E^+_{tx}(\mathcal{J}^0)$ is $V_-(t,x)$ by the definition (4.4) of lower value. This does not agree with the risk sensitive limit in Theorem 7.2 if $V_-(t,x) < V_+(t,x)$. In Section 9, we will define a smaller class $\Gamma_{\mathcal{S}}$ of strictly progressive strategies. By Theorem 9.1, $V_+(t,x)$ is the infimum of $E^+_{tx}(\mathcal{J}^0)$ among strategies $\alpha \in \Gamma_{\mathcal{S}}$.

XI.8 Time discretizations

In this section and in Section 9, we return to the differential game formulation in Section 3, with game dynamics (3.1) and (3.3). The main result is a characteriztion of the upper value function V_+ in terms of strictly progressive strategies for the minimizing controller (Theorem 9.1). In preparation, we first give another characterization of the upper value, using piecewise constant controls $u(s)$ for the minimizer. Let $\pi = \{r_0, r_1, \cdots, r_N\}$ with $t_0 = r_0 < r_1 < \cdots < r_N = t_1$. For initial time $t = r_i$ in π, let

$$\mathcal{U}_\pi(t,t_1) = \big\{u(\cdot) \in \mathcal{U}(t,t_1) : \ u(s) = u(r_j) \text{ for } s \in [r_j, r_{j+1}), j = i, \cdots, N-1\big\}.$$

Each such control is a piecewise constant function. Let

$$V_+^\pi(t,x) = \sup_{\beta \in \Delta} \inf_{u(\cdot) \in \mathcal{U}_\pi} P(t,x;u,\beta(u)), \tag{8.1}$$

where as in (4.2) we let $\Delta = \Delta(t,t_1)$, $\mathcal{U}_\pi = \mathcal{U}_\pi(t,t_1)$.

Lemma 8.1. *For $t = r_i$, $r = r_j$ with $i < j$, let $\mathcal{U}_{\pi_1} = \mathcal{U}_\pi(t,r)$, $\Delta_1 = \Delta(t,r)$. Then*

(8.2)
$$V_+^\pi(t,x) = \sup_{\beta_1\in\Delta_1} \inf_{u_1\in\mathcal{U}_{\pi_1}} \left[\int_t^r L(s,x(s),u_1(s),\beta_1(u_1(s)))ds + V_+^\pi(r,x(r)) \right].$$

Lemma 8.1 is a discrete time version of the dynamic programming principle for differential games. It is proved in exactly the same way as Theorem 5.1, taking $t = r_i, r = r_j$ and considering only controls $u(\cdot) \in \mathcal{U}_\pi(t,t_1)$, $u_1(\cdot) \in \mathcal{U}_\pi(t,r)$, $u_2(\cdot) \in \mathcal{U}_\pi(r,t)$.

If $\pi \subset \tilde{\pi}$, then $\mathcal{U}_\pi \subset \mathcal{U}_{\tilde{\pi}}$ and hence $V_+^{\tilde{\pi}} \le V_+^\pi$. Let

(8.3)
$$\hat{V}_+(t,x) = \lim_{\substack{\|\pi\|\to 0\\ r_i\to t}} V_+^\pi(r_i,x)$$

where $\|\pi\| = \max_i (r_{i+1} - r_i)$.

Theorem 8.1. $\hat{V}_+ = V_+$.

Proof. Since $\mathcal{U}_\pi \subset \mathcal{U}$, $V_+ \le V_+^\pi$ for every π. Hence $V_+ \le \hat{V}_+$. From Lemma 7.1 and the same proof as for Theorem 5.2, V_+^π is bounded and Lipschitz continuous, with the same constants as in (4.3), (5.2), (5.3) which do not depend on π. Hence $\hat{V}_+$ is bounded and Lipschitz continuous. Moreover, $\hat{V}_+(t_1,x) = V_+(t_1,x) = \psi(x)$. To show that $\hat{V}_+ \le V_+$, it suffices to show that $\hat{V}_+$ is a viscosity subsolution of the upper Isaacs PDE (3.6) and to use the comparison principle Theorem II.9.1.

Suppose that $w \in C^1(Q_0)$ and that $\hat{V}_+ - w$ has a strict local maximum at $(\bar{t},\bar{x}) \in Q_0$. We must show that

(8.4)
$$\frac{\partial}{\partial t} w(\bar{t},\bar{x}) + \min_{u\in U}\max_{z\in Z} F(\bar{t},\bar{x},u,z) \ge 0$$

with F as in (6.2). Suppose not. Then there exist $\theta > 0$, $\delta > 0$ and $u_0 \in U$ such that

(8.5)
$$\frac{\partial}{\partial t} w(s,y) + F(s,y,u_0,z) \le -\theta$$

for all $z \in Z$ and $(s,y) \in N_\delta$, where $N_\delta = \{(s,y) \in Q_0 : |s-\bar{t}|+|+|y-\bar{x}| < \delta\}$. Since V_+^π has a uniform Lipschitz bound, the limit in (8.3) is uniform on compact sets. Hence, there exist (t_π, x_π) tending to $(\bar{t},\bar{x})$ as in $\|\pi\| \to 0$, with $t_\pi \in \pi$, such that

(8.6)
$$V_+^\pi(r,y) - w(r,y) \le V_+^\pi(t_\pi,x_\pi) - w(t_\pi,x_\pi)$$

for all $(r,y) \in N_\delta$ such that $r \in \pi$. Choose $r \in \pi$ such that $t_\pi < r$ and $2(r - t_\pi) < \delta(1 + \|G\|)^{-1}$. Let $u_0(s) = u_0$ and consider any $\beta \in \Delta(t_\pi, r)$. Let

$x(s)$ be the corresponding solution to (3.1) with $x(t_\pi) = x_\pi$. For small enough $\|\pi\|$, $(s, x(s)) \in N_\delta$ for $t_\pi \le s \le r$. By applying the fundamental theorem of calculus to $w(r, x(r)) - w(t_\pi, x_\pi)$ and using (8.6) with $y = x(r)$,

$$
\begin{aligned}
(8.7) \qquad & V_+^\pi(r, x(r)) + \int_{t_\pi}^{r} L(s, x(s), u_0, \beta(u_0)(s))ds - V_+^\pi(t_\pi, x_\pi) \\
& \qquad < \int_{t_\pi}^{r} \left[\frac{\partial}{\partial t} w(s, x(s)) + F(s, x(s), u_0, \beta(u_0)(s)) \right] ds \\
& \qquad < -\theta(r - t_\pi).
\end{aligned}
$$

The left side of (8.7) is decreased when u_0 is replaced by the inf over $u(\cdot)$. Since (8.7) is true for all β, we obtain from Lemma 8.1 a contradiction. Thus $\hat{V}_+$ is a viscosity subsolution. □

In the next section, we will give another characterization of V_+^π in terms of strategies for the minimizing controller. This is important for our discussion there of strictly progressive strategies.

XI.9 Strictly progressive strategies

We now shift attention to strategies for the minimizing controller. The lower value $V_-(t, x)$ is obtained when the inf in (4.4) is taken among progressive strategies $\alpha \in \Gamma$. In this section, we will show that the upper value $V_+(t, x)$ is obtained when α is restricted to a smaller class of strategies Γ_S which we call strictly progressive. At an intuitive level, restricting α to Γ_S corresponds to denying the minimizer current information about his opponent's choice. Hence, the minimizer loses the information advantage.

A progressive strategy α for the minimizing controller is called *strictly progressive* if: for every progressive strategy β for the maximizing controller, the equations

$$(9.1) \qquad u = \alpha(z), \qquad z = \beta(u)$$

have a solution $\hat{u}(\cdot) \in \mathcal{U}(t, t_1)$, $\hat{z}(\cdot) \in \mathcal{Z}(t, t_1)$. This implies that $\hat{u}(\cdot)$ is a fixed point of the composite map $\alpha \circ \beta$.

Example 9.1. Consider any partition of $[t_0, t_1]$ into subintervals $[r_j, r_{j+1})$, with endpoints in a finite set π as in Section 8. For $t = r_i$, $\mathcal{U}_\pi(t, t_1)$ consists of functions $u(\cdot)$ constant on each subinterval with $i < j$. Let $\Gamma_\pi(t, t_1)$ consist of those progressive strategies α such that $\alpha(z) \in \mathcal{U}_\pi(t, t_1)$ for every $z(\cdot) \in \mathcal{U}(t, t_1)$. Then every $\alpha \in \Gamma_\pi(t, t_1)$ is strictly progressive. To see this, given any $\beta \in \Delta(t, t_1)$, choose $u_i \in U$ arbitrarily and let $\hat{u}(s) = u_i$, $\hat{z}(s) = \beta(\hat{u})(s)$ for

$s \in [r_i, r_{i+1})$. Proceeding by induction on j, for $j > i$ first choose $\hat{u}(s)$ and then $\hat{z}(s)$ on $[r_j, r_{j+1})$ so that

$$\hat{u}(s) = \hat{u}(r_j) = \alpha(\hat{z})(r_j), \qquad \hat{z}(s) = \beta(\hat{u})(s).$$

Note that $\alpha(\hat{z})(r_j)$ depends only on $\hat{z}(s)$ for $s \in [t, r_j)$ since α is progressive. Then $\hat{u}(\cdot), \hat{z}(\cdot)$ chosen by this stepwise procedure satisfy (9.1).

Example 9.2. Suppose that there is a "time delay" $\delta > 0$ such that $z(s) = \tilde{z}(s)$ for almost all $s \in [t, r - \delta]$ implies that $\alpha(z)(s) = \alpha(\tilde{z})(s)$ for almost all $s \in [t, r]$. A stepwise procedure similar to that in Example 9.1 shows that α is strictly progressive. Note that in this example, $\hat{u}(s)$ is chosen on an initial interval $[t, t + \delta)$ with no information about $z(\cdot)$.

Example 9.3. In this example, α is progressive but not strictly progressive. Let $\alpha(z)(s) = \phi(z(s))$ where ϕ is a non constant Borel measurable function from Z into U. There is a partition $U = U_1 \cup U_2$ into Borel sets such that $Z_1 = \phi^{-1}(U_1)$ and $Z_2 = \phi^{-1}(U_2)$ are both nonempty. Choose $z_i \in Z_i$ for $i = 1, 2$ and χ such that $\chi(u) = z_2$ if $u \in U_1$, $\chi(u) = z_1$ if $u \in U_2$. The composite $\phi \circ \chi$ has no fixed point. Let $\beta(u)(s) = \chi(u(s))$. Then $(\alpha \circ \beta)(z)(s) = (\phi \circ \chi)(z(s))$. If (9.1) holds, then $\hat{u}(s)$ is a fixed point of $\phi \circ \chi$ for almost all s, which is a contradiction. Thus, α is not strictly progressive.

Let $\Gamma_{\mathcal{U}} = \Gamma_{\mathcal{U}}(t, t_1)$ denote the class of all strictly progressive strategies α, and let

$$W(t, x) = \inf_{\alpha \in \Gamma_{\mathcal{S}}} \sup_{z \in \mathcal{Z}} P(t, x; \alpha(z), z). \tag{9.2}$$

The main result of this section is:

Theorem 9.1. *$W = V_+$, where V_+ is the upper value function.*

The following argument shows that $V_+ \le W$. Given $\theta > 0$ there exists $\beta \in \Delta$ such that (see (4.2))

$$P(t, x; u, \beta(u)) \ge V_+(t, x) - \theta \tag{9.3}$$

for every $u(\cdot) \in \mathcal{U}$. Given any $\alpha \in \Gamma_{\mathcal{S}}$, let $\hat{u}(\cdot), \hat{z}(\cdot)$ be a solution to (9.1). Then

$$\sup_{z \in \mathcal{Z}} P(t, x; \alpha(z), z) \ge P(t, x; \alpha(\hat{z}), \hat{z}) = P(t, x; \hat{u}, \beta(\hat{u})) \ge V_+(t, x) - \theta.$$

Since $\alpha \in \Gamma_{\mathcal{S}}$ and θ are arbitrary, $V_+ \le W$.

In order to show that $W \le V_+$, we first prove two lemmas about the functions V_+^π defined by (8.2).

Lemma 9.1. *For $t = r_i$, $i = 0, 1, \cdots, N - 1$,*

$$V_+^\pi(r_i, x) = \inf_{u \in U} \sup_{z_1 \in \mathcal{Z}_1} \left[\int_{r_i}^{r_{i+1}} L(s, x(s), u, z_1(s)) ds + V_+^\pi(r_{i+1}, x(r_{i+1})) \right], \tag{9.4}$$

where $\mathcal{Z}_1 = \mathcal{Z}(r_i, r_{i+1})$.

Proof. Denote by $(*)$ the expression in brackets. In Lemma 8.1, take $t = r_i$, $r = r_{i+1}$. Since $u(s) = u$ is a constant on $[r_i, r_{i+1})$, chosen without knowing $z_1(\cdot)$

$$V_+^\pi(r_i, x) = \sup_{\beta_1 \in \Delta_1} \inf_{u \in U} [(*)] = \inf_{u \in U} \sup_{z_1 \in \mathcal{Z}_1} [(*)]$$

where the second equality is by (2.3). □

Let $\Gamma_\pi = \Gamma_\pi(t, t_1)$ denote the class of progressive strategies α such that α maps $\mathcal{Z}(t, t_1)$ into $\mathcal{U}_\pi(t, t_1)$. By Example 9.1, every $\alpha \in \Gamma_\pi$ is strictly progressive.

Lemma 9.2. *For* $t = r_i,\ i = 0, 1, \cdots, N-1$,

$$V_+^\pi(t, x) = \inf_{\alpha \in \Gamma_\pi} \sup_{z \in \mathcal{Z}} P(t, x; \alpha(z), z). \tag{9.5}$$

Proof. Denote the right side of (9.5) by $W^\pi(t, x)$. Then $V_+^\pi \le W^\pi$ by the argument used above to show that $V_+ \le W$. We show that $W^\pi \le V_+^\pi$ as follows. Given (t, x) and $\theta > 0$, with $t = r_i$, we define $\alpha_\theta \in \Gamma_\pi(t, t_1)$ as follows. By Lemma 9.1, for each $\xi \in \mathbb{R}^n$ there exists $u_j(\xi) \in U$ such that

(9.6)

$$\sup_{z_j \in \mathcal{Z}_j} \left[\int_{r_j}^{r_{j+1}} L(s, x_j(s), u_j(\xi), z_j(s)) ds + V_+^\pi\big(r_{j+1}, x_j(r_{j+1})\big) \right] < V_+^\pi(r_j, \xi) + \frac{\theta}{N},$$

where $\mathcal{Z}_j = \mathcal{Z}(r_j, r_{j+1})$ and $x_j(s)$ is the solution of (3.1) on $[r_j, r_{j+1}]$ with $x_j(r_j) = \xi$, $u(s) = u_j(\xi)$ and $z(s) = z_j(s)$. For initial data $x(t) = x$ and $z(\cdot) \in \mathcal{Z}(t, t_1)$, we define $\alpha_\theta(z)$ by

$$\alpha_\theta(z)(s) = u_j(x(r_j)), \qquad r_j \le s < r_{j+1}$$

where $x(s)$ is the solution to (3.1) on $[t, t_1]$ with $x(t) = x$, defined on successive intervals as follows. For $r_j \le r < r_{j+1}$, $u(s) = u_j(x(r_j))$. Then

$$P(t, x; \alpha_\theta(z), z) = \sum_{j=i}^{N-1} \int_{r_j}^{r_{j+1}} L(s, x(s), u_j(x(r_j)), z(s)) ds + \psi(x(t_1))$$

$$< \sum_{j=i}^{N-1} [V_+^\pi(r_j, x(r_j)) - V_+^\pi(r_{j+1}, x(r_{j+1}))] + \psi(x(t_1)) + \theta.$$

The last term is $V_+^\pi(t, x) + \theta$. Hence

$$W^{\pi}(t,x) \leq \sup_{z\in\mathcal{Z}} P(t,x;\alpha_{\theta}(z),z) \leq V_{+}^{\pi}(t,x) + \theta.$$

Since θ is arbitrary, $W^{\pi} \leq V_{+}^{\pi}$. □

Proof of Theorem 9.1. We already showed that $V_{+} \leq W$. By (9.2) and Lemma 9.2, $W \leq V_{+}^{\pi}$ since $\Gamma_{\pi} \subset \Gamma_{\mathcal{S}}$. By Theorem 8.1, $W \leq V_{+}$. □

Remark 9.1. Let $\bar{\Gamma}_{\mathcal{S}}$ be the closure of $\Gamma_{\mathcal{S}}$ in the uniform norm. Thus, $\alpha \in \bar{\Gamma}_{\mathcal{S}}$ if there exists $\alpha \in \Gamma_{\mathcal{S}}$ such that $|\alpha_n(z)(s) - \alpha(z)(s)| \leq \eta_n$ for all $z(\cdot) \in \mathcal{Z}$, where $\eta_n \to 0$ as $n \to \infty$. Theorem 9.1 implies that

$$V_{+}(t,x) = \inf_{\alpha\in\bar{\Gamma}_{\mathcal{S}}} \sup_{z\in\mathcal{Z}} P(t,x;\alpha(z),z). \tag{9.7}$$

In particular, if $\underline{u}$ is a Lipschitz continuous control policy for the minimizer, let $\alpha(z)(s) = u(s)$ where $x(s)$ is the solution to (3.1)-(3.2) with $u(s) = \underline{u}(s,x(s))$. Then $\alpha \in \bar{\Gamma}_{\mathcal{S}}$. This is seen by approximating α by $\alpha_n \in \Gamma_{\mathcal{S}}$, where α_n is defined similarly with a time delay δ_n tending to 0 as $n \to \infty$. In fact, $\alpha_n(z)(s) = u_n(s)$, where for some $u_0 \in \mathcal{U}$, $u_n(s) = u_0$ if $s \in [t, t+\delta_n)$ and $u_n(s) = \underline{u}(s, x_n(s-\delta_n))$ if $s \in [t+\delta_n, t_1]$, with $x_n(s)$ the corresponding solution to (3.1)-(3.2).

XI.10 Historical remarks

The theory of two-controller, zero sum differential games was initiated by Isaacs [Is]. He did not have a mathematically rigorous theory of differential game value. However, he articulated the basic framework of differential game theory, including what is now called the Isaacs PDE. He also solved many interesting examples, using the method of characteristics. Early rigorous definitions of differential game value made use of time discretizations. See for example [F7][Fr2][PZ]. These were superseded by the more convenient Elliott-Kalton differential game values [ElKa]. See Section 4. Evans and Souganidis [ES1] characterized the upper and lower Elliott-Kalton value functions as unique viscosity solutions of the corresponding Isaacs PDEs (Section 6). Souganidis [Sou1] showed using viscosity solution methods that the Elliott-Kalton value functions are the same as those defined using time discretizations.

Differential game methods have a natural role in nonlinear H-infinity control. See [BCD][BB][Sor]. The small noise risk sensitive control limit in Theorem 7.2 was proved in [FM1][Jam1], under some additional assumptions. For related results, on a finite or infinite time horizon, see [BN2][FM2][KN1,2]. The discretization technique in Section 8 is based on an idea of Nisio. The definition of strictly progressive strategy in Section 9 is taken from [F5]. See [KSh2] for another proof of Theorem 9.1 which avoids time discretizations.

A

Duality Relationships

In this Appendix we outline proofs of statements made in Section I.8, in particular formulas (8.5)–(8.7). Let $\xi = (t,x)$ and let $L \in C^3(\overline{Q}_0 \times I\!R^n)$ satisfy (see I(8.3))

$$(A.1) \qquad L_{vv}(\xi, v) > 0$$

$$(A.2) \qquad \frac{L(\xi, v)}{|v|} \to +\infty \quad \text{as} \quad |v| \to \infty.$$

The dual function $H(\xi, p)$, defined by I(8.4), satisfies

$$(A.3) \qquad H(\xi, p) \geq -v \cdot p - L(\xi, v), \ \forall v \in I\!R^n.$$

Equality holds if and only if v maximizes the right side. By (A.1) and (A.2), given (ξ, p) the maximum occurs at the unique $v \in I\!R^n$ such that

$$(A.4) \qquad p = -L_v(\xi, v).$$

We rewrite (A.3) as

$$(A.5) \qquad L(\xi, v) \geq -v \cdot p - H(\xi, p), \ \forall p \in I\!R^n.$$

Given (ξ, v) choose p according to (A.4). Then equality holds in (A.3). This gives the dual formula I(8.5) for H.

The argument above shows that the mapping $v \to -L_v(\xi, v)$ is one-one and onto $I\!R^n$, for each $\xi \in \overline{Q}_0$. Using (A.1) and the implicit function theorem, there exists an inverse $\Gamma \in C^2(\overline{Q}_0 \times I\!R^n)$, such that

$$(A.6) \qquad p = -L_v(\xi, \Gamma(\xi, p)),$$

$$(A.7) \qquad H(\xi, p) = -\Gamma(\xi, p) \cdot p - L(z, \Gamma(\xi, p)).$$

Thus $H \in C^2(\overline{Q}_0 \times I\!R^n)$. By taking the gradient with respect to ξ in (A.7) and using (A.6), we get I(8.7a). By taking the gradient with respect to p in (A.7), we get $H_p = -\Gamma$, which is the second formula in I(8.6). The remaining formulas I(8.7b,c) then follow by further differentiations.

B

Dynkin's Formula for Random Evolutions with Markov Chain Parameters

In this Appendix, we prove a version of Dynkin's formula cited in Section III.4(c) and in Example III.9.2. Let $z(s)$ be a finite state continuous time Markov chain with state space Z. Given an initial state $z = z(t)$, let $\tau_0 = t$ and let $\tau_1 < \tau_2 < \cdots$ be the successive jump times for $z(s)$. Let $z_i = z(\tau_i^+)$ and let $x(s)$ be a continuous, bounded solution to the differential equation (see III(4.5))

(B.1)
$$\frac{dx}{ds} = f(s, x(s), z_i),\ \tau_i \le s < \tau_{i+1},$$

with initial data $x(t) = x$. For each $\Phi(t, x, z)$ such that the partial derivatives $\Phi_t, \Phi_{x_i}, i = 1, \cdots, n$, are continuous let (see III(4.6))

(B.2)
$$A\Phi = A^0\Phi - G_z\Phi,$$

$$A^0\Phi(t, x, z) = \Phi_t(t, x, z) + f(t, x, z) \cdot D_x\Phi(t, x, z),$$

$$G_z\Phi(t, x, z) = -\sum_{\zeta \neq z} \rho(t, x, z)\left[\Phi(t, x, \zeta) - \Phi(t, x, z)\right].$$

Let us show that (see III(2.7))

(B.3)
$$E_{txz}\Phi(s, x(s), z(s)) - \Phi(t, x, z) = E_{txz}\int_t^s A\Phi(r, x(r), z(r))dr.$$

To prove (B.3) we write

$$\begin{aligned}
&\Phi(s, x(s), z(s)) - \Phi(t, x, z) \\
&\quad = \sum_{i \ge 0}\left[\Phi(s \wedge \tau_{i+1}, x(s \wedge \tau_{i+1}), z_i) - \Phi(s \wedge \tau_i, x(s \wedge \tau_i), z_i)\right] \\
&\quad + \sum_{i \ge 0}\left[\Phi(s \wedge \tau_{i+1}, x(s \wedge \tau_{i+1}), z_{i+1}) - \Phi(s \wedge \tau_{i+1}, x(s \wedge \tau_{i+1}), z_i)\right] \\
&\quad = (1) + (2).
\end{aligned}$$

From the Fundamental Theorem of Calculus

$$E_{txz}(1) = E_{txz} \int_t^s A^0 \Phi(r, x(r), z(r)) dr.$$

Set $\xi_i = (\tau_i, z_i), \eta_i = (\tau_i, z_i, \tau_{i+1})$,

$$R(t,z) = \sum_{\zeta \neq z} \rho(t, z, \zeta).$$

Then τ_{i+1} conditioned on ξ_i is exponentially distributed on $[\tau_i, \infty)$ with rate $R(r, z_i)$. Therfore, a straightforward calculation yields

$$E_{\xi_i} \phi(s \wedge \tau_{i+1}) = E_{\xi_i} \int_{s\wedge\tau_i}^{s\wedge\tau_{i+1}} R(r, z_i) \phi(r) dr,$$

for any continuous ϕ. Now, by conditioning, we obtain

$$E_{\xi_i} \ [\Phi(s \wedge \tau_{i+1}, x(s \wedge \tau_{i+1}), z_{i+1}) - \Phi(s \wedge \tau_{i+1}, x(s \wedge \tau_{i+1,}), z_i)]$$

$$= E_{\xi_i} E_{\eta_i} [\Phi(s \wedge \tau_{i+1}, x(s \wedge \tau_{i+1}), z_{i+1}) - \Phi(s \wedge \tau_{i+1}, x(s \wedge \tau_{i+1}), z_i)]$$

$$= -E_{\xi_i} G_z \Phi(s \wedge \tau_{i+1}, x(s \wedge \tau_{i+1}), z_i) / R(s \wedge \tau_{i+1}, z_i)$$

$$= -E_{\xi_i} \int_{s\wedge\tau_i}^{s\wedge\tau_{i+1}} G_z \Phi(r, x(r), z_i) dr$$

By using the strong Markov property of $z(s)$ we then get

$$E_{txz}(2) = -E_{txz} \int_t^s G_z \Phi(r, x(r), z(r)) dr.$$

This proves (B.3).

In Example III.9.2 we take

$$f(x,z) = \underline{u}^*(x,z) - z,$$

where $\underline{u}^*$ is as in formula III(9.22). As noted in the discussion there, equation (B.1) with initial data $x(0) = x$ has a unique bounded continuous solution $x(s)$ for $s \geq 0$.

C

Extension of Lipschitz Continuous Functions; Smoothing

In Section VI.5 we used a result about Lipschitz continuous extensions of functions. Let $K \subset I\!R^n$ and let $g : K \to I\!R^m$ be Lipschitz continuous, with Lipschitz constant λ:

$$|g(x) - g(y)| \leq \lambda |x - y|, \ \forall x, y \in K.$$

For each $x \in I\!R^n$, let

$$\tilde{g}(x) = \inf_{y \in K} \left[g(y) + \lambda |x - y| \right]. \tag{C.1}$$

If $x \in K, y \in K$, then

$$g(x) \leq g(y) + \lambda |x - y|$$

with equality when $x = y$. Thus, $\tilde{g}(x) = g(x)$ for all $x \in K$. Moreover,

$$|\tilde{g}(x_1) - \tilde{g}(x_2)| \leq \lambda |x_1 - x_2|, \ \forall x_1, x_2 \in I\!R^n. \tag{C.2}$$

Thus, $\tilde{g}$ is a Lipschitz continuous extension of g. If K is bounded, we obtain a bounded Lipschitz continuous extension $\bar{g}$ of g, by taking $\bar{g} = \alpha \tilde{g}$, where α is real valued and Lipschitz continuous on $I\!R^n$ with $\alpha(x) = 1$ for $x \in K$ and $\alpha(x) = 0$ for $x \notin K_1$ for some compact set K_1.

Now let $K = \overline{O}$, with O bounded and open. We apply the above construction for each fixed t to $g(\cdot) = \underline{z}^*(t, \cdot)$ with $\underline{z}^*(t, x)$ as in VI(4.5) when $x \in \overline{O}$. The above construction gives an extension

$$\tilde{\underline{z}}^*(t, x) = \alpha(x) \min_{y \in \overline{O}} [\underline{z}^*(t, y) + \lambda |x - y|] \tag{C.3}$$

with the desired properties.

Smoothing. In the proof of the dynamic programming property (Theorem IV.7.1) and also in Section IX.4 we used the following smoothing technique. Let g be bounded and uniformly continuous on $I\!R^m$. For $k = 1, 2, \cdots$ let $\zeta_k \in C^\infty(I\!R^m)$ be such that

$$\zeta_k(\eta) \geq 0, \quad \int_{\mathbb{R}^m} \zeta_k(\eta)d\eta = 1$$

$$\zeta_k(\eta) = 0 \text{ if } |\eta| \geq k^{-1}.$$

We let

$$(C.4) \qquad g_k(\xi) = \int_{\mathbb{R}^m} g(\eta)\zeta_k(\xi - \eta)d\eta, \ \forall \xi \in \mathbb{R}^m,$$

Then $g_k \in C^\infty(\mathbb{R}^m)$ for each $k = 1, 2, \cdots$ and $\|g_k - g\| \to 0$ as $k \to \infty$. The operation in (C.4) is a convolution, and is also denoted by $g_k = g * \zeta_k$. The function g_k is often called a *mollification* of g. See proof of Theorem V.5.1.

In Section IX.4 we take $g(x) = \psi(x)$. In Step 2 of the proof Theorem IV.7.1 let $\xi = (t, x)$ and $g = f, \sigma$ or L. Then g depends on ξ, v. By, IV(6.1) and the assumption in Step 2 that f, L have compact support, g is bounded and uniformly continuous. We make a bounded uniformly continuous extension of g from $\overline{Q}_0 \times U$ to $\mathbb{R}^{n+1} \times U$. Then we define as in (C.4)

$$(C.5) \qquad g_k(\xi, v) = \int_{\mathbb{R}^{n+1}} g(\eta, v)\zeta_k(\xi - \eta)d\eta.$$

Then g_k is continuous, $g_k(\cdot, v) \in C^\infty(\mathbb{R}^{n+1})$ and $\|g_k - g\| \to 0$ as $k \to \infty$.

Smoothing on Q. In Section II.6 we take $w \in C^{1,2}(Q)$, where as before $Q = [t_0, t_1) \times O$. Then there is $Q^* \supset Q$ such that $w \in C^{1,2}(Q^*)$. Let $\tilde{w}$ be an extension of w to $\mathbb{R}^{n+1}$ that is equal to 0 on the complement of Q^*. We define $w^k(t, x)$ as in (C.4),

$$w^k(t, x) = \int_{\mathbb{R}^{n+1}} \tilde{w}(s, y)\zeta_k(t - s, x - y)dsdy.$$

Then $w^k \in C^\infty(\mathbb{R}^{n+1})$. Moreover for any multi-index $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n)$

$$D^\alpha w^k(t, x) = \int_{\mathbb{R}^{n+1}} \tilde{w}(s, y)D^\alpha \zeta_k(t - s, x - y)dsdy,$$

where

$$D^\alpha \phi = \frac{\partial^{|\alpha|}}{\partial^{\alpha_0} t \partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}\phi,$$

$|\alpha| = \alpha_0 + \alpha_1 + \cdots + \alpha_n$. Since $\tilde{w} = w$ on Q^* and $w \in C^{1,2}(Q^*)$, we can show that

$$w_t^k, w_{x_i}^k, w_{x_i x_j}^k \to w_t, w_{x_i}, w_{x_i x_j},$$

as $k \to \infty$, uniformly on compact subsets of Q^*, hence on compact subsets of Q.

D

Stochastic Differential Equations: Random Coefficients

In this Appendix we review some results about Ito-sense stochastic differential equations, with random (progressively measurable) coefficients. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $x(s) = x(s, \omega)$ be a Σ - valued stochastic processes defined on $[t, t_1] \times \Omega$. Σ is a complete separable metric space. In this book, usually $\Sigma \subset I\!R^m$ for some m. This process is called *measurable* if $x(\cdot, \cdot)$ is measurable with respect to $\mathcal{B}([t, t_1]) \times \mathcal{F}$ and $\mathcal{B}(\Sigma)$, where $\mathcal{B}(\Sigma)$ is the Borel σ - algebra.

Let $\{\mathcal{F}_s\}$ be an increasing family of σ - algebras for $t \le s \le t_1$ with $\mathcal{F}_{t_1} \subset \mathcal{F}$. Then the process is called $\mathcal{F}_s$ - *progressively measurable* if the map $(r, \omega) \to x(r, \omega)$ from $[t, s] \times \Omega$ into Σ is $\mathcal{B}_s \times \mathcal{F}_s$ measurable, where $\mathcal{B}_s = \mathcal{B}([t, s])$. The process is called $\mathcal{F}_s$ - *adapted* if $x(s, \cdot)$ is a $\mathcal{F}_s$ - measurable, Σ - valued random variable for each $s \in [t, t_1]$. Every $\mathcal{F}_s$ - progressively measurable process is $\mathcal{F}_s$ - adapted. A sufficient condition for a $\mathcal{F}_s$ - adapted process to be $\mathcal{F}_s$ - progressively measurable is that the sample paths $x(\cdot, \omega)$ be right continuous (or left continuous.) See [El, p. 15]. In particular, if the sample paths $x(\cdot, \omega)$ are continuous on $[t, t_1]$, then $\mathcal{F}_s$ - adapted is equivalent to $\mathcal{F}_s$ - progressively measurable.

A random variable θ, with values in the interval $[t, t_1]$ is called a $\mathcal{F}_s$ - stopping time if the event $\theta \le s$ is $\mathcal{F}_s$ - measurable for each $s \in [t, t_1]$.

We begin with some estimates for moments, for which we refer to [Kr1, Sec. 2.5]. Using the same notation as in Section IV.2, let $\nu = (\Omega, \{\mathcal{F}_s\}, P, w)$ be a reference probability system. Thus $(\Omega, \mathcal{F}_{t_1}, P)$ is a probability space, $\{\mathcal{F}_s\}$ is an increasing family of σ - algebras $(t \le s \le t_1)$ and w is a brownian motion of dimension d. The brownian motion w is $\mathcal{F}_s$ - adapted in the sense that the increments $w(r) - w(s)$ are independent of $\mathcal{F}_s$ for $s \le r \le t_1$.

Let $Q_0 = [t_0, t_1) \times I\!R^n$. Given $t \in [t_0, t_1)$, let $b(s, y, \omega), \gamma(s, y, \omega)$ be functions on $\overline{Q}_0 \times \Omega$ which are respectively $I\!R^n$ -valued and $n \times d$ - matrix valued. Moreover, for each $y \in I\!R^n$, the stochastic processes $b(\cdot, y, \cdot),\ \sigma(\cdot, y, \cdot)$ are $\mathcal{F}_s$ - progressively measurable on $[t, t_1]$.

Let $x(s)$ denote a $I\!R^n$ - valued, $\mathcal{F}_s$ - progressively measurable process which satisfies the stochastic differential equation

$$(D.1) \qquad dx = b(s,x(s))ds + \gamma(s,x(s))dw(s),\ t \le s \le t_1,$$

with initial data $x(t) = x^0$ ($x^0 \in I\!R^n$). In the notation we have now omitted the dependence on $\omega \in \Omega$ of b, σ, as well as of $x(s)$ and $w(s)$. We make the following assumptions about b and γ. There exist a constant K and stochastic processes $g(s), h(s)$, such that:

$$(D.2) \qquad \begin{cases} |\gamma(s,x) - \gamma(s,y)| \le K|x-y| \\ \\ |b(s,x) - b(s,y)| \le K^2|x-y|, \end{cases}$$

$$(D.3) \qquad \begin{cases} \frac{1}{2}|\gamma(s,y)|^2 \le g(s)^2 + K^2|y|^2 \\ \\ \qquad |b(s,y)| \le h(s) + K^2|y| \end{cases}$$

$$E\int_t^{t_1} [g(s)^m + h(s)^m]ds < \infty,\ \forall m > 0.$$

(In this Appendix we write $E_{tx^0} = E$, omitting dependence on the initial data tx^0.) By a standard successive approximation method, (D.1) with the initial data $x(t) = x^0$ has a pathwise unique solution. See, for example [GS2, Secs 6,7]. Let $\|\ \|$ denote the sup norm on $[t, t_1]$. From [Kr1, p.85], for every $m \ge 2$ there exists $N = N(m, K)$ such that:

$$(D.4) \qquad E\|x(\cdot) - x^0\|^m$$

$$\le N(t_1 - t)^{\frac{m}{2}-1} e^{N(t_1-t)} E\int_t^{t_1} [|x^0|^m + g(s)^m + h(s)^m]ds$$

$$(D.5) \qquad E\|x(\cdot)\|^m$$

$$\le N|x^0|^m + N(t_1 - t)^{\frac{m}{2}-1} e^{N(t_1-t)} E\int_t^{t_1} \left[|x^0|^m + g(s)^m + h(s)^m\right] ds.$$

In particular, consider $x(s)$ which satisfies the stochastic differential equation IV(2.1), where $u(\cdot)$ is a $\mathcal{F}_s$ - progressively measurable process ($u(\cdot) \in \mathcal{A}_\nu$). Let

$$(D.6) \qquad b(s,y,\omega) = f(s,y,u(s,\omega)),\ \gamma(s,y,\omega) = \sigma(s,y,u(s,\omega)).$$

Then (D.2) is immediate from IV(2.2a) if $K \ge \max(C, 1)$. If we require that $2K^2 \ge 3C^2$ and take

$$g(s) = (\tfrac{3}{2})^{\frac{1}{2}} C|u(s)|,\ h(s) = C|u(s)|$$

we get (D.3). In view of IV(2.3), (D.5) implies that $E\|x(\cdot)\|^m < \infty$ for each $m \ge 2$ (and hence also for $0 < m < 2$.)

In case the control space U is compact, as in Sections IV.6–10, recall assumptions IV(6.1). Using the inequality

$$|f(t,x,v)| \leq |f(t,x,0)| + |f(t,x,v) - f(t,x,0)|$$
$$\leq C_2 + C_1|x|,$$

assumption IV(6.1c) implies IV(2.2) with $C = \max(C_1, C_2)$. Since $u(s) \in U$ and U is compact, $|u(s)|$ is bounded. By (D.5) for every $m \geq 2$ we have, if U is compact,

$$(D.7) \qquad E\|x(\cdot)\|^m \leq B_m(1 + |x^0|^m)$$

where B_m depends on C_1, C_2 as well as $t_1 - t_0$. Also, by Hölder's inequality, for each $m \in (0,2]$ we have

$$E \| x(\cdot) \|^m \leq (E \| x(\cdot) \|^2)^{m/2} \leq (B_2(1 + |x^0|^2))^{m/2} \leq B_m(1 + |x^0|^m)$$

for suitable constant B_m. Hence (D.7) holds for every $m > 0$.

Let us next consider also $\tilde{x}(s)$ satisfying

$$((\tilde{D}.1) \qquad d\tilde{x} = \tilde{b}(s, \tilde{x}(s))ds + \gamma(s, \tilde{x}(s))dw(s), \; t \leq s \leq t_1,$$

with initial data $\tilde{x}(t) = \tilde{x}^0 (\tilde{x}^0 \in I\!R^n)$. By [Kr1, p.83], for every $m \geq 2$ there exists $N = N(m, K)$ such that

$$E\|x(\cdot) - \tilde{x}(\cdot)\|^m \leq N e^{N(t_1 - t)} |x^0 - \tilde{x}^0|^m$$

$$(D.8) \qquad + N(t_1 - t)^{\frac{m}{2} - 1} e^{N(t_1 - t)} E \int_t^{t_1} [|b(s, \tilde{x}(s)) - \tilde{b}(s, \tilde{x}(s))|^m$$
$$+ |\gamma(s, \tilde{x}(s)) - \tilde{\gamma}(s, \tilde{x}(s))|^m] ds.$$

In particular, let b, γ be as in (D.6) and $\tilde{b}(s, y, \omega) = \tilde{f}(s, y, u(s, \omega))$, $\tilde{\gamma}(s, y, \omega) = \tilde{\sigma}(s, y, u(s, \omega))$. If $x^0 = \tilde{x}^0$, the first term on the right side of (D.8) disappears. We have (see IV(6.11))

$$(D.9) \qquad E\|x(\cdot) - \tilde{x}(\cdot)\|^m \leq \bar{B}_m [\|f - \tilde{f}\|^m + \|\sigma - \tilde{\sigma}\|^m]$$

for $m \geq 2$ where $\bar{B}_m$ depends on C_1, C_2 as well as $t_1 - t_0$. As in (D.7), the case $m \in (0,2)$ is proved using Hölder's inequality.

In the proof of Lemma IV.8.1, we need an estimate for the case $\tilde{x}^0 = x^0 + h\xi, |\xi| = 1$ and $\tilde{b} = b, \tilde{\gamma} = \gamma$ as in (D.6). Let $\Delta x(s) = h^{-1}[\tilde{x}(s) - x(s)]$. Then

$$\Delta x(s) = \xi + \frac{1}{h} \int_t^s [f(r, \tilde{x}(r), u(r)) - f(r, x(r), u(r))] dr$$
$$+ \frac{1}{h} \int_t^s [\sigma(r, \tilde{x}(r), u(r)) - \sigma(r, x(r), u(r))] dw(r).$$

If we denote the last terms by (i) and (ii), then

$$|\Delta x(s)|^2 \le 3(|\xi|^2 + |(\mathrm{i})|^2 + |(\mathrm{ii})|^2)$$

By IV(6.1c), $f(s,\cdot,u(s))$ and $\sigma(s,\cdot,u(s))$ are Lipschitz continuous with constant C_1. By Cauchy - Schwartz and elementary properties of stochastic integrals,

$$E|\Delta x(s)|^2 \le 3[|\xi|^2 + (s-t)C_1^2 E\int_t^s |\Delta x(r)|^2 dr$$
$$+C_1^2 E\int_t^s |\Delta x(r)|^2 dr].$$

Since $|\xi| = 1$, Gronwall's inequality then implies

$$(D.10) \qquad E|\Delta x(s)|^2 \le B,$$

where the constant B depends on C_1 and $t_1 - t_0$.

In the proof of Lemma IV.7.1 we used an estimate for the oscillation of $x(s)$ on subintervals of $[t, t_1]$. Assume now that b, γ are bounded, and consider an interval $[\tau_1, \tau_2)$ with $t \le \tau_1 < \tau_2 \le t_1$ and $\tau_2 - \tau_1 \le 1$. Let

$$\zeta(s) = \int_{\tau_1}^s \gamma(r, x(r))dw(r), \ \tau_1 \le s \le \tau_2.$$

By using the Ito differential rule and induction, one can show that for each $\ell = 1, 2, \cdots$ there exists a constant α_ℓ such that

$$(D.11) \qquad E\|\zeta(\cdot)\|^{2\ell} \le \alpha_\ell \|\gamma\|^{2\ell}(\tau_2 - \tau_1)^\ell.$$

Then, for $\tau_1 \le s \le \tau_2$,

$$|x(s) - x(\tau_1)| \le |s - \tau_1| \ \|b\| + |\zeta(\cdot)\|,$$

$$E\|x(\cdot) - x(\tau_1)\|^{2\ell} \le \beta_\ell(\tau_2 - \tau_1)^\ell$$

for suitable constant β_ℓ, which depends on $\|b\|$ and $\|\gamma\|$. Therefore, for each $\rho > 0$

$$P(\max_{[\tau_1,\tau_2]} |x(s) - x(\tau_1)| \ge \rho) \le \rho^{-2\ell}\beta_\ell(\tau_1 - \tau_1)^\ell.$$

For the proof of Lemma IV.7.1. we take b, γ as in (D.6), $\ell = 2, \rho = \gamma/2, \tau_1 = s_j, \ \tau_2 = s_{j+1}, I_j = [s_j, s_{j+1})$ to get

$$(D.12) \qquad P(\max_{I_j} |x(s) - x(s_j)| \ge \frac{\gamma}{2}) \le \frac{C(s_{j+1} - s_j)^2}{\gamma^4}$$

with $C = 16\beta_2$.

Instead of (D.3), let us now assume

$$(D.3') \qquad \begin{cases} \gamma(s,y) \text{ is bounded} \\ |b(s,y)| \leq K(1+|y|). \end{cases}$$

Then there exists $k > 0$ such that

$$(D.13) \qquad E[\exp(k \parallel x(\cdot) \parallel^2)] < \infty.$$

This inequality is used in Section VI.3. To obtain (D.13), let

$$\eta(s) = \int_t^s \gamma(r, x(r))dw(r).$$

From (D.1) and Gronwall's inequality,

$$\parallel x(\cdot) \parallel \leq K_1(1 + |x^0| + \parallel \eta(\cdot) \parallel)$$

for some constant K_1. Hence, (D.13) holds for small enough $k > 0$ if there exists $k_1 > 0$ such that

$$(D.14) \qquad E[\exp(k_1 \parallel \eta(\cdot) \parallel^2)] < \infty.$$

Let $Q(\lambda) = P(\parallel \eta(\cdot) \parallel > \lambda)$. By an exponential martingale inequality [StV,p. 87]

$$Q(\lambda) \leq 2n \exp(-k_2 \lambda^2)$$

for some $k_2 > 0$, which depends on $t_1 - t$ and $\parallel \gamma \parallel$. Then

$$\begin{aligned} E[\exp(k_1 \parallel \eta(\cdot) \parallel^2] &= -\int_0^\infty e^{k_1 \lambda^2} dQ(\lambda) \\ &= 2k_1 \int_0^\infty \lambda e^{k_1 \lambda^2} Q(\lambda) d\lambda, \end{aligned}$$

which is finite if $k_1 < k_2$.

Feynman - Kac formula. We conclude this section of the Appendix by reviewing the following Feynman - Kac formula (D.15) for the case of solutions to the stochastic differential equation (D.1). Let $W \in C^{1,2}(\overline{Q})$, where $Q = [t_0, t_1) \times O$ with $O \subset I\!R^n$ an open bounded set. Let

$$\Gamma(s) = \exp \int_t^s c(r)dr$$

where $c(\cdot)$ is a $\mathcal{F}_s$ - progressively measurable process and $c(r) \leq M < \infty$. Let θ be any $\{\mathcal{F}_s\}$ - stopping time, such that $t \leq \theta \leq \tau$, where τ is the exit time of $(s, x(s))$ from Q. Then

$$(D.15) \qquad W(t, x^0) = E\{-\int_t^\theta \Gamma(s)(AW(s, x(s)) + c(s)W(s, x(s)))ds$$

$$+\Gamma(\theta)W(\theta, x(\theta))\},$$

$$AW(s,x(s)) = W_t(s,x(s)) + \frac{1}{2}\mathrm{tr}\,(\gamma\gamma')(s,x(s))D_x^2W(s,x(s)) \\ +b(s,x(s))\cdot D_xW(s,x(s)).$$

To obtain (D.15), we apply the Ito differential rule to $\Gamma(s)W(s,x(s))$:

$$d(\Gamma(s)W(s,x(s)) = W(s,x(s))d\Gamma(s) + \Gamma(s)dW(s,x(s)).$$

By integrating from t to θ and using $\Gamma(t) = 1$,

$$\Gamma(\theta)W(\theta, x(\theta) - W(t, x^0)$$

$$= \int_t^\theta \Gamma(s)[AW(s,x(s))+c(s)W(s,x(s))]ds + \int_t^\theta \Gamma(s)(D_xW\cdot\gamma)(s,x(s))dw(s).$$

The expectation of the last term is 0, which gives (D.15).

In Chapter IV (formula IV(3.20)) we take $c(s) = l(s,x(s),u(s))$, where $u(\cdot)$ is a $\mathcal{F}_s$ -progressively measurable control process. In Chapter VI (formula IV(3.18)) we take $c(s) = l(s,x(s))$.

Progressive measurability. As in Section VIII.3, let $z(s)$ be a process of bounded variation which is progressively measurable with respect to filtration $\mathcal{F}_s$. We assume that $\mathcal{F}_s$ is right continuous. Then the total variation of $z(\cdot)$ on $[0,t)$ is given by

$$\xi(t) = \sup\{\sum_{i=1}^n |z(t_i) - z(t_{i-1})| : 0 = t_0 < t_1 < \cdots < t_n = t\}.$$

Hence $\xi(s)$ is also $\mathcal{F}_s$ - progressively measurable. We continue by constructing a $\mathcal{F}_s$ - progressively measurable Radon–Nikodym derivative $\hat{u}(s)$ of $z(s)$ with respect to $d\xi(s)$. Set $z(s) = 0$ for $s \le 0$. Then $\xi(s) = 0$ for $s \le 0$. For $\varepsilon > 0$ define,

$$d^\varepsilon(s) = \xi(s+\varepsilon) - \xi(s-\varepsilon)$$

$$u^\varepsilon(s) = \begin{cases} [z(s+\varepsilon) - z(s-\varepsilon)](d^\varepsilon(s))^{-1}\ , & d^\varepsilon(s) \ne 0 \\ u_0 & , \ d^\varepsilon(s) = 0, \end{cases}$$

where u_0 is any element of U with $|u_0| \le 1$. Then $u^\varepsilon(s)$ is progressively measurable with respect to $\mathcal{F}_s^\varepsilon = \mathcal{F}_{s+\varepsilon}$. Observe that by the definition of $\xi(s), |u^\varepsilon(s)| \le 1$. Finally we define

$$\hat{u}(s) = \limsup_{\varepsilon\downarrow 0} u^\varepsilon(s).$$

Then $\hat{u}(s)$ is progressively measurable with respect to $\mathcal{F}_s^\varepsilon$ for every $\varepsilon > 0$. Since $\mathcal{F}_s$ is right continuous, $\hat{u}(s)$ is $\mathcal{F}_s$ - progressively measurable. Moreover by a differentiation theorem for Radon measures [EG, Theorem 2 page 40], we have

$$z(t) = \int_{[0,t)} \hat{u}(s)d\xi(s), t > 0.$$

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