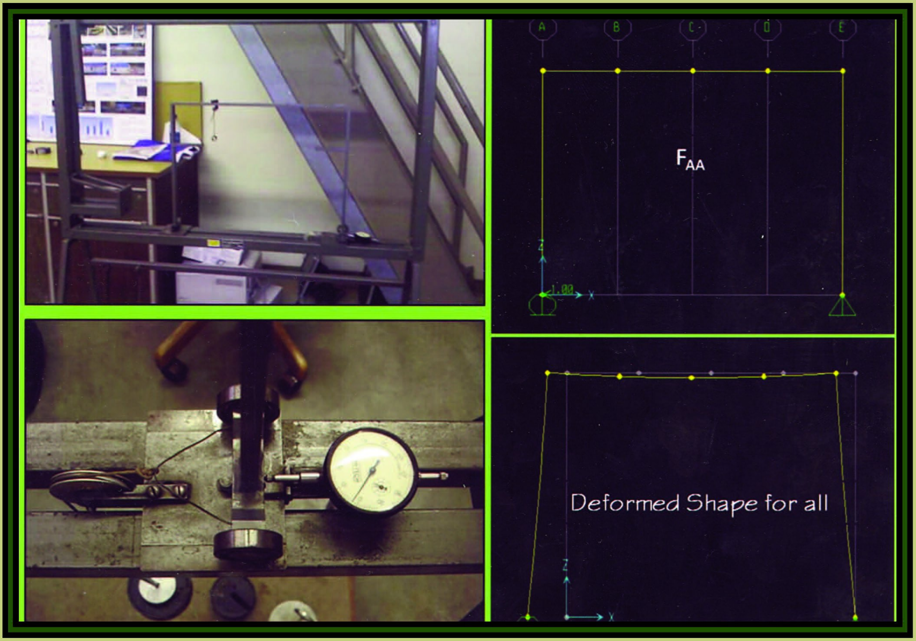
INDETERMINATE STRUCTURAL ANALYSIS



Indeterminate Structural Analysis

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Preface

The title of this book is "Indeterminate Structural Analysis", not "Structural Analysis" as most of the books on this subject are titled. Many textbooks have been written on structural analysis over the past several years with a twofold composition. They essentially deal with analysis of statically determinate structures followed by analysis of statically indeterminate structures using the force method, displacement methods (classical methods such as slope-deflection and moment distribution) and the stiffness method. Thus, the material covered in existing textbooks on structural analysis contains more than what is necessary to learn indeterminate structural analysis. As a result, these books become bulky and all their material cannot, and need not, be covered in a single course on indeterminate structural analysis. Moreover, these books rarely include an as-needed discussion of the unit load method, which is arguably the best method to calculate deflections when solving problems by the force method. Hence, the authors set out to create this book.

This book covers the analysis of indeterminate structures by force method, displacement method and stiffness method in a total of six chapters. The first chapter deals with application of the force method to analysis of beam, frame and truss structures. The unit load method is discussed with reference to the analysis of statically indeterminate structures. A few examples are discussed to illustrate these concepts. The second and third chapters deal with analysis of indeterminate structures by displacement methods. In the second chapter, concepts of slope-deflection method are developed and applied to beam and frame structures. The third chapter deals with developments of concepts of the moment distribution method. These concepts are then applied to beam and frame structures. The fourth chapter develops the concepts of the stiffness method. These are subsequently applied to beam structures. The fifth and sixth chapters deal with application of the stiffness method to frame and truss structures. Throughout the book, few but illustrative examples are discussed under each method. The intent is to cover as much material as is needed conceptually with minimal, yet sufficient, examples so the student can understand indeterminate structural analysis methods without being overwhelmed. This way, the book is kept less bulky compared to existing books on structural analysis. In addition, keeping the textbook concise will reduce the price far below that of existing textbooks, saving money for students. We believe this will be a big selling point because the amount of material covered is not compromised in covering the material in a concise manner. This is in addition to the fact that, this book is written by three Professors of Civil Engineering who have had vast experience in teaching and research in the area of structural analysis.

It is hoped that this experience is reflected in the write-up of this book so that it serves our twofold objective. The first objective is that we hope the instructor following this book as a textbook for his/her course on indeterminate structural analysis feels that all the required material is indeed covered in this textbook. Secondly, we hope that the students taking this course find the book and material covered easy to understand.

The authors are thankful to Mr. Kyle Anderson and Mr. AnhDuong Le, former graduate students in the Department of Civil and Environmental Engineering at California State University, Fullerton for going through the manuscript and making constructive comments. We also appreciate the editing work done by Mr. Alexander Motzny, undergraduate student in the Department of Civil and Environmental Engineering at California State University, Fullerton.

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Introduction

In structural analysis, there are three basics types of methods used for analyzing indeterminate structures. They are:

1. Force Method (Method of Consistent Deformation)
2. Displacement Methods (Slope-Deflection and Moment Distribution)
3. Stiffness Method

General idea about these methods:

The force method of analysis is an approach in which the reaction forces are found directly for a given statically indeterminate structure. These forces are found using compatibility requirements. This method will be discussed with more detail in Chapter 1.

The displacement methods use equilibrium requirements in which the displacements are solved for and are then used to find the forces through force-displacement equations. More on these methods can be found in Chapters 2 and 3.

The stiffness method is also considered a displacement method because the unknowns are displacements, however the forces and displacements are solved for directly. In this book, it will be considered separately due to procedural differences from the other displacement methods. The stiffness method is very powerful, versatile, and commonly used. This method will be discussed in Chapters 4, 5, and 6.

Chapter 1

Analysis of Statically Indeterminate Structures by the Force Method (Flexibility Method or Method of Consistent Deformation)

1.1 Basic Concepts of the Force Method

The force method (which is also called the flexibility method or the method of consistent deformations) uses the concept of structural Static Indeterminacy (SI). It is very conceptual in nature. The force method becomes cumbersome when the Static Indeterminacy of a structure is large. The results obtained by solving the problem using the force method, are all the unknown forces (such as reactions at the supports).

If one is interested in finding rotational or translational displacements of an indeterminate structure, they must be obtained separately using any methods of finding displacements (unit load method, moment area method or conjugate beam method for example).

This method is applicable for any kind of structure: beam, frame or truss. It is to be noted that beam and frame structures are predominantly bending (flexure) structures while trusses are predominantly direct stress structures (tension or compression) in nature. The truss members are not subjected to bending. In other words, all loads are axial.

1.1.1 List of Symbols and Abbreviations Used in the Force Method

Symbols and terms are defined along with equations. However, some are not in equations so they are defined below:

$(\Delta_A)_L$: Deflection at point A due to applied loading

$(\Delta_A)_R$: Deflection at A due to redundant loading $\equiv R_A * \delta_{aa}$

α_{aa}: Rotational deflection at A due to a unit load at A

θ_A: Rotational deflection at A due to applied loading

δ_{aa}: Deflection at A due to a unit load at A

1.2 Static Indeterminacy

The Static Indeterminacy (SI) for beams and frames is defined as,

$$SI = n_u - n_e \tag{1.1}$$

Where, n_u = Number of unknown support reactions

n_e = Number of equations of equilibrium

In general for a two-dimensional structure, there are three equations of equilibrium $(n_e = 3)$ and for a three-dimensional structure there are six $(n_e = 6)$. The static indeterminacy refers to the number of reactions that are unsolvable using basic statics.

This implies that a structure is statically determinate if $SI = 0$. An example of this would be a simply supported beam (one end pinned and the other having a roller support). This structure would have three unknowns, the reactions at the pin in both the x and y directions and the reaction at the roller in the y direction. The number of equilibrium equations would be three $(\sum F_x = 0, \sum F_y = 0, \text{ and } \sum M = 0)$. Therefore $SI = 0$.

If $SI \geq 1$, the structure is said to be statically indeterminate to that degree (value of SI), therefore the degree of Static Indeterminacy is equal to the value of $(n_u - n_e)$. It can be also said that the structure has "SI" number of redundants.

To solve a statically indeterminate beam (or frame) using the force method we will make use of redundant forces. A *redundant force* is one, which cannot be solved using static equilibrium equations alone. The forces will be taken out and reapplied so that the considered structure is always statically determinate. Additionally, the *principle of superposition* is applied and deflection will be found as an intermediate step to solving for a given redundant. This process will be further explained in the text along with examples.

For a truss, static indeterminacy involves both external and internal indeterminacy because of the internal members in a truss.

Static indeterminacy (SI) in the case of a truss is defined as,

$$SI = b + r - 2j \tag{1.2}$$

Where, b = Number of members in the truss

r = Number of reactions at the supports

j = Number of joints in the truss

The analysis of an indeterminate structure is split into a series of determinate structures acted on by applied loads (in the original structure) and acted on by redundant force(s). In both cases the deflections need to be found. Hence, the unit load method for finding deflection will be discussed briefly for a determinate structure.

1.3 Basic Concepts of the Unit Load Method for Deflection Calculation

The unit load method is also referred to as the method of virtual work. The basic equation to calculate displacement (whether translational or rotational) at a given point of a beam or frame is given as,

$$\Delta = \int \frac{M\, m\, dx}{EI} \tag{1.3}$$

Where, M = Moment at any point in a structure due to applied loads

m = Moment at any point in a structure due to the unit load (force or moment) at the point of interest corresponding to the parameter of interest (deflection or rotation)

E = Modulus of elasticity

I = Moment of inertia of the cross section of a member

Note: The above equation has been derived using energy principles.

To find m, the applied loads are removed and a unit load (force or moment) is applied at the point of interest, or redundancy, in a structure. If one is interested in finding a vertical deflection at a point in a structure, then a unit vertical force is applied (as it corresponds to vertical deflection). If one is interested in finding horizontal deflection at a point, then a unit horizontal force is applied at that point as it corresponds to horizontal deflection. Similarly, rotational displacement at a point in a structure is found by applying a unit moment as it corresponds to rotation.

The basic expression for finding displacement at a given point on a truss is given as,

$$\Delta = \sum \frac{NnL}{AE} \tag{1.4}$$

Where, N = Force in a truss member due to applied loads

n = Force in a truss member due to unit load applied at the point where the deflection is to be obtained

L = Length of a truss member

A = Cross sectional area of a truss member

Note: The summation in Eq. 1.4 includes all truss members.

To find n in a truss, the applied loads are removed and a unit load is applied at the point of interest. For example, if one is interested in finding a horizontal deflection at a point in a truss, a horizontal unit load is applied at that point. If vertical deflection at a point in a truss is of interest, then a unit vertical load is applied at that point. In the force method for a truss, whole members are taken as redundant.

1.4 Maxwell's Theorem of Reciprocal Deflections

This theorem states that,

$$\delta_{AB} = \delta_{BA} \tag{1.5}$$

where, δ_{AB} = Deflection at A due to a unit load applied at B

δ_{BA} = Deflection at B due to a unit load applied at A

Maxwell's theorem reduces the work needed to solve a statically indeterminate structure as it relieves several computations of deflection. For more details, the reader is advised to read the books by Chajes (1983), Wang (1953) and Hibbeler (2012).

1.5 Application of Force Method to Analysis of Indeterminate Beams

1. Calculate the Static Indeterminacy (SI) of the structure using Eq. 1.1 or Eq.1.2 depending upon whether the structure is beam, frame, or truss.
2. Choose one of the reaction forces (or internal members of the truss) as the redundant force. One at a time if there are multiple redundancies.
3. Split the statically indeterminate structure into a determinate structure (acted upon by applied loads on the structure) and determinate structure(s) acted on by the redundant forces (one at a time).
4. Analyze the determinate structures by the unit load method to find the displacement Δ_L, which is the displacement for the applied loading and redundant removed. Then find δ, which is the displacement for the unit load only, at the point of redundancy. If a moment is taken as redundant, the corresponding displacements will be θ and α.
5. Finally, formulate equation(s) of displacement compatibility at the support(s) (in the case of beams and frames). In the case of trusses, displacement compatibility of truss bars will be used.
6. Solve these equation(s) to get the redundant force(s).
7. Calculate all the reactions at the supports (in addition to the redundant force already determined in step 6) using principles of statics.

A total of 3 examples are solved in this chapter: one for a statically indeterminate beam, one for a statically indeterminate frame, and one for statically indeterminate truss. While any of the methods for finding deflection (double integration, moment area method, conjugate beam method, unit load method, or any other existing method) can be used to find displacements (translation or rotation), the authors recommend use of the unit load method because it is conceptually straight forward and easy to use.

1.5.1 Sign Convention

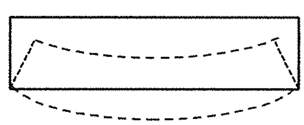
The following sign convention will be used for the force method:

- Counter-clockwise moments and displacements are positive

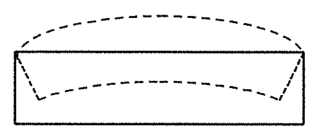
This is often referred to as the right hand rule.

When a member undergoes bending:

- Compression on a member's top fiber is positive bending
- Compression on a member's bottom fiber is negative bending.



a) Positive bending – top fiber compression



b) Negative bending – bottom fiber compression

Figure 1.1: Bending sign convention

1.5.2 Example of an Indeterminate Beam

An example dealing with the analysis of a statically indeterminate beam using the force method is solved below.

Example 1.5.2.1

Determine the reactions at the supports for the statically indeterminate structure shown in Fig. 1.2 by the force method. Use R_B as the redundant. Take E = 29000 ksi and I = 446 in^4.

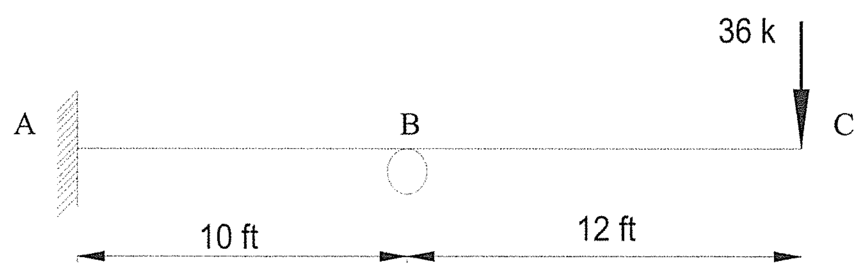


Figure 1.2: Statically indeterminate beam

Solution:

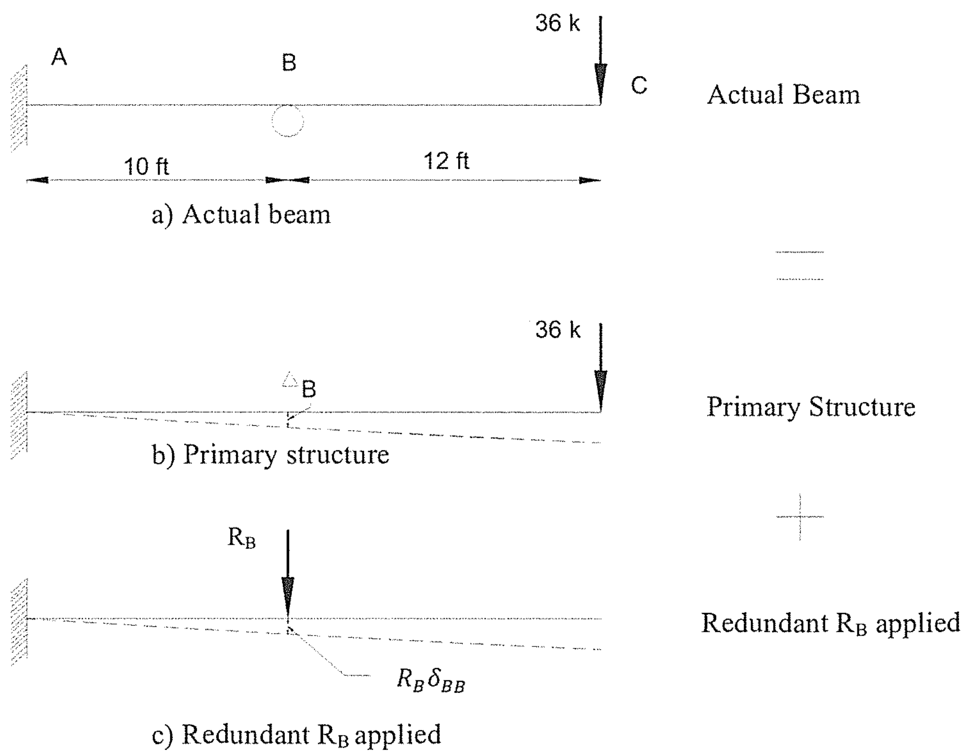


Figure 1.3: Two determinate structures

The given indeterminate structure is split into two determinate structures as shown in Fig. 1.3b and 1.3c choosing R_B as the redundant force. The basic equation used is as given in Eq. 1.3. This is stated again below:

$$\Delta = \int \frac{M\, m\, dx}{EI} \tag{1.3}$$

The procedure will be followed as it is stated earlier in this section.

Step 1.

$$SI = n_u - n_e = 4 - 3 = 1$$

Step 2

Choose R_B as the redundant force (in the problem statement).

Step 3

The two determinate structures are shown below with Fig. 1.4a acted on by the applied loading, and Fig. 1.4c acted on by the redundant force R_B (unit load). The deflection at B for the statically determinate structure $(\Delta_B)_L$ due to applied loads can be obtained from Eq. 1.3 using the values of M and m. Figures 1.4a – 1.4d are used to calculate M and m.

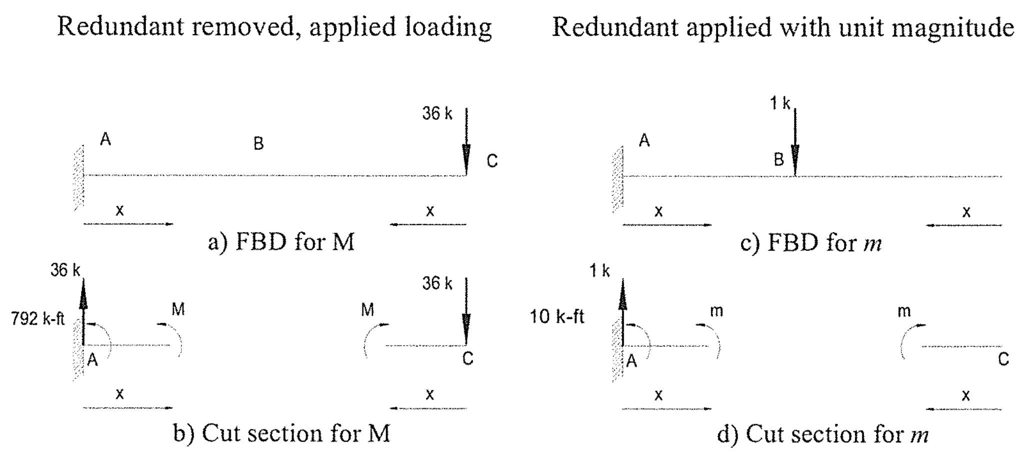


Figure 1.4: FBD for M and m (Δ_B) and cut sections

The values for M and m will depend on the origin chosen and the corresponding change in limits. In doing this, it may simplify the integration and the final value of deflection will be the same.

Step 4

Deflection at B due to the applied loads

The required values for calculation of deflection are tabulated in Table 1.1.

Table 1.1: Calculation of deflection $(\Delta_B)_L$ using Fig. 1.3a and b

Portion of the beam	AB	BC
Origin	A	C
Limit	$x = 0 \; to \; x = 10$	$x = 0 \; to \; x = 12$
M	$36x - 792$	$-36x$
m	$x - 10$	0

Calculate M

Portion AB:

Reaction forces: $R_A = \mathbf{36\ k}$ and

$$M_A = 36 * (10 + 12) = \mathbf{792}^{k-ft}$$

Equilibrium equation:

$$M + 792 - 36x = 0 \rightarrow \boldsymbol{M = 36x - 792}$$

Portion BC:

Equilibrium equation: $-M - 36x = 0 \rightarrow \boldsymbol{M = -36x}$

Calculate m

Portion AB:

Reaction forces: $R_A = 1k$ and $M_A = 1 * 10 = \mathbf{10}^{k-ft}$

Equilibrium equation: $m + 10 - x = 0 \rightarrow \boldsymbol{m = x - 10}$

Portion BC:

Equilibrium equation: $\boldsymbol{m = 0}$

It is to be noted that the value of M (shown in Table 1.1) is calculated using Fig. 1.4a & 1.4b while the value of *m* is calculated from Fig. 1.4c & d. The determinate structure shown in Fig. 1.4c is the same determinate structure as shown in Fig. 1.4a but acted on by a unit downward load at B (with no given applied loads) as it is assumed that the vertical deflection at B is downward. If at the end of the calculation, the deflection at B comes out to be positive, that means the actual deflection is downward. On the other hand, if the final deflection at B comes out to be negative, it means that the actual deflection at B is upward.

Substituting the values of M and *m* (from Table 1.1) in Eq. 1.3, the deflection $(\Delta_B)_L$ (deflection at B due to the applied loads) is calculated as,

$$\Delta_B = \sum \int \mathrm{Mm} \frac{dx}{EI} = \frac{1}{EI} \int_0^{10} (36x - 792)(x - 10)dx \qquad (1.6)$$

$$\Delta_B = \frac{33600}{EI} \tag{1.7}$$

The *deflection at B due to a unit value of the redundant force* R_B (δ_{bb}) is obtained from Fig. 1.5 as shown in Table 1.2 below.

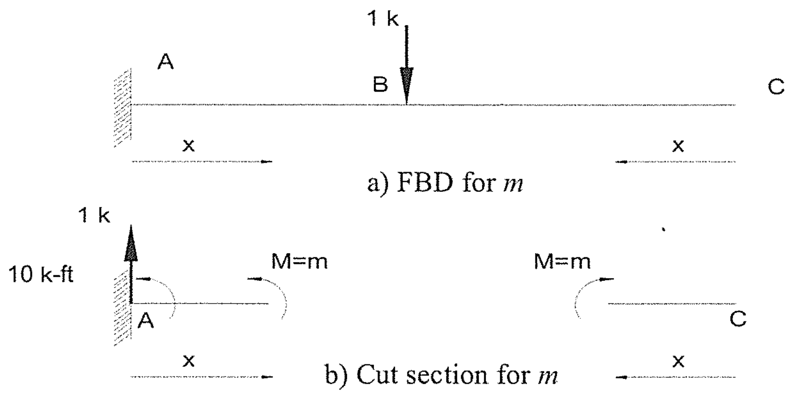


Figure 1.5: FBD for *m* (δ_{bb}) and cut section

Table 1.2: Calculation of deflection (δ_{bb}) using Fig. 1.4

Portion of the beam	AB	BC
Origin	A	C
Limit	x = 0 to x = 10	x = 0 to x = 12
M = m	$x - 10$	0

Substituting the values of M and m $(M = m)$ in eq. 1.3,

$$\delta_{bb} = \sum \int \mathrm{Mm}\frac{dx}{EI} = \frac{1}{EI}\int_0^{10}(x-10)(x-10)dx \tag{1.8}$$

The deflection at B due to a unit value of the redundant force (R_B) is obtained as,

$$\delta_{bb} = \frac{1000}{3EI} \tag{1.9}$$

Step 5

Equation of compatibility of displacement at joint B requires that,

$$(\Delta_B)_L - (\Delta_B)_R = 0 \tag{1.10}$$

Where,

$$(\Delta_B)_R = R_B * (\delta_{bb})$$

This equation is essentially saying, the total vertical displacement at B has to be zero as it is a roller joint.

Substituting the values of $(\Delta_B)_L$ and (δ_{bb}) calculated above, Eq. 1.10 can be rewritten as,

$$\frac{33600}{EI} + R_B \times \frac{1000}{3EI} = 0 \tag{1.11}$$

Step 6

Solving Eq. 1.11 above, R_B can be obtained as,

$$R_B = \mathbf{-100.8}\boldsymbol{k} \uparrow$$

This shows that R_B is upward, not downward, as assumed in Fig. 1.3c.

Step 7

Once the redundant force (R_B) is obtained, then the remaining reactions at A (R_A and M_A) can easily be obtained from equilibrium equations.

They are calculated using Fig. 1.6 as,

$$\overset{+}{\rightarrow} \Sigma F_x = 0 \rightarrow \boldsymbol{A_x = 0}$$

$$+\uparrow \Sigma F_y = 0 \rightarrow A_y + 100.8 - 36 = 0 \rightarrow \boldsymbol{A_y = -64.8\ k} \downarrow$$

$$\overset{+}{\circlearrowleft} \Sigma M_A = 0 \rightarrow M_A + 100.8(10) - 36(22) = 0 \rightarrow \boldsymbol{M_A = -216}^{\ k\text{-}ft} \circlearrowright$$

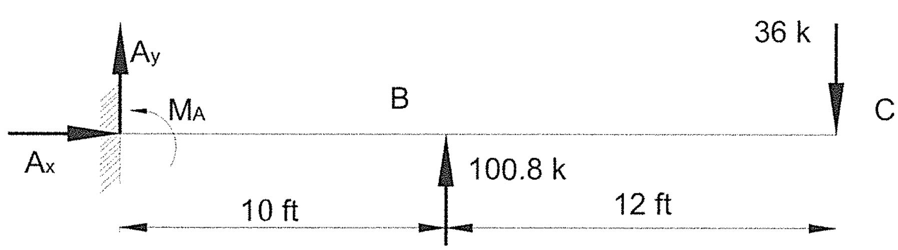


Figure 1.6: Final reactions for the indeterminate beam

It has been shown above by solving a simple example that when solving a statically indeterminate structure by the force method; first, write the correct expressions for M and m, and then integrate the expression to solve for deflection within the specified limits (consistent with the chosen origin).

1.5.3 Structures with Several Redundant Forces

As stated earlier, it is to be noted that if a structure has several redundant forces (i.e. $SI \geq 1$), then indeterminate structural analysis of the structure would involve obtaining redundant forces through solution of simultaneous equations. This will be followed by obtaining the remaining reactions at the supports (other than the redundant forces) through principles of statics as done in Ex. 1.5.2.1.

The reader is advised to see other literature for detailed information such as those found in the references of this book.

1.6 Application of the Force Method to Indeterminate Frames

The basic procedure for analysis of statically indeterminate frames essentially remains the same as outlined in Sec. 1.5, and as illustrated for a beam in Example 1.5.2.1 in Sec. 1.5.

Although the analysis of an indeterminate frame is, conceptually, very much similar to that of the beam, a frame consists of beams and columns so the analysis is slightly more complicated. After following the example below, it will be clear how to apply the force method to indeterminate frames.

1.6.1 Examples of an Indeterminate Frame

A structural analysis dealing with a statically indeterminate frame by the force method is shown below in Example 1.6.1.

Example 1.6.1.1

Determine the reactions at the supports of the frame shown in Fig. 1.7 using the force method. A = 100 in^2, E = 29000 ksi and I = 833 in^4.

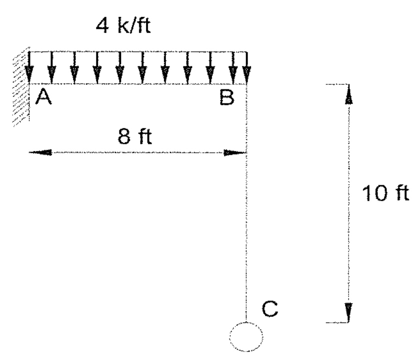


Figure 1.7: Statically indeterminate frame

Solution

The procedure followed is as stated in Sec. 1.5.

Step 1

$$SI = n_u - n_e = 4 - 3 = 1$$

Step 2

Choose M_A as the redundant moment.

Step 3

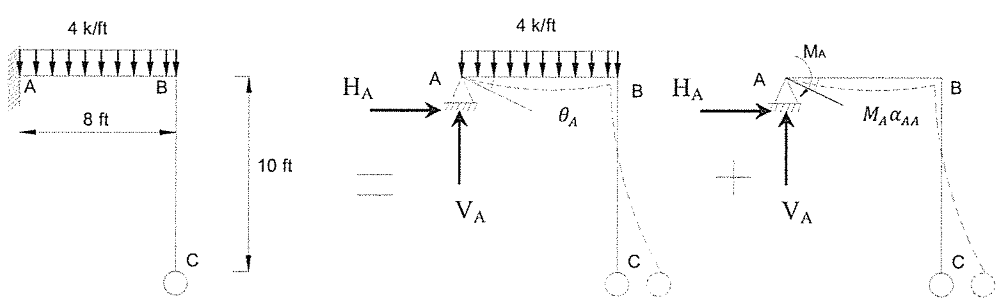
The given statically indeterminate structure is split into two determinate structures as shown in Fig. 1.8, with the redundant moment removed and with the applied loading as shown in Fig. 1.8b. In Fig. 1.8c, the frame is acted on by the redundant moment M_A. The rotational deflection at A due to applied loads is θ_A, and due to the unit load, M_A, is

$M_A \times \alpha_{AA}$. These rotations can be obtained from Eq. 1.3 by finding the values of M using Fig. 1.8b and respective values of m using Fig. 1.8c. In both cases H_A and V_A are found using static equilibrium equations.

Note on symbols: In general, (δ_{HAHA}) represents the horizontal deflection at A due to a unit horizontal unit load at A (i.e. $H_A = 1$). Similarly, (δ_{VAHA}) represents the vertical deflection at A due to a unit horizontal unit load at A. Along the same lines, (δ_{HAVA}) and (δ_{VAVA}) represent the horizontal and vertical deflection at A respectively due to a unit vertical load at A (i.e. $V_A = 1$).

Note: The values for M and m will depend on the origin chosen (with the corresponding change in limits). As can be expected, the final value of deflection will be the same irrespective of how it is done.

This frame is statically indeterminate to the first degree. Since we chose the moment reaction at A as the redundant, the support at A will become a pin as seen in Fig. 1.8.



a) Actual frame b) Primary structure c) Redundant M_A applied

Figure 1.8: Given indeterminate and corresponding determinate structures

Applying the principle of superposition to the frame yields:

$$-\theta_A - M_A * \alpha_{AA} = 0 \tag{1.12a}$$

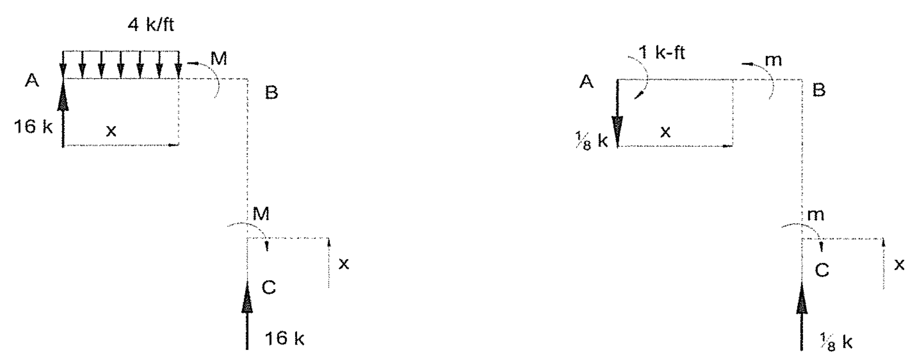
In this case, both θ_A and $M_A * \alpha_{AA}$ are negative because they both create a clockwise rotation at joint A. This is negative by the sign convention defined in section 1.5.1. Equation 1.12 can also be written as:

$$\theta_A + M_A * \alpha_{AA} = 0 \tag{1.12b}$$

Eq. 1.12b can also be found by considering that both θ_A and $M_A * \alpha_{AA}$ create compression at the top fiber of member AB.

Step 4

Use the unit load method to calculate θ_A:



a) Applied loads

b) Unit load M_A

Figure 1.9: Bending moments due to applied and unit loads

$$\rightarrow \theta_A = \sum \int Mm_\theta \frac{dx}{EI} = \frac{1}{EI} = \int_0^8 (16x - 2x^2)\left(1 - \frac{x}{8}\right) dx = \frac{256}{3EI}$$

Use virtual work (unit load method) to calculate α_{AA}:

Table 1.3: Deflection calculation for θ_A

Portion of the beam	AB	BC
Origin	A	C
Limit	x = 0 to x = 8	x = 0 to x = 10
M	$16x - 2x^2$	0
m_θ	$1 - \frac{x}{8}$	0

Table 1.4: Deflection calculation for α_{AA}

Portion of the beam	AB	BC
Origin	A	C
Limit	x = 0 to x = 8	x = 0 to x = 10
M = m_θ	$1 - \frac{x}{8}$	0

$$\rightarrow \alpha_{AA} = \sum \int \mathrm{m_\theta m_\theta} \frac{dx}{EI} = \frac{1}{EI} = \int_0^8 \left(1 - \frac{x}{8}\right)^2 dx = \frac{8}{3EI}$$

Step 5

Equation of compatibility:

$$\theta_A + M_A * \alpha_{AA} = 0$$

Step 6

Plugging in the values for deflection:

$$\frac{256}{3EI} + M_A\left(\frac{8}{3EI}\right) = 0 \rightarrow \boldsymbol{M_A = -32^{k-ft}} \circlearrowleft$$

Here, M_A is negative, which indicates that the moment is opposite to clockwise assumed direction of M_A in Fig. 1.8c.

Step 7

Use static equilibrium equations to calculate the remaining support reactions:

$$\overset{+}{\rightarrow} \Sigma F_x = 0 \rightarrow \boldsymbol{H_A = 0}$$

$$\overset{+}{\circlearrowleft} \Sigma M_A = 0:$$

$$M_A - 32(4) + V_C(8) = 0 \rightarrow \boldsymbol{V_C = 12\ k} \uparrow$$

$$+\uparrow \Sigma F_y = 0:$$

$$V_A + 12 - 32 = 0 \rightarrow \boldsymbol{V_A = 20\ k} \uparrow$$

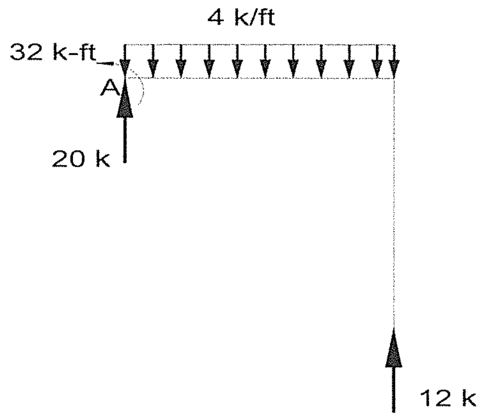


Figure 1.10: Final reactions and moments for the indeterminate frame

This completes solution of the problem.

1.7 Application of Force Method to Analysis of Indeterminate Trusses

The analysis procedure for a statically indeterminate truss follows the same lines of beams and frames, discussed in Sec. 1.3. The basic equation used for calculating deflection is given by Eq. 1.4 and stated here again as,

$$\Delta = \sum \frac{NnL}{AE} \tag{1.4}$$

An example dealing with the analysis of a statically indeterminate truss is solved in Example 1.7.1.

Example 1.7.1

Determine the reactions at the supports of the truss shown in Fig. 1.11 using the force method. AE is constant.

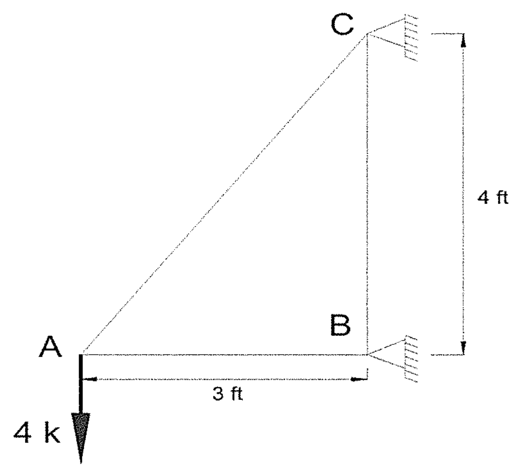


Figure 1.11: Statically indeterminate truss

Solution

Step 1

Degree of indeterminacy $= b + r - 2j$

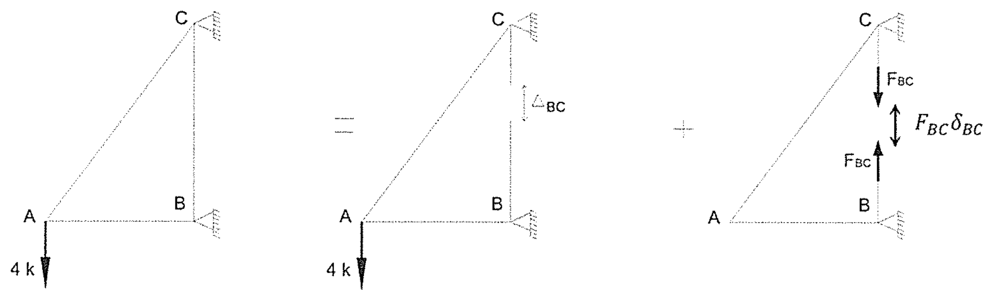
$$= 3 + 4 - 2(3) = 1$$

Step 2

Choosing BC as the redundant, this member will be "cut" to make the truss statically determinate.

Step 3

The given statically indeterminate structure is split into two determinate structures as shown in Fig. 1.12. Fig. 1.12b shows the structure under the given loading, and Fig. 1.12c shows the truss with the redundant unit load applied.



a) Actual Truss b) Primary structure c) Redundant F_{BC} applied

Figure 1.12: Statically indeterminate and corresponding determinate trusses

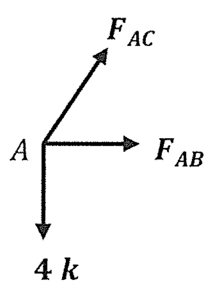
Applying the principle of superposition to the truss yields:

$$\Delta_{BC} + F_{BC} * \delta_{BC} = 0 \tag{1.13}$$

Step 4

Use the unit load method to calculate Δ_{BC}:

Calculate N and n for each member in both cases: real load and virtual unit load as is shown in Fig. 1.13.



$$+\uparrow \Sigma F_y = 0:$$

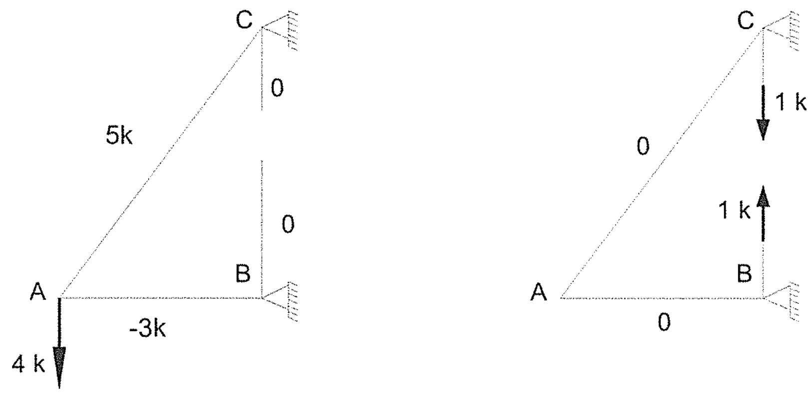
$$-4 + \boldsymbol{F_{AC}}\left(\frac{4}{5}\right) = 0 \rightarrow \boldsymbol{F_{AC} = 5\ k\ (T)}$$

$$\overset{+}{\rightarrow} \Sigma F_x = 0:$$

$$\boldsymbol{F_{AB}} + \boldsymbol{F_{AC}}\left(\frac{3}{5}\right) = 0$$

$$\boldsymbol{F_{AB} = -3\ k\ (C)}$$

a) Calculation of $\boldsymbol{F_{AB}}$ and $\boldsymbol{F_{AC}}$



b) Calculation of N c) Calculation of n

Figure 1.13: Calculation of forces for N and n

$$\Delta_{BC} = \Sigma \frac{nNL}{AE} = \frac{0(5)5}{AE} + \frac{0(-3)3}{AE} + \frac{1(0)4}{AE} = 0 \quad (1.14)$$

Use the unit load method to calculate δ_{BC}:

$$\delta_{BC} = \Sigma \frac{n^2 L}{AE} = \frac{0^2(5)}{AE} + \frac{0^2(3)}{AE} + \frac{1^2(4)}{AE} = \frac{4}{AE} \quad (1.15)$$

Step 5

The compatibility equation given by Eq. 1.13 is repeated below.

$$\Delta_{BC} + F_{BC} * \delta_{BC} = 0 \quad (1.13)$$

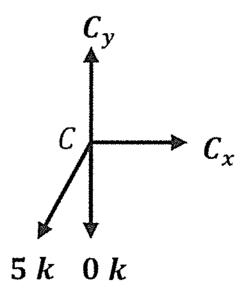
Step 6

From equation (1.13) $\rightarrow 0 + F_{BC} * \frac{4}{AE} = 0 \rightarrow \boldsymbol{F_{BC} = 0}$

Using this result, the forces in other members and the support reactions can be calculated easily using the method of joints.

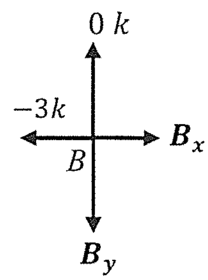
Step 7

The method of joints for B and C along with the final reactions are shown below in Fig. 1.14:



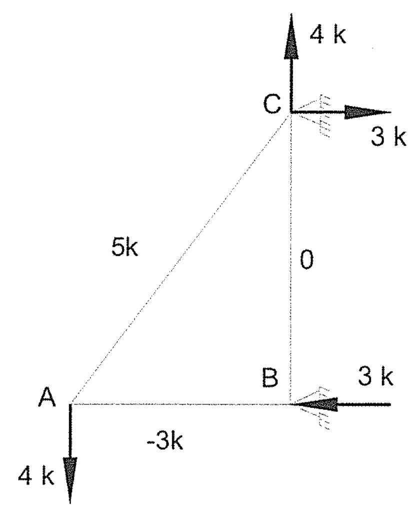
$$+\uparrow \Sigma F_y = 0:$$
$$C_y + 5\left(\tfrac{4}{5}\right) = 0$$
$$\boldsymbol{C_y = 4\ k} \uparrow$$
$$\overset{+}{\rightarrow} \Sigma F_x = 0:$$
$$C_x + 5\left(\tfrac{3}{5}\right) = 0$$
$$\boldsymbol{C_x = 3\ k} \rightarrow$$

a) Calculation of $\boldsymbol{C_x}$ and $\boldsymbol{C_y}$



$$+\uparrow \Sigma F_y = 0:$$
$$-B_y + 0 = 0$$
$$\boldsymbol{B_y = 0}$$
$$\overset{+}{\rightarrow} \Sigma F_x = 0:$$
$$B_x - (-3) = 0$$
$$\boldsymbol{B_x = -3\ k} \leftarrow$$

b) Calculation of $\boldsymbol{B_x}$



c) Final reactions

Figure 1.14: Final reactions and internal forces for the indeterminate truss

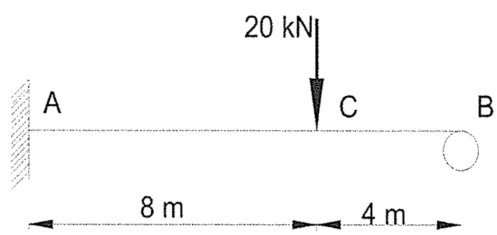
1.8 Summary

In this chapter, the basic concept of the force method is explained briefly but succinctly. This is followed by application of the force method to a set of problems dealing with structural analysis of an indeterminate beam, frame and truss. It is to be noted that force method uses the concept of Static Indeterminacy (SI) and involves a large number of deflection calculations. Hence, knowledge of the prerequisite courses dealing with deflection calculations is paramount to a strong understanding of this approach.

Problems

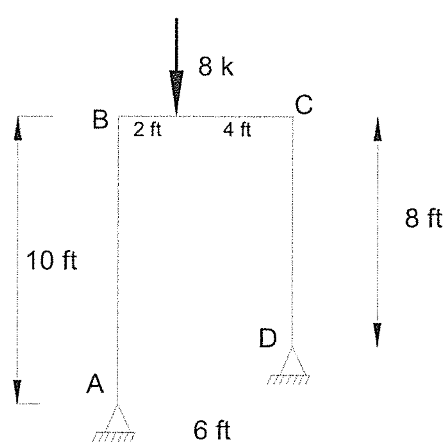
Analyze the Problems from 1.1 to 1.3 for all the unknown reactions using the force method.

Problem 1.1 Determine the reactions at the supports of the beam shown in this figure. EI is constant.



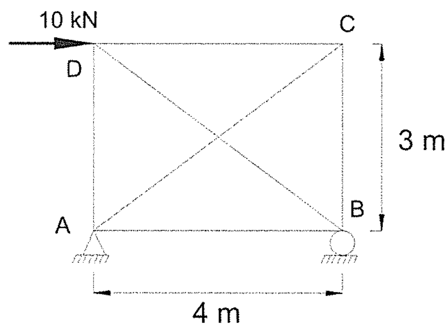
Problem 1.1

Problem 1.2 Determine the reactions at the supports of the frame shown in this figure. EI is constant.



Problem 1.2

Problem 1.3 Determine the reactions at the supports of the truss shown in this figure. AE is constant.



Problem 1.3

Chapter 2

Displacement Method of Analysis: Slope-Deflection Method

2.1 Basic Concepts of the Displacement Method

The displacement method refers to the general approach of solving indeterminate structural analysis problems with displacements as the primary variables. Two displacement methods that will be explained in this book are classical methods called slope-deflection and moment distribution. The displacement method uses the concept of structural Kinematic Indeterminacy (KI). The formula for this is:

$$\text{K.I.} = \sum \text{(degrees of freedom at all supports in the given structure)} \qquad (2.1)$$

Where: Degrees of freedom are unrestrained motions of a joint/support. This means a fixed support has zero degrees of freedom and a pin has one (rotation).

The results obtained using the slope-deflection method are the end moments (internal moments) at the supports of the structure. These are found through a two-step process of first finding the rotations (slopes), and second finding the end moments. In contrast, the moment distribution method, which will be discussed in Chapter 3, gives end moments directly as a result of the procedure. After finding the end moments, the reactions at various supports can be determined using principles of statics.

In the slope-deflection method, the unknown displacements are usually rotational displacements of a pin or roller support. The displacements are written in terms of the loads using the load-displacement relationships, also known as slope-deflection equations. The resulting equations are then solved for the displacements. Therefore, the main intermediate output resulting from the slope-deflection method is displacements. The final output is end moments.

2.2 Basic Procedure of the Slope-Deflection Method

2.2.1 Slope-Deflection Equations

Before the actual procedure is discussed, it is important to introduce the slope-deflection equations, which are key to the slope-deflection method. Derivation of the slope-deflection equations will not be shown; these are done with great detail in the books listed in the references section.

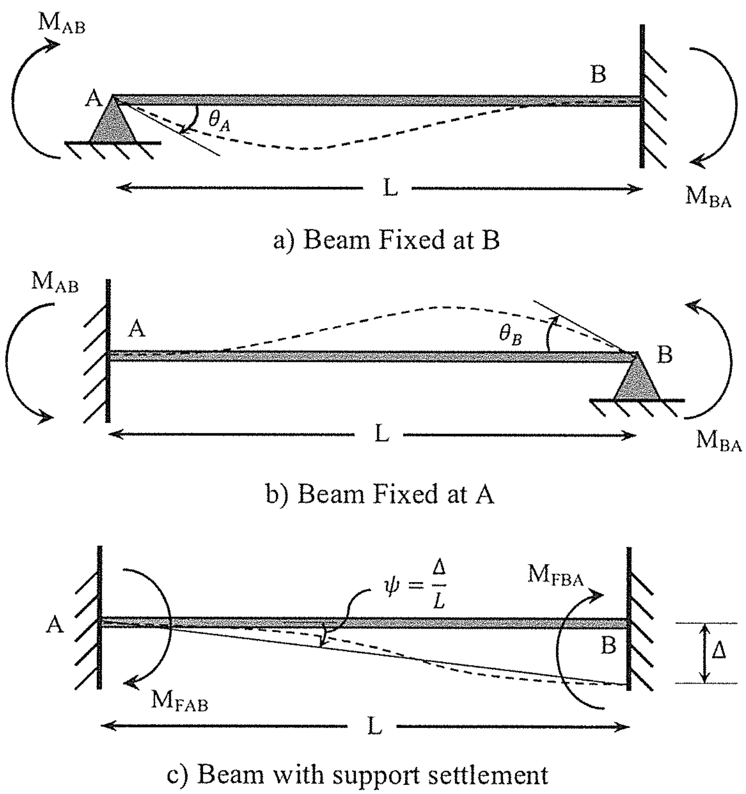


Figure 2.1: Moments and displacements on typical indeterminate beams

With respect to Fig. 2.1, the slope-deflection equations can be written (without support settlement) as,

$$M_{AB} = M_{FAB} + \frac{2EI}{L}(2\theta_A + \theta_B) \tag{2.2}$$

$$M_{BA} = M_{FBA} + \frac{2EI}{L}(2\theta_B + \theta_A) \tag{2.3}$$

Wang (1953) advocates using relative stiffness factors instead of actual stiffness factors to simplify the calculations. Modifying Eq. 2.2 and Eq. 2.3 to include stiffness factors yields,

$$M_{AB} = M_{FAB} + K_{AB}(2\theta_A + \theta_B) \tag{2.2a}$$

$$M_{BA} = M_{FBA} + K_{BA}(2\theta_B + \theta_A) \tag{2.2b}$$

Where,

M_{AB} = Moment at joint A of member AB
M_{BA} = Moment at joint B of member AB
M_{FAB} = Fixed-end moment at end A of member AB due to applied loading
M_{FBA} = Fixed-end moment at end B of member AB due to applied loading
θ_A = Slope at joint A
θ_B = Slope at joint B

2.2.2 Sign Convention for Displacement Methods

- Clockwise moments are positive
- Counterclockwise moments are negative
- Clockwise rotations are positive
- Counterclockwise rotations are negative

2.2.3 Fixed-End Moments

Fixed-end moments are reactionary moments of a single span beam having fixed supports for a given loading. Table 1A give fixed-end moment values for various load types.

2.3 Analysis of Continuous Beams by the Slope-Deflection Method

Before discussing examples, the calculation procedure will be outlined below:

1. Calculate all the fixed-end moments due to applied loads at the end of each span using Table 1A found in the appendix.
2. Calculate the Kinematic Indeterminacy (KI) of the structure. It is expressed as,

   K.I. = $\sum$ (degrees of freedom at all supports in the given structure)

   Degrees of freedom are unrestrained motions of a joint/support. This means a fixed support has zero degrees of freedom, a pin has one (rotation), and a frame's joint has one (rotation).
3. Formulate all the slope-deflection equations for each member of the continuous beam using Eq. 2.2 and Eq. 2.3. These equations are in terms of the unknown rotations at the supports.
4. Formulate simultaneous equilibrium equations at the joints (not fixed) using the basic premise that the sum of the end moments at the support (for all the members joining at the support) is zero. The number of unknown rotations in the problem is equal to the number of simultaneous equations to be solved as well as the KI found in step 2.
5. Solve the simultaneous equations formulated in Step 4 and obtain rotations at the supports.

6. Compute end moments by substituting rotations back into the slope-deflection equations.
7. Depending on the statement of the problem, calculate all the reactions.
8. Draw shear and moment diagrams for the continuous beam as needed.

Example 2.3.1

Determine the reactions at the supports for the statically indeterminate beam shown in Fig. 2.2 by the slope-deflection method. Take E = 29000 ksi and I = 446 in^4.

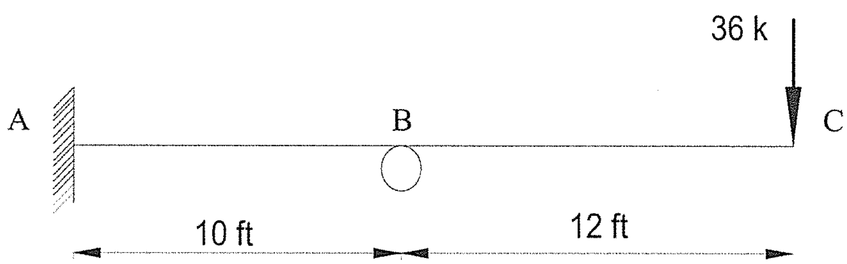


Figure 2.2: Statically indeterminate beam

Solution

Step 1

Calculate the fixed-end moments using Table 1A found in the appendix. The fixed-end moments for AB and BA are both zero because there is no loading on the span of member AB.

$$M_{FAB} = 0 \text{ and } M_{\mathrm{FBA}} = 0$$

Step 2

$$KI = 1$$

The unknown displacement is θ_B. Although the other unknown displacements (θ_C and Δ_C) exist, these displacements are unnecessary to solve the problem because they do not occur at a support where specific unknowns need to be found (B_y). θ_B must be found so that we can find the reaction at B. In contrast, solving for θ_c would not give us any information about the reactions of the structure.

Step 3

Slope-deflection equations are formed using Eq. 2.2 and Eq. 2.3 as,

$$M_{AB} = 0 + 2E\,\frac{I}{10}[2\theta_A + \theta_B] = \frac{EI}{5}\theta_B \qquad (2.4)$$

(Note: $\theta_A = 0$ due to the fixed support at A)

$$M_{BA} = 0 + 2E\,\frac{I}{10}[2\theta_B + \theta_A] = \frac{2EI}{5}\theta_B \qquad (2.5)$$

Similarly, M_{BC} can be written as,

$$M_{BC} = -36 * 12 = -432^{k-ft}$$

Note: M_{BC} is negative because the internal moment caused by the loading, acts in the counterclockwise direction (opposite to the external moment at that point).

Step 4

Since KI = 1 for this problem, there is only one unknown, which is θ_B. Hence, there is only one joint equilibrium equation to be solved. This is given as,

$$M_{BA} + M_{BC} = 0 \qquad (2.6)$$

Step 5

Substituting the expressions for M_{BA} and M_{BC} from step 3, we have,

$$\frac{2EI}{5}\theta_B - 432 = 0 \;\rightarrow\; \theta_B = \frac{1080}{EI} \qquad (2.7)$$

Step 6

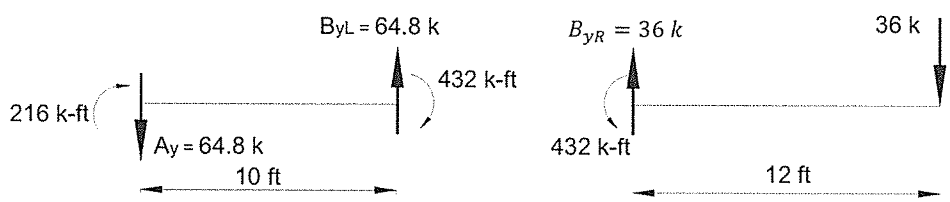
Substituting the value of the rotation back into the slope-deflection equations found in step 3, the end moments can be expressed as,

$$M_{AB} = 216^{k\text{-}ft} \circlearrowright \tag{2.8}$$

$$M_{BA} = 432^{k\text{-}ft} \circlearrowright \tag{2.9}$$

Step 7

The reactions at A (A_y) and at B (B_y) are calculated from principles of statics as shown in Fig. 2.3 below:



a) Reactions for AB

b) Reactions for BC

Figure 2.3: Reaction calculation

$$\overset{+}{\rightarrow} \Sigma F_x = 0 \rightarrow \boldsymbol{A_x = 0}$$

$$\overset{+}{\circlearrowright} \Sigma M_B = 0 : \text{Member AB}$$

$$216 + 432 - A_y * 10 = 0$$

$$A_y = \frac{216+432}{10} = \boldsymbol{64.8\,k} \downarrow$$

$$+\uparrow \Sigma F_y = 0\text{: Whole beam}$$

$$-A_y + B_y - 36 = 0$$

$$-64.8 + B_y - 36 = 0$$

$$\boldsymbol{B_y = 100.8\,k} \uparrow$$

Step 8

The bending moment diagram (BMD) can be drawn as needed.

2.4 Analysis of Continuous Beams with Support Settlements by the Slope-Deflection Method

The slope-deflection equations including settlement with respect to Fig. 2.4 are given as,

$$M_{AB} = M_{FAB} + \frac{2EI}{L}(2\theta_A + \theta_B - 3\Psi_{AB}) \quad (2.10)$$

$$M_{BA} = M_{FBA} + \frac{2EI}{L}(2\theta_B + \theta_A - 3\Psi_{BA}) \quad (2.11)$$

Where,

M_{AB} = Moment at joint A of member AB
M_{BA} = Moment at joint B of member AB
M_{FAB} = Fixed-end moments at the end A of member AB due to applied loading
M_{FAB} = Fixed-end moments at the end B of member AB due to applied loading
θ_A = Slope at joint A
θ_B = Slope at joint B
Ψ_{AB} = Rotation of the member AB due to translation (settlement) of joint B perpendicular to member AB

$$\Psi_{AB} = \Delta/L \quad (2.12)$$

Where,

Δ = Translation (settlement) of joint B perpendicular to axis of member AB
L = Length of member AB

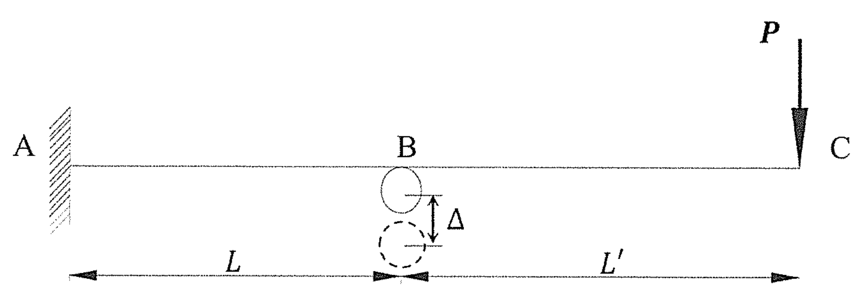


Figure 2.4: Statically indeterminate beam with support settlement at B

It is to be noted that Ψ is treated positive when the rotation is clockwise, consistent with the sign convention stated in Sec. 2.2.2.

Equations 2.10 and 2.11 can be rewritten using the relative stiffness factors and relative Ψ_{AB} values as,

$$M_{AB} = M_{FAB} + K_{AB}(2\theta_A + \theta_B - 3\Psi_{rel}) \tag{2.10a}$$

$$M_{BA} = M_{FBA} + K_{BA}(2\theta_B + \theta_A - 3\Psi_{rel}) \tag{2.11a}$$

The *relative stiffness factors* (K_{AB} and K_{BA}) for any general member AB can be expressed as $2EI/L$ and Ψ_{rel} as $\Delta\,/L$.

The procedure for solving continuous beams where joints are subjected to vertical translation amounting to settlement of supports remains the same as discussed in Sec. 2.3.

Example 2.4.1

Determine the reactions at the supports for the statically indeterminate beam shown in Fig. 2.4 by the slope-deflection method. Take E = 29000 ksi and I = 446 in^4. <u>The support at B is displaced downward 1 in.</u>

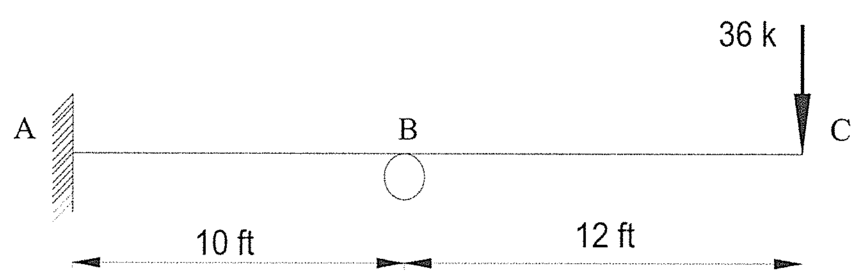


Figure 2.4(repeated): Statically indeterminate beam with support settlement at B

Solution

Step 1

In the slope-deflection method, *fixed end moments due to support settlement are not considered* because support settlement is accounted for using Ψ.

$$M_{FAB} = 0 \text{ and } M_{FBA} = 0$$

Step 2

$$KI = 1$$

Since Ψ is known, the only unknown displacement is θ_B. Moreover, due to downward displacement at B, it can be seen that the cord of span AB rotates clockwise, thus Ψ is positive.

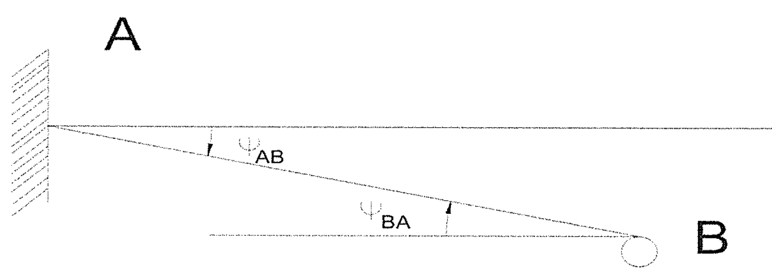


Figure 2.5: Effect of displacement at B

$$\Psi_{AB} = \Psi_{BA} = \frac{1\,in}{10(12)\,in} = 0.00833\,rad$$

Step 3

Slope-deflection equations are formed using Eq. 2.10 and Eq. 2.11.

$$M_{AB} = 0 + 2E\frac{I}{10}[2\theta_A + \theta_B - 3\Psi_{AB}] = \frac{EI}{5}(\theta_B - 3 * 0.00833)$$

$$M_{AB} = \frac{EI}{5}(\theta_B - 0.025)$$

$$M_{BA} = 0 + 2E\frac{I}{10}[2\theta_B + \theta_A - 3\Psi_{BA}] = \frac{EI}{5}(2\theta_B - 3 * 0.00833)$$

$$M_{BA} = \frac{EI}{5}(2\theta_B - 0.025)$$

In this problem, the fixed support at A inhibits rotation at the joint, therefore $\theta_A = 0$. This will be true for all fixed supports, even if there is joint translation; the member rotation is accounted for using Ψ.

From statics:

$$M_{BC} = -36 * 12 = -432^{k-ft}$$

Note: M_{BC} is negative because the internal moment caused by the loading, acts in the counterclockwise direction (opposite of the external moment at that point).

Step 4

The only joint equilibrium equation is for joint B and since it is a roller,

$$M_{BA} + M_{BC} = 0$$

Step 5

Substituting the expressions for M_{BA} and M_{BC} from step 3 we have,

$$\rightarrow \frac{EI}{5}(2\theta_B - 0.025) - 432 = 0$$

Solving this equation to find θ_B gives,

$$E = 29000\, ksi * 144 = 4176000\, ksf$$

$$I = 446\, in^4 = \frac{446}{12^4} ft^4 = 0.0215\, ft^4$$

$$\rightarrow \theta_B = 0.02453\, rad$$

Step 6

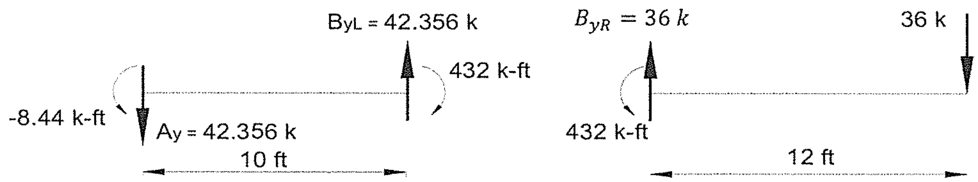
Substituting the value of the rotation back into expressions for end moments calculated in step 3, the end moments can be expressed as,

$$M_{AB} = -8.44^{k\text{-}ft} \circlearrowleft$$

$$M_{BA} = 432^{k\text{-}ft} \circlearrowright$$

Step 7 and Step 8

The reactions at A (R_A) and B (R_B) are calculated from principles of statics as can be seen below.



a) Reactions for AB

b) Reactions for BC

Figure 2.6: Reaction calculation

$\overset{+}{\rightarrow} \Sigma F_x = 0 \rightarrow \boldsymbol{A_x = 0}$

$\overset{+}{\circlearrowleft} \Sigma M_B = 0$: Member AB

$-8.44 + 432 - A_y * 10 = 0$

$A_y = \frac{432-8.44}{10} = \boldsymbol{42.356\ k} \downarrow$

$+\uparrow\ \Sigma F_y = 0$: Whole beam

$-A_y + B_y - 36 = 0$

$-42.356 + B_y - 36 = 0$

$\boldsymbol{B_y = 78.356\ k} \uparrow$

If needed, the bending moment diagrams (BMD) can be drawn.

2.5 Application of the Slope-Deflection Method to Analysis of Frames Without Joint Movement

The procedure for solving a statically indeterminate frame is the same as a statically indeterminate beam, which was explained in Sec.2.3. An example is provided below to clarify the concept and procedure.

Example 2.5.1

Determine the moments at each joint of the frame shown in Fig 2.7. E = 29000 ksi, A = 16 in^2, and I = 446 in^4 for all members.

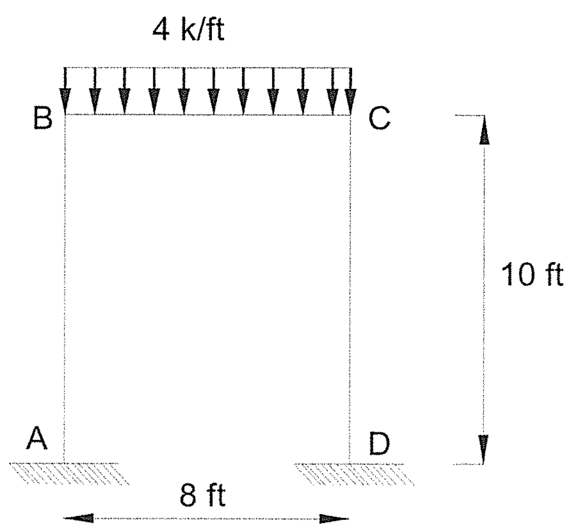


Figure 2.7: Indeterminate frame (no side sway)

Solution

Step 1

Since the loading is only on the span BC there will only be fixed-end moments in members BC and CB.

$$M_{FBC} = -\frac{wL^2}{12} = -\frac{4(8)^2}{12} = -21.33^{k-ft}$$

$$M_{FCB} = \frac{wL^2}{12} = \frac{4(8)^2}{12} = 21.33^{k-ft}$$

Step 2

$$KI = 2$$

There are two unknown displacements in this problem, which are θ_B and θ_C. They are unknown because these frame joints will rotate as the members bend due to the applied loading. The rotations θ_A and θ_D are zero because of the fixed supports at A and D. Due to symmetrical loading, there will be no side sway in the frame, therefore $\Psi = 0$.

Step 3

The slope-deflection equations are formulated below using Eq. 2.2 and Eq. 2.3 as,

$$M_{AB} = 2E\frac{I}{10}[2\theta_A + \theta_B] \qquad = \frac{EI}{5}\theta_B$$

$$M_{BA} = 2E\frac{I}{10}[2\theta_B + \theta_A] \qquad = \frac{2EI}{5}\theta_B$$

$$M_{BC} = -21.33 + 2E\frac{I}{8}[2\theta_B + \theta_C] = -21.33 + \frac{EI}{4}(2\theta_B + \theta_C)$$

$$M_{CB} = 21.33 + 2E\frac{I}{8}[2\theta_C + \theta_B] \quad = 21.33 + \frac{EI}{4}(\theta_B + 2\theta_C)$$

$$M_{CD} = 2E\frac{I}{10}[2\theta_C + \theta_D] \qquad = \frac{2EI}{5}\theta_C$$

$$M_{DC} = 2E\frac{I}{10}[2\theta_D + \theta_C] \qquad = \frac{EI}{5}\theta_C$$

Step 4

The corresponding joint equilibrium equations are written as,

$$M_{BA} + M_{BC} = 0 \quad and \quad M_{CB} + M_{CD} = 0$$

Step 5

Substituting the expressions for M_{BA}, M_{BC}, M_{CB}, and M_{CD} from step 3, we have,

$$\frac{2EI}{5}\theta_B - 21.33 + \frac{EI}{4}(2\theta_B + \theta_C) = 0 \qquad (1)$$

$$21.33 + \frac{EI}{4}(\theta_B + 2\theta_C) + \frac{2EI}{5}\theta_C = 0 \qquad (2)$$

Simplifying these equations to isolate θ_B and θ_C gives:

From (1) → $$0.9EI\theta_B + 0.25EI\theta_C = 21.33$$

From (2) → $$0.25EI\theta_B + 0.9EI\theta_C = -21.33$$

Solving (1) and (2) yields:

$$\theta_B = \frac{32.815}{EI} \quad and \quad \theta_C = -\frac{32.815}{EI}$$

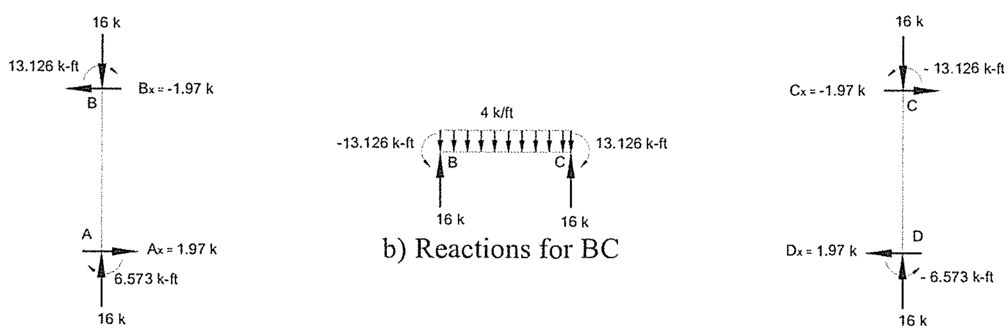
This step of solving the simultaneous equations can be greatly simplified by using a calculator with this capability. Otherwise, hand calculations can be used, but these will not be shown in the text.

Step 6

Substituting the value of the rotation back into expressions for end moments calculated in step 3, the end moments can be expressed as,

$$M_{AB} = 6.563^{k\text{-}ft}$$
$$M_{BA} = 13.126^{k\text{-}ft}$$
$$M_{BC} = -13.126^{k\text{-}ft}$$
$$M_{CB} = 13.126^{k\text{-}ft}$$
$$M_{CD} = -13.126^{k\text{-}ft}$$
$$M_{DC} = -6.563^{k\text{-}ft}$$

Based on the moments of each joints, we can easily compute the reactions at support:



a) Reactions for AB

b) Reactions for BC

c) Reactions for CD

Figure 2.8: Reactions for the frame

Alternatively, the relative stiffness factors could have been used to solve this problem. If one were to use this concept, the relative stiffness factors for AB and BC would be as follows:

$$K_{AB-rel} = \frac{2*I}{10}(20) = 4$$

$$K_{BC-rel} = \frac{2*I}{8}(20) = 5$$

Here, the relative stiffness factors $\left(\frac{2I}{L}\right)$ have been multiplied by the LCM (Least Common Multiple) to simplify the calculations. Also, E and I are not included in the relative stiffness factors because they must be constant in all members to use K_{rel}. The rotations obtained using this concept are different than those found using the actual stiffness factors because they are modified according to the LCM. The point to be noted is that, the final end moments remain the same and calculation is facilitated.

If one was to use these relative stiffness factors and modified slope-deflection equations, 2.2a and 2.2b, the value of θ_B comes out to be 1.6408. However, the actual end moments remain the same.

$$M_{AB} = \frac{EI}{5}\theta_B = \frac{EI}{5} * \frac{32.815}{EI} = 6.563^{k-ft}$$

$$M_{AB-rel} = 4[\theta_B] = 4 * 1.6408 = 6.563^{k-ft}$$

2.6 Derivation of Shear Condition for Frames (With Joint Movement)

When analyzing frames with joint movement, an extra unknown (Δ or Ψ) is added to the usual unknown displacements. This means an extra equation is needed. The equation is obtained from what is known as the "shear condition" at the base supports of the frame.

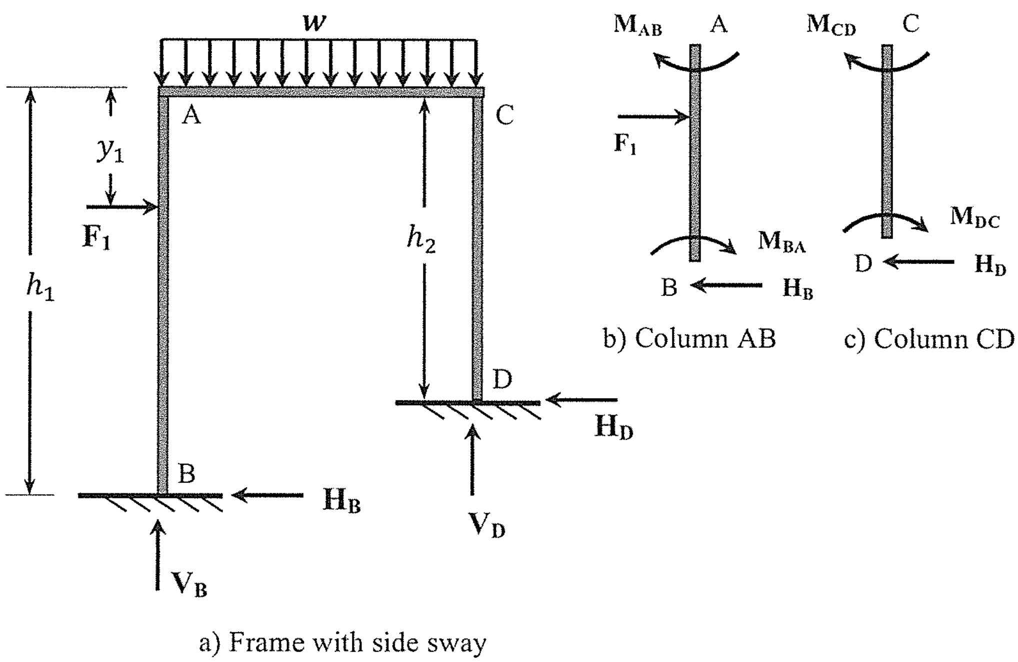


Figure 2.9: Frame with sidesway – Basic concept illustration

For a typical frame (Fig. 2.9), the shear condition obtained from the basic equation $\Sigma F_x = 0$ is given as,

$$F_1 - H_B - H_D = 0 \tag{2.13}$$

Where H_B and H_D can be found by taking $\sum M_A = 0$ and $\sum M_C = 0$ using Figures 2.9b and 2.9c.

This would yield the flowing two equations.

$$H_B = \frac{F_1 y_1}{h_1} - \frac{M_{AB}+M_{BA}}{h_1} \tag{2.13a}$$

$$H_D = -\frac{M_{CD}+M_{DC}}{h_2} \tag{2.13b}$$

Equations 2.13a and 2.13b are written with the assumption that the end moments of a column are clockwise (positive). Figure 2.9b and 2.9c show the free body diagrams for columns AB and CD with which the expression for H_B and H_D are derived. Equation 2.13 has to be solved in addition to the other joint equilibrium equations.

The rest of the procedure remains same as outlined in Sec. 2.3.

2.7 Application of the Slope-Deflection Method to Analysis of Frames With Joint Movement

An example is solved below to illustrate the analysis of a frame with sidesway using the slope-deflection method.

Example 2.7.1

Determine the reactions at the supports of the frame shown in Fig. 2.10. A = 100 in^2, E = 29000 ksi and I = 833 in^4.

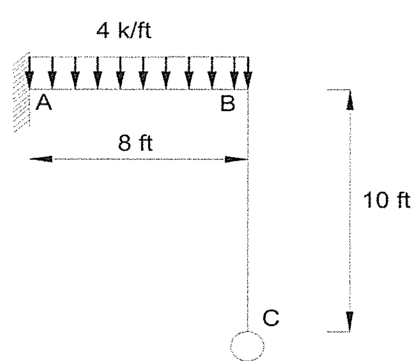


Figure 2.10: Indeterminate Frame (sidesway)

Solution

Step 1

The fixed-end moments are calculated using Table 1A found in the appendix.

$$M_{FAB} = -\frac{4*8^2}{12} = -21.33^{k-ft}$$

$$M_{FBA} = \frac{4*8^2}{12} = 21.33^{k-ft}$$

Step 2

The unknown displacements are θ_B, θ_C and Ψ_{BC}. $KI = 3$

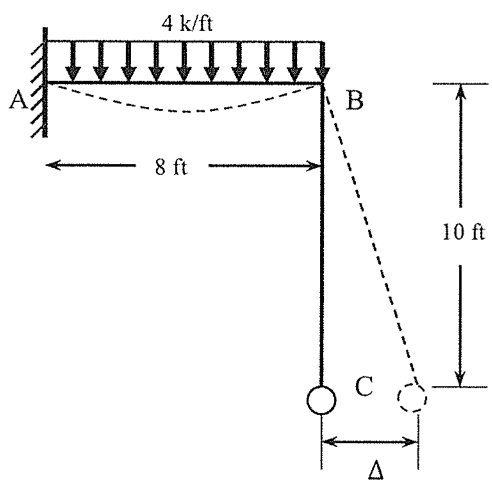


Figure 2.11: Bending of frame in Ex. 2.7.1

Step 3

The slope-deflection equations for this structure can be written as,

$$M_{AB} = 2E\frac{I}{8}[2(0) + \theta_B] - 21.33 = \frac{EI}{4}\theta_B - 21.33 \quad (1)$$

(Note: $\theta_A = 0$ due to fixed support at A)

$$M_{BA} = 2E\frac{I}{8}[2\theta_B + 0] + 21.33 = \frac{EI}{2}\theta_B + 21.33 \quad (2)$$

$$M_{BC} = 2E\frac{I}{10}[2\theta_B + \theta_C - \Psi_{BC}] \quad (3)$$

$$M_{CB} = 2E\frac{I}{10}[2\theta_C + \theta_B - \Psi_{BC}] \quad (4)$$

Step 4

Moment equilibrium required: $M_{BA} + M_{BC} = 0$ (5)

Roller support at C → $M_{CB} = 0$ (6)

Due to symmetrical loading there is no moment in member BC, but the structure will still sidesway. Hence,

$$M_{BC} = 0 \tag{7}$$

Step 5

From Equations (5) and (7) → $M_{BA} = 0$

Substitute in (2) →

$$\frac{EI}{2}\theta_B + 21.33 = 0 \rightarrow \boldsymbol{\theta_B} = -\frac{\mathbf{42.66}}{\boldsymbol{EI}}$$

Step 6

Substituting the value of the rotation back into expressions for end moments calculated in step 3, the end moments can be expressed as,

$$\boldsymbol{M_{AB} = -\,32^{k\text{-}ft}}$$

Step 7 and Step 8

The reactions at A (A_x and A_y) and C (C_y) are calculated from principles of statics:

$$\overset{+}{\rightarrow} \Sigma F_x = 0 \rightarrow \boldsymbol{A_x = 0}$$

$$\Sigma M_A = 0:$$

$$-32 + 32 * 4 - Cy * 8 = 0 \rightarrow \boldsymbol{C_y = 12k} \ (\uparrow)$$

$$+\uparrow \Sigma F_y = 0:$$

$$A_y + 12 - 32 = 0 \rightarrow \boldsymbol{A_y = 20k} \ (\uparrow)$$

After the reactions are obtained, the Bending Moment Diagram (BMD) and the Shear Force Diagram (SFD) can be drawn as needed.

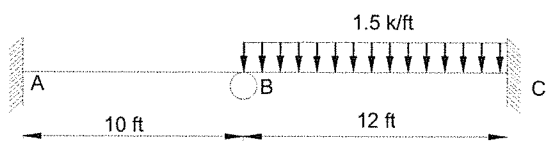
2.8 Summary

In this chapter, the fundamentals of a classical method, called the slope-deflection method, were discussed. This was followed by examples applying it to beams and frames. The slope-deflection method essentially consists of solving a set of simultaneous equations where the unknown values are displacements. Finally, end moments are calculated using these displacements. This method is easy to use; and unlike the force method, does not require knowing how to do deflection calculations.

Problems

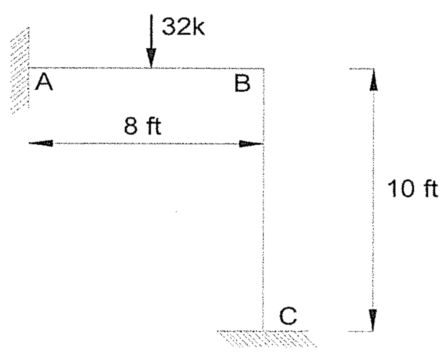
Analyze Problems 2.1 to 2.3 using the slope-deflection method.

Problem 2.1 Determine the reactions at the supports of the beam shown in this figure. EI is constant.



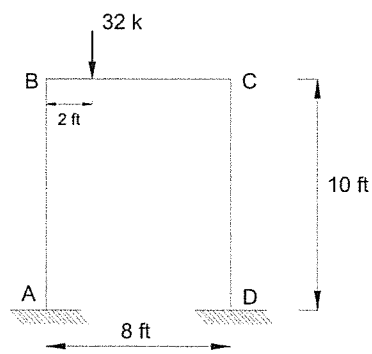
Problem 2.1

Problem 2.2 Determine the reactions at the supports of the frame shown in this figure. $A = 100$ in^2, $E = 29000$ ksi and $I = 833$ in^4.



Problem 2.2

Problem 2.3 Determine the reactions at the supports of the frame shown in this figure. $A = 100\ in^2$, $E = 29000$ ksi and $I = 833\ in^4$



Problem 2.3

Chapter 3

Displacement Method of Analysis: Moment Distribution Method

3.1 Basic Concepts of Moment Distribution Method

Hardy Cross originally developed the moment distribution method in 1930. It is a classical and iterative method. It essentially consists of locking and unlocking each joint consistent with the actual boundary conditions. This means the whole procedure of moment distribution is carried out in such a way that at the end of it, the final end moments for a hinge (pin) joint should be zero while a fixed joint can have any amount of moment. Analysis of a structure essentially involves finding the end moments for each member. It will be interesting to compare the moment distribution method with another classical method – the slope-deflection method (discussed in Ch. 2). In the case of the slope-deflection method, finding end moments of members is a two-step process. The first step is finding the slopes at each joint and the second is finding end moments for each member. On the other hand, the moment distribution method directly gives the end moments for each member. The moment distribution method, like the slope-deflection method, uses fixed-end moments and stiffness factors. Additionally, the moment distribution method uses distribution factors. It is through the distribution factors that the moment distribution is essentially carried out because they dictate how much moment a specific joint will transfer. Distribution factors are obtained using the stiffness factors for each member in such a way that it reflects the property of the joint. Thus, since the total moment at a hinge joint is zero, the distribution factor at a hinge joint is one. Similarly, the distribution factor at a fixed joint is zero as the fixed joint can carry any amount of moment. The distribution factor will be discussed with more detail in section 3.2.3.

3.2 Stiffness Factor, Carry-Over Factor and Distribution Factor

Three important factors used in the moment distribution method are the stiffness factor (K), Carry-Over factor (CO) and the Distribution Factor (DF). These will be described in the following sections.

3.2.1 Stiffness Factor

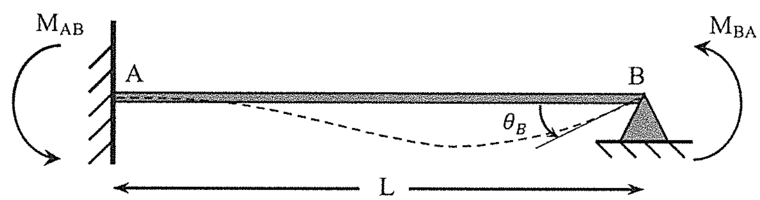


Figure 3.1: Beam with moment applied at B

Fig. 3.1 shows a beam with a moment applied at B. It can be proven that,

$$M_{BA} = \frac{4EI}{L}\theta_B \tag{3.1}$$

Or,

$$M_{BA} = K\theta_B \tag{3.2}$$

Where,

$$K = \frac{4EI}{L} \tag{3.3}$$

In Eq. 3.3, K is the *stiffness factor* for member AB, which is defined as the amount of moment needed at B to induce a unit rotation ($\theta_B = 1$ rad).

Other books sometimes modify the stiffness factors based on support conditions, but in this book the authors will advocate using the stiffness factor $K = 4EI/L$ for all members. Using $K = 4EI/L$ for all members will simplify the analysis and provide the same answers.

3.2.2 Carry-Over Factor (CO)

In Fig 3.1, it can be proven that the moment induced at A is,

$$M_{AB} = \frac{2EI}{L}\theta_B \tag{3.4}$$

From Eq. 3.1 and Eq. 3.4 it can be seen that the carry-over moment, moment induced at A, is 1/2 of the applied moment at B. This implies that the carry-over factor, which is the ratio of M_A to M_B, is 0.5. Thus it can be stated that for a beam simply supported at one end and fixed at the other, the CO is 0.5. This concept will be applied in the moment distribution procedure.

3.2.3 Distribution Factor (DF)

$$(DF)_{member} = \frac{K_{member}}{\sum K_{member}} \tag{3.5}$$

Where, $\sum K_{member}$ includes all members connected to the joint considered.

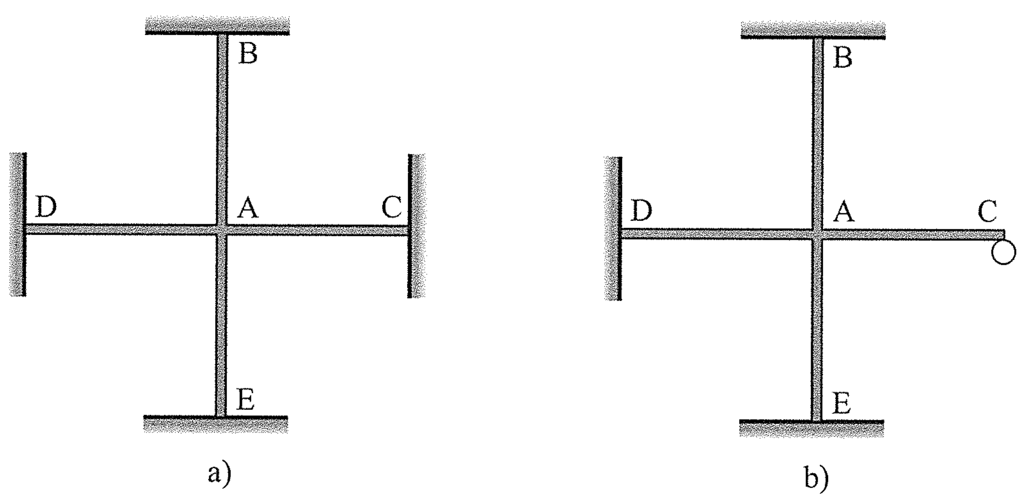


Figure 3.2: Concept of distribution factors

Using Fig. 3.2a, the distribution factors can be defined as,

$$(DF)_{AB} = K_{AB}/(K_{AB} + K_{AC} + K_{AD} + K_{AE}) \quad (3.5a)$$

$$(DF)_{AC} = K_{AC}/(K_{AB} + K_{AC} + K_{AD} + K_{AE}) \quad (3.5b)$$

$$(DF)_{AD} = K_{AD}/(K_{AB} + K_{AC} + K_{AD} + K_{AE}) \quad (3.5c)$$

$$(DF)_{AE} = K_{AE}/(K_{AB} + K_{AC} + K_{AD} + K_{AE}) \quad (3.5d)$$

The distribution factor at a fixed support is *zero* because it "absorbs" moments rather than distributing them. Applying Eq. 3.5 at joint E proves this as can be seen below:

$$DF_{EA} = \frac{K_{EA}}{K_{EA} + \infty} = 0 \qquad \text{(Fixed support)}$$

In theory, a fixed support is "infinitely" stiff because it could take a moment of any size. This makes the denominator of the above equation ∞, therefore $DF = 0$ for all fixed supports. Similarly, $DF = 1$ for pin and roller support at the end of a beam. Considering joint C in Fig. 3.2b, the distribution factor would be calculated as follows:

$$DF_{CA} = \frac{K_{CA}}{K_{CA}} = 1 \qquad \text{(End-pin support)}$$

Here, you can see that since there is only one member attached to end joint C, the stiffness factor is 1. This is true for all pin and roller supports at the ends of continuous beams.

3.3 Analysis of Continuous Beams by Moment Distribution Method

The basic procedure for solving problems containing continuous beams using the moment distribution methods will be explained first followed by an example.

3.3.1 Basic Procedure for Moment Distribution

The general procedure for analysis of beams and frames is the same. Therefore, the procedure listed below is applicable to beams and frames (i.e. structures essentially in flexure or bending):

1. Calculate the stiffness factors (K) for each span using the following equation:

$$K = \frac{4EI}{L} \tag{3.3}$$

2. Calculate the Distribution Factor (DF) for each member using the following relation:

$$DF_{member} = \frac{K_{member}}{\sum K_{member}} \tag{3.5}$$

   Where, $\sum K_{member}$ includes all members connected to the joint considered.

   Note: Distribution factors that are unknown must be solved using Equation 3.5, but those that are known, fixed supports and end-pin supports, can be found immediately.

3. Calculate the fixed-end moments using Table 1A in the Appendix. This step means locking all the joints. The sign convention used is: *clockwise moments and rotations are considered positive.*

4. Set up the moment distribution table by entering the calculated fixed-end moments for each member and the distribution factors for each joint. The

table will need to include all the members, and will look similar to the following depending on the amount of joints:

Joint Member DF	A AB	B BA	 BC	C CB	 CD	D DC
M_F Bal						
CO Bal						
↓	↓	↓	↓	↓	↓	↓
Final						

5. Start the 1st cycle of moment distribution by unlocking each joint. Sum up the M_F for all members connected at each joint to get the unbalanced moment ($M_{(unbalanced)}$). Multiply this moment by the respected DF and invert the sign to get the Bal for that member. The following relationship can be utilized:

$$Bal = -M_{(unbalanced)} * DF$$

   At a pin support, when summing the M_F and Bal of each cycle, both members will be delivering the same moment with opposite direction (sign) to the joint. This means the joint is balanced. Fixed supports will have a residual moment.

6. Find the CO by carrying Bal values across members (from joint to joint) with a factor of ½. Then get the new unbalancing moments by using the CO. Step 5 and 6 involve locking and unlocking the joints. The first locking moments are due to fixed moments (caused by the given loads). The successive locking moments are obtained through CO.
7. Continue the balancing until the final unbalance at each joint is about 1% of the initial unbalance moment at any joint.

3.3.2 Example for a Continuous Beam

The procedure discussed above will now be applied to a multi-span continuous beam.

Example 3.3.2.1:

Analyze the continuous beam shown in Fig. 3.3 using the moment distribution method.

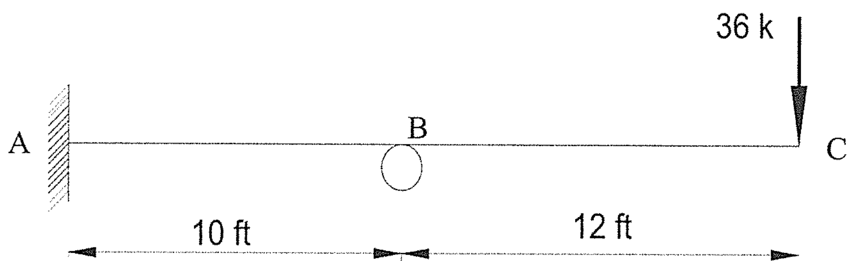


Figure 3.3: Continuous beam

Solution

Step 1

Stiffness Factors are not needed here because all the distribution factors are known.

Step 2

$DF_{AB} = 0$ (Fixed support)

$DF_{BA} = 1$ (End-pin support)

$DF_{BC} = 0$ (No moment is transferred from B to C)

Step 3

$$M_{FAB} = 0\,, \quad M_{FBA} = 0\,, \quad M_{FBC} = -36 * 12 = -432\ ^{k-ft}$$

Note: M_{FBC} is negative because the internal moment caused by the loading, acts in the counterclockwise direction (opposite to the external moment at that point).

Step 4 – 7

The moment distribution table is set up as shown in Table 3.1. First, the joints A and B are added to the table along with the corresponding members. Joint C is not part of the table because there is no support. Then, the distribution factors of each member are added based on their support type, and whether they are intermediate or end supports. In this case, joints A and B are end supports because the cantilever portion can be simplified into a moment acting at the BC location. Members AB and BA have no fixed-end moments because there is no loading on the span of the member. The balance for the first cycle in member BA needs to be 432 to satisfy joint equilibrium at joint B. This number can be found either by inspection $(-432 + x = 0)$ or by the standard procedure of:

$$Bal = -M_{(unbalanced)} * DF$$

$$Bal_{BA} = -(M_{BC} + M_{BA}) * 1$$

$$Bal_{BA} = -(-432) * 1 = 432^{ft-k}$$

The carry over only occurs from joint B to A because moments are not carried to C (cantilever), or from joint A because it is fixed $(DF = 0)$. In this problem, the process is repeated only one more time because the Bal in the second iteration was all zero. Other problems will require more iteration so that the Bal is 1% of the initial unbalanced moment.

Table 3.1: Moment distribution table

Joint	A	B	
Member	AB	BA	BC
DF	0	1	0
M_F	0	0	-432
Bal	0	432	0
CO	216	0	0
Bal	0	0	0
Final	**216**	**432**	**-432**

The answers found here, $M_{AB} = 216^{ft-k}$ ↻, $M_{BA} = 432^{ft-k}$ ↻, and $M_{BC} = -432^{ft-k}$ ↺ are the same as found by the force method in Chapter 1 (Ex. 1.5.2.1) and by the slope-deflection method in Chapter 2 (Ex. 2.3.1), but with less work. This is a good example of how powerful the moment distribution method is; yet the true power of this method will be seen once more complicated examples are solved.

3.4 Analysis of a Continuous Beam with Support Settlement by Moment Distribution Method

The procedure for moment distribution discussed in Sec. 3.1 is now applied to a continuous beam with support settlements and no other load. The fixed-end moments at each end are obtained using Equation 3.6a or 3.6b.

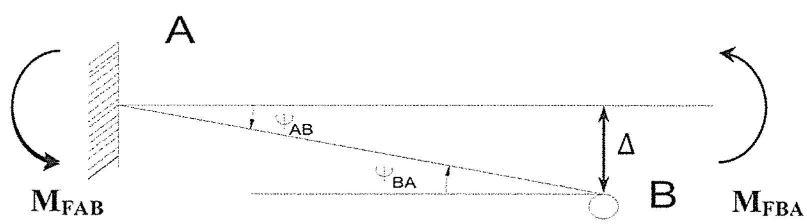


Figure 3.4: Effect of displacement at B

From the right column of Table 1A, when considering the far end pinned then,

$$M_{FAB} = \frac{-3EI\Delta}{L^2} \tag{3.6a}$$

From Table 1A, when considering both ends fixed then,

$$M_{FAB} = M_{FBA} = \frac{-6EI\Delta}{L^2} \tag{3.6b}$$

When solving problems where the far end is pinned, it is possible to take advantage of the right column of Table 1A, which gives fixed-end moments for a structure where the far end is pinned. An example of a far end pinned member is member AB. Although this method can reduce the number of calculations in the moment distribution table, it is not the only way to solve the problem. Alternatively, the standard fixed-end moments can be used for all joints, no matter the given support condition (fixed, pin, or roller). The difference between the two methods is that assuming all joints are fixed (using the left side of Table 1A) can be easier to set up, but it will often involve more iteration in the moment distribution table. In the next example both methods will be used to show that both methods provide the same answer without a great deal of difference in procedure.

Example 3.4.1:

Determine the reactions at the supports of the beam shown in Fig. 3.5 by the moment distribution method. Take E = 29,000 ksi and I = 446 in^4. The support at B is displaced downward 1 in.

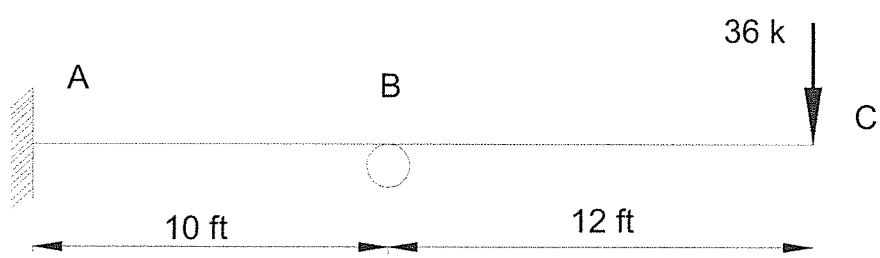


Figure 3.5: Statically indeterminate beam with support settlement

Solution

Method 1

Step 1

Stiffness Factors are not needed here because the distribution factors are known.

Step 2

$DF_{AB} = 0$ (Fixed support)

$DF_{BA} = 1$ (End-pin support)

$DF_{BC} = 0$ (No moment is transferred from B to C)

Step 3

Find M_{FAB} due to settlement downward of support B

$$\rightarrow M_{FAB} = -\frac{3EI\Delta}{L^2} = -\frac{3(4176000)(0.0215)\left(\frac{1}{12}\right)}{(10)^2} = -224.46^{k-ft}$$

$$M_{FBC} = -36(12) = -432^{k-ft}$$

Note: M_{FBC} is negative because the internal moment caused by the loading, acts in the counterclockwise direction (opposite to the external moment at that point).

Step 4 – 7

Now the moment distribution table is filled in using the distribution factors and fixed-end moments found in steps 1 through 3. In this moment distribution table, the unbalanced moments are equal to the fixed-end moments at each joint. Just as was done in the first problem, the unbalanced moment is multiplied by the respected DF and the Bal is found. Only one carry over takes place, which is from BA to AB. After only two cycles the moment distribution table is finished because all Bal values are zero.

Table 3.2: Moment distribution table (method 1)

Joint	A	B	
Member	AB	BA	BC
DF	0	1	0
M_F	-224.46	0	-432
Bal	0	432	0
CO	216	0	0
Bal	0	0	0
ΣM	**-8.46**	**432**	**-432**

The answers found here, $M_{AB} = -8.46^{ft-k}$ ↺, $M_{BA} = 432^{ft-k}$ ↻, and $M_{BC} = -432^{ft-k}$ ↺ are the same as found by the force method in Chapter One (Ex. 1.5.2.1) and by the slope-deflection method in Chapter Two (Ex. 2.3.1), but with less work.

Method 2

Step 1

Stiffness Factors are not needed because the distribution factors are known.

Step 2

$$DF_{AB} = 0 \qquad \text{(Fixed support)}$$

$$DF_{BA} = 1 \qquad \text{(End-pin support)}$$

$$DF_{BC} = 0 \qquad \text{(No moment is transferred from B to C)}$$

Step 3

$$M_{FAB} = M_{FBA} = -\frac{6EI\Delta}{L^2} = -\frac{6(4176000)(0.0215)\left(\frac{1}{12}\right)}{(10)^2} = -488.92^{k-ft}$$

Step 4

Since the left column fixed-end moments are used, member BA now has the same moment as member AB. The unbalanced moment for joint B is found by summing up M_{FBA} and M_{FBC}. This value is then multiplied by the distribution factor of members BA and BC as seen below:

$$Bal = -M_{(unbalanced)} * DF$$

$$Bal_{BA} = -(M_{BC} + M_{BA}) * 1$$

$$Bal_{BA} = -(-432 + (-448.92)\,) * 1 = 880.92^{ft-k}$$

Next, the carry-over factor $({}^{1}/{}_{2})$ is applied from joint B to Joint A, as indicated by the arrows. From here all balances are zero so the moments are summed up and the table is complete.

Table 3.3: Moment distribution table (method 2)

Joint	A	B	
Member	AB	BA	BC
DF	0	1	0
M_F	-448.92	-448.92	-432
Bal	0	880.92	0
CO	440.46	0	0
Bal	0	0	0
ΣM	**-8.46**	**432**	**-432**

When comparing the two methods, the final moments are exactly the same, and found in only two cycles. This proves that both methods can be used for a given problem based on preference. The authors of this book prefer method 2 due to the simplicity of finding the fixed-end moments. It should also be noted that the results are **identical** to those obtained using the slope-deflection method in Ex. 2.4.1 which can be seen in the chapter summary in Section 3.7.

3.5 Application of Moment Distribution to Analysis of Frames Without Sidesway

The analysis of frames without sidesway is similar to that of continuous beams. The procedure described in Sec. 3.3 will be used to analyze frames without side sway. An example is discussed below to illustrate this concept.

Example 3.5.1:

Determine the moments at each joint of the frame shown in Fig. 3.6 by the moment distribution method. E = 29,000 ksi, A = 16 in^2 and I = 446 in^4 for all members.

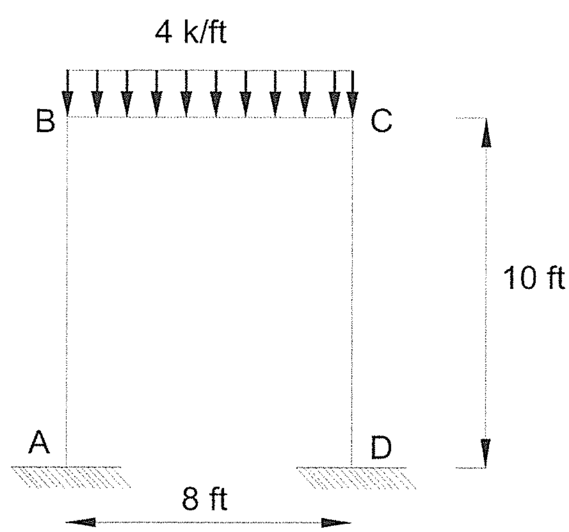


Figure 3.6: Indeterminate frame (no sidesway)

Solution

Step 1

$$K_{AB} = \frac{4EI}{10}, \quad K_{BC} = \frac{4EI}{8}, \text{ and } K_{CD} = \frac{4EI}{10}$$

Step 2

In this problem we must find the distribution factors for the members at joints B and C using Equation 3.5 because they are unknown. At joint B,

$$\rightarrow DF_{BA} = \frac{\frac{4EI}{10}}{\frac{4EI}{10}+\frac{4EI}{8}} = 0.444$$

$$\rightarrow DF_{BC} = 1\text{-}\ 0.444 = 0.556$$

Similarly at joint C,

$$\rightarrow DF_{CB} = \frac{\frac{4EI}{8}}{\frac{4EI}{10}+\frac{4EI}{8}} = 0.556$$

$$\rightarrow DF_{CD} = 1\text{-}\ 0.556 = 0.444$$

Step 3

The fixed-end moments are found using Table 1A.

$$M_{FBC} = -\frac{wL^2}{12} = -\frac{4(8)^2}{12} = -21.33^{k-ft}$$

$$M_{FCB} = \frac{4(8)^2}{12} = 21.33^{k-ft}$$

Step 4-7

All the joints, members, distribution factors, and fixed-end moments are filled in based on steps 1 – 3. A sample calculation for the balance of the first cycle for members BA and BC is given below:

$$Bal = -M_{(unbalanced)} * DF$$

$$Bal_{BA} = -(M_{BC} + M_{BA}) * 0.444$$

$$Bal_{BA} = -(-21.33 + 0\,) * 0.444 = 9.48^{ft-k}$$

$$Bal_{BC} = -(M_{BC} + M_{BA}) * 0.556$$

$$Bal_{BC} = -(-21.33 + 0\,) * 0.556 = 11.85^{ft-k}$$

Carry-over is applied between joints B and C, from B to A, and from C to D. This process is repeated until the balance is about 1% of the original unbalanced moment. Lastly, values for the internal end moments of each member are found by summing up all the entries in each members column starting with the fixed-end moment.

Table 3.4: Moment distribution table

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	0	0.444	0.556	0.556	0.444	0
M_F	0	0	-21.33	21.33	0	0
Bal	0	9.48	11.85	-11.85	-9.48	0
CO	4.74	0	-5.925	5.925	0	-4.74
Bal	0	2.633	3.292	-3.292	-2.633	0
CO	1.317	0	-1.646	1.646	0	-1.317
Bal	0	0.731	0.914	-0.914	-0.731	0
CO	0.366	0	-0.457	0.457	0	-0.366
Bal	0	0. 203	0.254	-0.254	-0.203	0
CO	0.102	0	-0.127	0.127	0	-0.102
Bal	0	0.056	0.071	-0. 071	-0.056	0
CO	0.028	0	-0.036	0.036	0	-0.028
Bal	0	0.016	0.02	-0.02	-0.016	0
ΣM	**6.55**	**13.12**	**-13.12**	**13.12**	**-13.12**	**-6.55**

The answers for the reactions at the base would be the moments, $M_{AB} = 6.55^{k-ft}$ ↻ and $M_{DC} = -6.55^{k-ft}$ ↺ . These values of end moments are very similar to the results of slope-deflection method.

Note: M_{AB} is the internal moment at joint A in member AB and is also the reaction at the support, M_A. M_A has the same magnitude and direction as M_{AB}

The vertical reactions at the base can be found using basic statics because the loading is symmetrical (half of the total distributed load force applied to each support acting upward). Also the horizontal forces at the base can be found in the following fashion:

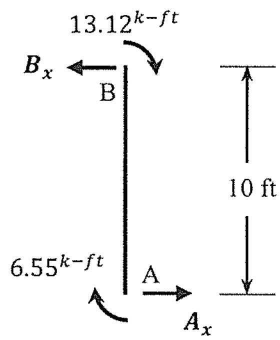


Figure 3.6.1: Member AB

$\overset{+}{\circlearrowright} \Sigma M_B = 0:$

$$13.12 + 6.55 - 10A_x = 0$$
$$A_x = 1.97\ k \ \rightarrow$$

For the whole frame:

$\overset{+}{\rightarrow} \Sigma F_x = 0:$

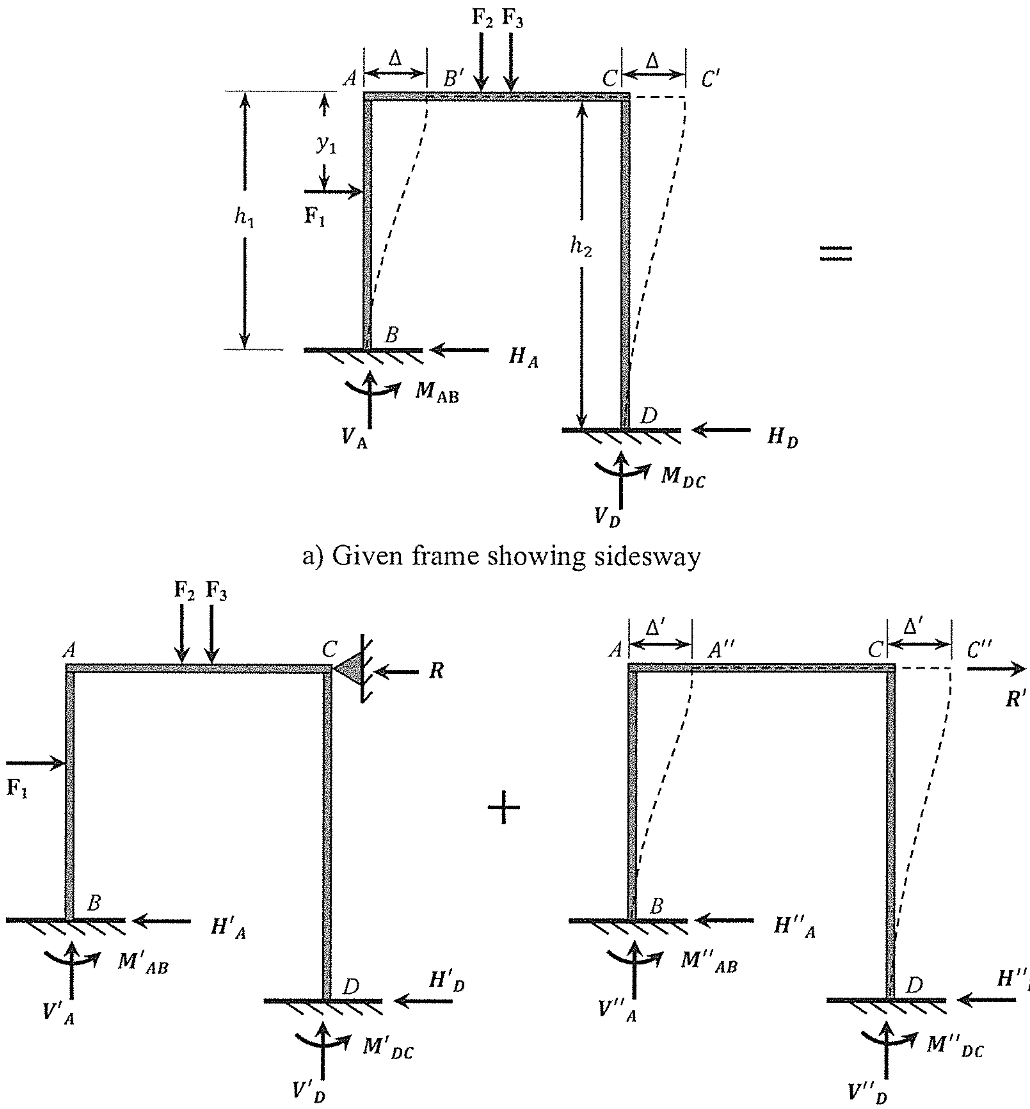
$$A_x + D_x = 0$$
$$D_x = -A_x = -1.97\ k \ \leftarrow$$

3.6 Application of Moment Distribution to Analysis of Frames with Sidesway

In this section, the basic concept involved in analysis of a frame with sidesway by moment distribution will be discussed followed by an example.

3.6.1 Basic Concepts: Application of Moment Distribution to Analysis of Frames with Sidesway

To solve a frame with side sway, the principle of superposition will be utilized, Fig. 3.7. This analysis involves two steps: 1) Analyze the frame with sidesway being restrained and with the applied loading, see Fig. 3.7b and 2) Analyze the frame with only sidesway and no applied loads, Fig. 3.7c. In both steps the moment distribution table will need to be used.



a) Given frame showing sidesway

b) Sidesway restrained

c) Sidesway allowed

Figure 3.7: Principle of superposition applied to a frame with sidesway

The basic superposition equation for the moment at can be derived as,

$$M_{NF} = M'_{NF} + kM''_{NF} \tag{3.7}$$

Where,

$$k = \ R/R' \tag{3.8}$$

$N =$ Near joint of a member

$M =$ Far joint of a member

The single prime denotes the moments on the restrained frame, Fig. 3.7b, and the double prime denotes the moments due to sidesway only, Fig. 3.7c. The expressions for sidesway moments are given in Eq. 3.6a and 3.6b.

The solution of frames with sidesway using the moment distribution methods essentially involves solving two moment distribution tables. The first table is for the frame restrained against sidesway having the given loading, and the second table is for the frame with sidesway only and no applied loads. The last portion of the superposition is relating the two sets of results through the k factor, Eq. 3.8. The process of filling in the moment distribution table will not change for frames with sidesway.

3.6.2 Example of Moment Distribution: Analysis of Frames with Sidesway

An example in which a frame with sidesway, or joint translation, is solved using the moment distribution method will be discussed below in Example 3.6.2.1

Example 3.6.2.1:

Analyze the frame shown in Fig. 3.8 using the moment distribution method. Determine the reactions at the supports of the frame shown in this figure. A = 100 in^2, E = 29000 ksi and I = 833 in^4.

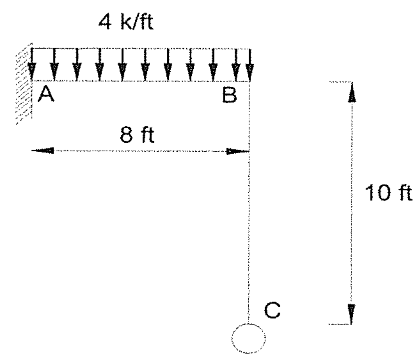


Figure 3.8: Frame with sidesway

Solution

Step 1

Here we need to calculate the stiffness factors of the members connected to joint B, because we do not know DF_{BA} or DF_{BC}.

$$K_{AB} = K_{BA} = \frac{4EI}{8} = 0.5EI$$

$$K_{BC} = K_{CB} = \frac{4EI}{10} = 0.4EI$$

Step 2

The distribution factors for members BA and BC are calculated below. DF_{BC} is found by $1 - DF_{BA}$ because the sum of the distribution factors at joint B must be 1. Also, since C is an end-pin support, DF_{CB} is 1.

$$DF_{BA} = \frac{0.5EI}{0.5EI + 0.4EI} = 0.556$$

$$DF_{BC} = 1 - 0.556 = 0.444$$

$$DF_{CB} = 1 \text{ (end-pin support)}$$

Step 3 – Restrained frame

The restrained frame includes an artificial support to inhibit sidesway at joint C. R is the reaction force at the artificial joint, C, due to the given loading. This is the first part of the principle of superposition applied to the frame.

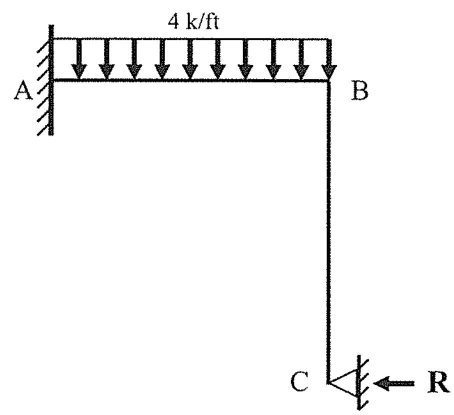


Figure 3.8: Restrained frame (no sidesway)

The fixed-end moments for member AB are calculated using Table 1A and are shown below. There are no fixed-end moments on member BC because there is no external loading on the member's span.

$$M_{FAB} = -\frac{4*8^2}{12} = -21.33^{k-ft} \ and \ M_{FBA} = \frac{4*8^2}{12} = 21.33^{k-ft}$$

Step 4

Table 3.5, the moment distribution table for the restrained frame, is set up using the distribution factors and the fixed-end moments found in the first three steps. Then, iteration of Bal and CO are carried out, making sure that joint equilibrium is satisfied in each cycle. The carry-over factor of $^1/_2$ is applied between joints where possible. Summing up each column yields the moments at each end of the members.

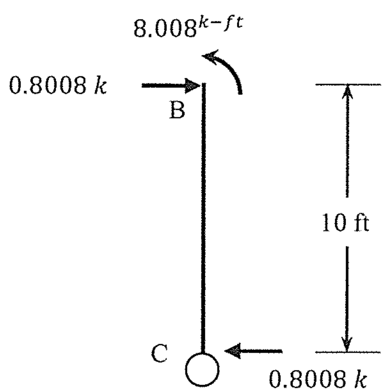
Table 3.5: Moment distribution table for the restrained frame

Joint	A	B		C
Member	AB	BA	BC	CB
DF	0	0.556	0.444	1
M_F	-21.33	21.33	0	0
Bal	0	-11.859	-9.471	0
CO	-5.930	0	0	-4.736
Bal	0	0	0	4.736
CO	0	0	2.368	0
Bal	0	-1.316	-1.051	0
CO	-0.658	0	0	-0.526
Bal	0	0	0	0.526
CO	0	0	0.263	0
Bal	0	-0.146	-0.117	0
Final	**-27.918**	**8.008**	**-8.008**	**0**

The results obtained here are, $M'_{AB} = -27.918^{k-ft}$ ↺, $M'_{BA} = 8.008^{k-ft}$ ↻, and $M'_{BC} = -8.008^{k-ft}$ ↺.

Calculate R:

Using the internal moments found for the restrained frame in Table 3.5, we can now find the force R.



$\overset{+}{\circlearrowright} \Sigma M_B = 0:$

$$-8.008 + 10C_x = 0$$
$$C_x = R = 0.8008\,k \leftarrow$$

Figure 3.9: Calculation of R using member BC

Using Figure 3.9, the following calculation can be made:

$$Required\ couple = \frac{8.008^{ft-k}}{10ft} = 0.8008\ k$$

In order to satisfy equilibrium on this member, the couple must be acting in the opposite direction as the M_{BC}, therefore:

$$R = 0.8008\ k$$

Note: R is positive because it is acting in the direction assumed, Fig. 3.8.

Frame with sidesway and artificial joint removed

Again, the fixed-end moments on the members need to be found. Member BC will have a fixed-end moment due to deflection at C. Typically this would involve using Table 1A found in the appendix, but if the columns have the same displacement (Δ'), E, I, and L, then an arbitrary value can be used with the correct direction. In this problem there is only one column and one moment so no problems will be encountered using an arbitrary moment of 100^{k-ft} applied clockwise at B.

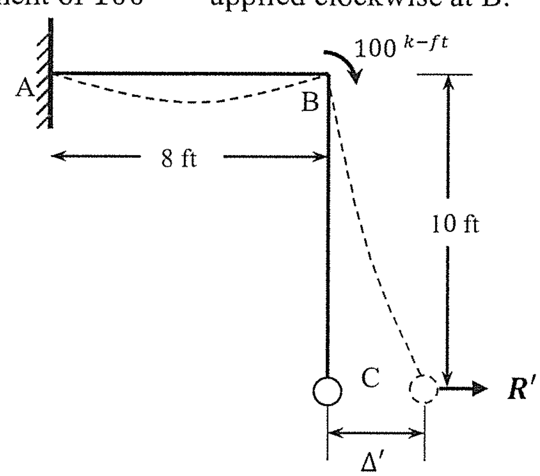


Figure 3.10: Frame with sidesway and artificial joint removed

To set up Table 3.6, all we need are the distribution factors and the single M_F for member BC. Then, the typical procedure of Bal and CO are used to fill in the rest of the table and find the end moments at each member.

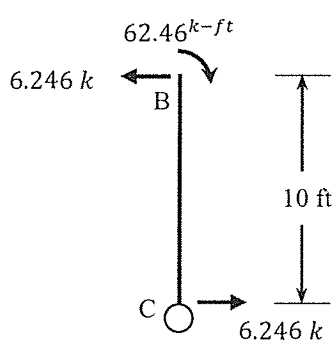
Table 3.6: Moment distribution table for the frame with side sway

Joint	A	B		C
Member	AB	BA	BC	CB
DF	0	0.556	0.444	1
M_F	0	0	100	0
Bal	0	-55.6	-44.4	0
CO	-27.8	0	0	-22.2
Bal	0	0	0	22.2
CO	0	0	11.1	0
Bal	0	-6.172	-4.928	0
CO	-3.086	0	0	-2.464
Bal	0	0	0	2.464
CO	0	0	1.232	0
Bal	0	-0.685	-0.547	0
Final	**-30.886**	**-62.457**	**62.457**	**0**

The results obtained here are, $M''_{AB} = -30.886^{k-ft}$ ↺, $M''_{BA} = -62.457^{k-ft}$ ↺, and $M''_{BC} = 62.457^{k-ft}$ ↻.

Calculate R':

Now we will find the value of R', which is the amount of force required to make the displacement, Δ', at joint C using the moments found in Table 3.6. This is the second part of the principal of superposition for the frame.



$\overset{+}{\circlearrowright}\Sigma M_B = 0:$

$$62.46 - 10C_x = 0$$
$$C_x = R' = 6.246\ k\ \rightarrow$$

Figure 3.11: Calculation of R' using member BC

Using the figure above, the following calculation can be made:

$$Required\ couple = \frac{62.46^{ft-k}}{10ft} = 6.246\ k$$

In order to satisfy equilibrium on this member, the couple must be acting in the opposite direction as the M_{BC}, therefore:

$$R' = 6.246\ k$$

Note: R' is positive because it acts in the direction assumed in Fig. 3.10

Final moments:

Using the ratio of $R/_{R'}$ we can complete the superposition and find the amount of moment that needs to be added or subtracted from the original moments found for the restrained frame.

$$M_{AB} = M'_{AB(restrained)} + \frac{R}{R'} M''_{AB(sidesway)}$$

$$M_{AB} = -27.918 + \frac{0.8008}{6.246}(-30.886) = -\mathbf{32}^{k-ft}$$

$$M_{BA} = 8.008 + \frac{0.8008}{6.246}(-62.46) = \mathbf{0}^{k-ft}$$

$$M_{BC} = -8.008 + \frac{0.8008}{6.246}(62.46) = \mathbf{0}^{k-ft}$$

The reactions at A (A_x and A_y), at C (C_y) are calculated from principles of statics:

$$\overset{+}{\rightarrow} \Sigma F_x = 0 \rightarrow \boldsymbol{A_x = 0}$$

$$\overset{+}{\circlearrowright} \Sigma M_A = 0 : -32 + 32 * 4 - C_y * 8 = 0$$

$$\boldsymbol{C_y = 12k}(\uparrow)$$

$$+\uparrow \Sigma F_y = 0 \text{: } A_y + 12 - 32 = 0$$

$$\boldsymbol{A_y = 20\ k}\ (\uparrow)$$

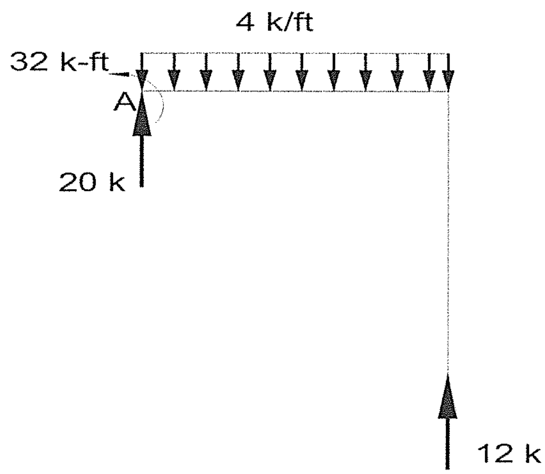


Figure 3.12: Final reactions for the frame

3.7 Summary

A concise description of the concept of the moment distribution is presented in this chapter. This is followed by few problems dealing with continuous beams, frame without joint movements (sidesway) and with joint movements. It should be noted that moment distribution method directly gives moments.

Table 3.7 shows a summary of the solutions to example problems from Chapter 1 to 3. The answers for each method are very close if not exactly the same, showing that any method may be used to analyze an indeterminate structure. The moment distribution method was the only method that showed slight differences. These can usually be minimized with more iteration.

Table 3.7: Comparison of example problem solutions from Ch. 1 - 3

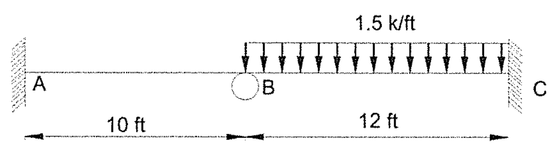
	Chapter 1 (Force Method)	Chapter 2 (Slope-Deflection)	Chapter 3 (Moment Distribution)
Beam	$B_y = 100.8\ k\ \uparrow$ $A_y = -64.8\ k\ \downarrow$ $M_A = 216^{k-ft}\ \circlearrowright$	$B_y = 100.8\ k\ \uparrow$ $A_y = -64.8\ k\ \downarrow$ $M_{AB} = 216^{k-ft}\ \circlearrowright$	$B_y = 100.8\ k\ \uparrow$ $A_y = -64.8\ k\ \downarrow$ $M_{AB} = 216^{k-ft}\ \circlearrowright$
Beam With Settlement		$B_y = 78.356\ k\ \uparrow$ $A_y = -43.356\ k\ \downarrow$ $M_{AB} = 8.44^{k-ft}\ \circlearrowleft$	$B_y = 78.354\ k\ \uparrow$ $A_y = -43.354\ k\ \downarrow$ $M_{AB} = 8.46^{k-ft}\ \circlearrowleft$
Frame No sidesway		$M_{AB} = 6.573^{k-ft}\ \circlearrowright$ $M_{AB} = 6.573^{k-ft}\ \circlearrowleft$ $A_y = D_y = 16\ k\ \uparrow$	$M_{AB} = 6.55^{k-ft}\ \circlearrowright$ $M_{AB} = 6.55^{k-ft}\ \circlearrowleft$ $A_y = D_y = 16\ k\ \uparrow$
Frame With Sidesway	$A_y = 20\ k\ \uparrow$ $C_y = 12\ k\ \downarrow$ $M_A = 32^{k-ft}\ \circlearrowleft$	$A_y = 20\ k\ \uparrow$ $C_y = 12\ k\ \downarrow$ $M_{AB} = 32^{k-ft}\ \circlearrowleft$	$A_y = 20\ k\ \uparrow$ $C_y = 12\ k\ \downarrow$ $M_{AB} = 32^{k-ft}\ \circlearrowleft$

Problems

Analyze Problems 3.1 to 3.3 using the moment distribution method.

Problem 3.1 Solve *Problem 2.1* by using the moment distribution method

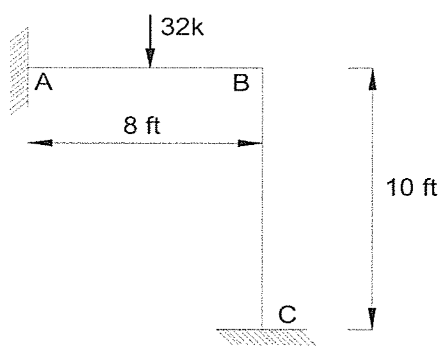
Problem 2.1 repeated Determine the reactions at the supports of the beam shown in this figure. EI is constant.



Problem 3.1

Problem 3.2 Solve *Problem 2.2* by using the moment distribution method

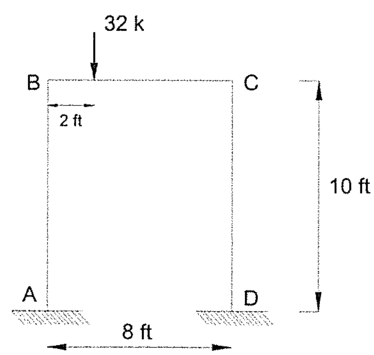
Problem 2.2 repeated Determine the reactions at the supports of the frame shown in this figure. A = 100 in^2, E = 29000 ksi and I = 833 in^4.



Problem 3.2

Problem 3.3 Solve *Problem 2.3* by using the moment distribution method

Problem 2.3 repeated Determine the reactions at the supports of the frame shown in this figure. A = 100 in^2, E = 29000 ksi and I = 833 in^4



Problem 3.3

Chapter 4

Direct Stiffness Method: Application to Beams

4.1 Basic Concepts of the Stiffness Method

The stiffness method is the most powerful method used for analysis of structures. Almost all the computer codes written to analyze structures use the stiffness method. One reason for the wide use is because the general procedure of the stiffness method can be applied to any type of structure – beam, frame, truss, or any structure for that matter. Also, it is very easy to use and can be coded for analysis of entire structures.

4.2 Kinematic Indeterminacy

A structure's Kinematic Indeterminacy (KI) must be established before solving a problem by the stiffness method. Again, Kinematic Indeterminacy is defined as the total number of degrees of freedom for all the joints in a given structure. Another method similar to the stiffness methods is the flexibility method. The flexibility method is a matrix equivalent of the force method.

4.3 Relation Between Stiffness Method and Direct Stiffness Method

While the stiffness method and the direct stiffness method are essentially the same, well-known authors have drawn a distinction (Weaver and Gere, 1990). Although the distinction is slight, it is important that this be explained, especially for the undergraduate students in civil engineering for whom this book is aimed at. In the stiffness method, the elements of a *stiffness matrix are derived* from the basic principles of engineering mechanics corresponding to the unknown displacements in the structure. In the case of the direct stiffness matrix, the *standard stiffness matrix* for each element (whether beam element, truss element or frame element) is used to assemble a structure stiffness matrix. This matrix is then used to solve for displacements. Thus, the direct stiffness method is more mechanical, to put it in plain terms, and is very easy to use. For this reason, the direct stiffness method is very popular and widely used for analysis of any type of structure. The details of the direct stiffness method will be discussed in the text and also applied to beam, frame and truss structures with specific examples of each in this Chapter, Chapter 5, and Chapter 6 respectively.

4.4 Derivation/Explanation of the Beam-Element Stiffness Matrix

A typical beam element, or member, is shown in Fig. 4.1. As can be seen in Fig. 4.1, the action $\{A\}$ and the displacement $\{D\}$ are shown at the ends of a typical beam element. $\{D\}$ represents a generalized displacement (translation or rotation) and $\{A\}$ represents a generalized force (force or moment). In the stiffness method, all loads will be distributed to the nodes. An assembly of the various actions of a structure will constitute an action vector. Similarly, an assembly of the various displacements of a structure will constitute a displacement vector. The reactions at the supports will also be considered part of the generalized action vector. In Fig. 4.1, the actions and displacements share a common number. To simplify things, one number and symbol are often written to represent both entities (See Fig 4.3).

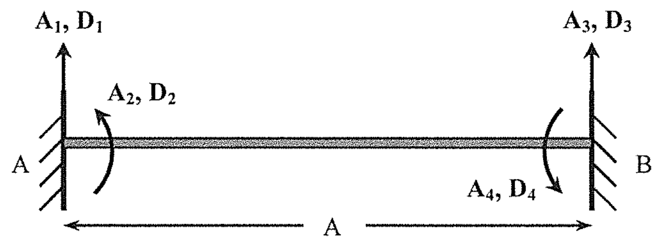


Figure 4.1: Typical Beam Element

Note: The sign convention used in the stiffness method follows the right hand rule. This means when curling the fingers on your right hand so that they point in the direction of the moment or rotation, if your thumb points up, or at you, then it is positive. Likewise, if it points down, or away from you, then it is negative.

The basic definition of stiffness can easily be obtained from the following equation which is the basic relationship used in the stiffness method,

$$\{A\} = [K]\{D\} \tag{4.1}$$

Where, $\{A\}$ = Action vector

$\{D\}$ = Displacement vector

$[K]$ = Stiffness matrix

From Eq. 4.1, if $\{D\} = 1$ then $[K] = \{A\}$. This implies that the force required to cause a unit displacement, is the stiffness. **This is a very important definition.** This basic relation is used in the analysis of every structure. If there is only one displacement, then $[K]$ will be an element instead of a matrix. $[K]$ will be different from element to element when the length (L), material (E), or cross section (I) are changed .

The matrix $[K]$, used to relate the actions at joint A (near end) and joint B (far end) to the displacements at joint A and B for a beam element, can be expressed as (Weaver and Gere, 1990),

$$[K] = \begin{array}{c c} \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} & \\ \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} & \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \end{array} \tag{4.2}$$

Note: $[K]$ is a symmetric matrix.

The action and displacement vectors for the beam element shown in Fig. 4.1 can be expressed as,

$$\{A\} = \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} \quad \text{and} \quad \{D\} = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix} \qquad \text{(4.3a \& 4.3b)}$$

The $[K]$ matrix in Eq. 4.2 is written in structure coordinates (global x, y, z axes), which is the same as member axes for a beam element. The member axis changes with respect to the angle of the member. For example, the member axis of a column (vertical member) would be rotated 90° counterclockwise from the structure axis. Frame and truss elements that are not horizontal will have different member and global axis. Hence, $[K]$ can also be called $[K_{\mathrm{MSi}}]$, which means stiffness matrix of the element in structure (global) coordinates. It is to be noted that the way the stiffness matrix $[K]$ is written in Eq. 4.2 has to properly correspond with the action and displacement vectors in

Eq. 4.3. As can be seen by the markings D_1 through D_4 in Eq. 4.2, the rows and columns of $[K]$ correspond to the translational and rotational displacements of a given member. The pattern used in Eq. 4.2 is not arbitrary. The first row corresponds to the translational displacement at the near (typically left) end of the member and the second row corresponds to the rotational displacement at the near end of the member. The third and fourth rows correspond to the translation and rotation of the far (typically right) end of the member respectively. This pattern must be followed for all members. Otherwise, the $[K]$ will be wrong and erroneous results will be obtained. The displacement vector has to consist of translations and rotations at the near end followed by translations and rotations at the far end. The same order is to be followed for the corresponding actions when determining the action vector.

Another point to be noted is that some of the displacements in the displacement vector will be unknown because they are free to displace. These will be called $\{D_F\}$, because they are the free displacements. On the other hand, some displacements will be known because they are zero, or move a certain amount (support settlement). These are designated as $\{D_R\}$ because they are the restrained displacements. The free displacements are the ones to be obtained. The actions corresponding to the free displacements are denoted $\{A_F\}$. They are known because they are the given applied loads. The unknown required forces (such as reactions) are designated as $\{A_R\}$ because they correspond to the restrained displacements.

To summarize, we want to find the unknown free displacements $\{D_F\}$ and the unknown required forces $\{A_R\}$ using the known restrained displacements $\{D_R\}$ and the known given loads $\{A_F\}$.

4.4.1 Global/Structure Stiffness Matrix

In addition to the element stiffness matrix $[K]$, there is the structure stiffness matrix $[K_J]$, which links each individual member to the whole structure. To make the member stiffness matrix for a given structure, the following formula will need to be used:

$$K_{Jij} = \sum_1^n K_{A_i-D_j} \tag{4.4}$$

Where, n = Number of terms with the same i-j location.
$K_{A_i-D_j}$ = Value in the element stiffness matrix $[K]$ corresponding to A_i and D_j

This formula is used to find each term of $[K_J]$ using the i-j location in the element stiffness matrix.

4.5 Application of the Direct Stiffness Method to a Continuous Beam

Here, the basic analysis procedure of the direct stiffness method for continuous beams will be explained followed by an example.

4.5.1 Basic Procedure of the Direct Stiffness Method for Beams

1. Number the joints.
2. Number and define the members (with respect to how the member is connected and to which joints). Follow the same order when defining the displacements and the corresponding actions.
3. Determine KI (Kinematic Indeterminacy) and identify the unknown displacements for the given structure. Then, number the displacements starting from the free (unknown) displacements followed by the

restrained displacements. Within the free displacements any order can be followed.

4. Write down the element stiffness matrix $[K]$ from Eq. 4.2 connecting the action vector and the displacement vector (from Eq.4.3) for all the members in the given structure. Again, make sure that the translational displacements are to be followed by rotational displacements (follow the same corresponding order for action vector as well).
5. Assemble the structure stiffness matrix $[K_J]$ by combining the elements of the same kind using the following equation.

$$K_{Jij} = \sum_1^n K_{A_i - D_j} \tag{4.4}$$

Where, n = Number of terms with the same i-j location

$K_{A_i-D_j}$ = Value in the element stiffness matrix $[K]$ corresponding to A_i and D_j.

$K_{A_i-D_j}$ corresponds to all elements of the member stiffness matrix $[K]$ that relate to a specific action, A_i, and displacement, D_j. The above equation generates all the elements of the structure stiffness matrix $[K_J]$ by combining all the elements corresponding to suffix i-j for all the members.

If the numbering of the displacements is done starting from the free displacements, the constitution of $[K_J]$, the joint structure stiffness matrix, as it relates to structure action and displacement will be as shown in Eq. 4.5 below.

$$\begin{Bmatrix} A_F \\ \hdashline A_R \end{Bmatrix} = \left[\begin{array}{c:c} K_{FF} & K_{FR} \\ \hdashline K_{RF} & K_{RR} \end{array}\right] \begin{Bmatrix} D_F \\ \hdashline D_R \end{Bmatrix} \tag{4.5}$$

Where: K_{FF} = Portion of the structure stiffness matrix containing the known actions $\{A_F\}$ and the corresponding free displacements $\{D_F\}$.

K_{FR} = Portion of the structure stiffness matrix containing the known actions $\{A_F\}$ and the restrained displacements $\{D_R\}$.

K_{RF} = Portion of the structure stiffness matrix containing the unknown forces $\{A_R\}$ and the free displacements $\{D_F\}$.

K_{RR} = Portion of the structure stiffness matrix containing the unknown forces $\{A_R\}$ and the restrained displacements $\{D_R\}$.

6. The unknown displacements $\{D_F\}$ can be obtained from the following equation (which is derived from Eq.4.5),

$$\{D_F\} = [K_{FF}]^{-1}\{A_{FN}\} \tag{4.6}$$

Where $\{A_{FN}\}$ denotes the net actual and equivalent joint loads corresponding to free displacements. Equivalent joint loads are member loads that are distributed to the joints.

7. The unknown reactions $\{A_R\}$ can be obtained from the following equation:

$$\{A_R\} = [K_{RF}]\{D_F\} - \{A_{RN}\} \tag{4.7}$$

Where $\{A_{RN}\}$ denotes the net actual and equivalent joint loads corresponding to restrained displacements. Using the above equation, one can easily obtain $\{A_R\}$ knowing $[K]$ and $\{D\}$ obtained in previous steps.

4.5.2 Example of a Continuous Beam Using the Stiffness Method

A continuous beam is solved below using the stiffness method described above.

Example 4.5.2.1

Determine the reactions at the supports of the beam shown in Fig. 4.2 using stiffness method. E = 29000 ksi, I = 446 in^4.

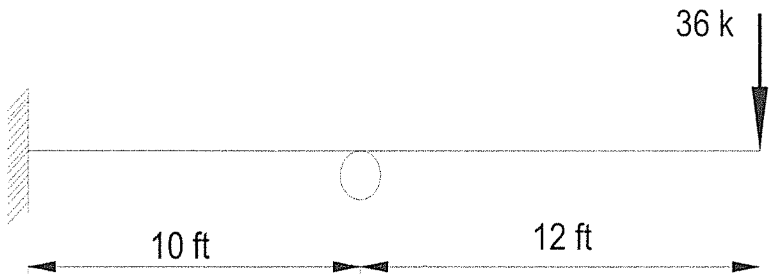


Figure 4.2: Continuous Beam

Solution

Steps 1-3

Fig. 4.3 shows the beam after executing steps 1 through 3. Member 1 is defined as node number 1-2 and member 2 is defined as node number 2-3. This order needs to be used when writing the $[K]$ matrix using the action and displacement vector using Eq. 4.2 and Eq. 4.3 respectively. Note that it could have been defined differently; like member 1 could have been defined as 2-1.

The Kinematic Indeterminacy of this structure is 3 and the unknown displacements are D_1, D_2 and D_3 (translation at joint 1 and rotations at joint 2 and 3). Figure 4.3 shows the displacement numbering is done starting with the free displacements followed by the restrained displacements. In this problem D_1, D_2 and D_3 are free displacements (unknowns) while D_4 through D_6 are the restrained displacements, which are zero in this problem. As explained earlier, the actions $\{A_F\}$ corresponding to $\{D_F\}$ are known (which can easily be obtained from the fixed end moments). The $\{A_R\}$ corresponding to the known $\{D_R\}$ are the unknowns.

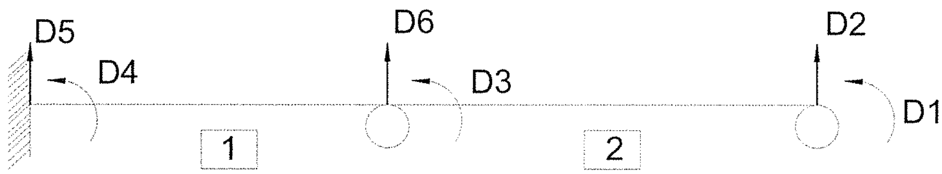


Figure 4.3: Beam showing displacements

Step 4

The stiffness matrix $[K_1]$ for member 1 is obtained from Eq. 4.2 as using the given properties of the beam element in the given problem.

Table 4.1: Stiffness matrix for member 1

$$[K_1] = \begin{array}{cccc} 5 & 4 & 6 & 3 \end{array}$$
$$\left[\begin{array}{cccc} 89.82 & 5389.17 & -89.82 & 5389.17 \\ 5389.17 & 431133.33 & -5389.17 & 215566.67 \\ -89.82 & -5389.17 & 89.82 & -5389.17 \\ 5389.17 & 215566.67 & -5389.17 & 431133.33 \end{array}\right] \begin{array}{c} 5 \\ 4 \\ 6 \\ 3 \end{array}$$

The above matrix, $[K_1]$, relates the action vector $\{A_5, A_4, A_6, A_3\}^t$ to the displacement vector $\{D_5, D_4, D_6, D_3\}^t$. As pointed out earlier in the basic procedure, displacements numbering follows the order of translation followed by rotations. The same order is followed by the corresponding actions as well. This is very important as noted earlier.

A derivation of some terms used in calculating the stiffness matrix $[K]$ is shown below.

$$K_{55} = K_{66} = -K_{65} = -K_{56} = \frac{12\,EI}{L^3} = \frac{12\,(29000)(446)}{(10*12)^3} = 89.82$$

Similarly, other terms in the stiffness matrix $[K_1]$ can be calculated. In exactly same way, $[K_2]$, the stiffness matrix for member 2 can be assembled using Eq. 4.2 and Eq. 4.3 as,

Table 4.2: Stiffness matrix for member 2

$[K_2] =$

6	3	2	1	
51.98	3742.48	-51.98	3742.48	6
3742.48	359277.78	-3742.48	179638.89	3
-51.98	-3742.48	51.98	-3742.48	2
3742.48	179638.89	-3742.48	359277.78	1

The above $[K]$ matrix relates the action vector $\{A_6, A_3, A_2, A_1\}^t$ to the displacement vector $\{D_6, D_3, D_2, D_1\}^t$. Again, translations are numbered first followed by rotations.

Step 5

The structure stiffness matrix for this problem $[K_J]$ can be assembled using Eq. 4.4 and the assembled matrices $[K_1]$ and $[K_2]$.

Table 4.2: Global/Structure stiffness matrix

$[K_J]=$

1	2	3	4	5	6	
359277.8	-3742.5	179638.9	0	0	3742.5	1
-3742.5	51.98	-3742.5	0	0	-51.98	2
179638.9	-3742.5	790411.1	215566.7	5389.17	-1646.7	3
0	0	215566.7	431133.3	5389.17	-5389.2	4
0	0	5389.2	5389.2	89.82	-89.82	5
3742.5	-51.98	-1646.7	-5389.2	-89.82	141.8	6

From the given condition, we can calculate,

$$\{A_{FN}\} = \begin{array}{c} \left\{\begin{matrix} 0 \\ -36 \\ 0 \end{matrix}\right\} \end{array}\begin{matrix} 4 \\ 5 \\ 6 \end{matrix}$$

From the above, the $[K_{FF}]$ can be written as,

$$[K_{FF}] = \begin{bmatrix} 359277.8 & -3742.5 & 179638.9 \\ -3742.5 & 51.98 & -3742.5 \\ 179638.9 & -3742.5 & 790411.1 \end{bmatrix}$$

We can solve the displacement:

$$\{D_F\} = [K_{FF}]^{-1} * \{A_{FN}\} = \begin{bmatrix} -0.04088 \\ -4.50183 \\ -0.01202 \end{bmatrix}$$

Calculate the reaction forces:

$$\begin{bmatrix} A_4 \\ A_5 \\ A_6 \end{bmatrix} = [K_{RF}]\,\{D_F\} = \begin{bmatrix} -2592^{k-in} \\ -64.8\ k \\ 100.8\ k \end{bmatrix} = \begin{bmatrix} -216^{k-ft} \\ -64.8\ k \\ 100.8\ k \end{bmatrix}$$

4.6 Summary

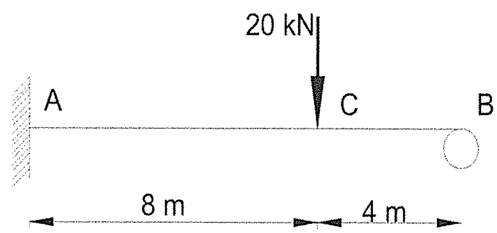
This chapter described the basic concepts of one of the most powerful methods of structural analysis – the direct stiffness method. A general procedure for solution of problems by the direct stiffness method is described with application to continuous beams. The method, as described in this chapter is general enough so that it can be applied to truss and frame as well.

Problems

Analyze Problem 1.1 in Chapter 1 and Problem 2.1 Chapter 2 using the direct stiffness method.

Problem 4.1 Solve *Problem 1.1* using the direct stiffness method.

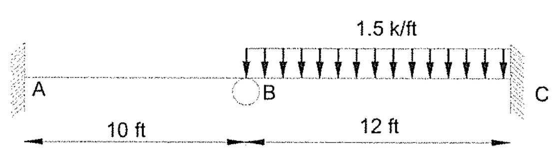
Problem 1.1 repeated Determine the reactions at the supports of the beam shown in this figure. Take EI constant.



Problem 4.1

Problem 4.2 Solve *Problem 2.1* using the direct stiffness method.

Problem 2.1 repeated Determine the reactions at the supports of the beam shown in this figure. EI is constant.



Problem 4.2

Chapter 5

Direct Stiffness Method: Application to Frames

5.1 Derivation/Explanation of the Stiffness Matrix for a Frame Element

The basic procedure of the direct stiffness method has been explained in Chapter 4. The explanation is essentially a general procedure, which is also applicable to frames. The only difference is that in the case of a frame, the element stiffness matrix $[K]$ for a frame element has to be used instead of for a beam element. This is explained below.

Consider the general frame element shown in Fig. 5.1

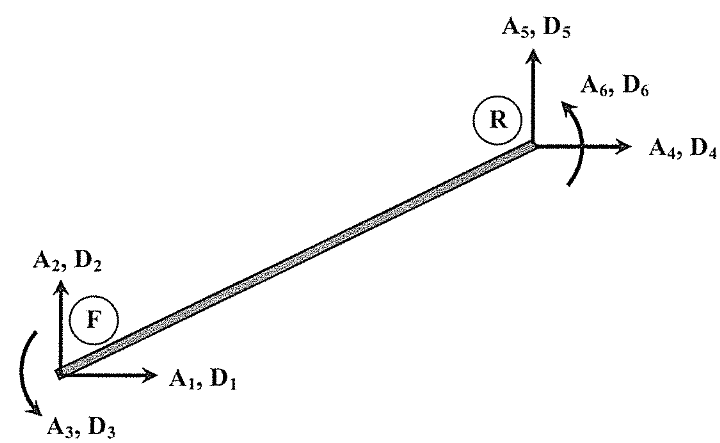


Figure 5.1: Typical frame element with free and restrained supports

The member stiffness matrix for the frame element in structural (global) coordinates, $[K_{MSi}]$ is given as,

Equation 5.1: Frame element stiffness matrix

$$[K_{MSi}]= \begin{array}{c} \begin{matrix} \quad 1 & \qquad 2 & \qquad 3 & \qquad 4 & \qquad 5 & \qquad 6 \end{matrix} \\ \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \end{array} \begin{matrix} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \qquad (5.1)$$

The global element stiffness matrix $[K_{MSi}]$ shown in Eq. 5.1 relates the action vector $\{A_1, A_2, A_3, A_4, A_5, A_6\}^t$ to displacement vector $\{D_1, D_2, D_3, D_4, D_5, D_6\}^t$.

Again, like in the case of the beam, some of the elements of the displacement vector will be zero. These will correspond to the restrained displacement vector $\{D_R\}$, which is subset of the $\{D\}$ vector. Similarly, the unknown, or free, displacements will be part of the free displacement vector $\{D_F\}$, which is also a subset of the total displacement vector $\{D\}$.

5.2 Application of the Direct Stiffness Method to a Frame

The process used to solve for the reactions of an indeterminate frame using the direct stiffness method is explained below using Example 5.2.1.

Example 5.2.1

Determine the reactions at the supports of the frame shown in Fig. 5.2 using the direct stiffness method. A = 100 in^2, E = 29000 ksi and I = 833 in^4.

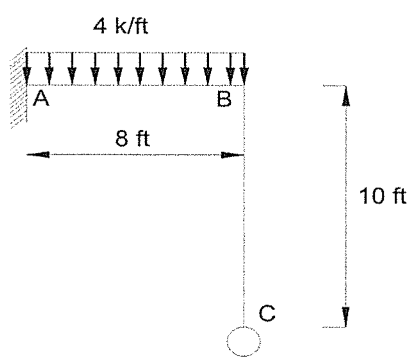


Figure 5.2: Indeterminate frame

Solution

Before starting the solution of the problem, it is to be noted that this frame has side sway (joint translation). However, an important point is that in the case of stiffness method, the procedure of analysis of a frame with or without joint translation is the same except that when the frame has side sway, the Kinematic Indeterminacy increases. This will, in essence, involve more equations to be solved.

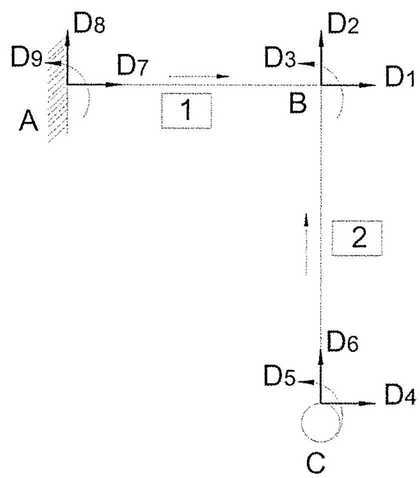


Figure 5.3: Frame showing displacements

Step 1-3

The numbering process is shown in Fig. 5.3. As can be seen from Fig. 5.3, KI (Kinematic Indeterminacy) of this structure is 5. So, the $\{D_F\}$ vector is: $\{D_1, D_2, D_3, D_4, D_5\}$. In this problem again, as in the beam problem, the numbering of the displacements is done from the free displacements. The restrained displacement vector, $\{D_R\}$, for this problem is $\{D_6, D_7, D_8, D_9\}$. Only the free displacements are to be obtained, from $[K_{FF}]$ and $\{A_{FN}\}$, as a solution to the problem. Then, the forces (actions) corresponding to the $\{D_R\}$ vector need to be calculated.

Step 4

The member stiffness matrix $[K_1]$ for member 1 is obtained using Eq. 5.1. It is given as,

Table 5.1: Stiffness matrix for member 1

$$
[K_1]= \begin{array}{c}
\begin{array}{cccccc} 7 & 8 & 9 & 1 & 2 & 3 \end{array} \\
\left[\begin{array}{cccccc}
30208 & 0 & 0 & -30208 & 0 & 0 \\
0 & 327.7 & 15727 & 0 & -327.7 & 15727 \\
0 & 15727 & 1006542 & 0 & -15727 & 503271 \\
-30208 & 0 & 0 & 30208 & 0 & 0 \\
0 & -327.7 & -15727 & 0 & 327.7 & -15727 \\
0 & 15727 & 503271 & 0 & -15727 & 1006542
\end{array}\right]
\end{array}
\begin{array}{c} \\ 7 \\ 8 \\ 9 \\ 1 \\ 2 \\ 3 \end{array}
$$

Similarly, the stiffness matrix for member 2 $[K_2]$ is given as,

Table 5.2: Stiffness matrix for member 2

$$
[K_2]= \begin{array}{c}
\begin{array}{cccccc} 4 & 6 & 5 & 1 & 2 & 3 \end{array} \\
\left[\begin{array}{cccccc}
167.8 & 0 & -10065 & -167.8 & 0 & -10065 \\
0 & 24167 & 0 & 0 & -24167 & 0 \\
-10065 & 0 & 805233 & 10065 & 0 & 402617 \\
-167.8 & 0 & 10065 & 167.8 & 0 & 10065 \\
0 & -24167 & 0 & 0 & 24167 & 0 \\
-10065 & 0 & 402617 & 10065 & 0 & 805233
\end{array}\right]
\end{array}
\begin{array}{c} \\ 4 \\ 6 \\ 5 \\ 1 \\ 2 \\ 3 \end{array}
$$

Step 5

Since only the unknown displacements are to be obtained in this problem, we need only $[K_{FF}]$ and $\{A_{FN}\}$ corresponding to the free displacements.

Here, $[K_{FF}]$ can be written as,

Table 5.3: Global/Structure stiffness matrix of the frame

$[K_J]=$

1	2	3	4	5	6	7	8	9	
30376.1	0	10065.4	-167.8	10065.4	0	-30208.3	0	0	1
0	24494.3	-15727	0	0	-24166.7	0	-327.7	-15727	2
10065.4	-15727.2	1811775	-10065.4	402617	0	0	15727	503271	3
-167.8	0	-10065.4	167.8	-10065.4	0	0	0	0	4
10065.4	0	402617	-10065.4	805233	0	0	0	0	5
0	-24166.7	0	0	0	24166.7	0	0	0	6
-30208.3	0	0	0	0	0	30208.3	0	0	7
0	-327.7	15727	0	0	0	0	327.7	15727	8
0	-15727.2	503271	0	0	0	0	15727	1006542	9

The $\{A_{FN}\}$ can be written as,

$$\{A_{FN}\} = \begin{array}{c} \left\{\begin{matrix} 0 \\ -16 \\ 256 \\ 0 \\ 0 \end{matrix}\right\} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}$$

In this problem there is a uniformly distributed load on element 1. If there are member loads acting on a particular structure (the way we had in the beam problem in Ch.4), then fixed end moments have to be calculated and the corresponding $\{A_{FN}\}$ vector can be assembled without any problem. The fixed end moments will be reverse (opposite sign).

Solving the above matrix (which essentially consists of solving 5 simultaneous equations), all the unknown displacements can be obtained in the following way as,

$$\{D_F\} = [K_{FF}]^{-1} * \{A_{FN}\} = \begin{bmatrix} 0 \\ -4.95\,x10^{-4} \\ 2.47\,x10^{-4} \\ 0.02959 \\ 2.47\,x10^{-4} \end{bmatrix}$$

Calculate the reaction forces

$$\begin{bmatrix} A_6 \\ A_7 \\ A_8 \\ A_9 \end{bmatrix} = [K_{RF}]\{D_F\} + \begin{bmatrix} 0 \\ 0 \\ 16\,k \\ 256^{k-in} \end{bmatrix} = \begin{bmatrix} 11.96\,k \\ 0 \\ 20.04\,\,k \\ 387.9^{k-in} \end{bmatrix} = \begin{bmatrix} \mathbf{11.96}\,\boldsymbol{k} \\ \mathbf{0} \\ \mathbf{20.04}\,\,\boldsymbol{k} \\ \mathbf{32.325}^{\boldsymbol{k-ft}} \end{bmatrix}$$

This completes solution of the given problem.

5.3 Summary

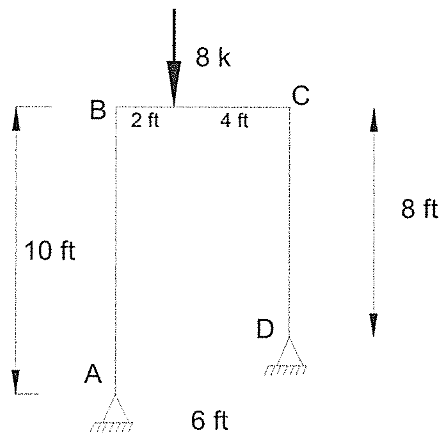
In this chapter, the powerful method called the direct stiffness method was applied to a frame problem. The beauty of the direct stiffness method, as applied to frames, is that no distinction is to be made for frames with or without joint movements. The only difference is that the problem has to be solved for additional unknowns, but the procedure remains same.

Problems

Analyze Problem 1.2 in Chapter 1 and Problems 2.3 and 2.4 in Chapter 2 using the direct stiffness method.

Problem 5.1 Solve *Problem 1.2* using the direct stiffness method.

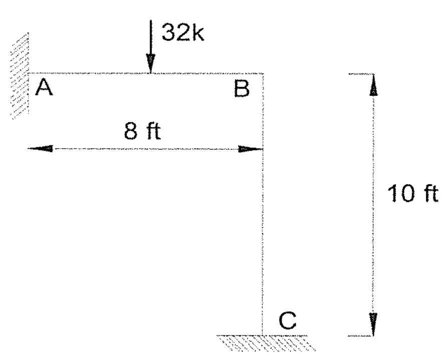
Problem 1.2 repeated Determine the reactions at the supports of the frame shown in this figure. EI constant.



Problem 5.1

Problem 5.2 Solve P*roblem 2.2* using the direct stiffness method.

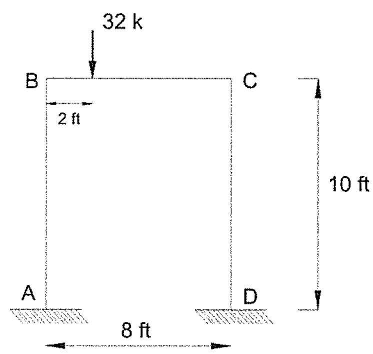
Problem 2.2 repeated Determine the reactions at the supports of the frame shown in this figure. A = 100 in^2, E = 29000 ksi and I = 833 in^4.



Problem 5.2

Problem 5.3 Solve *Problem 2.3* using the direct stiffness method.

Problem 2.3 repeated Determine the reactions at the supports of the frame shown in this figure. A = 100 in^2, E = 29000 ksi and I = 833 in^4



Problem 5.3

Chapter 6

Direct Stiffness Method: Application to Trusses

6.1 Derivation/Explanation of the Stiffness Matrix for a Truss Element

As stated earlier, the basic procedure of direct stiffness method has been explained in Chapter 4. The explanation is essentially a general procedure, which is also applicable to trusses. The only difference is that in the case of a truss, the element stiffness matrix $[K]$ for a truss element has to be used instead of that for a beam element. It should also be noted that the members of a truss are subjected to tension or compressive forces only as all the loads on the truss are nodal loads and not member loads. This means that the members of the truss are not subjected to any bending. While this doesn't make any difference in application of the direct stiffness method, it is an important point to be noted. This concept will be clearer once an example of a truss is solved using the direct stiffness method. Consider the general truss element shown in Fig. 6.1.

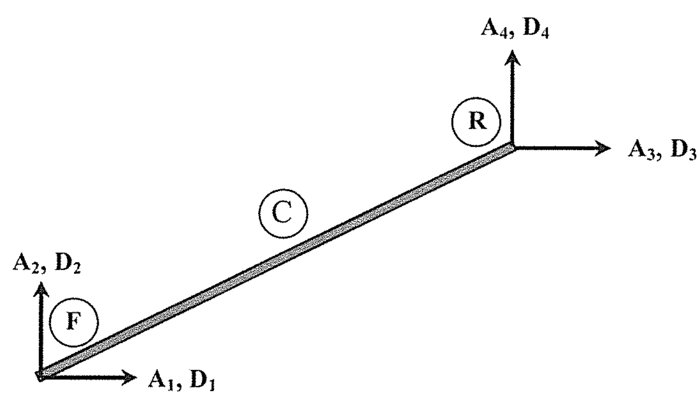


Figure 6.1: Typical truss element with free and restrained supports

The member stiffness matrix for the truss element in structural (global) coordinates, $[K_{MSi}]$ is given in Eq. 6.1 as,

Equation 6.1: Truss element stiffness matrix

$$[K_{MSi}] = \frac{AE}{L} \begin{array}{c} \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{bmatrix} C_x^{\,2} & C_x C_y & -C_x^{\,2} & -C_x C_y \\ C_x C_y & C_y^{\,2} & -C_x C_y & -C_y^{\,2} \\ -C_x^{\,2} & -C_x C_y & C_x^{\,2} & C_x C_y \\ -C_x C_y & -C_y^{\,2} & C_x C_y & C_y^{\,2} \end{bmatrix} \end{array} \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \tag{6.1}$$

Where, C_x and C_y are the direction cosines of the members given as (see Fig. 6.1 for reference),

$$C_x = \frac{(x_k - x_j)}{L} \tag{6.2}$$

$$C_y = \frac{(y_k - y_j)}{L} \tag{6.3}$$

EA is the usual axial rigidity of the truss member. The global stiffness matrix $[K_{MSi}]$ shown in Eq. 6.1 relates the action vector $\{A_1, A_2, A_3, A_4\}^t$ to the displacement vector $\{D_1, D_2, D_3, D_4\}^t$.

Again, like in the case of the beam, some of the elements of the displacement vector will be zero and will correspond to $\{D_R\}$, the restrained displacement vector, which is subset of the $\{D\}$ vector. Similarly, the unknown, or free, displacements will be part of the free displacement vector $\{D_F\}$, which is also a subset of the total displacement vector $\{D\}$.

6.2 Application of the Direct Stiffness Method to a Truss

The process used to solve for the reactions of an indeterminate truss using the direct stiffness method is explained below using Example 6.2.1.

Example 6.2.1

Determine the reactions at the support of the truss shown in Fig. 6.2 using direct stiffness method. AE is constant.

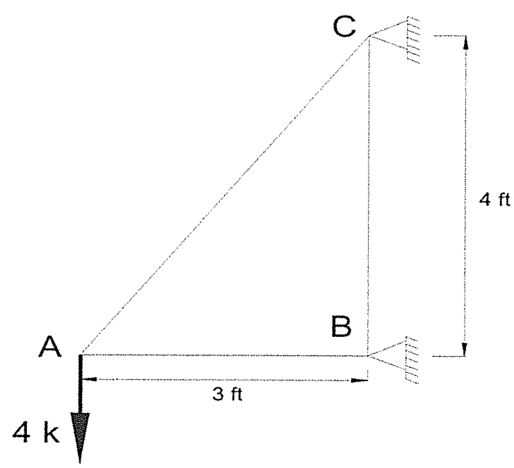


Figure 6.2: Indeterminate truss

Step 1-3

These steps are shown in Fig. 6.3. Figure 6.3 shows that the KI (Kinematic Indeterminacy) of this structure is 2. Thus, the $\{D_F\}$ vector is: $\{D_1, D_2\}$. In this problem, just as in the beam and frame problem, the numbering of the displacements is done starting with the free displacements. The restrained displacement vector for this problem $\{D_R\}$ is $\{D_3, D_4, D_5, D_6, D_7, D_8\}$. Only the free displacements are to be obtained from $[K_{FF}]$ and the $\{A_{FN}\}$ as a solution of the problem.

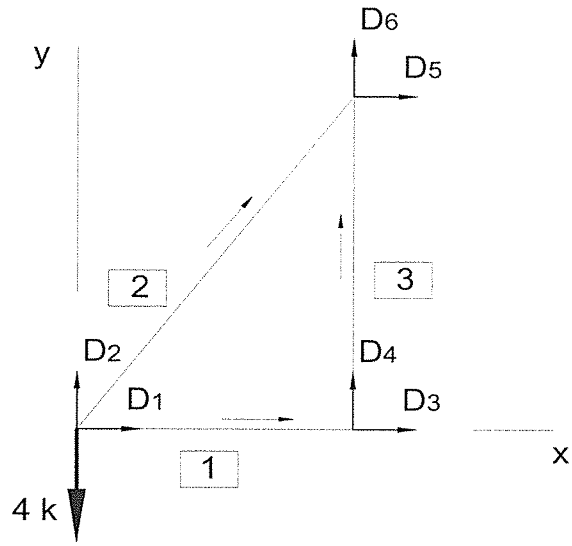


Figure 6.3: Truss showing displacements

Step 4

The member stiffness matrix $[K_1]$ for member 1 is obtained using Eq. 6.1. It is given as,

Table 6.1: Truss member stiffness matrix for member 1

$$[K_1] = AE \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.333 & 0 & -0.333 & 0 \\ 0 & 0 & 0 & 0 \\ -0.333 & 0 & 0.333 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Similarly, the stiffness matrix for member 2 $[K_2]$ is given as,

Table 6.2: Truss member stiffness matrix for member 2

$$[K_2] = AE \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \begin{bmatrix} 0.072 & 0.096 & -0.072 & -0.096 \\ 0.096 & 0.128 & -0.096 & -0.128 \\ -0.072 & -0.096 & 0.072 & 0.096 \\ -0.096 & -0.128 & 0.096 & 0.128 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array}$$

Finally, the stiffness matrix for member 3 $[K_3]$ is given as,

Table 6.3: Truss member stiffness matrix for member 3

$$[K_3] = AE \begin{array}{cccc|c} 3 & 4 & 5 & 6 & \\ \hline \end{array}$$

$$[K_3] = AE \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & -0.25 \\ 0 & 0 & 0 & 0 \\ 0 & -0.25 & 0 & 0.25 \end{array}\right] \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Combining the three member stiffness matrices yields the structure stiffness matrix given below in Table 6.4.

Table 6.4: Truss structure stiffness matrix

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ [K_J] = AE & \left[\begin{array}{cc|cccc} 0.405 & 0.096 & -0.333 & 0 & -0.072 & -0.096 \\ 0.096 & 0.128 & 0 & 0 & -0.096 & -0.128 \\ \hline -0.333 & 0 & 0.333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.25 & 0 & -0.25 \\ -0.072 & -0.096 & 0 & 0 & 0.072 & 0.096 \\ -0.096 & -0.128 & 0 & -0.25 & 0.096 & 0.378 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

Step 5

Since just the unknown displacements are to be obtained in this problem, we only need $[K_{FF}]$ and $\{A_{FN}\}$ corresponding to the free displacements.

From the given condition, we can calculate:

$$\{A_{FN}\} = \begin{Bmatrix} 0 \\ -4 \end{Bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

From $[K]$, we can derive $[K_{FF}]$:

$$[K_{FF}] = AE\begin{bmatrix} 0.405 & 0.096 \\ 0.096 & 0.128 \end{bmatrix}$$

We can solve the displacement:

$$\{D_F\} = [K_{FF}]^{-1} * \{A_{FN}\} = \frac{1}{AE}\begin{bmatrix} 9 \\ -38 \end{bmatrix}$$

Calculate the reaction forces:

$$\begin{bmatrix} A_3 \\ A_4 \\ A_5 \\ A_6 \end{bmatrix} = [K_{RF}]\{D_F\} = \begin{bmatrix} \boldsymbol{-3\ kips} \\ \boldsymbol{0} \\ \boldsymbol{3\ kips} \\ \boldsymbol{4\ kips} \end{bmatrix}$$

In this problem there is only one nodal load (at joint 4). So, the assembly of $\{A_{FN}\}$ is rather simple. However, if there are other loads at other nodes or if there are some support settlements, the corresponding $\{A_{FN}\}$ vector should be assembled accordingly.

This completes the solution of the problem.

6.3 Summary

In this chapter, the powerful method called the direct stiffness method has been applied to a truss problem. The point to be noted about the direct stiffness method is that it can be applied to structures that are mainly in flexure (bending) like beams and frames, as well as structures in tension or compression (like a trusses). The method essentially is the same except that the truss element stiffness matrix is used in analysis of trusses as opposed to a beam or frame element stiffness matrix.

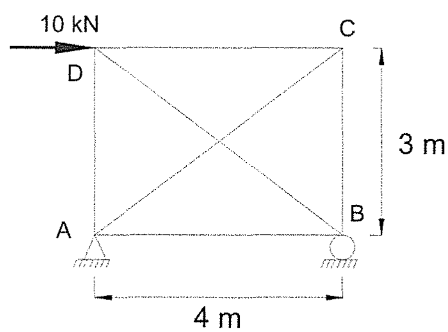
Hence, the direct stiffness method is so powerful and popular that it is used for almost all the computer codes prevalent in structural analysis.

Problems

Analyze Problem 1.3 in Chapter 1 using the direct stiffness method.

Problem 6.1 Solve *Problem 1.3* using the direct stiffness method.

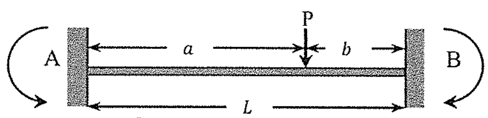
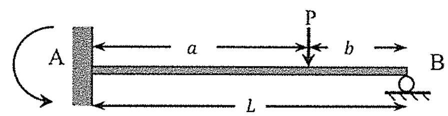
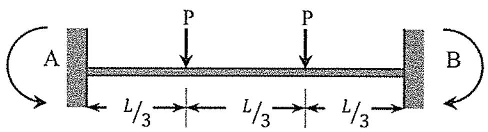
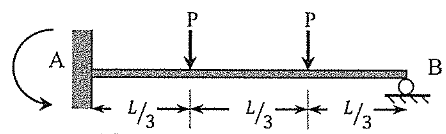
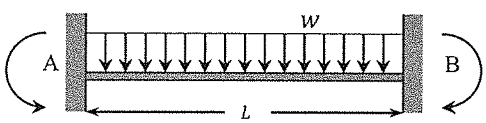
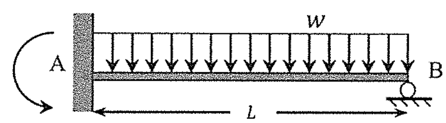
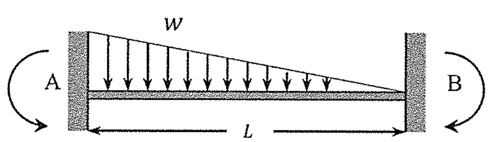
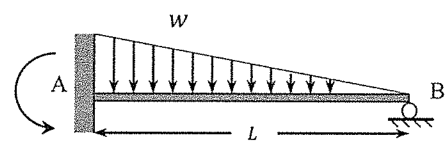
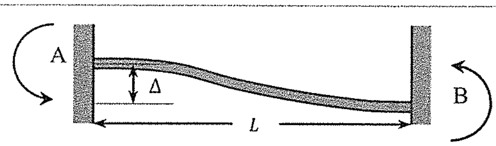
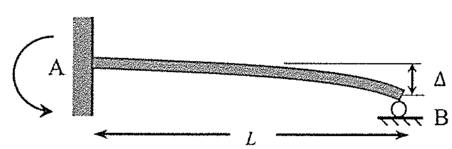
Problem 1.3 repeated Determine the reactions at the supports of the truss shown in this figure. AE is constant.



Problem 6.1

Appendix – A

Table 1A: Fixed-End Moments

Both Ends Fixed	Far End Pinned
 $M_{FAB} = \frac{Pab^2}{L^2}$ $M_{FBA} = \frac{Pa^2b}{L^2}$	 $M_{FAB} = \left(\frac{P}{L^2}\right)\left(b^2a + \frac{a^2b}{2}\right)$
 $M_{FAB} = \frac{2PL}{9}$ $M_{FBA} = \frac{2PL}{9}$	 $M_{FAB} = \frac{PL}{3}$
 $M_{FAB} = \frac{wL^2}{12}$ $M_{FBA} = \frac{wL^2}{12}$	 $M_{FAB} = \frac{wL^2}{8}$
 $M_{FAB} = \frac{wL^2}{20}$ $M_{FBA} = \frac{wL^2}{30}$	 $M_{FAB} = \frac{wL^2}{15}$
 $M_{FAB} = \frac{6EI\Delta}{L^2}$ $M_{FBA} = \frac{6EI\Delta}{L^2}$	 $M_{FAB} = \frac{3EI\Delta}{L^2}$

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