

Basic Theory of Structures

by

J. S. C. BROWNE B. SC. (ENG.)

Dundee College of Technology

Lecturer in Civil Engineering

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Preface

THERE are many excellent textbooks on the subject of 'Theory of Structures', but their very comprehensiveness often bewilders the newcomer to the subject. It is the purpose of this book to present to the student in an easily digested form a graded course intended to extend over a period of one year covering a sufficiently wide area of the subject to stimulate his interest and at the same time provide him with a sound foundation of structural theory upon which to build in subsequent years. For this reason, considerable space has been devoted to explaining basic concepts of structural behaviour, since once they are firmly grasped economy of thought is achieved and much needless error avoided.

The book has been written with the needs of the part-time student following a National Certificate course in civil engineering particularly in mind, although full-time Diploma and Degree students should equally find it of value.

The questions at the ends of the chapters have been carefully chosen and graded to test the thoroughness of the student's understanding of the fundamental principles concerned and to provide typical examination questions. The student is strongly recommended to attempt as many as possible. Answers to the numerical questions are provided.

Although there is no possibility of confusing units of mass with units of force in this textbook, since we are here only concerned with forces, I have, in the interests of standardisation adopted the recommendation of British Standard Specification 1991 that an "f" should be added to all units of *force* to

distinguish them from units of *mass*. This means loads are in *lbf* or *tonf* and stresses in *tonf/in²*, etc.

Grateful acknowledgement is due to the Institution of Civil Engineers for permission to reproduce those questions which are marked I.C.E., or in the case of the more recent questions which are selected from the Engineering Institutions Examination Part I, marked Eng. Inst.

J. S. C. BROWNE

CHAPTER 1

Equilibrium of Structures

AN ESSENTIAL requirement of any structure is that it must be capable of resisting a variety of loadings without changing its shape, other than by the tiny deformations due to straining of the material of which it is made. A mechanism on the other hand, which is also an assemblage of rigid parts designed to transmit forces, is free to alter its shape by sliding or rotation between the individual parts.

Structures are subjected to forces external to themselves, such as weights placed on them, the deadweight of the structure itself, wind or water pressure, and reactions exerted by the ground on which the structure rests; to name but a few of those most commonly met in civil engineering. If the structure is at rest, or to be more precise, not being accelerated, which is usually the case with structures attached to the ground, discounting vibrational and earthquake movements, the loads and reactions form a system of forces in equilibrium satisfying the laws of statics.

Before an engineer can design a structure he must be able to determine all the forces acting on it at any one time. These forces can be grouped under two headings, (i) loads, (ii) reactions at points of support, as shown diagrammatically in Fig. 1.1. As the loads and reactions together form a system of forces in equilibrium, the resultant of the loads will be equal and opposite to, and act along the same line of action as the resultant of the support reactions.

Loads may be subdivided into (a) dead loads, due to the self weight of the structure, and which are always present, (b) imposed loads, which embrace all loads not included in (a), for example; weight of machinery, stores, vehicles, cranes, and people, as well as loading due to wind or water pressure. Unlike dead loads, imposed loads may, or may not, be present.

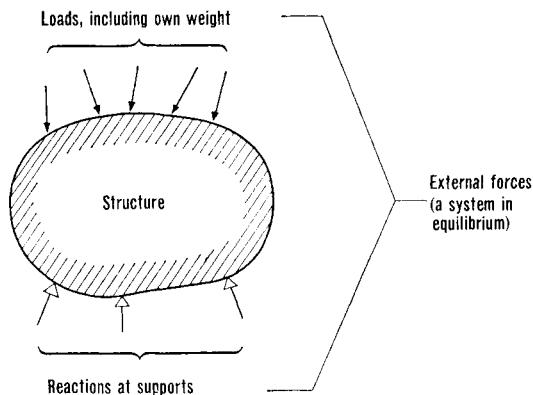


FIG. 1.1 Generalized diagram of the forces acting on a structure

Reactions at supports are just as much forces acting on the structure helping to maintain it in equilibrium as are the loads. They differ from the latter only by virtue of the fact that loads are those forces applied to the structure by active agents, such as gravity, wind or mechanical means; whereas reactions are those forces applied to the structure due to the resistance of a supporting medium, such as the ground. Think, for example, of a piece of cast iron resting on the floor. The lump of cast iron is in equilibrium under the action of two equal and opposite *forces*. One is the *load*, which is the weight of the piece of cast iron; that is the mutual gravitational attraction existing between it and the earth; and the other force is the *reaction* provided by the floor. The load is under direct con-

trol in the sense that a larger or a smaller piece of iron can be substituted, whereas the reaction of the floor adjusts itself to variations in the load, so that equilibrium is maintained. This automatic adjustment of the reaction to the applied load is similar to the way in which a spring stretches more when subjected to a larger pull and so exerts a reaction which is bigger by just the same amount.

Loading on Structures

It is not intended to deal here with the determination of the loading on various types of structures, which belongs more properly to the subject of design of structures, but to draw the student's attention to the fact that even arriving at the *design loading* involves considering quite a large number of factors.

Structures are designed to withstand, without developing excessive stresses or deflections, loadings specified by Building Regulations and Codes of Practice or by the person or Authority for whom the structure is being erected. The loading may be due to moving vehicles, in the case of a bridge, or to people and furniture in the case of a block of flats, or to water pressure in the case of a dam or water tank; to name but a few examples. In each case the engineer will have to estimate as accurately as he can the most severe form of loading to which the structure may be subjected. His task is simplified by reference to *Chapter V—Loading* of the British Standard Code of Practice *CP3* on the functional requirements of buildings.

This chapter gives the loads which it is recommended should be taken into account in the design of buildings and other structures. It specifies the minimum imposed loads for floors according to the use to which the building is put. Whether, for example the floor is part of an office block and relatively lightly loaded, or a heavily loaded warehouse floor.

Chapter V also enables wind loads on buildings and chimney shafts to be calculated, by providing a table of basic wind pressures, and means of determining how the wind pressure is modified by the shape of the building.

The *British Standard Specification B. S. 153* deals in a similar way with the design of *steel girder bridges*. It lists the forces which shall be taken into account when calculating the stresses developed and gives standard loadings for both highway and railway bridges.

British Standard Specifications and Codes of Practice are not, of course, part of the law of the land, but are drawn up by the *British Standards Institution* to simplify design and construction, eliminate waste of time and material, and yet ensure a safe structure. It only becomes obligatory to comply with them where a contract for the construction of a structure insists on it. What the designer must ensure is that his structure satisfies the local building Bye-laws and relevant Ministry Regulations.

In addition to imposed loads a structure must also carry its own weight. The dead load of a structure can only be roughly estimated in the first place, because the required size of the members is not known until the final design is arrived at. Because considerable judgement is needed to estimate the dead load of a structure in the early stages of design, there is a tendency to by-pass dead load calculations in questions of an academic nature, but this should not blind students to the importance of taking dead loads into account in any real design.

Returning to the consideration of a structure as a whole, in equilibrium under the action of a system of external forces, it will be seen that certain conditions must be fulfilled by the system of forces.

Conditions of Equilibrium

If a structure is in equilibrium then there can be no resultant force or couple acting on it. The resultant force will only vanish if the vector sum of all the forces acting on the structure is zero. In other words, if the forces are represented by lines drawn parallel to their lines of action, whose lengths are proportional to the forces, and joined together in sequence so that they follow one another, nose to tail, they must return to their starting point, forming a closed figure. This will be true even if the forces are not all in one plane, although in that case a *space polygon* would be formed which could be difficult to visualise, without a three dimensional model in stiff wire. Note that for coplanar forces the *polygon of forces* rule is satisfied.

In addition to the force vectors having to form a closed figure, the algebraic sum of the moments of the forces about every axis must be zero. This will be so if the algebraic sum of the moments of the forces about three mutually perpendicular axes vanish.

It is often more convenient to deal with the components of forces rather than the forces themselves. In fact it becomes essential to resolve forces along chosen directions when a structural problem is being solved analytically, that is, by calculation.

Consider, for example, a rigid body under the action of n forces $F_1, F_2, F_3, \dots, F_n$. Let F be any one of the forces and F_x, F_y and F_z its components parallel to a set of three coordinate axes x, y and z at right angles to each other. Also let M_x, M_y and M_z be the moments of F about the three axes. The conditions that the body shall be in equilibrium are: there shall be no resultant force in any direction and no resultant couple about any axis. Expressed in terms of components, these become: *the algebraic sum of the components of all the forces parallel to the x axis is zero*, and *the algebraic sum of the moments of all the forces about the x axis is zero*; and similarly for the y

and z axes. This can be more neatly expressed in the *general equations of equilibrium*.

$$\Sigma F_x = 0,$$

$$\Sigma F_y = 0,$$

$$\Sigma F_z = 0.$$

$$\Sigma M_x = 0,$$

$$\Sigma M_y = 0,$$

$$\Sigma M_z = 0,$$

where the Greek letter sigma means the algebraic sum of all the terms like the one which follows Σ .

These six equations cover the general case of any structure in equilibrium under the action of forces in three dimensions. For the simpler case of *coplanar forces*, the six equations reduce to the following three

$$\Sigma F_x = 0,$$

$$\Sigma F_y = 0,$$

$$\Sigma M = 0,$$

where the x and y axes are chosen in the plane of the forces. Note that one of the general force equations, $\Sigma F_z = 0$ and two of the general moment equations, $\Sigma M_x = 0$ and $\Sigma M_y = 0$ are no longer required, and that it is the equation $\Sigma M_z = 0$ which is preserved, although it is no longer necessary to continue using the suffix z . However, it is worth pointing out that equation $\Sigma M = 0$, which is a mathematical shorthand for the *principle of moments*, means the algebraic sum of the moments of all the forces about any arbitrarily chosen *axis* at right angles to the plane of the forces is zero; although to save words, we refer to moments of forces about a *point*, because the *trace* of the axis on the plane is in fact a point.

Three Forces in Equilibrium

When a structure is in equilibrium under the action of only three forces, it can be shown by an application of the general equations of equilibrium that they must be coplanar. Further, the forces must either all be *parallel* or *concurrent*. If parallel, the algebraic sum of the forces must be zero, and the principle of moments must be satisfied; if concurrent, the *free vectors* representing the forces must form a *triangle of forces*. The term *free vector* means that the vector is not confined to the line of action of the force it represents; although it must, of course, remain parallel to that line. Often the adjective *free* is omitted without ambiguity. The triangle of forces is merely a special case of the polygon of forces.

The Link Polygon

The conditions of equilibrium for a set of coplanar forces can be expressed thus: (i) the force vectors must form a closed figure, (ii) the principle of moments must be satisfied. Condition (i) is the polygon of forces rule, and is a graphical construction. Condition (ii) can be replaced by the graphical condition that it must be possible to construct a *link polygon* joining the forces. That is, it must be possible to draw a closed figure on the *space diagram*, with its vertices on the lines of action of the forces and its sides parallel to rays radiating from a pole to corresponding vertices in the *force diagram*. The physical significance of the link or *funicular polygon* is that it could, if made of cord, replace the rigid body on which the forces act, without disturbing their equilibrium. There are an infinite number of different link polygons for every set of balanced forces, since the pole can be anywhere on the force diagram, and the link polygon can be made to pass through any point on the space diagram, but provided one can be drawn, condition (ii) will

be satisfied. For further details on methods of constructing link polygons, students are referred to textbooks of Applied Mechanics.

Equilibrant and Resultant of a System of Forces

By definition, the *equilibrant* of a system of forces is the single force which will just balance them. The *resultant* of the system is the single force which will replace the forces and produce the same effect. It is evident then, that the equilibrant and resultant of a system of forces are equal and opposite and lie along the same line of action, since they will hold each other in equilibrium.

It will be seen that, for a system of forces in equilibrium, any one force of the system is the equilibrant of the remainder. If the equilibrant is reversed in direction, the resultant of the system is obtained. Thus the *polygon of forces* may be used to determine the resultant of an unbalanced system of forces.

Structures supported in a statically determinate manner

Civil engineering structures must possess stability as well as rigidity, so that they can resist a variety of loadings without overturning. For example, the *built-in* support A , of the projecting beam, or cantilever, in Fig. 1.2a must provide (i) an upward vertical reaction V , equal to the total downward component of the loading, (ii) a horizontal reaction H acting to the right, equal to the horizontal component of the loading acting to the left, (iii) an anticlockwise couple M , equal to the total clockwise moment produced by the loading about A . The loading and constraints acting on the cantilever are shown in the *free body diagram* Fig. 1.2b.

Take as another example, the roof truss shown in Fig. 1.2c. It is pinned to the ground at A and supported on rollers at B .

The hinge must provide a horizontal reaction H , equal to the total horizontal component of the loading, since the roller support cannot offer horizontal restraint. The vertical reactions V_A and V_B at A and B must be such that their sum equals the total vertical component of the loading, and the moment of either vertical reaction about the opposite support equals the total moment due to the loading about the same point. The *free body diagram* Fig. 1.2d helps to make this clear.

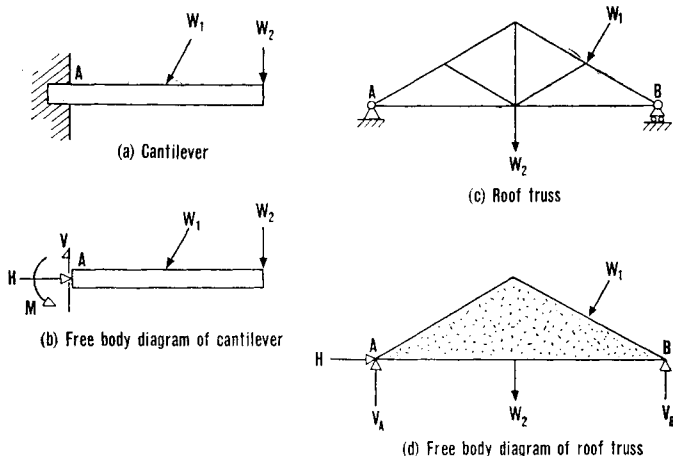


FIG. 1.2 Structures supported in a statically determinate manner

For both the cantilever and the roof truss, and indeed for any plane structure capable of maintaining its own *shape* (without the help of supports), only three reactions are necessary and sufficient to ensure equilibrium of the loaded structure. It is no coincidence that the number of independent equations of equilibrium which have to be satisfied by such a structure is also three. Hence, there are just sufficient equations to enable the three unknown reactions at the supports of a loaded structure to be determined. If less than three reactions are provided by the supports the structure will be unstable under all

but a few special cases of loading. If more than three reactions are provided it is impossible to determine them all by statics alone, since there will be more unknown reactions than there are independent equations for solving them. Such a structure is called *statically indeterminate* or *hyperstatic*. It may also be referred to as a redundant structure, since it possesses redun-

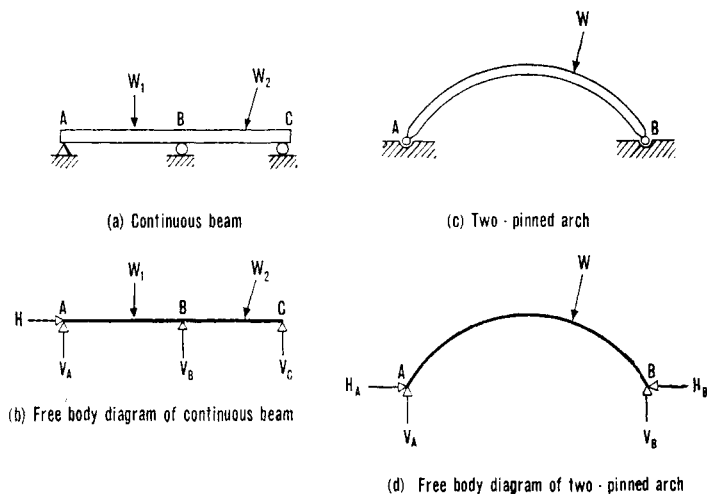


FIG. 1.3 Hyperstatically supported structures

dant supports additional to those just sufficient to ensure stability. An example of a hyperstatic structure is the continuous beam in Fig. 1.3a supported on a knife edge or pin at A , and rollers at B and C . The four reactions are shown in the free body diagram Fig. 1.3b. Another example is the two-pinned arch in Fig. 1.3c, pinned to the ground at A and B . The free body diagram, Fig. 1.3d, shows the four reactions; a horizontal and a vertical reaction at each pin. For neither of these statically indeterminate structures can the reactions be deter-

mined by pure statics. It is necessary to know how the structures deform under load, before the manner in which the loads are shared between the various supports can be determined.

Transmissibility of Force

In order to completely specify a force it is necessary to give its line of action, its magnitude and sense, and its point of application. Where we are dealing with an ideal rigid body (one that does not alter its dimensions under load) there is no need to specify the point of application of a force, since the behaviour of the body will not be affected, provided the line of action remains the same. However, in the case of real bodies or structures, which can never be perfectly rigid, the point of application of the force will affect the way in which it is diffused through the body to the supports, but if the body is supported in a statically determinate manner the reactions will be unaffected by transferring the point of application of a load to another point on the same line of action.

Principle of Superposition

The principle of superposition is of wide application, but is subjected to a number of limitations, and is consequently difficult to formulate in a few words. Virtually, it states that when you are dealing with a structure for which the deflections at every point are proportional to the loads causing them, then the effect produced by a combination of forces is equal to the sum of the effects produced by each force acting separately; the word *effect*, in this context, meaning stress, strain, deflection, internal force, shearing force, or bending moment.

Determination of Support Reactions

Simple beam with vertical loading

If a beam rests on two supports A and B , and carries loading, as in Fig. 1.4, it is a simple matter to determine the reac-

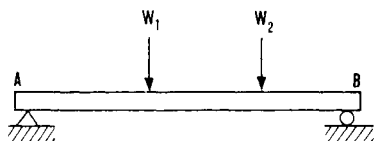


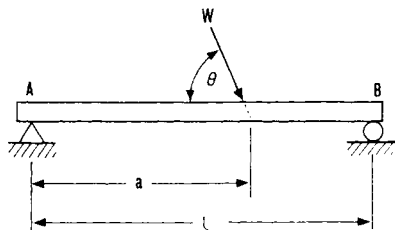
FIG. 1.4 Simple beam with vertical loading

tions at A and B . Because the loading in this example has no horizontal component, the reactions at A and B will be vertical. As the direction of the reaction at B is known, its magnitude can be calculated by imagining the beam to be pivoted at the other support; equating the anticlockwise moment about A , due to the reaction at B , to the clockwise moment about A , due to the external loads. Similarly, the reaction at A can be calculated by again applying the principle of moments, this time taking moments about the support B . Alternatively, the reaction at A can be obtained by equating the sum of the two vertical reactions at A and B to the total load. This is equivalent to stating that since the beam is in equilibrium under the action of forces which are all parallel, the algebraic sum of these forces is zero.

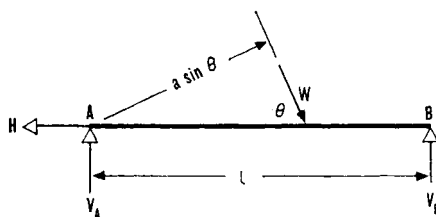
Simple beam with oblique loading

Consider next the case where the beam carries an inclined load, as in Fig. 1.5a. Again, we have a system of coplanar forces in equilibrium, so:

- (i) the algebraic sum of the horizontal components of all the forces must vanish,
- (ii) the algebraic sum of the vertical components of all the forces must vanish,
- (iii) the algebraic sum of the moments of all the forces about any arbitrarily chosen point must vanish.



(a) Loading.



(b) Free body diagram

FIG. 1.5 Simple beam with oblique loading

End B rests on a roller, which is assumed frictionless and therefore incapable of developing a horizontal force. The reaction at B must therefore be vertical. The support at A , on the other hand, must provide horizontal restraint, and this is indicated by showing a knife edge at A in the diagram. Note that if both supports were knife edges it would be impossible to determine the horizontal reactions at either A or B purely by

statics, since the beam would have an additional redundant restraint, and would become statically indeterminate.

The next step is to draw the *free body diagram*, Fig. 1.5b, which depicts the beam isolated from its surroundings, and showing all the forces acting on it and maintaining it in equilibrium. The roller at B has been replaced by the vertical reaction V_B and the knife edge at A has been replaced by a pair of forces, V_A the vertical reaction, and H the horizontal reaction. There are three unknown quantities to be determined V_A , V_B and H , but as there are three conditions of equilibrium it is possible to write down three independent equations of equilibrium involving these three unknowns, and hence to solve them as a set of three simultaneous equations in three unknowns.

The choice of axes or directions along which the forces are resolved is entirely arbitrary. Any two different directions in the plane of the forces can be chosen, not necessarily at right angles, and moments can be taken about any point in this plane; but to save unnecessary algebra in solving the equations, the axes should be chosen so as to reduce to a minimum the number of unknowns occurring in each equation. In this problem, by resolving horizontally, the two unknowns V_A and V_B are eliminated, and we are left with

$$W \cos \theta - H = 0$$

By resolving vertically H is eliminated from the equation and we get

$$V_A + V_B - W \sin \theta = 0$$

and by taking moments about A , the intersection of the lines of action of V_A and H these two forces are eliminated from the equation, resulting in

$$Wa \sin \theta - V_B l = 0$$

In a particular case, the load W and the geometry, a , l and θ would be given, and hence the reactions V_A , V_B and H could all be calculated. Additional loads will not increase the number of unknown reactions, so that the same basic procedure can be used when the beam is subjected to two or more loads.

Example 1.1 AC is a uniform beam 10 ft long, weighing

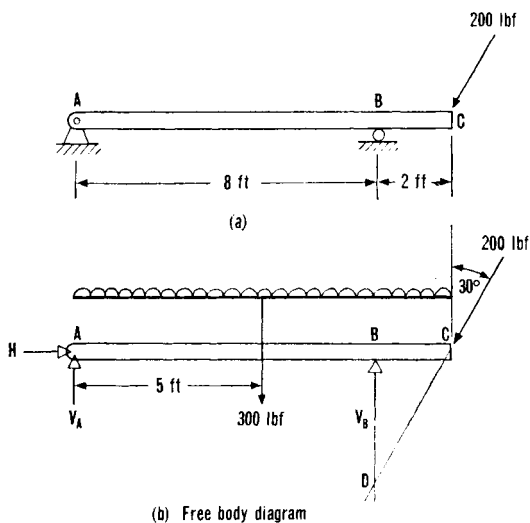


FIG. 1.6

300 lbf, hinged to the ground at A , and resting on a roller at B . It carries a load of 200 lbf, inclined at 30° to the vertical, at C , as shown in Fig. 1.6a. Determine the reactions at A and B .

Referring to the free body diagram of the beam, Fig. 1.6b, we notice that (as far as the reactions are concerned) the uniformly distributed load, due to the weight of the beam, can be replaced by a concentrated load, equal to the weight of the beam, at its centre of gravity.

Resolving horizontally:

V_A and V_B have no horizontal components, and so do not appear in the equation of equilibrium for horizontal forces, hence

$$H = 200 \sin 30^\circ$$

$$H = 100 \text{ lbf}$$

Taking moments about A:

H and V_A have no moment about A , and so do not appear in the equation of equilibrium for moments about A , hence

$$8V_B = 5 \times 300 + 10 \times 200 \cos 30^\circ$$

or $V_B = 404 \text{ lbf}$

Resolving vertically:

H has no vertical component, and so does not appear in the equation of equilibrium for vertical forces, hence

$$\begin{aligned} V_A + V_B &= 300 + 200 \cos 30^\circ \\ &= 473 \end{aligned}$$

$$V_A = 69 \text{ lbf}$$

Note that the problem can also be solved by taking moments about three points not in the same line. In other words, it is possible to replace the two force equations by an extra pair of moment equations.

Taking moments about A:

gives $V_B = 404 \text{ lbf}$, as before.

Taking moments about B:

H and V_B are eliminated.

$$8V_A = 3 \times 300 - 2 \times 200 \cos 30^\circ$$

$$= 554$$

$$V_A = 69 \text{ lbf}$$

Taking moments about D:

D is the intersection of the lines of action of V_B and the 200 lbf force. Theoretically any point would do which was not on a line with the two previous points used, but it is important to choose a point which keeps the algebra and trigonometry to a minimum.

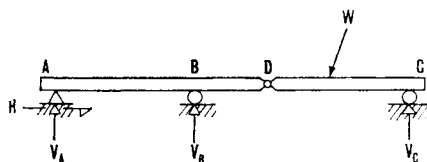
$$2H \cot 30^\circ + 8V_A = 3 \times 300$$

so $3.46H = 900 - 554$

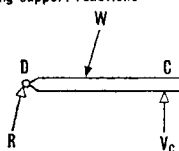
and $H = 100 \text{ lbf}$

Compound Beams

A compound beam is one which is composed of two or more sections pinned together and supported in a statically determinate manner. Figure 1.7a shows a compound beam consist-



(a) Compound beam, showing support reactions



(b) Free body diagram of part DC

FIG. 1.7 Compound beam on three supports

ing of two parts pinned together at D and carried on three supports, one of which is a knife edge and capable of exerting a horizontal reaction. The reactions can all be determined by statics (unlike the continuous beam in Fig. 1.3a). We can write down three equations of equilibrium for the complete compound beam which express the facts that:

- (i) the horizontal reaction H , at A , is equal to the horizontal component of the load,
- (ii) the sum of the vertical reactions at A , B and C is equal to the vertical component of the load,
- (iii) the sum of the moments about A , due to the reactions, is equal to the moment of the load about A .

The above equations are insufficient to enable all four reactions to be calculated. However, as there is a pin at D , it is possible to obtain a fourth equation of equilibrium by considering the equilibrium of just one part of the compound beam; for example, the portion DC , for which Fig. 1.7b is the free body diagram. R is the resultant reaction of the portion AD on DC , and is of course another unknown force, but we can avoid introducing R into the equation of equilibrium by taking moments about the pin D . Hence V_c can be obtained directly.

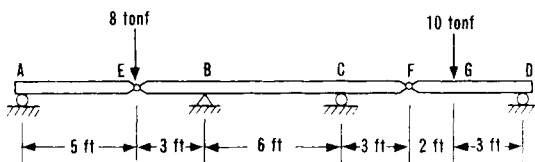


FIG. 1.8 Compound beam on four supports

Example 1.2. Figure 1.8 shows a compound beam. Calculate the reactions at all four supports due to the loads at E and G .

There are four unknown vertical reactions, one at each support, but there is no horizontal reaction at the knife edge

support at B , as the loading is entirely vertical. The latter fact disposes of the equation of equilibrium relating to horizontal forces, leaving only two equations of equilibrium for the beam, as a whole, to satisfy. However, there are two additional facts about the beam, which can be used to provide another two equations of equilibrium, and hence enable the problem to be solved; they are, that there is a pin joint at E and another at F .

Moments about E , for the portion AE

It is immediately obvious that there is no vertical reaction at A .

$$V_A = 0$$

Moments about F for the portion FD :

$$5V_D = 2 \times 10$$

$$V_D = 4 \text{ tonf}$$

Moments about B for the whole beam:

$$6V_G = 11 \times 10 - 3 \times 8 - 14V_D + 8V_A$$

$$V_G = 5 \text{ tonf}$$

Resolve vertically for the whole beam:

$$V_A + V_B + V_G + V_D = 8 + 10$$

$$V_B = 9 \text{ tonf}$$

Three-pinned Arches and Portals

The familiar semicircular masonry arch was an important feature in Roman architecture for spanning openings. It enables heavy loads to be transmitted out sideways to the arch

supports, or to the abutments in the case of a bridge, as essentially compressive forces; so avoiding tensile stresses developing, which masonry is incapable of sustaining. But the masonry arch is a statically indeterminate structure, as will now be shown.

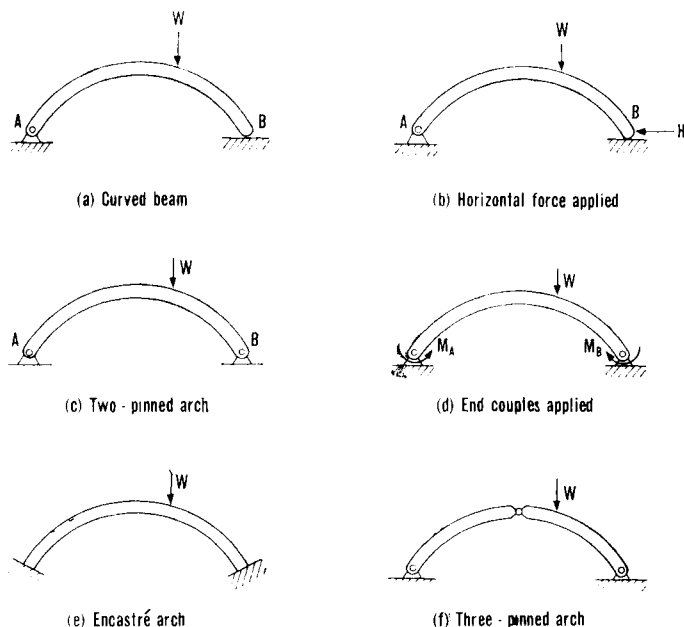


FIG. 1.9 Types of arches

Figure 1.9a represents a curved beam, hinged to the ground at A , and resting on smooth ground at B . When the beam is loaded the end B will splay outwards slightly as the beam deflects. However, the sliding end B can be brought back to its original position by applying a horizontal restraint or reaction H at B , as in Fig. 1.9b. This will automatically develop an equal horizontal reaction H at A . The curved beam with the correct horizontal thrust to prevent the ends from moving out-

wards under load is then equivalent to the two-pinned arch of Fig. 1.9c. In this state the ends A and B will rotate slightly when the load is applied. If now, two further restraints, M_A and M_B , are imposed, as in Fig. 1.9d, such that the ends of the beam are twisted back to their original, unloaded position, we have, in effect, an *encastré* or *built-in* arch, like the masonry arch in Fig. 1.9e. In other words, the *encastré* arch has three redundant constraints H , M_A and M_B , additional to the constraints required to just maintain equilibrium.

The two-pinned arch has one redundant constraint, the horizontal thrust H , and is also a hyperstatic structure. Like the continuous beam on three supports, in Fig. 1.3a, which can be converted into a statically determinate compound beam by the addition of a pin, so the two-pinned arch can be converted into a statically determinate arch by the addition of a third pin, which is usually placed at the crown, as in Fig. 1.9f.

Remember that the curved beam which we started with (Fig. 1.9a) is not an arch, as it is not subjected to a horizontal thrust at each end.

Unsymmetrical three-pinned arch with oblique loading

Most arches are symmetrical and carry vertical loads only, but the general case of an unsymmetrical arch with oblique loading provides a more complete illustration of the application of the laws of statics to three-pinned arches.

Consider the arch in Fig. 1.10a. It helps to focus attention on the relevant statics of the problem if we draw the free body diagram for the arch, Fig. 1.10b, as this shows *all* the forces acting on the arch, holding it in equilibrium. A study of the free body diagram then enables us to write down the equations of equilibrium from which the support reactions are determined. W is the external load, and H_A , H_B , V_A and V_B are the horizontal and vertical reactions at A and B . Thus there are

four unknown reactions, requiring four equations of equilibrium to enable them to be determined. By considering the equilibrium of the arch as a whole we can obtain three equations of equilibrium. A fourth equation can be written down by considering the equilibrium of one of the two parts into

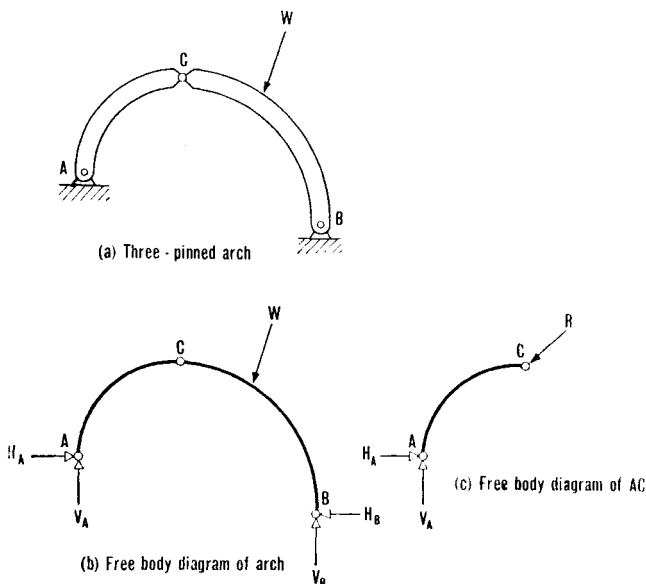


FIG. 1.10 Unsymmetrical three-pinned arch

which the arch is divided by the pin at the crown. For instance, consider the equilibrium of the portion AC, for which the free body diagram is Fig. 1.10c. R is the reaction exerted by the removed portion BC. Neither its magnitude nor direction need concern us if moments are taken about the crown pin C, since C is on the line of action of R .

Equation 1. Resolving vertically for the whole arch:

$$V_A + V_B = \text{vertical component of the external load.}$$

Equation 2. Resolving horizontally for the whole arch:

$H_A - H_B$ = horizontal component of the external load acting to the left.

Equation 3. Taking moments about A for the whole arch:

The sum of the anticlockwise moments due to the reactions at B, about A, equals the clockwise moment due to the loading, about A.

Equation 4. Taking moments about C for the portion AC:

The anticlockwise moment due to H_A equals the clockwise moment due to V_A , since there is no external loading on AC.

Given a particular case, in which the geometry of the arch and the loading on it is known, these four equations would enable the four reactions to be calculated.

Example 1.3. Figure 1.11a shows a three-pinned arch carrying a uniformly distributed load of 1 tonf/ft of horizontal projection over the portion AC, and an inclined load of 3 tonf on the portion CB. Calculate the reactions at A and B.

Examine the free body diagram of the arch, Fig. 1.11b.

Resolve vertically for the whole arch:

$$V_A + V_B = 4 + 3 \sin 60^\circ = 6.60 \quad (1)$$

Resolve horizontally for the whole arch:

$$H_A - H_B = 3 \cos 60^\circ = 1.50 \quad (2)$$

Take moments about B for the whole arch:

$$3H_A + 13V_A = 4 \times 11 + 3 \times 9 \sin 60^\circ = 67.4 \quad (3)$$

Take moments about C for the portion AC:

$$6H_A - 4V_A = -4 \times 2 = -8$$

or

$$3H_A - 2V_A = -4 \quad (4)$$

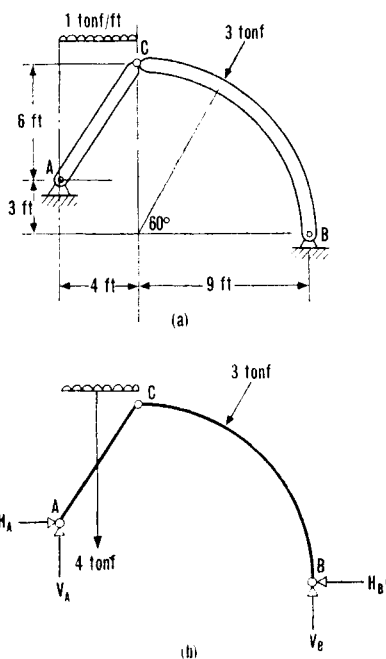


FIG. 1.11

From (3) and (4)

$$15V_A = 71.4$$

$$V_A = 4.75 \text{ tonf}$$

Hence

$$3H_A = 2 \times 4.75 - 4 = 5.50$$

$$H_A = 1.83 \text{ tonf}$$

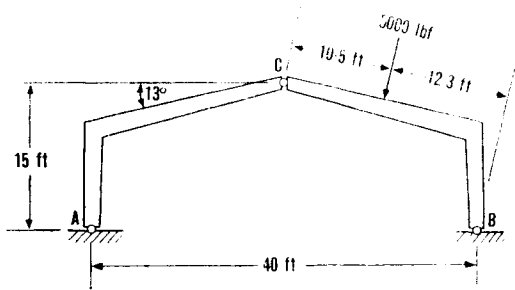
$$H_B = 1.83 - 1.50 = 0.33 \text{ tonf}$$

$$V_B = 6.60 - 4.75 = 1.85 \text{ tonf}$$

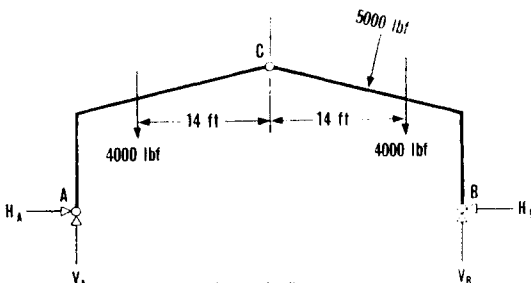
Three-pinned portal frame

It will be noted from the previous example that the reactions at the supports A and B depend only on the relative position of the three pins A , B and C , and the loading, and that the shape of the arch between A and C , and between C and B does not affect the value of these reactions. A portal frame differs from an arch, in that arches consist of curved members whilst portal frames consist of rectangular or cranked members.

Example 1.4 A factory building consists of symmetrical three-pinned portal frames, as shown in Fig. 1.12a, spaced at 10ft



(a) Portal frame and wind load



(b) Free body diagram

FIG. 1.12 Three-pinned portal frame

intervals. The dead load of each half-portal and its associated roof is 4000 lbf and acts at a distance of 14 ft from the centre line of the portal frames. The resultant of the wind pressure on each portal is a force of 5000 lbf acting normal to the roof, i.e. inclined at 13° to the vertical, and applied at the position shown in Fig. 1.12a. Calculate the reactions at the support pins.

Resolve vertically for whole portal:

$$\begin{aligned} V_A + V_B &= 4000 + 4000 + 5000 \cos 13^\circ \\ &= 12,880 \text{ lbf} \end{aligned}$$

Resolve horizontally for whole portal:

$$H_A - H_B = 5000 \sin 13^\circ = 1130 \text{ lbf}$$

Take moments about B for whole arch:

$$\begin{aligned} 40V_A &= 4000 \times 34 + 4000 \times 6 + 5000 \times 12.3 \\ V_A &= \mathbf{5530 \text{ lbf}} \end{aligned}$$

Take moments about C for portion AC:

$$\begin{aligned} V_A \times 20 - 15H_A &= 4000 \times 14 = 56,000 \\ H_A &= \mathbf{3640 \text{ lbf}} \end{aligned}$$

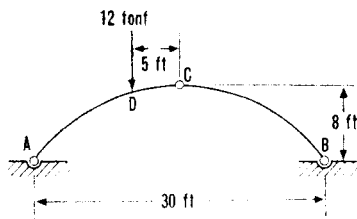
$$\text{and} \quad H_B = H_A - 1130 = \mathbf{2510 \text{ lbf}}$$

$$\text{and} \quad V_B = 12880 - V_A = \mathbf{7350 \text{ lbf}}$$

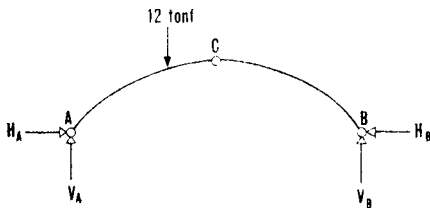
Example 1.5 The symmetrical three-pinned arch in Fig. 1.13a carries a single vertical load of 12 tonf at *D*. Determine the reactions at the supports.

Analytical solution:

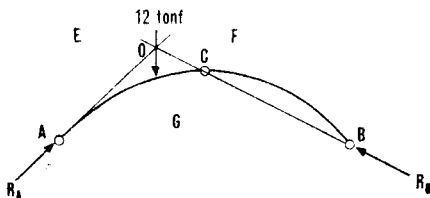
Free body diagram Fig. 1.13b. By considering the equilibrium of the whole arch and taking moments about *B* we get $V_A = 8$ tonf, and similarly, by taking moments about *A* we



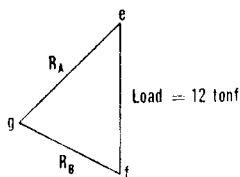
(a) Space diagram



(b) Free body diagram for analytical solution



(c) Free body diagram for graphical solution



(d) Triangle of forces

FIG. 1.13 Symmetrical three-pinned arch

find $V_B = 4$ tonf. Resolving vertically we can check that $V_A + V_B = 12$ tonf. Resolving horizontally we get $H_A = H_B$. Considering the equilibrium of just the right hand half of the arch, and taking moments about C we get $8H_B = 15V_B$, or $H_B = 7.50$ tonf $= H_A$. The total or resultant reaction at A

$$R_A = \sqrt{(H_A^2 + V_A^2)} = 10.98 \text{ tonf}$$

and similarly

$$R_B = 8.50 \text{ tonf}$$

Graphical solution:

Free body diagram Fig. 1.13c. We notice that the arch is subjected to just three forces; the 12 tonf load, the total reaction at A , and the total reaction at B , hence they must be concurrent. The resultant thrust in the right hand half of the arch must lie along BC , since it is pinned at both ends and is only subjected to forces at its ends, therefore the line of action of the total reaction at A must pass through O , the point of intersection of the lines of action of the load and the total reaction at B . The directions of all three forces acting on the arch is now known, and the magnitude of one of them, the load, is also known; hence a triangle can be constructed, representing the forces in direction and magnitude, Fig. 1.13d. ef is the vector representing the 12 tonf load, fg is the vector representing the total reaction R_B and scales 8.5 tonf, whilst R_A scales 11.0 tonf, agreeing with the analytical solution.

Particles Subjected to a Set of Forces in Equilibrium

If a particle is held in equilibrium by only three forces, the forces must obviously be concurrent, in order to act on the particle. The forces must also be coplanar. This is easily shown to be so, since a plane can be constructed to contain the lines of

action of any two of the forces, if the third force does not lie in this plane it will have a component perpendicular to the plane. But there is no force to counteract such a component and the particle could not be in equilibrium. Hence the third force cannot have a component perpendicular to the plane containing the other two forces. Thus all three forces must be coplanar. In addition, the three forces must satisfy the triangle of forces. If, therefore, a particle is in equilibrium under the action of three forces whose directions are known and the magnitude of one of the forces is also known, then by constructing the triangle of forces, the magnitude of the other two forces can be determined. Alternatively, the two unknown forces can be calculated by employing the equations of equilibrium. This is equivalent to solving the triangle of forces by trigonometry as may be seen in the following example.

Example 1.6. A load of 120 lbf hangs by a cord from a point P , which is supported by two light, inextensible cords attached to

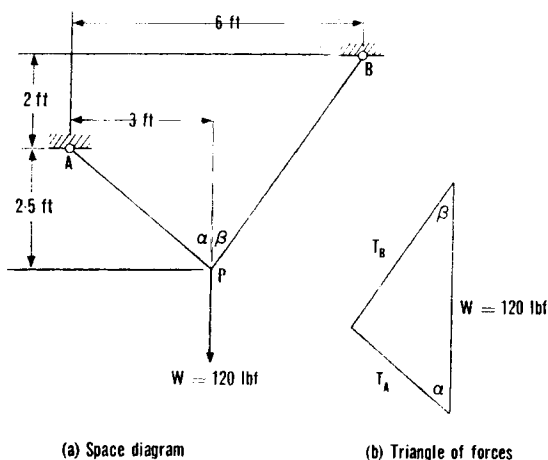


FIG. 1.14 Particle hanging from two cords

fixed points A and B , located as shown in Fig. 1.14a. Determine the forces in the two cords.

Graphical solution:

By constructing to scale the triangle of forces for the three forces acting on the particle, Fig. 1.14b, the tensions in the cord are found to be; in AP , 67 lbf and in BP , 93 lbf.

Alternatively, by resolution of forces:

Let α and β be the angles that AP and BP make with the vertical.

Resolving vertically:

$$T_A \cos \alpha + T_B \cos \beta = 120 \text{ lbf} \quad (1)$$

Resolving horizontally:

$$T_A \sin \alpha = T_B \sin \beta \quad (2)$$

Now $AP = \sqrt{(3^2 + 2.5^2)} = 3.9 \text{ ft}$

and $BP = \sqrt{(3^2 + 4.5^2)} = 5.4 \text{ ft}$

hence from equation (1)

$$\frac{2.5}{3.9} T_A + \frac{4.5}{5.4} T_B = 120$$

and from equation (2)

$$\frac{3.0}{3.9} T_A = \frac{3.0}{5.4} T_B$$

Solving this pair of simultaneous equations we get,

$$T_A = 67.0 \text{ lbf and } T_B = 92.7 \text{ lbf}$$

Equations (1) and (2) can also be obtained by solving the triangle of forces by trigonometry.

We are not restricted to resolving vertically and horizontally, nor need the two directions be at right angles to each other. The above problem may be solved by resolving thus:

Resolve perpendicularly to BP (eliminating T_B from the equation of equilibrium):

$$T_A \sin (\alpha + \beta) = W \sin \beta$$

$$\text{but} \quad \alpha = \tan^{-1} \frac{3.0}{2.5} = 50.2^\circ$$

$$\text{and} \quad \beta = \tan^{-1} \frac{3.0}{4.5} = 33.7^\circ$$

$$\text{Therefore} \quad T_A = 67.0 \text{ lbf}$$

Resolve perpendicularly to AP (eliminating T_A):

$$T_B \sin (\alpha + \beta) = W \sin \alpha$$

$$T_B = 92.7 \text{ lbf}$$

Non-coplanar forces

If a particle is held in equilibrium by forces acting in three dimensions there must be at least four forces, since if there were only three they would have to be coplanar. Taking the minimum number of four forces, whose directions are all known, if we know the magnitude of one of them, then the magnitudes of the other three can be determined by resolving along three different directions, not in the same plane, and obtaining a set of three simultaneous equations in three unknowns.

Example 1.7. A weight of 240 lbf hangs from a knot P which is suspended by three cords from points A , B and C at different levels. Determine the tension in each cord.

It is first of all necessary to measure the positions of A , B , C and P , and this can only be done after we have chosen a

system of reference axes to measure from. If we imagine the weight to be hanging above a horizontal floor then the floor will make a convenient reference plane. The points A , B , C and P can be projected vertically down on to the floor by a plumb bob, and marked A' , B' , C' , and P' . The heights of A , B , C and P above the floor can also be measured, being the z co-ordinates of these points. Before the x and y co-ordinates of the points can be measured, it will be necessary to arbitrarily rule two straight lines, intersecting at right angles, on the floor to act as the x and y axes. Imagine now that the x , y and z co-ordinates of A , B , C and P have been measured in inches and that they are A (3, 5, 36), B (6, 36, 44), C (40, 19, 54) and P (20, 20, 18).

The component of the force in a cord parallel to any axis is the tension in the cord multiplied by the cosine of the angle between the cord and any line parallel to the axis under consideration which intersects the cord. The cosine of this angle is the ratio of the projected length of the cord along the particular axis, to the length of the cord.

Let T_A , T_B and T_C be the tensions in the cords AP , BP and CP , and let a , b and c be their respective lengths which can be measured, or calculated from the square root of the sum of the squares of the three projected lengths.

$$a = \sqrt{[(3-20)^2 + (5-20)^2 + (36-18)^2]} = 28.9 \text{ in}$$

$$b = \sqrt{[(6-20)^2 + (36-20)^2 + (44-18)^2]} = 33.6 \text{ in}$$

$$c = \sqrt{[(40-20)^2 + (19-20)^2 + (54-18)^2]} = 41.2 \text{ in}$$

Resolving vertically along z axis:

$$\frac{(36-18)}{28.9} T_A + \frac{(44-18)}{33.6} T_B + \frac{(54-18)}{41.2} T_C = 240$$

$$\text{or} \quad 0.623T_A + 0.774T_B + 0.873T_C = 240 \quad (1)$$

Resolving horizontally along x axis:

$$\frac{(3-20)}{28.9} T_A + \frac{(6-20)}{33.6} T_B + \frac{(40-20)}{41.2} T_C = 0$$

$$-0.588T_A - 0.417T_B + 0.485T_C = 0 \quad (2)$$

Resolving horizontally along y axis:

$$\frac{(5-20)}{28.9} T_A + \frac{(36-20)}{33.6} T_B + \frac{(19-20)}{41.2} T_C = 0$$

$$-0.519T_A + 0.476T_B - 0.024T_C = 0 \quad (3)$$

Solving equations (1), (2) and (3) we get

$$T_A = 67.9 \text{ lbf}, T_B = 82.2 \text{ lbf} \text{ and } T_C = 153.1 \text{ lbf}$$

Exercise 1

1. Explain precisely the difference between loads and reactions.

A uniform lever AB , 6 ft long and weighing 12 lbf is pinned to the ground at A and rests on a smooth round peg at C so placed that the lever is inclined at 30° to the horizontal and the distance of C from A is 4 ft. If a vertical load of 6 lbf is applied at the free end B , determine completely the reactions at A and C .

Ans: Reaction at peg = 15.6 lbf

Vertical reaction at A = 4.5 lbf

Horizontal reaction at A = 7.8 lbf

2. Three forces act on a body and maintain it in equilibrium. State the conditions that they must satisfy (a) if the forces are parallel, (b) if the forces are not parallel.

A uniform ladder rests on rough horizontal ground and leans against a smooth vertical wall. If the coefficient of friction between the ground and the ladder is μ show that the greatest inclination that the ladder can have to the vertical, without slipping, is $\tan^{-1} 2\mu$.

3. If four or more coplanar forces act on a structure, holding it in equilibrium, state the conditions that they must satisfy.

The truss in Fig. 1.15 is pinned to the ground at A and rests on rollers at B and carries two loads, as shown. All the horizontal and vertical bars are of the same length. Calculate the reactions at the supports.

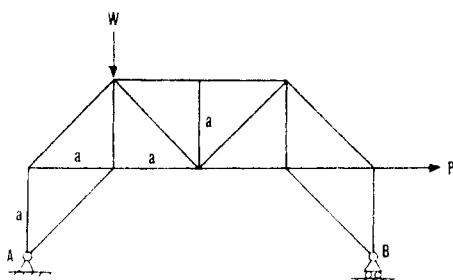


FIG. 1.15

Ans: Horizontal reaction at $A = P$

$$\text{Vertical reaction at } A = \frac{3}{4}W - \frac{P}{4}$$

$$\text{Vertical reaction at } B = \frac{W}{4} + \frac{P}{4}$$

4. What is a link polygon, and to what uses may it be put?

Determine graphically the support reactions for the loaded roof truss shown in Fig. 1.16.

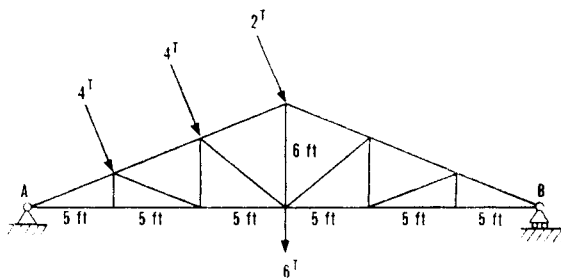


FIG. 1.16

Ans: $H_A = 3.7 \text{ tonf}$

$V_A = 9.0 \text{ tonf}$

$V_B = 6.3 \text{ tonf}$

5. A symmetrical three-pinned arch has a span l and a rise d . Show that the horizontal thrust in the arch, when it carries a single load W distance x from the left abutment, is $Wx/2d$, provided x is less than $l/2$.

If $l = 40$ ft and $d = 10$ ft and the arch carries loads of 3 tonf and 4 tonf at 10 ft and 25 ft from the left abutment, calculate the thrust in the arch, making use of the principle of superposition.

Ans: 4.5 tonf

6. Figure 1.17 shows an unsymmetrical arch made of two quadrants AC and BC of 4 ft and 6 ft radius respectively. It is pinned to the ground

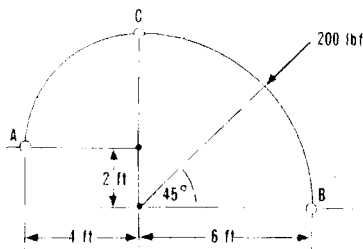


FIG. 1.17

at A and B and there is a third pin at C . It carries a 200 lbf load inclined at 45° . Calculate completely the reactions at A and B .

Ans: Both reactions equal 100 lbf, parallel to load.

7. A symmetrical three-pinned parabolic arch has a span of 60 ft and a rise of 20 ft. The dead load is equivalent to $\frac{1}{4}$ tonf/ft run, measured horizontally. The arch carries a concentrated load of 10 tonf at 15 ft from the left end. Calculate the horizontal thrust in the arch.

Ans: 9.37 tonf

8. Figure 1.18 shows a three-hinged portal frame for a factory building. If each half of the portal weighs 1550 lbf, and is assumed to be of

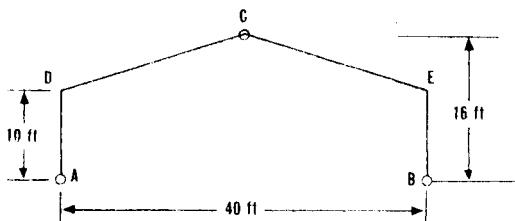


FIG. 1.18

uniform section, calculate the horizontal thrust in the portal. Calculate also the horizontal and vertical reactions at each of the lower pins due to a wind load of 1000 lbf assumed to be a load acting normal to the sloping portion and concentrated half-way along DC .

$$\begin{aligned} \text{Ans: } H &= 655 \text{ lbf} \\ V_A &= 625 \text{ lbf} \\ H_A &= 125 \text{ lbf} \\ V_B &= 330 \text{ lbf} \\ H_B &= 412 \text{ lbf} \end{aligned}$$

9. A semi-circular, three-pinned arch with a span of 40 ft has loads of 2 tonf and 4 tonf at 10 ft from the left- and right-hand abutments respectively. Calculate the horizontal thrust in the arch and the magnitude and direction of the force transmitted by the pin at the crown.

What effect, if any, will a rise in temperature have on the forces in the arch?

$$\begin{aligned} \text{Ans: } H &= 1.5 \text{ tonf} \\ 1.58 \text{ tonf at } \tan^{-1} \frac{1}{3} &\text{ to horizontal} \\ \text{No effect} \end{aligned}$$

10. A three-pinned segmental arch has a span of 60 ft and a rise of 20 ft. Calculate the horizontal thrust in the arch due to (a) a load of 6 tonf at 20 ft from the left-hand support, (b) a uniformly distributed load of $\frac{1}{2}$ tonf/ft over the left-hand half of the arch, (c) a u.d.l. of $\frac{1}{2}$ tonf/ft over the whole span, (d) the load in (a) together with an equal upward force at 20 ft from the right hand support.

$$\begin{aligned} \text{Ans: (a) } 3 \text{ tonf} \quad \text{(b) } 5.63 \text{ tonf} \\ \text{(c) } 11.25 \text{ tonf} \quad \text{(d) } 0 \end{aligned}$$

11. The structure depicted in Fig. 1.19 consists of two bars AB , BC each 6 ft long, pinned to a wall at A and to the ground at C , and carrying a 2 tonf load at the mid-point of AB which is perpendicular to the bar, and a vertical load of 3 tonf at B . Determine the horizontal and vertical reactions at A and C .

$$\begin{aligned} \text{Ans: At } A &\rightarrow 0.25 \text{ tonf } \uparrow 1.30 \text{ tonf} \\ \text{At } C &\rightarrow 1.25 \text{ tonf } \uparrow 3.43 \text{ tonf} \end{aligned}$$

12. A compound beam consists of two parts AD and DC pinned together at D resting on rollers at B and C , and attached by a pin to the ground at A , similarly to Fig. 1.7a. $AB = 8$ ft, $BD = 2$ ft and $DC = 4$ ft. AD carries a uniform loading of intensity 20 lbf/ft, whilst an oblique load

of 400 lbf is applied mid-way along DC , inclined at 60° to the horizontal. Determine all the support reactions.

Ans: $H_A = 200$ lbf, $V_A = 32$ lbf, $V_B = 341$ lbf, $V_C = 173$ lbf

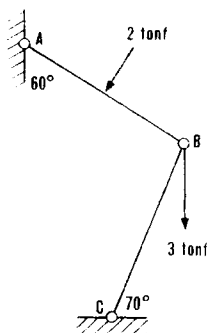


FIG. 1.19

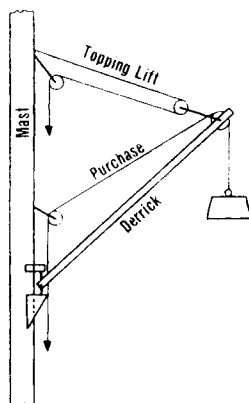


FIG. 1.20

13. Figure 1.20 shows a ship's hoisting gear. The derrick, purchase and topping lift are inclined at 48° , 60° and 70° to the vertical, respectively. If the derrick weighs 2 tonf and is of uniform section, find the average thrust in it and the tension in the topping lift wire when a load of 15 tonf is being raised.

Ans: 30 tonf; 5 tonf

14. Explain what is meant by the transmissibility of force.

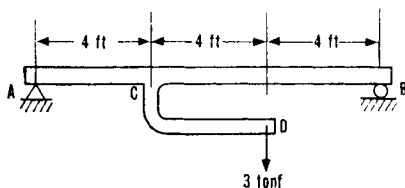


FIG. 1.21

Determine the support reactions for the beam in Fig. 1.21 loaded through a rigid crank CD .

Ans: $V_A = 1$ tonf, $V_B = 2$ tonf

15. Figure 1.22 shows a pair of cantilevers supporting a suspended simple span BC which is carrying a point load of 250 lbf. The dead load

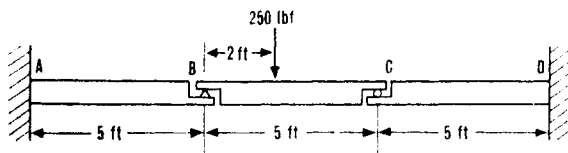


FIG. 1.22

of the cantilevers is 20 lbf/ft and that of the simple span is 10 lbf/ft. Calculate the restraints at the supports.

Ans: $V_A = 275$ lbf, $M_A = 1125$ lbf ft
 $V_D = 225$ lbf, $M_D = 875$ lbf ft

16. State the conditions of equilibrium relating to a particle under the action of a system of non-coplanar forces.

A tripod consisting of three equal legs of length l rests with its feet on rough horizontal ground. The figure formed by joining the feet is an equilateral triangle whose sides are also of length l . A load W is hung from the apex of the tripod. Calculate the force in each member.

Ans: $W/6$

17. The derrick in Fig. 1.23 has a vertical post AC tied back to the ground by stay wires CD and CE inclined at 45° to the vertical. A , D and

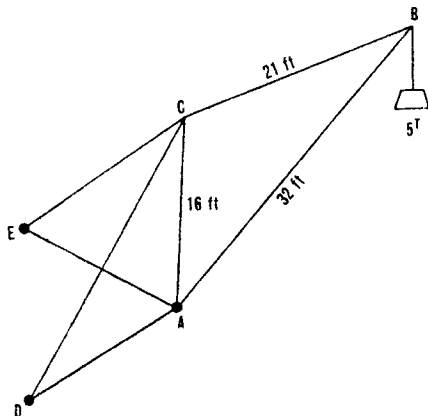


FIG. 1.23

E are in a horizontal plane and the plane of the jib and the tie bisects the angle DAE , which is a right angle. Determine the forces in the stays and in the post.

Ans: 5.9 tonf; 5.0 tonf (compr.)

18. Three members AD , BD and CD are connected together at D and are attached to a vertical wall at A , B and C . This framework forms a

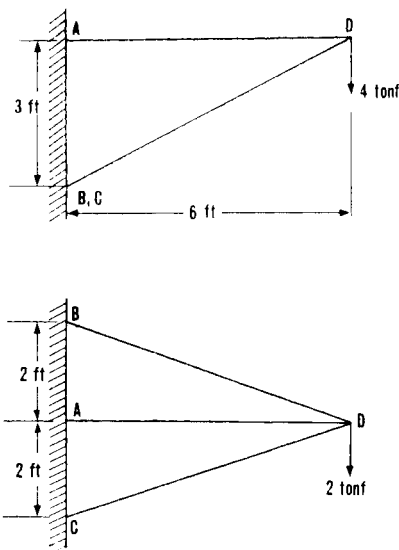


FIG. 1.24

bracket which supports a vertical load of 4 tonf and a horizontal load of 2 tonf as shown in Fig. 1.24. Assuming that all the connections are pin-joints, determine the magnitude and sense of the force in each member of the framework.

(I.C.E. Apr. 1952)

Ans: Force in AD +8 tonf, BD -1.17 tonf
 CD -8.17 tonf

19. Figure 1.25 shows a plan and elevation of three cords attached to a particle D carrying a load of 20 lbf. The cords are attached to rigid supports at A , B and C . Determine the forces in all three cords.

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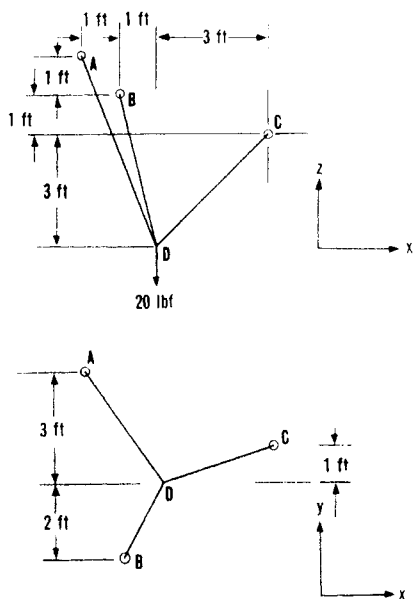


FIG. 1.25

Ans: Tension in AD 6.8 lbf
 BD 11.2 lbf
 CD 6.8 lbf

CHAPTER 2

Shearing Force and Bending Moment

Internal Forces

In the previous chapter the application of the laws of statics to structures was discussed, principally as a means of determining the external reactions induced at supports, due to the loading. Next we have to consider the manner in which the loads are transmitted through the structure, as *internal forces*, to the supports. Internal forces or stresses exist at every section of a loaded body, but they are not apparent to the senses; we cannot *see* them. If, however, we were to commence sawing through a member carrying an internal force it would not be long before we became aware that it was loaded. If the structure was not to collapse we should have to replace the severed internal force by a pair of equal and opposite forces; one acting on each cut face. Quite obviously, it is unnecessary to actually cut a structure in two, in order to determine the internal forces acting there; we need only imagine the effect of cutting it through. Wherever an internal force is severed it will have to be replaced by a pair of equal and opposite forces acting on the cut faces, of such a value that the equilibrium of each portion of the structure is preserved. What we have done, in effect, is to divide the original structure into two separate structures, each subjected to its own system of forces. Only one of the two parts need be considered, it does not really matter which; it is in equilibrium under the action of the

loads and support reactions acting on it, together with the forces acting on the cut face. The loads are known, and the reactions will have been determined by the methods given in Chapter 1; hence the only unknown forces are those acting on the cut face. Provided we can form as many independent equations of equilibrium as there are unknown forces acting on the cut face, we can always determine what the values of these internal forces are.

Bar subjected to a pull

To take a very simple example, let us consider the tie-rod AB , in Fig. 2.1a, clamped to a rigid support at A , and subjected to a pull P at B . What is the internal force acting in the tie-rod at the section X ?

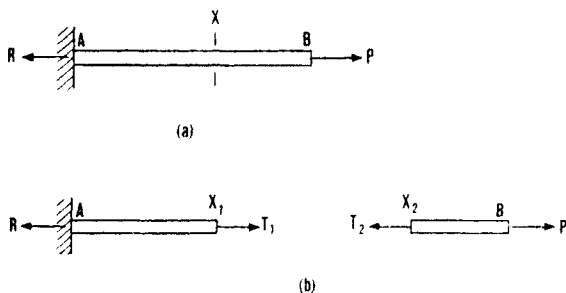


FIG. 2.1 Tie rod in pure tension

Thinking of all the forces which must act on the rod to hold it in equilibrium, we realise that there must be a reaction R at A , which is equal and opposite to the applied load P . As we want to know the value of the internal force at X , we imagine a cut to be made there, dividing the rod into two parts, AX_1 and X_2B , Fig. 2.1b. We must also introduce a pair of equal and opposite forces T_1 and T_2 acting on the cut faces X_1 and X_2 , to replace the severed internal force. It is immediately apparent,

by considering the equilibrium of the portion AX_1 , that T_1 is equal to R , and by considering the equilibrium of the portion X_2B , that T_2 is equal to P . Hence T_1 and T_2 are both equal to P , and the internal force at X is a tension P , as the reader will, no doubt, already have realised. The reason for going through this example in such detail is to present a complete sequence of logical steps for an elementary problem, which can be used as a basis for more complicated cases. It is worth noting that T_1 is the reaction of the right-hand portion on the left-hand portion of the rod, and T_2 is the reaction of the left-hand portion on the right-hand portion.

Cantilever subjected to an oblique load

Next, let us consider a bar capable of resisting bending, as well as an axial force, such as the cantilever AB in Fig. 2.2a.

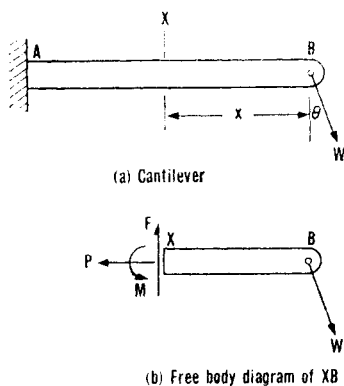


FIG. 2.2 Internal forces in a beam

What are the internal forces transmitted across the section at X ? The support must clamp the beam, since it has to prevent rotation.

Imagine the beam to be cut at X . Figure 2.2b is the free body diagram for the portion of the beam to the right of the cut. There must be an axial force P , acting on the cut face, to resist the axial component of the loading. There must also be a tangential force F , acting vertically upwards on the cut face, to resist the downward transverse component of the loading; and there must be a couple M , acting on the cut face, to resist the moment about X , due to the loading. So we see that for coplanar loading, the internal forces at any section can be resolved into three components, (i) an axial force, (ii) a transverse or shearing force, and (iii) a couple or bending moment; although one or more of these actions may vanish. Note that it is legitimate to classify a bending moment as an internal force, since a couple is statically equivalent to a pair of equal and opposite parallel forces at an appropriate distance apart.

Referring again to the above example, we see, from a study of Fig. 2.2, that $P = W \sin \theta$; $F = W \cos \theta$; and $M = Wx \cos \theta$.

Axial Force

The axial force at any cross-section of a beam (or column) is the component of the internal force transmitted across the section which is parallel to the length of the member, and its line of action passes through the centroid of the cross-section. The axial force will, in fact, be equal to the algebraic sum of the *axial components* of all the forces (not forgetting the support reactions) acting on the member to one side of the section.

Shearing Force

The shearing force at any cross-section of a beam is the component of the internal force transmitted across the section which lies in the plane of the cross-section i.e. perpendicular to the length of the member.

ular to the length of the beam. The s.f. at any section of a beam is thus the component of the internal force there trying to slice the beam in two. Shearing forces are of a tangential nature, like frictional forces. Figure 2.3a shows, in exaggerated manner, the parallelogram type of distortion produced by a shearing force. Figure 2.3b is a diagrammatic representation of a cantilever slit up into a number of vertical slices each of which slides a little with respect to adjacent slices, under the action of the shearing force F . If the slices were made thin enough the distortion would become indistinguishable from that in Fig. 2.3a. The last diagram, Fig. 2.3c, just shows a

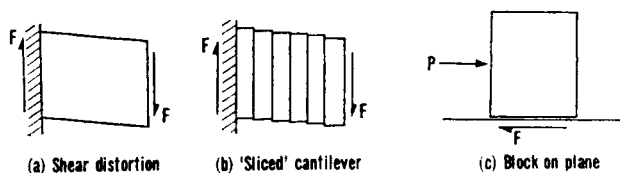


FIG. 2.3 Shearing action

block resting on a rough horizontal plane subjected to a horizontal push P . A friction force F equal and opposite to P will develop at the interface. The last diagram is included to show that shearing forces resemble friction forces.

The shearing force at any section of a beam is equal to the algebraic sum of the vertical components of all the forces acting on the part of the beam to one side of the section, assuming, of course, the beam to be horizontal.

Bending Moment

The bending moment at any cross-section of a beam subjected to coplanar loading is the couple transmitted across the section; or, what amounts to the same thing, the couple that

the portion of the beam to one side of the section exerts on the other portion. The b.m. at any section of a beam is thus the couple there trying to snap the beam in two. This means that the state of stress which exists at each section of a beam must be capable of developing a couple or *moment of resistance* to oppose the bending moment applied there. Figure 2.4 shows a short length of rectangular beam subjected to a sagging bending moment. If the beam is thought of as a number of longi-

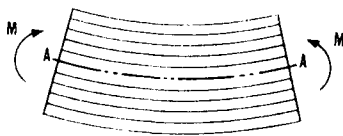


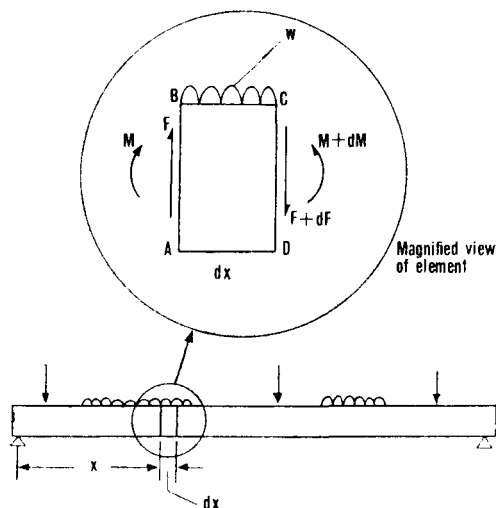
FIG. 2.4 Bending action on the fibres of a beam

tudinal fibres or layers, then the central layer *AA* will not change its length and remains unstrained. Fibres above this layer, or *neutral axis* as it is called, will be shortened and therefore in compression, whilst fibres below the neutral axis will be stretched and therefore in tension, creating a couple resisting the applied bending moment *M* in much the same way that the tension in a stretched wire resists the pull applied to its ends.

Referring back to Fig. 2.2, the section of the cantilever to the right of *X* can be thought of as a lever. Due to the loading on it, the lever *BX* will exert a couple on the left hand portion of the beam, which is the bending moment at *X*. It will thus be seen that the bending moment at any section of a beam is equal to the algebraic sum of the moments of all the forces acting on the portion of the beam to one side of the section about the centroid of the section.

Relationship between Loading and Shearing Force

Visualise a horizontal beam loaded and constrained by purely vertical coplanar forces. Let w be the intensity of loading (load per unit length), which may vary from point to point along the beam. If we consider a vertical slice of the beam of length dx at a distance x from the left end (Fig. 2.5) the load-



Beam subjected to any form of transverse loading

FIG. 2.5 Forces acting on a vertical slice of a beam

ing on it can be considered as uniformly distributed, provided dx is sufficiently small. There will be a shearing force F on the face AB and a shearing force $F + dF$ acting in the opposite direction on the face CD . The choice of *up on the left* and *down on the right* as positive shearing force is purely arbitrary, but has advantages when it comes to constructing shearing force diagrams, as will be seen later. In addition to the shearing forces, there will be a bending moment M acting on the face

AB and a bending moment of opposite sense $M + dM$ acting on the face CD . The choice of sagging bending moment as positive is also purely arbitrary.

It will be seen from Fig. 2.5 that $F + dF + w dx = F$ or $dF = -w dx$. Hence $dF/dx = -w$; or *the slope of the shear force diagram is equal to the intensity of loading*. Inversely, by integrating the expression $dF = -w dx$, between $x = x_1$, where the shearing force is F_1 , and $x = x_2$, where the shearing force is F_2 , we obtain the expression

$$F_1 - F_2 = \int_{x_1}^{x_2} w dx$$

Thus *the change in shearing force between two sections of a beam is equal to the area under the intensity of loading diagram between the same two sections*.

Relationship between Shearing Force and Bending Moment

Referring again to Fig. 2.5, and taking moments clockwise about D we obtain the equation

$$M - w dx \left(\frac{dx}{2} \right) + F dx = M + dM \quad \text{or} \quad F dx = dM$$

since dx can be made vanishingly small. Hence $F = dM/dx$, and so *the shearing force at any section of a beam is equal to the rate of change of bending moment there*. Inversely, since

$$\int dM = \int F dx$$

between corresponding limits,

$$M_2 - M_1 = \int_{x_1}^{x_2} F dx$$

thus *the change in bending moment between two sections of a beam is equal to the area under the shearing force diagram between the same two sections*.

Shearing Force and Bending Moment Diagrams

As the shearing force and bending moment vary from section to section along a beam the most convenient form for displaying this variation is by shearing force and bending moment diagrams, which are graphs showing the value of the shearing force or bending moment for each position along the beam. Such diagrams are, of course, only applicable for one particular loading and if the loading is moved or changed new diagrams must be constructed.

Shearing force and bending moment diagrams have a number of uses, and the most obvious is that they enable the engineer to see at a glance where the shearing or bending is greatest and so simplify the task of checking that the beams have adequate strength at every section to resist the applied shearing forces and bending moments.

Shearing force diagrams usually provide the quickest means of obtaining bending moment diagrams for statically determinate beams, by using the area under them to determine the bending moment.

Example 2.1. Beam with non-central load. The simple beam in Fig. 2.6 carries a load W placed at a distance a from the left end. Sketch the s.f. and b.m. diagrams.

The first step is to determine the reactions at the supports. By taking moments about B , we find that the reaction at A is Wb/l and similarly by taking moments about A the reaction at B is found to be Wa/l . It is worth checking at this stage that the sum of the two reactions is equal to the total load W . The shearing force between A and C is constant, and is merely equal to the reaction at A ; similarly the shearing force between C and B is equal to the reaction at B . Hence, to construct the shearing force diagram we start at the left-hand end of the base line AB , and go upwards by an amount equal to the force applied at A , i.e. the reaction Wb/l ; run along horizont-

ally until we meet the load, drop downwards by the amount of the load W , and continue horizontally until we meet the right-hand reaction, where we move upwards by an amount Wa/l , which brings us back on to the base line and completes the shearing force diagram. Because we have adopted the sign convention of *up on the left* and *down on the right*, (Fig. 2.3a) for positive shearing force we can work from left to right

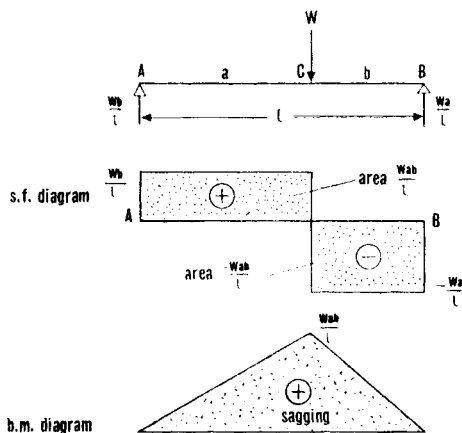


FIG. 2.6 Beam with non-central load

across the beam and follow the direction of the force vectors, without violating the convention that ordinates above the base line are positive. Next the areas of the separate parts of the shearing force diagram are calculated, e.g. the area under the shearing force diagram between A and C is Wab/l and this is equal to the change of bending moment between A and C ; but as there is no bending moment at A , since it is a simple support, the bending moment at C is Wab/l , and the bending moment diagram is triangular.

Example 2.2. Beam with distributed loading. The simple beam AB in Fig. 2.7a carries a uniform load of intensity w distributed over the left-hand half of the beam. The reactions R_A

and R_B are therefore $\frac{3}{8}wl$ and $\frac{1}{8}wl$. The shearing force diagram, Fig. 2.7b, is constructed by commencing at the left-hand end of the beam, and drawing an ordinate vertically upwards equal to the reaction there. The diagram then continues as a straight line sloping downwards towards the right until the end of the load is reached, by which time the line has drop-

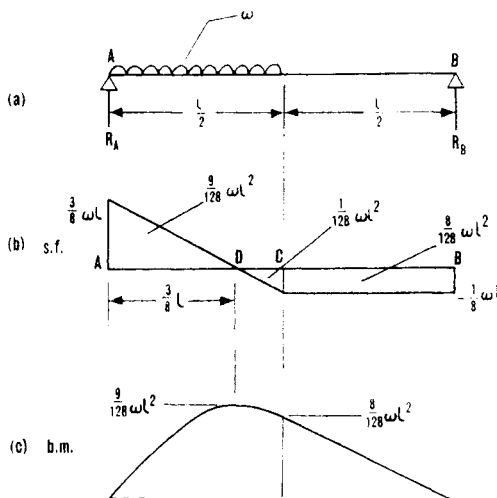
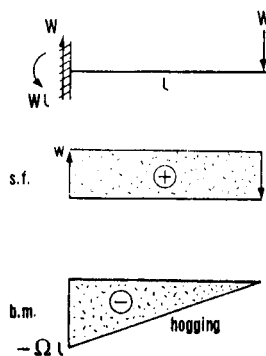


FIG. 2.7

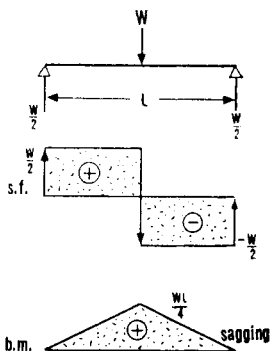
ped by an amount equal to the total load. After this, the diagram continues as a horizontal line to the end of the beam, where it abruptly returns to the base line on meeting the reaction R_B , completing the s.f. diagram.

The b.m. diagram, Fig. 2.7c, is a parabola over the left-hand half, where the uniform load is applied, and straight over the right-hand half. By dividing the space between the s.f. diagram and the base line into two triangles and a rectangle, and calculating their areas, taking account of whether they are positive (above the base line) or negative (below the base line), we

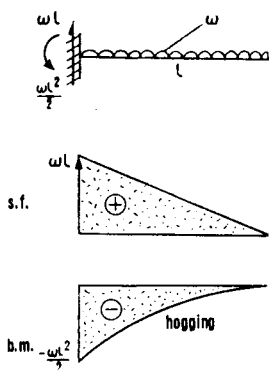
obtain the values of the b.m. at D and C . Explaining the procedure in more detail, the area $\frac{9}{128}wl^2$ under the s.f. diagram, between A and D , is equal to the difference between the bending moments at A and D . But, as the b.m. at A is zero, then the b.m. at D is equal to $\frac{9}{128}wl^2$. The area of the next portion is $-\frac{1}{128}wl^2$, and so the b.m. at C is $\frac{8}{128}wl^2$. The last por-



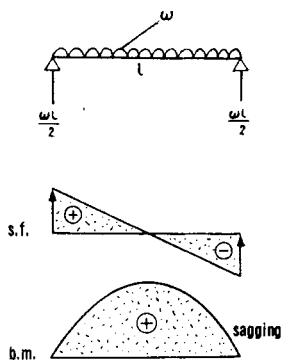
(a) Cantilever with end load



(b) Simple beam with central load



(c) Cantilever with uniform loading



(d) Simple beam with uniform loading

FIG. 2.8 S.f. and b.m. diagrams for some simple loadings

tion of the s.f. diagram has a negative area of $\frac{8}{128}wl^2$. Subtracting this amount from the b.m. at C we get zero, agreeing with the b.m. at B . Between A and D the area under the s.f. diagram is increasing at a slower and slower rate, until at D , the area ceases to increase, and hence the total area and the b.m. are a maximum.

The s.f. and b.m. diagrams for some common loadings on simple beams and cantilevers are given in Fig. 2.8.

General remarks on s.f. and b.m. diagrams

The s.f. diagram for a beam carrying concentrated loads will be a series of horizontal steps, and the b.m. diagram will be polygonal.

If a beam carries only uniform loading (not necessarily continuous over the whole span) the s.f. diagram will be a sloping straight line under the uniformly distributed load. If any part of the beam is not loaded, the s.f. diagram under that portion will be horizontal. The b.m. diagram will be parabolic over that portion carrying uniformly distributed load, and straight elsewhere, any straight portions running tangentially into the parabola. The slope of the s.f. diagram at any point is equal to the intensity of the loading there, since $dF/dx = -w$. In this connection, it should be noted that at a concentrated load the intensity of loading is infinite, so that under a concentrated load the s.f. diagram is vertical and a "step" is present, as stated in the previous paragraph.

If a beam carries both uniformly distributed and concentrated loads, and here it is worth reminding ourselves that every beam has to carry its own weight, which is a distributed weight that is usually uniform, the s.f. diagram will be a series of sloping steps, and the b.m. diagram will be a series of parabolic curves with sudden changes of slope at every concentrated

load (Fig. 2.11). It is helpful to make use of the principle of superposition and consider the loading as the sum of the concentrated loads acting alone, and the uniformly distributed load acting alone.

An important conclusion, which follows from the mathematical relation proved earlier, that $dM/dx = F$, is that the s.f. vanishes where the b.m. is a maximum. As design engineers are chiefly concerned with the maximum bending moments in structures, s.f. diagrams are often used to locate positions of maximum bending moment. It is, of course, important to remember that we may meet a beam for which the b.m. passes through a *minimum* value where the s.f. is zero, as dM/dx will be zero at such points as well, (Fig. 2.10).

Link polygon method of constructing s.f. and b.m. diagrams

Any problem in statics which can be solved by analytical methods (calculation) can also be solved by graphical means. Sometimes an analytical solution is to be preferred, as being more accurate, convenient or quicker, whilst for other problems a graphical construction may be better. The choice depends to some extent on personal preference.

The link polygon forms the basis of a graphical construction for s.f. and b.m. diagrams. The method is only applicable to concentrated loads, although distributed loading can be approximated, by dividing the distributed loading into discrete loads. Only a brief résumé of the method will be given for the example below, since the method is usually adequately dealt with in the subject of Applied Mechanics.

Example 2.3. A simple beam, 14 ft long, carries loads of 4, 6 and 3 tonf at distances of 4, 7 and 11 ft respectively from the left-hand end. Construct the s.f. and b.m. diagrams.

The first step is to construct to scale the space diagram, Fig. 2.9a, and letter the spaces between the lines of action of the forces according to Bow's notation. Next, the load line ad is drawn to scale, and a pole o chosen with rays radiating from it to the extremities a, b, c and d of the load vectors. Following this, lines OA, OB, OC, OD are drawn parallel to corresponding rays crossing the spaces A, B, C and D , and the link polygon is completed by the line OE to give the b.m. diagram, Fig. 2.9c.

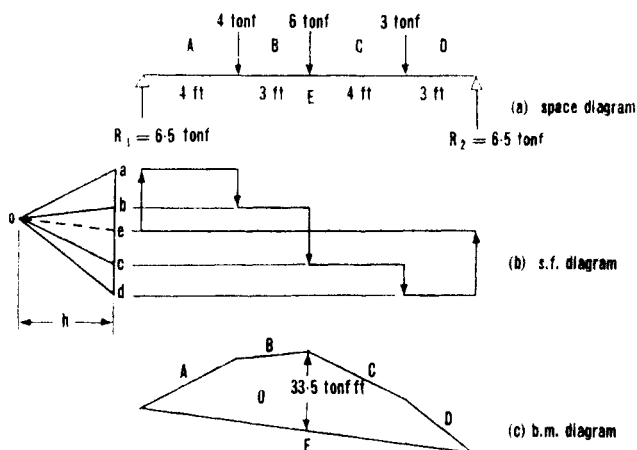


FIG. 2.9 Graphical construction of s.f. and b.m. diagrams

The support reactions ea, de are obtained by drawing oe parallel to OE to cut the load line at e . The s.f. diagram can then be constructed by projecting horizontally from the load line across to the lines of action of the corresponding forces. The scale of the s.f. diagram is that of the load line. The scale of the b.m. diagram is the product of the scale of space diagram, the scale of the load line, and the polar distance h , e.g. if the scale of the space diagram is 1 in = 4 ft and the scale of the load line is 1 in = 10 tonf, and the polar distance is 1 in,

then the scale of the b.m. diagram is

$$4 \frac{\text{ft}}{\text{in}} \times 10 \frac{\text{tonf}}{\text{in}} \times \text{lin} = 40 \frac{\text{tonf ft}}{\text{in}}$$

Notice, that in general the base line of the b.m. diagram is not horizontal, but ordinates of b.m. must still be measured vertically.

Example 2.4. Make dimensional sketches of the s.f. and b.m. diagrams for the overhanging beam shown in Fig. 2.10a.

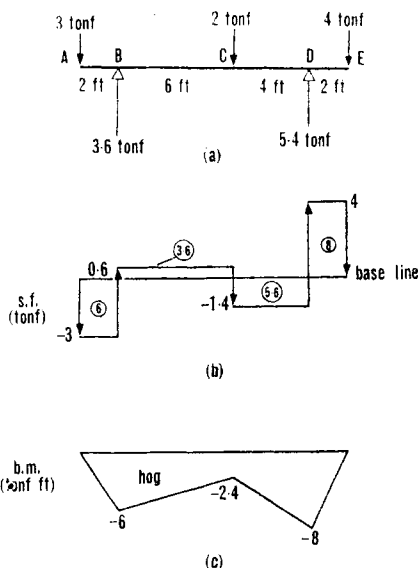


FIG. 2.10 Overhanging beam

The reactions at *B* and *D* are calculated in the usual way, by taking moments about *D* and *B* respectively. The s.f. diagram, Fig. 2.10b is constructed by starting at the left, and following the force vectors, as indicated by the arrowheads, moving horizontally to the right between the line of action of

each force. The ringed figures are the areas enclosed by the s.f. diagram, and are negative, when below the base line. Notice that the s.f. diagram crosses the base line three times, at *B*, *C* and *D*. It will be seen from the b.m. diagram, Fig. 2.10c, that the b.m. is hogging throughout, and has maximum values over the supports at *B* and *D*, and a minimum value under the load at *C*. A greater load at *C* could just reduce the b.m. there to zero, and an even greater load would produce a maximum sagging b.m. under *C*.

Example 2.5. A uniform beam 12 ft long, and weighing $\frac{1}{4}$ tonf per ft, is simply supported at its ends, and carries loads of 6 tonf

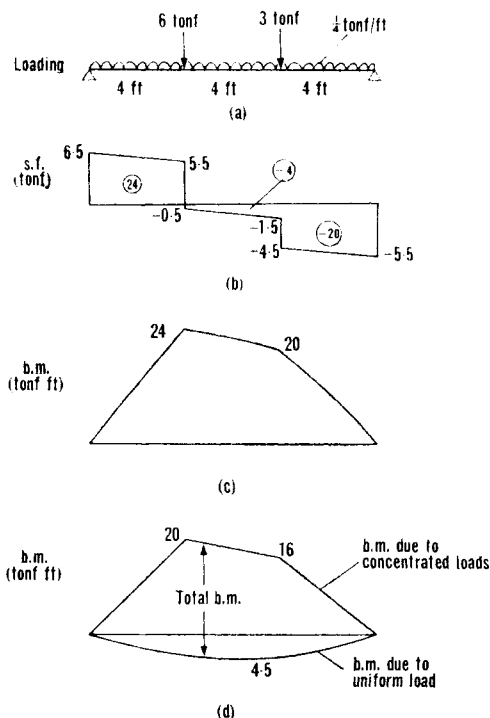


FIG. 2.11

and 3 tonf at 4 ft and 8 ft from the left-hand end, respectively. Construct the s.f. and b.m. diagrams.

Having calculated the reactions, the s.f. diagram is drawn under the load diagram in Fig. 2.11. The s.f. diagram is then divided into convenient sections, and the area of each trapezium calculated. These are the ringed figures in the s.f. diagram.

The b.m. diagram, Fig. 2.11c, is then drawn under the s.f. diagram using these areas. The b.m. diagram can also be constructed by drawing the b.m. diagram for the concentrated loads above the base line and the b.m. diagram for the u.d. loading below the base line, as in Fig. 2.11d. The total b.m. at any section is then given by measuring the ordinate between the two diagrams.

Position of Supports for Uniformly-Loaded Beam to give Maximum Load-carrying Capacity

The maximum intensity of uniformly distributed loading w that a given uniform beam of length l can carry is determined by its strength and the position of its supports, assuming that it rests on two simple supports. The strength of a given beam cannot be altered, but the position of the supports can. The maximum hogging moments will occur over the supports, and the maximum sagging moment will occur somewhere between the supports. The beam will have its greatest load carrying capacity when the numerical values of the bending moments at these three sections are equal (assuming that the bending strength or maximum moment of resistance of the beam is the same for hogging and sagging). Thus the supports must be symmetrically placed and in such a way that the hogging b.m. at the supports is equal to the sagging b.m. at mid-span, and the b.m. diagram will be as shown in Fig. 2.12c. It will be seen

from the s.f. diagram, Fig. 2.12b, that the hogging b.m. at the supports is equal to the area A , and that the sagging b.m. at mid-span is equal to $(B-A)$. As the latter must also be equal to A , then $B = 2A$. As the triangles A and B are similar, their

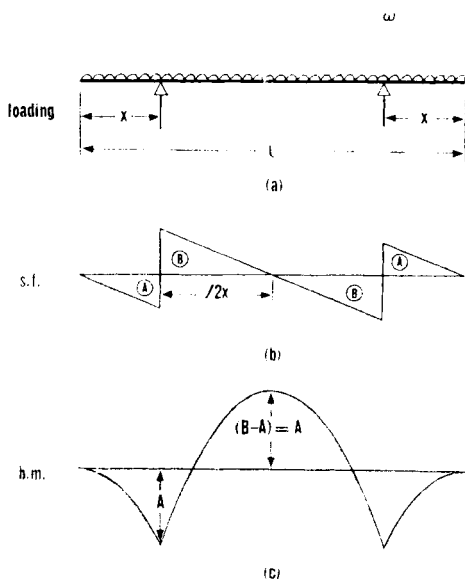


FIG. 2.12 Position of supports for least bending

areas must be proportional to the squares of their bases. Hence, as the triangle A has a base of length x , the base of the triangle B must be $\sqrt{2}x$. But $2(x + \sqrt{2}x) = l$, therefore

$$x = \frac{l}{2(1 + \sqrt{2})} = 0.207 l.$$

Beam with Oblique Loading

Example 2.6. A uniform beam 10 ft long, and weighing 40 lbf per ft, is pinned to a rigid support at the left-hand end, and rests on a smooth roller, placed 2 ft from the other end. A load of 500 lbf, inclined at 30° to the vertical, is applied to the beam 6 ft from the pinned end. Construct the s.f. and b.m. diagrams.

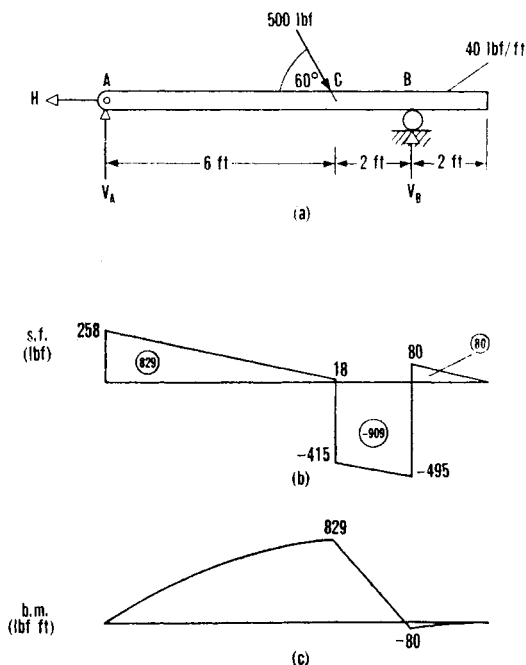


FIG. 2.13

First of all we make a sketch of the beam, Fig. 2.13a. The reaction at the roller V_B will be vertical, and is determined by taking moments about the pin.

$$8V_B = 40 \times 10 \times 5 + 500 \times 6 \cos 30^\circ,$$

hence

$$V_B = 575 \text{ lbf}$$

The vertical component of the reaction at the pin, V_A is determined by taking moments about the roller.

$$8V_A = 40 \times 10 \times 3 + 500 \times 2 \cos 30^\circ$$

hence

$$V_A = 258 \text{ lbf}$$

It is worth checking at this stage, that the sum of V_A and V_B , which is 833 lbf, is equal to the total vertical loading $500 \cos 30^\circ + 400 = 833$ lbf. The s.f. diagram Fig. 2.13b can now be constructed in the usual manner, taking into account only those forces or components acting perpendicularly to the beam. The b.m. diagram, Fig. 2.13c, is obtained from the area under the s.f. diagram.

In addition to the vertical reaction at the pin, there will also be a horizontal reaction $H = 500 \sin 30^\circ$ acting to the left. Hence between A and the point C , where the load is applied the beam will carry a axial pull of this amount.

Three-pinned Arch

Example 2.7. A symmetrical, three-pinned, parabolic arch has a span of 40 ft and a rise of 10 ft. It carries a load of 15 tonf at 12 ft from the left abutment. Calculate the b.m. in the arch under the load. Make also a dimensioned sketch of the b.m. diagram.

The free body diagram for the arch is shown in Fig. 2.14a. A straightforward application of the equations of equilibrium shows that $V_A = 10.5$ tonf, $V_B = 4.5$ tonf and $H = 9.0$ tonf. The b.m. at D , under the 15 tonf load, is equal to the algebraic sum of the moments of all the forces acting on the portion of the arch AD , i.e. it is equal to the sagging moment, due to the vertical reaction V_A less the hogging moment, due to the horizontal thrust. Hence the b.m. at D is equal to $12V_A - Hy$, where y is the height of D above the line AB . The distance y

is easily calculated, as the shape of the arch is a parabola.

Hence
$$\frac{10-y}{10} = \left(\frac{20-12}{20} \right)^2$$

therefore
$$y = 8.4 \text{ ft}$$

and so the b.m. at $D = 50.4 \text{ tonf ft (sag)}$.

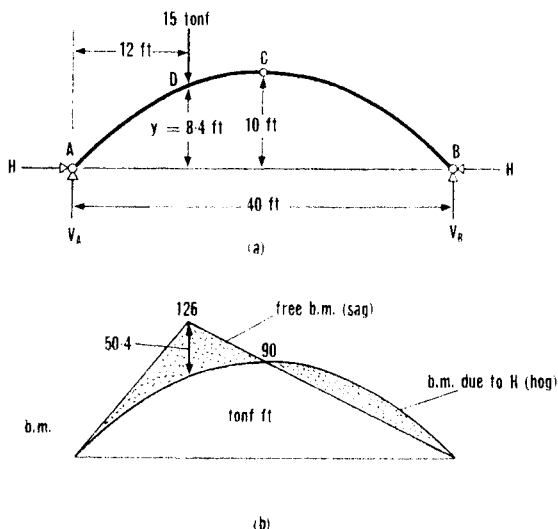


FIG. 2.14

The principle of superposition simplified the construction of the b.m. diagram. The b.m. at any point in the arch is equal to the sagging moment there, due to the vertical forces, less the hogging moment there, due to the horizontal thrust. Hence, if we superimpose the b.m. diagram for H on the b.m. diagram for a simple beam carrying the same loading, the difference between the two diagrams will give the b.m. in the

arch. The b.m. due to H is H_y , therefore, as H is constant throughout the arch, the shape of the b.m. diagram will be similar to that of the arch. In fact, if we choose an appropriate scale for our b.m. diagram, a scale diagram of the arch will also serve as the b.m. diagram due to H . Now, as there cannot be any b.m. at the crown pin, the two b.m. diagrams must coincide there.

Construction of b.m. diagram. Make a scale drawing of the arch. This is the b.m. diagram due to H , and its scale is such that the ordinate at the crown pin is equal to the sagging b.m. there, due to the vertical loading. The *free bending moment diagram* for a simple beam carrying the same loading as the arch can then be superimposed, as in Fig. 2.14b.

Linear Arch

The resultant thrust is not constant throughout an arch, but changes in magnitude and direction wherever it meets a load. Following the line of action of the resultant thrust through the arch, we get the *linear arch*. Its shape is, therefore, that of the link polygon for the given loading, which passes through all three pins of the three-pinned arch. Now, a link polygon for a given loading on a beam is similar in shape to the corresponding b.m. diagram (see Example 2.3), therefore the free b.m. diagram in Fig. 2.14b is also a scale drawing of the corresponding linear arch. Because the linear arch and the actual arch stand in the same relationship as the free b.m. diagram and the b.m. diagram due to horizontal thrust, any departure of the linear arch from the real arch indicates the presence of bending. On the other hand, if the linear arch and the actual arch are identical in shape the arch will be in pure compression, and there will be no bending anywhere. In practice, it is impossible to ensure that arch bridges are subjected to com-

pression only, because the live loads have to move across the bridge, and the linear arch will change shape as the load moves, whereas the shape of the real arch is fixed.

Beam with Varying Intensity of Loading

Example 2.8. A simple beam of length l carries a distributed load which varies uniformly in intensity from nothing at the left end to w_0 at the right-hand end. Derive mathematical expressions for the s.f. and b.m. at any point P , distance x from the left end and determine the position of maximum b.m. and its magnitude.

Figure 2.15a shows the intensity of loading. The intensity of loading at P , which we can call w , equals w_0x/l . The total load on the beam equals the average intensity of loading times the length of the beam $= w_0l/2$, and its centre of gravity will be at the centroid of the intensity of loading diagram, i.e. $2l/3$ from the left end. Hence, the reaction at A is $w_0l/6$ and the reaction at B is $w_0l/3$.

Referring to the free body diagram for the portion of the beam AP , Fig. 2.15b, we see that the s.f. at P is

$$R_A - \frac{wx}{2} = \frac{w_0l}{6} - \frac{w_0x^2}{2l}$$

This is the equation of a parabola, and is shown in Fig. 2.15c.

$$\text{The b.m. at } P = R_Ax - \frac{wx}{2} \cdot \frac{x}{3} = \frac{w_0lx}{6} - \frac{w_0x^3}{6l}$$

This is the equation of a cubic parabola, and is shown in Fig. 2.15d. The b.m. will be a maximum where the s.f. is zero. The value of x for which the s.f. vanishes is given by the equation

$$\frac{w_0l}{6} - \frac{w_0x^2}{2l} = 0, \text{ i.e. } x = \frac{l}{\sqrt{3}}$$

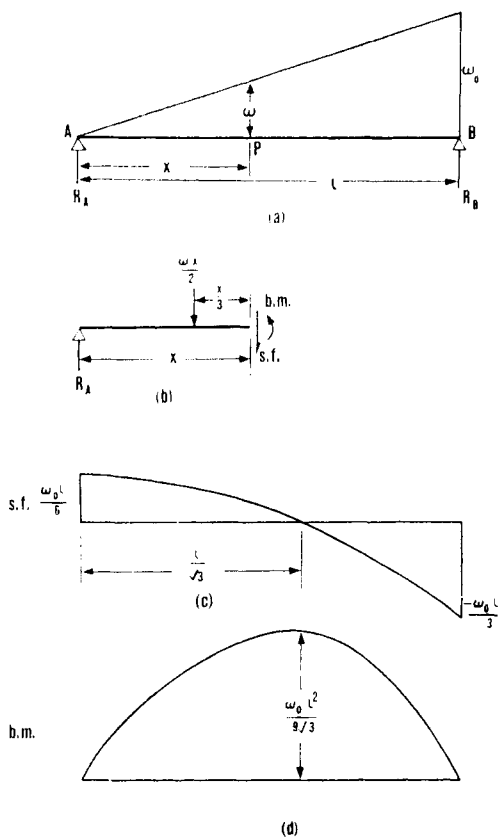


FIG. 2.15 Beam with varying intensity of loading

thus the maximum b.m. is given by substituting this value of x into the general expression for the b.m.; hence

$$\text{maximum b.m.} = \frac{w_0 l^2}{9\sqrt{3}} \text{ (sag)}$$

Exercise 2

1. Explain precisely what is meant by (a) the shearing force, (b) the bending moment, at a given section of a beam. How is each calculated from the loading on the beam?

A horizontal canopy over a hotel entrance projects 10 ft out from the building. Assuming that its weight of 3 tonf is uniformly distributed and that an additional load of $\frac{3}{4}$ tonf acts at the outer edge, calculate the maximum shearing force and bending moment in the cantilever, stating where they occur. Determine also the ratio of the s.f. and b.m. halfway along the cantilever to that at the fixed end.

Ans: 3.75 tonf, 22.5 tonf ft, $\frac{3}{5}$, $\frac{1}{3}$

2. Construct dimensioned sketches of the s.f. and b.m. diagrams for the loadings shown in Fig. 2.16 and read off the maximum b.m. in each case.

Ans: Max b.m. tonf ft, (a) 36, (b) 48, (c) 24, (d) 18, (e) 9 at B, (f) 8 (constant), (g) 12 immediately to left of point of application of couple, (h) $V_A = 4.16$ tonf, $V_B = 6.24$ tonf, max b.m. 25 tonf ft

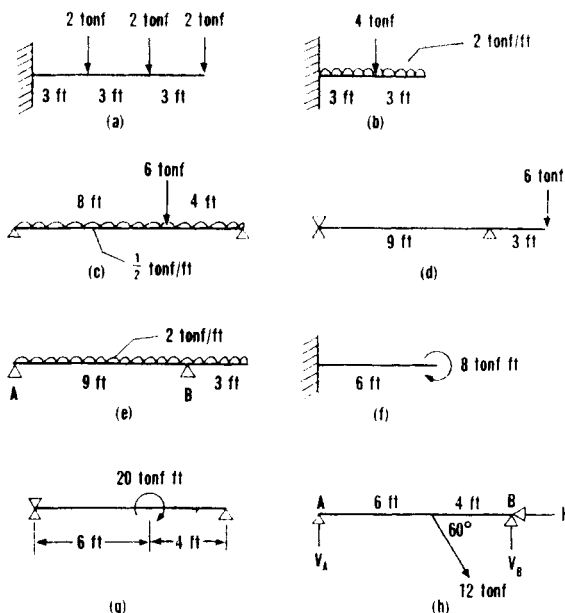


FIG. 2.16.

3. For the inclined lever in question 1 of Exercise 1 calculate the b.m. at *C*.

Ans: $8\sqrt{3}$ lbf ft

4. Calculate the b.m. in the unsymmetrical, three-pinned arch of question 6, Exercise 1, under the point of application of the load.

Ans: $300\sqrt{2}$ lbf ft

5. Sketch the b.m. diagram for the parabolic three-pinned arch of question 7, Exercise 1, and calculate the b.m. under the 10 tonf load.

Ans: 56 tonf ft (sag)

6. Determine the b.m. at the haunches *D* and *E* of the three-hinged portal frame of question 8, Exercise 1, due to the dead load plus the wind load.

Ans: 7,800 lbf ft; 10,670 lbf ft

7. Construct to scale the linear arch for the three-pinned, semi-circular arch of question 9, Exercise 1 and hence, or otherwise, determine the b.m. under the 2 tonf and 4 tonf loads.

Ans: 1.0 tonf ft (hog); 9.0 tonf ft (sag)

8. What are the values of the maximum bending moments in the bars *AB* and *BC* of the structure of question 11, Exercise 1.

Ans: 3 tonf ft; nil

9. Calculate the b.m. over the support *B* and under the point of application of the oblique load for the compound beam of question 12, Exercise 1.

Ans: 384 lbf ft (hog); 346 lbf ft (sag)

10. Make dimensioned sketches of the s.f. and b.m. diagrams for the beam *AB* in Fig. 1.21 loaded through a cranked bar, and read off the maximum b.m.

Ans: 16 tonf ft

11. Construct to scale the s.f. and b.m. diagrams for question 15, Exercise 1.

12. A board fence 6 ft high is supported by posts sunk into the ground every 10 ft. If the maximum wind pressure on the fence is 15 lbf/ft² calculate the maximum b.m. in each post.

Ans: 2700 lbf ft

13. Construct to scale the b.m. diagrams for the main beam AB and the auxiliary beam CD in Fig. 2.17 and write down the values of the maximum b.m. in each beam.

Ans: 48 tonf ft, 16 tonf ft

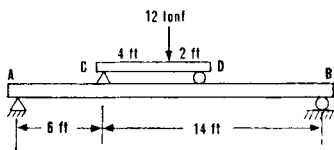


FIG. 2.17

14. A symmetrical parabolic arched rib, of 120 ft span and 25 ft central rise, is hinged at the crown and at each springing point. It carries concentrated vertical loads of 10, 20 and 20 tonf at horizontal distances from the left-hand hinge of 12, 30 and 48 ft respectively. Determine the magnitude of the horizontal thrust in the arch, and the magnitudes of the bending moments in the rib at the quarter-span points.

(I.C.E. Oct. 1952)

Ans: 33.6 tonf, 270 tonf ft (sagging) at 30 ft from L.H. hinge; 210 tonf ft (hogging) at 90 ft from the L.H. hinge

15. A beam AB , of total length 28 ft is simply supported at A and at a point C , 20 ft from A . The beam carries loads as shown in Fig. 2.18.

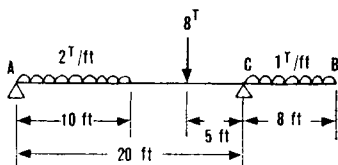


FIG. 2.18

Draw the s.f. and b.m. diagrams for this beam, marking on them all important values.

Determine (a) the position and magnitude of the maximum bending moment and (b) the position of the point of contraflexure.

(I.C.E. April 1952)

Ans: (a) 59.29 tonf ft at 7.7 ft from A

(b) 17.46 ft from A

16. Figure 2.19 gives the shearing forces diagram for a loaded beam simply supported at points R_1 and R_2 . The weight of the beam has been neglected.

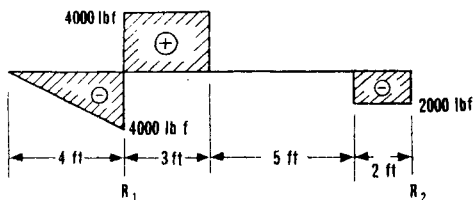


FIG. 2.19

- Draw the corresponding loading diagram for the beam, showing the magnitude and disposition of the loads.
- Sketch the bending moment diagram for the beam, giving the values of the bending moment at salient points.
- Determine the position of the point of contraflexure of the beam, with reference to R_1 .

(Eng. Inst. Oct. 1958)

Ans: (a) Loads of 1000 lbf/ft u.d.; 4000 lbf and 2000 lbf

(b) B.M. (left to right): at 0 ft: 0; 4 ft: 8000 lbf ft (hog); 7 ft and 12 ft: 4000 lbf ft (sag); 14 ft: 0

(c) 2 ft to right of R_1

CHAPTER 3

Bending Stresses in Beams

BEAMS are essentially structural members for carrying transverse loads; that is, loads acting perpendicularly to the beam, although they may sometimes be called upon to carry axial forces as well. A *simple beam* is one which is carried on a simple support at each end. Simple supports prevent deflection of the beam at the points where they are situated, but allow the beam freedom to rotate as it flexes. In other words, they provide a transverse reaction, but no couple, unlike the *built-in* or *encastré* support at the clamped end of a cantilever, which has to provide a fixing moment as well as a transverse reaction.

At every section of a loaded beam there is a bending moment due to the leverage of the external forces acting on the portion of the beam to one side of the section. This couple is opposed by the moment of resistance set up as a result of the curving of the beam as it deflects. Hence, when a beam made of a material for which stress is proportional to strain — steel is such a material, provided the yield stress is not exceeded — the beam bends so that the curvature at every section is proportional to the bending moment there. Curvature, remember, is the reciprocal of radius of curvature. Thus, if a uniform beam is subjected to a bending moment which is the same over its entire length, it will bend into an arc of a circle. For example, the portion *CD* of the beam in Fig. 3.1 will bend into an arc of a circle, since that part of it is subjected to a uniform bending moment Wa .

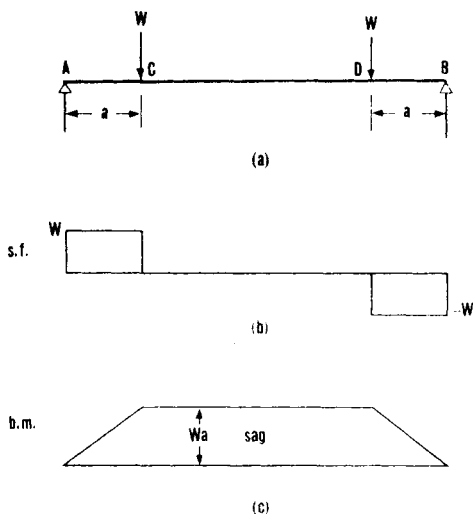
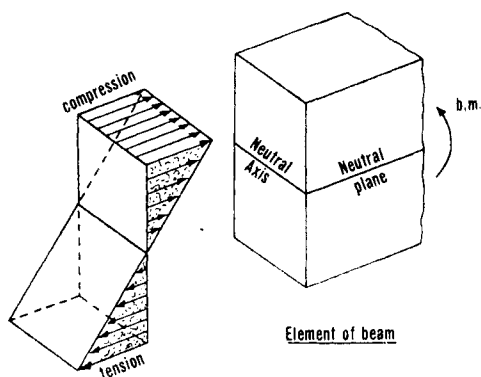


FIG. 3.1 Beam with uniform bending over central portion

Mechanism of Bending

As an aid to visualising the mechanism of bending, we can think of a beam as made up of a bundle of longitudinal fibres, (Fig. 2.4). There will be layer of unstrained fibres lying in the *neutral plane*. If, at a particular section, there is a sagging b.m., the fibres below the neutral plane will be stretched, and therefore in tension, whilst the fibres above the neutral plane will be compressed. The strain in a fibre, and hence the stress in it, is proportional to its distance from the neutral plane (Fig. 3.2). It is more common to refer to the *neutral axis* rather than the neutral plane which is the line formed by the intersection of the neutral plane with the plane of the cross-section.

The strain at any point is also proportional to the curvature of the beam at that section.



Stress distribution

FIG. 3.2 Stress distribution due to bending

Moment of Resistance

At every section of a loaded beam there is a bending moment due to the leverage of the forces acting on the portion of the beam to one side of the section being held in balance by the leverage of the forces acting on the other portion. The beam is prevented from breaking at the section by the internal forces carried by the stressed fibres. Fibres on opposite sides of the neutral axis carry opposing stresses, which increase in intensity from zero at the neutral axis to a maximum at the outer fibres, thus providing a couple or *moment of resistance* opposing the *attacking* bending moment. The position of the neutral axis is determined by the fact that the sum of the tensile forces in the stretched fibres to one side of the neutral axis must equal the sum of the compressive forces in the fibres on the other side of the neutral axis. It follows that the neutral axis passes through the centroid of the cross-section.

The simple theory of bending is developed in elementary text books of Applied Mechanics and gives us the important and widely used formulae

$$\frac{M}{I} = \frac{f}{y} = \frac{E}{R}$$

where M = moment of resistance of beam.

I = second moment of area of the cross-section of beam about the neutral axis.

f = fibre stress at a distance y from the neutral axis.

E = modulus of elasticity of the material of which beam is made.

R = radius of curvature to which beam is bent at the section considered.

The commonest source of error in the application of this pair of formulae is the use of wrong units. It is essential to use a consistent set of units, remembering, for example, that it is necessary to convert bending moments in tonf ft to lbf in if the stress and Young's modulus are in lbf/in²; in which case I , y and R should all be in inch units.

First Moments of Area

Figure 3.3 represents a plane figure of any shape. OX and OY are the reference axes. The total area A may be broken down into a very large number of tiny elements. Selecting one of them at random, its distance from the y axis will be x , and distance from the x axis will be y . Its area can be designated by δA (a little bit of A). Hence $A = \Sigma \delta A$, which is the mathematical shorthand for *the total area is the sum of all the little bits of area*. If the co-ordinates (position with reference

to the axes) of the centroid are $(\bar{x}, \bar{y})$ then the first moments of area of the figure about the x and y axes are, respectively

$$A\bar{y} = \Sigma y \delta A$$

and

$$A\bar{x} = \Sigma x \delta A$$

These two equations enable $\bar{x}$ and $\bar{y}$ to be determined, provided $\Sigma x \delta A$ and $\Sigma y \delta A$ can be evaluated.

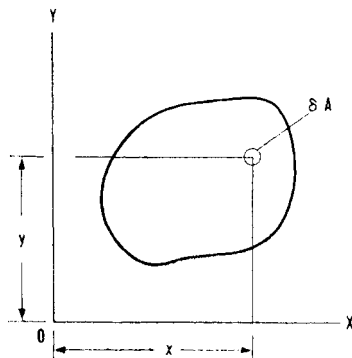


FIG. 3.3 Moments of area

Second Moments of Area

The second moment of area of the plane figure in Fig. 3.3, about the x axis is $\Sigma y^2 \delta A$. We give the symbol I to second moments of area, and add suffixes to indicate about which axis moments are taken, e.g. $I_{xx} = \Sigma y^2 \delta A$. The summation symbol Σ means that we add together the second moments of area for all the elements which go to make up the figure. It is often convenient to replace second moments of area by *radii of gyration*. The radius of gyration k is defined by the equation $I = Ak^2$, in other words, it is the distance at which the whole

area would have to be, to produce the same second moment of area as the figure itself.

There are two important theorems commonly used in calculating second moments of area.

(i) Parallel axis theorem

This theorem states that the second moment of area of a plane figure about any axis in its own plane is equal to the second moment of area of the figure about an axis through its centroid, parallel to the first axis, plus the product of the area of the figure and the square of the distance separating the two axes,

$$\text{i.e. } I_{xx} = I_{GG} + Ah^2$$

where GG is the axis through the centroid and h is the distance between it and the axis xx .

It follows therefore, that the second moment of area of a figure will be least when taken about an axis through its centroid.

(ii) Equimomental theorem or spot theorem

This theorem applies to the moments of area of a triangle about any axis in its own plane. If the triangle is replaced by a third of its area at the mid-point of each side, then the sum of the moments of area of the three portions or "spots" about the chosen axis is equal to the moment of area of the whole triangle about the same axis. The theorem is true for both first and second moments of area.

Principal Axes of Bending

Consider the second moment of area of any given cross-section, about an axis through its centroid. In general, if we alter the direction of the axis (keeping it in the plane of the cross-section) the value of the second moment will vary; rising to a maximum for one particular direction and falling gradually to a minimum for a direction at right angles. These two directions through the centroid are called the *principal axes* of the section. For beam sections with an axis of symmetry, which is the usual case, principal axes are easily found, since the axis of symmetry is one; the other one will be at right angles, through the centroid.

The simple theory of bending only applies to bending about a principal axis.

Addition of Second Moments of Area

Second moments of area can only be added together if they are about the same axis. If they are not all about the same axis, then the parallel axis theorem can be used to adjust the second moments, so that they are about a common axis, provided that the axes were all originally parallel. It is important, when applying the parallel axis theorem, to remember that the reference axis is *through the centroid*.

Holes, or parts missing from the cross-section, can be treated as negative areas, for the purposes of calculating moments of area.

Modulus of Section

The maximum moment of resistance of a beam depends upon two factors; the geometry of its cross-section, and the strength of the material of which it is made. The second factor is represented by the maximum permissible stress p and the

first factor, by the second moment of area of the beam divided by the distance of the outer fibres from the neutral axis. This ratio I/y_0 is given the symbol Z and called the modulus of the section. It follows from the bending formula that the maximum moment of resistance is pZ . If the neutral axis of the beam, about which bending is taking place, is not equidistant from the top and bottom fibres, there will be two values of Z . The one to use will be that which gives the lower moment of resistance when multiplied by the appropriate permissible stress, remembering that the permissible stress in compression may differ from that for tension, either because the material is weaker in tension than in compression, like cast iron; or because the stress in a slender compression flange has to be restricted to forestall the possibility of it buckling.

Materials for Beams

Beams can be made of solid timber, laminated timber, light alloy, steel, reinforced concrete, or pre-stressed concrete. Reinforced concrete beams consist of two materials with different elastic moduli, and pre-stressed concrete beams carry axial forces, so that the simple method of analysis used in this chapter is not sufficient to deal with them. Solid timber beams, which are always square or rectangular, are rather rare in this country in sizes larger than those used for house building, on account of the high cost of timber in relation to its strength. Large timber beams are mainly used in temporary work, where they can be salvaged and used again. Laminated timber uses the available wood from a tree far more economically, and produces a material of higher and more uniform strength, and beams made of laminated timber are becoming increasingly popular for medium span and even large span roofs. Beams of light alloys, such as those of aluminium, do not have

a general application in civil engineering construction on account of their high cost, but they may be used for special purposes where a saving of weight is particularly valuable, e.g. in bascule bridges. Most steel beams are rolled from structural mild steel, although steels of a higher tensile strength are being increasingly used, particularly for large span bridges.

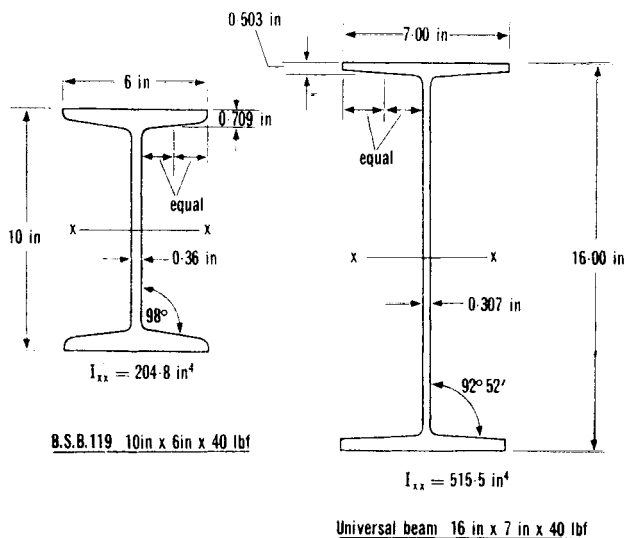


FIG. 3.4 Rolled steel beams

British steel manufacturers roll two basic types of I beam, the *British Standard Beam*, which ranges in size from 24 in $\times$ 7½ in down to 3 in $\times$ 1½ in, and more recently, the *Universal Beam*, which ranges in size from 36 in $\times$ 16½ in down to 8 in $\times$ 5½ in. Figure 3.4 shows the profiles of a 10 in $\times$ 6 in $\times$ 40 lbf B.S.B. and a 16 in $\times$ 7 in $\times$ 40 lbf Universal Beam. Both weigh the same per foot run, but the Universal Beam has a second moment of area about the neutral axis, which is two and a half times as great as that of the B.S.B.

Rolled steel beams can be strengthened by riveting or welding on extra flange plates. This is usually done only if there is insufficient headroom to use a deeper standard beam. Where beams bigger than the largest rolled section are required they can be fabricated by riveting or welding plates and angles to form a *plate web girder*.

Curtailment of Flanges

One of the advantages of built up rolled steel beams and plate girders is that the extra bending strength given by additional flange plates is only provided over that length where it is required. For example, if the b.m. diagram for a simple beam is as shown in Fig. 3.5, it will be necessary to reinforce

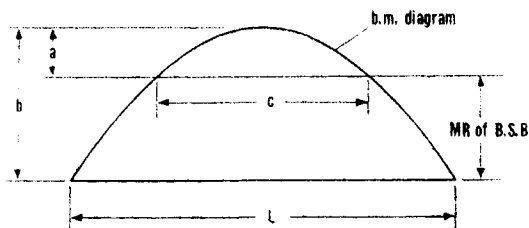


FIG. 3.5 Curtailment of the flange

the B.S.B. over the central length c , where the b.m. exceeds the maximum moment of resistance M_R of the plain beam. The additional flange area provided must be sufficient to ensure that the maximum moment of resistance of the reinforced beam is at least as great as the maximum bending moment encountered.

Moment of Resistance of a B.S.B.

Example 3.1. The following data are for an 8 in $\times$ 6 in $\times$ 35 lbf B.S.B. Calculate the approximate second moment of area about $x-x$ and the maximum moment of resistance of the beam. What is the greatest span over which the beam can simply support itself without the stress exceeding 8 tonf/in² in tension or compression?

Mean thickness of flanges	0.648 in
Thickness of web	0.35 in

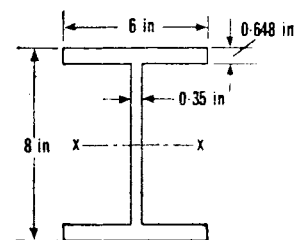


FIG. 3.6 Assumed shape of 8 in $\times$ 6 in B.S.B.

The approximate value of I_{xx} can be obtained by considering the flanges to be parallel, as in Fig. 3.6.

$$I_{xx} = \frac{6 \times 8^3}{12} - \frac{(6 - 0.35)(8 - 2 \times 0.648)^3}{12} = 115 \text{ in}^4$$

Compare this with the accurate value of 115.06 in⁴, from a *Steelmaker's Handbook* or *Engineer's Diary*.

The section modulus about $x-x$ will be $115 \div 4 = 28.7 \text{ in}^3$, and the maximum moment of resistance will be $28.7 \times 8 = 230 \text{ tonf in}$ or **43,000 lbf ft**.

The limiting span l will be given by equating the maximum b.m. to this moment of resistance.

$$\frac{35 \times l^2}{8} = 43,000$$

hence

$$l = 99.2 \text{ ft}$$

Compound Girder

Example 3.2 A compound girder spanning 25 ft is built up by riveting a 9 in $\times \frac{3}{4}$ in plate to each flange of a 10 in $\times$ 5 in $\times$ 30 lbf B.S.B. Calculate the maximum uniformly distributed load that the beam can carry in addition to its own weight if the stresses are not to exceed 10 tonf/in² in tension or compression. Determine the minimum length of the flange plates. Allow one $\frac{3}{4}$ in diameter rivet hole in each flange. The thickness of the flange of the B.S.B. is 0.552 in. $I_{xx} = 146.2 \text{ in}^4$.

$$M_R \text{ of B.S.B.} = \frac{146.2}{5} \times 10 = 292 \text{ tonf in} = 24.4 \text{ tonf ft}$$

I_{xx} of B.S.B. less $\frac{3}{4}$ in hole in each flange

$$146.2 - 2 \times \frac{3}{4} \times 0.552 \left(5 - \frac{0.552}{2} \right)^2 = 127.8 \text{ in}^4$$

(the I of the holes about their own centroids is negligible)

I_{xx} of compound girder

$$\begin{aligned} &= 127.8 + 2 \left[\frac{8\frac{1}{4} \times (\frac{3}{4})^3}{12} + 8\frac{1}{4} \times \frac{3}{4} \times 5.38^2 \right] \\ &= 127.8 + 2[0.3 + 179.2] = 487 \text{ in}^4 \end{aligned}$$

$$\begin{aligned} M_R \text{ of compound beam} &= \frac{487}{5.75} \times 10 = 846 \text{ tonf in} \\ &= 70.5 \text{ tonf ft} \end{aligned}$$

$$\text{Max. b.m.} \quad \frac{wl^2}{8} = \frac{25^2 w}{8} = 70.5 \text{ tonf ft}$$

Therefore $w = 0.904 \text{ tonf/ft} = 2030 \text{ lbf/ft}$,

so that the beam can carry $2030 - 30 = 2000 \text{ lbf/ft}$

The minimum length c of the extra flange plates can be obtained from the b.m. diagram and the moment of resistance of the beam (Fig. 3.5). The shape of the b.m. diagram for a uniformly distributed load is parabolic, and from the properties of a parabola we know that

$$\left(\frac{c}{l}\right)^2 = \frac{a}{b}$$

so
$$\left(\frac{c}{l}\right)^2 = \frac{70.5 - 24.4}{70.5}$$

therefore
$$c = 20.2 \text{ ft}$$

The minimum length of the flange plates would have to be slightly greater in practice, in order to allow extra length at each end, to accommodate sufficient rivets or welds to transfer enough force into the ends of the flange plates; so that, from the theoretical cut off onwards, the flange plates are able to carry their share of the bending moment.

Simplified Plate Girder Design

Example 3.3. A plate girder is required to resist a maximum b.m. of 300 tonf ft. If the web depth is to be 3 ft and the bending stresses are not to exceed 9 tonf/in², determine the minimum flange area required.

In this approximate method the web is considered to carry the whole of the shearing force, and the flanges, the whole of the bending moment. The tensile force in the bottom flange and the compressive force in the top flange together form a couple equal to, and opposing, the applied b.m. The flange force is equal to the area of the flange times the stress in it; and the lever arm of the couple is the distance apart of the flanges; which is approximately equal to the web depth. Hence

the required flange force is $300/3 = 100$ tonf. Therefore the area of each flange should be $100/9 = 11.11 \text{ in}^2$. In a welded design this might be provided by a single 12 in wide plate $\frac{15}{16}$ in thick, fillet welded to the web. In a riveted design it would be necessary to provide a pair of flange angles for each flange to connect the flange plates to the web. In which case, the sum of the net cross-sectional areas of the flange plates and the flange angles, after deducting the area of any rivet holes, should equal **11.11 in²**.

Example 3.4. A light bar rests on two knife edge supports placed 3 ft apart and overhangs each support by 1 ft. Loads of 15 lbf are placed at each end of the beam. The cross-section of the beam is an equilateral triangle of 1 in side, standing on its base. Calculate the maximum tensile and compressive stresses developed, the radius of curvature of the beam between the supports, and the central deflection. Young's modulus for the bar is 10^7 lbf/in^2 .

The s.f. and b.m. diagrams for the beam are shown in Fig. 3.7, from which it will be seen that the b.m. is constant between the supports *A* and *B* and is equal to 15 lbf ft or 180 lbf in.

The second moment of area of the cross-section can be calculated by the equimomental theorem. Referring to Fig. 3.7d, the neutral axis *x-x* passes through the centroid. The cross-sectional area $= \frac{1}{2} \times 0.866 = 0.433 \text{ in}^2$.

$$I_{xx} = \frac{0.433}{3} [0.288^2 + 2 \times (0.433 - 0.288)^2] = 0.0181 \text{ in}^4$$

Note in this case, since the cross-section is not symmetrical about the neutral axis, the value of the stresses in the top and bottom fibres will not be equal and it is probably better to avoid the use of section moduli in this problem. The tensile stress in the *top* fibres $= 180 \times 0.578/0.0181 = \mathbf{5750 \text{ lbf/in}^2}$

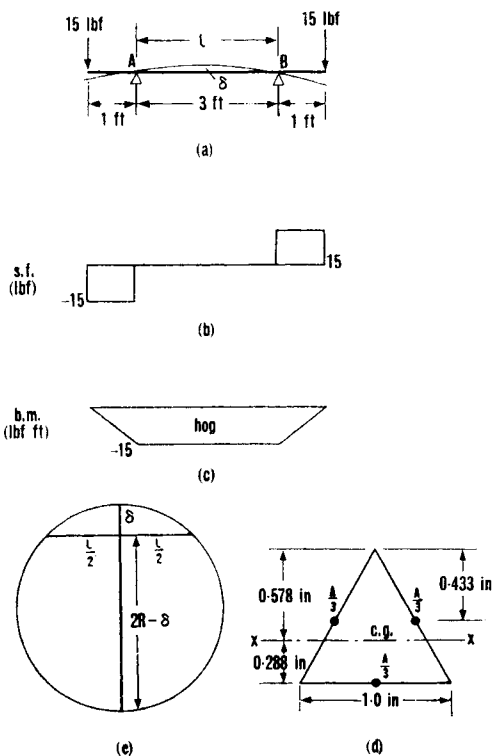


FIG. 3.7

The compressive stress in the *bottom* fibres

$$= 180 \times 0.288 / 0.0181 = 2870 \text{ lbf/in}^2$$

The beam will be bent to the arc of a circle of radius R between the supports, since the b.m. there is uniform. R is given by $M/I = E/R$, hence

$$R = EI/M = 10^7 \times \frac{0.0181}{180} = 1007 \text{ in}$$

By using the properties of intersecting chords of a circle, Fig. 3.7e, we see that

$$\left(\frac{l}{2}\right)^2 = (2R - \delta)\delta$$

or, since δ is very small compared with $2R$,

$$\delta = l^2/8R = 36^2/(8 \times 1007) = 0.161 \text{ in}$$

Exercise 3

1. Explain what is meant by the following terms: *neutral plane, neutral axis, fibre stress*.

A rolled steel Tee bar, 20 ft long, has a flange 4 in wide and $\frac{1}{2}$ in thick and is 6 in deep overall. The web is $\frac{3}{8}$ in thick. Determine the height of the neutral axis above the bottom of the flange.

If the bar is supported on knife edges at its ends, calculate the maximum tensile and compressive extreme fibre stresses in the bar due to its own weight. Steel weighs 490 lbf/ft³.

Ans: 1.77 in; 1010 lbf/in²; 2420 lbf/in²

2. Calculate the minimum diameter to which 16 S.W.G. steel strip (0.064 in) can be coiled if the stress in it is not to exceed 13 tonf/in. The modulus of elasticity of steel is 13000 tonf/in².

Ans: 64 in

3. Explain the difference between the moment of resistance exerted by a beam at a given section and the bending moment applied there.

A timber joist 3 in wide by 6 in deep is simply supported on a span of 12 ft and carries a load of 240 lbf uniformly distributed over the span together with a load of 240 lbf placed 4 ft from one support. Draw the s.f. diagram and hence determine the maximum bending stress in the joist.

Ans: 640 lbf/in²

4. A timber cantilever projects 6 ft and carries a 1200 lbf load at its free end and a uniformly distributed load of 200 lbf/ft. The beam is 6 in wide throughout and varies uniformly in depth from 6 in at the free to 12 in at the built in end. Construct the s.f. and b.m. diagrams for the

cantilever and calculate the maximum fibre stress at the support and at a section half-way along the cantilever. Neglect the deadweight of the cantilever.

Ans: 900 lbf/in²; 666 lbf/in²

5. A casting has the trapezoidal cross-section shown in Fig. 3.8 and is 12 ft long. Calculate the maximum tensile stress developed due to its

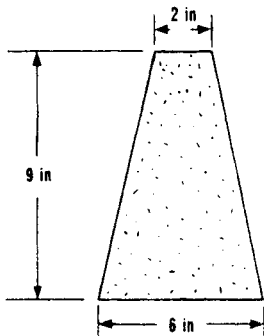


FIG. 3.8

own weight, if it is picked up by a crane (with the 2 in side uppermost) (a) at its ends, (b) centrally. In both cases make dimensioned sketches of the s.f. and b.m. diagrams. Cast iron weighs 500 lbf/ft³.

Ans: 455 lbf/in²; 635 lbf/in²

6. Figure 3.9 shows the cross-section of a cast iron girder weighing 100 lbf/ft run. If the tensile and compressive stresses are not to exceed

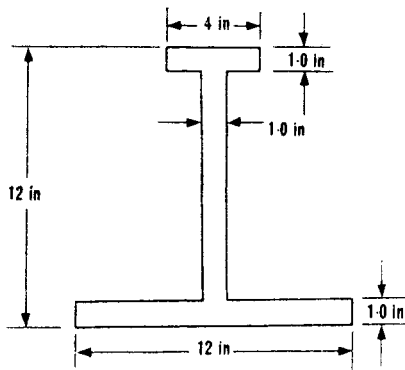


FIG. 3.9

2 tonf/in² and 6 tonf/in² respectively, determine the maximum concentrated load the girder can carry at the centre of a simply supported span of 30 ft.

Ans: 4220 lbf

7. A beam 12 ft long, of rectangular cross-section, is simply supported at its left-hand end and at 2 ft from the right-hand end. It carries a uniformly distributed load of 0.5 tonf/ft over its full length together with a concentrated load of 6 tonf mid-way between the supports. Construct the s.f. diagram and hence determine the maximum b.m. in the beam. If the beam is twice as deep as it is wide determine its depth if the maximum stress in it is 10 tonf/in². Calculate also the minimum radius of curvature of the beam if the modulus of elasticity of the material is 13000 tonf/in².

Ans: 20.8 tonf ft; 6.7 in; 363 ft

8. An I beam is 6 in wide and 10 in deep and has flanges 1 in thick and a web $\frac{1}{2}$ in thick. It carries a central load of 14 tonf on a simply supported span of 20 ft. Calculate the length over which the beam must be reinforced with additional flange plates, if the stresses are not to exceed 10 tonf/in². Neglect the deadweight of the beam.

Ans: 8.8 ft

9. Calculate the maximum central load that a 10 in $\times$ 6 in B.S.B. can carry on a simple span of 30 ft, when reinforced with an 8 in $\times$ $\frac{1}{2}$ in plate on each flange, if the maximum stress is not to exceed 8 tonf/in².

Determine also the minimum length of the flange plates required. Details of B.S.B: Flange thickness 0.709 in. Web thickness 0.360 in. Neglect the effect of rivet holes and self weight.

Ans: 6.85 tonf; 14.1 ft

10. A compound girder consisting of an 8 in $\times$ 6 in $\times$ 35 lbf B.S.B. with a 10 in $\times$ $\frac{1}{2}$ in plate riveted to each flange spans a distance of 14 ft and carries a uniformly distributed load of 28 tonf, including its own weight. Allowing one $\frac{3}{4}$ in diameter rivet hole in each flange, calculate the maximum stress in the girder at midspan.

If the working stress is limited to 10 tonf/in² calculate also the minimum length of the reinforcing flange plates.

I_{xx} of 8 in $\times$ 6 in $\times$ 35 lbf B.S.B. is 115 in⁴.

Flange thickness of B.S.B. is 0.65 in.

Ans: 9.25 tonf/in²; 10.9 ft

B/B.T.S. 7

11. A rolled steel joist has equal flanges, each $6 \text{ in} \times \frac{7}{8} \text{ in}$ and a web $12\frac{1}{4} \text{ in} \times \frac{1}{2} \text{ in}$. Find the total uniformly distributed load, inclusive of its own weight, which the girder can carry over a simply supported span of 24 ft, if flange plates $10 \text{ in} \times 1 \text{ in}$ are welded on each flange over the central portion. The maximum bending stress is not to exceed 9 tonf/in^2 . Find also the minimum central length over which the additional plates are required.

Ans: 51.8 tonf; 19.1 ft

12. A single track, plate girder bridge consisting of two main girders has a span of 60 ft, a web depth of 3 ft and weighs 45 tonf. The weight of a train may be taken as 1.25 tonf/ft . The safe working stresses for the steel are 7.5 tonf/in^2 in tension or compression and 5 tonf/in^2 in shear. Assuming that the web resists all the shear and the flanges all the bending, and that $4 \text{ in} \times 4 \text{ in} \times \frac{3}{8} \text{ in}$ flange angles are used, calculate the following:

(a) web thickness

(b) number, width and thickness of flange plates.

Neglect the effect of rivet holes.

Ans: 0.167 in; say $2N^\circ$. $\frac{3}{4} \text{ in} \times 10 \text{ in}$ plates

13. The cross-section of a gantry girder is as shown in Fig. 3.10. Determine the maximum tensile and compressive bending stresses in the girder

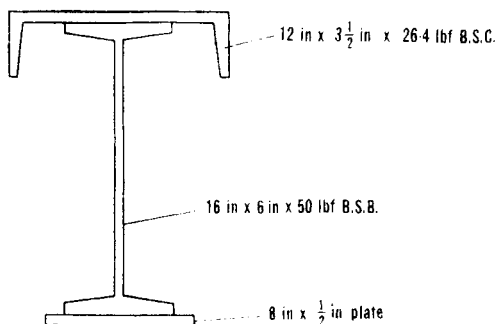


FIG. 3.10

when it is subjected to a vertical bending moment of 105 tonf ft . Details of the sections should be obtained from standard section tables. Neglect the effect of rivet holes.

(I.C.E. Apr. 1951)

Ans: 7.10 tonf/in^2 tension; 9.07 tonf/in^2 compression

14. The built-up section shown in Fig. 3.11 is composed of two 8 in $\times$ 3 in British Standard Channel sections and a 12 in $\times$ $\frac{1}{2}$ in plate. Ignoring the effect of rivet holes, calculate the second moment of area of the section about its neutral axis, and also the moment of resistance about this axis when the maximum stress in the metal is 9 tonf/in².

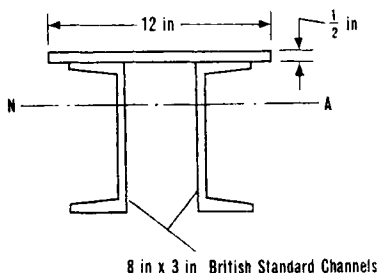


FIG. 3.11

For an 8 in $\times$ 3 in channel section:

$$I_{xx} = 51 \text{ in}^4$$

$$\text{Area} = 5.49 \text{ in}^2$$

Ans: 282 tonf in

(Eng. Inst. Apr. 1959)

15. A compound beam consists of two 14 in $\times$ 6 in $\times$ 46 lbf B.S.B. side by side with continuous 14 in $\times$ 1 in plates riveted across both flanges with two rows of $\frac{3}{4}$ in diameter rivets ($\frac{13}{16}$ in dia. holes) in each flange plate. The total self weight of the beam is 189.9 lbf/ft. This beam is simply supported on a span of 22 ft and carries a uniformly distributed load of 1.75 tonf/ft over the whole span.

Determine (a) the section modulus of the compound beam,

(b) the maximum additional concentrated load which may be applied at mid-span if the bending stress is limited to 10 tonf/in².

For one 14 in $\times$ 6 in B.S.B. $I_{xx} = 442.57 \text{ in}^4$, and the mean flange thickness 0.698 in.

(I.C.E. Apr. 1952)

Ans: (a) 272.5 in³; (b) 21.1 tonf

16. A timber floor is carried on 8 in $\times$ 2 in joists at 15 in centres which span 12 ft between the centres of supports. Calculate the maximum load

per square foot (inclusive of dead load) that the floor can carry if the timber stress is not to exceed 1000 lbf/in^2 .

Ans: 79 lbf/ft^2

17. A timber floor carries an inclusive load of 60 lbf/ft^2 . The joists are 2 in wide by 6 in deep and span 10 ft. Calculate the required spacing of the joists if the timber stress is not to exceed 1000 lbf/in^2 .

Ans: 16 in

18. Figure 3.12 is a plan view of a floor supported by a main beam and 6 in $\times$ 2 in joists spanning between the walls and the beam. Calculate the maximum inclusive load in lbf/ft^2 that the floor can carry if the stress

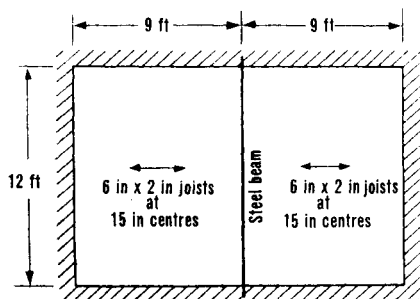


FIG. 3.12

in the timber is not to exceed 1000 lbf/in^2 . Determine also the minimum section modulus required for the main beam if the stress in it is not to exceed 10 tonf/in^2 . Neglect the weight of the main beam itself.

Ans: 99 lbf/ft^2 ; 8.56 in^3

19. Design a see-saw for a children's playground using timber with a tensile strength of 2000 lbf/in^2 . State what assumptions you make in arriving at the principal dimensions.

CHAPTER 4

Complex Stress

WHEN a body is subjected to external forces, internal forces are set up its various parts, as it were linking up the external forces and resulting in the material of the body being stressed. Now stress is directional, since it is the force transmitted per unit area; therefore to specify a stress, it is necessary to state its intensity (on a plane perpendicular to the stress) and direction; or, as is more usual, give the components of the stress perpendicular and tangential to a chosen plane. The perpendicular component is called the normal stress, and the tangential component, the shear stress. Hence, before a stress can be given a value, it is essential to specify the plane upon which it acts. In the case of simple tension or compression, where the direction of the stress is immediately obvious, the plane on which the stress acts is assumed to be at right angles to the stress.

The need to specify the plane on which a stress acts is emphasised by realising that the resultant stress acting on a plane varies as the inclination of the plane is altered; in much the same way as the intensity of sunlight falling on the earth varies with the altitude of the sun above the horizon, even though the radiant energy remains the same. Figure 4.1a shows a rectangular bar of cross-sectional area a subjected to tension by a pair of forces P - P . The tension in the bar is P/a (acting on a plane at right angles to P). Supposing, however, if instead of considering the stress on a cross-section, we take an oblique

section AB (Fig. 4.1b) inclined at an angle θ to the cross-section, the internal force transmitted across AB is still P , thus r the resultant stress on AB is given by $r = P/a \sec \theta = P \cos \theta / a$, and varies with θ . The direction of the resultant stress is

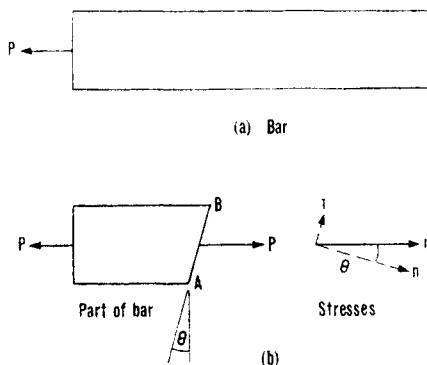


FIG. 4.1 Dependence of stress components upon the choice of reference plane

parallel to the internal force P . As θ varies r rises to a maximum value of P/a , when $\theta = 0$. The normal and tangential components of r are obtained by resolving r . The normal stress

$$n = r \cos \theta = \frac{P(1 + \cos 2\theta)}{2a}$$

and the shear stress

$$t = r \sin \theta = \frac{P \sin 2\theta}{2a}$$

It will be noted that when $\theta = 0$ there is no shear stress and the normal stress is P/a and is a maximum.

Frequently however, the state of stress is not uniaxial or *simple*, but *complex*. Consider, for example, the square plate, shown in Fig. 4.2, subjected to a tension p_x on one pair

of opposite edges and a tension p_y on the other pair of edges. Such a state of stress is called *biaxial tension*. It occurs, for example, in the stretched rubber of an inflated balloon, or in the shell of a boiler under pressure. The effect of curvature

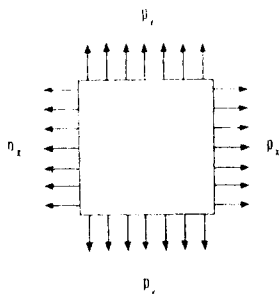


FIG. 4.2 Biaxial tension

is negligible provided the shell is relatively thin, since by choosing a sufficiently small element it can be made to approximate to a plane as closely as we like.

Complementary Shear Stress

The state of *pure shear* arises when a rectangular plate is subjected to shear stresses acting on its edges. It is a simple matter to prove that the intensity of shear stress must be the same on all four edges, and that at each corner the arrows indicating the sense of the shear stress must either both point towards each other or away from each other, as in Fig. 4.3. In other words $q_{xy} = q_{yx}$. This is universally true, even when normal stresses are acting as well. *Wherever a shear stress acts on a plane there will be a complementary shear stress of equal intensity acting on an adjacent face at right angles.* If the stress is varying from point to point, it will of course be necessary to keep the two planes very small.

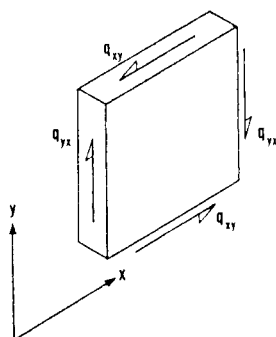


FIG. 4.3 Complementary shear stresses

The General Case of Plane Stress

A lamina, loaded by forces in its own plane, will be in a state of plane stress. The stress at any point P can be completely defined by isolating a square element surrounding the point, whose orientation is arbitrarily chosen. There will be a normal stress on each edge, arranged in equal and opposite pairs on opposing faces, and complementary shear stresses on all four edges, as shown in Fig. 4.4, so that, having chosen the orien-

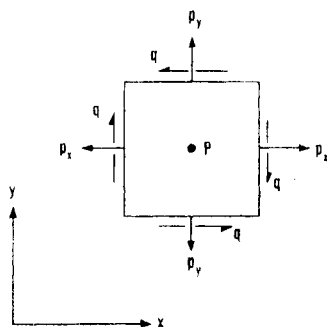


FIG. 4.4 General case of plane stress

tation of the element, the state of stress at P is designated by three stresses p_x , p_y and q . As the orientation of the element is varied, i.e. the inclination of the x and y axes, so the values of p_x , p_y and q will alter, although the state of stress at P does not vary, of course; but, for one particular orientation p_x will be a maximum, whilst p_y will be a minimum, and q will vanish. These particular values of p_x and p_y are called the *principal stresses*, and the planes on which they act, *principal planes*. Hence, any state of plane stress can be represented by a pair of principal stresses at right angles to each other. Each may be tensile or compressive, depending upon the state of stress. It is probably easier, when visualising a state of stress, to think of it primarily as the principal stresses, and then the normal and shear stresses, on planes other than principal planes, are merely the appropriate components of the principal stresses; although it is important to remember that frequently the direction of the principal stresses is not known in the first instance.

Simple stress then becomes a special case of plane stress, where one of the principal stresses is zero. *Pure shear* is equivalent to a pair of principal stresses of equal intensity, one tensile and the other compressive. The magnitude of the principal stresses is equal to that of the original shear stress, and their directions are inclined at 45° to the shear stresses, as shown in Fig. 4.5. In fact it is always true, in plane stress, that

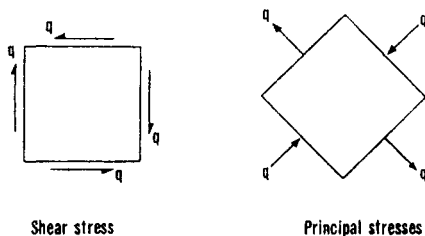


FIG. 4.5 Pure shear

the *maximum shear stress occurs on planes inclined at 45° to the principal planes.*

The most complex state of stress involves three dimensions, but it can always be reduced to three mutually perpendicular principal stresses. The simplest instance of such a state of stress is *hydrostatic pressure*, where all three principal stresses are equal and compressive, so that no shear stresses occur, whatever plane is chosen.

Distribution of Shear Stress across a Solid Rectangular Beam

The shearing force on a cross-section of any beam is distributed as a shear stress which varies from zero at the outer fibres to a maximum at the neutral axis. In beams of rectangular cross-section the distribution is parabolic, thus the maximum shear stress at the neutral axis is one and a half times the mean shear stress obtained by dividing the shearing force by the cross-sectional area, see Fig. 4.6.

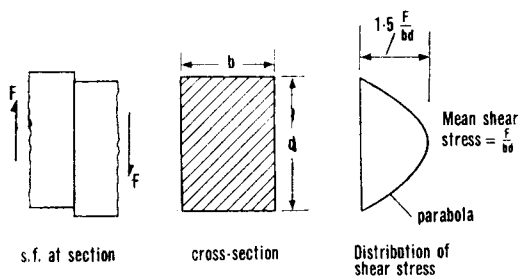


FIG. 4.6 Distribution of shear stress for a rectangular beam

It is interesting to compare the distribution of shear stress with the distribution of fibre stress due to bending. The fibre stress, or bending stress is a normal stress varying from zero

at the neutral axis to a maximum at the outer fibres. Thus, at the neutral axis, where the shear stress is a maximum, the bending stress is zero, and at the outer fibres, where the bending stress is a maximum, the shear stress is zero. Another point worth noting, is that, for a simple beam carrying a uniform load, the shearing force is zero at mid-span, where the bending moment is a maximum, and at the supports, where the shearing force is a maximum, the bending moment is zero.

Exercise 4

1. Contrast and compare the following pairs of stresses;
 - (a) direct stress and shear stress,
 - (b) direct stress and bending stress,
 - (c) shear stress and complementary shear stress.
2. Explain what is meant by the principal stresses at a point in a material subjected to plane stress.
3. For a material subjected to plane stress, explain the relationship between an arbitrarily selected cutting plane, principal planes, and planes on which the shear stress is a maximum.
4. A Universal Beam is simply supported at its ends and carries a uniformly distributed load. Describe the state of stress, giving the directions of the principal stresses, at the following points:
 - (a) on the neutral axis at mid-span,
 - (b) on the bottom flange at mid-span,
 - (c) on the neutral axis at the supports,
 - (d) on the top flange, above the supports.

CHAPTER 5

Statically Determinate Plane Trusses

THE structures we are going to deal with in this chapter are plane, triangulated, statically determinate frameworks, such as are used for roof trusses and bridge girders. The chief assumption which has to be made, in order to analyse such frames, i.e. determine the forces in the bars, is that the trusses are made up of separate bars which are connected together at their ends by frictionless pins. This seems a gross oversimplification, when we remember that the boom members running along the top and bottom of a girder are probably continuous at the joints, and that the intersecting members are riveted or welded to the members they join, either directly or by gusset plates. Fortunately, the constraint due to joint rigidity does not greatly modify the forces transmitted by the bars (although it does introduce bending), so that the assumption of pinned joints allows the forces in the bars to be determined with reasonable accuracy. Without this assumption it would be impossible to analyse a truss purely by statics.

The simplest possible plane frame is one consisting of three bars joined together to form a triangle. We can build upon this triangle to form a truss. Each additional joint will require two extra bars; thus the total number of bars in such a frame will be three less than twice the number of joints. A truss built up in this way will be self contained, in the sense that it can preserve its *shape* without independent support. In order to preserve its *position* it will require three *constraints* or reac-

tions which are not all parallel or concurrent. They are usually provided by a pin at one support (providing a vertical and horizontal reaction) and a roller at a second support (providing just a vertical reaction), as in Fig. 5.1a. A glance at Fig. 5.1d will show why three parallel constraints are unsuitable, since the frame is free to move at right angles to the constraining bars, similarly, Fig. 5.1e shows that three concurrent con-

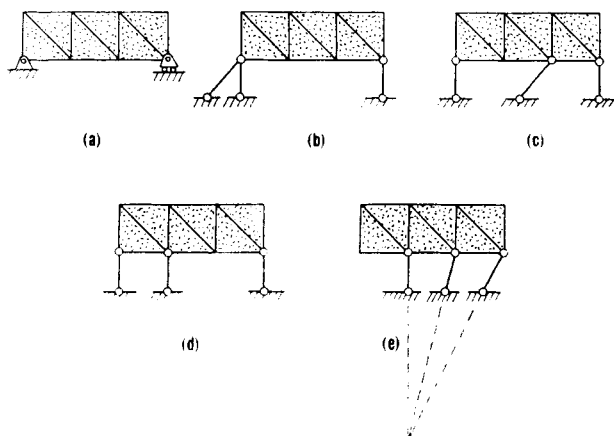


FIG. 5.1 Satisfactory and unsatisfactory forms of constraint

straining forces allow (limited) rotation about their point of intersection. The constraints in Fig. 5.1b and c are entirely satisfactory, and are statically determinate. The two unsatisfactory arrangements of constraining bars are, in fact, statically indeterminate, although they contain the right number of constraints.

Perfect Frames

A truss which has just the right number and arrangement of bars to provide a rigid structure, that can resist any form of loading (in its own plane, if it is a plane frame) is called a *just*

stiff or *perfect* frame; fewer bars would result in a truss that is *deficient* and almost certain to collapse; on the other hand, extra or redundant bars will produce a truss that is *hyperstatic*, and which cannot be analysed by statics alone. Figure 5.2a is an example of a perfect frame; Fig. 5.2b is deficient; Fig. 5.2c

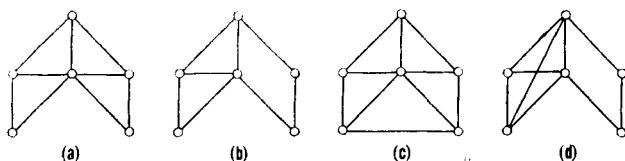


FIG. 5.2 Perfect, deficient and hyperstatic frames

is hyperstatic, since it has a redundant bar; and Fig. 5.2d, although it has the right number of bars, is not a perfect frame, since the arrangement of the bars is unsatisfactory. The left hand part is hyperstatic, whilst the right hand part is deficient.

A plane truss which is built up, joint by joint, from a triangle, by adding two bars at a time, will produce a perfect frame. Such a truss is called a *simple truss*.

Methods of Analysis

All methods of determining the internal forces in the bars of a truss depend upon applying the conditions of equilibrium. Having determined the support reactions, there are two basic ways in which this is done, either by considering the equilibrium of each joint in turn, i.e. the *method of joints*, or by imagining the structure to be cut in two and considering the equilibrium of one portion, i.e. the *method of sections*.

Method of joints

Each joint is in equilibrium under the action of the internal and external forces acting on it. The internal forces are the forces in the bars and the external forces are loads or reactions acting on the joint; together they form a concurrent, coplanar system of forces in equilibrium. Provided the forces in not more than two of the bars are unknown, they can be determined, either by resolving the forces acting on the joint along two different directions, so obtaining a pair of simultaneous equations, enabling the two unknowns to be solved; or by constructing a vector diagram for the forces acting on the joint. As the *directions* of the the forces are known and only the magnitudes of two of the forces are unknown it is always possible to do this. It is best to arrange for the vectors of the two unknown forces to be adjacent. In analysing a complete truss by the method of joints, one starts from a joint where only two bars meet, usually a point of support, and proceeds to work from joint to joint until the forces in all the bars are determined. One can start from more than one joint, if need be, and never meet a joint which is impossible to analyse, provided it is a *simple* truss. The usual practice, when using the graphical method is to construct a combined *force diagram* instead of drawing separate polygons of force for each joint.

Force Diagrams

Bow's notation is indispensable for designating the external forces and the internal forces, of bars. It is essential to remember always to work one way around the joints; the accepted direction is clockwise. The *spaces* between forces (external and internal) are given capital letters. The vectors representing the forces are designated by the corresponding lower case let-

ters; the sense of a force being given by the order of the pair of letters which denote it.

For every bar and external force there will be a corresponding vector parallel to it in the force diagram. Hence, force diagrams are also called *reciprocal figures*. First of all it is best to determine the support reactions, by the methods of Chapter 1, and construct a polygon for the external forces alone, then hang the rest of the force diagram, for the internal forces, on it. This usually makes it easier to position the force diagram and decide on a scale which is neither too small, nor so big that the diagram runs off the edge of the paper.

The force diagram is built up by adding on pairs of intersecting lines parallel to intersecting bars of the truss, proceeding from joint to joint in such a way that not more than two fresh bars are encountered at each joint. It is important to letter the vectors as the diagram grows. The internal forces in the bars are obtained by scaling off the appropriate vectors.

The *sense* of the forces is often immediately apparent. For example, the lower boom members of a simply supported truss under gravity loads are in tension, whilst the top boom members are in compression. Where it is not obvious whether a bar is in tension or compression, study a joint at one end of the bar. It will be denoted by the pair of letters to either side of the bar, taken in clockwise order round the joint. Note the direction of the corresponding vector in the force diagram; this is the direction of the force exerted by the bar on that joint. If the bar pulls on the joint it must be in tension, and is therefore a *tie*; if the bar pushes on the joint it must be in compression, and is therefore a *strut*.

Graphical Analysis of a Roof Truss

Example 5.1. Determine graphically the support reactions and the forces in the bars of the simple roof truss shown in Fig. 5.3a.

As a graphical construction is called for, we make use of a link polygon to determine the support reactions. The right-

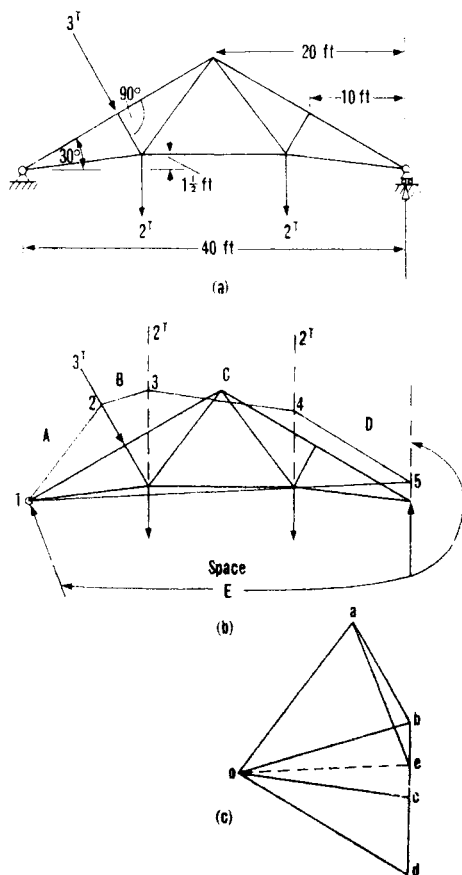


FIG. 5.3

hand reaction due to the roller must be vertical, but the direction of the total reaction at the hinge is unknown. The sequence of the operations is as follows:

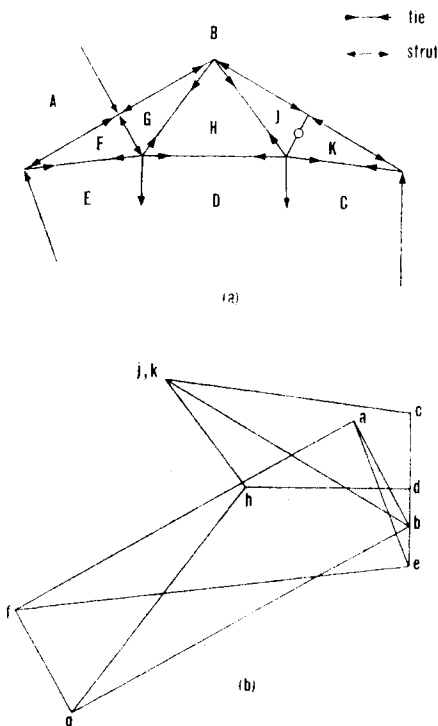


FIG. 5.4

(a) Construct the space diagram to scale, continuing the lines of action of the two forces on the lower beam upwards, so all the loads are effectively transferred to the top of the truss, and letter the spaces for the external forces according to Bow's notation — there is no need to letter the bars at this stage — Fig. 5.3b.

(b) Construct the load line $abcd$ in the vector diagram, Fig. 5.3c.

(c) Choose a pole o , and draw the rays oa , ob , oc and od .

(d) Starting from the hinge at the left support (because this is the only point which we know lies on the line of action of the reaction there), draw 12 parallel to oa across space A ; 23 parallel to ob across space B ; and so on, finishing at 5 on the line of action of the right-hand reaction.

(e) Join 15 and draw a ray oe parallel to it in Fig. 5.3c. Then de is the vector representing the reaction at the roller and ea is the vector for the inclined reaction at the hinge. Scaling de and ea we obtain **2.86 tonf** and **4.02 tonf** respectively.

(f) A fresh space diagram, Fig. 5.4a, has been drawn for determining the internal forces in the truss, as it is necessary to re-letter the external forces, and a separate drawing avoids confusion.

(g) The vector diagram $abcdea$ for the external forces is then redrawn in Fig. 5.4b, and the vectors for the internal forces are then built on, two intersecting lines at a time, starting with af and ef . The forces in the bars can then be scaled off, for example the tension in HD is **4.3 tonf**. The bars in the space diagram are then marked with arrowheads to indicate whether they are struts or ties.

Method of sections

The method of sections is particularly suitable for determining the forces in selected bars of a truss. The procedure is to divide the structure into two completely separate parts, by a plane cutting through the bar to be analysed. Either one of the parts is then treated as a *free body* in equilibrium, under the action of the loads and support reactions acting on it, together with the forces replacing the severed internal forces. Provided not more than three of the bars of the truss have been cut, in

order to part the truss in two, the forces in the cut bars can be determined, since it is possible to write down three independent equations of equilibrium for the chosen portion of truss, involving the unknown forces in the cut bars. The usual practice, where three bars have been cut, is to take moments about the intersection of the lines of action of two of the unknown forces, to solve for the third force. If two of the bars are parallel, the force in the other bar is most conveniently found by resolving perpendicularly to the parallel bars.

If one is uncertain whether a particular bar is a strut or a tie, assume it to be in tension. If subsequently the answer comes out negative, the bar is in compression.

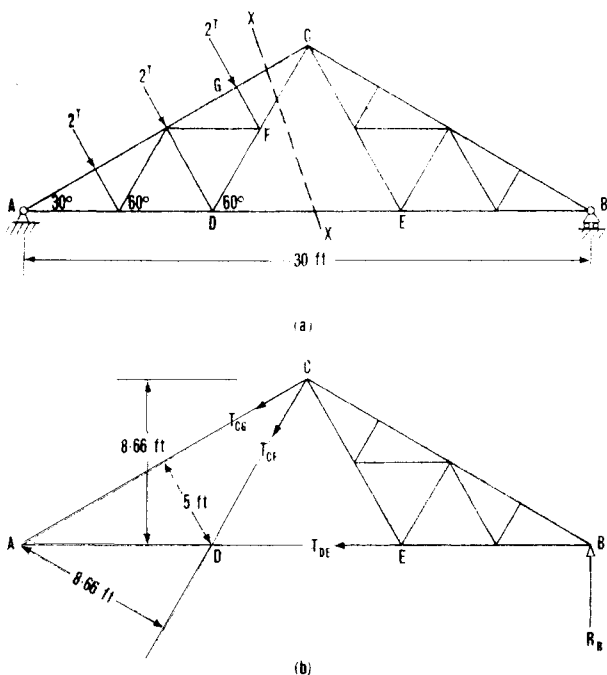


FIG. 5.5

Example 5.2. Determine the forces in the bars DE , CF and CG of the roof truss in Fig. 5.5a.

By making the cut XX the three above mentioned bars are all severed and the truss divided into two parts. Figure 5.5b is the free body diagram for the right-hand portion. The reaction R_B at the roller is vertical and its magnitude is determined by taking moments, for the whole truss, about the hinge A .

$$30R_B = 2 \times 5 \cos 30^\circ + 2 \times 10 \cos 30^\circ + 2 \times 15 \cos 30^\circ$$

$$R_B = 1.73 \text{ tonf}$$

Assume that the forces in the cut bars are all tensile and designate them by T_{DE} , etc. Taking moments about C eliminates T_{CG} and T_{CF} and gives $8.66 T_{DE} = 15 R_B$

hence $T_{DE} = 3.00 \text{ tonf tension}$

Taking moments about A eliminates T_{DE} and T_{CG} and gives

$$8.66 T_{CF} = 30 R_B$$

hence $T_{CF} = 6.00 \text{ tonf tension}$

Taking moments about D eliminates T_{DE} and T_{CF} and gives

$$5 T_{CG} + 20 R_B = 0$$

hence $T_{CG} = -6.94 \text{ tonf} = 6.94 \text{ tonf compression}$

Note that this truss is not a simple truss; it cannot be built up two bars at a time, from a triangle; and hence special methods are required when analysing it graphically. It will be noticed that the truss is really made up of two simple trusses, ACD and BCE , connected together by the pin at C and the bar DE . It is therefore a *compound truss*.

Solution by s.f. and b.m. diagrams

This is a method for determining the forces in the bars of parallel boom girders, based on the method of sections. It is

particularly suitable where a truss has to be completely analysed.

The s.f. diagram for a truss is a series of horizontal steps, since the loading is always applied at the joints or assumed to be concentrated at the joints, in the case of the deadweight of the members. The shearing force in each bay of the truss is carried by the diagonal member in that bay, and so *the vertical component of the force in a diagonal bar is equal to the s.f. in the bay.*

When dealing with an N girder, as in the following example, the force in a vertical bar is equal to the shearing force in the adjoining bay. To enable us to determine which of the two adjoining bays to use, we note which end of the vertical member does not carry an external load, the bay crossed by the diagonal bar joined to this end is the one required. Alternatively, you can work from the end-verticals towards the middle, in which case it is quite obvious which bay to take in each case.

The b.m. diagram for a truss is polygonal, since the loading is concentrated at the joints. The bending moment at every section is resisted by forces in the boom members, so that to determine the force in a boom member we divide the b.m. at the appropriate section by the depth of the truss. Each boom member forms the “base” of a triangle (although the triangles will be upside-down for the top boom) and the b.m. opposite the apex of this triangle is the appropriate value to use.

Example 5.3. Construct the s.f. and b.m. diagrams for the truss in Fig. 5.6a and hence determine the forces in the bars.

The support reactions are calculated in the usual way, and the s.f. and b.m. diagrams constructed — Figs. 5.6b and c.

The forces in the web members are determined by reference to the s.f. diagram, e.g. the force in AP is equal to the s.f. in the first bay and is therefore 10 tonf, and compressive, since it has to resist the push of the 10 tonf reaction at A . The force

in BP must be such that its vertical component is equal to the s.f. in the bay which it crosses. Hence $T_{BP} \sin \theta = 10$ tonf, so $T_{BP} = 10 \operatorname{cosec} \theta = 12.8$ tonf tension. In other words,

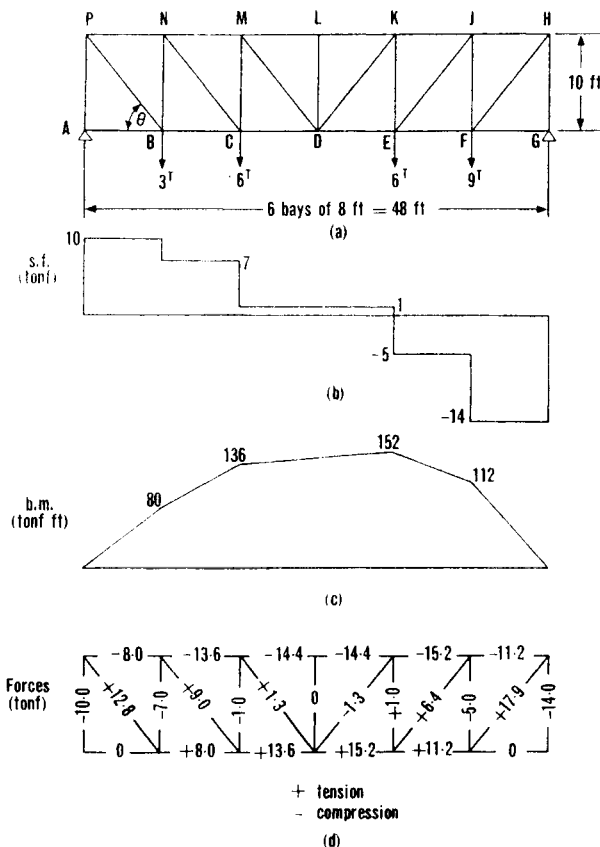


FIG. 5.6

the force in an inclined bar is equal to the s.f. in the bay, multiplied by the length of the inclined bar divided by the depth of the truss.

The force in the vertical bar BN is equal to the s.f. in the bay CN and the force in CM is equal to the s.f. in bay DM . The bar DL is a special case, since no inclined bar is joined to L , but if we examine the equilibrium of the joint L , we see immediately that there can be no force in DL — if there were, there would be nothing to balance it. Notice that the diagonals BP , CN , DM , FH and EJ are all in tension, but DK is in compression; this is due to the s.f. in bay DK remaining positive, although this bar is sloping the other way. Where a diagonal bar is in tension the vertical member connected to it at the unloaded joint is in compression; and vice versa.

The forces in the boom members are derived from the b.m. diagram, e.g. the force in NP is equal to the b.m. at B divided by the depth of the truss, and is compressive. The force in BC is equal to the b.m. at N (or B) divided by the depth of the truss, and is tensile; and so on. Bars AB and FG are *inactive*, that is, they carry no load, as can be seen by applying the same reasoning that we applied to the bar DL .

The forces in the bars are shown in Fig. 5.6d.

Exercise 5

1. Construct the reciprocal figure for the roof truss shown in Fig. 5.7 and write down the forces in the bars on the space diagram, denoting ties by a plus sign and struts by a negative sign.

Also determine the force in the central tie by the method of sections.

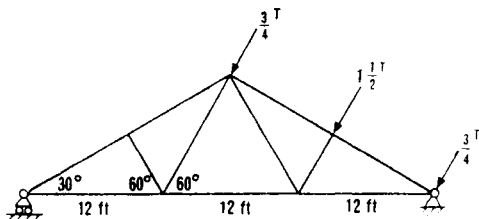


FIG. 5.7

Ans: 1.5 tonf

2. Determine graphically the magnitude and direction of the reactions at the supports and construct the reciprocal figure for the roof truss shown in Fig. 5.8 and hence determine the force in member GH .

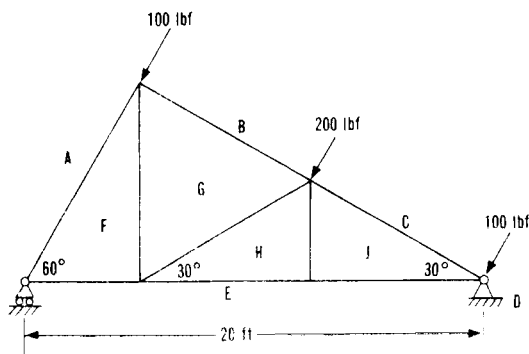


FIG. 5.8

Calculate the force in this bar also by the method of sections, making a separate sketch of the relevant portion of the truss and the forces acting on it, in order to show clearly your method of calculation.

Ans: 230 lbf compression

3. Explain what is meant by a *free body diagram*.

Calculate the forces in the bars CF , CG and DE of the roof truss shown in Fig. 5.9.

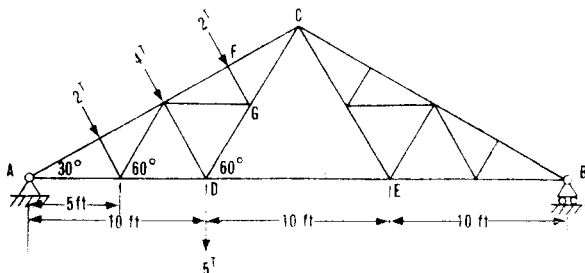


FIG. 5.9

Ans: 15.9 tonf compression; 13.8 tonf tension; 6.9 tonf tension

4. The truss of a deck bridge is loaded as shown in Fig. 5.10. Determine the forces in the members *P*, *R*, *S* and *T*, stating whether they are tensile or compressive.

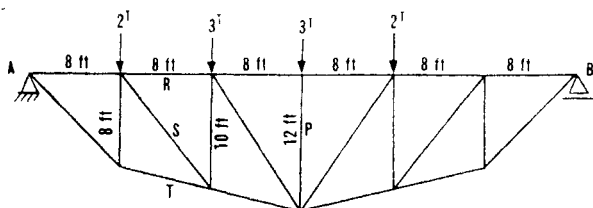


FIG. 5.10

Ans: 3.00 tonf compression; 7.72 tonf compression; 3.03 tonf tension; 6.00 tonf tension

5. The bridge truss shown in Fig. 5.11 is simply supported at *A* and *B* and carries the loads shown.

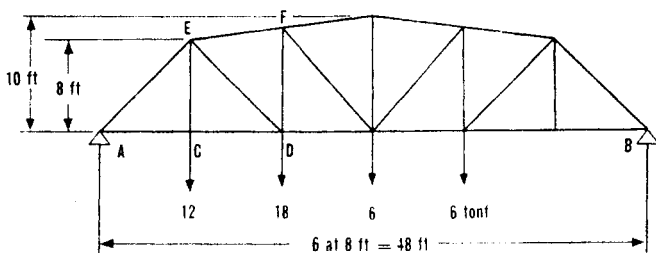


FIG. 5.11

Determine by the method of sections, the forces in the members *CD*, *ED* and *EF*, stating whether they are in tension or compression.

Ans: 27.0 tonf tension; 14.6 tonf tension; 37.6 tonf compression

6. Construct the s.f. and b.m. diagrams for the *N* girder in Fig. 5.12 and hence determine the force in each bar. Write its value against a diagram of the truss, indicating which bars are struts and which are ties.

What important assumption is made in order to enable the truss to be analysed?

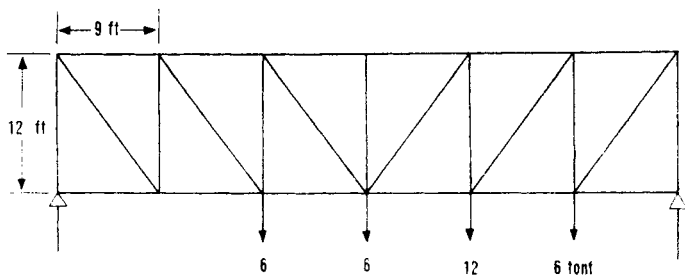


FIG. 5.12

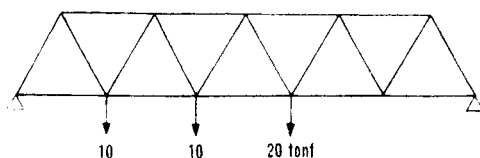


FIG. 5.13

7. The Warren girder in Fig. 5.13 is simply supported at its ends and carries the loads shown. If all members are the same length, analyse the truss to find the force in each bar.

8. What assumptions are made when analysing a just stiff framework?

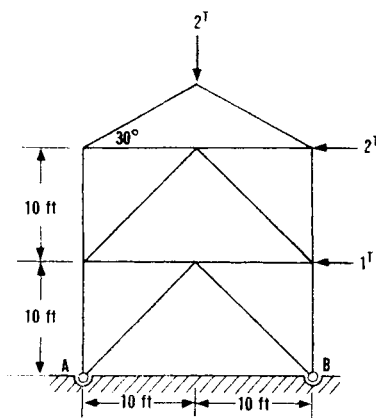


FIG. 5.14

Analyse the forces in the framework shown in Fig. 5.14 and determine the support reactions at *A* and *B*.

Ans: $V_A = 3.5$ tonf; $H_A = 1.5$ tonf; $V_B = 1.5$ tonf downwards; $H_B = 1.5$ tonf

9. Determine the reactions at *A* and *B* and the forces in all the bars of the truss shown in Fig. 5.15.

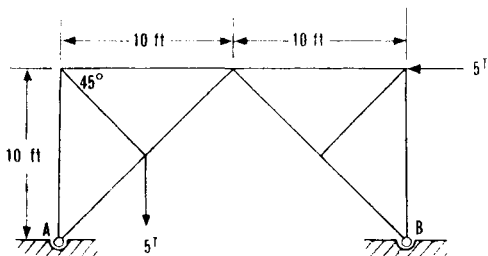


FIG. 5.15

Ans: $V_A = 6.25$ tonf; $H_A = 3.75$ tonf; $V_B = 1.25$ tonf downwards; $H_B = 1.25$ tonf

10. Show that the framework in Fig. 5.16, although it has the correct number of bars, is not a perfect frame, whereas the framework in Fig. 5.17 is a perfect frame. Prove that the reaction at *B* in Fig. 5.17 is hori-

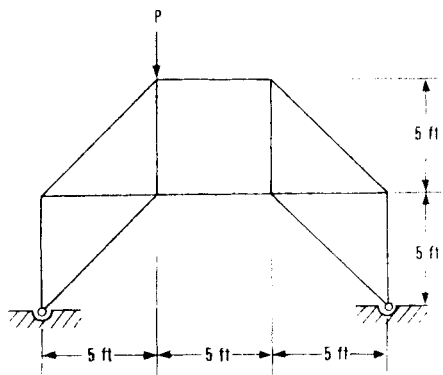


FIG. 5.16

zontal and proceed to determine the forces in all the bars of the second framework.

11. Figure 5.18 shows a pylon loaded by two inclined 10 ton forces. Determine graphically the magnitude and direction of the reactions at the ground and construct the force diagram for the pylon. Indicate on your space diagram whether the bars are struts or ties.

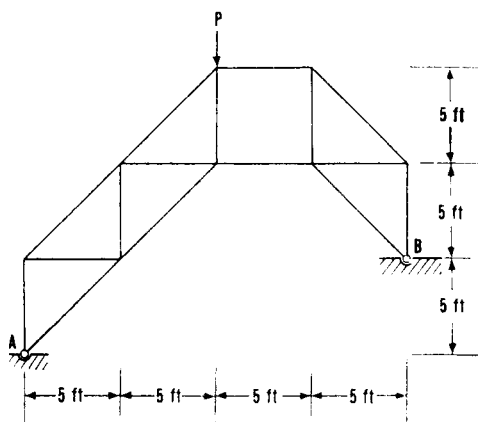


FIG. 5.17

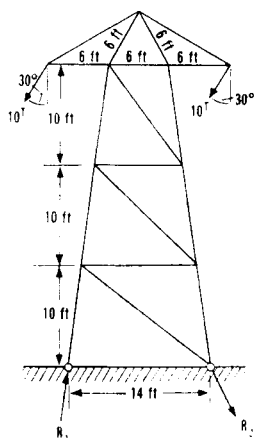


FIG. 5.18

12. Analyse the truss in Fig. 1.15 when $P = 1$ tonf and $W = 2$ tonf
13. Analyse the truss in Fig. 1.16.
14. The deck bridge truss in Fig. 5.19 has a span of 80 ft and is 10 ft

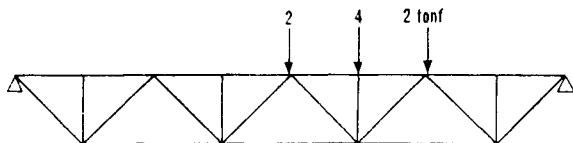


FIG. 5.19

deep. Mark the inactive bars and determine the forces in the remaining bars analytically.

15. The dimensions and loading of a framework are shown in Fig. 5.20, the member DE is vertical and the member BD is horizontal.

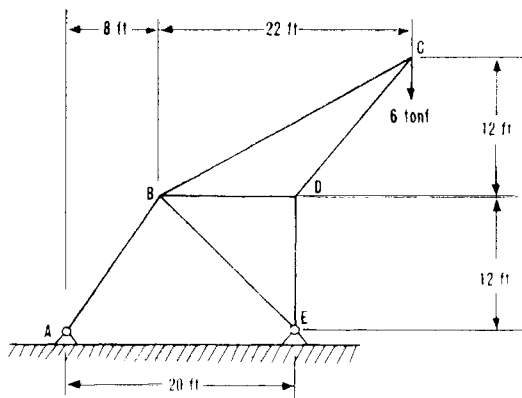


FIG. 5.20

Determine, analytically or graphically, the force in each member and show clearly whether the force is tensile or compressive.

(Eng. Inst. Apr. 1959)

Ans: $AB = 3.6$ tonf tension; $BC = 10.4$ tonf tension;
 $CD = 14.3$ tonf compression; $DE = 11$ tonf compression;
 $BD = 9.2$ tonf compression; $BE = 2.8$ tonf tension

16. Determine, by a force diagram or otherwise, the magnitude of the forces in the members marked X , Y and Z on the pin-jointed truss shown in Fig. 5.21. State whether the forces are tensile or compressive.

(I.C.E. Apr. 1950)

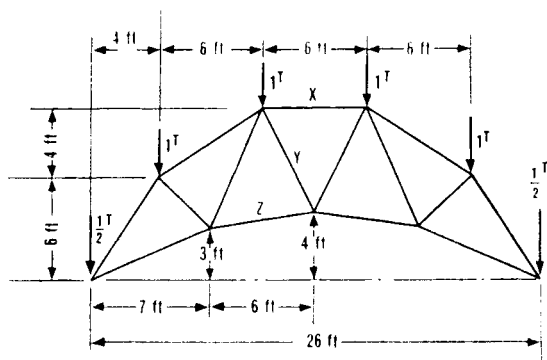


FIG. 5.21

Ans: $X = 2.33$ tonf compression
 $Y = 0.40$ tonf tension
 $Z = 2.18$ tonf tension

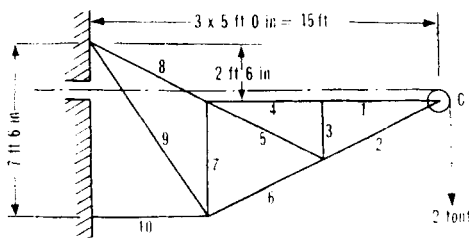


FIG. 5.22

17. Figure 5.22 shows the elevation of a warehouse crane in which the hoisting rope is passed over a pulley at C and thence returns horizontally to the hoisting drum in the building.

Draw the force diagram for this structure and tabulate the forces in the members using the reference numbers given. Distinguish between tension and compression.

(I.C.E. Apr. 1951)

Ans:

Member	1	2	3	4	5	6	7	8	9	10
Force (tonf)	+2	-4.47	0	+2	0	-4.47	+1	+2.24	+1.2	-4.67

The sign + indicates tension, - compression.

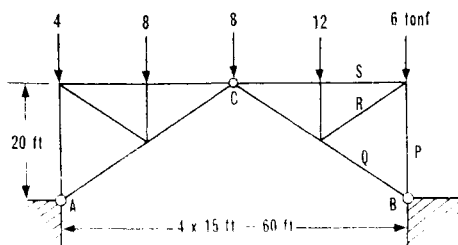


FIG. 5.23

18. Figure 5.23 shows the dimensions and loading of a braced three-hinged arch, the hinges being at *A*, *B*, and *C*.

Determine (a) the horizontal thrust at the abutments.

(b) the magnitudes of the forces in the members marked *P*, *Q*, *R*, and *S* and state whether they are tensile or compressive.

(I.C.E. Apr. 1952)

Ans: (a) 13.5 tonf

(b) *P* = 12 tonf compression; *Q* = 16.21 tonf compression;
R = 10.81 tonf tension; *S* = 9 tonf compression

19. Determine the nature and magnitude of the forces in the members marked 1, 2, 3, 4 and 5 of the pin-jointed K-truss shown in Fig. 5.24.

(I.C.E. Oct. 1952)

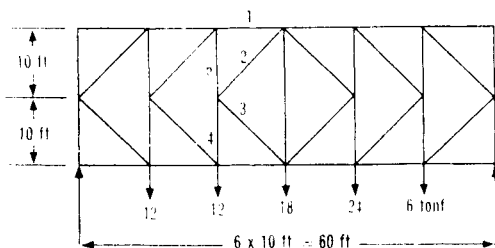


FIG. 5.24

Ans: (1) 30 tonf compression (2) 8.48 tonf compression
 (3) 8.48 tonf tension (4) no load (5) 12 tonf tension

20. The plane frame shown in Fig. 5.25 is pin-jointed at A , B , C , D , E and F . Determine the forces in all members under the given loading.

(Eng. Inst. Oct. 1958)

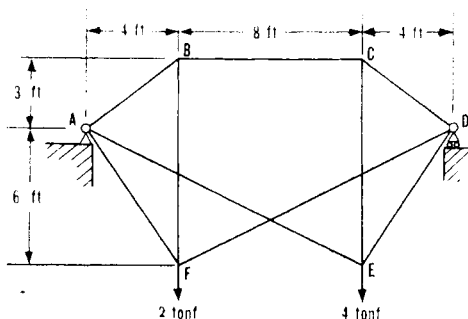


FIG. 5.25.

Ans: $BC = 1.71$ tonf; $AB = CD = 2.14$ tonf;
 $BF = CE = 1.29$ tonf; $AF = 0.64$ tonf; $AE = 1.52$ tonf

CHAPTER 6

Rolling Loads

THE function of a bridge is to provide for the passage of vehicles and people across it; the live or moving load thus successively occupies every position along the bridge. So far, in analysing beams and trusses, we have only considered loads in fixed positions. We now have to consider the effect of changing the position of the load. It would be extremely tedious to analyse each structure with the load occupying a series of positions, in order to determine the maximum stress in each member; so a different approach is used; we consider the *effect* produced at a given point in the structure by a unit load rolling across the bridge. The *effect* may be the shearing force, or bending moment, at some section of a beam; or again, it may be the internal force in a bar of the truss. Whatever it is, its value is plotted as an *influence line* against the position that the unit load happens to occupy at that instant. Thus the influence line of bending moment at a given section of a beam shows the value of the bending moment there, for a unit load successively occupying every position along the beam. In other words, such an influence line shows how the bending moment at the chosen section varies as a unit load rolls across the beam. Note that the influence line of bending moment is for one particular cross-section of the beam, whilst a bending moment diagram is for one particular stationary loading.

One of the most important uses of influence lines is for determining the position which a train of rolling loads must occupy

to produce a maximum effect. Without the help of influence lines this could be a laborious task.

Although influence lines are always drawn for a *unit load* crossing a structure, it is a simple matter to determine the effect of a train of rolling loads from the unit influence line. A train of loads usually consists of a number of concentrated (point) loads linked together, so that their spacing does not change, e.g. the axle loads of a vehicle or railway train; but if the axle loads are all equal and evenly spaced, at a distance apart which is much smaller than the span of the bridge, it is sufficiently accurate to treat the loading as uniformly distributed over the length of the train, thereby simplifying the calculations.

For a *train of concentrated loads*, the ordinate of the influence line diagram opposite each load is multiplied by the value of the corresponding load. The total effect is then the sum of these products. In the case of *uniformly distributed loads*, the area under the influence line, along the length of the beam occupied by the load, is multiplied by the intensity of the uniformly distributed load to obtain the total effect. This is because

$$\begin{aligned}\text{total effect} &= \text{total load} \times \text{average influence line ordinate} \\ &= \text{intensity of loading} \times \text{length of load} \times \text{average ordinate} \\ &= \text{intensity of loading} \times \text{area under influence line}.\end{aligned}$$

Example 6.1. A beam AB , 10 ft long, is hinged to a vertical wall at A , and supported horizontally by a wire CD inclined at 30° to the horizontal, as shown in Fig. 6.1a. Obtain the influence line for the tension in the wire as a unit load runs along the beam. Determine the maximum tension in the wire for (i) the rolling loads shown in Fig. 6.1b, (ii) a uniformly distributed load of 0.3 tonf/ft, 5 ft long. Determine also the

tension in the wire due to the dead weight of the beam, if it weighs 0.2 tonf/ft.

Figure 6.1c is the free body diagram for the beam with a unit load at a distance x from A . By taking moments about

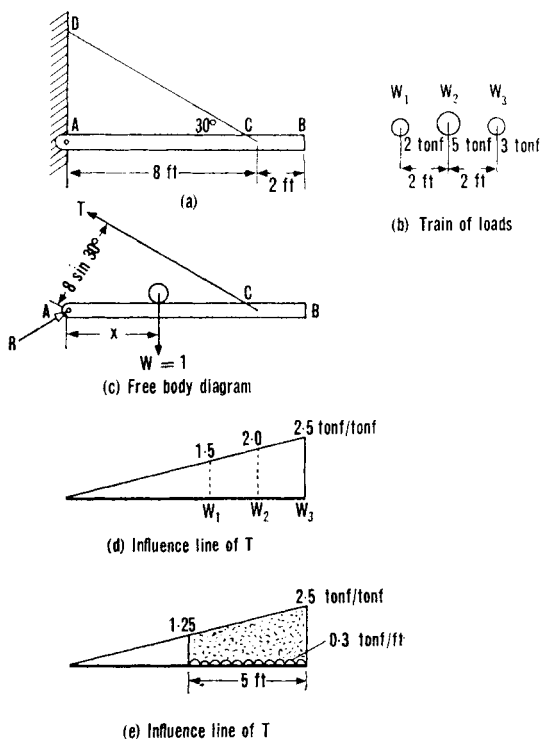


FIG. 6.1

At A we eliminate the unknown reaction at A and find that $8T \sin 30^\circ = Wx$, or $T = x/4$, since $W = 1$. Thus the influence line of T , the tension in the wire, is the triangular shape shown in Fig. 6.1d. The tension will obviously be a maximum when the load W_3 is at the end B , since the influence line ordinates opposite the three rolling loads will then be greatest. Multi-

plying each ordinate by the corresponding loads and adding, we get the maximum value of

$$T = 2 \times 1.5 + 5 \times 2.0 + 3 \times 2.5 = 20.5 \text{ tonf.}$$

For the uniformly distributed rolling load, T will be a maximum when the load just extends to the end of the beam, as this gives the greatest area under the influence line, see Fig. 6.1e. The maximum value of T in this case is equal to the stippled area times the intensity of the live load, therefore

$$T = \frac{5}{2} (1.25 + 2.5) \times 0.3 = 2.82 \text{ tonf.}$$

The tension in the wire due to the dead load can also be obtained from the influence line. T = area of whole diagram times the intensity of the dead load.

$$T = \frac{2.5}{2} \times 10 \times 0.2 = 2.5 \text{ tonf.}$$

Influence Lines for Simple Beams

AB is a simple beam of length l , carrying a unit rolling load W , Fig. 6.2a. The reactions at A and B are R_A and R_B . If x is the distance of the load from A at any instant, then $R_B = Wx/l$, so that the *influence line of reaction at B* is triangular, as shown in Fig. 6.2b, and the influence line of R_A is similar, but reversed, Fig. 6.2c.

The shearing force at a specified point P , distance a from A , is equal to the reaction at B , provided W is between A and P ; so that the *influence line of s.f. at P* consists of the influence line of R_B for $x < a$ and the influence line of R_A , for $x > a$, as shown in Fig. 6.2d. Note that the influence line between A and P is negative, since the s.f. at P is negative, according to our sign convention, when the load is between A and P . At P

the influence line suddenly jumps up by unity, and strictly speaking when the unit load is exactly at P , the s.f. at P is indeterminate, but provided W is not directly over P (although

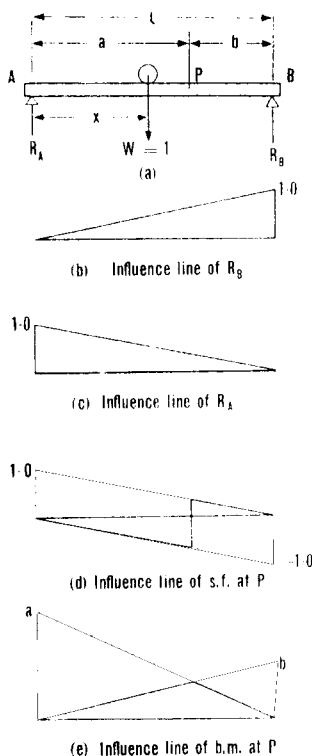


FIG. 6.2 Influence lines of reaction, s.f. and b.m. for a simple beam

it can be made as close to P as you like) there is no uncertainty as to which value of shearing force to take.

The bending moment at P equals $R_B b$ where b is the distance of P from the right-hand support, provided $x < a$; so the *influence line of b.m. at P* , Fig. 6.2e, consists of two straight lines. Between A and P the line is obtained by multiplying the

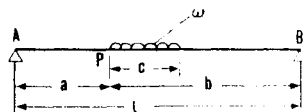
ordinates of the influence line of R_B by b ; whilst between B and P the line is obtained by multiplying the ordinates of the influence line of R_A by a .

Note that the influence line has a peak value ab/l at P , and is all of one sign (positive) since the b.m. is always sagging. It is only coincidental that the influence line of b.m. at P is identical with the b.m. diagram for a unit load at P .

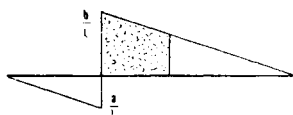
Position of a Train of Loads to give a Maximum s.f. at a Given Section

(a) Uniformly distributed load

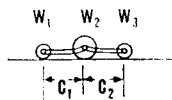
The intensity of the load is designated by w and its length by c . The positive s.f. at the chosen section P will be greatest when the leading end of the train has just reached P , in mov-



(a)



(b) Influence line of s.f. at P



(c)

FIG. 6.3

ing from right to left across the beam, because this position of the load will give the maximum (positive) area under the influence line diagram for the s.f. at P , Fig. 6.3b. The s.f. at P will have a maximum negative value, i.e. a minimum positive value, when the leading end of the load has just reached P approaching from the opposite direction. Usually we are only interested in finding the greatest numerical value of the s.f. at P and are not concerned with its sign.

(b) *Concentrated loads*

We can letter the loads in the train W_1, W_2, W_3 , etc., starting from the left-hand load, as in Fig. 6.3c. If the train advances from right to left across the beam the loads will arrive at the section P in the same order as we have lettered them. The distances between successive loads are denoted by c_1, c_2 , etc. If the loads were all equal, and evenly spaced, it is obvious that the (positive) s.f. at P would be a maximum when the first load of the train was at P , or rather, to be strictly accurate, immediately to the right of P . If however the first load is small compared with subsequent ones, or precedes the others by a considerable distance, the s.f. at P may be greater when the second load is about to arrive at P . The criterion for determining which load should be at P , in order to produce a maximum s.f. at P , is *if the first load divided by the distance by which it precedes the next load is less than the sum of all the rolling loads on the beam divided by the total span, then the s.f. at P will be greater when the second load is at P than when the first is there.* Expressed in mathematical terms this becomes

$$\text{If } \frac{W_1}{c_1} < \frac{\Sigma W}{l} \quad \text{then} \quad F_2 > F_1$$

where F denotes the s.f. at P and the suffixes indicate which load is at P . Of course the second load may not produce great-

est s.f. at P so the criterion is repeated;

$$\text{if } \frac{W_2}{c_2} < \frac{\Sigma W}{l} \quad \text{then} \quad F_3 > F_2$$

and so on, until on dividing one of the loads by the distance by which it precedes the next we obtain a value which is greater than the average intensity of loading on the span. Notice that the position of P does not affect the issue, provided none of the rolling loads is off the span.

Example 6.2. The train of loads in Fig. 6.4 crosses a beam with a span of 60 ft. What load must be immediately to the right of P (20 ft from the left support) to produce a maximum s.f. at P .

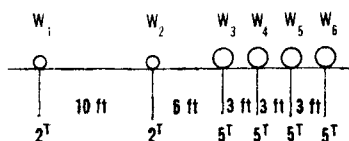


FIG. 6.4

The average intensity of loading = $24/60 = 0.4$ tonf/ft.

$$W_1/c_1 = 2/10 = 0.2 \text{ tonf/ft, therefore } F_2 > F_1$$

$$W_2/c_2 = 2/6 = 0.33 \text{ tonf/ft, therefore } F_3 > F_2$$

$$W_3/c_3 = 5/3 = 1.66 \text{ tonf/ft, therefore } F_4 < F_3$$

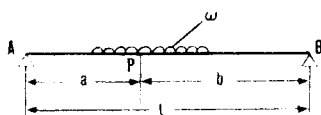
Therefore F_3 is the greatest value and W_3 must be at P to produce a maximum s.f.

Position of a Train to give a Maximum b.m. at a Given Section

(a) Uniformly distributed load

The given section P must divide the load in the same ratio as it divides the span, in order to produce the greatest b.m. at P , due to the passage of a uniformly distributed load.

Figure 6.5a is a diagram of the beam with the load in such a position that P divides it in the ratio a to b ; hence the ordinates of the influence line, Fig. 6.5b, at each end of the train are equal. It is easy to see that moving the load either to the left or the right will decrease the stippled area under the influence



(a)

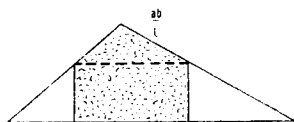
(b) Influence line of b.m. at P

FIG. 6.5 Position of a uniform rolling load to produce a maximum b.m. at P

line, and so the area must be a maximum. It follows that this position of the load will produce the greatest b.m. at P . It is also clear that the average intensity of loading on the portion of the beam to the left of P is equal to the average intensity of loading on the portion of the beam to the right of P . In other words, the loading on AP divided by the distance AP is equal to the loading on PB divided by the distance PB .

(b) Concentrated loads

The b.m. at P will be a maximum when one of the loads of the train is at P . The particular load is that one which “*tips the balance*”. If this load is included with the loads to the left,

then the average intensity of loading to the left of P is greater than that to the right; whereas if it is included with the loads to the right, the average intensity is greater to the right of P . The "average intensity of loading to the left of P " means the loading on this portion of the beam divided by the length AP ; and similarly for PB . Notice that, unlike the previous case, it is rarely possible to achieve a perfect "balance" and so it is necessary to resort to a pair of inequalities as our criterion for determining the position that the load must occupy to produce a maximum b.m. at P .

Example 6.3. Determine which load in Example 6.2 must be at P to produce a maximum b.m. at P .

The particular load which must be at P is W_3 , Fig. 6.4, as this is the one which "tips the balance"; since if W_3 is included with the loads to the left of P the average intensity of loading to that side is $9/20$ tonf/ft, which is greater than the intensity to the right of P ($15/40$ tonf/ft); whereas if W_3 is included with the loads to the right of P the average intensities to the left and to the right are $4/20$ and $20/40$ tonf/ft respectively. It is, of course, coincidental that W_3 also produces a maximum s.f. at P .

Position of a Train to give a Maximum b.m. under a Given Load

The previous criteria enable the position of a train of loads to give a maximum b.m. at a specified section of a simply supported beam, to be determined. Other sections along the beam will experience different maximum bending moments, so that a criterion is needed, which will enable the position of a train, to produce a maximum b.m. beneath a given load, to be determined. The maximum b.m. underneath a given load occurs when *mid-span bisects the distance between the centre of gravity of the train and that load*. Notice that the maximum

b.m. does not occur at mid-span, unless the chosen load is at the centre of gravity of the train, but, of course, the greatest possible b.m. is going to occur somewhere near mid-span, in fact it will occur under one or other of the loads immediately to either side of the c.g. of the train, when that load and the c.g. of the train are equidistant from mid-span. It is a matter of trying both to find out which gives the greater bending moment.

Example 6.4. At what position must the train of loads in Example 6.2 be, in order to produce the greatest possible b.m. anywhere in the beam?

The c.g. of the train occurs 1.08 ft to the left of W_4 , so that the maximum b.m. will occur under W_4 when W_4 is **0.54 ft to the right of mid-span** and the c.g. is 0.54 ft to the left of mid-span.

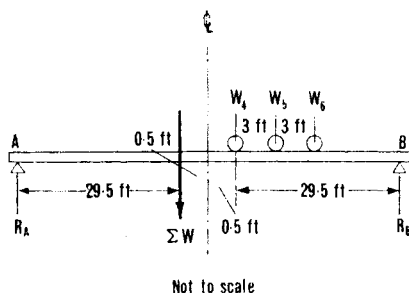


FIG. 6.6

The value of this maximum b.m. is readily calculated. Refer to Fig. 6.6. The total load is 24 tonf and the distance of the c.g. from $A = 29.5$ ft, so $R_B = 24 \times 29.5/60 = 11.8$ tonf. Therefore the b.m. under W_4 is $11.8 \times 29.5 - 5 \times 3 - 5 \times 6 = 303$ tonf ft. This is the greatest b.m. to which the beam is subjected and it would have to be designed to have a moment of resistance at least as great.

Influence Lines for Simply Supported Trusses

Each bar of a truss is subjected to an axial force which fluctuates as a load rolls across the truss. The varying part of the force is called the *live load* in the member, as opposed to the dead load, due to the weight of the structure itself, which is present all the time. An influence line can be constructed for each bar of a truss, showing the variation of axial force in the bar, as a unit load crosses the truss. This can be done by obtaining an expression for the force in the member as a function of x , the distance of the unit load from one end, using the method of sections or any other convenient analytical method. However, it is normally much quicker to make use of the influence lines of s.f. and b.m., treating the truss as a simple beam, with the proviso that loads are only transferred to the truss at panel points.

Most open web girder bridges consist of a pair of trusses, with the live load rolling between them on rail bearers, or on a deck supported by stringers running parallel to the main girders. At each panel point cross-girders span between the two trusses and carry the rail bearers or stringers. Each main truss consists of an upper and a lower boom running the length of the bridge, connected by vertical or inclined web members. In a *through bridge* the cross-girders are connected to the panel points or joints in the bottom boom; whereas in a *deck bridge* the cross-girders are connected to the panel points in the top boom.

Effect of stringers on the s.f. and b.m. at the section P

Figure 6.7 shows a simple beam AB , which is crossed by a unit load W . The load does not roll directly on the beam but on a series of stringers, only one of which, CD , spanning the

bay in which P is situated, is shown. The presence of the other stringers does not affect the s.f. or b.m. at P in whatever position the load is. The s.f. in the bay CD of the main beam is uniform (Fig. 6.7b) since the s.f. can only change where the

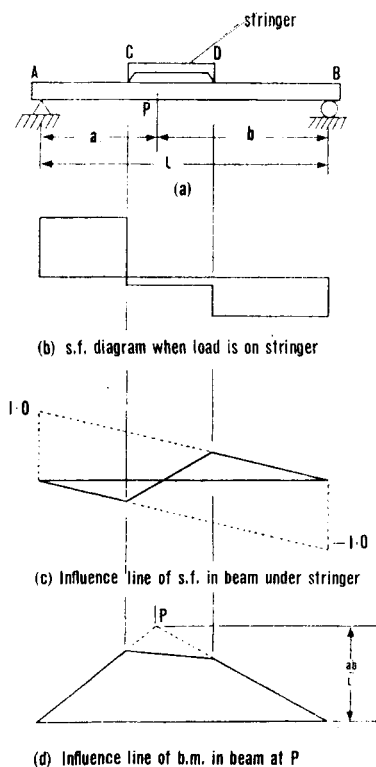


FIG. 6.7 Effect of stringers on influence lines of s.f. and b.m.

stringers transfer load to the beam. Hence, the influence line of s.f. in the panel CD is the same shape as the influence line of s.f. for any point in this panel, but with an oblique line joining the positive and negative portions, spanning the length CD , as shown in Fig. 6.7c.

Likewise, the influence line of b.m. at P is the same shape as the influence line of b.m. at P , if no stringers were present, but with the vertex chopped off, by a single cut, over the length CD , as in Fig. 6.7d.

Web members in parallel boom trusses

These members carry the shearing forces and so the force in such members is obtained by constructing the influence line of s.f. for the panel associated with that member. The vertical component of the force in a web member is equal to the s.f. there, provided the truss is of such a shape that a vertical section through the truss only cuts one web member.

Example 6.5. Figure 6.8a shows a Pratt truss or N girder of a through bridge. If we wish to find how the force in the diagonal bar EG , of the third panel from the left, varies as a unit load crosses the truss we construct the influence line of s.f. for this panel, Fig. 6.8b. The force in EG is then given by multiplying the ordinates of this influence line by the ratio of the length of the bar EG to the depth of the truss. The force in EG is tensile where the sign of the influence line is positive, as may be seen from a study of Figs 6.8a, b and c.

The vertical bar DG is associated with the diagonal bar EG (as may be seen by working across from the left-hand support, or by noting that their common joint G never carries an external load) so that Fig. 6.8b is also the influence line of force in DG , if we remember to reverse the signs, since when the diagonal EG is in tension DG will be in compression.

The corresponding influence line for the s.f. in the first panel, Fig. 6.8c, gives the forces in CF and AF . Note that the diagram is entirely positive and is, in fact, a "chopped" influence line for the left-hand reaction.

The central vertical bar EH does not carry any live load,

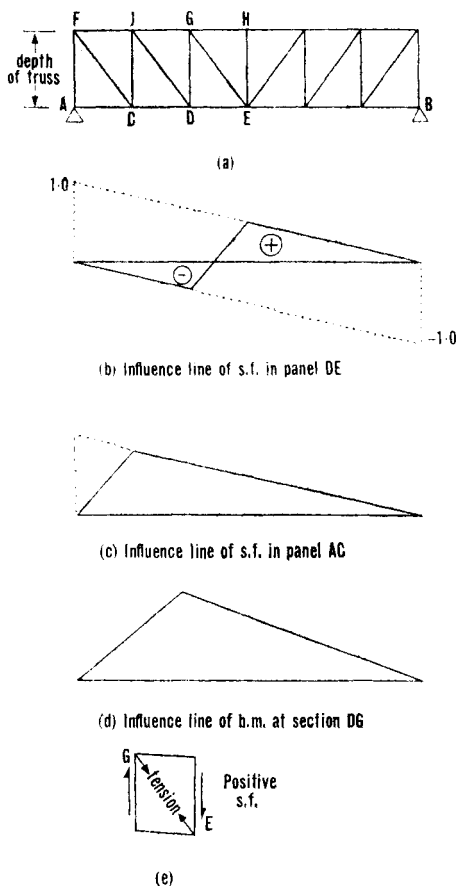


FIG. 6.8 Pratt truss

Example 6.6. Figure 6.9a shows a Warren truss. Again this is a “through” bridge and so the live load is transmitted to the bottom panel points. The influence line of s.f. in the second panel CD is shown in Fig. 6.9b. The forces in CE and DE are both equal to the s.f. in the second panel, multiplied by the ratio of the length of the inclined bars to the depth of the truss;

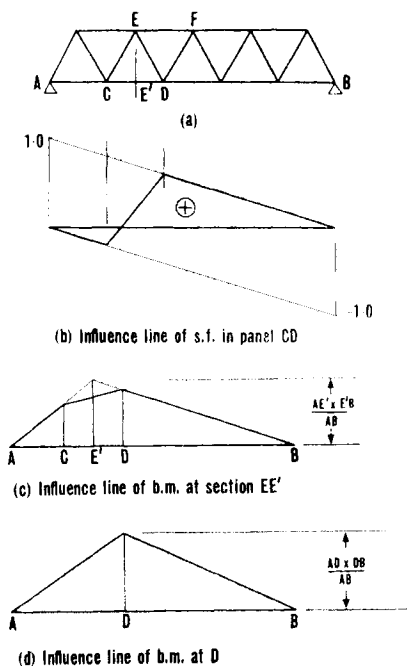


FIG. 6.9 Warren truss

but whereas the force in *DE* is tensile when the s.f. is positive, the force in *CE* is compressive when the s.f. is positive.

Example 6.7. Figure 6.10a shows a deck bridge, and Fig. 6.10b is the influence line of s.f. in the panel *CD*. Thus the force in the diagonal *CF* is obtained by multiplying the influence line ordinates by $\sec \theta$, the ratio of *CF* to *CE*. The force in *CF* is tensile where the s.f. is positive. Note that the vertical bar *CE* is associated with the second panel, not the third.

The influence line of force in the central vertical bar is shown in Fig. 6.10c. It will be seen that the force in this bar is zero except when the load is within the two central panels and that the force is unity when the unit load is directly above it.

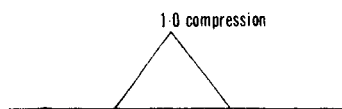
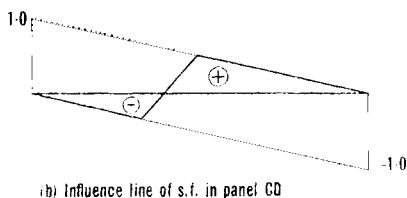
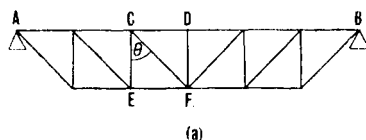


FIG. 6.10 Deck bridge

Boom members

The top and bottom bars of a parallel boom truss carry the bending moment at every section, by virtue of the couples they provide due to the moment of the force in a boom member about the joint opposite. Hence, the force in such a member is obtained by dividing the ordinates of the influence line of b.m. for the section through the appropriate joint, by the depth of the truss; bearing in mind that, if the bar in question is spanned by a stringer, the influence line will be “chopped” in the case of trusses where the joints on the two booms are not opposite each other.

Example 6.8. Referring again to the Pratt truss in Fig. 6.8a, if we want the force in *GJ* or *DE* the appropriate diagram is the influence line of b.m. at the section *DG*, Fig. 6.8d, whose

maximum ordinate is $AD \times DB/AB$. The upper member is in compression and the lower in tension. The lower bar is spanned by a stringer, but nothing can be "chopped" from the diagram by taking a short cut over the length of this stringer, since the top panel points are opposite the bottom ones.

Example 6.9. Referring once more to the Warren truss in Fig. 6.9a, the tension in the lower boom member CD is obtained by dividing the ordinates of the influence line of b.m. at section EE' by the depth of the truss. In this case, since CD is paralleled by a stringer, the peak of the diagram is clipped off over the length CD , Fig. 6.9c.

The compression in the top boom member EF is obtained by dividing the ordinates of the influence line of b.m. at D , Fig. 6.9d, by the depth of the truss. In this case the influence line is not clipped, since there is no stringer spanning between E and F .

Influence line of Horizontal Thrust in a Three-pinned Arch

Figure 6.11a is a free body diagram of a symmetrical, three-pinned arch of span l and rise d , carrying a unit rolling load W . The horizontal thrust H in the arch is equal to the moment about C due to V_B divided by the rise, provided the load is between A and C ; thus the influence line of H is the influence line of b.m. at C , due to the vertical forces, divided by d , as shown in Fig. 6.11b.

Envelope Diagrams of s.f. and b.m.

An envelope diagram of bending moment is a graph showing the maximum b.m. attained at each section along a beam, due to the passage of a specified train of loads. Such a diagram

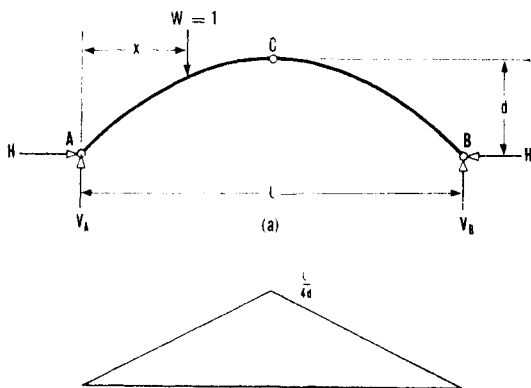
(b) Influence line of H

FIG. 6.11 Three-pinned arch

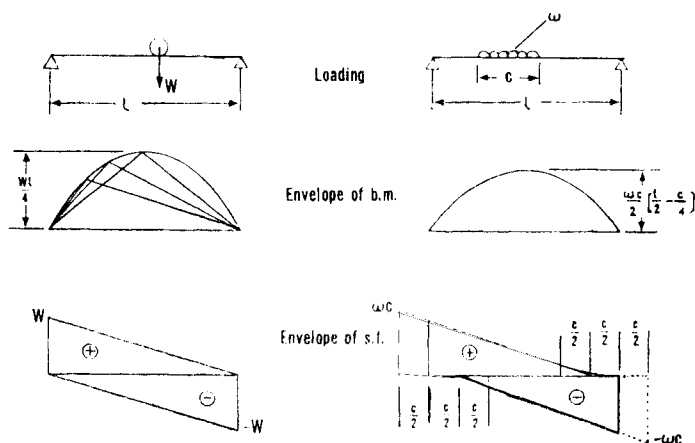


FIG. 6.12 Simple examples of envelope diagrams

could be constructed by drawing the envelope to a set of b.m. diagrams drawn for the loading in a series of positions along the beam, but such a procedure would be tedious, and it is

usually less effort to determine which load must occupy a specified position on the beam to produce a maximum b.m. there, and then calculate its value, repeating this for several positions along the beam. Probably the only occasion when it is necessary to go to the trouble of producing envelope diagrams, is where a beam is of non-uniform cross-section, whose moment of resistance at each section has to be matched to the maximum b.m. applied there.

Figure 6.12 shows envelope diagrams of s.f. and b.m. for a single concentrated rolling load W and for a uniform load of intensity w and length c .

Exercise 6

1. Explain what is meant by an influence line of bending moment. How does it differ from a b.m. diagram?

A cantilever is 6 ft long. Determine from first principles (a) the influence line of b.m. at the built-in end, (b) the influence line of b.m. half-way along the cantilever, (c) the influence line of s.f. at the built-in end. Hence calculate the maximum s.f. and b.m. at the built-in end due to a uniformly distributed load of 3 tonf, 2 ft long, crossing the cantilever.

Ans: 3 tonf; 15 tonf ft

2. Obtain from first principles the influence line of force in the strut CD in Fig. 6.13 as a unit load rolls across AB . Hence determine the maxi-

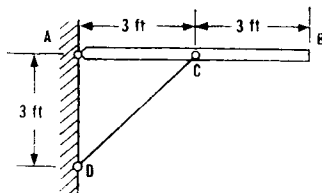


FIG. 6.13

imum force in CD due to the passage of a uniformly distributed load of 3 tonf, 2 ft long.

Ans: $5\sqrt{2}$ tonf compression

3. A girder 60 ft long is crossed by the following train of rolling loads.

Load (tonf)	14	18	15	10
Spacing (ft)	6	6	6	

Determine (a) the maximum b. m. at mid-span, (b) the position at which the greatest b.m. occurs, and its value.

Ans: (a) 708 tonf ft

(b) 710 tonf ft at 1.1 ft to left of mid-span when 18 tonf load is there

4. A single-carriageway bridge, simply supported on a span of 80 ft, consists of a pair of plate girders. The heaviest load which can cross the bridge is made up of the following train of axle loads.

Load (tonf)	20	40	20	20	20	20
Spacing (ft)	10	12	8	12	8	

Assuming that the girders share the load equally, determine the position at which the b.m. in each girder, due to the live load, will be a maximum, and calculate its value.

Determine also the maximum total b.m. in each girder at a section 20 ft from the left-hand abutment. The dead load of the bridge is 3 tonf/ft.

Ans: 890 tonf ft at 0.7 ft to left of mid-span; 1615 tonf ft

5. Figure 6.14 shows a compound beam on three supports *A*, *B* and *C*, with a pin at *D*. Deduce the influence lines of reaction for each support.

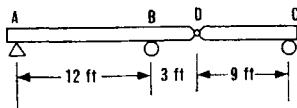


FIG. 6.14

Determine the maximum values of the three reactions due to the passage of a uniformly distributed load of 2 tonf/ft, longer than the total length of the compound beam.

Ans: $A = 12$ tonf; $B = 30$ tonf; $C = 9$ tonf

6. Five concentrated loads, at constant distances apart, roll from left to right across a simply supported beam of 40 ft span, the 8 tonf load leading.

Load (tonf)	5	5	7	10	8
Spacing (ft)	6	4	6	6	

Determine (a) the maximum bending moment which occurs at a point 15 ft from the left-hand support, (b) the maximum bending moment under the 10 tonf load.

(I.C.E. Apr. 1952)

Ans: (a) 226 tonf ft; (b) 242.8 tonf ft

7. The following train of loads rolls across a girder, simply supported on a span of 100 ft.

Load (tonf)	20	5	10	20
Spacing (ft)		5	5	5

If the girder weighs $\frac{1}{2}$ tonf/ft determine the maximum b.m. (a) at a point 40 ft from the left end, (b) anywhere on the beam.

Ans: (a) 1760 tonf ft; (b) 1840 tonf ft

8. A simple girder 30 ft long is crossed by the following train of loads.

Load (lbf)	800	500	600	700
Spacing (ft)	3	3	3	

Determine the maximum live b.m. produced in the beam. Calculate also the *equivalent u.d.l.* that is, that load which when uniformly distributed over the entire span produces the same maximum b.m. in the beam as the given train of loads.

Ans: 15,400 lbf ft; 4,100 lbf

9. A uniform girder *AB*, 120 ft long, and weighing 60 tonf is simply supported at *A* and *C*, points 80 ft apart. A uniformly distributed load of 40 tonf, 20 ft long, crosses the beam. Draw the influence lines of s.f. and b.m. for a point mid-way between *A* and *C* and find the maximum and minimum shearing force and bending moment at this point.

Ans: +10 tonf; -20 tonf; 900 tonf ft (sag); 400 tonf ft (hog)

10. Explain what is meant by an envelope diagram of bending moment for a particular train of loads.

Construct to scale an envelope diagram of b.m. for the train of loads in question 7 when crossing a simply supported bridge with a span of 50 ft, taking ordinates every 10 ft.

Construct on the same diagram the parabola of the b.m. diagram for the equivalent u.d.l.

11. A model three-pinned arch of segmental shape has a span of 40 in and a rise of 10 in. Determine from first principles the influence line of horizontal thrust in the arch for a unit load crossing it, and hence determine the thrust in the arch due to a concentrated load of 8 lbf placed 10 in from the left-hand end, together with a uniformly distributed load of intensity $\frac{1}{2}$ lbf/in occupying the whole span.

Determine also the b.m. in the arch under 8 lbf load.

Ans: 14 lbf; 23 lbf in

12. A lorry with axle centres of 12 ft has front and rear axle loads of 3 tonf and 6 tonf respectively. The lorry crosses a bridge girder simply supported over a span of 30 ft. Determine from first principles, the position of the loads for greatest bending moment on the span and evaluate its magnitude.

(I.C.E. Apr. 1953)

Ans: 50.7 tonf ft when front axle has moved 10 ft beyond mid-span

13. (a) Draw the influence lines of shearing force and bending moment for a point 20 ft from one support of a girder which is simply supported on a span of 50 ft.

(b) This girder is crossed by a train of ten similar wagons which give a series of concentrated wheel loads of 5 tonf each, uniformly spaced at 8 ft centres. Using the influence lines determine the maximum shearing force and the maximum bending moment which will occur at that point.

(I.C.E. Apr. 1950)

Ans: 7.2 tonf; 192 tonf ft

14. A bridge with two main girders, as shown in Fig. 6.15, has cross-girders at the lower panel points. A load of 4 tonf/ft (longer than the

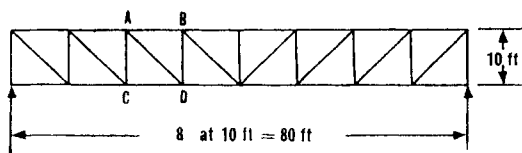


FIG. 6.15

span) crosses the bridge. Find the maximum forces produced in the members *AB*, *CD* and *AC*.

(Eng. Inst. Oct. 1958)

Ans: 150 tonf compression; 120 tonf tension; 35.7 tonf compression and 5.71 tonf tension

15. A transporter girder has the dimensions shown in Fig. 6.16 and is carried by vertical gantry legs at *A* and *C*. Draw influence lines for force in members *DG* and *JC*. Determine the maximum tensile and compressive forces in these members due to the passage of a trolley having a 4 ft wheel base and applying 2 tonf wheel loads to the girder. The load is

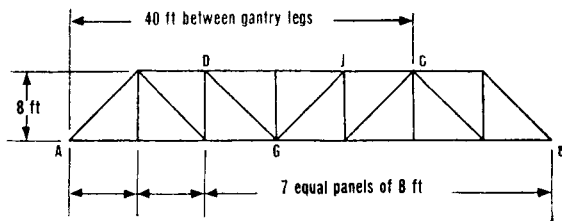


FIG. 6.16

applied to the lower panel points by means of cross-girders and may travel between *A* and *B*.

(I.C.E. Apr. 1953)

Ans: *DG*: 1.98 tonf tens. 1.98 tonf compr.

JC: 5.60 tonf tens. 3.00 tonf compr.

16. All the members of the four bay Warren girder in Fig. 6.17 are 10 ft long. If the dead load of the girder is $\frac{1}{2}$ tonf/ft and it is crossed by a

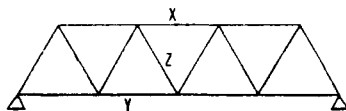


FIG. 6.17

uniformly distributed load of 20 tonf, 10 ft long, calculate the maximum tensile and compressive forces in the bars marked *X*, *Y* and *Z*.

Ans: 31.8 tonf compression; 26.4 tonf tension; 12.3 tonf tension; and 0.7 tonf compression

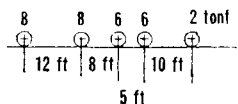
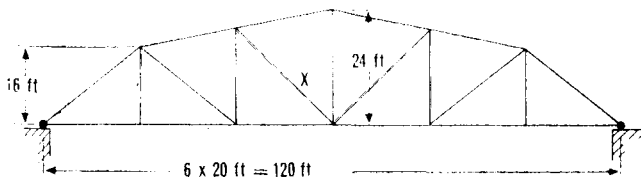


FIG. 6.18

17. Figure 6.18 shows the elevation of one of the two main girders of an emergency road bridge. The deck loads are transmitted equally to the main girders by cross-girders attached at the lower panel points.

Draw the influence line for the force in the member marked X and from it determine the maximum compressive and tensile forces developed in this member during the passage across the span of a heavy lorry and trailer having the axle loads as shown.

(I.C.E. Apr. 1951)

Ans: 6.95 tonf compression; 4.02 tonf tension

18. Figure 6.19 shows the dimensions of a three-hinged, spandrel-braced arch. Sketch the influence line for the force in the member marked X . From it determine the maximum compressive and tensile

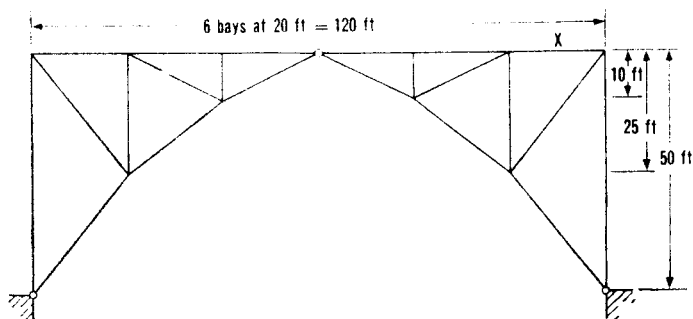


FIG. 6.19

force produced in the member when the arch is crossed by a uniformly distributed load, of intensity 1 tonf/ft, longer than the arch span.

(Eng. Inst. Apr. 1959)

Ans: 7.2 tonf tension; 11.2 tonf compression

CHAPTER 7

Combined Bending and Direct Stress

THE stress distribution across an axially loaded member is uniform; so that if a uniform bar, of cross-sectional area A , is subjected to a pair of equal and opposite external forces of magnitude P , applied at the centroid of the cross-section at each end of the bar, the stress over each cross-section will everywhere be P/A , and the stress would be referred to as *direct stress*.

The stress distribution across a beam, within the elastic limit, subjected to transverse loads only, varies linearly from nothing at the neutral axis to a maximum tensile stress and to a maximum compressive stress at opposite outer fibres. If the cross-section of the beam is symmetrical about the axis of bending, then the two opposing outer fibre stresses will be equal in magnitude. The stress in the outer fibres is called the *bending stress*, and like direct stress it is a *normal stress* — in the sense of being perpendicular to the cross-section.

Many structures carry not only axial forces, but also bending moments; and it is with such structures that this chapter is concerned. We can classify the chief types under four headings:

- (i) Beams with longitudinal loads.
- (ii) Eccentrically loaded stocky struts or ties.
- (iii) Structures which depend upon their weight to resist overturning; such as mass concrete, masonry, or brick chimneys and retaining walls, and gravity dams.
- (iv) Pre-stressed concrete beams.

In each of these structures there are bending stresses superimposed upon direct stresses. As both are normal stresses, and act parallel to each other, the total (normal) stress in each fibre at every section is equal to the algebraic sum of the direct and bending stresses there.

Beams with Longitudinal Loads

This group also includes axially loaded struts or ties with lateral loading, but it is assumed, in all cases, that any deflection is insufficient to introduce the type of "strut action" considered in the next chapter.

Example 7.1 A rectangular timber beam, whose cross-section is 6 in wide by 10 in deep, is subjected to a maximum b.m. of 5000 lbf ft about the major principal axis $x-x$ (Fig. 7.1a)

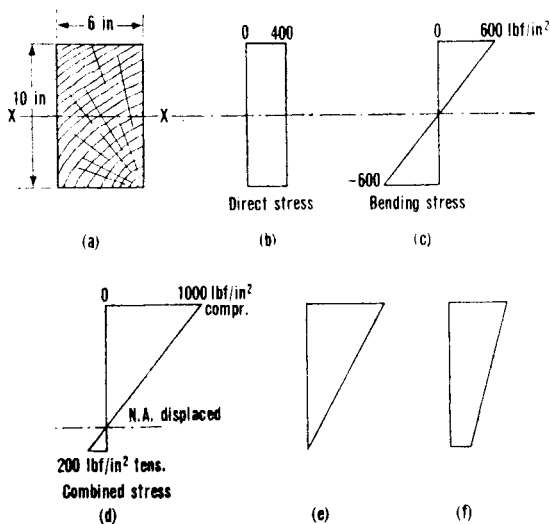


FIG. 7.1 Stress distribution in beam subjected to an axial force

and an axial compressive load of 24,000 lbf. Determine the stress distribution at the critical section.

$$\begin{aligned}\text{Direct stress} &= \frac{\text{Axial load}}{\text{c.s.a.}} \\ &= 24,000/60 = 400 \text{ lbf/in}^2\end{aligned}$$

$$\begin{aligned}\text{Bending stress (outer fibres)} &= \frac{\text{b.m.}}{\text{modulus of section}} \\ &= \frac{60,000 \times 6}{6 \times 10^2} = 600 \text{ lbf/in}^2\end{aligned}$$

The stress components and the combined stress are shown in Fig. 7.1. Note that the effect of the compressive axial load is to lower the neutral axis of a beam subjected to a sagging b.m. With a greater axial load it would be possible to bring the neutral axis down to the bottom fibres and produce a triangular stress distribution like Fig. 7.1e. An even greater axial load would produce a trapezoidal stress distribution like that in Fig. 7.1f.

Eccentrically Loaded Stocky Struts or Ties

As with the previous group only members which are sufficiently stout to be unaffected by flexibility are considered. Bending stresses arise in these members due to the line of action of the longitudinal load not coinciding with the centroid of the cross-section. The longitudinal load may be eccentric to both principal axes, in which case the total stress in any longitudinal fibre will be the algebraic sum of three stress components; the direct stress, the bending stress due to bending about one principal axis, and the bending stress due to bending about the other principal axis.

If A is the cross-sectional area of the member; I_{xx} and I_{yy} are the second moments of area about the principal axes; and P is

the longitudinal load. P is applied at a distance x_1 from the y axis and at a distance y_1 from the x axis. We therefore have:

$$\text{Direct stress} = \frac{P}{A}$$

Bending moment about the principal axis xx

$$= \text{Load} \times \text{eccentricity}$$

$$= Py_1$$

Bending stress in a fibre at a distance y from xx due to this b.m. is

$$\frac{Py_1 y}{I_{xx}}$$

Similarly, the bending stress in a fibre at a distance x from the principal axis yy due to bending about this axis is

$$\frac{Px_1 x}{I_{yy}}$$

Therefore the total stress in a fibre whose co-ordinates are (x, y) is

$$\frac{P}{A} + \frac{Px_1 x}{I_{yy}} + \frac{Py_1 y}{I_{xx}}$$

Note that it is important to use the correct signs for the four co-ordinates (x, y) and (x_1, y_1) .

Example 7.2. Figure 7.2a shows the cross-section of a brick pier which carries a longitudinal load of 100,000 lbf applied at Q . Determine the stress distribution over the cross-section.

$$\text{Direct stress } f_d = \frac{100,000}{12 \times 18} = 462 \text{ lbf/in}^2$$

Stress at (x, y) due to bending about xx :

$$\begin{aligned} f_{xx} &= \frac{100,000 \times 1.5 \times 12y}{12 \times 18^3} \\ &= 25.7y \text{ lbf/in}^2 \end{aligned}$$

Stress at (x,y) due to bending about yy :

$$f_{yy} = \frac{100,000 \times 1 \times 12x}{18 \times 12^3} = 38.5 \text{ lbf/in}^2$$

The stress distribution would best be shown by a three-dimensional model. It would have a horizontal base representing the cross-section, on which vertical ordinates would represent to scale the total stress in the fibres at each point. As the fibre stress varies linearly the surface representing the stress will be a plane. Thus it is only necessary to determine the stress at three points on the cross-section, which are not all in one line, in order to define the stress distribution. However, to give a more complete picture the stress has been evaluated for each corner and is tabulated below.

Corner	x in	y in	f_d lbf/in ²	f_{xx} lbf/in ²	f_{yy} lbf/in ²	Total stress lbf/in ² (compressive)
<i>A</i>	6	9	462	231	231	924
<i>B</i>	6	-9	462	-231	231	462
<i>C</i>	-6	9	462	231	-231	462
<i>D</i>	-6	-9	462	-231	-231	0

Negative stresses indicate tension

The stress distribution on a cross-section is shown in isometric projection in Fig. 7.2b. Notice that, in this example, the stress at corner *D* happens to be zero, whilst at the diagonally opposite corner the stress is double what it is at the other two corners.

It is perhaps important to point out that if the load is applied as a concentrated force directly to the brick pier (instead of through an extremely rigid distribution plate) the stress distribution in the neighbourhood of the point of application of the

concentrated load will be very much more complicated. However, within a short distance, say about twice the width of the pier, the stress distribution will have settled down to the simple linear one shown in Fig. 7.2b.

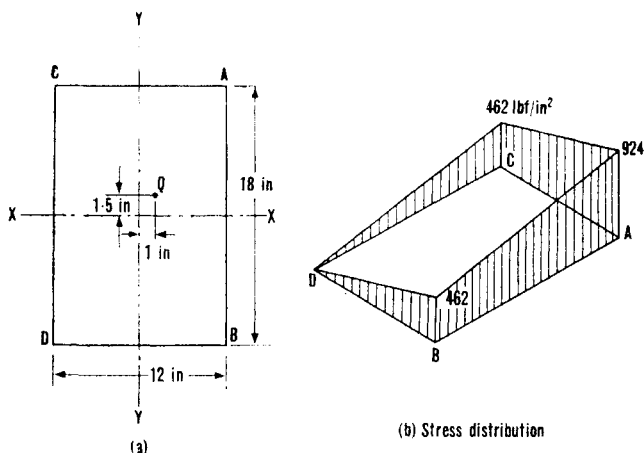


FIG. 7.2 Eccentrically loaded stocky column

Structures Relying on Gravity for Their Stability

Structures in this group can be thought of as cantilevers, sticking up from the ground and subjected to lateral loading, due to wind, water or earth pressure. The “cantilever” is not built-in at the ground, but held there and prevented from overturning by the pressure due to the weight of the structure above; so that every horizontal section through the structure is subjected to a direct stress, due to the weight of the structure above the section, and a bending stress, due to the eccentricity of the weight of the structure and to the lateral pressure. For continuous structures, such as retaining walls and dams,

it is only necessary to consider a unit slice, and neglect the complications due to the support of the end abutments.

In the case of structures built of materials of low tensile strength, such as brickwork, masonry, mass or plain concrete, it is necessary to ensure that tensile stresses do not develop, otherwise, cracks will probably occur, and this leads not only

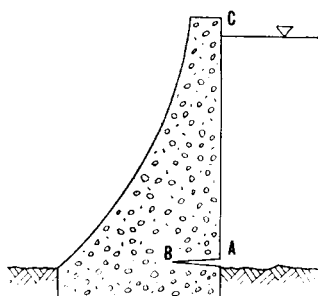


FIG. 7.3 Uplift pressure on a dam

to reduced strength and stability, but allows water to penetrate, which is particularly dangerous in the case of dams, where the high water pressure produces an additional overturning effect due to "uplift". The upper face of the crack at the base of the dam in Fig. 7.3 is subjected to the full water pressure, acting upwards, producing an overturning moment in addition to that due to the water pressure on the upstream face AC.

Stability of a Chimney

Example 7.3. A cylindrical brick chimney with internal and external diameters of 4.5 and 6 ft is 80 ft tall. Calculate the limiting value of the wind pressure if tension is not to develop. Assume that the wind pressure is uniform on a vertical pro-

jection of the chimney. Take the weight of brickwork as 120 lbf/ft³.

Let p be the limiting wind pressure in lbf/ft². The *critical section*, i.e. the one subjected to the greatest bending stress, is that where the chimney joins its base.

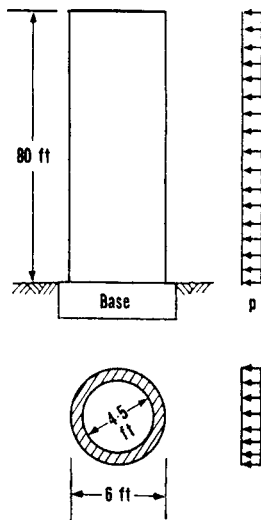


FIG. 7.4 Wind pressure on a chimney

Direct stress on base, due to weight of chimney

$$= 80 \times 120 = 9600 \text{ lbf/ft}^2$$

$$\text{b.m. at base} = 80 \times 6p \times 40 = 19,200p \text{ lbf ft}$$

$$\text{Modulus of section} = \frac{\pi}{64} \frac{(6^4 - 4.5^4)}{3} = 14.5 \text{ ft}^3$$

$$\text{bending stress at base} = \frac{19200p}{14.5} = 1324p \text{ lbf/ft}^2$$

If no tension is to be developed then the bending stress must not exceed the direct stress. Hence $1324p \geq 9600$

$$\text{or } p \geq 7.25 \text{ lbf/ft}^2$$

Note that the maximum pressure on the base, when this wind pressure is acting, will be double the direct stress. If the wind should exceed the above limiting value, the chimney would not immediately topple over, but cracking would probably occur at the base, on the windward side, whilst on the opposite side, the brickwork would be subjected to increasingly larger compressive stresses, which would eventually lead to crushing of the brickwork. This would happen whilst the line of action of the resultant thrust due to the weight of the chimney and the wind pressure was still within the perimeter of the chimney.

Stresses on the base of a dam

Example 7.4. A dam with a base 18 ft wide and a vertical upstream face retains water to a depth of 30 ft. A transverse slice of the dam, 1 ft thick, weighs 50,000 lbf, and its c.g. is 6 ft back from the upstream face. Determine the stress distribution on the base. Water weighs 62.4 lbf/ft³.

The water pressure on the vertical upstream face varies linearly from nothing at the free surface to $30 \times 62.4 \text{ lbf/ft}^2$ at the bottom. The total water pressure P on a 1 ft slice is therefore, $\frac{1}{2} \times 30 \times 62.4 \times 30 = 28,100 \text{ lbf}$, acting at the centre of pressure, 10 ft above the base.

The axis about which bending takes place is O , halfway along the base, in Fig. 7.5a. The modulus of the *critical section* is $18^2/6 = 54 \text{ ft}^3$ since it is rectangular and 1 ft wide.

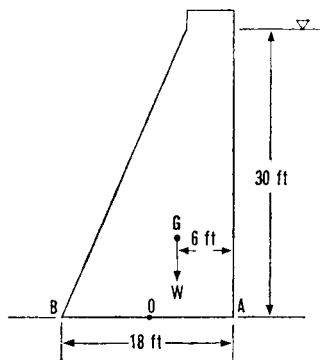
M , the b.m. at the critical section, comprises two opposing terms (as shown in Fig. 7.5b), the total water pressure term, and that due to the eccentricity of the weight of the dam.

$$M = 28,100 \times 10 - 50,000 \times 3 = 131,000 \text{ lbf ft}$$

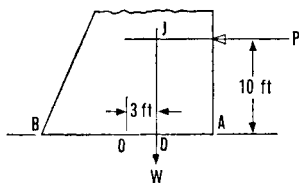
$$\text{Hence bending stress} = \frac{131,000}{54} = 2430 \text{ lbf/ft}^2$$

$$\text{direct stress} = \frac{50,000}{18} = 2780 \text{ lbf/ft}^2$$

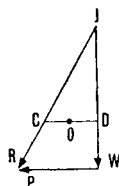
The final stress distribution thus varies linearly from $2780 + 2430 = 5210 \text{ lbf/ft}^2$ at the toe, to $2780 - 2430 =$



(a)



(b)



(c)

FIG. 7.5 Stability of a dam

350 lbf/ft² at the heel (upstream). In both cases the stresses are compressive, so that the dam would comply with the requirements of no tension.

Alternative method; based on the resultant thrust. R is the resultant thrust, due to the water pressure P and the weight of the dam W . Its line of action will pass through J , Fig. 7.5b, the intersection of W and P , and will cut the base at some point C , given by the triangle of forces for P and W ; i.e. from Fig. 7.5c:

$$CD : JD = P : W$$

$$\text{therefore} \quad CD = \frac{28,100 \times 10}{50,000} = 5.62 \text{ ft}$$

Therefore eccentricity of $R = OC = 2.62 \text{ ft}$

$$\begin{aligned} \text{Resultant b.m.} &= W \times 2.62 \\ &= 131,000 \text{ lbf ft, as previously.} \end{aligned}$$

Middle Third Rule

The rule is restricted to structures of solid, rectangular section, subjected to bending about a principal axis. Provided the resultant thrust falls within the *middle third* no tension will be developed. If it falls outside the middle third tension will occur. The middle third is that part of the cross-section extending one-sixth of the way either side of the principal axis about which bending takes place.

It will be seen that in the above example the resultant thrust is within the middle third, since the eccentricity of the resultant thrust where it cuts the base is $2.62/18$ or $1/6.86$ of the base.

Stability of a Retaining Wall

Example 7.5. A mass concrete retaining wall 12 ft high, of trapezoidal cross-section and vertical back, is 2 ft thick at the top. If it retains non-cohesive soil, level with its top, determine the minimum width required at the base. The concrete has a density of 144 lb/ft³ and the soil a density of 110 lb/ft³ and an angle of internal friction of 30°. Assume that no tension is allowed to develop in the concrete.

The theory of earth pressure is a branch of *soil mechanics* and a subject in its own right; it is sufficient here to say that as we are dealing with a non-cohesive soil the earth pressure on the wall is $K_a wh$ at a depth h ; where w is the density (or more accurately, the specific weight of the soil) and K_a is the coefficient of active earth pressure. K_a is related to the angle of internal friction of the soil ϕ , and is given by

$$K_a = \frac{1 - \sin \phi}{1 + \sin \phi}$$

In this case, $K_a = 1/3$ as $\sin \phi = 1/2$. The coefficient of active earth pressure is the ratio of the active earth pressure acting on the wall to the pressure which would exist if the soil were a fluid of the same density.

The maximum earth pressure acting on the vertical face of the wall $= \frac{1}{3} \times 110 \times 12 = 440$ lbf/ft². The average pressure is half this, so that P , the total earth pressure on a 1 ft slice, is $440 \times 12/2 = 2640$ lbf, and it acts at the centre of pressure, one-third of the way up the wall from the base.

Let the width of the base be b ft. Then from Fig. 7.6 it will be seen that W , the weight of a 1 ft slice of the wall is

$$W = \frac{1}{2} (b+2) \times 12 \times 144 = 864(b+2) \text{ lbf}$$

The distance $\bar{x}$ of the c.g. of the slice from the back of the wall

is given by

$$\bar{x} = \frac{2 \times 12 \times 1 + \frac{(b-2)}{2} \times 12 \left\{ \frac{(b-2)}{3} + 2 \right\}}{\frac{b+2}{2} \times 12} = \frac{b^2 + 2b + 4}{3(b+2)}$$

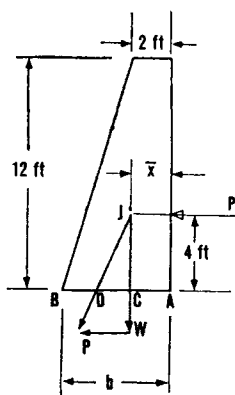


FIG. 7.6

From similar triangles in Fig. 7.6

$$\frac{CD}{JC} = \frac{P}{W}$$

Therefore $CD = \frac{2640 \times 4}{864(b+2)} = \frac{12.22}{(b+2)}$

But D is at the edge of the middle third, hence

$$AD = \frac{2}{3}b = \bar{x} + CD = \frac{b^2 + 2b + 4}{3(b+2)} + \frac{12.22}{(b+2)}$$

This reduces to the quadratic

$$b^2 + 2b - 40.6 = 0$$

the positive root of which gives $b = 5.45$ ft

Pre-stressed Concrete Beams

It is not intended to deal with the theory of pre-stressed concrete beams, beyond stating that here is another case of combined bending and direct stress. Steel tendons running the length of the beam are tensioned by hydraulic jacks, and the reaction to this tension is carried as a compression in the concrete. If the centre of gravity of the steel tendons coincides with the centroid of the concrete section the pre-stress in the concrete will be a uniform direct stress. If, as is more usual, the c.g. of the steel is not concentric with the centroid of the concrete then the pre-stress will consist of two components, a direct stress P/A , and a bending stress Pey/I , where: P is the total pre-stressing force (or tension in the tendons after removing the hydraulic jacks); e is the eccentricity, or distance of the c.g. of the tendons from the centroid of the concrete; A is the area of the concrete, and I its second moment of area about the appropriate principal axis; and y is the distance to the outer fibres from the principal axis.

In addition to the pre-stress, the concrete will be subjected to bending stresses due to the loading, both dead load and live or imposed load. These stresses are calculated using the methods of Chapter 3, treating the beam as made of concrete alone, capable of resisting tensile as well as compressive stresses. Actually, the combined stress in the concrete is never allowed to become tensile, or if it is, it is restricted to a very low value. The compressive pre-stress must be sufficient to neutralise any tensile bending stresses due to the loading.

Example 7.6. A pre-stressed concrete beam of rectangular cross-section is 12 in wide by 18 in deep and is subjected to a sagging b.m. of 1,296,000 lbf in due to the loading. The total pre-stressing force is 216,000 lbf, acting 12 in from the top of the beam, on the vertical centre line of the cross-section, i.e. this is the position of the c.g. of the steel.

Calculate the stresses in the top and bottom fibres due to (a) the pre-stressing force alone, (b) the pre-stressing force and the loading.

Area of cross-section (neglecting area of steel)

$$= 12 \times 18 = 216 \text{ in}^2$$

$$I \text{ about horizontal principal axis} = 12 \times \frac{18^3}{12} = 5840 \text{ in}^4$$

$$\text{Eccentricity of pre-stressing force} = 12 - 9 = 3 \text{ in}$$

$$\text{Direct stress} = \frac{216000}{216} = 1000 \text{ lbf/in}^2$$

$$\begin{aligned} \text{Bending stress due to pre-stressing force} &= \frac{216000 \times 3 \times 9}{5840} \\ &= 1000 \text{ lbf/in}^2 \end{aligned}$$

$$\begin{aligned} \text{therefore pre-stress at top fibres} &= 1000 - 1000 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{Pre-stress at bottom fibres} &= 1000 + 1000 \\ &= 2000 \text{ lbf/in}^2 \end{aligned}$$

We have a triangular stress distribution because the section is rectangular, and the pre-stressing force is applied at the bottom of the middle third.

Bending stress due to loading

$$= \frac{1,296,000}{5840} \times 9 = 2000 \text{ lbf/in}^2$$

The final stresses are

$$\text{at top} \quad 0 + 2000 = 2000 \text{ lbf/in}^2$$

$$\text{at bottom} \quad 2000 - 2000 = 0 \text{ lbf/in}^2$$

Exercise 7

(Specific weight of water 62.4 lbf/ft³)

1. The tie beam of a roof is 24 ft long and due to the thrust of the rafters there is a tension of 12,000 lbf in it. Calculate the maximum stress in the tie beam allowing for its own weight, if it is 6 in deep by 3 in wide. Timber weighs 50 lbf/ft³.

Ans: 967 lbf/in²

2. A masonry chimney 4 ft square externally and 2 ft square internally is 80 ft high and built of stone weighing 130 lbf/ft³. What is the maximum uniformly distributed wind pressure acting normal to one face of the chimney that the chimney can withstand without developing tension? What then will be the maximum stress in the stonework?

Ans: 8.12 lbf/ft²; 20,800 lbf/ft²

3. A brick chimney 240 ft high is of circular cross-section and tapers uniformly from 24 ft diameter at the base to 12 ft diameter at the top. The internal diameter at the base is 10 ft and the chimney weighs 1000 tonf. Determine the maximum wind pressure in lbf/ft² of projected area that the chimney can withstand without developing tension in the brickwork.

Ans: 17 lbf/ft²

4. A brick chimney 100 ft high has external diameters of 12 ft at the base 8 ft at the top. The total weight of the chimney is 175 tonf and the thickness of the brickwork at the base is 2 ft 3 in. The intensity of horizontal wind pressure p lbf/ft² on the projected area at height h ft above the base is given by $p = 10 + 0.15h$.

Determine the maximum and minimum intensities of vertical pressure at the base of the chimney.

(I.C.E. Apr. 1952)

Ans: 5.43 tonf/ft² compression; 0.346 tonf/ft² tension

5. Explain what principal axes of bending are.

A short timber post 8 in. square carries a longitudinal load of 12,000 lbf which is eccentric to both principal axes by 1 in. Calculate the maximum and minimum stresses that are developed.

Ans: 750 lbf/in² compression; 374 lbf/in² tension

6. The Tee bar shown in Fig. 7.7 is used as a tie to transmit an eccentric load. If the stress is not to exceed 10 tonf/in² calculate the maximum load that can be applied at Q .

Ans: 22.2 tonf

7. A 10 in $\times$ 6 in $\times$ 40 lbf B.S.B. stanchion carries the vertical loads shown in Fig. 7.8. Determine the greatest and least stresses, and where they occur.

Details of B.S.B. $A = 11.77$ in²; $I_{xx} = 205$ in⁴;
 $I_{yy} = 21.8$ in⁴

Ans: 13.12 tonf/in² compression at C ;
 2.24 tonf/in² tension at A

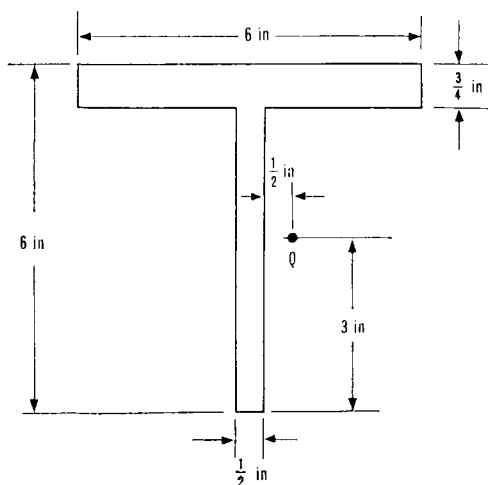


FIG. 7.7

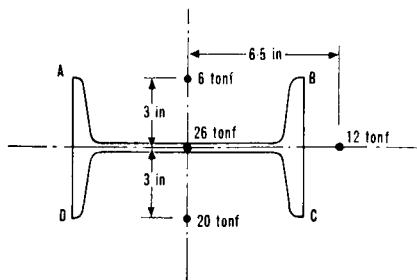


FIG. 7.8

8. A masonry dam is of trapezoidal section 32 ft high, 3 ft wide at the top and 18 ft wide at the base, with a vertical water face. The depth of water is 30 ft.

Determine:

- the point at which the resultant thrust cuts the base
- the greatest and least pressures on the base

Weight of masonry 150 lbf/ft³

State whether the dam is stable, and if not, indicate how this can be rectified.

Ans: (a) 11.72 ft from heel; (b) 2.38 tonf/ft²; 0.12 tonf/ft²; stable

9. Determine the maximum intensity of pressure on the ground due to the concrete dam shown in Fig. 7.9 when the reservoir is empty.

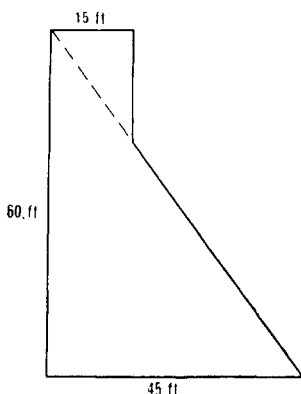


FIG. 7.9

Determine also whether tension will develop at the heel of the dam when the water level has risen to 3 ft below the crest. What then will be the maximum pressure on the ground? Specific weight of concrete is 150 lbf/ft³.

Ans: 10,330 lbf/ft²; No; 5370 lbf/ft²

10. A trapezoidal concrete dam 32 ft high retains water to within 2 ft of the crest. The upstream face is vertical and the width of the crest is one-third the width of the base. Determine the minimum width of the base if the dam is not to develop tension. Specific weight of concrete is 150 lbf/ft³.

Ans: 17 ft

11. Figure 7.10 shows the cross-section of a gravity dam built of concrete of density 150 lb/ft³. It has a vertical face and retains water to a depth of 33 ft. Determine the stresses at the heel and toe of the dam and explain whether you consider the dam is safe or not.

Ans: 810 lbf/ft² tension; 6750 lbf/ft² compression; unsafe

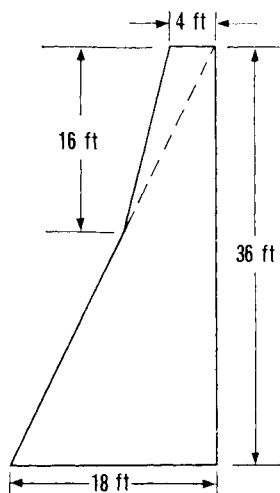


FIG. 7.10

12. A retaining wall with a vertical back has a trapezoidal section. Using the following data find the resultant thrust of the wall on its foundation and calculate the intensity of pressure on the base of the wall at the heel and toe. State the criterion used in deciding whether the wall is safe.

Width of wall at top	2 ft
Width of wall at base	4 ft
Height of wall	9 ft
Density of retained gravel, which is level with top of wall	90 lb/ft ³
Angle of repose of gravel	40°
Density of wall	140 lb/ft ³

Ans: 3850 lbf at 12° to vertical; 661 lbf/ft²; 1229 lbf/ft²

13. A concrete retaining wall has the section shown in Fig. 7.11. It retains cohesionless soil weighing 90 lb/ft³ with an angle of internal friction of 30°. Determine the resultant thrust per foot run due to the combined action of the earth pressure on the face *AB* and to the weight of the wall and hence calculate the distribution of pressure along the base *BC*.

Ans: 5270 lbf at 24.2° to vertical; 1240 lbf/ft² at toe and 360 lbf/ft² at heel

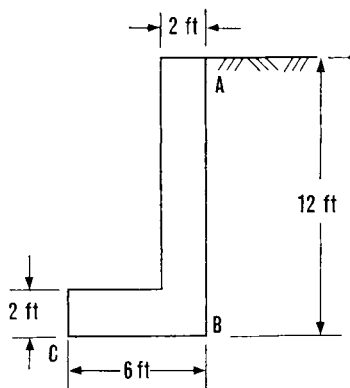


FIG. 7.11

14. Figure 7.12 is a diagrammatic representation of a flying buttress. It is constructed in masonry weighing 150 lbf/ft^3 and is 2 ft thick throughout its height. Determine the greatest value that P , the thrust trans-

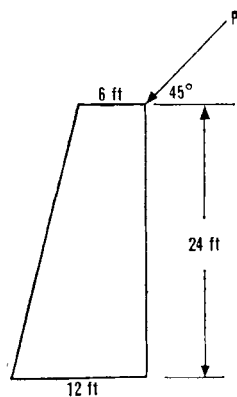


FIG. 7.12

mitted by the arch portion of the buttress, can have if no tension is to develop at the base of the buttress.

Ans: 19,000 lbf

15. A masonry dam of trapezoidal cross-section has a vertical upstream face. The total height of the dam is 26 ft and the maximum water-level is 24 ft above the base of the dam. If the top width is 5 ft determine the minimum thickness of the dam at the base if it is stipulated that no tension shall exist in the masonry. If you use a formula for this thickness prove it. Specific weight of masonry is 150 lbf/ft³.

(I.C.E. Apr. 1950)

Ans: 13.4 ft

16. A concrete beam is 8 in wide and 12 in deep and is pre-stressed by an axial force of 96,000 lbf. Calculate the greatest b.m. that the beam can carry (a) if no tension develops in the concrete, (b) if a tensile stress of 200 lbf/in² is allowed in the concrete.

If the tendon is moved down by 2 in what b.m. will the beam be able to resist without developing tension. What now will be the maximum stress in the concrete?

Ans: (a) 16,000 lbf ft; (b) 19,200 lbf ft; 32,000 lbf ft; 2000 lbf/in²

CHAPTER 8

Deflection of Beams

WHENEVER a beam is subjected to transverse loads a bending moment is set up at each cross-section, which is opposed by the moment of resistance created there by the straining of the fibres. This straining of the material of which the beam is made produces curvature of the beam, as was explained in the chapter on bending stresses.

In order to calculate the slope and deflection of beams it is necessary to treat a beam as a single line. If we consider what happens to the axis of a beam as it is loaded, we realise that it follows the deformations of the beam. We therefore use this line to represent the beam and we call it the *elastic line* of the beam.

Curvature

The curvature of a line at any position along it is equal to the rate of change of direction there, $d\theta/ds$; where θ is the inclination of the tangent to the curve and s the distance measured along the curve from an arbitrary starting point, see Fig. 8.1. P and Q are two points on a curve separated by a short distance δ_s along the curve. The tangent at P makes an angle θ with the reference axis and the tangent at Q makes an angle $\theta + \delta\theta$. C is the centre of curvature for this portion of the curve, which is sufficiently small to consider it as a part of a circle of radius R . The angle between the two tangents is $\delta\theta$ and is

the change in direction of the curve in going from P to Q . Therefore, the rate of change of direction at P is the limit of $\delta\theta/\delta s$ as δs tends to zero, i.e. the curvature equals $d\theta/ds$. But $\delta s = R\delta\theta$, therefore $R = ds/d\theta$. Hence curvature is the reciprocal of radius of curvature. Now, as we are dealing with a very flat curve in bent beams ds is very nearly equal to dx , provided the axis of the beam is taken as the x axis. Also θ is very nearly equal to the slope dy/dx .

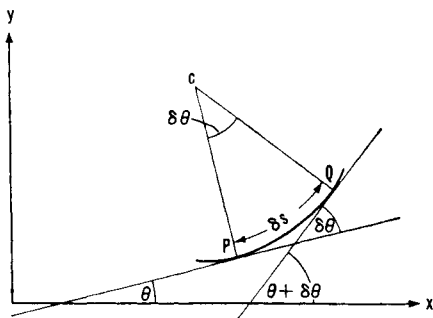


FIG. 8.1 Curvature

Hence

$$\frac{d\theta}{ds} \simeq \frac{d}{ds} \left(\frac{dy}{dx} \right) \simeq \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2y}{dx^2}$$

In other words, for initially straight beams, their curvature, when bent, is $\frac{d^2y}{dx^2}$. But as curvature is equal to $1/R$ and R is related to the bending moment at the section by the bending formula $M/I = E/R$ we obtain the equation

$$EI \frac{d^2y}{dx^2} = M$$

This is the differential equation for bending.

The slope of a bent beam varies along its length, hence the slope at any particular section is defined as the slope of the tangent to the elastic line at that section, and is equal to dy/dx .

Calculation of Beam Deflections

If we are to determine the deflection y , at a distance x along the beam we must be able to solve the second-order differential equation

$$EI \frac{d^2y}{dx^2} = M$$

This involves two successive integrations of M , with respect to x , which can only be done by expressing M as a function of x . In the following problems it is assumed that EI , the flexural rigidity of the beam, is uniform.

Cantilevers and Beams with Simple Loadings

Cantilever with end load

The origin for the x and y axes is chosen at the clamped end of the cantilever, which is placed for convenience on the left in Fig. 8.2. Note that the positive direction of y is upwards.

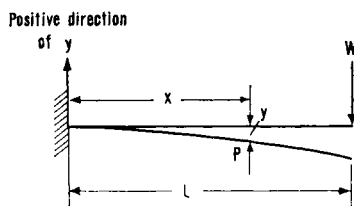


FIG. 8.2 Cantilever with end load

We then obtain a general expression for the b.m. in the cantilever by taking a representative point P on the beam distance x from the origin. The hogging bending moment at P is then

$W(l-x)$; and this expression will give the b.m. anywhere along the cantilever, by substituting the appropriate value of x .

$$\begin{aligned} EI \frac{d^2y}{dx^2} &= M \\ &= -W(l-x) \end{aligned}$$

(a minus sign because the b.m. is hogging)

$$= -Wl + Wx.$$

Integrating this expression with respect to x , and remembering that EI , W and l are constants, we get

$$EI \frac{dy}{dx} = -Wlx + \frac{Wx^2}{2} + A$$

where A is the constant of integration, and is found by substituting in this expression the *boundary condition* that at the clamped end the slope of the beam is always zero, i.e. at $x = 0$, $dy/dx = 0$. Hence $A = 0$. Thus the general expression for the slope anywhere along the cantilever is

$$EI \frac{dy}{dx} = -Wlx + \frac{Wx^2}{2}$$

Integrating this expression with respect to x we obtain

$$EIy = -\frac{Wlx^2}{2} + \frac{Wx^3}{6} + B$$

where B is the second constant of integration and is obtained by substituting in this expression, the *boundary condition* that at the clamped end the deflection is nothing, i.e. at $x = 0$, $y = 0$. Hence $B = 0$. So the general expression for deflection along the cantilever is

$$EIy = -\frac{Wlx^2}{2} + \frac{Wx^3}{6}$$

If we want, for example, the deflection Δ under the load we substitute $x = l$ in the last expression.

Hence
$$EI \Delta = -\frac{Wl^3}{2} + \frac{Wl^3}{6}$$

or
$$\Delta = -\frac{Wl^3}{3EI}$$

(the minus sign indicating downward deflection).

Cantilever with uniform loading

The cantilever in Fig. 8.3 carries a uniformly distributed load of intensity w . The general expression for the b.m. is

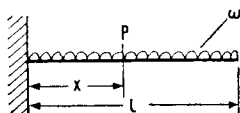


FIG. 8.3 Cantilever with uniform load

obtained by calculating the b.m. at the section P in terms of x , its distance from the clamped end.

$$M = -\frac{w}{2} (l-x)^2.$$

Hence the differential equation for deflection becomes

$$EI \frac{d^2y}{dx^2} = -\frac{wl^2}{2} + wlx - \frac{wx^2}{2}$$

Integrating with respect to x

$$EI \frac{dy}{dx} = -\frac{wl^2x}{2} + \frac{wlx^2}{2} - \frac{wx^3}{6} + A$$

The constant of integration A is obtained by substituting $dy/dx = 0$ when $x = 0$. Hence $A = 0$.

Therefore the general expression for the slope is:

$$EI \frac{dy}{dx} = -\frac{wl^2x}{2} + \frac{wlx^2}{2} - \frac{wx^3}{6}$$

Integrating again

$$EIy = -\frac{wl^2x^2}{4} + \frac{wlx^3}{6} - \frac{wx^4}{24} + B$$

The constant B is shown to be zero by substituting $y = 0$ at $x = 0$. Hence the general expression for deflection is

$$EIy = -\frac{wl^2x^2}{4} + \frac{wlx^3}{6} - \frac{wx^4}{24}$$

The end deflection Δ is obtained by putting $x = l$ into this expression,

$$EI\Delta = -\frac{wl^4}{4} + \frac{wl^4}{6} - \frac{wl^4}{24}$$

therefore
$$\Delta = -\frac{wl^4}{8EI}$$

Simple beam with uniform loading

The beam AB in Fig. 8.4 is simply supported at its ends and carries a uniformly distributed load of intensity w . In order to derive a general expression for the b.m. along the beam we

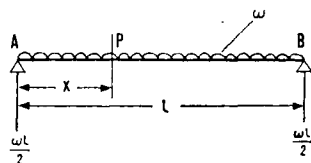


FIG. 8.4 Simple beam with uniform loading

consider a point P , distance x from the left-hand end A . The b.m. at P is the algebraic sum of the sagging bending moment due to the reaction at A and the hogging bending moment, due to the loading between A and P .

$$EI\frac{d^2y}{dx^2} = \frac{(wl)x}{2} - \frac{wx.x}{2}$$

Integrating with respect to x

$$EI \frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} + A$$

Since we do not know the slope at the support we proceed to integrate again, to obtain an expression for y ,

$$EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} + Ax + B$$

Note that the first arbitrary constant of integration A is also integrated. By substituting the boundary conditions, that at $x = 0$, $y = 0$, we obtain $B = 0$, and that at $x = l$, $y = 0$.

Therefore
$$0 = \frac{wl^4}{12} - \frac{wl^4}{24} + Al + 0$$

hence
$$A = -\frac{wl^3}{24}$$

Therefore the general expression for deflection is

$$EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24}$$

If we want the deflection at any particular distance along the beam, all we do is to substitute this distance for x in the general expression. For example, the deflection Δ at the centre is given by putting $x = l/2$.

$$EI\Delta = \frac{wl}{12} \left(\frac{l}{2}\right)^3 - \frac{w}{24} \left(\frac{l}{2}\right)^4 - \frac{wl^3}{24} \left(\frac{l}{2}\right)$$

therefore
$$\Delta = -\frac{5wl^4}{384EI}$$

Simple beam with central load

In Fig. 8.5 the expression for the b.m. at a general point P is $Wx/2$, but is only true provided x is less than $l/2$. However,

since the load is central, we only need concern ourselves with the left-hand half of the beam, since the right-hand half behaves symmetrically.

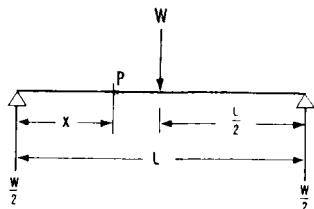


FIG. 8.5 Simple beam with central load

The differential equation is

$$EI \frac{d^2y}{dx^2} = \frac{Wx}{2}$$

Integrating, we arrive at the slope equation

$$EI \frac{dy}{dx} = \frac{Wx^2}{4} + A$$

Integrating again with respect to x

$$EIy = \frac{Wx^3}{12} + Ax + B$$

As $y = 0$ at $x = 0$, $B = 0$, but we cannot put $y = 0$ at $x = l$; since the expression for the b.m. is only true for $x \leq l/2$. However, from the symmetry of the deflected beam we see that the slope at the centre (where the deflection is the greatest) is zero. Hence substituting $dy/dx = 0$ at $x = l/2$ in the general expression for the slope of the beam we get

$$0 = \frac{W}{4} \left(\frac{l}{2} \right)^2 + A$$

therefore

$$A = -\frac{Wl^2}{16}$$

Hence the general expression for the deflection of the beam is

$$EIy = \frac{Wx^3}{12} - \frac{Wl^2x}{16}$$

Remember x must not exceed $l/2$.

The central deflection, Δ , is obtained by substituting $x = l/2$ into the above expression giving $\Delta = -\frac{Wl^3}{48EI}$

The results of these four special, simple cases are summarised in Fig. 8.6. The maximum deflection in each case can be ob-

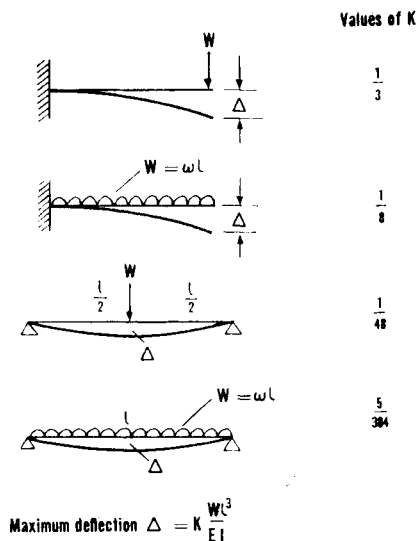


FIG. 8.6

tained by substituting the appropriate constant K into the expression $\Delta = KWl^3/EI$, where W is the total load, which in the case of uniform loading equals wl . These four results are in common use and worth remembering.

The above four loadings exhaust the types of loading that can be dealt with in the elementary way just demonstrated,

except that, of course, we can combined the end load on a cantilever, with a uniform loading; and the central concentrated load on a simple beam, with uniform loading.

Example 8.1. Calculate the maximum deflection of a beam of flexural rigidity $EI = 13,000 \times 80 \text{ tonf in}^2$, whose dead load is 95 lbf/ft, when simply supported on a span of 20 ft and carrying a central load of 2 tonf.

Maximum deflection due to dead load (in)

$$\begin{aligned} &= \frac{5(wl)^3}{384EI} \frac{\text{tonf} \times \text{in}^3}{\text{tonf in}^2} \\ &= \frac{5}{384} \times \frac{95 \times 20}{2240} \times \frac{(20 \times 12)^3}{13000 \times 80} \\ &= 0.147 \text{ in} \end{aligned}$$

Maximum deflection due to central load

$$\begin{aligned} &= \frac{Wl^3}{48EI} \\ &= \frac{2 \times (20 \times 12)^3}{48 \times 13000 \times 80} \\ &= 0.554 \text{ in} \end{aligned}$$

Therefore, by the principle of superposition, the total central deflection $= 0.554 + 0.147 = 0.701 \text{ in.}$

This is equivalent to $0.701/240 = 1/342$ of the span.

Macaulay's Method

Except for the simple cases dealt with above, the expression for the bending moment will be different for the various portions of the beam and so a special technique is needed to deal with this situation. Consider the simple beam in Fig. 8.7 car-

rying a non-central load W . The b.m. at the section P , distance x from the left hand support A is $Wbx/l - W[x-a]$. The term in square brackets is only included if $x > a$, because if P is

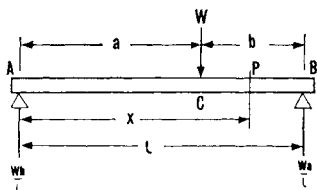


FIG. 8.7

between A and C the expression for b.m. reduces to Wbx/l . Hence the differential equation for deflection becomes

$$EI \frac{d^2y}{dx^2} = \frac{Wbx}{l} - W[x-a]$$

remembering to include the final term only when $[x-a]$ is positive. It is important to realise that the terms in square brackets, in Macaulay's method, must remain intact and must not be multiplied out, until a particular value of x is substituted.

Integrating in the usual way for the term without square brackets, and integrating the term in square brackets with respect to $(x-a)$, we obtain

$$EI \frac{dy}{dx} = \frac{Wbx^2}{2l} - \frac{W}{2}[x-a]^2 + A$$

Repeating the process, we get

$$EIy = \frac{Wbx^3}{6l} - \frac{W}{6}[x-a]^3 + Ax + B$$

The constants of integration are obtained, as before, by substituting the appropriate boundary conditions, but only including the terms in square brackets when they are positive.

At $x = 0$, $y = 0$, therefore $B = 0$.

At $x = l$, $y = 0$,

$$\text{therefore} \quad 0 = \frac{Wbl^3}{6l} - \frac{W}{6}(l-a)^3 + Al + 0$$

But $l-a = b$. Note that once a particular value has been substituted for x , the square brackets are dispensed with.

$$A = -\frac{Wab}{6l}(l+b)$$

Hence the general expression for the deflection of the beam is

$$EIy = \frac{Wbx^3}{6l} - \frac{Wab}{6l}(l+b)x - \frac{W}{6}[x-a]^3$$

the final term only being included when $[x-a]$ is positive. Hence the deflection Δ under the load is given by

$$EI\Delta = \frac{Wa^3b}{6l} - \frac{Wa^2b}{6l}(l+b)$$

$$\text{therefore} \quad \Delta = \frac{Wa^2b^2}{3EI}$$

Example 8.2. A beam AB , 12 ft long is simply supported at its ends and carries loads of 6 tonf and 12 tonf at C and D , 4 ft and 8 ft from the left end. If the flexural rigidity of the

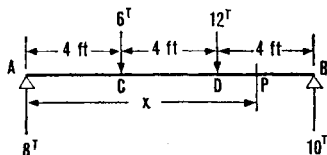


FIG. 8.8

beam is $13,000 \times 150$ tonf in², calculate the deflection at the centre of the span.

The beam and the forces acting on it are shown in Fig. 8.8. The origin of the reference axes is taken at *A* the left-hand end of the beam and the general expression for the b.m. is obtained by calculating the b.m. at the section *P* situated in the extreme right-hand portion of the beam *DB*.

M = algebraic sum of clockwise moments of forces acting on *AP*, about *P*

$$= 8x - 6[x - 4] - 12[x - 8]$$

Therefore

$$EI \frac{d^2y}{dx^2} = 8x - 6[x - 4] - 12[x - 8]$$

$$EI \frac{dy}{dx} = 4x^2 - 3[x - 4]^2 - 6[x - 8]^2 + A$$

$$EIy = \frac{4x^3}{3} - [x - 4]^3 - 2[x - 8]^3 + Ax + B$$

$$\text{At } x = 0, y = 0, \text{ therefore } B = 0.$$

$$\text{At } x = 12, y = 0$$

$$\text{therefore } 0 = \frac{4}{3} \times 12^3 - 8^3 - 2 \times 4^3 + 12A$$

$$A = -138.7 \text{ tonf ft}^2$$

Central deflection Δ in inches is given by

$$\begin{aligned} \frac{13000 \times 150}{12^2} \times \frac{\Delta}{12} &= \frac{4}{3} \times 6^3 - (6 - 4)^3 - 138.7 \times 6 \\ &= -552 \end{aligned}$$

$$\text{Therefore } \Delta = -0.488 \text{ in}$$

This is slightly less than the maximum deflection. The deflection will be a maximum where the slope is zero. Now dy/dx

$= 0$ when $x = 6.2$ ft, as can be checked by substituting this value into the general slope equation above. Putting $x = 6.2$ ft into the general deflection equation above, we find that the maximum deflection is 0.490 in. The difference between this value and the central deflection is negligible in this case.

Overhanging Beam

Example 8.3. Notice that for the overhanging beam in Fig. 8.9 we have placed the overhang on the right, to simplify the algebra. Distances are measured from the left-hand end and

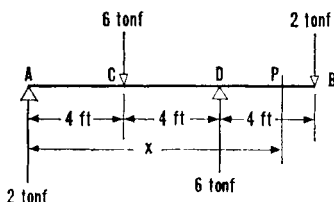


FIG. 8.9

the b.m. at P , near the right-hand end, is considered, as in the previous examples.

$$EI \frac{d^2y}{dx^2} = 2x - 6[x - 4] + 6[x - 8]$$

$$EI \frac{dy}{dx} = x^2 - 3[x - 4]^2 + 3[x - 8]^2 + A$$

$$EIy = \frac{x^3}{3} - [x - 4]^3 + [x - 8]^3 + Ax + B$$

At $x = 0$, $y = 0$, therefore $B = 0$.

At $x = 8$, $y = 0$,

therefore
$$0 = \frac{8^3}{3} - 4^3 + 8A$$

$$A = -13.3$$

The deflection y_B of the free end is given by:

$$EIy_B = \frac{12^3}{3} - 8^3 + 4^3 - 13.3 \times 12 = -32$$

If the flexural rigidity EI of the beam is 3×10^5 tonf in²,

$$\frac{3 \times 10^5}{12^2} \times \frac{y_B}{12} = -32$$

$$y_B = -0.184 \text{ in}$$

Exercise 8

(E for steel = 13000 tonf/in²)

1. A metre stick, 1 in. wide and $\frac{1}{4}$ in thick, is clamped at one end and projects out horizontally a distance of 30 in. If a weight of 0.5 lbf is hung at the free end determine the deflection half-way along the cantilever and at its end. Take $E = 1.5 \times 10^6$ lbf/in².

Ans: 0.72 in. 2.30 in

2. Calculate the safe inclusive uniformly distributed load for a 10 in $\times$ 6 in B.S.B. if the maximum permissible bending stress is 10.5 tonf/in² and the maximum permissible deflection is $1/325$ of the span when simply supported on a span of (a) 10 ft, (b) 20 ft. I_{xx} of B.S.B. = 205 in⁴.

Ans: (a) 28.6 tonf, (b) 11.0 tonf

3. Show that for a simply supported beam of span l carrying a uniformly distributed load the maximum deflection δ is given by:

$$\frac{\delta}{l} = \frac{5}{48} \times \frac{f}{E} \times \frac{l}{y}$$

where f is the maximum fibre stress, y is the distance of the outer fibres from the neutral axis, and E is Young's modulus.

Determine the ratio of the depth of a beam to its span to satisfy the following conditions:

The beam is uniformly loaded and simply supported at its ends. The maximum bending stress is to be 10 tonf/in² and E is 13,000 tonf/in². The maximum deflection is to be 1/325 of the span. The cross-section of the beam is symmetrical about the neutral axis.

Ans: 1:19·2

4. A uniform beam of weight W and length $3a$ is simply supported at its left-hand end and at a distance a from the other end. It carries a concentrated load W mid-way between the supports. Determine the deflection at the right-hand end.

Ans: $\frac{5}{24} \frac{Wa^3}{EI}$ upwards

5. A load-measuring device is to consist of a square section steel bar simply supported at its ends 6 in. apart. The load, which is limited to 2 tonf, is to be applied centrally and the deflection at the centre of the bar is to be measured by a dial gauge graduated in 0·0001 inch divisions. Determine the required cross-section of the bar if each division of the dial is to correspond to 0·05 tonf. Any expression for the deflection of the bar must be derived from the basic differential equation for bending.

Calculate also the maximum stress developed in the bar.

Ans: 1·2 in. 10·4 tonf/in²

6. A simply supported beam spans 12·5 ft and carries a 6 tonf load 9 ft from the left-hand end. Starting from

$$EI \frac{d^2y}{dx^2} = M$$

determine the position of maximum deflection and calculate its value in inches. $EI = 13000 \times 150$ tonf in².

Ans: 0·165 in at 6·92 ft

7. A simple beam with a second moment of area of 90 in⁴ spans 20 ft and carries loads of 3 tonf and 2 tonf at 8 ft and 14 ft from the left-hand end. Calculate the deflection at mid-span.

Ans: 1·09 in

8. Figure 8.10a shows a simple beam AB carrying a load W at C . Show that the deflection at D is

$$-\frac{Wc}{6EI} (l-d)(2ld-c^2-d^2)$$

If the load is transferred from C to D , as in Fig. 8.10b, show that the deflection at C has the same value, i.e. the deflection at D due to a load W at C is equal to the deflection at C when the load W is placed at D .

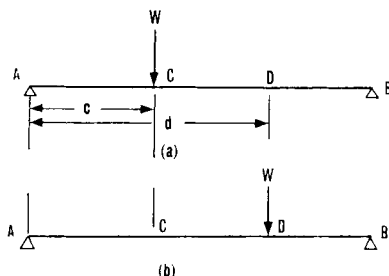


FIG. 8.10

9. A uniform cantilever of length l carries a uniform loading of intensity w . Calculate the value of the upward force to be applied at the free end so that the deflection there is nothing.

Ans: $\frac{3}{8} wl$

10. If the cantilever in question 9 is propped at a distance $l/3$ from its free end so that there is no deflection at the prop calculate the reaction carried by the prop.

Ans: $\frac{17}{24} wl$

11. A horizontal cantilever of length L and constant flexural rigidity, EI , is subjected to a load of $wL/2$ at its free end and a uniform distributed load of intensity w per unit length over that half of the cantilever nearer the fixed end, as shown in Fig. 8.11. Determine the deflection of the free end of the cantilever.

(Eng. Inst. Oct. 1958)

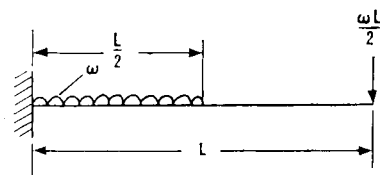


FIG. 8.11

$$\text{Ans: } \frac{71}{384} \frac{wL^4}{EI}$$

12. Prove that the vertical deflection at C of the uniform cantilever shown in Fig. 8.12 is $27WL^3/128EI$.

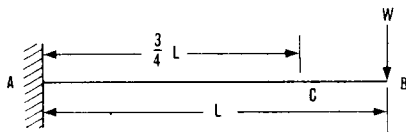


FIG. 8.12

Hence determine the magnitude of the force in a prop inserted at C which will reduce the deflection at B to zero.

(I.C.E. Apr. 1952)

$$\text{Ans: } 1.58W$$

13. A horizontal cantilever of constant flexural rigidity and length 12 ft carries vertical loads of W at the free end and $2W/3$ at mid-length. If the member consists of a 12 in. $\times$ 6 in. B.S.B. ($I_{xx} = 376 \text{ in}^4$) determine the value of W :

(a) if the maximum stress due to bending is not to exceed 9 tonf/in²

(b) if the deflection at the free end due to bending is not to exceed $\frac{1}{2}$ in.

The weight of the member itself is to be neglected.

(I.C.E. Apr. 1953)

$$\text{Ans: (a) } 2.94 \text{ tonf; (b) } 2.04 \text{ tonf}$$

CHAPTER 9

Buckling of Struts

It is vital, when considering the equilibrium of structural members subjected to *compression*, to study the stability of that equilibrium. A member may, according to calculation, be in equilibrium, but if that equilibrium is unstable, the member would collapse, as the slightest disturbance would upset its equilibrium. The reason why the problem only arises with members in compression, is that a member subjected to a pair of equal and opposite tensile forces will straighten (if not already straight) in stretching, and thus reduce any eccentricity of loading. A member subjected to a pair of compressive forces can allow its ends to approach each other by bending, thus increasing the eccentricity of the load and the stresses present over its central portion.

Before considering the buckling of struts let us examine this question of stability.

Stability of Equilibrium

A body in equilibrium yet free to move or deflect in certain ways may be perfectly *stable*. Such a body if given a small displacement, immediately returns to its equilibrium position because this position corresponds to a minimum potential energy, such as the ball resting in a concave hollow shown in Fig. 9.1a. On the other hand the body may be in *unstable*

equilibrium, so that when given a small displacement it continues to move or deform, because the equilibrium state corresponds to a position of maximum potential energy, like the ball on top of a convex hump in Fig. 9.1c.

There is a third kind of equilibrium, *neutral equilibrium*. A body in this state, if given a small displacement, will neither try to return to its original position, nor continue to move. It

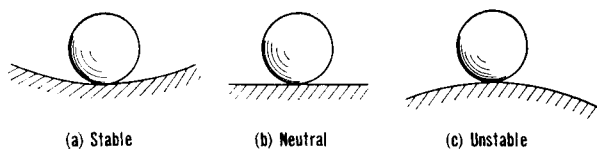


FIG. 9.1 Forms of equilibrium

will merely remain in its displaced position. This is represented by the ball on a horizontal plane in Fig. 9.1b. During any small displacement the potential energy of this system does not change.

Buckling of struts

A mechanical analogy was chosen to illustrate the three states of equilibrium but these states also apply to struts. Consider the initially straight, slender strut, in Fig. 9.2a, subjected to an axial load P . It is of uniform section with a knife edge at each end resting in a Vee groove. The groove at the bottom is fixed, whilst the one at the top is constrained to slide vertically. When the load P is small, the strut will be in stable equilibrium. If the strut is given a small lateral displacement near its centre, it will return to its initial, straight position when the lateral force is removed. However, with a large axial force, the strut would immediately collapse if given a tiny displacement. The equilibrium is unstable. For one particular

load the strut will be in a state of neutral equilibrium, so that if given a small lateral displacement, it will stay in its deflected position. The axial load which puts the strut into a state of neutral equilibrium is called the critical, buckling load. With real struts it is impossible to exceed the critical (buckling) load

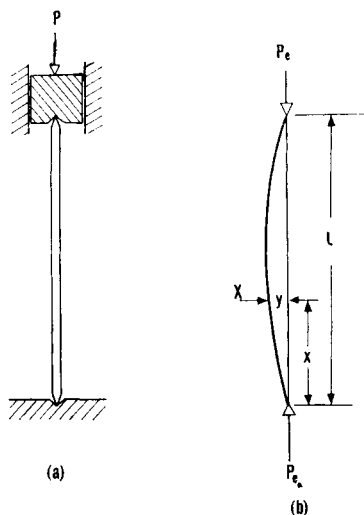


FIG. 9.2 Slender strut

and reach a state of unstable equilibrium, due to imperfections in the strut, which prevent the load ever being perfectly axial. It is rather like trying to balance a pencil on its point.

It is important to realise that this type of buckling has nothing to do with exceeding permissible stresses. The strut can still be within the elastic limit when the critical load is reached.

Determination of Critical Load

The following theory is based on that of the Swiss mathematician Euler (1707–1783). Let P_e be the critical or Euler buckling load. This is the axial load which puts the strut into a state of neutral equilibrium, so that once given a deflection it will stay that way. Figure 9.2b shows the elastic line of the strut in this state. The *elastic line* is the deflected centre line of the member. If y is the deflection at the point X , distance x from the lower end, then the b.m. at X is $P_e y$. The moment of resistance opposing this b.m. is $EI \times$ curvature of strut at X . EI is called the flexural rigidity of the strut and is the product of the modulus of elasticity of the material and the second moment of area of the cross-section of the strut. Curvature is, of course, the reciprocal of radius of curvature and is equal to the rate of change of direction d^2y/dx^2 , since the departure from straightness is quite small. Hence

$$EI \frac{d^2y}{dx^2} = -P_e y$$

The negative sign is accounted for by the fact that d^2y/dx^2 and y are always of opposite sign.

Now a sine curve will satisfy the above differential equation, as can be seen by substituting $y = c \sin (\pi x/l)$; where c is the value of y at the centre, where $x = l/2$, provided

$$P_e = \frac{\pi^2 EI}{l^2}$$

$$\text{Since} \quad \text{L.H.S.} = -EIc \frac{\pi^2}{l^2} \sin \frac{\pi x}{l}$$

$$\text{R.H.S.} = -P_e c \sin \frac{\pi x}{l}$$

$P_e = \pi^2 EI/l^2$ is the Euler formula for the critical load of a strut of length l , loaded through knife edges, pins or balls,

which are unable to transmit any bending moment. If we write $p_e = P_e/A$ for the Euler buckling stress, then we can rewrite the Euler formula in the form

$$p_e = \frac{\pi^2 E}{(l/k)^2}$$

where k = radius of gyration of the cross-section = $\sqrt{I/A}$.

Therefore, the direct stress that will cause elastic buckling depends only upon the modulus of elasticity of the material of the strut and the slenderness ratio l/k .

If buckling is not confined to a single plane, but can take place in any direction, the value of k (or I) must be the least value for the cross-section.

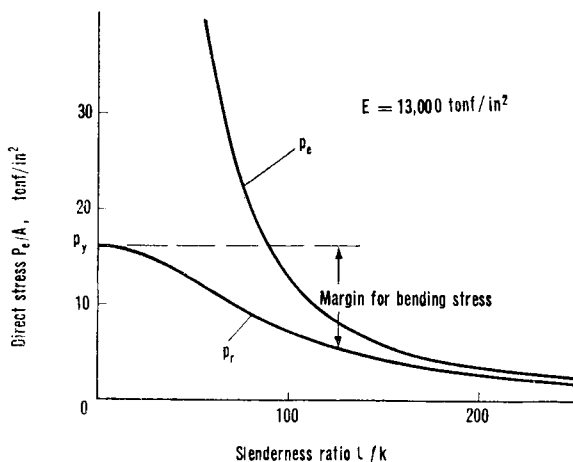


FIG. 9.3 Effect of slenderness ratio on the direct stress to cause buckling

A graph showing the relationship between p_e and l/k for steel struts is shown in Fig. 9.3. It will be seen that once the slenderness ratio has fallen to 90 the direct stress to cause elastic buckling has risen to about 16 tonf/in^2 which is equal to the minimum guaranteed yield strength of mild steels to

B.S.15. And do not forget that this is only the mean stress on the cross-section. There are bound to be bending stresses of unknown value superimposed, so that a steel strut with a slenderness ratio of 90 would fail due to plastic deformation somewhat before the direct stress reached the Euler value, because locally the total stress was greater than the yield strength of the material.

Obviously, if a strut is not to fail the direct stress must not exceed either the Euler stress or the yield stress p_y . In order to allow a margin for bending stresses, so that the total stress does not exceed p_y , we can use the "Rankine" buckling stress p_r . This stress is determined by the relationship

$$\frac{1}{p_r} = \frac{1}{p_y} + \frac{1}{p_e}$$

and is also plotted in Fig. 9.3. The curve intersects the vertical axis at the yield stress — 16 tonf/in² — and merges into the curve of Euler stress at large values of l/k .

The above expression for p_r can be rewritten in the form

$$p_r = \frac{p_y}{1 + (p_y/\pi^2 E)(l/k)^2}$$

which closely resembles the empirical formula developed by Lord Rankine

$$p_R = \frac{p_c}{1 + a(l/k)^2}$$

Unfortunately, the value of 21 tonf/in² adopted for p_c in the case of mild steel is higher than the yield stress of such steels; nor does his constant a equal $p_c/\pi^2 E$, so that p_r and p_R do not agree exactly. However, the agreement is sufficiently close to justify calling p_r the Rankine buckling stress. In any case, the point is somewhat academic because the maximum stresses in struts permitted by the British Standard Specification 449 on

“The Use of Structural Steel in Building” are based on the somewhat complicated Perry–Robertson formula which takes account of the *probable* initial lack of straightness of struts, obtained by a statistical analysis of the results of loading tests on numerous struts carried out by Robertson.

Factor of Safety and Load Factor

No engineer builds anything without leaving a margin of strength to allow for unpredictable overloads, occasional poor material or workmanship, and the fact that the strength of a structure can only be calculated if it is assumed to behave in a simplified manner. There are two ways in which the engineer can provide this margin of safety; either he can ensure that the maximum calculated stress developed anywhere in the structure does not exceed a certain value — called the maximum permissible stress — which is some arbitrary fraction of the stress which will cause the material to fail in the tensile test. The ratio of the “failure stress” to the maximum permissible stress is called the *factor of safety*.

The failure stress in the tensile test is usually taken as the yield stress of the material, if it is mild steel, rather than the tensile strength, since a structure has to all intents failed once its deformations become excessive due to yielding of the material at some point. The yield stress is that stress at which the material ceases to behave elastically, and deforms plastically without appreciable increase of stress. In materials which do not possess a definite yield point the 0.1 per cent proof stress can be chosen as the specified failure stress. The 0.1 per cent proof stress is that stress which will produce a *permanent strain* of 0.1 per cent.

The second method of providing a reserve of strength in a structure to cope with departures from the idealised conditions

assumed to exist when calculating its strength, is to design the structure so that it just fails when subjected to a loading which is a multiple of the design loading. The ratio of the loading to produce failure, to the design loading, is known as the *load factor*.

The load factor method is the more logical one to use, since a load factor of two means that a structure designed to such a load factor should not fail until the loading on it is double that for which it is designed. On the other hand the stresses in a structure designed with a factor of safety of two may be double the permissible stresses long before the loading on it has been doubled. Such a situation, however, only occurs in *non-linear* structures i.e. structures in which the deformations are not proportional to the load. For example, a strut behaves in a non-linear way when it starts to buckle; the deflections, and hence the bending stresses, increase at a more rapid rate than the load. So it is essential to use the load factor method when designing struts. Beams, within the elastic limit, behave as linear elastic structures, in other words their deflections are proportional to loads producing them, so a factor of safety of two means a load factor of two as well. As it is often easier to think in terms of safe stresses rather than fractions of the failure load, the factor of safety approach is the one most commonly used in the design of beams and structures which are not subject to buckling.

Example 9.1. Determine the required modulus of section for a beam to carry an inclusive load of 1225 lbf/ft run on a simple span of 16 ft made of steel with a minimum yield strength of 16 tonf/in² (a) using a factor of safety of 1.525 (b) using a load factor of 1.525.

$$(a) \text{ Allowable bending stress} = \frac{16}{1.525} = 10.5 \text{ tonf/in}^2$$

$$\text{Maximum b.m.} = 1225 \times \frac{16^2}{8} \text{ lbf ft}$$

$$= 210 \text{ tonf in}$$

$$\text{Required modulus of section} = \frac{210}{10.5}$$

$$= 20 \text{ in}^3$$

$$(b) \text{ Maximum b.m.} = 1.525 \times 1225 \times \frac{16^2}{8} \text{ lbf ft}$$

$$= 320 \text{ tonf in}$$

$$\text{Maximum bending stress to be } 16 \text{ tonf/in}^2$$

$$\text{Required modulus of section} = \frac{320}{16}$$

$$= 20 \text{ in}^3$$

Both methods give the same result, since we are dealing with a beam within the elastic range of behaviour and it is assumed that the compression flange of the beam is restrained against lateral buckling.

Example 9.2. Calculate the safe axial load that a $\frac{1}{2}$ in diameter steel rod 30 in long, with ball end constraints can carry if it is to have a load factor of 2 against collapse. The yield strength of the steel is 16 tonf/in² and its modulus of elasticity is 13,000 tonf/in².

$$\text{Radius of gyration of rod, } k = \frac{1}{4} \times \text{diameter}$$

$$= \frac{1}{8} \text{ in}$$

$$\text{cross-sectional area} = \frac{\pi}{4} \left(\frac{1}{2}\right)^2 = 0.196 \text{ in}^2$$

$$\text{therefore slenderness ratio } l/k = \frac{30}{1/8} = 240$$

The Rankine buckling stress p_r is given by

$$p_r = \frac{p_y}{1 + \frac{p_y}{\pi^2 E} \left(\frac{l}{k} \right)^2} = \frac{16}{1 + \frac{16 \times 240^2}{\pi^2 \times 13000}} = 1.95 \text{ tonf/in}^2$$

$$\text{Axial load to cause failure} = 1.95 \times 0.196$$

$$= 0.382 \text{ tonf}$$

$$\text{safe axial load} = \frac{0.382}{\text{load factor}}$$

$$= 0.191 \text{ tonf}$$

Note that it would be wrong to apply a *factor of safety* of 2 to the yield stress and substitute a value of 8 tonf/in² into the Rankine formula in place of p_y . This would give an axial load of 0.34 tonf and the load factor would be only $0.382/0.34 = 1.12$; not 2.

Effect of Different End Constraints

The Euler and Rankine strut formulae apply to so-called *pin-ended* struts; that is, struts whose ends are not allowed to move laterally, but are free to rotate, without developing end moments. Additional end constraint, such as rigidly connecting the end of a strut to a beam, has the effect of increasing the load required to buckle the strut. Conversely, reducing the end constraint, say, by allowing the top to sway over, will reduce the buckling load. A simple way of allowing for various conditions of end constraint, without altering the formula to be used, is to substitute an *effective length* for the actual length of the strut. The following examples will show how it is possible to do this.

Strut with built-in ends

The fixed ended strut in Fig. 9.4a has its lower end rigidly clamped to the ground. The top end is guided and rotation is prevented, so that the only possible movement at the top is a

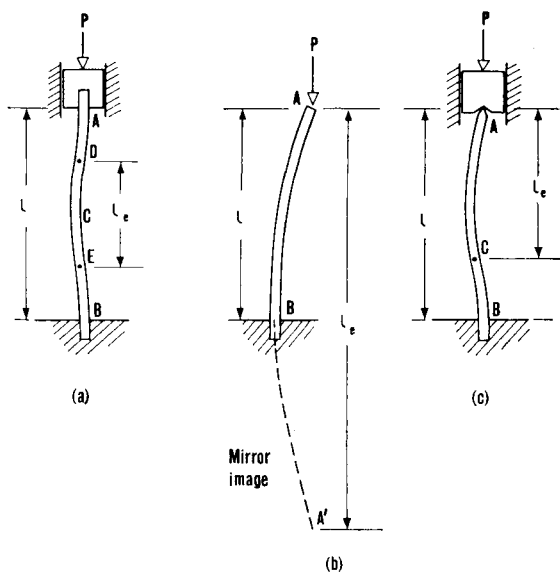


FIG. 9.4 Effective lengths of struts

vertical one. Points of contraflexure — where the curvature and hence the b.m. are zero, occur at D and E which, as AD is similar to CD, are a distance $l/2$ apart. Therefore the portion DE will behave as a pin-ended strut, and its buckling load will be the buckling load for the complete strut. Hence the buckling load of the fixed-ended strut is given by substituting an effective length of $l/2$ in the Euler or Rankine formulae. The Euler buckling load for a strut with fixed ends is thus four times as great as when the strut is pin-ended.

Strut with one end free

A study of Fig. 9.4b will show that the effective length of the strut is twice its actual length, thus giving an Euler buckling load of only one quarter that of the corresponding pin-ended strut.

Strut with one end fixed and the other pinned

A point of contraflexure C will occur somewhere along the strut, nearer the built-in end, as in Fig. 9.4c. It is not at a distance of $l/4$ from the fixed end, as in the case of the strut with both ends built-in, but approximately $0.3l$ from the fixed end, giving an effective length of approximately $0.7l$. The result can only be obtained by a mathematical analysis and is not obvious, as it is in the two previous cases. The Euler buckling load for a *pinned-fixed* strut is approximately double that for a strut which is pinned at both ends.

Columns in buildings

The end conditions dealt with above are idealisations. Not many structures incorporate pins or ball ends; and if they did they would still provide some bending restraint due to friction, even if lubricated. Again, it is impracticable to provide perfect knife edges to the ends of the struts; and any blunting would lead to bending restraint. The perfect built-in end would require an infinitely rigid support — another impossibility. Therefore the effective lengths of columns in buildings are determined empirically, in the case of steel columns, by reference to B.S. 449, which gives the effective lengths to be used for various forms of connection between columns and intersecting beams. The strut formulae are not applied to reinforced concrete columns, which are always very much more stocky than

steel ones, but the Code of Practice for the use of reinforced concrete, C.P. 114, gives a table of arbitrarily determined reduction coefficients for loads on long columns.

Example 9.3. A steel bar 30 in long has a rectangular cross-section 1.5 in $\times$ 1.0 in. It carries an axial load which is transmitted through a hardened steel ball at each end of the strut. Calculate the Euler buckling load and estimate the probable collapse load if the steel has a yield stress of 16 tonf/in². $E = 13,000$ tonf/in².

The minimum second moment of area about a principal axis

$$= \frac{1.5 \times 1.0^3}{12} = 0.125 \text{ in}^4$$

$$\begin{aligned} \text{Euler buckling load } P_e &= \frac{\pi^2 EI}{l^2} \\ &= \frac{\pi^2 \times 13000 \times 0.125}{30^2} \\ &= \mathbf{17.8 \text{ tonf}} \end{aligned}$$

Crushing load of a short length of bar

$$\begin{aligned} &= \text{cross-sectional area} \times \text{yield stress} \\ &= 1.5 \times 1.0 \times 16 \\ &= \mathbf{24 \text{ tonf}} \end{aligned}$$

Rankine buckling load P_r is given by

$$\frac{1}{P_r} = \frac{1}{24} + \frac{1}{17.8} = 0.0977$$

$P_r = \mathbf{10.25 \text{ tonf}}$, which is the probable collapse load. Notice that load is less than both the Euler buckling load and the crushing load for a short length.

Example 9.4. Determine the diameter of a steel rod 25 in long to carry an axial load of 2.8 tonf through pinned ends. The yield stress of the steel is 16 tonf/in². Allow a load factor of 2.

This problem is more difficult than the inverse one of finding the buckling load of a given strut. It can be solved, however, by forming an equation in d , the required diameter of the rod.

$$\text{Now } \frac{1}{p_r} = \frac{1}{p_y} + \frac{1}{p_e}$$

$$\begin{aligned} \text{But } p_r &= \frac{\text{load factor} \times \text{safe axial load}}{\text{cross-sectional area}} \\ &= \frac{2 \times 2.8}{(\pi/4)d^2} = \frac{7.13}{d^2} \text{ tonf/in}^2 \end{aligned}$$

$$p_y = 16 \text{ tonf/in}^2$$

$$\begin{aligned} \text{and } p_e &= \frac{\pi^2 E}{(l/k)^2} \\ &= \frac{13000\pi^2}{(25 \times 4/d)^2} = 12.8d^2 \end{aligned}$$

$$\text{since } k = d/4.$$

$$\begin{aligned} \text{Therefore } \frac{d^2}{7.13} &= \frac{1}{16} + \frac{1}{12.8d^2} \\ d^4 - 0.445d^2 - 0.556 &= 0 \end{aligned}$$

Solving this quadratic in d^2 gives

$$d^2 = 1.0$$

$$d = 1.0 \text{ in}$$

Example 9.5. A compound strut consists of two 10 in $\times$ 3 in $\times$ 19.28 lbf British Standard channels placed back to back and held together by battens welded to the flanges. Determine what the distance between the channels should be, if the radius of gyration about both principal axes is to be the same. Assuming a load factor of 2 and a yield stress for the steel of 16 tonf/in², determine the maximum safe concentric load that the strut can carry, if it has an effective length of 20 ft.

10 in $\times$ 3 in $\times$ 19.28 lbf Channel

Area	I_{xx}	I_{yy}	Distance of centroid from back of channel
5.67 in ²	82.7 in ⁴	4.0 in ⁴	0.74 in

Refer to Fig. 9.5 for the pair of channels

$$I_{xx} = 2 \times 82.7 = 165.4 \text{ in}^4$$

$$k_{xx} = \sqrt{\left(\frac{165.4}{2 \times 5.67} \right)} = 3.82 \text{ in}$$

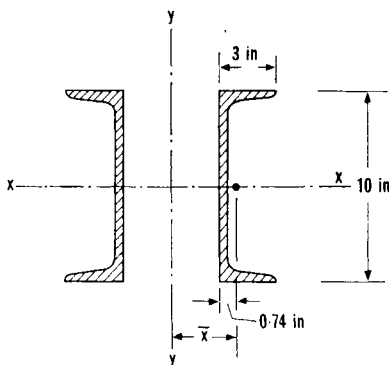


FIG. 9.5

Therefore $k_{yy} = 3.82 \text{ in}$

But $I_{yy} = 2\{4 + 5.67(\bar{x})^2\}$, where $\bar{x}$ is the distance of the c.g. of channel from yy axis.

$$\text{Therefore } k_{yy}^2 = \frac{2\{4 + 5.67(\bar{x})^2\}}{2 \times 5.67} = 3.82^2$$

$$\bar{x} = 3.72 \text{ in}$$

So channels must be $2(3.72 - 0.74) = 5.96 \text{ in}$ apart, say 6 in

$$\text{Slenderness ratio } \frac{l}{k} = \frac{20 \times 12}{3.82} = 62.8$$

$$\text{Therefore Euler buckling stress} = \frac{13000\pi^2}{62.8^2} = 32.5 \text{ tonf/in}^2$$

The Rankine buckling stress p_r is given by

$$\frac{1}{p_r} = \frac{1}{16} + \frac{1}{32.5}$$

$$p_r = 10.7 \text{ tonf/in}^2$$

$$\begin{aligned} \text{Rankine buckling load} &= 10.7 \times 2 \times 5.67 \\ &= 121.5 \text{ tonf} \end{aligned}$$

Dividing by the load factor,

$$\begin{aligned} \text{maximum safe load} &= \frac{121.5}{2} \\ &= 60.8 \text{ tonf} \end{aligned}$$

Exercise 9

(E for steel = 13,000 tonf/in²)

1. The slenderness ratio of a strut is defined as the ratio of the length to the least radius of gyration of the cross-section. For a mild-steel strut with pinned ends and of uniform cross-section use the appropriate Euler formula and plot a curve showing the critical direct stress as ordinate against the slenderness ratio as abscissa in the range of slenderness ratios from 50–200. Indicate how you need to modify this curve to give the failing loading over the whole range. The yield stress of the steel is 16 tonf/in².

(Eng. Inst. Oct. 1958)

2. A steel bar whose cross-section is $\frac{3}{4}$ in $\times$ $\frac{1}{2}$ in is axially loaded through ball end constraints. State the probable mode of failure and estimate the failure load if the bar is (a) 2 in long, (b) 30 in long, (c) 12 in long. Yield stress of the steel is 18 tonf/in².

Ans: (a) 6.75 tonf (b) 1.11 tonf (c) 3.43 tonf

3. A column 10 ft long is made of two 6 in $\times$ 3 in steel channels welded to form a box section. The thickness of the flanges of the channels is 0.5 in and of the webs 0.4 in. The load is transmitted axially to the column through ball seatings. Calculate the Euler buckling load.

B/.B.T.S. 14

If the yield stress of the steel is 16 tonf/in² estimate the probable buckling load, carefully explaining how you arrive at your estimate.

Ans: 440 tonf; 118 tonf

4. A central load of 176 lbf on a $\frac{1}{2}$ in diameter steel rod, 10 in long and simply supported at its ends, just causes permanent deformation of the beam. Calculate the yield stress of the steel in tonf/in².

If the same rod had been used as a strut, axially loaded through ball constraints, calculate at what load you would expect it to fail.

Ans: 16 tonf/in²; 1.76 tonf

5. A pair of shear legs consists of two steel tubes 10 ft long of $2\frac{1}{2}$ in outside diameter and 2 in internal diameter, pinned together at the top and inclined to each other at an angle of 50°. The plane containing the tubes is inclined at 70° to the horizontal. The legs are pinned to the ground and are prevented from tipping by a horizontal cable attached to the top of the legs, meeting them symmetrically.

Determine the greatest load that the shear legs can lift, allowing a load factor of four on the Euler buckling load.

Ans: 4.25 tonf

6. A column is built up of two 6 in $\times$ 3 in steel channels back to back. How far apart should the backs of the channels be placed to make the radii of gyration about both principal axes of inertia equal?

If the column is 24 ft long and encastré at both ends determine the Euler and Rankine buckling loads.

Rankine constants: $p_e = 21$ tonf/in²; $a = 1/7500$ for hinged ends.

Properties of 6 in $\times$ 3 in channel. $A = 3.65$ in²;

$I_{xx} = 21.3$ in⁴; $I_{yy} = 2.8$ in⁴; distance of centroid from back of channel = 0.89 in.

Ans: 2.72 in; 260 tonf; 104 tonf

7. Describe the difference between the behaviour of slender struts and stocky struts when loaded to failure.

A column in a building is 15 ft long and consists of two 8 in $\times$ 3 in steel channels back to back. The channels have $I_{xx} = 46.7$ in⁴, $I_{yy} = 3.58$ in⁴, $A = 5.49$ in² and their centres of area are 0.83 in from the back of the channels. If it is assumed that the columns are ball ended, calculate the maximum safe load from the Euler formula, using a load factor of 8.

Do you think this is a good arrangement of the channels? Give your reasons.

Ans: 7.30 tonf

8. A mild steel column 20 ft long and fixed at its ends is made of an I section 8 in deep and 6 in wide. The flanges and web are both $\frac{1}{2}$ in thick. Calculate the Euler and Rankine crippling loads. Explain why they differ.

Rankine constants: $p_e = 21 \text{ tonf/in}^2$; $a = 1/7500$ for pinned ends.

Ans: 161 tonf; 99.3 tonf

9. A compound column, 20 ft long, with fixed ends, is to consist of two 15 in $\times$ 4 in B.S. channels arranged as shown in Fig. 9.6. The two

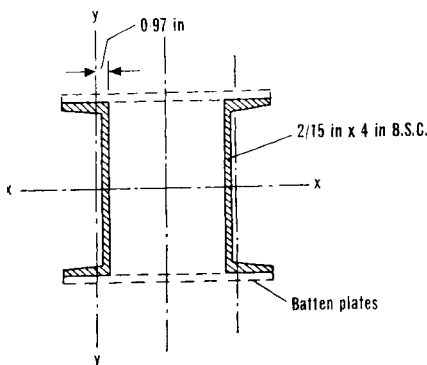


FIG. 9.6

channels are to be connected together by narrow transverse batten plates welded to the flanges. Determine (a) the minimum distance between the backs of the channels, and (b) the maximum distance between the centres of consecutive batten plates so that the section develops its maximum resistance to buckling. Estimate the safe axial load on the column by any method with which you are familiar.

For one 15 in $\times$ 4 in B.S. Channel $I_{xx} = 349.1 \text{ in}^4$;

$I_{yy} = 13.34 \text{ in}^4$; area = 10.7 in^2

(I.C.E. Apr. 1950)

Ans: (a) 9.26 in; (b) 23.55 in; (c) 153 tonf

10. Calculate the diameter of a solid, circular steel strut with an effective length of 10 ft to carry an axial load of 36 tonf with a load factor of two against failure. $p_y = 16 \text{ tonf/in}^2$.

Ans: 4 in

CHAPTER 10

Deflection of Trusses — Strain Energy Method

Strain Energy

When a structure is loaded it deflects and work is done by the loads in moving their points of application. Provided the structure remains within the elastic limit, this work is recoverable and is stored as strain energy in the stretched or compressed “fibres” of the material of which the structure is made, in the same way as a clock spring or archer’s bow stores energy when distorted.

If one makes the usual assumption that a structure rests on rigid supports, which do not “give” under load, then no energy is absorbed by the supports and the whole of the work done by the loads is stored as strain energy in the various parts of the structure.

If we consider a structure with just *one concentrated load* on it and assume, as is usually the case, that the structure deflects in proportion to the load, then the work done by the load, and hence the strain energy in the structure will be equal to half the product of the load and the deflection of the load point.

$$U = \frac{1}{2} W\Delta$$

where U is the strain energy. The reason for the factor of a half, is that the average force exerted is only $W/2$, since the load

starts from nothing, when the deflection is zero, and builds up to its final value of W , when the deflection has become Δ . This can also be seen from the load-deflection graph in Fig. 10.1. The area under the graph, $\frac{1}{2}W\Delta$ is the work done by the load during the deformation, thus, if we can determine the

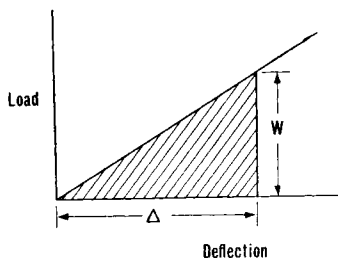


FIG. 10.1 Work done during elastic deformation

value of the strain energy stored in a structure, carrying a single load, we can calculate the deflection under the load from:

$$\Delta = \frac{U}{\frac{1}{2}W}$$

Deflection of a truss

The simplest case to consider is that of a pin-jointed truss carrying a single load. Each bar of the truss will be subjected to purely axial loading, so that if we consider a typical bar of the truss and represent the internal force in it by P , and its stretch, or shortening, by δ , then the strain energy in the bar will be $\frac{1}{2}P\delta$, and the total strain energy U in the truss will be $\frac{1}{2}\Sigma P\delta$, the sum of the strain energies in the individual bars.

Hence

$$\Delta = \frac{\Sigma P\delta}{W}$$

The change in length of a bar δ is given by:

$$\delta = \frac{PL}{AE}$$

where L = length of the bar,

A = cross-sectional area of bar,

E = modulus of elasticity for the bar.

Therefore
$$\Delta = \frac{1}{WE} \sum \frac{P^2 L}{A}$$

provided E is the same for all the bars of the truss, which is normally the case.

Steps in the strain energy method of analysis for trusses

- 1) Analyse the truss and determine the forces in all the bars.
- 2) Draw up a table, listing all the bars, and the lengths and cross-sectional areas.

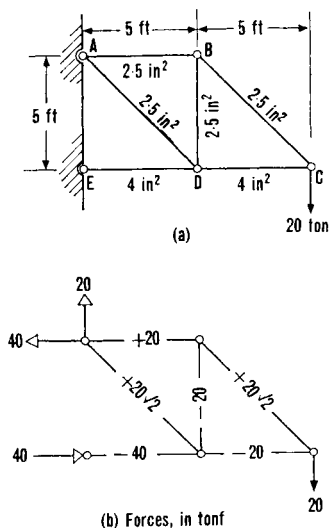


FIG. 10.2

(3) Evaluate P^2L/A for each bar and obtain the total, $\Sigma(P^2L/A)$

(4) Calculate

$$\Delta = \frac{1}{WE} \Sigma \frac{P^2L}{A}$$

taking care to ensure the correct units for each term.

N.B. This method can only be used to obtain the deflection directly under a load, when this is the only load on the structure.

Example 10.1 Determine the deflection of the cantilever truss in Fig. 10.2a, under the load. The cross-sectional areas of the bars are given on the figure. $E = 13000 \text{ tonf/in}^2$.

The forces in the bars are easily determined and are shown in Fig. 10.2b, the + sign indicating tension (although, as P is squared, the signs of the forces in the bars do not matter).

Table for evaluating $\Sigma \frac{P^2L}{A}$

Bar	L ft	A in ²	P tonf	P^2L/A tonf ² ft/in ²
<i>AB</i>	5	2.5	+20	800
<i>BC</i>	$5\sqrt{2}$	2.5	$+20\sqrt{2}$	2260
<i>CD</i>	5	4	-20	500
<i>DE</i>	5	4	-40	2000
<i>AD</i>	$5\sqrt{2}$	2.5	$+20\sqrt{2}$	2260
<i>BD</i>	5	2.5	-20	800
Σ				8620

$$\text{Strain energy} = \frac{1}{2} \cdot \frac{8620.12}{13000} \text{ tonf in}$$

Work done by 20 tonf load = $\frac{1}{2} \cdot 20\Delta$ tonf in if Δ is in inches.

$$\text{Therefore } \Delta = \frac{8620.12}{13,000.20} = 0.397 \text{ in}$$

Exercise 10

$$(E = 13,000 \text{ tonf/in}^2)$$

1. The truss shown in Fig. 10.3 is hinged to the ground at *A* and supported on rollers at *C* and carries a central load of 5 tonf. All the members have a cross-sectional area of $\frac{3}{4}$ in². Calculate the vertical deflection under the load.

Ans: 0.18 in

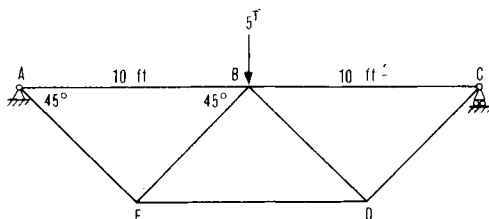


FIG. 10.3

2. The pin-jointed frame illustrated in Fig. 10.4 is pinned to a vertical wall at *A* and *B* and carries a vertical load of 2000 lbf at *D*. The cross-sectional area of each bar is $\frac{1}{4}$ in². Calculate the vertical deflection of *D*.

Ans: 0.056 in

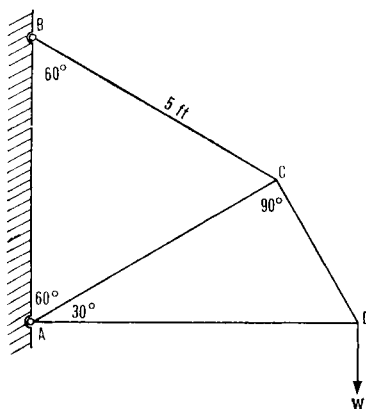


FIG. 10.4

3. Figure 10.5 shows a truss cantilevered from a vertical wall and carrying a single load of 4 tonf at D . The ringed figures against each bar are the cross-sectional areas of the members in in^2 . Calculate the vertical deflection of the load point.

Ans: 0.172 in

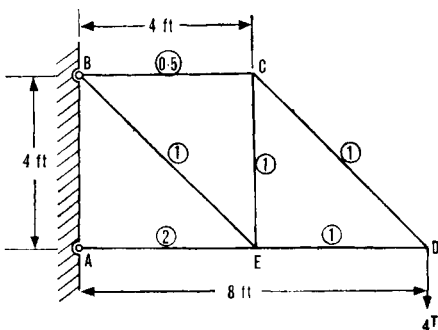


FIG. 10.5

4. Figure 10.6 represents a four bay N girder. Each bay is 6 ft long by 6 ft deep. The top boom members have a cross-sectional area of 5 in^2 , all other members have a cross-sectional area of 2.5 in^2 . If the truss carries a vertical load of 40 tonf at J calculate the vertical deflection there.

Ans: 0.50 in

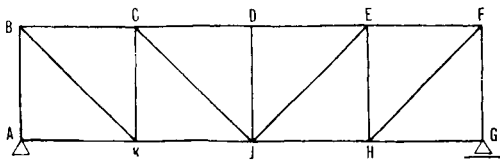


FIG. 10.6

5. If, in question 4, the 40 tonf load were placed at H calculate the vertical deflection of the load.

Ans: 0.36 in

6. The roof truss in Fig. 10.7 carries a single vertical load of 4 tonf at F . Calculate the vertical deflection of F if all the members have a cross-section of 0.2 in^2 .

Ans: 0.81 in

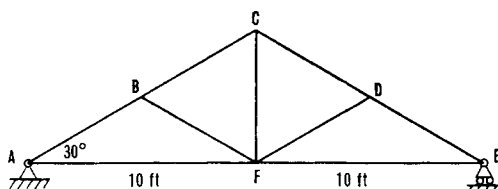


FIG. 10.7

7. If, in question 6, the 4 tonf load had been applied at *B*, calculate the deflection there.

Ans: 0.693 in

8. In the simply supported, pin-jointed frame of Fig. 10.8 all the members have a cross-sectional area of 0.25 in². Find the vertical deflection of the point *G* when the frame carries a vertical load of 4 tonf at *G*.

Ans: 0.43 in

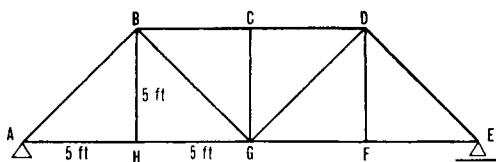


FIG. 10.8

9. Find the vertical deflection of point *F* if, in question 8, the load had been placed at *F* instead of *G*.

Ans: 0.286 in

10. Calculate the vertical component of the displacement in question 1 of Exercise 11.

Ans: 0.202 in

CHAPTER 11

Deflection of Trusses — Graphical Method

IN THE previous chapter it was shown that the deflection of a pin-jointed truss at the load point, when carrying a single, concentrated load, can be found by equating the work done by the load to the strain energy stored in the bars of the truss. A method will now be described for determining the displacement of each joint of a statically determinate, pin-jointed, plane frame, under any system of loads, in its own plane. The method is graphical and consists essentially of two parts (for all but the simplest of trusses). First, a *Williot diagram* has to be constructed to determine the displacement of each joint on the assumption that one of the bars of the truss which is pinned to the ground remains fixed in direction. As this is not, in fact, the case with most trusses, it is necessary to construct a second diagram, called the *Mohr diagram*, to allow for the rotation of this bar. In order to simplify the explanation it is best to deal with the Williot and Mohr diagrams separately before bringing them together in a single example.

Displacement of a Loaded Particle Supported by Two Cords

The particle *C*, in Fig. 11.1a, carries a load *W*, and is supported by two cords, *AC* and *BC*. The cords will thus have tensile forces acting in them, which can be determined by construct-

ing the triangle of forces for the particle C , or by any other convenient means. The tension in the cords will cause them to stretch by amounts which can be calculated, provided the elastic properties, or *stretchability*, of the cords is known. Let us assume that AC stretches by an amount CC_a and that

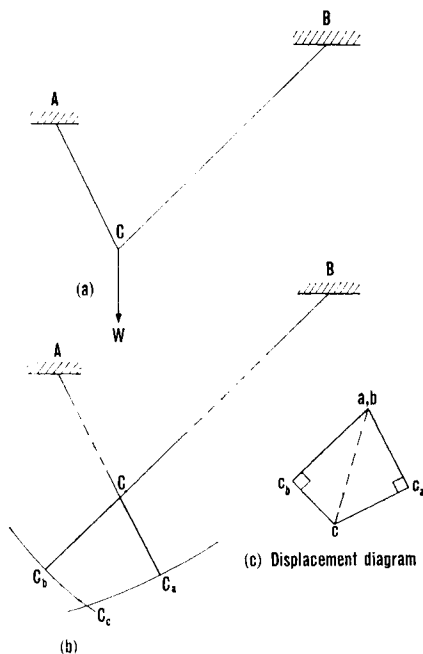


FIG. 11.1

BC stretches by an amount CC_b , as in Fig. 11.1b, then the displaced position of C will be C_c , obtained by swinging an arc of radius AC_a with centre A , and an arc of radius BC_b , with centre B and taking their intersection. Now as the extensions CC_a and CC_b are very small compared with the original lengths of the cords, it would be impracticable to use compasses to perform this construction. The extensions of the cords are

drawn to an enlarged scale and the arc C_aC_c is replaced by a straight line at right angles to the radius AC_a , and the arc C_bC_c by a straight line at right angles to BC_b . Hence the displacement of C can be obtained as a vector (giving both the horizontal and vertical movement of C) by constructing a displacement diagram, Fig. 11.1c. The stretch of AC is represented by a vector $\mathbf{ac}_a$ whose length is proportional to the extension of AC and drawn parallel to AC . Similarly, the extension of BC is represented by the vector $\mathbf{bc}_b$, noting that a and b are the same point, since neither suffers displacement. Lines are then drawn at right angles to these vectors to intersect at c . This diagram is, of course, the same shape as the "enlarged view" in Fig. 11.1b, but notice the notation is different. Lower case letters are used, as in all vector diagrams, and the order of the pair of letters designating a vector, gives its sense. The displacement of C when loaded is thus represented by the vector $\mathbf{ac}$ (shown dashed) and can be resolved into horizontal and vertical components, if so desired.

Construction of Williot Diagram for a Cantilever Truss

Figure 11.2a shows a pin-jointed truss pinned to a vertical wall at A and E . It carries loads, (not shown) as a result of which the various bars are stretched or shortened by amounts which can be calculated in a particular case. We note that joint D is linked by a pair of bars to fixed points A and E . Hence the displacement of D can be determined as in the example above. The starting point of the Williot or displacement diagram is the pair of coincident points a, e representing the fixed points A and E in the truss. If we assume the bar ED to be in compression, then the joint D is going to approach E , and we draw a vector $\mathbf{ed}_e$ in the direction DE . Similarly, if we assume

AD to be in tension, D will move away from A , and the vector ad_a representing the extension of AD is drawn parallel to AD . The thin lines $d_e d$ and $d_a d$ are drawn at right angles to the vectors representing the changes in length of the bars and intersect at d , to give the displaced position of D . The displaced position of B can now be determined, since it is linked

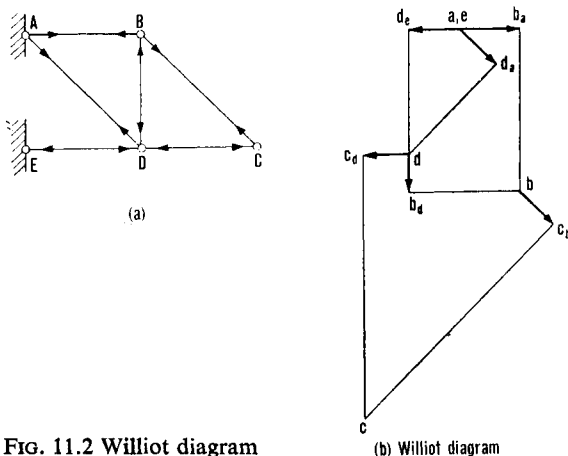


FIG. 11.2 Williot diagram

directly to A and D . A is a fixed point, of course, and the displaced position of D is now known, so, by drawing a vector ab_a to represent the stretch of AB (assuming it to be a tie) and a vector db_d to represent the shortening of DB (assuming it to be a strut), the displaced position of B is given by the intersection of the lines perpendicular to the last two vectors. Finally, the displaced position of C is obtained, locating it by the stretch of BC — vector bc_b , and the shortening of DC — vector dc_a , together with the lines at right angles to them, through their extremities. The displacement of any joint is given by the vector joining a , the fixed reference point, to the point identified by the same letter as the joint concerned. For example, the deflection of C when the frame is loaded is

represented in magnitude and direction by the vector joining a to c . If only the vertical deflection is required, the vertical component of ac is measured.

The arrowed vectors represent to scale, in a particular case, the changes in length of the bars, or the movement of the far end of a bar with respect to an end whose position is fixed,

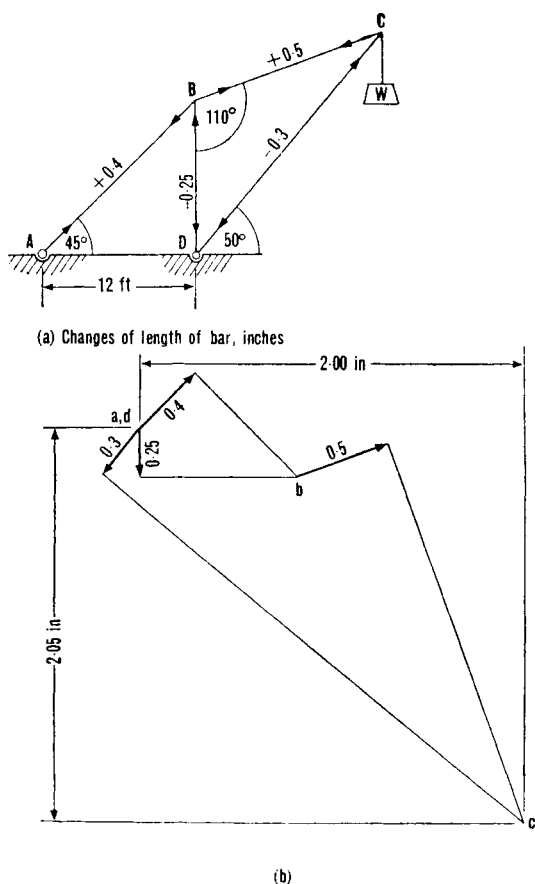


FIG. 11.3

or has just been determined. Hence the arrowheads point in the same direction as the arrowheads denoting the direction of the force exerted by a bar on the joint at the end whose position has already been determined (see Fig. 11.2a).

Example 11.1. The jib crane shown in Fig. 11.3a carries a load W , which produces changes in length of the various members whose values are shown against the appropriate bars. The units are in inches and a positive sign indicates stretch. The arrows indicating tension or compression have been added to Fig. 11.3a as they simplify the task of constructing the Williot diagram.

A and D are fixed, and therefore we start the displacement diagram from the point a, d , locating first b and then c . The vertical deflection of C is 2.05 in downwards, C also moves 2.00 in to the right, these values being the vertical and horizontal components of the vector given by joining a and c . The displacement of joint B is given by the vector ab , i.e. 0.25 in vertically downwards and 0.81 in to the right. The simple strain energy method of Chapter 10 would only give the *vertical* deflection of the load point C .

Effect of small rotations of a truss

Figure 11.4a represents a truss which is given a small clockwise rotation about a point O . If rays are drawn from the centre or pole O to each joint then the movement of each joint will be perpendicular to its ray, and the magnitude of the displacement will be proportional to length of the ray. Thus we can represent the displacement of each joint by vectors oa, ob etc. radiating from a single point o , each at right angles to OA, OB , etc., and proportional in length to OA, OB , etc., as in Fig. 11.4b. If we join ab, bc, cd , etc., as in Fig. 11.4c we get a figure similar to the truss. In fact Fig. 11.4c is in all respects

similar to Fig. 11.4a merely being to a different scale and turned clockwise through 90° .

Figure 11.4c is called a Mohr diagram. Drawn in isolation like this its scale is not important since we have not specified the amount of rotation and we are only interested in the ratio of the displacements, not their absolute values.

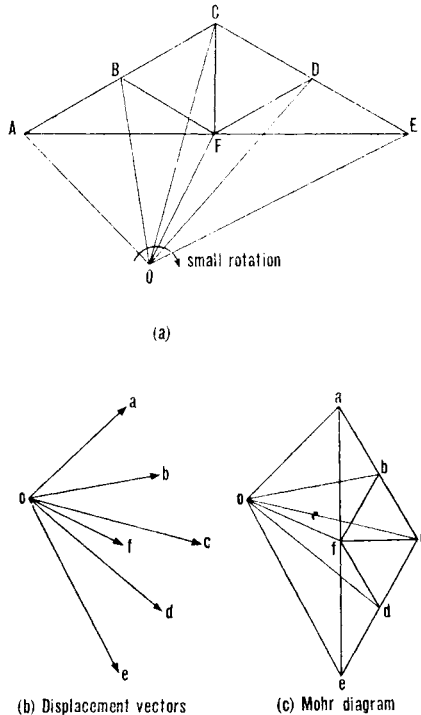


FIG. 11.4 Mohr diagram

Combined Williot-Mohr Displacement Diagram

We have seen how a Williot diagram will give the displacements of the joints of a loaded truss which is pinned to a pair of rigid supports, with one of the joints of the truss connected

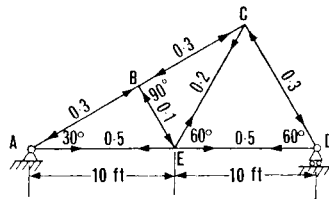
directly to both supports. If, however, a truss is only pinned to a rigid support at one end and rests on rollers at the other support we can still construct a Williot diagram for it by imagining that one of the bars connected to the pinned end is fixed in direction and does not rotate about the support pin as the truss deflects. It will now be found that the joint at the roller support has a displacement that lifts it off the roller track. To put matters right we now imagine that the whole truss is given a rotation about the support pin in order to bring the roller back to its correct level on the track. This is where the Mohr diagram comes in, for this diagram gives the displacements of the joints produced by this rotation. The final displacements of the joints is then the difference between the Williot and Mohr displacements. The Mohr diagram is a replica of the space diagram of the truss pivoted on the fixed reference point of the Williot diagram and turned through 90° relative to the truss. The scale of the Mohr diagram is adjusted so that the displacement of the roller is parallel to the roller track. As the true displacements of the joints are given by the vectors starting from the appropriate points on the Mohr diagram and finishing on the corresponding points on the Williot diagram, the size of the Mohr diagram is determined by the fact that the displacement of the roller in the Mohr diagram must be equal to the component of the displacement of the roller in the Williot diagram which is perpendicular to the roller track.

Deflection of Simply Supported Truss

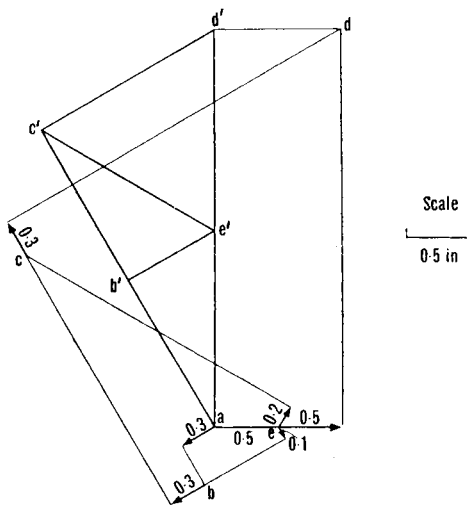
Example 11.2. Figure 11.5a shows the outline of a truss. The loading on it produces changes of length in the various bars, which are noted against each bar, the dimensions being in inches. The arrows on the bars indicate whether the bars are

in tension or compression and show the direction of the forces exerted by the bars on the joints.

Start the displacement diagram from a , since A is hinged to the ground. There is no joint adjacent to A which is positively



(a)



(b)

FIG. 11.5 Williot-Mohr diagram

located as with the cantilever trusses, so in order to proceed, it is necessary to assume either, that the bar AB does not rotate, or that the bar AE does not rotate. In this example we assume

that AE is fixed in direction; hence the displaced position of E is e , where $ae = 0.5$ in. Having located e we can proceed in turn to b , c , and d , by obtaining the intersections of pairs of "cranked" vectors. The term *cranked vector* means the vector representing the displacement of the remote end of a bar relative to the end whose position is already determined, together with the line at right angles to it through its extremity.

We thus have an apparent displacement of the roller support D , given by the vector ad — it is not necessary to join ad by a line to see this — in other words D would move upwards by an amount equal to the vertical component of ad . But D cannot rise off the roller track, it can only move horizontally. We can bring D back to its correct level again by giving a small clockwise rotation to the whole truss about the hinge A . Superimposing on the Williot diagram a Mohr diagram the same shape as the truss and pivoted on a , but set at right angles to the truss and of such a size that the vector $d'd$ is horizontal, completes the displacement diagram. The Mohr diagram carries the same letters as the truss, except that lower case letters with a dash are used. The displacement of any joint is given by the vector from the joint on the Mohr diagram to the corresponding point on the Williot diagram. Thus the displacement of D is given by $d'd$ and is entirely horizontal. The vertical deflection of E is the vertical component of $e'e$ and is 1.55 in.

The advantages of this graphical method are that it can deal with any number of loads on the frame and the displacement of every joint is determined. It is usually necessary to make a preliminary sketch of the Williot diagram, so that a suitable scale can be chosen, and the diagram placed conveniently on the drawing paper. If this is not done, the diagrams have a tendency to run off the edge of the sheet, as they rapidly expand in size in working from joint to joint.

Of course, the first step in a complete problem, as with the strain energy method, is to analyse the truss under the given

loading and determine the forces in all the bars. Secondly, the change in length of each bar is calculated from PL/AE , where P is the force in the bar, L its length, A its cross-sectional area and E the modulus of elasticity of the material used.

Exercise 11

$$(E = 13,000 \text{ tonf/in}^2)$$

1. A point load is carried by two similar wires, one 60 in long and the other 80 in long from two points at the same level 100 in apart. If a 100 in length of the wire stretches 0.2 in under a 100 lbf load, determine the vertical and horizontal components of the displacement of the loaded point when the load is increased by 150 lbf.

Ans: 0.20 in; 0.03 in

2 & 3. Solve questions 2 and 3 of Exercise 10 by constructing the Williot diagrams.

4-9. Solve questions 4 to 9 of Exercise 10 by constructing the Williot-Mohr diagrams.

10. All the bars of the truss in Fig. 10.8 have a cross-sectional area of 0.25 in^2 . Determine graphically the vertical deflection of G when the truss carries loads of 4, 4 and 8 tonf at F , G and H respectively.

Ans: 1.19 in

11. Apply the principle of superposition to the Williot-Mohr diagrams for questions 8 and 9 to solve question 10.

12. The Warren girder shown in Fig. 11.6 is supported at A and rests on a roller at G . All members are 150 in long, 2 in^2 in cross-section. Determine the vertical deflection of the joint J .

(Eng. Inst. Apr. 1959)

Ans: 0.248 in

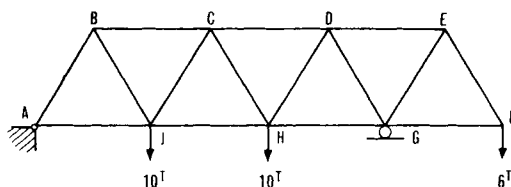


FIG. 11.6

CHAPTER 12

Reinforced Concrete

Factors Affecting the Strength of Concrete

Concrete consists of stone fragments or gravel and sand bonded together with Portland cement to form a solid material of high compressive strength. Although it resembles stone when hardened, it is fluid when freshly mixed and can be formed into any desired shape by the use of suitable moulds or shuttering.

Concrete is cheaper, weight for weight, than other building materials, but because its strength can be seriously reduced by a number of factors, it is necessary to control the ingredients, mixing, placing and curing with care, if the resulting structure is to possess a good finish and be safely and economically built.

High strength concrete depends upon using an aggregate of high crushing strength, but it depends even more upon obtaining a *dense* concrete. Spaces or voids in the concrete not only reduce the amount of solid material available for carrying stress, but, more important, introduce stress concentrations because wherever a hole exists the lines of stress have to make a detour around it and get bunched up locally in doing so. These points of high stress are potential sources for the initiation of a crack.

Each particle of aggregate should be coated with cement paste to bind the separate particles together into a solid lump.

The amount of cement required to adequately coat each particle is therefore proportional to the total surface area of all the aggregate and thus increases as the size of aggregate is reduced. Therefore, to reduce the cost of the concrete to a minimum the aggregate should be as large as possible, as it will then require less cement; although it must be borne in mind that the aggregate must be suitably graded and not all of one size, so that the spaces between the largest lumps are nearly filled by smaller lumps, leaving the voids around the latter to be filled by the smallest particles. The necessity for using clean aggregate, uncontaminated by dust or clay should be apparent. The maximum size of aggregate is dictated chiefly by the clearance between reinforcing bars and the shuttering, and is usually $\frac{3}{4}$ in for buildings. Larger aggregate can be used in mass concrete if the concrete mixer is capable of handling it.

The next important factor affecting strength is the water: cement ratio. 3 lb of water are sufficient to hydrate — that is, satisfy the chemical demand of — 10 lb of cement, i.e. a water: cement ratio of 0.3. Additional water results in a neat cement of lower strength. However, this amount of water is insufficient to render a concrete “fluid” or workable, so more water must be added until the concrete can be worked into every corner of the shuttering and around every reinforcing bar without leaving voids. Vibration can make fresh concrete flow more easily, with the result that a lower water: cement ratio can be used and yet attain the maximum degree of compaction, thus producing a concrete of greater strength.

Finally, there is the curing of the concrete. Chemical reactions continue in the cement long after the concrete has “set”, resulting in increased strength, provided there is sufficient moisture in the concrete. It is necessary therefore to prevent the concrete from drying out too rapidly, usually by keeping it covered with damp sacking for perhaps a week.

Like masonry or brickwork, plain concrete is weak in tension. The crushing strength of a typical 1:2:4 concrete at 28 days might be 3000 lbf/in², whereas its tensile strength would be only about one-tenth of this. This means that if concrete is to be used in beams, it must either have steel bars embedded in the concrete below the neutral axis (for beams subjected to sagging bending moments) to carry the tensile stresses; or the beam must be subjected to a longitudinal compressive force (usually applied by tensioned wires passing through the beam) sufficient to neutralise the tensile bending stresses developed. The second method mentioned is *pre-stressed concrete* and the basic principles are briefly discussed in Chapter 7 on combined bending and direct stress. The first method is *reinforced concrete*.

Reinforced Concrete Beams

Reinforced concrete depends upon the ability of concrete to firmly grip steel bars embedded in it. Figure 12.1 is a diagram of a reinforced concrete beam showing in exaggerated form the effect of sagging bending. If we imagine that the loading is gradually increased from nothing, the flexing of the beam

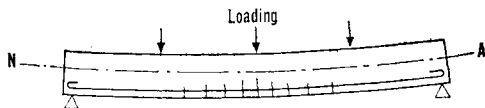


FIG. 12.1 Tension cracking in reinforced concrete beam

increases and so also, the tensile stresses in the concrete below the neutral axis until they exceed the tensile strength of the concrete, at which point the concrete cracks, leaving practically the whole of the tension to be carried by the steel bars. Notice that the cracks are longest at the centre of the beam where the b.m. is greatest, where they extend nearly to the

neutral axis. These fine cracks can be seen on any beam carrying its full safe load, provided the surface is reasonably smooth. A coat of whitewash and good illumination helps to show them up.

When calculating the moment of resistance of a reinforced concrete beam the tensile strength of the concrete is therefore neglected, and only the concrete above the neutral axis, which

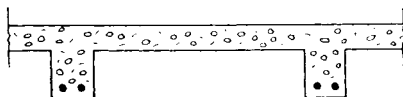


FIG. 12.2 How Tee beams usually occur

is in compression, is taken into account. Of course, we cannot dispense entirely with the concrete in the tensile zone, as some is necessary to hold the steel in position and to resist shearing forces, although it is quite common to reduce the amount of concrete below the neutral axis by using Tee beams, which are usually in the form of floor slabs reinforced with rectangular ribs on the underside, as in Fig. 12.2

Strength of rectangular R.C. beams

Quite evidently, the strength, or moment of resistance, of a rectangular, reinforced concrete beam depends upon the size of its cross-section, the amount of steel and its position, as well as the strengths of the steel and the concrete. It also depends upon the elastic behaviour of the steel and concrete since they have to strain in combination. Referring to Fig. 12.3a, we can summarise the factors affecting the moment of resistance under these headings.

Geometry of cross-section:

b , the breadth of the beam;

d , the effective depth of the beam, which is measured from the top face of the beam to the centre of gravity of the steel;

A , area of steel.

Strength of the concrete and steel:

p_c , the permissible compressive stress in the concrete;

p_t , the permissible tensile stress in the steel.

Elastic properties of the steel and concrete:

m , the modular ratio, or ratio of the moduli of elasticity of steel and concrete.

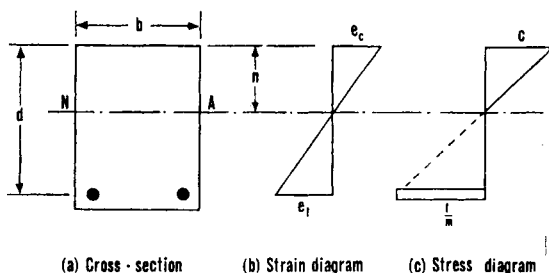


FIG. 12.3 Reinforced concrete beam

If we make the assumption that cross-sections remain plane during bending — as we did in Chapter 3, for beams made of a single material — then the strain in the concrete is proportional to its distance from the neutral axis, and the strain in the steel is equal to that in the surrounding concrete — see Fig. 12.3b. Above the neutral axis the stress in the concrete increases from zero to a maximum compressive stress of c at the outer fibres; below the neutral axis, the tensile strain does not give rise to tensile stress in the concrete, since it cracks, but the strain in the steel produces a tensile stress of t in it. If e_t is the tensile strain in the steel then t is obtained by multiplying

this strain by the modulus of elasticity of steel; so we have $t = E_s e_t$. Similarly, $c = E_c e_c$. Therefore if we multiply the strains by E_c we arrive at the stress diagram in Fig. 12.3c, since

$$E_c e_t = \frac{E_c}{E_s} E_s e_t = \frac{t}{m}$$

Note that it is the same shape as the strain diagram. It is important to realise that c and t are the *actual* stresses developed in the beam and not necessarily equal to the *permissible* stresses.

Position of the Neutral Axis

The position of N.A. is determined by equating the total compression in the concrete above the neutral axis to the total tension in the steel.

Compressive force in concrete:

$C =$ area of concrete above N.A. $\times$ average compressive stress in it

$$= bn \cdot \frac{c}{2}$$

$T =$ Tensile force in steel

$$= At$$

Therefore $At = \frac{nbc}{2}$ thus $\frac{c}{t} = \frac{2A}{nb}$

Also from the similar triangles of the stress diagram we have

$$\frac{c}{t/m} = \frac{n}{d-n} \quad \text{thus} \quad \frac{c}{t} = \frac{n}{m(d-n)}$$

Hence

$$\frac{2A}{nb} = \frac{n}{m(d-n)}$$

$$\text{and } mA(d-n) = nb \cdot \frac{n}{2}$$

In other words, the moment of m times the area of steel about the neutral axis is equal to the moment of the area of the effective concrete about the neutral axis. The rule for finding N.A. is, therefore, *equate moments of effective area of concrete about the neutral axis* — remembering that an area A of steel becomes an area mA when converted to an equivalent area of concrete. Note that the position of the neutral axis is determined by the geometry of the cross-section, not by the permissible stresses.

Moment of resistance

The moment of resistance M_R is equal to the couple provided by the pair of equal and opposite forces C and T , separated by a lever arm, a . The total compressive force C on the

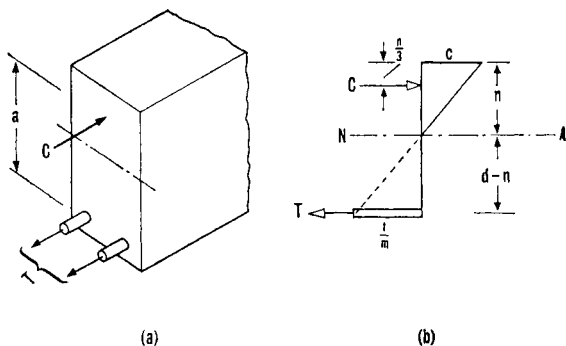


FIG. 12.4

concrete acts at the centre of pressure of the wedge-shaped stress distribution for the compressive stresses, and is thus at $n/3$ below the top face — see Fig. 12.4b.

The lever arm
and

$$a = d - n/3$$

$$M_R = Ca \text{ or } Ta$$

$$C = \frac{nbc}{2}$$

$$T = At$$

By substituting the permissible stresses in the last two expressions we obtain values for C and T . If they are not equal then only the *smaller* value can be used to determine M_R .

Determining the Moment of Resistance of a Given Beam

Example 12.1. A concrete beam is 10 in wide and 18 in deep overall. It is reinforced by two $\frac{7}{8}$ in diameter bars, with a cover of 1 in. The stresses in the steel and concrete are not to exceed 20,000 and 1000 lbf/in² respectively. Assume the usual value of 15 for the modular ratio of normal reinforced concrete.

$$\text{Area of steel} = 2 \cdot \frac{\pi}{4} \left(\frac{7}{8} \right)^2 = 1.2 \text{ in}^2$$

$$\text{Effective depth} = 18 - 1 - \frac{1}{2} \cdot \frac{7}{8} = 16.56 \text{ in}$$

Position of N.A.:

Equating moments of "concrete" area about N.A.

$$10n \cdot \frac{n}{2} = 15 \times 1.2(16.56 - n)$$

this is a quadratic in n which, when solved in the usual way, gives

$$n = 6.13 \text{ in}$$

The second root of the equation is negative and inadmissible.

$$\text{Therefore lever arm} = 16.56 - \frac{6.13}{3} = 14.52 \text{ in}$$

Moment of resistance:

$$\begin{aligned} \text{Compression in concrete cannot exceed } \frac{1000}{2} \times 10 \times 6.13 \\ = 30,650 \text{ lbf} \end{aligned}$$

$$\begin{aligned}\text{Tension in steel cannot exceed} & \quad 20,000 \times 1.2 \\ & = 24,000 \text{ lbf}\end{aligned}$$

$$\text{Therefore} \quad M_R = 24,000 \times 14.52 = 348,000 \text{ lbf in}$$

Determining the stresses set up in a given beam under a specified loading

The fundamental approach to problems of this type is to calculate the maximum b.m. in the beam due to the loading. The depth of the neutral axis can be determined from the cross-section and hence the lever arm will be known. Dividing the b.m. by the lever arm gives the force in the concrete or the steel. Hence, knowing their areas, the stresses in the concrete and steel can be calculated. The following numerical example follows this procedure.

Example 12.2. A reinforced concrete beam of rectangular section 9 in. wide has two $\frac{3}{4}$ in. diameter steel bars at an effective depth of 12 in. Calculate the maximum stresses in the concrete and steel at a section where the bending moment is 175,000 lbf in. $m = 15$

$$\text{Area of steel} = 2 \times \frac{\pi \left(\frac{3}{4}\right)^2}{4} = 0.884 \text{ in}^2$$

$$\text{To find } n: \frac{9n^2}{2} = 15 \times 0.884(12 - n)$$

$$\text{hence } n = 4.65 \text{ in}$$

To find stresses:

$$\text{Lever arm} = 12 - \frac{4.65}{3} = 10.45 \text{ in}$$

$$\begin{aligned}\text{Therefore force in concrete} &= \text{force in steel} = \frac{\text{b.m.}}{\text{lever arm}} \\ &= \frac{175,000}{10.45} = 16,770 \text{ lbf}\end{aligned}$$

$$\begin{aligned}\text{Average stress in concrete} &= \frac{16,770}{9 \times 4.65} \\ &= 400 \text{ lbf/in}^2\end{aligned}$$

$$\begin{aligned}\text{Maximum stress in concrete} &= \text{twice the average stress} \\ &= 800 \text{ lbf/in}^2\end{aligned}$$

$$\begin{aligned}\text{Stress in steel} &= \frac{16,770}{0.884} \\ &= 19,000 \text{ lbf/in}^2.\end{aligned}$$

Slabs Simply Supported on Two Opposite Edges

The main reinforcing bars span between the supports, but it is also necessary to provide transverse reinforcement at right angles to the main bars and resting on top of them, in order to tie the slab together and make it behave as a monolithic whole, instead of a series of separate planks. The amount of this *distribution steel* or *temperature steel*, as it is sometimes called, because it also resists strains set up due to changes of temperature, is determined by empirical rules, as there is no simple way of calculating from first principles what is required. For example, the British Standard Code of Practice for the use of reinforced concrete, C.P. 114, 1957 recommends that the minimum reinforcement should not be less in either direction than 0.15 per cent of the gross cross-sectional area of the concrete.

Provided a slab is uniformly loaded over its whole area it is sufficiently accurate to consider just a 1 ft wide strip, spanning between the supports for determining the strength of the slab.

Example 12.3. A concrete slab 6 in thick is supported on two opposite edges 12 ft apart and carries a uniformly distributed load of 40 lbf/ft². It is reinforced by $\frac{1}{2}$ in diameter rods at

4 in pitch with $\frac{1}{2}$ in cover. Calculate the maximum stresses in the steel and concrete. Reinforced concrete weighs 150 lbf/ft³. $m = 15$.

$$\text{Effective depth} = 6 - \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} = 5.25 \text{ in}$$

$$\begin{aligned}\text{Area of steel per foot width} &= \frac{\pi}{4} \left(\frac{1}{2} \right)^2 \cdot \frac{12}{4} \\ &= 0.589 \text{ in}^2\end{aligned}$$

$$\text{Weight of slab} = \frac{1}{2} \times 150 = 75 \text{ lbf/ft}^2$$

$$\begin{aligned}\text{Maximum b.m. per ft width} &= (75 + 40) \frac{12^2}{8} \text{ lbf ft} \\ &= 24,800 \text{ lbf in}\end{aligned}$$

To find n :

$$\frac{12n^2}{2} = 15 \times 0.589(5.25 - n)$$

$$\text{hence} \quad n = 2.14 \text{ in}$$

$$\text{lever arm} = 5.25 - \frac{2.14}{3} = 4.54 \text{ in}$$

$$\text{Therefore force} = \frac{24,800}{4.54} = 5450 \text{ lbf}$$

$$\text{stress in steel} = \frac{5450}{0.589} = 9250 \text{ lbf/in}^2$$

$$\text{max. stress in concrete} = \frac{5450}{12 \times 2.14} \times 2 = 424 \text{ lbf/in}^2$$

Economic Percentage of Steel

The position of the neutral axis of a rectangular beam of given breadth and effective depth is determined by the amount of steel, and as the area of steel is increased the neutral axis is

drawn lower. The depth of the neutral axis and the geometric properties of the cross-section are related by the equation

$$mA(d-n) = nb \cdot \frac{n}{2}$$

obtained by equating moments of equivalent concrete area about the neutral axis. If we define p , the percentage of steel, as the ratio of the area of steel to the gross cross-sectional area down to the centre of the steel, then $p = 100A/bd$, and the above equation can be expressed in the form

$$n^2 + \frac{mpd}{50} n - \frac{mpd^2}{50} = 0$$

which is a quadratic in n . In other words, for given values of b , d and m , the depth of the neutral axis is determined by the percentage of steel. By transforming the last equation we can obtain the percentage of steel as a function of n/d , the relative depth of the neutral axis.

$$p = \frac{50(n/d)^2}{m[1 - (n/d)]}$$

This non-dimensional expression has been plotted in Fig. 12.5 for a modular ratio of 15, and shows that for a rectangular

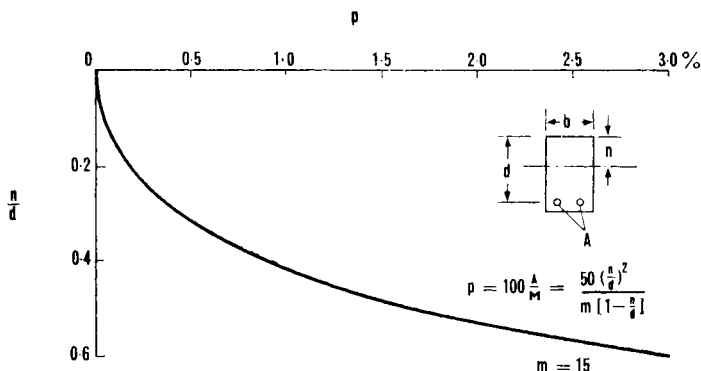


FIG. 12.5 Relative depth of neutral axis versus percentage of steel

beam of any shape and size the ratio of the depth of the neutral axis to the effective depth depends only on the percentage of steel, provided a single value of m is used.

Now the ratio of the actual stresses in the concrete and steel is obtained from the stress diagram (Fig. 12.3c),

$$\frac{c}{t/m} = \frac{n}{d-n}$$

$$\frac{c}{t} = \frac{n/d}{m[1-(n/d)]}$$

This stress ratio will only be equal to the ratio of the *permissible stresses* in the concrete and steel for one particular value of n/d . The neutral axis corresponding to this value of n/d is called the *critical neutral axis*. There is only one percentage of steel which will make the actual neutral axis coincide with the critical neutral axis and this is known as the *economic percentage of steel*. This is economic in the sense that the permissible stresses are developed simultaneously, although a beam incorporating the economic percentage of steel may not be the cheapest one to build, since, for example, if there is a rise in the cost of steel, without a proportionate rise in the cost of concrete it may be cheaper to reduce the amount of steel and use a bigger cross-section.

Simple Design Problems

The problem of design is basically — what size beam to provide to resist a given bending moment, using steel and concrete with specified permissible stresses and a stated value of m ? It is not a question of there being one correct solution, all others being wrong; there are usually an indefinite number of equally valid, practical designs which would meet the specifica-

tion. Design is still very much a matter of individual judgment. The usual approach is to rough out a design and then analyse it to see whether it satisfies the requirements; if it does not, then it is necessary to modify the design and repeat the process. The student, lacking experience in reinforced concrete design, needs a criterion to reduce the number of arbitrary decisions that have to be taken in designing a reinforced concrete beam, and the simplest one is to use the *economic section*, i.e. a section using the economic percentage of steel. It will still be necessary to arbitrarily decide the breadth of the beam, unless this is fixed by other considerations, such as matching up with the size of supporting columns.

Example 12.4. Determine the reinforcement and overall depth of a concrete beam 6 in wide to carry an imposed load of 600 lbf/ft on a simple span of 10 ft, using permissible stresses of 20,000 and 1000 lbf/in² for the steel and concrete.

Assuming a depth of 12 in to obtain the dead load.

$$\text{Dead load} = \frac{6 \times 12}{144} \times 150 = 75 \text{ lbf/ft}$$

therefore total load = 75 + 600 = 675 lbf/ft

$$\text{max. b.m.} = \frac{675 \times 10^2}{8} \times 12 = 101,300 \text{ lbf in}$$

To find the relative position of the critical N.A.:

Using the stress diagram, Fig. 12.3c and noting that the permissible stresses will be developed simultaneously.

$$\frac{c}{t/m} = \frac{n}{d-n}$$

$$\text{therefore} \quad \frac{1000.15}{20,000} = \frac{n/d}{1-(n/d)} = \frac{3}{4}$$

$$n/d = \frac{3}{7}$$

$$\text{Relative length of lever arm} = \frac{\text{lever arm}}{\text{effective depth}}$$

$$\frac{a}{d} = \frac{d - n/3}{d}$$

$$= 1 - \frac{1}{3} \frac{n}{d}$$

$$= \frac{6}{7}$$

$$\begin{aligned} \text{Compressive force in concrete} &= \frac{\text{bending moment}}{\text{lever arm}} \\ &= \frac{101,300}{\frac{6}{7}d} = \frac{118,200}{d} \end{aligned}$$

Also, compressive force in concrete = mean concrete stress
× area of concrete above N.A.

$$= .4 \times 1000 \times 6 \times \frac{3}{7} d$$

$$= 1285d$$

$$\text{Therefore} \quad 1285d = \frac{118,200}{d}$$

$$d = 9.6 \text{ in}$$

$$\text{Tensile force in steel} = \frac{118,200}{d}$$

$$= 12,360 \text{ lbf}$$

$$\text{So that area of steel required} = \frac{12,360}{20,000}$$

$$= 0.618 \text{ in}^2$$

say two No. $\frac{5}{8}$ in. diam. bars.

Therefore overall depth = $9.6 + \frac{1}{2} \times \frac{5}{8} + 1.0$ cover say **11 in.**
so the effective depth would actually be 9.7 in.

The dead load need not be recalculated as our original "guesstimate" of 12 in for the overall depth is sufficiently close.

Example 12.5. Carry out the (bending) design calculations for a floor slab to carry a uniform loading of 80 lbf/ft², when simply supported on two edges, 12 ft apart using permissible stresses of 20,000 and 750 lbf/in² for the steel and concrete.

Assume a 7 in slab for dead weight calculations, and carry out all calculations for a 1 ft wide strip.

$$\text{Dead load} = \frac{7 \times 12}{144} \times 150 = 88 \text{ lbf/ft}^2$$

$$\text{Therefore total load} = 88 + 80 = 168 \text{ lbf/ft}^2$$

$$\begin{aligned} \text{Maximum b.m.} &= \frac{168 \times 12^2}{8} \times 12 \\ &= 36,300 \text{ lbf in} \end{aligned}$$

Relative depth of critical N.A.:

$$\frac{15 \times 750}{20,000} = \frac{n}{d-n} = 0.562$$

$$\text{therefore} \quad \frac{n}{d} = 0.36$$

$$\frac{a}{d} = 1 - \frac{0.36}{3} = 0.88$$

$$\text{Total compression} = \frac{36,300}{0.88d} = \frac{750}{2} \times 12 \times 0.36d$$

$$\text{Therefore} \quad d = 5.04 \text{ in}$$

$$\text{Area of steel required} = \frac{36,300}{0.88 \times 5.04 \times 20,000} = 0.408 \text{ in}^2$$

Say $\frac{3}{8}$ in bars at $3\frac{1}{4}$ in pitch (0.407 in²)

Exercise 12

($m = 15$ except where specified otherwise. Reinforced concrete weighs 150 lbf/ft³)

1. A reinforced concrete lintel for a garage door is 4.5 in wide and 6 in deep overall and is reinforced by two $\frac{3}{8}$ in diameter steel bars with 1 in cover. Determine the depth of the neutral axis.

If the maximum stress in the steel is 15,000 lbf/in² what is (a) the moment of resistance of the beam, (b) the maximum stress in the concrete.

Ans: 2.02 in; 13,750 lbf in; 724 lbf/in²

2. A reinforced concrete beam 8 in wide has an effective depth of 12 in. The reinforcement consists of four 9/16 in diameter bars. If the stresses in the steel and the concrete are not to exceed 18,000 and 750 lbf/in² respectively, calculate the maximum bending moment that the beam can withstand.

Ans: 157,500 lbf in

3. A concrete beam, simply supported on a span of 18 ft is 6 in wide and 14 in deep overall and is reinforced by four $\frac{1}{2}$ in diameter steel bars with $1\frac{1}{2}$ in cover. The beam carries a uniformly distributed load of 230 lbf/ft in addition to its own weight. Calculate the maximum stresses developed in the steel and in the concrete.

Ans: 18,800 lbf/in²; 940 lbf/in²

4. A concrete slab is 6 in thick and has a span of 13 ft. It is reinforced with $\frac{1}{2}$ in diameter steel bars at 5 in pitch with $\frac{1}{2}$ in cover. Determine the maximum uniformly distributed load in lbf/ft², in addition to its own weight, that the slab can carry if the steel and concrete stresses are not to exceed 20,000 and 1000 lbf/in².

Ans: 95 lbf/ft²

5. Explain the need for steel bars in a reinforced concrete beam. What are the advantages and disadvantages of reinforced concrete for engineering structures compared with structural steelwork?

A concrete beam 6 in wide by 12 in deep overall is simply supported on a span of 12 ft. It is reinforced with three $\frac{1}{2}$ in diameter steel bars with a cover of $\frac{3}{4}$ in. The beam carries a uniformly distributed load, including its own weight, of 300 lbf/ft run. Calculate the maximum steel and concrete stresses in the beam.

Ans: 12,800 lbf/in²; 604 lbf/in²

6. A 6 in concrete floor slab is simply supported on a span of 20 ft. It is reinforced by $\frac{5}{8}$ in diameter bars at 4 in pitch with $\frac{1}{2}$ in cover. If

the maximum permissible steel and concrete stresses are 18,000 and 1000 lbf/in² determine the maximum imposed load in lbf/ft² that the slab can carry.

Ans: 33 lbf/ft²

7. A simply supported reinforced concrete beam 25 ft long is 6 in wide and 14 in deep overall and is reinforced by four bars of $\frac{5}{8}$ in diameter with 1½ in cover. What is the maximum concentrated load that the beam can carry at mid-span if the steel and concrete stresses are not to exceed 18,000 and 750 lbf/in². Use a modular ratio of 18 in this problem.

Ans: 840 lbf

8. State the assumptions made in the theory of reinforced concrete beams.

A reinforced concrete beam of rectangular section is 12 in wide with an effective depth of 20 in. If the stresses in the steel and concrete are limited to 20,000 and 1000 lbf/in² respectively, find, from first principles, the economic area of tensile reinforcement required and the safe uniformly distributed load (including its own weight) which the beam could carry when simply supported on a span of 20 ft.

Ans: 29,300 lbf

9. A reinforced concrete slab, simply supported on two edges 16 ft apart, is required to carry an inclusive load of 120 lbf/ft². Using maximum stresses of 750 and 20,000 lbf/in² in the concrete and steel, determine the required effective depth of the steel and the area of steel needed per foot width of slab.

Ans: 5.68 in; 0.462 in²

10. An underground pipe duct is to be covered by removable pre-cast reinforced concrete slabs. Each slab is to be 18 in wide and will be simply supported on an effective span of 8 ft 6 in. The slabs are to be designed to carry a load of 220 lbf/ft² including their own weight.

Determine the necessary thickness of the slabs and the number of $\frac{1}{2}$ in diameter bars required as main reinforcement in each slab. The maximum permissible stresses in steel and concrete are 18,000 and 1000 lbf/in² respectively.

(I.C.E. Apr. 1951)

Ans: 4½ in ($d = 3.21$ in); 4 bars

11. A reinforced concrete slab is simply supported on two edges on an effective span of 10 ft. The slab is 7 in thick overall and is reinforced with $\frac{1}{2}$ in diameter bars at 5 in centres, the centres of the bars being 5½ in below the upper surface of the slab. Calculate the safe imposed load per

square foot on this slab, in addition to its own weight, if the maximum stresses in the concrete and steel are limited to 900 and 18,000 lbf/in² respectively. Take the weight of the slab as 144 lbf/ft³.

(I.C.E. Apr. 1950)

Ans: 189 lbf/ft²

12. A reinforced concrete beam of rectangular section, with tension reinforcement only, is required to carry a uniformly distributed load of 1825 lbf/ft, including its own weight, when simply supported on an effective span of 15 ft. Determine the minimum cross-sectional dimensions of this beam, and the area of tension steel required at mid-span, assuming the breadth to be $\frac{2}{5}$ of the effective depth. The maximum stresses in steel and concrete are 18,000 and 1,000 lbf/in² respectively.

(I.C.E. Apr. 1952)

Ans: 20 in effective depth, say 22 in $\times$ 8 in; 2.02 in²

13. A reinforced concrete retaining wall, of inverted Tee section, is 15 ft high above the top of the base slab and retains earth level with the top of the wall. The earth weighs 110 lbf/ft³ and has an angle of repose of 32°. Calculate the minimum thickness of the base of the vertical leg of the wall assuming the reinforcement to be embedded 2 in to the centre of the bars. The permitted stresses in concrete and steel are 1,000 and 18,000 lbf/in² respectively. Determine also the spacing of the vertical bars assuming them to be $\frac{7}{8}$ in in diameter.

(I.C.E. Apr. 1950)

Ans: 10.94 in; 4.8 in

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