Control Engineering

Hebertt Sira-Ramírez

Sliding Mode Control

The Delta-Sigma Modulation Approach

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To the memory of my beloved mother,
Ilia Aurora Ramírez Espinel,

to María Elena Gozaine Mendoza,
with love and gratitude.

Preface

Sliding mode control is a well-known discontinuous feedback control technique, which has been extensively explored in several books and many journal articles by diverse authors. The technique is naturally suited for the regulation of *switched controlled* systems, such as power electronics devices, and a nonempty class of mechanical and electromechanical systems such as motors, satellites, and robots. Sliding mode control was studied primarily by Russian scientists in the former Soviet Union. We epitomize the pioneering work on switched controlled systems by the fundamental book by Tsypkin [30] which, to this day, continues being a source of inspiration to researchers and students in the field. A complete account of the history and fundamental results on sliding modes, or sliding regimes, is found in excellent books, such as those by Utkin (see Utkin [31], Utkin [32], and also Utkin *et al.* [33]). In Utkin's most recent book [33], the discontinuous feedback control of a rather complete collection of physical mechanical and electromechanical systems is addressed along with remarkable laboratory implementation results. In that book, there is some detailed attention devoted to the control and stabilization of DC to DC power converters. A more recent book on sliding modes, mainly devoted to the area of linear systems, with a terse and very clear exposition of the topics along with some interesting laboratory and industry applications, is that of Edwards and Spurgeon [4]. A well-documented book with some chapters on sliding mode control is that of Slotine and Li [23]. A book containing a rigorous exposition of Sliding Mode Control and interesting symbolic computation techniques is that of Kwatny and Blankenship [15]. Orlov [28] contains a lucid exposition of Sliding Mode Control in continuous and discrete systems while devoting attention in detail to the case of infinite dimensional systems and applications to electromechanical systems. A book by Shtessel *et al.* [22] contains the most recent developments in Sliding Mode Control and Sliding Mode Observers, including the, so-called, second order and higher order sliding mode approach to, both, control and observation problems. For the reader interested in an important relation between sliding mode control and optimization in uncertain systems, the book by Fridman, Poznyak, and Bejarano [11] is a source of interesting results. In particular, those pertaining to Output Integral Sliding Mode Control, a new development, with great potential for applications, whose original developments were formulated by Utkin and Shi in [34].

In this book, we provide an introduction to the sliding mode control of switch regulated nonlinear systems. Chapter 1 gives a tutorial introduction to sliding mode control on the basis of simple, physically oriented examples. Many examples are linear but, in order to early introduce nonlinear systems, we delve into several, simple, nonlinear examples. In this first chapter, we review some of the advantages of sliding mode control, pertaining to the robustness issue, and emphasize the need to familiarize the reader with the elements of sliding mode control. Some simple linear plant examples exhibiting a lack of global sliding mode existence, on linear sliding manifolds, are freed from such restriction by considering appropriate nonlinear sliding surfaces.

Chapter 2 is devoted to the sliding mode control of Single-Input Single-Output switched nonlinear systems case. Here, we formalize, in the language of elementary Differential Geometry, the elements of sliding mode control intuitively presented in the first chapter. Necessary as well as necessary and sufficient conditions are given for the local existence of sliding motions on given smooth sliding manifolds. Robustness of sliding mode controlled plants with respect to matched perturbations is specifically treated through the use of projection operators on tangent subspaces of sliding manifolds. The chapter illustrates the concepts with some physically oriented examples from aerospace, power electronics, and mechanical engineering with some attention to robotics.

Chapter 3 is devoted to $\Delta - \Sigma$ modulation, addressed from now on as "Delta-Sigma modulation." This key issue is explored in connection with a natural, and idealized, translation of smooth analog input signals to infinite frequency output switched signals whose average value precisely reproduces the input signal. This simple mechanism allows to make available the entire field of nonlinear control systems design to the class of switched systems. The chapter explores a generalization of Delta modulation and its applications in state estimation. Also, multilevel Delta-Sigma modulation is proposed as a means to fraction switch amplitudes and reduce the corresponding induced chattering. This development has a natural implication in the switched control of mechanical systems.

Chapter 4 deals with the Multiple Input Multiple Output (MIMO) case. The fundamental issues of MIMO sliding mode control, pertaining to the difficulties associated with a sound definition of sliding modes in the intersection of a finite number of sliding manifolds as well as the reachability problem are examined and a number of examples provided for enhancing the intuitive understanding of the material. The use of Delta-Sigma modulation sidesteps the difficult problem of sliding mode existence in the MIMO case. A more systematic design procedure is obtained in the last chapter via the exploitation of flatness.

Chapter 5 explores a fundamental possibility of defining sliding regimes on the basis only of available input and output signals for linear systems. Some extensions of input-output sliding mode control are shown to be possible in the nonlinear case. The key concept to be used is that of Generalized PI control. The GPI control approach and the Delta-Sigma modulation alternative are shown to provide a systematic and rather natural sliding mode design tool. GPI control enjoys interesting implications not only on the control of linear finite dimensional system but also in linear delay systems (see Fliess, Marquez, and Mounier [8]). Some of these topics, as well as some recent research topics, are treated in a rather tutorial fashion in this chapter.

Chapter 6 studies the advantages of combining Differential Flatness and Sliding Mode control in nonlinear single-input single-output (SISO) systems and MIMO systems control. The sliding surface design problem is trivialized, thanks to the exploitation of the flatness concept. For SISO systems the flat output is simply the linearizing output and sliding motions are naturally induced in appropriate linear combinations of the phase variables associated with the flat output. In MIMO nonlinear systems flatness systematically detects the need for dynamical feedback and how to go about it in specific instances through appropriate input extensions (see Sluis [24], Charlet *et al.* [2], Rouchon [19]).

In this chapter, we also provide a solution to a long-standing problem in sliding mode control theory.

The writing of this book owes recognition to many many people. First of all, the author is indebted to his former students who, throughout some years, endured post-graduate work in the field of discontinuous feedback control under my not always pleasant supervision. Marco Tulio Prada Rizzo, Miguel Rios-Bolívar, Pablo Lischinsky-Arenas, Orestes Llanes-Santiago, and Richard Márquez-Contreras, all of them at the Universidad de Los Andes in Mérida-Venezuela; students at some time, colleagues and friends ever since. The author has enjoyed, for many years, the friendship of pioneers in the area of sliding mode control: Alan S.I. Zinober, V. Utkin, S. Spurgeon, Chris Edwards, and L. Fridman. He has always learned from them something new and exciting out of informal conversations or heated discussions. Many years ago, the author started a new life, away from his beloved venezuelan Andean Mountains, in the megapolis of México City. The experience has been a most rewarding one, thanks to the beauty of the country, the wealth of its culture, the kindness of its people, and the rich flavor of its foods, wines, and drinks. The author is most indebted to his colleagues and numerous students at the Mechatronics Section of the Electrical Engineering Department of Cinvestav, a generous, first class, Research Institution in México. This book has been written during two sabbatical leaves of the author. One at the Industrial Engineering Department of the Universidad de Castilla-La Mancha in the Ciudad Real Campus, Spain. The generosity, kindness, and advice of Professor Vicente Feliu Battle is gratefully acknowledged. The second sabbatical was a most pleasurable stay at the Universidad Tecnológica de la Mixteca

in Huajuapan, Oaxaca, México. The kind support of Dr. Jesus Linares Flores and his enormous ability and leadership for transforming theoretical results in superb experiments is gratefully acknowledged. Last but not least, the author would like to pay special recognition and tribute to the memories of Prof. Thomas A. W. Dwyer III and of Prof. Fred C. Schweppe. Tom was a dear friend who wisely guided the author through his first publications in sliding mode control, applied to challenging aerospace problems, during his one year stay at the University of Illinois at Urbana-Champaign in 1986. Fred was an inspired thesis director, back at MIT in the glorious 70s, from which the author learned, as a student, everything there was to learn at the time in Automatic Control, except Sliding Mode control.

My wife Maria Elena has been a superb, reliable, enthusiastic, and kind spiritual and affective support throughout the many years in which sliding mode control seemed to me most important than anything else in life. Fortunately, I have come to realize that I was completely wrong, but, now, more than ever, I know she was not entirely mistaken.

México, D.F., Mexico Hebertt Sira-Ramírez

Contents

1

Introduction

1.1 Generalities about sliding mode control

This book is devoted to an exposition of sliding mode control of switch regulated nonlinear systems. Implications are explored in the feedback control of some physical system models exhibiting one or multiple, independent, switches as commanded inputs.

In this chapter, we begin by presenting the constitutive elements of sliding mode control in rather general terms. We then present some elementary illustrative examples of controlling, towards desired behaviors, low dimensional plants by means of sliding modes. The control objectives include achieving trajectory stabilization around constant equilibria as well as trajectory tracking tasks. The several examples have the intention of serving as a tutorial introduction to sliding mode control and some of its most important features. We include simple mechanical systems, some electrical circuits, electromechanical devices, and elementary hydraulic systems. The idea is to convey the feeling that if we know a good description of the switched system in the form of a mathematical model and we have a sound control objective, then we can always turn the problem of achieving such a control objective into an equivalent one of suitably creating a, so-called, sliding regime on some smooth manifold of the state space of the system, addressed as the *sliding surface*.

1.2 The elements of sliding mode control

The first fundamental element of the sliding mode control technique is *The plant*, which is the dynamical system that needs regulation towards a specific *control objective*. The plant, in the most general case treated in this book, is assumed to be described by a finite dimensional state space model of nonlinear nature. As such, the plant is provided with *outputs* representing the measurable variables of interest, *the states*, constituting a finite collection of

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variables, summarizing the past history of the system, whose knowledge allows us to predict its future behavior under precise knowledge of the system *inputs*. In our setting, the inputs are assumed to be represented by the commanded variables influencing the system behavior. These are assumed to take values in discrete sets. Inputs are thus represented by functions such as *switch position functions*. The control objectives adopt various forms: one may be interested in letting the output signal, or signals, of the plant to accurately track pre-specified output trajectories. In other instances, one may be interested in stabilization of the outputs, or states, trajectories to a region around a constant equilibrium point. One may also be interested in having the entire state vector of the system track a given trajectory.

It is assumed that we can translate the desired control objectives into one, or several, scalar *state constraints* on the state vector of the system. We restrict ourselves to only consider as many state constraints as there are control inputs driving the system state trajectory. If we manage to force the state trajectories of the system, independently of their initial value, to satisfy one, or several, given constraints, then, as a result of our control efforts, it is assumed that we will obtain a desirable behavior of the outputs of the system, or of the states of the system. This controlled behavior most precisely matches the pre-specified desired behavior. Such state constraints will often represent either a smooth surface or a set of non-redundant, independent smooth surfaces in the state space of the system with nonempty intersection. Smooth surfaces are completely specified by smooth scalar functions of the state vector. These functions will be termed *sliding surface coordinate functions* and they measure the distance to the *zero level set* defining the sliding surface or *sliding manifold*. In the case where the control objective demands that multiple independent constraints be simultaneously satisfied, it is then their intersection manifold that represents the desired state constraint. The smooth set of points where the simultaneous validity of the given set of state constraints is satisfied is also addressed as the *sliding surface*. Our designed *switching policy* or *switching strategy* will be responsible to drive, by means of the limited available control actions, the state of the system towards the surface representing the desired system behavior defined by the sliding surface. Generally speaking, the switching policy will be executed as a state *feedback law* of a discontinuous nature. Once the state trajectory “hits” the sliding surface, i.e., once the state constraints become simultaneously valid at an instant of time, it is mandatory to keep the state trajectory indefinitely evolving on such a surface. We concentrate, in this chapter, on the single input (switch) single (output) sliding surface case.

The evolution of the sliding mode controlled trajectories on a single smooth sliding surface is easily idealized by considering a *virtual* feedback control action that renders the sliding manifold an active constraint to the controlled state trajectories. This state evolution on the sliding surface assumes, of course, that the initial state was located, precisely, on such a sliding manifold. It is easy to envision that if we were given the possibilities of smooth valued control actions, instead of binary valued control actions, then, one could, at least locally and hopefully globally, determine the required smooth feedback control actions that ideally keeps the controlled system trajectory evolving on the given sliding surface. If such a smooth feedback control exists, we say that this control is responsible for making the sliding surface *invariant* with respect to controlled motions starting on such a surface. This special, virtual control action has a proper name, introduced by Utkin in [31]: *the equivalent control*. The existence of the equivalent control is a crucial feature in the assessment of the feasibility of the sliding regime existence on the given sliding surface.

A key element in sliding mode control is constituted by the *plant perturbations*. Their presence constitutes an inescapable feature of Automatic Control Engineering. Sliding Mode control is a discontinuous feedback control technique that is remarkably robust, with respect to additive plant perturbations in state space models, under certain structural restrictions known as *matching conditions*. We fully explore the several cases of additive vector perturbations in nonlinear systems and find the ubiquity of the matching conditions in the, so-called, drift field perturbation case, the control input field perturbation case and when the two types of perturbations are present. An input-output formulation of Sliding Mode Control demonstrates that matching conditions are trivially satisfied within that formulation.

In the following section, we present several elementary illustrative examples, which in detail explain the creation of sliding motions that render a desired behavior of the controlled plant.

1.3 A switch commanded RC circuit

Consider the following switch-commanded RC circuit, shown in Figure 1.1, fed by a current source of constant value I which can be switched "ON" and "OFF" from the circuit as indicated in the figure.

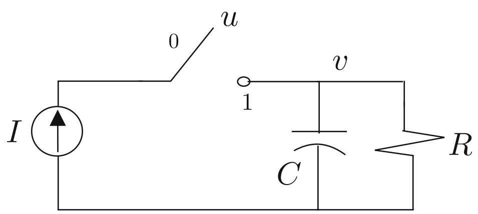

Fig. 1.1. Switched controlled RC circuit

The equation describing the behavior of the only state variable of the system, represented by the capacitor voltage v, is written as:

$$C\frac{dv(t)}{dt} = u(t)I - \frac{v(t)}{R} \tag{1.1}$$

where u is the control input, represented by a switch position function, taking values in the discrete set $\{0, 1\}$. The function u takes the value *zero* when the switch is open (i.e., it is OFF) and the source is detached from the circuit. The switch takes the value *one* (i.e., it is ON) when the switch is closed and the source is applied to the circuit. We denote the nature of the control input by writing

$$u \in \{0, 1\} \tag{1.2}$$

Suppose it is desired to obtain a given constant voltage, $v(t) = V_d$, at the capacitor output, for all times. Given the limited control actions we have at our disposal: u is either 1 or 0, the feasibility of the control objective must be examined pertaining to the achievable values of V_d.

A common procedure in switched systems to determine if there are any limitations in our control objective consists in separately examining the behaviors exhibited by the system with each one of the two possible control options. This is particularly simple on first and second order systems. The task may become extremely complex for systems of dimension 3 and higher.

When the control input u is sustained at the value $u = 1$, we obtain the following differential equation,

$$C\frac{dv}{dt} = I - \frac{v}{R} \quad \text{or} \quad \frac{dv}{dt} = \frac{I}{C} - \frac{v}{RC} \tag{1.3}$$

whose solution starting from an arbitrary initial value $v(0) = v_0$ at time $t = 0$, is given by

$$\begin{aligned} v(t) &= e^{-\frac{1}{RC}t}v_0 + \int_0^t e^{-\frac{1}{RC}(t-\sigma)}\frac{I}{C}d\sigma \\ &= e^{-\frac{1}{RC}t}v_0 + IR(1 - e^{-\frac{1}{RC}t}) \end{aligned} \tag{1.4}$$

The equilibrium point, which may be defined as $\lim_{t\to\infty} v(t)$ and denoted by $\overline{v}$, is clearly given by

$$\overline{v} = IR = V_{\text{ss}} \tag{1.5}$$

where V_{ss} stands for the constant steady state voltage.

Irrespective of the initial conditions, all trajectories starting below V_{ss} grow towards this positive value. All trajectories starting above V_{ss} decrease towards the same value. We address V_{ss} as V_{max}

On the other hand, when the switch is permanently held at the open position, $u = 0$, the system is described by the simpler linear differential equation:

$$C\frac{dv}{dt} = -\frac{v}{R}, \quad \text{or} \quad \frac{dv}{dt} = -\frac{v}{RC} \tag{1.6}$$

whose solution, for an initial condition $v(0) = v_0$, is simply

$$v(t) = e^{-\frac{1}{RC}t}v_0 \tag{1.7}$$

The equilibrium point is now obtained as

$$\overline{v} = 0 \tag{1.8}$$

In this situation, and regardless of the initial condition value, all trajectories starting above the equilibrium value $\overline{v} = 0$ decrease towards this value. All trajectories starting below $\overline{v} = 0$ increase towards the zero value.

The plots in Figure 1.2 depict the nature of the two time responses associated with the permanent switch positions $u = 0$ and $u = 1$.

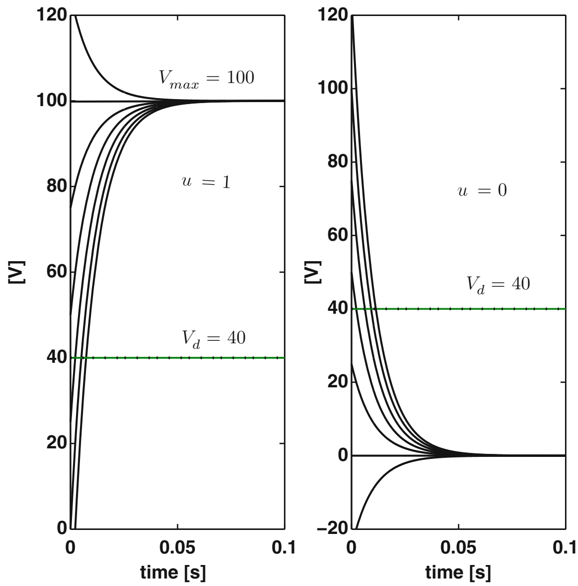

Fig. 1.2. Voltage trajectories for $u = 1$ and $u = 0$.

It is clear that, given the *binary valued* nature of the control input $u \in \{0, 1\}$, then, for an arbitrary initial condition v_0 in the real line, it becomes *impossible* to achieve a constant voltage value $v(t)$, which lies outside the closed region, $[0, V_{\max}]$, i.e., V_d should not be larger than V_{max} nor less than 0. The only region where the two sets of controlled trajectories pass through a possible desired voltage line while exhibiting opposite growth with each control option is, precisely, that represented by the interval $[0, V_{max}]$. Outside this interval, both sets of controlled trajectories pass through the desired voltage line but exhibit the same growth for either switch position. The crisscross nature of the two sets of controlled trajectories guarantees that, depending on the value of the initial condition, one of the switch positions will certainly drive the trajectory towards $v = V_d$. If the trajectory overshoots this line, the alternative control action can immediately correct by forcing the trajectory back towards this line. This is clearly portrayed in Fig 1.3.

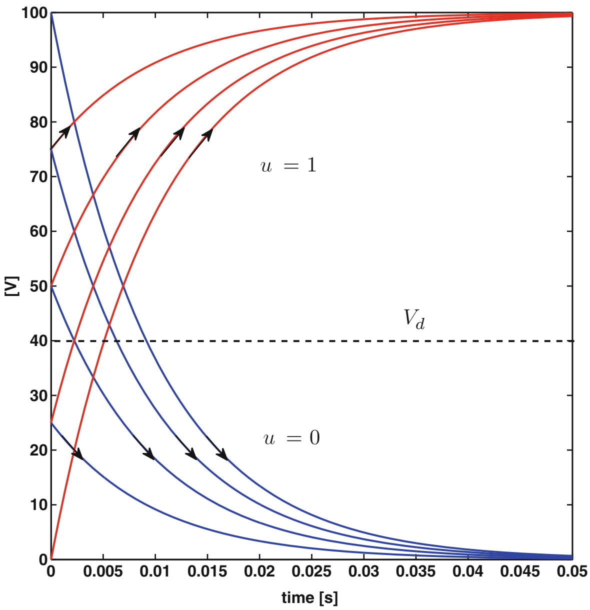

Fig. 1.3. Region of existence of a sliding regime for a constant voltage

We pose ourselves the following question: Is it possible, starting from arbitrary initial conditions, to indefinitely *reach and sustain*, by means of a suitable control switching policy an either constant, or, otherwise, time-varying, voltage signal which is bounded by the interval $[0, V_{\max}]$ of the real line?

We begin by considering the case in which it is desired to achieve a constant capacitor voltage of value V_d satisfying,

$$0 < V_d < IR = V_{\max} \tag{1.9}$$

For simplicity, let us assume that the initial condition is arbitrary but restricted to satisfy $v_0 \in [0, V_{\max}]$. The analysis equally applies when such an initial state restriction is not satisfied.

Consider the voltage error $\sigma = v - V_d$. Clearly, using (1.1) and the fact that v may also be written as $v = \sigma + V_d$, we have

$$\dot{\sigma} = \frac{IRu - (\sigma + V_d)}{RC} = \frac{V_{\max}u - \sigma - V_d}{RC} \tag{1.10}$$

Our interest is in reaching, and sustaining, the condition $\sigma = 0$ indefinitely in time, for in such a case v coincides with the desired constant voltage value V_d. The first phase of the problem solution, which we refer to as the *reaching phase*, will seek a switching policy that guarantees the approach of the condition $\sigma = 0$ from whatever initial condition for $v \in [0, V_{\max}]$. Due to the limited nature of the available control actions and the previous analysis, the reaching phase will require a switching policy that if indefinitely exercised will result in the values $v = V_{\max}$ or $\sigma = 0$. The crucial point is that using the appropriate value of the control input, the condition $v = V_d$, or $\sigma = 0$ will indeed be visited at some instant t_h, addressed as the "hitting time," since, in either case, the error function σ will change its initial sign. Therefore, we will necessarily have a *finite time reachability* of the desired condition $\sigma = 0$.

We are then led to consider the two only possible cases regarding the nature of σ at the beginning of the experiment. Either the value of σ, say at time $t = 0$, denoted by $\sigma(0)$, is positive or it is negative. Suppose that $\sigma(0)$ is *negative*, i.e., the voltage v of the capacitor is then initially smaller than the desired positive value V_d. We must then choose a control value which guarantees the growth of σ, from negative values, towards the value of zero. Since the control input u only influences the first time derivative of σ, the value of u must be chosen so that $\dot{\sigma}$ is then guaranteed to be positive. Clearly, we should set u to the value $u = 1$ for, in such a case, the numerator of $\dot{\sigma}$ is incremented in the positive quantity $IR = V_{\max}$. The controlled motions of σ, hence, satisfy the differential equation

$$\dot{\sigma} = \frac{V_{\max} - \sigma - V_d}{RC} = -\frac{1}{RC}\left[\sigma - (V_{\max} - V_d)\right] \tag{1.11}$$

Since σ is negative around the initial instant of time and, by hypothesis, $V_{\max} > V_d$ we have that indeed $\dot{\sigma}$ is positive as long as $u = 1$. Indeed

$$\dot{\sigma} = \frac{(V_{\max} - V_d) + |\sigma|}{RC} \tag{1.12}$$

Note that $\dot{\sigma}$ is still strictly positive at the moment when σ becomes zero at $t = t_h$, i.e.

$$\dot{\sigma}(t_h) = \frac{V_{\max} - V_d}{RC} > 0 \tag{1.13}$$

In fact, the linear differential equation describing the motions of σ from initially negative values predicts that the value of σ exponentially asymptotically converges towards the positive value $V_{\max} - V_d$. Hence, the value $\sigma = 0$ will be reached at some finite time t_h.

On the other hand, if σ is initially positive, then the other only possible choice is that of choosing $u = 0$. This is, indeed, the correct choice since now the evolution of σ is ruled by the following differential equation:

$$\dot{\sigma} = \frac{-\sigma - V_d}{RC} = -\frac{1}{RC}\left(\sigma + V_d\right) \tag{1.14}$$

The control input choice guarantees that the time derivative of σ is negative and hence the initial positive value of σ can only decrease as time goes on. Since the linear differential equation for σ predicts that σ converges exponentially towards the negative value $-V_d$ the condition $\sigma = 0$ will be reached in some finite time, say t_h.

In summary the switching policy

$$u = \begin{cases} 1 \text{ for } \sigma < 0 \\ 0 \text{ for } \sigma > 0 \end{cases} \quad \text{or} \quad u = \frac{1}{2}(1 - \text{sign}(\sigma)) \tag{1.15}$$

guarantees that the motions of the system approach, independently of the initial state value[1], the sliding surface represented by

$$\mathcal{S} = \{ v \ : \sigma = v - V_d = 0 \ \} \tag{1.16}$$

is reachable in finite time.

The second phase, that of the *sustaining task*, requires the sliding motion to be effectively kept on $\sigma = 0$ by fast switchings that corrects the small overshoots that may be due to a non-infinitely fast switch. Clearly the reaching control strategy equally accomplishes the sustaining phase by forcing the incipient values of the error σ to go back, through the sliding surface $\sigma = 0$. The trajectory of σ is again pushed back by the second control action causing a very fast zigzagged motion around $\sigma = 0$. We term this motion a *sliding regime*. The ideal frequency of this motion is, theoretically, infinite but in practice, with a real sensor and a real switch, the scheme accomplishes a highly oscillatory motion in the immediate vicinity of the sliding surface $\mathcal{S}$. We must idealize this *chattering* behavior of the capacitor voltage around $v = V_d$ by assuming that both our sensor and switch are infinitely fast. In practice, we know that the voltage evolution makes small excursions into the regions where the switch position will change as advised by the sensor measuring σ.

A sliding regime is then obtained for our switched RC circuit which keeps σ close to zero and, hence, v close to V_d. It is illustrative to find the smooth virtual control action, or the equivalent control, that would be responsible for such motions if the system were allowed to have continuous valued control inputs.

The equivalent control, denoted by u_{eq}, is obtained from the following *invariance conditions* :

$$\sigma = 0, \quad \dot{\sigma} = 0 \tag{1.17}$$

Using (1.9) one has

$$\dot{\sigma} = \left. \frac{V_{\max} u - \sigma - V_d}{RC} \right|_{\sigma = 0} = 0 \tag{1.18}$$

i.e.,

[1] We should add, even if such initial state v_0 is outside the region $[0, V_{\max}]$.

$$u_{\rm eq} = \frac{V_d}{V_{\rm max}} \tag{1.19}$$

The equivalent control is, in this case, positive and bounded above by 1, since, by hypothesis, $V_d < V_{\rm max}$, i.e.,

$$0 \; < \; u_{eq} \; < \; 1 \quad \text{when} \quad 0 < V_d < V_{max} \tag{1.20}$$

This restriction on the average control input u is consistent with the fact that the desired voltage, V_d, should not be negative and should not exceed the value V_{max}.

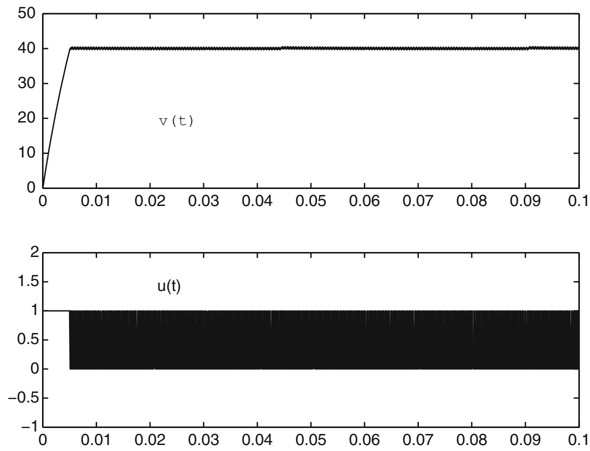

Fig. 1.4. Sliding mode controlled responses of switched RC circuit. Voltage response $v(t)$ and switch position function $u(t)$.

The upper part of Figure 1.4 depicts the reaching phase and the sliding mode sustaining phase. The reaching phase starts from an initial value $v(0) = v_0 = 0$ (i.e., $\sigma(0) = -V_d < 0$), which according to the switching strategy initially demands the control action $u = 1$. Once the sliding surface is reached, a rapid switching action, depicted in the lower part of Figure 1.4, is obtained. This control input behavior, characteristic of sliding modes, keeps the value of $v(t)$ in a small neighborhood of $\mathcal{S}$. In this case we have chosen:

$$R = 100\Omega, \quad C = 100\mu\,\text{F}, \quad I = 1\text{ A}, \quad V_d = 40\text{Volt}, \quad V_{\rm max} = 100\text{Volt}.$$

In the above example, the plant equations are linear in the state v and in the control input u. The limitations in the control actions, $u \in \{0, 1\}$, lead to a limitation in the possibility of accomplishing the objective of a desired constant voltage, limiting this voltage to a bounded interval of the one-dimensional state space. Even for this simple linear example, sliding may not take place on an arbitrary constant desired voltage line. Sliding regimes may only be locally possible in the state space.

Exercise 1.1. In simulations, it is often convenient to *normalize* the model of the plant. In the previous switched controlled RC circuit, define the capacitor voltage as $\vartheta = v/IR = v/V_{max}$. Via an appropriate time coordinate transformation (or, time normalization) show that the controlled system may be written as

$$\frac{d}{d\tau}\vartheta = u - \vartheta \tag{1.21}$$

where "τ" stands for the normalized time. The analysis of existence of sliding motions on the normalized model is considerably simpler.

1.4 The effect of unknown perturbations

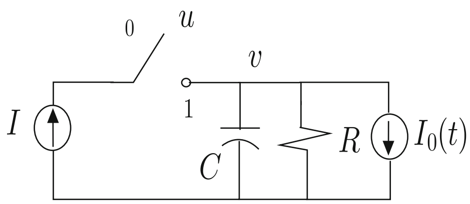

Fig. 1.5. Perturbed switched controlled RC circuit

Consider the perturbed RC circuit shown in Fig. 1.5 where now a current demand, denoted by $I_0(t)$, of a time-varying nature may be either draining current from the capacitor, thus discharging it, or else, injecting current into the capacitor thereby increasing its charge. Suppose such a varying demand is only known to satisfy the amplitude constraint : $\sup_t |I_0(t)| < K$, but it is otherwise completely unknown.

The perturbed differential equation governing the system is written as

$$\dot{v} = \frac{I}{C}u - \frac{v}{RC} - \frac{1}{C}I_0(t) \tag{1.22}$$

The analysis of existence of a sliding regime on $\sigma = 0$ with $\sigma = v - V_d$ entails examining the feasibility of having the virtual control action $u_{eq}(t)$ satisfy the condition $0 < u_{eq}(t) < 1$. This now is traduced into the following inequality for all times,

$$0 < u_{eq} = \frac{V_d}{RI} + \frac{I_0(t)}{I} < 1 \tag{1.23}$$

The virtual control action u_{eq} therefore explicitly depends on the perturbation input in a manner that implies an exact cancelation of the disturbance.

This last inequality is guaranteed to be valid for all perturbation realizations whenever we take the most adverse values for the perturbation signal values $I_0(t)$. In other words,

$$0 < \frac{K}{I} < \frac{V_d}{RI} < 1 - \frac{K}{I} \tag{1.24}$$

The effect of the perturbation is then translated into further reducing the feasibility region for the desired constant voltage value; i.e., recalling that $RI = V_{max}$, one has

$$\frac{K}{I}V_{max} < V_d < (1 - \frac{K}{I})V_{max} \tag{1.25}$$

This double inequality makes sense as long as the positive amplitude perturbation bound K on the current $I_0(t)$ is below 50% of the input source amplitude I. Under this circumstance, the interval of existence of a sliding regime is definitely reduced, with respect to the same interval in the unperturbed case, in direct proportion to the uncertainty present in the perturbation current amplitude.

We state then that unknown disturbance signals to the plant directly affect the region of existence of a sliding regime reducing it in accordance with the level of uncertainty associated with the disturbance.

1.5 Trajectory tracking in a switched RC circuit

The developments of the previous section may be slightly generalized to the case in which it is desired to have the capacitor voltage actually track a given, sufficiently smooth[2], reference trajectory $v^*(t)$. From the lessons learned in the previous example, it seems intuitively clear that one should only pursue the tracking of reference trajectories which are bounded within the interval $[0, V_{\max}]$. This is so, given that the system is the same and that the control input limitations are identical. The only change lies in the nature of the voltage error signal specifying the sliding line.

Take the tracking error, or sliding surface coordinate function, as $\sigma = v - v^*(t)$. The dynamics of σ obeys the time-varying controlled differential equation,

$$\begin{aligned}\dot{\sigma} &= \dot{v} - \dot{v}^*(t) \\ &= \frac{1}{RC}\left[V_{\max}u - \sigma - (v^*(t) + RC\dot{v}^*(t))\right]\end{aligned} \tag{1.26}$$

[2] Actually, we only need that the reference trajectory, $v^*(t)$, be continuous and differentiable over the real line, i.e. $v^*(t) \in C^1[0, \infty)$.

The analysis for the reaching phase is not entirely different to the one in the constant desired voltage case. Indeed, if σ is initially negative, one must now have a positive time derivative for σ. As before, this entitles using the cooperation of the term $V_{\max}$ in the numerator of the general expression for $\dot{\sigma}$ in (1.26). We set $u = 1$ as long as σ is negative. We obtain

$$\dot{\sigma} = \frac{1}{RC}\left[V_{\max} + |\sigma| - (v^*(t) + RC\dot{v}^*(t))\right] \tag{1.27}$$

The time derivative $\dot{\sigma}$ is guaranteed to remain positive for any negative value of σ whenever the reference trajectory *uniformly* satisfies the following condition:

$$V_{\max} \; > \; v^*(t) + RC\,\dot{v}^*(t), \qquad \forall t \tag{1.28}$$

On the other hand, if $\sigma > 0$, then $\dot{\sigma}$ should be negative. The most we can do is to annihilate the influence of the positive summand V_d from the numerator of the dynamics associated with σ. We thus set $u = 0$ and obtain

$$\dot{\sigma} = -\frac{1}{RC}\left[\sigma + (v^*(t) + RC\dot{v}^*(t))\right] \tag{1.29}$$

The time derivative $\dot{\sigma}$ is guaranteed to remain negative for any positive value of σ whenever the reference trajectory *uniformly* satisfies the following condition:

$$0 \; < \; v^*(t) + RC\,\dot{v}^*(t) \;\; \forall t \tag{1.30}$$

We therefore need that the reference trajectory $v^*(t)$ uniformly complies with the restriction

$$0 \; < \; v^*(t) + RC\,\dot{v}^*(t) \; < \; V_{\max}\; \forall t \tag{1.31}$$

The equivalent control is readily obtained from the expression (1.26), by letting the invariance conditions: $\sigma = 0$, $\dot{\sigma} = 0$ be valid and solving for the control u as u_{eq}. One obtains

$$u_{\text{eq}}(t) = \frac{1}{V_{\max}}\left[v^*(t) + RC\dot{v}^*(t)\right] \tag{1.32}$$

Note that the equivalent control coincides with the *nominal open loop control input* $u^*(t)$ corresponding with the given output reference trajectory, $v^*(t)$, obtained by *system inversion*[3]. The existence condition (1.31) is equivalent to

[3] Note that, in an average sense, the system dynamics is described by $\dot{v} = -(1/RC)\,v + (V_{max}/RC)\,u$, with u being a smooth control input. This justifies the statement just made which rests on the Internal Model principle (see Francis and Wohnam [10]).

the following one, obtained by dividing the inequality relation by the positive constant $V_{\max} = IR$,

$$0 \ < \frac{1}{V_{\max}} [\ v^*(t) + RC\ \dot{v}^*(t)] \ < 1 \ \forall t \tag{1.33}$$

which is seen to actually represent the well-known sliding mode existence condition in terms of the equivalent control.

$$0 \ < \ u_{\text{eq}}(t) \ < \ 1, \quad \forall t \tag{1.34}$$

Note that when $v^*(t) = V_d$ is a constant, the expression (1.33) reproduces the condition found in the previous example, (1.20), in the form $0 < u_{\text{eq}} = V_d/V_{\max} < 1$.

Suppose now that $v^*(t)$ is a biased sinusoid function of the form:

$$v^*(t) = \frac{V_{\max}}{2} + A\sin(\omega t) \tag{1.35}$$

centered around the line $v = V_{\max}/2$ and with an amplitude A yet to be determined in order to comply with the sliding mode existence conditions. These conditions, in turn, guaranteeing the accurate tracking of $v^*(t)$ by the system output voltage v.

The equivalent control, $u_{eq}(t)$, is found to be

$$\begin{aligned} u_{\text{eq}}(t) &= \frac{1}{V_{\max}} \left[\frac{V_{\max}}{2} + A\sin(\omega t) + RC\omega A\cos(\omega t) \right] \\ &= \frac{1}{2} + \frac{A}{V_{\max}} \sqrt{1 + (RC\ \omega)^2} \sin(\omega t + \phi) \end{aligned}$$

where ϕ is a frequency dependent angular shift, given by

$$\phi = \arctan\left(RC\omega\right) \tag{1.36}$$

The sliding mode existence condition $0 < u_{\text{eq}}(t) < 1$ leads to a frequency-amplitude tradeoff on the part of the polarized sinusoidal reference signal $v^*(t)$. Indeed one must have

$$-\frac{1}{2} < \frac{A}{V_{\max}} \sqrt{1 + (RC\ \omega)^2} < \frac{1}{2} \tag{1.37}$$

Assuming the amplitude A is always positive, one has

$$A \ < \ \frac{V_{\max}}{2\sqrt{1 + (RC\ \omega)^2}} < \frac{V_{\max}}{2} \tag{1.38}$$

For a given allowable amplitude: $A < V_{\max}/2$, the angular frequency ω of the voltage reference signal $v^*(t)$ must satisfy the following amplitude-frequency tradeoff,

$$\omega < \frac{1}{RC}\sqrt{\left(\frac{V_{\max}/2}{A}\right)^2 - 1} \tag{1.39}$$

For a given sinusoid amplitude A, the *bandwidth* of the controlled system is thus limited to frequencies satisfying the above inequality.

Figure 1.6 depicts the performance of the sliding mode controller when the amplitude-frequency inequality 1.38 is satisfied.

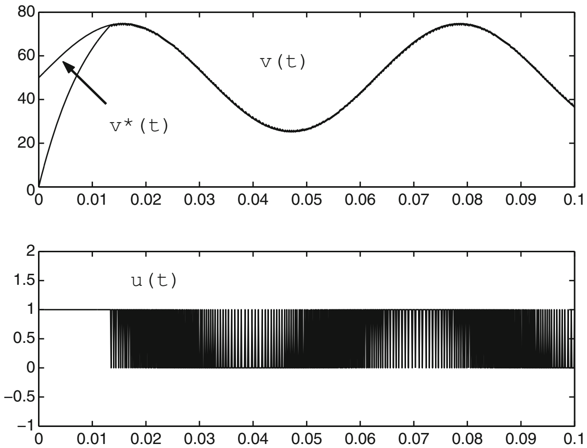

Fig. 1.6. Trajectory tracking task in sliding mode controlled RC circuit.

Figure 1.7 shows the system response when the bandwidth limitations are violated by the desired voltage reference trajectory. The figures depict the lack of uniform existence of sliding motions and the equivalent control signal exceeding the limitations of the interval $[0, 1]$.

Aside from the locality of the existence of sliding regimes, regardless of the reference voltage defining the sliding line being constant or not, the tracking of time-varying desired reference signals entitles an additional limitation represented by the time variability, or frequency content, of the desired reference voltage signal. For the specific case of sinusoidal signals, the conditions for the existence of a sliding regime explicitly reveals a bandwidth limitation, i.e., a compromise between the desired voltage amplitude and its frequency of oscillation.

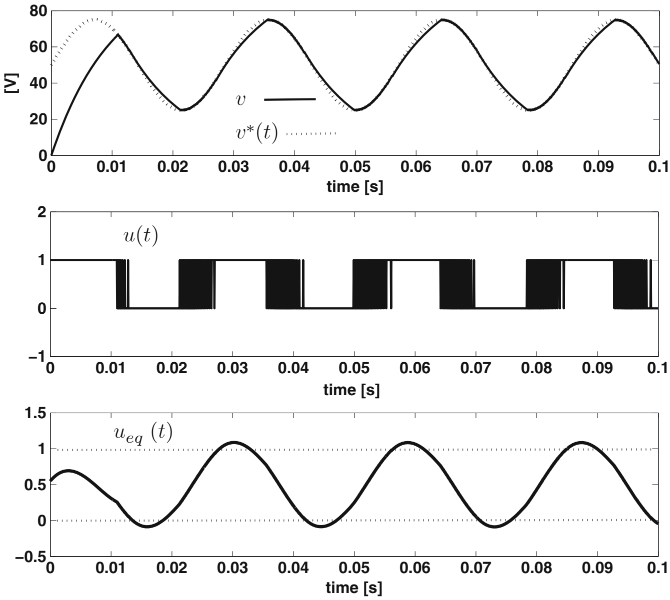

Fig. 1.7. Trajectory tracking under violation of bandwidth limitations.

1.6 A water tank system example

Consider the water tank shown in Figure 1.8, where u is the control input representing the valve position function and U volume per unit time entering the tank. As before, the control input is only allowed to take two possible values in the binary set $\{0, 1\}$.

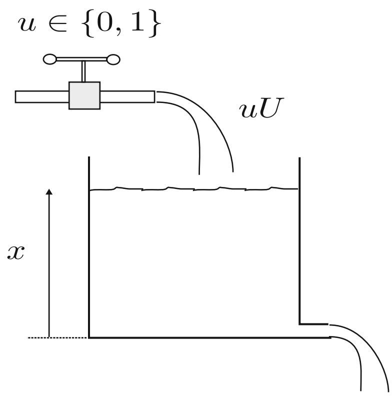

Fig. 1.8. Tank system

The system is described by the following nonlinear first order controlled differential equation,

$$\dot{x} = -\frac{c}{A}\sqrt{x} + \frac{U}{A}\,u, \quad u \in \{0, 1\} \tag{1.40}$$

where x represents the height of the liquid in the tank, c is a known coefficient, and A is the area of the base of the tank. It is desired to keep the liquid height at a constant value $x = X$.

When $u = 0$ the motions of the system, for an arbitrary initial condition on the liquid height, $x(0) = x_0 > 0$, at time $t = 0$, are governed by the solution to the following initial value problem:

$$\dot{x} = -\frac{c}{A}\sqrt{x}, \quad x(0) = x_0 \tag{1.41}$$

Using separation of variables, the solution of the uncontrolled differential equation (1.41) is given by

$$x(t) = \left(\sqrt{x_0} - \frac{c}{2A}t\right)^2 \tag{1.42}$$

From any positive initial condition, $x(t_0) = x_0 > 0$, the solutions for x reach zero in finite time, after which the model no longer has a physical meaning. The model is thus valid for only $x \geq 0$. Clearly, the tank will empty by itself in a finite time, T_e, given by

$$T_e = \frac{2A}{c}\sqrt{x_0} \tag{1.43}$$

On the other hand, if the control input is allowed to permanently take the alternative value, $u = 1$, then the system is governed by

$$\dot{x} = -\frac{c}{A}\sqrt{x} + \frac{U}{A} \tag{1.44}$$

The solution of (1.44) cannot be written in explicit form. This solution has the implicit relation:

$$-\frac{2A}{c}(\sqrt{x} - \sqrt{x_0}) + \frac{2AU}{c^2}\ln\left(\frac{U - c\sqrt{x_0}}{U - c\sqrt{x}}\right) = t \tag{1.45}$$

Note, however, that for $u = 1$, the equilibrium solution of equation 1.44 is given by $\overline{x} = U^2/c^2$. A tangent linearization of the nonlinear system around this equilibrium value shows that such an equilibrium is stable. In fact, all trajectories globally converge towards this equilibrium value from any initial conditions satisfying $x_0 \geq 0$.

Figure 1.9 shows the local responses of the tank system for sustained values (0 or 1) of the valve position function u.

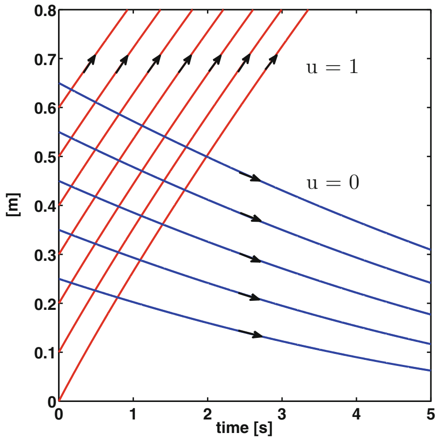

Fig. 1.9. Time responses for $u = 0$, $u = 1$.

Take the stabilization error, or sliding surface coordinate function as: $\sigma = x - X$. The dynamics of the sliding surface coordinate function is now given by the nonlinear equation

$$\dot{\sigma} = -\frac{c}{A}\sqrt{\sigma + X} + \frac{U}{A}u \tag{1.46}$$

The virtual control input, u_{eq}, that would achieve the desired constant height, X, of the liquid, provided one starts the liquid height evolution precisely at this value ($x(0) = X$), is characterized by the enforcement of the invariance conditions: $\sigma = 0$, $\dot{\sigma} = 0$. This leads to

$$u_{eq} = \frac{c}{U}\sqrt{X} \tag{1.47}$$

The existence of a sliding motion on $\sigma = 0$ is feasible whenever $0 < u_{eq} < 1$. This means that, $0 < X < U^2/c^2$. The existence of a sliding regime on the zero level set of the sliding surface coordinate function: $\sigma = x - X$, is, again, a local possibility in the state space for the desired constant liquid heights.

If σ is initially negative, i.e., the liquid height is smaller than the desired one, we have to strive to make $\dot{\sigma} > 0$. This may be accomplished using $u = 1$, i.e., by completely opening the valve. We have

$$\dot{\sigma} = -\frac{c}{A}\sqrt{\sigma + X} + \frac{U}{A} \tag{1.48}$$

Thus, provided the input flow U is such that

$$U \;>\; c\sqrt{X} \tag{1.49}$$

then, for all $\sigma < 0$, the time derivative of σ is guaranteed to be positive and σ grows towards zero, the desired error value.

On the other hand, if σ is initially positive, i.e., the liquid level is above the desired constant value, then we only have the choice of letting $u = 0$ and let the liquid level diminish by the assumed draining. The differential equation satisfied by the error height is given by

$$\dot{\sigma} = -\frac{c}{A}\sqrt{\sigma + X} \tag{1.50}$$

Then, the time derivative of σ is negative for any positive value of σ and, hence, σ decreases towards the desired value of zero.

A sliding regime is thus created on the condition, $\sigma = 0$, in finite time and the liquid height x is sustained via active valve switching around the desired value $x = X$. Clearly, a physical valve is incapable of sustaining a large *ON-OFF* switching action, much less, to sustain an ideal infinite frequency switching. Sliding mode control is severely limited in the regulation and control of hydraulic systems governed by valves. Something similar can be stated for mechanical systems. However, mechanical valves may be actuated by electric motors, a class of devices that is usually driven by sophisticated ON-OFF switches.

According to the system model (1.40), the largest available input flow is given by U when $u = 1$. The control input $u = 1$ is larger than the ideal, virtual, equilibrium input represented by u_{eq}. If this were not the case, a sliding regime could not have been created on $\sigma = 0$ since the valve fully open would not have been capable of increasing the level of the liquid. The desired

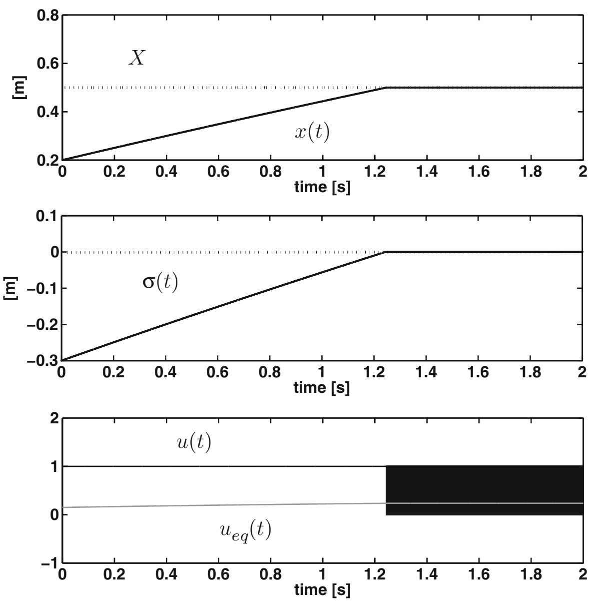

Fig. 1.10. Sliding mode controlled liquid height position

condition would only be instantaneously visited by the state trajectory from initial liquid heights located above the desired one and it would never be reached from lower heights.

The simulations shown in Figure 1.10 used the following numerical values:

$$c = 0.1, \quad X = 0.5 \text{ m}, \quad A = 1 \text{ m}^2, \quad U = 0.3\text{m}^3/\text{s}.$$

1.7 Trajectory tracking for the tank water height

It is in our interest to emphasize that trajectory tracking problems are not, essentially, more complicated than stabilization problems but they do exhibit bandwidth limitations.

Consider again the water tank system explained in the previous section. Suppose it is desired to have the liquid height $x(t)$ to follow a rather smooth *rest-to-rest* reference trajectory, denoted by $x^*(t) > 0$. In other words, we would like the height of the liquid, $x(t)$, to pass from a pre-specified initial equilibrium position towards a final equilibrium position, during a finite interval of time, specified by t_{init} and t_{final}. Evidently, the control algorithm should be good enough to first set up the desired initial equilibrium position, $x^*(t_{init})$, for whatever initial state, $x(0)$, the tank variable $x(t)$ happens to be located at time $t = 0 < t_{init}$.

The sliding surface coordinate function σ may be defined to be the tracking error $e = x - x^*(t)$, i.e.,

$$\sigma = x - x^*(t) \tag{1.51}$$

the tracking error time derivative satisfies thus the following first order differential equation,

$$\dot{\sigma} = -\frac{c}{A}\sqrt{\sigma + x^*(t)} + \frac{U}{A}u - \dot{x}^*(t) \tag{1.52}$$

The sliding mode existence condition for initially positive values of σ demands that $\dot{\sigma} < 0$. We must choose $u = 0$, for this is the only control option that makes σ actually decrease. On the other hand, initial negative values of σ demand that we set $u = 1$. Naturally, from (1.52), to have the condition $\dot{\sigma} > 0$ fulfilled in this case, the reference trajectory, $x^*(t)$, must be specified in such a manner that, for all $t \in [t_{init}, t_{final}]$, one has

$$c\sqrt{x^*(t)} + A\dot{x}^*(t) \; < \; U, \quad \text{i.e.} \quad \sup_{t \in [t_{init}, t_{final}]} \left[c\sqrt{x^*(t)} + A\dot{x}^*(t) \right] \; < \; U \tag{1.53}$$

The switching law is then specified as $u = 0$ for $\sigma > 0$ and $u = 1$ for $\sigma < 0$.

The equivalent control u_{eq} clearly coincides with the open loop nominal control input $u^*(t)$ corresponding to the given reference signal $x^*(t)$ and found by system inversion. The existence condition: $0 < u_{eq} < 1$ is seen to represent a *bandwidth* limitation of the control system due to the presence of the term, $\dot{x}^*$. Rapidly varying references may violate the tracking capabilities of the control system.

For simulation purposes, we consider the following rest-to-rest trajectory, defined with the help of an interpolating Bézier polynomial, of order 10 (see [26]):

$$x^*(t) = x^*_{init} + \left(x^*(t_{final}) - x^*(t_{init})\right) \psi(t, t_{init}, t_{final}) \tag{1.54}$$

where the function $\psi(t, t_{init}, t_{final})$ is defined as

$$\psi(t, t_{init}, t_{final}) =$$

$$\begin{cases} 0 & \text{for } t \ < \ t_{init} \\ \left[\frac{t-t_{init}}{t_{final}-t_{init}}\right]^5 \left[r_1 - r_2\left(\frac{t-t_{init}}{t_{final}-t_{init}}\right) + \cdots - r_6\left(\frac{t-t_{init}}{t_{final}-t_{init}}\right)^5\right] & \\ & \text{for } t \ \in [t_{init}, t_{final}] \\ 1 & \text{for } t \ > \ t_{final} \end{cases}$$

where

$$r_1 = 252, \ r_2 = 1050, \ r_3 = 1800, \ r_4 = 1575, \ r_5 \ 700, \ r_6 = 126$$

Figure 1.11 shows the sliding mode controlled tank variables and the smooth increase of the water height from an initial value towards a final value within a given finite time interval. We have set the same system parameters as in the stabilization case. The reference trajectory $x^*(t)$ defining parameters was chosen to be

$$t_{init} = 0.5, \ t_{final} = 3, \ x^*_{init} = 0.3, \ x^*_{final} = 0.5$$

In this figure, we have also plotted the equivalent control just to check that the sliding mode existence conditions are not violated with the specified trajectory. Notice that multiplying throughout by the positive quantity U the existence condition $0 < u_{eq} < 1$ leads to the equivalent condition: $0 < u_{eq}U < U = 0.3$

In this example it is interesting to assess the effects of a violation of the existence condition $0 < u_{eq}U < U$. For instance, if we required the rest-to-rest maneuver to be accomplished in a substantially smaller time (say, 1.5 units of time instead of 2.5 as in the previous simulation), then the time derivative of the reference trajectory is accordingly increased during the maneuver. Figure 1.12 shows that the existence condition is violated during an open interval of time between t_{init} and the new $t_{final} = 2.0$. The sliding mode is no longer sustained during this open interval and, as a consequence, the tracking of the reference trajectory is temporarily lost.

The previous examples show that designing for a sliding regime, which enforces a desired control objective, is rather simple in first order systems. The limitations entail: local existence of sliding regimes, i.e., conditions representing a desired objective achievable via sliding regimes may be limited to

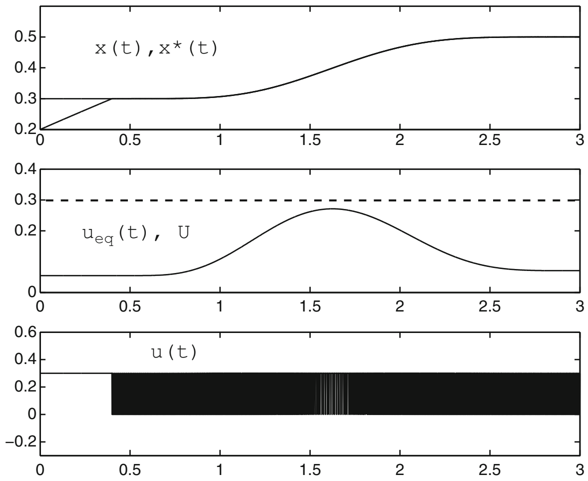

Fig. 1.11. Reference trajectory tracking for the tank system

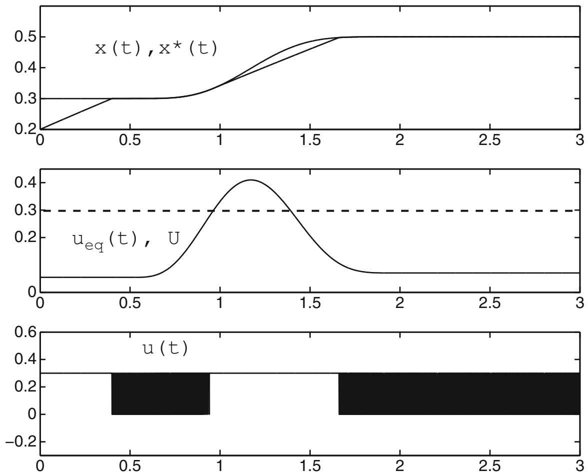

Fig. 1.12. Effects of violation of the bandwidth in the sliding mode existence conditions

certain regions of the state space. The sliding mode existence problem for a second order system exhibits further limitations while demanding more care, as the following examples will now demonstrate.

1.8 A Bilinear DC to DC Converter

Consider the differential equations describing a "boost" converter consisting of a voltage source of value E, an inductor of value L, and a capacitor of value C (Fig. 1.13). The action of a switch ($u = 0$) energizes the inductor, thus

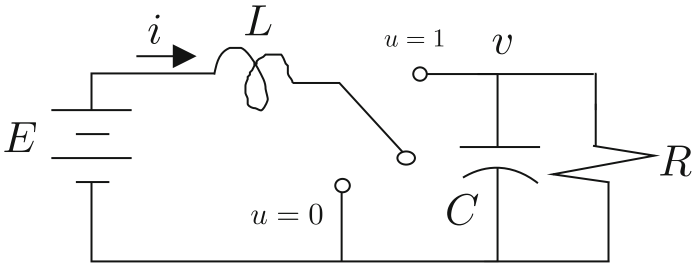

Fig. 1.13. A Boost Converter Circuit

storing energy in its magnetic field and then the switch ($u = 1$) allows this stored energy to be transferred to the output circuit consisting of a parallel connection of a capacitor and a load resistor of value R. The stored energy produces a voltage potential. These energy loading- energy transfer- energy storing cycles may take place at considerable speed. The coupled differential equations of the circuit are

$$L\frac{di}{dt} = -uv + E, \quad C\frac{dv}{dt} = ui - \frac{v}{R} \tag{1.55}$$

where $u \in \{0, 1\}$ is a switch position function and the state variables, i and v, respectively, represent the inductor current and the capacitor voltage. A magnitude and time normalization of the equations may be readily obtained by setting:

$$x_1 = \frac{1}{E}\sqrt{\frac{L}{C}}, \quad x_2 = \frac{v}{E}, \quad d\tau = dt/\sqrt{LC} \tag{1.56}$$

The normalized system is described by the simpler (bi-linear) system:

$$\frac{dx_1}{d\tau} = -ux_2 + 1, \quad \frac{dx_2}{d\tau} = ux_1 - \frac{x_2}{Q} \tag{1.57}$$

where $Q = R\sqrt{\frac{C}{L}}$.

Let “ $\dot{}$ ” abusively stand for normalized time differentiation ($d/d\tau$). Suppose the inductor is initially energized while the capacitor exhibits some nonzero potential. When u is set to the value $u = 0$, the system is described by the following set of differential equations

$$\dot{x}_1 = 1, \quad \dot{x}_2 = -\frac{x_2}{Q} \tag{1.58}$$

The normalized current x_1 becomes unstable (the inductor current grows without limit), while the normalized voltage x_2 is seen to be exponentially asymptotically stable to zero (i.e., the capacitor discharges its stored energy through the resistor load if the switch position is indefinitely held at $u = 0$.)

When the switch position is held at $u = 1$, the differential equations describing the circuit are given by:

$$\dot{x}_1 = -x_2 + 1, \quad \dot{x}_2 = x_1 - \frac{x_2}{Q} \tag{1.59}$$

The system exhibits the constant equilibrium point $x_1 = \frac{1}{Q}$, $x_2 = 1$. It is easy to verify that this equilibrium is globally asymptotically, exponentially, stable. Indeed, the characteristic polynomial of the linear map defining the system is given by

$$p(\lambda) = \lambda^2 + \frac{1}{Q}\lambda + 1 \tag{1.60}$$

whose roots are real and strictly negative for $0 < Q < 1/2$ and they are complex conjugate with strictly negative real parts for $Q > 1/2$. Typical trajectories are illustrated in Figure 1.14 for each value of u.

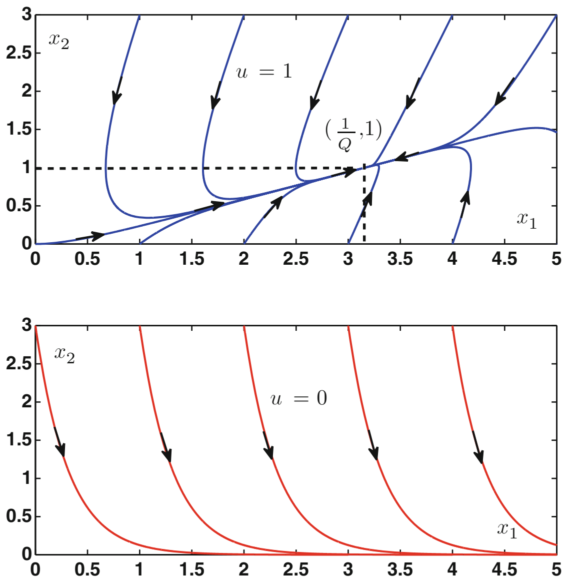

Fig. 1.14. State trajectories of the boost converter circuit for $u = 0$, and $u = 1$, switch positions.

1.8.1 Switching on a Desired Constant Voltage Line

Suppose it is desired to achieve a constant normalized voltage $x_2 = V_d$. The algebraic condition: $\sigma = x_2 - V_d = 0$, faithfully represents the constant output voltage control objective. We examine the feasibility of reaching this condition and indefinitely sustaining it. We restrict our considerations to the physically plausible region: $x_1 > 0$, $x_2 > 0$.

Ideally, when $x_2 = V_d$, the corresponding equivalent control is just, $u_{eq} = \frac{1}{V_d}$. The meaning of this constant control input is to be interpreted as the non-switching control input that would be required to permanently sustain the condition: $\sigma = 0$. Such a control input represents a virtual, continuous, control input that replaces, on $\sigma = 0$, the infinite frequency switchings. The equivalent control was derived from the condition $\dot{\sigma} = 0$. Notice that the corresponding dynamics for x_2 leads to $\dot{x}_2 = x_1/V_d - V_d/Q = 0$, i.e., $x_1 = V_d^2/Q$, which is the corresponding equilibrium value of the normalized inductor current x_1.

The time derivative of σ is just $\dot{\sigma} = ux_1 - x_2/Q$. Thus, setting $u = 0$ for $\sigma > 0$ yields $\dot{\sigma} = -x_2/Q < 0$, i.e., $\sigma\dot{\sigma} < 0$. The initially positive value of σ decreases towards the zero value. Setting $u = 1$ for $\sigma < 0$ leads to $\dot{\sigma} = x_1 - x_2/Q$ which is positive as long as $x_1 > (x_2/Q)$ i.e., whenever x_2 lies below the line: $x_2 = Qx_1$. This line has positive slope, contains the origin, and contains the equilibrium point $(1/Q, 1)$ of the system when u is fixed to $u = 1$. The intersection of this line with $\sigma = 0$ occurs at $(V_d/Q, V_d)$ which is to the left of the desired equilibrium point: $(V_d^2/Q, V_d)$ (i.e., $V_d/Q < V_d^2/Q$). Sliding motions occur to the right of $(V_d/Q, V_d)$. In this region, $\sigma\dot{\sigma} < 0$, so σ grows from initially negative values towards the value zero. The switching strategy: $u = (1/2)(1 - \text{sign}(\sigma))$ may then, indeed, lead to a sliding regime.

Now suppose $\sigma = 0$ i.e., $x_2 = V_d$, then, ideally, $\dot{\sigma} = ux_1 - V_d/Q = 0$. A continuous equivalent control law which sustains the condition $\sigma = 0$ is just $u_{eq} = V_d/x_1 Q$ which substituted on the corresponding dynamics for x_1 yields: $\dot{x}_1 = -V_d^2/(x_1 Q) + 1$. The equilibrium point of this equation coincides with the previously found equilibrium point. To the right of the equilibrium point we have: $\dot{x}_1 > 0$, so the normalized inductor current grows without limit. However, to the left of the equilibrium point a sliding regime does not exist on $\sigma = 0$, since, now, we are above the line: $x_2 = (Q/V_d)x_1$ and $\sigma\dot{\sigma} > 0$. The sliding regime on $\sigma = 0$ locally exists but it is not feasible and unsustainable.

1.8.2 Switching on a Desired Constant Current Line

Consider now the switching line $\sigma = x_1 - V_d^2/Q$. $\sigma = 0$ represents a vertical line in the state space (x_1, x_2). To the right of $\sigma = 0$, σ is positive and to the left of $\sigma = 0$, σ is negative. Initially, let $\sigma > 0$. The time derivative of σ is just $\dot{\sigma} = -ux_2 + 1$. Setting $u = 1$ guarantees a negative time derivative of σ provided $x_2 > 1$. Hence, above the line $x_2 = 1$ the condition: $\sigma\dot{\sigma} < 0$ is valid to the right of $\sigma = 0$. When $\sigma < 0$, the control $u = 0$ yields $\dot{\sigma} = 1$, i.e., $\sigma\dot{\sigma} < 0$. A sliding regime thus exists on the vertical line: $x_1 = V_d^2/Q$, above the normalized voltage line $x_2 = 1$.

Let $\sigma = 0$, i.e., $x_1 = V_d^2/Q$. The condition: $\dot{\sigma} = 0$ yields $\dot{\sigma} = -ux_2 + 1 = 0$. The equivalent control is then $u_{eq} = 1/x_2$. The closed loop dynamics for x_2 on $\sigma = 0$ is, hence, governed by $\dot{x}_2 = (V_d^2/(x_2 Q)) - x_2/Q$. This nonlinear dynamics exhibits the equilibria: $x_2 = \pm V_d$. Since there is no sliding regime below $x_2 = 1$, the physically feasible equilibrium point is $x_2 = V_d > 1$. The equilibrium, $x_2 = V_d$, is asymptotically stable as demonstrated by the Lyapunov function candidate: $V(x_2) = (1/2)(x_2 - V_d)^2$. Indeed, along the ideal state trajectories on the sliding line, one has: $\dot{V}(x_2) = (x_2 - V_d)[(V_d^2/(x_2Q) - x_2/Q] = [(x_2 - V_d)/(x_2Q)][V_d^2 - x_2^2] = -[1/(x_2Q)][(x_2 - V_d)^2(V_d + x_2)] < 0$. Thus indicating that the positive quantity $V(x_2)$ is constantly decreasing until reaching the condition $x_2 = V_d$, where now $V(x_2) = 0$ and $\dot{V}(x_2) = 0$.

Simulations were performed on a normalized boost circuit model with $Q = 0.31622$ and desired voltage: $V_d = 2$, with a corresponding equilibrium current $V_d^2/Q = 12.649$. Figure 1.15 depicts the controlled system trajectories in the state space of coordinates (x_1, x_2). The equilibrium point on the constant voltage sliding line is unstable and the sliding motion ceases to exist at some point when the current decreases below its equilibrium value. The equilibrium point on the constant current sliding line is asymptotically stable and the sliding motion takes place in the region of the state space where the boost converter amplifies, at the output, the normalized source voltage value, i.e., whenever $x_2 > 1$.

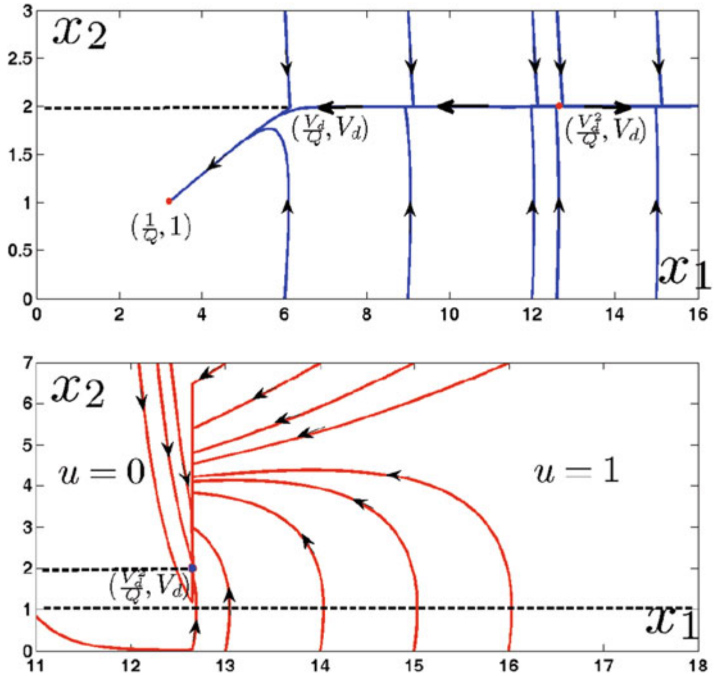

Fig. 1.15. State trajectories of the boost converter circuit seeking sliding motions on constant voltage and constant current lines.

In second order systems, where sliding surface coordinate functions may be a function of a single state variable or both state variables, it is seen that sliding modes face a new concern. Sliding surfaces defined as a function of a single state variable induce closed loop dynamics on the second state variable that may or may not be convenient. This phenomenon, known as the stability of the *zero dynamics* is a typical concern when the variable σ is considered as a regulated variable of the system. Its zeroing leaves unobserved dynamics that may exhibit convenient (possibly global, but generally local) stability features around equilibrium points; the sliding surface coordinate function σ is then a *minimum phase* output variable, otherwise it is addressed as a *non-minimum phase* variable. Non-minimum phase behavior is to be avoided at all costs by suitable consideration of an alternative, minimum phase, variable (See Di Benedetto and Grizzle [3]).

1.9 A Second Order System Example

Consider the following switch controlled second order system, consisting of a pure second order integrator system. We describe such a system via

$$\ddot{y} = 1 - 2u \tag{1.61}$$

with $u \in \{0, 1\}$. Suppose it is desired to control the system, from any reasonable arbitrary initial condition $(y(0), \dot{y}(0)) = (y_0, \dot{y}_0)$, towards a given final constant value for y, given by $y = Y$.

Consider the sliding line, $\sigma(y, \dot{y}) = \dot{y} + \alpha(y - Y)$ with $\alpha > 0$. This sliding line prescription is compatible with the control objective: $\lim_{t\to\infty} y(t) = Y$, i.e., stabilizing the variable y to a constant value Y. Notice that under ideal sliding conditions, on $\sigma = 0$, the constrained state evolves satisfying the linear, asymptotically, exponentially stable dynamics: $\dot{y} = -\alpha(y - Y)$. We deem this behavior as desirable and, hence, it represents a proper control objective.

For $u = 0$, the system is simply, $\ddot{y} = 1$ and the state trajectories, starting from an arbitrary initial state $y(0) = y_0$, $\dot{y}(0) = \dot{y}_0$, are given by

$$\dot{y}(t) = t + \dot{y}_0, \quad y(t) = \frac{1}{2}t^2 + \dot{y}_0 t + y_0 \tag{1.62}$$

The velocity $\dot{y}$ grows linearly in time, while the output y grows quadratically in time. This set of parametric equations corresponds to the following expression for the graph of the state trajectory in the phase plane $(y, \dot{y})$,

$$\begin{aligned} y &= \frac{1}{2}(\dot{y} - \dot{y}_0)^2 + \dot{y}_0(\dot{y} - \dot{y}_0) + y_0 \\ &= \frac{1}{2}\left[(\dot{y})^2 - 2\dot{y}_0^2 + 2y_0\right] \end{aligned}$$

Similarly, when $u = 1$, the system obeys, $\ddot{y} = -1$ and, the parametric equations describing the state trajectory from arbitrary initial states, are given by

$$\dot{y} = -t + \dot{y}(0), \quad y(t) = -\frac{1}{2}t^2 + \dot{y}(0)t + y(0) \tag{1.63}$$

Eliminating the variable t from the last two expressions, we obtain the equation for the graph described by the state trajectory in the phase space

$$y = -\frac{1}{2}\left[(\dot{y})^2 - \dot{y}_0^2 - 2y_0\right] \tag{1.64}$$

Figure 1.16 depicts typical phase trajectories for the two possible controlled systems, with fixed switch positions $u = 1$ or $u = 0$.

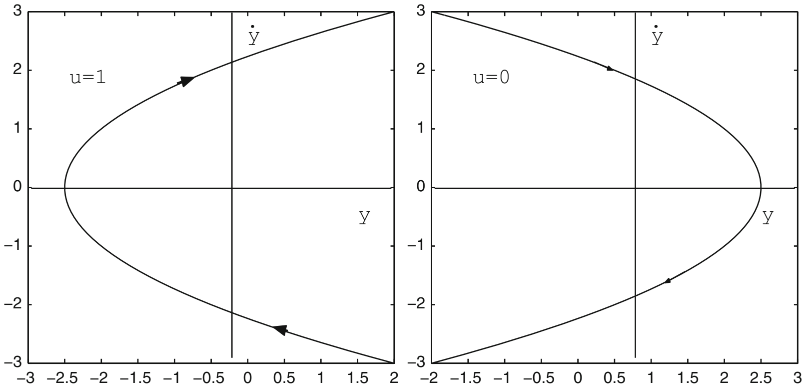

Fig. 1.16. Phase plane trajectories for $u = 1$ and $u = 0$

The time derivative of σ, along the controlled solutions of the second order plant, is obtained as

$$\dot{\sigma} = 1 - 2u + \alpha\dot{y} \tag{1.65}$$

When $\sigma > 0$ the most we can do is to set $u = 1$, so as to contribute as much as possible to the negativity of $\dot{\sigma}$. Conversely, when $\sigma < 0$, we must set $u = 0$ in order not to contribute with a negative additive term to the value of $\dot{\sigma}$ which should be as positive as possible. The switching strategy is circumscribed to: $u = 1$ for $\sigma > 0$ and $u = 0$ for $\sigma < 0$. In other words, $u = \frac{1}{2}(1 + \text{sign}\sigma)$.

A sliding regime exists provided σ and $\dot{\sigma}$ exhibit opposite signs in the vicinity of the sliding line: $S = \{(y, \dot{y}) \in \mathbb{R}^2 \mid \sigma(y, \dot{y}) = 0\}$. The product $\sigma\dot{\sigma}$ should be strictly negative close enough to S. When $\sigma > 0$, $\dot{\sigma} = -1 + \alpha\dot{y}$. It follows that $\dot{y} < 1/\alpha$. On the other hand, when $\sigma < 0$, $\dot{\sigma} = 1 + \alpha\dot{y}$, hence $\dot{y} > -1/\alpha$. The band: $-1/\alpha < \dot{y} < 1/\alpha$, shown in Figure 1.17, generates on the sliding line S the corresponding existence region: $Y - 1/\alpha^2 < y < Y + 1/\alpha^2$ for the y coordinate.

As the ideal sliding motions $\dot{y} = -\alpha(y - Y)$ are required to converge faster towards the equilibrium point $(Y, 0)$, by choosing a larger value of α, the rectangular sliding region on the phase space further shrinks thus having a smaller region of existence of the sliding regime.

Figure 1.18 depicts a particular controlled trajectory converging towards a desired equilibrium state $y = Y = 1$, $\dot{y} = 0$ and the corresponding discontinuous control input actions.

The local nature of sliding regimes may be overcome, in this particular instance, by proposing a nonlinear sliding line in the phase plane. This is examined next.

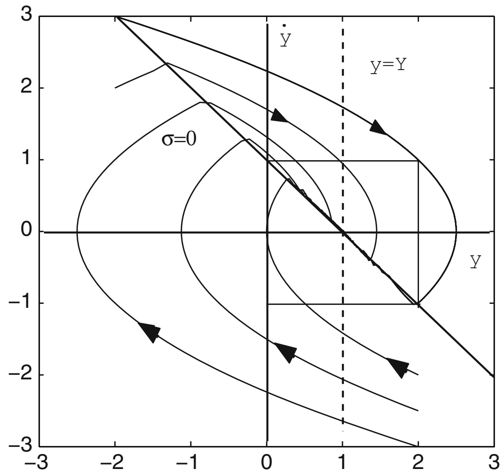

Fig. 1.17. Region of existence of sliding motions and controlled trajectories in the phase plane

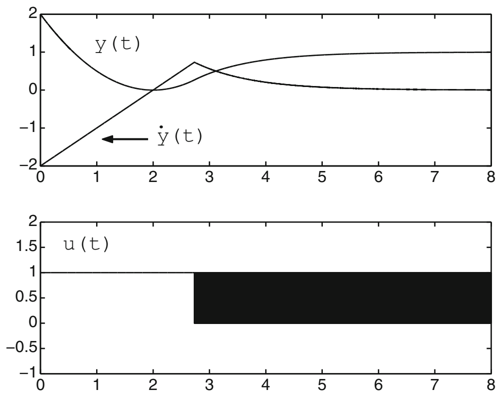

Fig. 1.18. Controlled trajectories for the phase variables and control input for second order system

1.10 The quest for global sliding motions

The previous example illustrates a typical situation where our desire to have simple controlled motions, of the linear kind, on the sliding surface is paid by a local validity of the sliding mode existence. One may then wonder if slightly more complex sliding surfaces, and hence slightly more complicated ideal controlled motions, may bestow us with a global nature of the existence of sliding motions. This would have a particular advantage regarding the quality of the performance of the feedback controlled responses.

In the previous example there are two particularly interesting trajectories in the phase space, generated by the available controls, $u = 0$ and $u = 1$, that precisely pass through the desired equilibrium point, $y = Y$, $\dot{y} = 0$, in the phase space. These trajectories are depicted in Figure 1.19.

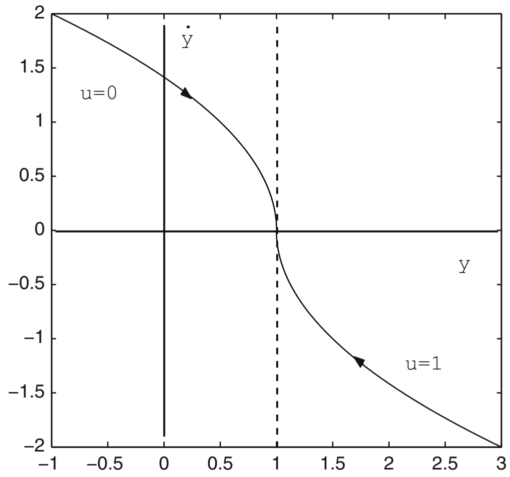

Fig. 1.19. Controlled motions leading to the desired equilibrium point

Being that these trajectories are obtained by actual available control inputs, they do not themselves qualify as sliding surfaces. Somehow sliding surfaces are generated by "average" control actions. The mathematical expressions of each branch of the parabola representing these particular trajectories are the following:

$$u = 0, \quad \dot{y} = +\sqrt{2(Y - y)}, \quad y < Y \tag{1.66}$$

and

$$u = 1, \quad \dot{y} = -\sqrt{2(y - Y)}, \quad y > Y \tag{1.67}$$

Consider then scaled versions of the arcs of parabolas (1.66), (1.67) leading towards the point $(Y,0)$, and form sliding surfaces with the resulting expressions. For this let λ be a positive scalar satisfying $0 \ < \ \lambda \ < \ 1$. Let $\mathcal{S}$ be defined as follows:

$$\mathcal{S} = \mathcal{S}^+ \cup \mathcal{S}^- \tag{1.68}$$

where the "$\cup$" symbol stands for union in the set theoretic sense and, also,

$$\begin{aligned} \mathcal{S}^+ &= \{(y,\dot{y}) \mid \sigma(y,\dot{y}) = \dot{y} - \lambda\sqrt{2(Y-y)} = 0, \quad y \ < \ Y\} \\ \mathcal{S}^- &= \{(y,\dot{y}) \mid \sigma(y,\dot{y}) = \dot{y} + \lambda\sqrt{2(y-Y)} = 0, \quad y \ > \ Y\} \end{aligned} \tag{1.69}$$

One of the advantages of these sliding arcs rests on the global nature of the sliding motions that can be induced on their union and moreover, once the sliding surface is reached the motions reach the desired equilibrium in finite time.

To demonstrate the validity of this last statement, consider the motions ideally taking place along the arc denoted by $\mathcal{S}^+$, on the portion of the phase space determined by $\dot{y} > 0$ and $y < Y$. Suppose that the initial condition on such a line is given by $y(0) = y_0$. The differential equation governing the closed loop system is given by

$$\dot{y} = \lambda\sqrt{2(Y-y)}, \quad y(0) = y_0 < Y \tag{1.70}$$

The solution of this nonlinear separable equation is given by

$$y(t) = Y - \frac{1}{2}\left[t - \sqrt{2(Y-y_0)}\right]^2 \tag{1.71}$$

Clearly, at time $t = T = \sqrt{2(Y-y_0)}$ the point $y(T) = Y$, $\dot{y}(T) = 0$ is reached.

Similarly, motions taking place on $\mathcal{S}^-$ from an initial condition $y(0) = y_0$ are governed by

$$\dot{y} = -\lambda\sqrt{2(y-Y)}, \quad y(0) = y_0 > Y \tag{1.72}$$

The solution of this initial value problem is given by

$$y(t) = Y + \frac{1}{2}\left[-t + \sqrt{2(y_0-Y)}\right]^2 \tag{1.73}$$

At time $t = T = \sqrt{2(y_0-Y)}$ the desired point $y(T) = Y$, $\dot{y}(T) = 0$ is reached.

To show that sliding motions on $\mathcal{S}$ exist globally consider first the branch of the sliding surface denoted by $\mathcal{S}^+$. The time derivative of the sliding surface coordinate function, $\sigma = \dot{y} - \lambda\sqrt{2(Y-y)}$, along the controlled motions of the system is given by

$$\begin{aligned} \dot{\sigma} &= -1 + 2u + \lambda\frac{\dot{y}}{\sqrt{2(Y-y)}} \\ &= -1 + 2u + \lambda\left[\frac{\sigma + \lambda\sqrt{2(Y-y)}}{\sqrt{2(Y-y)}}\right] \end{aligned} \tag{1.74}$$

For small, negligible, values of σ, whether positive or negative the time derivative of σ around $\mathcal{S}^+$ is governed by

$$\dot{\sigma} = -1 + 2u + \lambda \tag{1.75}$$

When $\sigma < 0$ our choice is $u = 1$. The time derivative of σ is given by $\dot{\sigma} = 1 + \lambda > 0$. If, on the other hand, $\sigma > 0$, then the choice $u = 0$ leads to $\dot{\sigma} = -1 + \lambda$. Since $\lambda \in (0, 1)$ the derivative is negative. Notice that these two conditions are valid everywhere in the immediate vicinity of $\mathcal{S}^+$. A similar analysis shows that sliding motions are global around $\mathcal{S}^-$.

Figures 1.20 and 1.21 depict the sliding mode controlled responses of the second order system when the nonlinear sliding surface guaranteeing finite time stabilization is used.

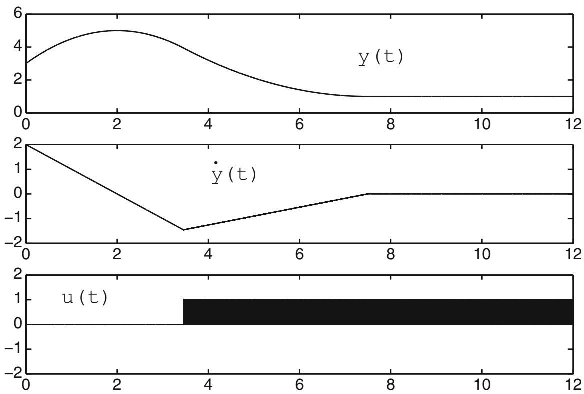

Fig. 1.20. Sliding mode controlled response of second order system with global, finite time, stabilization surface.

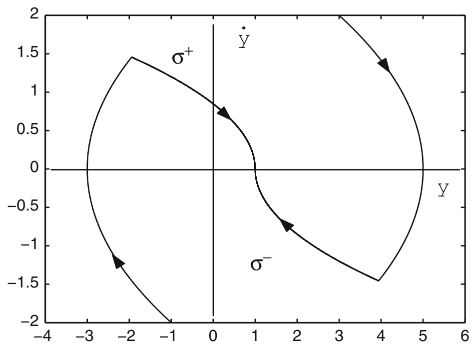

Fig. 1.21. Trajectories achieving global, finite time, stabilization on nonlinear sliding surface in phase plane.

Sliding regimes are, generally speaking, local. Global sliding regimes may be highly desirable. Achieving this feature is part of a design problem with no clear-cut general guidelines. The problem of achieving global sliding regimes is relatively easy to tackle for one and two dimensional systems. As dimensions increase, the intuitive feeling becomes blurred.

Exercise 1.2. Consider the mechanical system shown in Figure 1.22. Assume the control torque input is only capable of producing a fixed torque value in either sense (clockwise or counterclockwise). This translates into one of two possible applied force values, say F and $-F$ to the mass.

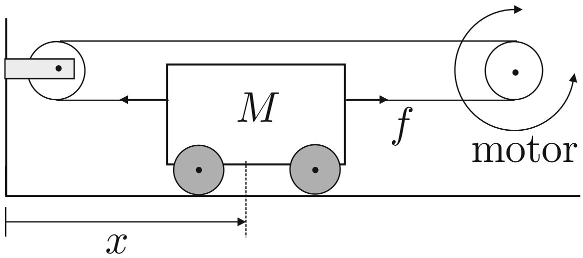

Fig. 1.22. Control of a mass position

The system model, including viscous friction forces, is given by the following second order system,

$$M\ddot{x} = -B\dot{x} + f, \quad f \in \{F, -F\} \tag{1.76}$$

It is desired to control the system, from any reasonable arbitrary initial position and initial velocity, to a given final constant position given by $x = X$. Work out the details. Propose a sliding surface, containing $x = X$, on which one may create a global sliding regime.

1.11 A nasty perturbation

Consider the following perturbed state space system,

$$\dot{x}_1 = x_2 + \epsilon \sin(\omega t), \quad \dot{x}_2 = W(1 - 2u) \tag{1.77}$$

with ϵ and ω being completely unknown, while $W > 0$ is a sufficiently large gain. Suppose it is desired to stabilize the system to the origin of coordinates $(x_1, x_2) = (0, 0)$.

A sliding surface of the form:

$$\mathcal{S} = \{(x_1, x_2) \mid \sigma(x) = x_2 + \alpha x_1 = 0, \quad \alpha > 0\} \tag{1.78}$$

may be reached via the switching control strategy: $u = \frac{1}{2}(1 + \text{sign}\sigma)$. Indeed,

$$\dot{\sigma} = W(1 - 2u) + \alpha(x_2 + \epsilon \sin(\omega t)) \tag{1.79}$$

Clearly, when $\sigma > 0$, $\dot{\sigma} = -W + \alpha x_2 + \alpha\epsilon \sin(\omega t)$ and when $\sigma < 0$, $\dot{\sigma} = W + \alpha x_2 + \alpha\epsilon \sin(\omega t)$. A sliding regime exists on the time-varying band of the state space,

$$-\frac{W}{\alpha} + \epsilon < x_2 < \frac{W}{\alpha} - \epsilon \tag{1.80}$$

which is a feasible region provided $\epsilon < W/\alpha$.

On $\mathcal{S}$, the evolution of the controlled system satisfies:

$$\dot{x}_1 = -\alpha(x_1 - \frac{\epsilon}{\alpha}\sin(\omega t)), \quad x_2 = -\alpha x_1 \tag{1.81}$$

and neither x_1, nor x_2, converge to zero as desired. The closed loop system is affected by the perturbation signal and even though its presence does not preclude the existence of a sliding regime, the ideal sliding dynamics is definitely affected by such perturbation inasmuch as the control objective is not achieved. The perturbation input directly affects the dynamics of the state x_1 while the control directly affects the dynamics of x_2. The perturbation is said to be *non-matched*.

Notice, however, that if the perturbed system is of the form,

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = W(1 - 2u) + \epsilon \sin(\omega t). \tag{1.82}$$

A sliding regime exists on the same sliding surface $\mathcal{S}$ specified above, with a rather similar region of existence:

$$-\frac{W}{\alpha} + \frac{\epsilon}{\alpha} < x_2 < \frac{W}{\alpha} - \frac{\epsilon}{\alpha} \tag{1.83}$$

where, now, it must be assumed that $\epsilon < W$.

On $\mathcal{S}$ the system is governed by

$$\dot{x}_1 = -\alpha x_1, \quad x_2 = -\alpha x_1 \tag{1.84}$$

and now both x_1 and x_2 asymptotically exponentially converge to zero.

In the second case, we say that the closed loop system is *robust* with respect to the perturbation signal since the ideal sliding dynamics is completely independent of such signal. The perturbation input, which now affects directly the evolution of the state x_2, similarly to the control input, is said to be *matched*.

1.12 Some lessons learned from the examples

The essential feature of the sliding mode control problem in a switched system is that of prescribing a suitable state restriction on the state variables, represented as a smooth manifold, called the sliding surface, for which the available binary valued control input can 1) guarantee, even if locally, an approach in finite time of the state trajectory representative point to the sliding surface while inducing opposite behaviors of the state trajectories in the immediate vicinity of the switching manifold. Such opposite behaviors refer to an *oblique* (as opposed to *tangential*) approach of the state trajectories to the sliding surface, from each one of the two regions in the state space delimited by the sliding manifold. This is tied to the potential of actually crossing such a boundary from each side, under the action of one of the fixed control inputs. Such a crossing is to occur in a direction opposite to that which can be achieved from the "other side" with the other (only) available control input. 2) The motion, constrained to the given surface, can be in principle indefinitely sustained, by using the same feedback control law that achieved the reaching of the sliding surface in finite time from any of the two "sides" of the sliding manifold. For this, the nature of the controlled trajectories around the surface is such that they are always pointing towards the surface in its immediate vicinity. The sliding motion will be lost, in a certain region of the state space, where the two kinds of controlled state trajectories no longer point towards each other around the geometric boundary represented by the sliding manifold. In general, the sliding motions are, generally speaking, expected to be only locally sustainable. Idealized, virtual, feedback control strategies which smoothly guarantee the evolution of the controlled trajectories on the given sliding manifold, for trajectories starting on it, are essential in assessing the validity of local, or global, existence conditions of the induced sliding regime. This virtual feedback control constitutes the equivalent control. The existence conditions of a sliding regime on a given sliding manifold simply demand that the equivalent control must be bounded within the interval determined by the two extreme numerical values assigned to the switch position function. The equivalent control law must be realizable as an intermediate smooth feedback control action with respect to the available switching control extremes. Such virtual control actions are responsible for sliding surface invariance, a concept that is intimately related to the ideal controlled behavior of the system on the restriction, or sliding, manifold. For trajectory tracking tasks, the existence conditions assessed in terms of the bounded nature of the equivalent control are tantamount to natural limitations on the bandwidth of the control system. The frequency content of the reference trajectories to be tracked is necessarily limited. In linear systems it is not difficult to show that such a frequency content limitation refers to a low frequency band. Finally, state coordinate and input coordinate transformations, locally or globally invertible, may help in visualizing a simpler solution to the sliding mode creation problem. Unfortunately, the physical significance of the system variables may be lost in the framework of transformed coordinates.

Let $\sigma(x)$ represent the sliding surface coordinate function of the system state vector x, with the sliding manifold being represented by:

$$\mathcal{S} = \{ \ x \mid \sigma(x) = 0 \ \} \tag{1.85}$$

A sliding regime exists on $\mathcal{S}$ if the following two conditions are satisfied ([31]):

$$\lim_{\sigma \to 0^+} \dot{\sigma} < 0 \quad \text{and} \quad \lim_{\sigma \to 0^-} \dot{\sigma} > 0 \tag{1.86}$$

This existence condition clearly depicts the opposite nature of the controlled motions around the condition $\sigma(x) = 0$. Indeed, approaching the surface from negative values of the sliding surface coordinate function this function must *grow* towards zero. When approaching the surface from positive values of $\sigma(x)$ this function must *decrease* towards zero. The two conditions above can be summarized into a single one, namely,

$$\lim_{\sigma \to 0} \sigma\dot{\sigma} < 0 \tag{1.87}$$

i.e., the sliding surface coordinate function $\sigma(x)$ and its time derivative $\dot{\sigma}(x)$, computed along the controlled trajectories, exhibit opposite signs on each small neighborhood of the sliding surface.[4].

The actual sliding motions taking place on $\mathcal{S}$, thanks to the active, infinite frequency control input switchings, can be idealized into smoothly controlled motions satisfying the following sliding surface *invariance conditions*:

$$\sigma(x) = 0, \quad \dot{\sigma}(x) = 0 \tag{1.88}$$

The invariance conditions describe the fact that motions take place precisely on the sliding surface and never abandon it. The virtual smooth feedback control action that would be responsible for the evolution of the state trajectory on the sliding manifold, very closely tracks the actual sliding motions without exhibiting small high frequency chattering, characteristic of realistic sliding motions, is addressed as the *equivalent control*.

The equivalent control is an *intermediate* control action lying between the two extreme values of the discrete set of available controls, which we have so far limited to switch position functions taking values in the discrete set $\{0, 1\}$. The equivalent control is therefore bounded by these two extreme values

$$0 \ < \ u_{\text{eq}}(x) \ < \ 1 \tag{1.89}$$

[4] Clearly condition (1.87) is a consequence of demanding that the positive (semi-definite) function $V(\sigma) = \frac{1}{2}\sigma^2$ locally exhibits a negative time derivative: $\dot{V} = \sigma\dot{\sigma}$ on any arbitrary small vicinity of $\mathcal{S}$. This Lyapunov like argument actually constitutes a necessary and sufficient condition for the local existence of sliding motions on $\mathcal{S}$ for the case of single input nonlinear systems (see Utkin[31]).

If the equivalent control violates these bounding restrictions, then it cannot be virtually synthesized as the result of the active switchings of the control input between the extreme values 0 and 1. It follows that the system becomes incapable of sustaining the controlled motions on the sliding manifold. The existence of the equivalent control between the hard bounds, defined by the control input set, is therefore tantamount to the definition of the *region of existence* of a sliding regime on the given manifold.

We list the limitations found so far of the sliding mode control methodology

- Sliding motions for switched systems are, generally speaking, not global. Global sliding motions demand sliding surfaces to be in a class "similar" to that of the integral manifolds of the controlled trajectories.
- In trajectory tracking problems, the bandwidth limitations directly affect the existence of sliding regimes accomplishing the trajectory tracking task. Tradeoffs can be easily established, via analytic techniques, in the case of linear systems.
- The presence of unknown perturbations limits the region of existence of sliding motions. Sliding motions may be eased in those instances where estimation techniques are available for the unknown disturbance signals.
- Robustness of the sliding mode controlled system to unknown disturbances seems to be ruled by rigid structural constraints that require control inputs and disturbances to affect the state dynamics through the same "channel." This reveals a severe limitation of the state space formulation of sliding regimes, known under the general name of *matching conditions*.
- The formulation of sliding regimes seems to be unavoidably linked to the state space formulation and to the need of measuring, or estimating, the unavailable states of the system. It is one of our tasks in this book to illustrate techniques where sliding motions can be obtained when input-output descriptions of the plant are available. We shall also explore non-traditional means of estimating states of the system via knowledge of inputs and outputs alone and use them in sliding regime creation under suitable additional compensation.

2

Single-input single-output sliding mode control

2.1 Introduction

In this chapter we formalize, in a rather direct, and elementary, mathematical language, the ideas presented in the previous chapter. We introduce all the previously discussed *elements* of sliding mode control theory and illustrate, through several design examples mainly drawn from the switched power electronics area, the use and features of this important control design methodology.

Using the language of elementary differential geometry, we formulate in a rather general setup the sliding mode creation problem for a given switched system. We revisit the single switch case (i.e., the *single-input-single output* case). We examine the most salient features and theoretical elements of sliding mode control, namely: the sliding surface meaning, its accessibility, or reachability, problem, the definition of the equivalent control and its corresponding ideal sliding dynamics. Finally, we address the robustness of closed loop sliding mode responses with respect to additive perturbation fields satisfying the so-called *matching condition*. The approach naturally allows to relate these important features with well-known concepts of nonlinear geometric control such as: invariance, zero dynamics, minimum and non-minimum phase outputs, projection operators (over tangent subspaces along, or parallel to, the span of a given set of input vector fields or, equivalently, to the span of the input matrix), and local stability in the sense of Lyapunov. After the theoretical introductions to sliding mode control, for the SISO case, we then center our attention on the sliding mode control of some of the most popular DC to DC power converters written, for ease of treatment, in normalized form. This practice not only greatly facilitates the algebraic manipulation and the computer simulation tasks, it is also a good guide and handy check for actual implementation of feedback controllers in many areas of Power Electronics.

H. Sira-Ramírez, *Sliding Mode Control,* Control Engineering,
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2.2 Variable structure systems

A variable structure system is a system in which the current dynamic model, or system structure, heavily depends on the region of the state space where the operation of the system is circumstantially found. The discontinuous nature of the model is characteristic and the structural changes occur due to either a voluntary action on the part of the operator, or due to the automatic activation of one or more switches present in the system, or, also, due to a sudden change in the temporary values of certain system parameters.

The class of described systems is quite wide for its detailed study and its interest in applications is somewhat limited. For this reason, we shall study variable structure systems regulated by one or several switches. The position of the switches constitutes our only set of available control inputs.

Additionally, we restrict ourselves to the class of systems where its several descriptions, or structures, have in common the invariance of the dimension of the resulting systems, as well as the nature of the describing state of the system.

2.3 Control of single switch regulated systems

We study the control of systems represented by nonlinear state space models of the following form:

$$\dot{x} = f(x) + g(x)u, \quad y = \sigma(x) \tag{2.1}$$

where $x \in \mathbb{R}^n$, $u \in \{0, 1\}$, $y \in \mathbb{R}$. The vector functions $f(x)$ and $g(x)$ represent smooth *vector fields*, i.e. infinitely differentiable vector fields, defined over the tangent space to $\mathbb{R}^n$. The output function, $\sigma(x)$, is a smooth scalar function of x taking values in the real line $\mathbb{R}$. We refer to x as the *state* of the system. The variable u is addressed as the *control input*, or simply as *the control*. The variable y is the *output* of the system. We usually refer to $f(x)$ as the *drift vector field* and to $g(x)$ as the *control input field*.

The main feature of the systems to be studied is the *binary* valued nature of the control input variable. Without loss of generality we assume that the control input takes values on the discrete set $\{0, 1\}$. Note that if the set of possible values for the scalar control input u were the discrete set $\{W_1, W_2\}$ with $W_i \in R, i = 1, 2$, then the following invertible input coordinate transformation:

$$v = (u - W_2)/(W_1 - W_2), \quad u = W_2 + v(W_1 - W_2) \tag{2.2}$$

makes the new control input v a binary valued control input function with values in the set $\{0, 1\}$

Example 2.1. The circuit in Figure 2.1 represents a DC to DC power converter, known as the "boost" converter, controlled by a single switch.

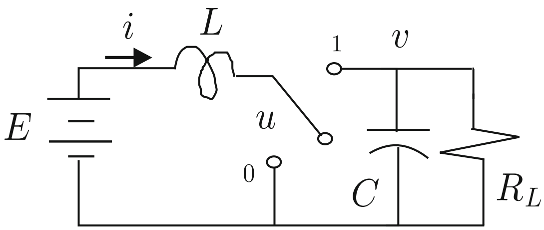

Fig. 2.1. Boost converter circuit

The controlled differential equations describing the system are given by

$$L\frac{di}{dt} = -uv + E$$
$$C\frac{dv}{dt} = ui - \frac{1}{R_L}v \tag{2.3}$$

where i is the input inductor current, v is the output voltage, and u is the switch position function satisfying ($u \in \{0, 1\}$).

In matrix terms, the mathematical description of the "boost" converter is given by

$$\frac{d}{dt}\begin{bmatrix} i \\ v \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{R_L C} \end{bmatrix}\begin{bmatrix} i \\ v \end{bmatrix} + \begin{bmatrix} -\frac{v}{L} \\ \frac{i}{L} \end{bmatrix} u + \begin{bmatrix} \frac{E}{L} \\ 0 \end{bmatrix} \tag{2.4}$$

Letting $x = [x_1 \ \ x_2]^T = [i \ \ v]^T$, we have

$$f(x) = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{R_L C} \end{bmatrix} x + \begin{bmatrix} \frac{E}{L} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{E}{L} \\ \\ -\frac{x_2}{R_L C} \end{bmatrix} \tag{2.5}$$

and

$$g(x) = \begin{bmatrix} -\frac{x_2}{L} \\ \\ \frac{x_1}{C} \end{bmatrix} \tag{2.6}$$

2.4 Switching between continuous feedback laws

The description we have adopted for the study of sliding regimes is general enough to include a class of switched systems that entitles the switching between two available feedback laws.

Indeed consider the nonlinear system

$$\dot{x} = F(x, v) \tag{2.7}$$

where F is smooth function of its arguments (x, v) and v is a scalar control input.

Suppose that the scalar control input v may be chosen to be one of two possible smooth feedback controls laws in accordance with the sign of a decision function $\sigma(x)$ as follows:

$$v = \begin{cases} u^+(x) \text{ for } \sigma(x) < 0 \\ u^-(x) \text{ for } \sigma(x) > 0 \end{cases} \tag{2.8}$$

Clearly the system may be represented by a switched system with controls taking values in the discrete set $\{0, 1\}$. Indeed consider the system

$$\dot{x} = F(x, u^-(x)) + u\left[F(x, u^+(x)) - F(x, u^-(x))\right] \tag{2.9}$$

with the switching law

$$u = \begin{cases} 1 \text{ for } \sigma < 0 \\ 0 \text{ for } \sigma > 0 \end{cases} \tag{2.10}$$

The control problem has now been cast into one in which the system is of the form: $\dot{x} = f(x) + ug(x)$ with $u \in \{0, 1\}$.

Example 2.2. An interesting case of switched feedback control of a linear system consists in controlling the system by being able to switch *the gain* of a linear output feedback control law. For this, consider a linear second order system which is a simpler version of one treated in Utkin's book ([31])

$$\ddot{y} = -\Psi\, y, \quad \Psi \in \{\alpha, -\alpha\}, \quad \alpha > 0 \tag{2.11}$$

We can rewrite the system in state space switched form as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(2u - 1)\alpha x_1 \\ y &= x_1 \end{aligned}$$

with $u \in \{0, 1\}$.

In this case

$$f(x) = \begin{bmatrix} x_2 \\ \alpha x_1 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ -2\alpha x_1 \end{bmatrix}, \quad u \in \{0, 1\} \tag{2.12}$$

We will be examining, later on, the possibilities of inducing a stable sliding motion towards the origin on the switching surfaced defined by the coordinate function: $\sigma = x_2 + \lambda x_1$, with $\lambda > 0$.

2.5 Sliding surface

In the context of single switch controlled n-dimensional systems, a sliding surface, denoted by $\mathcal{S}$, is represented by the set of state vectors x in $\mathbb{R}^n$, where the state restriction, $\sigma(x) = 0$, is satisfied, with σ being a smooth scalar output function.

The main assumption is the following:

The restriction $\sigma(x) = 0$ (which is represented by a smooth manifold, $\mathcal{S}$, of dimension $n-1$ in $\mathbb{R}^n$) locally satisfied by the state trajectory, $x(t)$, ideally produces a desired behavior for the state of the controlled system which represents the control objective. The constrained evolution of the state on $\mathcal{S}$ is locally, or globally, to be accomplished thanks to appropriate switched control input actions satisfying: $u \in \{0, 1\}$.

The restriction, $\sigma(x) = 0$, on the state vector, x, defines a smooth $n-1$-dimensional manifold in $\mathbb{R}^n$. We denote such a manifold by $\mathcal{S}$, i.e.

$$\mathcal{S} = \{x \in \mathbb{R}^n \mid \sigma(x) = 0\} \tag{2.13}$$

One of the primordial features in the design of feedback control laws for switch regulated systems is represented by the fact that the specification of the smooth scalar function $\sigma(x)$ is an *integral part of the design problem.* The choice of the output function $y = \sigma(x)$ and, hence, of the smooth manifold $\mathcal{S}$ entirely depends upon our *control objective.*

Example 2.3. In the previous "boost" converter example, a sliding surface may be proposed to be :

$$\sigma(x) = v - V_d = x_2 - V_d \tag{2.14}$$

where V_d is the average desired output equilibrium voltage. If one succeeds in forcing $\sigma(x)$ to be zero, along the controlled trajectories of the system, then the output voltage ideally coincides with the desired voltage.

Another sliding surface that one may consider is given by

$$\sigma(x) = i - I_d = x_1 - I_d \tag{2.15}$$

where $I_d = V_d^2/(ER_L)$ represents an average equilibrium input current which corresponds with the desired average output equilibrium voltage V_d.

As it was shown in Chapter 1, even though both sliding surfaces represent a desired behavior of the output, only one of them is actually feasible due to internal closed loop stability considerations.

2.6 Notation

Let $f(x)$ and $g(x)$ be smooth vector fields locally defined on the tangent space to $\mathbb{R}^n$ at x, here denoted by $T_x\mathbb{R}^n$. Let $\sigma(x)$ be a smooth scalar function taking values on $\mathbb{R}$. We denote by the triple (f, g, σ) the nonlinear system

$$\dot{x} = f(x) + g(x)u, \quad y = \sigma(x) \tag{2.16}$$

with state $x \in \mathbb{R}^n$, *drift* vector field $f(x)$, *input* vector field $g(x)$, control input $u \in \{0, 1\}$ and $\sigma(x)$ plays the role of a smooth scalar output function, customarily referred to as a sliding surface coordinate function .

We define the *directional derivative* of $\sigma(x)$ in the direction of $f(x)$ as the scalar quantity: $\left(\partial\sigma/\partial x^T\right) f(x)$, and we denote it by means of $L_f\sigma(x)$. Similarly, we refer to $L_g\sigma(x)$ as the directional derivative of $\sigma(x)$ in the direction of the vector field $g(x)$.

In local coordinates we have:

$$\frac{\partial\sigma}{\partial x^T} = \left(\frac{\partial\sigma}{\partial x_1} \; \frac{\partial\sigma}{\partial x_2} \cdots \frac{\partial\sigma}{\partial x_n} \right), \quad f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} \tag{2.17}$$

and

$$L_f\sigma(x) = \sum_{i=1}^{n} \frac{\partial\sigma}{\partial x_i} f_i(x) \tag{2.18}$$

Gradients of scalar functions are usually addressed as *differentials* of such functions, the expression $d\sigma$ stands for a row vector (in the dual to the tangent space of vector fields, or cotangent space). Thus, the directional derivative $L_f\sigma$ may also be denoted as

$$L_f\sigma = \langle d\sigma, f \rangle \tag{2.19}$$

with the operation "$\langle \; , \rangle$" being understood in the sense of the scalar product of the differential, $d\sigma$, and the vector field f. Row vectors with smooth functions as components are also called *co-vectors*, *cotangent vectors*, and *1-forms.* Differentials are particular cases of co-vectors.

2.7 The transversal condition

The existence of a sliding regime on $\sigma = 0$ entitles the opposite growth of the sliding surface coordinate function σ on a sufficiently large open vicinity of $\mathbb{R}^n$, locally intersecting $\mathcal{S}$. We denote this open vicinity of $\mathcal{S}$ by $\mathcal{N}$. If x is located in such a vicinity, $\mathcal{N}$, so that $\sigma(x) > 0$, the time derivative $\dot{\sigma}$ must be negative and, hence σ decreases. On the other hand, if $x \in \mathcal{N}$ is such that $\sigma(x) < 0$, the time derivative $\dot{\sigma}$ must be positive and σ increases towards its zero value. In other words, $\sigma\dot{\sigma} < 0$ remains locally valid on the considered open vicinity $\mathcal{N}$. We shall be abusively referring to "above the surface" to the region where locally the condition $\sigma > 0$ is valid. Contrarily, we use "below the surface" to indicate that $\sigma(x) < 0$. This is made only in the interest of intuition and the statement is clearly devoid of any mathematical meaning. We may assume without loss of generality that the following is valid *above* the surface, i.e., when $\sigma > 0$:

$$\dot{\sigma}(x) = L_f\sigma(x) = \langle d\sigma, f\rangle < 0 \tag{2.20}$$

where the control input has been set to $u = 0$. i.e., the vector field f "points" towards $\mathcal{S}$ and locally creates state trajectories that tend to cross the sliding surface $\mathcal{S}$ with σ constantly decreasing. When $\sigma(x)$ is negative, the control $u = 1$ is enforced, the existence of a sliding regime entitles

$$\dot{\sigma} = L_{f+g}\sigma(x) = \langle d\sigma, f + g\rangle = L_f\sigma + L_g\sigma > 0 \tag{2.21}$$

These two conditions are valid on $\mathcal{N}$ and, hence, locally valid on $\mathcal{S}$.

We have the following immediate result:

Proposition 2.4. *The condition $L_g\sigma(x) > 0$, $x \in S$ is a necessary condition for the local existence of a sliding regime. This condition is addressed as the* transversal condition *of the control input field g with respect to the sliding manifold $\mathcal{S}$.*

Proof The proof is rather simple. Suppose a sliding regime locally exists on $\mathcal{S}$ then on the open vicinity $\mathcal{N}$, $L_f\sigma(x) < 0$ and $L_f\sigma(x) + L_g\sigma(x) > 0$. Clearly, $L_g\sigma(x) > 0$ on $\mathcal{N}$. In particular, $L_g\sigma(x) > 0$ on $S \cap \mathcal{N}$.

We remark that if $\mathcal{S}$ is such that, locally, $L_g\sigma < 0$. The simple sliding surface coordinate change $\sigma = -\tilde{\sigma}$ makes all the previous assumptions valid. There is no loss of generality, then, in locally taking the transversal condition as: $L_g\sigma > 0$.

Henceforth, all our statements about existence of a sliding regime are meant in a *local* sense. This locality assumes that its validity occurs in a sufficiently large region of $\mathcal{S}$ for guaranteeing an effective, sustainable, controlled motion of the system on the sliding surface.

2.8 Equivalent control and ideal sliding dynamics

Let us assume that thanks to the use of an appropriate switching law, we manage to make the state x locally evolve restricted to the smooth manifold, $\mathcal{S}$. In other words, we achieve, by means of appropriate commutations, the *invariance* of $\mathcal{S}$ with respect to the trajectories of the state of the system. It is our assumption that while the condition $x \in \mathcal{S}$ is satisfied, we are complying with some specific control objective.

We define the *equivalent control* as the smooth feedback control law, denoted by $u_{eq}(x)$ which locally sustains the evolution of the state trajectory ideally restricted to the smooth manifold $\mathcal{S}$ when the initial state of the system $x(t_0) = x_0$ is located precisely on the manifold $\mathcal{S}$, i.e., when $\sigma(x_0) = 0$. In other words, the equivalent control is the smooth control that locally renders the sliding manifold $\mathcal{S}$ invariant.

The sliding surface coordinate function, $\sigma(x)$, satisfies, under ideal sliding motions on $\sigma(x) = 0$, the following set of *invariance conditions*, $\sigma = 0$, $\dot{\sigma} = 0$. Explicitly,

$$\dot{\sigma}(x) = \frac{\partial \sigma}{\partial x}(f(x) + g(x)u_{eq}(x))\bigg|_{\sigma=0} = 0 \tag{2.22}$$

where we are using the fact that $L_g\sigma \neq 0$, i.e., the vector field g is not tangential to $\mathcal{S}$. In other words,

$$L_f\sigma(x) + [L_g\sigma(x)]u_{eq}(x)\bigg|_{\sigma=0} = 0 \tag{2.23}$$

and therefore, the equivalent control, on $\sigma = 0$, is expressed in a unique fashion as the quotient:

$$u_{eq}(x) = -\frac{L_f\sigma(x)}{L_g\sigma(x)}\bigg|_{\sigma=0} = -\frac{\langle d\sigma, f\rangle}{\langle d\sigma, g\rangle}\bigg|_{\sigma=0} \tag{2.24}$$

The controlled vector field, $f(x) + g(x)u_{eq}(x)$, and the corresponding evolution over the smooth manifold $\mathcal{S}$ of the state trajectories of the system, is expressed as

$$\dot{x} = \left[f(x) - g(x)\frac{L_f\sigma(x)}{L_g\sigma(x)}\right]\bigg|_{\sigma=0} \tag{2.25}$$

and it will be addressed as the *ideal sliding dynamics* .

Note that any other initial condition which is not over the smooth manifold $\mathcal{S}$ evolves in such a manner that the function $\sigma(x)$ remains constant. Such a constant value only adopts the value of zero when the initial state x_0 is located on S. The closed loop system, fed back by the equivalent control, on $\mathcal{S}$, may be alternatively described as follows:

$$\dot{x} = \left[I - \frac{1}{L_g\sigma(x)}g(x)\frac{\partial \sigma}{\partial x^T}\right]f(x)\bigg|_{\sigma=0} = \mathcal{M}(x)f(x)\bigg|_{\sigma=0} \tag{2.26}$$

Proposition 2.5. *The square $n \times n$ matrix $\mathcal{M}(x)$ is a* **projection operator***, onto the tangent space to $\mathcal{S}$, along the* span$\{g(x)\}$*. The operator $\mathcal{M}(x)$ projects any smooth vector field defined in the tangent space of $\mathbb{R}^n$ onto the tangent subspace to the manifold $\mathcal{S}$ in a parallel fashion to the* span$\{g(x)\}$*, or* in the direction *of the control input field $g(x)$.*

Indeed, let v be a vector field in the tangent space to $\mathbb{R}^n$ such that $v \in$ span $g(x)$ i.e., $v(x)$ may be expressed as $v(x) = g(x)\alpha(x)$ where $\alpha(x)$ is a smooth scalar function. We then have

$$\begin{aligned}\mathcal{M}(x)v(x) &= \left\{ I - \frac{1}{L_g\sigma(x)} g(x) \frac{\partial \sigma}{\partial x} \right\} g(x)\alpha(x) \\ &= \left\{ g(x) - \frac{1}{L_g\sigma(x)} g(x) \frac{\partial \sigma}{\partial x} g(x) \right\} \alpha(x) \\ &= \left\{ g(x) - \frac{1}{L_g\sigma(x)} g(x) L_g\sigma(x) \right\} \alpha(x) \\ &= [g(x) - g(x)]\,\alpha(x) = 0 \end{aligned} \tag{2.27}$$

Additionally, the n-th dimensional co-vector: $\partial\sigma/\partial x^T$ annihilates the image under $\mathcal{M}(x)$ of the vector fields lying in the tangent space of $\mathbb{R}^n$. For this, it is enough to show that any 1-form in the span of $d\sigma$ annihilates all the column vectors of $\mathcal{M}(x)$.

A 1-form in the span of $d\sigma = \partial\sigma/\partial x^T$ is written as : $\xi(x)(\partial\sigma/\partial x^T)$, where $\xi(x)$ is a completely arbitrary nonzero scalar function. Indeed:

$$\begin{aligned}\xi(x)\frac{\partial \sigma}{\partial x^T}\mathcal{M}(x) &= \xi(x)\frac{\partial \sigma}{\partial x^T}\left\{ I - \frac{1}{L_g\sigma(x)} g(x) \frac{\partial \sigma}{\partial x^T} \right\} \\ &= \xi(x)\left[\frac{\partial \sigma}{\partial x^T} - L_g\sigma(x)[L_g\sigma(x)]^{-1}\frac{\partial \sigma}{\partial x^T} \right] \\ &= \xi(x)\left[\frac{\partial \sigma}{\partial x^T} - \frac{\partial \sigma}{\partial x^T} \right] = 0 \end{aligned} \tag{2.28}$$

The image under $\mathcal{M}(x)$ of any vector field in the tangent space to $\mathbb{R}^n$ is in the null space of $\partial\sigma/\partial x^T$. In other words, they are in the tangent subspace to $\mathcal{S}$.

Clearly, $\mathcal{M}^2(x) = \mathcal{M}(x)$ given that $\mathcal{M}(x)g(x) = 0$ or that $d\sigma M(x) = 0$.

Exercise 2.6. Show that the operator $I - \mathcal{M}(x)$ is a projection operator onto the span$\{g\}$ along the tangent subspace to $\mathcal{S}$ at x, $T_x\mathcal{S}$.

Exercise 2.7. Show that the tangent subspace of $\mathcal{S}$ coincides with the image of the operator $\mathcal{M}$. Proceed by establishing a contradiction.

The projection operator reveals an immediate property of the corresponding ideal sliding dynamics: The equivalent control idealization is invariant with respect to time coordinate transformations, even if these are state dependent. This property reveals that the equivalent control is an infinite frequency idealization associated with the actual input switching strategy and hence, invariant with respect to finite time coordinate transformations (See Fliess *et al.* [7]).

Let $\mu(x)$ be a strictly positive scalar function of the state x and define a transformed time scale τ for the system as $d\tau = dt/\mu(x)$. The underlying switch controlled system is represented as

$$\frac{dx}{d\tau} = \mu(x)f(x) + \mu(x)g(x)u = \tilde{f}(x) + \tilde{g}(x)u \tag{2.29}$$

Proposition 2.8. *The equivalent control, $u_{eq}(x)$, associated with the sliding surface $S = \{x \mid \sigma(x) = 0\}$, is independent of any time scaling exercised on the system via a time scaling factor $\mu(x) > 0$.*

Proof

$$\tilde{u}_{eq}(x) = -\frac{L_{\tilde{f}}\sigma(x)}{L_{\tilde{g}}\sigma(x)} = -\frac{L_f\sigma(x)}{L_g\sigma(x)} = u_{eq}(x) \tag{2.30}$$

The state dependent projection operator, $\mathcal{M}(x)$, is, therefore, independent of the time scaling factor $\mu(x)$. The ideal sliding dynamics is just $dx/d\tau = \mathcal{M}(x)\tilde{f}(x)$. The ideal sliding motions can be locally, artificially, accelerated, or slowed down, but the equivalent control is always the same function of the state x.

2.9 Accessibility of the sliding surface

Let x be a representative point of a state trajectory, located in an open neighborhood $\mathcal{N}$ of the manifold $\mathcal{S}$ (This neighborhood strictly contains its intersection with the sliding manifold). Assume, without loss of generality that, at this point x, the surface coordinate function $\sigma(x)$ of the manifold $\mathcal{S}$ is strictly positive, i.e., $\sigma(x) > 0$. We may conventionally say that we are located *above* the surface $\mathcal{S}$. Our objective is to prescribe an appropriate control action which guarantees that the trajectory of the system reaches and crosses the manifold $\mathcal{S}$. The time derivative of $\sigma(x)$ at the point x is given by

$$\frac{d}{dt}\sigma(x) = \frac{\partial\sigma}{\partial x}(f(x) + g(x)u) = L_f\sigma(x) + [L_g\sigma(x)]u \tag{2.31}$$

If we assume that $L_g\sigma(x) > 0$ in a neighborhood of $\mathcal{S}$ (i.e., $L_g\sigma(x)$ is strictly positive, *above* and *below* $\mathcal{S}$ in the vicinity of this surface), then we require that the time derivative of $\sigma(x)$ in (2.31) be strictly negative at x.

Since $L_g\sigma(x) > 0$, we must choose the control input that annihilates the positive incremental effect that this term has over the derivative of σ, we must then set $u = 0$. The sliding surface coordinate function time derivative coincides then with the value $L_f\sigma(x)$. It follows that being $L_g\sigma > 0$ in an open neighborhood of $\mathcal{S}$, it is necessary that $L_f\sigma(x)$ be strictly negative in a neighborhood of $\mathcal{S}$.

If we now assume that the point x is located "below" the surface, i.e., $\sigma(x) < 0$, then it is easy to see that for the trajectories to reach, and have the potential to cross, the sliding manifold $\mathcal{S}$, the time derivative of $\sigma(x)$ must be strictly positive. In other words, $L_f\sigma(x)+[L_g\sigma(x)]u > 0$. Since $L_g(x) > 0$ and $L_f\sigma(x) < 0$, we must choose $u = 1$, so as to magnify the positive incremental effect of $L_g\sigma(x)$ over the time derivative of $\sigma(x)$, but, besides, it is necessary that this positive term be of such magnitude that it also overcomes the effect of the negative increment represented by $L_f\sigma(x)$ over the time derivative.

We conclude that, assuming $L_g\sigma(x) > 0$, in an open neighborhood of $\mathcal{S}$, the necessary condition for the existence of a sliding regime on $\mathcal{S}$ is that $L_g\sigma(x) > -L_f\sigma(x) > 0$. In other words, dividing this inequality by the strictly positive quantity $L_g\sigma(x)$, it is necessary that

$$1 > -\frac{L_f\sigma(x)}{L_g\sigma(x)} > 0 \tag{2.32}$$

Note that this inequality must be valid in an open neighborhood of $\mathcal{S}$. In particular, if this inequality is locally valid for $x \in \mathcal{S}$, then it is also valid in an open neighborhood of $\mathcal{S}$ given the smooth character of the involved vector fields and of the surface coordinate function $\sigma(x)$.

Under the assumption that $L_g\sigma(x) > 0$ around $\mathcal{S}$, it is easy to see that the previously discussed existence condition is also sufficient.

Indeed, if the representative point is located, say, *above* the sliding manifold $\mathcal{S}$, the above inequality tells us that $L_f\sigma(x) < 0$ and then it suffices to take $u = 0$ and since then $\dot{\sigma}(x) < 0$ in any open neighborhood of $\mathcal{S}$. The state trajectory thus approaches, and crosses, the manifold $\mathcal{S}$ from any neighboring point located above the surface. If the representative point is located *below* $\mathcal{S}$, then, the inequality establishes that $L_f(x) + L_g\sigma(x) > 0$ and, therefore, the choice, $u = 1$, forces the condition: $\dot{\sigma}(x) > 0$ for any point in an open neighborhood of $\mathcal{S}$. This says that the state trajectory approaches the manifold $\mathcal{S}$.

Note that if we locally had $L_g\sigma(x) < 0$, we then should have $L_f\sigma(x) > 0$ in any neighborhood of $\mathcal{S}$. The changes in the previous arguments for surface reachability are reduced to the choice of u in each case. In this case, we would choose $u = 1$ when x is located "above" $\mathcal{S}$ and we should set $u = 0$ when we are "below" the sliding surface.

Nevertheless, and in order to avoid confusion, we note that if locally, $L_g\sigma(x) < 0$, we may always redefine $\mathcal{S}$ taking as a sliding surface coordinate function, $-\sigma(x)$, and now all the previous analysis becomes valid.

The condition $L_g\sigma(x) > 0$ is particularly important and it determines the switching policy that locally achieves a sliding regime over the sliding manifold $\mathcal{S}$. We address this condition as the *transversal condition* of the control input field $g(x)$ in relation to the sliding manifold $\mathcal{S}$. Note that if $L_g\sigma(x) = 0$ on an open set around the sliding manifold, the system is not controllable and the quantity $\dot{\sigma}(x)$ cannot be made to change its sign in such a vicinity of $\mathcal{S}$. Therefore, the transversal condition is a *necessary condition* for the local existence of a sliding regime.

By virtue of the fact that the quantity $-L_f\sigma(x)/L_g\sigma(x)$ coincides with the *equivalent control* on $\mathcal{S}$, we conclude as follows.

Theorem 2.9. *The necessary and sufficient condition for the local existence of a sliding regime over the smooth manifold $\mathcal{S} = \{\, x \mid \sigma(x) = 0\}$ is that the equivalent control satisfies*

$$0 \;<\; u_{eq}(x) \;<\; 1, \quad x \in \mathcal{S} \tag{2.33}$$

The transversal condition $L_g\sigma(x) > 0$, or, more generally : $L_g\sigma(x) \neq 0$, tells us that if the sliding surface coordinate function $\sigma(x)$, is considered as a *system output* function, $y = \sigma(x)$, then, this function must be, necessarily, locally *relative degree equals to 1* around the value $y = 0$. Note that for $y = 0$ the output *zero dynamics* entirely coincides with the *ideal sliding dynamics* redundantly given by

$$\dot{x} = f(x) - g(x)\frac{L_f\sigma(x)}{L_g\sigma(x)}\bigg|_{y=0} = f(x) + g(x)u_{eq}(x)\bigg|_{y=0} \tag{2.34}$$

We dwell, in more detail and generality, on the issues associated with the *relative degree* and the *zero dynamics* concepts further ahead in this chapter.

Under the assumption that the transversal condition adopts the form:

$$L_g\sigma(x) > 0 \tag{2.35}$$

in a sufficiently large open neighborhood of the sliding surface $\mathcal{S}$, the control law, that locally forces the state trajectories to reach the sliding surface and these acquire the possibility of "crossing" this surface, is given by

$$u = \begin{cases} 1 \text{ if } \sigma(x) \;<\; 0 \\ 0 \text{ if } \sigma(x) \;>\; 0 \end{cases}, \quad u = \frac{1}{2}\left[1 - \operatorname{sign}\sigma(x)\right] \tag{2.36}$$

Evidently, any incipient incursion of the state trajectory to the "other side" of the sliding manifold causes an immediate control reaction commanding the switch to change its position to the other only available realization. As a consequence, the trajectory is forced to return towards the surface possibly crossing it again with a corresponding new change in the switch control position. The resulting motion taking place around an arbitrarily small neighborhood of the sliding surface is characterized by a "zigzag" motion whose frequency is, theoretically speaking, infinitely large and widely known as a *sliding regime* or a *sliding motion*.

The transversal condition may undergo a singularity situation by which $L_g\sigma(x)$ changes sign on the sliding surface $\mathcal{S}$. In other words, there may exist a sub-manifold of $\mathcal{S}$ on which $L_g\sigma(x) = 0$. In spite of the fact that the equivalent control may not exist at a certain time intervals, or points, of the sliding motions, outside this singularity, it is still possible to locally maintain a sliding regime. Consider then,

$$\dot{\sigma} = L_f\sigma + uL_g\sigma \tag{2.37}$$

The switching policy, modified to the following one:

$$u = \frac{1}{2}\left[1 - \text{sign}(\sigma L_g\sigma)\right], \tag{2.38}$$

locally reaches the sliding surface $\mathcal{S}$ irrespective of the local sign of $L_g\sigma$. It guarantees the satisfaction of the necessary and sufficient condition for the existence of a sliding regime in open regions of $\mathcal{S}$ outside the sub-manifold: $\{x \in \mathcal{S} \mid L_g\sigma(x) = 0\}$. Notice that such a singularity arises from a tangency of the control vector field, $g(x)$, with the sliding surface tangent space at x. In other words, $L_g\sigma = 0$, if and only if, $g(x) \in T_x\mathcal{S} \subset T_x\mathbb{R}^n$.

Example 2.10. Consider the second order system linear system, with a switching output feedback gain, in example 2.2. We write this system in the form $\dot{x} = f(x) + ug(x)$, as follows:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(2u-1)\alpha x_1, \qquad u \in \{0,1\} \\ y &= x_1 \end{aligned} \tag{2.39}$$

where the parameter α is assumed to be strictly positive. For this system, we have

$$f(x) = \begin{bmatrix} x_2 \\ \alpha x_1 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ -2\alpha x_1 \end{bmatrix}, \quad \sigma(x) = x_2 + \lambda x_1 \tag{2.40}$$

We compute

$$L_g\sigma(x) = [\lambda \quad 1]\, g(x) = -2\alpha x_1, \quad L_f\sigma(x) = \lambda x_2 + \alpha x_1 \tag{2.41}$$

The equivalent control, in $\sigma(x) = 0$, is found to be

$$u_{eq}(x) = -\frac{L_f\sigma}{L_g\sigma} = \frac{\lambda x_2 + \alpha x_1}{2\alpha x_1} = \frac{(\alpha - \lambda^2)}{2\alpha} \tag{2.42}$$

Note that $L_g\sigma$ changes sign in accordance with the variable x_1. Hence, in order to obtain the right switching law, we will have to take into account, besides the sign of σ, also, the sign of x_1. So, let us first consider $x_1 < 0$ which yields $L_g\sigma > 0$ as hypothesized in previous sections.

The existence condition: $0 < u_{\mathrm{eq}} < 1$ yields the following two inequalities

$$\begin{aligned} \alpha + \lambda^2 &> 0 \\ \alpha - \lambda^2 &> 0 \end{aligned} \tag{2.43}$$

The first condition being always valid from the hypothesis that $\alpha > 0$, we have then, necessarily, the following existence condition

$$\lambda < \sqrt{\alpha} \tag{2.44}$$

Thus, in the fourth quadrant, the sliding line is below the stable eigen-line of the closed loop system $\ddot{y} = \alpha y$ and in the second quadrant the sliding line is above the stable eigen-line. The sliding motion is therefore globally sustained on the sliding surface

$$\mathcal{S} = \{(x_1, x_2) \; | \sigma(x) = \; x_2 + \lambda x_1 = 0, \quad \lambda < \sqrt{\alpha}\} \tag{2.45}$$

The switching law has to now take into account the sign of σ and x_1 since the nature of $L_g\sigma$ changes beyond the origin of coordinates along the sliding line. Clearly the switching law

$$u = \frac{1}{2}(1 - \mathrm{sign}(\sigma x_1)) \tag{2.46}$$

yields the desired global stable sliding motion on $\mathcal{S}$.

Figure 2.2 depicts the phase space trajectories of the switch regulated second order linear system where the output feedback gain acts as a switching control input.

Example 2.11. A slight modification in the above example leads to quite different realities. Consider, for instance, the following third order integrator system with switched output feedback,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -(2u - 1)\alpha x_1 \\ y &= x_1 \end{aligned}$$

A stabilizing sliding surface may be proposed via the following definition of the sliding surface coordinate function:

$$\sigma(x) = x_3 + 2\xi\omega_n x_2 + \omega_n^2 x_1 \tag{2.47}$$

with $\xi > 0$ and $\omega_n > 0$.

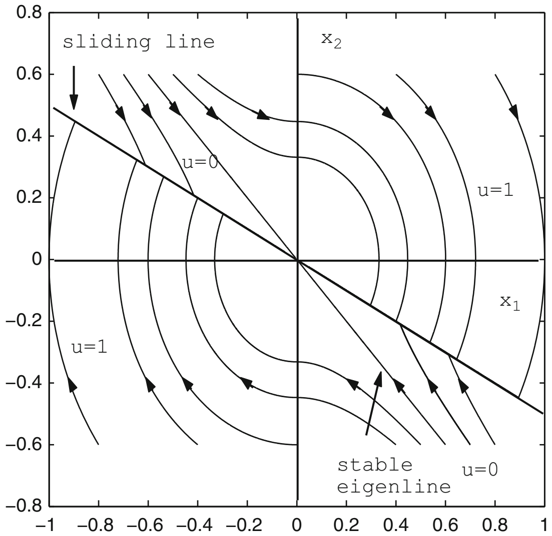

Fig. 2.2. Sliding mode control by switchings of the output feedback gain in a second order integrator system.

The ideal sliding dynamics, valid on $\mathcal{S} = \{x \mid \sigma(x) = 0\}$, is given by the following "traditional" asymptotically stable second order closed loop system

$$\ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2 y = 0 \tag{2.48}$$

We have

$$f(x) = \begin{bmatrix} x_2 \\ x_3 \\ \alpha x_1 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 0 \\ -2\alpha x_1 \end{bmatrix} \tag{2.49}$$

The equivalent control, on $\mathcal{S}$, is given by

$$u_{\mathrm{eq}}(x) = -\frac{\omega_n^2 x_2 + 2\xi\omega_n x_3 + \alpha x_1}{-2\alpha x_1} = \frac{1}{2} - \frac{\xi\omega_n^3}{\alpha} + \left[\frac{\omega_n^2(1-4\xi^2)}{2\alpha}\right]\frac{x_2}{x_1} \tag{2.50}$$

Note that, contrary to the previous case the equivalent control is now singular at the origin. It is worthwhile to examine the ideal sliding motions on the plane $x_1 - x_2$ in order to assess the behavior of the closed loop system in the vicinity of the origin.

The ideal sliding dynamics is governed by the linear state space model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.51}$$

whose characteristic polynomial is given by

$$p(s) = s^2 + 2\xi\omega_n s + \omega_n^2 \tag{2.52}$$

The positive choices for ξ and ω_n render an asymptotically stable origin as an equilibrium point. The approach of the controlled trajectory determines the nature of the equivalent control and the limit behavior of its singular character in the vicinity of the origin. It is clear that we have to impose some dominant eigenvalue behavior so that the controlled trajectories approach the origin along a convenient straight eigen-line. We should therefore not allow stable complex eigenvalues but, instead, real and widely separated. This is achieved by letting $\xi \geq 1$ and ω_n sufficiently large.

The eigenvalues of the closed loop linear dynamics are characterized by the set of straight lines

$$x_2 = \lambda x_1$$
$$x_2 = -\frac{\omega_n^2}{2\xi\omega_n - \lambda} x_1$$

with

$$\lambda = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1} \tag{2.53}$$

Both eigen-lines have negative slope in the plane $x_1 - x_2$. The dominant eigen-line is that corresponding to the largest eigenvalue.

2.10 A Lyapunov approach to surface reachability

Consider the scalar quantity:

$$V(y) = \frac{1}{2}y^2 = \frac{1}{2}\sigma^2(x) \geq 0 \tag{2.54}$$

This quantity represents a certain instantaneous energy, or quadratic magnitude, of the sliding surface coordinate function y with respect to its zero value, which defines the smooth manifold $\mathcal{S}$.

A plausible policy for reaching the desired condition $\sigma(x) = 0$, from any open vicinity of $\mathcal{S}$, is to adopt switching actions for the control input $u \in \{0, 1\}$ that result in a strict decrease of the positive semi-definite function $V(\sigma(x))$.

This can be locally achieved influencing over the system in such a manner that the speed of variation of $V(\sigma(x))$ be strictly negative on a sufficiently small region locally containing the zero level set of the function $\sigma(x)$. In other words,

$$\lim_{\sigma\to 0} \frac{d}{dt}\left(V(\sigma(x))\right) = \lim_{\sigma\to 0} \frac{1}{2}\frac{d}{dt}\left(\sigma^2(x)\right) = \lim_{\sigma\to 0} \sigma(x)\frac{d\sigma(x)}{dt} < 0 \tag{2.55}$$

Using the relation $\dot{\sigma}(x) = L_f\sigma(x) + L_g\sigma(x)u$ and realizing that $L_f\sigma(x) + L_g\sigma(x)u_{eq} = 0$ for any $x \notin \mathcal{S}$, we have, adding and subtracting the quantity $L_g\sigma(x)u_{eq}$ to the time derivative expression of the function $\sigma(x)$, the following relations,

$$\begin{aligned}\sigma(L_f\sigma(x)+L_g\sigma(x)u) &= \sigma\left(L_f\sigma(x)+L_g\sigma(x)(u-u_{eq})+L_g\sigma(x)u_{eq}\right)\\ &= \sigma L_g\sigma(x)(u-u_{eq})<0 \end{aligned} \quad (2.56)$$

We may assume, without any loss of generality, that the transversal condition $L_g\sigma(x) > 0$ is satisfied over an open neighborhood of the representative point x located on the immediate vicinity of the sliding surface $\mathcal{S}$. However, in the interest of some generality we let the sign of $L_g\sigma$ to be locally either positive or negative.

A choice for the switched control input, $u \in \{0,1\}$, which guarantees the validity of the above condition, regardless of the sign of the product $\sigma L_g\sigma$, is given by

$$u = \begin{cases} 1 \text{ if } \sigma(x)L_g\sigma(x) < 0 \\ 0 \text{ if } \sigma(x)L_g\sigma(x) > 0 \end{cases} \quad (2.57)$$

In other words,

$$u = \frac{1}{2}\left[1 - \text{sign }\left(\sigma(x)L_g\sigma(x)\right)\right] \quad (2.58)$$

When the transversal condition $L_g\sigma > 0$ is particularly valid, or enforced, then the switching control law is simply: $u = \frac{1}{2}\left[1 - \text{sign }\sigma(x)\right]$.

2.11 Control of the Boost converter

We revisit the Boost converter example performing the following *normalization* of the state variables and the time scale of the system:

$$x_1 = \frac{i}{E}\sqrt{\frac{L}{C}},\ x_2 = \frac{v}{E} \quad (2.59)$$

$$\tau = \frac{t}{\sqrt{LC}},\quad Q_L = R_L\sqrt{\frac{C}{L}} \quad (2.60)$$

The normalized model is then given by

$$\begin{aligned}\dot{x}_1 &= -ux_2 + 1\\ \dot{x}_2 &= ux_1 - \frac{1}{Q}x_2\end{aligned} \quad (2.61)$$

In the context of the previously defined notation we have

$$f(x) = \begin{bmatrix} 1 \\ -\frac{1}{Q}x_2 \end{bmatrix},\quad g(x) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \quad (2.62)$$

2.11.1 Direct control

The control objective is to drive the normalized average voltage x_2 to a desired equilibrium value X_2. We first try with the following sliding surface coordinate function

$$\sigma(x) = x_2 - \overline{X}_2 \tag{2.63}$$

Driving the output function $\sigma(x)$ to zero by means of discontinuous control means that the output voltage coincides with the desired average equilibrium output voltage. Nevertheless, we wish to establish the nature and the stability of the corresponding remaining internal dynamics, or zero dynamics. In our case we have

$$\begin{aligned} L_f\sigma(x) &= \frac{\partial \sigma}{\partial x^T} f(x) = -\frac{1}{Q} x_2 \\ L_g\sigma(x) &= \frac{\partial \sigma}{\partial x^T} g(x) = x_1 \end{aligned} \tag{2.64}$$

The equivalent control is found to be

$$u_{eq}(x) = -\frac{L_f\sigma(x)}{L_g\sigma(x)} = \frac{1}{Q}\left(\frac{x_2}{x_1}\right) \tag{2.65}$$

The ideal sliding dynamics occurs when $u_{eq}(x)$ acts on the system as a feedback function while the system is ideally satisfying the condition $x_2 = \overline{X}_2$. We then have

$$\dot{x}_1 = -\frac{1}{Q}\left(\frac{\overline{X}_2^2}{x_1}\right) + 1 \tag{2.66}$$

The above dynamics exhibits an unstable equilibrium point. We may establish this fact via several approaches:

Via approximate linearization:

This technique will provide us with the local nature of the stability around the equilibrium point of the zero dynamics corresponding with $\sigma(x) = 0$. The incremental model (or tangent linearization model) of the normalized inductor current is given by

$$\frac{d}{dt} x_{1\delta} = \left(\frac{1}{Q}\frac{\overline{X}_2^2}{\overline{X}_1^2}\right) x_{1\delta} \tag{2.67}$$

where $x_{1\delta} = x_1 - \overline{X}_1$, $x_{2\,\delta} = x_2 - X_2$.

The equilibrium point is clearly unstable in view of the fact that the linearized zero dynamics exhibits a characteristic polynomial with a zero in the right half part of the complex plane.

Via Lyapunov stability theory

We rewrite the zero dynamics corresponding to $\sigma(x) = 0$ as

$$\frac{dx_1}{d\tau} = \frac{1}{x_1}\left(x_1 - \frac{\overline{X}_2^2}{Q}\right) \tag{2.68}$$

Consider the following Lyapunov function candidate in the x_1 variable space

$$\mathcal{V} = \frac{1}{2}\left(x_1 - \frac{\overline{X}_2^2}{Q}\right)^2 \tag{2.69}$$

The derivative of this function, taking into account that $x_1 > 0$ is given by

$$\dot{\mathcal{V}} = \frac{1}{x_1}\left(x_1 - \frac{\overline{X}_2^2}{Q}\right)^2 \geq 0 \tag{2.70}$$

By means of the phase diagram (see Figure 2.3)

2.11.2 Indirect control

The alternative is then to use as a sliding surface coordinate function a function that, when set to zero, reproduces the desired equilibrium value of the input inductor current, in correspondence with the desired output equilibrium voltage.

$$\sigma(x) = x_1 - \overline{X}_1 \tag{2.71}$$

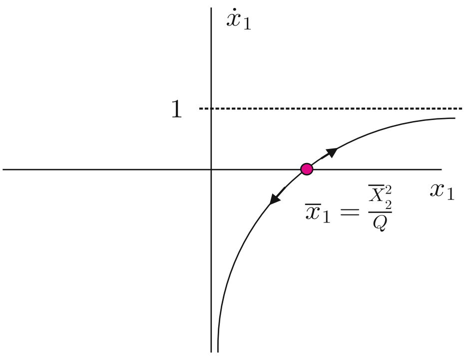

Fig. 2.3. Non-minimum phase character of the output voltage

To specify this function we compute the equilibrium point of the system under ideal sliding conditions. We write the equilibrium value of the current in terms of the equilibrium value of the output voltage.

$$\overline{X_1} = \frac{1}{Q}\overline{X}_2^2 \tag{2.72}$$

We now have

$$L_f\sigma(x) = 1, \quad L_g\sigma(x) = -x_2 \tag{2.73}$$

The equivalent control is then given by

$$u_{eq}(x) = \frac{1}{x_2} \tag{2.74}$$

The ideal sliding dynamics corresponding to $x_1 = \overline{X}_1$ is given by:

$$\dot{x}_2 = \frac{\overline{X}_2^2}{Qx_2} - \frac{x_2}{Q} \tag{2.75}$$

It is easy to see that the unique equilibrium point of the zero dynamics is an asymptotically stable equilibrium point. Indeed, consider the following Lyapunov function candidate in the x_2 space

$$V(x_2) = \frac{1}{2}(x_2 - \overline{X_2})^2 \tag{2.76}$$

The time derivative of this function is given by

$$\begin{aligned} \dot{V}(x_2) &= \frac{1}{Qx_2}(x_2 - \overline{X}_2)(\overline{X}_2^2 - x_2^2) \\ &= -\frac{1}{Qx_2}(x_2 - \overline{X}_2)^2(\overline{X}_2 + x_2) \end{aligned} \tag{2.77}$$

Evidently, the last expression is negative definite around the equilibrium point $\overline{X}_2$, given that $x_2 > 0$ around the equilibrium.

According to the developed theory, the sliding surface is reachable, or accessible, by means of the following switching policy,

$$u = \begin{cases} 1 \text{ if } (x_1 - \overline{X}_1) > 0 \\ 0 \text{ if } (x_1 - \overline{X}_1) < 0 \end{cases} \tag{2.78}$$

2.11.3 Simulations

We consider a boost converter with the following parameter values:

$$L = 0.01 \text{ [H]}, \quad C = 10^{-4} \text{ [F]},$$

$$R_L = 10 \ [\Omega], \quad E = 30 \text{ [V]}$$

It is desired to regulate the output voltage to the average equilibrium value

$$V_d = 119.92$$

The equilibrium value of the corresponding average normalized inductor current is, approximately, given by

$$I_d = 48.00 \text{ [A]}$$

Figure 2.4 depicts the sliding mode controlled responses of the chosen boost converter from zero initial conditions. Initially the control $u = 1$ is sustained until the inductor current reaches and slightly overshoots the desired constant value. After this the sliding mode is triggered letting the corresponding zero dynamics take over the output capacitor voltage making it increase towards its desired value. The figure also depicts the evolution of the Lyapunov function in its convergence towards the value of zero.

2.12 Control of the “Buck-Boost” converter

The circuit shown in Figure 2.5 represents a DC to DC power converter controlled by a switch. This system is better known as the “buck-boost” converter.

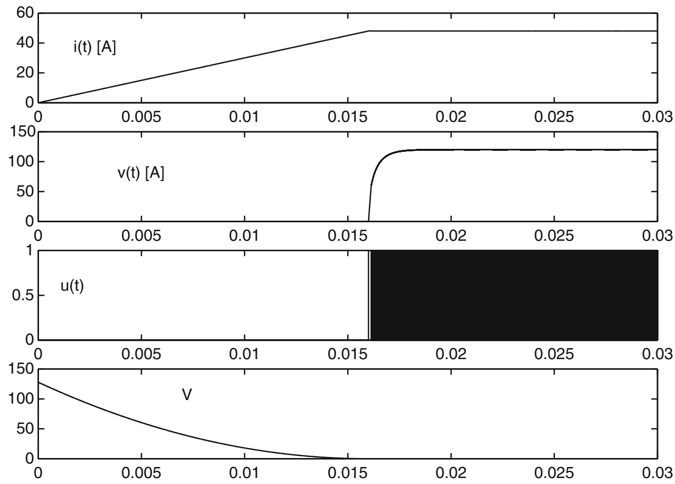

Fig. 2.4. Sliding mode controlled responses of the boost converter

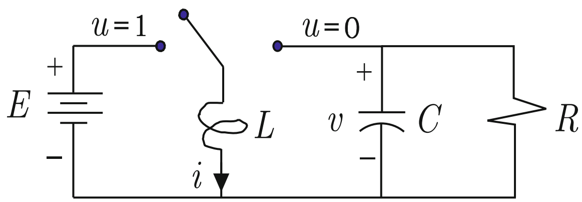

Fig. 2.5. Buck-Boost circuit

The differential equations describing this system are given by

$$L\frac{di}{dt} = (1-u)v + uE$$
$$C\frac{dv}{dt} = -(1-u)i - \frac{v}{R} \tag{2.79}$$

where i represents the input inductor current and v is the output voltage.

Performing the following normalization of the state variables and of the time variable:

$$x_1 = \frac{i}{E}\sqrt{\frac{L}{C}},\; x_2 = \frac{v}{E},$$
$$\tau = \frac{t}{\sqrt{LC}},\quad Q = R\sqrt{\frac{C}{L}}$$

we obtain the normalized average model of the DC to DC power converter

$$\frac{dx_1}{d\tau} = (1-u)x_2 + u$$
$$\frac{dx_2}{d\tau} = -(1-u)x_1 - \frac{1}{Q}x_2 \tag{2.80}$$

In the vector field notation introduced earlier, we specifically have

$$f(x) = \begin{bmatrix} x_2 \\ -x_1 - \frac{1}{Q}x_2 \end{bmatrix},\quad g(x) = \begin{bmatrix} 1 - x_2 \\ x_1 \end{bmatrix} \tag{2.81}$$

2.12.1 Direct control

The control objective is to have the normalized average voltage x_2 to converge towards the desired equilibrium value $\overline{X}_2$. We try first with the following sliding surface coordinate function

$$\sigma(x) = x_2 - \overline{X}_2 \tag{2.82}$$

Clearly, if $\sigma(x)$ is forced to be zero, the output capacitor voltage coincides with the desired value. As before, we must establish the stability features of the corresponding internal, or zero, dynamics of this output function.

In our case we have

$$L_f\sigma(x) = \frac{\partial\sigma}{\partial x^T}f(x) = -x_1 - \frac{1}{Q}x_2$$
$$L_g\sigma(x) = \frac{\partial\sigma}{\partial x^T}g(x) = x_1 \tag{2.83}$$

and the equivalent control is then given by

$$u_{eq}(x) = -\frac{L_f \sigma(x)}{L_g \sigma(x)} = 1 + \frac{1}{Q}\left(\frac{x_2}{x_1}\right) \tag{2.84}$$

The ideal sliding dynamics takes place when the control input is regarded as a smooth input and the equivalent control $u_{eq}(x)$ is used on the system dynamics. Of course this cannot be made in practice on a switched system. If furthermore the system is assumed to be initially on the sliding manifold, one ideally has $x_2(t) = \overline{X}_2$. We have

$$\dot{x}_1 = \frac{\left(1 - \bar{X}_2\right)\bar{X}_2}{Q}\left(\frac{1}{x_1}\right) + 1 \tag{2.85}$$

It will be shown that this dynamics has a unique equilibrium point which is unstable. We show this fact via several approaches.

Via approximate linearization:

The linear incremental model (or the tangent linearization model) of the normalized average current is given, after defining the incremental variables as: $x_{1\delta} = x_1 - \overline{X}_1$, $x_{2\delta} = x_2 - X_2$, by :

$$\frac{d}{d\tau}x_{1\delta} = \left[\frac{Q}{\left(\bar{X}_2 - 1\right)\bar{X}_2}\right]x_{1\delta} \tag{2.86}$$

which has an unstable equilibrium point due to the fact that the characteristic polynomial of the linearized dynamics exhibits a zero in the right-hand side of the complex plane. This is established from the fact that $\bar{X}_2 < 0$.

Via Lyapunov stability theory

We rewrite the zero dynamics corresponding to $\sigma(x) = 0$ as:

$$\frac{dx_1}{d\tau} = \frac{1}{x_1}\left(x_1 - \left(\bar{X}_2 - 1\right)\frac{\bar{X}_2}{Q}\right) \tag{2.87}$$

and consider the positive definite Lyapunov function in the x_1 space

$$\mathcal{V} = \frac{1}{2}\left(x_1 - \left(\bar{X}_2 - 1\right)\frac{\bar{X}_2}{Q}\right)^2 \tag{2.88}$$

By virtue of the fact that $x_1 > 0$, the time derivative of this function is positive semi-definite. Indeed,

$$\dot{\mathcal{V}} = \frac{1}{x_1}\left(x_1 - \left(\bar{X}_2 - 1\right)\frac{\bar{X}_2}{Q}\right)^2 \geq 0 \tag{2.89}$$

2.12.2 Indirect control

The alternative is then to use the sliding surface coordinate function a function which reproduces for the variable x_1 the desired equilibrium current, in correspondence with the desired average normalized voltage,

$$\sigma(x) = x_1 - \bar{X}_1 \tag{2.90}$$

To specify this function we calculate the equilibrium point of the system under ideal sliding conditions, writing the equilibrium current in terms of the average equilibrium normalized output voltage

$$\bar{X}_1 = -\left(1 - \bar{X}_2\right)\frac{\bar{X}_2}{Q} \tag{2.91}$$

We, then, have

$$L_f\sigma(x) = x_2, \quad L_g\sigma(x) = 1 - x_2 \tag{2.92}$$

The equivalent control is therefore given by

$$u_{eq}(x) = -\frac{x_2}{1 - x_2} \tag{2.93}$$

The ideal sliding dynamics corresponding to the zero value of the output function $\sigma(x)$, yielding $x_1 = \bar{X}_1$, is, after some algebraic manipulations, given by

$$\dot{x}_2 = -\frac{x_2 - \overline{X}_2}{Q}\left[1 - x_2 - \overline{X}_2\right] \tag{2.94}$$

Note that the factor $1 - x_2 - \overline{X}_2$ is strictly positive due to the fact that $x_2 < 0$ and $\overline{X}_2 < 0$. It is easy to verify that the unique equilibrium point of this zero dynamics, or ideal sliding dynamics, is asymptotically stable.

Indeed take as a Lyapunov function candidate the function

$$\mathcal{V}(x_2) = \frac{1}{2}\left(x_2 - \overline{X}_2\right)^2 \tag{2.95}$$

which is globally strictly positive except at $x_2 = \overline{X}_2$ where it is zero. The time derivative of this function, along the trajectories of the zero dynamics is given by

$$\dot{\mathcal{V}}(x_2) = -\frac{1}{Q}(x_2 - \overline{X}_2)^2\left[1 - x_2 - \overline{X}_2\right] \tag{2.96}$$

This quantity is zero at $x_2 = \overline{X}_2$ and strictly negative in the operating region of the converter $x_2 < 0$. The equilibrium point $x_2 = \overline{X}_2$ is asymptotically stable.

According to the developed theory, the sliding surface is reachable or accessible and the sliding motion is feasible due to internal stability reasons. The switching policy which reaches the sliding surface and sustains the sliding motion on this manifold is given by

$$u = \begin{cases} 1 \text{ if } (x_1 - \bar{X}_1) < 0 \\ 0 \text{ if } (x_1 - \bar{X}_1) > 0 \end{cases} \tag{2.97}$$

2.12.3 Simulations

Taking as the converter parameters the following ones,

$$L = 20\ [mH], \quad C = 20\ [\mu F],$$

$$R = 30\ [\Omega], \quad E = 15\ [V]$$

we obtain after normalization,

$$Q = 0.9487, \quad \sqrt{LC} = 6.3246 \times 10^{-4} \quad [s]$$

It is desired to control the average output voltage to the following desired equilibrium value

$$V_d = -22.5 \quad [V]$$

The corresponding equilibrium current is just

$$I_d = 1.875 \quad [A]$$

Figure 2.6 depicts the sliding mode controlled responses of the buck-boost converter from zero initial conditions. The inductor current grows initially towards its desired value with a fixed position of the switch. Meanwhile he capacitor voltage remains constant at the value zero. Once the sliding surface is reached, the ideal sliding dynamics commands the output capacitor voltage to negatively rise towards the desired constant value. The sliding mode is sustained from there on while the switch characteristically exhibits "bang-bang" behavior (of, theoretically, infinite frequency).

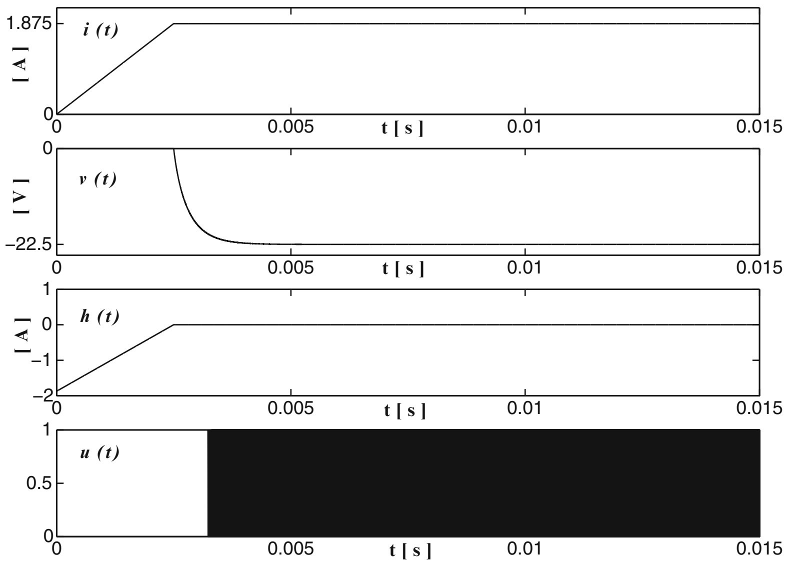

Fig. 2.6. Sliding mode controlled responses of the buck-boost converter

2.13 Sliding on a circle

The possibilities of sliding regime creation may be analyzed in the natural state space of the system. The system, however, may be described using inconvenient state coordinates. The sliding mode creation problem may be considerably simplified in transformed coordinates that exploit the most salient features of the system. The following example considers this issue.

Consider the following nonlinear system,

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1(x_1^2 + x_2^2 - R^2)\\ \dot{x}_2 &= -x_1 + x_2(x_1^2 + x_2^2 - R^2)\end{aligned}$$

where the control variable $R > 0$ takes one of two possible values $R \in \{R_1, R_2\}$ with $R_1 > R_2$. Let $R^2 = (R_1^2 + u(R_2^2 - R_1^2))$, so that $R = R_1$ when $u = 0$ and $R = R_2$ when $u = 1$. We first analyze the sliding mode creation problem in the state space of cartesian coordinates (x_1, x_2). Then we proceed with the same analysis using the more natural, polar coordinates: $(\rho = \sqrt{x_1^2 + x_2^2})$, $\theta = \arctan(x_2/x_1)$.

For $u = 0$, the system exhibits an unstable limit cycle consisting of a circle of radius R_1, centered at the origin of coordinates. Similarly, a second unstable limit cycle is obtained when u is set to permanently adopt the value $u = 1$.

Suppose the control objective is to achieve a desired constant state vector magnitude R_d i.e., it is desired to have: $x_1^2 + x_2^2 = R_d^2$ in finite time. The sliding surface interpreting the control objective is given by

$$S = \{(x_1, x_2) \in \mathbb{R}^2 \mid \sigma(x) = x_1^2 + x_2^2 - R_d^2 = 0\ \} \tag{2.98}$$

The vector fields involved are given by

$$f(x) = \begin{bmatrix} x_2 + x_1(x_1^2 + x_2^2 - R_1^2) \\ -x_1 + x_2(x_1^2 + x_2^2 - R_1^2) \end{bmatrix}, \quad g(x) = \begin{bmatrix} -x_1(R_2^2 - R_1^2) \\ -x_2(R_2^2 - R_1^2) \end{bmatrix} \tag{2.99}$$

In this case,

$$L_f\sigma(x) = 2(x_1^2 + x_2^2 - R_1^2)(x_1^2 + x_2^2), \quad L_g\sigma(x) = -2(x_1^2 + x_2^2)(R_2^2 - R_1^2) \tag{2.100}$$

Clearly $L_f\sigma > 0$ since $R_2 < R_1$. The equivalent control defined on the circle, $\sigma(x) = 0$, or equivalently on, $x_1^2 + x_2^2 = R_d^2$, is found to be given by

$$u_{eq} = -\frac{L_f\sigma}{L_g\sigma} = \frac{(R_d^2 - R_1^2)R_d^2}{R_d^2(R_2^2 - R_1^2)} = \frac{R_1^2 - R_d^2}{R_1^2 - R_2^2} \tag{2.101}$$

A sliding regime exists for those values of R_d such that

$$0 < \frac{R_1^2 - R_d^2}{R_1^2 - R_2^2} < 1 \tag{2.102}$$

which implies that $R_2^2 < R_d^2 < R_1^2$. A sliding regime is locally confined to exist on the annular region: $\{(x_1, x_2) \in \mathbb{R}^2 \mid R_1^2 < x_1^2 + x_2^2 < R_2^2\}$.

In polar coordinates the system is described by

$$\dot{\rho} = \rho(\rho^2 - R_1^2 - u(R_2^2 - R_1^2)), \quad \dot{\theta} = -1 \tag{2.103}$$

The sliding surface coordinate function is now linear in ρ, given by $\sigma(\rho, \theta) = \rho - R_d$. The problem becomes a one-dimensional problem. The vector fields involved are $f(\rho, \theta) = \rho(\rho^2 - R_1^2)\partial/\partial\rho$, $g(\rho\theta) = -\rho(R_2^2 - R_1^2)\partial/\partial\rho$. The quantities $L_f\sigma$ and $L_g\sigma$ are computed as

$$L_f\sigma = R_d(R_d^2 - R_1^2) \quad L_g\sigma = -R_d(R_2^2 - R_1^2)) \tag{2.104}$$

The equivalent control is

$$u_{eq} = \frac{R_d^2 - R_1^2}{R_2^2 - R_1^2} = \frac{R_1^2 - R_d^2}{R_1^2 - R_2^2} \tag{2.105}$$

just the same as before.

2.14 Trajectory tracking

When the control objective, in a switched system of the form $\dot{x} = f(x) + g(x)u$, $x \in \mathbb{R}^n$, $u \in \{0, 1\}$ is one of having a certain variable in the system to track a given smooth reference trajectory, the control objective is usually synthesized in terms of a *time-varying* sliding surface, i.e. one which explicitly depends on time.

$$\mathcal{S}(t) = \{x \in \mathbb{R}^n \mid \sigma(x, t) = 0, \ \forall\, t\} \tag{2.106}$$

Consider the space of vectors of the form $z^T = [x^T, t]$. The system equation adopts then the following form

$$\dot{z} = \frac{d}{dt}\begin{bmatrix} x \\ t \end{bmatrix} = \begin{bmatrix} f(x) \\ 1 \end{bmatrix} + \begin{bmatrix} g(x) \\ 0 \end{bmatrix} u = \tilde{f}(z) + \tilde{g}(z)u \tag{2.107}$$

Note that $L_{\tilde{g}}\sigma(z) = L_g\sigma(x, t)$ and $L_{\tilde{f}}\sigma(z) = L_f\sigma(x, t) + \frac{\partial\sigma(x,t)}{\partial t}$.
The equivalent control on $\sigma(z) = 0$ is then found to be

$$u_{\text{eq}}(x, t) = -\frac{L_f\sigma(x, t)}{L_g\sigma(x, t)} - \left[\frac{1}{L_g\sigma(x, t)}\right]\frac{\partial\sigma(x, t)}{\partial t} \tag{2.108}$$

The sliding mode existence condition $0 < u_{\text{eq}}(x,t) < 1$ is specified by taking the worst possible case on the values that the partial time derivative of $\sigma(x,t)$ may take

$$\sup_{(t,x\in\mathcal{S})}\left|\left[\frac{1}{L_g\sigma(x,t)}\right]\frac{\partial\sigma(x,t)}{\partial t}\right| < -\frac{L_f\sigma(x,t)}{L_g\sigma(x,t)}$$
$$< 1-\sup_{(t,x\in\mathcal{S})}\left|\left[\frac{1}{L_g\sigma(x,t)}\right]\frac{\partial\sigma(x,t)}{\partial t}\right|$$

Generally speaking, the region of existence of a sliding mode depends on the magnitude of the velocity of variation of the sliding surface coordinate function. The faster this variation, the smaller the region of existence. The time dependence of the sliding surface coordinate function, introduced by the trajectory tracking control objective, is translated into a limitation of the region of existence. Thus, the variation of the trajectory to be tracked must be limited to smaller values in order to guarantee the existence of the sliding regime. This is an indication of a frequency bandwidth of the controlled system.

Example 2.12. Consider the normalized model of a buck converter

$$\dot{x}_1 = -x_2 + u, \quad \dot{x}_2 = x_1 - \frac{x_2}{Q}, \quad y = x_2 \tag{2.109}$$

In customary applications, the converter is a step down converter reducing the constant voltage value of the source (here normalized to the value 1) to a constant fraction voltage at the output. Consider the trajectory tracking problem where a reference output voltage trajectory is given in the form of a biased sinusoidal signal centered around the constant voltage $x_2 = 0.5$, with amplitude, A, and normalized angular frequency ω:

$$y^*(\tau) = \frac{1}{2}(1 + A\sin(\omega\tau)) \tag{2.110}$$

Naturally, the corresponding nominal reference signal for the current x_1 is just obtained from the second equation of the converter as

$$x_1^*(\tau) = \dot{y}^*(\tau) + \frac{1}{Q}y^*(\tau) \tag{2.111}$$

the nominal control input $u^*(t)$ is given, from the first equation of the converter, by

$$u^*(\tau) = \dot{x}_1^*(\tau) + y^*(\tau) = \ddot{y}^*(\tau) + \frac{1}{Q}\dot{y}^*(\tau) + y^*(\tau) \tag{2.112}$$

It is not difficult to establish the relation between the equivalent control and the nominal control input in a suitable sliding mode scheme. They are equal in steady state. For instance, if an indirect sliding mode approach is adopted, a suitable time-varying sliding surface may be proposed in terms of the nominal current signal

$$\sigma(x_1,\tau) = x_1 - x_1^*(\tau) = x_1 - (\dot{y}^*(\tau) + \frac{1}{Q}y^*(\tau)) \tag{2.113}$$

Hence,

$$\dot{\sigma} = -x_2 + u - (\ddot{y}^*(\tau) + \frac{1}{Q}\dot{y}^*(\tau)) \tag{2.114}$$

Under ideal sliding conditions $\sigma = 0$, $\dot{\sigma} = 0$, one has

$$u_{eq}(\tau) = \ddot{y}^*(\tau) + \frac{1}{Q}\dot{y}^*(\tau) + y^*(\tau) \tag{2.115}$$

The existence condition: $0 < u_{eq}(\tau) < 1$ actually represents the bandwidth limitations. Indeed, for the specific desired signal $y^*(\tau)$, one finds

$$0 < \frac{1}{2} + \frac{A}{2}\left[(1-\omega^2)\sin(\omega\tau) + \frac{\omega}{Q}\cos(\omega\tau)\right] < 1 \tag{2.116}$$

Letting $\phi = \arctan(\omega/(Q(1-\omega^2)))$, the above restriction is equivalent to

$$-1 < A\sqrt{(1-\omega^2)^2 + (\frac{\omega}{Q})^2}\sin(\omega t + \phi) < 1 \tag{2.117}$$

i.e., the following amplitude frequency relation is obtained:

$$\left|A\sqrt{(1-\omega^2)^2 + (\frac{\omega}{Q})^2}\right| < 1 \tag{2.118}$$

Notice that a value of A larger than 1 violates the existence condition. Given a desired normalized frequency ω, the amplitude A is necessarily restricted by the relation

$$A < \frac{1}{\sqrt{(1-\omega^2)^2 + (\frac{\omega}{Q})^2}} \tag{2.119}$$

Conversely, given a desired value for A, such that $A < 1$, the range of normalized frequencies, for which a sliding regime exists and output reference trajectory tracking is feasible, is limited by the corresponding relation,

$$(1-\omega^2)^2 + (\frac{\omega}{Q})^2 < \frac{1}{A^2} \tag{2.120}$$

Exercise 2.13. In the previously considered buck converter example, a time-varying sliding surface coordinate function: $\sigma(x_2, x_1, \tau)$, inducing an asymptotic, exponential closed loop convergence of the converter voltage to the desired output voltage signal, may be prescribed as follows:

$$\sigma = \left(x_1 - \frac{x_2}{Q} - \frac{A\omega}{2}\cos(\omega\tau) \right) + \lambda \left(x_2 - \frac{1}{2}(1 + A\sin(\omega\tau)) \right) \tag{2.121}$$

with $\lambda > 0$. Show that, in steady state, the bandwidth limitation is precisely the same as the one derived in the example.

Explicit determination of the frequency bandwidth limitations in the case of nonlinear systems is particularly complex and not straightforward even for the simplest of examples. Graphical assessment is always a much simpler route.

Exercise 2.14. Consider the liquid height evolution on a tank of area A with liquid losses at the bottom and filed either at a constant rate, U $[m^3/s]$ or 0.

$$\dot{x} = -\frac{c}{A}\sqrt{x} + \left(\frac{U}{A}\right) u, \quad u \in \{0, 1\} \tag{2.122}$$

The sliding surface coordinate function $\sigma = x - H^*(t)$ is used, along with the switching policy $u = (1/2)(1 - \text{sign}\sigma)$. The equivalent control is given by

$$u_{eq} = \left[\frac{c}{U}\sqrt{H^*(t)} + \frac{A}{U}\dot{H}^*(t) \right] \tag{2.123}$$

For $H^*(t) = H + \alpha\sin(\omega t)$ with $H > \alpha > 0$, assess amplitude-frequency tradeoffs in the space (α, ω), for the existence of a sliding regime that lets the control liquid height, x, track the biased sinusoidal signal $H^*(t)$.

2.15 Invariance conditions under matched perturbations

One of the main features of the sliding regimes is their robustness with respect to certain external perturbation inputs affecting the system behavior. In this section, we explore what conditions should be satisfied for such perturbations to be automatically rejected from the ideal sliding dynamics.

2.15.1 Drift field perturbation

Consider the nonlinear additively perturbed system: $\dot{x} = f(x) + g(x)u + \xi(x,t)$, controlled by a single switch and let $\mathcal{S}$ be a smooth sliding surface over which we may create a local sliding regime in spite of the presence of the time-varying perturbation. The perturbation field $\xi(x,t)$ is an unknown function of the state and time and it is assumed that its values are uniformly bounded.

Suppose then that it is possible to create a sliding regime over the sliding surface, $\mathcal{S} = \{x \mid \sigma(x) = 0\}$, in spite of the presence of the perturbation field $\xi(x,t)$. The existence of such a sliding regime implies the existence of an equivalent control, $u_{eq}(x,t)$, which ideally, locally, sustains the state trajectories on the smooth manifold $\mathcal{S}$. The equivalent control is, necessarily, a function of the unknown perturbation field $\xi(x,t)$ and it is given by

$$u_{eq}(x,t) = -\frac{L_f\sigma(x) + L_\xi\sigma(x)}{L_g\sigma(x)} \tag{2.124}$$

The ideal sliding dynamics, with $x \in \mathcal{S}$, is then obtained to be

$$\begin{aligned}\dot{x} &= f(x) - g(x)\frac{L_f\sigma(x) + L_\xi\sigma(x)}{L_g\sigma(x)} + \xi(x,t)\\ &= \left[I - \frac{1}{L_g\sigma(x)}g(x)\frac{\partial\sigma}{\partial x^T}\right]f(x)\\ &\quad + \left[I - \frac{1}{L_g\sigma(x)}g(x)\frac{\partial\sigma}{\partial x^T}\right]\xi(x,t)\end{aligned} \tag{2.125}$$

The projection operator $\mathcal{M}(x)$ over the tangent space to $\mathcal{S}$, along the span of $g(x)$, acts over the addition of the vector fields: $f(x)+\xi(x,t)$, in the creation of the local sliding regime on $\mathcal{S}$.

Clearly, the ideal sliding dynamics is totally independent of the influence of the perturbation vector $\xi(x,t)$ if the vector field $\xi(x,t)$ is in the null space of $\mathcal{M}(x)$ for all t, i.e.,

$$\left[I - \frac{1}{L_g\sigma(x)}g(x)\frac{\partial\sigma}{\partial x^T}\right]\xi(x,t) = 0 \tag{2.126}$$

Recall that the null space of $\mathcal{M}(x)$ coincides with the *span* $g(x)$. In other words, the sliding motions are invariant with respect to the perturbation if the vector field $\xi(x,t)$ is in the span of $g(x)$ for all t, i.e. there exists a nonzero scalar function $\alpha(x,t)$ such that

$$\xi(x,t) = \alpha(x,t)g(x) \tag{2.127}$$

We have then that $\mathcal{M}(x)\xi(x,t) = \alpha(x,t)\mathcal{M}(x)g(x) = 0$. The perturbation field $\xi(x,t)$ does not affect the ideal sliding dynamics as long as it is uniformly *aligned* with the control vector field $g(x)$. Such perturbations receive the name of *matched perturbations* and the previous condition is known as the *matching condition* .

We can also establish that this alignment, or matching, condition is a necessary condition to have the ideal sliding motions completely independent of the perturbation field $\xi(x,t)$. Indeed, let $\xi(x,t)$ be such that

$$\left[I - \frac{1}{L_g\sigma(x)}g(x)\frac{\partial\sigma}{\partial x^T}\right]\xi(x,t) = 0 \tag{2.128}$$

Then we can write $\xi(x,t)$ in the following manner

$$\xi(x,t) = \frac{L_\xi \sigma(x)}{L_g \sigma(x)} g(x) \tag{2.129}$$

i.e. $\xi(x,t)$ is in the span of $g(x)$.

We have thus demonstrated the validity of the following theorem:

Theorem 2.15. *Let the drift field perturbed system: $\dot{x} = f(x)+g(x)u+\xi(x,t)$, with $\xi(x,t)$ being a uniformly bounded time-varying vector field, exhibit a local, or global, sliding motion on the sliding surface $\mathcal{S} = \{x \mid \sigma(x) = 0\}$. Then, the ideal sliding dynamics is locally, or globally, uniformly invariant with respect to $\xi(x,t)$ if and only if*

$$\xi(x,t) \;\in\; span\; g(x), \quad \text{for all } t \tag{2.130}$$

Example 2.16. Consider the normalized description of a pendulum system, including an external torque perturbation input of the Coulomb friction type.

$$\ddot{\theta} = -\sin\theta + \rho(2u-1) - \nu \text{ sign } \dot{\theta} \tag{2.131}$$

where $\nu > 0$ is the unknown amplitude of the perturbation torque term denoted by τ.

The state space description of the system is readily obtained by defining the state variables as the phase variables as follows: $x_1 = \theta$, $x_2 = \dot{\theta}$. We have

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\sin x_1 - \rho \end{bmatrix} + \begin{bmatrix} 0 \\ 2\rho \end{bmatrix} u + \begin{bmatrix} 0 \\ -\nu \text{ sign } x_2 \end{bmatrix} \tag{2.132}$$

The matching condition is clearly satisfied in this case, as the vector field characterizing the perturbation torque is aligned with the control input field. In our notation we have

$$\xi(x) = \begin{bmatrix} 0 \\ \nu \text{ sign } x_2 \end{bmatrix}, \quad g(x) = g = \begin{bmatrix} 0 \\ 2\rho \end{bmatrix} \tag{2.133}$$

A sliding surface coordinate function $\sigma(x)$ is given by our objective of linearizing the ideal sliding motions towards a desired constant equilibrium, $x_1 = \Theta$, on the sliding surface $\mathcal{S} = \{x \in \mathbb{R}^2 \mid \sigma(x) = 0\}$. We have

$$\sigma(x) = x_2 + \lambda(x_1 - \Theta), \quad \lambda > 0 \tag{2.134}$$

The reachability of the sliding surface is now clearly affected by the unknown perturbation torque amplitude in the following manner:

$$\dot{\sigma} = -\sin x_1 + \rho(2u-1) - \nu \text{ sign } x_2 + \lambda x_2 \tag{2.135}$$

The equivalent control is given by an expression that includes the unknown perturbation torque input. Indeed,

$$u_{\rm eq}(x) = \frac{1}{2} + \frac{1}{2\rho}\left[\sin x_1 + \nu \text{ sign } x_2 - \lambda x_2\right] \qquad (2.136)$$

The ideal sliding dynamics, given by $\dot{x}_1 = -\lambda(x_1 - \Theta)$ is clearly unaffected by the torque perturbation input. The sliding mode existence conditions, $0 < u_{\rm eq} < 1$, must be assessed under the worst possible circumstances of influence of the unknown friction torque summand. A sliding regime is guaranteed to exist in the region delimited by

$$-\rho + \nu \ < \ \sin x_1 - \lambda x_2 \ < \ \rho - \nu \qquad (2.137)$$

which is to be contrasted against the region defined by the existence conditions when no perturbation exists, which is recalled to be

$$-\rho \ < \ \sin x_1 - \lambda x_2 \ < \ \rho \qquad (2.138)$$

Evidently, the region of existence in the phase space for the perturbed system is diminished with respect to that of the unperturbed case. Assuming $\rho > \nu$, i.e., the available torque input is capable of overcoming the friction effects, then a sliding motion exists which drives the system towards the desired equilibrium point, $x_1 = \Theta, x_2 = 0$. In the simulations we have set $\Theta = 2.5$ rad (Fig. 2.7).

Example 2.17. The most typical example where the matching condition is not satisfied, and, hence, the ideal sliding motions cannot be made independent of the perturbation input τ_L, is represented by the classical DC motor system.

Consider the following model of a dc motor subject to unknown load torque perturbations

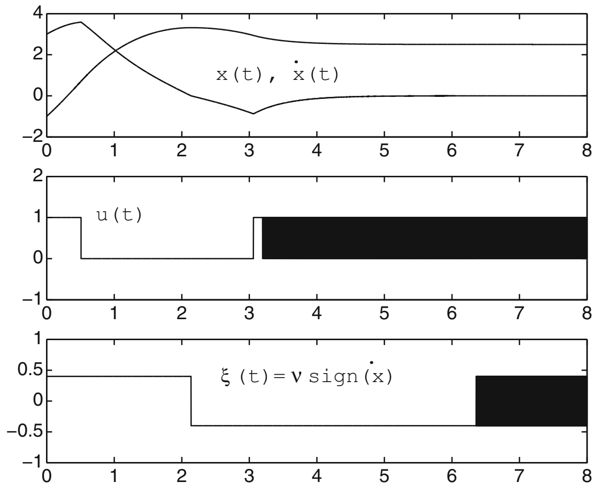

Fig. 2.7. Sliding mode control of pendulum perturbed by unknown Coulomb friction torque

$$L\frac{di}{dt} = u - Ri - K\omega$$
$$J\frac{d\omega}{dt} = Ki - \tau_L \tag{2.139}$$

where i is the armature circuit current, ω is the angular velocity of the motor shaft, u, the control input is representing the armature circuit input voltage, and τ_L is the load torque perturbation input, of unknown nature.

The state space description of the system is readily obtained by letting $x_1 = i$, $x_2 = \omega$. We have

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{R}{L}x_1 - \frac{K}{L}x_2 \\ \frac{K}{J}x_1 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -\frac{1}{J} \end{bmatrix} \tau_L \tag{2.140}$$

The matching condition is, evidently, not satisfied.

A sliding surface that reflects our desire to achieve a constant angular velocity is given by

$$\sigma(x) = [Kx_1 + \lambda J(x_2 - \Omega)], \quad \lambda > 0 \tag{2.141}$$

This choice, however, leads to the ideal sliding motions for the angular velocity which depend on the unknown input torque

$$\dot{\omega} = -\lambda(\omega - \Omega) - \frac{\tau_L}{J} \tag{2.142}$$

For constant load torques, the ideal equilibrium point depends on τ_L since, on the sliding surface, $\omega \to \Omega - \tau_L/(\lambda J)$.

2.15.2 Control field perturbations

We consider now control field perturbed switched systems of the form,

$$\dot{x} = f(x) + [g(x) + \xi(x,t)]u \tag{2.143}$$

We find a strikingly similar result to that of the previous theorem.

Theorem 2.18. *Consider the perturbed switched system* $\dot{x} = f(x) + [g(x) + \xi(x,t)]u$*, with* $\xi(x,t)$ *being an uniformly bounded, smooth, time-varying vector field. Suppose that the controlled system exhibits a local, or global, sliding regime on the sliding surface* $\mathcal{S} = \{x \in \mathbb{R}^n \mid \sigma(x) = 0\}$*. Then, the ideal sliding dynamics is independent of* $\xi(x,t)$*, if and only if, locally or globally,*

$$\xi(x,t) \ \in \ span\ g(x), \quad \text{for all } t \tag{2.144}$$

The ideal sliding dynamics associated with $\mathcal{S}$ is given now by the expression

$$\dot{x} = \left[I - \frac{1}{L_{g+\xi}\sigma} \left(g(x) + \xi(x,t)\right) \frac{\partial \sigma}{\partial x^T} \right] f(x) \tag{2.145}$$

Let $\xi(x,t) \in span\ g(x)$, then, there exists a non zero smooth bounded scalar signal $\alpha(x,t)$ such that $\xi(x,t) = \alpha(x,t)g(x)$. The ideal sliding dynamics turns out to be given by

$$\dot{x} = \left[I - \frac{(1+\alpha)}{L_{(1+\alpha)g}\sigma} g(x) \frac{\partial \sigma}{\partial x^T}\right] f(x) = \left[I - \frac{1}{L_g\sigma} g(x) \frac{\partial \sigma}{\partial x^T}\right] f(x) \quad (2.146)$$

i.e., the ideal sliding dynamics is completely independent of the function $\alpha(x,t)$ and, hence, it is independent of $\xi(x,t)$.

To prove necessity, suppose that the time-varying control perturbation field $\xi(x,t)$ is such that the ideal sliding dynamics is independent of it, i.e.,

$$\dot{x} = \left[I - \frac{1}{L_{g+\xi}\sigma}(g(x) + \xi(x,t)) \frac{\partial \sigma}{\partial x^T}\right] f(x) = \left[I - \frac{1}{L_g\sigma} g(x) \frac{\partial \sigma}{\partial x^T}\right] f(x)$$

We are led to the following equality

$$\xi(x,t) = \left[\frac{L_\xi\sigma(x)}{L_g\sigma(x)}\right] g(x) \quad (2.147)$$

i.e., $\xi(x,t)$ is uniformly contained in the span of $g(x)$.

2.15.3 Control and drift fields perturbations

Consider now the following switched regulated perturbed system

$$\dot{x} = f(x) + [g(x) + \xi(x,t)]u + \zeta(x,t) \quad (2.148)$$

and assume there exists a local sliding regime on the sliding surface, $\mathcal{S} = \{x \mid \sigma(x) = 0\}$, in spite of the presence of the uniformly bounded, but otherwise unknown, smooth, time-varying vector fields $\xi(x,t)$ and $\zeta(x,t)$.

It is easy to see that the invariance of the ideal sliding dynamics, with respect to the perturbation fields $\xi(x,t)$ and $\zeta(x,t)$, is verified if and only if the following matching conditions are satisfied,

$$\zeta(x,t) \ \in \ span\ g(x), \qquad \xi(x,t) \ \in \ span\ g(x), \quad \text{for all } t \quad (2.149)$$

Using Theorem 2.15, we find that the ideal sliding motions on $\mathcal{S}$ are unaffected by $\zeta(x,t)$ if and only if

$$\zeta(x,t) \ \in \ span\ [g(x) + \xi(x,t)] \quad (2.150)$$

Thus, the system is now, for some bounded scalar function $\alpha(x,t)$, of the form

$$\dot{x} = f(x) + [g(x) + \xi(x,t)]u + \alpha(x,t)[g(x) + \xi(x,t)] \tag{2.151}$$

and the ideal sliding dynamics on $\mathcal{S}$ is independent of $\alpha(x,t)$. In other words, the ideal sliding motions are identically described by the ideal sliding motions of the following system,

$$\dot{x} = f(x) + [g(x) + \xi(x,t)]u \tag{2.152}$$

It follows, using Theorem 2.18, that the ideal sliding dynamics of the previous system, on the manifold $\mathcal{S}$, is invariant with respect to the perturbation $\xi(x,t)$ if and only if

$$\xi(x,t) \in \; span \; g(x) \quad \text{for all } t. \tag{2.153}$$

Then, under this condition we have that also $\zeta(x,t) \in span\; g(x)$ for all t.

2.15.4 The equivalent control of a perturbed system

We examine a particularly simple issue related to the interpretation of the ideal sliding motions invariance condition with respect to a matched perturbation field in a drift vector field perturbed system.

Consider the switched nonlinear system (f, g, σ_0) with $\sigma_0(x)$ a given sliding surface coordinate function on which a local sliding regime exists for an appropriate switching policy $u \in \{0, 1\}$. Denote the equivalent control by $u^0_{eq}(x) = -L_f\sigma_0/L_f\sigma_0$. Clearly $0 < u^0_{eq}(x) < 1$ in the region of existence of the sliding regime. Let us address such an equivalent control, the *nominal* equivalent control.

Consider now the perturbed system

$$\dot{x} = f(x) + g(x)u + \xi(x,t), \quad \xi(x,t) = g(x)u_h(x,t) \tag{2.154}$$

i.e., $\xi(x,t)$ is a time-varying vector field which satisfies the matching condition for all t. $u_h(x,t)$ plays the role of a bounded disturbance control input given by an unknown state feedback law. We assume boundedness of the disturbance input, i.e., a strictly positive scalar constant, K_h, exists, such that $|u_h(x,t)| \leq K_h$, for all t and x. Assume, also, that the transversal condition, $L_g\sigma_0 > 0$, is locally valid on a region containing the region of existence of the sliding regime.

Suppose a sliding regime is accomplished by the switching law $u = \frac{1}{2}(1 - \text{sign}\sigma_0(x))$ in spite of the presence of the disturbance field $\xi(x,t)$.

From the invariance condition, $\dot{\sigma}_0 = 0$, on $\sigma_0 = 0$ we have

$$\dot{\sigma}_0(x) = L_f\sigma_0 + L_g\sigma_0 u + L_g\sigma_0 u_h(x,t) = 0. \tag{2.155}$$

The equivalent control is found to be

$$u_{eq} = -\frac{L_f \sigma_0}{L_g \sigma_0} - u_h(x,t) = u^0_{eq}(x) - u_h(x,t) \tag{2.156}$$

The equivalent control u_{eq} exhibits a counteracting effect, $-u_h(x,t)$, to the unknown disturbance input in the form of an additive input to the nominal equivalent control annihilating the effects of the disturbance input. This explains the origin of the robustness of the sliding regime with respect to matched perturbation inputs.

However, the region of existence of a sliding regime on the sliding surface, $\mathcal{S} = \{x|\sigma_0(x) = 0\}$, is clearly reduced thanks to the possible adverse effect of the unknown disturbance input. Notice that we still have: $0 < u_{eq}(x) < 1$ which, in terms of the nominal equivalent control, $u^0_{eq}(x)$, reads:

$$K_h < u^0_{eq}(x) < 1 - K_h \tag{2.157}$$

The region of existence of a sliding regime, on $\mathcal{S}$, is thus reduced by the presence of the bounded disturbance input. Necessarily, $K_h < 0.5$, since, otherwise, the region of existence of a sliding regime may become empty. Disturbances, even if matched, may prevent the existence of a sliding regime on a region of the sliding surface where one may have existed when no perturbation was present.

2.16 Sliding surface design

Before discussing some general aspects of sliding mode controller design in multi-variable systems, we revisit some structural aspects of SISO systems which may be generalized to the multi-variable case and, furthermore, provide a systematic procedure for sliding surface design in switched systems whose natural output stabilization, or tracking, errors do not directly conform to appropriate sliding surfaces thanks to a lack of explicit dependence on the control input of the sliding surface coordinate function first order time derivative $\dot{\sigma}$.

Consider the smooth SISO system

$$\dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^n, \quad u \in \{0,1\}$$
$$y = h(x), \quad y \in \mathbb{R}$$

Notice that we now emphasize that $y = h(x)$ is the measured output of the system which needs to be zeroed in order to satisfy a control objective (output stabilization errors and even output trajectory tracking errors can be handled in a similar fashion). The generality to be addressed next assumes that the first order time derivative of y may not explicitly depend on the control input u. The output function $h(x)$ is not *per se* a sliding surface coordinate function and, hence, the transversal condition, $L_g h(x) \neq 0$, may no longer be locally valid for $h(x)$. The above system is addressed as the triple (f, g, h). The following considerations and definitions are based on the geometric approach masterfully laid out and clearly explained in Isidori's book [13].

Definition 2.19. *Let r be a positive integer such that $r \leq n$, where n is the order of the system. We say the system (f, g, h) is relative degree r around a point $x^0 \in \mathbb{R}^n$, if the following two conditions are satisfied*

$$
\begin{aligned}
&1)\ \ L_g h(x) = L_g L_f h(x) = \cdots = L_g L_f^{r-2} h(x) = 0, \\
&\qquad \forall x \in \mathcal{N}(x^0) \\
&2)\ \ L_g L_f^{r-1} h(x^0) \neq 0
\end{aligned}
$$

where $\mathcal{N}(x^0)$ stands for an open neighborhood around the point x^0.

The previous statement is completely equivalent to saying that the time derivatives of the output function $y = h(x)$, along the trajectories of the controlled system do not exhibit an explicit dependence upon the control input in the first $r-1$ time derivatives, computed around the point x^0, but only the r-th time derivative does exhibit such a dependence.

Indeed, the time derivatives of y, up to order $r-1$, may be seen to coincide with a scalar functions of x alone;

$$
\begin{aligned}
y &= h(x) \\
\dot{y} &= \frac{\partial h}{\partial x^T}\dot{x} = L_f h(x) + \underbrace{(L_g h(x))}_{=0} u = L_f h(x) \\
\ddot{y} &= \frac{\partial L_f h(x)}{\partial x^T}\dot{x} = L_f(L_f h(x)) + \underbrace{L_g(L_f h(x))}_{=0} u \\
&= L_f^2 h(x) \\
y^{(i)} &= \frac{\partial L_f^{i-1} h(x)}{\partial x^T}\dot{x} = L_f^i h(x) + \underbrace{L_g(L_f^{i-1} h(x))}_{=0} u \\
&= L_f^i h(x) \\
&\vdots \\
y^{(r-1)} &= L_f^{r-1} h(x) \\
y^{(r)} &= \frac{\partial L_f^{r-1} h(x)}{\partial x^T}\dot{x} = L_f^r h(x) + [L_g L_f^{r-1} h(x)]u
\end{aligned}
$$

The input-output, state dependent, relation

$$
y^{(r)} = L_f^r h(x) + (L_g L_f^{r-1} h(x))u \tag{2.158}
$$

serves as the basis for the sliding surface design inasmuch as driving y to 0 is concerned (hence our surface specification procedure still has to resolve the issue of having a convenient, stable, corresponding zero dynamics).

Consider therefore the sliding surface coordinate function, written in terms of the output function time derivatives:

$$\sigma = y^{(r-1)} + \alpha_{r-2} y^{(r-2)} + \cdots + \alpha_0 y \tag{2.159}$$

Evidently, this specification can also be written in terms of the state x of the system as the following nonlinear scalar function:

$$\sigma(x) = L_f^{r-1} h(x) + \alpha_{r-2} L_f^{r-2} h(x) + \cdots + \alpha_1 L_f h(x) + \alpha_0 h(x) \tag{2.160}$$

It should be clear why the choice of such $\sigma(x)$ as the sliding surface coordinate function.

The invariance condition, $\sigma = 0$, yields the following closed loop dynamics for the output y

$$y^{(r-1)} + \alpha_{r-2} y^{(r-2)} + \cdots + \alpha_0 y = 0 \tag{2.161}$$

hence an appropriate choice of the parameter set $\{\alpha_0, \cdots, \alpha_{r-2}\}$ causes the differential equation satisfied by the output y to exhibit the origin of the subspace coordinatized by $y, \dot{y}, \cdots, y^{r-1}$ to be asymptotically exponentially stable. This is achieved by setting the parameter set to conform a set of *Hurwitz coefficients*.

To obtain the equivalent control, $u_{eq}(x)$, we compute $\dot{\sigma}$, in terms of the state variables, and obtain

$$\dot{\sigma}(x) = L_f^r h(x) + (L_g L_f^{r-1} h(x))u + \sum_{i=0}^{r-2} \alpha_i L_f^{i+1} h(x)$$

which on $\sigma(x) = 0$ yields, with $L_f^{r-1} h(x) = -\alpha_{r-2} L_f^{r-2} h(x) - \cdots - \alpha_0 h(x)$

$$\dot{\sigma}(x) = L_f^r h(x) + (L_g L_f^{r-1} h(x)) u_{eq}(x) + \sum_{i=0}^{r-2} (\alpha_{i-1} - \alpha_i \alpha_{r-2}) L_f^i h(x) = 0$$

with $\alpha_{-1} = 0$.

We have

$$u_{eq}(x) = -\frac{L_f^r h(x) + \sum_{i=0}^{r-2} (\alpha_{i-1} - \alpha_i \alpha_{r-2}) L_f^i h(x)}{L_g L_f^{r-1} h(x)}$$

Notice that from the definition of $\sigma(x)$ and the assumption that the system is relative degree r, we have

$$\begin{aligned} L_g\sigma(x) &= L_g L_f^{r-1}h(x) \\ L_f\sigma(x) &= L_f^r + \alpha_{r-2}L_f^{r-1}h(x) + \cdots + \alpha_0 L_f h(x) \end{aligned}$$

Hence, on $\sigma(x) = 0$ we have, as it is known to be true in general terms,

$$u_{eq}(x) = -\left.\frac{L_f\sigma(x)}{L_g\sigma(x)}\right|_{\sigma=0} \tag{2.162}$$

For switched systems, we may readily establish the switching policy for local sliding surface reachability. Consider the product $\sigma\dot{\sigma}$,

$$\sigma(x)\dot{\sigma}(x) = \sigma(x)L_f^r h(x) + \sigma(x)(L_g L_f^{r-1}h(x))u + \sigma(x)\left[\sum_{i=0}^{r-2}\alpha_i L_f^{i+1}h(x)\right]$$

The switching policy

$$u = \begin{cases} 1 \text{ for } & \sigma(x)(L_g L_f^{r-1}h(x)) < 0 \\ 0 \text{ for } & \sigma(x)(L_g L_f^{r-1}h(x)) > 0 \end{cases} \tag{2.163}$$

represents the feasible control actions geared towards obtaining a negative value for the product $\sigma(x)\dot{\sigma}(x)$, along the trajectories of the state in the immediate vicinity of $\sigma(x) = 0$. Clearly, the sliding motion is, at best, locally valid.

2.17 Some further geometric aspects

Before establishing the ideal sliding dynamics corresponding with the condition $h(x) = 0$, reachable in an asymptotically exponential fashion, we establish some classical formulae following [13]:

Consider the quantity

$$L_f L_g h(x) - L_g L_f h(x) \tag{2.164}$$

commuting the compositions of the operators L_f and L_g. This expression is clearly an expression involving iterations of directional derivatives and can be regarded as a difference of "second order" directional derivative. The interesting fact is that such an expression is a "first order" directional derivative for some vector field arising from $f(x)$ and $g(x)$.

We first write $L_f L_g h$ explicitly as

$$L_f L_g h = \frac{\partial}{\partial x^T}\left[\frac{\partial h}{\partial x^T} g\right] f$$
$$= g^T \frac{\partial^2 h}{\partial x^T \partial x} f + \frac{\partial h}{\partial x^T}\frac{\partial g}{\partial x^T} f$$

and, clearly

$$L_g L_f h = \frac{\partial}{\partial x^T}\left[\frac{\partial h}{\partial x^T} f\right] g$$
$$= f^T \frac{\partial^2 h}{\partial x^T \partial x} g + \frac{\partial h}{\partial x^T}\frac{\partial f}{\partial x^T} g$$

In both expressions the first summands are identical since, being scalar quantities, one is formally obtained from the other by transposing the triple products and also the involved Hessian matrix is clearly symmetric. Hence, only the second summands are involved in the difference:

$$L_f L_g h - L_g L_f h = \frac{\partial h}{\partial x^T}\frac{\partial g}{\partial x^T} f - \frac{\partial h}{\partial x^T}\frac{\partial f}{\partial x^T} g$$
$$= \frac{\partial h}{\partial x^T}\left[\frac{\partial g}{\partial x^T} f - \frac{\partial f}{\partial x^T} g\right]$$

We address the vector:

$$\left[\frac{\partial g}{\partial x^T} f - \frac{\partial f}{\partial x^T} g\right] = [f, g] \tag{2.165}$$

as the **Lie bracket** of f and g and denote it by either $[f, g]$ or by the operator $ad_f g$.

We have then that

$$L_f L_g h - L_g L_f h = \frac{\partial h}{\partial x^T}[f, g] = L_{[f,g]} h = L_{ad_f g} h \tag{2.166}$$

The composite Lie bracket

$$[f, ad_f g] = [f, [f, g]] \tag{2.167}$$

can be written as $ad_f^2 g$. The motivation for this notation comes from,

$$[f, ad_f g] = [f, [f, g]] = ad_f(ad_f g) = ad_f^2 g \tag{2.168}$$

We can recursively establish that

$$[f, ad_f^{j-1} g] = ad_f^j g, \quad j = 1, 2, \ldots \tag{2.169}$$

with $ad_f^0 g = g$.

Note that

$$\begin{aligned} L_{ad_f^2 g}h = L_{[f,ad_f g]}h &= L_f L_{ad_f g}h - L_{ad_f g}L_f h \\ &= L_f \left(L_f L_g h - L_g L_f h\right) \\ &\quad -L_f L_g(L_f h) + L_g L_f(L_f h) \\ &= L_f^2 L_g h - 2L_f L_g L_f h + L_g L_f^2 h \end{aligned}$$

Note that if $L_g h = 0$ locally around x^0 then from the equality

$$L_f L_g h - L_g L_f h = L_{ad_f g}h \tag{2.170}$$

we have that

$$L_{ad_f g}h = -L_g L_f h \tag{2.171}$$

and, therefore, if $L_g L_f h = 0$, then

$$L_{ad_f g}h = 0 \tag{2.172}$$

Similarly, from the fact that $L_g h = 0$ and $L_g L_f h = 0$ it follows from the equality

$$L_{ad_f^2 g}h = L_f^2 L_g h - 2L_f L_g L_f h + L_g L_f^2 h \tag{2.173}$$

that

$$L_{ad_f^2 g}h = L_g L_f^2 h \tag{2.174}$$

and, therefore $L_g L_f^2 h = 0$ implies that $L_{ad_f^2 g}h = 0$.

Thus, the set of conditions $L_g h = L_g L_f h = \cdots = L_g L_f^{r-2}h = 0$ imply that

$$L_g h = L_{ad_f g}h = L_{ad_f^2 g}h = \cdots = L_{ad_f^{r-2} g}h = 0 \tag{2.175}$$

and the fact that $L_g L_f^{r-1}h \neq 0$ implies that

$$L_{ad_f^{r-1} g}h \neq 0 \tag{2.176}$$

The first set of conditions simply says that

$$\frac{\partial h}{\partial x^T}\left[g, ad_f g, ad_f^2 g, \cdots, ad_f^{r-2} g\right] = 0 \tag{2.177}$$

The set of vectors $\{g, ad_f g, \ldots ad_f^{r-2} g\}$ conforms a set of linearly independent vectors. To verify this consider, contrary to what we want to prove that there exists a linear combination of this set of vectors which renders the vector zero as a result.

Let there exist a set of nonzero scalars $\{\alpha_0, \cdots \alpha_{r-2}\}$ such that

$$\alpha_0 g(x) + \alpha_1 ad_f g(x) + \cdots + \alpha_{r-2} ad_f^{r-2} g(x) = 0$$

Take the Lie bracket of this linear combination of vector fields with the vector field $f(x)$ to obtain:

$$\left[f, \alpha_0 g(x) + \cdots + \alpha_{r-2} ad_f^{r-2} g(x)\right] \tag{2.178}$$

to obtain:

$$\alpha_0 ad_f g + \alpha_1 ad_f^2 g + \cdots + \alpha_{r-2} ad_f^{r-1} g = 0 \tag{2.179}$$

Pre-multiplying the entire previous expression by the gradient of $h(x)$, we obtain

$$\alpha_0 L_{ad_f g} h + \cdots + \alpha_{r-3} L_{ad_f^{r-2} g} h + \alpha_{r-2} L_{ad_f^{r-1} g} h = \alpha_{r-2} L_{ad_f^{r-2} g} h = 0$$

Since $L_{ad_f^{r-1} g} h \neq 0$ around x^0, it follows that, necessarily, $\alpha_{r-2} = 0$. The linear combination is then reduced in one summand and we have from the previous expression

$$\alpha_0 ad_f g + \alpha_1 ad_f^2 g + \cdots + \alpha_{r-3} ad_f^{r-2} g = 0 \tag{2.180}$$

Taking the Lie bracket with f and pre-multiplying again by $\partial h/\partial x^T$, we obtain by virtue of the relative degree assumption:

$$\alpha_{r-3} L_{ad_f^{r-1} g} h = 0 \tag{2.181}$$

from where it follows that $\alpha_{r-3} = 0$. Continuing in this fashion we conclude that all the constants are zero. This is a contradiction and the only linear combination that renders the sum identically zero is the trivial linear combination.

The gradient vector of h simultaneously annihilates the set of independent vector fields: $\{g,\ ad_f g, \ldots, ad_f^{r-2} g\}$. It is easy to see that this gradient also annihilates the Lie bracket of any two vector fields in this collection. Indeed, consider $[ad_f^i g, ad_f^j h]$ for $0 \leq i, j \leq r-2$. We have

$$\begin{aligned} L_{[ad_f^i g, ad_f^j g]} h &= L_{ad_f^i g}(L_{ad_f^j g} h) - L_{ad_f^j g}(L_{ad_f^i g} h) \\ &= L_{ad_f^i g} 0 - L_{ad_f^j g} 0 = 0 \end{aligned}$$

i.e.

$$\frac{\partial h}{\partial x^T}\left[ad_f^i g, ad_f^j g\right] = 0 \tag{2.182}$$

The collection of vectors

$$\mathcal{R}_{r-2} = \{g, ad_f g, ad_f^2 g, \ldots, ad_f^{r-3} g, ad_f^{r-2} g\} \tag{2.183}$$

conforms an *integrable distribution* whose integral is the scalar function h. The collection is also said to be an *involutive distribution*.

The vector $ad_f^{r-1}g$ has a nonzero projection along the gradient vector $dh(x)$, i.e.,

$$L_{ad_f^{r-1}g}h = \left\langle dh, ad_f^{r-1}g \right\rangle = b(x) \neq 0 \tag{2.184}$$

Since the co-vector dh (or the gradient $\partial h/\partial x^T$) annihilates all the vectors in the distribution $\mathcal{R}_{r-2}$, then the vector $ad_f^{r-1}g$ cannot be in the span of this collection of vectors. In other words, $ad_f^{r-1}g$ is a locally linearly independent of the rest of vector fields in the distribution $\mathcal{R}_{r-2}$ around the point x^0

The set of vectors

$$\{g, ad_f g, ad_f^2 g, \cdots, ad_f^{r-2}g, ad_f^{r-1}g\} \tag{2.185}$$

is a set of linearly independent vectors locally around x^0. Arranged as an $n \times r$ matrix the set of column vectors constitutes a locally rank r matrix,

$$\left[g, ad_f g, ad_f^2 g, \cdots, ad_f^{r-1}g\right] \tag{2.186}$$

We may conclude this part by saying that the system (f, g, h) has relative degree r if and only if the following two conditions are satisfied:

1. $\{g, ad_f g, \cdots, ad_f^{r-2}g\}$ is an involutive distribution
2. $\{g, ad_f g, \cdots, ad_f^{r-2}g, ad_f^{r-1}g\}$ constitutes a set of linearly independent vectors.

Using this result we examine the rank of the matrix product

$$\begin{bmatrix} dh \\ dL_f h \\ \vdots \\ dL_f^{r-1}h \end{bmatrix} [g, ad_f g, \cdots, ad_f^{r-1}g] \tag{2.187}$$

We have, using the definition of $b(x)$ previously introduced:

$$\begin{bmatrix} dh \\ dL_f h \\ \vdots \\ dL_f^{r-1}h \end{bmatrix} [g, ad_f g, \cdots, ad_f^{r-1}g] = \begin{bmatrix} 0 & 0 \cdots & 0 & b(x) \\ 0 & 0 \cdots & b(x) & \star \\ \vdots & \vdots \ddots & \vdots & \vdots \\ b(x) & \star \cdots & \star & \star \end{bmatrix} \tag{2.188}$$

Since the rank of the product is r, due to its triangular structure, and nonzero character of the anti-diagonal elements, it follows that

$$\text{rank} \begin{bmatrix} dh \\ dL_f h \\ \vdots \\ dL_f^{r-1}h \end{bmatrix} = r \tag{2.189}$$

This implies, in turn, that the set of functions $h, L_f h, \cdots, L_f^{r-1} h$ are all functionally independent and their row gradients conform a maximum rank, r, matrix. The map

$$\Phi_r(x) = \begin{bmatrix} h(x) \\ \vdots \\ L_f^{r-1} h(x) \end{bmatrix} = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_r(x) \end{bmatrix} \tag{2.190}$$

is, therefore, full rank r.

Let $\phi_{r+1}(x)$, $\phi_{r+2}(x)$, $\cdots$, $\phi_n(x)$ be a set of locally functionally independent scalar functions (their gradient row vectors are all linearly independent around x^0) so that the set of co-vector fields $\{d\phi_1, \cdots, d\phi_r, d\phi_{r+1}, \cdots, d\phi_n\}$ are locally linearly independent.

Under this construction, the map

$$z = \Phi(x) = \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_r(x) \\ \phi_{r+1}(x) \\ \vdots \\ \phi_n(x) \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{r-1} h(x) \\ \phi_{r+1}(x) \\ \vdots \\ \phi_n(x) \end{bmatrix} \tag{2.191}$$

is a locally full rank, n, map, which qualifies as a locally invertible state coordinate transformation, $z = \Phi(x)$. Hence, uniquely, and locally around x^0, $x = \Phi^{-1}(z)$.

The transformed state coordinates $\xi = [z_1, \cdots, z_r]^T = [\xi_1, \cdots, \xi_r]$, satisfy:

$$\begin{aligned} \dot{z}_1 &= L_f h(x) = z_2 \\ \dot{z}_2 &= L_f^2 h(x) = z_3 \\ &\vdots \\ \dot{z}_{r-1} &= L_f^{r-1} h(x) = z_r \\ \dot{z}_r &= L_f^r h(\Phi^{-1}(z)) + [L_g L_f^{r-1} h(\Phi(z))] u \end{aligned}$$

while the remaining coordinates $\eta = [z_{r+1}, \cdots, z_n]^T$ do not exhibit any particularly special structure in their time derivative, except for the fact of, possibly, being linear in u. i.e.

$$\dot{\eta} = q(z) + p(z) u = q(\xi, \eta) + p(\xi, \eta) u \tag{2.192}$$

We, therefore, have that the transformed system has the following structure in ξ and η coordinates,

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_{r-1} &= \xi_r \\
\dot{\xi}_r &= L_f^r h(\Phi^{-1}(\xi,\eta)) + [L_g L_f^{r-1} h(\Phi(\xi,\eta))]u \\
\dot{\eta} &= q(\xi,\eta) + p(\xi,\eta)u \\
y &= \xi_1
\end{aligned}$$

We can also choose the coordinate functions $\eta = \{\phi_{r+1}(x), \cdots, \phi_n(x)\}$ in such a manner that, locally around x^0, $L_g\phi_j(x) = 0$ for $j = r+1, \cdots, n$, i.e., such that any control input influence is blocked from the last $n-r$ transformed coordinates. In such a case, the transformed system has the simpler structure

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_{r-1} &= \xi_r \\
\dot{\xi}_r &= L_f^r h(\Phi^{-1}(\xi,\eta)) + [L_g L_f^{r-1} h(\Phi^{-1}(\xi,\eta))]u \\
\dot{\eta} &= q(\xi,\eta) \\
y &= \xi_1
\end{aligned}$$

which is addressed as *Isidori's canonical form* (see [13]).

Let y and its first r time derivatives be identically zero on an open interval of time. The dynamics corresponding to such a (possibly forced) condition is evidently ruled by the nonlinear equation:

$$\dot{\eta} = q(0,\eta) = \theta(\eta) \tag{2.193}$$

Let $\eta = \overline{\eta}$ be an isolated equilibrium point for the above uncontrolled dynamics. We say that the system output y is locally minimum phase at the given equilibrium point, $\overline{\eta}$, if the trajectories of the above differential equation locally converge towards the equilibrium point. In other words, locally around the given equilibrium point, the linearized dynamics with $\eta_\delta = \eta - \overline{\eta}$:

$$\dot{\eta}_\delta = \left.\frac{\partial \theta}{\partial \eta^T}\right|_{\eta=\overline{\eta}} \eta_\delta \tag{2.194}$$

has all its eigenvalues located in the left half of the complex plane. If, on the other hand, the given local equilibrium point is unstable, then the system output is said to be locally non-minimum phase at such an equilibrium point.

In matrix notation the system exhibits the following structure:

$$\dot{\xi} = A\xi + \psi(\xi,\eta) + \rho(\xi,\eta)u, \;\; \dot{\eta} = q(\xi,\eta), \;\; y = c^T\xi \tag{2.195}$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \psi(z) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ L_f^r h(\Phi^{-1}(z)) \end{bmatrix}, \quad \rho(z) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ L_g L_f^{r-1} h(\Phi(z)) \end{bmatrix}, \tag{2.196}$$

$$c^T = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \tag{2.197}$$

As before, we let $a(z) = L_f^r h(\Phi^{-1}(z))$ and $b(z) = L_g L_f^{r-1} h(\Phi(z))$

The previously proposed sliding surface coordinate can now be expressed as a transformed linear surface in terms of the coordinates ξ,

$$\sigma(\xi) = \xi_r + \alpha_{r-2}\xi_{r-1} + \cdots + \alpha_0\xi_1 \tag{2.198}$$

On $\sigma = 0$, the transformed closed loop system is linear and governed by:

$$\frac{d}{dt}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{r-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_0 & -\alpha_1 & -\alpha_3 & \cdots & -\alpha_{r-2} \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{r-1} \end{bmatrix} \tag{2.199}$$

and, on $\sigma = 0$, the zero dynamics corresponding to the asymptotic exponential convergence to zero of the output $y = \xi_1$ and of its first r time derivatives, is represented by the following nonlinear dynamics.

$$\dot{\eta} = q(0,\eta) = \theta(\eta) \tag{2.200}$$

It is clear that, in general, it may not be entirely trivial to assess the local stability of the zero dynamics associated with the $n-r$ dimensional system characterized by the reduced vector field $\theta(\eta)$ in transformed coordinates. The fundamental limitation is that, as it was previously seen in some DC to DC converter examples, intuitively natural output errors may indeed lead to closed loop system instability due to the non-minimum phase character of the chosen output. There are, however, instances where the instability of the nonzero dynamics can be easily cut-off.

It should also be clear that for switched systems where $u \in \{0,1\}$, or any other finite set, the creation of a sliding motion on a given sliding surface may not be possible at all, from arbitrary initial conditions.

2.18 A soft landing example

Consider the model of a thrusted vehicle attempting a soft landing on a planet characterized by gravity acceleration g. The dynamics of the vehicle indeed corresponds to that of a switched system when regarded as controlled by a motor which can be turned on and off.

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= g - \left(\frac{\alpha W}{x_3}\right) u\\ \dot{x}_3 &= -Wu\end{aligned}$$

The coordinate x_1 represents the height to the ground, measured negatively from the zero level at the ground. x_2 is the corresponding velocity which is positive when the vehicle goes downwards and x_3 is the total mass of the spacecraft including the mass fuel that is spent when $u = 1$, and saved when $u = 0$. The thrust spends W Kg$/s$ of fuel mass when the motor is "on."

It is desired to softly approach a small height $H < 0$, typically 1 m, above the ground and hover over it for a while until a safe landing is attained by switching the motor off, and letting the spacecraft to land under a free fall condition from the small height. Clearly, an output error naturally associated with the system operation is represented by $y = x - H$.

Consider the following invertible state coordinate transformation

$$\begin{array}{ll} z_1 = x_1 & x_1 = z_1\\ z_2 = x_2 & x_2 = z_2\\ z_3 = x_2 - \alpha \ln x_3 & x_3 = \exp\left(\frac{z_2 - z_3}{\alpha}\right)\end{array} \tag{2.201}$$

The Isidori's canonical form of the system is just

$$\begin{aligned}\dot{z}_1 &= z_2\\ \dot{z}_2 &= g - W\alpha \exp\left(\frac{z_3 - z_2}{\alpha}\right) u\\ \dot{z}_3 &= g\end{aligned}$$

Clearly, the output $y = z_1 - H$ is a non-minimum phase output, since the variable z_3 is unstable and grows without limit. This instability causes the steady state mass behavior to evolve as

$$x_3 = \exp(-z_3/\alpha) \tag{2.202}$$

i.e., the sustained hovering depletes the fuel mass in a finite time.

Naturally, the motor may be shut off much before the residual fuel mass, needed for safely returning to the mother ship, is depleted. The mass remains constant from thereon and the short free fall landing is executed.

In this example, a natural sliding surface coordinate is clearly provided by

$$\sigma = z_2 + \lambda(z_1 - H), \quad \lambda > 0 \tag{2.203}$$

which causes a desired exponentially stable closed loop dynamics:

$$\dot{z}_1 = -\lambda(z_1 - H), \quad z_1 \to H \tag{2.204}$$

The time derivative of the sliding surface coordinate function σ is just

$$\dot{\sigma} = g - \alpha W \exp\left(\frac{z_3 - z_2}{\alpha}\right) u + \lambda z_2 \tag{2.205}$$

and the switching policy is clearly given by

$$u = \frac{1}{2}(1 + \text{sign }(\sigma)) \tag{2.206}$$

The equivalent control is, in this case, obtained as

$$u_{eq}(z) = \frac{\left(g - \lambda^2(z_1 - H)\right)}{\alpha W} \exp\left(-\frac{\lambda(z_1 - H) + z_3}{\alpha}\right) \tag{2.207}$$

As the hovering altitude is reached, $z_1 \to H$ while z_3 is growing. The equivalent control tends towards the value:

$$u_{eq} \to \left(\frac{g}{\alpha W}\right) e^{-\frac{z_3}{\alpha}} \tag{2.208}$$

The equivalent control ultimately tends to the constant value $g/(\alpha W)$. The non-minimum phase behavior is stopped by shutting off the main thruster of the landing vehicle.

Exercise 2.20. The following are the nonlinear equations of an orbiting satellite:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_1 x_4^2 - \frac{k}{x_1^2} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\frac{2x_2x_4}{x_1} + \left(\frac{1}{Mx_1^2}\right) u
\end{aligned}$$

where x_1 is the distance r from the center of the earth, x_2 is the corresponding radial velocity, $x_3 = \alpha$ is the angular displacement with respect to an arbitrary fixed direction. x_4 is the corresponding angular velocity. M is the mass of the satellite and k is the gravitational constant. The control u is the tangential thrust force acting on the satellite. Design a controller that stabilizes the system around an orbit characterized by $x_1 = R$, $x_3 = \omega t$, $x_4 = \omega$ with $u \in \{-W, 0, +W\}$. Assume $\omega^2 = k/R^3$

2.19 The exactly linearizable case

Clearly, when the relative degree of the system (f, g, h), at x^0, equals the dimension, n, of the state, we have

$$L_g h(x) = L_g L_f h(x) = L_g L_f^{n-2} h(x) = 0,\ x \in \mathcal{N}(x^0)$$

and

$$L_g L_f^{n-1} h(x^0) \neq 0 \tag{2.209}$$

In this case, the equivalent conditions yield (see Isidori [13])

1. The distribution $\{g, ad_f g, \cdots, ad_f^{n-2} g\}$ is involutive:
2. The set of vector fields:

$$\{g, ad_f g, \cdots, ad_f^{n-2} g, ad_f^{n-1} g\}$$

is linearly independent.

The full rank (invertible) state coordinate transformation map that exhibits the integration structure between the input and the output of the system is just given by

$$z = \xi = \Phi(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix} \tag{2.210}$$

and the transformed system is now given by

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_{n-1} &= \xi_n \\ \dot{\xi}_n &= L_f^n h(\Phi^{-1}(\xi)) + L_g L_f^{n-1} h(\Phi^{-1}(\xi))\, u \\ y &= \xi_1 \end{aligned}$$

Suppose the output $y = h(x) = \xi_1$ represents a variable (or a stabilization error) that needs to be zeroed. Since, in this case, the output y has maximal relative degree, there is no need for concern about the corresponding zero dynamics because, simply, it does not exist.

In this case, we call y a *linearizing output* or simply a *flat output* (See Chapter 6).

A natural sliding surface may be designed on the basis of the transformed system:

$$\sigma(\xi) = \xi_n + \alpha_{n-2}\xi_{n-1} + \cdots + \alpha_0 \xi_1 \tag{2.211}$$

yielding a closed loop system governed by

$$y^{(n-1)} + \alpha_{n-2}y^{(n-2)} + \cdots + \alpha_0 y = 0 \tag{2.212}$$

Back in original coordinates, we are synthesizing a nonlinear sliding surface given by:

$$\sigma(x) = L_f^{n-1}h(x) + \alpha_{n-2}L_f^{n-2}h(x) + \cdots + \alpha_0 h(x) \tag{2.213}$$

The switching policy may be determined from the expression of $\sigma\dot{\sigma}$ as follows:

$$\sigma(x)\dot{\sigma}(x) = \sigma(x)\left[L_f^n h(x) + L_g L_f^{n-1}h(x)u + \sum_{i=0}^{n-2}\alpha_i L_f^{i+1}h(x)\right]$$

We must choose

$$u = \frac{1}{2}\left(1 - \text{sign}\left(\sigma L_g L_f^{n-1}h(x)\right)\right) \tag{2.214}$$

On $\sigma = 0$, the equivalent control is, upon substitution of $L_f^{n-1}h(x)$ by the expression:

$$L_f^{n-1}h(x) = -\alpha_{n-2}L_f^{n-2}h(x) - \cdots - \alpha_0 h(x) \tag{2.215}$$

given by

$$u_{eq}(x) = -\frac{L_f^n h(x) + \sum_{j=0}^{n-2}(\alpha_{i-1} - \alpha_i\alpha_{n-2})L_f^i h(x)}{L_g L_f^{n-1}h(x)} \tag{2.216}$$

with $a_{-1} = 0$.

Exercise 2.21. Consider the following normalized controlled version of the famous Chua's circuit

$$\begin{aligned}
\dot{x}_1 &= p(-x_1 + x_2 - f(x_1)) \\
\dot{x}_2 &= x_1 - x_2 - x_3 \\
\dot{x}_3 &= -qx_2 + u \\
y &= x_1
\end{aligned}$$

where p and q are known parameters and $f(x_1)$ is in fact a piecewise linear function depicting the fact that the circuit locally exhibits a negative resistance. We may, however, use a smooth polynomial function for $f(x_1)$ such as, $f(x_1) = x_1(1 - x_1^2)$.

Control the system from $u \in \{-A, 0, A\}$ so as to exhibit a sustained sinusoidal output signal with constant amplitude and fixed frequency. Determine a convenient value for A

3

Delta-Sigma Modulation

3.1 Introduction

Delta modulation was actively developed for communication purposes in the sixties and seventies. It has also had applications in consumer audio equipment, industrial electronics devices, and precision measurement devices (see Jarman [14]). It consisted in an underlying analog-to-fixed sampling frequency pulse-width-modulation encoding transformation of the signal to be transmitted. Delta modulation was actually used for voice encoding and transmission in the first manned space flights. Its implications in analog to digital conversion schemes were recognized early and a rapid development followed in terms of suitable electronic circuits in countless applications of early computer oriented control of processes. A complete classical account of Delta-modulators, and their simplest modification: Delta-Sigma modulators, extensively used in analog signal encoding, which never benefited from the theoretical basis of sliding mode control, is found in the classical book by Steele [29] and in the excellent book by Norsworthy *et al.* [17]. A rather complete and informative survey on Delta-Sigma modulation, rich in tutorial material and perspectives, may be found in Reiss [18].

We may address this variant of Delta modulation as the *digital idealization* which has long served a magnificent purpose in Analog to Digital conversion circuits in computer based signal processing and computer based control of dynamic systems. However, a second idealization, *the analog idealization* is still possible and it naturally contains interesting interpretations and a reformulation of sliding mode control in a large class of switched systems. It is our purpose to explore such idealization in the realm of devising feedback controllers for switched controlled systems, analyze their implications in their underlying sliding motions, and explore the possibilities of using such modulators as suitable translators of average, continuous, controller designs to the, otherwise, restricted possibilities of switch controlled dynamics.

H. Sira-Ramírez, *Sliding Mode Control,* Control Engineering,
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In this second variant of Delta modulation, and, specifically, in its associated Delta-Sigma modulation scheme, we first dispense of the finite, periodic, sampling process, typical of the digital idealization. We allow this sampling to approach an infinite sampling rate idealization i.e., continuous signals will stay continuous. Secondly, the switch associated with the Delta modulator is idealized to be an infinitely fast switch. These idealizations allow us to quickly envision sliding regimes on a suitably extended state space of the system where sliding motions occur under relatively mild assumptions. The underlying closed loop dynamics is then interpreted as an average zero dynamics corresponding with the sliding motion created on the one-dimensional state extension. In essence, all continuous feedback controller design techniques become readily available, and easily implementable, for the control of switched systems. The theoretical implications demonstrate that the Delta-Sigma modulation approach is largely equivalent to the recently introduced integral sliding mode control approach, but far simpler and more natural.

A warning: Delta-Sigma modulation is also called Sigma Delta modulation. Here I quote a rather clever observation by U. Beis in his internet article [1]:

> Delta Sigma Converters or Sigma Delta Converters? Mankind does not seem to agree on one notation. Both notations are used equally often when you search via Google. I decided to stay with that guy who told he is living in the Mississippi Delta, so deltas mean something to him - and for him only the Sigma River may have a Sigma Delta... good point. Later I found out that the original name "Delta Sigma" was coined by the inventors Inose and Yasuda and "Sigma Delta" is actually not correct. I was lucky...

3.2 Delta Modulation

An idealized Delta modulator is a system which accepts, as an input, an analog signal, here denoted by $\xi(t)$, which needs to be, somehow, encoded and transmitted to a remote decoder in the form of an infinite frequency pulsed signal q. The pulses are all of the same amplitude. The pulsed signal q is viewed as the output of the local encoding system. The encoding process consists in a *quantization* device that assigns, depending on the sign, a certain quantization level with values on the discrete set $\{-W, W\}$ to the error signal e. The error signal is conformed by a *comparator* establishing the difference between the incoming analog signal $\xi(t)$ and the integral of the produced encoding signal, q, here denoted by x. The integrator is addressed as the local decoder. Delta modulation is based on the behavior of the circuit shown in Fig. 3.1.

The fundamental idea is to have a tracking of the incoming signal $\xi(t)$ by the feedback signal x, also called the locally decoded signal. Once the error is driven to zero the signal q exhibits an active (ideally infinite frequency) switching behavior which may be transmitted to a remote decoder consisting, much as the local decoder, on a time integration process. Thus, the obtained remotely decoded signal ideally coincides with the local feedback signal x.

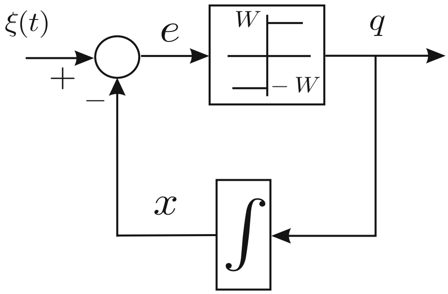

Fig. 3.1. Delta-modulator

The equations describing the Delta modulator circuit are given by

$$\begin{aligned} e &= \xi(t) - x \\ \dot{x} &= q \\ q &= W \text{sign} e \end{aligned} \tag{3.1}$$

While the output of the system is the signal q, it resembles the behavior of a control input undergoing sliding mode behavior. The error e may be identified with a sliding surface coordinate function, $\sigma = e$, which is to be driven to zero in finite time by the negative feedback of the integrated quantized output. The output of the quantization process takes values on the discrete set: $\{W, -W\}$. It may be, undoubtedly, related a switch position function, u, taking values in $\{0, 1\}$ by setting $u = \frac{1}{2W}[q + W]$ i.e., $q = W(2u - 1)$. All the elements of a sliding mode controlled system are therefore present in the Delta modulator dynamics. Indeed, rewrite the equations 3.1 in the more compact manner,

$$\dot{\sigma} = \dot{\xi}(t) - \dot{x} = \dot{\xi}(t) - q = \dot{\xi}(t) - W \text{sign}\sigma \tag{3.2}$$

If $\sigma > 0$, the controlled evolution of σ satisfies $\dot{\sigma} = \dot{\xi}(t) - W$. To have σ approach the desired objective, $\sigma = 0$, then one should have that $\dot{\sigma} < 0$. This translates into the condition $\dot{\xi}(t) < W$ for the incoming signal $\xi(t)$. On the other hand, if $\sigma < 0$, then $\dot{\sigma} = \dot{\xi} + W$. The growth of σ implies that one should have $-W < \dot{\xi}(t)$. The existence of a sliding motion on $\sigma = 0$ is summarized then in the following limitation for the time derivative of $\xi(t)$:

$$-W < \dot{\xi}(t) < W \tag{3.3}$$

Under ideal sliding motions ($\sigma = 0$, $\dot{\sigma} = 0$), the average value of the signal q (called for consistency q_{eq}) satisfies: $q_{eq} = \dot{\xi}$. On the average, the

Delta modulator produces the time derivative of the incoming signal. Delta modulation constitutes a time differentiator of analog inputs, with a binary coded differentiation signal at the output.

The amplitude limitation condition, (3.3), on $\dot{\xi}(t)$ induces an equivalent condition to be imposed on the equivalent control, u_{eq}, as obtained from the invariance conditions: $\sigma = 0$, $\dot{\sigma} = 0$, in the light of (3.2). Indeed,

$$\dot{\xi}(t) - W(2u_{eq} - 1) = 0, \quad u_{eq} = \frac{1}{2W}\left[W + \dot{\xi}(t)\right] \tag{3.4}$$

The time derivative of the incoming signal $\xi(t)$ determines the equivalent control and it should be limited to the open interval $(-W, W)$ implying that $0 < u_{eq} < 1$.

Naturally, the condition $\sigma\dot{\sigma} < 0$ is satisfied for arbitrary nonzero values of σ provided $-W < \dot{\xi}(t) < W$ when u is chosen as $u = \frac{1}{2W}(q + W\text{sign}\sigma)$. Indeed,

$$\begin{aligned} \sigma\dot{\sigma} &= \sigma(\dot{\xi}(t) - W\text{sign}\sigma) = \sigma\dot{\xi}(t) - W|\sigma| \\ &= -W|\sigma|(1 - \frac{\dot{\xi}(t)}{W}\text{sign}\sigma) < 0 \end{aligned} \tag{3.5}$$

Exercise 3.1. The normalized equations for the Delta modulator entitle scaling the incoming signal $\xi(t)$ by the quantization level W and letting the quantized signal q take values on the set $\{-1, 1\}$. The remote decoding process simply multiplies the pulsed signal $q \in \{-1, 1\}$ by the quantization level W. Work out the details.

Example 3.2. Consider a second order integration system $\ddot{y} = u$ which is to be stabilized to the equilibrium point: $y = Y$, $\dot{y} = 0$. A proportional derivative (PD) controller of the form: $u = -2\zeta\omega_n\dot{y} - \omega_n^2(y - Y)$ with $\zeta, \omega_n > 0$, may be synthesized with the help of a Delta modulator, acting as a differentiator for the output signal y. Thus a Delta modulator based PD controller is given by

$$u = -2\zeta\omega_n q - \omega_n^2(y - Y), \quad q = W\text{sign}\sigma, \ \sigma = y - x, \ \dot{x} = q \tag{3.6}$$

Clearly, this scheme is feasible thanks to the low pass filtering features of the given system. The above controller yields a discontinuous control input u. A customary continuous substitution for the sign function is provided by the following high gain continuous saturation function:

$$q = W\frac{\sigma}{|\sigma| + \epsilon}, \tag{3.7}$$

with ϵ being a very small positive constant ($\epsilon << 1$).

3.3 Second order Delta modulation

Second order Delta modulation is represented in the block diagram of Figure 3.2. The analysis of this circuit, under ideal sliding motions, allows for the derivation of its properties and the possibilities of using it as a differentiator.

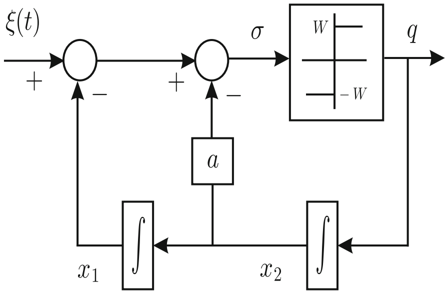

Fig. 3.2. Second order Delta-modulator

The equations describing the second order Delta modulator, with $a > 0$, are given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= q \\ q &= W \text{sign} \sigma \\ \sigma &= \xi(t) - (x_1 + a x_2) \end{aligned} \tag{3.8}$$

The output signal, q, takes values on the discrete set, $\{-W, +W\}$, and it plays an analogous role to a switched control input u in sliding mode control. One may view the modulator circuit as one in which σ needs to reach the zero value with $q = u \in \{-W, W\}$ being the available switched control input signal, helping to accomplish this invariance condition. Notice that

$$\dot{\sigma} = \dot{\xi}(t) - x_2 - au \tag{3.9}$$

Hence,

$$\dot{\sigma} = (\dot{\xi}(t) - x_2) - aW \text{sign} \sigma \tag{3.10}$$

A sliding motion exists on $\sigma = 0$ provided

$$-aW < \dot{\xi}(t) - x_2 < aW \tag{3.11}$$

The equivalent control, obtained from: $\dot{\sigma} = 0$ on $\sigma = 0$, yields

$$u_{eq} = \frac{1}{a}(\dot{\xi}(t) - x_2) \tag{3.12}$$

and its time derivative yields

$$\dot{u}_{eq} = \frac{1}{a}(\ddot{\xi}(t) - u_{eq}) = -\frac{1}{a}(u_{eq} - \ddot{\xi}(t)) \tag{3.13}$$

The equivalent control (or, more properly, equivalent output) is a low pass filtering of the second order time derivative of $\xi(t)$.

Under the ideal sliding condition, $\sigma = 0$ one has that $x_2 = \frac{1}{a}(\xi(t) - x_1)$ and, hence, $\dot{x}_1 = -\frac{1}{a}(x_1 - \xi(t))$. In other words, x_1 is the low pass filtering of the input signal $\xi(t)$. Since x_2 is the time derivative of x_1, it follows that x_2 is the low pass filtering of the signal $\dot{\xi}(t)$.

Let $\mathcal{L}[\psi(t)]$ and $\hat{\psi}(s)$, both, denote the Laplace transform of the signal $\psi(t)$. In the frequency domain, one obtains the following set of relations

$$x_1(s) = \frac{1/a}{s + 1/a}\mathcal{L}[\xi(t)], \quad x_2(s) = \frac{1/a}{s + 1/a}\mathcal{L}[\dot{\xi}(t)], \quad u_{eq}(s) = \frac{1/a}{s + 1/a}\mathcal{L}[\ddot{\xi}(t)] \tag{3.14}$$

All these are unit gain low pass filters with cut-off frequency given by $\omega_c = 1/a$. The choice of the parameter a depends on the frequency content of the signal $\xi(t)$ when the modulator is to be used as a differentiator. The common low pass filter transfer function is taken to represent the characterization of the second order modulator in the frequency domain.

From equation (3.8) and under ideal sliding conditions the input signal $\xi(t)$ can be obtained as a linear combination of the modulator variables x_1 and x_2. We address this linear combination as an estimate of $\xi(t)$ (a redundant one) and denote it by $\hat{\xi}(t)$. Similarly, the invariance conditions: $\dot{\sigma} = 0$ and $\sigma = 0$ yield $\dot{\xi}(t) = x_2 + au_{eq}$. One further differentiation leads to $\ddot{\xi}(t) = u_{eq} + a\dot{u}_{eq}$. These are all estimates of the consecutive time derivatives of the input signal. Summarizing

$$\hat{\xi}(t) = (x_1 + ax_2), \quad \hat{\dot{\xi}}(t) = x_2 + au_{eq}, \quad \hat{\ddot{\xi}}(t) = u_{eq} + a\dot{u}_{eq} \tag{3.15}$$

The equivalent output signal $q_{eq} = u_{eq}$ needs to be estimated from the switched output signal q. An ideal low pass filtering of the infinite frequency switched signal q would exactly render the equivalent signal q_{eq}. A realizable approximation is thus necessary. Suppose we want to obtain q_{eq} from a low pass filter that coincides with the second order modulator fundamental transfer function. Let, in equation (3.14) an estimate of $u_{eq}(s)$ be approximated by

$$u_{eq}(s) = \frac{1/a}{s + 1/a}q(s) \tag{3.16}$$

and take this relation as valid, i.e.,

$$u_{eq}(s) = \frac{1/a}{s + 1/a}q(s) = \frac{1/a}{s + 1/a}\mathcal{L}[\ddot{\xi}(t)] \tag{3.17}$$

The low pass filtering of the second order time derivative of the input to the modulator coincides with the same low pass filtering of the switched output signal. Based on this engineering justification, we advocate, in general, for the estimation of q_{eq}, the use of low pass filters which are coincident with the fundamental transfer function characterizing the second order modulator.

The use of this differentiator has some interest in the average control of second order systems with available output $y = \xi(t)$ and no measurement of the velocity signal $\dot{y}$.

Example 3.3. Suppose it is desired to regulate the second order linear system,

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = u, \quad y = z_1 \tag{3.18}$$

so that the output y of the plant stabilizes at the constant value $Y = 1$.

Using the second order delta modulator, in equation (3.8), as a state estimator, with the input signal to the modulator, $\xi(t)$, provided by the plant system output $y(t)$, one synthesizes the needed output time derivative term $\dot{y}$ in the linear classical feedback control law:

$$u = -2\zeta\omega_n \dot{y} - \omega_n^2 (y - Y), \ \zeta, \omega_n > 0, \tag{3.19}$$

The estimate of the velocity, $\dot{y}$, is obtained with the help of equation (3.15) and of a low pass filter of the second order Delta modulator output switched signal q, as follows:

$$\hat{\dot{y}}(t) = x_2 + a q_{eq}, \quad \dot{q}_{eq} = -\frac{1}{a}(q_{eq} - q) \tag{3.20}$$

A full state feedback control law entitles the use of the measured velocity $z_2 = \dot{y}$ in (3.19).

Figure 3.3 depicts the controlled response under, both, a full feedback control law and a feedback control law of the form 3.19 for different values of the cut-off frequency $\omega_c = 1$ [rad/s], $\omega_c = 10$ [rad/s] and $\omega_c = 100$ [rad/s]. As the bandwidth of the Delta modulator is increased the performance of the controller is found to be better and sufficiently close to the performance under full state feedback. Increased bandwidth makes the estimation scheme sensitive to output noise measurements.

3.4 Higher order Delta modulation

The preceding Delta modulation scheme may be generalized in a rather direct manner, as depicted in Figure 3.4. The system is described by the following set of equations,

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= q = W \operatorname{sign} \sigma \\
\sigma &= \xi(t) - (x_1 + a_1 x_2 + \cdots + a_{n-1} x_n)
\end{aligned} \tag{3.21}$$

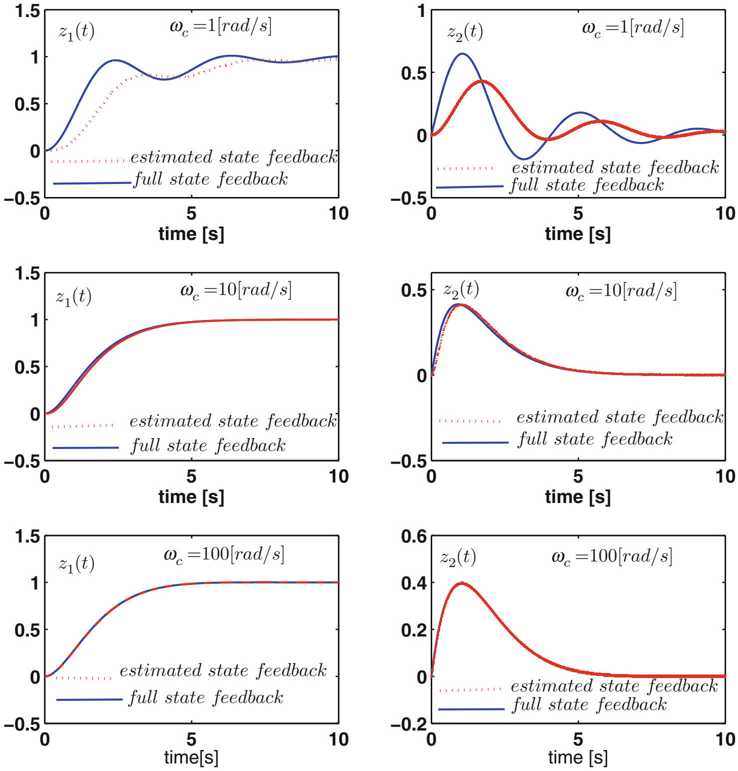

Fig. 3.3. Performance of second order delta modulation estimator in a feedback loop for several cut-off frequencies.

The time derivative of σ is given by

$$\dot{\sigma} = \dot{\xi} - (x_2 + a_1 x_3 + \cdots + a_{n-2} x_n + a_{n-1} W \text{sign}(\sigma)) \tag{3.22}$$

A sliding regime exists on $\sigma = 0$ if and only if

$$-a_{n-1} W < \dot{\xi} - (x_2 + a_1 x_3 + \cdots + a_{n-2} x_n) < a_{n-1} W \tag{3.23}$$

The equivalent output $q_{eq}(t)$ is just

$$q_{eq}(t) = \frac{1}{a_{n-1}} \left(\dot{\xi} - (x_2 + a_1 x_3 + \cdots + a_{n-2} x_n) \right) \tag{3.24}$$

Let $\mathcal{L}[\xi^{(j)}(t)]$ denote the Laplace transform of the j-th time derivative of $\xi(t)$. Taking $n-1$ time derivatives in (3.24), it follows that all the Delta modulator states satisfy the following relation in the frequency domain

$$\hat{q}_{eq}(s) = \frac{\mathcal{L}[\xi^{(n)}(t)]/a_{n-1}}{s^{n-1} + \left(\frac{a_{n-2}}{a_{n-1}}\right) s^{n-2} + \cdots + \left(\frac{a_1}{a_{n-1}}\right) s + \left(\frac{1}{a_{n-1}}\right)} \tag{3.25}$$

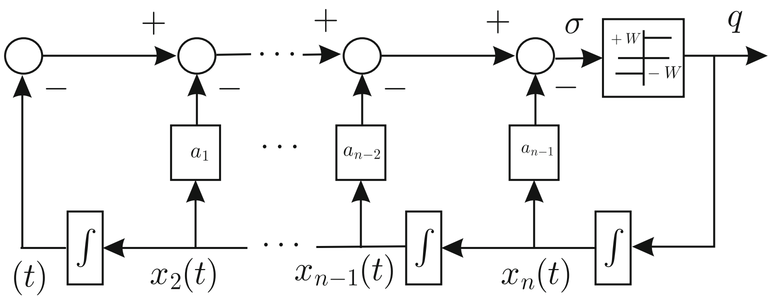

Fig. 3.4. An n-th order Delta modulator.

Using the modulator's state equations and the expression for $q_{eq}(t)$ in (3.24) one establishes the validity of the following expressions:

$$\hat{x}_j(s) = \frac{\mathcal{L}[\xi^{(j-1)}(t)]/a_{n-1}}{s^{n-1} + \left(\frac{a_{n-2}}{a_{n-1}}\right) s^{n-2} + \cdots + \left(\frac{a_1}{a_{n-1}}\right) s + \left(\frac{1}{a_{n-1}}\right)}, \; j = 1, \ldots, n \tag{3.26}$$

The modulator states x_j,$j = 1, 2, \ldots, n$ represent the same, unit gain, low pass filtering operation on the $(j-1)$-th time derivatives of the input signal $\xi(t)$ for $j = 1, 2, \ldots, n$. The ideal average output signal, $q_{eq}(t)$, represents the low pass filtering of the n-th order time derivative of the input signal $\xi(t)$.

Example 3.4. In this example, we propose a third order Delta modulator to generate the first and second order time derivatives of the output of a third order system. These time derivatives will be used as part of the feedback law. The system, or plant, to be considered is a normalized linear compartmental model of a heating system representing

$$\begin{aligned}
\dot{x}_1 &= x_2 - x_1 \\
\dot{x}_2 &= x_1 - 2x_2 + x_3 \\
\dot{x}_3 &= x_2 - 2x_3 + u \\
y &= x_1
\end{aligned} \tag{3.27}$$

The output $y = x_1$ is a privileged output in the sense that all variables may be expressed in terms of y and its time derivatives $\dot{y}$, $\ddot{y}$. Indeed by inspection on the set of equations (3.27), we have

$$x_1 = y, \quad x_2 = \dot{y} + y, \quad x_3 = \ddot{y} + 3\dot{y} + y, \quad u = y^{(3)} + 5\ddot{y} + 6\dot{y} + y \tag{3.28}$$

It is desired to have the output, $y = x_1$, track a desired given rest-to-rest sufficiently smooth temperature profile $y^*(t)$ by means of the continuous control input u. An output trajectory tracking feedback law may be synthesized by generating the first and second order time derivatives of y by means of a, say, third order Delta modulator. From the input-output relation in (3.28), a suitable full state feedback tracking controller, in terms of the tracking error $e_y = y - y^*(t)$, is then given by

$$u = u^*(t) - (5 + k_2)\ddot{e}_y - (6 + k_1)\dot{e}_y - (1 + k_0)e_y \tag{3.29}$$

with $u^*(t) = [y^*(t)]^{(3)} + 5\ddot{y}^*(t) + 6\dot{y}^*(t) + y^*(t)$. This yields a closed loop tracking error dynamics characterized by

$$e_y^{(3)} + k_2\ddot{e}_y + k_1\dot{e}_y + k_0 e_y = 0 \tag{3.30}$$

A classical, stabilizing, choice for the gains $\{k_2, k_1, k_0\}$ is just

$$k_2 = 2\zeta_c\omega_{nc} + p_c, \quad k_1 = \omega_{nc}^2 + 2\zeta_c\omega_{nc}p_c, \quad k_0 = \omega_{nc}^2 p_c$$

with ζ_c and ω_{nc} and p_c to be chosen in accordance with the bandwidth of the modulator and of the low pass filter generating the equivalent output signal q_{eq}.

Consider now the third order Delta modulator with input signal represented by the output y of the plant.

$$\begin{aligned} &\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = q, \quad q = W\text{sign}\sigma \\ &\sigma = y - (x_1 + a_1x_2 + a_2x_3) \end{aligned} \tag{3.31}$$

Under ideal sliding mode conditions, the invariance conditions $\sigma = \dot{\sigma} = 0$ translate into

$$y = x_1 + ax_2 + a_2x_3, \quad \dot{y} = x_2 + a_1x_3 + a_2q_{eq}, \quad \ddot{y} = x_3 + a_1q_{eq} + a_2\dot{q}_{eq} \tag{3.32}$$

where the average signal q_{eq} and its time derivative $\dot{q}_{eq}$ will be generated via the dynamics of the low pass filter built using the characterizing fundamental modulator transfer function under ideal sliding dynamics:

$$G(s) = \frac{\left(\frac{1}{a_2}\right)}{s^2 + \left(\frac{a_1}{a_2}\right)s + \left(\frac{1}{a_2}\right)}$$

We have then the following state space representation of the unit gain low pass filter dynamics

$$\dot{q}_{1eq} = q_{2eq}, \quad \dot{q}_{2eq} = -2\zeta_f\omega_{nf}q_{2eq} - \omega_{nf}^2(q_{1eq} - q), \quad q_{eq} = q_{1eq}. \tag{3.33}$$

i.e., $a_2 = 1/\omega_{nf}^2$ and $a_1 = 2\zeta_f/\omega_{nf}$.

The expressions in (3.32) will serve as estimators of the phase variables y, $\dot{y}$, and $\ddot{y}$. The first one being redundantly generated. So, we have

$$\hat{y} = x_1 + ax_2 + a_2x_3, \quad \hat{\dot{y}} = x_2 + a_1x_3 + a_2q_{1eq}, \quad \hat{\ddot{y}} = x_3 + a_1q_{1eq} + a_2q_{2eq} \tag{3.34}$$

Similarly one may generate the estimates of the plant states, z_1, z_2, z_3 via the expressions (3.28). We have

$$x_1 = y, \quad \hat{x}_2 = \hat{\dot{y}} + y, \quad \hat{x}_3 = \hat{\ddot{y}} + 3\hat{\dot{y}} + y, \tag{3.35}$$

Figure 3.5 depicts the performance of a 3d order Delta modulator based trajectory tracking controller for the previously described third order heating system described via Newton's heat propagation model in a compartmental fashion. The first set of graphs depicts the estimation of the phase variables associated with the system output. The second set of graphs in Figure 3.6 depicts the evolution of the states representing each compartment of the plant. The estimates of these states are also shown for comparison of the performance of the third order Delta modulator as a state estimator.

The parameters for the Delta modulator and the low pass filter, used in the simulations, were set as follows:

$$W = 3.0, \quad \omega_{nf} = 12.0 \text{ [rad/s]}, \quad \zeta_f = 1.0,$$

The parameters of the controller were chosen as

$$\zeta_c = 1.0 \quad \omega_{nc} = 2.0 \text{ [rad/s]}, \quad p_c = 2 \text{ [rad/s]},$$

A rest-to-rest maneuver plays the rôle of the output reference signal $y^*(t)$. The specification of this signal was made using a classical Bèzier interpolating polynomial, with initial (normalized) temperature $z_{1,init} = 1.0$ before $t = t_1 = 10$ (time units), and final temperature $z_{1,final} = 3$ after time $t = t_2 = 20$ (time units).

3.5 Delta-Sigma modulation

Delta-Sigma modulation is an important tool that will allow us to translate continuous (i.e., average) feedback controller design options into implementable switch controlled strategies with practically the same closed loop behavior.

The switched output signal, $q(t)$, of a Delta modulator reproduces, on the average, the first order time derivative of the input signal $\xi(t)$ provided the encoding condition: $-W < \dot{\xi}(t) < W$ is satisfied. Clearly, if the input signal to the Delta modulator undergoes a time integration before being processed by a Delta modulator, the switched output now reproduces, in an average sense, the original input signal. The equations governing the system are

$$\begin{aligned} \dot{z} &= \xi \\ \dot{x} &= q = u = W \,\text{sign}\sigma \\ \sigma &= z - x \end{aligned}$$

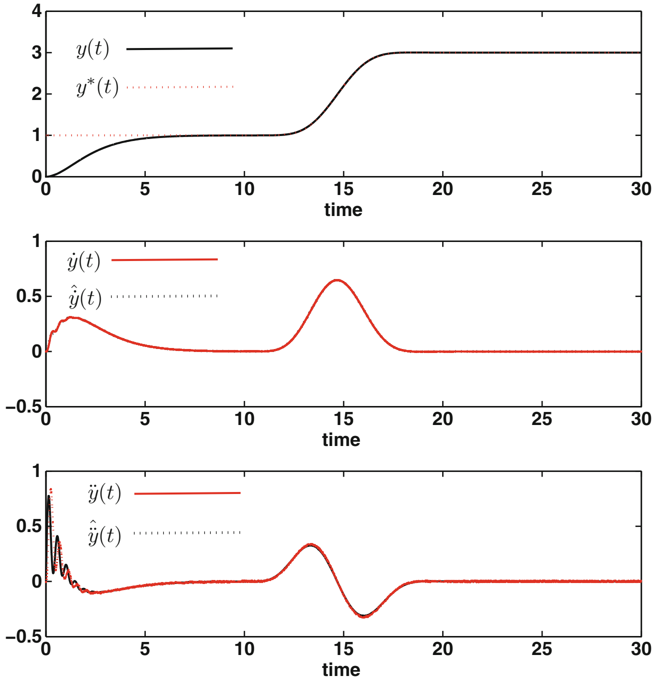

Fig. 3.5. Performance of 3d order Delta modulator based trajectory tracking controller for a compartmental control problem. Phase variables estimation.

Hence, $\dot{\sigma} = \xi(t) - W\mathrm{sign}(\sigma)$ and a sliding regime exists on $\sigma = 0$ provided $-W < \xi(t) < W$. The ideal sliding dynamics is given by $\sigma = 0$, i.e., $z(t) = x(t)$ and $\dot{\sigma} = 0$, which is equivalent to $\xi(t) = u_{eq}(t)$. The average output in an equivalent control sense equals the original input $\xi(t)$.

The integrator affecting the input signal $\xi(t)$ and the feedback integrator of the local decoder, producing the feedback signal $x(t)$, may be merged into a single integrator placed in the forward path just before the quantization unit representing the ideal switch. Also, in order to relate the Delta-Sigma modulator to average feedback designs in switched governed systems, we may use as quantization levels the values 0 and 1 instead of the values $-W$ and W. This gives rise to the block diagram given in Figure 3.7.

Consider the basic block diagram of Figure 3.7, reminiscent of a traditional Delta-Sigma modulator block used in early communications systems theory and analog to digital conversion schemes, but this time provided with a binary valued forward nonlinearity, taking values in the discrete set $\{0, 1\}$. The following theorem summarizes the relation of the considered modulator with sliding mode control while establishing the basic features of its input-output performance.

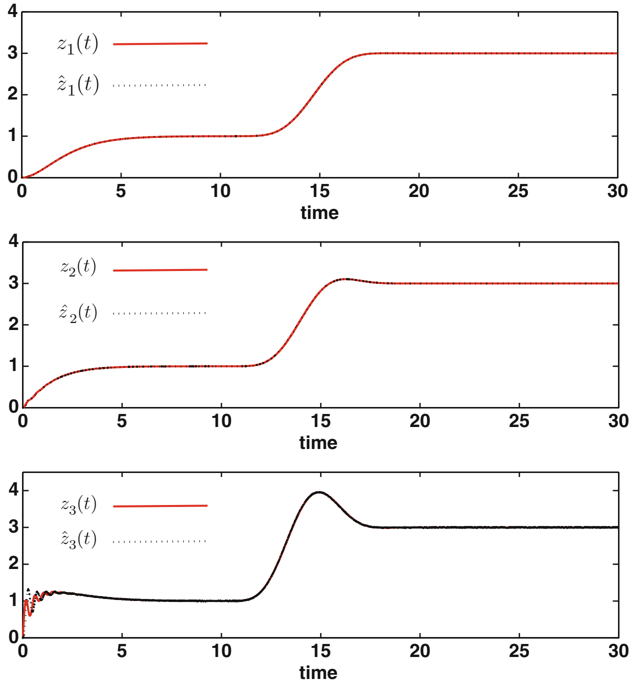

Fig. 3.6. Performance of 3d order Delta modulator based trajectory tracking controller for a compartmental control problem. State variables estimation.

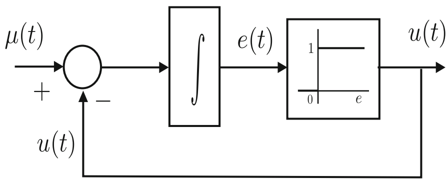

Fig. 3.7. Delta-Sigma modulator

Theorem 3.5. *In reference to the Delta-Sigma modulator of Figure 3.7. Given a sufficiently smooth, bounded, signal* $\mu(t)$*, then the integral error signal,* $e(t)$*, converges to zero in a finite time,* t_h*, and, moreover, from any arbitrary initial value,* $e(t_0)$*, a sliding motion exists on the perfect encoding condition surface, represented by* $e = 0$*, for all* $t > t_h$*, provided the following* encoding condition *is satisfied for all* t*,*

$$0 < \mu(t) < 1 \tag{3.36}$$

Proof. From the figure, the variables in the Delta-Sigma modulator satisfy the following relations:

$$\dot{e} = \mu(t) - u, \quad u = \frac{1}{2}\left[1 + \mathrm{sign}(e)\right] \tag{3.37}$$

The quantity $e\dot{e}$ is given by

$$e\dot{e} = e\left[\mu - \frac{1}{2}(1 + \text{sign}(e))\right] = -|e|\left[\frac{1}{2}(1 + \text{sign}(e)) - \mu\text{sign}(e)\right] \tag{3.38}$$

For $e > 0$ we have $e\dot{e} = -e(1 - \mu)$, which, according with to assumption in (3.36), leads to $e\dot{e} < 0$. On the other hand, when $e < 0$, we have $e\dot{e} = -|e|\mu < 0$. A sliding regime exists then on $e = 0$ for all time t after the hitting time t_h (see [31]). Under ideal sliding, or encoding, conditions, $e = 0, \dot{e} = 0$, we have that the, so-called, equivalent value of the switched output signal, u, denoted by $u_{eq}(t)$ satisfies $u_{eq}(t) = \mu(t)$.

An estimate of the hitting time t_h is obtained by examining the modulator system equations with the worst possible bound for the input signal $\mu(t)$ in each of the two conditions: $e > 0$ and $e < 0$, along with the corresponding value of u. Consider then $e(t_0) > 0$ at time $t = t_0$. We have for all $t_0 < t \le t_h$,

$$\begin{aligned} e(t) = e(t_0) + \int_{t_0}^{t} (\mu(\sigma) - u(\sigma))d\sigma &\le e(t_0) + (t - t_0)\left[\sup_{t\in[0,t]} \mu(t) - 1\right] \\ &< e(t_0) + (t_h - t_0)\left[\sup_t \mu(t) - 1\right]. \end{aligned} \tag{3.39}$$

Since $e(t_h) = 0$, we have

$$t_h \le t_0 + \frac{e(t_0)}{1 - \sup_t \mu(t)} \tag{3.40}$$

Similarly, for $e(t_0) < 0$ one obtains

$$t_h \le t_0 + \frac{|e(t_0)|}{\inf_t \mu(t)} \tag{3.41}$$

The expressions of the last two estimates of the hitting time t_h may be condensed into a single expression, regardless of the sign of the initial condition, $e(t_0)$, as follows:

$$t_h \le t_0 + \frac{2|e(t_0)|}{[1 - \sup_t \mu(t)](1 + \text{sign}\, e(t_0)) + \inf_t \mu(t)(1 - \text{sign}\, e(t_0))} \tag{3.42}$$

□

The average Delta-Sigma modulator output u_{eq} ideally yields the modulator's input signal $\mu(t)$ in an *equivalent control* sense ([31]).

To illustrate, by means of simulations, the feature just stated about Delta-Sigma modulation, we let $\mu(t) = 0.5(1 + A\sin(\omega t))$ with $A = 0.8$, $\omega = 3$ rad/s. At the output of the modulator we place a second order low pass filter of the form,

$$y = \left[\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right] u \tag{3.43}$$

with $\zeta = 0.81$ and $\omega_n = 30$. We may compare the filter output signal $y(t)$ with the input signal $\mu(t)$: modulo a small time delay and modulo the second order transient of the filter response starting from zero initial conditions. The low pass filtering of the switched output signal, $u(t)$, represented by the variable $y(t)$, approximately reproduces the sinusoidal input to the modulator. Figure 3.8 depicts the results.

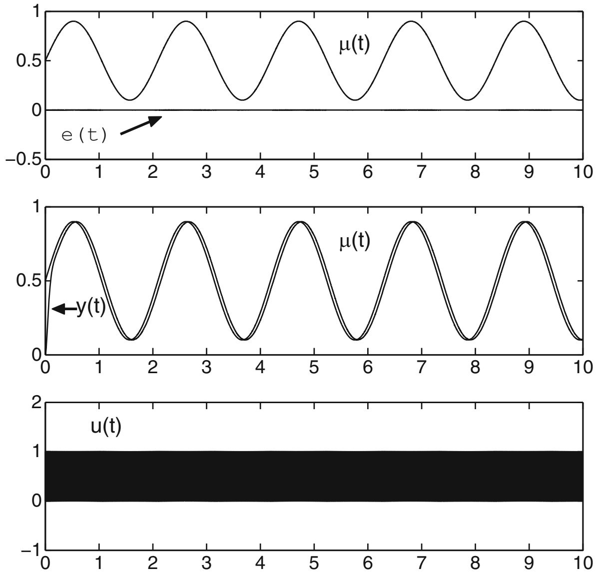

Fig. 3.8. Performance of Delta-Sigma modulator and tracking properties of the low pass filtered switched output.

The rôle of the above described Delta-Sigma modulator in sliding mode control schemes, avoiding full state measurements, and using average based controllers will be clear from its relations to integral sliding mode control schemes.

3.5.1 Two simple properties of Delta-Sigma modulators

Consider a Delta-Sigma modulator with symmetric quantization levels around the origin, i.e., $q \in \{-W, W\}$.

Property 1 Notice that in the local encoding nonlinearity, producing the Delta-Sigma modulator output $q = W\text{sign}\sigma$, the amplitude W may be a time-varying signal $W(t)$. Sliding motions are not precluded to exist, as long as this time-varying switch amplitude modulation signal remains strictly positive and the input signal $\xi(t)$ continues to satisfy: $-W(t) < \xi(t) < W(t)$ for all t. Indeed,

$$\dot{\sigma} = \mu(t) - W(t)\text{sign}\sigma \tag{3.44}$$

Since $W(t) > 0$ for all t, consider the time scaling $d\tau = W(t)dt$. One has

$$\frac{d}{d\tau}\sigma = \tilde{\mu}(t) - \text{sign}\sigma \tag{3.45}$$

for $\tilde{\mu}(t) = \mu/W(t)$. A sliding regime exists on $\sigma = 0$ provided $-1 < \tilde{\mu}(t) < 1$, i.e., $-W(t) < \mu(t) < W(t)$.

Property 2 A similar property holds if a block, characterized by the strictly positive gain function $a(x) > 0$, is located in the forward integration block of the Delta-Sigma modulator. In this case,

$$\dot{\sigma} = a(x)[\mu(t) - u], \quad u = W\text{sign}\sigma \tag{3.46}$$

The time scaling, $d\tau = a(x)dt$, yields

$$\frac{d\sigma}{d\tau} = \mu(\tau) - W\text{sign}\sigma \tag{3.47}$$

A sliding regime exists on $\sigma = 0$, provided $-W < \mu(\tau) < W$ for all τ.

3.5.2 High gain Delta-Sigma modulation

In non-switched controlled systems provided with saturation controllers, average feedback controller designs may also be implemented using high gain versions of the switched governed Delta-Sigma modulator. Here we examine two classical high gain continuous replacements of the switching nonlinearity characterizing a Delta-Sigma modulator.

Consider then the underlying encoding error dynamics,

$$\dot{\sigma} = u_{av} - \frac{1}{2}(1 + \text{sign}\sigma) = -\frac{1}{2}\text{sign}\sigma + \left(u_{av} - \frac{1}{2}\right) \tag{3.48}$$

A switch function of the form: $u = \frac{1}{2}(1 + \text{sign}\sigma)$ may be smoothed by an approximation of the sign function via a piecewise linear function characterized by a high slope on the ‘boundary layer”.‘$|\sigma| < \epsilon$, with $\epsilon > 0$ being an arbitrarily small parameter satisfying $\epsilon << 1$. Indeed,

$$u = \begin{cases} \frac{1}{2}(1 + \text{sign}\sigma) & \text{for} \quad |\sigma| > \epsilon \\ \frac{1}{2}(1 + \frac{\sigma}{\epsilon}) & \text{for} \quad |\sigma| \leq \epsilon \end{cases} \tag{3.49}$$

For $|\sigma| > \epsilon$ the output u of the modulator satisfies exactly the same dynamics as the switched modulator. The condition $\sigma\dot{\sigma} < 0$, viewed as the time derivative of the Lyapunov function candidate $V(\sigma) = \sigma^2/2$, implies that the function σ globally uniformly converges towards the boundary layer, described by $|\sigma| \leq \epsilon$, provided the encoding condition $0 < u_{av}(t) < 1$ is uniformly valid.

For $|\sigma| < \epsilon$ the governing equations of the Delta-Sigma modulator are given by

$$u = \frac{1}{2}(1 + \frac{\sigma}{\epsilon}), \quad \dot{\sigma} = u_{av} - u$$

i.e.,

$$\dot{\sigma} = -\frac{1}{2\epsilon}\sigma + \left(u_{av} - \frac{1}{2}\right) \tag{3.50}$$

Taking $V(\sigma) = \sigma^2/2$ as a natural Lyapunov function candidate, its time derivative along the solution trajectory of the differential equation for σ yields

$$\begin{aligned} \dot{V}(\sigma) &= -\frac{1}{2\epsilon}\sigma^2 + \sigma\left(u_{av} - \frac{1}{2}\right) \\ &\leq -\frac{1}{2\epsilon}\sigma^2 + \frac{1}{2}\left(\sigma^2 + (u_{av} - \frac{1}{2})^2\right) \\ &= -\frac{1}{2}\left(\frac{1-\epsilon}{\epsilon}\right)\sigma^2 + \frac{1}{2}\left(u_{av} - \frac{1}{2}\right)^2 \end{aligned} \tag{3.51}$$

where we have used Young's inequality (also known as the "Peter Paul" inequality) from the first line to the second. Thus, provided $u_{eq}(t) \in (0,1)$ for all t, it is not difficult to see that $\dot{V}(\sigma)$ is strictly negative for all σ outside the interval,

$$|\sigma| \leq \frac{1}{2}\sqrt{\frac{\epsilon}{1-\epsilon}} \tag{3.52}$$

The trajectories of σ are stably attracted towards a neighborhood of the origin but they do not converge to $\sigma = 0$.

A second possible replacement of the sign function is constituted by a continuous function, characterized also by a small but strictly positive parameter ϵ

$$\text{sign}\sigma \approx \frac{\sigma}{|\sigma| + \epsilon} \tag{3.53}$$

A disadvantage of this approximation is that it never quite adopts the saturating values -1 or $+1$.

Under this last approximation, the dynamics for σ in the Delta-Sigma modulator is globally governed by

$$\dot{\sigma} = -\frac{1}{2}\left(\frac{\sigma}{|\sigma| + \epsilon}\right) + \left(u_{av} - \frac{1}{2}\right) \tag{3.54}$$

Exercise 3.6. Estimate the region of attraction of the origin, $\sigma = 0$, for the above differential equation whenever $0 < u_{av}(t) < 1$.

3.6 Two level Delta-Sigma modulation

It is possible to extend the previously presented Delta-Sigma modulation scheme to deal with switch control inputs u taking values in the discrete set $\{-1, 0, 1\}$. To this category of switched systems can be reduced the great majority of switched controlled mechanical systems as well as some power electronics devices provided with "double bridges." The theoretical details of sliding mode existence in cases where the switch is characterized by $u \in \{-W, 0, +W\}$ are left as an exercise.

Figure 3.9 depicts a redundant arrangement of two Delta-Sigma modulators, one with a single positive encoding level and the other with a symmetrically negative encoding level. Clearly, such an arrangement is justified by the result in Theorem 3.7. A definite arrangement is shown in Figure 3.10.

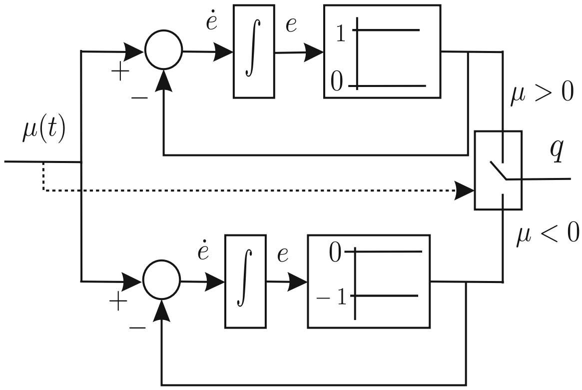

Fig. 3.9. Two level Delta-Sigma modulation (redundant arrangement)

Suppose that the continuous input signal, $\mu(t)$, takes values on the closed interval $[-1, 1]$ of the real line. The system equations are governed by

$$\dot{e} = \mu(t) - u$$
$$u = \begin{cases} \frac{1}{2}(1 + \text{sign}e) & \text{for } \mu > 0 \\ -\frac{1}{2}(1 - \text{sign}(e)) & \text{for } \mu < 0 \end{cases} \tag{3.55}$$

A sliding regime exists on $e = 0$ in any of the two cases ($\mu > 0$ and $\mu < 0$), and the two level Delta-Sigma modulation equations may be summarized as follows:

$$\dot{e} = \mu(t) - u, \quad u = \frac{1}{2}\left(\text{sign}\mu(t) + \text{sign}e\right) \tag{3.56}$$

Figure 3.11 shows a typical response of a two level Delta-Sigma modulator to an input signal of varying polarity, like a sinusoid function. The switchings actively commute between 0 and 1 when the input signal μ is positive and between 0 and -1 when the input signal μ is negative.

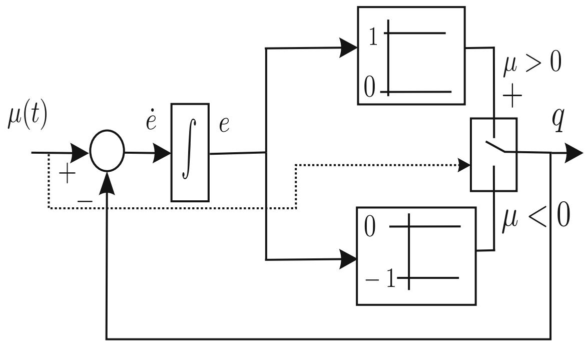

Fig. 3.10. Two level Delta-Sigma modulation

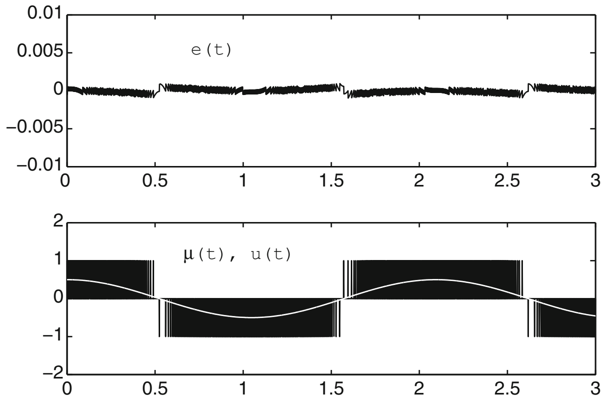

Fig. 3.11. Two level Delta-Sigma modulation circuit responses

3.6.1 Delta-Sigma modulation with an arbitrary quantization levels

In the preceding considerations, the output signal of the Delta-Sigma modulator is constituted by an (infinite frequency) succession of pulses of values 1 and 0. The considerations for describing the behavior of a Delta-Sigma modulator with output signal represented by pulses of arbitrary values: $\{W_{\min}, W_{\max}\}$, with $W_{\max} > W_{\min}$ are not too different (see Figure 3.12). Indeed, in such a case, we have

$$\dot{e} = \mu(t) - q, \quad q = u(W_{\max} - W_{\min}) + W_{\min} \tag{3.57}$$

with u representing a switch position function taking values in the set $\{0, 1\}$. The switching strategy is aimed at creating a sliding regime on $e = 0$. We now investigate the properties under which such a sliding regime exists.

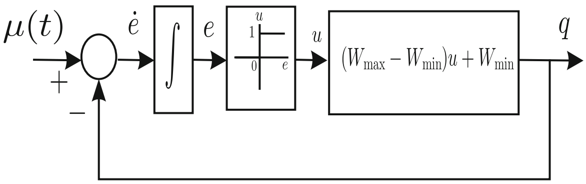

Fig. 3.12. Delta-Sigma modulator with arbitrary quantization levels

Consider the product $e\dot{e}$,

$$e\dot{e} = e\left(\mu(t) - [u(W_{\max} - W_{\min}) + W_{\min}]\right) \tag{3.58}$$

The switching strategy $u = 1/2(1+\text{sign}(e))$ produces the desired effect: $e\dot{e} < 0$ provided $\mu(t)$ satisfies the intermediacy condition: $W_{\min} < \mu(t) < W_{\max}$ for all t.

Indeed, if $e > 0$, then $e\dot{e} = e(\mu(t) - W_{\max})$. This expression is negative as long as $\mu(t) < W_{\max}$. On the other hand, if $e < 0$, then $e\dot{e} = (\mu(t) - W_{\min})$. This expression is negative provided $\mu(t) > W_{\min}$. Thus $W_{\min} < \mu(t) < W_{\max}$.

Under ideal sliding motions, the invariance condition, $\dot{e} = 0$, implies $\mu(t) = q_{eq}(t)$ where q_{eq} is the average value of the output signal q.

The equivalent control $u_{eq}(t)$ for the switch position function is obtained from the invariance conditions: $\dot{e} = 0$, $e = 0$. We have

$$\mu(t) - [u_{eq}(W_{\max} - W_{\min}) + W_{\min}] = 0 \tag{3.59}$$

and, therefore

$$u_{eq} = \frac{\mu(t) - W_{\min}}{W_{\max} - W_{\min}} \tag{3.60}$$

Since the existence of a sliding regime demands that the necessary and sufficient condition $0 < u_{eq} < 1$ be satisfied, the conditions on the input signal $\mu(t)$ follow.

We have demonstrated the following result:

Theorem 3.7. *Given the Delta-Sigma modulator with input signal $\mu(t)$ with arbitrary quantization levels: $W_{\max}$, $W_{\min}$ satisfying, $W_{\max} > W_{\min}$. Then, a sliding regime exists on $e = 0$ if and only if the input signal $\mu(t)$ satisfies*

$$W_{\min} \ < \ \mu(t) \ < \ W_{\max}.$$

The average value $q_{eq}(t)$ of the modulator output signal $q(t)$ taking values in the binary set: $\{W_{\min}, W_{\max}\}$, entirely coincides with $\mu(t)$.

The result in Theorem 3.7 is independent of whether $W_{\min}$ is positive or negative. This result conforms the basis for multilevel Delta-Sigma modulation to be explored next.

3.7 Multilevel Delta-Sigma modulation

A further generalization of the Delta-Sigma modulation encoding technique can be obtained by considering several "levels of coding" or "levels of digital quantization." Suppose we would like to have N positive levels of discontinuous encoding of a strictly positive signal $\xi(t)$. In other words, let W be a fixed positive real number representing a quantization, or granularity, level. Assume, moreover that the given positive signal $\xi(t)$ satisfies the following bound $\max_t \xi(t) \leq NW$ for some finite integer N. We would like to produce a discontinuous signal taking values on the finite set $\{0, W, 2W, \cdots, (N-1)W, NW\}$ and which switches between two adjacent values $(j-1)W$ and jW when the signal $\xi(t) \in [(j-1)W, jW]$ for every j.

The following generalization of the Delta-Sigma modulator produces an N level quantization, of width W, of a strictly positive signal bounded between 0 and NW.

$$
\begin{aligned}
\dot{e} &= \xi(t) - y \\
y &= \frac{W}{4}\left\{\sum_{j=1}^{N} [2j-1+\text{sign}(e)]\,[\text{sign}\,(\xi(t)-(j-1)W) - \text{sign}\,(\xi(t)-jW)]\right\}
\end{aligned}
$$

The idea behind this formula is quite elementary. Consider a signal f_j defined by

$$
f_j = \frac{1}{2}\,[(\text{sign}\,(\xi(t)-(j-1)W) - \text{sign}\,(\xi(t)-jW))] \tag{3.61}
$$

This signal takes the value 1, only when the signal $\xi(t)$ lies in the interval $[(j-1)W, jW]$, otherwise, it takes the value 0. We consider then functions of the form $\sum_j y_j f_j$. Once the proper summand is activated and the rest are inhibited, it is necessary to create a sliding regime on the manifold $e = 0$. Notice that $e\dot{e} = e(\xi - \sum_k^N y_k f_k) = e(\xi - y_j)$ is guaranteed to be always negative, whenever $\xi(t) \in [(j-1)W, jW]$, provided we choose the following switching strategy for the signal y_j, using as binary inputs the real numerical values in the discrete set, $\{(j-1)W, jW\}$:

$$
y_j = \begin{cases} (j-1)W & \text{for } e < 0 \\ jW & \text{for } e > 0 \end{cases}, \tag{3.62}
$$

which may also be synthesized as

$$
\begin{aligned}
y_j &= j\frac{W}{2}(1+\text{sign}(e)) + (j-1)\frac{W}{2}(1-\text{sign}(e)) \\
&= \frac{W}{2}\,[2j-1+\text{sign}(e)]
\end{aligned}
$$

This last switching policy creates a sliding regime on $e = 0$ while the signal $\xi(t)$ takes values in the interval $[(j-1)W, jW]$.

Clearly, the sums of products of y_j and f_j, in the expression $y = \sum_j y_j f_j$, yield the proposed formula.

Figure 3.13 depicts in its upper graph a six-level sliding mode quantization of a biased sinusoidal signal of amplitude 1.5 centered around 1.5, with levels of quantization of 0.5. The lower graph depicts the low pass filtering $y_f(t)$ of the signal $y(t)$ in comparison with the original signal. The low pass filter is a Butterworth second order filter with a damping factor of 0.8 and a cut-off frequency of 100 [rad/s].

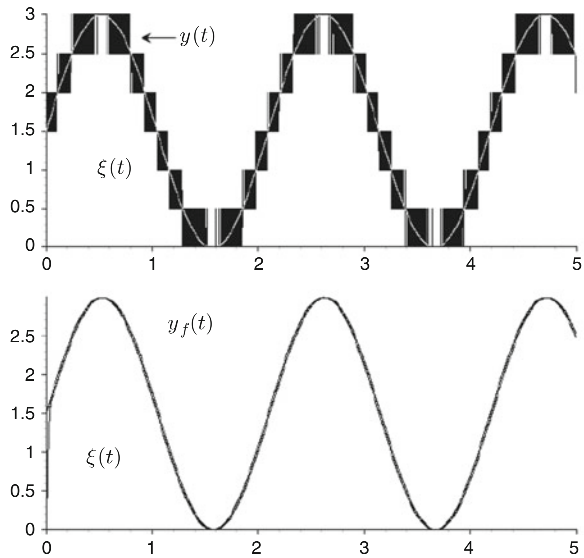

Fig. 3.13. Six-level Delta-Sigma modulation encoding of a positive signal

Finally, we leave it to the reader to show that for any signal bounded within the interval $[-NW, NW]$, the following multilevel Delta-Sigma modulation scheme renders a complete quantization of the system into $2N$ levels of width W with switchings taking place between the numerical values of the bounding levels of the quantization intervals.

$$\dot{e} = \xi(t) - y$$

$$y = \frac{W}{4}\left\{ \sum_{-N+1}^{N} [2j - 1 + \mathrm{sign}(e)] \left[\mathrm{sign}\left(\xi(t) - (j-1)W\right) - \mathrm{sign}\left(\xi(t) - jW\right)\right] \right\}$$

3.8 Average feedbacks and Delta-Sigma-Modulation

Suppose we have a smooth nonlinear system of the form $\dot{x} = f(x) + ug(x)$ with u being a (continuous) control input signal that, due to some physical limitations, requires to be bounded by the closed interval $[0, 1]$. Suppose, furthermore, that we have been able to specify a dynamic output feedback controller of the form $u = -\kappa(y, \xi), \dot{\zeta} = \varphi(y, \xi)$, with desirable closed loop performance features. Assume, furthermore, that for some reasonable set of initial states of the system (and of the dynamic controller), the values of the generated feedback signal function, $u(t)$, are uniformly strictly bounded by the closed interval $[0, 1]$.

If an additional implementation requirement entitles now that the control input u of the system no longer be allowed to continuously take values within the interval $[0, 1]$, but that it may only take values in the *discrete set*, $\{0, 1\}$, the natural question is: How can we now use, in view of the newly imposed actuator restriction, the derived continuous controller, so that we can recover, possibly in an *average sense*, its desirable features?

The answer is clearly given by the average differentiating features of the input signals in the previously considered Delta modulator. Recall, incidentally, that the output signal of such a modulator is restricted to take values, precisely, in the discrete set $\{0, 1\}$.

Thus, if the *time integral* of the output of the designed continuous controller, call it $u_{av}(t)$, is fed into the proposed Delta modulator, the output signal of the modulator reproduces, on the average, the required control signal u_{av}. Figure 3.14 shows the switch based implementation of an output feedback controller, through a Delta-Sigma modulator, which reproduces, in an average sense, the features of a designed continuous controller.

Theorem 3.8. *Consider the following smooth nonlinear single input, n-dimensional system: $\dot{x} = f(x) + ug(x)$, with the smooth scalar output map, $y = \sigma(x)$. Assume the dynamic smooth output feedback controller $u = -\kappa(y, \zeta), \dot{\zeta} = \varphi(y, \zeta)$, with $\zeta \in R^p$, locally (globally, semi-globally) asymptotically stabilizes the system towards a desired constant equilibrium state, denoted by X. Assume, furthermore, that the control signal, u, is uniformly strictly bounded by the closed interval* $[0, 1]$ *of the real line. Then the closed loop system:*

$$\begin{aligned}
\dot{x} &= f(x) + ug(x) \\
y &= \sigma(x) \\
u_{av}(y, \zeta) &= -\kappa(y, \zeta, X) \\
\dot{\zeta} &= \varphi(y, \zeta, X) \\
u &= \frac{1}{2}\left[1 + \text{sign } z\right] \\
\dot{z} &= u_{av}(y, \zeta) - u
\end{aligned}$$

exhibits an ideal sliding dynamics *which is locally (globally, semi-globally) asymptotically stable to the same constant state equilibrium point, X, of the system.*

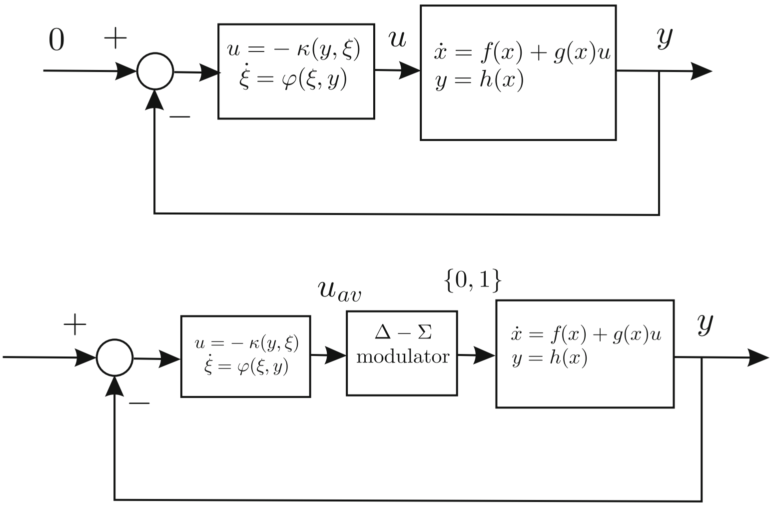

Fig. 3.14. Sliding mode implementation of a designed continuous output feedback controller through a Delta-modulator with integrated input

Proof

The proof of this theorem is immediate upon realizing that under the hypothesis on the average control input, u_{av}, the previous theorem establishes that a sliding regime exists on the manifold $z = 0$. Under the *invariance conditions*, $z = 0, \dot{z} = 0$, which characterize ideal sliding motions (see Sira-Ramírez [25]), the corresponding *equivalent control*, u_{eq}, associated with the system satisfies: $u_{eq}(t) = u_{av}(t)$. The *ideal sliding dynamics* is then represented by

$$\begin{aligned} \dot{x} &= f(x) + u_{av} g(x) \\ y &= \sigma(x) \\ u_{av}(y, \zeta) &= -\kappa(y, \zeta, X) \\ \dot{\zeta} &= \varphi(y, \zeta, X) \end{aligned}$$

which is assumed to be locally (globally, semi-globally) asymptotically stable towards the desired equilibrium point.

Note that the Delta-Sigma modulator state, z, can be initialized at the value $z(t_0) = 0$. This implies that the induced sliding regime exists uniformly for all times after t_0. Hence, *no reaching time* of the sliding surface, $z = 0$, is required. This practical feature is adopted throughout this book.

3.8.1 Control of the Double Bridge Buck Converter

Consider the normalized buck converter equations, given in 2.109

$$\dot{x}_1 = -x_2 + u, \quad x_2 = x_1 - \frac{x_2}{Q}, \quad y = x_2 \tag{3.63}$$

The factor of 1 multiplying the switch control input u represents the normalized (positive) constant external source voltage. As a consequence of the sign of the source the achievable voltages are necessarily positive and so are the inductor currents. If the voltage of the source is allowed to change its sign (appropriately flipping the source terminals), the normalized equations are simply

$$\dot{x}_1 = -x_2 - u, \quad x_2 = x_1 - \frac{x_2}{Q}, \quad y = x_2 \tag{3.64}$$

and now the achievable output voltages and currents are, both, negative. The double bridge buck converter allows this with the use of a diode bridge. The control input switch function, u, takes values now on the set $\{-1, 0, 1\}$. As a consequence of this, output reference signals $y^*(\tau)$ of changing polarity can now be tracked. For signals of positive polarity, sliding motions exist provided the well-known limitation: $0 < u_{eq} < 1$, is valid, on the equivalent control. For the tracking of signals with negative polarity, sliding motions exist whenever $-1 < u_{eq} < 0$. For signals of varying polarity, the equivalent control of this converter yields the existence condition: $-1 < u_{eq} < 1$, or $|u_{eq}(\tau)| < 1$. Recall that this necessary and sufficient condition for existence of sliding regimes determines the bandwidth limitations.

The Delta-Sigma modulation approach allows us to translate any continuous tracking controller design in a rather simple manner by inserting the Delta-Sigma modulator between the output of the controller and the switched input channel of the plant. For illustrative purposes, let us carry out a classical continuous output feedback controller design with the reference signal: $y^*(\tau) = A\sin(\omega\tau)$. We first establish the bandwidth limitations.

The nominal control input $u^*(t)$ satisfies the input-output relation

$$u^*(\tau) = \ddot{y}^*(\tau) + \frac{1}{Q}\dot{y}^*(\tau) + y^*(\tau) \tag{3.65}$$

If the output tracking is achievable via sliding mode control, the equivalent control coincides with the nominal control in steady state conditions (after the time-varying surface is reached and the tracking error transients die out). Hence, using $\phi = \arctan((\omega/Q)/(1-\omega^2))$, we have

$$\begin{aligned} |u^*(\tau)| \; &= \; A\sin(\omega\tau + \phi)\sqrt{(1-\omega^2)^2 + \left(\frac{\omega}{Q}\right)^2} \\ &\leq A\sqrt{(1-\omega^2)^2 + \left(\frac{\omega}{Q}\right)^2} \; < \; 1 \end{aligned} \tag{3.66}$$

or simply, given a desired normalized frequency ω for the sinusoidal output reference voltage, the allowable sinusoid signal amplitude A must satisfy

$$A \; < \; \frac{1}{\sqrt{(1-\omega^2)^2 + \left(\frac{\omega}{Q}\right)^2}} \tag{3.67}$$

Not surprisingly, the frequency-amplitude tradeoff relation of the buck converter coincides with that of the double bridge buck converter (see equation 2.118).

Consider the continuous, average, lead tracking error compensator, abusively described in a combination of time domain and frequency domain signals,

$$u_{av}(s) = u^*(\tau) - \left[\frac{ms+n}{s+p}\right](y - y^*(\tau)) \tag{3.68}$$

with

$$m = \frac{1}{Q}(\frac{1}{Q} - K_2) + (1 - K_1), \quad n = (\frac{1}{Q} - K_2) + K_0, \quad p = K_2 - \frac{1}{Q}$$

Let $e_y = y - y^*(t)$. The linear closed loop tracking error dynamics coincides with

$$e_y^{(3)} + K_2\ddot{e}_y + K_1\dot{y} + K_0 e_y = 0 \tag{3.69}$$

The choice $p > 0$ ensures a stable lead compensator while m and n either both positive or both negative, ensure a minimum phase behavior of the controller. This requirement is easily achieved. The choice of the parameters: K_0, K_1, K_2 must be made satisfying, $K_2 > 0$, $K_0 > 0$, and $K_2K_1 > K_0$. This demand ensures exponential asymptotic convergence of the tracking error e_y to zero. To obtain a characteristic polynomial, in the complex variable s, with known stable roots location, one equates the characteristic polynomial of (3.69) to the product of the following two stable polynomials: $s^2 + 2\zeta\omega_n s + \omega_n^2$ and $s + r$. We then set $K_2 = 2\zeta\omega_n + r$, $K_1 = \omega_n^2 + 2\zeta\omega_n r$, and $K_0 = \omega_n^2 r$.

The dynamic average output tracking controller is easily synthesized, in a state space form, as,

$$\dot{\theta} = -p\theta + e_y, \quad u = u^*(t) + (mp - n)\theta - me_y \tag{3.70}$$

with θ being an auxiliary variable with arbitrary initial conditions.

Figure 3.15 depicts the responses of the normalized buck system to the above designed two level Delta-Sigma modulation based classical output tracking feedback control. The chosen sinusoid signal, with amplitude $A = 0.6$ and normalized angular frequency $\omega = 0.3$, respects the normalized bandwidth limitation of the sliding mode existence condition. We used the following parameters for the lead controller,

$$m = -0.914 \; n = -0.493 \; p = 0.583$$

The average controller is characterized by $K_2 = 0.9$, $K_1 = 0.27$, and $K_0 = 0.09$. These parameters correspond to the choice $\zeta = 1$, $\omega_n = 0.3$, $r = 0.3$

Exercise 3.9. Let $W > 0$ be a constant parameter. Let m, g, and L denote constants representing, respectively, the mass of a pendulum bob, the acceleration of gravity, and the length of the pendulum inelastic rod. Consider the problem of controlling the actuated pendulum,

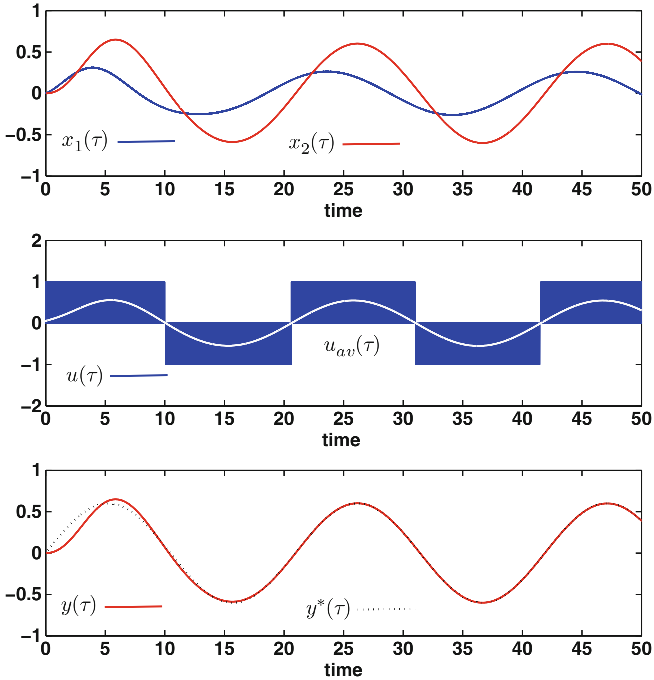

Fig. 3.15. Two level Delta Sigma based control of the buck converter

$$\ddot{\theta} = -mgL\sin\theta + W\tau \tag{3.71}$$

with a switched control input, τ, taking values in the discrete set, $\{-1, 0, 1\}$, as it corresponds to the possibilities of using a positive fixed torque, its opposite value, or no torque at all.

Device a sliding mode controller that allows the accurate tracking of a pre-specified smooth reference trajectory for the angular position θ.

Exercise 3.10. Consider the mechanical system shown in Figure 3.16 constituted by a mass and a spring. It is assumed that the friction forces on the wheels are negligible.

The force input, f, is only allowed to take values in the discrete set $f \in \{0, F\}$. We assume that x_0 is the equilibrium position for which the force in the spring is zero when $f = 0$.

- Verify that the equations of motion are given by

$$M\ddot{x} + k(x - x_0) = f \tag{3.72}$$

- Assume that a safety region for spring elongation is to be enforced beyond which the spring characteristics are permanently damaged. This region is given by

$$-L_{\min} < x - x_0 < L_{\max} \tag{3.73}$$

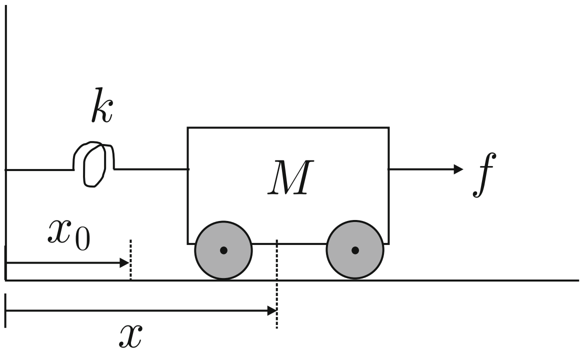

Fig. 3.16. Mass-spring system

- Devise a feedback switching controller that drives and sustains the motions of the mass at a feasible constant position value X.
- Discuss the nature of the ideal control that would smoothly sustain the motions at the constant value $x = X$.
- Devise a trajectory tracking controller for an arbitrary smooth reference position trajectory $x^*(t)$ respecting the spring elongation constraints.
- Device a feedback switching controller that uses a sliding surface coordinate function expression entitling only measurements of the input and the output and their integrals, but which does not use the measurement of the mass velocity.

3.9 Multilevel Sliding Mode control of mechanical systems

Multilevel Delta-Sigma modulation was seen to reproduce, in a convenient quantized switched manner, any continuous signal provided at the input of the modulator. This feature has important implications in the control of a mechanical system. As it is widely known, sliding mode control of mechanical systems is limited due to the chattering phenomenon. When the switching levels of forces and torques (usually opposite in sign and of symmetric nature) are widely separated, the induced chattering is severely increased. Switching between reduced force or torque levels makes the induced chattering slightly more acceptable. We illustrate this by means of a simulation example on a rather simple one degree of freedom controlled pendulum.

Example 3.11. Consider the one degree of freedom controlled pendulum, of bob mass m, length L, with negligible link mass described by,

$$mL^2\ddot{\theta} = -mgL\sin\theta + \tau \tag{3.74}$$

where τ is the applied torque input. A feedback linearizing controller is easily devised as

$$\tau = mL^2[\frac{g}{L}\sin\theta - 2\zeta\omega_n\dot{\theta} - \omega_n^2(\theta - \Theta)] \tag{3.75}$$

where Θ is the desired angular position.

Figure 3.17 depicts, in three columns, the evolutions of the angular position and the angular velocity responses, along with the applied designed torque input. In the first column the available torques take values in the set $\{-T, +T\}$. In the second column a two level Delta-Sigma modulation complements the average feedback controller with switched torques taking values in the set $\{-T, 0, +T\}$. The third column corresponds to a 10 level Delta-Sigma modulator implementing the average control torque input τ_{av}. In this case the switch takes values in the set:

$$\{-T, -(\frac{8}{10})T, \ldots, -(\frac{2}{10})T, 0, (\frac{2}{10})T, \ldots, (\frac{8}{10})T, T\}$$

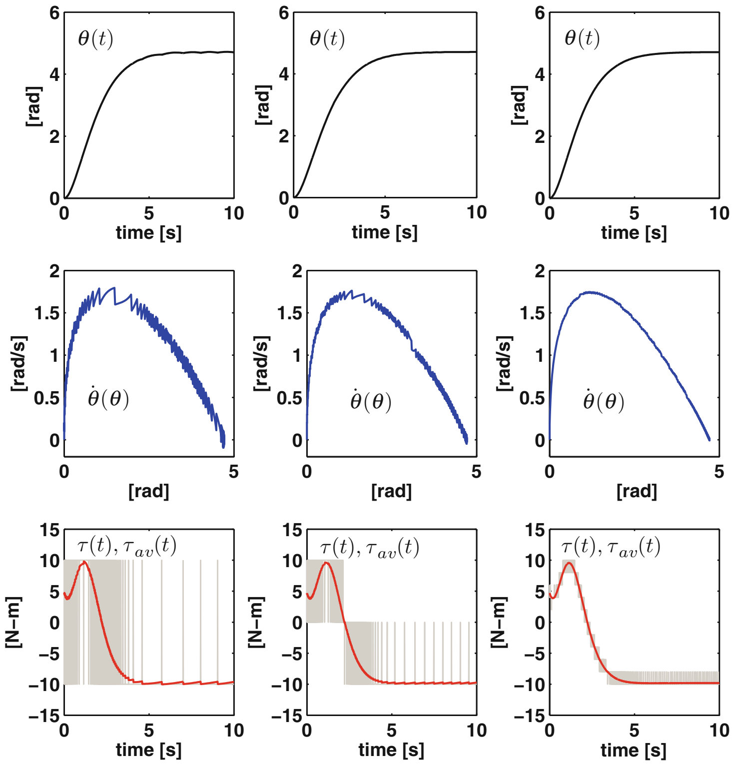

Fig. 3.17. Chattering reduction on a mechanical system via multilevel sliding mode control

The chattering reduction is quite significant, as seen from the phase plots in the second row.

In this example, the following numerical values were used for the simulations:

$$m = 1 \text{ [Kg]}, \quad L = 1 \text{ [mt]}, \quad g = 9.8 \text{ [m/s}^2\text{]}, \quad \Theta = \frac{3\pi}{2} \text{ [rad]}$$

The controller design parameters were set to be

$$\zeta = 1, \quad \omega_n = 1 \text{ [rad/s]}, \quad T = 10 \text{ [N} - \text{m]}$$

3.10 Second order Delta-Sigma modulation

Second and higher order Delta-Sigma modulation have been found useful in several applications due to their enhanced signal to noise ratio features (see Jarman [14]) . In our sliding mode context, second order Delta-Sigma modulators may be analyzed by establishing a relationship with second order Delta modulators. The general relation between higher order Delta-Sigma modulations and the corresponding higher order Delta modulators may be easily established.

Consider the second order Delta-Sigma modulator shown in Figure 3.18, with input signal $\xi(t)$ and output signal $q(t) \in \{-W, W\}$. The positive constant factor $a > 0$ will be important in determining a low pass filter bandwidth.

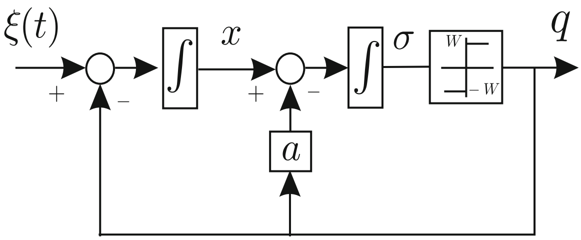

Fig. 3.18. Second order Delta-Sigma modulator

The equations characterizing the system are easily obtained from the block diagram

$$\begin{aligned} \dot{\sigma} &= x - aq \\ \dot{x} &= \xi(t) - q \\ q &= W \text{sign}\sigma \end{aligned} \tag{3.76}$$

From the first equation in (3.76), it is clear that a sliding regime exists on $\sigma = 0$ provided $-aW < x < aW$. The average output signal q_{eq} is simply given by $q_{eq} = x/a$ and the average evolution of x is, according to the second equation in (3.76), governed by

$$\dot{x} = -\frac{1}{a}(x - a\xi(t)) \tag{3.77}$$

On average, the signal x is a low pass filtering of the signal $a\xi(t)$. However, given that the average output signal q_{eq} satisfies $q_{eq} = x/a$, and using (3.77), the time evolution of q_{eq} is given by

$$\dot{q}_{eq} = -\frac{1}{a}(q_{eq} - \xi(t)) \tag{3.78}$$

i.e., the average output signal is a unity gain low pass filtering of the input signal. In other words:

$$\hat{q}_{eq}(s) = \frac{1/a}{s + 1/a}\hat{\xi}(s) \tag{3.79}$$

The average output of the second order Delta-Sigma modulator represents a second order integration of that produced by the second order Delta modulator (see equation 3.14).

The previous statement is easily proved by a rather straightforward block diagram manipulation. It should be clear that the Delta modulator (which is an ideal differentiator) preceded by a single integration operator is transformed into a Delta-Sigma modulator. The following Figure 3.19 depicts the transformation of a double Delta-Sigma modulator into a double Delta integrator preceded by two integration operators.

The first step is to pull out the integrator producing the variable x and replace it by two integrators, one located on the input path of the modulator and the other in the outer feedback loop leading to the first comparator. Next, the integrator producing the variable σ is replaced by two integrators, one in the forward path leading to the second comparator and the other in the inner feedback loop of the modulator, leading towards the second comparator. Now the two integrators leading to the two comparators are replaced by a single integrator in the feedback loop emerging from the output variable q. Finally, the integrator remaining in the forward path leading to the second comparator is replaced by two integrators, one located in the input path to the modulator and the other in the outer feedback loop path leading to the first comparator.

Exercise 3.12. Show that an n-th order Delta-Sigma modulator is equivalent to an n-th order Delta modulator preceded by n integration operators at the input.

Example 3.13. The use of higher order Delta-Sigma modulators in sliding mode control is to serve as analog to switched converters with some command on the noise attenuation capabilities of the underlying low pass filter describing the average behavior of the modulator.

Consider then a simple illustrative example, consisting of a single link actuated pendulum with a bob mass m, of length L, controlled by a torque u taking values in the set $\{W, -W\}$. The controlled pendulum is described by

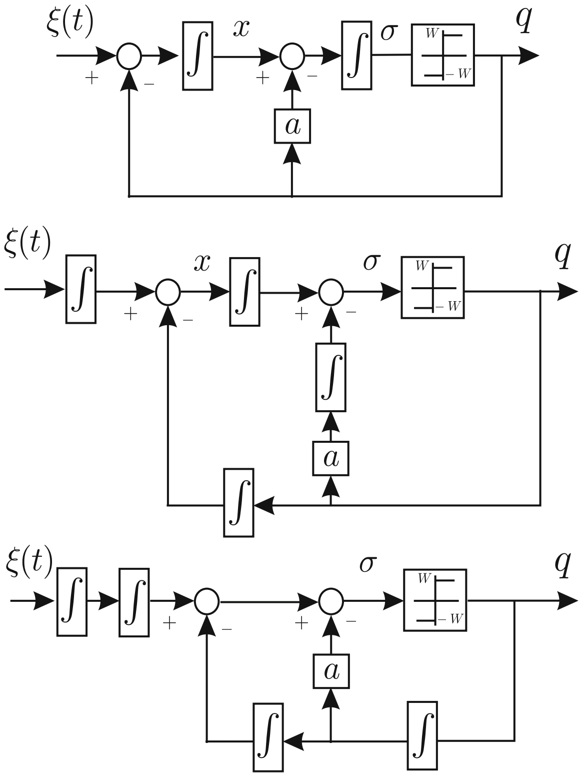

Fig. 3.19. Equivalence of second order Delta-Sigma modulator with a second order Delta modulator including two input integrations.

$$\ddot{\theta} = -\frac{g}{L}\sin\theta + \frac{u}{mgL} \tag{3.80}$$

Suppose it is desired to stabilize the pendulum, from its stable equilibrium position, around a desired constant value Θ.

An average feedback linearizing controller is readily designed to be,

$$u_{av} = mgL\left[\frac{g}{L}\sin\theta - 2\zeta_c\omega_c\dot{\theta} - \omega_c^2(\theta - \Theta)\right] \tag{3.81}$$

However, when u_{av} is produced some piecewise constant noise signal process $s(t)$ represented by a sequence of independent gaussian random variables is added due to output measurement noises, or production of the time derivative of the angular position, friction phenomena, etc. The average controller is implemented through a second order Delta-Sigma modulator with cut-off frequency $\omega = 1/a$.

Figure 3.20 depicts the deterioration of the controlled response for low bandwidth filtering while the quality of the responses increases with the enlargement of the cut-off frequency, ω, characterizing the low pass filter present in the second order Delta-Sigma modulator. The chattering associated with the modulator's sliding surface coordinate evolution, $\sigma(t)$, is substantially reduced as the cut-off frequency increases. However, the cut-off frequency cannot be arbitrarily increased since then the bandwidth limitation of the second order Delta-Sigma modulator is violated.

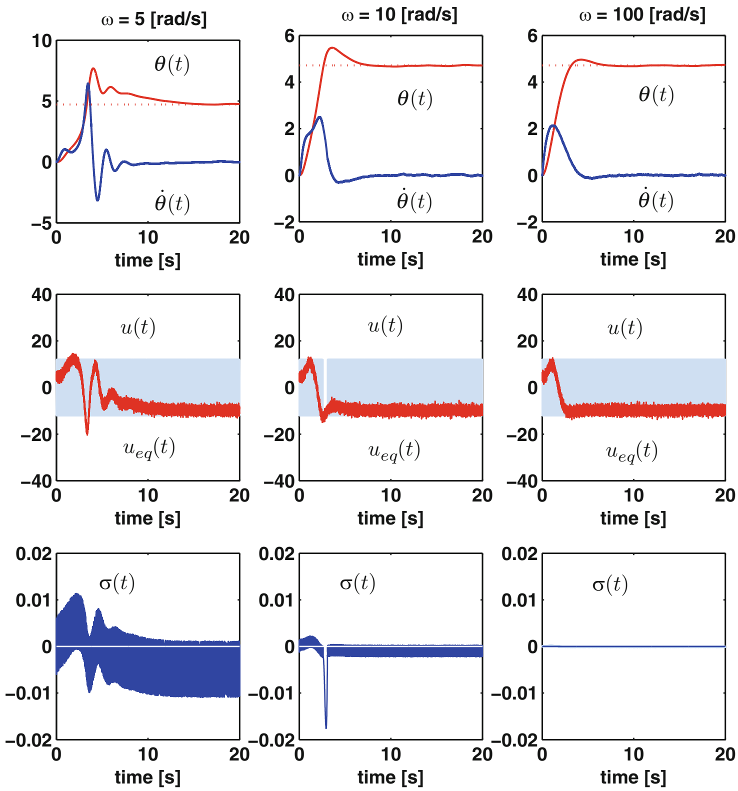

Fig. 3.20. Performance, under various cut-off frequencies ω, of switched controlled pendulum, using the low pass filtering capabilities of a second order Delta-Sigma modulator.

The parameters for the pendulum, the controller and the second order Delta-Sigma modulator were set to be:

$$m = 1 \quad [\mathrm{Kg}], \quad L = 1.0 \ [\mathrm{m}], \quad g = 9.8 \ [\mathrm{m/s^2}], \ W = 12 \ [\mathrm{N-m}]$$

$$\zeta_c = 0.707, \quad \omega_c = 1.0 \ [\mathrm{rad/s}], \quad \Theta = \frac{3\pi}{2} \ [\mathrm{rad}]$$

3.11 Delta-Sigma modulation and integral sliding mode control

3.11.1 Sliding on the integral control input error

We first explore a variant of sliding mode control centered around the idea of forcing the actual average control input to ideally behave as a nominal control input found in correspondence with a nominal desired state trajectory defined on a unperturbed system. (See Orlov [28] for a similar development detached from Delta-Sigma modulation considerations.)

Consider the following general switched controlled system described in state space,

$$\dot{x} = f(x, u), \; x \in \mathbb{R}^n, \; u \in \{0, 1\} \tag{3.82}$$

It is desired to have the state x track a given smooth trajectory $x^*(t)$. Suppose that, under the temporary assumption that u is a continuous control input, an average full state feedback controller design u_{av} is worked out which guarantees the accurate asymptotic tracking of $x^*(t)$ while u_{av} takes values in the compact interval $[0, 1]$ of the real line. The average control input, corresponding to the desired state reference trajectory is denoted by $u^*(t)$. The average feedback controller is, generally speaking, of the form

$$u_{av} = \varphi(x, x^*(t), u^*(t)) \in [0, 1] \tag{3.83}$$

Consider the sliding surface coordinate function σ given by

$$\sigma = \int_0^t [\varphi(x(\lambda), x^*(\lambda), u^*(\lambda)) - u] \, d\lambda \tag{3.84}$$

This choice of a sliding surface, and its corresponding switched controller design leading to a sliding regime is related to what is known in the literature as *integral sliding* mode. We deal with the original formulation further below.

The invariance conditions $\sigma = 0$, $\dot{\sigma} = 0$ define the equivalent control u_{eq} from the expression

$$\dot{\sigma} = \varphi(x(t), x^*(t), u^*(t)) - u_{eq} = 0 \tag{3.85}$$

i.e., the equivalent control, as an ideal feedback control law, entirely coincides with the average feedback controller design, u_{av}, i.e., $u_{eq} = u_{av}$.

Notice that, in general, before the sliding motions occur, one has

$$\dot{\sigma}\sigma = (\varphi(x, x^*(t), u^*(t)) - u)\sigma \tag{3.86}$$

with u being either 0 or 1. Clearly, a sliding mode exists on $\sigma = 0$ with the switched control policy:

$$u = \frac{1}{2}(1 + \text{sign}\sigma) \tag{3.87}$$

since under such a control switching policy, $\sigma\dot{\sigma} < 0$ is satisfied. Indeed, with $u_{av} = u_{eq} = \varphi \in [0,1]$, and $u \in \{0,1\}$, whenever $\sigma > 0$, then $u = 1$ and $\sigma\dot{\sigma} = (\varphi - 1)\sigma < 0$. Similarly, when $\sigma < 0$, then $u = 0$ and $\sigma\dot{\sigma} = \varphi\sigma < 0$.

The controller can be summarized as follows:

$$\begin{aligned}\dot{\sigma} &= u_{av} - u \\ u &= \frac{1}{2}(1 + \text{sign}\sigma)\end{aligned}$$

these equations coincide with those of a Delta-Sigma modulator with an analog input $\mu(t)$ given by the average control input signal, $u_{av}(t)$, the switched output coinciding with $u \in \{0,1\}$ and the error signal given by the difference between the input and the switched output.

3.11.2 Integral sliding modes: surface and control modification

Integral sliding modes were introduced by Utkin and Shi in [34]. Further developments can be found in the literature and the references in the book by Fridman *et al.* (see [11] and also in the book by Shtessel *et al.* [22]) . A more complete formulation of integral sliding mode controls is now examined under the switched control input restriction in a control input affine representation of the controlled system. We warn the reader that this class of sliding mode control technique is more suitable for systems with continuous although amplitude limited control inputs.

Consider then unperturbed nonlinear system given by

$$\dot{x} = f(x) + g(x)u, \quad y = \sigma_0(x) \quad x \in \mathbb{R}^n, \; u \in \{0,1\} \tag{3.88}$$

Let $\sigma_0(x)$ be a nominal sliding surface coordinate function for which a sliding regime exists on an open region of the sliding surface, $\mathcal{S} = \{x \in \mathbb{R}^n | \sigma_0(x) = 0\}$. The corresponding equivalent control, denoted by $u^0_{eq}(x) = -L_f\sigma_0/L_g\sigma_0$, defined on $\sigma_0 = 0$, satisfies: $0 < u^0_{eq}(x) < 1$, thus determining the region of existence of a sliding regime on $\mathcal{S}$. Notice that if an additive matched perturbation, $\phi(x,t) = g(x)u_h(x,t)$ arises in the system and a sliding regime still exists on $\sigma_0 = 0$, the equivalent control, is defined as

$$u_{eq}(x) = -\frac{L_f\sigma_0(x)}{L_g\sigma_0(x)} - u_h(x,t)\bigg|_{\sigma_0=0} = u^0_{eq}(x) - u_h(x,t) \tag{3.89}$$

Thus, although the invariance conditions $\sigma_0 = 0$, $\dot{\sigma}_0 = 0$, and the ideal sliding dynamics are unaffected by the perturbation, the equivalent control virtually cancels, or counteracts, the (unknown) disturbance input $u_h(x)$ in an automatic manner. The region of existence of the sliding regime, nominally characterized by $0 < u^0_{eq}(x) < 1$, is now modified by

$$0 < u^0_{eq}(x) - u_h(x,t) < 1, \quad \text{or} \quad u_h(x,t) < u^0_{eq}(x) < 1 + u_h(x,t) \tag{3.90}$$

Integral sliding mode control attempts to make the nominal existence conditions to remain invariant with respect to the disturbance input.

Consider then a perturbed nonlinear system given by

$$\dot{x} = f(x) + g(x)u + \xi(x,t), \quad x \in \mathbb{R}^n, \; u \in \{0,1\} \tag{3.91}$$

The disturbance vector $\xi(x,t)$ is assumed to satisfy the *matching condition* i.e., $\xi(x,t) \in \text{span}\{g\}$ for all t. There exists then a scalar disturbance input, $u_h(x,t)$, such that $\xi(x,t) = g(x)u_h(x,t)$.

In order to deal with the perturbed system, and above all to free the nominal region of existence from effects of perturbations induced modifications, consider the following sliding surface coordinate function $\sigma(x) = \sigma_0(x) + \sigma_1(x)$ where $\sigma_1(x)$ is an auxiliary sliding surface coordinate function, yet to be determined, aimed at absorbing the counterproductive control input arising from the perturbation field.

Additionally, the equivalent control corresponding to the sliding surface coordinate function $\sigma(x)$, for the perturbed system, is assumed to be of the form: $u_{eq}(x) = u^0_{eq}(x) + u_{1eq}(x)$. The actual control input is synthesized as $u = u^0_{eq}(x) + u_1$.

The invariance conditions $\sigma = 0$, $\dot{\sigma} = 0$ demand that, for all x and t,

$$\sigma_1(x) = -\sigma_0(x), \quad \dot{\sigma}(x) = L_f\sigma_0 + (L_g\sigma_0)u^0_{eq}(x) + (L_g\sigma_0)u_h(x,t) + \dot{\sigma}_1 = 0 \tag{3.92}$$

Let the following equality be enforced,

$$\dot{\sigma}_1 = -\left[L_f\sigma_0 + (L_g\sigma_0)u^0_{eq}\right], \quad \sigma_1(x(0)) = -\sigma_0(x_0) \tag{3.93}$$

i.e.,

$$\sigma_1(x(t)) = -\sigma_0(x_0) - \int_0^t \left[L_f\sigma_0(x(\lambda)) + (L_g\sigma_0)u^0_{eq}(x(\lambda))\right] d\lambda, \tag{3.94}$$

The sliding surface coordinate function for the perturbed system ideally satisfies

$$\dot{\sigma}(x) = [L_g\sigma_0(x)](u_{1eq} + u_h(x,t))\Big|_{\sigma=0} = 0 \tag{3.95}$$

If $\sigma(x)$ and $\dot{\sigma}(x)$ are both driven to zero by an appropriate switching strategy on $u \in \{0,1\}$, while the transversal condition, $L_g\sigma_0(x) > 0$, is being necessarily respected, then under ideal sliding mode conditions, on $\sigma(x) = 0$, $u_{1eq}(x(t)) = -u_h(x,t)$ for all t. The equivalent control $u_{1eq}(x)$, associated with the sliding surface characterized by $\sigma(x)$, ideally cancels the unknown disturbance input. The nominal equivalent control $u^0_{eq}(x)$ remains largely unaffected.

The advantages of integral sliding mode control entitle the preservation of the region of existence of a sliding regime on the sliding surface $\mathcal{S}$.

It is easy to reinterpret Integral Sliding Modes in terms of a nominal design (yielding the signal σ_1) plus a Delta-Sigma modulation controller, defined on $\sigma(x)$, watching over the unexpected appearance of the matched disturbances. Recall that the above analysis is particularized under ideal sliding motions on $\sigma = 0$ and that $u_h(x,t)$ is unknown.

With the help of the defined σ_1, the dynamics sliding surface coordinate, $\sigma(x)$, satisfies, in general:

$$\dot{\sigma}(x) = (L_g\sigma_0(x))u = (L_g\sigma_0(x))(u^0_{eq}(x) + u_1) \tag{3.96}$$

The signal $u^0_{eq}(x(t)$ is the input to the Delta-Sigma modulator and the switching signal $-u_1$ is the locally decoded signal being subtracted from $u^0_{eq}(x)$. Since $L_g\sigma_0(x) > 0$ it follows that Property 2 applies (see equation (3.46)) and, therefore, (3.96) indeed represents a Delta-Sigma modulation process.

4

Multi-variable sliding mode control

4.1 Introduction

Sliding mode control of multi-variable nonlinear systems has arisen special interest due to its potential for many applications in interesting technological areas such as robotics, aerospace, power electronics, and multi-phase drive position control. The essential difficulties of sliding mode in multi-variable (addressed here as MIMO) systems arise from the fact that several smooth surfaces have to be considered. In fact, there should exist as many sliding surfaces as independent control inputs. This fact already creates a first decision problem. Should each control input try to induce sliding motions on a particular one of the surfaces in a decoupled fashion? If the answer is yes, which one of the surfaces in the given finite set? If the answer is no, should then some 'teaming" of controllers be necessary to induce sliding motions in a particular sliding surface? What if this cannot be readily accomplished with the binary valued control input available to each commanded switch? Moreover, what if the achieved sliding motions, on each manifold, do not lead the trajectory towards the intersection of the set of sliding surfaces where all the desired state restrictions are satisfied? To make things worse, it is widely known that a sliding motion may exist on the intersection of a number of smooth sliding surfaces without necessarily locally existing on some, or any, of the sliding surfaces.

Many of these imprecisely formulated problems occupied the minds of control theorists, working in sliding mode control, for a long, long, time. Some very interesting techniques emerged as highly reasonable, but with some lack of systematic procedural recipes for the several applications areas which were in line waiting for clear theoretical results. This was the case of the

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"method of the hierarchy of controls" originally proposed by Utkin in [31] and which enjoyed popularity specially in initial applications to the robotics area. The picture has become clearer in recent years with the advent of, both, geometric and algebraic theories of nonlinear systems. The understanding of the structural aspects of nonlinear systems is the key that has allowed to propose and solve, with relative simplicity, the MIMO nonlinear sliding mode control problem. The concept of flatness, to our belief, completely answers the fundamental questions for a class of widespread, and particularly interesting, switched nonlinear systems. This development will be examined in detail in Chapter 6.

In this chapter, we use some elementary notions of differential geometry to address the problem of sliding mode creation in MIMO systems. We provide a number of illustrative examples, of physical significance, to which the technique can be applied.

4.2 Multiple Input Multiple Output case

The general description of systems controlled by multiple independent switches corresponds, within the framework of the state space representation, to the following form:

$$\dot{x} = f(x) + G(x)u, \quad y = \sigma(x) \tag{4.1}$$

where $x \in \mathbb{R}^n$, $u \in \{0,1\}^m$, and $y \in \mathbb{R}^m$. The notation $u \in \{0,1\}^m$ indicates that each component u_i, $i = 1, \ldots, m$, of the input vector u takes values in the binary set $\{0,1\}$ representing an independent switch position function. The function $f(x)$ is a smooth vector field defined over the tangent space to $\mathbb{R}^n$ and usually addressed as the *drift vector field*. $G(x)$ is a matrix whose entries are smooth functions of the state x of the system and its dimensions are $n \times m$, i.e., n rows and m columns. The columns of $G(x)$, denoted by means of $g_i(x), i = 1, 2, \ldots, m$ also represent smooth vector fields. The matrix $G(x)$ is called the *input matrix*. The output function $\sigma(x)$ is a smooth map taking values in $\mathbb{R}^m$. We refer to the point $x \in \mathbb{R}^n$ as the *state vector* of the system, while u is the *input vector* and y is the *output vector*.

Example 4.1. The following circuit shown in Figure 4.1 represents a *multi-input* DC to DC power converter controlled by two switches and known as the "boost-boost" converter. This converter clearly has two stages, each one controlled by means of an independent switch position function,

The differential equations describing the system are the following:

$$L_1 \frac{di_1}{dt} = -u_1 v_1 + E$$
$$C_1 \frac{dv_1}{dt} = u_1 i_1 - \frac{1}{R_1} v_1 - i_2$$

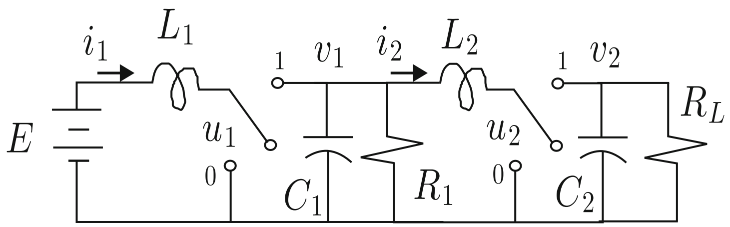

Fig. 4.1. Boost-Boost converter circuit

$$L_2 \frac{di_2}{dt} = -u_2 v_2 + v_1$$
$$C_2 \frac{dv_2}{dt} = u_2 i_2 - \frac{1}{R_L} v_2 \tag{4.2}$$

where i_1 is the input current, v_1 is the output voltage of the first stage, i_2 is the input current to the second stage, and v_2 represents the output voltage of the second stage.

In matrix terms, the mathematical description of the system is given by

$$\frac{d}{dt}\begin{bmatrix} i_1 \\ v_1 \\ i_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{R_1 C_1} & -\frac{1}{C_1} & 0 \\ 0 & \frac{1}{L_2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{R_L C_2} \end{bmatrix} \begin{bmatrix} i_1 \\ v_1 \\ i_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} -\frac{v_1}{L_1} & 0 \\ \frac{i_1}{L_1} & 0 \\ 0 & -\frac{v_2}{L_2} \\ 0 & \frac{i_2}{C_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \frac{E}{L_1} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Here, evidently, letting $x = [i_1 \ v_1 \ i_2 \ v_2]^T$ yields the following expressions for the drift vector field $f(x)$ and the input matrix $G(x)$,

$$f(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{R_1 C_1} & -\frac{1}{C_1} & 0 \\ 0 & \frac{1}{L_2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{R_L C_2} \end{bmatrix} x + \begin{bmatrix} \frac{E}{L_1} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad G(x) = \begin{bmatrix} -\frac{x_2}{L_1} & 0 \\ \frac{x_1}{L_1} & 0 \\ 0 & -\frac{x_4}{L_2} \\ 0 & \frac{x_3}{C_2} \end{bmatrix} \tag{4.3}$$

4.3 Sliding surfaces

In the context of n dimensional controlled systems regulated by m independent switches and where m sliding surface coordinate functions are defined as system outputs, a sliding surface is represented by the simultaneous satisfaction of m smooth algebraic state restrictions summarized in the equation: $\sigma(x) = 0$ which represents the intersection manifold:

$$\mathcal{S} = \{x \in \mathbb{R}^n \mid \sigma_i(x) = 0, \quad i = 1, 2, \ldots, m\} = \bigcap_{i=1}^{m} \mathcal{S}_i \tag{4.4}$$

The fundamental assumption is the following: *The simultaneous satisfaction of the m restrictions, $\sigma_i(x) = 0, i = 1, 2, \cdots, m$, on the part of the controlled state vector trajectory $x(t)$, ideally produces a desired closed loop behavior for the system. These restrictions are represented by a smooth intersection manifold, $\mathcal{S}$, locally of dimension $n - m$. The condition $x \in \mathcal{S}$ is achieved thanks to the control actions which, in turn, are restricted by: $u \in \{0, 1\}^m$, i.e., $u_i \in \{0, 1\}$ for $i = 1, 2, \cdots, m$.*

The smooth algebraic restrictions, $\sigma_i(x) = 0$, $i = 1, \ldots, m$, define m smooth manifolds in $\mathbb{R}^n$, each one of dimension $n-1$. We denote each smooth manifold by $\mathcal{S}_i$, and define it as

$$\mathcal{S}_i = \{x \in \mathbb{R}^n \mid \sigma_i(x) = 0\} \tag{4.5}$$

The intersection of the m smooth manifolds $\mathcal{S}_i$ is denoted by $\mathcal{S}$ and it is defined as follows:

$$\mathcal{S} = \{x \in \mathbb{R}^n \mid x \in \mathcal{S}_i, \ i = 1, 2, \cdots, m\} \tag{4.6}$$

One of the primordial facets of the design of feedback control laws for MIMO switch regulated systems is given by the fact that the *m smooth functions, $sigma_i(x)$, constitute an integral part of the control design problem. The choice of the outputs and, therefore, of the restrictions $\sigma_i(x) = 0$, i.e., of $\mathcal{S} = \bigcap \mathcal{S}_i$, depend entirely on our* control objective.

In order to avoid parallelism between the zero level sets of the sliding surfaces, we enforce the assumption that the map $\sigma : \mathbb{R}^n \to \mathbb{R}^m$ is full rank m. In other words, the set of row gradients $\partial\sigma_i(x)/\partial x^T$, $i = 1, 2, \cdots, m$, are locally linearly independent around an arbitrary operating point, x^0.

4.4 Some notation

Let $f(x)$ be a smooth vector field, defined on the tangent space to $\mathbb{R}^n$ and let $G(x)$ be a smooth matrix constituted by m columns representing smooth vector fields, $g_i(x), i = 1, 2, \cdots, m$. We assume that *dim* $\{\text{span } G(x)\} = m$, i.e., that span $G(x)$ is a proper m-dimensional subspace of the tangent space to $\mathbb{R}^n$. Also, we say that the *range* of $G(x)$ is m. Let $\sigma(x)$ be a smooth m dimensional map, i.e., one taking values in $\mathbb{R}^m$. We represent $\sigma(x)$ as an m vector of components $\sigma_i(x)$, $i = 1, 2, \cdots, m$. The matrix of row gradient vectors $\partial\sigma/\partial x^T$ is an $m \times n$ matrix and assumed to be locally full rank m.

We define the *directional derivative of a smooth vector function $\sigma(x)$, along the direction of the vector field $f(x)$,* as the m column vector quantity: $(\partial\sigma/\partial x^T)f(x)$. Each entry of the preceding column vector is of the form: $(\partial\sigma_i(x)/\partial x^T)f(x)= L_f\sigma_i(x)$.

$$L_f\sigma(x) = \begin{bmatrix} L_f\sigma_i(x) \\ \vdots \\ L_f\sigma_m(x) \end{bmatrix} \tag{4.7}$$

Similarly, we denote by $L_G\sigma(x)$, the following $m \times m$ matrix:

$$L_G\sigma(x) = \frac{\partial \sigma}{\partial x^T} G(x) = \frac{\partial \sigma}{\partial x^T} \left[g_1(x), \cdots, g_m(x)\right]$$
$$= \left[L_{g_1}\sigma(x), L_{g_2}\sigma(x), \cdots, L_{g_m}\sigma(x)\right] \qquad (4.8)$$

Notice that

$$L_G\sigma(x) = \begin{bmatrix} L_{g_1}\sigma_1(x) & L_{g_2}\sigma_1(x) & \cdots & L_{g_m}\sigma_1(x) \\ L_{g_1}\sigma_2(x) & L_{g_2}\sigma_2(x) & \cdots & L_{g_m}\sigma_2(x) \\ \vdots & \vdots & \ddots & \vdots \\ L_{g_1}\sigma_m(x) & L_{g_2}\sigma_m(x) & \cdots & L_{g_m}\sigma_m(x) \end{bmatrix} \qquad (4.9)$$

The simplest possible inputs-to-sliding surface coordinate functions relation occurs when the $m \times m$ matrix, $L_G\sigma(x)$ is locally invertible. This means that the first order time derivatives of each one of the sliding surface vector components: $\sigma_j(x)$, $j = 1, 2, \cdots, m$, depends on at least one, or some, of the control inputs but in a linearly independent manner. The invertibility assumption on $L_G\sigma(x)$ corresponds to a slight generalization of the "relative degree one" assumption in the scalar case. In the MIMO case, we say that the *vector relative degree* of $\sigma(x)$ is the m vector: $\mathbf{1}_m = (1, 1, \ldots, 1)$.

Example 4.2. Consider the following simplified model of an n-link robotic manipulator

$$M(q)\ddot{q} = C(q, \dot{q})\dot{q} + \tau \qquad (4.10)$$

where $q \in \mathbb{R}^n$ are the link positions and $\dot{q} \in \mathbb{R}^n$ are the link velocities. $M(q) \in \mathbb{R}^{n \times n}$ is a symmetric, positive definite matrix (i.e., it is globally invertible). The vector τ is the vector of independent control inputs $\tau = (\tau_1, \ldots, \tau_n)$. Let $\sigma(q, \dot{q})$ be given by the n vector $\sigma = \dot{q} + \Lambda(q - q^*)$ with Λ a full rank $n \times n$ positive definite symmetric matrix and q^* is a desired constant link position vector.

Clearly $x = (q^T, \dot{q}^T)^T = (x_1^T, x_2^T)^T$, $u = \tau$, $\sigma = x_2 + \Lambda(x_1 - q^*)$ and,

$$f(x) = \begin{pmatrix} x_2 \\ M^{-1}(x_1)C(x_1, x_2)x_2 \end{pmatrix}, \quad G(x) = \begin{pmatrix} 0 \\ M^{-1}(x_1) \end{pmatrix} \qquad (4.11)$$

The gradient matrix of the vector σ is $\partial\sigma/\partial x^T = [\Lambda; I]$. The column vector $L_f\sigma$ and the matrix $L_G\sigma$ are given by

$$L_f\sigma = \left(\Lambda + M^{-1}(x_1)C(x_1, x_2)\right)x_2, \quad L_G\sigma = [\Lambda; I]\begin{pmatrix} 0 \\ M^{-1}(x_1) \end{pmatrix} = M^{-1}(x_1) \qquad (4.12)$$

i.e., $L_G\sigma$ is invertible and σ is, globally, vector relative degree: $\mathbf{1}_n = (1, 1, \ldots, 1)$.

4.5 Equivalent control and ideal sliding dynamics

Let us assume that, somehow, through the application of suitable feedback control laws that define the position of the switches, we manage to make the condition: $x \in \mathcal{S}$ valid, even if this is only a local achievement. In other words, by appropriate commutations, we force the state x to evolve on the intersection of all the smooth manifolds $\mathcal{S}_i$, that represent the desired algebraic state restrictions which force the system to satisfy the specified control objectives.

We define the *equivalent control* as the smooth feedback control law, denoted by $u_{eq}(x)$, which, ideally, locally sustains the state evolution on the smooth manifold $\mathcal{S}$, provided the initial state of the system happens to be located on $\mathcal{S}$. The equivalent control enforces upon any state trajectory starting on $\mathcal{S}$ the invariance conditions: $\sigma = 0$, $\dot{\sigma} = 0$.

Let $L_G\sigma(x)$ be locally invertible. The vector of sliding surface coordinate functions, $\sigma(x)$, satisfies, on $\sigma = 0$, then:

$$\dot{\sigma}(x) = \frac{\partial \sigma}{\partial x^T}(f(x) + G(x)u_{eq}(x)) = 0 \tag{4.13}$$

i.e.,

$$L_f\sigma(x) + [L_G\sigma(x)]u_{eq}(x)\Big|_{\sigma=0} = 0 \tag{4.14}$$

and, therefore, the equivalent control is expressed, in a unique fashion as:

$$u_{eq}(x) = -[L_G\sigma(x)]^{-1}L_f\sigma(x)\Big|_{\sigma=0} \tag{4.15}$$

The closed loop controlled vector field evolving on the manifold $\mathcal{S}$ is expressed as:

$$\dot{x} = f(x) - G(x)[L_G\sigma(x)]^{-1}L_f\sigma(x) \tag{4.16}$$

Note that for any other initial condition which is not located on the smooth manifold $\mathcal{S}$, the state of the system, under the influence of the control $u = -[L_G\sigma(x)]^{-1}L_f\sigma(x)$, evolves in such a manner that $\sigma(x)$ remains a constant vector function. Clearly, this constant value adopts the value 0 only when the initial state x_0 satisfies $x_0 \in \mathcal{S}$. The closed loop system, virtually controlled by the equivalent control, may be alternatively written as

$$\dot{x} = \left\{ I - G(x)[L_G\sigma(x)]^{-1}\frac{\partial \sigma}{\partial x^T} \right\} f(x)\Big|_{\sigma=0} = \mathcal{M}(x)f(x)\Big|_{\sigma=0} \tag{4.17}$$

Proposition 4.3. *The square $n \times n$, matrix $\mathcal{M}(x)$, is a* **projection operator***, onto the tangent space of $\mathcal{S}$, whose null space is represented by the* span $G(x)$. *In other words, $\mathcal{M}(x)$ projects any smooth vector field lying in the tangent space to $\mathbb{R}^n$ onto the tangent subspace to $\mathcal{S}$ along the span of $G(x)$, or in a parallel fashion to* span$G(x)$.

Indeed, let v be a vector field defined in the tangent space to $\mathbb{R}^n$ such that $v \in$ span $G(x)$ i.e. , v may be expressed as $v(x) = G(x)\alpha(x)$ for a certain m-dimensional smooth vector field $\alpha(x)$. Then,

$$\begin{aligned}\mathcal{M}(x)v(x) &= \left\{ I - G(x)[L_G\sigma(x)]^{-1}\frac{\partial\sigma}{\partial x^T} \right\} G(x)\alpha(x) \\ &= \left\{ G(x) - G(x)[L_G\sigma(x)]^{-1}\frac{\partial\sigma}{\partial x^T}G(x) \right\} \alpha(x) \\ &= \left\{ G(x) - G(x)[L_G\sigma(x)]^{-1}L_G\sigma(x) \right\} \alpha(x) \\ &= [G(x) - G(x)]\,\alpha(x) = 0 \end{aligned} \tag{4.18}$$

Additionally, the n-dimensional row vectors of the matrix $\frac{\partial\sigma}{\partial x^T}$ are all orthogonal to the images under $\mathcal{M}(x)$ of the vector fields lying in the tangent space to $\mathbb{R}^n$. To see this, it is enough to demonstrate that any 1-form lying in the span de $\frac{\partial\sigma}{\partial x^T}$ annihilates all the vector fields constituting the matrix, $\mathcal{M}(x)$.

A 1-form in the span of $\frac{\partial\sigma}{\partial x^T}$ is written as: $\xi^T(x)\frac{\partial\sigma}{\partial x^T}$, where $\xi^T(x)$ is a nonzero, completely arbitrary, m-dimensional vector field.

Indeed:

$$\begin{aligned}\xi^T(x)\frac{\partial\sigma}{\partial x^T}\mathcal{M}(x) &= \xi^T(x)\frac{\partial\sigma}{\partial x^T}\left\{ I - G(x)[L_G\sigma(x)]^{-1}\frac{\partial\sigma}{\partial x^T} \right\} \\ &= \xi^T(x)\left[\frac{\partial\sigma}{\partial x^T} - L_G\sigma(x)[L_G\sigma(x)]^{-1}\frac{\partial\sigma}{\partial x^T}\right] \\ &= \xi^T(x)\left[\frac{\partial\sigma}{\partial x^T} - \frac{\partial\sigma}{\partial x^T}\right] = 0 \end{aligned} \tag{4.19}$$

The images under $\mathcal{M}(x)$ of any vector lying in the tangent space of $\mathbb{R}^n$ is in the null space of $\frac{\partial\sigma}{\partial x^T}$. In other words, they are in the tangent subspace to $\mathcal{S}$.

It is clear that $\mathcal{M}^2(x) = \mathcal{M}(x)$ given that $\mathcal{M}(x)G(x) = 0$.

Example 4.4. In the n link manipulator example described above, on $\sigma = 0$, one has $x_2 = -\Lambda(x_1 - q^*)$. The equivalent control is given by

$$u_{eq}(x) = -(L_G\sigma)^{-1}L_f\sigma = -(M(x_1)\Lambda + C(x_1, x_2))x_2 \tag{4.20}$$

and the ideal sliding dynamics is found to be

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \left(I - \begin{pmatrix} 0 \\ M^{-1}(x_1) \end{pmatrix} M(x_1)[\Lambda; I] \right) \begin{pmatrix} x_2 \\ M^{-1}(x_1)C(x_1, x_2)x_2 \end{pmatrix} \\ &= \begin{pmatrix} x_2 \\ M^{-1}(x_1)C(x_1, x_2)x_2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \Lambda & I \end{pmatrix} \begin{pmatrix} x_2 \\ M^{-1}(x_1)C(x_1, x_2)x_2 \end{pmatrix} \\ &= \begin{pmatrix} x_2 \\ -\Lambda x_2 \end{pmatrix} = \begin{pmatrix} -\Lambda(x_1 - q^*) \\ -\Lambda x_2 \end{pmatrix} \end{aligned} \tag{4.21}$$

i.e., on $\sigma(x) = 0$, the link positions are ideally governed by

$$\dot{x}_1 = -\Lambda(x_1 - q^*), \quad x_2 = -\Lambda(x_1 - q^*) \tag{4.22}$$

The equilibrium position vector, q^*, is a globally asymptotically exponentially stable equilibrium in $\mathbb{R}^n$.

4.6 Invariance with respect to matched perturbations

Consider the treated MIMO nonlinear system, additively perturbed by an unknown, possibly state dependent, vector field, of unknown nature denoted by, $\xi(x)$, affecting the system as follows: $\dot{x} = f(x) + G(x)u + \xi(x)$. The system is assumed to be controlled by a set of m independent switches acting as control inputs. Let $\mathcal{S}$ be a sliding surface, obtained as the intersection of m smooth manifolds represented by the algebraic conditions: $h_i(x) = 0$ for $i = 1, 2, \cdots, m$. Over this sliding surface, $\mathcal{S}$, we want to induce a forced trajectory of the system state as that obtained through the creation of a sliding regime, even if this achievement is only locally valid. The perturbation field $\xi(x)$ is assumed to be a *bounded* function of the state of the system.

Assume we may create a sliding motion on the sliding surface: $\mathcal{S}$ in spite of the presence of the perturbation field $\xi(x)$. The existence of such a sliding regime implies the existence of a smooth control, the perturbed equivalent control, still denoted by: u_{eq}, which in an ideal fashion would maintain the trajectories of the system constrained to the manifold $\mathcal{S}$.

Necessarily, the equivalent control in this case is a function of the unknown vector field $\xi(x)$ and it would be given by

$$u_{eq}(x) = -[L_G\sigma(x)]^{-1}\left(L_f\sigma(x) + L_\xi\sigma(x)\right) \tag{4.23}$$

The corresponding ideal sliding dynamics is given by

$$\begin{aligned}\dot{x} &= f(x) - G(x)[L_G\sigma(x)]^{-1}\left(L_f\sigma(x) + L_\xi\sigma(x)\right) \\ &= \left[I - G(x)[L_G\sigma(x)]^{-1}\frac{\partial\sigma}{\partial x^T}\right] f(x) \\ &\quad + \left[I - G(x)[L_G\sigma(x)]^{-1}\frac{\partial\sigma}{\partial x^T}\right]\xi(x)\end{aligned} \tag{4.24}$$

The projection operator $\mathcal{M}(x)$ over the tangent space to $\mathcal{S}$, parallel to the span of $G(x)$, acts over the sum of vector fields $f(x) + \xi(x)$, in the creation of a sliding regime on $\mathcal{S}$.

Clearly, the ideal sliding dynamics is totally independent of the perturbation input vector $\xi(x)$, if and only if the vector field $\xi(x)$ lies in the null space of $\mathcal{M}(x)$, i.e.,

$$\left[I - G(x)[L_G\sigma(x)]^{-1}\frac{\partial\sigma}{\partial x^T}\right]\xi(x) = 0 \tag{4.25}$$

The ideal sliding dynamics is invariant with respect to the perturbation field if and only if the vector $\xi(x)$ belongs to the span of $G(x)$. There exists then a nonzero vector function taking values in R^m, denoted by $\alpha(x)$, such that

$$\xi(x) = G(x)\alpha(x) \tag{4.26}$$

The perturbation field $\xi(x)$ is contained in the span of the columns of $G(x)$. Such perturbations receive the name of *matched perturbations* and the previous condition is known as the *matching condition*.

4.7 Reachability of the sliding surface

Consider the scalar quantity:

$$V(y) = \frac{1}{2}y^T y = \frac{1}{2}\sigma^T(x)\sigma(x) \geq 0 \tag{4.27}$$

This quantity represents a sort of instantaneous sliding surface "output error energy" quadratically measuring the distance from the representative point x in the state space to the smooth manifold $\mathcal{S}$. The quantity $V(y)$ is identically zero precisely over the manifold $\mathcal{S}$ and it represents a positive semi-definite function of the multi-variable sliding surface coordinate function $y = \sigma$.

Therefore, a plausible strategy to reach the sliding surface from a neighborhood of the manifold $\mathcal{S}$ which allows us to satisfy the desired restriction $\sigma(x) = 0$ is to exercise control actions $u \in \{0,1\}^m$ that result in a strict decrease of the quantity $V(\sigma(x))$.

This is achieved influencing the system in such a manner that the velocity of variation of $V(\sigma(x))$ be strictly negative. This means

$$\frac{d}{dt}\left(V(\sigma(x))\right) = \frac{1}{2}\frac{d}{dt}\left(\sigma^T(x)\sigma(x)\right) = \sigma^T(x)\dot{\sigma}(x) < 0 \tag{4.28}$$

Using the relation, $\dot{\sigma}(x) = L_f\sigma(x) + L_G\sigma(x)u$ and realizing that $L_f\sigma(x) + L_G\sigma(x)u_{eq} = 0$ for any $x \notin \mathcal{S}$ and further adding and subtracting the quantity: $L_G\sigma(x)u_{eq}$ to the first order time derivative of $\sigma(x)$ in the previous expression, we have the following relations:

$$\begin{aligned}\sigma^T(L_f\sigma(x) + L_G\sigma(x)u) &= \sigma^T\left(L_f\sigma(x) + L_G\sigma(x)(u - u_{eq})\right.\\ &\quad \left.+ L_G\sigma(x)u_{eq}\right)\\ &= \sigma^T L_G\sigma(x)(u - u_{eq}) < 0\end{aligned} \tag{4.29}$$

This inequality may be expressed in the following manner:

$$\begin{aligned}&\sigma^T[L_{g_1}\sigma]u_1 + \sigma^T[L_{g_2}h]u_2 + \cdots \sigma^T[L_{g_m}\sigma]u_m <\\ &\sigma^T[L_{g_1}\sigma]u_{1eq} + \sigma^T[L_{g_2}\sigma]u_{2eq} + \cdots \sigma^T[L_{g_m}\sigma]u_{meq}\end{aligned} \tag{4.30}$$

A sufficient condition to achieve this last inequality is to apply one of the two possible values for $u_j, j = 1, \ldots, m$, according to the sign of the factor multiplying the control input u_j represented by $\sigma^T L_{g_j} h$. We use then

$$u_j = \begin{cases} 1 \text{ if } \sigma^T L_{g_j} \sigma(x) < 0 \\ 0 \text{ if } \sigma^T L_{g_j} \sigma(x) > 0 \end{cases} \tag{4.31}$$

In other words,

$$u_j = \frac{1}{2} \left[1 - \text{sign} \left(\sigma^T L_{g_j} \sigma(x) \right) \right] \tag{4.32}$$

As usual, let $\mathbf{1}_m$ be an m dimensional column vector constituted by 1's, i.e., $\mathbf{1}_m = [1, 1, \ldots, 1]^T$. The suggested control law is written as follows:

$$u = \frac{1}{2} \left[\mathbf{1}_m - \text{SIGN} \left(\sigma^T L_G \sigma(x) \right)^T \right] \tag{4.33}$$

4.8 Control of the Boost-Boost converter

We retake the multi-variable "Boost-Boost" example with the following simplification: $L_1 = L_2 = L$ y $C_1 = C_2 = C$. We also carry out the following *normalization* of the state variables, the time and a redefinition of the resistances

$$x_1 = \frac{i_1}{E}\sqrt{\frac{L}{C}}, \; x_2 = \frac{v_1}{E}, \; x_3 = \frac{i_2}{E}\sqrt{\frac{L}{C}}, \; x_4 = \frac{v_2}{E},$$

$$\tau = \frac{t}{\sqrt{LC}}, \; Q = R\sqrt{\frac{C}{L}}, \; Q_L = R_L\sqrt{\frac{C}{L}}$$

The normalized model results then in the following set of differential equations

$$\begin{aligned} \dot{x}_1 &= -u_1 x_2 + 1 \\ \dot{x}_2 &= u_1 x_1 - \frac{1}{Q} x_2 - x_3 \\ \dot{x}_3 &= -u_2 x_4 + x_2 \\ \dot{x}_4 &= u_2 x_3 - \frac{x_4}{Q_L} \end{aligned} \tag{4.34}$$

Using the previously introduced notation, of vector fields, we have

$$f(x) = \begin{bmatrix} 1 \\ -x_3 - \frac{1}{Q} x_2 \\ x_2 \\ -\frac{1}{Q_L} x_4 \end{bmatrix}, \quad G(x) = \begin{bmatrix} -x_2 & 0 \\ x_1 & 0 \\ 0 & -x_4 \\ 0 & x_3 \end{bmatrix} \tag{4.35}$$

The control objective is to have the normalized average capacitor voltages x_2 and x_4 to adopt the following desired values $\overline{X}_2$, $\overline{X}_4$, respectively. We try the following sliding surface coordinate functions:

$$\sigma_1(x) = x_2 - \overline{X}_2, \quad \sigma_2(x) = x_4 - \overline{X}_4 \tag{4.36}$$

Clearly, forcing to zero the vector of sliding surface coordinate functions means that the capacitor voltages reach the desired equilibrium values. We must nevertheless establish the nature and stability of the corresponding zero dynamics.

In our case we have

$$\begin{aligned} L_f\sigma(x) &= \frac{\partial \sigma}{\partial x^T} f(x) = \begin{bmatrix} -x_3 - \frac{1}{Q}x_2 \\ -\frac{1}{Q_L}x_4 \end{bmatrix}, \\ L_G\sigma(x) &= \frac{\partial \sigma}{\partial x^T} G(x) = \begin{bmatrix} x_1 & 0 \\ 0 & x_3 \end{bmatrix} \end{aligned} \tag{4.37}$$

and the equivalent control is given by

$$u_{eq}(x) = -[L_G\sigma(x)]^{-1} L_f\sigma(x) = \begin{bmatrix} \dfrac{x_3 + (1/Q)x_2}{x_1} \\ \dfrac{1}{Q_L}\left(\dfrac{x_4}{x_3}\right) \end{bmatrix} \tag{4.38}$$

The ideal sliding dynamics occurs when $u_{eq}(x)$ acts over the system and this satisfies the conditions: $x_2 = \overline{X}_2$ and $x_4 = \overline{X}_3$. We then have

$$\begin{aligned} \dot{x}_1 &= -\left(\frac{x_3 + (1/Q)\overline{X}_2}{x_1}\right)\overline{X}_2 + E \\ \dot{x}_3 &= -\frac{1}{Q_L}\left(\frac{\overline{X}_4^2}{x_3}\right) + \overline{X}_2 \end{aligned} \tag{4.39}$$

It is not difficult to see that these set of dynamics is unstable around the desired equilibrium point.

The alternative is then to use as coordinate functions of the sliding surfaces other functions which stably reproduce the desired output voltages when forced to be zero. These alternative functions are represented by the stabilization errors of the input inductor currents in each stage of the cascaded system.

$$\sigma_1(x) = x_1 - \overline{X}_1, \quad \sigma_2(x) = x_3 - \overline{X}_3 \tag{4.40}$$

To specify these functions we compute the state and input equilibrium points in terms of the desired average output equilibrium voltages,

$$\overline{X_1} = \frac{1}{Q}\overline{X}_2^2 + \frac{1}{Q_L}\overline{X}_4^2, \quad \overline{X}_3 = \frac{1}{Q_L}\left(\frac{\overline{X}_4^2}{\overline{X}_2}\right) \tag{4.41}$$

We now have

$$L_f\sigma(x) = \begin{bmatrix} 1 \\ x_2 \end{bmatrix}, \quad L_G\sigma(x) = \begin{bmatrix} -x_2 & 0 \\ 0 & -x_4 \end{bmatrix} \tag{4.42}$$

The equivalent control is thus given by

$$u_{eq}(x) = \begin{bmatrix} \dfrac{1}{x_2} \\ \\ \dfrac{x_2}{x_4} \end{bmatrix} \tag{4.43}$$

The ideal sliding dynamics corresponding to $X_1 = \overline{X}_1$, $x_3 = \overline{X}_3$ is given by

$$\begin{aligned} \dot{x}_2 &= \frac{\overline{X}_1}{x_2} - \frac{x_2}{Q} - \overline{X_3} \\ \dot{x}_4 &= \frac{x_2}{x_4}\overline{X_3} - \frac{x_4}{Q_L} \end{aligned} \tag{4.44}$$

It is easy to verify that the obtained zero dynamics have the desired average output equilibrium voltages as asymptotically stable equilibria.

According to the developed theory, the intersection of the sliding surfaces is reachable by means of the following switching policy:

$$\begin{aligned} u_1 &= \begin{cases} 1 \text{ if } (x_1 - \overline{X}_1)x_2 > 0 \\ 0 \text{ if } (x_1 - \overline{X}_1)x_2 < 0 \end{cases} \\ u_2 &= \begin{cases} 1 \text{ if } (x_3 - \overline{X}_3)x_4 > 0 \\ 0 \text{ if } (x_3 - \overline{X}_3)x_4 < 0 \end{cases} \end{aligned} \tag{4.45}$$

4.8.1 Simulations

We take a typical converter with the following parameters

$$L_1 = L_2 = L = 0.01 \text{ [H]}, \quad C_1 = C_2 = 10^{-4} \text{ [F]}, \quad \mathrm{R_1} = \mathrm{R_L} = 100 \text{ [}\Omega\text{]}$$

It is desired to control the capacitor voltages to the values:

$$V_{1d} = 60.03 \text{ [V]}, \quad V_{2d} = 238.80 \text{ [V]}.$$

The equilibrium values of the average input inductor currents to each stage correspond approximately to the following values:

$$I_{1d} = 20.40 \text{ [A]}, \quad I_{2d} = 9.59 \text{ [A]}$$

Figures 4.2 and 4.3 depict the sliding mode controlled responses of the multi-variable boost-boost converter.

4.9 Control of the double buck-boost converter

Consider the composite converter constituted by the cascade connection of two stages of the "buck-boost" converter, which we address as the *double buck-boost* converter, shown in the Figure 4.4. This is clearly a MIMO converter regulated by two independent switches.

The set of differential equations describing the converter dynamics is readily obtained from the use of *Kirchoff's laws*, considering the four possible cases

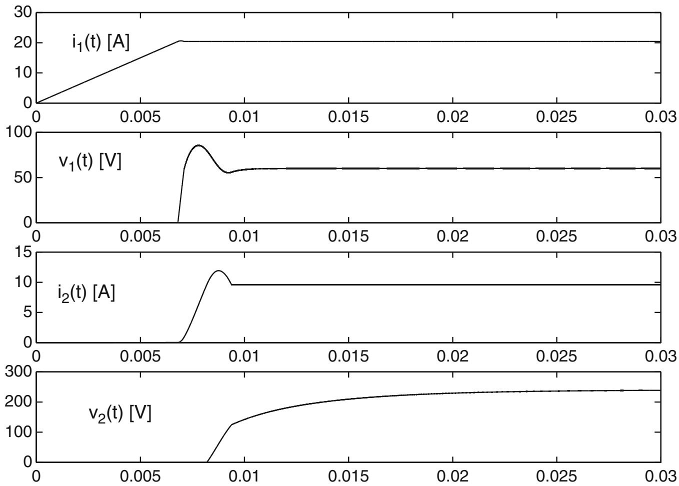

Fig. 4.2. State variable responses of sliding mode controlled boost-boost converter

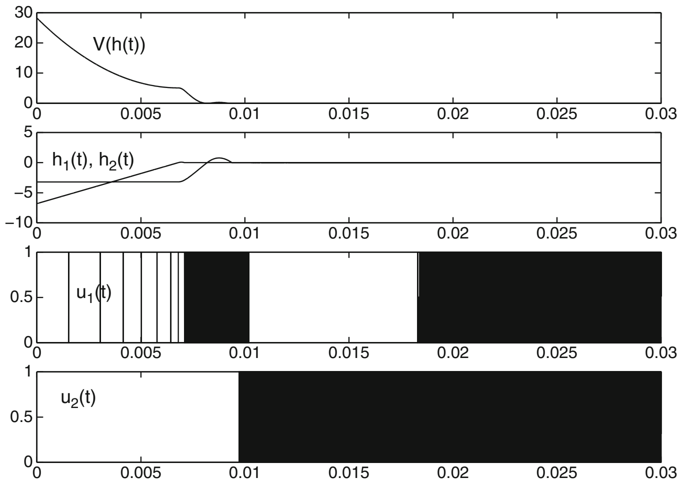

Fig. 4.3. Sliding surfaces, control inputs trajectories of sliding mode controlled boost-boost converter

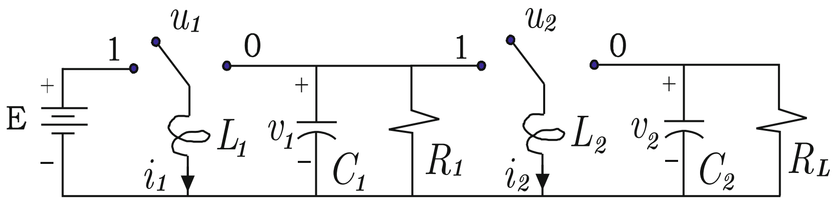

Fig. 4.4. Double buck-boost converter circuit

for the constant values of the control inputs: (u_1, u_2) and proceeding to obtain a traditional state model combining the four possibilities. The switched model is then given by

$$
\begin{aligned}
L_1 \frac{di_1}{dt} &= (1 - u_1)\, v_1 + u_1 E \\
C_1 \frac{dv_1}{dt} &= -\,(1 - u_1)\, i_1 - \frac{v_1}{R_1} \\
L_2 \frac{di_2}{dt} &= u_2 v_1 + (1 - u_2)\, v_2 \\
C_2 \frac{dv_2}{dt} &= -\,(1 - u_2)\, i_2 - \frac{v_2}{R_L}
\end{aligned}
\tag{4.46}
$$

where i_1 is the current in the first stage input inductor. v_1 is the output capacitor voltage at the first stage. i_2 is the input current to the second stage inductor and v_2 is the second stage output capacitor voltage.

We carry out, as usual, the following *normalization* of the state variables and the time variable:

$$
x_1 = \frac{i_1}{E}\sqrt{\frac{L_1}{C_1}},\ x_2 = \frac{v_1}{E},\ x_3 = \frac{i_2}{E}\sqrt{\frac{L_1}{C_1}},\ x_4 = \frac{v_2}{E} \tag{4.47}
$$

$$
\tau = \frac{t}{\sqrt{L_1 C_1}},\ Q_{1,L} = R_{1,L}\sqrt{\frac{C_1}{L_1}},\ \alpha_1 = \frac{L_2}{L_1},\ \alpha_2 = \frac{C_2}{C_1} \tag{4.48}
$$

The normalized model results then in

$$
\begin{aligned}
\dot{x}_1 &= (1 - u_1)\, x_2 + u_1 \\
\dot{x}_2 &= -\,(1 - u_1)\, x_1 - \frac{1}{Q_1} x_2 \\
\alpha_1 \dot{x}_3 &= u_2 x_2 + (1 - u_2)\, x_4 \\
\alpha_2 \dot{x}_4 &= -\,(1 - u_2)\, x_3 - \frac{1}{Q_L} x_4
\end{aligned}
\tag{4.49}
$$

In steady state conditions $u_1 = U_1$ and $u_2 = U_2$ we obtain the following operating, or state equilibrium, point

$$
\bar{X}_1 = \frac{1}{Q_1}\frac{U_1}{(1 - U_1)^2}, \qquad \bar{X}_2 = -\frac{U_1}{(1 - U_1)} \tag{4.50}
$$

where

$$\bar{X}_3 = -\frac{1}{Q_L}\frac{U_1U_2}{(1-U_1)(1-U_2)^2}$$
$$\bar{X}_4 = \frac{U_1U_2}{(1-U_1)(1-U_2)} \qquad (4.51)$$

In terms of vector fields and input matrices we clearly have the following identification:

$$f(x) = \begin{bmatrix} x_2 \\ -x_1 - \frac{1}{Q_1}x_2 \\ \frac{1}{\alpha_1}x_4 \\ -\frac{1}{\alpha_2}\left(x_3 + \frac{1}{Q_L}x_4\right) \end{bmatrix}$$

$$G(x) = \begin{bmatrix} 1-x_2 & 0 \\ x_1 & 0 \\ 0 & \frac{1}{\alpha_1}(x_2 - x_4) \\ 0 & \frac{1}{\alpha_2}x_3 \end{bmatrix}$$

4.9.1 Direct control

The control objective consists in stably regulating the average normalized output voltages, υ_2 and υ_4, towards the desired equilibrium values: $\bar{X}_2$ and $\bar{X}_4$ respectively. We first try the following sliding surface coordinate functions:

$$\sigma_1(x) = x_2 - \bar{X}_2, \qquad \sigma_2(x) = x_4 - \bar{X}_4 \qquad (4.52)$$

Forcing these functions to zero means that the output capacitor voltages coincide with the desired values. We must establish the nature of the stability of the corresponding zero dynamics.

For this system we have

$$L_f\sigma(x) = \frac{\partial\sigma}{\partial x^T}f(x) = \begin{bmatrix} -\left(x_1 + \frac{1}{Q_1}x_2\right) \\ -\frac{1}{\alpha_2}\left(x_3 + \frac{1}{Q_L}x_4\right) \end{bmatrix}$$
$$L_G\sigma(x) = \frac{\partial\sigma}{\partial x^T}G(x) = \begin{bmatrix} x_1 & 0 \\ 0 & \frac{x_3}{\alpha_2} \end{bmatrix} \qquad (4.53)$$

The equivalent control is then given by

$$u_{eq}(x) = -\left[L_G\sigma(x)\right]^{-1}L_f\sigma(x) = \begin{bmatrix} 1 + \dfrac{1}{Q_1}\dfrac{x_2}{x_1} \\ 1 + \dfrac{1}{Q_L}\dfrac{x_4}{x_3} \end{bmatrix} \qquad (4.54)$$

The ideal sliding dynamics occurs whenever $u_{eq}(x)$ acts on the system and this is satisfying the sliding conditions: $x_2 = \overline{X}_2$ and $x_4 = \overline{X}_4$. We then have

$$\dot{x}_1 = -\frac{1}{Q_1}\left(\frac{\bar{X}_2 - 1}{x_1}\right)\bar{X}_2 + 1$$
$$\alpha_1 \dot{x}_3 = -\frac{1}{Q_L}\left(\frac{\bar{X}_4 - \bar{X}_2}{x_3}\right)\bar{X}_4 + \bar{X}_2 \tag{4.55}$$

It is not difficult to see that this dynamics is unstable around the equilibrium point. We show this fact below by means of approximate linearization.

The ideal sliding dynamics, or the zero dynamics, represents a decoupled system in the state variable x_1. We carry out the stability analysis for this variable around the equilibrium point. We have

$$\bar{X}_1 = \left(\bar{X}_2 - 1\right)\frac{\bar{X}_2}{Q_1} \tag{4.56}$$

The incremental model, or tangent linearization model, of the normalized input current is derived to be

$$\dot{x}_\delta = \frac{Q_1}{\left(\bar{X}_2 - 1\right)\bar{X}_2} x_\delta \tag{4.57}$$

where $x_\delta = x_1 - \bar{X}_1$. The linearized system is evidently unstable for being a linear system with a characteristic polynomial with a zero in the right hand of the complex plane, given that $\bar{X}_2 < 0$. The zero dynamics is therefore unstable regardless of the stability characteristics of the variable x_3.

4.9.2 Indirect control

The alternative is then to use as sliding surface coordinate functions which reproduce the desired values of the inductor currents when they become zero.

$$h_1(x) = x_1 - \bar{X}_1, \quad h_2(x) = x_3 - \bar{X}_3 \tag{4.58}$$

To specify these functions we compute the equilibrium points of the system under ideal sliding conditions, rewriting the corresponding currents $\bar{X}_1$ and $\bar{X}_3$ in terms of the desired output voltage values at the stages 1 and 2.

$$\bar{X}_1 = \left(\bar{X}_2 - 1\right)\frac{\bar{X}_2}{Q_1}, \quad \bar{X}_3 = \left(\frac{\bar{X}_4}{\bar{X}_2} - 1\right)\frac{\bar{X}_4}{Q_L} \tag{4.59}$$

We now have

$$L_f\sigma(x) = \begin{bmatrix} x_2 \\ \frac{1}{\alpha_1}x_4 \end{bmatrix}, \quad L_G\sigma(x) = \begin{bmatrix} 1 - x_2 & 0 \\ 0 & \frac{1}{\alpha_1}\left(x_2 - x_4\right) \end{bmatrix}$$

The equivalent control is then given by

$$u_{eq}(x) = -\left[L_G\sigma(x)\right]^{-1} L_f\sigma(x) = \begin{bmatrix} \frac{x_2}{x_2-1} \\ \frac{x_4}{x_4-x_2} \end{bmatrix} \tag{4.60}$$

In this case, the ideal sliding dynamics corresponding to $x_1 = \bar{X}_1$, $x_3 = \bar{X}_3$ is given by

$$\begin{aligned} \frac{dx_2}{d\tau} &= \left(\frac{1-\bar{X}_2}{1-x_2}\right)\frac{\bar{X}_2}{Q_1} - \frac{x_2}{Q_1} \\ \alpha_2\frac{dx_4}{d\tau} &= -\left(\frac{x_2}{x_2-x_4}\right)\left(\frac{\bar{X}_4}{\bar{X}_2} - 1\right)\frac{\bar{X}_4}{Q_L} - \frac{x_4}{Q_L} \end{aligned} \tag{4.61}$$

It is easy to verify that the equilibrium points of this zero dynamics are asymptotically stable.

According to the developed theory, the intersection of the sliding surfaces is reachable by means of the following switching policy

$$u_1 = \begin{cases} 1 & \text{if} \quad \left(x_1 - \bar{X}_1\right)(1-x_2) < 0 \\ 0 & \text{if} \quad \left(x_1 - \bar{X}_1\right)(1-x_2) > 0 \end{cases}$$

$$u_2 = \begin{cases} 1 & \text{if} \quad \left(x_3 - \bar{X}_3\right)(x_2-x_4) < 0 \\ 0 & \text{if} \quad \left(x_3 - \bar{X}_3\right)(x_2-x_4) > 0 \end{cases} \tag{4.62}$$

In other words, the control policy is given by

$$\begin{aligned} u_1 &= \frac{1}{2}\left[1 - \text{sign}\left(\left(x_1 - \bar{X}_1\right)(1-x_2)\right)\right] \\ u_2 &= \frac{1}{2}\left[1 - \text{sign}\left(\left(x_3 - \bar{X}_3\right)(x_2-x_4)\right)\right] \end{aligned} \tag{4.63}$$

4.9.3 Simulations

Simulations were carried out with the following design parameter values:

$$L_1 = 20\ [mH], \quad C_1 = 20\ [\mu F], \quad L_2 = 20\ [mH],$$

$$C_2 = 10\ [\mu F], \ R_1 = R_L = 30\ [\Omega], \ E = 15\ [V].$$

which implies

$$Q_1 = Q_L = 0.9487, \quad \sqrt{L_1 C_1} = 6.3246 \times 10^{-4}\ [s]$$

$$\alpha_1 = 1, \quad \alpha_2 = 1$$

It is desired to regulate the voltage variables to the values

$$V_{1d} = -22.5\ [V], \quad V_{2d} = 22.5\ [V]$$

The corresponding equilibrium currents are given by

$$I_{1d} = 1.875\ [A], \quad I_{2d} = -1.5\ [A]$$

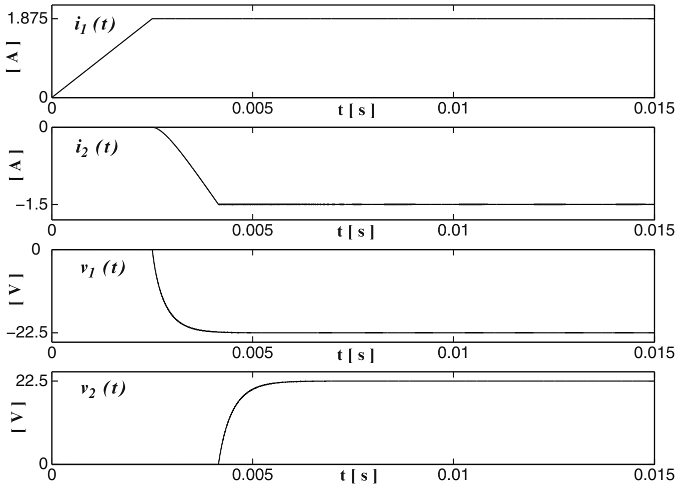

Fig. 4.5. Sliding mode controlled state variable responses of double buck-boost converter

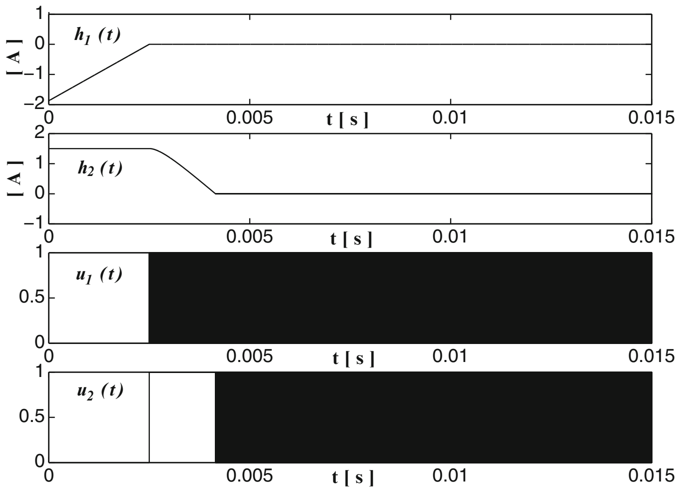

Fig. 4.6. Sliding surfaces and control inputs trajectories in sliding mode controlled double buck-boost converter

Figures 4.5 and 4.6 depict the sliding mode controlled responses of the multi-variable double buck-boost converter circuit.

4.10 The fully actuated rigid body

Consider the dynamic model of a fully actuated rigid body classically related to the Euler equations:

$$
\begin{aligned}
I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 + u_1 \\
I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 + u_2 \\
I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 + u_3
\end{aligned}
$$

where the ω variables stand for the angular velocities around the principal axes of inertia (Fig. 4.7). I_j, $j = 1, 2, 3$, correspond to the principal moments of inertia. The torque input variables are represented by $u_j \in \{0, 1\}$, $j = 1, 2, 3$. These are of the form:$u_j = W(2v_j - 1)$ with $v_j \in \{0, 1\}$, $j = 1, 2, 3$.

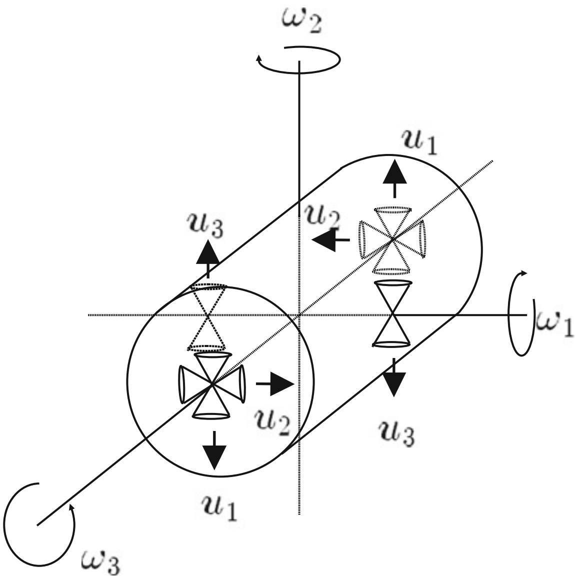

Fig. 4.7. The fully actuated rigid body

Suppose it is desired to bring the angular velocities of the rigid body to a complete rest. This maneuver is known as "de-tumbling." Let ω stand for the vector of angular velocities. A plausible vector of sliding surfaces may be chosen to be

$$
\sigma = \begin{bmatrix} \sigma_1(\omega) \\ \sigma_2(\omega) \\ \sigma_3(\omega) \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{4.64}
$$

Thus $\sigma = 0$ represents the desired stabilizing (de-tumbling) control objective.

Consider the quadratic function

$$V(\sigma) = \frac{1}{2}\sigma^T \sigma = \frac{1}{2}\left[\omega_1^2 + \omega_2^2 + \omega_3^2\right] \tag{4.65}$$

Therefore,

$$\dot{V} = \omega_1 \dot{\omega}_1 + \omega_2 \dot{\omega}_2 + \omega_3 \dot{\omega}_3 = \omega_1 u_1 + \omega_2 u_2 + \omega_3 u_3$$

The multi-variable torque input policy can therefore be synthesized as

$$u_1 = -W_1 \operatorname{sign}(\omega_1), \quad u_2 = -W_2 \operatorname{sign}(\omega_2), \quad u_3 = -W_3 \operatorname{sign}(\omega_3)$$

i.e., the switching policies are just

$$v_j = \frac{1}{2}(1 - \operatorname{sign}\omega_j), \quad j = 1, 2, 3. \tag{4.66}$$

This policy renders

$$\dot{V} = -W_1|\omega_1| - W_2|\omega_2| - W_3|\omega_3| \leq 0 \tag{4.67}$$

The choice of this policy makes use of the controllers full range of torque amplitude. Simulations next page show the performance of such a discontinuous control policy.

4.10.1 Simulations

For the simulations in Fig. 4.8, we used the following parameter values:

$$I_1 = 1 \text{ [N} - \text{m} - \text{s}^2], \quad I_2 = 0.5 \text{ [N} - \text{m} - \text{s}^2], \quad I_3 = 0.2 \text{ [N} - \text{m} - \text{s}^2]$$

$$W_1 = W_2 = W_3 = 0.3 \text{ [N} - \text{m]},$$

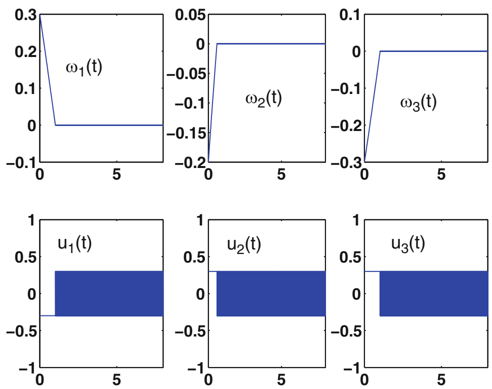

Fig. 4.8. Sliding mode controlled responses of rigid body angular velocities

4.10.2 A computed torque controller via $\boldsymbol{\Delta} - \boldsymbol{\Sigma}$ modulation

The fully actuated average rigid body system is *linearizable* via static state feedback, with the three linearizing outputs being the angular velocities, ω_1, ω_2, and ω_3.

Under the assumption of perfect knowledge of the moments of inertia, I_1, I_2, and I_3, a stabilizing, or de-tumbling, multi-variable feedback strategy is given by the following prescription of a control law, which includes integral compensation terms counteracting a possible, unknown, constant moment perturbation vector.

$$u_1 = -(I_2 - I_3)\omega_2\omega_3 + I_1\left(-\lambda_{11}\omega_1 - \lambda_{01}\int_0^t \omega_1(\sigma)d\sigma\right)$$
$$u_2 = -(I_3 - I_1)\omega_3\omega_1 + I_2\left(-\lambda_{12}\omega_2 - \lambda_{02}\int_0^t \omega_2(\sigma)d\sigma\right)$$
$$u_3 = -(I_1 - I_2)\omega_1\omega_2 + I_3\left(-\lambda_{13}\omega_3 - \lambda_{03}\int_0^t \omega_3(\sigma)d\sigma\right)$$

The closed loop system evolves in accordance with the following set of linear decoupled dynamics,

$$\dot{\omega}_1 = -\lambda_{11}\omega_1 - \lambda_{01}\int_0^t \omega_1(\sigma)d\sigma$$
$$\dot{\omega}_2 = -\lambda_{12}\omega_2 - \lambda_{02}\int_0^t \omega_2(\sigma)d\sigma$$
$$\dot{\omega}_2 = -\lambda_{13}\omega_3 - \lambda_{03}\int_0^t \omega_3(\sigma)d\sigma$$

which can be made to have the origin as an asymptotically exponentially stable equilibrium point under suitable choice of the controller design parameters $\lambda_{1i}, \lambda_{0,i},\ i = 1, 2, 3.$

4.10.3 Simulations

The numerical values, used in the simulations, for the moments of inertia, and for the design parameters were set to be

$$I_1 = 1\ [\mathrm{N-m-s^2}],\quad I_2 = 0.5\ [\mathrm{N-m-s^2}],\quad I_3 = 0.2\ [\mathrm{N-m-s^2}]$$
$$\lambda_{1i} = 2\zeta_i\omega_{ni},\quad \lambda_{0i} = \omega_{ni}^2,\quad \zeta_i = 0.707,\ \omega_{ni} = 0.5,\ i = 1, 2, 3$$

The performance of the proposed average feedback controller, addressed as the *computed torque controller*, is depicted in Fig. 4.9.

A two level multi-variable $\Sigma - \Delta$ modulation implementation of the average feedback control law is achieved via (see Fig. 4.10)

$$u_i = \frac{W_i}{2}\left[\mathrm{sign}(u_{iav}) + \mathrm{sign}(\mathrm{z_i})\right],\quad \dot{z}_i = u_{iav} - u_i \tag{4.68}$$

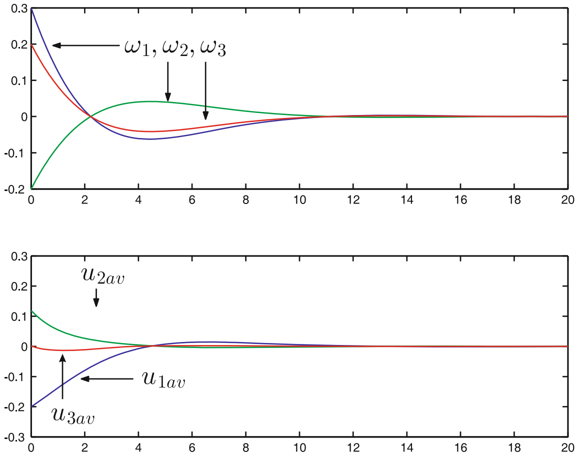

Fig. 4.9. Closed loop response of average controlled rigid body

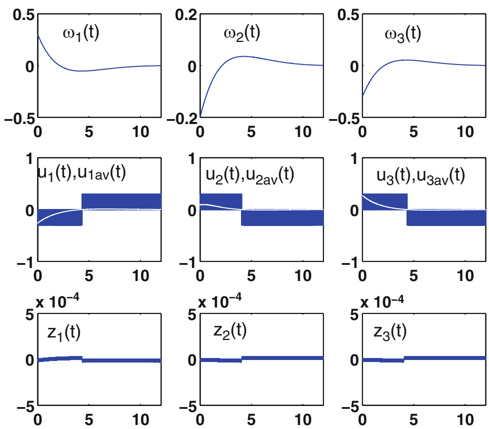

Fig. 4.10. Closed loop response of switched controlled rigid body via two level $\Delta - \Sigma$ modulation implementation of average computed torque control law

4.11 The multi-variable relative degree

For ease of reference, we denote by (f, G, H) the *square* system,

$$\dot{x} = f(x) + G(x)u, \; x \in \mathbb{R}^n, u \in \{0,1\}^m,$$
$$y = H(x), \quad y \in R^m$$

with $G(x) = [g_1(x), \cdots, g_m(x)]$ and $H(x) = [h_1(x), h_2(x), \ldots, h_m(x)]^T$.

Since we will be generalizing below the idea of vector relative degree for a multi-variable system, the relative degree $\mathbf{1}_m$ case is circumscribed primarily to sliding surface coordinate functions in their simplest possible input-output relationship. In general, we consider output functions $y_i = h_i(x)$ which will not necessarily qualify *per se* as sliding surface functions.

Definition 4.5. *(Isidori[13]) A system* (f, G, H) *has a* vector *relative degree* $(r_1, r_2, \cdots, r_m)$ *at* x^0 *if*

1. $L_{g_j} L_f^k h_i(x) = 0, \quad 1 \leq j \leq m, \; k < r_i - 1,$
   $1 \leq i \leq m, \; \forall\, x \in \mathcal{N}(x^0)$
2. *The* $m \times m$ *matrix*

$$\begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \cdots & L_{g_m} L_f^{r_1-1} h_1(x) \\ L_{g_1} L_f^{r_2-1} h_2(x) & \cdots & L_{g_m} L_f^{r_2-1} h_2(x) \\ \vdots & \vdots & \vdots \\ L_{g_1} L_f^{r_m-1} h_m(x) & \cdots & L_{g_m} L_f^{r_m-1} h_m(x) \end{bmatrix} \tag{4.69}$$

is non-singular at x^0*.*

It is clear that this definition generalizes that of SISO systems. Each integer r_i is tied to the i-th output. The dependence upon any one of the components of u of the successive derivatives of this output does not become manifest until r_i time derivatives have been taken. Thus

$$y_i^{(k_i)} = L_f^{k_i} h_i(x) \tag{4.70}$$

whenever $k_i < r_i - 1$.

For $k_i = r_i - 1$,

$$\begin{aligned} y_i^{(r_i)} &= L_f^{r_i} h_i(x) + L_{g_1} L_f^{r_i-1} h_i(x) u_1 + \cdots \\ &\quad \cdots + L_{g_2} L_f^{r_i-1} h_i(x) u_2 + \cdots + L_{g_m} L_f^{r_i-1} h_i(x) u_m \\ &= L_f^{r_i} h_i + \left[L_{g_1} L_f^{r_1-1} h_i \; L_{g_2} L_f^{r_i-1} h_i \; \cdots \; L_{g_m} L_f^{r_i-1} h_i \right] u \\ &= L_f^{r_i} h_i(x) + \left[L_G L_f^{r_i-1} h_i \right] u \end{aligned} \tag{4.71}$$

where it is implied that at least one of the factors $L_{g_j} L_f^{r_i-1} h_i(x)$ is nonzero for $1 \leq j \leq m$ and that the rows $L_G L_f^{r_i} h_i$ are linearly independent of all other rows of the form $L_G L_f^{r_j} h_j$, $j \neq i$.

In our simplified notation, things resemble the scalar case in a manner that may not be devoid of confusion. Let $\mathbf{r}$ denote the multi-index: $\mathbf{r} = (r_1, r_2, \ldots, r_m)$ or, simply, the vector of integer indices r_i and let $\mathbf{r} - \mathbf{1} = (r_1 - 1, \ldots, r_m - 1)$. The expression

$$y^{(\mathbf{r})} = L_f^{\mathbf{r}} H(x) + L_G L_f^{\mathbf{r-1}} H(x) u \tag{4.72}$$

is a shorthand notation for the more complex expression,

$$\begin{bmatrix} y_1^{(r_1)} \\ y_2^{(r_2)} \\ \vdots \\ y_m^{(r_m)} \end{bmatrix} = \begin{bmatrix} L_f^{r_1} h_1(x) \\ L_f^{r_2} h_2(x) \\ \vdots \\ L_f^{r_m} h_m(x) \end{bmatrix} + \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \cdots & L_{g_m} L_f^{r_1-1} h_1(x) \\ L_{g_1} L_f^{r_2-1} h_2(x) & \cdots & L_{g_m} L_f^{r_2-1} h_2(x) \\ \vdots & \vdots & \vdots \\ L_{g_1} L_f^{r_m-1} h_m(x) & \cdots & L_{g_m} L_f^{r_m-1} h_m(x) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

Example 4.6. The system represents a kinematic model of an object moving on the cartesian plane $(x_1, x_2)^T$,

$$\dot{x}_1 = \cos(x_3)u_1, \quad \dot{x}_2 = \sin(x_3)u_1, \quad \dot{x}_3 = u_2 \tag{4.73}$$

with u_1 being the forward velocity and u_2 being the turning rate. x_3 is the orientation angle with respect to the x_1 axis. Set the outputs $y = H(x) = [y_1 \ y_2] = [x_1 \ x_2]$. These outputs have an ill-defined vector relative degree. Since the first order time derivatives of the components of y already depend on the control input u_1, but they do not depend on u_2, the matrix $L_G H(x)$ is not invertible. Notice that $f(x) = 0$ in this case.

Example 4.7. It will be shown further ahead that the previous third order example is equivalent, via dynamic feedback, to the following sixth order system

$$\begin{aligned} \dot{x}_{11} &= x_{12} \\ \dot{x}_{12} &= \cos(x_3)v_1 - z_1 \sin(x_3)v_2 \\ \dot{z}_1 &= v_1 \\ \dot{x}_{21} &= x_{22} \\ \dot{x}_{22} &= \sin(x_3)v_1 + z_1 \cos(x_3)v_2 \\ \dot{x}_3 &= v_2 \end{aligned}$$

where v_1 and v_2 are the control inputs (in fact, $v_1 = \dot{u}_1$, $v_2 = u_2$, and $z_1 = u_1$ is a new state). Setting the output vector to be $y = H(x) = (x_{11}, x_{22})^T$, one finds that

$$L_G H(x) = \begin{bmatrix} \cos(x_3) & -z_1 \sin(x_3) \\ \sin(x_3) & z_1 \cos(x_3) \end{bmatrix} \tag{4.74}$$

The determinant of $L_G H(x)$ is just z_1, the forward velocity. As long as this velocity is nonzero, the vector relative degree of the output y is $\mathbf{r} = (2, 2)$

4.12 Sliding surface vector design

The previous input-output relation, along with the invertibility of the matrix $L_G L_f^{\mathbf{r-1}} H(x)$, motivates the following choices of the components of the vector of sliding surface coordinate functions:

$$\sigma(x) = \begin{bmatrix} \sigma_1(x) \\ \sigma_2(x) \\ \vdots \\ \sigma_m(x) \end{bmatrix} = \begin{bmatrix} L_f^{r_1-1}h_1 + \alpha_{r_1-2}^1 L_f^{r_1-2}h_1 + \cdots + \alpha_1^1 L_f h_1 + \alpha_0^1 h_1 \\ L_f^{r_2-1}h_2 + \alpha_{r_2-2}^2 L_f^{r_2-2}h_2 + \cdots + \alpha_1^2 L_f h_2 + \alpha_0^2 h_2 \\ \vdots \\ L_f^{r_m-1}h_m + \alpha_{r_m-2}^m L_f^{r_m-2}h_m + \cdots + \alpha_0^m h_m \end{bmatrix}$$

This choice guarantees that in closed loop, with $\sigma(x) = 0$ being permanently sustained, the output vector components satisfy the following set of linear time-invariant differential equations,

$$\begin{array}{c} {y_1}^{(r_1-1)} + \alpha_{r_1-2}^1 y_1^{(r_1-2)} + \cdots + \alpha_1^1 \dot{y}_1 + \alpha_0^1 y_1 = 0 \\ {y_2}^{(r_2-1)} + \alpha_{r_2-2}^2 y_2^{(r_2-2)} + \cdots + \alpha_1^2 \dot{y}_2 + \alpha_0^2 y_2 = 0 \\ \vdots \\ {y_m}^{(r_m-1)} + \alpha_{r_m-2}^m y_m^{(r_m-2)} + \cdots + \alpha_1^m \dot{y}_m + \alpha_0^m y_m = 0 \end{array} \tag{4.75}$$

The choice of the coefficients $\{a_{r_j-2}^j, \cdots a_1^j, a_0^j\}$ for $1 \leq j \leq m$ as *Hurwitz coefficients* guarantees the asymptotic convergence of the outputs y_j to zero.

The time derivative of the vector of surface coordinate functions $\dot{\sigma}$ is given by

$$\dot{\sigma} = \begin{bmatrix} L_f^{r_1}h_1 + \alpha_{r_1-2}^1 L_f^{r_1-1}h_1 + \cdots + \alpha_1^1 L_f^2 h_1 + \alpha_0^1 L_f h_1 \\ L_f^{r_2}h_2 + \alpha_{r_2-2}^2 L_f^{r_2-1}h_2 + \cdots + \alpha_1^2 L_f^2 h_2 + \alpha_0^2 L_f h_2 \\ \vdots \\ L_f^{r_m}h_m + \alpha_{r_m-2}^m L_f^{r_m-1}h_m + \cdots + \alpha_1^m L_f^2 h_m + \alpha_0^m L_f h_m \end{bmatrix}$$

$$+ \begin{bmatrix} L_{g_1}L_f^{r_1-1}h_1(x) & \cdots & L_{g_m}L_f^{r_1-1}h_1(x) \\ L_{g_1}L_f^{r_2-1}h_2(x) & \cdots & L_{g_m}L_f^{r_2-1}h_2(x) \\ \vdots & \vdots & \vdots \\ L_{g_1}L_f^{r_m-1}h_m(x) & \cdots & L_{g_m}L_f^{r_m-1}h_m(x) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

Notice that, from the assumption made about the output vector relative degree, the quantities $L_{g_j}L_f^{r_i-1}h_i$ coincide with $L_{g_j}\sigma_i$ for all j and i. We have

$$\dot{\sigma} = L_f\sigma + (L_G\sigma)\ u \tag{4.76}$$

where $L_G\sigma = [L_{g_1}\sigma, \cdots, L_{g_m}\sigma]$.

Since, according to the vector relative degree definition, $L_G\sigma$ is invertible we have

$$L_G\sigma = \begin{bmatrix} L_{g_1}\sigma_1 & L_{g_2}\sigma_1 & \cdots & L_{g_m}\sigma_1 \\ L_{g_1}\sigma_2 & L_{g_2}\sigma_2 & \cdots & L_{g_m}\sigma_2 \\ \vdots & \vdots & \ddots & \vdots \\ L_{g_1}\sigma_m & L_{g_2}\sigma_m & \cdots & L_{g_m}\sigma_m \end{bmatrix} \tag{4.77}$$

The invertibility of $L_G\sigma$ implies that each row of the matrix $L_G\sigma$ contains, at least, a nonzero element. This clearly says that the vector relative degree of the chosen sliding surface coordinate functions vector is, therefore, $(1, 1, \cdots, 1)$.

On $\sigma = 0$ the time derivative of σ is just

$$\dot{\sigma} = \begin{bmatrix} L_f^{r_1} h_1 + \sum_{i=0}^{r_1-2} \left(\alpha_{i-1}^1 - \alpha_i^1 \alpha_{r_1-2}^1\right) L_f^i h_1(x) \\ L_f^{r_2} h_2 + \sum_{i=0}^{r_2-2} \left(\alpha_{i-1}^2 - \alpha_i^2 \alpha_{r_2-2}^2\right) L_f^i h_2(x) \\ \vdots \\ L_f^{r_m} h_m + \sum_{i=0}^{r_m-2} \left(\alpha_{i-1}^m - \alpha_i^m \alpha_{r_m-2}^m\right) L_f^i h_m(x) \end{bmatrix}$$

$$+ \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \cdots & L_{g_m} L_f^{r_1-1} h_1(x) \\ L_{g_1} L_f^{r_2-1} h_2(x) & \cdots & L_{g_m} L_f^{r_2-1} h_2(x) \\ \vdots & \vdots & \vdots \\ L_{g_1} L_f^{r_m-1} h_m(x) & \cdots & L_{g_m} L_f^{r_m-1} h_m(x) \end{bmatrix} \begin{bmatrix} u_{1eq} \\ u_{2eq} \\ \vdots \\ u_{meq} \end{bmatrix}$$

$$= \begin{bmatrix} a_1(x) \\ a_2(x) \\ \vdots \\ a_m(x) \end{bmatrix} + \begin{bmatrix} b_{11}(x) & \cdots & b_{1m}(x) \\ b_{21}(x) & \cdots & b_{2m}(x) \\ \vdots & \ddots & \vdots \\ b_{m1}(x) & \cdots & b_{mm}(x) \end{bmatrix} u_{eq}$$

$$= a(x) + B(x) u_{eq}$$

and, therefore,

$$u_{eq} = \begin{bmatrix} u_{1\ eq} \\ u_{2\ eq} \\ \vdots \\ u_{m\ eq} \end{bmatrix} = - \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \cdots & L_{g_m} L_f^{r_1-1} h_1(x) \\ L_{g_1} L_f^{r_2-1} h_2(x) & \cdots & L_{g_m} L_f^{r_2-1} h_2(x) \\ \vdots & \vdots & \vdots \\ L_{g_1} L_f^{r_m-1} h_m(x) & \cdots & L_{g_m} L_f^{r_m-1} h_m(x) \end{bmatrix}^{-1} \times$$

$$\begin{bmatrix} L_f^{r_1} h_1 + \sum_{i=0}^{r_1-2} \left(\alpha_{i-1}^1 - \alpha_i^1 \alpha_{r_1-2}^1\right) L_f^i h_1(x) \\ L_f^{r_2} h_2 + \sum_{i=0}^{r_2-2} \left(\alpha_{i-1}^2 - \alpha_i^2 \alpha_{r_2-2}^2\right) L_f^i h_2(x) \\ \vdots \\ L_f^{r_m} h_m + \sum_{i=0}^{r_m-2} \left(\alpha_{i-1}^m - \alpha_i^m \alpha_{r_m-2}^m\right) L_f^i h_m(x) \end{bmatrix}$$

$$= -(L_G\sigma)^{-1} L_f \sigma = -B^{-1}(x) a(x)$$

The previous developments point to the fact that it is always possible to choose sliding surface coordinate functions, $\sigma_i(x)$, $i = 1, \cdots, m$, whose joint zero level sets result in an asymptotically exponential convergence of the output vector components to zero (this also applies to output error vectors with respect to constant reference values.)

As before, the inherent possible limitations are twofold: The first one is the determination of the appropriate switching actions, on the part of the individual control inputs, to guarantee local reachability of the intersection manifold: $\sigma(x) = 0$, and the second limitation is represented by the nature of the zero dynamics corresponding to the simultaneous zeroing of the output vector components.

Example 4.8. Consider the permanent magnet synchronous motor (PMSM) model,

$$\begin{aligned} L\frac{di_a}{dt} &= v_a - Ri_a + K_m\omega\sin(N_r\theta) \\ L\frac{di_b}{dt} &= v_b - Ri_b - K_m\omega\cos(N_r\theta) \\ J\frac{d\omega}{dt} &= K_m i_b\cos(N_r\theta) - K_m i_a\sin(N_r\theta) - B\omega \\ \frac{d\theta}{dt} &= \omega \end{aligned}$$

Let $x = (i_a, i_b, \omega, \theta)^T$, $u_1 = v_a$, $u_2 = v_b$. It is readily seen that

$$f(x) = \begin{bmatrix} (-Rx_1 + K_m x_3\sin(N_r x_4))/L \\ (-Rx_2 - K_m x_3\cos(N_r x_4))/L \\ (K_m x_2\cos(N_r x_4) - K_m x_1\sin(N_r x_4) - Bx_3)/J \\ x_3 \end{bmatrix} \tag{4.78}$$

and

$$g_1(x) = \begin{bmatrix} 1/L \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ g_2(x) = \begin{bmatrix} 0 \\ 1/L \\ 0 \\ 0 \end{bmatrix}, \quad G(x) = \begin{bmatrix} 1/L & 0 \\ 0 & 1/L \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{4.79}$$

Consider as the outputs of the PMSM: the angular position $y_1 = \theta = x_4$ and one of the currents, say $y_2 = i_a = x_1$.

Then, the input-output relation satisfies

$$\begin{aligned} y_1^{(3)} &= \frac{K_m}{JL}v_b\cos(N_r x_4) - \frac{K_m}{JL}v_a\sin(N_r x_4) + \xi(x_1, x_2, x_3, x_4) \\ \dot{y}_2 &= \frac{1}{L}(v_a - Rx_1 + K_m x_3\sin(N_r x_4)) \end{aligned}$$

where $\xi(x_1, x_2, x_3, x_4)$ is a function of the state vector alone. The system is of the form:

$$\begin{bmatrix} y_1^{(3)} \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \xi(x_1, x_2, x_3, x_4) \\ \frac{R}{L}x_1 + \frac{K_m}{L}x_3\sin(N_r x_4) \end{bmatrix} + \begin{bmatrix} -\frac{K_m}{JL}\sin(N_r x_4) & \frac{K_m}{JL}\cos(N_r x_4) \\ \frac{1}{L} & 0 \end{bmatrix}\begin{bmatrix} v_a \\ v_b \end{bmatrix}$$

The system has a well-defined vector relative degree, $\mathbf{r} = (3, 1)$, in the open region: $-\pi/(2N_r) < \theta < \pi/(2N_r)$.

Assume we are interested in tracking a given reference trajectory, $\theta^*(t)$, for the rotor angular position $y_1 = \theta = x_4$, while keeping the current, $y_2 = i_a = x_1$, say, at the constant value I_a. Let $e_1 = y_1 - \theta^*(t)$ be the tracking error associated with the angular position $y_1 = \theta$. A set of sliding surfaces, compatible with the control objective, may be proposed as follows:

$$\sigma_1 = \ddot{e}_1 + a_1^1\dot{e}_1 + a_0^1 e_1 = \frac{K_m}{J}x_2\cos(N_r x_4) - \frac{K_m}{J}x_1\sin(N_r x_4) - \frac{B}{J}x_3 - \ddot{\theta}^*(t)$$
$$+ a_1^1(x_3 - \dot{\theta}^*(t)) + a_0^1(x_4 - \theta^*(t))$$
$$\sigma_2 = x_1 - I_a$$

Considering now the currents, $y_1 = i_a$ and $y_2 = i_b$, as the outputs, these also have a well-defined vector relative degree since,

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R}{L}x_1 + \frac{K_m}{L}x_3\sin(N_r x_4) \\ -\frac{R}{L}x_2 - \frac{K_m}{L}x_3\cos(N_r x_4) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & \frac{1}{L} \end{bmatrix} \begin{bmatrix} v_a \\ v_b \end{bmatrix} \tag{4.80}$$

Notice that while in the previous case the vector relative degree is $\mathbf{r} = (3, 1)$ in a constrained region of the state space, then no zero dynamics is associated with the outputs $y = (x_4, x_1)^T = (\theta, i_a)^T$ since the sum of the components of $\mathbf{r}$ equals the order of the system. However, in the last case where $y = (x_1, x_2)^T = (i_a, i_b)^T$, the vector relative degree is $\mathbf{r} = (1, 1)$ and the sum of its components is 2, i.e., smaller than the order of the system. A second order zero dynamics exists. Notice that $\mathbf{r}$ is globally well defined.

4.13 Further notation

From the definitions in Chapter 2, it is not difficult to show that the iterated directional derivative $L_f L_{g_i} h_j(x) - L_{g_i} L_f h_j(x)$ may be written as a directional derivative in the direction of the *Lie bracket* of the vector fields $f(x)$ and $g_j(x)$, i.e.

$$L_f L_{g_i} h_j(x) - L_{g_i} L_f h_j(x) = L_{[f, g_i]} h_j(x) \tag{4.81}$$

where

$$[f, g_i] = \frac{\partial g_i}{\partial x^T} f(x) - \frac{\partial f}{\partial x^T} g_i(x). \tag{4.82}$$

We extend this definition to the case of the iterated vectors of directional derivatives of the form $L_f L_G h_j(x) - L_G L_f h_j(x)$ with $G = [g_1, \ldots, g_m]$ an $n \times m$ matrix.

Consider $L_f L_G h_j(x) = [L_f L_{g_1} h_j(x), \ldots, L_f L_{g_m} h_j(x)]$ while $L_G L_f h_j(x)$ is given by $[L_{g_1} L_f h_j(x), \ldots, L_{g_m} L_f h_j(x)]$. Then, ignoring the argument x,

$$L_fL_Gh_j - L_GL_fh_j = [L_fL_{g_1}h_j - L_{g_1}L_fh_j, \ldots, L_fL_{g_m}h_j - L_{g_m}L_fh_j]$$
$$= [L_{[f,g_1]}h_j, \ldots, L_{[f,g_m]}h_j] = L_{[f,G]}h_j$$

The row vector, $L_{[f,G]}h_j(x)$, may also be denoted as: $L_{ad_fG}h_j(x)$, with the identification: $[f(x), g_i(x)] = ad_fg_i(x)$, and $[f(x), G(x)] = ad_fG(x)$.

The iteration of Lie brackets, as in $[f, [f, g_j]]$ is denoted by $ad_f^2g_j$. The iteration $[f, [f, G]]$ is then just given by $[f, ad_fG] = ad(ad_fG) = ad_f^2G$. In general we define

$$[f, ad_f^{k-1}G] = ad_f^kG, \quad \text{with} \;\; ad_f^0G = G \tag{4.83}$$

Further extending the above notation, we may include expressions such as: $L_fL_GH(x) - L_GL_fH(x)$ in the form $L_{[f,G]}H(x)$ with H an m vector of smooth functions. Clearly

$$L_fL_GH(x) - L_GL_fH(x) = \begin{bmatrix} L_{[f,G]}h_1(x) \\ \vdots \\ L_{[f,G]}h_m(x) \end{bmatrix} = L_{[f,G]}H(x) \in \mathbb{R}^{m\times m} \tag{4.84}$$

Similarly we denote $L_{[f,G]}H(x)$ as $L_{ad_fG}H(x)$

The possibility of determining a state coordinate transformation, where the underlying multi-variable integration structure of the system is clearly exhibited, requires establishing the functional independence of the proposed new state coordinates. To this respect, we have the following proposition (see Isidori [13]):

Proposition 4.9. *Suppose that the system (f, G, H) has vector relative degree $\mathbf{r} = \{r_1, r_2, \cdots, r_m\}$ at x^0. Then, the matrix of row gradients:*

$$Q(x) = \begin{bmatrix} dh_1(x) \\ dL_fh_1(x) \\ \vdots \\ dL_f^{r_1-1}h_1(x) \\ dh_2(x) \\ \vdots \\ dL_f^{r_2-1}h_2(x) \\ \vdots \\ dL_fh_m(x) \\ \vdots \\ dL_f^{r_m}h_m(x) \end{bmatrix} \tag{4.85}$$

is full rank $r = r_1 + r_2 + \cdots + r_m \leq n$ around x^0.

The proof of this proposition may be found in the book by Isidori [13]

Suppose the system has a vector relative degree $\{r_1, \cdots, r_m\}$ at the point x^0. Then the set of coordinate functions

$$\begin{aligned} \phi_1^i(x) &= h_i(x) \\ \phi_2^i(x) &= L_f h_i(x) \\ &\vdots \\ \phi_{r_i}^i(x) &= L_f^{r_i-1} h_i(x) \end{aligned} \tag{4.86}$$

for $1 \leq i \leq m$, with $r = r_1 + r_2 + \cdots + r_m < n$ qualifies as a partial state coordinate transformation map which can be complemented with $n - \sum r_i = n - r$ additional functionally independent maps $\phi_{r+1}, \cdots \phi_n$ such that

$$\Phi(x) = \begin{bmatrix} \phi_1^1(x) \\ \vdots \\ \phi_{r_1}^1(x) \\ \vdots \\ \phi_1^2(x) \\ \vdots \\ \phi_{r_m}^m(x) \\ \phi_{r+1}(x) \\ \vdots \\ \phi_n(x) \end{bmatrix} \tag{4.87}$$

has a full rank Jacobian matrix at x^0. Moreover, if the input distribution span$\{g_1, \cdots, g_m\}$ is involutive around x^0, we can block-off the presence of the control inputs from the last $n - r$ transformed equations.

Let, for $1 \leq i \leq m$

$$\xi^i = \begin{bmatrix} \xi_1^i \\ \xi_2^i \\ \vdots \\ \xi_{r_i}^i \end{bmatrix} = \begin{bmatrix} \phi_1^i(x) \\ \phi_2^i(x) \\ \vdots \\ \phi_{r_i}^i(x) \end{bmatrix}, \quad \text{and } \xi = \begin{bmatrix} \xi^1 \\ \xi^2 \\ \vdots \\ \xi^m \end{bmatrix} \tag{4.88}$$

while

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{n-r} \end{bmatrix} = \begin{bmatrix} \phi_{r+1}(x) \\ \phi_{r+2}(x) \\ \vdots \\ \phi_n(x) \end{bmatrix} \tag{4.89}$$

Then, defining: $b_i(\xi, \eta) = L_f^{r_i} h_i(\Phi^{-1}(\xi, \eta))$

$$a_{ij}(\xi, \eta) = L_{g_j} L_f^{r_i-1} h_i(\Phi^{-1}(\xi, \eta)), \quad 1 \leq i, j \leq m \tag{4.90}$$

The transformed system equations, for $1 \le i \le m$, are then of the form:

$$\begin{aligned}
\dot{\xi}_1^i &= \xi_2^i \\
&\vdots \\
\dot{\xi}_{r_i-1}^i &= b_i(\xi, \eta) + \sum_{j=1}^{m} a_{ij}(\xi, \eta) u_j \\
y_i &= \xi_1^i
\end{aligned}$$

The rest of the $n - \sum r_i = n - r$ equations are of the form

$$\dot{\eta} = q(\xi, \eta) \tag{4.91}$$

provided the distribution $\{g_1, \cdots, g_m\}$ is involutive. Otherwise, the differential equation for the components of η exhibits the influence of the control inputs in an affine manner:

$$\dot{\eta} = q(\xi, \eta) + p(\xi, \eta) u \tag{4.92}$$

4.14 The under-actuated rigid body

Consider now the same dynamic model of a rigid body which lacks a control input for the third axis (Fig. 4.11)

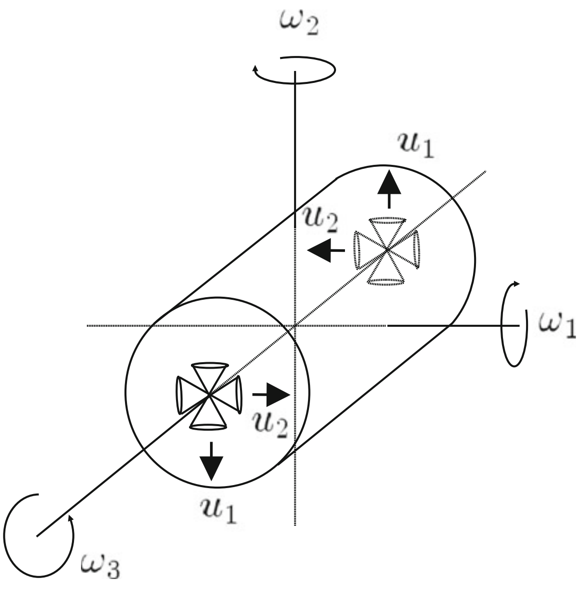

Fig. 4.11. The under-actuated rigid body.

$$
\begin{aligned}
I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 + u_1 \\
I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 + u_2 \\
I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2
\end{aligned}
$$

As before, the available control inputs u_i are assumed to take values in $\{-W_i, 0, +W_i\}$, $i = 1, 2$. They are the pulsed control input torques.

If we consider as the system outputs $y_1 = \omega_1$ and $y_2 = \omega_3$, the vector relative degree $\mathbf{r}$ of the outputs is

$$\mathbf{r} = \{1, 2\}$$

Since the sum of the components of the relative degree vector adds up to the system dimension, the system can be put in normal form without any zero dynamics.

Indeed, let $\xi_1^1 = \omega_1$ and $\xi_1^2 = \omega_3$. We have, by virtue of

$$\xi_2^2 = \dot{\omega}_3 = (\frac{I_1 - I_2}{I_3})\omega_1\omega_2 \tag{4.93}$$

that the following map qualifies as a full state coordinate transformation, away from $\omega_1 = 0$

$$\begin{bmatrix} \xi_1^1 \\ \\ \xi_1^2 \\ \\ \xi_2^2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_3 \\ (\frac{I_1 - I_2}{I_3})\omega_1\omega_2 \end{bmatrix} \tag{4.94}$$

The inverse transformation is just

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \xi_1^1 \\ \left(\frac{I_3}{I_1 - I_2}\right)\frac{\xi_2^2}{\xi_1^1} \\ \xi_1^2 \end{bmatrix} \tag{4.95}$$

The transformed system can be written in Isidori's canonical form (see [13]), as

$$
\begin{aligned}
\dot{\xi}_1^1 &= \left[\frac{(I_2 - I_3)I_3}{I_1(I_1 - I_2)}\right]\frac{(\xi_2^2)^2}{\xi_1^1} + u_1 \\
\dot{\xi}_1^2 &= \xi_2^2 \\
\dot{\xi}_2^2 &= \left[\frac{(I_2 - I_3)I_3}{I_1(I_1 - I_2)}\right]\frac{(\xi_2^2)^4}{(\xi_1^1)^2}(\xi_1^1 + \xi_1^2) + \left[\left(\frac{I_3}{I_1 - I_2}\right)\frac{\xi_2^2}{\xi_1^1}\right]u_1 + \xi_1^1 u_2 \\
y_1 &= \xi_1^1, \quad y_2 = \xi_1^2
\end{aligned}
$$

Suppose it is desired to bring the three angular velocities of the rigid body to a complete rest. Let, as before, ω stand for the vector of angular velocities.

Given that the new system only has two control inputs, u_1, u_2, we should only have two sliding surfaces.

Evidently the previously used Lyapunov function does not provide a solution to our problem since

$$V(e) = \frac{1}{2}\left[I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right] \tag{4.96}$$

leads to

$$\dot{V} = \omega_1 u_1 + \omega_2 u_2 \tag{4.97}$$

The same policy as before $u_1 = -W_1\text{sign}(\omega_1)$ and $u_2 = -W_2\text{sign}(\omega_2)$ yields

$$\dot{V} = -W_1|\omega_1| - W_2|\omega_2| \leq 0 \tag{4.98}$$

which is not negative definite, but only negative semi-definite. The asymptotic stability of the origin of the sliding surface space is not assured.

Consider the set of rather "natural" sliding surfaces:

$$\begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix} = \begin{bmatrix}\omega_1\\ \dot{\omega}_3 + \lambda_2\omega_3\end{bmatrix} \tag{4.99}$$

We have

$$\begin{aligned}
\dot{\sigma}_1 &= \left(\frac{I_2 - I_3}{I_1}\right)\omega_2\omega_3 + \frac{u_1}{I_1}\\
\dot{\sigma}_2 &= \left(\frac{I_1 - I_2}{I_3}\right)(\dot{\omega}_1\omega_2 + \omega_1\dot{\omega}_2) + \lambda_2\left(\frac{I_1 - I_2}{I_3}\right)\omega_1\omega_2\\
&= \left(\frac{I_1 - I_2}{I_3}\right)\left[\left(\frac{I_2 - I_3}{I_1}\right)\omega_2^2\omega_3 + \left(\frac{I_3 - I_1}{I_2}\right)\omega_1^2\omega_3\right]\\
&\quad + \lambda_2\left(\frac{I_1 - I_2}{I_3}\right)\omega_1\omega_2 + \left(\frac{I_1 - I_2}{I_3}\right)\left[\frac{u_1}{I_1}\omega_2 + \frac{u_2}{I_2}\omega_1\right]
\end{aligned}$$

Since $\sigma_1 = \omega_1$ is relative degree 1, the control policy $u_1 = -W_1\text{sign}\sigma_1$ drives ω_1 to zero in finite time whenever W_1 overcomes $(I_2 - I_3)\omega_2\omega_3$. This ideally results in a blocking of the control actions due to u_2 and, hence, σ_2 remains uncontrolled from there on.

The problem is then that of driving σ_2 to zero faster than σ_1. In other words, if ω_3 and $\dot{\omega}_3$ converged to zero faster than ω_1, then ω_2 would also converge to zero. In this case however, the control actions due to u_1 would be blocked from the dynamics of σ_2.

Besides the above inconvenient situation, the obstacle resides in the fact that ω_3 can only approach zero in an exponential manner which, theoretically, takes infinite time. The scheme will surely not produce the desired result as it can be verified from Figure 4.12.

In order to bestow some exponential decay on ω_1, which could be controlled through the imposed eigenvalue on the closed loop dynamics of ω_1 we propose the rather "unnatural" set of sliding surface coordinate functions

$$\sigma_1 = \dot{\omega}_1 + \lambda_1 \omega_1$$
$$\sigma_2 = \dot{\omega}_3 + \lambda_2 \omega_3$$

The first sliding surface coordinate function time derivative immediately involves the first order time derivative of the control input u_1. We consider then the extension of the control variable u_1, $\dot{u}_1$, as the actual control input while u_1 plays the role of an additional available state of the system.

We use the switching policy

$$\dot{u}_1 = -W_1 \text{sign}(\sigma_1), \quad u_2 = -W_2 \text{sign}(\sigma_2) \tag{4.100}$$

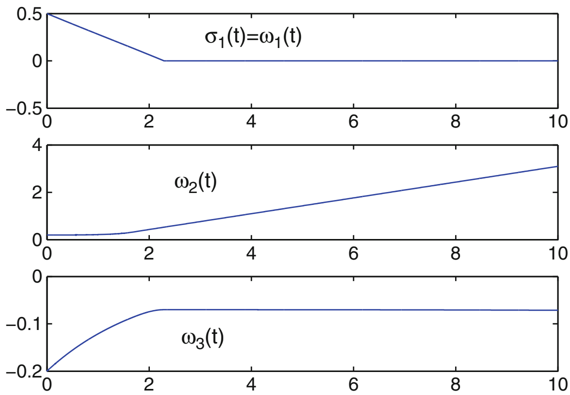

Fig. 4.12. Stabilization of ω_1 and ω_3 via static feedback

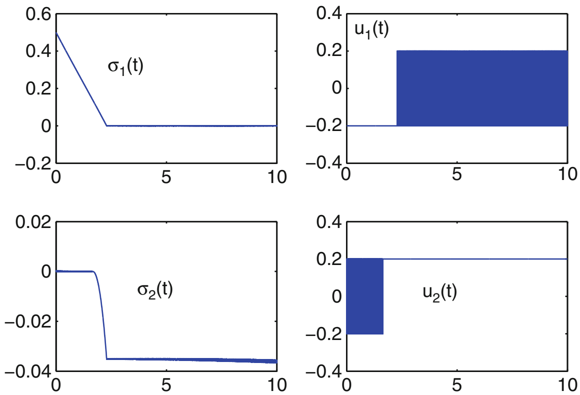

Fig. 4.13. Sliding surface coordinate functions and control inputs corresponding to the stabilization of ω_1 and ω_3

The evolution of the sliding surfaces coordinate functions and that of the corresponding control inputs is shown in Figure 4.13.

The switch controlled angular velocities are depicted in Figure 4.14. The steady state values for the angular velocities obtained in the simulations turned out to be

$$\omega_1(\infty) = 5.07 \times 10^{-4}, \quad \omega_2(\infty) = -4.53 \times 10^{-5}, \quad \omega_3(\infty) = 7.72 \times 10^{-8}$$

i.e., they are practically zero.

From the physical viewpoint, the above solution is not entirely satisfactory either. The control input u_1 actually represents a binary-valued torque. Since the extended control input, $\dot{u}_1$, was unjustifiably allowed to be binary valued, then u_1 results in a continuous signal which violates the initial assumption. The control of this system requires a more precise approach. The Delta-Sigma

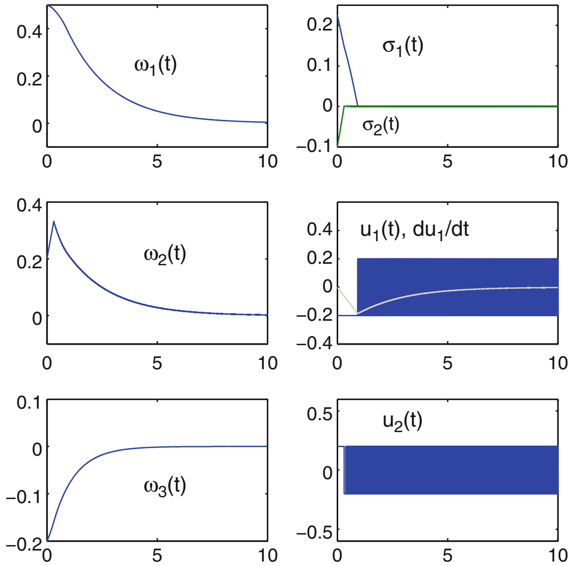

Fig. 4.14. Stabilization of under-actuated rigid body via dynamic extension

modulation approach is clearly suitable since an average (smooth) stabilizing policy is entirely possible by means of linearizing feedback which guarantees any desirable slow stabilization for ω_1 while respecting the suitably bounded character of the average input torques u_1 and u_2. Delta-Sigma modulation translates then this average designed control feedback law into a set of corresponding switching inputs for the actual u_1 and u_2. We leave the details as an exercise for the reader.

4.15 Two cascaded buck converters

As we have already seen, the buck converter is a SISO system whose average model is linear. However, the cascade connection of two independently controlled buck converters conforms an interesting nonlinear multi-variable system. Consider the normalized average model of two buck converters connected in cascade.

$$\begin{aligned}\frac{d}{dt}i_1 &= -v_1 + u_1\\ \frac{d}{dt}v_1 &= i_1 - \frac{v_1}{Q_1} - u_2 i_2\\ \frac{d}{dt}i_2 &= -v_2 + u_2 v_1\\ \frac{d}{dt}v_2 &= i_2 - \frac{v_2}{Q_2}\end{aligned}$$

Let the outputs of the system be given by $y_1 = v_1$, $y_2 = v_2$, i.e., the system outputs coincide with the output voltages.

Note that

$$\begin{bmatrix}\dot{v}_1\\ \ddot{v}_2\end{bmatrix} = \begin{bmatrix} i_1 - \frac{v_1}{Q_1}\\ -(1-\frac{1}{Q_2^2})v_2 - \frac{1}{Q_2}i_2\end{bmatrix} + \begin{bmatrix}0 & -i_2\\ 0 & v_1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} \tag{4.101}$$

The vector relative degree of $(y_1, y_2) = (v_1, v_2)$ is $(1, 2)$ and it is clearly ill defined since the decoupling matrix is singular. A suitable dynamic extension of the system yields

$$\begin{bmatrix}\ddot{v}_1\\ v_2^{(3)}\end{bmatrix} = \begin{bmatrix}\xi_1(i_1, v_1, i_2, v_2, u_2)\\ \xi_2(i_1, v_1, i_2, v_2, u_2)\end{bmatrix} + \begin{bmatrix}1 & -i_2\\ 0 & v_1\end{bmatrix}\begin{bmatrix}u_1\\ \dot{u}_2\end{bmatrix}$$

$$= \begin{bmatrix}-v_1 - \frac{1}{Q_1}(i_1 - \frac{v_1}{Q_1} - u_2 i_2) - u_2(-v_2 + u_2 v_1)\\ (i_2 - \frac{v_2}{Q_2})(1 - \frac{1}{Q_2^2}) + u_2(i_1 - \frac{v_1}{Q_1} - u_2 i_2) - \frac{1}{Q_2}(-v_2 + u_2 v_1)\end{bmatrix} + \begin{bmatrix}1 & -i_2\\ 0 & v_1\end{bmatrix}\begin{bmatrix}u_1\\ \dot{u}_2\end{bmatrix}$$

The extended input-output system has a well-defined relative degree where the condition $v_1 \neq 0$ is satisfied (in fact the voltage v_1 is strictly positive due to physical reasons).

A traditional approach to sliding mode control of the above extended system fails for a simple reason. The time derivative, $\dot{u}_2$, of the control input u_2, is the formal control input to the extended system. The actual input u_2 should be a switched signal taking values in $\{0, 1\}$. This is possible only via a complex impulsive control prescription for $\dot{u}_2$. The traditional sliding mode control approach is therefore not feasible. A practical solution rests on considering the extended system as an average system and proceeding to specify an average feedback control law, via Delta-Sigma modulation, that respects the bounded character of the average control input u_1, and of the newly created state u_2. In fact, the average restriction: $u_2 \in [0, 1]$ is an extended state restriction which is far easier to handle by the methods of Chapter 6.

Suppose we are interested in tracking the given pair of smooth voltage reference signals: $(v_1^*(t), v_2^*(t))$. An average feedback control for u_1 and $\dot{u}_2$ is readily synthesized as

$$\begin{bmatrix} u_1 \\ \dot{u}_2 \end{bmatrix} = \frac{1}{v_1} \begin{bmatrix} v_1 & i_2 \\ 0 & 1 \end{bmatrix} \left(- \begin{bmatrix} \xi_1(i_1, v_1, i_2, v_2, u_2) \\ \xi_2(i_1, v_1, i_2, v_2, u_2) \end{bmatrix} + \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} \right) \tag{4.102}$$

with

$$\begin{aligned} \vartheta_1 &= \ddot{v}_1^*(t) - \lambda_1(\dot{v}_1 - \dot{v}_1^*(t)) - \lambda_0(v_1 - v_1^*(t)) \\ \vartheta_2 &= [v_2^*(t)]^{(3)} - \gamma_2(\ddot{v}_3 - \ddot{v}_2^*(t)) - \gamma_1(\dot{v}_2 - \dot{v}_2^*(t)) - \gamma_0(v_2 - v_2^*(t)) \end{aligned} \tag{4.103}$$

The procedure consists in specifying output reference trajectories, $v_1^*(t)$ and $v_2^*(t)$, that result in suitably bounded average control input signals $u_1, u_2 \in [0, 1]$. While the first restriction is an input restriction, the second is a state restriction in the extended state space. Again, the methods of Chapter 6 easily handle the trajectory planning required to respect physical input and state constraints.

5

An Input-Output approach to Sliding Mode Control

5.1 Introduction

One of the disadvantages of classical sliding mode control resides in the need to feedback the entire state to synthesize the controller. Traditionally, state observers are used for this requirement. However, the lack of a separation principle in nonlinear systems makes it difficult to assess the closed loop stability of an observer based sliding mode controlled system. Even in the case of linear sliding mode controlled systems, the use of a linear observer may be subject to careful analysis and noise bandwidth considerations.

In the following pages we introduce Generalized Proportional Integral (GPI) control. A technique introduced by M. Fliess and his colleagues in Fliess *et al.* [9] for linear systems. The fundamental idea is to produce structural estimates of the states via finite linear combinations of iterated integrations of input and output variables. The result is valid for mono-variable and multi-variable linear systems. However, some class of nonlinear systems may benefit from the possibility of integral reconstructors (namely those systems where the nonlinearities are functions of the output variables alone). These reconstructors are, invariably, faulty with respect to the actual state values since the technique purposefully neglects the initial conditions. The effect of the neglected initial conditions propagates through the input-output dynamics in the form of classical time polynomial perturbations (constant, ramps, quadratic, etc. functions of time). The effect of this perturbation is easily compensated via suitable iterated integrals of the output error.

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5.2 GPI control of chains of integrators

In this section, we discuss the output feedback control of a particular class of linear systems constituted by finite chains of integrators. The problem is basic in the control of systems via linear output time-varying feedback control laws. The basic tool is constituted by integral reconstructors which also constitute the basic element in GPI control. The technique is explored here in the context of output trajectory tracking problems.

5.2.1 A double integrator

In order to introduce the fundamental ideas, we start by considering a second order integrator of the form

$$\ddot{y} = u \tag{5.1}$$

where it is desired to track a smooth trajectory $y^*(t)$. Clearly, if there exists an open loop control input, u^*, that ideally achieves the tracking of $y^*(t)$ for suitable initial conditions, it satisfies the relation

$$\ddot{y}^*(t) = u^*(t) \tag{5.2}$$

The tracking error $e_y = y - y^*(t)$ satisfies then the corresponding second order dynamics

$$\ddot{e}_y = e_u \tag{5.3}$$

where $e_u = u - u^*(t)$.

A feedback controller that exponentially regulates to zero the tracking error e_y, and it is robust with respect to unknown constant perturbations, is given by

$$e_u = -k_2 \dot{e}_y - k_1 e_y - k_0 \int_0^t e_y(\sigma) d\sigma \tag{5.4}$$

where k_2, k_1, and k_0 are design constants to be determined so that the closed loop characteristic polynomial $p(s) = s^3 + k_2 s^2 + k_1 s + k_0$ is a Hurwitz polynomial.

However, the controller requires the time derivative of e which is not available for measurement.

Consider the following relation obtained by integrating once the tracking error dynamics

$$\dot{e}_y = \dot{e}_y(0) + \int_0^t e_u(\sigma) d\sigma \tag{5.5}$$

We define the integral reconstructor of $\dot{e}$ as the integral of the control input

$$\widehat{\dot{e}}_y = \int_0^t e_u(\sigma) d\sigma \tag{5.6}$$

The integral reconstructor of $\dot{e}$ differs from the actual value of $\dot{e}_y$ by an unknown constant. Moreover, such an integral reconstructor is quite easy to synthesize as it only requires the integration of the known input signal. Since the proposed controller is robust with respect to unknown constant perturbations we can use such a faulty estimate of $\dot{y}$ in the feedback control law. We propose then the following feedback control law

$$\begin{aligned} e_u &= -k_2\widehat{\dot{e}}_y - k_1 e_y - k_0 \int_0^t e_y(\sigma)d\sigma \\ &= -\int_0^t \left[k_2 e_u(\sigma) + k_0 e_y(\sigma)\right] d\sigma - k_1 e_y \end{aligned} \tag{5.7}$$

Naturally, the closed loop dynamics of the system, controlled by the feedback law (5.7), after using (5.5), is given by

$$\ddot{e}_y + k_2\dot{e}_y + k_1 e_y + k_0 \int_0^t e_y(\sigma)d\sigma = k_2\dot{e}_y(0) \tag{5.8}$$

which clearly has the origin as an asymptotically, exponentially stable equilibrium point, independently of the values of the initial conditions of the system. To see this, simply define

$$\rho = \int_0^t e_y(\sigma)d\sigma - \frac{k_2}{k_0}e_y(0) \tag{5.9}$$

to obtain the linear system,

$$\begin{aligned} \ddot{e}_y + k_2\dot{e}_y + k_1 e_y &= -k_0\rho \\ \dot{\rho} &= e_y \end{aligned} \tag{5.10}$$

which has as characteristic polynomial, precisely, the Hurwitz polynomial $p(s)$.

Note that the proposed controller

$$e_u = -\int_0^t \left[k_2 e_u(\sigma) + k_0 e_y(\sigma)\right] d\sigma - k_1 e_y \tag{5.11}$$

can be written, in terms of Laplace transforms in the following manner.

$$\left(1 + \frac{k_2}{s}\right) e_u(s) = -\left(k_1 + \frac{k_0}{s}\right) e_y(s) \tag{5.12}$$

Combining, with an abuse of notation, the time domain expressions with the operational calculus notation, we write the controller as

$$u = u^*(t) - \left[\frac{k_1 s + k_0}{s + k_2}\right](y - y^*(t)) \tag{5.13}$$

It is instructive to examine the nature of the classical compensation filter constituting the GPI controller. From the Hurwitz nature of the polynomial $p(s) = s^3 + k_2 s^2 + k_1 s + k_0$ and the use of the Routh-Hurwitz stability criterion it follows that the coefficients must satisfy

$$k_2 k_1 - k_0 \ > \ 0, \ k_2 > 0, \ k_0 > 0 \tag{5.14}$$

It follows that $k_2 > \frac{k_0}{k_1}$ and, hence, $-k_2 < -\frac{k_0}{k_1}$. This means that the zero of the compensation network transfer function is closer to the imaginary axis than its pole. The compensation network is then a classical *lead compensator* (see Figure 5.1).

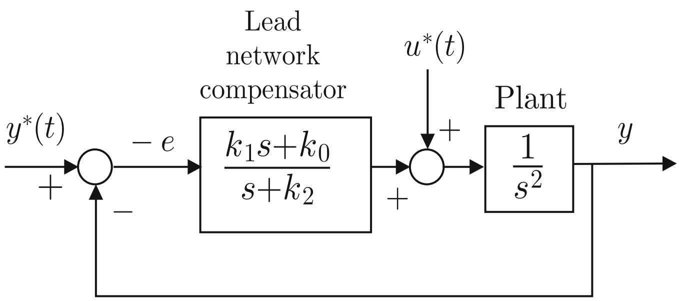

Fig. 5.1. GPI control of double integrator plant as a lead compensator for trajectory tracking

Notice that the implicit nature of the expression (5.11) admits the interpretation of the feedback scheme shown in Figure 5.2.

5.2.2 A third order integrator

Consider the third order system:

$$y^{(3)} = u$$

which admits the following integral input-output parametrization

$$\widehat{\ddot{y}} = \int_0^t u(\tau) d\tau, \ \ \widehat{\dot{y}} = \int_0^t \int_0^\tau u(\lambda) d\lambda$$

The relation between the structural estimates and the actual values of the states of the system are given by

$$\begin{aligned} \ddot{y} &= \widehat{\ddot{y}} + \ddot{y}_0 \\ \dot{y} &= \widehat{\dot{y}} + \dot{y}_0 + \ddot{y}_0 \, t \end{aligned}$$

Suppose that the problem is to have the system output, $y(t)$, track a given output reference signal, $y^*(t)$.

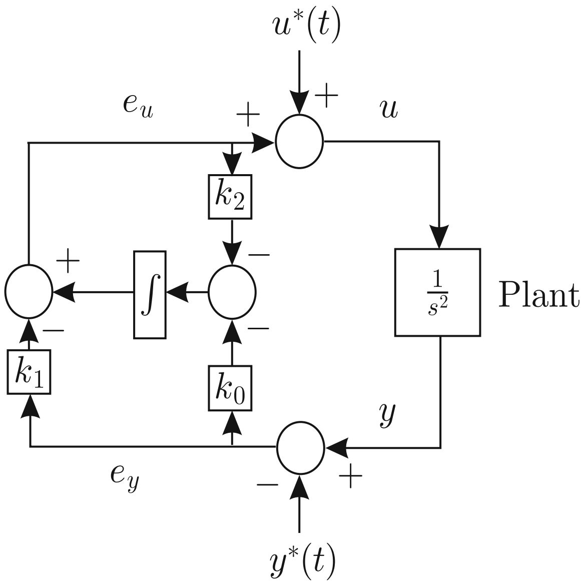

Fig. 5.2. An alternative interpretation for GPI control of second order plant

An output feedback controller achieving the trajectory tracking task, when all phase variables are known, is given by

$$
\begin{aligned}
u = & [y^*(t)]^{(3)} - k_4(\ddot{y} - \ddot{y}^*(t)) - k_3(\dot{y} - \dot{y}^*(t)) \\
& -k_2(y - y^*(t))
\end{aligned}
$$

If we use the structural estimates, plus integral error compensation in the controller synthesis, we obtain

$$
\begin{aligned}
u = & [y^*(t)]^{(3)} - k_4(\widehat{\ddot{y}} - \ddot{y}^*(t)) - k_3(\widehat{\dot{y}} - \dot{y}^*(t)) \\
& -k_2(y - y^*(t)) - k_1 \int_0^t (y - y^*(\tau))d\tau - \\
& k_0 \int_0^t \int_0^\tau (y - y^*(\lambda))d\lambda d\tau
\end{aligned}
$$

The closed loop tracking error, $e(t) = y - y^*(t)$, is then governed by

$$
\begin{aligned}
& e^{(3)} + k_4\ddot{e} + k_3\dot{e} + k_2 e + k_1 \int_0^t e(\tau)d\tau \\
& +k_0 \int_0^t \int_0^\tau e(\lambda)d\lambda d\tau = k_4\ddot{y}_0 + k_3(\dot{y}_0 + \ddot{y}_0\, t)
\end{aligned}
$$

Define

$$\xi_2 = -\frac{k_3}{k_0}\ddot{y}_0 + \int_0^t e(\tau)d\tau \tag{5.15}$$

$$\xi_1 = \int_0^t e(\tau)d\tau + \frac{k_0}{k_1}\int_0^t \xi_2(\tau)d\tau - \frac{k_4\ddot{y}_0 + k_3\dot{y}_0}{k_1} \tag{5.16}$$

Clearly, the closed loop tracking error $e(t) = y - y^*(t)$, obeys,

$$e^{(3)} + k_4\ddot{e} + k_3\dot{e} + k_2 e = -k_1\xi_1$$

$$\dot{\xi}_1 = e + \frac{k_0}{k_1}\xi_2, \quad \xi_1(0) = -\frac{k_4\ddot{y}_0 + k_3\dot{y}_0}{k_1}$$
$$\dot{\xi}_2 = e, \quad \xi_2(0) = -\frac{k_3\ddot{y}(0)}{k_0}$$

which exhibits the following characteristic polynomial,

$$p(s) = s^5 + k_4 s^4 + k_3 s^3 + k_2 s^2 + k_1 s + k_0 = 0$$

Let us reconsider the proposed controller:

$$u = [y^*(t)]^{(3)} - k_4(\widehat{\ddot{y}} - \ddot{y}^*(t)) - k_3(\widehat{\dot{y}} - \dot{y}^*(t))$$
$$-k_2(y - y^*(t)) - k_1\int_0^t (y - y^*(\tau))d\tau -$$
$$k_0\int_0^t\int_0^\tau (y - y^*(\lambda))d\lambda d\tau$$

Note that, from the model; $u^*(t) = [y^*(t)]^{(3)}$. Define: $e_u = u - u^*(t)$ y $e_y = y - y^*(t)$. Then we have

$$u = u^* - k_4\int_0^t e_u(\tau)d\tau - k_3\int_0^t\int_0^\tau e_u(\lambda)d\lambda d\tau$$
$$-k_2 e_y - k_1\int_0^t e_y(\tau)d\tau - k_0\int_0^t\int_0^\tau e_y(\lambda)d\lambda d\tau$$

In the frequency domain, the proposed controller satisfies the following relation:

$$\left(1 + \frac{k_4}{s} + \frac{k_3}{s^2}\right)e_u = -\left(k_2 + \frac{k_1}{s} + \frac{k_0}{s^2}\right)e_y \tag{5.17}$$

This frequency expression is readily interpreted as the block diagram shown in Figure 5.3.

The GPI controller also admits the following expression,

$$u = u^*(t) - \left[\frac{k_2 s^2 + k_1 s + k_0}{s^2 + k_4 s + k_3}\right](y - y^*(t)) \tag{5.18}$$

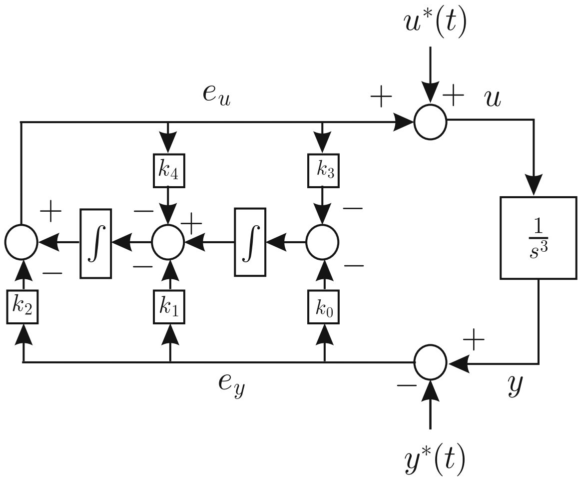

Fig. 5.3. GPI control of a third order unperturbed integrating plant

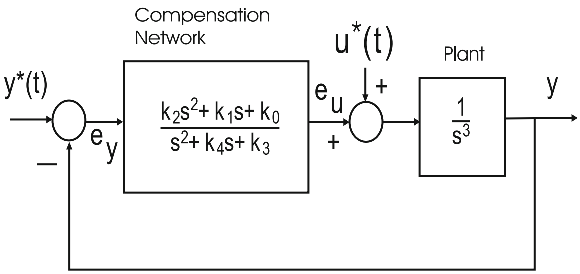

Fig. 5.4. Alternative interpretation of GPI control of a third order unperturbed integrating plant

which is interpreted as in Figure 5.4.

The origin of the tracking error space $e = y - y^*(t)$ is an exponentially asymptotically stable equilibrium point, for the closed loop system, provided the design gains, $k_4, \cdots, k_0$, are chosen so that the closed loop characteristic polynomial

$$p(s) = s^5 + k_4 s^4 + k_3 s^3 + k_2 s^2 + k_1 s + k_0 \tag{5.19}$$

becomes a Hurwitz polynomial.

Exercise 5.1. The derived GPI controller is not robust with respect to constant perturbation inputs. Suitably modify the compensation network to have a robust output feedback control scheme with respect to this simple class of perturbations.

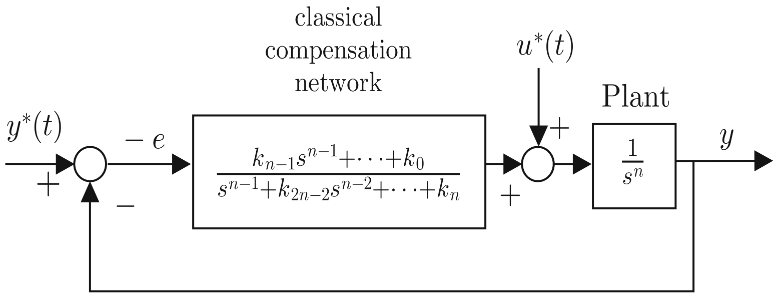

Fig. 5.5. GPI control of an n-th order unperturbed integrating plant

5.2.3 N-th order integrator

It is not difficult to see that an unperturbed n-th order integrator plant can be controlled to asymptotically exponentially track a given smooth reference trajectory $y^*(t)$, using the following classical compensation network (Fig. 5.5):

$$u = u^*(t) - \left[\frac{k_{n-1}s^{n-1} + k_{n-2}s^{n-2} + \cdots + k_0}{s^{n-1} + k_{2n-2}s^{n-2} + \cdots + k_n}\right](y - y^*(t)) \tag{5.20}$$

where $u^*(t) = [y^*(t)]^{(n)}$, and the set of design coefficients:

$$\{k_{2n-2}, k_{2n-3}, \ldots, k_0\},$$

are chosen so that the closed loop characteristic polynomial,

$$p(s) = s^{2n-1} + k_{2n-2}s^{2n-2} + \cdots + k_1 s + k_0 \tag{5.21}$$

has all its roots in the left portion of the complex plane (i.e., it is a Hurwitz polynomial).

Exercise 5.2. In the next subsection we address the problem of GPI control under classical perturbations. Show that if a constant perturbation affects the previously considered n-th order integrator, then an integral action is required for global exponential asymptotic stability. Show that the compensator is then of the form;

$$u = u^*(t) - \left[\frac{k_n s^n + k_{n-1}s^{n-1} + \cdots + k_1 s + k_0}{s(s^{n-1} + k_{2n-1}s^{n-2} + \cdots + k_{n+1})}\right](y - y^*(t)) \tag{5.22}$$

The set of design coefficients, $\{k_{2n-1}, k_{2n-2}, \ldots, k_0\}$, are chosen so that the closed loop characteristic polynomial,

$$p(s) = s^{2n} + k_{2n-1}s^{2n-1} + \cdots + k_1 s + k_0 \tag{5.23}$$

is a Hurwitz polynomial.

5.2.4 Robustness with respect to classical perturbations

The previous GPI control schemes are, unfortunately, not robust with respect to the presence of classical perturbation inputs. To see how the scheme may be modified to exhibit the required robustness we examine the first previous example but subject now to a constant, unknown, perturbation input.

5.2.5 A perturbed double integrator plant

Consider the following perturbed second order integrator system,

$$\ddot{y} = u + \zeta \tag{5.24}$$

where ζ is an unknown constant perturbation signal. A feedback controller of the PID type as the one initially proposed above would be sufficient to overcome the influence of the constant perturbation thanks to the included integral correction action. However, the estimation of the unavailable signal $\dot{y}$ requires one integration of the perturbed dynamics. This yields an error between the actual value of the velocity and the estimated value (which is computed neglecting the initial conditions and the perturbation input) which grows linearly in time, with unknown slope. To overcome this type of perturbation the controller requires an iterated integral tracking error action, i.e., a double integration of the tracking error.

The output tracking error e is now governed by

$$\ddot{e}_y = e_u + \zeta \tag{5.25}$$

where, as before, $e_u = u - u^*(t) = u - \ddot{y}^*(t)$. We propose then the following feedback controller,

$$e_u = -k_3\widehat{\dot{e}} - k_2 e_y - k_1 \int_0^t e_y(\sigma)d\sigma - k_0 \int_0^t \int_0^{\sigma_1} e_y(\sigma_2)d\sigma_2 d\sigma_1 \tag{5.26}$$

where the estimate $\widehat{\dot{e}}_y$ is computed, as before, in the form:

$$\widehat{\dot{e}}_y = \int_0^t e_y(\sigma)d\sigma \tag{5.27}$$

which is related to the actual value of $\dot{e}_y$ by the relation

$$\dot{e}_y = \widehat{\dot{e}}_y + \zeta t + \dot{e}_y(0) \tag{5.28}$$

The closed loop system using the proposed controller satisfies the following integro-differential equation

$$\ddot{e}_y + k_3\dot{e}_y + k_2 e_y + k_1 \int_0^t e_y(\sigma)d\sigma + k_0 \int_0^t \int_0^{\sigma_1} e_y(\sigma_2)d\sigma_2 d\sigma_1 = k_3\dot{e}_y(0) + k_3\zeta t \tag{5.29}$$

Define

$$\rho_1 = \int_0^t e_y(\sigma)d\sigma + \int_0^t \rho_2(\sigma)d\sigma$$
$$\rho_2 = \frac{k_0}{k_1}\int_0^t e_y(\sigma)d\sigma - \frac{k_3}{k_1}\zeta \tag{5.30}$$

It follows that the closed loop system can be written as a linear system with unknown initial conditions as follows:

$$\ddot{e}_y + k_3\dot{e}_y + k_2 e_y = -k_1\rho_1$$
$$\dot{\rho}_1 = e_y + \rho_2, \quad \rho_1(0) = -\frac{k_3}{k_1}\dot{e}_y(0)$$
$$\dot{\rho}_2 = \frac{k_0}{k_1}e_y, \quad \rho_2(0) = -\frac{k_3}{k_1}\zeta \tag{5.31}$$

The characteristic polynomial of the closed loop system is then given by the following fourth order polynomial

$$p(s) = s^4 + k_3 s^3 + k_2 s^2 + k_1 s + k_0 \tag{5.32}$$

whose roots are assignable at will, by proper choice of the design coefficients $\{k_3, k_2, k_1, k_0\}$.

We rewrite the output feedback controller (5.26) as follows:

$$e_u = -k_3\int_0^t e_u(\sigma)d\sigma - k_2 e_y - k_1\int_0^t e_y(\sigma)d\sigma - k_0\int_0^t\int_0^{\sigma_1} e_y(\sigma_2)d\sigma_2 d\sigma_1 \tag{5.33}$$

In Laplace transform terms we obtain the following relation for the proposed output error feedback controller

$$\left(1 + \frac{k_3}{s}\right)e_u(s) = -\left[k_2 + \frac{k_1}{s} + \frac{k_0}{s^2}\right]e_y(s) \tag{5.34}$$

The transfer function relating $e(s)$ and $e_u(s)$ is therefore given by

$$e_u(s) = -\left[\frac{k_2 s^2 + k_1 s + k_0}{s(s+k_3)}\right]e_y(s) \tag{5.35}$$

The controller is then given, in a combined time and frequency domain notation, by

$$u = u^*(t) - \left[\frac{k_2 s^2 + k_1 s + k_0}{s(s+k_3)}\right](y(t) - y^*(t)) \tag{5.36}$$

The new controller clearly exhibits an integral action which is characteristic of compensator networks that robustly perform against unknown constant perturbation inputs.

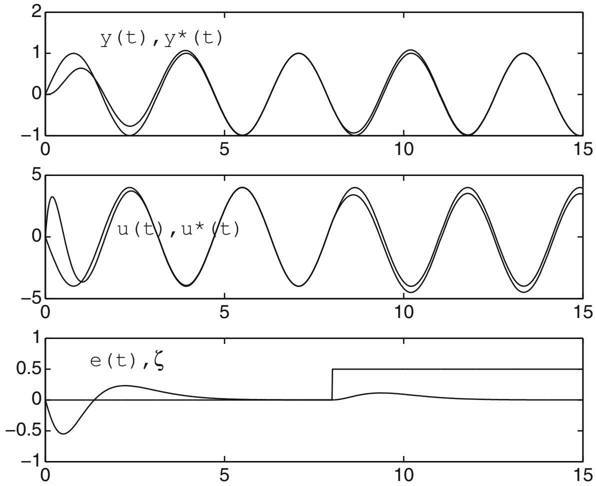

Fig. 5.6. Closed loop responses of perturbed second order integrator and GPI compensator with integral action

Figure 5.6 depicts the controlled trajectories of the perturbed second order integrator under the actions of the designed GPI feedback controller. The reference signal was taken to be a sinusoid $y^*(t) = sin(\omega t)$ with $\omega = 2$ rad/s. The designed constants were obtained from the desired characteristic polynomial $p(s) = (s + p)^4$ with $p = 2$. A constant perturbation input ζ appears at time $t = 8$. The controller is shown to be robust with respect to this kind of perturbation inputs.

5.2.6 The presence of noises

Consider the following noise perturbed system,

$$\begin{aligned} \dot{x}_1 &= x_2 + \eta_1 \\ \dot{x}_2 &= u + \eta_2 \\ y &= x_1 + \eta_0 \end{aligned} \tag{5.37}$$

with η_0 being a zero-mean normally distributed noise, and η_1 and η_2, possibly, being biased noises.

Suppose it is desired to execute an output reference trajectory tracking task involving a finite time rest-to-rest maneuver for the output variable y.

A GPI controller may be proposed to be of a form motivated by the presence of constant biases in the noisy perturbation inputs;

$$u = -k_2 y - \int_0^t [k_3 u(\tau) + k_1 y(\tau)]\, d\tau - k_0 \int_0^t \int_0^\sigma y(\lambda) d\lambda \tag{5.38}$$

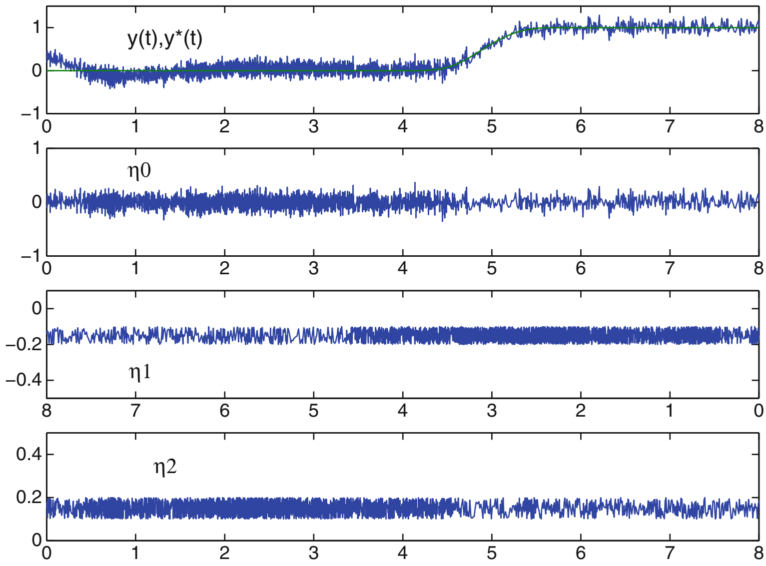

Fig. 5.7. Performance of GPI controller on a highly noise contaminated second order integrator plant.

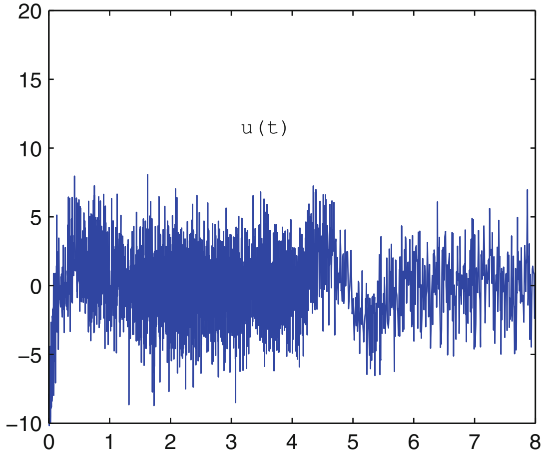

Fig. 5.8. GPI synthesized control input for a noisy second order integrator plant.

This control scheme can be proven to work reasonably well *on the average*. as the following simulations depict on Figure 5.7

The noisy control input evolution is shown in Figure 5.8.

5.2.7 A perturbed third order integration plant

Consider the following perturbed third order integrator system

$$y^{(3)} = u + \zeta, \tag{5.39}$$

where ζ is an unknown, constant, perturbation input signal.

The estimation of the unavailable signals $\dot{y}$, $\ddot{y}$ require two and one integrations, respectively, of the perturbed dynamics. This yields an error between the actual value of the velocity and the estimated values (which is computed neglecting the initial conditions and the perturbation input) which grows quadratically in time. To overcome this type of perturbation the average GPI controller requires a suitable iterated integral tracking error action, i.e., a triple integration of the tracking error.

The output tracking error e is now governed by

$$e^{(3)} = e_u + \zeta \tag{5.40}$$

where, as before, $e_u = u - u^*(t) = u - [y^*(t)]^{(3)}$. We propose then the following feedback controller,

$$\begin{aligned} e_u = & -k_5\widehat{\ddot{e}} - k_4\widehat{\dot{e}} - k_3 e - k_2\int_0^t e(\sigma)d\sigma - k_1\int_0^t\int_0^{\sigma_1} e(\sigma_2)d\sigma_2 d\sigma_1 \\ & -k_0\int_0^t\int_0^{\sigma_1}\int_0^{\sigma_2} e(\sigma_3)d\sigma_3 d\sigma_2 d\sigma_1 \end{aligned} \tag{5.41}$$

where the estimates $\widehat{\dot{e}}$ and $\widehat{\ddot{e}}$ are computed in the form:

$$\widehat{\ddot{e}} = \int_0^t e_u(\sigma)d\sigma, \quad \widehat{\dot{e}} = \int_0^t\int_0^{\sigma_1} e_u(\sigma_1)d\sigma_1 d\sigma. \tag{5.42}$$

These are related to the actual value of $\dot{e}$ and $\ddot{e}$ by the relations

$$\begin{aligned} \dot{e} &= \widehat{\dot{e}} + \frac{1}{2}\zeta t^2 + \ddot{e}(0)t + \dot{e}(0) \\ \ddot{e} &= \widehat{\ddot{e}} + \zeta t + \ddot{e}(0) \end{aligned}$$

The closed loop system using the proposed controller satisfies the following integro-differential equation

$$\begin{aligned} & e^{(3)} + k_5\ddot{e} + k_4\dot{e} + k_3 e + k_2\int_0^t e(\sigma)d\sigma + k_1\int_0^t\int_0^{\sigma_1} e(\sigma_2)d\sigma_2 d\sigma_1 \\ & +k_0\int_0^t\int_0^{\sigma_1}\int_0^{\sigma_2} e(\sigma_3)d\sigma_3 d\sigma_2 d\sigma_1 \\ & = k_5[\zeta t + \ddot{e}(0)] + k_4[\frac{1}{2}\zeta t^2 + \ddot{e}(0)t + \dot{e}(0)] \end{aligned}$$

whose characteristic polynomial is just given by

$$p(s) = s^6 + k_5 s^4 + k_4 s^3 + k_2 s^2 + k_1 s + k_0$$

A suitable choice of the set of coefficients $\{k_5, \ldots, k_0\}$ provides an asymptotically exponentially stable tracking design.

The feedback controller is then given by

$$e_u = -k_5 \left(\int_0^t e_u(\sigma) d\sigma \right) - k_4 \left(\int_0^t \int_0^{\sigma_1} e_u(\sigma_1) d\sigma_1 d\sigma \right) - k_3 e - k_2 \int_0^t e(\sigma) d\sigma$$
$$-k_1 \int_0^t \int_0^{\sigma_1} e(\sigma_2) d\sigma_2 d\sigma_1 - k_0 \int_0^t \int_0^{\sigma_1} \int_0^{\sigma_2} e(\sigma_3) d\sigma_3 d\sigma_2 d\sigma_1 \quad (5.43)$$

Taking Laplace transforms one obtains

$$\left(1 + \frac{k_5}{s} + \frac{k_4}{s^2}\right) e_u(s) = -\left(k_3 + \frac{k_2}{s} + \frac{k_1}{s^2} + \frac{k_0}{s^3}\right) e \quad (5.44)$$

i.e.,

$$e_u = -\left[\frac{k_3 s^3 + k_2 s^2 + k_1 s + k_0}{s(s^2 + k_5 s + k_4)}\right] e$$

or, combining frequency domain notations with time domain quantities:

$$u = u^*(t) - \left[\frac{k_3 s^3 + k_2 s^2 + k_1 s + k_0}{s(s^2 + k_5 s + k_4)}\right] (y - y^*(t))$$

The controller clearly exhibits an integral action which is characteristic of compensator networks that robustly perform against unknown constant perturbation inputs.

A state space realization of the controller may be proposed to be

$$\begin{aligned} u &= u^*(t) - (k_2 - k_5 k_3) z_3 - (k_1 - k_4 k_3) z_2 \\ &\quad - k_0 z_1 - k_3 (y - y^*(t)) \\ \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= -k_5 z_3 - k_4 z_2 + (y - y^*(t)) \end{aligned} \quad (5.45)$$

Figure 5.9 depicts the controlled trajectories of the perturbed third order integrator under the actions of the designed GPI feedback controller. The reference signal was taken to be a sinusoid $y^*(t) = A \sin(\omega t)$ with $\omega = 1.885$ rad/s, $A = 0.1$. The designed constants were obtained from the desired characteristic polynomial $p(s) = (s^2 + 2\zeta_n \omega_n s + \omega_n^2)^3$ with $\zeta_n = 0.81$ and $\omega_n = 2$. A constant perturbation input $\zeta = K = 0.5$ appears at time $t = 8$. The tracking controller is shown to be robust with respect to this kind of perturbation inputs.

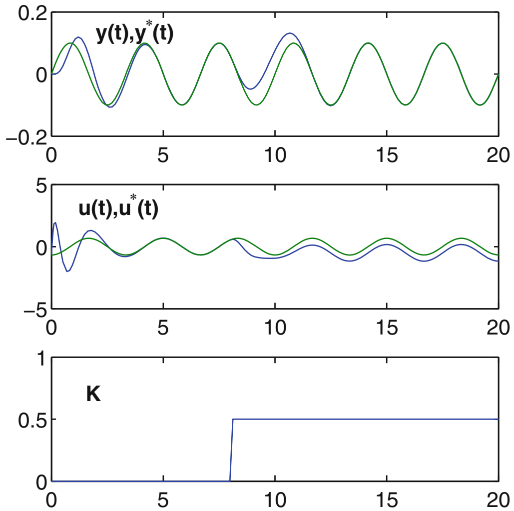

Fig. 5.9. Closed loop responses of perturbed third order integrator and GPI compensator with integral action

Exercise 5.3. Show that, in general, the GPI control procedure yields a compensator of the form:

$$u = u^*(t) - \left[\frac{k_{n+r}s^{n+r} + k_{n+r-1}s^{n+r-1} + \cdots + k_1 s + k_0}{s^{r+1}(s^{n-1} + k_{2n+r-1}s^{n-2} + \cdots + k_{n+r+1})}\right](y - y^*(t)) \tag{5.46}$$

for an n-th order integrator plant which needs to be robust with respect to unknown *time polynomial perturbation inputs of order up to* r. The controller design coefficients $\{k_{2n+r-1}, \cdots, k_1, k_0\}$ are chosen so that the closed loop characteristic polynomial

$$p(s) = s^{2n+r} + k_{2n+r-1}s^{2n+r-1} + \cdots + k_1 s + k_0 \tag{5.47}$$

is a Hurwitz polynomial.

5.3 Relations with classical compensator design

Evidently, the GPI control based on integral reconstructors of the state is intimately related to traditional, or classical, compensating network design. In the case of pure integration systems this is particularly easy to demonstrate in rather general terms.

Consider the block diagram of Figure 5.10 depicting a classical compensation scheme for the plant system represented by the transfer function $G(s)$.

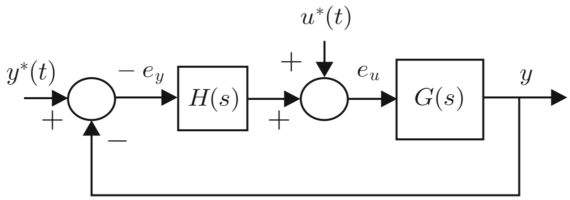

Fig. 5.10. Classical compensator design scheme

In our particular case, consider $G(s) = s^{-n}$ to be an n-th order integrator and $H(s)$ is a compensator to be designed. Let $e_y = y - y^*(t)$ and $e_u = u - u^*(t)$. The closed loop system expression is obtained from the following relations, where we abusively combine time domain expressions with frequency domain expressions,

$$y = G(s)u = G(s)(e_u + u^*(t)) = G(s)e_u + y^*(t),$$
$$e_u = H(s)(y^*(t) - y) = -H(s)e_y$$

We have, that the output tracking error, e_y, is governed by

$$(1 + G(s)H(s))e_y = 0 \tag{5.48}$$

The exponentially asymptotic convergence of e_y to zero is guaranteed as long as the numerator of the rational transfer function expression: $(1 + G(s)H(s))$ is a Hurwitz polynomial.

Assume we desire a compensator, with integral action, of the form

$$H(s) = \frac{k_n s^n + \cdots + k_1 s + k_0}{s(s^{n-1} + k_{2n-1}s^{n-2} + \cdots + k_{n+1})} \tag{5.49}$$

The numerator of the rational transfer function expression $1 + G(s)H(s)$ is given by

$$s^{2n} + k_{2n-1}s^{2n-1} + \cdots + k_{n+1}s^{n+1} + k_n s^n + \cdots + k_1 s + k_0 \tag{5.50}$$

The compensator design problem boils down to locate the poles of a $2n$-th order polynomial.

Both design procedures yield the same controller structure, depending on the type of perturbations one desires to reject in the closed loop operation of the system and they both boil down to locate the poles of a higher order characteristic polynomial.

It is still instructive to go through the details of a GPI compensator when the system is not a pure integration system. We propose to view this, and the equivalence with classical compensator designs, via a simple, second order, example.

Example 5.4. Consider the following second order plant

$$\frac{y(s)}{u(s)} = \frac{1}{s^2 + a_1 s + a_0} \tag{5.51}$$

It is desired to track a smooth reference signal $y^*(t)$. The system and the tracking error system may be written, in the time domain, as

$$\ddot{y} + a_1\dot{y} + a_0 y = u, \quad \ddot{e}_y + a_1\dot{e}_y + a_0 e_y = e_u \tag{5.52}$$

where $e_y = y - y^*(t)$ and $e_u = u - u^*(t)$ are, respectively, the output tracking error and the input tracking error. A PID feedback controller, specifying the input tracking error, is given by

$$e_u = (a_1 - k_2)\dot{e}_y + (a_0 - k_1)e_y - k_0 \int_0^t e_y(\sigma)d\sigma \tag{5.53}$$

This yields, evidently, a closed loop system represented by an integro differential equation for the output tracking error given by

$$\ddot{e}_y + k_2\dot{e}_y + k_1 e_y + k_0 \int_0^t e_y(\sigma)d\sigma = 0 \tag{5.54}$$

The characteristic polynomial, associated with this equation is easily shown to be

$$p(s) = s^3 + k_2 s^2 + k_1 s + k_0 \tag{5.55}$$

Thus, the design problem reduces to an appropriate choice of the feedback controller gains so as to make the above polynomial Hurwitz.

The signal $\dot{e}_y$ needed in the controller is, unfortunately, not available. We resort to an integral reconstructor of such a signal, aware of the fact that such a reconstructor differs from the actual signal by an unknown constant quantity fixed by the unchangeable initial conditions. Our provisions for an integral term in the compensator counteracts this constant estimation bias in the velocity tracking error. We propose, based on the plant error dynamics:

$$\widehat{\dot{e}_y} = -a_1 e_y + \int_0^t (e_u(\sigma) - a_0 e_y(\sigma))d\sigma \tag{5.56}$$

Substituting this expression in the proposed PID controller and collecting the terms on e_u on the left-hand side and those of e_y in the right-hand side, we obtain, after some elementary algebraic manipulations, the following expression for the compensator written in the frequency domain

$$e_u(s) = -\left[\frac{(k_1 - a_0 - a_1(k_2 - a_1))s + (k_0 - a_0(k_2 - a_1)}{s + k_2 - a_1}\right] e_y(s) \tag{5.57}$$

The control input to the system is then obtained as

$$u(t) = u^*(t) - \left[\frac{(k_1 - a_0 - a_1(k_2 - a_1))s + (k_0 - a_0(k_2 - a_1)}{s + k_2 - a_1}\right](y - y^*(t)) \tag{5.58}$$

The compensator, based on integral reconstructors and GPI, is of the form

$$\begin{aligned} e_u &= -\left[\frac{\beta_1 s + \beta_0}{s + \beta_2}\right] e_y \\ \beta_2 &= k_2 - a_1, \quad \beta_1 = k_1 - a_0 - a_1(k_2 - a_1), \quad \beta_0 = k_0 - a_0(k_2 - a_1) \end{aligned}$$

The classical compensator procedure may be carried out proposing a lead network of the form

$$H(s) = \frac{\beta_1 s + \beta_0}{s + \beta_2} \tag{5.59}$$

The stability condition on the closed loop expression $(1 + G(s)H(s))$ leads to the following characteristic polynomial

$$p(s) = s^3 + (a_1 + \beta_2)s^2 + (a_0 + a_1\beta_2 + \beta_1)s + a_0\beta_2 + \beta_0 = 0 \tag{5.60}$$

equating the corresponding coefficients of the closed loop characteristic polynomial $p(s)$ with those of a desired third order Hurwitz polynomial, given by $s^3 + k_2 s^2 + k_1 s + k_0$, we obtain that the compensator is given by

$$\begin{aligned} e_u &= -\left[\frac{\beta_1 s + \beta_0}{s + \beta_2}\right] e_y \\ \beta_2 &= k_2 - a_1, \quad \beta_1 = k_1 - a_0 - a_1(k_2 - a_1), \quad \beta_0 = k_0 - a_0(k_2 - a_1) \end{aligned}$$

i.e., exactly the same compensator found before by GPI control procedures.

5.4 A DC motor controller design example

Consider the following model of a DC motor

$$\begin{aligned} L\frac{di}{dt} &= -Ri - K\dot{\theta} + v \\ J\frac{d\dot{\theta}}{dt} &= -B\dot{\theta} + Ki \end{aligned}$$

where $\dot{\theta}$ denotes the angular velocity, i is the armature current, and v is the armature voltage input. The output of the system is regarded to be the angular position θ.

The input-output model of the motor is readily obtained as

$$\theta^{(3)} + \left(\frac{B}{J} + \frac{R}{L}\right)\ddot{\theta} + \left(\frac{RB}{LJ} + \frac{K^2}{LJ}\right)\dot{\theta} = \frac{K}{LJ}v \tag{5.61}$$

The transfer function description of the system between the input error $e_v = v - v^*(t)$ and the output tracking error $e_\theta = \theta - \theta^*(t)$ is given by

$$e_\theta(s) = \left[\frac{\frac{K}{LJ}}{s^3 + \left(\frac{B}{J} + \frac{R}{L}\right)s^2 + \left(\frac{RB}{LJ} + \frac{K^2}{LJ}\right)s}\right] e_v(s) \tag{5.62}$$

where $v^*(t)$ is the nominal control input corresponding to a desired angular position trajectory $\theta^*(t)$, given by

$$v^*(t) = \frac{LJ}{K}[\theta^*(t)]^{(3)} + \left(\frac{LB}{K} + \frac{RJ}{K}\right)\ddot{\theta}^*(t) + \left(\frac{RB}{K} + K\right)\dot{\theta}^*(t) \tag{5.63}$$

We propose a compensation network, with integral action, specified by

$$H(s) = \frac{e_v(s)}{e_\theta(s)} = \frac{k_3 s^3 + k_2 s^2 + k_1 s + k_0}{s(s^2 + k_5 s + k_4)} \tag{5.64}$$

where the design coefficients $\{k_5, k_4, \ldots, k_0\}$ will be determined by equating the characteristic polynomial, $p(s)$, of the closed loop system with a desired polynomial of the form:

$$\begin{aligned} p_d(s) &= (s^2 + 2\xi\omega_n s + \omega_n^2)^3 \\ &= s^6 + 6\xi\omega_n s^5 + \omega_n^2\left(3 + 12\xi^2\right)s^4 + \omega_n^3\left(12\xi + 8\xi^3\right)s^3 \\ &\quad + \omega_n^4\left(3 + 12\xi^2\right)s^2 + 6\omega_n^5\xi s + \omega_n^6 \end{aligned} \tag{5.65}$$

We have the following closed loop characteristic polynomial

$$\begin{aligned} p(s) = s^6 &+ \left(\frac{B}{J} + \frac{R}{L} + k_5\right)s^5 + \left[k_4 + \left(\frac{B}{J} + \frac{R}{L}\right)k_5 + \frac{K^2}{LJ} + \frac{BR}{JL}\right]s^4 \\ &+ \left[\frac{K}{JL}k_3 + \left(\frac{B}{J} + \frac{R}{L}\right)k_4 + \left(\frac{K^2}{LJ} + \frac{BR}{LJ}\right)k_5\right]s^3 \\ &+ \left[\frac{K}{LJ}k_2 + \left(\frac{K^2}{LJ} + \frac{BR}{LJ}\right)k_4\right]s^2 + \frac{K}{LJ}k_1 s + \frac{K}{LJ}k_0 \end{aligned} \tag{5.66}$$

We obtain the following semi-implicit expressions for the output error feedback controller design parameters

$$\begin{aligned} k_5 &= -\left(\frac{B}{J} + \frac{R}{L}\right) + 6\xi\omega_n \\ k_4 &= -\left(\frac{BR}{JL} + \frac{K^2}{JL}\right) - \left(\frac{B}{J} + \frac{R}{L}\right)\left[6\xi\omega_n - \left(\frac{B}{J} + \frac{R}{L}\right)\right] \end{aligned}$$

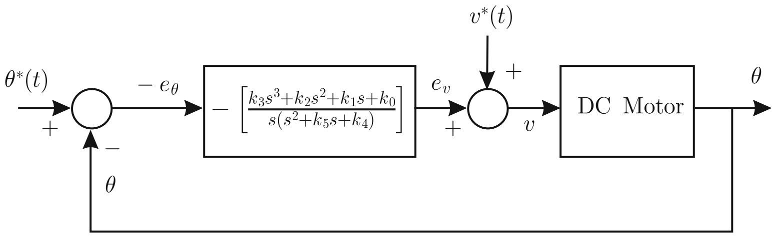

Fig. 5.11. Classical compensator design for DC motor tracking task

$$
\begin{aligned}
&+\omega_n^2(3+12\xi^2)\\
k_3 &= \frac{LJ}{K}\left[-\left(\frac{B}{J}+\frac{R}{L}\right)k_4-\left(\frac{K^2}{LJ}+\frac{BR}{LJ}\right)k_5+\omega_n^3(12\xi+8\xi^3)\right]\\
k_2 &= \frac{LJ}{K}\left[-\left(\frac{K^2}{LJ}+\frac{BR}{JL}\right)k_4+\omega_n^4(3+12\xi^2)\right]\\
k_1 &= 6\frac{LJ}{K}\omega_n^5\xi\\
k_0 &= \frac{LJ}{K}\omega_n^6
\end{aligned}
$$

We used the previously designed output feedback controller in a DC motor (Fig. 5.11) characterized by the following parameters:

$$L = 0.71\ \mathrm{H}\ \ R = 10\ \Omega,\ \ K = 77.5355\ \mathrm{V-s/rad},\ \ J = 0.0550\ \mathrm{N-m-s^2/rad},$$

$$B = 0.2203\ \mathrm{N-m-s/rad}$$

with design parameters chosen so that the closed loop characteristic polynomial exhibited a damping factor, $\xi = 1$, and a natural frequency $\omega_n = 18$.

It is desired to achieve a rest-to-rest maneuver from the initial position of 0 rad towards the end position of 2.5 rad in a time interval of 1 second, starting at $t = 1.0$ s. As depicted in Figure 5.12, the closed loop trajectories exhibit an accurate trajectory tracking feature while being robust with respect to sudden load torques. In the figure a constant torque $\tau = 0.25$ N-m was devised at time $t = 2.0$ s.

GPI control can be used as an average feedback control for a switched plant. The following design example illustrates this possibility in a popular two level switched system.

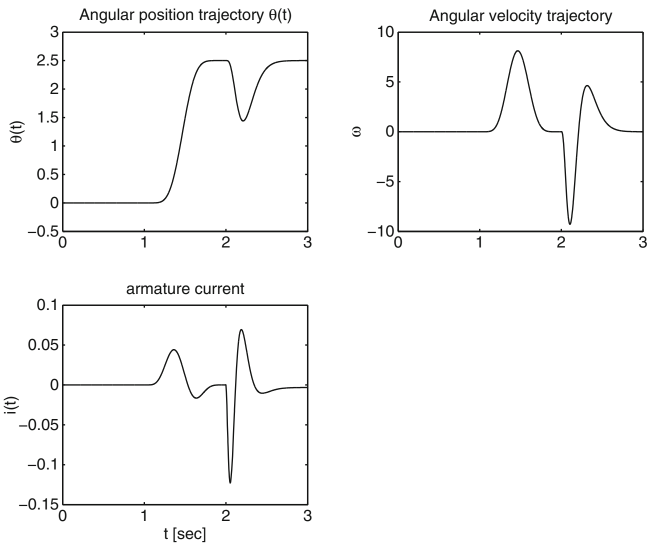

Fig. 5.12. Closed loop performance of perturbed GPI controlled DC motor.

5.5 Control of the Double Bridge Buck Converter

In reference to Figure 5.13 the several electronic switches take position values according to

$$\begin{cases} u = 1 \quad , S_1 = ON, S_2 = ON, S_3 = OFF, S_4 = OFF \\ u = -1 \ , S_1 = OFF, S_2 = OFF, S_3 = ON, S_4 = ON \\ u = 0 \quad , S_1 = S_4 = ON, S_2 \text{ (or } S_3) = OFF \end{cases} \tag{5.67}$$

Consider the following average model of a double bridge buck-converter:

$$\begin{aligned} L\dot{x}_1 &= -x_2 + u_{av}E \\ C\dot{x}_2 &= x_1 - \frac{x_2}{R} \\ y &= x_2 \end{aligned}$$

where x_1 is the inductor current, x_2 represents the capacitor voltage, u_{av} is the average control input assumed to take values in the closed interval: $u \in [-1, 1]$ of the real line. The parameters L,C,R, and E are assumed to be known.

An input-output model of the system is obtained by simply eliminating the state variable x_1 from the system equations, we have

$$LC\ddot{y} + \frac{L}{R}\dot{y} + y = u_{av}E \tag{5.68}$$

This relation may be written as

$$\ddot{y} + \frac{1}{RC}\dot{y} + \frac{1}{LC}y = u_{av}\frac{E}{LC} \tag{5.69}$$

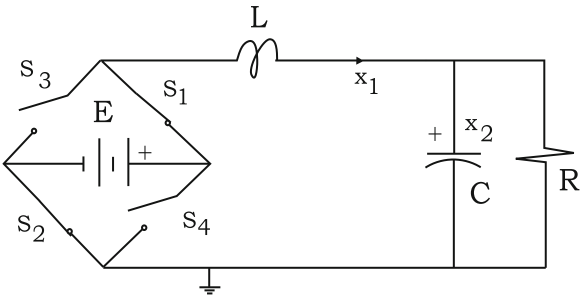

Fig. 5.13. The double bridge buck converter.

We rewrite the system as follows:

$$\ddot{y} + \gamma_2 \dot{y} + \gamma_1 y = \gamma_3 u_{av} \tag{5.70}$$

and we proceed to design a GPI tracking controller under the assumption that the parameters γ_1, γ_3, are all perfectly known ($\gamma_2 = 1/RC$, $\gamma_1 = 1/LC$, $\gamma_3 = E/LC$)

The average system is represented in transfer function form as

$$y = \left[\frac{\gamma_3}{s^2 + \gamma_2 s + \gamma_1} \right] u_{av} \tag{5.71}$$

We formulate the problem as follows:

Given a desired output voltage signal $y^*(t)$, it is required to design an output feedback controller, possibly of dynamic nature, which induces an exponentially asymptotic convergence of the output signal y towards the desired reference signal $y^*(t)$.

In other words, we want

$$y \to y^*(t) \;\; \text{exponentially} \tag{5.72}$$

The nominal input-output relation is clearly written as

$$y^*(s) = \left[\frac{\gamma_3}{s^2 + \gamma_2 s + \gamma_1} \right] u^*(s) \tag{5.73}$$

and, hence, defining $e = y - y^*$ and $e_u = u_{av} - u^*$

$$e(s) = \left[\frac{\gamma_3}{s^2 + \gamma_2 s + \gamma_1} \right] e_u(s) \tag{5.74}$$

We propose the following GPI controller,

$$e_u = -\frac{1}{\gamma_3} \left[\frac{k_2 s^2 + k_1 s + k_0}{s(s + k_3)} \right] e \tag{5.75}$$

A state space realization of such a GPI controller is readily obtained as

$$
\begin{aligned}
u_{av} &= u^*(t) - \left(\frac{k_1}{\gamma_3} - \frac{k_3 k_2}{\gamma_3}\right) z_2 - \frac{k_0}{\gamma_3} z_1 - \frac{k_2}{\gamma_3}(y - y^*) \\
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -k_3 z_2 + (y - y^*(t))
\end{aligned}
\tag{5.76}
$$

The closed loop characteristic polynomial governing the average tracking error is given, after following the classical compensation design methodology, by

$$
p(s) = s^4 + (\gamma_2 + k_3)\, s^3 + (\gamma_1 + k_3\gamma_2 + k_2)\, s^2 + (k_3\gamma_1 + k_1)\, s + k_0
$$

By equating, term by term, this closed loop characteristic polynomial to a desired characteristic polynomial of the form,

$$
\begin{aligned}
p_d(s) &= (s^2 + 2\zeta\omega_n s + \omega_n^2)^2 \\
&= s^4 + (4\zeta\omega_n)\, s^3 + \left(4\zeta^2\omega_n^2 + 2\omega_n^2\right) s^2 + \left(4\zeta\omega_n^3\right) s + \omega_n^4
\end{aligned}
\tag{5.77}
$$

we obtain the following output feedback controller design:

$$
\begin{aligned}
k_3 &= 4\zeta\omega_n - \gamma_2 \\
k_2 &= 4\zeta^2\omega_n^2 + 2\omega_n^2 - \gamma_1 - \gamma_2(4\zeta\omega_n - \gamma_2) \\
k_1 &= 4\zeta\omega_n^3 - \gamma_1\,(4\zeta\omega_n - \gamma_2) \\
k_0 &= \omega_n^4
\end{aligned}
$$

Simulations were performed with the following set of parameters:

$$
L = 10^{-3}\text{H},\ C = 1.0\ \mu F,\ R = 39.52\ \text{Ohm},\ E = 30\ \text{V}
$$

It is desired to track a rest-to-rest output voltage reference trajectory smoothly changing between two different constant values over a finite time interval. The smooth transition trajectory is specified by means of a Bézier polynomial.

We first simulate the average system behavior of the GPI controlled model with u_{av} taking values in $[-1, 1]$ (Fig. 5.14).

A two level Delta-Sigma modulation was used for the implementation of the average output feedback controller (Fig. 5.15). We used the following two level Delta-Sigma modulator

$$
u = \frac{1}{2}\left[\text{sign}(u_{av}(t)) + \text{sign} z\right], \quad \dot{z} = u_{av} - u
\tag{5.78}
$$

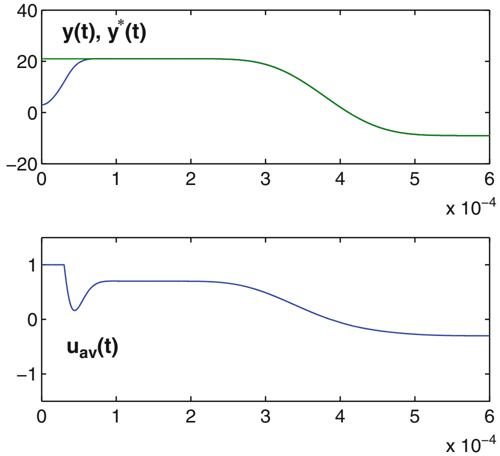

Fig. 5.14. Closed loop average converter response to GPI controller.

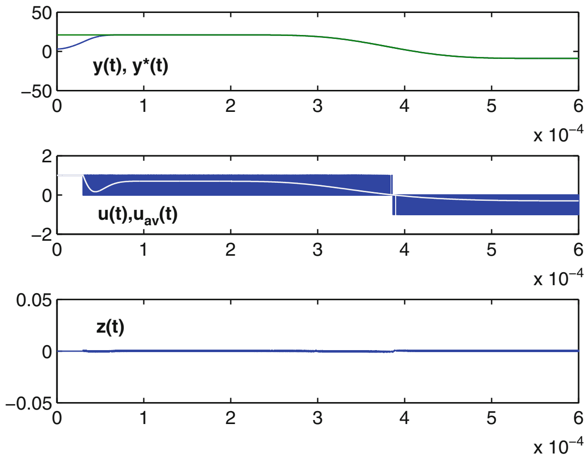

Fig. 5.15. Closed loop switched converter response

5.6 GPI control for systems in State Space Form

Direct derivation of integral state reconstruction is also possible in linear systems and some nonlinear systems written in the traditional state space form with the benefit of GPI control and a GPI based sliding mode control option for the particular class of switched systems. We start by a simple illustrative example considering the integral input-output parametrization of a normalized state model representing a heating system in compartmental form.

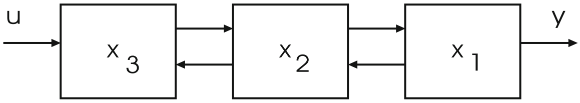

Fig. 5.16. Schematic Diagram of heating system in compartmental form

Example 5.5. Consider the compartmental model of a normalized heating system (Fig. 5.16),

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1 \\ \dot{x}_2 &= x_1 - 2x_2 + x_3 \\ \dot{x}_3 &= x_2 - 2x_3 + u \end{aligned} \tag{5.79}$$

The system is controllable and it is also observable from $y = x_1$. We easily obtain the following differential parametrization of all system state variables,

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} + y \\ x_3 &= \ddot{y} + 3\dot{y} + y \end{aligned} \tag{5.80}$$

From the observability property with respect to the chosen output, we directly obtain the following integral parametrization of the state variables:

$$\begin{aligned} \left(\int\int x_1\right) &= \left(\int\int y\right) \\ \left(\int\int x_2\right) &= \left(\int y\right) + \left(\int\int y\right) \\ \left(\int\int x_3\right) &= y + 3\left(\int y\right) + \left(\int\int y\right) \end{aligned} \tag{5.81}$$

From the system state equations we obtain

$$\begin{aligned} x_1 &= \left(\int x_2\right) - \left(\int x_1\right) \\ x_2 &= \left(\int x_1\right) - 2\left(\int x_2\right) + \left(\int x_3\right) \\ x_3 &= \left(\int x_2\right) - 2\left(\int x_3\right) + \left(\int u\right) \end{aligned} \tag{5.82}$$

Iterating once:

$$\begin{aligned} x_1 &= 2(\int\int x_1) - 3(\int\int x_2) + (\int\int x_3) \\ x_2 &= -3(\int\int x_1) + 6(\int\int x_2) - 4(\int\int x_3) + (\int\int u) \end{aligned}$$

$$x_3 = (\int\int x_1) - 4(\int\int x_2) + 5(\int\int x_3) - 2(\int\int u) + (\int u) \quad (5.83)$$

Eliminating the double integrals in the previous expression with the corresponding expressions in (5.81), we obtain the following integral reconstructors of the state variables :

$$\begin{aligned}
\hat{x}_1 &= y \\
\hat{x}_2 &= -4y - 6(\int y) - 5(\int\int y) + (\int\int u) \\
\hat{x}_3 &= 5y + 2(\int y) + 11(\int\int y) - 2(\int\int u) + (\int u)
\end{aligned} \quad (5.84)$$

which is an integral input-output parametrization of the state vector components, modulo initial conditions and their effects.

Notice that $\hat{x}_2$ and $\hat{x}_3$ are off by a linear function of time with respect to the actual values of the corresponding state variables x_2 and x_3. Hence, any feedback controller based on these integral reconstructors requires second order iterated integral output error compensation.

To design a GPI stabilizing controller, it is necessary to establish first the equilibrium state of the system. From (5.79) it is clear that the equilibrium values for the variables, x_1, x_2, and x_3 coincide, i.e., $\overline{x}_1 = \overline{x}_2 = \overline{x}_3 = \overline{x}$. Moreover, $\overline{u} = \overline{x}$.

Exercise 5.6. Consider a traditional stabilizing state feedback controller of the form: $u = \overline{u} - k_2(x_1 - \overline{x}) - k_3(x_2 - \overline{x}) - k_4(x_3 - \overline{x})$. This controller, in terms of the state vector reconstruction variables, needs to be additionally (integrally) compensated as follows:

$$\begin{aligned}
u = \overline{u} - k_2(\hat{x}_1 - \overline{x}) - k_3(\hat{x}_2 - \overline{x}) - k_4(\hat{x}_3 - \overline{x}) - k_1 \int_0^t (y(\lambda) - \overline{x}) d\lambda \\
-k_0 \int_0^t \int_0^\lambda (y(\rho) - \overline{x}) d\rho d\lambda
\end{aligned} \quad (5.85)$$

Determine a set of suitable constant gains $\{k_0, k_1, \ldots, k_4\}$ to achieve a stabilization of the state vector towards a desired equilibrium. Simulate the state responses and evaluate the performance of the proposed GPI state controller.

5.7 Generalization to MIMO linear systems

Consider the observable, time-invariant, linear system of m inputs and p outputs:

$$\dot{x} = Ax + Bu, \; x(0) = x_0, \quad y = cx, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad u \in \mathbb{R}^p \quad (5.86)$$

Taking Laplace transforms and integrating the system expression, one obtains

$$x(s) = A\frac{x(s)}{s} + B\frac{u(s)}{s} \tag{5.87}$$

Iterating on this functional relation we have

$$x(s) = A^2\frac{x(s)}{s^2} + AB\frac{u(s)}{s^2} + B\frac{u(s)}{s}$$

Iterating $n-1$ times, we have

$$x(s) = A^{n-1}\left(\frac{x(s)}{s^{n-1}}\right) + \sum_{i=1}^{n-1} A^{i-1}B\frac{u(s)}{s^i}$$

On the other hand, consider the output y and its successive derivatives, written also in the Laplace transform domain:

$$\begin{pmatrix} I \\ sI \\ \vdots \\ s^{(n-1)}I \end{pmatrix} y(s) = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} x(s) + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ CB & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ CA^{n-2}B & \cdots & \cdots & CB \end{pmatrix} \begin{pmatrix} I \\ sI \\ \vdots \\ s^{(n-2)}I \end{pmatrix} u(s)$$

Integrating $n-1$ times, we have

$$\begin{pmatrix} \frac{I}{s^{n-1}} \\ \frac{I}{s^{n-2}} \\ \vdots \\ I \end{pmatrix} y(s) = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} \frac{x(s)}{s^{n-1}} + \mathcal{M} \begin{pmatrix} \frac{I}{s^{n-1}} \\ \frac{I}{s^{n-2}} \\ \vdots \\ \frac{I}{s} \end{pmatrix} u(s) \tag{5.88}$$

Thanks to the observability of the system, one has

$$\frac{x(s)}{s^{n-1}} = [\mathcal{O}^T\mathcal{O}]^{-1}\mathcal{O}^T \left[\begin{pmatrix} \frac{I}{s^{n-1}} \\ \frac{I}{s^{n-2}} \\ \vdots \\ I \end{pmatrix} y(s) - \mathcal{M} \begin{pmatrix} \frac{I}{s^{n-1}} \\ \frac{I}{s^{n-2}} \\ \vdots \\ \frac{I}{s} \end{pmatrix} u(s) \right] \tag{5.89}$$

We can now combine this expression with the preceding one to obtain

$$x(s) = A^{n-1}\left(\frac{x(s)}{s^{n-1}}\right) + \sum_{i=1}^{n-1} A^{i-1}B\frac{u(s)}{s^i} \tag{5.90}$$

Finally, we have, combining frequency domain notation and time domain notation, that

$$x(t) = \mathcal{P}(s^{-1})y(t) + \mathcal{Q}(s^{-1})u(t) \tag{5.91}$$

We address this expression, which does not take into account the influence of the initial states, as the *integral state reconstructor* based on iterated integrals of inputs and outputs.

The integral state reconstructor may be used, in principle, on any linear state feedback control law $u = -k^T x(t)$ as long as it is complemented with additional compensation which counteracts the effect of the neglected initial conditions in the integral reconstructors,

$$u = -k^T\left[\mathcal{P}(s^{-1})y(t) + \mathcal{Q}(s^{-1})u(t)\right] + v$$

Such a compensator only requires iterated integrations of the outputs (or output tracking errors) and of the inputs (or of the input tracking errors).

Example 5.7. Consider the nonlinear model of the Planar Vertical Take-off and Landing Aircraft (PVTOL)

$$\begin{aligned}
\ddot{x} &= -u_1\sin\theta + \epsilon u_2\cos\theta\\
\ddot{z} &= u_1\cos\theta + \epsilon u_2\sin\theta - g\\
\ddot{\theta} &= u_2
\end{aligned}$$

Linearization around the equilibrium:

$$x = \overline{x}, \quad z = \overline{z}, \quad \theta = 0, \quad u_1 = g, \quad u_2 = 0 \tag{5.92}$$

yields

$$\ddot{x}_\delta = -g\theta_\delta + \epsilon u_{2\delta}, \quad \ddot{z}_\delta = u_{1\delta}, \quad \ddot{\theta}_\delta = u_{2\delta} \tag{5.93}$$

Notice that x_δ is a non-minimum phase output. Indeed, it is not difficult to establish the following unstable zero dynamics for x_δ acting as an output

$$\ddot{\theta}_\delta = (g/\epsilon)\theta_\delta \tag{5.94}$$

The linearized system outputs are given by

$$F = x_\delta - \epsilon\theta_\delta, \quad L = z_\delta \tag{5.95}$$

The system is equivalent to the following two independent chains of integrations,

$$u_{1\delta} = \ddot{L}, \quad u_{2\delta} = -\frac{1}{g}F^{(4)} \tag{5.96}$$

A pole assignment based compensator is readily suggested as

$$\begin{aligned} u_{1\delta} &= -\gamma_2 \dot{L} - \gamma_1 L \\ u_{2\delta} &= -\frac{1}{g}\left[-k_5 F^{(3)} - k_4 \ddot{F} - k_3 \dot{F} - k_2 F\right] \end{aligned}$$

Consider the system outputs :

$$y_{1\delta} = x_\delta, \quad y_{2\delta} = z_\delta \tag{5.97}$$

and the following integral state reconstructors associated with the chosen outputs,

$$\begin{aligned} \theta_\delta &= (\int\int u_{2\delta}), \quad x_\delta = y_{1\delta}, \quad z_\delta = y_{2\delta} \\ \dot{\theta}_\delta &= (\int u_{2\delta}), \quad \dot{x}_\delta = -g(\int\int\int u_{2\delta}) + \epsilon(\int u_2), \\ \dot{z}_\delta &= (\int u_{1\delta}) \end{aligned}$$

One proceeds to propose the following control law based on integral reconstructors involving only inputs, outputs, and iterated integrals of inputs and outputs,

$$\begin{aligned} u_{1\delta} &= -\gamma_2(\int u_{1\delta}) - \gamma_1 y_{2\delta} - \gamma_0(\int y_{2\delta}) \\ u_{2\delta} &= -\frac{1}{g}\Big[- k_5 \widehat{F}^{(3)} - k_4 \widehat{\ddot{F}} - k_3 \widehat{\dot{F}} - k_2 \widehat{F} - k_1(\int y_{1\delta}) - k_0(\int\int y_{1\delta})\Big] \end{aligned}$$

Using the reconstructed outputs and their reconstructed time derivatives, the GPI controller, with due compensation of the effects of the neglected initial states, is then given by

$$\begin{aligned} u_{1\delta} &= -\int [\gamma_2 u_{1\delta} - \gamma_0 y_{2\delta}] - \gamma_1 y_{2\delta} \\ u_{2\delta} & \quad -\frac{1}{g}\Bigg[(k_5 g u_{2\delta} - k_1 y_{1\delta}) - k_2 y_{1\delta} + \int\int [(k_4 g + k_2\epsilon) u_{2\delta} - k_0 y_{1\delta}] \\ & \quad + \int\int\int (k_3 g u_2 - k_2 y_{1\delta})\Bigg] \end{aligned} \tag{5.98}$$

Figure 5.17 depicts the performance of the proposed GPI controller on the incremental state and control input variables when the controller acts on the full nonlinear system model.

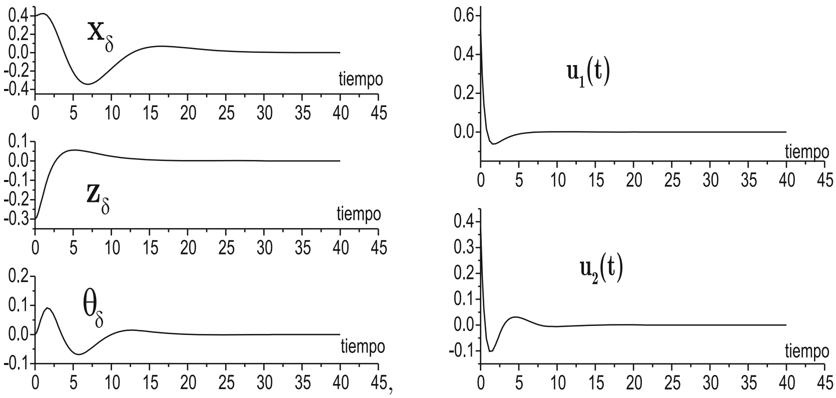

Fig. 5.17. Performance of GPI based controller for the PVTOL system

5.8 GPI and Sliding Mode Control

The philosophy of GPI control establishes that integral reconstructors of the states are sufficient to obtain an asymptotically stable unperturbed controlled system provided a suitable iterated integral error compensation is employed in the feedback law. This scheme is particularly appropriate for linear systems and a rather restricted class of nonlinear systems. Sliding mode control of linear systems makes the closed loop system nonlinear although its average features may be still deduced from a linear ideal sliding dynamic system.

Here we suitably modify GPI control so as to obtain a sliding mode control design for switched systems. An alternative, of course, is to resort to Delta-Sigma modulation after an average design has been achieved. We explore a direct extension of GPI control for sliding mode creation in linear plants.

5.8.1 Compensated sliding surface coordinate functions based on integral reconstructors

We begin by the following elementary example. Consider a second order plant, with inaccessible velocity variable, of the form

$$\ddot{y} = u \tag{5.99}$$

with $u \in \{-W, W\}$ with $W > 0$. Suppose it is desired to stabilize y to zero. An integral reconstructor for the velocity variable is simply synthesized as

$$\widehat{\dot{y}} = \int_0^t u(\lambda) d\lambda \tag{5.100}$$

A traditional sliding surface, achieving the desired objective, is, as it is by now well known, given by

$$S = \{(y, \dot{y}) \in \mathbb{R}^2 \mid \sigma = \dot{y} + k_1 y = 0\} \tag{5.101}$$

A sliding mode exists on the intersection of S with the rectangular region:

$$\{(y, \dot{y}) \in \mathbb{R}^2 \mid W/k_1 < \dot{y} < W/k_1,\ W/k_1^2 < y < W/k_1^2\} \tag{5.102}$$

The basic idea is to replace the sliding surface coordinate function σ by a suitably integral compensated structural estimate of such switching function. In other words, we propose

$$\hat{\sigma} = \int_0^t u(\lambda)d\lambda + k_1 y + k_0 \int_0^t y(\lambda)d\lambda = 0 \tag{5.103}$$

i.e.,

$$\hat{\sigma} = \int_0^t \left[u(\lambda) + k_0 y(\lambda)\right] d\lambda + k_1 y = 0 \tag{5.104}$$

In view of the fact that $\dot{y} = \widehat{\dot{y}} + \dot{y}(0)$, the invariance condition, $\hat{\sigma} = 0$, is equivalent to the following expression:

$$\hat{\sigma} = \dot{y} + k_1 y + k_0 \int_0^t y(\lambda)d\lambda - \dot{y}(0) = 0 \tag{5.105}$$

This perturbed dynamics is, evidently, exponentially asymptotically stable to zero provided the coefficients k_1 and k_0 are Hurwitz coefficients. The characteristic polynomial of this ideal dynamics is none other than $p(s) = s^2 + k_1 s + k_0$. The second invariance condition $\dot{\hat{\sigma}} = 0$ results in the following equivalent control:

$$u_{eq} = -k_1 \dot{y} - k_0 y \tag{5.106}$$

A sliding mode exists on $\hat{\sigma} = 0$ if and only if the following (bandwidth) limitation is satisfied,

$$-W \ < \ k_1 \dot{y} + k_0 y \ < \ W \tag{5.107}$$

The region of existence of sliding motions, projected onto the plane $(y, \dot{y})$, extends now to infinity with longitudinal axis represented by the straight line, $\dot{y} = -(k_0/k_1)y$, and characterized by the band:

$$-\frac{W}{k_1} - \left(\frac{k_0}{k_1}\right) y \ < \ \dot{y} \ < \ \frac{W}{k_1} - \left(\frac{k_0}{k_1}\right) y \tag{5.108}$$

Figure 5.18 depicts the GPI control scheme rendering a stabilizing sliding motion for the second order plant with switching control inputs. Only input and output measurements are thus required.

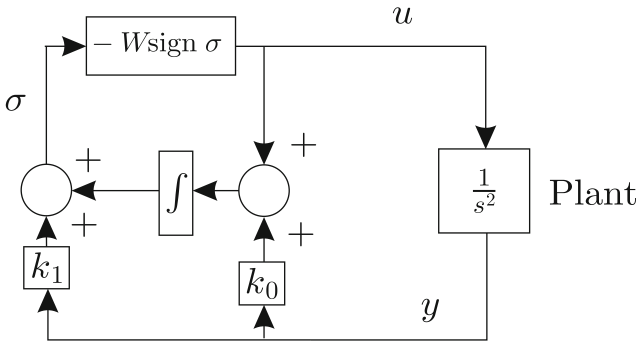

Fig. 5.18. GPI sliding mode stabilization of a second order switched integrator plant

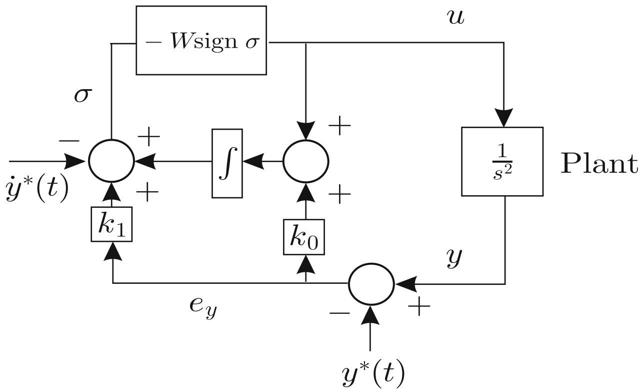

Fig. 5.19. GPI sliding mode output reference trajectory tracking scheme for a second order switched integrator plant

Exercise 5.8. Consider the previous illustrative example and assume it is desired to have the output y of the plant asymptotically track a given reference trajectory, $y^*(t)$, with the limited control action: $u \in \{-W, W\}$. Suitably modify the previous stabilizing control scheme to accomplish the desired objective. Show that the diagram in Fig. 5.19 is the suitable modification.

Execute a computer simulation program and verify that the obtained closed loop responses conform to those shown in Figure 5.20. The numerical values used in the simulation correspond to

$$k_0 = 2\zeta\omega_n, \; k_1 = \omega_n^2, \; \zeta = 0.707, \quad \omega_n = 1.0, \quad W = 2, \quad y^*(t) = \sin(t)$$

5.8.2 A GPI based sliding mode control of a perturbed system

Consider the elementary second order perturbed linear system,

$$\ddot{y} = u + \zeta \tag{5.109}$$

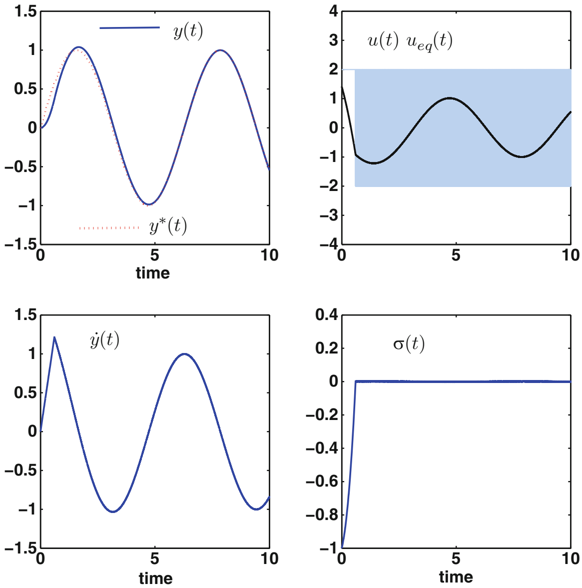

Fig. 5.20. Second order plant responses to GPI sliding mode output reference trajectory tracking

where ζ is an unknown constant perturbation input and u takes values in the set {-1,+1}. It is desired to robustly track the given trajectory $y^*(t)$ for an indefinite period of time.

The nominal (unperturbed) system satisfies the unperturbed dynamics

$$\ddot{y}^*(t) = u^*(t) \tag{5.110}$$

Hence, the tracking error $e = y - y^*(t)$ is obtained via the perturbed dynamics controlled by the control input error $e_u = u - u^*(t)$.

$$\ddot{e} = e_u + \zeta \tag{5.111}$$

A traditional sliding surface coordinate function, in the error space, is proposed as

$$\sigma = \dot{e} + k_2 e, \quad k_2 > 0 \tag{5.112}$$

However, since the integral reconstructor of the tracking error time derivative $\dot{e}$ is given by

$$\widehat{\dot{e}} = \int_0^t e_u(\tau)d\tau \tag{5.113}$$

and due to the presence of the unknown constant perturbation input ζ, and the presence of possible initial conditions, the error velocity estimation error is of the form:

$$\dot{e} = \widehat{\dot{e}} + \dot{e}(0) + Kt \tag{5.114}$$

Thus, the use of an estimated sliding surface, synthesized in terms of the integral reconstruction of $\dot{e}$, induces a polynomial error of first degree in t which needs to be counteracted via a compensation including up to a second order output error integration

$$\widehat{\sigma} = \widehat{\dot{e}} + k_2 e + k_1 \int_0^t e(\tau)d\tau + k_0 \int_0^t \int_0^\tau e(\tau_1)d\tau_1 d\tau \tag{5.115}$$

Under ideal sliding conditions we have that $\widehat{\sigma} = \dot{\widehat{\sigma}} = 0$. From the invariance condition, $\dot{\widehat{\sigma}} = 0$, we obtain, after replacing in the expression for $\widehat{\sigma}$ the term $\widehat{\dot{e}}$ by the integral term,

$$e_u + k_2 \dot{e} + k_1 e + k_0 \int_0^t e(\tau)d\tau = 0 \tag{5.116}$$

The equivalent control error is thus ideally given by

$$e_u = -k_2 \dot{e} - k_1 e - k_0 \int_0^t e(\tau)d\tau \tag{5.117}$$

The corresponding ideal sliding dynamics is then governed by

$$\ddot{e} = -k_2 \dot{e} - k_1 e - k_0 \int_0^t e(\tau)d\tau + \zeta, \tag{5.118}$$

whose characteristic polynomial is given by

$$p(s) = s^3 + k_2 s^2 + k_1 s + k_0 \tag{5.119}$$

This shows that the ideal sliding dynamics is robust with respect to the unknown, constant, perturbation input ζ.

If a Delta-Sigma modulation based control approach needs to be avoided, we can still synthesize a discontinuous feedback control law based on the sliding surface coordinate function expression involving the integral reconstructor. We have

$$\frac{d}{dt}\widehat{\sigma} = u - u^*(t) + k_2 \dot{e} + k_1 e + k_0 \int_0^\tau e(\tau)d\tau \tag{5.120}$$

A suitable switching policy is given then by

$$u = -\text{sign}\, \widehat{\sigma} \tag{5.121}$$

Clearly, it is assumed that $u^*(t)$, the nominal average control input, is strictly bounded within the interval $[-1, 1]$, i.e., the demanded trajectory $y^*(t)$ satisfies $\ddot{y}^*(t) \in [-1, 1]$. This condition is only necessary but not sufficient to guarantee existence of a sliding regime during the transient phase of the control process.

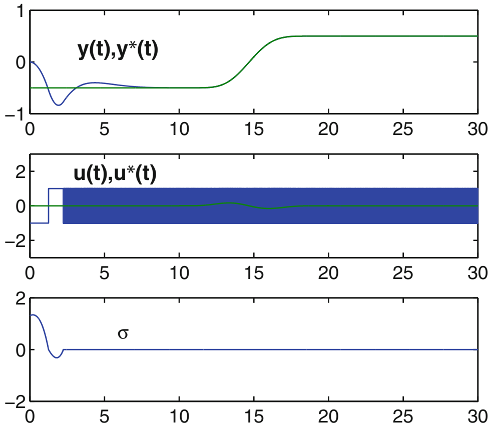

Fig. 5.21. GPI sliding mode control of a second order perturbed integrating plant.

Simulations were performed under the assumption that it is desired to track a rest-to-rest trajectory, specified by means of a suitable Bézier polynomial, defined on the output space which takes the output signal from an initial value of $y(t_{init}) = -0.5$ towards a final value of $y(t_{final}) = 0.5$, within a time interval of 10 s. i.e., with $t_{init} = 10$, $t_{final} = 20$ (Fig. 5.21). The ideal sliding dynamics characteristic polynomial is set to be given by the third order polynomial:

$$p(s) = (s^2 + 2\zeta_n \omega_n s + \omega_n^2)(s + p_n) \tag{5.122}$$

with $\zeta_n = 0.81$, $\omega_n = 1$, $p_n = 1$. i.e.,

$$k_2 = 2\zeta_n \omega_n + p_n, \quad k_1 = \omega_n^2 + 2\zeta_n \omega_n p_n, \quad k_0 = \omega_n^2 p_n \tag{5.123}$$

Notice a small overshoot of the sliding surface coordinate function around zero and the corresponding change in the control action. A sliding regime exists after the transient response involves small excursions from the initial stabilizing reference value thanks to the extra room conceded by the nominal control input $u^*(t) = 0$ during the transient.

Example 5.9. Consider the following benchmark example proposed by Wie and Bernstein [35] (see Figure 5.22). The perturbed differential equations describing the system are given by

$$\begin{aligned} m_1 \ddot{x}_1 &= -k(x_1 - x_2 + L) + u \\ m_2 \ddot{x}_2 &= k(x_1 - x_2 + L) + \omega \\ y &= x_2 \end{aligned} \tag{5.124}$$

where x_1 and x_2 are, respectively, the positions of the first and second mass, v is the control input force with limited amplitude values given by $u = W(2v-1)$ with $W > 0$ and $v \in \{0, 1\}$, i.e., $u \in \{-W, +W\}$. The measured displacement

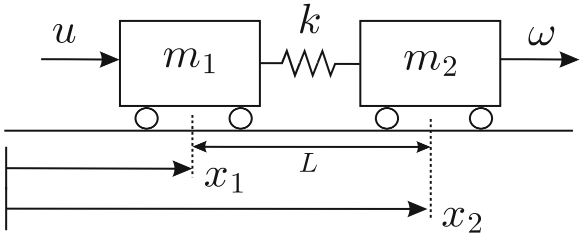

Fig. 5.22. Mass spring benchmark system.

is given by $y = x_2$. The input ω is an unknown disturbance and L is the length of the spring which produces no contraction, or expansion, forces.

It is desired to stabilize the system around the equilibrium value $x_1 = 0$, $x_2 = L$, after the system has subject to a unit impulse, through the force input ω, at time $t = 0$.

Thanks to the linearity of the system we may first proceed to design a GPI controller on the basis of integral reconstructors for the unperturbed system. The compensation in the controller will take into account the effect of the neglected disturbance and those of the initial states, with due consideration of the number of integrations that would have endured, both, the perturbation and the initial conditions, if they had been present in the manipulated expressions. The system is evidently controllable and observable from $y = x_2$.

The unperturbed input-output relation in the system is readily obtained as

$$y^{(4)} = \frac{k}{m_1 m_2} \left[u - (m_1 + m_2)\ddot{y} \right] \tag{5.125}$$

The perturbed input-output relation is, however, given by

$$y^{(4)} = \frac{k}{m_1 m_2} \left[u - (m_1 + m_2)\ddot{y} \right] + \frac{1}{m_2}\ddot{\omega} + \frac{k}{m_1 m_2}\omega \tag{5.126}$$

An uncompensated sliding surface coordinate function would be traditionally defined, in this case, as a suitable (Hurwitz) linear combination of the output and its time derivatives up to third order. However, the synthesis of such a sliding surface coordinate function entitles the use of integral reconstructors for all the involved sequence of time derivatives of y. The integral reconstruction requiring the largest number of integrations, of the perturbed input-output relation, is evidently the first order time derivative of y. Three integrations of the impulse disturbance will generate a quadratic polynomial in time. The effect of neglected initial conditions will also produce at most a second degree time polynomial error in the reconstruction effort. The sliding surface coordinate function thus requires a linear combination of iterated integral errors up to the third order of integration. We have

$$\hat{\sigma} = \widehat{y^{(3)}} + k_5\hat{\ddot{y}} + k_4\hat{\dot{y}} + k_3(y-L) + k_2\int_0^t (y(\lambda)-L)d\lambda$$
$$+k_1\int_0^t\int_0^\lambda (y(\rho)-L)d\rho + k_0\int_0^t\int_0^\lambda\int_0^\rho (y(\theta)-L)d\theta \quad (5.127)$$

The integral reconstructors of the phase variables $\dot{y}$, $\dot{y}$, $y^{(3)}$ are obtained as

$$\widehat{y^{(3)}} = \frac{k}{m_1m_2}(\int u) - \frac{k^2(m_1+m_2)}{m_1^2m_2^2}(\int^{(3)} u) + \frac{k^2(m_1+m_2)^2}{m_1^2m_2 r}(\int y)$$
$$\hat{\ddot{y}} = \frac{k}{m_1m_2}(\int^{(2)} u) - \frac{k(m_1+m_2)}{m_1m_2}y$$
$$\hat{\dot{y}} = \frac{k}{m_1m_2}(\int^{(3)} u) - \frac{k(m_1+m_2)}{m_1m_2}(\int y) \quad (5.128)$$

The switching control policy is given by $u = -W\mathrm{sign}\hat{\sigma}$, or, $u = W(2v-1)$ with $v = 0.5(1-\mathrm{sign}\hat{\sigma})$.

The invariance conditions: $\hat{\sigma} = 0$ and $\frac{d}{dt}\hat{sigma} = 0$, respectively, define the closed loop dynamics for y and the corresponding dynamics satisfied by the equivalent control input.

We specify the set of coefficients, $\{k_5,\ldots,k_0\}$, by equating, term by term, the closed loop characteristic polynomial, $p(s)= s^6+k_5s^5+\cdots+k_1s+k_0$, to a desired characteristic polynomial of the form: $(s^2+2\zeta\omega_n s+\omega_n^2)^3$. One obtains

$$\begin{aligned} k_5 &= 6\zeta\omega_n \\ k_4 &= 3\omega_n^2 + 12\zeta^2\omega_n^2 \\ k_3 &= 8\zeta^3\omega_n^3 + 12\zeta\omega_n^3 \\ k_2 &= 12\zeta^2 * \omega_n^4 + 3\omega_n^4 \\ k_1 &= 6\zeta\omega_n^5 \\ k_0 &= \omega_n^6 \end{aligned} \quad (5.129)$$

Figure 5.23 depicts the stabilization of the mass-spring benchmark system towards its equilibrium point $(x_1, x_3, x_2, x_4) = (0, 0, 0.1, 0)$ after an impulsive disturbance input is applied to the second car in the system at time $t = 0$. The accomplished settling time is below the 15 [s] demanded on the performance specification of the benchmark. The switching force amplitude does not exceed 1 [N].

The following parameter values, suggested in [35], were used for the simulations:

$$m_1 = m_2 = 1.0\ [Kg], \quad k = 1.0\ [N/m], \quad L = 0.1\ [m]$$

The sliding mode controller parameters were set to be

$$\zeta = 0.707, \quad \omega_n = 1.05\ [rad/s], \quad W = 1\ N$$

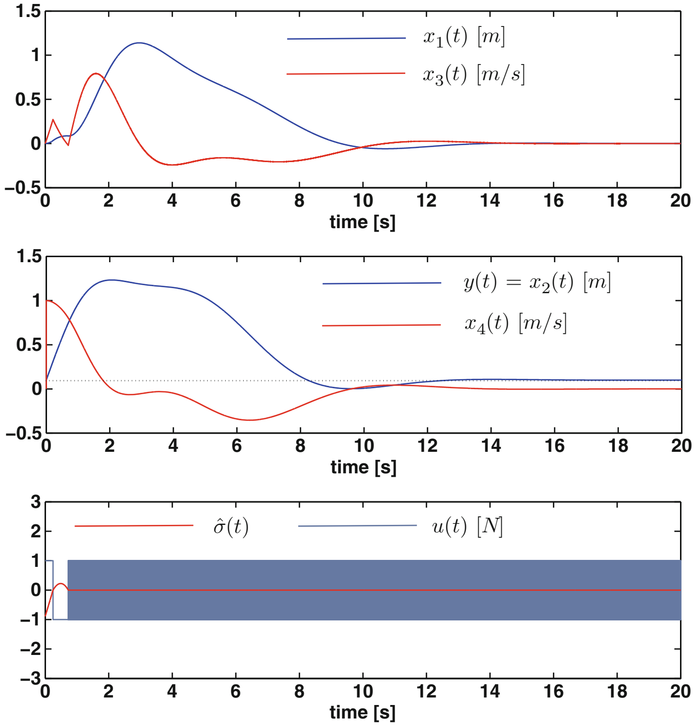

Fig. 5.23. Performance of GPI sliding mode control stabilization of the benchmark example.

5.9 GPI control of some nonlinear systems

In this section, we apply the GPI sliding mode control method for the trajectory tracking problem in two challenging nonlinear systems describing electric motors.

5.9.1 A permanent magnet stepper motor

A nonlinear dynamical system model describing a stepper motor is represented by the following set of differential equations,

$$
\begin{aligned}
L\frac{d}{dt}i_a &= -Ri_a + K_m\omega\sin(n_p\theta) + v_a \\
L\frac{d}{dt}i_b &= -Ri_b - K_m\omega\cos(n_p\theta) + v_b \\
J\frac{d}{dt}\omega &= -K_m i_a\sin(N_r\theta) + K_m i_b\cos(n_p\theta) + \tau \\
\frac{d}{dt}\theta &= \omega
\end{aligned}
\tag{5.130}
$$

where i_a and i_b are the phase currents, v_a and v_b are the voltages acting as the control inputs, θ is the angular position, and ω is the corresponding angular velocity.

We make the following assumptions:

- All motor parameters are known.
- Motor system is initially at rest on $\theta = 0$.
- τ is the unknown but constant torque. It appears unexpectedly.
- No measurements are available for the angular position and the angular velocity.

A *sensorless control scheme*, avoiding the measurement of the mechanical variables θ, and ω, is then required.

In view of the integrability of the nonlinear part of the current equations, one may readily propose an integral reconstructor for the position variable θ. Indeed, integrating the phase current equations and neglecting the effect of the initial conditions, one obtains

$$\begin{aligned} Li_a &= \int_0^t (v_a - Ri_a)d\sigma - \frac{K_m}{n_p}\left[\cos(n_p\theta) - 1\right] \\ Li_b &= \int_0^t (v_b - Ri_b)d\sigma - \frac{K_m}{n_p}\sin(n_p\theta) \end{aligned} \tag{5.131}$$

The set of expressions, (5.131), directly leads to the following integral reconstructor,

$$\widehat{\theta} = \theta = \frac{1}{n_p}\arctan\left[\frac{\int_0^t \left[v_b(\sigma) - Ri_b(\sigma)\right]d\sigma - Li_b}{\int_0^t \left[v_a(\sigma) - Ri_a(\sigma)\right]d\sigma - Li_a + \frac{K_m}{n_p}}\right] \tag{5.132}$$

Figure 5.24 depicts a block diagram for the integral reconstructor of the angular position in the stepper motor.

Note that the sine and cosine functions of the angular position variable are given by the following linear expressions:

$$\begin{aligned} \cos(n_p\theta) &= \frac{n_p}{K_m}\left[\int_0^t \left[v_a(\sigma) - Ri_a(\sigma)\right]d\sigma - Li_a + \frac{K_m}{n_p}\right] \\ \sin(n_p\theta) &= \frac{n_p}{K_m}\left[\int_0^t \left[v_b(\sigma) - Ri_b(\sigma)\right]d\sigma - Li_b\right] \end{aligned} \tag{5.133}$$

Using the expressions in (5.133) in the angular velocity equation of the machine leads to the following nonlinear integral reconstructor of the angular velocity,

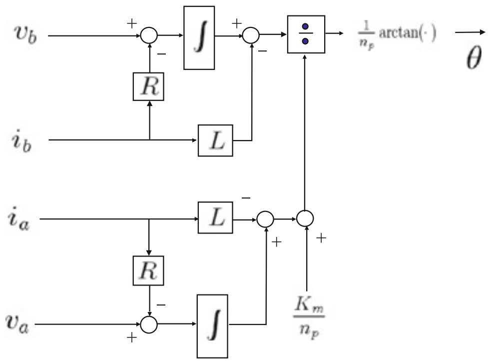

Fig. 5.24. Block diagram for the nonlinear integral reconstructor of the angular position in a stepper motor.

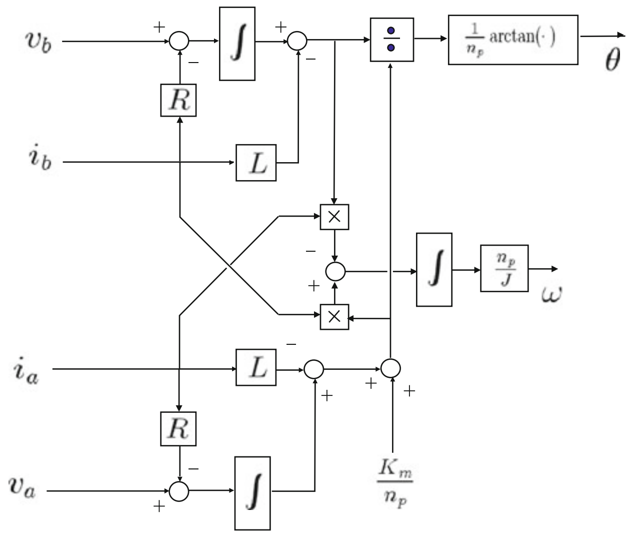

Fig. 5.25. Block diagram of nonlinear integral reconstructor of the angular velocity and the angular position in a stepper motor

$$\widehat{\omega} = \omega = \frac{n_p}{J} \int_0^t \Bigg[- i_a(\sigma) \int_0^\sigma (v_b(\rho) - R i_b(\rho))\, d\rho + i_b(\sigma) \left(\int_0^\sigma (v_a(\rho) - R i_a(\rho))\, d\rho + \frac{K_m}{n_p} \right) \Bigg] d\sigma \tag{5.134}$$

A diagram for the nonlinear integral reconstructor of the angular velocity is shown in Figure 5.25.

Note that if the initial angular displacement, $\theta(0)$, is nonzero, and nonzero initial values of the current are observed, with no external torque being applied to the motor axis, the initial angle can be computed, modulo an integer number of $\pi/2$ radians, as follows:

$$\widehat{\theta_0} = \frac{1}{n_p} \arctan\left(\frac{i_b(0)}{i_a(0)}\right) \tag{5.135}$$

If the angle is nonzero while zero currents are initially observed, then the external torque is necessarily zero, and the initial angle may be taken as the new zero reference.

We now develop a GPI-sliding mode control approach for the tracking of a given angular position reference trajectory.

Let v be an auxiliary input voltage. Assume that the field currents are given by the following *field oriented* expressions

$$\begin{bmatrix} i_a \\ i_b \end{bmatrix} = -\frac{J}{K_m} \begin{bmatrix} \sin(n_p\theta) \\ -\cos(n_p\theta) \end{bmatrix} v \tag{5.136}$$

Then, the angular velocity is seen to satisfy the following closed loop (possibly perturbed) linear dynamics,

$$\frac{d}{dt}\omega = \frac{d^2\theta}{dt^2} = v + \frac{1}{J}\tau \tag{5.137}$$

We proceed to force, via sliding mode control, the phase currents, i_a and i_b, to satisfy the linearizing algebraic restriction given by equation (5.136).

Define, as sliding surface coordinate functions, the following expressions,

$$\begin{aligned} \sigma_a &= i_a + \frac{J}{K_m} \sin(n_p\theta)v \\ \sigma_b &= i_b - \frac{J}{K_m} \cos(n_p\theta)v \end{aligned} \tag{5.138}$$

Using the linear expressions for the integral reconstructors for the sine and cosine functions, given in equation (5.133), we have

$$\begin{aligned} \sigma_a &= i_a + \frac{Jn_p}{K_m^2} \left[\int_0^t [v_b(\sigma) - Ri_b(\sigma)]\, d\sigma - Li_b \right] v \\ \sigma_b &= i_b - \frac{Jn_p}{K_m^2} \left[\int_0^t [v_a(\sigma) - Ri_a(\sigma)]\, d\sigma - Li_a + \frac{K_m}{n_p} \right] v \end{aligned} \tag{5.139}$$

The discontinuous feedback control policies

$$\begin{aligned} v_a &= -W_a \text{ sign } (\sigma_a), \quad W_a \ > \ 0 \\ v_b &= -W_b \text{ sign } (\sigma_b), \quad W_b \ > \ 0 \end{aligned} \tag{5.140}$$

locally create a sliding regime on the sliding surfaces $\sigma_a = 0$ and $\sigma_b = 0$, thus imposing the desired linearizing algebraic restrictions:

$$i_a = -\frac{J}{K_m} \sin(n_p\theta)v$$

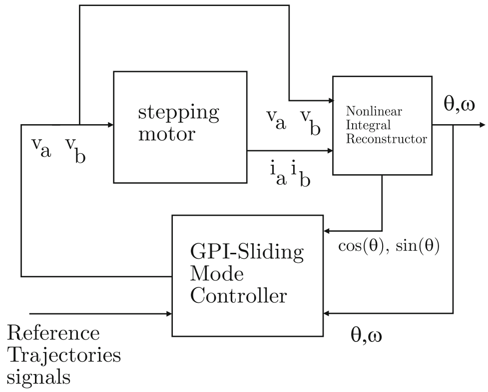

Fig. 5.26. GPI sliding mode control scheme for the permanent magnet stepping motor

$$i_b = \frac{J}{K_m}\cos(n_p\theta)v$$

The auxiliary control input v is specified as the following GPI controller,

$$v = \ddot{\theta}^*(t) - k_3\left(\hat{\omega} - \dot{\theta}^*(t)\right) - k_2(\hat{\theta} - \theta^*(t))$$
$$-k_1\int_0^t (\theta(\sigma) - \theta^*(\sigma))\,d\sigma - k_0\int_0^t\int_0^\sigma \left(\hat{\theta}(\lambda) - \theta^*(\lambda)\right)d\lambda d\sigma$$

with $\hat{\omega}$ and $\hat{\theta}$, respectively, representing the proposed integral reconstructors of the position and angular velocity variables.

The GPI controller counteracts the possible ramp function in the angular velocity reconstruction error, appearing when an unknown external torque is present (Fig. 5.26).

Robustness of the proposed feedback control scheme was tested by using the proposed GPI sliding mode controller on the following perturbed model, including an un-modeled *detent torque* term,

$$\begin{aligned}
L\frac{d}{dt}i_a &= -Ri_a + K_m\omega\sin(n_p\theta) + v_a \\
L\frac{d}{dt}i_b &= -Ri_b - K_m\omega\cos(n_p\theta) + v_b \\
J\frac{d}{dt}\omega &= -K_m i_a\sin(N_r\theta) + K_m i_b\cos(n_p\theta) - B\omega - K_D\sin(4n_p\theta) + \tau \\
\frac{d}{dt}\theta &= \omega
\end{aligned} \tag{5.141}$$

The desired reference trajectory, $\theta^*(t)$, was specified as a rest-to-rest maneuver entitling an increase of the angular position from an initial zero value towards a final constant value θ_{final}, during a finite interval of time

$[t_0, t_f]$. The varying portion of the angular position reference trajectory, in $[t_0, t_f]$, was specified by means of the following 10-th order Bèzier polynomial:

$$\theta^*(t) = \left(\frac{t-t_0}{t_f-t_0}\right)^5 \left[r_1 - r_2\left(\frac{t-t_0}{t_f-t_0}\right) + \cdots - r_6\left(\frac{t-t_0}{t_f-t_0}\right)^5\right]\theta_{final} \tag{5.142}$$

with

$$r_1 = 252,\ r_2 = 1050,\ r_3 = 1800,\ r_4 = 1575,\ r_5 = 700,\ r_6 = 126$$

The system parameters for the motor were set to be

$$R = 19.1388,\ L = 0.040,\ K_m = 0.1349$$
$$J = 4.1295 \times 10^{-4},\ B = 13 \times 10^{-4},\ n_p = 50,$$
$$K_D = 0.07 K_m i_{bd},\ \theta_{final} = 0.015 \text{ [rad]}$$

The ideal fourth order closed loop characteristic polynomial was chosen to be of the form,

$$p(s) = (s^2 + 2\zeta\omega_n s + \omega_n^2)^2 \tag{5.143}$$

with: $\zeta = 0.707$, $\omega_n = 80$ and $W_a = 10$, $W_b = 10$. [V]

To avoid excessive chattering a smoothed (high-gain) controller was implemented, instead of the sign function based sliding mode controller,

$$u_a = -W_a \frac{\sigma_a}{\mid \sigma_a \mid + \epsilon}, \quad u_b = -W_b \frac{s_b}{\mid s_b \mid + \epsilon}$$

with $\epsilon << 1$.

Figure 5.27 depicts the performance of the sensorless GPI sliding mode controller for the prescribed output reference trajectory tracking task.

5.9.2 An induction motor

Consider the following $a - b$ model of the induction motor:

$$\begin{aligned}
J\frac{d}{dt}\omega &= \frac{n_p M}{L_r}(\Psi_a i_b - \Psi_b i_a) - \tau_L \\
L_r \frac{d}{dt}\Psi_a &= -R_r \Psi_a - L_r n_p \omega \Psi_b + R_r M i_a \\
L_r \frac{d}{dt}\Psi_b &= -R_r \Psi_b + L_r n_p \omega \Psi_a + R_r M i_b \\
\sigma L_s \frac{d}{dt} i_a &= \frac{M R_r}{L_r^2}\Psi_a + \frac{n_p M}{L_r}\omega \Psi_b - \left(\frac{M^2 R_r + L_r^2 R_s}{L_r^2}\right) i_a + u_a
\end{aligned}$$

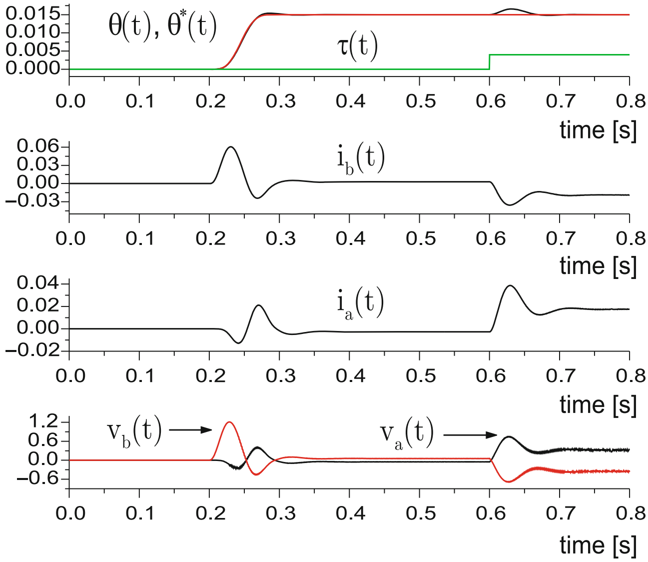

Fig. 5.27. Performance of the GPI sliding mode controller on a trajectory tracking task for a permanent magnet stepping motor.

$$\sigma L_s \frac{d}{dt} i_b = \frac{M R_r}{L_r^2} \Psi_b - \frac{n_p M}{L_r} \omega \Psi_a - \left(\frac{M^2 R_r + L_r^2 R_s}{L_r^2} \right) i_b + u_b \quad (5.144)$$

where i_a and i_b represent the phase currents while Ψ_a and Ψ_b represent the unmeasured fluxes. The control inputs are the voltages u_a and u_b. It is assumed that the angular velocity is not measured. In other words, we wish to establish a *sensorless control scheme* for the angular velocity reference trajectory tracking on an induction motor described by the above model.

The basis of the GPI sliding mode control approach to the output reference trajectory tracking problem in this system consists in proposing a linear integral reconstructor for the flux variables.

Simple algebraic manipulations of the motor equations lead to the following linear integral reconstructor for the rotor fluxes, which are independent of the rotor resistance R_r,

$$\Psi_a(t) = -\frac{\sigma L_s L_r}{M} i_a(t) + \frac{L_r}{M} \int_0^t \left[u_a(\lambda) - R_s i_a(\lambda) \right] d\lambda \quad (5.145)$$

and

$$\Psi_b(t) = -\frac{\sigma L_s L_r}{M} i_b(t) + \frac{L_r}{M} \int_0^t \left[u_b(\lambda) - R_s i_b(\lambda) \right] d\lambda \quad (5.146)$$

The flux integral reconstructors are *exact* for zero initial conditions. Otherwise, they are biased by an unknown constant. Using the flux integral reconstructors into the angular velocity equation leads to a nonlinear integral reconstructor for the angular velocity. Indeed,

$$\omega = \frac{n_p}{J}\int_0^t \left[i_b(\lambda)\int_0^\lambda [u_a(\rho) - R_s i_a(\rho)]\, d\rho - i_a(\lambda)\int_0^\lambda [u_b(\rho) - R_s i_b(\rho)]\, d\rho \right] d\lambda \tag{5.147}$$

The above integral reconstructors may be used in the feedback control of the induction motor. In such an endeavor high gain, or sliding mode control, proves to be rather useful.

We use sliding surfaces motivated by the following *field oriented control* scheme,

$$\begin{bmatrix} \sigma_a \\ \sigma_b \end{bmatrix} = \begin{bmatrix} i_b \\ i_a \end{bmatrix} + \left(\frac{J L_r}{n_p M} \right) \frac{1}{\Psi_a^2 + \Psi_b^2} \begin{bmatrix} -\Psi_a \\ \Psi_b \end{bmatrix} v \tag{5.148}$$

Note that if $\sigma_a = \sigma_b = 0$ then, ideally,

$$\frac{d\omega}{dt} = v - \frac{\tau}{J} \tag{5.149}$$

For the auxiliary control input v, we propose a PI controller of the form:

$$v = -k_2 \left(\hat{\omega} - \omega^*(t) \right) - k_1 \int_0^t \left(\hat{\omega}(\lambda) - \omega^*(\lambda) \right) d\lambda \tag{5.150}$$

where $\hat{\omega}$ is the integral reconstructor for the angular velocity variable.

A sliding mode control is readily obtained as

$$u_a = -W_a \text{sign}\ (\sigma_a)\,, \quad u_b = -W_b \text{sign}\,(\sigma_b) \tag{5.151}$$

with $W_a > 0$, $W_b > 0$.

Note that since $\Psi_a^2 + \Psi_b^2 = 0$, for zero initial conditions the singularity makes the controller momentarily undefined if the motor is started from rest conditions. As customary, an open loop “starter” must be used in connection with the proposed controller. The flux integral reconstructors are left “on” from the beginning (Fig. 5.28).

Simulations were performed to assess the performance of the sensor-less GPI sliding mode controller for a reference trajectory tracking task on the part of the unmeasured angular velocity. Sudden constant torques, of opposite signs, were provided, respectively, at two different time instants: $t = 2.0$ [sec] and $t = 8$ [sec]. Figure 5.29 depicts the rest-to-rest reference trajectory, one of the phase currents (i_a) and one of the integrally reconstructed fluxes Ψ_a. The control input u_a is also shown. The effect of the permanent disturbance torque inputs on the angular velocity tracking task is also depicted in the figure.

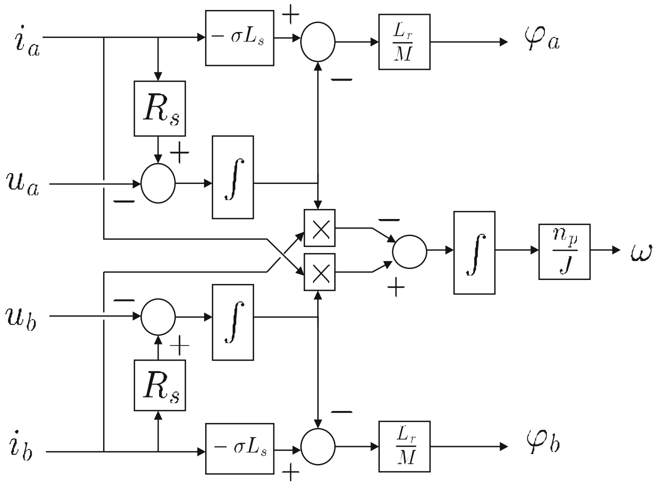

Fig. 5.28. A nonlinear integral reconstructor for the angular velocity

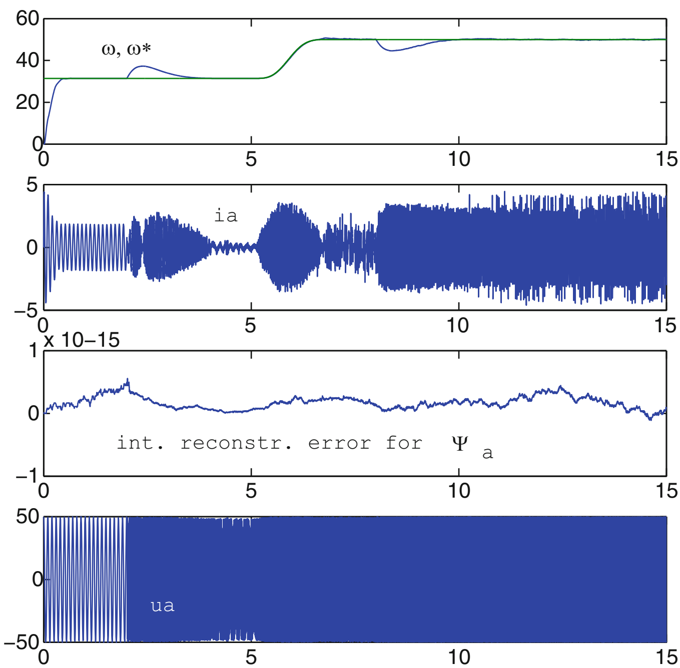

Fig. 5.29. Performance of GPI sliding mode controller for sensorless tracking in an induction motor subject to constant load torque inputs

6

Differential flatness and sliding mode control

6.1 Introduction

A system is said to be *differentially flat* if there exists a set of independent *differential functions* of the state (i.e., they do not satisfy any differential equations and, additionally, they are functions of the state and of a *finite* number of their time derivatives), called the *flat outputs*. The set of flat outputs exhibits the same number of elements as that found on the input set. The nature of the flat outputs is such that *all* variables in the system: i.e., states, outputs, and inputs, are, in turn, expressible as differential functions of the flat output. Flatness was introduced, by M. Fliess and his colleagues in a series of remarkable articles ([5–7]) where the reader is referred for theoretical issues and many illustrative examples. Contrary to unwarranted belief, flatness is not just another way to do feedback linearization. For SISO systems, indeed, flatness and feedback linearizability are equivalent but flatness goes beyond feedback linearization, specially in the MIMO case. Generally speaking, flatness is, in fact, a *structural property* of the system that allows one to establish *all* the salient features which are needed for the application of a particular feedback controller design technique (like back-stepping, passivity, sliding, and, of course, feedback linearization). Thus flatness is also an analysis tool naturally related to equilibria, control limitations, state restrictions, and singularity avoidance. Flatness, in its more popular conception, is a property that readily *trivializes* the exact linearization problem in a nonlinear system, whether or not the system is mono-variable. Moreover, flatness may be present on *any* type of nonlinear controlled system, regardless of the nonlinear, or affine, nature of the control inputs in the system equations. Flatness, thanks to its relations with invertibility, immediately yields the required open loop (nominal) behavior of the system for a particular desired trajectory tracking task. It is, therefore, most suitable for trajectory planning, controller saturation avoid-

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ance, the handling of state restrictions and predictive control, specially for those cases involving non-minimum phase outputs (see [27] and [9]). One of the distinctive features of flatness lies in the possibilities of differentially parameterizing all system variables. States, inputs, actual system's (non-flat) outputs are all expressible as functions of the flat outputs and a finite number of their time derivatives. In mono-variable cases, this allows for a natural specification of the sliding surface coordinate function in terms of a stable linear differential polynomial acting on the flat output. In multi-variable systems, flatness naturally leads to inputs-to-flat outputs decoupling via static or dynamic feedback (see Charlet *et al.* [2], Rouchon [9]). Flat outputs are, generally speaking, physically meaningful variables in the system. Thus, their control to specific values or reference time functions is immediately related to a control objective whose feasibility may be readily assessed. We shall assume that the flat output variables are all measurable for feedback purposes.

In this chapter, we examine the relations between differential flatness and sliding mode control for MIMO nonlinear systems. We do not pay special attention to the SISO case since it can be viewed as a particular case with far less complications, as evidenced by some examples here treated. The interested reader is referred to the articles by Fliess *et al.* [5, 6] and [7], or the rigorous book by Levine [16] and that by Sira-Ramírez and Agrawal [26]. Excellent books, devoted to infinite dimensional systems, are those of Rudolph [20] and Rudolph *et al.* [21].

In this chapter, we explore two possibilities of designing a sliding mode feedback control strategy on a given finite dimensional nonlinear MIMO flat system. If a switched system *formally* exhibits the flatness property, then the differential parametrization of the inputs directly leads to a decoupled design of the sliding surface coordinate functions. The term *formally* above suggests the second design route. If the system is found to be flat, the underlying temporary assumption is that the control inputs are not limited to take values on discrete sets, but, rather, they are assumed to take values on cartesian products of a collection of compact sets of the real line. Thus flatness may be immediately exploited on an average feedback design strategy complemented by a Delta-Sigma modulation implementation. Delta-Sigma modulation and average feedback designs are then directly related to the use of the flatness property.

6.2 Flatness in Multi-variable Nonlinear systems

We consider systems of the general form,

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \tag{6.1}$$

where $f = (f_1, \dots, f_n)$ is a *regular* function of x and u and the rank of the Jacobian matrix with respect to u, $\partial f / \partial u$ is maximal, i.e. it is m.

Definition 6.1. *We say, in general, that* ϕ *is a* differential function *of* x *if*

$$\phi = \phi(x, \dot{x}, \ddot{x}, \ldots, x^{(\beta)}) \tag{6.2}$$

where $\beta = (\beta_1, \ldots \beta_n)$ *is a multi-index, i.e., it is a vector of finite integers, each one depicting the order of differentiation of the corresponding component of the vector* x*. Thus,*

$$x^{(\beta)} = [x_1^{(\beta_1)}, \ldots, x_n^{(\beta_n)}]^T \tag{6.3}$$

The expression $x^{(\beta+\mathbf{1})}$ *is understood to imply the addition of the vector* β *with a vector of* 1*'s of the same dimension.*

If x is governed by a set of controlled differential equations of the form (6.1) then, necessarily, the higher order time differentiation specified in the definition of a *differential function* leads to consider derivatives of the components of the control input vector u. In other words,

$$\phi(x, \dot{x}, \ldots, x^{(\beta)}) = \psi(x, u, \dot{u}, \ddot{u}, \ldots, u^{(\beta+\mathbf{1})}) \tag{6.4}$$

Definition 6.2. *A system of the form (6.1) is said to be differentially flat if there exist* m differentially independent *functions*[1] *denoted by the vector* y*, constituted by a set of differential functions of the state vector* x

$$y = h(x, u, \dot{u}, \ddot{u}, \ldots, u^{(\alpha)}) \tag{6.5}$$

such that the inverse system, expressing the vector of inputs $u = (u_1, \ldots, u_m)$ *in terms of the vector of outputs,* $y = (y_1, \ldots, y_m)$*, does not exhibit any dynamics. In other words, the inputs* u *are determined solely in terms of the outputs* y *and a finite number of their time derivatives, with no need for solving differential equations for* u*. The quantities constituting the components of the vector* y *are referred to as the "flat outputs."*

Generally speaking, the set of flat outputs does not coincide with the set of outputs of the system. These last are addressed as the *actual outputs* and they are denoted by $z \in \mathbb{R}^p$. Usually, $z = \theta(x)$, in some cases, $z = \theta(x, u)$ or even $z = \theta(x, u, \dot{u}, \ldots, u^{(\mu)})$. We seldom consider outputs which are not purely functions of the state, or outputs which are non-flat outputs.

Example 6.3. The simple linear system, $\ddot{y} = u$, is evidently flat since y qualifies as a flat output. The inverse system $u = \ddot{y}$ with y as input and u as the output clearly does not have any dynamics. No differential equations have to be solved to find u under perfect knowledge of y. Notice that the velocity variable $\dot{y}$ which is one of the states of the system is trivially differentially parameterized by the flat output y.

[1] i.e., they do not satisfy any algebraic restrictions nor any set of differential equations.

In flat system of the form $\dot{x} = f(x,u), z = \theta(x)$, all variables (i.e., states, inputs, actual outputs) may be written as differential functions of the components of the flat output vector y, i.e.,

$$x = F(y, \dot{y}, \ldots, y^{(\gamma)}),\ u = G(y, \dot{y}, \ldots, y^{(\gamma+\mathbf{1})}),\ z = \theta(y, \dot{y}, \ldots, y^{(\gamma)}) \quad (6.6)$$

where γ is a multi-index. We refer to the above expressions as the *differential parametrization* of the system variables in terms of the flat outputs.

Example 6.4. Consider the following nonlinear SISO example

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin(x_1) - b(x_1 - x_3) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= c(x_1 - x_3) + du \end{aligned} \quad (6.7)$$

where a, b, c, and d are constant parameters.

The system is found to be differentially flat, as the variable $y = x_1$ differentially parameterizes all the variables in the system. Indeed, it is not difficult to see that

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \\ x_3 &= \frac{1}{b}\left(\ddot{y} + a\sin y\right) + y \\ x_4 &= \frac{1}{b}\left(y^{(3)} + a\dot{y}\cos y\right) + \dot{y} \\ u &= \frac{1}{d}\left[\frac{1}{b}\left(y^{(4)} + a\ddot{y}\cos y - a(\dot{y})^2\sin y\right) + \ddot{y} + \frac{c}{b}(\ddot{y} + a\sin y)\right] \end{aligned} \quad (6.8)$$

The fundamental relation between the flat output highest derivative, $y^{(4)}$, and the control input u is of the form

$$y^{(4)} = (bd)u + q(y, \dot{y}, \ddot{y}) \quad (6.9)$$

with $q(\cdot)$ representing all the nonlinearities affecting the flat output dynamics. This quantity may be either known or unknown. The preferred route in sliding mode control is to overcome this quantity under the assumption, or educated assessment, that such a quantity is globally or at least locally bounded in the region of interest. This type of relation is fundamental in sliding mode creation problems for a control objective defined on the basis of the flat output.

If a stabilization is desired around an equilibrium point $y = \Theta$, $\dot{y} = \ddot{y} = y^{(3)} = 0$, a sliding surface may be readily proposed to be

$$\sigma(y, \dot{y}, \ddot{y}, y^{(3)}) = y^{(3)} + \kappa_2\ddot{y} + \kappa_1\dot{y} + \kappa_0(y - \Theta) \quad (6.10)$$

with the coefficients κ_i,$i = 0, 1, 2$ chosen so that the associated characteristic polynomial : $p(s) = s^3 + \kappa_2 s^2 + \kappa_1 s + \kappa_0$ is a Hurwitz polynomial.

Notice that the following state dependent input coordinate transformation (written for simplicity in terms of y and its derivatives),

$$u = \frac{1}{d}\left[\frac{1}{b}\left(v + a\ddot{y}\cos y - a(\dot{y})^2 \sin y\right) + \ddot{y} + \frac{c}{b}(\ddot{y} + a\sin y)\right] \tag{6.11}$$

yields the following simple linear controllable system in *Brunovsky's canonical form* (see Isidori [13]),

$$y^{(4)} = v \tag{6.12}$$

The relation of single input flat systems with systems linearizable by means of static state feedback and state dependent input coordinates transformations (static state feedback linearizable in short, or even shorter; *feedback linearizable*) is quite interesting: they are equivalent.

Example 6.5. The kinematic model of a mono-cycle is given by

$$\dot{x}_1 = x_4 \cos x_3, \quad \dot{x}_2 = x_4 \sin x_3, \quad \dot{x}_3 = u_2, \quad \dot{x}_4 = u_1 \tag{6.13}$$

where (x_1, x_2) represent the position of the point of contact of the wheel with the coordinate plane (x_1, x_2), The variable x_3 is the angle of orientation of the plane of the wheel with respect to the x_1 axis. The wheel is assumed to be always perpendicular to the plane (x_1, x_2). u_1 is the forward acceleration and u_2 is the turning rate.

The flat outputs are just: $y_1 = x_1$, $y_2 = x_2$. Indeed,

$$x_4 = \sqrt{\dot{y}_1^2 + \dot{y}_2^2}, \quad x_3 = \arctan\left(\frac{\dot{y}_2}{\dot{y}_1}\right), \; u_2 = \frac{\ddot{y}_2\dot{y}_1 - \dot{y}_2\ddot{y}_1}{\dot{x}_1^2 + \dot{x}_2^2}, \; u_1 = \frac{\dot{y}_1\ddot{y}_1 + \dot{y}_2\ddot{y}_2}{\sqrt{\dot{y}_1^2 + \dot{y}_2^2}}$$

Knowledge of y_1, y_2 allows for the computation of the control inputs u_1, u_2 without solving differential equations. The inverse system has no dynamics.

The input-to flat output relation and the controlled dynamics for the flat outputs are obtained as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\dot{y}_1}{\sqrt{\dot{y}_1^2 + \dot{y}_2^2}} & \dfrac{\dot{y}_2}{\sqrt{\dot{y}_1^2 + \dot{y}_2^2}} \\ -\dfrac{\dot{y}_2}{\dot{y}_1^2 + \dot{y}_2^2} & \dfrac{\dot{y}_1}{\dot{y}_1^2 + \dot{y}_2^2} \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix},$$

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\dot{y}_1}{\sqrt{\dot{y}_1^2 + \dot{y}_2^2}} & -\dot{y}_2 \\ \dfrac{\dot{y}_2}{\sqrt{\dot{y}_1^2 + \dot{y}_2^2}} & \dot{y}_1 \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} \cos x_3 & -x_4 \sin x_3 \\ \sin x_3 & x_4 \cos x_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Clearly, $x_4 = 0$, blocks out all control efforts and it corresponds with zero forward velocity of the point of contact of the wheel with the plane

(x_1, x_2). Trajectory tracking of a desired path in the plane: $y_1^*(t), y_2^*(t)$ may be accomplished with the following set of sliding surfaces defined in terms of the reference tracking errors: $e_1 = y_1 - y_1^*(t)$, $e_2 = y_2 - y_2^*(t)$

$$\sigma_1 = \dot{e}_1 + \kappa_1 e_1, \quad \sigma_2 = \dot{e}_2 + \kappa_2 e_2 \tag{6.14}$$

Exercise 6.6. Notice that a more traditional kinematic model of the monocycle involves only forward velocity and turning rate as control input variables. The model is then customarily given by

$$\dot{x}_1 = u_1 \cos x_3, \quad \dot{x}_2 = u_1 \sin x_3, \quad \dot{x}_3 = u_2 \tag{6.15}$$

Work out the details and convince yourself that a *a dynamic extension* of the control input u_1 is necessary to obtain a proper input-output dynamics.

6.3 The rolling penny

A homogeneous disk rolls vertically over a horizontal plane without slipping, skidding, or tilting (Fig. 6.1). In spite of the non-physical nature of the example it constitutes a good example to study a multi-variable system subject to non-holonomic restrictions.

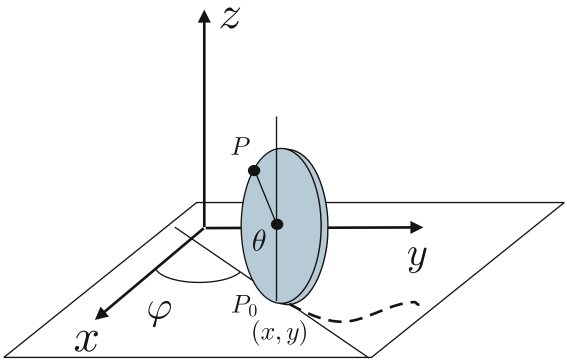


Fig. 6.1. The rolling penny

The configuration space is constituted by

$$q = (x, y, \varphi, \theta) \in R^2 \times S^1 \times S^1 \tag{6.16}$$

which describes the position of the contact point with the plane, the angle of rotation, and the orientation of the disk.

We assume the mass of the disk is m, the radius is R, and the moments of inertia I and J, respectively, with respect to the axes perpendicular to the plane of the disk which passes through its center and with respect to a vertical axis containing also the center of the disk. There are two control inputs u_θ and u_φ which regulate the motions of the disk on the plane.

The Lagrangian of the disk is given by

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2 \tag{6.17}$$

The non-holonomic restrictions of rolling without slipping are just

$$\dot{x} = R(\cos\varphi)\dot{\theta}, \quad \dot{y} = R(\sin\varphi)\dot{\theta} \tag{6.18}$$

These two equations conform the restriction functions of the vector type. They are thus given by

$$\begin{bmatrix} \widehat{f}_1(\dot{x}, \varphi, \dot{\theta}) \\ \widehat{f}_2(\dot{y}, \varphi, \dot{\theta}) \end{bmatrix} = \begin{bmatrix} \dot{x} - R(\cos\varphi)\dot{\theta} \\ \dot{y} - R(\sin\varphi)\dot{\theta} \end{bmatrix} = 0 \tag{6.19}$$

The system dynamics, in general terms, is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \begin{bmatrix} 0 \\ 0 \\ u_\varphi \\ u_\theta \end{bmatrix} + \lambda_1 \frac{\partial \widehat{f}_1}{\partial \dot{q}} + \lambda_2 \frac{\partial \widehat{f}_2}{\partial \dot{q}} \tag{6.20}$$

In specific terms, we have:

$$\begin{aligned} m\frac{d}{dt}\left(R\cos\varphi\dot{\theta}\right) &= \lambda_1 \\ m\frac{d}{dt}\left(R\sin\varphi\dot{\theta}\right) &= \lambda_2 \\ J\ddot{\varphi} &= u_\varphi \\ I\ddot{\theta} &= u_\theta - \lambda_1 R\cos\varphi - \lambda_2 R\sin\varphi \end{aligned}$$

Using the last two equations in the restrictions we have

$$\lambda_1 = m\frac{d}{dt}\left(R\cos\varphi\dot{\theta}\right), \quad \lambda_2 = m\frac{d}{dt}\left(R\sin\varphi\dot{\theta}\right), \tag{6.21}$$

Note that

$$-\lambda_1 R\cos\varphi - \lambda_2 R\sin\varphi = -mR^2\ddot{\theta} \tag{6.22}$$

We now have, using the restrictions:

$$\begin{aligned} \dot{x} &= R(\cos\varphi)\dot{\theta} \\ \dot{y} &= R(\sin\varphi)\dot{\theta} \\ J\ddot{\varphi} &= u_\varphi \\ (I + mR^2)\ddot{\theta} &= u_\theta \end{aligned}$$

The first two equations represent the *kinematics* of the system and the last two *the dynamics* of the system. Note that θ does not intervene in the equations, only the angular rolling rate $\dot{\theta}$.

The typical problem consists in controlling the system from the torque control inputs: u_φ, u_θ, in such a manner that the point of contact of the disk with the horizontal plane follows a pre-specified trajectory, given by: $x^*(t)$, $y^*(t)$.

The *independent* variables: x e y, play an important role in the understanding of the structure of this non-holonomic system and in the design of a feedback controller for the tracking of the smooth trajectories specified on the plane for the coordinates x, y.

Certainly, all system variables are expressible in terms of x, y and a finite number of its time derivatives. Indeed,

$$\begin{aligned}
\dot{\theta} &= \frac{1}{R}\sqrt{\dot{x}^2+\dot{y}^2} \\
\varphi &= \arctan\left(\frac{\dot{y}}{\dot{x}}\right) \\
u_\theta &= \frac{(I+mR^2)}{R}\left[\frac{\dot{x}\ddot{x}+\dot{y}\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right] \\
u_\varphi &= J\left[\frac{y^{(3)}\dot{x}-\dot{y}x^{(3)}}{(\dot{x})^2+(\dot{y})^2} - 2\frac{\ddot{y}\ddot{x}(\dot{x})^2+\dot{x}\dot{y}((\ddot{y})^2-(\ddot{x})^2)-(\dot{y})^2\ddot{x}\ddot{y}}{((\dot{x})^2+(\dot{y})^2)^2}\right]
\end{aligned}$$

The relation between the inputs and the highest order derivatives of x and y is not invertible. Clearly a dynamic extension of first order is needed on u_θ to achieve a well-defined input to flat outputs highest derivative relation. We have

$$\begin{bmatrix} u_\varphi \\ \dot{u}_\theta \end{bmatrix} = \begin{bmatrix} -J\dfrac{\dot{y}}{(\dot{x})^2+(\dot{y})^2} & J\dfrac{\dot{x}}{(\dot{x})^2+(\dot{y})^2} \\ \left(\dfrac{I+mR^2}{R}\right)\dfrac{\dot{x}}{\sqrt{(\dot{x})^2+(\dot{y})^2}} & \left(\dfrac{I+mR^2}{R}\right)\dfrac{\dot{y}}{\sqrt{(\dot{x})^2+(\dot{y})^2}} \end{bmatrix}\begin{bmatrix} x^{(3)} \\ y^{(3)} \end{bmatrix}$$

$$+\begin{bmatrix} -2J\left[\dfrac{\ddot{y}\ddot{x}(\dot{x})^2+\dot{x}\dot{y}((\ddot{y})^2-(\ddot{x})^2)-(\dot{y})^2\ddot{x}\ddot{y}}{((\dot{x})^2+(\dot{y})^2)^2}\right] \\ \dfrac{I+mR^2}{R}\left[\dfrac{(\ddot{x})^2+(\ddot{y})^2}{\sqrt{(\dot{x})^2+(\dot{y})^2}} - \dfrac{(\dot{x}\ddot{x}+\dot{y}\ddot{y})^2}{((\dot{x})^2+(\dot{y})^2)^{\frac{3}{2}}}\right] \end{bmatrix}$$

The Hagenmeyer Delaleau controller design procedure [12] calls for a linear controller design on the set of independent chains of integrators,

$$x^{(3)} = v_x, \quad y^{(3)} = v_y \tag{6.23}$$

and the use of the time-varying relation:

$$\begin{bmatrix} u_\varphi \\ \dot{u}_\theta \end{bmatrix} = \begin{bmatrix} -J\dfrac{\dot{y}^*}{(\dot{x}^*)^2+(\dot{y}^*)^2} & J\dfrac{\dot{x}^*}{(\dot{x}^*)^2+(\dot{y}^*)^2} \\ \left(\frac{I+mR^2}{R}\right)\dfrac{\dot{x}^*}{\sqrt{(\dot{x}^*)^2+(\dot{y}^*)^2}} & \left(\frac{I+mR^2}{R}\right)\dfrac{\dot{y}^*}{\sqrt{(\dot{x}^*)^2+(\dot{y}^*)^2}} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

$$+ \begin{bmatrix} -2J\left[\dfrac{\dot{y}^*\ddot{x}^*(\dot{x}^*)^2 + \dot{x}^*\dot{y}^*((\ddot{y}^*)^2-(\ddot{x}^*)^2) - (\dot{y}^*)^2\ddot{x}^*\ddot{y}^*}{((\dot{x}^*)^2+(\dot{y}^*)^2)^2}\right] \\ \frac{I+mR^2}{R}\left[\dfrac{(\ddot{x}^*)^2+(\ddot{y}^*)^2}{\sqrt{(\dot{x}^*)^2+(\dot{y}^*)^2}} - \dfrac{(\dot{x}^*\ddot{x}^*+\dot{y}^*\ddot{y}^*)^2}{((\dot{x}^*)^2+(\dot{y}^*)^2)^{\frac{3}{2}}}\right] \end{bmatrix}$$

where x^* and $y*$ stand, respectively, by $x^*(t)$ and $y^*(t)$, the desired reference trajectories for x and y.

The controllers for the chains of integrators are simply

$$v_x = [x^*(t)]^{(3)} - \left[\frac{k_3^x s^3 + k_2^x s^2 + k_1^x s + k_0^x}{s(s^2 + k_5^x s + k_4^x)}\right](x - x^*(t))$$

$$v_y = [y^*(t)]^{(3)} - \left[\frac{k_3^y s^3 + k_2^y s^2 + k_1^y s + k_0^y}{s(s^2 + k_5^y s + k_4^y)}\right](y - y^*(t))$$

Substitution of these expression into the expressions for $u_\varphi, \dot{u}_\theta$ leads to

$$\begin{bmatrix} u_\varphi \\ \dot{u}_\theta \end{bmatrix} = \begin{bmatrix} u_\varphi^*(t) \\ \dot{u}_\theta^*(t) \end{bmatrix}$$
$$+ \begin{bmatrix} -J\dfrac{\dot{y}^*(t)}{(\dot{x}^*(t))^2+(\dot{y}^*(t))^2} & J\dfrac{\dot{x}^*(t)}{(\dot{x}^*(t))^2+(\dot{y}^*(t))^2} \\ \left(\dfrac{I+mR^2}{R}\right)\dfrac{\dot{x}^*(t)}{\sqrt{(\dot{x}^*(t))^2+(\dot{y}^*(t))^2}} & \left(\frac{I+mR^2}{R}\right)\dfrac{\dot{y}^*(t)}{\sqrt{(\dot{x}^*(t))^2+(\dot{y}^*(t))^2}} \end{bmatrix} \times$$
$$\begin{bmatrix} -\left[\dfrac{k_3^x s^3 + k_2^x s^2 + k_1^x s + k_0^x}{s(s^2 + k_5^x s + k_4^x)}\right](x - x^*(t)) \\ -\left[\dfrac{k_3^y s^3 + k_2^y s^2 + k_1^y s + k_0^y}{s(s^2 + k_5^y s + k_4^y)}\right](y - y^*(t)) \end{bmatrix}$$

Exercise 6.7. Under the assumption of suitable binary-valued torque inputs, implement the derived average control laws by means of an appropriate Delta-Sigma modulation scheme. Assume that the unicycle is to move forwards always (i.e., no backing up).

6.4 Single axis car

Consider the model of a cart provided with a single axis joining two wheels. Each one of the wheels is independently actuated by means of a motor that provides the required torque. We name these torques τ_1 and τ_2 and they are considered as control inputs (Fig. 6.2).

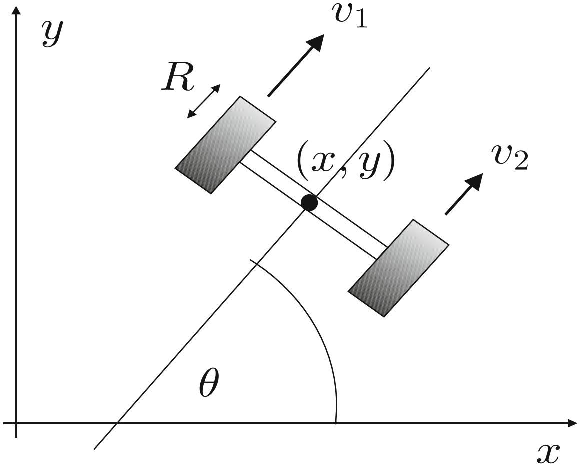


Fig. 6.2. A single axis car

Wheel No. 1 rolls with angular velocity $\dot{\phi}_1$ while the second wheel does it with angular velocity $\dot{\phi}_2$. This causes the linear tangential velocities of each wheel to be

$$v_1 = R\dot{\phi}_1, \quad v_2 = R\dot{\phi}_2 \tag{6.24}$$

The velocity of the point (x, y), which is located at the center of mass of the car, is given then by

$$v = \frac{v_1 + v_2}{2} = \frac{R}{2}\left[\dot{\phi}_1 + \dot{\phi}_2\right] \tag{6.25}$$

The projected velocities of the center of mass, along each cartesian coordinate axes, are consequently:

$$\dot{x} = \frac{R}{2}\left[\dot{\phi}_1 + \dot{\phi}_2\right]\cos\theta, \quad \dot{y} = \frac{R}{2}\left[\dot{\phi}_1 + \dot{\phi}_2\right]\sin\theta \tag{6.26}$$

Let ρ be the turning radius of the slowest wheel (say, wheel No. 2 in accordance with the figure). The lengths of the arcs covered by wheel No. 2 and wheel No. 1, during a differential interval of time dt, are given by:

$$ds_2 = \rho d\theta, \quad ds_1 = (\rho + L)d\theta, \quad ds = (\rho + \frac{L}{2})d\theta \tag{6.27}$$

where ds is the arc length covered by the center of mass. But,

$$ds_2 = v_2 dt, \quad ds_1 = v_1 dt, \quad ds = v dt \tag{6.28}$$

i.e.,

$$ds_2 = R\dot{\phi}_2 dt, \quad ds_1 = R\dot{\phi}_1 dt, \quad ds = \frac{R}{2}(\dot{\phi}_1 + \dot{\phi}_2) dt \tag{6.29}$$

We have then

$$\rho\dot{\theta} = R\dot{\phi}_2, \quad (\rho + L)\dot{\theta} = R\dot{\phi}_1$$
$$\frac{\rho}{\rho + L} = \frac{\dot{\phi}_2}{\dot{\phi}_1} \Rightarrow \rho = L\frac{\dot{\phi}_2}{\dot{\phi}_1 - \dot{\phi}_2}$$

and

$$\dot{\theta} = \frac{R}{L}\left(\dot{\phi}_1 - \dot{\phi}_2\right) \tag{6.30}$$

We count with three non-holonomic restrictions,

$$\widehat{f}_1 = \dot{x} - \frac{R}{2}\left[\dot{\phi}_1 + \dot{\phi}_2\right]\cos\theta = 0$$
$$\widehat{f}_2 = \dot{y} - \frac{R}{2}\left[\dot{\phi}_1 + \dot{\phi}_2\right]\sin\theta = 0$$
$$\widehat{f}_3 = \dot{\theta} - \frac{R}{L}\left(\dot{\phi}_1 - \dot{\phi}_2\right) = 0$$

The Lagrangian of the system is given by

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}J_0\dot{\phi}_1^2 + \frac{1}{2}J_0\dot{\phi}_2^2 + \frac{1}{2}I\dot{\theta}^2$$
$$+\tau_1\phi_1 + \tau_2\phi_2$$

The equations of motion of the system are obtained from the expression:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \lambda_1\frac{\partial \widehat{f}_1}{\partial \dot{q}} + \lambda_2\frac{\partial \widehat{f}_2}{\partial \dot{q}} + \lambda_3\frac{\partial \widehat{f}_3}{\partial \dot{q}} \tag{6.31}$$

where $q = (x, y, \theta, \phi_1, \phi_2)^T$.

We have then the following expressions:

$$m\frac{d}{dt}\left[\frac{R}{2}(\dot{\phi}_1 + \dot{\phi}_2)\cos\theta\right] = \lambda_1$$
$$m\frac{d}{dt}\left[\frac{R}{2}(\dot{\phi}_1 + \dot{\phi}_2)\sin\theta\right] = \lambda_2$$
$$\frac{IR}{L}\left[\ddot{\phi}_2 - \ddot{\phi}_1\right] = \lambda_3$$
$$J_0\ddot{\phi}_1 = \tau_1 - \lambda_1\frac{R}{2}\cos\theta - \lambda_2\frac{R}{2}\sin\theta + \lambda_3\frac{R}{L}$$
$$J_0\ddot{\phi}_2 = \tau_2 - \lambda_1\frac{R}{2}\cos\theta - \lambda_2\frac{R}{2}\sin\theta - \lambda_3\frac{R}{L}$$

Eliminating the λ's and taking the restrictions as system equations that must be satisfied we obtain the following complete model of the system:

$$\dot{x} = \frac{R}{2}(\dot{\phi}_1 + \dot{\phi}_2)\cos\theta$$
$$\dot{y} = \frac{R}{2}(\dot{\phi}_1 + \dot{\phi}_2)\sin\theta$$
$$\dot{\theta} = \frac{R}{L}(\dot{\phi}_2 - \dot{\phi}_1)$$
$$\begin{bmatrix} J_0 + \frac{mR^2}{4} + \frac{IR^2}{L^2} & \frac{mR^2}{4} - \frac{IR^2}{L^2} \\ \frac{mR^2}{4} - \frac{IR^2}{L^2} & J_0 + \frac{mR^2}{4} + \frac{IR^2}{L^2} \end{bmatrix} \begin{bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

The set of (privileged) outputs, or flat outputs, is constituted by the center of mass of the wheel system at the center of the axis, i.e., the coordinates x and y.

We have

$$\theta = \arctan\left(\frac{\dot{y}}{\dot{x}}\right), \quad \dot{\theta} = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2} \tag{6.32}$$

In order to find the parametrization of the coordinates $\dot{\phi}_1$ and $\dot{\phi}_2$ we use the parameterizations for $\dot{\phi}_2 - \dot{\phi}_1$ and the one for $\dot{\theta}$:

$$\dot{\phi}_2 + \dot{\phi}_1 = \frac{2}{R}\sqrt{(\dot{x})^2 + (\dot{y})^2}, \quad \dot{\phi}_2 - \dot{\phi}_1 = \frac{L}{R}\left(\frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{(\dot{x})^2 + (\dot{y})^2}\right)$$

We have

$$\dot{\phi}_1 = \frac{1}{R}\sqrt{(\dot{x})^2 + (\dot{y})^2} - \frac{L}{2R}\left[\frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{(\dot{x})^2 + (\dot{y})^2}\right]$$
$$\dot{\phi}_2 = \frac{1}{R}\sqrt{(\dot{x})^2 + (\dot{y})^2} + \frac{L}{2R}\left[\frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{(\dot{x})^2 + (\dot{y})^2}\right]$$

$$\ddot{\phi}_1 = \frac{1}{R}\frac{\dot{x}\ddot{x} + \dot{y}\ddot{y}}{\sqrt{(\dot{x})^2 + (\dot{y})^2}} - \frac{L}{2R}\left[\frac{y^{(3)}\dot{x} - \dot{y}x^{(3)}}{(\dot{x})^2 + (\dot{y})^2} - 2\frac{(\ddot{y}\dot{x} - \dot{y}\ddot{x})(\dot{x}\ddot{x} + \dot{y}\ddot{y})}{((\dot{x})^2 + (\dot{y})^2)^2}\right]$$
$$\ddot{\phi}_2 = \frac{1}{R}\frac{\dot{x}\ddot{x} + \dot{y}\ddot{y}}{\sqrt{(\dot{x})^2 + (\dot{y})^2}} + \frac{L}{2R}\left[\frac{y^{(3)}\dot{x} - \dot{y}x^{(3)}}{(\dot{x})^2 + (\dot{y})^2} - 2\frac{(\ddot{y}\dot{x} - \dot{y}\ddot{x})(\dot{x}\ddot{x} + \dot{y}\ddot{y})}{((\dot{x})^2 + (\dot{y})^2)^2}\right]$$

Exercise 6.8. Under the assumption of suitable binary-valued torque inputs τ_1, τ_2, implement the derived average control laws by means of an appropriate Delta-Sigma modulation scheme.

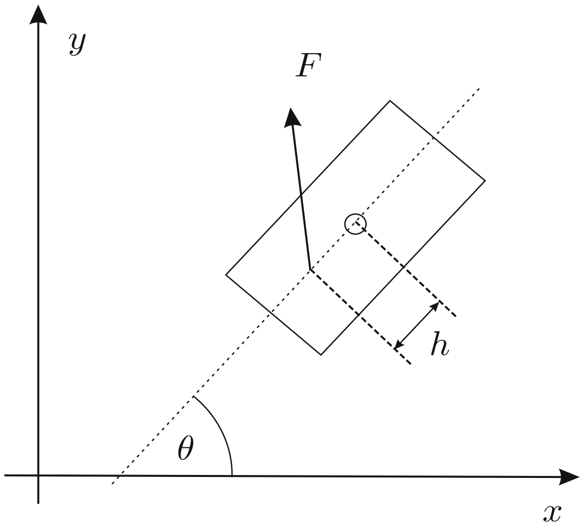


Fig. 6.3. The planar rigid body

6.5 The planar rigid body

This example is a simplified version of the hovercraft and is modeled as follows:

$$\begin{aligned} J\ddot{\theta} &= -hu_2 \\ m\ddot{x} &= u_1 \cos\theta - u_2 \sin\theta \\ m\ddot{y} &= u_1 \sin\theta + u_2 \cos\theta \end{aligned}$$

where x and y are the coordinates of the center of mass (Fig. 6.3). The variable θ describes the orientation of the main axis of the body with respect to the x axis. The control inputs u_1 and u_2 are the components of the applied force, respectively, along the main axis and its transversal axis.

Consider the following special outputs

$$F = x + \alpha\cos\theta, \qquad L = y + \alpha\sin\theta \tag{6.33}$$

The successive time derivatives of F and L result in

$$\begin{aligned} \dot{F} &= \dot{x} - \alpha\dot{\theta}\sin\theta \\ \ddot{F} &= \ddot{x} - \alpha\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) \\ &= (\frac{u_1}{m} - \alpha\dot{\theta}^2)\cos\theta - (1 - \alpha\frac{hm}{J})\frac{u_2}{m}\sin\theta \\ \dot{L} &= \dot{y} + \alpha\dot{\theta}\cos\theta \\ \ddot{L} &= \ddot{y} + \alpha(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \\ &= (\frac{u_1}{m} - \alpha\dot{\theta}^2)\sin\theta - (\alpha\frac{hm}{J} - 1)\frac{u_2}{m}\cos\theta \end{aligned}$$

Choosing $\alpha = J/(hm)$, we get

$$\ddot{F} = (\frac{u_1}{m} - \frac{J}{hm}\dot{\theta}^2)\cos\theta, \quad \ddot{L} = (\frac{u_1}{m} - \frac{J}{hm}\dot{\theta}^2)\sin\theta \tag{6.34}$$

We let

$$\xi = \frac{u_1}{m} - \frac{J}{hm}\,\dot{\theta}^2 \tag{6.35}$$

and, hence,

$$\ddot{F} = \xi\cos\theta, \quad \ddot{L} = \xi\sin\theta \tag{6.36}$$

Further differentiating the expressions for $\ddot{F}$, $\ddot{L}$, we find

$$\begin{aligned}
F^{(3)} &= \dot{\xi}\cos\theta - \dot{\theta}\xi\sin\theta \\
F^{(4)} &= \ddot{\xi}\cos\theta - 2\dot{\xi}\dot{\theta}\sin\theta - \ddot{\theta}\xi\sin\theta - \dot{\theta}^2\xi\cos\theta \\
&= \ddot{\xi}\cos\theta + \frac{J}{h}\xi\sin\theta u_2 - 2\dot{\xi}\dot{\theta}\sin\theta - \dot{\theta}^2\xi\cos\theta \\
L^{(3)} &= \dot{\xi}\sin\theta + \xi\dot{\theta}\cos\theta \\
L^{(4)} &= \ddot{\xi}\sin\theta + 2\dot{\xi}\dot{\theta}\cos\theta + \xi\ddot{\theta}\cos\theta - \xi\dot{\theta}^2\sin\theta \\
&= \ddot{\xi}\sin\theta - \frac{J}{h}\xi\cos\theta u_2 + 2\dot{\xi}\dot{\theta}\cos\theta - \xi\dot{\theta}^2\sin\theta
\end{aligned}$$

$$\begin{bmatrix} F^{(4)} \\ L^{(4)} \end{bmatrix} = \begin{bmatrix} \cos\theta & \frac{J}{h}\xi\sin\theta \\ \sin\theta & -\frac{J}{h}\xi\cos\theta \end{bmatrix} \begin{bmatrix} \ddot{\xi} \\ u_2 \end{bmatrix} + \begin{bmatrix} -2\dot{\xi}\dot{\theta}\sin\theta - \dot{\theta}^2\xi\cos\theta \\ 2\dot{\xi}\dot{\theta}\cos\theta - \dot{\theta}^2\xi\sin\theta \end{bmatrix}$$

The auxiliary controls v_F and v_L control the two independent fourth order chains of integrations $F^{(4)} = v_F$, $L^{(4)} = V_L$ by means of classical compensation networks

$$v_F = [F^\star(t)]^{(4)} - \left[\frac{k_4^F s^4 + k_3^F s^3 + k_2^F s^2 + k_1^F s + k_0^F}{s(s^3 + k_7^F s^2 + k_6^F s + k_5^F)}\right](F - F^*(t)) \tag{6.37}$$

$$v_L = [L^\star(t)]^{(4)} - \left[\frac{k_4^L s^4 + k_3^L s^3 + k_2^L s^2 + k_1^L s + k_0^L}{s(s^3 + k_7^L s^2 + k_6^L s + k_5^L)}\right](L - L^*(t)) \tag{6.38}$$

Note that F and L parameterize all the system variables. Indeed,

$$\theta = \arctan\left(\frac{\ddot{F}}{\ddot{L}}\right), \quad \xi = \sqrt{(\ddot{F})^2 + (\ddot{L})^2},$$

$$u_1 = m\xi + \frac{J}{h}\dot{\theta}^2 = m\sqrt{(\ddot{F})^2 + (\ddot{L})^2} + \frac{J}{h}\left(\frac{F^{(3)}\ddot{L} - \ddot{F}L^{(3)}}{(\ddot{F})^2 + (\ddot{L})^2}\right)^2$$

$$x = F - \frac{J}{hm}\left[\frac{\ddot{F}}{\sqrt{(\ddot{F})^2 + (\ddot{L})^2}}\right], \quad y = L - \frac{J}{hm}\left[\frac{\ddot{L}}{\sqrt{(\ddot{F})^2 + (\ddot{L})^2}}\right]$$

$$u_2 = -\frac{J}{h}\left\{\frac{F^{(4)}\ddot{L} - \ddot{F}L^{(4)}}{[\ddot{F}]^2 + [\ddot{L}]^2}\right\} - 2\frac{J}{h}\left\{\frac{F^{(3)}\ddot{L} - \ddot{F}L^{(3)}}{\left[[\ddot{F}]^2 + [\ddot{L}]^2\right]^2}\right\}\left(\ddot{F}F^{(3)} + \ddot{L}L^{(3)}\right)$$

A possibility for devising the required flat output based linear time-varying feedback control is as follows:

From the relations

$$\begin{aligned}
F^{(4)} &= \ddot{\xi}\cos\theta + \frac{J}{h}\xi\sin\theta u_2 - 2\dot{\xi}\dot{\theta}\sin\theta - \dot{\theta}^2\xi\cos\theta \\
L^{(4)} &= \ddot{\xi}\sin\theta - \frac{J}{h}\xi\cos\theta u_2 + 2\dot{\xi}\dot{\theta}\cos\theta - \xi\dot{\theta}^2\sin\theta
\end{aligned}$$

we let $F^{(4)} = v_F$ and $L^{(4)} = v_L$, as before, and eliminate $\ddot{\xi}$ multiplying out the first expression by $\sin\theta$, and subtracting from it the second equation multiplied by $\cos\theta$. We find, using the expression for ξ,

$$\begin{aligned}
u_1 &= m\xi + \frac{J}{h}\dot{\theta}^2 \\
u_2 &= \frac{h}{J\xi}\left[v_F\sin\theta - v_L\cos\theta + 2\dot{\xi}\dot{\theta}\right]
\end{aligned}$$

Multiplying now the first by $\cos\theta$ and the second by $\sin\theta$ we find a differential equation for ξ,

$$\ddot{\xi} = (v_F\cos\theta + v_L\sin\theta + \dot{\theta}^2\xi) \tag{6.39}$$

A linear time-varying controller, known as a GPI controller (see details on this class of controllers in Chapter 5), is synthesized as follows:

$$\begin{aligned}
u_1 &= m\xi + \frac{J}{h}[\dot{\theta}^*(t)]^2 \\
u_2 &= u_2^*(t) - \frac{h}{J\xi^*(t)}\sin\theta^*(t)\left[\frac{k_4^F s^4 + k_3^F s^3 + k_2^F s^2 + k_1^F s + k_0^F}{s(s^3 + k_7^F s^2 + k_6^F s + k_5^F)}\right](F - F^*(t)) \\
&\quad + \frac{h}{J\xi^*(t)}\cos\theta^*(t)\left[\frac{k_4^L s^4 + k_3^L s^3 + k_2^L s^2 + k_1^L s + k_0^L}{s(s^3 + k_7^L s^2 + k_6^L s + k_5 L)}\right](L - L^*(t)) \\
u_2^*(t) &= \frac{h}{J\xi^*(t)}\left\{[F^*(t)]^{(4)}\sin\theta^*(t) - [L^*(t)]^{(4)}\cos\theta^*(t) + 2\dot{\xi}^*(t)\dot{\theta}^*(t)\right\}
\end{aligned}$$

with ξ being the solution of the linear time-varying system (Fig. 6.4)

$$\begin{aligned}
\ddot{\xi} &= \ddot{\xi}^*(t) \\
&\quad - \cos\theta^*(t)\left[\frac{k_4^F s^4 + k_3^F s^3 + k_2^F s^2 + k_1^F s + k_0^F}{s(s^3 + k_7^F s^2 + k_6^F s + k_5^F)}\right](F - F^*(t)) \\
&\quad - \sin\theta^*(t)\left[\frac{k_4^L s^4 + k_3^L s^3 + k_2^L s^2 + k_1^L s + k_0^L}{s(s^3 + k_7^L s^2 + k_6^L s + k_5 L)}\right](L - L^*(t)) \\
&\quad + [\dot{\theta}^*(t)]^2(\xi - \xi^*(t))
\end{aligned}$$

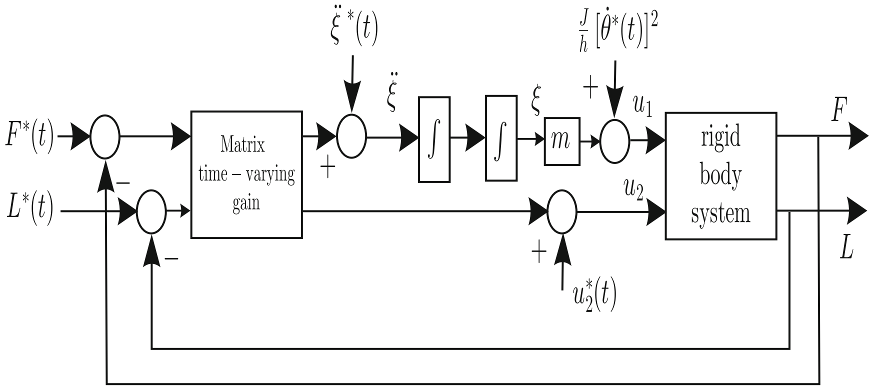


Fig. 6.4. Dynamic feedback control scheme for the planar rigid body

Exercise 6.9. Under the assumption of suitable binary-valued force components u_1, u_2, implement the derived average control laws by means of an appropriate Delta-Sigma modulation scheme.

6.5.1 The Rocket Example

This example is included to illustrate the flatness property in a classical and challenging nonlinear example with no relation to sliding mode control. The developments parallel those in Fliess *et al.* [6] and may also be found in Sira-Ramírez and Agrawal [26]. Consider the following simplified and normalized model of a rocket flying in the plane xy, without considering the fuel mass expenditure.

$$\begin{aligned}\ddot{x} &= u_1\\ \ddot{y} &= u_2\\ \epsilon\ddot{\theta} &= -u_1\cos\theta + (u_2+1)\sin\theta\end{aligned}$$

where x and y are, respectively, the horizontal and the vertical coordinates of the rocket center of mass. The angle θ is the inclination of the rocket's main axis with respect to the vertical direction. The control inputs u_1 and u_2 are the components of the thrust force, respectively, in the x and y directions, with u_2 already including the control of the gravity effect.

We hypothesize that the flat outputs are represented by coordinates of a point located on the main longitudinal axis of the rocket.

$$F = x + \alpha\sin\theta, \qquad L = y + \alpha\cos\theta \tag{6.40}$$

Taking successive time derivatives of this quantity we find

$$\begin{aligned}\dot{F} &= \dot{x} + \alpha\dot{\theta}\cos\theta\\ \ddot{F} &= u_1 + \frac{\alpha}{\epsilon}(-u_1\cos\theta + (u_2+1)\sin\theta)\cos\theta - \alpha\dot{\theta}^2\sin\theta\\ &= u_1(1-\frac{\alpha}{\epsilon}\cos^2\theta) + \frac{\alpha}{\epsilon}(u_2+1)u_1\cos\theta\sin\theta - \alpha\dot{\theta}^2\sin\theta\end{aligned}$$

$$\begin{aligned}
\dot{L} &= \dot{y} - \alpha\dot{\theta}\sin\theta \\
\ddot{L} &= u_2 - \frac{\alpha}{\epsilon}(-u_1\cos\theta + (u_2+1)\sin\theta)\sin\theta - \alpha\ddot{\theta}^2\cos\theta \\
&= (1 - \frac{\alpha}{\epsilon}\sin^2\theta)(u_2+1) + \frac{\alpha}{\epsilon}u_1\sin\theta\cos\theta - \alpha\dot{\theta}^2\cos\theta - 1
\end{aligned}$$

Letting $\alpha = \epsilon$ and

$$\zeta = u_1\sin\theta + (1+u_2)\cos\theta - \epsilon\dot{\theta}^2 \tag{6.41}$$

we find that

$$\ddot{F} = \zeta\sin\theta, \quad \ddot{L} = \zeta\cos\theta$$

Taking further derivatives of F and L one obtains

$$\begin{aligned}
F^{(3)} &= \dot{\zeta}\sin\theta + \zeta\dot{\theta}\cos\theta \\
F^{(4)} &= \ddot{\zeta}\sin\theta + 2\dot{\zeta}\dot{\theta}\cos\theta + \zeta\ddot{\theta}\cos\theta - \zeta\dot{\theta}^2\sin\theta \\
L^{(3)} &= \dot{\zeta}\cos\theta - \zeta\dot{\theta}\sin\theta \\
L^{(4)} &= \ddot{\zeta}\cos\theta - 2\dot{\zeta}\dot{\theta}\sin\theta - \zeta\ddot{\theta}\sin\theta - \zeta\dot{\theta}^2\cos\theta
\end{aligned}$$

Let

$$F^{(4)} = v_F \quad \text{and} \quad L^{(4)} = v_L \tag{6.42}$$

Eliminating $\ddot{\zeta}$ first and then eliminating $\ddot{\theta}$ we find

$$\begin{aligned}
v_F\cos\theta - v_L\sin\theta &= 2\dot{\zeta}\dot{\theta} + \zeta\ddot{\theta} \\
v_F\sin\theta + v_L\cos\theta &= \ddot{\zeta} - \zeta\dot{\theta}^2
\end{aligned}$$

Substituting $\ddot{\theta}$ from the system equations, in the previous equations, and using the expression for ζ, we obtain the following system of equations for the control inputs u_1 and $u_2 + 1$:

$$\begin{aligned}
-u_1\cos\theta + (1+u_2)\sin\theta &= \frac{\epsilon}{\zeta}\left[v_F\cos\theta - v_L\sin\theta - 2\dot{\zeta}\dot{\theta}\right] \\
u_1\sin\theta + (1+u_2)\cos\theta &= \zeta + \epsilon\dot{\theta}^2
\end{aligned}$$

with ζ being the solution of:

$$\ddot{\zeta} = v_F\sin\theta + v_L\cos\theta + \zeta\dot{\theta}^2 \tag{6.43}$$

6.6 Sliding surface design for flat systems

Flat outputs are known to have the distinctive property of representing physically meaningful variables. These statement, with some notable exception, is made on the basis of a rather large collection of physically oriented examples (see Sira-Ramírez and Agrawal [26]). A reasonable assumption is then that the flat output is available for measurement and that a sliding mode control scheme would primarily be devoted to regulate this set of variables in trajectory tracking or in stabilization tasks. If, on the contrary, system variables, other than the flat outputs, are required to be directly regulated so as to satisfy a property that may be translated into a sliding mode control design, then such an objective can be equally handled in terms of the flat output via a suitable use of the differential parametrization relating those system variables with the flat outputs.

Suppose that the system $\dot{x} = f(x,u)$, $u \in \mathbb{R}^m$ is flat, with flat output variables given by the vector y. Then, the differential parametrization of the input vector u is given by

$$u = B(y, \dot{y}, \cdots, y^{(\gamma+\mathbf{1})}) \tag{6.44}$$

It is important to realize the *multi-index* nature of γ, i.e. $\gamma = (\gamma_1, \cdots, \gamma_m)$. Thus, modulo some reordering of the flat outputs, we can write the following set of relations:

$$\begin{aligned} u_1 &= B_1(y, \dot{y}, \cdots, y_1^{(\gamma_1+1)}) \\ &\vdots \\ u_m &= B_m(y, \dot{y}, \cdots, y_m^{(\gamma_m+1)}) \end{aligned} \tag{6.45}$$

A set of independent sliding surfaces can then be readily proposed with the idea of achieving an exact linearization of the ideal sliding dynamics. Such linearization may be induced in the stabilization error dynamics or in a tracking error dynamics context. In the stabilization case, one proceeds to define the sliding surface coordinate functions as

$$\begin{aligned} \sigma_1 &= \sum_{j=0}^{\gamma_1} \alpha_{1,j} y_1^{(j)} \\ &\vdots \\ \sigma_m &= \sum_{j=0}^{\gamma_m} \alpha_{m,j} y_m^{(j)} \end{aligned}$$

with the coefficients, for each one of the linear combinations above, chosen so that the corresponding closed loop characteristic polynomials,

$$p_k(s) = s^{\gamma_k} + \alpha_{k,\gamma_k-1} s^{\gamma_k-1} + \cdots + \alpha_{k,0} \tag{6.46}$$

are guaranteed to be Hurwitz polynomials.

We present some illustrative examples so as to clarify the little specificity, which allows us, the generality of the previous formulae.

Being a structural property of the system, flatness is to be examined when the system is free of disturbances. However, in some instances, it may be convenient to carry along the perturbations in the system description as if they were seemingly known functions. This allows one to obtain a perturbed differential parametrization of all system variables with a clear assessment of the distribution of the effects of the disturbances on the state space variables, on the actual outputs and on the input space components.

6.6.1 A stepping motor example

Consider the following, rather popular, model of a stepping motor

$$\begin{aligned}
L\frac{d}{dt}i_a &= -Ri_a + K_m\omega\sin(n_p\theta) + v_a \\
L\frac{d}{dt}i_b &= -Ri_b - K_m\omega\cos(n_p\theta) + v_b \\
J\frac{d}{dt}\omega &= -K_m i_a\sin(n_p\theta) + K_m i_b\cos(n_p\theta) - \tau(t) \\
\frac{d}{dt}\theta &= \omega
\end{aligned}$$

where i_a and i_b represent measurable phase currents, ω is the angular velocity of the axis of the motor while θ represents the angular position. The control input variables are represented by the phase voltages v_a and v_b. We assume that all motor parameters are perfectly known and that the motor system is initially at rest on $\theta = 0$. The signal $\tau(t)$ represents the unknown but constant load torque. It appears quite unexpectedly and it constitutes an unknown perturbation.

The system is differentially flat, with flat outputs given by the angular position θ and one of the phase currents, say i_a. Indeed, if θ and i_a are known, we can compute the rest of the system variables in terms of these two variables and a finite number of their time derivatives. We have, for $\tau = 0$

$$\begin{aligned}
\omega &= \dot{\theta} \\
v_a &= L\frac{di_a}{dt} + Ri_a - K\dot{\theta}\sin(n_p\theta) \\
i_b &= \frac{1}{K_m\cos(n_p\theta)}\left[J\ddot{\theta} + K_m i_a\sin(n_p\theta)\right] \\
v_b &= \frac{L}{K_m}\left[\frac{\left(J\theta^{(3)} + K_m\frac{di_a}{dt}\sin(n_p\theta) + K_m i_a n_p\cos^2(n_p\theta)\right)}{\cos(n_p\theta)}\right. \\
&\quad \left. + \frac{\left(J\ddot{\theta} + K_m i_a\sin(n_p\theta)\right)n_p\sin(n_p\theta)}{\cos^2(n_p\theta)}\right] \\
&\quad + \frac{R}{K_m\cos(n_p\theta)}\left[J\ddot{\theta} + K_m i_a\sin(n_p\theta)\right] + K_m\dot{\theta}\cos(n_p\theta) \qquad (6.47)
\end{aligned}$$

Given a set of nominal trajectories for θ and i_a, all nominal evolutions of the state variables and inputs are readily determined from the above expressions.

Before prescribing a set of sliding surfaces to accomplish a desired objective, consider the perturbed differential parametrization of the system by including the presence of the load torque perturbation signal $\tau(t)$. We have

$$
\begin{aligned}
\omega &= \dot{\theta} \\
v_a &= L\frac{di_a}{dt} + Ri_a - K\dot{\theta}\sin(n_p\theta) \\
i_b &= \frac{1}{K_m\cos(n_p\theta)}\left[J\ddot{\theta} + K_m i_a \sin(n_p\theta) + \tau(t)\right] \\
v_b &= \frac{L}{K_m}\left[\frac{\left(J\theta^{(3)} + K_m\frac{di_a}{dt}\sin(n_p\theta) + K_m i_a n_p\cos^2(n_p\theta) + \dot{\tau}(t)\right)}{\cos(n_p\theta)}\right. \\
&\quad \left. + \frac{\left(J\ddot{\theta} + K_m i_a\sin(n_p\theta) + \tau(t)\right)n_p\sin(n_p\theta)}{\cos^2(n_p\theta)}\right] \\
&\quad + \frac{R}{K_m\cos(n_p\theta)}\left[J\ddot{\theta} + K_m i_a\sin(n_p\theta) + \tau(t)\right] + K_m\dot{\theta}\cos(n_p\theta)
\end{aligned}
\tag{6.48}
$$

The expressions describing the perturbed dynamics for the flat outputs, in a simplified manner, are of the form:

$$
\begin{aligned}
\frac{di_a}{dt} &= \left(\frac{1}{L}\right)v_a + q_{i_a}(i_a, \theta, \dot{\theta}) \\
\theta^{(3)} &= \left(\frac{K_m}{LJ}\right)v_b\cos(n_p\theta) + q_\theta(\theta, \dot{\theta}, i_a, \frac{di_a}{dt}, \tau, \dot{\tau})
\end{aligned}
$$

where $q_\theta(\cdot)$ and $q_{i_a}(\cdot)$ are considered as perturbation inputs. Only the angular position dynamics exhibits an explicit dependence on the load perturbation input signal $\tau(t)$. A sliding regime created on a set of sliding surfaces, seeking either stabilization or trajectory tracking for the flat outputs, will have to face this combination of endogenous (state) and exogenous (load torque) uncertainties. The presence of the factor $\cos(n_p\theta)$, affecting the control input v_b in a multiplicative manner, represents a limitation on the feasible angular position trajectories or angular position equilibria. To avoid singularities, the angle θ is to be restricted to the open interval:

$$
-\frac{\pi}{2n_p} < \theta < \frac{\pi}{2n_p} \tag{6.49}
$$

Flatness reveals the structure of the problem and the basic limitations associated therewith.

Assume one is interested in a stabilization of the current i_a around a given equilibrium value, I_a and, simultaneously, a stabilization of the motor shaft position around a constant angular position $\theta = \Theta$. Assume the voltages v_a and v_b may take values, respectively, in the discrete sets $\{-V_a, V_a\}$ and $\{-V_b, V_b\}$, where V_a, V_b represent maximum allowable magnitude voltage inputs to the motor. As discussed in the introduction of this chapter, there are, generally speaking, two procedures for the sliding mode control of nonlinear switched systems which are specially endowed with flatness. We call these methods, the "direct method" and the "average based method," which is based on Delta-Sigma modulation.

A direct method would try to directly create, via switched inputs v_a and v_b a sliding regime on the prescribed sliding surfaces reflecting the control objectives. For this one would prescribe the sliding surface coordinate functions:

$$\sigma_1 = i_a - I_a, \quad \sigma_2 = \ddot{\theta} + \kappa_1 \dot{\theta} + \kappa_0(\theta - \Theta) \tag{6.50}$$

Clearly, ideal sliding regimes on the surfaces

$$\mathcal{S}_1 = \{x \in \mathbb{R}^4 \mid \sigma_1 = i_a - I_a = 0\}, \ \mathcal{S}_2 = \{x \in \mathbb{R}^4 \mid \sigma_2 = \ddot{\theta} + \kappa_1 \dot{\theta} + \kappa_0(\theta - \Theta) = 0\} \tag{6.51}$$

with $x = (i_a, i_b, \dot{\theta}, \theta)$, accomplishes the desired objective, provided a sliding regime exists on these surfaces. Here the fundamental assumption is that the time-varying quantities q_i and q_θ, defined above, are uniformly absolutely bounded

$$\sup_t |q_i(i_a(t), \theta(t), \dot{\theta}(t))| \leq K_1, \ \sup_t |q_\theta(\theta(t), \dot{\theta}(t), i_a(t), \frac{di_a(t)}{dt}, \tau(t), \dot{\tau}(t))| \leq K_2 \tag{6.52}$$

In general, these bounds are not easy to establish in an analytic fashion. In this particular case, given that $q_\theta(\cdot)$ explicitly depends on an unknown input signal represented by the load torque $\tau(t)$ and its time derivative, the task of producing an adjusted estimate of K_2 is specially difficult. Flatness allows, however, an assessment of the nominal trajectories of this quantity via off-line simulations using typical load torque realizations (constant, periodic, etc.).

Additionally, notice that the load torque perturbations affect the sliding surface coordinate function σ_2 itself. Indeed, if only states are measurable from the system (so that the need to produce the angular acceleration, $\ddot{\theta}$, is sidestepped), then σ_2 is synthesized as

$$\sigma_2 = -\frac{K_m}{J} i_a \sin(n_p \theta) + \frac{K_m}{J} i_b \cos(n_p \theta) - \frac{1}{J}\tau(t) + \kappa_1 \omega + \kappa_0(\theta - \Theta) \tag{6.53}$$

The scheme results in a perturbation dependent sliding surface which may be inconvenient. One option is to ignore such a perturbation dependence and relegate the load torque effects to a larger switching gain authority. This may result in excessive chattering. Another option, which is becoming quite popular, is to attempt an on-line disturbance estimation scheme, based on the mechanical part of the system dynamics and the knowledge of the state variables, in order to adaptively inject this (asymptotic or finite time convergent) estimate into the sliding surface expression. This requires knowledge of a substantial set of parameters in the system (K_m, L, J).

The switching policy is of the form

$$v_a = -\frac{1}{2} L W_{i_a} \mathrm{sign}\sigma_1, \quad v_b = -\frac{LJ}{K_m} W_\theta \mathrm{sign}\sigma_2 \tag{6.54}$$

with W_{i_a} chosen so that $LW_{i_a} > K_1$ and W_{i_θ} is chosen so that $(LJ/K_m \cos(n_p\theta))W_\theta > K_2$.

The average approach is based on feedback linearization for the current and the angular position dynamics.

6.7 The feed-forward controller

Hagenmeyer and Delaleau (see [12]), have proposed a class of controllers for nonlinear systems addressed as *feed-forward linearizing controllers.* The controllers, for the most common case of affine in the control systems, turn out to be linear, time-varying controllers which are to be used in combination with some complementary feedback method designed for controlling a chain of integrators of the same order as the system. The developments are based on the following ideas which, we show, bring about a rather popular control method for nonlinear systems, the approximate linearization method, but with a twist.

Suppose a given SISO nonlinear system of the form $\dot{x} = f(x,u)$ is differentially flat, with flat output given by the variable y. The results we expose below extend, quite directly, to the MIMO case as well. The system, being flat, has then the following differential parametrization of the control input u, in terms of a differential function of y,

$$u = B(y, \dot{y}, \ddot{y}, \ldots, y^{(n)}) \tag{6.55}$$

In [12], the authors propose the following control scheme for tracking any smooth trajectory $y^*(t)$, with corresponding nominal control input given by $u^*(t) = B(y^*(t), \ldots, [y^*(t)]^{(n)})$:

Set:

$$u = B(y^*(t), \dot{y}^*, \ldots, [y^*(t)]^{(n-1)}, v) \tag{6.56}$$

with v being a GPI controller that renders the origin of the tracking error space $e = y - y^*(t)$ into an exponentially asymptotically stable equilibrium point for the n-th order integrator system

$$y^{(n)} = v \tag{6.57}$$

Using the results in the previous section about GPI control of chains of integrators, the Hagenmeyer-Delaleau controller [12] yields the following feedback control scheme for the system $\dot{x} = f(x,u)$

$$\begin{aligned} u(t) &= B(y^*(t), \dot{y}^*(t), \ldots, [y^*(t)]^{(n-1)}, v) \\ v &= [y^*(t)]^{(n)} - \left[\frac{k_{n-1}s^{n-1} + k_{n-2}s^{n-2} + \cdots + k_0}{s^{n-1} + k_{2n-2}s^{n-2} + \cdots + k_n}\right](y - y^*(t)) \end{aligned} \tag{6.58}$$

In particular, for the most common case of systems which are *affine* in the control input, i.e. of the form: $\dot{x} = f(x) + g(x)u$, it is easy to show that the control input differential parametrization, provided by the flatness property, is also affine in the highest order time derivative of the flat output. We have

$$u = \phi(y, \dot{y}, \cdots, y^{(n-1)}) + \left[\eta(y, \dot{y}, \cdots, y^{(n-1)})\right] y^{(n)} \tag{6.59}$$

For these systems, the output feedback controller is clearly linear and time-varying. Indeed, we have

$$\begin{aligned} u &= \phi(y^*(t), \dot{y}^*(t), \cdots, [y^*(t)]^{(n-1)}) + \left[\eta(y^*(t), \dot{y}^*(t), \cdots, [y^*(t)]^{(n-1)})\right] v \\ &= \alpha(t) + \beta(t)v \\ v &= [y^*(t)]^{(n)} - \left[\frac{k_{n-1}s^{n-1} + k_{n-2}s^{n-2} + \cdots + k_0}{s^{n-1} + k_{2n-2}s^{n-2} + \cdots + k_n}\right](y - y^*(t)) \end{aligned} \tag{6.60}$$

It is rigorously shown in [12] that such an output feedback controller locally stabilizes the controlled flat output trajectory towards the desired given reference trajectory $y^*(t)$. Indeed, if we substitute the GPI controller expression for v into the first equation in (6.60), we readily obtain

$$u = u^*(t) - \beta(t)\left[\frac{k_{n-1}s^{n-1} + k_{n-2}s^{n-2} + \cdots + k_0}{s^{n-1} + k_{2n-2}s^{n-2} + \cdots + k_n}\right](y - y^*(t)) \tag{6.61}$$

i.e., we are controlling the nonlinear system with a variant of the classical *approximate linearization* philosophy which injects the nominal control input $u^*(t)$ and complements it with the output of a classical compensation network controller acting on the incremental output tracking error. The only difference with the classical feedback control scheme of approximate linearization based control is the presence of the time-varying modulation factor $\beta(t)$ affecting the incremental classical controller.

This result, which has been known for several decades and used extensively in the control of industrial processes in various forms, is at the heart of the possibilities of designing linear, time-varying, output feedback controllers for switched systems whose models are, formally, differentially flat. Naturally, the idea is to use a Delta-Sigma modulation control scheme on the basis of the *average* control design represented by (6.61).

6.8 Flatness guided design in switched systems

The implications of flatness and the possibility of designing a suitable sliding surface for the system are rather direct but it requires some caution if the underlying actual system, from whose formal model flatness was established, is switched or not switched. If continuous controls are available, we defend the viewpoint by which sliding modes are rendered unnecessary or even superfluous. The interesting situation is when the system is switched, the model happens to be flat, and flatness has guided our average design. We examine these issues in the context of another example which besides illustrating the flatness property allows us to come up with a switched implementation of the flatness based controller design.

Example 6.10. Consider the normalized switch regulated boost converter circuit, given by

$$\begin{aligned}\dot{z}_1 &= -uz_2 + 1\\ \dot{z}_2 &= uz_1 - \frac{1}{Q}z_2\end{aligned} \tag{6.62}$$

where z_1 represents the normalized inductor current, z_2 is the normalized capacitor voltage, and u, the control input, is a switch position function taking values in $\{0, 1\}$. For the moment, we disregard the binary valued nature of the control input and treat the model as an average model with no restrictions whatsoever on u.

The total normalized stored energy, given by

$$y = \frac{1}{2}\left(z_1^2 + z_2^2\right) \tag{6.63}$$

qualifies as a flat output. The state coordinate transformation $(z_1, z_2) = (y, \dot{y})$ places the system in controllable canonical form.

The flat output and its time derivative are given by

$$y = \frac{1}{2}\left(z_1^2 + z_2^2\right), \quad \dot{y} = z_1 - \frac{z_2^2}{Q} \tag{6.64}$$

Notice that the flat output y completely differentially parameterizes the system variables z_1, z_2, and u. Indeed,

$$\begin{aligned}z_1 &= -\frac{Q}{2} + \sqrt{\frac{Q^2}{4} + 2y + Q\dot{y}}\\ z_2 &= \sqrt{-Q\dot{y} - \frac{Q^2}{2} + \sqrt{\frac{Q^4}{4} + Q^2\left(2y + Q\dot{y}\right)}}\\ u &= \frac{Q}{(2z_1(y,\dot{y}) + Q)z_2(y,\dot{y})}\left[1 + \frac{2}{Q^2}z_2^2(y,\dot{y}) - \ddot{y}\right]\end{aligned} \tag{6.65}$$

Suppose it is desired to have the normalized output capacitor voltage z_2 to track a given trajectory specified by the smooth function $z_2^*(t)$. Suppose, furthermore, that the desired trajectory for z_2 is a rest-to-rest maneuver starting at some initial time t_{init} and ending at some later time t_{final} and taking the value of z_2 from an initial equilibrium $\overline{z_2}(t_{initi})$ to the final value $\overline{z_2}(t_{final})$. The flat output y exhibits a corresponding behavior that transfers the initial value

$$\overline{y}_{init} = \frac{1}{2}\overline{z_2}^2(t_{init})\left(1 + \frac{\overline{z_2}^2(t_{init})}{Q^2}\right) \tag{6.66}$$

towards the final value

$$\overline{y}_{final} = \frac{1}{2}\overline{z_2}^2(t_{final})\left(1 + \frac{\overline{z_2}^2(t_{final})}{Q^2}\right) \tag{6.67}$$

Knowledge of the required initial and final equilibrium points for the flat output allows us to prescribe an arbitrary smooth trajectory for y that joins these two equilibrium points while prescribing constant values, coinciding with the computed equilibria, valid after t_{final} and before t_{init}. Let this off-line computed trajectory be denoted by $y^*(t)$ with $y^*(t) = \overline{y}_{init}$ for $t < t_{init}$, $y^*(t) = \overline{y}_{final}$ for $t > t_{final}$, and, during the equilibrium to equilibrium transfer interval $[t_{init}, t_{final}]$, the pre-computed flat output $y^*(t)$ is given by

$$y^*(t) = \overline{y}_{init} + \psi(t, t_{final}, t_{init})[\overline{y}_{final} - \overline{y}_{init}] \tag{6.68}$$

with $\psi(t, t_{final}, t_{init})$ being a smooth interpolating polynomial of the Bézier type with. $\psi(t_{init}, t_{final}, t_{init}) = 0$ and $\psi(t_{final}, t_{final}, t_{init}) = 1$.

We propose the following time-varying feed-forward controller of the GPI type:

$$u = \frac{Q}{(2z_1(y^*(t), \dot{y}^*(t)) + Q)z_2(y^*(t), \dot{y}^*(t))}\left[1 + \frac{2}{Q^2}z_2^2(y^*(t), \dot{y}^*(t)) - v\right] \tag{6.69}$$

where

$$\begin{aligned} z_1(y^*(t), \dot{y}^*(t)) &= -\frac{Q}{2} + \sqrt{\frac{Q^2}{4} + 2y^*(t) + Q\dot{y}^*(t)} \\ z_2(y^*(t), \dot{y}^*(t)) &= \sqrt{-Q\dot{y}^*(t) - \frac{Q^2}{2} + \sqrt{\frac{Q^4}{4} + Q^2\left(2y^* + Q\dot{y}^*(t)\right)}} \end{aligned} \tag{6.70}$$

and v is synthesized as a GPI controller of the form

$$v = \ddot{y}^*(t) - \left[\frac{k_1 s + k_0}{s + k_2}\right](y - y^*(t)) \tag{6.71}$$

Figure 6.5 depicts the performance of the feed-forward flatness based controller for a desired equilibrium to equilibrium transfer for the average normalized capacitor voltage.

The parameter values used in the simulation were set to be

$$Q = 1.0, \quad k_2 = 2\zeta\omega_n p, \quad k_1 = 2\zeta\omega_n + p, \quad k_0 = \omega_n^2 p$$

with $\zeta = 0.707$, $\omega_n = 1$, $p = 1.5$. The initial and final times were set to be $t_{init} = 10$, $t_{final} = 30$ while $\overline{z_2}(t_{init}) = 1.5$ and $\overline{z_2}(t_{final}) = 2.5$

A switched implementation of the average designed controller is possible using a $\Sigma - \Delta$ modulator, as follows:

$$\dot{z} = u_{av} - u$$
$$u = \frac{1}{2}\left(1 + \text{sign} z\right) \tag{6.72}$$

where u_{av} is the control input produced by the flatness based controller that we have just designed in (6.68) and u is the actual switched input applied to the converter. The state z of the $\Sigma - \Delta$ modulator may be conveniently initialized in $z(0) = 0$. In this fashion the modulator starts exhibiting a sliding regime on the "extended space," $\mathcal{S} = \{z \in R \mid z = 0\}$.

The simulations depicting the responses of the switch controlled system are shown in Figure 6.6.

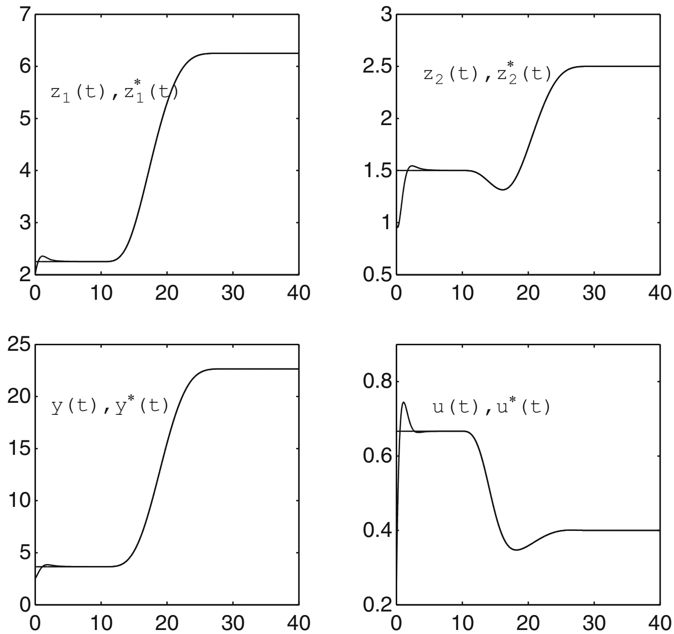


Fig. 6.5. Average performance of feed-forward flatness and GPI based smooth controller

A second illustration of the use of flatness based feedback controller in connection with sliding modes in switched systems, consider the following tank system constituted by three identical tanks.

Example 6.11. Consider the following model of a three-tank system, assumed for simplicity to be identical.

$$\dot{x}_1 = -\frac{c}{A}\sqrt{x_1} + \frac{1}{A}\vartheta$$
$$\dot{x}_2 = -\frac{c}{A}\sqrt{x_2} + \frac{c}{A}\sqrt{x_1}$$

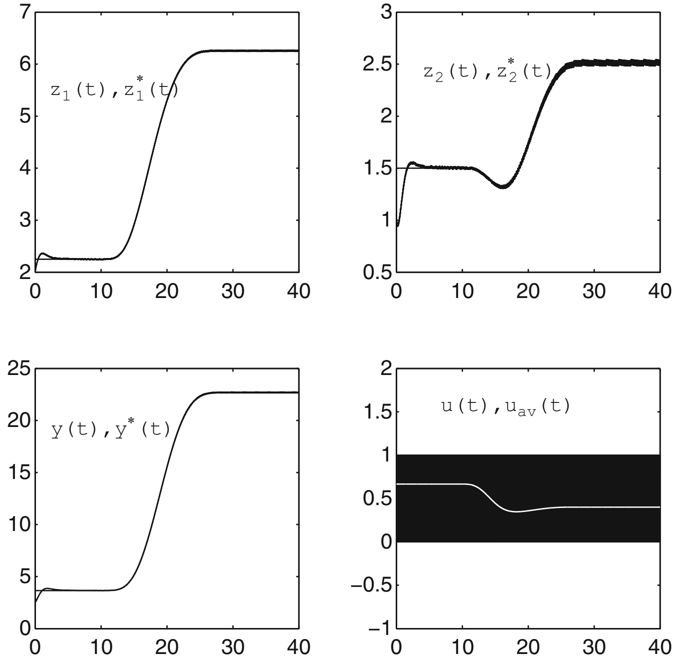


Fig. 6.6. Performance of feed-forward flatness and GPI based average controller implemented through a $\Sigma - \Delta$ modulator

$$\begin{aligned} \dot{x}_3 &= -\frac{c}{A}\sqrt{x_3} + \frac{c}{A}\sqrt{x_2} \\ y &= x_3 \end{aligned} \qquad (6.73)$$

where x_1, x_2, and x_3 are the liquid heights in the tanks, ϑ is the control input, assumed to take values only in the discrete set $\{0, U\}$, the constant coefficients c and A are assumed to be known and they represent a friction coefficient and the area of the bottom of any of the tanks.

A normalization of the system is easily accomplished by introducing a time scale and a redefinition of the control input as follows:

$$t' = t\left(\frac{c}{A}\right), \quad \vartheta = \left(\frac{U}{c}\right)u \qquad (6.74)$$

The new control input u takes values in the discrete set $\{0, 1\}$. The normalized system is written as

$$\begin{aligned} \dot{x}_1 &= -\sqrt{x_1} + \frac{U}{c}u \\ \dot{x}_2 &= -\sqrt{x_2} + \sqrt{x_1} \\ \dot{x}_3 &= -\sqrt{x_3} + \sqrt{x_2} \\ y &= x_3 \end{aligned} \qquad (6.75)$$

The system is flat, with flat output $F = x_3 = y$. The differential parametrization of all system variables and the control input is given by

$$
\begin{aligned}
x_3 &= y \\
x_2 &= (\dot{y} + \sqrt{y})^2 \\
x_1 &= \left[(\dot{y} + \sqrt{y}) + 2\,(\dot{y} + \sqrt{y}) \left(\ddot{y} + \frac{\dot{y}}{2\sqrt{y}} \right) \right]^2 \\
&= (\dot{y} + \sqrt{y})^2 \left[1 + 2 \left(\ddot{y} + \frac{\dot{y}}{2\sqrt{y}} \right) \right]^2 \\
u &= \frac{c}{U} \left[(\dot{y} + \sqrt{y}) \left(1 + 2 \left(\ddot{y} + \frac{\dot{y}}{2\sqrt{y}} \right) \right) \right] \times \\
& \left\{ \left(1 + 2 \left[\ddot{y} + \frac{\dot{y}}{2\sqrt{y}} \right] \right)^2 + 2(\dot{y} + \sqrt{y}) \left(2y^{(3)} + \frac{2y\ddot{y} - (\dot{y})^2}{2y\sqrt{y}} \right) \right\}
\end{aligned}
\tag{6.76}
$$

The previous differential parametrization allows one to parameterize all equilibria in terms of the equilibrium value of the flat output

$$
\overline{x}_1 = \overline{x_2} = \overline{x_3} = \overline{y}, \quad \overline{u} = \frac{c}{U} \sqrt{\overline{y}} \tag{6.77}
$$

Thus, the three tanks have exactly the same height as equilibrium height.

Suppose it is desired to transfer the system from an initial equilibrium to a final equilibrium. We then prescribe a nominal trajectory $y^*(t)$ smoothly taking the flat output, during a finite time interval of the form $[t_{init}, t_{final}]$, from an initial value $y^*(t_{init}) = \overline{y}_{init}$ to a final desired value $y^*(t_{final}) = \overline{y}_{final}$. We set, for instance,

$$
y^*(t) = \overline{y}_{init} + (\overline{y}_{final} - \overline{y}_{initial}) \psi(t, t_{final}, t_{init}) \tag{6.78}
$$

with $\psi(t, t_{final}, t_{init})$ being a Bézier polynomial smoothly interpolating between 0 and 1 in the time interval $[t_{init}, t_{final}]$. We choose a 16th order Bézier polynomial:

$$
\psi(t, t_{init}, t_{final}) =
$$

$$
\begin{cases}
0 & \text{for } t < t_{init} \\
\left[\dfrac{t - t_{init}}{t_{final} - t_{init}} \right]^8 \left[r_1 - r_2 \left(\dfrac{t - t_{init}}{t_{final} - t_{init}} \right) + \cdots + r_9 \left(\dfrac{t - t_{init}}{t_{final} - t_{init}} \right)^8 \right] & \text{for } t \in [t_{init}, t_{final}] \\
1 & \text{for } t > t_{final}
\end{cases}
$$

where

$$r_1 = 12870,\ r_2 = 91520,\ r_3 = 288288,\ r_4 = 524160,\ r_5 = 600600,$$

$$r_6 = 443520,\ r_7 = 205920,\ r_8 = 54912,\ r_9 = 6435$$

The feed-forward linearizing controller design technique for this system results in the following linear time-varying controller:

$$u = \frac{c}{U}\left[\left(\dot{y}^*(t) + \sqrt{y^*(t)}\right)\left(1 + 2\left(\ddot{y}^*(t) + \frac{\dot{y}^*(t)}{2\sqrt{y^*(t)}}\right)\right)\right] \times$$
$$\left\{\left(1 + 2\left[\ddot{y}^*(t) + \frac{\dot{y}^*(t)}{2\sqrt{y^*(t)}}\right]\right)^2 + 2(\dot{y}^*(t) + \sqrt{y^*(t)}) \times\right.$$
$$\left.\left(2v + \frac{2y^*(t)\ddot{y}^*(t) - (\dot{y}^*(t))^2}{2y^*(t)\sqrt{y^*(t)}}\right)\right\}$$

where v, which evidently satisfies $v = y^{(3)}$, is synthesized as a GPI controller for trajectory tracking given by

$$v = [y^*]^{(3)} - \left[\frac{k_2 s^2 + k_1 s + k_0}{s^2 + k_4 s + k_3}\right](y - y^*(t)) \qquad (6.79)$$

The set of design coefficients $\{k_4, k_3, \cdots, k_0\}$ are chosen so as to render the closed loop characteristic polynomial,

$$p(s) = s^5 + k_4 s^4 + k_3 s^3 + k_2 s^2 + k_1 s + k_0 \qquad (6.80)$$

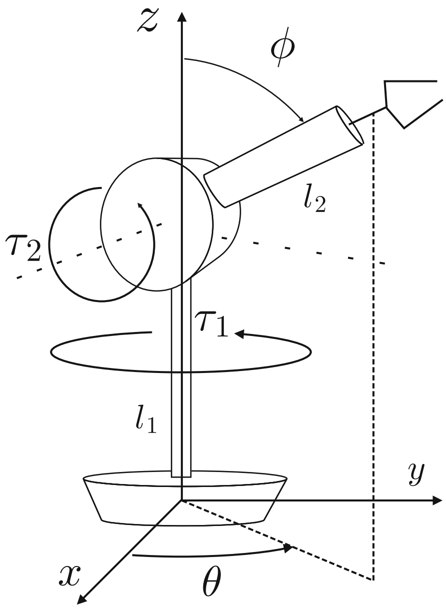
a Hurwitz polynomial.

Figure 6.7 shows the closed loop performance of the proposed linear time-varying controller in the trajectory tracking task. The maneuver interval has been adjusted so that the control input is fitted into the interval $[0, 1]$. As a desired characteristic polynomial, used for establishing the desired design parameters, we have used $p(s) = (s+p)^5$ with $p = 2$. The values of the system parameters were set to be $U = 0.3$, $c = 0.15$ and $t_{init} = 2$, $t_{final} = 25$.

The $Delta-Sigma$ implementation of the designed average feedback controller is shown in Figure 6.8. Note that the combination of a GPI controller with the Hagenmeyer-Delaleau controller allows the flatness based controller to be a truly linear time-varying output feedback average controller.

6.8.1 Control of a two degrees of freedom robot

Consider the mechanism shown in figure 5.6. We use as generalized coordinates the angular displacements, θ and ϕ shown in the figure.



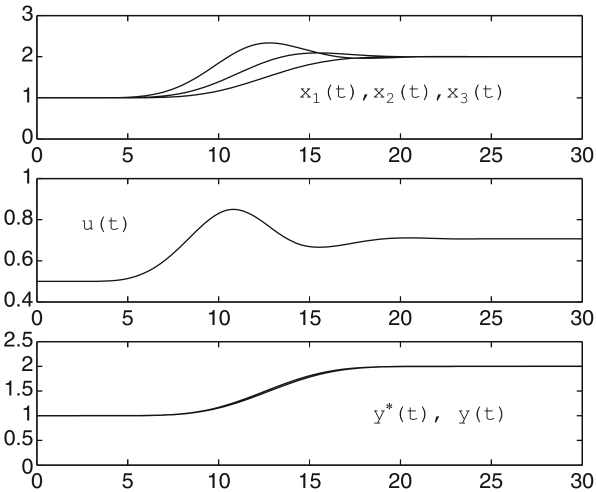


Fig. 6.7. Average closed loop response of the three tanks system to Hagenmeyer-Delaleau feed-forward controller combined with GPI controller

Two control inputs τ_1 and τ_2 are considered which take values in the discrete sets $\{T_1, 0, -T_1\}$ and $\{T_2, 0, -T_2\}$, respectively. For reasons that will become apparent we restrict T_1 and T_2 to be of the form

$$T_i = W_i(m_2 g l_2), \quad W_i > 1 \tag{6.81}$$

The first torque input acts over the angular position θ and the second over the angular displacement ϕ (Figs. 6.9, 6.10 and 6.11).

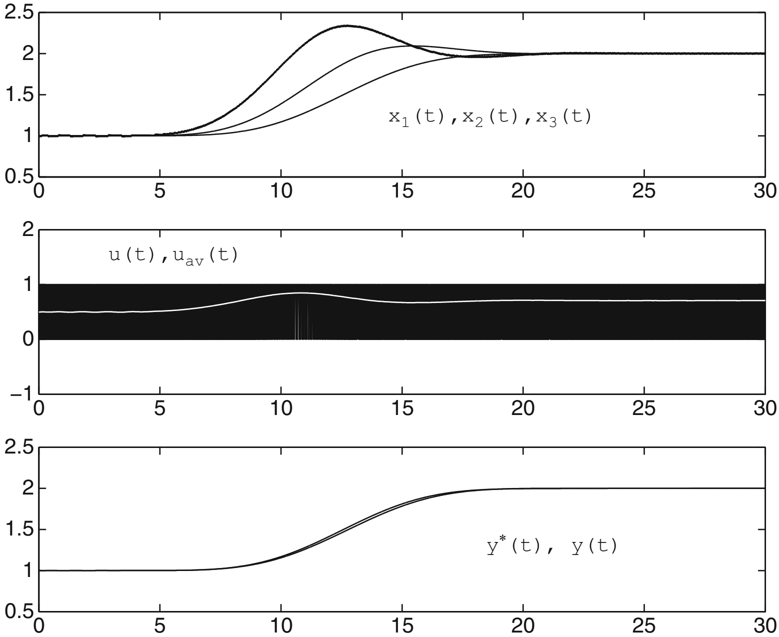


Fig. 6.8. Closed loop response of the three tanks system to Hagenmeyer-Delaleau feed-forward controller combined with GPI controller and $\Sigma - \Delta$ modulator

The natural coordinates for defining the arm movement are the spherical coordinates with a fixed radius of value l_2. The kinetic energy of the system is given by

$$K = \frac{1}{2}m_2\left(\frac{dl}{dt}\right)^2 = \frac{1}{2}m_2 l_2^2\left(\dot{\phi}^2 + \sin^2\phi\,\dot{\theta}^2\right) \tag{6.82}$$

The potential energy, on the other hand, is just

$$U = m_2 g(l_1 + l_2\cos\phi) \tag{6.83}$$

We then have

$$L = \frac{1}{2}m_2 l_2^2\left(\dot{\phi}^2 + \sin^2\phi\,\dot{\theta}^2\right) - m_2 g(l_1 + l_2\cos\phi) \tag{6.84}$$

The equations of motion are

$$\begin{bmatrix} m_2 l_2^2\sin^2\phi & 0 \\ 0 & m_2 l_2^2 \end{bmatrix}\begin{bmatrix}\ddot{\theta} \\ \ddot{\phi}\end{bmatrix}$$
$$+\begin{bmatrix} m_2 l_2^2\dot{\phi}\sin\phi\cos\phi & m_2 l_2^2\dot{\theta}\sin\phi\cos\phi \\ -m_2 l_2^2\dot{\theta}\sin\phi\cos\phi & 0 \end{bmatrix}\begin{bmatrix}\dot{\theta} \\ \dot{\phi}\end{bmatrix}$$
$$+\begin{bmatrix} 0 \\ -m_2 g l_2\sin\phi \end{bmatrix} = \begin{bmatrix}\tau_1 \\ \tau_2\end{bmatrix}$$

The normalized equations of the system can be obtained from the normalized Lagrangian obtained by dividing L by the quantity $m_2 l_2^2$. We get

$$L' = \frac{1}{2}\left(\dot{\phi}^2 + \sin^2\phi\,\dot{\theta}^2\right) - \left(\frac{l_1}{l_2} + \cos\phi\right) \tag{6.85}$$

Using the Euler Lagrange equations we obtain

$$\begin{aligned}&\begin{bmatrix}\sin^2\phi & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\ddot{\theta}\\ \ddot{\phi}\end{bmatrix}\\ &+\begin{bmatrix}\dot{\phi}\sin\phi\cos\phi & \dot{\theta}\sin\phi\cos\phi\\ -\dot{\theta}\sin\phi\cos\phi & 0\end{bmatrix}\begin{bmatrix}\dot{\theta}\\ \dot{\phi}\end{bmatrix}\\ &+\begin{bmatrix}0\\ -\sin\phi\end{bmatrix} = \begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix}\end{aligned}$$

where time differentiation is taken with respect to the normalized time t' and the control inputs are given by

$$t' = t\sqrt{\frac{g}{l_2}}, \quad \vartheta_i = \frac{\tau_i}{m_2 g l_2}, \quad i = 1, 2 \tag{6.86}$$

The normalized control inputs $\vartheta_i, \ i = 1, 2$ take values now on the sets, $\{W_1, 0, -W_1\}$ and $\{W_2, 0, -W_2\}$, respectively. We write the normalized system equations in the following manner:

$$\begin{aligned}\sin^2\phi\,\ddot{\theta} + 2\dot{\phi}\dot{\theta}\sin\phi\cos\phi &= W_1 u_1\\ \ddot{\phi} - \dot{\theta}^2\sin\phi\cos\phi - \sin\phi &= W_2 u_2\end{aligned}$$

where $\vartheta_i = W_i u_i, \ u_i \in \{-1, 0, 1\}, \ i = 1, 2.$

The system is flat with flat outputs given by ϕ and θ. Neglecting for the time being the switched character of the normalized and scaled control inputs u_1 and u_2 we proceed to design a feedback controller as if these control signals were continuously valued.

From the system equations, we readily obtain the following set of differential parametrization of the (average) inputs u_1, u_2 in terms of the flat outputs

$$\begin{aligned}u_1 &= \frac{1}{W_1}\left[\sin^2\phi\,\ddot{\theta} + 2\dot{\phi}\dot{\theta}\sin\phi\cos\phi\right]\\ u_2 &= \frac{1}{W_2}\left[\ddot{\phi} - \dot{\theta}^2\sin\phi\cos\phi - \sin\phi\right]\end{aligned}$$

Given desired trajectories for the position variables θ and ϕ, in the form, $\theta^*(t)$ and $\phi^*(t)$, we can immediately propose an average feedback control laws of a multi-variable version of the Hagenmeyer-Delaleau feedback controllers combined with GPI controllers:

$$\begin{aligned}u_{1\ \mathrm{av}} &= \frac{1}{W_1}\left[\sin^2\phi^*(t)\ v_1 + 2\dot{\phi}^*(t)\dot{\theta}^*(t)\sin\phi^*(t)\cos\phi^*(t)\right]\\ v_1 &= \ddot{\theta}^*(t) - \left[\frac{k_{1\,\theta}s + k_{0\,\theta}}{s + k_{2\,\theta}}\right](\theta - \theta^*(t))\end{aligned}$$

$$u_{2\,\mathrm{av}} = \frac{1}{W_2}\left[v_2 - \left(\dot{\theta}^*(t)\right)^2 \sin\phi^*(t)\cos\phi^*(t) - \sin\phi^*(t)\right]$$
$$v_2 = \ddot{\phi}^*(t) - \left[\frac{k_{1\,\phi}s + k_{0\,\phi}}{s + k_{2\,\phi}}\right](\phi - \phi^*(t)) \tag{6.87}$$

where the design coefficients: $k_{2\,\theta}, k_{1\,\theta}, k_{0\,\theta}$ and $k_{2\,\phi}, k_{1\,\phi}, k_{0\,\phi}$ are chosen so that the closed loop characteristic polynomials

$$p_\theta(s) = s^3 + k_{2\,\theta}s^2 + k_{1\,\theta}s + k_{0\,\theta}$$
$$p_\phi(s) = s^3 + k_{2\,\phi}s^2 + k_{1\,\phi}s + k_{0\,\phi} \tag{6.88}$$

are both Hurwitz.

Note that the normalized coordinates of the end effector, (x, y, z), as a function of the angles θ, ϕ are given by

$$x_n = \sin\phi\sin\theta$$
$$y_n = \sin\phi\cos\theta$$
$$z_n = \epsilon + \cos\phi$$

where $x_n = x/l_2$, $y_n = y/l_2$, $z_n = z/l_2$ y $\epsilon = \frac{l_1}{l_2}$.

From the previous relations, any trajectory of the angular positions of the system given by the evolution of the angular trajectories: $\theta^*(t)$ and $\phi^*(t)$, necessarily satisfies, at each instant t, the equation of the surface of a sphere:

$$x^{*2}(t) + y^{*2}(t) + (z^*(t) - \frac{l_1}{l_2})^2 = l \tag{6.89}$$

As an example, we propose the task of designing a switched controller that allows one to draw over the normalized sphere, centered around $z_n = 1$ and of unit radius, a spiral that starts close to the north pole and evolves towards the south pole with a constant angular velocity around the origin of the plane x_n, y_n.

We propose the following desired trajectory in the robot angular position coordinates,

$$\theta^*(t) = \omega t$$
$$\phi^*(t) = \phi_{init} + (\phi_{final} - \phi_{init})\varphi(t, t_0, T)$$

with $\varphi(t, t_0, T)$ a polynomial function smoothly interpolating between 0 and 1.

We take $\phi_{init} = 0.1$ [rad], $\phi_{final} = 3.04159\ldots$, $t_0 = 0$, $T = 40$ [t.u.], $\omega = 2$ [r/t.u.], ([t.u.] where "[t.u.]" stands for "time units."

The desired closed loop characteristic polynomials were chosen to be equal for the two controllers and of the form

$$p(s) = s^3 + (2\zeta\omega_n + p)s + (2\zeta\omega_n p + \omega_n^2)s + \omega_n^2 p \tag{6.90}$$

with $\zeta = 0.81$, $\omega_n = 0.4$ and $p = 0.4$.

The switching control inputs are synthesized by means of "two-sided" $\Sigma - \Delta$ modulation schemes,

$$\begin{aligned}
\dot{e}_\theta &= u_{1\text{ av}} - u_1 \\
u_1 &= \begin{cases} \frac{1}{2}(1+\text{sign}(e_\theta)) & \text{for } \ u_{1\text{ av}} > 0 \\ -\frac{1}{2}(1-\text{sign}(\text{e}_\theta)) & \text{for } u_{1\text{ av}} < 0 \end{cases} \quad = \frac{1}{2}(\text{sign}(u_{1\,\text{av}}) + \text{sign}(e_\theta)) \\
\dot{e}_\phi &= u_{2\text{ av}} - u_2 \\
u_2 &= \begin{cases} \frac{1}{2}(1+\text{sign}(e_\phi)) & \text{for } \ u_{2\text{ av}} > 0 \\ -\frac{1}{2}(1-\text{sign}(\text{e}_\phi)) & \text{for } u_{2\text{ av}} < 0 \end{cases} \quad = \frac{1}{2}(\text{sign}(u_{2\,\text{av}}) + \text{sign}(e_\phi)
\end{aligned} \tag{6.91}$$

where $e_\theta = \theta - \theta^*(t)$, and $e_\phi = \phi - \phi^*(t)$.

6.8.2 A "chained" mass-spring system

Consider the following chain of n cascaded identical moving masses attached through ideal springs of elasticity constant K as shown in Figure 6.12.

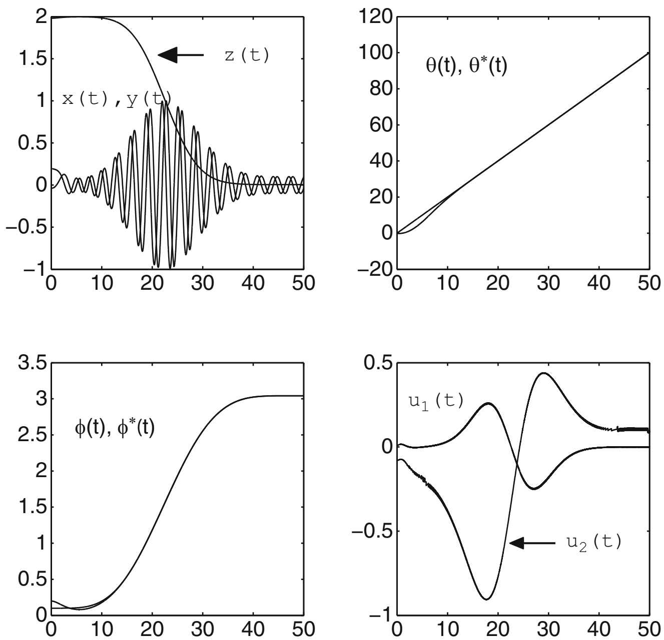


Fig. 6.9. Controlled trajectories of a two degree of freedom robot

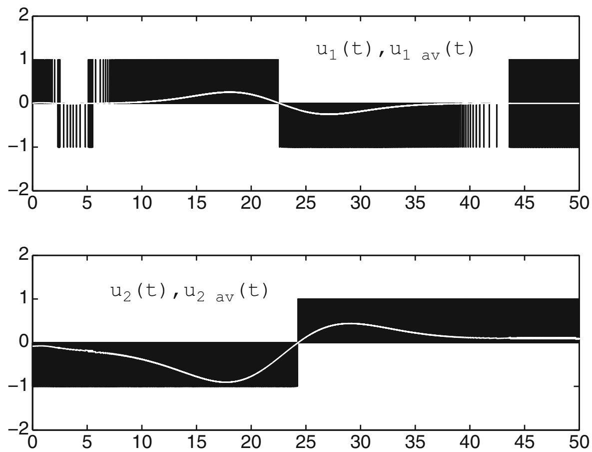


Fig. 6.10. Average control input signals and switch position functions

$$
\begin{aligned}
M\ddot{x}_1 &= K(x_2 - x_1) + f \\
M\ddot{x}_2 &= -K(x_2 - x_1) + K(x_3 - x_2) \\
M\ddot{x}_3 &= -K(x_3 - x_2) + K(x_4 - x_3) \\
&\vdots \\
M\ddot{x}_{n-1} &= -K(x_{n-1} - x_{n-2}) + K(x_n - x_{n-1}) \\
M\ddot{x}_n &= -K(x_n - x_{n-1})
\end{aligned}
$$

The system may be rewritten, after a time scale and input coordinate transformation, as the following normalized system:

$$
\begin{aligned}
\ddot{x}_1 &= -x_1 + x_2 + u \\
\ddot{x}_2 &= x_1 - 2x_2 + x_3 \\
\ddot{x}_3 &= x_2 - 2x_3 + x_4 \\
&\vdots \\
\ddot{x}_{n-1} &= x_{n-2} - 2x_{n-1} + x_n \\
\ddot{x}_n &= x_{n-1} - x_n
\end{aligned}
$$

where $u = f/K$ and the "dot" notation now stands for differentiation with respect to

$$\tau = t\sqrt{\frac{M}{K}} \tag{6.92}$$

Clearly, the flat output is constituted by the position of the n-th car, x_n. The differential parametrization of some of the state variables is given by

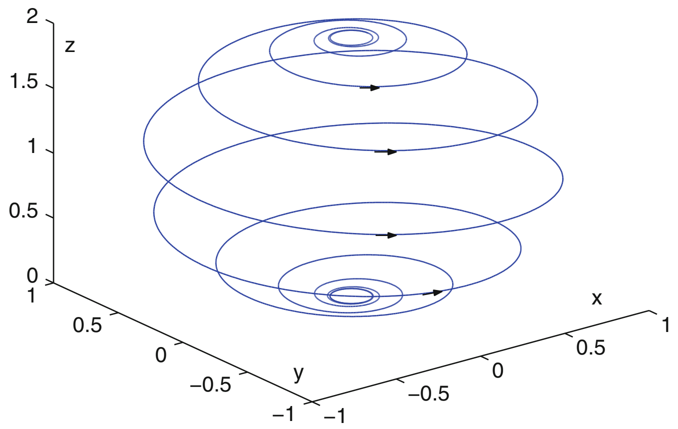


Fig. 6.11. Controlled trajectory in the work space

$$
\begin{aligned}
x_{n-1} &= \ddot{x}_n + x_n \\
x_{n-2} &= x_n^{(4)} + 3\ddot{x}_n + x_n \\
x_{n-3} &= x_n^{(6)} + 5x_n^{(4)} + 6\ddot{x}_n + x_n \\
x_{n-4} &= x_n^{(8)} + 7x_n^{(6)} + 15x_n^{(4)} + 10\ddot{x}_n + x_n \\
&\vdots
\end{aligned}
$$

The differential parametrization of the control input, in terms of the flat output, is given by an expression of the form:

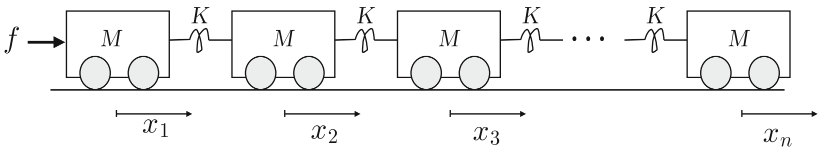


Fig. 6.12. Chained mass-spring system

$$u = x_n^{(2n)} + \alpha_n x_{2n-2}^{(2n-1)} + \cdots + \alpha_1 \ddot{x}_n + \alpha_0 x_n$$

The important fact of this parametrization relies on the fact that for trajectory tracking purposes, we can use the previous ideas of exact feed-forward linearization to propose the following flat output feedback controller:

$$u = u^* - \left[\frac{k_{2n-1}s^{2n-1} + \cdots + k_1 s + k_0}{s^{2n-1} + k_{4n-2}s^{2n-2} + \cdots + k_{2n}} \right] (x_n - x_n^*(t)) \qquad (6.93)$$

The coefficients of the controller are chosen so that the polynomial

$$p(s) = s^{4n-1} + k_{4n-2}s^{4n-2} + \cdots + k_{2n}s^{2n} + k_{2n-1}s^{2n-1} + \cdots + k_1 s + k_0 \quad (6.94)$$

is a Hurwitz polynomial.

If an integral control action is deemed necessary, we use instead

$$u = u^* - \left[\frac{k_{2n}s^{2n} + \cdots + k_1 s + k_0}{s(s^{2n-1} + k_{4n-1}s^{2n-2} + \cdots + k_{2n+1})} \right] (x_n - x_n^*(t)) \quad (6.95)$$

and choose the coefficients of the polynomial:

$$p(s) = s^{4n} + k_{4n-1}s^{4n-1} + \cdots + k_{2n}s^{2n} + k_{2n-1}s^{2n-1} + \cdots + k_1 s + k_0 \quad (6.96)$$

so that it becomes a Hurwitz polynomial.

In the simulations shown in Figure 6.13 we used a three mass example and set the normalized model to perform, for the last mass, a rest-to-rest maneuver from an initial 0 [m] position towards a final position of 0.10 [m] in 7.5 normalized time units. The controller parameters were chosen using the following polynomial:

$$p(s) = (s^2 + 2\zeta\omega_n s + \omega_n^2)^6 \quad (6.97)$$

i.e.,

$$\begin{aligned}
k_{11} &= 12\zeta\omega \\
k_{10} &= 60\zeta^2\omega^2 + 6\omega^2 \\
k_9 &= 160\zeta^3\omega^3 + 60\zeta\omega^3 \\
k_8 &= 240\zeta^4\omega^4 + 240\zeta^2\omega^4 + 15\omega^4 \\
k_7 &= 480\zeta^3\omega^5 + 120\zeta\omega^5 + 192\zeta^5\omega^5 \\
k_6 &= 20\omega^6 + 64\zeta^6\omega^6 + 360\zeta^2\omega^6 + 480\zeta^4\omega^6 \\
k_5 &= 120\zeta\omega^7 + 192\zeta^5\omega^7 + 480\zeta^3\omega^7 \\
k_4 &= 15\omega^8 + 240\zeta^2\omega^8 + 240\zeta^4\omega^8 \\
k_3 &= 60\zeta\omega^9 + 160\zeta^3\omega^9 \\
k_2 &= (6\omega^10 + 60\zeta^2\omega^{10} \\
k_1 &= 12\zeta\omega^{11} \\
k_0 &= \omega^{12}
\end{aligned}$$

with $\zeta = 0.81$, $\omega_n = 1$. The nominal control input was found to be

$$u^*(t) = x_3^{(6)}(t) + 4x_3^{(4)}(t) + 3\ddot{x}_3(t) \quad (6.98)$$

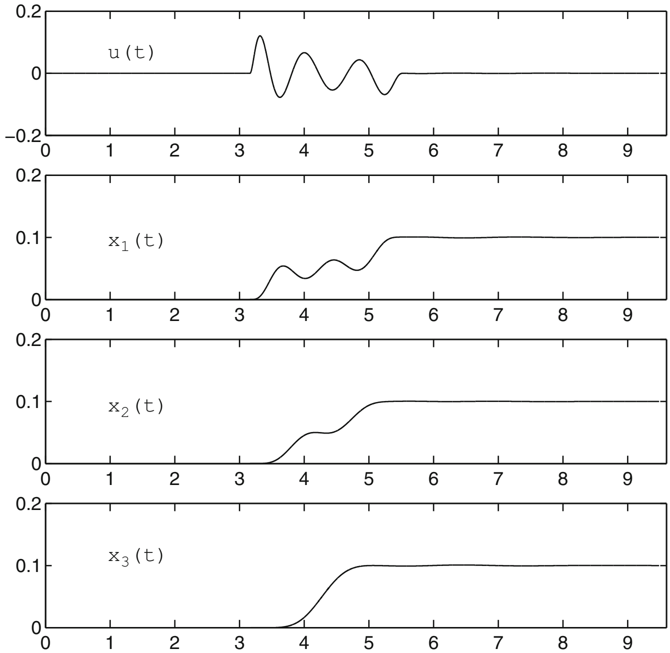


Fig. 6.13. Closed loop performance of mass-spring system

6.8.3 A single link-DC motor system

Consider the following combined model of a rigid link manipulator and a DC motor actuator.

$$
\begin{aligned}
[J + ml_{c1} + I]\,\ddot{\theta} + ml_{c1} g \sin\theta &= k_t i \\
L\frac{di}{dt} &= uV - Ri - \lambda_0 \dot{\theta}
\end{aligned}
\tag{6.99}
$$

It is desired to track a pre-specified angular position trajectory given by $\theta^*(t)$. The applied control input voltage to the motor's armature circuit takes values in the discrete set $\{-V, 0, V\}$. Thus, the switched control signal u takes values in $\{-1, 0, 1\}$. We first design an average GPI controller combined with a Hagenmeyer-Delaleau exact feed-forward controller assuming u is continuous valued over the real line.

The system is flat, with flat output given by the angular position θ. The input to flat output relation is obtained as

$$
\begin{aligned}
u = {} & \frac{L}{Vk_t}\,[J + ml_{c1} + I]\,\theta^{(3)} + \frac{R}{Vk_t}\,[J + ml_{c1} + I]\,\ddot{\theta} + \left[\frac{ml_{c1} g L}{Vk_t}\right]\dot{\theta}\cos\theta + \\
& + \left[\frac{mgl_{c1} R}{Vk_t}\right]\sin\theta + \frac{\lambda_0}{V}\dot{\theta}
\end{aligned}
\tag{6.100}
$$

A linear time-varying output feedback controller producing the average control is thus given by

$$\begin{aligned} u_{av} &= \frac{L}{Vk_t}\left[J+ml_{c1}+I\right]v+\left[\frac{ml_{c1}gL}{Vk_t}\right]\dot{\theta}^*(t)\cos\theta^*(t) \\ &\quad +\frac{R}{Vk_t}\left[J+ml_{c1}+I\right]\ddot{\theta}^*(t)+\left[\frac{ml_{c1}gR}{Vk_t}\right]\sin\theta^*(t)+\frac{\lambda_0}{V}\dot{\theta}^*(t) \\ v &= [\theta^*(t)]^{(3)}-\left[\frac{k_2s^2+k_1s+k_0}{s^2+k_4s+k_3}\right](\theta-\theta^*(t)) \end{aligned} \tag{6.101}$$

This controller is expressed in a simpler form as follows:

$$u_{av} = u^*(t)+\left[\frac{k_2s^2+k_1s+k_0}{s^2+k_4s+k_3}\right](\theta^*(t)-\theta) \tag{6.102}$$

The switched control is implemented via a two-sided $\Sigma-\Delta$ modulator

$$\begin{aligned} \dot{e} &= u_{av}(t)-u \\ u &= \frac{1}{2}\left(\text{sign}(u_{av}(t))+\text{sign}(e)\right) \end{aligned} \tag{6.103}$$

For the computer simulation, shown in Figure 6.14, we used the following parameter values for the robot and for the motor. The DC motor parameters were taken from Utkin *et al.* [33].

$$I=0.02,\ l_{c1}:0.3,\ g=9.8,\ m=0.5,\ L=10^{-3},\ R=0.5,$$

$$J=10^{-3},\ k_t=8\times10^{-3},\ \lambda_0=10^{-3}$$

6.8.4 A multilevel Buck DC to DC converter controller design

Consider the following normalized model of a double bridge multilevel Buck converter

$$\begin{aligned} \dot{x}_1 &= -x_2+u \\ \dot{x}_2 &= x_1-\frac{x_2}{Q} \\ y &= x_2 \end{aligned} \tag{6.104}$$

where the switched control input u takes values in the discrete set $\{-3W,-2W,-W,0,W,2W,3W\}$ with W being a fixed real number satisfying: $3W\leq 1$ and, hence $-3W\geq -1$, which represents the granularity of the multilevel switch u.

It is desired to produce, by means of an output voltage trajectory tracking task, a sinusoidal wave of amplitude bounded by the normalized voltage 1 and of a suitable frequency which does not cause saturation of the hard limits imposed on the switching controller.

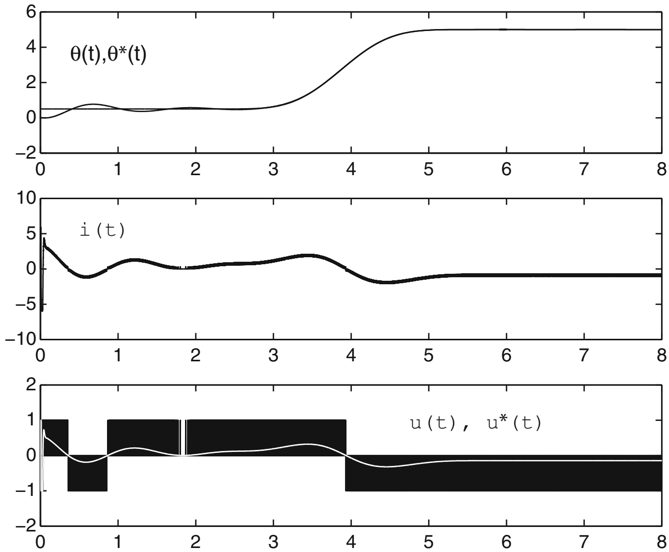


Fig. 6.14. Sliding mode controlled DC motor-link system

We tackle the problem in an average context first by assuming the control input u_{av} takes values continuously in the interval $[-1, 1]$ and consider the average system with the obvious abuse of notation:

$$\begin{aligned} \dot{x}_1 &= -x_2 + u_{av} \\ \dot{x}_2 &= x_1 - \frac{x_2}{Q} \\ y &= x_2 \end{aligned} \tag{6.105}$$

Once the trajectory tracking problem is solved, without average control input saturations, we proceed to implement the average feedback law by means of a multilevel $\Sigma - \Delta$ modulator

The average input-output normalized description of the system is given by

$$\ddot{y} + \frac{1}{Q}\dot{y} + y = u \tag{6.106}$$

Let $u^*(\tau)$ denote the average nominal control input given by

$$u^*(\tau) = \ddot{y}^*(\tau) + \frac{1}{Q}\dot{y}^*(\tau) + y^*(\tau) \tag{6.107}$$

For the particular case of a sinusoidal signal of the form $y^*(t) = A\sin(\omega\tau)$ the nominal average control input, $u^*_{av}(\tau)$, is computed to be

$$u^*_{av}(\tau) = A(1 - \omega^2)\sin(\omega\tau) + \frac{A\omega}{Q}\cos(\omega\tau) = M\sin(\omega\tau + \phi) \tag{6.108}$$

where

$$M = A\sqrt{(1-\omega^2)^2 + \omega^2/Q^2} \tag{6.109}$$

and

$$\phi = \arctan\left(\frac{\omega}{Q(1-\omega^2)}\right) \tag{6.110}$$

As usual in this DC to AC power conversion schemes, we have the following tradeoff between the normalized output voltage amplitudes and the desired normalized angular frequency of the desired sinusoidal output:

$$A^2\left[(1-\omega^2)^2 + \frac{\omega^2}{Q^2}\right] \;<\; 1 \tag{6.111}$$

The choice of the normalized amplitude, A, and the normalized angular frequency, ω, satisfying the above restriction, leads to a nominal average input signal u_{av}^* which does not saturate the controller in steady state operation (Fig. 6.15).

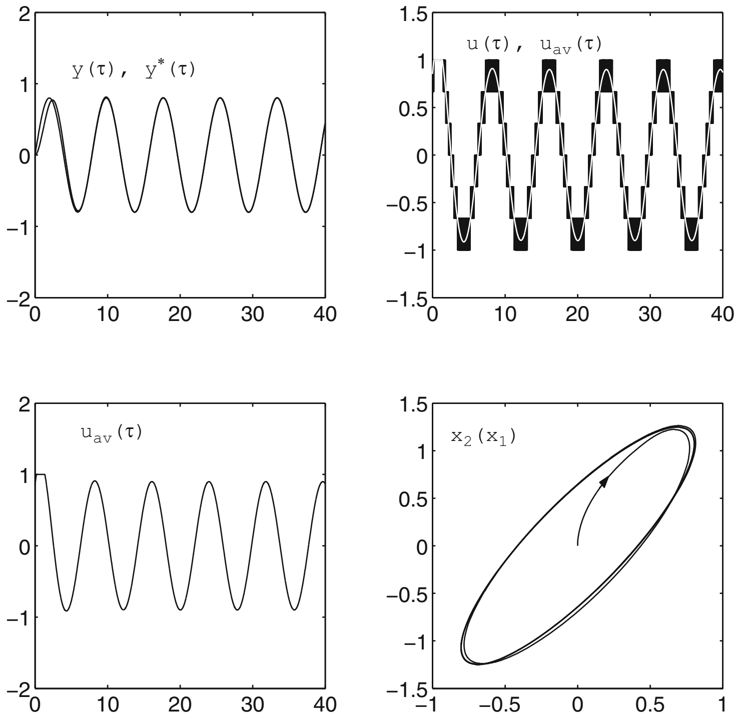


Fig. 6.15. GPI average controlled multilevel DC to AC Buck converter using a six-level $\Sigma - \Delta$ modulator for implementation

We propose the following GPI controller

$$u_{av} = u_{av}^*(\tau) - \left[\frac{k_2 s^2 + k_1 s + k_0}{s(s+k_3)}\right](y - y^*(\tau)) \tag{6.112}$$

The characteristic polynomial associated with the average closed loop system is found to be

$$p(s) = s^4 + (k_3 + \frac{1}{Q})s^3 + (k_2 + \frac{k_3}{Q} + 1)s^2 + (k_3 + k_1)s + k_0 \tag{6.113}$$

One readily obtains the design coefficients $\{k_3, k_2, k_1, k_0\}$ by a term by term identification of this polynomial with the desired polynomial:

$$p_d(s) = (s^2 + 2\zeta\omega_n s + \omega_n^2)^2 = s^4 + 4\zeta\omega_n s^3 + (2\omega_n^2 + 4\zeta^2\omega_n^2)s^2 + 4\omega_n^3\zeta s + \omega_n^4 \tag{6.114}$$

A six-level, two-phase, $\Sigma-\Delta$ modulator is proposed for the implementation of the switched input u on the basis of the GPI designed average feedback control input u_{av}. We use

$$\begin{aligned} \dot{e} &= u_{av}(\tau) - u \\ u &= \frac{W}{2}\left\{\sum_{-2}^{3}\left[2j - 1 + \mathrm{sign}(e)\right] f_j(\tau)\right\} \\ f_j &= \frac{1}{2}\left[\mathrm{sign}\left(u_{av}(\tau) - (j-1)W\right) - \mathrm{sign}\left(u_{av}(\tau) - jW\right)\right] \end{aligned} \tag{6.115}$$

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