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*Ravi P. Agarwal: To my wife Sadhna.*

*Erdal Karapınar: To my wife Senem Pınar and our child Ulaş Ege.*

*Donal O'Regan: To my wife Alice and our children Aoife, Lorna, Daniel, and Niamh.*

*Antonio F. Roldán-López-de-Hierro: To my wife María José, our daughters Ana and Sofía, and my parents María Dolores and Antonio.*

# Preface

Fixed-point theory is one of the major research areas in nonlinear analysis. This is partly due to the fact that in many real-world problems, fixed-point theory is the basic mathematical tool used to establish the existence of solutions to problems which arise naturally in applications. As a result, fixed-point theory is an important area of study in pure and applied mathematics, and it is a flourishing area of research. As the title states, this is a book on metric fixed-point theory where the basic ideas come from metric space topology. We present a self-contained account of the theory (techniques and results) in metric-type spaces (in particular in $G$-metric spaces).

The book consists of 12 chapters. The first three chapters present some preliminaries and historical notes on metric spaces (in particular $G$-metric spaces) and on mappings. A variety of Banach-type contraction theorems in metric-type spaces are established in Chaps. 4, 6, 7, and 8. Fixed-point theory in partially ordered $G$-metric spaces is discussed in Chaps. 5 and 8. Fixed-point theory for expansive mappings in metric-type spaces is presented in Chap. 9. The final three chapters discuss generalizations and present results and techniques in a very general abstract setting and framework.

We would like to express our thanks to our family and friends.

Ravi P. Agarwal, Erdal Karapınar, Donal O'Regan and Antonio F. Roldán-López-de-Hierro.

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# Special Symbols

| Symbol | Name |
|---|---|
| $X$ | A nonempty set |
| $X \subseteq Y$ | $X$ is a subset of $Y$ |
| . $X \subset Y$ | $X$ is a subset of $Y$ and $X$ is different from $Y$ |
| . $\mathbb{N} = \{0, 1, 2, \ldots\}$ | Set of nonnegative integer numbers |
| $\mathbb{R}$ | Set of all real numbers |
| $[0, \infty)$ | Interval of all nonnegative real numbers |
| $(0, \infty)$ | Interval of positive real numbers |
| $[a, b]$ | Interval of all real numbers $x$ such that $a \leq x \leq b$ |
| $f : X \to Y$ | $f$ is a mapping from $X$ into $Y$ |
| . $\circ$ | Composition |
| $\{x_n\} \subseteq X$ | A sequence in $X$ |
| $\{x_n\} \to x$ | The sequence $\{x_n\}$ converges to $x$ |
| . $G$ | $G$-metric |
| $\mathrm{Fix}(T)$ | Set of all fixed points of $T : X \to X$ |
| $\mathrm{Co}(T, g)$ | Set of all coincidence points of $T, g : X \to X$ |

# Chapter 1
# Introduction with a Brief Historical Survey

In 1906, Fréchet [78] gave the formal definition of the distance[1] by introducing a function $d$ that assigns a nonnegative real number $d(x, y)$ (the distance between $x$ and $y$) to every pair $(x, y)$ of elements (points) of a nonempty set $X$. It is assumed that this function satisfies the following conditions:

$(d1)$ $d(x, y) = 0$ if $x$ and $y$ coincide;
$(d2)$ $d(x, y) > 0$ if $x$ and $y$ are distinct;
$(d3)$ $d(x, y) = d(y, x)$ for all $x$ and $y$ in $X$;
$(d4)$ $d(x, y) \leq d(x, z) + d(z, y)$ for all $x$, $y$ and $z$ in $X$.

The pair $(X, d)$ is called a metric space.

## 1.1 2-Metric Spaces

In the sixties, the notion of a 2-metric space was introduced by Gähler [79, 80] in a series of papers which he claimed to be a generalization of ordinary metric spaces. This structure is as follows:

Let $X$ be a nonempty set. A function $d : X \times X \longrightarrow \mathbb{R}_+$ is said to be a 2-metric on $X$ if it satisfies the following properties:

$(t1)$ For distinct points $x, y \in X$, there is a point $z \in X$ such that $d(x, y, z) \neq 0$,
$(t2)$ $d(x, y, z) = 0$ if any two elements of the triplet $(x, y, z)$ are equal,
$(t3)$ $d(x, y, z) = d(x, z, y) = \ldots$, (symmetry),
$(t4)$ $d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z)$ for all $x, y, z \in X$, (triangle inequality).

[1] It was called *metric* first by F. Hausdorff [92].



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A nonempty set $X$ together with a 2-metric $d$ is called a 2-metric space.

In [79], Gähler claimed that a 2-metric function is a generalization of an ordinary metric function. Ha et al. in [90] showed that a 2-metric need not be a continuous function of its variables. In particular the contraction mapping theorem in metric spaces and in 2-metric spaces are unrelated. Dhage [64] introduced a new structure of a generalized metric space called a $D$-metric space.

## 1.2 *D*-Metric Spaces

**Definition 1.2.1.** A nonempty set $X$, together with a function $D : X \times X \times X \to [0, \infty)$ is called a $D$-metric space, denoted by $(X, D)$ if $D$ satisfies

$(i)$ $D(x, y, z) = 0$ if and only if $x = y = z$, (coincidence),
$(ii)$ $D(x, y, z) = D(px, y, z)$, where $p$ is a permutation of $x, y, z$ (*symmetry*),
$(iii)$ $D(x, y, z) \le D(x, y, a) + D(x, a, z) + D(a, y, z)$ for all $x, y, z, a \in X$, (*tetrahedral inequality*).

The nonnegative real function $D$ is called a $D$-metric on $X$. The set $X$ together with such a generalized metric $D$ is called a generalized metric space, or $D$-metric space, and denoted by $(X, D)$.

An additional property sometimes imposed on a $D$-metric (see [65]) is,

$$D(x, y, y) < D(x, z, z) + D(z, y, y) \quad \text{for all } \ x, y, z \in X.$$

If $D(x, x, y) = D(x, y, y)$ for all $x, y \in X$, then $D$ is referred to as a symmetric $D$-metric.

The perimeter of a triangle of vertices $x, y, z$ in $\mathbb{R}^2$ provided the typical example of a $D$-metric. Dhage [64] also gave the following examples of $D$-metrics:

$$(E_s) \quad D_s(d)(x, y, z) = \frac{1}{3}(d(x, y) + d(y, z) + d(z, x)), \qquad \textit{and}$$

$$(E_m) \quad D_m(d)(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\},$$

where $(X, d)$ is a metric space and $x, y, z \in X$.

**Definition 1.2.2.** In a $D$-metric space $(X, D)$, three possible notions for the convergence of a sequence $\{x_n\}$ to a point $x$ suggest themselves:

$$(C1) \qquad x_n \to x \quad \text{if} \quad D(x_n, x, x) \to 0 \quad \text{as} \quad n \to \infty,$$

$$(C2) \qquad x_n \to x \quad \text{if} \quad D(x_n, x_n, x) \to 0 \quad \text{as} \quad n \to \infty,$$

$$(C3) \qquad x_n \to x \quad \text{if} \quad D(x_m, x_n, x) \to 0 \quad \text{as} \quad m, n \to \infty,$$

Clearly, $(C3) \to (C2)$ and if $D$ is symmetric then $(C1) \leftrightarrow (C2)$. No other implications are true in general. For more details, see the works of Mustafa and Sims [142, 154].

In [64], Dhage also defined Cauchy sequences in a $D$-metric space as follows.

**Definition 1.2.3.** A sequence $\{x_n\} \subseteq X$ is said to be $D$-Cauchy if, for each $\varepsilon > 0$, there exists a positive integer $n_0$ such that, for all $m > n > p \geq n_0, D(x_m, x_n, x_p) < \varepsilon$.

In [64], Dhage mentioned the possibility of defining two topologies, denoted by $\tau^*$ and $\tau$, in any $D$-metric space, with convergence in the sense of $(C3)$ corresponding to convergence in the $\tau$-topology. More details were presented in two subsequent papers, [68] and [65].

## 1.3 Some Problems with $D$-Metric Spaces

In [64], the $\tau^*$-topology is generated by the family of open balls of the form

$$(B1) \quad B^*(x,r) := \{y \in X, \ \ D(x,y,y) < r\}$$

where $x \in X$ and $r > 0$.

The convergence of a sequence in the $\tau^*$-topology is equivalent to its $(C2)$ convergence. However, in [65], where the $\tau^*$-topology was discussed, $D$-metric convergence of a sequence is taken to mean that it converges in both the sense of $(C2)$ and $(C1)$, and it is claimed that the $D$-metric topology (here the $\tau^*$-topology) is the same as the topology of $D$-metric convergence of sequences in $X$ (in the sense of $(C1)$ and $(C2)$ convergence). This claim is not true; see Mustafa and Sims [154] where some examples are presented to affirm this assertion. Thus, the notion of a $D$-metric convergence is stronger than convergence in the $\tau^*$-topology. However, Mustafa and Sims [154] tried to correct this by taking convergence to mean only in the sense of $(C2)$, but they encountered a new problem; namely, they constructed a sequence $\{x_n\}$ which is convergent (in the sense of $(C2)$), but is not $D$-Cauchy.

The first attempt to define a $\tau$-topology [64] is as follows.

$$(B2) \quad B(x,r) := \cap_{z \in X}\{y, z \in X, \ \ D(x,y,z) < r\}$$

where $x \in X$ and $r > 0$. A second attempt [68] in order to modify the definition of $\tau$-topology is

$$(B2)' \quad B(x,r) := \{y, z \in X, \ \ D(x,y,z) < r\}.$$

Dhage (Theorem 6.1.2 in [68]) claimed that if a $D$-Cauchy sequence of points in a $D$-metric space contains a convergent (in the sense of $(C3)$) subsequence, then the sequence is itself convergent. However, Mustafa and Sims [154] provided a concrete example showing that this is not generally valid.

Moreover, Dhage [68] took the distance between a point $x$ and a subset $A$ of $(X, D)$ to be

$$d(x, x, A) =: \inf\{D(x, x, a), \quad a \in X\}$$

and claimed that the function $f(x) := d(x, x, A)$ is continuous in both the $\tau$-topology [68], and the $\tau^*$-topology [65]. However, the proofs of Lemma 5.1 in [68] and Lemma 1.2 in [65] rely on the continuity of $D$ in the respective topologies and also contain errors; for more details see [142, 154]. Mustafa and Sims [154] showed also that even a symmetric $D$-metric arising from a semi-metric, need not be a continuous function of its variables with respect to convergence in the sense of $(C3)$, contrary to the claim in [64], Lemma 2.1.

# Chapter 2
# Preliminaries

In this section we present fundamental definitions and elementary results (see Apostol [23], Bourbaki [51], and Schweizer and Sklar [186]).

## 2.1 Sets, Mappings and Sequences

In the sequel, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ denotes the set of all nonnegative integers, $\mathbb{R}$ denotes the set of all real numbers and $[0, \infty)$ (respectively, $(0, \infty)$) denotes the interval of nonnegative (respectively, positive) reals. The *absolute value* $|x|$ of a real number $x$ is the maximum between $x$ and $-x$, that is, $|x| = \max\{x, -x\}$.

Henceforth, $X$ and $Y$ will denote nonempty sets. Elements of $X$ are usually called *points*. Given a positive integer $n$, we use $X^n$ to denote the *nth Cartesian power of* $X$, that is, $X \times X \times \ldots \times X$ ($n$ times).

Let $f : X \to Y$ be a mapping. The *domain of* $f$ is $X$ and it is denoted by $\mathrm{Dom} f$. Its *range*, that is, the set of values of $f$ in $Y$, is denoted by $f(X)$ or by $\mathrm{Ran} f$. A mapping $f$ is completely characterized by its domain, its range, and the manner in which each *origin* $x \in \mathrm{Dom} f$ is applied on its *image* $f(x) \in f(X)$. For any set $X$, we denote the *identity mapping on* $X$ by $I_X : X \to X$, which is defined by $I_X(x) = x$ for all $x \in X$.

A mapping $f : X \to Y$ is said to be:

- *injective* (or *one to one*) if $x = y$ for all $x, y \in X$ such that $f(x) = f(y)$;
- *injective on a subset* $U \subseteq X$ if $x = y$ for all $x, y \in U$ such that $f(x) = f(y)$;
- *surjective* (or *onto*) *on a subset* $V \subseteq Y$ if for all $y \in V$, there exists $x \in X$ such that $f(x) = y$;
- *surjective* (or *onto*) if for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$;
- *bijective* if it is both injective and surjective.

**Proposition 2.1.1.** *If $T : X \to X$ is onto, then there exists a mapping $T' : X \to X$ such that $T \circ T'$ is the identity mapping on $X$.*



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*Proof.* For any point $x \in X$, let $y_x \in X$ be any point such that $T(y_x) = x$. Let $T'(x) = y_x$ for all $x \in X$. Then $T(T'(x)) = T(y_x) = x$ for all $x \in X$. □

Given two mappings $f : X \to Y$ and $g : Y \to Z$, the *composite of f and g* is the mapping $g \circ f : X \to Z$ given by

$$(g \circ f)(x) = g(f(x)) \quad \text{for all } x \in \operatorname{Dom} f.$$

We say that two self-mappings $f, g : X \to X$ are *commuting* if $f(g(x)) = g(f(x))$ for all $x \in X$ (that is, $f \circ g = g \circ f$).

The *iterates* of a self-mapping $f : X \to X$ are the mappings $\{f^n : X \to X\}_{n \in \mathbb{N}}$ defined by

$$f^0 = I_X, \quad f^1 = f, \quad f^2 = f \circ f, \quad f^{n+1} = f \circ f^n \quad \text{for all } n \geq 2.$$

If $f : X \to X$ is a self-mapping, the *orbit $O_f(x)$ of a point $x \in X$* is

$$O_f(x) = \{f^n(x) : n \in \mathbb{N}\} = \{x, f(x), f^2(x), f^3(x), \ldots\}.$$

The following facts are basic notions and properties about sequences. A *sequence in the set X* is a function $x : \mathbb{N} \to X$. The point $x(n) \in X$ will be denoted by $x_n$, and the sequence $x$ will be denoted by $\{x_n\}_{n \in \mathbb{N}}$ or, simply, by $\{x_n\}$ (we will use both notations). We will write $\{x_n\} \subseteq X$ to clarify that $\{x_n\}$ is a sequence whose terms are points of $X$. A *subsequence of* $\{x_n\}_{n \in \mathbb{N}}$ is a sequence $\{x_{m(k)}\}_{k \in \mathbb{N}}$, where $m : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function, that is, $m(k) < m(k+1)$ for all $k \in \mathbb{N}$. Given $n_0 \in \mathbb{N}$, the sequence $\{x_n\}_{n \geq n_0}$ is the subsequence $y : \mathbb{N} \to X$ defined by $y(n) = x(n + n_0)$ for all $n \in \mathbb{N}$.

A *sequence of real numbers* is a sequence $\{a_n\} \subset \mathbb{R}$ (symbol $\subset$ is used to denote that $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$ but we know that it is impossible that $\{a_n : n \in \mathbb{N}\}$ is the whole set $\mathbb{R}$). We will say that:

- $\{a_n\}$ is a *Cauchy sequence* if for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - a_m| \leq \varepsilon$ for all $n, m \geq n_0$:
- $\{a_n\}$ *converges to* $L \in \mathbb{R}$ (and we will denote it by $\{a_n\} \to L$) if for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| \leq \varepsilon$ for all $n \geq n_0$.

One of the most useful properties in analysis is known as the *squeeze theorem* or the *sandwich lemma*.

**Lemma 2.1.1.** *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three sequences of real numbers such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. If there exists $L \in \mathbb{R}$ such that $\{a_n\} \to L$ and $\{c_n\} \to L$, then $\{b_n\} \to L$.*

*Proof.* Let $\varepsilon > 0$ be arbitrary. Since $\{a_n\} \to L$, there exists $n_1 \in \mathbb{N}$ such that $|a_n - L| \leq \varepsilon$ for all $n \geq n_1$. Similarly, as $\{c_n\} \to L$, there exists $n_2 \in \mathbb{N}$ such that $|c_n - L| \leq \varepsilon$ for all $n \geq n_2$. Let $n_0 = \max\{n_1, n_2\} \in \mathbb{N}$. Therefore, if $n \in \mathbb{N}$ is such that $n \geq n_0$, then

$$b_n - L \leq c_n - L \leq |c_n - L| \leq \varepsilon \quad \text{and}$$
$$L - b_n \leq L - a_n \leq |a_n - L| \leq \varepsilon.$$

As a consequence,

$$|b_n - L| = \max\{b_n - L, L - b_n\} \leq \varepsilon$$

for all $n \geq n_0$, which means that $\{b_n\} \to L$. □

**Corollary 2.1.1.** *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of non-negative real numbers such that $a_n \leq b_n$ for all $n \in \mathbb{N}$. If $\{b_n\} \to 0$, then $\{a_n\} \to 0$.*

**Corollary 2.1.2.** *Let $\{a_n\}, \{b_n\}, \{c_n\} \subset [0, \infty)$ be three sequences of non-negative real numbers such that $\{\max(a_n, b_n)\} \to 0$ and $\{\max(a_n, b_n, c_n)\} \to L$, where $L \in [0, \infty)$. Then $\{c_n\} \to L$.*

*Proof.* Notice that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
0 \leq |L - c_n| &\leq |L - \max(a_n, b_n, c_n)| + |\max(a_n, b_n, c_n) - c_n| \\
&\leq |L - \max(a_n, b_n, c_n)| + \max(a_n, b_n, c_n) - c_n \\
&\leq |L - \max(a_n, b_n, c_n)| + \max(a_n, b_n) + c_n - c_n \\
&= |L - \max(a_n, b_n, c_n)| + \max(a_n, b_n).
\end{aligned}$$

Since $\{|L - \max(a_n, b_n, c_n)| + \max(a_n, b_n)\}_{n\in\mathbb{N}} \to 0$, then Corollary 2.1.1 implies that $\{|L - c_n|\}_{n\in\mathbb{N}} \to 0$, which means that $\{c_n\}_{n\in\mathbb{N}} \to L$. □

**Corollary 2.1.3.** *Let $\{a_n^1\}_{n\in\mathbb{N}}, \ldots, \{a_n^N\}_{n\in\mathbb{N}} \subset [0, \infty)$ be $N$ sequences of non-negative real numbers such that $\{\max(a_n^1, \ldots, a_n^N)\}_{n\in\mathbb{N}} \to 0$. Then $\{a_n^i\}_{n\in\mathbb{N}} \to 0$ for all $i \in \{1, 2, \ldots, N\}$.*

*Proof.* It follows from Corollary 2.1.1 taking into account that $0 \leq a_n^i \leq b_n$ for all $i \in \{1, 2, \ldots, N\}$ and all $n \in \mathbb{N}$, where $b_n = \max(a_n^1, \ldots, a_n^N)$ for all $n \in \mathbb{N}$. □

If the maximum does not necessarily converge to zero, then we have the following statement.

**Lemma 2.1.2.** *Let $\{a_n^1\}_{n\in\mathbb{N}}, \ldots, \{a_n^N\}_{n\in\mathbb{N}}$ be $N$ real lower bounded sequences such that $\{\max(a_n^1, \ldots, a_n^N)\}_{n\in\mathbb{N}} \to \delta$. Then there exists $i_0 \in \{1, 2, \ldots, N\}$ and a subsequence $\{a_{n(k)}^{i_0}\}_{k\in\mathbb{N}}$ such that $\{a_{n(k)}^{i_0}\}_{k\in\mathbb{N}} \to \delta$.*

*Proof.* Let $b_n = \max(a_n^1, a_n^2, \ldots, a_n^N)$ for all $n \in \mathbb{N}$. As $\{b_n\}$ is convergent, it is bounded. As $a_n^i \leq b_n$ for all $n \in \mathbb{N}$ and $i \in \{1, 2, \ldots, N\}$, then every $\{a_n^i\}$ is bounded. As $\{a_n^1\}_{n\in\mathbb{N}}$ is a real bounded sequence, it has a convergent subsequence $\{a_{\sigma_1(n)}^1\}_{n\in\mathbb{N}} \to a_1$. Consider the subsequences $\{a_{\sigma_1(n)}^2\}_{n\in\mathbb{N}}, \{a_{\sigma_1(n)}^3\}_{n\in\mathbb{N}}, \ldots, \{a_{\sigma_1(n)}^N\}_{n\in\mathbb{N}}$, that are $N-1$ real bounded sequences, and the sequence $\{b_{\sigma_1(n)}\}_{n\in\mathbb{N}}$ that also converges to $\delta$. As $\{a_{\sigma_1(n)}^2\}_{n\in\mathbb{N}}$ is a real bounded sequence, it has a convergent subsequence $\{a^2_{\sigma_2\sigma_1(n)}\}_{n\in\mathbb{N}} \to a_2$. Then the sequences $\{a^3_{\sigma_2\sigma_1(n)}\}_{n\in\mathbb{N}}$, $\{a^4_{\sigma_2\sigma_1(n)}\}_{n\in\mathbb{N}}$, ..., $\{a^N_{\sigma_2\sigma_1(n)}\}_{n\in\mathbb{N}}$ also are $N-2$ real bounded sequences and $\{a^1_{\sigma_2\sigma_1(n)}\}_{n\in\mathbb{N}} \to a_1$ and $\{b_{\sigma_2\sigma_1(n)}\}_{n\in\mathbb{N}} \to \delta$. Repeating this process $N$ times, we can find $N$ subsequences $\{a^1_{\sigma(n)}\}_{n\in\mathbb{N}}$, $\{a^2_{\sigma(n)}\}_{n\in\mathbb{N}}, \ldots, \{a^n_{\sigma(n)}\}_{n\in\mathbb{N}}$ (where $\sigma = \sigma_n \ldots \sigma_1$) such that $\{a^i_{\sigma(n)}\}_{n\in\mathbb{N}} \to a_i$ for all $i \in \{1,2,\ldots,N\}$. Now $\{b_{\sigma(n)}\}_{n\in\mathbb{N}} \to \delta$. Note

$$\{b_{\sigma(n)}\}_{n\in\mathbb{N}} = \{\max(a^1_{\sigma(n)},\ldots,a^N_{\sigma(n)})\}_{n\in\mathbb{N}} \to \max(a_1,\ldots,a_N),$$

so $\delta = \max(a_1,\ldots,a_N)$ and there exists $i_0 \in \{1,2,\ldots,N\}$ such that $a_{i_0} = \delta$. Therefore, there exists $i_0 \in \{1,2,\ldots,N\}$ and a subsequence $\{a^{i_0}_{\sigma(n)}\}_{n\in\mathbb{N}}$ such that $\{a^{i_0}_{\sigma(n)}\}_{n\in\mathbb{N}} \to a_{i_0} = \delta$. □

**Lemma 2.1.3.** *Let* $\{a^1_n\}_{n\in\mathbb{N}}, \{a^2_n\}_{n\in\mathbb{N}}, \ldots, \{a^N_n\}_{n\in\mathbb{N}} \subset [0,\infty)$ *be* $N$ *sequences of nonnegative real numbers and assume that there exists* $\lambda \in [0,1)$ *such that*

$$a^1_{n+1} + a^2_{n+1} + \ldots + a^N_{n+1} \le \lambda\left(a^1_n + a^2_n + \ldots + a^N_n\right) \quad \textit{for all } n \in \mathbb{N}.$$

*Then* $\{a^i_n\}_{n\in\mathbb{N}} \to 0$ *for all* $i \in \{1,2,\ldots,N\}$.

*Proof.* Let $b_n = a^1_n + a^2_n + \ldots + a^N_n$ for all $n \in \mathbb{N}$. Then we have that $b_n \le \lambda b_{n-1} \le \lambda^2 b_{n-2} \le \ldots \le \lambda^n b_0$. If $b_0 = 0$, then $b_n = 0$ for all $n \in \mathbb{N}$ and, in particular, $a^i_n = 0$ for all $n \in \mathbb{N}$ and all $i \in \{1,2,\ldots,N\}$. Hence $\{a^i_n\} \to 0$ for all $i \in \{1,2,\ldots,N\}$. Suppose that $b_0 > 0$ and let $\varepsilon > 0$ be arbitrary. As $\lambda \in [0,1)$, the geometric sequence $\{\lambda^n\}$ converges to zero. Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$\lambda^{n_0} \le \frac{\varepsilon}{b_0}.$$

Therefore, for all $n \in \mathbb{N}$ such that $n \ge n_0$, we have that

$$0 \le b_n \le \lambda^n b_0 \le \lambda^{n_0} b_0 \le \frac{\varepsilon}{b_0} b_0 = \varepsilon.$$

Hence, $\{b_n\} \to 0$. Now as $0 \le a^i_n \le b_n$ for all $n \in \mathbb{N}$ and all $i \in \{1,2,\ldots,N\}$, we conclude that $\{a^i_n\} \to 0$ for all $i \in \{1,2,\ldots,N\}$. □

**Corollary 2.1.4.** *Let* $\{a_n\} \subset [0,\infty)$ *be a sequence and assume that there exists* $\lambda \in [0,1)$ *such that* $a_{n+1} \le \lambda a_n$ *for all* $n \in \mathbb{N}$. *Then* $\{a_n\} \to 0$.

In the sequel, we will use sequences that depends on two natural numbers, so we introduce the following notation. A *double sequence of nonnegative real numbers* is a function $A : \mathbb{N} \times \mathbb{N} \to [0,\infty)$. Given a number $L \in [0,\infty)$, we will write:

- $\lim_{n,m\to\infty} A(n,m) = L$ if for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|A(n,m) - L| \le \varepsilon$ for all $n, m \in \mathbb{N}$ verifying $n \ge n_0$ and $m \ge n_0$;

- $\lim_{n,m\to\infty,\ n\leq m} A(n,m) = L$ if for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|A(n,m) - L| \leq \varepsilon$ for all $n, m \in \mathbb{N}$ verifying $m \geq n \geq n_0$;
- $\lim_{n,m\to\infty,\ n<m} A(n,m) = L$ if for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|A(n,m) - L| \leq \varepsilon$ for all $n, m \in \mathbb{N}$ verifying $m > n \geq n_0$.

For simplicity, we will use the notation "$n, m \geq n_0$" when "$n \geq n_0$ and $m \geq n_0$". If $L = 0$, then the previous notions can be written as follows.

- $\lim_{n,m\to\infty} A(n,m) = 0$ if for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $A(n,m) \leq \varepsilon$ for all $n, m \in \mathbb{N}$ verifying $n, m \geq n_0$;
- $\lim_{n,m\to\infty,\ n\leq m} A(n,m) = 0$ if for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $A(n,m) \leq \varepsilon$ for all $n, m \in \mathbb{N}$ verifying $m \geq n \geq n_0$;
- $\lim_{n,m\to\infty,\ n<m} A(n,m) = 0$ if for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $A(n,m) \leq \varepsilon$ for all $n, m \in \mathbb{N}$ verifying $m > n \geq n_0$.

## 2.2 Fixed, Coincidence and Common Fixed Points

In this section we present these well known concepts from the literature.

**Definition 2.2.1.** Given a self-mapping $T : X \to X$, we will say that a point $x \in X$ is a *fixed point of T* if $Tx = x$. We will denote by $\mathrm{Fix}(T)$ the set of all fixed points of $T$.

Similarly, given two mappings $T, g : X \to X$, we will say that a point $x \in X$ is a *coincidence point of T and g* if $Tx = gx$, and it is a *common fixed point of T and g* if $Tx = gx = x$. We will denote by $\mathrm{Co}(T, g)$ the set of all coincidence points of $T$ and $g$.

A coincidence point of two mappings $T$ and $g$ is a solution of the nonlinear equation $Tx = gx$. In this book we will present some sufficient conditions to guarantee existence and, in some cases, uniqueness, of fixed, coincidence or common fixed points in the setting of $G$-metric spaces.

In the past, many conditions have been introduced in order to guarantee existence of coincidence points. One of the most simple, but useful, property is the following one.

**Lemma 2.2.1.** *Let $T, g : [a, b] \to \mathbb{R}$ be two continuous functions such that $Ta < ga$ and $Tb > gb$. Then $T$ and $g$ have, at least, a coincidence point $c \in (a, b)$.*

*Remark 2.2.1.* If $T$ and $g$ are commuting and $x$ is a coincidence point of $T$ and $g$, then $y = Tx$ is also a coincidence point of $T$ and $g$. It follows from $Ty = Tgx = gTx = gy$.

## 2.3 Control Functions

This section introduces examples of functions that are usually involved in establishing fixed (or coincidence) point results.

Next, we list a collection of properties on a mapping $\phi : [0, \infty) \to [0, \infty)$.

($\mathcal{P}_1$) $\phi$ is non-decreasing, that is, if $0 \leq t \leq s$, then $\phi(t) \leq \phi(s)$.
($\mathcal{P}_2$) $\phi$ is increasing, that is, if $0 \leq t < s$, then $\phi(t) < \phi(s)$.
($\mathcal{P}_3$) $\phi(t) = 0$ if, and only if, $t = 0$ (this is equivalent to say that $\phi^{-1}(\{0\}) = \{0\}$).
($\mathcal{P}_4$) $\phi$ is continuous.
($\mathcal{P}_5$) $\phi$ is right-continuous.
($\mathcal{P}_6$) $\phi$ is left-continuous.
($\mathcal{P}_7$) $\phi$ is lower semi-continuous.
($\mathcal{P}_8$) $\phi$ is upper semi-continuous.
($\mathcal{P}_9$) There exist $k_0 \in \mathbb{N}$, $\lambda \in (0, 1)$ and a convergent series $\sum_{k \geq 1} v_k$ of non-negative real numbers such that

$$\phi^{k+1}(t) \leq \lambda \phi^k(t) + v_k \quad \text{for all } t > 0 \text{ and all } k \geq k_0.$$

($\mathcal{P}_{10}$) The series $\sum_{n \geq 1} \phi^n(t)$ converges for all $t > 0$.
($\mathcal{P}_{11}$) $\lim_{n \to \infty} \phi^n(t) = 0$ for all $t > 0$.
($\mathcal{P}_{12}$) $\phi(t) < t$ for all $t > 0$.
($\mathcal{P}_{13}$) $\lim_{t \to 0^+} \phi(t) = 0$.
($\mathcal{P}_{14}$) $\lim_{s \to t^+} \phi(s) < t$ for all $t > 0$.
($\mathcal{P}_{15}$) $\lim_{s \to t} \phi(s) > 0$ for all $t > 0$.
($\mathcal{P}_{16}$) $\phi(t) < 1$ for all $t \geq 0$, that is, $\phi : [0, \infty) \to [0, 1)$.
($\mathcal{P}_{17}$) If $\{t_n\} \subset [0, \infty)$ is a sequence such that $\{\phi(t_n)\} \to 1$, then $\{t_n\} \to 0$.
($\mathcal{P}_{18}$) $\phi$ is *subadditive*, that is, $\phi(t + s) \leq \phi(t) + \phi(s)$ for all $t, s \geq 0$.

In Table 2.1, we give some of the families we will use, together with the name of some of those functions.

*Remark 2.3.1.* We have the following implications.

- ($\mathcal{P}_1$) $\Rightarrow$ ($\mathcal{P}_2$).
- ($\mathcal{P}_4$) implies ($\mathcal{P}_5$), ($\mathcal{P}_6$), ($\mathcal{P}_7$) and ($\mathcal{P}_8$).
- ($\mathcal{P}_9$) $\Leftrightarrow$ ($\mathcal{P}_{10}$) (see [37]).
- ($\mathcal{P}_{10}$) $\Rightarrow$ ($\mathcal{P}_{11}$).
- ($\mathcal{P}_{12}$) $\Rightarrow$ ($\mathcal{P}_{13}$).

**Table 2.1** Some families of control functions

| Name | Family | Properties |
|---|---|---|
| Comparison function (Matkowski [135]) | $\mathcal{F}_{\text{com}}$ | $(\mathcal{P}_1) + (\mathcal{P}_{11})$ |
| (c)-Comparison function | $\mathcal{F}_{\text{com}}^{(c)}$ | $(\mathcal{P}_1) + (\mathcal{P}_9)$ |
| Altering distance function [123] | $\mathcal{F}_{\text{alt}}$ | $(\mathcal{P}_1) + (\mathcal{P}_4) + (\mathcal{P}_3)$ |
| Associated to altering distance function (I) | $\mathcal{F}'_{\text{alt}}$ | $(\mathcal{P}_3) + (\mathcal{P}_7)$ |
| Associated to altering distance function (II) | $\mathcal{F}''_{\text{alt}}$ | $(\mathcal{P}_3) + (\mathcal{P}_{13}) + (\mathcal{P}_{15})$ |
| Geraghty function | $\mathcal{F}_{\text{Ger}}$ | $(\mathcal{P}_{16}) + (\mathcal{P}_{17})$ |
| Boyd-Wong function | $\mathcal{F}_{\text{BW}}$ | $(\mathcal{P}_8) + (\mathcal{P}_{12})$ |
| Mukerjea function | $\mathcal{F}_{\text{Muk}}$ | $(\mathcal{P}_5) + (\mathcal{P}_{12})$ |
| Ćirić function | $\mathcal{F}_{\text{Cir}}$ | $(\mathcal{P}_{12}) + (\mathcal{P}_{14})$ |
| Browder function | $\mathcal{F}_{\text{Br}}$ | $(\mathcal{P}_1) + (\mathcal{P}_5)$ |
| Krasnoselskii function | $\mathcal{F}_{\text{Kr}}$ | $(\mathcal{P}_3) + (\mathcal{P}_4)$ |
| Auxiliary functions | $\mathcal{F}_{\text{A}}$ | $(\mathcal{P}_1) + (\mathcal{P}_3)$ |

### *2.3.1 Comparison Functions*

In [135], Matkowski considered functions satisfying $(\mathcal{P}_1)$, $(\mathcal{P}_{11})$ and $(\mathcal{P}_{12})$. Notice that, in general, there is no relationship between $(\mathcal{P}_{11})$ and $(\mathcal{P}_{12})$. For example, the function $\phi : [0, \infty) \to [0, \infty)$ defined by

$$\phi(t) = \begin{cases} 0, \text{ if } t \neq 1, \\ 2, \text{ if } t = 1, \end{cases}$$

satisfies $(\mathcal{P}_{11})$ but it does not satisfy $(\mathcal{P}_{12})$. Conversely, the function

$$\phi(t) = \begin{cases} 0, & \text{if } t \leq 1, \\ \dfrac{1+t}{2}, & \text{if } t > 1, \end{cases}$$

satisfies $(\mathcal{P}_{12})$ but it does not satisfy $(\mathcal{P}_{11})$ (notice that the sequence $\{t_n\}$, defined by $t_0 = 2$ and $t_{n+1} = \phi(t_n)$ for all $n \in \mathbb{N}$, converges to 1). However, when $\phi$ is non-decreasing, we have the following relationship.

**Proposition 2.3.1 (Matkowski [135]).** $(\mathcal{P}_1) + (\mathcal{P}_{11}) \Rightarrow (\mathcal{P}_{12})$.

*Proof.* Assume that $(\mathcal{P}_{12})$ is false. Then, there exists $t_0 > 0$ such that $t_0 \leq \phi(t_0)$. As $\phi$ is non-decreasing, then $\phi(t_0) \leq \phi(\phi(t_0))$, which implies that $t_0 \leq \phi(t_0) \leq \phi^2(t_0)$. By induction, it can be proved that $t_0 \leq \phi^n(t_0)$ for all $n \in \mathbb{N}$. Then $(\mathcal{P}_{11})$ cannot hold. □

Although functions satisfying $(\mathcal{P}_1)$ and $(\mathcal{P}_{11})$ (and, consequently, also $(\mathcal{P}_{12})$) could be called *Matkowski functions*, in the literature these functions are known as *comparison functions* (see, for example, [36–38]).

**Definition 2.3.1.** A *comparison function* is a non-decreasing function $\phi : [0,\infty) \to [0,\infty)$ such that $\{\phi^n(t)\} \to 0$ for all $t > 0$. Let $\mathcal{F}_{\text{com}}$ denote the family of all comparison functions.

Matkowski also pointed out the following partial converse using functions belonging to $\mathcal{F}_{\text{Muk}}$.

**Proposition 2.3.2 (Matkowski [135]).** $(\mathcal{P}_5) + (\mathcal{P}_{12}) \Rightarrow (\mathcal{P}_{11})$.

*Proof.* Let $\phi$ be a function verifying $(\mathcal{P}_5)$ and $(\mathcal{P}_{12})$. As $\phi$ is right-continuous at $t = 0$ and $\phi(t) < t$ for all $t > 0$, then $\phi(0) = 0$. Let $t_0 > 0$ be arbitrary and let $t_n = \phi^n(t_0)$ for all $n \in \mathbb{N}$. We distinguish two cases.

*Case 1.* $t_n > 0$ *for all* $n \in \mathbb{N}$. In this case, $0 < t_{n+1} = \phi(t_n) < t_n$ for all $n \in \mathbb{N}$. Then, $\{t_n\}$ is a bounded below, decreasing sequence of real numbers. Hence, it is convergent. Let $L \geq 0$ be its limit. As $\phi$ is right-continuous,

$$\phi(L) = \lim_{t \to L^+} \phi(t) = \lim_{n\to\infty} \phi(t_n) = \lim_{n\to\infty} t_{n+1} = L.$$

Thus, $L = 0$ and $\{\phi^n(t_0)\} \to 0$.

*Case 2.* *There exists* $n_0 \in \mathbb{N}$ *such that* $t_{n_0} = 0$. In this case, $t_{n_0+1} = \phi(t_{n_0}) = \phi(0) = 0$. By induction, $t_n = 0$ for all $n \geq n_0$, so $\{\phi^n(t_0)\} = \{t_n\} \to 0$. □

*Remark 2.3.2.* In the proof of Theorem 2.2 in [127], the authors announced another converse statement: $(\mathcal{P}_{12})$ and $(\mathcal{P}_{14})$ implied $(\mathcal{P}_{11})$, that is, if $\phi \in \mathcal{F}_{\text{Cir}}$, then $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$. Unfortunately, this is false. For example, if $\phi : [0,\infty) \to [0,\infty)$ is defined by

$$\phi(t) = \begin{cases} 1, & \textit{if } t = 0, \\ 0, & \textit{if } t > 0, \end{cases}$$

then $\phi \in \mathcal{F}_{\text{Cir}}$ but $\lim_{n\to\infty} \phi^n(1)$ does not exist because $\{\phi^n(1)\}_{n\in\mathbb{N}}$ is the alternated sequence $\{1, 0, 1, 0, 1, 0, \ldots\}$.

**Lemma 2.3.1 ([179]).** *If* $\phi : [0,\infty) \to [0,\infty)$ *is a comparison function, then:*

1. *each iterate* $\phi^n$ *is also a comparison function;*
2. $\phi(t) < t$ *for all* $t > 0$*;*
3. $\phi$ *is continuous at* $t = 0$ *and* $\phi(0) = 0$.

For practical reasons, Berinde introduced in [37] the notion of *(c)-comparison function* as follows.

**Definition 2.3.2.** A *(c)-comparison function* is a non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ such that there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k\geq 1} v_k$ verifying

$$\phi^{k+1}(t) \leq a\phi^k(t) + v_k \quad \text{for all } k \geq k_0 \text{ and all } t \geq 0.$$

In some sources, (c)-comparison functions are called *Bianchini-Grandolfi gauge functions* (see e.g. [45, 166, 167])

**Lemma 2.3.2 (Berinde [36, 37]).** *If $\phi : [0, \infty) \to [0, \infty)$ is a (c)-comparison function, then the following properties hold:*

1. *$\phi$ is a comparison function;*
2. *$\phi(t) < t$ for all $t \in (0, \infty)$;*
3. *$\phi$ is continuous at $t = 0$ and $\phi(0) = 0$;*
4. *the series $\sum_{n\geq 1}\phi^n(t)$ converges for all $t \in [0, \infty)$;*
5. *$\{\phi^n(t)\}_{n\in\mathbb{N}}$ converges to $0$ as $n \to \infty$ for all $t \in (0, \infty)$;*
6. *the function $\varphi_\phi : [0, \infty) \to [0, \infty)$ defined by*

$$\varphi_\phi(t) = \sum_{k=0}^{\infty} \phi^k(t) \quad \textit{for all } t \geq 0$$

*is non-decreasing and continuous at $t = 0$.*

### 2.3.2 Altering Distance Functions and Associated Functions

**Definition 2.3.3 (Khan et al. [123]).** An *altering distance function* is a continuous, non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t) = 0$ if, and only if, $t = 0$. Let $\mathcal{F}_{\text{alt}}$ denote the family of all altering distance functions.

$$\mathcal{F}_{\text{alt}} = \{\phi : [0, \infty) \to [0, \infty) : \phi \text{ is continuous, non-decreasing and } \phi^{-1}(\{0\}) = \{0\}\}.$$

As we shall see, many fixed point theorems involve a contractivity condition in which two functions, $\psi \in \mathcal{F}_{\text{alt}}$ and $\phi \in \mathcal{F}'_{\text{alt}}$, play a key role.

**Proposition 2.3.3.** $\mathcal{F}_{\text{alt}} \subset \mathcal{F}_{\text{A}}$, $\quad \mathcal{F}_{\text{alt}} \subset \mathcal{F}_{\text{Kr}}$.

The following results can be found in the literature, but we recall them here for the sake of completeness.

**Proposition 2.3.4.** *If $\phi : [0, \infty) \to [0, \infty)$ is a non-decreasing function and $\{a_n\} \subset [0, \infty)$ is a sequence such that $\phi(a_{n+1}) < \phi(a_n)$ for all $n \in \mathbb{N}$, then $a_{n+1} < a_n$ for all $n \in \mathbb{N}$. In particular, $\{a_n\}$ is convergent and $L < a_n$ for all $n \in \mathbb{N}$ (where $L$ is the limit of $\{a_n\}$).*

*Proof.* If there exists some $n_0 \in \mathbb{N}$ such that $a_{n_0} \leq a_{n_0+1}$, then $\phi(a_{n_0}) \leq \phi(a_{n_0+1}) < \phi(a_{n_0})$, which is impossible. □

**Lemma 2.3.3.** *If $\phi \in \mathcal{F}_{\mathrm{A}}$ and $\{a_n\} \subset [0,\infty)$ is a sequence such that $\{\phi(a_n)\} \to 0$, then $\{a_n\} \to 0$.*

*Proof.* Assume that $\{\phi(a_n)\} \to 0$ but $\{a_n\}$ does not converge to zero. This means that there exists $\varepsilon_0 > 0$ such that, for all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ with $m > n$ and $a_m \geq \varepsilon_0$. In particular, $\{a_n\}$ has a subsequence $\{a_{n(k)}\}$ such that $a_{n(k)} \geq \varepsilon_0$ for all $k \in \mathbb{N}$. Since $\phi^{-1}(\{0\}) = \{0\}$, we have that $\phi(\varepsilon_0) > 0$. Furthermore, as $\phi$ is non-decreasing, we have that $0 < \phi(\varepsilon_0) \leq \phi(a_{n(k)})$ for all $k \in \mathbb{N}$. However, $\{\phi(a_{n(k)})\}$ is a subsequence of $\{\phi(a_n)\}$ which converges to zero. This contradiction shows that necessarily $\{a_n\} \to 0$. □

The previous lemma is false if we replace monotonicity by continuity. For example, if

$$\phi(t) = \begin{cases} t, & \text{if } 0 \leq t \leq 1, \\ 1/t, & \text{if } t > 1, \end{cases}$$

then $\{\phi(n)\}_{n \in \mathbb{N}} \to 0$ but $\{n\}_{n \in \mathbb{N}} \to \infty$.

Many fixed point theorems use a contractivity condition involving a difference $\psi - \phi$ where $\psi \in \mathcal{F}_{\mathrm{alt}}$ and $\phi \in \mathcal{F}'_{\mathrm{alt}}$.

**Lemma 2.3.4.** *Let $\psi, \phi : [0,\infty) \to [0,\infty)$ be two functions such that $\psi$ is non-decreasing and $\phi^{-1}(\{0\}) = \{0\}$, and let $t, s, r \in [0,\infty)$.*

1. *If $\psi(t) \leq \psi(s) - \phi(r)$, then $t < s$ or $r = 0$.*
2. *If $\psi$ also verifies $\psi^{-1}(\{0\}) = \{0\}$ and $\psi(t) \leq (\psi - \phi)(s)$, then $t < s$ or $t = s = 0$. In any case, $t \leq s$.*

*Proof.* (**1**) Assume that $t \geq s$ and we have to prove that $r = 0$. Indeed, as $\psi$ is non-decreasing, $\psi(s) \leq \psi(t)$. Therefore,

$$\psi(t) \leq \psi(s) - \phi(r) \leq \psi(s) \leq \psi(t).$$

As a consequence, $\psi(t) = \psi(s)$ and $\phi(r) = 0$. Therefore $r = 0$.

(**2**) Next, assume that $\psi(t) \leq (\psi - \phi)(s)$ and $t \geq s$. By item (1), $s = 0$. Therefore, $0 \leq \psi(t) \leq \psi(0) - \phi(0) = 0$, so $\psi(t) = 0$ and $t = 0$. □

**Corollary 2.3.1.** *Let $\psi, \phi : [0,\infty) \to [0,\infty)$ be two functions such that $\psi$ is non-decreasing and $\psi^{-1}(\{0\}) = \phi^{-1}(\{0\}) = \{0\}$, and let $\{t_n\}, \{s_n\} \subset [0,\infty)$ be two sequences such that $\{s_n\} \to 0$.*

1. *If $\psi(t_n) \leq \psi(s_n) - \phi(s_n)$ for all $n \in \mathbb{N}$, then $\{t_n\} \to 0$.*
2. *If $\psi \in \mathcal{F}_{\mathrm{alt}}$ and $\psi(t_n) \leq \psi(s_n)$ for all $n \in \mathbb{N}$, then $\{t_n\} \to 0$.*

*Proof.* **(1)** By item 1 of Lemma 2.3.4, we have that, for all $n \in \mathbb{N}$, either $t_n < s_n$ or $t_n = s_n = 0$. In any case, $0 \le t_n \le s_n$. Therefore, Corollary 2.1.1 implies that $\{t_n\} \to 0$.

**(2)** As $\psi$ is continuous, it follows that $\{\psi(s_n)\} \to \psi(0) = 0$. Corollary 2.1.1 guarantees that $\{\psi(t_n)\} \to 0$ and Lemma 2.3.3 concludes that $\{t_n\} \to 0$. □

**Lemma 2.3.5.** *Let $\psi, \phi : [0, \infty) \to [0, \infty)$ be two functions such that $\psi$ is continuous and $\phi \in \mathcal{F}'_{\text{alt}}$. Let $\{t_n\}, \{s_n\} \subset [0, \infty)$ be two sequences that converge to the same limit $L \in [0, \infty)$ and satisfy*

$$\psi(t_n) \le \psi(s_n) - \phi(s_n) \quad \textit{for all } n \in \mathbb{N}. \tag{2.1}$$

*Then $L = 0$ and $\{\phi(s_n)\} \to 0$.*

*Proof.* By (2.1), we have that $0 \le \phi(s_n) \le \psi(s_n) - \psi(t_n)$ for all $n \in \mathbb{N}$. As $\psi$ is continuous, then

$$\lim_{n\to\infty} \psi(t_n) = \lim_{n\to\infty} \psi(s_n) = \psi(L).$$

Therefore, $\{\phi(s_n)\} \to 0$. Since $\phi$ is lower semi-continuous and $\{\phi(s_n)\}$ is convergent,

$$0 \le \phi(L) \le \liminf_{n\to\infty} \phi(s_n) = \lim_{n\to\infty} \phi(s_n) = 0.$$

Hence $\phi(L) = 0$, which implies that $L = 0$. □

**Corollary 2.3.2.** *Let $\{t_n\}, \{s_n\} \subset [0, \infty)$ be two sequences that converge to the same limit $L \in [0, \infty)$. Assume that there exist two functions $\psi \in \mathcal{F}_{\text{alt}}$ and $\phi \in \mathcal{F}'_{\text{alt}}$ such that*

$$\psi(t_n) \le \psi(s_n) - \phi(s_n) \quad \textit{for all } n \in \mathbb{N}.$$

*Then $L = 0$ and $\{\phi(s_n)\} \to 0$.*

*Proof.* It is a particular case of Lemma 2.3.5. □

**Corollary 2.3.3.** *Let $\phi \in \mathcal{F}'_{\text{alt}}$ and let $\{t_n\}, \{s_n\} \subset [0, \infty)$ be two sequences that converge to the same limit $L \in [0, \infty)$ and satisfy*

$$t_n \le s_n - \phi(s_n) \quad \textit{for all } n \in \mathbb{N}.$$

*Then $L = 0$ and $\{\phi(s_n)\} \to 0$.*

*Proof.* This is the particular case of Lemma 2.3.5 in which $\psi(t) = t$ for all $t \ge 0$. □

**Lemma 2.3.6.** *Let $\psi \in \mathcal{F}_{\mathrm{alt}}$, $\phi \in \mathcal{F}'_{\mathrm{alt}}$ and let $\{t_n\} \subset [0, \infty)$ be a sequence such that*

$$\psi(t_{n+1}) \leq \psi(t_n) - \phi(t_n) \quad \textit{for all } n \in \mathbb{N}. \tag{2.2}$$

*Then $\{t_n\} \to 0$.*

*Proof.* We distinguish two cases.

*Case 1.* *Assume that there exists some $n_0 \in \mathbb{N}$ such that $t_{n_0} \leq t_{n_0+1}$.* Since $\psi$ is non-decreasing,

$$\psi(t_{n_0+1}) \leq \psi(t_{n_0}) - \phi(t_{n_0}) \leq \psi(t_{n_0}) \leq \psi(t_{n_0+1}).$$

Therefore $\psi(t_{n_0}) = \psi(t_{n_0+1})$ and $\phi(t_{n_0}) = 0$. As $\phi \in \mathcal{F}'_{\mathrm{alt}}$, we deduce that $t_{n_0} = 0$. By (2.2), $\psi(t_{n_0+1}) \leq \psi(0) - \phi(0) = 0$, so $\psi(t_{n_0+1}) = 0$ and also $t_{n_0+1} = 0$. Repeating this argument, we deduce that $t_n = 0$ for all $n \geq n_0$. In particular, $\{t_n\} \to 0$.

*Case 2.* *Assume that $t_{n+1} < t_n$ for all $n \in \mathbb{N}$.* Then $\{t_n\}$ is a strictly decreasing sequence of nonnegative real numbers. Then, there exists $L \geq 0$ such that $\{t_n\} \to L$. Since $0 \leq \phi(t_n) \leq \psi(t_n) - \psi(t_{n+1})$ for all $n \in \mathbb{N}$ and $\psi$ is continuous, then $\{\phi(t_n)\} \to 0$. Using the same argument of the proof of Lemma 2.3.5,

$$0 \leq \phi(L) \leq \liminf_{n\to\infty} \phi(s_n) = \lim_{n\to\infty} \phi(s_n) = 0,$$

which implies that $L = 0$.

□

If $\psi(t) = t$ for all $t \geq 0$ in the previous lemma, then we get the following result.

**Corollary 2.3.4.** *Let $\phi \in \mathcal{F}'_{\mathrm{alt}}$ and let $\{t_n\} \subset [0, \infty)$ be a sequence such that*

$$t_{n+1} \leq t_n - \phi(t_n) \quad \textit{for all } n \in \mathbb{N}.$$

*Then $\{t_n\} \to 0$.*

Given $\psi \in \mathcal{F}_{\mathrm{alt}}$ and $\phi \in \mathcal{F}'_{\mathrm{alt}}$, if $s \leq t$, then $\psi(s) \leq \psi(t)$. However, we do not know the relationship between $\psi(t) - \phi(t)$ and $\psi(s) - \phi(s)$. The following result is an approach to this case.

**Lemma 2.3.7.** *Let $\psi, \phi : [0, \infty) \to [0, \infty)$ be two functions such that $\phi(0) = 0$ and $\psi^{-1}(\{0\}) = \{0\}$. Let $\{t_n\}, \{s_n\} \subset [0, \infty)$ be two sequences such that, for all $n \in \mathbb{N}$,*

$$s_n \leq t_n, \quad \psi(t_{n+1}) \leq \max\{\psi(t_n) - \phi(t_n), \psi(s_n) - \phi(s_n)\}.$$

*Then the following properties hold.*

1. *If there exists some* $n_0 \in \mathbb{N}$ *such that* $t_{n_0} = 0$*, then* $t_n = 0$ *for all* $n \geq n_0$*. In particular,* $\{t_n\} \to 0$ *and* $\{s_n\} \to 0$*.*
2. *If* $\psi$ *is non-decreasing,* $\phi^{-1}(\{0\}) = \{0\}$ *and* $t_n > 0$ *for all* $n \in \mathbb{N}$*, then* $t_{n+1} \leq t_n$ *for all* $n \in \mathbb{N}$*.*
3. *If* $\psi \in \mathcal{F}_{\text{alt}}$ *and* $\phi \in \mathcal{F}'_{\text{alt}}$*, then the sequences* $\{t_n\}$*,* $\{s_n\}$*,* $\{\phi(t_n)\}$ *and* $\{\phi(s_n)\}$ *converge to zero.*

*Proof.* **(1)** Since $0 \leq s_{n_0} \leq t_{n_0} = 0$, then $t_{n_0} = s_{n_0} = 0$. Therefore

$$\begin{aligned} 0 &\leq \psi(t_{n_0+1}) \leq \max\{\psi(t_{n_0}) - \phi(t_{n_0}), \psi(s_{n_0}) - \phi(s_{n_0})\} \\ &= \psi(0) - \phi(0) = 0. \end{aligned}$$

As $\psi^{-1}(\{0\}) = \{0\}$, then $t_{n_0+1} = 0$. By induction, the same argument proves that $t_n = 0$ for all $n \geq n_0$.

**(2)** Assume that $\psi$ is non-decreasing, $\phi^{-1}(\{0\}) = \{0\}$ and $t_n > 0$ for all $n \in \mathbb{N}$. Since $s_n \leq t_n$, then $\psi(s_n) \leq \psi(t_n)$. Therefore, for all $n \in \mathbb{N}$,

$$\begin{aligned} \psi(t_{n+1}) &\leq \max\{\psi(t_n) - \phi(t_n), \psi(s_n) - \phi(s_n)\} \\ &\leq \max\{\psi(t_n), \psi(s_n)\} = \psi(t_n). \end{aligned} \tag{2.3}$$

To prove that $t_{n+1} \leq t_n$ for all $n \in \mathbb{N}$, we reason by contradiction. Assume that there exists some $n \in \mathbb{N}$ such that $t_n < t_{n+1}$. In such a case,

$$\psi(t_n) \leq \psi(t_{n+1}) \leq \max\{\psi(t_n) - \phi(t_n), \psi(s_n) - \phi(s_n)\} \leq \psi(t_n).$$

Therefore,

$$\psi(t_n) = \psi(t_{n+1}) = \max\{\psi(t_n) - \phi(t_n), \psi(s_n) - \phi(s_n)\}. \tag{2.4}$$

Depending on the maximum, we distinguish two cases to get a contradiction. If

$$\max\{\psi(t_n) - \phi(t_n), \psi(s_n) - \phi(s_n)\} = \psi(t_n) - \phi(t_n),$$

then $\psi(t_n) = \psi(t_n) - \phi(t_n)$, so $\phi(t_n) = 0$. As $\phi^{-1}(\{0\}) = \{0\}$, then $t_n = 0$, which contradicts the fact that $t_n > 0$. In the other case, if

$$\max\{\psi(t_n) - \phi(t_n), \psi(s_n) - \phi(s_n)\} = \psi(s_n) - \phi(s_n), \tag{2.5}$$

then $\psi(t_n) = \psi(s_n) - \phi(s_n)$. Therefore $\psi(t_n) = \psi(s_n) - \phi(s_n) \leq \psi(t_n) - \phi(s_n) \leq \psi(t_n)$, so $\psi(t_n) = \psi(s_n)$ and $\phi(s_n) = 0$. As $\phi^{-1}(\{0\}) = \{0\}$, then $s_n = 0$. Hence, (2.4) and (2.5) prove that

$$\begin{aligned} \psi(t_{n+1}) &= \max\{\psi(t_n) - \phi(t_n), \psi(s_n) - \phi(s_n)\} = \psi(s_n) - \phi(s_n) \\ &= \psi(0) - \phi(0) = 0, \end{aligned}$$

so $t_{n+1} = 0$, which also contradicts the fact that $t_{n+1} > 0$. As a consequence, in any case we get a contradiction. Then the case $t_n < t_{n+1}$ is impossible and we conclude that $t_{n+1} \le t_n$ for all $n \in \mathbb{N}$.

**(3)** If there exists some $n_0 \in \mathbb{N}$ such that $t_{n_0} = 0$, item (1) guarantees that $t_n = 0$ for all $n \ge n_0$. As $0 \le s_n \le t_n = 0$, then $s_n = 0$ and $\phi(s_n) = 0$ for all $n \ge n_0$. In particular, all sequences $\{t_n\}$, $\{s_n\}$ and $\{\phi(s_n)\}$ converge to zero.

Next, assume that $t_n > 0$ for all $n \in \mathbb{N}$. By item (2), $t_{n+1} \le t_n$ for all $n \in \mathbb{N}$, so $\{t_n\}$ is a non-increasing, bounded below sequence. Then, it is convergent. Let $L = \lim_{n\to\infty} t_n$. By (2.3), for all $n \in \mathbb{N}$,

$$\psi(t_{n+1}) \le \max\{\psi(t_n) - \phi(t_n), \psi(s_n) - \phi(s_n)\} \le \psi(t_n).$$

As $\psi$ is continuous, letting $n \to \infty$ in the previous inequality, we deduce that

$$\lim_{n\to\infty} \max\{\psi(t_n) - \phi(t_n), \psi(s_n) - \phi(s_n)\} = \psi(L).$$

From Lemma 2.1.2 we deduce that there exists a subsequence of one of the sequences $\{\psi(t_n) - \phi(t_n)\}$ and $\{\psi(s_n) - \phi(s_n)\}$ that converge to $\psi(L)$. Then, we distinguish two cases.

Let $\{\psi(t_{n(k)}) - \phi(t_{n(k)})\}_{k\in\mathbb{N}}$ be a subsequence of $\{\psi(t_n) - \phi(t_n)\}_{n\in\mathbb{N}}$ such that $\{\psi(t_{n(k)}) - \phi(t_{n(k)})\}_{k\in\mathbb{N}} \to \psi(L)$. Then, as $\psi$ is continuous and $\{t_{n(k)}\} \to L$, it follows that

$$\begin{aligned}\lim_{k\to\infty} \phi\left(t_{n(k)}\right) &= \lim_{k\to\infty} \left[\psi\left(t_{n(k)}\right) - \left(\psi\left(t_{n(k)}\right) - \phi\left(t_{n(k)}\right)\right)\right] \\ &= \psi(L) - \psi(L) = 0.\end{aligned}$$

As $\phi$ is lower semi-continuous at $t = L$, then

$$0 \le \phi(L) \le \liminf_{t\to L} \phi(t) \le \lim_{k\to\infty} \phi\left(t_{n(k)}\right) = 0.$$

Then, $\phi(L) = 0$, so $L = 0$, which proves that $\{t_n\} \to L = 0$.

For the other case, let $\{\psi(s_{n(k)}) - \phi(s_{n(k)})\}_{k\in\mathbb{N}}$ be a subsequence of $\{\psi(s_n) - \phi(s_n)\}_{n\in\mathbb{N}}$ such that

$$\left\{\psi\left(s_{n(k)}\right) - \phi\left(s_{n(k)}\right)\right\}_{k\in\mathbb{N}} \to \psi(L).$$

As $0 \le s_{n(k)} \le t_{n(k)}$ for all $k \in \mathbb{N}$ and $\{t_{n(k)}\} \to L$, then $\{s_{n(k)}\}$ is a bounded sequence of real numbers. As a consequence, it has a convergent subsequence. Let $\{s_{n'(k)}\}_{k\in\mathbb{N}}$ be a convergent subsequence of $\{s_{n(k)}\}$. Then, there exists $L' \ge 0$ such that

$$\{s_{n'(k)}\}_{k\in\mathbb{N}} \to L' \quad \text{and} \quad \{\psi(s_{n'(k)}) - \phi(s_{n'(k)})\}_{k\in\mathbb{N}} \to \psi(L). \tag{2.6}$$

Since $0 \le s_{n'(k)} \le t_{n'(k)}$ for all $k \in \mathbb{N}$ and $\{t_{n(k)}\} \to L$, then $0 \le L' \le L$. As $\psi$ is non-decreasing, then $\psi(L') \le \psi(L)$. As $\psi$ is continuous, then

$$\begin{aligned} 0 \le \lim_{k\to\infty} \phi\left(s_{n'(k)}\right) &= \lim_{k\to\infty} \left[\, \psi\left(s_{n'(k)}\right) - \left(\psi\left(s_{n'(k)}\right) - \phi\left(s_{n'(k)}\right)\right) \right] \\ &= \psi\left(L'\right) - \psi(L) \le 0. \end{aligned}$$

In particular,

$$\lim_{k\to\infty} \phi\left(s_{n'(k)}\right) = 0. \tag{2.7}$$

As $\phi$ is lower semi-continuous at $t = L'$, then

$$0 \le \phi\left(L'\right) \le \liminf_{t\to L'} \phi(t) \le \lim_{k\to\infty} \phi\left(s_{n'(k)}\right) = 0.$$

Hence, $\phi(L') = 0$, so $L' = 0$. In particular, $\{s_{n'(k)}\}_{k\in\mathbb{N}} \to L' = 0$. By (2.6), (2.7) and the continuity of $\psi$, we deduce that

$$\psi(L) = \lim_{k\to\infty} \left[\, \psi\left(s_{n'(k)}\right) - \phi\left(s_{n'(k)}\right) \right] = \psi(0) - 0 = 0.$$

It follows that $L = \lim_{n\to\infty} t_n = 0$. In any case, we have just proved that $\{t_n\} \to 0$.

Since $0 \le s_n \le t_n$ for all $n \in \mathbb{N}$, then $\{s_n\} \to 0$. Furthermore, for all $n \in \mathbb{N}$,

$$\begin{aligned} \psi(t_{n+1}) &\le \max\{\psi(t_n) - \phi(t_n), \psi(s_n) - \phi(s_n)\} \\ &\le \psi(t_n) - \phi(t_n) + \psi(s_n) - \phi(s_n), \end{aligned}$$

which implies that

$$0 \le \max\{\phi(t_n), \phi(s_n)\} \le \phi(t_n) + \phi(s_n) \le \psi(t_n) + \psi(s_n) - \psi(t_{n+1}).$$

As $\psi$ is continuous, we deduce that $\{\max\{\phi(t_n), \phi(s_n)\}\} \to 0$, so the sequences $\{\phi(t_n)\}$ and $\{\phi(s_n)\}$ converge to zero. □

In the next result, we employ convergent sequences.

**Lemma 2.3.8.** *Let $\psi \in \mathcal{F}_{\mathrm{alt}}$ and $\phi \in \mathcal{F}'_{\mathrm{alt}}$ be two functions and let $\{t_n\}, \{s_n\}, \{r_n\} \subset [0, \infty)$ be three sequences such that, for all $n \in \mathbb{N}$,*

$$r_n \le s_n, \quad \psi(t_n) \le \max\{\psi(s_n) - \phi(s_n), \psi(r_n) - \phi(r_n)\}.$$

*If there exists some $L \in [0, \infty)$ such that $\{t_n\} \to L$ and $\{s_n\} \to L$, then $L = 0$.*

*Proof.* Since $\psi$ is non-decreasing, $\psi(r_n) \leq \psi(s_n)$. Then, for all $n \in \mathbb{N}$,

$$\begin{aligned}\psi(t_n) &\leq \max\{\psi(s_n) - \phi(s_n), \psi(r_n) - \phi(r_n)\}\\ &\leq \max\{\psi(s_n), \psi(r_n)\} \leq \psi(s_n).\end{aligned}$$

As $\psi$ is continuous and $\{t_n\} \to L$ and $\{s_n\} \to L$, Lemma 2.1.1 guarantees that

$$\lim_{n\to\infty} \max\{\psi(s_n) - \phi(s_n), \psi(r_n) - \phi(r_n)\} = \psi(L).$$

From Lemma 2.1.2 we deduce that there exists a subsequence of one of the sequences $\{\psi(s_n) - \phi(s_n)\}$ and $\{\psi(r_n) - \phi(r_n)\}$ that converge to $\psi(L)$. Then, we distinguish two cases.

Let $\{\psi(s_{n(k)}) - \phi(s_{n(k)})\}_{k\in\mathbb{N}}$ be a subsequence of $\{\psi(s_n) - \phi(s_n)\}_{n\in\mathbb{N}}$ such that $\{\psi(s_{n(k)}) - \phi(s_{n(k)})\}_{k\in\mathbb{N}} \to \psi(L)$. Then, as $\psi$ is continuous and $\{s_{n(k)}\} \to L$, it follows that

$$\begin{aligned}\lim_{k\to\infty} \phi(s_{n(k)}) &= \lim_{k\to\infty} \left[\psi(s_{n(k)}) - \left(\psi(s_{n(k)}) - \phi(s_{n(k)})\right)\right]\\ &= \psi(L) - \psi(L) = 0.\end{aligned}$$

As $\phi$ is lower semi-continuous at $t = L$, then

$$0 \leq \phi(L) \leq \liminf_{t\to L} \phi(t) \leq \lim_{k\to\infty} \phi(s_{n(k)}) = 0.$$

Then, $\phi(L) = 0$, so $L = 0$.

For the other case let $\{\psi(r_{n(k)}) - \phi(r_{n(k)})\}_{k\in\mathbb{N}}$ be a subsequence of $\{\psi(r_n) - \phi(r_n)\}_{n\in\mathbb{N}}$ such that

$$\left\{\psi(r_{n(k)}) - \phi(r_{n(k)})\right\}_{k\in\mathbb{N}} \to \psi(L).$$

As $0 \leq r_{n(k)} \leq s_{n(k)}$ for all $k \in \mathbb{N}$ and $\{s_{n(k)}\} \to L$, then $\{r_{n(k)}\}$ is a bounded sequence of real numbers. As a consequence, it has a convergent subsequence. Let $\{r_{n'(k)}\}_{k\in\mathbb{N}}$ be a convergent subsequence of $\{r_{n(k)}\}$. Then, there exists $L' \geq 0$ such that

$$\{r_{n'(k)}\}_{k\in\mathbb{N}} \to L' \quad \text{and} \quad \{\psi(r_{n'(k)}) - \phi(r_{n'(k)})\}_{k\in\mathbb{N}} \to \psi(L). \tag{2.8}$$

Since $0 \leq r_{n'(k)} \leq s_{n'(k)}$ for all $k \in \mathbb{N}$ and $\{s_{n(k)}\} \to L$, then $0 \leq L' \leq L$. As $\psi$ is non-decreasing, then $\psi(L') \leq \psi(L)$. As $\psi$ is continuous, then

$$\begin{aligned}0 \leq \lim_{k\to\infty} \phi(r_{n'(k)}) &= \lim_{k\to\infty} \left[\psi(r_{n'(k)}) - \left(\psi(r_{n'(k)}) - \phi(r_{n'(k)})\right)\right]\\ &= \psi(L') - \psi(L) \leq 0.\end{aligned}$$

In particular,

$$\lim_{k\to\infty} \phi(r_{n'(k)}) = 0. \tag{2.9}$$

As $\phi$ is lower semi-continuous at $t = L'$, then

$$0 \le \phi\left(L'\right) \le \liminf_{t\to L'} \phi\left(t\right) \le \lim_{k\to\infty} \phi(r_{n'(k)}) = 0.$$

Hence, $\phi\left(L'\right) = 0$, so $L' = 0$. In particular, $\{r_{n'(k)}\}_{k\in\mathbb{N}} \to L' = 0$. By (2.8), (2.9) and the continuity of $\psi$, we deduce that

$$\psi(L) = \lim_{k\to\infty} \left[\,\psi(r_{n'(k)}) - \phi(r_{n'(k)})\,\right] = \psi(0) - 0 = 0.$$

As $\psi \in \mathcal{F}_{\text{alt}}$, condition $\psi\left(L\right) = 0$ implies that $L = 0$. In any case, we have just proved that $L = 0$. □

*Remark 2.3.3.* As we have mentioned before, many fixed point theorems use a contractivity condition involving a difference $\psi - \phi$ where $\psi \in \mathcal{F}_{\text{alt}}$ and $\phi \in \mathcal{F}'_{\text{alt}}$. As we shall see, most of them are also valid using $\phi \in \mathcal{F}''_{\text{alt}}$. However, there is no relationship of inclusion between the classes $\mathcal{F}'_{\text{alt}}$ and $\mathcal{F}''_{\text{alt}}$ of functions associated to altering distance functions. For example, on the one hand, the function

$$\phi\left(t\right) = \begin{cases} 0, \text{ if } t = 0, \\ 1, \text{ if } t > 0, \end{cases}$$

belongs to $\mathcal{F}'_{\text{alt}}$, but it does not satisfy $(\mathcal{P}_{13})$. Hence, $\phi \in \mathcal{F}'_{\text{alt}} \setminus \mathcal{F}''_{\text{alt}}$. On the other hand, the function

$$\phi\left(t\right) = \begin{cases} t, \text{ if } 0 \le t < 1, \\ 2, \text{ if } t = 1, \\ 1, \text{ if } t > 1, \end{cases}$$

belongs to $\mathcal{F}''_{\text{alt}}$, but it does not satisfy $(\mathcal{P}_7)$. Hence, $\phi \in \mathcal{F}''_{\text{alt}} \setminus \mathcal{F}'_{\text{alt}}$.

Next, we repeat Lemmas 2.3.2 and 2.3.6 using $\phi \in \mathcal{F}''_{\text{alt}}$.

**Lemma 2.3.9.** *Let $\{t_n\}, \{s_n\} \subset [0,\infty)$ be two sequences that converge to the same limit $L \in [0,\infty)$. Assume that there exist two functions $\psi \in \mathcal{F}_{\text{alt}}$ and $\phi \in \mathcal{F}''_{\text{alt}}$ such that*

$$\psi\left(t_n\right) \le \psi\left(s_n\right) - \phi\left(s_n\right) \quad \textit{for all } n \in \mathbb{N}. \tag{2.10}$$

*Then $L = 0$.*

*Proof.* By contradiction, assume that $L > 0$. As $\phi$ satisfies axiom $(\mathcal{P}_{15})$, then $\ell = \lim_{s\to L} \phi(s) > 0$. Since $\psi$ is continuous, taking the limit in (2.10) as $n \to \infty$, we deduce that $\psi(L) \leq \psi(L) - \ell$, which is impossible because $\ell > 0$. Hence, $L = 0$. □

**Lemma 2.3.10.** *Let $\psi \in \mathcal{F}_{\text{alt}}$, $\phi \in \mathcal{F}''_{\text{alt}}$ and let $\{t_n\} \subset [0, \infty)$ be a sequence such that*

$$\psi(t_{n+1}) \leq \psi(t_n) - \phi(t_n) \quad \textit{for all } n \in \mathbb{N}. \tag{2.11}$$

*Then $\{t_n\} \to 0$.*

*Proof.* By item 2 of Lemma 2.3.4, $t_{n+1} \leq t_n$ for all $n \in \mathbb{N}$. Then, $\{t_n\}$ is a non-increasing sequence of non-negative real numbers. Hence, it is convergent. If we denote its limit by $L$, then Lemma 2.3.9 guarantees that $L = 0$. Thus, $\{t_n\} \to 0$. □

### *2.3.3 Ćirić Functions*

Inspired by Boyd and Wong [52] and Mukherjea [141], Lakshmikantham and Ćirić considered in [127] functions satisfying axioms $(\mathcal{P}_{12})$ and $(\mathcal{P}_{14})$.

**Lemma 2.3.11.** *Let $\phi \in \mathcal{F}_{\text{Cir}}$ be a function and let $\{a_m\} \subset [0, \infty)$ be a sequence.*

1. *If $M \geq 0$, then $\phi(t) \leq \max(\phi(0), M)$ for all $t \in [0, M]$. In particular, $\phi(t) \leq \max(\phi(0), t)$ for all $t \geq 0$.*
2. *If $a_{m+1} \leq \phi(a_m)$ for all $m \in \mathbb{N}$, then $a_{m+k} \leq \max(\phi(0), a_m)$ for all $m, k \geq 0$.*
3. *If $a_{m+1} \leq \phi(a_m)$ and $a_m \neq 0$ for all $m \in \mathbb{N}$, then $\{a_m\} \to 0$.*
4. *If there exists $L \geq 0$ such that $\{a_m\} \to L$ and satisfying $L \leq \phi(a_m)$ for all $m \in \mathbb{N}$, then $L = 0$.*
5. *Let $\{b_m\} \subset [0, \infty)$ be a sequence such that $b_m \leq \phi(a_m)$ for all $m \in \mathbb{N}$ and verifying the following condition:*

$$\textit{if there exists some } m_0 \in \mathbb{N} \textit{ such that } a_{m_0} = 0, \textit{ then } b_{m_0} = 0. \tag{2.12}$$

   *Then $b_m \leq a_m$ for all $m \in \mathbb{N}$. As a result, if $\{a_m\} \to 0$, then $\{b_m\} \to 0$.*
6. *Assume that $a_{m+1} \leq \phi(a_m)$ for all $m \in \mathbb{N}$ and the following property holds: if there exists some $m_0 \in \mathbb{N}$ such that $a_{m_0} = 0$, then $a_{m_0+1} = 0$. Then $\{a_m\} \to 0$.*
7. *If $\phi(0) = 0$, then $\phi$ is continuous at $t = 0$.*
8. *If $\phi(0) = 0$ and $\{b_m\} \subset [0, \infty)$ is a sequence verifying $a_m \leq \phi(b_m)$ for all $m$ and $\{b_m\} \to 0$, then $\{a_m\} \to 0$.*
9. *If $\phi(0) = 0$ and $a_{m+1} \leq \phi(a_m)$ for all $m$, then $\{a_m\} \to 0$ and $\{\phi(a_m)\} \to 0$.*

*Proof.* (**1**) Fix $M \geq 0$ and let $t \in [0, M]$ arbitrary. If $t = 0$, it is obvious. If $t > 0$, then $\phi(t) < t \leq M$.

(**2**) If $k = 0$ the result is evident for all $m$. If $k = 1$, the result follows from $a_{m+1} \leq \phi(a_m) \leq \max(\phi(0), a_m)$ for all $m$. Suppose, by induction, that for some $k \geq 1$, the condition "$a_{m+k} \leq \max(\phi(0), a_m)$ for all $m \geq 0$" holds and we are going to prove that it is also true for $k+1$. Indeed, $a_{m+(k+1)} = a_{(m+1)+k} \leq \max(\phi(0), a_{m+1}) \leq \max(\phi(0), \max(\phi(0), a_m)) = \max(\phi(0), a_m)$.

(**3**) Since $a_m \neq 0$, condition (I) implies that $a_{m+1} \leq \phi(a_m) < a_m$ for all $m$. Therefore, $\{a_m\}$ is a decreasing, bounded below sequence of real numbers, so it is convergent, that is, there is $L \geq 0$ such that $\{a_m\} \to L$. We prove that $L = 0$ reasoning by contradiction. Indeed, if $L > 0$, then $0 < L \leq a_{m+1} \leq \phi(a_m) < a_m$ for all $m$. This proves two facts: $\{\phi(a_m)\} \to L$ and $\{a_m\}$ is a strictly decreasing sequence. Hence, by (II),

$$L = \lim_{m \to \infty} \phi(a_m) = \lim_{t \to L^+} \phi(t) < L, \tag{2.13}$$

which is a contradiction. Thus $L = 0$.

(**4**) Assume that $L > 0$ and we will get a contradiction. Since $\{a_m\} \to L$, then there exists $m_0 \in \mathbb{N}$ such that $a_m > L/2 > 0$ for all $m \geq m_0$. As $\phi \in \mathcal{F}_{\mathrm{Cir}}$ and $a_m > 0$ for all $m \geq m_0$, then

$$L \leq \phi\,(a_m) < a_m \quad \text{for all } m \geq m_0. \tag{2.14}$$

Taking the limit as $n \to \infty$, we deduce that $\lim_{m \to \infty} \phi\,(a_m) = L$. Furthermore, (2.14) means that $\{a_{m+m_0}\} \to L^+$. Therefore, since $\phi \in \mathcal{F}_{\mathrm{Cir}}$, we have that

$$L = \lim_{m \to \infty} \phi\,(a_m) = \lim_{m \to \infty} \phi\,(a_{m+n_0}) = \lim_{s \to L^+} \phi\,(s) < L,$$

which is a contradiction. Then, necessarily $L = 0$.

(**5**) We claim that $b_m \leq a_m$ for all $m \in \mathbb{N}$. We prove it by distinguishing two cases. Let $m \in \mathbb{N}$ be arbitrary. If $a_m \neq 0$, then $b_m \leq \phi\,(a_m) < a_m$. On the other case, if $a_m = 0$, then, by hypothesis, $b_m = 0$, so $b_m = 0 = a_m$. In any case, $b_m \leq a_m$ for all $m \in \mathbb{N}$. As a result, if $\{a_m\} \to 0$, then also $\{b_m\} \to 0$.

(**6**) We distinguish two cases.

*Case 1. Suppose that there exists some* $m_0 \in \mathbb{N}$ *such that* $a_{m_0} = 0$. In this case, by hypothesis, $a_{m_0+1} = 0$. Applying again the hypothesis, $a_{m_0+2} = 0$. Thus by induction, $a_m = 0$ for all $m \geq m_0$. In particular, $\{a_m\} \to 0$.

*Case 2. Suppose that* $a_m \neq 0$ *for all* $m \in \mathbb{N}$. In this case, item 3 implies that $\{a_m\} \to 0$.

(**7**) Let $\{b_m\} \subset [0, \infty)$ be a sequence such that $\{b_m\} \to 0$. By item 1 applied to $M = b_m$, we deduce that $0 \leq \phi(b_m) \leq b_m$, so $\{\phi(b_m)\} \to 0$. Therefore, $\phi$ is continuous at $t = 0$.

(**8**) Since $\phi$ is continuous at $t = 0$, then $\{\phi(b_m)\} \to \phi(0) = 0$ and, therefore, $\{a_m\} \to 0$.

**(9)** By item 2, $a_{m+1} \leq \max(\phi(0), a_m) = a_m$ for all $m \geq 0$. Since $\{a_m\}$ is a non-increasing, bounded below sequence of real numbers, it is convergent, that is, there is $L \geq 0$ such that $\{a_m\} \to L$. We prove that $L = 0$ reasoning by contradiction. Indeed, if $L > 0$, then $0 < L \leq a_m$. Hence item 3 shows that $\{a_m\} \to 0$, which contradicts $L > 0$. □

Item 5 of Lemma 2.3.11 would not be valid if we avoid condition (2.12). For example, let $\phi : [0, \infty) \to [0, \infty)$ be defined by $\phi(0) = 1$ and $\phi(t) = 0$ for all $t > 0$. Then $\phi \in \mathcal{F}_{\text{Cir}}$. If we consider the sequences

$$a_m = \begin{cases} 0, \text{ if } m \text{ is even,} \\ 1, \text{ if } m \text{ is odd,} \end{cases} \quad \text{and} \quad b_m = \begin{cases} 1, \text{ if } m \text{ is even,} \\ 0, \text{ if } m \text{ is odd,} \end{cases}$$

then $b_m \leq \phi(a_m)$ for all $m \in \mathbb{N}$. However, condition $b_m \leq a_m$ is false when $m$ is even.

**Lemma 2.3.12.** *Let $\phi \in \mathcal{F}_{\text{Cir}}$ be a function and let $\{t_n\} \subset [0, \infty)$ be a sequence such that $t_{n+1} \leq \phi(t_n)$ for all $n \in \mathbb{N}$. Also assume that the following condition holds:*

$$\text{if there exists some } n_0 \in \mathbb{N} \text{ such that } t_{n_0} = 0, \text{ then } t_{n_0+1} = 0. \tag{2.15}$$

*Then $\{t_n\} \to 0$.*

*Proof.* We distinguish two cases.

*Case 1.* *There exists $n_0 \in \mathbb{N}$ such that $t_{n_0} = 0$.* In this case, by hypothesis (2.15), $t_{n_0+1} = 0$. In fact, $t_n = 0$ for all $n \geq n_0$. In particular, $\{t_n\} \to 0$.

*Case 2.* *$t_n > 0$ for all $n \in \mathbb{N}$.* In this case an easy standard argument guarantees the result. □

**Lemma 2.3.13.** *Let $\varphi \in \mathcal{F}_{\text{Cir}}$ be a function and let $\{t_n\}, \{s_n\} \subset [0, \infty)$ be two sequences that converge to the same limit $L \in [0, \infty)$ and satisfying $t_n \leq \varphi(s_n)$ and $L < s_n$ for all $n \in \mathbb{N}$. Then $L = 0$.*

*Proof.* Assume that $L > 0$ and we will get a contradiction. As $\{s_n\} \to L$, there exists $n_0 \in \mathbb{N}$ such that $s_n \geq L/2 > 0$ for all $n \geq n_0$. Moreover, as $\varphi \in \mathcal{F}_{\text{Cir}}$ and $s_n \neq 0$ for all $n \geq n_0$, then we have

$$t_n \leq \varphi(s_n) < s_n \quad \text{for all } n \geq n_0.$$

Therefore, by Lemma 2.1.1,

$$\lim_{n \to \infty} \varphi(s_n) = L.$$

However, as $\varphi \in \mathcal{F}_{\text{Cir}}$ and $\{s_n\} \to L^+$, we have that

$$L = \lim_{n\to\infty} \varphi(s_n) = \lim_{s\to L^+} \varphi(s) < L,$$

which is a contradiction. As a consequence, $L = 0$. □

**Lemma 2.3.14.** *Let $\varphi \in \mathcal{F}_{\text{Cir}}$ be a function and let $\{t_n\}, \{s_n\} \subset [0,\infty)$ be two sequences that converge to the same limit $L \in [0,\infty)$ and satisfying $L < t_n \leq \varphi(s_n)$ for all $n \in \mathbb{N}$. Then $L = 0$.*

*Proof.* Assume that $L > 0$ and we will get a contradiction. As $\{s_n\} \to L$, there exists $n_0 \in \mathbb{N}$ such that $s_n \geq L/2 > 0$ for all $n \geq n_0$. Moreover, as $\varphi \in \mathcal{F}_{\text{Cir}}$ and $s_n \neq 0$ for all $n \geq n_0$, then we have

$$L < t_n \leq \varphi(s_n) < s_n \quad \text{for all } n \geq n_0.$$

Therefore, by Lemma 2.1.1,

$$\lim_{n\to\infty} \varphi(s_n) = L.$$

However, as $\varphi \in \mathcal{F}_{\text{Cir}}$ and $\{s_n\} \to L^+$, we have that

$$L = \lim_{n\to\infty} \varphi(s_n) = \lim_{s\to L^+} \varphi(s) < L,$$

which is a contradiction. As a consequence, $L = 0$. □

**Lemma 2.3.15.** *Let $\varphi \in \mathcal{F}_{\text{BW}}$ and let $\{a_m\} \subset [0,\infty)$ be a sequence. If $a_{m+1} \leq \varphi(a_m)$ and $a_m \neq 0$ for all $m$, then $\{a_m\} \to 0$.*

### 2.3.4 Properties of Control Functions

In this subsection, we point out some basic facts that the reader can observe.

*Remark 2.3.4.* A strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ satisfies the following property: given $t, s \in [0,\infty)$,

$$\phi(t) \leq \phi(s) \;\Rightarrow\; t \leq s. \tag{2.16}$$

However, non-decreasing functions do not necessarily satisfy (2.16). This is the case of altering distance functions. For example, the function

$$\phi(t) = \begin{cases} t, & \text{if } 0 \leq t \leq 1, \\ 1, & \text{if } t > 1, \end{cases}$$

is an altering distance function but, using $t = 2$ and $s = 1$, it is clear that $\phi$ does not verify (2.16). To use property (2.16) in the context of non-decreasing functions (such as altering distance functions) is a mistake that can be found sometimes in the literature.

*Remark 2.3.5.* Conditions $(\mathcal{P}_{13})$ and $(\mathcal{P}_{15})$ are not strong enough to guarantee property $(\mathcal{P}_3)$. For example, let $N$ be a positive integer and let

$$A_N = \left\{ \frac{k}{N} \in [0,\infty) : k \in \mathbb{N} \right\} = \left\{ 0, \frac{1}{N}, \frac{2}{N}, \frac{3}{N}, \ldots \right\}.$$

Let $\phi : [0,\infty) \to [0,\infty)$ be the function

$$\phi(t) = \begin{cases} 0, & \text{if } t \in A_N, \\ t, & \text{if } 0 < t < \frac{1}{N}, \\ 1, & \text{otherwise.} \end{cases}$$

Then $\phi$ satisfies conditions $(\mathcal{P}_{13})$ and $(\mathcal{P}_{15})$. However, it takes the value zero at infinitely many points.

**Proposition 2.3.5.** *If $\psi : [0,\infty) \to [0,\infty)$ is a subadditive function, then*

$$\frac{1}{n}\,\psi(t) \leq \psi\left(\frac{t}{n}\right) \quad \text{for all } n \in \mathbb{N} \setminus \{0\} \text{ and all } t \in [0,\infty).$$

*Proof.* *Let $n \in \mathbb{N} \setminus \{0\}$ and let $t \in [0,\infty)$. By induction, we prove that*

$$\begin{aligned} \psi(t) &= \psi\left(n\,\frac{t}{n}\right) = \psi\left(\frac{t}{n} + \frac{t}{n} + \overset{(n)}{\cdots} + \frac{t}{n}\right) \\ &\leq \psi\left(\frac{t}{n}\right) + \psi\left(\frac{t}{n}\right) + \overset{(n)}{\cdots} + \psi\left(\frac{t}{n}\right) = n\,\psi\left(\frac{t}{n}\right). \end{aligned}$$

*Hence, the conclusion holds.* □

**Definition 2.3.4.** A *Geraghty function* is a function $\phi : [0,\infty) \to [0,1)$ such that if $\{t_n\} \subset [0,\infty)$ and $\{\phi(t_n)\} \to 1$, then $\{t_n\} \to 0$. Let $\mathcal{F}_{\mathrm{Ger}}$ denote the family of all Geraghty functions.

## 2.4 Metric Structures

**Definition 2.4.1.** A *metric* (or a *distance function*) on a nonempty set $X$ is a mapping $d : X \times X \to [0,\infty)$ satisfying the following conditions: for all $x, y, z \in X$,

- $d(x, x) = 0;$ (2.17)
- $d(x, y) > 0 \quad \text{if } x \neq y;$ (2.18)
- $d(x, y) = d(y, x);$ (2.19)
- $d(x, y) \leq d(x, z) + d(z, y).$ (2.20)

In such a case, the pair $(X, d)$ is called a *metric space*.

It is easy to show that a mapping $d : X \times X \to \mathbb{R}$ is a metric on $X$ if, and only if, it satisfies the following two conditions:

$$(i) \quad d(x, y) = 0 \text{ if, and only if, } x = y; \qquad (ii) \quad d(x, y) \leq d(z, x) + d(z, y)$$

for all $x, y, z \in X$.

*Example 2.4.1.* If $X$ is a nonempty subset of $\mathbb{R}$, the *Euclidean* (or *usual*) *metric on* $X$ is $d(x, y) = |x - y|$ for all $x, y \in X$.

*Example 2.4.2.* If $X$ is a nonempty subset of $\mathbb{R}^n$, the *Euclidean* (or *usual*) *metric on* $X$ is

$$d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2}$$

for all $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in X$.

*Example 2.4.3.* The Euclidean metric on $X \subseteq \mathbb{R}^n$ is a particular case of the distance function

$$d_p(x, y) = (|x_1 - y_1|^p + |x_2 - y_2|^p + \ldots + |x_n - y_n|^p)^{1/p}$$

for all $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in X$, which can be defined for all $p > 0$.

*Example 2.4.4.* Letting $p \to \infty$, we have the metric

$$d_\infty(x, y) = \max \{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}$$

for all $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in X \subseteq \mathbb{R}^n$.

*Example 2.4.5.* If $X$ is an arbitrary nonempty set, the *discrete metric on* $X$ is

$$d(x, y) = \begin{cases} 0, & \textit{if } x = y, \\ 1, & \textit{if } x \neq y. \end{cases}$$

*Example 2.4.6.* If $Y$ is a nonempty subset of $X$ and $d$ is a metric on $X$, then the restriction of $d$ to $Y \times Y$ is also a metric on $Y$.

**Definition 2.4.2.** A mapping $d : X \times X \to [0, \infty)$ is called:

- a *quasi-metric* (or a *nonsymmetric metric*) if it satisfies (2.17), (2.18) and (2.20);
- a *semi-metric* if it satisfies (2.17), (2.18) and (2.19);
- a *pseudo-metric* if it satisfies (2.17), (2.19) and (2.20);
- a *pseudo-quasi-metric* if it satisfies (2.17) and (2.20);
- an *extended real-valued metric* if it is allowed to assume the value $\infty$;
- an *ultrametric* if, instead of (2.20), it satisfies the stronger condition

$$d(x, y) \leq \max\{d(x, z), d(z, y)\} \quad \text{for all } x, y, z \in X. \tag{2.21}$$

If $d$ is a semi-metric on $X$, we say that $(X, d)$ is a *semi-metric space* (and similarly when using other metric structures).

## 2.5 Quasi-metric Spaces

It is of interest to discuss quasi-metrics because these are precisely the metric structure that we obtain when repeating two arguments of a $G$-metric (see Lemma 3.3.1). Therefore, we introduce convergent and Cauchy sequences, and completeness, in the framework of quasi-metric spaces (that include the class of metric spaces). First of all, we recall here the notion of quasi-metric and the notation we will use.

**Definition 2.5.1.** A *quasi-metric on* $X$ is a function $q : X \times X \to [0, \infty)$ satisfying the following properties:

$(q_1)$ $q(x, y) = 0$ if and only if $x = y$;
$(q_2)$ $q(x, y) \leq q(x, z) + q(z, y)$ for any points $x, y, z \in X$.

In such a case, the pair $(X, q)$ is called a *quasi-metric space*.

**Definition 2.5.2.** Let $(X, q)$ be a quasi-metric space, $\{x_n\}$ be a sequence in $X$, and $x \in X$. We will say that:

- $\{x_n\}$ *converges to* $x$ (we denote it by $\{x_n\} \xrightarrow{q} x$) if $\lim_{n\to\infty} q(x_n, x) = \lim_{n\to\infty} q(x, x_n) = 0$;
- $\{x_n\}$ is a *Cauchy sequence* if for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $q(x_n, x_m) < \varepsilon$ for all $n, m \geq n_0$.

The quasi-metric space is said to be *complete* if every Cauchy sequence is convergent.

As $q$ is not necessarily symmetric, some authors distinguished between left/right Cauchy/convergent sequences and completeness.

**Definition 2.5.3 (Jleli and Samet [97]).** Let $(X, q)$ be a quasi-metric space, $\{x_n\}$ be a sequence in $X$, and $x \in X$. We say that:

- $\{x_n\}$ *right-converges to* $x$ if $\lim_{n\to\infty} q(x_n, x) = 0$;
- $\{x_n\}$ *left-converges to* $x$ if $\lim_{n\to\infty} q(x, x_n) = 0$;
- $\{x_n\}$ is a *right-Cauchy sequence* if for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $q(x_n, x_m) < \varepsilon$ for all $m > n \geq n_0$;
- $\{x_n\}$ is a *left-Cauchy sequence* if for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $q(x_m, x_n) < \varepsilon$ for all $m > n \geq n_0$.

*Remark 2.5.1.* 1. The limit of a sequence in a quasi-metric space, if it exists, is unique. However, this is false if we consider right-limits or left-limits.
2. If a sequence $\{x_n\}$ has a right-limit $x$ and a left-limit $y$, then $x = y$, $\{x_n\}$ converges and it has one limit (from the right and from the left). However, it is possible that a sequence has two different right-limits when it has no left-limit.

*Example 2.5.1.* Let $X$ be a subset of $\mathbb{R}$ containing $[0, 1]$ and define, for all $x, y \in X$,

$$q(x, y) = \begin{cases} x - y, & \textit{if } x \geq y, \\ 1, & \textit{otherwise}. \end{cases}$$

Then $(X, q)$ is a quasi-metric space. Note $\{q(1/n, 0)\} \to 0$ but $\{q(0, 1/n)\} \to 1$. Therefore, $\{1/n\}$ right-converges to 0 but it does not converge from the left. We also point out that this quasi-metric satisfies the following property: if a sequence $\{x_n\}$ has a right-limit $x$, then it is unique.

**Definition 2.5.4.** Let $(X, q)$ be a quasi-metric space and let $T : X \to X$ be a mapping. We will say that $T$ is *right-continuous* if $\{q(Tx_n, Tu)\} \to 0$ for all sequence $\{x_n\} \subseteq X$ and all $u \in X$ such that $\{q(x_n, u)\} \to 0$.

To take advantage of some unidimensional results, we need to extend quasi-metrics on $X$ to the product space $X^2$. The following is an easy way to consider quasi-metrics on $X^2$ *via* quasi-metrics on $X$.

**Lemma 2.5.1 (Agarwal et al. [14]).** *Let* $q : X^2 \to [0, \infty)$ *and* $Q_s^q, Q_m^q : X^4 \to [0, \infty)$ *be three mappings verifying*

$$Q_s^q((x_1, x_2), (y_1, y_2)) = q(x_1, y_1) + q(x_2, y_2) \quad \textit{and}$$

$$Q_m^q((x_1, x_2), (y_1, y_2)) = \max(q(x_1, y_1), q(x_2, y_2)) \quad \textit{for all } x_1, x_2, y_1, y_2 \in X.$$

*Then the following conditions are equivalent.*

**(a)** *$q$ is a quasi-metric on $X$.*
**(b)** *$Q_s^q$ is a quasi-metric on $X^2$.*
**(c)** *$Q_m^q$ is a quasi-metric on $X^2$.*

*In such a case, the following properties hold.*

1. *Every sequence* $\{(x_n, y_n)\} \subseteq X^2$ *verifies:*

$$\{(x_n, y_n)\} \xrightarrow{Q_s^q} (x, y) \iff \{(x_n, y_n)\} \xrightarrow{Q_m^q} (x, y)$$
$$\iff \left[ \{x_n\} \xrightarrow{q} x \text{ and } \{y_n\} \xrightarrow{q} y \right].$$

2. $\{(x_n, y_n)\} \subseteq X^2$ *is* $Q_s^q$*-Cauchy* $\iff$ $\{(x_n, y_n)\}$ *is* $Q_m^q$*-Cauchy* $\iff$ $[\{x_n\}$ *and* $\{y_n\}$ *are q-Cauchy*$]$.
3. *Items 1 and 2 are valid from the right and from the left.*
4. $(X, q)$ *is right-complete* $\iff$ $(X^2, Q_s^q)$ *is right-complete* $\iff$ $(X^2, Q_m^q)$ *is right-complete.*
5. $(X, q)$ *is left-complete* $\iff$ $(X^2, Q_s^q)$ *is left-complete* $\iff$ $(X^2, Q_m^q)$ *is left-complete.*
6. $(X, q)$ *is complete* $\iff$ $(X^2, Q_s^q)$ *is complete* $\iff$ $(X^2, Q_m^q)$ *is complete.*
7. *The following conditions are equivalent.*

**(7.1)** *Each right-convergent sequence in* $(X, q)$ *has an unique right-limit.*
**(7.2)** *Each right-convergent sequence in* $(X^2, Q_s^q)$ *has an unique right-limit.*
**(7.3)** *Each right-convergent sequence in* $(X^2, Q_m^q)$ *has an unique right-limit.*

## 2.6 Topological Structures

**Definition 2.6.1.** A *topology on* $X$ is a family $\tau = \{A_i\}_{i\in\Lambda}$ of subsets of $X$ that includes both $X$ and $\emptyset$, and is closed under arbitrary unions and finite intersections, that is,

- $X, \emptyset \in \tau$,
- if $\Lambda' \subseteq \Lambda$, then $\cup_{i\in\Lambda'} A_i \in \tau$,
- if $n \in \mathbb{N}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \Lambda$, then $A_{\lambda_1} \cap A_{\lambda_2} \cap \ldots \cap A_{\lambda_n} \in \tau$.

In such a case, we say that $(X, \tau)$ is a *topological space*. A subset $A$ of $X$ is $\tau$*-open* (or *open relative to* $\tau$) if $A \in \tau$, and it is $\tau$*-closed* if its complement $X \diagdown A$ is $\tau$-open.

A subset $U \subseteq X$ is a $\tau$*-neighborhood of a point* $x \in X$ if there exists $A \in \tau$ such that $x \in A \subseteq U$. A topology $\tau$ on $X$ satisfies the *Hausdorff separation property* if it satisfies the following condition:

"for all distinct points $x, y \in X$, there exists a $\tau$-neighborhood $U$ of $x$ and a $\tau$-neighborhood $V$ of $y$ such that $U \cap V = \emptyset$".

If $\tau$ is a topology on $X$, the family $\beta_x$ of all $\tau$-open subsets of $X$ that contains $x$ is a neighborhood system at $x$. The following lemma provides an easy way to consider a topology on a set $X$ generated by the balls of a metric structure.

**Lemma 2.6.1.** *Let $X$ be a set and, for all $x \in X$, let $\beta_x$ be a non-empty family of subsets of $X$ verifying:*

1. *$x \in B$ for all $B \in \beta_x$.*
2. *For all $B_1, B_2 \in \beta_x$, there exists $B_3 \in \beta_x$ such that $B_3 \subseteq B_1 \cap B_2$.*
3. *For all $B \in \beta_x$, there exists $B' \in \beta_x$ such that for all $y \in B'$, there exists $B'' \in \beta_y$ verifying $B'' \subseteq B$.*

*Then there exists a unique topology $\tau$ on $X$ such that $\beta_x$ is a neighborhood system at $x$.*

# Chapter 3
# $G$-Metric Spaces

In this chapter we introduce the concept of *G-metric* on a set $X$, and we show some of its basic properties. We provide any $G$-metric space with a Hausdorff topology in which the notions of convergent and Cauchy sequences will be a key tool in almost all proofs. Later, we will study the close relationships between $G$-metrics and quasi-metrics.

## 3.1 $G$-Metric Spaces

In 2003, Mustafa and Sims [154] proved that most of the claims concerning the topological properties of $D$-metrics were incorrect. In order to repair these drawbacks, they gave a more appropriate notion of *generalized metrics*, called *G-metrics*. Mustafa provided many examples of $G$-metric spaces in [142] and developed some of their properties. For example, he proved that $G$-metric spaces are provided with a Hausdorff topology which allows us to consider, among other topological notions, convergent sequences, limits, Cauchy sequences, continuous mappings, completeness and compactness. He also developed further topics in $G$-metric spaces such as the properties of ordinary metrics derived from a $G$-metric, and he investigated the properties of $G$-metrics derived from ordinary metrics.

**Definition 3.1.1 (Mustafa and Sims [154]).** A *G-metric space* is a pair $(X, G)$ where $X$ is a nonempty set and $G : X \times X \times X \to [0, \infty)$ is a function such that, for all $x, y, z, a \in X$, the following conditions are fulfilled:

$$(G_1) \quad G(x,y,z) = 0 \quad \text{if } x = y = z; \tag{3.1}$$

$$(G_2) \quad G(x,x,y) > 0 \quad \text{for all } x, y \in X \text{ with } x \neq y; \tag{3.2}$$

$$(G_3) \quad G(x,x,y) \leq G(x,y,z) \quad \text{for all } x, y, z \in X \text{ with } z \neq y; \tag{3.3}$$



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$$(G_4) \quad G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots (\textit{symmetry in all 3}); \tag{3.4}$$

$$(G_5) \quad G(x,y,z) \le G(x,a,a) + G(a,y,z) \quad (\textit{rectangle inequality}). \tag{3.5}$$

In such a case, the function $G$ is called a *$G$-metric on $X$*.

The previous properties may be easily interpreted in the setting of metric spaces. Let $(X,d)$ be a metric space and define $G : X \times X \times X \to [0,\infty)$ by

$$G(x,y,z) = d(x,y) + d(x,z) + d(y,z) \quad \text{for all } x,y,z \in X.$$

Then $(X,G)$ is a $G$-metric space. In this case, $G(x,y,z)$ can be interpreted as the perimeter of the triangle of vertices $x$, $y$ and $z$. For example, $(G_1)$ means that with one point we cannot have a positive perimeter, and $(G_2)$ is equivalent to the fact that the distance between two different points cannot be zero. Furthermore, as the perimeter of a triangle cannot depend on the order in which we consider its vertices, we have $(G_4)$, and $(G_5)$ is an extension of the triangle inequality using a fourth vertex. Maybe, the most controversial axiom is $(G_3)$, which has an obvious geometric interpretation: the length of an edge of a triangle is less than or equal to its semiperimeter, that is,

$$d(x,y) \le \frac{d(x,y) + d(y,z) + d(z,x)}{2}.$$

*Example 3.1.1.* If $X$ is a non-empty subset of $\mathbb{R}$, then the function $G : X \times X \times X \to [0,\infty)$, given by

$$G(x,y,z) = |x-y| + |x-z| + |y-z| \quad \text{for all } x,y,z \in X,$$

is a $G$-metric on $X$.

*Example 3.1.2.* Every non-empty set $X$ can be provided with the *discrete $G$-metric*, which is defined, for all $x,y,z \in X$, by

$$G(x,y,z) = \begin{cases} 0, & \text{if } x = y = z, \\ 1, & \text{otherwise.} \end{cases}$$

*Example 3.1.3.* Let $X = [0,\infty)$ be the interval of nonnegative real numbers and let $G$ be defined by:

$$G(x,y,z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x,y,z\}, & \text{otherwise.} \end{cases}$$

Then $G$ is a complete $G$-metric on $X$.

*Example 3.1.4.* If $G$ is a $G$-metric on $X$ and $\alpha > 0$, then $G_\alpha$, defined by $G_\alpha(x,y,z) = \alpha\, G(x,y,z)$ for all $x,y,z \in X$, is another $G$-metric on $X$.

*Example 3.1.5.* If $G$ is a $G$-metric on $X$, then $G' : X^3 \to [0, \infty)$, given by

$$G'(x, y, z) = \frac{G(x, y, z)}{1 + G(x, y, z)} \quad \text{for all } x, y, z \in X,$$

is another $G$-metric on $X$.

### 3.1.1 Basic Properties

One of the most useful properties of $G$-metrics is the following one.

**Lemma 3.1.1.** *If $(X, G)$ is a G-metric space, then*

$$G(x, y, y) \le 2G(y, x, x) \quad \text{for all } x, y \in X. \tag{3.6}$$

*Proof.* By the rectangle inequality (3.5) together with the symmetry (3.4), we have

$$G(x, y, y) = G(y, y, x) \le G(y, x, x) + G(x, y, x) = 2G(y, x, x).$$

□

**Corollary 3.1.1.** *Let $\{x_n\}$ and $\{y_n\}$ be two sequences of a G-metric space $(X, G)$. Then*

$$\lim_{n\to\infty} G(x_n, x_n, y_n) = 0 \text{ if, and only if, } \lim_{n\to\infty} G(x_n, y_n, y_n) = 0.$$

*Proof.* It follows from the fact that, by using $(G_4)$ and Lemma 3.1.1, for all $n \in \mathbb{N}$,

$$0 \le G(x_n, x_n, y_n) \le 2G(x_n, y_n, y_n) \le 4G(x_n, x_n, y_n).$$

Therefore, Lemma 2.1.1 is applicable. □

The following lemma can be derived easily from the definition of a $G$-metric space.

**Lemma 3.1.2 (See, e.g., [154]).** *Let $(X, G)$ be a G-metric space. Then, for any $x, y, z, a \in X$, the following properties hold.*

1. $G(x, y, z) \le G(x, x, y) + G(x, x, z)$.
2. $G(x, y, z) \le G(x, a, a) + G(y, a, a) + G(z, a, a)$.
3. $|G(x, y, z) - G(x, y, a)| \le \max\{G(a, z, z), G(z, a, a)\}$.
4. *If $n \ge 2$ and $x_1, x_2, \ldots, x_n \in X$, then*

$$G(x_1, x_n, x_n) \le \sum_{i=1}^{n-1} G(x_i, x_{i+1}, x_{i+1}) \quad \text{and} \tag{3.7}$$

$$G(x_1, x_1, x_n) \leq \sum_{i=1}^{n-1} G(x_i, x_i, x_{i+1}).$$

5. *If* $G(x, y, z) = 0$*, then* $x = y = z$*.*
6. $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$*.*
7. $G(x, y, z) \leq \frac{2}{3}[G(x, y, a) + G(x, a, z) + G(a, y, z)]$*.*
8. *If* $x \in X \setminus \{z, a\}$*, then* $|G(x, y, z) - G(x, y, a)| \leq G(a, x, z)$*.*
9. $G(x, y, y) \leq 2\, G(x, y, z)$*.*

*Remark 3.1.1.* The reader may observe that properties 1, 2, 3 and 4 can be proved without using axiom $(G_3)$.

*Proof.* **(1)** Applying $(G_4)$ and $(G_5)$ using $a = x$, we have that

$$\begin{aligned} G(x, y, z) = G(y, x, z) &\leq G(y, x, x) + G(x, x, z) \\ &= G(x, x, y) + G(x, x, z). \end{aligned}$$

**(2)** By using $(G_5)$ twice and also $(G_4)$,

$$\begin{aligned} G(x, y, z) \leq G(x, a, a) + G(a, y, z) &= G(x, a, a) + G(y, a, z) \\ &\leq G(x, a, a) + G(y, a, a) + G(a, a, z). \end{aligned}$$

**(3)** By $(G_4)$ and $(G_5)$,

$$\begin{aligned} &G(x, y, z) = G(z, y, x) \leq G(z, a, a) + G(a, y, x), \\ &G(a, y, x) \leq G(a, z, z) + G(z, y, x). \end{aligned}$$

Therefore,

$$\begin{aligned} &G(x, y, z) - G(a, y, x) \leq G(z, a, a) \quad \text{and} \\ &G(a, y, x) - G(x, y, z) \leq G(a, z, z). \end{aligned}$$

Hence, $|G(x, y, z) - G(x, y, a)| \leq \max\{G(a, z, z), G(z, a, a)\}$.

**(4)** If $n = 2$, it is obvious, and if $n = 3$, then (3.7) is property $(G_5)$ using $x = x_1$, $a = x_2$ and $y = z = x_3$. By induction, if (3.7) holds for some $n \geq 3$, then it is also valid for $n + 1$ because, also by $(G_5)$ and the hypothesis of induction,

$$\begin{aligned} G(x_1, x_{n+1}, x_{n+1}) &\leq G(x_1, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}) \\ &\leq \sum_{i=1}^{n-1} G(x_i, x_{i+1}, x_{i+1}) + G(x_n, x_{n+1}, x_{n+1}) \\ &= \sum_{i=1}^{n} G(x_i, x_{i+1}, x_{i+1}). \end{aligned}$$

**(5)** Assume that $G(x,y,z)=0$. We claim that if $y\neq z$, then $x=y$. Indeed, by $(G_3)$, $0\leq G(x,x,y)\leq G(x,y,z)=0$, so $G(x,x,y)=0$. If $x\neq y$, then $G(x,x,y)>0$ by $(G_2)$, so condition $G(x,x,y)=0$ implies that $x=y$. As $G$ is symmetric on its variables, we have also proved that if $z\neq y$, then $x=z$. Hence, $y=x=z$, which is a contradiction with the hypothesis $y\neq z$. Then, all arguments must be equal ($x=y=z$).

**(6)** If $a=y$ or $a=x$, the result is obvious. Assume that $a\neq x$ and $a\neq y$. If $a=z$, then, by $(G_5)$,

$$\begin{aligned} G(x,y,z) &= G(x,y,a)\leq G(x,a,a)+G(a,y,a)\\ &\leq G(x,a,z)+G(a,y,z)\,. \end{aligned}$$

Next, assume that $a\neq z$. Then, by $(G_5)$ and $(G_3)$,

$$G(x,y,z)\leq G(x,a,a)+G(a,y,z)\leq G(x,a,z)+G(a,y,z)\,.$$

**(7)** By item (6) and $(G_4)$,

$$\begin{aligned} G(x,y,z) &\leq G(x,a,z)+G(a,y,z)\,,\\ G(x,y,z) &= G(y,z,x)\leq G(y,a,x)+G(a,z,x)\,,\\ G(x,y,z) &= G(z,x,y)\leq G(z,a,y)+G(a,x,y)\,. \end{aligned}$$

Adding the previous inequalities and using $(G_4)$,

$$3\,G(x,y,z)\leq 2\,[G(x,y,a)+G(x,a,z)+G(a,y,z)\,]\,.$$

**(8)** By item (3), $|\,G(x,y,z)-G(x,y,a)\,|\leq\max\{G(a,z,z),G(z,a,a)\}$. Then, using $(G_3)$,

$$\begin{aligned} x\neq a &\quad\Rightarrow\quad G(z,a,a)\leq G(z,a,x);\\ x\neq z &\quad\Rightarrow\quad G(a,z,z)\leq G(a,z,x). \end{aligned}$$

Then, by $(G_4)$, we conclude that $\max\{G(a,z,z),G(z,a,a)\}\leq G(x,a,z)$.

**(9)** We distinguish two cases. If $y=z$ then, $G(x,y,y)=G(x,y,z)\leq 2\,G(x,y,z)$. On the contrary case, if $y\neq z$, using Lemma 3.1.1 and axiom $(G_3)$, it follows that $G(x,y,y)\leq 2\,G(x,x,y)\leq 2\,G(x,y,z)$. □

### 3.1.2 Some Relationships Between Metrics and G-Metrics

Every metric on $X$ induces $G$-metrics on $X$ in different ways.

**Lemma 3.1.3 ([154]).** *If $(X, d)$ is a metric space, then the functions $G_m^d, G_s^d : X^3 \to [0, \infty)$, defined by*

$$G_m^d(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}, \tag{3.8}$$

$$G_s^d(x, y, z) = d(x, y) + d(y, z) + d(z, x) \tag{3.9}$$

*for all $x, y, z \in X$, are G-metrics on $X$. Furthermore,*

$$G_m^d(x, y, z) \leq G_s^d(x, y, z) \leq 3\, G_m^d(x, y, z) \quad \textit{for all } x, y, z \in X.$$

Conversely, a $G$-metric on $X$ also induces some metrics on $X$.

**Lemma 3.1.4.** *If $(X, G)$ is a G-metric space, then the functions $d_m^G, d_s^G : X^2 \to [0, \infty)$ defined by*

$$d_m^G(x, y) = \max\{G(x, y, y), G(y, x, x)\} \quad \textit{and}$$

$$d_s^G(x, y) = G(x, y, y) + G(y, x, x)$$

*for all $x, y \in X$, are metrics on $X$. Furthermore, the following properties hold.*

1. *$d_m^G(x, y) \leq d_s^G(x, y) \leq 2\, d_m^G(x, y)$ for all $x, y \in X$.*
2. *$d_m^G$ and $d_s^G$ are equivalent metrics on $X$ and they generate the same topology on $X$.*

The following result collects some basic relations between metrics and $G$-metrics involved in Lemmas 3.1.3 and 3.1.4.

**Lemma 3.1.5.** *If $d$ is a metric on $X$, then, for all $x, y \in X$,*

$$d_m^{G_m^d}(x, y) = d(x, y), \quad d_s^{G_m^d}(x, y) = d_m^{G_s^d}(x, y) = 2\, d(x, y),$$

$$d_s^{G_s^d}(x, y) = 4\, d(x, y),$$

$$d_s^{G_s^d} = 2\, d_s^{G_m^d} = 2\, d_m^{G_s^d} = 4\, d_m^{G_m^d} = 4\, d.$$

*Conversely, if $G$ is a G-metric on $X$, then, for all $x, y, z \in X$,*

$$G_m^{d_m^G}(x, y, z) = \max\{G(x, y, y), G(y, x, x), G(y, z, z), G(z, y, y), G(z, x, x), G(x, z, z)\},$$

$$G_s^{d_s^G}(x,y,z) = G(x,y,y) + G(y,x,x) + G(y,z,z) + G(z,y,y) + G(z,x,x) + G(x,z,z).$$

*In particular,* $G_m^{d_m^G} \le G_s^{d_s^G} \le 6\, G_m^{d_m^G}$.

*Proof.* We have the following straightforward calculations:

$$\begin{aligned} d_m^{G_m^d}(x,y) &= \max\left\{G_m^d(x,y,y), G_m^d(y,x,x)\right\} \\ &= \max\{\max(d(x,y), d(x,y), d(y,y)), \max(d(y,x), d(y,x), d(x,x))\} \\ &= d(x,y); \end{aligned}$$

$$\begin{aligned} d_s^{G_m^d}(x,y) &= G_m^d(x,y,y) + G_m^d(y,x,x) \\ &= \max(d(x,y), d(x,y), d(y,y)) + \max(d(y,x), d(y,x), d(x,x)) \\ &= 2\, d(x,y); \end{aligned}$$

$$\begin{aligned} d_m^{G_s^d}(x,y) &= \max\left\{G_s^d(x,y,y), G_s^d(y,x,x)\right\} \\ &= \max\{d(x,y) + d(x,y) + d(y,y),\ d(y,x) + d(y,x) + d(x,x)\} \\ &= 2\, d(x,y); \end{aligned}$$

$$\begin{aligned} d_s^{G_s^d}(x,y) &= G_s^d(x,y,y) + G_s^d(y,x,x) \\ &= d(x,y) + d(x,y) + d(y,y) + d(y,x) + d(y,x) + d(x,x) \\ &= 4\, d(x,y). \end{aligned}$$

Conversely,

$$\begin{aligned} G_m^{d_m^G}(x,y,z) &= \max\left\{d_m^G(x,y), d_m^G(y,z), d_m^G(z,x)\right\} \\ &= \max\{\max\{G(x,y,y), G(y,x,x)\}, \\ &\qquad \max\{G(y,z,z), G(z,y,y)\}, \\ &\qquad \max\{G(z,x,x), G(x,z,z)\}\} \\ &= \max\{G(x,y,y), G(y,x,x), G(y,z,z), G(z,y,y), \\ &\qquad G(z,x,x), G(x,z,z)\}; \end{aligned}$$

and

$$\begin{aligned} G_s^{d_s^G}(x,y,z) &= d_m^G(x,y) + d_m^G(y,z) + d_m^G(z,x) \\ &= G(x,y,y) + G(y,x,x) + G(y,z,z) + G(z,y,y) \\ &\quad + G(z,x,x) + G(x,z,z). \end{aligned}$$

□

### 3.1.3 Symmetric G-Metric Spaces

A $G$-metric space $(X, G)$ is called *symmetric* if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

The mappings given in Examples 3.1.1, 3.1.2 and 3.1.3 are symmetric G-metrics. There also exist $G$-metric spaces that are not symmetric, as we see in the following example.

*Example 3.1.6.* Let $X = \{0, 1, 2\}$ and let $G : X \times X \times X \to [0, \infty)$ be the function given by the following table.

| $(x, y, z)$ | $G(x, y, z)$ |
|---|---|
| $(0,0,0)$, $(1,1,1)$, $(2,2,2)$ | 0 |
| $(0,0,1)$, $(0,1,0)$, $(1,0,0)$, $(0,1,1)$, $(1,0,1)$, $(1,1,0)$ | 1 |
| $(1,2,2)$, $(2,1,2)$, $(2,2,1)$ | 2 |
| $(0,0,2)$, $(0,2,0)$, $(2,0,0)$, $(0,2,2)$, $(2,0,2)$, $(2,2,0)$ | 3 |
| $(1,1,2)$, $(1,2,1)$, $(2,1,1)$, $(0,1,2)$, $(0,2,1)$, $(1,0,2)$ | 4 |
| $(1,2,0)$, $(2,0,1)$, $(2,1,0)$ | 4 |

Then $G$ is a $G$-metric on $X$, but it is not symmetric because $G(1, 1, 2) = 4 \neq 2 = G(2, 2, 1)$.

**Lemma 3.1.6.** *If $(X, d)$ is a metric space, then $G_m^d$ and $G_s^d$ (defined in Lemma 3.1.4) are symmetric $G$-metrics on $X$. In fact,*

$$G_s^d(x, y, y) = 2G_m^d(x, y, y) = 2d(x, y) \quad \textit{for all } x, y \in X.$$

*Conversely, if $(X, G)$ is a symmetric $G$-metric space and $d_G : X \times X \to [0, \infty)$ is defined by*

$$d_G(x, y) = G(x, y, y) \quad \textit{for all } x, y \in X,$$

*then $(X, d_G)$ is a metric space.*

Given two $G$-metric spaces $(X_1, G_1)$ and $(X_2, G_2)$, the function $G : X \times X \times X \to [0, \infty)$ given, for all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X = X_1 \times X_2$, by

$$G((x_1, x_2), (y_1, y_2), (z_1, z_2)) = G_1(x_1, y_1, z_1) + G_2(x_2, y_2, z_2)$$

is not necessarily a $G$-metric on the Cartesian product $X$ because, although it satisfies conditions $(G_1)$, $(G_2)$, $(G_4)$ and $(G_5)$, axiom $(G_3)$ is not guaranteed. This only holds when the factors are symmetric.

**Theorem 3.1.1 ([154]).** *Let $\{(X_i, G_i)\}_{i=1}^n$ be a finite family of G-metric spaces and let $X = X_1 \times X_2 \times \ldots \times X_n$ be the Cartesian product. Consider the mappings $G_m, G_s : X \times X \times X \to [0, \infty)$ defined by*

$$G_m(A, B, C) = \max_{1 \leq i \leq n} G_i(a_i, b_i, c_i) \quad \text{and}$$

$$G_s(A, B, C) = \sum_{i=1}^{n} G_i(a_i, b_i, c_i),$$

*for all $A = (a_1, a_2, \ldots, a_n), B = (b_1, b_2, \ldots, b_n), C = (c_1, c_2, \ldots, c_n) \in X$. Then the following conditions are equivalent.*

1. *$(X_i, G_i)$ is a symmetric G-metric space for all $i \in \{1, 2, \ldots, n\}$.*
2. *$(X, G_m)$ is a symmetric G-metric space.*
3. *$(X, G_s)$ is a symmetric G-metric space.*

*Proof.* We only prove the equivalence between the first two items (the other equivalence is similar). First of all, we claim that $(X, G_m)$ satisfies the axioms $(G_1)$, $(G_2)$, $(G_4)$ and $(G_5)$.

$(G_1)$ $G_m(A, A, A) = \max_{1 \leq i \leq n} G_i(a_i, a_i, a_i) = \max_{1 \leq i \leq n} 0 = 0$.

$(G_2)$ Assume that $A \neq B$. Then, there exists $j \in \{1, 2, \ldots, n\}$ such that $a_j \neq b_j$. Since $G_j$ is a $G$-metric on $X_j$, then $G_j(a_j, a_j, b_j) > 0$. Hence, $G_m(A, A, B) = \max_{1 \leq i \leq n} G_i(a_i, a_i, b_i) \geq G_j(a_j, a_j, b_j) > 0$.

$(G_4)$ It follows from the fact that each $G_i$ is symmetric in its three variables.

$(G_5)$ Let $A = (a_1, a_2, \ldots, a_n), B = (b_1, b_2, \ldots, b_n), C = (c_1, c_2, \ldots, c_n), D = (d_1, d_2, \ldots, d_n) \in X$ be arbitrary. Then

$$\begin{aligned} G_m(B, C, D) &= \max_{1 \leq i \leq n} G_i(b_i, c_i, d_i) \\ &\leq \max_{1 \leq i \leq n} [G_i(b_i, a_i, a_i) + G_i(a_i, c_i, d_i)] \\ &\leq \max_{1 \leq i \leq n} G_i(b_i, a_i, a_i) + \max_{1 \leq i \leq n} G_i(a_i, c_i, d_i) \\ &= G_m(B, A, A) + G_m(A, C, D). \end{aligned}$$

Next, we prove the equivalence between (1) and (2).

(1) $\Rightarrow$ (2). Assume that $(X_i, G_i)$ is a symmetric $G$-metric space for all $i \in \{1, 2, \ldots, n\}$. We claim that $(X, G_m)$ verifies $(G_3)$. Let $A, B, C \in X$ be such that $B \neq C$. Let $i \in \{1, 2, \ldots, n\}$ be arbitrary. If $b_i = c_i$, then, using that $(X_i, G_i)$ is symmetric, $G_i(a_i, a_i, b_i) = G_i(b_i, b_i, a_i) = G_i(c_i, b_i, a_i) = G_i(a_i, b_i, c_i)$. If $b_i \neq c_i$, then $G_i(a_i, a_i, b_i) \leq G_i(a_i, b_i, c_i)$ by the axiom $(G_3)$ in $(X_i, G_i)$. In any case, we have just proved that $G_i(a_i, a_i, b_i) \leq G_i(a_i, b_i, c_i)$ for all $i \in \{1, 2, \ldots, n\}$. Therefore,

$$G_m(A, A, B) = \max_{1 \leq i \leq n} G_i(a_i, a_i, b_i) \leq \max_{1 \leq i \leq n} G_i(a_i, b_i, c_i) = G_m(A, B, C).$$

Hence, $(X, G_m)$ is a $G$-metric space. Moreover, it is symmetric because each factor is symmetric, that is, for all $A, B \in X$,

$$G_m(A, A, B) = \max_{1 \le i \le n} G_i(a_i, a_i, b_i) = \max_{1 \le i \le n} G_i(b_i, b_i, a_i) = G_m(B, A, A).$$

(2) $\Rightarrow$ (1). Assume that $(X, G_m)$ is a symmetric $G$-metric space. Fix a point $P = \left(p_1^0, p_2^0, \ldots, p_n^0\right) \in X$ and, for all $i \in \{1, 2, \ldots, n\}$ and all $a_i \in X_i$, let

$$A_P^i = (p_1^0, p_2^0, \ldots, p_{i-2}^0, p_{i-1}^0, a_i, p_{i+1}^0, p_{i+2}^0, \ldots, p_n^0) \in X.$$

Notice that for all $i \in \{1, 2, \ldots, n\}$ and all $a_i, b_i, c_i \in X_i$, we have that

$$G_i(a_i, b_i, c_i) = G_m(A_P^i, B_P^i, C_P^i). \tag{3.10}$$

Therefore, for all $i \in \{1, 2, \ldots, n\}$ and all $a_i, b_i \in X_i$,

$$G_i(a_i, a_i, b_i) = G_m(A_P^i, A_P^i, B_P^i) = G_m(B_P^i, B_P^i, A_P^i) = G_i(b_i, b_i, a_i).$$

Thus, each factor $(X_i, G_i)$ is symmetric. □

## 3.2 Topology of a *G*-Metric Space

In this section we introduce the canonical Hausdorff topology of a $G$-metric space and we present its corresponding topological notions.

**Definition 3.2.1 ([142]).** The *open ball of center $x \in X$ and radius $r > 0$* in a $G$-metric space $(X, G)$ is the subset $B_G(x, r) = \{y \in X : G(x, y, y) < r\}$. Similarly, the *closed ball of center $x \in X$ and radius $r > 0$* is $\overline{B}_G(x, r) = \{y \in X : G(x, y, y) \le r\}$.

Clearly, $x \in B_G(x, r) \subseteq \overline{B}_G(x, r)$.

**Proposition 3.2.1.** *If $(X, G)$ is a G-metric space and $d_m^G$ and $d_s^G$ are the metrics on $X$ defined in Lemma 3.1.4, then*

$$B_{d_s^G}(x, r) \subseteq B_{d_m^G}(x, r) \subseteq B_G(x, r) \subseteq B_{d_m^G}(x, 2r) \subseteq B_G(x, 2r)$$

*for all $x \in X$ and all $r > 0$.*

*Proof.* Let $y \in B_{d_m^G}(x, r)$. Then $\max\{G(x, y, y), G(y, x, x)\} = d_m^G(x, y) < r$. In particular, $G(x, y, y) < r$, so $y \in B_G(x, r)$. This proves that $B_{d_m^G}(x, r) \subseteq B_G(x, r)$. In a similar way, $B_{d_m^G}(x, 2r) \subseteq B_G(x, 2r)$. Now let $y \in B_G(x, r)$. Then $G(x, y, y) < r$. By Lemma 3.1.1, it follows that

$$G(y, x, x) \le 2G(x, y, y) < 2r.$$

Therefore, $d_m^G(x,y) = \max\{G(x,y,y), G(y,x,x)\} < 2r$, so $y \in B_{d_m^G}(x,2r)$. This proves that $B_G(x,r) \subseteq B_{d_m^G}(x,2r)$. □

*Example 3.2.1 ([142]).* Let $(X,d)$ be a metric space and let $G_m^d$ and $G_s^d$ be the $G$-metrics on $X$ defined in Lemma 3.1.3. Then, for all $x_0 \in X$ and all $r > 0$, we have the following properties:

$$B_{G_m^d}(x_0,r) = B_d(x_0,r), \quad B_{G_s^d}(x_0,r) = B_d\left(x_0,\frac{r}{2}\right),$$
$$\overline{B}_{G_m^d}(x_0,r) = \overline{B}_d(x_0,r) \quad \text{and} \quad \overline{B}_{G_s^d}(x_0,r) = \overline{B}_d\left(x_0,\frac{r}{2}\right).$$

*Example 3.2.2 ([142]).* Let $X$ be a nonempty set, and let $G_{dis}$ be the discrete $G$-metric on $X$ (see Example 3.1.2). For any $x_0 \in X$ and all $r > 0$, we have the following properties:

1. if $r < 1$, then $B_{G_{dis}}(x_0,r) = \overline{B}_{G_{dis}}(x_0,r) = \{x_0\}$;
2. if $r = 1$, then $B_{G_{dis}}(x_0,r) = \{x_0\}$ and $\overline{B}_{G_{dis}}(x_0,r) = X$; and
3. if $r > 1$, then $B(x_0,r) = \overline{B}(x_0,r) = X$.

The family of all open balls permit us to consider a topology on $X$.

**Theorem 3.2.1.** *There exists a unique topology $\tau_G$ on a $G$-metric space $(X,G)$ such that, for all $x \in X$, the family $\beta_x$ of all open balls centered at $x$ is a neighbourhood system at $x$. Furthermore, $\tau_G$ is metrizable because it is the metric topology on $X$ generated by the equivalent metrics $d_m^G$ and $d_s^G$ (defined in Lemma 3.1.4). In particular, $\tau_G$ satisfies the Hausdorff separation property.*

*Proof.* We use Lemma 2.6.1. The first two properties are trivial because $B(x,r_1) \cap B(x,r_2) = B(x,\min\{r_1,r_2\})$. Let $B = B(x,r) \in \beta_x$ be an open ball and let $B' = B \in \beta_x$. We have to prove that for all $y \in B$, there exists $B'' \in \beta_y$ satisfying $B'' \subseteq B$. Indeed, fix $y \in B = B(x,r)$. Therefore $G(x,y,y) < r$. Let $s$ and $\delta > 0$ be arbitrary numbers such that $G(x,y,y) < s < s+\delta < r$. We claim that $B'' = B(y,\delta) \subseteq B = B(x,r)$. To prove it, let $z \in B(y,\delta)$ be arbitrary, that is, $G(y,z,z) < \delta$. Then, by axiom $(G_5)$, it follows that $G(x,z,z) \leq G(x,y,y) + G(y,z,z) < s+\delta < r$. Hence $z \in B(x,r)$. Lemma 2.6.1 guarantees that there exists a unique topology $\tau_G$ on a $G$-metric space $(X,G)$ such that, for all $x \in X$, the family $\beta_x$ of all open balls centered at $x$ is a neighbourhood system at $x$.

Proposition 3.2.1 guarantees that, for all $x \in X$ and all $r > 0$,

$$B_{d_m^G}(x,r) \subseteq B_G(x,r) \subseteq B_{d_m^G}(x,2r) \subseteq B_G(x,2r).$$

This means that, for all $x \in X$, the family $\beta'_x = \left\{B_{d_m^G}(x,r) : r > 0\right\}$ is a neighbourhood system at $x$ equivalent to $\beta_x$, that is, they generate the same topology. Therefore, $\tau_G = \tau_{d_m^G}$, which implies that $\tau_G$ is metrizable and it satisfies the Hausdorff separation property. □

The following notions can be considered on a topological space (see [23, 51]), but we particularize them to the case of the topology $\tau_G$.

- A subset $U \subseteq X$ is a *G-neighborhood of a point* $x \in X$ if there is $r > 0$ such that $B_G(x, r) \subseteq U$.
- A subset $U \subseteq X$ is *G-open* if either it is empty or it is a $G$-neighborhood of all its points.
- A subset $U \subseteq X$ is *G-closed* if its complement $X \setminus U$ is $G$-open.
- An *adherent point* (also *closure point* or *point of closure*) of a subset $U \subseteq X$ is a point $x \in X$ such that every $G$-open set containing $x$ also contains, at least, one point of $U$, that is, for all $\varepsilon > 0$ we have that $B_G(x, \varepsilon) \cap U \neq \emptyset$.
- The *G-closure* $\overline{U} = \mathrm{cl}_G(U)$ of a subset $U \subseteq X$ is the family of all its adherent points. Clearly, $x \in \overline{U}$ if, and only if, $B_G(x, \varepsilon) \cap U \neq \emptyset$ for all $\varepsilon > 0$. In particular, $U \subseteq \overline{U}$. Moreover, $U$ is $G$-closed if, and only if, $U = \overline{U}$.
- The *G-interior* $\mathring{U} = \mathrm{int}_G(U)$ of a subset $U \subseteq X$ is the complement $X \setminus \overline{U}$. An *interior point of* $U$ is a point $x \in U$ such that there exists $r > 0$ verifying $B_G(x, r) \subseteq U$. In particular, $\mathring{U} \subseteq U$. Moreover, $U$ is $G$-open if, and only if, $\mathring{U} = U$.

For simplicity, we will omit the prefix $G$- in the previous notions.

### 3.2.1 Convergent and Cauchy Sequences

In this subsection, we introduce the notions of convergent sequence and Cauchy sequence using the topology $\tau_G$.

**Definition 3.2.2.** Let $(X, G)$ be a $G$-metric space, let $x \in X$ be a point and let $\{x_n\} \subseteq X$ be a sequence. We say that:

- $\{x_n\}$ *G-converges to* $x$, and we write $\{x_n\} \xrightarrow{G} x$ or $\{x_n\} \to x$, if $\lim_{n,m\to\infty} G(x_n, x_m, x) = 0$, that is, for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ satisfying $G(x_n, x_m, x) \leq \varepsilon$ for all $n, m \in \mathbb{N}$ such that $n, m \geq n_0$ (in such a case, $x$ is the *G-limit* of $\{x_n\}$);
- $\{x_n\}$ is *G-Cauchy* if $\lim_{n,m,k\to\infty} G(x_n, x_m, x_k) = 0$, that is, for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ satisfying $G(x_n, x_m, x_k) \leq \varepsilon$ for all $n, m, k \in \mathbb{N}$ such that $n, m, k \geq n_0$.
- $(X, G)$ is *complete* if every $G$-Cauchy sequence in $X$ is $G$-convergent in $X$.

The following two properties are well known.

**Proposition 3.2.2.** *The limit of a G-convergent sequence in a G-metric space is unique.*

*Proof.* Let $(X, G)$ be a $G$-metric space and let $\{x_n\} \subseteq X$ be a sequence that converges, at the same time, to $x \in X$ and to $y \in X$. We claim that $G(x, y, y) < \varepsilon$ for all $\varepsilon > 0$. Indeed, let $\varepsilon > 0$ be arbitrary. By definition, there exist natural numbers $n_1, n_2 \in \mathbb{N}$ such that

$$G(x_n, x_m, x) \leq \frac{\varepsilon}{3} \quad \text{for all } n, m \geq n_1;$$

$$G(x_n, x_m, y) \leq \frac{\varepsilon}{3} \quad \text{for all } n, m \geq n_2.$$

Let $n_0 = \max(n_1, n_2)$. Then, by $(G_5)$ and Lemma 3.1.1, we have that, for all $n \geq \max(n_1, n_2)$,

$$\begin{aligned} G(x, y, y) &\leq G(x, x_n, x_n) + G(x_n, y, y) \\ &\leq G(x_n, x_n, x) + 2G(x_n, x_n, y) \leq \frac{\varepsilon}{3} + 2\frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Consequently, we have that $G(x, y, y) = 0$ and, by $(G_2)$, we conclude that $x = y$. □

**Proposition 3.2.3.** *Every convergent sequence in a G-metric space is a Cauchy sequence.*

*Proof.* Let $(X, G)$ be a $G$-metric space and let $\{x_n\} \subseteq X$ be a sequence that converges to $x \in X$. Let $\varepsilon > 0$ be arbitrary. By definition, there exists $n_0 \in \mathbb{N}$ such that

$$G(x_n, x_m, x) \leq \frac{\varepsilon}{3} \quad \text{for all } n, m \geq n_0.$$

By $(G_4)$, $(G_5)$ and Lemma 3.1.1, we have that, for all $n, m, k \geq n_0$,

$$\begin{aligned} G(x_n, x_m, x_k) &\leq G(x_n, x, x) + G(x, x_m, x_k) \\ &\leq 2\,G(x_n, x_n, x) + G(x_m, x_k, x) \leq 2\frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence in $(X, G)$. □

*Remark 3.2.1.* The reader can observe that axiom $(G_3)$ is not necessary in the proofs of many results. For example, it was not used in the proofs of Propositions 3.2.2 and 3.2.3. In the same way, assumption $(G_3)$ will not be involved in the equivalences between conditions (a) to (h) in Lemma 3.2.1, or in the equivalences of Lemma 3.2.2.

Next, we characterize convergent and Cauchy sequences. In the following result, we use the notation introduced at the end of Sect. 2.1.

**Lemma 3.2.1.** *Let $(X, G)$ be a G-metric space, let $\{x_n\} \subseteq X$ be a sequence and let $x \in X$. Then the following conditions are equivalent.*

**(a)** *$\{x_n\}$ G-converges to $x$.*
**(b)** $\lim_{n \to \infty} G(x_n, x_n, x) = 0$, *that is, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in B_G(x, \varepsilon)$ for all $n \geq n_0$.*

**(c)** $\lim_{n\to\infty} G(x_n, x, x) = 0.$
**(d)** $\lim_{n,m\to\infty,\ m\geq n} G(x_n, x_m, x) = 0.$
**(e)** $\lim_{n\to\infty} G(x_n, x_n, x) = 0$ *and* $\lim_{n\to\infty} G(x_n, x_{n+1}, x) = 0.$
**(f)** $\lim_{n\to\infty} G(x_n, x, x) = 0$ *and* $\lim_{n\to\infty} G(x_n, x_{n+1}, x) = 0.$
**(g)** $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$ *and* $\lim_{n\to\infty} G(x_n, x_{n+1}, x) = 0.$
**(h)** $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$ *and* $\lim_{n,m\to\infty,\ m>n} G(x_n, x_m, x) = 0.$
**(i)** $\lim_{n,m\to\infty,\ m>n} G(x_n, x_m, x) = 0.$

*Proof.* (a)$\Rightarrow$(b) It is obvious using $m = n$.
(b)$\Rightarrow$(c). It follows from Lemma 3.1.1 because

$$G(x_n, x, x) \leq 2\, G(x_n, x_n, x)$$

for all $n \in \mathbb{N}$.
(c)$\Rightarrow$(a). It follows from the fact that, by $(G_5)$ and $(G_4)$, for all $n, m \in \mathbb{N}$,

$$\begin{aligned} G(x_n, x_m, x) &\leq G(x_n, x, x) + G(x, x_m, x) \\ &= G(x_n, x, x) + G(x_m, x, x) \end{aligned}$$

The implications (a)$\Rightarrow$(d)$\Rightarrow$(b), (a)$\Rightarrow$(e)$\Rightarrow$(b) and (a)$\Rightarrow$(f)$\Rightarrow$(c) are trivial.
(a)$\Rightarrow$(h) By Proposition 3.2.3, $\{x_n\}$ is a Cauchy sequence. Then, using $m = k = n + 1$ in the definition of Cauchy sequence, we deduce that $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$. Moreover, (a) trivially implies that

$$\lim_{n,m\to\infty,\ m>n} G(x_n, x_m, x) = 0.$$

(h)$\Rightarrow$(g). It is obvious using $m = n + 1$.
(g)$\Rightarrow$(b). By $(G_5)$ and $(G_4)$, for all $n \in \mathbb{N}$,

$$\begin{aligned} G(x_n, x_n, x) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_n, x) \\ &= G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x). \end{aligned}$$

Moreover, (a)$\Rightarrow$(i) is also apparent. The only implication in which we will use axiom $(G_3)$ is the following one.
(i)$\Rightarrow$(b). It follows from item 9 of Lemma 3.1.2 (which needs axiom $(G_3)$), because $G(x_n, x_n, x) \leq 2\, G(x_n, x, x_{n+1}) = 2\, G(x_n, x_{n+1}, x)$ for all $n \in \mathbb{N}$. $\square$

**Lemma 3.2.2.** *If $(X, G)$ is a G-metric space and $\{x_n\} \subseteq X$ is a sequence, then the following conditions are equivalent.*

**(a)** *$\{x_n\}$ is G-Cauchy.*
**(b)** $\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0.$

**(c)** $\lim_{n,m\to\infty,\ m\geq n} G(x_n,x_m,x_m)=0.$
**(d)** $\lim_{n,m\to\infty,\ m>n} G(x_n,x_m,x_m)=0.$
**(e)** $\lim_{n,m\to\infty} G(x_n,x_n,x_m)=0.$
**(f)** $\lim_{n,m\to\infty,\ m\geq n} G(x_n,x_n,x_m)=0.$
**(g)** $\lim_{n,m\to\infty,\ m>n} G(x_n,x_n,x_m)=0.$
**(h)** $\lim_{n\to\infty} G(x_n,x_{n+1},x_{n+1})=0$ *and* $\lim_{n,m\to\infty,\ m>n} G(x_n,x_{n+1},x_m)=0.$

*Proof.* (a)$\Rightarrow$(b)$\Rightarrow$(c) are obvious.

(c)$\Rightarrow$(a). Let $\varepsilon>0$ be arbitrary. By condition (b), there exists $n_0\in\mathbb{N}$ such that

$$G(x_n,x_m,x_m)\leq\frac{\varepsilon}{2}\quad\text{for all } m\geq n\geq n_0.$$

Let $n,m,k\in\mathbb{N}$ be such that $n,m,k\geq n_0$. Let $n'=\min\{n,m,k\}$, $k'=\max\{n,m,k\}$ and $m'=\{n,m,k\}\diagdown\{n',k'\}$. Then $\{n,m,k\}=\{n',m',k'\}$ and $n'\leq m'\leq k'$. Therefore, by $(G_5)$ and $(G_4)$,

$$\begin{aligned}G(x_n,x_m,x_k)&=G(x_{n'},x_{m'},x_{k'})\leq G(x_{n'},x_{k'},x_{k'})+G(x_{k'},x_{m'},x_{k'})\\&=G(x_{n'},x_{k'},x_{k'})+G(x_{m'},x_{k'},x_{k'})\leq\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence.

(c)$\Rightarrow$(d). It is obvious.

(d)$\Rightarrow$(c). If $n=m$, then $G(x_n,x_m,x_m)=G(x_n,x_n,x_n)=0$, and if $m>n$, then we can get $G(x_n,x_m,x_m)\leq\varepsilon$.

The equivalences (b)$\Leftrightarrow$(e), (c)$\Leftrightarrow$(f) and (d)$\Leftrightarrow$(g) follow from Lemma 3.1.1, because $G(x_n,x_m,x_m)\leq 2\,G(x_n,x_n,x_m)\leq 4\,G(x_n,x_m,x_m)$ for all $n,m\in\mathbb{N}$.

It is clear that (a)$\Rightarrow$(h).

(h)$\Rightarrow$(g). For all $n,m\in\mathbb{N}$ such that $m>n$, we have that

$$\begin{aligned}G(x_n,x_n,x_m)&\leq G(x_n,x_{n+1},x_{n+1})+G(x_{n+1},x_n,x_m)\\&=G(x_n,x_{n+1},x_{n+1})+G(x_n,x_{n+1},x_m).\end{aligned}$$

□

Taking into account that the topologies $\tau_G$, $\tau_{d_m^G}$ and $\tau_{d_s^G}$ coincide, it is convenient to highlight that they have the same Cauchy sequences and the same convergent sequences, converging to the same limits.

**Lemma 3.2.3.** *Given a $G$-metric space $(X,G)$, let $d_m^G$ and $d_s^G$ be the metrics on $X$ defined in Lemma 3.1.4. Then a sequence $\{x_m\}\subseteq X$ is $G$-convergent to $x\in X$ if, and only if, it is $d_m^G$-convergent (or $d_s^G$-convergent) to $x$.*

*Furthermore, $\{x_m\}\subseteq X$ is $G$-Cauchy if, and only if, it is $d_m^G$-Cauchy (or $d_s^G$-Cauchy).*

Closedness can be characterized using convergent sequences.

**Proposition 3.2.4 ([142]).** *Let $(X, G)$ be a G-metric space and let $U \subseteq X$ be a nonempty subset of X. Then U is G-closed if, and only if, for any G-convergent sequence $\{x_n\}$ in U with G-limit $x \in X$, one has that $x \in U$.*

### *3.2.2 Continuity of Mappings Between G-Metric Spaces*

**Definition 3.2.3.** Let $(X, G)$ be a $G$-metric space. We say that:

- a mapping $T : X \to X$ is *G-continuous* at $x \in X$ if $\{Tx_m\} \xrightarrow{G} Tx$ for all sequence $\{x_m\} \subseteq X$ such that $\{x_m\} \xrightarrow{G} x$;
- a mapping $F : X^n \to X$ is *G-continuous* at $(x_1, x_2, \ldots, x_n) \in X^n$ if

$$\{F\left(x_1^m, x_2^m, \ldots, x_n^m\right)\} \xrightarrow{G} F\left(x_1, x_2, \ldots, x_n\right)$$

  for all sequence $\{\left(x_1^m, x_2^m, \ldots, x_n^m\right)\} \subseteq X^n$ such that $\{x_i^m\} \xrightarrow{G} x_i$ for all $i \in \{1, 2, \ldots, n\}$;
- a mapping $H : X^n \to X^m$ is *G-continuous* at $(x_1, x_2, \ldots, x_n) \in X^n$ if $\pi_i^m \circ H : X^n \to X$ is $G$-continuous at $(x_1, x_2, \ldots, x_n)$ for all $i \in \{1, 2, \ldots, m\}$, where $\pi_i^m : X^m \to X$ is the $i$th-projection of $X^m$ onto $X$ (that is, $\pi_i^m\left(a_1, a_2, \ldots, a_m\right) = a_i$ for all $(a_1, a_2, \ldots, a_m) \in X^m$).

By Lemma 3.2.3, convergence of sequences on $X$ with respect to $G$, $d_m^G$ and $d_m^G$ coincide.

**Lemma 3.2.4.** *Let $(X, G)$ be a G metric space. Then a mapping $T : X \to X$ is G-continuous if, and only if, it is $d_m^G$-continuous ($d_m^G$-continuous). Similarly, a mapping $F : X^n \to X$ is G-continuous if, and only if, it is $d_m^G$-continuous ($d_m^G$-continuous).*

**Theorem 3.2.2 ([154]).** *If $(X, G)$ is a G-metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables, that is, if $x, y, z \in X$ and $\{x_n\}, \{y_n\}, \{z_n\} \subseteq X$ are sequences in X such that $\{x_m\} \xrightarrow{G} x$, $\{y_m\} \xrightarrow{G} y$ and $\{z_m\} \xrightarrow{G} z$, then $\{G\left(x_m, y_m, z_m\right)\} \to G\left(x, y, z\right)$.*

*Proof.* Applying the axiom $(G_5)$ three times,

$$\begin{aligned} G\left(x_m, y_m, z_m\right) &\leq G\left(x_m, x, x\right) + G\left(x, y_m, z_m\right) \\ &\leq G\left(x_m, x, x\right) + G\left(y_m, y, y\right) + G\left(y, x, z_m\right) \\ &\leq G\left(x_m, x, x\right) + G\left(y_m, y, y\right) + G\left(z_m, z, z\right) + G\left(z, x, y\right). \end{aligned}$$

In a similar way,

$$G(x,y,z) \le G(x,x_m,x_m) + G(y,y_m,y_m) + G(z,z_m,z_m) + G(z_m,x_m,y_m).$$

In particular, for all $m \in \mathbb{N}$,

$$\begin{aligned} G(x,y,z) - G(x,x_m,x_m) - G(y,y_m,y_m) - G(z,z_m,z_m) &\le G(x_m,y_m,z_m) \\ &\le G(x_m,x,x) + G(y_m,y,y) + G(z_m,z,z) + G(z,x,y). \end{aligned}$$

Letting $m \to \infty$ and using Lemmas 2.1.1 and 3.2.1, we conclude that $\{G(x_m,y_m,z_m)\} \to G(x,y,z)$. □

## 3.3 *G*-Metrics and Quasi-metrics

In this section, we analyze the close relationship between $G$-metrics and quasi-metrics.

**Lemma 3.3.1.** *Let $(X,G)$ be a G-metric space and let define $q_G, q'_G : X^2 \to [0,\infty)$ by*

$$q_G(x,y) = G(x,x,y) \quad \text{and} \quad q'_G(x,y) = G(x,y,y) \quad \text{for all } x,y \in X.$$

*Then the following properties hold.*

1. *$q_G$ and $q'_G$ are quasi-metrics on $X$. Moreover*

$$q_G(x,y) \le 2\,q'_G(x,y) \le 4\,q_G(x,y) \quad \text{for all } x,y \in X. \tag{3.11}$$

2. *In $(X,q_G)$ and in $(X,q'_G)$, a sequence is right-convergent (respectively, left-convergent) if, and only if, it is convergent. In such a case, its right-limit, its left-limit and its limit coincide.*
3. *In $(X,q_G)$ and in $(X,q'_G)$, a sequence is right-Cauchy (respectively, left-Cauchy) if, and only if, it is Cauchy.*
4. *In $(X,q_G)$ and in $(X,q'_G)$, every right-convergent (respectively, left-convergent) sequence has a unique right-limit (respectively, left-limit).*
5. *If $\{x_n\} \subseteq X$ and $x \in X$, then*

$$\{x_n\} \xrightarrow{G} x \iff \{x_n\} \xrightarrow{q_G} x \iff \{x_n\} \xrightarrow{q'_G} x.$$

6. *If $\{x_n\} \subseteq X$, then $\{x_n\}$ is G-Cauchy $\iff$ $\{x_n\}$ is $q_G$-Cauchy $\iff$ $\{x_n\}$ is $q'_G$-Cauchy.*
7. *$(X,G)$ is complete $\iff$ $(X,q_G)$ is complete $\iff$ $(X,q'_G)$ is complete.*

*Proof.* (1) Axiom $(q_1)$ follows from $(G_1)$ and $(G_2)$ and condition $(q_2)$ holds because of properties $(G_4)$ and $(G_5)$ since, for all $x,y,z \in X$,

$$\begin{aligned} q_G(x,y) &= G(x,x,y) = G(y,x,x) \leq G(y,z,z) + G(z,x,x) \\ &= G(x,x,z) + G(z,z,y) = q_G(x,z) + q_G(z,y); \\ q'_G(x,y) &= G(x,y,y) \leq G(x,z,z) + G(z,y,y) = q'_G(x,z) + q'_G(z,y). \end{aligned}$$

Inequalities (3.11) follow from Lemma 3.1.1. **(2)** It follows from Lemma 3.2.1. **(3)** It follows directly from the definitions. **(4)** It follows from item 2 and Remark 2.5.1. Other items are straightforward exercises. □

*Remark 3.3.1.* Notice that $q_G$ and $q'_G$ can be different quasi-metrics. For example, $q'_G$ is a quasi-metric even if $G$ does not verify axiom $(G_4)$, but $q_G$ needs that property.

# Chapter 4
# Basic Fixed Point Results in the Setting of $G$-Metric Spaces

The *Banach contractive mapping principle* is the most celebrated result in fixed point theory. The simplicity of its proof and the possibility of attaining the fixed point by using successive approximations makes it a useful tool in analysis and in applied mathematics. In this chapter, we present a variety of fixed (and coincidence) point results in the context of $G$-metric spaces.

## 4.1 The Banach Procedure

Almost all contractive type fixed point results follow the same technique. In this section, we describe this process in a very general context.

### *4.1.1 The Banach Procedure*

Let $X$ be a non-empty set and let $T, g : X \to X$ be two self-mappings.

**Part I. Existence of a fixed (or coincidence) point**

**Step 1. Construction of an iterative sequence.**
A sequence $\{x_n\}_{n\geq 0} \subseteq X$ is a *Picard sequence of* $T$ if

$$x_{n+1} = Tx_n \quad \text{for all } n \in \mathbb{N}.$$

Some authors say that this sequence is *based* on the initial point $x_0$. If this sequence contains a point $x_{n_0}$ such that $x_{n_0+1} = x_{n_0}$, then $x_{n_0}$ is a fixed point of $T$, and the existence of a fixed point is guaranteed. Therefore, it is usual to assume that



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$x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. In this case, the main objective is to prove that this sequence converges to a fixed point of $T$.

In the coincidence case, a sequence $\{x_n\}_{n\geq 0} \subseteq X$ is a *Picard sequence of* $(T, g)$ if

$$gx_{n+1} = Tx_n \quad \text{for all } n \in \mathbb{N}.$$

If there exists $n_0 \in \mathbb{N}$ such that $gx_{n_0+1} = gx_{n_0}$, then $gx_{n_0} = gx_{n_0+1} = Tx_{n_0}$, so $x_{n_0}$ is a coincidence point of $T$ and $g$. Therefore, it is usual to assume that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$.

**Lemma 4.1.1.** *If $T(X) \subseteq g(X)$, then there exists a Picard sequence of $(T, g)$ based on any $x_0 \in X$.*

*Proof.* Let $x_0 \in X$ be arbitrary. Since $Tx_0 \in T(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $gx_1 = Tx_0$. Analogously, since $Tx_1 \in T(X) \subseteq g(X)$, there exists $x_2 \in X$ such that $gx_2 = Tx_1$. Repeating this argument by the induction methodology, we can find a Picard sequence of $(T, g)$ based on $x_0$. □

**Step 2. To prove that $\{gx_n\}$ is asymptotically regular.**

A sequence $\{z_n\}$ in a quasi-metric space $(X, q)$ is *asymptotically regular* if

$$\lim_{n\to\infty} q(z_n, z_{n+1}) = \lim_{n\to\infty} q(z_{n+1}, z_n) = 0.$$

In the case of a $G$-metric space, it is only necessary to prove that

$$\lim_{n\to\infty} G(z_n, z_{n+1}, z_{n+1}) = 0$$

because, in such a case, Corollary 3.1.1 ensures that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = \lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = 0. \tag{4.1}$$

**Step 3. To prove that $\{gx_n\}$ is Cauchy.**

This is usually the key step of the proof and usually in the literature the argument involves reasoning by contradiction. The methodology we will follow is described in Sect. 4.1.2.

**Lemma 4.1.2.** *Let $\{x_n\}$ be a sequence in a $G$-metric space $(X, G)$ and assume that there exist a function $\varphi \in \mathcal{F}_{\mathrm{KR}}$ and $n_0 \in \mathbb{N}$ such that, at least, one of the following conditions holds:*

**(a)** $G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \varphi(G(x_n, x_{n+1}, x_{n+1}))$ *for all $n \geq n_0$;*
**(b)** $G(x_{n+1}, x_{n+1}, x_{n+2}) \leq \varphi(G(x_n, x_n, x_{n+1}))$ *for all $n \geq n_0$.*

*Then $\{x_n\}$ is a Cauchy sequence in $(X, G)$.*

As $\mathcal{F}_{\mathrm{com}}^{(c)} \subset \mathcal{F}_{\mathrm{KR}}$, the previous result is also valid when $\varphi \in \mathcal{F}_{\mathrm{com}}^{(c)}$.

*Proof.* Assume that condition (a) holds and let $y_n = x_{n+n_0}$ for all $n \in \mathbb{N}$. We claim that $\{y_n\}$ is a Cauchy sequence in $(X, G)$. Indeed, using the fact that $\varphi$ is non-decreasing, for all $n \geq 0$,

$$\begin{aligned} G\left(y_{n+1}, y_{n+2}, y_{n+2}\right) &= G\left(x_{n+n_0+1}, x_{n+n_0+2}, x_{n+n_0+2}\right) \\ &\leq \varphi\left(G\left(x_{n+n_0}, x_{n+n_0+1}, x_{n+n_0+1}\right)\right) = \varphi\left(G\left(y_n, y_{n+1}, y_{n+1}\right)\right). \end{aligned}$$

Repeating this argument, it follows that for all $n \geq 0$,

$$\begin{aligned} G\left(y_n, y_{n+1}, y_{n+1}\right) &\leq \varphi\left(G\left(y_{n-1}, y_n, y_n\right)\right) \\ &\leq \varphi^2\left(G\left(y_{n-2}, y_{n-1}, y_{n-1}\right)\right) \leq \ldots \leq \varphi^n\left(G\left(y_0, y_1, y_1\right)\right). \end{aligned} \tag{4.2}$$

If $G(y_0, y_1, y_1) = 0$, then $G(y_n, y_{n+1}, y_{n+1}) = 0$ for all $n \in \mathbb{N}$, which means that $y_{n+1} = y_n$ for all $n \in \mathbb{N}$. Then, the sequence $\{y_n\}$ is constant, that is, $y_n = y_0$ for all $n \in \mathbb{N}$. In particular, $\{y_n\}$ is a Cauchy sequence in $(X, G)$ because $G(y_n, y_m, y_\ell) = 0$ for all $n, m, \ell \in \mathbb{N}$.

Next, assume that $t_0 = G(y_0, y_1, y_1) > 0$ and let $\varepsilon > 0$ arbitrary. Since $\varphi \in \mathcal{F}_{\mathrm{KR}}$, the series $\sum_{n\in\mathbb{N}} \varphi^n(t_0)$ converges. In particular, there exists $n_1 \in \mathbb{N}$ such that

$$\sum_{k=n_1}^{\infty} \varphi^k(t_0) < \varepsilon.$$

Now, let $n, m \in \mathbb{N}$ be such that $n, m \geq n_1$. Without loss of generality, assume that $n < m$. From item 4 of Lemma 3.1.2 and using (4.2), we have that

$$\begin{aligned} G\left(y_n, y_m, y_m\right) &\leq \sum_{k=n}^{m-1} G\left(y_k, y_{k+1}, y_{k+1}\right) \leq \sum_{k=n}^{m-1} \varphi^k(t_0) \\ &\leq \sum_{k=n_1}^{\infty} \varphi^k(t_0) < \varepsilon. \end{aligned}$$

Therefore, by Lemma 3.2.2, $\{y_n\}$ is Cauchy in $(X, G)$. This argument also proves that $\{x_n\}$ is a Cauchy sequence because for all $m > n \geq n_0 + n_1$ we have that $m - n_0 > n - n_0 \geq n_1$, so

$$G\left(x_n, x_m, x_m\right) = G\left(y_{n-n_0}, y_{m-n_0}, y_{m-n_0}\right) < \varepsilon.$$

Case (b) is similar. □

If we take $\varphi_\lambda(t) = \lambda\, t$ for all $t \in [0, \infty)$, where $\lambda \in [0, 1)$, then $\varphi_\lambda \in \mathcal{F}_{\mathrm{com}}^{(c)}$ and the previous result can be stated as follows.

**Corollary 4.1.1.** *Let $\{x_n\}$ be a sequence in a G-metric space $(X, G)$ and assume that there exist a constant $\lambda \in [0, 1)$ and $n_0 \in \mathbb{N}$ such that, at least, one of the following conditions holds:*

**(a)** $G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \lambda\, G(x_n, x_{n+1}, x_{n+1})$ *for all* $n \geq n_0$*;*
**(b)** $G(x_{n+1}, x_{n+1}, x_{n+2}) \leq \lambda\, G(x_n, x_n, x_{n+1})$ *for all* $n \geq n_0$.

*Then* $\{x_n\}$ *is a Cauchy sequence in* $(X, G)$.

**Step 4. To prove that the limit of $\{gx_n\}$ is a coincidence point of $T$ and $g$.**
As $\{gx_n\}$ is a Cauchy sequence in a complete space (for example, in $X$ or in $g(X)$), then it is convergent. Its limit, $z \in X$, is usually a coincidence point of $T$ and $g$. To prove it, it is necessary to apply the contractivity condition using $gx_n$ and $z$. If $z \in g(X)$ (for example, when $g(X)$ is complete), then there exists $u \in X$ such that $z = gu$, so the contractivity condition can be applied to $gx_n$ and $gu$.

**Part II. Uniqueness**

Once we have proved the existence of a coincidence (or a fixed) point, it is of interest to discuss uniqueness if it is possible. Reasoning by contradiction and under additional hypotheses if necessary, sometimes we can deduce a contradiction assuming that $T$ and $g$ have two different coincidence points.

The following definition can be applied to any space provided with a notion of convergence (metric, quasi-metric, $G$-metric or topological space).

**Definition 4.1.1.** We say that $T : X \to X$ is a *Picard operator* if for all initial point $x_0 \in X$, the Picard sequence of $T$ based on $x_0$ converges to a fixed point of $T$.

### *4.1.2 About Asymptotically Regular Sequences that are not Cauchy*

In this section we describe some necessary conditions that must be verified by any asymptotically regular sequence if we suppose that it is not Cauchy.

In the following result, given a fixed integer number $p \in \mathbb{Z}$, we will consider the subsequence $\{G(x_{n+p}, y_n, z_n)\}_{n \geq |p|}$, and we consider the limit of this sequence when $n \to \infty$.

*Remark 4.1.1.* Throughout this subsection, we shall not use axiom $(G_3)$.

**Lemma 4.1.3.** *Let* $\{x_n\}$*,* $\{y_n\}$ *and* $\{z_n\}$ *be three sequences in a G-metric space* $(X, G)$*. Suppose that* $\{x_n\}$ *is asymptotically regular and that there exists* $L \in [0, \infty)$ *such that* $\lim_{n\to\infty} G(x_n, y_n, z_n) = L$*. Then, for all given* $p \in \mathbb{Z}$*,*

$$\lim_{n\to\infty} G(x_{n+p}, y_n, z_n) = L. \tag{4.3}$$

*Proof.* As $\{x_n\}$ is asymptotically regular, by (4.1), we have that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = \lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = 0. \tag{4.4}$$

If $p = 0$, property (4.3) holds by hypothesis. Assume that (4.3) holds for some $p \in \mathbb{N}$, and we will show that it also holds for $p+1$. Notice that, for all $n \in \mathbb{N}$,

$$G(x_{n+p+1}, y_n, z_n) \leq G(x_{n+p+1}, x_{n+p}, x_{n+p}) + G(x_{n+p}, y_n, z_n)$$

and

$$G(x_{n+p}, y_n, z_n) \leq G(x_{n+p}, x_{n+p+1}, x_{n+p+1}) + G(x_{n+p+1}, y_n, z_n).$$

Joining the last two inequalities,

$$\begin{aligned} G(x_{n+p}, y_n, z_n) - G(x_{n+p}, x_{n+p+1}, x_{n+p+1}) &\leq G(x_{n+p+1}, y_n, z_n) \\ &\leq G(x_{n+p+1}, x_{n+p}, x_{n+p}) + G(x_{n+p}, y_n, z_n). \end{aligned}$$

Using (4.4), the hypothesis of induction (4.3) and Lemma 2.1.1, we deduce that

$$\lim_{n\to\infty} G(x_{n+p+1}, y_n, z_n) = L,$$

which completes the proof. The case in which $p < 0$ can be proved similarly. □

In the following corollary, given $p_1, p_2, p_3 \in \mathbb{Z}$, we will consider the limit of the sequence

$$\left\{ G(x_{n+p_1}, y_{n+p_2}, z_{n+p_3}) \right\}_{n \geq \max\{|p_1|, |p_2|, |p_3|\}}.$$

**Corollary 4.1.2.** *Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be three asymptotically regular sequences in a G-metric space $(X, G)$ and assume that there exists $L \in [0, \infty)$ such that $\lim_{n\to\infty} G(x_n, y_n, z_n) = L$. Then, for all given $p_1, p_2, p_3 \in \mathbb{Z}$,*

$$\lim_{n\to\infty} G(x_{n+p_1}, y_{n+p_2}, z_{n+p_3}) = L. \tag{4.5}$$

*Proof.* From Lemma 4.1.3, we know that, for all fixed $p_1 \in \mathbb{Z}$,

$$\lim_{n\to\infty} G(x_{n+p_1}, y_n, z_n) = L.$$

Taking into account that $G(x_{n+p_1}, y_n, z_n) = G(y_n, x_{n+p_1}, z_n)$ for all $n$ and $p_1$, we can again apply Lemma 4.1.3 to deduce that, for all fixed $p_1, p_2 \in \mathbb{Z}$,

$$\lim_{n\to\infty} G(y_{n+p_2}, x_{n+p_1}, z_n) = L.$$

Repeating this argument, we conclude (4.5). □

Notice that we cannot deduce that a subsequence $\{x_{n(k)}\}$ of an asymptotically regular sequence $\{x_n\}$ is also asymptotically regular. For example, consider the sequence $\{x_n\}_{n\geq 1} \subset \mathbb{R}$ given by

$$x_n = \sum_{k=1}^{n} \frac{1}{k} \quad \text{for all } n \geq 1.$$

Then $\{x_n\}$ is asymptotically regular if we consider on $\mathbb{R}$ the $G$-metric

$$G(x, y, z) = \max\{|x-y|, |x-z|, |y-z|\}$$

for all $x, y, z \in \mathbb{R}$. However, as $\{x_n\}$ is strictly increasing and $\{x_n\} \to \infty$, it contains a subsequence $\{x_{n(k)}\}$ such that $x_{n(k+1)} \geq x_{n(k)} + k$ for all $k \in \mathbb{N}$, and this subsequence is not asymptotically regular.

**Lemma 4.1.4.** *Let $\{x_{n(k)}\}$, $\{x_{m(k)}\}$ and $\{x_{\ell(k)}\}$ be three subsequences of the same asymptotically regular sequence $\{x_n\}$ in a $G$-metric space $(X, G)$ and assume that there exists $L \in [0, \infty)$ such that*

$$\lim_{k\to\infty} G(x_{n(k)}, x_{m(k)}, x_{\ell(k)}) = L.$$

*Then, for all given $p_1, p_2, p_3 \in \mathbb{Z}$,*

$$\lim_{k\to\infty} G(x_{n(k)+p_1}, x_{m(k)+p_2}, x_{\ell(k)+p_3}) = L. \tag{4.6}$$

*Proof.* As $\{x_n\}$ is asymptotically regular, by (4.1), we have that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = \lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = 0. \tag{4.7}$$

Firstly, we show, by induction on $p_1$, that

$$\text{for all } p_1 \in \mathbb{N}, \quad \lim_{k\to\infty} G(x_{n(k)+p_1}, x_{m(k)}, x_{\ell(k)}) = L. \tag{4.8}$$

If $p_1 = 0$, then (4.8) holds by hypothesis. Assume that (4.8) holds for some $p_1 \in \mathbb{N}$, and we show that (4.8) also holds for $p_1 + 1$. Indeed, notice that, for all $k \in \mathbb{N}$,

$$\begin{aligned} &G(x_{n(k)+p_1+1}, x_{m(k)}, x_{\ell(k)}) \\ &\qquad \leq G(x_{n(k)+p_1+1}, x_{n(k)+p_1}, x_{n(k)+p_1}) + G(x_{n(k)+p_1}, x_{m(k)}, x_{\ell(k)}) \end{aligned}$$

and

$$\begin{aligned} &G(x_{n(k)+p_1}, x_{m(k)}, x_{\ell(k)}) \\ &\qquad \leq G(x_{n(k)+p_1}, x_{n(k)+p_1+1}, x_{n(k)+p_1+1}) + G(x_{n(k)+p_1+1}, x_{m(k)}, x_{\ell(k)}). \end{aligned}$$

Joining the last two inequalities,

$$\begin{aligned}
&G(x_{n(k)+p_1}, x_{m(k)}, x_{\ell(k)}) - G(x_{n(k)+p_1}, x_{n(k)+p_1+1}, x_{n(k)+p_1+1}) \\
&\quad \leq G(x_{n(k)+p_1+1}, x_{m(k)}, x_{\ell(k)}) \\
&\quad \leq G(x_{n(k)+p_1+1}, x_{n(k)+p_1}, x_{n(k)+p_1}) + G(x_{n(k)+p_1}, x_{m(k)}, x_{\ell(k)})).
\end{aligned}$$

Using (4.7), the hypothesis of induction (4.8) and Lemma 2.1.1, we deduce that

$$\lim_{k\to\infty} G(x_{n(k)+p_1+1}, x_{m(k)}, x_{\ell(k)}) = L,$$

which completes the proof. Then (4.8) holds.

The case in which $p_1 < 0$ can be proved similarly by induction on $-p_1$, so (4.8) holds for all $p_1 \in \mathbb{Z}$.

Now, using that $G(x_{n(k)+p_1}, x_{m(k)}, x_{\ell(k)}) = G(x_{m(k)}, x_{n(k)+p_1}, x_{\ell(k)})$ for all $k \in \mathbb{N}$, we can apply what we have just proved in order to deduce that

$$\text{for all } p_1, p_2 \in \mathbb{Z}, \quad \lim_{k\to\infty} G(x_{m(k)+p_2}, x_{n(k)+p_1}, x_{\ell(k)}) = L.$$

Similarly, in another step, we conclude that (4.6) holds. □

**Proposition 4.1.1.** *Let $\{x_n\}$ be a sequence in a G-metric space $(X, G)$.*

1. *If the following condition holds:*

$$\begin{gathered}
\textit{for all } \varepsilon > 0, \textit{ there exists } n_0 \in \mathbb{N} \textit{ such that} \\
G(x_n, x_m, x_m) \leq \varepsilon \textit{ for all } m > n \geq n_0,
\end{gathered} \tag{4.9}$$

   *then $\{x_n\}$ is a Cauchy sequence in $(X, G)$.*
2. *If the following condition holds:*

$$\begin{gathered}
\textit{for all } \varepsilon > 0, \textit{ there exists } n_0 \in \mathbb{N} \textit{ such that} \\
G(x_n, x_n, x_m) \leq \varepsilon \textit{ for all } m > n \geq n_0,
\end{gathered}$$

   *then $\{x_n\}$ is a Cauchy sequence in $(X, G)$.*

*Proof.* As $\{x_n\}$ is asymptotically regular, by (4.1), we have that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = \lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = 0.$$

Assume that condition (4.9) holds. Let $\varepsilon > 0$ be arbitrary and let $n_0 \in \mathbb{N}$ satisfy (4.9) for $\varepsilon/2 > 0$. If $\ell \geq m \geq n \geq n_0$, then

$$\begin{aligned}
G(x_n, x_m, x_\ell) &\leq G(x_n, x_m, x_m) + G(x_m, x_m, x_\ell) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence in $(X,G)$. The other condition yields the same conclusion using Lemma 3.1.1. □

**Theorem 4.1.1.** *Let $\{x_n\}$ be an asymptotically regular sequence in a G-metric space $(X,G)$ and suppose that $\{x_n\}$ is not Cauchy. Then the following properties hold.*

1. *There exists a positive real number $\varepsilon_1 > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that, for all $k \in \mathbb{N}$,*

$$k \leq n(k) < m(k) < n(k+1),$$
$$G\left(x_{n(k)}, x_{m(k)-1}, x_{m(k)-1}\right) \leq \varepsilon_1 < G\left(x_{n(k)}, x_{m(k)}, x_{m(k)}\right)$$

*and also, for all given $p_1, p_2, p_3 \in \mathbb{Z}$,*

$$\lim_{k\to\infty} G\left(x_{n(k)+p_1}, x_{m(k)+p_2}, x_{m(k)+p_3}\right) = \varepsilon_1. \tag{4.10}$$

2. *There exists a positive real number $\varepsilon_2 > 0$ and two subsequences $\{x_{r(k)}\}$ and $\{x_{s(k)}\}$ of $\{x_n\}$ such that, for all $k \in \mathbb{N}$,*

$$k \leq r(k) < s(k) < r(k+1),$$
$$G\left(x_{r(k)}, x_{r(k)}, x_{s(k)-1}\right) \leq \varepsilon_2 < G\left(x_{r(k)}, x_{r(k)}, x_{s(k)}\right)$$

*and also, for all given $p_1, p_2, p_3 \in \mathbb{Z}$,*

$$\lim_{k\to\infty} G\left(x_{r(k)+p_1}, x_{r(k)+p_2}, x_{s(k)+p_3}\right) = \varepsilon_2.$$

*Proof.* **(1)** As $\{x_n\}$ is asymptotically regular, by (4.1), we have that

$$\lim_{n\to\infty} G\left(x_n, x_{n+1}, x_{n+1}\right) = 0. \tag{4.11}$$

Taking into account that $\{x_n\}$ is not Cauchy, condition (4.9) cannot hold. If we deny that condition, we can find a positive real number $\varepsilon_1 > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that, for all $k \in \mathbb{N}$,

$$k \leq n(k) < m(k) < n(k+1), \quad \varepsilon_1 < G\left(x_{n(k)}, x_{m(k)}, x_{m(k)}\right).$$

For each $k \in \mathbb{N}$, it is possible to choose $m(k)$ as the lowest integer, greater than $n(k)$, verifying the previous condition. As $m(k)-1$ does not verify it, then $G\left(x_{n(k)}, x_{m(k)-1}, x_{m(k)-1}\right) \leq \varepsilon_1$ for all $k \in \mathbb{N}$. We claim that

$$\lim_{k\to\infty} G\left(x_{n(k)}, x_{m(k)}, x_{m(k)}\right) = \varepsilon_1. \tag{4.12}$$

Indeed, notice that, for all $k \in \mathbb{N}$,

$$\begin{aligned}\varepsilon_1 &< G\left(x_{n(k)}, x_{m(k)}, x_{m(k)}\right)\\ &\leq G\left(x_{n(k)}, x_{m(k)-1}, x_{m(k)-1}\right) + G\left(x_{m(k)-1}, x_{m(k)}, x_{m(k)}\right)\\ &\leq \varepsilon_1 + G\left(x_{m(k)-1}, x_{m(k)}, x_{m(k)}\right).\end{aligned}$$

Taking into account (4.11) and letting $k \to \infty$ in the previous inequality, we deduce, using Lemma 2.1.1, that (4.12) holds. Finally, using Lemma 4.1.4, we conclude that for all given $p_1, p_2, p_3 \in \mathbb{Z}$,

$$\lim_{k\to\infty} G\left(x_{n(k)+p_1}, x_{m(k)+p_2}, x_{m(k)+p_3}\right) = \varepsilon_1.$$

The proof of the other item is similar. □

**Lemma 4.1.5.** *Let $\{x_n\}$ be an asymptotically regular sequence in a G-metric space $(X, G)$ and suppose that $\{x_n\}$ is not Cauchy. Then there exists a positive real number $\varepsilon_0 > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that, for all $k \in \mathbb{N}$,*

$$\begin{aligned}&k \leq n(k) < m(k) < n(k+1),\\ &G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)-1}\right) \leq \varepsilon_0 < G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)}\right)\end{aligned}$$

*and also, for all given $p_1, p_2, p_3 \in \mathbb{Z}$,*

$$\lim_{k\to\infty} G\left(x_{n(k)+p_1}, x_{m(k)+p_2}, x_{m(k)+p_3}\right) = \varepsilon_0. \tag{4.13}$$

*Proof.* Under the asymptotically regular condition, we show that the assumption

$$\begin{gathered}\text{for all } \varepsilon > 0, \text{ there exists } n_0 \in \mathbb{N} \text{ such that}\\ G\left(x_n, x_{n+1}, x_m\right) \leq \varepsilon \text{ for all } m \geq n \geq n_0,\end{gathered} \tag{4.14}$$

implies that $\{x_n\}$ is a Cauchy sequence. Indeed, by $(G_4)$ and $(G_5)$, we have that

$$\begin{aligned}G\left(x_n.x_n, x_m\right) &\leq G\left(x_n, x_{n+1}, x_{n+1}\right) + G\left(x_{n+1}, x_n, x_m\right)\\ &= G\left(x_n, x_{n+1}, x_{n+1}\right) + G\left(x_n, x_{n+1}, x_m\right).\end{aligned}$$

As $G\left(x_n, x_{n+1}, x_{n+1}\right)$ and $G\left(x_n, x_{n+1}, x_m\right)$ are as small as we wish for $m \geq n \geq n_0$ and $n_0$ large enough, then Lemma 3.2.2 guarantees that $\{x_n\}$ is a Cauchy sequence. If we assume that $\{x_n\}$ is not a Cauchy sequence, then condition (4.14) is false. Then, there exists $\varepsilon_0 > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that, for all $k \in \mathbb{N}$, $k \leq n(k) < m(k) < n(k+1)$ and $G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)}\right) > \varepsilon_0$. If $m(k)$ is the smallest integer, greater that $n(k)$, such that this condition holds, then

$$G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)-1}\right) \leq \varepsilon_0 < G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)}\right)$$

for all $k \in \mathbb{N}$. As a result,

$$\begin{aligned}\varepsilon_0 &< G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)}\right) = G\left(x_{m(k)}, x_{n(k)}, x_{n(k)+1}\right) \\ &\leq G\left(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}\right) + G\left(x_{m(k)-1}, x_{n(k)}, x_{n(k)+1}\right) \\ &\leq G\left(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}\right) + \varepsilon_0.\end{aligned}$$

Since $\{x_n\}$ is asymptotically regular, then

$$\lim_{k\to\infty} G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)}\right) = \varepsilon_0.$$

Lemma 4.1.4 guarantees that (4.13) holds. □

## 4.2 Basic Fixed Point Theorems in the Context of *G*-Metric Spaces

We start this section by proving a fixed point theorem on $G$-metric spaces, which were is to Mustafa [142].

### *4.2.1 Banach Contractive Mapping Principle in G-Metric Spaces*

The following one can be considered as the first generalization of the Banach contractive mapping principle to the context of $G$-metric spaces.

**Theorem 4.2.1 ([142]).** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be a mapping such that there exists $\lambda \in [0, 1)$ satisfying*

$$G(Tx, Ty, Tz) \leq \lambda\, G(x, y, z) \quad \text{for all } x, y, z \in X. \tag{4.15}$$

*Then T has a unique fixed point. In fact, T is a Picard operator.*

*Proof.* Let $x_0$ be an arbitrary point of $X$ and let $\{x_n\}_{n\geq 0}$ be the Picard sequence of $T$ based on $x_0$, that is, $x_{n+1} = Tx_n$ for all $n \geq 0$. If there exists some $n_0$ such that $x_{n_0+1} = x_{n_0}$, then $x_{n_0}$ is a fixed point of $T$, and the existence of a fixed point is guaranteed. Therefore, assume that

$$x_{n+1} \neq x_n \quad \text{for all } n \geq 0.$$

By taking $x = x_n$ and $y = z = x_{n+1}$ in the contractive condition (4.15) of the theorem, we have that, for all $n \geq 0$,

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+2}) &= G(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ &\leq \lambda\, G(x_n, x_{n+1}, x_{n+1}). \end{aligned}$$

From Corollary 4.1.1, $\{x_n\}$ is a Cauchy sequence. As $(X, G)$ is complete, it is convergent, so there exists $z \in X$ such that $\{x_n\} \to z$. We assert that $z$ is a fixed point of $T$. By utilizing (4.15), we have that, for all $n \geq 0$,

$$G(x_{n+1}, Tz, Tz) = G(Tx_n, Tz, Tz) \leq \lambda\, G(x_n, z, z).$$

Letting $n \to \infty$ and using the fact that the metric $G$ is continuous (see Theorem 3.2.2), we get that

$$G(z, Tz, Tz) \leq \lambda\, G(z, z, z) = 0.$$

Hence, we conclude that $z = Tz$ by item 5 of Lemma 3.1.2. We shall show that $z$ is the unique fixed point of $T$. Suppose, on the contrary, that there exists another fixed point $w \in X$. If $w \neq x$, then $G(w, w, z) > 0$. From (4.15) and $\lambda < 1$ we have that

$$G(w, w, z) = G(Tw, Tw, Tz) \leq \lambda\, G(w, w, z) < G(w, w, z),$$

which is a contradiction. Hence, $z$ is the unique fixed point of $T$. □

If we carefully read the previous proof, we will notice that the contractivity condition was only used when two arguments of $G$ are equal. Then, the following contractivity condition, which is weaker than (4.15), leads to the same conclusion.

**Theorem 4.2.2.** *Let $(X, G)$ be a complete $G$-metric space and let $T : X \to X$ be a mapping such that there exists $\lambda \in [0, 1)$ satisfying*

$$G(Tx, Ty, Ty) \leq \lambda\, G(x, y, y) \quad \textit{for all } x, y, z \in X. \tag{4.16}$$

*Then $T$ has a unique fixed point. In fact, $T$ is a Picard operator.*

The proof of Theorem 4.2.2 is the same as the proof of Theorem 4.2.1. We omit the proof to avoid repetition.

*Remark 4.2.1.* Condition (4.15) implies condition (4.16). The converse is only true if $\lambda \in [0, 1/2)$. To prove it, assume that $0 \leq \lambda < 1/2$ and let $\lambda' = 2\lambda \in [0, 1)$. Let $x, y, z \in X$ arbitrary. If $x = y$ or $y = z$, then condition (4.16) implies condition (4.15). Assume that $x \neq y$ and $y \neq z$. Then, using (4.16) and axioms $(G_3)$ and $(G_5)$,

$$\begin{aligned} G(Tx, Ty, Tz) &\leq G(Tx, Ty, Ty) + G(Ty, Ty, Tz) \\ &\leq \lambda\, G(x, y, y) + \lambda\, G(y, y, z) \\ &\leq \lambda\, G(x, y, z) + \lambda\, G(x, y, z) \\ &= \lambda' G(x, y, z). \end{aligned}$$

### 4.2.2 Fixed Point Theorems Using Altering Distance Functions

In this section we weaken the contractive conditions on the map under consideration. In 1969, Boyd and Wong [52] defined the concept of $\Phi$-contraction. Later, in 1997, Alber and Guerre-Delabriere [17] defined the notion of weak $\phi$-contractions on Hilbert spaces and proved a fixed point theorem regarding such contractions. Specifically, a map $T : X \to X$ on a metric space $(X, d)$ into itself is called a *weak $\phi$-contraction* if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) \quad \text{for all } x, y \in X.$$

These types of contractions were discussed in the literature (see e.g. [111, 173, 200]).

In this subsection we present some fixed point results in the framework of $G$-metric spaces involving altering distance functions in the contractivity condition. To do that, recall that

$$\begin{aligned}\mathcal{F}_{\text{alt}} = \{\phi : [0,\infty) \to [0,\infty) :\ &\phi \text{ continuous, non-decreasing,}\\ &\phi\,(t) = 0 \Leftrightarrow t = 0\,\},\end{aligned}$$

$$\mathcal{F}'_{\text{alt}} = \{\phi : [0,\infty) \to [0,\infty) :\ \phi \text{ lower semi-continuous, } \phi\,(t) = 0 \Leftrightarrow t = 0\}\,.$$

**Theorem 4.2.3.** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be a self-mapping. Assume that there exist two functions $\psi \in \mathcal{F}_{\text{alt}}$ and $\phi \in \mathcal{F}'_{\text{alt}}$ such that*

$$\psi\,(G\,(Tx, Ty, Ty)) \leq \psi\,(G\,(x, y, y)) - \phi\,(G\,(x, y, y)) \tag{4.17}$$

*for all $x, y \in X$. Then $T$ has a unique fixed point. Furthermore, $T$ is a Picard operator.*

*Proof.* Let $x_0 \in X$ be an arbitrary point and let $\{x_n\}_{n\in\mathbb{N}}$ be the Picard sequence of $T$ based on $x_0$, that is, $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then $x_{n_0}$ if a fixed point of $T$, and the existence part is finished. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. In such a case, using the contractivity condition (4.17), for all $n \in \mathbb{N}$,

$$\begin{aligned}\psi\,(G\,(x_{n+1}, x_{n+2}, x_{n+2})) &= \psi\,(G\,(Tx_n, Tx_{n+1}, Tx_{n+1}))\\ &\leq \psi\,(G\,(x_n, x_{n+1}, x_{n+1})) - \phi\,(G\,(x_n, x_{n+1}, x_{n+1}))\,.\end{aligned}$$

From Lemma 2.3.6,

$$\lim_{n\to\infty} G\,(x_n, x_{n+1}, x_{n+1}) = 0,$$

which means that $\{x_n\}$ is an asymptotically regular sequence on $(X,G)$. Next, we will prove that $\{x_n\}$ is a Cauchy sequence in $(X,G)$ reasoning by contradiction. Suppose that $\{x_n\}$ is not Cauchy. Then, by Theorem 4.1.1, there exists a positive real number $\varepsilon_0>0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that, for all $k\in\mathbb{N}$,

$$\begin{aligned}&k\leq n(k)<m(k)<n(k+1),\\&G\left(x_{n(k)},x_{m(k)-1},x_{m(k)-1}\right)\leq\varepsilon_0<G\left(x_{n(k)},x_{m(k)},x_{m(k)}\right)\end{aligned}$$

and also, for $p_1=p_2=p_3=-1\in\mathbb{Z}$,

$$\lim_{k\to\infty}G\left(x_{n(k)},x_{m(k)},x_{m(k)}\right)=\lim_{k\to\infty}G\left(x_{n(k)-1},x_{m(k)-1},x_{m(k)-1}\right)=\varepsilon_0. \tag{4.18}$$

Using the contractivity condition (4.17), for all $k\in\mathbb{N}$,

$$\begin{aligned}\psi\left(G\left(x_{n(k)},x_{m(k)},x_{m(k)}\right)\right)&=\psi\left(G\left(Tx_{n(k)-1},Tx_{m(k)-1},Tx_{m(k)-1}\right)\right)\\&\leq\psi\left(G\left(x_{n(k)-1},x_{m(k)-1},x_{m(k)-1}\right)\right)\\&\qquad-\phi\left(G\left(x_{n(k)-1},x_{m(k)-1},x_{m(k)-1}\right)\right).\end{aligned}$$

From (4.18), note

$$\{t_k=G\left(x_{n(k)},x_{m(k)},x_{m(k)}\right)\}_{k\in\mathbb{N}},\ \ \{s_k=G\left(x_{n(k)-1},x_{m(k)-1},x_{m(k)-1}\right)\}_{k\in\mathbb{N}}$$

are two sequences in $[0,\infty)$ converging to the same limit $L=\varepsilon_0$. Then, it follows from Corollary 2.3.2 that $\varepsilon_0=0$, which is a contradiction. As a consequence, we have that $\{x_n\}$ is a Cauchy sequence in $(X,G)$. Taking into account that $(X,G)$ is complete, there exists $u\in X$ such that $\{x_n\}\to u$. In particular,

$$\lim_{n\to\infty}G\left(x_n,u,u\right)=0.$$

By using the contractivity condition (4.17), we deduce that, for all $n\in\mathbb{N}$,

$$\begin{aligned}\psi\left(G\left(x_{n+1},Tu,Tu\right)\right)&=\psi\left(G\left(Tx_n,Tu,Tu\right)\right)\\&\leq\psi\left(G\left(x_n,u,u\right)\right)-\phi\left(G\left(x_n,u,u\right)\right)\\&\leq\psi\left(G\left(x_n,u,u\right)\right).\end{aligned}$$

From item 2 of Corollary 2.3.1,

$$\lim_{n\to\infty}G\left(x_{n+1},Tu,Tu\right)=0.$$

Using the fact that $G$ is continuous on each variable (see Theorem 3.2.2), Lemma 2.3.3 guarantees that

$$G(u, Tu, Tu) = \lim_{n\to\infty} G(x_{n+1}, Tu, Tu) = 0.$$

As a consequence, by using $(G_2)$, we conclude that $Tu = u$.

Finally, we claim that $T$ has a unique fixed point. Let $u, v \in \operatorname{Fix}(T)$ be arbitrary fixed points of $T$. If $u \neq v$, then $G(u, v, v) > 0$ and, as $\phi \in \mathcal{F}'_{\text{alt}}$, then $\phi(G(u, v, v)) > 0$. Using the contractivity condition (4.17), we deduce that

$$\begin{aligned}\psi(G(u, v, v)) &= \psi(G(Tu, Tv, Tv)) \\ &\leq \psi(G(u, v, v)) - \phi(G(u, v, v)) \\ &< \psi(G(u, v, v)),\end{aligned}$$

which is a contradiction. Then $u = v$ and $T$ has a unique fixed point. □

The following contractivity condition, using three arbitrary arguments, is stronger than (4.17).

**Corollary 4.2.1.** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be a self-mapping. Assume that there exist two functions $\psi \in \mathcal{F}_{\text{alt}}$ and $\phi \in \mathcal{F}'_{\text{alt}}$ such that*

$$\psi(G(Tx, Ty, Tz)) \leq \psi(G(x, y, z)) - \phi(G(x, y, z))$$

*for all $x, y, z \in X$. Then $T$ has a unique fixed point. Furthermore, $T$ is a Picard operator.*

In the following result, we take $\psi$ as the identity mapping on $[0, \infty)$, obtaining a version of the Alber and Guerre-Delabriere's result (see [17]).

**Corollary 4.2.2.** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be a self-mapping. Assume that there exists a function $\phi \in \mathcal{F}'_{\text{alt}}$ such that*

$$G(Tx, Ty, Ty) \leq G(x, y, y) - \phi(G(x, y, y))$$

*for all $x, y \in X$. Then $T$ has a unique fixed point. Furthermore, $T$ is a Picard operator.*

In fact, we can suppose that $\phi$ is continuous, obtaining a version of Rhoades' theorem.

**Corollary 4.2.3.** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be a self-mapping. Assume that there exists a continuous function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t) = 0$ if, and only if, $t = 0$, and satisfies*

$$G(Tx, Ty, Ty) \leq G(x, y, y) - \phi(G(x, y, y))$$

*for all $x, y \in X$. Then $T$ has a unique fixed point. Furthermore, $T$ is a Picard operator.*

**Corollary 4.2.4.** *Let $(X, G)$ be a complete $G$-metric space and let $T : X \to X$ be a self-mapping. Assume that there exists a continuous function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t) = 0$ if, and only if, $t = 0$, and satisfies*

$$G(Tx, Ty, Tz) \leq G(x, y, z) - \phi(G(x, y, z))$$

*for all $x, y, z \in X$. Then $T$ has a unique fixed point. Furthermore, $T$ is a Picard operator.*

Finally, if we use $\phi_\lambda(t) = (1 - \lambda)t$ for all $t \in [0, \infty)$, where $\lambda \in [0, 1)$, then $\phi_\lambda \in \mathcal{F}'_{\mathrm{alt}}$ and we have the following consequence.

**Corollary 4.2.5.** *Theorem 4.2.1 is an immediate consequence of Corollary 4.2.4.*

*Proof.* Notice that, for all $x, y, z \in X$, we have that

$$\begin{aligned} G(Tx, Ty, Tz) &\leq \lambda\, G(x, y, z) \\ &= G(x, y, z) - (1 - \lambda)\, G(x, y, z) \\ &= G(x, y, z) - \phi_\lambda(G(x, y, z)), \end{aligned}$$

so Corollary 4.2.4 is applicable. □

The previous results are also valid if we employ $\phi \in \mathcal{F}''_{\mathrm{alt}}$.

**Theorem 4.2.4.** *If we replace the condition $\phi \in \mathcal{F}'_{\mathrm{alt}}$ by the assumption $\phi \in \mathcal{F}''_{\mathrm{alt}}$, then Theorem 4.2.3 (and its subsequent corollaries) also holds.*

*Proof.* Repeat the argument in the proof of Theorem 4.2.3 using Lemmas 2.3.9 and 2.3.10 rather than Lemma 2.3.6 and Corollaries 2.3.2 and 2.3.3. □

### *4.2.3 Jachymski's Equivalent Contractivity Conditions*

In 2011, Jachymski proved in [94], in the context of metric spaces, that the contractivity condition (4.17) can be expressed equivalently in a wide range of different ways.

Given functions $\psi, \eta : [0, \infty) \to [0, \infty)$, we set

$$E_{\psi,\phi} = \{ (t, u) \in [0, \infty) \times [0, \infty) : \psi(u) \leq \psi(t) - \phi(t) \}$$

and

$$E_\psi = \{ (t, u) \in [0, \infty) \times [0, \infty) : u \leq \psi(t) \}.$$

**Theorem 4.2.5 (Jachymski [94], Lemma 1).** *Let $D$ be a subset of $[0,\infty)^2 = [0,\infty) \times [0,\infty)$. Then the following statements are equivalent.*

**(i)** *there exist functions $\psi, \phi \in \mathcal{F}_{\text{alt}}$ such that $D \subseteq E_{\psi,\phi}$;*
**(ii)** *there exist $\psi \in \mathcal{F}_{\text{alt}}$ and a non-decreasing function $\phi : [0,\infty) \to [0,\infty)$ such that $\phi^{-1}(\{0\}) = \{0\}$ and $D \subseteq E_{\psi,\phi}$;*
**(iii)** *there exist $\psi \in \mathcal{F}_{\text{alt}}$ and a lower semi-continuous function $\phi : [0,\infty) \to [0,\infty)$ such that $\phi^{-1}(\{0\}) = \{0\}$, $\liminf_{t\to\infty} \phi(t) > 0$ and $D \subseteq E_{\psi,\phi}$;*
**(iv)** *there exist $\psi \in \mathcal{F}_{\text{alt}}$ and a function $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(0) = 0$ and for any sequence $\{t_n\}$ of positive reals, $\{\eta(t_n)\} \to 0$ implies $\{t_n\} \to 0$, and $D \subseteq E_{\psi,\phi}$;*
**(v)** *there exists $\psi \in \mathcal{F}_{\text{alt}}$ satisfying the condition: for any $\varepsilon > 0$, there exist $\delta > 0$ and $\gamma \in (0,\varepsilon)$ such that for any $(t,u) \in D$, $\psi(t) < \varepsilon + \delta$ implies $\psi(u) \leq \gamma$;*
**(vi)** *for any $\alpha \in (0,1)$, there exists $\psi \in \mathcal{F}_{\text{alt}}$ such that for any $(t,u) \in D$, $\psi(u) \leq \alpha\,\psi(t)$;*
**(vii)** *there exist $\alpha \in (0,1)$ and $\psi \in \mathcal{F}_{\text{alt}}$ such that for any $(t,u) \in D$, $\psi(u) \leq \alpha\,\psi(t)$;*
**(viii)** *there exists a continuous and non-decreasing function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) < t$ for any $t > 0$, and $D \subseteq E_\varphi$;*
**(ix)** *there exists a lower semi-continuous function $\eta : [0,\infty) \to [0,\infty)$ such that $\eta^{-1}(\{0\}) = \{0\}$, and $D \subseteq E_\varphi$, where $\varphi(t) = t - \eta(t)$ for all $t \in [0,\infty)$;*
**(x)** *there exists a function $\beta : [0,\infty) \to [0,1]$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\{\beta(t_n)\} \to 1$ implies $\{t_n\} \to 0$, and $D \subseteq E_\varphi$, where $\varphi(t) = t\,\beta(t)$ for all $t \in [0,\infty)$;*
**(xi)** *there exist $\psi \in \mathcal{F}_{\text{alt}}$ and a non-decreasing, right continuous function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) < t$ for all $t > 0$, and for any $(t,u) \in D$, $\psi(u) \leq \varphi(\psi(t))$;*
**(xii)** *there exist $\psi \in \mathcal{F}_{\text{alt}}$ with $\lim_{t\to\infty} \psi(t) = \infty$, and a lower semi-continuous function $\phi : [0,\infty) \to [0,\infty)$ such that $\phi^{-1}(\{0\}) = \{0\}$ and $D \subseteq E_{\psi,\phi}$.*

As a consequence of the previous result, he proved the following result.

**Theorem 4.2.6.** *Let $T$ be a selfmap of a metric space $(X,d)$. The following statements are equivalent.*

**(i)** *There exist functions $\psi, \phi \in \mathcal{F}_{\text{alt}}$ such that, for any $x, y \in X$,*

$$\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \phi(d(x,y)). \tag{4.19}$$

**(ii)** *There exist $\alpha \in [0,1)$ and $\psi \in \mathcal{F}_{\text{alt}}$ such that, for any $x, y \in X$,*

$$\psi(d(Tx,Ty)) \leq \alpha\,\psi(d(x,y)).$$

**(iii)** *There exists a continuous and non-decreasing function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) < t$ for any $t > 0$ and for any $x, y \in X$,*

$$d(Tx,Ty) \leq \varphi(d(x,y)).$$

**(iv)** *There exist* $\psi \in \mathcal{F}_{\text{alt}}$ *and a non-decreasing function* $\phi : [0,\infty) \to [0,\infty)$ *such that* $\phi^{-1}(\{0\}) = \{0\}$ *and (4.19) holds.*

**(v)** *There exist* $\psi \in \mathcal{F}_{\text{alt}}$ *and a lower semi-continuous function* $\phi : [0,\infty) \to [0,\infty)$ *such that* $\phi^{-1}(\{0\}) = \{0\}$*,* $\liminf_{t\to\infty} \phi(t) > 0$ *and (4.19) holds.*

Notice that in item (v), the function $\phi$ belongs to $\mathcal{F}'_{\text{alt}}$. Following exactly Jachymski's argument, it is easy to prove the following characterization.

**Theorem 4.2.7.** *Let* $T$ *be a selfmap of a G-metric space* $(X,G)$*. The following statements are equivalent.*

**(i)** *There exist functions* $\psi, \phi \in \mathcal{F}_{\text{alt}}$ *such that, for any* $x, y \in X$*,*

$$\psi\left(G\left(Tx,Ty,Ty\right)\right) \leq \psi\left(G\left(x,y,y\right)\right) - \phi\left(G\left(x,y,y\right)\right). \tag{4.20}$$

**(ii)** *There exist* $\alpha \in [0,1)$ *and* $\psi \in \mathcal{F}_{\text{alt}}$ *such that, for any* $x, y \in X$*,*

$$\psi\left(G\left(Tx,Ty,Ty\right)\right) \leq \alpha\, \psi\left(G\left(x,y,y\right)\right).$$

**(iii)** *There exists a continuous and non-decreasing function* $\varphi : [0,\infty) \to [0,\infty)$ *such that* $\varphi(t) < t$ *for any* $t > 0$ *and for any* $x, y \in X$*,*

$$G\left(Tx,Ty,Ty\right) \leq \varphi\left(G\left(x,y,y\right)\right).$$

**(iv)** *There exist* $\psi \in \mathcal{F}_{\text{alt}}$ *and* $\phi \in \mathcal{F}'_{\text{alt}}$ *such that (4.20) holds.*

**(v)** *There exist* $\psi \in \mathcal{F}_{\text{alt}}$ *and* $\phi \in \mathcal{F}'_{\text{alt}}$ *such that* $\liminf_{t\to\infty} \phi(t) > 0$ *and (4.20) holds.*

Following the same arguments given in the proof of Theorem 4.2.3, it is possible to prove that any self-mapping $T : X \to X$ from a complete $G$-metric space into itself, satisfying any of the previous conditions, is a Picard operator.

### *4.2.4 Ćirić's Fixed Point Theorems*

Consider the family

$$\mathcal{F}_{\text{Cir}} = \left\{ \varphi : [0,\infty) \to [0,\infty) : \ \phi\left(t\right) < t \text{ and } \lim_{s\to t^+} \phi\left(s\right) < t \text{ for all } t > 0 \right\}.$$

**Theorem 4.2.8.** *Let* $(X,G)$ *be a complete G-metric space and let* $T : X \to X$ *be a self-mapping. Assume that there exists a function* $\varphi \in \mathcal{F}_{\text{Cir}}$ *such that*

$$G\left(Tx,Ty,Ty\right) \leq \varphi\left(G\left(x,y,y\right)\right) \tag{4.21}$$

*for all* $x, y \in X$*. Then* $T$ *has a unique fixed point. Furthermore,* $T$ *is a Picard operator.*

*Proof.* Let $x_0 \in X$ be an arbitrary point and let $\{x_n\}_{n\in\mathbb{N}}$ be the Picard sequence of $T$ based on $x_0$, that is, $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then $x_{n_0}$ if a fixed point of $T$, and the existence part is finished. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. In such a case, using the contractivity condition (4.21), for all $n \in \mathbb{N}$,

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+2}) &= G(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ &\leq \varphi\left(G(x_n, x_{n+1}, x_{n+1})\right). \end{aligned}$$

From item 3 of Lemma 2.3.11,

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0,$$

which means that $\{x_n\}$ is an asymptotically regular sequence on $(X, G)$. Next, we will prove that $\{x_n\}$ is a Cauchy sequence in $(X, G)$ reasoning by contradiction. Suppose that $\{x_n\}$ is not Cauchy. Then, by Theorem 4.1.1, there exists a positive real number $\varepsilon_0 > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that, for all $k \in \mathbb{N}$,

$$\begin{aligned} &k \leq n(k) < m(k) < n(k+1), \\ &G\left(x_{n(k)}, x_{m(k)-1}, x_{m(k)-1}\right) \leq \varepsilon_0 < G\left(x_{n(k)}, x_{m(k)}, x_{m(k)}\right) \end{aligned}$$

and also, for $p_1 = p_2 = p_3 = -1 \in \mathbb{Z}$,

$$\begin{aligned} \lim_{k\to\infty} G\left(x_{n(k)}, x_{m(k)}, x_{m(k)}\right) &= \lim_{k\to\infty} G\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}\right) \\ &= \varepsilon_0. \end{aligned} \tag{4.22}$$

Using the contractivity condition (4.21), for all $k \in \mathbb{N}$,

$$\begin{aligned} \varepsilon_0 &< G\left(x_{n(k)}, x_{m(k)}, x_{m(k)}\right) = G\left(Tx_{n(k)-1}, Tx_{m(k)-1}, Tx_{m(k)-1}\right) \\ &\leq \varphi\left(G\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}\right)\right). \end{aligned}$$

From (4.22),

$$\{t_k = G\left(x_{n(k)}, x_{m(k)}, x_{m(k)}\right)\}_{k\in\mathbb{N}}, \quad \{s_k = G\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}\right)\}_{k\in\mathbb{N}}$$

are two sequences in $[0, \infty)$ converging to the same limit $L = \varepsilon_0$, and $L = \varepsilon_0 < t_k \leq \varphi(s_k)$ for all $k \in \mathbb{N}$. Then, it follows from Lemma 2.3.14 that $\varepsilon_0 = 0$, which is a contradiction. As a consequence, we have that $\{x_n\}$ is a Cauchy sequence in $(X, G)$.

Taking into account that $(X, G)$ is complete, there exists $u \in X$ such that $\{x_n\} \to u$. In particular,

$$\lim_{n\to\infty} G(x_n, u, u) = 0.$$

By using the contractivity condition (4.21), we deduce that, for all $n \in \mathbb{N}$,

$$\begin{aligned} G(x_{n+1}, Tu, Tu) &= G(Tx_n, Tu, Tu) \\ &\leq \varphi(G(x_n, u, u)). \end{aligned}$$

Notice that if there exists $n_0 \in \mathbb{N}$ such that $G(x_{n_0}, u, u) = 0$, then $x_{n_0} = u$, so $x_{n_0+1} = Tx_{n_0} = Tu$ and $G(x_{n_0+1}, Tu, Tu) = 0$. From item 5 of Lemma 2.3.11, we have that

$$\lim_{n\to\infty} G(x_{n+1}, Tu, Tu) = 0.$$

Using the fact that $G$ is continuous on each variable (see Theorem 3.2.2), it follows that

$$G(u, Tu, Tu) = \lim_{n\to\infty} G(x_{n+1}, Tu, Tu) = 0.$$

As a consequence, by using $(G_2)$, we conclude that $Tu = u$.

Finally, we claim that $T$ has a unique fixed point. Let $u, v \in \operatorname{Fix}(T)$ be arbitrary fixed points of $T$. If $u \neq v$, then $G(u, v, v) > 0$ and, as $\varphi \in \mathcal{F}_{\text{Cir}}$, then $\phi(G(u, v, v)) < G(u, v, v)$. Using the contractivity condition (4.21), we deduce that

$$\begin{aligned} G(u, v, v) &= G(Tu, Tv, Tv) \\ &\leq \varphi(G(u, v, v)) < G(u, v, v), \end{aligned}$$

which is a contradiction. Then $u = v$ and $T$ has a unique fixed point. □

**Corollary 4.2.6.** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be a self-mapping. Assume that there exists a function $\varphi \in \mathcal{F}_{\text{Cir}}$ such that*

$$G(Tx, Ty, Tz) \leq \varphi(G(x, y, z))$$

*for all $x, y, z \in X$. Then $T$ has a unique fixed point. Furthermore, $T$ is a Picard operator.*

## 4.3 Basic Common Fixed Point Theorems in the Context of $G$-Metric Spaces

In this section, we extend the previous results to the case in which we have two nonlinear operators $T, g : X \to X$, and we describe sufficient conditions to guarantee existence of coincidence points ($Tx = gx$) or common fixed points ($\omega = T\omega = g\omega$).

### *4.3.1 Basic Common Fixed Points Theorems in G-Metric Spaces*

We first state the following theorem concerning the existence and uniqueness of common fixed points which can be considered as a generalization of Theorem 4.2.1.

**Theorem 4.3.1.** *Let $(X, G)$ be a $G$-metric space and let $T, g : X \to X$ be two mappings. Suppose that there exists $\lambda \in [0, 1)$ such that*

$$G(Tx, Ty, Ty) \leq \lambda\, G(gx, gy, gy) \quad \text{for all } x, y \in X. \tag{4.23}$$

*Also assume that $T$ and $g$ satisfy the following conditions.*

$(A_1)$ $T(X) \subseteq g(X)$,
$(A_2)$ $(X, G)$ *is complete,*
$(A_3)$ $g$ *is $G$-continuous and commutes with $T$.*

*Then $T$ and $g$ have a unique common fixed point, that is, there is a unique $x \in X$ such that $gx = Tx = x$.*

*Proof.* Let $x_0 \in X$. By assumption $(A_1)$ and Lemma 4.1.1, there exists a Picard sequence $\{x_n\} \subseteq X$ of $(T, g)$, that is,

$$gx_{n+1} = Tx_n \quad \text{for all } n \in \mathbb{N}.$$

If there exists $n_0 \in \mathbb{N}$ such that $gx_{n_0+1} = gx_{n_0}$, then $gx_{n_0} = gx_{n_0+1} = Tx_{n_0}$, so $x_{n_0}$ is a coincidence point of $T$ and $g$. On the contrary case, assume that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. In particular, $G(gx_0, gx_1, gx_1) > 0$. Due to (4.23), we have that, for all $n \geq 0$,

$$\begin{aligned} G(gx_{n+1}, gx_{n+2}, gx_{n+2}) &= G(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ &\leq \lambda\, G(gx_n, gx_{n+1}, gx_{n+1}). \end{aligned}$$

From Corollary 4.1.1, $\{gx_n\}$ is a $G$-Cauchy sequence in $g(X) \subseteq X$. Since $(X, G)$ is complete, then there exists $z \in X$ such that $\{gx_n\} \to z$. Since $g$ is $G$-continuous, we have $\{ggx_n\} \to gz$. On the other hand, since $g$ and $T$ commute, we have that $ggx_{n+1} = gTx_n = Tgx_n$ for all $n \geq 0$. Thus,

$$G(ggx_{n+1}, Tz, Tz) = G(Tgx_n, Tz, Tz) \leq \lambda\, G(ggx_n, gz, gz)$$

for all $n \geq 0$. Letting $n \to \infty$ and using the fact that the metric $G$ is continuous, we get that

$$G(gz, Tz, Tz) \leq \lambda\, G(gz, gz, gz) = 0.$$

Hence $gz = Tz$. Furthermore, for all $n \geq 0$,

$$G(gx_{n+1}, gz, gz) = G(Tx_n, Tz, Tz) \leq \lambda\, G(gx_n, gz, gz).$$

Letting $n \to \infty$ and using the fact that $G$ is continuous, we obtain that

$$G(z, gz, gz) \leq \lambda\, G(z, gz, gz).$$

Hence we have $z = gz = Tz$. We now show that $z$ is the unique common fixed point of $T$ and $g$. Suppose that, contrary to our claim, there exists another common fixed point $w \in X$ with $w \neq z$. From (4.23) we have

$$G(z, w, w) = G(Tz, Tw, Tw) \leq \lambda\, G(z, w, w)$$

which is a contradiction since $\lambda < 1$. Hence, the common fixed point of $T$ and $g$ is unique. □

**Lemma 4.3.1.** *Let $(X, G)$ be a $G$-metric space and let $T : X \to X$ and $g : X \to X$ be two mappings such that there exists $\lambda \in [0, 1)$ satisfying*

$$G(Tx, Ty, Ty) \leq \lambda\, G(gx, gy, gy) \quad \textit{for all } x, y \in X.$$

*If $g$ is $G$-continuous at $\omega \in X$, then $T$ is also $G$-continuous at $\omega$. In particular, if $g$ is $G$-continuous, then $T$ is also $G$-continuous.*

*Proof.* Let $\{x_n\} \subseteq X$ be a sequence such that $\{x_n\} \to \omega$. As $g$ is $G$-continuous at $x$, then $\{gx_n\} \to g\omega$, that is,

$$\lim_{n\to\infty} G\,(gx_n, g\omega, g\omega) = 0.$$

Applying the contractivity condition, we have that

$$0 \leq \lim_{n\to\infty} G\,(Tx_n, T\omega, T\omega) \leq \lim_{n\to\infty} \lambda\, G\,(gx_n, g\omega, g\omega) = 0.$$

Then $\lim_{n\to\infty} G\,(Tx_n, T\omega, T\omega) = 0$, which means that $\{Tx_n\} \to T\omega$. Therefore, $T$ is $G$-continuous at $\omega$. □

The same argument used to prove Theorem 4.2.2 (which follows the proof of Theorem 4.2.1) is useful to obtain the following result.

**Corollary 4.3.1.** *Let $(X, G)$ be a G-metric space and let $T : X \to X$ and $g : X \to X$ be two mappings such that there exists $\lambda \in [0, 1)$ satisfying*

$$G(Tx, Ty, Tz) \leq \lambda G(gx, gy, gy) \quad \textit{for all } x, y \in X. \tag{4.24}$$

*Assume that T and g satisfy the following conditions.*

$(A_1)$ *$T(X) \subseteq g(X)$,*
$(A_2)$ *$(X, G)$ is complete,*
$(A_3)$ *$g$ is G-continuous and commutes with $T$.*

*Then T and g have a unique common fixed point, that is, there is a unique $x \in X$ such that $gx = Tx = x$.*

## 4.3.2 Common Fixed Point Theorems Using Altering Distance Functions

In this subsection we present a common fixed point theorem for nonlinear operators $T, g : X \to X$ using the following contractivity condition: for all $x, y \in X$,

$$\psi\left(G\left(Tx, Ty, Ty\right)\right) \leq \psi\left(G\left(gx, gy, gy\right)\right) - \phi\left(G\left(gx, gy, gy\right)\right), \tag{4.25}$$

where $\psi \in \mathcal{F}_{\text{alt}}$ and $\phi \in \mathcal{F}'_{\text{alt}}$.

**Lemma 4.3.2.** *Let $T, g : X \to X$ be two self-mappings on a G-metric space $(X, G)$ such that (4.25) holds, where $\psi \in \mathcal{F}_{\text{alt}}$. Then T is G-continuous at every point in which g is G-continuous.*

*Proof.* Assume that $g$ is $G$-continuous at a point $\omega \in X$ and let $\{x_n\} \subseteq X$ be a sequence such that $\{x_n\} \to \omega$. As $g$ is $G$-continuous at $\omega$, then $\{gx_n\} \to g\omega$, that is,

$$\lim_{n\to\infty} G\left(gx_n, g\omega, g\omega\right) = 0.$$

Applying the contractivity condition (4.25), we have that

$$\begin{aligned}\psi\left(G\left(Tx_n, T\omega, T\omega\right)\right) &\leq \psi\left(G\left(gx_n, g\omega, g\omega\right)\right) - \phi\left(G\left(gx_n, g\omega, g\omega\right)\right)\\ &\leq \psi\left(G\left(gx_n, g\omega, g\omega\right)\right).\end{aligned}$$

From item 2 of Corollary 2.3.1,

$$\lim_{n\to\infty} G\left(Tx_n, T\omega, T\omega\right) = 0,$$

which means that $\{Tx_n\} \to T\omega$. Therefore, $T$ is $G$-continuous at $\omega$. □

The following result is an extension of Theorem 4.2.3 to the coincidence case. We recall that we denote by $\operatorname{Co}(T, g)$ the family of all coincidence point of $T$ and $g$.

**Theorem 4.3.2.** *Let $(X, G)$ be a G-metric space and let $T, g : X \to X$ be two self-mappings. Assume that the following conditions are fulfilled:*

**(i)** *$(X, G)$ is complete.*
**(ii)** *$T(X) \subseteq g(X)$.*
**(iii)** *there exist two functions $\psi \in \mathcal{F}_{\text{alt}}$ and $\phi \in \mathcal{F}'_{\text{alt}}$ such that, for all $x, y \in X$,*

$$\psi\left(G\left(Tx, Ty, Ty\right)\right) \leq \psi\left(G\left(gx, gy, gy\right)\right) - \phi\left(G\left(gx, gy, gy\right)\right) \tag{4.26}$$

**(iv)** *$g$ is continuous and commutes with $T$.*

*Then $T$ and $g$ have a unique common fixed point $\omega$, that is, a point satisfying $\omega = T\omega = g\omega$. In fact, for any coincidence point $x$ of $T$ and $g$, we have that $\omega = Tx$. In particular, $gx = gy$ for all $x, y \in$ $\operatorname{Co}(T, g)$.*

*Proof.* First we prove that $T$ and $g$ have, at least, a coincidence point. Let $x_0 \in X$ be an arbitrary point. From Lemma 4.1.1, there exists a Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ of $(T, g)$, that is,

$$gx_{n+1} = Tx_n \quad \text{for all } n \in \mathbb{N}.$$

If there exists some $n_0 \in \mathbb{N}$ such that $gx_{n_0+1} = gx_{n_0}$, then $gx_{n_0} = gx_{n_0+1} = Tx_{n_0}$, so $x_{n_0}$ if a coincidence point of $T$ and $g$, and the existence part is finished. Assume that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$, that is,

$$G\left(gx_n, gx_{n+1}, gx_{n+1}\right) > 0 \quad \text{for all } n \in \mathbb{N}.$$

In such a case, using the contractivity condition (4.26), for all $n \in \mathbb{N}$,

$$\begin{aligned}\psi\left(G\left(gx_{n+1}, gx_{n+2}, gx_{n+2}\right)\right) &= \psi\left(G\left(Tx_n, Tx_{n+1}, Tx_{n+1}\right)\right)\\ &\leq \psi\left(G\left(gx_n, gx_{n+1}, gx_{n+1}\right)\right) - \phi\left(G\left(gx_n, gx_{n+1}, gx_{n+1}\right)\right).\end{aligned}$$

From Lemma 2.3.6,

$$\lim_{n\to\infty} G\left(gx_n, gx_{n+1}, gx_{n+1}\right) = 0,$$

which means that $\{gx_n\}$ is an asymptotically regular sequence on $(X, G)$. Next, we will prove that $\{gx_n\}$ is a Cauchy sequence in $(X, G)$ reasoning by contradiction. Suppose that $\{gx_n\}$ is not Cauchy. Then, by Theorem 4.1.1, there exists a positive real number $\varepsilon_0 > 0$ and two subsequences $\{gx_{n(k)}\}$ and $\{gx_{m(k)}\}$ of $\{gx_n\}$ such that, for all $k \in \mathbb{N}$,

$$\begin{aligned}&k \leq n(k) < m(k) < n(k+1),\\ &G\left(gx_{n(k)}, gx_{m(k)-1}, gx_{m(k)-1}\right) \leq \varepsilon_0 < G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right)\end{aligned}$$

and also, for $p_1 = p_2 = p_3 = -1 \in \mathbb{Z}$,

$$\lim_{k\to\infty} G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right) = \lim_{k\to\infty} G\left(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}\right)$$
$$= \varepsilon_0. \tag{4.27}$$

Using the contractivity condition (4.26), for all $k \in \mathbb{N}$,

$$\begin{aligned}\psi\left(G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right)\right) &\\ = \psi\left(G\left(Tx_{n(k)-1}, Tx_{m(k)-1}, Tx_{m(k)-1}\right)\right) &\\ \leq \psi\left(G\left(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}\right)\right) &\\ - \phi\left(G\left(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}\right)\right). &\end{aligned}$$

From (4.27), (with $k \in \mathbb{N}$),

$$\{t_k = G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right)\}, \ \{s_k = G\left(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}\right)\}$$

are two sequences in $[0, \infty)$ converging to the same limit $L = \varepsilon_0$ and satisfying

$$\psi(t_k) \leq \psi(s_k) - \phi(s_k) \quad \text{for all } n \in \mathbb{N}.$$

Then, it follows from Corollary 2.3.2 that $\varepsilon_0 = 0$, which is a contradiction. As a consequence, we have that $\{gx_n\}$ is a Cauchy sequence in $(X, G)$.

Taking into account that $(X, G)$ is complete, there exists $z \in X$ such that $\{gx_n\} \to z$. As $g$ is $G$-continuous, then $\{ggx_n\} \to gz$. In particular,

$$\lim_{n\to\infty} G(ggx_n, gz, gz) = 0.$$

On the other hand, since $g$ and $T$ commute, we have that

$$ggx_{n+1} = gTx_n = Tgx_n \quad \text{for all } n \geq 0.$$

Thus, by using the contractivity condition (4.26), we deduce that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\psi\left(G(ggx_{n+1}, Tz, Tz)\right) &= \psi\left(G(Tgx_n, Tz, Tz)\right)\\ &\leq \psi\left(G(ggx_n, gz, gz)\right) - \phi\left(G(ggx_n, gz, gz)\right)\\ &\leq \psi\left(G(ggx_n, gz, gz)\right).\end{aligned} \tag{4.28}$$

From item 2 of Corollary 2.3.1,

$$\lim_{n\to\infty} G(ggx_{n+1}, Tz, Tz) = 0,$$

which means that $\{ggx_n\} \to Tz$. However, as $\{ggx_n\} \to gz$, the uniqueness of the limit in a $G$-metric space concludes that $Tz = gz$, that is, $z$ is a coincidence point of $T$ and $g$.

Next, we claim that

$$gx = gy \text{ for all } x, y \in \operatorname{Co}(T, g). \tag{4.29}$$

Assume that $x$ and $y$ are two coincidence points of $T$ and $g$. By the contractivity condition (4.26),

$$\begin{aligned}\psi\left(G(gx, gy, gy)\right) &= \psi\left(G(Tx, Ty, Ty)\right)\\ &\leq \psi\left(G(gx, gy, gy)\right) - \phi\left(G(gx, gy, gy)\right)\\ &\leq \psi\left(G(gx, gy, gy)\right).\end{aligned}$$

Therefore, $\phi\left(G(gx, gy, gy)\right) = 0$, so $G(gx, gy, gy) = 0$ and $gx = gy$. This proves that (4.29) holds.

Next we show that, for all coincidence points $x$ of $T$ and $g$, the point $\omega = Tx$ is a common fixed point of $T$ and $g$. Let $x \in X$ be an arbitrary coincidence point of $T$ and $g$ and let $\omega = Tx = gx$. As $T$ and $g$ commutes, Remark 2.2.1 guarantees that $\omega = Tx$ is also a coincidence point of $T$ and $g$. Then, $T\omega = g\omega$. Moreover, by (4.29), we have that $gx = g\omega$. In particular, $T\omega = g\omega = gx = Tx = \omega$. As a result, $\omega$ is a common fixed point of $T$ and $g$.

Finally, we prove that $T$ and $g$ have a unique common fixed point. Let $\omega$ and $z$ be two common fixed points of $T$ and $g$, that is, $\omega = T\omega = g\omega$ and $z = Tz = gz$. By (4.29), we have that $g\omega = gz$, so $\omega = g\omega = gz = z$. □

The previous result is also valid if we consider $\phi \in \mathcal{F}''_{\text{alt}}$.

**Theorem 4.3.3.** *If we replace the condition $\phi \in \mathcal{F}'_{\text{alt}}$ by the assumption $\phi \in \mathcal{F}''_{\text{alt}}$, then Theorem 4.3.2 also holds.*

*Proof.* Repeat the argument in the proof of Theorem 4.3.2 using Lemmas 2.3.9 and 2.3.10 rather than Lemma 2.3.6 and Corollaries 2.3.2 and 2.3.1. □

### *4.3.3 Jachymski's Equivalent Contractivity Conditions*

Using Theorem 4.2.5, it is easy to express the contractivity condition (4.26) in several ways as follows.

**Theorem 4.3.4.** *Let $T$ and $g$ be two selfmaps of a $G$-metric space $(X, G)$. The following statements are equivalent.*

**(i)** *There exist functions $\psi, \phi \in \mathcal{F}_{\text{alt}}$ such that, for any $x, y \in X$,*

$$\psi\left(G\left(Tx, Ty, Ty\right)\right) \leq \psi\left(G\left(gx, gy, gy\right)\right) - \phi\left(G\left(gx, gy, gy\right)\right). \tag{4.30}$$

**(ii)** *There exist* $\alpha \in [0,1)$ *and* $\psi \in \mathcal{F}_{\text{alt}}$ *such that, for any* $x, y \in X$,

$$\psi\left(G\left(Tx, Ty, Ty\right)\right) \leq \alpha\, \psi\left(G\left(gx, gy, gy\right)\right).$$

**(iii)** *There exists a continuous and non-decreasing function* $\varphi : [0,\infty) \to [0,\infty)$ *such that* $\varphi(t) < t$ *for any* $t > 0$ *and for any* $x, y \in X$,

$$G\left(Tx, Ty, Ty\right) \leq \varphi\left(G\left(x, y, y\right)\right).$$

**(iv)** *There exist* $\psi \in \mathcal{F}_{\text{alt}}$ *and a non-decreasing function* $\phi : [0,\infty) \to [0,\infty)$ *such that* $\phi^{-1}(\{0\}) = \{0\}$ *and (4.30) holds.*

**(v)** *There exist* $\psi \in \mathcal{F}_{\text{alt}}$ *and* $\phi \in \mathcal{F}'_{\text{alt}}$ *such that* $\liminf_{t\to\infty} \phi(t) > 0$ *and (4.30) holds.*

### *4.3.4 Ćirić's Common Fixed Point Theorems*

In this section, given two mappings $T, g : X \to X$, we study the contractivity condition

$$G\left(Tx, Ty, Ty\right) \leq \varphi\left(G\left(gx, gy, gy\right)\right) \quad \text{for all } x, y \in X, \tag{4.31}$$

where $\varphi \in \mathcal{F}_{\text{Cir}}$. Unlike Lemma 4.3.2, the continuity of $g$ does not imply the continuity of $T$.

*Example 4.3.1.* Let $X = [0,1]$ endowed with the complete $G$-metric $G(x,y,z) = \max\{|x-y|, |x-z|, |y-z|\}$ for all $x, y, z \in X$. Define the mappings $T, g : X \to X$ and $\varphi : [0,\infty) \to [0,\infty)$ as

$$Tx = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x > 0; \end{cases} \qquad gx = 0, \qquad \varphi(t) = \begin{cases} 1, & \text{if } t = 0, \\ 0, & \text{if } t > 0. \end{cases}$$

Then $g$ is $G$-continuous and $\varphi \in \mathcal{F}_{\text{Cir}}$. We now show that (4.31) holds. Let $x, y \in X$ be arbitrary. If $Tx = Ty$, then (4.31) trivially holds. Assume that $Tx \neq Ty$. In this case, as $T(X) = \{0,1\}$, then $\{Tx, Ty\} = \{0,1\}$ and $G\left(Tx, Ty, Ty\right) = 1$. Therefore,

$$G\left(Tx, Ty, Ty\right) = 1 = \varphi(0) = \varphi\left(G(0,0,0)\right) = \varphi\left(G\left(gx, gy, gy\right)\right).$$

Although $g$ is $G$-continuous, $T$ is not continuous at $x = 0$.

**Theorem 4.3.5.** *Let* $(X, G)$ *be a complete* $G$*-metric space and let* $T, g : X \to X$ *be two self-mappings. Suppose that* $T(X) \subseteq g(X)$ *and* $g$ *is continuous and commutes with* $T$. *Also assume that there exists a function* $\varphi \in \mathcal{F}_{\text{Cir}}$ *such that*

$$G\left(Tx, Ty, Ty\right) \leq \varphi\left(G\left(gx, gy, gy\right)\right) \tag{4.32}$$

*for all $x, y \in X$. If*

$$T \text{ is } G\text{-continuous} \quad or \quad \varphi(0) = 0,$$

*then $T$ and $g$ have a unique common fixed point $\omega$. In fact, if $u \in \operatorname{Co}(T, g)$ is a coincidence point of $T$ and $g$, then $Tu = \omega$.*

*Proof.* Let $x_0 \in X$ be an arbitrary point and let $\{x_n\}_{n\in\mathbb{N}}$ be a Picard sequence of $(T, g)$ based on any $x_0 \in X$, that is, $gx_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ (see Lemma 4.1.1). If there exists some $n_0 \in \mathbb{N}$ such that $gx_{n_0+1} = gx_{n_0}$, then $x_{n_0}$ is a coincidence point of $T$ and $g$, and the existence part is finished. Assume that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. In such a case, $G(gx_n, gx_{n+1}, gx_{n+1}) > 0$ and using the contractivity condition (4.32), for all $n \in \mathbb{N}$,

$$\begin{aligned} G(gx_{n+1}, gx_{n+2}, gx_{n+2}) &= G(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ &\leq \varphi\left(G(gx_n, gx_{n+1}, gx_{n+1})\right). \end{aligned}$$

From item 3 of Lemma 2.3.11,

$$\lim_{n\to\infty} G(gx_n, gx_{n+1}, gx_{n+1}) = 0,$$

which means that $\{gx_n\}$ is an asymptotically regular sequence on $(X, G)$. Next, we will prove that $\{gx_n\}$ is a Cauchy sequence in $(X, G)$ reasoning by contradiction. Suppose that $\{gx_n\}$ is not Cauchy. From Theorem 4.1.1, there exists a positive real number $\varepsilon_0 > 0$ and two subsequences $\{gx_{n(k)}\}$ and $\{gx_{m(k)}\}$ of $\{gx_n\}$ such that, for all $k \in \mathbb{N}$,

$$\begin{aligned} &k \leq n(k) < m(k) < n(k+1), \\ &G\left(gx_{n(k)}, gx_{m(k)-1}, gx_{m(k)-1}\right) \leq \varepsilon_0 < G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right) \end{aligned}$$

and also, for $p_1 = p_2 = p_3 = -1 \in \mathbb{Z}$,

$$\begin{aligned} \lim_{k\to\infty} G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right) &= \lim_{k\to\infty} G\left(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}\right) \\ &= \varepsilon_0. \end{aligned} \tag{4.33}$$

Using the contractivity condition (4.32), for all $k \in \mathbb{N}$,

$$\begin{aligned} \varepsilon_0 < G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right) &= G\left(Tx_{n(k)-1}, Tx_{m(k)-1}, Tx_{m(k)-1}\right) \\ &\leq \varphi\left(G\left(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}\right)\right). \end{aligned}$$

From (4.33) (with $k \in \mathbb{N}$),

$$\{t_k = G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right)\}, \ \{s_k = G\left(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}\right)\}$$

are two sequences in $[0, \infty)$ converging to the same limit $L = \varepsilon_0$, and $L = \varepsilon_0 < t_k \leq \varphi(s_k)$ for all $k \in \mathbb{N}$. Then, it follows from Lemma 2.3.14 that $\varepsilon_0 = 0$, which is a contradiction. As a consequence, we have that $\{gx_n\}$ is a Cauchy sequence in $(X, G)$.

Taking into account that $(X, G)$ is complete, there exists $u \in X$ such that $\{gx_n\} \to u$. As $g$ is continuous, $\{ggx_n\} \to gu$. Moreover, as $T$ and $g$ commute,

$$\{Tgx_n\} = \{gTx_n\} = \{ggx_{n+1}\} \to gu.$$

Next, we distinguish two cases.

*Case 1.* *$T$ is G-continuous*. In this case, $\{Tgx_n\} \to Tu$. By the uniqueness of the limit, $Tu = gu$.

*Case 2.* $\varphi(0) = 0$. In this case, by using the contractivity condition (4.32), we deduce that, for all $n \in \mathbb{N}$,

$$G(Tgx_{n+1}, Tu, Tu) \leq \varphi(G(ggx_n, gu, gu)).$$

As $\varphi(0) = 0$, item 8 of Lemma 2.3.11 guarantees that $\{G(Tgx_{n+1}, Tu, Tu)\}$ converges to zero, that is, $\{Tgx_n\} \to Tu$. Again, by the uniqueness of the limit, $Tu = gu$.

In any case, we have just proved that $T$ and $g$ have, at least, a coincidence point. Now, we claim that

$$gu = gv \quad \text{for all } u, v \in \mathrm{Co}(T, g). \tag{4.34}$$

Indeed, let $u, v \in \mathrm{Co}(T, g)$ be two coincidence points of $T$ and $g$. If we suppose that $gu \neq gv$, then $G(gu, gv, gv) > 0$. As a consequence,

$$\begin{aligned} G(gu, gv, gv) &= G(Tu, Tv, Tv) \leq \varphi(G(gu, gv, gv)) \\ &< G(gu, gv, gv), \end{aligned}$$

which is a contradiction. Then $gu = gv$ and (4.34) holds.

Next, let $u \in \mathrm{Co}(T, g)$ be an arbitrary coincidence point of $T$ and $g$ and let $\omega = Tu = gu$. We claim that $\omega$ is the unique common fixed point of $T$ and $g$. Firstly, as $T$ and $g$ commute, $T\omega = Tgu = gTu = g\omega$, so $\omega$ is another coincidence point of $T$ and $g$. Using (4.34), $\omega = gu = g\omega$, so $\omega$ is a common fixed point of $T$ and $g$. If $z \in X$ is another common fixed point of $T$ and $g$, that is, $z = Tz = gz$, then, it follows from (4.34) that $z = gz = g\omega = \omega$, so $\omega$ is the unique common fixed point of $T$ and $g$. Finally, if $v \in \mathrm{Co}(T, g)$ is another arbitrary coincidence point of $T$ and $g$, then, also by (4.34), $gv = g\omega = \omega$. □

# Chapter 5
# Fixed Point Theorems in Partially Ordered $G$-Metric Spaces

In [168], Ran and Reurings established a fixed point theorem that extends the Banach contraction principle to the setting of partially ordered metric spaces (see Theorem A.1.1). In their original version, Ran and Reurings used a continuous function. Nieto and Rodríguez-López established a similar result replacing the continuity of the nonlinear operator by a property on the partially ordered metric space (see Theorem A.1.2). In this chapter, we present some fixed point theorems in the setting of partially ordered $G$-metric spaces. In particular, we will use a binary relation weaker than a partial order.

## 5.1 Binary Relations on a Set

We present here some notions and basic facts about binary relations like partial orders.

**Definition 5.1.1.** A *binary relation on* $X$ is a nonempty subset $\mathcal{R}$ of $X \times X$.

For simplicity, we let $x \preccurlyeq y$ if $(x, y) \in \mathcal{R}$, and we will say that $\preccurlyeq$ is the binary relation on $X$. Write $x \prec y$ when $x \preccurlyeq y$ and $x \neq y$. We write $y \succcurlyeq x$ when $x \preccurlyeq y$. We shall use $\preccurlyeq$ and $\preceq$ to denote binary relations on $X$.

**Definition 5.1.2.** A binary relation $\preccurlyeq$ on $X$ is

- *reflexive* if $x \preccurlyeq x$ for all $x \in X$;
- *transitive* if $x \preccurlyeq z$ for all $x, y, z \in X$ such that $x \preccurlyeq y$ and $y \preccurlyeq z$;
- *antisymmetric* if $x \preccurlyeq y$ and $y \preccurlyeq x$ imply $x = y$.

A reflexive and transitive relation on $X$ is a *preorder* (or a *quasiorder*) on $X$. In such a case, $(X, \preccurlyeq)$ is a *preordered space*. If a preorder $\preccurlyeq$ is also antisymmetric, then $\preccurlyeq$ is called a *partial order*, and $(X, \preccurlyeq)$ is a *partially ordered space* (or a *partially ordered set*).



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*Example 5.1.1.* The usual order on the set of all real numbers $\mathbb{R}$ is denoted by $\leq$. In fact, this partial order can be induced on any non-empty subset $A \subseteq \mathbb{R}$.

*Example 5.1.2.* Let $\preceq$ be the binary relation on $\mathbb{R}$ given by

$$x \preceq y \;\Leftrightarrow\; \quad (\, x = y \quad \text{or} \quad x < y \leq 0 \,)\,.$$

Then $\preceq$ is a partial order on $\mathbb{R}$, but it is different from $\leq$.

*Example 5.1.3.* Any equivalence relation is a preorder.

*Example 5.1.4.* Let $X$ be an arbitrary set and let $x_1$ and $x_2$ be two different points of $X$. If we define

$$x \preceq y \;\Leftrightarrow\; \quad (\, x = y \quad \text{or} \quad (x,y) = (x_1,x_2)\,)\,,$$

then $\preceq$ is a partial order on $X$. In fact, the relationship $\preceq$ only has two different comparable points, which are $x_1$ and $x_2$, being $x_1 \prec x_2$.

We consider fixed point theory in $G$-metric spaces provided with a partial order. In many cases, it is not necessary to consider a partial order: a preorder is enough. The main advantage of preorders if that the binary relation $\preceq_0$ on $X$, defined by

$$x \preceq_0 y \quad \text{for all } x, y \in X \tag{5.1}$$

is a preorder on $X$ (but it is not a partial order). Some of the contractive conditions we shall use are:

$$(a) \quad G(Tx,Ty,Ty) \leq \lambda\, G(x,y,y) \quad \text{for all } x,y \in X; \qquad \text{and}$$
$$(b) \quad G(Tx,Ty,Ty) \leq \lambda\, G(x,y,y) \quad \text{for all } x,y \in X \text{ such that } x \preccurlyeq y,$$

(where, in $(b)$, $\preccurlyeq$ is a partial order on $X$) can be treated in an unified way as the unique condition:

$$(c) \quad G(Tx,Ty,Ty) \leq \lambda\, G(x,y,y) \quad \text{for all } x,y \in X \text{ such that } x \preceq y,$$

where $\preceq$ is a preorder on $X$.

One of the most important hypothesis that we shall use in the results of this chapter is the monotonicity of the involved mappings.

**Definition 5.1.3.** Let $\preccurlyeq$ be a binary relation on $X$ and let $T, g : X \to X$ be mappings. We say that $T$ is:

- $(g,\preccurlyeq)$-*non-decreasing* if $Tx \preccurlyeq Ty$ for all $x,y \in X$ such that $gx \preccurlyeq gy$;
- $(g,\preccurlyeq)$-*non-increasing* if $Tx \succcurlyeq Ty$ for all $x,y \in X$ such that $gx \preccurlyeq gy$;
- $(g,\preccurlyeq)$-*increasing* (or *strictly increasing*) if $Tx \prec Ty$ for all $x,y \in X$ such that $gx \prec gy$;

- $(g, \preccurlyeq)$-*decreasing* (or *strictly decreasing*) if $Tx \succ Ty$ for all $x, y \in X$ such that $gx \prec gy$.

  If $g$ is the identity mapping on $X$, we say that $T$ is:

- $\preccurlyeq$-*non-decreasing* if $Tx \preccurlyeq Ty$ for all $x, y \in X$ such that $x \preccurlyeq y$;
- $\preccurlyeq$-*non-increasing* if $Tx \succcurlyeq Ty$ for all $x, y \in X$ such that $x \preccurlyeq y$;
- $\preccurlyeq$-*increasing* (or *strictly increasing*) if $Tx \prec Ty$ for all $x, y \in X$ such that $x \prec y$;
- $\preccurlyeq$-*decreasing* (or *strictly decreasing*) if $Tx \succ Ty$ for all $x, y \in X$ such that $x \prec y$.

Notice that if $\preccurlyeq$ is a partial order on $X$ and $T$ is $(g, \preccurlyeq)$-non-decreasing, then the condition $gx = gy$ implies that $Tx = Ty$. In particular, in such a case, if $T$ is injective, then $g$ is also injective.

**Definition 5.1.4.** An *ordered G-metric space* is a triple $(X, G, \preccurlyeq)$ where $(X, G)$ is a $G$-metric space and $\preccurlyeq$ is a partial order on $X$. If $\preccurlyeq$ is a preorder on $X$, then $(X, G, \preccurlyeq)$ is a *preordered G-metric space*.

## 5.2 Fixed Point Theorems in Preordered $G$-Metric Spaces

The following result can be considered as the natural extension of Ran and Reurings' result to the setting of $G$-metric spaces.

**Theorem 5.2.1.** *Let $(X, G, \preccurlyeq)$ be a preordered G-metric space and let $T : X \to X$ be a mapping. Suppose that the following conditions hold:*

**(i)** *$(X, G)$ is G-complete;*
**(ii)** *$T$ is non-decreasing (with respect to $\preccurlyeq$);*
**(iii)** *$T$ is G-continuous;*
**(iv)** *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;*
**(v)** *there exists a constant $\lambda \in [0, 1)$ such that, for all $x, y \in X$ with $x \succcurlyeq y$,*

$$G(Tx, Ty, Ty) \leq \lambda\, G(x, y, y). \tag{5.2}$$

*Then $T$ has a fixed point. Moreover, if for all $(x, y) \in X \times X$ there exists $w \in X$ such that $x \preccurlyeq w$ and $y \preccurlyeq w$, we obtain uniqueness of the fixed point.*

*Proof.* Let $x_0 \in X$ be a point satisfying (iii), that is, $x_0 \preccurlyeq Tx_0$. We define a sequence $\{x_n\}$ in $X$ as follows:

$$x_n = Tx_{n-1} \quad \text{for} \quad n \geq 1. \tag{5.3}$$

Regarding that $T$ is a non-decreasing mapping together with (5.3), we have $x_0 \preccurlyeq Tx_0 = x_1$ implies $x_1 = Tx_0 \preccurlyeq Tx_1 = x_2$. Inductively, we obtain

$$x_0 \preccurlyeq x_1 \preccurlyeq x_2 \preccurlyeq \ldots \preccurlyeq x_{n-1} \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq \ldots \tag{5.4}$$

Assume that there exists $n_0$ such that $x_{n_0} = x_{n_0+1}$. Since $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, then $x_{n_0}$ is the fixed point of $T$, which completes the existence part of the proof. Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Thus, by (5.4) we have

$$x_0 \prec x_1 \prec x_2 \prec \ldots \prec x_{n-1} \prec x_n \prec x_{n+1} \prec \ldots$$

Put $x = x_n$ and $y = x_{n-1}$ in (5.2). Then, for all $n \geq 1$,

$$G(x_{n+1}, x_{n+1}, x_n) = G(Tx_n, Tx_n, Tx_{n-1}) \leq \lambda\, G(x_n, x_n, x_{n-1}).$$

From Corollary 4.1.1, $\{x_n\}$ is a $G$-Cauchy sequence. As $(X, G)$ is complete, there exist $z \in X$ such that $\{x_n\} \to z$, that is,

$$\lim_{n\to\infty} G(x_n, x_n, z) = \lim_{n\to\infty} G(x_n, z, z) = 0.$$

We show now that $z$ is a fixed point of $T$. From the $G$-continuity of $T$, the sequence $\{Tx_n\} = \{x_{n+1}\}$ converges to $Tz$. By Proposition 3.2.2, the $G$-limit of a sequence is unique, so $z = Tz$.

To prove uniqueness, we assume that $y \in X$ is another fixed point of $T$ such that $z \neq y$. By hypothesis, there exists $w \in X$ such that $y \preccurlyeq w$ and $z \preccurlyeq w$. Let $\{w_n\}$ be the Picard sequence of $T$ based on $w_0 = w$. As $T$ is $\preccurlyeq$-non-decreasing, $y = Ty \preccurlyeq Tw = w_1$ and $z = Tz \preccurlyeq Tw = w_1$. By induction, $y \preccurlyeq w_n$ and $z \preccurlyeq w_n$ for all $n \geq 0$. Applying the contractivity condition (5.2), we have that, for all $n \geq 0$,

$$G(w_{n+1}, w_{n+1}, y) = G(Tw_n, Tw_n, Ty) \leq \lambda\, G(w_n, w_n, y) \quad \text{and}$$
$$G(w_{n+1}, w_{n+1}, z) = G(Tw_n, Tw_n, Tz) \leq \lambda\, G(w_n, w_n, z).$$

Hence, for all $n \geq 0$,

$$G(w_n, w_n, y) \leq \lambda^n G(w_0, w_0, y) \quad \text{and} \quad G(w_n, w_n, z) \leq \lambda^n G(w_0, w_0, z).$$

Letting $n \to \infty$ we deduce that $\{w_n\} \xrightarrow{G} y$ and $\{w_n\} \xrightarrow{G} z$, and the uniqueness of the limit concludes that $z = y$, so $T$ has a unique fixed point. □

The main advantage of the contractivity condition (5.2) versus (4.16) is that (5.2) only requires the inequality to hold for comparable points, that is, for all $x, y \in X$ such that $x \succcurlyeq y$. In the following example, (4.16) is false but (5.2) holds.

*Example 5.2.1.* Let $X$ be the set of all real numbers $\mathbb{R}$ endowed with the $G$-metric $G(x, y, z) = \max\{|x-y|, |x-z|, |y-z|\}$ for all $x, y, z \in \mathbb{R}$. Consider on $\mathbb{R}$ the partial order

$$x \preceq y \;\Leftrightarrow\; (\,x = y \quad \text{or} \quad x < y \leq 0\,).$$

Define $T : \mathbb{R} \to \mathbb{R}$ by

$$Tx = \begin{cases} \dfrac{x}{2}, & \text{if } x \leq 0, \\ 2x, & \text{if } x > 0. \end{cases}$$

Let $x, y \in X$ be such that $x \succeq y$. If $x = y$, then (5.2) trivially holds. Assume that $x \neq y$. Then $y < x \leq 0$. Hence

$$G(Tx, Ty, Ty) = G\left(\frac{x}{2}, \frac{y}{2}, \frac{y}{2}\right) = \left|\frac{x}{2} - \frac{y}{2}\right| =$$
$$= \frac{1}{2}\,|x - y| = \frac{1}{2}\,G(x, y, y).$$

Hence, (5.2) holds. However, (4.16) is false in this case because if $x = 1$ and $y = 2$, then

$$G(T1, T2, T2) = G\,(2, 4, 4) = 2 = 2\,G\,(1, 2, 2)\,.$$

Although Theorem 4.2.2 is not applicable, Theorem 5.2.1 guarantees that $T$ has a unique fixed point, which is $u = 0$.

In the following result, we use a contractivity condition involving three variables.

**Corollary 5.2.1.** *Let $(X, G, \preccurlyeq)$ be a preordered $G$-metric space and let $T : X \to X$ be a mapping. Suppose that the following conditions hold:*

- **(i)** *$(X, G)$ is $G$-complete;*
- **(ii)** *$T$ is non-decreasing (with respect to $\preccurlyeq$);*
- **(iii)** *$T$ is $G$-continuous;*
- **(iv)** *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;*
- **(v)** *there exists a constant $\lambda \in [0, 1)$ such that, for all $x, y, z \in X$ with $x \succcurlyeq y \succcurlyeq z$,*

$$G(Tx, Ty, Tz) \leq \lambda\, G(x, y, z). \tag{5.5}$$

*Then $T$ has a fixed point. Moreover, if for all $(x, y) \in X \times X$ there exists $w \in X$ such that $x \preccurlyeq w$ and $y \preccurlyeq w$, we obtain uniqueness of the fixed point.*

*Proof.* It follows from the fact that (5.5) implies (5.2). □

After the appearance of the Ran and Reurings' theorem [168], Nieto and Rodríguez-López [158] changed the continuity of the mapping $T$ with the following condition on the ordered metric space $(X, d, \preccurlyeq)$:

- if $x \in X$ and $\{x_n\} \subseteq X$ is a sequence in $X$ such that $\{x_n\} \xrightarrow{d} x$ and $x_n \preccurlyeq x_{n+1}$ for all $n \in \mathbb{N}$, then $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$.

Next, we present this notion in preordered $G$-metric spaces and we show an equivalent version of the Nieto and Rodríguez-López's result. In the following definition, we only assume that $\preceq$ is a binary relation on $X$. Later, we will use this notion when $\preceq$ is a partial order or a preorder.

**Definition 5.2.1.** Let $(X, G)$ be a $G$-metric space, let $A \subseteq X$ be a non-empty subset and let $\preceq$ be a binary relation on $X$. Then $(A, G, \preceq)$ is said to be:

- *non-decreasing-regular* if for all sequence $\{x_m\} \subseteq A$ such that $\{x_m\} \to a \in A$ and $x_m \preceq x_{m+1}$ for all $m \in \mathbb{N}$, we have that $x_m \preceq a$ for all $m \in \mathbb{N}$;
- *non-increasing-regular* if for all sequence $\{x_m\} \subseteq A$ such that $\{x_m\} \to a \in A$ and $x_m \succeq x_{m+1}$ for all $m \in \mathbb{N}$, we have that $x_m \succeq a$ for all $m \in \mathbb{N}$;
- *regular* if it is both non-decreasing-regular and non-increasing-regular.

**Theorem 5.2.2.** *Let $(X, \preccurlyeq)$ be a preordered set endowed with a G-metric and let $T : X \to X$ be a mapping. Suppose that the following conditions hold:*

**(i)** *$(X, G)$ is G-complete;*
**(ii)** *$T$ is non-decreasing (with respect to $\preccurlyeq$);*
**(iii)** *$(X, G, \preccurlyeq)$ is non-decreasing-regular;*
**(iv)** *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;*
**(v)** *there exists a constant $\lambda \in [0, 1)$ such that, for all $x, y \in X$ with $x \succcurlyeq y$,*

$$G(Tx, Ty, Ty) \leq \lambda\, G(x, y, y). \tag{5.6}$$

*Then $T$ has a fixed point. Moreover, if for all $(x, y) \in X \times X$ there exists $w \in X$ such that $x \preccurlyeq w$ and $y \preccurlyeq w$, we obtain uniqueness of the fixed point.*

*Proof.* Following the proof of Theorem 5.2.1, we have a $\preccurlyeq$-non-decreasing sequence $\{x_n\}$ which is $G$-convergent to $z \in X$. Due to (iii), we have that $x_n \preccurlyeq z$ for all $n$. We now show that $z$ is a fixed point of $T$. Suppose, on the contrary, that $z \neq Tz$, that is, $d_s^G(z, Tz) > 0$. Regarding (5.6) with $x = x_n$ and $y = Tz$, we have that

$$\begin{aligned} d_s^G(x_{n+1}, Tz) &= G(x_{n+1}, Tz, Tz) + G(Tz, x_{n+1}, x_{n+1}) \\ &= G(Tx_n, Tz, Tz) + G(Tz, Tx_n, Tx_n) \\ &\leq \lambda\, [G(x_n, z, z) + G(z, x_n, x_n)] \leq 3\lambda\, G(x_n, z, z). \end{aligned}$$

Letting $n \to \infty$, we get $d_s^G(z, Tz) = 0$, which is a contradiction. Hence, $Tz = z$. Uniqueness of $z$ can be observed as in the proof of Theorem 5.2.1. □

**Corollary 5.2.2.** *Let $(X, \preccurlyeq)$ be an ordered set endowed with a G-metric and let $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

**(i)** *$(X, G)$ is G-complete;*
**(ii)** *$T$ is $\preccurlyeq$-non-decreasing (with respect to $\preccurlyeq$);*
**(iii)** *$(X, G, \preccurlyeq)$ is non-decreasing-regular;*
**(iv)** *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;*
**(v)** *there exists a constant $\lambda \in [0, 1)$ such that, for all $x, y, z \in X$ with $x \succcurlyeq y \succcurlyeq z$,*

$$G(Tx, Ty, Tz) \leq \lambda\, G(x, y, z). \tag{5.7}$$

*Then $T$ has a fixed point. Moreover, if for all $(x, y) \in X \times X$ there exists $w \in X$ such that $x \preccurlyeq w$ and $y \preccurlyeq w$, we obtain uniqueness of the fixed point.*

Notice that, due to the symmetry of $G$, it is equivalent to assuming the contractivity condition (5.7) for all $x, y, z \in X$ such that $x \preccurlyeq y \preccurlyeq z$.

**Corollary 5.2.3.** *Theorem 4.2.2 follows immediately from Theorem 5.2.2.*

*Proof.* It is only necessary to consider on $X$ the preorder $\preceq_0$ defined in (5.1). Then, all the hypotheses of Theorem 5.2.2 are satisfied. □

Note one could repeat almost all the results of Sect. 4.2 in the context of preordered $G$-metric spaces.

## 5.3 Common Fixed Point Theorems in Preordered *G*-Metric Spaces

In this section, we prove some common fixed point theorems in the context of preordered $G$-metric spaces under different contractivity conditions.

### *5.3.1 Common Fixed Point Theorems in Preordered G-Metric Spaces Using Altering Distance Functions*

The following is one of the two main results of this subsection.

**Theorem 5.3.1.** *Let $(X, G, \preccurlyeq)$ be a preordered $G$-metric space and let $T, g : X \to X$ be two mappings. Suppose that the following conditions hold:*

- **(i)** *$(X, G)$ is $G$-complete;*
- **(ii)** *there exists $x_0 \in X$ such that $gx_0 \preccurlyeq Tx_0$;*
- **(iii)** *$T(X) \subseteq g(X)$;*
- **(iv)** *$T$ is $(g, \preccurlyeq)$-non-decreasing;*
- **(v)** *$g$ is $G$-continuous and commutes with $T$;*
- **(vi)** *there exist two functions $\psi \in \mathcal{F}_{\text{alt}}$ and $\phi \in \mathcal{F}'_{\text{alt}}$ such that, for all $x, y \in X$ with $gx \preccurlyeq gy$,*

$$\psi\left(G(Tx, Ty, Ty)\right) \leq \psi(G(gx, gy, gy)) - \phi(G(gx, gy, gy)); \tag{5.8}$$

- **(vii)** *$T$ is $G$-continuous.*

*Then $T$ and $g$ have, at least, a coincidence point, that is, there exists $z \in X$ such that $Tz = gz$.*

*Furthermore, assume that for all $x, y \in \mathrm{Co}(T, g)$, there exists $w \in X$ such that $gx \preccurlyeq gw$ and $gy \preccurlyeq gw$. Then:*

- *$gx = gy$ for all $x, y \in \mathrm{Co}(T, g)$, and*
- *$T$ and $g$ have a unique common fixed point $\omega$, which is $\omega = Tx$ where $x \in \mathrm{Co}(T, g)$ is arbitrary.*

In particular, if $g$ (or $T$) is injective on the set of all coincidence points of $T$ and $g$, then $T$ and $g$ have a unique coincidence point, which is also the common fixed point of $T$ and $g$.

*Proof.* Let $x_0 \in X$ such that $gx_0 \preccurlyeq Tx_0$. Since $T(X) \subseteq g(X)$, Lemma 4.1.1 guarantees the existence of a Picard sequence $\{x_n\}$ of $(T, g)$, that is,

$$gx_{n+1} = Tx_n, \quad \text{for all } n \geq 0.$$

Regarding that $T$ is a $(g, \preccurlyeq)$-non-decreasing mapping, we observe that

$$gx_0 \preccurlyeq Tx_0 = gx_1 \quad \text{implies} \quad gx_1 = Tx_0 \preccurlyeq Tx_1 = gx_2.$$

Inductively, we obtain

$$gx_0 \preccurlyeq gx_1 \preccurlyeq gx_2 \preccurlyeq \ldots \preccurlyeq gx_{n-1} \preccurlyeq gx_n \preccurlyeq gx_{n+1} \preccurlyeq \ldots \tag{5.9}$$

If there exists $n_0$ such that $gx_{n_0} = gx_{n_0+1}$, then $gx_{n_0} = gx_{n_0+1} = Tx_{n_0}$, that is, $T$ and $g$ have a coincidence point, which completes the existence part of the proof. On the contrary case, assume that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$, that is, $G(gx_n, gx_{n+1}, gx_{n+1}) > 0$ for all $n \geq 0$. Regarding (5.9), we set $x = x_n$ and $y = x_{n+1}$ in (5.8). Then we get, for all $n \in \mathbb{N}$,

$$\begin{aligned}\psi\left(G(gx_{n+1}, gx_{n+2}, gx_{n+2})\right) &= \psi\left(G(Tx_n, Tx_{n+1}, Tx_{n+1})\right)\\ &\leq \psi(G(gx_n, gx_{n+1}, gx_{n+1})) - \phi(G(gx_n, gx_{n+1}, gx_{n+1})).\end{aligned}$$

From Lemma 2.3.6, we deduce that

$$\lim_{n\to\infty} G(gx_n, gx_{n+1}, gx_{n+1}) = 0,$$

that is, $\{gx_n\}$ is an asymptotically regular sequence. Next, we will prove that $\{gx_n\}$ is a Cauchy sequence in $(X, G)$ reasoning by contradiction. Suppose that $\{gx_n\}$ is not Cauchy. Then, by Theorem 4.1.1, there exists a positive real number $\varepsilon_0 > 0$ and two subsequences $\{gx_{n(k)}\}$ and $\{gx_{m(k)}\}$ of $\{gx_n\}$ such that, for all $k \in \mathbb{N}$,

$$\begin{aligned}&k \leq n(k) < m(k) < n(k+1),\\ &G\left(gx_{n(k)}, gx_{m(k)-1}, gx_{m(k)-1}\right) \leq \varepsilon_0 < G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right)\end{aligned}$$

and also, for $p_1 = p_2 = p_3 = -1 \in \mathbb{Z}$,

$$\lim_{k\to\infty} G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right) = \lim_{k\to\infty} G\left(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}\right) = \varepsilon_0. \tag{5.10}$$

Notice that as $\preccurlyeq$ is transitive, then $gx_{n(k)-1} \preccurlyeq gx_{m(k)-1}$ for all $k \in \mathbb{N}$. Using the contractivity condition (5.8), for all $k \in \mathbb{N}$,

$$\begin{aligned}
&\psi\left(G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right)\right)\\
&\quad = \psi\left(G\left(Tx_{n(k)-1}, Tx_{m(k)-1}, Tx_{m(k)-1}\right)\right)\\
&\quad \le \psi\left(G\left(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}\right)\right)\\
&\qquad\qquad - \phi\left(G\left(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}\right)\right).
\end{aligned}$$

From (5.10) (with $k \in \mathbb{N}$),

$$\{t_k = G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right)\},\ \{s_k = G\left(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}\right)\}$$

are two sequences in $[0,\infty)$ converging to the same limit $L = \varepsilon_0$ and satisfying

$$\psi(t_k) \le \psi(s_k) - \phi(s_k) \quad \text{for all } n \in \mathbb{N}.$$

Then, it follows from Corollary 2.3.2 that $\varepsilon_0 = 0$, which is a contradiction. As a consequence, we have that $\{gx_n\}$ is a Cauchy sequence in $(X, G)$.

Taking into account that $(X, G)$ is complete, there exists $z \in X$ such that $\{gx_n\} \to z$. As $g$ and $T$ are $G$-continuous, then $\{ggx_n\} \to gz$ and $\{Tgx_n\} \to Tz$. On the other hand, since $g$ and $T$ commute, we have that

$$ggx_{n+1} = gTx_n = Tgx_n \quad \text{for all } n \ge 0.$$

Therefore, by the uniqueness of the limit of a convergent sequence in a $G$-metric space, we conclude that $gz = Tz$, that is, $z$ is a coincidence point of $T$ and $g$.

Now, assume that for all coincidence points $x$ and $y$ of $T$ and $g$, there exists $w \in X$ such that $gx \preccurlyeq gw$ and $gy \preccurlyeq gw$. We claim that

$$gx = gy \text{ for all } x, y \in \mathrm{Co}(T, g). \tag{5.11}$$

Assume that $x$ and $y$ are two coincidence points of $T$ and $g$ and let $w \in X$ be such that $gx \preccurlyeq gw$ and $gy \preccurlyeq gw$. Let $\{w_n\}$ be a Picard sequence of $(T, g)$ based on the point $w_0 = w$ (exists from Lemma 4.1.1). As $x \preccurlyeq w$ and $y \preccurlyeq w$ and $T$ is a $(g, \preccurlyeq)$-non-decreasing mapping, then $gx = Tx \preccurlyeq Tw_0 = gw_1$ and $gy = Ty \preccurlyeq Tw_0 = gw_1$. Similarly, by induction, it is easy to prove that $gx \preccurlyeq gw_n$ and $gy \preccurlyeq gw_n$ for all $n \in \mathbb{N}$. Applying the contractivity condition (5.8), for all $k \in \mathbb{N}$,

$$\begin{aligned}\psi\left(G\left(gx, gw_{n+1}, gw_{n+1}\right)\right) &= \psi\left(G\left(Tx, Tw_n, Tw_n\right)\right) \\ &\leq \psi\left(G\left(gx, gw_n, gw_n\right)\right) - \phi\left(G\left(gx, gw_n, gw_n\right)\right).\end{aligned}$$

Again by Lemma 2.3.6, we deduce that

$$\lim_{n\to\infty} G\left(gx, gw_n, gw_n\right) = 0,$$

so $\{gw_n\} \to gx$. Similarly, it can be proved that $\{gw_n\} \to gy$. As a consequence, $gx = gy$ and (5.11) holds.

Next we show that, for all coincidence point $x$ of $T$ and $g$, the point $\omega = Tx$ is a common fixed point of $T$ and $g$. Let $x \in X$ be an arbitrary coincidence point of $T$ and $g$ and let $\omega = Tx = gx$. As $T$ and $g$ commutes, Remark 2.2.1 guarantees that $\omega = Tx$ is also a coincidence point of $T$ and $g$. Then, $T\omega = g\omega$. Moreover, by (5.11), we have that $gx = g\omega$. In particular, $T\omega = g\omega = gx = Tx = \omega$. As a result, $\omega$ is a common fixed point of $T$ and $g$.

Finally, we prove that $T$ and $g$ have a unique common fixed point. Let $\omega$ and $z$ be two common fixed points of $T$ and $g$, that is, $\omega = T\omega = g\omega$ and $z = Tz = gz$. By (5.11), we have that $g\omega = gz$, so $\omega = g\omega = gz = z$. □

When the contractivity condition (5.8) is satisfied for all $x, y \in X$, then the continuity of $g$ implies the continuity of $T$ (recall Lemma 4.3.2). However, in the setting of preordered $G$-metric spaces, the contractivity condition (5.8) is not strong enough to guarantee this property. This is why, in the previous result, we assumed that both $T$ and $g$ are continuous.

*Example 5.3.1.* Let $X = \mathbb{R}$ endowed with the $G$-metric $G(x, y, z) = \max\{|x-y|, |x-z|, |y-z|\}$ for all $x, y, z \in X$ and the partial order $\preceq$ given by

$$x \preceq y \;\Leftrightarrow\; \left(\, x = y \quad \text{or} \quad x < y \leq 0\,\right).$$

Define $T : \mathbb{R} \to \mathbb{R}$ by

$$Tx = \begin{cases} \dfrac{x}{2}, & \text{if } x \leq 0, \\ 2x + 1, & \text{if } x > 0. \end{cases}$$

If $g$ is the identity mapping on $X$, we now show that (5.8) holds using $\psi(t) = t$ and $\phi(t) = t/2$ for all $t \in [0, \infty)$. Indeed, let $x, y \in X$ be such that $x = gx \preceq gy = y$. If $x = y$, then (5.8) trivially holds. Assume that $x \neq y$. Then $x < y \leq 0$. Hence

$$\begin{aligned}G(Tx, Ty, Ty) &= G\left(\frac{x}{2}, \frac{y}{2}, \frac{y}{2}\right) = \left|\frac{x}{2} - \frac{y}{2}\right| = \frac{1}{2}\,|x-y| \\ &= \frac{1}{2}\,G(x, y, y) = (\psi - \phi)\left(G(gx, gy, gy)\right).\end{aligned}$$

However, although $g$ is $G$-continuous on $\mathbb{R}$, $T$ is not continuous at $x = 0$ because $\{1/n\} \to 0$ but $\{T(1/n)\} \to 1 \neq 0 = T0$.

**Corollary 5.3.1.** *Let $(X, G, \preccurlyeq)$ be an ordered G-metric space and let $T, g : X \to X$ be two mappings. Suppose that the following conditions hold:*

- **(i)** *$(X, G)$ is G-complete;*
- **(ii)** *there exists $x_0 \in X$ such that $gx_0 \preccurlyeq Tx_0$;*
- **(iii)** *$T(X) \subseteq g(X)$;*
- **(iv)** *$T$ is $(g, \preccurlyeq)$-non-decreasing;*
- **(v)** *$T$ and $g$ are G-continuous and commuting;*
- **(vi)** *there exist two functions $\psi \in \mathcal{F}_{\text{alt}}$ and $\phi \in \mathcal{F}'_{\text{alt}}$ such that, for all $x, y \in X$ with $gx \preccurlyeq gy$,*

$$\psi\,(G(Tx, Ty, Ty)) \leq \psi(G(gx, gy, gy)) - \phi(G(gx, gy, gy)).$$

*Then $T$ and $g$ have, at least, a coincidence point, that is, there exists $z \in X$ such that $Tz = gz$.*

*Furthermore, assume that for all coincidence points $x$ and $y$ of $T$ and $g$, there exists $w \in X$ such that $gx \preccurlyeq gw$ and $gy \preccurlyeq gw$. Then $T$ and $g$ have a unique common fixed point.*

In the following result, we use a contractivity condition involving three variables.

**Corollary 5.3.2.** *Let $(X, G, \preccurlyeq)$ be a preordered G-metric space and let $T, g : X \to X$ be two mappings. Suppose that the following conditions hold:*

- **(i)** *$(X, G)$ is G-complete;*
- **(ii)** *there exists $x_0 \in X$ such that $gx_0 \preccurlyeq Tx_0$;*
- **(iii)** *$T(X) \subseteq g(X)$;*
- **(iv)** *$T$ is $(g, \preccurlyeq)$-non-decreasing;*
- **(v)** *$T$ and $g$ are G-continuous and commuting;*
- **(vi)** *there exist two functions $\psi \in \mathcal{F}_{\text{alt}}$ and $\phi \in \mathcal{F}'_{\text{alt}}$ such that, for all $x, y, z \in X$ with $gx \preccurlyeq gy \preccurlyeq gz$,*

$$\psi\,(G(Tx, Ty, Tz)) \leq \psi(G(gx, gy, gz)) - \phi(G(gx, gy, gz)).$$

*Then $T$ and $g$ have, at least, a coincidence point, that is, there exists $z \in X$ such that $Tz = gz$.*

*Furthermore, assume that for all coincidence points $x$ and $y$ of $T$ and $g$, there exists $w \in X$ such that $gx \preccurlyeq gw$ and $gy \preccurlyeq gw$. Then $T$ and $g$ have a unique common fixed point.*

If we use in $X$ the preorder $\preceq_0$ given by (5.1), we have the following consequence.

**Corollary 5.3.3.** *Theorem 4.3.2 follows from Theorem 5.3.1.*

**Corollary 5.3.4.** *Let $(X, G, \preccurlyeq)$ be a preordered G-metric space and let $T, g : X \to X$ be two mappings. Suppose that the following conditions hold:*

**(i)** *$(X, G)$ is G-complete;*
**(ii)** *there exists $x_0 \in X$ such that $gx_0 \preccurlyeq Tx_0$;*
**(iii)** *$T(X) \subseteq g(X)$;*
**(iv)** *$T$ is $(g, \preccurlyeq)$-non-decreasing;*
**(v)** *$T$ and $g$ are G-continuous and commuting;*
**(vi)** *there exists a function $\phi \in \mathcal{F}'_{\text{alt}}$ such that, for all $x, y \in X$ with $gx \preccurlyeq gy$,*

$$G(Tx, Ty, Ty) \leq G(gx, gy, gy) - \phi(G(gx, gy, gy)).$$

*Then $T$ and $g$ have, at least, a coincidence point, that is, there exists $z \in X$ such that $Tz = gz$.*

*Furthermore, assume that for all coincidence points $x$ and $y$ of $T$ and $g$, there exists $w \in X$ such that $gx \preccurlyeq gw$ and $gy \preccurlyeq gw$. Then $T$ and $g$ have a unique common fixed point.*

The previous result also holds if we additionally assume that $\phi$ is continuous rather than lower semi-continuous. Finally, if we use $\phi(t) = (1 - \lambda)\, t$ for all $t \geq 0$, where $\lambda \in [0, 1)$, we have the following result.

**Corollary 5.3.5.** *Let $(X, G, \preccurlyeq)$ be a preordered G-metric space and let $T, g : X \to X$ be two mappings. Suppose that the following conditions hold:*

**(i)** *$(X, G)$ is G-complete;*
**(ii)** *there exists $x_0 \in X$ such that $gx_0 \preccurlyeq Tx_0$;*
**(iii)** *$T(X) \subseteq g(X)$;*
**(iv)** *$T$ is $(g, \preccurlyeq)$-non-decreasing;*
**(v)** *$T$ and $g$ are G-continuous and commuting;*
**(vi)** *there exists a constant $\lambda \in [0, 1)$ such that, for all $x, y \in X$ with $gx \preccurlyeq gy$,*

$$G(Tx, Ty, Ty) \leq \lambda\, G(gx, gy, gy).$$

*Then $T$ and $g$ have, at least, a coincidence point, that is, there exists $z \in X$ such that $Tz = gz$.*

*Furthermore, assume that for all coincidence points $x$ and $y$ of $T$ and $g$, there exists $w \in X$ such that $gx \preccurlyeq gw$ and $gy \preccurlyeq gw$. Then $T$ and $g$ have a unique common fixed point.*

In the next theorem, we replace the continuity of $T$ by the non-decreasing-regularity of the preordered $G$-metric space.

**Theorem 5.3.2.** *Let $(X, G, \preccurlyeq)$ be a preordered G-metric space and let $T, g : X \to X$ be two mappings. Suppose that the following conditions hold:*

**(i)** *$(g(X), G)$ is complete;*
**(ii)** *there exists $x_0 \in X$ such that $gx_0 \preccurlyeq Tx_0$;*
**(iii)** *$T(X) \subseteq g(X)$;*
**(iv)** *$T$ is $(g, \preccurlyeq)$-non-decreasing;*
**(v)** *$g$ is G-continuous and commutes with $T$;*

**(vi)** *there exist two functions* $\psi \in \mathcal{F}_{\text{alt}}$ *and* $\phi \in \mathcal{F}'_{\text{alt}}$ *such that, for all* $x, y \in X$ *with* $gx \preccurlyeq gy$,

$$\psi\,(G(Tx, Ty, Ty)) \leq \psi(G(gx, gy, gy)) - \phi(G(gx, gy, gy)); \tag{5.12}$$

**(vii)** $(X, G, \preccurlyeq)$ *is non-decreasing-regular.*

*Then $T$ and $g$ have, at least, a coincidence point, that is, there exists $z \in X$ such that $Tz = gz$.*

*Furthermore, assume that for all $x, y \in \mathrm{Co}(T, g)$, there exists $w \in X$ such that $gx \preccurlyeq gw$ and $gy \preccurlyeq gw$. Then:*

- *$gx = gy$ for all $x, y \in \mathrm{Co}(T, g)$, and*
- *$T$ and $g$ have a unique common fixed point $\omega$, which is $\omega = Tx$ where $x \in \mathrm{Co}(T, g)$ is arbitrary.*

In particular, if $g$ (or $T$) is injective on the set of all coincidence points of $T$ and $g$, then $T$ and $g$ have a unique coincidence point, which is also the common fixed point of $T$ and $g$.

*Proof.* Repeating the argument in the proof of Theorem 5.3.1, we get that the $\preccurlyeq$-non-decreasing sequence $\{gx_n\}$ is Cauchy in $(g(X), G)$. As $(g(X), G)$ is complete, there exists $z \in g(X)$ such that $\{gx_n\} \to z$. Let $u \in X$ be such that $gu = z$. Since $(X, G, \preccurlyeq)$ is non-decreasing-regular, then $gx_n \preccurlyeq gu$ for all $n \in \mathbb{N}$. As $T$ and $g$ commutes,

$$ggx_{n+1} = gTx_n = Tgx_n \quad \text{for all } n \in \mathbb{N}.$$

By the contractivity condition (5.12), we have that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\psi\,(G(gx_{n+1}, Tu, Tu)) &= \psi\,(G(Tx_n, Tu, Tu)) \\ &\leq \psi(G(gx_n, gu, gu)) - \phi(G(gx_n, gu, gu)) \\ &\leq \psi(G(gx_n, gu, gu)).\end{aligned}$$

Since $\{gx_n\} \to gu$, item 2 of Corollary 2.3.1 guarantees that

$$\lim_{n\to\infty} G(gx_{n+1}, Tu, Tu) = 0$$

and $\{gx_n\} \to Tu$. By the uniqueness of the limit, we conclude that $Tu = gu$, that is, $T$ and $g$ have, at least, a coincidence point. The rest of the proof is similar to the proof of Theorem 5.3.1. □

The previous results are also valid if we employ $\phi \in \mathcal{F}''_{\text{alt}}$.

**Theorem 5.3.3.** *If we replace the condition $\phi \in \mathcal{F}'_{\text{alt}}$ by the assumption $\phi \in \mathcal{F}''_{\text{alt}}$, then Theorem 5.3.1 (and its subsequent corollaries) also holds.*

*Proof.* Repeat the argument in the proof of Theorem 5.3.1 using Lemmas 2.3.9 and 2.3.10 rather than Lemmas 2.3.6 and 2.3.2. □

All corollaries we have deduced from Theorem 5.3.1 can now also be repeated here.

### 5.3.2 Common Fixed Point Theorems for Compatible Mappings

If we revise in detail the proofs of Theorems 5.3.1 and 5.3.2, we notice that it is not very difficult to weaken some hypotheses. Note the following:

- The condition $T(X) \subseteq g(X)$ is only used to guarantee, by Lemma 4.1.1, that there exists a Picard sequence $\{x_n\}$ of $(T, g)$, that is, satisfying $gx_{n+1} = Tx_n$ for all $n \geq 0$.
- The condition "$gx \preccurlyeq gw$ and $gy \preccurlyeq gw$" which we have used to prove the uniqueness can be replaced by the condition that "$gx$ and $gy$ are, at the same time, comparable with $gw$"
- The continuity of $g$ will not be necessary if we assume that $g(X)$ is complete.
- As $T(X) \subseteq g(X) \subseteq X$ and $\{gx_{n+1} = Tx_n\} \subseteq T(X)$, then it is only necessary to assume that, at least, one of these subsets is complete.
- When $g$ is not the identity mapping on $X$, the commutativity between $T$ and $g$ is a very restrictive condition.

**Definition 5.3.1.** Let $(X, G)$ be a $G$-metric space endowed with a binary relation $\preceq$ and let $T, g : X \to X$ be two mappings. We will say that $(T, g)$ is an $(O, \preceq)$-*compatible pair* if we have that

$$\lim_{n\to\infty} G\left(gTx_n, Tgx_n, Tgx_n\right) = 0 \tag{5.13}$$

whenever $\{x_n\}$ is a sequence in $X$ such that $\{gx_n\}$ is $\preceq$-monotone and

$$\lim_{m\to\infty} Tx_n = \lim_{n\to\infty} gx_n \in X.$$

If $X$ is not endowed with a partial order, we have the following definition.

**Definition 5.3.2.** Let $(X, G)$ be a $G$-metric space and let $T, g : X \to X$ be two mappings. We will say that $(T, g)$ is an *O-compatible pair* if we have that

$$\lim_{n\to\infty} G\left(gTx_n, Tgx_n, Tgx_n\right) = 0$$

whenever $\{x_n\}$ is a sequence in $X$ such that

$$\lim_{m\to\infty} Tx_n = \lim_{n\to\infty} gx_n \in X.$$

Notice that if $T$ and $g$ are commuting, then the pair $(T, g)$ is $O$-compatible and $(O, \preceq)$-compatible whatever binary relation $\preceq$ on $X$. Also notice that, by Corollary 3.1.1, condition (5.13) is equivalent to its symmetric property:

$$\lim_{n\to\infty} G\left(gTx_n, gTx_n, Tgx_n\right) = 0.$$

**Theorem 5.3.4.** *Let $(X, G, \preccurlyeq)$ be a preordered $G$-metric space and let $T, g : X \to X$ be two mappings. Suppose that the following conditions hold.*

**(i)** *$T$ is $(g, \preccurlyeq)$-non-decreasing.*
**(ii)** *At least, one of the following conditions holds:*

**(ii.1)** *$T(X) \subseteq g(X)$ and there exists $x_0 \in X$ such that $gx_0 \preccurlyeq Tx_0$;*
**(ii.2)** *there exists a Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $gx_0 \preccurlyeq Tx_0$.*

**(iii)** *At least, one of the following conditions holds:*

**(iii.1)** *$X$ (or $g(X)$ or $T(X)$) is $G$-complete, $T$ and $g$ are $G$-continuous and $(T, g)$ is a $(O, \preccurlyeq)$-compatible pair;*
**(iii.2)** *$X$ (or $g(X)$ or $T(X)$) is $G$-complete and $T$ and $g$ are $G$-continuous and commuting;*
**(iii.3)** *$(g(X), G)$ is complete and $(X, G, \preccurlyeq)$ is non-decreasing-regular;*
**(iii.4)** *$(X, G)$ is complete, $g(X)$ is closed and $(X, G, \preccurlyeq)$ is non-decreasing-regular.*

**(iv)** *At least, one of the following conditions holds:*

**(iv.1)** *There exist two functions $\psi \in \mathcal{F}_{\text{alt}}$ and $\phi \in \mathcal{F}'_{\text{alt}}$ such that, for all $x, y \in X$ with $gx \preccurlyeq gy$,*

$$G(Tx, Ty, Ty) \leq \psi(G(gx, gy, gy)) - \phi(G(gx, gy, gy)). \tag{5.14}$$

**(iv.2)** *There exist two functions $\psi \in \mathcal{F}_{\text{alt}}$ and $\phi \in \mathcal{F}'_{\text{alt}}$ such that, for all $x, y, z \in X$ with $gx \preccurlyeq gy \preccurlyeq gz$,*

$$G(Tx, Ty, Tz) \leq \psi(G(gx, gy, gz)) - \phi(G(gx, gy, gz)).$$

*Then $T$ and $g$ have, at least, a coincidence point.*

Notice that under conditions (iii.3) and (iii.4), $T$ and $g$ does not need any kind of continuity nor commutativity.

*Proof.* Notice that (ii.1) implies (ii.2), so we can suppose (ii.2). Similarly, (iv.2) implies (iv.1), so we can assume (iv.1). Let $\{x_n\}_{n\in\mathbb{N}}$ be any Picard sequence of $(T, g)$ such that $gx_0 \preccurlyeq Tx_0$. Repeating the arguments in the proof of Theorem 5.3.1, we get that the $\preccurlyeq$-non-decreasing sequence $\{gx_n\}$ is Cauchy in $(X, G)$. At this point, we distinguish four cases.

*Case (iii.1) Assume that $X$ (or $g(X)$ or $T(X)$) is $G$-complete and $T$ and $g$ are $G$-continuous and $(T, g)$ is a $(O, \preccurlyeq)$-compatible pair.* As $\{gx_{n+1} = Tx_n\}$ is a Cauchy sequence that is contained in $X$, in $g(X)$ and in $T(X)$, and some of these subsets is complete, then there exists $z \in X$ such that $\{gx_n\} \to z$. As $T$ and $g$ are continuous, then $\{ggx_n\} \to gz$ and $\{Tgx_n\} \to Tz$. Furthermore, as $\{gx_n\}$ is $\preccurlyeq$-non-decreasing and

$$\lim_{m\to\infty} Tx_n = \lim_{n\to\infty} gx_n = z \in X,$$

then the $(O, \preccurlyeq)$-compatibility of the pair $(T, g)$ implies that

$$\lim_{n\to\infty} G\left(gTx_n, Tgx_n, Tgx_n\right) = 0.$$

As $\{gTx_n = ggx_{n+1}\} \to gz$ and $\{Tgx_n\} \to Tz$, the continuity of $G$ (see Theorem 3.2.2) implies that

$$G\left(gz, Tz, Tz\right) = \lim_{n\to\infty} G\left(gTx_n, Tgx_n, Tgx_n\right) = 0.$$

Therefore, $gz = Tz$ and $z$ is a coincidence point of $T$ and $g$.

*Case (iii.2) Assume that $X$ (or $g(X)$ or $T(X)$) is $G$-complete and $T$ and $g$ are $G$-continuous and commuting.* In this case, item (iii.1) is applicable because the commutativity of $T$ and $g$ implies that $(T, g)$ is an $(O, \preccurlyeq)$-compatible pair.

*Case (iii.3) Assume that $(g(X), G)$ is complete and $(X, G, \preccurlyeq)$ is non-decreasing-regular.* In this case, the $\preccurlyeq$-non-decreasing sequence $\{gx_n\}$ is Cauchy in $g(X)$, which is $G$-complete. Then, there exists $z \in g(X)$ such that $\{gx_n\} \to z$. Let $u \in X$ be any point such that $gu = z$. Taking into account that $(X, G, \preccurlyeq)$ is non-decreasing-regular, it follows that $gx_n \preccurlyeq gu$ for all $n \in \mathbb{N}$. The contractivity condition (5.14) yields, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\psi\left(G(gx_{n+1}, Tu, Tu)\right) &= \psi\left(G(Tx_n, Tu, Tu)\right) \\
&\leq \psi(G(gx_n, gu, gu)) - \phi(G(gx_n, gu, gu)) \\
&\leq \psi(G(gx_n, gu, gu)).
\end{aligned}$$

As $\{gx_n\} \to gu$ and $\psi$ is continuous, we deduce that

$$\lim_{n\to\infty} \psi\left(G(gx_{n+1}, Tu, Tu)\right) = 0,$$

so

$$\lim_{n\to\infty} G(gx_{n+1}, Tu, Tu) = 0,$$

which means that $\{gx_n\} \to Tu$. By the uniqueness of the limit, we conclude that $Tu = gu$, that is, $u$ is a coincidence point of $T$ and $g$.

*Case (iii.4) Assume that $(X, G)$ is complete, $g(X)$ is closed and $(X, G, \preccurlyeq)$ is non-decreasing-regular.* In this case, we can apply item (iii.3) because any closed subset of a complete $G$-metric space is also complete.

In any case we have just proved that $T$ and $g$ has, at least, a coincidence point. □

*Remark 5.3.1.* Notice that, by the symmetry of $G$, condition $gx \preccurlyeq gy \preccurlyeq gz$ in (iv.2) may be replaced by $gx \succcurlyeq gy \succcurlyeq gz$. Nevertheless, the roles of $x$ and $y$ in (5.14) are not equivalent. However, the reader can obtain a similar result replacing (5.14) by the alternative condition:

$$G(Tx, Tx, Ty) \leq \psi(G(gx, gx, gy)) - \phi(G(gx, gx, gy))$$

for all $x, y \in X$ such that $gx \preccurlyeq gy$.

In the next result, we study the uniqueness of the coincidence point.

**Theorem 5.3.5.** *Under the hypotheses of Theorem 5.3.4, also assume that the following properties are fulfilled.*

- **(v)** $T(X) \subseteq g(X)$.
- **(vi)** *for all coincidence points $x$ and $y$ of $T$ and $g$, there exists $w \in X$ such that $gw$ is, at the same time, $\preccurlyeq$-comparable to $gx$ and to $gy$.*

*Then:*

- *$gx = gy$ for all coincidence points $x$ and $y$ of $T$ and $g$, and*
- *$T$ and $g$ have a unique common fixed point.*

*Proof.* The proof of Theorem 5.3.1 can be followed to deduce the stated statements. Notice that if $gw \preccurlyeq gx$, then the Picard sequence $\{w_n\}$ of $(T, g)$ based on $w_0 = w$ (which exists by Lemma 4.1.1) also satisfies $gw_n \preccurlyeq gx$ for all $n \in \mathbb{N}$ because $T$ is $(g, \preccurlyeq)$-non-decreasing. Hence, the contractivity condition (5.14) is applicable. □

### *5.3.3 Ćirić's Common Fixed Point Theorems in Preordered G-Metric Spaces*

Following the work of Ćirić et al. [61], we generalize the above-mentioned results by introducing a function $g$.

**Theorem 5.3.6.** *Let $(X, \preccurlyeq)$ be a preordered set endowed with a G-metric and $T : X \to X$ and $g : X \to X$ be given mappings. Suppose that the following conditions hold:*

- *(i) $(X, G)$ is G-complete;*
- *(ii) $T$ is G-continuous;*
- *(iii) $T$ is $(g, \preccurlyeq)$-non-decreasing;*
- *(iv) there exists $x_0 \in X$ such that $gx_0 \preccurlyeq Tx_0$;*
- *(v) $T(X) \subseteq g(X)$ and $g$ is G-continuous and commutes with $T$;*
- *(vi) there exists a function $\varphi \in \mathcal{F}_{\mathrm{Cir}}$ such that, for all $x, y, z \in X$ with $gx \succcurlyeq gy \succcurlyeq gz$,*

$$G(Tx, Ty, Tz) \leq \varphi(G(gx, gy, gz)). \tag{5.15}$$

*Then $T$ and $g$ have a coincidence point, that is, there exists $w \in X$ such that $gw = Tw$.*

*Proof.* Let $x_0 \in X$ such that $gx_0 \preccurlyeq Tx_0$. Since $T(X) \subseteq g(X)$, Lemma 4.1.1 guarantees the existence of a Picard sequence $\{x_n\}$ of $(T, g)$, that is,

$$gx_{n+1} = Tx_n, \quad \text{for all } n \geq 0. \tag{5.16}$$

Regarding that $T$ is a $(g, \preccurlyeq)$-non-decreasing mapping together with (5.16), we observe that

$$gx_0 \preccurlyeq Tx_0 = gx_1 \text{ implies } gx_1 = Tx_0 \preccurlyeq Tx_1 = gx_2.$$

Inductively, we obtain

$$gx_0 \preccurlyeq gx_1 \preccurlyeq gx_2 \preccurlyeq \ldots \preccurlyeq gx_{n-1} \preccurlyeq gx_n \preccurlyeq gx_{n+1} \preccurlyeq \ldots \tag{5.17}$$

If there exists $n_0$ such that $gx_{n_0} = gx_{n_0+1}$, then $gx_{n_0} = gx_{n_0+1} = Tx_{n_0}$, that is, $T$ and $g$ have a coincidence point, which completes the proof. Assume that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$, that is, $G(gx_{n+1}, gx_{n+1}, gx_n) > 0$ for all $n \geq 0$. Regarding (5.17), we set $x = y = x_{n+1}$ and $z = x_n$ in (5.15). Then we get

$$\begin{aligned} G(gx_{n+2}, gx_{n+2}, gx_{n+1}) &= G(Tx_{n+1}, Tx_{n+1}, Tx_n) \\ &\leq \varphi(G(gx_{n+1}, gx_{n+1}, gx_n)), \end{aligned}$$

which implies, taking into account that $\varphi \in \mathcal{F}_{\mathrm{Cir}}$,

$$\begin{aligned} G(gx_{n+2}, gx_{n+2}, gx_{n+1}) &\leq \varphi(G(gx_{n+1}, gx_{n+1}, gx_n)) \\ &< G(gx_{n+1}, gx_{n+1}, gx_n). \end{aligned} \tag{5.18}$$

Let $t_n = G(gx_{n+1}, gx_{n+1}, gx_n)$. Then, $\{t_n\}$ is a non-increasing sequence of positive real numbers. Thus, there exists $L \geq 0$ such that

$$\lim_{n\to\infty} t_n = L \quad \text{and} \quad L < t_n \text{ for all } n \geq 0.$$

We now show that $L = 0$. Suppose that, contrary to our claim, $L > 0$. Letting $n \to \infty$ in (5.18) and taking into account that $\varphi \in \mathcal{F}_{\mathrm{Cir}}$, we get

$$L = \lim_{n\to\infty} t_{n+1} \leq \lim_{n\to\infty} \varphi(t_n) = \lim_{t\to \ell+} \varphi(t) < L,$$

which is a contradiction. Hence, we have

$$\lim_{n\to\infty} G(gx_{n+1}, gx_{n+1}, gx_n) = \lim_{n\to\infty} t_n = L = 0.$$

We now show that $\{gx_n\}$ is a $G$-Cauchy sequence. Suppose on the contrary, that the sequence $\{gx_n\}$ is not $G$-Cauchy. From Theorem 4.1.1, there exists $\varepsilon_0 > 0$ and sequences of natural numbers $\{m(k)\}$ and $\{\ell(k)\}$ such that, for each natural number $k, k \leq \ell(k) < m(k) < \ell(k+1)$,

$$G\left(x_{\ell(k)}, x_{m(k)-1}, x_{m(k)-1}\right) \leq \varepsilon_0 < G\left(x_{\ell(k)}, x_{m(k)}, x_{m(k)}\right),$$

and also

$$\lim_{k\to\infty} G\left(x_{\ell(k)}, x_{m(k)}, x_{m(k)}\right) = \lim_{k\to\infty} G\left(x_{\ell(k)+1}, x_{m(k)+1}, x_{m(k)+1}\right) = \varepsilon_0.$$

By the contractivity condition (5.15) applied to $x = y = x_{m(k)}$ and $z = x_{\ell(k)}$, we have that

$$\begin{aligned} G(gx_{m(k)+1}, gx_{m(k)+1}, gx_{\ell(k)+1}) &= G(Tx_{m(k)}, Tx_{m(k)}, Tx_{\ell(k)}) \\ &\leq \varphi(G(gx_{m(k)}, gx_{m(k)}, gx_{\ell(k)})). \end{aligned}$$

Since $\{G(gx_{m(k)}, gx_{m(k)}, gx_{\ell(k)})\} \searrow \varepsilon_0^+$, we deduce, using $\varphi \in \mathcal{F}_{\mathrm{Cir}}$, that

$$\begin{aligned} \varepsilon_0 &= \lim_{k\to\infty} G\left(x_{\ell(k)+1}, x_{m(k)+1}, x_{m(k)+1}\right) \leq \lim_{k\to\infty} \varphi(G(gx_{m(k)}, gx_{m(k)}, gx_{\ell(k)})) \\ &= \lim_{t\to\varepsilon_0^+} \varphi(t) < \varepsilon_0, \end{aligned}$$

which is a contradiction. Hence, $\{gx_n\}$ is a Cauchy sequence in the $G$-metric space $(X, G)$. Since $(X, G)$ is complete, there exists $w \in X$ such that $\{gx_n\}$ is convergent to $w$. From Lemma 3.2.1, we have

$$\lim_{n\to\infty} G(gx_n, gx_n, w) = \lim_{n\to\infty} G(gx_n, w, w) = 0.$$

The continuity of $g$ implies that the sequence $\{ggx_n\}$ is convergent to $gw$, that is,

$$\lim_{n\to\infty} G(ggx_n, ggx_n, gw) = \lim_{n\to\infty} G(ggx_n, gw, gw) = 0. \tag{5.19}$$

On the other hand, due to the commutativity of $T$ and $g$, we can write

$$ggx_{n+1} = gTx_n = Tgx_n \quad \text{for all } n \geq 0,$$

and the continuity of $T$ implies that the sequence $\{Tgx_n\} = \{ggx_{n+1}\}$ converges to $Tw$, so that,

$$\lim_{n\to\infty} G(Tgx_n, Tgx_n, Tw) = \lim_{n\to\infty} G(Tgx_n, Tw, Tw) = 0. \tag{5.20}$$

By the uniqueness of the limit, the expressions (5.19) and (5.20) yield that $gw = Tw$. □

In the next theorem, the $G$-continuity of $T$ is no longer required. However, we require the non-decreasing-regularity of $X$.

**Theorem 5.3.7.** *Let $(X, \preccurlyeq)$ be an ordered set endowed with a $G$-metric and $T : X \to X$ and $g : X \to X$ be given mappings. Suppose that the following conditions hold:*

*(i) $(g(X), G)$ is $G$-complete;*
*(ii) $(X, G, \preccurlyeq)$ is non-decreasing-regular;*
*(iii) $T$ is $(g, \preccurlyeq)$-non-decreasing;*
*(iv) there exists $x_0 \in X$ such that $gx_0 \preccurlyeq Tx_0$;*
*(v) $T(X) \subseteq g(X)$ and $g$ is $G$-continuous and commutes with $T$;*
*(vi) there exists a function $\varphi \in \mathcal{F}_{\mathrm{Cir}}$ such that for all $x, y, z \in X$ with $gx \succcurlyeq gy \succcurlyeq gz$,*

$$G(Tx, Ty, Tz) \le \varphi(G(gx, gy, gz)). \tag{5.21}$$

*Then $T$ and $g$ have a coincidence point, that is, there exists $w \in X$ such that $gw = Tw$.*

*Proof.* Following the proof in Theorem 5.3.6, we consider a $\preccurlyeq$-non-decreasing sequence $\{gx_n\}$ and conclude that it is a $G$-Cauchy sequence in the $G$-complete, $G$-metric space $(g(X), G)$. Thus, there exists $w \in X$ such that $\{gx_n\}$ is $G$-convergent to $gw$. Since $\{gx_n\}$ is non-decreasing and $(X, G, \preccurlyeq)$ is non-decreasing-regular, we have that $gx_n \preccurlyeq gw$ for all $n \in \mathbb{N}$. If $gw = gx_{n_0}$ for some natural number $n_0$, then $x_{n_0}$ is a coincidence point of $T$ and $g$ because $gw = gx_{n_0} \preccurlyeq gx_{n_0+1} \preccurlyeq gw$ and, as $\preccurlyeq$ is a partial order, $gx_{n_0} = gx_{n_0+1} = Tx_{n_0}$. Suppose that $gw \neq gx_n$ for all $n \in \mathbb{N}$. By the rectangle inequality together with the inequality (5.21), we have

$$\begin{aligned} G(Tw, gw, gw) &\le G(Tw, gx_{n+1}, gx_{n+1}) + G(gx_{n+1}, gw, gw) \\ &\le G(Tw, Tx_n, Tx_n) + G(gx_{n+1}, gw, gw) \\ &\le \varphi(G(gw, gx_n, gx_n)) + G(gx_{n+1}, gw, gw) \\ &< G(gw, gx_n, gx_n) + G(gx_{n+1}, gw, gw). \end{aligned}$$

Letting $n \to \infty$ in the inequality above, we get that $G(Tw, gw, gw) = 0$. Hence $Tw = gw$. □

If we take $\varphi(t) = \lambda\, t$, where $\lambda \in [0, 1)$, in Theorems 5.3.6 and 5.3.7 we deduce the following corollaries respectively.

**Corollary 5.3.6.** *Let $(X, \preccurlyeq)$ be a preordered set endowed with a G-metric and $T : X \to X$ and $g : X \to X$ be given mappings. Suppose that the following conditions hold:*

*(i)* $(X, G)$ *is G-complete;*
*(ii)* $T$ *is G-continuous;*
*(iii)* $T$ *is g-non-decreasing (with respect to* $\preccurlyeq$*);*
*(iv) there exists* $x_0 \in X$ *such that* $gx_0 \preccurlyeq Tx_0$*;*
*(v)* $T(X) \subseteq g(X)$ *and g is G-continuous and commutes with* $T$*;*
*(vi) there exists* $\lambda \in [0, 1)$ *such that for all* $x, y, z \in X$ *with* $gx \succcurlyeq gy \succcurlyeq gz$*,*

$$G(Tx, Ty, Tz) \leq \lambda\, G(gx, gy, gz). \tag{5.22}$$

*Then $T$ and $g$ have a coincidence point, that is, there exists $w \in X$ such that $gw = Tw$.*

**Corollary 5.3.7.** *Let $(X, \preccurlyeq)$ be an ordered set endowed with a G-metric and $T : X \to X$ and $g : X \to X$ be given mappings. Suppose that the following conditions hold:*

*(i)* $(g(X), G)$ *is G-complete;*
*(ii)* $(X, G, \preccurlyeq)$ *is non-decreasing-regular;*
*(iii)* $T$ *is g-non-decreasing (with respect to* $\preccurlyeq$*);*
*(iv) there exists* $x_0 \in X$ *such that* $gx_0 \preccurlyeq Tx_0$*;*
*(v)* $T(X) \subseteq g(X)$ *and g is G-continuous and commutes with* $T$*;*
*(vi) there exists* $\lambda \in [0, 1)$ *such that for all* $x, y, z \in X$ *with* $gx \succcurlyeq gy \succcurlyeq gz$*,*

$$G(Tx, Ty, Tz) \leq \lambda\, G(gx, gy, gz). \tag{5.23}$$

*Then $T$ and $g$ have a coincidence point, that is, there exists $w \in X$ such that $gw = Tw$.*

If we take $z = y$ in Theorems 5.3.6 and 5.3.7 we obtain the following particular cases.

**Corollary 5.3.8.** *Let $(X, \preccurlyeq)$ be an ordered set endowed with a G-metric and $T : X \to X$ and $g : X \to X$ be given mappings. Suppose that the following conditions hold:*

*(i)* $(X, G)$ *is G-complete;*
*(ii)* $T$ *is G-continuous;*
*(iii)* $T$ *is g-non-decreasing;*
*(iv) there exists* $x_0 \in X$ *such that* $gx_0 \preccurlyeq Tx_0$*;*
*(v)* $T(X) \subseteq g(X)$ *and g is G-continuous and commutes with* $T$*;*
*(vi) there exists a function* $\varphi \in \mathcal{F}_{\mathrm{Cir}}$ *such that for all* $x, y \in X$ *with* $gx \succcurlyeq gy$*,*

$$G(Tx, Ty, Ty) \leq \varphi(G(gx, gy, gy)). \tag{5.24}$$

*Then $T$ and $g$ have a coincidence point, that is, there exists $w \in X$ such that $gw = Tw$.*

**Corollary 5.3.9.** *Let $(X, \preccurlyeq)$ be an ordered set endowed with a G-metric and $T : X \to X$ and $g : X \to X$ be given mappings. Suppose that the following conditions hold:*

*(i)* $(g(X), G)$ *is G-complete;*
*(ii)* $(X, G, \preccurlyeq)$ *is non-decreasing-regular;*
*(iii)* $T$ *is g-non-decreasing;*
*(iv) there exists* $x_0 \in X$ *such that* $gx_0 \preccurlyeq Tx_0$*;*
*(v)* $T(X) \subset g(X)$ *and* $g$ *is G-continuous and commutes with* $T$*;*
*(vi) there exists a function* $\varphi \in \mathcal{F}_{\text{Cir}}$ *such that for all* $x, y \in X$ *with* $gx \succcurlyeq gy$,

$$G(Tx, Ty, Ty) \leq \varphi(G(gx, gy, gy)). \tag{5.25}$$

*Then $T$ and $g$ have a coincidence point, that is, there exists $w \in X$ such that $gw = Tw$.*

Finally, we let $g = I_X$ in Theorems 5.3.6 and 5.3.7.

**Theorem 5.3.8.** *Let $(X, \preccurlyeq)$ be an ordered set endowed with a G-metric and $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

*(i)* $(X, G)$ *is G-complete;*
*(ii)* $T$ *is G-continuous,*
*(iii)* $T$ *is non-decreasing (with respect to* $\preccurlyeq$*);*
*(iv) there exists* $x_0 \in X$ *such that* $x_0 \preccurlyeq Tx_0$*;*
*(v) there exists a function* $\varphi \in \mathcal{F}_{\text{Cir}}$ *such that for all* $x, y, z \in X$ *with* $x \succcurlyeq y \succcurlyeq z$,

$$G(Tx, Ty, Tz) \leq \varphi(G(x, y, z)). \tag{5.26}$$

*Then $T$ has a fixed point.*

**Theorem 5.3.9.** *Let $(X, \preccurlyeq)$ be an ordered set endowed with a G-metric and $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

*(i)* $(X, G)$ *is G-complete;*
*(ii)* $(X, G, \preccurlyeq)$ *is non-decreasing-regular;*
*(iii)* $T$ *is non-decreasing;*
*(iv) there exists* $x_0 \in X$ *such that* $x_0 \preccurlyeq Tx_0$*;*
*(v) there exists a function* $\varphi \in \mathcal{F}_{\text{Cir}}$ *such that for all* $x, y, z \in X$ *with* $x \succcurlyeq y \succcurlyeq z$,

$$G(Tx, Ty, Tz) \leq \varphi(G(x, y, z)). \tag{5.27}$$

*Then $T$ has a fixed point.*

### 5.3.4 Geraghty's Fixed Point Theorems in Preordered G-Metric Spaces

Let $\mathcal{F}_{\mathrm{Ger}}$ be the family of all Geraghty functions, that is, functions $\beta : [0, \infty) \to [0, 1)$ satisfying the condition

$$\{\beta(t_n)\} \longrightarrow 1 \quad \text{implies} \quad \{t_n\} \longrightarrow 0. \tag{5.28}$$

In 2007, Jachymski and Jóźwik [95] proved that the classes $\mathcal{F}_{\mathrm{Ger}}$ and $\mathcal{F}_{\mathrm{Cir}}$ generate equivalent conditions in the sense that, given an operator $T : X \to X$, there exists $\beta \in \mathcal{F}_{\mathrm{Ger}}$ such that

$$d\,(Tx, Ty) \leq \beta\,(d\,(x, y))\; d\,(x, y) \quad \text{for all } x, y \in X,$$

if, and only if, there exists $\phi \in \mathcal{F}_{\mathrm{Cir}}$ such that

$$d\,(Tx, Ty) \leq \phi\,(d\,(x, y)) \quad \text{for all } x, y \in X.$$

However, the following examples show that this relationship is not trivial. If $\beta \in \mathcal{F}_{\mathrm{Ger}}$, the function $\phi_\beta(t) = \beta(t)\, t$ for all $t \in [0, \infty)$ does not necessarily belong to $\mathcal{F}_{\mathrm{Cir}}$. Conversely, if $\phi \in \mathcal{F}_{\mathrm{Cir}}$, the function

$$\beta_\phi(t) = \begin{cases} 0, & \text{if } t = 0, \\ \dfrac{\phi(t)}{t}, & \text{if } t > 0, \end{cases} \tag{5.29}$$

does not necessarily belong to $\mathcal{F}_{\mathrm{Ger}}$.

*Example 5.3.2.* Let $\beta : [0, \infty) \to [0, 1)$ be the function

$$\beta\,(t) = \begin{cases} 0, & \text{if } t = 0, \\ \dfrac{1}{t+1}, & \text{if } 0 < t \leq 1, \\ \dfrac{1}{2} + \dfrac{1}{4}\,\sin\left(\dfrac{1}{t-1}\right), & \text{if } t > 1. \end{cases}$$

Since $1/4 \leq \beta(t) \leq 3/4$ for all $t > 1$, it is clear that $\beta$ is a Geraghty function in the sense of property (5.28). However, $\lim_{t\to 1^+} \beta(t)$ does not exist. As a consequence, it we consider the function $\phi_\beta : [0, \infty) \to [0, \infty)$ given by $\phi_\beta\,(t) = \beta(t)\, t$ for all $t \geq 0$, it is clear that $\phi_\beta(t) < t$ for all $t > 0$. Nevertheless,

$$\lim_{t\to 1^+} \phi_\beta(t)$$

does not exist, so $\phi_\beta$ cannot verify the condition

$$\lim_{t\to 1^+} \phi_\beta(t) < 1,$$

which must be satisfied by any function $\phi \in \mathcal{F}_{\text{Cir}}$.

*Example 5.3.3.* Let $\phi : [0,\infty) \to [0,1)$ be the function

$$\phi(t) = \frac{\pi\, t + 2\, t \arctan t}{2\pi} \quad \text{for all } t \in [0,\infty)\,.$$

Then $\phi \in \mathcal{F}_{\text{Cir}}$. The function $\beta_\phi : [0,\infty) \to [0,\infty)$ defined by (5.29) satisfies the condition $\beta_\phi(t) < 1$ for all $t > 0$. However, $\beta_\phi$ does not belong to $\mathcal{F}_{\text{Ger}}$ because the sequence $\{t_n = n\}_{n\in\mathbb{N}}$ satisfies $\{\beta_\phi(t_n)\} \to 1$ but $\{t_n\}$ does not converge to zero.

Under monotone conditions, there exists an inclusion $\mathcal{F}_{\text{Ger}} \hookrightarrow \mathcal{F}_{\text{Cir}}$.

**Proposition 5.3.1.** *If $\beta \in \mathcal{F}_{\text{Ger}}$ is non-increasing in $(0,\infty)$, then the function $\phi_\beta : [0,\infty) \to [0,\infty)$, given by $\phi_\beta(t) = \beta(t)\, t$ for all $t \geq 0$, belongs to $\mathcal{F}_{\text{Cir}}$.*

*Proof.* As $\beta(t) < 1$ for all $t \geq 0$, then $\phi_\beta(t) = \beta(t)\, t < t$ for all $t > 0$. Since $\beta$ is non-increasing, it has limit from the right at any point of $(0,\infty)$. Let $\{t_n\}$ be an strictly decreasing sequence converging to $\delta > 0$. Therefore, $0 < \delta < t_{n+1} < t_n$ and $\beta(t_n) \leq \beta(t_{n+1}) \leq \beta(\delta) < 1$. Hence, $\{\beta(t_n)\}$ is a non-decreasing, upper-bounded sequence. Thus, it is convergent and

$$\lim_{t\to\delta^+} \beta(t) = \lim_{n\to\infty} \beta(t_n) \leq \beta(\delta)\,.$$

As a consequence, the following limit exists and it satisfies

$$\lim_{t\to\delta^+} \phi_\beta(t) = \lim_{n\to\infty} \beta(t_n)\; t_n \leq \beta(\delta)\; \delta < \delta.$$

Hence, $\phi_\beta \in \mathcal{F}_{\text{Cir}}$. □

*Remark 5.3.2.* If $\beta(t) < 1$ for all $t > 0$, then $\beta(t)\, t \leq t$ for all $t \geq 0$.

Taking into account the relationships introduced by Jachymski and Jóźwik in [95], the following results are equivalent to those given in Sect. 5.3.3.

**Theorem 5.3.10.** *Let $(X,\preccurlyeq)$ be a preordered set endowed with a G-metric $G$ and let $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

*(i) $(X,G)$ is G-complete;*
*(ii) $T$ is G-continuous;*
*(iii) $T$ is non-decreasing (with respect to $\preccurlyeq$);*
*(iv) there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;*
*(v) there exists a function $\beta \in \mathcal{F}_{\text{Ger}}$ such that, for all $x, y \in X$ with $x \succcurlyeq y$,*

$$G(Tx, Ty, Ty) \leq \beta(G(x, y, y))\, G(x, y, y). \tag{5.30}$$

*Then $T$ has a fixed point.*

*Proof.* Let $x_0 \in X$ be such that $x_0 \preccurlyeq Tx_0$ and let $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Regarding that $T$ is a non-decreasing mapping, we have that $x_0 \preccurlyeq Tx_0 = x_1$ implies $x_1 = Tx_0 \preccurlyeq Tx_1 = x_2$. Inductively, we obtain

$$x_0 \preccurlyeq x_1 \preccurlyeq x_2 \preccurlyeq \ldots \preccurlyeq x_{n-1} \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq \ldots \tag{5.31}$$

Assume that there exists $n_0$ such that $x_{n_0} = x_{n_0+1}$. Since $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, then $x_{n_0}$ is the fixed point of $T$, which completes the existence part of the proof. Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Thus, by (5.31) we have

$$x_0 \prec x_1 \prec x_2 \prec \ldots \prec x_{n-1} \prec x_n \prec x_{n+1} \prec \ldots$$

In particular, $G(x_{n+1}, x_n, x_n) > 0$ for all $n \in \mathbb{N}$. Applying the contractivity condition (5.30) to $x = x_{n+1}$ and $y = x_n$, we obtain that, for all $n \in \mathbb{N}$,

$$\begin{aligned} G(x_{n+2}, x_{n+1}, x_{n+1}) &= G(Tx_{n+1}, Tx_n, Tx_n) \\ &\leq \beta(G(x_{n+1}, x_n, x_n))\, G(x_{n+1}, x_n, x_n) \\ &< G(x_{n+1}, x_n, x_n). \end{aligned}$$

Hence, the sequence $\{G(x_{n+1}, x_n, x_n)\}$ is convergent. Let $L \in [0, \infty)$ be its limit. To prove that $L = 0$, assume that $L > 0$. In such a case,

$$\begin{aligned} 1 &< \frac{G(x_{n+2}, x_{n+1}, x_{n+1})}{L} \leq \beta(G(x_{n+1}, x_n, x_n))\, \frac{G(x_{n+1}, x_n, x_n)}{L} \\ &< \frac{G(x_{n+1}, x_n, x_n)}{L}. \end{aligned}$$

Letting $n \to \infty$, Lemma 2.1.1 guarantees that $\{\beta(G(x_{n+1}, x_n, x_n))\} \to 1$. As $\beta \in \mathcal{F}_{\mathrm{Ger}}$, then $\{G(x_{n+1}, x_n, x_n)\} \to 0$, which contradicts the fact that $L > 0$. Thus, $L = 0$ and $\{x_n\}$ is an asymptotically regular sequence. We show that it is Cauchy reasoning by contradiction. In such a case, Theorem 4.1.1 ensures that there exists a positive real number $\varepsilon_0 > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that, for all $k \in \mathbb{N}$,

$$\begin{aligned} &k \leq n(k) < m(k) < n(k+1), \\ &G\left(x_{n(k)}, x_{n(k)-1}, x_{m(k)-1}\right) \leq \varepsilon_0 < G\left(x_{n(k)}, x_{n(k)}, x_{m(k)}\right), \end{aligned}$$

and also

$$\lim_{k\to\infty} G\left(x_{n(k)}, x_{n(k)}, x_{m(k)}\right) = \lim_{k\to\infty} G\left(x_{n(k)-1}, x_{n(k)-1}, x_{m(k)-1}\right) = \varepsilon_0. \tag{5.32}$$

As $\preccurlyeq$ is transitive, then $x_n \preccurlyeq x_m$ for all $n < m$. Then, by Remark 5.3.2,

$$\begin{aligned}\frac{G\left(x_{n(k)}, x_{n(k)}, x_{m(k)}\right)}{\varepsilon_0} &= \frac{G\left(Tx_{m(k)-1}, Tx_{n(k)-1}, Tx_{n(k)-1}\right)}{\varepsilon_0} \\ &\leq \beta(G(x_{m(k)-1}, x_{n(k)-1}, x_{n(k)-1}))\,\frac{G(x_{m(k)-1}, x_{n(k)-1}, x_{n(k)-1})}{\varepsilon_0} \\ &\leq \frac{G(x_{m(k)-1}, x_{n(k)-1}, x_{n(k)-1})}{\varepsilon_0}.\end{aligned}$$

As a consequence, letting $k \to \infty$,

$$\lim_{k\to\infty} \beta(G(x_{m(k)-1}, x_{n(k)-1}, x_{n(k)-1})) = 1.$$

As $\beta \in \mathcal{F}_{\mathrm{Ger}}$, then $\{G(x_{m(k)-1}, x_{n(k)-1}, x_{n(k)-1})\} \to 0$, which contradicts (5.32) because $\varepsilon_0 > 0$. Then $\{x_n\}$ is a Cauchy sequence. As $(X, G)$ is complete, there exists $u \in X$ such that $\{x_n\} \to u$, and as $T$ is continuous, $Tu = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = u$. Hence, $T$ has a fixed point. □

**Theorem 5.3.11.** *Let $(X, \preccurlyeq)$ be a preordered set endowed with a G-metric $G$ and let $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

*(i) $(X, G)$ is G-complete;*
*(ii) $(X, G, \preccurlyeq)$ is non-decreasing-regular;*
*(iii) $T$ is non-decreasing (with respect to $\preccurlyeq$);*
*(iv) there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;*
*(v) there exists a function $\beta \in \mathcal{F}_{\mathrm{Ger}}$ such that, for all $x, y \in X$ with $x \succcurlyeq y$,*

$$G(Tx, Ty, Ty) \leq \beta(G(x, y, y))\,G(x, y, y). \tag{5.33}$$

*Then $T$ has a fixed point.*

*Proof.* Following the proof in Theorem 5.3.10, we get that $\{x_n\} \to u \in X$. As $(X, G, \preccurlyeq)$ is non-decreasing-regular, it follows that $x_n \preccurlyeq u$ for all $n \in \mathbb{N}$. Applying the contractivity condition (5.33) and Remark 5.3.2, for all $n \in \mathbb{N}$,

$$\begin{aligned}G(Tu, x_{n+1}, x_{n+1}) &= G(Tu, Tx_n, Tx_n) \\ &\leq \beta(G(u, x_n, x_n))\,G(u, x_n, x_n) \\ &\leq G(u, x_n, x_n).\end{aligned}$$

Hence $Tu = \lim_{n\to\infty} x_{n+1} = u$. □

The two corollaries below are immediate consequences of Theorems 5.3.10 and 5.3.11

**Corollary 5.3.10.** *Let $(X, \preccurlyeq)$ be a preordered set endowed with a $G$-metric $G$ and let $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

*(i)* $(X, G)$ *is G-complete;*
*(ii)* $T$ *is G-continuous,*
*(iii)* $T$ *is non-decreasing (with respect to* $\preccurlyeq$*);*
*(iv) there exists* $x_0 \in X$ *such that* $x_0 \preccurlyeq Tx_0$*;*
*(v) there exists a function* $\beta \in \mathcal{F}_{\mathrm{Ger}}$ *such that, for all* $x, y, z \in X$ *with* $x \succcurlyeq y \succcurlyeq z$*,*

$$G(Tx, Ty, Tz) \leq \beta(G(x, y, z))\, G(x, y, z). \tag{5.34}$$

*Then $T$ has a fixed point.*

**Corollary 5.3.11.** *Let $(X, \preccurlyeq)$ be a preordered set endowed with a $G$-metric $G$ and let $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

*(i)* $(X, G)$ *is G-complete;*
*(ii)* $(X, G, \preccurlyeq)$ *is non-decreasing-regular;*
*(iii)* $T$ *is non-decreasing (with respect to* $\preccurlyeq$*);*
*(iv) there exists* $x_0 \in X$ *such that* $x_0 \preccurlyeq Tx_0$*;*
*(v) there exists a function* $\beta \in \mathcal{F}_{\mathrm{Ger}}$ *such that, for all* $x, y, z \in X$ *with* $x \succcurlyeq y \succcurlyeq z$*,*

$$G(Tx, Ty, Tz) \leq \beta(G(x, y, z))\, G(x, y, z). \tag{5.35}$$

*Then $T$ has a fixed point.*

The uniqueness of the fixed point can be obtained using the same additional assumption in Theorem 5.2.1.

# Chapter 6
# Further Fixed Point Results on $G$-Metric Spaces

In this chapter we present some fixed point theorems in the context of $G$-metric spaces.

## 6.1 A New Approach to Express Contractivity Conditions

In the contractivity conditions we have presented in the previous chapters, the mapping $T$ only appears in the left-hand term of the inequality (see, for example, (4.15), (4.23) and (5.2)).

**Theorem 6.1.1 ([24]).** *Let $(X, G)$ be a complete $G$-metric space and let $T : X \to X$ be a mapping. Suppose that there exists $\lambda \in [0, 1)$ such that*

$$G(Tx, Ty, Ty) \leq \lambda\, G(x, Tx, y) \quad \text{for all } x, y \in X. \tag{6.1}$$

*Then $T$ has a unique fixed point. In fact, $T$ is a Picard operator.*

*Proof.* Let $x_0 \in X$ be an arbitrary point and define the sequence $\{x_n\}$ by $x_n = T^n x_0$ for all $n \geq 0$. From (6.1), we have that, for all $n \geq 0$,

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+2}) &= G(Tx_n, Tx_{n+1}, Tx_{n+1}) \leq \lambda\, G(x_n, Tx_n, x_{n+1}) \\ &= \lambda\, G(x_n, x_{n+1}, x_{n+1}). \end{aligned}$$

From Corollary 4.1.1, $\{x_n\}$ is a Cauchy sequence in $(X, G)$. Due to the completeness of $(X, G)$, there exists $u \in X$ such that $\{x_n\}$ is $G$-convergent to $u$. We prove that $u$ is a fixed point of $T$. Indeed, for all $n \geq 0$, we have that



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$$G(x_{n+1}, Tu, Tu) = G(Tx_n, Tu, Tu) \leq \lambda\, G(x_n, x_{n+1}, u). \tag{6.2}$$

Letting $n \to \infty$ and using the fact that the metric $G$ is continuous (see Theorem 3.2.2), we get that

$$G(u, Tu, Tu) \leq \lambda\, G(u, u, u) = 0,$$

so $G(u, Tu, Tu) = 0$ and $Tu = u$. Finally, we claim that $T$ has a unique fixed point (which is $u$). Assume that $x, y \in \mathrm{Fix}\,(T)$ are two fixed points of $T$. Then, applying (6.1), we deduce that

$$G\,(x, y, y) = G\,(Tx, Ty, Ty) \leq \lambda\, G(x, x, y) \quad \text{and}$$
$$G\,(x, x, y) = G\,(Ty, Tx, Tx) \leq \lambda\, G(y, y, x).$$

As a consequence,

$$G\,(x, y, y) \leq \lambda\, G(x, x, y) \leq \lambda^2\, G(x, y, y).$$

If $G\,(x, y, y) > 0$, the previous inequality is false because $\lambda \in [0, 1)$. Hence, $G\,(x, y, y) = 0$ and $x = y$. □

*Example 6.1.1.* Let $X = [0, \infty)$ be the interval of nonnegative real numbers and let $G$ the complete $G$-metric on $X$ defined by

$$G(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x, y, z\}, & \text{otherwise.} \end{cases}$$

Define $T : X \to X$ by $Tx = x/5$ for all $x \in X$. Then, all the hypotheses of Theorem 6.1.1 hold. In fact,

$$G(Tx, Ty, Ty) = \frac{1}{5} \max\{x, y\} \leq \max\{x, y\} = G(x, Tx, y)$$

for all $x, y \in X$. Then $T$ has a unique fixed point on $X$, which is $u = 0$.

Based on Theorem 6.1.1, the following result can be easily proved.

**Corollary 6.1.1 ([24]).** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be a mapping. Suppose that there exist real numbers $a, b \in \mathbb{R}$, verifying $a + b < 1$, such that*

$$G(Tx, Ty, Tz) \leq a G(x, Tx, z) + b G(x, Tx, y) \quad \textit{for all } x, y, z \in X.$$

*Then $T$ has a unique fixed point. In fact, $T$ is a Picard operator.*

*Proof.* Let $\lambda = \max\{0, a+b\} \in [0,1)$. Then, for all $x, y \in X$, it follows that

$$\begin{aligned} G(Tx,Ty,Ty) &\le a\,G(x,Tx,y) + b\,G(x,Tx,y) \\ &= (a+b)\;G(x,Tx,y) \le \lambda\,G(x,Tx,y). \end{aligned}$$

Then we can apply Theorem 6.1.1. □

In the following result, we employ $T$, $T^2$ and $T^3$ in the contractivity condition.

**Theorem 6.1.2 ([24]).** *Let $(X,G)$ be a complete G-metric space and let $T: X \to X$ be a mapping. Assume that there exist nonnegative real numbers a, b, c and d, with $a+b+c+d<1$, such that, for all $x, y \in X$,*

$$\begin{aligned} G(Tx,Ty,T^2y) &\le a\,G(x,Tx,T^2x) + b\,G(y,Ty,T^2y) \\ &\quad + c\,G(x,Tx,Ty) + d\,G(y,Ty,T^3x) \end{aligned} \tag{6.3}$$

*Then T has a unique fixed point.*

*Proof.* Let $x_0 \in X$ be an arbitrary point and let $\{x_n\}_{n\ge 0}$ be the Picard sequence of $T$ based on $x_0$, that is,

$$x_{n+1} = Tx_n \text{ for all } n \ge 0.$$

If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, so $x_{n_0}$ is a fixed point of $T$. On the contrary case, assume that

$$x_n \neq x_{n+1} \quad \text{for all } n \in \mathbb{N}. \tag{6.4}$$

From (6.3) with $x = x_{n-1}$ and $y = x_n$ we have that, for all $n \ge 1$,

$$\begin{aligned} G\left(x_n,x_{n+1},x_{n+2}\right) &= G(Tx_{n-1},Tx_n,T^2x_n) \\ &\le a\,G(x_{n-1},Tx_{n-1},T^2x_{n-1}) + b\,G(x_n,Tx_n,T^2x_n) \\ &\quad + c\,G(x_{n-1},Tx_{n-1},Tx_n) + d\,G(x_n,Tx_n,T^3x_{n-1}) \\ &= a\,G(x_{n-1},x_n,x_{n+1}) + b\,G(x_n,x_{n+1},x_{n+2}) \\ &\quad + c\,G(x_{n-1},x_n,x_{n+1}) + d\,G(x_n,x_{n+1},x_{n+2}). \end{aligned}$$

As a result, for all $n \ge 1$

$$(1-b-d)\;G(x_n,x_{n+1},x_{n+2}) \le (a+c)\;G(x_{n-1},x_n,x_{n+1}),$$

which implies that

$$G(x_n,x_{n+1},x_{n+2}) = \lambda\,G(x_{n-1},x_n,x_{n+1}) \quad \text{for all } n \ge 1,$$

where

$$\lambda = \frac{a+c}{1-b-d} < 1.$$

Repeating the previous argument, we deduce that

$$G(x_n, x_{n+1}, x_{n+2}) \leq \lambda^n G(x_0, x_1, x_2) \quad \text{for all } n \geq 0. \tag{6.5}$$

Notice that, from $(G_3)$ and (6.4), we know that

$$G(x_n, x_n, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+2}) \quad \text{for all } n \geq 0.$$

From Lemma 3.1.1, it follows that, for all $n \geq 0$,

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\leq 2G(x_n, x_n, x_{n+1}) \\ &\leq 2G(x_n, x_{n+1}, x_{n+2}) \leq 2\lambda^n G(x_0, x_1, x_2). \end{aligned}$$

As a consequence, for all $n, m \in \mathbb{N}$ such that $n < m$, item 4 of Lemma 3.1.2 yields

$$\begin{aligned} G(x_n, x_m, x_m) &\leq \sum_{i=n}^{m-1} G(x_i, x_{i+1}, x_{i+1}) \leq \sum_{i=n}^{m-1} 2\lambda^i G(x_0, x_1, x_1) \\ &= 2(\lambda^n + \lambda^{n+1} + \lambda^{n+2} + \ldots + \lambda^{m-1}) G(x_0, x_1, x_1) \\ &\leq \frac{2\lambda^n}{1-\lambda} G(x_0, x_1, x_1). \end{aligned}$$

In particular, the same argument of Theorem 4.2.1 guarantees that

$$\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0,$$

and $\{x_n\}$ is Cauchy sequence in $(X, G)$. Due to the completeness of $(X, G)$, there exists $z \in X$ such that $\{x_n\}$ is $G$-convergent to $z$. We claim that $Tz$ is a fixed point of $T$. Indeed, for all $n \geq 0$, from (6.3) with $x = x_n$, we have,

$$\begin{aligned} G(x_{n+1}, Tz, T^2z) &= G(Tx_n, Tz, T^2z) \\ &\leq aG(x_n, Tx_n, T^2x_n) + bG(z, Tz, T^2z) \\ &\quad + cG(x_n, Tx_n, Tz) + dG(z, Tz, T^3x_n) \\ &= aG(x_n, x_{n+1}, x_{n+2}) + bG(z, Tz, T^2z) \\ &\quad + cG(x_n, x_{n+1}, Tz) + dG(z, Tz, x_{n+3}). \end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, we have that

$$G(z, Tz, T^2z) \le \frac{c+d}{1-b} G(z, z, Tz),$$

where $(c+d)/(1-b) < 1$. If we suppose that $Tz \ne T^2z$ then, by $(G_3)$, we get that

$$G(z, Tz, T^2z) \le \frac{c+d}{1-b} G(z, z, Tz) \le \frac{c+d}{1-b} G(z, Tz, T^2z) < G(z, Tz, T^2z),$$

which is a contradiction. Then, necessarily, $Tz = T^2z$, which means that $Tz$ is a fixed point of $T$.

To prove that $T$ has a unique fixed point, let $x, y \in \mathrm{Fix}(T)$ be two fixed points of $T$. Using (6.3), we have that

$$\begin{aligned} G(x, y, y) &\le aG(x, x, x) + bG(y, y, y) + cG(x, x, y) + dG(y, y, x) \\ &= cG(x, x, y) + dG(y, y, x). \end{aligned}$$

Therefore,

$$G(x, y, y) \le \mu G(x, x, y), \quad \text{where } \mu = \frac{c}{1-d} < 1.$$

Changing the roles of $x$ and $y$, we also have that $G(x, x, y) \le \mu G(x, y, y)$. Hence, $G(x, y, y) \le \mu G(x, x, y) \le \mu^2 G(x, y, y)$, which is not possible when $G(x, y, y) > 0$ because $\mu^2 < 1$. As a consequence, $G(x, y, y) = 0$ and $x = y$. □

Next, we show some results in which we combine these contractivity conditions (in which $T$ appears in both sides of the inequality) and control functions. Recall that

$$\begin{aligned} \mathcal{F}_{\text{alt}} = \{\phi : [0, \infty) \to [0, \infty) :\ &\phi \text{ continuous, non-decreasing,} \\ &\phi(t) = 0 \Leftrightarrow t = 0\}, \end{aligned}$$

$$\mathcal{F}'_{\text{alt}} = \{\phi : [0, \infty) \to [0, \infty) :\ \phi \text{ lower semi-continuous, } \phi(t) = 0 \Leftrightarrow t = 0\}.$$

**Theorem 6.1.3 ([24]).** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be a mapping. Assume that there exist $\psi \in \mathcal{F}_{\text{alt}}$ and $\phi \in \mathcal{F}'_{\text{alt}}$ such that, for all $x, y \in X$,*

$$\psi(G(Tx, T^2x, Ty)) \le \psi(G(x, Tx, y)) - \phi(G(x, Tx, y)). \tag{6.6}$$

*Then $T$ has a unique fixed point.*

*Proof.* Let $x_0 \in X$ be an arbitrary point and let $\{x_n\}_{n\ge0}$ be the Picard sequence of $T$ based on $x_0$, that is,

$$x_{n+1} = Tx_n \text{ for all } n \ge 0.$$

If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, so $x_{n_0}$ is a fixed point of $T$. On the contrary case, assume that

$$x_n \neq x_{n+1} \quad \text{for all } n \in \mathbb{N}. \tag{6.7}$$

From (6.6) with $x = x_n$ and $y = x_{n+1}$, we have that, for all $n \geq 0$,

$$\begin{aligned} \psi(G(x_{n+1}, x_{n+2}, x_{n+2})) &= \psi(G(Tx_n, T^2x_n, Tx_{n+1})) \\ &\leq \psi(G(x_n, Tx_n, x_{n+1})) - \phi(G(x_n, Tx_n, x_{n+1})) \\ &= (\psi - \phi)\,(G(x_n, x_{n+1}, x_{n+1})). \end{aligned}$$

Using Lemma 2.3.6, we deduce that:

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0, \tag{6.8}$$

and by Lemma 3.1.1,

$$\lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = 0. \tag{6.9}$$

Next, we show that $\{x_n\}_{n=0}^{\infty}$ is a $G$-Cauchy sequence. Suppose, to the contrary, that $\{x_n\}$ is not Cauchy in $(X, G)$. Then, by Theorem 4.1.1, there exist a positive real number $\varepsilon_0 > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that, for all $k \in \mathbb{N}$,

$$\begin{aligned} &k \leq n(k) < m(k) < n(k+1), \\ &G\left(x_{n(k)}, x_{m(k)-1}, x_{m(k)-1}\right) \leq \varepsilon_0 < G\left(x_{n(k)}, x_{m(k)}, x_{m(k)}\right), \\ &\lim_{k\to\infty} G\left(x_{n(k)}, x_{m(k)}, x_{m(k)}\right) = \lim_{k\to\infty} G(x_{m(k)}, x_{m(k)+1}, x_{n(k)}) \\ &\qquad = \lim_{k\to\infty} G(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1}) = \varepsilon_0. \end{aligned} \tag{6.10}$$

Therefore, using (6.6), it follows that, for all $k \in \mathbb{N}$,

$$\begin{aligned} &\psi(G(x_{m(k)}, x_{m(k)+1}, x_{n(k)})) = \psi(G(Tx_{m(k)-1}, T^2x_{m(k)-1}, Tx_{n(k)-1})) \\ &\quad \leq \psi(G(x_{m(k)-1}, Tx_{m(k)-1}, x_{n(k)-1})) - \phi(G(x_{m(k)-1}, Tx_{m(k)-1}, x_{n(k)-1})) \\ &\quad = \psi(G(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1})) - \phi(G(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1})) \end{aligned}$$

Using Lemma 2.3.5 applied to the sequences $\{t_k = G(x_{m(k)}, x_{m(k)+1}, x_{n(k)})\}$ and $\{s_k = G(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1})\}$, we conclude that $\varepsilon_0 = 0$, which is a contradiction. This contradiction proves that $\{x_n\}$ is a Cauchy sequence in $(X, G)$. Since, $(X, G)$ is complete, then there exist $z \in X$ such that $\{x_n\} \to z$. From (6.6) with $x = x_n$ and $y = z$ we have,

$$\begin{aligned}\psi(G(x_{n+1},x_{n+2},Tz)) &= \psi(G(Tx_n,T^2x_n,Tz))\\ &\le \psi(G(x_n,Tx_n,z)) - \phi(G(x_n,Tx_n,z))\\ &= \psi(G(x_n,x_{n+1},z)) - \phi(G(x_n,x_{n+1},z))\\ &\le \psi(G(x_n,x_{n+1},z)).\end{aligned}$$

Using that $\psi$ and $G$ are continuous, and taking the limit as $n \to \infty$ we get that

$$\psi(G(z,z,Tz)) \le \psi(0) = 0.$$

Then $G(z,z,Tz) = 0$, i.e., $z = Tz$. To prove uniqueness, suppose that $x, y \in \mathrm{Fix}(T)$ are two fixed points of $T$. Now, by (6.6) we get

$$\begin{aligned}\psi(G(x,x,y)) &= \psi(G(Tx,T^2x,Ty))\\ &\le \psi(G(x,Tx,y)) - \phi(G(x,Tx,y))\\ &= \psi(G(x,x,y)) - \phi(G(x,x,y)),\end{aligned}$$

which is impossible unless $G(x,x,y)) = 0$, that is, $x = y$. □

**Corollary 6.1.2.** *Let $(X,G)$ be a complete G-metric space and let $T : X \to X$ be a mapping. Assume that there exists a function $\phi \in \mathcal{F}'_{\mathrm{alt}}$ such that, for all $x, y \in X$,*

$$G(Tx,T^2x,Ty) \le G(x,Tx,y) - \phi(G(x,Tx,y)).$$

*Then $T$ has a unique fixed point.*

If $\phi(t) = (1-\lambda)\,t$ for all $t \in [0,\infty)$, where $0 \le \lambda < 1$, we have the following result.

**Corollary 6.1.3 ([24]).** *Let $(X,G)$ be a complete G-metric space and let $T : X \to X$ be a mapping. Assume that there exists $\lambda \in [0,1)$ such that, for all $x, y \in X$,*

$$G(Tx,T^2x,Ty) \le \lambda\, G(x,Tx,y).$$

*Then $T$ has a unique fixed point.*

*Example 6.1.2 ([24]).* Let $X = [0,\infty)$ and let

$$G(x,y,z) = \begin{cases} 0, & \text{if } x = y = z,\\ \max\{x,y\} + \max\{y,z\} + \max\{x,z\}, & \text{otherwise.}\end{cases}$$

Then $G$ is a complete $G$-metric on $X$. Define, $T : X \to X$ by $Tx = x/4$ for all $x \in X$. Then, for all $x, y \in X$,

$$G(x, Tx, y) = \begin{cases} 0, & \text{if } x = y = 0, \\ x + \max\{x, y\} + \max\left\{\frac{x}{4}, y\right\}, & \textit{otherwise}, \end{cases}$$

and

$$G(Tx, T^2x, Ty) = \begin{cases} 0, & \text{if } x = y = 0, \\ \frac{1}{4}\left(x + \max\{x, y\} + \max\left\{\frac{x}{4}, y\right\}\right), & \textit{otherwise}. \end{cases}$$

Therefore,

$$G(Tx, T^2x, Ty) \le \frac{1}{4} G(x, Tx, y)$$

for all $x, y \in X$ and all the conditions of Corollary 6.1.3 (and also Theorem 6.1.3) hold. Hence, $T$ has a unique fixed point, which is $u = 0$.

**Corollary 6.1.4 ([24]).** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be a mapping. Suppose that there exist $a, b \in \mathbb{R}$, where $0 \le a + b < 2$, such that, for all $x, y, z \in X$,*

$$G(Tx, T^2x, Ty) + G(Tx, T^2x, Tz) \le aG(x, Tx, y) + bG(x, Tx, z).$$

*Then $T$ has a unique fixed point.*

*Proof.* By taking $y = z$, we get that

$$G(Tx, T^2x, Ty) \le \frac{a + b}{2} G(x, Tx, y)$$

where $\lambda = \frac{a+b}{2} \in [0, 1)$. That is, the conditions of Theorem 6.1.3 hold (where $\psi(t) = t$ and $\phi(t) = (1 - \lambda)t$ for all $t \ge 0$) and $T$ has a unique fixed point. □

Jachymski [94] proved the equivalence of the so-called distance functions (see Lemma 1 in [94]). Inspired by this result, we state the following theorem.

**Theorem 6.1.4.** *Let $(X, \preccurlyeq)$ be an ordered set endowed with a G-metric and $T$ be a self-map on a G-complete partially ordered G-metric space $(X, G)$. The following statements are equivalent:*

($i$) *there exist functions $\psi, \eta \in \Phi_2^w$ such that*

$$\psi(G(Tx, Ty, Tz)) \le \psi(G(x, y, z)) - \eta(G(x, y, z)), \tag{6.11}$$

($ii$) *there exist $\alpha \in [0, 1)$ and a function $\psi \in \Phi_2^w$ such that*

$$\psi(G(Tx, Ty, Tz)) \le \alpha\psi(G(x, y, z)), \tag{6.12}$$

$(iii)$ *there exists a continuous and non-decreasing function* $\alpha : [0,\infty) \to [0,\infty)$ *such that* $\alpha(t) < t$ *for all* $t > 0$ *such that*

$$G(Tx,Ty,Tz) \le \alpha(G(x,y,z)),$$

$(iv)$ *there exist function* $\psi \in \Phi_2^w$ *and a non-decreasing function* $\eta : [0,\infty) \to [0,\infty)$ *with* $\eta^{-1}(0) = 0$ *such that*

$$\psi(G(Tx,Ty,Tz)) \le \psi(G(x,y,z)) - \eta(G(x,y,z)), \tag{6.13}$$

$(iv)$ *there exist function* $\psi \in \Phi_2^w$ *and a lower semi-continuous function* $\eta : [0,\infty) \to [0,\infty)$ *with* $\eta^{-1}(0) = 0$ *and* $\liminf_{t\to\infty} \eta(t) > 0$ *such that*

$$\psi(G(Tx,Ty,Tz)) \le \psi(G(x,y,z)) - \eta(G(x,y,z)), \tag{6.14}$$

*for any* $x,y,z \in X$ *with* $x \succcurlyeq y \succcurlyeq z$.

**Corollary 6.1.5.** *Let* $(X,\preccurlyeq)$ *be an ordered set endowed with a G-metric and T be a self-map on a G-complete partially ordered G-metric space* $(X,G)$. *The following statements are equivalent:*

$(i)$ *there exist functions* $\psi,\eta \in \Phi_2^w$ *such that*

$$\psi(G(Tx,Ty,Ty)) \le \psi(G(x,y,y)) - \eta(G(x,y,y)), \tag{6.15}$$

$(ii)$ *there exist* $\alpha \in [0,1)$ *and a function* $\psi \in \Phi_2^w$ *such that*

$$\psi(G(Tx,Ty,Ty)) \le \alpha\psi(G(x,y,y)), \tag{6.16}$$

$(iii)$ *there exists a continuous and non-decreasing function* $\alpha : [0,\infty) \to [0,\infty)$ *such that* $\alpha(t) < t$ *for all* $t > 0$ *such that*

$$G(Tx,Ty,Ty) \le \alpha(G(x,y,y)), \tag{6.17}$$

$(iv)$ *there exist function* $\psi \in \Phi_2^w$ *and a non-decreasing function* $\eta : [0,\infty) \to [0,\infty)$ *with* $\eta^{-1}(0) = 0$ *such that*

$$\psi(G(Tx,Ty,Ty)) \le \psi(G(x,y,y)) - \eta(G(x,y,y)), \tag{6.18}$$

$(iv)$ *there exist function* $\psi \in \Phi_2^w$ *and a lower semi-continuous function* $\eta : [0,\infty) \to [0,\infty)$ *with* $\eta^{-1}(0) = 0$ *and* $\liminf_{t\to\infty} \eta(t) > 0$ *such that*

$$\psi(G(Tx,Ty,Ty)) \le \psi(G(x,y,y)) - \eta(G(x,y,y)), \tag{6.19}$$

*for any* $x,y \in X$ *with* $x \preccurlyeq y$.

## 6.2 Fixed Point Theorems Using Contractivity Conditions Involving a Unique Variable

As a $G$-metric is a function in three variables, it is usual that fixed point theorems involve contractivity conditions using three different arguments $x, y, z \in X$. In this section, we show some results in which a unique variable plays the key role.

**Theorem 6.2.1.** *Let $(X, G, \preceq)$ be a preordered $G$-metric space and let $T, g : X \to X$ be two mappings. Suppose that the following conditions are fulfilled.*

**(a)** *There exists $x_0 \in X$ such that $gx_0 \preceq Tx_0$.*
**(b)** *$T(X) \subseteq g(X)$.*
**(c)** *$T$ is $(g, \preceq)$-non-decreasing.*
**(d)** *There exists a function $\varphi \in \mathcal{F}_{\text{com}}^{(c)}$ such that*

$$G\left(Tgx, T^2x, T^2x\right) \leq \varphi\left(G\left(ggx, gTx, gTx\right)\right) \tag{6.20}$$

*for all $x \in X$ such that $gx \preceq Tx$.*
**(e)** *$T$ and $g$ are $G$-continuous and commuting.*
**(f)** *$X$ (or $g(X)$ or $T(X)$) is $G$-complete.*

*Then $T$ and $g$ have, at least, a coincidence point.*

*Proof.* Using (a) and (b), Lemma 4.1.1 guarantees that there exists a Picard sequence $\{x_n\} \subseteq X$ of $(T, g)$ based on the point $x_0$, that is, a sequence satisfying

$$gx_{n+1} = Tx_n \quad \text{for all } n \in \mathbb{N}.$$

Using the initial condition $gx_0 \preceq Tx_0 = gx_1$, we have that

$$gx_n \preccurlyeq gx_{n+1} = Tx_n \quad \text{for all } n \in \mathbb{N}$$

because $T$ is $(g, \preceq)$-non-decreasing. Now, applying the contractivity condition (6.20) and the commutativity between $T$ and $g$, we have that, for all $n \in \mathbb{N}$,

$$\begin{aligned} G\left(ggx_{n+1}, ggx_{n+2}, ggx_{n+2}\right) &= G\left(gTx_n, gTx_{n+1}, gTx_{n+1}\right) \\ &= G\left(Tgx_n, Tgx_{n+1}, Tgx_{n+1}\right) = G\left(Tgx_n, TTx_n, TTx_n\right) \\ &\leq \varphi\left(G\left(ggx_n, gTx_n, gTx_n\right)\right) = \varphi\left(G\left(ggx_n, ggx_{n+1}, ggx_{n+1}\right)\right). \end{aligned}$$

Therefore, Lemma 4.1.2 implies that $\{ggx_n\}$ is a Cauchy sequence in $(X, G)$. As

$$ggx_{n+1} = gTx_n = Tgx_n \quad \text{for all } n \in \mathbb{N},$$

then $\{ggx_n\} \subseteq T(X) \subseteq g(X) \subseteq X$ for all $n \geq 1$. Taking into account that one of these subsets is $G$-complete, there exists $z \in X$ such that $\{ggx_n\} \to z$. Since $T$ and $g$ are $G$-continuous, then

$$\{gggx_n\} \to gz \quad \text{and} \quad \{Tggx_n\} \to Tz.$$

However, as $T$ and $g$ are commuting, it follows that, for all $n \in \mathbb{N}$,

$$Tggx_n = gTgx_n = ggTx_n = gggx_{n+1}.$$

By the uniqueness of the limit, we conclude that $Tz = gz$, that is, $z$ is a coincidence point of $T$ and $g$. □

**Corollary 6.2.1.** *Let $(X, G, \preceq)$ be a preordered G-metric space and let $T : X \to X$ be a $\preceq$-non-decreasing, continuous mapping. Assume that there exists a function $\varphi \in \mathcal{F}_{\text{com}}^{(c)}$ such that*

$$G\left(Tx, T^2x, T^2x\right) \leq \varphi\left(G\left(x, Tx, Tx\right)\right)$$

*for all $x \in X$ such that $x \preceq Tx$. If $X$ (or $T(X)$) is G-complete, then $T$ has, at least, a fixed point provided that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$.*

Theorem 6.2.1 is also interesting when $\preceq$ is a partial order on $X$ (we leave to the reader to change the word *preordered* by *ordered*). Furthermore, assume that we use the partial order $\preceq_0$ given in (5.1). In such a case, the following result is obtained.

**Corollary 6.2.2.** *Let $(X, G)$ be a G-metric space and let $T, g : X \to X$ be two mappings. Suppose that the following conditions are fulfilled.*

**(a)** *$T(X) \subseteq g(X)$.*
**(b)** *There exists a function $\varphi \in \mathcal{F}_{\text{com}}^{(c)}$ such that, for all $x \in X$,*

$$G\left(Tgx, T^2x, T^2x\right) \leq \varphi\left(G\left(ggx, gTx, gTx\right)\right).$$

**(c)** *$T$ and $g$ are G-continuous and commuting.*
**(d)** *$X$ (or $g(X)$ or $T(X)$) is G-complete.*

*Then $T$ and $g$ have, at least, a coincidence point.*

Finally, the following one is a version of Theorem 6.2.1 using the function $\varphi_\lambda(t) = \lambda\, t$ for all $t \in [0, \infty)$, where $\lambda \in [0, 1)$.

**Corollary 6.2.3.** *Let $(X, G, \preceq)$ be a preordered G-metric space and let $T, g : X \to X$ be two mappings. Suppose that the following conditions are fulfilled.*

**(a)** *There exists $x_0 \in X$ such that $gx_0 \preceq Tx_0$.*
**(b)** *$T(X) \subseteq g(X)$.*
**(c)** *$T$ is $(g, \preceq)$-non-decreasing.*
**(d)** *There exists a constant $\lambda \in [0, 1)$ such that*

$$G\left(Tgx, T^2x, T^2x\right) \leq \lambda\, G\left(ggx, gTx, gTx\right)$$

*for all $x \in X$ such that $gx \preceq Tx$.*
**(e)** *$T$ and $g$ are G-continuous and commuting.*
**(f)** *$X$ (or $g(X)$ or $T(X)$) is G-complete.*

*Then $T$ and $g$ have, at least, a coincidence point.*

## 6.3 Generalized Cyclic Weak $\phi$-Contractions on $G$-Metric Spaces

In this section we present the notion of a cyclic map and some fixed point theory for cyclic maps in $G$-metric spaces.

### *6.3.1 Cyclic Mappings on G-Metric Spaces*

We begin with the definition of a cyclic mapping.

**Definition 6.3.1.** A self-mapping $T : X \to X$ is *cyclic* if there exist non-empty subsets $A_0, A_1, \ldots, A_{p-1} \subseteq X$ such that

$$A_0 \cup A_1 \cup \ldots \cup A_{p-1} = X \quad \text{and}$$
$$T(A_i) \subseteq A_{i+1} \quad \text{for all } i \in \{0, 1, 2, \ldots, p-1\} \text{ (where } A_p = A_0).$$

In such a case, we say that $\{A_i\}_{i=0}^{p-1}$ is a *cyclic representation of* $(X, T)$.

If $X$ is endowed with a topology, we say that $\{A_i\}_{i=0}^{p-1}$ is *closed* if each $A_i$ is closed for all $i \in \{0, 1, 2, \ldots, p-1\}$.

It is usual to define $A_p = A_0$, $A_{p+1} = A_1$, $A_{p+2} = A_2$, etc. In other words,

$$\text{if } n, m \in \mathbb{N} \text{ and } m \equiv n \pmod{p}, \text{ then } A_m = A_n.$$

Using this agreement, notice that $\{A_i\}_{i=1}^{p} = \{A_1, A_2, A_3, \ldots, A_p\}$ is, indeed, the same cyclic representation of $T$. We will use this representation in the statements of theorems, but the initial point $x_0$ will belong to $A_0$.

**Lemma 6.3.1.** *If a cyclic self-mapping $T : X \to X$ has a fixed point $z$, then $z \in \cap_{i=1}^{p} A_i$ whatever the cyclic representation $\{A_i\}_{i=1}^{p}$ of $(X, T)$.*

*Proof.* As $z \in X = A_1 \cup A_2 \cup \ldots \cup A_p$, there exists $i \in \{1, 2, \ldots, p\}$ such that $z \in A_i$. Then $z = Tz \in T(A_i) \subseteq A_{i+1}$. By repeating this argument, $z \in A_j$ for all $j \in \{1, 2, \ldots, p\}$. □

If $\{x_m\}_{m \geq 0}$ is a Picard sequence of a cyclic operator $T : X \to X$ such that $x_0 \in A_0$, then $x_m \in A_m$ for all $m \in \mathbb{N}$. Furthermore,

$$x_{mp+i} \in A_i \quad \text{for all } m \in \mathbb{N} \text{ and all } i \in \{1, 2, \ldots, p\}.$$

Therefore, each $A_i$ contains a partial subsequence of $\{x_m\}$.

**Lemma 6.3.2.** *Let $X$ be a Hausdorff topological space and let $\{A_i\}_{i=1}^p$ be a closed cyclic representation of $(X,T)$, where $T : X \to X$ is a self-mapping. Then, the limit of any convergent Picard sequence of $T$ belongs to $\cap_{i=1}^p A_i$.*

*Proof.* As each $A_i$ contains a partial subsequence of the Picard sequence, then, its limit $z$ (which is unique by the Hausdorff property) belongs to the closure of $A_i$. But as each $A_i$ is closed, then $z \in A_i$ for all $i \in \{1,2,\ldots,p\}$. □

*Remark 6.3.1.* Let $\{A_i\}_{i=1}^p$ be a cyclic representation of $(X,T)$, where $T : X \to X$ is a self-mapping, and let $\{x_m\}_{m\geq 0}$ be a Picard sequence of $T : X \to X$ such that $x_0 \in A_0$. Let $n,m \in \mathbb{N}$ and $i,j \in \{1,2,\ldots,p\}$ be natural numbers. Then

$$x_n \in A_i, \quad m > n, \quad m - n \equiv j \pmod{p} \quad \Rightarrow \quad x_m \in A_{i+j}.$$

In particular,

$$\begin{aligned} &x_n \in A_i, \quad m > n, \quad m \equiv n \pmod{p} \quad \Rightarrow \quad x_m \in A_i; \\ &x_n \in A_i, \quad m > n, \quad m - n \equiv 1 \pmod{p} \quad \Rightarrow \quad x_m \in A_{i+1}. \end{aligned} \tag{6.21}$$

The following result extends Banach theorem to cyclic mappings.

**Theorem 6.3.1 ([119]).** *Let $(X,G)$ be a complete $G$-metric space and let $\{A_i\}_{i=1}^p$ be a closed cyclic representation of $(Y,T)$, where $Y \subseteq X$ is a non-empty subset and $T : Y \to Y$ is a mapping. If there exists $\lambda \in [0,1)$ such that*

$$G(Tx,Ty,Ty) \leq \lambda\, G(x,y,y) \tag{6.22}$$

*for all $x \in A_j$ and $y \in A_{j+1}$ (where $j \in \{1,2,\ldots,p\}$ is arbitrary), then $T$ has a unique fixed point, which belongs to $\cap_{i=1}^p A_i$. In fact, $T$ is a Picard operator.*

*Remark 6.3.2.* In the previous result, we assume that $T : Y \to Y$ is a cyclic mapping, where $(X,G)$ is complete and $\{A_i\}_{i=1}^p$ is a closed cyclic representation of $(Y,T)$. Since each $A_i$ is closed, the finite union $Y = \cup_{i=1}^p A_i$ is also closed. As $Y$ is closed in the complete space $(X,G)$, then $(Y,G)$ is also complete. Therefore, we have a mapping $T : Y \to Y$ from a complete $G$-metric space into itself.

*Proof.* We first prove the existence part. Take an arbitrary $x_0 \in Y$. Without loss of generality, assume that $x_0 \in A_0$. Let $\{x_n\}_{n\geq 0}$ be the Picard sequence of $T$ based on $x_0$. Since $T$ is cyclic, $x_n \in A_n$ for all $n \geq 0$. If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then $x_{n_0}$ is a fixed point of $T$. Assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Put $x = x_{n+1}$ and $y = x_{n+2}$ in (6.22). Then, for all $n \geq 0$,

$$G(x_{n+1},x_{n+2},x_{n+2}) = G(Tx_n,Tx_{n+1},Tx_{n+1}) \leq \lambda\, G(x_n,x_{n+1},x_{n+1}).$$

It follows from Corollary 4.1.1 that $\{x_n\}$ is Cauchy in $(Y, G)$. Since the space $(Y, G)$ is complete, then $\{x_n\}$ is convergent in $(Y, G)$. Let $u \in Y$ be the limit of $\{x_n\}$. From Lemma 6.3.2, $u \in \cap_{i=1}^{p} A_i$. Now we show that $u$ is a fixed point of $T$, that is, $u = Tu$. Indeed, for all $n$,

$$G(x_{n+1}, Tu, Tu) = G(Tx_n, Tu, Tu) \leq \lambda G(x_n, u, u).$$

Letting $n \to \infty$, we deduce that $G(u, Tu, Tu) = 0$, that is, $Tu = u$. Finally, in order to prove the uniqueness, we assume that $v \in Y$ is another fixed point of $T$. By Lemma 6.3.1, both $u$ and $v$ belong to $\cap_{i=1}^{p} A_i$. Thus, we can substitute $x = u$ and $y = v$ in the contractivity condition (6.22), which yields $G(u, v, v) = G(Tv, Tu, Tu) \leq \lambda G(u, v, v)$. As $\lambda < 1$, then $G(u, v, v) = 0$, and $u = v$. □

**Corollary 6.3.1 ([119]).** *Let $(X, G)$ be a G-metric space and let $\{A_i\}_{i=1}^{p}$ be a closed cyclic representation of $(Y, T)$, where $Y \subseteq X$ is complete and $T : Y \to Y$ is a mapping. If there exists $\lambda \in [0, 1)$ such that*

$$G(Tx, Ty, Tz) \leq \lambda G(x, y, z)$$

*for all $x \in A_j$ and $y, z \in A_{j+1}$ (where $j \in \{1, 2, \ldots, p\}$ is arbitrary), then $T$ has a unique fixed point, which belongs to $\cap_{i=1}^{p} A_i$. In fact, $T$ is a Picard operator.*

### 6.3.2 Generalized Cyclic Weak $\phi$-Contractions (Type I)

Recall that $\mathcal{F}_{\mathrm{Kr}}$ is the family of all continuous mappings $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t) = 0$ if, and only if, $t = 0$.

**Theorem 6.3.2 ([119]).** *Let $T : X \to X$ be a mapping from a complete G-metric space $(X, G)$ into itself and let $\{A_i\}_{i=1}^{p}$ be a closed cyclic representation of $(X, T)$. Assume that there exists a function $\phi \in \mathcal{F}_{\mathrm{Kr}}$ such that*

$$G(Tx, Ty, Ty) \leq M(x, y) - \phi(M(x, y)) \tag{6.23}$$

*for all $x \in A_i$ and $y \in A_{i+1}$ ( $i \in \{1, 2, \ldots, p\}$ arbitrary), where*

$$M(x, y) = \max\{G(x, y, y), G(x, Tx, Tx), G(y, Ty, Ty)\}. \tag{6.24}$$

*Then $T$ has a unique fixed point, which belongs to $\cap_{i=1}^{p} A_i$. In fact, $T$ is a Picard operator.*

*Proof.* To prove the existence part, we construct a sequence of Picard iterations as usual. Take an arbitrary $x_0 \in A_0$ and define the sequence $\{x_n\}$ as

$$x_{n+1} = Tx_n \quad \text{for all } n \in \mathbb{N}.$$

Since $T$ is cyclic, $x_n \in A_n$ for all $n \in \mathbb{N}$. If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then $x_{n_0}$ is a fixed point of $T$. Assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$, that is,

$$G(x_n, x_{n+1}, x_{n+1}) > 0 \quad \text{for all } n \in \mathbb{N}. \tag{6.25}$$

Let $x = x_n$ and $y = x_{n+1}$ in (6.23). Then

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+2}) &= G(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ &\leq M(x_n, x_{n+1}) - \phi(M(x_n, x_{n+1})), \end{aligned} \tag{6.26}$$

where

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\{G(x_n, x_{n+1}, x_{n+1}), G(x_n, Tx_n, Tx_n), \\ &\qquad G(x_{n+1}, Tx_{n+1}, Tx_{n+1})\} \\ &= \max\{G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2})\}. \end{aligned} \tag{6.27}$$

If there exists some $n \in \mathbb{N}$ such that

$$G(x_n, x_{n+1}, x_{n+1}) \leq G(x_{n+1}, x_{n+2}, x_{n+2}),$$

then (6.26) and (6.27) yield

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+2}) &= M(x_n, x_{n+1}) - \phi(M(x_n, x_{n+1})) \\ &= G(x_{n+1}, x_{n+2}, x_{n+2}) - \phi\left(G(x_{n+1}, x_{n+2}, x_{n+2})\right), \end{aligned}$$

which implies that $\phi(G(x_{n+1}, x_{n+2}, x_{n+2})) = 0$ and, as $\phi \in \mathcal{F}_{\mathrm{Kr}}$,

$$G(x_{n+1}, x_{n+2}, x_{n+2}) = 0,$$

which contradicts (6.25). Therefore, we must have $G(x_{n+1}, x_{n+2}, x_{n+2}) < G(x_n, x_{n+1}, x_{n+1})$ for all $n \in \mathbb{N}$, that is, $M(x_n, x_{n+1}) = G(x_n, x_{n+1}, x_{n+1})$. From (6.26), for all $n \in \mathbb{N}$,

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+2}) &\leq G(x_n, x_{n+1}, x_{n+1}) - \phi(G(x_n, x_{n+1}, x_{n+1})) \\ &\leq G(x_n, x_{n+1}, x_{n+1}). \end{aligned} \tag{6.28}$$

Thus, the sequence $\{G(x_n, x_{n+1}, x_{n+1})\}$ is a non-increasing sequence of nonnegative real numbers, which converges to $L \geq 0$. As $\phi$ is continuous, letting $n \to \infty$ in (6.28), we get

$$L \leq L - \phi(L). \tag{6.29}$$

It follows that $\phi(L) = 0$, so $L = 0$, that is,

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{6.30}$$

From Lemma 3.1.1, we also have

$$\lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = 0. \tag{6.31}$$

Next, we claim that $\{x_n\}$ is a $G$-Cauchy sequence in $(X, G)$. Assume the contrary, that is, $\{x_n\}$ is not $G$-Cauchy. Then, according to Theorem 4.1.1, there exist $\varepsilon > 0$ and corresponding subsequences $\{n(k)\}$ and $\{\ell(k)\}$ of $\mathbb{N}$ satisfying $n(k) > \ell(k) > k$ for which

$$G(x_{\ell(k)}, x_{n(k)-1}, x_{n(k)-1}) \leq \varepsilon < G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}), \tag{6.32}$$

$$\lim_{k\to\infty} G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) = \lim_{k\to\infty} G(x_{\ell(k)+1}, x_{n(k)+1}, x_{n(k)+1}) = \varepsilon. \tag{6.33}$$

Observe that for every $k \in \mathbb{N}$ there exists $s(k)$ satisfying $0 \leq s(k) < p$ such that

$$n(k) - \ell(k) + s(k) = n(k) - (\ell(k) - s(k)) \equiv 1 \ (\mathrm{mod}\ p). \tag{6.34}$$

As $\{\ell(k)\} \to \infty$, for large enough values of $k$ we have that $r(k) = \ell(k) - s(k) > 0$ and, by (6.21), $x_{r(k)}$ and $x_{n(k)}$ lie in consecutive sets $A_{j_k}$ and $A_{j_k+1}$, respectively, for some $0 \leq j_k < p$. We next substitute $x = x_{r(k)}$ and $y = x_{n(k)}$ in (6.23) to obtain

$$\begin{aligned} G(x_{r(k)+1}, x_{n(k)+1}, x_{n(k)+1}) &= G(Tx_{r(k)}, Tx_{n(k)}, Tx_{n(k)}) \\ &\leq M(x_{r(k)}, x_{n(k)}) - \phi(M(x_{r(k)}, x_{n(k)})), \end{aligned} \tag{6.35}$$

where

$$\begin{aligned} M(x_{r(k)}, x_{n(k)}) = \max \big\{ &G(x_{r(k)}, x_{n(k)}, x_{n(k)}), G(x_{r(k)}, x_{r(k)+1}, x_{r(k)+1}), \\ &G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) \big\}. \end{aligned} \tag{6.36}$$

If $s(k) = 0$, then $G(x_{r(k)}, x_{n(k)}, x_{n(k)}) = G(x_{\ell(k)}, x_{n(k)}, x_{n(k)})$, and when $s(k) > 0$, by item 3 of Lemma 3.1.2,

$$\begin{aligned} &|G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) - G(x_{r(k)}, x_{n(k)}, x_{n(k)})| \\ &\quad \leq \max\{G(x_{r(k)}, x_{r(k)}, x_{\ell(k)}), G(x_{r(k)}, x_{\ell(k)}, x_{\ell(k)})\} \\ &\quad \leq 2\, G(x_{r(k)}, x_{r(k)}, x_{\ell(k)} \leq 2 \sum_{i=r(k)}^{\ell(k)-1} G(x_i, x_i, x_{i+1}). \end{aligned} \tag{6.37}$$

Notice that $\ell(k) - r(k) = s(k) < p$, so the sum on the right-hand-side of (6.37) consists of a finite number of terms, and due to (6.30) and (6.31), each term of this sum tends to 0 as $k \to \infty$. Therefore, by (6.33),

$$\lim_{k\to\infty} G(x_{r(k)}, x_{n(k)}, x_{n(k)}) = \lim_{k\to\infty} G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) = \varepsilon. \tag{6.38}$$

Repeating the previous arguments and Lemma 4.1.4, it can be deduced from (6.33) that

$$\lim_{k\to\infty} G(x_{r(k)+1}, x_{n(k)+1}, x_{n(k)+1}) = \varepsilon. \tag{6.39}$$

Now, passing to the limit as $k \to \infty$ in (6.35) and using (6.30), (6.38) and (6.39), we get

$$\varepsilon \le \max\{\varepsilon, 0, 0\} - \phi(\max\{\varepsilon, 0, 0\}) = \varepsilon - \phi(\varepsilon),$$

and, hence $\phi(\varepsilon) = 0$. We conclude that $\varepsilon = 0$, which contradicts the assumption that $\{x_n\}$ is not $G$-Cauchy. Thus, the sequence $\{x_n\}$ is $G$-Cauchy. Since $(X, G)$ is $G$-complete, it is $G$-convergent to a limit, say $w \in X$. By Lemma 6.3.2, $w \in \cap_{i=1}^{p} A_i$.

To show that the limit of the Picard sequence is the fixed point of $T$, that is, $w = Tw$, we employ (6.23) with $x = x_n$ and $y = w$. This leads to

$$G(x_{n+1}, Tw, Tw) = G(Tx_n, Tw, Tw) \le M(x_n, w) - \phi(M(x_n, w))$$

where

$$M(x_n, w) = \max\{G(x_n, w, w), G(x_n, x_{n+1}, x_{n+1}), G(w, Tw, Tw)\}.$$

As a consequence,

$$\lim_{n\to\infty} M(x_n, w) = G(w, Tw, Tw).$$

Passing to the limit as $n \to \infty$, we get

$$G(w, Tw, Tw) \le G(w, Tw, Tw) - \phi(G(w, Tw, Tw)).$$

Thus, $\phi(G(w, Tw, Tw)) = 0$ and, hence, $G(w, Tw, Tw) = 0$, that is, $w = Tw$.

Finally, we prove that the fixed point is unique. Assume that $v \in X$ is another fixed point of $T$. Then, since both $v$ and $w$ belong to $\cap_{i=1}^{p} A_i$, we set $x = v$ and $y = w$ in (6.23), which yields

$$G(v, w, w) = G(Tv, Tw, Tw) \le M(v, w) - \phi(M(v, w)), \tag{6.40}$$

where

$$M(v,w) = \max\{G(v,w,w), G(v,Tv,Tv), G(w,Tw,Tw)\} = G(v,w,w).$$

Then (6.40) becomes $G(v,w,w) \leq G(v,w,w) - \phi(G(v,w,w))$ and, clearly, $G(v,w,w) = 0$, so we conclude that $v = w$, i.e., the fixed point of $T$ is unique. □

**Corollary 6.3.2 ([119]).** *Let $T : X \to X$ be a mapping from a complete G-metric space $(X,G)$ into itself and let $\{A_i\}_{i=1}^{p}$ be a closed cyclic representation of $(X,T)$. Assume that there exists a function $\phi \in \mathcal{F}_{\mathrm{Kr}}$ such that*

$$G(Tx,Ty,Tz) \leq M(x,y,z) - \phi(M(x,y,z))$$

*for all $x \in A_i$ and $y,z \in A_{i+1}$ ( $i \in \{1,2,\ldots,p\}$ arbitrary), where*

$$M(x,y,z) = \max\{G(x,y,z), G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz)\}.$$

*Then $T$ has a unique fixed point, which belongs to $\cap_{i=1}^{p} A_i$. In fact, $T$ is a Picard operator.*

For particular choices of the function $\phi$ we obtain the following corollaries. We employ Remark 6.3.2 to give the following versions.

**Corollary 6.3.3 ([119]).** *Let $(X,G)$ be a G-complete G-metric space and let $\{A_i\}_{i=1}^{p}$ be a family of nonempty G-closed subsets of $X$ with $Y = \cup_{i=1}^{p} A_i$. Let $T : Y \to Y$ be a map satisfying*

$$T(A_j) \subseteq A_{j+1} \quad \text{for all } j \in \{1,2,\ldots p\} \quad (\text{where } A_{p+1} = A_1).$$

*Suppose that there exists $\lambda \in [0,1)$ such that the map $T$ satisfies*

$$G(Tx,Ty,Tz) \leq \lambda M(x,y,z) \tag{6.41}$$

*for all $x \in A_j$ and all $y,z \in A_{j+1}$ (for $j \in \{1,2,\ldots m\}$), where*

$$M(x,y,z) = \max\{G(x,y,z), G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz)\}. \tag{6.42}$$

*Then $T$ has a unique fixed point, which belongs to $\cap_{i=1}^{p} A_i$.*

*Proof.* The proof is obvious by choosing the function $\phi$ in Theorem 6.3.2 as $\phi(t) = (1-\lambda)\,t$. □

**Corollary 6.3.4 ([119]).** *Let $(X,G)$ be a G-complete G-metric space and let $\{A_i\}_{i=1}^{p}$ be a family of nonempty G-closed subsets of $X$ with $Y = \cup_{i=1}^{p} A_i$. Let $T : Y \to Y$ be a map satisfying*

$$T(A_j) \subseteq A_{j+1} \quad \text{for all } j \in \{1,2,\ldots p\} \quad (\text{where } A_{p+1} = A_1).$$

*Suppose that there exist constants $a, b, c$ and $d$, with $a + b + c + d < 1$, such that the map $T$ satisfies*

$$G(Tx, Ty, Tz) \leq aG(x, y, z)+bG(x, Tx, Tx)+cG(y, Ty, Ty)+dG(z, Tz, Tz) \qquad (6.43)$$

*for all $x \in A_j$ and all $y, z \in A_{j+1}$ (for $j \in \{1, 2, \dots m\}$). Then $T$ has a unique fixed point, which belongs to $\cap_{i=1}^{p} A_i$.*

*Proof.* Clearly we have,

$$\begin{aligned} &aG(x, y, z) + bG(x, Tx, Tx) + cG(y, Ty, Ty) \\ &\qquad + dG(z, Tz, Tz) \leq (a + b + c + d)M(x, y, z), \end{aligned}$$

where

$$M(x, y, z) = \max\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\}. \qquad (6.44)$$

By Corollary 6.3.3 using $\lambda = \max\{a + b + c + d, 0\} \in [0, 1)$, the map $T$ has a unique fixed point. □

*Example 6.3.1 ([119]).* Let $X = [-1, 1]$ and let $T : X \to X$ be given as $Tx = -\frac{x}{3}$ for all $x \in X$. Let $A = [-1, 0]$ and $B = [0, 1]$. Define the function $G : X \times X \times X \to [0, \infty)$, for all $x, y, z \in X$, as

$$G(x, y, z) = |x^3 - y^3| + |y^3 - z^3| + |z^3 - x^3|.$$

Clearly, the function $G$ is a $G$-metric on $X$. Define also $\phi : [0, \infty) \to [0, \infty)$ as $\phi(t) = \frac{2t}{3}$ for all $t \geq 0$. Notice that $t - \phi(t) = t/3$ for all $t \geq 0$. It can be easily shown that the map $T$ satisfies condition (6.23). Indeed, note that

$$G(Tx, Ty, Tz) = \frac{1}{27}\left(|x^3 - y^3| + |y^3 - z^3| + |z^3 - x^3|\right)$$

and

$$\begin{aligned} M(x, y, z) = \max\Big\{ &|x^3 - y^3| + |y^3 - z^3| + |z^3 - x^3|, \frac{56}{27}|x^3|, \\ &\frac{56}{27}|y^3|, \frac{56}{27}|z^3|\Big\} \end{aligned}$$

for all $x, y, z \in X$. As $M(x, y, z) - \phi(M(x, y, z)) = \frac{1}{3}M(x, y, z)$, we have that, for all $x, y, z \in X$,

$$\begin{aligned} G(Tx, Ty, Tz) &= \frac{1}{27}\left(|x^3 - y^3| + |y^3 - z^3| + |z^3 - x^3|\right) \\ &\leq \frac{1}{3}\left(|x^3 - y^3| + |y^3 - z^3| + |z^3 - x^3|\right) \leq \frac{1}{3}M(x, y, z). \end{aligned}$$

Hence, by Theorem 6.3.2, $T$ has a unique fixed point, which belong to $A \cap B = \{0\}$.

Cyclic maps satisfying integral type contractive conditions are common applications of fixed point theorems.

**Corollary 6.3.5 ([119]).** *Let $(X, G)$ be a $G$-complete $G$-metric space and let $\{A_i\}_{i=1}^{p}$ be a family of nonempty $G$-closed subsets of $X$ with $Y = \cup_{i=1}^{p} A_i$. Let $T : Y \to Y$ be a map satisfying*

$$T(A_j) \subseteq A_{j+1} \quad \text{for all } j \in \{1, 2, \ldots p\} \quad (\text{where } A_{p+1} = A_1).$$

*Suppose also that*

$$\int_0^{G(Tx,Ty,Tz)} ds \leq \int_0^{M(x,y,z)} ds - \phi\left(\int_0^{M(x,y,z)} ds\right),$$

*where $\phi \in \mathcal{F}_{\mathrm{Kr}}$ and*

$$M(x, y, z) = \max\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\}$$

*for all $x \in A_j$ and $y, z \in A_{j+1}$ (for $j \in \{1, 2, \ldots p\}$). Then $T$ has a unique fixed point, which is in $\cap_{i=1}^{p} A_i$.*

**Corollary 6.3.6 ([119]).** *Let $(X, G)$ be a $G$-complete $G$-metric space and let $\{A_i\}_{i=1}^{p}$ be a family of nonempty $G$-closed subsets of $X$ with $Y = \cup_{i=1}^{p} A_i$. Let $T : Y \to Y$ be a map satisfying*

$$T(A_j) \subseteq A_{j+1} \quad \text{for all } j \in \{1, 2, \ldots p\} \quad (\text{where } A_{p+1} = A_1).$$

*Suppose also that*

$$\int_0^{G(Tx,Ty,Tz)} ds \leq \lambda \int_0^{M(x,y,z)} ds,$$

*where $\lambda \in [0, 1)$ and*

$$M(x, y, z) = \max\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\}$$

*for all $x \in A_j$ and $y, z \in A_{j+1}$ (for $j \in \{1, 2, \ldots p\}$). Then $T$ has a unique fixed point, which is in $\cap_{i=1}^{p} A_i$.*

Taking into account the equivalence between different classes of auxiliary functions due to Jachymski [94] (recall Theorem 4.2.5) we state the following result.

**Theorem 6.3.3 ([119]).** *Let $T$ be a self-map on a G-complete G-metric space $(X,G)$ and let $\{A_i\}_{i=1}^{p}$ be a family of nonempty G-closed subsets of $X$ with $Y = \cup_{i=1}^{p} A_i$. Let $T : Y \to Y$ be a map satisfying*

$$T(A_j) \subseteq A_{j+1} \quad \text{for all } j \in \{1,2,\dots p\} \quad (\text{where } A_{p+1} = A_1).$$

*Assume that*

$$M(x,y,z) = \max\{G(x,y,z), G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz)\}.$$

*Then the following statements are equivalent.*

$(i)$ *There exist functions $\psi, \phi \in \mathcal{F}_{\text{alt}}$ such that*

$$\psi(G(Tx,Ty,Tz)) \leq \psi(M(x,y,z)) - \phi(M(x,y,z)),$$

*for any $x \in A_j$ and $y,z \in A_{j+1}$ (for $j \in \{1,2,\dots p\}$).*

$(ii)$ *There exists a function $\beta : [0,\infty) \to [0,1]$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\{\beta(t_n)\} \to 1$ implies $\{t_n\} \to 0$, and*

$$G(Tx,Ty,Tz) \leq \beta(M(x,y,z))\,\psi(M(x,y,z)),$$

*for any $x \in A_j$ and $y,z \in A_{j+1}$ (for $j \in \{1,2,\dots p\}$).*

$(iii)$ *There exists a continuous function $\eta : [0,\infty) \to [0,\infty)$ such that $\eta^{-1}(\{0\}) = 0$ and*

$$G(Tx,Ty,Tz) \leq M(x,y,z) - \eta(M(x,y,z))$$

*for any $x \in A_j$ and $y,z \in A_{j+1}$ (for $j \in \{1,2,\dots p\}$).*

$(iv)$ *there exists function $\psi \in \mathcal{F}_{\text{alt}}$ and a non-decreasing, right continuous function $\varphi : [0,\infty) \to [0,\infty)$, with $\varphi(t) < t$ for all $t > 0$, with*

$$\psi(G(Tx,Ty,Tz)) \leq \varphi(\psi(M(x,y,z)))$$

*for any $x \in A_j$ and $y,z \in A_{j+1}$ (for $j \in \{1,2,\dots p\}$).*

$(v)$ *There exists a continuous and non-decreasing function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) < t$ for all $t > 0$, with*

$$\psi(G(Tx,Ty,Tz)) \leq \varphi(M(x,y,z))$$

*for any $x \in A_j$ and $y,z \in A_{j+1}$ (for $j \in \{1,2,\dots p\}$).*

### 6.3.3 Generalized Cyclic Weak $\phi$-Contractions (type II)

We start this subsection by recalling some sets of auxiliary functions. Note

$$\begin{aligned}\mathcal{F}_{\text{alt}} = \{\phi : [0,\infty) \to [0,\infty) :\ &\phi \text{ continuous, non-decreasing,}\\ &\phi(t) = 0 \Leftrightarrow t = 0\},\end{aligned}$$

$$\mathcal{F}'_{\text{alt}} = \{\phi : [0,\infty) \to [0,\infty) :\ \phi \text{ lower semi-continuous, } \phi(t) = 0 \Leftrightarrow t = 0\}\,.$$

Using these control functions, we present the following results, which were inspired from [46].

**Theorem 6.3.4.** *Let $T : X \to X$ be a mapping from a complete G-metric space $(X,G)$ into itself and let $\{A_i\}_{i=1}^p$ be a closed cyclic representation of $(X,T)$. Assume that there exist functions $\psi \in \mathcal{F}_{\text{alt}}$ and $\phi \in \mathcal{F}'_{\text{alt}}$ such that the map $T$ satisfies the inequality*

$$\psi(G(Tx,Ty,Ty)) \le \psi(M(x,y)) - \phi(M(x,y)) \tag{6.45}$$

*for all $x \in A_j$ and all $y \in A_{j+1}$ ( $i \in \{1,2,\ldots,p\}$ arbitrary), where*

$$\begin{aligned}M(x,y) = \max\Bigg\{&G(x,y,y), G(x,Tx,Tx), G(y,Ty,Ty),\\ &\frac{G(x,y,Tx)}{2}, \frac{2G(x,Ty,Ty) + G(y,Tx,Tx)}{4},\\ &\frac{G(x,Ty,Ty) + 2G(y,Tx,Tx)}{5}\Bigg\}\,.\end{aligned} \tag{6.46}$$

*Then $T$ has a unique fixed point, which belongs to $\cap_{i=1}^p A_i$. In fact, $T$ is a Picard operator.*

*Proof.* First, we show the existence of a fixed point of the map $T$. For this purpose, let $\{x_n\}$ be the Picard sequence of $T$ based on an arbitrary point $x_0 \in A_0$, that is,

$$x_{n+1} = Tx_n \quad \text{for all } n \in \mathbb{N}. \tag{6.47}$$

Since $T$ is cyclic, $x_n \in A_n$ for all $n \in \mathbb{N}$. If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then $x_{n_0}$ is a fixed point of $T$. Assume that $x_{n+1} \ne x_n$ for all $n \in \mathbb{N}$, that is,

$$G(x_n, x_{n+1}, x_{n+1}) > 0 \quad \text{for all } n \in \mathbb{N}. \tag{6.48}$$

Let $x = x_n$ and $y = x_{n+1}$ in (6.45). Then

$$\begin{aligned}\psi(G(x_{n+1}, x_{n+2}, x_{n+2})) &= \psi(G(Tx_n, Tx_{n+1}, Tx_{n+1})) \\ &\leq \psi(M(x_n, x_{n+1})) - \phi(M(x_n, x_{n+1})),\end{aligned} \tag{6.49}$$

where

$$\begin{aligned}M(x_n, x_{n+1}) = \max\Bigg\{&G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2}), \\ &\frac{G(x_n, x_{n+1}, x_{n+1})}{2}, \frac{2G(x_n, x_{n+2}, x_{n+2})}{4}, \frac{G(x_n, x_{n+2}, x_{n+2})}{5}\Bigg\}.\end{aligned} \tag{6.50}$$

Since

$$\begin{aligned}\frac{G(x_n, x_{n+2}, x_{n+2})}{5} &\leq \frac{2G(x_n, x_{n+2}, x_{n+2})}{4} \\ &\leq \frac{G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2})}{2} \\ &\leq \max\left\{G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2})\right\},\end{aligned}$$

then (6.50) becomes

$$M(x_n, x_{n+1}) = \max\left\{G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2})\right\}$$

for all $n \in \mathbb{N}$. If there exists some $n \in \mathbb{N}$ such that $G(x_n, x_{n+1}, x_{n+1}) \leq G(x_{n+1}, x_{n+2}, x_{n+2})$, then (6.49) yields

$$\psi(G(x_{n+1}, x_{n+2}, x_{n+2})) \leq \psi(G(x_{n+1}, x_{n+2}, x_{n+2})) - \phi(G(x_{n+1}, x_{n+2}, x_{n+2})),$$

which implies that $\phi(G(x_{n+1}, x_{n+2}, x_{n+2})) = 0$ and, as $\phi \in \mathcal{F}'_{\text{alt}}$,

$$G(x_{n+1}, x_{n+2}, x_{n+2}) = 0,$$

which contradicts (6.48). Therefore, we must have $G(x_{n+1}, x_{n+2}, x_{n+2}) < G(x_n, x_{n+1}, x_{n+1})$ for all $n \in \mathbb{N}$, that is, $M(x_n, x_{n+1}, x_{n+1}) = G(x_n, x_{n+1}, x_{n+1})$. From (6.49),

$$\psi(G(x_{n+1}, x_{n+2}, x_{n+2})) \leq \psi(G(x_n, x_{n+1}, x_{n+1})) - \phi(G(x_n, x_{n+1}, x_{n+1})).$$

From Lemma 2.3.6,

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0, \tag{6.51}$$

that is, $\{x_n\}$ is an asymptotically regular sequence.

Next, we show that $\{x_n\}$ is a $G$-Cauchy sequence in $(X, G)$. Suppose, on the contrary, that $\{x_n\}$ is not $G$-Cauchy. Reasoning as in the proof of Theorem 6.3.2, there exists $\varepsilon > 0$ and corresponding subsequences $\{n(k)\}$ and $\{\ell(k)\}$ of $\mathbb{N}$ satisfying $n(k) > \ell(k) > k$ for which (see Theorem 4.1.1)

$$\begin{aligned}
&G(x_{\ell(k)}, x_{n(k)-1}, x_{n(k)-1}) \leq \varepsilon < G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}),\\
&\lim_{k\to\infty} G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) = \varepsilon, && (6.52)\\
&0 \leq s(k) < p, \quad n(k) - \ell(k) + s(k) \equiv 1 \ (\mathrm{mod}\ p). && (6.53)
\end{aligned}$$

As $\{\ell(k)\} \to \infty$, for large enough values of $k$ we have that $r(k) = \ell(k) - s(k) > 0$. Applying item 3 of Lemma 3, we have that

$$\begin{aligned}
&\left| G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) - G(x_{r(k)}, x_{n(k)}, x_{n(k)}) \right|\\
&\quad \leq \max\{G(x_{\ell(k)}, x_{r(k)}, x_{r(k)}), G(x_{\ell(k)}, x_{\ell(k)}, x_{r(k)})\}\\
&\quad \leq 2\, G(x_{r(k)}, x_{\ell(k)}, x_{\ell(k)})\\
&\quad \leq 2 \sum_{i=r(k)}^{\ell(k)-1} G(x_i, x_{i+1}, x_{i+1}).
\end{aligned}$$

Taking into account that $\ell(k) - r(k) = s(k) \in \{0, 1, \ldots, p-1\}$, the previous sum has not more than $p$ terms. As $\{x_n\}$ is asymptotically regular, the right-hand term of the previous inequality tends to zero as $k \to \infty$. Therefore, using (6.52), we deduce that

$$\lim_{k\to\infty} G(x_{r(k)}, x_{n(k)}, x_{n(k)}) = \varepsilon.$$

It follows from Lemma 4.1.4 that, for all given $p_1, p_2, p_3 \in \mathbb{Z}$,

$$\lim_{k\to\infty} G(x_{r(k)+p_1}, x_{n(k)+p_2}, x_{n(k)+p_3}) = \varepsilon. \tag{6.54}$$

By (6.53) and (6.21), $x_{r(k)}$ and $x_{n(k)}$ lie in consecutive sets $A_{j_k}$ and $A_{j_k+1}$, respectively, for some $0 \leq j_k < p$. We next substitute $x = x_{r(k)}$ and $y = x_{n(k)}$ in (6.45) to obtain

$$\begin{aligned}
\psi\left(G(x_{r(k)+1}, x_{n(k)+1}, x_{n(k)+1})\right) &= \psi\left(G(Tx_{r(k)}, Tx_{n(k)}, Tx_{n(k)})\right)\\
&\leq \psi\left(M(x_{r(k)}, x_{n(k)})\right) - \phi(M(x_{r(k)}, x_{n(k)})), && (6.55)
\end{aligned}$$

where

$$M(x_{r(k)}, x_{n(k)}) = \max\Bigg\{G(x_{r(k)}, x_{n(k)}, x_{n(k)}), G(x_{r(k)}, x_{r(k)+1}, x_{r(k)+1}),$$
$$G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}), \frac{G(x_{r(k)}, x_{n(k)}, x_{r(k)+1})}{2},$$
$$\frac{2G(x_{r(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)}, x_{r(k)+1}, x_{r(k)+1})}{4},$$
$$\frac{G(x_{r(k)}, x_{n(k)+1}, x_{n(k)+1}) + 2\,G(x_{n(k)}, x_{r(k)+1}, x_{r(k)+1})}{5}\Bigg\}.$$

The first term of the previous maximum tends to $\varepsilon$ as $k \to \infty$. Notice also

$$\begin{aligned}
&\frac{G(x_{r(k)}, x_{n(k)}, x_{r(k)+1})}{2} \\
&\quad\le \frac{G(x_{r(k)}, x_{r(k)+1}, x_{r(k)+1}) + G(x_{r(k)+1}, x_{r(k)+1}, x_{n(k)})}{2} \\
&\quad\le \frac{G(x_{r(k)}, x_{r(k)+1}, x_{r(k)+1}) + 2\,G(x_{r(k)+1}, x_{n(k)}, x_{n(k)})}{2} \to \frac{0 + 2\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Also note

$$\begin{aligned}
&\frac{2G(x_{r(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)}, x_{r(k)+1}, x_{r(k)+1})}{4} \\
&\quad\le \frac{2G(x_{r(k)}, x_{n(k)+1}, x_{n(k)+1}) + 2\,G(x_{r(k)+1}, x_{n(k)}, x_{n(k)})}{4} \\
&\quad\to \frac{2\varepsilon + 2\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\frac{G(x_{r(k)}, x_{n(k)+1}, x_{n(k)+1}) + 2\,G(x_{n(k)}, x_{r(k)+1}, x_{r(k)+1})}{5} \\
&\quad\le \frac{G(x_{r(k)}, x_{n(k)+1}, x_{n(k)+1}) + 4\,G(x_{r(k)+1}, x_{n(k)}, x_{n(k)})}{5} \to \frac{\varepsilon + 4\varepsilon}{5} = \varepsilon.
\end{aligned}$$

Therefore,

$$\lim_{k\to\infty} M(x_{r(k)}, x_{n(k)}) = \varepsilon.$$

This proves that the sequences

$$\left\{ t_k = G(x_{r(k)+1}, x_{n(k)+1}, x_{n(k)+1}) \right\}_{k\in\mathbb{N}} \quad \text{and}$$
$$\left\{ s_k = M(x_{r(k)}, x_{n(k)}) \right\}_{k\in\mathbb{N}}$$

converge to the same limit $L = \varepsilon$, and they satisfy $\psi(t_k) \leq \psi(s_k) - \phi(s_k)$ for all $k \in \mathbb{N}$ (at least, for large enough values of $k$). As a consequence, Corollary 2.3.2 guarantees that $\varepsilon = 0$, which is a contradiction. This contradicts the assumption that $\{x_n\}$ is not $G$-Cauchy. As a result, the sequence $\{x_n\}$ is Cauchy in $(X, G)$. Since $(X, G)$ is complete, it is $G$-convergent to a limit, say $w \in X$. By Lemma 6.3.2, $w \in \cap_{i=1}^{p} A_i$.

To show that the limit of the Picard sequence is the fixed point of $T$, that is, $w = Tw$, we can employ (6.45) with $x = x_n$ and $y = w$. This leads to

$$\begin{aligned}\psi(G(x_{n+1}, Tw, Tw)) &= \psi(G(Tx_n, Tw, Tw)) \\ &\leq \psi(M(x_n, w)) - \phi(M(x_n, w)),\end{aligned} \tag{6.56}$$

where

$$\begin{aligned}M(x_n, w) = \max\Bigg\{ & G(x_n, w, w), G(x_n, x_{n+1}, x_{n+1}), G(w, Tw, Tw), \\ & \frac{G(x_n, w, x_{n+1})}{2}, \frac{2G(x_n, Tw, Tw) + G(w, x_{n+1}, x_{n+1})}{4}, \\ & \frac{G(x_n, Tw, Tw) + 2G(w, x_{n+1}, x_{n+1})}{5} \Bigg\}.\end{aligned}$$

Since $G$ is continuous,

$$\lim_{n\to\infty} M(x_n, w) = G(w, Tw, Tw).$$

Using Corollary 2.3.2 again applied to the sequences

$$\{ t_k = G(x_{n+1}, Tw, Tw) \}_{k\in\mathbb{N}}, \ \{ s_k = M(x_n, w) \}_{k\in\mathbb{N}}$$

we deduce that $G(w, Tw, Tw) = 0$, that is, $Tw = w$.

Finally, we prove that the fixed point of $T$ is unique. Assume that $v \in X$ is another fixed point of $T$ such that $v \neq w$. Then, since both $v$ and $w$ belong to $\bigcap_{j=1}^{m} A_j$, we set $x = v$ and $y = w$ in (6.45) which yields

$$\begin{aligned}\psi(G(v, w, w)) &= \psi(G(Tv, Tw, Tw)) \\ &\leq \psi(M(v, w)) - \phi(M(v, w)),\end{aligned} \tag{6.57}$$

where

$$
\begin{aligned}
M(v,w) = \max \Bigg\{ & G(v,w,w), G(v,Tv,Tv), G(w,Tw,Tw), \\
& \frac{G(v,w,Tv)}{2}, \frac{2G(v,Tw,Tw)+G(w,Tv,Tv)}{4}, \\
& \frac{G(v,Tw,Tw)+2G(w,Tv,Tv)}{5} \Bigg\} \\
= \max \Bigg\{ & G(v,w,w), \frac{G(v,w,v)}{2}, \frac{2G(v,w,w)+G(w,v,v)}{4} \\
& \frac{G(v,w,w)+2G(w,v,v)}{5} \Bigg\} \\
\leq \max & \{G(v,w,w), G(w,v,v)\}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\psi(G(w,v,v)) &= \psi(G(Tw,Tv,Tv)) \\
&\leq \psi(M(w,v)) - \phi(M(w,v)),
\end{aligned}
$$

where

$$
\begin{aligned}
M(w,v) = \max \Bigg\{ & G(w,v,v), \frac{G(w,v,w)}{2}, \\
& \frac{2G(w,v,v)+G(v,w,w)}{4}, \frac{G(w,v,v)+2G(v,w,w)}{5} \Bigg\} \\
\leq \max & \{G(w,v,v), G(v,w,w)\}.
\end{aligned}
$$

As $\psi$ is non-decreasing,

$$
\begin{aligned}
& \psi\{\max\{G(v,w,w), G(w,v,v)\}\} \\
&\quad = \max\{\psi(G(v,w,w)), \psi(G(w,v,v))\} \\
&\quad \leq \max\{\psi(M(v,w)), \psi(M(w,v))\} \\
&\qquad\qquad - \min\{\phi(M(v,w)), \phi(M(w,v))\} \\
&\quad \leq \psi\{\max\{G(v,w,w), G(w,v,v)\}\}.
\end{aligned}
$$

As a result,

$$
\min\{\phi(M(v,w)), \phi(M(w,v))\} = 0,
$$

so $\phi(M(v,w)) = 0$ or $\phi(M(w,v)) = 0$. Hence, $M(v,w) = 0$ or $M(w,v) = 0$. In any case, we deduce that $v = w$, that is, $T$ has a unique fixed point. ◻

To illustrate Theorem 6.3.4, we give the following example.

*Example 6.3.2 ([46]).* Let $X = [-1, 1]$ and let $T : X \to X$ be given as $Tx = \frac{-x}{8}$ for all $x \in X$. Let $A = [-1, 0]$ and $B = [0, 1]$. Let $G : X \times X \times X \to [0, \infty)$ be the $G$-metric

$$G(x, y, z) = |x - y| + |y - z| + |z - x| \quad \text{for all } x, y, z \in X. \tag{6.58}$$

Define also $\psi, \phi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{t}{2}$ and $\phi(t) = \frac{t}{8}$ for all $t \in [0, \infty)$. Obviously, the map $T$ has a unique fixed point, which is $u = 0 \in A \cap B$.

It can be easily shown that the map $T$ satisfies the contractivity condition (6.45). Indeed, for all $x, y \in X$,

$$G(Tx, Ty, Ty) = 2\,|Tx - Ty| = \frac{|y - x|}{4},$$

which yields that

$$\psi(G(Tx, Ty, Ty)) = \frac{|y - x|}{8}. \tag{6.59}$$

Moreover, as $G(x, y, y) = 2\,|x - y|$, then

$$M(x, y) \geq G(x, y, y) = 2\,|x - y|.$$

On the other hand, we have the following inequality

$$\begin{aligned} \psi(M(x, y)) - \phi(M(x, y)) &= \frac{M(x, y)}{2} - \frac{M(x, y)}{8} \\ &= \frac{3M(x, y)}{8}. \end{aligned} \tag{6.60}$$

From an elementary calculation, we conclude that

$$\frac{3\,|x - y|}{4} \leq \frac{3\,M(x, y)}{8} = \psi(M(x, y)) - \phi(M(x, y)). \tag{6.61}$$

Combining the expressions (6.59) and (6.60) we obtain that

$$\begin{aligned} \psi(G(Tx, Ty, Ty)) &= \frac{|y - x|}{8} \leq \frac{3\,|x - y|}{4} \leq \frac{3\,M(x, y)}{8} \\ &= \psi(M(x, y)) - \phi(M(x, y)). \end{aligned} \tag{6.62}$$

Hence, all the conditions of Theorem 6.3.4 are satisfied. Notice that $u = 0$ is the unique fixed point of $T$.

For particular choices of the functions $\phi$ and $\psi$, we obtain the following corollaries.

**Corollary 6.3.7.** *Let $(X, G)$ be a G-complete G-metric space and let $\{A_i\}_{i=1}^p$ be a family of nonempty G-closed subsets of X with $Y = \cup_{i=1}^p A_i$. Let $T : Y \to Y$ be a map satisfying*

$$T(A_i) \subseteq A_{i+1} \quad \text{for all } i \in \{1, 2, \dots, p\} \quad (\text{where } A_{p+1} = A_1). \tag{6.63}$$

*Suppose that there exist a constant $\lambda \in [0, 1)$ such that the map T satisfies*

$$G(Tx, Ty, Ty) \leq \lambda\, M(x, y) \tag{6.64}$$

*for all $x \in A_j$ and $y \in A_{j+1}$ (for some $j \in \{1, 2, \dots, p\}$), where*

$$\begin{aligned} M(x, y) = \max \Bigg\{ & G(x, y, y), G(x, Tx, Tx), G(y, Ty, Ty), \\ & \frac{G(x, y, Tx)}{2}, \frac{2G(x, Ty, Ty) + G(y, Tx, Tx)}{4}, \\ & \frac{G(x, Ty, Ty) + 2G(y, Tx, Tx)}{5} \Bigg\}. \end{aligned} \tag{6.65}$$

*Then T has a unique fixed point, which belongs to $\cap_{i=1}^p A_i$.*

*Proof.* This is a particular case of Theorem 6.3.4 choosing the functions $\psi$ and $\phi$ as $\psi(t) = t$ and $\phi(t) = (1 - \lambda)\, t$ for all $t \in [0, \infty)$. □

**Corollary 6.3.8.** *Let $(X, G)$ be a G-complete G-metric space and let $\{A_i\}_{i=1}^p$ be a family of nonempty G-closed subsets of X with $Y = \cup_{i=1}^p A_i$. Let $T : Y \to Y$ be a map satisfying*

$$T(A_i) \subseteq A_{i+1} \quad \text{for all } i \in \{1, 2, \dots, p\} \quad (\text{where } A_{p+1} = A_1).$$

*Suppose that there exist real constants a, b, c, d, e and f, with $a+b+c+d+e+f < 1$, such that the map T satisfies the inequality*

$$\begin{aligned} G(Tx, Ty, Ty) \leq\ & a\, G(x, y, y) + b\, G(x, Tx, Tx) + c\, G(y, Ty, Ty) \\ & + \frac{d}{2}\, G(x, y, Tx) + \frac{e}{4}\, (2G(x, Ty, Ty) + G(y, Tx, Tx)) \\ & + \frac{f}{5}\, (G(x, Ty, Ty) + 2G(y, Tx, Tx)) \end{aligned}$$

*for all $x \in A_j$ and $y \in A_{j+1}$ (for some $j \in \{1, 2, \dots, p\}$). Then T has a unique fixed point, which belongs to $\cap_{i=1}^p A_i$.*

*Proof.* Let $\lambda = \max\{a+b+c+d+e+f, 0\} \in [0, 1)$. Clearly we have that

$$\begin{aligned} G(Tx, Ty, Ty) &\leq a\, G(x, y, y) + b\, G(x, Tx, Tx) + c\, G(y, Ty, Ty) \\ &\quad + \frac{d}{2}\, G(x, y, Tx) + \frac{e}{4}\, (2G(x, Ty, Ty) + G(y, Tx, Tx)) \\ &\quad + \frac{f}{5}\, (G(x, Ty, Ty) + 2G(y, Tx, Tx)) \\ &\leq (a+b+c+d+e+f)\ M(x, y) \leq \lambda\, M(x, y) \end{aligned}$$

where $M(x, y)$ was given in (6.65). By Corollary 6.3.7, the map $T$ has a unique fixed point. □

**Corollary 6.3.9.** *Let $(X, G)$ be a G-complete G-metric spaces and let $\{A_i\}_{i=1}^p$ be a family of nonempty G-closed subsets of $X$ with $Y = \cup_{i=1}^p A_i$. Let $T : Y \to Y$ be a map satisfying*

$$T(A_i) \subseteq A_{i+1} \quad \text{for all } i \in \{1, 2, \ldots, p\} \quad (\text{where } A_{p+1} = A_1).$$

*Suppose that there exist functions $\psi \in \mathcal{F}_{\text{alt}}$ and $\phi \in \mathcal{F}'_{\text{alt}}$ such that the map $T$ satisfies the inequality*

$$\psi(G(Tx, Ty, Tz)) \leq \psi(M(x, y, z)) - \phi(M(x, y, z))$$

*for all $x \in A_j$ and $y, z \in A_{j+1}$ (for some $j \in \{1, 2, \ldots, p\}$), where*

$$\begin{aligned} M(x, y, z) = \max \Big\{ &G(x, y, y), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz), \\ &\frac{G(x, y, Tx)}{2}, \frac{G(x, Ty, Ty) + G(x, Tz, Tz) + G(y, Tx, Tx)}{4}, \\ &\frac{G(x, Ty, Ty) + G(y, Tx, Tx) + G(z, Tx, Tx)}{5} \Big\}. \end{aligned} \tag{6.66}$$

*Then $T$ has a unique fixed point, which belongs to $\cap_{i=1}^p A_i$.*

*Proof.* The expression (6.66) coincides with the expression (6.46) when $y = z$. Following the proof in Theorem 6.3.4, by letting $x = x_n$ and $y = z = x_{n+1}$, we get the desired result. □

**Corollary 6.3.10.** *Let $(X, G)$ be a G-complete G-metric space and let $\{A_i\}_{i=1}^p$ be a family of nonempty G-closed subsets of $X$ with $Y = \cup_{i=1}^p A_i$. Let $T : Y \to Y$ be a map satisfying*

$$T(A_i) \subseteq A_{i+1} \quad \text{for all } i \in \{1, 2, \ldots, p\} \quad (\text{where } A_{p+1} = A_1).$$

*Suppose also that there exist functions $\psi \in \mathcal{F}_{\rm alt}$ and $\phi \in \mathcal{F}'_{\rm alt}$ such that the map $T$ satisfies*

$$\psi\left(\int_0^{G(Tx,Ty,Ty)} ds\right) \leq \psi\left(\int_0^{M(x,y)} ds\right) - \phi\left(\int_0^{M(x,y)} ds\right),$$

*for all $x \in A_j$ and $y \in A_{j+1}$ (for some $j \in \{1, 2, \ldots, p\}$), where*

$$\begin{aligned} M(x,y) = \max\Bigg\{ & G(x,y,y), G(x,Tx,Tx), G(y,Ty,Ty), \\ & \frac{G(x,y,Tx)}{2}, \frac{2G(x,Ty,Ty)+G(y,Tx,Tx)}{4}, \\ & \frac{G(x,Ty,Ty)+2G(y,Tx,Tx)}{5}\Bigg\}. \end{aligned}$$

*Then $T$ has a unique fixed point, which belongs to $\cap_{i=1}^{p} A_i$.*

**Corollary 6.3.11.** *Let $(X, G)$ be a $G$-complete $G$-metric space and let $\{A_i\}_{i=1}^{p}$ be a family of nonempty $G$-closed subsets of $X$ with $Y = \cup_{i=1}^{p} A_i$. Let $T : Y \to Y$ be a map satisfying*

$$T(A_i) \subseteq A_{i+1} \quad \text{for all } i \in \{1, 2, \ldots, p\} \quad (\text{where } A_{p+1} = A_1).$$

*Suppose also that there exists a constant $\lambda \in [0, 1)$ such that the map $T$ satisfies*

$$\int_0^{G(Tx,Ty,Ty)} ds \leq \lambda \int_0^{M(x,y)} ds,$$

*for all $x \in A_j$ and $y \in A_{j+1}$ (for some $j \in \{1, 2, \ldots, p\}$), where*

$$\begin{aligned} M(x,y) = \max\Bigg\{ & G(x,y,y), G(x,Tx,Tx), G(y,Ty,Ty), \\ & \frac{G(x,y,Tx)}{2}, \frac{2G(x,Ty,Ty)+G(y,Tx,Tx)}{4}, \\ & \frac{G(x,Ty,Ty)+2G(y,Tx,Tx)}{5}\Bigg\}. \end{aligned}$$

*Then $T$ has a unique fixed point, which belongs to $\cap_{i=1}^{p} A_i$.*

## 6.4 On Common Fixed Points in $G$-Metric Spaces Using the (E.A) Property

In 2002, Aamri and El Moutawakil [2] defined the *(E.A) property* (which generalizes the concept of compatible mappings) and proved some common fixed point theorems. In this section, we consider types of commuting mappings on $G$-metric spaces (called $G$-weakly commuting of type $G_T$ and $G$-$\lambda$-weakly commuting of type $G_T$) and we obtain several common fixed point results using the (E.A) property.

### 6.4.1 *Weakly Compatibility and Weakly Commutativity*

First of all, we recall here the notion of weakly compatibility.

**Definition 6.4.1.** A pair $(T, g)$ of self-mappings $T, g : X \to X$ is called *weakly compatible* if they commute at their coincidence points, that is,

$$Tx = gx \quad \Rightarrow \quad Tgx = gTx. \tag{6.67}$$

We note that we should not confuse the notions of *coincidence point* and *point of coincidence*.

**Definition 6.4.2.** Following Abbas and Rhoades [10]:

- a *coincidence point* of two self-mappings $T$ and $g$ is a point $x \in X$ such that $Tx = gx$;
- if $x$ is a coincidence point of $T$ and $g$, then $w = Tx = gx$ is a *point of coincidence of* $T$ and $g$.

In this sense, we have the following property.

**Proposition 6.4.1 ([10]).** *Let $T$ and $g$ be weakly compatible self-maps of a set $X$. If $T$ and $g$ have a unique point of coincidence $\omega = Tx = gx$, then $\omega$ is the unique common fixed point of $T$ and $g$.*

*Proof.* First, we prove that $\omega$ is a common fixed point of $T$ and $g$. Using (6.67), $T\omega = Tgx = gTx = g\omega$. Then, $\omega' = T\omega = g\omega$ is another point of coincidence of $T$ and $g$. As we assume that $\omega$ is the unique point of coincidence of $T$ and $g$, then $\omega = \omega'$, which means that $\omega = \omega' = T\omega = g\omega$. Hence, $\omega$ is a common fixed point of $T$ and $g$. Assume that $z \in X$ is another common fixed point of $T$ and $g$, that is, $z = Tz = gz$. As $z$ is a coincidence point of $T$ and $g$, then $z = Tz = gz$ is a point of coincidence of $T$ and $g$. As it is unique, then $z = \omega$, so $\omega$ is unique. $\square$

We introduce two different kinds of commutativity between mappings in the context of $G$-metric spaces.

**Definition 6.4.3 ([147]).** A pair $(T, g)$ of self-mappings $T, g : X \to X$ from a $G$-metric space $(X, G)$ into itself is said to be *$G$-weakly commuting of type $G_T$* if

$$G(Tgx, gTx, TTx) \leq G(Tx, gx, Tx) \quad \text{for all } x \in X. \tag{6.68}$$

**Definition 6.4.4 ([147]).** Given $\lambda \geq 0$, a pair $(T, g)$ of self-mappings $T, g : X \to X$ from a G-metric space $(X, G)$ into itself is said to be *$G$-$\lambda$-weakly commuting of type $G_T$* if

$$G(Tgx, gTx, TTx) \leq \lambda\, G(Tx, gx, Tx) \quad \text{for all } x \in X. \tag{6.69}$$

It is clear that a $G$-weakly commuting pair of type $G_T$ is also a $G$-1-weakly commuting of type $G_T$. If $\lambda \leq 1$, then every $G$-$\lambda$-weakly commuting pair of type $G_T$ is also a $G$-weakly commuting pair of type $G_T$. Notice that the roles of $T$ and $g$ in the previous notions are not symmetric, and pairs of type $G_T$ are different to pairs of type $G_g$.

*Remark 6.4.1.* Notice that if $g$ is the identity mapping on $X$, then all pairs $(T, I_X)$ are weakly compatible. However, they are not necessarily $G$-weakly commuting of type $G_T$ nor $G$-$\lambda$-weakly commuting of type $G_T$.

*Example 6.4.1 ([147]).* Let $X = [0, 2]$ be endowed with the $G$-metric

$$G(x, y, z) = |x - y| + |y - z| + |x - z|$$

for all $x, y, z \in X$. Define $Tx = 2 - x$ and $gx = x$ for all $x \in X$. Then, from an easy calculation, one can show that $G(Tgx, gTx, TTx) = 4|x - 1|$ and $G(Tx, gx, Tx) = 4|x - 1|$. Hence, the pair $(T, g)$ is $G$-weakly commuting of type $G_T$ and $G$-1-weakly commuting of type $G_T$.

*Example 6.4.2.* Let $X = [1, 3]$ be endowed with the $G$-metric

$$G(x, y, z) = |x - y| + |y - z| + |x - z|$$

for all $x, y, z \in X$. Define $Tx = \frac{1}{2}x + 1$ and $gx = \frac{2}{3}x + 1$ for all $x \in X$. Then, for $x = 1$, we see that $G(Tgx, gTx, TTx) = \frac{1}{2}$ and $G(Tx, gx, Tx) = \frac{1}{3}$. Therefore, the pair $(T, g)$ is not $G$-weakly commuting of type $G_T$. However, it is $G$-$\lambda$-weakly commuting of type $G_T$ for $\lambda \geq \frac{3}{2}$.

The following example shows that a $G$-weakly commuting pair of type $G_T$ does not need to be $G$-weakly commutative of type $G_g$.

*Example 6.4.3 ([147]).* Let $X = [0, 1]$ be endowed with the $G$-metric

$$G(x, y, z) = \max\{|x - y, |y - z|, |x - z|\}$$

for all $x, y, z \in X$. Define $Tx = \frac{1}{4}x^2$ and $gx = x^2$ for all $x \in X$. Then

$$G(Tgx, gTx, TTx) = \frac{15}{64}x^4 \leq \frac{3}{4}x^2 = G(Tx, gx, Tx)$$

for all $x \in X$, so $(T, g)$ is $G$-weakly commuting of type $G_T$. However, for $x = 1$, we have

$$G(gT1, Tg1, gg1) = \frac{15}{16} > \frac{3}{4} = G(g1, T1, g1),$$

which means that the pair $(T, g)$ is not $G$-weakly commuting of type $G_g$.

**Lemma 6.4.1 ([147]).** *If $(T, g)$ is a $G$-weakly commuting pair of type $G_T$ (or a $G$-$\lambda$-weakly commuting pair of type $G_T$), then $T$ and $g$ are weakly compatible.*

*Proof.* Let $x$ be a coincidence point of $T$ and $g$, that is, $Tx = gx$. If the pair $(T, g)$ is $G$-weakly commuting of type $G_T$, then we have

$$G(Tgx, gTx, Tgx) = G(Tgx, gTx, TTx) \leq G(Tx, gx, Tx) = 0.$$

It follows that $Tgx = gTx$, so $T$ and $g$ are weakly compatible. If $(T, g)$ is $G$-$\lambda$-weakly commuting of type $G_T$, then

$$G(Tgx, gTx, Tgx) = G(Tgx, gTx, TTx) \leq \lambda\, G(Tx, gx, Tx) = 0,$$

and the same conclusion holds. □

The converse of Lemma 6.4.1 fails (for the case of $G$-weakly commutativity). The following example confirms this statement.

*Example 6.4.4 ([147]).* Let $X = [1, \infty)$ and $G(x, y, z) = |x-y| + |y-z| + |x-z|$ for all $x, y, z \in X$. Define $T, g : X \to X$ by $Tx = 2x - 1$ and $gx = x^2$ for all $x \in X$. We can see that $x = 1$ is the only coincidence point and, at this point, $Tg1 = T1 = 1$ and $gT1 = g1 = 1$. Therefore, $T$ and $g$ are weakly compatible. However, using $x = 2$ we have that

$$G(Tg2, gT2, TT2) = 8 > 2 = G(T2, g2, T2).$$

Therefore, $T$ and $g$ are not $G$-weakly commuting of type $G_T$.

**Definition 6.4.5.** Given a sequence $S = \{x_n\}_{n\geq 0}$ of elements of a set $X$, let, for all $n, m \in \mathbb{N}$,

$$O(x_n, m, S) = \{x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+m}\},$$
$$O(x_n, \infty, S) = \{x_n, x_{n+1}, x_{n+2}, \ldots\}.$$

We avoid using the notation $O(x_0, \infty)$ because it is confusing: the reader could think that $O(x_0, \infty)$ only depends on $x_0$, but this is not true since it may be necessary to involve all the sequence. For example, when we consider a Picard sequence of $(T, g)$, we define $x_{n+1}$ as a point (which is not necessarily unique) satisfying $gx_{n+1} = Tx_n$. In such a case, the notation $O(gx_0, \infty)$ is confusing because the sequence $\{gx_n\}$ is not uniquely determined. We prefer using $O(x_n, \infty, S)$. In some contexts (for example, when a unique sequence $\{x_n\}$ is considered throughout the work), it is possible to use the notation $O(x_0, \infty)$.

**Definition 6.4.6 ([147]).** Let $(X, G)$ be a $G$-metric space and let $T : X \longrightarrow X$ be a mapping. The *diameter* $\delta(A)$ of a non-empty subset $A \subseteq X$ is

$$\delta(A) = \sup\{G(x, y, z) : x, y, z \in A\}.$$

Notice that if $O(x_0, \infty, S)$ is $G$-bounded, then $\{\delta(O(x_0, n, S))\}_{n\geq 0}$ is a non-decreasing sequence of real numbers converging to $\delta(O(x_0, \infty, S))$.

Recall that a comparison function (or Matkowski function) is a non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ such that $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$. This kind of mappings must also satisfy $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$.

**Theorem 6.4.1.** *Let $(X, G)$ be a G-metric space and let $T, g : X \to X$ be mappings satisfying the following conditions.*

1. *There exists a Picard sequence $\{x_n\}_{n\geq 0}$ of $(T, g)$ such that*

$$\delta(O(gx_0, \infty, S)) < \infty$$

   *(where $S = \{gx_n\}_{n\geq 0}$).*
2. *$g(X)$ (or $T(X)$) is a G-complete subset of X.*
3. *there exists a continuous comparison function $\phi \in \mathcal{F}_{\mathrm{com}}$ such that*

$$G(Tx, Ty, Tz) \leq \phi(M^g(x, y, z)) \quad \textit{for all } x, y, z \in X, \tag{6.70}$$

   *where*

$$M^g(x, y, z) = \max\left\{\begin{array}{c} G(gx, gy, gz), G(gx, gx, Ty), G(gx, gx, Tz), \\ G(gy, gy, Tx), G(gy, gy, Tz), \\ G(gz, gz, Tx), G(gz, gz, Ty) \end{array}\right\}. \tag{6.71}$$

*Then T and g have, at least, a coincidence point.*

*Furthermore, if $(T, g)$ is a G-weakly commuting pair of type $G_T$ and $x \in X$ is any coincidence point of T and g, then $\omega = Tx = gx$ is the unique common fixed point of T and g.*

*Proof.* First we prove that $T$ and $g$ have, at least, a coincidence point. By hypothesis, let $\{x_n\}_{n\geq 0}$ be a Picard sequence of $(T, g)$ such that

$$\delta(O(gx_0, \infty, S)) < \infty$$

(where $S = \{gx_n\}_{n\geq 0}$). Let $y_n = gx_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ (so $S = \{y_n\}_{n\geq 0}\}$). If there exists some $n_0 \in \mathbb{N}$ such that $y_{n_0+1} = y_{n_0}$, then $Tx_{n_0+1} = y_{n_0+1} = y_{n_0} = gx_{n_0+1}$, so $x_{n_0+1}$ is a coincidence point of $T$ and $g$. Now, assume that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. In particular,

$$\delta\left(O\left(y_m, n, S\right)\right) > 0 \quad \text{for all } m \geq 0 \text{ and } n \geq 1.$$

Claim (1): for all $m, n \geq 0$ we have

$$\delta(O(y_m, n, S)) \leq \phi^m(\delta(O(y_0, n+m, S))). \tag{6.72}$$

If $n = 0$, there is nothing to prove. Assume that $n \geq 1$. We proceed by induction on $m$. If $m = 0$, then both members are equal (here $\phi^0$ stands for the identity mapping on $X$). If $m = 1$ and $n$ is arbitrary, then

$$\begin{aligned} O(y_1, n, S) &= \{y_1, y_2, \ldots, y_{n+1}\} \quad \text{and} \\ O(y_0, n+1, S) &= \{y_0, y_1, y_2, \ldots, y_{n+1}\}. \end{aligned}$$

Let $i, j, \ell \in \{1, 2, \ldots\}$ be indexes such that $1 \leq i \leq j \leq \ell$. Therefore, by the contractivity condition,

$$G(y_i, y_j, y_\ell) = G(Tx_i, Tx_j, Tx_\ell) \leq \phi(M^g(x_i, x_j, x_\ell)),$$

where

$$\begin{aligned} M^g(x_i, x_j, x_\ell) &= \max \left\{ \begin{array}{c} G(gx_i, gx_j, gx_\ell), G(gx_i, gx_i, Tx_j), \\ G(gx_i, gx_i, Tx_\ell), G(gx_j, gx_j, Tx_i), \\ G(gx_j, gx_j, Tx_\ell), G(gx_\ell, gx_\ell, Tx_i), \\ G(gx_\ell, gx_\ell, Tx_j) \end{array} \right\} \\ &= \max \left\{ \begin{array}{c} G(y_{i-1}, y_{j-1}, y_{\ell-1}), G(y_{i-1}, y_{i-1}, y_j), \\ G(y_{i-1}, y_{i-1}, y_\ell), G(y_{j-1}, y_{j-1}, y_i), \\ G(y_{j-1}, y_{j-1}, y_\ell), G(y_{\ell-1}, y_{\ell-1}, y_i), \\ G(y_{\ell-1}, y_{\ell-1}, y_j) \end{array} \right\}. \end{aligned} \tag{6.73}$$

Notice that all indexes in (6.73) are lower or equal to $\ell$. This means that if $1 \leq i \leq j \leq \ell \leq n+1$, then

$$\begin{aligned} M^g(x_i, x_j, x_\ell) &\leq \max\left(G(x_s, x_r, x_p) : s, r, p \in \{0, 1, 2, \ldots, n+1\}\right) \\ &\leq \delta(O(y_0, n+1, S) \leq \delta(O(y_0, \infty, S)) < \infty. \end{aligned}$$

As $\phi$ is non-decreasing, then

$$\begin{aligned} \delta(O(y_1, n, S)) &= \max\left\{G(y_i, y_j, y_\ell) : i, j, \ell \in \{1, 2, \ldots, n+1\}\right\} \\ &\leq \max\left\{\phi(M^g(x_i, x_j, x_\ell)) : i, j, \ell \in \{1, 2, \ldots, n+1\}\right\} \end{aligned}$$

$$
\begin{aligned}
&= \phi\left(\max\left\{M^g(x_i, x_j, x_\ell) : i, j, \ell \in \{1, 2, \ldots, n+1\}\right\}\right) \\
&\leq \phi\left(\delta(O(y_0, n+1, S)\right).
\end{aligned}
$$

This proves that (6.72) holds for $m = 1$ and arbitrary $n \in \mathbb{N}$. Assume that (6.72) holds for $m$, and we are going to prove it for $m+1$. Also taking into account (6.73) using $m+1 \leq i \leq j \leq \ell \leq m+1+n$, we have that

$$
\begin{aligned}
\delta(O(y_{m+1}, n, S)) &= \delta(\{y_{m+1}, y_{m+2}, \ldots, y_{m+1+n}\}) \\
&= \max\left\{G(y_i, y_j, y_\ell) : i, j, \ell \in \{m+1, m+2, \ldots, m+1+n\}\right\} \\
&\leq \max\left\{\phi(M^g(x_i, x_j, x_\ell)) : i, j, \ell \in \{m+1, m+2, \ldots, m+1+n\}\right\} \\
&= \phi\left(\max\left\{M^g(x_i, x_j, x_\ell) : i, j, \ell \in \{m+1, m+2, \ldots, m+1+n\}\right\}\right) \\
&\leq \phi\left(\delta(O(y_m, n+1, S)\right).
\end{aligned}
$$

Applying the hypothesis of induction (6.72) for $m$ and taking into account that $\phi$ is non-decreasing, it follows that

$$
\begin{aligned}
\delta(O(y_{m+1}, n, S)) &\leq \phi\left(\delta(O(y_m, n+1, S)\right) \leq \phi\left(\phi^m(\delta(O(y_0, n+1+m, S)))\right) \\
&= \phi^{m+1}(\delta(O(y_0, m+1+n, S))),
\end{aligned}
$$

which completes the induction. Hence, (6.72) holds for all $m, n \geq 0$.

Let $t_0 = \delta(O(y_0, \infty, S)) > 0$. Taking into account that, for all $m, n \geq 0$,

$$
\begin{aligned}
\delta(\{y_m, y_{m+1}, \ldots, y_{m+n}\}) = \delta(O(y_m, n, S)) &\leq \phi^m(\delta(O(y_0, n+m, S))) \\
&\leq \phi^m(\delta(O(y_0, \infty, S))) = \phi^m(t_0),
\end{aligned}
$$

the condition $\lim_{m\to\infty} \phi^m(t) = 0$ for all $t > 0$ implies that $\{y_n\} = \{gx_{n+1}\} = \{Tx_n\}$ is a Cauchy sequence in $(X, G)$. In fact, it is Cauchy in the $G$-complete subset $g(X)$ (or $T(X)$). Hence, there exists $z \in g(X)$ such that $\{gx_n\} \to z$. Let $u \in X$ be any point such that $gu = z$. We claim that $u$ is a coincidence point of $T$ and $g$.

We argue by contradiction. Assume that $Tu \neq gu$, that is, $G(gu, gu, Tu) > 0$. Using the contractivity condition (6.70),

$$
\begin{aligned}
G(Tu, gx_{n+1}, gx_{n+1}) &= G(Tu, Tx_n, Tx_n) \\
&\leq \phi\left(\max\{G(gu, gx_{n+1}, gx_{n+1}), G(gu, gu, gx_{n+2}),\right. \\
&\qquad\left. G(gx_{n+1}, gx_{n+1}, gx_{n+2}), G(gx_{n+1}, gx_{n+1}, Tu))\}\right).
\end{aligned}
$$

From the continuity of $\phi$ and letting $n \to \infty$, we deduce that

$$
G(Tu, gu, gu) \leq \phi\left(G(gu, gu, Tu)\right) < G(gu, gu, Tu),
$$

which is a contradiction. As a consequence, $Tu = gu$, and $u$ is a coincidence point of $T$ and $g$.

In the sequel, let $x \in X$ be an arbitrary coincidence point of $T$ and $g$, and let $\omega = Tx = gx$. Since the pair $(T, g)$ is $G$-weakly commuting of type $G_T$, then

$$G(Tgx, gTx, TTx) \leq G(Tx, gx, Tx) = 0.$$

Thus, $Tgx = gTx$. In particular,

$$T\omega = gTx = g\omega,$$

so $\omega$ is another coincidence point of $T$ and $g$. Moreover, by the contractivity condition (6.70),

$$G(T\omega, \omega, \omega) = G(T\omega, Tx, Tx) \leq \phi\left(M^g(\omega, x, x)\right) \quad \text{and}$$
$$G(T\omega, T\omega, \omega) = G(T\omega, T\omega, Tx) \leq \phi\left(M^g(\omega, \omega, x)\right),$$

where

$$\begin{aligned} M^g(\omega, x, x) = \max\{&G(g\omega, gx, gx), G(g\omega, g\omega, Tx), G(g\omega, g\omega, Tx), \\ &G(gx, gx, T\omega), G(gx, gx, Tx), G(gx, gx, T\omega), G(gx, gx, Tx)\} \\ = \max\{&G(T\omega, \omega, \omega), G(T\omega, T\omega, \omega)\} \quad \text{and} \end{aligned}$$

$$\begin{aligned} M^g(\omega, \omega, x) = \max\{&G(g\omega, g\omega, gx), G(g\omega, g\omega, T\omega), G(g\omega, g\omega, Tx), \\ &G(g\omega, g\omega, T\omega), G(g\omega, g\omega, Tx), G(gx, gx, T\omega), G(gx, gx, T\omega)\} \\ = \max\{&G(T\omega, \omega, \omega), G(T\omega, T\omega, \omega)\}. \end{aligned}$$

Hence,

$$\max\{G(T\omega, \omega, \omega), G(T\omega, T\omega, \omega)\} \leq \phi\left(\max\{G(T\omega, \omega, \omega), G(T\omega, T\omega, \omega)\}\right)$$

which is only possible when $\max\{G(T\omega, \omega, \omega), G(T\omega, T\omega, \omega)\} = 0$. This proves that $\omega = T\omega = g\omega$, so $\omega$ is a common fixed point of $T$ and $g$.

To prove the uniqueness, suppose that $v \in X$ is another common fixed point of $T$ and $g$, that is, $v = Tv = gv$. Therefore, the contractivity condition (6.70) ensures that

$$G(\omega, v, v) = G(T\omega, Tv, Tv) \leq \phi(M^g(\omega, v, v)) \quad \text{and}$$
$$G(\omega, \omega, v) = G(T\omega, T\omega, Tv) \leq \phi(M^g(\omega, \omega, v)),$$

where

$$\begin{aligned} M^g(\omega, v, v) = \max\{&G(g\omega, gv, gv), G(g\omega, g\omega, Tv), G(g\omega, g\omega, Tv), \\ &G(gv, gv, T\omega), G(gv, gv, Tv), G(gv, gv, T\omega), G(gv, gv, Tv)\} \\ = \max\{&G(\omega, v, v), G(\omega, \omega, v)\} \quad \text{and} \end{aligned}$$

$$\begin{aligned}M^g(\omega,\omega,v) &= \max\{G(g\omega,g\omega,gv),G(g\omega,g\omega,T\omega),G(g\omega,g\omega,Tv),\\&\qquad G(g\omega,g\omega,T\omega),G(g\omega,g\omega,Tv),G(gv,gv,T\omega),G(gv,gv,T\omega)\}\\&= \max\{G(\omega,v,v),G(\omega,\omega,v)\}.\end{aligned}$$

Hence,

$$\max\{G(\omega,v,v),G(\omega,\omega,v)\} \le \phi(\max\{G(\omega,v,v),G(\omega,\omega,v)\}),$$

which is only possible when $\max\{G(\omega,v,v),G(\omega,\omega,v)\} = 0$, that is, $\omega = v$, and $T$ and $g$ have a unique common fixed point. ☐

**Corollary 6.4.1.** *Let $(X,G)$ be a complete G-metric space and $S = \{x_n\}_{n\geq 0}$ be a Picard sequence of an operator $T : X \to X$ such that $\delta(O(x_0,\infty,S)) < \infty$. Assume that there exists a continuous comparison function $\phi \in \mathcal{F}_{\text{com}}$ such that*

$$G(Tx,Ty,Tz) \le \phi(M(x,y,z)) \quad \textit{for all } x,y,z \in X,$$

*where*

$$\begin{aligned}M(x,y,z) = \max\{&G(x,y,z),G(x,x,Ty),G(x,x,Tz),\\&G(y,y,Tx),G(y,y,Tz),G(z,z,Tx),G(z,z,Ty)\}.\end{aligned}$$

*Then $T$ has a unique fixed point.*

### *6.4.2 (E.A) Property*

In the proof of Theorem 6.4.1, we show that there exists a sequence $\{x_n\} \subseteq X$ and a point $z \in g(X)$ such that $\{gx_n\} \to z$ and $\{Tx_n\} \to z$.

**Definition 6.4.7 ([2]).** Let $T, g : X \to X$ be two self mappings of a metric space $(X,d)$. We say that $T$ and $g$ satisfy the *(E.A) property* if there exists a sequence $\{x_n\} \subseteq X$ and a point $z \in X$ such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} gx_n = z.$$

**Definition 6.4.8 ([147]).** Let $T, g : X \to X$ be two self mappings of a $G$-metric space $(X,G)$. We say that $T$ and $g$ satisfy the *(E.A) property* if there exists a sequence $\{x_n\} \subseteq X$ and a point $z \in X$ such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} gx_n = z.$$

In other words, thanks to Lemma 3.2.1,

$$\lim_{n\to\infty} G(Tx_n, Tx_n, z) = \lim_{n\to\infty} G(gx_n, gx_n, z) = 0.$$

In the following example, we show that the (E.A) property does not imply that $(T, g)$ is $G$-weakly commuting of type $G_T$.

*Example 6.4.5 ([147]).* We return to Example 6.4.4, in which $(T, g)$ is not a $G$-weakly commuting pair of type $G_T$. Let $x_n = 1 + \frac{1}{n}$ for all $n \geq 1$. Then, we have $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} gx_n = 1 \in X = [1, \infty)$, so $T$ and $g$ satisfy the (E.A) property.

**Theorem 6.4.2 ([147]).** *Let $(X, G)$ be a $G$-metric space and let $T, g : X \to X$ be two mappings such that the following conditions are fulfilled.*

1. *$T$ and $g$ satisfy the (E.A) property.*
2. *$g(X)$ is closed in $(X, G)$.*
3. *There exists a continuous comparison function $\phi \in \mathcal{F}_{\text{com}}$ such that*

$$G(Tx, Ty, Tz) \leq \phi(M^g(x, y, z)) \quad \textit{for all } x, y, z \in X, \tag{6.74}$$

*where*

$$M^g(x, y, z) = \max \left\{ \begin{array}{c} G(gx, Ty, Ty),\ G(gx, Tz, Tz), \\ G(gy, Tx, Tx),\ G(gy, Tz, Tz), \\ G(gz, Tx, Tx),\ G(gz, Ty, Ty) \end{array} \right\}.$$

*Then $T$ and $g$ have, at least, a coincidence point.*

*Furthermore, if $(T, g)$ is a $G$-weakly commuting pair of type $G_T$ and $x \in X$ is any coincidence point of $T$ and $g$, then $\omega = Tx = gx$ is the unique common fixed point of $T$ and $g$.*

*Proof.* From the (E.A) property, there exists a sequence $\{x_n\} \subseteq X$ and a point $z \in X$ such that $\{Tx_n\} \to z$ and $\{gx_n\} \to z$. Since $g(X)$ is a closed subset of $(X, G)$, then $z \in g(X)$, so there exists $u \in X$ such that $gu = z$. We claim that $u$ is a coincidence point to $T$ and $g$. Notice that, for all $n \in \mathbb{N}$,

$$G(Tu, Tu, Tx_n) \leq \phi(M^g(u, u, x_n)),$$

where

$$\begin{aligned} M^g(u, u, x_n) = \max \{&G(gu, Tu, Tu), G(gu, Tx_n, Tx_n), G(gu, Tu, Tu), \\ &G(gu, Tx_n, Tx_n), G(gx_n, Tu, Tu), G(gx_n, Tu, Tu)\}. \end{aligned}$$

Letting $n \to \infty$ we obtain that

$$\lim_{n\to\infty} M^g(u, u, x_n) = G(gu, Tu, Tu).$$

Since $\phi$ is continuous,

$$\begin{aligned} G\left(Tu,Tu,gu\right) &= \lim_{n\to\infty} G\left(Tu,Tu,Tx_n\right) \leq \lim_{n\to\infty} \phi(M^g(u,u,x_n)) \\ &= \phi\left(\lim_{n\to\infty} M^g(u,u,x_n)\right) = \phi\left(G(gu,Tu,Tu)\right), \end{aligned}$$

and this is only possible when $G\left(Tu,Tu,gu\right) = 0$, that is, $Tu = gu$.

Next, assume that $(T,g)$ is a $G$-weakly commuting pair of type $G_T$ and let $x \in X$ be any coincidence point of $T$ and $g$. Define $\omega = Tx = gx$. We claim that $\omega$ is the only common fixed point of $T$ and $g$. Indeed, since $T$ and $g$ are $G$-weakly commuting of type $G_T$, then

$$G(Tgx,gTx,TTx) \leq G(Tx,gx,Tx) = 0.$$

Thus, $TTx = Tgx = gTx$. In particular,

$$T\omega = TTx = gTx = g\omega,$$

so $\omega$ is another coincidence point of $T$ and $g$. Moreover, by the contractivity condition (6.74), we have that

$$\begin{aligned} G\left(T\omega,\omega,\omega\right) &= G\left(T\omega,Tx,Tx\right) \leq \phi\left(M^g(\omega,x,x)\right) \quad \text{and} \\ G\left(T\omega,T\omega,\omega\right) &= G\left(T\omega,T\omega,Tx\right) \leq \phi\left(M^g(\omega,\omega,x)\right), \end{aligned}$$

where

$$\begin{aligned} M^g\left(\omega,x,x\right) = \max\{&G(g\omega,Tx,Tx),G(g\omega,Tx,Tx),G(gx,T\omega,T\omega), \\ &G(gx,Tx,Tx),G(gx,T\omega,T\omega),G(gx,Tx,Tx)\} \\ = \max\{&G(T\omega,\omega,\omega),G(T\omega,T\omega,\omega)\} \end{aligned}$$

and

$$\begin{aligned} M^g\left(\omega,\omega,x\right) = \max\{&G(g\omega,T\omega,T\omega),G(g\omega,Tx,Tx),G(g\omega,T\omega,T\omega), \\ &G(g\omega,Tx,Tx),G(gx,T\omega,T\omega),G(gx,T\omega,T\omega)\} \\ = \max\{&G(T\omega,\omega,\omega),G(T\omega,T\omega,\omega)\} = M^g\left(\omega,x,x\right). \end{aligned}$$

Therefore,

$$\max\{G(T\omega,\omega,\omega),G(T\omega,T\omega,\omega)\} \leq \phi\left(\max\{G(T\omega,\omega,\omega),G(T\omega,T\omega,\omega)\}\right),$$

which is only possible when $\max\{G(T\omega,\omega,\omega),G(T\omega,T\omega,\omega)\} = 0$. In such a case, $\omega = T\omega = g\omega$ and this proves that $\omega$ is a common fixed point of $T$ and $g$.

To prove the uniqueness, suppose that $v \in X$ is another common fixed point of $T$ and $g$, that is, $v = Tv = gv$. Therefore, the contractivity condition (6.74) ensures that

$$G(\omega, v, v) = G(T\omega, Tv, Tv) \leq \phi(M^g(\omega, v, v)) \quad \text{and}$$
$$G(\omega, \omega, v) = G(T\omega, T\omega, Tv) \leq \phi(M^g(\omega, \omega, v)),$$

where

$$\begin{aligned} M^g(\omega, v, v) &= \max\{G(g\omega, Tv, Tv), G(g\omega, Tv, Tv), G(gv, T\omega, T\omega), \\ &\qquad G(gv, Tv, Tv), G(gv, T\omega, T\omega), G(gv, Tv, Tv)\} \\ &= \max\{G(\omega, v, v), G(\omega, \omega, v)\} \end{aligned}$$

and

$$\begin{aligned} M^g(\omega, \omega, v) &= \max\{G(g\omega, T\omega, T\omega), G(g\omega, Tz, Tz), G(g\omega, T\omega, T\omega), \\ &\qquad G(g\omega, Tz, Tz), G(gz, T\omega, T\omega), G(gz, T\omega, T\omega)\} \\ &= \max\{G(\omega, v, v), G(\omega, \omega, v)\} = M^g(\omega, v, v). \end{aligned}$$

Hence,

$$\max\{G(\omega, v, v), G(\omega, \omega, v)\} \leq \phi(\max\{G(\omega, v, v), G(\omega, \omega, v)\}),$$

which is only possible when $\max\{G(\omega, v, v), G(\omega, \omega, v)\} = 0$, that is, $\omega = v$, and $T$ and $g$ have a unique common fixed point. □

In the following result, we slightly change the contractivity condition and we replace the $G$-weakly commutative pair by a weakly compatible pair.

**Theorem 6.4.3 ([147]).** *Let $(X, G)$ be a G-metric space and let $T, g : X \to X$ be two mappings such that the following conditions are fulfilled.*

1. *$T$ and $g$ satisfy the (E.A) property.*
2. *$g(X)$ is closed in $(X, G)$.*
3. *There exists a continuous comparison function $\phi \in \mathcal{F}_{\text{com}}$ such that*

$$G(Tx, Ty, Tz) \leq \phi(M^g(x, y, z)) \quad \text{for all } x, y, z \in X, \tag{6.75}$$

*where*

$$M^g(x, y, z) = \max\left\{\begin{array}{l} G(gx, gy, gz), G(gx, Tx, gz), \\ G(gz, Tz, gz), G(gy, Ty, gz) \end{array}\right\}.$$

*Then $T$ and $g$ have, at least, a coincidence point.*

*Furthermore, if $(T, g)$ is a weakly compatible pair of type $G_T$ and $x \in X$ is any coincidence point of $T$ and $g$, then $\omega = Tx = gx$ is the unique common fixed point of $T$ and $g$.*

*Proof.* Following the same argument in the proof of Theorem 6.4.2, it is easy to prove that, if $\{Tx_n\} \to gu$ and $\{gx_n\} \to gu$, then $Tu = gu$.

Assume that $(T, g)$ is a weakly compatible pair of type $G_T$ and let $x \in X$ be any coincidence point of $T$ and $g$. Define $\omega = Tx = gx$. Then, by the weakly compatibility, $Tx = gx$ implies that $Tgx = gTx$. In particular, $T\omega = Tgx = gTx = g\omega$, so $\omega$ is also a coincidence point of $T$ and $g$. We can also follow the rest of the proof of Theorem 6.4.2 in order to conclude that $\omega$ is the unique common fixed point of $T$ and $g$. □

*Example 6.4.6 ([147]).* Let $X = [0, \infty)$ provided with the $G$-metric $G(x, y, z) = |x-y| + |y-z| + |x-z|$ for all $x, y, z \in X$. Consider $T, g : X \to X$ and $\phi : [0, \infty) \to [0, \infty)$ defined by

$$Tx = \frac{x}{8}, \quad gx = \frac{x}{2} \quad \text{for all } x, y, z \in X,$$
$$\phi(t) = \frac{2t}{3} \quad \text{for all } t \geq 0.$$

The only coincidence point of $T$ and $g$ is $x = 0$, where $Tg0 = gT0 = 0$, so $T$ and $g$ are weakly compatible. Let $x_n = 1/n$ for all $n \in \mathbb{N}$. Then $\{Tx_n\} \to 0$ and $\{gx_n\} \to 0$, so $T$ and $g$ satisfy the (E.A.) property. We also have that, for all $x, y, z \in X$,

$$\begin{aligned} G(Tx, Ty, Tz) &= \frac{1}{8}(|x-y| + |y-z| + |x-z|) \\ &\leq \frac{1}{3}(|x-y| + |y-z| + |x-z|) \\ &= \frac{2}{3}\left(\frac{1}{2}(|x-y| + |y-z| + |x-z|)\right) = \phi(G(gx, gy, gz)) \\ &\leq \phi(\max\{G(gx, gy, gz), G(gx, Tx, gz), \\ &\qquad G(gz, Tz, gz), G(gy, Ty, gz)\}). \end{aligned}$$

Hence, all the conditions of Theorem 6.4.3 are satisfied and $\omega = 0$ is the unique common fixed point of $T$ and $g$.

*Example 6.4.7.* Let $X = [0, 1]$ provided with the $G$-metric $G(x, y, z) = |x-y| + |y-z| + |x-z|$ for all $x, y, z \in X$. Consider $T, g : X \to X$ and $\phi : [0, \infty) \to [0, \infty)$ defined by

$$Tx = \frac{x^2}{4}, \quad gx = x^2 \quad \text{for all } x, y, z \in X,$$
$$\phi(t) = \frac{t}{3} \quad \text{for all } t \geq 0.$$

As in the previous example, it is easy to show that $T$ and $g$ are weakly compatible and verify the (E.A) property (by taking the sequence $x_n = 1/n$). We also have that, for all $x, y, z \in X$,

$$\begin{aligned} G(Tx, Ty, Tz) &= \frac{1}{4} \max\{|x^2 - y^2|, |y^2 - z^2|, |x^2 - z^2|\} \\ &\leq \frac{1}{3} \max\{|x^2 - y^2|, |y^2 - z^2|, |x^2 - z^2|\} \\ &= \phi(G(gx, gy, gz)) \\ &\leq \phi(\max\{G(gx, gy, gz), G(gx, Tx, gz), \\ &\qquad G(gz, Tz, gz), G(gy, Ty, gz)\}). \end{aligned}$$

Hence, all the conditions of Theorem 6.4.3 are satisfied and $\omega = 0$ is the unique common fixed point of $T$ and $g$.

*Example 6.4.8 ([147]).* Let $X = [2, 20]$ provided with the $G$-metric $G(x, y, z) = |x-y| + |y-z| + |x-z|$ for all $x, y, z \in X$. Consider $T, g : X \to X$ and $\phi : [0, \infty) \to [0, \infty)$ defined, for all $x, y, z \in X$ and all $t \geq 0$, by

$$Tx = \begin{cases} 2, \text{ if } x = 2, \\ 6, \text{ if } 2 < x \leq 5, \\ 2, \text{ if } 5 < x \leq 20; \end{cases} \quad g(x) = \begin{cases} 2, & \text{if } x = 2, \\ 14, & \text{if } 2 < x \leq 5, \\ \frac{4x+10}{15}, & \text{if } 5 < x \leq 20; \end{cases} \quad \phi(t) = \frac{t}{2}.$$

It is clear that $g(X) = [2, 6] \cup \{14\}$ is a closed subset of $X$ and $T$ and $g$ are weakly compatible. If we consider the sequence $\{x_n\} = \{5 + \frac{1}{n}\}$, then $\{Tx_n\} \to 2$ and $\{gx_n\} \to 2$ as $n \to \infty$. Thus, $T$ and $g$ satisfy the (E.A) property. On the other hand, a simple calculation gives that

$$G(Tx, Ty, Tz) \leq \phi(G(gx, gy, gz)) \quad \text{for all } x, y, z \in X,$$

so, in particular (6.75) holds. As a consequence, all the hypotheses of Theorem 6.4.3 are satisfied and $\omega = 2$ is the unique common fixed point of $T$ and $g$.

Note that the main result of Mustafa [142] is not applicable in this case. Indeed, for $y = z = \frac{5}{2}$ and $x = 2$, we have that

$$G\left(T(2), T\left(\frac{5}{2}\right), T\left(\frac{5}{2}\right)\right) = 8 > \lambda = \lambda G\left(2, \frac{5}{2}, \frac{5}{2}\right)$$

whenever $\lambda \in [0, 1)$.

**Theorem 6.4.4 ([147]).** *Let $(X, G)$ be a $G$-metric space and let $T, g : X \to X$ be two mappings such that the following conditions are fulfilled.*

1. *T and g satisfy the (E.A) property.*
2. *$g(X)$ is closed in $(X, G)$.*
3. *There exist nonnegative real constants $\alpha$ and $\beta$ with $0 \le \alpha + 2\beta < 1$ such that, for all $x, y, z \in X$,*

$$G(Tx, Ty, Tz) \le \alpha\, G(gx, gy, gz) + \beta\, (G(gx, Tx, Tx) + G(gy, Ty, Ty) + G(gz, Tz, Tz))\,. \tag{6.76}$$

*Then T and g have, at least, a coincidence point.*

*Furthermore, if $(T, g)$ is G-$\lambda$-weakly commuting of type $G_T$ (for some $\lambda > 0$) and $x \in X$ is any coincidence point of T and g, then $\omega = Tx = gx$ is the unique common fixed point of T and g.*

*Proof.* From the (E.A) property, there exists a sequence $\{x_n\} \subseteq X$ and a point $z \in X$ such that $\{Tx_n\} \to z$ and $\{gx_n\} \to z$. Since $g(X)$ is a closed subset of $(X, G)$, then $z \in g(X)$, so there exists $u \in X$ such that $gu = z$. We claim that $u$ is a coincidence point of $T$ and $g$. Notice that, for all $n \in \mathbb{N}$,

$$G(Tu, Tu, Tx_n) \le \alpha\, G(gu, gu, gx_n) + \beta\, (G(gu, Tu, Tu) + G(gu, Tu, Tu) + G(gx_n, Tx_n, Tx_n))\,.$$

Letting $n \to \infty$ we deduce that

$$G(Tu, Tu, gu) \le 2\beta\, G(gu, Tu, Tu).$$

If $gu \neq Tu$, then

$$G(Tu, Tu, gu) \le 2\beta\, G(gu, Tu, Tu) \le (\alpha + 2\beta)\, G(Tu, Tu, gu) < G(Tu, Tu, gu),$$

which is a contradiction. Therefore $gu = Tu$.

Now, assume that the pair $(T, g)$ is $G$-$\lambda$-weakly commuting of type $G_T$ (for some $\lambda > 0$). Let $x \in X$ be any coincidence point of $T$ and $g$ and let $\omega = Tx = gx$. We claim that $\omega$ is the only common fixed point of $T$ and $g$. Indeed, since the pair $(T, g)$ is $G$-$\lambda$-weakly commuting of type $G_T$, then

$$G(Tgx, gTx, TTx) \le G(Tx, gx, Tx) = 0.$$

Thus, $TTx = Tgx = gTx$. In particular,

$$T\omega = TTx = gTx = g\omega,$$

so $\omega$ is another coincidence point of $T$ and $g$. Moreover, by the contractivity condition (6.76), we have that

$$
\begin{aligned}
G(T\omega,\omega,\omega) = G(T\omega,Tx,Tx) &\le \alpha G(g\omega,gx,gx) \\
&\quad + \beta\,(G(g\omega,T\omega,T\omega) + G(gx,Tx,Tx) + G(gx,Tx,Tx)) \\
&= \alpha G(T\omega,\omega,\omega) \le (\alpha+2\beta)\,G(T\omega,\omega,\omega).
\end{aligned}
$$

If $T\omega \neq \omega$, then

$$G(T\omega,\omega,\omega) \le (\alpha+2\beta)\,G(T\omega,\omega,\omega) < G(T\omega,\omega,\omega),$$

which is a contradiction. Therefore, $\omega = T\omega = g\omega$, so $\omega$ is a common fixed point of $T$ and $g$.

To prove the uniqueness, suppose that $v \in X$ is another common fixed point of $T$ and $g$, that is, $v = Tv = gv$. Therefore, the contractivity condition (6.76) ensures that

$$
\begin{aligned}
G(\omega,v,v) = G(T\omega,Tv,Tv) &\le \alpha G(g\omega,gv,gv) \\
&\quad + \beta\,(G(g\omega,T\omega,T\omega) + G(gv,Tv,Tv) + G(gv,Tv,Tv)) \\
&= \alpha G(\omega,v,v).
\end{aligned}
$$

If $\omega \neq v$, then

$$G(\omega,v,v) \le \alpha G(\omega,v,v) \le (\alpha+2\beta)\,G(\omega,v,v) < G(\omega,v,v),$$

which is a contradiction. Then $\omega = v$ and $T$ and $g$ have a unique common fixed point. □

*Example 6.4.9 ([147]).* Let $X = [1,\infty)$ be endowed with the $G$-metric

$$G(x,y,z) = |x-y| + |y-z| + |x-z|$$

for all $x,y,z \in X$. Define $T,g : X \to X$ by $Tx = 2x-1$ and $gx = 3x-2$ for each $x \in X$. Set $\alpha = \frac{3}{4}$ and $\beta = 0$. It is clear that the mappings $T$ and $g$ are $G$-$\lambda$-weakly commuting of type $G_T$ (with $\lambda = 2$) and satisfy the following: (i) $T$ and $g$ satisfy the (E.A) property (by taking $x_n = 1 + \frac{1}{n}$ and $z = 1$), and (ii) $g(X)$ is a closed subspace of $X$. Moreover, for all $x,y,z \in X$ we have

$$
\begin{aligned}
G(Tx,Ty,Tz) &= 2[|x-y| + |x-z| + |y-z|] \\
&\le \frac{9}{4}[|x-y| + |x-z| + |y-z|] = \alpha G(gx,gy,gz) \\
&\quad + \beta\,(G(gx,Tx,Tx) + G(gy,Ty,Ty) + G(gz,Tz,Tz)).
\end{aligned}
$$

Thus, all the conditions of Theorem 6.4.4 are satisfied and $\omega = 1$ is the unique common fixed point of $T$ and $g$.

Note that the main result of Mustafa [142] is not applicable in this case. Indeed, for $y = z = 1$ and $x = 2$,

$$G(T2, T1, T1) = 4 > 2\lambda = \lambda\, G(2, 1, 1)$$

whenever $\lambda \in [0, 1)$. Also, the Banach principle [34] is not applicable using the Euclidean distance $d(x, y) = |x - y|$ for all $x, y \in X$. In such a case, we have, for $x \neq y$,

$$d(Tx, Ty) = 2\,|x - y| > \lambda\,|x - y|$$

whenever $\lambda \in [0, 1)$.

**Corollary 6.4.2 ([147]).** *Theorems 6.4.2, 6.4.3 and 6.4.4 remain true if we replace, respectively, G-weakly commutativity of type $G_T$, weakly compatibility and G-$\lambda$-weakly commutativity of type $G_T$ by any one of them (retaining the rest of hypothesis).*

Some corollaries could be derived from Theorems 6.4.1, 6.4.2, 6.4.3 and 6.4.4 by taking $z = y$ or $g$ as the identity mapping on $X$.

## 6.5 Generalized Meir–Keeler Type Contractions on *G*-Metric Spaces

In this section, we present the notion of a *generalized Meir–Keeler type contraction* on $G$-metric spaces. We will distinguish between whether the $G$-metric space is endowed with a partial order or not.

### *6.5.1 Generalized Meir–Keeler Type Contractions on G-Metric Spaces*

We begin this subsection by introducing the definition of a generalized Meir–Keeler type contraction.

**Definition 6.5.1 ([148]).** Let $(X, G)$ be a G-metric space and $T$ be a self map on $X$. Then $T$ is called a *generalized Meir–Keeler type contraction* whenever for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq M(x, y, z) < \varepsilon + \delta \quad \Rightarrow \quad G(Tx, Ty, Tz) < \varepsilon, \tag{6.77}$$

where

$$M(x, y, z) = \max\{G(x, y, z), G(Tx, x, x), G(Ty, y, y), G(Tz, z, z)\}.$$

*Remark 6.5.1.* Notice that if $T$ is a generalized Meir–Keeler type contraction, then we have

$$G(Tx, Ty, Tz) < M(x, y, z) \tag{6.78}$$

for all $x, y, z \in X$ such that $G(x, y, z) > 0$.

In 1971, Ćirić [61] introduced the notion of *orbitally continuous maps* on metric spaces as follows.

**Definition 6.5.2.** Let $(X, d)$ be a metric space and let $x_0 \in X$. A mapping $T : X \to X$ is *orbitally continuous at* $x_0$ if $\lim_{k\to\infty} TT^{n_k}u = Tx_0$ for all $u \in X$ and all strictly increasing sequence $\{n_k\}_{k\geq 1}$ of non-negative integer numbers satisfying $\lim_{k\to\infty} T^{n_k}u = x_0$.

**Definition 6.5.3 ([148]).** Let $(X, G)$ be a $G$-metric space, let $T : X \to X$ be a self map and let $x_0 \in X$. We say that $T$ is *orbitally G-continuous at* $x_0$ whenever $\lim\limits_{k\to\infty} G(T^{n_k}u, x_0, x_0) = 0$ implies that $\lim\limits_{k\to\infty} G(TT^{n_k}u, Tx_0, Tx_0) = 0$ whatever $u \in X$ and a strictly increasing sequence $\{n_k\}_{k\geq 1} \subseteq \mathbb{N}$ of non-negative integer numbers.

It is clear that all $G$-continuous mappings are also orbitally $G$-continuous.

**Definition 6.5.4.** In a $G$-metric space, a *cluster* (or *accumulation*) *point of a sequence* $\{x_n\}$ is a point $x \in X$ such that for every neighbourhood $V$ of $x$ in $\tau_G$, there are infinitely many natural numbers $\{n_k\}_{k\geq 1}$ such that $x_{n_k} \in V$ for all $k \geq 1$.

We show that every generalized Meir–Keeler type contraction is asymptotically regular.

**Proposition 6.5.1 ([148]).** *If $T : X \to X$ is a generalized Meir–Keeler type contraction in a G-metric space $(X, G)$, then $\lim_{n\to\infty} G(T^{n+1}x, T^n x, T^n x) = 0$ for all $x \in X$.*

*Proof.* Let $x_0 \in X$ be arbitrary and let $\{x_n = T^n x_0\}_{n\geq 0}$ be the Picard sequence of $T$ based on $x_0$. If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then $x_{n_0}$ is a fixed point of $T$. In particular, $T^n x_0 = x_n = x_{n_0}$ for all $n \geq n_0$, so the sequence $\{x_n\}$ converges, and the proposition follows. On the contrary case, assume that $x_{n+1} \neq x_n$ for all $n \geq 0$. Consequently, we have $M(x_{n+1}, x_n, x_n) \geq G(x_{n+1}, x_n, x_n) > 0$ for every $n \geq 0$. Notice that, for all $n \geq 0$,

$$\begin{aligned} M(x_{n+1}, x_n, x_n) &= \max\{G(x_{n+1}, x_n, x_n), G(Tx_{n+1}, x_{n+1}, x_{n+1}), \\ &\qquad G(Tx_n, x_n, x_n), G(Tx_n, x_n, x_n)\} \\ &= \max\{G(x_{n+1}, x_n, x_n), G(x_{n+2}, x_{n+1}, x_{n+1})\}. \end{aligned} \tag{6.79}$$

From Remark 6.5.1, we get that, for all $n \geq 0$,

$$\begin{aligned} G(x_{n+2}, x_{n+1}, x_{n+1}) &= G(Tx_{n+1}, Tx_n, Tx_n) < M(x_{n+1}, x_n, x_n) \\ &= \max\{G(x_{n+1}, x_n, x_n), G(x_{n+2}, x_{n+1}, x_{n+1})\}. \end{aligned}$$

If there exists some $n \in \mathbb{N}$ such that $G(x_{n+2}, x_{n+1}, x_{n+1}) \geq G(x_{n+1}, x_n, x_n)$, we get a contradiction. Therefore,

$$G(x_{n+2}, x_{n+1}, x_{n+1}) < G(x_{n+1}, x_n, x_n) \quad \text{for all } n \geq 0.$$

Thus, $\{G(x_{n+1}, x_n, x_n)\}_{n\geq 0}$ is a decreasing sequence which is bounded below by 0. Hence, it converges to some $\varepsilon \in [0, \infty)$, that is,

$$\lim_{n\to\infty} G(x_{n+1}, x_n, , x_n) = \varepsilon \quad \text{and} \tag{6.80}$$

$$G(x_{n+1}, x_n, , x_n) > \varepsilon \quad \text{for all } n \geq 0. \tag{6.81}$$

In particular, by (6.79), we have

$$\lim_{n\to\infty} M(x_{n+1}, x_n, x_n) = \varepsilon. \tag{6.82}$$

We claim that $\varepsilon = 0$. Suppose on the contrary that $\varepsilon > 0$. Regarding (6.82) together with the assumption that $T$ is a generalized Meir–Keeler type contraction, for this $\varepsilon > 0$, there exists $\delta > 0$ such that (6.77) holds. By (6.79) and (6.82), there exists a natural number $n_0 \in \mathbb{N}$ such that

$$\varepsilon \leq M(x_{n+1}, x_n, x_n) < \varepsilon + \delta \quad \text{for all } n \geq n_0,$$

but, in this case, by (6.77), we have that

$$G(x_{n+2}, x_{n+1}, x_{n+1}) = G(Tx_{n+1}, Tx_n, Tx_n) < \varepsilon,$$

which contradicts (6.81).

Hence, $\lim_{n\to\infty} G(T^{n+1}x, T^n x, T^n x) = \lim_{n\to\infty} G(x_{n+1}, x_n, x_n) = \varepsilon = 0$. □

**Theorem 6.5.1 ([148]).** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be an orbitally continuous generalized Meir–Keeler type contraction. Then $T$ has a unique fixed point and $T$ is a Picard operator.*

In other words, if $\omega \in X$ is the only fixed point of $T$, then $\{T^n x\} \to \omega$ for all $x \in X$.

*Proof.* Take $x_0 \in X$ arbitrary and let $\{x_n\}_{n\geq 0}$ be the Picard sequence of $T$ based on $x_0$, that is, $x_n = T^n x_0$ for all $n \geq 0$. From Proposition 6.5.1, we have that

$$\lim_{n\to\infty} G(x_{n+1}, x_n, x_n) = 0. \tag{6.83}$$

We prove that $\{x_n\}$ is a Cauchy sequence in $(X, G)$ reasoning by contradiction. Suppose that $\{x_n\}$ is not a Cauchy sequence. By Theorem 4.1.1, there exists $\varepsilon_0 > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that, for all $k \geq 0$,

$$\begin{aligned} &k \le n(k) < m(k) < n(k+1), \\ &G\left(x_{n(k)}, x_{m(k)-1}, x_{m(k)-1}\right) \le \varepsilon_0 < G\left(x_{n(k)}, x_{m(k)}, x_{m(k)}\right). \end{aligned} \tag{6.84}$$

and

$$\lim_{k\to\infty} G\left(x_{n(k)}, x_{m(k)}, x_{m(k)}\right) = \lim_{k\to\infty} G\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}\right) = \varepsilon_0. \tag{6.85}$$

As $T$ is a generalized Meir–Keeler type contraction, for $\varepsilon_0 > 0$, there exists $\delta > 0$ such that

$$\varepsilon_0 \le M(x, y, z) < \varepsilon_0 + \delta \quad \Rightarrow \quad G(Tx, Ty, Tz) < \varepsilon_0. \tag{6.86}$$

Notice that

$$\begin{aligned} M(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}) &= \max\{G(x_{n(k)-1}, x_{m(k)-1}x_{m(k)-1}), \\ &\qquad G(Tx_{n(k)-1}, x_{n(k)-1}, x_{n(k)-1}), \\ &\qquad G(Tx_{m(k)-1}, x_{m(k)-1}, x_{m(k)-1})\} \\ &= \max\{G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}), \\ &\qquad G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}), \\ &\qquad G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1})\}. \end{aligned} \tag{6.87}$$

By (6.83) and (6.85), there exists $k_0 \in \mathbb{N}$ such that, for all $k \ge k_0$,

$$\begin{aligned} &\max\{G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}), G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1})\} \\ &\quad < \frac{\varepsilon_0}{2} < G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}) \end{aligned} \tag{6.88}$$

We now show, reasoning by contradiction, that

$$G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}) \ge \varepsilon_0 \quad \text{for all } k \ge k_0. \tag{6.89}$$

Assume that there exists some $k' \ge k_0$ such that $G(x_{n(k')-1}, x_{m(k')-1}, x_{m(k')-1}) < \varepsilon_0$. Define $\varepsilon_0' = G(x_{n(k')-1}, x_{m(k')-1}, x_{m(k')-1}) \in ]\varepsilon_0/2, \varepsilon_0[$. As $T$ is a generalized Meir–Keeler type contraction, for $\varepsilon_0' > 0$, there exists $\delta' > 0$ such that

$$\varepsilon_0' \le M(x, y, z) < \varepsilon_0' + \delta' \quad \Rightarrow \quad G(Tx, Ty, Tz) < \varepsilon_0'. \tag{6.90}$$

From (6.87) and (6.88), we have that

$$M(x_{n(k')-1}, x_{m(k')-1}, x_{m(k')-1}) = \max\{G(x_{n(k')-1}, x_{m(k')-1}, x_{m(k')-1}),$$

$$G(x_{n(k')}, x_{n(k')-1}, x_{n(k')-1}), G(x_{m(k')}, x_{m(k')-1}, x_{m(k')-1})\}$$
$$= G(x_{n(k')-1}, x_{m(k')-1}, x_{m(k')-1}) = \varepsilon_0'.$$

Using (6.90),

$$G(x_{n(k')}, x_{m(k')}, x_{m(k')}) = G(Tx_{n(k')-1}, Tx_{m(k')-1}, Tx_{m(k')-1}) < \varepsilon_0' < \varepsilon_0,$$

which contradicts (6.84). This contradiction proves that (6.89) holds. As a consequence, it follows from (6.87), (6.88) and (6.89) that, for all $k \geq k_0$,

$$M(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}) = G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}) \geq \varepsilon_0.$$

Using (6.85) and $\delta > 0$, there exists $k_1 \geq k_0$ such that

$$\varepsilon_0 \leq M(x_{n(k_1)-1}, x_{m(k_1)-1}, x_{m(k_1)-1}) < \varepsilon_0 + \delta,$$

and by (6.86),

$$G(x_{n(k_1)}, x_{m(k_1)}, x_{m(k_1)}) = G(Tx_{n(k_1)-1}, Tx_{m(k_1)-1}, Tx_{m(k_1)-1}) < \varepsilon_0$$

which contradicts (6.84). This contradiction proves that $\{x_n\}$ is a Cauchy sequence in $(X, G)$. Since $(X, G)$ is $G$-complete, the sequence $\{x_n\}$ converges to some $\omega \in X$. From Lemma 3.2.1, we have that

$$\lim_{n\to\infty} G(x_n, \omega, \omega) = \lim_{n\to\infty} G(x_n, x_n, \omega) = 0.$$

Next, we will show that $\omega$ is a fixed point of $T$. Since $T$ is orbitally continuous,

$$\lim_{n\to\infty} G(T^n x_0, \omega, \omega) = \lim_{n\to\infty} G(x_n, \omega, \omega) = 0$$
$$\Rightarrow \quad \lim_{n\to\infty} G(x_{n+1}, T\omega, T\omega) = \lim_{n\to\infty} G(TT^n x_0, T\omega, T\omega) = 0.$$

Thus, $\{x_{n+1}\}$ converges to $T\omega$ in $(X, G)$. By the uniqueness of the limit, we get $T\omega = \omega$.

Finally, we show that $T$ has a unique fixed point. Assume that $u \in X$ is another fixed point of $T$. If $u \neq \omega$, then $M(u, \omega, \omega) \geq G(u, \omega, \omega) > 0$. Using Remark 6.5.1, we derive that

$$\begin{aligned} 0 < G(u, \omega, \omega) &= G(Tu, T\omega, T\omega) < M(u, \omega, \omega) \\ &= \max\{G(u, \omega, \omega), G(u, Tu, Tu), G(\omega, \omega, T\omega)\} \\ &= G(u, \omega, \omega), \end{aligned}$$

which is a contradiction. Hence, $u = \omega$ and $T$ has a unique fixed point. In particular, we have proved that, for all $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to a fixed point of $T$, which can only be $\omega$. $\square$

*Example 6.5.1.* Let $X = [0, 10]$ endowed with the $G$-metric $G(x, y, z) = \max\{|x - y|, |y-z|, |z-x|\}$ for all $x, y, z \in X$. Then $(X, G)$ is $G$-complete. Define $T : X \to X$ by

$$Tx = \begin{cases} \frac{x}{3}, \text{ if } 0 \le x < 6 \\ 2, \text{ if } 6 \le x \le 10. \end{cases}$$

We claim that $T$ is a generalized Meir–Keeler type contraction. Let $\varepsilon > 0$ be arbitrary. Taking $\delta = 2\varepsilon > 0$, we claim that (6.77) holds. Indeed, let $x, y, z \in X$. By the symmetry of $G$, we assume that $x \le y \le z$ without loss of generality. We have the following cases.

- Case 1: $0 \le x \le y \le z < 6$. Here, we have

$$G(Tx, Ty, Tz) = \frac{z}{3} - \frac{x}{3} = \frac{1}{3}\,(z - x)\,,$$

and

$$M(x, y, z) = \max\left\{z - x, \frac{2}{3}x, \frac{2}{3}y, \frac{2}{3}z\right\} = \max\left\{z - x, \frac{2}{3}z\right\}.$$

If $\varepsilon \le M(x, y, z) < \varepsilon + \delta = 3\varepsilon$, then

$$\begin{aligned} G(Tx, Ty, Tz) &= \frac{1}{3}\,(z - x) \le \frac{1}{3}\max\left\{z - x, \frac{2}{3}z\right\} \\ &= \frac{1}{3} M(x, y, z) < \varepsilon. \end{aligned}$$

- Case 2: $0 \le x \le y < 6 \le z \le 10$. Here, we have

$$G(Tx, Ty, Tz) = G\left(\frac{x}{3}, \frac{y}{3}, 2\right) = 2 - \frac{x}{3}$$

and

$$M(x, y, z) = \max\{z - x, \frac{2}{3}x, \frac{2}{3}y, z - 2\} = \max\{z - x, z - 2\}.$$

If $\varepsilon \le M(x, y, z) < \varepsilon + \delta = 3\varepsilon$, then

$$G(Tx, Ty, Tz) = 2 - \frac{x}{3} = \frac{1}{3}\,(6 - x) \le \frac{1}{3}\,(z - x)$$

$$= \frac{1}{3} M(x, y, z) < \varepsilon.$$

- Case 3: $0 \leq x < 6 \leq y \leq z \leq 10$. Here, we have

$$G(Tx, Ty, Tz) = G\left(\frac{x}{3}, 2, 2\right) = 2 - \frac{x}{3}$$

and

$$M(x, y, z) = \max\{z - x, \frac{2}{3}x, y - 2, z - 2\} = \max\{z - x, z - 2\}.$$

The same argument proves that if $\varepsilon \leq M(x, y, z) < \varepsilon + \delta = 3\varepsilon$, then $G(Tx, Ty, Tz) < \varepsilon$.

In any case, (6.77) holds and $T$ is a generalized Meir–Keeler type contraction. Also, the mapping $T$ is continuous in $(X, G)$, so it is also orbitally $G$-continuous. All the hypotheses of Theorem 6.5.1 are satisfied and $\omega = 0$ is the unique fixed point of $T$.

*Example 6.5.2 ([148]).* Let $X = [0, 1]$ endowed with the $G$-metric $G(x, y) = \max\{|x - y|, |y - z|, |z - x|\}$ for all $x, y, z \in X$, which is complete on $X$. Consider $T : X \to X$ defined by

$$Tx = \frac{x^2}{8} \quad \text{for all } x \in X.$$

If $x, y, z \in X$ satisfy, without loss of generality, $x \leq y \leq z$, then

$$G(Tx, Ty, Tz) = \frac{z^2}{8} - \frac{x^2}{8} = \frac{(z + x)(z - x)}{8} \leq \frac{2(z - x)}{8} = \frac{z - x}{4}.$$

Also we get

$$M(x, y, z) = \max\left\{z - x, x - \frac{x^2}{8}, y - \frac{y^2}{8}, z - \frac{z^2}{8}\right\}.$$

Given $\varepsilon > 0$, let $\delta = 3\varepsilon > 0$. Then, if $\varepsilon \leq M(x, y, z) < \varepsilon + \delta = 4\varepsilon$, we deduce that

$$\begin{aligned} G(Tx, Ty, Tz) &\leq \frac{z - x}{4} \leq \frac{1}{4} \max\left\{z - x, x - \frac{x^2}{8}, y - \frac{y^2}{8}, z - \frac{z^2}{8}\right\} \\ &= \frac{1}{4} M(x, y, z) < \varepsilon. \end{aligned}$$

As $T$ is continuous, all the hypotheses of Theorem 6.5.1 are satisfied. In this case, $\omega = 0$ is the unique fixed point of $T$.

*Remark 6.5.2.* Theorem 6.5.1 remains true if we replace the hypothesis that $T$ is a generalized Meir–Keeler type contraction with:

► For each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq N(x,y,z) < \varepsilon + \delta \;\Rightarrow\; G(Tx,Ty,Tz) < \varepsilon, \tag{6.91}$$

where $N(x,y,z)$ is given by

$$\max\{G(x,y,z), G(Tx,Tx,x), G(Ty,Ty,y), G(Tz,Tz,z)\}.$$

Finally, in the following two examples, we consider some non-symmetric $G$-metrics.

*Example 6.5.3 ([148]).* Let $X = \{0,1\}$ be endowed with the $G$-metric:

$$G(0,0,0) = G(1,1,1) = 0,\ \ G(0,0,1) = 1,\ \ G(0,1,1) = 2$$

(extended by symmetry in its three variables). As $G(0,0,1) \neq G(0,1,1)$, $G$ is not symmetric. Take $T : X \to X$ given by

$$T0 = T1 = 0.$$

For all $x,y,z \in X$, we have $G(Tx,Ty,Tz) = 0$. Clearly (6.77) holds. Applying Theorem 6.5.1, $T$ has a unique fixed point, which is $\omega = 0$.

*Example 6.5.4.* Let $X = \{0,1,2\}$ be endowed with the $G$-metric:

$$\begin{aligned}
&G(0,0,0) = G(1,1,1) = G(2,2,2) = 0,\\
&G(0,0,1) = G(0,1,1) = 1, \quad G(1,2,2) = \frac{3}{2},\\
&G(0,0,2) = G(1,1,2) = G(0,2,2) = G(0,1,2) = 2
\end{aligned}$$

(extended by symmetry in its three variables). Note that $G$ is not symmetric because $G(1,2,2) \neq G(1,1,2)$. Define $T : X \to X$ by

$$T0 = T1 = 0 \quad \text{and} \quad T2 = 1.$$

Let $\varepsilon > 0$. Taking $\delta = \frac{\varepsilon}{2}$, property (6.77) holds. Indeed, to prove this assertion, we distinguish three cases.

- Case 1: If

$$\begin{aligned}
(x,y,z) \in \{&(0,0,0),(1,1,1),(2,2,2),(0,0,1),(0,1,0),\\
&(1,0,0),(1,1,0),(0,1,1),(1,0,1)\},
\end{aligned}$$

then by a simple calculation we get

$$G(Tx, Ty, Tz) = 0,$$

and so clearly (6.77) holds.

- Case 2: If

$$\begin{aligned}(x, y, z) \in \{&(0,0,2), (0,2,0), (2,0,0), (0,1,2), (0,2,1),\\ &(1,0,2), (1,2,0), (2,1,0), (2,0,1), (0,2,2),\\ &(2,2,0), (2,0,2), (1,1,2), (1,2,1), (2,1,1)\},\end{aligned}$$

it is easy to see that

$$G(Tx, Ty, Tz) = 1 \quad \text{and} \quad M(x, y, z) = 2.$$

By taking $\delta = \varepsilon/2$, property (6.77) is satisfied.

- Case 3: If $(x, y, z) \in \{(1,2,2), (2,2,1), (2,1,2)\}$, we have

$$G(Tx, Ty, Tz) = 1 \quad \text{and} \quad M(x, y, z) = \frac{3}{2}.$$

Similarly, property (6.77) is satisfied. Applying Theorem 6.5.1, the map $T$ has a unique fixed point, which is $\omega = 0$.

### 6.5.2 $\phi$-Asymmetric Meir–Keeler Contractive Mappings on G-Metric Spaces

In this subsection we introduce a slightly different notion of a Meir–Keeler contraction using the following control functions. Recall that $\mathcal{F}_{\text{alt}}$ denotes the family of all continuous, non-decreasing functions $\phi : [0, \infty) \to [0, \infty)$ such that $\phi^{-1}(\{0\}) = \{0\}$.

**Definition 6.5.5.** Let $(X, G)$ be a $G$-metric space and $\phi \in \mathcal{F}_{\text{alt}}$. Suppose that $T : X \to X$ is a self-mapping satisfying the following condition: for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \le \phi(G(x, Tx, y)) < \varepsilon + \delta \quad \Rightarrow \quad \phi(G(Tx, T^2x, Ty)) < \varepsilon \tag{6.92}$$

for all $x, y \in X$. Then $T$ is called a *$\phi$-Asymmetric Meir–Keeler contractive mapping*.

*Remark 6.5.3.* Notice that if $T : X \to X$ is a $\phi$-Asymmetric Meir–Keeler contractive mapping and $x, y \in X$ satisfy that $x \ne Tx$ or $x \ne y$, then

$$\phi(G(Tx, T^2x, Ty)) < \phi(G(x, Tx, y)). \tag{6.93}$$

**Theorem 6.5.2.** *Let $(X, G)$ be a G-complete G-metric space and let $\phi \in \mathcal{F}_{\text{alt}}$ be an altering distance function. Suppose that $T : X \to X$ is a $\phi$-Asymmetric Meir–Keeler contractive mapping. Then $T$ has a unique fixed point.*

*Proof.* Take $x_0 \in X$. We construct a sequence $\{x_n\}_{n=0}^{\infty}$ of points in $X$ in the following way:

$$x_{n+1} = Tx_n \text{ for all } n = 0, 1, 2, \ldots$$

Notice that if there is $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then obviously $T$ has a fixed point. Thus, suppose that

$$x_n \neq x_{n+1} \qquad \text{for all } n \geq 0. \tag{6.94}$$

From $(G_2)$, we have

$$G(x_n, x_{n+1}, x_{n+1}) > 0 \qquad \text{for all } n \geq 0. \tag{6.95}$$

Let $s_n = G(x_n, x_{n+1}, x_{n+1}) > 0$ for all $n \geq 0$. By (6.93), we observe that for all $n \geq 0$,

$$\begin{aligned}\phi(s_{n+1}) = \phi(G(x_{n+1}, x_{n+2}, x_{n+2})) &= \phi(G(Tx_n, T^2x_n, Tx_{n+1})) \\ &< \phi(G(x_n, Tx_n, x_{n+1})) = \phi(G(x_n, x_{n+1}, x_{n+1})) = \phi(s_n).\end{aligned} \tag{6.96}$$

Therefore, $\{\phi(s_n)\}$ is a decreasing sequence in $[0, \infty)$ and, thus, it is convergent. Let $L \in [0, \infty)$ its limit and we claim that $L = 0$. Suppose, on the contrary, that $L > 0$. Thus, we have

$$0 < L \leq \phi(s_n) = \phi(G(x_n, x_{n+1}, x_{n+1})) \qquad \text{for all } n \geq 0. \tag{6.97}$$

Let $\varepsilon = L > 0$. Since $T : X \to X$ is a $\phi$-Asymmetric Meir–Keeler contractive mapping, there exists $\delta > 0$ such that (6.92) holds. As $\{\phi(s_n)\} \to L$, there exists $n_0 \in \mathbb{N}$ such that $L \leq \phi(s_{n_0}) < L + \delta$. Therefore, (6.92) implies that

$$\begin{aligned}&\varepsilon \leq \phi(G(x_{n_0}, Tx_{n_0+1}, x_{n_0+1})) = \phi(s_{n_0}) < \varepsilon + \delta \\ &\quad\Rightarrow \quad \phi(s_{n_0+1}) = \phi(G(x_{n_0+1}, x_{n_0+2}, x_{n_0+2})) \\ &\qquad\qquad\qquad = \phi(G(Tx_{n_0}, T^2x_{n_0+1}, Tx_{n_0+1})) < \varepsilon = L,\end{aligned}$$

which contradicts (6.97). Therefore, $L = 0$, which means that $\{\phi(s_n)\} \to 0$. From Lemma 2.3.3, $\{G(x_n, x_{n+1}, x_{n+1})\} \to 0$.

Next, we show that $\{x_n\}_{n=0}^{\infty}$ is a $G$-Cauchy sequence reasoning by contradiction. Suppose that $\{x_n\}$ is not a $G$-Cauchy sequence. In such a case, Lemma 4.1.5 guarantees that there exist a positive real number $\varepsilon_0 > 0$ and two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that, for all $k \in \mathbb{N}$,

$$k \leq m(k) < n(k) < m(k+1),$$
$$G\left(x_{m(k)}, Tx_{m(k)}, x_{n(k)-1}\right) \leq \varepsilon_0 < G\left(x_{m(k)}, Tx_{m(k)}, x_{n(k)}\right) \tag{6.98}$$

and also

$$\lim_{k\to\infty} G\left(x_{m(k)}, Tx_{m(k)}, x_{n(k)}\right) = \lim_{k\to\infty} G\left(x_{m(k)-1}, Tx_{m(k)-1}, x_{n(k)-1}\right) = \varepsilon_0.$$

Let $\varepsilon_1 = \phi(\varepsilon_0) > 0$. Since $\phi$ is non-decreasing,

$$\varepsilon_1 = \phi(\varepsilon_0) \leq \phi\left(G\left(x_{m(k)}, Tx_{m(k)}, x_{n(k)}\right)\right) \quad \text{for all } k \in \mathbb{N}. \tag{6.99}$$

As $\phi$ is continuous,

$$\lim_{k\to\infty} \phi\left(G\left(x_{m(k)-1}, Tx_{m(k)-1}, x_{n(k)-1}\right)\right) = \phi(\varepsilon_0) = \varepsilon_1. \tag{6.100}$$

In particular, there exists $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$,

$$\frac{\varepsilon_1}{2} < \phi\left(G\left(x_{m(k)-1}, Tx_{m(k)-1}, x_{n(k)-1}\right)\right).$$

We claim that

$$\varepsilon_1 \leq \phi\left(G\left(x_{m(k)-1}, Tx_{m(k)-1}, x_{n(k)-1}\right)\right) \quad \text{for all } k \geq k_0. \tag{6.101}$$

To prove this, assume that there exists $k' \geq k_0$ such that

$$\frac{\varepsilon_1}{2} < \phi\left(G\left(x_{m(k')-1}, Tx_{m(k')-1}, x_{n(k')-1}\right)\right) < \varepsilon_1.$$

In this case, as $T$ is a $\phi$-Asymmetric Meir–Keeler contractive mapping, corresponding to $\varepsilon = \phi\left(G\left(x_{m(k')-1}, Tx_{m(k')-1}, x_{n(k')-1}\right)\right) > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq \phi(G(x, Tx, y)) < \varepsilon + \delta \quad \Rightarrow \quad \phi(G(Tx, T^2x, Ty)) < \varepsilon.$$

Using $x = x_{m(k')-1}$ and $y = x_{n(k')-1}$, we see that

$$\begin{aligned}\phi(G(x_{m(k')}, Tx_{m(k')}, x_{n(k')})) &= \phi(G(Tx_{m(k')-1}, T^2x_{m(k')-1}, Tx_{n(k')-1})) \\ &< \varepsilon < \varepsilon_1 = \phi(\varepsilon_0),\end{aligned} \tag{6.102}$$

which contradicts (6.102). This contradiction shows that (6.101) holds. As $T$ is a $\phi$-Asymmetric Meir–Keeler contractive mapping, corresponding to $\varepsilon_1 > 0$, there exists $\delta_1 > 0$ such that

$$\varepsilon_1 \leq \phi(G(x, Tx, y)) < \varepsilon_1 + \delta_1 \quad \Rightarrow \quad \phi(G(Tx, T^2x, Ty)) < \varepsilon_1. \tag{6.103}$$

By (6.100) and (6.101), there exists $k_1 \geq k_0$ such that, for all $k \geq k_1$,

$$\varepsilon_1 \leq \phi\left(G\left(x_{m(k)-1}, Tx_{m(k)-1}, x_{n(k)-1}\right)\right) < \varepsilon_1 + \delta_1.$$

From (6.103), it follows that

$$\begin{aligned}\phi(G(x_{m(k_1)}, Tx_{m(k_1)}, x_{n(k_1)})) &= \phi(G(Tx_{m(k_1)-1}, T^2x_{m(k_1)-1}, Tx_{n(k_1)-1})) \\ &< \varepsilon_1 = \phi\left(\varepsilon_0\right),\end{aligned}$$

which contradicts (6.99). This contradiction proves that $\{x_n\}$ is a $G$-Cauchy sequence. Since $(X, G)$ is $G$-complete, there exists $w \in X$ such that $\{x_n\} \to w$ as $n \to \infty$. Since $G$ is a continuous function, then we have

$$\lim_{n\to\infty} G(x_n, x_{n+1}, w) = G(w, w, w) = 0. \tag{6.104}$$

We assert that $Tw = w$. Regarding (6.93), we have that

$$\begin{aligned}\phi(G(x_{n+1}, x_{n+2}, Tw)) &= \phi(G(Tx_n, T^2x_n, Tw)) \\ &< \phi(G(x_n, Tx_n, w)) = \phi(G(x_n, x_{n+1}, w)).\end{aligned} \tag{6.105}$$

Since $\phi$ is a continuous mapping, letting $n \to \infty$ in (6.105), it follows that $\phi(G(w, w, Tw)) \leq \phi(G(w, w, w)) = 0$. Consequently, we have $G(w, w, Tw) = 0$. Hence, by $(G_2)$, we have $Tw = w$.

Now we show $w$ is the unique fixed point of $T$. By contradiction, if there exists $u \in X$ such that $u \neq w$ and $u = Tu$, then (6.93) implies that

$$\phi(G(u, w, w)) = \phi(G(Tw, T^2w, Tu)) < \phi(G(w, Tw, u)) = \phi(G(w, w, u))$$

which is a contradiction. Thus $w$ is the unique fixed point of $T$. □

In the next result, we use the altering distance function given by $\phi(t) = t$ for all $t \in [0, \infty)$.

**Corollary 6.5.1.** *Let $(X, G)$ be a $G$-complete $G$-metric space and let $T : X \to X$ be a self-mapping verifying that for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$x, y \in X, \quad \varepsilon \leq G(x, Tx, y) < \varepsilon + \delta \quad \Rightarrow \quad G(Tx, T^2x, Ty) < \varepsilon. \tag{6.106}$$

*Then $T$ has a unique fixed point.*

*Example 6.5.5.* Let $X = [0, \infty)$. Define $G : X^3 \to [0, \infty)$, for all $x, y, z \in X$, by

$$G(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max(x, y) + \max(y, z) + \max(x, z), & \textit{otherwise}. \end{cases}$$

Clearly, $(X, G)$ is a $G$-complete $G$-metric space. Define $T : X \to X$ by $Tx = x/4$ for all $x \in X$. Taking into account that

$$\max(Tx, Ty) = \max(x/4, y/4) = \max(x, y)/4,$$

it is not difficult to prove that

$$G(Tx, Ty, Tz) = G(x, y, z)/4$$

for all $x, y, z \in X$. In particular, $G(Tx, T^2x, Ty) = G(x, Tx, y)/4$ for all $x, y \in X$, which means that (6.106) holds (it is only necessary to take $\delta = \varepsilon/3$). Therefore, the conditions of Corollary 6.5.1 hold and $T$ has a unique fixed point.

The following result uses a contractivity condition in the orbit of a point.

**Theorem 6.5.3.** *Let $(X, G)$ be a $G$-metric space and let $T : X \to X$ be a self-mapping. Assume that there exists a point $x_0 \in X$ satisfying:*

- *the orbit $O_T(x_0)$ of $x_0$ with respect to $T$ has a cluster point $z \in X$;*
- *$T$ is orbitally $G$-continuous at $z$;*
- *there exists a mapping $\psi_{x_0} \in \mathcal{F}_{\text{alt}}$ with the following property: for all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\left.\begin{array}{l} x, Tx \in \overline{O_T(x_0)}, \quad y = Tx \neq x, \\ \varepsilon \leq \psi_{x_0}(G(x, Tx, y)) < \varepsilon + \delta \end{array}\right\} \Rightarrow \psi_{x_0}(G(Tx, T^2x, Ty)) < \varepsilon. \tag{6.107}$$

*Then $z$ is a fixed point of $T$ in $\overline{O_T(x_0)}$.*

*Proof.* Consider the sequence $\{x_n = T^n x_0\}_{n\geq 0}$. Following the proof in Theorem 6.5.2, we can reduce to the case in which $x_n \neq x_{n+1}$ for all $n \geq 0$, which yields $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$. Since $z \in X$ is a cluster point of $O_T(x_0)$, there is a strictly increasing sequence $\{n(k)\}_{k\geq 1}$ of non-negative integer numbers satisfying $\{x_{n(k)} = T^{n(k)}x_0\} \to z$, that is, $\lim_{k\to\infty} G(x_{n(k)}, z, z) = \lim_{k\to\infty} G(T^{n(k)}x_0, z, z) = 0$. Since $T$ is orbitally $G$-continuous at $z$, then

$$\lim_{k\to\infty} G(x_{n(k)+1}, Tz, Tz) = \lim_{k\to\infty} G(TT^{n(k)}x_0, Tz, Tz) = 0.$$

From the modified triangle inequality $(G_5)$ together with Lemma 3.1.1, we have

$$\begin{aligned} G(z, Tz, Tz) &\leq G(z, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)+1}, Tz, Tz) \\ &\leq G(z, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) \\ &\qquad + G(x_{n(k)+1}, Tz, Tz) \end{aligned}$$

$$\leq 2G(x_{n(k)}, z, z) + G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) \\ + G(x_{n(k)+1}, Tz, Tz).$$

Letting $k \to \infty$ in the previous inequality, we deduce that $G(z, Tz, Tz) = 0$ and, hence, $Tz = z$. □

**Corollary 6.5.2.** *Let $(X, G)$ be a G-metric space and let $T : X \to X$ be a self-mapping. Assume that there exists a point $x_0 \in X$ verifying:*

- *the orbit $O_T(x_0)$ of $x_0$ with respect to $T$ has a cluster point $z \in X$;*
- *$T$ is orbitally G-continuous at $z$;*
- *for all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\left.\begin{array}{l} x, Tx \in \overline{O_T(x_0)}, \quad y = Tx \neq x, \\ \varepsilon \leq G(x, Tx, y) < \varepsilon + \delta \end{array}\right\} \quad \Rightarrow \quad G(Tx, T^2x, Ty) < \varepsilon.$$

*Then $z$ is a fixed point of $T$ in $\overline{O_T(x_0)}$.*

To show the applicability of Theorem 6.5.2, we present some immediate consequences to guarantee the existence of fixed points of integral type nonlinear operators. For this purpose, let $\Omega$ be the family of all Lebesgue integrable mappings $\chi : [0, \infty) \to [0, \infty)$ such that $\int_0^{\varepsilon} \chi(t)\, dt > 0$ for each $\varepsilon > 0$.

**Theorem 6.5.4.** *Let $(X, G)$ be a G-complete G-metric space and let $\phi \in \mathcal{F}_{\mathrm{alt}}$ be non-decreasing. Suppose that $T : X \to X$ is a self-mapping satisfying the following condition: for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \leq \int_0^{\phi(G(x,Tx,y))} \chi(t)\, dt < \varepsilon + \delta \quad \Rightarrow \quad \int_0^{\phi\left(G\left(Tx,T^2x,Ty\right)\right)} \chi(t)\, dt < \varepsilon \tag{6.108}$$

*whatever $\chi \in \Omega$ and $x, y, z \in X$. Then $T$ has a unique fixed point.*

*Proof.* For $\chi \in \Omega$ , consider the function $\Lambda : [0, \infty) \to [0, \infty)$ defined by $\Lambda(x) = \int_0^x \chi(t)\, dt$ for all $x \in [0, \infty)$. We note that $\Lambda \in \mathcal{F}_{\mathrm{Kr}}$ and $\Lambda$ is non-decreasing. Thus the inequality (6.108) becomes: for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq \Lambda(\phi(G(x, Tx, y))) < \varepsilon + \delta \quad \Rightarrow \quad \Lambda\left(\phi\left(G\left(Tx, T^2x, Ty\right)\right)\right) < \varepsilon$$

whatever $x, y \in X$. Setting $\Lambda \circ \phi = \varphi$, we have that $\varphi \in \mathcal{F}_{\mathrm{alt}}$ and $T$ is $\varphi$-Asymmetric Meir-Keeler contractive. Hence, by using Theorem 6.5.2, $T$ has a unique fixed point. □

We have also the following result.

**Theorem 6.5.5.** *Let $(X, G)$ be a G-metric space and let $T : X \to X$ be a self-mapping. Assume that there exists a point $x_0 \in X$ satisfying:*

- *the orbit $O_T(x_0)$ of $x_0$ with respect to $T$ has a cluster point $z \in X$;*
- *$T$ is orbitally $G$-continuous at $z$;*
- *there exists $\psi \in \mathcal{F}_{\text{alt}}$ such that for all $\varepsilon > 0$, there exists $\delta > 0$ verifying*

$$\varepsilon \le \int_0^{\psi(G(x,Tx,y))} \chi(t)\,dt < \varepsilon + \delta \quad \Rightarrow \quad \int_0^{\psi\left(G\left(Tx,T^2x,Ty\right)\right)} \chi(t)\,dt < \varepsilon$$

*whatever $\chi \in \Omega$ and $x, y \in X$.*

*Then $z$ is a fixed point of $T$ in $\overline{O_T(x_0)}$.*

### *6.5.3 Generalized Meir-Keeler Type Contractions on Partially Ordered G-Metric Spaces*

We say that the tripled $(x, y, z) \in X^3$ is distinct if at least one of the following holds

$$(i)\;\; x \neq y, \qquad (ii)\;\; y \neq z, \qquad (iii)\;\; x \neq z.$$

The tripled $(x, y, z) \in X^3$ is called strictly distinct if all inequalities $(i) - (iii)$ hold.

**Definition 6.5.6 ([71]).** Let $(X, G, \preccurlyeq)$ be a partially ordered $G$-metric space. We say that a self-mapping $T : X \to X$ is

- *G-Meir-Keeler contractive* if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y, z \in X$ with $x \preccurlyeq y \preccurlyeq z$,

$$\varepsilon \le G(x, y, z) < \varepsilon + \delta \quad \Rightarrow \quad G(Tx, Ty, Tz) < \varepsilon. \tag{6.109}$$

- *G-Meir-Keeler contractive of second type* if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in X$ with $x \preccurlyeq y$,

$$\varepsilon \le G(x, y, y) < \varepsilon + \delta \quad \Rightarrow \quad G(Tx, Ty, Ty) < \varepsilon. \tag{6.110}$$

*Remark 6.5.4.* Notice that if $T : X \to X$ is $G$-Meir-Keeler contractive on a $G$-metric space $(X, G)$ then $T$ is contractive, that is,

$$G(Tx, Ty, Tz) < G(x, y, z), \tag{6.111}$$

for all distinct tripled $(x, y, z) \in X^3$ with $x \preccurlyeq y \preccurlyeq z$.

*Remark 6.5.5.* It is easy to see that a $G$-Meir-Keeler contraction must be $G$-Meir-Keeler contractive of second type. In addition, if $T : X \to X$ is $G$-Meir-Keeler contractive of second type on a partially ordered $G$-metric space $(X, G, \preccurlyeq)$, then

$$G(Tx, Ty, Ty) < G(x, y, y), \tag{6.112}$$

for all $(x, y) \in X^2$ with $x \prec y$. Moreover, we have

$$G(Tx, Ty, Ty) \leq G(x, y, y), \tag{6.113}$$

for all $(x, y) \in X^2$ with $x \preccurlyeq y$.

**Theorem 6.5.6 ([71]).** *Let $(X, \preccurlyeq)$ be a partially ordered set endowed with a G-metric $G$ and let $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

*(i) $(X, G)$ is G-complete;*
*(ii) $T$ is non-decreasing (with respect to $\preccurlyeq$);*
*(iii) there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;*
*(iv) $T$ is G-continuous*
*(v) $T : X \to X$ is G-Meir-Keeler contractive of second type.*

*Then $T$ has a fixed point. Moreover, if for all $x, y \in \mathrm{Fix}(T)$ there exists $w \in X$ such that $x \preccurlyeq w$ and $y \preccurlyeq w$, we obtain uniqueness of the fixed point.*

*Proof.* Take $x_0 \in X$ such that condition (iii) holds, that is, $x_0 \preccurlyeq Tx_0$. We construct an iterative sequence $\{x_n\}$ in $X$ as follows:

$$x_{n+1} = Tx_n \quad \text{for all } n \geq 0. \tag{6.114}$$

Taking into account that $T$ is a non-decreasing mapping together with (6.114), we have that $x_0 \preccurlyeq Tx_0 = x_1$ implies $x_1 = Tx_0 \preccurlyeq Tx_1 = x_2$. By induction, we get

$$x_0 \preccurlyeq x_1 \preccurlyeq x_2 \preccurlyeq \ldots \preccurlyeq x_{n-1} \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq \ldots \tag{6.115}$$

Suppose that there exists $n_0$ such that $x_{n_0} = x_{n_0+1}$. Since $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, then $x_{n_0}$ is the fixed point of $T$, which completes the existence part of the proof. Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Thus, by (6.115) we have

$$x_0 \prec x_1 \prec x_2 \prec \ldots \prec x_{n-1} \prec x_n \prec x_{n+1} \prec \ldots \tag{6.116}$$

From $(G_2)$, we have

$$G(x_n, x_{n+1}, x_{n+1}) > 0 \quad \text{for all } n \geq 0. \tag{6.117}$$

By Remark 6.5.5, we observe that, for all $n \geq 0$,

$$G(x_{n+1}, x_{n+2}, x_{n+2}) = G(Tx_n, Tx_{n+1}, Tx_{n+1}) < G(x_n, x_{n+1}, x_{n+1}). \tag{6.118}$$

Due to (6.118), the sequence $\{G(x_n, x_{n+1}, x_{n+1})\}$ is a (strictly) decreasing sequence in $[0, \infty)$ and, thus, it is convergent, say $L \in [0, \infty)$. We claim that $L = 0$. Suppose, on the contrary, that $L > 0$. Thus, we have

$$0 < L < G(x_n, x_{n+1}, x_{n+1}) \quad \text{for all } n \geq 0. \tag{6.119}$$

Assume $\varepsilon = L > 0$. As $T$ is a $G$-Meir-Keeler contraction of second type, there exists a convenient $\delta > 0$ such that

$$\varepsilon \leq G(x, y, y) < \varepsilon + \delta \quad \Rightarrow \quad G(Tx, Ty, Ty) < \varepsilon. \tag{6.120}$$

Since $\{G(x_n, x_{n+1}, x_{n+1})\} \searrow L$, there exists $n_0 \in \mathbb{N}$ such that

$$\varepsilon < G(x_{n_0}, x_{n_0+1}, x_{n_0+1}) < \varepsilon + \delta. \tag{6.121}$$

Taking the condition (6.120) into account, the expression (6.121) yields that

$$G(x_{n_0+1}, x_{n_0+2}, x_{n_0+2}) = G(Tx_{n_0}, Tx_{n_0+1}, Tx_{n_0+1}) < \varepsilon = L \tag{6.122}$$

which contradicts (6.119). Hence $L = 0$, that is,

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0.$$

We show that $\{x_n\}_{n\geq 0}$ is a $G$-Cauchy sequence. Let $\varepsilon > 0$ be arbitrary. As $T$ is $G$-Meir-Keeler contractive of second type, there exists $\delta > 0$ such that (6.110) holds. Without loss of generality, we assume $\delta < \varepsilon$. Since $L = 0$, there exists $n_0 \in \mathbb{N}$ such that

$$G(x_{n-1}, x_n, x_n) < \delta \quad \text{for all } n \geq n_0. \tag{6.123}$$

We assert that for any fixed $n \geq n_0$,

$$G(x_n, x_{n+m}, x_{n+m}) \leq \varepsilon \quad \text{for all } m \geq 0. \tag{6.124}$$

To prove the assertion, we use the method of induction. Regarding (6.123), the assertion (6.124) is satisfied for $m = 0$ and $m = 1$. Suppose the assertion (6.124) is satisfied for some $m \in \mathbb{N}$. For $m + 1$, with the help of $(G_5)$ and (6.123), we consider

$$G(x_{n-1}, x_{n+m}, x_{n+m}) \leq G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+m}, x_{n+m}) < \delta + \varepsilon. \tag{6.125}$$

Next, we distinguish three cases.

- If $G(x_{n-1}, x_{n+m}, x_{n+m}) \geq \varepsilon$, then, by (6.110), we get

$$G(x_n, x_{n+m+1}, x_{n+m+1}) = G(Tx_{n-1}, Tx_{n+m}, Tx_{n+m}) < \varepsilon. \tag{6.126}$$

  Hence (6.124) is satisfied.

- If $G(x_{n-1},x_{n+m},x_{n+m}) = 0$, then, by $(G_2)$, we derive that $x_{n-1} = x_{n+m}$ and, hence, $x_n = Tx_{n-1} = Tx_{n+m} = x_{n+m+1}$. From $(G_1)$, we have

$$G(x_n,x_{n+m+1},x_{n+m+1}) = G(x_n,x_n,x_n) = 0 < \varepsilon,$$

and, thus, (6.124) is satisfied.
- If $0 < G(x_{n-1},x_{n+m},x_{n+m}) < \varepsilon$, then by Remark 6.5.5,

$$\begin{aligned} G(x_n,x_{n+m+1},x_{n+m+1}) &= G(Tx_{n-1},Tx_{n+m},Tx_{n+m}) \\ &\leq G(x_{n-1},x_{n+m},x_{n+m}) < \varepsilon. \end{aligned}$$

Consequently, (6.124) is satisfied for $m+1$ and this completes the induction. Hence, $G(x_n,x_{n+m},x_{n+m}) \leq \varepsilon$ for all $n \geq n_0$ and $m \geq 0$, which means

$$G(x_n,x_m,x_m) < \varepsilon, \quad \text{for all } m \geq n \geq n_0. \tag{6.127}$$

As a consequence, $\{x_n\}$ is a $G$-Cauchy sequence. Since $(X,G)$ is $G$-complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} G(x_n,u,u) = 0. \tag{6.128}$$

We now show that $u \in X$ is a fixed point of $T$, that is, $u = Tu$. Since $T$ is $G$-continuous, the sequence $\{Tx_n\} = \{x_{n+1}\}$ converges to $Tu$, that is,

$$\lim_{n\to\infty} G(x_{n+1},Tu,Tu) = \lim_{n\to\infty} G(Tx_n,Tu,Tu) = 0. \tag{6.129}$$

Hence, $\{x_n\}$ also converges to $Tu$. By the uniqueness of the limit, we deduce that $u = Tu$, that is, $u$ is a fixed point of $T$.

To prove the uniqueness, let $x,y \in X$ be fixed points of $T$. From the additional assumption, we know that there exists $w \in X$ such that $x \preccurlyeq w$ and $y \preccurlyeq w$. We claim that the sequence $\{T^n w\}$ converges, at the same time, to $x$ and to $y$ (so we will deduce $x = y$). We only reason using $x$ (but the same is true for $y$). From Remark 6.5.5, we get

$$G(x,Tw,Tw) = G(Tx,Tw,Tw) \leq G(x,w,w).$$

Since $T$ is non-decreasing, $x = Tx \preccurlyeq Tw$. Again by Remark 6.5.5, we get

$$G(x,T^2w,T^2w) = G(Tx,TTw,TTw) \leq G(x,Tw,Tw).$$

Continuing in this way, we conclude

$$G(x,T^n w,T^n w) \leq \cdots \leq G(x,Tw,Tw) \leq G(x,w,w).$$

Hence $\{G(x, T^n w, T^n w)\}_{n\geq 0}$ is a non-increasing sequence bounded below by zero. Thus, there exists $L \geq 0$ such that

$$\lim_{n\to\infty} G(x, T^n w, T^n w) = L \quad \text{and}$$
$$L \leq G(x, T^n w, T^n w) \quad \text{for all } n \geq 0. \tag{6.130}$$

We claim that $L = 0$. Suppose, on the contrary, that $L > 0$. Choose $\varepsilon = L > 0$ and let $\delta > 0$ be such that (6.110) holds. Then, there exists $n_0$ such that $L \leq G(x, T^{n_0} w, T^{n_0} w) < L + \delta$ which implies

$$G(x, T^{n_0+1} w, T^{n_0+1} w) = G(Tx, T^{n_0+1} w, T^{n_0+1} w) < L,$$

which contradicts (6.130). Hence $\lim_{n\to\infty} G(x, T^n w, T^n w) = L = 0$, so $\{T^n w\} \to x$. Similarly, it can be proved that $\{T^n w\} \to y$, so $x = y$ and $T$ has a unique fixed point. □

As every $G$-Meir-Keeler contractive mapping of second type is also $G$-Meir-Keeler contractive, we deduce the following consequence.

**Corollary 6.5.3 ([71]).** *Let $(X, \preccurlyeq)$ be a partially ordered set endowed with a G-metric $G$ and let $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

*(i) $(X, G)$ is G-complete;*
*(ii) $T$ is non-decreasing (with respect to $\preccurlyeq$);*
*(iii) there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;*
*(iv) $T$ is G-continuous*
*(v) $T : X \to X$ is G-Meir-Keeler contractive.*

*Then $T$ has a fixed point. Moreover, if for all $x, y \in \mathrm{Fix}(T)$ there exists $w \in X$ such that $x \preccurlyeq w$ and $y \preccurlyeq w$, we obtain uniqueness of the fixed point.*

In the following result, we replace condition (iv) in Theorem 6.5.6 with the assumption that $(X, G, \preccurlyeq)$ is non-decreasing-regular (recall Definition 5.2.1 and the fact that some authors use the term *ordered complete* for *non-decreasing-regularity*), and we obtain a similar result.

**Theorem 6.5.7 ([71]).** *Let $(X, \preccurlyeq)$ be a partially ordered set endowed with a G-metric and let $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

*(i) $(X, G)$ is G-complete;*
*(ii) $T$ is non-decreasing (with respect to $\preccurlyeq$);*
*(iii) there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;*
*(iv) $(X, G, \preccurlyeq)$ is non-decreasing-regular;*
*(v) $T : X \to X$ is G-Meir-Keeler contractive of second type.*

*Then $T$ has a fixed point. Moreover, if for all $x, y \in \mathrm{Fix}(T)$ there exists $w \in X$ such that $x \preccurlyeq w$ and $y \preccurlyeq w$, we obtain uniqueness of the fixed point.*

*Proof.* Following the proof in Theorem 6.5.6, we can deduce that the Picard sequence $\{x_{n+1} = Tx_n\}$ converges to some $u \in X$. We only need to show $u = Tu$. Since $\{x_n\}$ is $\preccurlyeq$-non-decreasing and $(X, G, \preccurlyeq)$ is non-decreasing-regular, we conclude $x_n \preccurlyeq u$ for all $n$. Then, by Remark 6.5.5, $(G_5)$ and (6.128), we get

$$\begin{aligned} G(Tu, u, u) &\leq G(Tu, x_n, x_n) + G(x_n, u, u) \\ &= G(Tx_{n-1}, Tx_{n-1}, Tu) + G(x_n, u, u) \\ &\leq G(x_{n-1}, x_{n-1}, u) + G(x_n, u, u). \end{aligned}$$

Letting $n \to \infty$, we conclude that $G(Tu, u, u) = 0$, i.e., $Tu = u$. The rest of the proof is similar. □

**Corollary 6.5.4 ([71]).** *Let $(X, \preccurlyeq)$ be a partially ordered set endowed with a G-metric and $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

*(i)* *$(X, G)$ is G-complete;*
*(ii)* *$T$ is non-decreasing (with respect to $\preccurlyeq$);*
*(iii)* *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;*
*(iv)* *$(X, G, \preccurlyeq)$ is non-decreasing-regular;*
*(v)* *$T : X \to X$ is G-Meir-Keeler contractive.*

*Then $T$ has a fixed point. Moreover, if for all $x, y \in \mathrm{Fix}(T)$ there exists $w \in X$ such that $x \preccurlyeq w$ and $y \preccurlyeq w$, we obtain uniqueness of the fixed point.*

To finish the chapter, we will show a version of Theorem 6.5.6 using integral contractivity conditions.

**Lemma 6.5.1 ([71]).** *Let $(X, \preccurlyeq)$ be a partially ordered set endowed with a G-metric $G$ and let $T : X \to X$ be a given mapping. Suppose that there exists a function $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:*

**(F1)** *$\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$;*
**(F2)** *$\varphi$ is increasing and right continuous;*
**(F3)** *for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in X$ with $x \preccurlyeq y$,*

$$\varepsilon \leq \varphi(G(x, y, y)) < \varepsilon + \delta \quad \textit{implies} \quad \varphi(G(Tx, Ty, Ty)) < \varphi(\varepsilon). \tag{6.131}$$

*Then $T$ is a G-Meir-Keeler contractive mapping of second type.*

*Proof.* Let $\varepsilon > 0$ be arbitrary. Due to (F1), we have $\varepsilon' = \varphi(\varepsilon) > 0$. Thus there exists $\delta' > 0$ such that

$$\varphi(\varepsilon) \leq \varphi(G(x, y, y)) < \varphi(\varepsilon) + \delta' \quad \text{implies} \quad \varphi(G(Tx, Ty, Ty)) < \varphi(\varepsilon) \tag{6.132}$$

From the right continuity of $\varphi$, there exists $\delta > 0$ such that $\varphi(\varepsilon + \delta) < \varphi(\varepsilon) + \delta'$. Fix $x, y \in X$ with $x \preccurlyeq y$ such that $\varepsilon \leq G(x, y, y) < \varepsilon + \delta$. So we have

$$\varphi(\varepsilon) \leq \varphi(G(x, y, y)) \leq \varphi(\varepsilon + \delta) < \varphi(\varepsilon) + \delta'.$$

Hence, $\varphi(G(Tx, Ty, Ty)) < \varphi(\varepsilon)$. Thus, we have $G(Tx, Ty, Ty) < \varepsilon$ which completes the proof. □

Since a function $t \to \int_0^t f(s)ds$ is absolutely continuous, we derive the following corollary from Theorem 6.5.6 and Lemma 6.5.1.

**Corollary 6.5.5 ([71]).** *Let $(X, \preccurlyeq)$ be a partially ordered set endowed with a G-metric $G$, $T : X \to X$ be a given mapping, and $f$ be a locally integrable function from $[0, \infty)$ into itself satisfying $\int_0^t f(s)ds > 0$ for all $t > 0$. Assume that conditions (i)–(iv) of Theorem 6.5.6 hold, and for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \leq \int_0^{G(x,y,y)} f(s)ds < \varepsilon + \delta \quad \Rightarrow \quad \int_0^{G(Tx,Ty,Ty)} f(s)ds < \int_0^{\varepsilon} f(s)ds \tag{6.133}$$

*for all $x, y \in X$ with $x \preccurlyeq y$. Then $T$ has a fixed point. Moreover, if for all $x, y \in \mathrm{Fix}(T)$ there exists $w \in X$ such that $x \preccurlyeq w$ and $y \preccurlyeq w$, we obtain uniqueness of the fixed point.*

# Chapter 7
# Fixed Point Theorems via Admissible Mappings

In this chapter we explain how to use functions in order to extend the notion of partial order or, more precisely, how non-decreasing mappings can be interpreted involving certain classes of admissible functions. The results we present are inspired by Samet et al. [183].

Throughout this chapter, we will employ the family $\mathcal{F}_{\mathrm{KR}}$ of all non-decreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that, for all $t > 0$, the series $\sum_{n\geq 1} \psi^n(t)$ converges. Since $(\mathcal{P}_{10}) \Rightarrow (\mathcal{P}_{11}) \Rightarrow (\mathcal{P}_{12})$ (recall Subsection §2.3) and using the monotonicity, these functions also verify the following properties.

- $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$.
- $\psi(t) < t$ for all $t > 0$.
- $\psi(0) = 0$ and $\psi$ is continuous at 0.

In particular, every (c)-comparison function belongs to $\mathcal{F}_{\mathrm{KR}}$, that is, $\mathcal{F}_{\mathrm{com}}^{(c)} \subset \mathcal{F}_{\mathrm{KR}}$. As a consequence, all the following results can be particularized to the case in which $\psi \in \mathcal{F}_{\mathrm{com}}^{(c)}$.

## 7.1 Fixed Point Results in *G*-Metric Spaces via Admissible Mappings

In this section, we present the notion of an $\alpha$ - $\psi$ - contractive mapping in metric spaces due to Samet et al. [183] to $G$-metric spaces using control functions in $\mathcal{F}_{\mathrm{com}}^{(c)}$.

**Definition 7.1.1.** Let $(X,G)$ be a $G$-metric space and let $T : X \to X$ be a given mapping. We say that $T$ is a

- *$G$ - $\beta$ - $\psi$ - contractive mapping of type I* if there exist two functions $\beta : X \times X \times X \to [0,\infty)$ and $\psi \in \mathcal{F}_{\mathrm{com}}^{(c)}$ such that, for all $x,y,z \in X$, we have



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$$\beta\,(x, y, z)\; G\,(Tx, Ty, Tz) \leq \psi\,(G\,(x, y, z))\,. \tag{7.1}$$

- *$G$ - $\beta$ - $\psi$ - contractive mapping of type II* if there exist two functions $\beta : X \times X \times X \to [0, \infty)$ and $\psi \in \mathcal{F}_{\text{com}}^{(c)}$ such that for all $x, y \in X$, we have

$$\beta\,(x, y, y)\; G\,(Tx, Ty, Ty) \leq \psi\,(G\,(x, y, y))\,. \tag{7.2}$$

Clearly, any contractive mapping, that is, a mapping satisfying (4.15), is a $G$ - $\beta$ - $\psi$ - contractive mapping of type I with $\beta\,(x, y, z) = 1$ for all $x, y, z \in X$ and $\psi_\lambda\,(t) = \lambda\, t$ for all $t \geq 0$, where $\lambda \in [0, 1)$. Analogously, a mapping satisfying (4.16) is a $G$ - $\beta$ - $\psi$ - contractive mapping of type II using the same $\beta$ and $\psi_\lambda$ as before.

**Definition 7.1.2.** Let $X$ be a set and let $T : X \to X$ and $\beta : X \times X \times X \to [0, \infty)$ be two mappings. We say that $T$ is *$\beta$ - admissible* if, for all $x, y, z \in X$, we have

$$\beta\,(x, y, z) \geq 1 \quad \Longrightarrow \quad \beta\,(Tx, Ty, Tz) \geq 1.$$

*Example 7.1.1.* Let $X$ be a non-empty subset of $\mathbb{R}$ and define $\beta : X \times X \times X \to [0, \infty)$ as follows:

$$\beta(x, y, z) = \begin{cases} \mathrm{e}, \text{ if } \; x \geq y \geq z, \\ 0, \; \textit{otherwise}. \end{cases}$$

Then any non-decreasing mapping $T : X \to X$ is $\beta$ - admissible.

**Theorem 7.1.1.** *Let $(X, G)$ be a complete $G$-metric space and let $T : X \to X$ be a $G$ - $\beta$ - $\psi$ - contractive mapping of type II satisfying the following conditions:*

**(i)** *$T$ is $\beta$ - admissible;*
**(ii)** *there exists $x_0 \in X$ such that $\beta\,(x_0, Tx_0, Tx_0) \geq 1$;*
**(iii)** *$T$ is $G$-continuous.*

*Then there exists $u \in X$ such that $Tu = u$.*

*Proof.* Let $x_0 \in X$ be such that $\beta\,(x_0, Tx_0, Tx_0) \geq 1$ and let $\{x_n\}$ be the Picard sequence of $T$ based on $x_0$ (that is, $x_{n+1} = Tx_n$ for all $n \geq 0$). If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $u = x_{n_0}$ is a fixed point of $T$. On the contrary case, assume that $x_n \neq x_{n+1}$ for all $n$. Since $T$ is $\beta$ - admissible, we have

$$\begin{aligned} \beta\,(x_0, x_1, x_1) &= \beta\,(x_0, Tx_0, Tx_0) \geq 1 \\ &\Longrightarrow \beta\,(x_1, x_2, x_2) = \beta\,(Tx_0, Tx_1, Tx_1) \geq 1. \end{aligned}$$

Inductively, we have that

$$\beta\,(x_n, x_{n+1}, x_{n+1}) \geq 1 \quad \text{for all } n \geq 0. \tag{7.3}$$

From (7.2) and (7.3), it follows that for all $n \geq 1$, we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \beta(x_{n-1}, x_n, x_n)\, G(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \psi(G(x_{n-1}, x_n, x_n)). \end{aligned}$$

From Lemma 4.1.2, $\{x_n\}$ is a Cauchy sequence in the $G$-metric space $(X, G)$. Since $(X, G)$ is complete, there exists $u \in X$ such that $\{x_n\}$ is convergent to $u$. Since $T$ is $G$-continuous, it follows that $\{x_{n+1} = Tx_n\}$ is $G$-convergent, at the same time, to $u$ and to $Tu$. By the uniqueness of the limit, we get $u = Tu$, that is, $u$ is a fixed point of $T$. □

The following corollary follows from the fact that every $G$-$\beta$-$\psi$-contractive mapping of type I is also of type II.

**Corollary 7.1.1.** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be a $G$-$\beta$-$\psi$-contractive mapping of type I satisfying the following conditions:*

**(i)** *$T$ is $\beta$-admissible;*
**(ii)** *there exists $x_0 \in X$ such that $\beta(x_0, Tx_0, Tx_0) \geq 1$;*
**(iii)** *$T$ is G-continuous.*

*Then there exists $u \in X$ such that $Tu = u$.*

**Theorem 7.1.2.** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be a $G$-$\beta$-$\psi$-contractive mapping of type II satisfying the following conditions:*

**(i)** *$T$ is $\beta$-admissible;*
**(ii)** *there exists $x_0 \in X$ such that $\beta(x_0, Tx_0, Tx_0) \geq 1$;*
**(iii)** *if $\{x_n\}$ is a sequence in $X$ such that $\beta(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\{x_n\}$ is G-convergent to $x \in X$, then $\beta(x_n, x, x) \geq 1$ for all $n \in \mathbb{N}$.*

*Then there exists $u \in X$ such that $Tu = u$.*

*Proof.* Following the argument in the proof of Theorem 7.1.1, we deduce that the Picard sequence $\{x_n\}$ converges to some $u \in X$. From (7.3) and (iii), we have

$$\beta(x_n, u, u) \geq 1 \quad \text{for all } n \geq 0. \tag{7.4}$$

Using $(G_5)$, $(G_4)$, (7.2) and (7.4), we have that, for all $n \in \mathbb{N}$,

$$\begin{aligned} G(u, Tu, Tu) &\leq G(u, Tx_n, Tx_n) + G(Tx_n, Tu, Tu) \\ &\leq G(u, x_{n+1}, x_{n+1}) + \beta(x_n, u, u)\, G(Tx_n, Tu, Tu) \\ &\leq G(u, x_{n+1}, x_{n+1}) + \psi(G(x_n, u, u)). \end{aligned}$$

Taking into account that $\psi$ is continuous at $t = 0$, letting $n \to \infty$, it follows that $G(u, Tu, Tu) = 0$, so $u = Tu$. □

**Corollary 7.1.2.** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be a $G$ - $\beta$ - $\psi$ - contractive mapping of type I satisfying the following conditions:*

**(i)** *$T$ is $\beta$ - admissible;*
**(ii)** *there exists $x_0 \in X$ such that $\beta\,(x_0, Tx_0, Tx_0) \geq 1$;*
**(iii)** *if $\{x_n\}$ is a sequence in $X$ such that $\beta\,(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\{x_n\}$ is G-convergent to $x \in X$, then $\beta\,(x_n, x, x) \geq 1$ for all $n \in \mathbb{N}$.*

*Then there exists $u \in X$ such that $Tu = u$.*

With the following example, we will show that the hypotheses in Theorems 7.1.1 and 7.1.2 do not guarantee uniqueness.

*Example 7.1.2.* Let $X = [0, \infty)$ endowed with the $G$-metric $G(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y \in X$. Consider the self-mapping $T : X \to X$ and the mapping $\beta : X \times X \times X \to [0, \infty)$ given by

$$Tx = \begin{cases} 2x - \frac{7}{4}, & \text{if } x > 1, \\ \dfrac{x}{4}, & \text{if } 0 \leq x \leq 1; \end{cases}$$

$$\beta(x, y, z) = \begin{cases} 1, & \text{if } y = z \text{ and } x, y \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

We claim that $T$ is a $G$ - $\beta$ - $\psi$ - contractive mapping of type II with respect to the (c)-comparison function $\psi(t) = t/2$ for all $t \geq 0$. To prove it, we observe that, for all $x, y \in X$, we have that

$$\beta(x, y, y)\, G(Tx, Ty, Ty) \leq \frac{1}{2} G(x, y, y).$$

Furthermore, for $x_0 = 1$ we have that $\beta(1, T1, T1) = \beta(1, \frac{1}{4}, \frac{1}{4}) = 1$. As $T$ is continuous, to show that $T$ satisfies all hypothesis of Theorem 7.1.1, it is sufficient to observe that $T$ is $\beta$-admissible. For this purpose, let $x, y \in X$ be such that $\beta(x, y, y) \geq 1$. In this case, $x, y \in [0, 1]$. Hence $Tx = x/4 \in [0, 1]$ and $Ty = y/4 \in [0, 1]$, which implies that $\beta(Tx, Ty, Ty) \geq 1$. As a result, all the conditions of Theorem 7.1.1 are satisfied. Theorem 7.1.1 guarantees the existence of a fixed point of $T$, but not its uniqueness. In this example, 0 and $\frac{7}{4}$ are two fixed points of $T$.

Notice that Theorem 4.16, given by Mustafa as a characterization of the Banach fixed point theorem, cannot be applied in this case because

$$G(T1, T2, T2) = 4 > 2 = G(1, 2, 2).$$

In the following example, $T$ is not continuous.

*Example 7.1.3.* Let $(X, G)$ and $\beta$ be given as in Example 7.1.2, and let $T$ and $\psi$ be given by

$$Tx = \begin{cases} 2x - \frac{7}{4}, & \text{if } x > 1, \\ \dfrac{x}{3}, & \text{if } 0 \leq x \leq 1; \end{cases} \qquad \psi(t) = t/3.$$

It is easy to show that, for all $x, y \in X$ we have

$$\beta(x, y, y)\, G(Tx, Ty, Ty) \leq \frac{1}{2} G(x, y, y).$$

Therefore, $T$ is a $G$ - $\beta$ - $\psi$ -contractive mapping of type II. Furthermore, the point $x_0 = 1$ verifies $\beta(1, T1, T1) = 1$ and $T$ is $\beta$-admissible. However, $T$ is not continuous. In this case, we can prove hypothesis (iii) of Theorem 7.1.2. Indeed, let $\{x_n\}$ be a sequence such that $\beta(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\{x_n\} \to x \in X$. Since $\beta(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $x_n \in [0, 1]$, and as this interval is closed, we deduce that $x \in [0, 1]$. Thus, $\beta(x_n, x, x) \geq 1$ for all $n \in \mathbb{N}$. As a result, all the conditions of Theorem 7.1.2 are satisfied. Theorem 7.1.2 guarantees the existence of a fixed point of $T$, but not its uniqueness. In fact, 0 and $\frac{7}{4}$ are two fixed points of $T$.

*Example 7.1.4 ([19]).* Let $X = [0, \infty)$ be endowed with the $G$-metric:

$$G(x, y, z) = |x - y| + |y - z| + |z - x| \quad \text{for all } x, y, z \in X.$$

Define $T : X \to X$ by $Tx = 3x$ for all $x \in X$, and $\beta : X \times X \times X \to [0, \infty)$ in the following way:

$$\beta(x, y, z) = \begin{cases} 1, & \text{if } (x, y, z) = (0, 0, 0), \\ \dfrac{1}{9}, & \text{otherwise.} \end{cases}$$

One can easily show that

$$\beta(x, y, z)\, G(Tx, Ty, Tz) \leq \frac{1}{3} G(x, y, z) \quad \text{for all } x, y, z \in X.$$

Then $T$ is a $G$ - $\beta$ - $\psi$ -contractive mapping of type I with $\psi(t) = \frac{1}{3}t$ for all $t \in [0, \infty)$. Notice that $T$ is $\beta$-admissible because if $\beta(x, y, z) \geq 1$, then $x = y = z = 0$, so $\beta(Tx, Ty, Tz) = \beta(0, 0, 0) = 1$. Then, all the conditions of Corollary 7.1.1 are satisfied. Here, 0 is the fixed point of $T$.

Also notice that the Banach contraction mapping principle is not applicable using the Euclidean metric $d(x, y) = |x - y|$ for all $x, y \in X$. Indeed, if $x \neq y$, then $d(Tx, Ty) = 3\,|x - y| > \lambda\,|x - y|$ for all $\lambda \in [0, 1)$. Furthermore, by the same argument, Theorem 4.2.1 is not applicable in this case.

The uniqueness of the fixed point can be deduced from an additional assumption.

**Theorem 7.1.3.** *Under the hypotheses of Theorem 7.1.1 (respectively, Theorem 7.1.2), also assume the following condition:*

$(U)$ *For all* $x, y \in \operatorname{Fix}(T)$, *there exists* $z \in X$ *such that* $\beta(x, z, z) \geq 1$ *and* $\beta(y, z, z) \geq 1$.

*Then* $T$ *has a unique fixed point.*

*Proof.* Let $x, y \in X$ be two fixed points of $T$. By $(U)$, there exists $z \in X$ such that $\beta(x, z, z) \geq 1$ and $\beta(y, z, z) \geq 1$. We claim that the sequence $\{T^n z\}_{n\geq 0}$ converges, at the same time, to $x$ and to $y$ and, hence, we will deduce that $x = y$. The following argument only uses $x$, but it is also valid involving $y$. Since $T$ is $\beta$-admissible, we get

$$\beta(x, z, z) \geq 1 \quad \Rightarrow \quad \beta(x, Tz, Tz) = \beta(Tx, Tz, Tz) \geq 1,$$

and, by induction,

$$\beta(x, T^n z, T^n z) \geq 1 \quad \text{for all } n \in \mathbb{N}. \tag{7.5}$$

From (7.2) and (7.5), we have that, for all $n \geq 0$,

$$\begin{aligned} G\left(x, T^{n+1}z, T^{n+1}z\right) &= G(Tx, TT^n z, TT^n z) \\ &\leq \beta(x, T^n z, T^n z)\, G(Tx, TT^n z, TT^n z) \\ &\leq \psi\left(G(x, T^n z, T^n z)\right). \end{aligned}$$

Thus, we get, by induction, that

$$G(x, T^n z, T^n z) \leq \psi^n\left(G(x, z, z)\right), \qquad \text{for all } n \in \mathbb{N}.$$

Letting $n \to \infty$ and taking into account that $\psi \in \mathcal{F}_{\text{com}}^{(c)}$, we have that $\{G(T^n z, T^n z, x)\} \to 0$, so $\{T^n z\} \to x$. Similarly, $\{T^n z\} \to y$. As a result, $x = y$ by the uniqueness of the limit. □

**Corollary 7.1.3.** *Adding condition* $(U)$ *to the hypotheses of Corollary 7.1.1 (respectively, Corollary 7.1.2), we obtain uniqueness of the fixed point of* $T$.

*Proof.* It is sufficient to take $z = y$ in the proof of Theorem 7.1.3. □

## 7.2 Consequences

The following results are simple consequences of Theorem 7.1.3 and Corollary 7.1.3 using $\beta(x, y, y) = 1$ for all $x, y \in X$.

**Corollary 7.2.1.** *Let $T : X \to X$ be a mapping from a complete G-metric space $(X, G)$ into itself and suppose that there exists $\psi \in \mathcal{F}_{\text{com}}^{(c)}$ such that*

$$G(Tx, Ty, Ty) \leq \psi(G(x, y, y)) \quad \textit{for all } x, y \in X.$$

*Then $T$ has a unique fixed point.*

**Corollary 7.2.2.** *Let $T : X \to X$ be a mapping from a complete G-metric space $(X, G)$ into itself and suppose that there exists $\psi \in \mathcal{F}_{\text{com}}^{(c)}$ such that*

$$G(Tx, Ty, Tz) \leq \psi(G(x, y, z)) \quad \textit{for all } x, y, z \in X.$$

*Then $T$ has a unique fixed point.*

In fact, we can deduce that the main results in [142] are simple consequences of the previous corollaries, using the (c)-comparison function $\psi_\lambda(t) = \lambda t$ for all $t \geq 0$, where $\lambda \in [0, 1)$.

**Corollary 7.2.3.** *Theorem 4.2.2 is a consequence of Corollary 7.2.1.*

**Corollary 7.2.4.** *Theorem 4.2.1 is a consequence of Corollary 7.2.2.*

## *7.2.1 Fixed Point Theorems on G-Metric Spaces Endowed with a Partial Order*

Throughout this subsection, denote by $(X, G, \preccurlyeq)$ an ordered $G$-metric space, that is, $\preccurlyeq$ is a partial order on a $G$-metric space $(X, G)$. In some cases, we will employ non-decreasing-regular ordered $G$-metric spaces (recall Definition 5.2.1).

**Theorem 7.2.1.** *Let $(X, G, \preccurlyeq)$ be an ordered G-metric space such that $(X, G)$ is complete and let $T : X \to X$ be a non-decreasing mapping with respect to $\preccurlyeq$. Suppose that there exists a function $\psi \in \mathcal{F}_{\text{com}}^{(c)}$ such that*

$$G(Tx, Ty, Ty) \leq \psi(G(x, y, y)), \tag{7.6}$$

*for all $x, y \in X$ with $x \preccurlyeq y$. Suppose also that the following conditions hold:*

**(i)** *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;*
**(ii)** *$T$ is G-continuous or $(X, \preccurlyeq, G)$ is non-decreasing-regular.*

*Then there exists $u \in X$ such that $Tu = u$. Furthermore, under the following additional assumption:*

$(U')$ *For all $x, y \in \mathrm{Fix}(T)$, there exists $z \in X$ such that $x \preccurlyeq z$ and $y \preccurlyeq z$,*

*one has uniqueness of the fixed point.*

*Proof.* Define the mapping $\beta : X \times X \times X \to [0, \infty)$ by

$$\beta(x, y, z) = \begin{cases} 1, & \text{if } x \preccurlyeq y = z, \\ 0, & \text{otherwise.} \end{cases} \tag{7.7}$$

Distinguishing the cases $\beta\,(x, y, y) = 0$ and $\beta\,(x, y, y) = 1$, it can be proved, from (7.6), that

$$\beta\,(x, y, y)\; G\,(Tx, Ty, Ty) \leq \psi\,(G\,(x, y, y)) \quad \text{for all } x, y \in X,$$

that is, $T$ is a $G$ - $\beta$ - $\psi$ - contractive mapping of type II. From condition (i), we have $\beta\,(x_0, Tx_0, Tx_0) \geq 1$. Furthermore, since $T$ is a non-decreasing mapping with respect to $\preccurlyeq$, then $T$ is $\beta$ - admissible because

$$\begin{aligned} \beta\,(x, y, z) \geq 1 \quad &\Leftrightarrow \quad x \preccurlyeq y = z \quad \Rightarrow \quad Tx \preccurlyeq Ty = Tz \\ &\Leftrightarrow \quad \beta\,(Tx, Ty, Tz) \geq 1. \end{aligned}$$

If $T$ is $G$-continuous, then $T$ has a fixed point by Theorem 7.1.1. On the other hand, assume that $(X, G, \preccurlyeq)$ is non-decreasing-regular. To prove condition (iii) of Theorem 7.1.2, let $\{x_n\}$ be a sequence in $X$ such that $\beta\,(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\{x_n\}$ is $G$-convergent to $x \in X$. This means that $x_n \preccurlyeq x_{n+1}$ for all $n \in \mathbb{N}$. Hence, by the non-decreasing-regularity, $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$, which is equivalent to $\beta\,(x_n, x, x) \geq 1$ for all $n \in \mathbb{N}$. As a result, condition (iii) of Theorem 7.1.2 holds, and this guarantees that $T$ has a fixed point. The uniqueness follows from condition $(U)$ in Theorem 7.1.3, which is equivalent to condition $(U')$. □

The following result follows from using $\psi_\lambda\,(t) = \lambda\, t$ for all $t \geq 0$, where $\lambda \in [0, 1)$.

**Corollary 7.2.5.** *Let $(X, G, \preccurlyeq)$ be an ordered G-metric space such that $(X, G)$ is complete and let $T : X \to X$ be a non-decreasing mapping with respect to $\preccurlyeq$. Suppose that there exists a constant $\lambda \in [0, 1)$ such that*

$$G\,(Tx, Ty, Ty) \leq \lambda\, G\,(x, y, y) \quad \textit{for all } x, y \in X \textit{ such that } x \preccurlyeq y.$$

*Also assume that the following conditions hold:*

**(i)** *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;*
**(ii)** *$T$ is G-continuous or $(X, \preccurlyeq, G)$ is non-decreasing-regular.*

*Then there exists $u \in X$ such that $Tu = u$. Furthermore, under the following additional assumption:*

$(U')$ *For all $x, y \in \mathrm{Fix}(T)$, there exists $z \in X$ such that $x \preccurlyeq z$ and $y \preccurlyeq z$,*

*one has uniqueness of the fixed point.*

## 7.2.2 Cyclic Contractions

Now, we will prove our results for cyclic contractive mappings in $G$-metric spaces.

**Theorem 7.2.2 (See [119]).** *Let $A$ and $B$ be non-empty $G$-closed subsets of a complete $G$-metric space $(X, G)$. Suppose also that $Y = A \cup B$ and $T : Y \to Y$ is a given self-mapping satisfying*

$$T(A) \subseteq B \text{ and } T(B) \subseteq A. \tag{7.8}$$

*If there exists a function $\psi \in \mathcal{F}_{\text{com}}^{(c)}$ such that*

$$G(Tx, Ty, Ty) \leq \psi(G(x, y, y)) \tag{7.9}$$

*for all $x \in A$ and $y \in B$ or vice versa, then $T$ has a unique fixed point, which belongs to $A \cap B$.*

*Proof.* Notice that $(Y, G)$ is a complete $G$-metric space since $A$ and $B$ are closed subsets of the complete $G$-metric space $(X, G)$. We define $\beta : X \times X \times X \to [0, \infty)$ in the following way:

$$\beta(x, y, z) = \begin{cases} 1, & \text{if } y = z \text{ and } (x, y) \in (A \times B) \cup (B \times A), \\ 0, & \text{otherwise.} \end{cases}$$

From the definition of $\beta$ and assumption (7.9), we have that

$$\beta(x, y, y)\, G(Tx, Ty, Ty) \leq \psi(G(x, y, y)) \tag{7.10}$$

for all $x, y \in Y$. Hence, $T$ is a $G$-$\beta$-$\psi$-contractive mapping in $(Y, G)$. Next, we show that $T$ is $\beta$-admissible. Let $x, y \in Y$ be such that $\beta(x, y, y) \geq 1$. We have two cases. If $(x, y) \in A \times B$, then $x \in A$ and $y \in B$. By (7.8), $Tx \in B$ and $Ty \in A$. Hence $(Tx, Ty) \in B \times A$ and $\beta(Tx, Ty, Ty) \geq 1$. In the other case, if $(x, y) \in B \times A$, the argument is similar. In any case, $\beta(x, y, y) \geq 1$ implies $\beta(Tx, Ty, Ty) \geq 1$, so $T$ is $\beta$-admissible.

Now, we claim that condition (iii) of Theorem 7.1.2 holds in $(Y, G)$. Let $\{x_n\}$ be a sequence in $Y$ such that $\{x_n\} \to x \in Y$ and $\beta\,(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. This means that $(x_n, x_{n+1}) \in (A \times B) \cup (B \times A)$ for all $n \in \mathbb{N}$. We distinguish two cases.

- *Case 1: There exists $n_0 \in \mathbb{N}$ such that $x_n \in Y \diagdown (A \cap B)$ for all $n \geq n_0$.* In this case, we have that:

$$\begin{aligned} x_n \in A \diagdown B \quad &\Rightarrow \quad (x_n, x_{n+1}) \in A \times B \quad \Rightarrow \quad x_{n+1} \in B \\ &\Rightarrow \quad x_{n+1} \in B \diagdown A; \end{aligned}$$

$$x_n \in B \diagdown A \quad \Rightarrow \quad (x_n, x_{n+1}) \in B \times A \quad \Rightarrow \quad x_{n+1} \in A$$
$$\Rightarrow \quad x_{n+1} \in A \diagdown B.$$

In this case, the sequence $\{x_{n_0}, x_{n_0+1}, \ldots\}$ is alternating between $A$ and $B$. Therefore, it has a subsequence on $A$ and a subsequence on $B$. As $\{x_n\}$ converges to $x$, then $x$ belongs to the closure of $A$ and of $B$ but, as $A$ and $B$ are closed, then $x \in A \cap B$. Hence, in this case, $\beta(x_n, x, x) \geq 1$ for all $n \in \mathbb{N}$.

- *Case 2: For all $n_0 \in \mathbb{N}$, there exists $m \geq n_0$ such that $x_m \in A \cap B$.* In this case, $\{x_n\}$ has a partial subsequence $\{x_{n(k)}\}$ such that $x_{n(k)} \in A \cap B$ for all $k$. As $\{x_n\} \to x$, then $\{x_{n(k)}\} \to x$, so $x \in A \cap B$. As a result, $\beta(x_n, x, x) \geq 1$ for all $n \in \mathbb{N}$.

In any case, we have proved that $\beta(x_n, x, x) \geq 1$ for all $n \in \mathbb{N}$, so condition (iii) of Theorem 7.1.2 holds in $(Y, G)$. This guarantees that $T$ has a fixed point. Moreover, $\mathrm{Fix}(T) \subseteq A \cap B$ by Lemma 6.3.1. In particular, if $x \in \mathrm{Fix}(T)$, then $\beta(x, y, y) \geq 1$ for all $y \in Y$. Thus, condition $(U)$ of Theorem 7.1.3 is satisfied, so the fixed point is unique. □

## 7.3 Generalized $G$ - $\beta$ - $\psi$ - Contractive Mappings on $G$-Metric Spaces

In this section, we extend some previously presented results.

### *7.3.1 Generalized G - β - ψ - Contractive Mappings of Types I and II*

In the following definition, we extend the notion given in Definition 7.1.1.

**Definition 7.3.1 ([19]).** Let $(X, G)$ be a $G$-metric space and let $T : X \to X$ be a given mapping. We say that $T$ is a:

- *generalized $G$ - $\beta$ - $\psi$ - contractive mapping of type I* if there exist two functions $\beta : X \times X \times X \to [0, \infty)$ and $\psi \in \mathcal{F}_{\mathrm{com}}^{(c)}$ such that, for all $x, y, z \in X$, we have

$$\beta(x, y, z)\, G(Tx, Ty, Tz) \leq \psi(M(x, y, z)), \tag{7.11}$$

where

$$M(x, y, z) = \max \Big\{ G(x, y, z),\ G(x, Tx, Tx),\ G(y, Ty, Ty),\ G(z, Tz, Tz),\ \frac{G(x, Ty, Ty) + G(y, Tz, Tz) + G(z, Tx, Tx)}{3} \Big\}; \tag{7.12}$$

- *generalized $G$ - $\beta$ - $\psi$ - contractive mapping of type II* if there exist two functions $\beta : X \times X \times X \to [0, \infty)$ and $\psi \in \mathcal{F}_{\mathrm{com}}^{(c)}$ such that, for all $x, y \in X$, we have

$$\beta (x, y, y)\, G (Tx, Ty, Ty) \le \psi (M (x, y, y)), \tag{7.13}$$

where $M$ is given in (7.12), that is,

$$M (x, y, y) = \max \Big\{ G (x, y, y),\ G (x, Tx, Tx),\ G (y, Ty, Ty),\ \frac{G (x, Ty, Ty) + G (y, Ty, Ty) + G (y, Tx, Tx)}{3} \Big\}. \tag{7.14}$$

Clearly, any contractive mapping, that is, a mapping satisfying (4.15), is a generalized $G$ - $\beta$ - $\psi$ - contractive mapping of type I with $\beta (x, y, z) = 1$ for all $x, y, z \in X$ and $\psi_\lambda (t) = \lambda\, t$ for all $t \ge 0$, where $\lambda \in [0, 1)$. Analogously, a mapping satisfying (4.16) is a generalized $G$ - $\beta$ - $\psi$ - contractive mapping of type II with the same $\beta$ and $\psi_\lambda$ as before.

**Theorem 7.3.1 ([19]).** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be a generalized $G$ - $\beta$ - $\psi$ - contractive mapping of type II satisfying the following conditions:*

**(i)** *$T$ is $\beta$ - admissible;*
**(ii)** *there exists $x_0 \in X$ such that $\beta (x_0, Tx_0, Tx_0) \ge 1$;*
**(iii)** *$T$ is G-continuous.*

*Then there exists $u \in X$ such that $Tu = u$.*

*Proof.* Let $x_0 \in X$ be such that $\beta (x_0, Tx_0, Tx_0) \ge 1$ and let $\{x_n\}$ be the Picard sequence of $T$ based on $x_0$ (that is, $x_{n+1} = Tx_n$ for all $n \ge 0$). If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $u = x_{n_0}$ is a fixed point of $T$. On the contrary case, assume that $x_n \ne x_{n+1}$ for all $n$. Since $T$ is $\beta$ - admissible, we have

$$\beta (x_0, x_1, x_1) = \beta (x_0, Tx_0, Tx_0) \ge 1 \implies \beta (x_1, x_2, x_2) = \beta (Tx_0, Tx_1, Tx_1) \ge 1.$$

Inductively, we have that

$$\beta (x_n, x_{n+1}, x_{n+1}) \ge 1 \quad \text{for all } n \ge 0. \tag{7.15}$$

From (7.13) and (7.15), it follows that for all $n \ge 1$, we have

$$\begin{aligned} G (x_n, x_{n+1}, x_{n+1}) &= G (Tx_{n-1}, Tx_n, Tx_n) \\ &\le \beta (x_{n-1}, x_n, x_n)\, G (Tx_{n-1}, Tx_n, Tx_n) \\ &\le \psi (M (x_{n-1}, x_n, x_n)), \end{aligned} \tag{7.16}$$

where

$$
\begin{aligned}
M(x_{n-1}, x_n, x_n) &= \max \Big\{ G(x_{n-1}, x_n, x_n),\ G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), \\
&\quad G(x_n, Tx_n, Tx_n), \\
&\quad \frac{G(x_{n-1}, Tx_n, Tx_n) + G(x_n, Tx_n, Tx_n) + G(x_n, Tx_{n-1}, Tx_{n-1})}{3} \Big\} \\
&= \max \Big\{ G(x_{n-1}, x_n, x_n),\ G(x_n, x_{n+1}, x_{n+1}), \\
&\quad \frac{G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1})}{3} \Big\} \\
&\leq \max \Big\{ G(x_{n-1}, x_n, x_n),\ G(x_n, x_{n+1}, x_{n+1}), \\
&\quad \frac{G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1})}{3} \Big\} \\
&= \max \{ G(x_{n-1}, x_n, x_n),\ G(x_n, x_{n+1}, x_{n+1}) \}. \qquad (7.17)
\end{aligned}
$$

If there exists some $n_0 \in \mathbb{N}$ such that $G(x_{n_0}, x_{n_0+1}, x_{n_0+1}) \geq G(x_{n_0-1}, x_{n_0}, x_{n_0})$, then $M(x_{n_0-1}, x_{n_0}, x_{n_0}) = G(x_{n_0}, x_{n_0+1}, x_{n_0+1})$ and it follows from (7.16) that

$$G(x_{n_0}, x_{n_0+1}, x_{n_0+1}) \leq \psi(G(x_{n_0}, x_{n_0+1}, x_{n_0+1})),$$

which is impossible when $G(x_{n_0}, x_{n_0+1}, x_{n_0+1}) > 0$. Hence, we have that $G(x_n, x_{n+1}, x_{n+1}) < G(x_{n-1}, x_n, x_n)$ for all $n \geq 1$ and it follows from (7.16) that

$$G(x_n, x_{n+1}, x_{n+1}) \leq \psi(G(x_{n-1}, x_n, x_n)) \quad \text{for all } n \geq 1.$$

It follows from Lemma 4.1.2 that $\{x_n\}$ is a Cauchy sequence in the $G$- metric space $(X, G)$. Since $(X, G)$ is complete, there exists $u \in X$ such that $\{x_n\}$ is convergent to $u$. Since $T$ is $G$-continuous, it follows that $\{x_{n+1} = Tx_n\}$ is $G$-convergent, at the same time, to $u$ and to $Tu$. By the uniqueness of the limit, we get $u = Tu$, that is, $u$ is a fixed point of $T$. □

**Corollary 7.3.1 ([19]).** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be a generalized $G$-$\beta$-$\psi$-contractive mapping of type I satisfying the following conditions:*

- **(i)** *$T$ is $\beta$-admissible;*
- **(i)** *there exists $x_0 \in X$ such that $\beta(x_0, Tx_0, Tx_0) \geq 1$;*
- **(iii)** *$T$ is G-continuous.*

*Then there exists $u \in X$ such that $Tu = u$.*

**Theorem 7.3.2 ([19]).** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be a generalized $G$ - $\beta$ - $\psi$ - contractive mapping of type II for some right-continuous $\psi \in \mathcal{F}_{\text{com}}^{(c)}$ satisfying the following conditions:*

**(i)** *$T$ is $\beta$ - admissible;*
**(ii)** *there exists $x_0 \in X$ such that $\beta(x_0, Tx_0, Tx_0) \geq 1$;*
**(iii)** *if $\{x_n\}$ is a sequence in $X$ such that $\beta(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\{x_n\}$ is G-convergent to $x \in X$, then $\beta(x_n, x, x) \geq 1$ for all $n \in \mathbb{N}$.*

*Then there exists $u \in X$ such that $Tu = u$.*

*Proof.* Following the argument in the proof of Theorem 7.3.1, we obtain that the Picard sequence $\{x_n\}$ converges to some $u \in X$. We claim that $Tu = u$. From (7.15) and (iii), we have that

$$\beta(x_n, u, u) \geq 1 \quad \text{for all } n \geq 0. \tag{7.18}$$

Using $(G_5)$, $(G_4)$, (7.13) and (7.18), we have that, for all $n \in \mathbb{N}$,

$$\begin{aligned} G(u, Tu, Tu) &\leq G(u, Tx_n, Tx_n) + G(Tx_n, Tu, Tu) \\ &\leq G(u, x_{n+1}, x_{n+1}) + \beta(x_n, u, u)\, G(Tx_n, Tu, Tu) \\ &\leq G(u, x_{n+1}, x_{n+1}) + \psi(M(x_n, u, u)). \end{aligned} \tag{7.19}$$

In this case,

$$\begin{aligned} M(x_n, u, u) = \max \Bigg\{ & G(x_n, u, u),\ G(x_n, x_{n+1}, x_{n+1}),\ G(u, Tu, Tu), \\ & \frac{G(x_n, Tu, Tu) + G(u, Tu, Tu) + G(u, x_{n+1}, x_{n+1})}{3} \Bigg\}. \end{aligned} \tag{7.20}$$

Therefore

$$\lim_{n\to\infty} M(x_n, u, u) = G(u, Tu, Tu).$$

To prove that $Tu = u$, we distinguish two cases.

- *Case 1: There exists $n_0 \in \mathbb{N}$ such that $M(x_n, u, u) = G(u, Tu, Tu)$ for all $n \geq n_0$.* In this case, from (7.19), we have that

  $$G(u, Tu, Tu) \leq G(u, x_{n+1}, x_{n+1}) + \psi(G(u, Tu, Tu))$$

  for all $n \geq n_0$. Letting $n \to \infty$, we deduce that $G(u, Tu, Tu) \leq \psi(G(u, Tu, Tu))$, which is only possible when $G(u, Tu, Tu) = 0$, that is, $Tu = u$.

- *Case 2: For all $n \in \mathbb{N}$, there exists $m \geq n$ such that $M(x_m, u, u) \neq G(u, Tu, Tu)$.* As $G(u, Tu, Tu)$ is included in the maximum that defines $M(x_n, u, u)$, we can find a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$M\left(x_{n(k)}, u, u\right) > G(u, Tu, Tu) \quad \text{for all } k \geq 1.$$

Thus, $\{M\left(x_{n(k)}, u, u\right)\}_{k\geq 1}$ is a sequence of real numbers, greater than $G(u, Tu, Tu)$, that converges to $G(u, Tu, Tu)$. Since $\psi$ is right-continuous,

$$\lim_{k\to\infty} \psi\left(M\left(x_{n(k)}, u, u\right)\right) = \lim_{t\to G(u,Tu,Tu)^+} \psi(t) = \psi(G(u, Tu, Tu)).$$

From (7.19), we have that

$$G(u, Tu, Tu) \leq G\left(u, x_{n(k)+1}, x_{n(k)+1}\right) + \psi\left(M\left(x_{n(k)}, u, u\right)\right)$$

for all $k$, and letting $k \to \infty$, we deduce that $G(u, Tu, Tu) \leq \psi(G(u, Tu, Tu))$, which yields $G(u, Tu, Tu) = 0$, that is, $Tu = u$. □

*Remark 7.3.1.* Notice that the previous result improves Theorem 30 and Corollary 31 in [19] in the sense that we only assume that $\psi$ is right-continuous, but not necessarily continuous.

**Corollary 7.3.2 ([19]).** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be a generalized $G$ - $\beta$ - $\psi$ - contractive mapping of type I for some right-continuous $\psi \in \mathcal{F}_{\text{com}}^{(c)}$ satisfying the following conditions:*

**(i)** *$T$ is $\beta$ - admissible;*
**(ii)** *there exists $x_0 \in X$ such that $\beta(x_0, Tx_0, Tx_0) \geq 1$;*
**(iii)** *if $\{x_n\}$ is a sequence in $X$ such that $\beta(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\{x_n\}$ is G-convergent to $x \in X$, then $\beta(x_n, x, x) \geq 1$ for all $n \in \mathbb{N}$.*

*Then there exists $u \in X$ such that $Tu = u$.*

Notice that Theorem 7.1.1 and Corollary 7.1.1 are simple consequences of Theorem 7.3.1 and Corollary 7.3.1. However, Theorem 7.1.2 and Corollary 7.1.2 cannot be deduced from Theorem 7.3.2 and Corollary 7.3.2 because, in the last ones, we assume that $\psi$ is right-continuous.

With the following example, we will show that the hypotheses in Theorems 7.3.1 and 7.3.2 do not guarantee uniqueness.

*Example 7.3.1 ([19]).* Let $X = \{(1, 0), (0, 1)\} \subset \mathbb{R}^2$ endowed with the following $G$-metric

$$G((x, y), (u, v), (z, w)) = |x - u| + |u - z| + |z - x| + |y - v| + |v - w| + |w - y|$$

for all $(x, y), (u, v), (z, w) \in X$. Obviously, $(X, G)$ is a complete metric space. The mapping $T(x, y) = (x, y)$ is trivially continuous and satisfies, for any $\psi \in \mathcal{F}_{\text{com}}^{(c)}$,

$$\beta((x,y),(u,v),(z,w))G(T(x,y),T(u,v),T(z,w)) \\ \leq \psi(M((x,y),(u,v),(z,w))),$$

for all $(x,y),(u,v),(z,w) \in X$, where

$$\beta((x,y),(u,v),(z,w)) = \begin{cases} 1, & \text{if } (x,y)=(u,v)=(z,w), \\ 0, & \text{otherwise.} \end{cases}$$

Thus $T$ is a generalized $G$ - $\beta$ - $\psi$ - contractive mapping of type I. On the other hand, for all $(x,y),(u,v),(z,w) \in X$, we have

$$\beta((x,y),(u,v),(z,w)) \geq 1 \quad \Rightarrow \quad (x,y)=(u,v)=(z,w),$$

which yields that $\beta(T(x,y),T(u,v),T(z,w)) \geq 1$. Hence $T$ is $\beta$ - admissible. Moreover, for all $(x,y) \in X$, we have $\beta((x,y),T(x,y),T(x,y)) \geq 1$. So the assumptions of Theorem 7.3.1 are satisfied. In fact $T$ satisfies all the assumptions of Theorem 7.3.2. However, in this case, $T$ has two fixed points in $X$.

*Example 7.3.2.* Let $X \subseteq \mathbb{R}$ be a closed, bounded subset, non reduced to a single point (for instance, a compact interval), endowed with the $G$-metric $G(x,y,z) = |x-y|+|x-z|+|y-z|$ for all $x,y,z \in X$. Let $T$ be the identity mapping on $T$ and define

$$\beta(x,y,z) = \begin{cases} 1, & \text{if } x=y=z, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\beta(x,y,z)\,G(Tx,Ty,Tz) = 0 \leq \psi(M(x,y,z))$$

for all $x,y,z \in X$ and all $\psi \in \mathcal{F}_{\text{com}}^{(c)}$, so $T$ is a generalized $G$ - $\beta$ - $\psi$ - contractive mapping of type I. Since $\beta(x,Tx,Tx) = \beta(x,x,x) = 1$ for all $x \in X$, all the conditions of Theorems 7.3.1 and 7.3.2 are satisfied. However, every point of $X$ is a fixed point of $T$ (in particular, it has more than one).

We present the following condition in order to ensure uniqueness of the fixed point.

**Theorem 7.3.3.** *Under the hypotheses of Theorem 7.3.1 (respectively, Corollary 7.3.1, Theorem 7.3.2, Corollary 7.3.2), also assume the following condition:*

$(U'')$ *For all* $x,y \in \mathrm{Fix}(T)$ *we have that* $\max\{\beta(x,y,y),\beta(y,x,x)\} \geq 1$.

*Then $T$ has a unique fixed point.*

In [19], the authors assumed the stronger condition:

**(iv)** For all $x \in \mathrm{Fix}(T)$ we have that $\beta(x,z,z) \geq 1$ for all $z \in X$.

*Proof.* Let $x, y \in \mathrm{Fix}(T)$ be two fixed points of $T$. By hypothesis $(U'')$, we have that $\max\{\beta(x,y,y), \beta(y,x,x)\} \geq 1$. Without loss of generality, assume that $\beta(x,y,y) \geq 1$. From (7.13),

$$G(x,y,y) \leq \beta(x,y,y)\, G(Tx,Ty,Ty) \leq \psi(M(x,y,y)),$$

where

$$\begin{aligned} M(x,y,y) &= \max\Big\{G(x,y,y),\ G(x,Tx,Tx),\ G(y,Ty,Ty), \\ &\qquad \frac{G(x,Ty,Ty) + G(y,Ty,Ty) + G(y,Tx,Tx)}{3}\Big\} \\ &= \max\Big\{G(x,y,y),\ \frac{G(x,y,y) + G(y,x,x)}{3}\Big\} \\ &\leq \max\Big\{G(x,y,y),\ \frac{G(x,y,y) + 2G(x,y,y)}{3}\Big\} = G(x,y,y). \end{aligned}$$

Therefore, since $\psi$ is non-decreasing, $G(x,y,y) \leq \psi(G(x,y,y))$, which is only possible when $G(x,y,y) = 0$, that is, $x = y$, which proves that $T$ has a unique fixed point. □

### 7.3.2 Generalized *G*-$\beta$-$\psi$-Contractive Mappings of Type III

In this subsection, we present a new contractivity condition.

**Definition 7.3.2 ([19]).** Let $(X, G)$ be a $G$-metric space and let $T : X \to X$ be a given mapping. We say that $T$ is a *generalized $G$-$\beta$-$\psi$-contractive mapping of type III* if there exist two functions $\beta : X \times X \times X \to [0,\infty)$ and $\psi \in \mathcal{F}^{(c)}_{\mathrm{com}}$ such that, for all $x, y, z \in X$, we have

$$\beta(x,y,z)\, G(Tx,Ty,Tz) \leq \psi(N(x,y,z)), \tag{7.21}$$

where

$$N(x,y,z) = \max\{G(x,y,z), G(x,x,Tx), G(y,y,Ty), G(z,z,Tz)\}.$$

**Theorem 7.3.4 ([19]).** *Let $(X, G)$ be a complete $G$-metric space and let $T : X \to X$ be a generalized $G$-$\beta$-$\psi$-contractive mapping of type III satisfying the following conditions:*

**(i)** *$T$ is $\beta$-admissible;*
**(ii)** *there exists $x_0 \in X$ such that $\beta(x_0, Tx_0, Tx_0) \geq 1$ (or $\beta(x_0, x_0, Tx_0) \geq 1$);*
**(iii)** *$T$ is $G$-continuous.*

*Then there exists $u \in X$ such that $Tu = u$.*

*Proof.* We follow the argument in the proof of Theorem 7.3.1, replacing $M$ by $N$ and the inequality (7.17) by

$$\begin{aligned} &N(x_{n-1}, x_n, x_n) \\ &\quad = \max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_{n-1}, Tx_{n-1}), G(x_n, x_n, Tx_n)\} \\ &\quad = \max\{G(x_{n-1}, x_n, x_n),\ G(x_n, x_n, x_{n+1})\}. \end{aligned}$$

The rest of the proof is similar. □

**Theorem 7.3.5.** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be a $G$ - $\beta$ - $\psi$ - contractive mapping of type III for some right-continuous $\psi \in \mathcal{F}_{\mathrm{com}}^{(c)}$ satisfying the following conditions:*

- **(i)** *$T$ is $\beta$ - admissible;*
- **(ii)** *there exists $x_0 \in X$ such that $\beta(x_0, Tx_0, Tx_0) \geq 1$;*
- **(iii)** *if $\{x_n\}$ is a sequence in $X$ such that $\beta(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\{x_n\}$ is G-convergent to $x \in X$, then $\beta(x_n, x, x) \geq 1$ for all $n \in \mathbb{N}$.*

*Then there exists $u \in X$ such that $Tu = u$.*

On the one hand, the previous result is also valid if we replace (ii) and (iii) by the following ones.

- **(ii)** there exists $x_0 \in X$ such that $\beta(x_0, x_0, Tx_0) \geq 1$;
- **(iii)** if $\{x_n\}$ is a sequence in $X$ such that $\beta(x_n, x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\{x_n\}$ is $G$-convergent to $x \in X$, then $\beta(x_n, x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

On the other hand, a similar comment to Remark 7.3.1 can also be given here.

*Proof.* We follow the argument in the proof of Theorem 7.3.2, replacing $M$ by $N$ and (7.20) by

$$\begin{aligned} N(x_n, u, u) &= \max\{G(x_n, u, u),\ G(x_n, x_n, Tx_n),\ G(u, Tu, Tu)\} \\ &= \max\{G(x_n, u, u),\ G(x_n, x_n, Tx_n),\ G(u, Tu, Tu)\}, \end{aligned}$$

and distinguishing the same two cases. □

As in Example 7.3.2, the uniqueness of the fixed point is not guaranteed. We need an additional condition.

**Theorem 7.3.6.** *Under the hypotheses of Theorem 7.3.4 (respectively, Theorem 7.3.5), also assume the following condition:*

$(U'')$ *For all $x, y \in \mathrm{Fix}(T)$ we have that* $\max\{\beta(x, y, y), \beta(y, x, x)\} \geq 1$.

*Then $T$ has a unique fixed point.*

*Proof.* We repeat the proof of Theorem 7.3.3 taking into account that, for all $x, y \in \mathrm{Fix}(T)$,

$$\begin{aligned} N(u, v, v) &= \max\{G(u, v, v), G(u, u, Tu), G(v, v, Tv)\} \\ &= G(u, v, v). \end{aligned}$$

□

## 7.4 Consequences

The following two results correspond to Theorem 7.3.2 and Corollary 7.3.2 in the case in which $\beta(x,y,z) = 1$ for all $x,y,z \in X$, and also applying Theorem 7.3.3.

**Corollary 7.4.1.** *Let $(X,G)$ be a complete $G$-metric space and let $T : X \to X$ be a mapping. Assume that there exists a right-continuous function $\psi \in \mathcal{F}_{\text{com}}^{(c)}$ such that, for all $x,y \in X$,*

$$G(Tx,Ty,Ty) \leq \psi\left(\max\left\{G(x,y,y)\,,\; G(x,Tx,Tx)\,,\; G(y,Ty,Ty)\,,\; \frac{G(x,Ty,Ty)+G(y,Ty,Ty)+G(y,Tx,Tx)}{3}\right\}\right).$$

*Then $T$ has a unique fixed point.*

**Corollary 7.4.2.** *Let $(X,G)$ be a complete $G$-metric space and let $T : X \to X$ be a mapping. Assume that there exists a right-continuous function $\psi \in \mathcal{F}_{\text{com}}^{(c)}$ such that, for all $x,y,z \in X$,*

$$G(Tx,Ty,Tz) \leq \psi\left(\max\left\{G(x,y,z)\,,\; G(x,Tx,Tx)\,,\; G(y,Ty,Ty)\,,\; G(z,Tz,Tz)\,,\; \frac{G(x,Ty,Ty)+G(y,Tz,Tz)+G(z,Tx,Tx)}{3}\right\}\right).$$

*Then $T$ has a unique fixed point.*

We can avoid the right-continuity of $\psi$ when $T$ is continuous, applying Theorem 7.3.1, Corollary 7.3.1 and Theorem 7.3.3.

**Corollary 7.4.3.** *Let $(X,G)$ be a complete $G$-metric space and let $T : X \to X$ be a $G$-continuous mapping. Assume that there exists a function $\psi \in \mathcal{F}_{\text{com}}^{(c)}$ such that, for all $x,y \in X$,*

$$G(Tx,Ty,Ty) \leq \psi\left(\max\left\{G(x,y,y)\,,\; G(x,Tx,Tx)\,,\; G(y,Ty,Ty)\,,\; \frac{G(x,Ty,Ty)+G(y,Ty,Ty)+G(y,Tx,Tx)}{3}\right\}\right).$$

*Then $T$ has a unique fixed point.*

**Corollary 7.4.4.** *Let $(X,G)$ be a complete $G$-metric space and let $T : X \to X$ be a $G$-continuous mapping. Assume that there exists a function $\psi \in \mathcal{F}_{\text{com}}^{(c)}$ such that, for all $x,y,z \in X$,*

$$G(Tx, Ty, Tz) \leq \psi\left(\max\left\{G(x, y, z)\,,\ G(x, Tx, Tx)\,,\right.\right.$$
$$G(y, Ty, Ty)\,,\ G(z, Tz, Tz)\,,$$
$$\left.\left.\frac{G(x, Ty, Ty) + G(y, Tz, Tz) + G(z, Tx, Tx)}{3}\right\}\right).$$

*Then $T$ has a unique fixed point.*

Assume that $\psi_\lambda(t) = \lambda\, t$ for all $t \geq 0$. Then we have the following particular cases of Corollaries 7.4.1 and 7.4.2.

**Corollary 7.4.5.** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be a mapping. Assume that there exists a constant $\lambda \in [0, 1)$ such that, for all $x, y \in X$,*

$$G(Tx, Ty, Ty) \leq \lambda \max\left\{G(x, y, y)\,,\ G(x, Tx, Tx)\,,\ G(y, Ty, Ty)\,,\right.$$
$$\left.\frac{G(x, Ty, Ty) + G(y, Ty, Ty) + G(y, Tx, Tx)}{3}\right\}.$$

*Then $T$ has a unique fixed point.*

**Corollary 7.4.6.** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be a mapping. Assume that there exists a constant $\lambda \in [0, 1)$ such that, for all $x, y, z \in X$,*

$$G(Tx, Ty, Tz) \leq \lambda \max\left\{G(x, y, z)\,,\ G(x, Tx, Tx)\,,\right.$$
$$G(y, Ty, Ty)\,,\ G(z, Tz, Tz)\,,$$
$$\left.\frac{G(x, Ty, Ty) + G(y, Tz, Tz) + G(z, Tx, Tx)}{3}\right\}.$$

*Then $T$ has a unique fixed point.*

Taking into account that $\psi$ is non-decreasing and $G(x, y, z) \leq M(x, y, z)$ for all $x, y, z \in X$, then we also have the following consequences.

**Corollary 7.4.7.** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be a mapping. Suppose that there exists a function $\psi \in \mathcal{F}_{\text{com}}^{(c)}$ such that, for all $x, y \in X$,*

$$G(Tx, Ty, Ty) \leq \psi(G(x, y, y)).$$

*Also assume that $T$ is G-continuous or $\psi$ is right-continuous. Then $T$ has a unique fixed point.*

**Corollary 7.4.8.** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be a mapping. Suppose that there exists a function $\psi \in \mathcal{F}_{\text{com}}^{(c)}$ such that, for all $x, y, z \in X$,*

$$G(Tx, Ty, Tz) \leq \psi(G(x, y, z)).$$

*Also assume that $T$ is G-continuous or $\psi$ is right-continuous. Then $T$ has a unique fixed point.*

**Corollary 7.4.9 ([19]).** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be a given mapping. Suppose that there exist nonnegative real numbers $a, b, c$ and $d$, with $a + b + c + d < 1$, such that*

$$\begin{aligned} G(Tx, Ty, Ty) \leq{}& aG(x, y, y) + bG(x, Tx, Tx) + cG(y, Ty, Ty) \\ &+ \frac{d}{3}\left(G(x, Ty, Ty) + G(y, Ty, Ty) + G(y, Tx, Tx)\right), \end{aligned}$$

*for all $x, y \in X$. Then $T$ has a unique fixed point.*

*Proof.* For all $x, y \in X$ we have that

$$\begin{aligned} &aG(x, y, y) + bG(x, Tx, Tx) + cG(y, Ty, Ty) \\ &\qquad + \frac{d}{3}\left(G(x, Ty, Ty) + G(y, Ty, Ty) + G(y, Tx, Tx)\right) \\ &\leq aM(x, y, y) + bM(x, y, y) + cM(x, y, y) \\ &\qquad + dM(x, y, y) = (a + b + c + d)\, M(x, y, y). \end{aligned}$$

Therefore, we can apply Corollary 7.4.5 using $\lambda = \max\{a + b + c + d, 0\}$. □

**Corollary 7.4.10 ([19]).** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be a given mapping. Suppose that there exist nonnegative real numbers $a, b, c, d$ and $e$, with $a + b + c + d + e < 1$, such that*

$$\begin{aligned} G(Tx, Ty, Tz) \leq{}& aG(x, y, z) + bG(x, Tx, Tx) \\ &+ cG(y, Ty, Ty) + dG(z, Tz, Tz) \\ &+ \frac{e}{3}\left(G(x, Ty, Ty) + G(y, Tz, Tz) + G(z, Tx, Tx)\right), \end{aligned}$$

*for all $x, y, z \in X$. Then $T$ has a unique fixed point.*

### *7.4.1 Fixed Point Theorems on G-Metric Spaces Endowed with a Partial Order*

In this subsection, we apply the previous results to the case in which $X$ is endowed with a partial order.

**Theorem 7.4.1.** *Let $(X, G, \preccurlyeq)$ be an ordered G-metric space such that $(X, G)$ is complete, and let $T : X \to X$ be a non-decreasing mapping with respect to $\preccurlyeq$. Suppose that there exists a function $\psi \in \mathcal{F}_{\mathrm{com}}^{(c)}$ such that*

$$G(Tx, Ty, Ty) \leq \psi\left(\max\left\{G(x, y, y),\ G(x, Tx, Tx),\ G(y, Ty, Ty),\ \frac{G(x, Ty, Ty) + G(y, Ty, Ty) + G(y, Tx, Tx)}{3}\right\}\right). \tag{7.22}$$

*for all $x, y \in X$ with $x \prec y$. Also assume that the following conditions are fulfilled:*

**(i)** *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;*
**(ii)** *At least, one of the following conditions holds:*

  **(ii.1)** *$T$ is G-continuous,* *or*
  **(ii.2)** *$(X, \preccurlyeq, G)$ is non-decreasing-regular and $\psi$ is right-continuous.*

*Then there exists $u \in X$ such that $Tu = u$. Furthermore, under the following additional assumption:*

$(U'')$ *For all $x, y \in \mathrm{Fix}(T)$ we have that $x \preccurlyeq y$ or $y \preccurlyeq x$ (that is, all fixed points of $T$ are $\preccurlyeq$-comparable),*

*one has uniqueness of the fixed point.*

*Proof.* Define the mapping $\beta : X \times X \times X \to [0, \infty)$ by

$$\beta(x, y, z) = \begin{cases} 1, & \text{if } x \preccurlyeq y = z, \\ 0, & \text{otherwise.} \end{cases}$$

Distinguishing the cases $\beta(x, y, y) = 0$ and $\beta(x, y, y) = 1$, it can be proved, from (7.22), that

$$\beta(x, y, y)\, G(Tx, Ty, Ty) \leq \psi(M(x, y, y)) \quad \text{for all } x, y \in X$$

(the case $x = y$ is obvious), that is, $T$ is a $G$-$\beta$-$\psi$-contractive mapping of type II. From condition (i), we have $\beta(x_0, Tx_0, Tx_0) \geq 1$. Furthermore, since $T$ is a non-decreasing mapping with respect to $\preccurlyeq$, then $T$ is $\beta$-admissible because

$$\begin{aligned} \beta(x, y, z) \geq 1 \quad &\Leftrightarrow \quad x \preccurlyeq y = z \quad \Rightarrow \quad Tx \preccurlyeq Ty = Tz \\ &\Leftrightarrow \quad \beta(Tx, Ty, Tz) \geq 1. \end{aligned}$$

If $T$ is $G$-continuous, then $T$ has a fixed point by Theorem 7.3.1. On the other case, assume that $(X, G, \preccurlyeq)$ is non-decreasing-regular and $\psi$ is right-continuous. To prove condition (iii) of Theorem 7.3.2, let $\{x_n\}$ be a sequence in $X$ such that

$\beta(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\{x_n\}$ is $G$-convergent to $x \in X$. This means that $x_n \preccurlyeq x_{n+1}$ for all $n \in \mathbb{N}$. Hence, by the non-decreasing-regularity, $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$, which is equivalent to $\beta(x_n, x, x) \geq 1$ for all $n \in \mathbb{N}$. Hence, condition (iii) of Theorem 7.3.2 holds, and this guarantees that $T$ has a fixed point. The uniqueness follows from condition $(U'')$ in Theorem 7.3.3, which is equivalent to our condition $(U'')$. □

**Corollary 7.4.11.** *Let $(X, G, \preccurlyeq)$ be an ordered G-metric space such that $(X, G)$ is complete, and let $T : X \to X$ be a non-decreasing mapping with respect to $\preccurlyeq$. Suppose that there exists a function $\psi \in \mathcal{F}_{\text{com}}^{(c)}$ such that*

$$G(Tx, Ty, Tz) \leq \psi\left(\max\left\{G(x, y, z),\ G(x, Tx, Tx),\ G(y, Ty, Ty),\ G(z, Tz, Tz),\ \frac{G(x, Ty, Ty) + G(y, Tz, Tz) + G(z, Tx, Tx)}{3}\right\}\right).$$

*for all $x, y \in X$ with $x \prec y \preccurlyeq z$. Also assume that the following conditions are fulfilled:*

**(i)** *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;*
**(ii)** *at least, one of the following conditions holds:*

  **(ii.1)** *$T$ is G-continuous, or*
  **(ii.2)** *$(X, \preccurlyeq, G)$ is non-decreasing-regular and $\psi$ is right-continuous.*

*Then there exists $u \in X$ such that $Tu = u$. Furthermore, under the following additional assumption:*

$(U'')$ *For all $x, y \in \text{Fix}(T)$ we have that $x \preccurlyeq y$ or $y \preccurlyeq x$ (that is, all fixed points of $T$ are $\preccurlyeq$-comparable),*

*one has uniqueness of the fixed point.*

In the following result, we employ $\psi_\lambda(t) = \lambda t$ for all $t \geq 0$, which is continuous.

**Corollary 7.4.12.** *Let $(X, G, \preccurlyeq)$ be an ordered G-metric space such that $(X, G)$ is complete, and let $T : X \to X$ be a non-decreasing mapping with respect to $\preccurlyeq$. Suppose that there exists a constant $\lambda \in [0, 1)$ such that*

$$G(Tx, Ty, Ty) \leq \lambda \max\left\{G(x, y, y),\ G(x, Tx, Tx),\ G(y, Ty, Ty),\ \frac{G(x, Ty, Ty) + G(y, Ty, Ty) + G(y, Tx, Tx)}{3}\right\}.$$

*for all $x, y \in X$ with $x \prec y$. Also assume that the following conditions are fulfilled:*

**(i)** *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;*

**(ii)** *at least, one of the following conditions holds:*

**(ii.1)** *$T$ is $G$-continuous, or*
**(ii.2)** *$(X, \preccurlyeq, G)$ is non-decreasing-regular.*

*Then there exists $u \in X$ such that $Tu = u$. Furthermore, under the following additional assumption:*

$(U'')$ *For all $x, y \in \mathrm{Fix}(T)$ we have that $x \preccurlyeq y$ or $y \preccurlyeq x$ (that is, all fixed points of $T$ are $\preccurlyeq$-comparable),*

*one has uniqueness of the fixed point.*

Inspired by Corollary 7.4.9, we present the following one (which has the same proof).

**Corollary 7.4.13.** *Let $(X, G, \preccurlyeq)$ be an ordered $G$-metric space such that $(X, G)$ is complete, and let $T : X \to X$ be a non-decreasing mapping with respect to $\preccurlyeq$. Suppose that there exist nonnegative real numbers $a, b, c$ and $d$, with $a+b+c+d < 1$, such that*

$$G(Tx, Ty, Ty) \leq a\,G(x, y, y) + b\,G(x, Tx, Tx) + c\,G(y, Ty, Ty) + \frac{d}{3}\left(G(x, Ty, Ty) + G(y, Ty, Ty) + G(y, Tx, Tx)\right)$$

*for all $x, y \in X$ with $x \prec y$. Also assume that the following conditions are fulfilled:*

**(i)** *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;*
**(ii)** *at least, one of the following conditions holds:*

**(ii.1)** *$T$ is $G$-continuous, or*
**(ii.2)** *$(X, \preccurlyeq, G)$ is non-decreasing-regular.*

*Then there exists $u \in X$ such that $Tu = u$. Furthermore, under the following additional assumption:*

$(U'')$ *For all $x, y \in \mathrm{Fix}(T)$ we have that $x \preccurlyeq y$ or $y \preccurlyeq x$ (that is, all fixed points of $T$ are $\preccurlyeq$-comparable),*

*one has uniqueness of the fixed point.*

### *7.4.2 Cyclic Contraction*

Now, we will prove our results for cyclic contractive mappings in $G$-metric spaces.

**Theorem 7.4.2 (See [119]).** *Let $A$ and $B$ be two non-empty, $G$-closed subsets of a complete $G$-metric space $(X, G)$. Suppose also that $Y = A \cup B$ and $T : Y \to Y$ is a given self-mapping satisfying*

$$T(A) \subseteq B \text{ and } T(B) \subseteq A. \tag{7.23}$$

*Assume that there exists a right-continuous function* $\psi \in \mathcal{F}_{\text{com}}^{(c)}$ *such that*

$$G(Tx, Ty, Ty) \leq \psi(M(x, y, y)) \tag{7.24}$$

*for all* $x \in A$ *and* $y \in B$ *or viceversa, where*

$$M(x, y, y) = \max\left\{G(x, y, y),\ G(x, Tx, Tx),\ G(y, Ty, Ty),\ \frac{G(x, Ty, Ty) + G(y, Ty, Ty) + G(y, Tx, Tx)}{3}\right\}.$$

*Then* $T$ *has a unique fixed point, which belongs to* $A \cap B$.

*Proof.* We follow the argument in the proof of Theorem 7.2.2, using $M(x, y, y)$ rather than $G(x, y, y)$. □

# Chapter 8
# New Approaches to Fixed Point Results on $G$-Metric Spaces

Recently, Samet et al. [184], and Jleli and Samet [97], observed that some fixed point theorems in the context of $G$-metric space in the literature can be concluded from existence results in the setting of quasi-metric spaces. In fact, if the contractivity condition of the fixed point theorem on a $G$-metric space can be reduced to two variables instead of there variables, then one can construct an equivalent fixed point theorem in the setup of usual metric spaces. More precisely, in [97, 184], the authors noticed that $q(x, y) = G(x, y, y)$ forms a quasi-metric.

In this chapter, we notice that, although the techniques used in [97, 184] are valid if the contractivity condition in the statement of the theorem can be expressed in two variables, we can also consider other fixed point theorems in the context of $G$-metric spaces for which the techniques in [97, 184] are not applicable.

## 8.1 A New Approach to Express Fixed Point Contraction Mappings

**Theorem 8.1.1.** *Let $(X, G)$ be a complete $G$-metric space and let $T : X \to X$ be mapping. Suppose that there exists $\lambda \in \left[0, \frac{1}{2}\right)$ such that*

$$G(Tx, Ty, Tz) \leq \lambda M(x, y, z) \quad \textit{for all } x, y, z \in X, \tag{8.1}$$



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*where*

$$M(x,y,z) = \max \left\{ \begin{array}{l} G(x,y,z),\ G(x,Tx,Tx),\ G(x,Ty,Ty), \\ G(x,Tx,y),\ G(x,Tx,z),\ G(y,Ty,Ty), \\ G(y,Tx,Ty),\ G(y,T^2x,Ty),\ G(y,Tz,Tz), \\ G(z,Tx,Tx),\ G(z,Tz,Tz),\ G(z,Tx,Ty), \\ G(z,T^2x,Tz),\ G(Tx,T^2x,Ty),\ G(Tx,T^2x,Tz) \end{array} \right\}.$$

*Then there is a unique $x \in X$ such that $Tx = x$. In fact, $T$ is a Picard operator.*

*Proof.* Let $x_0 \in X$ be arbitrary and let $\{x_n\}$ be the Picard sequence of $T$ based on $x_0$, that is

$$x_{n+1} = Tx_n \quad \text{for all } n \in \mathbb{N}. \tag{8.2}$$

If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then $x_{n_0}$ is a fixed point of $T$. Assume that

$$x_n \neq x_{n+1} \quad \text{for all } n \in \mathbb{N}. \tag{8.3}$$

Taking $x = x_n$ and $z = y = x_{n+1}$ in (8.1), we find that, for all $n \geq 0$,

$$G(x_{n+1},x_{n+2},x_{n+2}) = G(Tx_n,Tx_{n+1},Tx_{n+1}) \leq \lambda M(x_n,x_{n+1},x_{n+1}), \tag{8.4}$$

where $M(x_n,x_{n+1},x_{n+1})$ takes the value

$$\max \left\{ \begin{array}{l} G(x_n,x_{n+1},x_{n+1}),\ G(x_n,Tx_n,Tx_n), \\ G(x_n,Tx_{n+1},Tx_{n+1}),\ G(x_n,Tx_n,x_{n+1}), \\ G(x_n,Tx_n,x_{n+1}),\ G(x_{n+1},Tx_{n+1},Tx_{n+1}), \\ G(x_{n+1},Tx_n,Tx_{n+1}),\ G(x_{n+1},T^2x_n,Tx_{n+1}), \\ G(x_{n+1},Tx_{n+1},Tx_{n+1}),\ G(x_{n+1},Tx_n,Tx_n), \\ G(x_{n+1},Tx_{n+1},Tx_{n+1}),\ G(x_{n+1},Tx_n,Tx_{n+1}), \\ G(x_{n+1},T^2x_n,Tx_{n+1}),\ G(Tx_n,T^2x_n,Tx_{n+1}), \\ G(Tx_n,T^2x_n,Tx_{n+1}) \end{array} \right\}$$

$$= \max \left\{ \begin{array}{l} G(x_n,x_{n+1},x_{n+1}),\ G(x_n,x_{n+1},x_{n+1}), \\ G(x_n,x_{n+2},x_{n+2}),\ G(x_n,x_{n+1},x_{n+1}), \\ G(x_n,x_{n+1},x_{n+1}),\ G(x_{n+1},x_{n+2},x_{n+2}), \\ G(x_{n+1},x_{n+1},x_{n+2}),\ G(x_{n+1},x_{n+2},x_{n+2}), \\ G(x_{n+1},x_{n+2},x_{n+2}),\ G(x_{n+1},x_{n+1},x_{n+1}), \\ G(x_{n+1},x_{n+2},x_{n+2}),\ G(x_{n+1},x_{n+1},x_{n+2}), \\ G(x_{n+1},x_{n+2},x_{n+2}),\ G(x_{n+1},x_{n+2},x_{n+2}), \\ G(x_{n+1},x_{n+2},x_{n+2}) \end{array} \right\}$$

$$= \max \left\{ \begin{matrix} G(x_n, x_{n+1}, x_{n+1}), & G(x_n, x_{n+2}, x_{n+2}), \\ G(x_{n+1}, x_{n+2}, x_{n+2}), & G(x_{n+1}, x_{n+1}, x_{n+2}) \end{matrix} \right\}. \tag{8.5}$$

Now, we have to examine four cases in (8.5).

- If $M(x_n, x_{n+1}, x_{n+1}) = G(x_{n+1}, x_{n+2}, x_{n+2})$, then (8.4) turns into

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \lambda\, G(x_{n+1}, x_{n+2}, x_{n+2})$$

  which is impossible because $G(x_{n+1}, x_{n+2}, x_{n+2}) > 0$ and $\lambda < 1/2$.
- If $M(x_n, x_{n+1}, x_{n+1}) = G(x_{n+1}, x_{n+1}, x_{n+2})$, then (8.4) and Lemma 3.1.1 imply that

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+2}) &\leq \lambda\, G(x_{n+1}, x_{n+1}, x_{n+2}) \\ &\leq 2\lambda\, G(x_{n+1}, x_{n+2}, x_{n+2}), \end{aligned}$$

  which is impossible because $G(x_{n+1}, x_{n+2}, x_{n+2}) > 0$ and $2\lambda < 1$.
- If $M(x_n, x_{n+1}, x_{n+1}) = G(x_n, x_{n+2}, x_{n+2})$, the inequality (8.4) leads to

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+2}) &\leq \lambda\, G(x_n, x_{n+2}, x_{n+2}) \\ &\leq \lambda \left[ G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \right]. \end{aligned}$$

  In this case, we deduce that

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \gamma\, G(x_n, x_{n+1}, x_{n+1}) \tag{8.6}$$

  where $\gamma = \frac{\lambda}{1-\lambda} < 1$ since $0 \leq \lambda < \frac{1}{2}$.
- If $M(x_n, x_{n+1}, x_{n+1}) = G(x_n, x_{n+1}, x_{n+1})$, then the inequality (8.4) gives

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \lambda\, G(x_n, x_{n+1}, x_{n+1}). \tag{8.7}$$

As a result, the first two cases are impossible and, in the last two cases, we have that

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \gamma\, G(x_n, x_{n+1}, x_{n+1}) \quad \text{for all } n \in \mathbb{N},$$

where $\gamma = \frac{\lambda}{1-\lambda} < 1$ (notice that $\lambda < \gamma$). Using the classical Banach argument, we prove that $\{x_n\}$ is a Cauchy sequence in $(X, G)$ because $G(x_n, x_{n+1}, x_{n+1}) \leq \gamma^n G(x_0, x_1, x_1)$ and

$$G(x_n, x_m, x_m) \leq \frac{\gamma^n}{1-\lambda} G(x_0, x_1, x_1)$$

for all $n, m \in \mathbb{N}$ with $m > n$. Since $(X, G)$ is complete, there exists $u \in X$ such that $\{x_n\} \to u$. We now show that $Tu = u$. Using (8.1) with $x = x_{n+1}$ and $y = z = u$, we have that

$$G(x_{n+1}, Tu, Tu) = G(Tx_n, Tu, Tu) \leq \lambda M(x_n, u, u) \tag{8.8}$$

where

$$M(x_n, u, u) = \max \left\{ \begin{array}{l} G(x_n, u, u),\ G(x_n, x_{n+1}, x_{n+1}),\ G(x_n, Tu, Tu), \\ G(x_n, x_{n+1}, u),\ G(u, Tu, Tu),\ G(u, x_{n+1}, Tu), \\ G(u, x_{n+2}, Tu),\ G(u, x_{n+1}, x_{n+1}), \\ G(x_{n+1}, x_{n+2}, Tu) \end{array} \right\}.$$

As $G$ is continuous on each argument (see Theorem 3.2.2), all terms of $M(x_n, u, u)$ converge to 0, or $G(u, u, Tu)$ or $G(u, Tu, Tu)$. From Lemma 3.1.1, $G(u, u, Tu) \leq 2G(u, Tu, Tu)$. Hence, by taking the limit as $n \to \infty$ in (8.8), we deduce that

$$\begin{aligned} G(u, Tu, Tu) &\leq \lambda \max \{G(u, u, Tu),\ G(u, Tu, Tu)\} \\ &\leq 2\lambda\, G(u, Tu, Tu). \end{aligned}$$

Since $2\lambda < 1$, the previous inequality can only hold when $G(u, Tu, Tu) = 0$, which proves that $Tu = u$.

Finally, we show that $T$ has a unique fixed. Suppose that $x, y \in \operatorname{Fix}(T)$ are two fixed points of $T$. Then, by (8.1), we deduce that

$$\begin{aligned} G(x, y, y) &= G(Tx, Ty, Ty) \leq \lambda M(x, y, y) \\ &= \lambda \max \left\{ \begin{array}{l} G(x, y, y),\ G(x, Tx, Tx),\ G(x, Ty, Ty), \\ G(x, Tx, y),\ G(x, Tx, y),\ G(y, Ty, Ty), \\ G(y, Tx, Ty),\ G(y, T^2x, Ty),\ G(y, Ty, Ty), \\ G(y, Tx, Tx),\ G(y, Ty, Ty),\ G(y, Tx, Ty), \\ G(y, T^2x, Ty),\ G(Tx, T^2x, Ty),\ G(Tx, T^2x, Ty) \end{array} \right\} \\ &= \lambda \max \{G(x, y, y),\ G(x, x, y)\} \\ &\leq 2\lambda\, G(x, y, y). \end{aligned}$$

As $2\lambda < 1$, the previous inequality can only hold when $G(x, y, y) = 0$, which proves that $x = y$. □

In Theorem 8.1.1, we can take $\lambda$ belonging to the whole interval $[0, 1)$ if we remove the terms for which we need to apply Lemma 3.1.1. Following exactly the same proof (we omit it here), it is possible to obtain the following result, which is also valid adding some terms by symmetry.

**Theorem 8.1.2.** *Let* $(X, G)$ *be a complete G-metric space and let* $T : X \to X$ *be mapping. Assume that there exists* $\lambda \in [0, 1)$ *such that*

$$G(Tx, Ty, Tz) \leq \lambda M(x, y, z) \quad \textit{for all } x, y, z \in X, \tag{8.9}$$

*where*

$$M(x,y,z) = \max \left\{ \begin{array}{l} G(x,y,z),\ G(x,Tx,Tx),\ G(x,Tx,y), \\ G(x,Tx,z),\ G(y,Ty,Ty),\ G(y,Tz,Tz), \\ G(y,T^2x,Ty),\ G(z,Tx,Tx),\ G(z,Tz,Tz), \\ G(z,T^2x,Tz),\ G(Tx,T^2x,Tz),\ G(Tx,T^2x,Ty) \end{array} \right\}.$$

*Then there is a unique* $x \in X$ *such that* $Tx = x$*. In fact,* $T$ *is a Picard operator.*

## 8.2 Revisited Fixed Point Results via Admissible Mappings

In this section we introduce some contractivity conditions very similar to those used in Definition 7.1 and in Sect. 7.3.1, with a very important difference: these new conditions cannot be reduced to quasi-metrics.

**Definition 8.2.1.** Let $(X, G)$ be a $G$-metric space and let $T : X \to X$ be a given mapping. We say that $T$ is a *$G$ - $\beta$ - $\psi$ - contractive mapping of type $A$* if there exist two functions $\beta : X \times X \times X \to [0, \infty)$ and $\psi \in \mathcal{F}_{\text{com}}^{(c)}$ such that, for all $x, y, z \in X$, we have

$$\beta(x, y, Tx)\, G(Tx, Ty, T^2x) \leq \psi(G(x, y, Tx)). \tag{8.10}$$

**Theorem 8.2.1.** *Let* $(X, G)$ *be a complete G-metric space and let* $T : X \to X$ *be a* $G$ - $\beta$ - $\psi$ *- contractive mapping of type A satisfying the following conditions:*

**(i)** *$T$ is $\beta$ - admissible;*
**(ii)** *there exists* $x_0 \in X$ *such that* $\beta(x_0, Tx_0, Tx_0) \geq 1$*;*
**(iii)** *$T$ is G-continuous.*

*Then there exists* $u \in X$ *such that* $Tu = u$*.*

*Proof.* Let $x_0 \in X$ be such that $\beta(x_0, Tx_0, Tx_0) \geq 1$ and let $\{x_n\}$ be the Picard sequence of $T$ based on $x_0$ (that is, $x_{n+1} = Tx_n$ for all $n \geq 0$). If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $u = x_{n_0}$ is a fixed point of $T$. On the contrary case, assume that $x_n \neq x_{n+1}$ for all $n$. Since $T$ is $\beta$ - admissible, we have

$$\begin{aligned} \beta(x_0, x_1, x_1) &= \beta(x_0, Tx_0, Tx_0) \geq 1 \\ &\implies \beta(x_1, x_2, x_2) = \beta(Tx_0, Tx_1, Tx_1) \geq 1. \end{aligned}$$

Inductively, we have that

$$\beta(x_n, x_{n+1}, x_{n+1}) \geq 1 \quad \text{for all } n \geq 0. \tag{8.11}$$

From (8.10) and (8.11), it follows that, for all $n \geq 1$, we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G\left(Tx_{n-1}, Tx_n, T^2x_{n-1}\right) \\ &\leq \beta(x_{n-1}, x_n, Tx_{n-1})\, G\left(Tx_{n-1}, Tx_n, T^2x_{n-1}\right) \\ &\leq \psi(G(x_{n-1}, x_n, Tx_{n-1})) = \psi(G(x_{n-1}, x_n, x_n)). \end{aligned}$$

It follows from Lemma 4.1.2 that $\{x_n\}$ is a Cauchy sequence in the $G$- metric space $(X, G)$. Since $(X, G)$ is complete, there exists $u \in X$ such that $\{x_n\}$ is convergent to $u$. Since $T$ is $G$-continuous, it follows that $\{x_{n+1} = Tx_n\}$ is $G$-convergent, at the same time, to $u$ and to $Tu$. By the uniqueness of the limit, we get $u = Tu$, that is, $u$ is a fixed point of $T$. □

In the following result, we do not need the continuity of $T$.

**Theorem 8.2.2.** *Let $(X, G)$ be a complete $G$-metric space and let $T : X \to X$ be a $G$ - $\beta$ - $\psi$ - contractive mapping of type A satisfying the following conditions:*

**(i)** *$T$ is $\beta$ - admissible;*
**(ii)** *there exists $x_0 \in X$ such that $\beta(x_0, Tx_0, Tx_0) \geq 1$;*
**(iii)** *if $\{x_n\}$ is a sequence in $X$ such that $\beta(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\{x_n\}$ is $G$-convergent to $x \in X$, then $\beta(x_n, x, x) \geq 1$ for all $n \in \mathbb{N}$.*

*Then there exists $u \in X$ such that $Tu = u$.*

*Proof.* Following the argument in the proof of Theorem 8.2.1, we obtain that the Picard sequence $\{x_n\}$ converges to some $u \in X$. From (8.11) and (iii), we have

$$\beta(x_n, u, u) \geq 1 \quad \text{for all } n \geq 0.$$

Using $(G_5)$, $(G_4)$, (8.10) and (8.11), we have that, for all $n \in \mathbb{N}$,

$$\begin{aligned} G(u, Tu, Tu) &\leq G(u, Tx_n, Tx_n) + G(Tx_n, Tu, Tu) \\ &\leq G(u, x_{n+1}, x_{n+1}) + \beta(x_n, u, u)\, G(Tx_n, Tu, Tu) \\ &\leq G(u, x_{n+1}, x_{n+1}) + \psi(G(x_n, u, u)). \end{aligned}$$

Taking into account that $\psi$ is continuous at $t = 0$, letting $n \to \infty$, it follows that $G(u, Tu, Tu) = 0$, so $u = Tu$. □

With the following examples, we will show that the hypotheses in Theorems 8.2.1–8.2.2 do not guarantee uniqueness.

*Example 8.2.1.* Let $X = [0, \infty)$ be endowed with the $G$-metric $G(x,y,z) = |x-y| + |y-z| + |z-x|$ for all $x, y \in X$. Consider the self-mapping $T : X \to X$ and the mapping $\beta : X \times X \times X \to [0, \infty)$ given by

$$Tx = \begin{cases} 2x - \dfrac{7}{4}, & \text{if } x > 1, \\ \dfrac{x}{4}, & \text{if } 0 \le x \le 1; \end{cases} \qquad \beta(x,y,z) = \begin{cases} 1, & \text{if } x,y,z \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

We claim that $T$ is a $G$-$\beta$-$\psi$-contractive mapping of type A with respect to the (c)-comparison function $\psi(t) = t/4$ for all $t \ge 0$. To prove it, let $x, y \in X$ be points such that $\beta(x,y,Tx) > 0$ (if $\beta(x,y,Tx) = 0$, condition (8.10) trivially holds). Then $\beta(x,y,Tx) = 1$, which means that $x, y, Tx \in [0,1]$. In particular, $Tx = x/4$, $T^2x = x/16$ and $Ty = y/4$. Hence,

$$\begin{aligned} \beta(x,y,Tx)\, G(Tx,Ty,T^2x) &= G\left(\frac{x}{4}, \frac{y}{4}, \frac{x}{16}\right) = \\ &= \left|\frac{x}{4} - \frac{y}{4}\right| + \left|\frac{x}{4} - \frac{x}{16}\right| + \left|\frac{x}{16} - \frac{y}{4}\right| \\ &= \frac{1}{4}\left(|x-y| + \left|x - \frac{x}{4}\right| + \left|\frac{x}{4} - y\right|\right) \\ &= \frac{1}{4} G(x,y,Tx) = \psi(G(x,y,Tx)). \end{aligned}$$

Furthermore, for $x_0 = 1$ we have that $\beta(1, T1, T1) = \beta(1, \frac{1}{4}, \frac{1}{4}) = 1$. As $T$ is continuous, to show that $T$ satisfies all the hypothesis of Theorem 8.2.1, it is sufficient to observe that $T$ is $\beta$-admissible. For this purpose, let $x, y \in X$ such that $\beta(x,y,z) \ge 1$. In this case, $x, y, z \in [0,1]$. Hence $Tx = x/4 \in [0,1]$, $Ty = y/4 \in [0,1]$ and $Tz = z/4 \in [0,1]$, which implies that $\beta(Tx,Ty,Tz) \ge 1$. As a result, all the conditions of Theorem 8.2.1 are satisfied. Theorem 8.2.1 guarantees the existence of a fixed point of $T$, but not its uniqueness. In this example, 0 and $\frac{7}{4}$ are two fixed points of $T$.

Notice that Theorem 4.16, given by Mustafa as a characterization of the Banach fixed point theorem, cannot be applied in this case because

$$G(T1, T2, T2) = 4 > 2 = G(1,2,2).$$

In the following example, $T$ is not continuous.

*Example 8.2.2.* Let $(X, G)$ and $\beta$ be given as in Example 7.1.2, and let $T$ and $\psi$ be given by

$$Tx = \begin{cases} 2x - \dfrac{7}{4}, & \text{if } x > 1, \\ \dfrac{x}{3}, & \text{if } 0 \le x \le 1; \end{cases} \qquad \psi(t) = t/3.$$

Following the previous arguments, it is easy to show that, for all $x, y \in X$,

$$\beta(x, y, Tx)\, G(Tx, Ty, T^2x) \leq \frac{1}{4} G(x, y, Tx) \leq \psi\left(G\left(x, y, Tx\right)\right).$$

Therefore, $T$ is a $G$ - $\beta$ - $\psi$ - contractive mapping of type A. Furthermore, the point $x_0 = 1$ satisfies $\beta(1, T1, T1) = 1$ and $T$ is $\beta$ - admissible. However, $T$ is not continuous. In this case, we can prove hypothesis (iii) of Theorem 7.1.2. Indeed, let $\{x_n\}$ be a sequence such that $\beta(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\{x_n\} \to x \in X$. Since $\beta(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $x_n \in [0, 1]$, and as this interval is closed, we deduce that $x \in [0, 1]$. Thus, $\beta(x_n, x, x) \geq 1$ for all $n \in \mathbb{N}$. As a result, all the conditions of Theorem 8.2.2 are satisfied. Notice that Theorem 8.2.2 can only guarantee the existence of a fixed point of $T$, but not its uniqueness. In fact, 0 and $\frac{7}{4}$ are two fixed points of $T$.

The uniqueness of the fixed point can be deduced from an additional assumption.

**Theorem 8.2.3.** *Under the hypotheses of Theorem 8.2.1 (respectively, Theorem 8.2.2), also assume the following condition:*

$(U)$ *For all $x, y \in \operatorname{Fix}(T)$, there exists $z \in X$ such that $\beta\left(x, z, x\right) \geq 1$ and $\beta\left(y, z, y\right) \geq 1$.*

*Then $T$ has a unique fixed point.*

Notice that the previous condition is different from what we introduced in Theorem 7.1.3.

*Proof.* Let $x, y \in \operatorname{Fix}(T)$ be two fixed points of $T$. By $(U)$, there exists $z \in X$ such that $\beta\left(x, z, x\right) \geq 1$ and $\beta\left(y, z, y\right) \geq 1$. We claim that the sequence $\{z_n = T^n z\}_{n\geq 0}$ converges, at the same time, to $x$ and to $y$ and, hence, we will deduce that $x = y$. The following argument only uses $x$, but it is also valid involving $y$. Since $T$ is $\beta$ - admissible, we get

$$\beta\left(x, z, x\right) \geq 1 \quad \Rightarrow \quad \beta\left(x, Tz, x\right) = \beta\left(Tx, Tz, Tx\right) \geq 1,$$

and, by induction,

$$\beta\left(x, T^n z, Tx\right) = \beta\left(x, T^n z, x\right) \geq 1 \quad \text{for all } n \in \mathbb{N}. \tag{8.12}$$

From (8.10) and (8.12), we have that, for all $n \geq 0$,

$$\begin{aligned} G\left(x, T^{n+1}z, x\right) &\leq \beta\left(x, T^n z, Tx\right) G\left(Tx, TT^n z, T^2 x\right) \\ &\leq \psi\left(G\left(x, T^n z, Tx\right)\right) = \psi\left(G\left(x, T^n z, x\right)\right). \end{aligned}$$

Thus, we get, by induction, that

$$G\left(x, T^n z, x\right) \leq \psi^n\left(G\left(x, z, x\right)\right), \quad \text{for all } n \in \mathbb{N}.$$

Letting $n \to \infty$ and taking into account that $\psi \in \mathcal{F}_{\text{com}}^{(c)}$, we have that $\{G(x, T^n z, x)\} \to 0$, so $\{T^n z\} \to x$. Similarly, $\{T^n z\} \to y$, so $x = y$ by the uniqueness of the limit. □

## 8.3 Modified $\alpha$ - $\phi$ - Asymmetric Meir-Keeler Contractive Mappings

In this section, we present some theorems inspired from Sect. 6.5, by introducing a mapping $\alpha : X \times X \to [0, \infty)$ on the contractivity condition. Recall that we denote by $\mathcal{F}_{\text{alt}}$ the family of continuous, non-decreasing functions $\phi : [0, +\infty) \to [0, +\infty)$ such that $\phi(t) = 0$ if, and only if, $t = 0$.

**Definition 8.3.1 ([182]).** Let $(X, G)$ be a $G$-metric space and let $\phi \in \mathcal{F}_{\text{alt}}$ and $\alpha : X \times X \to [0, \infty)$ be two functions. We say that $T : X \to X$ is a *modified* $\alpha$ - $\phi$ - *asymmetric Meir-Keeler contractive mapping* if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left.\begin{array}{r} x, y \in X, \quad \alpha(x, y) \geq 1, \\ \varepsilon \leq \phi(G(x, Tx, y)) < \varepsilon + \delta \end{array}\right\} \quad \Rightarrow \quad \phi(G(Tx, T^2 x, Ty)) < \varepsilon. \tag{8.13}$$

*Remark 8.3.1.* If $T : X \to X$ is a modified $\alpha$ - $\phi$ - asymmetric Meir-Keeler contractive mapping and $x, y \in X$ are such that $x \neq Tx$ and $\alpha(x, y) \geq 1$, then

$$\phi(G(Tx, T^2 x, Ty)) < \phi(G(x, Tx, y)). \tag{8.14}$$

**Definition 8.3.2.** A function $\alpha : X \times X \to [0, \infty)$ is *transitive* if, given $x, y, z \in X$,

$$\alpha(x, y) \geq 1, \quad \alpha(y, z) \geq 1 \quad \Rightarrow \quad \alpha(x, z) \geq 1.$$

A mapping $T : X \to X$ is said to be $\alpha$ - *admissible* if

$$\alpha(x, y) \geq 1 \quad \Rightarrow \quad \alpha(Tx, Ty) \geq 1.$$

*Remark 8.3.2.* If $\alpha(x, y) \geq 1$ for all $x, y \in X$, then any mapping $T : X \to X$ is $\alpha$ - admissible. In particular, this property holds when $\alpha(x, y) = 1$ for all $x, y \in X$.

**Lemma 8.3.1.** *Let $T : X \to X$ be an $\alpha$ - admissible mapping and let $\{x_n\}_{n \geq 0} \subseteq X$ be a Picard sequence of $T$ based on a point $x_0 \in X$. If $x_0$ satisfies $\alpha(x_0, Tx_0) \geq 1$, then $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Additionally, if $\alpha$ is transitive, then $\alpha(x_n, x_m) \geq 1$ for all $n, m \in \mathbb{N}$ such that $n < m$.*

*Proof.* The initial condition $\alpha(x_0, Tx_0) \geq 1$ means that $\alpha(x_0, x_1) \geq 1$. Since $T$ is an $\alpha$ - admissible mapping, we have that $\alpha(Tx_0, Tx_1) \geq 1$, which means that $\alpha(x_1, x_2) \geq 1$. By induction, we deduce that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$.

Now suppose that $\alpha$ is transitive, and let $n, m \in \mathbb{N}$ such that $n < m$. As

$$\alpha(x_n, x_{n+1}) \geq 1, \ \alpha(x_{n+1}, x_{n+2}) \geq 1, \ldots, \alpha(x_{m-1}, x_m) \geq 1,$$

we deduce that $\alpha(x_n, x_m) \geq 1$ by the transitivity of $\alpha$. □

In order to prove the main result of this section, we need the following version of Theorem 4.1.1.

**Lemma 8.3.2.** *Let $(X, G)$ be a G-metric space and let $\{x_n\} \subseteq X$ be a sequence such that* $\lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = 0$.

1. *If $\{x_n\}$ satisfies the following property:*

$$\text{for all } \varepsilon > 0, \text{ there exists } n_0 \in \mathbb{N} \text{ such that}$$
$$G(x_n, x_{n+1}, x_m) < \varepsilon \text{ for all } n, m \in \mathbb{N} \text{ with } m > n \geq n_0, \tag{8.15}$$

   *then $\{x_n\}$ is a Cauchy sequence in $(X, G)$.*
2. *If $\{x_n\}$ is not a Cauchy sequence in $(X, G)$, then there exist $\varepsilon_0 > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that, for all $k \in \mathbb{N}$,*

$$k < n(k) < m(k),$$
$$G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)-1}\right) \leq \varepsilon_0 < G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)}\right).$$

   *Furthermore,*

$$\lim_{k\to\infty} G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)}\right) = \lim_{k\to\infty} G\left(x_{n(k)-1}, x_{n(k)}, x_{m(k)-1}\right) = \varepsilon_0.$$

*Proof.* Since $\lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = 0$, Lemma 3.1.1 shows that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0.$$

**(1)** Fix $\varepsilon > 0$ arbitrary. Let $n_1 \in \mathbb{N}$ be such that

$$\max\{G(x_n, x_n, x_{n+1}), G(x_n, x_{n+1}, x_{n+1})\} \leq \frac{\varepsilon}{8} \quad \text{for all } n \geq n_1.$$

By hypothesis, let $n_2 \in \mathbb{N}$ be such that

$$G(x_n, x_{n+1}, x_m) < \frac{\varepsilon}{8} \quad \text{for all } n, m \in \mathbb{N} \text{ with } m > n \geq n_2.$$

Now let $n_0 = \max(n_1, n_2)$. Let $n, m, p \in \mathbb{N}$ be such that $n_0 \leq n < m < p$. Then

$$G\left(x_n, x_m, x_p\right) = G\left(x_p, x_m, x_n\right) \leq G\left(x_p, x_m, x_m\right) + G\left(x_m, x_m, x_n\right)$$

$$\begin{aligned}
&\leq G\left(x_m, x_m, x_p\right) + 2G\left(x_m, x_n, x_n\right) \\
&\leq G\left(x_m, x_{m+1}, x_{m+1}\right) + G\left(x_{m+1}, x_m, x_p\right) \\
&\qquad + 2\left[G\left(x_n, x_{n+1}, x_{n+1}\right) + G\left(x_{n+1}, x_n, x_m\right)\right] \\
&\leq \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + 2\left[\frac{\varepsilon}{8} + \frac{\varepsilon}{8}\right] = \frac{3\varepsilon}{4} < \varepsilon.
\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence in $(X, G)$.

**(2)** If $\{x_n\}$ is not a Cauchy sequence in $(X, G)$, then condition (8.15) cannot hold. Then, there exists $\varepsilon_0 > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that, for all $k \in \mathbb{N}$,

$$k \leq n(k) < m(k) \quad \text{and} \quad \varepsilon_0 < G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)}\right).$$

If we choose $m(k)$ as the smallest integer, greater than $n(k)$, satisfying this property, then $m(k) - 1$ does not verify it. Hence, $G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)-1}\right) \leq \varepsilon_0$ for all $k \in \mathbb{N}$. In particular, for all $k \in \mathbb{N}$,

$$\begin{aligned}
\varepsilon_0 &< G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)}\right) \\
&\leq G\left(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}\right) + G\left(x_{m(k)-1}, x_{n(k)}, x_{n(k)+1}\right) \\
&\leq G\left(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}\right) + \varepsilon_0.
\end{aligned}$$

Letting $k \to \infty$ we deduce that

$$\lim_{k\to\infty} G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)}\right) = \varepsilon_0.$$

From Lemma 4.1.4,

$$\lim_{k\to\infty} G\left(x_{n(k)-1}, x_{n(k)}, x_{m(k)-1}\right) = \varepsilon_0, \tag{8.16}$$

which finishes the proof. □

Now, we are ready to state and prove the main result of this section.

**Theorem 8.3.1 ([182]).** *Let $(X, G)$ be a G-complete G-metric space and let $\phi \in \mathcal{F}_{\mathrm{alt}}$ and $\alpha : X \times X \to [0, \infty)$ be two functions. Suppose that $T : X \to X$ is a modified $\alpha$-$\phi$-asymmetric Meir-Keeler contractive mapping such that the following conditions hold:*

**(i)** *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*
**(ii)** *$\alpha$ is transitive and $T$ is $\alpha$-admissible;*
**(iii)** *$T$ is continuous.*

*Then $T$ has, at least, a fixed point.*

*Proof.* From (i), let $x_0 \in X$ be a point such that $\alpha(x_0, Tx_0) \geq 1$ and let $\{x_n\}_{n\geq 0}$ be the Picard sequence of $T$ based on $x_0$, that is, $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then $x_{n_0}$ is a fixed point of $T$. On the contrary case, suppose that

$$Tx_n = x_{n+1} \neq x_n \quad \text{for all } n \in \mathbb{N},$$

that is,

$$G(x_n, x_{n+1}, x_{n+1}) > 0 \quad \text{for all } n \in \mathbb{N}. \tag{8.17}$$

From Lemma 8.3.1 we get

$$\alpha(x_n, x_m) \geq 1 \quad \text{for all } m, n \in \mathbb{N} \text{ such that } n < m. \tag{8.18}$$

By (8.14) and (8.18), we observe that, for all $n \in \mathbb{N}$,

$$\begin{aligned} \phi(G(x_{n+1}, x_{n+2}, x_{n+2})) &= \phi(G(Tx_n, T^2x_n, Tx_{n+1})) \\ &< \phi(G(x_n, Tx_n, x_{n+1})) = \phi(G(x_n, x_{n+1}, x_{n+1})) \end{aligned} \tag{8.19}$$

Therefore, $\{\phi(G(x_n, x_{n+1}, x_{n+1}))\}_{n\in\mathbb{N}}$ is a decreasing sequence of nonnegative real numbers. Hence, it is convergent. Let $L \geq 0$ be its limit. We claim that $L = 0$. Suppose, on the contrary, that $L > 0$. Thus, we have

$$0 < L < \phi(G(x_n, x_{n+1}, x_{n+1})) \quad \text{for all } n \in \mathbb{N}. \tag{8.20}$$

Assume $\varepsilon = L > 0$. As $T$ is a modified $\alpha$ - $\phi$ - asymmetric Meir-Keeler contractive mapping, there exists $\delta > 0$ such that (8.13) holds. On the other hand, as $\{\phi(G(x_n, x_{n+1}, x_{n+1}))\} \searrow L$, then there exists $n_0 \in \mathbb{N}$ such that

$$\varepsilon \leq \phi(G(x_{n_0}, x_{n_0+1}, x_{n_0+1})) = \phi(G(x_{n_0}, Tx_{n_0}, x_{n_0+1})) < \varepsilon + \delta. \tag{8.21}$$

Taking the condition (8.13) into account, the expression (8.21) yields that

$$\phi(G(x_{n_0+1}, x_{n_0+2}, x_{n_0+2})) = \phi(G(Tx_{n_0}, T^2x_{n_0}, Tx_{n_0+1})) < \varepsilon = L \tag{8.22}$$

which contradicts (8.20). Hence $L = 0$, that is,

$$\lim_{n\to\infty} \phi(G(x_n, x_{n+1}, x_{n+1}) = 0.$$

Since $\phi \in \mathcal{F}_{\text{alt}}$, we deduce, by Lemma 2.3.3, that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0.$$

Moreover, by Lemma 3.1.1,

$$\lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = 0.$$

In fact, by Proposition 2.3.4, we have that

$$G(x_{n+1}, x_{n+2}, x_{n+2}) < G(x_n, x_{n+1}, x_{n+1}) \quad \text{for all } n \in \mathbb{N}.$$

We now show that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $(X, G)$ reasoning by contradiction. Assume that $\{x_n\}$ is not a Cauchy sequence in $(X, G)$. By item 2 of Lemma 8.3.2, there exists $\varepsilon_0 > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that, for all $k \geq 0$,

$$\begin{aligned} & k \leq n(k) < m(k), \\ & G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)-1}\right) \leq \varepsilon_0 < G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)}\right), \end{aligned} \tag{8.23}$$

and

$$\lim_{k\to\infty} G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)}\right) = \lim_{k\to\infty} G\left(x_{n(k)-1}, x_{n(k)}, x_{m(k)-1}\right) = \varepsilon_0.$$

As $\phi$ is non-decreasing,

$$\phi\left(\varepsilon_0\right) \leq \phi(G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)}\right)) \quad \text{for all } k \in \mathbb{N}. \tag{8.24}$$

Let $k_0 \in \mathbb{N}$ be a number such that

$$\frac{\varepsilon_0}{2} < G\left(x_{n(k)-1}, x_{n(k)}, x_{m(k)-1}\right) \quad \text{for all } k \geq k_0. \tag{8.25}$$

Let $\varepsilon_1 = \phi\left(\varepsilon_0\right) > 0$. As $\phi$ is continuous,

$$\lim_{k\to\infty} \phi(G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)}\right)) = \phi\left(\varepsilon_0\right) = \varepsilon_1. \tag{8.26}$$

As $T$ is a modified $\alpha$ - $\phi$ - asymmetric Meir-Keeler contractive mapping, for $\varepsilon_1 > 0$, there exists $\delta_1 > 0$ such that

$$\begin{aligned} & \text{if } x, y \in X \text{ and } \alpha\left(x, y\right) \geq 1, \\ & \varepsilon_1 \leq \phi(G(x, Tx, y)) < \varepsilon_1 + \delta_1 \quad \Rightarrow \quad \phi(G(Tx, T^2x, Ty)) < \varepsilon_1 = \phi\left(\varepsilon_0\right). \end{aligned} \tag{8.27}$$

If there exists some $k' \in \mathbb{N}$, with $k' \geq k_0$, such that

$$G(x_{n(k')-1}, x_{n(k')}, x_{m(k')-1}) \leq \varepsilon_0,$$

it follows from (8.14), (8.18) and (8.25) that

$$\begin{aligned}
Tx_{n(k')-1} = x_{n(k')} \neq x_{n(k')-1}, \quad &\alpha(x_{n(k')-1}, x_{m(k')-1}) \geq 1 \\
\Rightarrow \quad &\phi(G(Tx_{n(k')-1}, T^2x_{n(k')-1}, Tx_{m(k')-1})) \\
&\quad < \phi(G(x_{n(k')-1}, Tx_{n(k')-1}, x_{m(k')-1})) \\
\Rightarrow \quad &\phi(G(x_{n(k')}, x_{n(k')+1}, x_{m(k')})) \\
&\quad < \phi(G(x_{n(k')-1}, x_{n(k')}, x_{m(k')-1})) \leq \phi\,(\varepsilon_0) = \varepsilon_1,
\end{aligned}$$

but this is impossible because by (8.23),

$$\begin{aligned}
&\varepsilon_0 < G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)}\right) \\
&\qquad \Rightarrow \quad \varepsilon_1 = \phi\,(\varepsilon_0) \leq \phi(G(x_{n(k')}, x_{n(k')+1}, x_{m(k')})).
\end{aligned}$$

As a result, such a $k'$ cannot exist, so

$$G(x_{n(k)-1}, x_{n(k)}, x_{m(k)-1}) > \varepsilon_0 \quad \text{for all } k \geq k_0.$$

As $\phi$ is non-decreasing,

$$\varepsilon_1 = \phi\,(\varepsilon_0) \leq \phi(G(x_{n(k)-1}, x_{n(k)}, x_{m(k)-1})) \quad \text{for all } k \geq k_0.$$

Moreover, by (8.26), there exists some $\kappa \geq k_0$ such that

$$\varepsilon_1 \leq \phi(G(x_{n(\kappa)-1}, Tx_{n(\kappa)-1}, x_{m(\kappa)-1})) < \varepsilon_1 + \delta_1.$$

Taking into account that $\alpha(x_{n(\kappa)-1}, x_{m(\kappa)-1}) \geq 1$ by (8.18) and using (8.27), we conclude that

$$\begin{aligned}
\phi(G(x_{n(\kappa)}, x_{n(\kappa)+1}, x_{m(\kappa)})) &= \phi(G(Tx_{n(\kappa)-1}, T^2x_{n(\kappa)-1}, Tx_{m(\kappa)-1})) \\
&< \varepsilon_1 = \phi\,(\varepsilon_0)\,,
\end{aligned}$$

but this is a contradiction with (8.24). This contradiction proves that $\{x_n\}$ is a Cauchy sequence in $(X, G)$. As $(X, G)$ is complete, there exists $u \in X$ such that $\{x_n\} \to u$. As $T$ is continuous, then $\{x_{n+1} = Tx_n\} \to Tu$, and the uniqueness of the limit in a $G$-metric space we conclude that $Tu = u$. □

In order to avoid the continuity of $T$ in the previous result, we introduce the following notion.

**Definition 8.3.3.** Let $(X, G)$ be a $G$-metric space and let $\alpha : X \times X \to [0, \infty)$ be a function. We say that $(X, G)$ is *$\alpha$-non-decreasing-regular* if $\alpha\,(x_n, u) \geq 1$ for all $n \in \mathbb{N}$ provided that $\{x_n\} \subseteq X$ and $u \in X$ are such that $\{x_n\} \to u$ and $\alpha\,(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$.

**Theorem 8.3.2 ([182]).** *Let $(X, G)$ be a G-complete G-metric space and let $\phi \in \mathcal{F}_{\rm alt}$ and $\alpha : X \times X \to [0,\infty)$ be two functions. Suppose that $T : X \to X$ is a modified $\alpha$ - $\phi$ - asymmetric Meir-Keeler contractive mapping such that the following conditions hold:*

**(i)** *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*
**(ii)** *$\alpha$ is transitive and $T$ is $\alpha$ - admissible;*
**(iii)** *$(X, G)$ is $\alpha$-non-decreasing-regular.*

*Then $T$ has, at least, a fixed point.*

*Proof.* Following the proof in Theorem 8.3.1, we may deduce that there exists $u \in X$ such that $\{x_{n+1} = Tx_n\} \to u$ and (8.18) holds. Since $(X, G)$ is $\alpha$-non-decreasing-regular, we have that

$$\alpha\,(x_n, u) \geq 1 \quad \text{for all } n \in \mathbb{N}.$$

Since $Tx_n = x_{n+1} \neq x_n$ for all $n$, by Remark 8.3.1 we have that

$$\begin{aligned}\phi(G(x_{n+1}, x_{n+2}, Tu)) &= \phi(G(Tx_n, T^2x_n, Tu)) \\ &< \phi(G(x_n, Tx_n, u)) = \phi(G(x_n, x_{n+1}, u)).\end{aligned}$$

As $G$ and $\phi$ are continuous, $\lim_{n\to\infty} \phi(G(x_n, x_{n+1}, u)) = \phi\,(G\,(u,u,u)) = \phi\,(0) = 0$, so

$$\phi(G(u,u,Tu)) = \lim_{n\to\infty} \phi(G(x_{n+1}, x_{n+2}, Tu)) = 0.$$

As $\phi \in \mathcal{F}_{\rm alt}$, we deduce that $G(u,u,Tu) = 0$, so $Tu = u$. □

*Example 8.3.1.* Let $X = [0,\infty)$ and define $G : X^3 \to [0,\infty)$ by

$$G(x,y,z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x,y\} + \max\{y,z\} + \max\{x,z\}, & \text{otherwise.} \end{cases}$$

Clearly, $(X, G)$ is a complete $G$-metric space. Define $T : X \to X$, $\alpha : X \times X \to [0,\infty)$ and $\phi : [0,\infty) \to [0,\infty)$ by

$$Tx = \begin{cases} \frac{1}{4}x, & \text{if } x \in [0,1], \\ x^2 + 2|x-2||x-3|\ln x, & \text{if } x > 1; \end{cases} \qquad \phi(t) = t$$

$$\alpha(x,y) = \begin{cases} 8, & \text{if } x,y \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Let $x, y \in X$ be such that $\alpha(x,y) \geq 1$. Then $x, y \in [0,1]$. At first, assume that $x \leq y$. Then,

$$G(x, Tx, y) = \max\{x, Tx\} + \max\{Tx, y\} + \max\{x, y\} = x + 2y$$

and

$$\begin{aligned} G(Tx, T^2x, Ty) &= \max\{Tx, T^2x\} + \max\{T^2x, Ty\} + \max\{Tx, Ty\} \\ &= \frac{1}{4}(x + 2y) = \frac{1}{4}G(x, Tx, y). \end{aligned}$$

Next, assume that, $y < x$. Then,

$$\begin{aligned} G(x, Tx, y) &= \max\{x, Tx\} + \max\{Tx, y\} + \max\{x, y\} \\ &= 2x + \max\left\{\frac{1}{4}x, y\right\}, \end{aligned}$$

and

$$\begin{aligned} G(Tx, T^2x, Ty) &= \max\{Tx, T^2x\} + \max\{T^2x, Ty\} + \max\{Tx, Ty\} \\ &= \frac{1}{4}\left(2x + \max\left\{\frac{1}{4}x, y\right\}\right) = \frac{1}{4}G(x, Tx, y). \end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. Using $\delta = 3\varepsilon > 0$, condition (8.13) holds. Again if $\alpha(x, y) \geq 1$, then $x, y \in [0, 1]$. On the other hand for all $w \in [0, 1]$, we have $Tw \leq 1$. Hence $\alpha(Tx, Ty) \geq 1$. Further, if $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$, then $x, y, z \in [0, 1]$. Thus $\alpha(x, z) \geq 1$. This implies that $T$ is an $\alpha$ - admissible mapping. Clearly, $\alpha(0, T0) \geq 1$.

Although $T$ is not continuous, we can apply Theorem 8.3.2. Indeed, let $\{x_n\}$ be a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\{x_n\} \to x$. Then $\{x_n\} \subseteq [0, 1]$ and, hence, $x \in [0, 1]$. This implies that $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. Thus, all the conditions of Theorem 8.3.2 hold and $T$ has a fixed point.

Notice that Theorems 6 and 8 of [116] cannot be applied for this example because, although $\varepsilon \leq \phi(d(0, 1)) = 1 < \varepsilon + \delta$ for $\varepsilon = 1$ and $\delta > 0$ arbitrary (where $d$ is a Euclidean metric on $X$), we have that

$$\alpha(0, 1)\phi(d(T0, T1)) = 8\frac{1}{4} = 2 > 1 = \phi(d(0, 1)) \geq \varepsilon.$$

From Theorem 8.3.1 we can deduce the following corollary, using $\phi(t) = t$ for all $t \geq 0$.

**Corollary 8.3.1 ([182]).** *Let $(X, G)$ be a complete G-metric space and let $\alpha : X \times X \to [0, \infty)$ be a transitive function. Suppose that $T : X \to X$ is an $\alpha$ - admissible mapping satisfying the following condition:*

*for each* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that*

$$\left.\begin{array}{l} x, y \in X, \quad \alpha(x,y) \geq 1, \\ \varepsilon \leq G(x,Tx,y) < \varepsilon + \delta \end{array}\right\} \quad \Rightarrow \quad G(Tx,T^2x,Ty) < \varepsilon.$$

*If there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $T$ *is continuous, then* $T$ *has, at least, a fixed point.*

From Theorem 8.3.2 we can deduce the following result.

**Corollary 8.3.2 ([182]).** *Let* $(X,G)$ *be a complete G-metric space and let* $\alpha : X \times X \to [0,\infty)$ *be a transitive function. Suppose that* $T : X \to X$ *is an* $\alpha$ *- admissible mapping satisfying the following condition:*

*for each* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that*

$$\left.\begin{array}{l} x, y \in X, \quad \alpha(x,y) \geq 1, \\ \varepsilon \leq G(x,Tx,y) < \varepsilon + \delta \end{array}\right\} \quad \Rightarrow \quad G(Tx,T^2x,Ty) < \varepsilon.$$

*If there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $(X,G)$ *is* $\alpha$*-non-decreasing-regular, then* $T$ *has, at least, a fixed point.*

By taking $\alpha(x,y) = 1$ for all $x, y \in X$, in the above corollary we deduce the following result.

**Corollary 8.3.3 ([182]).** *Let* $(X,G)$ *be a complete G-metric space and let* $T : X \to X$ *be a self-mapping satisfying the following condition:*

*for each* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that*

$$x, y \in X, \;\; \varepsilon \leq G(x,Tx,y) < \varepsilon + \delta \quad \Rightarrow \quad G(Tx,T^2x,Ty) < \varepsilon.$$

*Then* $T$ *has, at least, a fixed point.*

### *8.3.1 Fixed Point Results in Partially Ordered G-Metric Spaces*

In the following result we show how a binary relation $\preccurlyeq$ induces an appropriate function $\alpha_{\preccurlyeq}$.

**Corollary 8.3.4 ([182]).** *Let* $(X,G,\preccurlyeq)$ *be an ordered complete G-metric space, let* $T : X \to X$ *be a* $\preccurlyeq$*-non-decreasing mapping and let* $\phi \in \mathcal{F}_{\text{alt}}$*. Assume that, given* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that*

$$\left.\begin{array}{l} x, y \in X, \quad x \preccurlyeq y, \\ \varepsilon \leq \phi(G(x,Tx,y)) < \varepsilon + \delta \end{array}\right\} \quad \Rightarrow \quad \phi(G(Tx,T^2x,Ty)) < \varepsilon. \tag{8.28}$$

*Suppose that* $T$ *is continuous and there exists* $x_0$ *in* $X$ *such that* $x_0 \preccurlyeq Tx_0$*. Then* $T$ *has, at least, a fixed point.*

*Proof.* Define the mapping $\alpha_{\preccurlyeq} : X \times X \to [0, \infty)$ given, for all $x, y \in X$, by

$$\alpha_{\preccurlyeq}(x, y) = \begin{cases} 1, & \text{if } x \preccurlyeq y, \\ 0, & \text{otherwise.} \end{cases} \tag{8.29}$$

Since $T$ is $\preccurlyeq$-non-decreasing mapping, then

$$\alpha_{\preccurlyeq}(x, y) \geq 1 \ \Rightarrow \ x \preccurlyeq y \ \Rightarrow \ Tx \preccurlyeq Ty \ \Rightarrow \ \alpha_{\preccurlyeq}(Tx, Ty) \geq 1.$$

Hence, $T$ is $\alpha$-admissible. The point $x_0$ such that $x_0 \preccurlyeq Tx_0$ satisfies $\alpha_{\preccurlyeq}(x_0, Tx_0) \geq 1$. Moreover, as $\preccurlyeq$ is transitive, then $\alpha_{\preccurlyeq}$ is transitive. Finally, condition (8.28) means that $T : X \to X$ is a modified $\alpha_{\preccurlyeq}$-$\phi$-asymmetric Meir-Keeler contractive mapping. Therefore, all the hypotheses of Theorem 8.3.1 are satisfied and, hence, $T$ has, at least, a fixed point in $X$. □

In the following corollary, we replace the continuity of $T$ by the non-decreasing-regularity of $(X, G, \preccurlyeq)$ (recall Definition 5.2.1).

**Corollary 8.3.5 ([182]).** *Let $(X, G, \preccurlyeq)$ be an ordered complete G-metric space, let $T : X \to X$ be a $\preccurlyeq$-non-decreasing mapping and let $\phi \in \mathcal{F}_{\mathrm{alt}}$. Assume that, given $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\left.\begin{array}{c} x, y \in X, \quad x \preccurlyeq y, \\ \varepsilon \leq \phi(G(x, Tx, y)) < \varepsilon + \delta \end{array}\right\} \quad \Rightarrow \quad \phi(G(Tx, T^2x, Ty)) < \varepsilon.$$

*Suppose that $(X, G, \preccurlyeq)$ is non-decreasing-regular and there exists $x_0$ in $X$ such that $x_0 \preccurlyeq Tx_0$. Then $T$ has, at least, a fixed point.*

*Proof.* It is only necessary to consider the function $\alpha_{\preccurlyeq}$ given in (8.29). Then, Theorem 8.3.2 is applicable. □

We can also deduce the following corollaries from the above theorems, taking $\phi(t) = t$ for all $t \geq 0$.

**Corollary 8.3.6 ([182]).** *Let $(X, G, \preccurlyeq)$ be an ordered complete G-metric space and let $T : X \to X$ be a $\preccurlyeq$-non-decreasing mapping. Assume that, given $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \leq G(x, Tx, y) < \varepsilon + \delta \quad \Rightarrow \quad G(Tx, T^2x, Ty) < \varepsilon$$

*for all $x, y \in X$ with $x \preccurlyeq y$. Also suppose that $T$ is continuous and that there exists $x_0$ in $X$ such that $x_0 \preccurlyeq Tx_0$. Then $T$ has, at least, a fixed point.*

**Corollary 8.3.7 ([182]).** *Let $(X, G, \preccurlyeq)$ be an ordered complete G-metric space and let $T : X \to X$ be a $\preccurlyeq$-non-decreasing mapping. Assume that, given $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \leq G(x, Tx, y) < \varepsilon + \delta \quad \Rightarrow \quad G(Tx, T^2x, Ty) < \varepsilon$$

*for all $x, y \in X$ with $x \preccurlyeq y$. Also suppose that $(X, G, \preccurlyeq)$ is non-decreasing-regular and that there exists $x_0$ in $X$ such that $x_0 \preccurlyeq Tx_0$. Then $T$ has, at least, a fixed point.*

### 8.3.2 Fixed Point Results for Orbitally G-Continuous Mappings

Following the techniques given in Sect. 6.5.2, we can deduce the following results in the context of orbitally $G$-continuous mappings (recall Definition 6.5.3).

**Theorem 8.3.3.** *Let $(X, G)$ be a $G$-metric space and let $T : X \to X$ be a self-mapping. Assume that, given $\varepsilon > 0$, there exist $\phi \in \mathcal{F}_{\mathrm{alt}}$ and $\delta > 0$ such that*

$$\varepsilon \leq \phi(G(x, Tx, y)) < \varepsilon + \delta \quad \Rightarrow \quad \phi(G(Tx, T^2x, Ty)) < \varepsilon \tag{8.30}$$

*for all distinct points $x, y \in \overline{O_T(x)}$ with $Tx = y$. Suppose also that:*

($C$) *for some $x_0 \in X$, the orbit $O_T(x_0)$ of $x_0$ with respect to $T$ has a cluster point $z \in X$.*

*If $T$ is orbitally $G$-continuous at $z$, then $z$ is a fixed point of $T$ in $\overline{O_T(x_0)}$.*

**Corollary 8.3.8 ([182]).** *Let $(X, G)$ be a $G$-metric space and let $T : X \to X$ be a self-mapping. Assume that, given $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \leq G(x, Tx, y) < \varepsilon + \delta \quad \Rightarrow \quad G(Tx, T^2x, Ty) < \varepsilon$$

*for all distinct $x, y \in \overline{O_T(x)}$ with $Tx = y$. Suppose also that:*

($C$) *for some $x_0 \in X$, the orbit $O_T(x_0)$ of $x_0$ with respect to $T$ has a cluster point $z \in X$.*

*If $T$ is orbitally $G$-continuous at $z$, then $z$ is a fixed point of $T$ in $\overline{O_T(x_0)}$.*

# Chapter 9
# Expansive Mappings

In this chapter we present some fixed point theorems for expansive mappings.

## 9.1 Fixed Point Theorems for Expansive Mappings on $G$-Metric Spaces

In this section, we establish some fixed point results for expansive mappings in the frameworks of $G$-metric spaces.

**Definition 9.1.1.** A mapping $T : X \to X$ from a $G$-metric space $(X, G)$ into itself is said to be:

- *expansive of type I* if there exists $\lambda > 1$ such that

$$G(Tx, Ty, Tz) \geq \lambda\, G(x, y, z) \quad \text{for all } x, y, z \in X.$$

- *expansive of type II* if there exists $\lambda > 1$ such that

$$G(Tx, Tx, Ty) \geq \lambda\, G(x, x, y) \quad \text{for all } x, y \in X.$$

To prove the main result of the section, we recall that if $T : X \to X$ is onto, then there exists a mapping $T' : X \to X$ such that $T \circ T'$ is the identity mapping on $X$ (see Proposition 2.1.1).

**Theorem 9.1.1.** *Let $(X, G)$ be a complete G-metric space and let $T : X \to X$ be an onto mapping such that there exist $\psi \in \mathcal{F}_{\text{alt}}$ and $\phi \in \mathcal{F}'_{\text{alt}}$ satisfying*

$$\psi\,(G\,(x, x, y)) \leq \psi\,(G\,(Tx, Tx, Ty)) - \phi\,(G\,(Tx, Tx, Ty)) \tag{9.1}$$

*for all $x, y \in X$. Then $T$ has a unique fixed point.*



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*Proof.* From Proposition 2.1.1, since $T$ is onto, there exists a mapping $T' : X \to X$ such that $T \circ T'$ is the identity mapping on $X$. Let $x, y \in X$ be arbitrary points and let $z = T'x$ and $w = T'y$. By using (9.1) applied to $z$ and $w$, we have that

$$\psi\left(G\left(z,z,w\right)\right) \leq \psi\left(G\left(Tz,Tz,Tw\right)\right) - \phi\left(G\left(Tz,Tz,Tw\right)\right).$$

Since $Tz = TT'x = x$ and $Tw = TT'y = y$, then

$$\psi\left(G\left(T'x,T'x,T'y\right)\right) \leq \psi\left(G\left(x,x,y\right)\right) - \phi\left(G\left(x,x,y\right)\right).$$

From Theorem 4.2.3, $T'$ has a unique fixed point $u \in X$. In particular, $u$ is also a fixed point of $T$ because $T'u = u$ implies that $Tu = TT'u = u$. If $u, v \in \mathrm{Fix}(T)$ were two distinct fixed points of $T$, then we would get the contradiction

$$\begin{aligned}\psi\left(G\left(u,u,v\right)\right) &\leq \psi\left(G\left(Tu,Tu,Tv\right)\right) - \phi\left(G\left(Tu,Tu,Tv\right)\right)\\ &= \psi\left(G\left(u,u,v\right)\right) - \phi\left(G\left(u,u,v\right)\right) < \psi\left(G\left(u,u,v\right)\right),\end{aligned}$$

so the fixed point of $T$ is unique. □

*Remark 9.1.1.* If $T$ is not onto, the previous result is false. For example, consider $X = (-\infty, 1] \cup [1, \infty)$ endowed with the $G$-metric $G(x,y,z) = |x-y| + |x-z| + |y-z|$ for all $x, y, z \in X$, and let $T : X \to X$ be defined by $Tx = -2x$ for all $x \in X$. Then $T$ has no fixed point although it satisfies (9.1) when $\psi(t) = t$ and $\phi(t) = t/2$ for all $t \geq 0$.

**Corollary 9.1.1.** *Let $(X, q)$ be a complete quasi-metric space and let $T : X \to X$ be an onto mapping such that there exists $\phi \in \mathcal{F}'_{\mathrm{alt}}$ satisfying*

$$G\left(x,x,y\right) \leq G\left(Tx,Tx,Ty\right) - \phi\left(G\left(Tx,Tx,Ty\right)\right)$$

*for all $x, y \in X$. Then $T$ has a unique fixed point.*

*Proof.* It is only necessary to consider $\psi(t) = t$ for all $t \geq 0$ in Theorem 9.1.1. □

**Corollary 9.1.2.** *Let $(X, G)$ be a complete $G$-metric space and let $T : X \to X$ be an onto mapping such that there exist $\psi \in \mathcal{F}_{\mathrm{alt}}$ and $\phi \in \mathcal{F}'_{\mathrm{alt}}$ satisfying*

$$\psi\left(G\left(x,y,z\right)\right) \leq \psi\left(G\left(Tx,Ty,Tz\right)\right) - \phi\left(G\left(Tx,Ty,Tz\right)\right)$$

*for all $x, y, z \in X$. Then $T$ has a unique fixed point.*

*Proof.* It is only necessary to take $z = x$ and apply Theorem 9.1.1. □

Using $\psi(t) = t$ for all $t \geq 0$, we deduce the following particular case.

**Corollary 9.1.3.** *Let $(X, G)$ be a complete $G$-metric space and let $T : X \to X$ be an onto mapping such that there exists $\phi \in \mathcal{F}'_{\mathrm{alt}}$ satisfying*

$$G(x, y, z) \leq G(Tx, Ty, Tz) - \phi(G(Tx, Ty, Tz))$$

*for all $x, y, z \in X$. Then $T$ has a unique fixed point.*

**Corollary 9.1.4.** *Any onto, expansive mapping (of type I or type II) from a complete G-metric space into itself has a unique fixed point.*

*Proof.* It follows by taking $\phi(t) = (1 - \lambda)t$ for all $t \geq 0$ in Corollaries 9.1.1 and 9.1.3. □

Next, we combine expansive mappings with contractivity conditions in which the mapping $T$ appears in both sides of the inequality.

**Theorem 9.1.2 ([24]).** *Let $(X, G)$ be a complete G-metric space and $T : X \to X$ be a onto mapping. Suppose that there exists $\lambda > 1$ such that*

$$G(Tx, T^2x, Ty) \geq \lambda G(x, Tx, y) \quad \text{for all } x, y \in X. \tag{9.2}$$

*Then $T$ has a unique fixed point.*

*Proof.* Let $x_0 \in X$ be arbitrary. Since $T$ is onto, there exists $x_1 \in X$ such that $x_0 = Tx_1$. By continuing this process, we can find a sequence $\{x_n\}$ such that $x_n = Tx_{n+1}$ for all $n \in \mathbb{N}$. If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0+1}$ is a fixed point of $T$. Now assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. From (9.2) with $x = x_{n+1}$ and $y = x_n$ we have that, for all $n \geq 1$,

$$\begin{aligned} G(x_n, x_{n-1}, x_{n-1}) &= G(Tx_{n+1}, T^2x_{n+1}, Tx_n) \\ &\geq \lambda G(x_{n+1}, Tx_{n+1}, x_n) = \lambda G(x_{n+1}, x_n, x_n) \end{aligned}$$

which implies that

$$G(x_{n+1}, x_n, x_n) \leq hG(x_n, x_{n-1}, x_{n-1})$$

where $h = \dfrac{1}{\lambda} < 1$. Then we have,

$$G(x_{n+1}, x_n, x_n) \leq h^n G(x_0, x_1, x_1)$$

From Lemma 3.1.1 we get,

$$G(x_n, x_{n+1}, x_{n+1}) \leq 2G(x_{n+1}, x_n, x_n) \leq 2h^n G(x_0, x_1, x_1).$$

Following the proof of Theorem 6.1.1, we derive that $\{x_n\}$ is a Cauchy sequence. Since, $(X, G)$ is complete, there exists $z \in X$ such that $\{x_n\} \to z$. As $T$ is onto, there exists $w \in X$ such that $z = Tw$. From (9.2) with $x = x_{n+1}$ and $y = w$ we have that, for all $n \geq 1$,

$$\begin{aligned}G(x_n, x_{n-1}, z) &= G(Tx_{n+1}, T^2x_{n+1}, Tw)\\ &\geq \lambda\, G(x_{n+1}, Tx_{n+1}, w) = \lambda\, G(x_{n+1}, x_n, w).\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality we get,

$$G(z, z, w) = \lim_{n\to\infty} G(x_n, x_{n-1}, z) = 0,$$

that is, $z = w$. Then, $z$ is a fixed point of $T$ because $z = Tw = Tz$. To prove uniqueness, suppose that $u, v \in \mathrm{Fix}(T)$ are two fixed points of $T$. If $u \neq v$, again by (9.2), we get

$$G(u, u, v) = G(Tu, T^2u, Tv) \geq \lambda\, G(u, Tu, v) \geq \lambda\, G(u, u, v) > G(u, u, v)$$

which is a contradiction. Hence, $u = v$. □

**Theorem 9.1.3 ([24]).** *Let $(X, G)$ be a complete $G$-metric space and let $T : X \to X$ be an onto mapping. Suppose that there exists $\lambda > 1$ such that*

$$G(Tx, Ty, T^2y) \geq \lambda\, G(x, Tx, T^2x) \quad \textit{for all } x, y \in X. \tag{9.3}$$

*Then $T$ has, at least, a fixed point.*

*Proof.* As in the previous proof, given an arbitrary point $x_0 \in X$, let $\{x_n\}$ be a sequence such that $x_n = Tx_{n+1}$ for all $n \in \mathbb{N}$. If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0+1}$ is a fixed point of $T$. Now assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. From (9.3) with $x = x_{n+1}$ and $y = x_n$ we have

$$G(Tx_{n+1}, Tx_n, T^2x_n) \geq \lambda\, G(x_{n+1}, Tx_{n+1}, T^2x_{n+1}),$$

which implies

$$G(x_n, x_{n-1}, x_{n-2}) \geq \lambda\, G(x_{n+1}, x_n, x_{n-1}),$$

and so,

$$G(x_{n+1}, x_n, x_{n-1}) \leq h G(x_n, x_{n-1}, x_{n-2})$$

where $h = \dfrac{1}{\lambda} < 1$. Mimicing the proof of Theorem 6.1.1, we can show that $\{x_n\}$ is a Cauchy sequence. Since, $(X, G)$ is a complete $G$-metric space, there exists $z \in X$ such that $\{x_n\} \to z$. As $T$ is onto, there exists $w \in X$ such that $z = Tw$. From (9.3) with $x = w$ and $y = x_{n+1}$ we have,

$$G(z, x_n, x_{n-1}) = G(Tw, Tx_{n+1}, T^2x_{n+1}) \geq \lambda\, G(w, Tw, T^2w).$$

Taking the limit as $n \to \infty$ in the above inequality, we deduce that $G(w, Tw, T^2w) = 0$, that is, $w = Tw = T^2w$. □

## 9.2 Fixed Point Theorems for $(\alpha, \psi, \phi)$-Expansive Mappings

In this section, we present a new expansivity condition and we prove some new fixed point results, avoiding the condition "$T$ is onto".

If $T : X \to X$ is an onto mapping, based on each $x_0 \in X$, there exists a sequence $\{x_n\} \subseteq X$ such that

$$Tx_{n+1} = x_n \quad \text{for all } n \geq 0. \tag{9.4}$$

A sequence $\{x_n\}$ verifying (9.4) is not necessarily unique.

**Definition 9.2.1.** We say that a sequence $\{x_n\} \subseteq X$ is an *inverse Picard sequence of $T : X \to X$ based on $x_0$* if $Tx_{n+1} = x_n$ for all $n \geq 0$.

**Definition 9.2.2.** An operator $T : X \to X$ from a quasi-metric space $(X, q)$ into itself is said to be *inverse Picard-continuous* if for all convergent inverse Picard sequence $\{x_n\}$ of $T$ we have that

$$T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} Tx_n.$$

*Remark 9.2.1.* 1. If $T$ is continuous on $(X, q)$, then $T$ is inverse Picard-continuous.
2. An operator $T$ is inverse Picard-continuous if, and only if, the limit of any convergent inverse Picard sequence is a fixed point of $T$.

**Definition 9.2.3.** Let $(X, q)$ be a quasi-metric space and let $T : X \to X$ be a mapping. We say that $T$ is an $(\alpha, \psi, \phi)$*-expansive mapping* if there exist three functions $\alpha : X \times X \to [0, \infty)$, $\psi \in \mathcal{F}_{\text{alt}}$ and $\phi \in \Phi$ such that, for all $x, y \in X$,

$$\alpha(x, y) \Big( \psi(q(x, y)) + \phi(q(x, y)) \Big) \leq \psi(q(Tx, Ty)). \tag{9.5}$$

In the following result we do not suppose that $T$ is onto.

**Theorem 9.2.1.** *Let $(X, q)$ be a quasi-metric space and let $T : X \to X$ be an $(\alpha, \psi, \phi)$-expansive mapping. Assume that there exists an inverse Picard sequence $\{x_n\}$ of $T$ satisfying $\alpha(x_n, x_m) \geq 1$ for all $n, m \geq 1$ such that $n \neq m$. Then $\{x_n\}$ is a Cauchy sequence in $(X, q)$.*

*Furthermore, if $(X, q)$ is complete and $T$ is an inverse Picard-continuous mapping (or a continuous mapping), then $\{x_n\}$ converges to a fixed point of $T$. In particular, $T$ has a fixed point.*

*In addition to this, if $\alpha(u, v) \geq 1$ for all $u, v \in \mathrm{Fix}(T)$, then $T$ has a unique fixed point.*

*Proof.* If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0}$ is a fixed point of $T$. On the contrary case, assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. In particular, $q(x_n, x_{n+1}) > 0$ and $q(x_{n+1}, x_n) > 0$ for all $n \geq 0$. Applying (9.5) to $x = x_{n+2}$, we obtain

$$\begin{aligned}\psi(q(x_{n+1},x_n)) &= \psi(q(Tx_{n+2},T^2x_{n+2}))\\ &\geq \alpha\,(x_{n+2},Tx_{n+2})\;[\,\psi(q(x_{n+2},Tx_{n+2}))+\phi(q(x_{n+2},Tx_{n+2}))\,]\\ &= \alpha\,(x_{n+2},x_{n+1})\;[\,\psi(q(x_{n+2},x_{n+1}))+\phi(q(x_{n+2},x_{n+1}))\,]\\ &\geq \psi(q(x_{n+2},x_{n+1}))+\phi(q(x_{n+2},x_{n+1}))\\ &> \psi(q(x_{n+2},x_{n+1}))\end{aligned} \tag{9.6}$$

for all $n \geq 0$. Regarding the properties of the functions $\psi$ and $\phi$, we derive that

$$q(x_{n+2},x_{n+1}) \leq q(x_{n+1},x_n) \quad \text{for all } n \geq 1.$$

Therefore $\{q(x_{n+1},x_n)\}$ is a decreasing sequence in $(0,\infty)$ and, thus, it is convergent. Let $L \in (0,\infty)$ be its limit. We claim that $L = 0$. Suppose, on the contrary, that $L > 0$. Since $\phi$ is lower semi-continuous,

$$\phi(L) \leq \liminf_{n\to\infty} \phi(q(x_{n+2},x_{n+1})),$$

and taking into account that $\psi$ is continuous,

$$\begin{aligned}\liminf_{n\to\infty} \psi(q(x_{n+1},x_n)) &= \lim_{n\to\infty} \psi(q(x_{n+1},x_n)) = \psi(L),\\ \liminf_{n\to\infty} \psi(q(x_{n+2},x_{n+1})) &= \lim_{n\to\infty} \psi(q(x_{n+2},x_{n+1})) = \psi(L).\end{aligned}$$

Therefore, taking the limit inferior as $n \to \infty$ in (9.6) we get $\psi(L) \geq \psi(L)+\phi(L)$, which implies that $\phi(L) = 0$. Therefore, $L = 0$, which is a contradiction. Hence, we have that $\lim_{n\to\infty} q(x_{n+1},x_n) = 0$. As the expansive condition (9.5) is symmetric on $x$ and $y$, in the same way we can deduce that $\lim_{n\to\infty} q(x_n,x_{n+1}) = 0$. Therefore

$$\lim_{n\to\infty} q(x_{n+1},x_n) = \lim_{n\to\infty} q(x_n,x_{n+1}) = 0. \tag{9.7}$$

Next, we show that the sequence $\{x_n\}$ is left-Cauchy in $(X,q)$ reasoning by contradiction. Suppose, on the contrary, that $\{x_n\}$ is not left-Cauchy. Reasoning as in the proof of Theorem 4.1.1, there exists $\varepsilon > 0$ for which one can find two partial subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that

$$q(x_{n(k)-1},x_{m(k)}) \leq \varepsilon < q(x_{n(k)},x_{m(k)}), \quad n(k) > m(k) \geq k \quad \text{for all } k \geq 1, \tag{9.8}$$

$$\lim_{k\to\infty} q(x_{n(k)},x_{m(k)}) = \lim_{k\to\infty} q(x_{n(k)-1},x_{m(k)-1}) = \varepsilon. \tag{9.9}$$

Regarding (9.7) and the contractivity condition (9.5), we have that, for all $k$,

$$\begin{aligned}\psi(q(x_{n(k)-1},x_{m(k)-1})) &= \psi(q(Tx_{n(k)},Tx_{m(k)}))\\ &\geq \alpha(x_{n(k)},x_{m(k)})\,\Big(\,\psi(q(x_{n(k)},x_{m(k)}))+\phi(q(x_{n(k)},x_{m(k)}))\,\Big)\\ &\geq \psi(q(x_{n(k)},x_{m(k)}))+\phi(q(x_{n(k)},x_{m(k)})).\end{aligned} \tag{9.10}$$

Since $\psi$ is continuous and $\phi$ is lower semi-continuous, using (9.9),

$$\phi(\varepsilon) \leq \liminf_{n\to\infty} \phi(q(x_{n(k)}, x_{m(k)})),$$

$$\liminf_{n\to\infty} \psi(q(x_{n(k)}, x_{m(k)})) = \lim_{n\to\infty} \psi(q(x_{n(k)}, x_{m(k)})) = \psi(\varepsilon),$$

$$\liminf_{n\to\infty} \psi(q(x_{n(k)-1}, x_{m(k)-1})) = \lim_{n\to\infty} \psi(q(x_{n(k)-1}, x_{m(k)-1})) = \psi(\varepsilon).$$

Then, taking the limit inferior in (9.10) as $n \to \infty$, we deduce that $\psi(L) \geq \psi(L) + \phi(L)$, which implies that $\phi(L) = 0$. Therefore, $L = 0$, which is a contradiction. As a consequence, the sequence $\{x_n\}$ is left-Cauchy in $(X, q)$. Analogously, it can be proved that $\{x_n\}$ is a right-Cauchy sequence, so it is Cauchy.

Now assume that $(X, q)$ is complete and $T$ is an inverse Picard-continuous mapping. In this case, $\{x_n\}$ is a convergent sequence in $(X, q)$ and item 2 of Remark 9.2.1 guarantees that its limit is a fixed point of $T$. The uniqueness of the fixed point directly follows from (9.5) applied to $u, v \in \mathrm{Fix}(T)$, which leads to $\phi(q(u, v)) = 0$, so $u = v$. □

**Corollary 9.2.1.** *Let $(X, q)$ be a complete quasi-metric space and let $T : X \to X$ be an inverse Picard-continuous mapping such that there exist $\psi \in \mathcal{F}_{\mathrm{alt}}$ and $\phi \in \Phi$ satisfying, for all $x, y \in X$,*

$$\psi(q(x, y)) + \phi(q(x, y)) \leq \psi(q(Tx, Ty)).$$

*If there exists an inverse Picard sequence $\{x_n\}$ of $T$, then $\{x_n\}$ converges to a fixed point of $T$. In such a case, $T$ has a unique fixed point.*

**Corollary 9.2.2.** *Let $(X, q)$ be a complete quasi-metric space and let $T : X \to X$ be an inverse Picard-continuous mapping such that there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\phi \in \Phi$ satisfying, for all $x, y \in X$,*

$$\alpha(x, y) \Big( q(x, y) + \phi(q(x, y)) \Big) \leq q(Tx, Ty).$$

*If there exists an inverse Picard sequence $\{x_n\}$ of $T$ such that $\alpha(x_n, x_m) \geq 1$ for all $n, m \geq 1$ satisfying $n \neq m$, then $\{x_n\}$ converges to a fixed point of $T$. In such a case, $T$ has a fixed point.*

*In addition to this, if $\alpha(u, v) \geq 1$ for all $u, v \in \mathrm{Fix}(T)$, then $T$ has a unique fixed point.*

A simple way in which we can easily ensure that there exists an inverse Picard sequence of $T$ (a necessary condition in the last two corollaries) is to assume that $T$ is onto. In this case, we have that the limit of any inverse Picard sequence of $T$ is a fixed point of $T$.

## 9.3 Fixed Point Theorems on Quasi-metric Spaces Using Expansivity Conditions Depending on a Unique Variable

In this section we study existence of fixed points under expansivity conditions depending on a unique variable. Given an operator $T : X \to X$, we are interested in quasi-metrics satisfying the following property.

$(\mathcal{R}_T)$ Any inverse Picard sequence of $T$ which is left-Cauchy in $(X, q)$ is also right-Cauchy in $(X, q)$.

By item 1 of Lemma 3.3.1, examples of such quasi-metrics are the quasi-metrics $q = q_G$ (or $q = q'_G$) associated to $G$-metrics $G$.

**Theorem 9.3.1.** *Let $(X, q)$ be a quasi-metric space and let $T : X \to X$, $\alpha : X \times X \to [0, \infty)$ and $\varphi \in \mathcal{F}_{\text{com}}$ be three mappings such that*

$$\alpha(x, Tx)\, q(x, Tx) \leq \varphi(q(Tx, T^2x)) \quad \text{for all } x \in X. \tag{9.11}$$

*Suppose that there exists an inverse Picard sequence $\{x_n\}$ of $T$ such that $\alpha(x_{n+1}, x_n) \geq 1$ for all $n \geq 1$. Then $\{x_n\}$ is left-Cauchy in $(X, q)$.*

*Furthermore, assume that $(X, q)$ is complete, $T$ is an inverse Picard-continuous mapping and $q$ satisfies the condition $(\mathcal{R}_T)$. Then $\{x_n\}$ converges to a fixed point of $T$. In particular, $T$ has a fixed point.*

Recall that the existence of inverse Picard sequences $\{x_n\}$ of $T$ is guaranteed if $T$ is onto.

*Proof.* Apply the expansivity condition to $x = x_{n+2}$, obtaining, for all $n \geq 1$,

$$\begin{aligned} q(x_{n+2}, x_{n+1}) &\leq \alpha(x_{n+2}, x_{n+1})\, q(x_{n+2}, x_{n+1}) \\ &= \alpha(x_{n+2}, Tx_{n+2})\, q(x_{n+2}, Tx_{n+2}) \\ &\leq \varphi(q(Tx_{n+2}, T^2x_{n+2})) = \varphi(q(x_{n+1}, x_n)). \end{aligned}$$

Repeating the argument in the proof of Theorem 9.2.1, we deduce that $\{x_n\}$ is left-Cauchy in $(X, q)$. The second part is as follows: by condition $(\mathcal{R}_T)$, $\{x_n\}$ is a Cauchy sequence in $(X, q)$; by the completeness, $\{x_n\}$ is a convergent sequence in $(X, q)$; and the limit of $\{x_n\}$ is a fixed point of $T$ because it is inverse Picard-continuous. □

**Corollary 9.3.1.** *Let $(X, q)$ be a complete quasi-metric space and let $T : X \to X$ be an inverse Picard-continuous operator for which there exists $\varphi \in \mathcal{F}_{\text{com}}$ such that*

$$q(x, Tx) \leq \varphi(q(Tx, T^2x)) \quad \text{for all } x \in X.$$

*Suppose that $q$ satisfies the condition $(\mathcal{R}_T)$ and there exists an inverse Picard sequence $\{x_n\}$ of $T$. Then $\{x_n\}$ converges to a fixed point of $T$. In particular, $T$ has a fixed point.*

We particularize the previous result to $G$-metric spaces. For example, the following result can be found on [24].

**Theorem 9.3.2 ([24]).** *Let* $(X, G)$ *be a complete* $G$*-metric space and be* $T : X \to X$ *be an onto mapping satisfying the following condition for all* $x, y \in X$*:*

$$G(Tx, T^2x, Ty) \geq \lambda\, G(x, Tx, y) \tag{9.12}$$

*where* $\lambda > 1$. *Then* $T$ *has a unique fixed point.*

**Corollary 9.3.2.** *Theorem 9.3.2 follows from Corollary 9.3.1.*

*Proof.* The function $\varphi_{1/\lambda}$, defined by $\varphi_{1/\lambda}(t) = (1/\lambda)\, t$ for all $t \geq 0$, satisfies $\varphi_{1/\lambda} \in \mathcal{F}_{\mathrm{com}}$ because $1/\lambda \in (0, 1)$. Letting $y = Tx$ in the expansive condition (9.12), we have that, for all $x \in X$,

$$\lambda\, q_G(x, Tx) = \lambda\, G(x, Tx, Tx) \leq G(Tx, T^2x, T^2x) = q_G(Tx, T^2x),$$

so $q_G(x, Tx) \leq (1/\lambda)\, q_G(Tx, T^2x) = \varphi_{1/\lambda}(q_G(Tx, T^2x))$. Since $T$ is onto, there exists an inverse Picard sequence $\{x_n\}$ of $T$. We now show that $T$ is an inverse Picard-continuous mapping. Let $\{x_n\}$ be any inverse Picard sequence of $T$ converging to $u \in X$ and we claim that $u$ is a fixed point of $T$. Since $T$ is onto, there exists $z \in X$ such that $Tz = u$. Applying condition (9.12) to $x = x_{n+2}$ and $y = z$,

$$\begin{aligned} G(x_{n+1}, x_n, u) &= G(Tx_{n+2}, T^2x_{n+2}, Tz) \geq \lambda\, G(x_{n+2}, Tx_{n+2}, z) \\ &= \lambda\, G(x_{n+2}, x_{n+1}, z). \end{aligned}$$

Letting $n \to \infty$ in the previous inequality, we deduce that $0 = G(u, u, u) \geq \lambda$ $G(u, u, z) \geq 0$, so $z = u = Tz$ and $z$ is a fixed point of $T$. In particular, $Tu = Tz = u$, and $u$ is a fixed point of $T$. This concludes that $T$ is an inverse Picard-continuous mapping. Corollary 9.3.1 guarantees that $T$ has a fixed point. The uniqueness directly follows from (9.12). □

# Chapter 10
# Reconstruction of $G$-Metrics: $G^*$-Metrics

The main aim of the present chapter is to prove new unidimensional and multidimensional fixed point results in the framework of $G$-metric spaces provided with a partial preorder (not necessarily a partial order). However, we need to overcome the well-known fact that the usual product of $G$-metrics is not necessarily a $G$-metric unless they come from classical metrics. Hence, we will omit one of the axioms that define a $G$-metric and we consider a new class of metrics, called *$G^*$-metrics*. Notice that our main results are valid in the context of $G$-metric spaces.

## 10.1 The Antecedents of $G^*$-Metric Spaces

The original Mustafa and Sims' notion of $G$-metric space is as follows (recall Definition 3.1.1): A *$G$-metric space* is a pair $(X, G)$ where $X$ is a nonempty set and $G : X \times X \times X \to [0, \infty)$ is a function such that, for all $x, y, z, a \in X$, the following conditions are fulfilled:

$(G_1)$ $\quad G(x, y, z) = 0 \quad$ if $x = y = z$;

$(G_2)$ $\quad G(x, x, y) > 0 \quad$ for all $x, y \in X$ with $x \neq y$;

$(G_3)$ $\quad G(x, x, y) \leq G(x, y, z) \quad$ for all $x, y, z \in X$ with $z \neq y$;

$(G_4)$ $\quad G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (*symmetry in all three variables*); and

$(G_5)$ $\quad G(x, y, z) \leq G(x, a, a) + G(a, y, z) \quad$ (*rectangle inequality*).

In such a case, the function $G$ is called a *$G$-metric on $X$*. A classical example of $G$-metric comes from a metric space $(X, d)$, where $G(x, y, z) = d_{xy} + d_{yz} + d_{zx}$ measures the perimeter of a triangle. In this case, property $(G_3)$ has an obvious



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geometric interpretation: the length of an edge of a triangle is less than or equal to its semiperimeter, that is, $2d_{xy} \leq d_{xy} + d_{yz} + d_{zx}$. One of the most important properties of $G$-metric spaces is the following one (see item 5 of Lemma 3.1.2).

$$G(x, y, z) = 0 \quad \Rightarrow \quad x = y = z. \tag{10.1}$$

Property $(G_3)$ was used to establish different fixed point theorems. However, it has an important drawback: the product of $G$-metric spaces is not necessarily another $G$-metric space. In fact, this is only true when all factors are symmetric but, in this case, they are all classical metric spaces (see Theorem 3.1.1). We explain this fact in detail.

Given a finite family of $G$-metric spaces $\{(X_i, G_i)\}_{i=1}^n$, consider the product space $X = X_1 \times X_2 \times \ldots \times X_n$ and define $G^m$ and $G^s$ on $X^3$ by:

$$G^m(\mathsf{X}, \mathsf{Y}, \mathsf{Z}) = \max_{1 \leq i \leq n} G_i(x_i, y_i, z_i) \quad \text{and} \quad G^s(\mathsf{X}, \mathsf{Y}, \mathsf{Z}) = \sum_{i=1}^n G_i(x_i, y_i, z_i),$$

for all $\mathsf{X} = (x_1, x_2, \ldots, x_n), \mathsf{Y} = (y_1, y_2, \ldots, y_n), \mathsf{Z} = (z_1, z_2, \ldots, z_n) \in X$. Property $(G_3)$ implies that, in general, the major structures $G^m$ and $G^s$ are not necessarily $G$-metrics on $X_1 \times X_2 \times \ldots \times X_n$. Only when each $G_i$ is *symmetric* (that is, $G(x, x, y) = G(y, y, x)$ for all $x, y \in X$), the product is also a $G$-metric (see Theorem 3.1.1 or [154]). In this case, symmetric $G$-metrics can be reduced to usual metrics, which limits the interest to this kind of space.

The most important disadvantage of this fact is that multidimensional fixed point theorems (coupled, tripled, quadrupled, etc., as we will see in Chap. 11) cannot be proved using unidimensional results. As a consequence, a direct proof must be presented in each case, using an appropriate contractivity condition.

In order to overcome this drawback, in 2013, Roldán and Karapınar [175] considered spaces verifying the axioms $(G_1)$, $(G_2)$, $(G_4)$ and $(G_5)$, which has their own Hausdorff topology $\tau_G$. The problem we have recently found is that these spaces do not have to satisfy condition (10.1) when $x$, $y$ and $z$ are different, as we show in the following example.

*Example 10.1.1.* Let $X = \{0, 1, 2\}$ and define $G : X \times X \times X \to [0, \infty)$, for all $x, y, z \in X$, by:

$$G(x, y, z) = \begin{cases} 0, \text{ if } \quad x = y = z \quad \text{or} \quad \{x, y, z\} = \{0, 1, 2\}, \\ 1, \text{ otherwise.} \end{cases}$$

Clearly, $G$ verifies $(G_1)$, $(G_2)$ and $(G_4)$. However, it does not satisfy condition (10.1) because $G(0, 1, 2) = 0$. To prove $(G_5)$, let $x, y, z \in X$ be such that $G(x, y, z) > 0$. In this case, in $\{x, y, z\}$ there are only two different points. Assume, for example, that $x \neq y = z$. Then, one of the terms $G(x, a, a)$ or $G(a, y, y)$ has, exactly, two different points. Hence

$$G(x, y, y) = 1 \leq G(x, a, a) + G(a, y, y),$$

so $(G_5)$ holds. As a consequence, $(X, G)$ is an example of the spaces considered by Roldán and Karapınar in [175], but it does not satisfy condition (10.1).

In this chapter, we consider a definition of $G^*$-metric spaces avoiding property $(G_3)$. Omitting this property, we consider a class of spaces for which $G^m$ and $G^s$ have the same initial metric structure.

## 10.2 Definition of $G^*$-Metric

**Definition 10.2.1.** A $G^*$*-metric* on a set $X$ is a mapping $G : X \times X \times X \to [0, \infty)$ satisfying the following properties, for all $x, y, z, a \in X$.

$(G_4)$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (*symmetry in all three variables*);

$(G_5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ (*rectangle inequality*); and

$(G_6)$ $G(x, y, z) = 0 \quad \Leftrightarrow \quad x = y = z$;

**Lemma 10.2.1.** *1. Every G-metric space (in the sense of Mustafa and Sims) is a $G^*$-metric space.*
*2. Every $G^*$-metric space satisfies axioms $(G_1)$, $(G_2)$, $(G_4)$ and $(G_5)$.*
*3. A $G^*$-metric space is a G-metric space (in the sense of Mustafa and Sims) if, and only if, it verifies $(G_3)$.*

*Proof.* **(1)** Axioms $(G_4)$ and $(G_5)$ are common. Condition $(G_1)$ means that $G(x, x, x) = 0$ for all $x \in X$. Conversely, condition (10.1) follows from item 5 of Lemma 3.1.2.
**(2)** Properties $(G_1)$ and $(G_2)$ immediately follows from $(G_6)$.

□

Although each $G^*$-metric space satisfies axioms $(G_1)$, $(G_2)$, $(G_4)$ and $(G_5)$, it is not necessarily a $G$-metric space. In other word, the converse of the first item of Lemma 10.2.1 is false, as we show in the following example.

*Example 10.2.1.* Let $X = \{0, 1, 2\}$ and define $G : X \times X \times X \to [0, \infty)$, for all $x, y, z \in X$, by:

$$G(x, y, z) = \begin{cases} 0, \text{ if } x = y = z, \\ 4, \text{ if } x \neq y \neq z \neq x \quad \text{(that is, } \{x, y, z\} = \{0, 1, 2\}), \\ 5, \text{ if } \{x, y, z\} = \{0, 0, 2\}, \\ 3, \text{ otherwise.} \end{cases}$$

Clearly, $G$ does not satisfy axiom $(G_3)$ since

$$G(0,0,2) = 5 > 4 = G(0,1,2).$$

Hence, $(X, G)$ is not a $G$-metric space. However, we claim that $(X, G)$ is a $G^*$-metric space. Properties $(G_4)$ and $(G_6)$ are obvious. Condition $(G_5)$ follows from the fact that the sum of two numbers in the set $\{3, 4, 5\}$ is always greater than a third number in the same set. Then, $(G_5)$ holds.

*Remark 10.2.1.* All results given in Roldán and Karapınar [175] hold if we additionally assume condition (10.1), that is, for $G^*$-metric spaces in the sense of Definition 10.2.1.

### *10.2.1 Basic Properties of $G^*$-Metric Spaces*

One of the most useful properties of $G$-metrics is the following one.

**Lemma 10.2.2.** *If $(X, G)$ is a $G^*$-metric space, then*

$$G(x,y,y) \leq 2G(y,x,x) \quad \textit{for all } x,y \in X.$$

*Proof.* By the rectangle inequality $(G_5)$ together with the symmetry $(G_4)$, we have

$$G(x,y,y) = G(y,y,x) \leq G(y,x,x) + G(x,y,x) = 2G(y,x,x).$$

□

The following lemma can be derived easily from the definition of a $G^*$-metric space as in Lemma 3.1.2.

**Lemma 10.2.3.** *Let $(X, G)$ be a $G^*$-metric space. Then, for any $x, y, z, a \in X$, the following properties hold.*

1. $G(x,y,z) \leq G(x,x,y) + G(x,x,z)$.
2. $G(x,y,z) \leq G(x,a,a) + G(y,a,a) + G(z,a,a)$.
3. $|G(x,y,z) - G(x,y,a)| \leq \max\{G(a,z,z), G(z,a,a)\}$.
4. *If $n \geq 2$ and $x_1, x_2, \ldots, x_n \in X$, then*

$$G(x_1,x_n,x_n) \leq \sum_{i=1}^{n-1} G(x_i,x_{i+1},x_{i+1}) \quad \textit{and} \tag{10.2}$$

$$G(x_1,x_1,x_n) \leq \sum_{i=1}^{n-1} G(x_i,x_i,x_{i+1}).$$

5. *If* $\{x_n\}, \{y_n\} \subseteq X$ *are two sequences, then* $\lim_{n\to\infty} G(x_n, y_n, y_n) = 0$ *if, and only if,* $\lim_{n\to\infty} G(x_n, x_n, y_n) = 0$.
6. *If* $\{x_n\} \subseteq X$ *is a sequence, then* $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$ *if, and only if,* $\lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = 0$.

### 10.2.2 The Hausdorff Topology of a $G^*$-Metric Space

All definitions and results in Sect. 3.2 can be repeated here in the ambient of $G^*$-metric spaces because the proofs there did not use axiom $(G_3)$. In order to avoid repetition, we only highlight the most important definitions and facts.

The *open ball of center* $x \in X$ *and radius* $r > 0$ in a $G^*$-metric space $(X, G)$ is the subset $B_G(x, r) = \{y \in X : G(x, y, y) < r\}$. Similarly, the *closed ball of center* $x \in X$ *and radius* $r > 0$ is

$$\overline{B}_G(x, r) = \{y \in X : G(x, y, y) \leq r\}.$$

Clearly, $x \in B_G(x, r) \subseteq \overline{B}_G(x, r)$.

If $(X, G)$ is a $G^*$-metric space, then the functions $d_m^G, d_s^G : X \times X \to [0, \infty)$ defined by

$$d_m^G(x, y) = \max\{G(x, y, y), G(y, x, x)\} \quad \text{and}$$
$$d_s^G(x, y) = G(x, y, y) + G(y, x, x)$$

for all $x, y \in X$, are metrics on $X$. Furthermore, $d_m^G(x, y) \leq d_s^G(x, y) \leq 2d_m^G(x, y)$ for all $x, y \in X$. In fact, $d_m^G$ and $d_s^G$ are equivalent metrics on $X$ and they generate the same topology on $X$.

If $(X, G)$ is a $G^*$-metric space and $d_m^G$ and $d_s^G$ are the metric defined as before, then

$$B_{d_m^G}(x, r) \subseteq B_G(x, r) \subseteq B_{d_m^G}(x, 2r) \subseteq B_G(x, 2r)$$

for all $x \in X$ and all $r > 0$.

The family of all open balls permit us to consider a topology on $X$.

**Theorem 10.2.1.** *There exists a unique topology $\tau_G$ on a $G^*$-metric space $(X, G)$ such that, for all $x \in X$, the family $\beta_x$ of all open balls centered at $x$ is a neighbourhood system at $x$. Furthermore, $\tau_G$ is metrizable because it is the metric topology on $X$ generated by the equivalent metrics $d_m^G$ and $d_s^G$. In particular, $\tau_G$ satisfies the Hausdorff separation property.*

The following notions can be considered on each topological space (see [23, 51]), but we particularize them to the case of the topology $\tau_G$.

- A subset $U \subseteq X$ is a *G-neighborhood of a point* $x \in X$ if there is $r > 0$ such that $B_G(x, r) \subseteq U$.
- A subset $U \subseteq X$ is *G-open* if either it is empty or it is a $G$-neighborhood of all its points.
- A subset $U \subseteq X$ is *G-closed* if its complement $X \setminus U$ is $G$-open.
- An *adherent point* (also *closure point* or *point of closure*) of a subset $U \subseteq X$ is a point $x \in X$ such that every $G$-open set containing $x$ also contains, at least, one point of $U$, that is, for all $\varepsilon > 0$ we have that $B_G(x, \varepsilon) \cap U \neq \emptyset$.
- The *G-closure* $\overline{U} = \mathrm{cl}_G(U)$ of a subset $U \subseteq X$ is the family of all its adherent points. Clearly, $x \in \overline{U}$ if, and only if, $B_G(x, \varepsilon) \cap U \neq \emptyset$ for all $\varepsilon > 0$. In particular, $U \subseteq \overline{U}$. Moreover, $U$ is $G$-closed if, and only if, $U = \overline{U}$.
- The *G-interior* $\overset{\circ}{U} = \mathrm{int}_G(U)$ of a subset $U \subseteq X$ is the complement $X \setminus \overline{U}$. An *interior point of* $U$ is a point $x \in U$ such that there exists $r > 0$ verifying $B_G(x, r) \subseteq U$. In particular, $\overset{\circ}{U} \subseteq U$. Moreover, $U$ is $G$-open if, and only if, $\overset{\circ}{U} = U$.

For simplicity, we will omit the prefix $G$- in the previous notions.

Let $(X, G)$ be a $G^*$-metric space, let $x \in X$ be a point and let $\{x_n\} \subseteq X$ be a sequence. We say that:

- $\{x_n\}$ *G-converges to* $x$, and we write $\{x_n\} \xrightarrow{G} x$ or $\{x_n\} \to x$, if $\lim_{n,m\to\infty} G(x_n, x_m, x) = 0$, that is, for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ satisfying $G(x_n, x_m, x) \leq \varepsilon$ for all $n, m \in \mathbb{N}$ such that $n, m \geq n_0$ (in such a case, $x$ is the *G-limit* of $\{x_m\}$);
- $\{x_n\}$ is *G-Cauchy* if $\lim_{n,m,k\to\infty} G(x_n, x_m, x_k) = 0$, that is, for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ satisfying $G(x_n, x_m, x_k) \leq \varepsilon$ for all $n, m, k \in \mathbb{N}$ such that $n, m, k \geq n_0$.
- $(X, G)$ is *complete* if every $G$-Cauchy sequence in $X$ is $G$-convergent in $X$.

**Proposition 10.2.1.** *The limit of a G-convergent sequence in a $G^*$-metric space is unique.*

**Proposition 10.2.2.** *Every convergent sequence in a G-metric space is a Cauchy sequence.*

Next, we characterize convergent and Cauchy sequences.

**Lemma 10.2.4.** *Let $(X, G)$ be a $G^*$-metric space, let $\{x_m\} \subseteq X$ be a sequence and let $x \in X$. Then the following conditions are equivalent.*

**(a)** *$\{x_n\}$ G-converges to x.*
**(b)** $\lim_{n\to\infty} G(x_n, x_n, x) = 0$, *that is, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in B_G(x, \varepsilon)$ for all $n \geq n_0$.*
**(c)** $\lim_{n\to\infty} G(x_n, x, x) = 0$.
**(d)** $\lim_{n,m\to\infty,\ m\geq n} G(x_n, x_m, x) = 0$.
**(e)** $\lim_{n\to\infty} G(x_n, x_n, x) = 0$ *and* $\lim_{n\to\infty} G(x_n, x_{n+1}, x) = 0$.
**(f)** $\lim_{n\to\infty} G(x_n, x, x) = 0$ *and* $\lim_{n\to\infty} G(x_n, x_{n+1}, x) = 0$.

**(g)** $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$ *and* $\lim_{n\to\infty} G(x_n, x_{n+1}, x) = 0$.
**(h)** $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$ *and* $\lim_{n,m\to\infty,\ m>n} G(x_n, x_m, x) = 0$.

**Lemma 10.2.5.** *If $(X, G)$ is a $G^*$-metric space and $\{x_m\} \subseteq X$ is a sequence, then the following conditions are equivalent.*

**(a)** $\{x_n\}$ *is G-Cauchy.*
**(b)** $\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0$.
**(c)** $\lim_{n,m\to\infty,\ m\geq n} G(x_n, x_m, x_m) = 0$.
**(d)** $\lim_{n,m\to\infty,\ m>n} G(x_n, x_m, x_m) = 0$.
**(e)** $\lim_{n,m\to\infty} G(x_n, x_n, x_m) = 0$.
**(f)** $\lim_{n,m\to\infty,\ m\geq n} G(x_n, x_n, x_m) = 0$.
**(g)** $\lim_{n,m\to\infty,\ m>n} G(x_n, x_n, x_m) = 0$.
**(h)** $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$ *and* $\lim_{n,m\to\infty,\ m>n} G(x_n, x_{n+1}, x_m) = 0$.

### *10.2.3 Continuity of Mappings Between G*-Metric Spaces*

**Definition 10.2.2.** Let $(X, G)$ be a $G^*$-metric space. We say that:

- a mapping $T : X \to X$ is *G-continuous* at $x \in X$ if $\{Tx_m\} \xrightarrow{G} Tx$ for all sequence $\{x_m\} \subseteq X$ such that $\{x_m\} \xrightarrow{G} x$;
- a mapping $F : X^n \to X$ is *G-continuous* at $(x_1, x_2, \ldots, x_n) \in X^n$ if $\{F(x_1^m, x_2^m, \ldots, x_n^m)\} \xrightarrow{G} F(x_1, x_2, \ldots, x_n)$ for all sequence $\{(x_1^m, x_2^m, \ldots, x_n^m)\} \subseteq X^n$ such that $\{x_i^m\} \xrightarrow{G} x_i$ for all $i \in \{1, 2, \ldots, n\}$;
- a mapping $H : X^n \to X^m$ is *G-continuous* at $(x_1, x_2, \ldots, x_n) \in X^n$ if $\pi_i^m \circ H : X^n \to X$ is $G$-continuous at $(x_1, x_2, \ldots, x_n)$ for all $i \in \{1, 2, \ldots, m\}$, where $\pi_i^m : X^m \to X$ is the $i$th-projection of $X^m$ onto $X$ (that is, $\pi_i^m(a_1, a_2, \ldots, a_m) = a_i$ for all $(a_1, a_2, \ldots, a_m) \in X^m$).

From Lemma 3.2.3, convergence of sequences on $X$ with respect to $G$, $d_m^G$ and $d_s^G$ coincide.

**Lemma 10.2.6.** *Let $(X, G)$ be a G metric space. Then a mapping $T : X \to X$ is G-continuous if, and only if, it is $d_m^G$-continuous ($d_m^G$-continuous). Similarly, a mapping $F : X^n \to X$ is G-continuous if, and only if, it is $d_s^G$-continuous ($d_s^G$-continuous).*

The proof of the following result only needs properties $(G_4)$ and $(G_5)$ (see the proof of Theorem 3.2.2).

**Theorem 10.2.2.** *If $(X, G)$ is a $G^*$-metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables, that is, if $x, y, z \in X$ and $\{x_n\}, \{y_n\}, \{z_n\} \subseteq X$ are sequences in $X$ such that $\{x_m\} \xrightarrow{G} x$, $\{y_m\} \xrightarrow{G} y$ and $\{z_m\} \xrightarrow{G} z$, then $\{G(x_m, y_m, z_m)\} \to G(x, y, z)$.*

### 10.2.4 Some Relationships Between G*-Metrics and Quasi-metrics

Next, we show that there exists a similar relationships between $G^*$-metric spaces and quasi-metric spaces that we described in Lemma 3.3.1. The same proof is valid.

**Lemma 10.2.7.** *Let $(X, G)$ be a $G^*$-metric space and define $q_G, q'_G : X^2 \to [0, \infty)$ by*

$$q_G(x, y) = G(x, x, y) \quad \text{and} \quad q'_G(x, y) = G(x, y, y) \quad \text{for all } x, y \in X.$$

*Then the following properties hold.*

1. *$q_G$ and $q'_G$ are quasi-metrics on $X$. Moreover*
$$q_G(x, y) \leq 2q'_G(x, y) \leq 4q_G(x, y) \quad \text{for all } x, y \in X. \tag{10.3}$$
2. *In $(X, q_G)$ and in $(X, q'_G)$, a sequence is right-convergent (respectively, left-convergent) if, and only if, it is convergent. In such a case, its right-limit, its left-limit and its limit coincide.*
3. *In $(X, q_G)$ and in $(X, q'_G)$, a sequence is right-Cauchy (respectively, left-Cauchy) if, and only if, it is Cauchy.*
4. *In $(X, q_G)$ and in $(X, q'_G)$, every right-convergent (respectively, left-convergent) sequence has a unique right-limit (respectively, left-limit).*
5. *If $\{x_n\} \subseteq X$ and $x \in X$, then $\{x_n\} \xrightarrow{G} x \iff \{x_n\} \xrightarrow{q_G} x \iff \{x_n\} \xrightarrow{q'_G} x$.*
6. *If $\{x_n\} \subseteq X$, then $\{x_n\}$ is $G$-Cauchy $\iff \{x_n\}$ is $q_G$-Cauchy $\iff \{x_n\}$ is $q'_G$-Cauchy.*
7. *$(X, G)$ is complete $\iff (X, q_G)$ is complete $\iff (X, q'_G)$ is complete.*

### 10.2.5 Regularity of G*-Metric Spaces

Many results in fixed point theory assume the regularity of the space.

**Definition 10.2.3.** Let $(X, G)$ be a $G^*$-metric space and let $\preccurlyeq$ be a relation on $X$. The triple $(X, G, \preccurlyeq)$ is said to be *non-decreasing-regular* (respectively, *non-increasing-regular*) if for all sequence $\{x_m\} \subseteq X$ such that $\{x_m\} \to x$ and $x_m \preccurlyeq x_{m+1}$ (respectively, $x_m \succcurlyeq x_{m+1}$) for all $m \in \mathbb{N}$, we have that $x_m \preccurlyeq x$ (respectively, $x_m \succcurlyeq x$) for all $m \in \mathbb{N}$. Also $(X, G, \preccurlyeq)$ is said to be *regular* if it is both non-decreasing-regular and non-increasing-regular.

## 10.3 Product of $G^*$-Metric Spaces

The main advantage of $G^*$-metric spaces versus $G$-metric spaces is that the product of $G^*$-metric spaces is also a $G^*$-metric space.

**Lemma 10.3.1.** *Given a family $\{(X_i, G_i)\}_{i=1}^n$ of $G^*$-metric spaces, consider the product space $\mathbb{X} = X_1 \times X_2 \times \ldots \times X_n$ and define $G_n^{\max}$ and $G_n^{\text{sum}}$ on $\mathbb{X}^3$ by*

$$G_n^{\max}(\mathsf{X}, \mathsf{Y}, \mathsf{Z}) = \max_{1 \le i \le n} G_i(x_i, y_i, z_i) \quad \text{and} \quad G_n^{\text{sum}}(\mathsf{X}, \mathsf{Y}, \mathsf{Z}) = \sum_{i=1}^{n} G_i(x_i, y_i, z_i)$$

*for all $\mathsf{X} = (x_1, x_2, \ldots, x_n), \mathsf{Y} = (y_1, y_2, \ldots, y_n), \mathsf{Z} = (z_1, z_2, \ldots, z_n) \in \mathbb{X}$. Then the following statements hold.*

1. *$G_n^{\max}$ and $G_n^{\text{sum}}$ are $G^*$-metrics on $\mathbb{X}$.*
2. *$G_n^{\max} \le G_n^{\text{sum}} \le n\, G_n^{\max}$, that is, $G_n^{\max}$ and $G_n^{\text{sum}}$ are equivalent $G^*$-metrics on $\mathbb{X}$.*
3. *If $A_m = (a_m^1, a_m^2, \ldots, a_m^n) \in \mathbb{X}$ for all $m$ and $A = (a_1, a_2, \ldots, a_n) \in \mathbb{X}$, then $\{A_m\}$ $G_n^{\max}$-converges (respectively, $G_n^{\text{sum}}$-converges) to $A$ if, and only if, each $\{a_m^i\}$ $G_i$-converges to $a_i$.*
4. *$\{A_m\}$ is $G_n^{\max}$-Cauchy (respectively, $G_n^{\text{sum}}$-Cauchy) if, and only if, each $\{a_m^i\}$ is $G_i$-Cauchy.*
5. *$(\mathbb{X}, G_n^{\max})$ (respectively, $(\mathbb{X}, G_n^{\text{sum}})$) is complete if, and only if, every $(X_i, G_i)$ is complete.*
6. *For all $i$, let $\preceq_i$ be a preorder on $X_i$ and define*

$$\mathsf{X} \preceq \mathsf{Y} \;\Leftrightarrow\; x_i \preceq_i y_i \text{ for all } i \in \{1, 2, \ldots, n\}.$$

*Then $(X, G_n^{\max}, \preceq)$ is regular (respectively, non-decreasing-regular, non-increasing-regular) if, and only if, each factor $(X_i, G_i)$ is also regular (respectively, non-decreasing-regular, non-increasing-regular).*

*Proof.* Let $G = G_n^{\max}$. Taking into account that $G_n^{\max} \le G_n^{\text{sum}} \le n\, G_n^{\max}$, we will only present the proof using $G$.

**(1)** It is a straightforward exercise to prove the following statements.

- $G(\mathsf{X}, \mathsf{X}, \mathsf{X}) = \max_{1 \le i \le n} G_i(x_i, x_i, x_i) = \max_{1 \le i \le n} 0 = 0$. Moreover, if $G_n^{\max}(\mathsf{X}, \mathsf{Y}, \mathsf{Z}) = \max_{1 \le i \le n} G_i(x_i, y_i, z_i) = 0$, then $G_i(x_i, y_i, z_i) = 0$ for all $i \in \{1, 2, \ldots, n\}$, so $x_i = y_i = z_i$ for all $i \in \{1, 2, \ldots, n\}$.
- Symmetry in all three variables of $G$ follows from symmetry in all three variables of each $G_i$.

- We have that

$$\begin{aligned}G(\mathsf{X},\mathsf{Y},\mathsf{Z}) &= \max_{1\le i\le n} G_i(x_i,y_i,z_i) \le \max_{1\le i\le n}\left[G_i(x_i,a_i,a_i)+G_i(a_i,y_i,z_i)\right]\\ &\le \max_{1\le i\le n} G_i(x_i,a_i,a_i)+\max_{1\le i\le n} G_i(a_i,y_i,z_i)\\ &= G(\mathsf{X},\mathsf{A},\mathsf{A})+G(\mathsf{A},\mathsf{Y},\mathsf{Z}).\end{aligned}$$

Then $G$ is a $G^*$-metric on $\mathbb{X}$.

**(3)** We use Lemma 10.2.4. Suppose that $\{A_m\}$ $G$-converges to $A$ and let $\varepsilon > 0$. Then, for all $j \in \{1,2,\ldots,n\}$ and all $m$

$$G_j(a_j,a_j,a_m^j) \le \max_{1\le i\le n} G_i(a_i,a_i,a_m^i) = G(A,A,A_m).$$

Therefore, $\{a_m^j\}$ $G_j$-converges to $a_j$. Conversely, assume that each $\{a_m^i\}$ $G_i$-converges to $a_i$. Let $\varepsilon > 0$ and let $m_i \in \mathbb{N}$ be such that if $m \ge m_i$, then $G_i(a_i,a_i,a_m^i) < \varepsilon$. If $m_0 = \max(m_1,m_2,\ldots,m_n)$ and $m \ge m_0$, then $G(A,A,A_m) = \max_{1\le i\le n} G_i(a_i,a_i,a_m^i) < \varepsilon$, so $\{A_m\}$ $G$-converges to $A$.

**(4)** We use Lemma 10.2.5. Suppose that $\{A_m\}$ is $G$-Cauchy and let $\varepsilon > 0$. Then, for all $j \in \{1,2,\ldots,n\}$ and all $m,m'$

$$G_j(a_m^j,a_m^j,a_{m'}^j) \le \max_{1\le i\le n} G_i(a_m^i,a_m^i,a_{m'}^i) = G(A_m,A_m,A_{m'}).$$

Therefore, $\{a_m^j\}$ is $G_j$-Cauchy. Conversely, assume that each $\{a_m^i\}$ is $G_i$-Cauchy. Let $\varepsilon > 0$ and let $m_i \in \mathbb{N}$ be such that if $m,m' \ge m_i$, then $G_i(a_m^j,a_m^j,a_{m'}^j) < \varepsilon$. If $m_0 = \max(m_1,m_2,\ldots,m_n)$ and $m,m' \ge m_0$, then $G(A_m,A_m,A_{m'}) = \max_{1\le i\le n} G_i(a_m^i,a_m^i,a_{m'}^i) < \varepsilon$, so $\{A_m\}$ is $G$-Cauchy.

**(5)** It is an easy consequence of items 3 and 4 since

$$\begin{aligned}\{A_m\}\ G\text{-Cauchy} \ &\Leftrightarrow\ \text{each } \{a_m^i\}\ G\text{-Cauchy} \ \Leftrightarrow\ \text{each } \{a_m^i\}\ G\text{-convergent}\\ &\Leftrightarrow\ \{A_m\}\ G\text{-convergent}.\end{aligned}$$

**(6)** A sequence $\{A_m\}$ on $\mathbb{X}$ is $\preceq$-monotone non-decreasing if, and only if, each sequence $\{a_m^i\}$ is $\preceq$-monotone non-decreasing. Moreover, $\{A_m\}$ $G$-converges to $A = (a_1,a_2,\ldots,a_n) \in \mathbb{X}$ if, and only if, each $\{a_m^i\}$ $G_i$-converges to $a_i$. Finally, $A_m \preceq A$ if, and only if, $a_m^i \preceq_i a_i$, for all $i$. Therefore, $(X, G_n^{\max}, \preceq)$ is regular non-decreasing if, and only if, each factor $(X_i, G_i)$ is also regular non-decreasing. Other statements may be proved similarly. □

Taking $(X_i, G_i) = (X, G)$ for all $i$, we have the following result.

**Corollary 10.3.1.** *Let $(X,G)$ be a $G^*$-metric space and consider on the product space $X^n$ the mappings $G_n$ and $G'_n$ defined by*

$$G_n(\mathsf{X},\mathsf{Y},\mathsf{Z}) = \max_{1\le i\le n} G(x_i,y_i,z_i) \quad \textit{and} \quad G'_n(\mathsf{X},\mathsf{Y},\mathsf{Z}) = \sum_{i=1}^{n} G(x_i,y_i,z_i)$$

*for all* $\mathsf{X} = (x_1,x_2,\ldots,x_n), \mathsf{Y} = (y_1,y_2,\ldots,y_n), \mathsf{Z} = (z_1,z_2,\ldots,z_n) \in X^n$. *Then the following properties hold.*

1. $G_n$ *and* $G'_n$ *are* $G^*$*-metrics on* $X^n$.
2. *If* $A_m = (a_m^1, a_m^2, \ldots, a_m^n) \in X^n$ *for all* $m$ *and* $A = (a_1,a_2,\ldots,a_n) \in X^n$, *then* $\{A_m\}$ $G_n$*-converges (respectively,* $G'_n$*-converges) to* $A$ *if, and only if, each* $\{a_m^i\}$ $G$*-converges to* $a_i$.
3. $\{A_m\}$ *is* $G_n$*-Cauchy (respectively,* $G'_n$*-Cauchy) if, and only if, each* $\{a_m^i\}$ *is* $G$*-Cauchy.*
4. $(X^n, G_n)$ *(respectively,* $(X^n, G'_n)$*) is complete if, and only if,* $(X,G)$ *is complete.*

## 10.4 Fixed Point Theorems in Partially Preordered $G^*$-Metric Spaces

As an initial result in $G^*$-metric spaces, we prove the following statement using a preorder rather than a partial order.

### 10.4.1 Some Results Under $(\psi, \varphi)$-Contractivity Conditions

**Theorem 10.4.1.** *Let* $(X,G)$ *be a complete* $G^*$*-metric space, let* $\preceq$ *be a preorder on* $X$ *and let* $T : X \to X$ *be a* $\preceq$*-non-decreasing mapping. Suppose that there exist two functions* $\psi \in \mathcal{F}_{\mathrm{alt}}$ *and* $\varphi \in \mathcal{F}'_{\mathrm{alt}}$ *such that, for all* $x, y \in X$ *with* $x \preceq y$,

$$\psi(G(Tx,Ty,Ty)) \le (\psi - \varphi)(G(x,y,y)). \tag{10.4}$$

*Also assume that* $T$ *is* $G$*-continuous or* $(X,G,\preceq)$ *is non-decreasing-regular. If there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$, *then* $T$ *has, at least, a fixed point. Furthermore, if for all* $x, y \in \mathrm{Fix}(T)$ *there exists* $\omega \in X$ *such that* $x \preceq \omega$ *and* $y \preceq \omega$, *we obtain uniqueness of the fixed point.*

*Proof.* Let $x_0 \in X$ be such that $x_0 \preceq Tx_0$ and let $\{x_n\}_{n\ge 0}$ be the Picard sequence of $T$ based on $x_0$, that is, $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then $x_{n_0}$ is a fixed point of $T$. In the sequel, assume that $x_{n+1} \ne x_n$ for all $n \in \mathbb{N}$. In such a case, as $T$ is non-decreasing (w.r.t. $\preceq$), we have that

$$x_0 \preceq Tx_0 = x_1 \;\Rightarrow\; x_1 = Tx_0 \preceq Tx_1 = x_2.$$

By induction,

$$x_n \preceq x_{n+1} \text{ for all } n \ge 0. \tag{10.5}$$

Then, using the contractivity condition (10.4), we have that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\psi(G(x_{n+1}, x_{n+2}, x_{n+2})) &= \psi(G(Tx_n, Tx_{n+1}, Tx_{n+1})) \\ &\leq (\psi - \varphi)(G(x_n, x_{n+1}, x_{n+1})).\end{aligned}$$

Applying Lemma 2.3.6, $\{G(x_n, x_{n+1}, x_{n+1})\} \to 0$. Let us show that $\{x_n\}$ is $G$-Cauchy. Reasoning by contradiction, if $\{x_n\}$ is not $G$-Cauchy, by Theorem 4.1.1, there exist $\varepsilon_0 > 0$ and two partial subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ satisfying $k < n(k) < m(k) < n(k+1)$,

$$\begin{aligned}&G(x_{n(k)}, x_{m(k)-1}, x_{m(k)-1}) \leq \varepsilon_0 < G(x_{n(k)}, x_{m(k)}, x_{m(k)}) \quad \text{for all } k \in \mathbb{N}, \\ &\lim_{k\to\infty} G(x_{n(k)}, x_{m(k)}, x_{m(k)}) = \lim_{k\to\infty} G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}) = \varepsilon_0.\end{aligned} \tag{10.6}$$

As $\preceq$ is transitive and $n(k) < m(k)$, we deduce from (10.5) that $x_{n(k)-1} \preceq x_{m(k)-1}$ for all $k \in \mathbb{N}$. The contractivity condition (10.4) implies that, for all $k \in \mathbb{N}$,

$$\begin{aligned}\psi\left(G(x_{n(k)}, x_{m(k)}, x_{m(k)})\right) &= \psi\left(G(Tx_{n(k)-1}, Tx_{m(k)-1}, Tx_{m(k)-1})\right) \\ &\leq (\psi - \varphi)\left(G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1})\right).\end{aligned}$$

Taking into account (10.6) and Lemma 2.3.5, we conclude that $\varepsilon_0 = 0$, which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence in $(X, G)$. As $(X, G)$ is complete, there exist there exists $z_0 \in X$ such that $\{x_n\} \xrightarrow{G} z_0$.

Now suppose that $T$ is $G$-continuous. Then $\{x_{m+1}\} = \{Tx_m\} \xrightarrow{G} Tz_0$. By the uniqueness of the limit of a sequence in a $G^*$-metric space (see Proposition 10.2.1), $Tz_0 = z_0$ and $z_0$ is a fixed point of $T$.

On the other case, suppose that $(X, G, \preceq)$ is non-decreasing-regular. Since $\{x_m\} \xrightarrow{G} z_0$ and $\{x_m\}$ is monotone non-decreasing (w.r.t. $\preceq$), it follows that $x_n \preceq z_0$ for all $n \in \mathbb{N}$. Hence, for all $n \in \mathbb{N}$,

$$\begin{aligned}\psi(G(x_{n+1}, x_{n+1}, Tz_0)) &= \psi(G(Tx_n, Tx_n, Tz_0)) \\ &\leq (\psi - \varphi)(G(x_n, x_n, z_0)).\end{aligned}$$

Since $\{x_n\} \xrightarrow{G} z_0$, then $\{G(x_n, x_n, z_0)\} \to 0$. Taking the limit when $k \to \infty$ we deduce that $\{\psi(G(x_{n+1}, x_{n+1}, Tz_0))\} \to 0$. By Lemma 2.3.3, $\{G(x_{n+1}, x_{n+1}, Tz_0)\} \to 0$, so $\{x_{n+1}\} \xrightarrow{G} Tz_0$ and we also conclude that $z_0$ is a fixed point of $T$.

To prove the uniqueness, let $x, y \in \mathrm{Fix}(T)$ be two fixed points of $T$. By hypothesis, there exists $\omega \in X$ such that $x \preceq \omega$ and $y \preceq \omega$. We now show that $\{\omega_n = T^n\omega\} \xrightarrow{G} x$. Since $x \preceq \omega$ and $T$ is $\preceq$-non-decreasing, we have that $x = Tx \preceq T\omega$. By induction, $x \preceq T^n\omega = \omega_n$ for all $n \in \mathbb{N}$. Using the contractivity condition (10.4), we have that, for all $n \in \mathbb{N}$,

$$\psi(G(x,x,\omega_{n+1})) = \psi(G(Tx,Tx,T\omega_n)) \leq (\psi - \varphi)(G(x,x,\omega_n)).$$

From Lemma 2.3.6, we deduce that $\{G(x,x,\omega_n)\} \to 0$, that is, $\{\omega_n\} \overset{G}{\to} x$. The same reasoning proves that $\{\omega_n\} \overset{G}{\to} y$, so $x = y$. □

In the next two results, we show the main advantage of using preorders.

**Corollary 10.4.1.** *Let $(X,G)$ be a complete $G^*$-metric space, let $\preccurlyeq$ be a partial order on $X$ and let $T : X \to X$ be a $\preccurlyeq$-non-decreasing mapping. Suppose that there exist two functions $\psi \in \mathcal{F}_{\text{alt}}$ and $\varphi \in \mathcal{F}'_{\text{alt}}$ such that, for all $x, y \in X$ with $x \preccurlyeq y$,*

$$\psi(G(Tx,Ty,Ty)) \leq (\psi - \varphi)(G(x,y,y)).$$

*Also assume that $T$ is $G$-continuous or $(X,G,\preccurlyeq)$ is non-decreasing-regular. If there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$, then $T$ has, at least, a fixed point. Furthermore, if for all $x, y \in \operatorname{Fix}(T)$ there exists $\omega \in X$ such that $x \preceq \omega$ and $y \preceq \omega$, we obtain uniqueness of the fixed point.*

In the following result, we apply that $\preceq$, defined as "$x \preceq y$ for all $x, y \in X$" is a preorder on $X$ (but not a partial order).

**Corollary 10.4.2.** *Let $(X,G)$ be a complete $G^*$-metric space and let $T : X \to X$ be a mapping. Suppose that there exist two functions $\psi \in \mathcal{F}_{\text{alt}}$ and $\varphi \in \mathcal{F}'_{\text{alt}}$ such that, for all $x, y \in X$,*

$$\psi(G(Tx,Ty,Ty)) \leq (\psi - \varphi)(G(x,y,y)).$$

*Then $T$ has a unique fixed point.*

*Proof.* We only have to notice that $(X,G,\preceq)$ is non-decreasing-regular, and that any $x_0 \in X$ satisfies $x_0 \preceq Tx_0$. □

**Corollary 10.4.3.** *Let $(X,G)$ be a complete $G^*$-metric space, let $\preceq$ be a preorder on $X$ and let $T : X \to X$ be a $\preceq$-non-decreasing mapping. Suppose that there exist two functions $\psi \in \mathcal{F}_{\text{alt}}$ and $\varphi \in \mathcal{F}'_{\text{alt}}$ such that, for all $x, y, z \in X$ with $x \preceq y \preccurlyeq z$,*

$$\psi(G(Tx,Ty,Tz)) \leq (\psi - \varphi)(G(x,y,z)). \tag{10.7}$$

*Also assume that $T$ is $G$-continuous or $(X,G,\preceq)$ is non-decreasing-regular. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then $T$ has, at least, a fixed point. Furthermore, if for all $x, y \in \operatorname{Fix}(T)$ there exists $\omega \in X$ such that $x \preceq \omega$ and $y \preceq \omega$, we obtain uniqueness of the fixed point.*

*Proof.* It follows from the fact that (10.7) implies (10.4). □

If we take $\psi \in \mathcal{F}_{\text{alt}}$ as the identity mapping on $X$, we deduce the following statement.

**Corollary 10.4.4.** *Let $(X, G)$ be a complete $G^*$-metric space, let $\preceq$ be a preorder on $X$ and let $T : X \to X$ be a $\preceq$-non-decreasing mapping. Suppose that there exists a function $\varphi \in \mathcal{F}'_{\text{alt}}$ such that, for all $x, y \in X$ with $x \preceq y$,*

$$G(Tx, Ty, Ty) \le G(x, y, y) - \varphi(G(x, y, y)).$$

*Also assume that $T$ is $G$-continuous or $(X, G, \preceq)$ is non-decreasing-regular. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then $T$ has, at least, a fixed point. Furthermore, if for all $x, y \in \text{Fix}(T)$ there exists $\omega \in X$ such that $x \preceq \omega$ and $y \preceq \omega$, we obtain uniqueness of the fixed point.*

If $\varphi(t) = (1 - \lambda)t$ for all $t \ge 0$, where $\lambda \in [0, 1)$, we have the following version.

**Corollary 10.4.5.** *Let $(X, G)$ be a complete $G^*$-metric space, let $\preceq$ be a preorder on $X$ and let $T : X \to X$ be a $\preceq$-non-decreasing mapping. Suppose that there exists a constant $\lambda \in [0, 1)$ such that, for all $x, y \in X$ with $x \preceq y$,*

$$G(Tx, Ty, Ty) \le \lambda\, G(x, y, y).$$

*Also assume that $T$ is $G$-continuous or $(X, G, \preceq)$ is non-decreasing-regular. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then $T$ has, at least, a fixed point. Furthermore, if for all $x, y \in \text{Fix}(T)$ there exists $\omega \in X$ such that $x \preceq \omega$ and $y \preceq \omega$, we obtain uniqueness of the fixed point.*

One can obtain particular versions of Corollaries 10.4.3, 10.4.4 and 10.4.5 using, on the one hand, a partial order $\preccurlyeq$ on $X$ (as in Corollary 10.4.1) and, on the other hand, the preorder "$x \preceq y$ for all $x, y \in X$" (as in Corollary 10.4.2).

## *10.4.2 Some Results Under $\varphi$-Contractivity Conditions*

Next, we give another version of Theorem 10.4.1 using a different contractivity condition.

**Theorem 10.4.2.** *Let $(X, G)$ be a complete $G^*$-metric space, let $\preceq$ be a preorder on $X$ and let $T : X \to X$ be a $\preceq$-non-decreasing mapping. Suppose that there exists a function $\varphi \in \mathcal{F}_{\text{Cir}}$ such that, for all $x, y \in X$ with $x \preceq y$,*

$$G(Tx, Ty, Ty) \le \varphi(G(x, y, y)). \tag{10.8}$$

*Also assume that $T$ is $G$-continuous or $(X, G, \preceq)$ is non-decreasing-regular. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then $T$ has, at least, a fixed point. Furthermore, if for all $x, y \in \text{Fix}(T)$ there exists $\omega \in X$ such that $x \preceq \omega$ and $y \preceq \omega$, we obtain uniqueness of the fixed point.*

*Proof.* Let $x_0 \in X$ be such that $x_0 \preceq Tx_0$ and let $\{x_n\}_{n\geq 0}$ be the Picard sequence of $T$ based on $x_0$, that is, $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then $x_{n_0}$ is a fixed point of $T$. In the sequel, assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$, that is,

$$G(x_n, x_{n+1}, x_{n+1}) > 0 \quad \text{for all } n \in \mathbb{N}. \tag{10.9}$$

In such a case, as $T$ is non-decreasing (w.r.t. $\preceq$), we have that

$$x_0 \preceq Tx_0 = x_1 \;\Rightarrow\; x_1 = Tx_0 \preceq Tx_1 = x_2.$$

By induction,

$$x_n \preceq x_{n+1} \text{ for all } n \geq 0. \tag{10.10}$$

Then, using the contractivity condition (10.8), we have that, for all $n \in \mathbb{N}$,

$$G(x_{n+1}, x_{n+2}, x_{n+2}) = G(Tx_n, Tx_{n+1}, Tx_{n+1}) \leq \varphi(G(x_n, x_{n+1}, x_{n+1})).$$

Taking into account (10.9) and applying item 3 of Lemma 2.3.11, $\{G(x_n, x_{n+1}, x_{n+1})\} \to 0$. We now show that $\{x_n\}$ is $G$-Cauchy. Reasoning by contradiction, if $\{x_n\}$ is not $G$-Cauchy, by Theorem 4.1.1, there exist $\varepsilon_0 > 0$ and two partial subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ satisfying $k < n(k) < m(k) < n(k+1)$,

$$\begin{aligned} &G(x_{n(k)}, x_{m(k)-1}, x_{m(k)-1}) \leq \varepsilon_0 < G(x_{n(k)}, x_{m(k)}, x_{m(k)}) \quad \text{for all } k \in \mathbb{N}, \\ &\lim_{k\to\infty} G(x_{n(k)}, x_{m(k)}, x_{m(k)}) = \lim_{k\to\infty} G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}) = \varepsilon_0. \end{aligned} \tag{10.11}$$

As $\preceq$ is transitive and $n(k) < m(k)$, we deduce from (10.10) that $x_{n(k)-1} \preceq x_{m(k)-1}$ for all $k \in \mathbb{N}$. The contractivity condition (10.8) implies that, for all $k \in \mathbb{N}$,

$$\begin{aligned} \varepsilon_0 < G(x_{n(k)}, x_{m(k)}, x_{m(k)}) &= G(Tx_{n(k)-1}, Tx_{m(k)-1}, Tx_{m(k)-1}) \\ &\leq \varphi\left(G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1})\right). \end{aligned}$$

Taking into account (10.11), item 4 of Lemma 2.3.11, applied to $L = \varepsilon_0$ and $\{a_k = G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1})\}_{k\in\mathbb{N}}$, guarantees that $\varepsilon_0 = 0$, which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence in $(X, G)$. As $(X, G)$ is complete, there exist there exists $z_0 \in X$ such that $\{x_n\} \xrightarrow{G} z_0$.

Now suppose that $T$ is $G$-continuous. Then $\{x_{m+1}\} = \{Tx_m\} \xrightarrow{G} Tz_0$. By the uniqueness of the limit of a sequence in a $G^*$-metric space (see Proposition 10.2.1), $Tz_0 = z_0$ and $z_0$ is a fixed point of $T$.

On the other case, suppose that $(X, G, \preceq)$ is non-decreasing-regular. Since $\{x_m\} \xrightarrow{G} z_0$ and $\{x_m\}$ is monotone non-decreasing (w.r.t. $\preceq$), it follows that $x_n \preceq z_0$ for all $n \in \mathbb{N}$. Hence, for all $n \in \mathbb{N}$,

$$G(x_{n+1}, x_{n+1}, Tz_0) = G(Tx_n, Tx_n, Tz_0) \leq \varphi(G(x_n, x_n, z_0)).$$

Since $\{x_n\} \xrightarrow{G} z_0$, then $\{G(x_n, x_n, z_0)\} \to 0$. Using item 5 of Lemma 2.3.11 applied to $\{a_n = G(x_n, x_n, z_0)\}_{n \in \mathbb{N}}$ and $\{b_n = G(x_{n+1}, x_{n+1}, Tz_0)\}_{n \in \mathbb{N}}$ (notice that if $a_n = 0$, then $x_n = z_0$, so $x_{n+1} = Tx_n = Tz_0$ and $b_n = 0$), we deduce that

$$G(z_0, z_0, Tz_0) = \lim_{n \to \infty} G(x_{n+1}, x_{n+1}, Tz_0) = 0,$$

so $z_0$ is a fixed point of $T$.

To prove the uniqueness, let $x, y \in \operatorname{Fix}(T)$ be two fixed points of $T$. By hypothesis, there exists $\omega \in X$ such that $x \preceq \omega$ and $y \preceq \omega$. Let us show that $\{\omega_n = T^n \omega\} \xrightarrow{G} x$. Since $x \preceq \omega$ and $T$ is $\preceq$-non-decreasing, we have that $x = Tx \preceq T\omega$. By induction, $x \preceq T^n \omega = \omega_n$ for all $n \in \mathbb{N}$. Using the contractivity condition (10.4), we have that, for all $n \in \mathbb{N}$,

$$G(x, x, \omega_{n+1}) = G(Tx, Tx, T\omega_n) \leq \varphi(G(x, x, \omega_n)).$$

Again, using item 6 of Lemma 2.3.11, we deduce that $\{G(x, x, \omega_n)\} \to 0$, that is, $\{\omega_n\} \xrightarrow{G} x$. The same reasoning proves that $\{\omega_n\} \xrightarrow{G} y$, so $x = y$. □

The same arguments that we have used in the corollaries of Theorem 10.4.1 can now be applied to deduce the following results.

**Corollary 10.4.6.** *Let $(X, G)$ be a complete $G^*$-metric space, let $\preccurlyeq$ be a partial order on $X$ and let $T : X \to X$ be a $\preccurlyeq$-non-decreasing mapping. Suppose that there exists a function $\varphi \in \mathcal{F}_{\mathrm{Cir}}$ such that, for all $x, y \in X$ with $x \preccurlyeq y$,*

$$G(Tx, Ty, Ty) \leq \varphi(G(x, y, y)).$$

*Also assume that $T$ is $G$-continuous or $(X, G, \preccurlyeq)$ is non-decreasing-regular. If there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$, then $T$ has, at least, a fixed point. Furthermore, if for all $x, y \in \operatorname{Fix}(T)$ there exists $\omega \in X$ such that $x \preceq \omega$ and $y \preceq \omega$, we obtain uniqueness of the fixed point.*

**Corollary 10.4.7.** *Let $(X, G)$ be a complete $G^*$-metric space and let $T : X \to X$ be a mapping. Suppose that there exists a function $\varphi \in \mathcal{F}_{\mathrm{Cir}}$ such that, for all $x, y \in X$,*

$$G(Tx, Ty, Ty) \leq \varphi(G(x, y, y)).$$

*Then $T$ has a unique fixed point.*

**Corollary 10.4.8.** *Let $(X, G)$ be a complete $G^*$-metric space, let $\preceq$ be a preorder on $X$ and let $T : X \to X$ be a $\preceq$-non-decreasing mapping. Suppose that there exists a function $\varphi \in \mathcal{F}_{\mathrm{Cir}}$ such that, for all $x, y, z \in X$ with $x \preceq y \preccurlyeq z$,*

$$G(Tx, Ty, Tz) \leq \varphi(G(x, y, z)).$$

*Also assume that $T$ is G-continuous or $(X, G, \preceq)$ is non-decreasing-regular. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then $T$ has, at least, a fixed point. Furthermore, if for all $x, y \in \mathrm{Fix}(T)$ there exists $\omega \in X$ such that $x \preceq \omega$ and $y \preceq \omega$, we obtain uniqueness of the fixed point.*

**Corollary 10.4.9.** *Let $(X, G)$ be a complete $G^*$-metric space, let $\preceq$ be a preorder on $X$ and let $T : X \to X$ be a $\preceq$-non-decreasing mapping. Suppose that there exists a constant $\lambda \in [0, 1)$ such that, for all $x, y \in X$ with $x \preceq y$,*

$$G(Tx, Ty, Ty) \leq \lambda\, G(x, y, y).$$

*Also assume that $T$ is G-continuous or $(X, G, \preceq)$ is non-decreasing-regular. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then $T$ has, at least, a fixed point. Furthermore, if for all $x, y \in \mathrm{Fix}(T)$ there exists $\omega \in X$ such that $x \preceq \omega$ and $y \preceq \omega$, we obtain uniqueness of the fixed point.*

## 10.5 Further Fixed Point Theorems in Partially Preordered $G^*$-Metric Spaces

In this section, inspired by the results of the previous section, we prove some theorems in the setting of $G^*$-metric spaces using contractivity conditions that cannot be reduced to quasi-metric spaces, that is, involving three different values in the arguments of $G$.

**Theorem 10.5.1.** *Let $(X, G)$ be a complete $G^*$-metric space, let $\preceq$ be a preorder on $X$ and let $T : X \to X$ be a $\preceq$-non-decreasing mapping. Suppose that there exist two functions $\psi \in \mathcal{F}_{\mathrm{alt}}$ and $\varphi \in \mathcal{F}'_{\mathrm{alt}}$ such that, for all $x, y \in X$ with $x \preceq y$,*

$$\psi(G(Tx, Ty, T^2x)) \leq (\psi - \varphi)\,(G(x, y, Tx)). \tag{10.12}$$

*Also assume that $T$ is G-continuous or $(X, G, \preceq)$ is non-decreasing-regular. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then $T$ has, at least, a fixed point. Furthermore, if for all $x, y \in \mathrm{Fix}(T)$ there exists $\omega \in X$ such that $x \preceq \omega$ and $y \preceq \omega$, we obtain uniqueness of the fixed point.*

*Proof.* Let $x_0 \in X$ be such that $x_0 \preceq Tx_0$ and let $\{x_n\}_{n\geq 0}$ be the Picard sequence of $T$ based on $x_0$, that is, $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then $x_{n_0}$ is a fixed point of $T$. In the sequel, assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. In such a case, as $T$ is non-decreasing (w.r.t. $\preceq$), we have that

$$x_0 \preceq Tx_0 = x_1 \;\Rightarrow\; x_1 = Tx_0 \preceq Tx_1 = x_2.$$

By induction,

$$x_n \preceq x_{n+1} \text{ for all } n \geq 0. \tag{10.13}$$

Then, using the contractivity condition (10.12), we have that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\psi(G(x_{n+1}, x_{n+2}, x_{n+2})) &= \psi(G(Tx_n, Tx_{n+1}, T^2x_n)) \\ &\leq (\psi - \varphi)(G(x_n, x_{n+1}, Tx_n)) = (\psi - \varphi)(G(x_n, x_{n+1}, x_{n+1})).\end{aligned}$$

Applying Lemma 2.3.6, $\{G(x_n, x_{n+1}, x_{n+1})\} \to 0$. Let us show that $\{x_n\}$ is $G$-Cauchy. Reasoning by contradiction, if $\{x_n\}$ is not $G$-Cauchy, by Lemma 8.3.2, there exist $\varepsilon_0 > 0$ and two partial subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ satisfying $k < n(k) < m(k) < n(k+1)$,

$$\begin{aligned} &G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)-1}\right) \leq \varepsilon_0 < G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)}\right) \quad \text{for all } k \in \mathbb{N}, \\ &\lim_{k\to\infty} G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)}\right) = \lim_{k\to\infty} G\left(x_{n(k)-1}, x_{n(k)}, x_{m(k)-1}\right) = \varepsilon_0. \end{aligned} \tag{10.14}$$

As $\preceq$ is transitive and $n(k) < m(k)$, we deduce from (10.13) that $x_{n(k)-1} \preceq x_{m(k)-1}$ for all $k \in \mathbb{N}$. The contractivity condition (10.12) implies that, for all $k \in \mathbb{N}$,

$$\begin{aligned}\psi\left(G(x_{n(k)}, x_{n(k)+1}, x_{m(k)})\right) &= \psi\left(G(Tx_{n(k)-1}, Tx_{m(k)-1}, T^2x_{n(k)-1})\right) \\ &\leq (\psi - \varphi)\left(G(x_{n(k)-1}, x_{m(k)-1}, Tx_{n(k)-1})\right) \\ &= (\psi - \varphi)\left(G(x_{n(k)-1}, x_{n(k)}, x_{m(k)-1})\right).\end{aligned}$$

Taking into account that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$, we have that $G(x_{n(k)}, x_{n(k)+1}, x_{m(k)}) > 0$ and $G(x_{n(k)-1}, x_{n(k)}, x_{m(k)-1}) > 0$ for all $k \in \mathbb{N}$. Using (10.14) and Lemma 2.3.5, we conclude that $\varepsilon_0 = 0$, which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence in $(X, G)$. As $(X, G)$ is complete, there exist there exists $z_0 \in X$ such that $\{x_n\} \xrightarrow{G} z_0$.

Now suppose that $T$ is $G$-continuous. Then $\{x_{m+1}\} = \{Tx_m\} \xrightarrow{G} Tz_0$. By the uniqueness of the limit of a sequence in a $G^*$-metric space (see Proposition 10.2.1), $Tz_0 = z_0$ and $z_0$ is a fixed point of $T$.

On the other hand, suppose that $(X, G, \preceq)$ is non-decreasing-regular. Since $\{x_m\} \xrightarrow{G} z_0$ and $\{x_m\}$ is monotone non-decreasing (w.r.t. $\preceq$), it follows that $x_n \preceq z_0$ for all $n \in \mathbb{N}$. Hence, for all $n \in \mathbb{N}$,

$$\begin{aligned}\psi(G(x_{n+1}, x_{n+2}, Tz_0)) &= \psi(G(Tx_n, Tz_0, T^2x_n)) \\ &\leq (\psi - \varphi)(G(x_n, z_0, Tx_n)) = (\psi - \varphi)(G(x_n, x_{n+1}, z_0)).\end{aligned}$$

Since $\{x_n\} \xrightarrow{G} z_0$, then $\{G(x_n, x_{n+1}, z_0)\} \to 0$. Taking the limit when $k \to \infty$ we deduce that $\{\psi(G(x_{n+1}, x_{n+1}, Tz_0))\} \to 0$. By Lemma 2.3.3, $\{G(x_{n+1}, x_{n+2}, Tz_0)\} \to 0$. As $G$ is continuous on each argument (see Theorem 10.2.2), we deduce that

$$G(z_0, z_0, Tz_0) = \lim_{n\to\infty} G(x_{n+1}, x_{n+2}, Tz_0) = 0,$$

so $Tz_0 = z_0$ and $z_0$ is a fixed point of $T$.

To prove the uniqueness, let $x, y \in \mathrm{Fix}(T)$ be two fixed points of $T$. By hypothesis, there exists $\omega \in X$ such that $x \preceq \omega$ and $y \preceq \omega$. Let us show that $\{\omega_n = T^n\omega\} \xrightarrow{G} x$. Since $x \preceq \omega$ and $T$ is $\preceq$-non-decreasing, we have that $x = Tx \preceq T\omega$. By induction, $x \preceq T^n\omega = \omega_n$ for all $n \in \mathbb{N}$. Using the contractivity condition (10.12), we have that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\psi(G(x, x, \omega_{n+1})) &= \psi(G(Tx, T\omega_n, T^2x)) \\ &\leq (\psi - \varphi)\,(G(x, \omega_n, Tx)) = (\psi - \varphi)\,(G(x, x, \omega_n)).\end{aligned}$$

From Lemma 2.3.6, we deduce that $\{G(x, x, \omega_n)\} \to 0$, that is, $\{\omega_n\} \xrightarrow{G} z_1$. The same reasoning proves that $\{\omega_n\} \xrightarrow{G} y$, so $x = y$. □

The same arguments that we have used in the corollaries of Theorem 10.4.1 can now be applied to deduce the following results.

**Corollary 10.5.1.** *Let $(X, G)$ be a complete $G^*$-metric space, let $\preccurlyeq$ be a partial order on $X$ and let $T : X \to X$ be a $\preccurlyeq$-non-decreasing mapping. Suppose that there exist two functions $\psi \in \mathcal{F}_{\mathrm{alt}}$ and $\varphi \in \mathcal{F}'_{\mathrm{alt}}$ such that, for all $x, y \in X$ with $x \preccurlyeq y$,*

$$\psi(G(Tx, Ty, T^2x)) \leq (\psi - \varphi)\,(G(x, y, Tx)).$$

*Also assume that $T$ is $G$-continuous or $(X, G, \preccurlyeq)$ is non-decreasing-regular. If there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$, then $T$ has, at least, a fixed point. Furthermore, if for all $x, y \in \mathrm{Fix}(T)$ there exists $\omega \in X$ such that $x \preccurlyeq \omega$ and $y \preccurlyeq \omega$, we obtain uniqueness of the fixed point.*

**Corollary 10.5.2.** *Let $(X, G)$ be a complete $G^*$-metric space and let $T : X \to X$ be a mapping. Suppose that there exist two functions $\psi \in \mathcal{F}_{\mathrm{alt}}$ and $\varphi \in \mathcal{F}'_{\mathrm{alt}}$ such that, for all $x, y \in X$,*

$$\psi(G(Tx, Ty, T^2x)) \leq (\psi - \varphi)\,(G(x, y, Tx)).$$

*Then $T$ has a unique fixed point.*

**Corollary 10.5.3.** *Let $(X, G)$ be a complete $G^*$-metric space, let $\preceq$ be a preorder on $X$ and let $T : X \to X$ be a $\preceq$-non-decreasing mapping. Suppose that there exists a function $\varphi \in \mathcal{F}'_{\mathrm{alt}}$ such that, for all $x, y \in X$ with $x \preceq y$,*

$$G(Tx, Ty, T^2x) \leq G(x, y, Tx) - \varphi(G(x, y, Tx)).$$

*Also assume that $T$ is $G$-continuous or $(X, G, \preceq)$ is non-decreasing-regular. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then $T$ has, at least, a fixed point. Furthermore, if for all $x, y \in \mathrm{Fix}(T)$ there exists $\omega \in X$ such that $x \preceq \omega$ and $y \preceq \omega$, we obtain uniqueness of the fixed point.*

**Corollary 10.5.4.** *Let $(X, G)$ be a complete $G^*$-metric space, let $\preceq$ be a preorder on $X$ and let $T : X \to X$ be a $\preceq$-non-decreasing mapping. Suppose that there exists a constant $\lambda \in [0, 1)$ such that, for all $x, y \in X$ with $x \preceq y$,*

$$G(Tx, Ty, T^2x) \leq \lambda\, G(x, y, Tx).$$

*Also assume that $T$ is $G$-continuous or $(X, G, \preceq)$ is non-decreasing-regular. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then $T$ has, at least, a fixed point. Furthermore, if for all $x, y \in \operatorname{Fix}(T)$ there exists $\omega \in X$ such that $x \preceq \omega$ and $y \preceq \omega$, we obtain uniqueness of the fixed point.*

# Chapter 11
# Multidimensional Fixed Point Theorems on *G*-Metric Spaces

In this chapter we introduce several notions of multidimensional fixed points. To prove results, it is usual to consider a number of sequences equal to the dimension of the product space in which the main mapping is defined. Also, using the techniques described in Sect. 10.3, we will show that most of multidimensional results can be deduced from the corresponding unidimensional result in $G^*$-metric spaces.

Throughout this chapter and for simplicity, given a positive integer number $n$, we will use $X^n$ to denote the *nth Cartesian power of* $X$, that is, $X \times X \times \ldots \times X$ ($n$ times).

## 11.1 Different Notions of Multidimensional Fixed Point

The notion of fixed point of a self-mapping $T : X \to X$ can be seen as a solution of the nonlinear equation $Tx = x$. In this sense, if $F : X^n \to X^n$ is also a self-mapping, a fixed point of $F$ is a point $(x_1, x_2, \ldots, x_n) \in X^n$ such that

$$F(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n).$$

However, there is a different viewpoint about multidimensional fixed points. For example, when we handle a mapping $F : X^n \to X$, a *fixed point of* $F$ (in a cyclic sense) is a point $(x_1, x_2, \ldots, x_n) \in X^n$ satisfying the following system involving $n$ equalities:



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$$\begin{cases} F\left(x_1, x_2, x_3, \ldots, x_{n-1}, x_n\right) = x_1, \\ F\left(x_2, x_3, x_4, \ldots, x_n, x_1\right) = x_2, \\ \quad\vdots \\ F\left(x_i, x_{i+1}, x_{i+2}, \ldots, x_n, x_1, x_2, \ldots, x_{i-1}\right) = x_i, \\ \quad\vdots \\ F\left(x_n, x_1, x_2, \ldots, , x_{n-2}, x_{n-1}\right) = x_n. \end{cases}$$

In 2006, Gnana-Bhaskar and Lakshmikantham [84] studied the coupled case in order to guarantee the existence and uniqueness of a solution of a boundary value problem. In their work, they considered a nonlinear operator $F : X \times X \to X$ with two arguments in a partially ordered metric space $(X, d, \preccurlyeq)$, and they characterized any solution of the differential system as a *coupled fixed point of* $F$, that is, a point $(x, y) \in X$ such that

$$\begin{cases} F\left(x, y\right) = x, \\ F\left(y, x\right) = y. \end{cases}$$

In fact, this notion corresponds to a cyclic 2-dimensional fixed point of $F$. However, one of the most attractive hypotheses they introduced in their main results was the fact that $F$ must have the *mixed monotone property* (see Definition 11.3.1). When Berinde and Borcut [41, 50] tried to extend the coupled case to a third variable, they considered that the mixed monotone property should also be assumed. Then, in order to take advantage of this property, they did not consider the cyclic notion of tripled fixed point, but rather they introduced, for a nonlinear operator $F : X \times X \times X \to X$, the notion of *tripled fixed point* as a point $(x, y, z) \in X \times X \times X$ such that

$$\begin{cases} F\left(x, y, z\right) = x, \\ F\left(y, x, y\right) = y, \\ F\left(z, y, x\right) = z. \end{cases}$$

In this case, both the second and the third equations do not correspond to the cyclic notion of a fixed point. Especially attractive for researchers was the second condition, $y = F(y, x, y)$, in which the variable $y$ is repeated and $z$ does not appear.

Later, Karapınar [110] introduced the quadrupled notion as a extension of the two previous cases, defining

$$\begin{cases} F\left(x, y, z, \omega\right) = x, \\ F\left(y, z, \omega, x\right) = y, \\ F\left(z, \omega, x, y\right) = z, \\ F\left(\omega, x, y, z\right) = \omega, \end{cases}$$

which correspond to the cyclic case. He also proved some existence and uniqueness theorems assuming the mixed monotone property on $X$.

A notion of multidimensional fixed point was introduced by Berzig and Samet in [42] and, simultaneously by Roldán-López-de-Hierro et al. in [174].

We briefly describe the different notions of fixed and coincidence points we will use throught this chapter in the low dimensional case ($n \in \{2, 3, 4\}$).

**Definition 11.1.1.** Given two mappings $T, g : X \to X$, we will say that a point $x \in X$ is a:

- *fixed point of* $T$ if $Tx = x$;
- *coincidence point of* $T$ *and* $g$ if $Tx = gx$;
- *common fixed point of* $T$ *and* $g$ if $Tx = gx = x$.

We will denote by $\operatorname{Fix} T$ the set of all fixed points of $T$ and by $\operatorname{Coin}(T, g)$ the family of all coincidence points of $T$ and $g$.

Following Gnana-Bhaskar and Lakshmikantham (see [84]), given $F : X^2 \to X$ and $g : X \to X$, we will say that a point $(x, y) \in X^2$ is a

- *coupled fixed point of* $F$ if $F(x, y) = x$ and $F(y, x) = y$;
- *coupled coincidence point of* $F$ *and* $g$ if $F(x, y) = gx$ and $F(y, x) = gy$;
- *common coupled fixed point of* $F$ *and* $g$ if $F(x, y) = gx = x$ and $F(y, x) = gy = y$.

We will denote by $\operatorname{Fix} F$ the set of all coupled fixed points of $F$ and by $\operatorname{Coin}(F, g)$ the family of all coupled coincidence points of $F$ and $g$.

Following Berinde and Borcut (see [41, 50]), given $F : X^3 \to X$ and $g : X \to X$, we will say that a point $(x, y, z) \in X^3$ is a

- *tripled fixed point of* $F$ if $F(x, y, z) = x$, $F(y, x, y) = y$ and $F(z, y, x) = z$.
- *tripled coincidence point of* $F$ *and* $g$ if $F(x, y, z) = gx$, $F(y, x, y) = gy$ and $F(z, y, x) = gz$.
- *common tripled fixed point of* $F$ *and* $g$ if $F(x, y, z) = gx = x$, $F(y, x, y) = gy = y$ and $F(z, y, x) = gz = z$.

Following Karapınar (see [110, 117]), given $F : X^4 \to X$ and $g : X \to X$, we will say that a point $(x, y, z, t) \in X^4$ is a

- *quadrupled fixed point of* $F$ if $F(x, y, z, t) = x$, $F(y, z, t, x) = y$, $F(z, t, x, y) = z$ and $F(t, x, y, z) = t$.
- *quadrupled coincidence point of* $F$ *and* $g$ if $F(x, y, z, t) = gx$, $F(y, z, t, x) = gy$, $F(z, t, x, y) = gz$ and $F(t, x, y, z) = gt$.
- *common quadrupled fixed point of* $F$ *and* $g$ if $F(x, y, z, t) = gx = x$, $F(y, z, t, x) = gy = y$, $F(z, t, x, y) = gz = z$ and $F(t, x, y, z) = gt = t$.

## 11.2 Preliminaries

In this section we introduce some technical properties we will use throughout this chapter.

**Lemma 11.2.1.** *Let $\{x_n^1\}, \{x_n^2\}, \ldots, \{x_n^N\} \subseteq X$ be $N$ sequences on a $G^*$-metric space $(X, G)$ such that*

$$\lim_{n\to\infty} G\left(x_n^i, x_{n+1}^i, x_{n+1}^i\right) = 0 \quad \textit{for all } i \in \{1, 2, \ldots, N\}. \tag{11.1}$$

*Suppose that, at least, one of them is not Cauchy in $(X, G)$. Then there exist $\varepsilon_0 > 0$, $i_0 \in \{1, 2, \ldots, N\}$ and two sequences of natural numbers $\{n(k)\}_{k\in\mathbb{N}}$ and $\{m(k)\}_{k\in\mathbb{N}}$ such that*

$$\begin{aligned}
&k \le n(k) < m(k) < n(k+1),\\
&\max_{1\le i\le N}\left\{G(x_{n(k)}^i, x_{m(k)-1}^i, x_{m(k)-1}^i)\right\} \le \varepsilon_0 < \max_{1\le i\le N}\left\{G(x_{n(k)}^i, x_{m(k)}^i, x_{m(k)}^i)\right\},\\
&\lim_{k\to\infty}\left[\max_{1\le i\le N}\left\{G(x_{n(k)}^i, x_{m(k)}^i, x_{m(k)}^i)\right\}\right]\\
&\quad = \lim_{k\to\infty}\left[\max_{1\le i\le N}\left\{G(x_{n(k)-1}^i, x_{m(k)-1}^i, x_{m(k)-1}^i)\right\}\right] = \varepsilon_0,\\
&\lim_{k\to\infty} G(x_{n(k)}^{i_0}, x_{m(k)}^{i_0}, x_{m(k)}^{i_0}) = \lim_{k\to\infty} G(x_{n(k)-1}^{i_0}, x_{m(k)-1}^{i_0}, x_{m(k)-1}^{i_0}) = \varepsilon_0.
\end{aligned}$$

*Proof.* For all $n, m \in \mathbb{N}$, let

$$S(n, m) = \max_{1\le i\le N} G(x_n^i, x_m^i, x_m^i).$$

Using this notation, property (11.1) means that

$$\lim_{n\to\infty} S(n, n+1) = 0. \tag{11.2}$$

Furthermore, for all $n, m, p \in \mathbb{N}$ we have that

$$\begin{aligned}
S(n, m) &= \max_{1\le i\le N} G(x_n^i, x_m^i, x_m^i)\\
&\le \max_{1\le i\le N}\left\{G(x_n^i, x_p^i, x_p^i) + G(x_p^i, x_m^i, x_m^i)\right\}\\
&\le \max_{1\le i\le N} G(x_n^i, x_p^i, x_p^i) + \max_{1\le i\le N} G(x_p^i, x_m^i, x_m^i)\\
&= S(n, p) + S(p, m).
\end{aligned} \tag{11.3}$$

Consider the following condition.

$$\begin{gathered}
\text{For all } \varepsilon > 0, \text{ there exists } n_0 \in \mathbb{N} \text{ such that}\\
S(n, m) \le \varepsilon \text{ for all } m > n \ge n_0.
\end{gathered} \tag{11.4}$$

We claim that if (11.4) holds, then all sequences $\{x_m^1\}, \{x_m^2\}, \ldots, \{x_m^N\}$ are Cauchy. Indeed, let $\varepsilon > 0$ be arbitrary. Let $n_0 \in \mathbb{N}$ satisfy (11.4). Therefore, for all $m > n \geq n_0$ and for all $j \in \{1, 2, \ldots, N\}$ we have that

$$G(x_n^j, x_m^j, x_m^j) \leq \max_{1 \leq i \leq N} \left\{ G(x_n^i, x_m^i, x_m^i) \right\} = S(n, m) \leq \varepsilon.$$

By Lemma 10.2.5, $\{x_n^j\}$ is a Cauchy sequence in $(X, G)$ for all $j \in \{1, 2, \ldots, N\}$. As we are assuming that at least, one of the sequences $\{x_n^1\}, \{x_n^2\}, \ldots, \{x_n^N\}$ is not Cauchy in $(X, G)$, condition (11.4) cannot hold. Then, there exists $\varepsilon_0 > 0$ such that

$$\text{for all } n_0 \in \mathbb{N}, \text{ there exist } m > n \geq n_0 \text{ such that } S(n, m) > \varepsilon_0.$$

Using this property repeatedly, we can find two sequences of natural numbers $\{n(k)\}_{k \in \mathbb{N}}$ and $\{m(k)\}_{k \in \mathbb{N}}$ such that, for all $k \in \mathbb{N}$,

$$\begin{aligned} & k \leq n(k) < m(k) < n(k+1) \quad \text{and} \\ & \varepsilon_0 < S(n(k), m(k)) = \max_{1 \leq i \leq N} \left\{ G(x_{n(k)}^i, x_{m(k)}^i, x_{m(k)}^i) \right\}. \end{aligned} \tag{11.5}$$

If we choose $m(k)$ as the lowest integer, greater than $n(k)$, satisfying (11.5), then we can assume that $S(n(k), m(k) - 1) \leq \varepsilon_0$, that is, for all $k \in \mathbb{N}$,

$$\begin{aligned} & \max_{1 \leq i \leq N} \left\{ G(x_{n(k)}^i, x_{m(k)-1}^i, x_{m(k)-1}^i) \right\} \leq \varepsilon_0 \\ & \qquad < S(n(k), m(k)) = \max_{1 \leq i \leq N} \left\{ G(x_{n(k)}^i, x_{m(k)}^i, x_{m(k)}^i) \right\}. \end{aligned}$$

Moreover,

$$\begin{aligned} \varepsilon_0 &< \max_{1 \leq i \leq N} \left\{ G(x_{n(k)}^i, x_{m(k)}^i, x_{m(k)}^i) \right\} \\ &\leq \max_{1 \leq i \leq N} \left\{ G(x_{n(k)}^i, x_{m(k)-1}^i, x_{m(k)-1}^i) + G(x_{m(k)-1}^i, x_{m(k)}^i, x_{m(k)}^i) \right\} \\ &\leq \max_{1 \leq i \leq N} G(x_{n(k)}^i, x_{m(k)-1}^i, x_{m(k)-1}^i) + \max_{1 \leq i \leq N} G(x_{m(k)-1}^i, x_{m(k)}^i, x_{m(k)}^i) \\ &\leq \varepsilon_0 + \max_{1 \leq i \leq N} G(x_{m(k)-1}^i, x_{m(k)}^i, x_{m(k)}^i) = \varepsilon_0 + S(m(k) - 1, m(k)). \end{aligned}$$

Taking the limit as $k \to \infty$ and by (11.2), we deduce that

$$\lim_{k \to \infty} S(n(k), m(k)) = \lim_{k \to \infty} \left[ \max_{1 \leq i \leq N} \left\{ G(x_{n(k)}^i, x_{m(k)}^i, x_{m(k)}^i) \right\} \right] = \varepsilon_0. \tag{11.6}$$

From (11.3)

$$S\,(n(k)-1,m(k)) = S\,(n(k)-1,n(k)) + S\,(n(k),m(k)) \quad \text{and}$$
$$S\,(n(k),m(k)) \leq S\,(n(k),m(k)-1) + S\,(m(k)-1,m(k))\,.$$

Therefore,

$$\begin{aligned} S\,(n(k),m(k)) - S\,(m(k)-1,m(k)) &\leq S\,(n(k),m(k)-1) \\ &\leq S\,(n(k)-1,n(k)) + S\,(n(k),m(k))\,. \end{aligned} \tag{11.7}$$

From (11.2) and (11.6), we deduce that

$$\lim_{k\to\infty} S\,(n(k),m(k)-1) = \varepsilon_0. \tag{11.8}$$

Similarly, by (11.3),

$$S\,(n(k)-1,m(k)-1) = S\,(n(k)-1,n(k)) + S\,(n(k),m(k)-1) \quad \text{and}$$
$$S\,(n(k),m(k)-1) \leq S\,(n(k),n(k)-1) + S\,(n(k)-1,m(k)-1)\,.$$

Hence

$$\begin{aligned} S\,(n(k),m(k)-1) - S\,(n(k),n(k)-1) &\leq S\,(n(k)-1,m(k)-1) \\ &\leq S\,(n(k)-1,n(k)) + S\,(n(k),m(k)-1)\,. \end{aligned} \tag{11.9}$$

Taking the limit as $k \to \infty$ and using (11.2) and (11.8),

$$\begin{aligned} &\lim_{k\to\infty}\left[\max_{1\leq i\leq N}\left\{G(x^i_{n(k)-1},x^i_{m(k)-1},x^i_{m(k)-1})\right\}\right] \\ &\qquad = \lim_{k\to\infty} S\,(n(k)-1,m(k)-1) = \varepsilon_0. \end{aligned}$$

Next, we consider the $N$ sequences $\{a^i_k = G(x^i_{n(k)},x^i_{m(k)},x^i_{m(k)})\}_{k\in\mathbb{N}}$, where $i \in \{1,2,\ldots,N\}$. They are lower bounded and they satisfy

$$\lim_{k\to\infty}\left(\max_{1\leq i\leq N} a^i_k\right) = \lim_{k\to\infty}\left[\max_{1\leq i\leq N}\left\{G(x^i_{n(k)},x^i_{m(k)},x^i_{m(k)})\right\}\right] = \varepsilon_0.$$

By Lemma 2.1.2, there exists $i_0 \in \{1,2,\ldots,N\}$ and a partial subsequence $\{a^{i_0}_{p(k)} = G(x^i_{p(n(k))},x^i_{p(m(k))},x^i_{p(m(k))})\}_{k\in\mathbb{N}}$ such that $\{a^{i_0}_{p(k)}\}_{k\in\mathbb{N}} \to \varepsilon_0$. Identifying $p\circ n \equiv n$ and $p \circ m \equiv m$ in order to not complicate the notation, we have that

$$\lim_{k\to\infty} G(x^{i_0}_{n(k)},x^{i_0}_{m(k)},x^{i_0}_{m(k)}) = \varepsilon_0.$$

Finally, the property

$$\lim_{k\to\infty} G(x^{i_0}_{n(k)-1}, x^{i_0}_{m(k)-1}, x^{i_0}_{m(k)-1}) = \varepsilon_0$$

can be deduced using the same argument we have followed in (11.7), (11.8) and (11.9). This completes the proof. □

**Lemma 11.2.2.** *Let* $\{x^1_n\}, \{x^2_n\}, \ldots, \{x^N_n\} \subseteq X$ *be* $N$ *sequences on a* $G^*$*-metric space* $(X, G)$ *such that*

$$\lim_{n\to\infty} G\left(x^i_n, x^i_{n+1}, x^i_{n+1}\right) = 0 \quad \textit{for all } i \in \{1, 2, \ldots, N\}.$$

*Suppose that, at least, one of them is not Cauchy in* $(X, G)$*. Then there exist* $\varepsilon_0 > 0$ *and two sequences of natural numbers* $\{n(k)\}_{k\in\mathbb{N}}$ *and* $\{m(k)\}_{k\in\mathbb{N}}$ *such that, for all* $k \in \mathbb{N}$*,*

$$k \le n(k) < m(k) < n(k+1),$$
$$\sum_{i=1}^{N} G(x^i_{n(k)}, x^i_{m(k)-1}, x^i_{m(k)-1}) \le \varepsilon_0 < \sum_{i=1}^{N} G(x^i_{n(k)}, x^i_{m(k)}, x^i_{m(k)}),$$

*and also, for all given* $p_1, p_2, p_3 \in \mathbb{Z}$*,*

$$\lim_{k\to\infty} \left[\sum_{i=1}^{N} G(x^i_{n(k)+p_1}, x^i_{m(k)+p_2}, x^i_{m(k)+p_3})\right] = \varepsilon_0.$$

*Proof.* By Corollary 10.3.1, the mapping $G'_N : X^N \times X^N \times X^N \to [0, \infty)$, given by

$$G'_N(\mathsf{X}, \mathsf{Y}, \mathsf{Z}) = \sum_{i=1}^{N} G(x_i, y_i, z_i)$$

for all $\mathsf{X} = (x_1, x_2, \ldots, x_N), \mathsf{Y} = (y_1, y_2, \ldots, y_N), \mathsf{Z} = (z_1, z_2, \ldots, z_N) \in X^N$, is a $G^*$-metric on $X^N$. Consider the sequence $\{A_n = (x^1_n, x^2_n, \ldots, x^N_n)\}_{n\in\mathbb{N}} \subseteq X^N$, which satisfies

$$\lim_{n\to\infty} G'_N(A_n, A_{n+1}, A_{n+1}) = \lim_{n\to\infty} \sum_{i=1}^{N} G\left(x^i_n, x^i_{n+1}, x^i_{n+1}\right) = 0.$$

Then, $\{A_n\}$ is an asymptotically regular sequence of $\left(X^N, G'_N\right)$. As one of the sequences $\{x^1_n\}, \{x^2_n\}, \ldots, \{x^N_n\}$ is not Cauchy in $(X, G)$, then item 3 of Corollary 10.3.1 guarantees that $\{A_n\}$ is not a Cauchy sequence in $\left(X^N, G'_N\right)$. From Theorem 4.1.1 (which is also valid in $G^*$-metric spaces by Remark 4.1.1), there exists a positive real number $\varepsilon_0 > 0$ and two subsequences $\{A_{n(k)}\}$ and $\{A_{m(k)}\}$ of $\{A_n\}$ such that, for all $k \in \mathbb{N}$,

$$k \leq n(k) < m(k) < n(k+1),$$

$$G'_N\left(A_{n(k)}, A_{m(k)-1}, A_{m(k)-1}\right) \leq \varepsilon_0 < G'_N\left(A_{n(k)}, A_{m(k)}, A_{m(k)}\right)$$

and also, for all given $p_1, p_2, p_3 \in \mathbb{Z}$,

$$\lim_{k\to\infty} G'_N\left(A_{n(k)+p_1}, A_{m(k)+p_2}, A_{m(k)+p_3}\right) = \varepsilon_0.$$

The proof is complete. □

**Lemma 11.2.3.** *Let $\{a_m\}_{m\in\mathbb{N}}$ be a sequence of non-negative real numbers which has no subsequence converging to zero. Then, for all $\varepsilon > 0$, there exist $\delta \in ]0, \varepsilon[$ and $m_0 \in \mathbb{N}$ such that $a_m \geq \delta$ for all $m \geq m_0$.*

*Proof.* Suppose that the conclusion is not true. Then, there exists $\varepsilon_0 > 0$ such that, for all $\delta \in ]0, \varepsilon_0[$, there exists $m_0 \in \mathbb{N}$ satisfying $a_{m_0} < \delta$. Let $k_0 \in \mathbb{N}$ be such that $1/k_0 < \varepsilon_0$. For all $k \in \mathbb{N}$, take $\delta_k = 1/(k+k_0) \in ]0, \varepsilon_0[$. Then there exists $m(k) \in \mathbb{N}$ verifying $0 \leq a_{m(k)} < \delta_k = 1/(k+k_0)$. Taking the limit when $k \to \infty$, we deduce that $\lim_{k\to\infty} a_{m(k)} = 0$. Then $\{a_m\}$ has a subsequence converging to zero (maybe, reordering $\{a_{m(k)}\}$), but this is a contradiction. □

**Lemma 11.2.4.** *Let $\{a^1_m\}, \{a^2_m\}, \ldots, \{a^n_m\}, \{b^1_m\}, \{b^2_m\}, \ldots, \{b^n_m\} \subset [0, \infty)$ be $2n$ sequences of non-negative real numbers and suppose that there exist $\psi, \varphi \in \mathcal{F}_{\text{alt}}$ such that*

$$\psi(a^i_{m+1}) \leq (\psi - \varphi)(b^i_m) \quad \textit{for all } i \textit{ and all } m, \qquad \textit{and}$$

$$\psi\left(\max_{1\leq i\leq n} b^i_m\right) \leq \psi\left(\max_{1\leq i\leq n} a^i_m\right) \quad \textit{for all } m.$$

*Then $\{a^i_m\} \to 0$ for all $i$.*

*Proof.* Let $c_m = \max_{1\leq i\leq n} a^i_m$ for all $m$. Then, for all $m$,

$$\begin{aligned} \psi(c_{m+1}) &= \psi\left(\max_{1\leq i\leq n} a^i_{m+1}\right) = \max_{1\leq i\leq n} \psi\left(a^i_{m+1}\right) \leq \max_{1\leq i\leq n} \left[\,(\psi - \varphi)(b^i_m)\,\right] \\ &\leq \max_{1\leq i\leq n} \psi(b^i_m) = \psi\left(\max_{1\leq i\leq n} b^i_m\right) \leq \psi\left(\max_{1\leq i\leq n} a^i_m\right) = \psi(c_m). \end{aligned}$$

Therefore, $\{\psi(c_m)\}$ is a non-increasing, bounded below sequence. Then, it is convergent. Let $\Delta \geq 0$ be such that $\{\psi(c_m)\} \to \Delta$ and $\Delta \leq \psi(c_m)$. We show that $\Delta = 0$. Since

$$\left\{\max_{1\leq i\leq n} \psi(a^i_m)\right\} = \left\{\psi\left(\max_{1\leq i\leq n} a^i_m\right)\right\} = \{\psi\,(c_m)\} \to \Delta,$$

Lemma 2.1.2 guarantees that there exists $i_0 \in \{1, 2, \ldots, n\}$ and a partial subsequence $\{a^{i_0}_{m(k)}\}_{k\in\mathbb{N}}$ such that $\{\psi(a^{i_0}_{m(k)})\} \to \Delta$. Moreover,

$$0 \leq \psi(a^{i_0}_{m(k)}) \leq (\psi - \varphi)(b^{i_0}_{m(k)-1}) \quad \text{for all } k. \tag{11.10}$$

Consider the sequence $\{b^{i_0}_{m(k)-1}\}_{k\in\mathbb{N}}$. If this sequence has a partial subsequence converging to zero, then we can take the limit in (11.10) when $k \to \infty$ using that partial subsequence, and we deduce $\Delta = 0$. On the contrary, if $\{b^{i_0}_{m(k)-1}\}_{k\in\mathbb{N}}$ has no partial subsequence converging to zero, Lemma 11.2.3 assures us that there exist $\delta \in \,]0, 1[$ and $k_0 \in \mathbb{N}$ such that $b^{i_0}_{m(k)-1} \geq \delta$ for all $k \geq k_0$. Since $\varphi$ is non-decreasing, $-\varphi(b^{i_0}_{m(k)-1}) \leq -\varphi(\delta) < 0$. Then, by (11.10), for all $k \geq k_0$,

$$\begin{aligned}
0 &\leq \psi(a^{i_0}_{m(k)}) \leq (\psi - \varphi)(b^{i_0}_{m(k)-1}) = \psi(b^{i_0}_{m(k)-1}) - \varphi(b^{i_0}_{m(k)-1}) \\
&\leq \psi(b^{i_0}_{m(k)-1}) - \varphi(\delta) \leq \psi\left(\max_{1\leq i\leq n} b^{i}_{m(k)-1}\right) - \varphi(\delta) \\
&\leq \psi\left(\max_{1\leq i\leq n} a^{i}_{m(k)-1}\right) - \varphi(\delta) = \psi\left(c_{m(k)-1}\right) - \varphi(\delta).
\end{aligned}$$

Taking the limit as $k \to \infty$ we deduce $\Delta \leq \Delta - \varphi(\delta)$, which is impossible. This proves that $\Delta = 0$. Since $\{\psi(c_m)\} \to \Delta = 0$, Lemma 2.3.3 implies that $\{c_m\} \to 0$, which is equivalent to $\{a^i_m\} \to 0$ for all $i$. □

**Lemma 11.2.5.** *Let $(X, G)$ be a G-metric space and let $\{x^1_n\}, \{x^2_n\}, \ldots, \{x^N_n\} \subseteq X$ be $N$ sequences in $X$. Assume that there exists $\psi \in \mathcal{F}^{(c)}_{\text{com}}$ such that*

$$\sum_{i=1}^{N} G\left(x^i_{n+2}, x^i_{n+1}, x^i_{n+1}\right) \leq \psi\left(\sum_{i=1}^{N} G\left(x^i_{n+1}, x^i_n, x^i_n\right)\right) \tag{11.11}$$

*for all $n \in \mathbb{N}$. Then each sequence $\{x^i_n\}_{n\in\mathbb{N}}$ is Cauchy in $(X, G)$ for all $i \in \{1, 2, \ldots, N\}$.*

*Proof.* Let

$$t_0 = \sum_{i=1}^{N} G\left(x^i_1, x^i_0, x^i_0\right) \in [0, \infty)\,.$$

As $\psi$ is non-decreasing, by (11.11), for all $n \geq 1$,

$$\begin{aligned}
\sum_{i=1}^{N} G\left(x^i_{n+1}, x^i_n, x^i_n\right) &\leq \psi\left(\sum_{i=1}^{N} G\left(x^i_n, x^i_{n-1}, x^i_{n-1}\right)\right) \\
&\leq \psi^2\left(\sum_{i=1}^{N} G\left(x^i_{n-1}, x^i_{n-2}, x^i_{n-2}\right)\right) \leq \ldots \\
&\leq \psi^n\left(\sum_{i=1}^{N} G\left(x^i_1, x^i_0, x^i_0\right)\right) = \psi^n\left(t_0\right).
\end{aligned}$$

As a consequence, for all $j \in \{1, 2, \ldots, N\}$ and all $n, m \in \mathbb{N}$ such that $n < m$, we have that

$$\begin{aligned} G(x_m^j, x_n^j, x_n^j) &\leq \sum_{i=1}^{N} G\left(x_m^i, x_n^i, x_n^i\right) \leq \sum_{i=1}^{N} \sum_{k=n}^{m-1} G\left(x_{k+1}^i, x_k^i, x_k^i\right) \\ &= \sum_{k=n}^{m-1} \left( \sum_{i=1}^{N} G\left(x_{k+1}^i, x_k^i, x_k^i\right) \right) \leq \sum_{k=n}^{m-1} \psi^k\left(t_0\right). \end{aligned}$$

If $t_0 = 0$, then $x_m^j = x_n^j$ for all $j \in \{1, 2, \ldots, N\}$ and all $n, m \in \mathbb{N}$. In such a case, each sequence $\{x_n^j\}_{n \in \mathbb{N}}$ is constant, so it is also Cauchy. Assume that $t_0 > 0$. Let $\varepsilon > 0$ be arbitrary. Since $\psi \in \mathcal{F}_{\mathrm{com}}^{(c)}$, the series $\sum_{k \in \mathbb{N}} \psi^k\left(t_0\right)$ converges. Then, there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{k=n_0}^{\infty} \psi^k\left(t_0\right) < \varepsilon.$$

Therefore, for all $j \in \{1, 2, \ldots, N\}$ and all $n, m \in \mathbb{N}$ such that $m > n \geq n_0$, it follows that

$$G\left(x_m^j, x_n^j, x_n^j\right) \leq \sum_{k=n}^{m-1} \psi^k\left(t_0\right) \leq \sum_{k=n_0}^{\infty} \psi^k\left(t_0\right) < \varepsilon.$$

From Lemma 3.2.2, each sequence $\{x_n^j\}_{n \in \mathbb{N}}$ is Cauchy in $(X, G)$. □

**Lemma 11.2.6.** *Let $(X, G)$ be a G-metric space and let $\{x_n^1\}, \{x_n^2\}, \ldots, \{x_n^N\} \subseteq X$ be $N$ sequences in $X$. Assume that there exists $\psi \in \mathcal{F}_{\mathrm{com}}^{(c)}$ such that*

$$\max_{1 \leq i \leq N} G\left(x_{n+2}^i, x_{n+1}^i, x_{n+1}^i\right) \leq \psi\left(\max_{1 \leq i \leq N} G\left(x_{n+1}^i, x_n^i, x_n^i\right)\right)$$

*for all $n \in \mathbb{N}$. Then each sequence $\{x_n^i\}_{n \in \mathbb{N}}$ is Cauchy in $(X, G)$ for all $i \in \{1, 2, \ldots, N\}$.*

*Proof.* Repeat the argument in the proof of Lemma 11.2.5 replacing the sum by the maximum. In particular, if

$$t_0 = \max_{1 \leq i \leq N} G\left(x_1^i, x_0^i, x_0^i\right) > 0,$$

then

$$G(x_m^j, x_n^j, x_n^j) \leq \max_{1 \leq i \leq N} G\left(x_m^i, x_n^i, x_n^i\right) \leq \max_{1 \leq i \leq N} \left\{ \sum_{k=n}^{m-1} G\left(x_{k+1}^i, x_k^i, x_k^i\right) \right\}$$

$$\leq \sum_{k=n}^{m-1} \left( \max_{1 \leq i \leq N} G\left(x_{k+1}^i, x_k^i, x_k^i\right) \right) \leq \sum_{k=n}^{m-1} \psi^k\left(t_0\right),$$

and continue. □

**Corollary 11.2.1.** *Let $(X, G)$ be a $G$-metric space and let $\{x_n^1\}, \{x_n^2\}, \ldots, \{x_n^N\} \subseteq X$ be $N$ sequences in $X$. Assume that there exists $\lambda \in [0, 1)$ such that*

$$\max_{1 \leq i \leq N} G\left(x_{n+2}^i, x_{n+1}^i, x_{n+1}^i\right) \leq \lambda \max_{1 \leq i \leq N} G\left(x_{n+1}^i, x_n^i, x_n^i\right)$$

*for all $n \in \mathbb{N}$. Then each sequence $\{x_n^i\}_{n \in \mathbb{N}}$ is Cauchy in $(X, G)$ for all $i \in \{1, 2, \ldots, N\}$.*

*Proof.* It is only necessary to apply Lemma 11.2.6 using $\psi_\lambda(t) = \lambda\, t$ for all $t \in [0, \infty)$. □

## 11.3 Coupled Fixed Point Theory in *G*-Metric Spaces

In this section, we describe sufficient conditions to ensure that a mapping $F : X \times X \to X$ has a coupled fixed point, that is, a point $(x, y) \in X^2$ such that $F(x, y) = x$ and $F(y, x) = y$.

### *11.3.1 Gnana-Bhaskar and Lakshmikantham's Coupled Fixed Point Theory*

The notion of coupled fixed point was introduced by Guo and Lakshmikantham in [89]. Later, in [84], Gnana-Bhaskar and Lakshmikantham reconsidered this concept and introduced the mixed monotone property.

**Definition 11.3.1 ([84]).** Let $X$ be a non-empty set endowed with a binary relation $\preccurlyeq$. A mapping $F : X^2 \to X$ is said to have the *mixed $\preccurlyeq$-monotone property* if $F(x, y)$ is monotone $\preccurlyeq$-non-decreasing in $x$ and monotone $\preccurlyeq$-non-increasing in $y$, that is, for all $x, y \in X$ ,

$$x_1, x_2 \in X, \quad x_1 \preccurlyeq x_2 \quad \Rightarrow \quad F(x_1, y) \preccurlyeq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \preccurlyeq y_2 \quad \Rightarrow \quad F(x, y_1) \succcurlyeq F(x, y_2).$$

In their original definition, Guo and Lakshmikantham considered that $\preccurlyeq$ was a partial order on $X$. We will use preorders in the theorems we present in this chapter. When the binary relation $\preccurlyeq$ is implicitly considered, it is usual to refer to the previous property as the *mixed monotone property*.

**Lemma 11.3.1.** *Let $\preceq$ be a transitive binary relation on a set $X$ and let $F : X^2 \to X$ be a mapping having the mixed $\preceq$-monotone property. Assume that there exists $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Then the sequences $\{x_n\}$ and $\{y_n\}$, iteratively defined by*

$$x_{n+1} = F(x_n, y_n) \quad \text{and} \quad y_{n+1} = F(y_n, x_n) \quad \text{for all } n \in \mathbb{N}, \tag{11.12}$$

*verify $x_n \preceq x_{n+1}$ and $y_n \succeq y_{n+1}$ for all $n \in \mathbb{N}$.*

*Furthermore, if there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$ and $y_{n_0} = y_{n_0+1}$, then $(x_{n_0}, y_{n_0})$ is a coupled fixed point of $F$.*

*Proof.* We proceed by induction. For $n = 0$, we assume, by hypothesis, that $x_0 \preceq F(x_0, y_0) = x_1$ and $y_0 \succeq F(y_0, x_0) = y_1$. Suppose that $x_n \preceq x_{n+1}$ and $y_n \succeq y_{n+1}$ for some $n \in \mathbb{N}$. Then, as $F$ has the mixed $\preceq$-monotone property, then

$$\begin{aligned} x_{n+1} &= F(x_n, y_n) \preceq F(x_{n+1}, y_n) \preceq F(x_{n+1}, y_{n+1}) = x_{n+2}, \\ y_{n+1} &= F(y_n, x_n) \succeq F(y_{n+1}, x_n) \succeq F(y_{n+1}, x_{n+1}) = y_{n+2}. \end{aligned}$$

This completes the induction. Furthermore, if the exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$ and $y_{n_0} = y_{n_0+1}$, then

$$x_{n_0} = x_{n_0+1} = F(x_{n_0}, y_{n_0}) \quad \text{and} \quad y_{n_0} = y_{n_0+1} = F(y_{n_0}, x_{n_0}),$$

so $(x_{n_0}, y_{n_0})$ is a coupled fixed point of $F$. □

**Theorem 11.3.1.** *Let $(X, G)$ be a complete G-metric space endowed with a preorder $\preceq$ and let $F : X^2 \to X$ be a mapping having the mixed $\preceq$-monotone property. Suppose that there exists a constant $\lambda \in [0, 1)$ such that*

$$G\left(F\left(x,y\right),F\left(u,v\right),F\left(z,w\right)\right) \leq \frac{\lambda}{2}\left[G\left(x,u,z\right)+G\left(y,v,w\right)\right] \tag{11.13}$$

*for all $(x,y), (u,v), (z,w) \in X^2$ satisfying $x \preceq u \preceq z$ and $y \succeq v \succeq w$. Also assume that $F$ is continuous and there exists $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succcurlyeq F(y_0, x_0)$. Then $F$ has, at least, a coupled fixed point.*

The condition "$x \preceq u \preceq z$ and $y \succeq v \succeq w$" can be replaced by the condition "$x \succeq u \succeq z$ and $y \preceq v \preceq w$" because $G$ is commutative and

$$G\left(F\left(x,y\right),F\left(u,v\right),F\left(z,w\right)\right) = G\left(F\left(z,w\right),F\left(u,v\right),F\left(x,y\right)\right).$$

*Proof.* Starting from the points $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succcurlyeq F(y_0, x_0)$, consider the sequences $\{x_n\}$ and $\{y_n\}$ as in (11.12). By Lemma 11.3.1, the sequence $\{x_n\}$ is $\preceq$-non-decreasing and $\{y_n\}$ is $\preceq$-non-increasing. If $x_0 = x_1$ and $y_0 = y_1$, then $(x_0, y_0)$ is a coupled fixed point of $F$, and the existence part is finished. In the other case, assume that

$$G(x_0, x_1, x_1) + G(y_0, y_1, y_1) > 0.$$

Then, using the contractivity condition (11.13), for all $n \in \mathbb{N}$,

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+2}) &= G(F(x_n, y_n), F(x_{n+1}, y_{n+1}), F(x_{n+1}, y_{n+1})) \\ &\leq \frac{\lambda}{2} \left( G(x_n, x_{n+1}, x_{n+1}) + G(y_n, y_{n+1}, y_{n+1}) \right) \end{aligned}$$

and, taking into account that $y_{n+1} \preceq y_{n+1} \preceq y_n$ and $x_{n+1} \succeq x_{n+1} \succeq x_n$,

$$\begin{aligned} G(y_{n+2}, y_{n+2}, y_{n+1}) &= G(F(y_{n+1}, x_{n+1}), F(y_{n+1}, x_{n+1}), F(x_n, y_n)) \\ &\leq \frac{\lambda}{2} \left[ G(y_{n+1}, y_{n+1}, y_n) + G(x_{n+1}, x_{n+1}, x_n) \right]. \end{aligned}$$

Therefore, for all $n \in \mathbb{N}$,

$$\begin{aligned} &G(x_{n+1}, x_{n+2}, x_{n+2}) + G(y_{n+2}, y_{n+2}, y_{n+1}) \\ &\quad \leq \lambda \left[ G(x_n, x_{n+1}, x_{n+1}) + G(y_n, y_{n+1}, y_{n+1}) \right]. \end{aligned}$$

Repeating this argument, we have that, for all $n \in \mathbb{N}$,

$$\begin{aligned} &G(x_n, x_{n+1}, x_{n+1}) + G(y_n, y_{n+1}, y_{n+1}) \\ &\quad \leq \lambda \left[ G(x_{n-1}, x_n, x_n) + G(y_{n-1}, y_n, y_n) \right] \\ &\quad \leq \lambda^2 \left[ G(x_{n-2}, x_{n-1}, x_{n-1}) + G(y_{n-2}, y_{n-1}, y_{n-1}) \right] \\ &\quad \leq \ldots \leq \lambda^n \left[ G(x_0, x_1, x_1) + G(y_0, y_1, y_1) \right]. \end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. As $\lambda \in [0, 1)$, the series $\Sigma_{n \geq 1} \lambda^n$ converges. Then, there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0}^{\infty} \lambda^n < \frac{\varepsilon}{G(x_0, x_1, x_1) + G(y_0, y_1, y_1)}.$$

Let $n, m \in \mathbb{N}$ be such that $m > n \geq n_0$. Then

$$\begin{aligned} \max\{G(x_n, x_m, x_m), G(y_n, y_m, y_m)\} &\leq G(x_n, x_m, x_m) + G(y_n, y_m, y_m) \\ \leq \sum_{k=n}^{m-1} \left[ G(x_k, x_{k+1}, x_{k+1}) + G(y_k, y_{k+1}, y_{k+1}) \right] \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=n}^{m-1} \left[ \lambda^k \left( G\left(x_0, x_1, x_1\right) + G\left(y_0, y_1, y_1\right) \right) \right] \\
&= \left[ G\left(x_0, x_1, x_1\right) + G\left(y_0, y_1, y_1\right) \right] \sum_{k=n}^{m-1} \lambda^k \\
&\leq \left[ G\left(x_0, x_1, x_1\right) + G\left(y_0, y_1, y_1\right) \right] \sum_{k=n_0}^{\infty} \lambda^k \leq \varepsilon.
\end{aligned}$$

As a consequence, the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy in $(X, G)$. As $(X, G)$ is complete, there exists $x, y \in X$ such that $\{x_n\} \to x$ and $\{y_n\} \to y$. Moreover, taking into account that $F$ is continuous, we know that the sequence $\{x_{n+1} = F(x_n, y_n)\}$ converges, at the same time, to $x$ and to $F(x, y)$, and the sequence $\{y_{n+1} = F(y_n, x_n)\}$ converges, at the same time, to $y$ and to $F(y, x)$. By the uniqueness of the limit in a $G$-metric space, we conclude that $(x, y)$ is a coupled fixed point of $F$. □

In the following result, we replace the continuity of $F$ by the regularity of $(X, G, \preceq)$ (recall Definition 5.2.1).

**Theorem 11.3.2.** *Theorem 11.3.1 also holds if we replace the continuity of $F$ by the regularity of $(X, G, \preceq)$.*

*Proof.* Following the argument in the proof of Theorem 11.3.1, we deduce that there exists $x, y \in X$ such that $\{x_n\} \to x$ and $\{y_n\} \to y$. Since $\{x_n\}$ is $\preceq$-non-decreasing and $\{x_n\} \to x$, the regularity of $(X, G, \preceq)$ implies that $x_n \preceq x$ for all $n \in \mathbb{N}$. Similarly, $y_n \succeq y$ for all $n \in \mathbb{N}$. Then, the contractivity condition (11.13) yields

$$\begin{aligned}
G\left(x_{n+1}, F\left(x, y\right), F\left(x, y\right)\right) &= G\left(F\left(x_n, y_n\right), F\left(x, y\right), F\left(x, y\right)\right) \\
&\leq \frac{\lambda}{2} \left[G\left(x_n, x, x\right) + G\left(y_n, y, y\right)\right].
\end{aligned}$$

As $G$ is continuous (see Theorem 10.2.2), we deduce that

$$G\left(x, F\left(x, y\right), F\left(x, y\right)\right) = \lim_{n\to\infty} G\left(x_{n+1}, F\left(x, y\right), F\left(x, y\right)\right) = 0,$$

so $F(x, y) = x$. Similarly,

$$\begin{aligned}
G\left(F\left(y, x\right), F\left(y, x\right), y_{n+1}\right) &= G\left(F\left(y, x\right), F\left(y, x\right), F\left(y_n, x_n\right)\right) \\
&\leq \frac{\lambda}{2} \left[G\left(y, y, y_n\right) + G\left(x, x, x_n\right)\right],
\end{aligned}$$

so $F(y, x) = y$ and $(x, y)$ is a coupled fixed point of $F$. □

**Theorem 11.3.3.** *Under the hypotheses of Theorem 11.3.1 (respectively, Theorem 11.3.2), also assume the following condition:*

$(U)$ *for all two coupled fixed points $(x, y)$ and $(x', y')$ of $F$, there exists $(z, w) \in X^2$ such that $x \preceq z$, $x' \preceq z$, $y \succeq w$ and $y' \succeq w$.*

*Then $F$ has a unique coupled fixed point.*

*Proof.* Let $(x, y)$ and $(x', y')$ be two arbitrary coupled fixed points of $F$. By hypothesis, there exists $(z_0, w_0) \in X^2$ such that $x \preceq z_0$, $x' \preceq z_0$, $y \succeq w_0$ and $y' \succeq w_0$. Define the sequences

$$z_{n+1} = F(z_n, w_n) \quad \text{and} \quad w_{n+1} = F(w_n, z_n) \quad \text{for all } n \in \mathbb{N}.$$

We claim that $\{z_n\} \to x$ and $\{w_n\} \to y$. The same argument will show that $\{z_n\} \to x'$ and $\{w_n\} \to y'$, so we will deduce $(x, y) = (x', y')$.

Since $x \preceq z_0$ and $y \succeq w_0$, and $F$ has the mixed $\preceq$-monotone property, then

$$x = F(x, y) \preceq F(z_0, y) \preceq F(z_0, w_0) = z_1 \quad \text{and}$$
$$y = F(y, x) \succeq F(w_0, x) \succeq F(w_0, z_0) = w_1.$$

Repeating this argument, we deduce, by induction, that $x \preceq z_n$ and $y \succeq w_n$ for all $n \in \mathbb{N}$. Therefore, using the contractivity condition (11.13), it follows that

$$\begin{aligned} G(x, x, z_{n+1}) &= G(F(x, y), F(x, y), F(z_n, w_n)) \\ &\leq \frac{\lambda}{2} \left[ G(x, x, z_n) + G(y, y, w_n) \right] \end{aligned}$$

and

$$\begin{aligned} G(w_{n+1}, y, y) &= G(F(w_n, z_n), F(y, x), F(y, x)) \\ &\leq \frac{\lambda}{2} \left[ G(w_n, y, y) + G(z_n, x, x) \right]. \end{aligned}$$

Joining the last two inequalities, for all $n \in \mathbb{N}$,

$$G(x, x, z_{n+1}) + G(y, y, w_{n+1}) \leq \lambda \left[ G(x, x, z_n) + G(y, y, w_n) \right].$$

From Lemma 2.1.3, we deduce that $\{G(x, x, z_{n+1})\} \to 0$ and $\{G(y, y, w_{n+1})\} \to 0$. Then $\{z_n\} \to x$ and $\{w_n\} \to y$. □

**Corollary 11.3.1.** *Let $(X, G)$ be a complete $G$-metric space endowed with a partial order $\preccurlyeq$ and let $F : X^2 \to X$ be a mapping having the mixed $\preccurlyeq$-monotone property. Suppose that there exists a constant $\lambda \in [0, 1)$ such that*

$$G\left(F\left(x,y\right),F\left(u,v\right),F\left(z,w\right)\right)\leq\frac{\lambda}{2}\left[G\left(x,u,z\right)+G\left(y,v,w\right)\right]$$

*for all* $(x,y),(u,v),(z,w)\in X^2$ *satisfying* $x\preccurlyeq u\preccurlyeq z$ *and* $y\succcurlyeq v\succcurlyeq w$. *Also assume that, at least, one of the following conditions holds:*

**(i)** *$F$ is continuous, or*
**(ii)** $(X,G,\preccurlyeq)$ *is regular.*

*If there exists* $x_0,y_0\in X$ *such that* $x_0\preccurlyeq F(x_0,y_0)$ *and* $y_0\succcurlyeq F(y_0,x_0)$, *then $F$ has, at least, a coupled fixed point.*

*Furthermore, if we additionally assume the following condition:*

$(U)$ *for all two coupled fixed points* $(x,y)$ *and* $(x',y')$ *of $F$, there exists* $(z,w)\in X^2$ *such that* $x\preccurlyeq z$, $x'\preccurlyeq z$, $y\succcurlyeq w$ *and* $y'\succcurlyeq w$;

*then $F$ has a unique coupled fixed point.*

In the next corollary, we use the special preorder $\preceq_0$ on $X$ given by "$x\preceq_0 y$ for all $x,y\in X$". In such a case, $(X,G,\preceq_0)$ is regular and $(U)$ trivially holds.

**Corollary 11.3.2.** *Let* $(X,G)$ *be a complete G-metric space and let* $F:X^2\to X$ *be a mapping. Suppose that there exists a constant* $\lambda\in[0,1)$ *such that*

$$G\left(F\left(x,y\right),F\left(u,v\right),F\left(z,w\right)\right)\leq\frac{\lambda}{2}\left(G\left(x,u,z\right)+G\left(y,v,w\right)\right)$$

*for all* $(x,y),(u,v),(z,w)\in X^2$. *Then $F$ has a unique coupled fixed point.*

### *11.3.2 Choudhury and Maity's Coupled Fixed Point Theorem in G-Metric Spaces*

In [58], Choudhury and Maity gave a version of Corollary 11.3.1 assuming the following contractivity condition: there exists a constant $\lambda\in[0,1)$ such that

$$G\left(F\left(x,y\right),F\left(u,v\right),F\left(z,w\right)\right)\leq\frac{\lambda}{2}\left(G\left(x,u,z\right)+G\left(y,v,w\right)\right)\tag{11.14}$$

for all $x,y,u,v,z,w\in X$ satisfying $x\succcurlyeq u\succcurlyeq z$ and $y\preccurlyeq v\preccurlyeq w$ where either $u\neq z$ or $v\neq w$. However, the proof given by the authors is false because the condition "either $u\neq z$ or $v\neq w$" is very restrictive. Let us review the lines of their proof.

Based on the points $x_0,y_0\in X$ such that $x_0\preccurlyeq F(x_0,y_0)$ and $F(y_0,x_0)\succcurlyeq y_0$ given by the hypothesis, the authors defined the sequences

$$x_{n+1}=F(x_n,y_n)\quad\text{and}\quad y_{n+1}=F(y_n,x_n)\quad\text{for all } n\in\mathbb{N},\ n\geq 0.$$

Using the mixed monotone property, they proved that

$$x_n \preccurlyeq x_{n+1} \quad \text{and} \quad y_n \succcurlyeq y_{n+1} \quad \text{for all } n \in \mathbb{N},\ n \geq 0.$$

If there exists some $n \in \mathbb{N}$ such that $(x_{n+1}, y_{n+1}) = (x_n, y_n)$, then $x_n = x_{n+1} = F(x_n, y_n)$ and $y_n = y_{n+1} = F(y_n, x_n)$, so $(x_n, y_n)$ is a coupled fixed point of $F$. On the contrary, assume that $(x_{n+1}, y_{n+1}) \neq (x_n, y_n)$ for all $n \geq 0$. Therefore, for all $n \geq 0$,

$$x_n \preccurlyeq x_{n+1} \quad \text{and} \quad y_n \succcurlyeq y_{n+1}, \qquad \text{but} \qquad x_n \neq x_{n+1} \quad \text{or} \quad y_{n+1} \neq y_n.$$

In this case, the authors could use the contractivity condition (11.14) to prove that

$$\begin{aligned} & x_1 \succeq x_1 \succeq x_0, \quad y_1 \preceq y_1 \preceq y_0, \quad x_0 \neq x_1 \quad \text{or} \quad y_0 \neq y_1 \\ & \quad \Rightarrow\ G(x_2, x_2, x_1) = G(F(x_1, y_1), F(x_1, y_1), F(x_0, y_0)) \\ & \qquad \leq \frac{\lambda}{2}\, [\, G(x_1, x_1, x_0) + G(y_1, y_1, y_0)\, ]. \end{aligned}$$

However, the corresponding inequality using $\{y_n\}$, that is,

$$G(y_2, y_2, y_1) \leq \frac{\lambda}{2}\, [\, G(y_1, y_1, y_0) + G(x_1, x_1, x_0)\, ] \tag{11.15}$$

cannot be proved because

$$G(y_2, y_2, y_1) = G(F(y_1, x_1), F(y_1, x_1), F(y_0, x_0))$$

but the previous conditions

$$\begin{cases} y_1 \succcurlyeq y_1 \succcurlyeq y_0, \\ x_1 \preccurlyeq x_1 \preccurlyeq x_0 \end{cases}$$

are not satisfied. In fact, we have the contrary inequalities: $x_1 \succcurlyeq x_0$ and $y_1 \preccurlyeq y_0$. Furthermore, it is not possible to use the symmetry of $G$ in its variables because the contractivity condition (11.14) requires that $x \succcurlyeq u \succcurlyeq w$ and $y \preccurlyeq v \preccurlyeq z$.

A version of Choudhury and Maity's result is the following one, which is also valid using a preorder $\preceq$ on $X$. In fact, the proof of Theorem 11.3.1 can be followed.

**Theorem 11.3.4.** *Let $(X, \preccurlyeq)$ be a partially ordered set and $G$ be a $G$-metric on $X$ such that $(X, G)$ is a complete $G$-metric space. Let $F : X \times X \to X$ be $G$-continuous mapping having the mixed $\preccurlyeq$-monotone property on $X$. Suppose that there exists a $\lambda \in [0, 1)$ such that*

$$G(F(x, y), F(u, v), F(z, w)) \leq \frac{\lambda}{2} [G(x, u, z) + G(y, v, w)] \tag{11.16}$$

*for all* $x, y, u, v, z, w \in X$ *with*

$$[\, x \succcurlyeq u \succcurlyeq z \text{ and } y \preccurlyeq v \preccurlyeq w \,] \quad \text{or} \quad [\, x \preccurlyeq u \preccurlyeq z \text{ and } y \succcurlyeq v \succcurlyeq w \,]\,,$$

*where either* $u \neq z$ *or* $v \neq w$. *If there exist* $x_0, y_0 \in X$ *such that* $x_0 \preccurlyeq F(x_0, y_0)$ *and* $F(y_0, x_0) \succcurlyeq y_0$, *then* $F$ *has a coupled fixed point.*

### 11.3.3 Berinde's Coupled Fixed Point Theory

In the setting of metric spaces, Berinde introduced in [40] a symmetric version of the contractivity condition (11.13).

**Theorem 11.3.5.** *Let* $(X, G)$ *be a complete G-metric space endowed with a preorder* $\preceq$ *and let* $F : X^2 \to X$ *be a mapping having the mixed* $\preceq$*-monotone property. Suppose that there exists a constant* $\lambda \in [0, 1)$ *such that*

$$\begin{aligned} &G\left(F\left(x,y\right),F\left(u,v\right),F\left(u,v\right)\right)+G\left(F\left(y,x\right),F\left(v,u\right),F\left(v,u\right)\right)\\ &\qquad \leq \lambda\left[G\left(x,u,u\right)+G\left(y,v,v\right)\right] \end{aligned} \tag{11.17}$$

*for all* $(x, y), (u, v) \in X^2$ *satisfying* $x \preceq u$ *and* $y \succeq v$. *Also assume that* $F$ *is continuous and there exists* $x_0, y_0 \in X$ *such that* $x_0 \preceq F(x_0, y_0)$ *and* $y_0 \succcurlyeq F(y_0, x_0)$. *Then* $F$ *has, at least, a coupled fixed point.*

Notice that the contractivity condition (11.13) implies (11.17).

*Proof.* Starting from the points $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succcurlyeq F(y_0, x_0)$, consider the sequences $\{x_n\}$ and $\{y_n\}$ as in (11.12). From Lemma 11.3.1, the sequence $\{x_n\}$ is $\preceq$-non-decreasing and $\{y_n\}$ is $\preceq$-non-increasing. If $x_0 = x_1$ and $y_0 = y_1$, then $(x_0, y_0)$ is a coupled fixed point of $F$, and the existence part is finished. In the other case, assume that

$$G\left(x_0, x_1, x_1\right) + G\left(y_0, y_1, y_1\right) > 0.$$

Then, using the contractivity condition (11.17), for all $n \in \mathbb{N}$,

$$\begin{aligned} &G\left(x_{n+1},x_{n+2},x_{n+2}\right)+G\left(y_{n+1},y_{n+2},y_{n+2}\right)\\ &\quad =G\left(F\left(x_n,y_n\right),F\left(x_{n+1},y_{n+1}\right),F\left(x_{n+1},y_{n+1}\right)\right)\\ &\qquad +G\left(F\left(y_n,x_n\right),F\left(y_{n+1},x_{n+1}\right),F\left(y_{n+1},x_{n+1}\right)\right)\\ &\quad \leq \lambda\left[G\left(x_n,x_{n+1},x_{n+1}\right)+G\left(y_n,y_{n+1},y_{n+1}\right)\right]. \end{aligned}$$

From Lemma 11.2.5, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $(X, G)$. Since $(X, G)$ is complete, there exists $u, v \in X$ such that $\{x_n\} \to u$ and $\{y_n\} \to v$. As $F$ is continuous, letting $n \to \infty$ in (11.12), we conclude that $F(u, v) = u$ and $F(v, u) = v$, so $(u, v)$ is a coupled fixed point of $F$. $\square$

**Theorem 11.3.6.** *If we replace the continuity of $F$ by the fact that $(X, G, \preceq)$ is regular, then Theorem 11.3.5 also holds.*

*Proof.* Following the proof of Theorem 11.3.5, we deduce that $\{x_n\}$ is $\preceq$-non-decreasing and $\{y_n\}$ is $\preceq$-non-increasing. Furthermore, there exists $u, v \in X$ such that $\{x_n\} \to u$ and $\{y_n\} \to v$. As $(X, G, \preceq)$ is regular, then $x_n \preceq u$ and $y_n \succeq v$ for all $n \in \mathbb{N}$. Then, from the contractivity condition (11.17),

$$\begin{aligned}
&G(x_{n+1}, F(u,v), F(u,v)) + G(y_{n+1}, F(v,u), F(v,u)) \\
&\quad = G(F(x_n, y_n), F(u,v), F(u,v)) + G(F(y_n, x_n), F(v,u), F(v,u)) \\
&\quad \leq \lambda\,[G(x_n, u, u) + G(y_n, v, v)]\,.
\end{aligned}$$

As a consequence, $u = \lim_{n\to\infty} x_{n+1} = F(u, v)$ and $v = \lim_{n\to\infty} y_{n+1} = F(v, u)$. $\square$

The following uniqueness result can be proved reasoning as in the proof of Theorem 11.3.3.

**Theorem 11.3.7.** *Under the hypothesis of Theorem 11.3.5 (respectively, Theorem 11.3.6), also assume the following condition:*

$(U)$ *for all two coupled fixed points $(x, y)$ and $(x', y')$ of $F$, there exists $(z, w) \in X^2$ such that $x \preceq z$, $x' \preceq z$, $y \succeq w$ and $y' \succeq w$.*

*Then $F$ has a unique coupled fixed point.*

### 11.3.4 G - β - ψ - Coupled Fixed Point Theorems in G-Metric Spaces

In this subsection, we have a $(c)$-comparison function in the contractivity condition.

**Theorem 11.3.8.** *Let $(X, G)$ be a $G$-complete $G$-metric space and let $F : X \times X \to X$ be a given mapping. Suppose there exist $\psi \in \mathcal{F}^{(c)}_{\text{com}}$ and $\beta : X^2 \times X^2 \times X^2 \to [0, \infty)$ such that*

$$\begin{aligned}
&\beta((x,y),(u,v),(u,v))\,G(F(x,y),F(u,v),F(u,v)) \\
&\quad \leq \frac{1}{2}\,\psi(G(x,u,u) + G(y,v,v)),
\end{aligned} \tag{11.18}$$

*for all $(x, y), (u, v) \in X^2$. Also assume that the following conditions hold.*

$(a)$ *For all* $(x, y), (u, v) \in X \times X$, *we have*

$$\beta((x, y), (u, v), (u, v)) \geq 1 \quad \Rightarrow \\ \beta((F(x, y), F(y, x)), (F(u, v), F(v, u)), (F(u, v), F(v, u))) \geq 1.$$

$(b)$ *There exists* $x_0, y_0 \in X$ *such that*

$$\beta((F(x_0, y_0), F(y_0, x_0)), (x_0, y_0), (x_0, y_0)) \geq 1 \text{ and}$$
$$\beta((F(y_0, x_0), F(x_0, y_0)), (y_0, x_0), (y_0, x_0)) \geq 1.$$

$(c)$ $F$ *is continuous.*

*Then $F$ has a coupled fixed point.*

*Proof.* Starting from $x_0, y_0 \in X$ as in $(b)$, let $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$ for all $n \in \mathbb{N}$. Condition $(b)$ means that

$$\beta((x_1, y_1), (x_0, y_0), (x_0, y_0)) \geq 1 \quad \text{and}$$
$$\beta((y_1, x_1), (y_0, x_0), (y_0, x_0)) \geq 1.$$

From hypothesis $(a)$,

$$\begin{aligned} &\beta((x_2, y_2), (x_1, y_1), (x_1, y_1)) \\ &\quad = \beta\Big((F(x_1, y_1), F(y_1, x_1)), (F(x_0, y_0), F(y_0, x_0)), \\ &\qquad\qquad (F(x_0, y_0), F(y_0, x_0))\Big) \geq 1 \quad \text{and} \\ &\beta((y_2, x_2), (y_1, x_1), (y_1, x_1)) \\ &\quad = \beta\Big((F(y_1, x_1), F(x_1, y_1)), (F(y_0, x_0), F(x_0, y_0)), \\ &\qquad\qquad (F(y_0, x_0), F(x_0, y_0))\Big) \geq 1. \end{aligned}$$

By induction, it can be proved that

$$\beta((x_{n+1}, y_{n+1}), (x_n, y_n), (x_n, y_n)) \geq 1 \quad \text{and} \tag{11.19}$$
$$\beta((y_{n+1}, x_{n+1}), (y_n, x_n), (y_n, x_n)) \geq 1. \tag{11.20}$$

Using the contractivity condition (11.18), for all $n \in \mathbb{N}$,

$$\begin{aligned} G(x_{n+2}, x_{n+1}, x_{n+1}) &= G(F(x_{n+1}, y_{n+1}), F(x_n, y_n), F(x_n, y_n)) \\ &\leq \beta((x_{n+1}, y_{n+1}), (x_n, y_n), (x_n, y_n)) \cdot \end{aligned}$$

$$\begin{aligned}&G\left(F\left(x_{n+1},y_{n+1}\right),F\left(x_n,y_n\right),F\left(x_n,y_n\right)\right)\\ &\leq\frac{1}{2}\,\psi\left(G(x_{n+1},x_n,x_n)+G(y_{n+1},y_n,y_n)\right).\end{aligned}$$

Similarly, for all $n\in\mathbb{N}$,

$$\begin{aligned}G\left(y_{n+2},y_{n+1},y_{n+1}\right)&=G\left(F\left(y_{n+1},x_{n+1}\right),F\left(y_n,x_n\right),F\left(y_n,x_n\right)\right)\\ &\leq\beta\left(\left(y_{n+1},x_{n+1}\right),\left(y_n,x_n\right),\left(y_n,x_n\right)\right)\cdot\\ &\qquad G\left(F\left(y_{n+1},x_{n+1}\right),F\left(y_n,x_n\right),F\left(y_n,x_n\right)\right)\\ &\leq\frac{1}{2}\,\psi\left(G(y_{n+1},y_n,y_n)+G(x_{n+1},x_n,x_n)\right).\end{aligned}$$

Joining the last two inequalities, for all $n\in\mathbb{N}$,

$$\begin{aligned}&G\left(x_{n+2},x_{n+1},x_{n+1}\right)+G\left(y_{n+2},y_{n+1},y_{n+1}\right)\\ &\quad\leq\psi\left(G(x_{n+1},x_n,x_n)+G(y_{n+1},y_n,y_n)\right).\end{aligned}$$

From Lemma 11.2.5, the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy in $(X,G)$. As $(X,G)$ is complete, there exists $u,v\in X$ such that $\{x_n\}\to u$ and $\{y_n\}\to v$. Finally, since $F$ is continuous, then $\{x_{n+1}=F\left(x_n,y_n\right)\}\to F\left(u,v\right)$ and $\{y_{n+1}=F\left(y_n,x_n\right)\}\to F\left(v,u\right)$, which means that $F\left(u,v\right)=u$ and $F\left(v,u\right)=v$. □

**Theorem 11.3.9.** *Let $(X,G)$ be a G-complete G-metric space and $F:X\times X\to X$ be a given mapping. Suppose there exist $\psi\in\mathcal{F}_{\mathrm{com}}^{(c)}$ and $\beta:X^2\times X^2\times X^2\to[0,\infty)$ such that*

$$\begin{aligned}&\beta((x,y),(u,v),(u,v))\,G(F(x,y),F(u,v),F(u,v))\\ &\qquad\leq\frac{1}{2}\,\psi\left(G(x,u,u)+G(y,v,v)\right),\end{aligned}\tag{11.21}$$

*for all $(x,y),(u,v)\in X\times X$. Also assume that the following conditions hold.*

$(a)$ *For all $(x,y),(u,v)\in X\times X$, we have*

$$\begin{aligned}&\beta((x,y),(u,v),(u,v))\geq 1\quad\Rightarrow\\ &\quad\beta((F(x,y),F(y,x)),(F(u,v),F(v,u)),(F(u,v),F(v,u)))\geq 1.\end{aligned}$$

$(b)$ *There exists $x_0,y_0\in X$ such that*

$$\begin{aligned}&\beta((F(x_0,y_0),F(y_0,x_0)),(x_0,y_0),(x_0,y_0))\geq 1\quad\textit{and}\\ &\beta((F(y_0,x_0),F(x_0,y_0)),(y_0,x_0),(y_0,x_0))\geq 1.\end{aligned}$$

(*c*) *If* $\{x_n\}, \{y_n\} \subseteq X$ *are sequences in* $X$*,* $G$*-convergent to* $u$ *and* $v$*, respectively, such that*

$$\beta\left(\left(x_{n+1}, y_{n+1}\right), \left(x_n, y_n\right), \left(x_n, y_n\right)\right) \geq 1 \quad \textit{and}$$
$$\beta\left(\left(y_{n+1}, x_{n+1}\right), \left(y_n, x_n\right), \left(y_n, x_n\right)\right) \geq 1 \quad \textit{for all } n \in \mathbb{N},$$

*then*

$$\beta((x_n, y_n), (u, v), (u, v)) \geq 1 \quad \textit{and} \tag{11.22}$$
$$\beta\left(\left(y_n, x_n\right), \left(v, u\right), \left(v, u\right)\right) \geq 1 \quad \textit{for all } n \in \mathbb{N}. \tag{11.23}$$

*Then* $F$ *has a coupled fixed point.*

*Proof.* Following the proof of Theorem 11.3.8, there exists $u, v \in X$ such that $\{x_n\} \to u$ and $\{y_n\} \to v$. Using (11.19)–(11.20), assumption (*c*) guarantees that (11.22)–(11.23). Applying the contractivity condition (11.21), it follows that

$$\begin{aligned} G\left(x_{n+1}, F\left(u, v\right), F\left(u, v\right)\right) &= G\left(F\left(x_n, y_n\right), F\left(u, v\right), F\left(u, v\right)\right) \\ &\leq \beta\left(\left(x_n, y_n\right), \left(u, v\right), \left(u, v\right)\right) \cdot \\ &\qquad G\left(F\left(x_n, y_n\right), F\left(u, v\right), F\left(u, v\right)\right) \\ &\leq \frac{1}{2}\, \psi(G(x_n, u, u) + G(y_n, v, v)). \end{aligned}$$

Similarly,

$$\begin{aligned} G\left(y_{n+1}, F\left(v, u\right), F\left(v, u\right)\right) &= G\left(F\left(y_n, x_n\right), F\left(v, u\right), F\left(v, u\right)\right) \\ &\leq \beta\left(\left(y_n, x_n\right), \left(v, u\right), \left(v, u\right)\right) \cdot \\ &\qquad G\left(F\left(y_n, x_n\right), F\left(v, u\right), F\left(v, u\right)\right) \\ &\leq \frac{1}{2}\, \psi(G(y_n, v, v) + G(x_n, u, u)). \end{aligned}$$

As $\psi$ is continuous at $t = 0$, letting $n \to \infty$ in the last two inequalities, we deduce that

$$F\left(u, v\right) = \lim_{n\to\infty} x_{n+1} = u \quad \text{and} \quad F\left(v, u\right) = \lim_{n\to\infty} y_{n+1} = v.$$

Therefore, $(u, v)$ is a coupled fixed point of $F$. □

**Theorem 11.3.10.** *Adding the following condition to the hypotheses of Theorem 11.3.8 (resp. Theorem 11.3.9) we obtain uniqueness of the coupled fixed point of* $F$*.*

**(d)** For all $(x,y),(u,v)\in X\times X$, , there exists $(z_1,z_2)\in X\times X$ such that

$$\beta\left((x,y),(z_1,z_2),(z_1,z_2)\right)\geq 1,\quad \beta\left((z_2,z_1),(y,x),(y,x)\right)\geq 1,$$
$$\beta\left((u,v),(z_1,z_2),(z_1,z_2)\right)\geq 1,\quad \beta\left((z_2,z_1),(v,u),(v,u)\right)\geq 1.$$

### *11.3.5 Some Coupled Coincidence Point Theorems*

In this subsection we show how to prove some coupled coincidence point theorems under a very general contractivity condition.

**Definition 11.3.2.** Let $X$ be a non-empty set endowed with a binary relation $\preccurlyeq$ and let $F:X^2\to X$ and $g:X\to X$ be two mappings. The mapping $F$ is said to have the *mixed* $(g,\preccurlyeq)$*-monotone property* if $F(x,y)$ is monotone $(g,\preccurlyeq)$-non-decreasing in $x$ and monotone $(g,\preccurlyeq)$-non-increasing in $y$, that is, for all $x,y\in X$,

$$x_1,x_2\in X,\quad gx_1\preccurlyeq gx_2\quad\Rightarrow\quad F(x_1,y)\preccurlyeq F(x_2,y)$$

and

$$y_1,y_2\in X,\quad gy_1\preccurlyeq gy_2\quad\Rightarrow\quad F(x,y_1)\succcurlyeq F(x,y_2).$$

**Lemma 11.3.2.** *Let $F:X^2\to X$ and $g:X\to X$ be two mappings such that $F(X^2)\subseteq g(X)$. Then, starting from any points $x_0,y_0\in X$, there exist two sequences $\{x_n\}$ and $\{y_n\}$ on $X$ such that*

$$gx_{n+1}=F(x_n,y_n)\quad\text{and}\quad gy_{n+1}=F(y_n,x_n)\quad\text{for all } n\in\mathbb{N}.\tag{11.24}$$

*Proof.* Let $x_0,y_0\in X$ be arbitrary. Since $F(x_0,y_0)\in F(X^2)\subseteq g(X)$, then there exists $x_1\in X$ such that $gx_1=F(x_0,y_0)$. Similarly, as $F(y_0,x_0)\in F(X^2)\subseteq g(X)$, then there exists $y_1\in X$ such that $gy_1=F(y_0,x_0)$. If we repeat the same argument using $x_1$ and $y_1$ rather than $x_0$ and $y_0$, we can find $x_2,y_2\in X^2$ such that $gx_2=F(x_1,y_1)$ and $gy_2=F(y_1,x_1)$. By induction, we may define the sequences $\{x_n\}$ and $\{y_n\}$ on $X$. □

**Definition 11.3.3.** Given two mappings $F:X^2\to X$ and $g:X\to X$, a *Picard* $(F,g)$*-sequence* is a sequence $\{(x_n,y_n)\}_{n\in\mathbb{N}}\subseteq X^2$ satisfying (11.24).

**Proposition 11.3.1.** *If $\{(x_n,y_n)\}_{n\in\mathbb{N}}\subseteq X^2$ is a Picard $(F,g)$-sequence of two mappings $F:X^2\to X$ and $g:X\to X$ and there exists $n_0\in\mathbb{N}$ such that $gx_{n_0}=gx_{n_0+1}$ and $gy_{n_0}=gy_{n_0+1}$, then $(x_{n_0},y_{n_0})$ is a coupled coincidence point of $F$ and $g$.*

*Proof.* If the exists $n_0\in\mathbb{N}$ such that $gx_{n_0}=gx_{n_0+1}$ and $gy_{n_0}=gy_{n_0+1}$, then

$$gx_{n_0}=gx_{n_0+1}=F(x_{n_0},y_{n_0})\quad\text{and}\quad gy_{n_0}=gy_{n_0+1}=F(y_{n_0},x_{n_0}),$$

so $(x_{n_0},y_{n_0})$ is a coupled coincidence point of $F$ and $g$. □

Given two mappings $F : X^2 \to X$ and $g : X \to X$, the condition $F(X^2) \subseteq g(X)$ is sufficient to guarantee that there exists a Picard $(F, g)$-sequence on $X$ based on any initial points $x_0, y_0 \in X$. However, it is not necessary.

**Lemma 11.3.3.** *Let $\preceq$ be a transitive binary relation on a set $X$ and let $F : X^2 \to X$ and $g : X \to X$ be two mappings such that the following conditions are fulfilled.*

**(i)** *There exists a Picard $(F, g)$-sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq X^2$.*
**(ii)** *$gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$.*
**(iii)** *$F$ has the mixed $(g, \preceq)$-monotone property.*

*Then $\{gx_n\}$ is $\preceq$-non-decreasing and $\{gy_n\}$ is $\preceq$-non-increasing (that is, $gx_n \preceq gx_{n+1}$ and $gy_n \succeq gy_{n+1}$ for all $n \in \mathbb{N}$).*

*Proof.* By (ii), we have that $gx_0 \preceq F(x_0, y_0) = gx_1$ and $gy_0 \succeq F(y_0, x_0) = gy_1$. Assume that there exist $n \in \mathbb{N}$ such that $gx_n \preceq gx_{n+1}$ and $gy_n \succeq gy_{n+1}$. Then, as $F$ has the mixed $(g, \preceq)$-monotone property, it follows that

$$gx_{n+1} = F(x_n, y_n) \preceq F(x_{n+1}, y_n) \preceq F(x_{n+1}, y_{n+1}) = gx_{n+2} \quad \text{and}$$
$$gy_{n+1} = F(y_n, x_n) \succeq F(y_{n+1}, x_n) \succeq F(y_{n+1}, x_{n+1}) = gy_{n+2}.$$

As $\preceq$ is transitive, then $gx_{n+1} \preceq gx_{n+2}$ and $gy_{n+1} \succeq gy_{n+2}$, and this completes the induction. □

In order to present a very general result, we introduce the following definitions.

**Definition 11.3.4.** Let $(X, G)$ be a $G^*$-metric space endowed with a binary relation $\preceq$ and let $F : X^2 \to X$ and $g : X \to X$ be two mappings. We will say that $(F, g)$ is an $(O, \preceq)$-*compatible pair* if we have that

$$\lim_{n\to\infty} G\left(gF(x_n, y_n), F(gx_n, gy_n), F(gx_n, gy_n)\right) = 0 \quad \text{and}$$
$$\lim_{n\to\infty} G\left(gF(y_n, x_n), F(gy_n, gx_n), F(gy_n, gx_n)\right) = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that $\{gx_n\}$ and $\{gy_n\}$ are $\preceq$-monotone and

$$\lim_{m\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n \in X \quad \text{and}$$
$$\lim_{m\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n \in X.$$

**Definition 11.3.5.** Let $(X, G)$ be a $G^*$-metric space and let $F : X^2 \to X$ and $g : X \to X$ be two mappings. We will say that $(F, g)$ is an *O-compatible pair* if we have that

$$\lim_{n\to\infty} G\left(gF(x_n, y_n), F(gx_n, gy_n), F(gx_n, gy_n)\right) = 0 \quad \text{and}$$

$$\lim_{n\to\infty} G\left(gF(y_n,x_n), F(gy_n,gx_n), F(gy_n,gx_n)\right) = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that

$$\lim_{m\to\infty} F(x_n,y_n) = \lim_{n\to\infty} gx_n \in X \quad \text{and}$$
$$\lim_{m\to\infty} F(y_n,x_n) = \lim_{n\to\infty} gy_n \in X.$$

*Remark 11.3.1.* If $F$ and $g$ are commuting, then $(F,g)$ is an $(O,\preceq)$-compatible pair and an $O$-compatible pair.

**Theorem 11.3.11.** *Let $(X,G)$ be a G-metric space endowed with a preorder $\preceq$ and let $F : X^2 \to X$ and $g : X \to X$ be two mappings such that $F$ has the mixed $(g,\preceq)$-monotone property. Assume that the following conditions hold.*

**(i)** *There exist two functions $\psi \in \mathcal{F}_{\mathrm{alt}}$ and $\varphi \in \mathcal{F}'_{\mathrm{alt}}$ such that*

$$\begin{aligned}&\psi\left(G\left(F(x,y), F(u,v), F(z,w)\right)\right)\\ &\qquad \le (\psi-\varphi)\Big(\max\left\{G(gx,gu,gz), G(gy,gv,gw)\right\}\Big) \qquad (11.25)\end{aligned}$$

*for all $(x,y),(u,v),(z,w) \in X^2$ for which $gx \preceq gu \preceq gz$ and $gy \succeq gv \succeq gw$.*

**(ii)** *At least, one of the following conditions holds.*

- **(ii.1)** *$F(X^2) \subseteq g(X)$ and there exists $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0,y_0)$ and $gy_0 \succeq F(y_0,x_0)$.*
- **(ii.2)** *There exists a Picard $(F,g)$-sequence $\{(x_n,y_n)\}_{n\in\mathbb{N}} \subseteq X^2$ such that $gx_0 \preceq F(x_0,y_0)$ and $gy_0 \succeq F(y_0,x_0)$.*

**(iii)** *At least, one of the following conditions holds.*

- **(iii.1)** *$(X,G)$ (or $(g(X),G)$ or $\big(F(X^2),G\big)$) is complete and $F$ and $g$ are continuous and $(O,\preceq)$-compatible.*
- **(iii.2)** *$(X,G)$ (or $(g(X),G)$ or $\big(F(X^2),G\big)$) is complete and $F$ and $g$ are continuous and commuting.*
- **(iii.3)** *$(g(X),G)$ is complete and $(X,G,\preceq)$ is regular.*

*Then $F$ and $g$ have, at least, a coupled coincidence point.*

*Proof.* By Lemmas 11.3.2 and 11.3.3, (ii.1)$\Rightarrow$(ii.2). We present the proof assuming (ii.2). From Lemma 11.3.3, $\{gx_n\}$ is $\preceq$-non-decreasing and $\{gy_n\}$ is $\preceq$-non-increasing. As $\preceq$ is transitive, we deduce that

$$gx_n \preceq gx_m \quad \text{and} \quad gy_n \succeq gy_m \quad \text{for all } n,m \in \mathbb{N} \text{ such that } n \le m.$$

By the contractivity condition (11.25), for all $n \in \mathbb{N}$ we have, taking into account that $gx_n \preceq gx_{n+1} \preceq gx_{n+1}$ and $gy_n \succeq gy_{n+1} \succeq gy_{n+1}$,

$$\begin{aligned}
&\psi\left(G\left(gx_{n+1}, gx_{n+2}, gx_{n+2}\right)\right) \\
&\quad= \psi\left(G\left(F\left(x_n, y_n\right), F\left(x_{n+1}, y_{n+1}\right), F\left(x_{n+1}, y_{n+1}\right)\right)\right) \\
&\quad\leq (\psi - \varphi)\left(\max\left\{G\left(gx_n, gx_{n+1}, gx_{n+1}\right), G\left(gy_n, gy_{n+1}, gy_{n+1}\right)\right\}\right),
\end{aligned}$$

and using that $gy_{n+1} \preceq gy_{n+1} \preceq gy_n$ and $gx_{n+1} \succeq gx_{n+1} \succeq gx_n$, we deduce that

$$\begin{aligned}
&\psi\left(G\left(gy_{n+2}, gy_{n+2}, gy_{n+1}\right)\right) \\
&\quad= \psi\left(G\left(F\left(y_{n+1}, x_{n+1}\right), F\left(y_{n+1}, x_{n+1}\right), F\left(y_n, x_n\right)\right)\right) \\
&\quad\leq (\psi - \varphi)\left(\max\left\{G\left(gy_{n+1}, gy_{n+1}, gy_n\right), G\left(gx_{n+1}, gx_{n+1}, gx_n\right)\right\}\right).
\end{aligned}$$

As $\psi$ is non-decreasing,

$$\begin{aligned}
&\psi\left(\max\left\{G\left(gx_{n+1}, gx_{n+2}, gx_{n+2}\right), G\left(gy_{n+1}, gy_{n+2}, gy_{n+2}\right)\right\}\right) \\
&\quad= \max\left\{\psi\left(G\left(gx_{n+1}, gx_{n+2}, gx_{n+2}\right)\right), \psi\left(G\left(gy_{n+1}, gy_{n+2}, gy_{n+2}\right)\right)\right\} \\
&\quad\leq (\psi - \varphi)\left(\max\left\{G\left(gx_n, gx_{n+1}, gx_{n+1}\right), G\left(gy_n, gy_{n+1}, gy_{n+1}\right)\right\}\right).
\end{aligned}$$

Applying Lemma 2.3.6 to the sequence

$$\{a_n = \max\left\{G\left(gx_n, gx_{n+1}, gx_{n+1}\right), G\left(gy_n, gy_{n+1}, gy_{n+1}\right)\right\}\},$$

we deduce that $\{a_n\} \to 0$ and, in particular,

$$\lim_{n\to\infty} G\left(gx_n, gx_{n+1}, gx_{n+1}\right) = \lim_{n\to\infty} G\left(gy_n, gy_{n+1}, gy_{n+1}\right) = 0.$$

Next, we show that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences on $(X, G)$. We reason by contradiction assuming that some of them is not Cauchy in $(X, G)$. In such a case, by Lemma 11.2.1, there exist $\varepsilon_0 > 0$ and two sequences of natural numbers $\{n(k)\}_{k\in\mathbb{N}}$ and $\{m(k)\}_{k\in\mathbb{N}}$ such that

$$\begin{aligned}
&k \leq n(k) < m(k) < n(k+1), \\
&\max_{1\leq i\leq N}\left\{G\left(gx_{n(k)}, gx_{m(k)-1}, gx_{m(k)-1}\right), G\left(gy_{n(k)}, gy_{m(k)-1}, gy_{m(k)-1}\right)\right\} \\
&\qquad\leq \varepsilon_0 < \max_{1\leq i\leq N}\left\{G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right), G\left(gy_{n(k)}, gy_{m(k)}, gy_{m(k)}\right)\right\}, \\
&\lim_{k\to\infty}\left[\max_{1\leq i\leq N}\left\{G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right), G\left(gy_{n(k)}, gy_{m(k)}, gy_{m(k)}\right)\right\}\right]
\end{aligned}$$

$$= \lim_{k\to\infty}\left[\max_{1\le i\le N}\left\{G\left(gx_{n(k)},gx_{m(k)-1},gx_{m(k)-1}\right),\right.\right.$$
$$\left.\left.G\left(gy_{n(k)},gy_{m(k)-1},gy_{m(k)-1}\right)\right\}\right] = \varepsilon_0.$$

Moreover, at least one of the following conditions holds:

$$\lim_{k\to\infty} G\left(gx_{n(k)},gx_{m(k)},gx_{m(k)}\right) = \lim_{k\to\infty} G\left(gx_{n(k)},gx_{m(k)-1},gx_{m(k)-1}\right) = \varepsilon_0,$$
$$\lim_{k\to\infty} G\left(gy_{n(k)},gy_{m(k)},gy_{m(k)}\right) = \lim_{k\to\infty} G\left(gy_{n(k)},gy_{m(k)-1},gy_{m(k)-1}\right) = \varepsilon_0.$$

Since $n(k) < m(k)$, we have that $gx_{n(k)-1} \preceq gx_{m(k)-1} \preceq gx_{m(k)-1}$ and $gy_{n(k)-1} \succeq gy_{m(k)-1} \succeq gy_{m(k)-1}$ for all $k \in \mathbb{N}$. By the contractivity condition (11.25),

$$\psi\left(G\left(gx_{n(k)},gx_{m(k)},gx_{m(k)}\right)\right)$$
$$= \psi\left(G\left(F\left(x_{n(k)-1},y_{n(k)-1}\right),F\left(x_{m(k)-1},y_{m(k)-1}\right),\right.\right.$$
$$\left.\left.F\left(x_{m(k)-1},y_{m(k)-1}\right)\right)\right)$$
$$\le (\psi-\varphi)\left(\max\left\{G\left(gx_{n(k)-1},gx_{m(k)-1},gx_{m(k)-1}\right),\right\}\right.$$
$$\left.G\left(gy_{n(k)-1},gy_{m(k)-1},gy_{m(k)-1}\right)\}\right).$$

Furthermore, as $gy_{m(k)-1} \preceq gy_{m(k)-1} \preceq gy_{n(k)-1}$ and $gx_{m(k)-1} \succeq gx_{m(k)-1} \succeq gx_{n(k)-1}$, then

$$\psi\left(G\left(gy_{m(k)},gy_{m(k)},gy_{n(k)}\right)\right)$$
$$= \psi\left(G\left(F\left(y_{m(k)-1},x_{m(k)-1}\right),F\left(y_{m(k)-1},x_{m(k)-1}\right),\right.\right.$$
$$\left.\left.F\left(y_{n(k)-1},x_{n(k)-1}\right)\right)\right)$$
$$\le (\psi-\varphi)\left(\max\left\{G\left(gy_{m(k)-1},gy_{m(k)-1},gy_{n(k)-1}\right),\right.\right.$$
$$\left.\left.G\left(gx_{m(k)-1},gx_{m(k)-1},gx_{n(k)-1}\right)\right\}\right).$$

Combining the last two inequalities and taking into account that $\psi$ is nondecreasing, if follows that

$$\psi\left(\max\left\{G\left(gx_{n(k)},gx_{m(k)},gx_{m(k)}\right),G\left(gy_{n(k)},gy_{m(k)},gy_{m(k)}\right)\right\}\right)$$
$$= \max\left\{\psi\left(G\left(gx_{n(k)},gx_{m(k)},gx_{m(k)}\right)\right),\right.$$

$$\psi\left(G\left(gy_{m(k)},gy_{m(k)},gy_{n(k)}\right)\right)\Big\}$$
$$\leq(\psi-\varphi)\Big(\max\left\{G\left(gx_{n(k)-1},gx_{m(k)-1},gx_{m(k)-1}\right),\right.$$
$$\left.G\left(gy_{n(k)-1},gy_{m(k)-1},gy_{m(k)-1}\right)\right\}\Big).$$

Applying Lemma 2.3.5 to the sequences

$$\{t_k=\max\left\{G\left(gx_{n(k)},gx_{m(k)},gx_{m(k)}\right),G\left(gy_{n(k)},gy_{m(k)},gy_{m(k)}\right)\right\}\}_{k\in\mathbb{N}}$$

and

$$\Big\{s_k=\max\Big\{G\left(gx_{n(k)-1},gx_{m(k)-1},gx_{m(k)-1}\right),$$
$$G\left(gy_{n(k)-1},gy_{m(k)-1},gy_{m(k)-1}\right)\Big\}\Big\}_{k\in\mathbb{N}},$$

we conclude that $\varepsilon_0=0$, which is a contradiction. As a consequence, $\{gx_n\}$ and $\{gy_n\}$ must be Cauchy sequences on $(X,G)$. To continue the proof, we distinguish some cases.

*Case (iii.1). Assume that* $(X,G)$ *(or* $(g(X),G)$ *or* $\left(F(X^2),G\right)$*) is complete and* $F$ *and* $g$ *are continuous and* $(O,\preceq)$*-compatible.* In such a case, there exist $z,\omega\in X$ such that $\{gx_n\}\to z$ and $\{gy_n\}\to\omega$. As $F$ and $g$ are continuous, we deduce that

$$\{ggx_n\}\to gz,\quad\{ggy_n\}\to g\omega,$$
$$\{F\left(gx_n,gy_n\right)\}\to F(z,\omega),\quad\{F\left(gy_n,gx_n\right)\}\to F(\omega,z),$$

Therefore, $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that $\{gx_n\}$ and $\{gy_n\}$ are $\preceq$-monotone and

$$\lim_{m\to\infty}F(x_n,y_n)=\lim_{n\to\infty}gx_n=z\in X\quad\text{and}$$
$$\lim_{m\to\infty}F(y_n,x_n)=\lim_{n\to\infty}gy_n=\omega\in X.$$

Since $F$ and $g$ are compatible, we have that

$$\lim_{n\to\infty}G\left(gF(x_n,y_n),F(gx_n,gy_n),F(gx_n,gy_n)\right)=0\quad\text{and}$$
$$\lim_{n\to\infty}G\left(gF(y_n,x_n),F(gy_n,gx_n),F(gy_n,gx_n)\right)=0.$$

In particular

$$G\left(gz,F\left(z,\omega\right),F\left(z,\omega\right)\right)=\lim_{n\to\infty}G\left(ggx_n,F(gx_n,gy_n),F(gx_n,gy_n)\right)$$
$$=\lim_{n\to\infty}G\left(gF(x_n,y_n),F(gx_n,gy_n),F(gx_n,gy_n)\right)=0$$

and

$$G\left(g\omega,F\left(\omega,z\right),F\left(\omega,z\right)\right)=\lim_{n\to\infty}G\left(ggy_n,F(gy_n,gx_n),F(gy_n,gx_n)\right)$$
$$=\lim_{n\to\infty}G\left(gF(y_n,x_n),F(gy_n,gx_n),F(gy_n,gx_n)\right)=0.$$

Hence, $gz=F\left(z,\omega\right)$ and $g\omega=F\left(\omega,z\right)$, so $(z,\omega)$ is a coupled coincidence point of $F$ and $g$.

*Case (iii.2). Assume that $(X,G)$ (or $(g(X),G)$ or $\left(F(X^2),G\right)$) is complete and $F$ and $g$ are continuous and commuting.* It follows from item (iii.1) because if $F$ and $g$ are commuting, then they are also $(O,\preceq)$-compatible.

*Case (iii.3). Assume that $(g(X),G)$ is complete and $(X,G,\preceq)$ is regular.* Since $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences on $(g(X),G)$, there exist $z,\omega\in g(X)$ such that $\{gx_n\}\to z$ and $\{gy_n\}\to\omega$. Let $u,v\in X$ be arbitrary points such that $gu=z$ and $gv=\omega$. As $(X,G,\preceq)$ is regular, we deduce that $gx_n\preceq gu$ and $gy_n\succeq gv$ for all $n\in\mathbb{N}$. Therefore, applying the contractivity condition (11.25) to $gx_n\preceq gu\preceq gu$ and $gy_n\succeq gv\succeq gv$, we obtain

$$\psi\left(G\left(gx_{n+1},F\left(u,v\right),F\left(u,v\right)\right)\right)=\psi\left(G\left(F\left(x_n,y_n\right),F\left(u,v\right),F\left(u,v\right)\right)\right)$$
$$\leq\left(\psi-\varphi\right)\left(\max\left\{G\left(gx_n,gu,gu\right),G\left(gy_n,gv,gv\right)\right\}\right).$$

As $\{gx_n\}\to z=gu$ and $\{gy_n\}\to\omega=gv$, then

$$\lim_{n\to\infty}\psi\left(G\left(gx_{n+1},F\left(u,v\right),F\left(u,v\right)\right)\right)=0.$$

Since $\psi\in F_{\text{alt}}$, Lemma 2.3.3 shows that

$$G\left(gu,F\left(u,v\right),F\left(u,v\right)\right)=\lim_{n\to\infty}G\left(gx_{n+1},F\left(u,v\right),F\left(u,v\right)\right)=0.$$

Therefore, $gu=F\left(u,v\right)$. Similarly, applying the contractivity condition (11.25) to $gv\preceq gv\preceq gy_n$ and $gu\succeq gu\succeq gx_n$, we obtain

$$\psi\left(G\left(F\left(v,u\right),F\left(v,u\right),gy_{n+1}\right)\right)=\psi\left(G\left(F\left(v,u\right),F\left(v,u\right),F\left(y_n,x_n\right)\right)\right)$$
$$\leq\left(\psi-\varphi\right)\left(\max\left\{G\left(gv,gv,gy_n\right),G\left(gu,gu,gx_n\right)\right\}\right).$$

As $\{gx_n\}\to z=gu$ and $\{gy_n\}\to\omega=gv$, then

$$\lim_{n\to\infty}\psi\left(G\left(F\left(v,u\right),F\left(v,u\right),gy_{n+1}\right)\right)=0.$$

Since $\psi\in F_{\text{alt}}$, Lemma 2.3.3 shows that

$$G\left(gv,F\left(v,u\right),F\left(v,u\right)\right)=\lim_{n\to\infty}G\left(F\left(v,u\right),F\left(v,u\right),gy_{n+1}\right)=0.$$

Therefore, $gv=F\left(v,u\right)$, and we conclude that $(u,v)$ is a coupled coincidence point of $F$ and $g$. □

Theorem 11.3.11 can be particularized in a wide range of different ways. For example, assume that $\preceq$ is the preorder on $X$ given by "$x \preceq y$ for all $x, y \in X$". Then, obviously, $F$ has the mixed $(g, \preceq)$-monotone property and $(X, G, \preceq)$ is regular.

**Corollary 11.3.3.** *Let $(X, G)$ be a $G$-metric space and let $F : X^2 \to X$ and $g : X \to X$ be two mappings. Assume that the following conditions hold.*

**(i)** *There exist two functions $\psi \in \mathcal{F}_{\text{alt}}$ and $\varphi \in \mathcal{F}'_{\text{alt}}$ such that, for all $(x, y), (u, v), (z, w) \in X^2$,*

$$\begin{aligned}&\psi\left(G\left(F\left(x, y\right), F\left(u, v\right), F\left(z, w\right)\right)\right)\\&\qquad\leq(\psi-\varphi)\Big(\max\left\{G\left(gx, gu, gz\right), G\left(gy, gv, gw\right)\right\}\Big).\end{aligned}$$

**(ii)** *At least, one of the following conditions holds.*

- **(ii.1)** $F(X^2) \subseteq g(X)$.
- **(ii.2)** *There exists in $X$ a Picard $(T, g)$-sequence.*

**(iii)** *At least, one of the following conditions holds.*

- **(iii.1)** $(X, G)$ *(or* $(g(X), G)$ *or* $\big(F(X^2), G\big)$*) is complete and $F$ and $g$ are continuous and $O$-compatible.*
- **(iii.2)** $(X, G)$ *(or* $(g(X), G)$ *or* $\big(F(X^2), G\big)$*) is complete and $F$ and $g$ are continuous and commuting.*
- **(iii.3)** $(g(X), G)$ *is complete.*

*Then $F$ and $g$ have, at least, a coupled coincidence point.*

Another way to particularize Theorem 11.3.11 occurs when $\preceq$ is a partial order. We do not include here such a statement because it is similar to Theorem 11.3.11, replacing the preorder $\preceq$ by a partial order $\preccurlyeq$. It is interesting to consider that case when $\psi$ is the identity mapping on $[0, \infty)$.

**Corollary 11.3.4.** *Let $(X, G)$ be a $G$-metric space endowed with a preorder $\preceq$ and let $F : X^2 \to X$ and $g : X \to X$ be two mappings such that $F$ has the mixed $(g, \preceq)$-monotone property. Assume that the following conditions hold.*

**(i)** *There exists a function $\varphi \in \mathcal{F}'_{\text{alt}}$ such that*

$$\begin{aligned}G\left(F\left(x, y\right), F\left(u, v\right), F\left(z, w\right)\right)&\leq\max\left\{G\left(gx, gu, gz\right), G\left(gy, gv, gw\right)\right\}\\&\qquad-\varphi\Big(\max\left\{G\left(gx, gu, gz\right), G\left(gy, gv, gw\right)\right\}\Big)\end{aligned}$$

*for all $(x, y), (u, v), (z, w) \in X^2$ for which $gx \preceq gu \preceq gz$ and $gy \succeq gv \succeq gw$.*

**(ii)** *At least, one of the following conditions holds.*

- **(ii.1)** $F(X^2) \subseteq g(X)$ *and there exists $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$.*

**(ii.2)** *There exists a Picard* $(F, g)$*-sequence* $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq X^2$ *such that* $gx_0 \preceq F(x_0, y_0)$ *and* $gy_0 \succeq F(y_0, x_0)$.

**(iii)** *At least, one of the following conditions holds.*

**(iii.1)** $(X, G)$ *(or* $(g(X), G)$ *or* $\big(F(X^2), G\big)$*) is complete and* $F$ *and* $g$ *are continuous and* $(O, \preceq)$*-compatible.*
**(iii.2)** $(X, G)$ *(or* $(g(X), G)$ *or* $\big(F(X^2), G\big)$*) is complete and* $F$ *and* $g$ *are continuous and commuting.*
**(iii.3)** $(g(X), G)$ *is complete and* $(X, G, \preceq)$ *is regular.*

*Then* $F$ *and* $g$ *have, at least, a coupled coincidence point.*

If we take $\varphi(t) = (1-\lambda)t$ for all $t \in [0, \infty)$, where $\lambda \in [0, 1)$, we deduce the following statement.

**Corollary 11.3.5.** *Let* $(X, G)$ *be a G-metric space endowed with a preorder* $\preceq$ *and let* $F : X^2 \to X$ *and* $g : X \to X$ *be two mappings such that* $F$ *has the mixed* $(g, \preceq)$*-monotone property. Assume that the following conditions hold.*

**(i)** *There exists a constant* $\lambda \in [0, 1)$ *such that*

$$G(F(x,y), F(u,v), F(z,w)) \\ \leq \lambda \max\{G(gx, gu, gz), G(gy, gv, gw)\}$$

*for all* $(x, y), (u, v), (z, w) \in X^2$ *for which* $gx \preceq gu \preceq gz$ *and* $gy \succeq gv \succeq gw$.
**(ii)** *At least, one of the following conditions holds.*

**(ii.1)** $F(X^2) \subseteq g(X)$ *and there exists* $x_0, y_0 \in X$ *such that* $gx_0 \preceq F(x_0, y_0)$ *and* $gy_0 \succeq F(y_0, x_0)$.
**(ii.2)** *There exists a Picard* $(F, g)$*-sequence* $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq X^2$ *such that* $gx_0 \preceq F(x_0, y_0)$ *and* $gy_0 \succeq F(y_0, x_0)$.

**(iii)** *At least, one of the following conditions holds.*

**(iii.1)** $(X, G)$ *(or* $(g(X), G)$ *or* $\big(F(X^2), G\big)$*) is complete and* $F$ *and* $g$ *are continuous and* $(O, \preceq)$*-compatible.*
**(iii.2)** $(X, G)$ *(or* $(g(X), G)$ *or* $\big(F(X^2), G\big)$*) is complete and* $F$ *and* $g$ *are continuous and commuting.*
**(iii.3)** $(g(X), G)$ *is complete and* $(X, G, \preceq)$ *is regular.*

*Then* $F$ *and* $g$ *have, at least, a coupled coincidence point.*

Finally, we particularize Theorem 11.3.11 to the case in which $g$ is the identity mapping on $X$.

**Theorem 11.3.12.** *Let* $(X, G)$ *be a complete G-metric space endowed with a preorder* $\preceq$ *and let* $F : X^2 \to X$ *be a* $\preceq$*-non-decreasing mapping. Assume that the following conditions hold.*

**(i)** *There exist two functions* $\psi \in \mathcal{F}_{\text{alt}}$ *and* $\varphi \in \mathcal{F}'_{\text{alt}}$ *such that*

$$\begin{aligned}\psi\left(G\left(F\left(x,y\right),F\left(u,v\right),F\left(z,w\right)\right)\right)\\ \leq\left(\psi-\varphi\right)\left(\max\left\{G\left(x,u,z\right),G\left(y,v,w\right)\right\}\right)\end{aligned}$$

*for all* $(x,y),(u,v),(z,w)\in X^2$ *for which* $x\preceq u\preceq z$ *and* $y\succeq v\succeq w$.

**(ii)** *There exists* $x_0,y_0\in X$ *such that* $x_0\preceq F(x_0,y_0)$ *and* $y_0\succeq F(y_0,x_0)$.

**(iii)** *At least, one of the following conditions holds.*

**(iii.1)** *$F$ is continuous.*

**(iii.2)** $(X,G,\preceq)$ *is regular.*

*Then $F$ has, at least, a coupled fixed point.*

Now we present some comments concerning a recent coupled fixed point theorem. In [140], Mohiuddine and Alotaibi announced a coupled fixed point theorem using an ordered $G$-metric space $(X,G,\preccurlyeq)$ and a contractivity condition as follows: there exist two functions $\psi\in\mathcal{F}_{\text{alt}}$ and $\varphi\in\mathcal{F}'_{\text{alt}}$ such that $\psi$ is subadditive and

$$\begin{aligned}&\psi\left(G\left(F\left(x,y\right),F\left(u,v\right),F\left(s,t\right)\right)\right)\\ &\quad\leq\frac{\psi\left(G\left(x,u,s\right)+G\left(y,v,t\right)\right)}{2}-\varphi\left(\frac{G\left(x,u,s\right)+G\left(y,v,t\right)}{2}\right)\end{aligned}$$

for all $x,y,u,v,s,t\in X$ with $x\succcurlyeq u\succcurlyeq s$ and $y\preccurlyeq v\preccurlyeq t$ where either $u\neq s$ or $v\neq t$. In this case, all comments given in Sect. 11.3.2 can now be repeated to show that their proof is not correct.

## 11.3.6 Aydi et al's Coupled Coincidence Point Theorems

In [30], Aydi et al. introduced a version of the following Ćirić-type result. In fact, they assumed that the function $\varphi\in\mathcal{F}_{\text{Cir}}$ also satisfied $\varphi^{-1}\left(\{0\}\right)=\{0\}$, but it was not necessary.

**Theorem 11.3.13.** *Let* $(X,\preceq)$ *be a preordered set and let $G$ be a $G$-metric on $X$. Let* $F:X^2\to X$ *and* $g:X\to X$ *be two mappings such that* $F(X^2)\subseteq g(X)$, *$F$ has the mixed* $(g,\preceq)$*-monotone property and $g$ is continuous and commutes with $F$. Suppose that there exist* $\varphi\in\mathcal{F}_{\text{Cir}}$ *such that*

$$G(F(x,y),F(u,v),F(w,z))\leq\varphi\left(\frac{G(gx,gu,gw)+G(gy,gv,gz)}{2}\right)\qquad(11.26)$$

*for all $x, y, u, v, w, z \in X$ with $gx \preceq gu \preceq gw$ and $gy \succeq gv \succeq gz$. Also assume that, at least, one of the following conditions holds.*

**(a)** *$F$ is $G$-continuous and $X$ (or $F(X^2)$ or $g(X)$) is $G$-complete.*
**(b)** *$\varphi(0) = 0$, $(g(X), G)$ is $G$-complete and $(X, G, \preceq)$ is regular.*

*If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then $F$ and $g$ have a coupled coincidence point.*

The previous result can be improved using weaker hypotheses as in Theorem 11.3.11. Furthermore, by the symmetry of $G$, (11.26) also holds if $gx \succeq gu \succeq gw$ and $gy \preceq gv \preceq gz$.

*Proof.* Define $\varphi' : [0, \infty) \to [0, \infty)$ by $\varphi'(t) = 2\varphi(t/2)$ for all $t \in [0, \infty)$. Clearly, $\varphi' \in \mathcal{F}_{\mathrm{Cir}}$. From Lemmas 11.3.2 and 11.3.3, there exists a Picard sequence $\{(x_n, y_n)\}$ of $(T, g)$ such that $\{gx_n\}$ is $\preceq$-non-decreasing and $\{gy_n\}$ is $\preceq$-non-increasing. As $\preceq$ is transitive, we deduce that

$$gx_n \preceq gx_m \quad \text{and} \quad gy_n \succeq gy_m \quad \text{for all } n, m \in \mathbb{N} \text{ such that } n \leq m. \tag{11.27}$$

If there exists some $n_0 \in \mathbb{N}$ such that $(gx_{n_0}, gy_{n_0}) = (gx_{n_0+1}, gy_{n_0+1})$, then $(x_{n_0}, y_{n_0})$ is a coupled coincidence point of $T$ and $g$, and the proof is finished. On the contrary case, assume that $(gx_n, gy_n) \neq (gx_{n+1}, gy_{n+1})$, that is, for all $n \in \mathbb{N}$,

$$G(gx_n, gx_{n+1}, gx_{n+1}) + G(gy_n, gy_{n+1}, gy_{n+1}) > 0.$$

By the contractivity condition (11.26), for all $n \in \mathbb{N}$ we have, taking into account that $gx_n \preceq gx_{n+1} \preceq gx_{n+1}$ and $gy_n \succeq gy_{n+1} \succeq gy_{n+1}$,

$$\begin{aligned} G(gx_{n+1}, gx_{n+2}, gx_{n+2}) &= G(F(x_n, y_n), F(x_{n+1}, y_{n+1}), F(x_{n+1}, y_{n+1})) \\ &\leq \varphi\left(\frac{G(gx_n, gx_{n+1}, gx_{n+1}) + G(gy_n, gy_{n+1}, gy_{n+1})}{2}\right) \\ &= \frac{1}{2}\varphi'(G(gx_n, gx_{n+1}, gx_{n+1}) + G(gy_n, gy_{n+1}, gy_{n+1})) \end{aligned}$$

and also

$$\begin{aligned} G(gy_{n+1}, gy_{n+2}, gy_{n+2}) &= G(F(y_{n+1}, x_{n+1}), F(y_{n+1}, x_{n+1}), F(y_n, x_n)) \\ &\leq \varphi\left(\frac{G(gy_{n+1}, gy_{n+1}, gy_n) + G(gx_{n+1}, gx_{n+1}, gx_n)}{2}\right) \\ &= \frac{1}{2}\varphi'(G(gx_n, gx_{n+1}, gx_{n+1}) + G(gy_n, gy_{n+1}, gy_{n+1})). \end{aligned}$$

Joining the last two inequalities, for all $n \in \mathbb{N}$,

$$\begin{aligned}&G\left(gx_{n+1}, gx_{n+2}, gx_{n+2}\right) + G\left(gy_{n+1}, gy_{n+2}, gy_{n+2}\right)\\&\qquad \leq \varphi'\left(G(gx_n, gx_{n+1}, gx_{n+1}) + G(gy_n, gy_{n+1}, gy_{n+1})\right).\end{aligned}$$

From item 3 of Lemma 2.3.11,

$$\lim_{n\to\infty}\left[G(gx_n, gx_{n+1}, gx_{n+1}) + G(gy_n, gy_{n+1}, gy_{n+1})\right] = 0,$$

so $\{gx_n\}$ and $\{gy_n\}$ are asymptotically regular sequences. To show that they are Cauchy, assume that, at least, one of them is not a Cauchy sequence. In such a case, Lemma 11.2.2 assures that there exist $\varepsilon_0 > 0$ and two sequences of natural numbers $\{n(k)\}_{k\in\mathbb{N}}$ and $\{m(k)\}_{k\in\mathbb{N}}$ such that, for all $k \in \mathbb{N}$,

$$\begin{aligned}&k \leq n(k) < m(k) < n(k+1),\\&G(gx_{n(k)}, gx_{m(k)-1}, gx_{m(k)-1}) + G(gy_{n(k)}, gy_{m(k)-1}, gy_{m(k)-1}) \leq \varepsilon_0\\&\qquad < G(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{m(k)}, gy_{m(k)}),\end{aligned}$$

and also

$$\begin{aligned}&\lim_{k\to\infty}\left[G(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}) + G(gy_{n(k)-1}, gy_{m(k)-1}, gy_{m(k)-1})\right]\\&\quad = \lim_{k\to\infty}\left[G(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{m(k)}, gy_{m(k)})\right] = \varepsilon_0.\end{aligned}$$

Using (11.27), $gx_{n(k)-1} \preceq gx_{m(k)-1}$ and $gy_{n(k)-1} \succeq gy_{m(k)-1}$ for all $k \in \mathbb{N}$. Therefore, the contractivity condition (11.26) yields

$$\begin{aligned}&G(gx_{n(k)}, gx_{m(k)}, gx_{m(k)})\\&\quad = G(F\left(x_{n(k)-1}, y_{n(k)-1}\right), F\left(x_{m(k)-1}, y_{m(k)-1}\right), F\left(x_{m(k)-1}, y_{m(k)-1}\right))\\&\quad \leq \varphi\left(\frac{1}{2}\left(G(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1})\right.\right.\\&\qquad\qquad \left.\left.+G(gy_{n(k)-1}, gy_{m(k)-1}, gy_{m(k)-1})\right)\right)\\&\quad = \frac{1}{2}\varphi'\left(G(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1})\right.\\&\qquad\qquad \left.+G(gy_{n(k)-1}, gy_{m(k)-1}, gy_{m(k)-1})\right).\end{aligned}$$

In a similar way,

$$\begin{aligned}&G(gy_{n(k)}, gy_{m(k)}, gy_{m(k)})\\&\quad = G\left(F\left(y_{m(k)-1}, x_{m(k)-1}\right), F\left(y_{m(k)-1}, x_{m(k)-1}\right),\right.\end{aligned}$$

$$\begin{aligned}
&F\left(y_{n(k)-1}, x_{n(k)-1}\right)\right) \\
&\leq \varphi\left(\frac{1}{2}\left(G(gy_{m(k)-1}, gy_{m(k)-1}, gy_{n(k)-1}\right)\right. \\
&\qquad \left.\left.+G(gx_{m(k)-1}, gx_{m(k)-1}, gx_{n(k)-1})\right)\right) \\
&= \frac{1}{2}\,\varphi'\left(G(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}\right) \\
&\qquad \left.+G(gy_{n(k)-1}, gy_{m(k)-1}, gy_{m(k)-1})\right).
\end{aligned}$$

Combining the last two inequalities, we deduce that, for all $k \in \mathbb{N}$,

$$\begin{aligned}
\varepsilon_0 &< G(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{m(k)}, gy_{m(k)}) \\
&\leq \varphi'\left(G(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}\right) \\
&\qquad \left.+G(gy_{n(k)-1}, gy_{m(k)-1}, gy_{m(k)-1})\right).
\end{aligned}$$

Applying Lemma 2.3.14 to the sequences

$$\begin{aligned}
&\left\{t_k = G(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{m(k)}, gy_{m(k)})\right\} \quad \text{and} \\
&\left\{s_k = G(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}) + G(gy_{n(k)-1}, gy_{m(k)-1}, gy_{m(k)-1})\right\},
\end{aligned}$$

that converge to $L = \varepsilon_0$, we conclude that $\varepsilon_0 = 0$, which is a contradiction. Therefore, $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $(X, G)$. Since these sequences are included in $F(X^2)$, in $g(X)$ and in $X$ and, at least, one of them is $G$-complete, there exists $u, v \in X$ such that $\{gx_n\} \to u$ and $\{gy_n\} \to v$. Moreover, as $g$ is continuous, $\{ggx_n\} \to gu$ and $\{ggy_n\} \to gv$. Notice that as $T$ and $g$ commute,

$$\begin{aligned}
&\{F(gx_n, gy_n)\} = \{gF(x_n, y_n)\} = \{ggx_{n+1}\} \to gu \quad \text{and} \\
&\{F(gy_n, gx_n)\} = \{gF(y_n, x_n)\} = \{ggy_{n+1}\} \to gv.
\end{aligned}$$

Next, we distinguish two cases.

*Case 1.* *$F$ is $G$-continuous and $X$ (or $F(X^2)$ or $g(X)$) is $G$-complete.* In this case, letting $n \to \infty$, we observe that

$$\begin{aligned}
gu &= \lim_{n\to\infty} F(gx_n, gy_n) = F(u, v) \quad \text{and} \\
gv &= \lim_{n\to\infty} F(gy_n, gx_n) = F(v, u),
\end{aligned}$$

so $(u, v)$ is a coupled coincidence point of $F$ and $g$.

*Case 2.* $\varphi(0) = 0$, *$(g(X), G)$ is $G$-complete and $(X, G, \preceq)$ is regular.* In this case, as $g(X)$ is complete, then $u, v \in g(X)$. Let $z, w \in X$ be arbitrary points such that

$gz = u$ and $gw = v$. Taking into account that $\{gx_n\}$ is $\preceq$-non-decreasing, $\{gy_n\}$ is $\preceq$-non-increasing and they are convergent to $gz = u$ and $gw = v$, respectively, we deduce that

$$gx_n \preceq gz \quad \text{and} \quad gy_n \succeq gw \quad \text{for all } n \in \mathbb{N}.$$

By the contractivity condition (11.26),

$$\begin{aligned}
G\left(gx_{n+1}, F\left(z,w\right), F\left(z,w\right)\right) &= G\left(F\left(x_n,y_n\right), F\left(z,w\right), F\left(z,w\right)\right) \\
&\leq \varphi\left(\frac{G(gx_n,gz,gz) + G(gy_n,gw,gw)}{2}\right) \\
&= \frac{1}{2}\,\varphi'\left(G(gx_n,gz,gz) + G(gy_n,gw,gw)\right)
\end{aligned}$$

and, similarly,

$$\begin{aligned}
G\left(gy_{n+1}, F\left(w,z\right), F\left(w,z\right)\right) &= G\left(F\left(w,z\right), F\left(w,z\right), F\left(y_n,x_n\right)\right) \\
&\leq \varphi\left(\frac{G(gw,gw,gy_n) + G(gz,gz,gx_n)}{2}\right) \\
&= \frac{1}{2}\,\varphi'\left(G(gx_n,gz,gz) + G(gy_n,gw,gw)\right).
\end{aligned}$$

Since $\varphi' \in \mathcal{F}_{\mathrm{Cir}}$ and $\varphi'(0) = 2\varphi(0) = 0$, letting $n \to \infty$ in the previous inequalities and applying item 8 of Lemma 2.3.11, we conclude that $gz = u = \lim_{n\to\infty} gx_{n+1} = F(z,w)$ and $gw = v = \lim_{n\to\infty} gy_{n+1} = F(w,z)$, that is, $(z,w)$ is a coupled coincidence point of $F$ and $g$.

□

**Theorem 11.3.14.** *Under the hypothesis of Theorem 11.3.13, also assume that* $\varphi(0) = 0$ *and that the following condition holds.*

$(U_2)$ *For all* $(x,y), (x',y') \in \mathrm{Co}(F,g)$, *there exists* $(u,v) \in X^2$ *such that, at least, one of the following properties is satisfied.*

- $F(x,y) \preceq F(u,v)$, $F(y,x) \succeq F(v,u)$, $F(x',y') \preceq F(u,v)$ *and* $F(y',x') \succeq F(v,u)$.
- $F(x,y) \preceq F(u,v)$, $F(y,x) \succeq F(v,u)$, $F(x',y') \succeq F(u,v)$ *and* $F(y',x') \preceq F(v,u)$.
- $F(x,y) \succeq F(u,v)$, $F(y,x) \preceq F(v,u)$, $F(x',y') \preceq F(u,v)$ *and* $F(y',x') \succeq F(v,u)$.
- $F(x,y) \succeq F(u,v)$, $F(y,x) \preceq F(v,u)$, $F(x',y') \succeq F(u,v)$ *and* $F(y',x') \preceq F(v,u)$.

*Then $F$ and $g$ have a unique common coupled fixed point $(z,\omega)$ (that is, a point satisfying $z = gz = F(z,\omega)$ and $\omega = g\omega = F(\omega,z)$). In fact, if $(x,y)$ is an arbitrary coupled coincidence point of $F$ and $g$, then $z = gx$ and $\omega = gy$.*

*Proof.* First of all, we claim that

$$gx = gx' \quad \text{and} \quad gy = gy' \quad \text{for all } (x,y), (x',y') \in \operatorname{Co}(F,g). \tag{11.28}$$

Indeed, let $(x,y), (x',y') \in \operatorname{Co}(F,g)$ be arbitrary coupled coincidence points of $F$ and $g$ and let $(u_0, v_0) \in X^2$ be the point that condition $(U_2)$ guarantees. As $F(X^2) \subseteq g(X)$, there exists a Picard sequence $\{(u_n, v_n)\}$ of $(T,g)$ (recall Lemma 11.3.2), that is,

$$gu_{n+1} = F(u_n, v_n) \quad \text{and} \quad gv_{n+1} = F(v_n, u_n) \quad \text{for all } n \in \mathbb{N}.$$

In order to prove (11.28), we are going to show that $\{gu_n\} \to gx$, $\{gu_n\} \to gx'$, $\{gv_n\} \to gy$ and $\{gy_n\} \to gy'$. Hence, by the uniqueness of the limit, we conclude that $gx = gx'$ and $gy = gy'$. We only reason using $(x,y)$, but the same arguments can be identically applied to $(x',y')$. Assume, for example, that the first bullet property holds (the other ones are similar). Therefore

$$\begin{aligned} gx &= F(x,y) \preceq F(u_0,v_0) = gu_1 \quad \text{and} \\ gy &= F(y,x) \succeq F(v_0,u_0) = gv_1. \end{aligned}$$

As $F$ has the mixed $(g,\preceq)$-monotone property, then

$$\begin{aligned} gx &= F(x,y) \preceq F(u_1,v_0) \preceq F(u_1,v_1) = gu_2 \quad \text{and} \\ gy &= F(y,x) \succeq F(v_1,u_0) \succeq F(v_1,u_1) = gv_2. \end{aligned}$$

Since $\preceq$ is transitive, then $gx \preceq gu_2$ and $gy \succeq gv_2$. By induction, it can be proved that $gx \preceq gu_n$ and $gy \succeq gv_n$ for all $n \in \mathbb{N}$. Using the contractivity condition (11.26),

$$\begin{aligned} G(gx, gu_{n+1}, gu_{n+1}) &= G(F(x,y), F(u_n,v_n), F(u_n,v_n)) \\ &\leq \varphi\left(\frac{G(gx,gu_n,gu_n) + G(gy,gv_n,gv_n)}{2}\right) \\ &= \frac{1}{2}\varphi'(G(gx,gu_n,gu_n) + G(gy,gv_n,gv_n)), \end{aligned}$$

and similarly,

$$\begin{aligned} G(gy, gv_{n+1}, gv_{n+1}) &= G(F(v_n,u_n), F(v_n,u_n), F(y,x)) \\ &\leq \varphi\left(\frac{G(gv_n,gv_n,gy) + G(gu_n,gu_n,gx)}{2}\right) \\ &= \frac{1}{2}\varphi'(G(gx,gu_n,gu_n) + G(gy,gv_n,gv_n)). \end{aligned}$$

Combining the last two inequalities, we derive that, for all $n \in \mathbb{N}$,

$$\begin{aligned} &G\left(gx, gu_{n+1}, gu_{n+1}\right) + G\left(gy, gv_{n+1}, gv_{n+1}\right) \\ &\qquad \leq \varphi'\left(G(gx, gu_n, gu_n) + G(gy, gv_n, gv_n)\right). \end{aligned}$$

Since $\varphi(0) = 0$, then $\varphi'(0) = 2\varphi(0) = 0$. Applying item 9 of Lemma 2.3.11, we deduce that

$$\{G\left(gx, gu_{n+1}, gu_{n+1}\right) + G\left(gy, gv_{n+1}, gv_{n+1}\right)\} \to 0.$$

Hence, $\{gu_n\} \to gx$ and $\{gv_n\} \to gy$. If we have used $(x', y')$, then we would have deduced that $\{gu_n\} \to gx'$ and $\{gv_n\} \to gy'$. Then $gx = gx'$ and $gy = gy'$ and we have just proved that (11.28) holds.

Next, let $(x, y) \in \mathrm{Co}(F, g)$ be an arbitrary coupled coincidence point of $F$ and $g$ and let $z = gx = F(x, y)$ and $\omega = gy = F(y, x)$. Since $F$ and $g$ commute,

$$\begin{aligned} &gz = gF(x, y) = F(gx, gy) = F(z, \omega) \quad \text{and} \\ &g\omega = gF(y, x) = F(gy, gx) = F(\omega, z). \end{aligned}$$

Thus, $(z, \omega)$ is another coupled coincidence point of $F$ and $g$. Applying (11.28) to $(x, y)$ and $(z, \omega)$, we deduce that

$$z = gx = gz \quad \text{and} \quad \omega = gy = g\omega.$$

Hence, $z = gz = F(z, \omega)$ and $\omega = g\omega = F(\omega, z)$, which means that $(z, \omega)$ is a common coupled fixed point of $F$ and $g$.

To prove the uniqueness, let $(z', \omega')$ be another common coupled fixed point of $F$ and $g$. Then $z' = gz' = F(z', \omega')$ and $\omega' = g\omega' = F(\omega', z')$. Applying (11.28), we deduce that $z = gz = gz' = z'$ and $\omega = g\omega = g\omega' = \omega'$. Then, $F$ and $g$ have a unique common coupled fixed point, which is $(z, \omega)$.

Finally, let $(u, v) \in \mathrm{Co}(F, g)$ be another arbitrary coupled coincidence point of $F$ and $g$. By using (11.28), $gu = g\omega = \omega$ and $gv = gz = z$. Therefore, we get the point $(z, \omega)$ starting from any coupled coincidence point of $F$ and $g$. □

Immediate corollaries can be derived in the following particular cases: (1) using $g$ as the identity mapping on $X$; (2) involving a partial order $\preccurlyeq$ on $X$; (3) using $\varphi(t) = \lambda\, t$ for all $t \in [0, \infty)$, where $\lambda \in [0, 1)$.

## 11.4 Berinde and Borcut's Tripled Fixed Point Theory

In [41], Berinde and Borcut presented the notion of a tripled fixed point of a mapping $F : X^3 \to X$, which is a point $(x, y, z) \in X^3$ such that

$$\begin{cases} F(x, y, z) = x, \\ F(y, x, y) = y, \\ F(z, y, x) = z. \end{cases}$$

In that paper, they proved some results to guarantee existence and uniqueness of such points in partially ordered metric spaces involving the contractivity condition

$$d(F(x, y, z), F(u, v, w)) \le \lambda_1 d(x, u) + \lambda_2 d(y, v) + \lambda_3 d(z, w)$$

for all $x, y, z, u, v, w \in X$ such that $gx \preceq gu$, $gy \succeq gv$ and $gz \preceq gw$, where $\lambda_1, \lambda_2, \lambda_3 \in [0, 1)$ verify $\lambda_1 + \lambda_2 + \lambda_3 < 1$. In a subsequent paper, the same authors extended the previous condition to the coincidence case involving two mappings $F : X^3 \to X$ and $g : X \to X$ (see [50]). In this section we present some tripled fixed/coincidence point theorems in the sense of Berinde and Borcut. The notion of mixed monotone property is common to all results.

**Definition 11.4.1.** Let $X$ be a non-empty set endowed with a binary relation $\preccurlyeq$ and let $F : X^3 \to X$ and $g : X \to X$ be two mappings. The mapping $F$ is said to have the *mixed* $(g, \preccurlyeq)$*-monotone property* if $F(x, y, z)$ is monotone $(g, \preccurlyeq)$-non-decreasing in $x$ and in $z$, and monotone $(g, \preccurlyeq)$-non-increasing in $y$, that is, for all $x, y, z \in X$,

$$\begin{aligned} x_1, x_2 \in X, \quad gx_1 \preccurlyeq gx_2 \quad &\Rightarrow \quad F(x_1, y, z) \preccurlyeq F(x_2, y, z), \\ y_1, y_2 \in X, \quad gy_1 \preccurlyeq gy_2 \quad &\Rightarrow \quad F(x, y_1, z) \succcurlyeq F(x, y_2, z) \quad \text{and} \\ z_1, z_2 \in X, \quad gz_1 \preccurlyeq gz_2 \quad &\Rightarrow \quad F(x, y, z_1) \preccurlyeq F(x, y, z_2). \end{aligned}$$

If $g$ is the identity mapping on $X$, then we say that $F$ has the *mixed* $\preccurlyeq$*-monotone property*.

The following properties will be useful throughout this section.

**Lemma 11.4.1.** *Let $F : X^3 \to X$ and $g : X \to X$ be two mappings such that $F(X^3) \subseteq g(X)$. Then, starting from any points $x_0, y_0, z_0 \in X$, there exist three sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ on $X$ such that*

$$\begin{aligned} &gx_{n+1} = F(x_n, y_n, z_n), \quad gy_{n+1} = F(y_n, x_n, y_n) \quad \text{and} \\ &gz_{n+1} = F(z_n, y_n, x_n) \end{aligned} \tag{11.29}$$

*for all $n \in \mathbb{N}$.*

*Proof.* Let $x_0, y_0, z_0 \in X$ be arbitrary. Since $F(x_0, y_0, z_0) \in F(X^2) \subseteq g(X)$, then there exists $x_1 \in X$ such that $gx_1 = F(x_0, y_0, z_0)$. Similarly, as $F(y_0, x_0, y_0) \in F(X^2) \subseteq g(X)$, then there exists $y_1 \in X$ such that $gy_1 = F(y_0, x_0, y_0)$. Again, as $F(z_0, y_0, x_0) \in F(X^2) \subseteq g(X)$, then there exists $z_1 \in X$ such that $gz_1 = F(z_0, y_0, x_0)$. If we repeat the same argument using $x_1$, $y_1$ and $z_1$ rather than $x_0$, $y_0$ and $z_0$, we can find $x_2, y_2, z_2 \in X^2$ such that $gx_2 = F(x_1, y_1, z_1)$, $gy_2 = F(y_1, x_1, y_1)$ and $gz_2 = F(z_1, y_1, x_1)$. By induction, we may define the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ on $X$. □

**Definition 11.4.2.** Given two mappings $F : X^3 \to X$ and $g : X \to X$, a *Picard $(F, g)$-sequence* is a sequence $\{(x_n, y_n, z_n)\}_{n\in\mathbb{N}} \subseteq X^3$ verifying (11.29).

**Proposition 11.4.1.** *If $\{(x_n, y_n, z_n)\}_{n\in\mathbb{N}} \subseteq X^3$ is a Picard $(F, g)$-sequence of two mappings $F : X^3 \to X$ and $g : X \to X$ and there exists $n_0 \in \mathbb{N}$ such that $gx_{n_0} = gx_{n_0+1}$, $gy_{n_0} = gy_{n_0+1}$ and $gz_{n_0} = gz_{n_0+1}$, then $(x_{n_0}, y_{n_0}, z_{n_0})$ is a tripled coincidence point of $F$ and $g$.*

*Proof.* If the exists $n_0 \in \mathbb{N}$ such that $gx_{n_0} = gx_{n_0+1}$, $gy_{n_0} = gy_{n_0+1}$ and $gz_{n_0} = gz_{n_0+1}$, then

$$gx_{n_0} = gx_{n_0+1} = F(x_{n_0}, y_{n_0}, z_{n_0}), \quad gy_{n_0} = gy_{n_0+1} = F(y_{n_0}, x_{n_0}, y_{n_0}) \quad \text{and}$$
$$gz_{n_0} = gz_{n_0+1} = F(z_{n_0}, y_{n_0}, x_{n_0}),$$

so $(x_{n_0}, y_{n_0}, z_{n_0})$ is a tripled coincidence point of $F$ and $g$. □

Given two mappings $F : X^3 \to X$ and $g : X \to X$, the condition $F(X^3) \subseteq g(X)$ is sufficient to guarantee that there exists a Picard $(F, g)$-sequence on $X$ based on any initial points $x_0, y_0, z_0 \in X$. However, it is not necessary.

**Lemma 11.4.2.** *Let $\preceq$ be a transitive binary relation on a set $X$ and let $F : X^3 \to X$ and $g : X \to X$ be two mappings such that the following conditions are fulfilled.*

**(i)** *There exists a Picard $(F, g)$-sequence $\{(x_n, y_n, z_n)\}_{n\in\mathbb{N}} \subseteq X^3$.*
**(ii)** *$gx_0 \preceq F(x_0, y_0, z_0)$, $gy_0 \succeq F(y_0, x_0, y_0)$ and $gz_0 \preceq F(z_0, y_0, x_0)$.*
**(iii)** *$F$ has the mixed $(g, \preceq)$-monotone property.*

*Then $\{gx_n\}$ and $\{gz_n\}$ are $\preceq$-non-decreasing and $\{gy_n\}$ is $\preceq$-non-increasing (that is, $gx_n \preceq gx_{n+1}$, $gy_n \succeq gy_{n+1}$ and $gz_n \preceq gz_{n+1}$ for all $n \in \mathbb{N}$).*

*Proof.* By (ii), we have that $gx_0 \preceq F(x_0, y_0, x_0) = gx_1$, $gy_0 \succeq F(y_0, x_0, y_0) = gy_1$ and $gz_0 \preceq F(z_0, y_0, x_0) = gz_1$. Assume that there exist $n \in \mathbb{N}$ such that $gx_n \preceq gx_{n+1}$, $gy_n \succeq gy_{n+1}$ and $gz_n \preceq gz_{n+1}$. Then, as $F$ has the mixed $(g, \preceq)$-monotone property, it follows that

$$\begin{aligned} gx_{n+1} &= F(x_n, y_n, z_n) \preceq F(x_{n+1}, y_n, z_n) \preceq F(x_{n+1}, y_{n+1}, z_n) \\ &\preceq F(x_{n+1}, y_{n+1}, z_{n+1}) = gx_{n+2}, \\ gy_{n+1} &= F(y_n, x_n, y_n) \succeq F(y_{n+1}, x_n, y_n) \succeq F(y_{n+1}, x_{n+1}, y_n) \end{aligned}$$

$$\begin{aligned}&\succeq F\left(y_{n+1}, x_{n+1}, y_{n+1}\right) = gy_{n+2} \quad \text{and}\\ gz_{n+1} &= F\left(z_n, y_n, x_n\right) \preceq F\left(z_{n+1}, y_n, x_n\right) \preceq F\left(z_{n+1}, y_{n+1}, x_n\right)\\ &\preceq F\left(z_{n+1}, y_{n+1}, x_{n+1}\right) = gz_{n+2}.\end{aligned}$$

As $\preceq$ is transitive, then $gx_{n+1} \preceq gx_{n+2}$, $gy_{n+1} \succeq gy_{n+2}$ and $gz_{n+1} \preceq gz_{n+2}$, and this completes the induction. □

### *11.4.1 Berinde and Borcut's Tripled Fixed Point Theorems in G-Metric Spaces*

In this subsection, we show corresponding versions, in the context of preordered $G$-metric spaces, of some fixed point results given in [41].

**Theorem 11.4.1.** *Let $(X, \preceq)$ be preordered set and let $(X, G)$ be a complete G-metric space. Let $F : X^3 \to X$ be a mapping having the mixed $\preceq$-monotone property on X. Suppose that there exists $\lambda \in [0, 1)$ such that*

$$\begin{aligned}&G\left(F\left(x, y, z\right), F\left(u, v, w\right), F\left(u, v, w\right)\right)\\ &\qquad \le \lambda \max\left\{G\left(x, u, u\right), G\left(y, v, v\right), G\left(z, w, w\right)\right\} \qquad (11.30)\end{aligned}$$

*for $x, y, z, u, v, w \in X$ with $x \preceq u$, $y \succeq v$ and $z \preceq w$. Also assume that either F is continuous or $(X, G, \preceq)$ is regular. If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, y_0)$ and $z_0 \preceq F(z_0, y_0, x_0)$, then F has a tripled fixed point in X, that is, there exist $x, y, z \in X$ such that*

$$F(x, y, z) = x, \quad F(y, x, y) = y \quad \textit{and} \quad F(z, y, x) = z.$$

*Proof.* Starting from the points $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, y_0)$ and $z_0 \preceq F(z_0, y_0, x_0)$, let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences defined by

$$\begin{aligned}&x_{n+1} = F(x_n, y_n, z_n), \quad y_{n+1} = F(y_n, x_n, y_n) \quad \text{and}\\ &z_{n+1} = F(z_n, y_n, x_n) \qquad (11.31)\end{aligned}$$

for all $n \in \mathbb{N}$. As $F$ has the mixed $\preceq$-monotone property, Lemma 11.4.2 assures that $\{x_n\}$ and $\{z_n\}$ are $\preceq$-non-decreasing and $\{y_n\}$ is $\preceq$-non-increasing. Applying the contrativity condition (11.30) to $x_n \preceq x_{n+1}$, $y_n \succeq y_{n+1}$ and $z_n \preceq z_{n+1}$, we obtain that

$$\begin{aligned}&G\left(x_{n+1}, x_{n+2}, x_{n+2}\right)\\ &\quad = G\left(F(x_n, y_n, z_n), F(x_{n+1}, y_{n+1}, z_{n+1}), F(x_{n+1}, y_{n+1}, z_{n+1})\right)\\ &\quad \le \lambda \max\left\{G\left(x_n, x_{n+1}, x_{n+1}\right), G\left(y_n, y_{n+1}, y_{n+1}\right), G\left(z_n, z_{n+1}, z_{n+1}\right)\right\}.\end{aligned}$$

Similarly, as $y_{n+1} \preceq y_n$ and $x_{n+1} \succeq x_n$,

$$\begin{aligned} &G\left(y_{n+1}, y_{n+2}, y_{n+2}\right) \\ &\quad = G\left(F(y_{n+1}, x_{n+1}, y_{n+1}), F(y_{n+1}, x_{n+1}, y_{n+1}), F(y_n, x_n, y_n)\right) \\ &\quad \leq \lambda \max\left\{G\left(y_{n+1}, y_{n+1}, y_n\right), G\left(x_{n+1}, x_{n+1}, x_n\right), G\left(y_{n+1}, y_{n+1}, y_n\right)\right\} \\ &\quad = \lambda \max\left\{G\left(x_n, x_{n+1}, x_{n+1}\right), G\left(y_n, y_{n+1}, y_{n+1}\right)\right\}. \end{aligned}$$

Repeating this argument, since $z_n \preceq z_{n+1}$, $y_n \succeq y_{n+1}$ and $x_n \preceq x_{n+1}$, we deduce that

$$\begin{aligned} &G\left(z_{n+1}, z_{n+2}, z_{n+2}\right) \\ &\quad = G\left(F(z_n, y_n, x_n), F(z_{n+1}, y_{n+1}, x_{n+1}), F(z_{n+1}, y_{n+1}, x_{n+1})\right) \\ &\quad \leq \lambda \max\left\{G\left(z_n, z_{n+1}, z_{n+1}\right), G\left(y_n, y_{n+1}, y_{n+1}\right), G\left(x_n, x_{n+1}, x_{n+1}\right)\right\}. \end{aligned}$$

Joining the last three inequalities, we conclude that, for all $n \in \mathbb{N}$,

$$\begin{aligned} &\max\left\{G\left(x_{n+1}, x_{n+2}, x_{n+2}\right), G\left(y_{n+1}, y_{n+2}, y_{n+2}\right), G\left(z_{n+1}, z_{n+2}, z_{n+2}\right)\right\} \\ &\quad \leq \lambda \max\left\{G\left(x_n, x_{n+1}, x_{n+1}\right), G\left(y_n, y_{n+1}, y_{n+1}\right), G\left(z_n, z_{n+1}, z_{n+1}\right)\right\}. \end{aligned}$$

From Corollary 11.2.1, the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are Cauchy. Since $(X, G)$ is complete, there exists $u, v, w \in X$ such that $\{x_n\} \to u$, $\{y_n\} \to v$ and $\{z_n\} \to w$. Next, we distinguish two cases.

*Case 1.* *$F$ is $G$-continuous.* In this case, letting $n \to \infty$ in (11.31), we deduce that $u = F(u, v, w)$, $v = F(v, u, v)$ and $w = F(w, v, u)$, that is, $(u, v, w)$ is a tripled fixed point of $F$.

*Case 2.* *$(X, G, \preceq)$ is regular.* As $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are $\preceq$-monotone, convergent sequences, the regularity implies that

$$x_n \preceq u, \quad y_n \succeq v \quad \text{and} \quad z_n \preceq w \quad \text{for all } n \in \mathbb{N}.$$

Hence, the contractivity condition (11.30) ensures that

$$\begin{aligned} &G\left(x_{n+1}, F\left(u, v, w\right), F\left(u, v, w\right)\right) \\ &\quad = G\left(F(x_n, y_n, z_n), F\left(u, v, w\right), F\left(u, v, w\right)\right) \\ &\quad \leq \lambda \max\left\{G\left(x_n, u, u\right), G\left(y_n, v, v\right), G\left(z_n, w, w\right)\right\}. \end{aligned}$$

Similarly,

$$\begin{aligned} &G\left(y_{n+1}, F\left(v, u, v\right), F\left(v, u, v\right)\right) \\ &\quad = G\left(F\left(v, u, v\right), F\left(v, u, v\right), F(y_n, x_n, y_n)\right) \\ &\quad \leq \lambda \max\left\{G\left(v, v, y_n\right), G\left(u, u, x_n\right)\right\}. \end{aligned}$$

Repeating this argument,

$$\begin{aligned}
&G\left(z_{n+1}, F\left(w, v, u\right), F\left(w, v, u\right)\right)\\
&\quad = G\left(F(z_n, y_n, x_n), F\left(w, v, u\right), F\left(w, v, u\right)\right)\\
&\quad \leq \lambda \ \max\left\{G\left(z_n, w, w\right), G\left(y_n, v, v\right), G\left(x_n, u, u\right)\right\}.
\end{aligned}$$

Letting $n \to \infty$ in the previous inequalities, we conclude that

$$F\left(u, v, w\right) = \lim_{n\to\infty} x_{n+1} = u$$

and, similarly, $v = F\left(v, u, v\right)$ and $w = F\left(w, v, u\right)$. Hence, $(u, v, w)$ is a tripled fixed point of $F$.

□

**Corollary 11.4.1.** *Let $(X, \preceq)$ be preordered set and let $(X, G)$ be a complete $G$-metric space. Let $F : X^3 \to X$ be a mapping having the mixed $\preceq$-monotone property on $X$. Suppose that there exists $\lambda \in [0, 1)$ such that*

$$\begin{aligned}
&G\left(F\left(x, y, z\right), F\left(u, v, w\right), F\left(a, b, c\right)\right)\\
&\quad \leq \lambda \ \max\left\{G\left(x, u, a\right), G\left(y, v, b\right), G\left(z, w, c\right)\right\}
\end{aligned} \tag{11.32}$$

*for $x, y, z, u, v, w, a, b, c \in X$ with $x \preceq u \preceq a$, $y \succeq v \succeq b$ and $z \preceq w \preceq c$. Also assume that either $F$ is continuous or $(X, G, \preceq)$ is regular. If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, y_0)$ and $z_0 \preceq F(z_0, y_0, x_0)$, then $F$ has a tripled fixed point in $X$.*

The following result is a version, in the setting of partially ordered $G$-metric spaces, of Theorems 7 and 8 in [41].

**Corollary 11.4.2.** *Let $(X, \preccurlyeq)$ be a partially ordered set and let $(X, G)$ be a complete $G$-metric space. Let $F : X^3 \to X$ be a mapping having the mixed $\preceq$-monotone property on $X$. Suppose that there exists $\lambda_1, \lambda_2, \lambda_3 \in [0, 1)$ such that $\lambda_1+\lambda_2+\lambda_3 < 1$ and verifying*

$$\begin{aligned}
&G\left(F\left(x, y, z\right), F\left(u, v, w\right), F\left(a, b, c\right)\right)\\
&\quad \leq \lambda_1 G\left(x, u, a\right) + \lambda_2 G\left(y, v, b\right) + \lambda_3 G\left(z, w, c\right)
\end{aligned} \tag{11.33}$$

*for $x, y, z, u, v, w, a, b, c \in X$ with $x \preccurlyeq u \preccurlyeq a$, $y \succcurlyeq v \succcurlyeq b$ and $z \preccurlyeq w \preccurlyeq c$. Also assume that either $F$ is continuous or $(X, G, \preceq)$ is regular. If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, y_0)$ and $z_0 \preceq F(z_0, y_0, x_0)$, then $F$ has a tripled fixed point in $X$.*

*Proof.* If $\lambda = \lambda_1 + \lambda_2 + \lambda_3 < 1$, then

$$\begin{aligned}
&G\left(F\left(x,y,z\right),F\left(u,v,w\right),F\left(a,b,c\right)\right)\\
&\quad\leq \lambda_1 G\left(x,u,a\right) + \lambda_2 G\left(y,v,b\right) + \lambda_3 G\left(z,w,c\right)\\
&\quad\leq \lambda_1 \max\left\{G\left(x,u,a\right),G\left(y,v,b\right),G\left(z,w,c\right)\right\}\\
&\qquad + \lambda_2 \max\left\{G\left(x,u,a\right),G\left(y,v,b\right),G\left(z,w,c\right)\right\}\\
&\qquad + \lambda_3 \max\left\{G\left(x,u,a\right),G\left(y,v,b\right),G\left(z,w,c\right)\right\}\\
&\quad= \lambda \max\left\{G\left(x,u,a\right),G\left(y,v,b\right),G\left(z,w,c\right)\right\}
\end{aligned}$$

for $x,y,z,u,v,w,a,b,c \in X$ with $x \preccurlyeq u \preccurlyeq a$, $y \succcurlyeq v \succcurlyeq b$ and $z \preccurlyeq w \preccurlyeq c$. Then, (11.33) implies (11.32). □

### 11.4.2 Aydi et al.'s Tripled Fixed Point Theorems in G-Metric Spaces

In the following result, which improves those given in [31], we will employ a comparison function $\phi \in \mathcal{F}_{\text{com}}$, that is, $\phi$ is non-decreasing and $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$. Recall that we also have that $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$. As a consequence, $\phi(t) \leq t$ for all $t \geq 0$, so $\phi$ is continuous at $t = 0$.

**Theorem 11.4.2.** *Let $(X,\preceq)$ be preordered set and let $(X,G)$ be a complete G-metric space. Let $F : X^3 \to X$ be a mapping having the mixed $\preceq$-monotone property on $X$. Suppose that there exists $\phi \in \mathcal{F}_{\text{com}}$ such that for $x,y,z,u,v,w \in X$, with $x \succeq u \succeq a$, $y \preceq v \preceq b$, and $z \succeq w \succeq c$, one has*

$$\begin{aligned}
&G\left(F\left(x,y,z\right),F\left(u,v,w\right),F\left(a,b,c\right)\right)\\
&\quad\leq \phi\left(\max\left\{G\left(x,u,a\right),G\left(y,v,b\right),G\left(z,w,c\right)\right\}\right).
\end{aligned} \tag{11.34}$$

*Also assume that either $F$ is continuous or $(X,G,\preceq)$ is regular. If there exist $x_0,y_0,z_0 \in X$ such that $x_0 \preceq F(x_0,y_0,z_0)$, $y_0 \succeq F(y_0,x_0,y_0)$ and $z_0 \preceq F(z_0,y_0,x_0)$, then $F$ has a tripled fixed point in $X$, that is, there exist $x,y,z \in X$ such that*

$$F(x,y,z) = x, \quad F(y,x,y) = y \quad \textit{and} \quad F(z,y,x) = z.$$

*Proof.* Starting from the points $x_0,y_0,z_0 \in X$, define the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by

$$x_{n+1} = F\left(x_n,y_n,z_n\right), \quad y_{n+1} = F\left(y_n,x_n,y_n\right) \quad \text{and} \quad z_{n+1} = F\left(z_n,y_n,x_n\right) \tag{11.35}$$

for all $n \in \mathbb{N}$. If we suppose that $x_0 = x_1$, $y_0 = y_1$ and $z_0 = z_1$, then $(x_0, y_0, z_0)$ is a tripled fixed point of $F$ because

$$x_0 = x_1 = F(x_0, y_0, z_0), \quad y_0 = y_1 = F(y_0, x_0, y_0) \text{ and}$$
$$z_0 = z_1 = F(z_0, y_0, x_0).$$

In this case, the existence part is finished. On the contrary case, assume that

$$\max\{G(x_1, x_0, x_0), G(y_1, y_0, y_0), G(z_1, z_0, z_0)\} > 0. \tag{11.36}$$

We claim that the sequences $\{x_n\}$ and $\{z_n\}$ are $\preceq$-non-decreasing, and $\{y_n\}$ is $\preceq$-non-increasing. Indeed, by hypothesis, $x_0 \preceq F(x_0, y_0, z_0) = x_1$, $y_0 \succeq F(y_0, x_0, y_0) = y_1$ and $z_0 \preceq F(z_0, y_0, x_0) = z_1$. Assume that, for some $n \in \mathbb{N}$, we have that $x_n \preceq x_{n+1}$, $y_n \succeq y_{n+1}$ and $z_n \preceq z_{n+1}$. Then, using the mixed $\preceq$-monotone property,

$$\begin{aligned}
x_{n+1} &= F(x_n, y_n, z_n) \preceq F(x_{n+1}, y_n, z_n) \preceq F(x_{n+1}, y_{n+1}, z_n) \\
&\preceq F(x_{n+1}, y_{n+1}, z_{n+1}) = x_{n+2}, \\
y_{n+1} &= F(y_n, x_n, y_n) \succeq F(y_{n+1}, x_n, y_n) \succeq F(y_{n+1}, x_{n+1}, y_n) \\
&\succeq F(y_{n+1}, x_{n+1}, y_{n+1}) = y_{n+2}, \\
z_{n+1} &= F(z_n, y_n, x_n) \preceq F(z_{n+1}, y_n, x_n) \preceq F(z_{n+1}, y_{n+1}, x_n) \\
&\preceq F(z_{n+1}, y_{n+1}, x_{n+1}) \preceq z_{n+2}.
\end{aligned}$$

As a consequence, $\{x_n\}$ and $\{z_n\}$ are $\preceq$-non-decreasing, and $\{y_n\}$ is $\preceq$-non-increasing. As $\preceq$ is transitive, then

$$x_n \preceq x_m, \quad y_n \succeq y_m \quad \text{and} \quad z_n \preceq z_m \quad \text{for all } n \leq m. \tag{11.37}$$

Applying the contractivity condition (11.34),

$$\begin{aligned}
&G(x_{n+2}, x_{n+1}, x_{n+1}) \\
&\quad = G(F(x_{n+1}, y_{n+1}, z_{n+1}), F(x_n, y_n, z_n), F(x_n, y_n, z_n)) \\
&\quad \leq \phi(\max\{G(x_{n+1}, x_n, x_n), G(y_{n+1}, y_n, y_n), G(z_{n+1}, z_n, z_n)\}),
\end{aligned}$$

and, similarly,

$$\begin{aligned}
&G(y_{n+2}, y_{n+1}, y_{n+1}) \\
&\quad = G(F(y_{n+1}, x_{n+1}, y_{n+1}), F(y_n, x_n, y_n), F(y_n, x_n, y_n)) \\
&\quad = G(F(y_n, x_n, y_n), F(y_n, x_n, y_n), F(y_{n+1}, x_{n+1}, y_{n+1})) \\
&\quad \leq \phi(\max\{G(y_n, y_n, y_{n+1}), G(x_n, x_n, x_{n+1})\}) \\
&\quad \leq \phi(\max\{G(x_{n+1}, x_n, x_n), G(y_{n+1}, y_n, y_n), G(z_{n+1}, z_n, z_n)\})
\end{aligned}$$

and

$$\begin{aligned}G(z_{n+2}, z_{n+1}, z_{n+1}) &= G(F(z_{n+1}, y_{n+1}, x_{n+1}), F(z_n, y_n, x_n), F(z_n, y_n, x_n)) \\ &\leq \phi(\max\{G(z_{n+1}, z_n, z_n), G(y_{n+1}, y_n, y_n), G(x_{n+1}, x_n, x_n)\}).\end{aligned}$$

As $\phi$ is non-decreasing, we deduce that for all $n \in \mathbb{N}$,

$$\begin{aligned}&\max\{G(x_{n+2}, x_{n+1}, x_{n+1}), G(y_{n+2}, y_{n+1}, y_{n+1}), G(z_{n+2}, z_{n+1}, z_{n+1})\} \\ &\quad\leq \phi(\max\{G(x_{n+1}, x_n, x_n), G(y_{n+1}, y_n, y_n), G(z_{n+1}, z_n, z_n)\}).\end{aligned} \tag{11.38}$$

Repeating (11.38) $n$ times and taking into account that $\phi$ is non-decreasing,

$$\begin{aligned}&\max\{G(x_{n+1}, x_n, x_n), G(y_{n+1}, y_n, y_n), G(z_{n+1}, z_n, z_n)\} \\ &\quad\leq \phi(\max\{G(x_n, x_{n-1}, x_{n-1}), G(y_n, y_{n-1}, y_{n-1}), G(z_n, z_{n-1}, z_{n-1})\}) \\ &\quad\leq \phi^2(\max\{G(x_{n-1}, x_{n-2}, x_{n-2}), G(y_{n-1}, y_{n-2}, y_{n-2}), \\ &\qquad\qquad G(z_{n-1}, z_{n-2}, z_{n-2})\}) \\ &\quad\leq \ldots \leq \phi^n(\max\{G(x_1, x_0, x_0), G(y_1, y_0, y_0), G(z_1, z_0, z_0)\}).\end{aligned} \tag{11.39}$$

From (11.36) and $\phi \in \mathcal{F}_{\text{com}}$, we deduce that

$$\lim_{n\to\infty} \phi^n(\max\{G(x_1, x_0, x_0), G(y_1, y_0, y_0), G(z_1, z_0, z_0)\}) = 0. \tag{11.40}$$

In particular,

$$\lim_{n\to\infty} G(x_{n+1}, x_n, x_n) = \lim_{n\to\infty} G(y_{n+1}, y_n, y_n) = \lim_{n\to\infty} G(z_{n+1}, z_n, z_n) = 0,$$

that is, the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are asymptotically regular.

Next, we show that, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\max\{G(x_n, x_n, x_m), G(y_n, y_n, y_m), G(z_n, z_n, z_m)\} \leq \varepsilon \tag{11.41}$$

for all $m \geq n \geq n_0$. Indeed, if $m = n$, then (11.41) trivially holds. Assume that $m > n$. Let $\varepsilon > 0$ be arbitrary. As $\phi(\varepsilon) < \varepsilon$, let $\delta = \varepsilon - \phi(\varepsilon) > 0$. From (11.40), there exists $n_0 \in \mathbb{N}$ such that

$$\phi^n(\max\{G(x_1, x_0, x_0), G(y_1, y_0, y_0), G(z_1, z_0, z_0)\}) < \delta = \varepsilon - \phi(\varepsilon)$$

for all $n \geq n_0$. Using (11.39),

$$\begin{aligned}&\max\{G(x_n, x_n, x_{n+1}), G(y_n, y_n, y_{n+1}), G(z_n, z_n, z_{n+1})\} \\ &\quad\leq \phi^n(\max\{G(x_1, x_0, x_0), G(y_1, y_0, y_0), G(z_1, z_0, z_0)\}) < \varepsilon - \phi(\varepsilon)\end{aligned} \tag{11.42}$$

for all $n \geq n_0$. This means that if $m = n + 1$, then (11.41) also holds. reasoning by induction, assume that (11.41) holds for some $m > n$, and we will prove it for $m + 1$. It follows from

$$\begin{aligned}
G(x_n, x_n, x_{m+1}) &= G(x_{m+1}, x_n, x_n) \\
&\leq G(x_{m+1}, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_n, x_n) \\
&< G\left(F\left(x_m, y_m, z_m\right), F\left(x_n, y_n, z_n\right), F\left(x_n, y_n, z_n\right)\right) + \varepsilon - \phi\left(\varepsilon\right) \\
&\leq \phi\left(\max\left\{G\left(x_m, x_n, x_n\right), G\left(y_m, y_n, y_n\right), G\left(z_m, z_n, z_n\right)\right\}\right) + \varepsilon - \phi\left(\varepsilon\right) \\
&\leq \phi\left(\varepsilon\right) + \varepsilon - \phi\left(\varepsilon\right) = \varepsilon.
\end{aligned}$$

Similarly, $G(y_n, y_n, y_{m+1}) \leq \varepsilon$ and $G(z_n, z_n, z_{m+1}) \leq \varepsilon$. As a consequence, (11.41) holds, and this guarantees that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are Cauchy sequences. As $(X, G)$ is complete, there exist $x, y, z \in X$ such that $\{x_n\} \to x$, $\{y_n\} \to y$ and $\{z_n\} \to z$. Next, we distinguish two cases.

If we assume that $F$ is continuous, letting $n \to \infty$ in (11.35), we deduce that $F(x, y, z) = x$, $F(y, x, y) = y$ and $F(z, y, x) = z$, that is, $(x, y, z)$ is a tripled fixed point of $F$. In the other case, assume that $(X, G, \preceq)$ is regular. Taking into account that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are convergent, $\preceq$-monotone sequences, we deduce that

$$x_n \preceq x, \quad y_n \succeq y \quad \text{and} \quad z_n \preceq z \quad \text{for all } n \in \mathbb{N}.$$

Using the contractivity condition (11.34), we have that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
G\left(F\left(x, y, z\right), x_{n+1}, x_{n+1}\right) &= G\left(F\left(x, y, z\right), F\left(x_n, y_n, z_n\right), F\left(x_n, y_n, z_n\right)\right) \\
&\leq \phi\left(\max\left\{G\left(x, x_n, x_n\right), G\left(y, y_n, y_n\right), G\left(z, z_n, z_n\right)\right\}\right).
\end{aligned}$$

As $\phi$ is continuous at $t = 0$, letting $n \to \infty$ in the previous inequality, we deduce that $F(x, y, z) = x$. Similarly, for all $n \in \mathbb{N}$,

$$\begin{aligned}
G\left(y_{n+1}, F\left(y, x, y\right), F\left(y, x, y\right)\right) &= G\left(F\left(y_n, x_n, y_n\right), F\left(y, x, y\right), F\left(y, x, y\right)\right) \\
&\leq \phi\left(\max\left\{G\left(y_n, y, y\right), G\left(x_n, x, x\right)\right\}\right),
\end{aligned}$$

so $F\left(y, x, y\right) = y$. In the same way, we can prove that $F(z, y, x) = z$, so $(x, y, z)$ is a tripled fixed point of $F$. □

**Corollary 11.4.3.** *Let $(X, \preceq)$ be preordered set and let $(X, G)$ be a complete $G$-metric space. Let $F : X^3 \to X$ be a mapping having the mixed monotone property on $X$. Suppose that there exists $\lambda \in [0, 1)$ such that for $x, y, z, u, v, w \in X$, with $x \succeq u \succeq a$, $y \preceq v \preceq b$, and $z \succeq w \succeq c$, one has*

$$\begin{aligned}
&G\left(F\left(x, y, z\right), F\left(u, v, w\right), F\left(a, b, c\right)\right) \\
&\qquad \leq \lambda \, \max\left\{G\left(x, u, a\right), G\left(y, v, b\right), G\left(z, w, c\right)\right\}.
\end{aligned}$$

*Also assume that either $F$ is continuous or $(X, G, \preceq)$ is regular. If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, y_0)$ and $z_0 \preceq F(z_0, y_0, x_0)$, then $F$ has a tripled fixed point in $X$, that is, there exist $x, y, z \in X$ such that*

$$F(x, y, z) = x, \quad F(y, x, y) = y \quad \text{and} \quad F(z, y, x) = z.$$

Taking into account that $t + s + r \leq 3 \max\{t, s, r\}$ for all $t, s, r \in \mathbb{R}$, we can also establish the following result.

**Corollary 11.4.4.** *Let $(X, \preceq)$ be preordered set and let $(X, G)$ be a complete $G$-metric space. Let $F : X^3 \to X$ be a mapping having the mixed monotone property on $X$. Suppose that there exists $\lambda \in [0, 1)$ such that for $x, y, z, u, v, w \in X$, with $x \succeq u \succeq a$, $y \preceq v \preceq b$, and $z \succeq w \succeq c$, one has*

$$\begin{aligned} &G\left(F\left(x, y, z\right), F\left(u, v, w\right), F\left(a, b, c\right)\right) \\ &\qquad \leq \frac{\lambda}{3} \left(G\left(x, u, a\right) + G\left(y, v, b\right) + G\left(z, w, c\right)\right). \end{aligned}$$

*Also assume that either $F$ is continuous or $(X, G, \preceq)$ is regular. If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, y_0)$ and $z_0 \preceq F(z_0, y_0, x_0)$, then $F$ has a tripled fixed point in $X$, that is, there exist $x, y, z \in X$ such that*

$$F(x, y, z) = x, \quad F(y, x, y) = y \quad \text{and} \quad F(z, y, x) = z.$$

*Example 11.4.1.* Let $X = \mathbb{R}$ be endowed with the complete $G$-metric $G(x, y, z) = \max\{|x - y|, |x - z|, |y - z|\}$ for all $x, y, z \in X$. If we define $F : X^3 \to X$ by

$$F(x, y, z) = \frac{6x - 6y + 6z + 5}{36} \quad \text{for all } x, y, z \in X,$$

then $F$ has the mixed $\leq$-monotone property. If $x, y, z, u, v, w, a, b, c \in X$ are such that $x \geq u \geq a$, $y \leq v \leq b$, and $z \geq w \geq c$, then

$$\begin{aligned} |x - a| &= x - a \geq \max\{x - u, u - a\} = \max\left\{|x - u|, |u - a|\right\}, \\ |b - y| &= b - y \geq \max\{b - v, v - y\} = \max\left\{|b - v|, |v - y|\right\}, \\ |z - c| &= z - c \geq \max\{z - w, w - c\} = \max\left\{|z - w|, |w - c|\right\}. \end{aligned}$$

Therefore

$$\begin{aligned} &G\left(F\left(x, y, z\right), F\left(u, v, w\right), F\left(a, b, c\right)\right) \\ &\quad = \left|\frac{6x - 6y + 6z + 5}{36} - \frac{6a - 6b + 6c + 5}{36}\right| \end{aligned}$$

$$= \left| \frac{6(x-a) - 6(y-b) + 6(z-c)}{36} \right| = \frac{6(x-a) + 6(b-y) + 6(z-c)}{36}$$
$$= \frac{1}{6} \left( (x-a) + (b-y) + (z-c) \right)$$
$$= \frac{1/2}{6} \left( G(x,u,a) + G(y,v,b) + G(z,w,c) \right).$$

If $\lambda = 1/2$, then all conditions of Corollary 11.4.4 are fulfilled (notice that $F$ is $G$-continuous). Therefore, $G$ has a tripled fixed point in $X$, which is $(1/6, 1/6, 1/6)$.

Another particular case occurs when $\preceq$ is a partial order on $X$.

**Corollary 11.4.5.** *Let $(X, \preceq)$ be partially ordered set and let $(X, G)$ be a complete $G$-metric space. Let $F : X^3 \to X$ be a mapping having the mixed monotone property on $X$. Suppose that there exists $\phi \in \mathcal{F}_{\text{com}}$ such that for $x, y, z, u, v, w \in X$, with $x \succeq u \succeq a$, $y \preceq v \preceq b$, and $z \succeq w \succeq c$, one has*

$$G(F(x,y,z), F(u,v,w), F(a,b,c)) \\ \leq \phi(\max\{G(x,u,a), G(y,v,b), G(z,w,c)\}).$$

*Also assume that either $F$ is continuous or $(X, G, \preceq)$ is regular. If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, y_0)$ and $z_0 \preceq F(z_0, y_0, x_0)$, then $F$ has a tripled fixed point in $X$, that is, there exist $x, y, z \in X$ such that*

$$F(x,y,z) = x, \quad F(y,x,y) = y \quad \textit{and} \quad F(z,y,x) = z.$$

Finally, if we use the preorder $\preceq_0$on $X$ given in (5.1), we deduce the following version.

**Corollary 11.4.6.** *Let $(X, G)$ be a complete $G$-metric space and let $F : X^3 \to X$ be a mapping. Suppose that there exists $\phi \in \mathcal{F}_{\text{com}}$ such that, for $x, y, z, u, v, w, a, b, c \in X$, one has*

$$G(F(x,y,z), F(u,v,w), F(a,b,c)) \\ \leq \phi(\max\{G(x,u,a), G(y,v,b), G(z,w,c)\}).$$

*Then $F$ has a tripled fixed point in $X$, that is, there exist $x, y, z \in X$ such that*

$$F(x,y,z) = x, \quad F(y,x,y) = y \quad \textit{and} \quad F(z,y,x) = z.$$

### 11.4.3 Aydi et al.'s Tripled Coincidence Point Theorems in G-Metric Spaces

In this subsection we prove and extend the main results given by Aydi, Karapınar and Shatanawi in [32]. In order to present a very general result, we introduce the following definitions.

**Definition 11.4.3.** Let $(X, G)$ be a $G^*$-metric space endowed with a binary relation $\preceq$ and let $F : X^3 \to X$ and $g : X \to X$ be two mappings. We will say that $(F, g)$ is an $(O, \preceq)$-*compatible pair* if we have that

$$\lim_{n\to\infty} G\left(gF(x_n, y_n, z_n), F(gx_n, gy_n, gz_n), F(gx_n, gy_n, gz_n)\right) = 0,$$
$$\lim_{n\to\infty} G\left(gF(y_n, x_n, y_n), F(gy_n, gx_n, gy_n), F(gy_n, gx_n, gy_n)\right) = 0 \quad \text{and}$$
$$\lim_{n\to\infty} G\left(gF(z_n, y_n, x_n), F(gz_n, gy_n, gx_n), F(gz_n, gy_n, gx_n)\right) = 0$$

whenever $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are sequences in $X$ such that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are $\preceq$-monotone and

$$\lim_{m\to\infty} F(x_n, y_n, z_n) = \lim_{n\to\infty} gx_n \in X,$$
$$\lim_{m\to\infty} F(y_n, x_n, y_n) = \lim_{n\to\infty} gy_n \in X \quad \text{and}$$
$$\lim_{m\to\infty} F(z_n, y_n, x_n) = \lim_{n\to\infty} gz_n \in X.$$

**Definition 11.4.4.** Let $(X, G)$ be a $G^*$-metric space and let $F : X^3 \to X$ and $g : X \to X$ be two mappings. We will say that $(F, g)$ is an *O-compatible pair* if we have that

$$\lim_{n\to\infty} G\left(gF(x_n, y_n, z_n), F(gx_n, gy_n, gz_n), F(gx_n, gy_n, gz_n)\right) = 0,$$
$$\lim_{n\to\infty} G\left(gF(y_n, x_n, y_n), F(gy_n, gx_n, gy_n), F(gy_n, gx_n, gy_n)\right) = 0 \quad \text{and}$$
$$\lim_{n\to\infty} G\left(gF(z_n, y_n, x_n), F(gz_n, gy_n, gx_n), F(gz_n, gy_n, gx_n)\right) = 0$$

whenever $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are sequences in $X$ such that

$$\lim_{m\to\infty} F(x_n, y_n, z_n) = \lim_{n\to\infty} gx_n \in X,$$
$$\lim_{m\to\infty} F(y_n, x_n, y_n) = \lim_{n\to\infty} gy_n \in X \quad \text{and}$$
$$\lim_{m\to\infty} F(z_n, y_n, x_n) = \lim_{n\to\infty} gz_n \in X.$$

*Remark 11.4.1.* If $F$ and $g$ are commuting, then $(F,g)$ is an $(O,\preceq)$-compatible pair and an $O$-compatible pair.

**Theorem 11.4.3.** *Let $(X,G)$ be a G-metric space endowed with a preorder $\preceq$ and let $F : X^3 \to X$ and $g : X \to X$ be two mappings such that $F$ has the mixed $(g,\preceq)$-monotone property. Assume that the following conditions hold.*

**(i)** *There exist two functions $\psi \in \mathcal{F}_{\text{alt}}$ and $\varphi \in \mathcal{F}'_{\text{alt}}$ such that*

$$\begin{aligned}\psi\left(G\left(F\left(x,y,z\right),F\left(u,v,w\right),F\left(a,b,c\right)\right)\right)\\ \leq (\psi-\varphi)\Big(\max\left\{G\left(gx,gu,ga\right),G\left(gy,gv,gb\right),G\left(gz,gw,gc\right)\right\}\Big)\end{aligned} \tag{11.43}$$

*for all $(x,y,z),(u,v,w),(a,b,c)\in X^3$ for which $gx\preceq gu\preceq ga$, $gy\succeq gv\succeq gb$ and $gz\preceq gw\preceq gc$.*

**(ii)** *At least, one of the following conditions holds.*

**(ii.1)** *$F(X^3)\subseteq g(X)$ and there exist $x_0,y_0,z_0\in X$ such that $gx_0\preceq F(x_0,y_0,z_0)$, $gy_0\succeq F(y_0,x_0,y_0)$ and $gz_0\preceq F(z_0,y_0,x_0)$.*

**(ii.2)** *There exists a Picard $(F,g)$-sequence $\{(x_n,y_n,z_n)\}_{n\in\mathbb{N}}\subseteq X^3$ such that $gx_0\preceq F(x_0,y_0,z_0)$, $gy_0\succeq F(y_0,x_0,y_0)$ and $gz_0\preceq F(z_0,y_0,x_0)$.*

**(iii)** *At least, one of the following conditions holds.*

**(iii.1)** *$(X,G)$ (or $(g(X),G)$ or $\big(F(X^3),G\big)$) is complete and $F$ and $g$ are continuous and $(O,\preceq)$-compatible.*

**(iii.2)** *$(X,G)$ (or $(g(X),G)$ or $\big(F(X^3),G\big)$) is complete and $F$ and $g$ are continuous and commuting.*

**(iii.3)** *$(g(X),G)$ is complete and $(X,G,\preceq)$ is regular.*

*Then $F$ and $g$ have, at least, a tripled coincidence point.*

*Proof.* By Lemmas 11.4.1 and 11.4.2, (ii.1)$\Rightarrow$(ii.2). We present the proof assuming (ii.2). From Lemma 11.4.2, $\{gx_n\}$ and $\{gz_n\}$ are $\preceq$-non-decreasing, and $\{gy_n\}$ is $\preceq$-non-increasing. As $\preceq$ is transitive, we deduce that $gx_n\preceq gx_m$, $gy_n\succeq gy_m$ and $gx_n\preceq gx_m$ for all $n,m\in\mathbb{N}$ such that $n\leq m$. If there exists some $n_0\in\mathbb{N}$ such that $(gx_{n_0},gy_{n_0},gz_{n_0})=(gx_{n_0+1},gy_{n_0+1},gz_{n_0+1})$, then

$$\begin{aligned} gx_{n_0} &= gx_{n_0+1} = F\left(x_{n_0},y_{n_0},z_{n_0}\right),\\ gy_{n_0} &= gy_{n_0+1} = F\left(y_{n_0},x_{n_0},y_{n_0}\right) \quad \text{and}\\ gz_{n_0} &= gz_{n_0+1} = F\left(z_{n_0},y_{n_0},x_{n_0}\right),\end{aligned}$$

so $(x_{n_0},y_{n_0},z_{n_0})$ is a tripled coincidence point of $F$ and $g$. Next, assume that

$$(gx_n,gy_n,gz_n)\neq(gx_{n+1},gy_{n+1},gz_{n+1}) \quad \text{for all } n\in\mathbb{N}. \tag{11.44}$$

For convenience, let define, for all $n, m \in \mathbb{N}$ such that $n < m$,

$$\begin{aligned}
T(n,m) &= \max\{G(gx_n, gx_m, gx_m),\\
&\qquad\qquad G(gy_n, gy_m, gy_m), G(gz_n, gz_m, gz_m)\},\\
S(n,m) &= \max\{G(gx_n, gx_m, gx_m), G(gy_n, gy_m, gy_m)\},\\
t_n &= T(n, n+1),\\
s_n &= S(n, n+1).
\end{aligned}$$

Condition (11.44) is equivalent to

$$t_n > 0 \quad \text{for all } n \in \mathbb{N}.$$

By the contractivity condition (11.43), for all $n \in \mathbb{N}$ we have, taking into account that $gx_n \preceq gx_{n+1} \preceq gx_{n+1}$, $gy_n \succeq gy_{n+1} \succeq gy_{n+1}$ and $gz_n \preceq gz_{n+1} \preceq gz_{n+1}$,

$$\begin{aligned}
&\psi(G(gx_{n+1}, gx_{n+2}, gx_{n+2}))\\
&\quad = \psi(G(F(x_n, y_n, z_n), F(x_{n+1}, y_{n+1}, z_{n+1}), F(x_{n+1}, y_{n+1}, z_{n+1})))\\
&\quad \le (\psi - \varphi)\Big(\max\{G(gx_n, gx_{n+1}, gx_{n+1}),\\
&\qquad\qquad G(gy_n, gy_{n+1}, gy_{n+1}), G(gz_n, gz_{n+1}, gz_{n+1})\}\Big)\\
&\quad = \psi(t_n) - \phi(t_n).
\end{aligned} \tag{11.45}$$

Using that $gy_{n+1} \preceq gy_{n+1} \preceq gy_n$ and $gx_{n+1} \succeq gx_{n+1} \succeq gx_n$, we deduce that

$$\begin{aligned}
&\psi(G(gy_{n+2}, gy_{n+2}, gy_{n+1}))\\
&\quad = \psi(G(F(y_{n+1}, x_{n+1}, y_{n+1}), F(y_{n+1}, x_{n+1}, y_{n+1}), F(y_n, x_n, y_n)))\\
&\quad \le (\psi - \varphi)\Big(\max\{G(gy_{n+1}, gy_{n+1}, gy_n), G(gx_{n+1}, gx_{n+1}, gx_n)\}\Big)\\
&\quad = \psi(s_n) - \phi(s_n).
\end{aligned} \tag{11.46}$$

In the same way, since $gz_n \preceq gz_{n+1}$, $gy_n \succeq gy_{n+1}$ and $gx_n \preceq gx_{n+1}$, we also have that

$$\begin{aligned}
&\psi(G(gz_{n+1}, gz_{n+2}, gx_{n+2}))\\
&\quad = \psi(G(F(z_n, y_n, x_n), F(z_{n+1}, y_{n+1}, x_{n+1}), F(z_{n+1}, y_{n+1}, x_{n+1})))\\
&\quad \le (\psi - \varphi)\Big(\max\{G(gz_n, gz_{n+1}, gz_{n+1}),
\end{aligned}$$

$$G\left(gy_{n}, gy_{n+1}, gy_{n+1}\right), G\left(gx_{n}, gx_{n+1}, gx_{n+1}\right)\}\Big)$$
$$= \psi\left(t_{n}\right) - \phi\left(t_{n}\right). \tag{11.47}$$

Notice that the right-hand term of inequality (11.46) is different from the right-hand terms of inequalities (11.45) and (11.47). However, as $\psi$ is non-decreasing, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\psi\left(t_{n+1}\right) = \psi\Big(&\max\left\{G\left(gx_{n+1}, gx_{n+2}, gx_{n+2}\right), G\left(gy_{n+1}, gy_{n+2}, gy_{n+2}\right),\right.\\
&\left.G\left(gz_{n+1}, gz_{n+2}, gx_{n+2}\right)\right\}\Big)\\
= \max\Big\{&\psi\left(G\left(gx_{n+1}, gx_{n+2}, gx_{n+2}\right)\right), \psi\left(G\left(gy_{n+1}, gy_{n+2}, gy_{n+2}\right)\right)\\
&\psi\left(G\left(gz_{n+1}, gz_{n+2}, gx_{n+2}\right)\right)\Big\}\\
\leq \max\{&\psi\left(t_{n}\right) - \phi\left(t_{n}\right), \psi\left(s_{n}\right) - \phi\left(s_{n}\right)\}.
\end{aligned}$$

From item (3) of Lemma 2.3.7, we deduce that $\{t_n\} \to 0$. In particular,

$$\begin{aligned}
&\{G\left(gx_{n}, gx_{n+1}, gx_{n+1}\right)\} \to 0, \quad \{G\left(gy_{n}, gy_{n+1}, gy_{n+1}\right)\} \to 0\\
&\text{and} \quad \{G\left(gz_{n}, gz_{n+1}, gz_{n+1}\right)\} \to 0,
\end{aligned}$$

that is, the sequences $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are asymptotically regular.

Next, we show that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences on $(X, G)$. We reason by contradiction assuming that one of them is not Cauchy in $(X, G)$. In such a case, by Lemma 11.2.1, there exist $\varepsilon_0 > 0$ and two sequences of natural numbers $\{n(k)\}_{k\in\mathbb{N}}$ and $\{m(k)\}_{k\in\mathbb{N}}$ such that, for all $k \in \mathbb{N}$,

$$\begin{aligned}
&k \leq n(k) < m(k) < n(k+1),\\
&\max\left\{G\left(gx_{n(k)}, gx_{m(k)-1}, gx_{m(k)-1}\right), G\left(gy_{n(k)}, gy_{m(k)-1}, gy_{m(k)-1}\right),\right.\\
&\qquad\left.G\left(gz_{n(k)}, gz_{m(k)-1}, gz_{m(k)-1}\right)\right\}\\
&\quad \leq \varepsilon_0 < \max\left\{G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right),\right.\\
&\qquad\left.G\left(gy_{n(k)}, gy_{m(k)}, gy_{m(k)}\right), G\left(gz_{n(k)}, gz_{m(k)}, gz_{m(k)}\right)\right\},
\end{aligned}$$

and also

$$\begin{aligned}
\lim_{k\to\infty}\Big[\max\Big\{&G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right), G\left(gy_{n(k)}, gy_{m(k)}, gy_{m(k)}\right),\\
&G\left(gz_{n(k)}, gz_{m(k)}, gz_{m(k)}\right)\Big\}\Big]
\end{aligned}$$

$$= \lim_{k\to\infty} \Big[ \max \Big\{ G\left(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}\right), \\ G\left(gy_{n(k)-1}, gy_{m(k)-1}, gy_{m(k)-1}\right), \\ G\left(gz_{n(k)-1}, gz_{m(k)-1}, gz_{m(k)-1}\right) \Big\} \Big] = \varepsilon_0. \tag{11.48}$$

Moreover, the number $i_0$ in Lemma 11.2.1 guarantees that, at least, one of the following conditions holds:

$$\lim_{k\to\infty} G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right) = \lim_{k\to\infty} G\left(gx_{n(k)}, gx_{m(k)-1}, gx_{m(k)-1}\right) = \varepsilon_0, \tag{11.49}$$

$$\lim_{k\to\infty} G\left(gy_{n(k)}, gy_{m(k)}, gy_{m(k)}\right) = \lim_{k\to\infty} G\left(gy_{n(k)}, gy_{m(k)-1}, gy_{m(k)-1}\right) = \varepsilon_0, \tag{11.50}$$

$$\lim_{k\to\infty} G\left(gz_{n(k)}, gz_{m(k)}, gz_{m(k)}\right) = \lim_{k\to\infty} G\left(gz_{n(k)}, gz_{m(k)-1}, gz_{m(k)-1}\right) = \varepsilon_0. \tag{11.51}$$

In order to apply Lemma 2.3.8, let consider the sequences $\{t_k\}$, $\{s_k\}$ and $\{r_k\}$ given by

$$t_k = T(n(k), m(k)), \quad s_k = T(n(k)-1, m(k)-1)$$
$$\text{and} \quad r_k = S(n(k)-1, m(k)-1) \quad \text{for all } k \in \mathbb{N}.$$

Clearly, $r_k \le s_k$ for all $k \in \mathbb{N}$. Moreover, if we define $L = \varepsilon_0$, conditions (11.48) means that $\{t_k\} \to L$ and $\{s_k\} \to L$. Since $n(k) < m(k)$, we have that $gx_{n(k)-1} \preceq gx_{m(k)-1} \preceq gx_{m(k)-1}$, $gy_{n(k)-1} \succeq gy_{m(k)-1} \succeq gy_{m(k)-1}$ and $gz_{n(k)-1} \preceq gz_{m(k)-1} \preceq gz_{m(k)-1}$ for all $k \in \mathbb{N}$. By the contractivity condition (11.43),

$$\psi\left(G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right)\right)$$
$$= \psi\Big(G\Big(F\left(x_{n(k)-1}, y_{n(k)-1}, z_{n(k)-1}\right), \\ F\left(x_{m(k)-1}, y_{m(k)-1}, z_{m(k)-1}\right), \\ F\left(x_{m(k)-1}, y_{m(k)-1}, z_{m(k)-1}\right)\Big)\Big)$$
$$\le (\psi - \varphi)\Big(\max\Big\{G\left(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}\right), \\ G\left(gy_{n(k)-1}, gy_{m(k)-1}, gy_{m(k)-1}\right), \\ G\left(gz_{n(k)-1}, gz_{m(k)-1}, gz_{m(k)-1}\right)\Big\}\Big)$$
$$= (\psi - \varphi)\left(T\left(n(k)-1, m(k)-1\right)\right) = \psi\left(s_k\right) - \varphi\left(s_k\right).$$

Furthermore, as $gy_{m(k)-1} \preceq gy_{m(k)-1} \preceq gy_{n(k)-1}$ and $gx_{m(k)-1} \succeq gx_{m(k)-1} \succeq gx_{n(k)-1}$, then

$$\begin{aligned}
&\psi\left(G\left(gy_{n(k)}, gy_{m(k)}, gy_{m(k)}\right)\right) \\
&\quad = \psi\Big(G\Big(F\left(y_{m(k)-1}, x_{m(k)-1}, y_{m(k)-1}\right), \\
&\qquad\qquad F\left(y_{m(k)-1}, x_{m(k)-1}, y_{m(k)-1}\right), \\
&\qquad\qquad F\left(y_{n(k)-1}, x_{n(k)-1}, y_{n(k)-1}\right)\Big)\Big) \\
&\quad \leq (\psi-\varphi)\Big(\max\Big\{G\left(gy_{n(k)-1}, gy_{m(k)-1}, gy_{m(k)-1}\right), \\
&\qquad\qquad G\left(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}\right)\Big\}\Big) \\
&\quad = (\psi-\varphi)\left(S\left(n(k)-1, m(k)-1\right)\right) = \psi\left(r_k\right) - \varphi\left(r_k\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\psi\left(G\left(gz_{n(k)}, gz_{m(k)}, gz_{m(k)}\right)\right) \\
&\quad = \psi\Big(G\Big(F\left(z_{n(k)-1}, y_{n(k)-1}, x_{n(k)-1}\right), \\
&\qquad\qquad F\left(z_{m(k)-1}, y_{m(k)-1}, x_{m(k)-1}\right), \\
&\qquad\qquad F\left(z_{m(k)-1}, y_{m(k)-1}, x_{m(k)-1}\right)\Big)\Big) \\
&\quad \leq (\psi-\varphi)\Big(\max\Big\{G\left(gz_{n(k)-1}, gz_{m(k)-1}, gz_{m(k)-1}\right), \\
&\qquad\qquad G\left(gy_{n(k)-1}, gy_{m(k)-1}, gy_{m(k)-1}\right), \\
&\qquad\qquad G\left(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}\right)\Big\}\Big) \\
&\quad = (\psi-\varphi)\left(T\left(n(k)-1, m(k)-1\right)\right) = \psi\left(s_k\right) - \varphi\left(s_k\right).
\end{aligned}$$

Combining the last three inequalities and taking into account that $\psi$ is non-decreasing, if follows that, for all $k \in \mathbb{N}$,

$$\begin{aligned}
\psi\left(t_k\right) &= \psi\Big(\max\Big\{G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right), G\left(gy_{n(k)}, gy_{m(k)}, gy_{m(k)}\right) \\
&\qquad G\left(gz_{n(k)}, gz_{m(k)}, gz_{m(k)}\right)\Big\}\Big) \\
&= \max\Big\{\psi\left(G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right)\right), \\
&\qquad \psi\left(G\left(gy_{m(k)}, gy_{m(k)}, gy_{n(k)}\right)\right),
\end{aligned}$$

$$\psi\left(G\left(gz_{n(k)}, gz_{m(k)}, gz_{m(k)}\right)\right)\Big\}$$

$$\leq \max\left\{\psi\left(s_k\right) - \varphi\left(s_k\right), \psi\left(r_k\right) - \varphi\left(r_k\right)\right\}.$$

Applying Lemma 2.3.8, we conclude that $\varepsilon_0 = L = 0$, which is a contradiction. As a consequence, $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ must be Cauchy sequences on $(X, G)$. To continue the proof, we distinguish some cases.

*Case (iii.1). Assume that* $(X, G)$ *(or* $(g(X), G)$ *or* $\left(F(X^3), G\right)$*) is complete and* $F$ *and* $g$ *are continuous and* $(O, \preceq)$*-compatible.* In such a case, as the sequences $\{gx_{n+1} = F(x_n, y_n, z_n)\}$, $\{gy_{n+1} = F(y_n, x_n, y_n)\}$ and $\{gz_{n+1} = F(z_n, y_n, x_n)\}$ belong to $F(X^3) \subseteq g(X) \subseteq X$, and one of these spaces is $G$-complete, there exist $u, v, w \in X$ such that $\{gx_n\} \to u$, $\{gy_n\} \to v$ and $\{gz_n\} \to w$. As $F$ and $g$ are continuous, we deduce that

$$\begin{aligned}
&\{ggx_n\} \to gu, \quad \{ggy_n\} \to gv, \quad \{ggz_n\} \to gw,\\
&\{F\left(gx_n, gy_n, gz_n\right)\} \to F(u, v, w), \quad \{F\left(gy_n, gx_n, gy_n\right)\} \to F(v, u, v),\\
&\{F\left(gz_n, gy_n, gx_n\right)\} \to F(w, v, u).
\end{aligned}$$

Therefore, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are sequences in $X$ such that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are $\preceq$-monotone and

$$\begin{aligned}
&\lim_{m\to\infty} F(x_n, y_n, z_n) = \lim_{n\to\infty} gx_n = u \in X,\\
&\lim_{m\to\infty} F(y_n, x_n, y_n) = \lim_{n\to\infty} gy_n = v \in X \quad \text{and}\\
&\lim_{m\to\infty} F(z_n, y_n, x_n) = \lim_{n\to\infty} gz_n = w \in X,
\end{aligned}$$

Since $F$ and $g$ are $(O, \preceq)$-compatible, we have that

$$\begin{aligned}
&\lim_{n\to\infty} G\left(gF(x_n, y_n, z_n), F(gx_n, gy_n, gz_n), F(gx_n, gy_n, gz_n)\right) = 0,\\
&\lim_{n\to\infty} G\left(gF(y_n, x_n, y_n), F(gy_n, gx_n, gy_n), F(gy_n, gx_n, gy_n)\right) = 0 \quad \text{and}\\
&\lim_{n\to\infty} G\left(gF(z_n, y_n, x_n), F(gz_n, gy_n, gx_n), F(gz_n, gy_n, gx_n)\right) = 0
\end{aligned}$$

In particular

$$\begin{aligned}
&G\left(gu, F\left(u, v, w\right), F\left(u, v, w\right)\right)\\
&\quad = \lim_{n\to\infty} G\left(ggx_{n+1}, F\left(gx_n, gy_n, gz_n\right), F\left(gx_n, gy_n, gz_n\right)\right)\\
&\quad = \lim_{n\to\infty} G\left(gF(x_n, y_n, z_n), F\left(gx_n, gy_n, gz_n\right), F\left(gx_n, gy_n, gz_n\right)\right) = 0.
\end{aligned}$$

It follows that $gu = F(u, v, w)$. In the same way,

$$\begin{aligned} G(gv, F(v, u, v), F(v, u, v)) &= \lim_{n\to\infty} G(ggy_{n+1}, F(gy_n, gx_n, gy_n), F(gy_n, gx_n, gy_n)) \\ &= \lim_{n\to\infty} G(gF(y_n, x_n, y_n), F(gy_n, gx_n, gy_n), F(gy_n, gx_n, gy_n)) = 0, \end{aligned}$$

so $gv = F(v, u, v)$. Finally, as

$$\begin{aligned} G(gw, F(w, v, u), F(w, v, u)) &= \lim_{n\to\infty} G(ggz_{n+1}, F(gz_n, gy_n, gx_n), F(gz_n, gy_n, gx_n)) \\ &= \lim_{n\to\infty} G(gF(z_n, y_n, x_n), F(gz_n, gy_n, gx_n), F(gz_n, gy_n, gx_n)) = 0, \end{aligned}$$

it follows that $gw = F(w, v, u)$. As a consequence, $(u, v, w)$ is a tripled coincidence point of $F$ and $g$.

*Case (iii.2).* *Assume that* $(X, G)$ *(or* $(g(X), G)$ *or* $\big(F(X^3), G\big)$*) is complete and* $F$ *and* $g$ *are continuous and commuting.* It follows from item (iii.1) because if $F$ and $g$ are commuting, then they are also $(O, \preceq)$-compatible.

*Case (iii.3).* *Assume that* $(g(X), G)$ *is complete and* $(X, G, \preceq)$ *is regular.* Since $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences on $(g(X), G)$, there exist $u', v', w' \in g(X)$ such that $\{gx_n\} \to u'$, $\{gy_n\} \to v'$ and $\{gz_n\} \to w'$. Let $u, v, w \in X$ be arbitrary points such that $gu = u'$, $gv = v'$ and $gw = w'$. As $(X, G, \preceq)$ is regular, and the sequences $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are convergent and $\preceq$-monotone, we deduce that $gx_n \preceq gu$, $gy_n \succeq gv$ and $gz_n \preceq gw$ for all $n \in \mathbb{N}$. Therefore, applying the contractivity condition (11.43) to $gx_n \preceq gu \preceq gu$, $gy_n \succeq gv \succeq gv$ and $gz_n \preceq gw \preceq gw$, we obtain

$$\begin{aligned} &\psi(G(gx_{n+1}, F(u, v, w), F(u, v, w))) \\ &\quad = \psi(G(F(x_n, y_n, z_n), F(u, v, w), F(u, v, w))) \\ &\quad \le (\psi - \varphi)\Big(\max\{G(gx_n, gu, gu), \\ &\qquad\qquad G(gy_n, gv, gv), G(gz_n, gw, gw)\}\Big) \\ &\quad \le \psi\Big(\max\{G(gx_n, gu, gu), G(gy_n, gv, gv), G(gz_n, gw, gw)\}\Big). \end{aligned}$$

As $\{gx_n\} \to gu$, $\{gy_n\} \to gv$, $\{gz_n\} \to gw$ and $\psi$ is continuous, we deduce that

$$\begin{aligned} &\psi(G(gu, F(u, v, w), F(u, v, w))) \\ &\quad = \lim_{n\to\infty} \psi(G(gx_{n+1}, F(u, v, w), F(u, v, w))) \\ &\quad \le \psi(\max\{0, 0, 0\}) = 0, \end{aligned}$$

so $gu = F(u, v, w)$. In the same way, applying the contractivity condition (11.43) to $gv \preceq gv \preceq gy_n$ and $gu \succeq gu \succeq gx_n$, we obtain

$$\begin{aligned}
&\psi\left(G\left(F\left(v,u,v\right),F\left(v,u,v\right),gy_{n+1}\right)\right) \\
&\quad= \psi\left(G\left(F\left(v,u,v\right),F\left(v,u,v\right),F\left(y_n,x_n,y_n\right)\right)\right) \\
&\quad\le \left(\psi-\varphi\right)\left(\max\left\{G\left(gv,gv,gy_n\right),G\left(gu,gu,gx_n\right)\right\}\right) \\
&\quad\le \psi\left(\max\left\{G\left(gv,gv,gy_n\right),G\left(gu,gu,gx_n\right)\right\}\right).
\end{aligned}$$

again, and letting $n \to \infty$ we deduce that

$$\begin{aligned}
&\psi\left(G\left(F\left(v,u,v\right),F\left(v,u,v\right),gv\right)\right) \\
&\quad= \lim_{n\to\infty} \psi\left(F\left(v,u,v\right),F\left(v,u,v\right),gy_{n+1}\right) \\
&\quad\le \psi\left(\max\{0,0\}\right) = 0,
\end{aligned}$$

and, therefore, $gv = F(v, u, v)$. Repeating the previous arguments, we can show that $gw = F(w, v, u)$. Thus, we conclude that $(u, v, w)$ is a tripled coincidence point of $F$ and $g$. □

We leave to the reader to particularize Theorem 11.4.3 as we did it in Sect. 11.3.5. We only include the following results, which can be considered as extensions of Borcut and Berinde's Theorem 4 in [50].

**Corollary 11.4.7.** *Let $(X, G)$ be a complete $G$-metric space endowed with a preorder $\preceq$ and let $F : X^3 \to X$ and $g : X \to X$ be two mappings such that $F(X^3) \subseteq g(X)$ and $F$ has the mixed $(g, \preceq)$-monotone property. Suppose that there exists $\lambda \in [0, 1)$ such that*

$$\begin{aligned}
&G\left(F\left(x,y,z\right),F\left(u,v,w\right),F\left(a,b,c\right)\right) \\
&\qquad\le \lambda\, \max\left\{G\left(gx,gu,ga\right),G\left(gy,gv,gb\right),G\left(gz,gw,gc\right)\right\}
\end{aligned} \tag{11.52}$$

*for all $(x, y, z), (u, v, w), (a, b, c) \in X^3$ for which $gx \preceq gu \preceq ga$, $gy \succeq gv \succeq gb$ and $gz \preceq gw \preceq gc$. Also assume that, at least, one of the following conditions holds.*

**(a)** *$(X, G)$ (or $(g(X), G)$ or $\left(F(X^3), G\right)$) is complete and $F$ and $g$ are continuous and commuting.*
**(b)** *$(g(X), G)$ is complete and $(X, G, \preceq)$ is regular.*

**Theorem 11.4.4.** *Then $F$ and $g$ have, at least, a tripled coincidence point.*

*Proof.* It follows from Theorem 11.4.3 using $\psi(t) = t$ and $\phi(t) = (1 - \lambda)\, t$ for all $t \in [0, \infty)$. □

**Corollary 11.4.8.** *Let $(X, G)$ be a complete G-metric space endowed with a partial order $\preccurlyeq$ and let $F : X^3 \to X$ and $g : X \to X$ be two mappings such that $F(X^3) \subseteq g(X)$ and $F$ has the mixed $(g, \preccurlyeq)$-monotone property. Suppose that there exist $\lambda_1, \lambda_2, \lambda_3 \in [0, 1)$ such that $\lambda_1 + \lambda_2 + \lambda_3 < 1$ and verifying*

$$\begin{aligned} &G\left(F\left(x,y,z\right),F\left(u,v,w\right),F\left(a,b,c\right)\right) \\ &\qquad \leq \lambda_1 G\left(gx,gu,ga\right) + \lambda_2 G\left(gy,gv,gb\right) + \lambda_3 G\left(gz,gw,gc\right) \end{aligned} \tag{11.53}$$

*for all $(x, y, z), (u, v, w), (a, b, c) \in X^3$ for which $gx \preccurlyeq gu \preccurlyeq ga$, $gy \succcurlyeq gv \succcurlyeq gb$ and $gz \preccurlyeq gw \preccurlyeq gc$. Also assume that, at least, one of the following conditions holds.*

**(a)** *$(X, G)$ (or $(g(X), G)$ or $\left(F(X^3), G\right)$) is complete and $F$ and $g$ are continuous and commuting.*
**(b)** *$(g(X), G)$ is complete and $(X, G, \preccurlyeq)$ is regular.*

**Theorem 11.4.5.** *Then $F$ and $g$ have, at least, a tripled coincidence point.*

*Proof.* It follows reasoning as in Corollary 11.4.2 because (11.53) implies (11.52). □

## 11.5 Karapınar's Quadrupled Fixed Point Theory

In [110], Karapınar introduced the notion of a quadrupled fixed point of a mapping $F : X^4 \to X$ as a point $(x, y, z, \omega) \in X^4$ such that

$$\begin{cases} F\left(x,y,z,\omega\right) = x, \\ F\left(y,z,\omega,x\right) = y, \\ F\left(z,\omega,x,y\right) = z, \\ F\left(\omega,x,y,z\right) = \omega. \end{cases}$$

All arguments given in Sects. 11.3 and 11.4, can now be repeated. For more details, see also [113, 117].

**Definition 11.5.1.** Let $X$ be a non-empty set endowed with a binary relation $\preccurlyeq$ and let $F : X^4 \to X$ and $g : X \to X$ be two mappings. The mapping $F$ is said to have the *mixed* $(g, \preccurlyeq)$*-monotone property* if $F(x, y, z, \omega)$ is monotone $(g, \preccurlyeq)$-non-decreasing in $x$ and in $z$, and monotone $(g, \preccurlyeq)$-non-increasing in $y$ and in $\omega$, that is, for all $x, y, z, \omega \in X$,

$$\begin{aligned} x_1, x_2 \in X, \quad gx_1 \preccurlyeq gx_2 \quad &\Rightarrow \quad F(x_1, y, z, \omega) \preccurlyeq F(x_2, y, z, \omega), \\ y_1, y_2 \in X, \quad gy_1 \preccurlyeq gy_2 \quad &\Rightarrow \quad F(x, y_1, z, \omega) \succcurlyeq F(x, y_2, z, \omega), \end{aligned}$$

$$z_1, z_2 \in X, \quad gz_1 \preccurlyeq gz_2 \quad \Rightarrow \quad F(x, y, z_1, \omega) \preccurlyeq F(x, y, z_2, \omega) \quad \text{and}$$
$$\omega_1, \omega_2 \in X, \quad g\omega_1 \preccurlyeq g\omega_2 \quad \Rightarrow \quad F(x, y, z, \omega_1) \succcurlyeq F(x, y, z, \omega_2).$$

If $g$ is the identity mapping on $X$, then we say that $F$ has the *mixed* $\preccurlyeq$*-monotone property*.

**Theorem 11.5.1.** *Let* $(X, G)$ *be a G-metric space endowed with a preorder* $\preceq$ *and let* $F : X^4 \to X$ *and* $g : X \to X$ *be two mappings such that* $F$ *has the mixed* $(g, \preceq)$*-monotone property. Assume that the following conditions hold.*

**(i)** *There exist two functions* $\psi \in \mathcal{F}_{\text{alt}}$ *and* $\varphi \in \mathcal{F}'_{\text{alt}}$ *such that*

$$\begin{aligned} &\psi \left(G\left(F\left(x, y, z, t\right), F\left(u, v, w, p\right), F\left(a, b, c, d\right)\right)\right) \\ &\quad \leq (\psi - \varphi)\left(\max\left\{G\left(gx, gu, ga\right), G\left(gy, gv, gb\right),\right.\right. \\ &\qquad \left.\left. G\left(gz, gw, gc\right), G\left(gt, gp, gd\right)\right\}\right) \end{aligned}$$

*for all* $(x, y, z, t), (u, v, w, p), (a, b, c, d) \in X^4$ *for which* $gx \preceq gu \preceq ga$*,* $gy \succeq gv \succeq gb$*,* $gz \preceq gw \preceq gc$ *and* $gt \succeq gp \succeq gd$*.*

**(ii)** *At least, one of the following conditions holds.*

- **(ii.1)** $F(X^4) \subseteq g(X)$ *and there exist* $x_0, y_0, z_0, t_0 \in X$ *such that* $gx_0 \preceq F(x_0, y_0, z_0, t_0)$*,* $gy_0 \succeq F(y_0, z_0, t_0, x_0)$*,* $gz_0 \preceq F(z_0, t_0, x_0, y_0)$ *and* $gt_0 \succeq F(t_0, x_0, y_0, z_0)$*.*
- **(ii.2)** *There exists a Picard* $(F, g)$*-sequence* $\{(x_n, y_n, z_n, t_n)\}_{n\in\mathbb{N}} \subseteq X^4$ *such that* $gx_0 \preceq F(x_0, y_0, z_0, t_0)$*,* $gy_0 \succeq F(y_0, z_0, t_0, x_0)$*,* $gz_0 \preceq F(z_0, t_0, x_0, y_0)$ *and* $gt_0 \succeq F(t_0, x_0, y_0, z_0)$*.*

**(iii)** *At least, one of the following conditions holds.*

- **(iii.1)** $(X, G)$ *(or* $(g(X), G)$ *or* $\left(F(X^4), G\right)$*) is complete and* $F$ *and* $g$ *are continuous and commuting.*
- **(iii.2)** $(g(X), G)$ *is complete and* $(X, G, \preceq)$ *is regular.*

*Then* $F$ *and* $g$ *have, at least, a quadrupled coincidence point.*

## 11.6 Roldán et al.'s Multidimensional Fixed Point Theory

Inspired by the previous notions of coupled, tripled and quadrupled fixed point, and the Berinde and Borcut's condition $F(y, x, y) = y$, in 2012, Roldán et al. [174] introduced the following notion of multidimensional fixed point using two mappings $F : X^n \to X$ and $g : X \to X$ (see also [178]).

### 11.6.1 The Notion of Multidimensional Fixed Point

Henceforth, let $\{\mathsf{A}, \mathsf{B}\}$ be a partition of $\Lambda_n = \{1, 2, \ldots, n\}$, that is, $\mathsf{A}$ and $\mathsf{B}$ are non-empty subsets of $\Lambda_n$ such that $\mathsf{A} \cup \mathsf{B} = \Lambda_n$ and $\mathsf{A} \cap \mathsf{B} = \varnothing$. From now on, let $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be a $n$-tuple of mappings from $\{1, 2, \ldots, n\}$ into itself.

**Definition 11.6.1 (Roldán et al. [174]).** Given two mappings $F : X^n \to X$ and $g : X \to X$, we say that a point $(x_1, x_2, \ldots, x_n) \in X^n$ is a

- $\Phi$-*fixed point of* $F$ if

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}) = x_i \quad \text{for all } i \in \{1, 2, \ldots, n\};$$

- $\Phi$-*coincidence point of* $F$ *and* $g$ if

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}) = gx_i \quad \text{for all } i \in \{1, 2, \ldots, n\};$$

- $\Phi$-*common fixed point of* $F$ *and* $g$ if

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}) = gx_i = x_i \quad \text{for all } i \in \{1, 2, \ldots, n\}.$$

If we represent a mapping $\sigma : \Lambda_n \to \Lambda_n$ throughout its ordered image, i.e., $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$, then:

- the Gnana-Bhaskar and Lakshmikantham's condition in $n = 2$ is $\sigma_1 = (1, 2)$ and $\sigma_2 = (2, 1)$;
- the Berinde and Borcut's condition in $n = 3$ is $\sigma_1 = (1, 2, 3)$, $\sigma_2 = (2, 1, 2)$ and $\sigma_2 = (3, 2, 1)$;
- the Karapınar's condition in $n = 4$ is $\sigma_1 = (1, 2, 3, 4)$, $\sigma_2 = (2, 3, 4, 1)$, $\sigma_3 = (3, 4, 1, 2)$ and $\sigma_4 = (4, 1, 2, 3)$;
- the cyclic condition is $\sigma_i = (i, i+1, \ldots, n, 1, 2, \ldots, i-1)$ for all $i \in \{1, 2, \ldots, n\}$.

Low dimensional cases consider $\mathsf{A}$ as the odd numbers in $\{1, 2, \ldots, n\}$ and $\mathsf{B}$ as its even numbers. Another definition was due to Berzig and Samet [42], who used $\mathsf{A} = \{1, 2, \ldots, m\}$, $\mathsf{B} = \{m+1, \ldots, n\}$ and arbitrary mappings between them.

As we shall see, Roldán et al. succeeded in proving existence and uniqueness of multidimensional fixed (or coincidence) points when $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a $n$-tuple of mappings from $\{1, 2, \ldots, n\}$ into itself satisfying $\sigma_i \in \Omega_{\mathsf{A},\mathsf{B}}$ if $i \in \mathsf{A}$ and $\sigma_i \in \Omega'_{\mathsf{A},\mathsf{B}}$ if $i \in \mathsf{B}$, where

$$\Omega_{\mathsf{A},\mathsf{B}} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(\mathsf{A}) \subseteq \mathsf{A} \text{ and } \sigma(\mathsf{B}) \subseteq \mathsf{B}\},$$

$$\Omega'_{\mathsf{A},\mathsf{B}} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(\mathsf{A}) \subseteq \mathsf{B} \text{ and } \sigma(\mathsf{B}) \subseteq \mathsf{A}\}.$$

In order to prove a multidimensional result, we need to extend the notion of the mixed monotone property.

Given a binary relation $\preccurlyeq$ on $X$ and $i \in \Lambda_n$, let denote by $\preccurlyeq_i$ the binary relation $\preccurlyeq$, if $i \in \mathsf{A}$, and the binary relation $\succcurlyeq$, if $i \in \mathsf{B}$. In other words, for all $x, y \in X$,

$$x \preccurlyeq_i y \quad \Leftrightarrow \quad \begin{cases} x \preccurlyeq y, \text{ if } i \in \mathsf{A}, \\ x \succcurlyeq y, \text{ if } i \in \mathsf{B}. \end{cases} \tag{11.54}$$

**Definition 11.6.2 (Roldán et al. [174]).** Let $\preccurlyeq$ be a binary relation on a set $X$ and let $F : X^n \to X$ and $g : X \to X$ be two mappings. We say that $F$ has the *mixed* $(g, \preccurlyeq)$*-monotone property (w.r.t.* $\{\mathsf{A}, \mathsf{B}\}$*)* if $F$ is monotone $(g, \preccurlyeq)$-non-decreasing in arguments of $\mathsf{A}$ and monotone $(g, \preccurlyeq)$-non-increasing in arguments of $\mathsf{B}$, i.e., for all $x_1, x_2, \ldots, x_n, y, z \in X$ and all $i$,

$$\begin{aligned} gy \preccurlyeq gz \quad \Rightarrow \quad & F(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \\ & \preccurlyeq_i F(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n). \end{aligned}$$

**Lemma 11.6.1.** *Let $F : X^n \to X$ and $g : X \to X$ be two mappings such that $F(X^n) \subseteq g(X)$ and let $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be a $n$-tuple of mappings from $\{1, 2, \ldots, n\}$ into itself. Then, starting from any points $x_0^1, x_0^2, \ldots, x_0^n \in X$, there exists a sequence $\{(x_m^1, x_m^2, \ldots, x_m^n)\}_{m \in \mathbb{N}}$ on $X^n$ such that*

$$\begin{aligned} gx_{m+1}^i = {} & F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}) \\ & \textit{for all } m \in \mathbb{N} \textit{ and all } i \in \{1, 2, \ldots, n\}. \end{aligned} \tag{11.55}$$

*Proof.* Let $x_0^1, x_0^2, \ldots, x_0^n \in X$ be arbitrary. Given $i \in \{1, 2, \ldots, n\}$, since $F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)}) \in F(X^n) \subseteq g(X)$, then there exists $x_1^i \in X$ such that $gx_1^i = F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)})$. Then, we have $n$ points $x_1^1, x_1^2, \ldots, x_1^n \in X$. Similarly, given $i \in \{1, 2, \ldots, n\}$, since $F(x_1^{\sigma_i(1)}, x_1^{\sigma_i(2)}, \ldots, x_1^{\sigma_i(n)}) \in F(X^n) \subseteq g(X)$, then there exists $x_2^i \in X$ such that

$$gx_2^i = F(x_1^{\sigma_i(1)}, x_1^{\sigma_i(2)}, \ldots, x_1^{\sigma_i(n)}).$$

If we repeat by induction this argument, we can define a sequence

$$\{(x_m^1, x_m^2, \ldots, x_m^n)\}_{m \in \mathbb{N}}$$

on $X^n$ satisfying (11.55). □

**Definition 11.6.3.** Given $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ and two mappings $F : X^n \to X$ and $g : X \to X$, a *Picard* $(F, g, \Phi)$*-sequence* is a sequence $\{(x_m^1, x_m^2, \ldots, x_m^n)\}_{m \in \mathbb{N}}$ on $X^n$ satisfying (11.55).

**Proposition 11.6.1.** *If* $\{(x_m^1, x_m^2, \ldots, x_m^n)\}_{m\in\mathbb{N}}$ *on* $X^n$ *is a Picard* $(F, g, \Phi)$*-sequence and there exists* $n_0 \in \mathbb{N}$ *such that* $gx_{n_0}^i = gx_{n_0+1}^i$ *for all* $i \in \{1, 2, \ldots, n\}$*, then* $(x_{n_0}^1, x_{n_0}^2, \ldots, x_{n_0}^n)$ *is a* $\Phi$*-coincidence point of* $F$ *and* $g$*.*

*Proof.* It follows from the fact that, for all $i \in \{1, 2, \ldots, n\}$, $gx_{n_0}^i = gx_{n_0+1}^i = F(x_{n_0}^{\sigma_i(1)}, x_{n_0}^{\sigma_i(2)}, \ldots, x_{n_0}^{\sigma_i(n)})$, so $(x_{n_0}^1, x_{n_0}^2, \ldots, x_{n_0}^n)$ is a $\Phi$-coincidence point of $F$ and $g$. □

Given two mappings $F : X^n \to X$ and $g : X \to X$, the condition $F(X^n) \subseteq g(X)$ is sufficient to guarantee that there exists a Picard $(F, g, \Phi)$-sequence on $X$ based on any initial points $x_0^1, x_0^2, \ldots, x_0^n \in X$. However, it is not necessary.

**Lemma 11.6.2.** *Let* $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ *be a n-tuple of mappings from* $\{1, 2, \ldots, n\}$ *into itself such that* $\sigma_i \in \Omega_{\mathsf{A},\mathsf{B}}$ *if* $i \in \mathsf{A}$ *and* $\sigma_i \in \Omega'_{\mathsf{A},\mathsf{B}}$ *if* $i \in \mathsf{B}$*. Let* $\preccurlyeq$ *be a transitive binary relation on a set* $X$ *and let* $F : X^n \to X$ *and* $g : X \to X$ *be two mappings such that the following conditions are fulfilled.*

**(i)** *There exists a Picard* $(F, g, \Phi)$*-sequence* $\{(x_m^1, x_m^2, \ldots, x_m^n)\}_{m\in\mathbb{N}} \subseteq X^n$*.*
**(ii)** $gx_0^i \preccurlyeq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)})$ *for all* $i \in \{1, 2, \ldots, n\}$*.*
**(iii)** $F$ *has the mixed* $(g, \preccurlyeq)$*-monotone property.*

*Then, for each* $i \in \{1, 2, \ldots, n\}$*, the sequence* $\{gx_m^i\}_{m\in\mathbb{N}}$ *is* $\preccurlyeq_i$*-non-decreasing (that is, it is* $\preccurlyeq$*-non-decreasing if* $i \in \mathsf{A}$ *and* $\preccurlyeq$*-non-increasing if* $i \in \mathsf{B}$*). In particular,*

$$gx_m^i \preccurlyeq_i gx_\ell^i \quad \text{for all } i \in \{1, 2, \ldots, n\} \text{ and all } m, \ell \in \mathbb{N} \text{ with } m \le \ell. \tag{11.56}$$

*Proof.* By (ii), we have that $gx_0^i \preccurlyeq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)}) = gx_1^i$ for all $i \in \{1, 2, \ldots, n\}$. Assume, by hypothesis of induction, that there exist $m \in \mathbb{N}$ such that $gx_m^i \preccurlyeq_i gx_{m+1}^i$ for all $i \in \{1, 2, \ldots, n\}$. This condition means that

$$\begin{cases} gx_m^j \preccurlyeq gx_{m+1}^j, \text{ if } j \in \mathsf{A}, \\ gx_m^j \succcurlyeq gx_{m+1}^j, \text{ if } j \in \mathsf{B}. \end{cases} \tag{11.57}$$

To complete the induction process, we have to prove that $gx_{m+1}^i \preccurlyeq_i gx_{m+2}^i$ for all $i \in \{1, 2, \ldots, n\}$, that is,

$$\begin{cases} gx_{m+1}^j \preccurlyeq gx_{m+2}^j, \text{ if } j \in \mathsf{A}, \\ gx_{m+1}^j \succcurlyeq gx_{m+2}^j, \text{ if } j \in \mathsf{B}. \end{cases} \tag{11.58}$$

We distinguishing two cases.

*Case 1:* $i \in \mathsf{A}$. In this case, $\sigma_i(\mathsf{A}) \subseteq \mathsf{A}$ and $\sigma_i(\mathsf{B}) \subseteq \mathsf{B}$. As $F$ has the mixed $g$-monotone property, we apply that $F$ is $g$-monotone $(g, \preccurlyeq)$-non-decreasing in $\mathsf{A}$-arguments with the first inequalities of (11.57) and we deduce that, for all $a_1, a_2, \ldots, a_n \in X$:

$$\text{if } j, s \in \mathsf{A}, \quad gx_m^j \preccurlyeq gx_{m+1}^j \;\Rightarrow$$
$$F(a_1, \ldots, a_{s-1}, x_m^j, a_{s+1}, \ldots, a_n) \preccurlyeq F(a_1, \ldots, a_{s-1}, x_{m+1}^j, a_{s+1}, \ldots, a_n),$$

and that $F$ is $g$-monotone $(g, \preccurlyeq)$-non-increasing in $\mathsf{B}$-arguments with the second inequalities of (11.57):

$$\text{if } j, s \in \mathsf{B}, \quad gx_m^j \succcurlyeq gx_{m+1}^j \;\Rightarrow$$
$$F(a_1, \ldots, a_{s-1}, x_m^j, a_{s+1}, \ldots, a_n) \preccurlyeq F(a_1, \ldots, a_{s-1}, x_{m+1}^j, a_{s+1}, \ldots, a_n).$$

In any case, it follows that, if $j, s \in \{1, 2, \ldots, n\}$ satisfy $j, s \in \mathsf{A}$ or $j, s \in \mathsf{B}$, then:

$$F(a_1, \ldots, a_{s-1}, x_m^j, a_{s+1}, \ldots, a_n) \preccurlyeq F(a_1, \ldots, a_{s-1}, x_{m+1}^j, a_{s+1}, \ldots, a_n)$$

for all $a_1, a_2, \ldots, a_n \in X$. As $\sigma_i \in \Omega_{\mathsf{A},\mathsf{B}}$:

$$\begin{aligned}
gx_{m+1}^i &= F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, x_m^{\sigma_i(3)}, \ldots, x_m^{\sigma_i(n)}) \\
&\qquad\qquad (1, \sigma_i(1) \in \mathsf{A} \text{ or } 1, \sigma_i(1) \in \mathsf{B}) \\
&\preccurlyeq F(x_{m+1}^{\sigma_i(1)}, x_m^{\sigma_i(2)}, x_m^{\sigma_i(3)}, \ldots, x_m^{\sigma_i(n)}) \\
&\qquad\qquad (2, \sigma_i(2) \in \mathsf{A} \text{ or } 2, \sigma_i(2) \in \mathsf{B}) \\
&\preccurlyeq F(x_{m+1}^{\sigma_i(1)}, x_{m+1}^{\sigma_i(2)}, x_m^{\sigma_i(3)}, \ldots, x_m^{\sigma_i(n)}) \\
&\qquad\qquad (3, \sigma_i(3) \in \mathsf{A} \text{ or } 3, \sigma_i(3) \in \mathsf{B}) \\
&\preccurlyeq \ldots \preccurlyeq F(x_{m+1}^{\sigma_i(1)}, x_{m+1}^{\sigma_i(2)}, x_{m+1}^{\sigma_i(3)}, \ldots, x_{m+1}^{\sigma_i(n)}) = gx_{m+2}^i.
\end{aligned}$$

Hence $gx_{m+1}^i \preccurlyeq gx_{m+2}^i$ when $i$ is in $\mathsf{A}$, so (11.58) holds if $i \in \mathsf{A}$.

*Case 2:* $i \in \mathsf{B}$. In this case, we apply that $F$ is $(g, \preccurlyeq)$-monotone non-decreasing in $\mathsf{A}$-arguments with the second inequalities of (11.57) and that $F$ is $(g, \preccurlyeq)$-monotone non-increasing in $\mathsf{B}$-arguments with the first inequalities of (11.57), and we deduce, for all $a_1, a_2, \ldots, a_n \in X$, that, if $j, s \in \{1, 2, \ldots, n\}$ satisfy $j \in \mathsf{A}$, $s \in \mathsf{B}$ or $j \in \mathsf{B}$, $s \in \mathsf{A}$, then

$$F(a_1, \ldots, a_{s-1}, x_m^j, a_{s+1}, \ldots, a_n) \succcurlyeq F(a_1, \ldots, a_{s-1}, x_{m+1}^j, a_{s+1}, \ldots, a_n).$$

Since $\sigma_i \in \Omega'_{\mathsf{A},\mathsf{B}}$, therefore:

$$\begin{aligned}
gx_{m+1}^i &= F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, x_m^{\sigma_i(3)}, \ldots, x_m^{\sigma_i(n)}) \\
&\qquad\qquad (1 \in \mathsf{A}, \sigma_i(1) \in \mathsf{B} \text{ or } 1 \in \mathsf{B}, \sigma_i(1) \in \mathsf{A}) \\
&\succcurlyeq F(x_{m+1}^{\sigma_i(1)}, x_m^{\sigma_i(2)}, x_m^{\sigma_i(3)}, \ldots, x_m^{\sigma_i(n)})
\end{aligned}$$

$$
\begin{aligned}
&\qquad (2 \in \mathsf{A}, \sigma_i(2) \in \mathsf{B} \text{ or } 2 \in \mathsf{B}, \sigma_i(2) \in \mathsf{A}) \\
&\succcurlyeq F(x_{m+1}^{\sigma_i(1)}, x_{m+1}^{\sigma_i(2)}, x_m^{\sigma_i(3)}, \ldots, x_m^{\sigma_i(n)}) \\
&\qquad (3 \in \mathsf{A}, \sigma_i(3) \in \mathsf{B} \text{ or } 3 \in \mathsf{B}, \sigma_i(3) \in \mathsf{A}) \\
&\succcurlyeq \ldots \succcurlyeq F(x_{m+1}^{\sigma_i(1)}, x_{m+1}^{\sigma_i(2)}, x_{m+1}^{\sigma_i(3)}, \ldots, x_{m+1}^{\sigma_i(n)}) = gx_{m+2}^i.
\end{aligned}
$$

Hence $gx_{m+1}^i \succcurlyeq gx_{m+2}^i$ when $i \in \mathsf{B}$.

In any case, we have proved that (11.58) holds. In particular, (11.56) holds because $\preccurlyeq$ is transitive. □

In order to present a very general result, we introduce the following definitions.

**Definition 11.6.4.** Let $(X, G)$ be a $G^*$-metric space endowed with a binary relation $\preccurlyeq$ and let $F : X^n \to X$ and $g : X \to X$ be two mappings. Given $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, we will say that $(F, g)$ is an $(O, \preccurlyeq, \Phi)$-*compatible pair* if we have that, for all $i \in \{1, 2, \ldots, n\}$,

$$
\begin{aligned}
\lim_{n\to\infty} G\big(&gF(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}), F(gx_m^{\sigma_i(1)}, gx_m^{\sigma_i(2)}, \ldots, gx_m^{\sigma_i(n)}), \\
&F(gx_m^{\sigma_i(1)}, gx_m^{\sigma_i(2)}, \ldots, gx_m^{\sigma_i(n)})\big) = 0
\end{aligned}
$$

whenever $\{(x_m^1, x_m^2, \ldots, x_m^n)\}_{m\in\mathbb{N}}$ is a sequence in $X^n$ such that each $\{gx_m^i\}$ is $\preccurlyeq$-monotone and

$$
\lim_{m\to\infty} F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}) = \lim_{n\to\infty} gx_m \in X \quad \text{for all } i \in \{1, 2, \ldots, n\}.
$$

**Definition 11.6.5.** Let $(X, G)$ be a $G^*$-metric space and let $F : X^n \to X$ and $g : X \to X$ be two mappings. Given $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, we will say that $(F, g)$ is an $(O, \Phi)$-*compatible pair* if we have that, for all $i \in \{1, 2, \ldots, n\}$,

$$
\begin{aligned}
\lim_{n\to\infty} G\big(&gF(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}), F(gx_m^{\sigma_i(1)}, gx_m^{\sigma_i(2)}, \ldots, gx_m^{\sigma_i(n)}), \\
&F(gx_m^{\sigma_i(1)}, gx_m^{\sigma_i(2)}, \ldots, gx_m^{\sigma_i(n)})\big) = 0
\end{aligned}
$$

whenever $\{(x_m^1, x_m^2, \ldots, x_m^n)\}_{m\in\mathbb{N}}$ is a sequence in $X^n$ such that

$$
\lim_{m\to\infty} F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}) = \lim_{n\to\infty} gx_m \in X \quad \text{for all } i \in \{1, 2, \ldots, n\}.
$$

*Remark 11.6.1.* If $F$ and $g$ are commuting, then $(F, g)$ is an $(O, \preccurlyeq, \Phi)$-compatible pair and an $(O, \Phi)$-compatible pair.

### *11.6.2 Existence of $\Phi$-Coincidence Points*

Next, we present one of the main results of the chapter.

**Theorem 11.6.1.** *Let $(X, G)$ be a G-metric space endowed with a preorder $\preccurlyeq$ and let $F : X^n \to X$ and $g : X \to X$ be two mappings such that $F$ has the mixed $(g, \preccurlyeq)$-monotone property. Let $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be a n-tuple of mappings from $\{1, 2, \ldots, n\}$ into itself satisfying $\sigma_i \in \Omega_{\mathsf{A,B}}$ if $i \in \mathsf{A}$ and $\sigma_i \in \Omega'_{\mathsf{A,B}}$ if $i \in \mathsf{B}$. Assume that the following conditions hold.*

**(i)** *There exist two functions $\psi, \varphi \in \mathcal{F}_{\text{alt}}$ such that*

$$\psi\left(G\left(F\left(\mathsf{X}\right), F\left(\mathsf{Y}\right), F\left(\mathsf{Z}\right)\right)\right) \le (\psi - \varphi)\left(\max_{1\le i\le n} G\left(gx_i, gy_i, gz_i\right)\right) \tag{11.59}$$

*for all $\mathsf{X}\left(x_1, x_2, \ldots, x_n\right), \mathsf{Y}\left(y_1, y_2, \ldots, y_n\right), \mathsf{Z}\left(z_1, z_2, \ldots, z_n\right) \in X^n$ for which*

$$gx_i \preccurlyeq_i gy_i \preccurlyeq_i gz_i \quad \textit{for all } i \in \{1, 2, \ldots, n\}.$$

**(ii)** *At least, one of the following conditions holds.*

- **(ii.1)** *$F(X^n) \subseteq g(X)$ and there exist $x_0^1, x_0^2, \ldots, x_0^n \in X$ such that $gx_0^i \preccurlyeq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)})$ for all $i \in \{1, 2, \ldots, n\}$.*
- **(ii.2)** *There exists a Picard $(F, g, \Phi)$-sequence $\{\left(x_m^1, x_m^2, \ldots, x_m^n\right)\}_{m\in\mathbb{N}} \subseteq X^n$ such that $gx_0^i \preccurlyeq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)})$ for all $i \in \{1, 2, \ldots, n\}$.*

**(iii)** *At least, one of the following conditions holds.*

- **(iii.1)** *$(X, G)$ (or $(g(X), G)$ or $(F(X^n), G)$) is complete and $F$ and $g$ are continuous and $(O, \preccurlyeq, \Phi)$-compatible.*
- **(iii.2)** *$(X, G)$ (or $(g(X), G)$ or $(F(X^n), G)$) is complete and $F$ and $g$ are continuous and commuting.*
- **(iii.3)** *$(g(X), G)$ is complete and $(X, G, \preceq)$ is regular.*

*Then $F$ and $g$ have, at least, a $\Phi$-coincidence point.*

*Proof.* By Lemma 11.6.1, (ii.1)$\Rightarrow$(ii.2). We present the proof assuming (ii.2). From Lemma 11.6.2,

$$gx_m^i \preccurlyeq_i gx_\ell^i \quad \text{for all } i \in \{1, 2, \ldots, n\} \text{ and all } m, \ell \in \mathbb{N} \text{ with } m \le \ell.$$

From Lemma 11.6.1, if there exists $n_0 \in \mathbb{N}$ such that $gx_{n_0}^i = gx_{n_0+1}^i$ for all $i \in \{1, 2, \ldots, n\}$, then $\left(x_{n_0}^1, x_{n_0}^2, \ldots, x_{n_0}^n\right)$ is a $\Phi$-coincidence point of $F$ and $g$, and the existence part is finished. On the contrary case, assume that

$$\max_{1\le j\le n} G\left(gx_{m+1}^j, gx_{m+2}^j, gx_{m+2}^j\right) > 0 \quad \text{for all } m \in \mathbb{N}.$$

Let for all $j \in \{1, 2, \ldots, n\}$ and all $m \in \mathbb{N}$,

$$\left\{a_m^j = G\left(gx_m^j, gx_{m+1}^j, gx_{m+1}^j\right)\right\}_{m\in\mathbb{N}} \quad \text{and}$$
$$\left\{b_m^j = \max_{1\le i\le n} G\left(gx_m^{\sigma_j(i)}, x_{m+1}^{\sigma_j(i)}, x_{m+1}^{\sigma_j(i)}\right)\right\}_{m\in\mathbb{N}}$$

By the contractivity condition (11.59), for all $m \in \mathbb{N}$ and all $j \in \{1, 2, \ldots, n\}$, we have, taking into account that $gx_{m+1}^j \preccurlyeq_j gx_{m+2}^j \preccurlyeq_j gx_{m+2}^j$,

$$\begin{aligned}
\psi\left(a_{m+1}^j\right) &= \psi\left(G\left(gx_{m+1}^j, gx_{m+2}^j, gx_{m+2}^j\right)\right) \\
&= \psi\Big(G\Big(F(x_m^{\sigma_j(1)}, x_m^{\sigma_j(2)}, \ldots, x_m^{\sigma_j(n)}), \\
&\qquad F(x_{m+1}^{\sigma_j(1)}, x_{m+1}^{\sigma_j(2)}, \ldots, x_{m+1}^{\sigma_j(n)}), F(x_{m+1}^{\sigma_j(1)}, x_{m+1}^{\sigma_j(2)}, \ldots, x_{m+1}^{\sigma_j(n)})\Big)\Big) \\
&\le (\psi - \varphi)\left(\max_{1\le i\le n} G\left(gx_m^{\sigma_j(i)}, x_{m+1}^{\sigma_j(i)}, x_{m+1}^{\sigma_j(i)}\right)\right) = (\psi - \varphi)\left(b_m^j\right).
\end{aligned}$$

Notice that

$$\begin{aligned}
\max_{1\le j\le n} b_m^j &= \max_{1\le j\le n}\left(\max_{1\le i\le n} G\left(gx_m^{\sigma_j(i)}, x_{m+1}^{\sigma_j(i)}, x_{m+1}^{\sigma_j(i)}\right)\right) \\
&\le \max_{1\le j\le n} G\left(gx_m^j, gx_{m+1}^j, gx_{m+1}^j\right) = \max_{1\le j\le n} a_m^j,
\end{aligned}$$

and as $\psi$ is non-decreasing,

$$\psi\left(\max_{1\le j\le n} b_m^j\right) \le \psi\left(\max_{1\le j\le n} a_m^j\right).$$

Applying Lemma 11.2.4, we deduce that

$$\lim_{m\to\infty} G\left(gx_m^j, gx_{m+1}^j, gx_{m+1}^j\right) = \lim_{m\to\infty} a_m^j = 0 \quad \text{for all } j \in \{1, 2, \ldots, n\}.$$

Next we show that each sequence $\{gx_m^j\}$ is Cauchy in $(X, G)$ reasoning by contradiction. If we suppose that, at least, one of them is not Cauchy in $(X, G)$, Lemma 11.2.1 guarantees that there exist $\varepsilon_0 > 0$, $i_0 \in \{1, 2, \ldots, n\}$ and two sequences of natural numbers $\{n(k)\}_{k\in\mathbb{N}}$ and $\{m(k)\}_{k\in\mathbb{N}}$ such that, for all $k \in \mathbb{N}$,

$$\begin{aligned}
&k \le n(k) < m(k) < n(k+1), \\
&\max_{1\le i\le n}\left\{G(gx_{n(k)}^i, gx_{m(k)-1}^i, gx_{m(k)-1}^i)\right\} \le \varepsilon_0 \\
&\qquad < \max_{1\le i\le n}\left\{G(gx_{n(k)}^i, gx_{m(k)}^i, gx_{m(k)}^i)\right\},
\end{aligned}$$

and also

$$\lim_{k\to\infty}\left[\max_{1\le i\le n}\left\{G(gx^i_{n(k)},gx^i_{m(k)},gx^i_{m(k)})\right\}\right]=\varepsilon_0, \tag{11.60}$$

$$\lim_{k\to\infty}\left[\max_{1\le i\le n}\left\{G(gx^i_{n(k)-1},gx^i_{m(k)-1},gx^i_{m(k)-1})\right\}\right]=\varepsilon_0, \tag{11.61}$$

$$\lim_{k\to\infty}G(gx^{i_0}_{n(k)},gx^{i_0}_{m(k)},gx^{i_0}_{m(k)})=\varepsilon_0. \tag{11.62}$$

Since $n(k)<m(k)$, we have that $gx^i_{n(k)-1}\preccurlyeq_i gx^i_{m(k)-1}\preccurlyeq_i gx^i_{m(k)-1}$ for all $i\in\{1,2,\ldots,n\}$ and all $k\in\mathbb{N}$. By the contractivity condition (11.59),

$$\begin{aligned}\psi\left(G\left(gx^{i_0}_{n(k)},gx^{i_0}_{m(k)},gx^{i_0}_{m(k)}\right)\right)&=\psi\left(G\left(F(x^{\sigma_{i_0}(1)}_{n(k)-1},x^{\sigma_{i_0}(2)}_{n(k)-1},\ldots,x^{\sigma_{i_0}(n)}_{n(k)-1}),\right.\right.\\&\qquad F(x^{\sigma_{i_0}(1)}_{m(k)-1},x^{\sigma_{i_0}(2)}_{m(k)-1},\ldots,x^{\sigma_{i_0}(n)}_{m(k)-1}),\\&\qquad \left.\left.F(x^{\sigma_{i_0}(1)}_{m(k)-1},x^{\sigma_{i_0}(2)}_{m(k)-1},\ldots,x^{\sigma_{i_0}(n)}_{m(k)-1})\right)\right)\\&\le(\psi-\varphi)\left(\max_{1\le s\le n}G\left(gx^{\sigma_{i_0}(s)}_{n(k)-1},x^{\sigma_{i_0}(s)}_{m(k)-1},x^{\sigma_{i_0}(s)}_{m(k)-1}\right)\right).\end{aligned}\tag{11.63}$$

From item 2 of Lemma 2.3.4, we have that, for all $k\in\mathbb{N}$,

$$G\left(gx^{i_0}_{n(k)},gx^{i_0}_{m(k)},gx^{i_0}_{m(k)}\right)<\max_{1\le s\le n}G\left(gx^{\sigma_{i_0}(s)}_{n(k)-1},x^{\sigma_{i_0}(s)}_{m(k)-1},x^{\sigma_{i_0}(s)}_{m(k)-1}\right)\quad\text{or}$$
$$G\left(gx^{i_0}_{n(k)},gx^{i_0}_{m(k)},gx^{i_0}_{m(k)}\right)=\max_{1\le s\le n}G\left(gx^{\sigma_{i_0}(s)}_{n(k)-1},x^{\sigma_{i_0}(s)}_{m(k)-1},x^{\sigma_{i_0}(s)}_{m(k)-1}\right)=0.$$

From (11.62), the second case is impossible for infinite values of $k$. Then, there exists $k_0\in\mathbb{N}$ such that

$$G\left(gx^{i_0}_{n(k)},gx^{i_0}_{m(k)},gx^{i_0}_{m(k)}\right)<\max_{1\le s\le n}G\left(gx^{\sigma_{i_0}(s)}_{n(k)-1},x^{\sigma_{i_0}(s)}_{m(k)-1},x^{\sigma_{i_0}(s)}_{m(k)-1}\right)$$

for all $k\ge k_0$. As a consequence,

$$\begin{aligned}G\left(gx^{i_0}_{n(k)},gx^{i_0}_{m(k)},gx^{i_0}_{m(k)}\right)&<\max_{1\le s\le n}G\left(gx^{\sigma_{i_0}(s)}_{n(k)-1},x^{\sigma_{i_0}(s)}_{m(k)-1},x^{\sigma_{i_0}(s)}_{m(k)-1}\right)\\&\le\max_{1\le s\le n}G(gx^s_{n(k)-1},gx^s_{m(k)-1},gx^s_{m(k)-1})\end{aligned}$$

for all $k\ge k_0$. Letting $k\to\infty$ and using (11.61) and (11.62), we deduce that

$$\lim_{k\to\infty}\left[\max_{1\le s\le n}G\left(gx^{\sigma_{i_0}(s)}_{n(k)-1},x^{\sigma_{i_0}(s)}_{m(k)-1},x^{\sigma_{i_0}(s)}_{m(k)-1}\right)\right]=\varepsilon_0. \tag{11.64}$$

Hence, the sequences

$$\left\{t_k = G\left(gx^{i_0}_{n(k)}, gx^{i_0}_{m(k)}, gx^{i_0}_{m(k)}\right)\right\}_{k\in\mathbb{N}} \quad \text{and}$$
$$\left\{s_k = \max_{1\leq s\leq n} G\left(gx^{\sigma_{i_0}(s)}_{n(k)-1}, x^{\sigma_{i_0}(s)}_{m(k)-1}, x^{\sigma_{i_0}(s)}_{m(k)-1}\right)\right\}_{k\in\mathbb{N}}$$

satisfy, by (11.63), that $\psi\,(t_k) \leq (\psi-\varphi)\,(s_k)$ for all $k\in\mathbb{N}$. Furthermore, they have the same limit $L=\varepsilon_0$ by (11.62) and (11.64). Lemma 2.3.5 yields $\varepsilon_0=0$, which is a contradiction. This contradiction proves that each $\{gx^i_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $(X,G)$. To continue the proof, we distinguish three cases.

*Case (iii.1). Assume that* $(X,G)$ *(or* $(g(X),G)$ *or* $(F(X^n),G)$*) is complete and* $F$ *and* $g$ *are continuous and* $(O,\preccurlyeq,\Phi)$*-compatible.* In such a case, there exist $z_1,z_2,\ldots,z_n\in X$ such that $\{gx^i_m\}_{m\in\mathbb{N}}\to z_i$ for all $i\in\{1,2,\ldots,n\}$. As $F$ and $g$ are continuous, we deduce that

$$\{ggx^i_m\}_{m\in\mathbb{N}}\to gz_i,$$
$$\{F(gx^{\sigma_i(1)}_m, gx^{\sigma_i(2)}_m,\ldots,gx^{\sigma_i(n)}_m)\}\to F(z_{\sigma_i(1)},z_{\sigma_i(2)},\ldots,z_{\sigma_i(2)})$$

for all $i\in\{1,2,\ldots,n\}$. Therefore, each $\{gx^i_m\}_{m\in\mathbb{N}}$ is a $\preccurlyeq$-monotone sequence and

$$\lim_{m\to\infty} F(x^{\sigma_i(1)}_m, x^{\sigma_i(2)}_m,\ldots,x^{\sigma_i(n)}_m) = \lim_{n\to\infty} gx^i_{n+1} = z_i\in X$$

for all $i\in\{1,2,\ldots,n\}$. As the pair $(F,g)$ is $(O,\preccurlyeq,\Phi)$-compatible, we deduce that, for all $i\in\{1,2,\ldots,n\}$,

$$\lim_{m\to\infty} G\left(gF(x^{\sigma_i(1)}_m, x^{\sigma_i(2)}_m,\ldots,x^{\sigma_i(n)}_m), F(gx^{\sigma_i(1)}_m, gx^{\sigma_i(2)}_m,\ldots,gx^{\sigma_i(n)}_m),\right.$$
$$\left.F(gx^{\sigma_i(1)}_m, gx^{\sigma_i(2)}_m,\ldots,gx^{\sigma_i(n)}_m)\right) = 0.$$

As a consequence, for all $i\in\{1,2,\ldots,n\}$,

$$\begin{aligned}
&G\left(gz_i, F(z_{\sigma_i(1)},z_{\sigma_i(2)},\ldots,z_{\sigma_i(2)}), F(z_{\sigma_i(1)},z_{\sigma_i(2)},\ldots,z_{\sigma_i(2)})\right)\\
&\quad=\lim_{m\to\infty} G\left(ggx^i_{m+1}, F(gx^{\sigma_i(1)}_m, gx^{\sigma_i(2)}_m,\ldots,gx^{\sigma_i(n)}_m),\right.\\
&\qquad\qquad\left.F(gx^{\sigma_i(1)}_m, gx^{\sigma_i(2)}_m,\ldots,gx^{\sigma_i(n)}_m)\right)\\
&\quad=\lim_{m\to\infty} G\left(gF(x^{\sigma_i(1)}_m, x^{\sigma_i(2)}_m,\ldots,x^{\sigma_i(n)}_m),\right.\\
&\qquad\qquad F(gx^{\sigma_i(1)}_m, gx^{\sigma_i(2)}_m,\ldots,gx^{\sigma_i(n)}_m),\\
&\qquad\qquad\left.F(gx^{\sigma_i(1)}_m, gx^{\sigma_i(2)}_m,\ldots,gx^{\sigma_i(n)}_m)\right) = 0,
\end{aligned}$$

so $F(z_{\sigma_i(1)}, z_{\sigma_i(2)}, \ldots, z_{\sigma_i(2)}) = gz_i$ for all $i \in \{1, 2, \ldots, n\}$, which means that $(z_1, z_2, \ldots, z_n) \in X^n$ is a $\Phi$-coincidence point of $F$ and $g$.

*Case (iii.2). Assume that* $(X, G)$ *(or* $(g(X), G)$ *or* $\left(F(X^2), G\right)$*) is complete and* $F$ *and* $g$ *are continuous and commuting.* It follows from item (iii.1) because if $F$ and $g$ are commuting, then they are also $(O, \preceq, \Phi)$-compatible.

*Case (iii.3). Assume that* $(g(X), G)$ *is complete and* $(X, G, \preceq)$ *is regular.* Since $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences on $(g(X), G)$, there exist $z_1, z_2, \ldots, z_n \in g(X)$ such that $\{gx_m^i\}_{m \in \mathbb{N}} \to z_i$ for all $i \in \{1, 2, \ldots, n\}$. Let $\omega_1, \omega_2, \ldots, \omega_n \in X$ be arbitrary points such that $g\omega_i = z_i$ for all $i \in \{1, 2, \ldots, n\}$. As $(X, G, \preceq)$ is regular and each sequence $\{gx_m^i\}$ is $\preccurlyeq_i$-non-decreasing and convergent, we deduce that

$$gx_m^i \preccurlyeq_i g\omega_i \quad \text{for all } m \in \mathbb{N}.$$

Therefore, applying the contractivity condition (11.59) to $gx_m^i \preccurlyeq_i g\omega_i \preccurlyeq_i g\omega_i$, we obtain

$$\begin{aligned}
&\psi\left(G(gx_{m+1}^i, F(\omega_{\sigma_i(1)}, \omega_{\sigma_i(2)}, \ldots, \omega_{\sigma_i(n)}), F(\omega_{\sigma_i(1)}, \omega_{\sigma_i(2)}, \ldots, \omega_{\sigma_i(n)}))\right) \\
&\quad = \psi\left(G\left(F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}), F(\omega_{\sigma_i(1)}, \omega_{\sigma_i(2)}, \ldots, \omega_{\sigma_i(n)}),\right.\right. \\
&\qquad\qquad \left.\left. F(\omega_{\sigma_i(1)}, \omega_{\sigma_i(2)}, \ldots, \omega_{\sigma_i(n)})\right)\right) \\
&\quad \leq (\psi - \varphi)\left(\max_{1 \leq j \leq n} G\left(gx_m^{\sigma_i(j)}, g\omega_{\sigma_i(j)}, g\omega_{\sigma_i(j)}\right)\right).
\end{aligned}$$

As $\{gx_m^{\sigma_i(j)}\} \to g\omega_{\sigma_i(j)}$ for all $i, j \in \{1, 2, \ldots, n\}$, and $G$, $\psi$ and $\varphi$ are continuous, we deduce that

$$\begin{aligned}
&\lim_{m \to \infty} \psi G(gx_{m+1}^i, F(\omega_{\sigma_i(1)}, \omega_{\sigma_i(2)}, \ldots, \omega_{\sigma_i(n)}), \\
&\qquad\qquad F(\omega_{\sigma_i(1)}, \omega_{\sigma_i(2)}, \ldots, \omega_{\sigma_i(n)}))) = 0.
\end{aligned}$$

Since $\psi \in \mathcal{F}_{\text{alt}}$, Lemma 2.3.3 shows that

$$\begin{aligned}
&G(g\omega_i, F(\omega_{\sigma_i(1)}, \omega_{\sigma_i(2)}, \ldots, \omega_{\sigma_i(n)}), F(\omega_{\sigma_i(1)}, \omega_{\sigma_i(2)}, \ldots, \omega_{\sigma_i(n)})) \\
&\quad = \lim_{m \to \infty} G(gx_{m+1}^i, F(\omega_{\sigma_i(1)}, \omega_{\sigma_i(2)}, \ldots, \omega_{\sigma_i(n)}), \\
&\qquad\qquad F(\omega_{\sigma_i(1)}, \omega_{\sigma_i(2)}, \ldots, \omega_{\sigma_i(n)})) = 0,
\end{aligned}$$

so $F(\omega_{\sigma_i(1)}, \omega_{\sigma_i(2)}, \ldots, \omega_{\sigma_i(n)}) = g\omega_i$ for all $i \in \{1, 2, \ldots, n\}$, which means that $(\omega_1, \omega_2, \ldots, \omega_n) \in X^n$ is a $\Phi$-coincidence point of $F$ and $g$. □

The previous results have many particularizations. For example, when $\preccurlyeq$ is a partial order on $X$ (we do not write such version because it is similar to Theorem 11.6.1). It is interesting to consider the preorder "$x \preccurlyeq y$ for all $x, y \in X$". In such a case, we have the following version.

**Corollary 11.6.1.** *Let $(X, G)$ be a G-metric space and let $F : X^n \to X$ and $g : X \to X$ be two mappings. Let $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be a n-tuple of mappings from $\{1, 2, \ldots, n\}$ into itself satisfying $\sigma_i \in \Omega_{\mathsf{A,B}}$ if $i \in \mathsf{A}$ and $\sigma_i \in \Omega'_{\mathsf{A,B}}$ if $i \in \mathsf{B}$. Assume that the following conditions hold.*

**(i)** *There exist two functions $\psi, \varphi \in \mathcal{F}_{\text{alt}}$ such that*

$$\psi\left(G\left(F\left(\mathsf{X}\right), F\left(\mathsf{Y}\right), F\left(\mathsf{Z}\right)\right)\right) \leq \left(\psi - \varphi\right)\left(\max_{1 \leq i \leq n} G\left(gx_i, gy_i, gz_i\right)\right)$$

*for all $\mathsf{X}\left(x_1, x_2, \ldots, x_n\right), \mathsf{Y}\left(y_1, y_2, \ldots, y_n\right), \mathsf{Z}\left(z_1, z_2, \ldots, z_n\right) \in X^n$.*

**(ii)** *At least, one of the following conditions holds.*

**(ii.1)** *$F(X^n) \subseteq g(X)$.*

**(ii.2)** *There exists a Picard $(F, g, \Phi)$-sequence*

$$\{\left(x_m^1, x_m^2, \ldots, x_m^n\right)\}_{m \in \mathbb{N}} \subseteq X^n.$$

**(iii)** *At least, one of the following conditions holds.*

**(iii.1)** *$(X, G)$ is complete and $F$ and $g$ are continuous and $(O, \Phi)$-compatible.*

**(iii.2)** *$(X, G)$ is complete and $F$ and $g$ are continuous and commuting.*

**(iii.3)** *$(g(X), G)$ is complete.*

*Then $F$ and $g$ have, at least, a $\Phi$-coincidence point.*

In the next result, we assume that $\psi$ is the identity mapping on $[0, \infty)$.

**Corollary 11.6.2.** *Let $(X, G)$ be a G-metric space endowed with a preorder $\preccurlyeq$ and let $F : X^n \to X$ and $g : X \to X$ be two mappings such that $F$ has the mixed $(g, \preccurlyeq)$-monotone property. Let $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be a n-tuple of mappings from $\{1, 2, \ldots, n\}$ into itself satisfying $\sigma_i \in \Omega_{\mathsf{A,B}}$ if $i \in \mathsf{A}$ and $\sigma_i \in \Omega'_{\mathsf{A,B}}$ if $i \in \mathsf{B}$. Assume that the following conditions hold.*

**(i)** *There exists a function $\varphi \in \mathcal{F}_{\text{alt}}$ such that*

$$\begin{aligned} & G\left(F\left(\mathsf{X}\right), F\left(\mathsf{Y}\right), F\left(\mathsf{Z}\right)\right) \\ & \qquad \leq \max_{1 \leq i \leq n} G\left(gx_i, gy_i, gz_i\right) - \varphi\left(\max_{1 \leq i \leq n} G\left(gx_i, gy_i, gz_i\right)\right) \end{aligned}$$

*for all $\mathsf{X}\left(x_1, x_2, \ldots, x_n\right), \mathsf{Y}\left(y_1, y_2, \ldots, y_n\right), \mathsf{Z}\left(z_1, z_2, \ldots, z_n\right) \in X^n$ for which*

$$gx_i \preccurlyeq_i gy_i \preccurlyeq_i gz_i \quad \text{for all } i \in \{1, 2, \ldots, n\}.$$

**(ii)** *At least, one of the following conditions holds.*

**(ii.1)** *$F(X^n) \subseteq g(X)$ and there exist $x_0^1, x_0^2, \ldots, x_0^n \in X$ such that $gx_0^i \preccurlyeq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)})$ for all $i \in \{1, 2, \ldots, n\}$.*

**(ii.2)** *There exists a Picard* $(F, g, \Phi)$*-sequence* $\{(x_m^1, x_m^2, \ldots, x_m^n)\}_{m\in\mathbb{N}} \subseteq X^n$ *such that* $gx_0^i \preccurlyeq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)})$ *for all* $i \in \{1, 2, \ldots, n\}$.

**(iii)** *At least, one of the following conditions holds.*

**(iii.1)** $(X, G)$ *(or* $(g(X), G)$ *or* $(F(X^n), G)$*) is complete and* $F$ *and* $g$ *are continuous and* $(O, \preccurlyeq, \Phi)$*-compatible.*
**(iii.2)** $(X, G)$ *(or* $(g(X), G)$ *or* $(F(X^n), G)$*) is complete and* $F$ *and* $g$ *are continuous and commuting.*
**(iii.3)** $(g(X), G)$ *is complete and* $(X, G, \preceq)$ *is regular.*

*Then* $F$ *and* $g$ *have, at least, a* $\Phi$*-coincidence point.*

If we take $\varphi(t) = (1 - \lambda)t$ for all $t \geq 0$, where $\lambda \in [0, 1)$, we obtain the following result.

**Corollary 11.6.3.** *Let* $(X, G)$ *be a G-metric space endowed with a preorder* $\preccurlyeq$ *and let* $F : X^n \to X$ *and* $g : X \to X$ *be two mappings such that* $F$ *has the mixed* $(g, \preccurlyeq)$*-monotone property. Let* $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ *be a n-tuple of mappings from* $\{1, 2, \ldots, n\}$ *into itself satisfying* $\sigma_i \in \Omega_{\mathsf{A},\mathsf{B}}$ *if* $i \in \mathsf{A}$ *and* $\sigma_i \in \Omega'_{\mathsf{A},\mathsf{B}}$ *if* $i \in \mathsf{B}$*. Assume that the following conditions hold.*

**(i)** *There exists a constant* $\lambda \in [0, 1)$ *such that*

$$G(F(\mathsf{X}), F(\mathsf{Y}), F(\mathsf{Z})) \leq \lambda \max_{1\leq i\leq n} G(gx_i, gy_i, gz_i)$$

*for all* $\mathsf{X}(x_1, x_2, \ldots, x_n), \mathsf{Y}(y_1, y_2, \ldots, y_n), \mathsf{Z}(z_1, z_2, \ldots, z_n) \in X^n$ *for which*

$$gx_i \preccurlyeq_i gy_i \preccurlyeq_i gz_i \quad \textit{for all } i \in \{1, 2, \ldots, n\}.$$

**(ii)** *At least, one of the following conditions holds.*

**(ii.1)** $F(X^n) \subseteq g(X)$ *and there exist* $x_0^1, x_0^2, \ldots, x_0^n \in X$ *such that* $gx_0^i \preccurlyeq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)})$ *for all* $i \in \{1, 2, \ldots, n\}$.
**(ii.2)** *There exists a Picard* $(F, g, \Phi)$*-sequence* $\{(x_m^1, x_m^2, \ldots, x_m^n)\}_{m\in\mathbb{N}} \subseteq X^n$ *such that* $gx_0^i \preccurlyeq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)})$ *for all* $i \in \{1, 2, \ldots, n\}$.

**(iii)** *At least, one of the following conditions holds.*

**(iii.1)** $(X, G)$ *(or* $(g(X), G)$ *or* $(F(X^n), G)$*) is complete and* $F$ *and* $g$ *are continuous and* $(O, \preccurlyeq, \Phi)$*-compatible.*
**(iii.2)** $(X, G)$ *(or* $(g(X), G)$ *or* $(F(X^n), G)$*) is complete and* $F$ *and* $g$ *are continuous and commuting.*
**(iii.3)** $(g(X), G)$ *is complete and* $(X, G, \preccurlyeq)$ *is regular.*

*Then* $F$ *and* $g$ *have, at least, a* $\Phi$*-coincidence point.*

A version of Theorem 11.6.1 using $g$ as the identity mapping is the following one.

**Corollary 11.6.4.** *Let $(X, G)$ be a complete G-metric space endowed with a preorder $\preccurlyeq$ and let $F : X^n \to X$ be a mapping having the mixed $\preccurlyeq$-monotone property. Let $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be a n-tuple of mappings from $\{1, 2, \ldots, n\}$ into itself satisfying $\sigma_i \in \Omega_{\mathsf{A},\mathsf{B}}$ if $i \in \mathsf{A}$ and $\sigma_i \in \Omega'_{\mathsf{A},\mathsf{B}}$ if $i \in \mathsf{B}$. Assume that the following conditions hold.*

**(i)** *There exist two functions $\psi, \varphi \in \mathcal{F}_{\text{alt}}$ such that*

$$\psi\left(G\left(F\left(\mathsf{X}\right), F\left(\mathsf{Y}\right), F\left(\mathsf{Z}\right)\right)\right) \leq \left(\psi - \varphi\right)\left(\max_{1 \leq i \leq n} G\left(x_i, y_i, z_i\right)\right)$$

*for all $\mathsf{X}\left(x_1, x_2, \ldots, x_n\right), \mathsf{Y}\left(y_1, y_2, \ldots, y_n\right), \mathsf{Z}\left(z_1, z_2, \ldots, z_n\right) \in X^n$ for which*

$$x_i \preccurlyeq_i y_i \preccurlyeq_i z_i \quad \textit{for all } i \in \{1, 2, \ldots, n\}.$$

**(ii)** *There exist $x_0^1, x_0^2, \ldots, x_0^n \in X$ such that $gx_0^i \preccurlyeq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)})$ for all $i \in \{1, 2, \ldots, n\}$.*

**(iii)** *At least, one of the following conditions holds.*

**(iii.1)** *$F$ is continuous, or*

**(iii.2)** *$(X, G, \preccurlyeq)$ is regular.*

*Then $F$ has, at least, a $\Phi$-fixed point.*

### 11.6.3 Uniqueness

Finally, we describe how we can ensure the uniqueness of the $\Phi$-coincidence point.

**Theorem 11.6.2.** *Under the hypotheses of Theorem 11.6.1, also assume that $F(X^n) \subseteq g(X)$. Let $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ be two $\Phi$-coincidence points of $F$ and $g$ for which there exists $(\omega_1, \omega_2, \ldots, \omega_n) \in X^n$ such that:*

$$gx_i \preccurlyeq_i g\omega_i \quad \textit{and} \quad gy_i \preccurlyeq_i g\omega_i \quad \textit{for all } i \in \{1, 2, \ldots, n\}.$$

*Then $gx_i = gy_i$ for all $i \in \{1, 2, \ldots, n\}$.*

*Proof.* Let $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ be two $\Phi$-coincidence points of $F$ and $g$ for which there exists $\left(\omega_0^1, \omega_0^2, \ldots, \omega_0^n\right) \in X^n$ such that

$$gx_i \preccurlyeq_i g\omega_0^i \quad \text{and} \quad gy_i \preccurlyeq_i g\omega_0^i \quad \text{for all } i \in \{1, 2, \ldots, n\}. \tag{11.65}$$

From Lemma 11.6.1, there exists a sequence $\{\left(\omega_m^1, \omega_m^2, \ldots, \omega_m^n\right)\}_{m \in \mathbb{N}}$ on $X^n$ such that

$$g\omega_{m+1}^i = F(\omega_m^{\sigma_i(1)}, \omega_m^{\sigma_i(2)}, \ldots, \omega_m^{\sigma_i(n)})$$

$$\text{for all } m \in \mathbb{N} \text{ and all } i \in \{1,2,\ldots,n\}\,.$$

We claim that, for all $m \in \mathbb{N}$ and all $i \in \{1,2,\ldots,n\}$, we have that

$$gx_i \preccurlyeq_i g\omega_m^i \quad \text{and} \quad gy_i \preccurlyeq_i g\omega_m^i. \tag{11.66}$$

We only show the first part (using $(x_1, x_2, \ldots, x_n)$) because the second one is similar. For $m = 0$, (11.66) holds by (11.65). Assume, by hypothesis of induction, that there exist $m \in \mathbb{N}$ such that $gx_i \preccurlyeq_i g\omega_m^i$ for all $i \in \{1,2,\ldots,n\}$. This condition means that

$$\begin{cases} gx_j \preccurlyeq g\omega_m^j, \text{ if } j \in \mathsf{A}, \\ gx_j \succcurlyeq g\omega_m^j, \text{ if } j \in \mathsf{B}. \end{cases} \tag{11.67}$$

To complete the induction process, we have to prove that $gx_i \preccurlyeq_i g\omega_{m+1}^i$ for all $i \in \{1,2,\ldots,n\}$, that is,

$$\begin{cases} gx_j \preccurlyeq g\omega_{m+1}^j, \text{ if } j \in \mathsf{A}, \\ gx_j \succcurlyeq g\omega_{m+1}^j, \text{ if } j \in \mathsf{B}. \end{cases} \tag{11.68}$$

We distinguishing two cases.

*Case 1:* $i \in \mathsf{A}$. In this case, $\sigma_i(\mathsf{A}) \subseteq \mathsf{A}$ and $\sigma_i(\mathsf{B}) \subseteq \mathsf{B}$. As $F$ has the mixed $g$-monotone property, we apply that $F$ is $g$-monotone $(g, \preccurlyeq)$-non-decreasing in $\mathsf{A}$-arguments with the first inequalities of (11.57) and we deduce that, for all $a_1, a_2, \ldots, a_n \in X$:

$$\text{if } j,s \in \mathsf{A}, \quad gx_j \preccurlyeq g\omega_m^j \;\Rightarrow$$
$$F(a_1,\ldots,a_{s-1},x_j,a_{s+1},\ldots,a_n) \preccurlyeq F(a_1,\ldots,a_{s-1},\omega_m^j,a_{s+1},\ldots,a_n),$$

and that $F$ is $g$-monotone $(g, \preccurlyeq)$-non-increasing in $\mathsf{B}$-arguments with the second inequalities of (11.67):

$$\text{if } j,s \in \mathsf{B}, \quad gx_j \succcurlyeq g\omega_m^j \;\Rightarrow$$
$$F(a_1,\ldots,a_{s-1},x_j,a_{s+1},\ldots,a_n) \preccurlyeq F(a_1,\ldots,a_{s-1},\omega_m^j,a_{s+1},\ldots,a_n).$$

In any case, it follows that, if $j,s \in \{1,2,\ldots,n\}$ satisfy $j,s \in \mathsf{A}$ or $j,s \in \mathsf{B}$, then:

$$F(a_1,\ldots,a_{s-1},x_j,a_{s+1},\ldots,a_n) \preccurlyeq F(a_1,\ldots,a_{s-1},\omega_m^j,a_{s+1},\ldots,a_n)$$

for all $a_1, a_2, \ldots, a_n \in X$. As $\sigma_i \in \Omega_{\mathsf{A},\mathsf{B}}$:

$$gx_i = F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, x_{\sigma_i(3)}, \ldots, x_{\sigma_i(n)}) \qquad (1, \sigma_i(1) \in \mathsf{A} \text{ or } 1, \sigma_i(1) \in \mathsf{B})$$

$$\preccurlyeq F(\omega_m^{\sigma_i(1)}, x_{\sigma_i(2)}, x_{\sigma_i(3)}, \ldots, x_{\sigma_i(n)}) \qquad (2, \sigma_i(2) \in \mathsf{A} \text{ or } 2, \sigma_i(2) \in \mathsf{B})$$
$$\preccurlyeq F(\omega_m^{\sigma_i(1)}, \omega_m^{\sigma_i(2)}, x_{\sigma_i(3)}, \ldots, x_{\sigma_i(n)}) \qquad (3, \sigma_i(3) \in \mathsf{A} \text{ or } 3, \sigma_i(3) \in \mathsf{B})$$
$$\preccurlyeq \ldots \preccurlyeq F(\omega_m^{\sigma_i(1)}, \omega_m^{\sigma_i(2)}, \omega_m^{\sigma_i(3)}, \ldots, \omega_m^{\sigma_i(n)}) = g\omega_{m+1}^i.$$

Hence $gx_i \preccurlyeq g\omega_{m+1}^i$ when $i$ is in $\mathsf{A}$, so (11.68) holds if $i \in \mathsf{A}$.

*Case 2:* $i \in \mathsf{B}$. In this case, we apply that $F$ is $(g, \preccurlyeq)$-monotone non-decreasing in $\mathsf{A}$-arguments with the second inequalities of (11.67) and that $F$ is $(g, \preccurlyeq)$-monotone non-increasing in $\mathsf{B}$-arguments with the first inequalities of (11.67), and we deduce, for all $a_1, a_2, \ldots, a_n \in X$, that, if $j, s \in \{1, 2, \ldots, n\}$ satisfy $j \in \mathsf{A}$, $s \in \mathsf{B}$ or $j \in \mathsf{B}$, $s \in \mathsf{A}$, then

$$F(a_1, \ldots, a_{s-1}, x_j, a_{s+1}, \ldots, a_n) \succcurlyeq F(a_1, \ldots, a_{s-1}, \omega_m^j, a_{s+1}, \ldots, a_n).$$

Since $\sigma_i \in \Omega'_{\mathsf{A},\mathsf{B}}$, therefore:

$$\begin{aligned} gx_i &= F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, x_{\sigma_i(3)}, \ldots, x_{\sigma_i(n)}) \\ &\qquad\qquad (1 \in \mathsf{A}, \sigma_i(1) \in \mathsf{B} \text{ or } 1 \in \mathsf{B}, \sigma_i(1) \in \mathsf{A}) \\ &\succcurlyeq F(\omega_m^{\sigma_i(1)}, x_{\sigma_i(2)}, x_{\sigma_i(3)}, \ldots, x_{\sigma_i(n)}) \\ &\qquad\qquad (2 \in \mathsf{A}, \sigma_i(2) \in \mathsf{B} \text{ or } 2 \in \mathsf{B}, \sigma_i(2) \in \mathsf{A}) \\ &\succcurlyeq F(\omega_m^{\sigma_i(1)}, \omega_m^{\sigma_i(2)}, x_{\sigma_i(3)}, \ldots, x_{\sigma_i(n)}) \\ &\qquad\qquad (3 \in \mathsf{A}, \sigma_i(3) \in \mathsf{B} \text{ or } 3 \in \mathsf{B}, \sigma_i(3) \in \mathsf{A}) \\ &\succcurlyeq \ldots \succcurlyeq F(\omega_m^{\sigma_i(1)}, \omega_m^{\sigma_i(2)}, \omega_m^{\sigma_i(3)}, \ldots, \omega_m^{\sigma_i(n)}) = g\omega_{m+1}^i. \end{aligned}$$

Hence $gx_i \succcurlyeq g\omega_{m+1}^i$ when $i \in \mathsf{B}$, that is, we also have that $gx_i \preccurlyeq_i g\omega_{m+1}^i$ when $i \in \mathsf{B}$. This completes the induction so (11.66) holds.

Next, let, for all $i \in \{1, 2, \ldots, n\}$ and all $m \in \mathbb{N}$,

$$\left\{a_m^i = G\left(gx_i, gx_i, g\omega_{m+1}^i\right)\right\}_{m\in\mathbb{N}} \quad \text{and}$$
$$\left\{b_m^i = \max_{1\le j\le n} G\left(gx_{\sigma_i(j)}, gx_{\sigma_i(j)}, g\omega_m^{\sigma_i(j)}\right)\right\}_{m\in\mathbb{N}}.$$

By the contractivity condition (11.59), for all $m \in \mathbb{N}$ and all $i \in \{1, 2, \ldots, n\}$, we have, taking into account that $gx_i \preccurlyeq_i gx_i \preccurlyeq_i g\omega_m^i$,

$$\begin{aligned} \psi\left(a_{m+1}^i\right) &= \psi\left(G\left(gx_i, gx_i, g\omega_{m+1}^i\right)\right) \\ &= \psi\left(G\left(F\left(x_{\sigma_i(1)}, x_{\sigma_i(2)}, x_{\sigma_i(3)}, \ldots, x_{\sigma_i(n)}\right),\right.\right. \end{aligned}$$

$$F\left(x_{\sigma_i(1)}, x_{\sigma_i(2)}, x_{\sigma_i(3)}, \ldots, x_{\sigma_i(n)}\right), F(\omega_m^{\sigma_i(1)}, \omega_m^{\sigma_i(2)}, \ldots, \omega_m^{\sigma_i(n)})))$$
$$\leq (\psi - \varphi)\left(\max_{1\leq j\leq n} G\left(gx_{\sigma_i(j)}, gx_{\sigma_i(j)}, g\omega_m^{\sigma_i(j)}\right)\right) = (\psi - \varphi)\left(b_m^i\right).$$

Notice that

$$\max_{1\leq i\leq n} b_m^i = \max_{1\leq i\leq n}\left(\max_{1\leq j\leq n} G\left(gx_{\sigma_i(j)}, gx_{\sigma_i(j)}, g\omega_m^{\sigma_i(j)}\right)\right)$$
$$\leq \max_{1\leq i\leq n} G\left(gx_i, gx_i, g\omega_m^i\right) = \max_{1\leq i\leq n} a_m^i,$$

and, as $\psi$ is non-decreasing,

$$\psi\left(\max_{1\leq i\leq n} b_m^i\right) \leq \psi\left(\max_{1\leq i\leq n} a_m^i\right).$$

Applying Lemma 11.2.4, we deduce that

$$\lim_{m\to\infty} G\left(gx_i, gx_i, g\omega_{m+1}^i\right) = \lim_{m\to\infty} a_m^i = 0 \quad \text{for all } i \in \{1, 2, \ldots, n\}.$$

Hence $\{g\omega_m^i\}_{m\in\mathbb{N}} \to gx_i$ for all $i \in \{1, 2, \ldots, n\}$. Using the same argument, we also have that $\{g\omega_m^i\}_{m\in\mathbb{N}} \to gy_i$ for all $i \in \{1, 2, \ldots, n\}$. By the uniqueness of the limit, we conclude that $gx_i = gy_i$ for all $i \in \{1, 2, \ldots, n\}$. □

We will say that $g$ is *injective on the set of all $\Phi$-coincidence points of $F$ and $g$* if for all $\Phi$-coincidence points $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ of $F$ and $g$ such that $gx_i = gy_i$ for all $i \in \{1, 2, \ldots, n\}$, we can deduce that $x_i = y_i$ for all $i \in \{1, 2, \ldots, n\}$.

**Corollary 11.6.5.** *Under the hypotheses of Theorem 11.6.1, also assume that $F(X^n) \subseteq g(X)$ and the following conditions:*

$(U)$ *For all $\Phi$-coincidence points $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ of $F$ and $g$, there exists $(\omega_1, \omega_2, \ldots, \omega_n) \in X^n$ such that*

$$gx_i \preccurlyeq_i g\omega_i \quad \text{and} \quad gy_i \preccurlyeq_i g\omega_i \quad \text{for all } i \in \{1, 2, \ldots, n\}.$$

$(U')$ *$g$ is injective on the set of all $\Phi$-coincidence points of $F$ and $g$.*

*Then $F$ and $g$ have a unique $\Phi$-coincidence point.*

*Proof.* Let $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ be two arbitrary $\Phi$-coincidence points of $F$ and $g$. From Theorem 11.6.2, $gx_i = gy_i$ for all $i \in \{1, 2, \ldots, n\}$, and as $g$ is injective on the set of all $\Phi$-coincidence points of $F$ and $g$, we conclude that $x_i = y_i$ for all $i \in \{1, 2, \ldots, n\}$. □

## 11.7 Reducing Multidimensional Results to Unidimensional Ones

Many authors have proved that coupled, tripled and quadrupled fixed point results can be deduced from their corresponding unidimensional version. This section describes how some of the previous multidimensional theorems can be easily concluded from simple unidimensional results.

### *11.7.1 The Low-Dimensional Reducing Technique*

Throughout this section, given $n \in \{2,3,4\}$ and two mappings $F : X^n \to X$ and $g : X \to X$, denote by $T_F^n, G^n : X^n \to X^n$ the mappings

$$\begin{cases} n = 2,\ T_F^2(x,y) = (F(x,y), F(y,x)), \\ n = 3,\ T_F^3(x,y,z) = (F(x,y,z), F(y,x,y), F(z,y,x)), \\ n = 4,\ T_F^4(x,y,z,t) = (F(x,y,z,t), F(y,z,t,x), F(z,t,x,y), \\ \qquad\qquad\qquad\qquad\qquad F(t,x,y,z)). \end{cases} \tag{11.69}$$

$$\begin{cases} n = 2,\ H_g^2(x,y) = (gx, gy), \\ n = 3,\ H_g^3(x,y,z) = (gx, gy, gz), \\ n = 4,\ H_g^4(x,y,z,t) = (gx, gy, gz, gt). \end{cases} \tag{11.70}$$

The following lemma guarantees that multidimensional notions of common/fixed/coincidence points can be interpreted in terms of $T_F^N$ and $G^N$.

**Lemma 11.7.1.** *Given $n \in \{2,3,4\}$, $F : X^n \to X$ and $g : X \to X$, a point $(x_1, x_2, \ldots, x_n) \in X^n$ is:*

1. *a coupled/tripled/quadrupled fixed point of $F$ if, and only if, it is a fixed point of $T_F^n$;*
2. *a coupled/tripled/quadrupled coincidence point of $F$ and $g$ if, and only if, it is a coincidence point of $T_F^n$ and $H_g^n$;*
3. *a coupled/tripled/quadrupled common fixed point of $F$ and $g$ if, and only if, it is a common fixed point of $T_F^n$ and $H_g^n$.*

*Proof.* For example, if $n = 2$, a point $(x,y) \in X^2$ is a coupled coincidence point of $F$ and $g$ if, and only if,

$$\left.\begin{array}{r} F(x,y) = gx, \\ F(y,x) = gy \end{array}\right\} \Leftrightarrow (F(x,y), F(y,x)) = (gx, gy)$$

$$\Leftrightarrow T_F^2(x,y) = H_g^2(x,y),$$

which means that $(x,y)$ is a coincidence point of $T_F^2$ and $H_g^2$. The other cases are similar. □

**Proposition 11.7.1.** *The mappings $F$ and $g$ commute if, and only if, $T_F^n$ and $H_g^n$ commute.*

*Proof.* Assume $n = 2$. If $F$ and $g$ commute, then, for all $(x, y) \in X^2$,

$$\begin{aligned} H_g^2 T_F^2 (x, y) &= H_g^2(F(x, y), F(y, x)) = (gF(x, y), gF(y, x)) \\ &= (F(gx, gy), F(gy, gx)) = T_F^2(gx, gy) = H_g^2 T_F^2 (x, y) , \end{aligned}$$

so $T_F^2$ and $H_g^2$ also commute. Conversely, if $T_F^2$ and $H_g^2$ commute, then

$$\begin{aligned} (gF(x, y), gF(y, x)) &= H_g^2(F(x, y), F(y, x)) = H_g^2 T_F^2 (x, y) = \\ &= H_g^2 T_F^2 (x, y) = T_F^2(gx, gy) = (F(gx, gy), F(gy, gx)) . \end{aligned}$$

In particular, $gF(x, y) = F(gx, gy)$ for all $x, y \in X$, so $F$ and $g$ commute. □

The continuity of $F$ and $g$ implies the continuity of $T_F^n$ and $H_g^n$.

**Lemma 11.7.2.** *If $(X, G)$ is a G-metric space and let $F : X^n \to X$ and $g : X \to X$ be two mappings.*

1. *The mapping $g$ is continuous if, and only if, the mapping $H_g^n : X^n \to X^n$ is also continuous (considering in $X^n$ the $G^*$-metric $G_n$ or $G'_n$ as in Lemma 10.3.1).*
2. *The mapping $F$ is continuous if, and only if, the mapping $T_F^n : X^n \to X^n$ is also continuous (considering in $X^n$ the $G^*$-metric $G_n$ or $G'_n$ as in Lemma 10.3.1).*

*Proof.* **(1)** Assume that $g$ is continuous and let $\{(x_m^1, x_m^2, \ldots, x_m^n)\}_{m\in\mathbb{N}} \subseteq X^n$ and $(a^1, a^2, \ldots, a^n) \in X^n$ be such that

$$\{(x_m^1, x_m^2, \ldots, x_m^n)\} \xrightarrow{G_n} (a^1, a^2, \ldots, a^n) .$$

From item 2 of Lemma 10.3.1, $\{x_m^i\} \xrightarrow{G} a^i$ for all $i \in \{1, 2, \ldots, n\}$. Since $g$ is continuous, then $\{gx_m^i\} \xrightarrow{G} ga^i$ for all $i \in \{1, 2, \ldots, n\}$. Again by item 2 of Lemma 10.3.1, we deduce that

$$\begin{aligned} \{H_g^n (x_m^1, x_m^2, \ldots, x_m^n)\} &= \{(gx_m^1, gx_m^2, \ldots, gx_m^n)\} \\ &\xrightarrow{G_n} (ga^1, ga^2, \ldots, ga^n) = H_g^n (a^1, a^2, \ldots, a^n) . \end{aligned}$$

Then, $H_g^n$ is $G_n$-continuous.

Conversely, assume that $H_g^n$ is $G_n$-continuous and let $\{x_m\}_{m\in\mathbb{N}} \subseteq X$ and $a \in X$ be such that $\{x_m\} \xrightarrow{G} a$. Therefore, $\{(x_m, x_m, \ldots, x_m)\} \xrightarrow{G_n} (a, a, \ldots, a)$. As $H_g^n$ is $G_n$-continuous, then

$$\{(gx_m, gx_m, \ldots, gx_m)\} = \{H_g^n (x_m, x_m, \ldots, x_m)\}$$

$$\xrightarrow{G_n} H_g^n(a,a,\ldots,a) = (ga,ga,\ldots,ga)\,.$$

As a result, $\{x_m\} \xrightarrow{G} a$, so $g$ is continuous.

**(2)** It is similar to the proof of the previous item.

□

Any binary relation $\preceq$ on $X$ can be induced on $X^n$ generating the binary relation $\sqsubseteq$ given by:

$$\begin{cases} n=2,\ (x,y)\sqsubseteq(u,v) \ \Leftrightarrow\ [x\preceq u \text{ and } y\succeq v\,]\,, \\ n=3,\ (x,y,z)\sqsubseteq(u,v,w) \ \Leftrightarrow\ [x\preceq u,\ y\preceq v \text{ and } z\succeq w\,]\,, \\ n=4,\ (x,y,z,t)\sqsubseteq(u,v,w,s) \ \Leftrightarrow\ [x\preceq u,\ y\preceq v,\ z\succeq w \text{ and } t\preceq s\,]\,. \end{cases} \tag{11.71}$$

Notice that $\sqsubseteq$ directly depends on $\preceq$. Also notice that $\sqsubseteq$ coincides with the binary relation defined in (11.79) when $\mathsf{A}$ is the subset of all odd numbers in $\{1,2,\ldots,n\}$ and $\mathsf{B}$ contains its even numbers.

**Lemma 11.7.3.** *The binary relation $\preccurlyeq$ on $X$ is reflexive (respectively, transitive, a preorder, a partial order) if, and only if, the binary relation $\sqsubseteq$ on $X^n$ is reflexive (respectively, transitive, a preorder, a partial order).*

Furthermore, it would be equivalent to consider the binary relation $\sqsubseteq'$ given by

$$(x,y)\sqsubseteq'(u,v) \ \Leftrightarrow\ [x\succeq u \text{ and } y\preceq v\,]\,. \tag{11.72}$$

We also have the following properties.

**Lemma 11.7.4.** *Let $\preceq$ be a binary relation on $X$ and, given $n\in\{2,3,4\}$, let $F: X^n\to X$ and $g: X\to X$ be two mappings.*

1. *If $F$ has the mixed $(g,\preceq)$-monotone property and $\preceq$ is transitive, then $T_F^n$ is a $\left(H_g^n,\sqsubseteq\right)$-non-decreasing mapping.*
2. *If $T_F^n$ is a $\left(H_g^n,\sqsubseteq\right)$-non-decreasing mapping and $\preceq$ is reflexive, then $F$ has the mixed $(g,\preceq)$-monotone property.*

*Proof.* We show the proof assuming that $n=2$. Let $(x,y)\,,(u,v)\in X^2$ be such that $H_g^2(x,y)\sqsubseteq H_g^2(u,v)$. In particular, $gx\preceq gu$ and $gy\succeq gv$. Therefore, as $F$ has the mixed $(g,\preceq)$-monotone property,

$$F(x,y)\preceq F(u,y)\preceq F(u,v) \quad \text{and} \quad F(y,x)\succeq F(v,x)\succeq F(v,u).$$

As $\preceq$ is transitive, $F(x,y)\preceq F(u,v)$ and $F(y,x)\succeq F(v,u)$. Hence,

$$T_F^2(x,y) = (F(x,y),F(y,x)) \sqsubseteq (F(u,v),F(v,u)) = T_F^2(u,v)\,.$$

This proves that $T_F^2$ is a $\left(H_g^2,\sqsubseteq\right)$-non-decreasing mapping.

Conversely, assume that $T_F^2$ is a $\left(H_g^2, \sqsubseteq\right)$-non-decreasing mapping. Let $x, y \in X$ be such that $gx \preceq gy$. Now let $a \in X$ arbitrary. As $\preceq$ is reflexive, then $ga \succeq ga$. Therefore, $(gx, ga) \sqsubseteq (gy, ga)$. In other words, $H_g^2(x, a) \sqsubseteq H_g^2(y, a)$. As $T_F^2$ is a $\left(H_g^2, \sqsubseteq\right)$-non-decreasing mapping, then

$$(F(x, a), F(a, x)) = T_F^2(x, a) \sqsubseteq T_F^2(y, a) = (F(y, a), F(a, y)).$$

Hence, $F(x, a) \preceq F(y, a)$ (and $F$ is $(g, \preceq)$-non-decreasing in its first argument) and $F(a, x) \succeq F(a, y)$ (so $F$ is $(g, \preceq)$-non-increasing in its second argument). Therefore, $F$ has the mixed $(g, \preceq)$-monotone property. □

**Corollary 11.7.1.** *Let $\preceq$ be a preorder on $X$ and, given $n \in \{2, 3, 4\}$, let $F : X^n \to X$ and $g : X \to X$ be two mappings. Then $F$ has the mixed $(g, \preceq)$-monotone property if, and only if, $T_F^n$ is a $\left(H_g^n, \sqsubseteq\right)$-non-decreasing mapping.*

If we had considered the binary relation $\sqsubseteq'$ on $X^n$ given in (11.72), then $T_F^n$ would have been a $\left(H_g^n, \sqsubseteq\right)$-non-increasing mapping.

**Lemma 11.7.5.** *Given a reflexive binary relation $\preceq$ on a $G$-metric space $(X, G)$ and $n \in \{2, 3, 4\}$, let $\sqsubseteq$ be the binary relation on $X^n$ given in (11.71) and let $G_n$ and $G'_n$ the $G^*$-metrics on $X^n$ defined in Lemma 10.3.1. Then the following conditions are equivalent (the same is valid for $G'_n$).*

- **(i)** *$(X, G, \preceq)$ is regular.*
- **(ii)** *$(X^n, G_n, \sqsubseteq)$ is regular.*
- **(iii)** *$(X^n, G_n, \sqsubseteq)$ is non-decreasing-regular.*
- **(iv)** *$(X^n, G_n, \sqsubseteq)$ is non-increasing-regular.*

Notice that the condition "$(X, G, \preceq)$ is non-decreasing-regular" is not strong enough to guarantee that $(X^n, G_n, \sqsubseteq)$ is regular nor $(X^n, G_n, \sqsubseteq)$ is non-decreasing-regular.

*Proof.* We show the proof in the coupled case (the other cases are similar).

(i)$\Rightarrow$(iii) Assume that $(X, G, \preceq)$ is regular and let $\{(x_m, y_m)\}_{m\in\mathbb{N}} \subseteq X^2$ and $(x, y) \in X^2$ be such that $(x_m, y_m) \sqsubseteq (x_{m+1}, y_{m+1})$ for all $m \in \mathbb{N}$ and $\{(x_m, y_m)\} \xrightarrow{G_2} (x, y)$. On the one hand, $x_m \preceq x_{m+1}$ and $y_m \succeq y_{m+1}$ for all $m \in \mathbb{N}$. By item 2 of Lemma 10.3.1, we have that $\{x_m\} \xrightarrow{G} x$ and $\{y_m\} \xrightarrow{G} y$. As $(X, G, \preceq)$ is both non-decreasing and non-increasing-regular, then $x_m \preceq x$ and $y_m \succeq y$ for all $m \in \mathbb{N}$. In particular, $(x_m, y_m) \sqsubseteq (x, y)$ for all $m \in \mathbb{N}$. This proves that $\left(X^2, G_2, \sqsubseteq\right)$ is non-decreasing-regular.

(iii)$\Rightarrow$(i) Assume that $\left(X^2, G_2, \sqsubseteq\right)$ is non-decreasing-regular and let $\{x_m\}_{m\in\mathbb{N}} \subseteq X$ and $x \in X$ be such that $x_m \preceq x_{m+1}$ for all $m \in \mathbb{N}$ and $\{x_m\} \xrightarrow{G} x$. Given an arbitrary point $a \in X$, let $y_m = a$ for all $m \in \mathbb{N}$. As $\preceq$ is reflexive, then $y_m \succeq y_{m+1}$ for all $m \in \mathbb{N}$. Moreover, $\{y_m\} \xrightarrow{G} a$. In particular, $(x_m, y_m) \sqsubseteq (x_{m+1}, y_{m+1})$ for all $m \in \mathbb{N}$. From item 2 of Lemma 10.3.1, $\{(x_m, y_m)\} \xrightarrow{G_2} (x, y)$. As $\left(X^2, G_2, \sqsubseteq\right)$ is non-decreasing-regular, then $(x_m, y_m) \sqsubseteq (x, a)$ for all $m \in \mathbb{N}$. In particular, $x_m \preceq x$ for all $m \in \mathbb{N}$, which proves that $(X, G, \preceq)$ is regular.

(i)⇔(iv) It is exactly the same proof of the previous two cases.

(i)⇔(ii) It follows from the equivalence (iii)⇔(i)⇔(iv). □

### 11.7.2 Reducing Coupled Fixed Point Theorems

In the following result, it is important to notice that $\left(X^2, G'_2\right)$ is a $G^*$-metric space, but it is not necessarily a $G$-metric space.

**Theorem 11.7.1.** *Theorems 11.3.1, 11.3.2 and 11.3.3 immediately follow from Corollary 10.4.5.*

*Proof.* Assume that $(X, G)$ is a complete $G$-metric space endowed with a preorder $\preceq$ and let $F : X^2 \to X$ be a mapping having the mixed $\preceq$-monotone property. Suppose that there exists a constant $\lambda \in [0, 1)$ such that

$$G\left(F\left(x,y\right),F\left(u,v\right),F\left(z,w\right)\right) \leq \frac{\lambda}{2}\left(G\left(x,u,z\right)+G\left(y,v,w\right)\right) \tag{11.73}$$

for all $(x, y), (u, v), (z, w) \in X^2$ satisfying $x \preceq u \preceq z$ and $y \succeq v \succeq w$. Also assume that either $F$ is continuous or $(X, G, \preceq)$ is regular, and that there exists $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succcurlyeq F(y_0, x_0)$.

As $G$ is a $G$-metric on $X$, then $G'_2$, defined by

$$G'_2\left(\left(x_1,y_1\right),\left(x_2,y_2\right),\left(x_3,y_3\right)\right)=G\left(x_1,x_2,x_3\right)+G\left(y_1,y_2,y_3\right)$$

for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X^2$, is a $G^*$-metric on $X^2$ (see item 1 of Corollary 10.3.1). Consider on $X^2$ the binary relation $\sqsubseteq$ defined in (11.71). Then, we have the following properties.

- By item 4 of Lemma 10.3.1, $\left(X^2, G'_2\right)$ is complete.
- By Lemma 11.7.3, $\sqsubseteq$ is a preorder on $X^2$.
- By Corollary 11.7.1, $T_F^2$ is a $\left(H_g^2, \sqsubseteq\right)$-non-decreasing mapping.
- As there exists $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succcurlyeq F(y_0, x_0)$, then $(x_0, y_0) \sqsubseteq T_F^2(x_0, y_0)$.
- By item 2 of Lemma 11.7.2, if $F$ is $G$-continuous, then $T_F^2 : X^2 \to X^2$ is also $G'_2$-continuous.
- By Lemma 11.7.5, if $(X, G, \preceq)$ is regular, then $\left(X^2, G'_2, \sqsubseteq\right)$ is also regular.

Let us show that the contractivity condition (11.73) implies that, for all $(x, y), (u, v) \in X^2$ with $(x, y) \sqsubseteq (u, v)$,

$$G'_2(T_F^2\left(x,y\right),T_F^2\left(u,v\right),T_F^2\left(u,v\right)) \leq \lambda\, G'_2(\left(x,y\right),\left(u,v\right),\left(u,v\right)).$$

To prove it, we notice that by (11.73), $x \preccurlyeq u \preccurlyeq u$ and $y \succcurlyeq v \succcurlyeq v$ implies that

$$G\left(F\left(x,y\right),F\left(u,v\right),F\left(u,v\right)\right) \leq \frac{\lambda}{2}\left(G\left(x,u,u\right)+G\left(y,v,v\right)\right),$$

and by $v \preccurlyeq v \preccurlyeq y$ and $u \succcurlyeq u \succcurlyeq x$ we deduce

$$G\left(F\left(v,u\right),F\left(v,u\right),F\left(y,x\right)\right) \leq \frac{\lambda}{2}\left(G\left(v,v,y\right)+G\left(u,u,x\right)\right).$$

Joining both inequalities, we conclude that

$$\begin{aligned}
&G_2'(T_F^2\left(x,y\right),T_F^2\left(u,v\right),T_F^2\left(u,v\right))\\
&\quad = G\left(F\left(x,y\right),F\left(u,v\right),F\left(u,v\right)\right)+G\left(F\left(v,u\right),F\left(v,u\right),F\left(y,x\right)\right)\\
&\quad \leq \lambda\left(G\left(x,u,u\right)+G\left(y,v,v\right)\right)\\
&\quad = \lambda\, G_2'(\left(x,y\right),\left(u,v\right),\left(u,v\right)).
\end{aligned}$$

This property is the contractivity condition in Corollary 10.4.5.

As a consequence, all hypotheses of Corollary 10.4.5, applied to $T_F^2$ on $\left(X^2, G_2', \sqsubseteq\right)$ hold. Then $T_F^2$ has, at least, a fixed point, which is a coupled fixed point of $F$ by item 1 of Lemma 11.7.1. Furthermore, the condition $(U)$ in Theorem 11.3.3 is equivalent to saying that for all fixed point $(x,y), (x',y') \in \mathrm{Fix}(T_F^2)$ (which are coupled fixed points of $F$), there exists $(z,w) \in X^2$ such that $(x,y) \sqsubseteq (z,w)$ and $(x',y') \sqsubseteq (z,w)$, which is precisely the uniqueness condition in Corollary 10.4.5.

□

### *11.7.3 Weakness of Some Coupled Fixed Point Results*

After the appearance of the reducing technique we have described in Sect. 11.7.2, many coupled fixed point results were reduced to the unidimensional case. In fact, such a procedure showed the weakness of some given statements. For instance, the following result was obtained by Shatanawi in [190] as a corollary of a previous coincidence point result.

**Theorem 11.7.2.** *Let $(X,G)$ be a complete G-metric space. Let $F: X\times X \to X$ be a mapping such that*

$$G(F(x,y),F(u,v),F(u,v)) \leq k\left(G(x,u,u)+G(y,v,v)\right) \tag{11.74}$$

*for all $x,y,u,v \in X$. If $k \in [0,1/2)$, then there is a unique $x$ in $X$ such that $F(x,x)=x$.*

In [13], Agarwal and Karapınar showed that the previous result can be deduced from Theorem 4.2.1. Furthermore, the following example illustrates a case in which Theorem 4.2.1 can be applied but Theorem 11.7.2 cannot be applied.

*Example 11.7.1.* Let $X = \mathbb{R}$ be endowed with the $G$-metric $G(x,y,z) = |x-y| + |x-z| + |y-z|$ for all $x,y,z \in X$, and let $F : X \times X \to X$ be the mapping given by $F(x,y) = (3x-y)/5$ for all $x,y \in X$. Then, for all $x,y,u,v,w,z \in X$, it follows that

$$
\begin{aligned}
G(F(x,y),F(u,v),F(w,z)) &= G\left(\frac{3x-y}{5},\frac{3u-v}{5},\frac{3w-z}{5}\right) \\
&= \left|\frac{3x-y}{5}-\frac{3u-v}{5}\right| + \left|\frac{3x-y}{5}-\frac{3w-z}{5}\right| \\
&\quad + \left|\frac{3u-v}{5}-\frac{3w-z}{5}\right| \\
&= \left|\frac{3}{5}(x-u)-\frac{1}{5}(y-v)\right| + \left|\frac{3}{5}(x-w)-\frac{1}{5}(y-z)\right| \\
&\quad + \left|\frac{3}{5}(u-w)-\frac{1}{5}(v-z)\right| \\
&\leq \frac{3}{5}\,|x-u| + \frac{1}{5}\,|y-v| + \frac{3}{5}\,|x-w| + \frac{1}{5}\,|y-z| \\
&\quad + \frac{3}{5}\,|u-w| + \frac{1}{5}\,|v-z| \\
&= \frac{3}{5}\,(|x-u|+|x-w|+|u-w|) \\
&\quad + \frac{1}{5}\,(|y-v|+|y-z|+|v-z|) \\
&\leq \frac{3}{5}\,(G(x,u,w)+G(y,v,z)).
\end{aligned}
$$

It is easy to see that there is no $k \in [0,1/2)$ verifying condition (11.74) because if $x = 1$ and $y = u = v = 0$, then

$$
\begin{aligned}
&G(F(1,0),F(0,0),F(0,0)) = G(3/5,0,0) = 2\,\frac{3}{5} = \frac{6}{5}, \\
&k\,(G(1,0,0)+G(0,0,0)) = k\,(2+0) = 2k < 1.
\end{aligned}
$$

However note that the mapping $T_F^2 : X^2 \to X^2$ satisfies

$$
\begin{aligned}
&G_2'\left(T_F^2(x,y),T_F^2(u,v),T_F^2(w,z)\right) \\
&\qquad = G_2'\left((F(x,y),F(y,x)),(F(u,v),F(v,u)),(F(w,z),F(z,w))\right) \\
&\qquad = G\left(F(x,y),F(u,v),F(w,z)\right) + G\left(F(y,x),F(v,u),F(z,w)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= G\left(\frac{3x-y}{5}, \frac{3u-v}{5}, \frac{3w-z}{5}\right) + G\left(\frac{3y-x}{5}, \frac{3v-u}{5}, \frac{3z-w}{5}\right) \\
&\leq \frac{3}{5}\,(\,|\,x-u\,| + |\,x-w\,| + |\,u-w\,|\,) \\
&\qquad + \frac{1}{5}\,(\,|\,y-v\,| + |\,y-z\,| + |\,v-z\,|\,) \\
&\qquad + \frac{3}{5}\,(\,|\,y-v\,| + |\,y-z\,| + |\,v-z\,|\,) \\
&\qquad + \frac{1}{5}\,(\,|\,x-u\,| + |\,x-w\,| + |\,u-w\,|\,) \\
&= \frac{4}{5}\,(\,|\,x-u\,| + |\,x-w\,| + |\,u-w\,| + |\,y-v\,| + |\,y-z\,| + |\,v-z\,|\,) \\
&= \frac{4}{5}\,(\,G\,(x,u,w) + G\,(y,v,z)\,) \\
&= \frac{4}{5}\,G_2'\,((x,y)\,,(u,v)\,,(w,z))\,.
\end{aligned}
$$

As the other hypothesis can be easily checked, Theorem 4.2.1 guarantees that $F$ has a unique fixed point, which is a coupled fixed point of $F$. In fact, the unique coupled fixed point of $F$ is $(0,0)$. However, here Theorem 11.7.2 is not applicable.

### *11.7.4 Choudhury and Maity's Coupled Fixed Point Results in G-Metric Spaces*

In Sect. 11.3.2, we showed why the technique used in the proof of main result in [58] was not suitable because the contractivity condition could not be applied to incomparable points. As a consequence, we gave a correct version of such a result (see Theorem 11.3.4). In this subsection we prove that this coupled result is a consequence of the following theorem.

**Theorem 11.7.3.** *Let $(X,G)$ be a complete G-metric space endowed with a partial order $\preccurlyeq$ and let $T : X \to X$ be a continuous, $\preccurlyeq$-non-decreasing self-mapping. Assume that there exists $\lambda \in [0,1)$ such that*

$$G(Tx,Tx,Ty) \leq \lambda\, G(x,x,y)$$

*for all $x,y \in X$ with $x \preccurlyeq y$ and $x \neq y$. If there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$, then $T$ has, at least, a fixed point.*

*Proof.* Let $\{x_n\}$ be the Picard sequence of $T$ based on $x_0$. As $x_0 \preccurlyeq Tx_0$ and $T$ is $\preccurlyeq$-non-decreasing, then $x_n \preccurlyeq x_{n+1}$ for all $n \in \mathbb{N}$. As $\preccurlyeq$ is a partial order, $x_n \preccurlyeq x_m$ for all $n, m \in \mathbb{N}$ with $n \leq m$. If there exist $n_0, m_0 \in \mathbb{N}$ with $n_0 < m_0$ such that $x_{n_0} = x_{m_0}$, then $x_{n_0} \preccurlyeq x_{n_0+1} \preccurlyeq \ldots \preccurlyeq x_{m_0} = x_{n_0}$, so $x_{n_0} = x_{n_0+1} = Tx_{n_0}$ and $x_{n_0}$ is a fixed point of $T$. In the contrary case, assume that $x_n \preccurlyeq x_m$ with $x_n \neq x_m$ for all $n, m \in \mathbb{N}$ with $n < m$. Then, the same argument in the proof of Theorem 5.2.1 concludes that $T$ has a fixed point. $\square$

As a consequence, we have the following result.

**Theorem 11.7.4.** *Theorem 11.3.4 immediately follows from Theorem 11.7.3.*

*Proof.* The same argument in the proof of Theorem 11.7.1 can be applied to $T_F^2$ in $(X^2, G_2', \sqsubseteq)$. We only describe the contractivity condition. Let $(x, y), (u, v) \in X^2$ be such that $(x, y) \sqsubseteq (u, v)$ and $(x, y) \neq (u, v)$. Then $x \preccurlyeq u$ and $y \succcurlyeq v$, but $x \neq u$ or $y \neq v$. By the contractivity condition (11.16),

$$G(F(x,y), F(x,y), F(u,v)) \leq \frac{\lambda}{2}[G(x,x,u) + G(y,y,v)].$$

Since (11.16) can also be applied when $u \succcurlyeq x$ and $v \preccurlyeq y$ (being $u \neq x$ or $v \neq y$), then

$$G(F(y,x), F(y,x), F(v,u)) \leq \frac{\lambda}{2}[G(y,y,v) + G(x,x,u)].$$

Hence,

$$\begin{aligned}
&G_2'(T_F^2(x,y), T_F^2(x,y), T_F^2(u,v)) \\
&\quad = G(F(x,y), F(x,y), F(u,v)) + G(F(y,x), F(y,x), F(v,y)) \\
&\quad \leq \lambda\, (G(x,x,u) + G(y,y,v)) \\
&\quad = \lambda\, G_2'((x,y), (x,y), (u,v)).
\end{aligned}$$

Theorem 11.7.3 implies that $T_F^2$ has a fixed point, which is a coupled fixed point of $F$. $\square$

### *11.7.5 Reducing Tripled Fixed Point Theorems*

The main aim of this subsection is to show how very recent tripled fixed (and coincidence) point results can be reduced to their corresponding unidimensional version. For example, in [31, Theorem 2.1], Aydi, Karapınar and Shatanawi[1] proved the following result.

[1]In their original paper, the authors omitted, by mistake, the completeness of the $G$-metric space.

**Theorem 11.7.5.** *Let $(X, \preccurlyeq)$ be a partially ordered set and let $(X, G)$ be a complete $G$-metric space. Let $F : X^3 \to X$ be a continuous mapping having the mixed $\preceq$-monotone property on $X$. Suppose that there exists $\phi \in \mathcal{F}_{\text{com}}$ such that for $x, y, z, a, b, c, u, v, w \in X$, with $x \succcurlyeq a \succcurlyeq u$, $y \preccurlyeq b \preccurlyeq v$ and $z \succcurlyeq c \succcurlyeq w$, one has*

$$G\left(F\left(x,y,z\right), F\left(a,b,c\right), F\left(u,v,w\right)\right) \\ \leq \phi\left(\max\left\{G\left(x,a,u\right), G\left(y,b,v\right), G\left(z,c,w\right)\right\}\right). \tag{11.75}$$

*If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \preccurlyeq F(x_0, y_0, z_0)$, $y_0 \succcurlyeq F(y_0, x_0, y_0)$ and $z_0 \preccurlyeq F(z_0, y_0, x_0)$, then $F$ has a tripled fixed point in $X$, that is, there exist $x, y, z \in X$ such that*

$$F(x,y,z) = x, \quad F(y,x,y) = y \quad \text{and} \quad F(z,y,x) = z.$$

This theorem is a particular version of Theorem 11.4.2, and we have already showed a direct proof. However, now we explain that this result is a direct consequence of the following result (which is a simple version of Theorem 5.3.6 in the context of partially ordered $G$-metric spaces).

**Theorem 11.7.6.** *Let $(X, G)$ be a complete $G^*$-metric space endowed with a preorder $\preceq$ and let $T : X \to X$ be a $\preceq$-non-decreasing self-mapping. Suppose that there exists $\phi \in \mathcal{F}_{\text{com}}$ such that, for all $x, y, z \in X$ with $x \succeq y \succeq z$,*

$$G(Tx, Ty, Tz) \leq \phi(G(x,y,z)). \tag{11.76}$$

*Also assume that, at least, one of the following conditions holds.*

- *$T$ is $G$-continuous, or*
- *$(X, G, \preceq)$ is regular.*

*If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then $T$ has, at least, a fixed point.*

The proof of the previous result is similar to the proof of Theorem 5.3.6 (when $g$ is the identity mapping on $X$), but it is useful to deduce the following consequence.

**Theorem 11.7.7.** *Theorem 11.7.5 immediately follows from Theorem 11.7.6.*

*Proof.* As $G$ is a $G$-metric on $X$, then $G_3$, defined by

$$G_3\left(\left(x_1,y_1,z_1\right), \left(x_2,y_2,z_2\right), \left(x_3,y_3,z_3\right)\right) \\ = \max\left\{G\left(x_1,x_2,x_3\right), G\left(y_1,y_2,y_3\right), G\left(z_1,z_2,z_3\right)\right\}$$

for all $\left(x_1,y_1,z_1\right), \left(x_2,y_2,z_2\right), \left(x_3,y_3,z_3\right) \in X^3$, is a $G^*$-metric on $X^3$ (see item 1 of Corollary 10.3.1). Consider on $X^3$ the binary relation $\sqsubseteq$ defined in (11.71) and the mappings $T_F^3, H_g^3 : X^3 \to X^3$ defined in (11.69)–(11.70). Then, we have the following properties.

- By item 4 of Lemma 10.3.1, $(X^3, G_3)$ is complete.
- By Lemma 11.7.3, $\sqsubseteq$ is a partial order on $X^3$.
- By Corollary 11.7.1, $T_F^3$ is a $\left(H_g^3, \sqsubseteq\right)$-non-decreasing mapping, where $g$ is the identity mapping on $X$.
- As there exists $x_0, y_0, z_0 \in X$ such that $x_0 \preccurlyeq F(x_0, y_0, z_0)$, $y_0 \succcurlyeq F(y_0, x_0, y_0)$ and $z_0 \preccurlyeq F(z_0, y_0, x_0)$, then $(x_0, y_0, z_0) \sqsubseteq T_F^3\,(x_0, y_0, z_0)$.
- By item 2 of Lemma 11.7.2, if $F$ is $G$-continuous, then $T_F^3 : X^3 \to X^3$ is also $G_3$-continuous.

Let us show that the contractivity condition (11.75) implies that, for all $(x, y, z)\,, (a, b, c), (u, v, w) \in X^3$ with $(x, y, z) \sqsupseteq (a, b, c) \sqsupseteq (u, v, w)$,

$$\begin{aligned} &G_3(T_F^3\,(x, y, z)\,, T_F^3\,(a, b, c)\,, T_F^3\,(u, v, w)) \\ &\qquad \leq \phi\,(G_3((x, y, z)\,, (a, b, c)\,, (u, v, w)))\,. \end{aligned}$$

Indeed, since $(x, y, z) \sqsupseteq (a, b, c) \sqsupseteq (u, v, w)$, then $x \succcurlyeq a \succcurlyeq u$, $y \preccurlyeq b \preccurlyeq v$ and $z \succcurlyeq c \succcurlyeq w$. On the one hand, (11.75) implies that

$$\begin{aligned} &G\,(F\,(x, y, z)\,, F\,(a, b, c)\,, F\,(u, v, w)) \\ &\qquad \leq \phi\,(\max\,\{G\,(x, a, u)\,, G\,(y, b, v)\,, G\,(z, c, w)\})\,. \end{aligned}$$

On the other hand, since $v \succcurlyeq b \succcurlyeq y$ and $u \preccurlyeq a \preccurlyeq x$, we deduce

$$\begin{aligned} &G\,(F\,(v, u, v)\,, F\,(b, a, b)\,, F\,(y, x, y)) \\ &\qquad \leq \phi\,(\max\,\{G\,(v, b, y)\,, G\,(u, a, x)\,, G\,(v, b, y)\}) \\ &\qquad = \phi\,(\max\,\{G\,(x, a, u)\,, G\,(y, b, v)\}) \\ &\qquad \leq \phi\,(\max\,\{G\,(x, a, u)\,, G\,(y, b, v)\,, G\,(z, c, w)\})\,. \end{aligned}$$

Finally, since $z \succcurlyeq c \succcurlyeq w$, $y \preccurlyeq b \preccurlyeq v$ and $x \succcurlyeq a \succcurlyeq u$, it follows that

$$\begin{aligned} &G\,(F\,(z, y, x)\,, F\,(c, b, a)\,, F\,(w, v, u)) \\ &\qquad \leq \phi\,(\max\,\{G\,(z, c, w)\,, G\,(y, b, v)\,, G\,(x, a, u)\}) \\ &\qquad = \phi\,(\max\,\{G\,(x, a, u)\,, G\,(y, b, v)\,, G\,(z, c, w)\})\,. \end{aligned}$$

Joining the last three inequalities, we conclude that

$$\begin{aligned} &G_3(T_F^3\,(x, y, z)\,, T_F^3\,(a, b, c)\,, T_F^3\,(u, v, w)) \\ &\qquad = G_3\Big((F(x, y, z), F(y, x, y), F(z, y, x)) \\ &\qquad\qquad\qquad (F(a, b, c), F(b, a, b), F(c, b, a)), \end{aligned}$$

$$
\begin{aligned}
&\left.(F(u,v,w),F(v,u,v),F(w,v,u))\right)\\
&= \max\left\{G\left(F(x,y,z),F(a,b,c),F(u,v,w)\right)\right.\\
&\qquad G(F(y,x,y),F(b,a,b),F(v,u,v)),\\
&\qquad \left.G(F(z,y,x),F(c,b,a),F(w,v,u))\right\}\\
&\leq \phi\left(\max\left\{G\left(x,a,u\right),G\left(y,b,v\right),G\left(z,c,w\right)\right\}\right).\\
&= \phi\left(G_3((x,y,z),(a,b,c),(u,v,w))\right).
\end{aligned}
$$

This property is the contractivity condition in Theorem 11.7.6. □

Also in [31, Theorems 2.1 and 2.4], the authors replaced the continuity of $F$ by the regularity of the partially ordered $G$-metric space, obtaining the following result (see [31, Theorems 2.1 and 2.4], Theorem 2.4).

**Theorem 11.7.8.** *Let $(X,\preccurlyeq)$ be a partially ordered set and let $(X,G)$ be a complete $G$-metric space such that $(X,G,\preceq)$ is regular. Let $F : X^3 \to X$ be a mapping having the mixed $\preceq$-monotone property on $X$. Suppose that there exists $\phi \in \mathcal{F}_{\text{com}}$ such that for $x,y,z,a,b,c,u,v,w \in X$, with $x \succeq a \succeq u$, $y \preceq b \preceq v$, and $z \succeq c \succeq w$, one has*

$$
\begin{aligned}
&G\left(F\left(x,y,z\right),F\left(a,b,c\right),F\left(u,v,w\right)\right)\\
&\qquad \leq \phi\left(\max\left\{G\left(x,a,u\right),G\left(y,b,v\right),G\left(z,c,w\right)\right\}\right).
\end{aligned}
$$

*If there exist $x_0,y_0,z_0 \in X$ such that $x_0 \preccurlyeq F(x_0,y_0,z_0)$, $y_0 \succcurlyeq F(y_0,x_0,y_0)$ and $z_0 \preccurlyeq F(z_0,y_0,x_0)$, then $F$ has a tripled fixed point in $X$, that is, there exist $x,y,z \in X$ such that*

$$F(x,y,z) = x, \quad F(y,x,y) = y \quad \text{and} \quad F(z,y,x) = z.$$

**Theorem 11.7.9.** *Theorem 11.7.5 immediately follows from Theorem 11.7.6.*

*Proof.* We follow the proof in Theorem 11.7.7, replacing the continuity of $F$ by the fact that if $(X,G,\preccurlyeq)$ is regular, then $(X^3,G_3,\sqsubseteq)$ is also regular (see Lemma 11.7.5). □

In [32, Theorem 15], the authors proved the following result (notice that they assumed that the function $\phi \in \mathcal{F}'_{\text{alt}}$ was non-decreasing).

**Theorem 11.7.10.** *Let $(X,\preccurlyeq)$ be a partially ordered set and let $(X,G)$ be a complete $G$-metric space. Let $F : X^3 \to X$ and $g : X \to X$ be two mappings. Suppose that there exists $\psi \in \mathcal{F}_{\text{alt}}$ and $\phi \in \mathcal{F}'_{\text{alt}}$ such that $\phi$ is non-decreasing and, for $x,y,z,a,b,c,u,v,w \in X$ with $gx \succcurlyeq ga \succcurlyeq gu$, $gy \preccurlyeq gb \preccurlyeq gv$ and $gz \succcurlyeq gc \succcurlyeq gw$, we have*

$$\begin{aligned}
&\psi\left(G\left(F\left(x,y,z\right),F\left(a,b,c\right),F\left(u,v,w\right)\right)\right)\\
&\quad\leq\psi\left(\max\left\{G\left(gx,ga,gu\right),G\left(gy,gb,gv\right),G\left(gz,gc,gw\right)\right\}\right)\\
&\qquad\qquad-\phi\left(\max\left\{G\left(gx,ga,gu\right),G\left(gy,gb,gv\right),G\left(gz,gc,gw\right)\right\}\right).
\end{aligned}\tag{11.77}$$

*Assume that $F$ and $g$ satisfy the following conditions:*

**(1)** $F(X^3)\subseteq g(X)$,
**(2)** *$F$ has the mixed $(g,\preceq)$-monotone property on $X$,*
**(3)** *$F$ is $G$-continuous,*
**(4)** *$g$ is continuous and commutes with $F$.*

*Suppose that there exist $x_0,y_0,z_0\in X$ such that $gx_0\preccurlyeq F(x_0,y_0,z_0)$, $gy_0\succcurlyeq F(y_0,x_0,y_0)$ and $gz_0\preccurlyeq F(z_0,y_0,x_0)$. Then $F$ and $g$ have a tripled coincidence point in $X$, i.e., there exist $x,y,z\in X$ such that*

$$F(x,y,z)=x,\quad F(y,x,y)=y\quad\text{and}\quad F(z,y,x)=z.$$

It seems that Theorem 11.7.10 might follow from Theorem 5.3.1. In fact, we can repeat the scheme of the proof in Theorem 11.7.7 using $G_3$, the binary relation $\sqsubseteq$ and the mappings $T_F^3,H_g^3:X^3\to X^3$. However, the contractivity condition (11.77) does not coincide with the following one:

$$\begin{aligned}
&\psi\left(G_3(T_F^3\left(x,y,z\right),T_F^3\left(a,b,c\right),T_F^3\left(u,v,w\right))\right)\\
&\quad\leq(\psi-\phi)\left(G_3(H_g^3\left(x,y,z\right),H_g^3\left(a,b,c\right),H_g^3\left(u,v,w\right))\right),
\end{aligned}$$

for all $(x,y,z),(a,b,c),(u,v,w)\in X^3$ with $H_g^3\left(x,y,z\right)\sqsupseteq H_g^3(a,b,c)\sqsupseteq H_g^3\left(u,v,w\right)$. The reason is hidden in the second Berinde's equation: since $H_g^3\left(x,y,z\right)\sqsupseteq H_g^3(a,b,c)\sqsupseteq H_g^3\left(u,v,w\right)$, then $gx\succcurlyeq ga\succcurlyeq gu$, $gy\preccurlyeq gb\preccurlyeq gv$ and $gz\succcurlyeq gc\succcurlyeq gw$. Then

$$\begin{aligned}
&\psi\left(G\left(F\left(x,y,z\right),F\left(a,b,c\right),F\left(u,v,w\right)\right)\right)\\
&\quad\leq\psi\left(\max\left\{G\left(gx,ga,gu\right),G\left(gy,gb,gv\right),G\left(gz,gc,gw\right)\right\}\right)\\
&\qquad\qquad-\phi\left(\max\left\{G\left(gx,ga,gu\right),G\left(gy,gb,gv\right),G\left(gz,gc,gw\right)\right\}\right)\\
&\quad=(\psi-\phi)\left(G_3(H_g^3\left(x,y,z\right),H_g^3\left(a,b,c\right),H_g^3\left(u,v,w\right))\right)
\end{aligned}$$

and

$$\begin{aligned}
&\psi\left(G\left(F\left(z,y,x\right),F\left(c,b,a\right),F\left(w,v,u\right)\right)\right)\\
&\quad\leq\psi\left(\max\left\{G\left(gz,gc,gw\right),G\left(gy,gb,gv\right),G\left(gx,ga,gu\right)\right\}\right)\\
&\qquad\qquad-\phi\left(\max\left\{G\left(gz,gc,gw\right),G\left(gy,gb,gv\right),G\left(gx,ga,gu\right)\right\}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \psi \left(\max \left\{G\left(gx, ga, gu\right), G\left(gy, gb, gv\right), G\left(gz, gc, gw\right)\right\}\right) \\
&\qquad - \phi \left(\max \left\{G\left(gx, ga, gu\right), G\left(gy, gb, gv\right), G\left(gz, gc, gw\right)\right\}\right) \\
&= \left(\psi - \phi\right) \left(G_3(H_g^3\left(x, y, z\right), H_g^3\left(a, b, c\right), H_g^3\left(u, v, w\right))\right).
\end{aligned}
$$

However, from $gv \succcurlyeq gb \succcurlyeq gy$ and $gu \preccurlyeq ga \preccurlyeq gx$, we deduce that

$$
\begin{aligned}
&\psi \left(G\left(F\left(v, u, v\right), F\left(b, a, b\right), F\left(y, x, y\right)\right)\right) \\
&\quad \leq \psi \left(\max \left\{G\left(gv, gb, gy\right), G\left(gu, ga, gx\right), G\left(gv, gb, gy\right)\right\}\right) \\
&\qquad - \phi \left(\max \left\{G\left(gv, gb, gy\right), G\left(gu, ga, gx\right), G\left(gv, gb, gy\right)\right\}\right) \\
&\quad = \psi \left(\max \left\{G\left(gx, ga, gu\right), G\left(gy, gb, gv\right)\right\}\right) \\
&\qquad - \phi \left(\max \left\{G\left(gx, ga, gu\right), G\left(gy, gb, gv\right)\right\}\right).
\end{aligned}
$$

The term $\phi \left(\max \left\{G\left(gx, ga, gu\right), G\left(gy, gb, gv\right)\right\}\right)$ is not strong enough to obtain $\left(\psi - \phi\right) \left(G_3(H_g^3\left(x, y, z\right), H_g^3\left(a, b, c\right), H_g^3\left(u, v, w\right))\right)$. As a consequence, this type of result would need a direct proof using Lemmas 2.3.7 and 2.3.8.

### *11.7.6 Reducing Quadrupled Fixed Point Theorems*

In this section, we show how to reduce to the unidimensional case a version Theorem 2.1 given in [145] by Mustafa. In the original theorem, given a $G$-metric space $(X, G)$ and two mappings $F : X^4 \to X$ and $g : X \to X$, the author studied the contractivity condition

$$
\begin{aligned}
&\psi \left(G\left(F\left(x, y, z, w\right), F\left(u, v, s, t\right), F\left(a, b, c, d\right)\right)\right) \\
&\quad \leq \frac{1}{4} \psi \Big(G\left(gx, gu, ga\right) + G\left(gy, gv, gb\right) \\
&\qquad\qquad + G\left(gz, gs, gc\right) + G\left(gw, gt, gd\right)\Big) \\
&\qquad - \phi \left(\frac{1}{4} \Big(G\left(gx, gu, ga\right) + G\left(gy, gv, gb\right)\right. \\
&\qquad\qquad \left. + G\left(gz, gs, gc\right) + G\left(gw, gt, gd\right)\Big)\right)
\end{aligned} \tag{11.78}
$$

for all $x, y, z, w, u, v, s, t, a, b, c, d \in X$ with $gx \succcurlyeq gu \succcurlyeq ga$, $gy \preccurlyeq gv \preccurlyeq gb$, $gz \succcurlyeq gs \succcurlyeq gc$ and $gw \preccurlyeq gt \preccurlyeq gd$. The control functions $\psi$ and $\phi$ belong to

$$
\begin{aligned}
\mathcal{F}_{\text{subalt}} &= \{\, \psi \in \mathcal{F}_{\text{alt}} : \psi \text{ is subadditive} \,\} \quad \text{and} \\
\mathcal{F}'_{\text{subalt}} &= \{\, \phi : [0, \infty) \to [0, \infty) : \phi \text{ verifies } (\mathcal{P}_{13}) + (\mathcal{P}_{15}) \,\},
\end{aligned}
$$

respectively. However, functions belonging to $\mathcal{F}'_{\text{subalt}}$ can take the value zero at infinitely many points (see Remark 2.3.5), which is a drawback to prove a fixed point theorem. Hence, in the following result, we shall employ $\phi \in \mathcal{F}''_{\text{alt}}$.

**Theorem 11.7.11.** *Let $(X, \preccurlyeq)$ be a partially ordered set and let $(X, G)$ be a G-metric space. Let $F : X^4 \to X$ and $g : X \to X$ be such that $F(X^4) \subseteq g(X)$, F has the mixed $(g, \preccurlyeq)$-monotone property and g is continuous and commutes with F. Assume that there exist $\psi \in \mathcal{F}_{\text{alt}}$ and $\phi \in \mathcal{F}''_{\text{alt}}$ such that $\psi$ is subadditive and satisfying inequality (11.78) for all $x, y, z, w, u, v, s, t, a, b, c, d \in X$ with $gx \succcurlyeq gu \succcurlyeq ga$, $gy \preccurlyeq gv \preccurlyeq gb$, $gz \succcurlyeq gs \succcurlyeq gc$ and $gw \preccurlyeq gt \preccurlyeq gd$. Also assume that, at least, one of the following conditions holds:*

**(a)** *$(X, G)$ is complete and F is continuous,* or
**(b)** *$(g(X), G)$ is complete and $(X, G, \preccurlyeq)$ is regular.*

*If there exist $x_0, y_0, z_0, w_0 \in X$ such that*

$$gx_0 \preccurlyeq F(x_0, y_0, z_0, w_0), \quad gy_0 \succcurlyeq F(y_0, z_0, w_0, x_0),$$
$$gz_0 \preccurlyeq F(z_0, w_0, x_0, y_0) \quad \text{and} \quad gw_0 \succcurlyeq F(w_0, x_0, y_0, z_0),$$

*then F and g have, at least, a quadrupled coincidence point.*

**Theorem 11.7.12.** *Theorem 11.7.11 immediately follows from Theorem 5.3.3.*

*Proof.* We follow the argument in the proof of Theorem 11.7.1 but using four variables. As $G$ is a $G$-metric on $X$, then $G'_4$, defined by

$$G'_2\left((x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2), (x_3, y_3, z_3, w_3)\right)$$
$$= G(x_1, x_2, x_3) + G(y_1, y_2, y_3) + G(z_1, z_2, z_3) + G(w_1, w_2, w_3)$$

for all $(x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2), (x_3, y_3, z_3, w_3) \in X^4$, is a $G^*$-metric on $X^4$ (see item 1 of Corollary 10.3.1). Consider on $X^4$ the binary relation $\sqsubseteq$ defined in (11.71). Then, we have the following properties.

- By item 4 of Lemma 10.3.1, $\left(X^4, G'_4\right)$ is complete.
- By Lemma 11.7.3, $\sqsubseteq$ is a preorder on $X^4$.
- By Corollary 11.7.1, $T_F^4$ is a $\left(H_g^4, \sqsubseteq\right)$-non-decreasing mapping.
- By Proposition 11.7.1, as $F$ and $g$ commute, then $T_F^4$ and $H_g^4$ also commute.
- As there exists $x_0, y_0, z_0, w_0 \in X$ such that $gx_0 \preccurlyeq F(x_0, y_0, z_0, w_0)$, $gy_0 \succcurlyeq F(y_0, z_0, w_0, x_0)$, $gz_0 \preccurlyeq F(z_0, w_0, x_0, y_0)$ and $gw_0 \succcurlyeq F(w_0, x_0, y_0, z_0)$, then $H_g^4(x_0, y_0, z_0, w_0) \sqsubseteq T_F^4(x_0, y_0, z_0, w_0)$.
- By item 2 of Lemma 11.7.2, if $F$ is $G$-continuous, then $T_F^4 : X^4 \to X^4$ is also $G'_4$-continuous.
- By Lemma 11.7.5, if $(X, G, \preceq)$ is regular, then $\left(X^4, G'_4, \sqsubseteq\right)$ is also regular.
- As $\phi \in \mathcal{F}''_{\text{alt}}$, then $\phi'$, defined by $\phi'(t) = 4\,\phi(t/4)$ for all $t \in [0, \infty)$, also belongs to $\mathcal{F}''_{\text{alt}}$.

Let us show that the contractivity condition (11.78) implies that, for all $(x,y,z,w),(u,v,s,t),(a,b,c,d)\in X^4$ with $(x,y,z,w)\sqsupseteq(u,v,s,t)\sqsupseteq(a,b,c,d)$,

$$\begin{aligned}&\psi\left(G_4'(T_F^4(x,y,z,w),T_F^4(u,v,s,t),T_F^4(a,b,c,d))\right)\\&\qquad\leq(\psi-\phi)\left(G_4'(H_g^4(x,y,z,w),H_g^4(u,v,s,t),H_g^4(a,b,c,d))\right).\end{aligned}$$

To prove it, we notice that by (11.78), $gx\succcurlyeq gu\succcurlyeq ga$, $gy\preccurlyeq gv\preccurlyeq gb$, $gz\succcurlyeq gs\succcurlyeq gc$ and $gw\preccurlyeq gt\preccurlyeq gd$ imply that

$$\begin{aligned}&\psi\left(G\left(F(x,y,z,w),F(u,v,s,t),F(a,b,c,d)\right)\right)\\&\leq\frac{1}{4}\,\psi\Big(G(gx,gu,ga)+G(gy,gv,gb)\\&\qquad+G(gz,gs,gc)+G(gw,gt,gd)\Big)\\&\quad-\phi\left(\frac{1}{4}\Big(G(gx,gu,ga)+G(gy,gv,gb)\right.\\&\qquad\left.+G(gz,gs,gc)+G(gw,gt,gd)\Big)\right)\\&=\frac{1}{4}\left(\psi-\phi'\right)\Big(G(gx,gu,ga)+G(gy,gv,gb)\\&\qquad+G(gz,gs,gc)+G(gw,gt,gd)\Big)\end{aligned}$$

and

$$\begin{aligned}&\psi\left(G\left(F(z,w,x,y),F(s,t,u,v),F(c,d,a,b)\right)\right)\\&\leq\frac{1}{4}\,\psi\Big(G(gz,gs,gc)+G(gw,gt,gd)\\&\qquad+G(gx,gu,ga)+G(gy,gv,gb)\Big)\\&\quad-\phi\left(\frac{1}{4}\Big(G(gz,gs,gc)+G(gw,gt,gd)\right.\\&\qquad\left.+G(gx,gu,ga)+G(gy,gv,gb)\Big)\right)\\&=\frac{1}{4}\left(\psi-\phi'\right)\Big(G(gx,gu,ga)+G(gy,gv,gb)\\&\qquad+G(gz,gs,gc)+G(gw,gt,gd)\Big).\end{aligned}$$

Similarly, using that $gb \succcurlyeq gv \succcurlyeq gy$, $gc \preccurlyeq gs \preccurlyeq gz$, $gd \succcurlyeq gt \succcurlyeq gw$ and $ga \preccurlyeq gu \preccurlyeq gx$, it follows that

$$\begin{aligned}
&\psi\left(G\left(F\left(b,c,d,a\right),F\left(v,s,t,u\right),F\left(y,z,w,x\right)\right)\right)\\
&\quad\leq \frac{1}{4}\,\psi\,\Big(G\left(gb,gv,gy\right)+G\left(gc,gs,gz\right)\\
&\qquad\qquad\qquad+G\left(gd,gt,gw\right)+G\left(ga,gu,gx\right)\Big)\\
&\qquad-\phi\left(\frac{1}{4}\Big(G\left(gb,gv,gy\right)+G\left(gc,gs,gz\right)\right.\\
&\qquad\qquad\qquad\left.+G\left(gd,gt,gw\right)+G\left(ga,gu,gx\right)\Big)\right)\\
&\quad=\frac{1}{4}\,\left(\psi-\phi'\right)\Big(G\left(gx,gu,ga\right)+G\left(gy,gv,gb\right)\\
&\qquad\qquad+G\left(gz,gs,gc\right)+G\left(gw,gt,gd\right)\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\psi\left(G\left(F\left(d,a,b,c\right),F\left(t,u,v,s\right),F\left(w,x,y,z\right)\right)\right)\\
&\quad\leq \frac{1}{4}\,\psi\,\Big(G\left(gd,gt,gw\right)+G\left(ga,gu,gx\right)\\
&\qquad\qquad\qquad+G\left(gb,gv,gy\right)+G\left(gc,gs,gz\right)\Big)\\
&\qquad-\phi\left(\frac{1}{4}\Big(G\left(gd,gt,gw\right)+G\left(ga,gu,gx\right)\right.\\
&\qquad\qquad\qquad\left.+G\left(gb,gv,gy\right)+G\left(gc,gs,gz\right)\Big)\right)\\
&\quad=\frac{1}{4}\,\left(\psi-\phi'\right)\Big(G\left(gx,gu,ga\right)+G\left(gy,gv,gb\right)\\
&\qquad\qquad+G\left(gz,gs,gc\right)+G\left(gw,gt,gd\right)\Big).
\end{aligned}$$

As $\psi$ is non-decreasing and subadditive, combining the last four inequalities, we deduce that

$$\begin{aligned}
&\psi\left(G_4'(T_F^4\left(x,y,z,w\right),T_F^4\left(u,v,s,t\right),T_F^4\left(a,b,c,d\right))\right)\\
&\quad=\psi\,\Big(G\left(F\left(x,y,z,w\right),F\left(u,v,s,t\right),F\left(a,b,c,d\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&\quad + G\left(F\left(y,z,w,x\right), F\left(v,s,t,u\right), F\left(b,c,d,a\right)\right) \\
&\quad + G\left(F\left(z,w,x,y\right), F\left(s,t,u,v\right), F\left(c,d,a,b\right)\right) \\
&\qquad + G\left(F\left(w,x,y,z\right), F\left(t,u,v,s\right), F\left(d,a,b,c\right)\right)\Big) \\
&\leq \psi\left(G\left(F\left(x,y,z,w\right), F\left(u,v,s,t\right), F\left(a,b,c,d\right)\right)\right) \\
&\quad + \psi\left(G\left(F\left(y,z,w,x\right), F\left(v,s,t,u\right), F\left(b,c,d,a\right)\right)\right) \\
&\quad + \psi\left(G\left(F\left(z,w,x,y\right), F\left(s,t,u,v\right), F\left(c,d,a,b\right)\right)\right) \\
&\quad + \psi\left(G\left(F\left(w,x,y,z\right), F\left(t,u,v,s\right), F\left(d,a,b,c\right)\right)\right) \\
&\leq 4\,\frac{1}{4}\,\left(\psi - \phi'\right)\Big(G\left(gx,gu,ga\right) + G\left(gy,gv,gb\right) \\
&\qquad\qquad + G\left(gz,gs,gc\right) + G\left(gw,gt,gd\right)\Big) \\
&\leq \left(\psi - \phi'\right)\left(G_4'(H_g^4\left(x,y,z,w\right), H_g^4\left(u,v,s,t\right), H_g^4\left(a,b,c,d\right))\right).
\end{aligned}
$$

This property is the contractivity condition in Theorem 5.3.3.

As a consequence, all the hypotheses of Theorem 5.3.3, applied to $T_F^4$ and $H_g^4$ on $\left(X^4, G_4', \sqsubseteq\right)$ hold. Then $T_F^4$ and $H_g^4$ have, at least, a coincidence point, which is a quadrupled coincidence point of $F$ and $g$ by item 3 of Lemma 11.7.1. □

We note that the existence and uniqueness of a common quadrupled coincidence point of $F$ and $g$ can be derived using the following additional condition (which is equivalent to that given in Theorem 5.3.2):

($U$) for all quadrupled coincidence points $(x,y,z,w)$ and $(u,v,s,t)$ of $F$ and $g$, there exists $(a,b,c,d) \in X^4$ such that $H_g^4\,(a,b,c,d)$ is $\sqsubseteq$-comparable, at the same time, to $H_g^4\,(x,y,z,w)$ and to $H_g^4\,(u,v,s,t)$.

## 11.8 Multidimensional $\Phi$-Fixed Point Results in Partially Preordered $G^*$-Metric Spaces

The technique we have shown in the previous section can be applied in the multidimensional case in order to obtain new fixed point theorems. This is the main aim of the present section. To do that, we introduce the following notation. Recall that $\{\mathsf{A},\mathsf{B}\}$ represents a partition of $\Lambda_n = \{1,2,\ldots,n\}$ and let $\Phi = (\sigma_1,\sigma_2,\ldots,\sigma_n)$ be a $n$-tuple of mappings from $\{1,2,\ldots,n\}$ into itself.

To start with, we extend a binary relation $\preccurlyeq$ on $X$ to a binary relation $\sqsubseteq$ on the product space $X^n$ as follows: for $\mathsf{X} = (x_1,x_2,\ldots,x_n), \mathsf{Y} = (y_1,y_2,\ldots,y_n) \in X^n$,

$$
\mathsf{X} \sqsubseteq \mathsf{Y} \quad \Leftrightarrow \quad x_i \preccurlyeq_i y_i, \text{ for all } i \in \Lambda_n, \tag{11.79}
$$

where $\preccurlyeq_i$ was defined in (11.54). Notice that $\sqsubseteq$ depends on $\mathsf{A}$ and $\mathsf{B}$. We say that two points $\mathsf{X}$ and $\mathsf{Y}$ are $\sqsubseteq$-*comparable* if $\mathsf{X} \sqsubseteq \mathsf{Y}$ or $\mathsf{X} \sqsupseteq \mathsf{Y}$.

**Lemma 11.8.1.** *The binary relation $\preccurlyeq$ is reflexive (respectively, transitive, a preorder, antisymmetric, a partial order) on $X$ if, and only if, $\sqsubseteq$ is reflexive (respectively, transitive, a preorder, antisymmetric, a partial order) on $X^n$.*

*Proof. (Transitivity)* Assume that $\preccurlyeq$ is transitive and let

$$\mathsf{X}\,(x_1, x_2, \ldots, x_n)\,, \mathsf{Y}\,(y_1, y_2, \ldots, y_n)\,, \mathsf{Z}\,(z_1, z_2, \ldots, z_n) \in X^n$$

be such that $\mathsf{X} \sqsubseteq \mathsf{Y} \sqsubseteq \mathsf{Z}$. Therefore, $x_i \preccurlyeq_i y_i \preccurlyeq_i z_i$ for all $i \in \Lambda_n$. If $i \in \mathsf{A}$, then $x_i \preccurlyeq y_i \preccurlyeq z_i$, so $x_i \preccurlyeq z_i$ because $\preccurlyeq$ is transitive. Then, $x_i \preccurlyeq_i z_i$. For the other case, if $i \in \mathsf{B}$, then $x_i \succcurlyeq y_i \succcurlyeq z_i$, so $x_i \succcurlyeq z_i$ because $\preccurlyeq$ is transitive. Then, $x_i \preccurlyeq_i z_i$. In any case, $x_i \preccurlyeq_i z_i$ for all $i \in \Lambda_n$, so $\mathsf{X} \sqsubseteq \mathsf{Z}$.

Conversely, assume that $\sqsubseteq$ is transitive and let $x, y, z \in X^n$ be such that $x \preccurlyeq y \preccurlyeq z$. Define $\mathsf{X}\,(x_1, x_2, \ldots, x_n)\,, \mathsf{Y}\,(y_1, y_2, \ldots, y_n)\,, \mathsf{Z}\,(z_1, z_2, \ldots, z_n) \in X^n$ by:

$$x_i = \begin{cases} \mathrm{x}, \text{ if } i \in \mathsf{A}, \\ \mathrm{z}, \text{ if } i \in \mathsf{B}; \end{cases} \qquad y_i = y; \quad z_i = \begin{cases} z, \text{ if } i \in \mathsf{A}, \\ x, \text{ if } i \in \mathsf{B}. \end{cases} \tag{11.80}$$

Therefore, $\mathsf{X} \sqsubseteq \mathsf{Y} \sqsubseteq \mathsf{Z}$. As $\sqsubseteq$ is transitive, then $\mathsf{X} \sqsubseteq \mathsf{Z}$. In particular, $x \preccurlyeq z$ and $\preccurlyeq$ is transitive.

*(Antisymmetry)* Assume that $\preccurlyeq$ is antisymmetric and let

$$\mathsf{X}\,(x_1, x_2, \ldots, x_n)\,, \mathsf{Y}\,(y_1, y_2, \ldots, y_n) \in X^n$$

be such that $\mathsf{X} \sqsubseteq \mathsf{Y} \sqsubseteq \mathsf{X}$. Therefore, $x_i \preccurlyeq_i y_i \preccurlyeq_i x_i$ for all $i \in \Lambda_n$. If $i \in \mathsf{A}$, then $x_i \preccurlyeq y_i \preccurlyeq x_i$, so $x_i = y_i$ because $\preccurlyeq$ is antisymmetric. Now if $i \in \mathsf{B}$, then $x_i \succcurlyeq y_i \succcurlyeq x_i$, so again $x_i = y_i$. In any case, $x_i = y_i$ for all $i \in \Lambda_n$. Then $\mathsf{X} = \mathsf{Y}$ and $\sqsubseteq$ is antisymmetric.

Conversely, assume that $\sqsubseteq$ is antisymmetric and let $x, z \in X$ be such that $x \preccurlyeq z \preccurlyeq x$. Define $\mathsf{X}\,(x_1, x_2, \ldots, x_n)\,, \mathsf{Z}\,(z_1, z_2, \ldots, z_n) \in X^n$ as in (11.80). Then $\mathsf{X} \sqsubseteq \mathsf{Y} \sqsubseteq \mathsf{X}$. As $\sqsubseteq$ is antisymmetric, then $\mathsf{X} = \mathsf{Z}$, so $x = z$ and $\preccurlyeq$ is antisymmetric. □

**Proposition 11.8.1.** *If $\mathsf{X} \sqsubseteq \mathsf{Y}$ and $\sigma \in \Omega_{\mathsf{A},\mathsf{B}} \cup \Omega'_{\mathsf{A},\mathsf{B}}$, then*

$$(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) \sqsubseteq (y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)}) \quad \text{if } \sigma \in \Omega_{\mathsf{A},\mathsf{B}},$$
$$(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) \sqsupseteq (y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)}) \quad \text{if } \sigma \in \Omega'_{\mathsf{A},\mathsf{B}}.$$

*In particular, $(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ and $(y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)})$ are $\sqsubseteq$-comparable.*

*Proof.* Suppose that $x_i \preccurlyeq_i y_i$ for all $i$. Hence $x_{\sigma(i)} \preccurlyeq_{\sigma(i)} y_{\sigma(i)}$ for all $i$. Fix $\sigma \in \Omega_{\mathsf{A},\mathsf{B}}$. If $i \in \mathsf{A}$, then $\sigma(i) \in \mathsf{A}$, so $x_{\sigma(i)} \preccurlyeq_{\sigma(i)} y_{\sigma(i)}$ implies that $x_{\sigma(i)} \preccurlyeq y_{\sigma(i)}$, which means that $x_{\sigma(i)} \preccurlyeq_i y_{\sigma(i)}$. If $i \in \mathsf{B}$, then $\sigma(i) \in \mathsf{B}$, so $x_{\sigma(i)} \preccurlyeq_{\sigma(i)} y_{\sigma(i)}$ implies that $x_{\sigma(i)} \succcurlyeq y_{\sigma(i)}$, which means that $x_{\sigma(i)} \preccurlyeq_i y_{\sigma(i)}$. In any case, if $\sigma \in \Omega_{\mathsf{A},\mathsf{B}}$, then $x_{\sigma(i)} \preccurlyeq_i y_{\sigma(i)}$ for all $i$.

It follows that $(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) \sqsubseteq (y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)})$. Now fix $\sigma \in \Omega'_{\mathsf{A},\mathsf{B}}$. If $i \in \mathsf{A}$, then $\sigma(i) \in \mathsf{B}$, so $x_{\sigma(i)} \preccurlyeq_{\sigma(i)} y_{\sigma(i)}$ implies that $x_{\sigma(i)} \succcurlyeq y_{\sigma(i)}$, which means that $x_{\sigma(i)} \succcurlyeq_i y_{\sigma(i)}$. If $i \in \mathsf{B}$, then $\sigma(i) \in \mathsf{A}$, so $x_{\sigma(i)} \preccurlyeq_{\sigma(i)} y_{\sigma(i)}$ implies that $x_{\sigma(i)} \preccurlyeq y_{\sigma(i)}$, which means that $x_{\sigma(i)} \succcurlyeq_i y_{\sigma(i)}$. $\square$

Given a mapping $F : X^n \to X$ and a $n$-tupled $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, we define $\mathbb{F}_\Phi : X^n \to X^n$ by

$$\mathbb{F}_\Phi(x_1, x_2, \ldots, x_n) = \Big(F(x_{\sigma_1(1)}, x_{\sigma_1(2)}, \ldots, x_{\sigma_1(n)}), \\ F(x_{\sigma_2(1)}, x_{\sigma_2(2)}, \ldots, x_{\sigma_2(n)}), \ldots, F(x_{\sigma_n(1)}, x_{\sigma_n(2)}, \ldots, x_{\sigma_n(n)})\Big)$$

and $F^2_\Phi = F \circ \mathbb{F}_\Phi : X^n \to X$ will be

$$F^2_\Phi(x_1, x_2, \ldots, x_n) = F\Big(F(x_{\sigma_1(1)}, x_{\sigma_1(2)}, \ldots, x_{\sigma_1(n)}), \\ F(x_{\sigma_2(1)}, x_{\sigma_2(2)}, \ldots, x_{\sigma_2(n)}), \ldots, F(x_{\sigma_n(1)}, x_{\sigma_n(2)}, \ldots, x_{\sigma_n(n)})\Big)$$

for all $\mathsf{X} = (x_1, x_2, \ldots, x_n) \in X^n$. Notice that $\mathbb{F}_\Phi$ depends on $n$, but we avoid the notation $\mathbb{F}^n_\Phi$ because $n$ is implicitly considered in $\Phi$ and because we will use the composition $\mathbb{F}^2_\Phi = \mathbb{F}_\Phi \circ \mathbb{F}_\Phi$.

Furthermore, given a mapping $g : X \to X$, we define $H^n_g : X^n \to X^n$, for all $(x_1, x_2, \ldots, x_n) \in X^n$, by

$$H^n_g(x_1, x_2, \ldots, x_n) = (gx_1, gx_2, \ldots, gx_n).$$

Using this notation, the following lemma is immediate.

**Lemma 11.8.2.** *Let $X$ be a non-empty set, let $\mathsf{Z} \in X^n$ be a point, let $F : X^n \to X$ and $g : X \to X$ be two mappings and let $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be a $n$-tuple of mappings from $\{1, 2, \ldots, n\}$ into itself.*

1. *$\mathsf{Z}$ is a $\Phi$-fixed point of $F$ if, and only if, $\mathsf{Z}$ is a fixed point of $\mathbb{F}_\Phi$ (that is, $\mathbb{F}_\Phi \mathsf{Z} = \mathsf{Z}$).*
2. *$\mathsf{Z}$ is a $\Phi$-coincidence point of $F$ and $g$ if, and only if, $\mathsf{Z}$ is a coincidence point of $\mathbb{F}_\Phi$ and $H^n_g$ (that is, $\mathbb{F}_\Phi \mathsf{Z} = H^n_g \mathsf{Z}$).*
3. *$\mathsf{Z} \in X^n$ is a common $\Phi$-fixed point of $F$ and $g$ if, and only if, $\mathsf{Z}$ is a common fixed point of $\mathbb{F}_\Phi$ and $H^n_g$ (that is, $\mathbb{F}_\Phi \mathsf{Z} = H^n_g \mathsf{Z} = \mathsf{Z}$).*

*Proof.* We only show the last property. We define that $\mathsf{Z} = (z_1, z_2, \ldots, z_n) \in X^n$ is a common $\Phi$-fixed point of $F$ and $g$ if

$$F(x_{\sigma_1(1)}, x_{\sigma_1(2)}, \ldots, x_{\sigma_1(n)}) = gx_i = x_i \quad \text{for all } i \in \{1, 2, \ldots, n\}.$$

This is equivalent to saying that

$$\Big(F(z_{\sigma_1(1)}, z_{\sigma_1(2)}, \ldots, z_{\sigma_1(n)}), F(z_{\sigma_2(1)}, z_{\sigma_2(2)}, \ldots, z_{\sigma_2(n)}), \ldots,$$
$$F(z_{\sigma_n(1)}, z_{\sigma_n(2)}, \ldots, z_{\sigma_n(n)})\Big) = (gz_1, gz_2, \ldots, gz_n)$$
$$= (z_1, z_2, \ldots, z_n) .$$

Using the previous notation, this is equivalent to saying that $\mathbb{F}_\Phi \mathsf{Z} = H_g^n \mathsf{Z} = \mathsf{Z}$, that is, $\mathsf{Z}$ is a common fixed point of $\mathbb{F}_\Phi$ and $H_g^n$. □

Next, we prove that the mixed $(g, \preccurlyeq)$-monotone property implies certain type of non-decreasingness.

**Lemma 11.8.3.** *Let $F : X^n \to X$ and $g : X \to X$ be two mappings, let $\preccurlyeq$ be a binary relation on $X$ and let $\sqsubseteq$ the binary relation on $X^n$ defined in (11.79) (depending on the partition $\{\mathsf{A}, \mathsf{B}\}$).*

1. *If the mapping $F$ has the mixed $(g, \preccurlyeq)$-monotone property, then $\mathbb{F}_\Phi$ is $\left(H_g^n, \sqsubseteq\right)$-non-decreasing.*
2. *If the mapping $F$ has the mixed $\preccurlyeq$-monotone property, then $\mathbb{F}_\Phi$ is $\sqsubseteq$-non-decreasing.*

*Proof.* Assume that $F$ has the mixed $(g, \preccurlyeq)$-monotone property and let $\mathsf{X}\,(x_1, x_2, \ldots, x_n)\,, \mathsf{Y}\,(y_1, y_2, \ldots, y_n) \in X^n$ be two points such that $H_g^n(\mathsf{X}) \sqsubseteq H_g^n(\mathsf{Y})$. Therefore, $gx_i \preccurlyeq_i gy_i$ for all $i \in \Lambda_n$. This means that

$$\begin{cases} gx_i \preccurlyeq gy_i, \text{ if } i \in \mathsf{A}, \\ gx_i \succcurlyeq gy_i, \text{ if } i \in \mathsf{B}. \end{cases} \tag{11.81}$$

In particular,

$$gx_{\sigma_i(j)} \preccurlyeq_{\sigma_i(j)} gy_{\sigma_i(j)} \quad \text{for all } i, j \in \Lambda_n. \tag{11.82}$$

We distinguish four cases, but the conclusion is the same.

- if $i, j \in \mathsf{A} \;\Rightarrow\; \sigma_i(j) \in \mathsf{A} \;\Rightarrow\; gx_{\sigma_i(j)} \preccurlyeq gy_{\sigma_i(j)}$
$$\Rightarrow\; F(a_1, \ldots, a_{j-1}, x_{\sigma_i(j)}, a_{j+1}, \ldots, a_n) \preccurlyeq F(a_1, \ldots, a_{j-1}, y_{\sigma_i(j)}, a_{j+1}, \ldots, a_n)$$
$$\Rightarrow\; F(a_1, \ldots, a_{j-1}, x_{\sigma_i(j)}, a_{j+1}, \ldots, a_n) \preccurlyeq_i F(a_1, \ldots, a_{j-1}, y_{\sigma_i(j)}, a_{j+1}, \ldots, a_n);$$
- if $i \in \mathsf{A}$ and $j \in \mathsf{B} \;\Rightarrow\; \sigma_i(j) \in \mathsf{B} \;\Rightarrow\; gx_{\sigma_i(j)} \succcurlyeq gy_{\sigma_i(j)}$
$$\Rightarrow\; F(a_1, \ldots, a_{j-1}, x_{\sigma_i(j)}, a_{j+1}, \ldots, a_n) \preccurlyeq F(a_1, \ldots, a_{j-1}, y_{\sigma_i(j)}, a_{j+1}, \ldots, a_n)$$

$$\Rightarrow\ F(a_1,\dots,a_{j-1},x_{\sigma_i(j)},a_{j+1},\dots,a_n)$$
$$\preccurlyeq_i F(a_1,\dots,a_{j-1},y_{\sigma_i(j)},a_{j+1},\dots,a_n);$$

- if $i \in \mathsf{B}$ and $j \in \mathsf{A}$ $\Rightarrow$ $\sigma_i(j) \in \mathsf{B}$ $\Rightarrow$ $gx_{\sigma_i(j)} \succcurlyeq gy_{\sigma_i(j)}$

$$\Rightarrow\ F(a_1,\dots,a_{j-1},x_{\sigma_i(j)},a_{j+1},\dots,a_n)$$
$$\succcurlyeq F(a_1,\dots,a_{j-1},y_{\sigma_i(j)},a_{j+1},\dots,a_n)$$
$$\Rightarrow\ F(a_1,\dots,a_{j-1},x_{\sigma_i(j)},a_{j+1},\dots,a_n)$$
$$\preccurlyeq_i F(a_1,\dots,a_{j-1},y_{\sigma_i(j)},a_{j+1},\dots,a_n);$$

- if $i,j \in \mathsf{B}$ $\Rightarrow$ $\sigma_i(j) \in \mathsf{A}$ $\Rightarrow$ $gx_{\sigma_i(j)} \preccurlyeq gy_{\sigma_i(j)}$

$$\Rightarrow\ F(a_1,\dots,a_{j-1},x_{\sigma_i(j)},a_{j+1},\dots,a_n)$$
$$\succcurlyeq F(a_1,\dots,a_{j-1},y_{\sigma_i(j)},a_{j+1},\dots,a_n)$$
$$\Rightarrow\ F(a_1,\dots,a_{j-1},x_{\sigma_i(j)},a_{j+1},\dots,a_n)$$
$$\preccurlyeq_i F(a_1,\dots,a_{j-1},y_{\sigma_i(j)},a_{j+1},\dots,a_n).$$

In any case, we deduce that, for all $i,j \in \Lambda_n$ and all $a_1,a_2,\dots,a_n \in X$,

$$F(a_1,\dots,a_{j-1},x_{\sigma_i(j)},a_{j+1},\dots,a_n) \preccurlyeq_i F(a_1,\dots,a_{j-1},y_{\sigma_i(j)},a_{j+1},\dots,a_n).$$

Therefore, for all $i \in \Lambda_n$,

$$\begin{aligned}
F\big(x_{\sigma_i(1)},x_{\sigma_i(2)},x_{\sigma_i(3)},\dots,x_{\sigma_i(n)}\big) &\preccurlyeq_i F\big(y_{\sigma_i(1)},x_{\sigma_i(2)},x_{\sigma_i(3)},\dots,x_{\sigma_i(n)}\big)\\
&\preccurlyeq_i F\big(y_{\sigma_i(1)},y_{\sigma_i(2)},x_{\sigma_i(3)},\dots,x_{\sigma_i(n)}\big) \preccurlyeq_i \dots\\
&\preccurlyeq_i F\big(y_{\sigma_i(1)},y_{\sigma_i(2)},y_{\sigma_i(3)},\dots,y_{\sigma_i(n)}\big).
\end{aligned}$$

As $\preccurlyeq_i$ is transitive, then

$$F\big(x_{\sigma_i(1)},x_{\sigma_i(2)},x_{\sigma_i(3)},\dots,x_{\sigma_i(n)}\big) \preccurlyeq_i F\big(y_{\sigma_i(1)},y_{\sigma_i(2)},y_{\sigma_i(3)},\dots,y_{\sigma_i(n)}\big)$$

for all $i \in \Lambda_n$, which means that

$$\begin{aligned}
\mathbb{F}_\Phi(x_1,x_2,\dots,x_n) = \Big(&F\big(x_{\sigma_1(1)},x_{\sigma_1(2)},\dots,x_{\sigma_1(n)}\big),\\
&F\big(x_{\sigma_2(1)},x_{\sigma_2(2)},\dots,x_{\sigma_2(n)}\big),\dots,F\big(x_{\sigma_n(1)},x_{\sigma_n(2)},\dots,x_{\sigma_n(n)}\big)\Big)\\
\sqsubseteq \Big(&F\big(y_{\sigma_1(1)},y_{\sigma_1(2)},\dots,y_{\sigma_1(n)}\big),
\end{aligned}$$

$$F(y_{\sigma_2(1)}, y_{\sigma_2(2)}, \ldots, y_{\sigma_2(n)}), \ldots, F(y_{\sigma_n(1)}, y_{\sigma_n(2)}, \ldots, y_{\sigma_n(n)})\Big)$$

$$\sqsubseteq \mathbb{F}_\Phi(y_1, y_2, \ldots, y_n).$$

Hence, $\mathbb{F}_\Phi$ is $(H_g^n, \sqsubseteq)$-non-decreasing. □

**Lemma 11.8.4.** *Let $F : X^n \to X$ and $g : X \to X$ be two mappings, let $\{\mathsf{X}_m\}_{m\in\mathbb{N}} \subseteq X^n$ be a sequence and let $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be a n-tuple of mappings from $\{1, 2, \ldots, n\}$ into itself. Then $\{\mathsf{X}_m\}_{m\in\mathbb{N}} \subseteq X^n$ is a Picard $(F, g, \Phi)$-sequence if, and only if, $H_g^n(\mathsf{X}_{m+1}) = \mathbb{F}_\Phi(\mathsf{X}_m)$ for all $m \in \mathbb{N}$.*

*Proof.* Suppose that $\mathsf{X}_m = (x_m^1, x_m^2, \ldots, x_m^n)$ for all $m \in \mathbb{N}$. Then

$$\{\mathsf{X}_m\}_{m\in\mathbb{N}} \subseteq X^n \text{ is a Picard } (F, g, \Phi)\text{-sequence}$$

$$\Leftrightarrow \quad gx_{m+1}^i = F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}) \quad \forall m \in \mathbb{N}, \forall i \in \Lambda_n$$

$$\Leftrightarrow \quad \big(gx_{m+1}^1, gx_{m+1}^2, \ldots gx_{m+1}^n\big) = \Big(F(x_m^{\sigma_1(1)}, x_m^{\sigma_1(2)}, \ldots, x_m^{\sigma_1(n)}),$$

$$F(x_m^{\sigma_2(1)}, x_m^{\sigma_2(2)}, \ldots, x_m^{\sigma_2(n)}), \ldots, F(x_m^{\sigma_n(1)}, x_m^{\sigma_n(2)}, \ldots, x_m^{\sigma_n(n)})\Big)$$

$$\forall m \in \mathbb{N}$$

$$\Leftrightarrow \quad H_g^n(\mathsf{X}_{m+1}) = \mathbb{F}_\Phi(\mathsf{X}_m) \quad \text{for all } m \in \mathbb{N}.$$

□

**Lemma 11.8.5.** *Let $(X, G)$ be a $G^*$-metric space and let $G_n$ and $G'_n$ be the $G^*$-metrics defined in Lemma 10.3.1.*

1. *If $g : X \to X$ is G-continuous, then $H_g^n : X^n \to X^n$ is $G_n$-continuous (and also $G'_n$-continuous).*
2. *If $F : X^n \to X$ is G-continuous, then $\mathbb{F}_\Phi : X^n \to X^n$ is $G_n$-continuous (and also $G'_n$-continuous) and $F_\Phi^2 = F \circ \mathbb{F}_\Phi : X^n \to X$ is G-continuous.*

*Proof.* **(1)** Let $\mathsf{Z} = (z^1, z^2, \ldots, z^n) \in X^n$ be a point and let $\{\mathsf{X}_m = (x_m^1, x_m^2, \ldots, x_m^n)\}_{m\in\mathbb{N}} \subseteq X^n$ be a sequence such that $\{\mathsf{X}_m\} \xrightarrow{G_n} \mathsf{Z}$. From item 2 of Lemma 10.3.1, we have that $\{x_m^i\} \xrightarrow{G} z^i$ for all $i \in \Lambda_n$. As $g$ is $G$-continuous, then $\{gx_m^i\} \xrightarrow{G} gz^i$ for all $i \in \Lambda_n$. Again by item 2 of Lemma 10.3.1, we deduce that

$$\big\{\big(gx_m^1, gx_m^2, \ldots, gx_m^n\big)\big\} \xrightarrow{G_n} \big(gz^1, gz^2, \ldots, gz^n\big),$$

which means that $\{H_g^n(\mathsf{X}_m)\} \xrightarrow{G_n} H_g^n(\mathsf{Z})$. Hence, $H_g^n : X^n \to X^n$ is $G_n$-continuous.

**(2)** Let $\mathsf{Z} = (z^1, z^2, \ldots, z^n) \in X^n$ be a point and let $\{\mathsf{X}_m = (x_m^1, x_m^2, \ldots, x_m^n)\}_{m\in\mathbb{N}} \subseteq X^n$ be a sequence such that $\{\mathsf{X}_m\} \xrightarrow{G_n} \mathsf{Z}$. From item 2 of Lemma 10.3.1, we have that $\{x_m^i\} \xrightarrow{G} z^i$ for all $i \in \Lambda_n$. In particular, $\{x_m^{\sigma_i(j)}\} \xrightarrow{G} z^{\sigma_i(j)}$ for all $i, j \in \Lambda_n$. As $F$ is $G$-continuous, then

$$\left\{F\left(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}\right)\right\}_{m\in\mathbb{N}} \xrightarrow{G_n} F\left(z^{\sigma_i(1)}, z^{\sigma_i(2)}, \ldots, z^{\sigma_i(n)}\right)$$

for all $i \in \Lambda_n$. Again, by item 2 of Lemma 10.3.1, it follows that

$$\begin{aligned}\Big\{\Big(&F(x_m^{\sigma_1(1)}, x_m^{\sigma_1(2)}, \ldots, x_m^{\sigma_1(n)}), F(x_m^{\sigma_2(1)}, x_m^{\sigma_2(2)}, \ldots, x_m^{\sigma_2(n)}),\\ &\qquad \ldots, F(x_m^{\sigma_n(1)}, x_m^{\sigma_n(2)}, \ldots, x_m^{\sigma_n(n)})\Big)\Big\}_{m\in\mathbb{N}}\\ \xrightarrow{G_n} \Big(&F(z^{\sigma_1(1)}, z^{\sigma_1(2)}, \ldots, z^{\sigma_1(n)}), F(z^{\sigma_2(1)}, z^{\sigma_2(2)}, \ldots, z^{\sigma_2(n)}),\\ &\qquad \ldots, F(z^{\sigma_n(1)}, z^{\sigma_n(2)}, \ldots, z^{\sigma_n(n)})\Big).\end{aligned}$$

In other words, $\{\mathbb{F}_\Phi(\mathsf{X}_m)\} \xrightarrow{G_n} \mathbb{F}_\Phi(\mathsf{Z})$. Hence, $\mathbb{F}_\Phi : X^n \to X^n$ is $G_n$-continuous. The second part follows from the fact that the composition $F_\Phi^2 = F \circ \mathbb{F}_\Phi$ of continuous mappings is also a continuous mapping. □

The following result is an extension of Lemma 11.7.5 and it can be proved similarly (using the fact that the partition $\{\mathsf{A}, \mathsf{B}\}$ has non-empty sets).

**Lemma 11.8.6.** *Let $(X, G)$ be a $G^*$-metric space and let $G_n$ and $G'_n$ be the $G^*$-metrics on $X^n$ defined in Lemma 10.3.1. Given a transitive binary relation $\preccurlyeq$ on $X$, let $\sqsubseteq$ be the binary relation on $X^n$ defined in (11.79). Then the following properties are equivalent (the same is valid for $G'_n$).*

- **(i)** *$(X, G, \preccurlyeq)$ is regular.*
- **(ii)** *$(X^n, G_n, \sqsubseteq)$ is regular.*
- **(iii)** *$(X^n, G_n, \sqsubseteq)$ is non-decreasing-regular.*
- **(iv)** *$(X^n, G_n, \sqsubseteq)$ is non-increasing-regular.*

### *11.8.1 A First Multidimensional Fixed Point Theorem*

In this subsection we apply Theorem 5.3.1 considering $T = \mathbb{F}_\Phi$ defined on $(X^n, G_n, \sqsubseteq)$. We notice that, joining with some of the previous results, we obtain the following consequences.

- If $(X, G)$ is complete, it follows from Corollary 10.3.1 that $(X^n, G_n)$ is also complete.
- By item 2 of Lemma 11.8.3, if $F$ has the mixed $\preccurlyeq$-monotone property, then $\mathbb{F}_\Phi$ is $\sqsubseteq$-monotone non-decreasing on $X^n$.

- By item 2 of Lemma 11.8.5, if $F$ is $G$-continuous, then $\mathbb{F}_\Phi : X^n \to X^n$ is $G_n$-continuous and $F_\Phi^2 = F \circ \mathbb{F}_\Phi : X^n \to X$ is $G$-continuous.
- If $(X, G, \preccurlyeq)$ is regular, it follows from Lemma 11.8.6 that $(X^n, G_n, \sqsubseteq)$ is also regular.
- If $x_0^1, x_0^2, \ldots, x_0^n \in X$ are such that $x_0^i \preccurlyeq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)})$ for all $i \in \Lambda_n$, then $\mathsf{X}_0 = (x_0^1, x_0^2, \ldots, x_0^n) \in X^n$ verifies $\mathsf{X}_0 \sqsubseteq \mathbb{F}_\Phi(\mathsf{X}_0)$.

We study how the contractivity condition

$$\psi(G_n(\mathbb{F}_\Phi\mathsf{X}, \mathbb{F}_\Phi\mathsf{Y}, \mathbb{F}_\Phi^2\mathsf{X})) \leq (\psi - \varphi)\,(G_n(\mathsf{X}, \mathsf{Y}, \mathbb{F}_\Phi\mathsf{X}))$$
$$\text{for all } \mathsf{X}, \mathsf{Y} \in X^n \text{ such that } \mathsf{X} \sqsubseteq \mathsf{Y}$$

may be equivalently established. Let $\mathsf{X} = (x_1, x_2, \ldots, x_n) \in X^n$ and let $z_i = F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}) \in X$ for all $i$. Then

$$\begin{aligned}
\mathbb{F}_\Phi^2\mathsf{X} &= \mathbb{F}_\Phi\left(F(x_{\sigma_1(1)}, x_{\sigma_1(2)}, \ldots, x_{\sigma_1(n)}), \ldots, F(x_{\sigma_n(1)}, x_{\sigma_n(2)}, \ldots, x_{\sigma_n(n)})\right)\\
&= \mathbb{F}_\Phi(z_1, z_2, \ldots, z_n)\\
&= (F(z_{\sigma_1(1)}, z_{\sigma_1(2)}, \ldots, z_{\sigma_1(n)}), \ldots, F(z_{\sigma_n(1)}, z_{\sigma_n(2)}, \ldots, z_{\sigma_n(n)}))\\
&= \left(F\left(F(x_{\sigma_{\sigma_1(1)}(1)}, \ldots, x_{\sigma_{\sigma_1(1)}(n)}), \ldots, F(x_{\sigma_{\sigma_1(n)}(1)}, \ldots, x_{\sigma_{\sigma_1(n)}(n)})\right),\right.\\
&\qquad F\left(F(x_{\sigma_{\sigma_2(1)}(1)}, \ldots, x_{\sigma_{\sigma_2(1)}(n)}), \ldots, F(x_{\sigma_{\sigma_2(n)}(1)}, \ldots, x_{\sigma_{\sigma_n(n)}(n)})\right),\\
&\qquad \ldots,\\
&\qquad \left.F\left(F(x_{\sigma_{\sigma_n(1)}(1)}, \ldots, x_{\sigma_{\sigma_n(1)}(n)}), \ldots, F(x_{\sigma_{\sigma_n(n)}(1)}, \ldots, x_{\sigma_{\sigma_n(n)}(n)})\right)\right)\\
&= \left(F_\Phi^2(x_{\sigma_1(1)}, x_{\sigma_1(2)} \ldots, x_{\sigma_1(n)}), \ldots, F_\Phi^2(x_{\sigma_n(1)}, x_{\sigma_n(2)} \ldots, x_{\sigma_n(n)})\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
G_n(\mathsf{X}, \mathsf{Y}, \mathbb{F}_\Phi\mathsf{X}) &= \max_{1 \leq i \leq n} G(x_i, y_i, F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)})) \qquad \text{and}\\
G_n(\mathbb{F}_\Phi\mathsf{X}, \mathbb{F}_\Phi\mathsf{Y}, \mathbb{F}_\Phi^2\mathsf{X}) &= \max_{1 \leq i \leq n} G(F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}),\\
&\qquad F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \ldots, y_{\sigma_i(n)}), F_\Phi^2(x_{\sigma_i(1)}, x_{\sigma_i(2)} \ldots, x_{\sigma_i(n)})).
\end{aligned}$$

Therefore, a possible version of Theorem 5.3.1 (using $g$ as the identity mapping on $X^n$) applied to $(X^n, G_n, \sqsubseteq)$ taking $T = \mathbb{F}_\Phi$ is the following.

**Theorem 11.8.1.** *Let $(X, G)$ be a complete $G^*$-metric space and let $\preccurlyeq$ be a preorder on $X$. Let $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be a n-tuple of mappings from $\{1, 2, \ldots, n\}$ into itself satisfying $\sigma_i \in \Omega_{\mathsf{A},\mathsf{B}}$ if $i \in \mathsf{A}$ and $\sigma_i \in \Omega'_{\mathsf{A},\mathsf{B}}$ if $i \in \mathsf{B}$. Let $F : X^n \to X$ be mapping verifying the mixed monotone property on $X$. Assume that there exist $\psi, \varphi \in \mathcal{F}_{\text{alt}}$ such that*

$$\max_{1\le i\le n}\psi\left(G\left(F(x_{\sigma_i(1)},x_{\sigma_i(2)},\ldots,x_{\sigma_i(n)}),F(y_{\sigma_i(1)},y_{\sigma_i(2)},\ldots,y_{\sigma_i(n)}),\right.\right.$$
$$\left.\left.F_\Phi^2(x_{\sigma_i(1)},x_{\sigma_i(2)}\ldots,x_{\sigma_i(n)})\right)\right)$$
$$\le(\psi-\varphi)\left(\max_{1\le i\le n}G(x_i,y_i,F(x_{\sigma_i(1)},x_{\sigma_i(2)},\ldots,x_{\sigma_i(n)}))\right) \tag{11.83}$$

*for which $x_i \preccurlyeq_i y_i$ for all i. Suppose either F is continuous or $(X,G,\preccurlyeq)$ is regular. If there exist $x_0^1,x_0^2,\ldots,x_0^n\in X$ verifying $x_0^i\preccurlyeq_i F(x_0^{\sigma_i(1)},x_0^{\sigma_i(2)},\ldots,x_0^{\sigma_i(n)})$ for all i, then F has, at least, one $\Phi$-fixed point.*

### *11.8.2 A Second Multidimensional Fixed Point Theorem*

In this section we introduce a slightly different contractivity condition that cannot be directly deduced by applying Theorem 5.3.1 to $(X,G_n,\sqsubseteq)$ taking $T=\mathbb{F}_\Phi$, because the contractivity condition is weaker.

**Theorem 11.8.2.** *Let $(X,G)$ be a complete $G^*$-metric space and let $\preccurlyeq$ be a partial preorder on X. Let $\Phi=(\sigma_1,\sigma_2,\ldots,\sigma_n)$ be a n-tuple of mappings from $\{1,2,\ldots,n\}$ into itself satisfying $\sigma_i\in\Omega_{\mathsf{A},\mathsf{B}}$ if $i\in\mathsf{A}$ and $\sigma_i\in\Omega'_{\mathsf{A},\mathsf{B}}$ if $i\in\mathsf{B}$. Let $F:X^n\to X$ be mapping verifying the mixed monotone property on X. Assume that there exist $\psi,\varphi\in\mathcal{F}_{\mathrm{alt}}$ such that*

$$\psi(G(F(x_1,x_2,\ldots,x_n),F(y_1,y_2,\ldots,y_n),F_\Phi^2(x_1,x_2,\ldots,x_n))$$
$$\le(\psi-\varphi)\left(\max_{1\le i\le n}G(x_i,y_i,F(x_{\sigma_i(1)},x_{\sigma_i(2)},\ldots,x_{\sigma_i(n)}))\right) \tag{11.84}$$

*for which $(x_1,x_2,\ldots,x_n),(y_1,y_2,\ldots,y_n)\in X^n$ are $\sqsubseteq$-comparable. Suppose either F is continuous or $(X,G,\preccurlyeq)$ is regular. If there exist $x_0^1,x_0^2,\ldots,x_0^n\in X$ verifying $x_0^i\preccurlyeq_i F(x_0^{\sigma_i(1)},x_0^{\sigma_i(2)},\ldots,x_0^{\sigma_i(n)})$ for all i, then F has, at least, one $\Phi$-fixed point.*

Notice that (11.83) and (11.84) are very different contractivity conditions. For example, (11.83) would be simpler if the image of all $\sigma_i$ are sets with a few points.

*Proof.* Define $\mathsf{X}_0=(x_0^1,x_0^2,\ldots,x_0^n)$ and let $x_1^i=F(x_0^{\sigma_i(1)},x_0^{\sigma_i(2)},\ldots,x_0^{\sigma_i(n)})$ for all $i$. If $\mathsf{X}_1=(x_1^1,x_1^2,\ldots,x_1^n)$, then $x_0^i\preccurlyeq_i x_1^i$ for all $i$ is equivalent to $\mathsf{X}_0\sqsubseteq\mathsf{X}_1=\mathbb{F}_\Phi(\mathsf{X}_0)$. By recurrence, define $x_{m+1}^i=F(x_m^{\sigma_i(1)},x_m^{\sigma_i(2)},\ldots,x_m^{\sigma_i(n)})$ for all $i$ and all $m$, and we have that $\mathsf{X}_m\sqsubseteq\mathsf{X}_{m+1}=\mathbb{F}_\Phi(\mathsf{X}_m)$. This means that the sequence $\{\mathsf{X}_{m+1}=\mathbb{F}_\Phi(\mathsf{X}_m)\}$ is $\sqsubseteq$-monotone non-decreasing. Since $(X^n,G_n,\sqsubseteq)$ is complete, it is only necessary to prove that $\{\mathsf{X}_m\}$ is $G_n$-Cauchy in order to deduce that it is $G_n$-convergent. From item 4 of Lemma 10.3.1, it will be sufficient to prove that each sequence $\{x_m^i\}$ is $G$-Cauchy. Firstly, notice that $\mathsf{X}_{m+1}=\mathbb{F}_\Phi(\mathsf{X}_m)$ means that

$$x_{m+1}^i=F(x_m^{\sigma_i(1)},x_m^{\sigma_i(2)},\ldots,x_m^{\sigma_i(n)})\qquad\text{for all } i \text{ and all } m.$$

Hence

$$\begin{aligned}
x_{m+2}^{i} &= F(x_{m+1}^{\sigma_i(1)}, x_{m+1}^{\sigma_i(2)}, \ldots, x_{m+1}^{\sigma_i(n)}) \\
&= F\Big(F(x_m^{\sigma_{\sigma_i(1)}(1)}, x_m^{\sigma_{\sigma_i(1)}(2)}, \ldots, x_m^{\sigma_{\sigma_i(1)}(n)}), \\
&\qquad\quad F(x_m^{\sigma_{\sigma_i(2)}(1)}, x_m^{\sigma_{\sigma_i(2)}(2)}, \ldots, x_m^{\sigma_{\sigma_i(2)}(n)}), \ldots, \\
&\qquad\quad F(x_m^{\sigma_{\sigma_i(n)}(1)}, x_m^{\sigma_{\sigma_i(n)}(2)}, \ldots, x_m^{\sigma_{\sigma_i(n)}(n)})\Big) \\
&= F_\Phi^2(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}).
\end{aligned}$$

Furthermore, for all $m$

$$\begin{aligned}
F_\Phi^2(\mathsf{X}_m) &= F_\Phi^2(x_m^1, x_m^2, \ldots, x_m^n) = \\
&= F(F(x_m^{\sigma_1(1)}, x_m^{\sigma_1(2)}, \ldots, x_m^{\sigma_1(n)}), F(x_m^{\sigma_2(1)}, x_m^{\sigma_2(2)}, \ldots, x_m^{\sigma_2(n)}), \\
&\qquad \ldots, F(x_m^{\sigma_n(1)}, x_m^{\sigma_n(2)}, \ldots, x_m^{\sigma_n(n)})) \\
&= F(x_{m+1}^1, x_{m+1}^2, \ldots, x_{m+1}^n) = F(\mathsf{X}_{m+1}).
\end{aligned} \tag{11.85}$$

Therefore, for all $i$ and all $m$

$$\begin{aligned}
&\psi(G(x_{m+1}^i, x_{m+2}^i, x_{m+2}^i)) \\
&= \psi(G(F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}), F(x_{m+1}^{\sigma_i(1)}, x_{m+1}^{\sigma_i(2)}, \ldots, x_{m+1}^{\sigma_i(n)}), \\
&\qquad\qquad F_\Phi^2(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)})) \\
&\leq (\psi - \varphi)\left(\max_{1\leq j\leq n} G(x_m^{\sigma_i(j)}, x_{m+1}^{\sigma_i(j)}, F(x_m^{\sigma_{\sigma_i(j)}(1)}, x_m^{\sigma_{\sigma_i(j)}(2)}, \ldots, x_m^{\sigma_{\sigma_i(j)}(n)}))\right) \\
&= (\psi - \varphi)\left(\max_{1\leq j\leq n} G(x_m^{\sigma_i(j)}, x_{m+1}^{\sigma_i(j)}, x_{m+1}^{\sigma_i(j)})\right).
\end{aligned}$$

Since $\psi$ is non-decreasing, for all $i$ and all $m$,

$$\psi\left(\max_{1\leq j\leq n} G(x_m^{\sigma_i(j)}, x_{m+1}^{\sigma_i(j)}, x_{m+1}^{\sigma_i(j)})\right) \leq \psi\left(\max_{1\leq j\leq n} G(x_m^j, x_{m+1}^j, x_{m+1}^j)\right).$$

Applying Lemma 11.2.4 using

$$a_m^i = G(x_m^i, x_{m+1}^i, x_{m+1}^i) \qquad \text{and} \qquad b_m^i = \max_{1\leq j\leq n} G(x_m^{\sigma_i(j)}, x_{m+1}^{\sigma_i(j)}, x_{m+1}^{\sigma_i(j)})$$

for all $i$ and all $m$, we deduce that

$$\{G(x^i_m, x^i_{m+1}, x^i_{m+1})\} \to 0 \quad \text{for all } i, \quad \text{that is,}$$
$$\{G_n(\mathsf{X}_m, \mathsf{X}_{m+1}, \mathsf{X}_{m+1})\} \to 0. \tag{11.86}$$

Next, we prove that every sequence $\{x^i_m\}$ is $G$-Cauchy reasoning by contradiction. Suppose that $\{x^{i_1}_m\}_{m\geq 0}, \ldots, \{x^{i_s}_m\}_{m\geq 0}$ are not $G$-Cauchy ($s \geq 1$) and $\{x^{i_{s+1}}_m\}_{m\geq 0}, \ldots, \{x^{i_n}_m\}_{m\geq 0}$ are $G$-Cauchy, with $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$. From Proposition 11.8.1, for all $r \in \{1, 2, \ldots, s\}$ there exists $\varepsilon_r > 0$ and subsequences $\{x^{i_r}_{n_r(k)}\}_{k\in\mathbb{N}}$ and $\{x^{i_r}_{m_r(k)}\}_{k\in\mathbb{N}}$ such that, for all $k \in \mathbb{N}$,

$$k < n_r(k) < m_r(k) < n_r(k+1), \quad G(x^{i_r}_{n_r(k)}, x^{i_r}_{n_r(k)+1}, x^{i_r}_{m_r(k)}) \geq \varepsilon_r,$$

and

$$G(x^{i_r}_{n_r(k)}, x^{i_r}_{n_r(k)+1}, x^{i_r}_{m_r(k)-1}) < \varepsilon_r.$$

Now, let $\varepsilon_0 = \max(\varepsilon_1, \ldots, \varepsilon_s) > 0$ and $\varepsilon'_0 = \min(\varepsilon_1, \ldots, \varepsilon_s) > 0$. Since $\{x^{i_{s+1}}_m\}_{m\geq 0}, \ldots, \{x^{i_n}_m\}_{m\geq 0}$ are $G$-Cauchy, for all $j \in \{i_{s+1}, \ldots, i_n\}$, there exists $n^j \in \mathbb{N}$ such that if $m, m' \geq n^j$, then $G(x^j_m, x^j_{m+1}, x^j_{m'}) < \varepsilon'_0/8$. Let $n_0 = \max_{j\in\{i_{s+1},\ldots,i_n\}}(n^j)$. Therefore, we have proved that there exists $n_0 \in \mathbb{N}$ such that if $m, m' \geq n_0$ then

$$G(x^j_m, x^j_{m+1}, x^j_{m'}) < \varepsilon'_0/4 \quad \text{for all } j \in \{i_{s+1}, \ldots, i_n\}. \tag{11.87}$$

Next, let $q \in \{1, 2, \ldots, s\}$ such that $\varepsilon_q = \varepsilon_0 = \max(\varepsilon_1, \ldots, \varepsilon_s)$. Let $k_1 \in \mathbb{N}$ such that $n_0 < n_q(k_1)$ and define $n(1) = n_q(k_1)$. Consider the numbers $n(1)+1, n(1)+2, \ldots, m_q(k_1)$ until we find the first positive integer $m(1) > n(1)$ satisfying

$$\max_{1\leq r\leq s} G(x^{i_r}_{n(1)}, x^{i_r}_{n(1)+1}, x^{i_r}_{m(1)}) \geq \varepsilon_0, \quad G(x^{i_j}_{n(1)}, x^{i_j}_{n(1)+1}, x^{i_j}_{m(1)-1}) < \varepsilon_0,$$

for all $j \in \{1, 2, \ldots, s\}$. Now let $k_2 \in \mathbb{N}$ such that $m(1) < n_q(k_2)$ and define $n(2) = n_q(k_2)$. Consider the numbers $n(2)+1, n(2)+2, \ldots, m_q(k_2)$ until we find the first positive integer $m(2) > n(2)$ satisfying

$$\max_{1\leq r\leq s} G(x^{i_r}_{n(2)}, x^{i_r}_{n(2)+1}, x^{i_r}_{m(2)}) \geq \varepsilon_0, \quad G(x^{i_j}_{n(2)}, x^{i_j}_{n(2)+1}, x^{i_j}_{m(2)-1}) < \varepsilon_0,$$

for all $j \in \{1, 2, \ldots, s\}$. Repeating this process, we can find sequences such that, for all $k \geq 1$,

$$n_0 < n(k) < m(k) < n(k+1), \quad \max_{1\leq r\leq s} G(x^{i_r}_{n(k)}, x^{i_r}_{n(k)+1}, x^{i_r}_{m(k)}) \geq \varepsilon_0,$$
$$G(x^{i_j}_{n(k)}, x^{i_j}_{n(k)+1}, x^{i_j}_{m(k)-1}) < \varepsilon_0, \quad \text{for all } j \in \{1, 2, \ldots, s\}.$$

Note that by (11.87), $G(x^{i_r}_{n(k)}, x^{i_r}_{n(k)+1}, x^{i_r}_{m(k)}), G(x^{i_r}_{n(k)}, x^{i_r}_{n(k)+1}, x^{i_r}_{m(k)-1}) < \varepsilon_0'/4 < \varepsilon_0/2$ for all $r \in \{s+1, s+2, \ldots, n\}$, so

$$\begin{aligned}
&\max_{1\le j\le n} G(x^{j}_{n(k)}, x^{j}_{n(k)+1}, x^{j}_{m(k)}) = \max_{1\le r\le s} G(x^{i_r}_{n(k)}, x^{i_r}_{n(k)+1}, x^{i_r}_{m(k)}) \ge \varepsilon_0,\\
&G(x^{i}_{n(k)}, x^{i}_{n(k)+1}, gx^{i}_{m(k)-1}) < \varepsilon_0
\end{aligned} \tag{11.88}$$

for all $i \in \{1, 2, \ldots, n\}$ and all $k \ge 1$. Next, for all $k$, let $i(k) \in \{1, 2, \ldots, s\}$ be an index such that

$$\begin{aligned}
G(x^{i(k)}_{n(k)}, x^{i(k)}_{n(k)+1}, x^{i(k)}_{m(k)}) &= \max_{1\le r\le s} G(x^{i_r}_{n(k)}, x^{i_r}_{n(k)+1}, x^{i_r}_{m(k)})\\
&= \max_{1\le j\le n} G(x^{j}_{n(k)}, x^{j}_{n(k)+1}, x^{j}_{m(k)}) \ge \varepsilon_0.
\end{aligned}$$

Notice that, applying ($G_5$) twice and (11.88), for all $k$ and all $j$,

$$\begin{aligned}
&G(x^{j}_{n(k)-1}, x^{j}_{n(k)}, x^{j}_{m(k)-1})\\
&\quad\le G(x^{j}_{n(k)-1}, x^{j}_{n(k)}, x^{j}_{n(k)}) + G(x^{j}_{n(k)}, x^{j}_{n(k)}, x^{j}_{m(k)-1})\\
&\quad\le G(x^{j}_{n(k)-1}, x^{j}_{n(k)}, x^{j}_{n(k)}) + G(x^{j}_{n(k)}, x^{j}_{n(k)+1}, x^{j}_{n(k)+1})\\
&\qquad\quad + G(x^{j}_{n(k)+1}, x^{j}_{n(k)}, x^{j}_{m(k)-1})\\
&\quad\le G(x^{j}_{n(k)-1}, x^{j}_{n(k)}, x^{j}_{n(k)}) + G(x^{j}_{n(k)}, x^{j}_{n(k)+1}, x^{j}_{n(k)+1}) + \varepsilon_0.
\end{aligned} \tag{11.89}$$

Applying Proposition 11.8.1 to guarantee that the following points are $\sqsubseteq$-comparable, the contractivity condition (11.84) assures us for all $k$

$$\begin{aligned}
0 &< \psi(\varepsilon_0) \le \psi\left(G(x^{i(k)}_{n(k)}, x^{i(k)}_{n(k)+1}, x^{i(k)}_{m(k)})\right) = \psi\left(G(x^{i(k)}_{n(k)}, x^{i(k)}_{m(k)}, x^{i(k)}_{n(k)+1})\right)\\
&= \psi(G(F(x^{\sigma_{i(k)}(1)}_{n(k)-1}, x^{\sigma_{i(k)}(2)}_{n(k)-1}, \ldots, x^{\sigma_{i(k)}(n)}_{n(k)-1}),\\
&\qquad\quad F(x^{\sigma_{i(k)}(1)}_{m(k)-1}, x^{\sigma_{i(k)}(2)}_{m(k)-1}, \ldots, x^{\sigma_{i(k)}(n)}_{m(k)-1}),\\
&\qquad\quad F^2_{\Phi}(x^{\sigma_{i(k)}(1)}_{n(k)-1}, x^{\sigma_{i(k)}(2)}_{n(k)-1}, \ldots, x^{\sigma_{i(k)}(n)}_{n(k)-1})))\\
&\le (\psi - \varphi)\left(\max_{1\le j\le n} G(x^{\sigma_{i(k)}(j)}_{n(k)-1}, x^{\sigma_{i(k)}(j)}_{m(k)-1}, F(x^{\sigma_{i(k)}(1)}_{n(k)-1}, x^{\sigma_{i(k)}(2)}_{n(k)-1}, \ldots, x^{\sigma_{i(k)}(n)}_{n(k)-1}))\right)\\
&= (\psi - \varphi)\left(\max_{1\le j\le n} G(x^{\sigma_{i(k)}(j)}_{n(k)-1}, x^{\sigma_{i(k)}(j)}_{m(k)-1}, x^{\sigma_{i(k)}(j)}_{n(k)})\right)\\
&= (\psi - \varphi)\left(\max_{1\le j\le n} G(x^{\sigma_{i(k)}(j)}_{n(k)-1}, x^{\sigma_{i(k)}(j)}_{n(k)}, x^{\sigma_{i(k)}(j)}_{m(k)-1})\right).
\end{aligned} \tag{11.90}$$

Consider the sequence:

$$\left\{\max_{1\leq j\leq n} G(x_{n(k)-1}^{\sigma_{i(k)}(j)}, x_{n(k)}^{\sigma_{i(k)}(j)}, x_{m(k)-1}^{\sigma_{i(k)}(j)})\right\}_{k\geq 1}. \tag{11.91}$$

If this sequence has a subsequence that converges to zero, then we can take the limit when $k \to \infty$ in (11.90) using this subsequence, so that we would have $0 < \psi(\varepsilon_0) \leq \psi(0) - \varphi(0) = 0$, which is impossible since $\varepsilon_0 > 0$. Therefore, the sequence (11.91) has no subsequence converging to zero. In this case, taking $\varepsilon_0 > 0$ in Lemma 11.2.3, there exist $\delta \in ]0, \varepsilon_0[$ and $k_0 \in \mathbb{N}$ such that $\max_{1\leq j\leq n} G(x_{n(k)-1}^{\sigma_{i(k)}(j)}, x_{n(k)}^{\sigma_{i(k)}(j)}, x_{m(k)-1}^{\sigma_{i(k)}(j)}) \geq \delta$, for all $k \geq k_0$. It follows that, for all $k \geq k_0$, $-\varphi(\max_{1\leq j\leq n} G(x_{n(k)-1}^{\sigma_{i(k)}(j)}, x_{n(k)}^{\sigma_{i(k)}(j)}, x_{m(k)-1}^{\sigma_{i(k)}(j)})) \leq -\varphi(\delta)$. Thus, by (11.90) and (11.89),

$$\begin{aligned}
0 < \psi(\varepsilon_0) &\leq \psi\left(\max_{1\leq j\leq n} G(x_{n(k)-1}^{\sigma_{i(k)}(j)}, x_{n(k)}^{\sigma_{i(k)}(j)}, x_{m(k)-1}^{\sigma_{i(k)}(j)})\right) \\
&\qquad - \varphi\left(\max_{1\leq j\leq n} G(x_{n(k)-1}^{\sigma_{i(k)}(j)}, x_{n(k)}^{\sigma_{i(k)}(j)}, x_{m(k)-1}^{\sigma_{i(k)}(j)})\right) \\
&\leq \psi\left(\max_{1\leq j\leq n} G(x_{n(k)-1}^{\sigma_{i(k)}(j)}, x_{n(k)}^{\sigma_{i(k)}(j)}, x_{m(k)-1}^{\sigma_{i(k)}(j)})\right) - \varphi(\delta) \\
&\leq \psi\left(\max_{1\leq j\leq n} G(x_{n(k)-1}^{j}, x_{n(k)}^{j}, x_{m(k)-1}^{j})\right) - \varphi(\delta) \\
&\leq \psi\left(\max_{1\leq j\leq n}\left(G(x_{n(k)-1}^{j}, x_{n(k)}^{j}, x_{n(k)}^{j}) + G(x_{n(k)}^{j}, x_{n(k)+1}^{j}, x_{n(k)+1}^{j})\right) + \varepsilon_0\right) \\
&\qquad - \varphi(\delta).
\end{aligned} \tag{11.92}$$

Taking the limit in (11.92) as $k \to \infty$ and taking into account (11.86), we deduce that $0 < \psi(\varepsilon_0) \leq \psi(\varepsilon_0) - \varphi(\delta)$, which is impossible. The previous reasoning proves that every sequence $\{x_m^i\}$ is $G$-Cauchy.

Corollary 10.3.1 guarantees that the sequence

$$\{\mathbb{F}_\Phi^m(\mathsf{X}_0) = \mathsf{X}_m = (x_m^1, x_m^2, \ldots, x_m^n)\}$$

is $G_n$-Cauchy. Since $(X^n, G_n)$ is complete (again by Corollary 10.3.1), there exists $\mathsf{Z} \in X^n$ such that $\{\mathsf{X}_m\} \xrightarrow{G_n} \mathsf{Z}$, that is, if $\mathsf{Z} = (z_1, z_2, \ldots, z_n)$ then

$$\{G(x_m^i, x_{m+1}^i, z_i)\} \to 0 \qquad \text{for all } i. \tag{11.93}$$

*Suppose that $F$ is $G$-continuous*. In this case, item 2 of Lemma 11.8.5 implies that $\mathbb{F}_\Phi$ is $G_n$-continuous, so $\{\mathsf{X}_m\} \xrightarrow{G_n} \mathsf{Z}$ and $\{\mathsf{X}_{m+1} = \mathbb{F}_\Phi(\mathsf{X}_m)\} \xrightarrow{G_n} \mathbb{F}_\Phi(\mathsf{Z})$. By the uniqueness of the $G_n$-limit, $\mathbb{F}_\Phi(\mathsf{Z}) = \mathsf{Z}$, which means that $\mathsf{Z}$ is a $\Phi$-fixed point of $F$.

*Suppose that* $(X, G, \preccurlyeq)$ *is regular*. In this case, by Corollary 10.3.1, $(X^n, G_n, \sqsubseteq)$ is also regular. Then, taking into account that $\{\mathsf{X}_m = \mathbb{F}_\Phi^m(\mathsf{X}_0)\}$ is a $\sqsubseteq$-monotone non-decreasing sequence such that $\{\mathsf{X}_m\} \xrightarrow{G_n} \mathsf{Z}$, we deduce that $\mathsf{X}_m \sqsubseteq \mathsf{Z}$ for all $m$. From Proposition 11.8.1, since $(x_m^1, x_m^2, \ldots, x_m^n) = \mathsf{X}_m \sqsubseteq \mathsf{Z} = (z_1, z_2, \ldots, z_n)$, then $(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)})$ and $(z_{\sigma_i(1)}, z_{\sigma_i(2)}, \ldots, z_{\sigma_i(n)})$ are $\sqsubseteq$-comparable for all $i$ and all $m$. Notice that for all $i$ and all $m$,

$$\begin{aligned}
&F(x_{m+1}^{\sigma_i(1)}, x_{m+1}^{\sigma_i(2)}, \ldots, x_{m+1}^{\sigma_i(n)}) \\
&\qquad = F(F(x_m^{\sigma_{\sigma_i(1)}(1)}, x_m^{\sigma_{\sigma_i(1)}(2)}, \ldots, x_m^{\sigma_{\sigma_i(1)}(n)}), \\
&\qquad\qquad\qquad \ldots, F(x_m^{\sigma_{\sigma_i(n)}(1)}, x_m^{\sigma_{\sigma_i(n)}(2)}, \ldots, x_m^{\sigma_{\sigma_i(n)}(n)})) \\
&\quad = F_\Phi^2(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}).
\end{aligned}$$

It follows from condition (11.84) and (11.85) that, for all $i$,

$$\begin{aligned}
&\psi(G(F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}), F(x_{m+1}^{\sigma_i(1)}, x_{m+1}^{\sigma_i(2)}, \ldots, x_{m+1}^{\sigma_i(n)}), \\
&\qquad\qquad F(z_{\sigma_i(1)}, z_{\sigma_i(2)}, \ldots, z_{\sigma_i(n)})) \\
&\quad = \psi(G(F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}), F(z_{\sigma_i(1)}, z_{\sigma_i(2)}, \ldots, z_{\sigma_i(n)}), \\
&\qquad\qquad F_\Phi^2(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)})) \\
&\quad \leq (\psi - \varphi)\Bigg(\max_{1 \leq j \leq n} G(x_m^{\sigma_i(j)}, z_{\sigma_i(j)}, \\
&\qquad\qquad F(x_m^{\sigma_{\sigma_i(j)}(1)}, x_m^{\sigma_{\sigma_i(j)}(2)}, \ldots, x_m^{\sigma_{\sigma_i(j)}(n)}))\Bigg) \\
&\quad = (\psi - \varphi)\left(\max_{1 \leq j \leq n} G(x_m^{\sigma_i(j)}, z_{\sigma_i(j)}, x_{m+1}^{\sigma_i(j)})\right) \\
&\quad \leq \psi\left(\max_{1 \leq j \leq n} G(x_m^{\sigma_i(j)}, x_{m+1}^{\sigma_i(j)}, z_{\sigma_i(j)})\right) \\
&\quad \leq \psi\left(\max_{1 \leq j \leq n} G(x_m^j, x_{m+1}^j, z_j)\right).
\end{aligned}$$

By (11.93) we deduce that

$$\left\{F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)})\right\} \to F(z_{\sigma_i(1)}, z_{\sigma_i(2)}, \ldots, z_{\sigma_i(n)}) \qquad \text{for all } i,$$

which means that

$$\begin{aligned}
&\left\{\mathbb{F}_\Phi \mathsf{X}_m = \left(F(x_m^{\sigma_1(1)}, x_m^{\sigma_1(2)}, \ldots, x_m^{\sigma_1(n)}), \ldots, F(x_m^{\sigma_n(1)}, x_m^{\sigma_n(2)}, \ldots, x_m^{\sigma_n(n)})\right)\right\} \\
&\quad \xrightarrow{G_n} \left(F(z_{\sigma_1(1)}, z_{\sigma_1(2)}, \ldots, z_{\sigma_1(n)}), \ldots, F(z_{\sigma_n(1)}, z_{\sigma_n(2)}, \ldots, z_{\sigma_n(n)})\right) = \mathbb{F}_\Phi \mathsf{Z}.
\end{aligned}$$

Since $\{\mathbb{F}_\Phi \mathsf{X}_m = \mathsf{X}_{m+1}\} \xrightarrow{G_n} \mathsf{Z}$, we conclude that $\mathbb{F}_\Phi \mathsf{Z} = \mathsf{Z}$, that is, $\mathsf{Z}$ is a $\Phi$-fixed point of $F$. □

If take $\psi(t) = t$ for all $t \in [0, \infty)$ in Theorem 11.8.2 then, we get the following results.

**Corollary 11.8.1.** *Let $(X, G)$ be a complete $G^*$-metric space and let $\preccurlyeq$ be a partial preorder on $X$. Let $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be a n-tuple of mappings from $\{1, 2, \ldots, n\}$ into itself verifying $\sigma_i \in \Omega_{\mathsf{A,B}}$ if $i \in \mathsf{A}$ and $\sigma_i \in \Omega'_{\mathsf{A,B}}$ if $i \in \mathsf{B}$. Let $F : X^n \to X$ be mapping satisfying the mixed monotone property on $X$. Assume that there exists $\varphi \in \mathcal{F}_{\text{alt}}$ such that*

$$\begin{aligned} G(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n), F_\Phi^2(x_1, x_2, \ldots, x_n) \\ \leq \max_{1 \leq i \leq n} G(x_i, y_i, F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)})) \\ - \varphi\left(\max_{1 \leq i \leq n} G(x_i, y_i, F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}))\right) \end{aligned}$$

*for which $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$ are $\sqsubseteq$-comparable. Suppose either $F$ is continuous or $(X, G, \preccurlyeq)$ is regular. If there exist $x_0^1, x_0^2, \ldots, x_0^n \in X$ verifying $x_0^i \preccurlyeq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)})$ for all i, then $F$ has, at least, one $\Phi$-fixed point.*

If take $\varphi(t) = (1 - \lambda)\, t$ for all $t \geq 0$, with $\lambda \in [0, 1)$, in Corollary 11.8.1 then, we derive the following result.

**Corollary 11.8.2.** *Let $(X, G)$ be a complete $G^*$-metric space and let $\preccurlyeq$ be a partial preorder on $X$. Let $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be a n-tuple of mappings from $\{1, 2, \ldots, n\}$ into itself satisfying $\sigma_i \in \Omega_{\mathsf{A,B}}$ if $i \in \mathsf{A}$ and $\sigma_i \in \Omega'_{\mathsf{A,B}}$ if $i \in \mathsf{B}$. Let $F : X^n \to X$ be mapping verifying the mixed monotone property on $X$. Assume that there exists $\lambda \in [0, 1)$ such that*

$$\begin{aligned} G(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n), F_\Phi^2(x_1, x_2, \ldots, x_n)) \\ \leq \lambda \max_{1 \leq i \leq n} G(x_i, y_i, F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)})) \end{aligned} \tag{11.94}$$

*for which $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$ are $\sqsubseteq$-comparable. Suppose either $F$ is continuous or $(X, G, \preccurlyeq)$ is regular. If there exist $x_0^1, x_0^2, \ldots, x_0^n \in X$ verifying $x_0^i \preccurlyeq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)})$ for all i, then $F$ has, at least, one $\Phi$-fixed point.*

*Example 11.8.1.* Let $X = \{0, 1, 2, 3, 4\}$ and let $G$ be the $G$-metric on $X$ given, for all $x, y, z \in X$, by $G(x, y, z) = \max(|x - y|, |x - z|, |y - z|)$. Then $(X, G)$ is complete and $G$ generates the discrete topology on $X$. Consider on $X$ the following partial order:

$$x, y \in X, \qquad x \preccurlyeq y \quad \Leftrightarrow \quad x = y \text{ or } (x, y) = (0, 2).$$

Define $F : X^n \to X$ by:

$$F(x_1, x_2, \dots, x_n) = \begin{cases} 0, \text{ if } x_1, x_2, \dots, x_n \in \{0, 1, 2\}, \\ 1, \text{ otherwise.} \end{cases}$$

Then the following statements hold.

1. $F$ is a $G$-continuous mapping.
2. If $y, z \in X$ satisfy $y \preccurlyeq z$, then either $y, z \in \{0, 1, 2\}$ or $y, z \in \{3, 4\}$. In particular, $F(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) = F(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n)$ and $F$ has the mixed monotone property on $X$.
3. If $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in X^n$ are $\sqsubseteq$-comparable, then $F(x_1, x_2, \dots, x_n) = F(y_1, y_2, \dots, y_n)$. In particular, (11.94) holds for $\lambda = 1/2$.
   For simplicity, henceforth, suppose that $n$ is even and let $\mathsf{A}$ (respectively, $\mathsf{B}$) be the set of all odd (respectively, even) numbers in $\{1, 2, \dots, n\}$.
4. For a mapping $\sigma : \Lambda_n \to \Lambda_n$, we use the notation $\sigma \equiv (\sigma(1), \sigma(2), \dots, \sigma(n))$ and consider

$$\sigma_i \equiv (i, i+1, \dots, n-1, n, 1, 2, \dots, i-1) \quad \text{for all } i.$$

   Then $\sigma_i \in \Omega_{\mathsf{A},\mathsf{B}}$ if $i$ is odd and $\sigma_i \in \Omega'_{\mathsf{A},\mathsf{B}}$ if $i$ is even. Let $\Phi = (\sigma_1, \sigma_2, \dots, \sigma_n)$.
5. Take $x_0^i = 0$ if $i$ is odd and $x_0^i = 2$ if $i$ is even. Then $x_0^i \preccurlyeq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \dots, x_0^{\sigma_i(n)})$ for all $i$.

Therefore, we can apply Corollary 11.8.2 to conclude that $F$ has, at least, one $\Phi$-fixed point. To finish, we prove the previous statements.

If $\{x_m\} \xrightarrow{G} x$, then there exists $m_0 \in \mathbb{N}$ such that $|x_m - x| = G(x, x, x_m) < 1/2$ for all $m \geq m_0$. Since $X$ is discrete, then $x_m = x$ for all $m \geq m_0$. This proves that $\tau_G$ is the discrete topology on $X$.

1. If $\{a_m^1\}, \{a_m^2\}, \dots, \{a_m^n\} \subseteq X$ are $n$ sequences such that $\{a_m^i\} \xrightarrow{G} a_i \in X$ for all $i$, then there exists $m_0 \in \mathbb{N}$ such that $a_m^i = a_i$ for all $m \geq m_0$ and all $i$. Then $\{F(a_m^1, a_m^2, \dots, a_m^n)\} \xrightarrow{G} F(a_1, a_2, \dots, a_n)$ and $F$ is $G$-continuous.
2. If $y, z \in X$ verify $y \preccurlyeq z$, the either $y = z$ (in this case, there is nothing to prove) or $(y, z) = (0, 2)$. Then either $y, z \in \{0, 1, 2\}$ or $y, z \in \{3, 4\}$. In particular,

$$\begin{aligned} &F(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) \\ &\quad = \left\{ \begin{array}{l} 0, \text{ if } x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n \in \{0, 1, 2\}, \\ 1, \text{ otherwise} \end{array} \right\} \\ &\quad = F(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_n). \end{aligned}$$

   Hence $F$ has the mixed monotone property on $X$.

3. Suppose that $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$ are $\sqsubseteq$-comparable and we claim that $F(x_1, x_2, \ldots, x_n) = F(y_1, y_2, \ldots, y_n)$. Indeed, assume, for example, that $x_i \preccurlyeq_i y_i$ for all $i$. By item 2, for all $i$, either $x_i, y_i \in \{0, 1, 2\}$ or $x_i, y_i \in \{3, 4\}$. Then

$$F(x_1, x_2, \ldots, x_n) = \left\{ \begin{array}{l} 0, \text{ if } x_1, x_2, \ldots, x_n \in \{0, 1, 2\}, \\ 1, \text{ otherwise} \end{array} \right\}$$

$$= \left\{ \begin{array}{l} 0, \text{ if } y_1, y_2, \ldots, y_n \in \{0, 1, 2\}, \\ 1, \text{ otherwise} \end{array} \right\} = F(y_1, y_2, \ldots, y_n).$$

If $x_i \succcurlyeq_i y_i$ for all $i$, the proof is similar. Next, we prove that (11.94) holds using $\lambda = 1/4$. If $(x_1, x_2, \ldots, x_n) \in X^n$, then $F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}) \in \{0, 1\} \subset \{0, 1, 2\}$. Therefore

$$\begin{aligned} &F_\Phi^2(x_1, x_2, \ldots, x_n) \\ &\quad = F(F(x_{\sigma_1(1)}, x_{\sigma_1(2)}, \ldots, x_{\sigma_1(n)}), F(x_{\sigma_2(1)}, x_{\sigma_2(2)}, \ldots, x_{\sigma_2(n)}), \\ &\qquad\qquad \ldots, F(x_{\sigma_n(1)}, x_{\sigma_n(2)}, \ldots, x_{\sigma_n(n)})) = 0. \end{aligned}$$

Suppose that $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$ are $\sqsubseteq$-comparable. It follows that

$$\begin{aligned} &G(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n), F_\Phi^2(x_1, x_2, \ldots, x_n)) \\ &\quad = \max(|F(x_1, x_2, \ldots, x_n) - F(y_1, y_2, \ldots, y_n)|, |F(x_1, x_2, \ldots, x_n) - 0|, \\ &\qquad\qquad |F(y_1, y_2, \ldots, y_n) - 0|) \\ &\quad = \max(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \\ &\quad = \begin{cases} 0, \text{ if } F(x_1, x_2, \ldots, x_n) = F(y_1, y_2, \ldots, y_n) = 0, \\ 1, \text{ otherwise.} \end{cases} \end{aligned}$$

It is clear that (11.94) holds if the previous number is 0. On the contrary, suppose that

$$G(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n), F_\Phi^2(x_1, x_2, \ldots, x_n)) = 1.$$

Then $F(x_1, x_2, \ldots, x_n) = 1$ or $F(y_1, y_2, \ldots, y_n) = 1$ (both cases are similar). Assume, for example, that $F(x_1, x_2, \ldots, x_n) = 1$. Then there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $x_{i_0} \in \{3, 4\}$. In particular

$$\left| x_{i_0} - F(x_{\sigma_{i_0}(1)}, x_{\sigma_{i_0}(2)}, \ldots, x_{\sigma_{i_0}(n)}) \right| \geq 3 - 1 = 2.$$

Therefore

$$\begin{aligned}\max_{1\le i\le n} & G(x_i, y_i, F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}))\\ &\ge G(x_{i_0}, y_{i_0}, F(x_{\sigma_{i_0}(1)}, x_{\sigma_{i_0}(2)}, \ldots, x_{\sigma_{i_0}(n)}))\\ &\ge \left|x_{i_0} - F(x_{\sigma_{i_0}(1)}, x_{\sigma_{i_0}(2)}, \ldots, x_{\sigma_{i_0}(n)})\right| \ge 2.\end{aligned}$$

This means that

$$\begin{aligned}G(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n), F_\Phi^2(x_1, x_2, \ldots, x_n)) = 1\\ = \frac{1}{2}\,2 \le \frac{1}{2}\max_{1\le i\le n} G(x_i, y_i, F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)})).\end{aligned}$$

Therefore, in this case, (11.94) also holds.

4. It is clear.
5. Since $x_0^i \in \{0, 1, 2\}$ for all $i$, then $F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)}) = 0$ for all $i$. If $i$ is odd, then $x_0^i = 0 \preccurlyeq_i 0 = F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)})$. If $i$ is even, then $x_0^i = 2 \succcurlyeq 0 = F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)})$, so $x_0^i \preccurlyeq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)})$.

## 11.9 Multidimensional Cyclic Fixed Point Theory

In this section, we show some sufficient conditions to guarantee that two mappings $F : X^n \to X$ and $g : X \to X$ have a *cyclic (multidimensional) coincidence point*, that is, a point $(x_1, x_2, \ldots, x_n) \in X^n$ such that

$$F(x_i, x_{i+1}, x_{i+2}, \ldots, x_n, x_1, x_2, \ldots, x_{i-1}) = gx_i \quad \text{for all } i \in \{1, 2, \ldots, n\}.$$

Given a point $(x_1, x_2, \ldots, x_n) \in X^n$, for simplicity, we denote

$$\hat{x}_i^n = (x_i, x_{i+1}, x_{i+2}, \ldots, x_n, x_1, x_2, \ldots, x_{i-1}) \in X^n.$$

Notice that we shall not involve the mixed monotone property.

**Theorem 11.9.1.** *Let $(X, G)$ be a complete $G$-metric space and let $F : X^n \to X$ and $g : X \to X$ be two mappings such that $F(X^n) \subseteq g(X)$ and $g$ is continuous and commutes with $F$. Suppose that exist two functions $\psi \in \mathcal{F}_{\text{alt}}$ and $\phi \in \mathcal{F}'_{\text{alt}}$ such that, for all $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$,*

$$\begin{aligned}\max_{1\le i\le N} & \psi\left(G\left(F(\hat{x}_i^n), F(\hat{y}_i^n), F(\hat{y}_i^n)\right)\right)\\ &\le (\psi - \phi)\left(\max_{1\le i\le N} G(gx_i, gy_i, gy_i)\right).\end{aligned} \tag{11.95}$$

*Then $F$ and $g$ have a unique common cyclic fixed point $(\omega_1, \omega_2, \ldots, \omega_n) \in X^n$, that is, a point satisfying*

$$F(\omega_i, \omega_{i+1}, \ldots, \omega_n, \omega_1, \omega_2, \ldots, \omega_{i-1}) = g\omega_i = \omega_i.$$

*for all $i \in \{1, 2, \ldots, n\}$.*

*Proof.* From Corollary 10.3.1, consider on $X^n$ the $G^*$-metric given by

$$G_n(\mathsf{X}, \mathsf{Y}, \mathsf{Z}) = \max_{1 \le i \le N} G(x_i, y_i, z_i)$$

for all $\mathsf{X} = (x_1, x_2, \ldots, x_n), \mathsf{Y} = (y_1, y_2, \ldots, y_n), \mathsf{Z} = (z_1, z_2, \ldots, z_n) \in X^n$. Define $\mathcal{H}, \mathcal{G} : X^n \to X^n$ by

$$\begin{aligned} \mathcal{H}(x_1, x_2, \ldots, x_n) &= \left(F(\hat{x}_1^n), F(\hat{x}_2^n), \ldots, F(\hat{x}_n^n)\right) \quad \text{and} \\ \mathcal{G}(x_1, x_2, \ldots, x_n) &= (gx_1, gx_2, \ldots, gx_n) \end{aligned}$$

for all $(x_1, x_2, \ldots, x_n) \in X^n$. Then $\mathcal{G}$ is $G_n$-continuous and it commutes with $\mathcal{H}$ because

$$\begin{aligned} F(\widehat{gx_i}^n) &= F(gx_i, gx_{i+1}, \ldots, gx_n, gx_1, \ldots, gx_{i-1}) \\ &= gF(x_i, x_{i+1}, \ldots, x_n, x_1, \ldots, x_{i-1}) = gF(\hat{x}_i^n) \end{aligned}$$

for all $i \in \{1, 2, \ldots, n\}$. Then

$$\begin{aligned} \mathcal{H}\mathcal{G}(x_1, x_2, \ldots, x_n) &= \mathcal{H}(gx_1, gx_2, \ldots, gx_n) \\ &= (F(\widehat{gx_1}^n), F(\widehat{gx_2}^n), \ldots, F(\widehat{gx_n}^n)) \\ &= \left(gF(\hat{x}_1^n), gF(\hat{x}_2^n), \ldots, gF(\hat{x}_n^n)\right) \\ &= \mathcal{G}\left(F(\hat{x}_1^n), F(\hat{x}_2^n), \ldots, F(\hat{x}_n^n)\right) \\ &= \mathcal{G}\mathcal{H}(x_1, x_2, \ldots, x_n). \end{aligned}$$

The contractivity condition (11.95) means that, for all $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$,

$$\begin{aligned} &G_n(\mathcal{H}(x_1, x_2, \ldots, x_n), \mathcal{H}(y_1, y_2, \ldots, y_n), \mathcal{H}(y_1, y_2, \ldots, y_n)) \\ &\quad = G_n\Big(\left(F(\hat{x}_1^n), F(\hat{x}_2^n), \ldots, F(\hat{x}_n^n)\right), \left(F(\hat{y}_1^n), F(\hat{y}_2^n), \ldots, F(\hat{y}_n^n)\right) \\ &\qquad\qquad \left(F(\hat{y}_1^n), F(\hat{y}_2^n), \ldots, F(\hat{y}_n^n)\right)\Big) \\ &\quad = \max_{1 \le i \le N} \psi\left(G\left(F(\hat{x}_i^n), F(\hat{y}_i^n), F(\hat{y}_i^n)\right)\right) \end{aligned}$$

$$\leq (\psi - \phi) \left( \max_{1 \leq i \leq N} G(gx_i, gy_i, gy_i) \right)$$
$$\leq (\psi - \phi) \left( G_n \left( \mathcal{G}(x_1, x_2, \ldots, x_n), \mathcal{G}(y_1, y_2, \ldots, y_n), \mathcal{G}(y_1, y_2, \ldots, y_n) \right) \right).$$

Applying Theorem 4.3.2 to $\mathcal{H}$ and $\mathcal{G}$ in $(X^n, G_n)$ (which is also valid for $G^*$-metric spaces), we deduce that $\mathcal{H}$ and $\mathcal{G}$ have a unique common fixed point $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in X^n$, that is, a point satisfying $\omega = \mathcal{H}\omega = \mathcal{G}\omega$. This condition shows that

$$(\omega_1, \omega_2, \ldots, \omega_n) = (g\omega_1, g\omega_2, \ldots, g\omega_n)$$
$$= \left( F(\hat{\omega}_1^n), F(\hat{\omega}_2^n), \ldots, F(\hat{\omega}_n^n) \right),$$

that is,

$$F(\hat{\omega}_i^n) = g\omega_i = \omega_i \quad \text{for all } i \in \{1, 2, \ldots, n\}.$$

Hence, $(\omega_1, \omega_2, \ldots, \omega_n)$ is the unique common cyclic fixed point of $F$ and $g$. □

**Corollary 11.9.1.** *Let $(X, G)$ be a complete G-metric space and let $F : X^n \to X$ and $g : X \to X$ be two mappings such that $F(X^n) \subseteq g(X)$ and $g$ is continuous and commutes with $F$. Suppose that exists a constant $\lambda \in [0, 1)$ such that, for all $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$,*

$$\max_{1 \leq i \leq N} G\left( F(\hat{x}_i^n), F(\hat{y}_i^n), F(\hat{y}_i^n) \right) \leq \lambda \max_{1 \leq i \leq N} G(gx_i, gy_i, gy_i).$$

*Then $F$ and $g$ have a unique common cyclic fixed point $(\omega_1, \omega_2, \ldots, \omega_n) \in X^n$, that is, a point satisfying*

$$F(\omega_i, \omega_{i+1}, \ldots, \omega_n, \omega_1, \omega_2, \ldots, \omega_{i-1}) = g\omega_i = \omega_i.$$

*for all $i \in \{1, 2, \ldots, n\}$.*

# Chapter 12
# Recent Motivating Fixed Point Theory

In this chapter, we present some recent fixed/coincidence point results. They show some current research, thoughts and directions on fixed point theory in metric type spaces. However, in order not to enlarge the present book we will not include their proofs. We give the references so that the interested reader can find the proofs.

## 12.1 Some Almost Generalized $(\psi, \phi)$-Contractions in $G$-Metric Spaces

In [28], Aydi, Amor and Karapınar proved the following results. Let $(X, G)$ be a $G$-metric space. First, we consider the following expressions:

$$M(x, y, z) = \max\big\{G(x, Tx, y), G(y, T^2x, Ty), G(Tx, T^2x, Ty), G(x, Tx, z)\\ G(z, T^2x, Tz), G(Tx, T^2x, Tz), G(x, y, z)\big\}$$

and

$$N(x, y, z) = \min\{G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\\ G(z, Tx, Tx), G(y, Tz, Tz)\}$$

for all $x, y, z \in X$.

**Theorem 12.1.1.** *Let $(X, G)$ be a complete $G$-metric space. Let $T : X \to X$ be a self-mapping. Suppose there exist $\psi \in \mathcal{F}_{\mathrm{alt}}$, $\phi \in \mathcal{F}'_{\mathrm{alt}}$ and $L \geq 0$ such that, for all $x, y, z \in X$,*

$$\psi(G(Tx, Ty, Tz)) \leq \psi(M(x, y, z)) - \phi(M(x, y, z)) + LN(x, y, z).$$

*Then $T$ has a unique fixed point.*



R.P. Agarwal et al., *Fixed Point Theory in Metric Type Spaces*,
DOI 10.1007/978-3-319-24082-4_12

In the next result, the authors used the following notation: for all $x, y \in X$,

$$M^*(x,y) = \max\left\{G(x,Tx,y), G(y,T^2x,Ty), G(Tx,T^2x,Ty),\right.$$
$$\left. G(x,Tx,Tx), G(Tx,T^2x,T^2x), G(x,y,Tx)\right\}$$

and

$$N^*(x,y) = \min\{G(x,Tx,Tx), G(y,Ty,Ty), G(y,Tx,Tx)\}.$$

**Theorem 12.1.2.** *Let $(X,G)$ be a complete G-metric space. Let $T : X \to X$ be a self-mapping. Suppose there exist $\psi \in \mathcal{F}_{\text{alt}}$, $\phi \in \mathcal{F}'_{\text{alt}}$ and $L \geq 0$ such that, for all $x, y \in X$,*

$$\psi\left(G\left(Tx,Ty,T^2x\right)\right) \leq \psi\left(M^*(x,y)\right) - \phi\left(M^*(x,y)\right) + LN^*(x,y).$$

*Then $T$ has a unique fixed point.*

## 12.2 Common Fixed Point for Two Pairs of Mappings Satisfying the *(E.A)* Property in *G*-Metric Spaces

In [129], Long, Abbas, Nazir and Radenović proved the following result, where they used the notion of weak compatibility introduced in Definition 6.4.1 and the concept of the point of coincidence given in Definition 6.4.2.

**Theorem 12.2.1.** *Let $X$ be a G-metric space and $f, g, S, T : X \to X$ be mappings with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$ such that*

$$\psi(G(fx,gy,gy)) \leq \psi(M(x,y,y)) - \phi(M(x,y,y))$$

*where*

$$M(x,y,y) = \max\left\{G(Sx,Ty,Ty), G(fx,Sx,Sx), G(Ty,gy,gy),\right.$$
$$\left.\frac{G(fx,Ty,Ty) + G(Sx,gy,gy)}{2}\right\},$$

*or*

$$\psi(G(fx,fx,gy)) \leq \psi(M(x,x,y)) - \phi(M(x,x,y))$$

*where*

$$M(x,x,y) = \max\left\{G(Sx,Sx,Ty), G(fx,fx,Sx), G(Ty,Ty,gy),\right.$$
$$\left.\frac{G(fx,fx,Ty) + G(Sx,Sx,gy)}{2}\right\},$$

*hold for all $x, y \in X$, where $\psi \in \mathcal{F}_{\mathrm{alt}}$ and $\phi \in \mathcal{F}'_{\mathrm{alt}}$. Suppose that one of the pairs $(f, S)$ and $(g, T)$ satisfies the* (E.A) *property and one of the subspace $f(X)$, $g(X)$, $S(X)$, $T(X)$ is closed in $X$. Assume that for every sequence $\{y_n\}$ in $X$, at least one of the following conditions holds:*

$(a)$ *$\{gy_n\}$ is bounded in the case when $(f, S)$ satisfies* (E.A) *property,*
$(b)$ *$\{fy_n\}$ is bounded in the case when $(g, T)$ satisfies* (E.A) *property.*

*Then, the pairs $(f, S)$ and $(g, T)$ have a common point of coincidence in $X$. Moreover, if the pairs $(f, S)$ and $(g, T)$ are weakly compatible, then $f$, $g$, $S$ and $T$ have a unique common fixed point.*

## 12.3 Coincidence Point Results Using Six Mappings

In [87], Gu and Yang proved the following result.

**Theorem 12.3.1.** *Let $(X, G)$ be a complete $G$-metric space, and let $f$, $g$, $h$, $A$, $B$ and $C$ be six mappings of $X$ into itself satisfying the following conditions:*

**(i)** *$f(X) \subseteq B(X)$, $g(X) \subseteq C(X)$, $h(X) \subseteq A(X)$;*
**(ii)** *For all $x, y, z \in X$,*

$$G(fx, gy, hz) \le \lambda \,\max\begin{pmatrix} G(Ax, By, Cz),\; G(Ax, fx, fx), \\ G(By, gy, gy),\; G(Cz, hz, hz), \\ G(Ax, gy, gy),\; G(Ax, hz, hz), \\ G(By, fx, fx),\; G(By, hz, hz), \\ G(Cz, fx, fx),\; G(Cz, gy, gy) \end{pmatrix}$$

*or*

$$G(fx, gy, hz) \le \lambda \,\max\begin{pmatrix} G(Ax, By, Cz),\; G(Ax, Ax, fx), \\ G(By, By, gy),\; G(Cz, Cz, hz), \\ G(Ax, Ax, gy),\; G(Ax, Ax, hz), \\ G(By, By, fx),\; G(By, By, hz), \\ G(Cz, Cz, fx),\; G(Cz, Cz, gy) \end{pmatrix}$$

*where $\lambda \in \left[0, \frac{1}{2}\right)$. If one of the following conditions is satisfied:*

**(a)** *Either $f$ or $A$ is $G$-continuous, the pair $(f, A)$ is weakly commuting, the pairs $(g, B)$ and $(h, C)$ are weakly compatible;*
**(b)** *Either $g$ or $B$ is $G$-continuous, the pair $(g, B)$ is weakly commuting, the pairs $(f, A)$ and $(h, C)$ are weakly compatible;*
**(c)** *Either $h$ or $C$ is $G$-continuous, the pair $(h, C)$ is weakly commuting, the pairs $(f, A)$ and $(g, B)$ are weakly compatible;*

*then*

**(I)** *one of the pairs* $(f, A)$, $(g, B)$ *and* $(h, C)$ *has a coincidence point in* $X$*;*
**(II)** *the mappings* $f$, $g$, $h$, $A$, $B$ *and* $C$ *have a unique common fixed point in* $X$.

In [86], Gu and Shatanawi introduced the following notion.

**Definition 12.3.1.** Let $(X, G)$ be a $G$-metric space and let $f, g, h, R, S, T : X \to X$ be six mappings. We say that the triple $(f, g, h)$ is a *generalized weakly G-contraction mapping of type A* with respect to the triple $(R, S, T)$ if for all $x, y, z \in X$, the following inequality holds:

$$\psi\left(G\left(fx, gy, hz\right)\right) \leq \psi\left(\frac{G(Rx, gy, gy) + G(Sy, hz, hz) + G(Tz, fx, fx)}{3}\right)$$
$$- \phi\left(G(Rx, gy, gy), G(Sy, hz, hz), G(Tz, fx, fx)\right),$$

where $\psi \in \mathcal{F}_{\text{alt}}$ and $\phi : [0, \infty)^3 \to [0, \infty)$ is a continuous function with $\phi\left(t, s, u\right) = 0$ if, and only if, $t = s = u = 0$.

**Definition 12.3.2.** Let $(X, G)$ be a $G$-metric space and let $A, B, S, T : X \to X$ be four self-maps on $X$. The pairs $(A, S)$ and $(B, T)$ are said to satisfy the *common (E.A) property* if there exist $t \in X$ and two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t.$$

**Theorem 12.3.2.** *Let* $(X, G)$ *be a G-metric space and let* $f, g, h, R, S, T : X \to X$ *be six mappings such that* $(f, g, h)$ *is a generalized weakly G-contraction mapping of type* A *with respect to* $(R, S, T)$*. If one of the following conditions is satisfied, then the pairs* $(f, R)$*,* $(g, S)$ *and* $(h, T)$ *have a common point of coincidence in* $X$*.*

**(i)** *The subspace RX is closed in X,* $f(X) \subseteq S(X)$*,* $g(X) \subseteq T(X)$*, and two pairs of* $(f, R)$ *and* $(g, S)$*satisfy the common* (E.A) *property;*
**(ii)** *The subspace SX is closed in X,* $g(X) \subseteq T(X)$*,* $h(X) \subseteq R(X)$*, and two pairs of* $(g, S)$ *and* $(h, T)$ *satisfy the common* (E.A) *property;*
**(iii)** *The subspace TX is closed in X,* $f(X) \subseteq S(X)$*,* $h(X) \subseteq R(X)$*, and two pairs of* $(f, R)$ *and* $(h, T)$ *satisfy the common* (E.A) *property.*

*Moreover, if the pairs* $(f, R)$*,* $(g, S)$ *and* $(h, T)$ *are weakly compatible, then* $f$*,* $g$*,* $h$*,* $R$*,* $S$ *and* $T$ *have a unique common fixed point in* $X$.

## 12.4 Common Fixed Point Theorems of Altman Integral Type Mappings in *G*-Metric Spaces

In 1975, Altman [22] proved a fixed point theorem for a mapping $T : X \to X$ which satisfies the condition

$$d\left(Tx, Ty\right) \leq Q\left(d\left(x, y\right)\right) \quad \text{for all } x, y \in X,$$

where $Q : [0, \infty) \to [0, \infty)$ is an increasing function satisfying the following conditions:

$(i)$ $0 < Q(t) < t$ for all $t \in (0, \infty)$;
$(ii)$ $p(t) = t/(t - Q(t))$ is a decreasing function;
$(iii)$ for some positive number $t_1$, there holds

$$\int_0^{t_1} p(t)\, dt < \infty.$$

From condition $(i)$ and the fact that $Q$ is increasing, we observe that $Q(0) = 0$ and $Q(t) = t$ if, and only if, $t = 0$. In [88], Gu and Ye introduced the following result. We denote by $\phi : [0, \infty) \to [0, \infty)$ a function satisfying $0 < \phi(t) < t$ for all $t > 0$.

**Theorem 12.4.1.** *Let $(X, G)$ be a complete G-metric space and let $S$, $T$, $R$, $f$, $g$, and $h$ be six mappings of $X$ into itself. If there exists an increasing function $Q : [0, \infty) \to [0, \infty)$ satisfying the conditions $(i)$, $(ii)$ and $(iii)$, and the following conditions:*

$(iv)$ $S(X) \subseteq g(X)$, $T(X) \subseteq h(X)$, $R(X) \subseteq f(X)$,
$(v)$ *for all $x, y, z \in X$,*

$$\int_0^{G(Sx,Ty,Rz)} \delta(t)\, dt \le \phi\left(\int_0^{Q(G(fx,gy,hz))} \delta(t)\, dt\right),$$

*where $\delta(t)$ is a Lebesgue integrable function which is summable nonnegative such that*

$$\int_0^{\varepsilon} \delta(t)\, dt > 0 \quad \textit{for all } \varepsilon > 0.$$

*Then*

$(a)$ *one of the pairs $(S, f)$, $(T, g)$ and $(R, h)$ has a coincidence point in $X$,*
$(b)$ *if $(S, f)$, $(T, g)$ and $(R, h)$ are three pairs of continuous $\phi$-weakly commuting mappings, then the mappings $S$, $T$, $R$, $f$, $g$ and $h$ have a unique common fixed point in $X$.*

# Appendix A
# Some Basic Definitions and Results in Metric Spaces

## A.1 First Results in Partially Ordered Metric Spaces

A function $\psi : [0,\infty) \longrightarrow [0,\infty)$ is upper semi-continuous provided that for each $t \geq 0$ and each sequence $\{t_n\}_{n\in\mathbb{N}}$ such that if $\lim_{n\to\infty} t_n = t$, it follows that

$$\limsup_{n\to\infty} \psi(t_n) \leq \psi(t).$$

A function $\psi : [0,\infty) \longrightarrow [0,\infty)$ is lower semi-continuous provided that for each $t \geq 0$ and each sequence $\{t_n\}_{n\in\mathbb{N}}$ such that if $\lim_{n\to\infty} t_n = t$, it follows that

$$\liminf_{n\to\infty} \psi(t_n) \geq \psi(t).$$

Let $X, Y$ be non empty sets and $T : X \to Y$ a given mapping: $T$ is said surjective (or onto) if for all $y \in Y$, there exists $x \in X$ such that $Tx = y$.
$T$ is said injective (or one to one) if for some $x, y \in X$ such that $Tx = Ty$, then $x = y$. For $x \in X$, define $T^2x = T(Tx)$. Inductively, we define for $m \geq 3$, $T^m(x) = T(T^{m-1}x)$.

**Theorem A.1.1 (Ran and Reurings [168]).** *Let $(X, \preccurlyeq)$ be an ordered set endowed with a metric $d$ and let $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

*(i)* $(X, d)$ *is complete;*
*(ii)* *$T$ is continuous and non-decreasing (with respect to $\preccurlyeq$);*
*(iii)* *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;*
*(iv)* *there exists a constant $\lambda \in (0,1)$ such that for all $x, y \in X$ with $x \succcurlyeq y$,*

$$d(Tx, Ty) \leq \lambda\, d(x, y).$$



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*Then $T$ has a fixed point. Moreover, if for all $(x, y) \in X \times X$ there exists $z \in X$ such that $x \preccurlyeq z$ and $y \preccurlyeq z$, we obtain uniqueness of the fixed point.*

**Definition A.1.1 ([84]).** An ordered metric space $(X, d, \preccurlyeq)$ is said to be *non-decreasing-regular* (respectively, *non-increasing-regular*) if for all sequence $\{x_m\} \subseteq X$ such that $\{x_m\} \to x$ and $x_m \preccurlyeq x_{m+1}$ (respectively, $x_m \succcurlyeq x_{m+1}$) for all $m$, we have that $x_m \preccurlyeq x$ (respectively, $x_m \succcurlyeq x$) for all $m$. Also $(X, d, \preccurlyeq)$ is said to be *regular* if it is both non-decreasing-regular and non-increasing-regular.

**Theorem A.1.2 (Nieto and Rodríguez-López [158]).** *Let $(X, \preccurlyeq)$ be an ordered set endowed with a metric $d$ and $T : X \to X$ be a mapping. Suppose that the following conditions hold:*

*(i)* $(X, d)$ *is complete;*
*(ii)* $(X, d, \preccurlyeq)$ *is non-decreasing-regular;*
*(iii)* *$T$ is $\preccurlyeq$-non-decreasing;*
*(iv)* *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;*
*(v)* *there exists a constant $\lambda \in (0, 1)$ such that for all $x, y \in X$ with $x \succcurlyeq y$,*

$$d(Tx, Ty) \leq \lambda\, d(x, y).$$

*Then $T$ has a fixed point. Moreover, if for all $(x, y) \in X \times X$ there exists $w \in X$ such that $x \preccurlyeq w$ and $y \preccurlyeq w$, we obtain uniqueness of the fixed point.*

## A.2 $\alpha$ - $\psi$ - Contractive Mappings on Metric Spaces

Recently, Samet et al. [183] introduced the following concepts.

**Definition A.2.1.** Let $(X, d)$ be a metric space and let $T : X \to X$ be a given mapping. We say that $T$ is an $\alpha$ - $\psi$ - *contractive mapping* if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \mathcal{F}_{\text{com}}^{(c)}$ such that

$$\alpha\,(x, y)\; d\,(Tx, Ty) \leq \psi\,(d\,(x, y)) \quad \text{for all } x, y \in X.$$

Clearly, any contractive mapping in the Banach sense (that is, verifying $d\,(Tx, Ty) \leq \lambda\, d\,(x, y)$) is an $\alpha$ - $\psi$ - contractive mapping with $\alpha\,(x, y) = 1$ for all $x, y \in X$ and $\psi\,(t) = \lambda t$ for all $t \geq 0$ and some $\lambda \in [0, 1)$.

In some cases, the function $\alpha$ will be intimately related with a partial order in the following sense: if $\preccurlyeq$ is a partial order on $X$, we will consider the mapping $\alpha_{\preccurlyeq} : X \times X \to X$ given, for all $x, y \in X$, by

$$\alpha_{\preccurlyeq}\,(x, y) = \begin{cases} 1, & \text{if } x \preccurlyeq y, \\ 0, & \text{otherwise.} \end{cases}$$

**Definition A.2.2.** Let $X$ be a set and let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be two mappings. We say that $T$ is $\alpha$ - *admissible* if, for all $x, y \in X$, we have

$$\alpha(x, y) \geq 1 \quad \Rightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Various examples of such mappings were presented in [183]. The main results in [183] are the following fixed point theorems.

**Theorem A.2.1.** *Let $(X, d)$ be a complete metric space and let $T : X \to X$ be an $\alpha$ - $\psi$ - contractive mapping. Suppose that*

**(i)** *$T$ is $\alpha$ - admissible;*
**(ii)** *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
**(iii)** *$T$ is continuous.*

*Then there exists $u \in X$ such that $Tu = u$.*

**Theorem A.2.2.** *Let $(X, d)$ be a complete metric space and let $T : X \to X$ be an $\alpha$ - $\psi$ - contractive mapping. Suppose that*

**(i)** *$T$ is $\alpha$ - admissible;*
**(ii)** *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
**(iii)** *if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\{x_n\} \to x \in X$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

*Then there exists $u \in X$ such that $Tu = u$.*

To obtain the uniqueness of the fixed point, an additional hypothesis can be considered.

**Theorem A.2.3.** *Under the hypotheses of Theorem A.2.1 (respectively, Theorem A.2.2), also assume the following condition:*

$(U_1)$ *For all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.*

*Then $T$ has a unique fixed point.*

Recently, Karapınar and Samet [118] introduced the following concept.

**Definition A.2.3.** Let $(X, d)$ be a metric space and let $T : X \to X$ be a given mapping. We say that $T$ is a *generalized $\alpha$ - $\psi$ - contractive mapping* if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \mathcal{F}_{\text{com}}^{(c)}$ such that

$$\alpha(x, y)\, d(Tx, Ty) \leq \psi(M(x, y)) \quad \text{for all } x, y \in X,$$

where

$$M(x, y) = \max\left\{ d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2} \right\}.$$

Clearly, since $\psi$ is non-decreasing, every $\alpha$ - $\psi$ - contractive mapping, presented in [183], is a generalized $\alpha$ - $\psi$ - contractive mapping.

**Theorem A.2.4.** *Let $(X, d)$ be a complete metric space and let $T : X \to X$ be a generalized $\alpha$ - $\psi$ - contractive mapping. Suppose that*

**(i)** *$T$ is $\alpha$ - admissible;*
**(ii)** *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
**(iii)** *$T$ is continuous.*

*Then there exists $u \in X$ such that $Tu = u$.*

**Theorem A.2.5.** *Let $(X, d)$ be a complete metric space and let $T : X \to X$ be a generalized $\alpha$ - $\psi$ - contractive mapping. Suppose that*

**(i)** *$T$ is $\alpha$ - admissible;*
**(ii)** *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
**(iii)** *if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\{x_n\} \to x \in X$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

*Then there exists $u \in X$ such that $Tu = u$.*

**Theorem A.2.6.** *Under the hypotheses of Theorem A.2.4 (respectively, Theorem A.2.5), also assume the following condition:*

$(U_2)$ *For all $x, y \in \mathrm{Fix}(T)$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.*

*Then $T$ has a unique fixed point.*

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