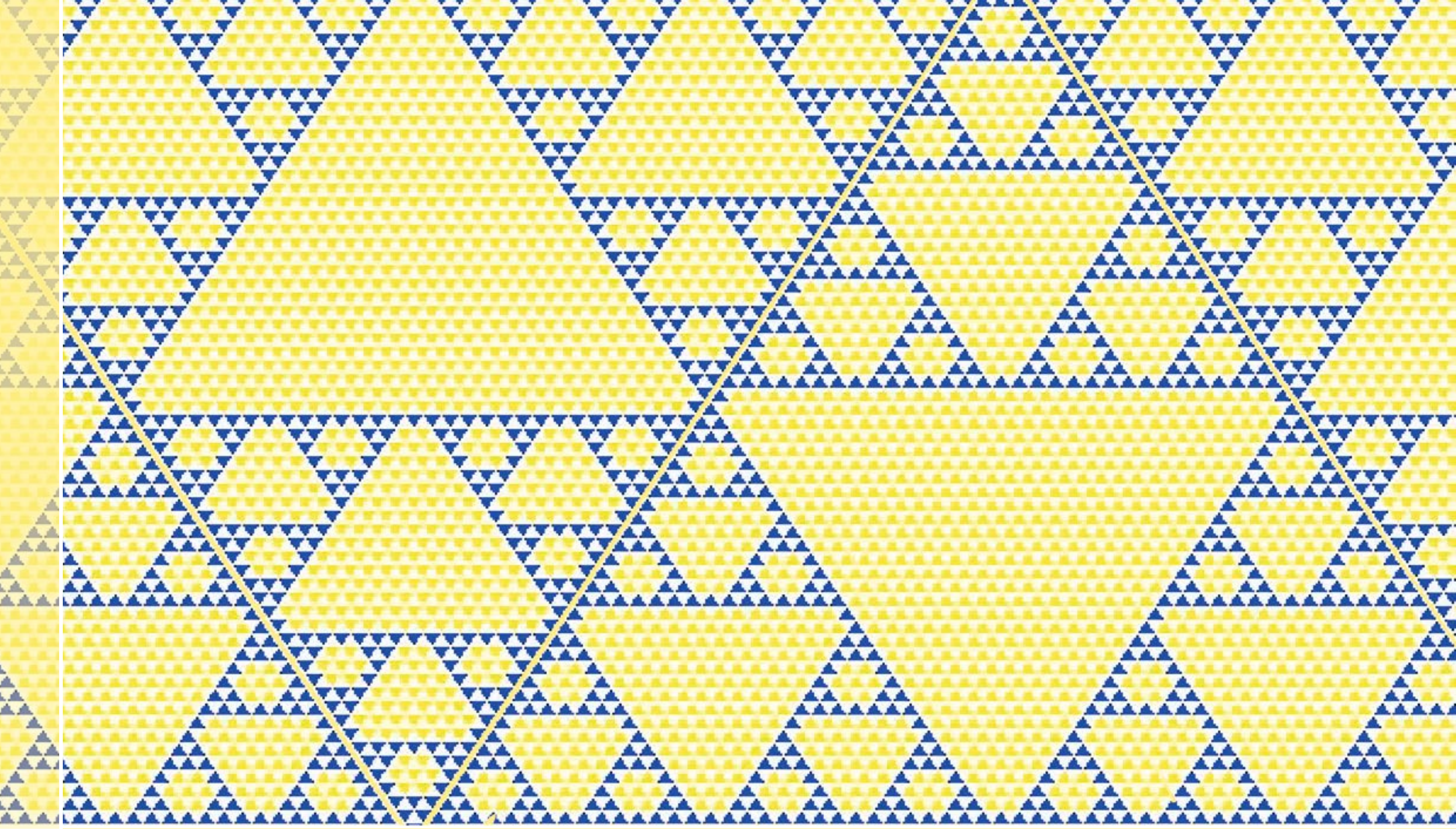

Thomas Koshy

Pell and Pell–Lucas Numbers with Applications

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Dedicated to

A good friend, colleague, an excellent problem-proposer and solver, and a Fibonacci and Pell enthusiast, Thomas Eugene Moore (1944–2014) of Bridgewater State University, Bridgewater, Massachusetts

Preface

A thing of beauty is a joy for ever:
Its loveliness increases;
It will never pass into nothingness.
– John Keats (1777–1855)

Two New Bright Stars

Like the well-known Fibonacci and Lucas numbers, Pell and Pell–Lucas numbers are two spectacularly bright stars on the mathematical firmament. They too continue to amaze the mathematical community with their splendid beauty, ubiquity, and applicability, providing delightful opportunities to experiment, explore, conjecture, and problem-solve. Pell and Pell–Lucas numbers form a unifying thread intertwining analysis, geometry, trigonometry, and various areas of discrete mathematics, such as number theory, graph theory, linear algebra, and combinatorics. They belong to an extended Fibonacci family, and are a powerful tool for extracting numerous interesting properties of a vast array of number sequences. Both families share numerous fascinating properties.

A First in the Field

Pell and Pell–Lucas numbers and their delightful applications appear widely in the literature, but unfortunately they are scattered throughout a multitude of periodicals. As a result, they remain out of reach of many mathematicians and amateurs. This vacuum inspired me to create this book, the first attempt to collect, organize, and present information about these integer families in a systematic and enjoyable fashion. It is my hope that this unique undertaking will offer a thorough introduction to one of the most delightful topics in discrete mathematics.

Audience

The book is intended for undergraduate/graduate students depending on the college or university and the instructors in those institutions. It will also engage the intellectually curious high schoolers and teachers at all levels. The exposition proceeds from the basics to more advanced topics, motivating with examples and exercises in a rigorous, systematic fashion. Like the Catalan and Fibonacci books, this will be an important resource for seminars, independent study, and workshops.

The professional mathematician and computer scientist will certainly profit from the exposure to a variety of mathematical skills, such as pattern recognition, conjecturing, and problem-solving techniques.

Through my Fibonacci and Catalan books, I continue to hear from a number of enthusiasts coming from a wide variety of backgrounds and interests, who express their rewarding experiences with these books. I now encourage all Pell and Pell–Lucas readers to also communicate with me about their experiences with the Pell family.

Prerequisites

This book requires a strong foundation in precalculus mathematics; users will also need a good background in matrices, determinants, congruences, combinatorics, and calculus to enjoy most of the material.

It is my hope that the material included here will challenge both the mathematically sophisticated and the less advanced. I have included fundamental topics such as the floor and ceiling functions, summation and product notations, congruences, recursion, pattern recognition, generating functions, binomial coefficients, Pascal's triangle, binomial theorem, and Fibonacci and Lucas numbers. They are briefly summarized in Chapter 1. For an extensive discussion of these topics, refer to my *Elementary Number Theory with Applications* and *Discrete Mathematics with Applications*.

Historical Background

The personalities and history behind the mathematics make up an important part of this book. The study of Pell's equation, continued fractions, and square-triangular numbers lead into the study of the Pell family in a logical and natural fashion. The book also contains an intriguing array of applications to combinatorics, graph theory, geometry, and mathematical puzzles.

It is important to note that Pell and Pell–Lucas numbers serve as a bridge linking number theory, combinatorics, graph theory, geometry, trigonometry, and analysis. These numbers occur, for example, in the study of lattice walks, and the tilings of linear and circular boards using unit square tiles and dominoes.

Pascal's Triangle and the Pell Family

It is well known that Fibonacci and Lucas numbers can be read directly from Pascal's triangle. Likewise, we can extract Pell and Pell–Lucas numbers also from Pascal's triangle, showing the close relationship between the triangular array and the Pell family.

A New Hybrid Family

The closely-related Pell and Fibonacci families are employed to construct a new hybrid Pell–Fibonacci family. That too is presented with historical background.

Opportunities for Exploration

Pell and Pell–Lucas numbers, like their closely related cousins, offer wonderful opportunities for high-school, undergraduate and graduate students to enjoy the beauty and power of mathematics, especially number theory. These families can extend a student's mathematical horizons, and offer new, intriguing, and challenging problems. To faculty and researchers, they offer the chance to explore new applications and properties, and to advance the frontiers of mathematical knowledge.

Most of the chapters end in a carefully prepared set of exercises. They provide opportunities for establishing number-theoretic properties and enhancement of problem-solving skills. Starred exercises indicate a certain degree of difficulty. Answers to all exercises can be obtained electronically from the publisher.

Symbols and Abbreviations

For quick reference, a list of symbols and a glossary of abbreviations is included. The symbols index lists symbols used, and their meanings. Likewise, the abbreviations list provides a gloss for the abbreviations used for brevity, and their meanings.

Salient Features

The salient features of the book include extensive and in-depth coverage; user-friendly approach; informal and non-intimidating style; plethora of interesting applications and properties; historical context, including the name and affiliation of every discoverer, and year of discovery; harmonious linkage with Pascal's triangle, Fibonacci and Lucas numbers, Pell's equation, continued fractions, square-triangular, pentagonal, and hexagonal numbers; trigonometry and complex numbers; Chebyshev polynomials and tilings; and the introduction of the brand-new Pell–Fibonacci hybrid family.

Acknowledgments

In undertaking this extensive project, I have immensely benefited from over 250 sources, a list of which can be found in the *References*. Although the information compiled here does not, of course, exhaust all applications and occurrences of the Pell family, these sources provide, to the best of my ability, a reasonable sampling of important contributions to the field.

I have immensely benefited from the constructive suggestions, comments, support, and cooperation from a number of well-wishers. To begin with, I am greatly indebted to the reviewers for their great enthusiasm and suggestions for improving drafts of the original version. I am also grateful to Steven M. Bairos of Data Translation, Inc. for his valuable comments on some early chapters of the book; to Margarite Landry for her superb editorial assistance and patience; to Jeff Gao for creating the Pascal's binary triangle in Figure 5.6, preparing the Pell, Pell–Lucas, Fibonacci, and Lucas tables in the Appendix, and for co-authoring with me several articles on the topic; to Ann Kostant, Consultant and Senior Advisor at Springer for her boundless enthusiasm and support for the project; and to Elizabeth Loew, Senior Editor at Springer along with her Springer staff for their dedication, cooperation, and interaction with production to publish the book in a timely fashion.

Framingham, Massachusetts
August, 2014

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If I have been able to see farther, it was only
because I stood on the shoulders of giants.
– Sir Isaac Newton (1643–1727)

Contents

List of Symbols

Symbol	Meaning
$\lfloor x \rfloor$	greatest integer $\le x$
$\lceil x \rceil$	least integer $\ge x$
■	end of an example, and for statements of theorems, lemmas and corollaries without proofs.
$\sum_{i=k}^{i=m} a_i = \sum_{i=k}^{m} a_i$	$a_k + a_{k+1} + \cdots + a_m$
$\prod_{i=k}^{i=m} a_i = \prod_{i=k}^{m} a_i$	$a_k a_{k+1} \cdots a_m$
$a \mid b$	a is a factor of b
$a \equiv b \pmod{m}$	a is congruent to b
$a \not\equiv b \pmod{m}$	a is not congruent to b
$n!$	$n(n-1) \cdot 3 \cdot 2 \cdot 1$
$\mathbf{W}$	set of whole numbers $0, 1, 2, \ldots$
$(a_1, a_2, \ldots, a_n)$	greatest common factor of $a_1, a_2, \ldots, a_n$
M_n	nth Mersenne number $2^n - 1$
$(a_1 a_2 \cdots a_n)_{\text{two}}$	binary number $a_1 a_2 \cdots a_n$
$\binom{n}{r}$	binomial coefficient
F_n	nth Fibonacci number
L_n	nth Lucas number
α	$(1 + \sqrt{5})/2$
β	$(1 - \sqrt{5})/2$
P_n	nth Pell number
L_n	nth Pell–Lucas number
γ	$1 + \sqrt{2}$
δ	$1 - \sqrt{2}$
$\lvert x \rvert$	absolute value of real number x
$A = (a_{ij})_{m \times n}$	matrix A of size m by n
I_n	identity matrix of size n by n
A^{-1}	inverse of square matrix A
$\lvert A \rvert$	determinant of square matrix A
$\lg x$	$\log_2 x$
$x^2 - dy^2 = 1$	Pell's equation
$x^2 - dy^2 = (-1)^n$	Pell's equation

Symbol	Meaning
$u = x + y\sqrt{d}$	quadratic surd
$\overline{u}$	$x - y\sqrt{d}$
$N(u)$	$x^2 - dy^2$
$\approx$	is approximately equal to
$\{a_n\}$	sequence with nth term a_n
$\overline{AB}$	line segment AB
$x^2 - dy^2 = k$	Pell's equation
$[a_0; a_1, a_2, \ldots, a_n]$	finite simple continued fraction
C_k	kth convergent
$[a_0; \overline{a_1, a_2, \ldots, a_n}]$	infinite simple continued fraction
x-y-z	Pythagorean triple
t_n	nth triangular number
$\boxed{?}$	unsolved problem
$\rho(N)$	digital root of N
$(\frac{a}{m})$	Jacobi symbol
p_n	nth Pentagonal number
λ	eigenvalue
$a \nmid b$	a is not a factor of b
$[a, b]$	least common multiple of a and b
i	$\sqrt{-1}$
$p_n(x)$	nth Pell polynomial in x
$q_n(x)$	nth Pell–Lucas polynomial in x
$\alpha(x)$	$(x + \sqrt{x^2 + 4})/2$
$\beta(x)$	$(x - \sqrt{x^2 + 4})/2$
$\gamma(x)$	$x + \sqrt{x^2 + 1}$
$\delta(x)$	$x - \sqrt{x^2 + 1}$
$T_n(x)$	nth Chebyshev polynomial of the first kind
$U_n(x)$	nth Chebyshev polynomial of the second kind

Abbreviations

Abbreviation	Meaning
LHRWCCs	linear homogeneous recurrence with constant coefficients
LNHRWCCs	linear nonhomogeneous recurrence with constant coefficients
RHS	right-hand side
LHS	left-hand side
PMI	principle of mathematical induction
rms	root-mean-square
AIME	American Invitational Mathematics Examinations
FSCF	finite simple continued fraction
LDE	linear diophantine equation
ISCF	infinite simple continued fraction

1

Fundamentals

1.1 Introduction

There is a vast array of integer sequences, many of which display interesting patterns and a number of fascinating properties. Quite a few of them are within reach of high school students and certainly number-theoretic enthusiasts. The Fibonacci and Lucas sequences, for example, are two of the most popular and delightful number sequences. Their beauty and ubiquity continue to amaze the mathematics community.

Our main focus here is on two such families of sequences: Pell and Pell–Lucas numbers. Before we turn to them, we will briefly introduce some fundamental concepts, techniques, and notations.

We begin with the floor and ceiling functions, which appear frequently in discrete mathematics, and consequently in computer science [127].

1.2 Floor and Ceiling Functions

The *floor* of a real number x, denoted by $\lfloor x \rfloor$, is the greatest integer $\leq x$; and the *ceiling* of x, denoted by $\lceil x \rceil$, is the least integer $\geq x$. The functions f and g, defined by $f(x) = \lfloor x \rfloor$ and $g(x) = \lceil x \rceil$, are the *floor* and *ceiling functions*, respectively. They are also called the *greatest integer function* and *least integer function*, respectively.

For example, $\lfloor -3.45 \rfloor = -4$, $\lfloor 3.45 \rfloor = 3$, $\lceil -3.45 \rceil = -3$, and $\lceil 3.45 \rceil = 4$.

The notations $\lfloor x \rfloor$ and $\lceil x \rceil$, and the names *floor* and *ceiling* were introduced in the early 1960s by the Canadian mathematician Kenneth Eugene Iverson (1920–2004). Both notations are slight variations of the original greatest integer notation $[x]$.

Both functions satisfy a number of properties. A few of them are listed in the following theorem. Their proofs are basic, so we omit them in the interest of brevity.

T. Koshy, *Pell and Pell–Lucas Numbers with Applications*, DOI 10.1007/978-1-4614-8489-9_1,

Theorem 1.1 *Let x be any real number and n any integer. Then*

(1) $\lfloor n \rfloor = n = \lceil n \rceil$ (2) $\lceil x \rceil = \lfloor x \rfloor + 1$, *where x is* not *an integer.*

(3) $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ (4) $\lceil x + n \rceil = \lceil x \rceil + n$

(5) $\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil = n$ (6) $\lceil x \rceil = -\lfloor -x \rfloor$. ■

Next we turn to the popular summation and product notations used throughout this book.

1.3 Summation Notation

Sums, such as $a_k + a_{k+1} + \cdots + a_m$, occur very often in mathematics. They can be rewritten in a concise form using the *summation symbol* Σ (uppercase Greek letter *sigma*):

$$\sum_{i=k}^{i=m} a_i = a_k + a_{k+1} + \cdots + a_m.$$

The summation notation was introduced in 1772 by the French mathematician Joseph Louis Lagrange (1736–1813).

The variable i is the *summation index*; and the values k and m are the *lower* and *upper limits*, respectively, of the index i. The "$i =$" above the $\sum$ is often omitted. Thus $\sum_{i=k}^{i=m} a_i = \sum_{i=k}^{m} a_i$.

The index i is a *dummy variable*; its choice has absolutely *no* bearing on the sum. For example, $\sum_{i=k}^{m} a_i = \sum_{r=k}^{m} a_r = \sum_{s=k}^{m} a_s$.

The following theorem lists some results useful in evaluating finite sums. They can be established using the principle of mathematical induction (PMI).

Theorem 1.2 *Let n be any positive integer, c any real number; and $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ any two number sequences. Then*

(1) $\sum_{i=1}^{n} c = nc$ (2) $\sum_{i=1}^{n} (ca_i) = c\left(\sum_{i=1}^{n} a_i\right)$

(3) $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$ (4) $\sum_{i=1}^{n} a_i = \sum_{i=1}^{k} a_i + \sum_{i=k+1}^{n} a_i$

(5) $\sum_{i=1}^{n} a_i = \sum_{j=k}^{n+k-1} a_{j-k+1}$ (6) $\sum_{i=1}^{n} \left(\sum_{j=1}^{i} a_{ij}\right) = \sum_{j=1}^{n} \left(\sum_{i=j}^{n} a_{ij}\right) = \sum_{1 \le j \le i \le n} a_{ij}$.

These results remain true for any integral lower limit. ■

The following summation formulas will come in handy in our later discussions. They too can be confirmed using PMI.

(1) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(2) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(3) $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

(4) $\sum_{i=1}^{n} ar^{i-1} = \frac{a(r^n-1)}{r-1}, \quad r \neq 1.$

Next we turn to product notation.

1.4 Product Notation

Using the *product symbol* $\prod$ (uppercase Greek letter *pi*), the product $a_k a_{k+1} \cdots a_m$ is written as $\prod_{i=k}^{i=m} a_i$. Thus $\prod_{i=k}^{i=m} a_i = a_k a_{k+1} \cdots a_m$. As before, the $''i\ ='' $ above the $\prod$ is often omitted. Thus $\prod_{i=k}^{i=m} a_i = \prod_{i=k}^{m} a_i$. Again, i is a dummy variable.

Next we present one of the most powerful and useful relations in mathematics: the *congruence relation*. The theory of congruences [130] was introduced in 1801 by the outstanding German mathematician Karl Friedrich Gauss (1777–1855), popularly known as the "prince of mathematics." It has marvelous applications to discrete mathematics and consequently to computer science [129].

1.5 Congruences

Let m be an integer ≥ 2. Then an integer a is *congruent to* an integer b *modulo* m if $m \mid (a-b)$. Symbolically, we then write $a \equiv b \pmod{m}$; m is the *modulus* of the congruence relation. If a is *not* congruent to b modulo m, then we write $a \not\equiv b \pmod{m}$.

For example, $6 \mid (17-5)$, so $17 \equiv 5 \pmod{6}$; since $8 \mid [12-(-4)]$, $12 \equiv -4 \pmod{8}$. Notice that $84 \equiv 0 \pmod{12}$, but $35 \not\equiv 5 \pmod{12}$.

We often use the *moduli* (plural of modulus) 7, 12, and 24 in our daily life. For example, they count the days of the week, hours on a 12-hour clock, and hours on a 24-hour clock.

The congruence relation satisfies a number of properties. Some of them are quite similar to those of the equality relation; so the use of the congruence symbol $\equiv$, introduced by Gauss, is quite suggestive. Some fundamental properties are summarized in the following theorem. We omit their proofs in the interest of brevity. But for the sake of clarity, we add:

- If r is the remainder when an integer a is divided by an integer $m \geq 2$, then, by the division algorithm, $0 \leq r < m$.
- (a, b) denotes the *greatest common divisor* (gcd) of positive integers a and b.

Theorem 1.3 *Let a, b, c, and d be arbitrary integers, n any positive integer, and m any integer ≥ 2. Let a mod m denote the remainder when a is divided by m. Then*

(1) $a \equiv a \pmod{m}$. *(Reflexive property)*
(2) *If* $a \equiv b \pmod{m}$, *then* $b \equiv a \pmod{m}$. *(Symmetric property)*
(3) *If* $a \equiv b \pmod{m}$ *and* $b \equiv c \pmod{m}$, *then* $a \equiv c \pmod{m}$. *(Transitive property)*
(4) $a \equiv b \pmod{m}$ *if and only if* $a = b + km$ *for some integer* k.
(5) $a \equiv b \pmod{m}$ *if and only if* $a \bmod m = b \bmod m$.
(6) *If* $a \equiv r \pmod{m}$ *and* $0 \leq r < m$, *then* $r = a \bmod m$.
(7) *If* $a \equiv b \pmod{m}$ *and* $c \equiv d \pmod{m}$, *then* $a + c \equiv b + d \pmod{m}$.
(8) *If* $a \equiv b \pmod{m}$ *and* $c \equiv d \pmod{m}$, *then* $ac \equiv bd \pmod{m}$
(9) *If* $a + c \equiv b + c \pmod{m}$, *then* $a \equiv b \pmod{m}$.
(10) *If* $a \equiv b \pmod{m}$, *then* $a^n \equiv b^n \pmod{m}$.
(11) *If* $ac \equiv bc \pmod{m}$ *and* $(c, m) = 1$, *then* $a \equiv b \pmod{m}$. ■

A few words about some of the properties may be useful. By property (5), if $a \equiv b \pmod{m}$ they leave the same remainder when divided by m; its converse is also true. By properties (7) and (8), two congruences can be added and multiplied, as in the case of equality. By property (9), the same number can be added to and subtracted from both sides of a congruence. Property (10) follows from (8) by PMI. Finally, by property (11), if $ac \equiv bc \pmod{m}$, we can cancel c from both sides only if $(c, m) = 1$.

Next we present *recursion*, one of the most elegant problem-solving techniques used in both discrete mathematics and computer science [127]. It may take a while to get used to recursion and to think recursively; but once you have mastered the art of thinking and solving problems recursively, you will appreciate its power and beauty.

1.6 Recursion

Suppose there are n guests at a party, where $n \geq 1$. Each person shakes hands with everybody else exactly once. We would like to define recursively the total number of handshakes $h(n)$ made by the guests. (This is the well-known *handshake problem*.)

To this end, we consider two cases:

Case 1 Suppose $n = 1$. Then $h(1) = 0$.

Case 2 Suppose $n \geq 2$; that is, there are at least two guests at the party. Let X be one of them, so there are $n - 1$ guests left at the party; see Figure 1.1. By definition, they can make $h(n - 1)$ handshakes among themselves. Now X can shake hands with each of the $n - 1$ guests; this way an additional $n - 1$ handshakes can be made.

Thus the total number of handshakes made by the n guests equals $h(n - 1) + (n - 1)$; that is, $h(n) = h(n - 1) + (n - 1)$, where $n \geq 2$.

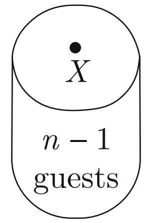

Figure 1.1.

This formula enables us to compute $h(n)$, provided we know the value of $h(n-1)$. For example, suppose $h(10) = 45$. Then $h(11) = h(10) + 10 = 45 + 10 = 55$.

The formula $h(n) = h(n-1) + (n-1)$ is called a *recurrence* (or *recursive formula*). This, coupled with the *initial condition* $h(1) = 0$, yields a recursive definition of $h(n)$:

$$\begin{aligned} h(1) &= 0 && \leftarrow \text{initial condition} \\ h(n) &= h(n-1) + (n-1), \quad n \geq 2. && \leftarrow \text{recurrence} \end{aligned}$$

Let's now explore the skeleton of the handshake problem. We are given a problem of size n (n guests at the party). In Case 2, we expressed it in terms of a smaller version (size $n-1$) of itself. So the original problem can be solved if the simpler version can be solved.

Thus $h(n)$ can be computed if we know its predecessor value $h(n-1)$. But $h(n-1)$ can be computed if we know $h(n-2)$. Continuing like this, we can compute $h(n)$ if $h(1)$ is known. But we do know the value of $h(1)$. Thus, by working backwards, we can compute $h(n)$. Such a definition is an *inductive definition*. It consists of three parts:

(1) The *basis clause* specifies some initial values $f(a), f(a+1), \ldots, f(a+k-1)$. Equations which define them are *initial conditions*.
(2) The *recursive clause* provides a formula for computing $f(n)$ using the k predecessor values $f(n-1), f(n-2), \ldots, f(n-k)$.
(3) The *terminal clause* ensures that the only valid values of f are obtained by steps 1 and 2. (The terminal clause is often omitted, for convenience.)

In general, in a *recursive definition*, the values used in the recursive clause do *not* need to be predecessor values; so a recursive definition does not need to be inductive.

The next three examples illustrate the recursive definition.

Example 1.1 Consider the familiar *factorial function* $f(n) = n!$, where $f(0) = 1$ and $n \geq 1$. When $n \geq 1$, $n! = n \cdot (n-1)!$; so $f(n) = n \cdot f(n-1)$. Thus the factorial function f can be defined recursively as follows:

$$\begin{aligned} f(0) &= 1 && \leftarrow \text{initial condition} \\ f(n) &= n \cdot f(n-1), n \geq 1. && \leftarrow \text{recurrence} \end{aligned}$$

■

Example 1.2 ([262]) Let $f : \mathbf{W} \to \mathbf{W}$ such that

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ f\left(\frac{n-1}{2}\right) & \text{otherwise,} \end{cases}$$

where $\mathbf{W}$ denotes the set of of whole numbers 0, 1, 2, 3, ... Compute $f(23)$ and $f\left(2^{2^n} + 1\right)$.

Solution.

(1) $f(23) = f(11) = f(5) = f(2) = 1$

(2) $f\left(2^{2^n} + 1\right) = f\left(\frac{2^{2^n}+1-1}{2}\right) = f\left(2^{2^n-1}\right) = 2^{2^n-2}$. ■

Numbers of the form $2^{2^n} + 1$ are *Fermat numbers*, named after the French mathematician and lawyer Pierre de Fermat (1601–1665).

Example 1.3 Suppose we would like to compute the gcd $(a_1, a_2, \ldots, a_n)$ of n positive integers $a_1, a_2, \ldots, a_n$, where $n \geq 3$. Since gcd is a binary operator, we need to apply recursion to compute their gcd: $(a_1, a_2, \ldots, a_n) = ((a_1, a_2, \ldots, a_{n-1}), a_n)$. This can be confirmed using divisibility properties [130]. Now that we know how to compute the gcd of two positive integers, we can compute the gcd of n positive integers using recursion.

For example, we have

$$\begin{aligned} (36,\ 60,\ 78,\ 165) &= ((36,\ 60,\ 78), 165) \\ &= (((36,\ 60), 78), 165) \\ &= ((12,\ 78), 165) \\ &= (6,\ 165) = 3. \end{aligned}$$

The next three examples illustrate how recursion is useful in the study of number patterns. To define a number sequence $\{a_n\}$ recursively, you must be good at discovering patterns. This may require a lot of patience, perseverance, and practice, depending on the complexity of the pattern. ■

Example 1.4 Define recursively the number sequence $1, 3, 7, 15, 31, 63, \ldots$.

Solution. Let M_n denote the nth term of the sequence. Clearly, $M_1 = 1$. So let $n \geq 2$. Each term M_n is one more than twice its predecessor M_{n-1}; that is, $M_n = 2M_{n-1} + 1$. Thus M_n can be defined recursively as follows:

$$\begin{aligned} M_1 &= 1 \\ M_n &= 2M_{n-1} + 1, \quad n \geq 2. \end{aligned}$$ ■

To explore this example a bit further, we apply iteration to conjecture an explicit formula for M_n:

$$\begin{aligned} M_n &= 2M_{n-1} + 1 & &= 2^1 M_{n-1} + 1 \\ &= 2(2M_{n-2} + 1) + 1 & &= 2^2 M_{n-2} + 2 + 1 \\ &= 2^2(2M_{n-3} + 1) + 2 + 1 & &= 2^3 M_{n-3} + 2^2 + 2 + 1 \\ & & &\vdots \\ & & &= 2^{n-1} M_1 + 2^{n-2} + \cdots + 2 + 1 \\ & & &= 2^{n-1} + 2^{n-2} + \cdots + 2 + 1 \\ & & &= 2^n - 1. \end{aligned}$$

We can confirm the formula $M_n = 2^n - 1$ using PMI.

In general, the same number sequence can exhibit more than one recursive pattern. For example, M_n can also be defined by $M_n = M_{n-1} + 2^{n-1}$, where $M_1 = 1$ and $n \geq 2$.

Since $2^n = 1 + 2 + 2^2 + \cdots + 2^{n-1}$, it follows that the binary expansion of $M_n = 2^n - 1$ consists of n ones: $2^n - 1 = \overbrace{11\cdots11}^{n \text{ ones}}{}_{\text{two}}$. For example, $7 = 111_{\text{two}}$ and $31 = 11111_{\text{two}}$.

The numbers $M_n = 2^n - 1$ are *Mersenne numbers*, after the French mathematician and Franciscan monk Marin Mersenne (1588–1648), who investigated them extensively. The name "Mersenne numbers"was given to them by W.W. Rouse Ball of Trinity College, Cambridge, England. They play a pivotal role in cryptography and in the study of *even perfect numbers* $2^{p-1}M_p = 2^{p-1}(2^p - 1)$, where p and M_p are primes [130].

The next two examples deal with defining a number sequence recursively. Although not obvious, both examples are closely related. We will encounter them in Chapters 6 and 7.

Example 1.5 Define recursively the number sequence $1, 6, 35, 204, 1189, \ldots$.

Solution. This time the pattern is not that obvious. When that is the case, try to rewrite the terms in such a way that a pattern can be created:

$$\begin{aligned} 1 &= 1 \\ 6 &= 6 \\ 35 &= 6 \cdot 6 - 1 \\ 204 &= 6 \cdot 35 - 6 \\ 1189 &= 6 \cdot 204 - 35 \\ &\vdots \end{aligned}$$

Clearly, a pattern emerges; so we can now define the nth term b_n of the sequence recursively:

$$\begin{aligned} b_1 &= 1, \quad b_2 = 6 \\ b_n &= 6b_{n-1} - b_{n-2}, \quad n \geq 3. \end{aligned}$$

(There are two initial conditions here. We will find an explicit formula for b_n shortly.) ■

Example 1.6 Define recursively the number sequence $1, 8, 49, 288, 1681, \ldots$.

Solution. This time, the pattern is a little more complicated, so we follow the technique in the previous example and rewrite the terms in such a way that a pattern will emerge:

$$\begin{aligned} 1 &= 1 \\ 8 &= 8 \\ 49 &= 6 \cdot 8 - 1 + 2 \\ 288 &= 6 \cdot 49 - 8 + 2 \\ 1681 &= 6 \cdot 288 - 49 + 2 \\ &\vdots \end{aligned}$$

Using this pattern, we can define the nth term a_n recursively:

$$\begin{aligned} a_1 &= 1, \quad a_2 = 8 \\ a_n &= 6a_{n-1} - a_{n-2} + 2, \quad n \geq 3. \end{aligned}$$

(We will find an explicit formula for a_n in Example 1.9.) ■

The next example is a simple application of recursion. It appeared in the 1990 *American High School Mathematics Examination* (AHSME).

Example 1.7 Let $R_n = \frac{1}{2}(a^n + b^n)$, where $a = 3 + 2\sqrt{2}$ and $b = 3 - 2\sqrt{2}$. Prove that R_n is a positive integer and the units digit of R_{12345} is 9.

Proof. Since $a + b = 6$ and $ab = 1$, we have

$$\begin{aligned} 2(a+b)R_n &= (a+b)(a^n + b^n) \\ &= (a^{n+1} + b^{n+1}) + ab(a^{n-1} + b^{n-1}) \\ 6R_n &= R_{n+1} + R_{n-1}. \end{aligned}$$

Thus R_n satisfies the recurrence $R_{n+1} = 6R_n - R_{n-1}$, where $R_0 = 1$, $R_1 = 3$, and $n \geq 2$. Since R_0 and R_1 are positive integers, it follows by PMI that every R_n is a positive integer. (The sequence $\{R_n\}$ is $1, 3, 17, 99, 577, \ldots$; we will revisit it in detail in Chapter 7.)

The units digits of the sequence $\{R_n\}_{n \geq 0}$ display an interesting pattern: $\underbrace{1, 3, 7, 9, 7, 3}, \underbrace{1, 3, 7, 9, 7, 3}, \ldots$. It again follows by PMI that $R_n \equiv R_{n+6} \pmod{10}$; so R_n and R_{n+6} end in the same digit. Since $12345 = 6 \cdot 2057 + 3$, it follows that R_{12345} ends in 9. ■

Next we pursue briefly the solving of recurrences.

1.7 Solving Recurrences

The recursive definition of a function f does not give an explicit formula for $f(n)$. *Solving a recurrence* for $f(n)$ means finding such a formula.

In Example 1.4, we employed iteration to predict an explicit formula for M_n. But this technique has only limited scope. So we will now briefly develop a method ([127] for a detailed discussion) for solving two large and important classes of recurrences.

1.7.1 LHRWCCs

A *kth-order linear homogeneous recurrence with constant coefficients* (LHRWCCs) is a recurrence of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}, \tag{1.1}$$

where each c_i is a real number and $c_k \neq 0$.

A few words of explanation may be helpful. The term *linear* means the power of every predecessor of a_n on the RHS[1] of equation (1.1) is at most one. A recurrence is *homogeneous* if every a_i has the same exponent. Since a_n depends on its k immediate predecessors, the *order* of the recurrence is k; consequently, we will need k initial conditions to solve the LHRWCCs.

For example, the recurrence $M_n = 2M_{n-1}+1$ in Example 1.4 is linear, but not homogeneous. The one in Example 1.6 is both linear and homogeneous; its order is 2.

In the interest of brevity and convenience, we will confine our discussion to the second-order LHRWCCs

$$a_n = aa_{n-1} + ba_{n-2}, \tag{1.2}$$

where a and b are nonzero real numbers. Suppose this recurrence has a nonzero solution of the form cr^n. Then $cr^n = acr^{n-1} + bcr^{n-2}$. Since $cr \neq 0$, this implies that r must be a solution of the *characteristic equation*

$$x^2 - ax - b = 0 \tag{1.3}$$

of recurrence (1.2). The solutions of this quadratic equation are the *characteristic roots* of the recurrence.

The next theorem provides a road map for solving recurrence (1.2). We will omit its proof for convenience.

Theorem 1.4 *Let r and s be the distinct (real or complex) characteristic roots of recurrence (1.2). Then the general solution of the recurrence is of the form $a_n = Ar^n + bs^n$, where A and B are constants.* ∎

The general solution is a linear combination of the *basic solutions* r^n and s^n of recurrence (1.2), which are linearly independent. The constants A and B can be determined using the two initial conditions.

Theorem 1.4 can be extended in an obvious way to any LHRWCCs with distinct characteristic roots. It has to be modified if the recurrence has repeated roots [127].

The next example illustrates the various steps involved in the theorem.

---

[1] RHS and LHS are abbreviations of right-hand side and left-hand side, respectively.

Example 1.8 Solve the recurrence in Example 1.5.

Solution. The recurrence $b_n = 6b_{n-1} - b_{n-2}$ is a second-order LHRWCCs. Its characteristic equation $x^2 - 6x + 1 = 0$ has two distinct solutions: $r = 3 + 2\sqrt{2}$ and $s = 3 - 2\sqrt{2}$. By Theorem 1.4, the general solution of the recurrence is $b_n = Ar^n + bs^n$, where A and B are constants to be determined.

Using the initial conditions $b_1 = 1$ and $b_2 = 6$, we get the following 2×2 linear system:

$$\begin{aligned} Ar + Bs &= 1 \\ Ar^2 + Bs^2 &= 6. \end{aligned}$$

Solving this, we get $A = \frac{1}{4\sqrt{2}} = -B$. Thus the general solution of the recurrence is $b_n = \frac{r^n - s^n}{4\sqrt{2}}$, where $n \geq 1$. (We will revisit this example in Chapter 6.) ■

The next example illustrates how to solve a *linear nonhomogenous recurrence* with constant coefficients (LNHRWCCs).

Example 1.9 Solve the LNHRWCCs in Example 1.6.

Solution. Solving this LNHRWCCs is slightly more complicated. Fortunately, we did most of the work in Example 1.8. The general solution of the linear homogeneous part of the recurrence $a_n = 6a_{n-1} - a_{n-2}$ is $a_n = Ar^n + Bs^n$, where $r = 3 + 2\sqrt{2}$, $s = 3 - 2\sqrt{2}$, and A and B are constants to be determined using the initial conditions.

To solve the nonhomogeneous part, we look for a particular solution of the form $a_n = C$, where C is a constant. Substituting for a_n in the recurrence, we get $C = 6C - C + 2$, so $C = -\frac{1}{2}$.

Combining the general solution of the homogeneous part with this particular solution yields the general solution of the given recurrence: $a_n = Ar^n + Bs^n - \frac{1}{2}$.

Now we must determine the unknowns A and B. The initial conditions yield a 2×2 linear system in A and B:

$$\begin{aligned} Ar + Bs &= \frac{3}{2} \\ Ar^2 + Bs^2 &= \frac{17}{2}. \end{aligned}$$

Solving this linear system, we get $A = \frac{1}{4} = B$.

Thus the desired solution is $a_n = \frac{1}{4}(r^n + s^n) - \frac{1}{2} = \frac{r^n + s^n - 2}{4}$, where $n \geq 1$. (We will revisit this example also in Chapter 6.) ■

More generally, we can employ the following strategy to transform a LNHRRWCCs to a LHRRWCCs. To see this, consider the second-order recurrence $x_{n+1} = ax_n + bx_{n-1} + c$. We can rewrite this as follows:

$$\begin{aligned} x_{n+1} &= (a+1)x_n + bx_{n-1} + c - x_n \\ &= (a+1)x_n + bx_{n-1} + c - (ax_{n-1} + bx_{n-2} + c) \end{aligned}$$

$$= (a+1)x_n + (b-a)x_{n-1} - bx_{n-2},$$

which is a LHRRWCCs. Example 1.11 illustrates this technique.

1.8 Generating Functions

Generating functions are a powerful tool for solving recurrences and combinatorial problems. They were invented by the French mathematician Abraham De Moivre (1667–1754). Generating functions are basically formal power series that keep track of the various coefficients. In other words, they are "clotheslines on which we hang up sequences of numbers for display" [259].

More formally, let $a_0, a_1, a_2, \cdots$ be any real numbers. Then the function

$$g(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots = \sum_{n=0}^{\infty} a_nx^n$$

is called the *generating function* of the sequence $\{a_n\}_{n=0}^{\infty}$. In the study of generating functions, we are *not* interested in the convergence of the series; x^n is simply a place-holder for the coefficient a_n.

Generating functions $f(x) = \sum_{n=0}^{\infty} a_nx^n$ and $g(x) = \sum_{n=0}^{\infty} b_nx^n$ can be added, subtracted, and multiplied, as can be expected:

$$f(x) \pm g(x) = \sum_{n=0}^{\infty}(a_n \pm b_n)x^n$$

$$f(x) \cdot g(x) = \sum_{n=0}^{\infty} c_nx^n,$$

where $c_n = \sum_{i=0}^{n} a_ib_{n-i}$. The sequence $\{c_n\}$ is the *convolution* of the sequences $\{a_n\}$ and $\{b_n\}$; its generating function is $f(x)g(x)$.

In particular, let $f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = g(x)$; so $a_n = 1 = b_n$ for every n. Then $c_n = \sum_{i=0}^{n} 1 \cdot 1 = n+1$; so every positive integer can be generated by the convolution of the sequence of 1s with itself:

$$\sum_{n=0}^{\infty}(n+1)x^n = \left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty} x^n\right) = \frac{1}{1-x} \cdot \frac{1}{1-x} = \frac{1}{(1-x)^2}.$$

Suppose we let $a_n = n+1$ and $b_n = 1$. Then

$$c_n = \sum_{i=0}^{n} a_ib_{n-i} = \sum_{i=0}^{n}(i+1)$$

$$= \frac{(n+1)(n+2)}{2}.$$

Thus the numbers[2] $t_{n+1} = \frac{(n+1)(n+2)}{2}$ can be generated by the functions $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n$ and $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$; [130] that is,

$$\sum_{n=0}^{\infty} t_{n+1}x^n = \frac{1}{(1-x)^2}\cdot\frac{1}{1-x} = \frac{1}{(1-x)^3}.$$

This can also be achieved by differentiating $\sum_{n=0}^{\infty}(n+1)x^n = \frac{1}{(1-x)^2}$ with respect to x.

The next two examples illustrate how a recursive definition of a sequence $\{a_n\}$ can be used to develop a generating function of the sequence.

Example 1.10 Find a generating function of the sequence $\{b_n\}$ in Example 1.5.

Solution. Let $g(x)$ be the generating function: $g(x) = x+6x^2+35x^3+204x^4+\cdots+b_nx^n+\cdots$. Then

$$\begin{aligned}
6xg(x) &= 6x^2 + 36x^3 + 210x^4 + \cdots + 6b_{n-1}x^n + \cdots\\
x^2g(x) &= \qquad\quad x^3 + \quad 6x^4 + \cdots + \ b_{n-2}x^n + \cdots\\
(1-6x+x^2)g(x) &= x\\
g(x) &= \frac{x}{1-6x+x^2}.
\end{aligned}$$

This is the desired generating function:

$$\frac{x}{1-6x+x^2} = 1 + 6x + 35x^2 + 204x^3 + 1189x^4 + \cdots .$$

(We will revisit this generating function in Chapter 6.) ■

Example 1.11 Find a generating function of the sequence $\{a_n\}$ in Example 1.6.

Solution. We have $a_n = 6a_{n-1} - a_{n-2} + 2$, where $a_1 = 1, a_2 = 8$, and $n \geq 3$. Since the recurrence is nonhomogeneous, we rewrite it to get a homogeneous one:

$$\begin{aligned}
a_n &= 6a_{n-1} - a_{n-2} + 2\\
&= (7a_{n-1} - a_{n-1}) - a_{n-2} + 2
\end{aligned}$$

[2] $t_n = \frac{n(n+1)}{2}$ is the nth *triangular number*.

$$
\begin{aligned}
&= 7a_{n-1} - (6n_{n-2} - a_{n-3} + 2) - a_{n-2} + 2 \\
&= 7a_{n-1} - 7a_{n-2} - a_{n-3}.
\end{aligned}
$$

This is a third-order LHRWCCs. Thus $\{a_n\}$ can be redefined as follows:

$$
\begin{aligned}
a_0 &= 0, \quad a_1 = 1, \quad a_2 = 8 \\
a_n &= 7a_{n-1} - 7a_{n-2} - a_{n-3}, \quad n \geq 3.
\end{aligned}
$$

Let $g(x) = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$ be the generating function of $\{a_n\}$. Then

$$
\begin{aligned}
7xg(x) &= 7a_1x^2 + 7a_2x^3 + 7a_3x^4 + \cdots \\
7x^2g(x) &= \qquad\qquad 7a_1x^3 + 7a_2x^4 + \cdots \\
x^3g(x) &= \qquad\qquad\qquad\qquad + \; a_1x^4 + \cdots
\end{aligned}
$$

So

$$
\begin{aligned}
(1 - 7x + 7x^2 - x^3)g(x) &= a_1x + (a_2 - 7a_1)x^2 \\
&= x + x^2 \\
g(x) &= \frac{x + x^2}{1 - 7x + 7x^2 - x^3} \\
&= \frac{x(1 + x)}{(1 - x)(1 - 6x + x^2)}.
\end{aligned}
$$

This gives the generating function of the sequence $\{a_n\}$:

$$
\frac{x(1 + x)}{(1 - x)(1 - 6x + x^2)} = x + 8x^2 + 49x^3 + 288x^4 + \cdots .
$$

(We will revisit this generating function also in Chapter 6.) ■

Next we present an abbreviated introduction to binomial coefficients. These are the coefficients that occur in the binomial expansion of $(x + y)^n$. Their earliest known occurrence is in a tenth-century commentary by the Indian mathematician Halayudha, on Pingla's *Chandas Shastra*. Bhaskara II (1114–1185?) gives a full discussion of binomial coefficients in his famous work *Leelavati*, written in 1150. However, the term "binomial coefficient"was coined by the German algebraist Michel Stifel (1486–1567).

1.9 Binomial Coefficients

Let n and r be nonnegative integers. The *binomial coefficient*[3] is defined as

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)}{r!}, \tag{1.4}$$

where $0 \le r \le n$. When $r > n$ or $r < 0$, $\binom{n}{r}$ is defined as 0. Read $\binom{n}{r}$ as "n choose r" to be consistent with the typesetting language *Latex*.

For example, $\binom{7}{4} = 35$, $\binom{7}{0} = 1 = \binom{7}{7}$, and $\binom{7}{9} = 0 = \binom{7}{-3}$.

Combinatorially, $\binom{n}{r}$ counts the number of r-member subcommittees that can be formed from an n-member committee. In particular, exactly $\binom{n}{2}$ line segments can be drawn using n points on a plane such that no three of them are collinear.

Interestingly, this special case with $r = 2$ is a geometric representation of the handshake problem we studied earlier. Geometrically, each point represents a guest at the party and each line segment a handshake. Thus $h(n) = \binom{n}{2} = \frac{n(n-1)}{2}$.

Since an n-member committee has $\binom{n}{r}$ r-member subcommittees, it follows that the committee has a total of $\sum_{r=0}^{n} \binom{n}{r}$ subcommittees; $\sum_{r=k}^{n} \binom{n}{r}$ of them consist of k or more members. We will shortly find a formula for the sum $\sum_{r=0}^{n} \binom{n}{r}$.

It follows from the definition that $\binom{n}{0} = 1 = \binom{n}{n}$, $\binom{n}{1} = 1 = \binom{n}{n-1}$, and $\binom{n}{r} = 1 = \binom{n}{n-r}$. These can be confirmed both algebraically and combinatorially.

1.9.1 Pascal's Identity

Binomial coefficients satisfy a multitude of properties. They include an extremely useful recurrence, called *Pascal's identity* after the French mathematician and philosopher Blaise Pascal (1623–1662): $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$. Although this can be verified algebraically, we will now give a simple combinatorial argument.

Consider an n-member committee S. It has $\binom{n}{r}$ r-member subcommittees. We will now count these subcommittees in a different way, by partitioning them into two disjoint families. To this end, suppose Bob is a member of S.

Case 1 Suppose Bob belongs to a subcommittee. The remaining $r - 1$ members of the subcommittee must be selected from the remaining $n - 1$ members of S. This can be done in $\binom{n-1}{r-1}$ ways; that is, there are $\binom{n-1}{r-1}$ r-member subcommittees that include Bob.

Case 2 Suppose Bob does not belong to any r-member subcommittee. So the r members of the subcommittee must be selected from the remaining $n - 1$ members of the committee. Exactly $\binom{n-1}{r}$ such subcommittees can be formed.

[3] The parenthesized bi-level notation was introduced by the German mathematician and physicist Andreas von Ettinghausen (1796–1878) in his book *Die Combinatorische Analysis*, published in 1826. It is also denoted by $C(n, r)$ and nCr.

Thus, by Cases 1 and 2, there is a total of $\binom{n-1}{r-1} + \binom{n-1}{r}$ r-member subcommittees. But the total number of such subcommittees is $\binom{n}{r}$. Thus $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$, as desired. ∎

For example, $\binom{11}{6} + \binom{11}{7} = 462 + 330 = 792 = \binom{12}{7}$.

Next we visit the well-known binomial theorem.

1.9.2 Binomial Theorem

Binomial coefficients play a central role in the development of the binomial expansion of $(x + y)^n$. Euclid knew the expansion for $n = 2$, and it appears in his classic work, *Elements*, written around 300 B.C. The Indian mathematician and astronomer Aryabhata (ca.476–ca.550) knew it for $n = 2$ and $n = 3$, while Brahmagupta (598?–670?) knew the expansion for $n = 3$. The binomial theorem for positive integral exponents was known to the Persian poet and mathematician Omar Khayyám (1048–1131). But the English mathematician and physicist Isaac Newton (1642–1727) is credited with the discovery of the theorem in its current form.

We can prove the binomial theorem using *Pascal's identity* and PMI. However, we will give a short and elegant combinatorial proof, employing the fundamental addition and multiplication principles from combinatorics. To this end, we let $|X|$ denote the number of ways task X can be done. Then

- If A and B are mutually exclusive events, then task A or B can take place in $|A|+|B|$ different ways (*addition principle*).
- Two tasks A and B can occur in that order in $|A|\cdot|B|$ different ways (*multiplication principle*).

Theorem 1.5 (*The Binomial Theorem*) *Let x and y be any real numbers, and n any nonnegative integer. Then* $(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$.

Proof. Notice that $(x+y)^n = \underbrace{(x + y)(x + y)\cdots(x + y)}_{n \text{ factors}}$. Every term in the expansion is of the form $Cx^{n-r}y^r$, where the constant C counts the number of times $x^{n-r}y^r$ occurs in the expansion and $0 \le r \le n$. An x in x^{n-r} can be selected from any of the $n - r$ factors on the RHS, and a y in y^r from any of the remaining r factors. So the $n - r$ x's can be selected in $\binom{n}{n-r}$ ways and the r y's in $\binom{r}{r}$ ways. So, by the multiplication principle, $C = \binom{n}{n-r}\binom{r}{r} = \binom{n}{r}$. Since this is true for every r, it follows by the addition principle that $(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$. ∎

For example, $(x+y)^4 = \sum_{r=0}^{4} \binom{4}{r} x^{4-r} y^r = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ and $(x-y)^5 = \sum_{r=0}^{5} \binom{5}{r} x^{5-r}(-y)^r = x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$.

Several interesting results follow from the binomial theorem. To begin with, $(1 + x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$. Letting $x = 1$, this yields $2^n = \sum_{r=0}^{n} \binom{n}{r}$; that is, the sum of the binomial coefficients

$\binom{n}{r}$ equals 2^n. In other words, an n-member committee has a total of $\sum_{r=0}^{n}\binom{n}{r} = 2^n$ subcommittees; they include the null subcommittee. For example, $\sum_{r=0}^{5}\binom{5}{r} = \binom{5}{0}+\binom{5}{1}+\binom{5}{2}+\binom{5}{3}+\binom{5}{4}+\binom{5}{5} = 2^5$.

Suppose we let $x = 1$ and $y = -1$ in the binomial theorem. Then we get $\sum_{r=0}^{n}\binom{n}{r}(-1)^r = 0$; that is, $\sum_{r\text{ even}}\binom{n}{r} = \sum_{r\text{ odd}}\binom{n}{r}$. In words, the sum of the binomial coefficients in "even" positions equals that in "odd" positions.

For example, consider the sum $\sum_{r=0}^{6}\binom{6}{r}$: Then

$$\begin{aligned}\sum_{r\text{ even}}\binom{6}{r} &= \binom{6}{0}+\binom{6}{2}+\binom{6}{4}+\binom{6}{6} = 32\\ &= \binom{6}{1}+\binom{6}{3}+\binom{6}{5} = \sum_{r\text{ odd}}\binom{6}{r}.\end{aligned}$$

The two sums are equal, as expected.

Since $\sum_{r=0}^{n}\binom{n}{r} = 2^n$, it follows that $\sum_{r\text{ even}}\binom{n}{r} + \sum_{r\text{ odd}}\binom{n}{r} = 2^n$. But we just found that the two sums are equal; so each sum equals 2^{n-1}. For example, $\sum_{r\text{ even}}\binom{7}{r} = \sum_{r\text{ odd}}\binom{7}{r} = 64 = 2^6$.

We highlight these properties in the following corollary.

Corollary 1.1 *Let x be any real number. Then*

$$\begin{aligned}(1+x)^n &= \sum_{r=0}^{n}\binom{n}{r}x^{n-r}\\ \sum_{r=0}^{n}\binom{n}{r} &= 2^n\\ \sum_{r\text{ even}}\binom{n}{r} &= \sum_{r\text{ odd}}\binom{n}{r}.\end{aligned}$$

■

In passing, we note that an abundant number of binomial identities can be developed from the binomial theorem using algebra and calculus.

We also note that by extending the definition of the binomial coefficient in (1.4) to negative integers and rational numbers, we can generalize the binomial theorem to include negative and rational exponents [137], as Newton did.

1.9.3 Pascal's Triangle

The binomial coefficients $\binom{n}{r}$ can be arranged as a triangular array, as in Figure 1.2. It is called *Pascal's triangle* after Pascal, who wrote about the array in his *Treatise on the Arithmetic Triangle* in 1653, but published posthumously in 1665. However, the array was known earlier in Germany, Italy, The Netherlands, and England.

$$
\begin{array}{ccccccccccccc}
 & & & & & & \mathbf{1} & & & & & & \leftarrow \text{row } 0\\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & \mathbf{2} & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & \mathbf{6} & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & \mathbf{20} & & 15 & & 6 & & 1\\
 & & & & & & \uparrow & & & & & & \\
 & & & & & & \text{CBCs} & & & & & &
\end{array}
$$

Figure 1.2. Pascal's Triangle

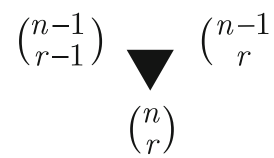

Figure 1.3.

It appeared four centuries earlier in the Chinese mathematician Shi-Kié's *The Precious Mirror of the Four Elements*. According to Shi-Kié, the triangle appeared even earlier, in a 1275 book by Yang Hui Yang. The Chinese and Japanese versions are quite similar. Omar Khayyám knew of the array around 1100, most probably from Indian sources.

Pascal's triangle possesses many interesting properties. For instance, each internal entry is the sum of the entries to its left and right in the previous row. This follows by virtue of Pascal's identity; see Figure 1.3.

By Corollary 1.1, $\sum_{r=0}^{k} \binom{k}{r} = 2^k$. Consequently, the sum of the numbers in row k equals 2^k, where the top row is labeled row 0. So the cumulative sum of the numbers in rows 0 through $n-1$ equals $\sum_{k=0}^{n-1} 2^k = 2^n - 1 = M_n$, the nth *Mersenne number*.

The binomial coefficients $\binom{2n}{n}$, which appear in the middle of the triangle, are the *central binomial coefficients* (CBCs). They are generated by the function $\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n$. They pop up in numerous places.

For example, they occur in the study of *Catalan numbers* [131] $C_n = \frac{1}{n+1}\binom{2n}{n}$, which are generated by the function $\frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} C_n x^n$.

We will encounter Pascal's triangle several times in the following chapters.

Next we present the celebrated Fibonacci and Lucas sequences [126]. These two bright shining stars on the mathematical horizon continue to charm professionals and amateurs alike. They are a delightful playground for exploring, conjecturing, and establishing fascinating properties, intriguing the mathematical community with their beauty and ubiquity. Like human twins, they enjoy strikingly similar properties.

1.10 Fibonacci and Lucas Numbers

Although Fibonacci numbers and their recursive formulation are named after the Italian mathematician Leonardo of Pisa (ca. 1170–ca.1250), they were known in India several centuries prior to Fibonacci. They were discovered by Virahanka between 600 and 800 A.D.; by Gopala before 1135 A.D.; and by Acharya Hemachandra around 1150 A.D. They also appear as a special case of a formula discovered by Narayana Pandit (1340–1400).

1.10.1 Fibonacci's Rabbits

There is an interesting puzzle illustrating Fibonacci numbers, which appears as a problem in Fibonacci's *Liber Abaci*, published in 1202. The puzzle runs like this:

> Suppose we have a mixed pair (one male and the other female) of newborn rabbits. Each pair takes a month to become mature. Starting at the beginning of the following month, each adult pair produces a mixed pair every month. Assuming that the rabbits are immortal, find the number of pairs of rabbits we will have at the end of the year.

For convenience, assume that the rabbits were born on January 1. They become mature on February 1, so we still have one pair in February. This pair is two months old on March 1, so it produces a mixed pair on March 1; thus there are 2 pairs of rabbits in March. On April 1, the adult-pair produces a new baby pair; the baby pair from March becomes an adult pair; so there are 3 pairs in April. Continuing like this, there will be 5 pairs in May, 8 pairs in June, 13 pairs in July, and so on. In December, there will be a total of 144 mixed pairs of rabbits.

1.10.2 Fibonacci Numbers

The numbers $1, 1, 2, 3, 5, 8, \ldots$ are the *Fibonacci numbers*, so called by the French mathematician François Edouard Anatole Lucas (pronounced 'Lucah') (1842–1891) in the 19th century. They manifest an intriguing pattern: Every Fibonacci number F_n, except the first two, is the sum of its two immediate predecessors, F_{n-1} and F_{n-2}. Consequently, Fibonacci numbers can be defined recursively:

$$\begin{aligned} F_1 &= 1 = F_2 \\ F_n &= F_{n-1} + F_{n-2}, \quad n \geq 3. \end{aligned}$$

The first six Fibonacci numbers are 1, 1, 2, 3, 5, and 8.

1.10.3 Lucas Numbers

Closely related to Fibonacci numbers are the *Lucas numbers* L_n, named after Lucas. They follow exactly the same pattern, except that $L_2 = 3$. So they have a nearly identical recursive definition:

$$\begin{aligned} L_1 &= 1, \quad L_2 = 3 \\ L_n &= L_{n-1} + L_{n-2}, \quad n \geq 3. \end{aligned}$$

The first six Lucas numbers are 1, 3, 4, 7, 11, and 18. Table T.1 in the Appendix lists the first 100 Fibonacci and Lucas numbers.

Using these recursive definitions, we can extend both sequences to zero and negative subscripts: $F_0 = 0$, $L_0 = 2$, $F_{-n} = (-1)^{n-1}F_n$, and $L_{-n} = (-1)^n L_n$.

1.10.4 Binet's Formulas

Although Fibonacci and Lucas numbers are often defined recursively for the sake of simplicity, they can be defined explicitly as well:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the solutions[4] of the quadratic equation $x^2 = x + 1$. These two formulas are called *Binet's formulas*, after the French mathematician Jacques Philippe Marie Binet (1788–1865). Both can be confirmed using PMI.

Binet found the formula for F_n in 1843. However, it was discovered in 1718 by De Moivre, who employed generating functions to develop it. It was also discovered independently in 1844 by another French mathematician, Gabriel Lamé (1795–1870).

1.10.5 Fibonacci and Lucas Identities

Fibonacci and Lucas numbers satisfy a vast array of identities. Some of them are listed in Table 1.1.

Table 1.1. Fibonacci and Lucas Identities

(1)	$F_{n+1} + F_{n-1} = L_n$	(2)	$L_{n+1} + L_{n-1} = 5F_n$
(3)	$F_{n+2} + F_{n-2} = 3F_n$	(4)	$F_{2n} = F_n L_n$
(5)	$F_{n+2} - F_{n-2} = L_n$	(6)	$L_{n+2} - L_{n-2} = 5F_n$
(7)	$F_{n+1}^2 + F_n^2 = F_{2n+1}$	(8)	$L_{n+1}^2 + L_n^2 = 5F_{2n+1}$
(9)	$F_{n+1}^2 - F_n^2 = F_{n-1}F_{n+2}$	(10)	$L_{n+1}^2 - L_n^2 = L_{n-1}L_{n+2}$
(11)	$F_{n+1}^2 - F_{n-1}^2 = F_{2n}$	(12)	$L_{n+1}^2 - L_{n-1}^2 = 5F_{2n}$
(13)	$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$	(14)	$L_{n+1}L_{n-1} - L_n^2 = 5(-1)^n$
(15)	$\sum_{i=1}^{n} F_i = F_{n+2} - 1$	(16)	$\sum_{i=1}^{n} L_i = L_{n+2} - 3$
(17)	$\sum_{i=1}^{n} F_{2i-1} = F_{2n}$	(18)	$\sum_{i=1}^{n} L_{2i-1} = L_{2n} - 2$
(19)	$\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1$	(20)	$\sum_{i=1}^{n} L_{2i} = L_{2n+1} - 1$
(21)	$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$	(22)	$\sum_{i=1}^{n} L_i^2 = L_n L_{n+1} - 2$
(23)	$\sum_{i=0}^{n} \binom{n}{i} F_{i+j} = F_{2n+j}$	(24)	$\sum_{i=0}^{n} \binom{n}{i} L_{i+j} = L_{2n+j}$

[4] The number α is the well-known *golden ratio*, $\alpha + \beta = 1$, and $\alpha\beta = -1$.

Identities (7), (11), and (15) were developed by Lucas in 1876. Identity (13), called *Cassini's formula*, was discovered in 1680 by the Italian mathematician, Giovanni Domenico Cassini (1625–1712).

Next we highlight a few interesting byproducts that follow from some of the identities:

(1) It follows from Cassini's formula that every two consecutive Fibonacci numbers are relatively prime; that is, $(F_n, F_{n-1}) = 1$. On the other hand, identity (14) implies that $(L_n, L_{n-1}) = 1$ or 5. Suppose $(L_n, L_{n-1}) = 5$. Then $L_n \equiv L_{n-1} \equiv 0 \pmod 5$. But this is impossible, since the Lucas numbers modulo 5 form a cyclic pattern: $\underbrace{1\,3\,4\,2}\ \underbrace{1\,3\,4\,2}\ \underbrace{1\,3\,4\,2}\ \ldots$. Thus every two consecutive Lucas numbers are also relatively prime.

(2) In 1964, J.H.E. Cohn established that $F_1 = 1$ and $F_{12} = 144$ are the only two distinct square Fibonacci numbers. Consequently, by identity (7), no two consecutive Fibonacci numbers can be the lengths of the legs of a right triangle with integral sides. Likewise, the same holds for any two consecutive Lucas numbers.

(3) Since $F_{2n} = F_n L_n$, it follows that every even-numbered Fibonacci number F_{2n} has a nontrivial factorization, where $n \geq 3$. For example, $F_{18} = 2584 = 34 \cdot 76 = F_9 L_9$.

(4) Identity (21) has an interesting geometric interpretation: Every $F_{n+1} \times F_n$ rectangle can be filled with $F_i \times F_i$ squares, where $1 \leq i \leq n$. For example, the 21×13 rectangle in Figure 1.4 can be filled with one 13×13 square, one 8×8 square, one 5×5 square, one 3×3 square, one 2×2 square, and two 1×1 squares.

On the other hand, suppose we try to cover an $L_{n+1} \times L_n$ rectangle with distinct $L_i \times L_i$ squares, where $1 \leq i \leq n$. Then, by identity (22), we will have two unit squares uncovered.

For example, consider the 18×11 rectangle in Figure 1.5. It can be covered with one 11×11 square, one 7×7 square, one 4×4 square, one 3×3 square, and one 1×1 square. Unfortunately, this leaves a 2×1 rectangle uncovered.

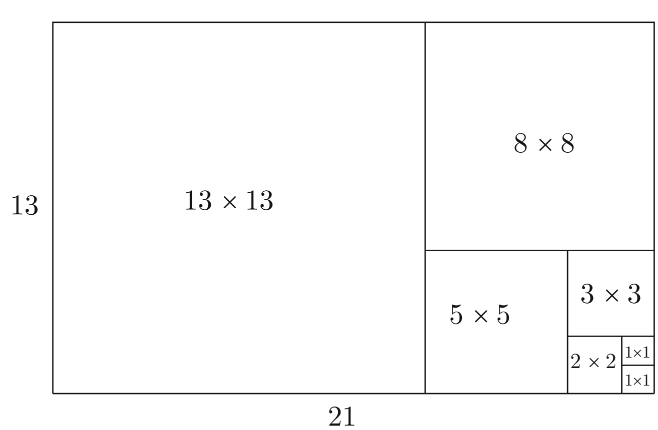

Figure 1.4.

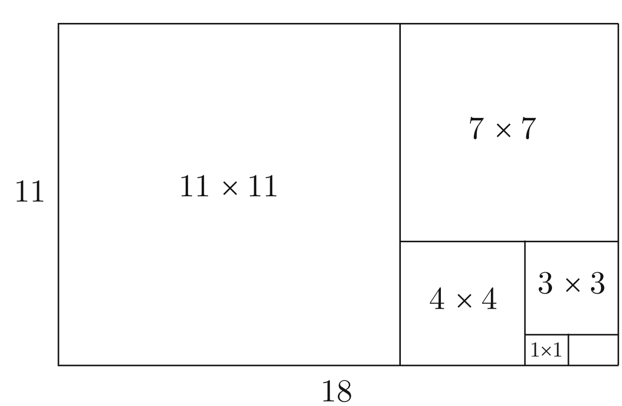

Figure 1.5.

(5) Identity (9) implies that if we remove an $F_n \times F_n$ square from an $F_{n+1} \times F_{n+1}$ square, the remaining area equals $F_{n-1}F_{n+2}$; see Figure 1.6. By identity (10), a similar result holds for an $L_{n+1} \times L_{n+1}$ square; see Figure 1.7.

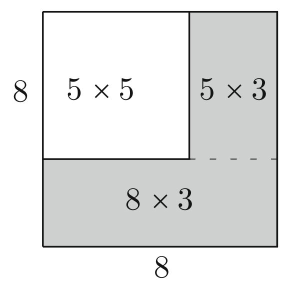

Figure 1.6. Shaded area = 13·3

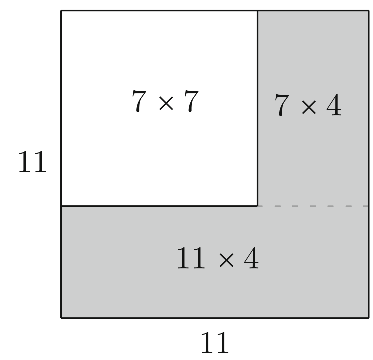

Figure 1.7. Shaded area = 18·4

(6) By Cassini's formula, we can form an area $F_{n+1}F_{n-1}$ by adding a unit area to an $F_n \times F_n$ area if n is even, and by deleting a unit area from it if n is odd; see Figures 1.8 and 1.9. There is a fascinating paradox based on *Cassini's formula* [126]. Identity (14) can be interpreted in a similar fashion.

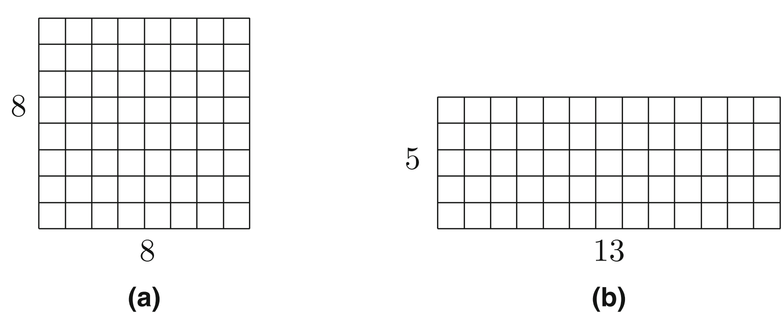

Figure 1.8. (**a**) Add One Unit Area

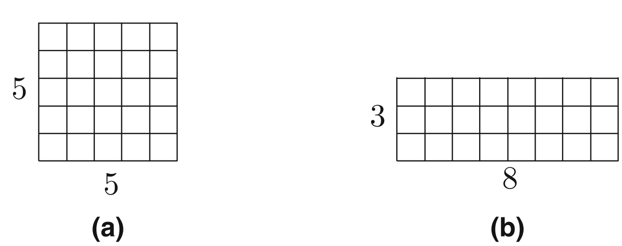

Figure 1.9. (**a**) Delete One Unit Area

1.10.6 Lucas' Formula for F_n

In 1876, Lucas developed an explicit formula for F_n:

$$F_n = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-k-1}{k}. \tag{1.5}$$

This can be confirmed using Pascal's identity and PMI [11] for a combinatorial argument.

The beauty of this formula lies in the fact that Fibonacci numbers can be computed by adding up the binomial coefficients along the northeast diagonals in Pascal's triangle. For example, $F_6 = \sum_{k=0}^{2} \binom{5-k}{k} = \binom{5}{0} + \binom{4}{1} + \binom{3}{2} = 1 + 4 + 3 = \textcircled{8}$; see Figure 1.10.

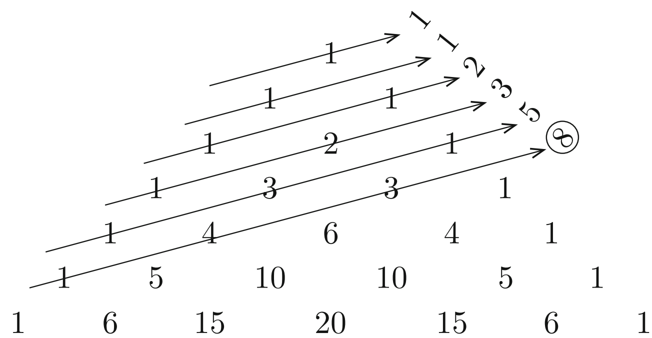

Figure 1.10.

By virtue of identity (1), Lucas numbers also can be computed from Pascal's triangle: Add pairs of alternate northeast diagonals; see Figure 1.11.

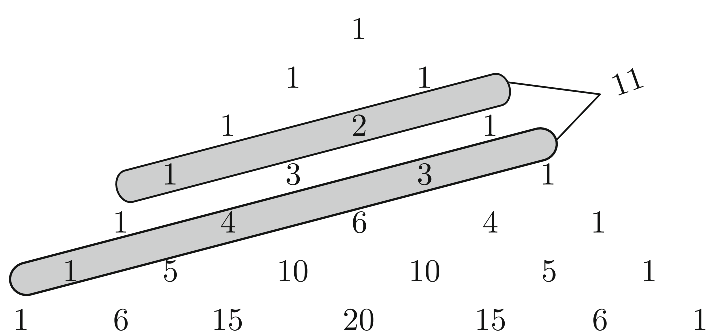

Figure 1.11.

Lucas' formula, coupled with the fact that $L_n = F_{n-1} + F_{n+1}$, can be used to develop an explicit formula for Lucas numbers:

$$L_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k}.$$

For example, $L_n = \sum_{k=0}^{2} \frac{5}{5-k} \binom{5-k}{k} = 1 + 5 + 5 = 11 = (1+2) + (1+4+3)$; see Figure 1.11.

For the sake of clarity and expediency, we next present a partial preview of Pell and Pell–Lucas numbers, the central figures in the development of this huge undertaking. This will help us see a number of their occurrences in different contexts in Chapters 2–6. We will study them in detail in Chapter 7.

1.11 Pell and Pell–Lucas Numbers: A Preview

Pell numbers are named after the English mathematician John Pell (1611–1685), since they occur in the study of *Pell's equation* $x^2 - dy^2 = (-1)^n$, where d is a positive nonsquare integer. Unfortunately, this attribution to Pell is an error; see Chapter 2. Pell–Lucas numbers, on the other hand, are named after him and Lucas, although neither had anything to do with them. Like Fibonacci and Lucas numbers, Pell and Pell–Lucas numbers are mathematical twins; they too are ubiquitous and share a number of similar properties. This is perhaps the only justification for the hyphenated name for the latter family.

Pell numbers P_n and *Pell–Lucas numbers* Q_n are also often defined recursively:

$$P_1 = 1, \quad P_2 = 2 \qquad\qquad Q_1 = 1, \quad Q_2 = 3$$
$$P_n = 2P_{n-1} + P_{n-2}, \quad n \geq 3; \qquad\qquad Q_n = 2Q_{n-1} + Q_{n-2}, \quad n \geq 3.$$

They both satisfy the same *Pell recurrence* $x_n = 2x_{n-1} + x_{n-2}$. The only difference between the two recursive definitions is in the second initial conditions: $P_2 = 2$, whereas $Q_2 = 3$.

The first six Pell numbers are 1, 2, 5, 12, 29, and 70; and the first six Pell–Lucas numbers are 1, 3, 7, 17, 41, and 99. Tables T.2 and T.3 in the Appendix list the first 100 Pell and Pell–Lucas numbers, respectively.

We will learn in Chapter 2 that the solutions of Pell's equation $x^2 - 2y^2 = (-1)^n$ are (Q_n, P_n). For example, $41^2 - 2 \cdot 29^2 = -1$ and $99^2 - 2 \cdot 70^2 = 1$.

Recall that Fibonacci and Lucas numbers can be defined by Binet's formulas. Likewise, Pell and Pell–Lucas numbers can be defined explicitly by similar-looking formulas.

1.11.1 Binet-like Formulas

The characteristic equation of the Pell recurrence is $t^2 - 2t - 1 = 0$; solving it, we get two distinct characteristic roots: $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$. Notice that $\gamma + \delta = 2, \gamma - \delta = 2\sqrt{2}$, and $\gamma\delta = -1$; we will be using these facts frequently. By Theorem 1.4, the general solution of the Pell recurrence is $P_n = A\gamma^n + B\delta^n$. The initial conditions $P_1 = 1$ and $P_2 = 2$ yield the equations $A\gamma + B\delta = 1$ and $A\gamma^2 + B\delta^2 = 2$. Solving these equations, we get $A = -B = \frac{1}{2\sqrt{2}} = \frac{1}{\gamma - \delta}$. So $P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}$. Similarly, $Q_n = \frac{\gamma^n + \delta^n}{2}$. Thus we have the following *Binet-like formulas* for P_n and Q_n:

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \quad \text{and} \quad Q_n = \frac{\gamma^n + \delta^n}{2}, \quad n \geq 1.$$

For example,

$$\begin{aligned} P_4 &= \frac{\gamma^4-\delta^4}{\gamma-\delta} = \frac{(\gamma^2-\delta^2)(\gamma^2+\delta^2)}{\gamma-\delta} \\ &= (\gamma+\delta)[(\gamma+\delta)^2-2\gamma\delta] = 2(2^2+2) = 12 \\ Q_3 &= \frac{\gamma^3+\delta^3}{2} = \frac{(\gamma+\delta)(\gamma^2+\delta^2-\gamma\delta)}{2} \\ &= (\gamma+\delta)^2-3\gamma\delta = 7. \end{aligned}$$

It follows from the Binet-like formulas that $Q_n+\sqrt{2}P_n=\gamma^n$ and $Q_n-\sqrt{2}P_n=\delta^n$.

Since $Q_n-\sqrt{2}P_n=\delta^n$, $|Q_n-\sqrt{2}P_n|=|\delta|^n<\frac{1}{2}$. So $\sqrt{2}P_n-\frac{1}{2}<Q_n<\sqrt{2}P_n+\frac{1}{2}$. Since Q_n is an integer, it follows that $Q_n=\lfloor\sqrt{2}P_n+\frac{1}{2}\rfloor$. This gives an explicit formula for Q_n in terms of P_n. Likewise, $P_n=\frac{1}{2}\lfloor\sqrt{2}Q_n+1\rfloor$.

For example, $Q_5=\lfloor\sqrt{2}P_5+\frac{1}{2}\rfloor=\lfloor 29\sqrt{2}+0.5\rfloor=\lfloor 41.512193\cdots\rfloor=41$ and $P_6=\frac{1}{2}\lfloor 99\sqrt{2}+1\rfloor=\frac{1}{2}\lfloor 141.007142\cdots\rfloor=70$, as expected.

1.11.2 Example 1.7 Revisited

Recall from Example 1.7 that $R_n=\frac{1}{2}(a^n+b^n)$ is a positive integer and $R_n\equiv 1,3,7,9,7,$ or $3 \pmod{10}$, where $a=3+2\sqrt{2}=\gamma^2$ and $b=3-2\sqrt{2}=\delta^2$. Consequently, $R_n=\frac{1}{2}(\gamma^{2n}+\delta^{2n})=Q_{2n}$.

Notice that the sequence $\{Q_n \pmod{10}\}$ is periodic with period 12: $\underbrace{1,3,7,7,1,9,9,7,3,3,9,1}$, $\underbrace{1,3,7,7,\ldots}\ldots$. So $Q_{m+12k}\equiv Q_m \pmod{10}$. Since $24,690\equiv 12\cdot 2057+6$, it follows that $R_{12345}\equiv Q_6\equiv 9 \pmod{10}$, as found earlier. (Notice that $\{Q_{2n} \pmod{10}\}$ is periodic with period 6: $\underbrace{3,7,9,7,3,1}$, $\underbrace{3,7,9,7,3,1}\ldots$.)

You will encounter numerous occurrences of Pell and Pell–Lucas numbers in a variety of unrelated contexts in Chapter 2–6. Enjoy them, and look for more in the literature.

Matrices and determinants play an important role in the study of the Pell family. We will take advantage of their considerable power and beauty. First, we briefly introduce these related mathematical structures. (See [6] for a detailed discussion.)

1.12 Matrices and Determinants

The foundation for the theory of matrices was laid by the English mathematicians Arthur Cayley (1821–1895) and James Joseph Sylvester (1814–1897). Matrices are a clean, compact way to store and study groups of data.

For example, suppose you bought 15 coconut donuts, 3 chocolate donuts, and 8 vanilla donuts from Shop I; and 6 coconut donuts, 12 chocolate donuts, and no vanilla donuts from Shop II. These data can be arranged in a compact form:

$$\begin{array}{c} \\ \text{Shop I} \\ \text{Shop II} \end{array} \begin{array}{c} \begin{array}{ccc} \text{coconut} & \text{chocolate} & \text{vanilla} \end{array} \\ \begin{bmatrix} 15 & 3 & 8 \\ 6 & 12 & 0 \end{bmatrix} \end{array}.$$

Suppose you remember that the first row refers to Shop I, second row to Shop II, first column to coconut donuts, and so on. Then we can delete the row and column headings. Let M denote the resulting array:

$$M = \begin{bmatrix} 15 & 3 & 8 \\ 6 & 12 & 0 \end{bmatrix}.$$

Such a rectangular arrangement is a matrix.

More generally, a *matrix* A is a rectangular array of numbers, called *elements*. It is often enclosed by brackets or parentheses to indicate its collective nature. A matrix with m rows and n columns is an $m \times n$ (read m by n) matrix. If $m = n$, then A is a *square matrix of order* n. If $m = 1$, A is a *row vector*; and if $n = 1$, it is a *column vector*.

For example, M is a 2×3 matrix; $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ is a square matrix of order 2; (1,2,5) is a row vector; and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is a column vector.

Let a_{ij} denote the element in row i and column j of an $m \times n$ matrix A, where $1 \le i \le m$ and $1 \le j \le n$. For convenience, we then write $A = (a_{ij})_{m\times n}$. If $a_{ij} = 0$ for every i and j, then A is a *zero matrix*. For example, (0,0,0) and $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ are zero matrices.

Two matrices $(a_{ij})_{m\times n}$ and $(b_{ij})_{r\times s}$ are *equal* if and only if $m = r, n = s$, and $a_{ij} = b_{ij}$ for every i and j. For example, let $\begin{bmatrix} w & 2 \\ 5 & z \end{bmatrix} = \begin{bmatrix} 12 & x \\ y & 29 \end{bmatrix}$. Then $w = 12, x = 2, y = 5$, and $z = 29$.

Just as we can add like things in ordinary algebra, we can add two matrices of the same size.

1.12.1 Matrix Addition

Let $A = (a_{ij})_{m\times n}$ and $(b_{ij})_{m\times n}$. Then $A + B = (a_{ij} + b_{ij})_{m\times n}$; that is, $A + B$ is obtained by adding the corresponding elements in A and B. For example,

$$\begin{bmatrix} 3 & 5 \\ 8 & 13 \end{bmatrix} + \begin{bmatrix} 5 & 8 \\ 13 & 21 \end{bmatrix} = \begin{bmatrix} 8 & 13 \\ 21 & 34 \end{bmatrix}.$$

1.12.2 Scalar Multiplication

Suppose each donut you bought cost 75 cents. Then the costs of each type of donuts at Shops I and II are given by the matrix

$$\begin{bmatrix} 75\cdot 15 & 75\cdot 3 & 75\cdot 8 \\ 75\cdot 6 & 75\cdot 12 & 75\cdot 0 \end{bmatrix} = \begin{bmatrix} 1125 & 225 & 600 \\ 450 & 900 & 0 \end{bmatrix}.$$

This matrix is denoted by $75M$.

More generally, let k be any scalar (real number) and $A = (a_{ij})_{m\times n}$. Then the *product* kA is defined by $kA = (ka_{ij})_{m\times n}$. In particular, $-A = (-1)A = (-a_{ij})_{m\times n}$ is the *negative* of A. Using the negative of a matrix, we can now define matrix subtraction: $A - B = A + (-B)$.

Next we turn to matrix multiplication.

1.12.3 Matrix Multiplication

Let $A = (a_{ij})_{m\times s}$ and $(b_{ij})_{s\times n}$. Their *product* AB is the matrix $C = (c_{ij})_{m\times n}$, where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{is}b_{sj}$. The product AB is defined if and only if the number of columns of A equals the number of rows of B.

For example,

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^2 &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1\cdot 1 + 2\cdot 1 & 1\cdot 2 + 2\cdot 1 \\ 1\cdot 1 + 1\cdot 1 & 1\cdot 2 + 1\cdot 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}; \\ \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 4 \end{bmatrix} &= \begin{bmatrix} 2\cdot 7 + 3\cdot 4 \\ 1\cdot 7 + 2\cdot 4 \end{bmatrix} = \begin{bmatrix} 26 \\ 15 \end{bmatrix}. \end{aligned}$$

A square matrix $A = (a_{ij})_{n\times n}$ is the *identity matrix* of order n if $a_{ij} = 1$ when $i = j$, and zero otherwise. It is denoted by I_n. For example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

1.12.4 Invertible Matrix

A square matrix A of order n is *invertible* if there is a matrix B (of the same size) such that $AB = I_n = BA$. Then $B = A^{-1}$, the *multiplicative inverse* of A.

For example, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \frac{1}{k}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $k = ad - bc \neq 0$. Then $AB = I_2 = BA$, so $B = A^{-1}$.

Unfortunately, *not* every square matrix is invertible. For instance, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not.

Next we turn to determinants, which are closely related to square matrices.

1.12.5 Determinants

Determinants were first discovered by the Japanese mathematician Seki Kōwa (1642–1708). However, the German mathematician Baron Gottfried Wilhelm Leibniz (1646–1716) is widely credited with the discovery, although it came ten years after Kōwa's. Interestingly, they both discovered determinants while solving linear systems of equations.

A *determinant* is a function from the set of square matrices $A = (a_{ij})_{n\times n}$ to the set of real numbers. The determinant of A is denoted by $\det A$ or $|A|$. The latter notation should not be confused with the absolute value of a real number, since the argument A in $|A|$ is a matrix, *not* a number.

The determinant of the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined by

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

In particular, $\begin{vmatrix} 5 & 12 \\ 12 & 29 \end{vmatrix} = 5 \cdot 29 - 12 \cdot 12 = 1$, and $\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = -1$.

We can evaluate the determinant of a square matrix using Laplace's expansion, named after the French mathematician Pierre-Simon Laplace (1749–1827).

1.12.6 Laplace's Expansion

Let $A = (a_{ij})_{n\times n}$ and A_{ij} denote the submatrix obtained by deleting row i and column j of A. Then

$$|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}|$$

is the *Laplace expansion* on $|A|$ by row i, where $1 \le i \le n$. (In fact, $|A|$ can be expanded with respect to any column also.)

For example, let

$$M = \begin{bmatrix} 1 & 2 & 5 \\ 12 & 29 & 70 \\ 169 & 408 & 985 \end{bmatrix}.$$

Expanding $|M|$ with respect to row 1, we get

$$\begin{aligned} |M| &= (-1)^{1+1} \begin{vmatrix} 29 & 70 \\ 408 & 985 \end{vmatrix} + (-1)^{1+2}\, 2 \begin{vmatrix} 12 & 70 \\ 169 & 985 \end{vmatrix} + (-1)^{1+3}\, 5 \begin{vmatrix} 29 & 29 \\ 169 & 408 \end{vmatrix} \\ &= 5 - 2(-10) + 5(-5) = 0. \end{aligned}$$

The *det* function satisfies a number of interesting properties, which can cleverly be used to simplify the task of evaluation.

Finally, we add that a square matrix A is invertible if and only if $|A| \neq 0$. For example, the matrix $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ is invertible, whereas the above matrix M is not.

Exercises 1

Evaluate each, where $\lg x = \log_2 x$.

1. $\sum_{k=3}^{n} (k-2)(k-1)k$.
2. $\sum_{k=1}^{n} \lfloor k/2 \rfloor$.
3. $\sum_{k=1}^{n} \lceil k/2 \rceil$.
4. $\sum_{k=1}^{n} \lg(1 + 1/k)$.
5. $\sum_{k=1}^{2^n} \lfloor \lg(1 + 1/k) \rfloor$.
6. $\sum_{k=1}^{2^n} \lceil \lg(1 + 1/k) \rceil$.
7. $\sum_{k=1}^{n} k \cdot k!$ (M.S. Klamkin, 1963). *Hint*: $k \cdot k! = (k+1)! - k!$.
8. $\prod_{k=2}^{n} \left(1 - \frac{2}{k^3+1}\right)$ (M.S. Klamkin, 1963).
9. $\prod_{k=0}^{n} \left(a^{2^k} + 1\right)$, where $a \neq 1$. (C.W. Trigg, 1965).
10. $\prod_{r=1}^{n} (k+r) \equiv 0 \pmod{n}$.
11. $n^5 \equiv n \pmod{30}$.

Define each sequence $\{a_n\}$ recursively.

12. $1, 2, 5, 26, 677, 458330, \ldots$.
13. $2, 12, 70, 408, 2378, \ldots$.
14. $3, 5, 17, 257, 65537, \ldots$.
15. $1, 1, 2, 4, 7, 13, \ldots$.

16. The 91-*function* f on **W**, invented by John McCarthy[5] (1927–2011), is defined recursively:

$$f(n) = \begin{cases} n - 10 & \text{if } n > 100 \\ f(f(n+11)) & \text{if } 0 \le n \le 100. \end{cases}$$

Compute $f(99)$ and $f(f(99))$.

17. Let u_n be an integer sequence such that $u_0 = 4$ and $u_n = f(u_{n-1})$, where f is a function defined by the following table and $n \geq 1$. Compute u_{99999}. (*Mathematics Teacher* 98 (2004))

x	1	2	3	4	5
$f(x)$	4	1	3	5	2

Hint: First show that $u_{4m+r} = u_r$, where $0 \le r \le 3$.

18. Let $\{a_n\}$ be a sequence defined by $a_n = -a_{n-1} - 2a_{n-2}$, where $a_1 = 1 = -a_2$ and $n \geq 3$. Prove that $2^{n+1} - 7a_{n-1}^2$ is a square. (E. Just, 1972)

Prove each, where $a|b$ means a is a factor of b and $n \geq 1$.

19. $(n+1) \mid \binom{2n}{n}$.
20. $\sum_{r=0}^{n} \binom{n}{r}^2 = \binom{2n}{n}$. (*Lagrange's identity*)
21. $\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$.
22. $F_{n+1} + F_{n-1} = L_n$.
23. $L_{n+1} + L_{n-1} = 5F_n$.
24. $F_n^2 + F_{n+1}^2 = F_{2n+1}$.
25. $L_n^2 + L_{n+1}^2 = 5F_{2n+1}$.
26. $P_n + P_{n-1} = Q_n$.
27. $Q_n + Q_{n+1} = 2P_{n+1}$.
28. $P_{n+1} - P_n = Q_n$.
29. $Q_{n+1} - Q_n = 2P_n$.
30. $P_{n+1} + P_{n-1} = 2Q_n$.
31. $Q_{n+1} + Q_{n-1} = 4P_n$.

Solve each recurrence using the corresponding initial conditions, where b is an integer and $n \geq 3$.

32. $a_n = a_{n-1} + a_{n-2}; a_1 = 1, a_2 = b$.
33. $a_n = 2a_{n-1} + a_{n-2}; a_1 = 1, a_2 = b$.
34. $a_n = a_{n-1} + a_{n-2} - b; a_1 = b + 1 = a_2$. *Hint*: Let $b_n = a_n - b$.

[5] McCarthy coined the term *artificial intelligence* while at Dartmouth College, New Hampshire.

35. $a_n = 2a_{n-1} + a_{n-2} - 2b; a_1 = b + 1, a_2 = 2b$. *Hint*: Let $b_n = a_n - b$.

Find a generating function for each sequence $\{a_n\}$, which is defined recursively.

36. $a_n = a_{n-1} + n; a_0 = 0$.
37. $a_n = 6a_{n-1} - a_{n-2}; a_0 = 0, a_1 = 6$.
38. $a_n = 6a_{n-2} - a_{n-4}; a_1 = 1, a_2 = 5, a_3 = 11, a_4 = 31$.
39. $a_n = 34a_{n-1} - a_{n-2} + 2; a_1 = 1, a_2 = 36$.

Let $Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

40. Prove that $Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$, where $n \geq 1$.
41. Deduce Cassini's formula for Fibonacci numbers. *Hint*: $|AB| = |A| \cdot |B|$, where A and B are square matrices of the same size.
42. Establish the addition formula $F_{m+n} = F_m F_{n+1} + F_{m-1} F_n$.
43. Is Q invertible? If yes, find Q^{-1}.
44. Find $(Q - I)^{-1}$, if possible.
45. Prove that $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$. *Hint*: $I + Q + Q^2 + \cdots + Q^n = Q^{n+2} - Q$.

Let $P = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$.

46. Prove that $P^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}$, where $n \geq 1$.
47. Deduce that $P_{n+1} P_{n-1} - P_n^2 = (-1)^n$.
48. Prove the addition formula $P_{m+n} = P_m P_{n+1} + P_{m-1} P_n$.
49. Is P invertible? If yes, find P^{-1}.

2

Pell's Equation

2.1 Introduction

In February, 1657, Fermat challenged the English mathematicians John Wallis (1616–1703) and Lord William V. Brouncker (1620–1684) to solve the non-linear diophantine equation $x^2 - dy^2 = 1$, where d is nonsquare and positive. The amateur French mathematician Bernard de Bessey (ca. 1605–1675) solved it for $d \leq 150$. This equation is now called *Pell's equation*, since the great Swiss mathematician Leonhard Euler (1707–1783) erroneously attributed Brouncker's work to John Pell (1611–1685) of England. In fact, Pell's contribution to the analysis of the equation is negligible, since he was "revising someone's translation [Wallis'] to someone else's algebra [10]."

Historically, Indian mathematicians knew how to solve the equation as early as 800 A.D. Around 650 A.D., Brahmagupta (ca. 598–ca. 670) wrote that "a person who can solve the equation to $x^2 - 92y^2 = 1$ within a year is a mathematician." Its least positive solution is $x = 1151, y = 120 : 1151^2 - 92 \cdot 120^2 = 1$. Acharya Jaydeva (ca. 1000) and Bhaskara II described a method for solving Pell's equation.

Unfortunately, the equation has been given multiple names. Some authors called it *Pellian equation*, some the *Pell equation*, and some the *Fermat equation*. In 1963, Clas-Olaf Selenius of the University of Uppsala called it the *Bhaskara–Pell equation*; four years later, the Indian mathematician C.N. Srinivasa Iyengar called it the *Brahmagupta–Bhaskara equation*. In 1975, Salenius changed his mind and wrote that "perhaps the *Jayadeva–Bhaskara* equation would be the best." The Museum of Science in Boston, Massachusetts, calls it the *Pell equation* in its display of the contributions of Bhaskara II.

The famous *cattle problem* [10] by the Greek mathematician Archimedes (287–212 B.C.) involves solving Pell's equation $x^2 - 4729494y^2 = 1$. Predictably, its solutions are so enormous, they are too large for all scientific calculators. In fact, many doubt whether the cattle problem was indeed proposed by Archimedes; even if he did propose it, he could not possibly have solved it. In 1768, Lagrange provided a proof of a method for solving Pell's equation using Euler's work on the topic and continued fractions.

T. Koshy, *Pell and Pell–Lucas Numbers with Applications*, DOI 10.1007/978-1-4614-8489-9_2,

Although studied by mathematicians for over 1300 years, Pell's equation continues to be an area of mathematical interest. H.C. Williams of the University of Calgary, Alberta, Canada, said at the 2000 *Millennial Conference on Number Theory,* held at the University of Illinois, Urbana, that over 100 articles had been published on Pell's equation in the 1990s. As part of algebraic number theory, Pell's equation has applications to computer science, factoring of large integers, and cryptography [153] and [260]. In 2003, E.J. Barbeau of the University of Toronto, Canada, wrote a book devoted to Pell's equation [9].

In 1991, James P. Jones of the University of Calgary and Y.V. Matijasevič of the Steklov Mathematical Institute, Leningrad, Russia (both logic number-theorists), employed Pell's equation $x^2 - dy^2 = 1$ to establish the recursive unsolvability of the Tenth Problem of the great German mathematician David Hilbert (1862–1943): Find an algorithm to determine the solvability of the diophantine polynomial equation $P(x_1, x_2, \ldots, x_n) = 0$. Hilbert proposed this problem in 1900. Although Matijasevič had proved its unsolvability in 1970, the 1991 proof is much shorter.

It may not seem obvious at first that there is an extremely close relationship between Pell's equation $x^2 - dy^2 = 1$ and continued fractions. But Chapter 3 will show that there is.

How do we solve Pell's equation? When $d = 0$, it has infinitely many solutions: $(\pm 1, y)$, where y is arbitrary. Suppose $d \leq 2$. Then $x^2 - dy^2 \geq 1$, except when $x = y = 0$. So the only two solutions are $(\pm 1, 0)$. If $d = -1$, then $x^2 + y^2 = 1$; it has exactly four solutions: $(\pm 1, 0), (0, \pm 1)$. These are *trivial solutions*.

Suppose d is a square D^2. Then Pell's equation becomes $x^2 - D^2y^2 = 1$; that is, $(x + Dy)(x - Dy) = 1$. So $x + Dy = x - Dy = \pm 1$. Solving these two linear systems, we will get a solution in each case.

Thus we know how to solve Pell equation when $d \leq 0$ or d is a square. So we turn to the case when d is nonsquare and positive; we will assume this throughout our discourse. Suppose (x, y) is a solution. Then $(x, -y)$ and $(-x, \pm y)$ are also solutions. Consequently, knowing all positive solutions will enable us to find all solutions. So we confine our pursuit to positive integral solutions.

The simplest Pell equation is $x^2 - 2y^2 = 1$; that is, $1 + 2y^2 = x^2$. By inspection, $(\alpha, \beta) = (3,\ 2)$ is the solution with the least positive value of x: $3^2 - 2 \cdot 2^2 = 1$; it is the *fundamental solution*. The next one is $(17,\ 12) : 17^2 - 2 \cdot 12^2 = 1$.

The following theorem shows how we can compute all solutions from the fundamental solution. We omit its proof [130] for the sake of brevity.

Theorem 2.1 *Let* (α, β) *be the fundamental solution of Pell's equation* $x^2 - dy^2 = 1$. *It has infinitely many solutions* (x_n, y_n)*:*

$$
\begin{aligned}
x_n &= \frac{1}{2}\left[(\alpha + \beta\sqrt{d})^n + (\alpha - \beta\sqrt{d})^n\right] \\
y_n &= \frac{1}{2\sqrt{d}}\left[(\alpha + \beta\sqrt{d})^n - (\alpha - \beta\sqrt{d})^n\right],
\end{aligned}
$$

where $(x_1, y_1) = (\alpha, \beta)$ *and* $n \geq 2$. ∎

The next two examples illustrate the theorem.

Example 2.1 Find six solutions of Pell's equation $x^2 - 2y^2 = 1$.

Solution. Since the fundamental solution is $(\alpha, \beta) = (3, 2)$, by Theorem 2.1, the general solution (x_n, y_n) is given by

$$x_n = \frac{1}{2}\left[(3+2\sqrt{2})^n + (3-2\sqrt{2})^n\right] = \frac{\gamma^{2n}+\delta^{2n}}{2} = Q_{2n}$$

$$y_n = \frac{1}{2\sqrt{2}}\left[(3+2\sqrt{2})^n - (3-2\sqrt{2})^n\right] = \frac{\gamma^{2n}-\delta^{2n}}{\gamma-\delta} = P_{2n}.$$

Then $(x_2, y_2) = (Q_4, P_4) = (17, 12)$ is a solution. Likewise, (99, 70) is a solution. The first six solutions are listed in Table 2.1.

Table 2.1.

n	x_n	y_n
1	3	2
2	17	12
3	99	70
4	577	408
5	3363	2378
6	19601	13860

■

The next example presents a geometric application of the square triangular problem, studied by M.E. Larsen of Denmark in 1987 [149]. We will study it in detail in Chapter 6.

Example 2.2 Identify the triangular arrays that contain a square number of bricks, as in Figure 2.1.

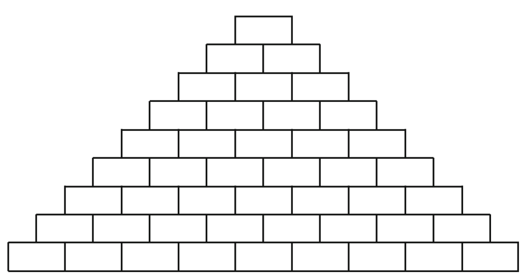

Figure 2.1.

Solution. Let x denote the number of rows of bricks in the triangular array. The number of bricks in the array equals $t_x = \frac{x(x+1)}{2}$. We want $t_x = y^2$ for some positive integer y. So $x(x+1) = 2y^2$. Letting $u = 2x+1$ and $v = 2y$, this yields Pell's equation $u^2 - 2v^2 = 1$:

$$\begin{aligned} u^2 - 2v^2 &= (2x+1)^2 - 2(2y)^2 \\ &= 4(x^2 + x - 2y^2) + 1 \\ &= 1. \end{aligned}$$

By Example 2.1, the solutions of this equation are given by $(u_n, v_n) = (Q_{2n}, P_{2n})$, where $(u_1, v_1) = (Q_2, P_2) = (3, 2)$ and $n \geq 1$. Correspondingly, $x_n = \frac{u_n - 1}{2}$ and $y_n = \frac{v_n}{2}$.

Table 2.2 shows the first five possible solutions (x_n, y_n) of the brick problem.

Table 2.2.

n	u_n	v_n	x_n	y_n
1	3	2	1	1
2	17	12	8	6
3	99	70	49	35
4	577	408	288	204
5	3363	2378	1681	1189

■

The next example will reappear in Chapter 6.

Example 2.3 Solve Pell's equation $x^2 - 8y^2 = 1$.

Solution. This equation can be rewritten as $x^2 - 2z^2 = 1$, where $z = 2y$ is even. By Example 2.1, its general solution is $(x_n, z_n) = (Q_{2n}, P_{2n})$. Consequently, the general solution of the given equation is $(x_n, y_n) = (Q_{2n}, \frac{1}{2}P_{2n})$. (This implies that P_{2n} is an even integer; see Chapter 7.)

In particular, $(x_1, y_1) = (3, 1)$ and $(x_2, y_2) = (17, 6)$ are two solutions of the given equation. ■

Example 2.4 Find three solutions of Pell's equation $x^2 - 7y^2 = 1$.

Solution. Solving the given equation amounts to solving $1 + 7y^2 = x^2$. By trial and error, we find that $(\alpha, \beta) = (8, 3)$ is the fundamental solution. The remaining solutions are given by:

$$\begin{aligned} x_n &= \frac{1}{2}\left[(8 + 3\sqrt{7})^n + (8 - 3\sqrt{7})^n\right] \\ y_n &= \frac{1}{2\sqrt{7}}\left[(8 + 3\sqrt{7})^n - (8 - 3\sqrt{7})^n\right], \quad n \geq 2. \end{aligned}$$

When $n = 2$:

$$\begin{aligned} x_2 &= \frac{1}{2}\left[(8 + 3\sqrt{7})^2 + (8 - 3\sqrt{7})^2\right] = 127 \\ y_2 &= \frac{1}{2\sqrt{7}}\left[(8 + 3\sqrt{7})^2 - (8 - 3\sqrt{7})^2\right] = 48. \end{aligned}$$

So $(x_2, y_2) = (127, 48)$ is a solution. Likewise, $(x_3, y_3) = (2024, 765)$ is also a solution. ■

As d gets larger, it is not easy for us to find the fundamental solution by inspection. We can resort to continued fractions, as we will see in Chapter 3.

The next example, *the problem of the square pyramid*, is also a geometric application of Pell's equation, again studied by Larsen [149]. It was originally studied by Lucas in 1875: "The number of cannon balls piled in a pyramid on a square base is a perfect square. Show that the number of balls on a side of the base is 24." The same problem is discussed by the well-known English puzzlist Henry E. Dudeney (1857–1930) in his 1917 book, *Amusements in Mathematics*. It was also investigated by the English mathematician George N. Watson (1886–1965) in 1919 [254].

Example 2.5 Consider the brick pyramid in Figure 2.2. Identify such pyramids that have the property that the total number of bricks in each pyramid is a square.

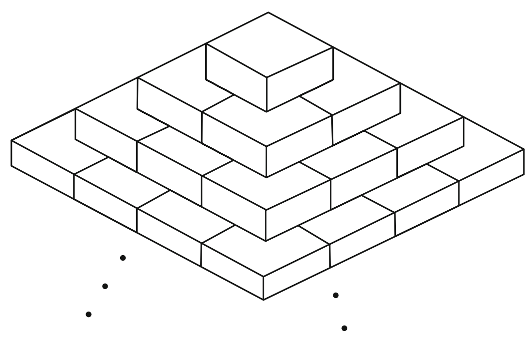

Figure 2.2.

Solution. Let x denote the number of layers of bricks in the pyramid. Since the number of bricks in each layer is a square, the total number of bricks in the pyramid equals $\sum_{i=1}^{x} i^2 = \frac{x(x+1)(2x+1)}{6}$. We want this to be a square y^2:

$$\frac{x(x+1)(2x+1)}{6} = y^2. \tag{2.1}$$

Clearly, $(x, y) = (1,\ 1)$ is a solution.

The product of two consecutive positive integers is never a square. One way of solving it is by letting $2x + 1 = u^2$ and $x + 1 = v^2$. Then the equation becomes $x = 6\left(\frac{y}{uv}\right)^2$; that is, $\frac{u^2-1}{2} = 6t^2$, where $t = \frac{y}{uv}$. So $u^2 - 3(2t)^2 = 1$; that is, $u^2 - 3w^2 = 1$, where $w = 2t = \frac{2y}{uv}$.

By Theorem 2.1, the solutions of this Pell's equation are given by

$$\begin{aligned} u_n &= \frac{1}{2}\left[(2+\sqrt{3})^n + (2-\sqrt{3})^n\right] \qquad (2.2)\\ w_n &= \frac{1}{2\sqrt{3}}\left[(2+\sqrt{3})^n - (2-\sqrt{3})^n\right]. \end{aligned}$$

Knowing u_n, we can compute the corresponding x_n from the substitution $u^2 = 2x+1$; so we will focus on u_n. Once we know x_n, we can return to equation (2.1) to compute the corresponding y_n. Notice that, since $u_n^2 = 2x_n + 1$, every u_n is odd.

From equation (2.2), $u_1 = \frac{1}{2}[(2+\sqrt{3}) + (2-\sqrt{3})] = 2$, which is not odd; so it is *not* an acceptable solution of $u^2 = 2x + 1$.

When $n = 2$, $u_2 = \frac{1}{2}[(2+\sqrt{3})^2 + (2-\sqrt{3})^2] = 7$. This is clearly valid. Correspondingly, $2x + 1 = 49$; so $x = 24$. Then $y^2 = \frac{24\cdot 25\cdot 49}{6} = 4,900$; so $y = 70$. Thus (24, 70) is a nontrivial solution of equation (2.1): $\frac{24(24+1)(2\cdot 24+1)}{6} = 70^2$.

Are there other nontrivial solutions? In 1919, Watson proved that this is the *only* nontrivial solution to the problem. We now add that since u_n is an integer and $0 < 2-\sqrt{3} < 1$, it follows that $u_n = \lceil (2+\sqrt{3})^n/2 \rceil$. ■

The following example is an interesting application to statistics; it shows the occurrence of Pell's equation in strange and unrelated places. It was proposed as a problem in the *American Mathematical Monthly* in 1989 by Jim Delany of California Polytechnic State University, San Luis Obispo [60].

Example 2.6 The *mean and standard deviation* of any seven consecutive positive integers are both integers. Find integers $n(\geq 2)$ that share this property with 7.

Solution. The mean μ (Greek letter *mu*) of n numbers $x_1, x_2, \ldots, x_n$ and their standard deviation σ (Greek letter *sigma*) are given by $\mu = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2}$, respectively.

We will investigate the general case by considering the arithmetic sequence $a, a+d, \ldots, a+(n-1)d$, where a is an arbitrary integer and d the common integral difference. Their mean is given by $\mu = \frac{1}{n}\left[na + \frac{(n-1)n}{2}d\right] = a + \frac{(n-1)d}{2}$; and their standard deviation by

$$\begin{aligned}
n\sigma^2 &= d^2\sum_{i=0}^{n-1}(i-t)^2 \\
&= d^2\sum_{i=0}^{n-1}(i^2 - 2it + t^2) \\
&= d^2\frac{n(n^2-1)}{12} \\
\sigma &= \frac{d}{2}\sqrt{\frac{n^2-1}{3}},
\end{aligned}$$

after some basic algebra, where $t = \frac{n-1}{2}$.

Case 1 Let n be odd. Then $n^2 \equiv 1 \pmod 8$ and $(n-1)d$ is even; so μ is integral. Consequently, σ is integral if and only if $\sqrt{\frac{n^2-1}{3}}$ is a positive even integer u; that is, if and only if (n, u) is a solution of Pell's equation $n^2 - 3u^2 = 1$, where u is even.

Case 2 Let n be even. Then μ is integral if and only if d is even. So σ is integral if and only if $\frac{n^2-1}{3}$ is a square u^2; this yields Pell's equation $n^2-3u^2 = 1$, where u is an integer, not necessarily even.

From the previous example, the solutions (n_k, u_k) of Pell's equation are given by

$$\begin{aligned} n_k &= \frac{1}{2}\left[(2+\sqrt{3})^k + (2-\sqrt{3})^k\right] \\ u_k &= \frac{1}{2\sqrt{3}}\left[(2+\sqrt{3})^k - (2-\sqrt{3})^k\right]. \end{aligned} \tag{2.3}$$

Since n_k is an integer and $0 < 2 - \sqrt{3} < 1$, it follows that $n_k = \lceil (2+\sqrt{3})^k/2 \rceil$, where $k \geq 1$.

It follows from (2.3) by the binomial theorem that

$$\begin{aligned} 2n_k &= \sum_{i=0}^{k} \binom{k}{i} 2^{k-i} \left[(\sqrt{3})^i + (-\sqrt{3})^i\right] \\ n_k &= \sum_{i=0}^{k-1} \binom{k}{i} 2^{k-i-1} \left[(\sqrt{3})^i + (-\sqrt{3})^i\right] + \frac{1}{2}\left[1 + (-1)^k\right] 3^{k/2} \\ &= \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2i} 2^{k-2i} 3^i + \frac{1}{2}\left[1 + (-1)^k\right] 3^{k/2} \\ &= A + \frac{1}{2}\left[1 + (-1)^k\right] 3^{k/2}, \end{aligned} \tag{2.4}$$

where A is even since $k - 2i \geq 1$.

It follows from (2.4) that n_k is odd if and only if k is even. When n_k is even, k is odd and d even. Let $k = 2j + 1$, where $j \geq 0$. The smallest five such values of n_{2j+1} are 2, 26, 362, 5042, and 70226.

Suppose d is odd. (In particular, d can be one, as in the given problem.) Then n_k is odd. So k is even, say $k = 2j$, where $j \geq 1$. Correspondingly, $n_{2j} = \lceil (7 + 4\sqrt{3})^j/2 \rceil$. The smallest five values of n_{2j} are 7, 97, 1351, 18817, and 262087. ∎

2.2 Pell's Equation $x^2 - dy^2 = (-1)^n$

Closely related to $x^2 - dy^2 = 1$ is Pell's equation $x^2 - dy^2 = -1$. *Not* every such equation is solvable. For example, suppose the equation $x^2 - 3y^2 = -1$ is solvable. Then $x^2 \equiv -1 \equiv 2 \pmod 3$. But this congruence is not solvable, so Pell's equation $x^2 - 3y^2 = -1$ is not solvable. But when $x^2 - dy^2 = -1$ is solvable, its infinitely many solutions can be generated from its fundamental solution, as the next theorem shows [130].

Theorem 2.2 *Let* (α, β) *be the fundamental solution of Pell's equation* $x^2 - dy^2 = -1$. *It has infinitely many solutions* (x_n, y_n), *given by*

$$\begin{aligned} x_n &= \frac{1}{2}\left[(\alpha + \beta\sqrt{d})^{2n-1} + (\alpha - \beta\sqrt{d})^{2n-1}\right] \\ y_n &= \frac{1}{2\sqrt{d}}\left[(\alpha + \beta\sqrt{d})^{2n-1} - (\alpha - \beta\sqrt{d})^{2n-1}\right], \end{aligned}$$

where $(x_1, y_1) = (\alpha, \beta)$ *and* $n \geq 2$. ∎

The formulas for the solution (x_n, y_n) in Theorems 2.1 and 2.2, although they look simple and elegant, are not practical. As n gets larger and larger, expanding $(\alpha \pm \beta\sqrt{d})^n$ gets more and more complicated, and simplifications become more and more tedious.

We will make this task far less cumbersome by developing a simple recursive formula for constructing all solutions of the equation $x^2 - dy^2 = (-1)^n$ from its fundamental solution. Its beauty lies in the fact that we do not have to deal with the binomial theorem or radicals. In addition, the recursive approach can easily be implemented with a computer. But first some new vocabulary.

2.3 Norm of a Quadratic Surd

Let x and y be rational numbers. Then the number $u = x + y\sqrt{d}$ is a *quadratic surd*. Its *conjugate* $\overline{u}$ is $x - y\sqrt{d}$. Its *norm* $N(u)$ is given by $N(u) = u\overline{u} = (x + y\sqrt{d})(x - y\sqrt{d}) = x^2 - dy^2$. Clearly, $N(u)$ is the LHS of the equation $x^2 - dy^2 = (-1)^n$.

For example, $N(3 + 2\sqrt{2}) = 3^2 - 2 \cdot 2^2 = 1$ and $N(17 - 6\sqrt{8}) = 17^2 - 8 \cdot 6^2 = 1$

The conjugate and norm functions satisfy several interesting properties; some are listed below, where u and v are quadratic surds:

(1)	$\overline{\overline{u}} = u$	(2)	$\overline{u \pm v} = \overline{u} \pm \overline{v}$
(3)	$\overline{uv} = \overline{u}\,\overline{v}$	(4)	$\overline{\left(\frac{u}{v}\right)} = \frac{\overline{u}}{\overline{v}}$, where $v \neq 0$
(5)	$\overline{N(u)} = N(u)$	(6)	$N(\overline{u}) = N(u)$
(7)	$N(uv) = N(u)N(v)$	(8)	$N\left(\frac{u}{v}\right) = \frac{N(u)}{N(v)}$, where $v \neq 0$.

We omit their proofs in the interest of brevity. [It follows from property (5) that $N(u)$ is a rational number.]

2.4 Recursive Solutions

The following theorem gives the desired recursive formula.

Theorem 2.3 *Let* (x_n, y_n) *be an arbitrary solution of* $x^2 - dy^2 = (-1)^n$*, where* x *and* y *are positive integers. Then*

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} \alpha & d\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix},$$

where $(x_1, y_1) = (\alpha, \beta)$ *denotes its fundamental solution and* $n \geq 2$.

Proof. Assume that (x_{n-1}, y_{n-1}) is a solution. Then

$$\begin{aligned} x_n + y_n\sqrt{d} &= (\alpha x_{n-1} + \beta d y_{n-1}) + (\beta x_{n-1} + \alpha y_{n-1})\sqrt{d} \\ &= (\alpha + \beta\sqrt{d})(x_{n-1} + y_{n-1}\sqrt{d}) \\ N(x_n + y_n\sqrt{d}) &= N(\alpha + \beta\sqrt{d}) \cdot N(x_{n-1} + y_{n-1}\sqrt{d}) \\ x_n^2 - dy_n^2 &= (\alpha^2 - d^2\beta^2)(x_{n-1}^2 - dy_{n-1}^2) \\ &= (-1) \cdot (-1)^{n-1} \\ &= (-1)^n. \end{aligned}$$

Since (x_1, y_1) is a solution, the result follows by PMI. ■

For example, $(\alpha, \beta) = (73,\ 12)$ is the fundamental solution of $x^2 - 37y^2 = 1 : 73^2 - 37 \cdot 12^2 = 1$. By Theorem 2.3,

$$\begin{aligned} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} &= \begin{bmatrix} 73 & 73 \cdot 12 \\ 12 & 73 \end{bmatrix} \begin{bmatrix} 73 \\ 12 \end{bmatrix} \\ &= \begin{bmatrix} 10567 \\ 1752 \end{bmatrix} \end{aligned}$$

is also a solution: $10567^2 - 37 \cdot 1752^2 = 1$.

Theorem 2.3 yields two interesting byproducts:

(1) Since $x_n = \alpha x_{n-1} + d\beta y_{n-1} > 1 \cdot x_{n-1} + d \cdot 0 \cdot y_{n-1} = x_{n-1}$ and $y_n = \beta x_{n-1} + \alpha y_{n-1} > 0 \cdot x_{n-1} + 1 \cdot y_{n-1} = y_{n-1}$, it follows that $x_n, y_n > 0$, and that both $\{x_n\}$ and $\{y_n\}$ are increasing sequences. Consequently, when Pell's equation is solvable, it has infinitely many solutions.

(2) When n is even, (x_n, y_n) is a solution of $x^2 - dy^2 = 1$; otherwise, it is a solution of $x^2 - dy^2 = -1$. That is, (x_{2n}, y_{2n}) is a solution of $x^2 - dy^2 = 1$ and (x_{2n-1}, y_{2n-1}) a solution of $x^2 - dy^2 = -1$.

2.4.1 A Second-Order Recurrence for (x_n, y_n)

Theorem 2.3, coupled with recursion and matrices, can be used to develop a second-order recurrence for the solution (x_n, y_n). To this end, first notice that

$$\begin{aligned}\begin{bmatrix}\alpha & d\beta\\ \beta & \alpha\end{bmatrix}^2 &= \begin{bmatrix}\alpha^2+d\beta^2 & 2d\alpha\beta\\ 2\alpha\beta & \alpha^2+d\beta^2\end{bmatrix}\\ &= \begin{bmatrix}2\alpha^2-(-1)^n & 2d\alpha\beta\\ 2\alpha\beta & 2\alpha^2-(-1)^n\end{bmatrix}\\ &= 2\alpha\begin{bmatrix}\alpha & d\beta\\ \beta & \alpha\end{bmatrix} - (-1)^n\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.\end{aligned}$$

Consequently, by Theorem 2.3, we have

$$\begin{aligned}\begin{bmatrix}x_n\\ y_n\end{bmatrix} &= \begin{bmatrix}\alpha & d\beta\\ \beta & \alpha\end{bmatrix}\begin{bmatrix}x_{n-1}\\ y_{n-1}\end{bmatrix}\\ &= \begin{bmatrix}\alpha & d\beta\\ \beta & \alpha\end{bmatrix}^2\begin{bmatrix}x_{n-2}\\ y_{n-2}\end{bmatrix}\\ &= \left(2\alpha\begin{bmatrix}\alpha & d\beta\\ \beta & \alpha\end{bmatrix} - (-1)^n\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}x_{n-2}\\ y_{n-2}\end{bmatrix}\\ &= 2\alpha\begin{bmatrix}x_{n-1}\\ y_{n-1}\end{bmatrix} - (-1)^n\begin{bmatrix}x_{n-2}\\ y_{n-2}\end{bmatrix}. \qquad (2.5)\end{aligned}$$

This is the desired second-order recurrence for (x_n, y_n).

2.5 Solutions of $x^2 - 2y^2 = (-1)^n$

Consider Pell's equation $x^2-2y^2 = -1$. Its fundamental solution is $(x_1, y_1) = (\alpha, \beta) = (1,\ 1)$. By Theorem 2.3, its remaining solutions are given by the recurrence

$$\begin{bmatrix}x_n\\ y_n\end{bmatrix} = \begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}x_{n-1}\\ y_{n-1}\end{bmatrix},$$

where n is odd. This implies that $x_n = x_{n-1} + 2y_{n-1}$ and $y_n = x_{n-1} + y_{n-1}$, where $n \geq 2$. It follows from these two recurrences that both x_n and y_n satisfy the Pell recurrence; so $(x_n, y_n) = (Q_n, P_n)$, where n is odd.

This follows from the second-order recurrence (2.5) also when n is odd:

$$\begin{bmatrix}x_n\\ y_n\end{bmatrix} = 2\begin{bmatrix}x_{n-1}\\ y_{n-1}\end{bmatrix} + \begin{bmatrix}x_{n-2}\\ y_{n-2}\end{bmatrix}.$$

Clearly, both x_n and y_n satisfy the Pell recurrence. Since $x_1 = 1 = Q_1, x_2 = 7 = Q_3, y_1 = 1 = P_1$, and $y_2 = 5 = P_2$, it follows that $(x_n, y_n) = (Q_n, P_n)$, where $n \geq 1$. Thus $(x_{2n}, y_{2n}) = (Q_{2n}, P_{2n})$ gives solutions of $x^2 - 2y^2 = 1$ and $(x_{2n-1}, y_{2n-1}) = (Q_{2n-1}, P_{2n-1})$ solutions of $x^2 - 2y^2 = -1$.

Since (Q_{2n}, P_{2n}) is a solution of $x^2 - 2y^2 = 1$, it follows by Theorem 2.3 that

$$\begin{bmatrix} Q_{2n} \\ P_{2n} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} Q_{2n-2} \\ P_{2n-2} \end{bmatrix}.$$

Thus $Q_{2n} = 3Q_{2n-2} + 4P_{2n-2}$ and $P_{2n} = 2Q_{2n-2} + 3P_{2n-2}$, where $n \geq 2$.

2.5.1 An Interesting Byproduct

Since (Q_n, P_n) is a solution of $x^2 - 2y^2 = (-1)^n$, it follows that $Q_n^2 - 2P_n^2 = (-1)^n$. (We will revisit this identity in Chapter 7.) For example, $Q_5^2 - 2P_5^2 = 41^2 - 2 \cdot 29^2 = (-1)^5$ and $Q_6^2 - 2P_6^2 = 99^2 - 2 \cdot 70^2 = (-1)^6$.

To digress a bit, consider the Pell equation $x^2 - 8y^2 = (-1)^n$[†]; that is, $x^2 - 2u^2 = (-1)^n$, where $u = 2y$ [7]. Every solution of $x^2 - 8y^2 = (-1)^n$ is of the form $(x_n, y_n) = (Q_n, \frac{1}{2}P_n)$, where $n \geq 1$. Suppose n is even, say, $n = 2m$. Then $(x_n, y_n) = (Q_{2m}, \frac{1}{2}P_{2m}) = (Q_{2m}, P_m Q_m)$, since $P_{2m} = 2P_m Q_m$ (see Chapter 7).

On the other hand, let $n = 2m + 1$. Then $(x_n, y_n) = (Q_{2m+1}, \frac{1}{2}P_{2m+1})$. Since P_{2m+1} has odd parity (see Chapter 7), the equation $x^2 - 8y^2 = (-1)^n$ has *no* integer solutions. The first three such solutions are $(1, \frac{1}{2})$, $(7, \frac{5}{2})$, and $(41, \frac{1}{29})$.

2.6 Euler and Pell's Equation $x^2 - dy^2 = (-1)^n$

Euler found that if (u, v) is a solution of $x^2 - dy^2 = -1$, then $(2u^2 + 1, 2uv)$ is a solution of $x^2 - dy^2 = 1$:

$$\begin{aligned}(2u^2 + 1)^2 - d(2uv)^2 &= 4u^2(u^2 + 1) + 1 - 4du^2v^2 \\ &= 4u^2(dy^2) + 1 - 4du^2v^2 \\ &= 1.\end{aligned}$$

Likewise, if (u, v) is a solution of $x^2 - dy^2 = 1$, then so is $(2u^2 - 1, 2uv)$.

In particular, let $d = 2$. Since (Q_{2n-1}, P_{2n-1}) is a solution of $x^2 - 2y^2 = -1$, it follows that $(2Q_{2n-1}^2 + 1, 2P_{2n-1}Q_{2n-1})$ is a solution of $x^2 - 2y^2 = 1$. So $(2Q_{2n-1}^2 + 1)^2 - 2(2P_{2n-1}Q_{2n-1})^2 = 1$. On the other hand, (Q_{2n}, P_{2n}) is a solution of $x^2 - 2y^2 = 1$; so is $(2Q_{2n}^2 - 1, 2P_{2n}Q_{2n})$. Thus $(2Q_{2n}^2 - 1)^2 - 2(2P_{2n}Q_{2n})^2 = 1$.

The next example illustrates the relevance of Pell's equation in geometry. It is well known that the Pythagorean triangle with sides 3, 4, and 5 has two interesting properties: The sides are

[†] This is a special case of the Pell equation $x^2 - (k^2 + 4)y^2 = (-1)^n$.

consecutive integers and the area is an integer, namely, 6 (a *perfect number*); see [130] for a discussion of perfect numbers. A triangle with sides 13, 14, and 15 also has both properties. Are there other such triangles? We will now show that there is an infinitude of such triangles.

Example 2.7 Prove that there are infinitely many triangles whose sides are consecutive integers, and whose area is an integer.

Proof. Let a, b, and c denote the lengths of the sides of the triangle, and s its semi-perimeter. Then, by *Heron's formula*, its area A is given by $A = \sqrt{s(s-a)(s-b)(s-c)}$.

In particular, let $a = t-1, b = t$, and $c = t+1$, where $t \geq 4$. Then

$$\begin{aligned} A &= \sqrt{\frac{3t}{2} \cdot \frac{t+2}{2} \cdot \frac{t}{2} \cdot \frac{t-2}{2}} \\ &= \frac{t}{4}\sqrt{3(t^2-4)}. \end{aligned}$$

Since A is an integer, t must be even, say $t = 2x$. Then $A = x\sqrt{3(x^2-1)}$. Furthermore, $3(x^2-1)$ must be a square, say, u^2. Then $3(x^2-1) = u^2$ implies that $3|u$. Letting $u = 3y$, this yields Pell's equation $x^2 - 3y^2 = 1$.

Since (2, 1) is its fundamental solution, by Theorem 2.2, this Pell's equation has infinitely many solutions, given by $x_n = 2x_{n-1} + 3y_{n-1}, y_n = x_{n-1} + 2y_{n-1}$, where $n \geq 2$. Consequently, there are infinitely many triangles with sides $2x_n - 1, 2x_n$, and $2x_n + 1$, and area $A = x_n\sqrt{3(x_n^2-1)} = x_n\sqrt{3(3y_n^2)} = 3x_n y_n$. ∎

The next two triangles with the desired properties have sides 51, 52, and 53; and 193, 194, and 195. Their areas are 1170 and 16296, respectively.

To pursue this example a bit further, it follows from the recurrences that $x_n \equiv y_{n-1} \pmod 2$ and $y_n \equiv x_{n-1} \pmod 2$. So $x_{n+2} \equiv x_n \pmod 2$ for every $n \geq 1$. But x_1 is even. Therefore, x_{2n-1} is even. Likewise, x_{2n} is odd. Thus $x_n \equiv n+1 \pmod 2$. Similarly, $y_n \equiv n \pmod 2$. Consequently, x_n and y_n have opposite parity. Thus the area of every triangle in this example is an even integer.

Another application of Theorem 2.3 is the *root-mean-square problem*, which appeared in the 1986 USA Mathematical Olympiad.

Example 2.8 The root-mean-square (rms) of n positive integers $a_1, a_2, \ldots, a_n$ is given by

$$\text{rms}(a_1, a_2, \cdots, a_n) = \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}.$$

Clearly, rms(1) is the integer 1. Find, if possible, an integer $n \geq 2$ such that $\text{rms}(1, 2, \ldots, n)$ is an integer.

Solution. Notice that rms(1, 2)≈ 1.58, rms(1,2,3)≈ 2.16, and rms(1,2,3,4)≈ 2.74; they are not integers. Clearly, this approach is *not* practical.

So we will let the power of solving Pell's equation $x^2 - dy^2 = 1$ do the job for us. To this end, suppose $\text{rms}(1, 2, \ldots, n) = m^2$ for some positive integer m; that is,

$$\frac{1^2 + 2^2 + 3^2 + \cdots + n^2}{n} = m^2.$$

Using summation formula (2), this yields $2n^2 + 3n + 1 - 6m^2 = 0$. Completing the square, the equation becomes $(4n + 3)^2 - 48m^2 = 1$; that is, $x^2 - 3y^2 = 1$, where $x = 4n + 3$ and $y = 4m$. Thus we are looking for a solution (x, y) of Pell's equation $x^2 - 3y^2 = 1$, where $x \equiv -1 \pmod 4$, $y \equiv 0 \pmod 4$, and $x \geq 13$.

Since $(\alpha, \beta) = (2, 1)$ is the fundamental solution of $x^2 - 3y^2 = 1$, by Theorem 2.3, the remaining solutions are given by the recursive formula

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix},$$

where $n \geq 2$. We will now use this formula to find a solution (x_n, y_n) such that $x_n \equiv -1 \pmod 4$, $y_n \equiv 0 \pmod 4$, and $x_n \geq 13$.

Using iteration, this matrix formula yields

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \end{bmatrix} = \begin{bmatrix} 26 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} x_4 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 26 \\ 15 \end{bmatrix} = \begin{bmatrix} 97 \\ 56 \end{bmatrix} \qquad \begin{bmatrix} x_5 \\ y_5 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 97 \\ 56 \end{bmatrix} = \begin{bmatrix} 362 \\ 209 \end{bmatrix}$$

$$\begin{bmatrix} x_6 \\ y_6 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 362 \\ 209 \end{bmatrix} = \begin{bmatrix} 1351 \\ 780 \end{bmatrix}.$$

Finally, a solution is at hand: $x_6 = 1351 \equiv -1 \pmod 4$ and $y_6 = 780 \equiv 0 \pmod 4$. Thus $(1351, 780)$ is a solution of $x^2 - 3y^2 = 1$, satisfying the three conditions.

Consequently, $4n + 3 = 1350$ and $n = 337$. (We will revisit this problem in Example 3.10.) ∎

The following theorem also can be used to solve the equation $x^2 - dy^2 = 1$. Again, we omit its proof for the sake of brevity.

Theorem 2.4 *Let (α, β) be the fundamental solution of Pell's equation $x^2 - dy^2 = 1$. Then all its solutions (x_n, y_n) are given by $x_n + y_n\sqrt{d} = (\alpha + \beta\sqrt{d})^n$, where $(x_1, y_1) = (\alpha, \beta)$ and $n \geq 1$.* ∎

The next example also employs the equation $u^2 - 3v^2 = 1$ from Example 2.6 and invokes Theorem 2.4. It was proposed as a problem in 1992 by D.M. Bloom of Brooklyn College, New

York City [21]. Three years later, R. Drnošek, then a student at the Institute of Mathematics, Physics, and Mechanics, Ljubljana, Slovenia, gave a simple solution to it [77].

Example 2.9 Prove that there are infinitely many positive integers a such that both $a + 1$ and $3a + 1$ are squares. In addition, if $\{a_n\}$ denotes the increasing sequence of such integers, then $a_n a_{n+1} + 1$ is also a square. See [63] for a related problem.

Proof. Let $a + 1 = x^2$ and $3a + 1 = y^2$. Then $3x^2 - y^2 = 2$; so x and y have the same parity. Suppose $x = 2a$ and $y = 2b$. Then $2(3a^2 - b^2) = 1$, which is a contradiction. So both x and y are odd.

The equation $3x^2 - y^2 = 2$ can be rewritten as

$$\begin{aligned} 6x^2 - 2y^2 &= 4 \\ (3x - y)^2 - 3(y - x)^2 &= 4 \\ \left(\frac{3x - y}{2}\right)^2 - 3\left(\frac{y - x}{2}\right)^2 &= 1 \\ u^2 - 3v^2 &= 1, \end{aligned}$$

where $u = (3x - y)/2$ and $v = (y - x)/2$. Since we are interested only in positive solutions, $x \le y \le 3x$.

Since (2, 1) is the fundamental solution of Pell's equation $u^2 - 3v^2 = 1$, by Theorem 2.4, all its solutions (u_n, v_n) are given by $u_n + v_n\sqrt{3} = (2 + \sqrt{3})^n$, where $n \ge 1$. Let $r = 2 + \sqrt{3}$ and $s = 2 - \sqrt{3}$, the solutions of the equation $t^2 - 4t + 1 = 0$. Since $u_n + v_n\sqrt{3} = r^n$ and $u_n - v_n\sqrt{3} = s^n$, it follows that $u_n = \frac{r^n + s^n}{2}$ and $v_n = \frac{r^n - s^n}{2\sqrt{3}}$. So $u_n + v_n = x_n$ and hence $a_n = x_n^2 - 1 = (u_n + v_n)^2 - 1 = \frac{1}{6}\left(r^{2n+1} + s^{2n+1} - 4\right)$. Since x_n is an integer, so is a_n.

Then

$$\begin{aligned} a_n a_{n+1} + 1 &= (x_n^2 - 1)(x_{n+1}^2 - 1) + 1 \\ &= \frac{1}{6}\left(r^{2n+1} + s^{2n+1} - 4\right) \cdot \frac{1}{6}\left(r^{2n+3} + s^{2n+3} - 4\right) + 1 \\ &= \left[\left(r^{2n+2} + s^{2n+2} - 8\right)/6\right]^2 \end{aligned}$$

is also a square integer, as desired. ∎

In particular, $a_1 + 1 = 3^2, a_2 + 1 = 11^2, 3a_1 + 1 = 3 \cdot 8 + 1 = 5^2$, and $a_1 a_2 + 1 = 8 \cdot 120 + 1 = 31^2$.

Next we present a geometric application of Pell's equation $x^2 - 2y^2 = -1$, studied by Larsen [149].

Example 2.10 Figure 2.3 shows two triangular arrays of bricks. They have the property that the number of bricks in one array equals twice that in the other. Identify such triangular arrays.

Figure 2.3.

Solution. Let x denote the number of rows of bricks in one triangular array and y that in the other array. Then array 1 contains $t_x = \frac{x(x+1)}{2}$ bricks and array 2 contains $t_y = \frac{y(y+1)}{2}$ bricks.

Suppose $2t_x = t_y$. Then $2 \cdot \frac{x(x+1)}{2} = \frac{y(y+1)}{2}$; that is, $2x(x+1) = y(y+1)$. We now make a convenient substitution: $u = 2y + 1$ and $v = 2x + 1$. Then

$$\begin{aligned} u^2 - 2v^2 &= (2y+1)^2 - 2(2x+1)^2 \\ &= [4y(y+1)] - 2[4x(x+1)+1] \\ &= -1. \end{aligned}$$

Clearly, its solutions are given by $(u_n, v_n) = (Q_{2n-1}, P_{2n-1})$, where $n \ge 1$.

Correspondingly, the given brick problem has infinitely many solutions, given by $(x_n, y_n) = \left(\frac{v_n - 1}{2}, \frac{u_n - 1}{2}\right)$, where $n \ge 1$. Table 2.3 shows five such solutions.

Table 2.3.

n	u_n	v_n	x_n	y_n	$2t_{x_n} = t_{y_n}$
1	1	1	0	0	0
2	7	5	2	3	6
3	41	29	14	20	210
4	239	169	84	119	7,140
5	1393	985	492	696	242,556

■

We now stretch this example a bit further. The sequences $\{u_n\}$ and $\{v_n\}$ satisfy the same recursive pattern, so they can be defined recursively:

$$u_1 = 1, \quad u_2 = 7$$
$$u_n = 6u_{n-1} - u_{n-2}, \quad n \ge 3;$$

$$v_1 = 1, \quad v_2 = 5$$
$$v_n = 6v_{n-1} - v_{n-2}, \quad n \ge 3.$$

For example, $u_4 = 6u_3 - u_2 = 6 \cdot 41 - 7 = 239$ and $v_4 = 6v_3 - v_2 = 6 \cdot 29 - 5 = 169$; see Table 2.3.

The next example shows an interesting algebraic identity satisfied by every solution (x, y) of Pell's equation $x^2 - 2y^2 = -1$. It was originally proposed as a problem by Lt. Col. Allan J.C. Cunningham (1842–1928) of the British Army, in the *Educational Times* in 1900 [56].

Example 2.11 Let (x, y) be a solution of Pell's equation $x^2 - 2y^2 = -1$. Prove that

$$1^3 + 3^3 + 5^3 + \cdots + (2y-1)^3 = x^2 y^2.$$

Proof. Let $S = 1^3 + 3^3 + 5^3 + \cdots + (2y-1)^3$. Then

$$\begin{aligned} S &= \sum_{k=1}^{2y-1} k^3 - \left[2^3 + 4^3 + \cdots + (2y-2)^3\right] \\ &= \sum_{k=1}^{2y-1} k^3 - 8\sum_{k=1}^{y-1} k^3 \\ &= \left[\frac{(2y-1)(2y)}{2}\right]^2 - 8\left[\frac{(y-1)y}{2}\right]^2 \\ &= y^2(2y-1)^2 - 2y^2(y-1)^2 \\ &= y^2(2y^2-1) \\ &= x^2y^2. \end{aligned}$$

∎

The next example, geometric in nature, appeared in the 1985 American Invitational Mathematics Examinations (AIME). Although it is not directly related to Pell's equation $x^2 - 2y^2 = -1$, it will lead us to the equation, as we will see shortly.

Example 2.12 A small square is constructed inside a unit square $ABCD$ by dividing each side into n equal parts and then connecting its vertices to the division points closest to the opposite vertices; see Figure 2.4. Find the value of n such that the area of the small square (see the shaded area) is exactly $\frac{1}{1985}$.

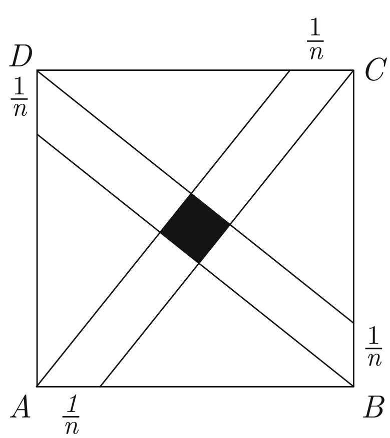

Figure 2.4.

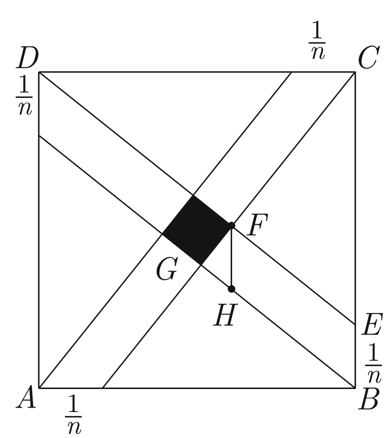

Figure 2.5.

Solution. Label the points as in Figure 2.5 such that $\overline{FH}$ is parallel to $\overline{EB}$. Clearly, $\triangle DCE$ and $\triangle FGH$ are similar, so the lengths of their corresponding sides are proportional. Consequently, $\frac{FG}{FH} = \frac{DC}{DE}$; so $FG^2 = \frac{DC^2}{DE^2} \cdot FH^2$. By the Pythagorean theorem, this implies that

$$\frac{1}{1985} = \frac{1^2}{1 + \left(1 - \frac{1}{n}\right)^2} \cdot \frac{1}{n^2}.$$

That is, $n^2 - n - 992 = 0$; that is, $(n - 32)(n + 31) = 0$. Thus $n = -31$ or 32. Since $n > 0$, it follows that $n = 32$. ■

The following example, based on this AIME problem, was proposed as a problem independently in 1987 by R.C. Gebhardt of the County College of Norris, Randolph, New Jersey, and C.H. Singer of Great Neck, New York [92]. The solution given here is based on the one by W.H. Pierce of Stonington, Connecticut, and is a fine application of Pell's equation $x^2 - 2y^2 = -1$ [174].

Example 2.13 Let s denote the length of a side of the small square in Figure 2.6. Find all positive integers n such that s is the reciprocal of an integer.

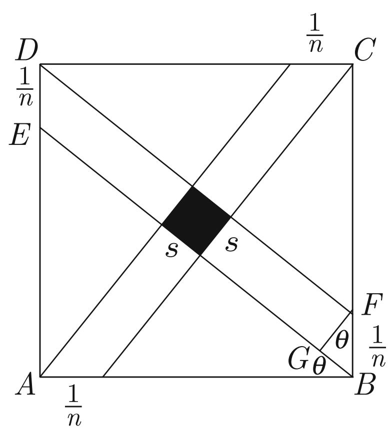

Figure 2.6.

Solution. Draw $\overline{FG}$ perpendicular to $\overline{BE}$; see Figure 2.6. Let $\angle BFG = \theta$. Then $\angle ABE = \theta$. Notice that $\triangle BFG$ and $\triangle ABE$ are right triangles. From $\triangle BFG$, $\cos\theta = \dfrac{FG}{BF} = \dfrac{s}{1/n} = ns$ and from $\triangle ABE$, $\cos\theta = \dfrac{AB}{BE} = \dfrac{1}{\sqrt{1 + \left(1 - \frac{1}{n}\right)^2}} = \dfrac{n}{\sqrt{n^2 + (n-1)^2}} = \dfrac{n}{\sqrt{2n^2 - 2n + 1}}$. So $ns = \dfrac{n}{\sqrt{2n^2 - 2n + 1}}$; that is, $s = \dfrac{1}{\sqrt{2n^2 - 2n + 1}}$. So $\sqrt{2n^2 - 2n + 1} = \frac{1}{s}$. But we want $\frac{1}{s}$ to be an integer, say, y. Then $2n^2 - 2n + 1 = y^2$. Multiplying both sides by 2 and then completing the square, this yields Pell's equation $x^2 - 2y^2 = -1$, where $x = 2n - 1$ and $y = \frac{1}{s}$.

The solutions (x_k, y_k) of this Pell's equation are $(x_k, y_k) = (Q_{2k-1}, P_{2k-1})$, where $k \geq 1$. The corresponding values n_k of n and s_k of s are given by $n_k = (x_k + 1)/2 = (Q_{2k-1} + 1)/2$ and $s_k = \frac{1}{y_k}$, respectively, as desired.

Table 2.4 gives the first ten values of x_k, y_k, and n_k.

Table 2.4.

k	1	2	3	4	5	6	7	8	9	10
x_k	1	7	41	239	1393	8119	47321	275807	1607521	9369319
y_k	1	5	29	169	985	5741	33461	195025	1136689	6625109
n_k	1	4	21	120	697	4060	23661	137904	803761	4684660

■

Additionally, we note that both x_k and y_k satisfy the same recurrence: $z_k = 6z_{k-1} - z_{k-2}$, where $z_1 = 1$ and

$$z_2 = \begin{cases} 7 & \text{if } z_2 = x_2 \\ 5 & \text{otherwise,} \end{cases}$$

and $k \geq 3$. On the other hand, n_k satisfies the recurrence $n_k = 6n_{k-1} - n_{k-2} - 2$, where $n_1 = 1, n_2 = 4$ and $k \geq 3$. For example, $n_5 = 6n_4 - n_3 - 2 = 6 \cdot 120 - 21 - 2 = 697$, as expected. You will see similar recurrences in Chapter 6.

We will encounter the sequences $\{x_k\}$ and $\{y_k\}$ a number of times in this chapter and the chapters that follow.

Next we investigate a close relationship between any two solutions of Pell's equation.

2.7 A Link Between Any Two Solutions of $x^2 - dy^2 = (-1)^n$

Suppose (x, y) and (X, Y) are any two solutions of the equation $x^2 - dy^2 = 1$. How are they related? This problem was studied by R.W.D. Christie in 1907 [44].

To see this relationship, since $x^2 - dy^2 = 1 = X^2 - dY^2$, $\dfrac{x^2-1}{y^2} = d = \dfrac{X^2-1}{Y^2}$. So $(x^2 - 1)Y^2 = (X^2 - 1)y^2$; that is, $(xY)^2 + y^2 = (Xy)^2 + Y^2$.

On the other hand, suppose (x, y) and (X, Y) are any two solutions of $x^2 - dy^2 = -1$. Then $\dfrac{x^2+1}{y^2} = \dfrac{X^2+1}{Y^2}$. So $(xY)^2 - y^2 = (Xy)^2 - Y^2$.

We can combine these two properties into one: $(xY)^2 + (-1)^n y^2 = (Xy)^2 + (-1)^n Y^2$.

For example, $(x, y) = (17,\ 12)$ and $(X, Y) = (99,\ 70)$ are two solutions of the equation $x^2 - 2y^2 = -1$; see Example 2.1. Then $(xY)^2 + y^2 = (17 \cdot 70)^2 + 12^2 = 1{,}416{,}244 = (99 \cdot 12)^2 + 70^2 = (Xy)^2 + Y^2$.

Likewise, $(x, y) = (18,\ 5)$ and $(X, Y) = (23382,\ 6485)$ are two solutions of the equation $x^2 - 13y^2 = -1$. Then $(xY)^2 - y^2 = (18 \cdot 6485)^2 - 5^2 = 13{,}625{,}892{,}875 = (23382 \cdot 5)^2 - 6485^2 = (Xy)^2 - Y^2$.

We now take a different approach to arrive at the sequences $\{Q_n\}$ and $\{P_n\}$ from Pell's equation $x^2 - 2y^2 = (-1)^n$, similar to what Larsen did in 1987 [149]. It yields some interesting dividends. To this end, first we factor the LHS:

$$(x + y\sqrt{2})(x - y\sqrt{2}) = (-1)^n. \tag{2.6}$$

Now raise both sides to the nth power:

$$(x + y\sqrt{2})^n (x - y\sqrt{2})^n = (-1)^n.$$

Using the binomial theorem, we can rewrite this as

$$(x_n + y_n\sqrt{2})(x_n - y_n\sqrt{2}) = (-1)^n, \tag{2.7}$$

where

$$\begin{aligned} x_n &= \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n}{2r} 2^r x^{n-2r} y^{2r}; \\ y_n &= \sum_{r=0}^{\lfloor (n+1)/2 \rfloor} \binom{n}{2r-1} 2^{r-1} x^{n-2r+1} y^{2r-1}; \end{aligned}$$

and $n \geq 1$. Since $(x_1, y_1) = (1,\ 1)$ is the fundamental solution of Pell's equation $x^2 - 2y^2 = (-1)^n$, these give us explicit formulas for Q_n and P_n:

$$Q_n = \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n}{2r} 2^r \tag{2.8}$$

$$P_n = \sum_{r=0}^{\lfloor (n+1)/2 \rfloor} \binom{n}{2r-1} 2^{r-1}. \tag{2.9}$$

For example, $Q_3 = \sum_{r=0}^{1} \binom{3}{2r} 2^r = \binom{3}{0}2^0 + \binom{3}{2}2^1 = 7$ and $P_4 = \sum_{r=1}^{2} \binom{4}{2r-1} 2^{r-1} = \binom{4}{1}2^0 + \binom{4}{3}2^1 = 12$. (We will revisit these two formulas in Chapter 9.)

It follows from formulas (2.8) and (2.9) that the values of Q_n and P_n can be read from Pascal's triangle with proper weights 2^k, where $k \geq 0$. For Q_n, we use alternate entries on row n beginning at $\binom{n}{0}$ and increasing weights $2^0, 2^1, 2^2, \ldots$; and for P_n, we use alternate entries on row $n+1$ beginning at $\binom{n+1}{1}$ and increasing weights $2^0, 2^1, 2^2, \ldots$.

For example, $Q_4 = \binom{4}{0}2^0 + \binom{4}{2}2^1 + \binom{4}{4}2^2 = 17$ (see the circled numbers in Figure 2.7) and $P_5 = \binom{5}{1}2^0 + \binom{5}{3}2^1 + \binom{5}{5}2^2 = 29$ (see the boxed numbers).

$$\begin{array}{ccccccccccccc} &&&&&& 1 &&&&&& \\ &&&&& 1 && 1 &&&&& \\ &&&& 1 && 2 && 1 &&&& \\ &&& 1 && 3 && 3 && 1 &&& \\ && \textcircled{1} && 4 && \textcircled{6} && 4 && \textcircled{1} && \\ & 1 && \boxed{5} && 10 && \boxed{10} && 5 && \boxed{1} & \\ 1 && 6 && 15 && 20 && 15 && 6 && 1 \end{array}$$

Figure 2.7.

Next we will show that the above recurrences for x_n and y_n can be recovered from formulas (2.6) and (2.7). To see this, notice that

$$(x_n + y_n\sqrt{2})(1 + \sqrt{2}) = (x_n + 2y_n) + (x_n + y_n)\sqrt{2}.$$

The RHS of this equation has exactly the same form as $x_n + y_n\sqrt{2}$; furthermore, we have

$$\begin{aligned}(x_n+2y_n)+(x_n+y_n)(x_n+2y_n)\sqrt{2}-(x_n+y_n)\sqrt{2} &= (x_n + 2y_n)^2 - 2(x_n + y_n)^2 \\ &= (x_n^2+4y_n^2+4x_n y_n)-2(x_n^2+y_n^2+2x_n y_n) \\ &= 2y_n^2 - x_n^2 \\ &= (-1)^{n+1}, \quad \text{by equation (2.7).}\end{aligned}$$

So, if (x_n, y_n) is a solution of equation (2.7), then so is $(x_{n+1}, y_{n+1}) = (x_n + 2y_n, x_n + y_n)$. Thus $x_{n+1} = x_n + 2y_n$ and $y_{n+1} = x_n + y_n$.

For example, $(x_4, y_4) = (17,\ 12)$ is a solution. Therefore, $(x_5, y_5) = (17+2\cdot 12, 17+12) = (41,\ 29)$ is also a solution.

Notice that

$$\begin{aligned}x_{n+1} &= x_n + 2y_n \\ &= x_n + 2(x_{n-1} + y_{n-1}) \\ &= x_n + x_{n-1} + (x_{n-1} + 2y_{n-1}) \\ &= x_n + x_{n-1} + x_n \\ &= 2x_n + x_{n-1}.\end{aligned}$$

Similarly, $y_{n+1} = 2y_n + y_{n-1}$. Thus, if (x_n, y_n) is a solution of $x^2 - 2y^2 = (-1)^n$, then so is $(x_{n+1}, y_{n+1}) = (2x_n + x_{n-1}, 2y_n + y_{n-1})$, where $n \geq 2$.

On the other hand, we can show that solutions obtained recursively this way exhaust all solutions of the equation $x^2-2y^2 = (-1)^n$. Consequently, its solutions (x_n, y_n) can be computed recursively:

$$\begin{aligned}&x_1 = 1, \quad x_2 = 3 && y_1 = 1, \quad y_2 = 2 \\ &x_n = 2x_{n-1} + x_{n-2}, \quad n \geq 3; && y_n = 2y_{n-1} + y_{n-2}, \quad n \geq 3.\end{aligned}$$

We will revisit these recursive definitions in Chapter 7; they define the central figures in the development of this book.

We found a bit earlier that $x_n = x_{n-1} + 2y_{n-1}$ and $y_n = x_{n-1} + y_{n-1}$, where $x_1 = 1 = y_1$ and $n \geq 2$. Since

$$\begin{aligned}x_n^2 - 2y_n^2 &= (x_{n-1} + 2y_{n-1})^2 - 2(x_{n-1} + y_{n-1})^2 \\ &= (x_{n-1}^2 + 4x_{n-1}y_{n-1} + 4y_{n-1}^2) - 2(x_{n-1}^2 + 2x_{n-1}y_{n-1} + y_{n-1}^2) \\ &= -(x_{n-1}^2 - 2y_{n-1}^2),\end{aligned}$$

and $x_1^2 - 2y_1^2 = -1$, it follows by PMI that $x_n^2 - 2y_n^2 = (-1)^n$ for every $n \geq 1$. Thus, if x_n, y_n satisfy the above recursive definitions, then (x_n, y_n) is a solution of Pell's equation $x_n^2 - 2y_n^2 = (-1)^n$.

These two sequences $\{x_n\}$ and $\{y_n\}$ provide a multitude of opportunities for exploration and fun, as you will see in Chapters 7–13.

Since every solution of $x^2 - 2y^2 = (-1)^n$ is (Q_n, P_n) and $(1, 1)$ is its fundamental solution, it follows by Theorem 2.3 that

$$\begin{bmatrix} Q_n \\ P_n \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}.$$

Then, by iteration, we have,

$$\begin{aligned} \begin{bmatrix} Q_n \\ P_n \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^2 \begin{bmatrix} Q_{n-2} \\ P_{n-2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^3 \begin{bmatrix} Q_{n-3} \\ P_{n-3} \end{bmatrix} \\ &\vdots \\ &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{n-1} \begin{bmatrix} Q_1 \\ P_1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

This can be confirmed using PMI.

For example,

$$\begin{aligned} \begin{bmatrix} Q_4 \\ P_4 \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 10 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 17 \\ 12 \end{bmatrix}. \end{aligned}$$

It also follows from the above discussion that

$$\begin{bmatrix} Q_n \\ P_n \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{n-k} \begin{bmatrix} Q_k \\ P_k \end{bmatrix}.$$

Notice that the matrix $M = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ is invertible and $M^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$. So

$$\begin{bmatrix} Q_k \\ P_k \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}^{n-k} \begin{bmatrix} Q_n \\ P_n \end{bmatrix}.$$

This matrix equation can be used to compute any predecessor solution (Q_k, P_k) from the solution (Q_n, P_n). So the process is completely reversible.

For example, $(Q_8, P_8) = (577,\ 408)$. Consequently,

$$\begin{aligned} \begin{bmatrix} Q_5 \\ P_5 \end{bmatrix} &= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}^3 \begin{bmatrix} 577 \\ 408 \end{bmatrix} \\ &= \begin{bmatrix} -7 & 10 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} 577 \\ 408 \end{bmatrix} \\ &= \begin{bmatrix} 41 \\ 29 \end{bmatrix}. \end{aligned}$$

2.8 A Preview of Chebyshev Polynomials

We can use the second-order recurrence (2.5) to introduce the family of Chebyshev polynomials, which we will explore in detail in Chapter 19. Let (x_n, y_n) be the nth solution of the equation $x^2 - dy^2 = 1$, where $n \geq 0$. Clearly, $(x_0, y_0) = (1,\ 0)$ is a solution and $(x_1, y_1) = (\alpha, \beta)$ is its fundamental solution. It follows from equation (2.5) that

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = 2\alpha \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} - \begin{bmatrix} x_{n-2} \\ y_{n-2} \end{bmatrix}.$$

So $x_n = 2\alpha x_{n-1} - x_{n-2}$ and $y_n = 2\alpha y_{n-1} - y_{n-2}$.

Then $x_0 = 1, x_1 = \alpha, x_2 = 2\alpha^2 - 1$, and $x_3 = 4\alpha^3 - 3\alpha$; and $y_0 = 0, y_1 = \beta, y_2 = 2\alpha\beta$, and $y_3 = \beta(4\alpha^2 - 1)$. More generally, the polynomials x_n and $\frac{y_n}{\beta}$ are the *Chebyshev polynomials* in α; again see Chapter 19.

Finally, we study briefly Pell's equation $x^2 - dy^2 = k$, where $k \neq 0$.

2.9 Pell's Equation $x^2 - dy^2 = k$

There is a close link between the solutions of the equations $x^2 - dy^2 = k$ and $x^2 - dy^2 = 1$, when they are solvable . To see this, let (α, β) be the fundamental solution of $x^2 - dy^2 = 1$ and (u, v) a solution of $x^2 - dy^2 = k$. Then $N[(\alpha u + d\beta v) + (\beta u + \alpha v)\sqrt{d}\,] = k$; so

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha & d\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

is a solution of $x^2 - dy^2 = k$. This recursive formula can be used to generate infinitely many solutions of $x^2 - dy^2 = k$ *associated* with (u, v); they belong to the *same* class of solutions.

The next example illustrates this technique.

Example 2.14 Consider the equations $x^2-2y^2 = 1$ and $x^2-2y^2 = 9$. We have $(\alpha, \beta) = (3,\ 2)$ and $(u, v) = (9,\ 6)$. Then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 51 \\ 36 \end{bmatrix}$$

is a solution of $x^2 - 2y^2 = 9$, associated with (9, 6): $51^2 - 2 \cdot 36^2 = 9$. So is (297, 210).

On the other hand, consider the equation $x^2 - 2y^2 = -7$. Here $(u, v) = (5,\ 4)$. Then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 31 \\ 22 \end{bmatrix}$$

is also a solution of $x^2 - 2y^2 = -7$, associated with (5, 4): $31^2 - 2 \cdot 22^2 = -7$. So is (181, 128). ■

In reality, there can be more than one fundamental solution of $x^2 - dy^2 = k$. Two solutions, (u, v) and (u', v'), belong to the *same* class if and only if $uu' \equiv dvv' \pmod{|k|}$ and $uv' \equiv vu' \pmod{|k|}$.

For example, consider the solutions $(u, v) = (5,\ 4)$ and $(u', v') = (31,\ 22)$ of $x^2 - 2y^2 = -7$. Then $uu' = 5 \cdot 31 \equiv 2 \cdot 4 \cdot 22 \equiv dvv' \pmod 7$ and $uv' = 5 \cdot 22 \equiv 4 \cdot 31 \equiv vu' \pmod 7$. So (5, 4) and (31, 22) belong to same class, as we already knew.

The next theorem provides bounds for the fundamental solution (u, v) of $x^2 - dy^2 = k$. Once again, we omit its proof [169] in the interest of brevity.

Theorem 2.5 *Let (α, β) be the fundamental solution of $x^2 - dy^2 = 1$ and (u, v) that of $x^2 - dy^2 = k$, where $k \neq 0$. Then*

(1) $0 < |u| \le \sqrt{\frac{k(\alpha+1)}{2}}$ *and* $0 \le v \le \beta\sqrt{\frac{k}{2(\alpha+1)}}$, *if* $k > 0$*;*

(2) $0 \le |u| \le \sqrt{\frac{|k|(\alpha-1)}{2}}$ *and* $0 < v \le \beta\sqrt{\frac{|k|}{2(\alpha-1)}}$, *otherwise.* ■

The next two examples illustrate this powerful theorem. We will revisit both in Chapter 5.

Example 2.15 Let (u, v) be the fundamental solution of $x^2 - 2y^2 = 9$. Recall that the fundamental solution of $x^2 - 2y^2 = 1$ is $(\alpha, \beta) = (3,\ 2)$. By Part 1 of Theorem 2.5, u and v must satisfy the inequalities $0 < |u| \le 3\sqrt{2}$ and $0 \le v \le \frac{3}{\sqrt{2}}$. So $-4 \le u \le 4$ and

$0 \le v \le 2$, except that $u \ne 0$. Exactly two of the possible 24 possible pairs (u, v) are solutions of $x^2 - 2y^2 = 9 : (\pm 3, 0)$. Since $(-3, 0)$ and (3, 0) belong to the same class, we choose $(u, v) = (3,\ 0)$.

Its successor solution (x_2, y_2) is given by

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \end{bmatrix},$$

as found in Example 2.14. The next three solutions are (51, 36), (297, 210), and (1731, 1224). ∎

Example 2.16 Let (u, v) be the fundamental solution of the equation $x^2 - 2y^2 = -7$ in Example 2.14. Again $(\alpha, \beta) = (3,\ 2)$. By Part 2 of Theorem 2.5, we must have $0 \le |u| \le \sqrt{7}$ and $0 \le v \le \sqrt{7}$; so $-2 \le u \le 2$ and $0 \le v \le 2$, where $u \ne 0$. Two of the 12 possible pairs (u, v) yield solutions of $x^2 - 2y^2 = -7 : (\pm 1, 2)$. They do not belong to the same class. Consequently, they generate distinct classes. Their immediate successors are:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 8 \end{bmatrix}.$$

They belong to the classes associated with $(-1, 2)$ and (1, 2), respectively.

Merging the two classes, we get the first six positive solutions: (1, 2), (5, 4), (11, 8), (31, 22), (65, 46), and (181, 128); they alternate between the two classes. ∎

Exercises 2

Let (x_n, y_n) be an arbitrary positive solution of the Pell's equation $x^2 - 48y^2 = 1$.

1. Find the fundamental solution (α, β) of the equation.
2. Using recursion and the solution $(x_3, y_3) = (1351,\ 195)$, find the next two solutions of the Pell's equation.
3. Using the solutions (x_3, y_3) and (x_4, y_4), and the second-order recurrence (2.5), compute (x_5, y_5) and (x_6, y_6).
4. Define x_n and y_n recursively.
5. Find a generating function for $\{x_n\}$.
6. Find a generating function for $\{y_n\}$.

Let (x_n, y_n) be an arbitrary positive solution of the Pell's equation $x^2 - 13y^2 = -1$.

7. Find the fundamental solution (α, β) of the equation.

8. Using recursion, find the next two solutions (x_2, y_2) and (x_3, y_3). *Hint*: Use Theorem 2.3.
9. Using the second-order recurrence (2.5), define (x_n, y_n) recursively.

Let (x_n, y_n) be an arbitrary solution of the Pell's equation $x^2 - 3y^2 = 6$, and (α, β) the fundamental solution of $x^2 - 3y^2 = 1$.

10. Find (α, β).
11. Find the fundamental solution (u, v) of $x^2 - 3y^2 = 6$, if it exists. *Hint*: Use Theorem 2.5.
12. Find an explicit formula for (x_n, y_n), if possible.

Use the Pell's equation $x^2 - 3y^2 = -11$ for Exercises 13–15.

13. Find the number of distinct classes of solutions. *Hint*: Use Theorem 2.5.
14. Find two new solutions belonging to each class.
15. Find an explicit formula for the arbitrary solution (x_n, y_n) in each class.
16. Find a generating function for the sequence $\{x_n\}$, where $x_n = 6x_{n-1} - x_{n-2}$, where $x_1 = 1$ and $x_2 = 7$.
17. Find a generating function for the sequence $\{n_k\}$, where $n_k = 6n_{k-1} - n_{k-2} - 2$, where $n_1 = 1$ and $n_2 = 4$.

3 Continued Fractions

3.1 Introduction

This chapter explores fractional expressions which most people do not use or see in their everyday life. Two such fractions are the multi-decked expressions

$$1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5}}}}$$

and

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}.$$

Such a multi-layered fraction is a *continued fraction*, a term coined by Wallis. The Indian mathematician-astronomer Aryabhata (ca. 476–ca. 550) used continued fractions to solve the linear diophantine equation (LDE) $ax + by = c$, where a, b, c, x, and y are integers and $(a, b) = 1$. The Italian mathematician Rafael Bombelli (1526–1573) used continued fractions to approximate $\sqrt{13}$ in his *L'Algebra Opera* (1572). In 1613, Pietro Antonio Cataldi (1548–1626), another Italian mathematician, employed them for approximating square roots of numbers. The Dutch physicist and mathematician Christiaan Huygens (1629–1695) used them in the design of a mathematical model for planets (1703).

T. Koshy, *Pell and Pell–Lucas Numbers with Applications*, DOI 10.1007/978-1-4614-8489-9_3,

The German mathematician Johann Heinrich Lambert (1728–1777), Euler, and Lagrange led the development of the modern theory of continued fractions. Most of Euler's work on continued fractions appears in his *De Fractionlous Continious* (1737). In 1759, Euler used them to solve the Pell's equation $x^2 - dy^2 = 1$. Lagrange's *Resolution des équations numériques* (1798) gives a method for approximating the real roots of equations using continued fractions and properties of periodic continued fractions. The Indian mathematical genius Srinivasa Aiyangar Ramanujan (1887–1920) also made significant contributions to continued fractions.

In 1931, Derrick H. Lehmer (1905–1991) of Stanford University and R.E. Powers of the Denver and Rio Grande Western Railroad developed an integer-factoring algorithm using continued fractions. Forty-three years later, Michael A. Morrison, a graduate student at Rice University, Houston, Texas, and John D. Brillhart of the University of Arizona exemplified its power by factoring the *Fermat number* $f_7 = 2^{2^7} + 1$.

Continued fractions can be used to solve the LDE $ax + by = c$ and the Pell equation $x^2 - dy^2 = (-1)^n$, and to factor large integers. Continued fractions also play a significant role in approximating the square roots of positive integers. This chapter will focus on solving the Pell equation. But first, we will briefly introduce continued fractions.

3.2 Finite Continued Fractions

A *finite continued fraction* is an expression of the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}, \tag{3.1}$$

where $a_0, a_1, \ldots, a_n$ are real numbers, and $a_1, a_2, \ldots, a_n$ are positive. The numbers $a_1, a_2, \ldots, a_n$ are the *partial quotients* of the continued fraction. It is a *simple* continued fraction if every a_i is an integer.

Because of the cumbersome notation for a continued fraction, it is often rewritten using the compact notation $[a_0; a_1, a_2, \ldots, a_n]$, where the semicolon separates the integral part $a_0 = \lfloor x \rfloor$ from the fractional part.

For example,

$$\begin{aligned}
[1; 2, 3, 4, 5] &= 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5}}}} \\
&= \frac{225}{157}.
\end{aligned}$$

Although it is fairly obvious from equation (3.1) that every *finite simple continued fraction* (FSCF) represents a rational number, this can be established using PMI. Is the converse true? That is, can every rational number be represented by a finite simple continued fraction? This was affirmed by Euler in 1737. The proof is an application of the well-known euclidean algorithm, which is often used to compute the gcd of two positive integers. In lieu of giving a proof, we will illustrate it using two simple examples.

Example 3.1 Represent $\frac{225}{157}$ as an FSCF.

Solution. Using the euclidean algorithm, we have

$$\begin{aligned} 225 &= 1 \cdot 157 + 68 \\ 157 &= 2 \cdot 68 \;+ 21 \\ 68 &= 3 \cdot 21 \;+ 5 \\ 21 &= 4 \cdot 5 \;\;+ 1 \\ 1 &= 5 \cdot 1 \;\;+ 0. \end{aligned}$$

Then

$$\begin{aligned} \frac{225}{157} &= 1 + \frac{68}{157} \qquad = 1 + \frac{1}{157/68} \\ &= 1 + \cfrac{1}{2 + \cfrac{21}{68}} \qquad = 1 + \cfrac{1}{2 + \cfrac{1}{68/21}} \\ &= 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{5}{21}}} = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{21/5}}} \\ &= 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5}}}} \\ &= [1; 2, 3, 4, 5], \end{aligned}$$

as expected. ■

The next example involves the ratio $\frac{F_{n+1}}{F_n}$ of two consecutive *Fibonacci numbers*. [Recall from Chapter 1 that $(F_{n+1}, F_n) = 1$.]

Example 3.2 Express $\dfrac{F_{n+1}}{F_n}$ as an FSCF.

Solution. By the Fibonacci recurrence and the *euclidean algorithm*, we have

$$\begin{aligned}
F_{n+1} &= 1 \cdot F_n \quad + F_{n-1}\\
F_n &= 1 \cdot F_{n-1} + F_{n-2}\\
F_{n-1} &= 1 \cdot F_{n-2} + F_{n-3}\\
&\vdots\\
F_4 &= 1 \cdot F_3 \quad + F_2\\
F_3 &= 1 \cdot F_2 \quad + F_1\\
F_2 &= 1 \cdot F_1 \quad + 0.
\end{aligned}$$

As in Example 3.1, we now continually substitute for the ratios $\dfrac{F_{k+1}}{F_k}$:

$$\begin{aligned}
\frac{F_{n+1}}{F_n} &= 1 + \frac{F_{n-1}}{F_n} && = 1 + \cfrac{1}{F_n/F_{n-1}}\\
&= 1 + \cfrac{1}{1 + \cfrac{F_{n-2}}{F_{n-1}}} && = 1 + \cfrac{1}{1 + \cfrac{1}{F_{n-1}/F_{n-2}}}\\
&= 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{F_{n-3}}{F_{n-2}}}} && = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{F_{n-2}/F_{n-3}}}}\\
&\vdots\\
&= 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots \, \cfrac{1}{1 + \cfrac{F_1}{F_2}}}}} && = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots \, \cfrac{1}{1 + \cfrac{1}{F_2/F_1}}}}}
\end{aligned}$$

$$= 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots \cfrac{1}{1 + \cfrac{1}{1}}}}}$$

$$= [\underbrace{1; 1, 1, 1, \ldots, 1}_{n \text{ ones}}].$$

The continued fraction representation of $\frac{F_{n+1}}{F_n}$ consists of n ones and hence $n-1$ partial quotients.

In particular, $\frac{F_9}{F_8} = \frac{34}{21} = [\underbrace{1; 1, 1, 1, 1, 1, 1, 1}_{8 \text{ ones}}]$. You may confirm this by direct computation.

■

Next we introduce the concept of a convergent of the continued fraction $x = [a_0; a_1, a_2, \ldots, a_n]$. Convergents can be used to approximate continued fractions. They are obtained by chopping the continued fraction immediately after each partial quotient.

3.2.1 Convergents of a Continued Fraction

The kth *convergent* C_k of the continued fraction $x = [a_0; a_1, a_2, \ldots, a_n]$ is given by $C_k = [a_0; a_1, a_2, \ldots, a_k]$, where $0 \le k \le n$. Thus $C_0 = a_0$ and $C_n = [a_0; a_1, a_2, \ldots, a_n]$.

For example, the continued fraction $\frac{225}{157} = [1; 2, 3, 4, 5]$ has five convergents:

$C_0 = 1$ $\qquad$ $C_1 = [1; 2] = \frac{3}{2} = 1.5$

$C_2 = [1; 2, 3] = \frac{10}{7} \approx 1.42857142857$ $\qquad$ $C_3 = [1; 2, 3, 4] = \frac{43}{30} \approx 1.43333333333$

$C_4 = [1; 2, 3, 4, 5] = \frac{225}{157} \approx 1.43312101911.$

An interesting observation: $C_0 < C_2 < C_4 < C_3 < C_1$. This is not just an observation, but the pattern holds for all even-numbered convergents C_{2k} and odd-numbered convergents C_{2k+1}, where $k \ge 0$: $C_0 < C_2 < \ldots < C_{2k} < \ldots < C_{2k+1} < \ldots < C_3 < C_1$.

As another example, the continued fraction $\frac{13}{8} = [1; 1, 1, 1, 1, 1]$ has six convergents:

$C_0 = 1 = \frac{1}{1}$ $\qquad$ $C_1 = [1; 1] = \frac{2}{1}$

$C_2 = [1; 1, 1] = \frac{3}{2}$ $\qquad$ $C_3 = [1; 1, 1, 1] = \frac{5}{3}$

$C_4 = [1; 1, 1, 1, 1] = \frac{8}{5}$ $\qquad$ $C_5 = [1; 1, 1, 1, 1, 1] = \frac{13}{8}.$

Here also $C_0 < C_2 < C_4 < C_5 < C_3 < C_1$.

3.2.2 Recursive Definitions of p_k and q_k

As k becomes larger and larger, the obvious and direct way of computing the convergent $C_k = \frac{p_k}{q_k}$ can become tedious and time-consuming. Fortunately, we can employ recursion to facilitate its computation. To see this, consider the continued fraction $[a_0; a_1, a_2, \ldots, a_n]$. Then

$$C_0 = a_0 = \tfrac{p_0}{q_0}; \text{ so } p_0 = a_0 \text{ and } q_0 = 1.$$
$$C_1 = a_0 + \frac{1}{a_1} = \frac{a_1 a_0 + 1}{a_1} = \frac{p_1}{q_1}; \text{ so } p_1 = a_1 a_0 + 1 \text{ and } q_1 = a_1.$$
$$C_2 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}} = \frac{a_2(a_1 a_0 + 1) + a_0}{a_2 a_1 + 1} = \frac{a_2 p_1 + p_0}{a_2 q_1 + q_0} = \frac{p_2}{q_2};$$

so $p_2 = a_2 p_1 + p_0$ and $q_2 = a_2 q_1 + q_0$.

More generally, it follows by PMI that $p_k = a_k p_{k-1} + p_{k-2}$ and $q_k = a_k q_{k-1} + q_{k-2}$, where $2 \le k \le n$. Consequently, the sequences $\{p_k\}$ and $\{q_k\}$ can be defined recursively:

$$\begin{aligned} p_0 &= a_0 & q_0 &= 1 \\ p_1 &= a_1 a_0 + 1 & q_1 &= a_1 \\ p_k &= a_k p_{k-1} + p_{k-2} & q_k &= a_k q_{k-1} + q_{k-2}, \end{aligned}$$

where $2 \le k \le n$.

These recurrences can be combined into a matrix equation in two different ways:

$$\begin{bmatrix} p_k & q_k \end{bmatrix} = \begin{bmatrix} a_k & 1 \end{bmatrix} \begin{bmatrix} p_{k-1} & q_{k-1} \\ p_{k-2} & q_{k-2} \end{bmatrix}$$
$$\begin{bmatrix} p_k & q_k \\ p_{k-1} & q_{k-1} \end{bmatrix} = \begin{bmatrix} a_k & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{k-1} & q_{k-1} \\ p_{k-2} & q_{k-2} \end{bmatrix}. \tag{3.2}$$

The recursive definitions will enable us to compute the convergents more rapidly, as the following example illustrates.

Example 3.3 Compute the convergents of the continued fraction [3;1,5,2,7].

Solution. We have $a_0 = 3, a_1 = 1, a_2 = 5, a_3 = 2$, and $a_4 = 7$. First, we compute p_k and q_k recursively, where $0 \le k \le 4$:

$$\begin{aligned} p_0 &= a_0 = 3 & q_0 &= 1 \\ p_1 &= a_1 a_0 + 1 = 1 \cdot 3 + 1 = 4 & q_1 &= a_1 = 1 \\ p_2 &= a_2 p_1 + p_0 = 5 \cdot 4 + 3 = 23 & q_2 &= a_2 q_1 + q_0 = 5 \cdot 1 + 1 = 6 \\ p_3 &= a_3 p_2 + p_1 = 2 \cdot 23 + 4 = 50 & q_3 &= a_3 q_2 + q_1 = 2 \cdot 6 + 1 = 13 \\ p_4 &= a_4 p_3 + p_2 = 7 \cdot 50 + 23 = 373 & q_4 &= a_4 q_3 + q_2 = 7 \cdot 13 + 6 = 97. \end{aligned}$$

Accordingly, the convergents are

$$C_0 = \frac{p_0}{q_0} = 3 \qquad C_1 = \frac{p_1}{q_1} = 4$$

$$C_2 = \frac{p_2}{q_2} = \frac{23}{6} \qquad C_3 = \frac{p_3}{q_3} = \frac{50}{13}$$

$$C_4 = \frac{p_4}{q_4} = \frac{373}{97} = [3; 1, 5, 2, 7].$$

■

To pursue this example a bit further, we can compute the values of p_k and q_k fairly easily using a table, such as Table 3.1. For example, $p_3 = 2 \cdot 23 + 4 = \textcircled{50}$ and $q_3 = 2 \cdot 6 + 1 = \boxed{13}$; see Table 3.1.

Table 3.1.

k	0	1	2	3	4
a_k	3	1	5	2	7
p_k	3	4	23	(50)	373
q_k	1	1	6	$\boxed{13}$	97

Let $C_k = \frac{p_k}{q_k}$ be the kth convergent of the FSCF $[a_0; a_1, a_2, \ldots, a_n]$. Then it follows by PMI that

$$p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}, \tag{3.3}$$

where $1 \leq k \leq n$.

For example, consider the convergents $\frac{p_k}{q_k}$ in Example 3.3. Notice that $p_3 q_2 - q_3 p_2 = \textcircled{50} \cdot 6 - \boxed{13} \cdot 23 = (-1)^{3-1}$ and $p_4 q_3 - q_4 p_3 = 373 \cdot 13 - 50 \cdot 97 = (-1)^{4-1}$; see Table 3.1.

The *Cassini-like* formula (3.3) follows from equation (3.2) also, using the fact that $|AB| = |A| \cdot |B|$, where $|M|$ denotes the determinant of the square matrix M. It implies that $(p_k, q_k) = 1$ for every k, and has an interesting byproduct: It can be used to solve the LDE $ax + by = c$.

3.3 LDEs and Continued Fractions

It is well known that the LDE $ax + by = c$ is solvable if and only if $d|c$, where $d = (a, b)$ [130]. If (x_0, y_0) is a *particular solution* of the LDE, then it has infinitely many solutions, given by $x = x_0 + (b/d)t$, $y = y_0 - (a/d)t$, where t is an arbitrary integer.

Suppose $(a, b) = 1$. Then the LDE $ax + by = 1$ is clearly solvable. Suppose (x_0, y_0) is a particular solution of the LDE. Then $a(cx_0) + b(cy_0) = c$; so (cx_0, cy_0) is a solution of the LDE $ax + by = c$. Conversely, if (x_0, y_0) is a solution of the LDE $ax + by = c$, then it can be shown that $(\frac{x_0}{c}, \frac{y_0}{c})$ is a solution of $ax + by = 1$. Consequently, we will focus on solving the LDE $ax + by = 1$ using the FSCF of the rational number $\frac{a}{b}$, where $(a, b) = 1$.

The technique behind this method hinges on the Cassini-like formula (3.3). In particular, the formula implies that $p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1}$, where $(p_n, q_n) = 1$. Since $\frac{p_n}{q_n} = \frac{a}{b}$ and

$(p_n, q_n) = 1 = (a, b)$, it follows that $p_n = a$ and $q_n = b$. Thus $aq_{n-1} - bp_{n-1} = (-1)^{n-1}$; that is, $aq_{n-1} + b(-p_{n-1}) = (-1)^{n-1}$. If n is odd, then $(x_0, y_0) = (q_{n-1}, -p_{n-1})$ is a particular solution of the LDE $ax + by = 1$; otherwise, $(x_0, y_0) = (-q_{n-1}, p_{n-1})$ is a particular solution. In either case, the *general solution* of the LDE $ax+by = 1$ is given by $x = x_0+bt$, $y = y_0-at$, where t is an arbitrary integer.

The next two examples illustrate this method.

Example 3.4 Find the general solution of the LDE $182x + 65y = 299$.

Solution. Notice that (182, 65) = 13 and 13|299. So the LDE $182x + 65y = 299$ is solvable. Dividing both sides of the LDE by 13 yields $14x + 5y = 23$.

First, we will find a particular solution of the LDE $14x + 5y = 1$. To this end, notice that $\frac{14}{5} = [2; 1, 4]$. So $C_0 = \frac{2}{1}, C_1 = \frac{3}{1}$, and $C_2 = \frac{14}{5}$. Since $p_2q_1 - q_2p_1 = 14 \cdot 1 - 5 \cdot 3 = -1$, $(x_0, y_0) = (-1, 3)$ is a particular solution of the LDE $14x + 5y = 1$. So a particular solution of the LDE $14x + 5y = 23$ is $(23x_0, 23y_0) = (-23, 69)$. Thus the general solution of the LDE $14x + 5y = 23$ is given by $x = -23 + 5t$, $y = 69 - 14t$, where t is an arbitrary integer. ■

Example 3.5 Solve the LDE $F_{n+1}x + F_n y = c$, where c is a positive integer.

Solution. Since $(F_{n+1}, F_n) = 1$, the LDE is solvable. By Cassini's formula, $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$. When n is even, $(x_0, y_0) = (F_{n-1}, -F_n)$ is a particular solution of the LDE $F_{n+1}x + F_n y = 1$; otherwise, $(x_0, y_0) = (-F_{n-1}, F_n)$ is a particular solution. So a particular solution of the LDE $F_{n+1}x + F_n y = c$ is (cx_0, cy_0). The general solution is given by $x = x_0 + F_n t$, $y = x_0 - F_{n+1}t$, where t is an arbitrary integer. ■

Next we turn to infinite simple continued fractions (ISCFs).

3.4 Infinite Simple Continued Fractions (ISCF)

Earlier, we found that a rational number represents a FSCF and vice versa. So how about irrational numbers? To answer this, we introduce the concept of an infinite continued fraction.

An *infinite continued fraction* is an expression of the form

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \dots}}}, \tag{3.4}$$

where $a_0, a_1, \dots$ and $b_1, b_2, \dots$ are real numbers.

The first recorded infinite continued fraction is the one for $\frac{4}{\pi}$, discovered by Brouncker in 1655. He discovered it by converting the famous Wallis' infinite product

$$\frac{4}{\pi} = \frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \dots}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \dots}$$

into a spectacular infinite continued fraction:

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \dots}}}}.$$

Although Brouncker did not provide a proof, Euler gave one in 1775.

There is an equally spectacular infinite continued fraction for its reciprocal as well:

$$\frac{\pi}{4} = 1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \cfrac{4^2}{9 + \dots}}}}.$$

Ramanujan discovered 200 infinite continued fractions. Two of the most astounding ones are

$$\left(\sqrt{\sqrt{5}\alpha - \alpha}\right) e^{2\pi/5} = \cfrac{1}{1 + \cfrac{e^{-2\pi}}{1 + \cfrac{e^{-4\pi}}{1 + \cfrac{e^{-6\pi}}{1 + \dots}}}}$$

and its reciprocal

$$\frac{e^{-2\pi/5}}{\sqrt{\sqrt{5}\alpha} - \alpha} = 1 + \cfrac{1}{1 + \cfrac{e^{-2\pi}}{1 + \cfrac{e^{-4\pi}}{1 + \cfrac{e^{-6\pi}}{1 + \dots}}}},$$

discovered in 1908, where α denotes the *golden ratio* $\frac{1+\sqrt{5}}{2}$.

Returning to the infinite continued fraction in (3.4), suppose a_0 is a nonnegative integer, $a_1, a_2, \dots$ are positive integers, and every $b_i = 1$. The corresponding *infinite simple continued fraction* (ISCF) has the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}. \tag{3.5}$$

Using the compact notation, we denote it by $[a_0; a_1, a_2, \dots]$.

Let $C_k = [a_0; a_1, a_2, \dots, a_k]$ denote the kth convergent of this continued fraction. The sequence $\{C_{2k}\}$ is a strictly increasing sequence, bounded above by C_1; and $\{C_{2k+1}\}$ is strictly decreasing, bounded below by C_0. So both converge, and converge to the same limit ℓ [130]. Thus $\{C_k\}$ converges to ℓ, and ℓ is the value of the ISCF $[a_0; a_1, a_2, \dots]$:

$$[a_0; a_1, a_2, \dots] = \lim_{k\to\infty} C_k.$$

For example, let C_k denote the kth convergent of the ISCF $[1; 1, 1, 1, \dots]$. Earlier, we found that $C_k = \frac{F_{k+1}}{F_k}$. Let $x = \lim\limits_{k\to\infty} \frac{F_{k+1}}{F_k}$. Then, using the Fibonacci recurrence, we have

$$\begin{aligned} x &= \lim_{k\to\infty} \frac{F_{k+1}}{F_k} \\ &= \lim_{k\to\infty} \frac{F_k + F_{k-1}}{F_k} \\ &= 1 + \cfrac{1}{\lim\limits_{k\to\infty} \cfrac{F_k}{F_{k-1}}} \\ &= 1 + \frac{1}{x}. \end{aligned}$$

So $x^2 - x - 1 = 0$. Solving this equation, we get $x = \frac{1\pm\sqrt{5}}{2}$. Since $x > 0$, it follows that $x = \frac{1+\sqrt{5}}{2}$, the *golden ratio*. Thus

$$[1; 1, 1, 1, \dots] = \frac{1+\sqrt{5}}{2}.$$

As another example, consider the ISCF $[1; 2, 2, 2, \dots]$. Its convergents converge to the limit ℓ, where $\ell = 1 + \frac{1}{1+\ell}$. Solving this equation, we get $\ell = \sqrt{2}$.

Earlier, we found that every FSCF represents a rational number. Is there a corresponding result for an ISCF? In fact, there is one: Every ISCF $[a_0; a_1, a_2, \dots]$ represents an irrational number; this can be confirmed using contradiction. Is its converse also true? That is, does every irrational number have an ISCF expansion? Fortunately, the answer is yes, as the following theorem shows. Its proof follows by PMI.

Theorem 3.1 *Let $x = x_0$ be an irrational number. Define a sequence $\{a_k\}$ of integers recursively as follows:*

$$a_k = \lfloor x_k \rfloor, \quad x_{k+1} = \frac{1}{x_k - a_k},$$

where $k \geq 0$. Then $x = [a_0; a_1, a_2, \ldots]$. ∎

The beauty of this theorem lies in the fact that it provides a constructive recursive algorithm for computing the ISCF that represents the irrational number x. The next example illustrates this algorithm. Have a good calculator handy to facilitate the computation.

Example 3.6 Express $\sqrt{19}$ as an ISCF.

Solution. By Theorem 3.4, we have $x = x_0 = \sqrt{19}$; so $a_0 = \lfloor \sqrt{19} \rfloor = 4$. Then

$$\begin{aligned}
x_1 &= \frac{1}{\sqrt{19}-4} = \frac{\sqrt{19}+4}{3}; \quad a_1 = \lfloor x_1 \rfloor = 2 \\
x_2 &= \frac{3}{\sqrt{19}-2} = \frac{\sqrt{19}+2}{5}; \quad a_2 = \lfloor x_2 \rfloor = 1 \\
x_3 &= \frac{5}{\sqrt{19}-3} = \frac{\sqrt{19}+3}{2}; \quad a_3 = \lfloor x_3 \rfloor = 3 \\
x_4 &= \frac{2}{\sqrt{19}-3} = \frac{\sqrt{19}+3}{3}; \quad a_4 = \lfloor x_4 \rfloor = 1 \\
x_5 &= \frac{5}{\sqrt{19}-2} = \frac{\sqrt{19}+2}{3}; \quad a_5 = \lfloor x_5 \rfloor = 2 \\
x_6 &= \frac{3}{\sqrt{19}-4} = \frac{\sqrt{19}+4}{1}; \quad a_6 = \lfloor x_6 \rfloor = 8 \\
x_7 &= \frac{1}{\sqrt{19}-4} = \frac{\sqrt{19}+4}{3}; \quad a_7 = \lfloor x_7 \rfloor = 2.
\end{aligned}$$

Since $x_7 = x_1$, clearly this pattern continues. Thus

$$\sqrt{19} = [4; \underbrace{2, 1, 3, 1, 2, 8}, \underbrace{2, 1, 3, 1, 2, 8}, \ldots].$$

The sequence of partial quotients in this ISCF shows an interesting pattern: It is *periodic* with *period* 6. Accordingly, we rewrite it as $\sqrt{19} = [4; \overline{2, 1, 3, 1, 2, 8}]$, using a bar over the smallest repeating block to indicate its periodicity. ∎

To pursue this example a bit further, we can compute the convergents of this ISCF using a table, such as Table 3.2. The first seven convergents $C_k = \frac{p_k}{q_k}$ are $\frac{4}{1}, \frac{9}{2}, \frac{13}{3}, \frac{48}{11}, \frac{61}{14}, \frac{170}{39}, \frac{1420}{326}$, and $\frac{3010}{691}$. Notice that $\frac{3010}{691} \approx 4.35600578871$, $\sqrt{19} \approx 4.35889894354$, and $\sqrt{19} - \frac{3010}{691} \approx 0.002893154829$.

Table 3.2.

k	0	1	2	3	4	5	6	7
a_k	4	2	1	3	1	2	8	2
p_k	4	9	13	48	61	170	1420	3010
q_k	1	2	3	11	14	39	326	691

Table 3.3 gives the ISCF of $\sqrt{d}$, where d is a positive nonsquare integer and $2 \le d \le 20$.

Table 3.3.

$\sqrt{2} = [1; \overline{2}]$	$\sqrt{3} = [1; \overline{1, 2}]$
$\sqrt{5} = [2; \overline{4}]$	$\sqrt{6} = [2; \overline{2, 4}]$
$\sqrt{7} = [2; \overline{1, 1, 1, 4}]$	$\sqrt{8} = [2; \overline{1, 4}]$
$\sqrt{10} = [3; \overline{6}]$	$\sqrt{11} = [3; \overline{3, 6}]$
$\sqrt{12} = [3; \overline{2, 6}]$	$\sqrt{13} = [3; \overline{1, 1, 1, 1, 6}]$
$\sqrt{14} = [3; \overline{1, 2, 1, 6}]$	$\sqrt{15} = [3; \overline{1, 6}]$
$\sqrt{17} = [4; \overline{8}]$	$\sqrt{18} = [4; \overline{4, 8}]$
$\sqrt{19} = [4; \overline{2, 1, 3, 1, 2, 8}]$	$\sqrt{20} = [4; \overline{2, 8}]$

It appears from the table that the ISCFs of $\sqrt{d}$ are always periodic; this is certainly true: $\sqrt{d} = [a_0; \overline{a_1, a_2, \ldots, a_n}]$, where d is a positive nonsquare integer. In addition, $a_n = 2a_0$. Since the period is n, it follows that $a_{mn} = 2a_0$ for every integer $m \ge 1$.

For example, consider $\sqrt{19} = [4; \overline{2, 1, 3, 1, 2, 8}]$. Notice that $a_6 = 8 = 2a_0$. Likewise, $\sqrt{13} = [3; \overline{1, 1, 1, 1, 6}]$; here $a_5 = 6 = 2a_0$.

3.5 Pell's Equation $x^2 - dy^2 = (-1)^n$ and ISCFs

We are now ready to reveal an open "secret:" We can successfully employ ISCFs to solve the Pell's equation $x^2 - dy^2 = (-1)^n$. This can be accomplished by invoking the next three results [37].

Theorem 3.2 *Let (x, y) be a solution of the Pell equation $x^2 - dy^2 = 1$, where d is a positive nonsquare integer. Then $\frac{x}{y}$ is a convergent of the ISCF of $\sqrt{d}$.* ∎

The following example illustrates this theorem.

Example 3.7 Consider the Pell's equation $x^2 - 23y^2 = 1$. Notice that $24^2 - 23 \cdot 5^2 = 1$, so $(24, 5)$ is a solution of the equation.

To see that $\frac{24}{5}$ is a convergent of $\sqrt{23}$, we have $\sqrt{23} = [4; \overline{1, 3, 1, 8}]$. We now compute its convergents C_k, using Table 3.4. It follows from the table that $\frac{24}{5}$ is indeed a convergent, as desired: $\frac{24}{5} = \frac{p_3}{q_3} = C_3$.

Table 3.4.

k	0	1	2a	3
a_k	4	1	3	1
p_k	4	5	19	24
q_k	1	1	4	5

Table 3.5.

k	0	1	2	3	4
a_k	5	2	1	1	2
p_k	5	11	16	27	70
q_k	1	2	3	5	13

Likewise, (70, 13) is a solution of $x^2 - 29y^2 = -1 : 70^2 - 29 \cdot 13^2 = -1$. Notice that $\frac{70}{13}$ is the fourth convergent of $\sqrt{29} = [5; \overline{2, 1, 1, 2, 10}]$; see Table 3.5. ∎

Theorem 3.3 *Let $\frac{p_n}{q_n}$ be a convergent of the ISCF of $\sqrt{d}$, where d is a positive non-square integer. Then (p_n, q_n) is a solution of one of the equations $x^2 - dy^2 = k$, where $k < 1 + 2\sqrt{d}$.* ∎

Example 3.8 Let $d = 2$. Then $|k| < 1+2\sqrt{2}$, so k can be $\pm 1, \pm 2, \pm 3$. The first six convergents of the ISCF of $\sqrt{2} = [1; \overline{2}]$ are $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}$, and $\frac{99}{70}$. Then

$$
\begin{aligned}
p_0^2 - 2q_0^2 &= -1 & \qquad p_1^2 - 2q_1^2 &= 1 \\
p_2^2 - 2q_2^2 &= -1 & \qquad p_3^2 - 2q_3^2 &= 1 \\
p_4^2 - 2q_4^2 &= -1 & \qquad p_5^2 - 2q_5^2 &= 1.
\end{aligned}
$$

So (3, 2) is the fundamental solution of $x^2 - 2y^2 = 1$ (see Example 2.1) and (1, 1) is the fundamental solution of $x^2 - 2y^2 = -1$ (see Example 2.10), as expected. ∎

Recall from Chapter 2 that the general solution of the Pell's equation $x^2 - 2y^2 = (-1)^n$ is (Q_n, P_n). Since $\frac{Q_{n+1}}{P_{n+1}}$ is the nth convergent of the ISCF of $\sqrt{2}$, it follows that $\lim_{n \to \infty} \frac{Q_{n+1}}{P_{n+1}} = \sqrt{2}$, where $n \geq 0$. This also follows from the property $Q_n^2 - 2P_n^2 = (-1)^n$.

We encountered the sequences $\{p_n\}$ and $\{q_n\}$ in Chapters 1 and 2. Following the lead of M.N. Khatri of the University of Baroda, India, in 1959, we can use them to construct two intriguing sequences $\{y_n\}$ and $\{m_n\}$, where $y_n = p_n q_n$ [122] and

$$
m_n = \begin{cases} 2q_n^2 & \text{if } n \text{ is even} \\ p_n^2 & \text{otherwise.} \end{cases}
$$

Table 3.6 shows the resulting sequences. Do you see anything special about them? Can you define them recursively? Can you develop explicit formulas for them? In any case, we will revisit them in Chapter 6.

Table 3.6.

n	y_n	m_n
1	$1 \cdot 1 = 1$	$1^2 = 1$
2	$2 \cdot 3 = 6$	$2 \cdot 2^2 = 8$
3	$5 \cdot 7 = 35$	$7^2 = 49$
4	$12 \cdot 17 = 204$	$2 \cdot 12^2 = 288$
5	$29 \cdot 41 = 1189$	$41^2 = 1681$
6	$70 \cdot 99 = 6930$	$2 \cdot 70^2 = 9800$
7	$169 \cdot 239 = 40391$	$239^2 = 57121$
8	$408 \cdot 577 = 235416$	$2 \cdot 408^2 = 332928$
9	$985 \cdot 1393 = 1372105$	$1393^2 = 1940449$
10	$2378 \cdot 3363 = 7997214$	$2 \cdot 2378^2 = 11309768$

Example 3.9 Let $d = 7$. Recall from Table 3.3 that $\sqrt{7} = [2; \overline{1, 1, 1, 4}]$. The period of the continued fraction is 4. To compute the convergents $\frac{p_n}{q_n}$, we construct Table 3.7. The first twelve convergents are $\frac{2}{1}, \frac{3}{1}, \frac{5}{2}, \frac{8}{3}, \frac{37}{14}, \frac{45}{17}, \frac{82}{31}, \frac{127}{48}, \frac{590}{223}, \frac{717}{271}, \frac{1307}{494}$, and $\frac{2024}{765}$.

Table 3.7.

k	0	1	2	3	4	5	6	7	8	9	10	11	12
a_k	2	1	1	1	4	1	1	1	4	1	1	1	4
p_k	2	3	5	$\boxed{8}$	37	45	82	$\boxed{127}$	590	717	1307	$\boxed{2024}$	9403
q_k	1	1	2	$\boxed{3}$	14	17	31	$\boxed{48}$	223	271	494	$\boxed{765}$	3554

Notice that

$$p_0^2 - 7q_0^2 = 2^2 - 7 \cdot 1^2 = -3$$
$$p_1^2 - 7q_1^2 = 3^2 - 7 \cdot 1^2 = 2$$
$$p_2^2 - 7q_2^2 = 5^2 - 7 \cdot 2^2 = -3$$
$$p_3^2 - 7q_3^2 = 8^2 - 7 \cdot 3^2 = \boxed{1}$$
$$p_4^2 - 7q_4^2 = 37^2 - 7 \cdot 14^2 = -3$$
$$p_5^2 - 7q_5^2 = 45^2 - 7 \cdot 17^2 = 2$$
$$p_6^2 - 7q_6^2 = 82^2 - 7 \cdot 31^2 = -3$$
$$p_7^2 - 7q_7^2 = 127^2 - 7 \cdot 48^2 = \boxed{1}$$
$$p_8^2 - 7q_8^2 = 590^2 - 7 \cdot 223^2 = -3$$
$$p_9^2 - 7q_9^2 = 717^2 - 7 \cdot 271^2 = 2$$
$$p_{10}^2 - 7q_{10}^2 = 1307^2 - 7 \cdot 494^2 = -3$$
$$p_{11}^2 - 7q_{11}^2 = 2024^2 - 7 \cdot 765^2 = \boxed{1}.$$

Clearly, $(p_3, q_3) = (8,\ 3)$, $(p_7, q_7) = (127,\ 48)$, and $(p_{11}, q_{11}) = (2024,\ 765)$ are solutions of Pell's equation $x^2 - 7y^2 = 1$.

An interesting observation: $(p_3, q_3) = (8,\ 3)$ is the fundamental solution of the equation $x^2 - 7y^2 = 1$. Every solution seems to be of the form (p_{4n-1}, q_{4n-1}), where $n \geq 1$. (Corollary 3.2 will indeed confirm this; also see Example 3.11.) ■

The next powerful result paves the way for finding solutions of the Pell equation $x^2 - dy^2 = (-1)^n$ using the continued fraction of $\sqrt{d}$.

Theorem 3.4 *Let n denote the period of the ISCF of $\sqrt{d}$. Then the convergent $\frac{p_{nk-1}}{q_{nk-1}}$ satisfies the equation $p_{nk-1}^2 - dq_{nk-1}^2 = (-1)^{kn}$, where $k \geq 1$.* ∎

Consequently, (p_{nk-1}, q_{nk-1}) is a solution of the equation $x^2 - dy^2 = (-1)^{kn}$.

Case 1 Suppose n is even. Then $(-1)^{kn} = 1$ for every k. So (p_{nk-1}, q_{nk-1}) is a solution of the Pell's equation $x^2 - dy^2 = 1$ for every $k \geq 1$.

It now follows that if n is even, the equation $x^2 - dy^2 = -1$ has *no* solutions.

Case 2 Suppose n is odd. If k is even, then also $(-1)^{kn} = 1$; so (p_{2nk-1}, q_{2nk-1}) is a solution of the equation $x^2 - dy^2 = 1$ for every $k \geq 1$.

On the other hand, suppose k is odd. Then $(-1)^{kn} = -1$. So every solution of $x^2 - dy^2 = -1$ is of the form (p_{nk-1}, q_{nk-1}), where both n and k are odd.

Thus we have the following result.

Corollary 3.1 *Let n be the period of the ISCF of $\sqrt{d}$. Then*

- *If n is even, then every solution of $x^2 - dy^2 = 1$ is of the form (p_{nk-1}, q_{nk-1}).*
- *If n is even, then $x^2 - dy^2 = -1$ has no solutions.*
- *If n is odd and k is even, then every solution of $x^2 - dy^2 = 1$ is of the form (p_{2nk-1}, q_{2nk-1}).*
- *If both n and k are odd, then every solution of $x^2 - dy^2 = -1$ is of the form (p_{nk-1}, q_{nk-1}).* ∎

The next result is an immediate consequence of this corollary.

Corollary 3.2 *Let n be the period of the ISCF of $\sqrt{d}$. Then*

- *If n is even, then the fundamental solution of $x^2 - dy^2 = 1$ is (p_{n-1}, q_{n-1}).*
- *If n is even, then $x^2 - dy^2 = -1$ has no solutions.*
- *If n is odd and k is even, then the fundamental solution of $x^2 - dy^2 = 1$ is (p_{2n-1}, q_{2n-1}).*
- *If both n and k are odd, then the fundamental solution of $x^2 - dy^2 = -1$ is (p_{n-1}, q_{n-1}).* ∎

In particular, let $d = 2$. Since $\sqrt{2} = [1; \overline{2}]$, the period of the continued fraction of $\sqrt{2}$ is $n = 1$. So, by Corollary 3.2, the fundamental solution of the equation $x^2 - 2y^2 = 1$ is given by $(p_1, q_1) = (3,\ 2)$, as expected.

The next three examples also illustrate these two corollaries. First, we will revisit Example 2.8.

Example 3.10 (Example 2.8 Revisited) We will use the ISCF of $\sqrt{3}$ to find a solution of Pell's equation $x^2 - 3y^2 = 1$, where $x \equiv -1 \pmod 4$, $y \equiv 0 \pmod 4$, and $x \geq 13$. To this end, notice that the continued fraction of $\sqrt{3}$ is periodic with period 2: $\sqrt{3} = [1; \overline{1, 2}]$. Table 3.8 shows the first eleven convergents of the continued fractions. The convergent $\frac{p_{11}}{q_{11}} = \frac{1351}{780}$ satisfies both conditions: $p_{11} = 1351 \equiv -1 \pmod 4$ and $q_{11} = 780 \equiv 0 \pmod 4$. So $(x_{11}, y_{11}) = (1351,\ 780)$ is a solution of Pell's equation satisfying all three conditions. As in Example 2.7, $x_{11} = 1351$ yields $n = 337$.

Table 3.8.

k	0	1	2	3	4	5	6	7	8	9	10	11
a_k	1	1	2	1	2	1	2	1	2	1	2	1
p_k	1	2	5	7	19	26	71	97	265	362	989	1351
q_k	1	1	3	4	11	15	41	56	153	209	571	780

■

Example 3.11 Find the first four solutions of Pell's equation $x^2 - 7y^2 = 1$. (This example is basically the same as Example 3.9.)

Solution. Recall from Example 3.9 that the continued fraction of $\sqrt{7}$ is periodic with period 4: $\sqrt{7} = [2; \overline{1, 1, 1, 4}]$. Since the period is even, by Corollary 3.2, the fundamental solution of the equation is $(p_3, q_3) = (8,\ 3)$; see Table 3.7.

By Corollary 3.1, the remaining solutions are given by (p_{4k-1}, q_{4k-1}), where $k \geq 2$. When $k = 2$, the solution is $(p_7, q_7) = (127,\ 48)$; when $k = 3$, the solution is $(p_{11}, q_{11}) = (2024,\ 765)$; and when $k = 3$, the solution is $(p_{15}, q_{15}) = (33257,\ 12192)$. ■

Example 3.12 Find the first two solutions of each of the Pell equations $x^2 - 13y^2 = \pm 1$.

Solution. Recall from Table 3.3 that the continued fraction of $\sqrt{13}$ is periodic with period 5: $\sqrt{13} = [3; \overline{1, 1, 1, 1, 3}]$. Table 3.9 shows the first 15 convergents of the continued fraction.

Table 3.9.

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a_k	3	1	1	1	1	6	1	1	1	1	6	1	1	1	1	6
p_k	3	4	7	11	18	119	137	256	393	649	4287	4936	9223	14159	23382	154451
q_k	1	1	2	3	5	33	38	71	109	180	1189	1369	2558	3927	6485	42837

Since the period is odd, every solution of $x^2 - 13y^2 = 1$ is of the form (p_{10k-1}, q_{10k-1}), where $k \geq 1$. When $k = 1$, $(p_9, q_9) = (649,\ 180)$ gives the fundamental solution: $649^2 - 13 \cdot 180^2 = 1$. The second solution is $(x_2, y_2) = (p_{19}, q_{19}) = (842401,\ 233640)$, corresponding to $k = 19$: $842401^2 - 13 \cdot 233640^2 = 1$.

Since p_k and q_k are getting larger and larger, we can also invoke Theorem 2.4 to find the solution (x_2, y_2). It is given by

$$
\begin{aligned}
x_2 + y_2\sqrt{13} &= (649 + 180\sqrt{13})^2 \\
&= 842401 + 233640\sqrt{13}.
\end{aligned}
$$

This yields $(x_2, y_2) = (842401,\ 233640)$. (Can you find the next solution?)

On the other hand, the solutions of $x^2 - 13y^2 = -1$ can be obtained when k is odd. Every one of the solutions is of the form (p_{5k-1}, q_{5k-1}), where k is odd. When $k = 1$, we get the fundamental solution $(p_4, q_4) = (18,\ 5)$: $18^2 - 13 \cdot 5^2 = -1$. The next solution $(p_{14}, q_{14}) = (23382,\ 6485)$ corresponds to $k = 3$: $23382^2 - 13 \cdot 6485^2 = -1$. (Can you find the next solution?) ■

Table 3.10 gives the fundamental solution (p_k, q_k) of the Pell equation $x^2 - dy^2 = 1$ for $2 \le d \le 24$, where $\frac{p_k}{q_k}$ is some convergent of the ISCF of $\sqrt{d}$. It follows by Corollary 3.1 and Table 3.3 that the equation $x^2 - dy^2 = -1$ is solvable for only five values of $d \le 25$: 2, 5, 10, 13, and 17. The corresponding fundamental solutions are given in Table 3.11.

Table 3.10.

k	d	p_k	q_k	k	d	p_k	q_k
1	2	3	2	3	14	15	4
1	3	2	1	1	15	4	1
1	5	9	4	1	17	33	8
1	6	5	2	1	18	17	4
3	7	8	3	5	19	170	39
1	8	3	1	1	20	9	2
1	10	19	6	5	21	55	12
1	11	10	3	5	22	197	42
1	12	7	2	3	23	24	5
9	13	649	180	1	24	5	1

Table 3.11.

k	d	p_k	q_k
0	2	1	1
0	5	2	1
0	10	3	1
4	13	18	5
0	17	4	1

You may recall that *not* every Pell's equation is solvable. For instance, consider $x^2 - 3y^2 = -1$. Since $\sqrt{3} = [1; \overline{1, 2}]$, its period is even. So, by Corollary 3.1, it is *not* solvable.

Finally, we add that *not* every irrational number has a periodic expansion. For example, the two Ramanujan's continued fractions given earlier are *not* periodic. Neither are the following continued fractions:

$$\pi = [3; 7, 15, 1, 292, 1, 1, 2, 1, 3, 1, \ldots] \qquad e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$$

$$\frac{e-1}{e+1} = [0; 2, 6, 10, 14, 18, \ldots] \qquad \frac{e^2-1}{e^2+1} = [0; 1, 3, 5, 7, 9, \ldots]$$

$$\log 2 = \cfrac{1}{1 + \cfrac{1^2}{1 + \cfrac{2^2}{1 + \cfrac{3^2}{1 + \ldots}}}} \qquad \pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{6 + \ldots}}}},$$

where $e \approx 2.71828182846$ and denotes the base of the natural logarithm. The expansions of e, $\frac{e-1}{e+1}$, and $\frac{e^2-1}{e^2+1}$ were discovered by Euler in 1737. The last expansion of π was discovered by L.J. Lange of the University of Missouri in 1999.

We close this chapter with a combinatorial interpretation of the nth convergent $c_n = \frac{p_n}{q_n}$ of the ISCF $[a_0; a_1, a_2, \ldots]$. It was studied by A.T. Benjamin and F.E. Su of Harvey Mudd College, Claremont, California, and J.J. Quinn of Occidental College, Los Angeles, California [14].

3.6 A Simple Continued Fraction Tiling Model

Let A_n denote the number of ways of tiling a $1 \times (n+1)$ linear board with 1×2 dominoes and unit square tiles, such that there are *no* gaps or overlappings. The square tiles can be stacked up; but *no* tiles, dominoes or squares, can be placed on top of a domino. No domino can be stacked on anything. Suppose the $n+1$ cells (square tiles) of the board are labeled 0 through n. Then cell i can be covered by a maximum of a_i square tiles, where $0 \le i \le n$.

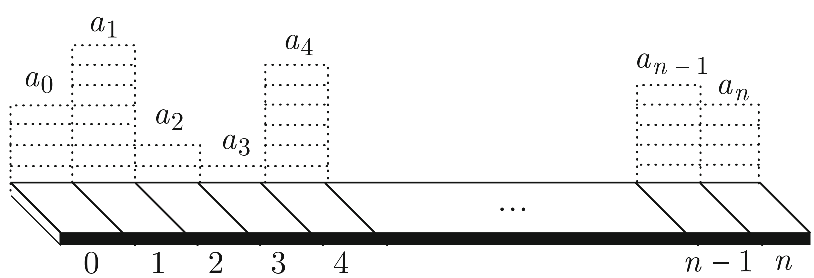

Figure 3.1.

Figure 3.1 shows an empty board with the maximum possible stack sizes $a_0, a_1, a_2, \ldots, a_n$. Figure 3.2 shows a valid tiling of a 1×10 board with stack sizes 5, 10, 4, 2, 5, 3, 8, 10, 2, and 4.

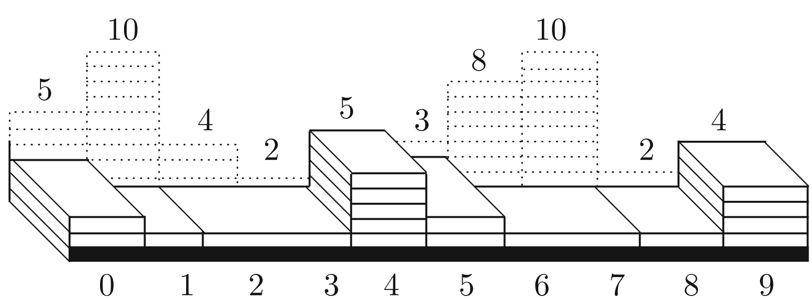

Figure 3.2. A Valid Tiling of a 1×10 Board

Next we will define A_n recursively. First, when $n = 0$, the board consists of a single cell, namely, cell 0. So we can stack up as many as a_0 square tiles at cell 0; see Figure 3.3. Hence $A_0 = a_0$.

Suppose $n = 1$. Then the board consists of two cells, cells 0 and 1. There are two cases to consider, as Figure 3.4 shows. If a tiling ends in a square tile, then square tiles must occupy both cells; see Figure 3.4a. There are $a_1 a_0$ such tilings. On the other hand, if a square tile does not occupy cell 1, then a domino must occupy both cells 0 and 1; see Figure 3.4b. There is exactly one such tiling. So by the addition principle, $A_1 = a_1 a_0 + 1$.

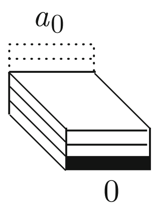

Figure 3.3.

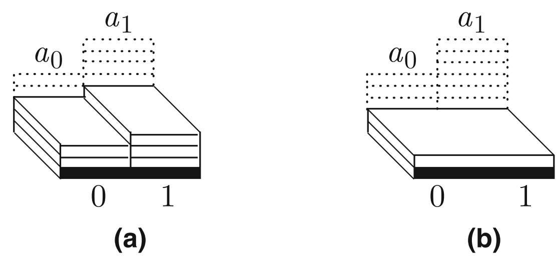

Figure 3.4.

Suppose $n \geq 2$. Suppose a tiling ends in a square tile. Since the stack size at cell n is a_n, there are $a_n A_{n-1}$ such tilings. On the other hand, suppose a domino occupies cell n (and hence cell $n-1$); so there is exactly one way of tiling cells n and $n-1$. By definition, there are A_{n-2} ways of tiling cells 0 through $n-2$. So there are $1 \cdot A_{n-2} = A_{n-2}$ tilings ending in a domino. Consequently, $A_n = a_n A_{n-1} + A_{n-2}$.

Thus A_n can be defined recursively:

$$\begin{aligned} A_0 &= a_0, \quad A_1 = a_1 a_0 + 1 \\ A_n &= a_n A_{n-1} + A_{n-2}, \quad n \geq 2. \end{aligned}$$

This is exactly the recursive definition of the numerator p_n of the nth convergent of the ISCF $[a_0; a_1, a_2, \ldots]$. So $A_n = p_n$.

To interpret q_n combinatorially, we consider the $1 \times n$ board in Figure 3.1, with cell 0 removed. Let B_n denote the number of ways of tiling this board with dominoes and stackable square tiles as before. Defining $B_0 = 1$ to denote the empty tiling, it follows by a similar argument that

$$\begin{aligned} B_0 &= a_0, \quad B_1 = a_1 \\ B_n &= a_n B_{n-1} + A_{n-2}, \quad n \geq 2. \end{aligned}$$

Clearly, $b_n = q_n$.

For example, consider the ISCF $\pi = [3; 7, 15, 1, 292, 1, 1, 2, \ldots]$; there are $p_0 = a_0 = 3$ ways of tiling cell 0; see Figure 3.5. There are $p_1 = a_1 a_0 + 1 = 3 \cdot 7 + 1 = 22$ ways of tiling cells 0 and 1; and $q_1 = a_1 = 7$ ways of tiling cell 1. This yields the first convergent $c_1 = \frac{p_1}{q_1} = \frac{22}{7}$. There are $p_2 = 15 \cdot 22 + 3 = 333$ ways of tiling cells 0, 1 and 2; and $q_2 = 15 \cdot 7 + 1 = 106$ ways of tiling cells 1 and 2. This yields the second convergent $c_2 = \frac{p_2}{q_2} = \frac{333}{106}$. Similarly, $c_3 = \frac{p_3}{q_3} = \frac{355}{113}$. Clearly, these computations can be continued indefinitely.

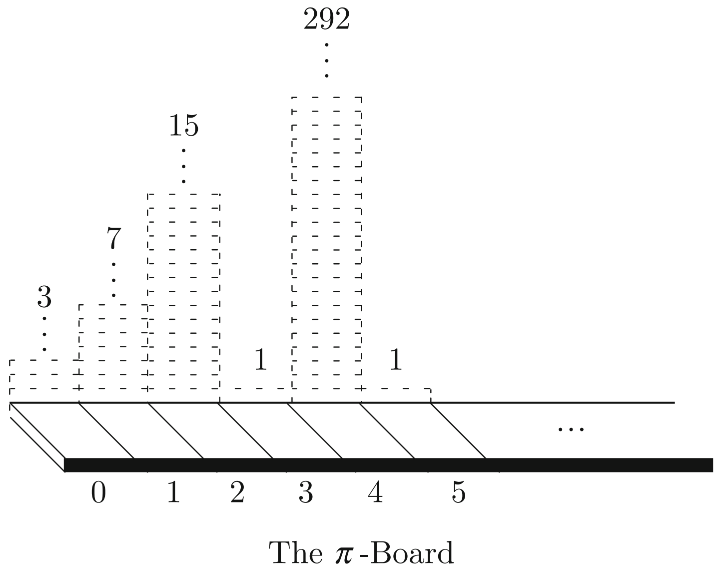

Figure 3.5.

Next we investigate two special cases of the tiling problem.

3.6.1 A Fibonacci Tiling Model

Suppose $a_i = 1$ for every $i \geq 0$; see Figure 3.6. Then $p_0 = 1, p_1 = 2$; and $p_n = p_{n-1} + p_{n-2}$, where $n \geq 2$. So $p_n = F_{n+1}$. Likewise, $q_n = F_n$. So F_n counts the number of tilings of a $1 \times n$ board with dominoes and stackable square tiles. Every convergent $\frac{F_{n+1}}{F_n}$ is an approximation of the golden ratio α.

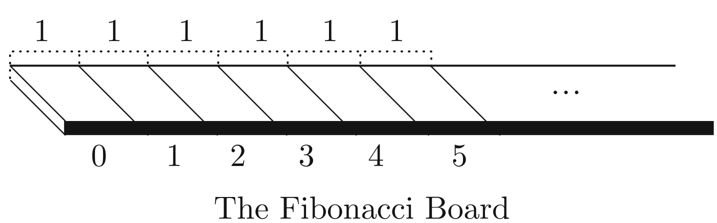

Figure 3.6.

3.6.2 A Pell Tiling Model

Suppose $a_0 = 1$ and $a_i = 2$ for every $i \geq 1$; see Figure 3.7. Then $p_0 = 1, p_1 = 3$; and $p_n = 2p_{n-1} + p_{n-2}$, where $n \geq 2$. Likewise, $q_0 = 1, q_1 = 2$; and $q_n = 2q_{n-1} + q_{n-2}$, where $n \geq 2$. Clearly, $p_n = Q_n$ and $q_n = P_n$. Consequently, Q_n counts the number of tilings of a $1 \times (n+1)$ board with dominoes and stackable square tiles, and P_n the number of such tilings of a $1 \times n$ board, where every stack size is 2.

Next we generalize the simple continued fraction tiling problem.

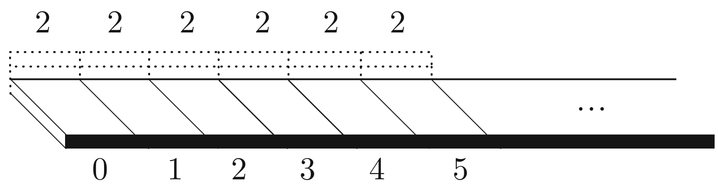

Figure 3.7. The Pell Board

3.7 A Generalized Continued Fraction Tiling Model

Suppose we allow both dominoes and square tiles to be stacked up, dominoes over dominoes and squares over squares. As before, suppose as many as a_i square tiles can be stacked up at cell i, where $i \geq 0$; and as many as b_i dominoes can be stacked up at cells $i-1$ and i, where $i \geq 1$. Let $\hat{p}_n$ denote the number of such tilings of a $1 \times (n+1)$ board and $\hat{q}_n$ the number of such tilings of a $1 \times n$ board, with cell 0 removed. Then, as above, it follows that

$$\begin{aligned} \hat{p}_n &= a_n \hat{p}_{n-1} + b_n \hat{p}_{n-2} \\ \hat{q}_n &= a_n \hat{q}_{n-1} + b_n \hat{q}_{n-2}, \end{aligned}$$

where $n \geq 2$, and $\hat{p}_0 = a_0, \hat{p}_1 = a_1 a_0 + b_1, \hat{q}_0 = 1, \hat{q}_1 = a_1$. Then $\frac{\hat{p}_n}{\hat{q}_n}$ is precisely the nth convergent of the continued fraction (3.4).

For example, consider the infinite continued fraction expansion for $\frac{\pi}{4}$ we encountered earlier. For the corresponding tiling problem, $a_0 = 1, a_i = 2i+1$, and $b_i = i^2$, where $i \geq 1$. Then

$$\begin{aligned} \hat{p}_0 &= 1, \quad \hat{p}_1 = 4 & \hat{q}_0 &= 1, \quad \hat{q}_1 = 3 \\ \hat{p}_n &= (2n+1)\hat{p}_{n-1} + n^2 \hat{p}_{n-2} & \hat{q}_n &= (2n+1)\hat{q}_{n-1} + n^2 \hat{q}_{n-2}, \end{aligned}$$

where $n \geq 2$. So $\frac{\hat{p}_n}{\hat{q}_n}$ is the nth convergent of the continued fraction expansion of $\frac{\pi}{4}$.

We will investigate additional tiling models in Chapters 16 and 20.

Exercises 3

Represent each rational number as a FSCF.

1. $\dfrac{239}{169}$.
2. $\dfrac{169}{70}$.
3. $\dfrac{577}{239}$.
4. $\dfrac{P_{n+1}}{P_n}$.
5. $\dfrac{Q_{n+1}}{Q_n}$.

Compute the convergents of each continued fraction.

6. $[2; 2, 2, 2, 2]$.
7. $[2; 2, 2, 2, 3]$.

Solve each LDE, if possible, where k is a positive integer. *Hint*: Use convergents.

8. $70x + 29y = 169$.
9. $99x + 41y = 181$.
10. $P_{2n+1}x + P_{2n}x = k$. *Hint*: $P_{n+1}P_{n-1} - P_n^2 = (-1)^n$.
11. $P_{2n+1}x + Q_{2n+1}x = k$. *Hint*: $P_n Q_{n-1} - Q_n P_{n-1} = (-1)^{n-1}$.

Using convergents, find three positive solutions of each Pell's equation.

12. $x^2 - 6y^2 = 1$.
13. $x^2 - 11y^2 = 1$.
14. $x^2 - 6y^2 = -2$. *Hint*: Use Exercise 12 and Theorem 2.5.
15. $x^2 - 11y^2 = -2$. *Hint*: Use Exercise 13 and Theorem 2.5.

4

Pythagorean Triples

4.1 Introduction

The *Pythagorean Theorem* is one the most elegant results in euclidean geometry: The sum of the squares of the lengths of the legs of a right triangle equals the square of the length of its hypotenuse. Using Figure 4.1, it can be restated as follows: Let $\triangle ABC$ be a right triangle, right-angled at C; then $AC^2 + BC^2 = AB^2$. The converse of the Pythagorean theorem is also true: If $AC^2 + BC^2 = AB^2$ in a triangle ABC, then it is a right triangle, right-angled at C. A right triangle whose sides have integral lengths is a *Pythagorean triangle*.

Numerous proofs of the Pythagorean theorem using various techniques can be found in the mathematical literature. For instance, E.S. Loomis' 1968 book, *The Pythagorean Proposition* [159], gives 230 different proofs.

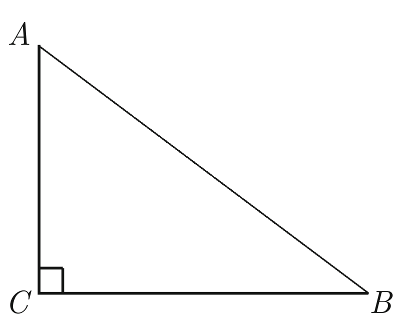

Figure 4.1.

Figure 4.2. A Greek Stamp illustrating the Pythagorean Theorem

T. Koshy, *Pell and Pell–Lucas Numbers with Applications*, DOI 10.1007/978-1-4614-8489-9_4,

4.2 Pythagorean Triples

The Pythagorean Theorem has ramifications beyond geometry. To see this, let x and y denote the lengths of the legs of the right triangle, and z the length of its hypotenuse. Then x, y and z satisfy the nonlinear diophantine equation $x^2 + y^2 = z^2$. Such a triple x-y-z is a *Pythagorean triple*. Every Pythagorean triple corresponds to a Pythagorean triangle, and vice versa.

The simplest Pythagorean triple is 3-4-5, depicted by the Greek stamp in Figure 4.2: $3^2 + 4^2 = 5^2$. This yields infinitely many Pythagorean triples $3n$-$4n$-$5n$: $(3n)^2 + (4n)^2 = (5n)^2$, where n is an arbitrary positive integer.

There are Pythagorean triples involving of four consecutive Fibonacci numbers. To see this, we let $x = F_n F_{n+3}$, $y = 2F_{n+1}F_{n+2}$, and $z = F_{n+1}^2 + F_{n+2}^2$. Then

$$\begin{aligned} x^2 + y^2 &= (F_n F_{n+3})^2 + (2F_{n+1}F_{n+2})^2 \\ &= [(F_{n+2} - F_{n+1})(F_{n+2} + F_{n+1})]^2 + (2F_{n+1}F_{n+2})^2 \\ &= (F_{n+2}^2 - F_{n+1}^2)^2 + 4F_{n+1}^2 F_{n+2}^2 \\ &= (F_{n+2}^2 + F_{n+1}^2)^2 \\ &= z^2. \end{aligned}$$

For example, let $n = 7$. Then $x = F_7 F_{10} = 13 \cdot 55 = 715$, $y = 2F_8 F_9 = 2 \cdot 21 \cdot 34 = 1428$, and $z = F_8^2 + F_9^2 = 21^2 + 34^2 = 1597$. Notice that $x^2 + y^2 = 715^2 + 1428^2 = 2,550,409 = 1597^2 = z^2$.

Likewise, $L_n L_{n+3}$ - $2L_{n+1}L_{n+2}$ - $(L_{n+1}^2 + L_{n+2}^2)$ is a Pythagorean triple. For example, again we let $n = 7$. Then $x = L_7 L_{10} = 29 \cdot 123 = 3567$, $y = 2L_8 L_9 = 2 \cdot 47 \cdot 76 = 7144$, and $z = L_8^2 + L_9^2 = 47^2 + 76^2 = 7985$. Again, notice that $x^2 + y^2 = 63,760,225 = z^2$.

More generally, let $\{G_n\}$ be any integer sequence such that $G_n = G_{n-1} + G_{n-2}$, where $n \geq 3$. Let $x = G_n G_{n+3}$, $y = 2G_{n+1}G_{n+2}$, and $z = G_{n+1}^2 + G_{n+2}^2$. Then x-y-z is a Pythagorean triple.

4.2.1 Primitive Pythagorean Triples

The Pythagorean triple 3-4-5 has the property that (3,4,5) = 1; that is, 3, 4, and 5 are pairwise relatively prime. Such a triple x-y-z is said to be *primitive*. For example, 5-12-13 and 119-120-169 are primitive Pythagorean triples.

Let x-y-z be an arbitrary Pythagorean triple and $(x, y, z) = d$. Then $x = du$, $y = dv$, and $z = dw$ for some positive integers u, v, and w, and $(u, v, w) = 1$. Since $x^2 + y^2 = z^2$, it follows that $u^2 + v^2 = w^2$, where $(u, v, w) = 1$; so u-v-w is a primitive Pythagorean triple. Thus every Pythagorean triple is a positive multiple of a primitive Pythagorean triple. Consequently, we confine our discussion to primitive Pythagorean triples.

Primitive Pythagorean triples can be characterized by the following elegant result. Again, we omit its proof [130] in the interest of brevity.

Theorem 4.1 *Let x, y and z be arbitrary positive integers, where y is even. Then x-y-z is a primitive Pythagorean triple if and only if there are positive integers m and n with $m > n$, $(m,n) = 1$, and $m \not\equiv n \pmod 2$ such that $x = m^2 - n^2$, $y = 2mn$, and $z = m^2 + n^2$.* ■

The integers m and n are the *generators* of the primitive Pythagorean triple x-y-z.

Table 4.1 shows the the primitive Pythagorean triples with $m \leq 10$. It follows from the table that the lengths of the sides of a Pythagorean triangle can be squares; see the circled numbers. But no two sides of a Pythagorean triangle can be squares.

Table 4.1.

Generators		Primitive Pythagorean Triples		
m	n	$x = m^2 - n^2$	$y = 2mn$	$z = m^2 + n^2$
2	1	3	(4)	5
3	2	5	12	13
4	1	15	8	17
4	3	7	24	(25)
5	2	[21]	[20]	29
5	4	(9)	40	41
6	1	35	12	37
6	5	11	60	61
7	2	45	28	53
7	4	33	56	65
7	6	13	84	85
8	1	63	(16)	65
8	3	55	48	73
8	5	39	80	89
8	7	15	112	113
9	2	77	(36)	85
9	4	65	72	97
9	8	17	(144)	145
10	1	99	20	101
10	3	91	60	109
10	7	51	140	149
10	9	19	180	181

Can the lengths of the legs of a primitive Pythagorean triangle be consecutive integers? Clearly, yes; 3-4-5 is the most obvious example. (It is the only Pythagorean triangle such that the lengths of all sides are consecutive integers; this can be confirmed using basic algebra.) Are there other primitive Pythagorean triangles such that the lengths of their legs are consecutive integers? Table 4.1 shows one more such triangle: [21]-[20]-29.

Are there any other such primitive Pythagorean triangles? If yes, how do we find them? To investigate such primitive Pythagorean triangles, we must have $y = x + 1$ or $x = y + 1$. So, by Theorem 4.1, we must have $2mn = m^2 - n^2 + 1$ or $m^2 - n^2 = 2mn + 1$; so $(m-n)^2 - 2n^2 = -1$ or $(m-n)^2 - 2n^2 = 1$. These two equations can be combined into the Pell's equation $u^2 - 2v^2 = (-1)^k$, where $u = m - n, v = n$, and $m > n$.

By Examples 2.1 and 2.10, the elements of the sequence $\{u_k\}$ are $1, 3, 7, 17, 41, 99, \ldots$. The corresponding values of the sequence $\{v_k\}$ are $2, 5, 12, 29, 70, 169\ldots$. Table 4.2 lists the first twelve pairs of generators (m, n) and the lengths $x = m^2 - n^2$, $y = 2mn$, and $z = m^2 + n^2$ of the sides of the corresponding primitive Pythagorean triangles. The lengths of the legs of these triangles are consecutive integers. Clearly, there are infinitely many such primitive Pythagorean triangles. In other words, the diophantine equation $x^2 + (x \pm 1)^2 = z^2$ has infinitely many solutions.

Table 4.2.

Index	Generators		Primitive Pythagorean Triangles With Consecutive Legs		
k	m	n	$x = m^2 - n^2$	$y = 2mn$	$z = m^2 + n^2$
1	2	1	3	4	5
2	5	2	21	20	29
3	12	5	119	120	169
4	29	12	697	696	985
5	70	29	4059	4060	5741
6	169	70	23661	23660	33461
7	408	169	137903	137904	195025
8	985	408	803761	803760	1136689
9	2378	985	4684659	4684660	6625109
10	5741	2378	27304197	27304196	38613965
11	13860	5741	159140519	159140520	225058681
12	33461	13860	927538921	927538920	1311738121

4.2.2 Some Quick Observations

We can now make some interesting and useful observations about the sequences $\{u_k\}, \{v_k\}$ and $\{m_k\}$, and Table 4.2.

(1) The sequences $\{u_k\}$ and $\{v_k\}$ follow exactly the same recursive Pell pattern.

(2) The sequences $\{v_k\}$ and $\{m_k\}$ are the same except that the initial seed v_1 is missing in the latter sequence.

(3) The legs alternate between larger and smaller lengths; that is, if $x_k < y_k$, then $x_{k+1} > y_{k+1}$; and if $x_k > y_k$, then $x_{k+1} < y_{k+1}$. These can be established using the recursive definition of the sequence $\{v_k\}$ and the fact that it is an increasing sequence. For example, if $v_k^2 - v_{k-1}^2 < 2v_k v_{k-1}$, then $v_{k+1}^2 - v_k^2 > 2v_{k+1}v_k$.

(4) The lengths z_k of the hypotenuses are the even-numbered values of m_k; that is, $z_k = m_{2k}$. This follows from the facts that $m_k = v_{k+1}, n_k = v_k$, and $v_{k+1}^2 + v_k^2 = v_{2k+1}$.

(5) The lengths z_k of the hypotenuses also satisfy a recursive pattern:

$$\begin{aligned} z_1 &= 5, \quad z_2 = 29 \\ z_k &= 6z_{k-1} - z_{k-2}, \quad k \geq 3. \end{aligned}$$

(We encountered this recurrence in Example 1.5.)

(6) Consider the smaller of the lengths of the legs for each k. Let a_k denote the resulting sequence: $3, 20, 119, 696, 4059, 23660, \ldots$. It also satisfies a recursive pattern:

$$\begin{aligned} a_1 &= 3, \ \ a_2 = 20 \\ a_k &= 6a_{k-1} - a_{k-2} + 2, \quad k \geq 3. \end{aligned}$$

(We saw a similar pattern in Example 1.6.)

(7) Suppose the pair (m_k, n_k) generates the kth primitive Pythagorean triple. Then $2m_k + m_{k-1} = m_{k+1}$ and $n_{k+1} = 2n_k + n_{k-1} = 2m_{k-1} + m_{k-2} = m_k$. Consequently, the pair $(2m_k + m_{k-1}, m_k)$ generates the $(k+1)$st primitive Pythagorean triple whose legs are consecutive integers, where $k > 2$.

This gives a quick algorithm to compute each generator from its predecessor and thus to compute the corresponding primitive Pythagorean triple.

For example, (29, 12) generates the fourth such primitive Pythagorean triple 697-696-985. So the fifth such triple is generated by $(2 \cdot 29 + 12,\ 29) = (70,\ 29)$; see Table 4.2.

4.3 A Recursive Algorithm

Next we develop a recursive algorithm for generating all primitive Pythagorean triples such that the lengths of the legs are consecutive integers. To this end, suppose x-y-z is such a primitive Pythagorean triple. Then $x - y = \pm 1$, so $x^2 + (x+1)^2 = z^2$. Let $u = 3x + 2z + 1, v = 3x + 2z + 2$, and $w = 4x + 3z + 2$. Then

$$\begin{aligned} u^2 + v^2 &= 18x^2 + 24xz + 8z^2 + 18x + 12z + 5 \\ &= (16x^2 + 8z^2 + 24xz + 16x + 12z + 4) + (2x^2 + 2x + 1) \\ &= (16x^2 + 8z^2 + 24xz + 16x + 12z + 4) + z^2 \\ &= 16x^2 + 9z^2 + 24xz + 16x + 12z + 4 \\ &= w^2. \end{aligned}$$

Consequently, u-v-w is a Pythagorean triple. It is easy to see that it is primitive also; thus it is a primitive Pythagorean triple. Knowing primitive Pythagorean triple x_n-y_n-z_n, this gives a recursive algorithm to construct a primitive Pythagorean triple x_{n+1}-y_{n+1}-z_{n+1} with the same property.

Since the simplest such triple is 3-4-5, this algorithm can be defined recursively:

$$\begin{aligned} x_1 &= 3, \ \ y_1 = 4, \ \ z_1 = 5 \\ x_{n+1} &= 3x_n + 2z_n + 1, \quad y_{n+1} = x_{n+1} + 1, \quad z_{n+1} = 4x_n + 3z_n + 2, \quad n \geq 1. \end{aligned} \tag{4.1}$$

For example, $x_2 = 3x_1 + 2z_1 + 1 = 3 \cdot 3 + 2 \cdot 5 + 1 = 20, y_2 = x_2 + 1 = 21$, and $z_2 = 4x_1 + 3z_1 + 2 = 4 \cdot 3 + 3 \cdot 5 + 2 = 29$; likewise, x_3-y_3-z_3 = 119-120-169; see Table 4.3.

Table 4.3. Primitive Pythagorean Triples Generated by Recursion

n	x_n	y_n	z_n
1	3	4	5
2	20	21	29
3	119	120	169
4	696	697	985
5	4059	4060	5741
6	23660	23661	33461
7	137904	137903	195025

Notice that in Table 4.2, the leg x_n is always odd and y_n is always even; but in Table 4.3, their parities alternate.

Notice also that the sequence $\{x_n\}$ can be defined recursively:

$$\begin{aligned} x_1 &= 3, \quad x_2 = 20 \\ x_n &= 6x_{n-1} - x_{n-2} + 2, \quad n \geq 3. \end{aligned}$$

For example, $x_4 = 6x_3 - x_2 + 2 = 6 \cdot 119 - 20 + 2 = 696$.

It follows from the recursive definition (4.1) that there are infinitely many primitive Pythagorean triples x_n-y_n-z_n with consecutive integral legs. Conversely, we can prove that every primitive Pythagorean triple with consecutive integral legs can be generated using the recursive definition (4.1) [233].

Finally, we note that the three recursive formulas in definition (4.1) can be combined into a single matrix equation:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 3 & 2 & 2 \\ 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \\ 1 \end{bmatrix}, \quad n \geq 1.$$

Exercises 4

Verify that each is a Pythagorean triple.

1. $(17 \cdot 41, 2 \cdot 12 \cdot 29, 985)$.
2. $(13860, 70^2 + 1, 3 \cdot 70^2 + 1)$.
3. Prove that 3-4-5 is the only primitive Pythagorean triple consisting of consecutive integers.
4. Prove that there are infinitely many Pythagorean triples.
5. Suppose x-y-z is a primitive Pythagorean triple with x even. Prove that both yz is odd.
6. Prove that the length of one leg of a Pythagorean triangle is a multiple of 3 (C.W. Trigg, 1970). *Hint:* Use Fermat's little theorem [130].

Prove that each is a Pythagorean triple.

7. $(2n + 1, 2n(n + 1), 2n(n + 1) + 1)$.

8. $(4n, 4n^2 - 1, 4n^2 + 1)$.
9. $(3(2n + 3), 2n(n + 3), 2n^2 + 6n + 9)$.
10. $(Q_n Q_{n+1}, 2P_n P_{n+1}, P_{2n+1})$.
11. $(P_{4n}, P_{2n}^2 + 1, 3P_{2n}^2 + 1)$. *Hint*: Use the Pythagorean triple $(2ab, a^2 - b^2, a^2 + b^2)$ with $a = Q_{2n}$ and $b = P_{2n}$, and the identities $P_{2n} = 2P_n Q_n$ and $Q_n^2 - 2P_n^2 = (-1)^n$.
12. The lengths of the legs of a Pythagorean triangle are $Q_n Q_{n+1}$ and $2P_n P_{n+1}$. Compute the area of the triangle. *Hint*: $P_{2k} = 2P_k Q_k$.

Develop a generating function for the sequence $\{a_n\}$ satisfying the given recurrence with the corresponding initial conditions.

13. $a_n = 6a_{n-1} - a_{n-2}; a_1 = 5, a_2 = 29$.
14. $a_n = 6a_{n-1} - a_{n-2} + 2; a_1 = 3, a_2 = 20$.

*15 The ratio of the area of a Pythagorean triangle to its semi-perimeter is p^m, where p is a prime and m a positive integer. Prove that there are $m + 1$ such triangles if $p = 2$, and $2m + 1$ such triangles otherwise. (H. Klostergaard, 1979)

5

Triangular Numbers

5.1 Introduction

The old Chinese proverb, "A picture is worth a thousand words," is true in mathematics. We frequently use geometric illustrations to clarify concepts and illustrate relationships in every branch of mathematics. *Figurate Numbers* provide such a link between number theory and geometry; they are positive integers that can be represented by geometric patterns. Although the Pythagoreans are usually given credit for their discovery, the ancient Chinese seem to have originated such representations about 500 years before Pythagoras. Many centuries later, in 1665, Pascal wrote a book on them, *Treatise on Figurate Numbers*.

Polygonal numbers are a special class of figurate numbers; they are positive integers that can be represented by regular n-gons, where $n \geq 3$. When $n = 3, 4$, and 5, they are called *triangular*, *square*, and *pentagonal numbers*, respectively. In this chapter, we will focus on triangular numbers.

5.2 Triangular Numbers

We often see triangular arrangements of objects in the real world. A set of the ten bowling pins are initially set up in a triangular shape. In the game of pool, the 15 balls are also initially set up as a triangular array. The white floral design in the tablecloth in Figure 5.1 represents the number 15; the gray spaces in between the designs represent the number 10. Grocery stores often stack fruits, such as apples and oranges, in triangular arrangements; see Figure 5.2. The numbers 3, 10, and 15 are triangular numbers.

More generally, a *triangular number* t_n is a positive integer that can be represented by an equilateral triangular array. The first four triangular numbers are $t_1 = 1$, $t_2 = 3$, $t_3 = 6$, and $t_4 = 10$. They are represented pictorially in Figure 5.3.

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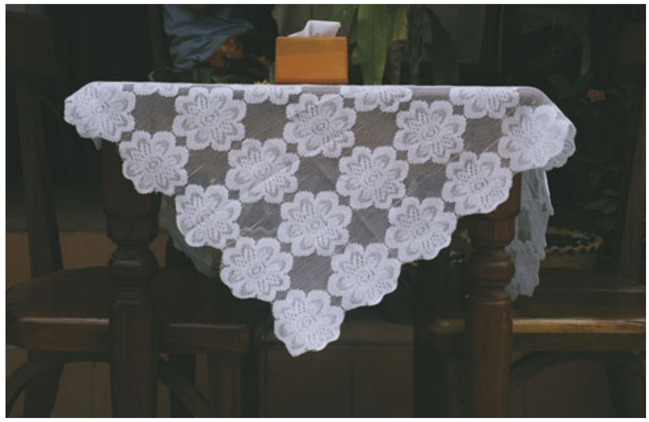

Figure 5.1. A Thai tablecloth

Figure 5.2. Oranges in Grocery Store

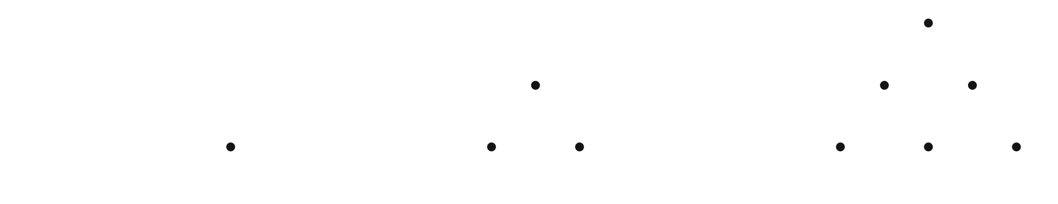

Figure 5.3. The First Four Triangular Numbers

Since row i in such an array contains i dots, it follows that $t_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

For example, $t_{36} = \frac{36 \cdot 37}{2} = 666$ (the *beastly number*[6]), and $t_{1681} = \frac{1681 \cdot 1682}{2} = 1,413,721 = 1189^2$[7], a square.

Since $t_n = \frac{n(n+1)}{2} = \binom{n+1}{2}$, triangular numbers can be read directly from Pascal's triangle; see the rising diagonal in Figure 5.4.

Since row n in the array contains n dots, t_n can be defined recursively:

---

6 See the Book of *Revelation* in the Bible.

7 For Fibonacci enthusiasts, we note that 11 is a Lucas number and 89 a Fibonacci number; there are 1189 chapters in the Bible, of which 89 are in the New Testament.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

(triangular numbers)

Figure 5.4. Pascal's Triangle

$$t_1 = 1$$
$$t_n = t_{n-1} + n, \quad n \geq 2.$$

Triangular numbers appear in a variety of situations. For example, we can find them in the famous carol "Twelve Days of Christmas" [130]: Suppose you send i gifts to your true love on the ith day of Christmas; how many gifts s_n would be sent on the nth day? Clearly, $s_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} = t_n$. In particular, $s_{12} = \frac{12 \cdot 13}{2} = 78 = t_{12}$, as in the carol.

5.3 Pascal's Triangle Revisited

There is an interesting relationship between triangular numbers and *Pascal's triangle*. To see this, replace each even number by 0 and each odd number by 1. Figure 5.5 shows the resulting *Pascal's binary triangle*.

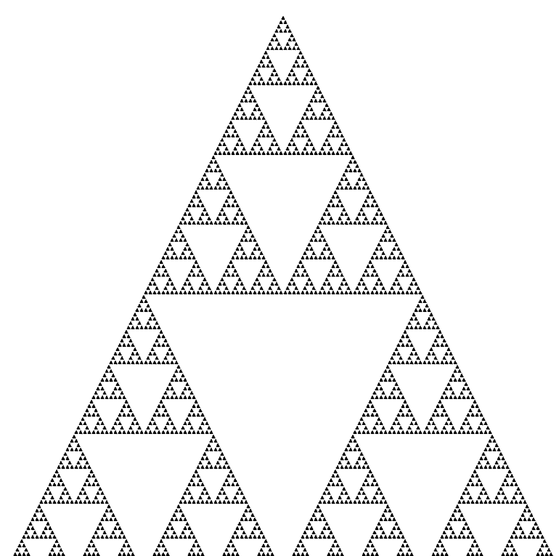

Figure 5.5. Pascal's Binary Triangle

Notice that some rows consist of 1s only; they are row 0 and rows labeled M_n, when $n \geq 1$. This fact was established in 1981 by R.M. Dacic of Belgrade, Serbia [130].

Pascal's binary triangle contains another treasure. To see this, consider the centrally located triangles ∇_n consisting of 0s; they point downward and have their bases on row 2^n, where $n \geq 1$. Since row n contains $(2^n + 1) - 2 = M_n$ zeros, ∇_n consists of $\frac{M_n(M_n+1)}{2} = 2^{n-1}M_n$ zeros. But $2^{n-1}M_n = t_{M_n}$. So triangle ∇_n represents the triangular number t_{M_n} for every $n \geq 1$.

Since every even perfect number is of the form $2^{p-1}M_p$ and $2^{p-1}M_p = t_{M_p}$, it follows that every *even perfect number* is represented by the triangle ∇_{M_p}, where M_p is a prime.

For example, $M_3 = 2^3 - 1 = 7$ and $t_{M_3} = t_7 = \frac{7 \cdot 8}{2} = 28$, the second perfect number. It is represented by ∇_{M_3}, the third centrally located triangle pointing downward.

5.4 Triangular Mersenne Numbers

Although every even perfect number is triangular, what is the case with Mersenne numbers M_n? Notice that $M_1 = t_1, M_2 = 3 = t_2, M_4 = 15 = t_5$, and $M_{12} = 4095 = t_{90}$ are all triangular numbers. But there are no additional *triangular Mersenne numbers* $\leq M_{30}$. So they appear to be sparsely populated.

In 1958, U.V. Satyanarayana of Andhra University, India, proved that there are infinitely many Mersenne numbers which are *not* triangular. This can be established fairly easily using congruence modulo 10. First, notice that every triangular number t_n is congruent to 1, 3, 5, 6, or 8 modulo 10. Secondly, let n be any positive integer. Then, by the division algorithm, $n \equiv 0, 1, 2$, or $3 \pmod 4$.

Case 1 Suppose $n \equiv 0 \pmod 4$, so $n = 4k$ for some integer $k \geq 0$. Then

$$M_n = 2^{4k} - 1 = (2^4)^k - 1 \equiv 6^k - 1 \equiv 6 - 1 \equiv 5 \pmod{10}.$$

Case 2 Suppose $n \equiv 1 \pmod 4$, so $n = 4k + 1$ for some integer $k \geq 0$. Then

$$M_n = 2^{4k+1} - 1 = 2 \cdot (2^4)^k - 1 \equiv 2 \cdot 6^k - 1 \equiv 2 \cdot 6 - 1 \equiv 1 \pmod{10}.$$

Likewise, it can be shown that

$$M_n = \begin{cases} 3 \pmod{10} & \text{if } n \equiv 2 \pmod 4 \\ 7 \pmod{10} & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Thus

$$M_n = \begin{cases} 5 \pmod{10} & \text{if } n \equiv 0 \pmod{4} \\ 1 \pmod{10} & \text{if } n \equiv 1 \pmod{4} \\ 3 \pmod{10} & \text{if } n \equiv 2 \pmod{4} \\ 7 \pmod{10} & \text{otherwise.} \end{cases}$$

Since $t_n \equiv 1, 3, 5, 6$, or 8 (mod 10), it follows that M_n cannot be triangular if $n \equiv 3 \pmod{4}$. But there are infinitely many *Mersenne numbers* of the form M_{4k+3}. Consequently, there are infinitely many non-triangular Mersenne numbers.

5.5 Properties of Triangular Numbers

Triangular numbers satisfy a vast array of interesting properties. The simplest and most obvious is the fact that the sum of any two consecutive triangular numbers is a square: $t_n + t_{n-1} = n^2$. The proof follows algebraically.

Figure 5.6 provides a geometric illustration of this fact when $n = 4$ and $n = 5$.

Figure 5.6.

Next we establish a criterion for two consecutive triangular numbers to have the same *parity* (oddness or evenness).

Theorem 5.1 *The triangular numbers t_n and t_{n+1} have the same parity if and only if n is odd.*

Proof. Suppose $t_n \equiv t_{n+1} \pmod{2}$. Then $t_n + t_{n+1} = (n+1)^2$ is even; so $n + 1$ is even. Hence n is odd.

Conversely, let n be odd. Then $n + 1$ and hence $(n+1)^2$ are even. So $t_n + t_{n+1}$ is even. Consequently, $t_n \equiv t_{n+1} \pmod{2}$. ∎

The next result, discovered by Diophantus (ca. 250), can also be established algebraically: One more than eight times a triangular number is a square.

Theorem 5.2 (Diophantus) $8t_n + 1 = (2n+1)^2$. ∎

Figure 5.7 shows a pictorial illustration of Diophantus' result, developed in 1985 by E.G. Landauer of the General Physics Corporation [148].

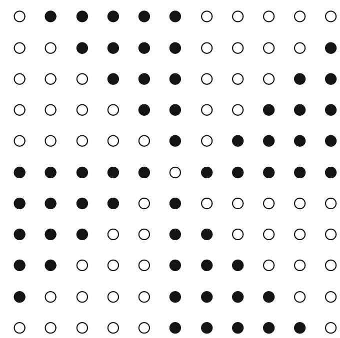

Figure 5.7. A Visual Proof of Diophantus' Result

The Diophantus Theorem yields an interesting byproduct. The square of every odd integer is congruent to 1 modulo 8; that is, $(2n+1)^2 \equiv 1 \pmod 8$. This theorem also implies that if $8N+1=(2n+1)^2$, then $N=\frac{n(n+1)}{2}=t_n$. See the geometric illustration in Figure 5.7.

As another byproduct, it can be used to determine whether or not a positive integer N is a triangular number t_n; if it is, then we can identify the value of n, as the next two examples illustrate.

Example 5.1 Determine whether or not $N = 1{,}983{,}036$ is a triangular number t_n. If it is, find n.

Solution. Assume N is a triangular number t_n. Then, by Theorem 5.2, $8N+1=(2n+1)^2$; so $n=\frac{\sqrt{8N+1}-1}{2}=\frac{\sqrt{8\cdot 1983036}-1}{2}$ must be a positive integer. Notice that $\frac{\sqrt{8\cdot 1983036}-1}{2}=1991$ is indeed an integer. So N is a triangular number and $N=t_{1991}=\frac{1991\cdot 1992}{2}$. ∎

Example 5.2 Determine whether or not $N = 1{,}967{,}139$ is a triangular number t_n. If it is, find n.

Solution. Again, assume N is a triangular number t_n. Then, as in Example 5.1, $n=\frac{\sqrt{8N+1}-1}{2}=\frac{\sqrt{8\cdot 1967139}-1}{2}$ must be an integer. But $\frac{\sqrt{8\cdot 1967139}-1}{2}\approx 1983.0015$, a contradiction. So N is *not* a triangular number. ∎

Table 5.1 shows a list of properties of triangular numbers; they all can be confirmed algebraically.

Table 5.1. Properties of Triangular Numbers

(1)	$t_n + t_{n-1} = n^2$	(2)	$t_{t_n} + t_{t_n - 1} = t_n^2$
(3)	$t_n^2 + t_{n-1}^2 = t_{n^2}$	(4)	$8t_n + 1 = (2n+1)^2$
(5)	$8t_{n-1} + 4n = (2n)^2$	(6)	$t_{2n} = 3t_n + t_{n-1}$
(7)	$t_{2n+1} = 3t_n + t_{n+1}$	(8)	$t_n^2 - t_{n-1}^2 = n^3$
(9)	$t_{2n} - 2t_n = n^2$	(10)	$t_{2n-1} - 2t_{n-1} = n^2$
(11)	$t_n^2 = t_n + t_{n-1}t_{n+1}$	(12)	$2t_n t_{n-1} = t_{n^2-1}$

Identity (8) has an interesting byproduct. It can be employed to establish the summation formula for $\sum_{i=1}^{n} i^3$; see Exercise 19.

The properties in Table 5.1 can be illustrated pictorially, providing an enjoyable exercise. This art has been popularized over the years by R.B. Nelsen of Lewis and Clark College, Portland, Oregon [170, 171]. For example, Figures 5.8, 5.9, 5.10, and 5.11 provide visual illustrations of properties (3), (5), (6), and (7), respectively.

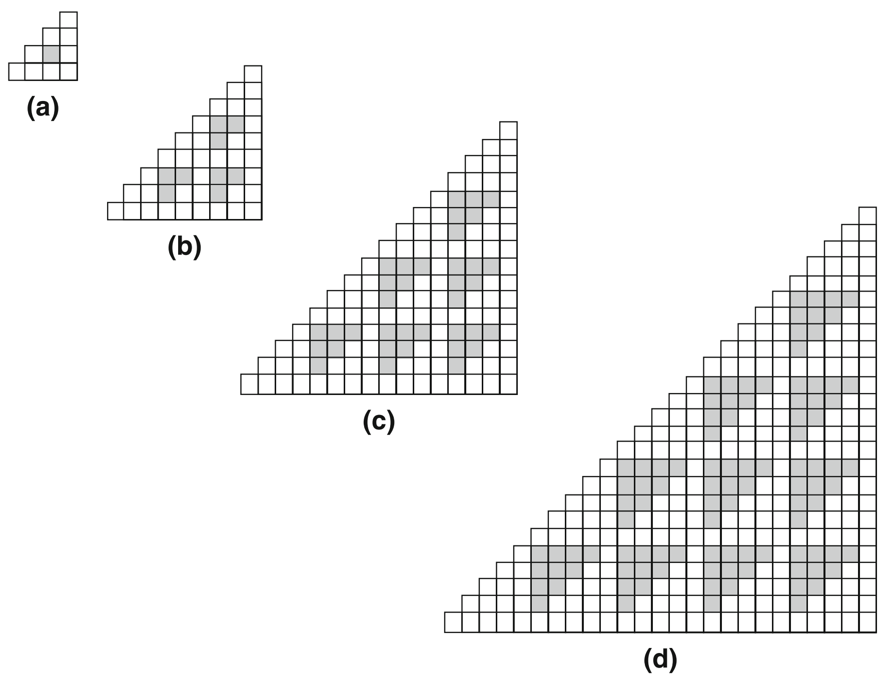

Figure 5.8. (**a**) $t_1^2 + t_2^2 = t_4$ (**b**) $t_2^2 + t_3^2 = t_9$ (**c**) $t_3^2 + t_4^2 = t_{16}$ (**d**) $t_4^2 + t_5^2 = t_{25}$

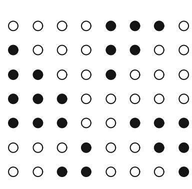

Figure 5.9. $8t_{n-1} + 4n = (2n)^2$

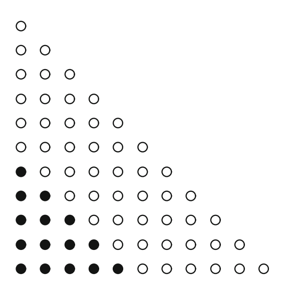

Figure 5.10. $3t_n + t_{n-1} = t_{2n}$

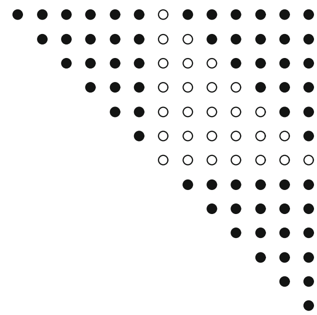

Figure 5.11. $3_{tn} + t_{n+1} = t_{2n+1}$

5.6 Triangular Fermat Numbers

Since

$$\begin{aligned} t_{3n} &= \frac{3n(3n+1)}{2} \equiv 2(3n)(3n+1) \equiv 0 \pmod 3 \\ t_{3n+1} &= \frac{(3n+1)(3n+2)}{2} \equiv 2(3n+1)(3n+2) \equiv 1 \pmod 3 \\ t_{3n+2} &= \frac{(3n+2)(3n+3)}{2} \equiv 2(3n+2)(3n+3) \equiv 0 \pmod 3, \end{aligned}$$

it follows that

$$t_m \equiv \begin{cases} 1 \pmod 3 & \text{if } m \equiv 1 \pmod 3 \\ 0 \pmod 3 & \text{otherwise.} \end{cases}$$

This congruence has an interesting consequence. To see this, consider the nth Fermat number $f_n = 2^{2^n} + 1$, where $n \ge 0$. Notice that $f_0 = 3$ is a triangular number. Suppose $n \ge 1$. Then $f_n \equiv (-1)^{2^n} + 1 \equiv 1 + 1 \equiv 2 \pmod 3$; so f_n cannot be a triangular number when $n \ge 1$. In other words, 3 is the only *triangular Fermat number*.

In 1987, S. Asadulla [5] established the same fact using PMI and the following facts, where $n \ge 1$: 1) $f_n \equiv 7 \pmod{10}$; and 2) $t_n \equiv 1, 5, 6, \text{ or } 8 \pmod{10}$.

5.7 The Equation $x^2 + (x+1)^2 = z^2$ Revisited

In Chapter 4 we found that the diophantine equation $x^2 + (x+1)^2 = z^2$ has infinitely many solutions. This fact has an interesting byproduct. To see this, let (x, z) be such a solution. Then

$$\begin{aligned} t_{2x}^2 + t_{2x+1}^2 &= \left[\frac{2x(2x+1)}{2}\right]^2 + \left[\frac{(2x+1)(2x+2)}{2}\right]^2 \\ &= (2x+1)^2\left[x^2 + (x+1)^2\right] \\ &= [(2x+1)z]^2. \end{aligned}$$

Since the equation $x^2 + (x + 1)^2 = z^2$ has infinitely many solutions, it follows that there are infinitely many primitive Pythagorean triangles whose legs are consecutive triangular numbers. For example, $t_6^2 + t_7^2 = (7 \cdot 5)^2, t_{40}^2 + t_{41}^2 = (41 \cdot 29)^2$, and $t_{238}^2 + t_{239}^2 = (239 \cdot 169)^2$ yield three such *Pythagorean triangles*.

Next we develop a generating function for triangular numbers using differentiation. This technique works since we are not interested in the convergence of the corresponding power series.

5.8 A Generating function For Triangular Numbers

We have

$$\begin{aligned} \frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\ \frac{1}{(1-x)^2} &= \sum_{n=0}^{\infty} (n+1)x^n \\ \frac{1}{(1-x)^3} &= \sum_{n=1}^{\infty} \frac{n(n+1)}{2} x^n. \end{aligned}$$

This is the desired generating function.

5.9 Triangular Numbers and Pell's Equation

There is a close link between triangular numbers and Pell's equation. The following example illustrates such a relationship. We will explore this further in the next chapter.

Example 5.3 There are triangular numbers t_n that differ from a square by 1; that is, $|t_n - m^2| = 1$, where m is a positive integer and $|x|$ denotes the absolute value of the number x. For example, $|t_2 - 2^2| = |3 - 2^2| = 1$ and $|t_4 - 3^2| = |10 - 3^2| = 1$. Find the next six such triangular numbers.

Solution. The condition $|t_n - m^2| = 1$ implies that $n(n + 1) - 2m^2 = \pm 2$; that is, $n^2 + n - 2m^2 = \pm 2$. Multiplying by 4 and then completing the square, we get two Pell's equations: $x^2 - 2y^2 = -7$ or 9, where $x = 2n + 1$ and $y = 2m$; so $x \equiv 1 \pmod 2$ and $y \equiv 0 \pmod 2$.

Case 1 Let $x^2 - 2y^2 = -7$. We solved this equation in Example 2.16. Table 5.2 gives the first four solutions (x, y), and the corresponding values of n, m, t_n and m^2. In each case, $|t_n - m^2| = 1$. (Note that t_0 also works, although it is not considered a triangular number.

Table 5.2.

x	y	n	m	t_n	m^2
1	2	0	1	0	1
5	4	2	2	3	4
11	8	5	4	15	16
31	22	15	11	120	121

Table 5.3.

x	y	n	m	t_n	m^2
3	0	1	0	1	0
9	6	4	3	10	9
51	36	25	18	325	324
297	210	148	105	11026	11025

∎

Case 2 Let $x^2 - 2y^2 = 9$. Using Example 2.15, we can construct a similar table; see Table 5.3. Again, $|t_n - m^2| = 1$ in each case. (Notice that t_1 is a possibility if we allow m to be zero.)

5.9.1 Two Interesting Dividends

The y-values found in Case 2 above follow an interesting pattern: $0, 6, 36, 210, 1224, \ldots$. This sequence $\{y_n\}$ can be defined recursively:

$$\begin{aligned} y_0 &= 0, \quad y_1 = 6 \\ y_n &= 6y_{n-1} - y_{n-2}, \quad n \geq 2. \end{aligned}$$

This is a homogeneous recurrence with constant coefficients. (We will encounter it in the next chapter also.) Using the standard technique, we can solve it to find an explicit formula for y_n. The general solution is $y_n = 3\left(\frac{\gamma^{2n}-\delta^{2n}}{2\sqrt{2}}\right) = 3P_{2n}$, where $n \geq 0$. For example, $y_2 = 3P_4 = 3 \cdot 12 = 36$.

A generating function for the sequence $\{y_n\}$ is

$$\frac{6x}{1-6x+x^2} = 6x + 36x^2 + 210x^3 + 1224x^4 + \ldots .$$

Next we investigate the $\{x_k\}$-values found in Case 1: $1, 5, 11, 31, 65, \ldots$. They too follow an interesting pattern. Can you find it without reading any further?

The sequence $\{x_k\}$ can be defined recursively:

$$\begin{aligned} x_1 &= 1, \quad x_2 = 5, \quad x_3 = 11, \quad x_4 = 31 \\ x_k &= 6x_{k-2} - x_{k-4}, \quad k \geq 5. \end{aligned} \tag{5.1}$$

For example, $x_6 = 6x_4 - x_2 = 6 \cdot 31 - 5 = 181$. Likewise, $x_7 = 379$. Table 5.4 gives the first eleven values of x_k and y_k, and the corresponding values n_k and m_k of n and m, respectively. Using the recursive definition (5.1), we can extend it to include the case $n = 0$: $x_4 = 6x_2 - x_0$; that is, $31 = 6 \cdot 5 - x_0$; so x_0 must be -1.

Accordingly, we can modify the recursive definition:

$$\begin{aligned} x_0 &= -1, \quad x_1 = 1, \quad x_2 = 5, \quad x_3 = 11 \\ x_k &= 6x_{k-2} - x_{k-4}, \quad k \geq 4. \end{aligned} \tag{5.2}$$

Table 5.4.

k	1	2	3	4	5	6	7	8	9	10	11
x_k	1	5	11	31	65	181	379	1055	2209	6149	12875
y_k	2	4	8	22	46	128	268	746	1562	4348	9104
n_k	0	2	5	15	32	90	189	527	1104	3074	6437
m_k	1	2	4	11	23	64	134	373	781	2174	4552

This will make it easier to find an explicit formula for x_k. To this end, we will solve the recurrence (5.2).

The recurrence (5.2) is a linear homogeneous one with constant coefficients. Its characteristic equation is $z^4 - 6z^2 + 1 = 0$. Solving it, we get $z^2 = 3 \pm 2\sqrt{2} = (1 \pm \sqrt{2})^2$; so $z = \pm 1 \pm \sqrt{2}$. Thus there are four characteristic roots: $r = \gamma, s = \delta, t = -\gamma$, and $u = -\delta$. So the general solution is of form $x_k = Ar^k + Bs^k + Ct^k + Du^k$, where the constants A, B, C, and D are to be determined.

The four initial conditions yield the following 4×4 linear system in A, B, C, and D:

$$\begin{aligned} A + B + C + D &= -1 \\ Ar + Bs + Ct + Du &= 1 \\ Ar^2 + Bs^2 + Ct^2 + Du^2 &= 5 \\ Ar^3 + Bs^3 + Ct^3 + Du^3 &= 11. \end{aligned} \tag{5.3}$$

Solving this is lengthy, tedious, and time-consuming. For the sake of brevity, we will omit the complicated computations and give only the key steps. The following values will come in handy in the process: $r - s = 2\sqrt{2}, r - t = 2 + 2\sqrt{2}, r - u = 2, r^2 = t^2 = 3 + 2\sqrt{2}, s^2 = u^2 = 3 - 2\sqrt{2}, r^3 - s^3 = 10\sqrt{2}, r^3 - t^3 = 14 + 10\sqrt{2}, r^3 - u^3 = 14$, and $r^3 + 11 = 18 + 5\sqrt{2}$.

These facts can be used to obtain a 3×3 linear system in B, C, and D:

$$\begin{aligned} \sqrt{2}B + (1 + \sqrt{2})C + D &= -\frac{2 + \sqrt{2}}{2} \\ B + D &= -\frac{4 + \sqrt{2}}{2\sqrt{2}} \\ 10\sqrt{2}B + (14 + 10\sqrt{2})C + 14D &= -18 - 5\sqrt{2}. \end{aligned} \tag{5.4}$$

This leads to the following linear system in B and C:

$$\begin{aligned} (\sqrt{2} - 1)B + (1 + \sqrt{2})C &= \frac{2 - \sqrt{2}}{2} \\ (5\sqrt{2} - 7)B + (7 + 5\sqrt{2})C &= \frac{18 - 11\sqrt{2}}{2\sqrt{2}}. \end{aligned}$$

Solving this linear system yields

$$B = -\frac{2-2\sqrt{2}}{4\sqrt{2}} = -\frac{\delta}{2\sqrt{2}} \quad \text{and} \quad C = \frac{6-4\sqrt{2}}{4\sqrt{2}} = \frac{\delta^2}{2\sqrt{2}}.$$

Using equation (5.4), we get $D = -\frac{6+4\sqrt{2}}{4\sqrt{2}} = -\frac{\gamma^2}{2\sqrt{2}}$. Using equation (5.3), we now can compute A: $A = \frac{2+2\sqrt{2}}{4\sqrt{2}} = \frac{\gamma}{2\sqrt{2}}$.

Substituting for A, B, C, and D, we get the desired general solution satisfying the initial conditions:

$$\begin{aligned} x_k &= \frac{\gamma}{2\sqrt{2}}\cdot\gamma^k - \frac{\delta}{2\sqrt{2}}\cdot\delta^k + \frac{\delta^2}{2\sqrt{2}}(-\gamma)^k - \frac{\gamma^2}{2\sqrt{2}}(-\delta)^k \\ &= \frac{\gamma^{k+1}-\delta^{k+1}}{2\sqrt{2}} + \frac{(-1)^k}{2\sqrt{2}}\left[3(\gamma^k-\delta^k) - 2\sqrt{2}(\gamma^k+\delta^k)\right] \\ &= P_{k+1} + (-1)^k(3P_k - 2Q_k), \text{where } k \geq 0. \end{aligned}$$

For example $x_2 = P_3 + (3P_2 - 2Q_2) = 5 + (3\cdot 2 - 2\cdot 3) = 5$, as expected. Similarly, $x_3 = 11$.

Computing these values by hand or even with the aid of a scientific calculator is extremely time-consuming, and a test of accuracy and patience. A computer algebra system such as *Mathematica*[8] can be extremely useful for such complex computations.

Using the power of matrices and *Mathematica*® to solve the above 4×4 linear system in A, B, C, and D, we can save hours of frustration.

5.9.2 The Matrix Method Using *Mathematica*®

The linear system can be written as a matrix equation:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ r & s & t & u \\ r^2 & s^2 & t^2 & u^2 \\ r^3 & s^3 & t^3 & u^3 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 5 \\ 11 \end{bmatrix}.$$

The coefficient matrix

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ r & s & t & u \\ r^2 & s^2 & t^2 & u^2 \\ r^3 & s^3 & t^3 & u^3 \end{bmatrix}$$

[8] *Mathematica*® is a registered trademark of Wolfram Research, Inc.

is the 4×4 *Vandermonde matrix*, named after the French mathematician *Charles Auguste Vandermonde* (1735–1796). Since $|M| = (r-s)(r-t)(r-u)(t-s)(t-u)(s-u)$ and no two characteristic roots are equal, it follows that $|M| \neq 0$, where $|M|$ denotes the determinant of the matrix M. Consequently, M is invertible. Using *Mathematica*®,

$$M^{-1} = \frac{1}{128}\begin{bmatrix} 32-24\sqrt{2} & -80+56\sqrt{2} & 8\sqrt{2} & 16-8\sqrt{2} \\ 32+24\sqrt{2} & -80-56\sqrt{2} & -8\sqrt{2} & 16+8\sqrt{2} \\ 32-24\sqrt{2} & 80-56\sqrt{2} & 8\sqrt{2} & -16+8\sqrt{2} \\ 32+24\sqrt{2} & 80+56\sqrt{2} & -8\sqrt{2} & -16-8\sqrt{2} \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \frac{1}{128}\begin{bmatrix} 32-24\sqrt{2} & -80+56\sqrt{2} & 8\sqrt{2} & 16-8\sqrt{2} \\ 32+24\sqrt{2} & -80-56\sqrt{2} & -8\sqrt{2} & 16+8\sqrt{2} \\ 32-24\sqrt{2} & 80-56\sqrt{2} & 8\sqrt{2} & -16+8\sqrt{2} \\ 32+24\sqrt{2} & 80+56\sqrt{2} & -8\sqrt{2} & -16-8\sqrt{2} \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 5 \\ 11 \end{bmatrix}$$

$$= \frac{1}{4\sqrt{2}}\begin{bmatrix} 2+2\sqrt{2} \\ -2+2\sqrt{2} \\ 6-4\sqrt{2} \\ -6-4\sqrt{2} \end{bmatrix}.$$

These are exactly the same values we got earlier for A, B, C, and D by the traditional method.

We can develop a generating function $g(t)$ for the sequence $\{x_k\}$ using the recursive definition (5.2):

$$\frac{5t^3 + 11t^2 + t - 1}{t^4 - 6t^2 + 1} = -1 + t + 5t^2 + 11t^3 + 31t^4 + \cdots + x_k t^k + \cdots .$$

5.9.3 Example 5.3 Revisited

The Pell equation $x^2 - 2y^2 = 9$ has an added byproduct. To see this, suppose we would like to find three consecutive triangular numbers whose product is a square; that is, find a positive integer n such that $t_{n-1}t_nt_{n+1}$ is a square. Since $\frac{1}{2}(n-1)n \cdot \frac{1}{2}n(n+1) \cdot \frac{1}{2}(n+1)(n+2) = \frac{n^2(n+1)^2}{4} \cdot \frac{(n-1)(n+2)}{2}$, this implies that $\frac{(n-1)(n+2)}{2} = m^2$ for some positive integer m. This yields the equation $x^2 - 2y^2 = 9$, where $x = 2n+1$ and $y = 2m$.

Since $y \neq 0$, the least positive solution is $(x_1, y_1) = (9, 6)$; then $n = 4$. Correspondingly, $t_3t_4t_5 = 6 \cdot 10 \cdot 15 = 30^2$. With $(x_2, y_2) = (51, 36)$, we get $n = 25$; then $t_{24}t_{25}t_{26} = 300 \cdot 325 \cdot 351 = 5850^2$. Since the Pell's equation has infinitely many solutions, it follows that there is an infinitude of triples of consecutive triangular numbers such that their product is a square.

We close this chapter with an unsolved problem involving triangular numbers.

5.10 An Unsolved Problem

In 1949, the Polish mathematician K. Zarankiewicz (1902–1959) asked whether or not there are Pythagorean triangles whose sides are triangular numbers. Interestingly, there is at least one: ? $t_{132}^2 + t_{143}^2 = 8778^2 + 10296^2 = 183,060,900 = 13,530^2 = t_{164}^2$. It is *not* known if there are any other such Pythagorean triangles.

Exercises 5

1. Prove Diophantus' theorem.

Determine if each integer is a triangular number t_n? When it is, find n.

2. 2,025,078.
3. 1,983,037.

Evaluate each.

4. $\sum_{k=1}^{n} \frac{1}{t_k}$.
5. $\sum_{k=1}^{\infty} \frac{1}{t_k}$.

This problem was proposed by Huygens to the German mathematician Gottfried Wilhelm Leibniz (1646–1716) during the former's stay in Paris at the invitation of Louis XIV. This problem led to the development of *Leibniz's harmonic triangle* [137].

6. Prove that $\{t_n \pmod{10}\}$ is periodic with period 20.

7–18. Prove identities 1–12 in Table 5.1.

19. Using identity (8) in Table 5.1, develop a summation formula for $\sum_{i=1}^{n} i^3$.
20. Solve the recurrence $y_n = 6y_{n-1} - y_{n-2}$, where $y_0 = 0$, $y_1 = 6$, and $n \geq 2$.
21. Using the recursive definition of y_n in Exercise 19, find a generating function for $\{y_n\}$.
22. Develop a generating function for the sequence $\{x_n\}$, where $x_n = 6x_{n-2} - x_{n-4}$, $x_1 = 1$, $x_2 = 5$, $x_3 = 11$, $x_4 = 31$, and $n \geq 5$.

6

Square-Triangular Numbers

6.1 Introduction

In the preceding chapter we introduced triangular numbers and some of their properties, including some visual representations. This chapter focuses on a special class of triangular numbers, which has major ramifications.

To begin with, we ask if there are triangular numbers that are also squares. Certainly. We saw two such triangular numbers in the previous chapter: $t_1 = 1 = 1^2$ and $t_{1681} = 1,413,721 = 1189^2$. So there are at least two *square-triangular numbers*. Are there any others? If yes, how many are there? How do we find them? We will answer these questions now.

Finding square-triangular numbers by trial and error is not practical, so we will take advantage of Diophantus' theorem: $8t_n + 1 = (2n+1)^2$. Thus every triangular number t_n has the property that $8t_n + 1$ is a square. In addition, we want t_n to be a square m^2; then $8m^2 + 1$ also must be a square. So we must select integers n and m such that Pell's equation $(2n+1)^2 - 8m^2 = 1$ is satisfied.

6.2 Infinitude of Square-Triangular Numbers

Recall from Example 2.3 that *Pell's equation* $x^2 - 8y^2 = 1$ has infinitely many solutions $(x_n, y_n) = (Q_{2n}, \frac{1}{2}P_{2n})$, where $n \geq 1$. Consequently, there are infinitely many square-triangular numbers y_k^2, as Euler discovered in 1730.

Table 6.1 gives fifteen solutions (x_k, y_k) of the equation, the first fifteen square-triangular numbers y_k^2, and their corresponding subscripts n_k.

T. Koshy, *Pell and Pell–Lucas Numbers with Applications*, DOI 10.1007/978-1-4614-8489-9_6,

Table 6.1.

k	x_k	y_k	Square-Triangular Number y_k^2	Corresponding Subscript n_k in t_{n_k}
1	3	1	1	1
2	17	6	36	8
3	99	35	1225	49
4	577	204	41616	288
5	3363	1189	1413721	1681
6	19601	6930	48024900	9800
7	114243	40391	1631432881	57121
8	665857	235416	55420693056	332928
9	3880899	1372105	1882672131025	1940449
10	22619537	7997214	63955431761796	11309768
11	131836323	46611179	2172602007770041	65918161
12	768398401	271669860	73804512832419600	384199200
13	4478554083	1583407981	2507180834294496361	2239277041
14	26102926097	9228778026	85170343853180456676	13051463048
15	152139002499	53789260175	2893284510173841030625	76069501249

The formula for the square-triangular number y_k^2 can be written in different ways:

$$\begin{aligned} y_k^2 &= \frac{1}{4}P_{2k}^2 = \frac{1}{32}\left[(1+\sqrt{2})^{4k} + (1-\sqrt{2})^{4k} - 2\right] \\ &= \frac{1}{32}\left[(3+2\sqrt{2})^{2k} + (3-2\sqrt{2})^{2k} - 2\right] = \frac{1}{16}(Q_{4k} - 1) \\ &= \frac{1}{32}\left[(17+12\sqrt{2})^{k} + (17-12\sqrt{2})^{k} - 2\right]. \end{aligned}$$

Knowing a square-triangular number y_k^2, how do we know which triangular number t_{n_k} it is? To answer this, again we return to *Diophantus' theorem*. If y_k^2 is the triangular number t_{n_k}, then we must have $8y_k^2 + 1 = (2n_k + 1)^2$; so $n_k = \frac{\sqrt{8y_k^2+1}-1}{2}$. Thus $y_k^2 = t_{\frac{\sqrt{8y_k^2+1}-1}{2}}$.

For example, consider the square-triangular number $y_3^2 = 1225$. Then $n_3 = \frac{\sqrt{8\cdot 1225+1}-1}{2} = 49$. So $1225 = t_{49}$, the 49th triangular number; see Table 6.1.

We now make some interesting observations about Table 6.1:

(1) The sequence $\{y_k\} = 1, 6, 35, 204, 1189, \ldots$ contains two hidden treasures. To see this, we can factor each y_k as $y_k = P_k Q_k$:

$$\begin{array}{rcl} 1 &=& 1\cdot 1 \\ 6 &=& 2\cdot 3 \\ 35 &=& 5\cdot 7 \\ 204 &=& 12\cdot 17 \\ 1189 &=& 29\cdot 41 \\ & \vdots & \uparrow \quad \uparrow \\ & & P_k \quad Q_k \end{array}$$

We encountered the sequences $\{P_n\}$ and $\{Q_n\}$ in a variety of contexts in Chapter 2–5 and will explore them at length in Chapters 7–15, where we will show that $(P_n, Q_n) = 1$.

(2) Recall from Example 1.5 that the sequence $\{y_k\}$ is defined recursively as follows:

$$\begin{aligned} y_1 &= 1, \quad y_2 = 6 \\ y_k &= 6y_{k-1} - y_{k-2}, \quad k \geq 3. \end{aligned}$$

It follows by Example 1.8 that $y_k = \dfrac{r^k - s^k}{4\sqrt{2}}$, where $r = 3 + 2\sqrt{2}$ and $s = 3 - 2\sqrt{2}$. But $r = \gamma^2$ and $s = \delta^2$; so

$$y_k = \frac{\gamma^{2k} - \delta^{2k}}{4\sqrt{2}} = \frac{1}{2}P_{2k}. \tag{6.1}$$

For example, $y_3 = \frac{1}{2}P_6 = \frac{1}{2}(70) = 35$, as expected; see Table 6.1.

(3) The ratios $\frac{y_{k+1}}{y_k}$ follow an interesting pattern; see Table 6.2. It appears that the ratios approach a finite limit as k approaches ∞. In fact, $\lim_{k\to\infty} \frac{y_{k+1}}{y_k} = \gamma^2$; see Exercise 3.

Table 6.2.

k	y_k	$\frac{y_{k+1}}{y_k}$
1	1	6.00000000000
2	6	5.83333333333
3	35	5.82857142857
4	204	5.82843137255
5	1189	5.82842724979
6	6930	5.82842712843
7	40391	5.82842712485
8	235416	5.82842712475
9	1372105	5.82842712475
10	7997214	5.82842712475

(4) The sequence $\{x_k\}$ in Column 1 follows exactly the same recursive pattern, but with different initial conditions:

$$\begin{aligned} x_1 &= 3, \quad x_2 = 17 \\ x_k &= 6x_{k-1} - x_{k-2}, \quad k \geq 3. \end{aligned}$$

Solving this LHRWCCs, we get

$$x_k = \frac{1}{2}\left[(3 + 2\sqrt{2})^k + (3 - 2\sqrt{2})^k\right] = Q_{2k}. \tag{6.2}$$

For example, $x_3 = Q_6 = 99$, as expected.

(5) Consider the sequence of subscripts of the square-triangular numbers in Column 5: $1, 8, 49, 288, 1681, \ldots$. This is the same sequence $\{n_k\}$ we studied in Example 1.6. It is defined by

$$\begin{aligned} n_1 &= 1, \quad n_2 = 8 \\ n_k &= 6n_{k-1} - n_{k-2} + 2, \quad k \geq 3. \end{aligned}$$

It follows by Example 1.9 that $n_k = \dfrac{r^k + s^k - 2}{4}$, where $r = 3 + 2\sqrt{2}$ and $s = 3 - 2\sqrt{2}$. Since $r = \gamma^2$ and $s = \delta^2$,

$$n_k = \frac{1}{4}\left(\gamma^{2k} + \delta^{2k} - 2\right) = \frac{1}{2}(Q_{2k} - 1). \tag{6.3}$$

For example, $n_4 = \frac{1}{2}(Q_8 - 1) = \frac{1}{2}(577 - 1) = 288$, as expected. That is, $t_{288} = 1,413,712$; see Table 6.1.

In 1972, D.C.D. Potter of Hillcroft School, London, England, found an interesting relationship between two successive members of the sequence $\{n_k\}$: $(n_k + n_{k-1} - 1)^2 = 8n_kn_{k-1}$, where $k \geq 2$ [177]. This can be confirmed using PMI.

For example, $(n_5 + n_4 - 1)^2 = (1681 + 288 - 1)^2 = 3,873,024 = 8 \cdot 1681 \cdot 288 = 8n_5n_4$. Using this property and PMI, Potter also showed that $n_k^2 + n_k = 2y_k^2$ from the recursive definitions of the sequences $\{n_k\}$ and $\{y_k\}$, a fact we already knew.

(6) The ratios $\frac{n_{k+1}}{n_k}$ also exhibit an interesting pattern: They too seem to approach a finite limit as k approaches ∞; see Table 6.3. As before, $\lim\limits_{k\to\infty} \frac{n_{k+1}}{n_k} = \gamma^2$; see Exercise 4.

Table 6.3.

k	y_k	$\frac{n_{k+1}}{n_k}$
1	1	8.00000000000
2	8	6.12500000000
3	49	5.87755102041
4	288	5.83680555556
5	1681	5.82986317668
6	9800	5.82867346939
7	57121	5.82846938954
8	332928	5.82843437620
9	1940449	5.82842836890
10	11309768	5.82842733821

(7) The ratios $\frac{n_k}{y_k}$ also display an interesting pattern: They approach $\sqrt{2} \approx 1.4142135623\cdots$; see Table 6.4 and Exercise 5.

Table 6.4.

k	n_k	y_k	$\frac{n_k}{y_k}$
1	1	1	1.0000000000
2	8	6	1.3333333333
3	49	35	1.4000000000
4	288	204	1.4117647058
5	1681	1189	1.4137931035
6	9800	6930	1.4142011834
7	57121	40391	1.4142011834
8	332928	235416	1.4142114385
9	1940449	1372105	1.4142131980
10	11309768	7997214	1.4142134999

6.2.1 An Alternate Method

Since $1 + 8y_k^2 = (2n_k + 1)^2$, it follows that

$$\begin{aligned} 2n_k + 1 &= \sqrt{1 + 8y_k^2} \\ \lim_{k\to\infty}\left(2\cdot\frac{n_k}{y_k} + \frac{1}{y_k}\right) &= \lim_{k\to\infty}\sqrt{\frac{1}{y_k^2} + 8} \\ 2\lim_{k\to\infty}\frac{n_k}{y_k} + 0 &= \sqrt{0+8} \\ \lim_{k\to\infty}\frac{n_k}{y_k} &= \sqrt{2}. \end{aligned}$$

(8) Now consider the square-triangular numbers $c_k = y_k^2$: $1, 36, 1225, 41616, 1413721, \ldots$. They too can be defined recursively:

$$\begin{aligned} c_1 &= 1, \quad c_2 = 36 \\ c_k &= 34c_{k-1} - c_{k-2} + 2, \quad k \geq 3. \end{aligned} \tag{6.4}$$

For example, $c_5 = 34 \cdot 41616 - 1225 + 2 = 1,413,721$, as expected.

We can solve this nonhomogeneous recurrence using the same technique as in (5), by splitting it into homogeneous and nonhomogeneous parts. It can be shown that

$$c_k = \frac{1}{32}\left(\gamma^{4k} + \delta^{4k} - 2\right) = \frac{1}{16}(Q_{4k} - 1) = \frac{1}{4}P_{2k}^2. \tag{6.5}$$

(We omit the details in the interest of brevity.)

As an example, $c_3 = \frac{1}{4}P_6^2 = \frac{1}{4}(70) = 1225$, as expected.

Since k is arbitrary, formula (6.5) also implies that there are infinitely many square-triangular numbers.

(9) The ratios of consecutive square-triangular numbers $\frac{c_{k+1}}{c_k}$ also manifest an interesting pattern: They approach $\gamma^4 = 17 + 12\sqrt{2} \approx 33.9705627485$; see Table 6.5 and Exercise 6. The fact that $\frac{c_{k+1}}{c_k} \approx 34$ follows easily from the recurrence (6.4).

Table 6.5.

k	c_k	$\frac{c_{k+1}}{c_k}$
1	1	36.0000000000
2	8	34.0277777778
3	49	33.9722448980
4	288	33.9706122645
5	1681	33.9705642061
6	9800	33.9705627914
7	57121	33.9705627497

(10) Finally, notice that the sequence $\{x_k\}$ is a subsequence of the sequence we encountered in Example 2.1, while solving Pell's equation $x^2 - 2y^2 = 1$.

Next we extract the ends of the numbers x_k, y_k, y_k^2, and n_k.

6.2.2 The Ends of x_k, y_k, y_k^2, and n_k

Returning to Table 6.1, we find that it contains four additional treasures:

(1) The sequence $\{x_k \pmod{10}\}$ shows an interesting periodic pattern with period 6:

$$\underbrace{3\ 7\ 9\ 7\ 3\ 1}\ \underbrace{3\ 7\ 9\ 7\ 3\ 1}\cdots$$

That is,

$$x_k \equiv \begin{cases} 3 \pmod{10} & \text{if } k \equiv 1 \pmod 6 \\ 7 \pmod{10} & \text{if } k \equiv 2 \pmod 6 \\ 9 \pmod{10} & \text{if } k \equiv 3 \pmod 6 \\ 7 \pmod{10} & \text{if } k \equiv 4 \pmod 6 \\ 3 \pmod{10} & \text{if } k \equiv 5 \pmod 6 \\ 1 \pmod{10} & \text{otherwise.} \end{cases}$$

(2) $$y_k = P_k Q_k \equiv \begin{cases} 1 \pmod{10} & \text{if } k \equiv 1 \pmod 6 \\ 6 \pmod{10} & \text{if } k \equiv 2 \pmod 6 \\ 5 \pmod{10} & \text{if } k \equiv 3 \pmod 6 \\ 4 \pmod{10} & \text{if } k \equiv 4 \pmod 6 \\ 9 \pmod{10} & \text{if } k \equiv 5 \pmod 6 \\ 0 \pmod{10} & \text{otherwise.} \end{cases}$$

$$
(3) \quad y_k^2 = P_k^2 Q_k^2 \equiv \begin{cases} 1 \pmod{10} & \text{if } k \equiv 1 \pmod 6 \\ 6 \pmod{10} & \text{if } k \equiv 2 \pmod 6 \\ 5 \pmod{10} & \text{if } k \equiv 3 \pmod 6 \\ 6 \pmod{10} & \text{if } k \equiv 4 \pmod 6 \\ 1 \pmod{10} & \text{if } k \equiv 5 \pmod 6 \\ 0 \pmod{10} & \text{otherwise.} \end{cases}
$$

$$
(4) \quad n_k \equiv \begin{cases} 1 \pmod{10} & \text{if } k \equiv 1 \pmod 6 \\ 8 \pmod{10} & \text{if } k \equiv 2 \pmod 6 \\ 9 \pmod{10} & \text{if } k \equiv 3 \pmod 6 \\ 8 \pmod{10} & \text{if } k \equiv 4 \pmod 6 \\ 1 \pmod{10} & \text{if } k \equiv 5 \pmod 6 \\ 0 \pmod{10} & \text{otherwise.} \end{cases}
$$

These properties can be confirmed using induction and the corresponding recursive definitions.

In 1999, D. Sengupta of Elizabeth City State University of New Jersey computed the first forty-nine square-triangular numbers [229]. One of them is the 26-digit number $t_{2584123765441}$, which does not contain a single zero: $t_{2584123765441} = 3,338,847,817,559,778,254,844,961$. On the other hand, the 33-digit square-triangular number $t_{17380816062160328} = 151,046,383,493,325,234,090,009,219,613,956$ contains five 0s, but no 7s.

Next we pursue another recurrence for y_k, developed by T. Cross of Wolverley High School, Wolverley, England, in 1991 [55]. It yields two interesting dividends.

6.2.3 Cross' Recurrence for y_k

Let $d_k = n_k - y_k$. Table 6.6 shows the first ten values of the sequences $\{n_k\}$, $\{y_k\}$, and $\{d_k\}$.

Table 6.6.

k	n_k	y_k	d_k
1	1	1	0
2	8	6	2
3	49	35	14
4	288	204	84
5	1681	1189	492
6	9800	6930	2870
7	57121	40391	16730
8	332928	235416	97512
9	1940449	1372105	568344
10	11309768	7997214	3312554

It follows from observations (1) and (4) above that

$$\begin{aligned} 2(n_{k+1} - n_k) &= (1 + \sqrt{2})^{2k+1} + (1 - \sqrt{2})^{2k+1} \\ &= 2(y_{k+1} + y_k). \end{aligned}$$

So $n_{k+1} - n_k = y_{k+1} + y_k$. Thus $n_{k+1} - y_{k+1} = n_k + y_k$; that is, $d_{k+1} = n_k + y_k$.

For example, $n_4 + y_4 = 288 + 204 = 492 = d_5$.

Since $t_{n_k} = y_k^2$, $1 + 8y_k^2 = (2n_k + 1)^2$; so $n_k = \frac{1}{2}\left(\sqrt{1 + 8y_k^2} - 1\right)$. Consequently, we have

$$\begin{aligned} d_{k+1} &= \frac{1}{2}\left(\sqrt{1 + 8y_k^2} - 1\right) + y_k \\ &= \frac{1}{2}\left[(2y_k - 1) + \sqrt{1 + 8y_k^2}\,\right]. \end{aligned} \tag{6.6}$$

For example,

$$\begin{aligned} \frac{1}{2}\left[(2y_5 - 1) + \sqrt{1 + 8y_5^2}\,\right] &= \frac{1}{2}\left(2 \cdot 1189 - 1 + \sqrt{1 + 8 \cdot 1189^2}\right) \\ &= 2870 = d_6. \end{aligned}$$

Since $t_{n_r} = y_r^2$, we have $\frac{1}{2}n_{k+1}(n_{k+1} + 1) = (n_{k+1} - d_{k+1})^2$. This yields the quadratic equation $n_{k+1}^2 - (1 + 4d_{k+1})n_{k+1} + 2d_{k+1}^2 = 0$. Solving this, we get

$$\begin{aligned} n_{k+1} &= \frac{1}{2}\left[1 + 4d_{k+1} \pm \sqrt{(1 + 4d_{k+1})^2 - 8d_{k+1}^2}\,\right] \\ &= \frac{1}{2}\left[1 + 4d_{k+1} \pm \sqrt{1 + 8d_{k+1} + 8d_{k+1}^2}\,\right]. \end{aligned}$$

Since $n_{k+1} > 0$, we choose the positive root:

$$n_{k+1} = \frac{1}{2}\left[1 + 4d_{k+1} + \sqrt{1 + 8d_{k+1} + 8d_{k+1}^2}\,\right]. \tag{6.7}$$

Substituting for d_{k+1} and n_{k+1} from equations (6.6) and (6.7) in the equation $y_{k+1} = n_{k+1} - d_{k+1}$, and after a lot of algebra, we get a recurrence relation for y_{k+1}:

$$y_{k+1} = y_k + \frac{1}{2}\sqrt{1 + 8y_k^2} + \frac{1}{2}\sqrt{1 + 24y_k^2 + 8y_k\sqrt{1 + 8y_k^2}}. \tag{6.8}$$

For example,

$$\begin{aligned} y_4 &= 35 + \frac{1}{2}\sqrt{1+8\cdot 35^2} + \frac{1}{2}\sqrt{1+24\cdot 35^2 + 8\cdot 35\sqrt{1+8\cdot 35^2}} \\ &= 35 + \frac{1}{2}\cdot 99 + \frac{1}{2}\cdot 239 = 204, \quad \text{as expected.} \end{aligned}$$

Formula (6.7) yields an interesting dividend. To see this, notice that $\lim_{k\to\infty} \frac{1}{y_k} = 0$. So

$$\begin{aligned} \lim_{k\to\infty} \frac{y_{k+1}}{y_k} &= \lim_{k\to\infty}\left(1 + \frac{1}{2}\sqrt{8+\frac{1}{y_k^2}} + \frac{1}{2}\sqrt{\frac{1}{y_k^2} + 24 + 8\sqrt{\frac{1}{y_k^2}+8}}\right) \\ &= 1 + \frac{1}{2}\sqrt{8+0} + \frac{1}{2}\sqrt{0+24+8\sqrt{0+8}} \\ &= 1+\sqrt{2}+\sqrt{2}(1+\sqrt{2}) \\ &= 3+2\sqrt{2} = \gamma^2. \end{aligned}$$

We also note the remarkable fact that the sequence $\left\{\frac{y_{k+1}}{y_k}\right\}$ converges to the limit $3+2\sqrt{2}$ very fast. For example, $\frac{y_8}{y_7} \approx 5.82842712485$, and $\frac{y_9}{y_8} \approx 5.82842712475$.

Consequently, y_{k+1} can also be defined by a much simpler recurrence:

$$\begin{aligned} y_1 &= 1 \\ y_k &= \lceil 3+2\sqrt{2}y_{k-1}\rceil, \quad k \ge 2. \end{aligned}$$

For example, $y_5 = \lceil 3 + 2\sqrt{2}\cdot 204\rceil = 1189$.

Since $1+8y_k^2 = (2n_k+1)^2$, formula (6.8) can be rewritten as follows:

$$y_{k+1} = y_k + \frac{1}{2}(2n_k+1) + \frac{1}{2}\sqrt{1+24y_k^2+8y_k(2n_k+1)}. \tag{6.9}$$

For example,

$$\begin{aligned} y_3 &= y_2 + \frac{1}{2}(2n_2+1) + \frac{1}{2}\sqrt{1+24y_2^2+8y_2(2n_2+1)} \\ &= 6 + \frac{1}{2}(2\cdot 8+1) + \frac{1}{2}\sqrt{1+24\cdot 36+8(2\cdot 8+1)\cdot 6} \\ &= 6 + \frac{17}{2} + \frac{41}{2} = 35. \end{aligned}$$

6.3 The Infinitude of Square-Triangular Numbers Revisited

We can establish the infinitude of square-triangular numbers without resorting to Pell's equation. First, let t_n be a square-triangular number. Then $4t_n$ is a square, and so is $8t_n + 1$ by Diophantus' theorem. Consequently, $4t_n(8t_n + 1)$ is also a square. But $4t_n(8t_n + 1) = \frac{(8t_n)(8t_n+1)}{2} = t_{8t_n}$ is also a triangular number. Thus, if t_n is a square, then so is t_{8t_n}. Since t_1 is a square-triangular number, it follows that there are infinitely many square-triangular numbers. (Notice that $t_{8t_n} = 4t_n(2n+1)^2$, by Diophantus' theorem.)

This short and elegant proof [245] was given in 1962 by A.V. Sylvester of the U.S. Naval Ordinance Laboratory, Corona, California, as a proof for a problem proposed in 1961 by J.L. Pietenpol of Columbia University [175].

In 1961, the Polish mathematician W. Sierpiński (1882–1969) proved that if $t_x = y^2$ is a square-triangular number, then $t_{3x+4y+1} = (2x + 3y + 1)^2$ is the next square-triangular number. Again, since t_1 is a square-triangular number, this formula also establishes the infinitude of square-triangular numbers. For example, $t_8 = 6^2$ is a square-triangular number; the next one is $t_{3\cdot 8+4\cdot 6+1} = t_{49} = 35^2$.

In 1962, E. Just of Bronx Community College, Bronx, New York, also gave a short and neat proof [119]: Recall that Pell's equation $x^2 - 2y^2 = 1$ has infinitely many solutions; that is, the equation $\frac{x^2-1}{2} = y^2$ has infinitely many solutions. In other words, there are infinitely many squares of the form $\frac{1}{2}(x^2 - 1)$. So there exist infinitely many squares of the form $\frac{(x^2-1)x^2}{2}$; that is, there are infinitely many square-triangular numbers t_{x^2-1}.

For example, recall again that (17, 12) is a solution of the Pell equation $x^2 - 2y^2 = 1$. So $t_{17^2-1} = t_{288} = \frac{288\cdot 289}{2} = 41616 = 204^2$ is a square-triangular number, as we already know.

6.4 A Recursive Definition of Square-Triangular Numbers

Sylvester's proof provides an elegant algorithm for computing square-triangular numbers $t_n^{(s)}$ recursively:

$$\begin{aligned} t_1^{(s)} &= 1 \\ t_n^{(s)} &= 4t_{n-1}^{(s)}\left(8t_{n-1}^{(s)} + 1\right), \quad n \geq 2. \end{aligned}$$

The first four such triangular numbers are

$$\begin{aligned} t_1^{(s)} &= 1 &&= 1^2 &&= t_1^2 \\ t_2^{(s)} &= 4\cdot 1(8\cdot 1 + 1) = 36 &&= 6^2 &&= t_{8\cdot 1}^2 \\ t_3^{(s)} &= 4\cdot 36(8\cdot 36 + 1) = 41{,}616 &&= 204^2 &&= t_{8\cdot 36}^2 \\ t_4^{(s)} &= 4\cdot 41{,}616(8\cdot 41{,}616 + 1) = 55{,}420{,}693{,}056 &&= 235{,}416^2 &&= t_{8\cdot 41{,}616}^2. \end{aligned}$$

In 1942, W. Ljunggren established that there are exactly two triangular numbers whose squares are also triangular numbers. They are t_1 and t_6: $t_1^2 = 1 = t_1$ and $t_6^2 = 36 = t_8$. He also showed that *no* triangular number is the fourth power of an integer [233].

6.5 Warten's Characterization of Square-Triangular Numbers

Next we present a different characterization of square-triangular numbers, developed by R.M. Warten in 1958, when he was an undergraduate at Brooklyn College, New York. The gist of his technique lies in the algorithm developed by the *Pythagoreans* for solving the Pell equations $2u^2 - v^2 = \pm 1$.

Before we can present it, we need two simple lemmas.

Lemma 6.1 *A positive integer d is square-triangular if and only if there are positive integers u and v such that $d = u^2v^2$ and $2u^2 - v^2 = \pm 1$.*

Proof. Suppose $d = u^2v^2$.

Case 1 Let $2u^2 - v^2 = 1$. Then $d = \frac{v^2(v^2+1)}{2}$ is clearly square-triangular.

Case 2 Let $2u^2 - v^2 = -1$. Then $d = \frac{(v^2-1)v^2}{2}$ is also square-triangular.

Thus, in both cases, d is square-triangular.

Conversely, suppose $d = \frac{n(n+1)}{2} = y^2$ is square-triangular.

Case 1 Suppose n is even. Then $\frac{n}{2}$ and $n+1$ are relatively prime; $\left(\frac{n}{2}, n+1\right) = 1$. Since d is a square, suppose $d = (q_1q_2\cdots q_k)^2$, where each q_i is a distinct prime-power $p_i^{e_i}$. Without loss of generality, we can assume that $\frac{n}{2} = (q_1q_2\cdots q_r)^2$ and $n+1 = (q_{r+1}q_{r+2}\cdots q_k)^2$. Let $u^2 = \frac{n}{2}$ and $v^2 = n+1$. Then $d = u^2v^2$ and $2u^2 - v^2 = n - (n+1) = -1$, as desired.

Case 2 Suppose n is odd. A similar argument will show that $d = u^2v^2$ and $2u^2 - v^2 = 1$.

Thus, d is square-triangular if and only if $d = u^2v^2$ and $2u^2 - v^2 = \pm 1$. ∎

For example, recall that $d = 36 = t_8$ is square-triangular: $d = 2^23^2$, where $2\cdot 2^2 - 3^2 = -1$. Likewise, $d = 1225 = t_{49}$ is square-triangular: $d = 5^27^2$, where $2\cdot 5^2 - 7^2 = 1$.

Lemma 6.2 *Let u and v be integers > 1 such that $2u^2 - v^2 = \pm 1$. Then $2u > v > u$.*

Proof (by contradiction). Assume $2u^2 - v^2 = \pm 1$.

(1) Suppose $u \geq v$. Then $u^2 \geq uv \geq v^2 > 1$. Therefore, $2u^2 \geq u^2 + v^2 > v^2 + 1$, so $2u^2 - v^2 > 1$. This is a contradiction; so $v > u$.

(2) Suppose $v \geq 2u$. Since $v > u$, $v \geq u+1$. Thus $v \geq 2u$ and $v \geq u+1$. Consequently, we have

$$\begin{aligned} v^2 &\geq 2u(u+1) \\ &= 2u^2 + 2u \\ &> 2u^2 + 1, \quad \text{since } u > 1. \end{aligned}$$

So $2u^2 - v^2 < -1$, again a contradiction. So $2u > v$.

Thus $2u > v > u$. ∎

We are now ready to present Warten's characterization of square-triangular numbers [253].

Theorem 6.1 *Let $\{N_k\}$ be a sequence of positive integers, defined recursively as follows:*

(1) $N_1 = u_1^2 v_1^2$, *where* $u_1 = 1 = v_1$.
(2) $N_k = u_k^2 v_k^2$, *where* $u_k = u_{k-1} + v_{k-1}, v_k = 2u_{k-1} + v_{k-1}$, *and* $k \geq 2$.

Then the sequence $\{N_k\}$ consists of all square-triangular numbers.

Proof. The proof consists of two parts. First, we will show that every N_k is a square-triangular number. We will then show that every square-triangular number belongs to the sequence $\{N_k\}$. (The proof of the second half is a bit long, and requires some patience to follow the logic.)

(1) We will establish this part using PMI. To this end, first notice that N_1 is square-triangular.
Now assume that N_k is a square-triangular number for an arbitrary integer $k \geq 1$. Then, by Lemma 6.1, there exist positive integers u_k and v_k such that $N_k = u_k^2 v_k^2$, where $2u_k^2 - v_k^2 = \pm 1$. Then

$$\begin{aligned} N_{k+1} &= u_{k+1}^2 v_{k+1}^2 \\ &= (u_k^2 + v_k^2)(2u_k + v_k)^2 \end{aligned}$$

and

$$\begin{aligned} 2(u_k + v_k)^2 - (2u_k + v_k)^2 &= 2(2u_k^2 + v_k^2 + 4u_k v_k) - (4u_k^2 + v_k^2 + 4u_k v_k) \\ &= -(2u_k^2 - v_k^2) \\ &= \mp 1. \end{aligned}$$

Consequently, by Lemma 6.1, N_{k+1} is also a square-triangular number. Thus, by PMI, N_k is a square-triangular for every integer $k \geq 1$.

(2) We will now prove the second-half using contradiction. To this end, suppose there is a square-triangular number S_1 that is *not* in the sequence $\{N_k\}$. Then, by Lemma 6.1, there are integers $a_1, b_1 > 1$ such that $S_1 = a_1^2 b_1^2$ and $2a_1^2 - b_1^2 = \pm 1$. By Lemma 6.2, $2a_1 > b_1 > a_1$.
Let $S_2 = a_2^2 b_2^2$, where $a_2 = b_1 - a_1$ and $b_2 = 2a_1 - b_1$. Then $2a_2^2 - b_2^2 = 2(b_1 - a_1)^2 - (2a_1 - b_1)^2 = -(2a_1^2 - b_1^2) = \mp 1$. Since $b_1 > a_1, a_2 \geq 1$; since $2a_1 > b_1$, it follows that $b_2 \geq 1$. Thus, by Lemma 6.1, S_2 is a square-triangular number. In addition, since $2a_1 > b_1$, $a_1 > b_1 - a_1$; so $a_1 > a_2$. Since $2b_1 > 2a_1, b_1 > 2a_1 - b_1$; that is, $b_1 > b_2$. Thus $a_1 > a_2$ and $b_1 > b_2$. Consequently, $S_2 = a_2^2 b_2^2$ is square-triangular and $S_1 > S_2 > 0$.
Continuing like this, we can generate a sequence of square-triangular numbers $\{S_i\}$ such that $S_1 > S_2 > \cdots > 0$.
Earlier we assumed that S_1 does *not* belong to the sequence $\{N_k\}$. Suppose S_2 belongs to the sequence $\{N_k\}$. We have $a_2 = b_1 - a_1$ and $b_2 = 2a_1 - b_1$. So $a_2 + b_2 = (b_1 - a_1) + 2a_1 - b_1 = a_1$ and $2a_2 + b_2 = 2(b_1 - a_1) + (2a_1 - b_1) = b_1$. Consequently, S_1 exists in

the sequence $\{N_k\}$, which is a contradiction. So S_2 does not belong to the sequence $\{N_k\}$. It now follows that no S_i belongs to the sequence $\{N_k\}$.

Let S_ℓ be the least positive term in the sequence $\{S_i\}$. Suppose $S_\ell \neq 1$. Then, by Lemma 6.1, there are positive integers a_ℓ and b_ℓ such that $S_\ell = a_\ell^2 b_\ell^2$, where $2a_\ell^2 - b_\ell^2 = \pm 1$. We now claim that both a_ℓ and b_ℓ must be greater than 1.

To see this, suppose $a_\ell = 1$. Then $2 - b_\ell^2 = \pm 1$, so $b_\ell^2 = 2 \mp 1$. Then either $b_\ell = \sqrt{3}$ or $b_\ell = 1$. Both cases are clearly unacceptable.

On the other hand, suppose $b_\ell = 1$. Then $2a_\ell^2 = 1 \pm 1$, so $a_\ell = 1$ or $a_\ell = 0$. Again, both these cases are unacceptable.

Thus $a_\ell, b_\ell > 1$. So they can be used to construct a new square-triangular number $S_{\ell+1}$ such that $S_\ell > S_{\ell+1} > 0$. But this violates the hypothesis that S_ℓ is the least term of the sequence $\{S_i\}$. So $S_\ell = 1$ and S_1 belongs to the sequence $\{N_k\}$.

Thus the sequence $\{N_k\}$ consists of all square-triangular numbers, as desired. ∎

Although there are infinitely many square-triangular numbers, there is only one triangular number that is also a cube, namely, 1. Euler proved this in 1738 [70].

We close this chapter with a short discussion of the *generating functions for square-triangular numbers* y_k^2 and their subscripts n_k, and $\{y_k\}$.

6.6 A Generating Function For Square-Triangular Numbers

The recursive definition (6.4) can be used to develop a generating function for square-triangular numbers $c_k = y_k^2$. Notice that recurrence (6.4) is a NHRWCCs. But we can modify it slightly to make it homogeneous:

$$c_k = 35c_{k-1} - 35c_{k-2} + c_{k-3}. \tag{6.10}$$

Thus the recursive definition (6.4) can be rewritten with a LHRWCCs:

$$\begin{aligned} c_0 &= 0, \quad c_1 = 1, \quad c_2 = 36 \\ c_k &= 35c_{k-1} - 35c_{k-2} + c_{k-3}. \end{aligned} \tag{6.11}$$

For example, $c_3 = 35c_2 - 35c_1 + c_0 = 35\cdot 36 - 35\cdot 1 + 0 = 1,225$ and $c_4 = 35c_3 - 35c_2 + c_1 = 35 \cdot 1225 - 35 \cdot 36 + 1 = 41,616$.

With these tools, we can develop a generating function for square-triangular numbers:

$$\frac{x(1+x)}{(1-x)(1-34x+x^2)} = x + 36x^2 + 1225x^3 + 41616x^4 + \cdots .$$

This was originally developed in 1992 by S. Plouffe of the University of Quebec, Canada. ∎

Next we find a generating function for the subscripts n_k of square-triangular numbers y_k^2.

6.6.1 A Generating Function For $\{n_k\}$

It follows by Example 1.11 that the generating function of the sequence $\{n_k\}$ is

$$\frac{x(1+x)}{(1-x)(1-6x+x^2)} = x + 8x^2 + 49x^3 + 288x^4 + \cdots .$$

Next we develop a generating function for the sequence $\{y_k\}$: $1, 6, 35, 204, 1189, \ldots$.

6.6.2 A Generating Function For $\{y_k\}$

Recall that $y_n = 6y_{n-1} - y_{n-2}$, where $y_1 = 1$ and $y_2 = 6$. Notice that $y_0 = 0$. It follows by Example 1.10 that the desired generating function is

$$\frac{x}{1-6x+x^2} = 1 + 6x + 35x^2 + 204x^3 + 1189x^4 + \cdots .$$

Exercises 6

Confirm that each is a square-triangular number.

1. 1,413,721
2. 48,024,900

Confirm each.

3. $\lim_{k\to\infty} \frac{y_{k+1}}{y_k} = \gamma^2$.
4. $\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = \gamma^2$.
5. $\lim_{k\to\infty} \frac{n_k}{y_k} = \sqrt{2}$.
6. $\lim_{k\to\infty} \frac{c_{k+1}}{c_k} = \gamma^4$.
7. Rewrite the NHRWCCs $c_k = 34c_{k-1} - c_{k-2} + 2$ as a LHRWCCs.
8. Solve the recurrence $x_n = 35x_{n-1} - 35x_{n-2} + x_{n-3}$, where $x_0 = 0, x_1 = 1$, and $x_2 = 36$.
9. Develop a generating function for the sequence $\{x_n\}$ in Exercise 8.
10. Define the sequence $\{d_n\}$ recursively. *Hint*: Use Table 6.6.
11. Re-define the sequence $\{d_n\}$ using a LHRWCCs.
12. Develop a generating function for the sequence $\{d_n\}$. *Hint*: Use Exercise 11.

7

Pell and Pell–Lucas Numbers

7.1 Introduction

Like Fibonacci and Lucas numbers, the Pell family is ubiquitous. Pell and Pell–Lucas numbers also provide boundless opportunities to experiment, explore, and conjecture; they are a lot of fun for inquisitive amateurs and professionals alike. In this chapter, we formally introduce the family, and cite their occurrences in earlier chapters, as well as some of their fundamental properties. In Chapter 12, we will find geometric interpretations of both Pell and Pell–Lucas numbers, and in Chapter 16 some combinatorial interpretations.

7.2 Earlier Occurrences

Recall that in Chapters 2–6, we found several coincidences. In our study of the Pell equation $x^2 - 2y^2 = 1$, we encountered two prominent number sequences: $\{Q_{2n}\}$ and $\{P_{2n}\}$. While solving the equation $x^2 - 2y^2 = -1$, we came across two related number sequences: $\{Q_{2n-1}\}$ and $\{P_{2n-1}\}$. Then we found that the solutions of the equation $x^2 - 2y^2 = (-1)^n$ are (Q_n, P_n).

Recall from Example 3.8 that the *convergents* $c_n = \frac{p_n}{q_n}$ of the ISCF for $\sqrt{2}$ are $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \ldots$. The numerators and denominators of the convergents are Q_{n+1}, and P_{n+1}, respectively, where $n \geq 0$.

When we investigated *primitive Pythagorean triangles* with consecutive legs, we encountered the sequence $5, 29, 169, 985, 5741, \ldots$; see Table 4.2. This is the same as the sequence $\{P_{2n+1}\}$.

In our study of square-triangular numbers, we came across the sequence $\{y_k\} = 1, 6, 35, 204, 1189, 6930, \ldots$. From Chapter 3, $y_k = P_k Q_k$, the product of the numerator and denominator of the kth convergent c_k of the continued fraction of $\sqrt{2}$ (see Table 3.6):

T. Koshy, *Pell and Pell–Lucas Numbers with Applications*, DOI 10.1007/978-1-4614-8489-9_7,

Table 7.1.

$$\begin{array}{rcl} 1 & = & 1\cdot 1 \\ 6 & = & 2\cdot 3 \\ 35 & = & 5\cdot 7 \\ 204 & = & 12\cdot 17 \\ 1189 & = & 29\cdot 41 \\ 6930 & = & 70\cdot 99 \\ & \vdots & \uparrow \quad \uparrow \\ & & P_k \quad Q_k \end{array}$$

The subscripts n_k of square-triangular numbers y_k^2 in Table 3.6 reveal another spectacular coincidence; see Table 7.2. Suppose k is odd. Then the numbers $\sqrt{n_k}$ generate the sequence $1, 7, 41, 239, 1393, \ldots$. They are the numerators of the even-numbered convergents c_{2i} of the continued fraction expansion of $\sqrt{2}$, where $i \geq 0$.

Table 7.2.

k	n_k			k	n_k		
1	1^2	=	1	2	$2\cdot 2^2$	=	8
3	7^2	=	49	4	$2\cdot 12^2$	=	288
5	41^2	=	1681	6	$2\cdot 70^2$	=	9800
7	239^2	=	57121	8	$2\cdot 408^2$	=	332928
9	1393^2	=	1940449	10	$2\cdot 2378^2$	=	11309768

On the other hand, suppose k is even. Then the numbers $\sqrt{\frac{n_k}{2}}$ also generate an interesting sequence: $2, 12, 70, 408, 2378, \ldots$. They are the denominators of the odd-numbered convergents c_{2i-1} of continued fraction expansion of $\sqrt{2}$, where $i \geq 1$.

Interestingly, each of these number sequences is either $\{P_n\}, \{Q_n\}$ or their subsequences. Because of the close-knit relationship between the sequence $\{P_n\}$ and the Pell equation $x^2 - 2y^2 = -1$, the numbers P_n are called *Pell numbers*; they correspond to *Fibonacci numbers*. On the other hand, the sequence $\{Q_n\}$ is closely related to the equation $x^2 - 2y^2 = 1$; the numbers Q_n are called *Pell–Lucas numbers*. Like Fibonacci and Lucas numbers, Pell and Pell–Lucas numbers are also very closely related, just like twins; so always look for similarities.

We now define Pell and Pell–Lucas numbers recursively.

7.3 Recursive Definitions

Recall from Chapters 1 and 2 that both P_n and Q_n satisfy the same Pell recurrence: $x_n = 2x_{n-1} + x_{n-2}$, where $n \geq 3$; they satisfy the same first initial condition also. The only distinction between the two definitions is in the second initial condition: $P_2 = 2$, but $Q_2 = 3$.

The *Pell recurrence* can be translated into a matrix equation:

$$\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ x_{n-2} \end{bmatrix},$$

where $n \geq 3$. We will take advantage of this matrix equation in the next chapter.

Table 7.3 gives the first twelve Pell and Pell–Lucas numbers.

Table 7.3.

n	1	2	3	4	5	6	7	8	9	10	11	12
P_n	1	2	5	12	29	70	169	408	985	2378	5741	13860
Q_n	1	3	7	17	41	99	239	577	1393	3363	8119	19601

As in the case of Fibonacci and Lucas numbers, the Pell families also can be extended to zero and negative subscripts: $P_0 = 0$ and $P_{-n} = (-1)^{n-1}P_n$; and $Q_0 = 1$ and $Q_{-n} = (-1)^n Q_n$.

Next we make some simple observations from Table 7.3:

(1) Every even-numbered Pell number has even parity.

(2) Every Pell–Lucas number has odd parity.

(3) Q_n can end in any odd digit, except 5. This can indeed be confirmed. Notice that the sequence $\{Q_n \pmod 5\}$ shows an interesting pattern:

$$\underbrace{1\;3\;2\;2\;1\;4\;4\;2\;3\;3\;4\;1}\;\underbrace{1\;3\;2\;2\;1\;4\;4\;2\;3\;3\;4\;1}\cdots$$

It is periodic, with period 12. Furthermore, no zero occurs in the first block and hence none in succeeding blocks; so $Q_n \not\equiv 0 \pmod 5$ for every n. Thus no Q_n ends in a 5.

(4) The siblings P_n and Q_n are relatively prime: $(P_n, Q_n) = 1$. This follows from the fact that $\frac{Q_{n+1}}{P_{n+1}}$ is the nth convergent of the ISCF of $\sqrt{2}$, and $(Q_n, P_n) = 1$ by formula (3.3). (We will reprove this also later.)

(5) $P_{3n} \equiv 0 \pmod 5$. We will give a simple and short proof later.

7.4 Alternate Forms for γ and δ

We now digress slightly to show how the numbers $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$ are related to continued fractions. To see this relationship, we let $b = [2; \overline{2}]$. Then $b = 2 + \frac{1}{b}$; so $b^2 - 2b - 1 = 0$ and hence $b = 1 \pm \sqrt{2}$. Since $b > 0$, $b = 1 + \sqrt{2} = \gamma$. Thus $\gamma = [2; \overline{2}]$.

Since $1 - \sqrt{2} = \delta = -\frac{1}{\gamma}$, it follows that $\delta = 1 - \sqrt{2} = -[0; \overline{2}]$.

The powers of γ reveal an interesting pattern:

$$\begin{aligned} \gamma &= 0 + 1\gamma \\ \gamma^2 &= 1 + 2\gamma \\ \gamma^3 &= 2 + 5\gamma \\ &\vdots \end{aligned}$$

More generally, it follows by PMI that $\gamma^n = P_{n-1} + P_n\gamma$; so $\gamma = \sqrt[n]{P_{n-1} + P_n\gamma}$.

Repeated substitution for γ yields a complex radical expression for γ:

$$\gamma = \sqrt[n]{P_{n-1} + P_n \sqrt[n]{P_{n-1} + P_n \sqrt[n]{P_{n-1} + \cdots}}}.$$

In particular, when $n = 2$, this gives

$$\gamma = \sqrt{1 + 2\sqrt{1 + 2\sqrt{1 + \cdots}}}.$$

Similarly, since $\gamma = 2 + \frac{1}{\gamma}$, again it follows by PMI that $\gamma^n = P_{n+1} + \frac{P_n}{\gamma}$, where $n \geq 1$. This gives rise to another cumbersome radical expression for γ:

$$\begin{aligned}\gamma &= \sqrt[n]{P_{n+1} + \frac{P_n}{\gamma}}\\ &= \sqrt[n]{P_{n+1} + \cfrac{P_n}{\sqrt[n]{P_{n+1} + \cfrac{P_n}{\sqrt[n]{P_{n+1} + \cdots}}}}}.\end{aligned}$$

In particular,

$$\gamma = \sqrt{5 + \cfrac{2}{\sqrt{5 + \cfrac{2}{\sqrt{5 + \cdots}}}}}.$$

More generally, let t be a positive real number. It follows from $\gamma^2 = 2\gamma + 1$ that $(t\gamma)^2 = 2t^2\gamma + t^2$; so $t\gamma = \sqrt{t^2 + 2t(t\gamma)}$ and $t\gamma = 2t + \frac{t^2}{t\gamma}$. Continued substitution of the latter for $t\gamma$ inside the radical yields the following radical continued fraction for $t\gamma$:

$$t\gamma = \sqrt{5t^2 + \cfrac{2t^3}{\sqrt{5t^2 + \cfrac{2t^3}{\sqrt{5t^2 + \cdots}}}}}.$$

Next, we give a geometric interpretation of the characteristic equation of the Pell recurrence.

7.5 A Geometric Confluence

Recall that the Pell recurrence yields the characteristic equation $t^2 = 2t + 1$; that is, $t(t-2) = 1$. So $\frac{t}{1} = \frac{1}{t-2}$. This equation has an interesting geometric interpretation.

To see this, consider a $t \times 1$ rectangle; see Figure 7.1. Now remove two unit squares from it. By virtue of the equality $\frac{t}{1} = \frac{1}{t-2}$, the remaining rectangle has the same ratio of length to its width as the original rectangle.

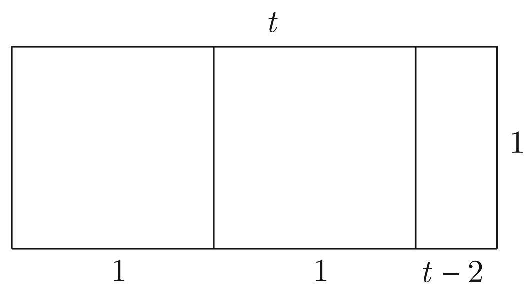

Figure 7.1.

7.6 Pell's equation $x^2 - 2y^2 = -1$ Revisited

In 1993, Z. Zaiming of Yuxi Teachers' College, Yuxi, Yunnan, China, proposed the next example as a problem [264]. It shows that Pell numbers can appear in unexpected places.

Example 7.1 Find all pairs (m, n) of positive integers such that $1 + 2 + \cdots + m = (m+1) + (m+2) + \cdots n$.

Solution. Using the summation formula (1), the given equation yields

$$\begin{aligned}
\frac{m(m+1)}{2} &= (n-m)m + \frac{(n-m)(n-m+1)}{2} \\
m(m+1) &= (n-m)(n-m+1) \\
2m(m+1) &= n(n+1) \\
u^2 - 2v^2 &= -1,
\end{aligned}$$

where $u = 2n + 1$ and $v = 2m + 1$.

Recall from Example 2.11 that the solutions of this Pell equation are given by $(u_k, v_k) = (Q_{2k-1}, P_{2k-1})$. Consequently, $n_k = \frac{1}{2}(Q_{2k-1} - 1)$ and $m_k = \frac{1}{2}(P_{2k-1} - 1)$, where $k \geq 2$.

For instance, $n_5 = \frac{1}{2}(Q_9 - 1) = 696$ and $m_5 = \frac{1}{2}(P_9 - 1) = 492$. Notice that $1 + 2 + \cdots + 492 = (492 + 1) + (492 + 2) + \cdots + 696 = 121,278$. ■

The next example is an application of the *Binet-like formulas*; P. Mana of the University of New Mexico, Albuquerque, New Mexico proposed it as a problem in 1970 [162]. The solution presented here is based on the one given in the following year by L. Carlitz of Duke University, Durham, North Carolina [40].

Example 7.2 Show that there is a sequence $\{A_k\}$ such that $P_{n+2k} = P_{n+k}A_k - (-1)^k P_n$, and define A_k recursively.

Solution. We have

$$\begin{aligned}(\gamma-\delta)[P_{n+2k}+(-1)^k P_n] &= (\gamma^{n+2k}-\delta^{n+2k})+(-1)^k(\gamma^n-\delta^n)\\ &= (\gamma^{n+2k}-\delta^{n+2k})+(\gamma\delta)^k(\gamma^n-\delta^n)\\ &= (\gamma^{n+2k}-\delta^{n+2k})+\gamma^{n+k}\delta^k-\gamma^k\delta^{n+k}\\ &= (\gamma^{n+k}-\delta^{n+k})(\gamma^k+\delta^k)\\ P_{n+2k}+(-1)^k P_n &= 2P_{n+k}Q_k,\end{aligned}$$

as desired. So $A_k = 2Q_k$.

Since $Q_1 = 1$ and $Q_2 = 3$, $A_1 = 2$ and $A_2 = 6$. Furthermore,

$$\begin{aligned}2A_{k-1}+A_{k-2} &= 4Q_{k-1}+2Q_{k-2}\\ &= 2(2Q_{k-1}+Q_{k-2})\\ &= 2Q_k = A_k.\end{aligned}$$

Thus A_k satisfies the Pell recurrence. So A_k can be defined recursively:

$$\begin{aligned}A_1 &= 2, \quad A_2 = 6\\ A_k &= 2A_{k-1}+A_{k-2}, \quad k \geq 3.\end{aligned}$$ ■

The next example is an interesting confluence of geometry and the Pell family. It appeared in the *International Mathematical Olympiad* in 1979.

Example 7.3 Suppose there is a frog at vertex A of the octagon $ABCDEFGH$ in Figure 7.2. From any vertex, except E, it can jump to either of the two adjacent vertices. When the frog reaches E, it stops and stays there. Find the number a_n of distinct paths with exactly n jumps ending at E.

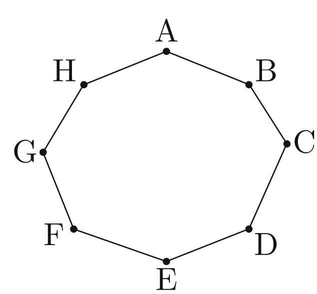

Figure 7.2.

Solution. The number of jumps needed from A to E is always even. So $a_{2n-1} = 0$ for every $n \geq 1$.

The frog cannot reach E from A in two steps; so $a_2 = 0$. From A, it can reach E in four steps in two different ways: $ABCDE$ and $AHGFE$; so $a_4 = 2$.

Next we will find a recurrence for a_{2n}, where $n \geq 3$. To this end, let b_n denote the number of distinct paths from C (or G) to E. There are four different moves the frog can make with two jumps: $A-B-A$, $A-H-A$, $A-B-C$, and $A-H-G$.

Suppose the frog returns to A after two jumps. Since there are $2n-2$ jumps left, it follows by definition and symmetry that there are $2a_{2n-2}$ such moves to reach E. Suppose it reaches C (or G) in two steps. Then it takes $2b_{2n-2}$ jumps to reach E. Consequently, by the addition principle,

$$a_{2n} = 2a_{2n-2} + 2b_{2n-2}. \tag{7.1}$$

We will now find another recurrence satisfied by both a_{2n} and b_{2n}. Suppose the frog is at C. There are three possible moves it can make, assuming it does not land at E: $A{-}B{-}C$, $C{-}D{-}C$, and $C-B-A$. This implies that

$$b_{2n} = 2b_{2n-2} + a_{2n-2}. \tag{7.2}$$

Since $b_{2n-2} = \frac{1}{2}(a_{2n} - 2a_{2n-2})$ from (7.1), this yields a second-order recurrence for a_{2n}:

$$a_{2n+2} = 4a_{2n} - 2a_{2n-2}. \tag{7.3}$$

This implies that $a_4 = 4a_2 - 2a_0$; so $a_0 = -1$.

The characteristic equation of recurrence (7.3) is $x^2 - 4x + 2 = 0$, with characteristic roots $\sqrt{2}\gamma$ and $\sqrt{2}\delta$. So the general solution of (7.3) is $a_{2n} = A(\sqrt{2}\gamma)^n + B(-\sqrt{2}\delta)^n$, where A and B are constants and $n \geq 0$.

Using the initial conditions $a_0 = -1$ and $a_2 = 0$, we get $A = -\frac{\delta}{2}$ and $A = -\frac{\gamma}{2}$. Thus we have

$$\begin{aligned} a_{2n} &= -\frac{\delta}{2}(\sqrt{2}\gamma)^n - \frac{\gamma}{2}(-\sqrt{2}\delta)^n \\ &= 2^{(n-2)/2}\left[\gamma^{n-1} - (-\delta)^{n-1}\right] \\ &= \begin{cases} 2^{(n+1)/2} P_{n-1} & \text{if } n \text{ is odd} \\ 2^{n/2} Q_{n-1} & \text{otherwise.} \end{cases} \end{aligned}$$

For example, $a_6 = 2^2 P_2 = 8$, and $a_8 = 2^2 Q_3 = 28$. ∎

Next, we present some fundamental identities satisfied by Pell and Pell–Lucas numbers. Most of them can be established using Binet-like formulas. We will prove a few of them and leave the others as straightforward exercises for Pell enthusiasts. You will find the following facts useful in the proofs: $\gamma + \delta = 2$, $\gamma\delta = -1$, $\gamma - \delta = 2\sqrt{2}$, $(\gamma - \delta)^2 = 8$, $\gamma + 1 = \sqrt{2}\gamma$, $\delta + 1 = -\sqrt{2}\delta$, $\gamma^2 + \delta^2 = (\gamma + \delta)^2 - 2\gamma\delta = 6$, $\gamma^3 = \gamma(2\gamma + 1) = 2\gamma^2 + \gamma = 2(2\gamma + 1) + \gamma = 5\gamma + 2$, and similarly, $\delta^3 = 5\delta + 2$.

7.7 Fundamental Pell Identities

(1) $P_n + P_{n-1} = Q_n$. Since $P_1 > 0$, it follows by PMI that $P_n < Q_n$ for every $n \geq 2$.

(2) $Q_n + Q_{n-1} = 2P_n$. Consequently, $Q_n \equiv Q_{n-1} \pmod 2$. But Q_1 is odd. So every Q_n is odd.

(3) $P_n + Q_n = P_{n+1}$.

(4) $2P_n + Q_n = Q_{n+1}$.

(5) $2Q_n + 3P_n = P_{n+2}$.

(6) $3Q_n + 4P_n = Q_{n+2}$.

(7) $Q_{n+1} - Q_n = 2P_n$.

(8) $P_{n+1} + P_{n-1} = 2Q_n$.

(9) $Q_{n+1} + Q_{n-1} = 4P_n$.

(10) $P_n + P_{n+1} + P_{n+3} = 3P_{n+2}$.

(11) $Q_n + Q_{n+1} + Q_{n+3} = 3Q_{n+2}$.

(12) $P_{n+1} - P_{n-1} = 2P_n$.

(13) $Q_{n+1} - Q_{n-1} = 2Q_n$.

(14) $P_{n+2} + P_{n-2} = 6P_n$.

(15) $Q_{n+2} + Q_{n-2} = 6Q_n$.

(16) $P_{n+2} - P_{n-2} = 4Q_n$.

(17) $Q_{n+2} - Q_{n-2} = 8P_n$.

(18) $P_{n+1}^2 + P_n^2 = P_{2n+1}$.

More generally, we have the following identity, discovered in 1992 by H.T. Freitag of Roanoke, Virginia [88].

(19) $P_{m+n}^2 - (-1)^n P_m^2 = P_{2m+n} P_n$.

Thus $P_{2m+n} P_n = \begin{cases} P_{m+n}^2 + P_m^2 & \text{if } n \text{ is odd} \\ P_{m+n}^2 - P_m^2 & \text{if } n \text{ is even.} \end{cases}$ Consequently, $P_{m+n}^2 + P_m^2$ is always factorable.

For example, $P_{5+7}^2 + P_5^2 = 192,100,441 = P_{2\cdot 5+7} P_7$ and $P_{5+6}^2 - P_5^2 = 32,958,240 = P_{2\cdot 5+6} P_6$.

Identity (19) has a counterpart for Pell–Lucas numbers, also discovered by Freitag in 1992. Its proof follows similarly.

(20) $Q_{m+n}^2 - (-1)^n Q_m^2 = 2P_{2m+n} P_n$. For example, $Q_{5+7}^2 + Q_5^2 = 384,200,882 = 2P_{2\cdot 5+7} P_7$ and $Q_{5+6}^2 - Q_5^2 = 65,916,480 = 2P_{2\cdot 5+6} P_6$.

It follows from identities (18) and (19) that $2P_{m+n}^2 - Q_{m+n}^2 = (-1)^n (2P_m^2 - Q_m^2)$.

(21) $Q_{n+1}^2 + Q_n^2 = 2P_{2n+1}$.

(22) $P_{n+1}^2 - P_n^2 = Q_{n+1} Q_n$.

(23) $Q_{n+1}^2 - Q_n^2 = 4P_{n+1} P_n$; that is, $Q_{n+1}^2 - Q_n^2 = Q_{2n+1} - (-1)^n$.

(24) $4(P_n^2 + Q_n^2) = 3Q_{2n} + (-1)^n$.

(25) $2P_n + Q_n = Q_{n+1}$. This also implies that every Q_n is odd.

(26) $2P_n + Q_{n+2} = 3Q_{n+1}$.

(27) $P_{n+1} + Q_{n-1} = 3P_n$.

(28) $P_{2n} = 2P_nQ_n$. This corresponds to the identity $F_{2n} = F_nL_n$.
It implies:

- P_{2n} has even parity; that is, $P_{2n} \equiv 0 \pmod 2$.
- P_{2n} is factorable, where $n \geq 2$.
- $P_{2^k n} = 2^k P_n Q_n Q_{2n} \cdots Q_{2^{k-1}n}$, where $n, k \geq 1$.
  Since P_{2n} is even and Q_{2n} is odd, it follows by the identity $P_{2n} + P_{2n-1} = Q_{2n}$ that P_{2n-1} is odd. Thus $P_n \equiv n \pmod 2$. Consequently, $Q_1^2 + Q_2^2 + \cdots + Q_n^2 \equiv P_n^2 \pmod 2$. More generally, $Q_1^k + Q_2^k + \cdots + Q_n^k \equiv P_n^k \pmod 2$, where k is a positive integer.

(29) $P_{3n} \equiv 0 \pmod 5$. Since $P_{2n} \equiv 0 \pmod 2$ and $P_{3n} \equiv 0 \pmod 5$, it follows that $P_{6n} \equiv 0 \pmod{10}$.

(30) $Q_n^2 + 2P_n^2 = Q_{2n}$.

(31) $Q_n^2 - 2P_n^2 = (-1)^n$; that is, $\begin{vmatrix} Q_n & P_n \\ 2P_n & Q_n \end{vmatrix} = (-1)^n$, where $|M|$ denotes the determinant of the square matrix M. This result reconfirms a fact we already learned in Chapter 3: $\frac{Q_{n+1}}{P_{n+1}}$ is the nth convergent of the ISCF of $\sqrt{2}$; that is, (Q_{n+1}, P_{n+1}) is a solution of the Pell equation $x^2 - 2y^2 = (-1)^n$, where $n \geq 0$.

This result has several additional interesting byproducts:

- The triangular numbers t_{1^2} and t_{41^2} share an interesting property: $t_{1^2} = 1^2$ and $t_{41^2} = 1189^2$. In fact, there are infinitely many square-triangular numbers with square subscripts n. To see this, let $t_n = m^2$ and $n = x^2$ for some positive integers m and x. Then $x^2 - 2t^2 = -1$, where $t = m/x$. Its solutions are given by $(x, t) = (Q_{2k-1}, P_{2k-1})$. So $m = P_{2k-1}Q_{2k-1} = \frac{1}{2}P_{2k-2}$. Thus $t_{Q_{2k-1}^2} = \frac{1}{4}P_{2k-2}^2$. For example, $t_{Q_7^2} = t_{239^2} = 1{,}631{,}432{,}881 = 40{,}391^2 = \frac{1}{4}P_{14}^2$.
- Since $2P_{2n+1}^2 = Q_{2n+1}^2 + 1$, we have

$$\begin{aligned}
&t_{Q_{2n+1}-2} + t_{Q_{2n+1}-1} + t_{Q_{2n+1}} + t_{Q_{2n+1}+1}\\
&\quad= \frac{(Q_{2n+1}-2)(Q_{2n+1}-1)}{2} + \frac{(Q_{2n+1}-1)Q_{2n+1}}{2}\\
&\qquad+ \frac{Q_{2n+1}(Q_{2n+1}+1)}{2} + \frac{(Q_{2n+1}+1)(Q_{2n+1}+2)}{2}\\
&\quad= \frac{1}{2}[(Q_{2n+1}-1)(2Q_{2n+1}-2) + (Q_{2n+1}+1)(2Q_{2n+1}+2)]\\
&\quad= (Q_{2n+1}-1)^2 + (Q_{2n+1}+1)^2\\
&\quad= 2Q_{2n+1}^2 + 2\\
&\quad= 4P_{2n+1}^2.
\end{aligned}$$

Thus $(2P_{2n+1})^2$ can be expressed as the sum of four consecutive triangular numbers, as K.S. Bhanu and M.N. Deshpande found in 2008 [18].

In particular, let $n = 5$. Then

$$\begin{aligned}
4P_{11}^2 &= 4 \cdot 5741^2 = 131,836,324 \\
&= 32,946,903 + 32,955,021 + 32,963,140 + 32,971,260 \\
&= t_{8117} + t_{8118} + t_{8119} + t_{8120} = t_{Q_{11}-2} + t_{Q_{11}-1} + t_{Q_{11}} + t_{Q_{11}+1}.
\end{aligned}$$

More generally, let x be an arbitrary integer ≥ 2. Then

$$\begin{aligned}
2\,(t_{x-2} + t_{x-1} + t_x + t_{x+1}) &= (x-2)(x-1) + (x-1)x + x(x+1) \\
&\quad +(x+1)(x+2) \\
&= 2[(x-1)^2 + (x+1)^2] = 4(x^2+1) \\
t_{x-2} + t_{x-1} + t_x + t_{x+1} &= 2(x^2+1).
\end{aligned}$$

Since $(x_n, y_n) = (Q_{2n+1}, P_{2n+1})$ is a solution of the equation $x^2 - 2y^2 = -1$, it follows that $y_n = P_{2n+1}$ has the property that $4y_n^2 = 4P_{2n+1}^2$ is the sum of four consecutive triangular numbers.

- Consider the diophantine equation

$$x^2(3x-1)^2 = 8y^2 + 4. \tag{7.4}$$

Letting $X = x(3x-1)/2$, it can be rewritten as $X^2 - 2y^2 = 1$. Since its solutions (x, y) are given by (Q_{2n}, P_{2n}), it follows that (x, y) is a solution of (7.4) if and only if $x(3x-1)/2 = Q_{2n}$ and $y = P_{2n}$ for some integer $n \geq 0$.

Solving the equation $3x^2 - x - 2Q_{2n} = 0$, we get $x = \frac{1 \pm \sqrt{1+24Q_{2n}}}{6}$. Clearly, the negative root is *not* acceptable, so $x = \frac{1+\sqrt{1+24Q_{2n}}}{6}$. We will establish later that $1 + 24Q_k$ is a square if and only if $k = 0$ or 1. When $n = 0$, $X = Q_0 = 1$, $y = P_0 = 0$; so (1, 0) is a nonnegative solution of the diophantine equation. When $n = 1$, $(X, y) = (Q_3, P_2) = (3,\ 2)$; but (3, 2) is *not* a solution. Thus (1, 0) is the only nonnegative solution of (7.4).

It follows similarly that (1, 1) is the only nonnegative solution of the diophantine equation $x^2(3x-1)^2 = 8y^2 - 4$.

- Consider the diophantine equation

$$2x^2 = y^2(3y-1)^2 - 2. \tag{7.5}$$

It can be rewritten as $x^2 - 2Y^2 = -1$, where $Y = y(3y-1)/2$. The solutions of $x^2 - 2Y^2 = -1$ are given by (Q_{2n+1}, P_{2n+1}); so (x, y) is a solution of (7.5) if and only if $x = Q_{2n+1}$ and $y(3y-1)/2 = P_{2n+1}$ for some integer $n \geq 0$.

Solving the equation $3y^2 - y - 2P_{2n+1} = 0$ yields $y = \frac{1+\sqrt{1+24P_{2n+1}}}{6}$ as before. Again, we will establish later that $1 + 24P_k$ is a square if and only if $k = 0, 1, 2, 3, 4$, or 6; so $1 + 24P_{2n+1}$ is a square if and only if $n = 0$ or 1. When $n = 0$, $(x, y) = (Q_1, (1 + \sqrt{1 + 24P_1})/6) = (1,\ 1)$. Likewise, when $n = 1$, $(x, y) = (7,\ 2)$. Thus the diophantine equation $2x^2 = y^2(3y - 1)^2 - 2$ has exactly two positive solutions: (1, 1) and (7, 2).

A similar argument will confirm that the diophantine equation $2x^2 = y^2(3y-1)^2 + 2$ has exactly two nonnegative solutions: (1, 0) and (17, 3).

- Since $Q_{2k}^2 = 1 + 2P_{2k}^2$, the subscript n_k of the square-triangular number $t_{n_k} = y_k^2$ (see Chapter 6) is given by $n_k = \frac{1}{2}\left(\sqrt{8y_k^2 + 1} - 1\right) = \frac{1}{2}\left(\sqrt{2P_{2k}^2 + 1} - 1\right) = \frac{1}{2}(Q_{2k} - 1)$. Consequently, $t_{n_k} = t_{\frac{Q_{2k}-1}{2}}$.
- Recall from Chapter 2 that if (x, y) and (X, Y) are two solutions of the Pell equation $x^2 - 2y^2 = (-1)^n$, then $(xY)^2 + (-1)^n y^2 = (Xy)^2 + (-1)^n Y^2$. Consequently, since (Q_k, P_k) and (Q_m, P_m) are solutions of $x^2 - 2y^2 = (-1)^n$, it follows that $(Q_k P_m)^2 + (-1)^n P_k^2 = (Q_m P_k)^2 + (-1)^n P_m^2$.

  For example, $(Q_4, P_4) = (17,\ 12)$ and $(Q_8, P_8) = (577,\ 408)$ are solutions of $x^2 - 2y^2 = 1$. Then $(Q_4P_8)^2 + P_4^2 = (17 \cdot 408)^2 + 12^2 = 48,108,240 = (577 \cdot 12)^2 + 408^2 = (Q_8P_4)^2 + P_8^2$. Likewise, $(Q_5P_9)^2 - P_5^2 = 1,630,947,384 = (Q_9P_5)^2 - P_9^2$.
- Since $Q_n^2 \equiv 1 \pmod 2$, the identity reconfirms the odd parity of every Q_n.
- We have $Q_n^2 = 2P_n^2 + (-1)^n$. So $2P_n^2 + (-1)^n$ is always a square.

(32) $Q_{2n} = 2Q_n^2 - (-1)^n$. We will use this identity in our study of pentagonal Pell–Lucas numbers later. It follows from this identity that $Q_{2n} \equiv (-1)^{n-1} \pmod{Q_n}$.

(33) $Q_{2n} = 4P_n^2 + (-1)^n$. This identity implies that $Q_{2n} \equiv (-1)^n \pmod 4$. It follows by PMI that $Q_n \equiv (-1)^{\lfloor n/2 \rfloor} \pmod 4$.

For example, $Q_7 = 239 \equiv -1 \equiv (-1)^{\lfloor 7/2 \rfloor} \pmod 4$ and $Q_8 = 577 \equiv 1 \equiv (-1)^{\lfloor 8/2 \rfloor} \pmod 4$.

This identity has an interesting byproduct. To see this, note that $4P_{2n}^2 = Q_{4n} - 1$; so $Q_{4n} - 1$ is a square. On the other hand, $4P_{2n-1}^2 = Q_{4n-2} + 1$; so $Q_{4n-2} + 1$ is also a square. Thus, if $4|n$, $Q_n - 1$ is a square; and if $2|n$ and $4 \nmid n$, then $Q_n + 1$ is a square. For example, $Q_8 - 1 = 577 - 1 = (2 \cdot 12)^2$ and $Q_6 + 1 = 99 + 1 = (2 \cdot 5)^2$.

(34) $P_{n+1}P_{n-1} - P_n^2 = (-1)^n$. This is the *Cassini-like* formula for Pell numbers. It has an interesting byproduct. To see this, let $d = (P_{n+1}, P_n)$. Then $d|P_{n+1}$ and $d|P_n$. So $d|(P_{n+1}P_{n-1} - P_n^2)$; that is, $d|(-1)^n$. So $d = 1$. Thus, as in the case of Fibonacci numbers, every two consecutive Pell numbers are relatively prime. For example, $(P_6, P_7) = (70,\ 169) = 1 = (985,\ 2378) = (P_{10}, P_{11})$.

(35) $Q_{n+1}Q_{n-1} - Q_n^2 = 2(-1)^{n-1}$. This is the *Cassini-like* formula for Pell–Lucas numbers. It follows from the formula that $d|2$, where $d = (Q_{n+1}, Q_n)$; so $d = 1$ or 2. But every Q_n is odd; so $d = 1$. Thus every two consecutive Pell–Lucas numbers are also relatively prime. For example, $(Q_6, Q_7) = (99,\ 239) = 1 = (3363,\ 8119) = (Q_{10}, Q_{11})$.

(36) $P_n Q_{n-1} - Q_n P_{n-1} = (-1)^{n-1}$. This formula has two immediate consequences: $(P_n, P_{n-1}) = 1 = (Q_n, Q_{n-1})$, as we already knew. Since $P_n + P_{n-1} = Q_n$, it also follows that $(P_n, Q_n) = 1$.
This formula looks strikingly similar to the one satisfied by the convergents $\frac{q_n}{p_n}$ of the ISCF of $\sqrt{N}$: $p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1}$.

(37) $P_n Q_{n-1} + Q_n P_{n-1} = P_{2n-1}$.

(38) $2(P_{n+1}^2 + P_n^2) = Q_{2n} + Q_{2n+1}$. Using identity (18), this implies that $Q_{2n} + Q_{2n+1} = 2P_{2n+1}$.

(39) $2(Q_{n+1}^2 + Q_n^2) = Q_{2n} + Q_{2n+2}$. Since $Q_{2n} + Q_{2n+2} = 4P_{2n+1}$, by identity (9), this implies that $Q_n^2 + Q_{n+1}^2 = 2P_{2n+1}$.

(40) $Q_n Q_{n+1} - 2P_n P_{n+1} = (-1)^n$.

(41) $P_n P_{n+3} - P_{n+1} P_{n+2} = 2(-1)^{n-1}$.

(42) $Q_n Q_{n+3} - Q_{n+1} Q_{n+2} = 4(-1)^n$.

(43) $P_{n+3}^2 + P_n^2 = 5P_{2n+3}$.

(44) $Q_{n+3}^2 + Q_n^2 = 10P_{2n+3}$.

(45) In Chapter 6, we learned that $\frac{1}{2}P_{2n} = P_n Q_n$ satisfies the recurrence $y_n = 6y_{n-1} - y_{n-2}$, $n \geq 3$.

(46) We also learned that $\frac{1}{2}(Q_{2n} - 1)$ satisfies the recurrence $x_n = 6x_{n-1} - x_{n-2} + 2$, where $n \geq 3$. Consequently, Q_{2n} satisfies the recurrence $Q_{2n} = 6Q_{2n-2} - Q_{2n-4}$, where $n \geq 2$; see Example 2.13.

(47) Since $Q_{2n} = 2Q_n^2 - (-1)^n$ (identity 32), this yields a recurrence for Q_n^2 : $Q_n^2 = 6Q_{n-1}^2 - Q_{n-2}^2 + 4(-1)^n$; again, see Chapter 8.

Identities (31) and (33) provide a delightful application[9]. To see this, let $A = \begin{bmatrix} Q_2 & P_2 \\ 2P_2 & Q_2 \end{bmatrix}$. Then it follows by PMI that $A^n = \begin{bmatrix} Q_{2n} & P_{2n} \\ 2P_{2n} & Q_{2n} \end{bmatrix}$ for every $n \geq 1$. (Since $|A|^\dagger = 1$, this implies that $|A^n| = |A|^n = 1$; that is, $Q_{2n}^2 - 2P_{2n}^2 = 1$, a fact that we already knew.) So $A^n - I = \begin{bmatrix} Q_{2n} - 1 & P_{2n} \\ 2P_{2n} & Q_{2n} - 1 \end{bmatrix}$.

Let $d_n = (Q_{2n} - 1, P_{2n}, 2P_{2n}, Q_{2n} - 1)$, the gcd of all the entries in the matrix $A^n - I$. Then $d_n = (Q_{2n} - 1, P_{2n})$.

Since $Q_{2n}^2 - 1 = 2P_{2n}^2$, $(Q_{2n} - 1)\left(\frac{Q_{2n}+1}{2}\right) = P_{2n}^2$. But every Q_k is odd; so this implies that $(Q_{2n} - 1) \mid P_{2n}^2$. Consequently, $d_n \geq \sqrt{Q_{2n} - 1}$. Thus, $d_n \to \infty$ as $n \to \infty$; that is, the gcd of all entries in $A^n - I$ approaches ∞ as $n \to \infty$.

The next example also illustrates the fact that Pell numbers can pop up in quite unexpected places. It employs identities (28), (31), and (33).

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9 Based on the 1994 William L. Putnam Mathematical Competition, *Mathematical Association of America* [4].

† $|M|$ denotes the determinant of the square matrix M.

Example 7.4 Find the positive integers n such that $\sum_{i=1}^{n}(4i-3)$ is a square.

Solution. Since $\sum_{i=1}^{n}(4i-3) = 4\cdot\frac{n(n+1)}{2} - 3n = 2n^2 - n$, we want $2n^2 - n = m^2$ for some positive integer m. This yields the familiar Pell's equation $x^2 - 2y^2 = 1$, where $x = 4n - 1$ and $y = 2m$.

Recall that this Pell's equation has infinitely many solutions $(x_k, y_k) = (Q_{2k}, P_{2k})$. Correspondingly, $4n_k - 1 = Q_{2k}$, and hence $n_k = \frac{Q_{2k}+1}{4}$, where $k \geq 1$. But n_k is an integer if and only if $Q_{2k} \equiv 3 \pmod 4$. By identity (32), $Q_{2k} \equiv 3 \pmod 4$ if and only if k is odd. So n_k is an integer if and only if k is odd. Consequently, if k is odd and $n_k = \frac{Q_{2k}+1}{4}$, then the corresponding sum S_k is a square. ∎

For example, we have

$$\begin{aligned}
n_1 &= \tfrac{3+1}{4} = 1 = 1^2; & S_1 &= 1^2 = (1\cdot 1)^2\\
n_3 &= \tfrac{99+1}{4} = 25 = 5^2; & S_3 &= 35^2 = (5\cdot 7)^2\\
n_5 &= \tfrac{3363+1}{4} = 841 = 29^2; & S_5 &= 1189^2 = (29\cdot 41)^2\\
n_7 &= \tfrac{114243+1}{4} = 28561 = 169^2; & S_7 &= 40391^2 = (169\cdot 239)^2
\end{aligned}$$

7.7.1 Two Interesting Byproducts

These two patterns hold for any odd positive integer k:

- By identity (33), $Q_{2k} + 1 = 4P_k^2$. So $n_k = P_k^2$.
- By identity (31), we have

$$\begin{aligned}
S_k &= n_k(2n_k - 1) = \frac{Q_{2k}+1}{4}\cdot\frac{Q_{2k}-1}{2}\\
&= \frac{Q_{2k}^2 - 1}{8} = \frac{P_{2k}^2}{4}\\
&= (P_k Q_k)^2, \quad \text{as observed above.}
\end{aligned}$$

The next example was proposed by B.A. Reznick of the University of Illinois at Urbana-Champagne in 1987 [182]. It provides a delightful bridge linking analytic geometry, calculus, combinatorics, and the Pell family. The featured proof is based on the one by the students of the 1987 Mathematical Olympiad team at United States Military Academy, West Point, New York [241].

Example 7.5 Let $\{C_n\}_{n\geq 0}$ be a sequence of circles on the cartesian plane such that:

(i) C_0 is the unit circle $x^2 + y^2 = 1$; and
(ii) the circle C_{n+1} lies in the upper half-plane, and is tangent to both C_n and the two branches of the hyperbola $x^2 - y^2 = 1$, where $n \geq 0$.

Let r_n denote the length of the radius of C_n. Show that r_n is an integer, and $r_n = Q_{2n}$.

Proof. Let $n \geq 1$. Symmetry guarantees that the center of C_n must lie on the y-axis. Let it be $(0, a_n)$; so the equation of C_n is $x^2 + (y - a_n)^2 = r_n^2$.

Since C_n is tangent to C_{n-1}, it follows that

$$a_n - a_{n-1} = r_n + r_{n-1}. \tag{7.6}$$

The y-coordinates y_n of the points of tangency between C_n and the hyperbola are given by $y^2 + 1 + (y - a_n)^2 = r_n^2$; that is,

$$2y^2 - 2a_n y + a_n^2 - r_n^2 + 1 = 0. \tag{7.7}$$

By symmetry, the two y-coordinates must be equal; that is, this equation has a double root. Consequently, its derivative with respect to y must be zero; so $y_n = \frac{a_n}{2}$.

Substituting this value of y in (7.7) yields the recurrence

$$a_n^2 = 2r_n^2 - 2. \tag{7.8}$$

This implies that $a_n^2 - a_{n-1}^2 = 2(r_n^2 - r_{n-1}^2)$. This, coupled with (7.6), yields $a_n + a_{n-1} = 2(r_n - r_{n-1})$. Adding this to (7.6) gives $2a_n = 3r_n - r_{n-1}$. Substituting for a_n in (7.6) yields the recurrence

$$r_n = 6r_{n-1} - r_{n-2}, \tag{7.9}$$

where $n \geq 2$. (We encountered this recurrence in Chapters 1, 2, 4, 5, and 6.) Since $a_0 = 0, r_0 = 1$, and $r_1 > 0$, it follows by (7.6) and (7.8) that $r_1 = 3$. Since both r_0 and r_1 are integers, it follows by (7.9) that r_n is an integer for every $n \geq 0$.

Using the initial conditions $r_0 = 1$, and $r_1 = 3$, it follows from (7.9) that $r_n = \frac{1}{2}(\gamma^{2n} + \delta^{2n}) = Q_{2n}$, as desired. ∎

As a byproduct, it follows by (7.8) and identity (31) that $a_n^2 = 2(Q_{2n}^2 - 1) = 2(2P_{2n}^2) = 4P_{2n}^2$; so $a_n = 2P_{2n}$ and hence $y_n = P_{2n}$ for $n \geq 0$.

Table 7.4 shows the values of a_n, y_n, and r_n for $0 \leq n \leq 10$.

Table 7.4.

n	0	1	2	3	4	5	6	7	8	9	10
a_n	0	4	48	280	1632	9512	55440	323128	1883328	10976840	63977712
y_n	0	1	12	70	408	2378	13860	80782	470832	2744210	15994428
r_n	1	3	17	99	577	3363	19601	114243	665857	3880899	22619537

7.8 Pell Numbers and Primitive Pythagorean Triples

We can use identities (28), (30), and (36) to generate a family of *primitive Pythagorean triples*. Recall from Chapter 4 that primitive Pythagorean triples are generated by $(2ab, a^2-b^2, a^2+b^2)$, where $(a, b) = 1$. By properties (28) and (30), we have $2Q_{2n}P_{2n} = P_{4n}$ and $Q_{2n}^2 - P_{2n}^2 = P_{2n}^2 + 1$; so we let $a = Q_{2n}$ and $b = P_{2n}$. By identity (36), $(Q_{2n}, P_{2n}) = 1$. Then $2ab = P_{4n}, a^2 - b^2 = P_{2n}^2 + 1$, and $a^2 + b^2 = Q_{2n}^2 + P_{2n}^2 = (2P_{2n}^2 + 1) + P_{2n}^2 = 3P_{2n}^2 + 1$. Thus, $(P_{4n}, P_{2n}^2 + 1, 3P_{2n}^2 + 1)$ is a primitive Pythagorean triple, as found by M. Wachtel of Zurich, Switzerland in 1989 [251].

For example, let $n = 2$. Then $(P_8, P_4^2 + 1, 3P_4^2 + 1) = (408, 145, 433)$ is a primitive Pythagorean triple: $433^2 = 408^2 + 145^2$, where the generators $a = Q_4 = 17$ and $b = P_4 = 12$ are relatively prime.

7.9 A Harmonic Bridge

Next we will study an interesting link between the two Pell families. To this end, first we need the identity $P_{2n} = 2P_{n+1}Q_{n-1} + 2(-1)^n$ (see Exercise 48).

This identity can be used to derive a simple formula for the harmonic mean of P_n and Q_n. The *harmonic mean* h of two positive numbers x and y is given by $\frac{1}{h} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right)$; that is, $h = \frac{2xy}{x+y}$.

Consequently, the harmonic mean of P_n and Q_n is given by

$$\begin{aligned} \frac{2P_nQ_n}{P_n + Q_n} &= \frac{P_{2n}}{P_{n+1}} \\ &= \frac{2P_{n+1}Q_{n-1} + 2(-1)^n}{P_{n+1}} \\ &= 2Q_{n-1} + \frac{2(-1)^n}{P_{n+1}}. \end{aligned}$$

For example, the harmonic mean of P_5 and Q_5 is $2Q_4 - \frac{2}{P_6} = 2 \cdot 17 - \frac{2}{70} = \frac{1189}{35} \approx 33.9714286$.

Next, we return to square-triangular numbers, and establish some interesting relationships within the Pell family.

7.10 Square-Triangular Numbers with Pell Generators

First, we will show that $(P_nQ_n)^2 = \frac{1}{4}P_{2n}^2$ is a triangular number. By identity (30), we have $2P_n^2 = Q_n^2 - (-1)^n$. So $(P_nQ_n)^2 = \dfrac{Q_n^2\left[Q_n^2 - (-1)^n\right]}{2}$. Thus $(P_nQ_n)^2$ is a triangular number for $n \geq 1$. (It is true even when $n = 0$.)

For example, $(P_9Q_9)^2 = (408 \cdot 1393)^2 = \frac{166464 \cdot 166465}{2}$ is a triangular number.

Since $Q_n^2 = 2P_n^2 + (-1)^n$, this formula can be rewritten as $(P_nQ_n)^2 = P_n^2\left[2P_n^2 + (-1)^n\right]$; that is, $P_{2n}^2 = 4P_n^2\left[2P_n^2 + (-1)^n\right]$. It now follows that the area of the right triangle with legs Q_n^2 and $2P_n^2 = Q_n^2 - (-1)^n$ is $(P_nQ_n)^2$.

We will now develop another explicit formula for the triangular number $(P_nQ_n)^2$. Notice that

$$4P_m^2 = Q_{2m} - (-1)^m. \tag{7.10}$$

This is true since $4P_m^2 = \frac{1}{2}\left[\gamma^{2m} + \delta^{2m} - 2(\gamma\delta)^m\right] = Q_{2m} - (-1)^m$, where $m \geq 0$.

In particular, $4P_{2n}^2 = Q_{4n} - 1$, so $(P_nQ_n)^2 = \dfrac{Q_{4n} - 1}{16}$, a second explicit formula for the triangular number $(P_nQ_n)^2 = \dfrac{1}{4}P_{2n}^2$.

For example,

$$\frac{Q_{12} - 1}{16} = 1225 = \frac{49 \cdot 50}{2} = (5 \cdot 7)^2.$$

and

$$\frac{Q_{16} - 1}{16} = 41,616 = \frac{288 \cdot 289}{2} = (12 \cdot 17)^2.$$

It follows from formula (7.10) that $Q_{4n} \equiv 1 \pmod{16}$. For example, $Q_{16} = 665,857 \equiv 1 \pmod{16}$.

Although $Q_{2n} \equiv 1 \pmod{16}$ if n is even, it can be shown by induction that $Q_{2n} \equiv 3 \pmod{16}$ if n is odd. Thus

$$Q_{2n} \equiv \begin{cases} 1 \pmod{16} & \text{if } n \text{ is even} \\ 3 \pmod{16} & \text{otherwise.} \end{cases}$$

For example, $Q_8 = 577 \equiv 1 \pmod{16}$ and $Q_{10} = 3363 \equiv 3 \pmod{16}$.

Next, we express Q_{2n} in terms of a triangular number, as the next theorem shows. To this end, we let

$$\Delta_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 3 & \text{otherwise.} \end{cases}$$

Using the Binet-like formulas, we can show that $R_k = \sum_{i=1}^{k} P_i = \frac{Q_{k+1}-1}{2}$; see Chapter 10.

Theorem 7.1 $Q_{2n} = 16t_{R_{n-1}} + \Delta_n$.

Proof. We have

$$\begin{aligned} t_{R_{n-1}} &= \frac{R_{n-1}(R_{n-1}+1)}{2} \\ &= \frac{\frac{Q_n-1}{2}\left(\frac{Q_n+1}{2}\right)}{2} \\ &= \frac{Q_n^2-1}{8}. \end{aligned}$$

Therefore,

$$\begin{aligned} 16t_{R_{n-1}} + \Delta_n &= 16 \cdot \frac{Q_n^2-1}{8} + \Delta_n \\ &= 2(Q_n^2-1) + \Delta_n \\ &= 2Q_n^2 - (-1)^n \\ &= Q_{2n}, \end{aligned}$$

as desired. ■

For example,

$$\begin{aligned} 16t_{R_6} + 3 &= 16t_{1+2+5+12+29+70} = 16t_{119} + 3 \\ &= 16 \cdot \frac{119 \cdot 120}{2} + 3 = 114,243 \\ &= Q_{14}. \end{aligned}$$

The next two corollaries follow from this theorem.

Corollary 7.1 $\frac{Q_{2n} - \Delta_n}{16}$ *is a triangular number. More specifically,* $\frac{Q_{4n}-1}{16} = t_{R_{2n-1}}$ *and* $\frac{Q_{4n-2}-3}{16} = t_{R_{2n-2}}$. ■

Corollary 7.2

$$Q_{2n} \equiv \begin{cases} 1 \pmod{16} & \text{if } n \text{ is even} \\ 3 \pmod{16} & \text{otherwise.} \end{cases}$$ ■

Next, we pursue a similar formula for $t_{S_{n-1}}$, where $S_k = \sum_{i=0}^{k} Q_i$. Again, using the Binet-like formulas, we can show that $S_k = P_{k+1}$. (This also follows from the fact that $P_n + Q_n = P_{n+1}$.)

Since $S_{n-1} = P_n$, it follows that

$$t_{S_{n-1}} = t_{P_n} = \frac{P_n\,(P_n + 1)}{2}.$$

But

$$\begin{aligned} 8P_n(P_n + 1) &= (\gamma^n - \delta^n)\,[(\gamma^n - \delta^n) + (\gamma - \delta)] \\ &= (\gamma^{2n} + \delta^{2n}) + (\gamma^{n+1} + \delta^{n+1}) + (\gamma^{n-1} + \delta^{n-1}) + 2(-1)^{n-1} \\ &= 2Q_{2n} + 2Q_{n+1} + 2Q_{n-1} - 2(-1)^n \\ t_{S_{n-1}} &= \frac{Q_{2n} + Q_{n+1} + Q_{n-1} - (-1)^n}{8} \\ t_{P_n} &= \frac{Q_{2n} + 4P_n - (-1)^n}{8}, \end{aligned} \tag{7.11}$$

since $Q_{n+1} + Q_{n-1} = 4P_n$.

For example, $t_{S_6} = t_{1+1+3+7+17+41} = t_{70} = \frac{Q_{12}+4P_6-1}{8} = \frac{19601+4\cdot 70-1}{8} = 2,485 = t_{P_6}$.

It follows from formula (7.11) that $Q_{2n} + Q_{n+1} + Q_{n-1} = Q_{2n} + 4P_n \equiv (-1)^n \pmod 8$.

Since $P_n + Q_n = P_{n+1}$, it follows from (7.11) that

$$t_{P_n} + Q_n = \frac{Q_{2n+2} + 4P_{n+1} + (-1)^n}{8}$$

and hence $Q_{2n+2} + 4P_{n+1} \equiv (1)^{n-1} \pmod 8$.

As above, it can be shown that $2Q_n(Q_n + 1) = Q_{2n} + 2Q_n + (-1)^n$, so

$$t_{Q_n} = \frac{Q_{2n} + 2Q_n + (-1)^n}{4}. \tag{7.12}$$

For example,

$$\frac{Q_6 + 2Q_3 - 1}{4} = \frac{99 + 2\cdot 7 - 1}{4} = 28 = t_7 = t_{Q_3}.$$

It follows from (7.12) that $Q_{2n} + 2Q_n \equiv (-1)^{n-1} \pmod 4$.

Finally, it is well known that $16t_n + 1$ can be a square. For example, $16\cdot 3 + 1$ is a square, but $16\cdot 6 + 1$ is not. Since $16(P_nQ_n)^2 + 1 = Q_{4n}$, it would be interesting to find those n for which Q_{4n} is a square. For example, Q_0 is a square, but Q_4, Q_8, Q_{12}, Q_{16}, Q_{20}, and Q_{24} are not.

In fact, since $Q_{2n} = 4P_n^2 + (-1)^n$, it follows that Q_{2n} is not a square, when $n \geq 0$. Consequently, $16(P_nQ_n)^2 + 1 = Q_{4n}$ is *not* a square, when $n \geq 1$.

Next, we return to primitive Pythagorean triples with consecutive legs, and investigate their relationship with Pell numbers.

7.11 Primitive Pythagorean Triples With Consecutive Legs Revisited

Recall from Chapter 3 that there are infinitely many primitive Pythagorean triples such that the lengths of their legs are consecutive integers. In other words, the diophantine equation $x^2+(x\pm 1)^2=z^2$ has infinitely many solutions. We found that the generators m and n of the first ten such triangles are the Pell numbers $2, 5, 12, \ldots, 5741$ and $1, 2, 5, \ldots, 2378$, respectively; the lengths of their hypotenuses are the Pell numbers $5, 29, 169, \ldots, 38613965$; see Table 4.2. We also found that the generators must satisfy Pell's equation $(m-n)^2-2n^2=\pm 1$; that is, $u^2-2v^2=\pm 1$.

This makes us curious. Are these just accidental coincidences? Or are they always the case? Fortunately, we have done all the necessary groundwork in Chapters 2 and 3 to answer this. All we have to do is pick up the right pieces and assemble them.

To this end, recall that the solutions of the equation $u^2-2v^2=\pm 1$ are given by $(u_k, v_k)=(Q_k, P_k)$, where $k\geq 1$. The corresponding generators m_k, n_k are given by $m_k-n_k=u_k=Q_k$ and $n_k=v_k=P_k$. So $m_k=n_k+Q_k=P_k+Q_k=P_{k+1}$. Thus the generators of the primitive Pythagorean triples with consecutive legs are given by the consecutive Pell numbers P_{k+1} and P_k, where $k\geq 1$, as we saw in Table 4.2.

The lengths x_k and y_k of the corresponding legs are given by $x_k=m_k^2-n_k^2=P_{k+1}^2-P_k^2=(P_{k+1}+P_k)(P_{k+1}-P_k)=Q_{k+1}Q_k$ and $y_k=2m_kn_k=2P_{k+1}P_k$; see columns 4 and 5 of Table 4.2. The length of the corresponding hypotenuse is given by $z_k=m_k^2+n_k^2=P_{k+1}^2+P_k^2=P_{2k+1}$; see Column 6 of Table 4.2. Consequently, we have the identity

$$(Q_kQ_{k+1})^2+(2P_kP_{k+1})^2=P_{2k+1}^2.$$

Thus, if $x_k=Q_{k+1}Q_k$ and $y_k=2P_{k+1}P_k$ are consecutive integer lengths of the primitive Pythagorean triple, then (u_k, v_k) is a solution of the Pell equation $u^2-2v^2=\pm 1$, where $u_k=m_k-n_k=Q_k$ and $v_k=n_k=P_k$.

Conversely, suppose $(u_k, v_k)=(Q_k, P_k)$ is a solution the Pell equation $u^2-2v^2=\pm 1$: $Q_k^2-2P_k^2=\pm 1$. Then $m_k-n_k=Q_k, n_k=P_k, m_k=n_k+Q_k=P_k+Q_k=P_{k+1}$, and $m_k+n_k=P_{k+1}+P_k=Q_{k+1}$. So

$$\begin{aligned} x_k &= m_k^2-n_k^2 \\ &= (m_k-n_k)(m_k+n_k) \\ &= Q_kQ_{k+1}, \end{aligned}$$

and $y_k=2m_kn_k=2P_kP_{k+1}$. Then $x_k-y_k=Q_kQ_{k+1}-2P_kP_{k+1}=(-1)^k$, by identity (40); that is, $|x_k-y_k|=1$. In other words, x_k and y_k are consecutive integral lengths, as we observed earlier.

The next example shows a totally unexpected occurrence of Pell–Lucas numbers. It was proposed as a problem in 1963 by Leo Moser of the University of Alberta, Alberta, Canada [168]. The proof given here is based on one given in the following year by Henry W. Gould of West Virginia University, Morgantown, West Virginia [95]. It employs the binomial expansion in Corollary 1.1.

Example 7.6 Prove that $\sum_{r=1}^{2n-1} \binom{4n-2}{2r} 2^{r-1}$ is a square.

Proof. Since $(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$, we have

$$(1+x)^{4n-2} + (1-x)^{4n-2} = 2\sum_{r=0}^{2n-1} \binom{4n-2}{2r} x^{2r}.$$

Letting $x = \sqrt{2}$, this sum yields

$$\begin{aligned}
\sum_{r=0}^{2n-1} \binom{4n-2}{2r} 2^r &= Q_{4n-2} \\
\sum_{r=1}^{2n-1} \binom{4n-2}{2r} 2^r &= Q_{4n-2} - 1 \\
\sum_{r=1}^{2n-1} \binom{4n-2}{2r} 2^{r-1} &= \frac{Q_{4n-2} - 1}{2} \\
&= Q_{2n-1}^2,
\end{aligned}$$

where we have used identity (32). Thus, not only that the given sum is a square, it is the square of the Pell–Lucas number Q_{2n-1}. (We will revisit this result in Chapter 11.) ■

For example, let $n = 4$. Then

$$\sum_{r=1}^{2n-1} \binom{14}{2r} 2^{r-1} = 57,121 = 239^2 = Q_7^2.$$

Next we investigate the square of a Pell Sum, studied by Díaz-Barerro of Barcelona, Spain, in 2007 [67].

7.12 Square of a Pell Sum

Let a and b be any real numbers. Then $a^4 + b^4 + (a+b)^4 = 2(a^2 + ab + b^2)^2$; this can be confirmed algebraically.

This algebraic identity has an interesting application to the Pell family. To see this, let $\{x_n\}$ be an integer sequence satisfying the Pell recurrence. Suppose we let $a = x_n$ and $b = 2x_{n+1}$. Then

$$x_n^4 + 16x_{n+1}^4 + x_{n+2}^4 = 2(x_n^2 + 2x_n x_{n+1} + 4x_{n+1}^2)^2.$$

In particular, this gives the following Pell identities:

$$P_n^4 + 16P_{n+1}^4 + P_{n+2}^4 = 2(P_n^2 + 2P_nP_{n+1} + 4P_{n+1}^2)^2$$

$$Q_n^4 + 16Q_{n+1}^4 + Q_{n+2}^4 = 2(Q_n^2 + 2Q_nQ_{n+1} + 4Q_{n+1}^2)^2.$$

For example, let $n = 5$. Then

$$\begin{aligned} P_5^4 + 16P_6^4 + P_7^4 &= 29^4 + 16\cdot 70^4 + 169^4 \\ &= 1,200,598,002 \\ &= 2(29^2 + 2\cdot 29\cdot 70 + 4\cdot 70^2)^2 \\ &= 2(P_5^2 + 2P_5P_6 + 4P_6^2)^2. \end{aligned}$$

Likewise, $Q_5^4+16Q_6^4+Q_7^4 = 4,802,588,018 = 2(49003)^2 = 2(41^2+2\cdot 41\cdot 99+4\cdot 239^2)^2 = 2(Q_5^2 + 2Q_5Q_6 + 4Q_7^2)^2$.

To digress a bit, suppose we let $a = F_n$ and $b = F_{n+1}$. Since $F_n^2 + F_{n+1}^2 = F_{2n+1}$ and $L_n^2 + L_{n+1}^2 = 5L_{2n+1}$, the identity yields the following Fibonacci and Lucas identities:

$$\begin{aligned} F_n^4 + F_{n+1}^4 + F_{n+2}^4 &= 2(F_nF_{n+1} + F_{2n+1})^2 \\ L_n^4 + L_{n+1}^4 + L_{n+2}^4 &= 2(L_nL_{n+1} + 5L_{2n+1})^2. \end{aligned}$$

The next example presents a congruence, developed by R. Fecke of North Texas State University, Denton, Texas, in 1973 [83]. It shows that the sum of every three consecutive Pell numbers P_k with weights 2^k is always divisible by 5. We will establish it using the strong version of PMI.

Example 7.7 Prove that $\sum_{k=1}^{n+2} 2^k P_k \equiv 0 \pmod 5$, where n is any positive integer.

Proof (by PMI). Since $2P_1 + 4P_2 + 8P_3 = 1\cdot 1 + 4\cdot 2 + 8\cdot 5 = 50 \equiv 0 \pmod 5$ and $4P_2 + 8P_3 + 16P_4 = 4\cdot 2 + 8\cdot 5 + 16\cdot 12 = 240 \equiv 0 \pmod 5$, the congruence is true when $n = 1$ and $n = 2$.

Assume it is true for all positive integers $< n$. Then

$$\begin{aligned} \sum_{k=1}^{n+2} 2^k P_k &= 2^n P_n + 2^{n+1}P_{n+1} + 2^{n+2}P_{n+2} \\ &= 4\left(2^{n-2}P_n + 2^{n-1}P_{n+1} + 2^nP_{n+2}\right) \\ &= 4\left[2^{n-2}(2P_{n-1} + P_{n-2}) + 2^{n-1}(2P_n + P_{n+1}) + 2^n(2P_{n+} + P_n)\right] \\ &= 4\left(2^{n-1}P_{n-1} + 2^nP_n + 2^{n+1}P_{n+1}\right) + 4\left(2^{n-2}P_{n-2} + 2^{n-1}P_{n-1} + 2^nP_n\right) \\ &= 4\cdot 0 + 4\cdot 0 \pmod 5, \text{ by the inductive hypothesis} \\ &= 0 \pmod 5. \end{aligned}$$

Consequently, it follows by PMI that the congruence is true for every integer $n \geq 1$. ■

In particular, let $n = 5$. Then

$$\begin{aligned} 2^5 P_5 + 2^6 P_6 + 2^7 P_7 &= 2^5 \cdot 29 + 2^6 \cdot 70 + 2^7 \cdot 169 \\ &= 2 \cdot (-1) + (-1) \cdot 0 + 3 \cdot (-1) \pmod 5 \\ &= 0 \pmod 5. \end{aligned}$$

7.13 The Recurrence $x_{n+2} = 6x_{n+1} - x_n + 2$ Revisited

Next we pursue a close relationship between the recurrence $x_{n+2} = 6x_{n+1} - x_n + 2$ that we studied in Chapter 6, and the Pell family. It was discovered by M.N. Deshpande of Nagpur, India, in 2007 [64]. The proof presented is based on the one by H.-E. Seiffert of Berlin, Germany, a prolific problem proposer and solver [221].

Example 7.8 Consider the integer sequence $\{x_n\}$, defined by $x_{n+2} = 6x_{n+1} - x_n + 2$, where $x_1 = 1, x_2 = 10$, and $n \geq 1$. Prove that $\dfrac{8x_n(x_n + 1) + 20}{(P_{2n} - P_{2n-2})^2} = 9$.

Proof. It follows by the Pell recurrence that $P_{2n+4} = 6P_{2n+2} - P_{2n}$ and $Q_{2n+4} = 6Q_{2n+2} - Q_{2n}$; see Exercise 49.

Let $a_n = 3P_{2n} - \frac{3}{2}Q_{2n} - \frac{1}{2}$. Then $a_1 = 3P_2 - \frac{3}{2}Q_2 - \frac{3}{2} = 3 \cdot 2 - \frac{1}{2} \cdot 3 - \frac{1}{2} = 1 = x_1$, and $a_2 = 3P_4 - \frac{3}{2}Q_4 - \frac{1}{2} = 3 \cdot 12 - \frac{3}{2} \cdot 17 - \frac{1}{2} = 10 = x_2$. Furthermore, we have

$$\begin{aligned} 6a_{n+1} - a_n + 2 &= 6\left(3P_{2n+2} - \frac{3}{2}Q_{2n+2} - \frac{1}{2}\right) - \left(3P_{2n} - \frac{3}{2}Q_{2n} - \frac{1}{2}\right) + 2 \\ &= 3(6P_{2n+2} - P_{2n}) - \frac{3}{2}(6Q_{2n+2} - Q_{2n}) - \frac{1}{2} \\ &= 3P_{2n+4} - \frac{3}{2}Q_{2n+4} - \frac{1}{2} \\ &= a_{n+2}. \end{aligned}$$

Consequently, a_n satisfies the same recursive definition as x_n; so $a_n = x_n$. Thus, $x_n = 3P_{2n} - \frac{3}{2}Q_{2n} - \frac{1}{2}$.

Then, by identities (2) and (31), we have

$$\begin{aligned} 8x_n(x_n + 1) &= \frac{8}{4}(6P_{2n} - 3Q_{2n} - 1)(6P_{2n} - 3Q_{2n} + 1) \\ &= 2\left[9(2P_{2n} - Q_{2n})^2 - 1\right] \\ 8x_n(x_n + 1) + 20 &= 2\left[9(2P_{2n} - Q_{2n})^2 + 9\right] \\ &= 18\left[(2P_{2n} - Q_{2n})^2 + 1\right] \end{aligned}$$

$$= 18\left(Q_{2n-1}^2 + 1\right)$$
$$= 18 \cdot 2P_{2n-1}^2.$$

Consequently,

$$\frac{8x_n(x_n+1)+20}{(P_{2n}-P_{2n-2})^2} = \frac{18 \cdot 2P_{2n-1}^2}{4P_{2n-1}^2} = 9, \quad \text{as desired.}$$

■

For a specific case, let $n = 5$. Since $x_5 = 2089$, we have

$$\frac{8x_5(x_5+1)+20}{(P_{10}-P_8)^2} = \frac{8 \cdot 2089 \cdot 2090 + 20}{(2378-408)^2} = \frac{99207716}{3880900} = 9.$$

As a byproduct of this example, it follows that $2x_n(x_n+1)+5$ is a square for every $n \geq 1$. For instance, $2x_6(x_6+1)+5 = 2 \cdot 12178 \cdot 12179 + 5 = 296,631,729 = 17,233^2$.

In this example, we found that $x_n = 3P_{2n} - \frac{3}{2}Q_{2n} - \frac{1}{2}$ satisfies the recurrence $x_{n+2} = 6x_{n+1} - x_n + 2$. Interestingly enough, $y_n = 3Q_{2n} - \frac{3}{2}P_{2n}$ satisfies the homogeneous portion of this recurrence: $y_{n+2} = 6y_{n+1} - y_n$. We encountered this recurrence also in Chapter 6.

As before, this can be established as follows:

$$\begin{aligned} 6y_{n+1} - y_n &= 6\left(3Q_{2n+2} - \frac{3}{2}P_{2n+2}\right) - \left(3Q_{2n} - \frac{3}{2}P_{2n}\right) \\ &= 3\left(6Q_{2n+2} - Q_{2n}\right) - \frac{3}{2}\left(6P_{2n+2} - P_{2n}\right) \\ &= 3Q_{2n+4} - \frac{3}{2}P_{2n+4} \\ &= y_{n+2}. \end{aligned}$$

To digress a bit, note that the first five elements of the sequence $\{y_n\}$ are 6, 33, 192, 1119, and 6522. Using the recurrence, we could define $y_0 = 3$.

Clearly, $3|y_n$ for every $n \geq 0$. It follows from the recurrence that $y_{n+2} \equiv -y_n \pmod 6$. So $6|y_{n+2}$ if and only if $6|y_n$. Since $6|y_1$, it follows by PMI that $6|y_{2n+1}$ for every $n \geq 0$.

Recall from Chapter 6 that $y_n = Ar^n + Bs^n$, where $r = 3+2\sqrt{2} = \gamma^2$ and $s = 3-2\sqrt{2} = \delta^2$. Using the initial conditions, we get $A = \frac{3(11-8\sqrt{2})}{4(4-3\sqrt{2})}$ and $B = \frac{3(5-4\sqrt{2})}{4(4-3\sqrt{2})}$. Thus

$$y_n = \frac{3}{4(4-3\sqrt{2})}\left[(11-8\sqrt{2})\gamma^{2n} + (5-4\sqrt{2})\delta^{2n}\right],$$

where $n \geq 0$.

It is well known that the ratios of consecutive Fibonacci and Lucas numbers approach the golden ratio: $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{L_{n+1}}{L_n} = \alpha = \frac{1+\sqrt{5}}{2}$. We will now explore the ratios of consecutive Pell and Pell–Lucas numbers.

7.14 Ratios of Consecutive Pell and Pell–Lucas Numbers

Let $\lim_{n\to\infty} \dfrac{P_{n+1}}{P_n} = \lambda$. Since $P_{n+1} = 2P_n + P_{n-1}$, it follows that

$$\begin{aligned} \frac{P_{n+1}}{P_n} &= 2 + \frac{1}{P_n/P_{n-1}} \\ \lim_{n\to\infty} \frac{P_{n+1}}{P_n} &= 2 + \frac{1}{\lim_{n\to\infty} (P_n/P_{n-1})} \\ \lambda &= 2 + \frac{1}{\lambda}. \end{aligned}$$

So $\lambda^2 - 2\lambda - 1 = 0$. Solving this, we get $\lambda = 1 \pm \sqrt{2}$. Since $\lambda > 0$, it follows that $\lambda = \gamma$. Thus $\lim_{n\to\infty} \dfrac{P_{n+1}}{P_n} = \gamma$. Similarly, $\lim_{n\to\infty} \dfrac{Q_{n+1}}{Q_n} = \gamma$. Thus the ratios of consecutive Pell and Pell–Lucas numbers approach the irrational number $\gamma \approx 2.4142135624$. Consequently, $\frac{P_{n+1}}{P_n} - 1$ and $\frac{Q_{n+1}}{Q_n} - 1$ are good approximations of $\sqrt{2}$, as n gets larger and larger.

For example, $\frac{P_{12}}{P_{11}} - 1 = \frac{13860}{5741} - 1 \approx 1.4142135517 \approx \sqrt{2}$ and $\frac{Q_{12}}{Q_{11}} - 1 = \frac{19601}{8119} - 1 \approx 1.4142135731 \approx \sqrt{2}$.

Next we investigate a close relationship between polygonal numbers and the Pell family. First, we look for Pell numbers that are also triangular numbers.

7.15 Triangular Pell Numbers

Clearly, $P_1 = 1 = t_1$ is a triangular number. Are there others? If there are, how many are there? How do we find them? We will answer these questions shortly.

Recall from Chapter 5 that a positive integer N is a triangular number if and only if $8N + 1$ is a square. So P_n is a triangular number if and only if $8P_n + 1$ is a square > 1. Consequently, it suffices to find those positive integers n such that $8P_n + 1$ is a square.

In 1996, W.L. McDaniel of the University of Missouri at St. Louis proved that $P_1 = 1$ is the only triangular Pell number [167]. His proof is based on the following identities and four lemmas:

$$\begin{aligned} P_{-n} &= (-1)^{n+1} P_n, \quad Q_{-n} = (-1)^n Q_n \\ P_{m+n} &= P_m P_{n+1} + P_{m-1} P_n && (7.13) \\ &= 2P_m Q_n - (-1)^n P_{m-n} && (7.14) \\ P_{2^t n} &= P_n(2Q_n)(2Q_{2n})(2Q_{4n})\cdots(2Q_{2^{t-1}n}) \end{aligned}$$

$$
\begin{aligned}
Q_n^2 &= 2P_n^2 + (-1)^n \\
Q_{2n} &= 2Q_n^2 - (-1)^n \\
&= 4P_n^2 + (-1)^n \qquad (7.15) \\
(P_m, Q_n) &= \begin{cases} Q_{(m,n)} & \text{if } \frac{m}{(m,n)} \text{ is even} \\ 1 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Suppose $t \geq 2$. Then, by identity (7.15), $Q_{2^t} = 4P_{2^{t-1}}^2 + 1$. Since $P_{2n} \equiv 0 \pmod 2$, this implies that $Q_{2^t} \equiv 1 \pmod 8$, when $t \geq 2$.

Using identity (7.14), we will now prove the following results.

Lemma 7.1 *Let n, k, and t be any nonnegative integers, and g any odd positive integer. Then (1) $P_{n+2kt} \equiv (-1)^{t(k+1)} P_n \pmod{Q_k}$ and (2) $P_{2kg} \equiv (-1)^{(g-1)/2} P_{2k} \pmod{Q_{2k}}$.*

Proof. (1) We will prove this by induction on t. The congruence is clearly true when $t = 0$. Since $P_{n+2k} = P_{(n+k)+k} = 2P_{n+k}Q_k - (-1)^k P_n \equiv (-1)^{k+1} P_n \pmod{Q_k}$, it is also true when $t = 1$.

Now assume it is true for all nonnegative integers $\leq t$. Then

$$
\begin{aligned}
P_{n+2k(t+1)} &= P_{(n+2kt)+2k} \\
&= 2P_{n+2kt}Q_{2k} - (-1)^{2k} P_{n+2k(t-1)} \\
&\equiv 2(-1)^{t(k+1)} P_n \left[2Q_k^2 - (-1)^k\right] - (-1)^{(t-1)(k+1)} P_n \pmod{Q_k} \\
&\equiv \left[2(-1)^{(t+1)(k+1)} - (-1)^{(t+1)(k+1)}\right] P_n \pmod{Q_k} \\
&\equiv (-1)^{(t+1)(k+1)} P_n \pmod{Q_k}.
\end{aligned}
$$

So the congruence is also true for $t + 1$. Thus, the result follows by the strong version of PMI.

(2) By identity (7.14), we have

$$
\begin{aligned}
P_{2kg} &= P_{2k(g-1)+2k} \\
&= 2P_{2k(g-1)}Q_{2k} - (-1)^{2k} P_{2k(g-2)} \\
&\equiv (-1)^1 P_{2k(g-2\cdot 1)} \pmod{Q_{2k}} \\
&\equiv (-1)^2 P_{2k(g-2\cdot 2)} \pmod{Q_{2k}} \\
&\equiv (-1)^3 P_{2k(g-2\cdot 3)} \pmod{Q_{2k}},
\end{aligned}
$$

and so on. More generally, $P_{2kg} \equiv (-1)^r P_{2k(g-2r)} \pmod{Q_{2k}}$. This process can be continued until $g - 2r = 1$. This yields $P_{2kg} \equiv (-1)^{(g-1)/2} P_{2k} \pmod{Q_{2k}}$, as desired. ∎

For example, let $n = 3 = t$ and $k = 2$. Then

(1) $P_{n+2kt} = P_{15} = 195{,}025 \equiv 1 \equiv (-1)^{3\cdot 3}5 \pmod{3}$.

(2) In addition, let $g = 7$. Then $P_{2k} = P_4 = 12$, $Q_{2k} = Q_4 = 17$, and $P_{2kg} = P_{28} = 18{,}457{,}556{,}052 \equiv 5 \equiv (-1)^{(7-1)/2}12 \pmod{17}$.

The proofs of the next two lemmas involve the *Jacobi symbol* $\left(\frac{a}{m}\right)$ [130]; so we omit their proofs in the interest of brevity.

Lemma 7.2 *Let $k = 2^t$, where $t \geq 1$. Then* $\left(\frac{8P_{2k}+1}{Q_{2k}}\right) = \left(\frac{-8P_{2k}+1}{Q_{2k}}\right)$. ▪

Lemma 7.3 *Let $k = 2^t$, where $t \geq 2$. Then* $\left(\frac{8P_k+Q_k}{33}\right) = -1$*; that is,* $\frac{8P_k+Q_k}{33}$ *is not a square.* ▪

For example, $8P_8 + Q_8 = 8 \cdot 408 + 577 = 3841$ is not a square.

Lemma 7.4 *If $n \equiv m \pmod{24}$, then $P_n \equiv P_m \pmod{9}$.*

Proof. Since $P_{24} = 543{,}339{,}720 \equiv 0 \pmod{9}$ and $P_{25} = 1{,}311{,}738{,}121 \equiv 1 \pmod{9}$, by identity (7.13), $P_{n+24} = P_nP_{25} + P_{n-1}P_{24} \equiv P_n + 0 \equiv P_n \pmod{9}$. By PMI, the desired result now follows. ▪

For example, let $n = 23$ and $m = 3$. Then $P_{27} = 7{,}645{,}370{,}045 \equiv 5 \equiv P_3 \pmod{9}$.

Using these lemmas and the Jacobi symbol, McDaniel proved that $8P_n + 1$ is *not* a square if $n > 1$. Since $8P_1 + 1 = 9$ is a square, it follows that $P_1 = 1$ is the only triangular Pell number.

Next we investigate Pell and Pell–Lucas numbers that are also pentagonal numbers. But before we do, we will give a very brief introduction to pentagonal numbers.

7.16 Pentagonal Numbers

Just like triangular numbers, *pentagonal numbers* p_n are polygonal numbers. They are positive integers that can be represented geometrically by regular pentagons. The first ten pentagonal numbers are 1, 5, 12, 22, 35, 51, 70, 92, 117, and 145. Figure 7.3 shows the geometric representations of the first four pentagonal numbers.

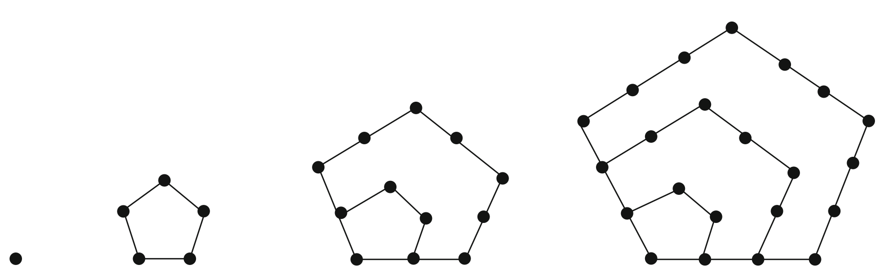

Figure 7.3.

Pentagonal numbers can also be defined recursively:

$$\begin{aligned} p_1 &= 1 \\ p_n &= p_{n-1} + 3(n-1) + 1, \quad n \geq 2. \end{aligned}$$

See Figure 7.4. Clearly, this recurrence can be rewritten as $p_n = p_{n-1} + 3n - 2$. Explicitly, $p_n = n(3n-1)/2$, where $n \geq 1$; this can be confirmed easily using PMI.

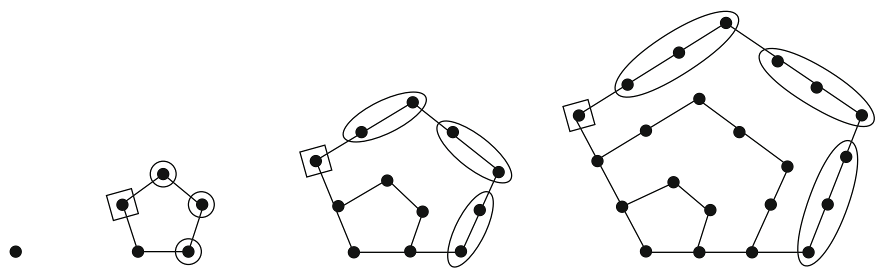

Figure 7.4.

For example, $p_7 = p_6 + 19 = 51 + 19 = 70$. Likewise, $p_8 = 92$.

We are now ready to search for Pell numbers that are also pentagonal.

7.17 Pentagonal Pell Numbers

Four of the first ten pentagonal numbers are also Pell numbers: (1), (5), (12), and (70). Are there others? If there are, how many are there? How do we find them? We will answer these questions shortly.

In 2002, V.S.R. Prasad and B.S. Rao of Osmania University, Hyderbad, India, established that 1, 5, 12, and 70 are the only pentagonal Pell numbers. The proof is fairly long and involves the Jacobi symbol, so in the interest of brevity, we will highlight just the key steps [178].

To begin with, we need a few additional properties. First, notice that identity (32) can be generalized:

$$Q_{m+n} = 2Q_m Q_n - (-1)^n Q_{m-n}. \tag{7.16}$$

This identity, discovered by Prasad and Rao, follows by the Binet-like formula for Q_k.

It follows by this identity, and identities (31) and (32) that

$$\begin{aligned} Q_{3n} &= 2Q_{2n}Q_n - (-1)^n Q_n \\ &= 2Q_n[2Q_n^2 - (-1)^n] - (-1)^n Q_n \\ &= Q_n^3 + 3Q_n[Q_n^2 - (-1)^n] \\ &= Q_n(Q_n^2 + 6P_n^2). \end{aligned} \tag{7.17}$$

The sequence $\{Q_{2k+1} \pmod 8\}$ is periodic: $\underbrace{1\ 7}\ \underbrace{1\ 7}\ldots$; $\{Q_{6k+3} \pmod 8\}$ is also periodic: $\underbrace{7\ 1}\ \underbrace{7\ 1}\ldots$. It now follows by equation (7.17) that

$$Q_m^2 + 6P_m^2 \equiv 7 \pmod 8, \tag{7.18}$$

where m is odd. When m is odd, we also have

$$P_m \equiv 1 \pmod 4 \quad \text{and} \quad Q_m \equiv \begin{cases} 1 \pmod 4 & \text{if } m \equiv 1 \pmod 4 \\ -1 \pmod 4 & \text{if } m \equiv 3 \pmod 4. \end{cases} \tag{7.19}$$

The identity $P_{2m} = 2P_mQ_m$ also can be generalized:

$$P_{m+n} = 2P_mQ_n - (-1)^n P_{m-n}. \tag{7.20}$$

This also follows by the Binet-like formula for P_k.

This identity has an interesting consequence, as the following lemma shows.

Lemma 7.5 *Let n, k, and m be nonnegative integers. Then* $P_{n+2km} \equiv (-1)^{(k+1)m}P_n \pmod{Q_k}$.

Proof (by PMI on m). Clearly, the congruence holds when $m = 0$. By identity (7.20), $P_{n+2k} = P_{(n+k)+k} = 2P_{n+k}Q_k - (-1)^k P_{(n+k)-k} \equiv (-1)^{k+1}P_n \pmod{Q_k}$. so it also true when $m = 1$.

Now assume that the congruence holds for all nonnegative integers $\le m$, where $m \ge 1$. Then, by identity (7.20) and the inductive hypothesis, we have

$$\begin{aligned} P_{n+2k(m+1)} &= P_{(n+2km)+2k} \\ &= 2P_{n+2km}Q_{2k} - (-1)^{2k}P_{n+2k(m-1)} \\ &\equiv 2(-1)^{(k+1)m}P_nQ_{2k} - (-1)^{(k+1)(m-1)}P_n \pmod{Q_k} \\ &\equiv 2(-1)^{(k+1)m}P_n(-1)^{k+1} - (-1)^{(k+1)(m-1)}P_n \pmod{Q_k}, \text{ by identity (32)} \\ &\equiv \left[2(-1)^{(k+1)(m+1)} - (-1)^{(k+1)(m+1)}\right]P_n \pmod{Q_k} \\ &\equiv (-1)^{(k+1)(m+1)}P_n \pmod{Q_k}. \end{aligned}$$

So the result holds for $m+1$ also.

Thus, by the strong version of PMI, the congruence holds for every $m \ge 0$. ∎

For example, let $n = 3, k = 4$, and $m = 3$. Then $P_3 = 5$, $Q_k = Q_4 = 17$, $P_{n+2km} = P_{27} = 7,645,370,045 \equiv 12 \equiv (-1)^{5\cdot 3}5 \pmod{17}$.

It follows from the explicit formula for pentagonal numbers that P_n is a pentagonal number $m(3m-1)/2$ if and only if $24P_n + 1 = (6m-1)^2$. Consequently, we need to identify those positive integers n such that $24P_n + 1$ is a square.

The following eight lemmas will pave the way for identifying such integers. The Jacobi symbol plays a pivotal role in their proofs.

Lemma 7.6 *Let* $n \equiv \pm 1 \pmod{2^2 \cdot 5}$. *Then* $24P_n + 1$ *is a square if and only if* $n = \pm 1$. ■

Lemma 7.7 *Let* $n \equiv \pm 3 \pmod{2^4}$. *Then* $24P_n + 1$ *is a square if and only if* $n = \pm 3$. ■

Lemma 7.8 *Let* $n \equiv 4 \pmod{2^2 \cdot 5}$. *Then* $24P_n + 1$ *is a square if and only if* $n = 4$. ■

Lemma 7.9 *Let* $n \equiv 2 \pmod{2^2 \cdot 5 \cdot 7}$. *Then* $24P_n + 1$ *is a square if and only if* $n = 2$. ■

Lemma 7.10 *Let* $n \equiv 6 \pmod{2^2 \cdot 3 \cdot 5 \cdot 7}$. *Then* $24P_n + 1$ *is a square if and only if* $n = 6$. ■

Lemma 7.11 *Let* $n \equiv 0 \pmod{2 \cdot 3 \cdot 7^2 \cdot 13}$. *Then* $24P_n + 1$ *is a square if and only if* $n = 0$. ■

Lemma 7.12 *Let* $n \equiv 0, \pm 1, 2, \pm 3, 4, or\ 6 \pmod{2^4 \cdot 3 \cdot 5 \cdot 7^2 \cdot 13}$. *Then* $24P_n + 1$ *is a square if and only if* $n = 0, \pm 1, 2, \pm 3, 4, or, 6$. ■

Lemma 7.13 *Suppose* $n \not\equiv 0, \pm 1, 2, \pm 3, 4, or\ 6 \pmod{2^4 \cdot 3 \cdot 5 \cdot 7^2 \cdot 13}$. *Then* $24P_n + 1$ *is not a square.* ■

Tying all the pieces together, we get the following delightful result.

Theorem 7.2 (Prasad and Rao, 2002) P_n *is a pentagonal number if and only if* $n = 1, 3, 4,$ *or* 6.

Proof. This follows by Lemmas 7.12 and 7.13. ■

7.18 Zeitlin's Identity

Recall that identity (7.20), and hence Lemma 7.5, played an important role in identifying pentagonal Pell numbers. As an added bonus, the identity can be used to develop another Pell identity, discovered by D. Zeitlin of Minneapolis, Minnesota, in 1996 [269]. The following example features it. The proof is based on the one given by Seiffert [209] in the following year.

Example 7.9 Find integers a, b, c, and d such that $P_n = P_{n-a} + 444P_{n-b} + P_{n-c} + P_{n-d}$, where $1 < a < b < c < d$.

Proof. By identity (7.20), we have $P_{r+s} = 2P_rQ_s - (-1)^s P_{r-s}$, where r and s are arbitrary integers. Using $r+s = m$, this can be rewritten as $P_m = 2P_{m-s}Q_s - (-1)^s P_{m-2s}$. Consequently, we have

$$\begin{aligned} P_n &= P_{(n-b)+b} = 2P_{n-b}Q_b - (-1)^b P_{n-2b} \\ P_{n-b+k} &= P_{(n-b)+k} = 2P_{n-b}Q_k - (-1)^k P_{n-b-k}. \end{aligned}$$

Subtracting, we get

$$P_n - P_{n-b+k} = 2P_{n-b}(Q_b - Q_k) - (-1)^b P_{n-2b} + (-1)^k P_{n-b-k}.$$

Choosing b to be odd and k even, this yields

$$P_n = P_{n-b+k} + 2P_{n-b}(Q_b - Q_k) + P_{n-b-k} + P_{n-2b}.$$

Now we need to make a clever choice for b and k. Since we want $Q_b - Q_k = 222 = 239 - 17$, we choose $b = 7$ and $k = 4$. We then let $a = b - k = 3$, $c = b + k = 11$, and $2b = 14$. Thus

$$P_n = P_{n-3} + 444P_{n-7} + P_{n-11} + P_{n-14}. \tag{7.21}$$

■

For example, let $n = 17$. Then

$$\begin{aligned} P_{14} + 444P_{10} + P_6 + P_3 &= 80782 + 444 \cdot 2378 + 70 + 5 \\ &= 1,136,689 \\ &= P_{17}. \end{aligned}$$

Next we investigate Pell–Lucas numbers that are also pentagonal.

7.19 Pentagonal Pell–Lucas Numbers

Returning to identity (7.16), it also has an interesting consequence, as the following lemma shows [179].

Lemma 7.14 (Prasad and Rao, 2001). *Let n and k be nonnegative integers, and m an even integer. Then $Q_{n+2km} \equiv (-1)^k Q_n \pmod{Q_m}$.*

Proof (by PMI on k). Clearly, the congruence holds when $k = 0$. By identity (7.16), $Q_{n+2m} = Q_{(n+m)+m} = 2Q_{n+m}Q_m - (-1)^m Q_n$; so the congruence is true when $k = 1$ also, since m is even.

Assume the result holds for all nonnegative integers $\leq k$, where $k \geq 1$. Then, by identity (7.16), we have

$$\begin{aligned} Q_{n+2(k+1)m} &= 2Q_{n+2km}Q_{2m} - Q_{n+2(k-1)m} \\ &\equiv 2(-1)^k Q_n Q_{2m} - (-1)^{k-1} Q_n \pmod{Q_m} \\ &\equiv (-1)^k (2Q_{2m} + 1) Q_n \pmod{Q_m} \end{aligned}$$

$$\equiv (-1)^k(-1)Q_n \pmod{Q_m}, \text{ by identity (32)}$$
$$\equiv (-1)^{k+1}Q_n \pmod{Q_m}.$$

So the result also holds for $k+1$.

Thus, by the strong version of PMI, the congruence holds for every $k \geq 0$. ∎

For example, let $n = 2, k = 3$, and $m = 4$. Then $Q_n = Q_5 = 41, Q_m = Q_4 = 17$, and $Q_{n+2km} = Q_{26} = 4,478,554,083 \equiv 14 \equiv -3 \equiv (-1)^3 Q_2 \pmod{Q_4}$, as expected.

As before, Q_n is a pentagonal number $m(3m-1)/2$ if and only if $24Q_n + 1$ is a square, where $n \geq 1$. We will show that $24Q_n + 1$ is a square only when $n = 1$ or 3. Its proof hinges on the next four lemmas; again, we omit their proofs in the interest of brevity [179].

Lemma 7.15 *Suppose* $n \equiv 0$ *or* $1 \pmod{2^2 \cdot 3^2}$. *Then* $24Q_n + 1$ *is a square if and only if* $n = 0$ *or* 1. ∎

Lemma 7.16 *Suppose* $n \equiv 3 \pmod{2^2 \cdot 3^2 \cdot 7}$. *Then* $24Q_n + 1$ *is a square if and only if* $n = 3$. ∎

Lemma 7.17 *Suppose* $n \equiv 0, 1$ *or* $3 \pmod{2^3 \cdot 3^2 \cdot 5 \cdot 7}$. *Then* $24Q_n + 1$ *is a square only if* $n = 0, 1,$ *or* 3. ∎

Lemma 7.18 $24Q_n + 1$ *is* not *a square if* $n \not\equiv 0, 1,$ *or* $3 \pmod{2^3 \cdot 3^2 \cdot 5 \cdot 7}$. ∎

With these tools, we can now establish the desired result.

Theorem 7.3 *Q_1 is the only pentagonal Pell–Lucas number.*

Proof. It follows by Lemmas 7.17 and 7.18 that $24Q_n + 1$ is a square only when $n = 1$ or 3. But $Q_3 = 7$ is not pentagonal. But $Q_1 = 1$ is pentagonal. Thus, Q_1 is the only pentagonal number. ∎

7.20 Heptagonal Pell Numbers

Next we investigate Pell numbers that are also heptagonal. A *heptagonal number* is a positive integer of the form $\frac{m(5m-3)}{2}$, where m is a positive integer. The first six heptagonal numbers are 1, 7, 18, 34, 55, and 81. Like triangular and pentagonal numbers, they also can be represented geometrically; see Figure 7.5.

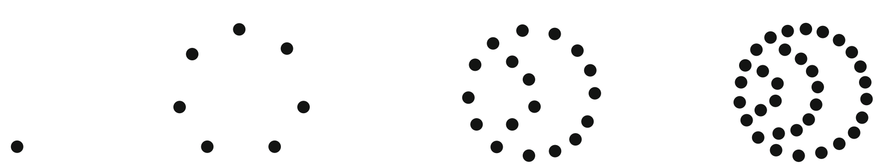

Figure 7.5.

A positive integer N is heptagonal if and only if $N = \frac{m(5m-3)}{2}$. Then $2N = 5m^2 - 3m$; so $40N + 9 = 100m^2 - 60m + 9 = (10m - 3)^2$. Thus N is heptagonal if and only if $40N + 9$ is a positive square. Consequently, P_n is heptagonal if and only if $40P_n + 9$ is a square > 1. Clearly, $P_1 = 1$ is heptagonal. Are there others? In 2005 Rao established that P_1 is the only such number. His proof employs three lemmas and the following fundamental properties, some of which we have already seen [181]:

$$Q_n^2 = 2P_n^2 + (-1)^n$$
$$Q_{3n} = Q_n(Q_n^2 + 6P_n^2)$$
$$P_{m+n} = 2P_m Q_n - (-1)^n P_{m-n}$$
$$P_{n+2kt} \equiv (-1)^{t(k+1)} P_n \pmod{Q_k}$$
$$2|P_n \text{ if and only if } 2|n\text{, and } 2 \nmid Q_n \text{ for any } n.$$
$$3|P_n \text{ if and only if } 4|n\text{, and } 3|Q_n \text{ if and only if } n \equiv 2 \pmod 4.$$
$$5|P_n \text{ if and only if } 3|n\text{, and } 5 \nmid Q_n \text{ for any } n.$$
$$9|P_n \text{ if and only if } 12|n\text{, and } 9|Q_n \text{ if and only if } n \equiv 6 \pmod{12}.$$

Let n be odd. Then

- $Q_m \equiv \pm 1 \pmod 4$ according as $m \equiv \pm 1 \pmod 4$;
- $P_m \equiv 1 \pmod 4$; and
- $Q_m^2 + 6P_m^2 \equiv 7 \pmod 8$.

The proofs of the following lemmas require the Jacobi symbol, so we will omit them for the sake of brevity.

Lemma 7.19 *Suppose* $n \equiv \pm 1 \pmod{2^2 \cdot 5}$. *Then* $40P_n + 9$ *is a square if and only if* $n = \pm 1$. ■

Lemma 7.20 *Suppose* $n \equiv 6 \pmod{2^2 \cdot 5^3 \cdot 7^2}$. *Then* $40P_n + 9$ *is a square if and only if* $n = 6$. ■

When $n = 6$, notice that $40P_n + 9 = 40P_6 + 9 = 40 \cdot 70 + 9 = 53^2$, a square.

Lemma 7.21 *Suppose* $n \equiv 0 \pmod{2 \cdot 7 \cdot 5^3}$. *Then* $40P_n + 9$ *is a square if and only if* $n = 0$. ■

When $n = 0$, notice that $40P_n + 9 = 40P_0 + 9 = 40 \cdot 0 + 9 = 3^2$, again a square.

The following result follows from these three lemmas.

Corollary 7.3 *Suppose* $n \equiv 0, \pm 1,$ *or* $6 \pmod{2^2 \cdot 5^3 \cdot 7^2}$. *Then* $40P_n + 9$ *is a square if and only if* $n = 0, \pm 1,$ *or* 6. ■

We will need one more lemma.

Lemma 7.22 *If* $n \not\equiv 0, \pm 1,$ *or* $6 \pmod{2^2 \cdot 5^3 \cdot 7^2}$*. Then* $40P_n + 9$ *is* not *a square.* ■

For example, let $n = 17$. Then $40P_n + 9 = 40P_{17} + 9 = 40 \cdot 1136689 + 9 = 45,467,569$ is *not* a square.

With these tools, we can now prove that P_1 is the only heptagonal Pell number.

It follows by Corollary 7.3 and Lemma 7.22 that $40P_n + 9$ is a square if and only if $n = 0, \pm 1$, or 6; but -1 and 0 are not acceptable, since $n \geq 1$. When $n = 6, 40P_n + 9 = 40P_6 + 9 = 40 \cdot 70 + 9 = 53^2$; this must equal to $(10m - 3)^2$ for some $m \geq 1$; but $(10m - 3)^2 = 53^2$ implies that $m = -5$, which is not admissible. This leaves $n = 1$; so P_1 is the only heptagonal Pell number; see Table 7.5.

Table 7.5.

n	0	± 1	6
P_n	0	1	70
$40P_n + 9$	3^2	7^2	53^2
m	0	(1)	-5

This discourse has an interesting consequence for the theory of diophantine equations. To this end, consider the diophantine equation $2x^2 = y^2(5y - 3)^2 - 2$, which can be rewritten as $x^2 = 2\left[\frac{y(5y-3)}{2}\right]^2 - 1$. Its solutions are given by (Q_k, P_k), where k is odd. From Table 7.1, the only admissible value of y is 1. Correspondingly, $x = \pm 1$. Thus, $(\pm 1, 1)$ are the only two solutions of the diophantine equation $2x^2 = y^2(5y - 3)^2 - 2$.

Likewise, the diophantine equation $2x^2 = y^2(5y - 3)^2 + 2$ can be rewritten as $x^2 = 2\left[\frac{y(5y-3)}{2}\right]^2 + 1$; its solutions are given by (Q_k, P_k), where k is even. From Table 7.1, $n = 0$ or 6. The corresponding solutions are $(\pm 1, 0)$ and $(\pm 99, -5)$.

Exercises 7

Prove the following identities.

1. $P_n + P_{n-1} = Q_n$.
2. $Q_n + Q_{n-1} = 2P_n$.
3. $P_n + Q_n = P_{n+1}$.
4. $2P_n + Q_n = Q_{n+1}$.
5. $2Q_n + 3P_n = P_{n+2}$.
6. $3Q_n + 4P_n = Q_{n+2}$.
7. $Q_{n+1} - Q_n = 2P_n$.
8. $P_{n+1} + P_{n-1} = 2Q_n$.
9. $Q_{n+1} + Q_{n-1} = 4P_n$.

10. $P_n + P_{n+1} + P_{n+3} = 3P_{n+2}$. *Hint*: Use the Pell recurrence.
11. $Q_n + Q_{n+1} + Q_{n+3} = 3Q_{n+2}$.
12. $P_{n+1} - P_{n-1} = 2P_n$.
13. $Q_{n+1} - Q_{n-1} = 2Q_n$.
14. $P_{n+2} + P_{n-2} = 6P_n$.
15. $Q_{n+2} + Q_{n-2} = 6Q_n$.
16. $P_{n+2} - P_{n-2} = 4Q_n$. *Hint*: Use Exercise 8.
17. $Q_{n+2} - Q_{n-2} = 8P_n$.
18. $P_{n+1}^2 + P_n^2 = P_{2n+1}$.
19. $P_{m+n}^2 - (-1)^n P_m^2 = P_{2m+n}P_n$. *Hint*: Use the Binet-like formula.
20. $Q_{m+n}^2 - (-1)^n Q_m^2 = 2P_{2m+n}P_n$.
21. $Q_{n+1}^2 + Q_n^2 = 2P_{2n+1}$. *Hint*: Use identity (20).
22. $P_{n+1}^2 - P_n^2 = Q_{n+1}Q_n$. *Hint*: Use identity (3).
23. $Q_{n+1}^2 - Q_n^2 = 4P_{n+1}P_n$.
24. $4(P_n^2 + Q_n^2) = 3Q_{2n} + (-1)^n$. *Hint*: Use the Binet-like formulas.
25. $2P_n + Q_n = Q_{n+1}$.
26. $2P_n + Q_{n+2} = 3Q_{n+1}$. *Hint*: Use identity (25) and Pell recurrence.
27. $P_{n+1} + Q_{n-1} = 3P_n$.
28. $P_{2n} = 2P_nQ_n$.
29. $Q_n^2 + 2P_n^2 = Q_{2n}$. *Hint*: Use the Binet-like formulas.
30. $Q_n^2 - 2P_n^2 = (-1)^n$.
31. $Q_{2n} = 2Q_n^2 - (-1)^n$.
32. $Q_{2n} = 4P_n^2 + (-1)^n$.
33. $P_{n+1}P_{n-1} - P_n^2 = (-1)^n$. *Hint*: Use matrices, PMI, or the Binet-like formula.
34. $Q_{n+1}Q_{n-1} - Q_n^2 = 2(-1)^{n-1}$.
35. $P_nQ_{n-1} - Q_nP_{n-1} = (-1)^{n-1}$.
36. $P_nQ_{n-1} + Q_nP_{n-1} = P_{2n-1}$.
37. $2(P_{n+1}^2 + P_n^2) = Q_{2n} + Q_{2n+1}$.
38. $2(Q_{n+1}^2 + Q_n^2) = Q_{2n} + Q_{2n+2}$.
39. $Q_nQ_{n+1} - 2P_nP_{n+1} = (-1)^n$.
40. $P_nP_{n+3} - P_{n+1}P_{n+2} = 2(-1)^{n-1}$. *Hint*: Use the Binet-like formula for P_n and the fact that $\gamma^3 + \delta^3 = 14$.
41. $Q_nQ_{n+3} - Q_{n+1}Q_{n+2} = 4(-1)^n$.
42. $P_{n+3}^2 + P_n^2 = 5P_{2n+3}$. *Hint*: Use the Pell recurrence and identity (19).
43. $Q_{n+3}^2 + Q_n^2 = 10P_{2n+3}$.

44. $P_n Q_n = 6P_{n-1}Q_{n-1} - P_{n-2}Q_{n-2}$.
45. $Q_{2n} = 6Q_{2n-2} - Q_{2n-4}$.
46. $Q_n^2 = 6Q_{n-1}^2 - Q_{n-2}^2 + 4(-1)^n$.
47. $P_{3n} \equiv 0 \pmod 5$.
48. $2P_{n+1}Q_{n-1} + 2(-1)^n = P_{2n}$.
49. $P_{2n+4} = 6P_{2n+2} - P_{2n}$.
50. $P_{m+n} = 2P_m Q_n - (-1)^n P_{m-n}$.
51. $Q_{m+n} = 2Q_m Q_n - (-1)^n Q_{m-n}$.
52. The number of digits in P_n equals $\lceil n \log \gamma - 1.5 \log 2 \rceil$.
53. The number of digits in Q_n equals $\lceil n \log \gamma - \log 2 \rceil$.

8

Additional Pell Identities

8.1 Introduction

In the preceding chapter, we studied some fundamental identities of the Pell family. We now present some additional ones. Again, for the sake of brevity, we will prove only some of them, but will add some comments on others when needed. We will revisit some of these results in Chapter 11, when we study generating functions for the Pell family.

We begin our pursuit with a common recurrence for even-numbered Pell and Pell–Lucas numbers. To this end, suppose x_n satisfies the Pell recurrence. Then

$$\begin{aligned} 6x_{2n-2} - x_{2n-4} &= 5x_{2n-2} + (x_{2n-2} - x_{2n-4}) \\ &= 5x_{2n-2} + 2x_{2n-3} \\ &= 5x_{2n-2} + 2(x_{2n-1} - 2x_{2n-2}) \\ &= x_{2n-2} + 2x_{2n-1} \\ &= x_{2n}. \end{aligned}$$

In particular, we have

(1) $P_{2n} = 6P_{2n-2} - P_{2n-4}$.

(2) $Q_{2n} = 6Q_{2n-2} - Q_{2n-4}$.
For example, $P_{10} = 2378 = 6 \cdot 408 - 70 = 6P_8 - P_6$ and $Q_{10} = 239 = 6 \cdot 41 - 7 = 6Q_8 - Q_6$.
Both identities will enable us to develop their generating functions in Chapter 11.

The recurrence for Q_{2n} has an interesting consequence. To see this, recall from Chapter 6 that the subscript n_k of the square-triangular number y_k^2 is given by $n_k = \frac{1}{2}(Q_{2k} - 1)$. Consequently, we have

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$$
\begin{aligned}
6n_{k-1} - n_{k-2} + 2 &= \frac{6}{2}(Q_{2k-2} - 1) - \frac{1}{2}(Q_{2k-4} - 1) + 2 \\
&= \frac{1}{2}(6Q_{2k-2} - Q_{2k-4} - 1) \\
&= \frac{1}{2}(Q_{2k} - 1) = n_k,
\end{aligned}
$$

as found in Chapter 6.

(3) $P_n^2 = 6P_{n-1}^2 - P_{n-2}^2 - 2(-1)^n$.

This can be established using the recurrence and the Cassini-like formula for P_n:

$$
\begin{aligned}
P_n^2 &= (2P_{n-1} + P_{n-2})^2 \\
&= 4P_{n-1}^2 + P_{n-2}^2 + 4P_{n-1}P_{n-2} \\
&= 4P_{n-1}^2 + P_{n-2}^2 + 2P_{n-2}(P_n - P_{n-2}) \\
&= 4P_{n-1}^2 - P_{n-2}^2 + 2P_nP_{n-2} \\
&= 6P_{n-1}^2 - P_{n-2}^2 + 2(P_nP_{n-2} - P_{n-1}^2) \\
&= 6P_{n-1}^2 - P_{n-2}^2 + 2(-1)^{n-1} \\
&= 6P_{n-1}^2 - P_{n-2}^2 - 2(-1)^n.
\end{aligned}
\tag{8.1}
$$

For example, $6P_5^2 - P_4^2 - 2(-1)^6 = 6 \cdot 29^2 - 12^2 - 2 = 4900 = 70^2 = P_6^2$.

Formula (8.1) shows that the squares of Pell numbers can be defined recursively:

$$
\begin{aligned}
P_1^2 &= 1, \quad P_2^2 = 4 \\
P_n^2 &= 6P_{n-1}^2 - P_{n-2}^2 - 2(-1)^n, \quad n \geq 3.
\end{aligned}
$$

8.2 An Interesting Byproduct

This recurrence has a delightful byproduct: It can be used to confirm the recurrence for the square-triangular numbers $c_k = y_k^2$ we studied in Chapter 6. By identity (8.1), we have

$$
\begin{aligned}
P_{2k}^2 &= 6P_{2k-1}^2 - P_{2k-2}^2 - 2 \\
&= 6(6P_{2k-2}^2 - P_{2k-3}^2 + 2) - P_{2k-2}^2 - 2 \\
&= 35P_{2k-2}^2 - 6P_{2k-3}^2 + 10 \\
&= 34P_{2k-2}^2 + (P_{2k-2}^2 - 6P_{2k-3}^2 + 2) + 8 \\
&= 34P_{2k-2}^2 - P_{2k-4}^2 + 8.
\end{aligned}
$$

Since $c_k = \frac{1}{2}P_{2k}$, this implies that $c_k = 34c_{k-1} - c_{k-2} + 2$, as desired.

The next result shows that Q_n enjoys a similar property. It can be established using a similar argument.

(4) $Q_n^2 = 6Q_{n-1}^2 - Q_{n-2}^2 + 4(-1)^n$.
Consequently, Q_n^2 also can be defined recursively:

$$\begin{aligned} Q_1^2 &= 1, \quad Q_2^2 = 9 \\ Q_n^2 &= 6Q_{n-1}^2 - Q_{n-2}^2 + 4(-1)^n, \quad n \geq 3. \end{aligned}$$

These two identities can be deduced from the formula

$$X_{n+2}^2 - (a^2 - 2b)X_{n+1}^2 + b^2 X_n^2 = 2b^{n+1}(X_1^2 - aX_1X_0 + bX_0^2),$$

where $\{X_n\}$ satisfies the recurrence $X_{n+2} = aX_{n+1} - bX_n$. This was established by D. Zeitlin of Minneapolis, Minnesota in 1965 [265]. With $a = 2, b = -1$, and $X_n = P_n$, identity (3) follows. Likewise, With $a = 2, b = -1$, and $X_n = Q_n$, identity (4) also follows.

The above recurrences for P_n^2 and Q_n^2 are not homogeneous. We will now take a different approach to derive a common homogeneous recurrence. To this end, suppose x_n satisfies the Pell recurrence. Then

$$\begin{aligned} x_{n+3}^2 &= (2x_{n+2} + x_{n+1})^2 \\ &= 4x_{n+2}^2 + 4x_{n+2}x_{n+1} + x_{n+1}^2 \\ &= 4x_{n+2}^2 + x_{n+1}^2 + 2x_{n+1}(2x_{n+1} + x_n) + x_{n+2}(x_{n+2} - x_n) \\ &= 5x_{n+2}^2 + 5x_{n+1}^2 - x_n(x_{n+2} - 2x_{n+1}) \\ &= 5x_{n+2}^2 + 5x_{n+1}^2 - x_n^2, \end{aligned} \tag{8.2}$$

where $n \geq 0$. Thus x_n^2 satisfies the third-order recurrence $s_{n+3} = 5s_{n+2} + 5s_{n+1} - s_n$, with characteristic roots $-1, \gamma^2$ and δ^2.

For example, let $x_n = P_n$. Then $s_0 = P_0^2 = 0, s_1 = P_1^2 = 1$, and $s_2 = P_2^2 = 4$. Consequently, $s_3 = P_3^2 = 5 \cdot 4 + 5 \cdot 1 - 0 = 5^2$. On the other hand, let $x_n = Q_n$. Then $s_4 = Q_4^2 = 5 \cdot 7^2 + 5 \cdot 3^3 - 162 = 17^2$.

We will invoke recurrence (8.2) for developing generating functions for the sequences $\{P_n^2\}$ and $\{Q_n^2\}$ in Chapter 11.

(5) $4(Q_nQ_{n+1} - P_nP_{n+1}) = Q_{2n+1} + 3(-1)^n$.

(6) $2Q_nQ_{n+1} = Q_{2n+1} + (-1)^n$.

(7) $P_n = P_kQ_{n-k} + Q_kP_{n-k}$, where $1 \leq k \leq n$.
In particular, this identity gives two dividends we already know: $P_{2n} = 2P_nQ_n$ and $P_n = P_{n-1} + Q_{n-1}$.

(8) We can use the recurrence for P_n to generate an interesting pattern:

$$\begin{aligned} P_n &= 1P_n \\ &= 2P_{n-1} + P_{n-2} = 2(2P_{n-2} + P_{n-3}) + P_{n-2} \\ &= 5P_{n-2} + 2P_{n-3} = 5(2P_{n-3} + P_{n-4}) + 2P_{n-3} \\ &= 12P_{n-3} + 5P_{n-4} \\ &\vdots \end{aligned}$$

More generally, we can conjecture that $P_n = P_{k+1}P_{n-k} + P_k P_{n-k-1}$. This can be confirmed using induction on k or using the Binet-like formula for P_n.

Using a similar argument, we can show that $Q_n = P_{k+1}Q_{n-k} + P_k Q_{n-k-1}$. For example, $P_4P_7 + P_3P_6 = 12 \cdot 169 + 5 \cdot 70 = 2,378 = P_{10}$ and $Q_4Q_7 + Q_3Q_6 = 12 \cdot 239 + 5 \cdot 99 = 3,363 = Q_{10}$.

(9) $4(P_n^2 + Q_n^2) = 3Q_{2n} + (-1)^n$.
The next two identities follow from this one.

(10) $4(3P_n^2 + Q_n^2) = 5Q_{2n} - (-1)^n$.

(11) $3Q_n^2 + 2P_n^2 = 2Q_{2n} + (-1)^n$.

(12) $P_{m+n}P_{m+r} - P_{m+n+r}P_m = (-1)^m P_n P_r$.
This was proposed as a problem in 1969 by M.N.S. Swamy of Nova Scotia Technical College, Halifax, Canada, and C.A. Vespe of the University of New Mexico, Albuquerque, New Mexico [244]. The proof follows algebraically in a straightforward fashion.

For example, we have

$$\begin{aligned} P_{4+7}P_{4+5} - P_{4+5+7}P_4 &= P_{11}P_9 - P_{16}P_4 = 5741 \cdot 985 - 470832 \cdot 12 \\ &= 4901 = (-1)^4 \cdot 29 \cdot 169 = (-1)^4 P_5 P_7. \end{aligned}$$

This formula has two interesting byproducts:

- Changing n to $n - m$ and letting $r = 1$, it yields $P_n P_{m+1} - P_{n+1}P_m = (-1)^m P_{n-m}$. Switching the variables m and n, we get

$$P_m P_{n+1} - P_{m+1} P_n = (-1)^n P_{m-n}. \tag{8.3}$$

  When $m = n - 1$, this yields the Cassini-like formula for Pell numbers.

  Identity (8.3) has a Fibonacci counterpart: $F_m F_{n+1} - F_{m+1} F_n = (-1)^n F_{m-n}$. It is called *d'Ocagne's identity*, after the French mathematician Philbert Maurice d'Ocagne (1862–1938); when $m = n - 1$, it yields Cassini's formula.

- When $r = -n$, the formula yields a generalization of the Cassini-like formula:

$$P_{m+n}P_{m-n} - P_m^2 = (-1)^{m+n-1} P_n^2, \tag{8.4}$$

  where we have used the fact that $P_{-k} = (-1)^{k-1} P_k$. In particular, this yields the Cassini-like formula $P_{m+1}P_{m-1} - P_m^2 = (-1)^m$.

It has two other interesting special cases:

$$P_{2m}^2 - P_{3m}P_m = (-1)^m P_m^2 \qquad (8.5)$$

$$P_{m+1}P_{n+1} - P_m P_n = P_{m+n+1}. \qquad (8.6)$$

We will revisit the latter identity later.

The next result is the counterpart of identity (12) for Pell–Lucas numbers. Its proof follows along the same lines.

(13) $Q_{m+n}Q_{m+r} - Q_{m+n+r}Q_m = 2(-1)^{m-1}P_nP_r$. For example, $Q_{5+7}Q_{5+3} - Q_{5+7+3}Q_5 = Q_{12}Q_8 - Q_{15}Q_5 = 19601 \cdot 577 - 275807 \cdot 41 = 1690 = 2 \cdot 169 \cdot 5 = 2(-1)^{5-1}P_7P_3$. In particular, we have

$$\begin{aligned} Q_mQ_{n+r} - Q_{m+r}Q_n &= 2(-1)^{n-1}P_{m-n}P_r \\ Q_mQ_{n+1} - Q_{m+1}Q_n &= 2(-1)^{n-1}P_{m-n} \\ Q_{m+n}Q_{m-n} - Q_m^2 &= 2(-1)^{m+n}P_n^2 \\ Q_{m+1}Q_{m-1} - Q_m^2 &= 2(-1)^{m-1} \\ Q_{m+1}Q_{n+1} - 2P_mP_n &= Q_{m+n+1}. \end{aligned}$$

(14) $P_n^2 + P_{n+3}^2 = 5P_{2n+3}$.
This result has a counterpart for Pell–Lucas numbers.

(15) $Q_n^2 + Q_{n+3}^2 = 10P_{2n+3}$.
For example, $Q_5^2 + Q_8^2 = 41^2 + 577^2 = 334,610 = 10 \cdot 33461 = 10P_{13}$.

(16) $4(4P_n^2 + P_{n-1}^2 + 2P_nP_{n-1}) = 2Q_{2n+1} + Q_{2n-2} - (-1)^n$.

Proof.

$$\begin{aligned} \text{LHS} &= 4[4P_n^2 + P_{n-1}(2P_n + P_{n-1})] = 4(4P_n^2 + P_{n+1}P_{n-1}) \\ &= 4[(P_{n+1} - P_{n-1})^2 + P_{n+1}P_{n-1}] = 4(P_{n+1}^2 + P_{n-1}^2 - P_{n+1}P_{n-1}) \\ &= 4[P_{n+1}(P_{n+1} - P_{n-1}) + P_{n-1}^2] = 4(P_{n+1} \cdot 2P_n + P_{n-1}^2) \\ &= 4(2P_nP_{n+1} + P_{n-1}^2) \\ 2(\text{LHS}) &= 2(\gamma^n - \delta^n)(\gamma^{n+1} - \delta^{n+1}) + (\gamma^{n-1} - \delta^{n-1})^2 \\ &= 2[\gamma^{2n+1} + \delta^{2n+1} - 2(-1)^n] + [\gamma^{2n-2} + \delta^{2n-2} + 2(-1)^n] \\ &= 4Q_{2n+1} + 2Q_{2n-2} - 2(-1)^n \\ \text{LHS} &= 2Q_{2n+1} + Q_{2n-2} - (-1)^n = \text{RHS}. \end{aligned}$$

■

For example, $4(4P_5^2 + P_4^2 + 2P_5P_4) = 4(4 \cdot 29^2 + 12^2 + 2 \cdot 29 \cdot 12) = 16,816 = 2 \cdot 8119 + 577 + 1 = 2Q_{11} + Q_8 - (-1)^5$.

The next identity can be proved similarly.

(17) $2(4Q_n^2+Q_{n-1}^2+2Q_nQ_{n-1}) = 2Q_{2n+1}+Q_{2n-2}+(-1)^n$. For example, $2(4Q_5^2+Q_4^2+2Q_5Q_4) = 2(4\cdot 41^2+17^2+2\cdot 41\cdot 17) = 16{,}814 = 2\cdot 8119+577-1 = 2Q_{11}+Q_8+(-1)^5$.

8.3 A Pell and Pell–Lucas Hybridity

In the book, *Pell's Equation* [9], the author gives an interesting number pattern and asks the reader to predict the underlying formula for the pattern and then establish it:

$$\begin{aligned}
3^4 - 5\cdot 4^2 &= 1\\
7^4 - 24\cdot 10^2 &= 1\\
17^4 - 145\cdot 24^2 &= 1\\
41^4 - 840\cdot 58^2 &= 1\\
&\vdots
\end{aligned}$$

Factoring the numbers $5, 24, 145, 840, \ldots$ in column 2 and the numbers $4, 10, 24, 58, \ldots$ in column 3 reveals the formula behind this fascinating pattern: $Q_{n+1}^4 - P_nP_{n+2}(2P_{n+1})^2 = 1$.

We could establish this property using the Binet-like formulas; but this approach will involve a lot of messy algebra. Instead, we will let identity (31) in Chapter 7 and determinants do the job for us; the resulting proof is both short and elegant.

Proof. We have

$$\begin{aligned}
Q_{n+1}^4 - P_nP_{n+2}(2P_{n+1})^2 &= \begin{vmatrix} Q_{n+1}^2 & 4P_nP_{n+2}\\ P_{n+1}^2 & Q_{n+1}^2\end{vmatrix}\\
&= \begin{vmatrix} Q_{n+1}^2+0 & 4[P_{n+1}^2+(-1)^{n+1}]\\ P_{n+1}^2 & Q_{n+1}^2\end{vmatrix}\\
&= \begin{vmatrix} Q_{n+1}^2 & 4P_{n+1}^2\\ P_{n+1}^2 & Q_{n+1}^2\end{vmatrix} + \begin{vmatrix} 0 & 4(-1)^{n+1}\\ P_{n+1}^2 & Q_{n+1}^2\end{vmatrix}\\
&= (Q_{n+1}^4 - 4P_{n+1}^4) - 4(-1)^{n+1}P_{n+1}^2\\
&= (Q_{n+1}^2 - 2P_{n+1}^2)(Q_{n+1}^2 + 2P_{n+1}^2) - 4(-1)^{n+1}P_{n+1}^2\\
&= (-1)^{n+1}(Q_{n+1}^2 + 2P_{n+1}^2) - 4(-1)^{n+1}P_{n+1}^2\\
&= (-1)^{n+1}(Q_{n+1}^2 - 2P_{n+1}^2)\\
&= (-1)^{n+1}\cdot(-1)^{n+1}\\
&= 1,
\end{aligned}$$

as desired. ∎

8.4 Matrices and Pell Numbers

As is the case with Fibonacci numbers, we can also use matrices to generate Pell numbers. Consequently, matrices are helpful in extracting properties of Pell numbers. To this end, consider the matrix

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$P^2 = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} = 2P + I,$$

where I denotes the 2×2 identity matrix. So P satisfies the matrix equation $M^2 = 2M + I$.

Notice also that

$$P = \begin{bmatrix} P_2 & P_1 \\ P_1 & P_0 \end{bmatrix} \text{ and } P^2 = \begin{bmatrix} P_3 & P_2 \\ P_2 & P_1 \end{bmatrix}.$$

More generally, we have the following result. We will prove it by PMI.

Theorem 8.1 *Let n be any positive integer. Then*

$$P^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}.$$

Proof. The result is clearly true when $n = 1$. Now assume that it is true for an arbitrary integer $k \geq 1$:

$$P^k = \begin{bmatrix} P_{k+1} & P_k \\ P_k & P_{k-1} \end{bmatrix}.$$

Then, using the Pell recurrence, we have

$$\begin{aligned} P^{k+1} &= P^k \cdot P \\ &= \begin{bmatrix} P_{k+1} & P_k \\ P_k & P_{k-1} \end{bmatrix} \begin{bmatrix} P_2 & P_1 \\ P_1 & P_0 \end{bmatrix} \\ &= \begin{bmatrix} 2P_{k+1} + P_k & P_{k+1} \\ 2P_k + P_{k-1} & P_k \end{bmatrix} \\ &= \begin{bmatrix} P_{k+2} & P_{k+1} \\ P_{k+1} & P_k \end{bmatrix}. \end{aligned}$$

So the result is true when $n = k + 1$. Thus, by PMI, the result is true for every integer $n \geq 1$. ∎

It follows by the theorem that the powers of the matrix P can be used to generate all Pell numbers.

An immediate consequence of Theorem 8.1 is the Cassini-like formula for P_n, which we derived earlier. We now rederive it by invoking the fact that $|A \cdot B| = |A| \cdot |B|$, where $|M|$ denotes the determinant of the square matrix M. Then

$$|P|^n = |P^n| = \begin{vmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{vmatrix} = P_{n+1}P_{n-1} - P_n^2.$$

But $|P| = -1$; so $|P|^n = (-1)^n$. Thus $P_{n+1}P_{n-1} - P_n^2 = (-1)^n$, as desired.

Theorem 8.1 has additional byproducts.

(1) First, we will prove the *addition formula*

$$P_{m+n} = P_m P_{n+1} + P_{m-1}P_n. \tag{8.7}$$

Proof. Since $P^{m+n} = P^m P^n$, by Theorem 8.1, we have

$$\begin{aligned} \begin{bmatrix} P_{m+n+1} & P_{m+n} \\ P_{m+n} & P_{m+n-1} \end{bmatrix} &= \begin{bmatrix} P_{m+1} & P_m \\ P_m & P_{m-1} \end{bmatrix} \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} P_{m+1}P_{n+1} + P_m P_n & P_{m+1}P_n + P_m P_{n-1} \\ P_m P_{n+1} + P_{m-1}P_n & P_m P_n + P_{m-1}P_{n-1} \end{bmatrix}. \end{aligned}$$

The addition formula follows by equating the lower left-hand elements from both sides. ∎

For example, $P_{10} = P_{4+6} = P_4 P_7 + P_3 P_6 = 12 \cdot 169 + 5 \cdot 70 = 2,378$.

Notice that Pell's recurrence follows from the addition formula; so the addition formula is a generalization of the Pell recurrence.

Formula (8.7) yields three interesting dividends:

(a) Suppose we let $m = n$. Then formula (8.7) yields

$$\begin{aligned} P_{2n} &= P_n P_{n+1} + P_{n-1}P_n = P_n(P_{n+1} + P_{n-1}) \\ &= P_n(2Q_n) = 2P_n Q_n, \end{aligned}$$

a fact we already knew.

(b) Next we claim that $13|P_{7n}$ for every integer $n \geq 1$.
We will prove this using PMI. First, notice that $P_7 = 169$ and $13|P_7$. Assume the claim holds for an arbitrary positive integer n. Then, by formula (8.7), $P_{7(n+1)} = P_{7n+7} = P_{7n}P_8 + P_{7n-1}P_7$. Since $13|P_{7n}$ by the inductive hypothesis, it follows that $13|P_{7(n+1)}$. Thus, by PMI, the result is true for every $n \geq 1$.

More generally, suppose $q|P_m$, where q is a prime. Then it follows by a similar argument that $q|P_{mn}$ for every integer n. For example, $17|P_8$ and $17|P_{16}$.

(c) Suppose we let $m = 2n$ in formula (8.7). Then

$$\begin{aligned} P_{3n} &= P_{2n}P_{n+1} + P_{2n-1}P_n \\ &= 2P_nQ_nP_{n+1} + P_{2n-1}P_n \\ &= (2Q_nP_{n+1} + P_{2n-1})P_n. \end{aligned}$$

So $P_n|P_{3n}$ for every positive integer n.

For example, $P_5|P_{15}$, where $P_5 = 29$ and $P_{15} = 195,025$.

(2) More generally, we have the following theorem, which we will establish using PMI.

Theorem 8.2 *Let n be any positive integer. Then $P_m|P_{mn}$.*

Proof. Since the statement is clearly true when $n = 1$, assume it is true for an arbitrary integer $n \geq 1$. Then, by formula (8.7), we have

$$\begin{aligned} P_{m(n+1)} &= P_{mn+m} \\ &= P_{mn}P_{m+1} + P_{mn-1}P_m. \end{aligned}$$

Since $P_m|P_{mn}$ by the inductive hypothesis, it follows that $P_m|P_{m(n+1)}$. Thus, by PMI, $P_m|P_{mn}$ for every integer $n \geq 1$. ■

Fortunately, the converse is also true, as the next theorem shows.

Theorem 8.3 *If $P_m|P_n$, then $m|n$.*

Proof. By the division algorithm, let $n = mk + r$, where $0 \leq r < m$. Then, by formula (8.7), we have

$$\begin{aligned} P_n &= P_{mk+r} \\ &= P_{mk}P_{r+1} + P_{mk-1}P_r. \end{aligned}$$

Since $P_m|P_n$ and $P_m|P_{mk}$, it follows that $P_m|P_{mk-1}P_r$. Since $(P_{mk}, P_{mk-1}) = 1$, it follows that $P_m \nmid P_{mk-1}$. Consequently, $P_m|P_r$. But this is impossible, unless r = 0. Thus $n = mk$ and hence $m|n$, as desired. ■

Combining Theorems 8.3 and 8.4, we have the following result.

Theorem 8.4 *$P_m|P_n$ if and only if $m|n$.* ■

(3) Next we will show how Theorem 8.1 and the convergents of the ISCF of $\sqrt{2}$ are closely related.

8.5 Convergents of the ISCF of $\sqrt{2}$ Revisited

It is well known that every square matrix M satisfies its *characteristic equation* $|M - \lambda I| = 0$, where λ denotes the *eigenvalue* of M and I the identity matrix of the same size as M. This is the celebrated *Cayley–Hamilton theorem*, named after the English mathematician Arthur Cayley (1821–1895) and the Irish mathematician William Rowan Hamilton (1805–1865).

The characteristic equation $|P^n - \lambda I| = 0$ of P^n gives us a surprising dividend. Substitute for P^n from Theorem 8.1:

$$\begin{aligned}
\begin{vmatrix} P_{n+1} - \lambda & P_n \\ P_n & P_{n-1} - \lambda \end{vmatrix} &= 0 \\
(P_{n+1} - \lambda)(P_{n+1} - \lambda) - P_n^2 &= 0 \\
\lambda^2 - (P_{n+1} + P_{n-1})\lambda + (P_{n+1}P_{n-1} - P_n^2) &= 0 \\
\lambda^2 - 2Q_n\lambda + (-1)^n &= 0.
\end{aligned}$$

Solving this quadratic equation, we get

$$\begin{aligned}
\lambda &= \frac{2Q_n \pm \sqrt{4Q_n^2 - 4(-1)^n}}{2} \\
&= Q_n \pm \sqrt{Q_n^2 - (-1)^n} \\
&= Q_n \pm \sqrt{2P_n^2} \\
&= Q_n \pm P_n\sqrt{2}.
\end{aligned}$$

But $Q_n + P_n\sqrt{2} = \frac{\gamma^n + \delta^n}{2} + \frac{\gamma^n - \delta^n}{2} = \gamma^n$, and similarly, $Q_n - P_n\sqrt{2} = \delta^n$. Thus the characteristic roots of the equation $|P^n - \lambda I| = 0$ are γ^n and δ^n.

Recall from Example 3.8 that $\frac{Q_{n+1}}{P_{n+1}}$ is the nth convergent of the ISCF of $\sqrt{2}$. Consequently, the nth convergent of the ISCF of $\sqrt{2}$ can be used to compute the characteristic roots of the equation $|P^n - \lambda I| = 0$.

The addition formula (8.7) can be used to derive a formula for P_{m-n}. Changing n to $-n$, it yields

$$\begin{aligned}
P_{m-n} &= P_m P_{-(n-1)} + P_{m-1}P_{-n} \\
&= (-1)^{n-2}P_m P_{n-1} + (-1)^{n-1}P_{m-1}P_n \\
&= (-1)^n(P_m P_{n-1} - P_{m-1}P_n). \qquad (8.8)
\end{aligned}$$

For example, $P_4 = P_{7-3} = (-1)^3(P_7P_2 - P_6P_3) = -(162 \cdot 2 - 70 \cdot 5) = 12$.

Formula (8.8) can be confirmed using the Binet-like formula for Pell numbers.

8.5.1 An Alternate Method

Interestingly, Theorem 8.1 can also be used to derive formula (8.8). As is well known, the 2×2 matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $|M| = ad - bc \neq 0$. When it is invertible, the inverse is given by

$$M^{-1} = \frac{1}{|M|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Since $|P| = -1 \neq 0$, the matrix P is invertible; so is P^n. The inverse P^{-n} is given by

$$P^{-n} = \frac{1}{|\Delta|} \begin{bmatrix} P_{n-1} & -P_n \\ -P_n & P_{n+1} \end{bmatrix},$$

where $\Delta = P_{n+1}P_{n-1} - P_n^2 = (-1)^n$. So

$$P^{-n} = (-1)^n \begin{bmatrix} P_{n-1} & -P_n \\ -P_n & P_{n+1} \end{bmatrix}.$$

Thus,

$$\begin{aligned} P^{m-n} &= P^m \cdot P^{-n} \\ &= (-1)^n \begin{bmatrix} P_{m+1} & P_m \\ P_m & P_{n-1} \end{bmatrix} \begin{bmatrix} P_{n-1} & -P_n \\ -P_n & P_{n+1} \end{bmatrix} \\ \begin{bmatrix} P_{m-n+1} & P_{m-n} \\ P_{m-n} & P_{m-n-1} \end{bmatrix} &= (-1)^n \begin{bmatrix} P_{m+1}P_{n-1} - P_m P_n & P_m P_{n+1} - P_{m+1} P_n \\ P_m P_{n-1} - P_{m-1} P_n & P_{m-1} P_{n+1} - P_m P_n \end{bmatrix}. \end{aligned}$$

This matrix equation yields two formulas for P_{m-n}:

$$\begin{aligned} P_{m-n} &= (-1)^n (P_m P_{n+1} - P_{m+1} P_n) \\ &= (-1)^n (P_m P_{n-1} - P_{m-1} P_n). \end{aligned}$$

Notice that these formulas result in the Cassini-like formula for Pell numbers when $m = n + 1$.

Next we investigate the powers of the matrix

$$Q = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{bmatrix}.$$

Although the powers of Q also follow an interesting pattern, the pattern is slightly more complicated:

$$Q^1 = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{bmatrix} \qquad Q^2 = 2\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} = 2^1\begin{bmatrix} P_3 & P_2 \\ P_2 & P_1 \end{bmatrix}$$

$$Q^3 = 2\begin{bmatrix} 17 & 7 \\ 7 & 3 \end{bmatrix} = 2^1\begin{bmatrix} Q_4 & Q_3 \\ Q_3 & Q_2 \end{bmatrix} \qquad Q^4 = 4\begin{bmatrix} 29 & 12 \\ 12 & 5 \end{bmatrix} = 2^2\begin{bmatrix} P_5 & P_4 \\ P_4 & P_3 \end{bmatrix}.$$

More generally, we claim that

$$Q^n = \begin{cases} 2^{n/2}\begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} & \text{if } n \text{ is even} \\ 2^{\lfloor n/2 \rfloor}\begin{bmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{bmatrix} & \text{otherwise.} \end{cases} \tag{8.9}$$

This can be confirmed using PMI as follows.

Proof (by PMI). Clearly, the formula works when $n = 1$ and $n = 2$. Assume it works for all positive integers $< n$, where n is an arbitrary integer ≥ 2.

Case 1 Let n be even. Then

$$\begin{aligned} Q^n &= Q^{n-1} \cdot Q \\ &= 2^{(n-2)/2}\begin{bmatrix} Q_n & Q_{n-1} \\ Q_{n-1} & Q_{n-2} \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \\ &= 2^{(n-2)/2}\begin{bmatrix} 3Q_n + Q_{n-1} & Q_n + Q_{n-1} \\ 3Q_{n-1} + Q_{n-2} & Q_{n-1} + Q_{n-2} \end{bmatrix} \\ &= 2^{(n-2)/2}\begin{bmatrix} Q_{n+1} + Q_n & Q_n + Q_{n-1} \\ Q_n + Q_{n-1} & Q_{n-1} + Q_{n-2} \end{bmatrix} \\ &= 2^{(n-2)/2}\begin{bmatrix} 2P_{n+1} & 2P_n \\ 2P_n & 2P_{n-1} \end{bmatrix} \\ &= 2^{n/2}\begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}. \end{aligned}$$

The case for odd n follows similarly; see Exercise 16. So the formula works for n also. Thus, by PMI, the result is true for all positive integers n. ■

Formula (8.9) has two delightful consequences. To see them, first we invoke a useful fact from the theory of matrices: Let $A = (a_{ij})_{m \times m}$ and $B = k(a_{ij})_{m \times m} = (ka_{ij})_{m \times m}$. Then $|B| = k^m|A|$, where $|M|$ denotes the determinant of the matrix M.

Case 1 Suppose n is even. By formula (8.9), we have

$$|Q^n| = \left(2^{\lfloor n/2 \rfloor}\right)^2 \begin{vmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{vmatrix}.$$

But $|Q^n| = |Q|^n = 2^n$. So

$$\begin{aligned} 2^n \begin{vmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{vmatrix} &= 2^n \\ P_{n+1}P_{n-1} - P_n^2 &= 1 = (-1)^n. \end{aligned}$$

Case 2 On the other hand, suppose n is odd. Then, as in Case 1, we have

$$\begin{aligned} 2^{n-1} \begin{vmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{vmatrix} &= 2^n \\ Q_{n+1}Q_{n-1} - Q_n^2 &= 2 = 2(-1)^{n-1}. \end{aligned}$$

8.6 Additional Addition Formulas

Formula (8.9) can be used to develop addition formulas for P_{m+n} and Q_{m+n} in special cases. To this end, we have $Q^{m+n} = Q^m \cdot Q^n$.

Case 1 Suppose both m and n are odd. Then, by formula (8.9), we have

$$\begin{aligned} 2^{(m+n)/2} \begin{bmatrix} P_{m+n+1} & P_{m+n} \\ P_{m+n} & P_{m+n-1} \end{bmatrix} &= 2^{(m-1)/2} 2^{(n-1)/2} \begin{bmatrix} Q_{m+1} & Q_m \\ Q_m & Q_{m-1} \end{bmatrix} \begin{bmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{bmatrix} \\ &= 2^{(m+n-2)/2} \begin{bmatrix} Q_{m+1}Q_{n+1} + Q_mQ_n & Q_{m+1}Q_n + Q_mQ_{n-1} \\ Q_mQ_{n+1} + Q_{m-1}Q_n & Q_mQ_n + Q_{m-1}Q_{n-1} \end{bmatrix}. \end{aligned}$$

This yields the formula

$$2P_{m+n} = Q_mQ_{n+1} + Q_{m-1}Q_n, \tag{8.10}$$

where both m and n are odd.

For example, $Q_3Q_6 + Q_2Q_5 = 7 \cdot 99 + 3 \cdot 41 = 816 = 2 \cdot 408 = 2P_8 = 2P_{3+5}$.

Case 2 Suppose m is odd and n is even. Then formula (8.9) yields

$$2^{(m+n-1)/2}\begin{bmatrix} Q_{m+n+1} & Q_{m+n} \\ Q_{m+n} & Q_{m+n-1} \end{bmatrix} = 2^{(m-1)/2}2^{n/2}\begin{bmatrix} Q_{m+1} & Q_m \\ Q_m & Q_{m-1} \end{bmatrix}\begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}$$

$$\begin{bmatrix} Q_{m+n+1} & Q_{m+n} \\ Q_{m+n} & Q_{m+n-1} \end{bmatrix} = 2^{(m+n-1)/2}\begin{bmatrix} Q_{m+1}P_{n+1}+Q_mP_n & Q_{m+1}P_n+Q_mP_{n-1} \\ Q_mP_{n+1}+Q_{m-1}P_n & Q_mP_n+Q_{m-1}P_{n-1} \end{bmatrix}.$$

This implies that

$$\begin{aligned} Q_{m+n} &= Q_{m+1}P_n + Q_mP_{n-1} \\ &= Q_mP_{n+1} + Q_{m-1}P_n, \end{aligned}$$

where m is odd and n is even.

For example, let $m = 3$ and $n = 6$. Then $Q_4P_6 + Q_3P_5 = 17 \cdot 70 + 7 \cdot 29 = 1393 = Q_9$. Likewise, $Q_3P_7 + Q_2P_6 = 1393 = Q_9$.

Case 3 Suppose m is even and n is odd. Then formula (8.9) yields

$$\begin{aligned} \begin{bmatrix} Q_{m+n+1} & Q_{m+n} \\ Q_{m+n} & Q_{m+n-1} \end{bmatrix} &= \begin{bmatrix} P_{m+1} & P_m \\ P_m & P_{m-1} \end{bmatrix}\begin{bmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} P_{m+1}Q_{n+1}+P_mQ_n & P_{m+1}Q_n+P_mQ_{n-1} \\ P_mQ_{n+1}+P_{m-1}Q_n & P_mQ_n+P_{m-1}Q_{n-1} \end{bmatrix}. \end{aligned}$$

This yields

$$\begin{aligned} Q_{m+n} &= P_{m+1}Q_n + P_mQ_{n-1} \\ &= P_mQ_{n+1} + P_{m-1}Q_n, \end{aligned}$$

where m is even and n is odd.

For example, let $m = 4$ and $n = 7$. Then $P_5Q_7 + P_4Q_6 = 29 \cdot 239 + 12 \cdot 99 = 8119 = Q_{11} = P_4Q_8 + P_3Q_7$.

Case 4 Suppose m and n are both even. This case yields to formula (8.7), found earlier.

8.6.1 Formula (8.10) Revisited

Although formula (8.10) is restricted to odd integers m and n, it is true for all nonnegative integers m and n. This follows by the Binet-like formulas:

$$\begin{aligned} 4(Q_mQ_{n+1} + Q_{m-1}Q_n) &= (\gamma^m + \delta^m)(\gamma^{n+1} + \delta^{n+1}) + (\gamma^{m-1} + \delta^{m-1})(\gamma^n + \delta^n) \\ &= \gamma^{m+n}\left(\gamma + \frac{1}{\gamma}\right) + \delta^{m+n}\left(\delta + \frac{1}{\delta}\right) + \gamma^m\delta^n\left(\delta + \frac{1}{\gamma}\right) \\ &\quad + \gamma^n\delta^m\left(\gamma + \frac{1}{\delta}\right) \end{aligned}$$

$$\begin{aligned} &= \gamma^{m+n}(\gamma-\delta) - \delta^{m+n}(\gamma-\delta) \\ &= (\gamma-\delta)(\gamma^{m+n} - \delta^{m+n}) = 8P_{m+n} \\ Q_mQ_{n+1} + Q_{m-1}Q_n &= 2P_{m+n}. \end{aligned} \tag{8.11}$$

For example, $Q_4Q_8 + Q_3Q_7 = 17 \cdot 577 + 7 \cdot 239 = 11,482 = 2 \cdot 5741 = 2P_{11} = 2P_{4+7}$.

Suppose we let $m = n + 1$ in identity (8.11). Then it yields $Q_{n+1}^2 + Q_n^2 = 2P_{2n+1}$, which is identity (21) in Chapter 7.

Changing n to $-n$, identity (8.11) yields

$$2P_{m-n} = (-1)^{n-1}(Q_mQ_{n-1} - Q_{m-1}Q_n). \tag{8.12}$$

For example, $Q_9Q_4 - Q_8Q_5 = 1393 \cdot 17 - 577 \cdot 41 = 24 = 2 \cdot 12 = 2 \cdot P_{9-5}$.

Notice that the Cassini-like formula for Pell–Lucas numbers follows from formula (8.12) by letting $m = n + 1$.

The formula $Q_{m+n} = Q_{m+1}P_n + Q_mP_{n-1}$ can be used to extract an interesting fact. To this end, we have

$$\begin{aligned} Q_{2n} &= Q_{n+1}P_n + Q_nP_{n-1} \\ &= (2Q_n + Q_{n-1})P_n + Q_nP_{n-1} \\ &= Q_n(2P_n + P_{n-1}) + P_nQ_{n-1} \\ &= Q_nP_{n+1} + P_nQ_{n-1}. \end{aligned}$$

Suppose $Q_n|Q_{2n}$. Then $Q_n|P_nQ_{n-1}$. But $(P_n, Q_n) = 1$. So $Q_n|Q_{n-1}$, which is impossible. Consequently, $Q_n \nmid Q_{2n}$.

The identity

$$2P_nQ_{n-k} = P_{2n-k-r}Q_r + Q_{2n-k-r}P_r + (-1)^{n-k}P_k \tag{8.13}$$

can be established fairly easily using the Binet-like formulas

Identity (8.13) has a number of special cases. When $k = -1, 0$, and 1, it yields

$$2P_nQ_{n+1} = P_{2n-r+1}Q_r + Q_{n-2r+1}P_r - (-1)^n \tag{8.14}$$

$$P_{2n} = P_{2n-r}Q_r + Q_{n-2r}P_r \tag{8.15}$$

$$2P_nQ_{n-1} = P_{2n-r-1}Q_r + Q_{n-2r-1}P_r - (-1)^n. \tag{8.16}$$

Suppose we let $r = 3$ in (8.14) and (8.16). Then we get

$$2P_nQ_{n+1} = 7P_{2n-2} + 5Q_{2n-2} - (-1)^n \tag{8.17}$$

$$2P_nQ_{n-1} = 7P_{2n-4} + 5Q_{2n-4} - (-1)^n. \tag{8.18}$$

In particular, these yield the following identities, discovered by K.S. Bhanu and M.N. Deshpande in 2008 [18, 19]:

$$\begin{aligned} 2P_{2n+2}Q_{2n+3} &= 7P_{4n+2} + 5Q_{4n+2} - 1 \\ 2P_{2n+2}Q_{2n+1} &= 7P_{4n} + 5Q_{4n} - 1. \end{aligned}$$

For example, let $n = 6$ in (8.17). Then $7P_{10} + 5Q_{10} - 1 = 7 \cdot 2378 + 5 \cdot 3363 - 1 = 33,460 = 2 \cdot 70 \cdot 239 = 2P_6Q_7$. Likewise, $7P_{12} + 5PQ_{12} + 1 = 195,026 = 2P_7Q_8$.

It is well known that Fibonacci numbers satisfy the property that $(F_m, F_n) = F_{(m,n)}$. Interestingly, the same property holds for Pell numbers as well; we will establish it shortly.

8.7 Pell Divisibility Properties Revisited

In Chapter 7, we found that $(P_m, P_{m-1}) = 1$. Using Theorem 8.2, we can generalize it, as the following lemma shows.

Lemma 8.1 $(P_{qn-1}, P_n) = 1$, *where q is a positive integer.*

Proof. Let $d = (P_{qn-1}, P_n)$. Then $d|P_{qn-1}$ and $d|P_n$. But, by Theorem 8.2, $P_n|P_{qn}$; so $d|P_{qn}$. Thus $d|P_{qn}$ and $d|P_{qn-1}$. But $(P_{qn}, P_{qn-1}) = 1$. So $d|1$ and hence $d = 1$. Thus $(P_{qn-1}, P_n) = 1$, as desired. ∎

We need one more lemma before we can prove the desired property.

Lemma 8.2 *Let $m = qn + r$, where $0 \leq r \leq n$. Then $(P_m, P_n) = (P_n, P_r)$.*

Proof. Using identity (8.7) and Lemma 8.1, we have

$$\begin{aligned} (P_m, P_n) &= (P_{qn+r}, P_n) \\ &= (P_{qn}P_{r+1} + P_{qn-1}P_r, P_n) \\ &= (P_{qn-1}P_r, P_n) = (P_r, P_n) \\ &= (P_n, P_r). \end{aligned}$$

∎

For example, let $m = 15$ and $n = 6$. Clearly, $q = 2$ and $r = 3$. Then $(P_{15}, P_6) = (195025, 70) = 5 = (70, 5) = (P_6, P_3)$.

We are now ready to establish the property mentioned earlier. The essence of the proof lies in the well-known euclidean algorithm [130].

Theorem 8.5 $(P_m, P_n) = P_{(m,n)}$.

Proof. Without loss of generality, we can assume that $m \geq n$. Then, by the euclidean algorithm, we get the following sequence of equations:

$$\begin{aligned} m &= q_0 n + r_1, && 0 \le r_1 < n \\ n &= q_1 r_1 + r_2, && 0 \le r_2 < r_1 \\ r_1 &= q_2 r_2 + r_3, && 0 \le r_3 < r_2 \\ &\vdots \\ r_{n-2} &= q_{n-1} r_{n-1} + r_n, && 0 \le r_n < r_{n-1} \\ r_{n-1} &= q_n r_n + 0. \end{aligned}$$

It follows from a repeated application of Lemma 8.2 that $(P_m, P_n) = (P_n, P_{r_1}) = (P_{r_1}, P_{r_2}) = \cdots = (P_{r_{n-2}}, P_{r_{n-1}}) = (P_{r_{n-1}}, P_{r_n})$. Since $r_{n-1} = q_n r_n, (P_{r_{n-1}}, P_{r_n}) = (P_{q_n r_n}, P_{r_n})$. By Theorem 8.2, $P_{r_n} | P_{q_n r_n}$. So $(P_{q_n r_n}, P_{r_n}) = P_{r_n}$. But $r_n = (m, n)$. So $(P_{q_n r_n}, P_{r_n}) = P_{(m,n)}$. Thus $(P_m, P_n) = P_{(m,n)}$, as desired. ∎

This theorem gives a quick and efficient algorithm for computing the gcd of any two Pell numbers. A scientific calculator such as TI-86 or higher will come in handy, because it has a built-in *gcd* function.

For example, $(P_{21}, P_{14}) = (38613965, 80782) = 169 = P_7 = P_{(21,14)}$.

It follows by Theorem 8.5 that the *least common denominator* (lcm) $[P_m, P_n]$ of P_m and P_n also can be computed quickly:

$$[P_m, P_n] = \frac{P_m P_n}{(P_m, P_n)} = \frac{P_m P_n}{P_{(m,n)}}.$$

For example, $[P_{15}, P_{10}] = [195025, 2378] = \frac{195025 \cdot 2378}{P_5} = \frac{195025 \cdot 2378}{29} = 15,992,050$.

When P_m and P_n are fairly small, we could invoke the built-in function *lcm* to compute their lcm.

We would like to emphasize that Theorem 8.5 does *not* hold for Pell–Lucas numbers: $(Q_m, Q_n) \neq Q_{(m,n)}$ in general. For example, $(Q_{12}, Q_6) = (19601, 99) = 1$, whereas $Q_{(12,6)} = Q_6 = 99$.

Theorem 8.5 has an immediate byproduct. To see this, suppose m and n are relatively prime; that is, $(m, n) = 1$. Then $(P_m, P_n) = P_{(m,n)} = P_1$; so P_m and P_n are relatively prime. Conversely, if P_m and P_n are relatively prime, then so are m and n. Thus we have the following result.

Corollary 8.1 $(P_m, P_n) = 1$ *if and only if* $(m, n) = 1$. ∎

This corollary can be used to reconfirm the infinitude of primes, first established by the Greek mathematician Euclid (ca. 330–275 B.C.), the father of number theory and geometry. The proof given next is adapted from the one given in 1965 by M. Wunderlich of the University of Colorado for Fibonacci numbers [263].

Corollary 8.2 *There are infinitely many primes.*

Proof. Suppose there are exactly k primes, $q_1, q_2, \ldots, q_k$. Consider the Pell numbers $P_{q_1}, P_{q_2}, \ldots, P_{q_k}$. Since $(q_i, q_j) = 1$, $(P_{q_i}, P_{q_j}) = 1$ by Corollary 8.1, where $i \neq j$. Since there are only k primes by our assumption, this implies that each P_{q_i} has exactly one prime factor. But this is a contradiction, since $P_{17} = 1136689 = 137 \cdot 8297$, where 137 and 8297 are primes. So P_{17} has two distinct prime factors. (Notice that $P_{23} = 225058681 = 229 \cdot 982789$ also has two distinct prime factors.) Since this is a contradiction, it follows that there are infinitely many primes. ∎

8.8 Additional Identities

Algebraic identities can be used to develop new Pell and Pell–Lucas identities. For example, consider the identity $(x+y)^3 - x^3 - y^3 = 3xy(x+y)$. Letting $x = 2P_n$ and $y = P_{n-1}$, it yields the identity

$$P_{n+1}^3 - 8P_n^3 - P_{n-1}^3 = 6P_{n+1}P_nP_{n-1}.$$

Similarly, we have

$$Q_{n+1}^3 - 8Q_n^3 - Q_{n-1}^3 = 6Q_{n+1}Q_nQ_{n-1}.$$

For example, $P_6^3 - 8P_5^3 - P_4^3 = 70^3 - 8\cdot 29^3 - 12^3 = 146,160 = 6\cdot 70\cdot 29\cdot 12 = 6P_6P_5P_4$ and $Q_6^3 - 8Q_5^3 - Q_4^3 = 99^3 - 8\cdot 41^3 - 17^3 = 414,018 = 6\cdot 99\cdot 41\cdot 17 = 6Q_6Q_5Q_4$.

The identities $(x+y)^5 - x^5 - y^5 = 5xy(x+y)(x^2+xy+y^2)$ and $(x+y)^7 - x^7 - y^7 = 7xy(x+y)(x^2+xy+y^2)$ can be employed to derive the following identities:

$$\begin{aligned}
P_{n+1}^5 - 32P_n^5 - P_{n-1}^5 &= \frac{5}{2}P_{n+1}P_nP_{n-1}[5Q_{2n} - (-1)^n] \\
Q_{n+1}^5 - 32Q_n^5 - Q_{n-1}^5 &= 10Q_{n+1}Q_nQ_{n-1}[2Q_{2n} + (-1)^n] \\
P_{n+1}^7 - 128P_n^7 - P_{n-1}^7 &= \frac{5}{8}P_{n+1}P_nP_{n-1}[5Q_{2n} - (-1)^n]^2 \\
Q_{n+1}^7 - 128Q_n^7 - Q_{n-1}^7 &= 10Q_{n+1}Q_nQ_{n-1}[5Q_{2n} + (-1)^n]^2.
\end{aligned}$$

Using the identities $(x+y)^2 + x^2 + y^2 = 2(x^2+y^2+xy)$ and $(x+y)^4 + x^4 + y^4 = 2(x^2+y^2+xy)^2$, it can be shown that

$$\begin{aligned}
P_{n+1}^2 + 4P_n^2 + P_{n-1}^2 &= \frac{1}{2}[2Q_{2n+1} + Q_{2n-2} - (-1)^n] \\
Q_{n+1}^2 + 4Q_n^2 + Q_{n-1}^2 &= 2Q_{2n+1} + Q_{2n-2} + (-1)^n] \\
P_{n+1}^4 + 16P_n^4 + P_{n-1}^4 &= \frac{1}{8}[2Q_{2n+1} + Q_{2n-2} - (-1)^n]^2 \\
Q_{n+1}^4 + 16P_n^2 + Q_{n-1}^2 &= \frac{1}{2}[2Q_{2n+1} + Q_{2n-2} + (-1)^n]^2.
\end{aligned}$$

8.9 Candido's Identity and the Pell Family

Let x and y be any two real numbers. Then

$$\left[x^2 + y^2 + (x+y)^2\right]^2 = 2\left[x^4 + y^4 + (x+y)^4\right].$$

This is *Candido's identity*, named after the Italian mathematician Giacomo Candido (1871–1941).

As can be predicted, Candido's identity has an interesting geometric interpretation. To see this, consider three line segments $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$ such that $AB = x^2$, $BC = y^2$ and $CD = (x+y)^2$. Now form the square $ADEF$; see Figure 8.1. Then

$$\begin{aligned} \text{Area } ADEF &= \left[x^2 + y^2 + (x+y)^2\right]^2 \\ &= 2\left[x^4 + y^4 + (x+y)^4\right] \\ &= 2(\text{sum of three shades areas}). \end{aligned}$$

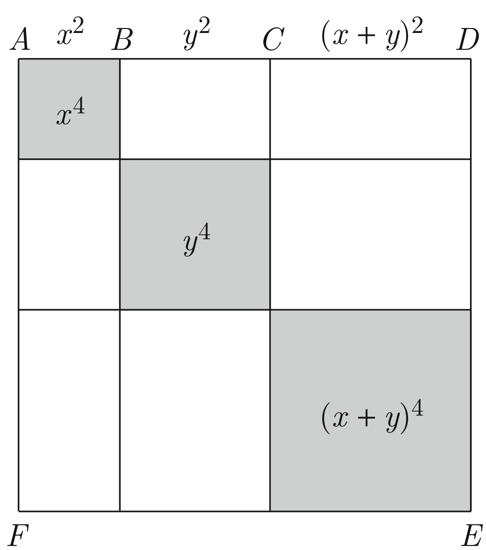

Figure 8.1. Pascal's Triangle

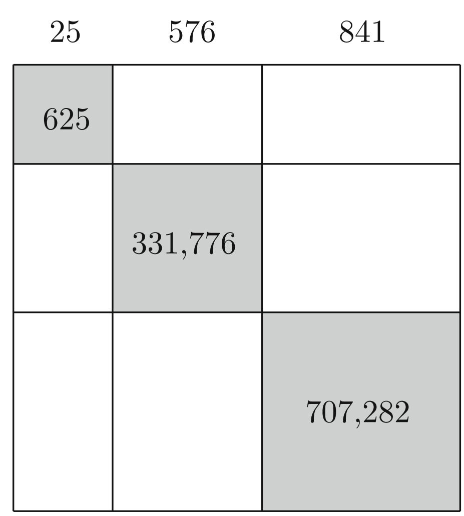

Figure 8.2.

Let $x = P_{n-2}$ and $y = 2P_{n-1}$. Then Candido's identity yields the Pell identity

$$\left(P_{n-2}^2 + 4P_{n-1}^2 + P_n^2\right)^2 = 2\left(P_{n-2}^4 + 16P_{n-1}^4 + P_n^4\right). \tag{8.19}$$

For example, let $n = 5$. Then

$$\begin{aligned} \text{LHS} &= (P_3^2 + 4P_4^2 + P_5^2)^2 = (5^2 + 4\cdot 12^2 + 29^2)^2 \\ &= 2,079,364 = 2(5^4 + 16\cdot 12^4 + 29^4) \\ &= 2(P_3^4 + 16P_4^4 + P_5^4) = \text{RHS}. \end{aligned}$$

See Figure 8.2.

Letting $x = Q_{n-2}$ and $y = 2Q_{n-1}$. Then Candido's identity yields the Pell–Lucas identity

$$\left(Q_{n-2}^2 + 4Q_{n-1}^2 + Q_n^2\right)^2 = 2\left(Q_{n-2}^4 + 16Q_{n-1}^4 + Q_n^4\right). \tag{8.20}$$

For example, let $n = 5$. Then LHS = $(Q_3^2 + 4Q_4^2 + Q_5^2)^2 = (7^2 + 4 \cdot 17^2 + 41^2)^2 = 8,328,996 = 2(7^4 + 16 \cdot 17^4 + 41^4) = 2(Q_3^4 + 16Q_4^4 + Q_5^4) = \text{RHS}$, as expected.

8.10 Pell Determinants

We can evaluate special determinants containing Pell and Pell–Lucas numbers by using basic algebra, and the Pell recurrence and identities. For example, suppose we would like to evaluate the determinant

$$D = \begin{vmatrix} P_{n+3} & P_{n+2} & P_{n+1} & P_n \\ P_{n+2} & P_{n+3} & P_n & P_{n+1} \\ P_{n+1} & P_n & P_{n+3} & P_{n+2} \\ P_n & P_{n+1} & P_{n+2} & P_{n+3} \end{vmatrix}.$$

This looks like the determinant

$$A = \begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}.$$

This determinant was first evaluated in 1866 [173]:

$$A = (a + b + c + d)(a + b - c - d)(a - b + c - d)(a - b - c + d).$$

Using this formula,

$$\begin{aligned} D = &(P_{n+3} + P_{n+2} + P_{n+1} + P_n)(P_{n+3} + P_{n+2} - P_{n+1} - P_n)(P_{n+3} - P_{n+2} + P_{n+1} - P_n) \\ &\times (P_{n+3} - P_{n+2} - P_{n+1} + P_n). \end{aligned}$$

For convenience, we will now simplify each factor separately:

$$\begin{aligned} P_{n+3} + P_{n+2} + P_{n+1} + P_n &= P_{n+2} + 3P_{n+2} = 4P_{n+2} \\ P_{n+3} + P_{n+2} - P_{n+1} - P_n &= 2P_{n+2} + 2P_{n+1} = 2Q_{n+2} \\ P_{n+3} - P_{n+2} + P_{n+1} - P_n &= 2Q_{n+2} - 2Q_{n+1} = 2(Q_{n+2} - Q_{n+1}) \\ &= 2(2P_{n+1}) = 4P_{n+1} \\ P_{n+3} - P_{n+2} - P_{n+1} - P_n &= 2P_{n+2} - P_{n+2} - P_n = P_{n+2} + P_n. \\ &= 2Q_{n+1}. \end{aligned}$$

Thus

$$\begin{aligned} D &= 4P_{n+2} \cdot 2Q_{n+2} \cdot 4P_{n+1} \cdot 2Q_{n+1} \\ &= 4(2P_{n+2}Q_{n+2}) \cdot 4(2P_{n+1}Q_{n+1}) \\ &= 4P_{2n+4} \cdot 4P_{2n+2} = 16P_{2n+2}P_{2n+4}. \end{aligned}$$

Similarly we have

$$\begin{aligned} \begin{vmatrix} Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n \\ Q_{n+2} & Q_{n+3} & Q_n & Q_{n+1} \\ Q_{n+1} & Q_n & Q_{n+3} & Q_{n+2} \\ Q_n & Q_{n+1} & Q_{n+2} & Q_{n+3} \end{vmatrix} &= 4Q_{n+2} \cdot 4P_{n+2} \cdot 4P_{n+1} \cdot 4Q_{n+1} \\ &= 8(2P_{n+2}Q_{n+2}) \cdot 8(2P_{n+1}Q_{n+1}) \\ &= 64P_{2n+2}P_{2n+4}. \end{aligned}$$

Exercises 8

Establish the following identities, where $\{x_n\}$ satisfies the Pell recurrence.

1. $Q_n^2 = 6Q_{n-1}^2 - Q_{n-2}^2 + 4(-1)^n$. *Hint*: Use the Pell recurrence and Cassini's formula for Q_n.
2. $4(Q_nQ_{n+1} - P_nP_{n+1}) = Q_{2n+1} + 3(-1)^n$.
3. $2Q_nQ_{n+1} = Q_{2n+1} + (-1)^n$.
4. $P_n = P_kQ_{n-k} + Q_kP_{n-k}$, where $1 \le k \le n$.
5. $P_n = P_{k+1}P_{n-k} + P_kP_{n-k-1}$. *Hint*: Use PMI or the Binet-like formula for P_n.
6. $Q_n = P_{k+1}Q_{n-k} + P_kQ_{n-k-1}$.
7. $4(P_n^2 + Q_n^2) = 3Q_{2n} + (-1)^n$.
8. $4(3P_n^2 + Q_n^2) = 5Q_{2n} - (-1)^n$.
9. $3Q_n^2 + 2P_n^2 = 2Q_{2n} + (-1)^n$.
10. $P_{m+n}P_{m+r} - P_{m+n+r}P_m = (-1)^mP_nP_r$.
11. $P_{m+n}P_{m-n} - P_m^2 = (-1)^{m+n-1}P_n^2$.
12. $Q_{m+n}Q_{m+r} - Q_{m+n+r}Q_m = 2(-1)^{m-1}P_nP_r$.
13. $P_n^2 + P_{n+3}^2 = 5P_{2n+3}$.
14. $Q_n^2 + Q_{n+3}^2 = 10P_{2n+3}$.
15. $2(4Q_n^2 + Q_{n-1}^2 + 2Q_nQ_{n-1}) = 2Q_{2n+1} + Q_{2n-2} + (-1)^n$. *Hint*: Use the Pell recurrence and the Binet-like formula for P_n.

16. $Q^n = 2^{\lfloor n/2 \rfloor} \begin{bmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{bmatrix}$, where $Q = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ and n is odd.
17. $x_{n+1}^3 - 8x_n^3 - x_{n-1}^3 = 6x_{n+1}x_n x_{n-1}$.
18. $(x_{n-2}^2 + 4x_{n-1}^2 + x_n^2)^2 = 2(x_{n-2}^4 + 16x_{n-1}^4 + x_n^4)$.
19. Candido's identity.
20. Evaluate the determinant $\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}$.

9

Pascal's Triangle and the Pell Family

9.1 Introduction

Recall from Chapter 1 that Fibonacci and Lucas numbers are given by the explicit formulas

$$F_n = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-j-1}{j} \quad \text{and} \quad L_n = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{n}{n-j} \binom{n-j}{j},$$

where $n \geq 1$. Furthermore, both families can be extracted from Pascal's triangle.

Correspondingly, there are explicit formulas for Pell and Pell–Lucas numbers as well. They too can be extracted from Pascal's triangle.

The following theorem gives an explicit formula for P_n, which resembles closely the Lucas formula for F_n. We will establish the formula using strong induction.

Theorem 9.1

$$P_n = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-j-1}{j} 2^{n-2j-1}. \tag{9.1}$$

Proof. Since

$$\sum_{j=0}^{0} \binom{0}{0} 2^0 = 1 = P_1 \quad \text{and} \quad \sum_{j=0}^{0} \binom{1-j}{j} 2^{1-2j} = 2 = P_2,$$

the formula works when $n = 1$ and $n = 2$.

T. Koshy, *Pell and Pell–Lucas Numbers with Applications*, DOI 10.1007/978-1-4614-8489-9_9,

Suppose the formula works for all positive integers $n \leq k$, where k is an arbitrary integer ≥ 2. We will now show that it works when $n = k + 1$.

By the Pell recurrence, we have

$$\begin{aligned} P_{k+1} &= 2P_k + P_{k-1} \\ &= 2\sum_{j=0}^{\lfloor (k-1)/2 \rfloor} \binom{k-j-1}{j} 2^{k-2j-1} + \sum_{j=0}^{\lfloor (k-2)/2 \rfloor} \binom{k-j-2}{j} 2^{k-2j-2}. \end{aligned} \tag{9.2}$$

Case 1 Let $k = 2m + 1$ be odd. Then, by (9.2) and Pascal's identity, we have

$$\begin{aligned} P_{k+1} &= \sum_{j=0}^{m} \binom{2m-j}{j} 2^{2m-2j+1} + \sum_{j=0}^{m-1} \binom{2m-j-1}{j} 2^{2m-2j-1} \\ &= \sum_{j=0}^{m} \binom{2m-j}{j} 2^{2m-2j+1} + \sum_{j=1}^{m} \binom{2m-j}{j-1} 2^{2m-2j+1} \\ &= \sum_{j=0}^{m} \binom{2m-j}{j} 2^{2m-2j+1} + \sum_{j=0}^{m} \binom{2m-j}{j-1} 2^{2m-2j+1} \\ &= \sum_{j=0}^{m} \left[\binom{2m-j}{j} + \binom{2m-j}{j-1}\right] 2^{2m-2j+1} \\ &= \sum_{j=0}^{m} \binom{2m-j+1}{j} 2^{2m-2j+1}. \end{aligned}$$

Case 2 Let $k = 2m$ be even. Then (9.2) yields

$$\begin{aligned} P_{k+1} &= \sum_{j=0}^{m-1} \binom{2m-j-1}{j} 2^{2m-2j} + \sum_{j=0}^{m-1} \binom{2m-j-2}{j} 2^{2m-2j-2} \\ &= \sum_{j=0}^{m-1} \binom{2m-j-1}{j} 2^{2m-2j} + \sum_{j=1}^{m} \binom{2m-j-1}{j-1} 2^{2m-2j} \\ &= \sum_{j=0}^{m} \binom{2m-j-1}{j} 2^{2m-2j} + \sum_{j=0}^{m} \binom{2m-j-1}{j-1} 2^{2m-2j} \\ &= \sum_{j=0}^{m} \left[\binom{2m-j-1}{j} + \binom{2m-j-1}{j-1}\right] 2^{2m-2j} \\ &= \sum_{j=0}^{m} \binom{2m-j}{j} 2^{2m-2j}. \end{aligned}$$

It follows by Cases 1 and 2 that the formula also works when $n = k+1$. Thus, by the strong version of PMI, the formula is true for every positive integer n. ∎

It follows by the theorem that P_n can be computed by multiplying the binomial coefficients $\binom{n-j-1}{j}$ along the northeast diagonal beginning at $\binom{n-1}{0}$ in row $n-1$ with weights 2^{n-2j-1} and then adding up the products.

For example,

$$\begin{aligned} P_7 &= \sum_{j=0}^{3} \binom{6-j}{j} 2^{6-2j} = \binom{6}{0}2^6 + \binom{5}{1}2^4 + \binom{4}{2}2^2 + \binom{3}{3}2^0 \\ &= \textcircled{1}\cdot 2^6 + \textcircled{5}\cdot 2^4 + \textcircled{6}\cdot 2^2 + \textcircled{1}\cdot 2^0 \\ &= 64 + 80 + 24 + 1 = 169, \text{ as expected.} \end{aligned}$$

See the circled numbers in Figure 9.1.

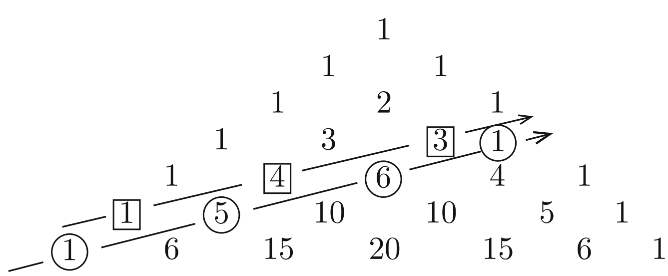

Figure 9.1.

Likewise, $P_6 = \binom{5}{0}2^5 + \binom{4}{1}2^3 + \binom{3}{2}2^1 = \boxed{1}\cdot 32 + \boxed{4}\cdot 8 + \boxed{3}\cdot 2 = 70$:

Northeast diagonal elements:	$\boxed{1}$	$\boxed{4}$	$\boxed{3}$
Weights:	2^5	2^3	2^1
Multiply:	32	32	6
Sum:	70		

See the boxed numbers in Figure 9.1.

Notice that formula (9.1) can be rewritten as

$$\sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-j-1}{j} = \frac{1}{2^{n-1}} P_n. \tag{9.3}$$

(We will use this result in Chapter 14 to develop yet another formula for P_n.)

Theorem 9.1 implies that Pell numbers can be extracted from the rising diagonals of a modified Pascal triangle. To this end, consider the triangular array in Figure 9.2; the elements on its ith descending diagonal are obtained by multiplying the elements of the corresponding diagonal in Pascal's triangle by 2^i, where $i \geq 0$.

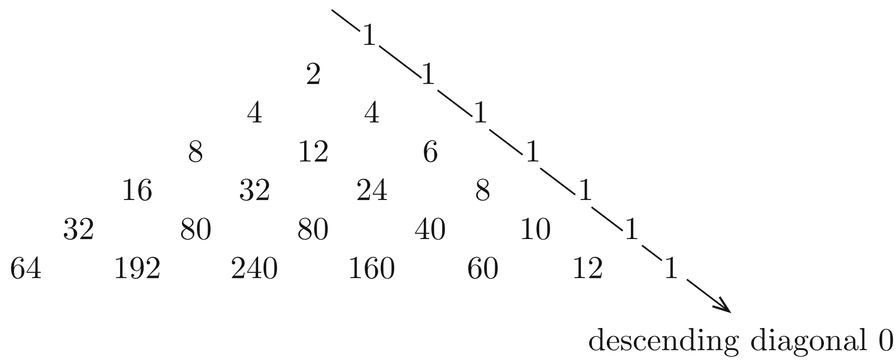

Figure 9.2.

Now, add the elements along the rising northeast diagonals; interestingly, each sum is a Pell number P_n. See Figure 9.3.

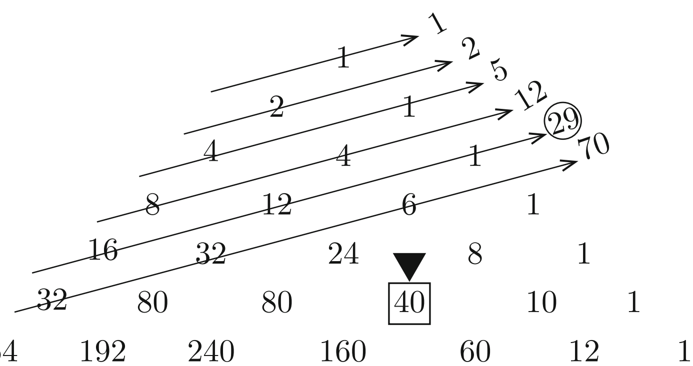

Figure 9.3.

For example, $P_5 = 16 + 12 + 1 = \textcircled{29}$ and $P_6 = 32 + 32 + 6 = 70$. Notice, for instance, that

$$\begin{aligned} P_7 &= 169 = 64 + 80 + 24 + 1 \\ &= 1 \cdot 2^6 + 5 \cdot 2^4 + 6 \cdot 2^2 + 1 \cdot 2^0 \\ &= \binom{6}{0} 2^6 + \binom{5}{1} 2^4 + \binom{4}{2} 2^2 + \binom{3}{3} 2^0. \end{aligned}$$

The array A in Figure 9.3 has two interesting properties:

1) Each element $A(n, r)$ can be defined recursively:

$$A(n,r) = \begin{cases} 2^n & \text{if } \ r = 0 \\ 2A(n-1,r) + A(n-1,r-1) & \text{if } \ 0 < r < n \\ 1 & \text{if } \ r = n. \end{cases}$$

For example, $2A(4,3) + A(4,2) = 2 \cdot 8 + 24 = \boxed{40} = A(5,3)$; see Figure 9.3.

2) The nth row sum is 3^n, where $n \geq 0$.
3) The elements in row n of array A are the coefficients in the binomial expansion of $(2+x)^n$: $(2+x)^n = \sum_{r=0}^{n} \binom{n}{r} 2^{n-r} x^r$.

9.2 An Alternate Approach

The odd-numbered binomial coefficients in row n with proper weights also can be used to compute Pell numbers, as D. Lind of Cambridge, England, did in 1970:

$$\begin{aligned}
(\gamma - \delta) P_n &= \gamma^n - \delta^n = (1+\sqrt{2})^n - (1-\sqrt{2})^n \\
&= \sum_{j=0}^{n} \binom{n}{j} \left[2^{j/2} - (-1)^j 2^{j/2}\right] = 2\sqrt{2} \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2r+1} 2^r \\
P_n &= \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2r+1} 2^r. \qquad (9.4)
\end{aligned}$$

For example, $P_5 = \sum_{r=0}^{2} \binom{5}{2r+1} 2^r = \binom{5}{1} 2^0 + \binom{5}{3} 2^1 + \binom{5}{5} 2^2 = \textcircled{5} \cdot 2^0 + \textcircled{10} \cdot 2^1 + \textcircled{1} \cdot 2^2 = 29$. See the circled numbers in row 5 in Figure 9.4.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & \textcircled{5} & & 10 & & \textcircled{10} & & 5 & & \textcircled{1} & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

Figure 9.4.

Using Pascal's identity, we can rewrite formula (9.4) in a different way:

$$\begin{aligned}
P_n &= \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2r+1} 2^r \\
&= \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} \left[\binom{n-1}{2r} + \binom{n-1}{2r+1}\right] 2^r \\
&= \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1}{2r} 2^r + \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1}{2r+1} 2^r
\end{aligned}$$

$$= \sum_{s \text{ even}} \binom{n-1}{s} 2^{s/2} + \sum_{s \text{ odd}} \binom{n-1}{s} 2^{(s-1)/2}$$

$$= \sum_{s \text{ even}} \binom{n-1}{s} 2^{\lfloor s/2 \rfloor} + \sum_{s \text{ odd}} \binom{n-1}{s} 2^{\lfloor s/2 \rfloor}$$

$$= \sum_{r=0}^{n-1} \binom{n-1}{r} 2^{\lfloor r/2 \rfloor}. \tag{9.5}$$

The beauty of this formula lies in the fact that we can compute P_n using the binomial coefficients $\binom{n-1}{r}$ in row $n-1$ of Pascal's triangle with weights $2^{\lfloor r/2 \rfloor}$.

For example,

$$P_5 = \sum_{r=0}^{4} \binom{4}{r} 2^{\lfloor r/2 \rfloor} = \binom{4}{0} 2^0 + \binom{4}{1} 2^0 + \binom{4}{2} 2^1 + \binom{4}{3} 2^1 + \binom{4}{4} 2^2$$

$$= \textcircled{1} \cdot 1 + \textcircled{4} \cdot 1 + \textcircled{6} \cdot 2 + \textcircled{4} \cdot 1 + \textcircled{1} \cdot 4 = \boxed{29}.$$

See Figure 9.5.

Row 4:	(1)	(4)	(6)	(4)	(1)
Weights:	2^0	2^0	2^1	2^1	2^2
Multiply:	1	4	12	8	4
Sum:	$\boxed{29}$				

Figure 9.5.

Next we develop an explicit formula for Q_n using its Binet-like version. We will then use the formula to compute Q_n from Pascal's triangle.

9.3 Another Explicit Formula for Q_n

Using the Binet-like formula for Q_n and Corollary 1.1, we can show that

$$Q_n = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} 2^j. \tag{9.6}$$

Thus Q_n can be computed using the even-numbered binomial coefficients $\binom{n}{2j}$ with weights 2^j, where $0 \le j \le \lfloor n/2 \rfloor$.

For example,

$$\begin{aligned} Q_6 &= \sum_{j=0}^{3} \binom{6}{2j} 2^j \\ &= \binom{6}{0} \cdot 2^0 + \binom{6}{2} \cdot 2^1 + \binom{6}{4} \cdot 2^2 + \binom{6}{6} \cdot 2^3 \\ &= \boxed{1} \cdot 1 + \boxed{15} \cdot 2 + \boxed{15} \cdot 4 + \boxed{1} \cdot 8 = 99. \end{aligned}$$

See the boxed numbers in row 6 in Figure 9.6.

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 1 & & 1 & & & & & \\ & & & & 1 & & 2 & & 1 & & & & \\ & & & 1 & & 3 & & 3 & & 1 & & & \\ & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ \boxed{1} & & 6 & & \boxed{15} & & 20 & & \boxed{15} & & 6 & & \boxed{1} \end{array}$$

Figure 9.6.

It follows from formula (9.6) that $Q_n = 1 + \sum_{j=1}^{\lfloor n/2 \rfloor} \binom{n}{2j} 2^j \equiv 1 \pmod 2$. So every Q_n is odd, a fact we already knew.

It also follows from formula (9.6) that

$$Q_{2n} = \sum_{j=0}^{n} \binom{2n}{2j} 2^j \tag{9.7}$$

$$Q_{2n-1} = \sum_{j=0}^{n-1} \binom{2n-1}{2j} 2^j. \tag{9.8}$$

Next we employ the Binet-like formula for P_n to find a recurrence for P_{2n}, as Lind did in 1970 [157].

9.4 A Recurrence for Even-numbered Pell Numbers

Using Corollary 1.1, we can establish that

$$P_{2n} = \sum_{j=0}^{n} \binom{n}{j} 2^j P_j. \tag{9.9}$$

Consequently, P_{2n} can be computed using the binomial coefficients $\binom{n}{j}$ in row n with weights $2^j P_j$, where $0 \le j \le n$.

For example,

$$\begin{aligned} P_8 &= \sum_{j=0}^{4} \binom{n}{j} 2^j P_j = \binom{4}{0} 2^0 P_0 + \binom{4}{1} 2^1 P_1 + \binom{4}{2} 2^2 P_2 + \binom{4}{3} 2^3 P_3 + \binom{4}{4} 2^4 P_4 \\ &= 0 + 4 \cdot 2 \cdot 1 + 6 \cdot 4 \cdot 2 + 4 \cdot 8 \cdot 5 + 1 \cdot 16 \cdot 12 = \boxed{408}. \end{aligned}$$

See Figure 9.7.

Row 4:	1	4	6	4	1
Weights:	$1 \cdot 0$	$2 \cdot 1$	$4 \cdot 2$	$8 \cdot 5$	$16 \cdot 12$
Multiply:	0	8	48	160	192
Sum:	$\boxed{408}$				

Figure 9.7.

We now develop an explicit formula for even-numbered Pell numbers.

9.5 Another Explicit Formula for P_{2n}

Using the Binet-like formula for P_n and Corollary 1.1, it follows that

$$P_{2n} = \sum_{j=1}^{n} \binom{2n}{2j-1} 2^{j-1}. \tag{9.10}$$

This gives an explicit formula for P_{2n} in terms of the odd-numbered binomial coefficients $\binom{2n}{2j-1}$ with weights 2^{j-1}.

For example,

$$\begin{aligned} P_6 &= \sum_{j=1}^{3} \binom{6}{2j-1} 2^{j-1} = \binom{6}{1} 2^0 + \binom{6}{3} 2^1 + \binom{6}{5} 2^2 \\ &= 6 \cdot 1 + 60 \cdot 2 + 6 \cdot 4 = 70. \end{aligned}$$

9.6 An Explicit Formula for P_{2n-1}

We can use formulas (9.10) and (9.6), coupled with identity (3) in Chapter 7, to extract a formula for odd-numbered Pell numbers:

$$\begin{aligned}
P_{2n-1} &= P_{2n-2} + Q_{2n-2} \\
&= \sum_{j=1}^{n-1} \binom{2n-2}{2j-1} 2^{j-1} + \sum_{j=0}^{n-1} \binom{2n-2}{2j} 2^j \\
&= \sum_{j=0}^{n-2} \left[\binom{2n-2}{2j+1} + \binom{2n-2}{2j} \right] 2^j + \binom{2n-2}{2n-2} 2^{n-1} \\
&= \sum_{j=0}^{n-2} \binom{2n-1}{2j+1} 2^j + \binom{2n-1}{2n-1} 2^{n-1} \\
&= \sum_{j=0}^{n-1} \binom{2n-1}{2j+1} 2^j. \qquad (9.11)
\end{aligned}$$

It follows from formula (9.11) that we can compute the odd-numbered Pell numbers P_{2n-1} using the odd-numbered binomial coefficients $\binom{2n-1}{2j+1}$ in row $2n-1$ with weights 2^j.

For example:

$$\begin{aligned}
P_7 &= \sum_{j=0}^{n-1} \binom{2n-1}{2j+1} 2^j = \binom{7}{1} 2^0 + \binom{7}{3} 2^1 + \binom{7}{5} 2^2 + \binom{7}{7} 2^3 \\
&= 7 + 35 \cdot 2 + 21 \cdot 4 + 8 = 169, \text{ as expected.}
\end{aligned}$$

Next we develop formulas for P_n^2 and Q_n^2, again using binomial coefficients.

9.7 Explicit Formulas for P_n^2 and Q_n^2

By identity (32) in Chapter 7 and formula (9.7), we have

$$\begin{aligned}
2Q_n^2 &= Q_{2n} + (-1)^n \\
&= \sum_{j=0}^{n} \binom{2n}{2j} 2^j + (-1)^n \\
&= \sum_{j=1}^{n} \binom{2n}{2j} 2^j + 1 + (-1)^n
\end{aligned}$$

$$Q_n^2 = \begin{cases} \sum_{j=1}^{n} \binom{2n}{2j} 2^{j-1} + 1 & \text{if } n \text{ is even} \\ \sum_{j=1}^{n} \binom{2n}{2j} 2^{j-1} & \text{otherwise.} \end{cases} \qquad (9.12)$$

For example, $Q_5^2 = \sum_{j=1}^{5} \binom{10}{2j} 2^{j-1} = 45 + 210 \cdot 2 + 210 \cdot 4 + 45 \cdot 8 + 1 \cdot 16 = 1681 = 41^2$. Likewise, using identity (31) in Chapter 7, we have

$$2P_n^2 = \begin{cases} \sum_{j=1}^{n} \binom{2n}{2j} 2^{j-1} & \text{if } n \text{ is even} \\ \sum_{j=1}^{n} \binom{2n}{2j} 2^{j-1} + 1 & \text{otherwise.} \end{cases} \tag{9.13}$$

For example, $2P_5^2 = \sum_{j=1}^{5} \binom{10}{2j} 2^{j-1} + 1 = 1681 + 1$; so $P_5^2 = 841 = 29^2$.

As byproducts of formulas (9.12) and (9.13), it follows that $\sum_{j=1}^{n} \binom{2n}{2j} 2^{j-1} + 1$ and $\sum_{j=1}^{n} \binom{2n}{2j} 2^{j}$ are squares if n is even; and so are $\sum_{j=1}^{n} \binom{2n}{2j} 2^{j-1}$ and $\frac{1}{2}\left[\sum_{j=1}^{n} \binom{2n}{2j} 2^{j-1} + 1\right]$ if n is odd.

For example, $\frac{1}{2}\left[\sum_{j=1}^{7} \binom{14}{2j} 2^{j-1} + 1\right] = \frac{1}{2}(57,122) = 28,561 = 169^2$ and $\sum_{j=1}^{7} \binom{14}{2j} 2^{j-1} = 57,121 = 239^2$.

It now follows that $\sum_{j=1}^{2n-1} \binom{4n-2}{2j} 2^{j-1}$ is always a square, as we saw in Example 7.6.

We will now develop another method for computing Pell–Lucas numbers and odd-numbered Pell numbers from Pascal's triangle. In the process, we will show how Lucas numbers and odd-numbered Fibonacci numbers can be extracted from the array. To this end, we need an identity that can be obtained from the binomial theorem.

9.8 Lockwood's Identity

Let x and y be arbitrary real numbers. Then, by the binomial theorem, we have

$$\begin{aligned} x + y &= (x + y) \\ x^2 + y^2 &= (x + y)^2 - 2xy \\ x^3 + y^3 &= (x + y)^3 - 3(xy)(x + y) \\ x^4 + y^4 &= (x + y)^4 - 4(xy)(x + y)^2 + 2(xy)^2 \\ x^5 + y^5 &= (x + y)^5 - 5(xy)(x + y)^3 + 5(xy)^2(x + y). \end{aligned}$$

In each case, the expression $x^n + y^n$ is expressed as a sum of $\lfloor n/2 \rfloor + 1$ terms in xy and $x + y$.

More generally, we have the following identity, developed by E.H. Lockwood in 1967 [15]:

$$x^n + y^n = (x+y)^n + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] (xy)^k (x+y)^{n-2k},$$

where $n \geq 1$. This identity can be confirmed using strong induction, Pascal's identity, and a lot of algebra:

Proof. When $n = 1$:

$$\begin{aligned} \text{RHS} &= (x+y)^1 + \sum_{k=1}^{0} (-1)^k \left[\binom{1-k}{k} + \binom{0-k}{k-1} \right] (xy)^k (x+y)^{1-2k} \\ &= (x+y) + 0 = \text{LHS}. \end{aligned}$$

When $n = 2$:

$$\begin{aligned} \text{RHS} &= (x+y)^2 + \sum_{k=1}^{1} (-1)^k \left[\binom{2-k}{k} + \binom{1-k}{k-1} \right] (xy)^k (x+y)^{2-2k} \\ &= (x+y)^2 - \left[\binom{1}{1} + \binom{0}{0} \right] (xy)(x+y)^0 = (x+y)^2 - 2xy \\ &= x^2 + y^2 = \text{LHS}. \end{aligned}$$

So the identity is true when $n = 1$ and $n = 2$.

Now assume that it is true for all positive integers $\leq n$, where n is an arbitrary integer ≥ 2; that is, assume that

$$(x+y)^n = x^n + y^n - \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] (xy)^k (x+y)^{n-2k}.$$

Then

$$\begin{aligned} (x+y)^{n+1} = x^{n+1} + y^{n+1} + x^n y + xy^n - \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] \\ \times (xy)^k (x+y)^{n+1-2k}. \end{aligned}$$

Case 1 Let n be even; say, $n = 2m$. Then,

$$\begin{aligned} (x+y)^{2m+1} &= x^{2m+1} + y^{2m+1} + x^{2m} y + xy^{2m} \\ &\quad - \sum_{k=1}^{m} (-1)^k \left[\binom{2m-k}{k} + \binom{2m-1-k}{k-1} \right] (xy)^k (x+y)^{2m+1-2k} \end{aligned}$$

$$
\begin{aligned}
&= x^{2m+1} + y^{2m+1} + x^{2m}y + xy^{2m} \\
&\quad -\sum_{k=1}^{m}(-1)^k\left[\binom{2m-k}{k} + \binom{2m-1-k}{k-1}\right](xy)^k(x+y)^{2m+1-2k} \\
&\quad -(xy)(x+y)^{2m-1} + (xy)(x+y)^{2m-1} \\
&= x^{2m+1} + y^{2m+1} \\
&\quad -\sum_{k=1}^{m}(-1)^k\left[\binom{2m-k}{k} + \binom{2m-1-k}{k-1}\right](xy)^k(x+y)^{2m+1-2k} + \\
&\quad \sum_{k=1}^{m-1}(-1)^k\left[\binom{2m-1-k}{k} + \binom{2m-2-k}{k-1}\right](xy)^{k+1}(x+y)^{2m-1-2k} \\
&\quad +(xy)(x+y)^{2m-1} \\
&= x^{2m+1} + y^{2m+1} + \left[\binom{2m-1}{1} + \binom{2m-2}{0} + 1\right](xy)(x+y)^{2m-1} \\
&\quad -\sum_{k=2}^{m}(-1)^k\left[\binom{2m-k}{k} + \binom{2m-1-k}{k-1}\right](xy)^k(x+y)^{2m+1-2k} \\
&\quad +\sum_{k=2}^{m}(-1)^k\left[\binom{2m-k}{k-1} + \binom{2m-1-k}{k-2}\right](xy)^k(x+y)^{2m+1-2k} \\
&= x^{2m+1} + y^{2m+1} + \left[\binom{2m}{1} + \binom{2m-1}{0}\right](xy)(x+y)^{2m-1} \\
&\quad +\sum_{k=2}^{m}(-1)^k\left\{\left[\binom{2m-k}{k} + \binom{2m-k}{k-1}\right]\right. \\
&\quad \left. + \left[\binom{2m-1-k}{k-1} + \binom{2m-1-k}{k-2}\right]\right\}(xy)^k(x+y)^{2m+1-2k} \\
&= x^{2m+1} + y^{2m+1} + \left[\binom{2m}{1} + \binom{2m-1}{0}\right](xy)(x+y)^{2m-1} \\
&\quad +\sum_{k=2}^{m}(-1)^k\left[\binom{2m+1-k}{k} + \binom{2m-k}{k-1}\right](xy)^k(x+y)^{2m+1-2k}
\end{aligned}
$$

$$= x^{2m+1} + y^{2m+1} + \sum_{k=1}^{m} (-1)^k \left[\binom{2m+1-k}{k} + \binom{2m-k}{k-1} \right] (xy)^k (x+y)^{2m+1-2k}$$

$$= x^{n+1} + y^{n+1} + \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (-1)^k \left[\binom{n+1-k}{k} + \binom{n-k}{k-1} \right] (xy)^k (x+y)^{n+1-2k}.$$

So the formula works for $n+1$, when n is even.

Similarly, we can show that the formula works when n is odd. Thus, by PMI, the identity works for all positive integers n. ∎

Lockwood's identity can be rewritten as follows:

$$x^n + y^n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] (xy)^k (x+y)^{n-2k}. \tag{9.14}$$

where $\binom{r}{-1} = 0$. It follows from equation (9.14), for example, that

$$x^7 + y^7 = (x+y)^7 - 7(xy)(x+y)^5 + 14(xy)^2(x+y)^3 - 7(xy)^3(x+y).$$

9.9 Lucas Numbers and Pascal's Triangle

Lockwood's identity yields several interesting dividends. First, we can extract Lucas numbers from Pascal's triangle. To see this, we let $x = \alpha$ and $y = \beta$ in (9.14). Then it yields

$$L_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] (-1)^k$$

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right]. \tag{9.15}$$

Consequently, L_n can be computed by adding up the elements along two alternate rising diagonals. For example, $L_7 = \sum_{k=0}^{3} \left[\binom{7-k}{k} + \binom{6-k}{k-1}\right] = \left[\binom{7}{0} + \binom{6}{-1}\right] + \left[\binom{6}{1} + \binom{5}{0}\right] + \left[\binom{5}{2} + \binom{4}{1}\right] + \left[\binom{4}{3} + \binom{3}{2}\right] = (1+0) + (6+1) + (10+4) + (4+3) = (0+1+4+3) + (1+6+10+4) = \textcircled{29}$. See the bold-faced numbers Figure 9.8.

Notice that formula (9.14) can also be written as follows:

$$x^n + y^n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (xy)^k (x+y)^{n-2k}. \tag{9.16}$$

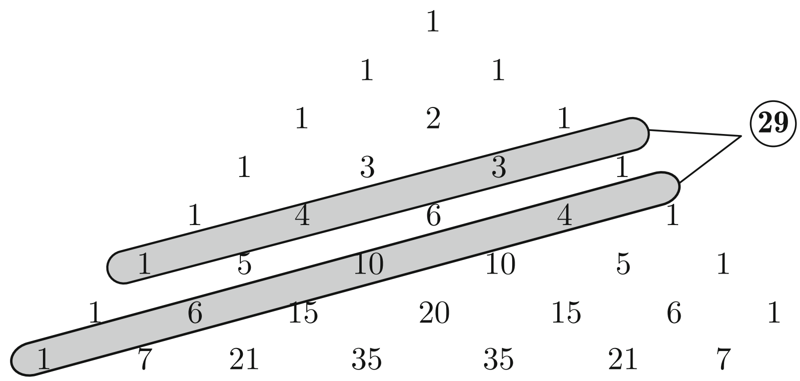

Figure 9.8.

Consequently,

$$L_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k}\binom{n-k}{k}. \tag{9.17}$$

Thus L_n can be computed using the elements on the rising diagonal beginning at $\binom{n}{0}$ with weights $\frac{n}{n-k}$. For example, $L_7 = \sum_{k=0}^{3} \frac{7}{7-k}\binom{7-k}{k} = \frac{7}{7}\binom{7}{0} + \frac{7}{6}\binom{6}{1} + \frac{7}{5}\binom{5}{2} + \frac{7}{4}\binom{4}{3} = 1 + 7 + 14 + 7 = 29$, as expected.

Formula (9.15), coupled with Lucas' formula, yields a well-known formula connecting Fibonacci and Lucas numbers:

$$\begin{aligned} L_n &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k-1}{k-1} + \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \\ &= \sum_{i=0}^{\lfloor (n-2)/2 \rfloor} \binom{n-i-2}{i} + \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \\ &= F_{n-1} + F_{n+1}. \end{aligned}$$

This can also be established using Binet's formulas; this approach is a lot simpler [126].

Next we turn to Pell and Pell–Lucas numbers.

9.10 Pell–Lucas Numbers and Pascal's Triangle

We can extract Pell–Lucas numbers also from Pascal's triangle with proper weights. To see this, we let $x = \gamma$ and $y = \delta$ in (9.14). Then:

$$Q_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \left[\binom{n-k}{k} + \binom{n-k-1}{k-1}\right] 2^{n-2k-1} \tag{9.18}$$

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k}\binom{n-k}{k} 2^{n-2k-1}. \tag{9.19}$$

For example, $Q_7 = \sum_{k=0}^{3}\left[\binom{7-k}{k}+\binom{6-k}{k-1}\right]2^{6-2k} = \left[\binom{7}{0}+\binom{6}{-1}\right]2^6 + \left[\binom{6}{1}+\binom{5}{0}\right]2^4 + \left[\binom{5}{2}+\binom{4}{1}\right]2^2 + \left[\binom{4}{3}+\binom{3}{2}\right]2^0 = (1+0)\cdot 2^6 + (6+1)\cdot 2^4 + (10+4)\cdot 2^2 + (4+3)\cdot 2^0 = 239$. Consequently, we can compute Q_7 by multiplying the sums of the entries inside the loops beginning at $\binom{7}{0}$ in Figure 9.9 by the weights $2^6, 2^4, 2^2$ and 2^0, respectively, and then adding up the products.

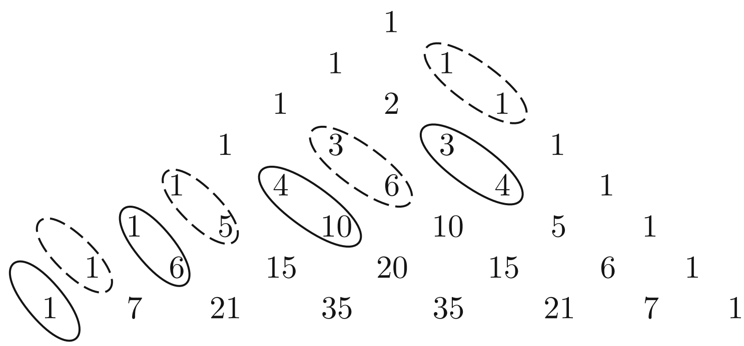

Figure 9.9.

Likewise, $Q_6 = (1+0)\cdot 2^5 + (5+1)\cdot 2^3 + (6+3)\cdot 2^1 + (1+1)\cdot 2^{-1} = 99$; see the dotted loops in Figure 9.9.

9.11 Odd-Numbered Fibonacci Numbers and Pascal's Triangle

Next we will show that odd-numbered Fibonacci numbers can be computed from Pascal's triangle in a different way. To this end, we let n be odd and change y to $-y$ in (9.14). Then

$$x^n - y^n = \sum_{k=0}^{(n-1)/2} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1}\right] (-xy)^k (x-y)^{n-2k}. \qquad (9.20)$$

Letting $x = \alpha$ and $y = \beta$, this yields

$$\begin{aligned}
(\alpha-\beta)F_n &= \sum_{k=0}^{(n-1)/2} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1}\right] (\alpha-\beta)^{n-2k} \\
F_n &= \sum_{k=0}^{(n-1)/2} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1}\right] 5^{(n-2k-1)/2} \qquad (9.21) \\
&= \sum_{k=0}^{(n-1)/2} (-1)^k \frac{n}{n-k}\binom{n-k}{k} 5^{(n-2k-1)/2},
\end{aligned}$$

where n is odd.

For example, $F_7 = \sum_{k=0}^{3}(-1)^k\left[\binom{7-k}{k}+\binom{6-k}{k-1}\right]5^{3-k} = \left[\binom{7}{0}+\binom{6}{-1}\right]5^3 - \left[\binom{6}{1}+\binom{5}{0}\right]5^2 + \left[\binom{5}{2}+\binom{4}{1}\right]5^1 - \left[\binom{4}{3}+\binom{3}{2}\right]5^0 = (1+0)\cdot 5^3 - (6+1)\cdot 5^2 + (10+4)\cdot 5^1 - (4+3)\cdot 5^0 = 13.$ See the solid loops in Figure 9.9.

9.12 Odd-Numbered Pell Numbers and Pascal's Triangle

Using formula (9.20), we can compute odd-numbered Pell numbers from Pascal's triangle. To see this, letting $x = \gamma$ and $y = \delta$, formula (9.20) yields

$$\begin{aligned}
(\gamma-\delta)P_n &= \sum_{k=0}^{(n-1)/2}(-1)^k\left[\binom{n-k}{k}+\binom{n-k-1}{k-1}\right](-\gamma\delta)^k(\gamma-\delta)^{n-2k} \\
P_n &= \sum_{k=0}^{(n-1)/2}(-1)^k\left[\binom{n-k}{k}+\binom{n-k-1}{k-1}\right](2\sqrt{2})^{n-2k-1} \\
&= \sum_{k=0}^{(n-1)/2}(-1)^k\left[\binom{n-k}{k}+\binom{n-k-1}{k-1}\right]8^{(n-2k-1)/2} \qquad (9.22) \\
&= \sum_{k=0}^{(n-1)/2}(-1)^k\frac{n}{n-k}\binom{n-k}{k}8^{(n-2k-1)/2},
\end{aligned}$$

where n is odd. Thus, P_n can be computed using the same loops for F_n, but with different weights, where n is odd.

For example, $P_7 = \sum_{k=0}^{3}(-1)^k\left[\binom{7-k}{k}+\binom{6-k}{k-1}\right]8^{3-k} = (1+0)\cdot 8^3 - (6+1)\cdot 8^2 + (10+4)\cdot 8^1 - (4+3)\cdot 8^0 = 169$. See the loops in Figure 9.9.

9.13 Pell Summation Formulas

Just as we developed Fibonacci and Lucas summation formulas using Corollary 1.1, we can develop similar Pell summation formulas.

Theorem 9.2

$$\sum_{i=0}^{n}\binom{n}{i}P_i = \begin{cases} 2^{n/2}P_n & \textit{if } n \textit{ is even} \\ 2^{(n-1)/2}Q_n & \textit{otherwise.} \end{cases} \qquad (9.23)$$

Proof. By Corollary 1.1, we have

$$\begin{aligned}
(\gamma-\delta)\cdot \text{LHS} &= \sum_{i=0}^{n}\binom{n}{i}(\gamma^i-\delta^i)\\
&= \sum_{i=0}^{n}\binom{n}{i}\gamma^i-\sum_{i=0}^{n}\binom{n}{i}\delta^i\\
&= (1+\gamma)^n-(1+\delta)^n\\
&= (\sqrt{2}\gamma)^n-(-\sqrt{2}\delta)^n\\
&= 2^{n/2}\left[\gamma^n-(-\delta)^n\right]\\
\text{LHS} &= \begin{cases} 2^{n/2}P_n & \text{if } n \text{ is even}\\ 2^{n/2}\frac{\gamma^n+\delta^n}{\gamma-\delta} & \text{otherwise}\end{cases}\\
&= \begin{cases} 2^{n/2}P_n & \text{if } n \text{ is even}\\ 2^{(n-1)/2}Q_n & \text{otherwise}\end{cases}\\
&= \text{RHS}.
\end{aligned}$$

■

For example, $\sum_{i=0}^{4}\binom{4}{i}P_i = \binom{4}{0}P_0+\binom{4}{1}P_1+\binom{4}{2}P_2+\binom{4}{3}P_3+\binom{4}{4}P_4 = 1\cdot 0+4\cdot 1+6\cdot 2+4\cdot 5+1\cdot 12 = 48 = 4P_4$; and similarly, $\sum_{i=0}^{3}\binom{3}{i}P_i = 14 = 2Q_3$.

As in Theorem 9.2, a quite similar formula for $\sum_{i=0}^{n}\binom{n}{i}Q_i$ can be developed:

$$\sum_{i=0}^{n}\binom{n}{i}Q_i = \begin{cases} 2^{n/2}Q_n & \text{if } n \text{ is even}\\ 2^{(n+1)/2}P_n & \text{otherwise.}\end{cases} \tag{9.24}$$

For example, $\sum_{i=0}^{4}\binom{4}{i}Q_i = \binom{4}{0}Q_0+\binom{4}{1}Q_1+\binom{4}{2}Q_2+\binom{4}{3}Q_3+\binom{4}{4}Q_4 = 1\cdot 1+4\cdot 1+6\cdot 3+4\cdot 7+1\cdot 17 = 68 = 2^2Q_4$; and similarly, $\sum_{i=0}^{5}\binom{5}{i}Q_i = 232 = 2^3P_5$.

Corollary 1.1 can also be employed to develop formulas for $\sum_{i=0}^{n}(-1)^i\binom{n}{i}P_i$ and $\sum_{i=0}^{n}(-1)^i\binom{n}{i}Q_i$:

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}P_i = \begin{cases} 0 & \text{if } n \text{ is even}\\ -2^{(n-1)/2} & \text{otherwise}\end{cases} \tag{9.25}$$

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}Q_i = \begin{cases} 2^{n/2} & \text{if } n \text{ is even} \\ 0 & \text{otherwise.}\end{cases} \tag{9.26}$$

For example, $\sum_{i=0}^{4}(-1)^i\binom{4}{i}P_i = 1\cdot 0-4\cdot 1+6\cdot 2-4\cdot 5+1\cdot 12=0$, and $\sum_{i=0}^{5}(-1)^i\binom{5}{i}Q_i = 1\cdot 1-5\cdot 1+10\cdot 3-10\cdot 7+5\cdot 17-1\cdot 41=0$.

The following theorem gives a summation formula for the numbers P_i^2 with the binomial coefficients $\binom{n}{i}$ as the corresponding weights.

Theorem 9.3

$$\sum_{i=0}^{n}\binom{n}{i}P_i^2 = \begin{cases} 2^{(3n-4)/2}Q_n & \textit{if } n \textit{ is even} \\ 2^{3(n-1)/2}P_n & \textit{otherwise.}\end{cases}$$

Proof. Again, by Corollary 1.1, we have

$$\begin{aligned}
8\sum_{i=0}^{n}\binom{n}{i}P_i^2 &= \sum_{i=0}^{n}\binom{n}{i}\left(\gamma^i-\delta^i\right)^2 \\
&= \sum_{i=0}^{n}\binom{n}{i}\left[\gamma^{2i}+\delta^{2i}-2(-1)^i\right] \\
&= \sum_{i=0}^{n}\binom{n}{i}\gamma^{2i}+\sum_{i=0}^{n}\binom{n}{i}\delta^{2i}-2\cdot 0 \\
&= \left(1+\gamma^2\right)^n+\left(1+\delta^2\right)^n \\
&= (2\sqrt{2}\gamma)^n+(-2\sqrt{2}\delta)^n \\
&= \begin{cases}(2\sqrt{2})^n\cdot 2Q_n & \text{if } n \text{ is even} \\ (2\sqrt{2})^n\cdot 2\sqrt{2}P_n & \text{otherwise}\end{cases} \\
\sum_{i=0}^{n}\binom{n}{i}P_i^2 &= \begin{cases}2^{(3n-4)/2}Q_n & \text{if } n \text{ is even} \\ 2^{3(n-1)/2}P_n & \text{otherwise.}\end{cases}
\end{aligned}$$

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Similarly, it can be shown that

$$\sum_{i=0}^{n}\binom{n}{i}Q_i^2 = \begin{cases} 2^{(3n-2)/2}Q_n & \text{if } n \text{ is even} \\ 2^{(3n-1)/2}P_n & \text{otherwise.}\end{cases}$$

For example, $\sum_{i=0}^{4}\binom{4}{i}P_i^2 = 272 = 16\cdot 17 = 2^{(3\cdot 4-4)/2}Q_4$, and $\sum_{i=0}^{4}\binom{4}{i}Q_i^2 = 544 = 32\cdot 17 = 2^{(3\cdot 4-2)/2}Q_4$.

Finally, using Corollary 1.1, we can develop formulas for $\sum_{i=0}^{n}(-1)^i\binom{n}{i}P_i^2$ and $\sum_{i=0}^{n}(-1)^i\binom{n}{i}Q_i^2$:

$$\begin{aligned}\sum_{i=0}^{n}(-1)^i\binom{n}{i}P_i^2 &= 2^{n-2}\left[(-1)^n Q_n - 1\right]\\ \sum_{i=0}^{n}(-1)^i\binom{n}{i}Q_i^2 &= 2^{n-1}\left[(-1)^n Q_n + 1\right].\end{aligned}$$

For example, $\sum_{i=0}^{5}(-1)^i\binom{5}{i}P_i^2 = -336 = 8(-41-1) = 2^{5-2}\left[(-1)^5Q_5 - 1\right]$ and $\sum_{i=0}^{4}(-1)^i\binom{4}{i}Q_i^2 = 144 = 8(17+1) = 2^{4-1}\left[(-1)^4Q_4 + 1\right]$.

Exercises 9

1. Establish the recurrence $A(n,r) = 2A(n-1,r) + A(n-1,r-1)$ for the array A in Figure 9.3.
2. Define $A(n,r)$ recursively.
3. Find an explicit formula for $A(n,r)$.

Prove each.

4. $\sum_{r=0}^{n} A(n,r) = 3^n$.
5. $A(n,0) = 2^n$, where $n \geq 0$.
6. $A(n,n) = A(n-1,n-1)$, where $n \geq 1$.
7. $A(n,n) = 1$, where $n \geq 1$.
8. $A(n,n-1) = 2n$, where $n \geq 1$.
9. Let D_n denote the nth rising diagonal sum of array A. Then $D_n = P_{n+1}$, where $n \geq 0$.

Prove the following Pell summation formulas.

10. $Q_n = \sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j}2^j$.
11. $P_{2n} = \sum_{j=0}^{n}\binom{n}{j}2^jP_j$.
12. $P_{2n} = \sum_{j=1}^{n}\binom{2n}{2j-1}2^{j-1}$.
13. $2P_n^2 = \begin{cases}\sum_{j=1}^{n}\binom{2n}{2j}2^{j-1} & \text{if } n \text{ is even}\\ \sum_{j=1}^{n}\binom{2n}{2j}2^{j-1}+1 & \text{otherwise.}\end{cases}$

14. $\sum_{i=0}^{n} \binom{n}{i} Q_i = \begin{cases} 2^{n/2} Q_n & \text{if } n \text{ is even} \\ 2^{(n+1)/2} P_n & \text{otherwise.} \end{cases}$

15. $\sum_{i=0}^{n} (-1)^i \binom{n}{i} P_i = \begin{cases} 0 & \text{if } n \text{ is even} \\ -2^{(n-1)/2} & \text{otherwise.} \end{cases}$

16. $\sum_{i=0}^{n} (-1)^i \binom{n}{i} Q_i = \begin{cases} 2^{n/2} & \text{if } n \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$

17. $\sum_{i=0}^{n} \binom{n}{i} Q_i^2 = \begin{cases} 2^{(3n-2)/2} Q_n & \text{if } n \text{ is even} \\ 2^{(3n-1)/2} P_n & \text{otherwise.} \end{cases}$

18. $\sum_{i=0}^{n} (-1)^i \binom{n}{i} P_i^2 = 2^{n-2} [(-1)^n Q_n - 1].$

19. $\sum_{i=0}^{n} (-1)^i \binom{n}{i} Q_i^2 = 2^{n-1} [(-1)^n Q_n + 1].$

10

Pell Sums and Products

10.1 Introduction

In this chapter we investigate some finite and infinite Pell and Pell–Lucas sums; some infinite sums involving the Fibonacci and Pell families; a Pell inequality; and then an infinite product involving Pell numbers. In Chapter 14, we will study additional Pell and Pell–Lucas sums.

10.2 Pell and Pell–Lucas Sums

Telescoping sums, the fundamental identities, Corollary 1.1, and PMI can be used to derive a number of Pell and Pell–Lucas summation formulas. Some of them are

$$\sum_{i=1}^{n} P_i = \frac{Q_{n+1}-1}{2} \tag{10.1}$$

$$\sum_{i=1}^{n} Q_i = P_{n+1}-1 \tag{10.2}$$

$$\sum_{i=1}^{n} P_{2i-1} = \frac{P_{2n}}{2} \tag{10.3}$$

$$\sum_{i=1}^{n} P_{2i} = \frac{P_{2n+1}-1}{2} \tag{10.4}$$

$$\sum_{i=1}^{n} Q_{2i-1} = \frac{Q_{2n}-1}{2} \tag{10.5}$$

$$\sum_{i=1}^{n} Q_{2i} = \frac{Q_{2n+1}-1}{2} \tag{10.6}$$

T. Koshy, *Pell and Pell–Lucas Numbers with Applications*, DOI 10.1007/978-1-4614-8489-9_10,

$$\sum_{i=1}^{n} P_i^2 = \begin{cases} \frac{2P_{2n+1}-Q_{2n}+1}{8} & \text{if } n \text{ is odd} \\ \frac{2P_{2n+1}-Q_{2n}-1}{8} & \text{otherwise} \end{cases} \tag{10.7}$$

$$\sum_{i=1}^{n} Q_i^2 = \begin{cases} \frac{2P_{2n+1}-Q_{2n}-3}{4} & \text{if } n \text{ is odd} \\ \frac{2P_{2n+1}-Q_{2n}-1}{4} & \text{otherwise} \end{cases} \tag{10.8}$$

$$\sum_{i=1}^{n} P_i P_{i+1} = \frac{Q_n^2-1}{4} \tag{10.9}$$

$$\sum_{i=1}^{n} Q_i Q_{i+1} = \begin{cases} \frac{Q_n^2-3}{4} & \text{if } n \text{ is odd} \\ \frac{Q_n^2-1}{4} & \text{otherwise.} \end{cases} \tag{10.10}$$

In the interest of brevity, we will not prove them; see Exercises 1–10.

Since $Q_{2n} = 4P_n^2 + (-1)^n$ and $Q_n^2 = 2P_n^2 + (-1)^n$, formula (10.5) can be rewritten as follows:

$$\begin{aligned} \sum_{k=0}^{n} Q_{2k+1} &= \frac{4P_{n+1}^2 - 1 - (-1)^n}{2} \\ &= \begin{cases} 2P_{n+1}^2 & \text{if } n \text{ is odd} \\ 2P_{n+1}^2 - 1 & \text{otherwise} \end{cases} \\ &= \begin{cases} 2P_{n+1}^2 & \text{if } n \text{ is odd} \\ Q_{n+1}^2 & \text{otherwise.} \end{cases} \end{aligned} \tag{10.11}$$

This formula reveals the interesting pattern we observed in Table 3.6:

$$\begin{aligned} 1 &= 1^2 \\ 1+7 &= 2\cdot 2^2 \\ 1+7+41 &= 7^2 \\ 1+7+41+239 &= 2\cdot 12^2 \\ 1+7+41+239+1393 &= 41^2 \\ 1+7+41+239+1393+8119 &= 2\cdot 70^2 \\ 1+7+41+239+1393+8119+47321 &= 239^2. \end{aligned}$$

It follows by the Binet-like formula for P_k and (10.11) that

$$4\sum_{k=0}^{n} P_k P_{k+1} = \sum_{k=0}^{n} Q_{2k+1} - \sum_{k=0}^{n} (-1)^k$$

$$= \begin{cases} 2P_{n+1}^2 & \text{if } n \text{ is odd} \\ Q_{n+1}^2 - 1 & \text{otherwise;} \end{cases}$$

$$\sum_{k=0}^{n} P_k P_{k+1} = \begin{cases} \frac{1}{2}P_{n+1}^2 & \text{if } n \text{ is odd} \\ \frac{Q_{n+1}^2-1}{4} & \text{otherwise.} \end{cases}$$

For example, $\sum_{k=0}^{5} P_k P_{k+1} = 2450 = \frac{1}{2}P_6^2$ and $\sum_{k=0}^{4} P_k P_{k+1} = 420 = \frac{Q_5^2-1}{4}$.
Since $2Q_k Q_{k+1} = Q_{2k+1} + (-1)^k$, it follows by (10.11) that

$$\sum_{k=0}^{n} Q_k Q_{k+1} = \begin{cases} P_{n+1}^2 & \text{if } n \text{ is odd} \\ \frac{Q_{n+1}^2+1}{2} & \text{otherwise.} \end{cases}$$

For example, $\sum_{k=0}^{3} Q_k Q_{k+1} = 144 = P_4^2$ and $\sum_{k=0}^{4} Q_k Q_{k+1} = 841 = \frac{Q_5^2+1}{2}$.

The following interesting problem was proposed in 2009 by Brian Bradie of Christopher Newport University, Newport News, Virginia [24].

Example 10.1 Let $a_n = \left(2\sum_{i=0}^{n} Q_i\right)^2 - 2\sum_{i=0}^{n} Q_{2i+1}$, where $i \geq 0$. Evaluate $\sum_{i=0}^{\infty} \frac{a_n}{n!}$.

Solution. Using formula (10.1), $\sum_{i=0}^{n} Q_i = P_{n+1}$. By formula (10.5), $2\sum_{i=0}^{n} Q_{2i+1} = Q_{2n+2} - 1$. Using identity (33) in Chapter 7, we can rewrite this as $2\sum_{i=0}^{n} Q_{2i+1} = 4P_{n+1}^2 - 1 - (-1)^n$. Thus,

$$\begin{aligned} a_n &= 4P_{n+1}^2 - \left[4P_{n+1}^2 - 1 - (-1)^n\right] \\ &= 1 + (-1)^n \\ \sum_{i=0}^{\infty} \frac{a_n}{n!} &= \sum_{i=0}^{\infty} \frac{1}{n!} + \sum_{i=0}^{\infty} \frac{(-1)^n}{n!} \\ &= e + \frac{1}{e} \\ &\approx 3.08616126963. \end{aligned}$$

■

10.3 Infinite Pell and Pell–Lucas Sums

Using the identities we have developed thus far, we can evaluate infinite sums involving members of the Pell family. The next two examples illustrate this.

Example 10.2 Evaluate the infinite sum $\sum_{n=1}^{\infty} \frac{Q_n}{P_{n+1}P_n}$.

Solution. Since $Q_n = P_{n+1} - P_n$, we have

$$\begin{aligned} \frac{Q_n}{P_{n+1}P_n} &= \frac{P_{n+1} - P_n}{P_{n+1}P_n} \\ &= \frac{1}{P_n} - \frac{1}{P_{n+1}} \end{aligned}$$

$$\begin{aligned} \sum_{n=1}^{k} \frac{Q_n}{P_{n+1}P_n} &= \sum_{n=1}^{k} \left(\frac{1}{P_n} - \frac{1}{P_{n+1}} \right) \\ &= 1 - \frac{1}{P_{k+1}} \\ \sum_{n=1}^{\infty} \frac{Q_n}{P_{n+1}P_n} &= \lim_{k\to\infty} \left(1 - \frac{1}{P_{k+1}} \right) \\ &= 1 - 0 \\ &= 1, \end{aligned}$$

where we have used the *telescoping sum* $\sum_{k=1}^{n} (a_k - a_{k-1}) = a_n - a_0$. ■

Similarly, we can show that

$$\sum_{n=1}^{\infty} \frac{P_n}{Q_{n+1}Q_n} = \frac{1}{2}.$$

Example 10.3 Evaluate the infinite sum $\sum_{n=2}^{\infty} \frac{P_n}{P_{n+1}P_{n-1}}$.

Solution. Let $S_n = \sum_{k=2}^{n} \frac{2P_k}{P_{k+1}P_{k-1}}$. Using the Pell recurrence, we have

$$\begin{aligned} S_n &= \sum_{k=2}^{n} \frac{P_{k+1} - P_{k-1}}{P_{k+1}P_{k-1}} \\ &= \sum_{k=2}^{n} \left(\frac{1}{P_{k-1}} - \frac{1}{P_{k+1}} \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=2}^{n}\left[\left(\frac{1}{P_{k-1}}-\frac{1}{P_k}\right)+\left(\frac{1}{P_k}-\frac{1}{P_{k+1}}\right)\right]\\
&= \sum_{k=2}^{n}\left(\frac{1}{P_{k-1}}-\frac{1}{P_k}\right)+\sum_{k=2}^{n}\left(\frac{1}{P_k}-\frac{1}{P_{k+1}}\right)\\
&= \left(1-\frac{1}{P_n}\right)+\left(\frac{1}{2}-\frac{1}{P_{n+1}}\right)\\
&= \frac{3}{2}-\frac{1}{P_n}-\frac{1}{P_{n+1}}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=2}^{\infty}\frac{P_n}{P_{n+1}P_{n-1}} &= \frac{1}{2}\lim_{n\to\infty} S_n\\
&= \frac{1}{2}\left(\frac{3}{2}-0-0\right)\\
&= \frac{3}{4}. \qquad \blacksquare
\end{aligned}$$

Likewise, using the identities $P_{n+1}^2 - P_n^2 = Q_{n+1}Q_n$ and $Q_{n+1}^2 - Q_n^2 = 4P_{n+1}P_n$, we can show that $\sum_{n=1}^{\infty} \frac{Q_{n+1}Q_n}{P_{n+1}^2 P_n^2} = 1$ and $\sum_{n=1}^{\infty} \frac{P_{n+1}P_n}{Q_{n+1}^2 Q_n^2} = \frac{1}{4}$.

The next infinite sum was studied by Br. J. Mahon of Australia in 2010 [160]. The solution presented is based on the one by Bruckman [36].

Example 10.4 Evaluate the infinite sum $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}P_{6k+3}}{P_{3k}^2 P_{3k+3}^2}$.

Solution. It follows by identity (43) in Chapter 7 that $P_{3k}^2 + P_{3k+3}^2 = 5P_{6k+3}$. Consequently, we have

$$\begin{aligned}
\sum_{k=1}^{n} \frac{(-1)^{k-1}P_{6k+3}}{P_{3k}^2 P_{3k+3}^2} &= \sum_{k=1}^{n}(-1)^{k-1}\frac{P_{3k}^2 + P_{3k+3}^2}{5P_{3k}^2 P_{3k+3}^2}\\
&= \frac{1}{5}\sum_{k=1}^{n}\left[\frac{(-1)^{k-1}}{P_{3k}^2} - \frac{(-1)^k}{P_{3k+3}^2}\right] \qquad (10.12)\\
&= \frac{1}{5}\left[\frac{1}{P_3^2} - \frac{(-1)^n}{P_{3n+3}^2}\right]\\
&= \frac{1}{125} - \frac{(-1)^n}{5P_{3n+3}^2},
\end{aligned}$$

where we have used the fact that the sum in equation (10.12) is a telescoping sum. Thus

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} P_{6k+3}}{P_{3k}^2 P_{3k+3}^2} = \frac{1}{125} - 0 = \frac{1}{125}.$$

(The limiting process is justified since the limit exists.) ■

Similarly, it follows that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} Q_{6k+3}}{Q_{3k}^2 P_{3k+3}^2} = \frac{1}{490}.$$

Next we pursue an example proposed as a problem in 1994 by R. Euler of Northwest Missouri State University, Maryville, Missouri [80].

Example 10.5 Evaluate the infinite sum $\sum_{n=0}^{\infty} \frac{n2^n Q_n}{5^n}$.

Solution. The sum implicitly hints that we investigate the infinite series $f(x) = \sum_{n=0}^{\infty} nQ_n x^n$. Recall from Chapter 1 that the power series $\sum_{n=0}^{\infty} nx^n$ is generated by the function $\frac{x}{(1-x)^2}$; it converges to the sum $\frac{x}{(1-x)^2}$ for $|x| < 1$. So the series $\sum_{n=0}^{\infty} n(\gamma x)^n$ converges to $\frac{\gamma x}{(1-\gamma x)^2}$ if $|x| < 1/\gamma$; that is, if $|x| < \sqrt{2} - 1$. Similarly, the series $\sum_{n=0}^{\infty} n(\delta x)^n$ converges to $\frac{\delta x}{(1-\delta x)^2}$ if $|x| < \sqrt{2} + 1$. Consequently, both series converge when $|x| < \sqrt{2} - 1$.

Since the sum of two convergent series is convergent, it follows that

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{n(\gamma^n + \delta^n)}{2} x^n \\ &= \frac{1}{2}\left[\sum_{n=0}^{\infty} n(\gamma x)^n + \sum_{n=0}^{\infty} n(\delta x)^n\right] \\ &= \frac{1}{2}\left[\frac{\gamma x}{(1-\gamma x)^2} + \frac{\delta x}{(1-\delta x)^2}\right] \\ &= \frac{\gamma x(1-\delta x)^2 + \delta x(1-\gamma x)^2}{2[(1-\gamma x)(1-\delta x)]^2} \\ &= \frac{x(1+2x-x^2)}{(1-2x-x^2)^2}, \end{aligned}$$

where $|x| < \sqrt{2} - 1$.

In particular, let $x = \frac{2}{5}$, which is less than $\sqrt{2} - 1$. Then $\sum_{n=0}^{\infty} \frac{n2^n Q_n}{5^n} = f(2/5) = 410.$ ■

10.4 A Pell Inequality

The next example features a Pell inequality, studied by J. Díaz-Barrero and J. Egozcue of Barcelona, Spain in 2003 [68]. Although the inequality looks a bit overwhelming, the proof is a straightforward application of the binomial theorem, and the power series $\frac{1}{(1-x)^{r+1}} = \sum_{n=0}^{\infty} \binom{n+r}{n} x^n$, which converges when $|x| < 1$.

Example 10.6 Let m and n be positive integers. Prove that

$$\sum_{k=0}^{n} \binom{m+k+1}{k+1} \left[\sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{k+1}{j} P_n^{j-k-1} \right] \le P_n^{m+1} - 1. \tag{10.13}$$

Proof. By the binomial theorem, we have

$$\begin{aligned} \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{k+1}{j} P_n^{j-k-1} &= \frac{(-1)^{k+1}}{P_n^{k+1}} \sum_{j=0}^{k+1} \binom{k+1}{j} (-P_n)^j \\ &= \frac{(-1)^{k+1}}{P_n^{k+1}} (1 - P_n)^{k+1} \\ &= (1 - 1/P_n)^{k+1}. \end{aligned}$$

Therefore, since $\frac{1}{P_n} \le 1$, we have

$$\begin{aligned} \sum_{k=0}^{n} \binom{m+k+1}{k+1} \left[\sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{k+1}{j} P_n^{j-k-1} \right] &= \sum_{k=0}^{n} \binom{m+k+1}{k+1} (1 - 1/P_n)^{k+1} \\ &= \sum_{r=1}^{n+1} \binom{m+r}{r} (1 - 1/P_n)^r \\ &\le \sum_{r=1}^{\infty} \binom{m+r}{r} (1 - 1/P_n)^r \\ &= \frac{1}{[1 - (1 - 1/P_n)]^{m+1}} - 1 \\ &= P_n^{m+1} - 1, \text{ as claimed.} \end{aligned}$$

Notice that equality holds in (10.13) when $n = 1$. ∎

In particular, let $m = 5$ and $n = 2$. Then

$$\begin{aligned}
\text{LHS} &= \sum_{k=0}^{2}\binom{6+k}{k+1}\left[\sum_{j=0}^{k+1}(-1)^{k+1-j}\binom{k+1}{j}2^{j-k-1}\right]\\
&= \binom{6}{1}\sum_{j=0}^{1}(-1)^{1-j}\binom{1}{j}2^{j-1}+\binom{7}{2}\sum_{j=0}^{2}(-1)^{2-j}\binom{2}{j}2^{j-2}+\binom{8}{3}\sum_{j=0}^{3}(-1)^{3-j}\binom{3}{j}2^{j-3}\\
&= 6\cdot\frac{1}{2}-21\cdot\frac{1}{4}+56\cdot\frac{1}{8}=\frac{19}{8}\\
&< 2^6-1,\ \text{as expected.}
\end{aligned}$$

Returning to inequality (10.13), we note that there is nothing sacred about the choice of P_n. Since the power series $\sum_{n=0}^{\infty}\binom{n+r}{n}x^n$ converges for every real number x where $|x| < 1$, the inequality holds for every such x, as K.B. Davenport of Frackville, Pennsylvania, observed in 2004 [58]. For example,

$$\sum_{k=0}^{n}\binom{m+k+1}{k+1}\left[\sum_{j=0}^{k+1}(-1)^{k+1-j}\binom{k+1}{j}Q_n^{j-k-1}\right]\le Q_n^{m+1}-1.$$

10.5 An Infinite Pell Product

The next example features an infinite Pell product, studied by M. Catalani of the University of Turin, Italy, in 2004 [42]. The solution employs the Binet-like formula for Q_n, and identity (31) from Chapter 7: $Q_n^2 = 2P_n^2 + (-1)^n$; so $Q_{2^k}^2 = 2P_{2^k}^2 + 1$, where $k \ge 1$.

Example 10.7 Evaluate $\prod_{k=1}^{\infty}\left(1+\frac{1}{\sqrt{2P_{2^k}^2+1}}\right)$, if it exists.

Solution. We have

$$\begin{aligned}
\frac{1}{\sqrt{2P_{2^k}^2+1}} &= \frac{1}{\sqrt{Q_{2^k}^2}}\\
&= \frac{1}{Q_{2^k}} = \frac{2}{\gamma^{2^k}+\delta^{2^k}}\\
&= \frac{2}{\gamma^{2^k}[1+\delta^{2^k}(-\delta)^{2^k}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2(-\delta)^{2^k}}{1+\delta^{2^{k+1}}}\\
1+\frac{1}{\sqrt{2P_{2^k}^2+1}} &= 1+\frac{2\delta^{2^k}}{1+\delta^{2^{k+1}}}\\
&= \frac{\left(1+\delta^{2^k}\right)^2}{1+\delta^{2^{k+1}}}\\
\prod_{k=1}^{n}\left(1+\frac{1}{\sqrt{2P_{2^k}^2+1}}\right) &= \frac{(1+\delta^2)^2}{1+\delta^4}\cdot\frac{(1+\delta^4)^2}{1+\delta^8}\cdots\frac{\left(1+\delta^{2^n}\right)^2}{1+\delta^{2^{n+1}}}\\
&= \frac{1+\delta^2}{1+\delta^{2^{n+1}}}\cdot(1+\delta^2)(1+\delta^4)\cdots(1+\delta^{2^n})\\
&= \frac{1+\delta^2}{1+\delta^{2^{n+1}}}\cdot\frac{1-\delta^{2^{n+1}}}{1-\delta^2}\\
&= \frac{1+\delta^2}{1-\delta^2}\cdot\frac{1-\delta^{2^{n+1}}}{1+\delta^{2^{n+1}}}\\
\prod_{k=1}^{\infty}\left(1+\frac{1}{\sqrt{2P_{2^k}^2+1}}\right) &= \frac{1+\delta^2}{1-\delta^2}\cdot\lim_{n\to\infty}\frac{1-\delta^{2^{n+1}}}{1+\delta^{2^{n+1}}}.
\end{aligned}
$$

Since $|\delta|<1$, $\lim\limits_{n\to\infty}\frac{1-\delta^{2^{n+1}}}{1+\delta^{2^{n+1}}}=\frac{1-0}{1+0}=1$; so the limit exists and hence the given infinite product converges. Consequently,

$$
\begin{aligned}
\prod_{k=1}^{\infty}\left(1+\frac{1}{\sqrt{2P_{2^k}^2+1}}\right) &= \frac{1+\delta^2}{1-\delta^2}\\
&= \frac{\delta^2(\delta-\gamma)}{\delta(-\gamma-\delta)}=\frac{\gamma-\delta}{\gamma+\delta}\\
&= \frac{2\sqrt{2}}{2}=\sqrt{2}.
\end{aligned}
$$

As a bonus, it follows from the solution that $\prod\limits_{k=1}^{\infty}\left(\frac{1+Q_{2^k}}{Q_{2^k}}\right)$ exists and equals $\sqrt{2}$. ■

Next we investigate the convergence of the power series $\sum_{n=0}^{\infty} P_n x^n$ and $\sum_{n=0}^{\infty} Q_n x^n$, and then evaluate each sum at two distinct positive rational numbers x. This will yield some surprising Pell dividends. In the process, we will encounter primitive Pythagorean triples and the Pell's equation $u^2 - 2n^2 = 1$ as well.

To this end, first we will identify the radii of convergence of both power series.

10.6 Radii of Convergence of the Series

To minimize our exposition, let $\{S_n\}$ denote an integer sequence satisfying the Pell recurrence. Then $S_n = A\gamma^n + B\delta^n$, where A and B are constants. So

$$\sum_{n=0}^{\infty} S_n x^n = A \sum_{n=0}^{\infty} \gamma^n x^n + B \sum_{n=0}^{\infty} \delta^n x^n.$$

We need to know exactly when the series on the LHS can be evaluated. The two series on the RHS converge if and only if $|x| < \frac{1}{|\gamma|}$ and $|x| < \frac{1}{|\delta|}$; that is, if and only if $|x| < \min\left(\frac{1}{|\gamma|}, \frac{1}{|\delta|}\right)$. Since $\min\left(\frac{1}{|\gamma|}, \frac{1}{|\delta|}\right) = -\delta$, it follows that the series $\sum_{n=0}^{\infty} S_n x^n$ converges if and only if $\delta < x < -\delta$. Consequently, the series $\sum_{n=0}^{\infty} P_n x^n$ and $\sum_{n=0}^{\infty} Q_n x^n$ converge if and only if $\delta < x < -\delta$.

10.6.1 Sum of the Series $\sum_{n=0}^{\infty} P_{nx}^{n}$

We now look for positive rational numbers x such that $f(x) = \sum_{n=0}^{\infty} P_n x^n$ is a rational number r, where $\delta < x < -\delta$. We then have

$$\begin{aligned} \frac{x}{1 - 2x - x^2} &= r \\ rx^2 + (2r+1)x - r &= 0 \\ x &= \frac{-(2r+1) \pm \sqrt{(2r+1)^2 + (2r)^2}}{2r}. \end{aligned}$$

Since x has to be rational, we will employ primitive Pythagorean triples to simplify the radicand. To this end, we choose $2r + 1 = m^2 - n^2$ and $2r = 2mn$, where $m > n \geq 1$ and $(m, n) = 1$. Then $(2r + 1)^2 + (2r)^2 = (m^2 + n^2)^2$. So

$$\begin{aligned} x &= \frac{-(m^2 - n^2) \pm (m^2 + n^2)}{2mn} \\ &= \frac{n}{m}, -\frac{m}{n}. \end{aligned}$$

Since we want $x > 0$, we will choose $x = \frac{n}{m}$.

The equation $m^2 - n^2 = 2mn + 1$ implies that $(m-n)^2 = 2n^2 + 1$. This yields the Pell equation $u^2 - 2n^2 = 1$, where $u = m - n$. Recall that its solutions are given by $(u_k, n_k) = (Q_{2k}, P_{2k})$, where $k \geq 0$. Then $m_k = n_k + Q_{2k} = P_{2k} + Q_{2k} = P_{2k+1}$.

Since

$$\begin{aligned}\frac{n_k}{m_k} &= \frac{P_{2k}}{P_{2k+1}}\\ &= \frac{\gamma^{2k} - \delta^{2k}}{\gamma^{2k+1} - \delta^{2k+1}} < \frac{\gamma^{2k} - \delta^{2k}}{\gamma^{2k+1}}\\ &< \frac{\gamma^{2k}}{\gamma^{2k+1}} = \frac{1}{\gamma}\\ &= -\delta,\end{aligned}$$

the series $\sum_{n=0}^{\infty} P_n x^n$ converges when $x = \frac{P_{2k}}{P_{2k+1}}$.

Using the Cassini-like formula for P_m, we then have

$$\begin{aligned}\sum_{n=0}^{\infty} P_n \left(\frac{P_{2k}}{P_{2k+1}}\right)^n &= \frac{\frac{P_{2k}}{P_{2k+1}}}{1 - 2\frac{P_{2k}}{P_{2k+1}} - \frac{P_{2k}^2}{P_{2k+1}^2}}\\ &= \frac{P_{2k}P_{2k+1}}{P_{2k+1}(P_{2k+1} - 2P_{2k}) - P_{2k}^2}\\ &= \frac{P_{2k}P_{2k+1}}{P_{2k+1}P_{2k-1} - P_{2k}^2}\\ &= \frac{P_{2k}P_{2k+1}}{(-1)^{2k}}\\ &= P_{2k}P_{2k+1}, \text{ an even integer.}\end{aligned}$$

For example, $\sum_{n=0}^{\infty} P_n(\frac{2}{5})^n = 2 \cdot 5 = 10$ and $\sum_{n=0}^{\infty} P_n(\frac{12}{29})^n = 12 \cdot 29 = 348$. Notice that $\sum_{n=0}^{86} P_n(\frac{2}{5})^n \approx 9.5$ and $\sum_{n=0}^{6445} P_n(\frac{12}{29})^n \approx 347.5$, so the convergence is very slow.

Suppose we let $x = \frac{Q_{2k-1}}{Q_{2k}}$. Then also $\delta < x < -\delta$. So $\sum_{n=0}^{\infty} P_n x^n$ converges when $x = \frac{Q_{2k-1}}{Q_{2k}}$ also. Consequently, it follows by the Cassini-like formula for Q_m that

$$\sum_{n=0}^{\infty} P_n \left(\frac{Q_{2k-1}}{Q_{2k}}\right)^n = \frac{Q_{2k-1}Q_{2k}}{2},$$

where $k \geq 1$. (Since every Q_n is odd, this sum is *not* an integer.)

For example, $\sum_{n=0}^{\infty} P_n\left(\frac{1}{3}\right)^n = \frac{1\cdot 3}{2} = 1.5$ and $\sum_{n=0}^{\infty} P_n\left(\frac{7}{17}\right)^n = \frac{7\cdot 17}{2} = 59.5$. Notice that $\sum_{n=0}^{23} P_n(\frac{1}{3})^n \approx 1.5$ and $\sum_{n=0}^{1195} P_n(\frac{7}{17})^n \approx 59.5$; again the convergence is extremely slow.

Next we evaluate the sum $\sum_{n=0}^{\infty} Q_n x^n$ at two special values of x.

10.6.2 Sum of the Series $\sum_{n=0}^{\infty} Q_{nx}^n$

Suppose we choose $x = \frac{Q_{2k-1}}{Q_{2k}}$. Since $\delta < x < -\delta$, it follows again by the Cassini-like formula for Q_m that

$$\sum_{n=0}^{\infty} Q_n \left(\frac{Q_{2k-1}}{Q_{2k}}\right)^n = P_{2k-1}Q_{2k},$$

where $k \geq 1$.

Likewise, we also have

$$\sum_{n=0}^{\infty} Q_n \left(\frac{P_{2k}}{P_{2k+1}}\right)^n = Q_{2k}P_{2k+1},$$

where $k \geq 1$.

For example, $\sum_{n=0}^{\infty} Q_n(\frac{7}{17})^n = 5\cdot 17 = 85$ and $\sum_{n=0}^{\infty} Q_n(\frac{41}{99})^n = 29\cdot 99 = 2871$; and $\sum_{n=0}^{\infty} Q_n(\frac{2}{5})^n = 3\cdot 5 = 15$ and $\sum_{n=0}^{\infty} Q_n(\frac{12}{29})^n = 17\cdot 29 = 493$.

Exercises 10

Prove the following summation formulas.

1. $\sum_{i=1}^{n} P_i = \frac{Q_{n+1}-1}{2}$. *Hint*: Use the identity $Q_{i+1} - Q_i = 2P_i$.
2. $\sum_{i=1}^{n} Q_i = P_{n+1} - 1$. *Hint*: Use the identity $P_{i+1} - P_i = Q_i$.
3. $\sum_{i=1}^{n} P_{2i-1} = \frac{P_{2n}}{2}$. *Hint*: Use the recurrence $P_{2i} - P_{2i-2} = 2P_{2i-1}$.
4. $\sum_{i=1}^{n} P_{2i} = \frac{P_{2n+1}-1}{2}$. *Hint*: Use the fact that $\sum_{i=1}^{n} P_{2i} = \sum_{i=1}^{2n} P_i - \sum_{i=1}^{n} P_{2i-1}$.
5. $\sum_{i=1}^{n} Q_{2i-1} = \frac{Q_{2n}-1}{2}$. *Hint*: Use the recurrence $Q_{2i} - Q_{2i-2} = 2Q_{2i-1}$.

6. $\sum_{i=1}^{n} Q_{2i} = \frac{Q_{2n+1}-1}{2}$. *Hint*: Use the fact that $\sum_{i=1}^{n} Q_{2i} = \sum_{i=1}^{2n} Q_i - \sum_{i=1}^{n} Q_{2i-1}$.
7. $\sum_{i=1}^{n} P_i^2 = \begin{cases} \frac{Q_{2n+1}+1}{8} & \text{if } n \text{ is odd} \\ \frac{Q_{2n+1}-1}{8} & \text{otherwise.} \end{cases}$ *Hint*: Use the identity $4P_i^2 = Q_{2i} - (-1)^i$.
8. $\sum_{i=1}^{n} Q_i^2 = \begin{cases} \frac{Q_{2n+1}-3}{4} & \text{if } n \text{ is odd} \\ \frac{Q_{2n+1}-1}{4} & \text{otherwise.} \end{cases}$ *Hint*: Use the identity $Q_i^2 = 2P_i^2 + (-1)^i$.
9. $\sum_{i=1}^{n} P_i P_{i+1} = \frac{Q_n^2-1}{4}$. *Hint*: Use the identity $4P_i P_{i+1} = Q_i^2 - Q_{i-1}^2$.
10. $\sum_{i=1}^{n} Q_i Q_{i+1} = \begin{cases} \frac{Q_n^2-3}{4} & \text{if } n \text{ is odd} \\ \frac{Q_n^2-1}{4} & \text{otherwise.} \end{cases}$ *Hint*: Use the identity $Q_i Q_{i+1} = 2P_i P_{i+1} + (-1)^i$.
11. $\sum_{n=1}^{\infty} \frac{P_n}{Q_{n+1} Q_n} = \frac{1}{2}$.
12. $\sum_{n=1}^{\infty} \frac{Q_{n+1} Q_n}{P_{n+1}^2 P_n^2} = 1$.
13. $\sum_{n=1}^{\infty} \frac{P_{n+1} P_n}{Q_{n+1}^2 Q_n^2} = \frac{1}{4}$.
14. $\sum_{k=1}^{\infty} \frac{(-1)^{k-1} P_{6k+3}}{Q_{3k}^2 Q_{3k+3}^2} = \frac{1}{490}$. *Hint*: Use identity 44 in Chapter 7.

11

Generating Functions for the Pell Family

11.1 Introduction

In this chapter we will develop generating functions for Pell and Pell–Lucas numbers; their squares; and odd- and even-numbered Pell and Pell–Lucas numbers. Along the way, we will study some interesting applications. We begin with the generating functions for Pell and Pell–Lucas numbers.

11.2 Generating Functions for the Pell and Pell–Lucas Sequences

To streamline the process, we introduce a larger integer family $\{A_n\}$, which satisfies the Pell recurrence, where $A_0 = a$, an arbitrary integer, and $A_1 = 1$.

Let $A(x)$ denote the generating function for sequence $\{A_n\}$. Then

$$\begin{aligned}
A(x) &= a + x + A_2x^2 + A_3x^3 + \cdots + A_nx^n + \cdots \\
2xA(x) &= 2ax + 2x^2 + 2A_2x^3 + \cdots + 2A_{n-1}x^n + \cdots \\
x^2A(x) &= ax^2 + x^3 + \cdots + A_{n-2}x^n + \cdots \\
(1-2x-x^2)A(x) &= a + (1-2a)x \\
A(x) &= \frac{a+(1-2a)x}{1-2x-x^2}. \qquad (11.1)
\end{aligned}$$

Case 1 Let $a = 0$, so $A_n = P_n$. Correspondingly, we have

$$\frac{x}{1-2x-x^2} = 1 + 2x + 5x^2 + 12x^3 + 29x^4 + \cdots + P_nx^n + \cdots .$$

T. Koshy, *Pell and Pell–Lucas Numbers with Applications*, DOI 10.1007/978-1-4614-8489-9_11,

Case 2 Let $a = 1$. Then $A_n = Q_n$. Then, by equation (11.1), we have

$$\frac{1-x}{1-2x-x^2} = 1 + 3x + 7x^2 + 17x^3 + 41x^4 + \cdots + Q_n x^n + \cdots .$$

The next example combines the generating function for Pell numbers and identity (18) in Chapter 7 without acknowledging the presence of Pell numbers. This example appeared in the 1999 William Lowell Putnam Mathematical Competition [106].

Example 11.1 Prove that the series

$$\frac{1}{1-2x-x^2} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

has the property that the sum of the squares of the coefficients of every two consecutive terms occurs later as the coefficient of a term in the series.

Proof. (Since the problem does not say anything about Pell numbers, we will ignore them and proceed accordingly. The proof simply reverses the process we developed to establish the desired identity.)

Let $1-2x-x^2 = (1-\gamma x)(1-\delta x)$, where $\gamma = 1+\sqrt{2}$, $\delta = 1-\sqrt{2}$, $\gamma+\delta = 2$, $\gamma-\delta = 2\sqrt{2}$ and $\gamma\delta = -1$. (We are using the same Greek symbols to avoid any possible confusion.) Since

$$\begin{aligned} \frac{1}{1-2x-x^2} &= \frac{1}{\gamma-\delta}\left(\frac{\gamma}{1-\gamma x} - \frac{\delta}{1-\delta x}\right) \\ &= \sum_{i=0}^{\infty} \frac{\gamma^{n+1}-\delta^{n+1}}{\gamma-\delta} x^n, \end{aligned}$$

it follows that

$$a_n = \frac{\gamma^{n+1}-\delta^{n+1}}{\gamma-\delta}.$$

Then

$$\begin{aligned} 8(a_n^2 + a_{n+1}^2) &= \left(\gamma^{n+1}-\delta^{n+1}\right)^2 + \left(\gamma^{n+2}-\delta^{n+2}\right)^2 \\ &= \left[\gamma^{2n+2}+\delta^{2n+2}-2(-1)^{n+1}\right] + \left[\gamma^{2n+4}+\delta^{2n+4}-2(-1)^{n+2}\right] \\ &= \gamma^{2n+3}\left(\gamma+\frac{1}{\gamma}\right) + \delta^{2n+3}\left(\delta+\frac{1}{\delta}\right) \\ &= \gamma^{2n+3}(\gamma-\delta) - \delta^{2n+3}(\gamma-\delta) = (\gamma-\delta)^2 a_{2n+2} \\ a_n^2 + a_{n+1}^2 &= a_{2n+2}, \end{aligned}$$

where $n \geq 0$; see identity (18) in Chapter 7. Thus the sum of the squares of every two consecutive coefficients in the given power series is also a coefficient of a subsequent term of the series and the generating function for the Pell numbers, where $a_0 = 1 = P_1$ and $a_1 = 2 = P_2$. ∎

Clearly, we can adapt this example to use the power series expansion of

$$\frac{1-x}{1-2x-x^2} = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots + b_nx^n + \cdots$$

to establish that $b_n^2 + b_{n+1}^2 = 2a_{n+2}$, as the following example shows.

Example 11.2 Using the power series expansion

$$\frac{1-x}{1-2x-x^2} = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots + b_nx^n + \cdots$$

show that the sum of the squares of the coefficients of every two consecutive terms in the series equals twice a coefficient in the expansion of $\frac{1}{1-2x-x^2}$ in the previous example.

Proof. As before, we let $1-2x-x^2 = (1-\gamma x)(1-\delta x)$. So

$$\begin{aligned}\sum_{n=0}^{\infty} b_n x^n &= (1+x)(1-\gamma x)^{-1}(1-\delta x)^{-1}\\ &= (1+x)\left(\sum_{i=0}^{\infty}\gamma^i x^i\right)\left(\sum_{j=0}^{\infty}\delta^j x^j\right).\end{aligned}$$

Equating the coefficients of x^n from both sides, we get

$$\begin{aligned}b_n &= \sum_{i+j=n}\gamma^i\delta^j + \sum_{i+j+1=n}\gamma^i\delta^{j+1} = \sum_{i=0}^{n}\gamma^i\delta^{n-i} + \sum_{i=0}^{n-1}\gamma^i\delta^{n-i-1}\\ &= \delta^n\sum_{i=0}^{n}(\gamma/\delta)^i + \delta^{n-1}\sum_{i=0}^{n-1}(\gamma/\delta)^i = \delta^n\left[\frac{1-(\gamma/\delta)^{n+1}}{1-\gamma/\delta}\right] + \delta^{n-1}\left[\frac{1-(\gamma/\delta)^n}{1-\gamma/\delta}\right]\\ &= \frac{\gamma^{n+1}-\delta^{n+1}}{\gamma-\delta} + \frac{\gamma^n-\delta^n}{\gamma-\delta} = \frac{\gamma^{n+1}\left(1+\frac{1}{\gamma}\right)-\delta^{n+1}\left(1+\frac{1}{\delta}\right)}{\gamma-\delta}\\ &= \frac{\gamma^{n+1}(1-\delta)-\delta^{n+1}(1-\gamma)}{\gamma-\delta} = \frac{\sqrt{2}(\gamma^{n+1}+\delta^{n+1})}{\gamma-\delta}\\ &= \frac{\gamma^{n+1}+\delta^{n+1}}{2}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}4\left(b_n^2+b_{n+1}^2\right) &= \left(\gamma^{n+1}+\delta^{n+1}\right)^2 + \left(\gamma^{n+2}+\delta^{n+2}\right)^2 = \gamma^{2n+2}+\delta^{2n+2}+\gamma^{2n+4}+\delta^{2n+4}\\ &= \gamma^{2n+3}\left(\gamma+\frac{1}{\gamma}\right)+\delta^{2n+2}\left(\delta+\frac{1}{\delta}\right) = \gamma^{2n+3}(\gamma-\delta)-\delta^{2n+3}(\gamma-\delta)\\ &= (\gamma-\delta)\left(\gamma^{2n+3}-\delta^{2n+3}\right)\\ b_n^2+b_{n+1}^2 &= 2\cdot\frac{\gamma^{2n+3}-\delta^{2n+3}}{\gamma-\delta} = 2a_{2n+2}, \quad \text{as desired.}\end{aligned}$$

It follows from the given generating function that $b_n = Q_{n+1}$ and $Q_{n+1}^2 + Q_{n+2}^2 = 2P_{2n+3}$; see identity (21) in Chapter 7. ■

Returning to the generating function (11.1), we can rewrite it as

$$A(x) = \frac{1}{\gamma - \delta}\left(\frac{C}{1-\gamma x} - \frac{D}{1-\delta x}\right) = \sum_{n=0}^{\infty} A_n x^n, \tag{11.2}$$

where $C = 1 - 2a + a\gamma$ and $D = 1 - 2a + a\delta$. When $a = 0$, $A_n = P_n$, and $C = 1 = D$; and when $a = 1$, $A_n = Q_n$, $CD = -2$, and $C^2 = 2 = D^2$.

Using the power of generating functions, we will now develop formulas for $\sum_{k=0}^{n} A_{2k}$, $\sum_{k=0}^{n} A_{2k+1}$, $\sum_{k=0}^{n} A_k A_{n-k}$, $\sum_{k=0}^{n} A_{2k} A_{2n-2k}$, $\sum_{k=0}^{n} A_{2k+1} A_{2n-2k+1}$, and $\sum_{k=0}^{n} A_{2k} A_{2n-2k+1}$.

11.3 Formulas for $\sum_{k=0}^{n} A_{2k}$ and $\sum_{k=0}^{n} A_{2k+1}$

Consider the even function

$$A_e(x) = \frac{A(x) + A(-x)}{2} = \sum_{n=0}^{\infty} A_{2n} x^{2n}.$$

By equation (11.1), we have

$$\begin{aligned} A(x) + A(-x) &= \frac{a + (1-2a)x}{1 - 2x - x^2} + \frac{a - (1-2a)x}{1 + 2x - x^2} \\ \sum_{n=0}^{\infty} A_{2n} x^{2n} &= \frac{a(1-x^2) + 2(1-2a)x^2}{1 - 6x^2 + x^4}. \end{aligned}$$

This yields the following generating functions:

$$\begin{aligned} \frac{2x^2}{1 - 6x^2 + x^4} &= \sum_{n=0}^{\infty} P_{2n} x^{2n} \\ \frac{1 - 3x^2}{1 - 6x^2 + x^4} &= \sum_{n=0}^{\infty} Q_{2n} x^{2n}. \end{aligned}$$

Similarly, using the odd function

$$A_o(x) = \frac{A(x) - A(-x)}{2} = \sum_{n=0}^{\infty} A_{2n+1} x^{2n+1} = \frac{2ax + (1-2a)(x - x^3)}{1 - 6x^2 + x^4},$$

we can find the generating functions for $\{P_{2n+1}\}$ and $\{Q_{2n+1}\}$:

$$\begin{aligned} \frac{x - x^3}{1 - 6x^2 + x^4} &= \sum_{n=0}^{\infty} P_{2n+1} x^{2n+1} \\ \frac{x + x^3}{1 - 6x^2 + x^4} &= \sum_{n=0}^{\infty} Q_{2n+1} x^{2n+1}. \end{aligned}$$

11.4 A Formula for $\sum_{k=0}^{n} A_k A_{n-k}$

From (11.2), we have

$$A^2(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} A_k A_{n-k} \right) x^n. \tag{11.3}$$

But we also have

$$\begin{aligned} A^2(x) &= \frac{1}{8}\left[\frac{C^2}{(1-\gamma x)^2} + \frac{D^2}{(1-\delta x)^2} - \frac{2CD}{(1-\gamma x)(1-\delta x)} \right] \\ &= \frac{1}{8}\left[\frac{C^2}{(1-\gamma x)^2} + \frac{D^2}{(1-\delta x)^2} - \frac{2CD}{2\sqrt{2}x} \left(\frac{1}{1-\gamma x} - \frac{1}{1-\delta x} \right) \right] \\ &= \frac{1}{8}\left[\sum_{n=0}^{\infty} (n+1)(C^2\gamma^n + D^2\delta^n)x^n - \frac{2CD}{x} \sum_{n=0}^{\infty} \frac{\gamma^n - \delta^n}{2\sqrt{2}} x^n \right] \\ &= \frac{1}{8}\left[\sum_{n=0}^{\infty} (n+1)(C^2\gamma^n + D^2\delta^n)x^n - 2CD \sum_{n=0}^{\infty} P_{n+1} x^n \right] \\ &= \frac{1}{8} \sum_{n=0}^{\infty} \left[(n+1)(C^2\gamma^n + D^2\delta^n)x^n - 2CDP_{n+1}x^n \right]. \end{aligned} \tag{11.4}$$

Equating the coefficients of x^n from (11.3) and (11.4), we get

$$8 \sum_{k=0}^{n} A_k A_{n-k} = (n+1)\left(C^2\gamma^n + D^2\delta^n\right) - 2CDP_{n+1}. \tag{11.5}$$

In particular, this implies that

$$\begin{aligned} 4\sum_{k=0}^{n} P_k P_{n-k} &= (n+1)Q_n - P_{n+1} \\ 2\sum_{k=0}^{n} Q_k Q_{n-k} &= (n+1)Q_n + P_{n+1}. \end{aligned}$$

For example, $4\sum_{k=0}^{4} P_k P_{5-k} = 176 = 6Q_5 - P_6$ and $4\sum_{k=0}^{5} Q_k Q_{5-k} = 316 = 6Q_5 + P_6$; see Figure 11.1.

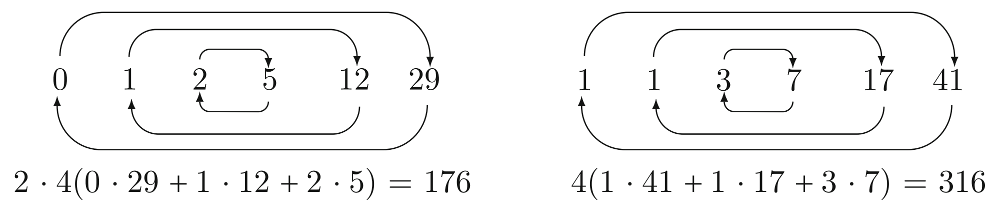

Figure 11.1.

Next we will develop a formula for $\sum_{k=0}^{n} A_{2k}A_{2n-2k}$.

11.5 A Formula for $\sum_{k=0}^{n} A_{2k}A_{2n-2k}$

Consider the even function

$$A_e(x) = \frac{A(x) + A(-x)}{2} = \sum_{n=0}^{\infty} A_{2n}x^{2n}.$$

Then

$$A_e^2(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} A_{2k}A_{2n-2k}\right)x^{2n}. \tag{11.6}$$

Since

$$\begin{aligned} A_e(x) &= \frac{1}{4\sqrt{2}}\left[\left(\frac{C}{1-\gamma x} - \frac{D}{1-\delta x}\right) + \left(\frac{C}{1+\gamma x} - \frac{D}{1+\delta x}\right)\right] \\ &= \frac{1}{2\sqrt{2}}\left(\frac{C}{1-\gamma^2x^2} - \frac{D}{1-\delta^2x^2}\right), \end{aligned}$$

we have

$$\begin{aligned}
A_e^2(x) &= \frac{1}{8}\left[\frac{C^2}{(1-\gamma^2x^2)^2}+\frac{D^2}{(1-\delta^2x^2)^2}-\frac{2CD}{(1-\gamma^2x^2)(1-\delta^2x^2)}\right]\\
&= \frac{1}{8}\left[\frac{C^2}{1-\gamma^2x^2}+\frac{D^2}{1-\delta^2x^2}-\frac{CD}{2\sqrt{2}}\left(\frac{1}{1-\gamma^2x^2}-\frac{1}{1-\delta^2x^2}\right)\right]\\
&= \frac{1}{8}\left[\sum_{n=0}^{\infty}(n+1)\left(C^2\gamma^{2n}+D^2\delta^{2n}\right)x^{2n}-\frac{CD}{x^2}\sum_{n=0}^{\infty}P_{2n}x^{2n}\right]\\
&= \frac{1}{8}\sum_{n=0}^{\infty}\left[(n+1)\left(C^2\gamma^{2n}+D^2\delta^{2n}\right)-CDP_{2n+2}\right]x^{2n}. \qquad (11.7)
\end{aligned}$$

Equating the coefficients of x^{2n} from (11.6) and (11.7), we get

$$8\sum_{k=0}^{n}A_{2k}A_{2n-2k} = (n+1)\left(C^2\gamma^{2n}+D^2\delta^{2n}\right)-CDP_{2n+2}. \qquad (11.8)$$

In particular, this yields

$$\begin{aligned}
8\sum_{k=0}^{n}P_{2k}P_{2n-2k} &= 2(n+1)Q_{2n}-P_{2n+2}\\
4\sum_{k=0}^{n}Q_{2k}Q_{2n-2k} &= 2(n+1)Q_{2n}+P_{2n+2}.
\end{aligned}$$

For example, $8\sum_{k=0}^{4}P_{2k}P_{8-2k} = 3,392 = 10\cdot 577 - 2378 = 2\cdot 5Q_8 - P_{10}$ and $4\sum_{k=0}^{4}Q_{2k}Q_{8-2k} = 8,148 = 10\cdot 577 + 2378 = 2\cdot 5Q_8 + P_{10}$; see Figure 11.2.

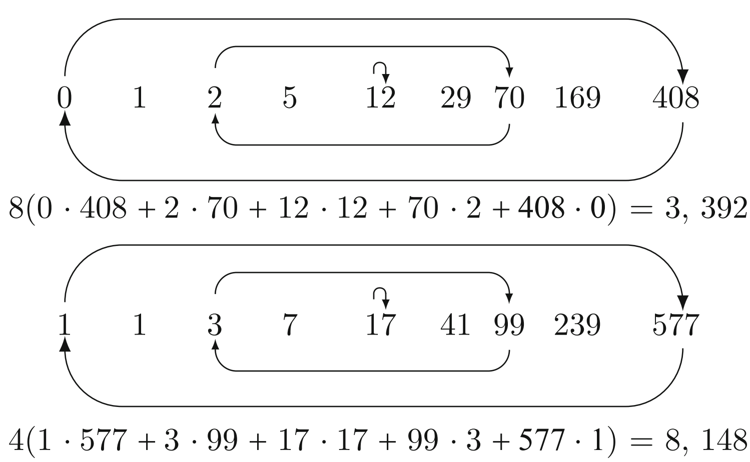

Figure 11.2.

We will now develop the corresponding formula for odd-numbered products $A_{2k+1}A_{2n-2k+1}$.

11.6 A Formula for $\sum_{k=0}^{n} A_{2k+1}A_{2n-2k+1}$

Consider the odd function

$$A_o(x) = \frac{A(x) - A(-x)}{2} = \sum_{n=0}^{\infty} A_{2n+1}x^{2n+1}.$$

Then

$$A_o^2(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} A_{2k+1}A_{2n-2k+1}\right)x^{2n+2}. \tag{11.9}$$

Since

$$\begin{aligned} A_o(x) &= \frac{1}{4\sqrt{2}}\left[\left(\frac{C}{1-\gamma x} - \frac{D}{1-\delta x}\right) - \left(\frac{C}{1+\gamma x} - \frac{D}{1+\delta x}\right)\right] \\ &= \frac{1}{2\sqrt{2}}\left(\frac{C\gamma x}{1-\gamma^2x^2} - \frac{D\delta x}{1-\delta^2x^2}\right), \end{aligned}$$

we also have

$$\begin{aligned} A_e^2(x) &= \frac{1}{8}\left[\frac{C^2\gamma^2x^2}{(1-\gamma^2x^2)^2} + \frac{D^2\delta^2x^2}{(1-\delta^2x^2)^2} + \frac{2CDx^2}{(1-\gamma^2x^2)(1-\delta^2x^2)}\right] \\ &= \frac{1}{8}\left[\frac{C^2\gamma^2x^2}{1-\gamma^2x^2} + \frac{D^2\delta^2x^2}{1-\delta^2x^2} + \frac{CD}{2\sqrt{2}}\left(\frac{1}{1-\gamma^2x^2} - \frac{1}{1-\delta^2x^2}\right)\right] \\ &= \frac{1}{8}\left[\sum_{n=0}^{\infty}(n+1)\left(C^2\gamma^{2n+2} + D^2\delta^{2n+2}\right)x^{2n+2} + CD\sum_{n=0}^{\infty}P_{2n}x^{2n}\right] \\ &= \frac{1}{8}\sum_{n=0}^{\infty}\left[(n+1)\left(C^2\gamma^{2n+2} + D^2\delta^{2n+2}\right) + CDP_{2n+2}\right]x^{2n+2}. \end{aligned} \tag{11.10}$$

From (11.9) and (11.10), we have

$$8\sum_{k=0}^{n} A_{2k+1}A_{2n-2k+1} = (n+1)(C^2\gamma^{2n+2} + D^2\delta^{2n+2}) + CDP_{n+1}. \tag{11.11}$$

This implies that

$$8\sum_{k=0}^{n} P_{2k+1}P_{2n-2k+1} = 2(n+1)Q_{2n+2} + P_{2n+2}$$

$$4\sum_{k=0}^{n} Q_{2k+1}Q_{2n-2k+1} = 2(n+1)Q_{2n+2} - P_{2n+2}.$$

For example, $8\sum_{k=0}^{4} P_{2k+1}P_{9-2k} = 36,008 = 10 \cdot 3363 + 2378 = 2 \cdot 5Q_{10} + P_{10}$; and $4\sum_{k=0}^{4} Q_{2k+1}Q_{9-2k} = 31,252 = 10 \cdot 3363 - 2378 = 2 \cdot 5Q_{10} - P_{10}$ (see Figure 11.3).

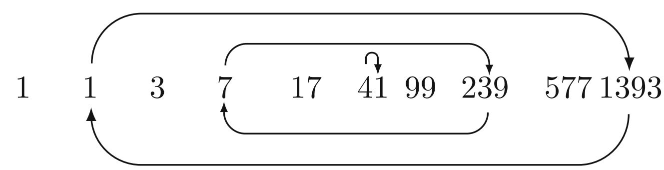

Figure 11.3.

Using the generating functions $A_e(x)$ and $A_o(x)$, we can develop a formula for the hybrid sum $\sum_{k=0}^{n} A_{2k}A_{2n-2k+1}$.

11.7 A Formula for the Hybrid Sum $\sum_{k=0}^{n} A_{2k}A_{2n-2k+1}$

Using the generating functions $A_e(x)$ and $A_o(x)$, we have

$$A_e(x)A_o(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} A_{2k}A_{2n-2k+1}\right)x^{2n+1}. \tag{11.12}$$

We also have

$$\begin{aligned}
A_e(x)A_o(x) &= \frac{1}{2\sqrt{2}}\left(\frac{C}{1-\gamma^2x^2} - \frac{D}{1-\delta^2x^2}\right) \cdot \frac{1}{2\sqrt{2}}\left(\frac{C\gamma x}{1-\gamma^2x^2} - \frac{D\delta x}{1-\delta^2x^2}\right)\\
&= \frac{1}{8}\left[\frac{C^2\gamma x}{(1-\gamma^2x^2)^2} + \frac{D^2\delta x}{(1-\delta^2x^2)^2} - \frac{2CDx}{(1-\gamma^2x^2)(1-\delta^2x^2)}\right]\\
&= \frac{1}{8}\left[\frac{C^2\gamma x}{(1-\gamma^2x^2)^2} + \frac{D^2\delta x}{(1-\delta^2x^2)^2} - \frac{CD}{2\sqrt{2}x}\left(\frac{1}{1-\gamma^2x^2} - \frac{1}{1-\delta^2x^2}\right)\right]
\end{aligned}$$

$$= \frac{1}{8}\left[\sum_{n=0}^{\infty}(n+1)\left(C^2\gamma^{2n+1}+D^2\delta^{2n+1}\right)x^{2n+1} - \frac{CD}{x}\sum_{n=0}^{\infty}P_{2n}x^{2n}\right]$$

$$= \frac{1}{8}\sum_{n=0}^{\infty}\left[(n+1)\left(C^2\gamma^{2n+1}+D^2\delta^{2n+1}\right) - CD\sum_{n=0}^{\infty}P_{2n+2}\right]x^{2n+1}. \qquad (11.13)$$

It now follows from (11.12) and (11.13) that

$$8\sum_{k=0}^{n}A_{2k}A_{2n-2k+1} = (n+1)(C^2\gamma^{2n+1}+D^2\delta^{2n+1}) - CDP_{n+2}. \qquad (11.14)$$

This yields the following results:

$$8\sum_{k=0}^{n}P_{2k}P_{2n-2k+1} = 2(n+1)Q_{2n+1} - P_{2n+2}$$

$$4\sum_{k=0}^{n}Q_{2k}Q_{2n-2k+1} = 2(n+1)Q_{2n+2} + P_{2n+2}.$$

For example, $8\sum_{k=0}^{4}P_{2k}P_{9-2k} = 11,552 = 10\cdot 1393 - 2378 = 2\cdot 5Q_9 - P_{10}$; and $4\sum_{k=0}^{4}Q_{2k}Q_{9-2k} = 16,308 = 10\cdot 1393 + 2378 = 2\cdot 5Q_9 + P_{10}$; see Figure 11.4.

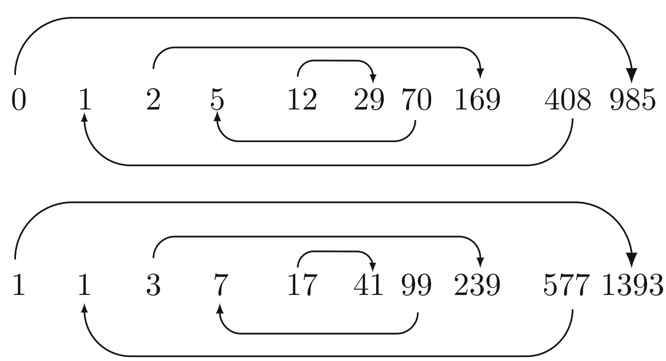

Figure 11.4.

Next we pursue generating functions for the squares of Pell and Pell–Lucas numbers.

11.8 Generating Functions for $\{P_n^2\}$ and $\{Q_n^2\}$

Since both $\{P_n^2\}$ and $\{Q_n^2\}$ satisfy the same recurrence (8.2), in the interest of brevity, we will first find a generating function $s(x)$ for the sequence $\{s_n\}$, where $\{s_n\}$ satisfies the recurrence $s_n = 5s_{n-1} + 5s_{n-2} - s_{n-3}$, and s_n equals $\{P_n^2\}$ or $\{Q_n^2\}$, depending on the context. To this end, let

$$
\begin{aligned}
s(x) &= s_0 + s_1x + s_2x^2 + s_3x^3 + \cdots + s_nx^n + \cdots \text{. Then}\\
5xs(x) &= 5s_0x + 5s_1x^2 + 5s_2x^3 + \cdots + 5s_{n-1}x^n + \cdots\\
5x^2s(x) &= 5s_0x^2 + 5s_1x^3 + \cdots + 5s_{n-2}x^n + \cdots\\
x^3s(x) &= s_0x^3 + \cdots + s_{n-3}x^n + \cdots\\
(1-5x-5x^2+x^3)s(x) &= s_0 + (s_1 - 5s_0)x + (s_2 - 5s_1 - 5s_0)x^2\\
s(x) &= \frac{s_0 + (s_1 - 5s_0)x + (s_2 - 5s_1 - 5s_0)x^2}{1-5x-5x^2+x^3}.
\end{aligned}
$$

Case 1 Suppose $s_n = P_n^2$. Then the generating function $g(x)$ for the sequence $\{P_n^2\}$ is given by

$$
\begin{aligned}
g(x) &= \frac{P_0^2 + (P_1^2 - 5P_0^2)x + (P_2^2 - 5P_1^2 - 5P_0^2)x^2}{1-5x-5x^2+x^3}\\
&= \frac{x - x^2}{(1+x)(1-6x+x^2)}.
\end{aligned}
$$

Thus

$$\frac{x-x^2}{(1+x)(1-6x+x^2)} = 1^2x + 2^2x^2 + 5^2x^3 + 12^2x^4 + \cdots + P_n^2x^n + \cdots .$$

Case 2 Suppose $s_n = Q_n^2$. Then the generating function $g(x)$ for the sequence $\{Q_n^2\}$ is given by

$$
\begin{aligned}
g(x) &= \frac{Q_0^2 + (Q_1^2 - 5Q_0^2)x + (Q_2^2 - 5Q_1^2 - 5Q_0^2)x^2}{(1+x)(1-6x+x^2)}\\
&= \frac{1 - 4x - x^2}{(1+x)(1-6x+x^2)}.
\end{aligned}
$$

Thus

$$\frac{1-4x-x^2}{(1+x)(1-6x+x^2)} = 1^2 + 1^2x + 3^2x^2 + 7^2x^3 + 17^2x^4 + \cdots + Q_n^2x^n + \cdots .$$

11.9 Generating Functions for $\{P_{2n+1}\}, \{Q_{2n}\}, \{Q_{2n+1}\}$, and $\{P_{2n}\}$ Revisited

It is worth noting that the generating functions for $\{P_{2n+1}\}, \{Q_{2n}\}, \{Q_{2n+1}\}$, and $\{P_{2n}\}$ can be found in other ways; see Exercises 5–8.

With the resources we already have, we can develop the generating functions for products of pairs of consecutive Pell and Pell–Lucas numbers.

11.10 Generating Functions for $\{P_nP_{n+1}\}$ and $\{Q_nQ_{n+1}\}$

Using the identity $4P_{n+1}P_n = Q_{n+1}^2 - Q_n^2$ and the generating function for $\{Q_k^2\}$, we can develop a generating function for $\{P_nP_{n+1}\}$:

$$\begin{aligned}
4P_{n+1}P_n &= Q_{n+1}^2 - Q_n^2 \\
\sum_{n=0}^{\infty} 4P_{n+1}P_n x^n &= \sum_{n=0}^{\infty} Q_{n+1}^2 x^n - \sum_{n=0}^{\infty} Q_n^2 x^n \\
&= \frac{1}{x}\left[\frac{1-4x-x^2}{(1+x)(1-6x+x^2)} - 1\right] - \frac{1-4x-x^2}{(1+x)(1-6x+x^2)} \\
&= \frac{8x}{(1+x)(1-6x+x^2)}.
\end{aligned}$$

So

$$\frac{2x}{(1+x)(1-6x+x^2)} = \sum_{n=0}^{\infty} P_nP_{n+1}x^n.$$

Likewise, we can show that

$$\begin{aligned}
\frac{(1-x)^2}{(1+x)(1-6x+x^2)} &= \sum_{n=0}^{\infty} Q_nQ_{n+1}x^n \\
\frac{1-5x}{(1+x)(1-6x+x^2)} &= \sum_{n=0}^{\infty} P_{n+1}P_{n-1}x^n \\
\frac{1+8x-3x^2}{(1+x)(1-6x+x^2)} &= \sum_{n=0}^{\infty} Q_{n+1}Q_{n-1}x^n;
\end{aligned}$$

see Exercises 9–11.

Next we develop yet another explicit formula for P_n.

11.11 Another Explicit Formula for P_n

In 1993, Seiffert developed yet another explicit formula P_n[200]:

$$P_n = \sum_{\substack{0\le k\le n-1 \\ 4\nmid(2n+k)}} (-1)^{\lfloor (3k+3-2n)/4\rfloor} 2^{\lfloor 3k/2\rfloor}\binom{n+k}{2k+1} \tag{11.15}$$

The proof featured below is based on the one given by Bruckman in the following year and is a bit long [31]. It uses a lot of algebra, complex numbers, and the generating function for Pell numbers.

Proof. Recall that the function $\frac{x}{f(x)} = \frac{x}{1-2x-x^2}$ generates the Pell numbers P_n.

Let S_n denote the expression on the RHS of (11.15) and $g(x) = \sum_{n=0}^{\infty} S_n x^n$. So $S_n = P_n$ if and only if $g(x) = \frac{x}{f(x)}$. Consequently, it suffices to show that $g(x) = \frac{x}{f(x)}$.

We have

$$g(x) = \sum_{n=1}^{\infty} \left[\sum_{\substack{0 \le k \le n-1 \\ 4 \nmid (2n+k)}} (-1)^{\lfloor (3k+3-2n)/4 \rfloor} 2^{\lfloor 3k/2 \rfloor} \binom{n+k}{2k+1} \right] x^n.$$

Letting $n = m + k + 1$, this becomes

$$g(x) = \sum_{\substack{k,m \ge 0 \\ 4 \nmid (2m+3k+2)}} (-1)^{\lfloor (k+1-2m)/4 \rfloor} 2^{\lfloor 3k/2 \rfloor} \binom{m+2k+1}{2k+1} x^{m+k+1}.$$

Considering the cases that m can be even (say, $m = 2u$) or odd (say, $m = 2u + 1$), this yields

$$\begin{aligned} g(x) &= \sum_{\substack{k,u \ge 0 \\ 4 \nmid (3k+2)}} (-1)^{\lfloor (k+1)/4 \rfloor + u} 2^{\lfloor 3k/2 \rfloor} \binom{2u+2k+1}{2u} x^{2u+k+1} + \\ &\quad \sum_{\substack{k,u \ge 0 \\ 4 \nmid 3k}} (-1)^{\lfloor (k-1)/4 \rfloor + u} 2^{\lfloor 3k/2 \rfloor} \binom{2u+2k+2}{2u+1} x^{2u+k+2}. \end{aligned}$$

Letting $j = k + 1$, this can be rewritten as

$$\begin{aligned} g(x) &= \sum_{\substack{j \ge 1 \\ 4 \nmid (j+1)}} (-1)^{\lfloor j/4 \rfloor} 2^{\lfloor 3(j-1)/2 \rfloor} x^j \sum_{u \ge 0} (-1)^u \binom{-2j}{2u} x^{2u} - \\ &\quad \sum_{\substack{j \ge 1 \\ 4 \nmid (j-1)}} (-1)^{\lfloor (j-2)/4 \rfloor} 2^{\lfloor 3(j-1)/2 \rfloor} x^j \sum_{u \ge 0} (-1)^u \binom{-2j}{2u+1} x^{2u+1}, \end{aligned} \tag{11.16}$$

where we have used the fact that $\binom{-n}{r} = (-1)^r \binom{n+r-1}{r}$.

Now let $e_\nu = \frac{1}{2}[1 + (-1)^\nu]$ and $z = 1 + ix$, where $i = \sqrt{-1}$. Since $(1+t)^{-n} = \sum_{r=0}^{\infty} \binom{-n}{r} t^r$, we have

$$\begin{aligned}
\sum_{u \ge 0} (-1)^u \binom{-2j}{2u} x^{2u} &= \sum_{\nu \ge 0} e_\nu \binom{-2j}{\nu} (ix)^\nu \\
&= \frac{1}{2}\left(z^{-2j} + \overline{z^{-2j}}\right) \\
&= \operatorname{Re}\left(z^{-2j}\right), \qquad (11.17)
\end{aligned}$$

where $\overline{w}$ denotes the conjugate of the complex number w and $\operatorname{Re}(w)$ its real part.

Letting $o_\nu = \frac{1}{2}[1 - (-1)^\nu]$, we have

$$\begin{aligned}
\sum_{u \ge 0} (-1)^u \binom{-2j}{2u+1} x^{2u+1} &= -i \sum_{\nu \ge 0} o_\nu \binom{-2j}{\nu} (ix)^\nu \\
&= \frac{1}{2i}\left(z^{-2j} - \overline{z^{-2j}}\right) \\
&= \operatorname{Im}\left(z^{-2j}\right), \qquad (11.18)
\end{aligned}$$

where $\operatorname{Im}(w)$ denotes the imaginary part of w.

Now let

$$U(x) = \sum_{\substack{j \ge 1 \\ 4 \nmid (j+1)}} (-1)^{\lfloor j/4 \rfloor} 2^{\lfloor 3(j-1)/2 \rfloor} x^j z^{-2j} \qquad (11.19)$$

$$V(x) = \sum_{\substack{j \ge 1 \\ 4 \nmid (j-1)}} (-1)^{\lfloor (j-2)/4 \rfloor} 2^{\lfloor 3(j-1)/2 \rfloor} x^j z^{-2j}. \qquad (11.20)$$

It then follows by equation (11.16) that

$$g(x) = \operatorname{Re}[U(x) + iV(x)].$$

We now let $j = 4r + s$, where $r \ge 0$ and $s = 1, 2,$ or 4 in (11.19). (Notice that $s \ne 3$.) Then $U(x)$ can be rewritten as

$$U(x) = \left(\frac{x}{z^2} + \frac{2x^2}{z^4} - \frac{16x^4}{z^8}\right) h(x),$$

where $h(x) = \sum_{r=0}^{\infty} (-1)^r 2^{6r} x^{4r} z^{-8r} = \left(1 + \frac{64x^4}{z^8}\right)^{-1} = \frac{z^8}{z^8 + 64x^4}$.

So

$$U(x) = \frac{x(z^6 + 2xz^4 - 16x^3)}{z^8 + 64x^4}$$

$$= \frac{x(z^4+4xz^2+8x^2)(z^2-2x)}{(z^4+4xz^2+8x^2)(z^4-4xz^2+8x^2)}$$
$$= \frac{x(z^2-2x)}{z^4-4xz^2+8x^2}.$$

Similarly, letting $s = 2, 3,$ or 4 in (11.20) yields

$$\begin{aligned}
V(x) &= \left(\frac{2x^2}{z^4}+\frac{8x^3}{z^6}+\frac{16x^4}{z^8}\right)h(x) \\
&= \frac{2x^2(z^4+4xz^2+8x^2)}{z^8+64x^4} \\
&= \frac{2x^2(z^4+4xz^2+8x^2)}{(z^4+4xz^2+8x^2)(z^4-4xz^2+8x^2)} \\
&= \frac{2x^2}{z^4-4xz^2+8x^2}.
\end{aligned}$$

But $z^2 = 1+2ix-x^2 = 1-2x-x^2+2(1+i)x = f+2(1+i)x$, where $f = f(x)$. So

$$\begin{aligned}
z^4 &= f^2+4(1+i)xf+8ix^2 \\
&= f[f+4(1+i)x]+8ix^2 \\
z^2-4xz^2+8x^2 &= f[f+4(1+i)x]+8ix^2-4x[f+2(1+i)x]+8x^2 \\
&= f(f+4ix).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
U(x)+iV(x) &= \frac{x(z^2-2x+2ix)}{f(f+4ix)} \\
&= \frac{x(f+4ix)}{f(f+4ix)} \\
&= \frac{x}{f}.
\end{aligned}$$

Thus, $U(x)+iV(x)$ is real; so $g(x) = \text{Re}[U(x)+iV(x)] = U(x)+iV(x) = \frac{x}{f(x)}$, as desired. ∎

For example, we have

$$P_5 = \sum_{\substack{0\le k\le 4 \\ 4\nmid(10+k)}} (-1)^{\lfloor(3k-7)/4\rfloor} 2^{\lfloor 3k/2\rfloor} \binom{5+k}{2k+1}$$

$$= \sum_{\substack{0\le k\le 4\\ 4\nmid(k+2)}} (-1)^{\lfloor (3k+1)/4\rfloor} 2^{\lfloor 3k/2\rfloor}\binom{5+k}{2k+1}$$

$$= 1\cdot\binom{5}{1} - 2\cdot\binom{6}{3} + 16\cdot\binom{8}{7} - 64\cdot\binom{9}{9}$$

$$= 5 - 40 + 128 - 64 = 29, \text{ as expected.}$$

11.12 Hoggatt's Array

The next example is interesting in its own right. Hoggatt proposed it as a problem in 1977 [103].

Example 11.3 Row 0 in the array in Figure 11.5 consists of the Pell numbers P_n and each succeeding row is obtained by taking the absolute differences of adjacent elements in the previous row. Let $\{d_n\}_{n=0}^{\infty}$ denote the leftmost diagonal sequence, where $d_0 = 1 = d_1, d_2 = 2 = d_3, d_4 = 4 = d_5, d_6 = 8 = d_7, \cdots$. Prove each:

(1) $d_{2n} = d_{2n+1} = 2^n$, where $n \ge 0$.

(2) $d(x) = \frac{1}{x} g\left(\frac{x}{1+x}\right)$, where $d(x)$ denotes the generating function of the sequence $\{d_n\}$, and $g(x)$ that of the Pell sequence.

```
1     2     5     12     29     70
   1     3     7     17     41
      2     4     10     24   ···
         2     6     14
            4     8
               4
                  ⋮
```

Figure 11.5.

Proof. Notice that Row 1 of the array consists of the Pell–Lucas numbers Q_n. This should not be a surprise, since $P_{n+1} - P_n = Q_n$.

We will now confirm a key observation: Every row of the array satisfies the same Pell recurrence. To see this, consider the sequence $\{x_n\}$ defined by the recurrence $x_{n+2} = ax_{n+1} + bx_n$, and $\{y_n\}$ a sequence defined by $y_n = x_{n+1} - x_n$. Then

$$\begin{aligned} ay_{n+1} + by_n &= a(x_{n+2} - x_{n+1}) + b(x_{n+1} - x_n) \\ &= (ax_{n+2} + bx_{n+1}) - (ax_{n+1} + bx_n) \end{aligned}$$

$$\begin{aligned} &= x_{n+3} - x_{n+2} \\ &= y_{n+2}. \end{aligned}$$

Consequently, the sequence $\{y_n\}$ satisfies the same recurrence as $\{x_n\}$.

In particular, let $a = 2$ and $b = 1$. Thus d_n satisfies the Pell recurrence.

We will now prove parts (1) and (2):

(1) Let $\{e_n\}_{n=0}^{\infty}$ denote the second southeast diagonal from the left: $2, 3, 4, 6, 8, \ldots$. We will prove by PMI that $d_{2n} = d_{2n+1} = 2^n$, $e_{2n} = 2d_{2n}$, and $e_{2n+1} = 3d_{2n+1}$ for every integer $n \geq 0$.

Clearly, $e_0 = 2 = 2d_0$ and $e_1 = 3 = 3d_1$; $e_2 = 4 = 2d_2$ and $e_3 = 6 = 3d_3$, the formulas work when $n = 0$ and $n = 1$.

Now assume that they are true for all nonnegative integers $< n$, where $n \geq 2$. Since $d_{2n-2} = d_{2n-1} = 2^{n-1}$, and $e_{2n-1} = 3d_{2n-1}$, $d_{2n} = e_{2n-1} - d_{2n-1} = 3d_{2n-1} - d_{2n-1} = 2d_{2n-1} = 2^n$ and $d_{2n+1} = e_{2n} - d_{2n} = 2 \cdot 2^n - 2^n = 2^n$. Thus, by PMI, the formulas hold for all integers $n \geq 0$.

To digress a bit, note that the elements of the $(2k)$th and $(2k + 1)$st rows of the array reveal an interesting pattern:

$$\begin{array}{ccccccccccc} 1 \cdot 2^k & & 2 \cdot 2^k & & 5 \cdot 2^k & & 12 \cdot 2^k & & 29 \cdot 2^k & \cdots \\ & 1 \cdot 2^k & & 3 \cdot 2^k & & 7 \cdot 2^k & & 17 \cdot 2^k & & \end{array}$$

Row $2k$ is made up of the sequence $\{2^k P_n\}$ and row $2k + 1$ is made up of the sequence $\{2^k Q_n\}$. These can be confirmed using the properties $P_{n+1} - P_n = Q_n$ and $Q_{n+1} - Q_n = 2P_n$.

(2) To prove part (2), recall that $g(x) = \frac{x}{1-2x-x^2}$ is the generating function of Pell numbers. Then

$$g\left(\frac{x}{1+x}\right) = \frac{x + x^2}{1 - 2x^2}.$$

We will now show that the function $\frac{1}{x} g\left(\frac{x}{1+x}\right)$ generates the sequence $\{d_n\}$.

To this end, let $d(x) = \sum_{n=0}^{\infty} d_n x^n$. Since $d_{2n} = 2^n = d_{2n+1}$, we have

$$\begin{aligned} d(x) &= \sum_{n=0}^{\infty} 2^n (x^{2n} + x^{2n+1}) \\ &= 1 + x + \sum_{n=1}^{\infty} 2^n (x^{2n} + x^{2n+1}) \\ 2x^2 d(x) &= \sum_{n=0}^{\infty} 2^{n+1} (x^{2n+2} + x^{2n+3}) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} 2^n (x^{2n} + x^{2n+1}) \\
(1-2x^2)d(x) &= 1 + x \\
d(x) &= \frac{1+x}{1-2x^2} \\
&= \frac{1}{x} g\left(\frac{x}{1+x}\right), \quad \text{as desired.}
\end{aligned}
$$

Thus the sequence $\{d_n\}$ is generated by the function $d(x) = \frac{1}{x} g\left(\frac{x}{1+x}\right) = \frac{1+x}{1-2x^2}$:

$$\frac{1+x}{1-2x^2} = 1 + x + 2x^2 + 2x^3 + 2^2x^4 + 2^2x^5 + 2^3x^6 + \cdots .$$ ■

Suppose row 1 in Figure 11.5 begins with the number $0 = P_0$. The resulting array, studied by Piero Filipponi in 1993, shows an additional pattern; see Figure 11.6.

$$
\begin{array}{ccccccccccccc}
0 & & 1 & & 2 & & 5 & & 12 & & 29 & & 70 \\
& 1 & & 1 & & 3 & & 7 & & 17 & & 41 & \\
& & 0 & & 2 & & 4 & & 10 & & 24 & & \\
& & & 2 & & 2 & & 6 & & 14 & & \cdots & \\
& & & & 0 & & 4 & & 8 & & & & \\
& & & & & 4 & & 4 & & & & & \\
& & & & & & 0 & & & & & & \\
& & & & & & & \vdots & & & & &
\end{array}
$$

Figure 11.6.

Let $\{a_n\}_{n=0}^{\infty}$ denote the left most diagonal sequence of this array. Since $a_k = e_k - 2d_k$, it follows that $a_{2n} = e_{2n} - 2d_{2n} = 2d_{2n} - 2d_{2n} = 0$ and $a_{2n+1} = e_{2n+1} - 2d_{2n+1} = 3d_{2n+1} - 2d_{2n+1} = 2^n$.

The sequence $\{a_n\}$ is generated by the function $\frac{x}{1-2x^2}$:

$$\frac{x}{1-2x^2} = x + 2x^3 + 2^2x^5 + 2^3x^7 + \cdots .$$

Exercises 11

Develop a generating function for each sequence $\{a_n\}$, where:

1. $a_n = P_{2n+1}$. *Hint*: Use the odd function $A_o(x)$.
2. $a_n = Q_{2n+1}$.
3. $a_n = P_{2n}$. *Hint*: Use the even function $A_e(x)$.
4. $a_n = Q_{2n}$.

Find a generating function for:

5. $\{P_{2n+1}\}$, using the the identity $P_{n+1}^2 + P_n^2 = P_{2n+1}$ and the generating function for $\{P_n^2\}$.
6. $\{Q_{2n}\}$, using the identity $Q_{2n} = 2Q_n^2 - (-1)^n$ and the generating function for $\{Q_n^2\}$.
7. $\{Q_{2n+1}\}$, using the identity $Q_{n+1} = 2P_{n+1} - Q_n$.
8. $\{P_{2n}\}$, using the generating functions for $\{Q_{2n}\}$ and $\{P_{2n+1}\}$, and the identity $P_n + Q_n = P_{n+1}$.

Develop a generating function for each sequence:

9. $\{Q_{n+1}Q_n\}$. *Hint*: $P_{n+1}^2 - P_n^2 = Q_{n+1}Q_n$.
10. $\{P_{n+1}P_{n-1}\}$. *Hint*: Use Cassini-like formula for P_n.
11. $\{Q_{n+1}Q_{n-1}\}$. *Hint*: Use Cassini-like formula for Q_n.

Using Seiffert's formula (11.15), compute each Pell number.

12. P_6.
13. P_{11}.

12

Pell Walks

12.1 Introduction

Like the Fibonacci family [126], the Pell family has delightful applications to combinatorics. This chapter presents several such applications. In the process, we will revisit Fibonacci numbers, and encounter a new family that arises in combinatorics.

To begin with, we introduce some basic vocabulary for clarity. A *lattice point* on the cartesian plane is a point (x, y) with integral coordinates x and y. For example, $(3, 5)$ and $(8, -13)$ are lattice points, whereas $(0, \sqrt{2})$ and $(\pi, -\pi)$ are not. A *lattice path* is a sequence of connected horizontal, vertical, or diagonal unit steps $\overline{P_i P_{i+1}}$, where both P_i and P_{i+1} are lattice points; it often emanates from the origin. The number of unit steps in the path is its *length*.

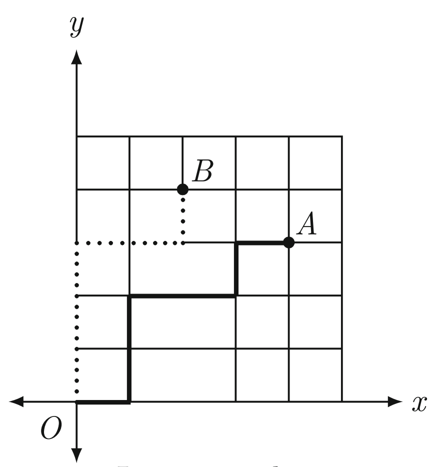

Figure 12.1.

For example, Figure 12.1 shows two lattice paths, one a solid path and the other a dotted path. The solid path, originating at O and ending at A, consists of one unit step in the easterly

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direction (E), followed by two in the northerly direction (N), two in the easterly direction (E), one in the northerly direction (N), and one in the easterly direction (E). Its length is seven and is denoted by the "word"$ENNEENE$. Likewise, the dotted path $NNNEEN$ from O to B is of length six.

The *height* of a lattice path is the height of its final step above the x-axis. For example, the height of the solid path in Figure 12.1 is three and that of the dotted path is four. Let n denote the length of a lattice path and h its height. Then $0 \le h \le n$.

The examples we will study shortly deal with applications of Pell and Pell–Lucas numbers to the study of lattice paths. The first example was studied by Richard P. Stanley of Massachusetts Institute of Technology, Cambridge, Massachusetts [238], and Asamoah Nkwanta and Louis W. Shapiro of Howard University, Washington, D.C. The next two were studied by Nkwanta and Shapiro, who called the paths *Pell walks* [172].

Example 12.1 Starting at the origin on the cartesian plane, suppose we walk n unit steps east (E), north (N), or west (W), such that no E-step follows immediately a W-step or vice versa; that is, *no* word can contain EW or WE as a subword. Find the number a_n of such lattice paths possible.

Let $q_{n,h}$ denote the number of such paths with height h. Clearly, $q_{0,0} = 1 = a_0 = Q_1$; see Figure 12.2. There are exactly two such paths of length $n \ge 1$ and height zero: $\underbrace{EE\cdots E}_{n\ E\text{s}}$ and $\underbrace{WW\cdots W}_{n\ W\text{s}}$; see Figure 12.3. So $q_{n,0} = 2$ for every $n \ge 1$. Consequently, $a_1 = 1 + 2 = 3 = Q_2 = q_{0,0} + q_{1,0}$. Furthermore, there is a unique path of height n: $\underbrace{NN\cdots N}_{n\ N\text{s}}$; so $q_{n,n} = 1$ for every $n \ge 0$; see Figure 12.4.

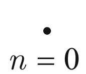

Figure 12.2.

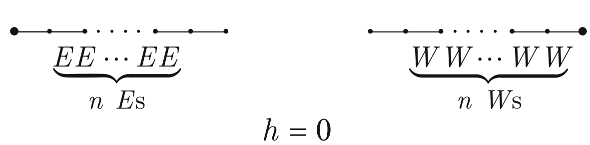

Figure 12.3.

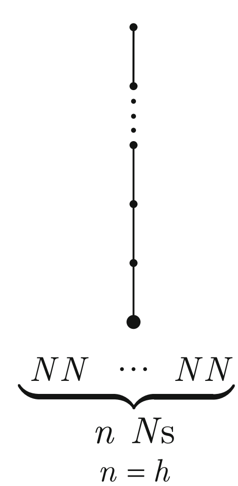

Figure 12.4.

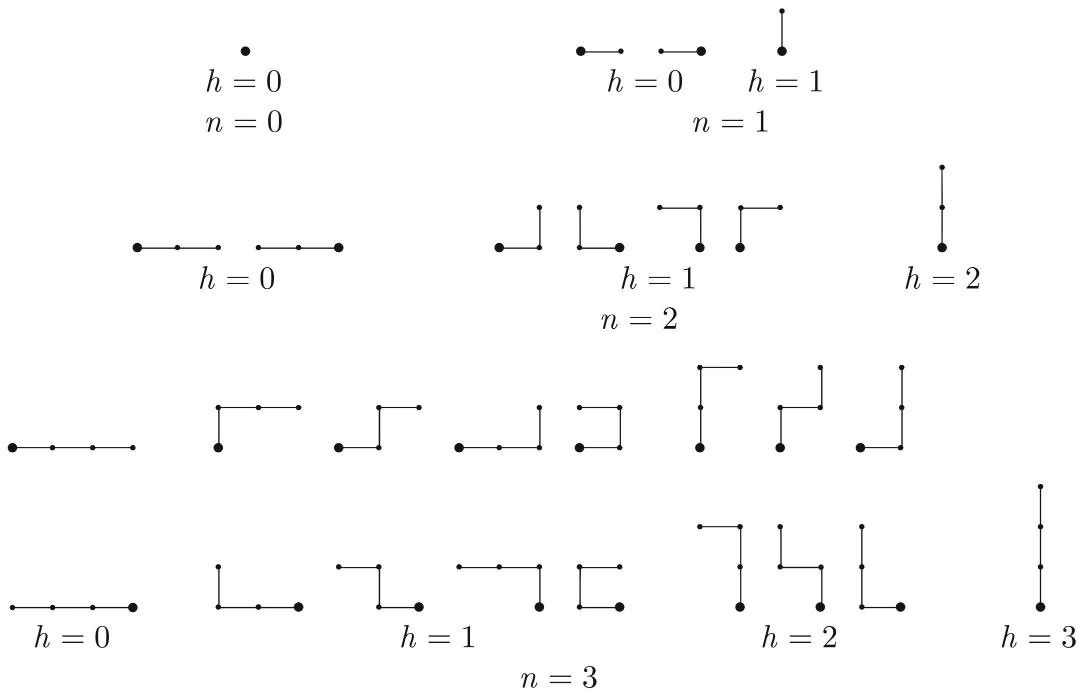

Figure 12.5.

Table 12.1.

$n \backslash h$	0	1	2	3	4	5	row sums
0	1						1
1	2	1					3
2	2	4	1				7
3	2	8	6	1			17
4	2	12	18	8	1		41
5	2	16	38	32	10	1	99
							↑ Q_{n+1}

Array Q

Figure 12.5 shows the possible paths of length n and height h, where a thick dot indicates the origin and $0 \le n \le 3$. The corresponding values of $q_{n,h}$ can be summarized in a table $Q = (q_{n,h})_{n,h\ge 0}$, as in Table 12.1, where $0 \le h \le n \le 5$. Nkwanta and Shapiro developed this array in 2005.

The row sums of this array exhibit an interesting pattern: The nth row sum is Q_{n+1}, where $n \ge 0$; that is, $\sum_{h=0}^{n} q_{n,h} = Q_{n+1}$. We will now establish this using strong induction.

Proof. Notice that a path of length $n+1$ and height h can be obtained in two different ways:

(1) Appending an N to a path of length n and height $h-1$.
(2) Appending an N to paths of length $\le n$ and height $h-1$, followed by an appropriate sequence of Es or Ws.

Consequently, the array elements $q_{n,h}$ can be defined recursively:

$$\begin{aligned} q_{0,0} &= 1 \\ q_{n+1,h} &= q_{n,h-1} + 2\sum_{j\ge1} q_{n-j,h-1}, \end{aligned} \tag{12.1}$$

where it is understood that $q_{n,h}=0$ if $h>n$, and $n\ge 0$.

For example, $q_{4,2} = q_{3,1} + 2\sum_{j\ge1} q_{3-j,1} = q_{3,1} + 2(q_{2,1}+q_{1,1}) = 8 + 2(4+1) = \boxed{18}$; see Table 12.1. Likewise, $q_{5,3} = q_{4,2} + 2(q_{3,2}+q_{2,2}) = 18 + 2(6+1) = \textcircled{32}$.

We are now ready to present the proof. Clearly, $a_0 = 1 = \sum_{h=0}^{0} q_{0,h} = Q_1$ and $a_1 = 3 = \sum_{h=0}^{1} q_{1,h} = Q_2$. So the formula works when $n=0$ and $n=1$.

Now assume that it works for all nonnegative integers $< n$, where n is an arbitrary integer ≥ 1. Using the recurrence (12.1), we then have

$$\begin{aligned} \sum_{h=0}^{n+1} q_{n+1,h} &= q_{n+1,0} + \sum_{h=1}^{n} q_{n+1,h} + q_{n+1,n+1} \\ &= q_{n,0} + \sum_{h=1}^{n} q_{n,h-1} + 2\sum_{h=1}^{n}\sum_{j\ge1} q_{n-j,h-1} + q_{n,n} \\ &= \left(\sum_{h=0}^{n-1} q_{n,h} + q_{n,n}\right) + 2\sum_{h=1}^{n}\sum_{j\ge1} q_{n-j,h-1} + q_{n,0} \\ &= \sum_{h=0}^{n} q_{n,h} + 2\sum_{j\ge1}\sum_{h=1}^{n} q_{n-j,h-1} + q_{n,0} \\ &= Q_{n+1} + 2\sum_{h=0}^{n-1} q_{n-1,h} + 2\sum_{j\ge2}\sum_{h=1}^{n} q_{n-j,h-1} + q_{n,0} \\ &= Q_{n+1} + \sum_{h=0}^{n-1} q_{n-1,h} + \sum_{h=1}^{n} q_{n-1,h-1} + 2\sum_{j\ge2}\sum_{h=1}^{n} q_{n-j,h-1} + q_{n,0} \end{aligned}$$

$$\begin{aligned}
&= Q_{n+1} + Q_n + \sum_{h=1}^{n}\left(q_{n-1,h-1} + 2\sum_{j\geq 2} q_{n-j,h-1}\right) + q_{n,0} \\
&= Q_{n+1} + Q_n + \sum_{h=1}^{n}\left(q_{n-1,h-1} + 2\sum_{k\geq 1} q_{n-1-k,h-1}\right) + q_{n,0} \\
&= Q_{n+1} + Q_n + \sum_{h=1}^{n} q_{n,h} + q_{n,0} = Q_{n+1} + Q_n + \sum_{h=0}^{n} q_{n,h} \\
&= Q_{n+1} + Q_n + Q_{n+1} = 2Q_{n+1} + Q_n \\
&= Q_{n+2}.
\end{aligned}$$

So the formula also works for $n + 1$.

Thus, by the strong version of PMI, it works for $n \geq 0$; that is, $a_n = Q_{n+1}$ for $n \geq 0$. ∎

This fact can be expressed as a matrix equation, where the blanks indicate zeros:

$$\begin{bmatrix} 1 & & & & \\ 2 & 1 & & & \\ 2 & 4 & 1 & & \\ 2 & 8 & 6 & 1 & \\ 2 & 12 & 18 & 8 & 1 \\ & & \cdots & & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \cdots \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 7 \\ 17 \\ 41 \\ \cdots \end{bmatrix}.$$

12.2 Interesting Byproducts

The fact that $a_n = Q_{n+1}$ has several interesting byproducts:

(1) Suppose we would like to find the number of paths b_n of length n ending in N. Figure 12.6 shows the possible paths for $1 \leq n \leq 4$. Notice that $b_1 = 1 = Q_1, b_2 = 3 = Q_2, b_3 = 7 = Q_3$, and $b_4 = 17 = Q_4$. So we conjecture that $b_n = Q_n$ for every $n \geq 1$.

To confirm this notice that every path of length n ending in N must be of the form $w_1w_2\cdots w_{n-1}N$, where w_{n-1} can be E, N, or W. Consequently, the number of such paths equals the number of paths of length $n-1$. By part (1), there are Q_n paths of length $n-1$; so $b_n = Q_n$. (Consequently, there are Q_n such paths beginning with N.) ∎

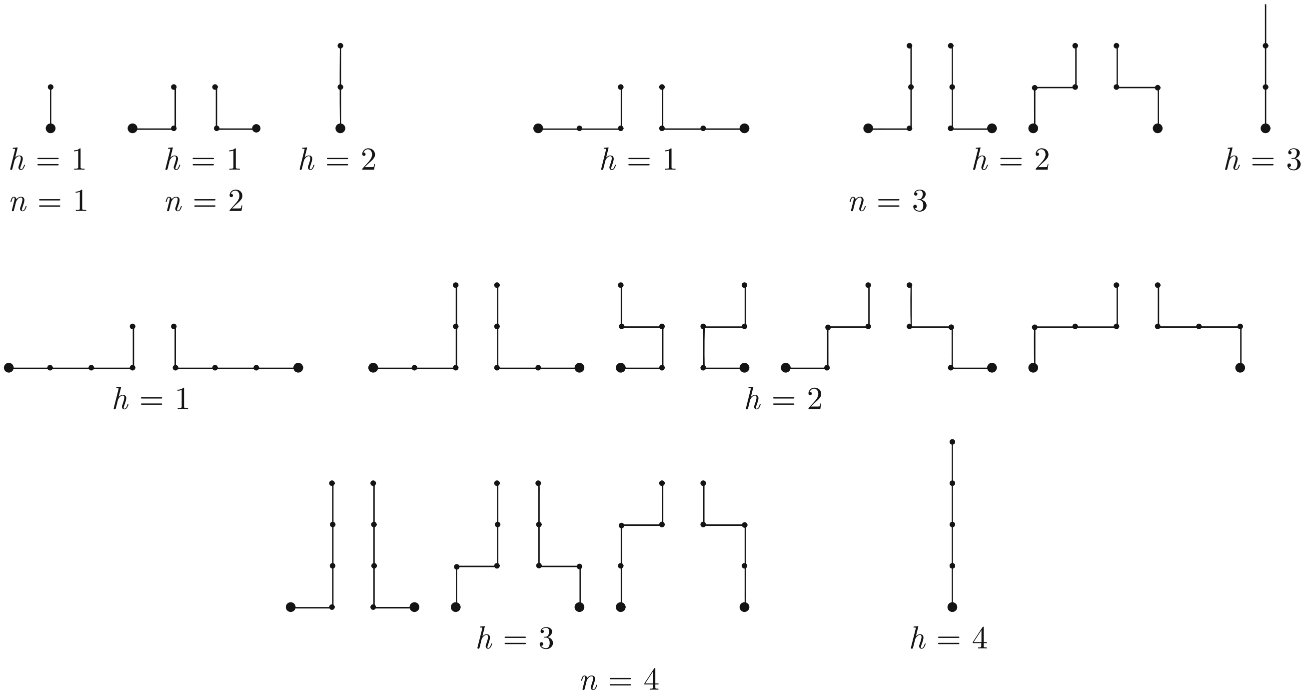

Figure 12.6.

For example, there are $Q_2 = 3$ such walks of length 2: EN, NN, and WN; and $Q_3 = 7$ such walks of length 3: $EEN, ENN, NEN, NNN, NWN, WNN$, and WWN; see Figure 12.5.

(2) Since every path of length n ending in NN is of the form $w_1w_2\cdots w_{n-2}NN$, it now follows that there are Q_{n-1} such Pell walks, where $n \geq 2$; see Figure 12.6.

(3) Suppose we would like to find the number of lattice paths c_n of length n ending in E. Figure 12.7 shows the possible paths, where $1 \leq n \leq 4$. It follows from the figure that $c_1 = 1 = P_1, c_2 = 2 = P_2, c_3 = 5 = P_3$, and $c_4 = 12 = P_4$. So we conjecture that $c_n = P_n$ for every $n \geq 1$.

Since there are $a_n = Q_{n+1}$ Pell paths of length n and $b_n = Q_n$ of them end in N, it follows that $2c_n = a_n - b_n = Q_{n+1} - Q_n = 2P_n$; so $c_n = P_n$, as desired. (This can also be confirmed using PMI.) ■

Therefore, there are P_n paths beginning with E. For instance, there are $P_3 = 5$ paths of length three that begin with E: EEE, EEN, ENE, ENN, and ENW; see Figure 12.5.

(4) It follows by property (3) that there are P_{n-1} paths $w_1w_2\cdots w_{n-2}EE$, ending in EE, where $n \geq 2$; see Figure 12.8. ■

(5) Since there are P_{n-1} paths of length n ending in EE, P_{n-1} paths ending in WW, and Q_{n-1} paths ending in NE, the total number of walks of length n ending in EE, WW, or NE is given by $P_{n-1} + (P_{n-1} + Q_{n-1}) = P_{n-1} + P_n = Q_n$; see Figure 12.9, where $n = 3$.

(6) Suppose we would like to find the number of walks beginning with N and ending in W. Such a path is of the form $Nw_2\cdots w_{n-2}w_{n-1}W$, where $w_{n-1} \neq E$. Since there are P_{n+1}

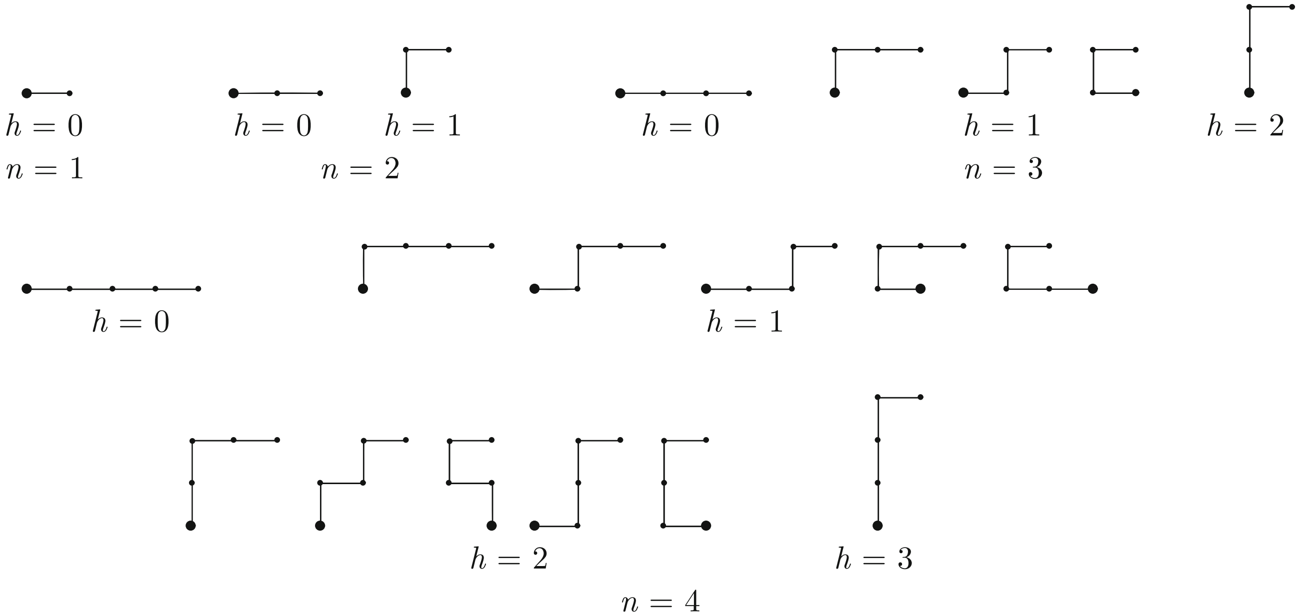

Figure 12.7.

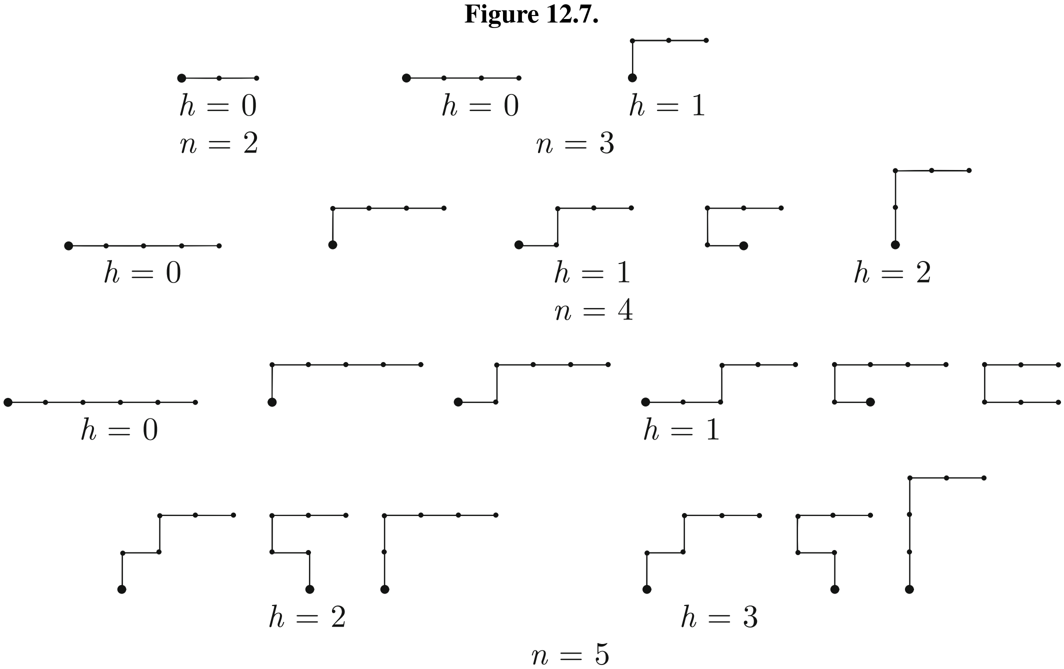

Figure 12.8.

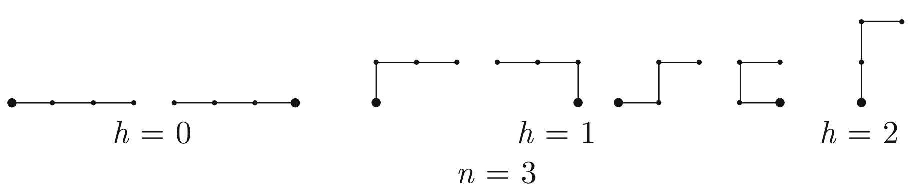

Figure 12.9.

paths *not* ending in W, it follows that there are $Q_{n+1} - P_{n+1} = P_n$ paths ending in W. They are of the form $w_1 \cdots w_{n-1} W$, where w_1 can be N, E, or W, and $w_{n-1} \neq E$.

Suppose $w_1 = N = w_{n-1}$. By property (2) above, there are Q_{n-2} such paths. On the other hand, let $w_{n-1} = W$. Such a path is of the form $Nw_2 \cdots w_{n-2}WW$. Since w_2 can be N, E, or W, there are P_{n-2} such paths. Thus there are $Q_{n-2} + P_{n-2} = P_{n-1}$ paths that begin with N and end in W, where $n \geq 2$.

For example, there are exactly $P_2 = 2$ paths of length three that begin with N and end in W: NNW and NWW; see Figure 12.5. Likewise, there are $P_3 = 5$ such paths of length five: $NENW$, $NNNW$, $NNWW$, $NWNW$, and $NWWW$.

(7) There are Q_n paths beginning with N, and P_{n-1} paths that begin with N and end in W. So there are $Q_n - P_{n-1} = P_n$ paths that begin with N and do not end in W.

For instance, there are $P_3 = 5$ such paths of length three: NEE, NEN, NNE, NNN, and NWN; see Figure 12.5.

(8) Recall that there are Q_n paths beginning with N, and Q_{n-1} paths beginning with and ending in N. Thus we have

$$\begin{aligned}\begin{pmatrix}\text{Number of walks beginning}\\ \text{with and not ending in } N\end{pmatrix} &= Q_n - Q_{n-1}\\ &= 2P_{n-1}.\end{aligned}$$

For example, there are $2P_{3-1} = 2P_2 = 4$ such walks of length three: NEE, NNE, NNW, and NWW.

(9) We have

$$\begin{aligned}\begin{pmatrix}\text{Number of walks}\\ \text{not ending in } W\end{pmatrix} &= \begin{pmatrix}\text{total number}\\ \text{of walks}\end{pmatrix} - \begin{pmatrix}\text{number of walks}\\ \text{ending in } W\end{pmatrix}\\ &= Q_{n+1} - P_n\\ &= P_{n+1}.\end{aligned}$$

Consequently, the number of paths that do not end in W, but begin with W, equals $P_n - P_{n-1} = Q_{n-1}$, where $n \geq 2$.

For instance, there are $P_{3+1} = P_4 = 12$ paths of length three that do not end in W: EEE, EEN, ENE, ENN, NEE, NEN, NNE, NNN, NWN, WNE, WNN, and WWN; exactly, $Q_2 = 3$ of them begin with W: WWN, WNE, and WNN.

(10) The number of paths of length n not ending in N is given by

$$\begin{aligned}\begin{pmatrix}\text{Total number}\\ \text{of paths}\end{pmatrix} - \begin{pmatrix}\text{number of paths}\\ \text{ending in } N\end{pmatrix} &= Q_{n+1} - Q_n\\ &= 2P_n.\end{aligned}$$

As an example, there are $2P_3 = 10$ Pell walks of length three that do not end in N: $EEE, NEE, ENE, ENW, NNE, NNW, NWW, WNE, WNW$, and WWW; see Figure 12.5.

(11) The number of Pell walks ending in W is given by

$$\begin{aligned}\begin{pmatrix}\text{Total number}\\ \text{of paths}\end{pmatrix} - \begin{pmatrix}\text{number of paths}\\ \text{not ending in } W\end{pmatrix} &= Q_{n+1} - P_{n+1}\\ &= P_n.\end{aligned}$$

For instance, there are exactly $P_3 = 5$ paths of length three that end in W: ENW, NNW, NWW, WNW, and WWW; see Figure 12.5.

(12) Every walk that begins with and end in NN is of the form $NNw_3 \cdots w_{n-2}NN$. Since both w_3 and w_{n-2} can be E, N, or W, there are exactly Q_{n-3} such paths, where $n \geq 4$.

For example, there are $Q_{6-3} = Q_3 = 7$ such paths of length six: $NNNNNN$, $NNNENN$, $NNNWNN$, $NNEENN$, $NNENNN$, $NNNENN$, and $NNNWNN$.

Next we develop a recurrence for the number of Pell walks that begin with and end in E.

12.3 Walks Beginning with and Ending in *E*

Let f_n denote the number Pell walks $Ew_2 \cdots w_{n-1}E$, where $w_2, w_{n-1} \neq W$. Clearly, $f_1 = 1 = f_2$.

Case 1 Let $w_{n-1} = E$. Such a path is of the form $\underbrace{Ew_2 \cdots E}_{l=n-1} E$. By definition, there are f_{n-1} such Pell walks.

Case 2 Let $w_{n-1} = N$. Such a Pell word is of the form $\underbrace{Ew_2 \cdots w_{n-2}}_{l=n-2} NE$, where w_{n-2} is arbitrary. There are P_{n-2} such paths that begin with E.

Combining the two cases yields a recurrence for f_n:

$$f_n = f_{n-1} + P_{n-2}, \tag{12.2}$$

where $f_1 = 1$ and $n \geq 2$.

For example, there are $f_5 = f_4 + P_3 = 4 + 5 = 9$ Pell walks of length 5 that begin with and end in *E: EEEEE, EEENE, EENEE, EENNE, ENEEE, ENENE, ENNEE,* $ENNNE$, and $ENWNE$. Likewise, there are $f_6 = 21$ such walks of length 6.

Interestingly, the first-order recurrence (12.2) can be rewritten as a second-order recurrence:

$$\begin{aligned} f_n &= (f_{n-2} + P_{n-3}) + P_{n-2} \\ &= f_{n-2} + (P_{n-3} + P_{n-2}) \\ &= f_{n-2} + Q_{n-2}, \end{aligned} \tag{12.3}$$

where $f_1 = 1 = f_2$ and $n \geq 3$.

For example, $f_5 = f_3 + Q_3 = 2 + 7 = 9$, as expected.

Recurrence (12.2), coupled with the summation formula for Pell numbers, can be used to develop an explicit formula for f_n. Since $P_0 = 0$, it follows by iteration from (12.2) that

$$\begin{aligned} f_n &= 1 + \sum_{k=1}^{n-2} P_k \\ &= 1 + \frac{Q_{n-1} - 1}{2} \\ &= \frac{Q_{n-1} + 1}{2}. \end{aligned} \tag{12.4}$$

For example, $f_6 = \frac{Q_5+1}{2} = 21$.

As a byproduct, it follows from (12.4) also that every Pell–Lucas number has odd parity.

Using the summation formulas for Pell–Lucas numbers, the explicit formula (12.4) can be obtained from (12.3) as well.

Recurrence (12.3) has an interesting consequence: it can be used to find a recurrence for the number of walks g_n of length n that begin with and end in EE : $EEw_3 \cdots w_{n-2}EE$. It follows from the recurrence that $g_n = g_{n-2} + Q_{n-4}$, where $g_4 = 1, g_5 = 2$, and $n \geq 4$.

For example, there are $g_6 = g_4 + Q_2 = 1 + 3 = 4$ such walks of length 6: $EEEEEE$, $EEENEE$, $EENEEE$, and $EENNEE$. Likewise, there are $g_7 = 9$ walks of length 7: $EEEEEEE$, $EEEENEE$, $EEENEEE$, $EEENNEE$, $EENEEEE$, $EENENEE$, $EENNEEE$, $EENNNEE$, and $EENWNEE$.

Next we find the number of paths beginning with E and ending in W.

12.4 Paths Beginning with *E* and Ending in *W*

Let x_n denote the number of paths $E\underbrace{w_2 \cdots w_{n-1}}_{l=n-2}W$, where $w_2 \neq W$ and $w_{n-1} \neq E$. Clearly, $x_2 = 0, x_3 = 1$, and $x_4 = 3$.

There are four cases that we need to investigate. In the interest of brevity, we will omit all details.

Suppose $w_2 = E$ and $w_{n-1} = N$. There are P_{n-3} paths $E\,\underbrace{Ew_3\cdots w_{n-2}}_{l=n-3}NW$; on the other hand, let $w_2 = E$ and $w_{n-1} = W$. By definition, there are x_{n-2} paths $E\,\underbrace{Ew_3\cdots w_{n-2}W}_{l=n-2}W$; Suppose $w_2 = N = w_{n-1}$. There are Q_{n-3} paths $EN\,\underbrace{w_3\cdots w_{n-2}}_{l=n-4}NW$; and if $w_2 = N$ and $w_{n-1} = W$, there are P_{n-3} paths $EN\,\underbrace{w_3\cdots w_{n-2}}_{l=n-4}WW$, where $w_{n-2} \neq E$.

Thus we have

$$\begin{aligned} x_n &= x_{n-2} + (2P_{n-3} + Q_{n-3}) \\ &= x_{n-2} + Q_{n-2}, \end{aligned} \tag{12.5}$$

where $n \geq 2$.

It follows from (12.5) by iteration, and the summation formulas for Pell–Lucas numbers, that $x_n = \frac{Q_{n-1}-1}{2}$, where $n \geq 2$.

For example, there are $x_5 = \frac{Q_4-1}{2} = 8$ Pell walks beginning with E and ending in W: $EEENW$, $EENNW$, $EENWW$, $ENENW$, $ENNNW$, $ENNWW$, $ENWNW$, and $ENWWW$.

Next we find the number of Pell walks that begin with E, but do not end in W.

12.5 Paths Beginning with *E*, but not Ending in *W*

Let y_n denote the number of Pell paths $Ew_2\cdots w_{n-1}w_n$, where $w_n = E$ or N.

Suppose $w_n = E$. There are $\frac{Q_{n-1}+1}{2}$ paths $Ew_2\cdots w_{n-1}E$. On the other hand, let $w_n = N$. Such a path is of the form $\underbrace{Ew_2\cdots w_{n-1}}_{l=n-1}N$, where w_{n-1} is arbitrary. By part (3), there are P_{n-1} such walks.

Thus, we have

$$\begin{aligned} y_n &= \frac{Q_{n-1}+1}{2} + P_{n-1} \\ &= \frac{(2P_{n-1} + Q_{n-1}) + 1}{2} \\ &= \frac{Q_n + 1}{2}. \end{aligned} \tag{12.6}$$

For example, there are $y_4 = \frac{Q_4+1}{2} = 9$ Pell walks of length 4 that begin with E, but do not end in W: $EEEE$, $EEEN$, $EENE$, $EENN$, $ENEE$, $ENEN$, $ENNE$, $ENNN$, and $ENWN$. Similarly, there are $y_5 = 21$ such walks of length five.

Finally, we find the number of Pell walks that do not begin with or end in E.

12.6 Paths not Beginning with or Ending in *E*

Let z_n denote the number of Pell walks $w_1w_2\cdots w_{n-1}w_n$, where $w_1, w_n \neq E$.

Briefly, there are Q_{n-1} paths $N\underbrace{w_2\cdots w_{n-1}}_{l=n-2}N$, where w_2 and w_{n-1} are arbitrary; by part (4), there are P_{n-1} paths $\underbrace{Nw_2\cdots w_{n-1}W}_{l=n}$, beginning with N and ending in W; there are $\frac{Q_{n-1}+1}{2}$ paths $\underbrace{Ww_2\cdots w_{n-1}W}_{l=n}$ that begin with and end in W; and by part (3), there are P_{n-1} paths $\underbrace{Ww_2\cdots w_{n-1}}_{l=n-1}N$.

Combining the four cases, we get

$$\begin{aligned} z_n &= Q_{n-1} + P_{n-1} + \frac{Q_{n-1}+1}{2} + P_{n-1} \\ &= (2P_{n-1} + Q_{n-1}) + \frac{Q_{n-1}+1}{2} \\ &= Q_n + \frac{Q_{n-1}+1}{2} \\ &= \frac{(2Q_n + Q_{n-1}) + 1}{2} \\ &= \frac{Q_{n+1}+1}{2}. \end{aligned} \tag{12.7}$$

For example, there are $z_3 = \frac{Q_4+1}{2} = 9$ walks of length three that do not begin with or end in E: $NEN, NNN, NNW, NWN, NWW, WNN, WNW, WWN$, and WWW. Similarly, there are $z_4 = 21$ such walks of length four.

The next example, studied by Nkwanta and Shapiro, also deals with Pell walks, but with an added condition.

Example 12.2 Suppose we add the restriction that *no* paths can end in a W. Figure 12.10 shows the possible paths, where $0 \leq n \leq 3$.

Summarizing the data for $0 \leq n \leq 4$ yields another Pascal-like triangle; see Table 12.2.

Let $d_{n,h}$ denote the number of Pell walks of length n and height h that do *not* end in W. For example, $d_{3,2} = 5$; see Figure 12.10. Since every path with height zero goes east, $d_{n,0} = 1$. As before, there is a unique path of height $h = n$; so $d_{n,n} = 1$.

The array $D = (d_{n,h})_{n,h\geq 0}$ in Table 12.2 also can be defined recursively:

$$\begin{aligned} d_{n,0} &= 1, \quad d_{n,n} = 1, \quad n \geq 0 \\ d_{n+1,h} &= d_{n,h} + d_{n,h-1} + d_{n-1,h-1}, \end{aligned} \tag{12.8}$$

where $1 \leq h \leq n$.

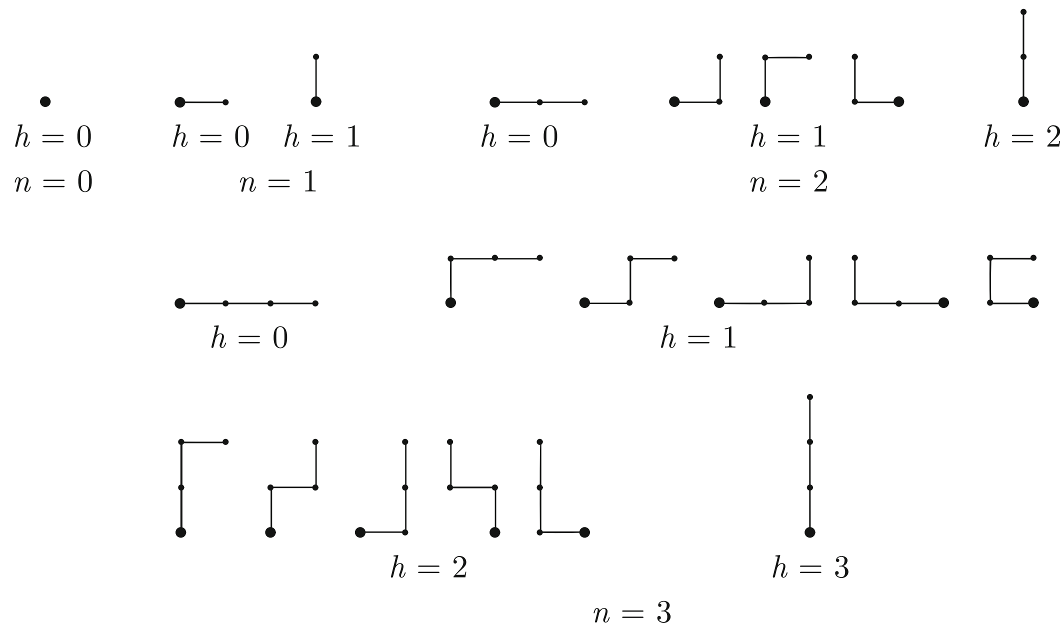

Figure 12.10.

Table 12.2.

$n \backslash h$	0	1	2	3	4	row sums
0	1					1
1	1	1				2
2	1	3	1			5
3	1	5	5	1		12
4	1	7	(13)	7	1	29
						↑ P_{n+1}

Array D

For example, $d_{4,2} = d_{3,2} + d_{3,1} + d_{2,1} = 5 + 5 + 3 = (13)$; see Table 12.2.

The row sums of array D also exhibit an interesting pattern: The nth row sum r_n equals P_{n+1}, where $n \geq 0$. We can prove this by strong induction and recurrence (12.8). But we will take a much shorter route:

$$\begin{aligned}\begin{pmatrix}\text{Number of Pell walks}\\ \text{not ending in } W\end{pmatrix} &= \begin{pmatrix}\text{number of Pell walks ending in } E\\ + \text{ number of Pell walks ending in } N\end{pmatrix}\\ &= P_n + Q_n\\ &= P_{n+1}.\end{aligned}$$

[This also follows by the fact that $a_n = Q_{n+1}$ and property (11) above.]

For example, let $n = 3$. Then there are $P_4 = 12$ Pell walks that do not end in W; they end in E or N; see Figure 12.5.

The fact that $r_n = P_{n+1}$ also can be expressed as a matrix equation:

$$\begin{bmatrix} \textcircled{1} & & & & \\ 1 & 1 & & & \\ 1 & \textcircled{3} & 1 & & \\ 1 & 5 & 5 & 1 & \\ 1 & 7 & \textcircled{13} & 7 & 1 \\ & & \cdots & & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \cdots \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \\ 12 \\ 29 \\ \cdots \end{bmatrix}.$$

We note, as an aesthetic curiosity, that array D in Table 12.2 can be rewritten as a symmetric triangle, as Figure 12.11 shows. By virtue of the recurrence (12.8) (see the diamond in the figure) and the initial conditions, it is relatively easy to construct this modified form. For example, $3 + 5 + 5 = \textcircled{13}$.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 1 & & & & \\
 & & & 1 & & 5 & \blacklozenge & 5 & & 1 & & & \\
 & & 1 & & 7 & & \textcircled{13} & & 7 & & 1 & & \\
 & 1 & & 9 & & 25 & & 25 & & 9 & & 1 & \\
1 & & 11 & & 41 & & \boxed{63} & & 41 & & 11 & & 1
\end{array}$$

Figure 12.11.

12.7 A Hidden Treasure

We return to array D, which contains a delightful treasure. The entries $d_{n,h}$ are the *Delannoy numbers*, named after the French mathematician Henri Auguste Delannoy (1833–1915). The central elements $d_{2n,n}$ are the *central Delannoy numbers* $1, 3, \textcircled{13}, \boxed{63}, 321, 1683, 8989, 48639, 265729, \cdots$. Combinatorially, they count the number of lattice paths from the origin to the lattice point (n, n) on the cartesian plane, using only the single steps N, NE, or E. Figure 12.12 shows all possible such paths, where $0 \le n \le 3$.

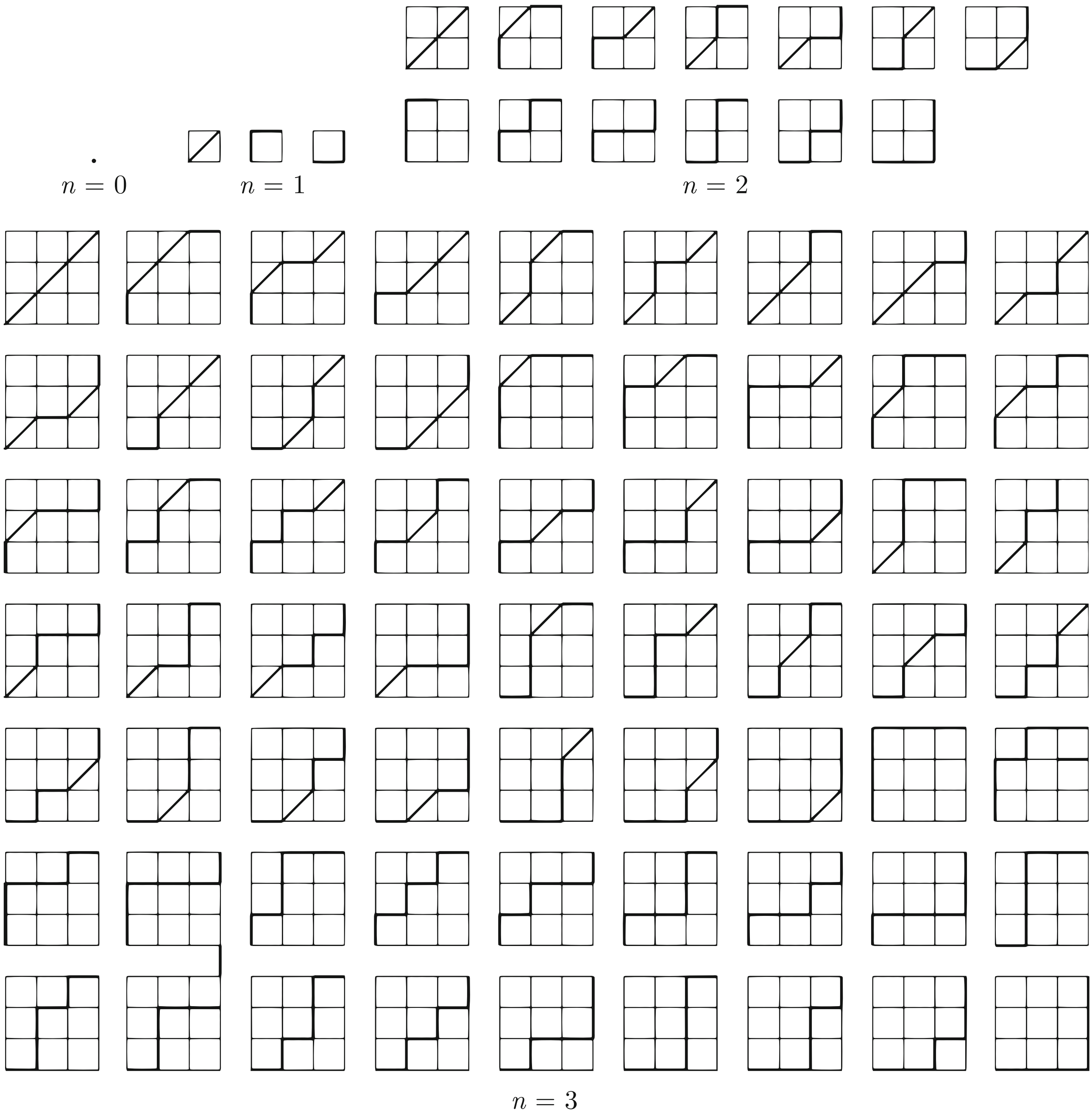

Figure 12.12.

The central Delannoy numbers $d_n = d_{2n,n}$ can also be computed as a sum of products of binomial coefficients [47]:

$$d_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k}. \tag{12.9}$$

For example,

$$\begin{aligned} d_3 &= \sum_{k=0}^{3} \binom{3}{k}\binom{3+k}{k} \\ &= \binom{3}{0}\binom{3}{0} + \binom{3}{1}\binom{4}{1} + \binom{3}{2}\binom{5}{2} + \binom{3}{3}\binom{6}{3} \\ &= 1\cdot 1 + 3\cdot 4 + 3\cdot 10 + 1\cdot 20 = \boxed{63}, \text{ as expected.} \end{aligned}$$

The sum (12.9) counts the number of lattice points from the origin to the point $(2n, 0)$ using the single steps NE, $EE = D$ and SE (southeast). Figure 12.13 shows the $d_2 = 13$ possible paths from the origin to the point $(4, 0)$.

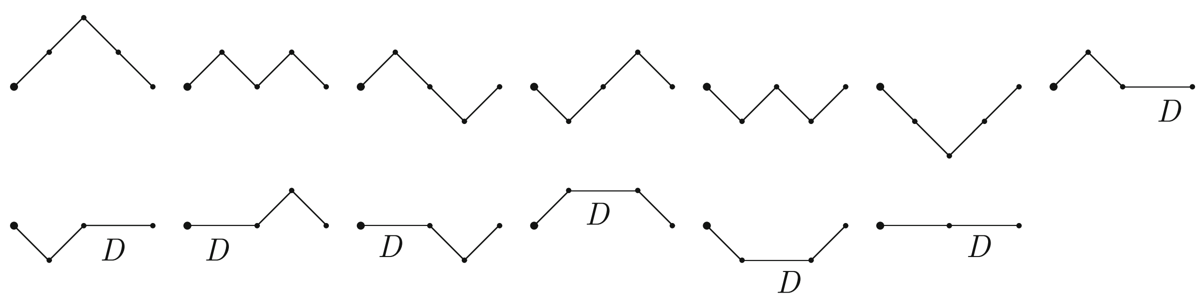

Figure 12.13.

The central Delannoy numbers can also be defined recursively [239]:

$$\begin{aligned} d_0 &= 1, \quad d_2 = 3 \\ d_n &= 3(2n-1)d_{n-1} - (n-1)d_{n-2}, \quad n \geq 3. \end{aligned}$$

For example, $d_4 = 3 \cdot 7d_3 - 3d_2 = 21 \cdot 63 - 3 \cdot 13 = 321$.
The Delannoy numbers d_n can be generated by the function $\frac{1}{\sqrt{1-6x+x^2}}$[239]:

$$\frac{1}{\sqrt{1-6x+x^2}} = 1 + 3x + 13x^2 + 63x^3 + 321x^4 + \cdots .$$

More generally, the *Delannoy number* $d_{m,n}$ counts the number of lattice paths from the origin to the lattice point (m, n) on the cartesian plane, using the single steps N, NE, or E. Table 12.3 shows the Delanny numbers $d_{m,n}$, where $0 \leq m \leq n \leq 5$; it is called the *Delannoy array*. Notice that the northeast diagonals of the Delannoy array are the same as the rows of array D in Table 12.3.

Table 12.3.

$m \backslash n$	0	1	2	3	4	5
0	**1**	1	1	1	1	1
1	1	**3**	5	7	9	11
2	1	5	**13**	25	41	61
3	1	7	25	**63**	129	231
4	1	9	41	129	**321**	377
5	1	11	61	231	681	**1683**

Delannoy *Array D*

Delannoy numbers also can be defined recursively [47]:

$$\begin{aligned} d_{0,0} &= 1 \\ d_{m,n} &= d_{m-1,n} + d_{m,n-1} + d_{m-1,n-1}, \quad m,n \geq 1. \end{aligned}$$

For example, $d_{4,3} = d_{3,3} + d_{4,2} + d_{3,2} = 63 + 41 + 25 = 129$; see Table 12.3.
The numbers $d_{m,n}$ can be computed using the following summation formulas [47] as well:

$$\begin{aligned} d_{m,n} &= \sum_{k=0}^{m} \binom{n}{k} \binom{m+n-k}{n} \\ &= \sum_{k=0}^{m} 2^k \binom{n}{k} \binom{m}{k}. \end{aligned}$$

For example,

$$\begin{aligned} d_{4,3} &= \sum_{k=0}^{3} \binom{3}{k} \binom{7-k}{3} \\ &= \binom{3}{0}\binom{7}{3} + \binom{3}{1}\binom{6}{3} + \binom{3}{2}\binom{5}{3} + \binom{3}{3}\binom{4}{3} \\ &= 1 \cdot 35 + 3 \cdot 20 + 3 \cdot 10 + 1 \cdot 4 = 129 \end{aligned}$$

and

$$\begin{aligned} \sum_{k=0}^{4} 2^k \binom{3}{k}\binom{4}{k} &= 2^0\binom{3}{0}\binom{4}{0} + 2^1\binom{3}{1}\binom{4}{1} + 2^2\binom{3}{2}\binom{4}{2} + 2^3\binom{3}{3}\binom{4}{3} + 2^4\binom{3}{4}\binom{4}{4} \\ &= 1 + 24 + 72 + 32 + 0 = 129 \\ &= d_{4,3}. \end{aligned}$$

Finally, the Delannoy numbers can be generated by a function in two variables x and y [47]:

$$\frac{1}{1-x-y-xy} = \sum_{m,n\geq 0} d_{m,n}x^m y^n.$$

12.8 Example 12.2 Revisited

Using Example 12.2, we can now determine the number of Pell walks u_n that do not begin with E or end in W:

$$\begin{aligned}
\text{Number of paths not ending in } W &= P_{n+1}\\
\begin{pmatrix}\text{Number of walks beginning}\\ \text{with } E \text{ and not ending in } W\end{pmatrix} &= \frac{Q_n+1}{2}\\
u_n &= P_{n+1} - \frac{Q_n+1}{2}\\
&= \frac{(2P_{n+1}-Q_n)-1}{2}\\
&= \frac{(Q_{n+2}-Q_{n+1})-Q_n-1}{2}\\
&= \frac{(Q_{n+1}+Q_n)-Q_n-1}{2}\\
&= \frac{Q_{n+1}-1}{2}.
\end{aligned}$$

For example, there are $u_3 = \frac{Q_4-1}{2} = 8$ walks that do not begin with E or end in W: $NEE, NEN, NNE, NNN, NWN, WNE, WNN$, and WWN.

Next we pursue a different family of Pell walks.

Example 12.3 Let a_n denote the number of lattice paths of length n, using unit steps N, E, or double E-steps (of length 2) which are denoted by D. Figure 12.14 shows such paths for $0 \leq n \leq 3$.

Summarizing the data for $0 \leq h \leq n \leq 4$ yields the triangular array S in Table 12.4.

The elements $s_{n,h}$ of array S can be defined recursively:

$$\begin{aligned}
s_{0,0} &= 1\\
s_{1,0} &= 1 = s_{1,1}\\
s_{n,h} &= s_{n-1,h} + s_{n-1,h-1} + s_{n-2,h}, \qquad (12.10)
\end{aligned}$$

where $0 \leq h \leq n$ and $n \geq 2$. Once again, it is understood that $s_{n,h} = 0$ if $h < 0$ or $h > n$.

For example, $s_{4,2} = s_{3,2} + s_{3,1} + s_{2,2} = 3 + 5 + 1 = \textcircled{9}$; see Table 12.4.

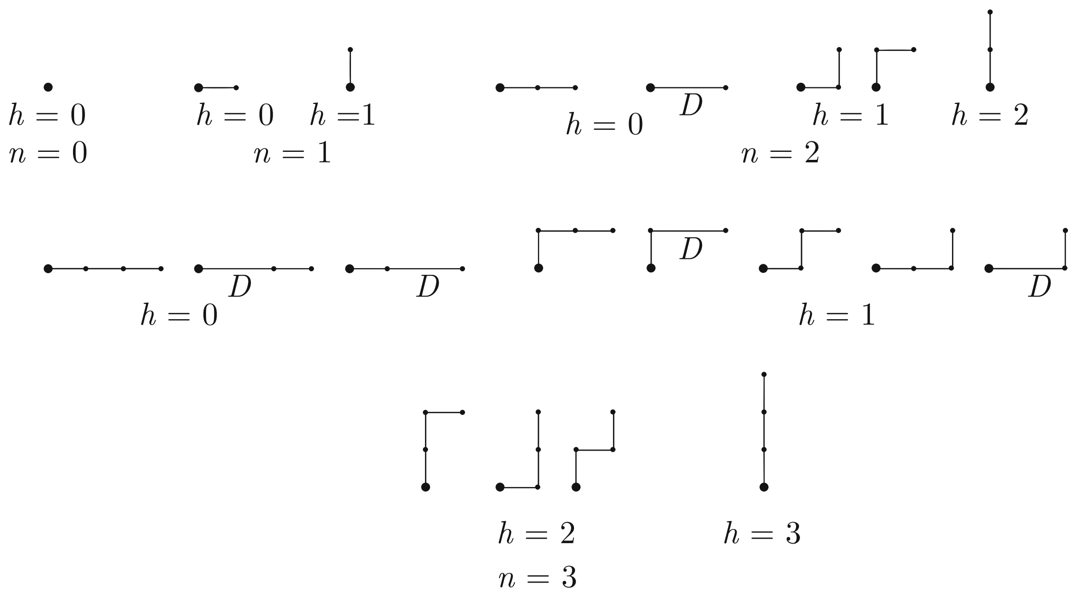

Figure 12.14.

Table 12.4.

n \ h	0	1	2	3	4	row sums
0	1					1
1	1	1				2
2	2	2	1			5
3	3	5	3	1		12
4	5	10	9	4	1	29
	↑ F_{n+1}					↑ P_{n+1}

Array S shows two delightful patterns:

(1) Column 0 consists of Fibonacci numbers: $s_{n,0} = F_{n+1}$.

(2) The nth row sum is P_{n+1}: $\sum_{h=0}^{n} s_{n,h} = P_{n+1}$.

We will now establish that both patterns do hold.

(1) Consider a lattice path of length n and height 0. The problem of using E-steps and EE-steps corresponds to climbing a ladder of n rungs, taking one rung or two rungs at each step, where $n \geq 0$. There are F_{n+1} such ways we can climb up the ladder [126]. Consequently, $s_{n,0} = F_{n+1}$, where $n \geq 0$. ■

(2) We will establish this pattern using strong induction. Since $\sum_{h=0}^{0} s_{0,h} = 1 = P_1$ and $\sum_{h=0}^{1} s_{1,h} = 2 = P_2$, the formula works for $n = 0$ and $n = 1$.

Now assume that it is true for all nonnegative integers $< n$, where n is an arbitrary integer ≥ 2. Then, by the recurrence (12.10), we have

$$\begin{aligned}
\sum_{h=0}^{n} s_{n,h} &= \sum_{h=0}^{n} s_{n-1,h} + \sum_{h=0}^{n} s_{n-1,h-1} + \sum_{h=0}^{n} s_{n-2,h} \\
&= \sum_{h=0}^{n-1} s_{n-1,h} + \sum_{h=0}^{n-1} s_{n-1,h} + \sum_{h=0}^{n-2} s_{n-2,h} \\
&= P_n + P_n + P_{n-1} = 2P_n + P_{n-1} \\
&= P_{n+1}.
\end{aligned}$$

So the formula works for n also.

Thus, by the strong version of PMI, $s_n = P_{n+1}$ for every integer $n \geq 0$. ∎

13

Pell Triangles

13.1 Introduction

Pell and Pell–Lucas numbers can be used to construct a Pascal-like *Pell triangle*, developed in 2005. The top northeast and southeast diagonals consist of Pell numbers, and the next northeast and southeast diagonals Pell–Lucas numbers. Each remaining element is obtained by adding twice its immediate predecessor in the same diagonal to the one immediately before that in the very same diagonal; see Figure 13.1. For example, $\textcircled{46} = 2 \cdot 19 + 8 = 2 \cdot 17 + 12$.

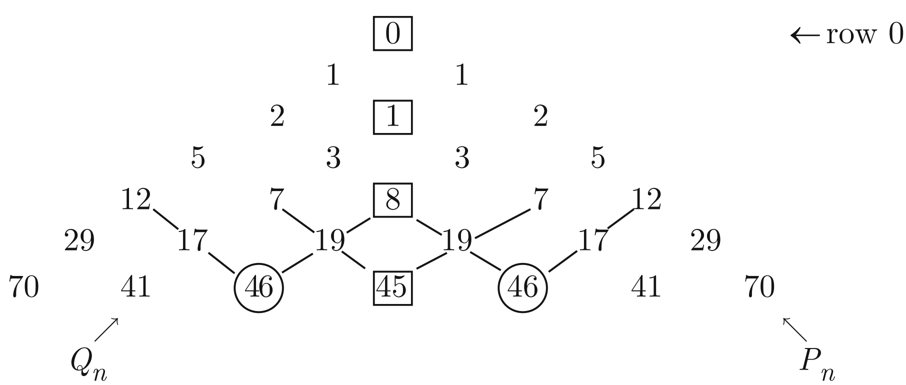

Figure 13.1.

Accordingly, the Pell triangle can be defined recursively:

$$\begin{aligned} g(0,0) &= 0, \quad g(1,0) = g(1,1) = g(2,1) = 1 \\ g(n,0) &= 2g(n-1,0) + g(n-2,0), \quad \text{if } n \geq 2 \\ g(n,n) &= 2g(n-1,n-1) + g(n-2,n-2), \quad \text{if } n \geq 2 \end{aligned}$$

T. Koshy, *Pell and Pell–Lucas Numbers with Applications*, DOI 10.1007/978-1-4614-8489-9_13,

$$\begin{aligned} g(n,j) &= 2g(n-1,j)+g(n-2,j), \quad \text{if } 1 \le j \le n-2 &(13.1)\\ &= 2g(n-1,j-1)+g(n-2,j-2), \quad \text{if } 2 \le j \le n-1. &(13.2) \end{aligned}$$

Notice that the Pell array is symmetric about the vertical line through the apex; that is, $g(n,j) = g(n,n-j)$. Also, $g(n,0) = P_n = g(n,n)$ for every $n \ge 0$.

Applying the recurrence (13.1) successively, we have

$$\begin{aligned} g(n,j) &= 2g(n-1,j)+g(n-2,j)\\ &= 5g(n-2,j)+2g(n-3,j)\\ &= 12g(n-3,j)+5g(n-4,j)\\ &= 29g(n-4,j)+12g(n-5,j). \end{aligned}$$

More generally,

$$g(n,j) = P_{k+1}\cdot g(n-k,j) + P_k \cdot g(n-k-1,j), \tag{13.3}$$

where $1 \le k \le n-j-1$. This can be confirmed by induction.

In particular, let $j = 1$ and $k = n-2$. Then

$$\begin{aligned} g(n,1) &= P_{n-1}\cdot g(2,1) + P_{n-2}\cdot g(1,1)\\ &= P_{n-1} + P_{n-2} = Q_{n-1} \end{aligned}$$

and $g(n,n-1) = g(n,1) = Q_{n-1}$ by symmetry. Accordingly, the second northeast and southeast diagonals consist of Pell–Lucas numbers; see Figure 13.1.

When $k = n-j-1$, we have

$$\begin{aligned} g(n,j) &= P_{n-j}\cdot g(j+1,j) + P_{n-j-1}\cdot g(j,j)\\ &= P_{n-j}\cdot Q_j + P_{n-j-1}P_j\\ &= P_{n-j}Q_j + P_{n-j-1}P_j. &(13.4) \end{aligned}$$

Using this property, we can compute any element in the array directly from Pell and Pell–Lucas numbers.

Next we investigate the central elements of the array.

13.2 Central Elements in the Pell Triangle

Let K_n denote the central element in row $2n$ in the Pell triangle; it is the nth central element, counting from the apex. Using formula (13.4),

$$K_n = P_n \cdot Q_n + P_{n-1}\cdot P_n$$

$$
\begin{aligned}
&= P_n(P_n + P_{n-1}) + P_{n-1}P_n \\
&= P_n^2 + 2P_nP_{n-1}. \qquad (13.5)
\end{aligned}
$$

This formula enables us to compute the central elements in the Pell triangle directly from Pell numbers; see the boxed numbers in Figure 13.1.

For example, $K_4 = P_4^2 + 2P_4P_3 = 144 + 2 \cdot 12 \cdot 5 = 264$.

Using Pell recurrence, we can rewrite formula (13.5) in an alternate way:

$$
\begin{aligned}
K_n &= P_n^2 + P_{n-1}(P_{n+1} - P_{n-1}) \\
&= P_n^2 - P_{n-1}^2 + P_{n-1}P_{n+1} \\
&= P_{n+1}P_{n-1} + Q_nQ_{n-1}.
\end{aligned}
$$

Using identities (22), (31), and (34) in Chapter 7, we can develop a shorter formula for K_n:

$$
\begin{aligned}
K_n &= [P_n^2 + (-1)^n] + (P_n^2 - P_{n-1}^2) \\
&= [2P_n^2 + (-1)^n] - P_{n-1}^2 \\
&= Q_n^2 - P_{n-1}^2.
\end{aligned}
$$

For example, $K_3 = 45 = 7^2 - 2^2 = Q_3^2 - P_2^2$.

We can use formula (13.4) to compute the two middle elements in row n, where n is odd. Letting $j = \lfloor n/2 \rfloor$, it yields

$$
\begin{aligned}
g(n, \lfloor n/2 \rfloor) &= P_{n-\lfloor n/2 \rfloor}Q_{\lfloor n/2 \rfloor} + P_{n-\lfloor n/2 \rfloor-1}P_{\lfloor n/2 \rfloor} \\
&= g(n, n - \lfloor n/2 \rfloor) \\
&= P_{\lfloor n/2 \rfloor}Q_{n-\lfloor n/2 \rfloor} + P_{\lfloor n/2 \rfloor-1}P_{n-\lfloor n/2 \rfloor}.
\end{aligned}
$$

13.3 An Alternate Formula for *g(n, j)*

We can use formula (13.4) to derive an alternate formula for $g(n, j)$:

$$
\begin{aligned}
g(n, j) &= P_{n-j}Q_j + P_{n-j-1}P_j \\
&= \frac{\gamma^{n-j} - \delta^{n-j}}{2\sqrt{2}} \cdot \frac{\gamma^j + \delta^j}{2} + \frac{\gamma^{n-j-1} - \delta^{n-j-1}}{2\sqrt{2}} \cdot \frac{\gamma^j - \delta^j}{2\sqrt{2}} \\
&= \frac{(\gamma^n - \delta^n) + (\gamma\delta)^j(\gamma^{n-2j} - \delta^{n-2j})}{4\sqrt{2}}
\end{aligned}
$$

$$+\frac{(\gamma^{n-1}+\delta^{n-1})-(\gamma\delta)^j(\gamma^{n-2j-1}+\delta^{n-2j-1})}{8}$$

$$= \frac{P_n+(-1)^jP_{n-2j}}{2}+\frac{Q_{n-1}-(-1)^jQ_{n-2j-1}}{4}. \qquad (13.6)$$

For example,

$$\begin{aligned} g(5,2) &= \frac{P_5+P_1}{2}+\frac{Q_4-Q_0}{8} \\ &= \frac{29+1}{2}+\frac{17-1}{4}=19. \end{aligned}$$

It follows from formula (13.6) that

$$\begin{aligned} Q_n &\equiv (-1)^jQ_{n-2j} \pmod 8 \\ &\equiv (-1)^{\lfloor n/2\rfloor}Q_{n-2\lfloor n/2\rfloor} \pmod 8. \end{aligned}$$

So $Q_{2n}\equiv(-1)^nQ_0\equiv 2(-1)^n \pmod 8$ and $Q_{2n+1}\equiv(-1)^nQ_1\equiv 2(-1)^n \pmod 8$. Thus $Q_n\equiv\pm 2 \pmod 8$ for every n.

Formula (13.6) can be used to develop an alternate formula for the central element:

$$\begin{aligned} K_n &= \frac{P_{2n}+(-1)^nP_0}{2}+\frac{Q_{2n-1}-(-1)^nQ_{-1}}{4} \\ &= \frac{1}{2}P_{2n}+\frac{Q_{2n-1}+(-1)^n}{4}. \qquad (13.7) \end{aligned}$$

For example,

$$\begin{aligned} K_3 &= \frac{1}{2}P_6+\frac{Q_5+(-1)^3}{4} \\ &= \frac{70}{2}+\frac{41-1}{4}=45. \end{aligned}$$

It follows from formulas (13.5) and (13.7) that

$$\begin{aligned} \frac{1}{2}P_{2n}+\frac{Q_{2n-1}+(-1)^n}{4} &= P_n^2+2P_nP_{n-1} \\ 2P_{2n}+Q_{2n-1} &= 4(P_n^2+2P_nP_{n-1})-(-1)^n. \end{aligned}$$

Using the identities $P_{m-1}+P_{m+1}=2Q_m$ and $Q_{m-1}+Q_{m+1}=4P_m$, we can rewrite this in two other ways:

$$\begin{aligned} 5P_{2n}+P_{2n-2} &= 8(P_n^2+2P_nP_{n-1})-2(-1)^n \\ Q_{2n+1}+3Q_{2n-1} &= 8(P_n^2+2P_nP_{n-1})-2(-1)^n. \end{aligned}$$

13.4 A Recurrence for K_n

We can use formula (13.1) to find a recurrence for K_n:

$$\begin{aligned} K_n &= g(2n, n) \\ &= 2g(2n-1, n-1) + g(2n-2, n-2) \\ &= 2g(2n-1, n-1) + 2g(2n-3, n-2) + g(2n-4, n-2) \\ &= 2[g(2n-1, n-1) + g(2n-3, n-2)] + K_{n-2}. \end{aligned} \tag{13.8}$$

Thus K_n can be computed by adding twice the sum of the southwest and northwest neighbors of $K_{n-1}$to compute K_{n-2}; see Figure 13.2.

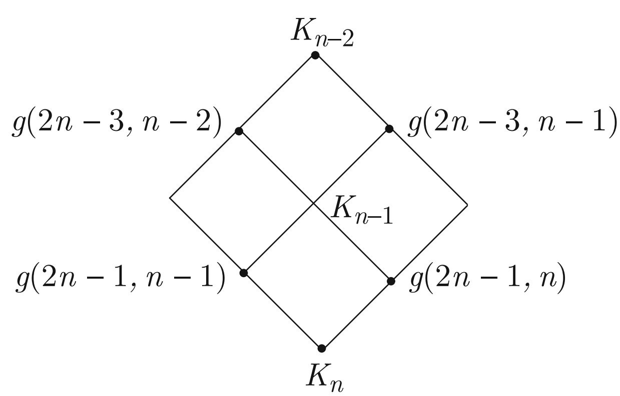

Figure 13.2.

Using (13.4), formula (13.8) yields

$$\begin{aligned} K_n &= 2\left(P_n Q_{n-1} + P_{n-1} Q_{n-2} + P_{n-1}^2 + P_{n-2}^2\right) + K_{n-2} \\ &= 2\left(P_n Q_{n-1} + P_{n-1} Q_{n-2} + P_{2n-3}\right) + K_{n-2}. \end{aligned} \tag{13.9}$$

For example,

$$\begin{aligned} K_4 &= 2(P_4 Q_3 + P_3 Q_2 + P_5) + K_2 \\ &= 2(12 \cdot 7 + 5 \cdot 3 + 29) + 8 = 264. \end{aligned}$$

It follows from formula (13.9) that twice the sum of the southwest and northwest neighbors of K_{n-1} is $2(P_n Q_{n-1} + P_{n-1} Q_{n-2} + P_{2n-3})$, where $n \geq 2$. For example, $2(19 + 3) = 2(5 \cdot 3 + 2 \cdot 1 + 5)$.

Finally, it follows from (13.5) and (13.9) that

$$2(P_n Q_{n-1} + P_{n-1} Q_{n-2} + P_{2n-3}) + K_{n-2} = P_n^2 + 2P_n P_{n-1}.$$

Thus

$$K_n = P_{n+2}^2 + 2P_{n+2}P_{n+1} - 2(P_{n+2}Q_{n+1} + P_{n+1}Q_n + P_{2n+1}).$$

For instance,

$$\begin{aligned} K_3 &= P_5^2 + 2P_5P_4 - 2(P_5Q_4 + P_4Q_3 + P_7) \\ &= 29^2 + 2 \cdot 29 \cdot 12 - 2(29 \cdot 17 + 12 \cdot 7 + 169) = 45, \quad \text{as desired.} \end{aligned}$$

13.5 DiDomenico's Triangles

In the early 1990s, Angelo S. DiDomenico of Milford, Massachusetts, developed the triangular array A in Figure 13.3 [72].

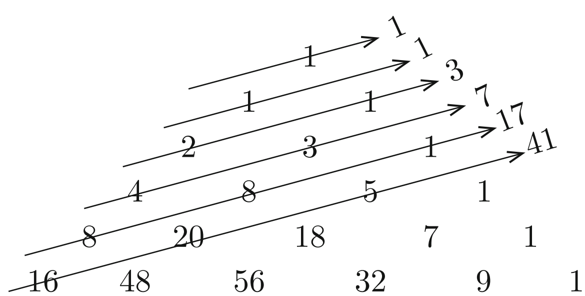

Figure 13.3.

The rising diagonal sums in Figure 13.3 yield the Pell numbers Q_n, where $n \geq 0$; see Exercise 2.

In 2005, DiDomenico developed two additional triangular arrays; see Figures 13.4 and 13.5. Clearly, both arrays can be defined recursively. They manifest several interesting properties. For example, the nth row sum in Figure 13.4 is P_n, and that in Figure 13.5 is Q_n.

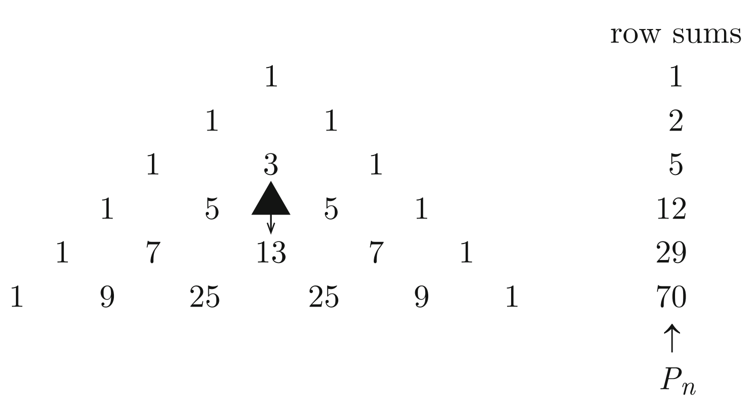

Figure 13.4.

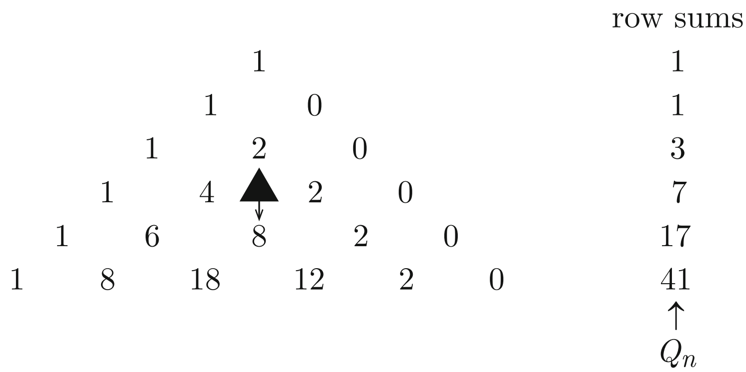

Figure 13.5.

Exercises 13

1. Define recursively the array A in Figure 13.3.
2. Let D_n denote the nth rising diagonal sum of array A. Prove that $D_n = Q_n$, where $n \geq 0$.
3. Let A_n denote the nth row sum of array A. Prove that $A_n = 2 \cdot 3^{n-1}$, where $n \geq 1$.
4. Define recursively the array B in Figure 13.4.
5. Let B_n denote the nth row sum of array B. Prove that $B_n = P_n$, where $n \geq 1$.
6. Define recursively the array C in Figure 13.4.
7. Let C_n denote the nth row sum of array C. Prove that $C_n = Q_n$, where $n \geq 0$.

14

Pell and Pell–Lucas Polynomials

14.1 Introduction

Pell numbers and Pell–Lucas numbers are specific values of *Pell polynomials* $p_n(x)$ and *Pell–Lucas polynomials* $q_n(x)$, respectively. Both families were studied extensively in 1985 by A.F. Horadam of the University of New England, Armidale, Australia, and Bro. J.M. Mahon of the Catholic College of Education, Sydney, Australia [108]. Both families are often defined recursively:

$$\begin{aligned} p_0(x) &= 0, \quad p_1(x) = 1 \\ p_n(x) &= 2xp_{n-1}(x) + p_{n-2}(x); \end{aligned} \qquad \begin{aligned} q_0(x) &= 2, \quad q_1(x) = 2x \\ q_n(x) &= 2xq_{n-1}(x) + q_{n-2}(x), \end{aligned}$$

where $n \geq 3$. The first ten Pell and Pell–Lucas polynomials are given in Table 14.1.

Table 14.1. Pell and Pell–Lucas Polynomials

n	$p_n(x)$	$q_n(x)$
1	1	$2x$
2	$2x$	$4x^2 + 2$
3	$4x^2 + 1$	$8x^3 + 6x$
4	$8x^3 + 4x$	$16x^4 + 16x^2 + 2$
5	$16x^4 + 12x^2 + 1$	$32x^5 + 40x^3 + 10x$
6	$32x^5 + 32x^3 + 6x$	$64x^6 + 96x^4 + 36x^2 + 2$
7	$64x^6 + 80x^4 + 24x^2 + 1$	$128x^7 + 224x^5 + 112x^3 + 14x$
8	$128x^7 + 192x^5 + 80x^3 + 8x$	$256x^8 + 512x^6 + 320x^4 + 64x^2 + 2$
9	$256x^8 + 448x^6 + 240x^4 + 40x^2 + 1$	$512x^9 + 1152x^7 + 864x^5 + 240x^3 + 18x$
10	$512x^9 + 1024x^7 + 672x^5 + 160x^3 + 10x$	$1024x^{10} + 2560x^8 + 2240x^6 + 800x^4 + 100x^2 + 2$

T. Koshy, *Pell and Pell–Lucas Numbers with Applications*, DOI 10.1007/978-1-4614-8489-9_14,

14.2 Special Cases

Fibonacci and Pell numbers are special values of Pell polynomials, and Lucas and Pell–Lucas numbers are special cases of Pell–Lucas polynomials: $p_n(1/2) = F_n, p_n(1) = P_n, q_n(1/2) = L_n$, and $q_n(1) = 2Q_n$.

More generally, the Fibonacci polynomials $f_n(x)$ and Lucas polynomials [126] $l_n(x)$ are special cases of $p_n(x)$ and $q_n(x)$, respectively. They were studied by the Belgian mathematician Eugene Charles Catalan (1814–1894) in 1883, and the German mathematician Ernst Jacobsthal (1882–1965). They were also studied extensively by M.N.S. Swamy of the University of Saskatchwan in Canada: $f_n(x) = p_n(x/2)$ and $l_n(x) = q_n(x/2)$. So $F_n = f_n(1) = p_n(1/2)$ and $L_n = l_n(1) = q_n(1/2)$. Since $f_n(-x) = (-1)^{n-1} f_n(x)$ and $l_n(-x) = (-1)^n l_n(x)$, it follows that $p_n(-x) = (-1)^{n-1} p_n(x)$ and $q_n(-x) = (-1)^n q_n(x)$. The Fibonacci and Lucas polynomials are often defined recursively:

$$\begin{aligned} f_1(x) &= 1, \quad f_2(x) = x; & l_1(x) &= x, \quad l_2(x) = x^2 + 2 \\ f_n(x) &= x f_{n-1}(x) + f_{n-2}(x), \quad n \geq 3; & l_n(x) &= x l_{n-1}(x) + l_{n-2}(x), \quad n \geq 3. \end{aligned}$$

They have their own Binet-like versions:

$$f_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} \quad \text{and} \quad l_n(x) = \alpha^n(x) + \beta^n(x),$$

where $\alpha(x) = \frac{x+\sqrt{x^2+4}}{2}$ and $\beta(x) = \frac{x-\sqrt{x^2+4}}{2}$ are the solutions of the equation $t^2 - xt - 1 = 0$. Clearly, $\alpha(1) = \alpha$ and $\beta(1) = \beta, \alpha(2) = \gamma$, and $\beta(2) = \delta$.

They can be defined explicitly as well:

$$f_n(x) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-j-1}{j} x^{n-2j-1} \quad \text{and} \quad l_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{n}{n-j} \binom{n-j}{j} x^{n-2j},$$

where $n \geq 1$. These can be confirmed using strong induction.

The first ten Fibonacci and Lucas polynomials are listed in Table 14.2.

Table 14.2. Fibonacci and Lucas Polynomials

n	$f_n(x)$	$l_n(x)$
1	1	x
2	x	$x^2 + 2$
3	$x^2 + 1$	$x^3 + 3x$
4	$x^3 + 2x$	$x^4 + 4x^2 + 2$
5	$x^4 + 3x^2 + 1$	$x^5 + 5x^3 + 5x$
6	$x^5 + 4x^3 + 3x$	$x^6 + 6x^4 + 9x^2 + 2$
7	$x^6 + 5x^4 + 6x^2 + 1$	$x^7 + 7x^5 + 14x^3 + 7x$
8	$x^7 + 6x^5 + 10x^3 + 4x$	$x^8 + 8x^6 + 20x^4 + 16x^2 + 2$
9	$x^8 + 7x^6 + 15x^4 + 10x^2 + 1$	$x^9 + 9x^7 + 27x^5 + 30x^3 + 9x$
10	$x^9 + 8x^7 + 21x^5 + 20x^3 + 5x$	$x^{10} + 10x^8 + 35x^6 + 50x^4 + 25x^2 + 2$

14.3 Gauthier's Formula

In 2012, Napoleon Gauthier of the Royal Military College, Kingston, Ontario, Canada, developed an interesting formula for P_n using the Fibonacci polynomial $f_n(z)$ of a complex variable z [91]. To see this, we let $\Delta = \Delta(z) = \sqrt{z^2+4}$. Clearly, $\Delta(2i) = 0, 2\alpha(z) = z + \Delta$, and $2\beta(z) = z - \Delta$, where $i = \sqrt{-1}$.

We will now evaluate $f_n(2i)$ using Binet's formula:

$$\begin{aligned} f_n(z) &= \frac{1}{2^n \Delta}[(z+\Delta)^n - (z-\Delta)^n] \\ &= \frac{1}{2^{n-1}} \sum_{r \text{ odd}} \binom{n}{r} z^{n-r} \Delta^{r-1} \\ f_n(2i) &= \frac{1}{2^{n-1}} \cdot n(2i)^{n-1} \cdot 1 + 0 \\ &= ni^{n-1}. \end{aligned}$$

Using the explicit formula for $f_n(z)$, we then have

$$\begin{aligned} f_n(2i) &= \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-j-1}{j} (2i)^{n-2j-1} \\ ni^{n-1} &= (2i)^{n-1} \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-j-1}{j} \frac{(-1)^j}{4^j} \\ \frac{n}{2^{n-1}} &= \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-j-1}{j} \frac{(-1)^j}{4^j} \end{aligned} \tag{14.1}$$

By formula (9.1), we have

$$\frac{1}{2^{n-1}} P_n = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-j-1}{j} \frac{1}{4^j}. \tag{14.2}$$

Adding formulas (14.1) and (14.2), we get

$$\begin{aligned} \frac{1}{2^{n-1}}(P_n + n) &= \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{4^j} \binom{n-j-1}{j} [1 + (-1)^j] \\ &= 2 \sum_{j \text{ even}} \binom{n-j-1}{j} \frac{1}{4^j}. \end{aligned}$$

This yields *Gauthier's formula*:

$$\frac{1}{2^n}(P_n + n) = \sum_{k=0}^{\lfloor (n-1)/4 \rfloor} \binom{n-2k-1}{2k} \frac{1}{16^k}. \tag{14.3}$$

Similarly, it follows from (14.1) and (14.2) that

$$\frac{1}{2^{n-2}}(P_n - n) = \sum_{k=0}^{\lfloor (n-3)/4 \rfloor} \binom{n-2k-2}{2k+1} \frac{1}{16^k}. \tag{14.4}$$

For example,

$$\begin{aligned} \frac{P_9 + 9}{512} &= \frac{497}{256} = \sum_{k=0}^{2} \binom{8-2k}{2k} \frac{1}{16^k} \\ \frac{P_8 - 8}{64} &= \frac{25}{4} = \sum_{k=0}^{1} \binom{6-2k}{2k+1} \frac{1}{16^k}. \end{aligned}$$

Using similar steps with Binet's formula and the explicit formula for $l_n(x)$, and formula (9.18) for Q_n, we can derive the corresponding formulas for Q_n:

$$\frac{1}{2^n}(Q_n + 1) = \sum_{k=0}^{\lfloor n/4 \rfloor} \frac{n}{n-2k} \binom{n-2k}{2k} \frac{1}{16^k} \tag{14.5}$$

$$\frac{1}{2^{n-2}}(Q_n - 1) = \sum_{k=0}^{\lfloor (n-2)/4 \rfloor} \frac{n}{n-2k-1} \binom{n-2k-1}{2k+1} \frac{1}{16^k}. \tag{14.6}$$

For example,

$$\begin{aligned} \frac{Q_9 + 1}{512} &= \frac{697}{256} = \sum_{k=0}^{2} \frac{9}{9-2k} \binom{9-2k}{2k} \frac{1}{16^k} \\ \frac{Q_6 - 1}{16} &= \frac{49}{8} = \sum_{k=0}^{1} \frac{6}{5-2k} \binom{5-2k}{2k+1} \frac{1}{16^k}. \end{aligned}$$

14.4 Binet-like Formulas

Using standard tools such as generating functions and solving recurrences, we can derive the Binet-like formulas for $p_n(x)$ and $q_n(x)$:

$$p_n(x) = \frac{\gamma^n - \delta^n}{\gamma - \delta} \quad \text{and} \quad q_n(x) = \gamma^n + \delta^n,$$

where $\gamma = \gamma(x) = x + \sqrt{x^2+1}$ and $\delta = \delta(x) = x - \sqrt{x^2+1}$ are the solutions of the quadratic equation $\lambda^2 = 2x\lambda + 1$. They can be confirmed using induction also. Notice that $\gamma + \delta = 2x, \gamma - \delta = 2\sqrt{x^2+1}$, and $\gamma\delta = -1$.

The next example features an interesting Pell polynomial identity, discovered by Seiffert in 2000 [213].

Example 14.1 Let x be any nonzero real number $\neq 1$ and n any positive integer. Then

$$\sum_{k=1}^{n} \binom{n}{k} (1-x)^{n-k} p_k(x) = x^{n-1} p_n(1/x). \tag{14.7}$$

Proof. Notice that $(1-x)+\gamma = 1+\sqrt{x^2+1}$ and $(1-x)+\delta = 1-\sqrt{x^2+1}$, where $\gamma = \gamma(x)$ and $\delta = \delta(x)$. By the Binet-like formula for $p_k(x)$ and the the binomial theorem, we have

$$\begin{aligned}
2\sqrt{x^2+1}\left[\sum_{k=1}^{n}\binom{n}{k}(1-x)^{n-k}p_k(x)\right] &= \sum_{k=1}^{n}\binom{n}{k}(1-x)^{n-k}(\gamma^k - \delta^k)\\
&= \sum_{k=1}^{n}\binom{n}{k}(1-x)^{n-k}\gamma^k - \sum_{k=1}^{n}\binom{n}{k}(1-x)^{n-k}\delta^k\\
&= [(1-x)+\gamma]^n - 1] - [(1-x)+\delta)^n - 1]\\
&= (1+\sqrt{x^2+1})^n - (1-\sqrt{x^2+1})^n\\
&= 2\sqrt{x^2+1}\cdot x^{n-1}p_n(1/x).
\end{aligned}$$

This yields the desired identity. ■

In particular, when $x = 2$, identity (14.7) yields an interesting Pell–Fibonacci identity:

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} p_k(2) = 2^{n-1} F_n. \tag{14.8}$$

For example, $\sum_{k=1}^{5} \binom{5}{k}(-1)^{5-k} p_k(2) = 5{\cdot}1 - 10{\cdot}4 + 10{\cdot}17 - 5{\cdot}72 + 1{\cdot}305 = 80 = 2^4{\cdot}5 = 2^4 F_5$, as expected.

A similar argument can be used to develop the following Pell–Lucas identity, where $q_0(x) = 2$:

$$\sum_{k=0}^{n} \binom{n}{k} (1-x)^{n-k} q_k(x) = x^n q_n(1/x). \tag{14.9}$$

In particular, this yields the following Pell–Lucas identity:

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}q_k(2) = 2^n L_n. \tag{14.10}$$

For instance, $\sum_{k=0}^{5}\binom{5}{k}(-1)^{5-k}q_k(2) = -1\cdot 2+5\cdot 4-10\cdot 18+10\cdot 76-5\cdot 322+1\cdot 1364 = 352 = 2^5\cdot 11 = 2^5 L_5$, as expected.

14.5 A Pell Divisibility Test

Next we study an interesting divisibility test for a small class of Pell numbers, developed by Seiffert in 2000 [211, 212]. To this end, we need the next two results.

Lemma 14.1 *Let q be a prime. Then $\binom{q-1}{r} \equiv (-1)^r \pmod{q}$, where $1 \le r < q$.*

Proof. It is well known that $q|\binom{q}{k}$, where $1 \le k \le q-1$. So, by Pascal's identity, $\binom{q-1}{r} \equiv -\binom{q-1}{r-1} \pmod{q}$. Since $\binom{q-1}{1} = q-1 \equiv -1 \equiv -\binom{q-1}{0} \pmod{q}$, the desired result follows by PMI. ∎

The following lemma gives an interesting formula for Q_n.

Lemma 14.2

$$Q_n = 2^{\lfloor -n/2\rfloor}\sum_{\substack{0\le k\le n\\ 4\nmid(n+2k+2)}}(-1)^{\lfloor (n-2k+1)/4\rfloor}\binom{2n}{2k}. \tag{14.11}$$

Proof. Let $z \ne 1$ be any complex number, $i = \sqrt{-1}$, and $x = 2i\cdot\frac{1+z}{1-z}$. Then

$$\begin{aligned} 2\alpha(x) &= 2i\cdot\frac{1+z}{1-z}+\frac{2}{1-z}\sqrt{(1-z)^2-(1+z)^2}\\ &= 2i\cdot\frac{1+z}{1-z}+\frac{2}{1-z}\sqrt{-4z} = \frac{2i(1+\sqrt{z})^2}{1-z}\\ \alpha(x) &= \frac{i(1+\sqrt{z})^2}{1-z}. \end{aligned}$$

Changing $\sqrt{z}$ to $-\sqrt{z}$, it follows that $\beta(x) = \frac{i(1-\sqrt{z})^2}{1-z}$.

By the Binet-like formula for $l_n(x)$ and the binomial theorem, we have

$$\begin{aligned} l_n\left(2i\cdot\frac{1+z}{1-z}\right) &= \frac{i^n}{(1-z)^n}\left[(1+\sqrt{z})^n+(1+\sqrt{z})^n\right]\\ &= \frac{2i^n}{(1-z)^n}\sum_{k=0}^{n}\binom{2n}{2k}z^k. \end{aligned}$$

Now let $z = -i$. Since $\frac{1-i}{1+i} = -i$, $\frac{1}{1+i} = \frac{1-i}{2}$, $-i = \frac{1}{i}$, and $l_n(2) = 2Q_n$, this formula implies that

$$Q_n = 2^{-n} \sum_{k=0}^{n} \binom{2n}{2k} i^{n-k} (1-i)^n.$$

But, by Euler's formula, $i = e^{i\pi/2}$; and by De Moivre's theorem[10], $1-i = \sqrt{2}e^{-i\pi/4}$. So this yields

$$\begin{aligned} Q_n &= 2^{-n} \sum_{k=0}^{n} \binom{2n}{2k} e^{i(n-k)\pi/2} \cdot 2^{n/2} e^{-in\pi/4} \\ &= 2^{-n/2} \sum_{k=0}^{n} \binom{2n}{2k} e^{i(n-2k)\pi/4}. \end{aligned}$$

Equating the real parts of both sides, we get

$$Q_n = 2^{-n/2} \sum_{k=0}^{n} \binom{2n}{2k} A_{n-2k},$$

where

$$\begin{aligned} A_j &= \cos(j\pi/4) \\ &= \begin{cases} (-1)^{\lfloor (j+1)/4 \rfloor} 2^{\lfloor j/2 \rfloor - j/2} & \text{if } j \not\equiv 2 \pmod 4 \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and j is an integer.

So

$$\begin{aligned} A_{n-2k} &= \begin{cases} (-1)^{\lfloor (n-2k+1)/4 \rfloor} 2^{\lfloor (n-2k)/2 \rfloor - (n-2k)/2} & \text{if } n+2k \not\equiv 2 \pmod 4 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} (-1)^{\lfloor (n-2k+1)/4 \rfloor} 2^{\lfloor n/2 \rfloor - n/2} & \text{if } n+2k \not\equiv 2 \pmod 4 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

---

10 $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, where $i = \sqrt{-1}$ and θ is any angle.

Thus

$$\begin{aligned} Q_n &= 2^{-n/2} \sum_{\substack{0\le k\le n \\ 4\nmid(n+2k+2)}} (-1)^{\lfloor (n-2k+1)/4\rfloor} \binom{2n}{2k} 2^{\lfloor n/2\rfloor - n/2} \\ &= 2^{\lfloor -n/2\rfloor} \sum_{\substack{0\le k\le n \\ 4\nmid(n+2k+2)}} (-1)^{\lfloor (n-2k+1)/4\rfloor} \binom{2n}{2k}, \end{aligned}$$

as desired. ∎

For example, we have

$$\begin{aligned} Q_3 &= 2^{\lfloor -3/2\rfloor} \sum_{\substack{0\le k\le 3 \\ 4\nmid(5+2k)}} (-1)^{\lfloor (4-2k)/4\rfloor} \binom{6}{2k} \\ &= 2^{-2}\left[(-1)^{-1}\binom{6}{0} + (-1)^0\binom{6}{2} + (-1)^0\binom{6}{4} + (-1)^{-1}\binom{6}{6}\right] \\ &= \frac{1}{4}(-1+15+15-1) = 7, \text{ as expected.} \end{aligned}$$

We are now ready to present the divisibility test.

Theorem 14.1 *(Seiffert, 2000) Let q be a prime such that $q \equiv 1 \pmod 8$. Then $q \mid P_{(q-1)/4}$ if and only if $2^{(q-1)/4} \equiv (-1)^{(q-1)/8} \pmod q$.*

Proof. Since $q \equiv 1 \pmod 8$, $q = 8j+1$ for some positive integer j. Letting $n = (q-1)/2$ in Lemma 14.2, we have

$$\begin{aligned} Q_{(q-1)/2} &= 2^{-2j} \sum_{\substack{0\le k\le 4j \\ 4\nmid(4j+2k+2)}} (-1)^{\lfloor j-(2k-1)/4\rfloor} \binom{q-1}{2k} \\ 2^{(q-1)/4} Q_{(q-1)/2} &\equiv \sum_{\substack{0\le k\le 4j \\ 4\nmid(4j+2k+2)}} (-1)^{\lfloor j-(2k-1)/4\rfloor} (-1)^{2k} \pmod q, \end{aligned}$$

by Lemma 14.1. But $4 \nmid (4j+2k+2)$ if and only if $4 \nmid 2(k+1)$; that is, if and ony if k is even. So this congruence can be rewritten as

$$2^{(q-1)/4} Q_{(q-1)/2} \equiv \sum_{\substack{0\le k\le 4j \\ k \text{ even}}} (-1)^{\lfloor j-(2k-1)/4\rfloor} \pmod q.$$

Let $k = 2r$. Then this becomes

$$\begin{aligned} 2^{(q-1)/4} Q_{(q-1)/2} &\equiv \sum_{r=0}^{2j} (-1)^{\lfloor j-r+\frac{1}{4} \rfloor} \pmod{q} \\ &\equiv \sum_{r=0}^{2j} (-1)^{j-r} \equiv (-1)^j \pmod{q}. \end{aligned}$$

That is,

$$2^{(q-1)/4} Q_{(q-1)/2} \equiv (-1)^{(q-1)/8} \pmod{q}. \tag{14.12}$$

But $4P_m^2 = Q_{2m} - (-1)^m$, by identity (33) in Chapter 7. Using $m = (q-1)/4$, this yields $Q_{(q-1)/2} = 4P_{(q-1)/4}^2 + (-1)^{(q-1)/4}$. Consequently, congruence (14.12) can be rewritten as

$$2^{(q+7)/4} P_{(q-1)/4}^2 \equiv (-1)^{(q-1)/8} - 2^{(q-1)/4} \pmod{q}.$$

Thus, $q | P_{(q-1)/4}$ if and only if $q | \left[(-1)^{(q-1)/8} - 2^{(q-1)/4} \right]$; that is, if and only if $2^{(q-1)/4} \equiv (-1)^{(q-1)/8} \pmod{q}$, as desired. ■

For example, let $q = 17$; so $q \equiv 1 \pmod{8}$. Then $P_{(q-1)/4} = P_4 = 12$, and $17 \nmid P_4$. Notice that $2^{(q-1)/4} = 2^4 = 16 \not\equiv 1 \equiv (-1)^2 \equiv (-1)^{(q-1)/8} \pmod{17}$.

On the other hand, let $q = 41 \equiv 1 \pmod{8}$. Then $2^{(q-1)/4} = 2^{10} = -1 \equiv (-1)^5 \equiv (-1)^{(q-1)/8} \pmod{41}$. Notice that $41 | P_{10}$, where $P_{10} = 2378$, as expected. The next such prime that works is $q = 113 = 8 \cdot 14 + 1$.

14.6 Generating Functions for $p_n(x)$ and $q_n(x)$

Using the standard techniques, we can find generating functions for the sequences $\{p_n(x)\}$ and $\{q_n(x)\}$:

$$\begin{aligned} \frac{y}{1 - 2xy - y^2} &= \sum_{n=0}^{\infty} p_n(x) y^n \\ \frac{2(1 - xy)}{1 - 2xy - y^2} &= \sum_{n=0}^{\infty} q_n(x) y^n. \end{aligned}$$

14.7 Elementary Properties of $p_n(x)$ and $q_n(x)$

Next we list a few elementary properties of $p_n(x)$ and $q_n(x)$. They can be established using the Binet-like formulas:

$$\begin{aligned}
p_{n+1}(x) + p_{n-1}(x) &= q_n(x) \\
2xp_n(x) + 2p_{n-1}(x) &= q_n(x) \\
q_{n+1}(x) + q_{n-1}(x) &= 4(x^2+1)p_n(x) && (14.13) \\
p_n(x)q_n(x) &= p_{2n}(x) && (14.14) \\
q_n^2(x) + 4(x^2+1)p_n^2(x) &= 2q_{2n}(x) \\
p_{n+1}(x)p_{n-1}(x) - p_n^2(x) &= (-1)^n && (14.15) \\
q_{n+1}(x)q_{n-1}(x) - q_n^2(x) &= 4(-1)^{n-1}(x^2+1) && (14.16) \\
p_{n+1}^2(x) - p_{n-1}^2(x) &= 2xp_{2n}(x) \\
4(x^2+1)p_n^2(x) - q_n^2(x) &= 4(-1)^{n-1}.
\end{aligned}$$

Formulas (14.15) and (14.16) are the Cassini-like formulas for Pell and Pell–Lucas polynomials, respectively.

14.8 Summation Formulas

We can use these fundamental properties to derive the following summation formulas:

$$\begin{aligned}
\sum_{i=1}^{n} p_{2i}(x) &= \frac{p_{2n+1}(x)-1}{2x} & \sum_{i=1}^{n} p_{2i-1}(x) &= \frac{p_{2n}(x)}{2x} \\
\sum_{i=1}^{n} p_i(x) &= \frac{p_{n+1}(x)+p_n(x)-1}{2x} & \sum_{i=1}^{n} q_{2i}(x) &= \frac{q_{2n+1}(x)-2x}{2x} \\
\sum_{i=1}^{n} q_{2i-1}(x) &= \frac{q_{2n}(x)-2}{2x} & \sum_{i=1}^{n} q_i(x) &= \frac{q_{n+1}(x)+q_n(x)-2x-2}{2x}.
\end{aligned}$$

For example, we have

$$\begin{aligned}
\sum_{i=1}^{3} p_{2i}(x) &= p_2(x) + p_4(x) + p_6(x) \\
&= 2x + (8x^3+4x) + (32x^5+32x^3+6x) = 32x^5+40x^3+12x \\
&= \frac{(64x^6+80x^4+24x+1)-1}{2x} = \frac{p_7(x)-1}{2x}; \\
\sum_{i=1}^{4} q_i(x) &= q_1(x) + q_2(x) + q_3(x) + q_4(x)
\end{aligned}$$

$$
\begin{aligned}
&= 2x + (4x^2 + 2) + (8x^3 + 6x) + (16x^4 + 16x^2 + 2) \\
&= 16x^4 + 8x^3 + 20x^2 + 8x + 4 \\
&= \frac{(32x^5 + 40x^3 + 10x) + (16x^4 + 16x^2 + 2) - 2x - 2}{2x} \\
&= \frac{q_5(x) + q_4(x) - 2x - 2}{2x}.
\end{aligned}
$$

14.9 Matrix Generators for $p_n(x)$ and $q_n(x)$

The matrix

$$P = \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}$$

can be used to generate Pell and Pell–Lucas polynomials, and we can use it to establish a number of properties of both families, just as Horadam and Mahon did [108].

Using induction, we can show that

$$P^n = \begin{bmatrix} p_{n+1}(x) & p_n(x) \\ p_n(x) & p_{n-1}(x) \end{bmatrix},$$

where $n \geq 1$. Consequently,

$$\begin{vmatrix} p_{n+1}(x) & p_n(x) \\ p_n(x) & p_{n-1}(x) \end{vmatrix} = |P^n| = |P|^n = (-1)^n.$$

This yields the *Cassini-like formula* for $p_n(x)$:

$$p_{n+1}(x)p_{n-1}(x) - p_n^2(x) = (-1)^n.$$

Since

$$\begin{bmatrix} p_{n+1}(x) \\ p_n(x) \end{bmatrix} = P^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

it follows that

$$p_n(x) = \begin{bmatrix} 1 & 0 \end{bmatrix} P^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Since

$$\begin{aligned} q_n(x) &= 2xp_n(x) + 2p_{n-1}(x) \\ &= \begin{bmatrix} p_n(x) \\ p_{n-1}(x) \end{bmatrix} \begin{bmatrix} 2x \\ 2 \end{bmatrix}, \end{aligned}$$

it follows that

$$\begin{aligned} \begin{bmatrix} q_{n+1}(x) \\ q_n(x) \end{bmatrix} &= \begin{bmatrix} p_{n+1}(x) & p_n(x) \\ p_n(x) & p_{n-1}(x) \end{bmatrix} \begin{bmatrix} 2x \\ 2 \end{bmatrix} \\ &= P^n \begin{bmatrix} 2x \\ 2 \end{bmatrix}. \end{aligned}$$

Consequently,

$$q_n(x) = \begin{bmatrix} 1 & 0 \end{bmatrix} P^{n-1} \begin{bmatrix} 2x \\ 2 \end{bmatrix}.$$

14.10 Addition Formulas

The matrix approach can be used effectively to derive an addition formula for $p_{m+n}(x)$:

$$p_{m+n}(x) = p_{m-1}(x)p_n(x) + p_m(x)p_{n+1}(x). \tag{14.17}$$

This can be achieved as follows:

$$\begin{aligned} \text{RHS} &= \begin{bmatrix} p_m(x) & p_{m-1}(x) \end{bmatrix} \begin{bmatrix} p_{n+1}(x) \\ p_n(x) \end{bmatrix} \\ &= \begin{bmatrix} p_m(x) & p_{m-1}(x) \end{bmatrix} P^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p_m(x) & p_{m-1}(x) \\ p_{m-1}(x) & p_{m-2}(x) \end{bmatrix} \cdot P^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} P^{m-1} \cdot P^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} P^{m+n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= p_{m+n}(x) = \text{LHS}. \end{aligned}$$

Similarly, it can be shown that

$$q_{m+n}(x) = p_{m-1}(x)q_n(x) + p_m(x)q_{n+1}(x). \tag{14.18}$$

It follows from identity (14.17) with $m = n + 1$ that

$$p_{n+1}^2(x) + p_n^2(x) = p_{2n+1}(x). \tag{14.19}$$

This can also be established using matrices:

$$\begin{aligned}
p_{n+1}^2(x) + p_n^2(x) &= \begin{bmatrix} p_{n+1}(x) & p_n(x) \end{bmatrix} \begin{bmatrix} p_{n+1}(x) \\ p_n(x) \end{bmatrix} \\
&= \begin{bmatrix} p_{n+1}(x) & p_n(x) \end{bmatrix} \cdot P^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} P^n \cdot P^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} P^{2n} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= p_{2n+1}(x), \text{ as claimed.}
\end{aligned}$$

Similarly,

$$q_{n+1}^2(x) + q_n^2(x) = 4(x^2 + 1)p_{2n+1}(x). \tag{14.20}$$

It follows from (14.19) and (14.20) that $P_n^2 + P_{n+1}^2 = P_{2n+1}$ and $Q_n^2 + Q_{n+1}^2 = 2P_{2n+1}$, as we learned in Chapter 7. [Recall that $q_k(1) = 2Q_k$.]

Table 14.3 lists some additional polynomial extensions of the identities we encountered in Chapter 7 and 8. They can be confirmed using the Binet-like formulas or matrices.

Next, we study a delightful Pell–Lucas congruence, developed by A. Dorp of Brooklyn, New York, in 2000 [73]. The featured solution is based on the one by Seiffert [214] and the identity

$$P_{(n+2)a+b} = 2Q_a P_{(n+1)a+b} + P_{na+b}. \tag{14.21}$$

This follows from the polynomial identity

$$p_{(n+2)a+b}(x) = p_{(n+1)a+b}(x) + (-1)^{a-1} p_{na+b}(x),$$

discovered by Horadam and Mahon in 1985, when a is odd [108].

Example 14.2 Find integers a, b, and m such that $L_n \equiv P_{na+b} \pmod{m}$, where n is an arbitrary integer.

Solution. Let a be an odd integer, and $m = (2Q_a - 1, P_b - 2, P_{a+b} - 1)$, where (x, y) denotes the greatest common divisor of x and y. (The reason for this choice of m would become clear

Table 14.3.

$$p_{m+n}(x)+p_{m-n}(x) = \begin{cases} p_m(x)q_n(x) & \text{if } n \text{ is even}\\ q_m(x)p_n(x) & \text{otherwise}\end{cases}$$

$$q_{m+n}(x)+q_{m-n}(x) = \begin{cases} q_m(x)q_n(x) & \text{if } n \text{ is even}\\ 4(x^2+1)p_m(x)p_n(x) & \text{otherwise}\end{cases}$$

$$p_{m+n}(x)-p_{m-n}(x) = \begin{cases} q_m(x)p_n(x) & \text{if } n \text{ is even}\\ p_m(x)q_n(x) & \text{otherwise}\end{cases}$$

$$q_{m+n}(x)-q_{m-n}(x) = \begin{cases} 4(x^2+1)p_m(x)p_n(x) & \text{if } n \text{ is even}\\ q_m(x)q_n(x) & \text{otherwise}\end{cases}$$

$$p^2_{m+n}(x)-p^2_{m-n}(x) = p_{2m}(x)p_{2n}(x)$$

$$q^2_{m+n}(x)-q^2_{m-n}(x) = 4(x^2+1)p_{2m}(x)p_{2n}(x)$$

$$p_{mn+r}(x) = \begin{cases} p_n(x)q_{(m-1)n+r}(x)+(-1)^n p_{(m-2)n+r}(x) & \text{if } n \text{ is even}\\ p_{(m-1)n+r}(x)q_n(x)+(-1)^{n-1} p_{(m-2)n+r}(x) & \text{otherwise}\end{cases}$$

$$q_{mn+r}(x) = q_{(m-1)n+r}(x)q_n(x)+(-1)^{n-1}q_{(m-2)n+r}(x)$$

$$p_{m+n}(x)p_{m-n}(x)-p^2_m(x) = (-1)^{m-n-1}p^2_n(x)$$

$$q_{m+n}(x)q_{m-n}(x)-q^2_m(x) = 4(-1)^{m-n}(x^2+1)p^2_n(x)$$

$$p_{m+n}(x)p_{m+k}(x)-p_m(x)p_{m+n+k}(x) = (-1)^m p_n(x)p_k(x)$$

$$q_{m+n}(x)q_{m+k}(x)-q_m(x)q_{m+n+k}(x) = 4(-1)^{m-1}(x^2+1)p_n(x)p_k(x)$$

$$p_{m+n}(x)q_{m+k}(x)-p_m(x)q_{m+n+k}(x) = (-1)^m p_n(x)q_k(x).$$

shortly.) Then $m|(2Q_a-1)$; so $2Q_a \equiv 1 \pmod{m}$. Consequently, by identity (14.21),

$$P_{(n+2)a+b} \equiv P_{(n+1)a+b}+P_{na+b} \pmod{m}. \tag{14.22}$$

Now let $A_n = L_n - P_{na+b}$. Then, by the Lucas recurrence and identity (14.22), we have

$$\begin{aligned} A_{n+2} &= L_{n+2}-P_{(n+2)a+b}\\ &\equiv (L_{n+1}+L_n)-\left[P_{(n+1)a+b}+P_{na+b}\right] \pmod{m}\\ &\equiv \left[L_{n+1}-P_{(n+1)a+b}\right]+(L_n-P_{na+b}) \pmod{m}\\ &\equiv A_{n+1}+A_n \pmod{m}. \end{aligned} \tag{14.23}$$

But $A_0 = L_0 - P_b = 2 - P_b \equiv 0 \pmod{m}$, since $m|(P_b - a)$. Likewise, $A_1 = 1 - P_{a+b} \equiv 0 \pmod{m}$.

Consequently, it follows by PMI and congruence (14.23) that $A_n \equiv 0 \pmod{m}$ for every integer n. Thus, $L_n \equiv P_{na+b} \pmod{m}$ for every integer n, as desired. ■

For example, let $a = 5$ and $b = 10$. Then $m = (2Q_5 - 1, P_{10} - 2, P_{15} - 1) = (81, 2376, 195024) = 3$; so $L_n \equiv P_{5n+10} \pmod{3}$ and $\{L_n \pmod{3}\}_{n\geq 0} = \{P_{5n+10} \pmod{3}\}_{n\geq 0} = \underbrace{2\,1\,0\,1\,1\,2\,0\,2}\ \underbrace{2\,1\,0\,1\,1\,2\,0\,2}\,2\,1\,0\,\ldots$.

Corresponding to the well-known De Moivre theorem in trigonometry, we have two Fibonacci–Lucas identities [102, 126]:

$$\left(\frac{L_m + \sqrt{5}F_n}{2}\right)^n = \frac{L_{mn} + \sqrt{5}F_{mn}}{2} \tag{14.24}$$

$$\left(\frac{L_m - \sqrt{5}F_n}{2}\right)^n = \frac{L_{mn} - \sqrt{5}F_{mn}}{2}. \tag{14.25}$$

Interestingly, these two identities are special cases of the following Pell polynomial identities, respectively:

$$\left[q_m(x) + 2\sqrt{x^2+1}\,p_m(x)\right]^n = 2^{n-1}\left[q_{mn}(x) + 2\sqrt{x^2+1}\,p_{mn}(x)\right] \tag{14.26}$$

$$\left[q_m(x) - 2\sqrt{x^2+1}\,p_m(x)\right]^n = 2^{n-1}\left[q_{mn}(x) - 2\sqrt{x^2+1}\,p_{mn}(x)\right]. \tag{14.27}$$

When $x = 1/2$, these two identities yield the Fibonacci–Lucas formulas (14.24) and (14.25), respectively.

Suppose we let $x = 1$ in identity (14.26). Then

$$\begin{aligned}
\left[q_m(1) + 2\sqrt{2}p_m(1)\right]^n &= 2^{n-1}\left[q_{mn}(1) + 2\sqrt{2}p_{mn}(1)\right]\\
(2Q_m + 2\sqrt{x^2+1}P_m)^n &= 2^{n-1}(2Q_m + 2\sqrt{2}P_m)^n\\
(Q_m + \sqrt{2}P_m)^n &= Q_{mn} + \sqrt{2}P_{mn}.
\end{aligned} \tag{14.28}$$

Likewise, identity (14.27) yields [or simply change $\sqrt{2}$ to $-\sqrt{2}$ in (14.28).]

$$(Q_m - \sqrt{2}P_m)^n = Q_{mn} - \sqrt{2}P_{mn}. \tag{14.29}$$

For example, let $m = 5$ and $n = 3$. We have $Q_5 = 41$, $P_5 = 29$, $Q_{15} = 275,807$, and $P_{15} = 195,025$. Then

$$\begin{aligned}
(Q_5 + \sqrt{2}P_5)^3 &= (41 + 29\sqrt{2})^3\\
&= 41^3 + 3\cdot 41^2\cdot\sqrt{2}\cdot 29 + 3\cdot 41\cdot 2\cdot 29^2 + 2\sqrt{2}\cdot 29^3\\
&= (41^3 + 3\cdot 41\cdot 2\cdot 29^2) + (3\cdot 41^2\cdot 29 + 2\cdot 29^3)\sqrt{2}\\
&= 275807 + 195025\sqrt{2} = Q_{15} + 2\sqrt{2}P_{15}, \text{ as expected.}
\end{aligned}$$

Consequently, $(Q_5 - \sqrt{2}P_5)^3 = 275807 - 195025\sqrt{2} = Q_{15} - \sqrt{2}P_{15}$. (You may verify this independently.)

14.11 Explicit Formulas for $p_n(x)$ and $q_n(x)$

Using strong induction, we can establish the following explicit formulas for $p_n(x)$ and $q_n(x)$:

$$p_n(x) = \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-r-1}{r} (2x)^{n-2r-1} \tag{14.30}$$

$$q_n(x) = \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{n}{n-r} \binom{n-r}{r} (2x)^{n-2r}. \tag{14.31}$$

For example, formula (14.30) is trivially true when $n = 1$ and when $n = 2$. Assume it is true for all positive integers $\le k$, where $k \ge 3$. Then

$$\begin{aligned} p_{k+1}(x) &= 2x p_k(x) + p_{k-1}(x) \\ &= \sum_{r=0}^{\lfloor (k-1)/2 \rfloor} \binom{k-r-1}{r} (2x)^{k-2r} + \sum_{r=0}^{\lfloor (k-2)/2 \rfloor} \binom{k-r-2}{r} (2x)^{k-2r-2}. \end{aligned}$$

Suppose k is even, say, $k = 2t$. Then, using Pascal's identity, we have

$$\begin{aligned} p_{k+1}(x) &= \sum_{r=0}^{t} \binom{2t-r-1}{r} (2x)^{2t-2r} + \sum_{r=0}^{t-1} \binom{2t-r-2}{r} (2x)^{2t-2r-2} \\ &= \sum_{r=0}^{t} \binom{2t-r-1}{r} (2x)^{2t-2r} + \sum_{r=1}^{t} \binom{2t-r-1}{r-1} (2x)^{2t-2r} \\ &= \binom{2t-1}{0} (2x)^{2t} + \sum_{r=1}^{t} \left[\binom{2t-r-1}{r} + \binom{2t-r-1}{r-1} \right] (2x)^{2t-2r} \\ &= \binom{2t}{0} (2x)^{2t} + \sum_{r=1}^{t} \binom{2t-r}{r} (2x)^{2t-2r} \\ &= \sum_{r=0}^{t} \binom{2t-r}{r} (2x)^{2t-2r} \\ &= \sum_{r=0}^{\lfloor k/2 \rfloor} \binom{k-r}{r} (2x)^{k-2r}. \end{aligned}$$

Thus formula (14.30) holds by strong induction when k is even. Similarly, it is also true when k is odd. Thus, it is true for every integer $n \geq 1$.

We omit the proof of (14.31) in the interest of brevity; see Exercise 32.

For example, we have

$$\begin{aligned} p_6(x) &= \sum_{r=0}^{2} \binom{5-r}{r} (2x)^{5-2r} = 32x^5 + 32x^3 + 6x \\ q_5(x) &= \sum_{r=0}^{2} \frac{5}{5-r} \binom{5-r}{r} (2x)^{5-2r} = 32x^5 + 40x^3 + 10x. \end{aligned}$$

As an application of formula (14.31), we now express $L_{k(2n+1)}$ as a polynomial in L_{2n+1}, where $k \geq 0$. The key in this endeavor is to show that $q_k(L_{2n+1}/2) = L_k(2n+1)$. We will establish this using PMI.

Since $q_0(L_{2n+1}/2) = 2 = L_0(2n+1)$ and $q_1(L_{2n+1}/2) = 2(L_{2n+1}/2) = L_{2n+1}$, the result is true when $k = 0$ and $k = 1$.

Now assume that it works for all nonnegative integers $\leq k$. Then

$$\begin{aligned} q_{k+1}(L_{2n+1}/2) &= 2(L_{2n+1}/2)q_k(L_{2n+1}/2) + q_{k-1}(L_{2n+1}/2) \\ &= L_{2n+1}L_{k(2n+1)} + L_{(k-1)(2n+1)}. \end{aligned}$$

Using the fact [126] that $L_{a+b} - L_{a-b} = L_a L_b$ when b is odd, this implies that $q_{k+1}(L_{2n+1}/2) = L_{(k+1)(2n+1)}$. Thus the formula works for every $k \geq 0$.

By (14.31), we now have

$$q_k(L_{2n+1}/2) = \sum_{r=0}^{\lfloor k/2 \rfloor} \frac{k}{k-r} \binom{k-r}{r} L_{2n+1}^{k-2r}.$$

That is,

$$L_{k(2n+1)/2} = \sum_{r=0}^{\lfloor k/2 \rfloor} \frac{k}{k-r} \binom{k-r}{r} L_{2n+1}^{k-2r}, \tag{14.32}$$

as discovered by Piero Filipponi in 1989.

In particular, we have

$$\begin{aligned} L_{2(2n+1)} &= L_{2n+1}^2 + 2 & L_{3(2n+1)} &= L_{2n+1}^3 + 3L_{2n+1} \\ L_{4(2n+1)} &= L_{2n+1}^4 + 4L_{2n+1}^2 + 2 & L_{5(2n+1)} &= L_{2n+1}^5 + 5L_{2n+1}^3 + 5L_{2n+1}. \end{aligned}$$

14.12 Pell Polynomials from Rising Diagonals

Interestingly, Pell polynomials can be generated from the rising diagonals of a triangular array. To see this, consider the array in Table 14.4, where row n represents the various terms in the expansion of $(2x+1)^n$:

$$(2x+1)^n = \sum_{r=0}^{n} \binom{n}{r} (2x)^{n-r}.$$

Table 14.4.

$n \backslash r$	0	1	2	3	4	5
0	1					
1	$2x$	1				
2	$4x^2$	$4x$	1			
3	$8x^3$	$12x^2$	$6x$	1		
4	$16x^4$	$32x^3$	$24x^2$	$8x$	1	
5	$32x^5$	$80x^4$	$80x^3$	$40x^2$	$10x$	1

We will now show that the rising diagonal sum $d_n(x)$ on the northeast diagonal n is the Pell polynomial $p_n(x)$.

To this end, we have $(2x+1)^n = \sum_{r=0}^{n} \binom{n}{r}(2x)^{n-r} = \sum_{r=0}^{n} a(n,r)x^{n-r}$, where $a(n,r) = \binom{n}{r}2^{n-r}$. So $d_n(x) = \sum_{r=0}^{\lfloor n/2 \rfloor} a(n-r,r)x^{n-2r} = \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n-r}{r}(2x)^{n-2r} = p_n(x)$, as desired.

For example, $d_5(x) = \sum_{r=0}^{2} \binom{5-r}{r}(2x)^{5-2r} = \binom{5}{0}(2x)^5 + \binom{4}{1}(2x)^3 + \binom{3}{2}(2x) = 32x^5 + 32x^3 + 6x = p_5(x)$.

14.13 Pell–Lucas Polynomials from Rising Diagonals

Next, we will show that Pell–Lucas polynomials also can be generated from the rising diagonals of another triangular array. To this end, consider the array in Table 14.5, where row n represents the various terms in the expansion of $(2x+1)^n(2x+2)$, where $n \geq 0$.

Table 14.5.

$n \backslash r$	0	1	2	3	4	5	6
0	$2x$	2					
1	$4x^2$	$6x$	2				
2	$8x^3$	$16x^2$	$10x$	2			
3	$16x^4$	$40x^3$	$36x^2$	$14x$	2		
4	$32x^5$	$96x^4$	$112x^3$	$64x^2$	$18x$	2	
5	$64x^6$	$224x^5$	$320x^4$	$240x^3$	$100x^2$	$22x$	2

Notice that

$$\begin{aligned}(2x+1)^n(2x+2) &= (2x+1)^{n+1}+(2x+1)^n\\ &= \sum_{r=0}^{n+1}\binom{n+1}{r}(2x)^{n+1-r}+\sum_{r=0}^{n}\binom{n}{r}(2x)^{n-r}\\ &= \sum_{r=0}^{n+1}\binom{n+1}{r}(2x)^{n-r+1}+\sum_{r=1}^{n+1}\binom{n}{r-1}(2x)^{n-r+1}\\ &= \sum_{r=0}^{n+1} b(n,r)x^{n-r+1},\end{aligned}$$

where

$$\begin{aligned}b(n,r) &= \left[\binom{n+1}{r}+\binom{n}{r-1}\right]2^{n-r+1}\\ &= \frac{(n+1)!}{r!(n+1-r)!}\left(1+\frac{r}{n+1}\right)2^{n-r+1}\\ &= \frac{n+r+1}{n+1}\binom{n+1}{r}2^{n-r+1}. \end{aligned} \tag{14.33}$$

Thus

$$(2x+1)^n(2x+2)=\sum_{r=0}^{n+1}\frac{n+r+1}{n+1}\binom{n+1}{r}(2x)^{n-r+1}.$$

For instance, $(2x+1)^3(2x+2)=\sum_{r=0}^{4}\frac{4+r}{4}\binom{4}{r}(2x)^{4-r}=16x^4+40x^3+36x^2+14x+2.$

Let S_n denote the nth northeast diagonal sum of the array in Table 14.5. Then, using formula (14.33), we have

$$\begin{aligned} S_n(x) &= \sum_{r=0}^{\lfloor (n+1)/2 \rfloor} b(n-r,r)x^{n-2r+1} \\ &= \sum_{r=0}^{\lfloor (n+1)/2 \rfloor} \frac{n+1}{n-r+1}\binom{n-r+1}{r}(2x)^{n-2r+1} \\ &= q_n(x), \end{aligned}$$

by formula (14.31). Thus every diagonal sum is a Pell–Lucas polynomial.

Next, we arrange the coefficients in the Pell polynomials $p_n(x)$ in increasing order of powers j of x, where $n \geq 1$ and $0 \leq j \leq n-1$. This results in the array in Table 14.6. Notice that

$$p_n(x) = \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-r-1}{r}(2x)^{n-2r-1}. \tag{14.34}$$

The change in the formula was necessitated by the relabeling of the rows in Table 14.4.

Table 14.6.

$n \backslash j$	0	1	2	3	4	5	6	7	8	9
1	1									
2	0	2								
3	1	0	4							
4	0	4	0	8						
5	1	0	12	0	16					
6	0	6	0	32	0	32				
7	1	0	24	0	80	0	64			
8	0	8	0	80	0	192	0	128		
9	1	0	40	0	240	0	448	0	256	
10	0	10	0	160	0	672	0	1024	0	512

Let $p(n, j)$ denote the entry in row n and column j. It satisfies several interesting properties:

$$p(2n-1, 0) = 1 \tag{14.35}$$

$$p(2n, 2j) = 0 \tag{14.36}$$

$$p(2n-1, 2j-1) = 0 \tag{14.37}$$

$$p(2n, 2j-1) = \binom{n+j-1}{n-j}2^{2j-1}, \quad 1 \leq j \leq n-1 \tag{14.38}$$

$$p(2n-1, 2j) = \binom{n+j-1}{n-j-1}2^{2j}, \quad 0 \leq j \leq n-1. \tag{14.39}$$

These properties can easily be established using formula (14.34).

It follows from properties (14.38) and (14.39) that $p(2n, 2n-1) = 2^{2n-1}$ and $p(2n-1, 2n-2) = 2^{2n-2}$; thus $p(n, n-1) = 2^{n-1}$, where $n \geq 1$. Consequently, the farthest southeast diagonal in Table 14.6 consists of the various powers of 2.

14.14 Summation Formulas

Using properties (14.35) through (14.39), we can derive the following summation formulas:

$$\sum_{j=0}^{n-1} p(2n-j-1, j) = 3^{n-1} \tag{14.40}$$

$$\sum_{i=1}^{2n} p(i, 2j-1) = \frac{1}{2} p(2n+1, 2j) \tag{14.41}$$

$$\sum_{i=1}^{2n} p(i, 2j) = \frac{1}{2} p(2n, 2j+1) \tag{14.42}$$

$$\sum_{i=1}^{2n-1} p(i, 2j-1) = \frac{1}{2} p(2n-1, 2j) \tag{14.43}$$

$$\sum_{i=1}^{2n-1} p(i, 2j) = \frac{1}{2} p(2n, 2j+1). \tag{14.44}$$

Table 14.7.

$n \backslash j$	0	1	2	3	4	5	6	7	8	9	10
1	0	2									
2	2	0	4								
3	0	6	0	8							
4	2	0	16	0	16						
5	0	10	0	40	0	32					
6	2	0	36	0	96	0	64				
7	0	14	0	112	0	224	0	128			
8	2	0	18	0	320	0	512	0	256		
9	0	18	0	148	0	864	0	1152	0	512	
10	2	0	54	0	616	0	2240	0	2560	0	1024

The coefficients of the Pell–Lucas polynomials $q_n(x)$ can also be arranged in a triangular array, as in Table 14.7, where the coefficients $q(n, j)$ of x^j are arranged in increasing powers j.

The array can be defined by the recurrence

$$q(n, j) = 2q(n-1, j-1) + q(n-2, j). \tag{14.45}$$

It follows from the property $q_n(x) = p_{n+1}(x) + p_{n-1}(x) = 2xp_n(x) + 2p_{n-1}(x)$ that

$$q(n, j) = p(n+1, j) + p(n-1, j) = 2p(n, j-1) + 2p(n-1, j). \tag{14.46}$$

For example, $p(7,4) + p(5,4) = 80 + 16 = 96 = q(6,4)$ and $2p(7,2) + 2p(6,3) = 2 \cdot 24 + 2 \cdot 32 = 112 = q(7,3)$.

It follows from the property $q_{n+1}(x) + q_{n-1}(x) = 4(x^2+1)p_n(x)$ that

$$q(n+1, j) + q(n-1, j) = 4p(n, j) + 4p(n, j-2). \tag{14.47}$$

For example, $q(5,3) + q(3,3) = 40 + 8 = 48 = 4 \cdot 8 + 4 \cdot 4 = 4p(4,3) + 4p(4,1)$.

Since $q(1,1) = 2$, it follows by induction and the recurrence (14.45) that $q(n,n) = 2^n$ for every $n \geq 1$. It also follows from property (14.46), since $p(n,n) = 2^{n-1}$ and $p(n-1,n) = 0$.

Using Pascal's identity, and properties (14.38), (14.39), and (14.46), we have

$$\begin{aligned} q(2n, 2j) &= 2p(2n, 2j-1) + 2p(2n-1, 2j) \\ &= \binom{n+j-1}{n-j}2^{2j} + 2 \cdot \binom{n+j-1}{n-j-1}2^{2j} \\ &= \left\{\left[\binom{n+j-1}{n-j} + \binom{n+j-1}{n-j-1}\right] + \binom{n+j-1}{n-j-1}\right\} 2^{2j} \\ &= \left[\binom{n+j}{n-j} + \binom{n+j-1}{n-j-1}\right]2^{2j}. \end{aligned} \tag{14.48}$$

For example,

$$\begin{aligned} q(8,6) &= \left[\binom{4+3}{4-3} + \binom{4+3-1}{4-3-1}\right]2^6 \\ &= 64\left[\binom{7}{1} + \binom{6}{0}\right] = 64(7+1) = 512. \end{aligned}$$

See Table 14.7.

Notice that $q(2n, 2j) = b(n+j-1, n-j)$.

Using Pascal's identity, and formulas (14.38), (14.39), and (14.46), we also have

$$q(2n-1, 2j+1) = 2p(2n-1, 2j) + 2p(2n-2, 2j+1)$$

$$\begin{aligned} &= 2\binom{n+j-1}{n-j-1}\cdot 2^{2j} + 2\binom{n+j-1}{n-j-2}\cdot 2^{2j+1} \\ &= \left[\binom{n+j}{n-j-1} + \binom{n+j-1}{n-j-2}\right] 2^{2j+1}. \end{aligned} \tag{14.49}$$

For example,

$$\begin{aligned} q(7,5) &= \left[\binom{4+2}{4-2-1} + \binom{4+2-1}{4-2-1}\right]\cdot 2^5 \\ &= \left[\binom{6}{1} + \binom{5}{0}\right]\cdot 2^5 = 7\cdot 32 = 224. \end{aligned}$$

See Table 14.7.

Notice that $q(2n-1, 2j+1) = b(n+j-1, n-j-1)$.

Next we investigate Pythagorean triples with Pell generators.

14.15 Pell Polynomials and Pythagorean Triples

Recall that $p_n(x)$ satisfies the Cassini-like formula $p_{n+1}(x)p_{n-1}(x) - p_n^2(x) = (-1)^n$; so $(p_n(x), p_{n+1}(x)) = 1$. In addition, $p_{n+1}^2(x) + p_n^2(x) = p_{2n+1}(x)$. We will need both properties shortly.

Let $g_n(x)$ be $p_n(x)$ or $q_n(x)$. It follows by basic algebra that

$$[g_{n+1}^2(x) - g_n^2(x)]^2 + [2g_{n+1}g_n(x)]^2 = [g_{n+1}^2(x) + g_n^2(x)]^2. \tag{14.50}$$

This implies that $[p_{n+1}^2(x) - p_n^2(x)]^2 + [2p_{n+1}p_n(x)]^2 = p_{2n+1}^2(x)$. Consequently a-b-c is a generalized Pythagorean triple, generated by $p_{n+1}(x)$ and $p_n(x)$, where $a = p_{n+1}^2(x) - p_n^2(x), b = 2p_{n+1}p_n(x)$, and $c = p_{2n+1}$.

Since $q_{n+1}^2(x) + q_n^2(x) = (x^2+1)[p_{n+1}^2(x) + p_n^2(x)] = (x^2+1)p_{2n+1}(x)$, if follows from (14.50) that the Pell–Lucas polynomial family $\{q_n(x)\}$ satisfies the Pythagorean identity

$$[q_{n+1}^2(x) - q_n^2(x)]^2 + [2q_{n+1}(x)q_n(x)]^2 = (x^2+1)^2 p_{2n+1}^2(x). \tag{14.51}$$

This generates another family of Pythagorean triples, generated by $q_{n+1}(x)$ and $q_n(x)$, where $(q_{n+1}(x), q_n(x)) = 1$.

14.16 Pythagorean Triples with Pell Generators

Since $P_{n+1}^2 - P_n^2 = (P_{n+1} + P_n)(P_{n+1} - P_n) = Q_{n+1}Q_n$ and $P_{n+1}^2 + P_n^2 = P_{2n+1}$, it follows from identity (14.51) that

$$(Q_n Q_{n+1})^2 + (2P_n P_{n+1})^2 = P_{2n+1}^2. \tag{14.52}$$

For example, $(239 \cdot 577)^2 + (2 \cdot 169 \cdot 408)^2 = 195,025^2$.

Obviously, $P_{n+1} > P_n$ and they have different parity. Furthermore, it follows by the Cassini-like formula $P_{n-1}P_{n+1} - P_n^2 = (-1)^n$ that $(P_n, P_{n+1}) = 1$. Consequently, a-b-c = $Q_n Q_{n+1}$-$2P_n P_{n+1}$-P_{2n+1} is a primitive Pythagorean triple, generated by P_{n+1} and P_n.

Table 14.8 lists the first ten such triplets.

Table 14.8.

n	1	2	3	4	5	6	7	8	9	10
a	3	21	119	697	4059	23661	137903	803761	4684659	27304197
b	4	20	120	696	4060	23660	137904	803760	4684660	27304196
c	5	29	169	985	5741	33461	195025	1136689	6625109	38613965

Since every Q_i is odd, and P_i and P_{i+1} have opposite parity, it follows that $Q_n Q_{n+1}$ is odd and $P_n P_{n+1}$ is even. Thus the triplet a-b-c has the property that a is odd and $4|b$. Consequently, c is odd.

The legs of these Pythagorean triangles manifest an interesting pattern: $a = Q_n Q_{n+1}$ and $b = 2P_n P_{n+1}$ are consecutive integers. This can be confirmed as follows:

$$\begin{aligned} 4(Q_n Q_{n+1} - 2P_n P_{n+1}) &= (\gamma^n + \delta^n)\left(\gamma^{n+1} + \delta^{n+1}\right) - (\gamma^n - \delta^n)\left(\gamma^{n+1} - \delta^{n+1}\right) \\ &= 2(\gamma + \delta)(-1)^n = 4(-1)^n \\ Q_n Q_{n+1} - 2P_n P_{n+1} &= (-1)^n. \end{aligned} \tag{14.53}$$

Thus, $Q_n Q_{n+1}$ and $2P_n P_{n+1}$ are consecutive integers; thus $(Q_n Q_{n+1}, 2P_n P_{n+1}) = 1$,

Consequently, the area of the Pythagorean triangle, given by $\frac{1}{2}ab = \frac{1}{2}(Q_n Q_{n+1})(2P_n P_{n+1})$ $= P_n P_{n+1} Q_n Q_{n+1}$, is a triangular number. Since area = $(P_n Q_n)(P_{n+1} Q_{n+1}) = \frac{1}{4} P_{2n} P_{2n+2}$, $2|P_{2k}$, and $4 \not| P_{2k}$ implies $4|P_{2k+2}$; so the area has even parity.

The Pythagorean triangle satisfy several other intriguing properties:

(1) Since the product of the lengths of the legs of a Pythagorean triangle is divisible by 12, it follows that $P_n Q_n P_{n+1} Q_{n+1} \equiv 0 \pmod 6$.

For example, $P_5 P_6 Q_5 Q_6 = 29 \cdot 41 \cdot 70 \cdot 99 \equiv 0 \pmod 6$.

(2) The product of the lengths of the sides of a Pythagorean triangle is divisible by 60; so $(Q_n Q_{n+1})(2P_n P_{n+1})\,(2P_{2n+1}) = P_{2n} P_{2n+1} P_{2n+2} \equiv 0 \pmod{60}$.

For example, $P_{10} P_{11} P_{12} = 2378 \cdot 5741 \cdot 13860 \equiv 0 \pmod{60}$.

(3) The area of a Pythagorean triangle cannot be an integral square. So the triangular number number $(P_n Q_n)(P_{n+1} Q_{n+1}) = \frac{1}{4} P_{2n} P_{2n+2}$ is not a square. Consequently, $P_{2n} P_{2n+2}$ is not a square.

For example, $P_5 Q_5 P_6 Q_6 = 29 \cdot 41 \cdot 70 \cdot 99 = 8,239,770$ is not a square.

(4) The difference $c - a$ is twice the square of a Pell number: $P_{2n+1} - Q_n Q_{n+1} = 2P_n^2$.

(5) The difference $c - b$ is the square of a Pell-Lucas number: $P_{2n+1} - 2P_n P_{n+1} = Q_n^2$.

(6) The difference $a - b$ is ± 1: $Q_n Q_{n+1} - 2P_n P_{n+1} = Q_n^2 - 2P_n^2 = (-1)^n$.
(7) The sum $a + b$ is a Pell-Lucas number: $2P_n P_{n+1} + Q_n Q_{n+1} = Q_{2n+1}$.
(8) The sum $b + c$ is the square of a Pell-Lucas number: $2P_n P_{n+1} + P_{2n+1} = Q_{n+1}^2$.
(9) The sum $c + a$ is twice the square of a Pell number: $Q_n Q_{n+1} + P_{2n+1} = 2P_{n+1}^2$.
(10) Perimeter $= 2P_{n+1}^2 + 2P_{n+1}P_n = 2P_{n+1}(P_{n+1} + P_n) = 2P_{n+1}Q_{n+1} = P_{2n+2}$.
(11) Let r denote the inradius of the triangle. Since area $= \frac{1}{2}(a + b + c)r$, it follows that $r = \frac{2(\text{area})}{a+b+c} = \frac{1}{2}P_{2n}$.
(12) The circumradius R of the triangle is given by $R = \frac{c}{2} = \frac{1}{2}P_{2n+1}$.

Since $Q_n Q_{n+1} = P_{n+1}^2 - P_n^2, 2P_n P_{n+1} = \frac{Q_{n+1}^2 - Q_n^2}{2}$, and $P_n^2 + P_{n+1}^2 = P_{2n+1}$, identity (14.52) can be rewritten as

$$4\left(P_{n+1}^2 - P_n^2\right)^2 + \left(Q_{n+1}^2 - Q_n^2\right)^2 = 4\left(P_{n+1}^2 + P_n^2\right)^2. \tag{14.54}$$

For example, $4\left(29^2 - 12^2\right)^2 + \left(41^2 - 17^2\right)^2 = 4\left(29^2 + 12^2\right)^2$; that is, $\left[2\left(29^2 - 12^2\right)\right]^2 + \left(41^2 - 17^2\right)^2 = \left[2\left(29^2 + 12^2\right)\right]^2$.

Suppose we choose $u = Q_{n+1}^2 - Q_n^2, v = 2Q_{n+1}Q_n$, and $w = Q_{n+1}^2 + Q_n^2$. Since $u = 2b, v = 2a$, and $w = 2c$, it follows that $(u, v, w) = (2b, 2a, 2c) = 2(b, a, c) = 2 \cdot 1 = 2$. Consequently, each triple u-v-w equals twice the corresponding triple b-a-c. For example, 8120-8118-11482 = 2(4060-4059-5741). Thus, although u-v-w is a Pythagorean triple, it is not primitive.

Since $Q_{n+1}^2 - Q_n^2 = Q_{2n+1} - (-1)^n$ and $Q_{n+1}^2 + Q_n^2 = 2P_{2n+1}$, the identity

$$\left(Q_{n+1}^2 - Q_n^2\right)^2 + (2Q_{n+1}Q_n)^2 = \left(Q_{n+1}^2 + Q_n^2\right)^2 \tag{14.55}$$

can be rewritten as

$$[Q_{2n+1} - (-1)^n]^2 + (2Q_{n+1}Q_n)^2 = 4P_{2n+1}^2. \tag{14.56}$$

For example, $(8119 + 1)^2 + (2 \cdot 99 \cdot 41)^2 = 4 \cdot 5741^2$.

Exercises 14

Establish each formula.

1. Formula (14.5).
2. Formula (14.8).
3. Formula (14.10).

Verify each for $n = 10$ and 11.

4. Formula (14.5).
5. Formula (14.8).
6. Formula (14.10).
7. Derive a generating function for $\{p_n(x)\}$.
8. Derive a generating function for $\{q_n(x)\}$.

Establish the following properties of $p_n(x)$ and $q_n(x)$. *Hint*: Use the Binet-like formulas.

9. $p_{n+1}(x) + p_{n-1}(x) = q_n(x)$.
10. $2xp_n(x) + 2p_{n-1}(x) = q_n(x)$.
11. $q_{n+1}(x) + q_{n-1}(x) = 4(x^2 + 1)p_n(x)$.
12. $p_n(x)q_n(x) = p_{2n}(x)$.
13. $q_n^2(x) + 4(x^2 + 1)p_n^2(x) = 2q_{2n}(x)$.
14. $p_{n+1}(x)p_{n-1}(x) - p_n^2(x) = (-1)^n$.
15. $q_{n+1}(x)q_{n-1}(x) - q_n^2(x) = 4(-1)^{n-1}(x^2 + 1)$.
16. $p_{n+1}^2(x) - p_{n-1}^2(x) = 2xp_{2n}(x)$.
17. $4(x^2 + 1)p_n^2(x) - q_n^2(x) = 4(-1)^{n-1}$.

Derive the following summation formulas.

18. $\sum_{i=1}^{n} p_{2i}(x) = \frac{p_{2n+1}(x) - 1}{2x}$.
19. $\sum_{i=1}^{n} p_{2i-1}(x) = \frac{p_{2n}(x)}{2x}$.
20. $\sum_{i=1}^{n} p_i(x) = \frac{p_{n+1}(x) + p_n(x) - 1}{2x}$.
21. $\sum_{i=1}^{n} q_{2i}(x) = \frac{q_{2n+1}(x) - 2x}{2x}$.
22. $\sum_{i=1}^{n} q_{2i-1}(x) = \frac{q_{2n}(x) - 2}{2x}$.
23. $\sum_{i=1}^{n} q_i(x) = \frac{q_{n+1}(x) + q_n(x) - 2x - 2}{2x}$.

Deduce a formula for each sum using the formulas in Exercises 18–23.

24. $\sum_{i=1}^{n} P_{2i}$.
25. $\sum_{i=1}^{n} P_{2i-1}$.
26. $\sum_{i=1}^{n} P_i$.

27. $\sum_{i=1}^{n} Q_{2i}$.

28. $\sum_{i=1}^{n} Q_{2i-1}$.

29. $\sum_{i=1}^{n} Q_{i}$.

Prove the following polynomial identities.

30. $q_{m+n}(x) = p_{m-1}(x)q_n(x) + p_m(x)q_{n+1}(x)$.

31. $q_{n+1}^2(x) + q_n^2(x) = 4(x^2+1)p_{2n+1}(x)$.

32. $q_n(x) = \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{n}{n-r}\binom{n-r}{r}(2x)^{n-2r}$. *Hint*: Use Exercise 10.

33–37. Identities (14.35) through (14.39).

38–42. Establish the summation formulas (14.40) through (14.44).

15

Pellonometry

15.1 Introduction

A number of properties bridge the Pell family with trigonometry. To study them, we will frequently need to rely on an important formula in trigonometry, *Euler's formula*: $e^{ix} = \cos x + i \sin x$, where x is any real number and $i = \sqrt{-1}$.

To establish the first link, we will need the following result.

Example 15.1 Prove that $2P_n P_{n+1}\left[Q_{2n+1} + (-1)^n\right] = Q_n Q_{n+1}\left[Q_{2n+1} - (-1)^n\right]$.

Proof. Using the Binet-like formulas, we have

$$\begin{aligned}
8(\text{LHS}) &= (\gamma^n - \delta^n)(\gamma^{n+1} - \delta^{n+1})\left[(\gamma^{2n+1} + \delta^{2n+1} + 2(-1)^n\right] \\
&= \left[\gamma^{2n+1} + \delta^{2n+1} - 2(-1)^n\right]\left[(\gamma^{2n+1} + \delta^{2n+1} + 2(-1)^n\right] \\
&= \left(\gamma^{2n+1} + \delta^{2n+1}\right)^2 - 4 \\
8(\text{RHS}) &= (\gamma^n + \delta^n)(\gamma^{n+1} + \delta^{n+1})\left[(\gamma^{2n+1} + \delta^{2n+1} - 2(-1)^n\right] \\
&= \left[\gamma^{2n+1} + \delta^{2n+1} + 2(-1)^n\right]\left[(\gamma^{2n+1} + \delta^{2n+1} - 2(-1)^n\right] \\
&= \left(\gamma^{2n+1} + \delta^{2n+1}\right)^2 - 4 = 8(\text{LHS}).
\end{aligned}$$

So LHS = RHS, as claimed. ∎

For example, let $n = 6$. Then

$$\begin{aligned}
\text{LHS} &= 2P_6 P_7(Q_{13} + 1) = 2 \cdot 70 \cdot 169 \cdot 47322 \\
&= 1,119,638,520 = 99 \cdot 239 \cdot 47320 \\
&= Q_6 Q_7(Q_{13} - 1) = \text{RHS}.
\end{aligned}$$

T. Koshy, *Pell and Pell–Lucas Numbers with Applications*, DOI 10.1007/978-1-4614-8489-9_15,

Likewise, when $n = 5$, LHS = $2P_5P_6(Q_{11} - 1) = 32,959,080 = Q_5Q_6(Q_{11} + 1)$ = RHS.

We are now ready to present two relationships connecting the Pell family and the inverse tangent function. They were developed by Robert W.D. Christie in 1906 [44].

Example 15.2 Prove the following identities:

$$2\tan^{-1}\frac{P_n}{P_{n+1}} + (-1)^n \tan^{-1}\frac{1}{Q_{2n+1}} = \frac{\pi}{4} \tag{15.1}$$

$$2\tan^{-1}\frac{Q_n}{Q_{n+1}} - (-1)^n \tan^{-1}\frac{1}{Q_{2n+1}} = \frac{\pi}{4}. \tag{15.2}$$

Proof. In the interest of brevity, we will prove identity (15.1), and leave the other for Pell enthusiasts to pursue.

To Prove Identity (15.1):

Case 1 Let n be even. Let $\mu_n = 2\tan^{-1}\frac{P_n}{P_{n+1}}$. Using the *double-angle formula* $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ and identity (22) in Chapter 7, we have

$$\begin{aligned}
\tan\mu_n &= \tan\left(2\tan^{-1}\frac{P_n}{P_{n+1}}\right)\\
&= \frac{\frac{2P_n}{P_{n+1}}}{1-\frac{P_n^2}{P_{n+1}^2}} = \frac{2P_nP_{n+1}}{P_{n+1}^2 - P_n^2}\\
&= \frac{2P_nP_{n+1}}{Q_nQ_{n+1}}.
\end{aligned}$$

Let $\theta_n = 2\tan^{-1}\frac{P_n}{P_{n+1}} + \tan^{-1}\frac{1}{Q_{2n+1}}$. Then, by the *sum formula* $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$ and Example 15.1, we have

$$\begin{aligned}
\tan\theta_n &= \frac{\frac{2P_nP_{n+1}}{Q_nQ_{n+1}} + \frac{1}{Q_{2n+1}}}{1 - \frac{2P_nP_{n+1}}{Q_nQ_{n+1}}\cdot\frac{1}{Q_{2n+1}}}\\
&= \frac{2P_nP_{n+1}Q_{2n+1} + Q_nQ_{n+1}}{Q_nQ_{n+1}Q_{2n+1} - 2P_nP_{n+1}}\\
&= 1\\
\theta_n &= \frac{\pi}{4}, \quad \text{as desired.}
\end{aligned}$$

Case 2 Let n be odd. Letting $\theta_n = 2\tan^{-1}\frac{P_n}{P_{n+1}} - \tan^{-1}\frac{1}{Q_{2n+1}}$, as above we get

$$\tan\theta_n = \frac{2P_nP_{n+1}Q_{2n+1} - Q_nQ_{n+1}}{Q_nQ_{n+1}Q_{2n+1} + 2P_nP_{n+1}}$$

$$
\begin{aligned}
&= 1 \\
\theta_n &= \frac{\pi}{4}, \quad \text{as desired.}
\end{aligned}
$$

Identity (15.2) follows by a similar argument. ■

15.2 Euler's and Machin's Formulas

Identities (15.1) and (15.2) have interesting consequences. For example, when $n = 1$, identity (15.1) yields *Euler's formula*: $2\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$.

When $n = 1$, identity (15.2) yields *Machin's formula*, discovered by the English mathematician John Machin (1680–1751): $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$.

It follows from these two formulas that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$.

When $n = 3$, identity (15.1) yields $2\tan^{-1}\frac{5}{12} - \tan^{-1}\frac{1}{239} = \frac{\pi}{4}$; that is, $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \frac{\pi}{4}$. Machin discovered this formula in 1706.

15.3 Identities (15.1) and (15.2) Revisited

Recall from Chapter 7 that $Q_n^2 - 2P_n^2 = (-1)^n$, and that $\frac{Q_{n+1}}{P_{n+1}}$ is the nth convergent of the ISCF of $\sqrt{2}$. Consequently, identities (15.1) and (15.2) can be stated in terms of the solutions of Pell's equation $x^2 - 2y^2 = (-1)^n$, or the convergents of ISCF of $\sqrt{2}$.

15.4 An Additional Byproduct of Example 15.2

Taking limits of both sides of identity (15.1) as $n \to \infty$, we get

$$
\begin{aligned}
2\tan^{-1}\left(\lim_{n\to\infty}\frac{P_n}{P_{n+1}}\right) + 0 &= \frac{\pi}{4} \\
\tan^{-1}\left(\frac{1}{\gamma}\right) &= \frac{\pi}{8} \\
\tan^{-1}(-\delta) &= \frac{\pi}{8},
\end{aligned}
$$

where we have used the fact that $\tan^{-1}$ is a continuous function in the interval $(-\pi/2, \pi/2)$. Formula (15.2) also yields the same result.

Next we develop a host of interesting relationships linking the Pell polynomial family and the inverse tangent function.

First, let $\theta_n = \tan^{-1}\frac{1}{p_{2n}(x)} - \tan^{-1}\frac{1}{p_{2n+2}(x)}$. Then, by the sum formula, $\tan(x-y) = \frac{\tan x - \tan y}{1+\tan x \tan y}$ and the Cassini-like formula (14.9), we have

$$\begin{aligned}
\tan\theta_n &= \frac{\frac{1}{p_{2n}(x)} - \frac{1}{p_{2n+2}(x)}}{1+\frac{1}{p_{2n}(x)}\cdot\frac{1}{p_{2n+2}(x)}} \\
&= \frac{p_{2n+2}(x) - p_{2n}(x)}{1+p_{2n}(x)p_{2n+2}(x)} \\
&= \frac{2xp_{2n+1}(x)}{p_{2n+1}^2(x)} \\
&= \frac{2x}{p_{2n+1}(x)} \\
\theta_n &= \tan^{-1}\frac{2x}{p_{2n+1}(x)}.
\end{aligned}$$

Thus, we have the following result.

Theorem 15.1 $\tan^{-1}\frac{1}{p_{2n}(x)} - \tan^{-1}\frac{1}{p_{2n+2}(x)} = \tan^{-1}\frac{2x}{p_{2n+1}(x)}$. ■

In particular, this implies that

$$\tan^{-1}\frac{1}{P_{2n}} - \tan^{-1}\frac{1}{P_{2n+2}} = \tan^{-1}\frac{2}{P_{2n+1}}$$

and

$$\tan^{-1}\frac{1}{F_{2n}} - \tan^{-1}\frac{1}{F_{2n+2}} = \tan^{-1}\frac{1}{F_{2n+1}}.$$

Theorem 15.1 has another interesting byproduct:

$$\begin{aligned}
\sum_{k=1}^{n}\tan^{-1}\frac{2x}{p_{2k+1}(x)} &= \sum_{k=1}^{n}\left(\tan^{-1}\frac{1}{p_{2k}(x)} - \tan^{-1}\frac{1}{p_{2k+2}(x)}\right) \\
&= \tan^{-1}\frac{1}{p_2(x)} - \tan^{-1}\frac{1}{p_{2n+2}(x)}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\sum_{k=1}^{\infty}\tan^{-1}\frac{2x}{p_{2k+1}(x)} &= \tan^{-1}\frac{1}{2x} - 0 \\
&= \tan^{-1}\frac{1}{2x}.
\end{aligned}$$

This gives us the next result.

Theorem 15.2 $\sum_{k=1}^{\infty} \tan^{-1} \frac{2x}{p_{2k+1}(x)} = \tan^{-1} \frac{1}{2x}$. ■

This yields the following two results:

$$\sum_{n=1}^{\infty} \tan^{-1} \frac{2}{P_{2n+1}} = \tan^{-1} \frac{1}{2} \approx 26.5650511771$$

$$\sum_{n=1}^{\infty} \tan^{-1} \frac{2}{F_{2n+1}} = \frac{\pi}{4}.$$

The American mathematician Derrick H. Lehmer (1905–1991) discovered this Fibonacci result [126] in 1936 when he was at Lehigh University, Pennsylvania.

We will now prove a formula for Pell–Lucas polynomials, similar to Theorem 15.1.

Theorem 15.3 $\tan^{-1} \frac{q_{2n+1}(x)}{q_{2n}(x)} + \tan^{-1} \frac{q_{2n+1}(x)}{q_{2n+2}(x)} = \tan^{-1} p_{4n+2}(x)$.

Proof. Let $\theta_n = \tan^{-1} \frac{q_{2n+1}(x)}{q_{2n}(x)} + \tan^{-1} \frac{q_{2n+1}(x)}{q_{2n+2}(x)}$. Then, by the sum formula, $\tan(x+y) = \frac{\tan x + \tan y}{1 + \tan x \tan y}$, and identities (14.7) and (14.8), we have

$$\begin{aligned}
\tan \theta_n &= \frac{\frac{q_{2n+1}(x)}{q_{2n}(x)} + \frac{q_{2n+1}(x)}{q_{2n+2}(x)}}{1 - \frac{q_{2n+1}^2(x)}{q_{2n}(x)q_{2n+2}(x)}} \\
&= \frac{[q_{2n}(x) + q_{2n+2}(x)]\, q_{2n+1}(x)}{q_{2n}(x)q_{2n+2}(x) - q_{2n+1}^2(x)} \\
&= \frac{4(x^2+1)\, p_{2n+1}(x)q_{2n+1}(x)}{4(x^2+1)} \\
&= p_{2n+1}(x)q_{2n+1}(x) \\
&= p_{4n+2}(x) \\
\theta_n &= \tan^{-1} p_{4n+2}(x), \quad \text{as claimed.}
\end{aligned}$$

■

This theorem has two interesting byproducts, as the following corollary shows. They follow by letting $x = 1$ and $x = 1/2$ in the theorem.

Corollary 15.1

$$\tan^{-1} \frac{Q_{2n+1}}{Q_{2n}} + \tan^{-1} \frac{Q_{2n+1}}{Q_{2n+2}} = \tan^{-1} P_{4n+2}$$

$$\tan^{-1} \frac{L_{2n+1}}{L_{2n}} + \tan^{-1} \frac{L_{2n+1}}{L_{2n+2}} = \tan^{-1} F_{4n+2}.$$

■

The next theorem and two corollaries follow by similar arguments. We omit their proofs for the sake of brevity.

Theorem 15.4

$$\begin{aligned}
\tan^{-1}\frac{p_{n+1}(x)}{p_{n+2}(x)} - \tan^{-1}\frac{p_n(x)}{p_{n+1}(x)} &= \tan^{-1}\frac{(-1)^n}{p_{2n+2}(x)}\\
\tan^{-1}\frac{q_n(x)}{q_{n+1}(x)} - \tan^{-1}\frac{q_{n+1}(x)}{q_{n+2}(x)} &= \tan^{-1}\frac{(-1)^n}{p_{2n+2}(x)}.
\end{aligned}$$

■

Corollary 15.2

$$\begin{aligned}
\tan^{-1}\frac{P_{n+1}}{P_{n+2}} - \tan^{-1}\frac{P_n}{P_{n+1}} &= \tan^{-1}\frac{(-1)^n}{P_{2n+2}}\\
\tan^{-1}\frac{F_{n+1}}{F_{n+2}} - \tan^{-1}\frac{F_n}{F_{n+1}} &= \tan^{-1}\frac{(-1)^n}{F_{2n+2}}\\
\tan^{-1}\frac{Q_n}{Q_{n+1}} - \tan^{-1}\frac{Q_{n+1}}{Q_{n+2}} &= \tan^{-1}\frac{(-1)^n}{P_{2n+2}}\\
\tan^{-1}\frac{L_n}{L_{n+1}} - \tan^{-1}\frac{L_{n+1}}{L_{n+2}} &= \tan^{-1}\frac{(-1)^n}{F_{2n}}.
\end{aligned}$$

■

Corollary 15.3

$$\begin{aligned}
\sum_{k=0}^{n}\tan^{-1}\frac{(-1)^k}{p_{2k+2}(x)} &= \tan^{-1}\tfrac{p_{n+1}(x)}{p_{n+2}(x)} & \sum_{k=0}^{n}\tan^{-1}\frac{(-1)^k}{P_{2k+2}} &= \tan^{-1}\tfrac{P_{n+1}}{P_{n+2}}\\
\sum_{k=0}^{\infty}\tan^{-1}\frac{(-1)^k}{P_{2k+2}} &= \tan^{-1}\tfrac{1}{\gamma} & \sum_{k=0}^{n}\tan^{-1}\frac{(-1)^k}{F_{2k+2}} &= \tan^{-1}\tfrac{F_{n+1}}{F_{n+2}}\\
\sum_{k=0}^{\infty}\tan^{-1}\frac{(-1)^k}{F_{2k+2}} &= \tan^{-1}\tfrac{1}{\alpha} & \sum_{k=0}^{n}\tan^{-1}\frac{(-1)^k}{p_{2k+2}(x)} &= \tan^{-1}\tfrac{1}{2x^2+1} - \tan^{-1}\tfrac{q_{n+1}(x)}{q_{n+2}(x)}\\
\sum_{k=0}^{\infty}\tan^{-1}\frac{(-1)^k}{P_{2k+2}} &= \tan^{-1}\tfrac{1}{3} & \sum_{k=0}^{\infty}\tan^{-1}\frac{(-1)^k}{F_{2k+2}} &= \tan^{-1}\tfrac{2}{3} - \tan^{-1}\tfrac{1}{\alpha}.
\end{aligned}$$

■

15.5 Shapiro's Formula

Next we study an interesting trigonometric formula for P_n:

$$P_n = 2^{\lfloor n/2\rfloor}\prod_{k=1}^{\lfloor (n-1)/2\rfloor}\left(3+\cos\frac{2k\pi}{n}\right), \tag{15.3}$$

where $n \geq 1$. It was discovered by Shapiro in 1994 while he was finding one for the special case P_{23} [231]. This special case was proposed as a problem in the previous year by C. Cooper and R.E. Kennedy of Central Missouri State University, Warrensburg, Missouri [53].

The proof of (15.3) uses the notion of a primitive nth root of unity, so we will first define it. A complex number w is an nth root of unity if $w^n = 1$; w is a *primitive nth* root of unity is $w^n = 1$, and $w^m \neq 1$ for $0 < m < n$. For example, the complex number i is a fourth root of unity, since $i^4 = 1$. It is also a primitive fourth root of unity since $i^m \neq 1$ for $0 < m < 4$. Clearly, $e^{2\pi i/n}$ is an nth root of unity.

We are now ready to present a proof of (15.3).

Proof. The proof employs the fact that the polynomial $x^n - y^n$ can be factored using primitive nth roots of unity:

$$x^n - y^n = \prod_{k=0}^{n-1}(x - w^k y),$$

where x and y are real numbers, and w is a primitive nth root of unity. In particular, let n be odd. Then

$$\begin{aligned} x^n - y^n &= (x-y)\prod_{k=1}^{n-1}(x - w^k y) \\ &= (x-y)\prod_{k=1}^{(n-1)/2}(x - w^k y)(x - w^{-k} y) \\ &= (x-y)\prod_{k=1}^{(n-1)/2}\left(x^2 - 2xy\cos\frac{2k\pi}{n} + y^2\right). \end{aligned}$$

Choosing $x = \gamma$ and $y = \delta$, this yields

$$\begin{aligned} P_n &= \prod_{k=1}^{(n-1)/2}\left(6 + 2\cos\frac{2k\pi}{n}\right) \\ &= 2^{(n-1)/2}\prod_{k=1}^{(n-1)/2}\left(3 + \cos\frac{2k\pi}{n}\right). \end{aligned}$$

It follows similarly that

$$P_n = 2^{n/2}\prod_{k=1}^{(n-2)/2}\left(3 + \cos\frac{2k\pi}{n}\right),$$

when n is even. Combining the two cases, we get formula (15.3). ∎

For example, $P_4 = 4(3 + \cos\frac{\pi}{2}) = 12$, and

$$\begin{aligned} P_5 &= 4\prod_{k=1}^{2}\left(3+\cos\frac{2k\pi}{5}\right) = 4\left(3+\cos\frac{2\pi}{5}\right)\left(3+\cos\frac{4\pi}{5}\right) \\ &= 4\left(3+\frac{-1+\sqrt{5}}{4}\right)\left\{3+\left[2\left(\frac{-1+\sqrt{5}}{4}\right)^2-1\right]\right\} \\ &= 29, \text{ as expected.} \end{aligned}$$

Shapiro's formula also follows from the following result, [266] discovered by D. Zeitlin in 1967:

$$Z_n = \prod_{k=1}^{n-1}\left(d - 2\sqrt{c}\cos\frac{k\pi}{n}\right),$$

where Z_n satisfies the recurrence $Z_{n+1} = dZ_{n+1} - cZ_n$, and $Z_0 = 0$ and $Z_1 = 1$.

15.6 Seiffert's Formulas

Next we will study two other spectacular formulas, one for odd-numbered Pell numbers, and one for even-numbered Pell–Lucas numbers; both were discovered by Seiffert in 2008 [224]:

$$P_{2n-1} = 2^{n-2}(4^{n-1}+1) - 2^{2-n}\sum_{k=0}^{\lfloor (n-3)/4\rfloor}\binom{4n-2}{2n-8k-5} \tag{15.4}$$

$$Q_{2n} = 2^{n-1}(2^{2n-1}+1) - 2^{2-n}\sum_{k=0}^{\lfloor (n-2)/4\rfloor}\binom{4n}{2n-8k-4}. \tag{15.5}$$

The proof that follows is a bit long; it is based on the one given by G.C. Greubel of Newport News, Virginia in 2010 [96]. In addition to Euler's formula, the proof employs the binomial theorem.

Proof. Consider the sums

$$S_m = \sum_{k=0}^{\lfloor (m-4)/8\rfloor}\binom{2m}{m-8k-4}$$

and

$$V_m(x) = \sum_{k=0}^{m}\binom{2m}{m-k}x^k = \sum_{k=0}^{m}\binom{2m}{k}x^{m-k} = \sum_{k=m}^{2m}\binom{2m}{k}x^{k-m},$$

where m is an arbitrary positive integer. Then

$$\begin{aligned} V_m(x) + V_m(x^{-1}) &= \sum_{k=0}^{m} \binom{2m}{k} x^{m-k} + \sum_{k=0}^{m} \binom{2m}{k} x^{k-m} \\ &= \sum_{k=0}^{m} \binom{2m}{k} x^{m-k} + \sum_{k=m}^{2m} \binom{2m}{k} x^{m-k} \\ &= \sum_{k=0}^{2m} \binom{2m}{k} x^{m-k} + \binom{2m}{m} \\ &= \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^{2m} + \binom{2m}{m}. \end{aligned} \tag{15.6}$$

By Euler's formula,

$$\sum_{r=0}^{7} e^{n\pi i/4} = \begin{cases} 0 & \text{if } 4 \nmid n \\ 8 & \text{otherwise.} \end{cases}$$

Consequently, we have

$$\begin{aligned} 8S_m &= \sum_{k=0}^{m} \binom{2m}{m-k} \sum_{r=0}^{7} e^{r(k-4)\pi/4} \\ &= \sum_{k=0}^{m} \binom{2m}{m-k} \sum_{r=0}^{7} e^{-r\pi i} e^{rk\pi i/4} \\ &= \sum_{k=0}^{m} \binom{2m}{m-k} \sum_{r=0}^{7} (-1)^r e^{rk\pi i/4} \\ &= \sum_{r=0}^{7} (-1)^r \left[\sum_{k=0}^{m} \binom{2m}{m-k} e^{rk\pi i/4}\right] \\ &= \sum_{r=0}^{7} (-1)^r V_m\left(e^{r\pi i/4}\right). \end{aligned} \tag{15.7}$$

A similar argument shows that

$$8S_m = \sum_{r=0}^{7} (-1)^r V_m\left(e^{-r\pi i/4}\right). \tag{15.8}$$

Adding equations (15.7) and (15.8), we get

$$\begin{aligned}
16S_m &= \sum_{r=0}^{7}(-1)^r\left[V_m\left(e^{r\pi i/4}\right)+V_m\left(e^{-r\pi i/4}\right)\right]\\
&= \sum_{r=0}^{7}(-1)^r\left[\left(e^{r\pi i/8}+e^{-r\pi i/8}\right)^{2m}+\binom{2m}{m}\right], \text{ by equation (15.6)}\\
&= 4^m\sum_{r=0}^{7}(-1)^r\cos^{2m}(r\pi/8)+\binom{2m}{m}\sum_{r=0}^{7}(-1)^r\\
&= 4^m\sum_{r=0}^{7}(-1)^r\cos^{2m}(r\pi/8)+\binom{2m}{m}\cdot 0\\
S_m &= 4^{m-2}\sum_{r=0}^{7}(-1)^r\cos^{2m}(r\pi/8).
\end{aligned}$$

From basic trigonometry, we have $\cos\pi/8=\frac{\sqrt{2+\sqrt{2}}}{2}=\frac{\sqrt{\sqrt{2}\gamma}}{2}$, $\cos 2\pi/8=\frac{1}{\sqrt{2}}$, $\cos 3\pi/8=\frac{\sqrt{2-\sqrt{2}}}{2}=\frac{\sqrt{-\sqrt{2}\delta}}{2}$, $\cos 4\pi/8=0$, $\cos 5\pi/8=-\frac{\sqrt{2-\sqrt{2}}}{2}=-\frac{\sqrt{-\sqrt{2}\delta}}{2}$, $\cos 6\pi/8=-\frac{1}{\sqrt{2}}$, and $\cos 7\pi/8=-\frac{\sqrt{2+\sqrt{2}}}{2}=-\frac{\sqrt{\sqrt{2}\gamma}}{2}$. So

$$\begin{aligned}
S_m &= 4^{m-2}\left\{1+2\left(\frac{1}{2^m}+\frac{1}{2^m}\right)-\frac{2}{2^{3m/2}}\left[\gamma^m+(-1)^m\delta^m\right]\right\}\\
&= 4^{m-2}\left\{1+\frac{1}{2^{m-1}}-\frac{1}{2^{3m/2-1}}\left[\gamma^m+(-1)^m\delta^m\right]\right\}.
\end{aligned}$$

Therefore,

$$\gamma^m+(-1)^m\delta^m=2^{\frac{3m}{2}-1}+2^{\frac{m}{2}}-2^{3-\frac{m}{2}}S_m. \tag{15.9}$$

Case 1 Let m be an odd integer, say, $m=2n-1$. Then equation (15.9) gives

$$\begin{aligned}
P_{2n-1} &= \frac{1}{2\sqrt{2}}\left(2^{3(2n-1)/2-1}+2^{n-1/2}\right)-2^{2-n}S_{2n-1}\\
&= 2^{3n-4}+2^{n-2}-2^{2-n}S_{2n-1}\\
&= 2^{n-2}\left(4^{n-1}+1\right)-2^{2-n}\sum_{k\ge 0}\binom{4n-2}{2n-8k-5}.
\end{aligned}$$

Since $8k \leq 2n-5$, $k \leq \lfloor (n-3)/4 \rfloor$. Thus

$$P_{2n-1} = 2^{n-2}(4^{n-1}+1) - 2^{2-n} \sum_{k=0}^{\lfloor (n-3)/4 \rfloor} \binom{4n-2}{2n-8k-5}, \text{ as claimed.}$$

Case 2 Let $m = 2n$ be an even integer. Then equation (15.9) gives

$$\begin{aligned} Q_{2n} &= 2^{3n-2} + 2^{n-1} - 2^{2-n} S_{2n} \\ &= 2^{n-1}\left(2^{2n-1}+1\right) - 2^{2-n} \sum_{k \geq 0} \binom{4n}{2n-8k-4}. \end{aligned}$$

Since $8k \leq 2n-4$, $k \leq \lfloor (n-2)/4 \rfloor$. Thus

$$Q_{2n} = 2^{n-1}(2^{2n-1}+1) - 2^{2-n} \sum_{k=0}^{\lfloor (n-2)/4 \rfloor} \binom{4n}{2n-8k-4},$$

as desired. ■

For example, let $n = 5$. Then $P_9 = 2^3(4^4+1) - 2^{-3}\binom{18}{5} = 985$, and $Q_{10} = 2^4(2^9+1) - 2^{-3}\binom{20}{6} = 3363$, as expected.

15.6.1 Additional Seiffert Formulas

In 2008, Seiffert discovered additional formulas for the Pell family [225, 226]:

$$P_{2n-1} = 2^{-n} \sum_{k=0}^{2n-1} (-1)^{\lfloor (2n-5k-5)/4 \rfloor} \binom{4n-1}{k} \tag{15.10}$$

$$Q_{2n-1} = 2^{1-n} \sum_{\substack{0 \leq k \leq 2n-1 \\ 2n-k \equiv 2,3,6,7 \pmod 8}} (-1)^{\lfloor (2n-5k-1)/4 \rfloor} \binom{4n-1}{k} \tag{15.11}$$

$$P_{2n} = 2^{-n} \sum_{\substack{0 \leq k \leq 2n \\ 2n-k \equiv 1,2,5,6 \pmod 8}} (-1)^{\lfloor (2n-5k+4)/4 \rfloor} \binom{4n+1}{k} \tag{15.12}$$

$$Q_{2n} = 2^{-n} \sum_{k=0}^{2n} (-1)^{\lfloor (2n-5k)/4 \rfloor} \binom{4n+1}{k}. \tag{15.13}$$

Their proofs employ Fibonacci polynomials $f_n(x)$, Euler's formula, and the identity

$$\sum_{k=0}^{n} \binom{2n+1}{n-k} f_{2k+1}(z) = (z^2+4)^n, \tag{15.14}$$

also discovered by Seiffert in 2003, where z is a complex variable [215].

We are now ready for the proof.

Proof. Let $z = i\sqrt{-\sqrt{2}\delta}$. Since $\sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2} = \frac{\sqrt{-\sqrt{2}\delta}}{2}$, $z = 2i\sin\frac{\pi}{8}$. Then $z^2+4 = 4-4\sin^2\frac{\pi}{8} = 4+\sqrt{2}\delta = \sqrt{2}\gamma$. We also have

$$\begin{aligned} 2\alpha(z) &= 2i\sin\frac{\pi}{8} + \sqrt{4-4\sin^2\frac{\pi}{8}} \\ \alpha(z) &= \cos\frac{\pi}{8} + i\sin\frac{\pi}{8} = e^{\pi i/8}. \end{aligned}$$

Similarly, $\beta(z) = -e^{\pi i/8}$. Thus, by Binet's formula, we have

$$\begin{aligned} f_n(z) &= \frac{\alpha^n(z)-\beta^n(z)}{\alpha(z)-\beta(z)} \\ &= \frac{e^{n\pi i/8}+e^{-n\pi i/8}}{2\cos\frac{\pi}{8}} = \frac{\cos\frac{n\pi}{8}}{\cos\frac{\pi}{8}}. \end{aligned}$$

Thus, by identity (15.14),

$$\sum_{k=0}^{n} \binom{2n+1}{n-k} a_k = (\sqrt{2}\gamma)^n, \tag{15.15}$$

where $a_k = \frac{\cos(2k+1)\pi/8}{\cos\pi/8}$.

The sequence $\{a_k\}_{k=0}^{\infty}$ satisfies the recurrence $a_{k+4} = -a_k$, where $a_0 = 1 = 1+\sqrt{2}\cdot 0$, $a_1 = \sqrt{2}-1 = -1+\sqrt{2}\cdot 1$, $a_2 = 1-\sqrt{2} = 1+\sqrt{2}\cdot(-1)$, and $a_3 = -1 = -1+\sqrt{2}\cdot 0$. So $a_k = (-1)^{\lfloor 5k/4\rfloor} + \sqrt{2}b_k$, where

$$b_k = \begin{cases} (-1)^{\lfloor (5k+4)/4\rfloor} & \text{if } k \equiv 1,2,5,6 \pmod 8 \\ 0 & \text{otherwise.} \end{cases}$$

Case 1 Replacing n with $2n-1$ in (15.15), we get

$$\sum_{k=0}^{2n-1} \binom{4n-1}{2n-1-k} a_k = (\sqrt{2}\gamma)^{2n-1}. \tag{15.16}$$

From the Binet-like formulas for P_k and Q_k, $\gamma^{2n-1} = \frac{1}{2}(Q_{2n-1} + 2\sqrt{2}P_{2n-1})$. Therefore,

$$\begin{aligned}\sum_{k=0}^{2n-1}\binom{4n-1}{2n-1-k}a_k &= 2^{(n-3)/2}(Q_{2n-1} + 2\sqrt{2}P_{2n-1})\\ &= 2^n P_{2n-1} + 2^{n-2}\sqrt{2}Q_{2n-1}.\end{aligned}$$

Equating the rational and irrational parts from both sides, we get

$$P_{2n-1} = 2^{-n}\sum_{k=0}^{2n-1}(-1)^{\lfloor 5k/4\rfloor}\binom{4n-1}{2n-1-k} \tag{15.17}$$

$$Q_{2n-1} = 2^{1-n}\sum_{\substack{0\le k\le 2n-1\\ 2n-k\equiv 2,3,6,7 \pmod 8}} b_{2n-1-k}\binom{4n-1}{2n-1-k}. \tag{15.18}$$

Replacing $2n-1-k$ with k, (15.17) yields

$$\begin{aligned}P_{2n-1} &= 2^{-n}\sum_{k=0}^{2n-1}(-1)^{\lfloor(10n-5k-5)/4\rfloor}\binom{4n-1}{k}\\ &= 2^{-n}\sum_{k=0}^{2n-1}(-1)^{\lfloor(2n-5k-5)/4\rfloor}\binom{4n-1}{k}.\end{aligned}$$

Similarly, (15.18) yields

$$Q_{2n-1} = 2^{1-n}\sum_{\substack{0\le k\le 2n-1\\ 2n-k\equiv 2,3,6,7 \pmod 8}}(-1)^{\lfloor(2n-5k-1)/4\rfloor}\binom{4n-1}{k}. \tag{15.19}$$

Case 2 Similarly, replacing n with $2n$ in (15.15), we get formulas (15.12) and (15.13). ∎

For example, let $n = 3$. Then

$$\begin{aligned}P_5 &= 2^{-3}\sum_{k=0}^{5}(-1)^{\lfloor(1-5k)/4\rfloor}\binom{11}{k} = 29\\ P_6 &= 2^{-3}\sum_{\substack{0\le k\le 6\\ k\equiv 0,1,4,5 \pmod 8}}(-1)^{\lfloor(2-5k)/4\rfloor}\binom{13}{k} = 70\end{aligned}$$

$$Q_5 = 2^{-2} \sum_{\substack{0 \le k \le 5 \\ k \equiv 0,3,4 \pmod 8}} (-1)^{\lfloor (5-5k)/4 \rfloor} \binom{11}{k} = 41$$

$$Q_6 = 2^{-3} \sum_{k=0}^{6} (-1)^{\lfloor (6-5k)/4 \rfloor} \binom{13}{k} = 99.$$

Next we will find four infinite series involving Fibonacci, Lucas, Pell, and Pell–Lucas numbers.

15.7 Roelants' Expansions of $\frac{\pi}{4}$

In 2008, H. Roelants of Leuven, Belgium, developed a delightful infinite series expansion of $\frac{\pi}{4}$ in terms of the elements of the sequence $\{u_n\}$, defined recursively:

$$u_0 = 0, \quad u_1 = 1$$
$$u_n = pu_{n-1} + qu_{n-2},$$

where $p, q > 0$ and $n \ge 2$:

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n u_{2n+1}}{(2n+1)(p^2+4q)^n} t^{2n+1}, \tag{15.20}$$

where $t = \frac{2}{1+\sqrt{\frac{p^2+8q}{p^2+4q}}}$.

We will establish this result using the technique employed by Greubel in 2010 [97]; Gregory's series[11] $\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{2n+1}$, which converges when $|x| < 1$; and the fact that $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$. We will also employ the fact that if the series $\sum_{n=0}^{\infty} a_n$ converges to A and $\sum_{n=0}^{\infty} b_n$ to B, then $\sum_{n=0}^{\infty} (a_n + b_n)$ converges to $A + B$.

Proof. Solving the recurrence for u_n, we get the Binet-like formula $u_n = \frac{r^n - s^n}{r-s}$, where $2r = p + \lambda, 2s = p - \lambda, \lambda = \sqrt{p^2+4q} = r - s$ and $rs = -q$.

Next we will show that the series in (15.20) converges. Let $\mu = \sqrt{p^2+8q}$; so $\mu > p$ and $t = \frac{2}{1+\mu/\lambda} = \frac{2\lambda}{\lambda+\mu}$. Let

$$a_n = \frac{(-1)^n u_{2n+1}}{(2n+1)(p^2+4q)^n} t^{2n+1}.$$

[11] The Scottish mathematician James Gregory (1638–1675) discovered this series in 1671.

Then

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{t^2}{\lambda^2}\cdot\frac{2n+1}{2n+3}\cdot\frac{u_{2n+3}}{u_{2n+1}}\\
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{t^2}{\lambda^2}\cdot\lim_{n\to\infty}\frac{2n+1}{2n+3}\cdot\lim_{n\to\infty}\frac{u_{2n+3}}{u_{2n+1}}\\
&= \frac{t^2}{\lambda^2}\cdot 1\cdot r^2 = \frac{t^2}{\lambda^2}\left(\frac{2\lambda}{\lambda+\mu}\right)^2\\
&= \left(\frac{2r}{\lambda+\mu}\right)^2 = \left(\frac{\lambda+p}{\lambda+\mu}\right)^2\\
&< 1, \text{ since } p<\mu.
\end{aligned}$$

Therefore, by the ratio test, the series converges to a limit S.

We will now show that $S=\frac{\pi}{4}$. Using the Binet-like formula for u_n above, we have

$$\begin{aligned}
S &= \sum_{n=0}^{\infty}\frac{(-1)^n\left(r^{2n+1}-s^{2n+1}\right)t^{2n+1}}{(2n+1)\lambda^{2n+1}}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{rt}{\lambda}\right)^{2n+1}-\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{st}{\lambda}\right)^{2n+1}\\
&= \tan^{-1}\left(\frac{rt}{\lambda}\right)-\tan^{-1}\left(\frac{st}{\lambda}\right)\\
&= \tan^{-1}\left(\frac{\frac{rt}{\lambda}-\frac{st}{\lambda}}{1+\frac{rt}{\lambda}\cdot\frac{st}{\lambda}}\right) = \tan^{-1}\left(\frac{\lambda^2 t}{\lambda^2-qt^2}\right).
\end{aligned}$$

Next we will evaluate the argument of $\tan^{-1}$:

$$\begin{aligned}
\frac{\lambda^2 t}{\lambda^2-qt^2} &= \frac{2\lambda^3(\lambda+\mu)}{\lambda^2(\lambda+\mu)^2-4q\lambda^2}\\
&= \frac{2\lambda(\lambda+\mu)}{\lambda^2+2\lambda\mu+(\mu^2-4q)}\\
&= \frac{2\lambda(\lambda+\mu)}{\lambda^2+2\lambda\mu+\lambda^2}=1.
\end{aligned}$$

Consequently, $S=\tan^{-1}1=\frac{\pi}{4}$, as desired. ■

Next we investigate four interesting special cases of formula (15.20).

15.7.1 Special Cases

(1) Let $p = 1 = q$. Then $t = \frac{\sqrt{5}}{\alpha^2}$. When $u_n = F_n$, formula (15.20) yields

$$\frac{\pi}{4} = \sqrt{5}\sum_{n=0}^{\infty}\frac{(-1)^n F_{2n+1}}{(2n+1)\alpha^{4n+2}}. \tag{15.21}$$

(2) Let $p = 2$ and $q = 1$. Then $t = \frac{2\sqrt{2}}{\sqrt{2}+\sqrt{3}}$. Suppose $u_n = P_n$. Then formula (15.20) gives

$$\frac{\pi}{4} = 2\sqrt{2}\sum_{n=0}^{\infty}\frac{(-1)^n P_{2n+1}}{(2n+1)(\sqrt{2}+\sqrt{3})^{2n+1}}. \tag{15.22}$$

Similar series exist for Lucas and Pell–Lucas numbers. But we need to be a bit careful, since $L_n = \alpha^n + \beta^n$ and $Q_n = \frac{1}{2}(\gamma^n + \delta^n)$. Then, using the formula $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, we have

$$\tan^{-1}\frac{1}{3} = \sum_{n=0}^{\infty}\frac{(-1)^n L_{2n+1}}{(2n+1)\alpha^{4n+2}} \tag{15.23}$$

$$\frac{\pi}{12} = \sum_{n=0}^{\infty}\frac{(-1)^n Q_{2n+1}}{(2n+1)(\sqrt{2}+\sqrt{3})^{2n+1}}. \tag{15.24}$$

15.8 Another Explicit Formula for P_n

In 1996, Seiffert discovered yet another explicit formula for P_n. To this end, he first developed the following formula for $f_{n+1}(x)$:

$$f_{n+1}(x) = \sum_{k=0}^{n}\binom{n+k+1}{2k+1}\Delta^k \cos\alpha_k, \tag{15.25}$$

where $\Delta = \sqrt{x^2+4}$ and $\alpha_k = (n-k)\pi/2 - k\arccos(x/\Delta)$. Its proof requires a knowledge of *Jacobi polynomials*, differentiation, Taylor's theorem, and Euler's formula; so we omit it in the interest of brevity [207, 208].

When $x = 2$, $\frac{x}{\Delta} = \frac{2}{\sqrt{4+4}} = \frac{1}{\sqrt{2}}$; $\arccos\left(\frac{x}{\Delta}\right) = \frac{\pi}{4}$. So $\alpha_k = (n-k)\frac{\pi}{2} - k\cdot\frac{\pi}{4} = (2n-3k)\frac{\pi}{4}$. Thus, when $x = 2$, formula (15.25) yields

$$
\begin{aligned}
P_{n+1} &= \sum_{k=0}^{n} \binom{n+k+1}{2k+1} 2^{3k/2} \cos{(2n-3k)}\frac{\pi}{4} \\
&= 2^n \sum_{k=0}^{n} \binom{n+k+1}{2k+1} 2^{(3k-2n)/2} \cos(3k-2n)\frac{\pi}{4} \\
&= 2^n \sum_{k=0}^{n} \binom{n+k+1}{2k+1} A_{3k-2n},
\end{aligned}
$$

where $A_j = 2^{j/2} \cos \frac{j\pi}{4}$ and j is an integer.

It follows by the addition formula for the cosine function that $A_{4r} = (-1)^r 2^{2r}$, $A_{4r+1} = (-1)^r 2^{2r}$, $A_{4r+2} = 0$, and $A_{4r+3} = (-1)^{r+1} 2^{2r+1}$; that is,

$$
A_j = \begin{cases} (-1)^{\lfloor (j+1)/4 \rfloor} 2^{\lfloor j/2 \rfloor} & \text{if } j \not\equiv 2 \pmod 4 \\ 0 & \text{otherwise.} \end{cases}
$$

Since $\lfloor (3k-2n)/2 \rfloor = \lfloor 3k/2 \rfloor - n$, we have

$$
\begin{aligned}
2^n A_{3k-2n} &= 2^n (-1)^{\lfloor (3k-2n+1)/4 \rfloor} 2^{\lfloor 3k/2 \rfloor - n} \\
&= (-1)^{\lfloor (3k-2n+1)/4 \rfloor} 2^{\lfloor 3k/2 \rfloor},
\end{aligned}
$$

where $3k - 2n \not\equiv 2 \pmod 4$. So

$$
P_{n+1} = \sum_{k=0}^{n} \binom{n+k+1}{2k+1} (-1)^{\lfloor (3k-2n+1)/4 \rfloor} 2^{\lfloor 3k/2 \rfloor}.
$$

Replacing n with $n-1$, we get another summation formula for P_n:

$$
P_n = \sum_{\substack{0 \le k \le n-1 \\ 3k \not\equiv 2n \pmod 4}} \binom{n+k}{2k+1} (-1)^{\lfloor (3k-2n+3)/4 \rfloor} 2^{\lfloor 3k/2 \rfloor}. \tag{15.26}
$$

For example, we have

$$
\begin{aligned}
P_4 &= \sum_{\substack{0 \le k \le 3 \\ k \not\equiv 0 \pmod 4}} \binom{4+k}{2k+1} (-1)^{\lfloor (3k-5)/4 \rfloor} 2^{\lfloor 3k/2 \rfloor} \\
&= \binom{5}{3} (-1)^{-1} 2^1 + \binom{6}{5} (-1)^0 2^3 - \binom{7}{7} (-1)^1 2^4 \\
&= -20 + 48 - 16 = 12.
\end{aligned}
$$

15.9 $p_n(x)$, $q_n(x)$, and Hyperbolic Functions

Both Pell polynomials and Pell–Lucas polynomials are related to the hyperbolic functions sinh and cosh. In 1963, P.F. Byrd showed [38] that $p_{2n}(x) = \frac{\sinh 2nt}{\cosh t}$ and $p_{2n+1}(x) = \frac{\cosh(2n+1)t}{\cosh t}$, where $x = \sinh t$.

Consequently, $q_{2n}(x) = p_{2n+1}(x) + p_{2n-1}(x) = \frac{\cosh(2n+1)t}{\cosh t} + \frac{\cosh(2n-1)t}{\cosh t} = 2\cosh 2nt$. Likewise, $q_{2n+1}(x) = 2\sinh(2n+1)t$.

Finally, we will see more close links between the Pell family and trigonometry in Chapter 18.

Exercises 15

Prove the following identities.

1. $2\tan^{-1}\dfrac{Q_n}{Q_{n+1}} - (-1)^n\tan^{-1}\dfrac{1}{Q_{2n+1}} = \dfrac{\pi}{4}$.
2. $\tan^{-1}\dfrac{p_{n+1}(x)}{p_{n+2}(x)} - \tan^{-1}\dfrac{p_n(x)}{p_{n+1}(x)} = \tan^{-1}\dfrac{(-1)^n}{p_{2n+2}(x)}$.
3. $\tan^{-1}\dfrac{q_n(x)}{q_{n+1}(x)} - \tan^{-1}\dfrac{q_{n+1}(x)}{q_{n+2}(x)} = \tan^{-1}\dfrac{(-1)^n}{p_{2n+2}(x)}$.
4. $\tan^{-1}\dfrac{P_{n+1}}{P_{n+2}} - \tan^{-1}\dfrac{P_n}{P_{n+1}} = \tan^{-1}\dfrac{(-1)^n}{P_{2n+2}}$.
5. $\tan^{-1}\dfrac{F_{n+1}}{F_{n+2}} - \tan^{-1}\dfrac{F_n}{F_{n+1}} = \tan^{-1}\dfrac{(-1)^n}{F_{2n+2}}$.
6. $\tan^{-1}\dfrac{Q_n}{Q_{n+1}} - \tan^{-1}\dfrac{Q_{n+1}}{Q_{n+2}} = \tan^{-1}\dfrac{(-1)^n}{P_{2n+2}}$.
7. $\tan^{-1}\dfrac{L_n}{L_{n+1}} - \tan^{-1}\dfrac{L_{n+1}}{L_{n+2}} = \tan^{-1}\dfrac{(-1)^n}{F_{2n}}$.
8. $\sum_{k=0}^{n}\tan^{-1}\dfrac{(-1)^k}{p_{2k+2}(x)} = \tan^{-1}\dfrac{p_{n+1}(x)}{p_{n+2}(x)}$.
9. $\sum_{k=0}^{n}\tan^{-1}\dfrac{(-1)^k}{P_{2k+2}} = \tan^{-1}\dfrac{P_{n+1}}{P_{n+2}}$.
10. $\sum_{k=0}^{\infty}\tan^{-1}\dfrac{(-1)^k}{P_{2k+2}} = \tan^{-1}\dfrac{1}{\gamma}$.

11. $\sum_{k=0}^{n} \tan^{-1} \frac{(-1)^k}{F_{2k+2}} = \tan^{-1} \frac{F_{n+1}}{F_{n+2}}.$

12. $\sum_{k=0}^{\infty} \tan^{-1} \frac{(-1)^k}{F_{2k+2}} = \tan^{-1} \frac{1}{\alpha}.$

13. $\sum_{k=0}^{n} \tan^{-1} \frac{(-1)^k}{p_{2k+2}(x)} = \tan^{-1} \frac{1}{2x^2+1} - \tan^{-1} \frac{q_{n+1}(x)}{q_{n+2}(x)}.$

14. $\sum_{k=0}^{\infty} \tan^{-1} \frac{(-1)^k}{P_{2k+2}} = \tan^{-1} \frac{\gamma - 3}{3\gamma + 1}.$

15. $\sum_{k=0}^{\infty} \tan^{-1} \frac{(-1)^k}{F_{2k+2}} = \tan^{-1} \frac{2}{3} - \tan^{-1} \frac{1}{\alpha}.$

16. Using Shapiro's formula (15.3), compute P_6 and P_8.

17. Using Seiffert's formula (15.4), compute P_5 and P_7.

18. Using Seiffert's formula (15.5), compute Q_6 and Q_8.

19. Using formula (15.17), compute P_5 and P_7.

20. Using formula (15.19), compute Q_3 and Q_7.

21. Using formula (15.26), compute P_5 and P_6.

16

Pell Tilings

16.1 Introduction

In Chapter 12 we studied some interesting applications of the Pell family to combinatorics, in particular, to the theory of lattice-walking. This chapter presents additional applications to combinatorics, including the theory of partitioning.

To begin with, we present a simple combinatorial interpretation of Fibonacci numbers. This will provide a smooth transition to the Pell applications.

16.2 A Combinatorial Model for Fibonacci Numbers

In 1974, Krishnaswami Alladi[12] (1955 –) of Vivekananda College, Madras (now Chennai), Tamil Nadu, India, and Vernon Emil Hoggatt, Jr. (1921–1980) of then San Jose State College, San Jose, California, studied ordered sums of 1s and 2s that yield the positive integer n [2]. Such sums are called *compositions*. For example, $1+2$ and $2+1$ are two different compositions of the integer 3.

Table 16.1 shows the compositions of the integers 1 through 6. It appears from the table that the number of compositions C_n of n is the Fibonacci number F_{n+1}. The following theorem, established by Alladi and Hoggatt in 1974, confirms this observation.

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12 Currently at the University of Florida at Gainesville.

T. Koshy, *Pell and Pell–Lucas Numbers with Applications*, DOI 10.1007/978-1-4614-8489-9_16,

Table 16.1.

n	Compositions of n	C_n
1	1	1
2	$1+1$, 2	2
3	$1+1+1$, $1+2$, $2+1$	3
4	$1+1+1+1$, $1+1+2$, $1+2+1$, $2+1+1$, $2+2$	5
5	$1+1+1+1+1$, $1+1+1+2$, $1+1+2+1$, $1+2+1+1$, $2+1+1+1$, $1+2+2$, $2+1+2$, $2+2+1$	8
6	$1+1+1+1+1+1$, $1+1+1+1+2$, $1+1+1+2+1$, $1+1+2+1+1$, $1+2+1+1+1$, $2+1+1+1+1$, $1+1+2+2$, $1+2+1+2$, $1+2+2+1$, $2+1+2+1$, $2+2+1+1$, $2+1+1+2$, $2+2+2$	13

Theorem 16.1 *The number of compositions C_n of the positive integer n is F_{n+1}, where $n \geq 1$.*

Proof. It follows from Table 16.1 that $C_1 = 1 = F_2$ and $C_2 = 2 = F_3$. So we let $n \geq 3$.

Case 1 Suppose the composition of n ends in 1. Deleting this 1 yields a composition of $n-1$:

$$\underbrace{\cdots}_{\text{A composition of } n-1} \quad + 1$$

By definition, there are C_{n-1} such compositions. Consequently, there are C_{n-1} compositions of n that end in 1. (Notice that there are $C_4 = 5 = F_5$ compositions of 5 ending in 1.)

Case 2 Suppose the composition of n ends in 2. Deleting this 2 yields a composition of $n-2$:

$$\underbrace{\cdots}_{\text{A composition of } n-2} \quad + 2$$

Again, by definition, there are C_{n-2} such compositions. So there are C_{n-2} compositions of n that end in 2. (Notice that there are $C_3 = 3 = F_4$ compositions of 5 ending in 2.)

Since every composition ends in 1 or 2, it follows by the addition principle that $C_n = C_{n-1} + C_{n-2}$, where $n \geq 3$. Since $C_1 = F_2$ and $C_2 = F_3$, and $C_n = C_{n-1} + C_{n-2}$, it follows that $C_n = F_{n+1}$, as desired. ∎

We can extend the definition of C_n to include the case $n = 0$. Since C_0 denotes the number of empty compositions of 0 and there is exactly one such composition, we define $C_0 = 1 = F_1$. So the theorem holds for every integer $n \geq 0$.

16.3 A Fibonacci Tiling Model

Interestingly, this theorem has a delightful geometric interpretation [11]. To this end, consider a $1 \times n$ board (an array of n unit squares). Suppose we would like to cover it with 1×1 tiles (unit squares) and 1×2 tiles (dominoes); such a process is called a *tilings* of the board.

Figure 16.1 shows the tilings of a $1 \times n$ board with square tiles and dominoes, where $1 \leq n \leq 5$. Clearly, they are geometric representations of the compositions in Table 16.1. Since this process is completely reversible, it follows by the theorem that there are F_{n+1} ways of tiling a $1 \times n$ board.

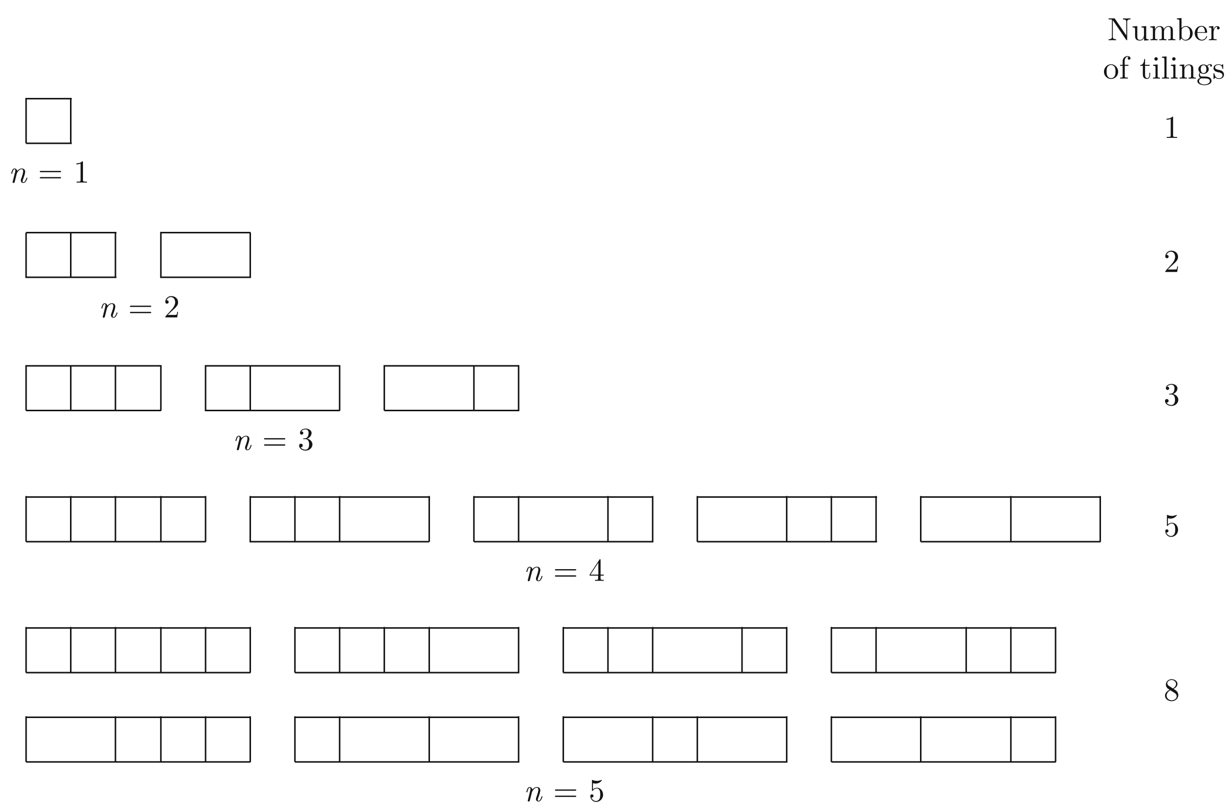


Figure 16.1.

We now turn to interpreting the Pell numbers P_n combinatorially.

16.4 A Combinatorial Model For Pell Numbers

Suppose, for argument's sake, a square tile costs twice as much as a domino. So we assign a *weight* of 2 to each square (tile) and a weight of 1 to each domino. The *weight of a tiling* is the product of the weights of its tiles. The weight of the empty tiling is defined as 1. Figure 16.2 shows the weights of the tiles, weights of the tilings, and the sums of the weights of all tilings of length n, where $0 \leq n \leq 5$.

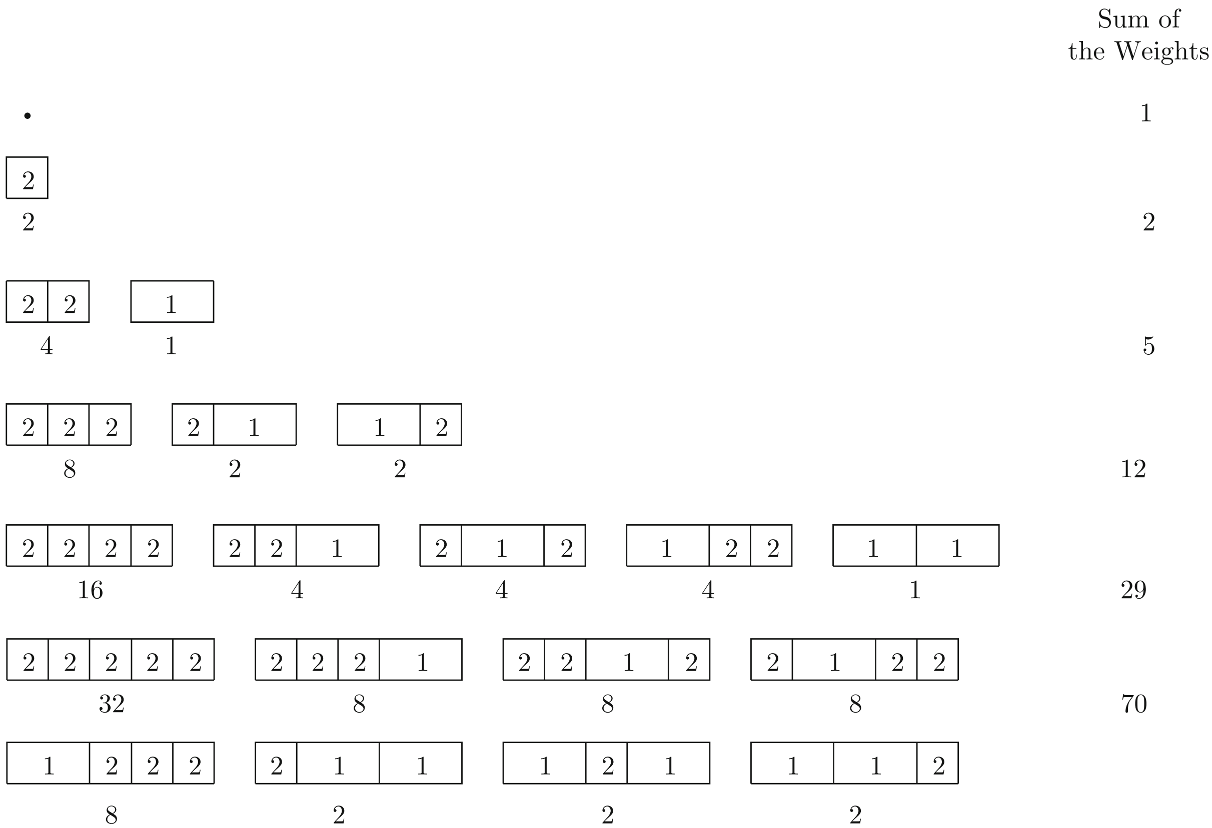


Figure 16.2.

We observe the Pell pattern in this model: The sums of weights of tilings of length n is P_{n+1}, where $n \geq 0$.

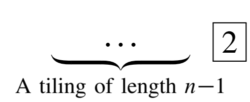
The following theorem confirms this interesting observation. Its proof follows basically the same argument as that of Theorem 16.1.

Theorem 16.2 *The sum of the weights of the tilings of a* $1 \times n$ *board with square tiles and dominoes is* P_{n+1}*, where* $n \geq 0$.

Proof. Let S_n denote the sum of the weights of the tilings of the board. Clearly, $S_0 = 1 = P_1$ and $S_1 = 2 = P_2$.

Consider an arbitrary tiling of length n, where $n \geq 2$.

Case 1 Suppose the tiling ends in a square. Such a tilings is composed of a tiling of length $n-1$, followed by a square of weight 2:



The sum of the weights of such tilings is $2S_{n-1}$.

Case 2 Suppose the tiling ends in a domino. It consists of a tiling of length $n-2$ and a domino:

$$\underbrace{\cdots}_{\text{A tiling of length } n-2}\quad \boxed{1}$$

Since the weight of the domino is 1, such tilings have a total weight of S_{n-2}.

So, by the addition principle, $S_n = 2S_{n-1} + S_{n-2}$. Thus S_n satisfies the Pell recurrence, with the initial conditions $S_0 = P_1$ and $S_1 = P_2$. So $S_n = P_{n+1}$, as desired. ∎

This proof provides a constructive algorithm for finding all tilings of length n from those of lengths $n-1$ and $n-2$: to the tilings of length $n-2$, append a domino of weight 1; to those of length $n-1$, append a square of weight 2. (Unfortunately, this will produce duplicate tilings.)

The next theorem gives a combinatorial proof of the explicit formula for P_{n+1} in Theorem 9.1: $P_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} 2^{n-2k}$.

Theorem 16.3 *Establish the formula* $P_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} 2^{n-2k}$, *using a combinatorial argument.*

Proof. We will establish this formula using weighted tilings of a $1 \times n$ board with square tiles and dominoes.

Suppose a tiling has exactly k dominoes. Then it has $n-2k$ squares, where $0 \le k \le \lfloor n/2 \rfloor$. So it has a weight of $(1^k)(2^{n-2k}) = 2^{n-2k}$.

Since there are exactly k dominoes in a tiling, it uses a total of $(n-2k)+k = n-k$ tiles. So the k dominoes can be placed in any k of the $n-k$ positions; that is, they can be placed in $\binom{n-k}{k}$ different ways. In other words, there are $\binom{n-k}{k}$ tilings, each containing exactly k dominoes. The weight of such a tiling is $\binom{n-k}{k} 2^{n-2k}$, where $0 \le k \le \lfloor n/2 \rfloor$. So the sum of the weights of all tilings of length n is $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} 2^{n-2k}$. By Theorem 16.2, the sum of the weights is P_{n+1}. Thus, $P_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} 2^{n-2k}$, as desired. ∎

In Chapter 8, we established the addition formula $P_{m+n} = P_m P_{n+1} + P_{m-1} P_n$; see identity (8.7). We now have the tools to confirm it combinatorially, as the next theorem shows.

To this end, first we introduce the concept of *breakability*. A tiling is *breakable* at cell k if a domino does *not* occupy cells k and $k+1$; otherwise, it is *unbreakable* at cell k. Thus a tiling is breakable at cell k if and only if it can be split up into two sub-tilings, one covering cells 1 through k and the other covering cells $k+1$ through n.

For example, the tiling in Figure 16.3 is not breakable at cell 4, whereas that in Figure 16.4 is breakable at cell 4.

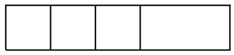


Figure 16.3.

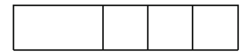


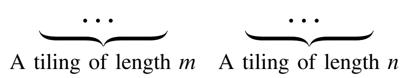
Figure 16.4.

Theorem 16.4 *Let* $m, n \geq 0$. *Then* $P_{m+n} = P_m P_{n+1} + P_{m-1} P_n$.

Proof. Consider a board of length $m + n$. By Theorem 16.2, the sum of the weights of its tilings equals P_{m+n+1}.

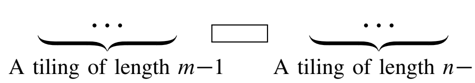
Consider an arbitrary tiling T of length $m + n$.

Case 1 Suppose it is breakable at cell m. This yields a sub-tiling of length m and a sub-tiling of length n:



By Theorem 16.2, the sum of their weights are P_{m+1} and P_{n+1}, respectively. So, by the multiplication principle, the sum of the weights of such tilings T equals $P_{m+1} P_{n+1}$.

Case 2 Suppose the tiling T is *not* breakable at cell m. So a domino occupies cells m and $m + 1$. This results in two tilings, one of length $m - 1$ and the other of length $n - 1$:



By the multiplication principle, the sum of the weights of such tilings T equals $+P_m P_n$.

Combining the two cases, we have $P_{m+n+1} = P_{m+1} P_{n+1} + P_m P_n$. Changing m to $m - 1$, the desired result follows. ■

Next we investigate Pell tilings, where square tiles are available in two different colors.

16.5 Colored Tilings

Suppose square tiles come in two colors, black and white. Earlier we assigned a weight of 2 to a square and a weight of 1 to a domino; but this time we assign the *same* weight 1 to each tile, square or domino. (So we can safely ignore their weights.) The weight of a colored tiling, as before, is the product of the weights of the tiles. Since each tile has weight 1, it follows that the weight of each colored tiling is also 1.

Figure 16.5 shows the resulting colored tilings of length n, and the number of such tilings, where $0 \leq n \leq 4$. The black square tiles are lightly shaded in the figure.

Based on the experimental data from Figure 16.5, we conjecture that the number of colored tilings of length n is P_{n+1}, where $n \geq 0$. The following theorem confirms this observation.

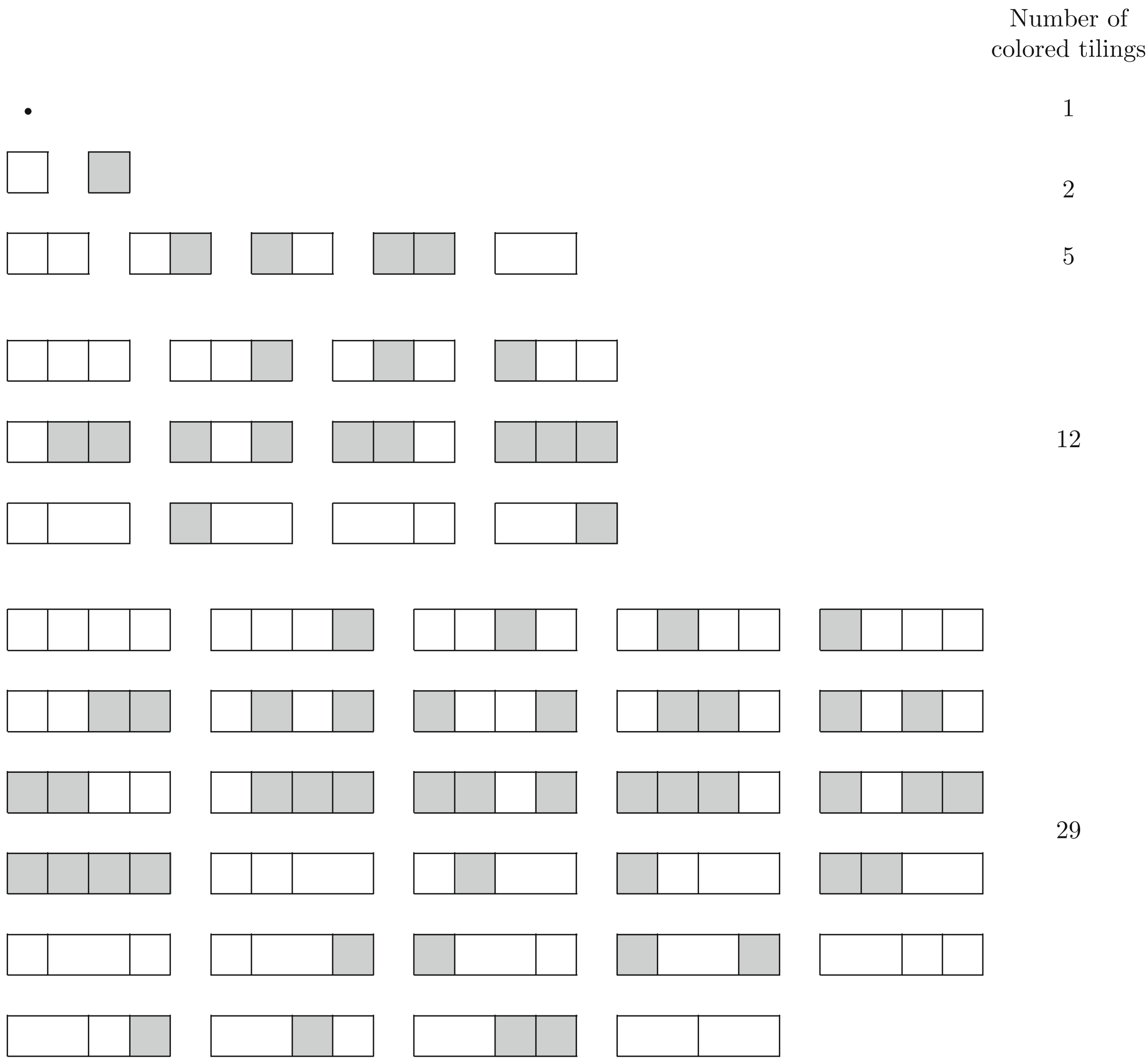


Figure 16.5.

Although it follows from Theorem 16.4, we will give an independent proof, using the same argument as in the proof of Theorem 16.4; but it is slightly longer because of colored tiles.

Theorem 16.5 *The number of colored tilings of length n is* P_{n+1}, *where* $n \geq 0$.

Proof. Let C_n denote the number of colored tilings of length n. It follows from Figure 16.5 that $C_0 = 1 = P_1$ and $C_1 = 2 = P_2$.

Consider an arbitrary colored tiling of length $n \geq 3$.

Case 1 Suppose the tiling ends in a square tile.

Subcase 1 Suppose the square is black. Deleting this black tile results in a colored tiling of length $n-1$:

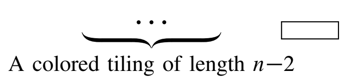
By definition, there are C_{n-1} such tilings. So there are C_{n-1} tilings ending in a black square tile.

Subcase 2 Suppose the square is white:

As in Subcase 1, there are C_{n-1} tilings ending in a white square tile.

So, by the addition principle, there are $C_{n-1} + C_{n-1} = 2C_{n-1}$ colored tilings of length n ending in a square tile, black or white. (Notice that in Figure 16.5, there are $12 = C_3$ tilings of length 4 ending in a black square and $12 = C_3$ tilings ending in a white square.)

Case 2 Suppose the colored tiling ends in a domino. Deleting this domino yields a colored tiling of length $n-2$:



There are C_{n-2} colored tilings of length $n-2$; so there are C_{n-2} colored tilings of length $n-2$ ending in a domino.

Thus, by the addition principle, there are $C_n = 2C_{n-1} + C_{n-2}$ colored tilings of length n. Since C_n satisfies the Pell recurrence with $C_0 = P_1$ and $C_1 = P_2$, it follows that $C_n = P_{n+1}$. ∎

Next we give three combinatorial interpretations of the Pell–Lucas number Q_n.

16.6 Combinatorial Models for Pell–Lucas Numbers

Recall that the recursive definitions of Pell and Pell–Lucas numbers differ only in the second initial condition: $P_2 = 2$, but $Q_2 = 3$. So in the uncolored tilings we investigated in Theorem 16.2, we keep the weight 2 for unit squares and 1 for dominoes, with one *major* exception: If a tiling begins with a square tile, it is assigned a weight of 1.

What can we say about the sum of the weights of such tilings of length n? Before we answer this, we will do some experiments, collect data, look for a pattern, and then make a conjecture.

Figure 16.6 shows such tilings of length n and their weights, where $0 \le n \le 5$. Notice that the tilings in Figure 16.2 and 16.6 are closely related. There is just one difference between the two: If the tiling begins with a square, then its weight in Figure 16.2 is 2 and that in Figure 16.6 is 1.

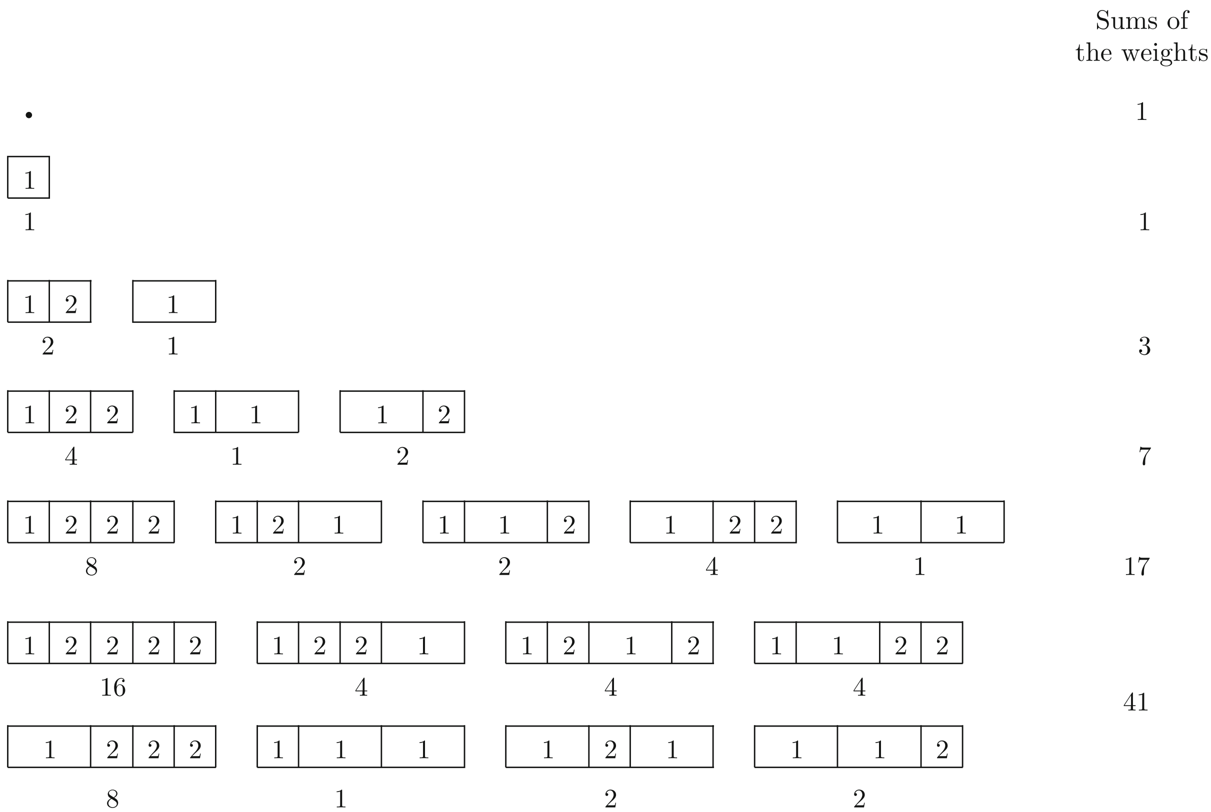


Figure 16.6.

Using the data from the figure, we conjecture that the sum of the weights of the tilings of length n is Q_n. The following theorem confirms this observation.

Theorem 16.6 *Suppose the uncolored tilings of a* $1 \times n$ *board are made up of squares and dominoes. Suppose the weight of a square is* 2 *and that of a domino is* 1*, except that if the tiling begins with a square, its weight is* 1*. The sum of the weights of the tilings of length n is* Q_n*, where* $n \geq 0$*.*

Proof. Let S_n denote the sum of the weights of the tilings of length n. Then, by Figure 16.6, $S_0 = 1 = Q_0$ and $S_1 = 1 = Q_1$.

Now consider an arbitrary tiling of length $n \geq 2$.

Case 1 Suppose the tiling ends in a square; the weight of this square is 2. Deleting this square yields a tiling of length $n - 1$:

$$\underbrace{\cdots}_{\text{A tiling of length } n-1}\ \boxed{2}$$

It follows by definition that the sum of the weights of such tilings equals $2S_{n-1}$.

Case 2 Suppose the tiling ends in a domino; its weight is 1:

$$\underbrace{\cdots}_{\text{A tiling of length } n-2} \ \boxed{1}$$

Deleting this domino results in a tiling of length $n-2$. Again, it follows by definition that the sum of the weights of such tilings equals $1 \cdot S_{n-2} = S_{n-2}$.

Thus, by the addition principle, $S_n = 2S_{n-1} + S_{n-2}$, where $n \geq 2$.

Since S_n satisfies exactly the same recursive definition as Q_n, it follows that $S_n = Q_n$, where $n \geq 0$. ■

It follows from the proof of this theorem that there are exactly F_{n+1} such tilings of length n; this follows by changing the weight of a square from 2 to 1. See Figure 16.1 also.

The concept of breakability, introduced earlier, can be employed to reconfirm the addition formula $Q_{m+n} = Q_{m+1}P_n + Q_m P_{n-1}$ that we developed in Chapter 8, as the following theorem shows.

Theorem 16.7 *Let $m, n \geq 0$. Then $Q_{m+n} = Q_{m+1}P_n + Q_m P_{n-1}$.*

Proof. Consider an uncolored tiling of length $m+n$, which is composed of squares and dominoes. By Theorem 16.6, there are Q_{m+n} such tilings.

Consider an arbitrary tiling of length $m+n$.

Case 1 Suppose it is breakable at cell m. This yields two sub-tilings, one of length m and the other of length n:

$$\underbrace{\cdots}_{\text{A tiling of length } m} \ \underbrace{\cdots}_{\text{A tiling of length } n}$$

By Theorem 16.6, the sum of the weights of tilings of length m is Q_m; and by Theorem 16.2, the sum of the weights of tilings of length n is P_{n+1}. So the sum of the weights of such tilings of length $m+n$ is $Q_m P_{n+1}$.

Case 2 Suppose the tiling is *not* breakable at cell m. So a domino occupies cells m and $m+1$. This creates a sub-tiling A of length $m-1$, followed by the domino and a subtiling B of length $n-1$:

$$\underbrace{\cdots}_{\text{A tiling of length } m-1} \ \boxed{1} \ \underbrace{\cdots}_{\text{A tiling of length } n-1}$$

By Theorem 16.6, the sum of the weights of tilings of type A is Q_{m-1}; and by Theorem 16.2, the sum of the weights of tilings of type B is P_n. So the sum of the weight of such tilings of length $m+n$ equals $Q_{m-1} \cdot 1 \cdot P_n = Q_{m-1}P_n$.

Combining the two cases, we get $Q_{m+n} = Q_{m-1}P_n + Q_m P_{n+1}$. Changing m to $m+1$ and n to $n-1$, the desired result follows. ■

We now present a second combinatorial interpretation of Pell–Lucas numbers, using colored tiles.

16.7 Colored Tilings Revisited

In this model, there are black and white square tiles, and dominoes. Each has weight 1, with one exception: If a tiling begins with a white square, then the tile has weight 2.

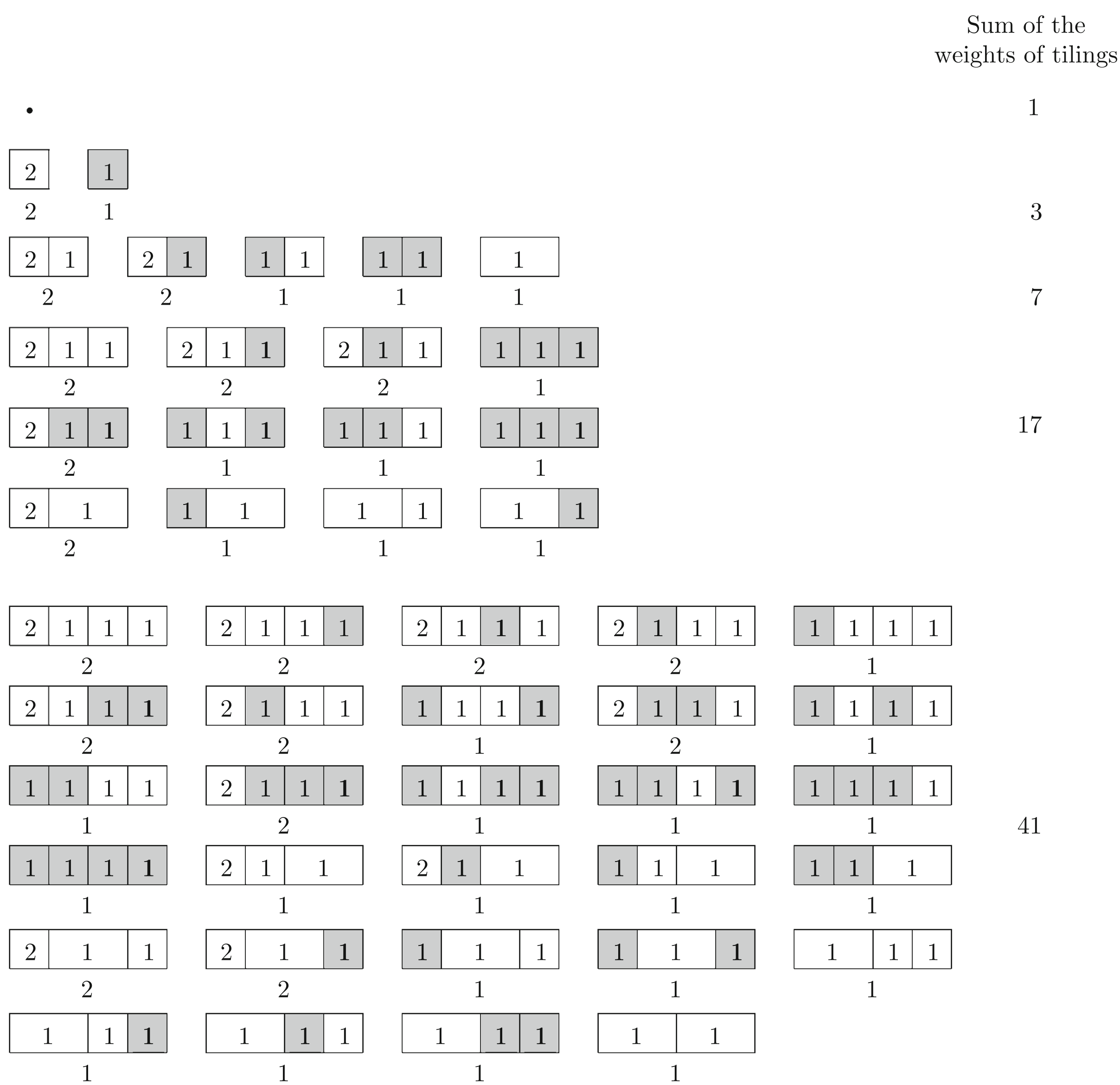


Figure 16.7.

Figure 16.7 shows such tilings and the sum of the weights for $0 \leq n \leq 4$, where black squares are shaded in light gray. Based on the experimental data we have collected, we conjecture that the sum of the weights of colored tilings of length n in this model is Q_{n+1}, where $0 \le n \le 4$. The following theorem establishes this observation.

Theorem 16.8 *Suppose the colored tiling of a* $1 \times n$ *board is made up of square tiles (black or white) and dominoes, where* $n \ge 0$*. Every square tile and domino has weight* 1*, with one exception: If a tiling begins with a white square tile, then the tile has weight* 2*. Then the sum of the weights of the tilings of length n is* Q_{n+1}*.*

Proof. Let S_n denote the sum of the weights of the tilings of length n. It follows from Figure 16.7 that $S_0 = 1 = Q_1$ and $S_1 = 3 = Q_2$.

Consider an arbitrary colored tiling of length $n \ge 2$.

Case 1 Suppose the tiling begins with a square tile.

Subcase 1 Suppose the tile is white. Then its weight is 2. Deleting this white square results in a sub-tiling of length $n - 1$:

$$\boxed{2} \quad \underbrace{\cdots}_{\text{A colored tiling of length } n-1}$$

By Theorem 16.5, there are P_n such sub-tilings; so the sum of the weights of tilings beginning with a white square tile is $2P_n$.

Subcase 2 Suppose the tile is black. Its weight is 1. Deleting this square yields a colored sub-tiling of length $n - 1$:

$$\boxed{1} \quad \underbrace{\cdots}_{\text{A colored tiling of length } n-1}$$

The weight of this sub-tiling is 1. Again by Theorem 16.5, since there are P_n such sub-tilings, the sum of the weights of such tilings is P_n.

Case 2 Suppose the tiling begins with a domino. Deleting this domino yields a colored sub-tiling of length $n - 2$ to its right:

$$\boxed{\quad 1 \quad} \quad \underbrace{\cdots}_{\text{A colored tiling of length } n-2}$$

It follows by Theorem 16.5, there are P_{n-1} such sub-tilings, each with weight 1. So the sum of the weights of tilings beginning a domino is P_{n-1}.

Thus, by Cases 1 and 2, and the addition principle, we have

$$\begin{aligned} S_n &= 2P_n + P_n + P_{n-1} = P_n + (2P_n + P_{n-1}) \\ &= P_n + P_{n+1} = Q_{n+1}, \end{aligned}$$

as desired. ■

Theorem 16.8 has an interesting byproduct. Before we discuss it, let's return to Figure 16.7 to see a fascinating fact: let $f(n)$ denote the number of colored tilings of length n. Notice that $f(0) = 1 = P_1$, $f(1) = 2 = P_2$, $f(2) = 5 = P_3$, $f(3) = 12 = P_4$, and $f(4) = 29 = P_5$. So we conjecture that $f(n) = P_{n+1}$. This can be confirmed fairly easily, as the following corollary shows.

Corollary 16.1 *The number of colored tilings of length n in Theorem 16.8 is* P_{n+1}.

Proof. Suppose we assign the weight 1 to every square and domino. Then the weight of every colored tiling is 1. So the sum of the weights of all colored tilings of length n equals the number of tilings of length n. So, by Cases 1, 2, and 3 in Theorem 16.8, we have:

Number of colored tilings of length n beginning with a white square	=	P_n
Number of colored tilings of length n beginning with a black square	=	P_n
Number of colored tilings of length n beginning with a domino	=	P_{n-1}

Therefore, the total number of colored tilings of length n equals $2P_n + P_{n-1} = P_{n+1}$. ■

For example, there are $5 = P_3$ colored tilings of length 2, and $12 = P_4$ colored tilings of length 3; see Figure 16.7.

Finally, we present a circular tiling model for the Pell–Lucas numbers Q_n.

16.8 Circular Tilings and Pell–Lucas Numbers

Consider a circular board of n cells (in lieu of a linear board of length n), often called a *bracelet*. Suppose the cells are ordered 1 through n in the counterclockwise direction. We would like to tile it. (Although squares and dominoes are *not* circular in reality, we will be using the same terminology for lack of a better one.) Every square has weight 2, and every domino 1. But the weight of the initial domino is 2.

Figure 16.8 shows the circular tilings of length n and the sum of their weights, where $1 \le n \le 4$.

It appears from the figure that the sum of the weights of the circular tilings of length n is $2Q_n$. The following theorem confirms this observation.

Theorem 16.9 *The sum of the weights of the circular tilings of length n is* $2Q_n$*, where the weight of a square is* 2 *and that of a domino is* 1*, except that the weight of the initial domino is 2.*

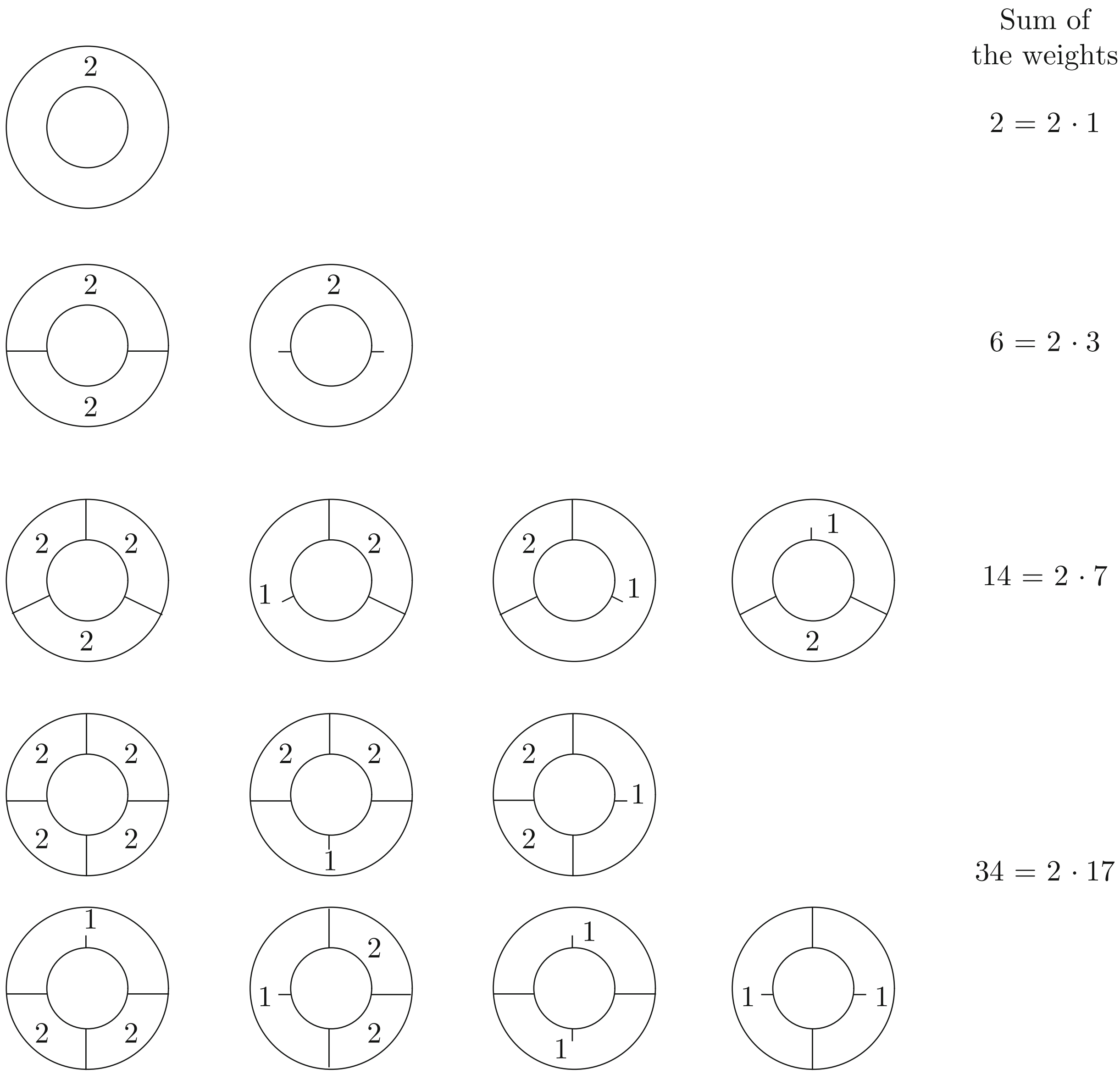


Figure 16.8.

Proof. Let S_n denote the sum of the weights of the circular tilings of length n. Then $S_1 = 2 = 2Q_1$ and $S_2 = 6 = 2Q_2$.

Consider an arbitrary circular tiling of length $n \geq 3$.

Case 1 Suppose the tiling begins with a square. It has weight 2. Deleting this square yields a circular tiling of length $n - 1$. Since the sum of the weights of tilings of length $n - 1$ is S_{n-1}, it follows that the sum of the weights of tilings of length n that begins with a square is $2S_{n-1}$.

Case 2 Suppose the tiling begins with a domino. Since its weight is 1, it follows as in Case 1 that the sum of the weights of such tilings of length n is $1 \cdot S_{n-2} = S_{n-2}$.

So, by the addition principle, $S_n = 2S_{n-1} + S_{n-2}$, where $n \geq 3$. Since S_n satisfies the Pell recurrence with $S_1 = 2 = 2Q_1$ and $S_2 = 6 = 2Q_2$, if follows that $S_n = 2Q_n$. ∎

This theorem has an interesting consequence. It can be used to develop the explicit formula (9.18) for Q_n, as the next theorem shows. The gist of its proof lies in counting the number of circular tilings with exactly k dominoes for every possible value of $k \geq 0$.

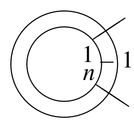
Theorem 16.10

$$Q_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k}\binom{n-k}{k} 2^{n-2k-1}.$$

Proof. Consider a circular tiling of length n. By Theorem 16.9, the sum of the weights of circular tilings of length n is $2Q_n$.

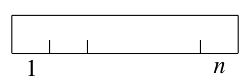
Suppose the tiling contains exactly k dominoes. So it contains $n - 2k$ square tiles.

Case 1 Suppose a domino occupies cells n and 1:



There are $(n-2k)+k-1 = n-k-1$ tiles covering cells 2 through $n-1$. So the remaining $k-1$ dominoes can be placed in $\binom{n-k-1}{k-1}$ different ways; that is, there are $\binom{n-k-1}{k-1}$ bracelets with a domino occupying cells n and 1.

Case 2 Suppose a domino does *not* occupy cells n and 1. Then the circular board can be considered a linear board of length n, containing exactly k dominoes:



Since it contains $n - 2k$ squares and k dominoes, it takes a total of $(n - 2k) + k = n - k$ tiles. So the k dominoes can be placed in $\binom{n-k}{k}$ different ways; that is, there are exactly $\binom{n-k}{k}$ bracelets without a domino in cells n and 1.

By Cases 1 and 2, there are $\binom{n-k-1}{k-1} + \binom{n-k}{k} = \frac{n}{n-k}\binom{n-k}{k}$ bracelets, each containing exactly k dominoes. Each such tiling contains $n - 2k$ squares. Since each square has weight 2 and each domino 1, the weight of such a tiling is $2^{n-2k} \cdot 1^k = 2^{n-2k}$. So the sum of the weights of circular tilings with exactly k domonies is $\frac{n}{n-k}\binom{n-k}{k}2^{n-2k}$. Consequently, the sum of the weights of circular tilings of length n is $\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k}\binom{n-k}{k}2^{n-2k}$.

But the cumulative sum of the weights is $2Q_n$; so

$$\begin{aligned} 2Q_n &= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k}\binom{n-k}{k} 2^{n-2k} \\ Q_n &= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k}\binom{n-k}{k} 2^{n-2k-1}, \end{aligned}$$

as desired. ∎

For example, consider the tilings of length $n = 4$ in Figure 16.8. There is $\frac{4}{4-0}\binom{4-0}{0} = 1$ tiling with 0 dominoes; its weight is $2^4 = 16$. There are $\frac{4}{4-1}\binom{4-1}{1} = 4$ tilings, each with exactly 1 domino; each tiling has weight 2^2; so the sum of the weights of the tilings with exactly 1 domino is $4 \cdot 2^2 = 16$. Finally, there are $\frac{4}{4-2}\binom{4-2}{2} = 2$ tilings, with exactly 2 dominoes each; each tiling has weight 1; so the sum of the weights of the tilings with exactly 2 dominoes is $2 \cdot 1 = 2$. Thus the cumulative sum of the weights is $16 + 16 + 2 = 34 = 2Q_4 = \sum_{k=0}^{2} \frac{4}{4-k}\binom{4-k}{k}2^{4-2k}$, as expected.

Theorem 16.10 can be used to compute the number of circular tilings of length n, as the following corollary shows.

Corollary 16.2 *The number of circular tilings of length n is L_n.*

Proof. Assign a weight 1 to every square in Theorem 16.10. Then the sum of the weights of the circular tilings of length n equals the number of circular tilings. From the proof of the theorem, it follows that the number of circular tilings of length n is $\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k}\binom{n-k}{k}$.

Recall from Chapter 10 that this sum is L_n. Thus the number of circular tilings of length n is L_n. ∎

For example, consider Figure 16.8. There are $4 = L_3$ circular tilings of three cells, and $7 = L_4$ tilings of 4 cells.

Next we will construct combinatorial models for the Pell polynomial family, by extending the ones we developed thus far for Pell and Pell–Lucas numbers. This is achieved by assigning suitable weights for the tiles.

16.9 Combinatorial Models for the Pell Polynomial $p_n(x)$

Suppose we would like to tile a $1 \times n$ linear board of n cells with square tiles and dominoes. We assign a weight of $2x$ to each square and 1 to each domino. As before, the weight of a tiling is the product of the weights of the tiles. The weight of the empty tiling is again defined as 1.

Figure 16.9 shows the resulting tilings of a $1 \times n$ board, the corresponding weights, and the sum of the weights of tilings of length n, where $0 \le n \le 5$.

The sum of the weights of the tilings of length n seems to be the Pell polynomial $p_{n+1}(x)$. The next theorem confirms this observation. Since its proof is nearly identical to that of Theorem 16.2, we omit it.

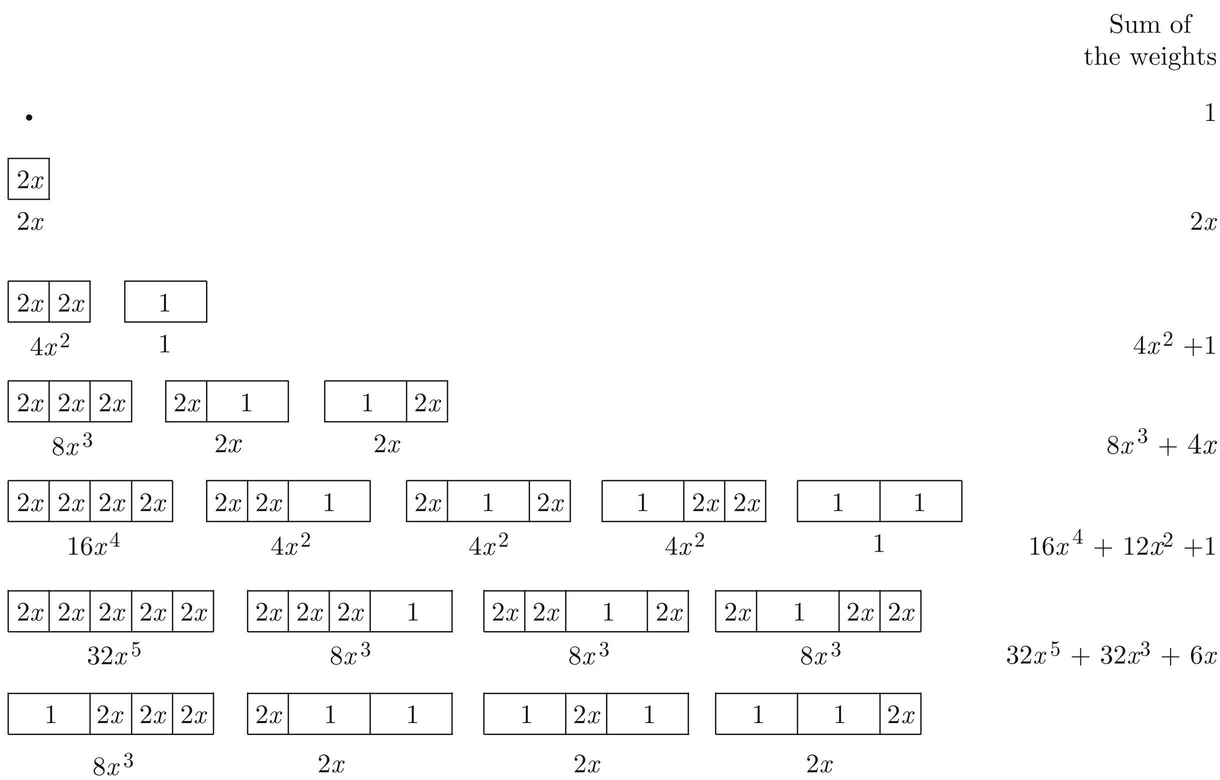


Figure 16.9.

Theorem 16.11 *The sum of the weights of the tilings of a* $1 \times n$ *board with square tiles and dominoes is the Pell polynomial* $p_{n+1}(x)$, *where the weight of a square is* $2x$ *and that of a domino is* 1, *where* $n \geq 0$. ■

Suppose a tiling has exactly k dominoes, where $k \geq 0$. Then it has $n - 2k$ squares and a weight of $(2x)^{n-2k} \cdot 1^k = (2x)^{n-2k}$. Since there are exactly $\binom{n-k}{k}$ such tilings, it follows that the sum of the weights of tilings of length n is $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}(2x)^{n-2k}$. This, coupled with Theorem 16.11, yields the explicit formula (14.24) for $p_{n+1}(x)$.

Theorem 16.12 *Let* $n \geq 0$. *Then*

$$p_{n+1}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}(2x)^{n-2k}.$$

■

As in the case of Theorem 16.14, the concept of breakability can be invoked to derive the addition formula for Pell polynomials. Since the reasoning is quite similar, again we omit its proof.

Theorem 16.13 *Let* $m, n \geq 0$. *Then* $p_{m+n}(x) = p_m(x)p_{n+1}(x) + p_{m-1}(x)p_n(x)$. ■

For the next combinatorial model for the Pell polynomial family, we turn to colored tilings.

16.10 Colored Tilings and Pell Polynomials

Suppose square tiles are available in two colors, black and white. We assign every square a weight x and every domino 1. Figure 16.10 shows such colored tilings of length n, where $0 \leq n \leq 4$.

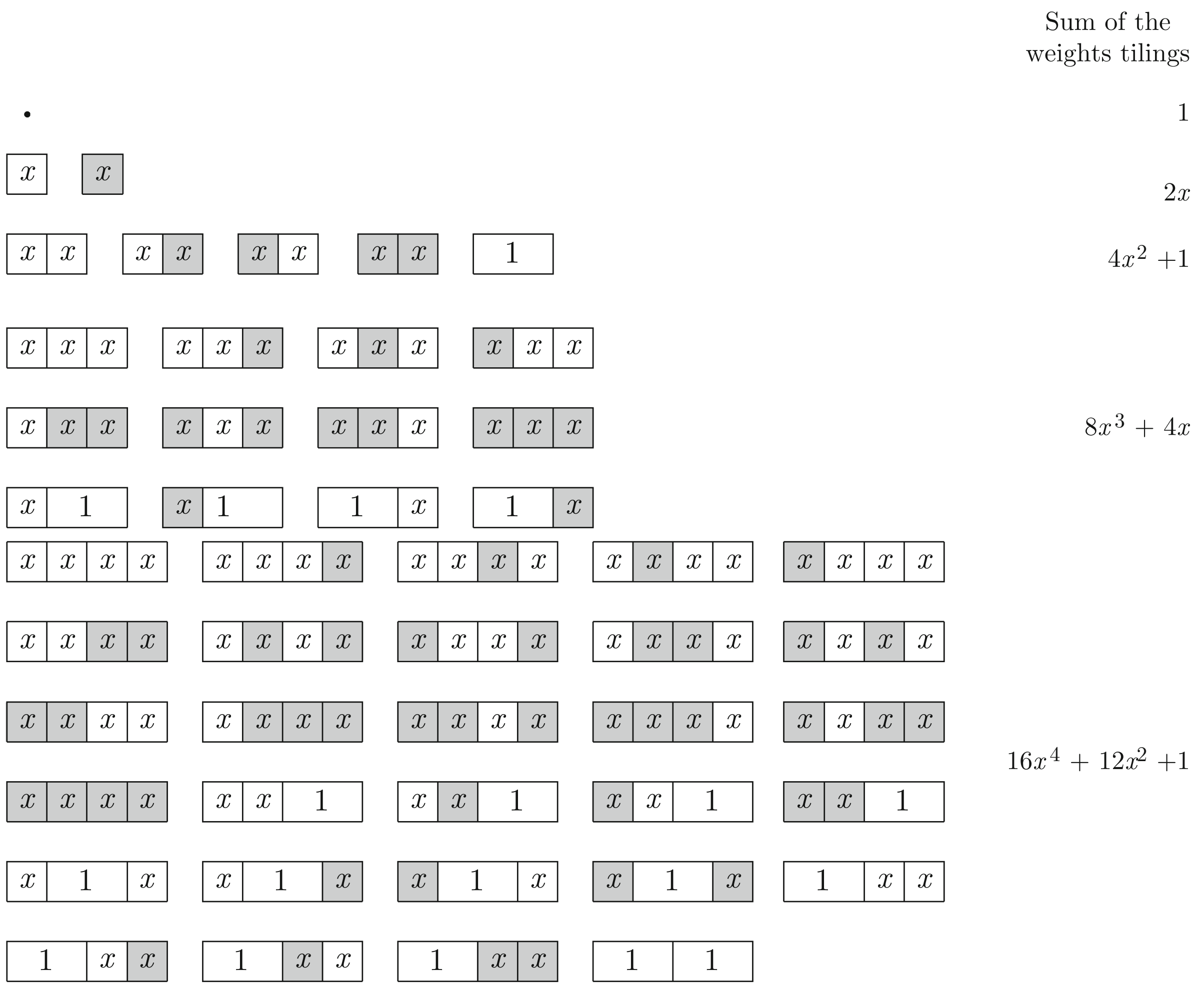


Figure 16.10.

Reasoning as in Theorem 16.5, we now establish that the sum of the weights of such colored tilings of length n is $p_{n+1}(x)$. Because of the close similarity of the proofs, we will give only the essence of its proof.

Theorem 16.14 *The sum of the weights of colored tilings of length n is $p_{n+1}(x)$, where the weight of a square tile is x and that of a domino is 1, and $n \geq 0$.*

Proof. Let $S_n(x)$ denote the sum of the weights of colored tilings of length n. Then, from Figure 16.10, $S_0(x) = 1 = p_1(x)$ and $S_1(x) = 2x = p_2(x)$.

Consider an arbitrary colored tiling of length $n \geq 3$.

Case 1 Suppose it ends in a square tile. Since it can be black or white, the sum of the weights of such tilings is $2xS_{n-1}(x)$.

Case 2 Suppose it ends in a domino. The sum of such tilings is $S_{n-2}(x)$.

Thus, by Cases 1 and 2, $S_n(x) = 2xS_{n-1}(x) + S_{n-2}(x)$, where $S_0(x) = 1 = p_1(x)$, $S_1(x) = 2x = p_2(x)$, and $n \geq 3$. So $S_n(x) = p_{n+1}(x)$, as desired. ■

Next we pursue combinatorial interpretations of Pell–Lucas polynomial $q_n(x)$.

16.11 Combinatorial Models for Pell–Lucas Polynomials

For the first such model, we return to the tilings in Theorem 16.11, where the weight of every square was $2x$. We now make one exception: If a tiling begins with a square tile, it is assigned a weight of x. The weight of a domino remains 1.

Figure 16.11 shows such tilings of length n, where $0 \leq n \leq 5$.

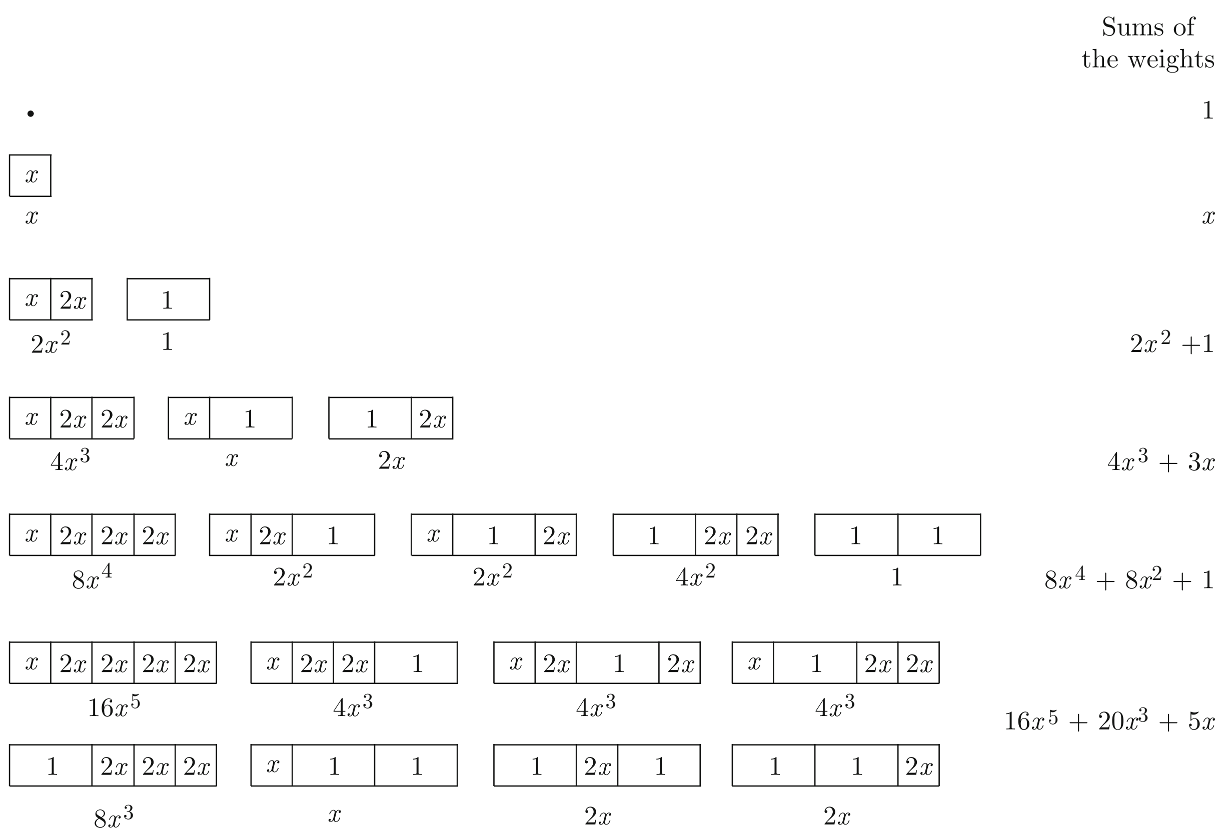


Figure 16.11.

Using the reasoning in the proof of Theorem 16.6, we can establish the following result.

Theorem 16.15 *Suppose the weight of a square is $2x$ and that of a domino is 1, except that if the tiling begins with a square tile, its weight is x. Then the sum of the weights of the tilings of a $1 \times n$ board is $\frac{1}{2}q_n(x)$, where $n \geq 0$.* ■

In particular, let $x = 1$. Then the sum of the weights is $\frac{1}{2}q_n(1) = Q_n$, as we found in Theorem 16.6.

Using the concept of breakability and the reasoning in the proof of Theorem 16.7, we can prove the addition formula (14.18) for Pell–Lucas polynomials.

Theorem 16.16 *Let $m, n \geq 0$. Then $q_{m+n}(x) = q_m p_{n-1}(x) + q_{m+1}(x)p_n(x)$.* ■

The next model consists of circular tilings with proper weights for the tiles.

16.12 Bracelets and Pell–Lucas Polynomials

In the circular tilings of bracelets, every square has weight $2x$ and every domino has weight 1. But the initial domino has weight 2. So does the empty tiling. Figure 16.12 shows such circular tilings of a circular board with n cells and the sum of their weights, where $0 \leq n \leq 4$.

The following theorem shows that the sum of the weights of the circular tilings of n cells with the assigned weights is $q_n(x)$. Its proof employs the same reasoning as in Theorem 16.9; so we again omit the proof in the interest of brevity.

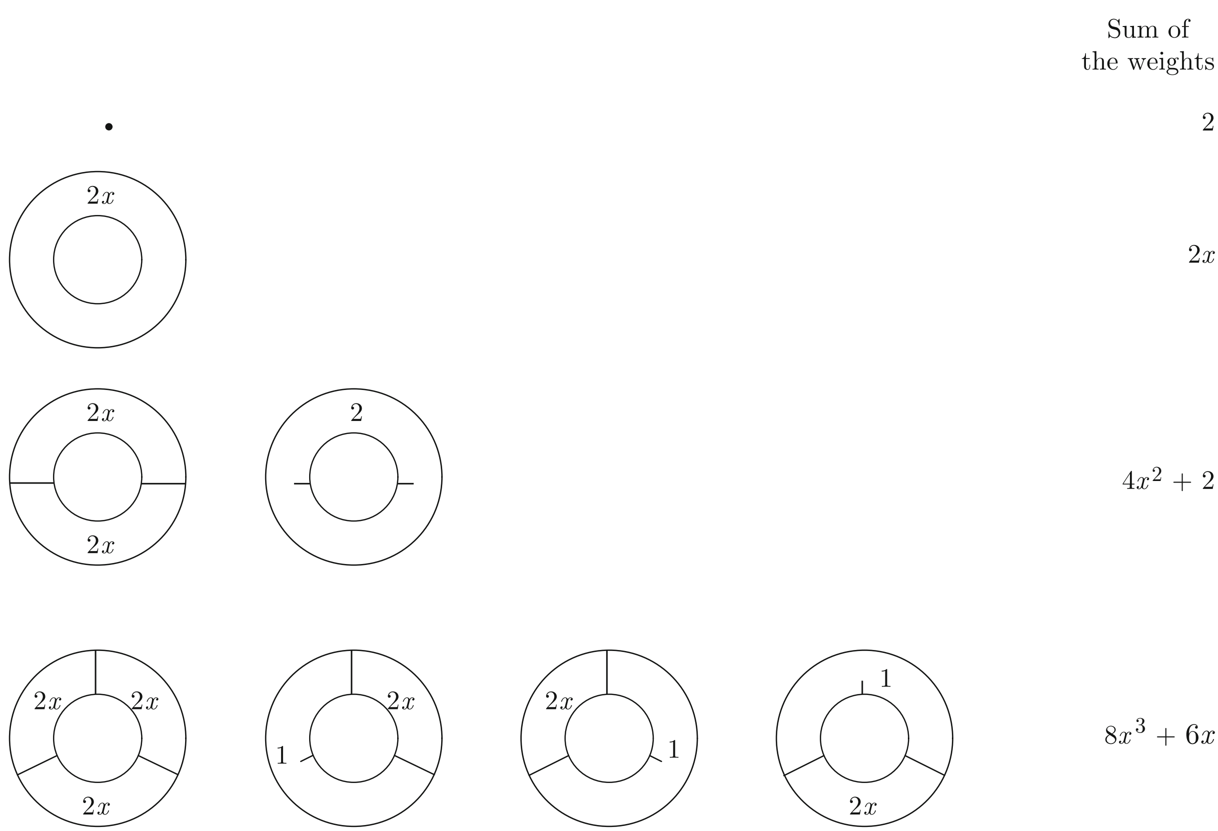


Figure 16.12. (continued)

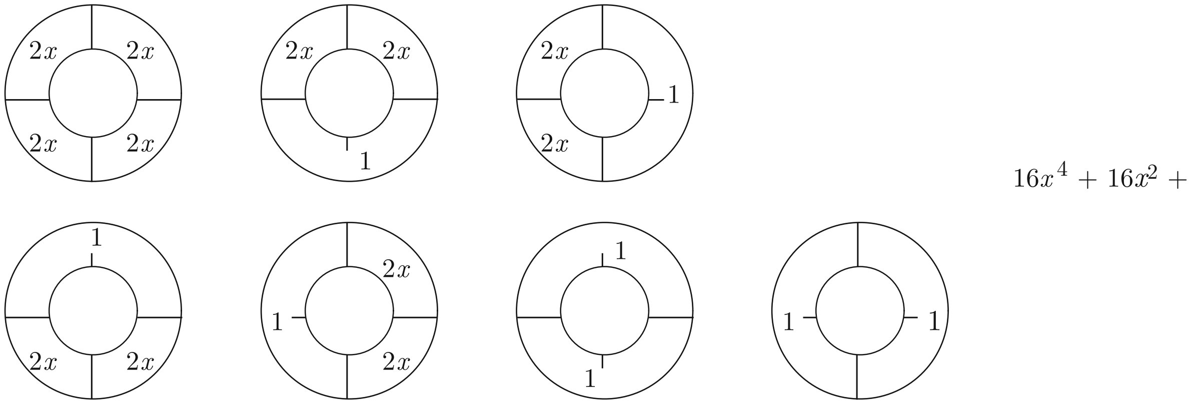


Figure 16.12.

Theorem 16.17 *The sum of the weights of the circular tilings of* n *cells is* $q_n(x)$*, where the weight of every square is* $2x$ *and that of a domino is* 1*, except that the initial domino has weight* 2.

Finally, this theorem can be employed to develop the explicit formula for $q_n(x)$ in Chapter 14. Its proof follows exactly the same argument as in Theorem 16.10. Although it would be a good exercise to derive it, we omit its proof also for the sake of brevity.

Theorem 16.18 *Let* $n \geq 0$. *Then* $q_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k}\binom{n-k}{k}(2x)^{n-2k}$. ■

Clearly, the explicit formula for Q_n in Theorem 16.10 follows from this result.

17

Pell–Fibonacci Hybridities

17.1 Introduction

The Pell and Fibonacci families coexist in perfect harmony, and share a number of charming properties. In this chapter we will study a number of their bridging relationships.

17.2 A Fibonacci Upper bound

First, we present an upper bound for P_n in terms of a suitable Fibonacci number. It was discovered by Seiffert in 1995 [205]. The proof, an interesting application of PMI, is based on the one given by P.S. Bruckman of Highwood, Illinois, in the following year [32].

Example 17.1 Prove that $P_n < F_{\lfloor(11n+2)/6\rfloor}$, where $n \geq 4$.

Proof. (Notice that the inequality does *not* hold if $n < 4$. For example, $P_3 = 5 = F_5 = F_{\lfloor(11 \cdot 3+2)/6\rfloor}$. But $P_4 = 12 < 13 = F_{\lfloor(11 \cdot 4+2)/6\rfloor}$.)

Since $\gamma = 1 + \sqrt{2}$, $\gamma^6 = 70\gamma + 29$; so $\gamma^{12} = 13860\gamma + 5741 = 198 \cdot 70\gamma + 5741 = 198(\gamma^6 - 29) + 5741 = 198\gamma^6 - 1$. Consequently, we have

$$\gamma^{n+12} = 198\gamma^{n+6} - \gamma^n \quad (17.1)$$

$$\delta^{n+12} = 198\delta^{n+6} - \delta^n. \quad (17.2)$$

It follows by equations (17.1) and (17.2) that $P_{n+12} = 198P_{n+6} - P_n$ for every $n \geq 1$. Likewise, it can be shown that $F_{n+22} = 199F_{n+11} + F_n$.

Next we make a very useful observation: $\left\lfloor \frac{11(n+6)+2}{6} \right\rfloor = \left\lfloor \frac{11n+2}{6} \right\rfloor + 11$.

Let S denote the set of integers ≥ 4 for which the inequality holds. It follows from Table 17.1 that it is true for $4 \leq n \leq 15$.

T. Koshy, *Pell and Pell–Lucas Numbers with Applications*, DOI 10.1007/978-1-4614-8489-9_17,

Table 17.1.

n	4	5	6	7	8	9	10	11	12	13	14	15
P_n	7	9	11	13	15	16	18	20	22	24	26	27
$\lfloor \frac{11(n+6)+2}{6} \rfloor$	12	29	70	169	408	985	2378	5741	13860	33461	80782	195025
$F_{\lfloor \frac{11(n+6)+2}{6} \rfloor}$	13	34	89	233	610	987	2584	6765	17711	46368	121393	196418

Suppose the inequality holds for an arbitrary integer $n \geq 4$ and $n+6$; that is, $P_n < F_{\lfloor(11n+2)/6\rfloor}$ and $P_{n+6} < F_{\lfloor[11(n+6)+2)/6]\rfloor} = F_{\lfloor(11n+2)/6\rfloor+11}$. Then

$$\begin{aligned} P_{n+12} &= 198P_{n+6} - P_n \\ &< 198P_{n+6} \\ &< 198F_{\lfloor(11n+2)/6\rfloor+11} \\ &< 199F_{\lfloor(11n+2)/6\rfloor+11} + F_{\lfloor(11n+2)/6\rfloor} \\ &< F_{\lfloor(11n+2)/6\rfloor+22} \\ &= F_{\lfloor[11(n+12)+2]\rfloor/6}. \end{aligned}$$

Thus, if $n, n+6 \in S$, then $n+12 \in S$. So, by PMI, the inequality holds for $n \geq 4$. ■

The next example also deals with an inequality linking Fibonacci and Pell numbers. It was proposed as a problem by M.J. DeLeon of Florida Atlantic University, Boca Raton, Florida [61]. The proof presented here is based on the one by David Zeitlin of Minneapolis, Minnesota [268].

Example 17.2 Prove that $P_{6n} < F_{11n}$, where $n \geq 1$.

Proof. We will now establish the inequality in six steps:

(a) Let r be a solution of the equation $x^2 = x + 1$. Then it follows by PMI that $r^m = rF_m + F_{m-1}$ for every integer $m \geq 1$ [126].

(b) Consequently, we have

$$\begin{aligned} r^{22} - 199r^{11} - 1 &= (rF_{22} + F_{21}) - 199(rF_{11} + F_{10}) - 1 \\ &= (F_{22} - 199F_{11})r + (F_{21} - 199F_{10} - 1) \\ &= (17711 - 199 \cdot 89)r + (10946 - 199 \cdot 55 - 1) \\ &= 0 \cdot r + 0 = 0. \end{aligned}$$

(c) Let s be a solution of the equation $x^2 = 2x + 1$. Then, it can be confirmed by PMI that $s^m = sP_m + P_{m-1}$, where $m \geq 1$. Consequently,

$$\begin{aligned} s^{12} - 198s^6 + 1 &= (sP_{12} + P_{11}) - 198(sP_6 + P_5) + 1 \\ &= (P_{12} - 198P_6)s + (P_{11} - 198P_5 + 1) \end{aligned}$$

$$\begin{aligned} &= (13860 - 198 \cdot 70)s + (5741 - 198 \cdot 29 + 1) \\ &= 0 \cdot s + 0 = 0. \end{aligned}$$

(d) Let $y_n = F_{11n}$. By Binet's formula for F_k, $y_n = \frac{\alpha^{11n} - \beta^{11n}}{\alpha - \beta}$. So,

$$\begin{aligned} (\alpha - \beta)(y_{n+2} - 199y_{n+1} - y_n) &= \left[\alpha^{11(n+2)} - \beta^{11(n+2)}\right] - 199\left[\alpha^{11(n+1)} - \beta^{11(n+1)}\right] \\ &\quad - \left[\alpha^{11n} - \beta^{11n)}\right] \\ &= \left(\alpha^{11n+22} - 199\alpha^{11n+11} - \alpha^{11n}\right) \\ &\quad - \left(\beta^{11n+22} - 199\beta^{11n+11} - \beta^{11n}\right) \\ &= \alpha^{11n}(\alpha^{22} - 199\alpha^{11} - 1) - \beta^{11n}(\beta^{22} - 199\beta^{11} - 1) \\ &= \alpha^{11n} \cdot 0 - \beta^{11n} \cdot 0 = 0. \end{aligned}$$

Since $\alpha \neq \beta$, this implies that $y_{n+2} - 199y_{n+1} - y_n = 0$.

(e) Next, we let $z_n = P_{6n} = \frac{\gamma^{6n} - \delta^{6n}}{\gamma - \delta}$. Then

$$\begin{aligned} (\gamma - \delta)(z_{n+2} - 198z_{n+1} + z_n) &= \left[\gamma^{6(n+2)} - \delta^{6(n+2)}\right] - 198\left[\gamma^{6(n+1)} + \delta^{6(n+1)}\right] \\ &\quad + \left[\gamma^{6n} - \delta^{6n)}\right] \\ &= \gamma^{6n}(\gamma^{12} - 198\gamma^{6} + 1) - \delta^{6n}(\delta^{12} - 198\delta^{6} + 1) \\ &= \gamma^{6n} \cdot 0 - \delta^{6n} \cdot 0 = 0. \end{aligned}$$

This implies that $z_{n+2} - 198z_{n+1} + z_n = 0$.

(f) Finally, we let $w_n = z_n - y_n = P_{6n} - F_{11n}$. Then

$$\begin{aligned} w_{n+2} &= z_{n+2} - y_{n+2} \\ &= 198(z_{n+1} - y_{n+1}) - y_{n+1} - y_n - z_n \\ w_{n+2} - 198w_{n+1} &= -y_{n+1} - y_n - z_n \\ &< 0. \end{aligned}$$

That is, $w_n < 198w_{n-1}$, where $n \geq 2$. This implies, by PMI, that $w_n < 198^{n-1}w_1$ for every $n \geq 2$.

But $w_1 = P_6 - F_{11} = 70 - 89 < 0$. Consequently, $w_n < 0$ for every $n \geq 1$. That is, $P_{6n} < F_{11n}$ for every integer $n \geq 1$, as desired. ■

The next example is a somewhat related problem, also proposed by DeLeon in the same year [62]. The featured argument is based on the one by P. Mana and W. Vucenic of the University of New Mexico, Albuquerque, New Mexico [164].

Example 17.3 Prove or disprove that $F_{11n} < P_{6n+1}$, where $n \geq 1$.

Solution. We will disprove this inequality. To this end, notice that $|\alpha| > |\beta|$ and $|\gamma| > |\delta|$. So

$$\begin{aligned}\lim_{n\to\infty}\frac{F_{11n}}{P_{6n+1}} &= \lim_{n\to\infty}\frac{\alpha^{11n}-\beta^{11n}}{\alpha-\beta}\cdot\frac{\gamma-\delta}{\gamma^{6n+1}-\delta^{6n+1}}\\ &= \frac{1-\delta/\gamma}{\alpha-\beta}\cdot\lim_{n\to\infty}\left(\frac{\alpha^{11}}{\gamma^6}\right)^n,\end{aligned}$$

where $\frac{1-\delta/\gamma}{\alpha-\beta}$ is a positive constant. But $\alpha^{11} = F_{11}\alpha + F_{10} = \frac{F_{11}(1+\sqrt{5})+2F_{10}}{2} = \frac{L_{11}+F_{11}\sqrt{5}}{2} = \frac{199+89\sqrt{5}}{2} \approx 199.0050$ and $\gamma^6 = P_6\gamma + P_5 = 70\gamma + 29 = 70(1+\sqrt{2}) + 29 = 99 + 70\sqrt{2} \approx 197.9949$. So $\alpha^{11} > \gamma^6$. Consequently, $\lim_{n\to\infty}\left(\frac{\alpha^{11}}{\gamma^6}\right)^n$ is infinite. Thus, when n is sufficiently large, $F_{11n} > P_{6n+1}$.

As a concrete counterexample, it has been found, with the aid of a computer, that when $n = 128$, $F_{11n} > 8 \times 10^{293} > P_{6n+1}$. ∎

17.3 Cook's Inequality

The next bridge is a cubic inequality linking Fibonacci, Lucas, and Pell numbers, studied by C.K. Cook of Sumter, South Carolina, in 2009, [51,52]

$$F_n^3 + L_n^3 + P_n^3 + 3F_nL_nP_n > 2(F_n + L_n)^2P_n, \tag{17.3}$$

where $n > 3$. The inequality fails when $n = 1, 2,$ or 3; however, it is true when $n = 0$. But it works when $n = 4$: $3^3 + 7^3 + 12^3 + 3\cdot 3\cdot 7\cdot 12 = 2854 > 2400 = 2\cdot(3+7)^2\cdot 12$.

Proof. First, we will prove by PMI that $P_n > 2L_n + F_n$, where $n \geq 5$. Clearly, this is the case when $n = 5$ and $n = 6$. Assume that is true for all integers $\leq n$, where $n \geq 6$. Then

$$\begin{aligned}P_{n+1} &= 2P_n + P_{n-1}\\ &> 2(2L_n + F_n) + (2L_{n-1} + F_{n-1})\\ &= 2(2L_n + L_{n-1}) + (2F_n + F_{n-1})\\ &= 2(L_n + L_{n+1}) + (F_n + F_{n+1})\\ &> 2L_{n+1} + F_{n+1}.\end{aligned}$$

Thus, by the strong version of PMI, $P_n > 2L_n + F_n$, for every $n \geq 5$.

Assume $n \geq 5$ for the rest of the proof. Since $P_n + 2L_n > L_n$ and $P_n - 2L_n > F_n$, it follows that $P_n^2 - 4L_n^2 > F_nL_n$; so $P_n^2 - F_nL_n > 4L_n^2$ and hence

$$P_n(P_n^2 - F_nL_n) > 4L_n^2P_n. \tag{17.4}$$

Since $L_nP_n > 0$, $L_nP_n + L_nF_n > L_nF_n$. But $P_n > F_n$ and $P_n > L_n$. So

$$\begin{aligned}
L_n(2P_n - L_n) + F_n(2P_n - F_n) &> L_nF_n \\
2(L_n + F_n)P_n &> L_n^2 + F_n^2 + L_nF_n \\
2(L_n + F_n)(L_n - F_n)P_n &> (L_n^2 + F_n^2 + L_nF_n)(L_n - F_n) \\
2(L_n^2 - F_n^2)P_n &> L_n^3 - F_n^3 \\
(2P_n - L_n)L_n^2 &> (2P_n - F_n)F_n^2 \\
2(2P_n - L_n)L_n^2 &> (2P_n - F_n)F_n^2 + (2P_n - L_n)L_n^2 \\
4P_nL_n^2 &> (2P_n - F_n)F_n^2 + (2P_n - L_n)L_n^2.
\end{aligned}$$

Using (17.4), this implies that

$$P_n(P_n^2 - F_nL_n) > 2P_n(F_n^2 + L_n^2) - F_n^3 - L_n^3.$$

This yields the desired inequality. ■

Interestingly, Cook's inequality also works for Pell–Lucas numbers:

$$F_n^3 + L_n^3 + Q_n^3 + 3F_nL_nQ_n > 2(F_n + L_n)^2Q_n, \tag{17.5}$$

where $n > 3$. Its proof follows the above argument, with P_n replaced with Q_n.

For example, $F_4^3 + L_4^3 + Q_4^3 + 3F_4L_4Q_4 = 6,354 > 3,400 = 2(F_4 + L_4)^2Q_4$ and $F_5^3 + L_5^3 + Q_5^3 + 3F_5L_5Q_5 = 77,142 > 20,992 = 2(F_5 + L_5)^2Q_5$.

The next two congruences were also discovered by Seiffert in 1994 [201].

Example 17.4 Prove that $P_{3n-1} \equiv F_{n+2} \pmod{13}$ and $P_{3n+1} \equiv (-1)^{\lfloor (n+1)/2 \rfloor} F_{4n-1} \pmod{7}$, where $n \ge 1$.

Proof. (1) First, notice that the sequence $\{P_n \pmod{13}\}$ is periodic with period 28:

$$\underbrace{1\ 2\ 5\ 12\ 3\ 5\ 0\ 5\ 10\ 12\ 8\ 2\ 12\ 0\ 12\ 11\ 8\ 1\ 10\ 8\ 0\ 8\ 3\ 1\ 5\ 11\ 1\ 0}\ \underbrace{1\ 2\ 5\ 12\cdots 1\ 0}\ \cdots$$

The sequence $\{P_{3n-1} \pmod{13}\}$ is also periodic with period 28:

$$\underbrace{2\ 3\ 5\ 8\ 0\ 8\ 8\ 3\ 11\ 1\ 12\ 0\ 12\ 12\ 11\ 10\ 8\ 5\ 0\ 5\ 5\ 10\ 2\ 12\ 1\ 0\ 1\ 1}\ \underbrace{2\ 3\ 5\cdots 1\ 1}\ \cdots$$

On the other hand, $\{F_{n+2} \pmod{13}\}$ is exactly the same periodic sequence with period 28:

$$\underbrace{2\ 3\ 5\ 8\ 0\ 8\ 8\ 3\ 11\ 1\ 12\ 0\ 12\ 12\ 11\ 10\ 8\ 5\ 0\ 5\ 5\ 10\ 2\ 12\ 1\ 0\ 1\ 1}\ \underbrace{2\ 3\ 5\cdots 1\ 1}\ \cdots$$

Since both sequences have exactly the same repeating cycle, it follows that $P_{3n-1} \equiv F_{n+2} \pmod{13}$.

(2) The sequence $\{P_{3n+1} \pmod 7\}$ is periodic with period 2: $\underbrace{5\ 1}\ \underbrace{5\ 1}\ldots$, whereas the sequence $\{F_{4n-1} \pmod 7\}$ is periodic with period 4: $\underbrace{2\ 6\ 5\ 1}\underbrace{2\ 6\ 5\ 1}\ldots$. So $\{(-1)^{\lfloor (n+1)/2\rfloor} F_{4n-1} \pmod 7\}$ is periodic with the repeating cycle $\underbrace{-2\ -6\ 5\ 1}$; but this is the same as $\underbrace{5\ 1}\ \underbrace{5\ 1}$ modulo 7. So the sequence $\{F_{4n-1} \pmod 7\}$ is also periodic with the same repeating cycle as $\{P_{3n+1} \pmod 7\}$. Thus $P_{3n+1} \equiv (-1)^{\lfloor (n+1)/2\rfloor} F_{4n-1} \pmod 7$, as desired. ■

Seiffert also found that $P_{6n-4} \equiv (-1)^{\lfloor (n-1)/2\rfloor} F_{5n+21} \pmod{11}$, where $n \geq 1$.

The next congruence, also studied by Seiffert, appeared in 1995 [204]. The proof here is based on the one given by L. Somer of the Catholic University of America, Washington, D.C., in 1996 [235]. We omit a few details for the sake of brevity.

Example 17.5 Prove that $\dfrac{F_{kn}}{F_k} \equiv \dfrac{P_{kn}}{P_k} \pmod{Q_k - L_k}$, where $n \geq 0$ and $k \geq 1$.

Proof. Since $F_k | F_{kn}$ and $P_k | P_{kn}$ (see Theorem 8.2), both $\frac{F_{kn}}{F_k}$ and $\frac{P_{kn}}{P_k}$ are integers.

We will prove a more general result and then deduce the desired result from it. To this end, consider the sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$, defined recursively as follows:

$$\begin{aligned} A_{n+2} &= aA_{n+1} - bA_n, \quad A_0 = 0, \quad A_1 = 1; \\ B_{n+2} &= cB_{n+1} - bB_n, \quad B_0 = 0, \quad B_1 = 1, \end{aligned}$$

where a, b, and c are nonzero integers. Assume that $A_n B_n \neq 0$ for every $n \geq 0$.

Let $\{C_n\}$ be a sequence satisfying the same recurrence as A_n, but with the initial conditions $C_0 = 2$ and $C_1 = a$. Let $\{D_n\}$ be a sequence satisfying the same recurrence as B_n, but with the initial conditions $D_0 = 2$ and $D_1 = c$. Then both $\frac{A_{kn}}{A_k}$ and $\frac{B_{kn}}{B_k}$ are integers. We will now establish by PMI that

$$\frac{A_{kn}}{A_k} \equiv \frac{B_{kn}}{B_k} \pmod{C_k - D_k} \tag{17.6}$$

for $n \geq 0$.

The sequences $\left\{\frac{A_{kn}}{A_k}\right\}$ and $\left\{\frac{B_{kn}}{B_k}\right\}$ satisfy the recurrences

$$x_{n+2} = C_k x_{n+1} - b^k x_k \tag{17.7}$$

and

$$y_{n+2} = D_k y_{n+1} - b^k y_k \tag{17.8}$$

respectively [151].

To establish congruence (17.6):
Clearly, $\frac{A_{k\cdot 0}}{A_k} = 0 \equiv \frac{B_{k\cdot 0}}{B_k} \pmod{C_k - D_k}$ and $\frac{A_{k\cdot 1}}{A_k} = 1 \equiv \frac{B_{k\cdot 1}}{B_k} \pmod{C_k - D_k}$. So (17.6) holds when $n = 0$ and $n = 1$.

Now assume it is true for every nonnegative integer $< n$. Then, by (17.7), (17.8), and the inductive hypothesis, we have

$$\begin{aligned}
\frac{A_{k(n+1)}}{A_k} &= \frac{C_k A_{kn} - b^k A_{k(n-1)}}{A_k}\\
\frac{B_{k(n+1)}}{B_k} &= \frac{D_k B_{kn} - b^k B_{k(n-1)}}{B_k}\\
&\equiv D_k \cdot \frac{A_{kn}}{A_k} - b^k \cdot \frac{A_{k(n-1)}}{A_k} \equiv \frac{C_k A_{kn} - b^k A_{k(n-1)}}{A_k} \pmod{C_k - D_k}\\
&\equiv \frac{A_{k(n+1)}}{A_k} \pmod{C_k - D_k},
\end{aligned}$$

where we have used the fact that $C_k \equiv D_k \pmod{C_k - D_k}$.

Thus, by the strong version of PMI, congruence (17.6) holds for every $n \ge 0$.

In particular, let $a = 2$ and $b = -1 = -c$. Then $A_{n+2} = 2A_{n+1} + A_n$ and $B_{n+2} = 2B_{n+1} + B_n$; so $A_n = P_n$ and $B_n = F_n$. Thus, the desired result follows. ■

17.4 Pell–Fibonacci Congruences

The next example presents a congruence linking Pell and Fibonacci families. It was found by Seiffert in 2007 [222]. The solution is based on the one by P.S. Bruckman [35].

Example 17.6 Let r, s, and n be arbitrary positive integers, and $m = (P_r, F_s, P_{r-1} - F_{s-1})$, where (a, b, c) denotes the gcd of the integers a, b, and c.

(1) Prove that $F_n P_{n+r} \equiv P_n F_{n+s} \pmod{m}$.
(2) Show that $F_n P_{n+8} \equiv P_n F_{n+18} \pmod{68}$.
(3) Find integers r and s such that $F_n P_{n+r} \equiv P_n F_{n+s} \pmod{13}$.

Solution.

(1) To establish the congruence, we will need the following addition formulas:

$$\begin{aligned}
P_{n+r} &= P_{n+1} P_r + P_n P_{r-1}\\
F_{n+s} &= F_{n+1} F_s + F_n F_{s-1}.
\end{aligned}$$

We then have

$$\begin{aligned}
F_n P_{n+r} - P_n F_{n+s} &= F_n (P_{n+1} P_r + P_n P_{r-1}) - P_n (F_{n+1} F_s + F_n F_{s-1})\\
&= F_n P_{n+1} P_r - P_n F_{n+1} F_s + F_n P_n (P_{r-1} - F_{s-1}).
\end{aligned}$$

Since $m|P_r, m|F_s$, and $m|(P_{r-1} - F_{s-1})$, it follows that $m|$RHS. So $m|$LHS. This yields the desired congruence.

(2) Choose $r = 8$, and $s = 18$. Then $m = (P_8, F_{18}, P_7 - F_{17}) = (408, 2584, 169-1597) = 68$. So $F_n P_{n+8} \equiv P_n F_{n+18} \pmod{68}$, by part (1).

(3) Clearly, $m = 13 = (P_r, F_s, P_{r-1} - F_{s-1})$ for some positive integers r and s. Now $13|P_r$ if and only if $7|r$; this follows by the addition formula and PMI. We have $13 = F_7$, and $F_7|F_s$ if and only if $7|s$. So $13|F_s$ if and only if $7|s$.

Let $r = 7$. Suppose we choose $s = 7$. Then $P_6 - F_6 = 70 - 6 = 62$; but $13 \nmid 62$. So $s \neq 7$. Likewise, $s \neq 14$. Now try $s = 21$. Then $P_6 - F_{20} = 70 - 6765 = -6695 = 13(-515)$; so $13|(P_6 - F_{20})$. Thus, $r = 7$ and $s = 21$ works by part (1). ■

The next congruence also links Pell and Fibonacci numbers.

Example 17.7 Prove that

$$F_{4n+m} P_{4n+m} \equiv F_m P_m + 3n F_m P_{m+2} + 6n F_{m+2} P_m \pmod{9}, \tag{17.9}$$

where m is an arbitrary integer.

Proof. Since $\alpha^4 = 3\alpha^2 - 1$ and $\beta^4 = 3\beta^2 - 1$, it follows by Binet's formula and the binomial theorem that

$$\begin{aligned}
\sqrt{5} F_{4n+m} &= \alpha^{4n+m} - \beta^{4n+m} \\
&= \alpha^m (3\alpha^2 - 1)^n - \beta^m (3\beta^2 - 1)^n \\
&= \alpha^m \sum_{r=0}^{n} (-1)^{n-r} 3^r \binom{n}{r} \alpha^{2r} - \beta^m \sum_{r=0}^{n} (-1)^{n-r} 3^r \binom{n}{r} \beta^{2r} \\
&= \sum_{r=0}^{n} (-1)^{n-r} 3^r \binom{n}{r} \left(\alpha^{2r+m} - \beta^{2r+m}\right) \\
F_{4n+m} &= \sum_{r=0}^{n} (-1)^{n-r} 3^r \binom{n}{r} F_{2r+m} \\
&\equiv (-1)^n \left(F_m - 3n F_{m+2}\right) \pmod{9}.
\end{aligned}$$

Similarly, since $\gamma^4 = 6\gamma^2 - 1$ and $\delta^4 = 6\delta^2 - 1$, we have

$$\begin{aligned}
2\sqrt{2} P_{4n+m} &= \sum_{r=0}^{n} (-1)^{n-r} 6^r \binom{n}{r} \left(\gamma^{2r+m} - \delta^{2r+m}\right) \\
P_{4n+m} &= \sum_{r=0}^{n} (-1)^{n-r} 6^r \binom{n}{r} P_{2r+m} \\
&\equiv (-1)^n \left(P_m - 6n P_{m+2}\right) \pmod{9}.
\end{aligned}$$

So

$$\begin{aligned} F_{4n+m} P_{4n+m} &\equiv (F_m - 3nF_{m+2})(P_m - 6nP_{m+2}) \pmod 9 \\ &\equiv F_m P_m + 3nF_m P_{m+2} + 6nF_{m+2} P_m \pmod 9, \end{aligned}$$

as claimed. ∎

For example, let $n = 2$ and $m = 3$. Then

$$\begin{aligned} F_3 P_3 + 6F_3 P_5 + 12F_5 P_3 &= 2 \cdot 5 + 6 \cdot 2 \cdot 29 + 12 \cdot 5 \cdot 5 \\ &\equiv 1 \equiv 89 \cdot 5741 \pmod 9 \\ &\equiv F_{11} P_{11} \pmod 9. \end{aligned}$$

Congruence (17.9) has four interesting byproducts. When $m = 0, 1, 2$, and 3, it yields the following congruences, where $W_n = F_n P_n$:

$$\begin{aligned} W_{4n} &\equiv 0 \pmod 9 & W_{4n+1} &\equiv 0 \pmod 9 \\ W_{4n+2} &\equiv 2 \pmod 9 & W_{4n+3} &\equiv 1 \pmod 9. \end{aligned}$$

Corresponding to (17.9), we have a Pell–Lucas congruence:

$$L_{4n+m} P_{4n+m} \equiv L_m P_m + 3nL_m P_{m+2} + 6nL_{m+2} P_m \pmod 9. \tag{17.10}$$

This was found by C. Georghiou of the University of Patras, Greece, in 1991 [93]. Its proof follows the same argument as before, so we omit it.

For example, let $n = 2$ and $m = 3$. Then

$$\begin{aligned} L_3 P_3 + 6L_3 P_5 + 12L_5 P_3 &= 4 \cdot 5 + 6 \cdot 4 \cdot 29 + 12 \cdot 11 \cdot 5 \\ &\equiv 8 \equiv 199 \cdot 5741 \pmod 9 \\ &\equiv L_{11} P_{11} \pmod 9. \end{aligned}$$

Congruence (17.9) also has four interesting byproducts; they correspond to $m = 0, 1, 2$, and 3, where $X_n = L_n P_n$:

$$\begin{aligned} X_{4n} &\equiv 3n \pmod 9 & X_{4n+1} &\equiv 3n + 1 \pmod 9 \\ X_{4n+2} &\equiv 3n + 6 \pmod 9 & X_{4n+3} &\equiv 3n + 2 \pmod 9. \end{aligned}$$

These were found by Seiffert in 1990 [196].

Congruences (17.9) and (17.10) have their counterparts for Pell–Lucas numbers. Since $Q_{4n+m} \equiv (-1)^n (Q_m + 3nQ_{m+2}) \pmod 9$, we have

$$\begin{aligned} F_{4n+m} Q_{4n+m} &\equiv (F_m - 3nF_{m+2})(Q_m + 3nQ_{m+2}) \pmod 9 \\ &\equiv F_m Q_m + 3nF_m Q_{m+2} + 6nF_{m+2} Q_m \pmod 9 \end{aligned} \tag{17.11}$$

$$\begin{aligned} L_{4n+m}Q_{4n+m} &\equiv (L_m - 3nL_{m+2})(Q_m + 3nQ_{m+2}) \pmod 9 \\ &\equiv L_mQ_m + 3nL_mQ_{m+2} + 6nL_{m+2}Q_m \pmod 9. \end{aligned} \tag{17.12}$$

Congruences (17.11) and (17.12) yield the following special cases, where $Y_n = F_nQ_n$ and $Z_n = L_nQ_n$:

$$\begin{aligned} Y_{4n} &\equiv 6n \pmod 9 & Y_{4n+1} &\equiv 6n+1 \pmod 9 \\ Y_{4n+2} &\equiv 6n+3 \pmod 9 & Y_{4n+3} &\equiv 6n+5 \pmod 9 \\ Z_{4n} &\equiv 2 \pmod 9 & Z_{4n+1} &\equiv 1 \pmod 9 \\ Z_{4n+2} &\equiv 0 \pmod 9 & Z_{4n+3} &\equiv 1 \pmod 9). \end{aligned}$$

17.4.1 A Generalization

Let $\{A_n\}$ and $\{B_n\}$ be two integer sequences satisfying the Fibonacci and Pell recurrences, respectively. Then, it follows by congruences (17.9)–(17.12) that

$$A_{4n+m}B_{4n+m} \equiv A_mB_m + 3nA_mB_{m+2} + 6nA_{m+2}B_m \pmod 9. \tag{17.13}$$

17.5 Israel's Congruence

The Pell–Fibonacci congruence

$$P_n \equiv (-1)^n \left[(18n^2 + 21n + 2)F_n + 12F_{n+1}\right] \pmod{27} \tag{17.14}$$

was discovered by R.B. Israel of the University of British Columbia, Canada, in 1991, where $n \geq 0$ [207]. We will now prove this by showing that $R_n = (-1)^n \left[(18n^2 + 21n + 2)F_n + 12F_{n+1}\right]$ satisfies the Pell recursive definition modulo 27. The proof is a bit tedious, and depends on the Fibonacci recurrence and modular arithmetic.

Proof. Since $P_0 = 0 = F_0$, the congruence is clearly true when $n = 0$. Since $-(18 + 21 + 2) \cdot 1 + 12 \cdot 1 = -26 \equiv 1 \pmod{27}$, it also true when $n = 1$.

We will now show that R_n satisfies the Pell recurrence. Since

$$\begin{aligned} 2R_{n+1} + R_n &= 2(-1)^{n+1}\left\{\left[18(n+1)^2 + 21(n+1) + 2\right]F_{n+1} + 12(n+1)F_{n+2}\right\} + \\ &\quad (-1)^n\left[(18n^2 + 21n + 2)F_n + 12F_{n+1}\right]. \end{aligned}$$

Then

$$\begin{aligned} (-1)^n[2R_{n+1} + R_n] &= -[36(n^2 + 2n + 1) + 42(n+1) + 4]F_{n+1} - 24(n+1)F_{n+2} + \\ &\quad (18n^2 + 21n + 2)F_n + 12nF_{n+1} \\ &= (-36F_{n+1} + 18F_n)n^2 - (72 + 42)nF_{n+1} - 24nF_{n+2} + 21nF_n + \\ &\quad 12nF_{n+1} + 26F_{n+1} - 24F_{n+2} + 2F_n \end{aligned}$$

$$\begin{aligned}
&\equiv 18n^2(F_{n+1} + F_n) + 21n(F_{n+1} + F_n) - 24nF_{n+2} + 12nF_{n+1} + \\
&\quad 26F_{n+1} + 3F_{n+2} + 2F_n \pmod{27} \\
&\equiv 18n^2F_{n+2} + 21nF_{n+2} - 24nF_{n+2} + 12nF_{n+1} + 26F_{n+1} + \\
&\quad 30F_{n+2} + 2F_n \pmod{27} \\
&\equiv 18n^2F_{n+2} + 24nF_{n+2} + 12nF_{n+1} + 24(F_{n+1} + F_{n+2}) + \\
&\quad 6F_{n+2} + 2(F_{n+1} + F_n) \pmod{27} \\
&\equiv 18n^2F_{n+2} + 12nF_{n+2} + 12n(F_{n+2} + F_{n+1}) + 24F_{n+3} \\
&\quad +8F_{n+2} \pmod{27} \\
&\equiv 18n^2F_{n+2} + 12nF_{n+2} + 12nF_{n+3} + 24F_{n+3} + 8F_{n+2} \pmod{27} \\
&\equiv (18n^2 + 12n + 8)F_{n+2} + 12(n+2)F_{n+3} \pmod{27} \\
&\equiv \left[18(n+2)^2 + 21(n+2) + 2)F_{n+2} + 12(n+2)F_{n+3}\right] \pmod{27} \\
&\equiv (-1)^{n+2}R_{n+2} \pmod{27}.
\end{aligned}$$

Thus R_n satisfies the same recursive definition as P_n, modulo 27. This establishes congruence (17.14). ■

For example, let $n = 5$. Then

$$\begin{aligned}
\text{RHS} &\equiv (-1)^5[(18\cdot 5^2 + 21\cdot 5 + 2)\cdot 5 + 12\cdot 5\cdot 8] \pmod{27} \\
&\equiv 2 \equiv 29 \pmod{27} \\
&\equiv P_{29} \pmod{27}.
\end{aligned}$$

17.6 Seiffert's Congruence

Using congruence (17.14), we will prove the following congruence, discovered by Seiffert in 1989 [195]:

$$6(n+1)P_{n-1} + P_{n+1} \equiv (-1)^{n+1}(9n^2 - 7)F_{n+1} \pmod{27}. \tag{17.15}$$

Proof. By congruence (17.14), we have

$$\begin{aligned}
(-1)^{n+1}(\text{LHS}) &\equiv 6(n+1)\left\{[18(n-1)^2 + 21(n-1) + 2]F_{n-1} + 12(n-1)F_n\right\} + \\
&\quad [18(n+1)^2 + 21(n+1) + 2]F_{n+1} + 12(n+1)F_{n+2} \pmod{27} \\
&\equiv [0 + 18(n^2-1) + 12(n+1)]F_{n-1} + 18(n^2-1)F_n + \\
&\quad [18(n+1)^2 + 21(n+1) + 2]F_{n+1} + 12(n+1)F_{n+2} \pmod{27}
\end{aligned}$$

$$
\begin{aligned}
&\equiv 18n^2(F_{n-1}+F_n+F_{n+1})+(12F_{n-1}+3F_{n+1}+12F_{n+2})n-\\
&\quad 6F_{n-1}-18F_n+14F_{n+1}+12F_{n+2} \pmod{27}\\
&\equiv 36n^2F_{n+1}+[12F_{n-1}+3F_{n+1}+12(F_{n+1}+F_n)]n-\\
&\quad 6(F_{n-1}+F_n)+14F_{n+1}+12(F_{n+2}-F_n) \pmod{27}\\
&\equiv 9n^2F_{n+1}+[12F_{n-1}+15(F_n+F_{n-1})+12F_n]n-\\
&\quad 6F_{n+1}+14F_{n+1}+12F_{n+1} \pmod{27}\\
&\equiv 9n^2F_{n+1}+0+20F_{n+1} \pmod{27}\\
&\equiv (9n^2-7)F_{n+1} \pmod{27}\\
\text{LHS} &\equiv (-1)^{n+1}(9n^2-7)F_{n+1} \pmod{27}, \quad \text{as desired.}
\end{aligned}
$$

■

For example, let $n = 10$. Then

$$\text{LHS} = 6\cdot 11\cdot P_9 + P_{11} = 66\cdot 985 + 5741 \equiv 11 \equiv (-1)^{11}(9\cdot 10^2-7)F_{11} \pmod{27}.$$

17.6.1 Israel's and Seiffert's Congruences Revisited

Since Lucas and Fibonacci numbers satisfy the same recurrence, it follows from the proof of congruence (17.14) that $(-1)^n\left[(18n^2+21n+2)L_n+12nL_{n+1}\right] \pmod{27}$ also satisfies the Pell recurrence. When $n=0$, this yields $4\equiv 4Q_0 \pmod{27}$. So the corresponding congruence for Pell–Lucas numbers is

$$4Q_n \equiv (-1)^n\left[(18n^2+21n+2)L_n+12nL_{n+1}\right] \pmod{27}.$$

For example, $(-1)^5\left[(18\cdot 5^2+21\cdot 5+2)L_5+12\cdot 5L_6\right]\equiv 4Q_5 \pmod{27}$.

This implies that Seiffert's congruence (17.15) also has a counterpart for Pell–Lucas numbers. It is obtained by replacing F_k with L_k, and P_k with $4Q_k$:

$$24(n+1)Q_{n-1}+4Q_{n+1} \equiv (-1)^{n+1}(9n^2-7)L_{n+1} \pmod{27}.$$

That is,

$$4Q_{n+1}-3(n+1)Q_{n-1} \equiv (-1)^{n+1}(9n^2-7)L_{n+1} \pmod{27}.$$

For example, $4Q_6-3\cdot 6Q_4\equiv 9\equiv (-1)^6(9\cdot 5^2-7)L_6 \pmod{27}$.

17.7 Pell–Lucas Congruences

The next example presents a Pell–Lucas congruence, found by Seiffert [188]. The solution presented here is based on the one by L.A.G. Dresel of the University of Reading, England [75].

Example 17.8 Prove that $P_{n+3} + P_{n+1} + P_n \equiv 3(-1)^n L_n \pmod 9$.

Proof. First, notice that

$$\begin{aligned} P_{n+3} + P_{n+1} + P_n &= (2P_{n+2} + P_{n+1}) + P_{n+1} + P_n \\ &= 2P_{n+2} + (2P_{n+1} + P_n) \\ &= 3P_{n+2}. \end{aligned}$$

Next we will show that $K_n = (-1)^n L_n$ satisfies the same recurrence as P_{n+2} modulo 3. To this end, recall that $L_n = L_{n-1} + L_{n-2}$; so $(-1)^n L_n = (-1)^n L_{n-1} + (-1)^n L_{n-2}$; that is, $K_n = -K_{n-1} + K_{n-2} \equiv 2K_{n-1} + K_{n-2} \pmod 3$. Since $P_n = 2P_{n-1} + P_{n-2} \pmod 3$, both P_n and K_n satisfy the same recurrence modulo 3. But $K_1 = -L_1 = -1 \equiv 2 \equiv P_3 \pmod 3$ and $K_2 = L_2 = 3 \equiv 12 \equiv P_4 \pmod 3$. So $P_{n+2} \equiv K_n \pmod 3$ for every integer $n \geq 1$. Consequently, $3P_{n+2} \equiv 3K_n \pmod 9$; this yields the desired congruence. ∎

For example, we have

$$\begin{aligned} P_{10} + P_8 + P_7 &= 2378 + 408 + 169 \\ &= 2995 \equiv 3 \pmod 9 \\ &\equiv -87 \equiv -3 \cdot 29 \pmod 9 \\ &\equiv 3(-1)^7 L_7 \pmod 9. \end{aligned}$$

Our next example investigates five congruences linking Lucas and Pell–Lucas numbers. They were originally studied by S. Rabinowitz of Westford, Massachusetts, in 1998 [180]. Their proofs are based on the ones given by D.M. Bloom of Brooklyn College, New York, in 1999 [22].

Example 17.9 Let n be any nonnegative integer.

(1) Prove that $2Q_{7n} \equiv L_n \pmod{159}$.
(2) Find an integer $m \geq 2$ such that $2Q_{11n} \equiv L_n \pmod m$.
(3) Find an integer a such that $2Q_{an} \equiv L_n \pmod{31}$.
(4) Find an integer $m \geq 2$ such that $2Q_{19n} \equiv L_n \pmod m$.
(5) Show that there is *no* integer a such that $2Q_{an} \equiv L_n \pmod 7$.

Proof. The proofs and solutions hinge on the identity

$$Q_{n+k} + (-1)^k Q_{n-k} = 2Q_n Q_k. \tag{17.16}$$

where $k \geq 0$. It can be established by PMI.

Using this identity, we will now prove the following result by PMI:

$$\text{If } a \text{ is odd and } 2Q_a \equiv 1 \pmod m, \text{ then } 2Q_{an} \equiv L_n \pmod m \text{ for every } n. \tag{17.17}$$

Clearly, the congruence is true when $n = 0$ and $n = 1$. Assume it is true for nonnegative integers $n \le j$, where $j \ge 1$. Then, by identity (17.16) and the inductive hypothesis, we have

$$\begin{aligned} 2Q_{a(j+1)} &= 2Q_{a(j-1)} + (2Q_{aj})(2Q_a) \\ &\equiv L_{j-1} + L_j \cdot 1 \equiv L_{j+1} \pmod{m}. \end{aligned}$$

So the congruence holds when $n = j + 1$. Thus, by the strong version of PMI, it holds for every $n \ge 0$.

We can now deduce all results from property (17.17):

(1) Let $a = 7$ and $m = 159$. Then $2Q_7 = 2 \cdot 239 = 478$; so $2Q_7 - 1 = 477 = 3 \cdot 159$. Consequently, $2Q_{7n} \equiv L_n \pmod{159}$.

(2) We need to choose an integer m such that $2Q_{11} \equiv 1 \pmod{m}$. Since $2Q_{11} = 2 \cdot 8119 = 2 \cdot 23 \cdot 353$, the smallest integer $m \ge 2$ that works is $m = 13$: $2Q_{11n} \equiv L_n \pmod{13}$.

(3) First, we will choose an odd integer a such that $2Q_a \equiv 1 \pmod{31}$. By trial and error, $a = 17$ works: $2Q_{17} = 2 \cdot 1607521 \equiv 1 \pmod{31}$. So, by congruence (17.17), $2Q_{17n} \equiv L_n \pmod{31}$.

(4) When $n = 1$, we must have $2Q_{19} \equiv 1 \pmod{m}$. Since $2Q_{19} = 2 \cdot 9369319 = 18,738,638$, we can choose $m = 2Q_{19} - 1 = 18,738,637$.

(5) Suppose there is an integer a such that $2Q_{an} \equiv L_n \pmod{7}$ for every n. In particular, $2Q_n \equiv L_1 \equiv 1 \pmod{7}$. But this is impossible, since the sequence $\{2Q_n \pmod{7}\}_{n=0}^{\infty}$ follows the pattern $\underbrace{2\,2\,6\,0\,6\,5}\ \underbrace{2\,2\,6\,0\,6\,5}\cdots$ and the repeating cycle does *not* contain the residue 1. ■

17.8 Seiffert's Pell–Lucas Congruences

The next two congruences link Pell and Lucas numbers modulo 5. It follows from Table 17.2 that the sequence $\{P_n \pmod{5}\}$ is periodic with period 12, whereas $\{L_n \pmod{5}\}$ is periodic with period 4. Since $[12,4] = 12$, it suffices to consider the first 12 terms in each sequence, where $[a, b]$ denotes the least common multiple (lcm) of a and b. Since $P_1 \equiv L_1 \pmod{5}$, $P_4 \equiv L_4 \pmod{5}$, $P_7 \equiv L_7 \pmod{5}$, and $P_{10} \equiv L_{10} \pmod{5}$ (see Table 17.2), it follows that $P_{3n+1} \equiv L_{3n+1} \pmod{5}$. Likewise, $P_{3n+2} \equiv 4L_{3n+2} \pmod{5}$. These two congruences were found by Seiffert in 1992 [198].

Table 17.2.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$P_n \pmod{5}$	0	(1)	2	0	(2)	4	0	(4)	3	0	(3)	1	0	1
$L_n \pmod{5}$	2	(1)	3	4	(2)	1	3	(4)	2	1	(3)	4	2	1

Table 17.3.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$P_n \pmod 5$	0	1	2	0	2	4	0	4	3	0	3	1	0	1	2
$F_n \pmod 5$	0	1	1	2	3	0	3	3	1	4	0	4	4	3	2
n	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
$P_n \pmod 5$	0	2	4	0	4	3	0	3	1	0	1	2	0	2	4
$F_n \pmod 5$	0	2	2	4	1	0	1	1	2	3	0	3	3	1	4
n	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44
$P_n \pmod 5$	0	4	3	0	3	1	0	1	2	0	2	4	0	4	3
$F_n \pmod 5$	0	4	4	3	2	0	2	2	4	1	0	1	1	2	3
n	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59
$P_n \pmod 5$	0	3	1	0	1	2	0	2	4	0	4	3	0	3	1
$F_n \pmod 5$	0	3	3	1	4	0	4	4	3	2	0	2	2	4	1

We can use a similar technique to extract congruences linking Pell and Fibonacci numbers modulo 5. It follows from Table 17.3 that $\{F_n \pmod 5\}$ is periodic with period 20. So it suffices to study only the first $60 = [12, 20]$ terms of the sequences $\{P_n \pmod 5\}$ and $\{F_n \pmod 5\}$ to extract such relationships. It follows from the table that

$$P_{15n+r} \equiv F_{15n+r} \pmod 5 \qquad P_{15n+4r} \equiv 4F_{15n+4r} \pmod 5$$
$$2P_{15n+2r} \equiv 4F_{15n+2r} \pmod 5 \qquad 2P_{15n+7r} \equiv 4F_{15n+7r} \pmod 5.$$

where $r = -1$, 0 or 1. These congruences repeat every 15 entries since $P_{n+15} \equiv 2P_n \pmod 5$ and $F_{n+15} \equiv 2F_n \pmod 5$; see Table 17.3.

17.9 Hybrid Sums

The next bridge deals with the sum $\sum_{k=0}^{n} A_k P_k$, where $\{A_k\}$ is an integer sequence satisfying the Fibonacci recurrence. We will prove by PMI that

$$3\sum_{k=0}^{n} A_k P_k = A_n P_{n+1} + A_{n+1} P_n - \mu, \tag{17.18}$$

where $\mu = \begin{cases} 0 & \text{if } A_k = F_k \\ 2 & \text{if } A_k = L_k. \end{cases}$

Proof. Suppose $n = 0$.

Case 1. Let $A_k = F_k$. Then LHS = 0 = $1 \cdot 1 + A_1 \cdot 0 - 0$ = RHS.

Case 2. Let $A_k = L_k$. Then LHS = 0 = $2 \cdot 1 + A_1 \cdot 0 - 2$ = RHS.

Thus, the formula works when $n = 0$.

Suppose it is true for an arbitrary nonnegative integer n. Then

$$\begin{aligned}3\sum_{k=0}^{n+1} A_k P_k &= 3\sum_{k=0}^{n} A_k P_k + 3A_{n+1}P_{n+1}\\ &= (A_n P_{n+1} + A_{n+1}P_n - \mu) + 3A_{n+1}P_{n+1}\\ &= A_{n+1}(2P_{n+1} + P_n) + (A_{n+1} + A_n) - \mu\\ &= A_{n+1}P_{n+2} + A_{n+2}P_{n+1} - \mu.\end{aligned}$$

So it also works for $n+1$.

Thus, by PMI, it is true for every $n \geq 0$. ■

For example, let $n = 4$. Then

$$3\sum_{k=0}^{4} F_k P_k = 3(0\cdot 0 + 1\cdot 1 + 1\cdot 2 + 2\cdot 5 + 3\cdot 12) = 147 = 3\cdot 29 + 5\cdot 12 = F_4P_5 + F_5P_4 - 0.$$

Likewise, $3\sum_{k=0}^{4} L_k P_k = 333 = L_4P_5 + L_5P_4 - 2$.

It follows from formula (17.18) that

$$A_n P_{n+1} + A_{n+1}P_n \equiv \mu \pmod 3. \tag{17.19}$$

Thus $F_nP_{n+1} + F_{n+1}P_n \equiv 0 \pmod 3$ and $L_nP_{n+1} + L_{n+1}P_n \equiv 2 \pmod 3$. We will employ these two congruences a bit later.

Interestingly, (17.18) has a companion formula for Pell–Lucas numbers:

$$3\sum_{k=0}^{n} A_k Q_k = A_n Q_{n+1} + A_{n+1}Q_n - \nu, \tag{17.20}$$

where $\nu = \begin{cases} 1 & \text{if } A_k = F_k \\ -3 & \text{if } A_k = L_k. \end{cases}$ Its proof also follows by PMI.

For example, let $n = 5$. Then

$$3\sum_{k=0}^{5} F_k Q_k = 3(0\cdot 1+1\cdot 1+1\cdot 3+2\cdot 7+3\cdot 17+5\cdot 41) = 822 = 5\cdot 99+8\cdot 41-1 = F_5Q_6+F_6Q_5-1.$$

Similarly, $3\sum_{k=0}^{5} L_k Q_k = 1830 = L_5Q_6 + L_6Q_5 + 3$.

It follows from formula (17.20) that $F_nQ_{n+1} + F_{n+1}Q_n \equiv 1 \pmod 3$ and $L_nQ_{n+1} + L_{n+1}Q_n \equiv 0 \pmod 3$. These two congruences also will come in handy shortly.

17.9.1 Weighted Hybrid Sums

Next we derive a summation formula for the weighted sum $\sum_{k=0}^{n} kA_kP_k$, where $\{A_k\}$ is again an integer sequence satisfying Fibonacci recurrence. To this end, first notice that

$$\begin{aligned}
A_{n+3}P_{n+3} &= (A_{n+2}+A_{n+1})(2P_{n+2}+P_{n+1})\\
&= 2A_{n+2}P_{n+2}+A_{n+2}P_{n+1}+2A_{n+1}P_{n+2}+A_{n+1}P_{n+1}\\
&= A_{n+2}P_{n+2}+A_{n+2}(P_{n+2}+P_{n+1})+2A_{n+1}P_{n+2}+A_{n+1}P_{n+1}\\
&= A_{n+2}P_{n+2}+A_{n+2}(3P_{n+1}+P_n)+2A_{n+1}P_{n+2}+A_{n+1}P_{n+1}\\
&= A_{n+2}P_{n+2}+3A_{n+2}P_{n+1}+A_{n+2}P_n+2A_{n+1}P_{n+2}+A_{n+1}P_{n+1}\\
&= A_{n+2}P_{n+2}+3(A_{n+2}P_{n+1}+A_{n+1}P_{n+2})-A_{n+1}P_{n+2}+\\
&\quad A_{n+2}P_n+A_{n+1}P_{n+1}\\
&= A_{n+2}P_{n+2}+3(A_{n+2}P_{n+1}+A_{n+1}P_{n+2})-A_{n+1}(2P_{n+1}+P_n)+\\
&\quad (A_n+A_{n+1})P_n+A_{n+1}P_{n+1}\\
&= A_{n+2}P_{n+2}+3(A_{n+2}P_{n+1}+A_{n+1}P_{n+2})-A_{n+1}P_{n+1}+A_nP_n.
\end{aligned}$$

Thus

$$A_{n+2}P_{n+2}+3(A_{n+2}P_{n+1}+A_{n+1}P_{n+2})+A_nP_n = A_{n+3}P_{n+3}+A_{n+1}P_{n+1}. \tag{17.21}$$

Using this identity and PMI, we will now prove that

$$9\sum_{k=0}^{n} kA_kP_k = 3(n+1)(A_nP_{n+1}+A_{n+1}P_n)-A_{n+2}P_{n+2}-A_nP_n+\lambda, \tag{17.22}$$

where $\lambda = \begin{cases} 2 & \text{if } A_k = F_k \\ 0 & \text{if } A_k = L_k. \end{cases}$ This formula was developed by Seiffert in 1988 [193].

Proof. Suppose $n = 0$. When $A_k = F_k$, RHS = 0 = LHS. On the other hand, when $A_k = L_k$, RHS = 0 = LHS. So the formula works when $n = 0$.

Now assume it is true for an arbitrary nonnegative integer n. Since

$$\begin{aligned}
A_nP_{n+1}+A_{n+1}P_n+3A_{n+1}P_{n+1} &= (A_nP_{n+1}+A_{n+1}P_{n+1})+A_{n+1}(2P_{n+1}+P_n)\\
&= A_{n+2}P_{n+1}+A_{n+1}P_{n+2},
\end{aligned}$$

we have

$$A_nP_{n+1}+A_{n+1}P_n = A_{n+2}P_{n+1}+A_{n+1}P_{n+2}-3A_{n+1}P_{n+1}.$$

Then, by the inductive hypothesis and formula (17.21), we have

$$\begin{aligned}
9\sum_{k=0}^{n+1} kA_kP_k &= 9\sum_{k=0}^{n} kA_kP_k + 9(n+1)A_{n+1}P_{n+1}\\
&= [3(n+1)(A_nP_{n+1} + A_{n+1}P_n) - A_{n+2}P_{n+2} - A_nP_n + \lambda] +\\
&\quad 9(n+1)A_{n+1}P_{n+1}\\
&= 3(n+1)(A_{n+1}P_{n+2} + A_{n+2}P_{n+1} - 3A_{n+1}P_{n+1}) - A_{n+2}P_{n+2} -\\
&\quad A_nP_n + 9(n+1)A_{n+1}P_{n+1} + \lambda\\
&= 3(n+1)(A_{n+1}P_{n+2} + A_{n+2}P_{n+1}) - A_{n+2}P_{n+2} - A_nP_n + \lambda\\
&= 3(n+2)(A_{n+1}P_{n+2} + A_{n+2}P_{n+1}) -\\
&\quad [3(A_{n+1}P_{n+2} + A_{n+2}P_{n+1}) + A_{n+2}P_{n+2} + A_nP_n] + \lambda\\
&= 3(n+2)(A_{n+1}P_{n+2} + A_{n+2}P_{n+1}) - A_{n+3}P_{n+3} - A_{n+1}P_{n+1}] + \lambda.
\end{aligned}$$

So the formula also works for $n + 1$.

Thus, by PMI, it works for every $n \geq 0$. ∎

For example, let $n = 5$ and $A_k = F_k$. Then

$$\begin{aligned}
\text{LHS} &= 9\sum_{k=0}^{5} kA_kP_k\\
&= 9(1\cdot 1\cdot 1 + 2\cdot 1\cdot 2 + 3\cdot 2\cdot 5 + 4\cdot 3\cdot 12 + 5\cdot 5\cdot 29) = 8,136\\
&= (3\cdot 6)(5\cdot 70 + 8\cdot 29) - 13\cdot 169 - 5\cdot 29 + 2\\
&= \text{RHS}.
\end{aligned}$$

Likewise, $9\sum_{k=0}^{5} kL_kP_k = 18,036 = (3\cdot 6)(L_5P_6 + L_6P_5) - L_7P_7 - L_5P_5$.

17.10 Congruence Byproducts

Formula (17.22) has interesting byproducts. Since $A_nP_{n+1} + A_{n+1}P_n \equiv \mu \pmod 3$ by formula (17.19), it follows by (17.22) that

$$F_{n+2}P_{n+2} + F_nP_n \equiv 2 \pmod 9 \tag{17.23}$$

$$L_{n+2}P_{n+2} + L_nP_n \equiv 6(n+1) \pmod 9. \tag{17.24}$$

Now multiply (17.23) by L_m and (17.24) by F_m, and add the resulting congruences, where m is any integer. Then

$$(F_mL_{n+2} + L_mF_{n+2})P_{n+2} + (F_mL_n + L_mF_n)P_n \equiv 2L_m + 6(n+1)F_m \pmod 9.$$

Using the addition formula [126] $F_m L_k + L_m F_k = 2F_{m+k}$, this becomes

$$2F_{m+n+2}P_{n+2} + 2F_{m+n}P_n \equiv 2L_m + 6(n+1)F_m \pmod 9.$$

Since 2 and 9 are relatively prime, this implies that

$$F_{m+n+2}P_{n+2} + F_{m+n}P_n \equiv 3(n+1)F_m L_m \pmod 9. \tag{17.25}$$

For example, let $m = 3$ and $n = 5$. Then

$$\begin{aligned} \text{LHS} &= 55 \cdot 169 + 21 \cdot 29 \equiv 4 \pmod 9 \\ &\equiv 3 \cdot 6 \cdot 2 + 4 \pmod 9 \\ &\equiv \text{RHS}. \end{aligned}$$

On the other hand, multiplying (17.23) by $5F_m$ and (17.24) by L_m, and then adding the resulting congruences, we get

$$(5F_m F_{n+2} + L_m L_{n+2})P_{n+2} + (5F_m F_n + L_m L_n)P_n \equiv 10F_m + 6(n+1)L_m \pmod 9.$$

Using the addition formula [126] $5F_m F_k + L_m L_k = 2L_{m+k}$, this yields

$$\begin{aligned} 2L_{m+n+2}P_{n+2} + 2L_{m+n}P_n &\equiv 10F_m + 6(n+1)L_m \pmod 9 \\ L_{m+n+2}P_{n+2} + L_{m+n}P_n &\equiv 5F_m + 3(n+1)L_m \pmod 9. \end{aligned} \tag{17.26}$$

For example, when $m = 3$ and $n = 5$,

$$L_{10}P_7 + L_8 P_5 = 123 \cdot 169 + 47 \cdot 29 \equiv 1 \equiv 5 \cdot 2 + 3 \cdot 6 \equiv 5F_3 + 3 \cdot 6L_3 \pmod 9.$$

17.10.1 Special Cases

Congruences (17.25) and (17.26) have intriguing special cases:

(1) Let $m = -n$ in (17.25). Since $F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^n L_n$, it implies that $P_{n+2} \equiv (-1)^{n+1}[3(n+1)F_n - L_n] \pmod 9$.

(2) Letting $m = -(n+1)$ in (17.25), we get $P_{n+2} + P_n \equiv 3(n+1)(-1)^{n+1}F_{n+1} + (-1)^{n+1}L_{n+1} \pmod 9$. That is, $Q_n \equiv (-1)^{n+1}(3nF_n - L_n) \pmod 9$.

(3) Letting $m = -(n+1)$ in (17.26), it yields $2P_{n+1} \equiv (-1)^{n+1}[3(n+1)L_{n+1} - 5F_{n+1}$ $\pmod 9$. That is, $2P_n \equiv (-1)^n(3nL_n - 5F_n) \pmod 9$.

(4) Let $m = -n$ in (17.26). Then it yields $3P_{n+2} + 2P_n \equiv (-1)^n[3(n+1)L_n - 5F_n] \pmod 9$.

17.11 A Counterpart for Pell–Lucas Numbers

Formula (17.22) has a similar-looking counterpart for Pell–Lucas numbers:

$$9\sum_{k=0}^{n} kA_kQ_k = 3(n+1)(A_nQ_{n+1} + A_{n+1}Q_n) - A_{n+2}Q_{n+2} - A_nQ_n + \mu, \tag{17.27}$$

where $\mu = \begin{cases} 0 & \text{if } A_k = F_k \\ 2 & \text{if } A_k = L_k. \end{cases}$ Its proof follows similarly.

For example,

$$\begin{aligned} 9\sum_{k=0}^{5} kF_kQ_k = 11,502 &= (3\cdot 6)(F_5Q_6 + F_6Q_5) - F_7Q_7 - F_5Q_5 + 0 \\ 9\sum_{k=0}^{5} kL_kQ_k = 25,506 &= (3\cdot 6)(L_5Q_6 + L_6Q_5) - L_7Q_7 - L_5Q_5 + 2. \end{aligned}$$

As can be expected, formula (17.27) also has interesting congruence consequences. First, recall from formula (17.20) that $F_nQ_{n+1} + F_{n+1}Q_n \equiv 1 \pmod 3$ and $L_nQ_{n+1} + L_{n+1}Q_n \equiv 0 \pmod 3$. So, it follows from (17.27) that

$$F_{n+2}Q_{n+2} + F_nQ_n \equiv 0 \pmod 3 \tag{17.28}$$

$$L_{n+2}Q_{n+2} + L_nQ_n \equiv 2 \pmod 3. \tag{17.29}$$

As before, using the addition formula for Fibonacci numbers, these two yield:

$$\begin{aligned} 2F_{m+n+2}Q_{n+2} + 2F_{m+n}Q_n &\equiv 0\cdot L_m + 2F_m \pmod 3 \\ F_{m+n+2}Q_{n+2} + 2F_{m+n}Q_n &\equiv F_m \pmod 3. \end{aligned} \tag{17.30}$$

For example, let $m = 3$ and $n = 5$. Then LHS = = $55\cdot 239 + 21\cdot 41 \equiv 2 \equiv F_3 \pmod 3$. It follows from (17.28) and (17.29) that

$$(5F_mF_{n+2} + L_mL_{n+2})Q_{n+2} + (5F_mF_n + L_mL_n)Q_n \equiv 0\cdot 5F_m + 2L_m \pmod 3.$$

Using the addition formula for Lucas numbers, this implies that

$$\begin{aligned} 2L_{m+n+2}Q_{n+2} + 2L_{m+n}Q_n &\equiv 2L_m \pmod 3 \\ L_{m+n+2}Q_{n+2} + L_{m+n}Q_n &\equiv L_m \pmod 3. \end{aligned} \tag{17.31}$$

For example, when $m = 3$ and $n = 5$, LHS = $123\cdot 239 + 47\cdot 41 \equiv 1 \equiv L_3 \pmod 3$.

17.11.1 Special Cases

As before, congruences (17.30) and (17.31) also have interesting special cases:

(1) Letting $m = -n$ in (17.30) yields $Q_{n+2} \equiv (-1)^{n-1} F_n \pmod 3$.
(2) Suppose we let $m = -(n+1)$ in (17.30). Then we get $Q_{n+2} + Q_n \equiv (-1)^n F_{n+1} \pmod 3$. That is,

$$\begin{aligned} 4P_n &\equiv (-1)^{n-1} F_n \pmod 3 \\ P_n &\equiv (-1)^{n-1} F_n \pmod 3. \end{aligned}$$

(3) Let $m = -(n+1)$ in (17.31). Then $Q_{n+2} - Q_n \equiv (-1)^{n+1} L_{n+1} \pmod 3$. This implies that $Q_n \equiv (-1)^{n-1} L_n \pmod 3$.
(4) Letting $m = -n$ in (17.31), we get $3Q_{n+2} + 2Q_n \equiv (-1)^{n-1} L_n \pmod 3$.

The next bridge is a generalization of a Pell–Fibonacci sum, studied by Seiffert in 1986, when he was a student [188, 189]. The proof, based on the one by Bruckman, illustrates a delightful technique in the theory of finite differences [26, 27].

Example 17.10 Let $\{A_n\}$ be an integer sequence satisfying Pell recurrence. Prove that

$$9\sum_{k=1}^{n} A_k F_k = A_{n+2}F_n + A_{n+1}F_{n+2} + A_n F_{n-1} - A_{n-1}F_{n+1} - \lambda, \tag{17.32}$$

where $\lambda = \begin{cases} 0 & \text{if } A_i = P_i \\ 4 & \text{if } A_i = Q_i. \end{cases}$

Proof. Let $R_n = A_{n+2}F_n + A_{n+1}F_{n+2} + A_n F_{n-1} - A_{n-1}F_{n+1}$. Using Pell and Fibonacci recurrences, we can simplify this sum:

$$\begin{aligned} R_n &= (2A_{n+1} + A_n)F_n + A_{n+1}(F_{n+1} + F_n) + A_n(F_{n+1} - F_n) - (A_{n+1} - 2A_n)F_{n+1} \\ &= 3(A_{n+1}F_n + A_n F_{n+1}). \end{aligned}$$

Let $\Delta R_n = R_{n+1} - R_n$. Then

$$\begin{aligned} \Delta R_n &= 3(A_{n+2}F_{n+1} + A_{n+1}F_{n+2}) - 3(A_{n+1}F_n + A_n F_{n+1}) \\ &= 3\left[(2A_{n+1} + A_n)F_{n+1} + A_{n+1}(F_{n+1} + F_n)\right] - 3(A_{n+1}F_n + A_n F_{n+1}) \\ &= 9A_{n+1}F_{n+1}. \end{aligned}$$

Now let S_n denote the sum on the LHS of equation (17.32): $S_n = 9\sum_{k=1}^{n} A_k F_k$. Then

$$\begin{aligned}\Delta S_n &= S_{n+1} - S_n \\ &= 9\left(\sum_{k=1}^{n+1} A_k F_k - \sum_{k=1}^{n} A_k F_k\right) \\ &= 9A_{n+1}F_{n+1}.\end{aligned}$$

Since $\Delta R_n = \Delta S_n$, it follows that $R_n = S_n + C$, where C is a constant independent of n. In particular, $R_1 = S_1 + C$.

Case 1. Let $A_i = P_i$. Since $P_0 = 0 = F_0$ and $P_1 = 1 = F_1$, $S_1 = 9P_1F_1 = 9$ and $R_1 = 9$. Consequently, $C = 0$.

Case 2. Let $A_i = Q_i$. Since $Q_0 = 1, F_0 = 0, Q_1 = 1 = F_1$, $S_1 = 9Q_1F_1 = 9$ and $R_1 = 13$. So $R_1 = S_1 + C$ implies that $C = 4$.
Combining these two cases, we get the desired formula. ■

The corresponding formula involving Lucas numbers is given by

$$9\sum_{k=1}^{n} A_k L_k = A_{n+2}L_n + A_{n+1}L_{n+2} + A_nL_{n-1} - A_{n-1}L_{n+1} - \mu, \tag{17.33}$$

where $\mu = \begin{cases} 6 & \text{if } A_i = P_i \\ 9 & \text{if } A_i = Q_i. \end{cases}$

This can be established using an argument similar to the one in Example 17.1. Using the same notations as before, we have $\Delta R_n = 9A_{n+1}L_{n+1} = \Delta S_n$ and $R_n = S_n + C'$, where C' is a constant independent of n. In particular, $R_1 = S_1 + C'$.

Case 1. Let $A_i = P_i$. Then $S_1 = 9P_1F_1 = 9$ and $R_1 = P_3L_1 + P_2L_3 + P_1L_0 - P_0L_2 = 5\cdot 1 + 2\cdot 4 + 1\cdot 2 - 0 = 15$; so $C' = 6$.

Case 2. Let $A_i = Q_i$. Then $S_1 = 9Q_1F_1 = 9$ and $R_1 = Q_3L_1 + Q_2L_3 + Q_1L_0 - Q_0L_2 = 7\cdot 1 + 3\cdot 4 + 1\cdot 2 - 1\cdot 3 = 18$. So $C' = 9$.

These two cases together give us formula (17.33).

As a byproduct, it follows from formula (17.33) that $A_{n+2}L_n + A_{n+1}L_{n+2} + A_nL_{n-1} - A_{n-1}L_{n+1} \equiv 0 \pmod 3$, where $A_i = P_i$ or Q_i. In particular, $Q_{n+2}L_n + Q_{n+1}L_{n+2} + Q_nL_{n-1} - Q_{n-1}L_{n+1} \equiv 0 \pmod 9$.

For example, let $A_i = P_i$ and $n = 5$. Then

$$\begin{aligned}\text{LHS} &= P_7L_5 + P_6L_7 + P_5L_4 - P_4L_6 \\ &= 169\cdot 11 + 70\cdot 29 + 29\cdot 7 - 12\cdot 18 \\ &= 3876 = 0 \pmod 3.\end{aligned}$$

Likewise, $Q_7L_5 + Q_6L_7 + Q_5L_4 - Q_4L_6 = 239 \cdot 11 + 99 \cdot 29 + 41 \cdot 7 - 17 \cdot 18 = 5481 \pmod 9$.

Next we investigate two infinite sums studied by Seiffert in 1994 [202]. They involve Fibonacci, Lucas, Pell, and Pell–Lucas numbers. Their proofs are a bit long, and are based on the ones given by N. Jensen of Kiel, Germany, in the following year [116].

Example 17.11 Prove that

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{F_{2^n} Q_{2^n}}{2(L_{2^n} P_{2^n})^2 - 5(F_{2^n} Q_{2^n})^2} = \frac{1}{6}$

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{L_{2^n} P_{2^n}}{2(L_{2^n} P_{2^n})^2 - 5(F_{2^n} Q_{2^n})^2} = \frac{8 - 3\sqrt{2}}{12}.$

Proof. We will establish both results in small steps. First, recall from Chapter 1 that $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_n = \alpha^n + \beta^n$, where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, $\alpha\beta = -1$, and $n \geq 1$; and from Chapter 7 that $\gamma^n = P_{n-1} + P_n\gamma$.

Next we will show that

$$\sum_{n=1}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}} = \frac{x^2}{1 - x^2}, \tag{17.34}$$

where $|x| < 1$. To this end, we will need the fact that the series $\sum_{k=1}^{\infty} x^{2k}$ is absolutely convergent with limit $\frac{x^2}{1-x^2}$, when $|x| < 1$. Consequently, we can add up the terms of this series in an arbitrary order, without affecting the convergence or the limit. So

$$\begin{aligned} \sum_{k=1}^{\infty} x^{2k} &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} x^{2^n(2m+1)} \\ &= \sum_{n=1}^{\infty} x^{2^n} \sum_{m=0}^{\infty} x^{2^{n+1}m} \\ &= \sum_{n=1}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}}. \end{aligned}$$

Therefore, the series $\displaystyle\sum_{k=1}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}}$ also converges and converges to the sum $\dfrac{x^2}{1 - x^2}$, as claimed.

We will find the following product also useful for the brevity of the proofs:

$$\begin{aligned}(\gamma^2-\alpha^2)(\gamma^2-\beta^2) &= \gamma^4-(\alpha^2+\beta^2)\gamma^2+(\alpha\beta)^2\\ &= (5+12\gamma)-3(1+2\gamma)+1\\ &= 3+6\gamma\\ &= 9+6\sqrt{2}.\end{aligned} \tag{17.35}$$

Next we will prove that

$$\sqrt{2}L_{2^n}P_{2^n}-\sqrt{5}F_{2^n}Q_{2^n}=(\gamma/\alpha)^{2^n}[1-(\alpha/\gamma)^{2^{n+1}}]. \tag{17.36}$$

We have

$$\begin{aligned}\sqrt{8}L_{2^n}P_{2^n} &= \left(\alpha^{2^n}+\beta^{2^n}\right)\left(\gamma^{2^n}-\delta^{2^n}\right)\\ &= (\alpha\gamma)^{2^n}-(\alpha\delta)^{2^n}+(\beta\gamma)^{2^n}-(\beta\delta)^{2^n}\end{aligned}$$

and

$$\begin{aligned}2\sqrt{5}F_{2^n}Q_{2^n} &= \left(\alpha^{2^n}-\beta^{2^n}\right)\left(\gamma^{2^n}+\delta^{2^n}\right)\\ &= (\alpha\gamma)^{2^n}+(\alpha\delta)^{2^n}-(\beta\gamma)^{2^n}-(\beta\delta)^{2^n}.\end{aligned}$$

So

$$\begin{aligned}\sqrt{8}L_{2^n}P_{2^n}-2\sqrt{5}F_{2^n}Q_{2^n} &= 2\left[(\beta\gamma)^{2^n}-(\alpha\delta)^{2^n}\right]\\ \sqrt{2}L_{2^n}P_{2^n}-\sqrt{5}F_{2^n}Q_{2^n} &= (\gamma/\alpha)^{2^n}\left[1-(\alpha/\gamma)^{2^{n+1}}\right],\end{aligned}$$

as desired.

Changing $\sqrt{5}$ to $-\sqrt{5}$, (17.36) yields the following result:

$$\sqrt{2}L_{2^n}P_{2^n}+\sqrt{5}F_{2^n}Q_{2^n}=(\gamma/\beta)^{2^n}[1-(\beta/\gamma)^{2^{n+1}}]. \tag{17.37}$$

With these tools at hand, we are now ready to confirm both results.

Replacing x with α/γ in formula (17.34) and using formula (17.36), we get

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{\sqrt{2}L_{2^n}P_{2^n}-\sqrt{5}F_{2^n}Q_{2^n}} &= \sum_{n=1}^{\infty}\frac{(\alpha/\gamma)^{2^n}}{1-(\alpha/\gamma)^{2^n}}\\ &= \frac{(\alpha/\gamma)^2}{1-(\alpha/\gamma)^2}\\ &= \frac{\alpha^2}{\gamma^2-\alpha^2}.\end{aligned} \tag{17.38}$$

Changing $\sqrt{5}$ to $-\sqrt{5}$, this yields

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2}L_{2^n}P_{2^n} + \sqrt{5}F_{2^n}Q_{2^n}} = \frac{\beta^2}{\gamma^2 - \beta^2}. \tag{17.39}$$

(1) Since the difference of two convergent series is also convergent, it follows from (17.38) and (17.39) that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{2\sqrt{5}F_{2^n}Q_{2^n}}{2(L_{2^n}P_{2^n})^2 - 5(F_{2^n}Q_{2^n})^2} &= \frac{\alpha^2}{\gamma^2 - \alpha^2} - \frac{\beta^2}{\gamma^2 - \beta^2} \\
&= \frac{(\alpha^2 - \beta^2)\gamma^2}{(\gamma^2 - \alpha^2)(\gamma^2 - \beta^2)} \\
&= \frac{\sqrt{5}(1 + 2\gamma)}{9 + 6\sqrt{2}} = \frac{\sqrt{5}(3 + 2\sqrt{2})}{9 + 6\sqrt{2}} \\
\sum_{n=1}^{\infty} \frac{F_{2^n}Q_{2^n}}{2(L_{2^n}P_{2^n})^2 - 5(F_{2^n}Q_{2^n})^2} &= \frac{3 + 2\sqrt{2}}{2(9 + 6\sqrt{2})} = \frac{1}{6}, \text{ as desired.}
\end{aligned}$$

(2) Adding the series (17.38) and (17.39), we get

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{2\sqrt{2}L_{2^n}P_{2^n}}{2(L_{2^n}P_{2^n})^2 - 5(F_{2^n}Q_{2^n})^2} &= \frac{\alpha^2}{\gamma^2 - \alpha^2} + \frac{\beta^2}{\gamma^2 - \beta^2} \\
&= \frac{(\alpha^2 + \beta^2)\gamma^2 - 2(\alpha\beta)^2)}{(\gamma^2 - \alpha^2)(\gamma^2 - \beta^2)} \\
&= \frac{3(1 + 2\gamma) - 2}{9 + 6\sqrt{2}} = \frac{1 + 6\gamma}{9 + 6\sqrt{2}} \\
&= \frac{7 + 6\sqrt{2}}{9 + 6\sqrt{2}} \\
\sum_{n=1}^{\infty} \frac{L_{2^n}P_{2^n}}{2(L_{2^n}P_{2^n})^2 - 5(F_{2^n}Q_{2^n})^2} &= \frac{1}{2\sqrt{2}} \cdot \frac{7 + 6\sqrt{2}}{9 + 6\sqrt{2}} \\
&= \frac{8 - 3\sqrt{2}}{12}, \text{ again as desired.}
\end{aligned}$$

■

17.12 Catalani's Identities

Next we study two identities linking Fibonacci and Lucas numbers with Pell numbers; they were discovered in 2004 by Mario S. Catalani of the University of Torino, Italy [43].

Let $U_n = F_{P_n}$ and $V_n = L_{P_n}$, where $n \geq 0$. Then $U_0 = F_{P_0} = 0$ and $U_1 = F_{P_1} = 1$. Likewise, $V_0 = 2$ and $V_1 = 1$.

We will now establish the following identities:

$$2U_{n+2} = U_n[V_{n+1}^2 - 2(-1)^{n+1}] + U_{n+1}V_{n+1}\sqrt{5U_n^2 + 4(-1)^n} \tag{17.40}$$

$$2V_{n+2} = V_n[V_{n+1}^2 - 2(-1)^{n+1}] + U_{n+1}V_{n+1}\sqrt{5V_n^2 - 20(-1)^n}. \tag{17.41}$$

Their proofs employ the following facts: the addition formulas $2F_{a+b} = F_aL_b + F_bL_a$ and $2L_{a+b} = L_aL_b + 5F_aF_b$; $P_n \equiv n \pmod 2$; $L_n^2 - 5F_n^2 = 4(-1)^n$; $F_nL_n = F_{2n}$; and $L_n^2 - 2(-1)^n = L_{2n}$ [126], as Bruckman did in 2005 [34]. Consequently, $5U_n^2 + 4(-1)^n = V_n^2, 5V_n^2 - 20(-1)^n = 25U_n^2, U_{n+1}V_{n+1} = F_{2P_{n+1}}$ and $V_{n+1}^2 - 2(-1)^{n+1} = L_{2P_{n+1}}$. Then

$$\begin{aligned}
U_n[V_{n+1}^2 - 2(-1)^{n+1}] + U_{n+1}V_{n+1}\sqrt{5U_n^2 + 4(-1)^n} &= F_{P_n}L_{2P_{n+1}} + F_{2P_{n+1}}L_{P_n}\\
&= 2F_{2P_{n+1}+P_n} = 2F_{P_{n+2}}\\
&= 2U_{n+2};\\
V_n[V_{n+1}^2 - 2(-1)^{n+1}] + U_{n+1}V_{n+1}\sqrt{5V_n^2 - 20(-1)^n} &= L_{P_n}L_{2P_{n+1}} + 5F_{2P_{n+1}}F_{P_n}\\
&= 2L_{2P_{n+1}+P_n} = 2L_{P_{n+2}}\\
&= 2V_{n+2}, \text{ as desired.}
\end{aligned}$$

■

For example, let $n = 3$. Then $U_3 = F_{P_3} = F_5 = 5, U_4 = F_{P_4} = F_{12} = 144, V_3 = L_{P_3} = L_5 = 11$, and $V_4 = L_{P_4} = L_{12} = 322$. Then, by identities (17.40) and (17.41), we have

$$\begin{aligned}
2U_5 &= U_3[V_4^2 - 2(-1)^4] + U_4V_4\sqrt{5U_3^2 + 4(-1)^3}\\
&= 5(322^2 - 2) + 144 \cdot 322\sqrt{5 \cdot 5^2 - 4}\\
&= 518,410 + 510,048 = 1,028,458\\
U_5 &= 514,229 = F_{P_5};\\
2V_5 &= V_3[V_4^2 - 2(-1)^4] + U_4V_4\sqrt{5V_3^2 - 20(-1)^3}\\
&= 11(322^2 - 2) + 144 \cdot 322\sqrt{5 \cdot 11^2 + 20}\\
&= 1,140,502 + 1,159,200 = 2,299,702\\
V_5 &= 1,149,851 = L_{P_5}, \text{ as expected.}
\end{aligned}$$

Clearly, identities (17.40) and (17.41) can be extended to the sequences $\{F_{g(m)}\}$ and $\{L_{g(m)}\}$, where the sequence $\{g(m)\}$ satisfies the Pell recurrence. For example, let $A_n = F_{Q_n}$ and $B_n = L_{Q_n}$. Then

$$\begin{aligned} 2A_{n+2} &= A_n\left[B_{n+1}^2 - 2(-1)^{n+1}\right] + A_{n+1}B_{n+1}\sqrt{5A_n^2 + 4(-1)^n} && (17.42)\\ 2B_{n+2} &= B_n\left[B_{n+1}^2 - 2(-1)^{n+1}\right] + A_{n+1}B_{n+1}\sqrt{5B_n^2 - 20(-1)^n}. && (17.43)\end{aligned}$$

For example, let $n = 3$. Then $A_3 = F_{Q_3} = F_7 = 13$, $A_4 = F_{Q_4} = F_{17} = 1597$, $B_3 = L_{Q_3} = L_7 = 29$, $B_4 = L_{Q_4} = L_{17} = 3571$. so

$$\begin{aligned} 2A_5 &= 13(3571^2 - 2) + 1597 \cdot 3571\sqrt{5 \cdot 13^2 - 4}\\ &= 331,160,230\\ A_5 &= 165,580,115;\\ 2B_5 &= 29(3571^2 - 2) + 1597 \cdot 3571\sqrt{5 \cdot 29^2 + 20}\\ &= 740,496,786\\ B_5 &= 370,248,393.\end{aligned}$$

17.13 A Fibonacci–Lucas–Pell Bridge

In 2001, J.L. Díaz-Barrero of Barcelona, Spain, developed an intriguing formula linking the Fibonacci, Lucas, and Pell families [65]:

$$\frac{F_n + L_nP_n}{(F_n - L_n)(F_n - P_n)} + \frac{L_n + F_nP_n}{(L_n - F_n)(L_n - P_n)} + \frac{P_n + F_nL_n}{(P_n - F_n)(P_n - L_n)} = 1,$$

where $n \ge 2$. Its proof requires a knowledge of operator theory, so we omit it [33]. But we will illustrate it with a simple numeric example.

Let $n = 5$. Then

$$\begin{aligned} \text{LHS} &= \frac{F_5 + L_5P_5}{(F_5 - L_5)(F_5 - P_5)} + \frac{L_5 + F_5P_5}{(L_5 - F_5)(L_5 - P_5)} + \frac{P_5 + F_5L_5}{(P_5 - F_5)(P_5 - L_5)}\\ &= \frac{5 + 11 \cdot 29}{(5 - 11)(5 - 29)} + \frac{11 + 5 \cdot 29}{(11 - 5)(11 - 29)} + \frac{29 + 5 \cdot 11}{(29 - 5)(29 - 11)}\\ &= \frac{324}{144} - \frac{156}{108} + \frac{84}{432} = 1, \text{ as expected.}\end{aligned}$$

17.14 Recurrences for $\{F_nP_n\}$, $\{L_nP_n\}$, $\{F_nQ_n\}$, and $\{L_nQ_n\}$

Using the sequences $\{F_n\}_{n=0}^{\infty}$ and $\{P_n\}_{n=0}^{\infty}$, we can form a hybrid sequence $\{A_n\} = \{F_nP_n\}$: $0, 1, 2, 10, 36, 145, \ldots$. Likewise, using Lucas and Pell numbers, we can form the hybrid sequence $\{B_n\} = \{L_nP_n\}$: $0, 1, 6, 20, 84, 319, \ldots$. Similarly, we can form two additional sequences $\{C_n\}$ and $\{D_n\}$:

$$\begin{aligned}\{C_n\} &= \{F_nQ_n\} = 0, 1, 3, 14, 51, 205, \ldots \\ \{D_n\} &= \{L_nQ_n\} = 2, 1, 9, 28, 119, 451, \ldots .\end{aligned}$$

We will now develop recurrences for the sequences $\{A_n\}, \{B_n\}, \{C_n\}$, and $\{D_n\}$ as special cases of a recurrence for $\{z_n\}$, where $z_n = x_ny_n$, and x_n and y_n satisfy the recurrences $x_{n+2} = ax_{n+1} + bx_n$, $y_{n+2} = cy_{n+1} + dy_n, ac \neq 0$, and $n \geq 0$. Then

$$\begin{aligned}z_{n+4} &= x_{n+4}y_{n+4} \\ &= (ax_{n+3} + bx_{n+2})(cy_{n+3} + dy_{n+2}) \\ &= acz_{n+3} + bdz_{n+2} + adx_{n+3}y_{n+2} + bcx_{n+2}y_{n+3} \\ &= acz_{n+3} + bdz_{n+2} + ady_{n+2}(ax_{n+2} + bx_{n+1}) + bcx_{n+2}(cy_{n+2} + dy_{n+1}) \\ &= acz_{n+3} + (a^2d + bc^2 + bd)z_{n+2} + abdx_{n+1}(cy_{n+1} + dy_n) \\ &\quad +bcdx_{n+2}\left(\frac{y_{n+2} - dy_n}{c}\right) \\ &= acz_{n+3} + (a^2d + bc^2 + 2bd)z_{n+2} + abcdz_{n+1} + abd^2x_{n+1}y_n \\ &\quad -bd^2y_n(ax_{n+1} + bx_n) \\ &= acz_{n+3} + (a^2d + bc^2 + 2bd)z_{n+2} + abcdz_{n+1} - b^2d^2z_n. \end{aligned} \tag{17.44}$$

Thus z_n satisfies a recurrence of order four.

Now choose a, b, c, and d cleverly: $a = b = d = 1$ and $c = 2$. Then $\{x_n\}$ satisfies the Fibonacci recurrence and $\{y_n\}$ the Pell recurrence. Thus we have

$$z_{n+4} = 2z_{n+3} + 7z_{n+2} + 2z_{n+1} - z_n, \tag{17.45}$$

as Seiffert discovered in 1988 [191]. This recurrence reappeared in a slightly different fashion in a problem proposed by J.L. Díaz-Barrero and J.L. Egozcue of Spain in 2003 [69]. We will revisit it later.

Since Fibonacci and Lucas numbers satisfy the same recurrence, it follows that the sequences $\{A_n\}$, $\{B_n\}$, $\{C_n\}$, and $\{D_n\}$ satisfy the same recurrence (17.45). The corresponding initial conditions are $A_0 = 0, A_1 = 1, A_2 = 2$, and $A_3 = 10$; and $B_0 = 0, B_1 = 1, B_2 = 6$, and $B_3 = 20$; $C_0 = 0, C_1 = 1, C_2 = 3$, and $C_3 = 14$; and $D_0 = 2, D_1 = 1, D_2 = 9$, and $D_3 = 28$, respectively.

Let $z_n = A_n, B_n, C_n$, or D_n. Since $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha = \lim_{n\to\infty} \frac{L_{n+1}}{L_n}$, and $\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \gamma = \lim_{n\to\infty} \frac{Q_{n+1}}{Q_n}$, it follows that $\lim_{n\to\infty} \frac{z_{n+1}}{z_n} = \alpha\gamma \approx 3.9062796000$.

For example, $\frac{A_{20}}{A_{19}} = \frac{6765 \cdot 15994428}{4181 \cdot 6625109} \approx 3.9062795383$. So the convergence of the sequence $\left\{\frac{A_{n+1}}{A_n}\right\}_{n=1}^{\infty}$ is extremely fast.

Since $\lim_{n\to\infty} \frac{z_n}{z_{n+1}} = \frac{1}{\alpha\gamma} = \beta\delta$, it follows by the ratio test that the series $\sum_{n=1}^{\infty} \frac{1}{z_n}$ converges, and converges to the limit $\beta\delta \approx 0.2589980601$.

The recurrence (17.45), coupled with the four sets of initial conditions, can be used to develop generating functions $a(x), b(x), c(x)$, and $d(x)$, for the sequences $\{A_n\}, \{B_n\}, \{C_n\}$, and $\{D_n\}$, respectively. The first two were also discovered by Seiffert in 1988.

17.15 Generating Functions for $\{A_n\}, \{B_n\}, \{C_n\}$, and $\{D_n\}$

Let $f(z) = \sum_{n=0}^{\infty} z_n z^n$ be a generating function of $\{z_n\}$, where z_n satisfies the recurrence (17.45). Then, by virtue of the recurrence (17.45), we have

$$\begin{aligned} (1 - 2z - 7z^2 - 2z^3 + z^4) f(z) &= z_0 + (z_1 - 2z_0)z + (z_2 - 2z_1 - 7z_0)z^2 \\ &\quad + (z_3 - 2z_2 - 7z_1 - 2z_0)z^3 \\ f(z) &= \frac{z_0 + (z_1 - 2z_0)z + (z_2 - 2z_1 - 7z_0)z^2 + (z_3 - 2z_2 - 7z_1 - 2z_0)z^3}{1 - 2z - 7z^2 - 2z^3 + z^4}. \end{aligned}$$

Using the initial values of the sequences $\{A_n\}, \{B_n\}, \{C_n\}$, and $\{D_n\}$, this yields the desired generating functions:

$$\begin{aligned} a(x) &= \frac{z - z^3}{1 - 2z - 7z^2 - 2z^3 + z^4} & b(x) &= \frac{z + 4z^2 + z^3}{1 - 2z - 7z^2 - 2z^3 + z^4} \\ c(x) &= \frac{z + z^2 + 5z^3}{1 - 2z - 7z^2 - 2z^3 + z^4} & d(x) &= \frac{2 - 3z - 7z^2 - z^3}{1 - 2z - 7z^2 - 2z^3 + z^4}. \end{aligned}$$

The next example is a generalization of a Pell–Fibonacci identity discovered by Díaz-Barrero in 2005 [66]. It builds another bridge between Pell and Fibonacci families. The featured proof is based on the one given for the special case by H. Kwong of the State University of New York at Fredonia, New York, in 2006 [146].

Example 17.12 Let $\{A_n\}$ be an integer sequence satisfying Pell recurrence, where $A_1 = 1$, $A_2 \geq 3$, and n be a positive integer. Prove that

$$\frac{\left| \frac{|F_n - L_n|}{F_{2n}} + \frac{2F_{n+1}}{F_{2n}} - \frac{2}{A_n} \right| + \frac{|F_n - L_n|}{F_{2n}} + \frac{2F_{n+1}}{F_{2n}} + \frac{2}{A_n}}{\max\left\{\frac{1}{F_n}, \frac{1}{L_n}, \frac{1}{A_n}\right\}} = 4.$$

Proof. It follows by the strong version of PMI that $F_n \le L_n \le A_n$ for every $n \ge 1$. Consequently, $\max\left\{\frac{1}{F_n}, \frac{1}{L_n}, \frac{1}{A_n}\right\} = \frac{1}{F_n}$.

Let R_n denote the fractional expression on the LHS. Then

$$R_1 = \left|\frac{|F_1 - L_1|}{F_2} + \frac{2F_2}{F_2} - \frac{2}{A_1}\right| + \frac{|F_1 - L_1|}{F_2} + \frac{2F_2}{F_2} + \frac{2}{A_1} = 4.$$

Similarly, $R_2 = 4$.

Now let $n \ge 3$. Then, using the fact [126] that $F_{k-1} + F_{k+1} = L_k$, we have

$$\begin{aligned}
\frac{|F_n - L_n|}{F_{2n}} + \frac{2F_{n+1}}{F_{2n}} &= \frac{L_n - F_n + 2F_{n+1}}{F_{2n}} \\
&= \frac{L_n + (F_{n+1} - F_n) + F_{n+1}}{F_{2n}} \\
&= \frac{L_n + (F_{n-1} + F_{n+1})}{F_{2n}} \\
&= \frac{2L_n}{F_nL_n} = \frac{2}{F_n} \\
&\ge \frac{2}{A_n}.
\end{aligned}$$

Consequently, $\dfrac{|F_n - L_n|}{F_{2n}} + \dfrac{2F_{n+1}}{F_{2n}} - \dfrac{2}{A_n} \ge 0$.

Thus

$$\begin{aligned}
\text{LHS} &= F_n\left(\frac{L_n - F_n}{F_{2n}} + \frac{2F_{n+1}}{F_{2n}} - \frac{2}{A_n} + \frac{L_n - F_n}{F_{2n}} + \frac{2F_{n+1}}{F_{2n}} + \frac{2}{A_n}\right) \\
&= \frac{2F_n}{F_{2n}} \cdot (L_n - F_n + 2F_{n+1}) = \frac{2F_n}{F_{2n}} \cdot 2L_n \\
&= 4, \quad \text{as desired.}
\end{aligned}$$

■

In particular, the result holds when $A_n = P_n$, as Díaz-Barrero found, and when $A_n = Q_n$.

The following two results are quite similar. Their proofs use the facts that $P_{2n} = 2P_nQ_n$, $Q_n - P_n = P_{n-1}$, and $P_{n+1} + P_{n-1} = 2Q_n$. Since they follow a parallel argument, we omit them for convenience:

$$\frac{\left|\frac{|P_n-Q_n|}{P_{2n}} + \frac{P_{n+1}}{P_{2n}} - \frac{2}{P_n}\right| + \frac{|P_n-Q_n|}{P_{2n}} + \frac{P_{n+1}}{P_{2n}} + \frac{2}{P_n}}{\max\left\{\frac{1}{P_n}, \frac{1}{Q_n}\right\}} = 2;$$

$$\frac{\left|\frac{|P_n-Q_n|}{P_{2n}} + \frac{P_{n+1}}{P_{2n}} - \frac{1}{Q_n}\right| + \frac{|P_n-Q_n|}{P_{2n}} + \frac{P_{n+1}}{P_{2n}} + \frac{1}{Q_n}}{\max\left\{\frac{1}{P_n}, \frac{1}{Q_n}\right\}} = 2.$$

The next example evaluates a complicated-looking Pell–Fibonacci determinant. It was studied by Cook in 2006 [49, 50].

Example 17.13 Evaluate the determinant $|M|$, where M denotes the matrix

$$\begin{bmatrix} F_n^2 + L_n^2 - P_n^2 - Q_n^2 & 2(L_nP_n - F_nQ_n) & 2(F_nP_n + L_nQ_n) \\ 2(F_nQ_n + L_nP_n) & F_n^2 - L_n^2 + P_n^2 - Q_n^2 & 2(P_nQ_n - F_nL_n) \\ 2(L_nQ_n - F_nP_n) & 2(F_nL_n + P_nQ_n) & F_n^2 - L_n^2 - P_n^2 + Q_n^2 \end{bmatrix}.$$

Solution. Let A and B be two square matrices of the same order. Then it is well known [6] that $|AB| = |A| \cdot |B|$. Also, $|A^T| = |A|$, where A^T denotes the transpose of A.

Let $K = F_n^2 + L_n^2 + P_n^2 + Q_n^2$. Using these two properties, we have

$$\begin{aligned} |M|^2 &= |M| \cdot |M^T| \\ &= |M \cdot M^T| \\ &= \begin{bmatrix} K^2 & 0 & 0 \\ 0 & K^2 & 0 \\ 0 & 0 & K^2 \end{bmatrix} \\ &= K^6. \end{aligned}$$

So $|M| = \pm K^3$.

To determine the correct sign, notice that when $n = 0$,

$$|M| = \begin{bmatrix} 0 & 0 & 2 \\ 0 & -2 & 0 \\ 2 & 0 & 0 \end{bmatrix} = 8$$

is positive. So $|M| = K^3 = (F_n^2 + L_n^2 + P_n^2 + Q_n^2)^3$, where $n \geq 0$. ■

The above determinant $|M|$ is a special case of the determinant

$$\Delta = \begin{vmatrix} a^2 + b^2 - c^2 - d^2 & 2bc - 2ad & 2bd + 2ac \\ 2bc + 2ad & a^2 - b^2 + c^2 - d^2 & 2cd - 2ab \\ 2bd - 2ac & 2cd + 2ab & a^2 - b^2 - c^2 + d^2 \end{vmatrix}$$

studied by C.W. Trigg of San Diego, California, in 1970 [247]. Using the same argument as in the example, $\Delta = (a^2 + b^2 + c^2 + d^2)^3$. Consequently, when $a = F_n, b = L_n, c = P_n$, and $d = Q_n$, $|M| = (F_n^2 + L_n^2 + P_n^2 + Q_n^2)^3$.

Next we investigate an ISCF which contains Fibonacci, Lucas, Pell, and Pell–Lucas numbers as special cases.

17.16 ISCF Revisited

Recall from Chapter 3 that the nth convergent $\frac{p_n}{q_n}$ of the ISCF $[a_0; a_1, a_2, a_3, \ldots]$ can be computed using the following recursive definitions, where $n \geq 2$:

$$\begin{aligned} p_0 &= a_0 & q_0 &= 1 \\ p_1 &= a_1 a_0 + 1 & q_1 &= a_1 \\ p_n &= a_n p_{n-1} + p_{n-2}; & q_n &= a_n q_{n-1} + q_{n-2}. \end{aligned}$$

In particular, let $a_{2k} = a$ and $a_{2k+1} = b$. (This special case was studied by Seiffert in 1992.) Then

$$\begin{aligned} p_0 &= a & q_0 &= 1 \\ p_1 &= ab + 1 & q_1 &= b \\ p_{2n} &= ap_{2n-1} + p_{2n-2} & q_{2n} &= a_n q_{2n-1} + q_{2n-2} \\ p_{2n+1} &= bp_{2n} + p_{2n-1}; & q_{2n+1} &= b_n q_{2n} + q_{2n-1}, \end{aligned}$$

where $n \geq 2$.

Then the sequences $\{p_{2n}\}$ and $\{q_{2n}\}$ can be defined recursively, where $n \geq 2$:

$$\begin{aligned} p_0 &= a & q_0 &= 1 \\ p_2 &= a(ab + 2) & q_2 &= ab + 1 \\ p_{2n} &= (ab + 2)p_{2n-2} - p_{2n-4}; & q_{2n} &= (ab + 2)q_{2n-2} - q_{2n-4}. \end{aligned}$$

So both p_{2n} and q_{2n} satisfy the same recurrence. Its characteristic equation is $x^2-(ab+2)x+1 = 0$, with characteristic roots $r = \frac{1}{2}(ab+2+\Delta)$ and $s = \frac{1}{2}(ab+2-\Delta)$, where $\Delta = \sqrt{ab(ab + 4)}$.

The general solution of the recurrence is $Ar^n + bs^n$. Using the two sets of initial conditions, the solutions are given by

$$p_{2n} = \frac{a}{\Delta}\left(r^{n+1} - s^{n+1}\right) \text{ and } q_{2n} = \frac{1}{\Delta}\left[(r-1)r^n - (s-1)s^n\right].$$

Since $r > s$, this implies that

$$\begin{aligned} \lim_{n\to\infty} \frac{p_{2n}}{q_{2n}} &= \lim_{n\to\infty} \frac{a\left(r^{n+1} - s^{n+1}\right)}{(r-1)r^n - (s-1)s^n} \\ &= \lim_{n\to\infty} \frac{a\left[1 - (s/r)^{n+1}\right]}{(r-1)/r - [(s-1)/r](s/r)^n} \\ &= \frac{ar}{r-1}. \end{aligned}$$

Since $\Delta p_{2n+1} = (ab + 1 - s)r^{n+1} + (r - ab - 1)s^{n+1}$ and $\Delta q_{2n+1} = b\left(r^{n+1} - s^{n+1}\right)$, it follows that

$$\lim_{n\to\infty} \frac{p_{2n+1}}{q_{2n+1}} = \frac{ab + 1 - s}{b} = \frac{a}{1-s} = \frac{ar}{r-1}.$$

Consequently, $\lim_{n\to\infty} \frac{p_n}{q_n} = \frac{ar}{r-1}$ irrespective of the parity of n. Thus the ISCF $[a; \overline{b, a}]$ converges to $\frac{ar}{r-1}$.

Since we have established the convergence of the ISCF, we can compute the limit in a different way. To this end, let $\ell = [a; \overline{b, a}]$. Then $b\ell^2 - ab\ell - a = 0$, so $\ell = \frac{ab \pm \sqrt{a^2+b^2+4ab}}{2b}$. But $\ell > 0$; so $\ell = \frac{a}{2}\left[1 + \sqrt{1 + \frac{4}{ab}}\right]$.

We will now study seven special cases.

17.16.1 Special Cases

(1) Let $a = P_n = b$. Then $[P_n; \overline{P_n}] = \frac{P_n}{2}\left[1 + \sqrt{1 + \frac{4}{P_n^2}}\right] = \frac{1}{2}\left[P_n + \sqrt{P_n^2 + 4}\right]$.

(2) Let $a = Q_n = b$. Then $[Q_n; \overline{Q_n}] = \frac{1}{2}\left[Q_n + \sqrt{Q_n^2 + 4}\right]$.

(3) Let $a = P_n$ and $b = Q_n$. Then we get $[P_n; \overline{Q_n, P_n}] = \frac{P_n}{2}\left[1 + \sqrt{1 + \frac{8}{P_{2n}}}\right]$.

(4) Let $a = Q_n$ and $b = P_n$. Then $[Q_n; \overline{P_n, Q_n}] = \frac{Q_n}{2}\left[1 + \sqrt{1 + \frac{8}{P_{2n}}}\right]$.

(5) Let $a = F_n = b$, we get $[F_n; \overline{F_n}] = \frac{1}{2}\left[F_n + \sqrt{F_n^2 + 4}\right]$.

(6) Let $a = F_n$ and $b = L_n$. Then $[F_n; \overline{L_n, F_n}] = \frac{F_n}{2}\left[1 + \sqrt{1 + \frac{4}{F_{2n}}}\right]$. (This continued fraction was studied by L. Kupiers of Sierre, Switzerland, in 1991 [143].)

(7) Finally, let $a = 1 = b$. Then we get $[1; \overline{1}] = \alpha$, the golden ratio, as we learned in Chapter 3.

Finally, we turn to a truly delightful application of Fibonacci and Pell numbers to the study of domino tilings of a graph. But before we begin, we present a brief introduction to basic terminology in graph theory.

17.17 Basic Graph-theoretic Terminology

Let V be a finite nonempty set, and E a set of unordered pairs $\{v, w\}$ of elements v and w from V. Then the ordered pair (V, E) is a *graph* G: $G = (V, E)$. The elements in V are the *vertices* (or *nodes*) of the graph, and the unordered pairs $\{v, w\}$ are its *edges*. The edge $\{v, w\}$ is also denoted by $v - w$. Since $\{v, w\} = \{w, v\}$, every edge is *undirected*. Vertices v and w are the *endpoints* of the edge $\{v, w\}$. A vertex v is *adjacent* to vertex w if there is an edge $v - w$.

Geometrically, G is a nonempty set of points (vertices), together with arcs or line segments joining them.

For example, the graph in Figure 17.1 has four vertices – A, B, C, and D – and seven edges. It has edges with the same endpoints; such edges are *parallel edges*. For instance, edges x and y are parallel edges. On the other hand, the graph in Figure 17.2 has *no* parallel edges. But it has an edge z with the same endpoint; such an edge is a *loop*.

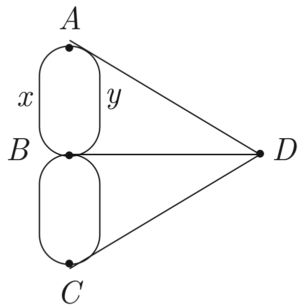


Figure 17.1.

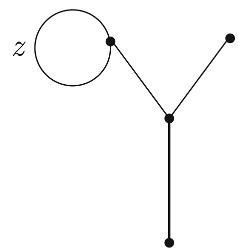


Figure 17.2.

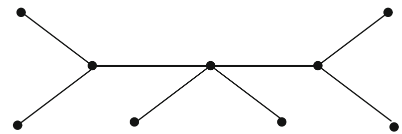


Figure 17.3.

A loop-free graph that contains no parallel edges is a *simple graph*. The graph in Figure 17.3 is a simple graph.

A *path graph* consists of n vertices v_i, and edges $v_i - v_{i+1}$, where $1 \leq i \leq n-1$; it is denoted by P_n. Figure 17.4 shows the path graphs P_1, P_2, P_3, and P_4.

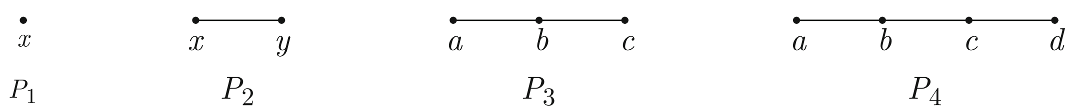


Figure 17.4.

A *cycle graph* C_n consists of n vertices v_i and edges $v_i - v_{i+1}$ such that $v_{n+1} = v_1$, where $1 \leq i \leq n-1$. So the vertices of a cycle graph can be placed on a circle. Figure 17.5 shows the cycle graphs C_3 and C_4.

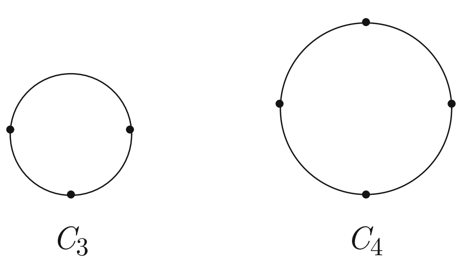


Figure 17.5.

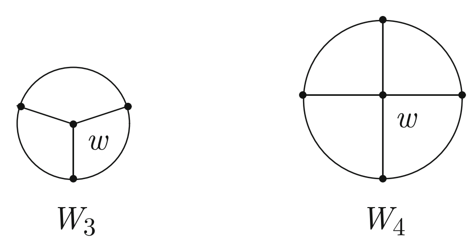


Figure 17.6.

A *wheel graph* W_n consists of n vertices v_i, a special vertex w, and edges $v_i - v_{i+1}$ and $v_i - w$ for every i. So W_n is the cycle graph C_n such that every vertex v_i is adjacent to the hub w. Figure 17.6 shows the wheel graphs W_3 and W_4.

Next we present the concept of the product of two graphs.

17.18 Cartesian Product of Two Graphs

The *cartesian product* $G_1 \times G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ has vertex set $V_1 \times V_2$ such that the vertex $(v_1, v_2) \in V_1 \times V_2$ is adjacent to the vertex $(w_1, w_2) \in V_1 \times V_2$ if and only if:

(1) $v_1 = w_1$ and v_2 is adjacent to w_2 in G_2; or
(2) $v_2 = w_2$ and v_1 is adjacent to w_1 in G_1.

It follows from the definition that $G_1 \times G_2$ contains exact copies of G_2 at each vertex of G_1 and those of G_1 at each vertex of G_2.

For example, consider the path graphs P_2 and P_3 in Figure 17.4. Figure 17.7 shows the cartesian products $P_2 \times P_2$ and $P_2 \times P_3$, where we have denoted the vertex (v, w) by vw for notational brevity; they are 2×2 and 2×3 grids, respectively. More generally, the cartesian product $P_m \times P_n$ is the $m \times n$ grid on the cartesian plane. Such graphs play an important role in the theory of communications.

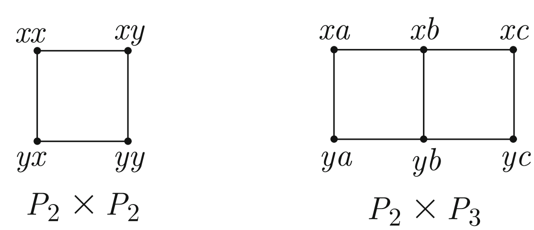


Figure 17.7.

Figure 17.8 shows the cartesian products $W_4 \times P_1$ and $W_4 \times P_2$. You may confirm both using the definition.

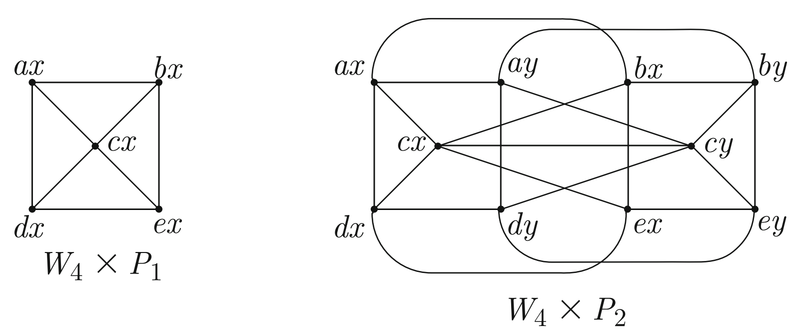


Figure 17.8.

17.19 Domino Tilings of $W_4 \times P_{n-1}$

In 1988, F.J. Faase [82] of the University of Twente, Netherlands, investigated the domino tilings of the product $W_4 \times P_{n-1}$, where $n \geq 2$. He found that the number of tilings C_n can be defined recursively:

$$\begin{aligned} C_2 &= 2, \quad C_3 = 10, \quad C_4 = 36, \quad C_5 = 145; \\ C_{n+4} &= 2C_{n+3} + 7C_{n+2} + 2C_{n+1} - C_n, \end{aligned} \tag{17.46}$$

where $n \geq 2$.

Although the numbers $2, 10, 36, 145, 560, \cdots$ do not appear to reveal any obvious pattern, they do contain a hidden gem. To see this treasure, notice that

$$\begin{aligned} 2 &= 1 \cdot 2 \\ 10 &= 2 \cdot 5 \\ 36 &= 3 \cdot 12 \\ 145 &= 5 \cdot 29 \\ 560 &= 8 \cdot 70 \\ & \quad \uparrow \quad \uparrow \\ & \quad F_n \quad P_n \end{aligned}$$

In each case, the first factor on the RHS is a Fibonacci number, and the second factor a Pell number.

More specifically, we have the following result, established in 2002 by James A. Sellers of Pennsylvania State University, University Park, Pennsylvania [228].

Theorem 17.1 *Let C_n denote the number of domino tilings of the cartesian product $W_4 \times P_{n-1}$, where $n \geq 2$. Then $C_n = F_n P_n$.*

Proof. Clearly, $F_n P_n$ satisfies the four initial conditions. So it suffices to verify that $F_n P_n$ satisfies recurrence (17.46). To this end, first notice that.

$$F_{n+2} = F_n + F_{n+1}, \quad F_{n+3} = F_n + 2F_{n+1}, \text{ and } F_{n+4} = 2F_n + 3F_{n+1}; \text{ and}$$

$$P_{n+2} = P_n + 2P_{n+1}, \quad P_{n+3} = 2P_n + 5P_{n+1}, \text{ and } P_{n+4} = 5P_n + 12P_{n+1}. \text{ Then}$$

$$\begin{aligned} 2C_{n+3} + 7C_{n+2} + 2C_{n+1} - C_n &= 2F_{n+3}P_{n+3} + 7F_{n+2}P_{n+2} + 2F_{n+1}P_{n+1} - F_nP_n \\ &= 2(F_n + 2F_{n+1})(2P_n + 5P_{n+1}) + 7(F_n + F_{n+1})(P_n + 2P_{n+1}) + \\ &\quad 2F_{n+1}P_{n+1} - F_nP_n \\ &= (4 + 7 - 1)F_nP_n + (8 + 7)F_{n+1}P_n + (10 + 14)F_nP_{n+1} + \\ &\quad (20 + 14 + 2)F_{n+1}P_{n+1} \\ &= 10F_nP_n + 15F_{n+1}P_n + 24F_nP_{n+1} + 36F_{n+1}P_{n+1} \end{aligned}$$

$$\begin{aligned} &= 5P_n(2F_n + 3F_{n+1}) + 12P_{n+1}(2F_n + 3F_{n+1}) \\ &= (2F_n + 3F_{n+1})(5P_n + 12P_{n+1}) \\ &= F_{n+4}P_{n+4} \\ &= C_{n+4}, \text{ as desired.} \end{aligned}$$

■

Exercises 17

1. Let x be a real number such that $x^2 = x + 1$. Prove that $x^n = xF_n + F_{n-1}$, where $n \geq 1$.
2. Let x be a real number such that $x^2 = 2x + 1$. Prove that $x^n = xP_n + P_{n-1}$, where $n \geq 1$.
3. Verify Cook's inequality (17.3) for $n = 5$ and 6.
4. Verify inequality (17.5) for $n = 5$ and 6.
5. Confirm inequality (17.5) for $n > 3$.
6. Establish congruence (17.10).
7. Verify congruence (17.10) for $m = 3 = n$.
8. Establish congruence (17.11).
9. Establish congruence (17.12).
10. Prove identity (17.16). *Hint*: Use PMI or the Binet-like formula for Q_n.

Prove the following Fibonacci–Lucas identities.

11. $F_m L_n + F_n L_m = 2F_{m+n}$.
12. $L_m L_n + 5F_m F_n = 2L_{m+n}$.
13. $L_n^2 - 5F_n^2 = 4(-1)^n$.
14. $L_n^2 - 2(-1)^n = L_{2n}$.

Define each integer sequence $\{z_n\}$ recursively.

15. $0, 1, 3, 14, 51, 205, 792, \ldots$.
16. $2, 1, 9, 28, 119, 451, 1782, \ldots$.

Deduce from formula (17.44) a recurrence satisfied by

17. F_n^2.
18. L_n^2.
19. P_n^2.
20. Q_n^2.
21. $P_n Q_n$.
22. P_{2n}.

18

An Extended Pell Family

18.1 Introduction

This chapter presents an extended Pell family of polynomial functions $g_n(x)$, which includes the Fibonacci, Lucas, Pell, and Pell–Lucas polynomials. Using the power of matrices and determinants, we will develop a Cassini-like formula for this extended family, from which the corresponding formulas for the four sub-families will follow fairly quickly.

We begin with the definition of $g_n(x)$.

18.2 An Extended Pell Family

The *extended Pell polynomial functions* $g_n(x)$ are defined recursively as follows:

$$\begin{aligned} g_0(x) &= a, \quad g_1(x) = b \\ g_n(x) &= 2xg_{n-1}(x) + g_{n-2}(x), \end{aligned}$$

where $a = a(x)$ and $b = b(x)$, and $n \geq 2$.

When $a(x) = 0$ and $b(x) = 1$, $g_n(x) = p_n(x)$; so $g_n(1) = p_n(1) = P_n$. When $a(x) = 2$ and $b(x) = 2x$, $g_n(x) = q_n(x)$; then $g_n(1) = q_n(1) = 2Q_n$. Recall from Chapter 14 that $p_n(x/2) = f_n(x)$ and $q_n(x/2) = l_n(x)$.

Notice that

$$\begin{aligned} g_2(x) = 2xb + a &= a \cdot 1 + b(2x) = ap_1(x) + bp_2(x) \\ g_3(x) = 2x(2bx + a) + b &= a(2x) + b(4x^2 + 1) = ap_2(x) + bp_3(x) \\ g_4(x) = 2x(4bx^2 + 2ax + b) &= a(4x^2 + 1) + b(8x^3 + 4x) = ap_3(x) + bp_4(x). \end{aligned}$$

More generally, we conjecture that $g_n(x) = ap_{n-1}(x) + bp_n(x)$, where $n \geq 0$. We will now establish this using strong induction. To this end, notice that $p_{-1}(x) = 1$.

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Lemma 18.1 *$g_n(x) = ap_{n-1}(x) + bp_n(x)$ for every $n \geq 0$.*

Proof (by PMI). When $n = 0$, RHS $= ap_{-1}(x) + bp_0(x) = a \cdot 1 + b \cdot 0 = a = g_0(x) =$ LHS. Likewise, when $n = 1$, RHS $= ap_0(x) + bp_1(x) = a \cdot 0 + b \cdot 1 = b = g_1(x) =$ LHS. So the formula works when $n = 0$ and $n = 1$.

Now assume that it works for all nonnegative integers $< n$, where n is an arbitrary integer ≥ 2. Then

$$\begin{aligned} g_n(x) &= 2xg_{n-1}(x) + g_{n-2}(x) \\ &= 2x[ap_{n-2}(x) + bp_{n-1}(x)] + [ap_{n-3}(x) + bp_{n-2}(x)] \\ &= a[2xp_{n-2}(x) + p_{n-3}(x)] + b[2xp_{n-1}(x) + p_{n-2}(x)] \\ &= ap_{n-1}(x) + bp_n(x). \end{aligned}$$

So the formula also works for n.

Thus, by the strong version of PMI, the result is true for all integers $n \geq 0$. ■

Interestingly, $g_n(x)$ satisfies a Binet-like formula. To establish this, we need the following result also.

Lemma 18.2

$$p_{n+1}(x)p_{n-2}(x) - p_n(x)p_{n-1}(x) = 2(-1)^{n-1}x.$$

Proof. Using the Binet-like formula for $p_k(x)$, we have

$$\begin{aligned} (\gamma - \delta)^2 \cdot \text{LHS} &= \left(\gamma^{n+1} - \delta^{n+1}\right)\left(\gamma^{n-2} - \delta^{n-2}\right) - (\gamma^n - \delta^n)\left(\gamma^{n-1} - \delta^{n-1}\right) \\ &= \left[-\gamma^3(\gamma\delta)^{n-2} - \delta^3(\gamma\delta)^{n-2}\right] - \left[-\gamma(\gamma\delta)^{n-1} - \delta(\gamma\delta)^{n-1}\right] \\ &= (-1)^{n-1}(\gamma^3 + \delta^3 + \gamma + \delta) \\ &= (-1)^{n-1}(\gamma + \delta)\left[(\gamma^2 + \delta^2) - \gamma\delta + 1\right] \\ &= (-1)^{n-1}(2x)\left[(\gamma + \delta)^2 + 4\right] \\ &= 2(-1)^{n-1}x(4x^2 + 4) \\ &= 8(-1)^{n-1}x(x^2 + 1) \\ \text{LHS} &= 2(-1)^{n-1}x, \text{ as desired.} \end{aligned}$$

■

It follows from this lemma that $P_{n+1}P_{n-2} - P_nP_{n-1} = 2(-1)^{n-1}$. For example, $P_7P_4 - P_6P_5 = 169 \cdot 12 - 70 \cdot 29 = -2 = 2(-1)^{6-1}$.

We will now use these two lemmas to establish a Cassini-like formula for $g_n(x)$.

Theorem 18.1

$$g_{n+1}(x)g_{n-1}(x) - g_n^2(x) = (-1)^{n-1}(a^2 - b^2 + 2abx). \tag{18.1}$$

Proof. Using Lemmas 18.1 and Lemmas 18.2, we have

$$\begin{aligned}
\text{LHS} &= [ap_n(x)+bp_{n+1}(x)][ap_{n-2}(x)+bp_{n-1}(x)]-[ap_{n-1}(x)+bp_n(x)]^2\\
&= a^2[p_n(x)p_{n-2}(x)-p_{n-1}^2(x)]+b^2[p_{n+1}(x)p_{n-1}(x)-p_n^2(x)]+\\
&\quad ab[p_{n+1}(x)p_{n-2}(x)-p_{n-1}(x)p_n(x)]\\
&= a^2(-1)^{n-1}+b^2(-1)^n+ab\left[2(-1)^{n-1}x\right]\\
&= (-1)^{n-1}(a^2-b^2+2abx)\\
&= \text{RHS, as desired.}
\end{aligned}$$

■

In particular, the following Cassini-like identities follow from formula (18.1):

$$\begin{aligned}
p_{n+1}(x)p_{n-1}(x)-p_n^2(x) &= (-1)^n\\
q_{n+1}(x)q_{n-1}(x)-q_n^2(x) &= 4(-1)^{n-1}(x^2+1)\\
f_{n+1}(x)f_{n-1}(x)-f_n^2(x) &= (-1)^n\\
l_{n+1}(x)l_{n-1}(x)-l_n^2(x) &= 5(-1)^{n-1}(x^2+4).
\end{aligned}$$

For example, we have

$$\begin{aligned}
q_5(x)q_3(x)-q_4^2(x) &= (32x^5+40x^3+10x)(8x^3+6x)-(16x^4+16x^2+2)^2\\
&= -4(x^2+1)=4(-1)^{4-1}(x^2+1).
\end{aligned}$$

These formulas yield the familiar Cassini-like identities for Pell, Pell–Lucas, Fibonacci, and Lucas numbers, respectively.

18.3 A Generalized Cassini-like Formula

Next, we pursue a generalized Cassini-like formula for $g_n(x)$. To this end, we need the next two results from the theory of matrices [125].

Theorem 18.2

(1) *Let A, B, and C be $n\times n$ identical matrices, except that their ith rows (or columns) are different, and that the ith row (or column) of C is the sum of the ith rows (or columns) of A and B. Then $|C|=|A|+|B|$, where $|M|$ denotes the determinant of the square matrix M.*

(2) *Let A and C be $n\times n$ identical matrices, except that the ith row (or column) of C is k times the ith row (or column) of A. Then $|C|=k|A|$.* ■

Combining these two properties, we have the following result.

Theorem 18.3 *Let A, B, and C be $n \times n$ identical matrices, except that their ith rows (or columns) are different, and that the ith row (or column) of C is the sum of k times the ith row (or column) of A and the ith row (or column) of B. Then $|C| = k|A| + |B|$.* ■

Next we introduce two families of matrices. First, let

$$A_{-1}(x) = \begin{bmatrix} g_n(x) & g_{n-1}(x) \\ g_{n+1}(x) & g_n(x) \end{bmatrix}$$

and

$$A_0(x) = \begin{bmatrix} g_n(x) & g_n(x) \\ g_{n+1}(x) & g_{n+1}(x) \end{bmatrix}.$$

Notice that $|A_0(x)| = 0$.

We now construct the matrix $A_i(x)$ recursively from $A_{i-1}(x)$ and $A_{i-2}(x)$ as follows: multiply column 2 of $A_{i-1}(x)$ by $2x$ and add column 2 of $A_{i-2}(x)$ to the resulting matrix. For example,

$$A_1(x) = \begin{bmatrix} g_n(x) & 2xg_n(x) + g_{n-1}(x) \\ g_{n+1}(x) & 2xg_{n+1}(x) + g_n(x) \end{bmatrix} = \begin{bmatrix} g_n(x) & g_{n+1}(x) \\ g_{n+1}(x) & g_{n+2}(x) \end{bmatrix}$$

and

$$A_2(x) = \begin{bmatrix} g_n(x) & 2xg_{n+1}(x) + g_n(x) \\ g_{n+1}(x) & 2xg_{n+2}(x) + g_{n+1}(x) \end{bmatrix} = \begin{bmatrix} g_n(x) & g_{n+2}(x) \\ g_{n+1}(x) & g_{n+3}(x) \end{bmatrix},$$

where $|A_1(x)| = (-1)^n(a^2 - b^2 + 2abx)$ by Theorem 18.1.

More generally, it follows by induction that

$$A_r(x) = \begin{bmatrix} g_n(x) & g_{n+r}(x) \\ g_{n+1}(x) & g_{n+r+1}(x) \end{bmatrix}.$$

By virtue of Theorem 18.3, $|A_r(x)| = 2x|A_{r-1}(x)| + |A_{r-2}(x)|$, where $|A_0(x)| = 0$ and $|A_1(x)| = (-1)^n(a^2-b^2+2abx)$. In other words, $|A_r(x)|$ satisfies the same recurrence as $g_r(x)$ does, with $a(x) = 0$ and $b(x) = (-1)^n(a^2 - b^2 + 2abx)$. Therefore, by Lemma 18.1, we have

$$\begin{aligned} |A_r(x)| &= 0 \cdot p_{r-1}(x) + \left[(-1)^n(a^2 - b^2 + 2abx)\right] p_r(x) \\ &= (-1)^n(a^2 - b^2 + 2abx)\, p_r(x). \end{aligned}$$

We now introduce the second family of matrices $B_s(x)$. To this end, we let

$$B_0(x) = \begin{bmatrix} g_n(x) & g_n(x) \\ g_{n-r}(x) & g_{n-r}(x) \end{bmatrix}$$

and

$$B_1(x) = \begin{bmatrix} g_{n+1}(x) & g_n(x) \\ g_{n-r+1}(x) & g_{n-r}(x) \end{bmatrix}.$$

Notice that $|B_0(x)| = 0$. The array $B_1(x)$ is obtained by changing n to $n-r$ in $A_r^T(x)$, and switching the rows and then the columns of the resulting matrix, where M^T denotes the *transpose* of the matrix M. So $|B_1(x)| = (-1)^{n-r}(a^2 - b^2 + 2abx)p_r(x)$.

Now construct $B_i(x)$ recursively from $B_{i-1}(x)$ and $B_{i-2}(x)$ as follows: multiply column 1 of $B_{i-1}(x)$ by $2x$ and add column 1 of $B_{i-2}(x)$ to the resulting matrix. For example,

$$B_2(x) = \begin{bmatrix} g_{n+2}(x) & g_n(x) \\ g_{n-r+2}(x) & g_{n-r}(x) \end{bmatrix}$$

and

$$B_3(x) = \begin{bmatrix} g_{n+3}(x) & g_n(x) \\ g_{n-r+3}(x) & g_{n-r}(x) \end{bmatrix}.$$

More generally, it follows by induction that

$$B_s(x) = \begin{bmatrix} g_{n+s}(x) & g_n(x) \\ g_{n-r+s}(x) & g_{n-r}(x) \end{bmatrix}.$$

Again, by Theorem 18.3, $|B_s(x)| = 2x|B_{s-1}(x)| + |B_{s-2}(x)|$, where $|B_0(x)| = 0$ and $|B_1(x)| = (-1)^{n-r}(a^2 - b^2 + 2abx)p_r(x)$. In other words, $|B_s(x)|$ satisfies the same recursive definition as $g_s(x)$ with $a(x) = 0$ and $b(x) = (-1)^{n-r}(a^2 - b^2 + 2abx)p_r(x)$. Consequently, by Lemma 18.1,

$$\begin{aligned} |B_s(x)| &= 0 \cdot p_{s-1}(x) + \left[(-1)^{n-r}(a^2 - b^2 + 2abx)p_r(x)\right] p_s(x) \\ &= (-1)^{n-r}(a^2 - b^2 + 2abx)p_r(x)p_s(x). \end{aligned}$$

That is,

$$g_{n+s}(x)g_{n-r}(x) - g_{n-r+s}(x)g_n(x) = (-1)^{n-r}(a^2 - b^2 + 2abx)p_r(x)p_s(x).$$

Let $m = n - r + s$. Then this becomes

$$g_{m+r}(x)g_{n-r}(x) - g_m(x)g_n(x) = (-1)^{n-r}(a^2 - b^2 + 2abx)p_r(x)p_{m-n+r}(x). \tag{18.2}$$

18.3.1 Two Interesting Special Cases

(1) Suppose $a(x) = 0$ and $b(x) = 1$. Then $g_n(x) = p_n(x)$ and formula (18.2) becomes

$$p_{m+r}(x)p_{n-r}(x) - p_m(x)p_n(x) = (-1)^{n-r-1}p_r(x)p_{m-n+r}(x).$$

Since $p_n(x/2) = f_n(x)$, this implies that

$$f_{m+r}(x)f_{n-r}(x) - f_m(x)f_n(x) = (-1)^{n-r-1}f_r(x)f_{m-n+r}(x).$$

In particular, these two formulas yield the following Pell and Fibonacci identities:

$$\begin{aligned} P_{m+r}P_{n-r} - P_mP_n &= (-1)^{n-r-1}P_rP_{m-n+r} \\ F_{m+r}F_{n-r} - F_mF_n &= (-1)^{n-r-1}F_rF_{m-n+r}. \end{aligned} \tag{18.3}$$

Suppose we let $r = 1$ and replace n with $n + 1$ in identity (18.3). We then get

$$F_{m+1}F_n - F_mF_{n+1} = (-1)^nF_{m-n}.$$

This is called *d'Ocagne's identity*, after the French mathematician Philbert Maurice d'Ocagne (1862–1938).

(2) On the other hand, suppose $a(x) = 2$ and $b(x) = 2x$. Then $g_n(x) = q_n(x)$ and formula (18.2) yields the following identities:

$$\begin{aligned} q_{m+r}(x)q_{n-r}(x) - q_m(x)q_n(x) &= 4(-1)^{n-r}(x^2+1)p_r(x)p_{m-n+r}(x) \\ l_{m+r}(x)l_{n-r}(x) - l_m(x)l_n(x) &= (-1)^{n-r}(x^2+4)f_r(x)f_{m-n+r}(x) \\ q_{m+r}q_{n-r} - q_mq_n &= 2(-1)^{n-r}P_rP_{q-n+r} \\ L_{m+r}L_{n-r} - L_mL_n &= 5(-1)^{n-r}F_rF_{m-n+r}. \end{aligned}$$

When $m = n$, the generalized Cassini-like formula (18.2) yields the following identity for the extended Pell family:

$$g_{n+r}(x)g_{n-r}(x) - g_n^2(x) = (-1)^{n-r}(a^2 - b^2 + 2abx)p_r^2(x).$$

This yields the following special cases:

$$\begin{aligned} p_{n+r}(x)p_{n-r}(x) - p_n^2(x) &= (-1)^{n-r-1}p_r^2(x) \\ q_{n+r}(x)q_{n-r}(x) - q_n^2(x) &= 4(-1)^{n-r}(x^2+1)p_r^2(x) \\ f_{n+r}(x)f_{n-r}(x) - f_n^2(x) &= (-1)^{n-r-1}f_r^2(x) \end{aligned} \tag{18.4}$$

$$
\begin{aligned}
l_{n+r}(x)l_{n-r}(x) - l_n^2(x) &= 4(-1)^{n-r}(x^2+4)f_r^2(x)\\
P_{n+r}P_{n-r} - P_n^2 &= (-1)^{n-r-1}P_r^2\\
Q_{n+r}Q_{n-r} - Q_n^2 &= 2(-1)^{n-r}P_r^2\\
F_{n+r}F_{n-r} - F_n^2 &= (-1)^{n-r-1}F_r^2\\
L_{n+r}L_{n-r} - L_n^2 &= 5(-1)^{n-r}F_r^2.
\end{aligned}
\tag{18.5}
$$

For example,

$$
\begin{aligned}
p_7(x)p_3(x) - p_5^2(x) &= (64x^6+80x^4+24x^2+1)(4x^2+1) - (16x^4+12x^2+1)^2\\
&= 4x^2 = (-1)^{5-2-1}p_2^2(x);
\end{aligned}
$$

and $P_{14}P_4 - P_9^2 = 80,782 \cdot 12 - 985^2 = -841 = -29^2 = (-1)^{9-5-1}P_5^2$;

$$
\begin{aligned}
q_6(x)q_2(x) - q_4^2(x) &= (64x^6+96x^4+36x^2+2)(4x^2+2) - (16x^4+16x^2+2)^2\\
&= 16x^2(x^2+1) = 4(x^2+1)(2x)^2\\
&= 4(-1)^{4-2}(x^2+1)p_2^2(x);
\end{aligned}
$$

and $Q_{10}Q_4 - Q_7^2 = 3363 \cdot 17 - 239^2 = 50 = 2 \cdot 5^2 = 2(-1)^{7-3}P_3^2$;

$$
\begin{aligned}
f_8(x)f_2(x) - f_5^2(x) &= (x^7+6x^5+10x^3+4x)x - (x^4+3x^2+1)^2\\
&= -(x^2+1)^2 = (-1)^{5-3-1}f_3^2(x);
\end{aligned}
$$

and $F_{11}F_5 - F_8^2 = 89 \cdot 5 - 21^2 = 4 = (-1)^{8-3-1}F_3^2$; and

$$
\begin{aligned}
l_8(x)l_2(x) - l_5^2(x) &= (x^8+8x^6+20x^4+16x^2+2)(x^2+2) - (x^5+5x^3+5x)^2\\
&= x^6+6x^4+9x^2+4 = (-1)^{5-3}(x^2+4)f_3^2(x);
\end{aligned}
$$

and $L_{11}L_5 - L_8^2 = 199 \cdot 11 - 47^2 = -20 = 5(-1)^{8-3}F_3^2$.

Identity (18.5) is *Catalan's identity*, named after the Belgian mathematician Eugene Charles Catalan (1814–1894). The ubiquitous Catalan numbers are named after him [131].

Letting $r = n$ in (18.4), we get the identity

$$
2q_{2n}(x) = q_n^2(x) + 4(x^2+1)p_n^2(x). \tag{18.6}
$$

For example,

$$
\begin{aligned}
q_3^2(x) + 4(x^2+1)p_3^2(x) &= (8x^3+6x)^2 + 4(x^2+1)(4x^2+1)^2\\
&= 128x^6+192x^4+72x^2+4 = 2q_6(x).
\end{aligned}
$$

In particular, it follows from (18.6) that $Q_{2n} = Q_n^2 + 2P_n^2$, as we saw in Chapter 7. For example, $Q_6^2 + 2P_6^2 = 99^2 + 2 \cdot 70^2 = 19,601 = Q_{12}$.

19

Chebyshev Polynomials

19.1 Introduction

Pell and Pell–Lucas polynomials are related to the well-known Chebyshev polynomials, named after the eminent Russian mathematician Pafnuty Lvovich Chebyshev (1821–1894). Just as the Pell polynomial family consists of two closely related sub-families $\{p_n(x)\}$ and $\{q_n(x)\}$, the Chebyshev family is made up of two closely related sub-families $\{T_n(x)\}$ and $\{U_n(x)\}$. (The letter T comes from the French transliteration, *Tchebycheff* or the German one, *Tschebyscheff*.) Chebyshev polynomials have applications to approximation theory, combinatorics, Fourier series, numerical analysis, geometry, graph theory, number theory, and statistics [184].

This chapter presents a brief introduction to Chebyshev polynomials of both kinds. They are closely related to the Pell equation $u^2 - dv^2 = 1$, trigonometry, and the tilings of $1 \times n$ linear and circular boards.

We begin our discussion with the Chebyshev polynomials $\{T_n(x)\}$.

19.2 Chebyshev Polynomials of the First Kind

Chebyshev polynomials of the first kind $T_n(x)$ are often defined recursively:

$$\begin{aligned} T_0(x) &= 1, \quad T_1(x) = x \\ T_n(x) &= 2xT_{n-1}(x) - T_{n-2}(x), \end{aligned} \tag{19.1}$$

where $n \geq 2$.

For example, $T_2(x) = 2xT_1(x) - T_0(x) = 2x \cdot x - 1 = 2x^2 - 1$ and $T_3(x) = 2x(2x^2 - 1) - x = 4x^3 - 3x$.

Table 19.1 shows the first ten Chebyshev polynomials of the first kind.

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Table 19.1.

$T_0(x) = 1$	$T_5(x) = 16x^5 - 20x^3 + 5x$
$T_1(x) = x$	$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$
$T_2(x) = 2x^2 - 1$	$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$
$T_3(x) = 4x^3 - 3x$	$T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$
$T_4(x) = 8x^4 - 8x^2 + 1$	$T_9(x) = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$

We will now show how the polynomials $T_n(x)$ and $q_n(x)$ are closely related. To this end, first notice that $q_1(x) = 2x = 2i^4x = 2(-i)(ix) = 2(-i)T_1(x)$; and $q_2(x) = 4x^2 + 2 = 2(2x^2+1) = 2i^4(2x^2+1) = 2(-i)^2[2(ix)^2 - 1] = 2(-i)^2T_2(ix)$, where i denotes the imaginary number $\sqrt{-1}$.

More generally, suppose $q_n(x) = 2(-i)^nT_n(ix)$ for all nonnegative integer $< n$, where n is an arbitrary integer ≥ 2. Then, by Chebyshev recurrence (19.1), we have

$$\begin{aligned}
T_n(ix) &= 2(ix)T_{n-1}(ix) - T_{n-2}(ix)\\
&= 2ix\left[\frac{q_{n-1}(x)}{2(-i)^{n-1}}\right] - \frac{q_{n-2}(x)}{2(-i)^{n-2}}\\
&= \frac{2xq_{n-1}(x)}{2(-i)^n} + \frac{q_{n-2}(x)}{2(-i)^n}\\
&= \frac{2xq_{n-1}(x) + q_{n-2}(x)}{2(-i)^n}\\
&= \frac{q_n(x)}{2(-i)^n}\\
2(-i)^nT_n(ix) &= q_n(x).
\end{aligned}$$

Thus, by the strong version of PMI, $q_n(x) = 2(-i)^nT_n(ix)$ for $n \geq 0$. ■

For example,

$$\begin{aligned}
2(-i)^5T_5(ix) &= -2i[16(ix)^5 - 20(ix)^3 + 5(ix)]\\
&= -2i(16ix^5 + 20ix^3 + 5ix)\\
&= 32x^5 + 40x^3 + 10x\\
&= q_5(x), \quad \text{as expected.}
\end{aligned}$$

19.3 Pell–Lucas Numbers Revisited

Since $q_n(1) = 2Q_n$, it now follows that $2Q_n = 2(-i)^nT_n(i)$; so $Q_n = |T_n(i)|$, where $|z|$ denotes the absolute value of the complex number z.

For example, $Q_5 = (-i)^5T_5(i) = -i(16i^5 - 20i^3 + 5i) = 16 + 20 + 5 = 41$. Similarly, $(-i)^6T_6(i) = 99 = Q_6$.

Next we develop an explicit formula for $T_n(x)$ from its recurrence.

19.4 An Explicit Formula for $T_n(x)$

The characteristic equation of recurrence (19.1) is $t^2 - 2xt + 1 = 0$. Its solutions are $\frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$. So the general solution of the recurrence is given by $T_n(x) = A(x + \sqrt{x^2-1})^n + B(x - \sqrt{x^2-1})^n$, where $A = A(x)$ and $B = B(x)$ are to be determined. Using the two initial conditions, we get $A = B = \frac{1}{2}$. Thus the desired explicit formula is

$$T_n(x) = \frac{(x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n}{2}, \qquad (19.2)$$

where $n \geq 0$.

For example,

$$\begin{aligned} 2T_3(x) &= (x + \sqrt{x^2-1})^3 + (x - \sqrt{x^2-1})^3 \\ &= 2[x^3 + 3x(x^2-1)] = 2(4x^3 - 3x) \\ T_3(x) &= 4x^3 - 3x. \end{aligned}$$

Formula (19.2) can also be used to prove that $|T_n(i)| = Q_n$:

$$\begin{aligned} 2T_n(i) &= (i + \sqrt{i^2-1})^n + (i - \sqrt{i^2-1})^n \\ &= (i + i\sqrt{2})^n + (i - i\sqrt{2})^n \\ &= i^n(\gamma^n + \delta^n) \\ |T_n(i)| &= \frac{\gamma^n + \delta^n}{2}, \quad \text{since } |i| = 1 \\ &= Q_n, \quad \text{as desired.} \end{aligned}$$

The next example shows an interesting property of the Chebyshev polynomials $T_n(x)$.

Example 19.1 Prove that $2T_m(x)T_n(x) = T_{m+n}(x) + T_{|m-n|}(x)$.

Proof. Suppose $m \geq n \geq 0$. For convenience, we let $\sqrt{x^2-1} = r$; so $x^2 - r^2 = 1$.

$$\begin{aligned} 4T_m(x)T_n(x) &= [(x+r)^m + (x-r)^m]\,[(x+r)^n + (x-r)^n] \\ &= \left[(x+r)^{m+n} + (x-r)^{m+n}\right] + (x+r)^m(x-r)^n + (x-r)^m(x+r)^n \\ &= 2T_{m+n}(x) + (x+r)^n(x-r)^n\,[(x+r)^{m-n} + (x-r)^{m-n}] \\ &= 2T_{m+n}(x) + (x^2-r^2)^n \cdot 2T_{m-n}(x) \\ &= 2T_{m+n}(x) + 2T_{m-n}(x) \\ 2T_m(x)T_n(x) &= T_{m+n}(x) + T_{m-n}(x). \end{aligned}$$

Similarly, it can be shown that $2T_m(x)T_n(x) = T_{m+n}(x) + T_{n-m}(x)$ when $n > m$. Thus the desired result follows by combining the two cases. ∎

For example, let $m = 5$ and $n = 3$. Then

$$\begin{aligned}
\text{RHS} &= T_8(x) + T_2(x) \\
&= (128x^8 - 256x^6 + 160x^4 - 32x^2 + 1) + (2x^2 - 1) \\
&= 128x^8 - 256x^6 + 160x^4 - 30x^2 \\
&= 2(16x^5 - 20x^3 + 5x)(4x^3 - 3x) \\
&= 2T_5(x)T_3(x).
\end{aligned}$$

Suppose we let $n = 1$ and assume that $m \geq 1$. Then the formula yields

$$\begin{aligned}
T_{m+1}(x) + T_{m-1}(x) &= 2T_m(x)T_1(x) \\
&= 2xT_m(x).
\end{aligned}$$

This is the Chebyshev recurrence (19.1). So the formula in Example 19.1 is a generalization of this recurrence.

Next we develop another explicit formula for $T_n(x)$ using formula (19.2) and the binomial theorem.

19.5 Another Explicit Formula for $T_n(x)$

Again, we let $\sqrt{x^2 - 1} = r$ for brevity and clarity. By the binomial theorem, we have

$$\begin{aligned}
2T_n(x) &= (x + r)^n + (x - r)^n \\
&= \sum_{k=0}^{n} [1 + (-1)^k] \binom{n}{k} x^{n-k} r^k \\
&= 2 \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} x^{n-2k} r^{2k} \\
T_n(x) &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (x^2 - 1)^k x^{n-2k}. \qquad (19.3)
\end{aligned}$$

For example,

$$T_3(x) = \sum_{k=0}^{1} \binom{3}{2k} (x^2 - 1)^k x^{3-2k}$$

$$
\begin{aligned}
&= \binom{3}{0}x^3+\binom{3}{2}(x^2-1)x \\
&= x^3+3x(x^2-1) \\
&= 4x^3-3x.
\end{aligned}
$$

We can derive an explicit formula for Q_n from (19.3):

$$
Q_n=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^k. \tag{19.4}
$$

This is the same as formula (9.6) in Chapter 9.

Formula (19.4) shows that the sum of the absolute values of the coefficients in $T_n(x)$ is Q_n. For example, consider $T_4(x)=8x^4-8x^2+1$. The sum of the absolute values of its coefficients equals $8+8+1=17=Q_4$.

19.5.1 Two Interesting Byproducts

We can extract two interesting properties of $T_n(x)$ from formula (19.3):

(1) The highest term in x from the factor $(x^2-1)^k$ is x^{2k}. So the highest power of x in $(x^2-1)^k x^{n-2k}$ equals $x^{2k+n-2k}=x^n$. Consequently, $T_n(x)$ is a polynomial of degree n.

(2) The coefficient of x^n in $T_n(x)$ is given by $\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}$. But by Corollary 1.1, $\sum_{r\text{ even}}\binom{n}{r}=\sum_{r\text{ odd}}\binom{n}{r}=2^{n-1}$. So the coefficient of x^n is 2^{n-1}; that is, the leading coefficient in $T_n(x)$ is 2^{n-1}.

For example, the leading coefficient of $T_5(x)$ equals $16=2^4$ and that of $T_6(x)$ is $32=2^5$.

Next we exhibit a close relationship between $T_n(x)$ and the Pell's equation $u^2-(x^2-1)v^2=1$.

19.6 $T_n(x)$ and the Pell Equation $u^2-(x^2-1)v^2=1$

Clearly, $(u_1,v_1)=(x,1)$ is the fundamental solution of the Pell equation $u^2-(x^2-1)v^2=1$, where $x^2-1>0$ and is nonsquare. Its solutions (u_n,v_n) are given by

$$
\begin{aligned}
\begin{bmatrix}u_n\\ v_n\end{bmatrix} &= \begin{bmatrix}x & x^2-1\\ 1 & x\end{bmatrix}\begin{bmatrix}u_{n-1}\\ v_{n-1}\end{bmatrix} \\
&= 2x\begin{bmatrix}u_{n-1}\\ v_{n-1}\end{bmatrix}-\begin{bmatrix}u_{n-2}\\ v_{n-2}\end{bmatrix},
\end{aligned}
$$

where $n \geq 2$. [Notice that $(u_0, v_0) = (1, 0)$ is also a solution.] It follows by formula (19.2) that $u_n = T_n(x)$. (We will revisit this Pell equation a bit later.)

Next we display an interestingly close relationship between the polynomials $T_n(x)$ and trigonometry.

19.7 Chebyshev Polynomials $T_n(x)$ and Trigonometry

Let u, v, and θ be any three angles, where $0 \leq \theta \leq \pi$. Using the *Euler's formula*, $e^{i\theta} = \cos\theta + i\sin\theta$, we have

$$\begin{aligned} e^{i(u+v)} &= e^{iu} \cdot e^{iv} \\ \cos(u+v) + i\sin(u+v) &= (\cos u + i\sin u)(\cos v + i\sin v) \\ &= (\cos u\cos v - \sin u\sin v) + i(\sin u\cos v + \cos u\sin v). \end{aligned}$$

Equating the real and imaginary parts, we get the *addition formulas* for the cosine and sine functions:

$$\cos(u+v) = \cos u\cos v - \sin u\sin v \qquad (19.5)$$

$$\sin(u+v) = \sin u\cos v + \cos u\sin v. \qquad (19.6)$$

Since $\sin^2\theta + \cos^2\theta = 1$, formula (19.5) yields the *double-angle formula* $\cos 2\theta = \cos(\theta+\theta) = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1$. It follows from formula (19.6) that $\sin 2\theta = 2\sin\theta\cos\theta$. Consequently,

$$\begin{aligned} \cos 3\theta &= \cos(2\theta+\theta) \\ &= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta \\ &= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta \\ &= 2\cos^3\theta - \cos\theta - 2\cos\theta(1-\cos^2\theta) \\ &= 4\cos^3\theta - 3\cos\theta. \end{aligned}$$

Similarly, $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$.

Table 19.2 shows the first eight *multiple-angle formulas* for the cosine function. They manifest an interesting pattern: The RHS of $\cos n\theta$ is a polynomial in $\cos\theta$ with exactly the same coefficients as in $T_n(x)$, where $0 \leq n \leq 7$; that is, $\cos n\theta = T_n(\cos\theta)$, where $0 \leq n \leq 7$.

Table 19.2.

$\cos 0\theta$	=	1	$\cos 4\theta$	=	$8\cos^4\theta - 8\cos^2\theta + 1$
$\cos 1\theta$	=	$\cos\theta$	$\cos 5\theta$	=	$16\cos^5\theta - 20\cos^3\theta + \cos\theta$
$\cos 2\theta$	=	$2\cos^2\theta - 1$	$\cos 6\theta$	=	$32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$
$\cos 3\theta$	=	$4\cos^3\theta - 3\cos\theta$	$\cos 7\theta$	=	$64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$

Is this true in general? That is, is $\cos n\theta = T_n(\cos\theta)$ for every $n \geq 0$? Fortunately, the answer is yes. We will confirm this using induction and the well-known *De Moivre theorem*:

First, notice that

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\cos\theta(\cos 1\theta) - \cos 0\theta\\ &= 2\cos\theta T_1(\cos\theta) - T_0(\cos\theta).\end{aligned}$$

Likewise,

$$\begin{aligned}\cos 3\theta &= 4\cos^3\theta - 3\cos\theta\\ &= 2\cos\theta(2\cos^2\theta - 1) - \cos\theta\\ &= 2\cos\theta\cos 2\theta - \cos\theta\\ &= 2\cos\theta T_2(\cos\theta) - T_1(\cos\theta).\end{aligned}$$

We are now ready to confirm the observation. Since $T_0(\cos\theta) = 1 = \cos 0\theta$ and $T_1(\cos\theta) = \cos\theta$, the statement is true when $n = 0$ and $n = 1$. It remains to show that $\cos n\theta$ satisfies the same recurrence as $T_n(\cos\theta)$.

Assume that it is true for all integers $< n$, where n is an arbitrary positive integer. Let $a = e^{i\theta}$ and $b = e^{-i\theta}$. Then $a + b = 2\cos\theta$ and $ab = 1$. By De Moivre's theorem, $a^k = e^{ik\theta}$ and $b^k = e^{-ik\theta}$; so $a^k + b^k = 2\cos k\theta$. Substituting for a and b in the algebraic identity

$$a^n + b^n = (a+b)(a^{n-1} + b^{n-1}) - ab(a^{n-2} + b^{n-2}),$$

we get

$$\begin{aligned}2\cos n\theta &= 2\cos\theta \cdot 2\cos(n-1)\theta - 1 \cdot 2\cos(n-2)\theta\\ \cos n\theta &= 2\cos\theta[\cos(n-1)\theta] - \cos(n-2)\theta\\ &= 2\cos\theta \cdot T_{n-1}(\cos\theta) - T_{n-2}(\cos\theta)\\ &= T_n(\cos\theta).\end{aligned}$$

Thus, by the strong version of PMI, $T_n(\cos\theta) = \cos n\theta$ for every $n \geq 0$. Thus $\cos n\theta$ is a Chebyshev polynomial function of $\cos\theta$.

Since $0 \leq \theta \leq \pi$, $-1 \leq \cos\theta \leq 1$; that is, $-1 \leq x \leq 1$. So $T_n(x) = \cos n\theta = \cos(n \arccos x)$, where $0 \leq \arccos x \leq \pi$.

19.8 Chebyshev Recurrence Revisited

Interestingly, the Chebyshev recurrence (19.1) can be obtained using the *sum identity*

$$\cos u + \cos v = 2\cos\frac{u+v}{2}\cos\frac{u-v}{2}.$$

To see this, let $u = n\theta$ and $v = (n-2)\theta$. Then

$$\begin{aligned}\cos n\theta + \cos(n-2)\theta &= 2\cos(n-1)\theta\cos\theta\\ T_n(x) + T_{n-2}(x) &= 2xT_{n-1}(x).\end{aligned}$$

This is the desired recurrence.

Let $x = \cos\theta$. Then the fact that $\cos n\theta = T_n(x)$ can be used to derive several properties of the Chebyshev polynomial $T_n(x)$, as the following three examples illustrate. We leave their proofs as exercises.

Example 19.2 Prove that $T_m(T_n(x)) = T_{mn}(x)$, where m and n are nonnegative integers. ■

For example, let $m = 3$ and $n = 2$. Then $T_2(x) = 2x^2 - 1$ and $T_3(x) = 4x^3 - 3x$. So

$$\begin{aligned}T_3(T_2(x)) &= 4(2x^2-1)^3 - 3(2x^2-1)\\ &= 4(8x^6 - 12x^4 + 6x^2 - 1) - 6x^2 + 3\\ &= 32x^6 - 48x^4 + 18x^2 - 1\\ &= T_6(x).\end{aligned}$$

Similarly, $T_2(T_3(x)) = 32x^6 - 48x^4 + 18x^2 - 1 = T_6(x)$.

The next example reconfirms the identity in Example 19.1 in a much quicker way, using a trigonometric identity.

Example 19.3 Reconfirm the identity $2T_m(x)T_n(x) = T_{m+n}(x) + T_{|m-n|}(x)$, where m and n are nonnegative integers. ■

The next example also shows the power and beauty of the trigonometric relationship in establishing properties of the Chebyshev polynomials.

Example 19.4 Prove that $[T_{m+n}(x) - 1][T_{m-n}(x) - 1] = [T_m(x) - T_n(x)]^2$. ■

19.9 A Summation Formula For $T_n(x)$

Using the binomial theorem and De Moivre's theorem, we can now derive a summation formula for $T_n(x)$:

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= \sum_{r=0}^{n} i^r \binom{n}{r} \cos^{n-r}\theta\sin^r\theta\\ \cos n\theta + i\sin n\theta &= \cos^n\theta + i\binom{n}{1}\cos^{n-1}\theta\sin\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \cdots\end{aligned}$$

Equating the real parts from both sides, we get

$$\begin{aligned}
\cos n\theta &= \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \binom{n}{4}\cos^{n-4}\theta\sin^2\theta + \cdots + \\
&\quad (-1)^{\lfloor n/2\rfloor}\binom{n}{2\lfloor n/2\rfloor}\cos^{n-2\lfloor n/2\rfloor}\theta\sin^{2\lfloor n/2\rfloor}\theta \\
&= \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{2k}\cos^{n-2k}\theta\sin^{2k}\theta \\
&= \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{2k}\cos^{n-2k}\theta(1-\cos^2\theta)^k \\
&= \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{2k}\cos^{n-2k}\theta\left[\sum_{j=0}^{k}(-1)^j\binom{k}{j}\cos^{2j}\theta\right] \\
&= \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\sum_{j=k}^{\lfloor n/2\rfloor}\binom{n}{2j}\binom{j}{k}\cos^{n-2k}\theta,
\end{aligned}$$

after a lot of algebra [184]. This implies that

$$T_n(x) = \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\sum_{j=k}^{\lfloor n/2\rfloor}\binom{n}{2j}\binom{j}{k}x^{n-2k}. \tag{19.7}$$

For example,

$$\begin{aligned}
T_3(x) &= \sum_{k=0}^{1}(-1)^k\sum_{j=k}^{1}\binom{3}{2j}\binom{j}{k}x^{3-2k} \\
&= \sum_{j=0}^{1}\binom{3}{2j}\binom{j}{0}x^3 - \sum_{j=1}^{1}\binom{3}{2}\binom{j}{1}x \\
&= (1+3)x^3 - 3x \\
&= 4x^3 - 3x.
\end{aligned}$$

Formula (19.7) also implies that $T_n(x)$ is a polynomial of degree n. Its leading coefficient, by Corollary 1.1, equals

$$\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j}\binom{j}{0} = \sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j} = 2^{n-1}.$$

19.9.1 A Summation Formula For Q_n

Formula (19.7) gives an interesting byproduct:

$$Q_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{j=k}^{\lfloor n/2 \rfloor} \binom{n}{2j}\binom{j}{k}. \tag{19.8}$$

Consequently, we can compute Q_n using the even-numbered binomial coefficients in row n of Pascal's triangle.

For example,

$$\begin{aligned} Q_5 &= \sum_{k=0}^{2}\sum_{j=k}^{2}\binom{5}{2j}\binom{j}{k} \\ &= \sum_{j=0}^{2}\binom{5}{2j}\binom{j}{0} + \sum_{j=1}^{2}\binom{5}{2j}\binom{j}{1} + \sum_{j=2}^{2}\binom{5}{2j}\binom{j}{2} \\ &= \left[\binom{5}{0}\binom{0}{0} + \binom{5}{2}\binom{1}{0} + \binom{5}{4}\binom{2}{0}\right] + \left[\binom{5}{2}\binom{1}{1} + \binom{5}{4}\binom{2}{1}\right] + \binom{5}{4}\binom{2}{2} \\ &= (1+10+5) + (10+10) + 5 = \boxed{41}\text{, as expected.} \end{aligned}$$

See Table 19.3 as well.

Table 19.3.

Even-numbered entries in row 5:	1	10	5	1	10	5	1	10	5
Weights:	$\binom{0}{0}$	$\binom{1}{0}$	$\binom{2}{0}$	·	$\binom{1}{1}$	$\binom{2}{1}$	·	·	$\binom{2}{2}$
Multiply:	1	10	5	·	10	10	·	·	5
Add:	16			20			5		
Cumulative sum:	(41)								

Likewise,

$$Q_4 = \sum_{k=0}^{2}\sum_{j=k}^{2}\binom{4}{2j}\binom{j}{k} = 8 + 8 + 1 = 17.$$

Next we present the closely-related Chebyshev polynomials of the second kind; they satisfy the same recurrence as $T_n(x)$.

19.10 Chebyshev Polynomials of the Second Kind

Chebyshev polynomials of the second kind $U_n(x)$ are also often defined recursively:

$$\begin{aligned} U_0(x) &= 1, \quad U_1(x) = 2x \\ U_n(x) &= 2xU_{n-1}(x) - U_{n-2}(x), \end{aligned} \tag{19.9}$$

where $n \geq 2$.

Table 19.4 shows the first ten Chebyshev polynomials of the second kind.

Table 19.4.

$U_0(x) = 1$	$U_5(x) = 32x^5 - 32x^3 + 6x$
$U_1(x) = 2x$	$U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1$
$U_2(x) = 4x^2 - 1$	$U_7(x) = 128x^7 - 192x^5 + 80x^3 - 8x$
$U_3(x) = 8x^3 - 4x$	$U_8(x) = 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1$
$U_4(x) = 16x^4 - 12x^2 + 1$	$U_9(x) = 512x^9 - 1024x^7 + 672x^5 - 160x^3 + 10x$

Just as Chebyshev polynomials of the first kind $T_n(x)$ and Pell–Lucas polynomials $q_n(x)$ are closely related, so are the Chebyshev polynomials of the second kind $U_n(x)$ and the Pell polynomials $p_n(x)$. To see this relationship, first notice that

$$\begin{aligned} p_1(x) &= 1 = (-i)^0 \cdot 1 = (-i)^0 U_0(ix) \\ p_2(x) &= 2x = (-i)(2ix) = (-i)^1 U_1(ix) \\ p_3(x) &= 4x^2 + 1 = (-i)^2[4(ix)^2 - 1] = (-i)^2 U_2(ix). \end{aligned}$$

More generally, we claim that $p_n(x) = (-i)^{n-1}U_{n-1}(ix)$, where $n \geq 1$. To this end, suppose that it is true for all positive integers $\leq n$, where n is an arbitrary integer ≥ 2. Then, by the recurrence (19.9), we have

$$\begin{aligned} U_n(ix) &= 2ixU_{n-1}(ix) - U_{n-2}(ix) \\ &= 2ix\left[\frac{p_n(x)}{(-i)^{n-1}}\right] - \frac{p_{n-1}(x)}{(-i)^{n-2}} \\ &= \frac{2xp_n(x)}{(-i)^n} + \frac{p_{n-1}(x)}{(-i)^n} \\ (-i)^n U_n(ix) &= 2xp_n(x) + p_{n-1}(x) \\ &= p_{n+1}(x). \end{aligned}$$

Thus, by the strong version of PMI, it follows that $p_n(x) = (-i)^{n-1}U_{n-1}(ix)$ for every $n \geq 1$. ■

For example,

$$\begin{aligned}(-i)^5U_5(ix) &= -i\left[32(ix)^5 - 32(ix)^3 + 6(ix)\right]\\ &= 32x^5 + 32x^3 + 6x\\ &= p_6(x).\end{aligned}$$

19.11 Pell Numbers Revisited

Since $P_n = p_n(1)$, it follows from the formula $p_n(x) = (-i)^{n-1}U_{n-1}(ix)$ that $P_n = (-i)^{n-1}U_{n-1}(i)$; so $P_n = |U_{n-1}(i)|$ = sum of the absolute values of the coefficients in $U_{n-1}(x)$.

For example, $P_5 = (-i)^4U_4(i) = 16i^4 - 12i^2 + 1 = 16 + 12 + 1 = 29$. Similarly, $P_7 = (-i)^6U_6(i) = 169$.

Next we find an explicit formula for $U_n(x)$ from the recurrence (19.9).

19.12 An Explicit Formula for $U_n(x)$

Since both $T_n(x)$ and $U_n(x)$ satisfy the same recurrence, it follows from our earlier discussion that

$$U_n(x) = A(x + \sqrt{x^2-1})^n + B(x - \sqrt{x^2-1})^n,$$

where $A = A(x)$ and $B = B(x)$ are to be determined. Using the initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$, we get

$$U_n(x) = \frac{(x + \sqrt{x^2-1})^{n+1} - (x - \sqrt{x^2-1})^{n+1}}{2\sqrt{x^2-1}}, \tag{19.10}$$

where $n \geq 0$.

For example,

$$\begin{aligned}U_3(x) &= \frac{(x + \sqrt{x^2-1})^4 - (x - \sqrt{x^2-1})^4}{2\sqrt{x^2-1}}\\ &= \frac{(2x)2\sqrt{x^2-1}\left\{\left[x^2 + (x^2-1) + 2x\sqrt{x^2-1}\right] + \left[x^2 + (x^2-1) - 2x\sqrt{x^2-1}\right]\right\}}{2\sqrt{x^2-1}}\\ &= 2x(4x^2-2)\\ &= 8x^3 - 4x,\end{aligned}$$

where we have used the fact that $a^4 - b^4 = (a+b)(a-b)(a^2+b^2)$.

Interestingly, formula (19.10) can also be used to establish that $P_n = |U_{n-1}(i)|$:

$$\begin{aligned}
2\sqrt{2}U_{n-1}(i) &= (i+\sqrt{i^2-1})^n - (i-\sqrt{i^2-1})^n \\
&= (i+i\sqrt{2})^n - (i-i\sqrt{2})^n \\
&= i^n(\gamma^n - \delta^n) \\
|U_{n-1}(i)| &= \frac{\gamma^n-\delta^n}{2\sqrt{2}} \\
&= P_n, \quad \text{as expected.}
\end{aligned}$$

19.13 Another Explicit Formula for $U_n(x)$

Formula (19.10), coupled with the binomial theorem, can be used to develop a second explicit formula for $U_n(x)$. For convenience, we let $r = \sqrt{x^2-1}$. Then

$$\begin{aligned}
2rU_n(x) &= (x+r)^{n+1} - (x-r)^{n+1} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} [1-(-1)^k] x^{n-k+1} r^k \\
&= 2\sum_{k \text{ odd}} \binom{n+1}{k} x^{n-k+1} r^k \\
&= 2\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} x^{n-2k} r^{2k+1}
\end{aligned}$$

$$U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} (x^2-1)^k x^{n-2k}. \tag{19.11}$$

For example,

$$\begin{aligned}
U_3(x) &= \sum_{k=0}^{1} \binom{4}{2k+1} (x^2-1)^k x^{3-2k} \\
&= \binom{4}{1} x^3 + \binom{4}{3} (x^2-1)x \\
&= 8x^3 - 4x.
\end{aligned}$$

19.13.1 An Explicit Formula for P_n

It follows from formula (19.11) that

$$P_n = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k+1} 2^k. \tag{19.12}$$

This is the same as formula (9.4) in Chapter 9.

Returning to formula (19.11), we note that it implies that $U_n(x)$ is a polynomial of degree n. Its leading coefficient is given by

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} = \sum_{r \text{ odd}} \binom{n+1}{r} = 2^n.$$

For example, $U_7(x)$ is a polynomial of degree 7, with leading coefficient $128 = 2^7$.

When $x = 1$, formula (19.11) yields

$$U_n(1) = \binom{n+1}{1} + \sum_{k \geq 2} \binom{n+1}{2k-1} \cdot 0 = n+1.$$

For example, $U_4(1) = 16 - 12 + 1 = 5$.

On the other hand, suppose we let $x = -1$. Then the formula yields

$$U_n(-1) = \binom{n+1}{1}(-1)^n + \sum_{k \geq 2} \binom{n+1}{2k-1} \cdot 0 = (n+1)(-1)^n.$$

For example, $U_4(-1) = 16(-1)^4 - 12(-1)^2 + 1 = 5 = (4+1)(-1)^4$; similarly, $U_5(-1) = -6 = (5+1)(-1)^5$.

Suppose we let $n = 2m$ in formula (19.11). Then

$$U_{2m}(x) = \sum_{k=0}^{m} \binom{2m+1}{2k+1} (x^2-1)^k x^{2m-2k}.$$

So we can obtain the constant term in $U_{2m}(x)$ when $k = m$, namely, $\binom{2m+1}{2m+1}(-1)^m = (-1)^m$. Thus the constant term in $U_n(x)$ is $(-1)^{n/2}$, when n is even.

For example, the constant term in $U_6(x)$ is $-1 = (-1)^3$ and that in $U_8(x)$ is $1 = (-1)^4$.

On the other hand, suppose we let $n = 2m+1$ in formula (19.11). Then

$$U_{2m+1}(x) = \sum_{k=0}^{m} \binom{2m+2}{2k+1} (x^2-1)^k x^{2m-2k+1}.$$

So we can obtain the coefficient of x in $U_{2m+1}(x)$ when $k = m$, namely, $\binom{2m+2}{2m+1}(-1)^m = (2m+2)(-1)^m$. Thus, when n is odd, the coefficient of x in $U_n(x)$ is $(n+1)(-1)^{(n-1)/2}$.

For example, the coefficient of x in $U_5(x)$ is $6 = (5+1)(-1)^{(5-1)/2}$ and that in $U_7(x)$ is $-8 = (7+1)(-1)^{(7-1)/2}$.

The next four examples establish some properties satisfied by the two Chebyshev families.

Example 19.5 Prove that $U_n(x) - U_{n-2}(x) = 2T_n(x)$, where $n \geq 2$. ∎

For example,

$$\begin{aligned} U_7(x) - U_5(x) &= (128x^7 - 192x^5 + 80x^3 - 8x) - (32x^5 - 32x^3 + 6x) \\ &= 2(64x^7 - 112x^5 + 56x^3 - 7x) \\ &= 2T_7(x). \end{aligned}$$

The following example will need the value of $U_{-1}(x)$; so we will find it now. It follows from the recurrence (19.9) that $U_1(x) = 2xU_0(x) - U_{-1}(x)$; that is, $2x = 2x \cdot 1 - U_{-1}(x)$. So $U_{-1}(x) = 0$.

Example 19.6 Prove that $U_n(x) - xU_{n-1}(x) = T_n(x)$, where $n \geq 0$. ∎

For example,

$$\begin{aligned} U_7(x) - xU_6(x) &= (128x^7 - 192x^5 + 80x^3 - 8x) - x(64x^6 - 80x^4 + 24x^2 - 1) \\ &= 64x^7 - 112x^5 + 56x^3 - 7x \\ &= T_7(x). \end{aligned}$$

19.14 Pell's Equation Revisited

Next we show that $(T_n(x), U_{n-1}(x))$ is a solution of the Pell equation $u^2 - (x^2-1)v^2 = 1$, where $x^2 - 1 > 0$ and is nonsquare.

Example 19.7 Show that $T_n^2(x) - (x^2-1)U_{n-1}^2(x) = 1$, where $n \geq 1$.

Proof. We will let the explicit formulas for $T_n(x)$ and $U_{n-1}(x)$ do the job for us. For convenience, once again we let $\sqrt{x^2-1} = r$; so $x^2 - r^2 = 1$. Then

$$\begin{aligned} 4T_n^2(x) &= [(x+r)^n + (x-r)^n]^2 \\ &= (x+r)^{2n} + (x-r)^{2n} + 2 \\ 4(x^2-1)U_{n-1}^2 &= [(x+r)^n - (x-r)^n]^2 \\ &= (x+r)^{2n} + (x-r)^{2n} - 2 \\ 4\left[T_n^2(x) - (x^2-1)U_{n-1}^2\right] &= 4 \\ T_n^2(x) - (x^2-1)U_{n-1}^2 &= 1. \end{aligned}$$

Consequently, $(T_n(x), U_{n-1}(x))$ is a solution of the Pell equation $u^2 - (x^2-1)v^2 = 1$. ■

For example, consider the polynomials $T_4(x) = 8x^4 - 8x^2 + 1$ and $U_3(x) = 8x^3 - 4x$. Then

$$\begin{aligned} T_4^2(x) - (x^2-1)U_3^2(x) &= (8x^4 - 8x^2 + 1)^2 - (x^2-1)(8x^3-4x)^2 \\ &= (64x^8 - 128x^6 + 80x^4 - 16x^2 + 1) - (64x^8 - 128x^6 + 80x^4 - 16x^2) \\ &= 1\text{, as expected.} \end{aligned}$$

Conversely, is every solution of the equation $u^2 - (x^2-1)v^2 = 1$ of the form $(T_n(x), U_{n-1}(x))$? To answer this, recall that every solution (u_n, v_n) is given by

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = 2x \begin{bmatrix} u_{n-1} \\ v_{n-1} \end{bmatrix} - \begin{bmatrix} u_{n-2} \\ v_{n-2} \end{bmatrix},$$

where $n \geq 2$. Clearly, $u_n = T_n(x)$. Is $v_n = U_n(x)$? Not quite! Although we might be tempted to think so, notice that $v_n = U_{n-1}(x)$, where $n \geq 1$.

Thus every solution of the Pell's equation $u^2-(x^2-1)v^2 = 1$ is of the form $(T_n(x), U_{n-1}(x))$.

Next we study the relationship between the polynomials $U_n(x)$ and trigonometry.

19.15 $U_n(x)$ and Trigonometry

Earlier, we studied the relationship between Chebyshev polynomial $T_n(x)$ and trigonometry by exploring the multiple-angle formulas for the cosine function. To see a similar relationship between the polynomial $U_n(x)$ and trigonometry, we can follow a hunch and explore the multiple-angle formulas for the sine function:

$$\begin{aligned} \sin 1\theta &= 1 \cdot \sin\theta \\ \sin 2\theta &= (2\cos\theta)\sin\theta \\ \sin 3\theta &= (4\cos^2\theta - 1)\sin\theta \\ \sin 4\theta &= (8\cos^3\theta - 4\cos\theta)\sin\theta \\ \sin 5\theta &= (16\cos^4\theta - 12\cos^2\theta + 1)\sin\theta \\ \sin 6\theta &= (32\cos^5\theta - 32\cos^3\theta + 6\cos\theta)\sin\theta. \end{aligned}$$

Substituting x for $\cos\theta$, these formulas yield the following equations

$$\begin{aligned} \frac{\sin 1\theta}{\sin\theta} &= 1 \\ \frac{\sin 2\theta}{\sin\theta} &= 2x \\ \frac{\sin 3\theta}{\sin\theta} &= 4x^2 - 1 \\ \frac{\sin 4\theta}{\sin\theta} &= 8x^3 - 4x \end{aligned}$$

$$\begin{aligned}\frac{\sin 5\theta}{\sin\theta} &= 16x^4 - 12x^2 + 1\\ \frac{\sin 6\theta}{\sin\theta} &= 32x^5 - 32x^3 + 6x.\end{aligned}$$

Our intuition seems to work: $\frac{\sin(n+1)\theta}{\sin\theta} = U_n(x)$, where $x = \cos\theta$ and $0 \le n \le 5$. We will now confirm this observation.

Using the addition formula for the sine function and Example 19.6, we have

$$\begin{aligned}\sin(n+1)\theta &= \sin n\theta\cos\theta + \cos n\theta\sin\theta\\ &= xU_{n-1}(x)\sin\theta + T_n(x)\sin\theta\\ \frac{\sin(n+1)\theta}{\sin\theta} &= xU_{n-1}(x) + T_n(x)\\ &= U_n(x).\end{aligned}$$

Since this result works when $n = 0$ and $n = 1$, by PMI it works for all integers $n \ge 0$.

19.16 Chebyshev Recurrence for $U_n(x)$ Revisited

We can now obtain the recurrence for $U_n(x)$ using the *sum identity* $\sin u + \sin v = 2\sin\frac{u+v}{2}\cos\frac{u-v}{2}$. To this end, let $u = (n+1)\theta$ and $v = (n-1)\theta$. Then

$$\sin(n+1)\theta + \sin(n-1)\theta = 2\sin n\theta\cos\theta.$$

Dividing both sides by $\sin\theta$, this yields the desired recurrence:

$$U_n(x) + U_{n-2}(x) = 2xU_{n-1}(x).$$

Using the trigonometric relationship, we can develop a number of identities satisfied by the Chebyshev families, as the following examples illustrate. Again, we leave their proofs as exercises.

Example 19.8 Prove that $T_{m+n}(x) - T_{m-n}(x) = 2(x^2-1)U_{m-1}(x)U_{n-1}(x)$, where $m \ge n$. ■

19.16.1 An Interesting Special Case

We will now explore this identity a bit further. Since $q_n(x) = 2(-i)^n T_n(ix)$, it yields an interesting Pell–Lucas polynomial identity:

$$\frac{q_{m+n}(x)}{2(-i)^{m+n}} - \frac{q_{m-n}(x)}{2(-i)^{m-n}} = 2[(ix)^2 - 1]\frac{p_m(x)}{(-i)^{m-1}}\cdot\frac{p_n(x)}{(-i)^{n-1}}$$

$$\begin{aligned}\frac{1}{2(-i)^{m+n}}[q_{m+n}(x)-(-1)^n q_{m-n}(x)] &= \frac{-2(x^2+1)p_m(x)p_n(x)}{(-i)^{m+n-2}}\\ q_{m+n}(x)-(-1)^n q_{m-n}(x) &= 4(x^2+1)p_m(x)p_n(x). \qquad (19.13)\end{aligned}$$

For example, let $m=5$ and $n=3$. Then

$$\begin{aligned}\text{LHS} &= q_8(x)+q_2(x)\\ &= (256x^8+512x^6+320x^4+64x^2+2)+(4x^2+2)\\ &= 256x^8+512x^6+320x^4+68x^2+4\\ &= 4(x^2+1)(16x^4+12x^2+1)(4x^2+1)\\ &= 4(x^2+1)p_5(x)p_3(x)=\text{RHS}.\end{aligned}$$

Identity (19.13) has two interesting byproducts:

(1) Suppose we let $x=1$. Since $q_k(1)=2Q_k$ and $p_k(1)=P_k$, it yields the hybrid Pell identity

$$Q_{m+n}-(-1)^n Q_{m-n}=4P_mP_n. \qquad (19.14)$$

For example, when $m=8$ and $n=5$, LHS = $Q_{13}+Q_3=47321+7=47328=4\cdot 408\cdot 29=4P_8P_5$ = RHS; and similarly, when $m=10$ and $n=4$, LHS = $Q_{14}-Q_6=114114=4P_{10}P_5$ = RHS.

Notice that identity (19.14) yields the identity $Q_{m+1}+Q_{m-1}=4P_m$, as we saw in Chapter 7.

(2) Suppose we let $x=1/2$ in identity (19.13). Since $q_k(1/2)=L_k$ and $p_k(1/2)=F_k$, it yields the Fibonacci–Lucas identity

$$L_{m+n}-(-1)^n L_{m-n}=5F_mF_n. \qquad (19.15)$$

For example, when $m=8$ and $n=5$, LHS = $L_{13}+L_3=521+4=525=5\cdot 21\cdot 5=5F_8F_5$ = RHS; and similarly, when $m=10$ and $n=4$, LHS = $L_{14}-L_6=825=5F_{10}F_4$ = RHS.

Example 19.9 Prove that $U_{m+n}(x)+U_{m-n}(x)=2U_m(x)T_n(x)$, where $m\ge n$. ■

Similarly, it can be shown that

$$U_{m+n}(x)-U_{m-n}(x)=2T_{m+1}(x)U_{n-1}(x). \qquad (19.16)$$

For example, let $m = 6$ and $n = 3$. Then

$$\begin{aligned}
U_9(x) - U_3(x) &= (512x^9 - 1024x^7 + 672x^5 - 160x^3 + 10x) - (8x^3 - 4x)\\
&= 512x^9 - 1024x^7 + 672x^5 - 168x^3 + 14x\\
&= 2(64x^7 - 112x^5 + 56x^5 - 7x)(4x^2 - 1)\\
&= 2T_7(x)U_2(x).
\end{aligned}$$

19.16.2 An Interesting Byproduct

Since $p_k(x) = (-i)^{k-1}U_{k-1}(ix)$ and $q_k(x) = (-i)^k T_k(ix)$, identity (19.16) can be used to derive an identity for Pell polynomials:

$$\begin{aligned}
\frac{p_{m+n+1}(x)}{(-i)^{m+n}} - \frac{p_{m-n+1}(x)}{(-i)^{m-n}} &= 2\frac{q_{m+1}(x)p_n(x)}{2(-i)^{m+1}(-i)^{n-1}}\\
\frac{1}{(-i)^{m+n}}[p_{m+n+1}(x) - (-1)^n p_{m-n+1}(x)] &= \frac{q_{m+1}(x)p_n(x)}{(-i)^{m+n}}\\
p_{m+n+1}(x) - (-1)^n p_{m-n+1}(x) &= q_{m+1}(x)p_n(x).
\end{aligned}$$

Changing m to $m-1$, this yields

$$p_{m+n}(x) - (-1)^n p_{m-n}(x) = q_m(x)p_n(x). \tag{19.17}$$

For example, let $m = 5$ and $n = 3$. Then

$$\begin{aligned}
p_8(x) + p_2(x) &= (128x^7 + 192x^5 + 80x^3 + 8x) + 2x\\
&= 128x^7 + 192x^5 + 80x^3 + 10x\\
&= (32x^5 + 40x^3 + 10x)(4x^2 + 1)\\
&= q_5(x)p_3(x).
\end{aligned}$$

Identity (19.17) has two interesting special cases:

(1) Suppose we let $x = 1$. Since $p_k(1) = P_k$ and $q_k(1) = 2Q_k$, it yields the hybrid Pell identity

$$P_{m+n} - (-1)^n P_{m-n} = 2Q_m P_n. \tag{19.18}$$

For example, when $m = 8$ and $n = 5$, LHS = $P_{13} + P_3 = 33{,}466 = 2Q_8P_5$ = RHS; and when $m = 10$ and $n = 4$, LHS = $P_{14} - P_6 = 80{,}712 = 2Q_{10}P_4$ = RHS.

(2) Let $x = 1/2$. Since $p_k(1/2) = F_k$ and $q_k(1/2) = L_k$, it yields the Fibonacci–Lucas identity

$$F_{m+n} - (-1)^n F_{m-n} = L_m F_n. \tag{19.19}$$

For example, when $m = 7$ and $n = 4$, LHS = $F_{11} + F_3 = 87 = L_7F_4$ = RHS; and when $m = 12$ and $n = 7$, LHS = $F_{19} + F_5 = 4186 = L_{12}F_7$ = RHS.

The following example is somewhat a counterpart of Example 19.2 for $U_n(x)$.

Example 19.10 Prove that $U_{m-1}(T_n(x))U_{n-1}(x) = U_{mn-1}(x)$, where $m, n \geq 1$. ∎

For example, let $m = 5$ and $n = 2$. Then

$$\begin{aligned} T_2(x) &= 2x^2 - 1 \\ U_4(x) &= 16x^4 - 12x^2 + 1 \\ U_4(T_2(x)) &= 16(2x^2-1)^4 - 12(2x^2-1)^2 + 1 \\ &= 256x^8 - 512x^6 + 336x^4 - 80x^2 + 5 \\ U_4(T_2(x))U_1(x) &= (256x^8 - 512x^6 + 336x^4 - 80x^2 + 5)(2x) \\ &= 512x^9 - 1024x^7 + 672x^5 - 160x^3 + 10x \\ &= U_{5\cdot 2-1}(x). \end{aligned}$$

Suppose we let $\hat{U}_k(x) = U_k(x)$. Then the identity in Example 19.10 can be rewritten as $\hat{U}_m(T_n(x))\hat{U}_n(x) = \hat{U}_{mn}(x)$, which looks better stylistically.

Next we establish the addition formulas for both Chebyshev polynomials. Again, we will let the sine and cosine functions do the job for us.

Example 19.11 Prove that $T_{m+n}(x) = T_m(x)U_n(x) - T_{m-1}(x)U_{n-1}(x)$, where $m, n \geq 1$.

Proof. Using the addition formulas (19.5) and (19.6), we have

$$\begin{aligned} \sin\theta \cdot \text{RHS} &= \cos m\theta \sin(n+1)\theta - \cos(m-1)\theta \sin n\theta \\ &= \cos m\theta(\sin n\theta \cos\theta + \cos n\theta \sin\theta) - \sin n\theta(\cos m\theta \cos\theta + \sin m\theta \sin\theta) \\ &= (\cos m\theta \cos n\theta - \sin m\theta \sin n\theta)\sin\theta \\ &= \cos(m+n)\theta \sin\theta \\ \text{RHS} &= \cos(m+n)\theta \\ &= T_{m+n}(x) \\ &= \text{LHS}. \end{aligned}$$

∎

For example, let $m = 3$ and $n = 5$. Then

$$\begin{aligned} T_3(x)U_5(x) - T_2(x)U_4(x) &= (4x^3 - 3x)(32x^5 - 32x^3 + 6x) - (2x^2 - 1)(16x^4 - 12x^2 + 1) \\ &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1 \\ &= T_{3+5}(x). \end{aligned}$$

Since $q_k(x) = 2(-i)^k T_k(ix)$ and $p_k(x) = (-i)^{k-1}U_{k-1}(ix)$, it follows from Example 19.11 that

$$\begin{aligned} \frac{q_{m+n}(x)}{2(-i)^{m+n}} &= \frac{q_m(x)}{2(-i)^m}\cdot\frac{p_{n+1}(x)}{(-i)^n} - \frac{q_{m-1}(x)}{2(-i)^{m-1}}\cdot\frac{p_n(x)}{(-i)^{n-1}} \\ q_{m+n}(x) &= q_m(x)p_{n+1}(x) + q_{m-1}(x)p_n(x). \end{aligned} \tag{19.20}$$

This is the *addition formula* for Pell–Lucas polynomials, found in Chapter 14.

For example, when $m = 3$ and $n = 5$, we have

$$\begin{aligned} q_3(x)p_6(x) + q_2(x)p_5(x) &= (8x^3+6x)(32x^5+32x^3+6x) + (4x^2+2)(16x^4+12x^2+1) \\ &= 256x^8 + 512x^6 + 320x^4 + 64x^2 + 1 \\ &= q_8(x). \end{aligned}$$

In particular, formula (19.20) yields the addition formula $Q_{m+n} = Q_m P_{n+1} + Q_{m-1}P_n$ for Pell–Lucas numbers, as we learned in Chapter 8.

The following example gives the *addition formula* for $U_k(x)$.

Example 19.12 Prove that $U_{m+n}(x) = U_m(x)U_n(x) - U_{m-1}(x)U_{n-1}(x)$, where $m, n \geq 1$.

Proof. Using the product formula $2\sin u \sin v = \cos(u - v) - \cos(u + v)$, we have

$$\begin{aligned} \sin^2\theta\cdot\text{RHS} &= \sin(m+1)\theta\sin(n+1)\theta - \sin m\theta\sin n\theta \\ &= \frac{1}{2}[\cos(m-n)\theta - \cos(m+n+2)\theta] - \frac{1}{2}[\cos(m-n)\theta - \cos(m+n)\theta] \\ &= \frac{1}{2}[\cos(m+n)\theta - \cos(m+n+2)\theta] \\ &= \sin(m+n+1)\theta\sin\theta \\ \text{RHS} &= \frac{\sin(m+n+1)\theta}{\sin\theta} \\ &= U_{m+n}(x) \\ &= \text{LHS}. \end{aligned}$$

■

For example, let $m = 3$ and $n = 5$. Then

$$\begin{aligned} U_3(x)U_5(x) - U_2(x)U_4(x) &= (8x^3-4x)(32x^5-32x^3+6x) - (4x^2-1)(16x^4-12x^2+1) \\ &= 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1 \\ &= U_{3+5}(x). \end{aligned}$$

Since $p_k(x) = (-i)^{k-1}U_{k-1}(ix)$, it follows from this example that

$$\begin{aligned}\frac{p_{m+n+1}(x)}{(-i)^{m+n}} &= \frac{p_{m+1}(x)}{(-i)^m}\cdot\frac{p_{n+1}(x)}{(-i)^n} - \frac{p_m(x)}{(-i)^{m-1}}\cdot\frac{p_n(x)}{(-i)^{n-1}}\\ p_{m+n+1}(x) &= p_{m+1}(x)p_{n+1}(x) + p_m(x)p_n(x).\end{aligned}$$

That is,

$$p_{m+n}(x) = p_m(x)p_{n+1}(x) + p_{m-1}(x)p_n(x), \tag{19.21}$$

which is the addition formula for Pell polynomials, as we found in Chapter 14.

In particular, this implies that $P_{m+n} = P_m P_{n+1} + P_{m-1}P_n$; see formula (8.7).

Interestingly, both Chebyshev polynomials also satisfy Cassini-like formulas. In the interest of brevity, we will leave their proofs as routine exercises.

19.17 Cassini-like Formulas for Chebyshev Polynomials

$$T_{n+1}(x)T_{n-1}(x) - T_n^2(x) = x^2 - 1 \tag{19.22}$$

$$U_{n+1}(x)U_{n-1}(x) - U_n^2(x) = -1. \tag{19.23}$$

For example,

$$\begin{aligned}T_5(x)T_3(x) - T_4^2(x) &= (16x^5 - 20x^3 + 5x)(4x^3 - 3x) - (8x^4 - 8x^2 + 1)^2\\ &= (64x^8 - 128x^6 + 80x^4 - 15x^2) - (64x^8 - 128x^6 + 80x^4 - 16x^2 + 1)\\ &= x^2 - 1;\\ U_5(x)U_3(x) - U_4^2(x) &= (16x^4 - 12x^2 + 1)(4x^2 - 1) - (8x^3 - 4x)^2\\ &= (64x^6 - 64x^4 + 16x^2 - 1) - (64x^6 - 64x^4 + 16x^2)\\ &= -1.\end{aligned}$$

19.18 Generating Functions for Chebyshev Polynomials

Finally, using standard techniques, it is relatively easy to develop generating functions for Chebyshev polynomials; see Exercises 20 and 21:

$$\begin{aligned}\frac{1 - xy}{1 - 2xy + y^2} &= \sum_{n=0}^{\infty} T_n(x)y^n.\\ \frac{1}{1 - 2xy + y^2} &= \sum_{n=0}^{\infty} U_n(x)y^n.\end{aligned}$$

Exercises 19

Prove each, where $T_n(x)$ and $U_n(x)$ denote Chebyshev polynomials of the first and second kinds, respectively.

1. $2T_m(x)T_n(x) = T_{m+n}(x) + T_{n-m}(x)$, where $m < n$. *Hint*: Let $r = \sqrt{x^2-1}$; use formula (19.2).
2. $Q_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 2^k$. *Hint*: Use formula (19.3).
3. $T_m(T_n(x)) = T_{mn}(x)$, where m and n are nonnegative integers. *Hint*: $T_k(x) = \cos k\theta$.
4. Confirm the identity in Example 19.1 using the fact that $T_n(x) = \cos n\theta$. *Hint*: Use the identity $\cos u + \cos v = 2\cos\frac{u+v}{2}\cos\frac{u-v}{2}$.
5. $[T_{m+n}(x) - 1][T_{m-n}(x) - 1] = [T_m(x) - T_n(x)]^2$.
6. $Q_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{j=k}^{\lfloor n/2 \rfloor} \binom{n}{2j}\binom{j}{k}$. *Hint*: Let $x = i$ in formula (19.7).
7. Derive the explicit formula (19.10) for $U_n(x)$. *Hint*: Using the initial conditions, find A and B.
8. Deduce the explicit formula (19.12) for P_n from (19.11).
9. $U_n(x) - U_{n-2}(x) = 2T_n(x)$, where $n \ge 2$. *Hint*: Show that $\frac{1}{2}[U_n(x) - U_{n-2}(x)]$ satisfies the recursive definition of $T_n(x)$.
10. $U_n(x) - xU_{n-1}(x) = T_n(x)$, where $n \ge 0$.
11. $T_{m+n}(x) - T_{m-n}(x) = 2(x^2-1)U_{m-1}(x)U_{n-1}(x)$, where $m \ge n$. *Hint*: Use the identity $\cos v - \cos u = 2\sin\frac{u+v}{2}\sin\frac{u-v}{2}$.
12. $U_{m+n}(x) + U_{m-n}(x) = 2U_m(x)T_n(x)$, where $m \ge n$. *Hint*: Use the identity $\sin u + \sin v = 2\sin\frac{u+v}{2}\cos\frac{u-v}{2}$.
13. $U_{m+n}(x) - U_{m-n}(x) = 2T_{m+1}(x)U_{n-1}(x)$, where $m \ge n$.
14. $U_{m-1}(T_n(x))U_{n-1}(x) = U_{mn-1}(x)$, where $m, n \ge 1$. *Hint*: Let $T_n(x) = \cos n\theta$ and $t = \cos\theta$. Then $U_{m-1}(t) = (1-t^2)^{-1/2}\sin(m \arccos t)$.
15. $T_n'(x) = nU_{n-1}(x)$, where the prime indicates differentiation with respect to x.

Let $y = T_n(x) = \cos n\theta$. Then:

16. $(1-x^2)y'' - xy' + n^2y = 0$. *Hint*: Differentiate $y' = \frac{\sin n\theta}{\sin\theta}$ with respect to x.
17. $(1-x^2)U_n''(x) - 3xU_n'(x) + n(n+2)U_n(x) = 0$.
18. Establish the Cassini-like formula $T_{n+1}(x)T_{n-1}(x) - T_n^2(x) = x^2 - 1$. *Hint*: Use the explicit formula for $T_n(x)$.

19. Establish the Cassini-like formula $U_{n+1}(x)U_{n-1}(x) - U_n^2(x) = -1$.

Develop a generating function for:

20. $T_n(x)$. *Hint*: Let $\{S_n(x)\}$ be a sequence of polynomial functions satisfying the Chebyshev recurrence.

21. $U_n(x)$.

20

Chebyshev Tilings

20.1 Introduction

In Chapter 16, we studied different combinatorial models for Pell and Pell–Lucas polynomials by constructing linear and circular tilings of boards with n cells. In each case, our success hinged on a clever assignment of weights to square tiles and dominoes. Since the Pell family is a subfamily of the Chebyshev family, we are tempted to ask whether the Pell tiling models can be extended to the larger family. Fortunately, the answer is yes. We will begin our investigation with the Chebyshev tiling models for the polynomials $U_n(x)$ of the second kind.

20.2 Combinatorial Models for $U_n(x)$

Recall from Chapter 19 that the nonzero constant term in a Chebyshev polynomial of each type can be 1 or -1. This implies that some tiles in the Chebyshev tilings must be assigned a weight of -1. More specifically, we assign to each square a weight of $2x$ and to each domino -1.

Figure 20.1 shows the resultant tilings of a linear board with n cells and the sum of their weights, where $0 \leq n \leq 5$. In each case, the sum of the weights of the tilings is the polynomial $U_n(x)$.

More generally, we have the following result, discovered by Shapiro in 1981 [230]. Since its proof follows the same reasoning as that of Theorem 16.2, we omit it for the sake of brevity. (In fact, most of the results in this chapter can be proved by following the argument of the corresponding result in Chapter 16. So we will omit many of the proofs, although proving each would be a good exercise in its own right.)

T. Koshy, *Pell and Pell–Lucas Numbers with Applications*, DOI 10.1007/978-1-4614-8489-9_20,

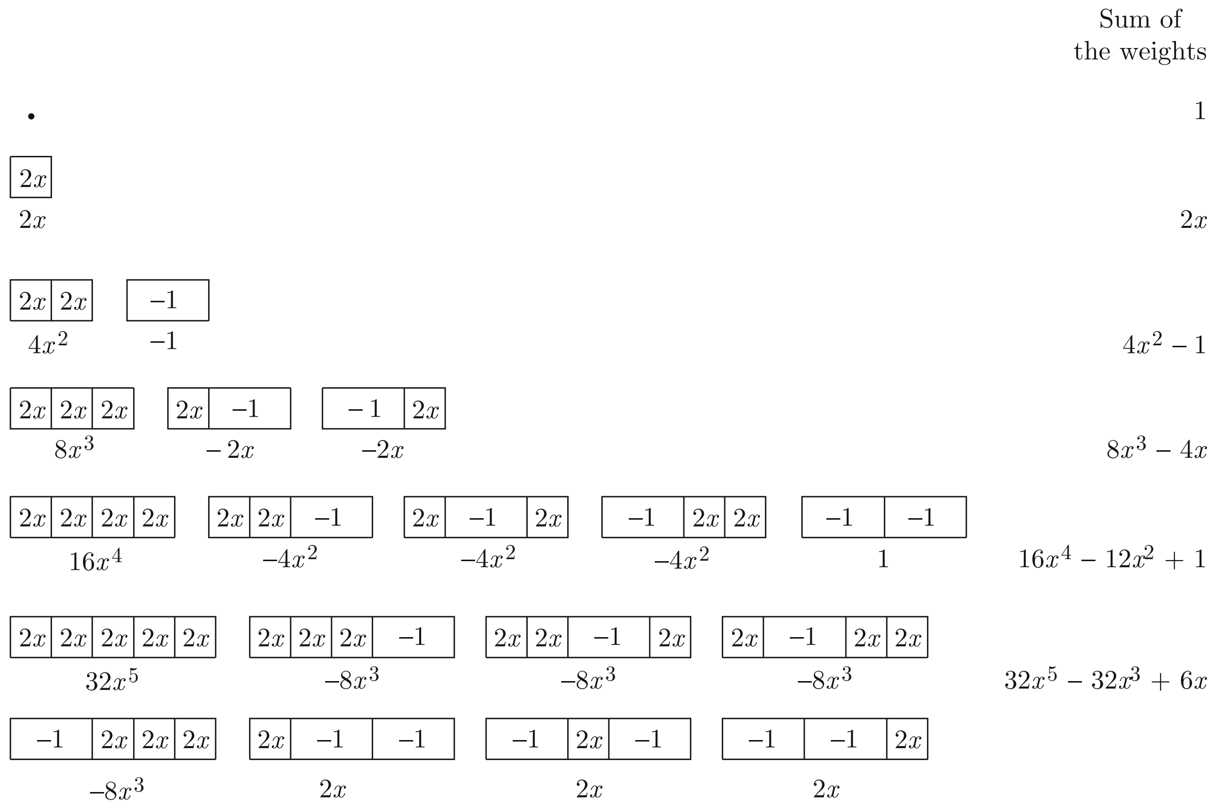


Figure 20.1.

Theorem 20.1 *The sum of the weights of the tilings of a* $1 \times n$ *board with square tiles and dominoes is* $U_n(x)$*, where the weight of a square tile is* $2x$ *and that of a domino is* -1*, and* $n \geq 0$. ■

Suppose a tiling has k dominoes. Then it has $n - 2k$ squares and takes a total of $n - k$ tiles. So there are $\binom{n-k}{k}$ tilings of a $1 \times n$ board, each containing exactly k dominoes. The weight of each such tiling is $(-1)^k(2x)^{n-2k}$. So the sum of the weights of tilings of a $1 \times n$ board is $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}(-1)^k(2x)^{n-2k}$. This fact, coupled with Theorem 20.1, yields an explicit formula for $U_n(x)$, as the following theorem shows.

Theorem 20.2 *Let* $n \geq 0$. *Then* $U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}(-1)^k(2x)^{n-2k}$. ■

For example,

$$
\begin{aligned}
U_3(x) &= \sum_{k=0}^{1} \binom{3-k}{k} (-1)^k (2x)^{3-2k} \\
&= (2x)^3 - 2(2x) \\
&= 8x^3 - 4x.
\end{aligned}
$$

Theorem 20.2 has interesting byproducts, as the next five corollaries show. We will leave their proofs as routine exercises.

Corollary 20.1 *$U_n(ix) = i^n p_{n+1}(x)$, where $p_k(x)$ denotes the kth Pell polynomial, $i = \sqrt{-1}$, and $n \ge 0$.* ▪

In particular, we have the following result.

Corollary 20.2 *$U_n(i) = i^n P_{n+1}$ and $|U_n(i)| = P_{n+1}$, where $n \ge 0$.* ▪

The following result also follows from Theorem 20.2.

Corollary 20.3 *$U_n(ix/2) = i^n f_{n+1}(x)$, $|U_n(ix/2)| = f_{n+1}(x)$, and $|U_n(i/2)| = F_{n+1}$, where $n \ge 0$.* ▪

For example, we have

$$\begin{aligned} U_5(x) &= 32x^5 - 32x^3 + 6x \\ U_5(ix/2) &= 32(ix/2)^5 - 32(ix/2)^3 + 6(ix/2) \\ &= i^5x^5 - 4i^3x^3 + 3ix = i(x^5 + 4x^3 + 3x) \\ &= i^5 f_6(x). \end{aligned}$$

So $|U_5(ix/2)| = x^5 + 4x^3 + 3x = f_6(x)$ and $|U_5(i/2)| = 8 = F_6$.

Corollary 20.4 *The sum of the weights of tilings of a $1 \times n$ board with an even number of dominoes minus that with an odd number of dominoes is $U_n(x)$; that is,*

$$U_n(x) = \sum_{k \text{ even}} \binom{n-k}{k}(2x)^{n-2k} - \sum_{k \text{ odd}} \binom{n-k}{k}(2x)^{n-2k}.$$ ▪

For example, consider the tilings of a 1×5 board in Figure 20.1. There is exactly one tiling with 0 dominoes; its weight is $32x^5$. There are three tilings with 2 dominoes each; the sum of their weights is $2x + 2x + 2x = 6x$. So the sum of the weights of tilings with an even number of dominoes is $32x^5 + 6x$.

On the other hand, there are four tilings with exactly one domino each; the sum of their weights is $8x^3 + 8x^3 + 8x^3 + 8x^3 = 32x^3$.

The difference of the two sums is $(32x^5 + 6x) - 32x^3 = 32x^5 - 32x^3 + 6x = U_5(x)$, as expected.

The next result follows from Corollary 20.4.

Corollary 20.5 *The sum of the weights of tilings of a* $1 \times n$ *board with an even number of dominoes minus that with an odd number of dominoes is* $U_n(x)$*, where the weight of a square is* 2 *and that of a domino is* -1*; that is,*

$$U_n(1) = \sum_{k \text{ even}} \binom{n-k}{k} 2^{n-2k} - \sum_{k \text{ odd}} \binom{n-k}{k} 2^{n-2k}. \qquad \blacksquare$$

For example, consider the tilings of the 1×5 board in Figure 20.1. From the previous paragraph, we have

$$\begin{aligned}
U_5(1) &= 32 - 32 + 6 = 6 \\
\sum_{k \text{ even}} \binom{5-k}{k} 2^{5-2k} &= \binom{5}{0} 2^5 + \binom{3}{2} 2 = 38 \\
\sum_{k \text{ odd}} \binom{5-k}{k} 2^{5-2k} &= \binom{4}{1} 2^3 = 32
\end{aligned}$$

$$\begin{aligned}
\sum_{k \text{ even}} \binom{5-k}{k} 2^{5-2k} - \sum_{k \text{ odd}} \binom{5-k}{k} 2^{5-2k} &= 38 - 32 = 6 \\
&= U_5(1).
\end{aligned}$$

Likewise, $\sum_{k \text{ even}} \binom{6-k}{k} 2^{6-2k} - \sum_{k \text{ odd}} \binom{6-k}{k} 2^{6-2k} = 88 - 81 = 7 = U_6(1)$.

We can use the concept of breakability to develop the addition formula for $U_k(x)$, as in Theorem 16.4.

Theorem 20.3 *Let* $m, n \geq 1$*. Then* $U_{m+n}(x) = U_m(x)U_n(x) - U_{m-1}(x)U_{n-1}(x)$. ■

For example, let $m = 5$ and $n = 2$. Then

$$\begin{aligned}
U_7(x) &= U_5(x)U_2(x) - U_4(x)U_1(x) \\
&= (32x^5 - 32x^3 + 6x)(4x^2 - 1) - (16x^4 - 12x^2 + 1)(2x) \\
&= 128x^7 - 192x^5 + 80x^3 - 8x.
\end{aligned}$$

Suppose we change x to ix in the addition formula. Then, by Corollary 20.1, we get

$$\begin{aligned}
U_{m+n}(ix) &= U_m(ix)U_n(ix) - U_{m-1}(ix)U_{n-1}(ix) \\
i^{m+n} p_{m+n+1}(x) &= i^{m+n} p_{m+1}(x) p_{n+1}(x) - i^{m+n-2} p_m(x) p_n(x) \\
p_{m+n+1}(x) &= p_{m+1}(x) p_{n+1}(x) + p_m(x) p_n(x).
\end{aligned}$$

Changing m to $m-1$, this yields the addition formula (14.17) for Pell polynomials:

$$p_{m+n}(x) = p_m(x)p_{n+1}(x) + p_{m-1}(x)p_n(x).$$

In particular, $P_{m+n} = P_m P_{n+1} + P_{m-1}P_n$, as we found in Chapter 8; see formula (8.7).

20.3 A Colored Combinatorial Model for $U_n(x)$

In the second model for $U_n(x)$, we assume that square tiles come in two colors, black and white. Each is assigned a weight of x, while the weight of a domino remains the same, namely -1, as in the previous model.

Figure 20.2 shows the resulting tilings of a $1 \times n$ board, where $0 \le n \le 4$.

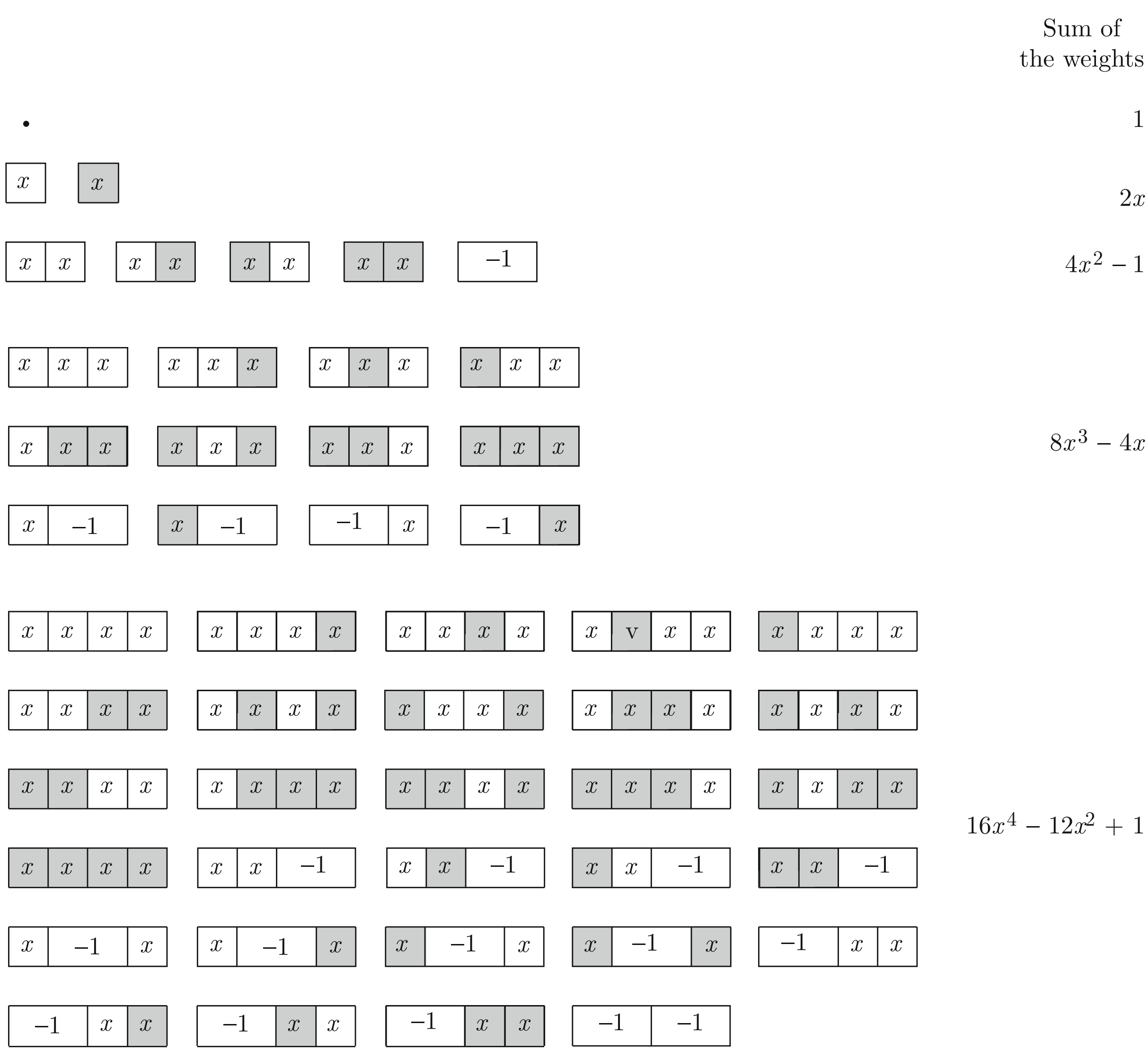


Figure 20.2.

Recall from Theorem 16.5 that there are P_{n+1} such colored tilings of a $1 \times n$ board. The next theorem shows that the sum of their weights is $U_n(x)$.

Theorem 20.4 *The sum of the weights of colored tilings of a $1 \times n$ board is $U_n(x)$, where the weight of a square (black or white) is x and that of a domino is -1, and $n \geq 0$.* ∎

This theorem has an interesting corollary, quite similar to Corollary 20.5. To this end, we let $x = 1$ in this model; that is, each square gets weight 1. So a colored tiling with exactly k dominoes has weight $(-1)^k$. Consequently, $U_n(1)$ counts the difference of the number of tilings with an even number of dominoes and that with an odd number of dominoes. Since $U_n(1) = n + 1$, this difference is $n + 1$. Thus we have the following result.

Corollary 20.6 *The difference of the number of tilings of a $1 \times n$ board with an even number of dominoes and that with an odd number of dominoes is $U_n(1) = n + 1$, where the weight of a square (black or white) is x and that of a domino is -1, and $n \geq 0$.* ∎

For example, consider the colored tilings of the 1×4 board in Figure 20.2. There are 17 tilings with an even number of dominoes, and 12 tilings with an odd number of dominoes. So the difference is $17 - 12 = 5 = U_4(1)$.

Next we present four combinatorial models for Chebyshev polynomials of the first kind.

20.4 Combinatorial Models for $T_n(x)$

In the first model for $T_n(x)$, we use uncolored tiles – squares and dominoes – to tile a linear board of n cells. As in Figure 20.1, we assign a weight of $2x$ to each square and -1 to each domino, but with one exception: if a tiling begins with a square, then the tile gets a weight of x.

Figure 20.3 shows the possible tilings of a $1 \times n$ board, the corresponding sum of their weights, and $0 \leq n \leq 5$. In each case, the sum is $T_n(x)$. The next theorem confirms this.

Theorem 20.5 *The sum of the weights of all uncolored tilings of a $1 \times n$ board is $T_n(x)$, where if the initial title is a square, its weight is x, and all other squares have weight $2x$; and every domino has weight -1.* ∎

As in Theorem 16.7, we can establish the addition formula for $T_n(x)$ (see Example 19.11) using the concept of breakability.

Theorem 20.6 *Let $m, n \geq 1$. Then $T_{m+n}(x) = T_m(x)U_n(x) - T_{m-1}(x)U_{n-1}(x)$.* ∎

Recall from Chapter 19 that $T_n(\cos\theta) = \cos n\theta$. In light of Theorem 20.5, we can now interpret this result combinatorially. By Theorem 20.5, $T_n(\cos\theta)$ is the sum of the weights of all tilings of a $1 \times n$ board, where every domino has weight -1 and every square has weight $2x = 2\cos\theta$; but if a tiling begins with a square, then its weight is $x = \cos\theta$. So $\cos n\theta$ is the sum of the weights of all tilings of a $1 \times n$ board.

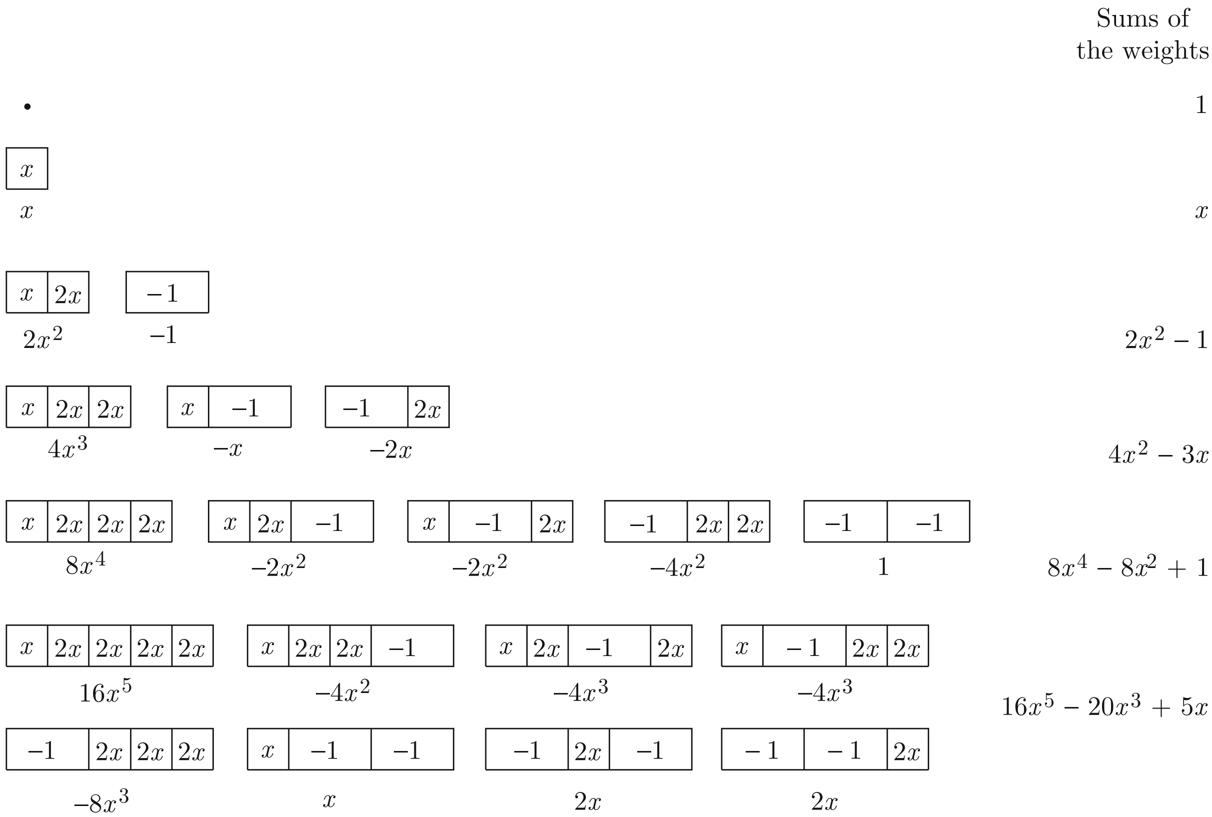


Figure 20.3.

In the second model for $T_n(x)$, we introduce colored square tiles. Every square has weight x and every domino -1. But *no* tiling can begin with a black square.

Figure 20.4 shows the possible colored tilings of a $1 \times n$ board and the sum of their weights, where $0 \le n \le 4$. Again, in each case, the sum of the weights is $T_n(x)$.

The next theorem generalizes this observation. We will give only a skeleton proof.

Theorem 20.7 *The sum of the weights of all colored tilings of a* $1 \times n$ *board is* $T_n(x)$*, where the weight of a square (black or white) is* x *and that of a domino is* -1*, and no tiling begins with a black square tile.*

Proof. Let $S_n(x)$ denote the sum of the weights of a $1 \times n$ board. Then $S_0(x) = 1 = T_0(x)$ and $S_1(x) = x = T_1(x)$.

Consider an arbitrary colored tiling of a $1 \times n$ board, where $n \ge 2$. Suppose it ends in a square. Since it can be black or white, the sum of the weights of such tiles is $2xS_{n-1}$.

On the other hand, suppose the tiling ends in a domino. The sum of the weights of such tilings is $-S_{n-2}$.

Thus, $S_n = 2xS_{n-1} - S_{n-2}$. This, along with the initial conditions, implies that $S_n(x) = T_n(x)$, as claimed. ∎

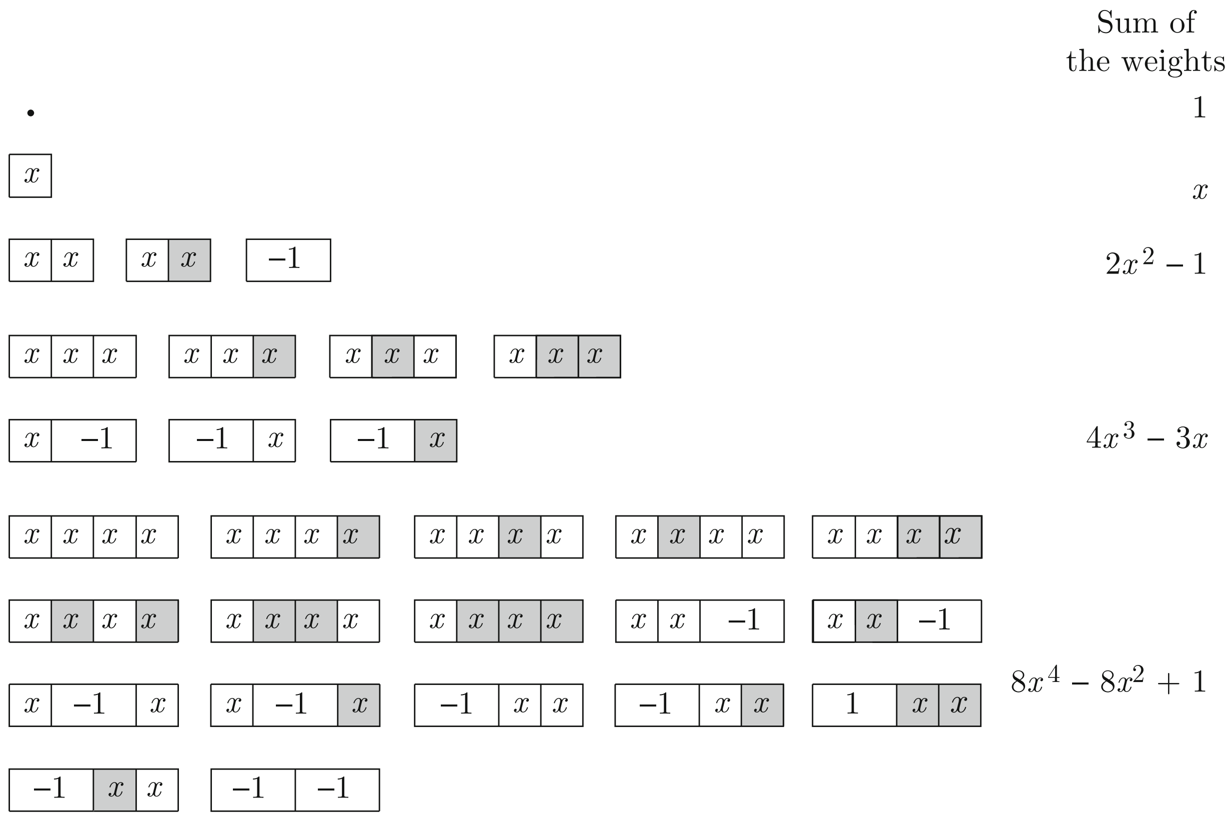


Figure 20.4.

The next theorem shows that the number of colored tilings of a $1 \times n$ board in this model is Q_n.

Theorem 20.8 *The number of colored tilings of a* $1 \times n$ *board is* Q_n*, where the weight of a square (black or white) is* x *and that of a domino is* -1*, no tiling begins with a black square tile, and* $n \geq 0$.

Proof. Let S_n denote the number of colored tilings of a $1 \times n$ board. Then $S_0 = 1 = Q_0$ and $S_1 = 1 = Q_1$.

Let the weight of a square tile be 1 and that of a domino be 1. Then each tiling has weight 1; so S_n is the number of tilings of length n.

Using the reasoning in Theorem 20.7, it follows that $S_n = 2S_{n-1} + S_{n-2}$, where $n \geq 3$. This, coupled with the two initial conditions, implies that $S_n = Q_n$. ∎

For example, consider the tilings in Figure 20.4. There are $7 = Q_3$ tilings of length 3 and $17 = Q_4$ tilings of length 4.

In the third model, we allow the initial square to be white or black. As before, the weight of each square is x, except that the initial square has weight $\frac{x}{2}$. The weight of a domino is -1. Figure 20.5 shows such tilings of $1 \times n$ board and the sum of their weights, where $0 \leq n \leq 3$. In each case, the sum is the Chebyshev polynomial $T_n(x)$.

More generally, we have the following result. Its proof follows as in Theorem 20.7.

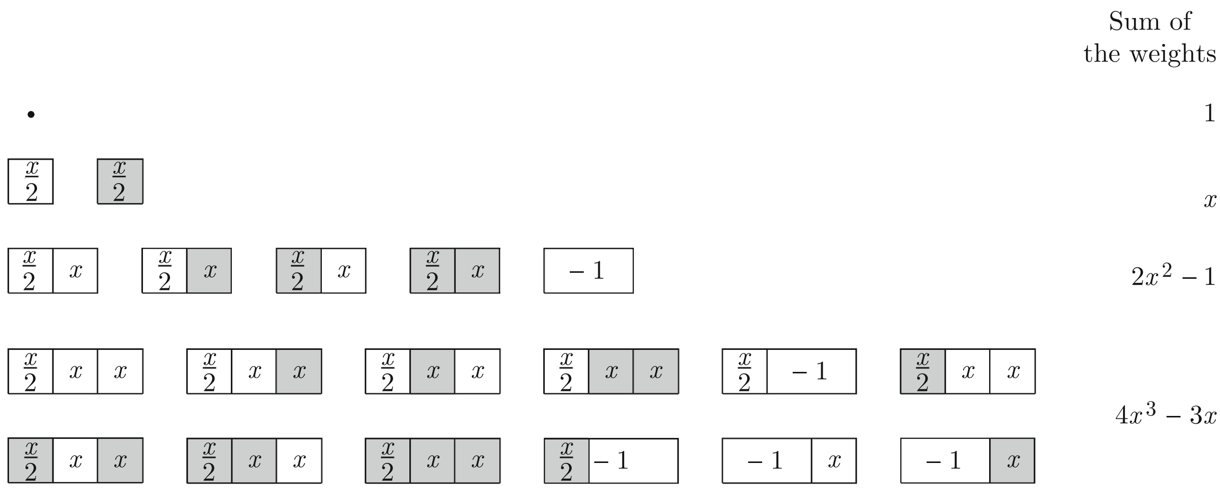


Figure 20.5.

Theorem 20.9 *The sum of the weights of colored tilings of a $1 \times n$ board is $T_n(x)$, where the weight of a domino is -1 and that of a square is x, except that the weight of the initial square is $\frac{x}{2}$.* ■

Next we establish combinatorially that $T_n(\cos\theta) = \cos n\theta$. It follows by Theorem 20.5 that $T_n(\cos\theta)$ is the sum of the weights of all uncolored tilings of length n, where a domino has weight -1; and a square has weight $2\cos\theta$, but with one exception: an initial square has weight $\cos\theta$.

By Euler's formula, $2\cos\theta = e^{i\theta} + e^{-i\theta}$. Consequently, we can assign the weight $e^{i\theta} + e^{-i\theta}$ to every square, except the initial one, which has weight $\frac{e^{i\theta}+e^{-i\theta}}{2}$.

20.5 A Combinatorial Proof[13] that $T_n(\cos\theta) = \cos n\theta$

We now introduce colored squares into the tiling scheme, with different weights for squares of opposite color. White squares have weight $e^{i\theta}$ and black squares have weight $e^{-i\theta}$. If the initial square is white, its weight is $\frac{e^{i\theta}}{2}$; otherwise, its weight is $\frac{e^{-i\theta}}{2}$. The weight of a domino remains -1.

For example, the tiling in Figure 20.6 has weight $\frac{1}{2}e^{i\theta}$.

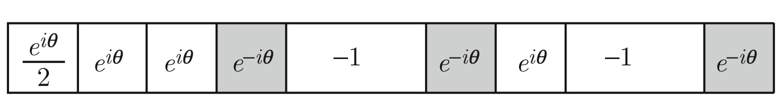


Figure 20.6.

[13] A.T. Benjamin and D. Walton developed this proof in 2007, the 300th anniversary of Euler's birth [13].

Figure 20.7 shows the colored tilings of length n and the corresponding weights, where $0 \leq n \leq 3$.

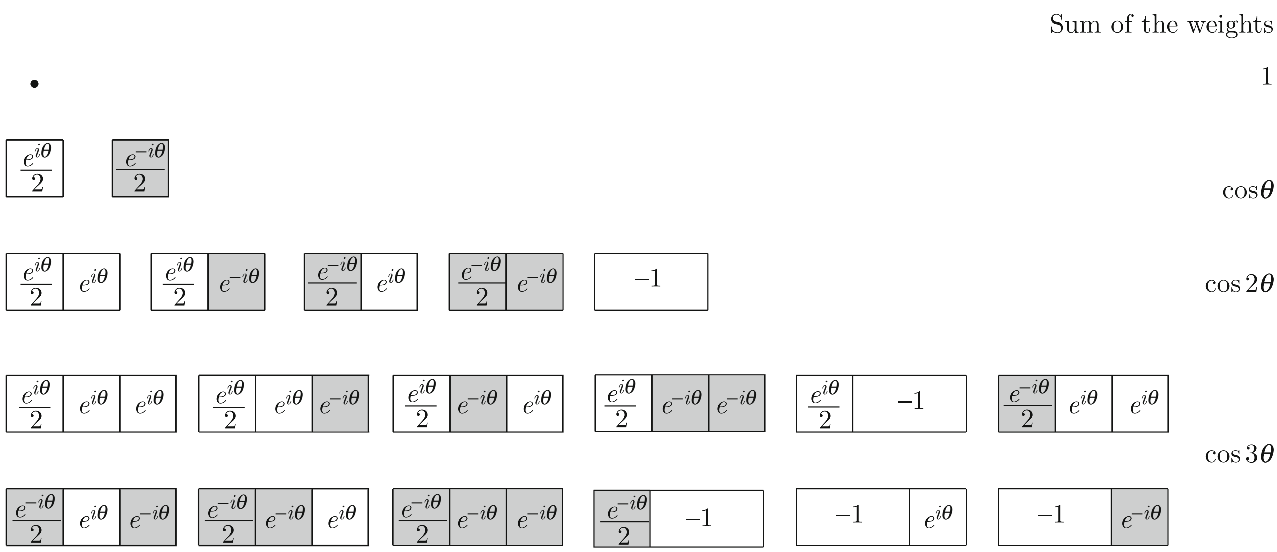


Figure 20.7.

More generally, by Theorem 20.9, $T_n(\cos\theta)$ is the sum of the weights of all such colored tilings of length n.

Next we introduce the concept of an impure tiling. A colored tiling is *impure* if it contains two adjacent squares of different colors or a domino; otherwise, it is *pure*. For example, the tiling in Figure 20.6 has impurities at cells 3 and 4; 5 and 6; 7 and 8; and 9 and 10.

We will now show that the sum of the weights of all impure tilings is zero. To this end, let X be an arbitrary impure tiling with its first impurity at cells k and $k+1$.

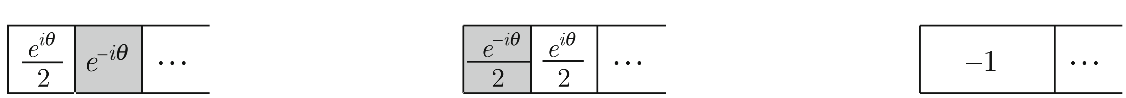
Case 1 Let $k \geq 2$.

Subcase 1 Suppose cells k and $k+1$ are occupied by squares of opposite color. Let X' be the tiling obtained by replacing these two squares with a domino, leaving the other tiles untouched. Since $e^{i\theta} \cdot e^{-i\theta} = 1$ and the weight of a domino is -1, it follows that weight(X) + weight(X') $= 0$.

Subcase 2 Suppose cells k and $k+1$ are covered by a domino. Let X' be the tiling obtained by replacing the domino with two squares of opposite color such that cells k and $k-1$ have the same color. Again, weight(X) + weight$(X') = 0$.

Thus, corresponding to every impure tiling X, there is a tiling X' such that the sum of their weights is zero, where $k \geq 2$.

Case 2 Let $k = 1$. So the first impurity in X occurs at cells 1 and 2; this involves three possibilities:



Consequently, given such a tiling X, we can create two tilings X_1 and X_2 such that $X = X_1 = X_2$, except for the first two cells. The sum of their weights is $\frac{1}{2}e^{i\theta} \cdot e^{-i\theta} \cdot w + \frac{1}{2}e^{-i\theta} \cdot e^{i\theta} \cdot w + (-1) \cdot w = 0$, where w denotes the weight of the subtiling in cells 3 through n.

Thus, by Cases 1 and 2, the sum of the weights of all impure tilings is zero.

Consequently, $T_n(\cos\theta)$ equals the sum of the weights of the two remaining pure tilings, one consisting of n white squares and the other consisting of n black squares:

$\frac{e^{i\theta}}{2}$	$e^{i\theta}$	$e^{i\theta}$	$\cdots$	$e^{i\theta}$	$e^{i\theta}$

$\frac{e^{-i\theta}}{2}$	$e^{-i\theta}$	$e^{-i\theta}$	$\cdots$	$e^{-i\theta}$	$e^{-i\theta}$

Their weights are $\frac{1}{2}e^{in\theta}$ and $\frac{1}{2}e^{-in\theta}$, respectively. Thus, $T_n(\cos\theta) = \frac{1}{2}\left(e^{in\theta} + e^{-in\theta}\right) = \cos n\theta$, as desired. ∎

For example, let $n = 3$. From Figure 20.7, there are six tilings with the first impurity at cells 1 and 2; the sum of their weights is $\frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta} + \frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta} - e^{i\theta} - e^{-i\theta} = 0$. There are four tilings with the first impurity at cells 2 and 3; the sum of their weights is $\frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta} - \frac{1}{2}e^{i\theta} - \frac{1}{2}e^{-i\theta} = 0$. The sum of the weights of the two pure tilings is $\frac{1}{2}e^{i\theta} \cdot e^{i\theta} \cdot e^{i\theta} + \frac{1}{2}e^{-i\theta} \cdot e^{-i\theta} \cdot e^{-i\theta} = \cos 3\theta$, as expected.

Finally, using the formula $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$, it can be shown that $U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$.

20.6 Two Hybrid Chebyshev Identities

Before we present the fourth model for $T_n(x)$, we will present two Chebyshev identities involving both polynomials. To this end, first notice that

$$\begin{aligned} xU_3(x) + T_4(x) &= x(8x^3 - 4x) + (8x^4 - 8x^2 + 1) \\ &= 16x^4 - 12x^2 + 1 \\ &= U_4(x). \end{aligned}$$

This result can be interpreted combinatorially. To see this, consider the colored tilings of the 1×4 board in Figure 20.2. There are exactly $12 (= P_4)$ tilings that begin with a black square; the sum of their weights is $8x^4 - 4x^2 = x(8x^3 - 4x) = xU_3(x)$. There are exactly $17 (= Q_4)$ tilings that do not begin with a black square; the sum of their weights is $8x^4 - 8x^2 + 1 = T_4(x)$. So the sum of the weights of all tilings of length 4 is $xU_3(x) + T_4(x) = 16x^4 - 12x^2 + 1 = U_4(x)$.

More generally, we claim that $xU_{n-1}(x) + T_n(x) = U_n(x)$, where $n \geq 1$. This can be confirmed using PMI, and the explicit formulas for $T_n(x)$ and $U_n(x)$. But we will now give a combinatorial proof, taking advantage of Theorems 20.4 and 20.7.

Theorem 20.9 *$xU_{n-1}(x) + T_n(x) = U_n(x)$, where $n \geq 1$.*

Proof. Recall from Theorem 20.4 that the sum of all colored tilings of a $1 \times n$ board is $U_n(x)$. We will now compute this sum in a different way, by considering two disjoint cases.

Case 1 Suppose the tiling begins with a black square. Dropping this yields a sub-tiling of a $1 \times (n-1)$ board, which may not begin with a black square:

By Theorem 20.4, the sum of the weights of such sub-tilings is $U_{n-1}(x)$; so the sum of the weights of all tilings of a $1 \times n$ board that begin with a black square is $xU_{n-1}(x)$.

Case 2 Suppose the tiling does not begin with a black square. By Theorem 20.7, the sum of the weights of such tilings of $1 \times n$ board is $T_n(x)$.

Thus, by the addition principle, $xU_{n-1}(x) + T_n(x) = U_n(x)$, as desired. ∎

Next notice that

$$\begin{aligned} xU_3(x) - U_2(x) &= x(8x^3 - 4x) - (4x^2 - 1) \\ &= 8x^4 - 8x^2 + 1 \\ &= T_4(x). \end{aligned}$$

This result also can be interpreted combinatorially. Again, consider the colored tilings of 1×4 board in Figure 20.4. The sum of their weights is $8x^4 - 8x^2 + 1 = T_4(x)$; see also Theorem 20.7.

We will now count this sum in a different way. There are exactly $10 (= 2P_3)$ tilings that begin with a square; the sum of their weights is $8x^4 - 4x^2$. There are $5 (= P_3)$ tilings that begin with a domino; the sum of their weights is $-4x^2 + 1$. So the sum of the weights of all tilings of a 1×4 board is $(8x^4 - 4x^2) + (-4x^2 + 1) = 8x^4 - 8x^2 + 1 = T_4(x)$, as expected.

This observation leads us to the next Chebyshev polynomial identity.

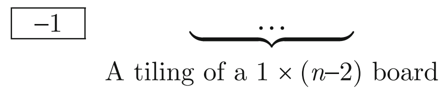
Theorem 20.10 *$T_n(x) = xU_{n-1}(x) - U_{n-2}(x)$, where $n \geq 1$.*

Proof. By Theorem 20.7, the sum of the weights of colored tilings of $1 \times n$ board is $T_n(x)$. Now consider an arbitrary tiling of a $1 \times n$ board, where $n \geq 2$.

Case 1 Suppose the tiling begins with a (black) square. Then it is followed by a sub-tiling of a $1 \times (n-1)$ board:

Since this sub-tiling can begin with a square (black or white) or a domino, it follows by Theorem 20.4 that the sum of the weights of such tilings is $xU_{n-1}(x)$.

Case 2 Suppose the tiling begins with a domino:



As in Case 1, the sum of the weights of such tilings is $-U_{n-2}(x)$.

The desired result now follows by the addition principle. ∎

Notice that this result follows quickly from Theorem 20.9:

$$\begin{aligned} T_n(x) &= U_n(x) - xU_{n-1}(x) \\ &= [2xU_{n-1}(x) - U_{n-2}(x)] - xU_{n-1}(x) \\ &= xU_{n-1}(x) - U_{n-2}(x). \end{aligned}$$

Next we turn to the fourth model for $T_n(x)$, where we count the circular tilings of bracelets of n cells. Every square has weight $2x$ and every domino -1, with one exception: The weight of the domino is -2 when $n = 2$.

Figure 20.8 shows all circular tilings of n cells and the sum of their weights, where $1 \le n \le 4$. Unlike the earlier models, the sum of the weights appears to be $2T_n(x)$. The next theorem confirms that this is indeed the case. The proof follows the same reasoning as in Theorem 16.9.

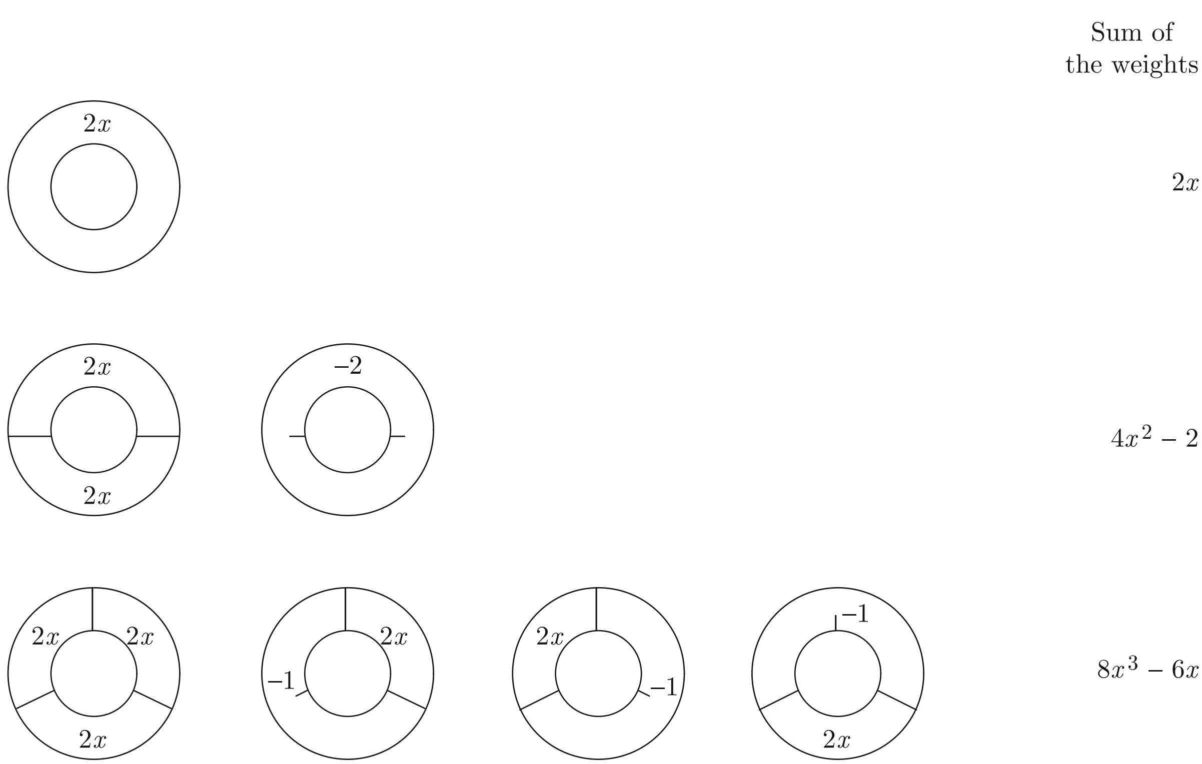


Figure 20.8. (continued)

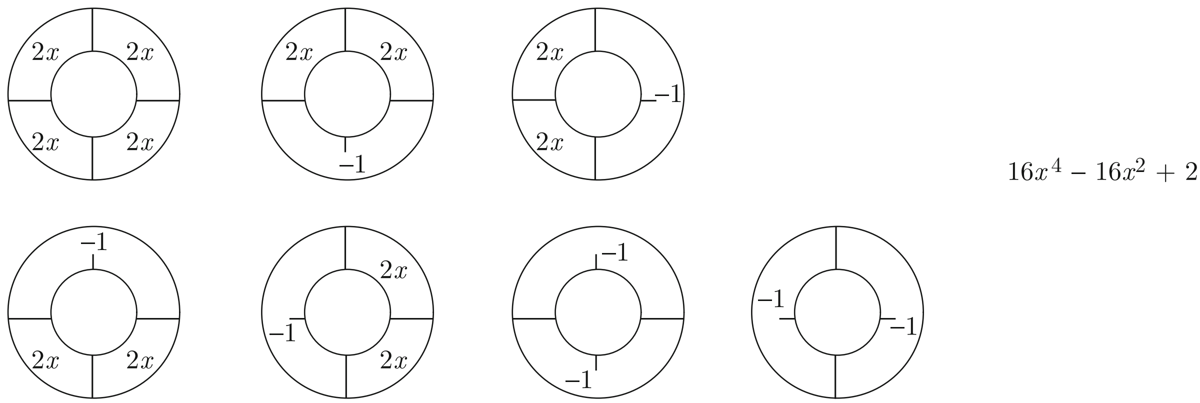


Figure 20.8.

Theorem 20.11 *The sum of the weights of tilings of a* $1\times n$ *bracelet is* $2T_n(x)$*, where the weight of a square is* $2x$ *and that of a domino is* -1*, except that the weight of the domino is* -2 *when* $n = 2$*.* ∎

The next result (see Example 19.5) is an interesting consequence of this theorem. Its proof follows by considering the cases where a domino may or may not occupy cells n and 1, and by invoking Theorem 20.1.

Theorem 20.12 *Let* $n \geq 2$*. Then* $2T_n(x) = U_n(x) - U_{n-2}(x)$*.* ∎

For example,

$$\begin{aligned} U_5(x) - U_3(x) &= (32x^5 - 32x^3 + 6x) - (8x^3 - 4x) \\ &= 2(16x^5 - 20x^3 + 5x) \\ &= 2T_5(x). \end{aligned}$$

Theorem 20.11 has another interesting byproduct. Counting the number of circular tilings with exactly k dominoes each for $k \geq 0$, we can develop an explicit formula for $T_n(x)$; use the same reasoning as in Theorem 16.10.

Theorem 20.13 *Let* $n \geq 1$*. Then*

$$2T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k}\binom{n-k}{k}(-1)^k(2x)^{n-2k}.$$ ∎

For example,

$$\begin{aligned} 2T_5(x) &= \sum_{k=0}^{2} \frac{5}{5-k}\binom{5-k}{k}(-1)^k(2x)^{5-2k} \\ &= (2x)^5 - 5(2x)^3 + 5(2x) \end{aligned}$$

$$\begin{aligned} &= 32x^5 - 40x^3 + 10x \\ T_5(x) &= 16x^5 - 20x^3 + 5x. \end{aligned}$$

As in the case of Theorem 20.1, this result also has several interesting byproducts, as the next four corollaries show.

Corollary 20.7 $2T_n(ix) = i^n q_n(x)$, *where* $n \geq 0$. ■

In particular, this implies the next result, as we found in Chapter 19.

Corollary 20.8 $T_n(i) = i^n Q_n$ *and* $|T_n(i)| = Q_n$, *where* $n \geq 0$. ■

The next result follows from this corollary.

Corollary 20.9 $2T_n(ix/2) = i^n l_n(x), 2|T_n(ix/2)| = l_n(x)$, *and* $2|T_n(i/2)| = L_n$, *where* $n \geq 0$. ■

For example,

$$\begin{aligned} T_5(ix/2) &= 16(ix/2)^5 - 20(ix/2)^3 + 5(ix/2) \\ &= \frac{i}{2}x^5 + \frac{5i}{2}x^3 + \frac{5i}{2}x = \frac{i}{2}(x^5 + x^3 + x) \\ 2|T_n(ix/2)| &= x^5 + 5x^3 + 5x = l_5(x) \\ 2|T_5(i/2)| &= 11 = L_5. \end{aligned}$$

Finally, notice that the formula in Theorem 20.13 can be rewritten as follows:

$$2T_n(x) = \sum_{k \text{ even}} \frac{n}{n-k}\binom{n}{n-k}(2x)^{n-2k} - \sum_{k \text{ odd}} \frac{n}{n-k}\binom{n}{n-k}(2x)^{n-2k}.$$

This can be re-stated in words, as the following corollary shows.

Corollary 20.10 $2T_n(x)$ *counts the sum of the weights of circular tilings of a* $1 \times n$ *bracelet with an even number of dominoes and with an odd number of dominoes.* ■

For example, consider the tilings of the 1×4 bracelet in Figure 20.8. The sum of the weights of the tilings with an even number of dominoes equals $16x^4 + 2$, and that with an odd number of dominoes equals $-16x^2$. Their sum is $(16x^4 + 2) + (-16x^2) = 16x^4 - 16x^2 + 2 = 2T_4(x)$, as expected.

Exercises 20

Prove each.

1. Corollary 20.1. *Hint*: Use Theorem 20.2.
2. Corollary 20.2. *Hint*: Use Theorem 20.2.
3. Corollary 20.3. *Hint*: Use Theorem 20.2.
4. Corollary 20.4. *Hint*: Use Theorem 20.2.
5. Corollary 20.5. *Hint*: Use Corollary 20.4.
6. Theorem 20.3.
7. Theorem 20.4.
8. Theorem 20.5.
9. Theorem 20.6.
10. Theorem 20.9.
11. Theorem 20.12.
12. Theorem 20.13.
13. Corollary 20.7. *Hint*: Use Theorem 20.13.
14. Corollary 20.8. *Hint*: Use Corollary 20.7.
15. Corollary 20.9. *Hint*: Use Theorem 20.13.

Appendix

Appendix A.1. The First 100 Pell and Pell–Lucas Numbers

n	P_n	Q_n
1	1	1
2	2	3
3	5	7
4	12	17
5	29	41
6	70	99
7	169	239
8	408	577
9	985	1,393
10	2,378	3,363
11	5,741	8,119
12	13,860	19,601
13	33,461	47,321
14	80,782	114,243
15	195,025	275,807
16	470,832	665,857
17	1,136,689	1,607,521
18	2,744,210	3,880,899
19	6,625,109	9,369,319
20	15994,428	22,619,537
21	38,613,965	54,608,393
22	93,222,358	131,836,323
23	225,058,681	318,281,039
24	543,339,720	768,398,401
25	1,311,738,121	1,855,077,841
26	3,166,815,962	4,478,554,083
27	7,645,370,045	10,812,186,007
28	18,457,556,052	26,102,926,097
29	44,560,482,149	63,018,038,201
30	107,578,520,350	152,139,002,499

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Appendix A.1. The First 100 Pell and Pell–Lucas Numbers (Continued)

n	P_n	Q_n
31	259,717,522,849	367,296,043,199
32	627,013,566,048	886,731,088,897
33	1,513,744,654,945	2,140,758,220,993
34	3,654,502,875,938	5,168,247,530,883
35	8,822,750,406,821	12,477,253,282,759
36	21,300,003,689,580	30,122,754,096,401
37	51,422,757,785,981	72,722,761,475,561
38	124,145,519,261,542	175,568,277,047,523
39	299,713,796,309,065	423,859,315,570,607
40	723,573,111,879,672	1,023,286,908,188,737
41	1,746,860,020,068,409	2,470,433,131,948,081
42	4,217,293,152,016,490	5,964,153,172,084,899
43	10,181,446,324,101,389	14,398,739,476,117,879
44	24,580,185,800,219,268	34,761,632,124,320,657
45	59,341,817,924,539,925	83,922,003,724,759,193
46	143 263 821 649 299 118	202,605,639,573,839,043
47	345,869,461,223,138,161	489,133,282,872,437,279
48	835,002,744,095,575,440	1,180,872,205,318,713,601
49	2,015,874,949,414,289,041	2,850,877,693,509,864,481
50	4,866,752,642,924,153,522	6,882,627,592,338,442,563
51	11,749,380,235,262,596,085	16,616,132,878,186,749,607
52	28,365,513,113,449,345,692	40,114,893,348,711,941,777
53	68,480,406,462,161,287,469	96,845,919,575,610,633,161
54	165,326,326,037,771,920,630	233,806,732,499,933,208,099
55	399,133,058,537,705,128,729	564,459,384,575,477,049,359
56	963,592,443,113,182,178,088	1,362,725,501,650,887,306,817
57	2,326,317,944,764,069,48,4905	3,289,910,387,877,251,662,993
58	5,616,228,332,641,321,147,898	7,942,546,277,405,390,632,803
59	13,558,774,610,046,711,780,701	19,175,002,942,688,032,928,599
60	32,733,777,552,734,744,709,300	46,292,552,162,781,456,490,001
61	79,026,329,715,516,201,199,301	111,760,107,268,250,945,908,601
62	190,786,436,983,767,147,107,902	269,812,766,699,283,348,307,203
63	460,599,203,683,050,495,415,105	651,385,640,666,817,642,523,007
64	1,111,984,844,349,868,137,938,112	1,572,584,048,032,918,633,353,217
65	2,684,568,892,382,786,771,291,329	3,796,553,736,732,654,909,229,441
66	6,481,122,629,115,441,680,520,770	9,165,691,521,498,228,451,812,099
67	15,646,814,150,613,670,132,332,869	22,127,936,779,729,111,812,853,639
68	37,774,750,930,342,781,945,186,508	53,421,565,080,956,452,077,519,377
69	91,196,316,011,299,234,022,705,885	128,971,066,941,642,015,967,892,393
70	220,167,382,952,941,249,990,598,278	311,363,698,964,240,484,013,304,163

Appendix A.1. The First 100 Pell and Pell–Lucas Numbers (Continued)

n	P_n	Q_n
71	531,531,081,917,181,734,0039,02,441	751,698,464,870,122,983,994,500,719
72	1,283,229,546,787,304,717,998,403,160	1,814,760,628,704,486,452,002,305,601
73	3,097,990,175,491,791,170,000,708,761	4,381,219,722,279,095,887,999,111,921
74	7,479,209,897,770,887,057,999,820,682	10,577,200,073,262,678,228,000,529,443
75	18,056,409,971,033,565,286,000,350,125	25,535,619,868,804,452,344,000,170,807
76	43,592,029,839,838,017,630,000,520,932	61,648,439,810,871,582,916,000,871,057
77	105,240,469,650,709,600,546,001,391,989	148,832,499,490,547,618,176,001,912,921
78	254,072,969,141,257,218,722,003,304,910	359,313,438,791,966,819,268,004,696,899
79	613,386,407,933,224,037,990,008,001,809	867,459,377,074,481,256,712,011,306,719
80	1,480,845,785,007,705,294,702,019,308,528	2,094,232,192,940,929,332,692,027,310,337
81	3,575,077,977,948,634,627,394,046,618,865	5,055,923,762,956,339,922,096,065,927,393
82	8,631,001,740,904,974,549,490,112,546,258	12,206,079,718,853,609,176,884,159,165,123
83	20,837,081,459,758,583,726,374,271,711,381	29,468,083,200,663,558,275,864,384,257,639
84	50,305,164,660,422,142,002,238,655,969,020	71,142,246,120,180,725,728,612,927,680,401
85	121,447,410,780,602,867,730,851,583,649,421	171,752,575,441,025,009,733,090,239,618,441
86	293,199,986,221,627,877,463,941,823,267,862	414,647,397,002,230,745,194,793,406,917,283
87	707,847,383,223,858,622,658,735,230,185,145	1,001,047,369,445,486,500,122,677,053,453,007
88	1,708,894,752,669,345,122,781,412,283,638,152	2,416,742,135,893,203,745,440,147,513,823,297
89	4,125,636,888,562,548,868,221,559,797,461,449	5,834,531,641,231,893,991,002,972,081,099,601
90	9,960,168,529,794,442,859,224,531,878,561,050	14,085,805,418,356,991,727,446,091,676,022,499
91	24,045,973,948,151,434,586,670,623,554,583,549	34,006,142,477,945,877,445,895,155,433,144,599
92	58,052,116,426,097,312,032,565,778,987,728,148	82,098,090,374,248,746,619,236,402,542,311,697
93	140,150,206,800,346,058,651,802,181,530,039,845	198,202,323,226,443,370,684,367,960,517,767,993
94	338,352,530,026,789,429,336,170,142,047,807,838	478,502,736,827,135,487,987,972,323,577,847,683
95	816,855,266,853,924,917,324,142,465,625,655,521	1,155,207,796,880,714,346,660,312,607,673,463,359
96	1,972,063,063,734,639,263,984,455,073,299,118,880	2,788,918,330,588,564,181,308,597,538,924,774,401
97	4,760,981,394,323,203,445,293,052,612,223,893,281	6,733,044,458,057,842,709,277,507,685,523,012,161
98	11,494,025,852,381,046,154,570,560,297,746,905,442	16,255,007,246,704,249,599,863,612,909,970,798,723
99	27,749,033,099,085,295,754,434,173,207,717,704,165	39,243,058,951,466,341,909,004,733,505,464,609,607
100	66,992,092,050,551,637,663,438,906,713,182,313,772	94,741,125,149,636,933,417,873,079,920,900,017,937

Appendix A.2. The First 100 Fibonacci and Lucas Numbers

n	F_n	L_n
1	1	1
2	1	3
3	2	4
4	3	7
5	5	11
6	8	18
7	13	29
8	21	47
9	34	76
10	55	123
11	89	199
12	144	322
13	233	521
14	377	843
15	610	1,364
16	987	2,207
17	1,597	3,571
18	2,584	5,778
19	4,181	9,349
20	6,765	15,127
21	10,946	24,476
22	17,711	39,603
23	28,657	64,079
24	46,368	103,682
25	75,025	167,761
26	121,393	271,443
27	196,418	439,204
28	317,811	710,647
29	514,229	1,149,851
30	832,040	1,860,498
31	1,346,269	3,010,349
32	2,178,309	4,870,847
33	3,524,578	7,881,196
34	5,702,887	12,752,043
35	9,227,465	20,633,239
36	14,930,352	33,385,282
37	24,157,817	54,018,521
38	39,088,169	87,403,803
39	63,245,986	141,422,324
40	102,334,155	228,826,127

Appendix A.2. The First 100 Fibonacci and Lucas Numbers (Continued)

n	F_n	L_n
41	165,580,141	370,248,451
42	267,914,296	599,074,578
43	433,494,437	969,323,029
44	701,408,733	1,568,397,607
45	1,134,903,170	2,537,720,636
46	1,836,311,903	4,106,118,243
47	2,971,215,073	6,643,838,879
48	4,807,526,976	10,749,957,122
49	7,778,742,049	17,393,796,001
50	12,586,269,025	28,143,753,123
51	20,365,011,074	45,537,549,124
52	32,951,280,099	73,681,302,247
53	53,316,291,173	119,218,851,371
54	86,267,571,272	192,900,153,618
55	139,583,862,445	312,119,004,989
56	225,851,433,717	505,019,158,607
57	365,435,296,162	817,138,163,596
58	591,286,729,879	1,322,157,322,203
59	956,722,026,041	2,139,295,485,799
60	1,548,008,755,920	3,461,452,808,002
61	2,504,730,781,961	5,600,748,293,801
62	4,052,739,537,881	9,062,201,101,803
63	6,557,470,319,842	14,662,949,395,604
64	10,610,209,857,723	23,725,150,497,407
65	17,167,680,177,565	38,388,099,893,011
66	27,777,890,035,288	62,113,250,390,418
67	44,945,570,212,853	100,501,350,283,429
68	72,723,460,248,141	162,614,600,673,847
69	117,669,030,460,994	263,115,950,957,276
70	190,392,490,709,135	425,730,551,631,123
71	308,061,521,170,129	688,846,502,588,399
72	498,454,011,879,264	1,114,577,054,219,522
73	806,515,533,049,393	1,803,423,556,807,921
74	1,304,969,544,928,657	2,918,000,611,027,443
75	2,111,485,077,978,050	4,721,424,167,835,364
76	3,416,454,622,906,707	7,639,424,778,862,807
77	5,527,939,700,884,757	12,360,848,946,698,171
78	8,944,394,323,791,464	20,000,273,725,560,978
79	14,472,334,024,676,221	32,361,122,672,259,149
80	23,416,728,348,467,685	52,361,396,397,820,127

Appendix A.2. The First 100 Fibonacci and Lucas Numbers (Continued)

n	F_n	L_n
81	37,889,062,373,143,906	84,722,519,070,079,276
82	61,305,790,721,611,591	137,083,915,467,899,403
83	99,194,853,094,755,497	221,806,434,537,978,679
84	160,500,643,816,367,088	358,890,350,005,878,082
85	259,695,496,911,122,585	580,696,784,543,856,761
86	420,196,140,727,489,673	939,587,134,549,734,843
87	679,891,637,638,612,258	1,520,283,919,093,591,604
88	1,100,087,778,366,101,931	2,459,871,053,643,326,447
89	1,779,979,416,004,714,189	3,980,154,972,736,918,051
90	2,880,067,194,370,816,120	6,440,026,026,380,244,498
91	4,660,046,610,375,530,309	10,420,180,999,117,162,549
92	7,540,113,804,746,346,429	16,860,207,025,497,407,047
93	12,200,160,415,121,876,738	27,280,388,024,614,569,596
94	19,740,274,219,868,223,167	44,140,595,050,111,976,643
95	31,940,434,634,990,099,905	71,420,983,074,726,546,239
96	51,680,708,854,858,323,072	115,561,578,124,838,522,882
97	83,621,143,489,848,422,977	186,982,561,199,565,069,121
98	135,301,852,344,706,746,049	302,544,139,324,403,592,003
99	218,922,995,834,555,169,026	489,526,700,523,968,661,124
100	354,224,848,179,261,915,075	792,070,839,848,372,253,127

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