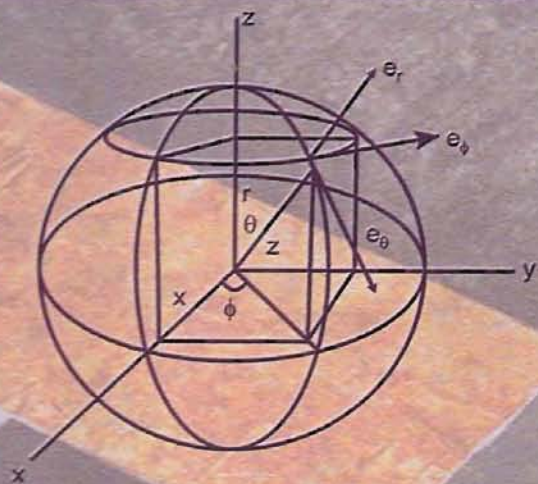


ENGINEERING MATHEMATICS I

Second Edition

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INCLUDES

- Solved Questions from University Question Papers
- Number of solved examples

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BS Publications

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SECOND EDITION

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1

Ordinary Differential Equations of First Order and First Degree

1.1 Introduction

Differential equations play an important role in many applications in the field of science and engineering, such as (i) problems relating to motion of particles (ii) problems involving bending of beams (iii) stability of electric system, etc. For example, Newton's law of cooling states that the rate of change of temperature of a body varies as the excess temperature of the body to that of its surroundings. If $\theta(t)$ is the temperature of the body at time 't' and θ_0 is the temperature of the room

in which the body is kept, then $\frac{d\theta}{dt}$ gives the rate of change of temperature with time.

$$\frac{d\theta}{dt} = K(\theta - \theta_0) ; K \text{ is constant}$$

Similarly Newton's second law of motion for a particle of mass m moving in a straight line can be written as

$$m \frac{d^2x}{dt^2} = F$$

Where m is the mass, x is the distance of the particle at time 't' measured from a fixed origin and F the external impressed force.

A differential equation is an equation involving an unknown function and its derivatives. If there is only one independent variable and one dependent variable the equation is called *an ordinary differential equation*.

If there are more than one independent variable the equation is called a *partial differential equation* as this involves partial derivatives.

For example :

$$4 \frac{d^3 y}{dx^3} + 3x \frac{dy}{dx} - y = e^x \quad \dots(a)$$

$$\left(\frac{d^3 y}{dx^3} \right)^4 + \left(\frac{d^2 y}{dx^2} \right)^8 + \left(\frac{dy}{dx} \right)^{12} + y^6 = x^8 \quad \dots(b)$$

$$\frac{d^2 y}{dx^2} + 4y = 0 \quad \dots(c)$$

$$\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^3 = 4 \frac{d^2 y}{dx^2} \quad \dots(d)$$

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial t^2} \quad \dots(e)$$

$$\frac{\partial u}{\partial t} = e^2 \frac{\partial^2 u}{\partial x^2} \quad \dots(f)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \left(\frac{\partial^2 u}{\partial t^2} \right) \quad \dots(g)$$

The first four equations (a), (b), (c) and (d) are ordinary differential equations and the remaining three are partial differential equations.

Order of a differential equation : The order of a differential equation is the order of the highest ordered derivative appearing in the equation.

Degree of a differential equation : The degree of a differential equation is the power to which the highest ordered derivative appears in the equation after clearing the radicals if any.

In the above examples :

Example : 1.1(a) is a differential equation of order 3 and degree 1.

Example : 1.1(b) is of third order and fourth degree differential equation.

Example : 1.1(c) is a second order, first degree differential equation.

Example : 1.1(d) is a second order, second degree differential equation.

1.2 Example

Formation of an ordinary D.E :

The differential equations are formed by eliminating all the arbitrary constants that are involved in the functional relationship between the dependent and independent variables.

For example :

$$y = cx^2 + c^2 \text{ where } c \text{ is an arbitrary constant. (1)}$$

To eliminate 'c' : (only one constant)

$$\text{From (1)} \quad \frac{dy}{dx} = c.2x + 0$$

$$c = \frac{1}{2x} \frac{dy}{dx}$$

Substitution of c in (1) gives

$$y = \frac{1}{2x} \frac{dy}{dx} x^2 + \frac{1}{4x^2} \left(\frac{dy}{dx} \right)^2$$

$$\left(\frac{dy}{dx} \right)^2 + 2x^3 \frac{dy}{dx} - 4x^2 y = 0$$

is the required D.E and $y = cx^2 + c^2$ is called the solution of the D.E.

Note :

Depending on the number of constants in the given equation differentiate it as many number of times successively. Then the elimination of the arbitrary constants from the resulting equations and the given equation gives the required differential equation whose order is equal to the number of constants.

1.3 Example

Eliminate the arbitrary constants a, b from $xy + x^2 = ae^x + be^{-x}$ and form the differential equation.

Solution :

The given equation is

$$xy + x^2 = ae^x + be^{-x} \quad \dots(1)$$

The number of arbitrary constants is two. Differentiating (1) w.r., to 'x' two times successively.

$$x \frac{dy}{dx} + y + 2x = ae^x - be^{-x} \quad \dots(2)$$

$$x \frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} + 2.1 = ae^x + be^{-x} \quad \dots (3)$$

From (1), (2) and (3) elimination of a, b gives the D.E.

From (1) and (3) we get

$$x \frac{d^2y}{dx^2} + \frac{2dy}{dx} + 2 = xy + x^2 \text{ is the required D.E.}$$

1.4 Example

Form the differential equation by eliminating the constants a and b from $ax^2 + by^2 = 1$

Solution :

Differentiating

$$ax^2 + by^2 = 1 \text{ w.r.t 'x'} \quad \dots (1)$$

$$2ax + 2by \frac{dy}{dx} = 0 \quad \dots (2)$$

Again differentiating w.r.t., 'x'

$$2a + 2by \frac{d^2y}{dx^2} + 2b \frac{dy}{dx} \cdot \frac{dy}{dx} = 0 \quad \dots (3)$$

Elimination of a, b from (1), (2) and (3) gives

$$\begin{vmatrix} x^2 & y^2 & -1 \\ x & yy_1 & 0 \\ 1 & (yy_2 + y_1^2) & 0 \end{vmatrix} = 0$$

Expanding the determinant we get

$$x \frac{d^2y}{dx^2} + x \left(\frac{dy}{dx} \right)^2 - y \left(\frac{dy}{dx} \right) = 0$$

1.5 Example

Form the differential equation by eliminating the constants from

$$y = a \sec x + b \tan x$$

Solution

Given equation is

$$y = a \sec x + b \tan x \quad \dots (1)$$

Differentiating w.r. to 'x'

$$\frac{dy}{dx} = a \sec x \tan x + b \sec^2 x \quad \dots (2)$$

$$\frac{dy}{dx} = \sec x [a \tan x + b \sec x] \quad \dots (3)$$

Further differentiation gives

$$\frac{d^2 y}{dx^2} = a \sec x \tan^2 x + a \sec^3 x + b 2 \sec^2 x \tan x$$

$$\text{i.e.,} \quad \frac{d^2 y}{dx^2} = a \sec x \tan^2 x + b \sec^2 x \tan x + b \sec^2 x \tan x + a \sec^3 x$$

$$\text{i.e.,} \quad \frac{d^2 y}{dx^2} = \sec x \tan x (a \tan x + b \sec x) + \sec^2 x (b \tan x + a \sec x) \dots (4)$$

Substituting

$$a \sec x + b \tan x = y \text{ from (1)}$$

$$\text{and} \quad a \tan x + b \sec x = \frac{\left(\frac{dy}{dx}\right)}{\sec x} \text{ from (2)}$$

in (3) we get

$$\left(\frac{d^2 y}{dx^2}\right) = \sec x \tan x \frac{\left(\frac{dy}{dx}\right)}{\sec x} + \sec^2 x (y)$$

$$\text{i.e.,} \quad \frac{d^2 y}{dx^2} - \tan x \frac{dy}{dx} - y \sec^2 x = 0$$

1.6 Example

Form the differential equation of all circles passing through the origin and having their centres on the x -axis.

Solution

Take any tangent to the circle as y -axis, the centre lies on x -axis. Let ' a ' be the radius of the circle.

Then centre is $(a, 0)$

$$\therefore \text{Equation of the circle is } (x - a)^2 + y^2 = a^2 \quad \dots (1)$$

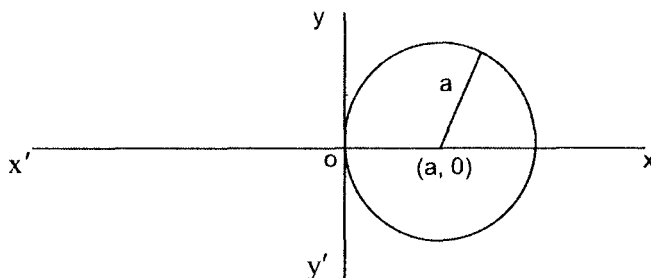


Fig. 1.1

Differentiating (1) w.r. to ' x '

$$2(x - a) + 2y \frac{dy}{dx} = 0$$

$$x - a = -y \frac{dy}{dx}$$

$$a = x + y \frac{dy}{dx} \quad \dots (2)$$

From (1) and (2)

$$\left[x - \left(x + y \frac{dy}{dx} \right) \right]^2 + y^2 = \left[x + y \frac{dy}{dx} \right]^2$$

$$y^2 \left(\frac{dy}{dx} \right)^2 + y^2 = x^2 + y^2 \left(\frac{dy}{dx} \right)^2 + 2xy \left(\frac{dy}{dx} \right)$$

$$2xy \frac{dy}{dx} + x^2 - y^2 = 0 \text{ is the required D.E}$$

1.7 Example

Form the differential equation of all central conics whose axes coincide with the axes of coordinates.

Solution

The equation of all central conics whose axes coincide with the axes is

$$ax^2 + by^2 = 1 \quad \dots (1)$$

Differentiating (1) w.r.t., x

$$2ax + 2by \frac{dy}{dx} = 0 \quad \dots (2)$$

Differentiating (2) w.r.t., ' x ' again

$$a + by \frac{d^2y}{dx^2} + b \left(\frac{dy}{dx} \right)^2 = 0 \quad \dots (3)$$

Eliminating a, b from (1), (2) (3)

$$\begin{vmatrix} x^2 & y^2 & 1 \\ x & y \frac{dy}{dx} & 0 \\ 1 & y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 & 0 \end{vmatrix} = 0$$

$$\Rightarrow xy \frac{d^2y}{dx^2} + x \left(\frac{dy}{dx} \right)^2 - y \left(\frac{dy}{dx} \right) = 0 \text{ is the required D.E.}$$

Exercise - 1(a)

1. Eliminate the arbitrary constants from the following and find the corresponding differential equation :

(i) $y = mx + c$ (m, c arbitrary constants)

$$[\text{Ans : } \frac{d^2y}{dx^2} = 0]$$

(ii) $y = a e^{2x} + b e^{-2x}$ (a, b arbitrary - constants)

$$[\text{Ans : } \frac{d^2y}{dx^2} - 4y = 0]$$

(iii) $y = ax + bx^2$ (a, b arb. constants)

$$[\text{Ans} : x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0]$$

(iv) $(x - h)^2 + (y - k)^2 = a^2$, (h, k a, b arb. constants)

$$[\text{Ans} : \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^3 = a^2 \left(\frac{d^2 y}{dx^2} \right)]$$

(v) $y = (a + bx)e^{-x}$, (a, b arb. constants)

$$[\text{Ans} : \frac{d^2 y}{dx^2} + \frac{2dy}{dx} + y = 0]$$

(vi) $y = a \sin x + b \cos x$ (a, b arb. constants)

$$[\text{Ans} : \frac{d^2 y}{dx^2} + y = 0]$$

2. Find the differential equation of all circles with centre on the line $y = x$ and having radius '1'.

$$[\text{Ans} : \left[1 + \frac{dy}{dx} \right]^2 = (x - y)^2 \left[1 + \left(\frac{dy}{dx} \right)^2 \right]]$$

3. Form the differential equation of all the circles with centre on the line $y = -x$ and passing through the origin.

$$[\text{Ans.} : (x^2 + y^2) \left(\frac{dy}{dx} - 1 \right) = 2(y - x) \left(x + y \frac{dy}{dx} \right)]$$

4. Find the differential equation of all the parabolas with vertex at the origin and foci on the x - axis.

$$[\text{Ans} : y^2 - 2xy \frac{dy}{dx} = 0]$$

5. Find the differential equation of all parabolas with the origin as focus and axis along x -axis.

$$[\text{Ans} : 2x \frac{dy}{dx} + y \left(\frac{dy}{dx} \right)^2 - y = 0]$$

Methods to Solve

1.8 The differential equations of the first order and of the First Degree :

1.8.1 Separation of Variables

Sometimes the differential equation

$$\frac{dy}{dx} = q(x, y)$$

can be written as

$$f(x) dx + g(y)dy = 0 \quad \dots (1)$$

if the variables can be separated.

Integration of (1) gives the solution of the equation.

$$\text{i.e.,} \quad \int f(x)dx + \int g(y) dy = c$$

where c is an arbitrary constant.

1.8.2 Example

Solve $e^x \tan y \, dx + (1 - e^x) \sec^2 y \, dy = 0$

Solution

The given equation

$$e^x \tan y \, dx + (1 - e^x) \sec^2 y \, dy = 0$$

can be rearranged as

$$\frac{e^x}{1 - e^x} dx + \frac{\sec^2 y}{\tan y} dy = 0$$

$$\text{Integrating} \quad \int \frac{e^x}{1 - e^x} dx + \int \frac{\sec^2 y}{\tan y} dy = c$$

$$-\log(e^x - 1) + \log \tan y = c$$

$$\text{i.e.,} \quad \frac{\tan y}{e^x - 1} = c \text{ or } \tan y = c(e^x - 1)$$

1.8.3 Example

Solve $\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$

Solution

The given differential equation can be written as

$$\frac{dy}{dx} = (1+x^2)(1+y^2)$$

$$\frac{dy}{1+y^2} = (1+x^2)dx \Rightarrow \tan^{-1} y = x + \frac{x^3}{3} + C$$

1.8.4 Example

Solve $\frac{dy}{dx} - 2xy = x$, where $y(0) = 1$

Solution

$$\frac{dy}{dx} = x(1+2y)$$

$$\frac{dy}{1+2y} = xdx$$

On integration

$$\frac{1}{2} \log(1+2y) = \frac{x^2}{2} + C \quad \dots (1)$$

Given $y = 1$ when $x = 0$

Substituting in (1)

$$\frac{1}{2} \log(3) = 0 + C$$

$$C = \frac{1}{2} \log 3 = \log(\sqrt{3})$$

Hence the solution is

$$\frac{1}{2} \log(1+2y) = \frac{x^2}{2} + \log \sqrt{3} \quad (\text{i.e.,}) \quad \log\left(\frac{1+2y}{3}\right) = x^2$$

1.8.5 Example

Solve $\frac{dy}{dx} = (4x + y + 1)^2$

Solution

$$\frac{dy}{dx} = (4x + y + 1)^2 \quad \dots (1)$$

Substituting $4x + y + 1 = t$ in (1)

$$4 + \frac{dy}{dx} = \frac{dt}{dx}$$

i.e., $\frac{dy}{dx} = \frac{dt}{dx} - 4$

$$\frac{dt}{dx} - 4 = t^2$$

Integrating

$$\int \frac{dt}{t^2 + 4} = \int dx \Rightarrow \frac{dt}{t^2 + 2^2} = x + c$$

i.e., $\frac{1}{2} \tan^{-1} \frac{t}{2} = x + c$

$$\tan^{-1} \left(\frac{4x + y + 1}{2} \right) = 2(x + c)$$

The solution can also be written as

$$4x + y + 1 = 2 \tan(2x + c)$$

where C is an arbitrary constant.

Exercise - 1(b)

1.9 Solve the Following Differential Equations

1. $\frac{dy}{dx} = e^{x+y}$ [Ans : $e^x + e^{-y} = c$]

2. $(2 - x)dy - (3 + y) dx = 0$ [Ans $(3 + y)(2 - x) = c$]

3. $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$ [Ans : $e^y = e^x + \frac{x^3}{3} + c$]

$$4. \quad y - x \frac{dy}{dx} = a \left(y^2 + \frac{dy}{dx} \right) \quad [\text{Ans : } yc = (a + x)(1 - ay)]$$

$$5. \quad 3e^x \tan y dx + (1 - e^x) \sec^2 y dy = 0 \quad [\text{Ans : } \tan y = c(1 - e^x)^3]$$

$$6. \quad (x-y)^2 \frac{dy}{dx} = a^2 \quad [\text{Ans : } (x-y) + \log \left(\frac{x-y-a}{x-y+a} \right) = x + c]$$

$$7. \quad \frac{dy}{dx} = (3x + 4y + 1)^2 \quad [\text{Ans : } 2(3x + 4y + 1) = \sqrt{3} \tan^{-1} 2\sqrt{3}x + c]$$

$$8. \quad \frac{dy}{dx} = (2x + y + 1)^2 \quad [\text{Ans : } \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{2x + y + 1}{\sqrt{2}} \right) = x + c]$$

$$9. \quad \frac{dy}{dx} = \tan(x + y) \quad [\text{Ans : } \log[\sin(x + y) + \cos(x + y)] = x - y + c]$$

$$10. \quad \frac{dy}{dx} = \frac{2}{(x + 2y - 3)} \quad [\text{Ans : } (x + 2y - 3) - 4 \log(x + 2y + 1) = x + c]$$

1.9.1 Homogeneous Equations

The differential equation of the form

$$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$$

where f, g are homogeneous functions of same degree in x, y is called a homogenous differential equation.

Such a differential equation can be written as

$$\frac{dy}{dx} = \frac{x^n \psi\left(\frac{y}{x}\right)}{x^n \phi\left(\frac{y}{x}\right)} = \frac{x^n \psi\left(\frac{y}{x}\right)}{x^n \phi\left(\frac{y}{x}\right)} \quad \dots (1)$$

Substituting $y = vx$

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

The D.E (1) becomes

$$v + x \frac{dv}{dx} = \frac{\psi(v)}{\phi(v)}$$

$$\frac{dx}{x} + \frac{\phi(v)dv}{v\phi(v) - \psi(v)} = 0$$

Integration yields the solution

$$\int \frac{dx}{x} + \int \frac{\phi(v)dv}{v\phi(v) - \psi(v)} = c \text{ where } v = \frac{y}{x}$$

1.9.2 Example

Solve $(x^2 + y^2) \frac{dy}{dx} = xy$

Solution

$$\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$$

Substituting $y = vx$, $\frac{dy}{dx} = v + x \frac{dv}{dx}$

$$\therefore v + x \frac{dv}{dx} = \frac{xvx}{x^2 + v^2x^2}$$

$$v + x \frac{dv}{dx} = \frac{v}{1 + v^2}$$

$$x \frac{dv}{dx} = \frac{v}{1 + v^2} - v$$

$$x \frac{dv}{dx} = \frac{-v^3}{1 + v^2}$$

$$\frac{dx}{x} + \frac{1 + v^2}{v^3} dv = 0$$

$$\int \frac{dx}{x} + \int \frac{1}{v^3} dv + \int \frac{1}{v} dv = c$$

$$\log x + \frac{v^{-2}}{-2} + \log v = c$$

$$\log xv - \frac{1}{2v^2} = c$$

$$\log x \left(\frac{y}{x} \right) - \frac{x^2}{2y^2} = c \Rightarrow 2y^2 \log y - x^2 = 2cy^2$$

i.e., $2y^2 \log y = 2cy^2 + x^2$

1.9.3 Example

Solve $x \frac{dy}{dx} + \frac{y^2}{x} = y$

Solution

$$x^2 \frac{dy}{dx} + y^2 = xy$$

i.e., $\frac{dy}{dx} = \frac{xy - y^2}{x^2}$

Substituting $y = vx$,

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

$\therefore v + x \frac{dv}{dx} = \frac{xvx - v^2 x^2}{x^2}$

$$v + x \frac{dv}{dx} = v - v^2$$

$$x \frac{dv}{dx} = -v^2$$

$$\frac{dx}{x} + \frac{dv}{v^2} = 0$$

Integration yields

$$\log x + \frac{v^{-2+1}}{-2+1} = c \Rightarrow \log x - \frac{1}{v} = c$$

$$\log x - \frac{x}{y} = c$$

$$y \log x - x = cy \quad \text{or} \quad y \log x = x + cy$$

1.9.4 Example

$$\text{Solve } \left[x + y \sin\left(\frac{y}{x}\right) \right] dx = x \sin\left(\frac{y}{x}\right) dy$$

Solution

The given differential equation can be written as

$$\frac{dy}{dx} = \frac{x + y \sin\left(\frac{y}{x}\right)}{x \sin\left(\frac{y}{x}\right)}$$

$$\text{Substituting } y = vx \Rightarrow \frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$$

$$\therefore v + x \frac{dv}{dx} = \frac{x + vx \sin v}{x \sin \sin(v)}$$

$$v + x \frac{dv}{dx} = \frac{1 + v \sin v}{\sin v}$$

$$x \frac{dv}{dx} = \frac{1 + v \sin v}{\sin v} - v = \frac{1}{\sin v}$$

$$\sin v \, dv = \frac{dx}{x}$$

Integrating

$$\Rightarrow -\cos v = \log x + c$$

$$\Rightarrow \log x + \cos\left(\frac{y}{x}\right) = c$$

Exercise - 1 (c)

1.10 Solve the Following Equations

$$1. \quad \frac{dy}{dx} = \frac{y^2}{xy - x^2} \quad [\text{Ans : } y = ce^{\frac{y}{x}}]$$

$$2. \quad (2 - 2xy)dx = (x^2 - 2xy)dy \quad [\text{Ans : } xy(y - x) = c]$$

$$3. \quad 2xy + (y^2 - x^2) \frac{dy}{dx} = 0 \quad [\text{Ans : } x^2 + y^2 = cy]$$

$$4. \quad (x + y)^2 dx = 2x^2 dy \quad [\text{Ans : } \log x = 2 \tan^{-1} \left(\frac{y}{x} \right) + c]$$

$$5. \quad xdy - ydx = \sqrt{x^2 + y^2} dx \quad [\text{Ans : } y + \sqrt{x^2 + y^2} = cx^2]$$

$$6. \quad x \sin \left(\frac{y}{x} \right) dy = \left(y \sin \frac{y}{x} - x \right) dx \quad [\text{Ans : } \cos \frac{y}{x} = \log cx]$$

$$7. \quad x \cos \left(\frac{y}{x} \right) (ydx + xdy) = y \sin \left(\frac{y}{x} \right) (xdy - ydx) \quad [\text{Ans : } \sec \left(\frac{y}{x} \right) = yxc]$$

$$8. \quad (x^2 y - x^3)dy - (y^3 + xy^2) dx = 0 \quad [\text{Ans : } y\sqrt{x^2 + y^2} = cx.e^{\tan^{-1} \left(\frac{y}{x} \right)}]$$

$$9. \quad (x^2 + y^2) dx = 2xydy \quad [\text{Ans : } (x^2 - y^2) = xc]$$

$$10. \quad x \frac{dy}{dx} = y[\log y - \log x + 1] \quad [\text{Ans : } y = xe^{cx}]$$

1.10.1 Non-homogeneous Differential Equations

The D.E of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C} \quad \dots (1)$$

where a, b, c, A, B, C are constants, is called a *non-homogeneous differential equation*.

Case (i)

$$\text{If } \frac{a}{A} \neq \frac{b}{B}$$

Substituting $x = X + h$, $Y = y + K$

where h, k are (constants) to be chosen so as to satisfy.

$$ah + bk + c = 0, Ah + Bk + C = 0$$

Solving these equations, values of h, k are obtained.

The given D.E then reduces to a homogeneous D.E.

$$\frac{dY}{dX} = \frac{aX + bY}{AX + BY}$$

which is then solved taking $Y = VX$ and then

substitute $X = x - h$ and $Y = y - k$ in the solution.

Case (ii)

$$\text{If } \frac{a}{A} = \frac{b}{B}$$

Then the differential equation will be of the form

$$\frac{dy}{dx} = \frac{(ax + by) + c}{m(ax + by) + C} \quad \dots (2)$$

since $Ax + By$ will be constant m times $ax + by$.

Now substitute

$$ax + by = t,$$

Differentiation gives

$$a + b \frac{dy}{dx} = \frac{dt}{dx}$$

$$\text{i.e., } \frac{dy}{dx} = \frac{\frac{dt}{dx} - a}{b}$$

D.E (2) then reduces to

$$\frac{\frac{dt}{dx} - a}{b} = \frac{t + c}{mt + C}$$

Then the solution is obtained by using the method of separation of variables.

1.10.2 Example

Solve $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$

Solution

$$\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3} \quad \dots (1)$$

Substituting

$$x = X + h, y = Y + k$$

where h, k are chosen to satisfy $h + 2k - 3 = 0$ and $2h + k - 3 = 0$

solving we get

$$h = 1, k = 1 \quad \dots (2)$$

i.e., we take $x = X + 1, y = Y + 1$

The D.E (1) reduces to

$$\frac{dY}{dX} = \frac{X-2Y}{2X+Y}$$

Substituting $Y = VX, \frac{dY}{dX} = V + X \frac{dV}{dX}$

$$V + X \frac{dV}{dX} = \frac{X + 2VX}{2X + VX}$$

$$V + X \frac{dV}{dX} = \frac{1 + 2V}{2 + V} \Rightarrow X \frac{dV}{dX} = \frac{1 - V^2}{2 + V}$$

$$\frac{2 + v}{1 - v} dv = \frac{dX}{X}$$

$$\frac{dX}{X} + \left(1 + \frac{3}{v-1}\right) dv = 0$$

$$\log X + v + 3 \log(v-1) = \log c$$

$$v + \log(v-1)^3 + \log X = \log c$$

$$v - \log(v-1)^3 X = \log c$$

$$\frac{Y}{X} + \log\left(\frac{Y-x}{X}\right)^3 X = \log c$$

i.e., $\log(Y-X)^3 + XY = X \log c$

$$\log(y-1-x+1)^3 + (x-1)(y-1) = (x-1) \log c$$

$$\log(y-x)^3 = (x-1)(\log c - y + 1)$$

1.10.3 Example

Solve $(2x + 3y + 1) dx + (2y - 3x + 5) dy = 0$

Solution

$$\frac{dy}{dx} = -\frac{2x+3y+1}{2y-3x+5}$$

Substituting $x = X + h, y = Y + k, \frac{dy}{dx} = \frac{dY}{dX}$

Choosing h, k so that $2h + 3k + 1 = 0, 3h - 3k - 5 = 0$

we get $h = 1, k = -1$

The given differential equation reduces to

$$\frac{dY}{dX} = \frac{2X+3Y}{3X-2Y}$$

Substituting

$$Y = VX$$

$$\frac{dY}{dX} = V + X \frac{dV}{dX}$$

$$V + X \frac{dV}{dX} = \frac{2X+3VX}{3X-2VX}$$

$$V + X \frac{dV}{dX} = \frac{2+3V}{3-2V}$$

$$X \frac{dV}{dX} = \frac{2(1+V^2)}{3-2V}$$

$$\left(\frac{3-2V}{1+V^2}\right)dV = \frac{2dX}{X}$$

Integrating

$$3 \int \frac{dv}{1+v^2} - \int \frac{2v}{1+v^2} dv = 2 \int \frac{dX}{X} + c$$

$$3 \tan^{-1}(V) - \log(1+v^2) = 2 \log X + c$$

$$3 \tan^{-1}\left(\frac{y}{x}\right) - \log\left(\frac{X^2 - Y^2}{X^2}\right) = 2 \log X + c$$

But

$$X = x - h = x - 1$$

$$Y = y - k = y + 1$$

$$3 \tan^{-1}\left(\frac{y+1}{x-1}\right) - \log\left[\frac{(x-1)^2 + (y+1)^2}{(x-1)^2}\right] = 2 \log(x-1) + c$$

1.10.4、 Example

Solve $\frac{dy}{dx} = \frac{x-y+1}{2x-2y}$

Solution

$$\frac{dy}{dx} = \frac{(x-y)+1}{2(x-y)}$$

Substituting $x-y = V$

$$\frac{dy}{dx} = 1 - \frac{dV}{dx}$$

$$1 - \frac{dV}{dx} = \frac{V+1}{2V}$$

$$\frac{dV}{dx} = 1 - \frac{V+1}{2V} = \frac{V-1}{2V}$$

$$\frac{2V}{V-1} dV = dx$$

Integrating

$$2 \int \frac{V-1+1}{V-1} dV = x + c$$

$$2[V + \log(V-1)] = x + c$$

$$2[x - y + \log(x - y - 1)] = x + c$$

$$2\log(x - y - 1) = 2y - x + c$$

or $(x - y - 1)2 = e^{2y-x} C$

1.10.5 Example

Solve $\frac{dy}{dx} = -\frac{4x+6y+1}{2x+3y-5}$

Solution

$$\frac{dy}{dx} = -\frac{2(2x+3y)+1}{(2x+3y)-5}$$

Substituting

$$2x + 3y = V$$

$$2+3 \frac{dy}{dx} = \frac{dV}{dx}$$

$$\frac{dy}{dx} = \frac{1}{3} \left[\frac{dV}{dx} - 2 \right]$$

The differential equation reduces to

$$\frac{1}{3} \left(\frac{dV}{dx} - 2 \right) = -\frac{-2V+1}{V-5}$$

$$\frac{dV}{dx} = 2 + \frac{6V+3}{5-V} = \frac{10-2V+6V+3}{5-V} = \frac{13+4V}{5-V}$$

$$\frac{5-V}{4V+13} dV = dx$$

$$\text{Integrating } \int \left(1 - \frac{33}{4V+13}\right) dv = -4 \int dx + c$$

$$v - \frac{33}{4} \log(4V+13) = -4x + c$$

$$(2x+3y) - \frac{33}{4} \log[4(2x+3y)+13] = -4xm + C$$

$$\text{i.e., } 4(6x+3y-c) = 33 \log[4(2x+3y)+13]$$

Exercise - 1(d)

1.11 Solve the Following Equations

$$1. \quad \frac{dy}{dx} + \frac{y-x+5}{y+x+3}$$

$$[\text{Ans : } \tan^{-1} \left(\frac{y+4}{x-1} \right) + \frac{1}{2} \log \left[\left(\frac{y+4}{x-1} \right)^2 + 1 \right] = \log(x-1) + \log c]$$

$$2. \quad \frac{dy}{dx} = \frac{x+2y-3}{2x+3y-5}$$

$$[\text{Ans : } (2+\sqrt{3}) \log \left[\frac{y-1}{x-1} - \frac{1}{\sqrt{3}} \right] - (2-\sqrt{3}) \log \left[\frac{y-1}{x-1} + \frac{1}{\sqrt{3}} \right] + \frac{2}{\sqrt{3}} \log x = \log c]$$

$$3. \quad \frac{dy}{dx} = \frac{2y-x-4}{y-3x+3}$$

$$[\text{Ans : } (x-2)^2 - 5(x-2)(y-3) + (y-3)^2 = c \left\{ \frac{2(y-3) - [5 + \sqrt{21}(x-2)]}{2(y-3) - (5 - \sqrt{21}(x-2))} \right\}]$$

$$4. \quad (2x+5y+1)dx - (5x+2y-1)dy = 0$$

$$[\text{Ans : } (x+y)^7 = c \left(x - y - \frac{2}{3} \right)^3]$$

$$5. \frac{dy}{dx} = \frac{y-x+1}{y+x+5}$$

$$[\text{Ans : } \tan^{-1}\left(\frac{y-2}{x-3}\right) + \frac{1}{2} \log [(y-2)^2 + (x-3)^2] = \log c]$$

$$6. \frac{dy}{dx} = \frac{3y+2x+4}{4x+6y+5}$$

$$[\text{Ans : } \frac{2}{7}(2x+3y) - \frac{9}{49}(14x+21y+22) = x + c]$$

$$7. (4x-6y-1) dx + (3y-2x-2)dy = 0$$

$$[\text{Ans : } x-y + \frac{3}{4} \log(8x-12y-5) = c]$$

$$8. \frac{dy}{dx} = \frac{x-y+3}{2x-2y+5}$$

$$[\text{Ans : } (x-y) + \log[2+x-y] = \frac{x}{2} + c]$$

1.12 Linear Differential Equations

A differential equation of the form $\frac{dy}{dx} + py = q$ where p, q are functions of 'x', alone or constants is said to be a linear differential equation of first order. Multiplying both sides of the equation by $e^{\int p dx}$ [called the *integrating factor (I.F)*] we get,

$$e^{\int p dx} \frac{dy}{dx} + e^{\int p dx} py = q.e^{\int p dx} \quad \dots (i)$$

The left hand side is the differential coefficient of $y.e^{\int p dx}$

$\therefore$ (i) can be written as

$$d\left(y.e^{\int p dx}\right) = q.e^{\int p dx}$$

Integrating

$$y.e^{\int p dx} = \int q.e^{\int p dx} dx + c$$

which gives the desired solution.

Note :

In some cases a differential equation can be reduced to the linear form by taking 'y' as independent variable and x as the dependent variable.

The D.E is written as

$$\frac{dx}{dy} + p_1x = q_1$$

p_1, q_1 are functions of y or constants

Now the I.F = $e^{\int p_1 dy}$

∴ Solution is

$$x \cdot e^{\int p_1 dy} = \int q_1 \cdot e^{\int p_1 dy} + c$$

1.12.1 Example

Solve $(1 + x^2) \frac{dy}{dx} + 2yx - 6x^2 = 0$

Solution

Rearranging the given differential equation to the form

$$\frac{dy}{dx} + py = q$$

We have

$$\frac{dy}{dx} + \frac{2x}{1+x^2} y = \frac{6x^2}{1+x^2}$$

Here

$$p = \frac{2x}{1+x^2}, q = \frac{6x^2}{1+x^2}$$

$$\text{I.F} = e^{\int p dx} = e^{\int \frac{2x}{1+x^2} dx}$$

$$\text{I.F} = e^{\log(1+x^2)} = (1+x^2)$$

Solution is given by

$$y(\text{I.F}) = \int p(\text{I.F}) dx + c$$

$$\therefore y(1+x^2) = \int \frac{6x^2}{1+x^2} \times (1+x^2) dx + c$$

$$y(1-x^2) = 6 \frac{x^3}{3} + c$$

$$\text{i.e., } y(1-x^2) = 2x^3 + c$$

1.12.2 Example

Solve $x \log x \frac{dy}{dx} + y = 2 \log x$

Solution

$$\frac{dy}{dx} + \frac{y}{x \log x} = \frac{2}{x}$$

Here $p = \frac{1}{x \log x}$ and $q = \frac{2}{x}$

$$\text{I.F.} = e^{\int p dx} = e^{\int \frac{1}{x \log x} dx} = e^{\log(\log x)} = \log x$$

Solution is

$$y \log x = \int \frac{2}{x} \log x dx + c$$

$$y \log x = 2 \frac{(\log x)^2}{2} + c$$

1.12.3 Example

Solve $x \cos x \frac{dy}{dx} + y(x \sin x + \cos x) = 1$

Solution

$$\frac{dy}{dx} + y \cdot \frac{x \sin x + \cos x}{x \cos x} = \frac{1}{x \cos x}$$

$$p = \frac{x \sin x + \cos x}{x \cos x}, q = \frac{1}{x \cos x}$$

$$\text{I.F} = e^{\int \frac{x \sin x + \cos x}{x \cos x} dx}$$

$$= e^{\int \left(\tan x + \frac{1}{x} \right) dx}$$

$$= e^{(\log(x \sec x))}$$

$$\text{I.F} = e^{\log(x \sec x)}$$

$$= x \sec x$$

Solution is

$$y(x \sec x) = \int \frac{1}{x \cos x} \times x \sec x dx + c$$

$$xy \sec x = \int \sec^2 x dx + c$$

$$xy \sec x = \tan x + c$$

1.12.5 Example

Solve $\frac{dy}{dx} + 2y \tan x = \sin x$ given that $y = 0$ where $x = \frac{\pi}{3}$,

Solution

$$\frac{dy}{dx} + 2y \tan x = \sin x$$

$$p = 2 \tan x$$

$$q = \sin x$$

$$\text{IF} = e^{\int 2 \tan x dx}$$

$$dx = e^{2 \log \sec x}$$

$$= \sec^2 x$$

$$y^{\text{IF}} = \int q \times \text{IF} . dx + c$$

$$y \sec^2 x = \int \sec x . \sec^2 x dx + c$$

$$y \sec^2 x = \int \sec x \tan x dx + c$$

$$y \sec^2 x = \sec x + c \quad \dots (1)$$

Given that $y = 0$ when $x = \frac{\pi}{3}$

$$\therefore 0 = \sec \frac{\pi}{3} + c \Rightarrow c = -2$$

Substituting $c = -2$ in (1)

$$y \sec^2 x = \sec x - 2$$

is the required solution

1.12.6 Example

$$\text{Solve } (x + 2y^2) \frac{dy}{dx} = y$$

Solution

$$x + 2y^2 = y \frac{dy}{dx}$$

$$\therefore \frac{dx}{dy} - \frac{x}{y} = 2y^2$$

is in the form of

$$\frac{dx}{dy} + p_1 y = q_1$$

$$p_1 = -\frac{1}{y}, q_1 = 2y^2$$

$$\text{IF} = e^{\int p_1 dy}$$

$$= e^{\int -\frac{1}{y} dy}$$

$$= \frac{1}{y}$$

$$x(IF) = \int q_1(IF) dy + c$$

$$x \frac{1}{y} = \int 2y^2 \times \frac{1}{y} dy + c$$

$$\frac{x}{y} = y^2 + c$$

1.13.7 Example

Solve $(x + y + 1) \frac{dy}{dx} = 1$

Solution

$$\frac{dy}{dx} - x = y + 1$$

$$P_1 = -1, Q_1 = y + 1$$

$$IF = e^{\int -1 dy} \\ = e^{-y}$$

Solution is given by

$$x(IF) = \int q_1(IF) dy + c$$

$$xe^{-y} = \int (y + 1)e^{-y} dy + c$$

$$xe^{-y} = -(y + 1)e^{-y} - e^{-y} + c$$

i.e., $e^{-y}[x + y + 2] = c$ or $x + y + 2 = ce^y$

Exercise 1(e)

1.13 Solve the Following Differential Equations

1. $(1 + y^2)dx = (\tan^{-1}y - x)dx$

[Ans : $xe^{\tan^{-1}y} = \tan^{-1}ye^{\tan^{-1}y} - e^{\tan^{-1}y} + c$]

2. $\cos^2x \frac{dy}{dx} + y = \tan x$

[Ans : $y = ce^{-\tan x} - \tan x - 1$]

$$3. \quad x \frac{dy}{dx} + 2y - x^2 \log x = 0$$

$$[\text{Ans : } y = \frac{c}{x^2} + \frac{1}{4} x^2 \log x - \frac{x^2}{16}]$$

$$4. \quad \frac{dy}{dx} + y \cot x = 4x \operatorname{cosec} x, \text{ if } y = 0, \text{ when } x = \frac{\pi}{2}$$

$$[\text{Ans : } y \sin x = 2x^2 - \frac{\pi^2}{2}]$$

$$5. \quad ye^y dx = (y + 2xe^y) dy$$

$$[\text{Ans : } xy^{-2} = c - e^{-y}]$$

$$6. \quad (x + 3y^3) \frac{dy}{dx} = y$$

$$[\text{Ans : } x = \frac{3}{2} y^3 + cy]$$

$$7. \quad (xy - 1) \frac{dy}{dx + y^3} + y^3 = 0$$

$$[\text{Ans : } x = ce^{\frac{1}{y}} + \frac{1}{y} + 1]$$

$$8. \quad \frac{dy}{dx} = x^3 - 2xy \text{ if } y = 2 \text{ when } x = 1$$

$$[\text{Ans : } 2y - x^2 + 1 = 4e^{1-x^2}]$$

$$9. \quad (1 + x^3) \frac{dy}{dx} + 6x^2y = e^x$$

$$\text{Ans : } y(1+x^3)^2 = e^x (x^3 - 3x^2 + 6x - 5) + c$$

$$10. \quad \frac{dy}{dx} = -\frac{x + y \cos x}{1 + \sin x}$$

$$[\text{Ans : } y(1+\sin x) = c - \frac{x^2}{2}]$$

1.13.1 Non-linear Differential Equation of First Order

Ber noulli's equation :

The differential equation of the form

$$\frac{dy}{dx} + py = qy^n \quad \dots (1)$$

where p, q are functions of x alone is said to be a Bernoulli's differential equation.

Dividing (1) throughout by y^n

$$y^{-n} \frac{dy}{dx} + py^{-n+1} = q \quad \dots (2)$$

Substituting y^{-n+1} (coefficient of p) = v

$$(1-n) \frac{1}{y^n} \frac{dy}{dx} = \frac{dv}{dx}$$

(2) reduces to

$$\frac{1}{(1-n)} \frac{dv}{dx} + pv = q$$

$$\frac{dv}{dx} + (1-n)p v = (1-n)q$$

which is linear in v .

The above D.E can be solved by using the method given in 1.12.1 example.

1.13.2 Example

Solve $\frac{dy}{dx} - y \tan x = y^2 \sec x$

Solution

$$\frac{dy}{dx} - y \tan x = y^2 \sec x \quad \dots (1)$$

$$y^{-2} \frac{dy}{dx} - \frac{1}{y} \tan x = \sec x$$

Substituting

$$-\frac{1}{y} = v,$$

$$+ \frac{1}{y^2} \frac{dy}{dx} = \frac{dv}{dx}$$

(1) reduces to

$$\frac{dv}{dx} + v \tan x = \sec x$$

is linear in v.

Here $p = \tan x$, $q = \sec x$

$$\begin{aligned} \text{IF} &= e^{\int p dx} \\ &= e^{\int \tan x dx} \\ &= e^{\log \sec x} \\ &= \sec x \end{aligned}$$

Solution of the D.E (1) is therefore

$$\begin{aligned} v(\text{IF}) &= \int q(\text{IF}) dx + c \\ v \cdot \sec x &= \int \sec x \cdot \sec x dx + c \\ v \sec x &= \int \sec^2 x dx + c \end{aligned}$$

Substituting

$$\begin{aligned} v &= -\frac{1}{y} \\ -\frac{1}{y} \sec x &= \tan x + c \end{aligned}$$

$$\text{i.e., } y(\tan x + c) + \sec x = 0$$

1.13.3 Example

Solve $(3xy + y^2) dx - 3x^2 dy = 0$

Solution

$$3x^2 \frac{dy}{dx} - 3xy = y^2$$

is in Bernoulli's form

$$\frac{1}{y^2} \frac{dy}{dx} - \frac{1}{xy} = \frac{1}{3x^2}$$

Substituting

$$\frac{-1}{y} = v$$

D.E reduces to

$$\frac{dv}{dx} - \frac{1}{xy} v = \frac{1}{3x^2} \quad \dots (1)$$

which is linear in v

Here $p = \frac{1}{x}, \quad q = \frac{1}{3x^2}$

$$\begin{aligned} \text{IF} &= e^{\int p dx} \\ &= e^{\int \frac{1}{x} dx} \\ &= x \end{aligned}$$

Solution of the D.E (1) is

$$\begin{aligned} v(\text{IF}) &= \int q(\text{IF}) dx + c \\ -\frac{1}{y} x &= \int \frac{1}{3x^2} \cdot x dx + c \\ -\frac{x}{y} &= \frac{1}{3} \log x + c \end{aligned}$$

i.e., $y \left(\frac{1}{3} \log x + c \right) + x = 0$

1.13.4 Example

Solve $\frac{dy}{dx} + (2x \tan^{-1} y - x^3) (1 + y^2) = 0$

Solution

$$\frac{1}{1+y^2} \frac{dy}{dx} = \frac{dv}{dx}$$

The given differential equation becomes

$$\frac{dv}{dx} + 2xv = x^3 \quad \dots (1)$$

which is linear in v

Here $p = 2x, q = x^3$

$$\text{IF} = e^{\int 2x dx} = e^{x^2}$$

Solution of the D.E (1) is

$$v(\text{IF}) = \int q(\text{IF}) dx + c$$

Writing $x^2 = t$

$$x dx = \frac{1}{2} dt$$

$$v \cdot e^{x^2} = \int t \cdot e^t \frac{dt}{2} + c$$

$$\tan^{-1} y \cdot e^{x^2} = \frac{1}{2} [t \cdot e^t - e^t] + c$$

$$e^{x^2} \tan^{-1} y = \frac{1}{2} [x^2 e^{x^2} - e^{x^2}] + c$$

1.14.5 Example

Solve $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^3 x$

Solution

$$\tan y \frac{dy}{dx} + \tan x = \cos y \cos^3 x$$

Dividing by $\cos y$ throughout

$$\sec y \tan y \frac{dy}{dx} + \sec y \tan x = \cos^3 x$$

Substituting $\sec y = v$, we get

$$\sec y \tan y \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dv}{dx} + v \cdot \tan x = \cos^3 x \quad \dots (1)$$

is linear in v

Here $p = \tan x$, $q = \cos^3 x$

$$\text{IF} = e^{\int p dx} = e^{\int \tan x dx} = \sec x$$

Solution of the D.E (1) is

$$v(\text{IF}) = \int q(\text{IF}) dx + c$$

$$v \cdot \sec x = \int \cos^3 x \times \sec x dx + c$$

$$\sec y \sec x = \int \cos^2 x dx + c$$

$$\sec x \sec y = \int \frac{1 + \cos 2x}{2} dx + c$$

$$\sec x \sec y = \frac{x}{2} + \frac{\sin 2x}{4} + c$$

1.13.6 Example

Solve $\frac{dy}{dx} = (\sin x - \sin y) \frac{\cos x}{\cos y}$

Solution

$$\cos y \frac{dy}{dx} + \sin y \cos x = \sin x \cos x$$

Substituting $\sin y = v$

$$\cos y \frac{dy}{dx} = \frac{dv}{dx}$$

The given equation reduces to

$$\frac{dv}{dx} + v \cos x = \sin x \cos x \quad \dots (1)$$

is linear in v

Here $p = \cos x$, $q = \sin x \cos x$

$$IF = e^{\int p dx} = e^{\int \cos x dx} = e^{\sin x}$$

Solution of the D.E (1) is

$$v \cdot e^{\sin x} = \int \sin x \cos x \cdot e^{\sin x} dx + c$$

write $\sin x = t \Rightarrow \cos x dx = dt$ in the RHS

$$v \cdot e^{\sin x} = \int t \cdot e^t dt + c$$

$$\sin y \cdot e^{\sin x} = [te^t - e^t] + c$$

$$\sin y e^{\sin x} = e^{\sin x} [\sin x - 1] + c$$

Exercise - 4(f)

1.14 Solve the Following Differential Equations

1. $(y \log x - 1)y dx = x dy$

[Ans : $\frac{1}{y} = 1 + \log x + cx$]

2. $\frac{dy}{dx} \cos x + y \sin x = \sqrt{y \sec x}$

[Ans : $2y \cdot y^{\frac{1}{2}} \sqrt{\sec x} = \tan x + 2c$]

3. $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

[Ans : $\frac{1}{y^2} = -\sin^2 x - \sin x - \frac{1}{2} + ce^{2 \sin x}$]

4. $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x) e^x \sec y$

[Ans : $\sin y = (1+x)(e^x + c)$]

$$5. \quad x \frac{dy}{dx} + y \log y = xye^x$$

$$[\text{Ans : } x \log y = (x - 1)e^x + c]$$

$$6. \quad 3 \frac{dy}{dx} + \frac{2}{x+1} y = \frac{x^3}{y^2}$$

$$[\text{Ans : } (x + 1)^2 y^3 = \frac{x^6}{6} + \frac{2x^5}{5} + \frac{x^4}{4} + c]$$

$$7. \quad \frac{dy}{dx} + y \tan x = y^3 \sec x$$

$$[\text{Ans : } \cos^2 x = y^2(c + 2 \sin x)]$$

$$8. \quad \frac{dy}{dx} = \frac{1}{x^2 y^3 + xy}$$

$$[\text{Ans : } -\frac{1}{x} e^{\frac{y^2}{2}} = 2 \left(\frac{y^2}{2} - 1 \right) e^{\frac{y^2}{2}} + c]$$

1.14.1 Exact Differential Equations

Let us consider the differential equation $Mdx + Ndy = 0$ where M, N are functions of x, y .

If this equation is to be exact, then it must have been derived by directly differentiating some function of x, y .

$$\text{Hence} \quad Mdx + Ndy = du, \text{ say} \quad \dots (1)$$

But from differential calculus

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \quad \dots (2)$$

From (1) and (2) we get

$$M = \frac{\partial u}{\partial x}, \quad N = \frac{\partial u}{\partial y}$$

$$\text{Now} \quad \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ is the condition for exactness.}$$

$\therefore$ The differential equation $Mdx + Ndy = 0$

$$\text{is exact if } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Then the solution is expressed in the form

$$\int Mdx + \int Ndy = c$$

(treating y as
constant integrate w.r.t x)

(integrate w.r.t y those
terms that are independent of x)

Note :

If N has no term independent of x then the solution is

$$\int Mdx = c$$

1.14.2 Example

$$\text{Solve } (x + 2y - 3) dy - (2x - y + 1) dx = 0$$

Solution

$$(x + 2y - 3)dy - (2x - y + 1) dx = 0$$

$$M = -(2x - y + 1)$$

$$N = (x + 2y - 3)$$

$$\frac{\partial M}{\partial y} = 1$$

$$\frac{\partial N}{\partial x} = 1$$

$\therefore$ The given differential equation is exact

The solution is

$$\therefore - \int (2x - y + 1) dx + \int (x + 2y - 3) dy = c$$

$$-\frac{2x^2}{2} + yx - x + \frac{2y^2}{2} - 3y = c$$

$$\Rightarrow y^2 - x^2 + xy - x - 3y = c$$

1.14.3 Example

Solve $(x^2 + y^2)dx + 2xy dy = 0$

Solution

$$(x^2 + y^2) dx + 2xy dy = 0$$

$$M = x^2 + y^2 \quad N = 2xy$$

$$\frac{\partial M}{\partial y} = 2y \quad \frac{\partial M}{\partial x} = 2y$$

$\therefore$ The given differential equation is exact

$\therefore$ Solution is $\int (x^2 + y^2) dx + \int 2xy dy = c$

$$\frac{x^3}{3} + y^2x = c$$

1.14.4 Example

Solve $(1 + e^{\frac{x}{y}})dx + e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right) dy = 0$

Solution

$$(1 + e^{\frac{x}{y}}) dx + e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right) dy = 0$$

$$M = 1 + e^{\frac{x}{y}}, N = e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right)$$

and $\frac{\partial N}{\partial x} = e^{\frac{x}{y}} \left(-\frac{1}{y}\right) + \left(1 - \frac{x}{y}\right) e^{\frac{x}{y}} \times \frac{1}{y}$

$$\frac{\partial M}{\partial y} = \frac{-x}{y^2} e^{\frac{x}{y}}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The given differential equation is exact

∴ . Solution is

$$\int \left(1 + e^{-x/y}\right) dx + \int e^{-x/y} \left(1 - \frac{x}{y}\right) dy = c$$

$$x + e^{-x/y} (y) = c$$

Exercise - 4(g)

1.15 Solve the Following Differential Equations

1. $(e^y + 1)\cos x dx + e^y \sin x dy = 0$

[Ans : $(e^y + 1)\sin x = c$]

2. $(y\cos x + \sin y + y)dx + (\sin x + x\cos y + x)dy = 0$

[Ans : $y\sin x + (\sin y + y)x = c$]

3. $(x^2 - ay)dx = (ax - y^2)dy$

Ans : $x^3 + y^3 - 3axy = c$]

4. $(ax + hy + g)dx + (hx + by + f)dy = 0$

[Ans : $\frac{ax^2}{2} + (hy + g)x + \frac{by^2}{2} + fy = c$]

5. $(x^2 + y^2 - a^2)xdx + (x^2 - y^2 - b^2)ydy = 0$

[Ans : $x^4 + 2x^2y^2 - 2a^2x^2 - y^4 - 2b^2y^2 = 4c$]

1.15.1 Integrating factors

If the differential equation $Mdx + Ndy = 0$ is not exact, it can be made exact by multiplying it with some function of x, y . Such a function is called an *integrating factor*.

Rules for finding the integrating factors :

1. Integrating factors found by inspection :

Example

Solve $x dy - y dx = 0$

Solution

$$x dy - y dx = 0$$

Dividing by x^2

$$\frac{xdy - ydx}{x^2} = 0$$

$$d\left(\frac{y}{x}\right) = 0$$

On integration

$$\frac{y}{x} = c$$

First method to find an integrating factor :

If the differential equation $Mdx + Ndy$ is not exact, but is homogenous and $Mx + Ny \neq 0$, then the integrating factor is $\frac{1}{Mx + Ny}$. Multiply the differential equation by IF. The DE becomes exact.

1.15.2 Example

Solve $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

Solution

$$(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0 \quad \dots (1)$$

The differential equation (1) is homogeneous

$$M = x^2y - 2xy^2 \quad N = -(x^3 - 3x^2y)$$

$$\frac{\partial M}{\partial y} = x^2 - 4xy \quad \frac{\partial N}{\partial x} = -3x^2 + 6xy$$

$\therefore$ The DE is not exact and

$$Mx + Ny = x^2y^2 \neq 0$$

$$\therefore \text{IF} = \frac{1}{Mx + Ny} = \frac{1}{x^2y^2}$$

Multiplying the DE by the integrating factor $\frac{1}{x^2y^2}$

$$\left(\frac{x^2y - 2xy^2}{x^2y^2}\right)dx - \left(\frac{x^3 - 3x^2y}{x^2y^2}\right)dy = 0 \quad \dots (2)$$

write $M_1 = \frac{1}{y} - \frac{2}{x} \quad N_1 = \frac{-x}{y^2} + \frac{3}{y}$

then $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} = -\frac{1}{y^2}$

$\therefore$ DE (2) is exact

$\therefore$ Solution is

$$\int \left(\frac{1}{y} - \frac{2}{x} \right) dx - \int \left(\frac{x}{y^2} - \frac{3}{y} \right) dy = c$$

$$\frac{x}{y} - 2\log x + 3\log y + c$$

i.e., $\frac{x}{y} = \log x^2 y^3 c$

or $x^2 y^3 c = e^{\frac{x}{y}}$

1.15.3 Second Method to Find the Integrating Factor

If the differential equation $Mdx + Ndy = 0$ is not exact and is of the form

$$f(xy)ydx + g(xy)x dy = 0$$

and $Mx - Ny \neq 0$

then $\frac{1}{Mx - Ny}$ is an integrating factor

1.15.4 Example

Solve $(x^2 y^2 + xy + 1)ydx + (x^2 y^2 - xy + 1)x dy = 0$

Solution

$$(x^2 y^2 + xy + 1)ydx + (x^2 y^2 - xy + 1)x dy = 0 \quad \dots (1)$$

$$M = x^2 y^3 + xy^2 + y, \quad N = x^2 y^2 - x^2 y + x$$

$$Mx - Ny = 2x^2 y^2 \neq 0$$

Hence the IF = $\frac{1}{Mx - Ny} = \frac{1}{2x^2 y^2}$

Multiplying (1) by IF

$$\frac{(x^2 y^2 + xy + 1)}{2x^2 y^2} + \frac{(x^2 y^2 - xy + 1)}{2x^2 y^2} = 0$$

$$\frac{1}{2} \left\{ y + \frac{1}{x} + \frac{2}{x^2 y} \right\} dx + \frac{1}{2} \left\{ x - \frac{1}{y} + \frac{1}{xy^2} \right\} dy = 0 \quad \dots (2)$$

$$\frac{\partial M_1}{\partial y} = \frac{1}{2} - \frac{1}{x^2 y^2}$$

$$\frac{\partial N_1}{\partial x} = \frac{1}{2} - \frac{1}{x^2 y^2}$$

$\therefore$ DE (2) is exact

Solution of the DE (1) is

$$\frac{1}{2} \int \left(y + \frac{1}{x} + \frac{1}{x^2 y} \right) dx + \frac{1}{2} \int \left(x - \frac{1}{y} + \frac{1}{xy^2} \right) dy = c$$

$$\frac{1}{2} \left[xy + \log x - \frac{1}{xy} \right] - \frac{1}{2} \log y = c$$

1.15.5 Example

Solve $(xysin xy + \cos xy)ydx + (xysin xy - \cos xy)x dy = 0$

Solution

$$(xysin xy + \cos xy)dx + (xysin xy - \cos xy)x dy = 0 \quad \dots (1)$$

$$M = (xysin xy + \cos xy)y, N = (xysin xy - \cos xy)x$$

$$M_x - N_y = 2xy \cos xy \neq 0$$

$$\therefore IF = \frac{1}{M_x - N_y} = \frac{1}{2xy \cos xy}$$

Multiplying the DE by $\frac{1}{2xy \cos xy}$

The DE reduces to

$$\frac{1}{2} \left(y \tan xy + \frac{1}{x} \right) dx + \frac{1}{2} \left(x \tan xy - \frac{1}{y} \right) dy = c \quad \dots (2)$$

which is exact (verify)

∴ Solution is

$$\frac{1}{2} \int \left(y \tan xy + \frac{1}{x} \right) dx + \frac{1}{2} \int \left(x \tan xy - \frac{1}{y} \right) dy = c$$

$$\frac{1}{2} \log \sec xy + \frac{1}{2} \log x - \frac{1}{2} \log y = c$$

i.e., $\log \sec xy = 2c + \log y$. Taking $2c$ as $\log A$

$$\Rightarrow x \sec xy = Ay$$

Exercise - 4(h)

1.16 Solve the Following Differential Equations

1. $(x^3y^3 + x^2y^2 + xy + 1)ydx + (x^3y^3 - x^2y^2 - xy + 1)x dy = 0$

$$[\text{Ans : } xy - \frac{1}{xy} - 2\log y = c]$$

2. $(x^2y^2 + xy + 1)ydx + (x^2y^2 - xy + 1)x dy = 0$

$$[\text{Ans : } xy + \log x - \log y - \frac{1}{xy} = c]$$

3. $(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)ydx = 0$

$$[\text{Ans : } \log x^2 - \log y - \frac{1}{xy} = c]$$

4. $(x^4y^4 + x^2y^2 + xy)y dx + (x^4y^4 - x^2y^2 + xy)x dy = 0$

$$[\text{Ans : } \frac{1}{2} x^2 y^2 - \frac{1}{xy} - \log x - \log y = c]$$

1.16.1 Third Method to Find the Integrating Factor

If $Mdx + Ndy = 0$ is not exact and $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of x alone say $f(x)$,

then the IF = $e^{\int f(x) dx}$

1.16.2 Example

Solve $(x^2 + y^3 + 6x)dx + y^2xdy = 0$

Solution

$$(x^2 + y^3 + 6x)dx + y^2xdy = 0 \quad \dots(1)$$

$$M = x^2 + y^3 + 6x \quad N = y^2x$$

$$\frac{\partial M}{\partial y} = 3y^2, \quad \frac{\partial N}{\partial x} = y^2$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3y^2 - y^2}{y^2x} = \frac{2}{x}$$

is a function of x alone

$$\therefore \text{IF} = e^{\int \frac{2}{x} dx} = x^2$$

Multiplying the given DE by x^2

$$x^2(x^2 + y^3 + 6x)dx + y^2x^3dy = 0 \quad \dots(2)$$

$$\frac{\partial M_1}{\partial y} = 3x^2y^2$$

$$\frac{\partial N_1}{\partial x} = 3x^2y^2$$

$\therefore$ DE (2) is exact

$\therefore$ Solution is

$$\int (x^4 + x^2y^3 + 6x^3)dx + \int y^2x^2dy = c$$

$$\frac{x^5}{5} + \frac{x^3}{3}y^3 + \frac{6x^4}{4} = c$$

1.16.3 Example

Solve $(x^2 + y^2)dx - 2xydy = 0$

Solution

$$(x^2 + y^2)dx - 2xydy = 0 \quad \dots(1)$$

$$M = x^2 + y^2, N = -2xy$$

$$\frac{\partial M}{\partial y} = 2y, \frac{\partial N}{\partial x} = -2y$$

$$\frac{\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}}{N} = \frac{2y + 2y}{-2xy} = \frac{-2}{x}$$

is a function of x alone

$$\therefore \quad \text{IF} = e^{\int \frac{-2}{x} dx} = \frac{1}{x^2}$$

Multiplying the DE by

$$\text{IF} = \frac{1}{x^2}$$

The given DE reduces to

$$\left(1 + \frac{y^2}{x^2}\right) dx - \frac{2y}{x} dy = 0 \quad \dots\dots(2)$$

$$\frac{\partial M_1}{\partial y} = \frac{2y}{x^2}, \quad \frac{\partial N_1}{\partial x} = \frac{2y}{x^2}$$

(2) is exact

Solution of (1) is

$$\int \left(1 + \frac{y^2}{x^2}\right) dx + \int -\frac{2y}{x} dy = c$$

$$x - \frac{y^2}{x} = c$$

1.16.4 Fourth Method to Find the Integrating Factor

If the differential equation $Mdx + Ndy = 0$ is not exact and $\left(\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} \right)$ is a

function of y alone, say $f(y)$

Then $IF = e^{\int f(y)dy}$

1.16.5 Example

Solve $(xy^3 + y) dx + 2(x^2y^2 + x + y^4)dy = 0$

Solution

$$(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$$

$$M = xy^3 + y \quad N = 2(x^2y^2 + x + y^4)$$

$$\frac{\partial M}{\partial y} = 3xy^2 + 1 \quad \frac{\partial N}{\partial x} = 4xy^2 + 2$$

$$\left(\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} \right) = \frac{(4xy^2 + 2) - (3xy^2 + 1)}{xy^3 + y}$$

$$= \frac{(xy^2 + 1)}{y(xy^2 + 1)} = \frac{1}{y}$$

is a function of y alone

$$F = e^{\int \frac{1}{y} dy} = y$$

Multiplying the differential equation by I.F = y

$$(xy^4 + y^2) dx + 2(x^2y^3 + xy + y^5)dy = 0 \quad \dots(1)$$

$$\frac{\partial M_1}{\partial y} = 4xy^3 + 2y \quad \frac{\partial N_1}{\partial x} = 4xy^3 + 2y$$

$\therefore$ The differential equation (1) is exact

Solution is $\int (xy^4 + y^2)dx + 2 \int (x^2y^3 + xy + y^5) dy = 0$

$$\frac{x^2y^4}{2} + y^2x + \frac{2 \cdot y^6}{6} = c$$

$$3x^2y^4 + 6y^2x + 2y^6 = c$$

1.16.5 Example

$$\text{Solve } (3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$$

Solution

$$(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$$

$$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x \qquad \frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{(6x^2y^3 - 2x) - (12x^2y^3 + 2x)}{3x^2y^4 + 2x} = \frac{6x^2y^3 - 4x}{y(3x^2y^3 + 2x)} = -\frac{2}{y}$$

is a function of y alone

$$\therefore \text{IF} = e^{-\int \frac{2}{y} dy} = \frac{1}{y^2}$$

Multiplying the DE by $\frac{1}{y^2}$, the DE reduces to

$$\left(3x^2y^2 + \frac{2x}{y} \right) dx + \left(2x^3y - \frac{x^2}{y^2} \right) dy = 0$$

$$\frac{\partial M_1}{\partial y} = 6x^2y - \frac{2x}{y^2} \qquad \frac{\partial N_1}{\partial x} = 6xy - \frac{2x}{y^2}$$

$\therefore$ DE (1) is exact

Solution of the DE is

$$\left(3x^2y^2 + \frac{2x}{y} \right) dx + \left(2x^3y - \frac{x^2}{y^2} \right) dy = 0$$

$$3 \frac{x^3}{3} y^2 + \frac{2x^2}{y} = c$$

$$xy^3 + x^2 = cy$$

Exercise - 4(i)

1.17 Solve the Following Differential Equations

1. $(x + y^3 + 1)dx + y^2dy = 0$

Ans : [**Ans :** $\frac{x^2 e^{3x}}{3} - \frac{2x}{9} e^{3x} + \frac{2}{9} \frac{e^{3x}}{3} + (y^3 + 1) \frac{e^{3x}}{3} = c$]

2. $(x^2 + y^2 + 2x)dx + 2ydy = 0$

[**Ans :** $(x^2 + y^2)e^x = c$]

3. $2xydy - (x^2 + y^2 + 1)dx = 0$

[**Ans :** $y^2 - x^2 + 1 = cx$]

4. $\left(y + \frac{y^3}{3} + \frac{x^2}{2} \right) dx + \frac{1}{4} (x + xy^2) dy = 0$

[**Ans :** $3x^4y + x^4y^3 + x^6 = c$]

5. $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$

[**Ans :** $xy + \frac{2x}{y^2} + y^2 = c$]

6. $(xy^2 - x^2)dx + (3x^2y^2 + x^2y - 2x^3)dy = 0$

[**Ans :** $\frac{x^2}{2} y^2 e^{6y} - e^{6y} \cdot \frac{x^3}{3} = c$]

1.17.1 Fifth Method for Finding Integrating Factor

If the differential equation

$$Mdx + Ndy = 0 \quad \dots (1)$$

is not exact and is of the form

$$x^a y^b (mydx + nxdy) + x^r y^s (pydx + qxdy) = 0$$

when a, b, r, s, m, n, p, q are constants

Then the IF = $x^h y^k$

where h, k are constants such that the equation (1) after multiplying with IF becomes exact.

1.17.2 Example

$$\text{Solve } xy^3(ydx + 2xdy) + 3ydx + 5xdy = 0$$

Solution

$$xy^3(ydx + 2xdy) + (3ydx + 5xdy) = 0 \quad \dots(1)$$

The IF = $x^h y^k$

Multiplying (1) by IF = $x^h y^k$ it must become exact

$$(x^{h+1} y^{k+1} + 3x^h y^{k+1})dx + (2x^{h+2} + y^{k+3} + 5x^{h+1} y^k)dy = 0 \quad \dots(2)$$

is exact if

$$\frac{\partial M}{\partial y} = (4 + k)x^{h+1}y^{k+3} + 3x^h(k + 1)y^k$$

$$\frac{\partial N}{\partial x} = 2(h + 2)x^{h+1}y^{k+3} + 5(h + 1)x^h y^k$$

For DE (2) to be exact

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$$

$$\therefore (4 + k)x^{h+1}y^{k+1} + 3x^h(k + 1)y^k = 2(h + 2)x^{h+1}y^{k+3} + 5(h+1)x^h y^k$$

Comparing the coefficients on both sides

$$2h - k = 0 \text{ and } 5h - 3k = -2$$

Solving $h = 2, k = 4$

Substituting $h = 2, k = 4$ in (1) DE (1) reduces to

$$(x^3y^8 + 3x^2y^5) dx + (2x^4y^7 + 5x^3y^4)dy = 0 \text{ which is exact}$$

$\therefore$ Solution is

$$\frac{x^4}{4}y^8 + \frac{3x^3}{y}y^5 = c \Rightarrow x^4y^8 + 4y^5x^3 = c$$

1.17.3 Example

$$\text{Solve } (3x + 2y^2)ydx + 2x(2x + 3y^2)dy = 0$$

Solution

$$(3x + 2y^2)ydx + 2x(2x + 3y^2)dy = 0 \quad \dots (1)$$

$$\text{IF} = x^h y^k$$

Multiplying (1) by IF

$$(x^h y^{k+2} + 3x^{h+2} y^{k+1})dx + (2x^{h+3} y^k - x^{h+1} y^{k+1})dy = 0 \quad \dots (2)$$

$$\frac{\partial M}{\partial x} = 3(k+2)x^h y^{k+1} + 2(k+1)x^{h+2} y^k$$

$$\frac{\partial N}{\partial x} = 2(h+3)x^{h+2} y^k - (h+1)x^h y^{k+1}$$

The DE(1) is to be exact

$$\therefore \frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$$

$$\therefore 2(k+1) = 2(h+3) \Rightarrow h = -\frac{5}{2} \quad k = -\frac{1}{2}$$

Comparing the coefficients

$$k+2 = -(h+1)$$

$$\text{and} \quad 2(k+1) = 2(h+3) \Rightarrow h = -\frac{5}{2} \quad k = -\frac{1}{2}$$

Substituting h, k values in (2)

$$\left(x^{-5/2} y^{3/2} + 2x^{-1/2} y^{1/2} \right) dx + \left(2x^{1/2} y^{-1/2} - x^{-3/2} y^{1/2} \right) dy = 0 \quad \dots (3)$$

$\therefore$ (3) is an exact DE

$$\text{Solution is } -\frac{2}{3} x^{-2/3} y^{3/2} + 4x^{1/2} y^{1/2} = c$$

Exercise - 4(j)

I Solve the following differential equations

1. $x(3ydx + 2xdy) + 8y^4(ydx + 3xdy) = 0$

[Ans : $x^2y^3(x + 4y^4) = c$]

2. $(3x^2y - y^4)dx + (6x^3 + xy^3)dy = 0$

[Ans : $\frac{3x^{\frac{4}{3}+1}y^{\frac{2}{3}}}{-\frac{4}{3}+1} - \frac{x^{\frac{10}{3}+1}}{-\frac{10}{3}+1}y^{\frac{7}{3}} = c$]

3. $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$

[Ans : $4x^{\frac{1}{2}}y^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}}y^{\frac{3}{2}} = cx$]

4. $(2x^2y - 3y^4)dx + (3x^3 + 2xy^3)dy = 0$

[Ans : $5x^{\frac{-36}{13}}y^{\frac{24}{13}} - 12x^{\frac{-10}{13}}y^{\frac{15}{13}} = c$]

1.17.4 Applications to geometry, law of natural growth and Newton's law of cooling :

When some action is applied on a quantity the changes and the action affects all the parts equally. The rate of change depends on the original quantity.

For example the total population 'p' of a country increases with respect to time

't', say, then its rate of change with time is $\frac{dp}{dt}$. Under ideal conditions the rate of change of the population will be proportional to the total population at any given time and is called the law of natural growth.

∴ The growth of the population satisfies the differential equation

$$\frac{dp}{dt} = kp$$

where $k > 0$ is a constant

Solving the differential equation we have

$$p = ce^{kt}$$

The rate at which a quantity x is decreasing is given by $\frac{dx}{dt}$ and this is negative.

This rate of decrease or decay is found to be proportional to x itself

$$\therefore \frac{dx}{dt} = -kx \quad (\text{where } k > 0 \text{ is a constant})$$

is the law of decomposition

Solving the DE $x = ce^{-kt}$

Newton's law of cooling :

Newton's law of cooling states that the rate of decrease of the temperature of a body is proportional to the difference of the temperature of the body and that of its surrounding medium.

Let θ be the temperature of the body at time 't' and θ_0 the temperature of its surrounding medium.

$\therefore$ Difference between the temperatures $= \theta - \theta_0$ ($\theta > \theta_0$) and the rate of decrease of the temperature of the body is $-\frac{d\theta}{dt}$, since the body is cooling $\propto$.

$$\text{Using Newton's law of cooling} \quad -\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

$$\text{or} \quad \frac{-d\theta}{dt} = k(\theta - \theta_0) \text{ where } k > 0 \text{ is a constant of proportionality}$$

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

$$\int \frac{d\theta}{\theta - \theta_0} = -k \int dt + c$$

$$\log(\theta - \theta_0) = -kt + c \quad \text{or} \quad \theta = \theta_0 + Ce^{-kt}$$

1.17.4 Example

The mass of crystalline deposit increases at a rate which is proportional to its mass at that time. The deposit has started around a crystal seed of 5 grams. Find an expression of its mass at time 't'. If in 30 minutes the mass of the deposit increases by '1' gram, what will be the mass of the deposit after 10 hours.

Solution

Let m be the mass of crystalline at time 't' then by law of growth

$$\frac{dm}{dt} = k; \quad \frac{dm}{km} = dt$$

on integration

$$\log m = kt + c \quad \dots (1)$$

initially when $t = 0$, $m = 5$

$$(1) \Rightarrow \log 5 = 0 + c$$

$$\therefore c = -\log 5$$

Substituting c value in (1)

$$\log m = kt - \log 5$$

$$\text{i.e.,} \quad kt = \log \left(\frac{m}{5} \right) \quad \dots (2)$$

When $t = 30$ minutes mass deposit increases by '1' gram

$$\therefore m = 5 + 1 = 6 \text{ grams}$$

Substituting

$$t = \frac{1}{2} \text{ (hours), } m = 6$$

$$k \left(\frac{1}{2} \right) = \log \left(\frac{6}{5} \right)$$

$$k = \log \left(\frac{6}{5} \right)^2$$

Substituting in (2)

$$\log \left(\frac{6}{5} \right)^2 t = \log \frac{m}{5} \quad \dots (3)$$

Now to find the mass after 10 hours (i.e $t = 10$)

from (3) we get

$$\log \left(\frac{6}{5} \right)^2 \times 10 = \log \frac{m}{5} \Rightarrow \log \left(\frac{m}{5} \right) = \log \left(\frac{6}{5} \right)^{20}$$

$$m = 5 \left(\frac{6}{5} \right)^{20} \text{ grams}$$

1.17.5 Example

The rate at which a certain substance decomposes in a certain solution at any instant is proportional to the amount of it present in the solution at that instant. Initially, there are 27 grams and three hours later, it is found that 8 grams are left. How much substance will be left after one more hour.

Solution

If m grams is the amount of the substance left in the solution at time 't', then the rate at which it decomposes is $\frac{dm}{dt}$, which is proportional to m .

∴ By law of decay

$$\frac{dm}{dt} = -km \quad (k > 0)$$

$$\int \frac{dm}{m} = -k \int dt + c$$

$$\log m = -kt + c \quad \dots (1)$$

Initially when $t = 0$, $m = 27$

From (1) we have

$$\log 27 = -k \cdot 0 + c$$

$$\Rightarrow c = \log 27$$

Substitution of 'c' in (1) gives

$$\log m = -kt + \log 27$$

$$\therefore \log \left(\frac{m}{27} \right) = -kt \quad \dots (2)$$

It is given that $m = 8$ when $t = 3$

∴ From (2)

$$\log \left(\frac{8}{27} \right) = -k \cdot 3$$

$$\Rightarrow -k = \log \left(\frac{8}{27} \right)^{\frac{1}{3}}$$

$$-k = \log \frac{2}{3}$$

Then (2) becomes

$$\log \frac{m}{27} = \log \left(\frac{2}{3} \right) t \quad \dots (3)$$

when $t = 4$

$$\log \left(\frac{m}{27} \right) = 4 \log \frac{2}{3}$$

$$\log \frac{m}{27} = \log \left(\frac{2}{3} \right)^4$$

$$m = 27 \times \left(\frac{2}{3} \right)^4 \text{ grams}$$

$$m = \frac{16}{3} \text{ grams.}$$

1.17.6 Example

The number x of bacteria in a culture grow at a rate proportional to x . The value of x was initially 50 and increased to 150 in one hour what will be the value of x after

$1\frac{1}{2}$ hour.

Solution

$$\frac{dx}{dt} = kx$$

$$\frac{dx}{x} = kdt$$

$$\log x = kt + c \quad \dots (1)$$

c is the constant of integration

when $t = 0, x = 50$

$$\therefore \log 50 = k.0 + c$$

$$\text{or } c = \log 50$$

$$(1) \Rightarrow \log x = kt + \log 50$$

$$\therefore \log \frac{x}{50} = kt$$

$$x = 150, \text{ when } t = 1$$

$$\therefore \log \frac{150}{50} = k.1$$

$$\text{or } k = \log 3$$

(2) then gives

$$\log \left(\frac{x}{50} \right) = t \log 3$$

we want to find x when $t = \frac{3}{2}$

$$\log \left(\frac{x}{50} \right) = \frac{3}{2} \log 3$$

$$\frac{x}{50} = (3)^{\frac{3}{2}}$$

$$x = 50(3)^{\frac{3}{2}} \text{ grams}$$

1.17.7 Example

The rate of cooling of a body is proportional to the difference between the temperature of the body and the surrounding air. If the air temperature is 20°C and the body cools for 20 minutes from 140°C to 80°C , find when the temperature will be 35°C .

Solution

If θ is the temperature of the body at time 't' then from Newton's law of cooling

$$-\frac{d\theta}{dt} \propto (\theta - 20) \Rightarrow \frac{d\theta}{dt} = -k(\theta - 20)$$

$$\therefore \int \frac{d\theta}{\theta - 20} = -k \int dt + c$$

$$\log(\theta - 20) = -kt + c \quad \dots (1)$$

Initially when

$$t = 0, \theta = 140$$

$$\log(\theta - 20) = 0 + c,$$

$\therefore c = \log(120)$, (1) reduces to

$$\log(140 - 20) = -kt + \log 120$$

$$\text{or} \quad +kt = \log(120) - \log(\theta - 20) \quad \dots (2)$$

It is given that $q = 80$ when $t = 20$ minutes

$$k \cdot 20 = \log 120 - \log(80 - 20)$$

$$k = \frac{1}{20} \log \left(\frac{120}{60} \right)$$

$$k = \frac{1}{20} \log 2$$

Substituting in (2)

$$\left(\frac{1}{20} \log 2 \right) t = \log 120 - \log(\theta - 20)$$

It is required to find t when

$$\theta = 35^\circ \text{C}$$

$$t = \frac{\log 120 - \log(35 - 20)}{\frac{1}{20} \log 2} = \frac{20 \log \left(\frac{120}{15} \right)}{\log 2}$$

$$t = 20 \frac{\log(8)}{\log 2} = 20 \frac{\log 2^3}{\log 2} = 60 \frac{\log 2}{\log 2} = 60 \text{ minutes}$$

Exercise - 4(k)

1. In a certain reaction, the rate of conversion of a substance at time 't' is proportional to the quantity of the substance still untransformed at that instant. At the end of one hour 60 grams while at the end of four hours 21 grams remain. How many grams of the first substance was there initially?

[Ans : 85 grams approximately]

2. The rate of growth of a bacteria is proportional to the number present. If initially there were 100 bacteria and the amount doubles in '1' hour, how many bacteria will be there after $2\frac{1}{2}$ hours.

[Ans : 564]

3. Under certain conditions cane sugar in water is converted into dextrose at a rate which is proportional to the amount unconverted at any time. If 75 grams was there at time $t = 0.0$ and 8 grams are converted during the first 30 minutes find the amount converted in $1\frac{1}{2}$ hour.

[Ans : 21.5 gms]

4. The rate of cooling of a body is proportional to the difference between the temperature of the body and the surrounding air. If the surrounding air is kept at 30°C and the body cools from 80°C to 60°C in 20 minutes. Find the temperature of the body after 40 minutes.

[Ans : 48°C]

5. If the air is maintained at 30°C and the temperature of the body cools from 80°C to 60°C in 20 minutes. Find the temperature of the body after 40 minutes.

[Ans : 48°C]

6. The rate at which a heated body cools in air is proportional to the difference between the temperature of the body and that of the surrounding air. A body originally at 80°C cools down to 60°C in 20 minutes the temperature of the air being 40°C what will be the temperature of the body after 40 minutes from the original temperature.

[Ans : 50°C]

7. The rate at which bacteria multiply is proportional to the instantaneous number present. If the original number doubles in 2 hours in how many hours will it triple.

[Ans : $\frac{2\log 3}{\log 2}$]

8. Water at temperature 100°C cools in 10 minutes to 80°C in a room of temperature 25°C . Find the temperature of water after 20 minutes.

[Ans : 65.5°C]

9. A cup of coffee at temperature 100°C is placed in a room whose temperature is 15°C and it cools to 60°C in 5 minutes, find its temperature after a further interval of 5 minutes.

[Ans : 38.8°C]

Orthogonal Trajectories

1.18.0 Definition (1)

A trajectory of a family of curves is a curve cutting all the members of the system according to some law. For example a curve cutting a family of curves at a constant angle is a trajectory.

Definition (2)

If a curve cuts every member of a given family of curves at right angles, it is called an orthogonal Trajectory. The orthogonal trajectories of a given family of curves themselves form a family of Curves. If the two families of curves are such that each member of either family cuts each member of the other family at right angles then the members of one family are known as the orthogonal trajectories of the other.

In two dimensional problems in the flow of heat, the curves along which the heat flow takes place and the isothermal curves or loci of points at the same temperature are orthogonal trajectories. In hydrodynamics, the flow of water from a lake into narrow channel produces a family of streamlines which are orthogonal trajectories to the curves of equal Velocity Potential. In the flow of electricity in thin conducting sheets, the paths along which the current flows are the orthogonal trajectories of the equipotential curves and vice – versa.

1.18.1 Orthogonal Trajectories : Cartesian Coordinates

Let $f(x, y, c) = 0$ (1)

be a family of curves, where c is a parameter . We can form a first order differential equation by eliminating ' c ' from (1)

i.e., $F\left(x, y, \frac{dy}{dx}\right) = 0$ (2)

is the differential equation whose general solution is (1)

If the two curves are orthogonal (curves intersecting at right angles) the product of the slopes of the tangents at their point of intersection must be equal to -1 .

Suppose (x, y) is the point of intersection of the curve (1) and its orthogonal trajectory.

At this point slope of the tangent to the curve (1) is $\frac{dy}{dx}$ and $-\frac{dx}{dy}$ is the slope of the tangent to the orthogonal trajectory. Therefore on replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (2), the equation thus obtained is the differential equation of the family of orthogonal trajectories of the family(1).

$$\therefore \phi \left(x, y, -\frac{dx}{dy} \right) = 0 \quad \text{.....(3)}$$

is the differential equation of the system of orthogonal trajectories, and its solution is the family of orthogonal trajectories of (1).

1.18.2 Orthogonal Trajectories – Polar Coordinates

Suppose $f(r, \theta, c) = 0$ (1)

is the family of curves where c is the arbitrary constant.

We can form a differential equation $F \left(r, \theta, \frac{dr}{d\theta} \right) = 0$ (2)

of the family (1), after elimination of the constant 'c'.

Let ϕ be the angle between the radius vector and the tangent at any point (r, θ) . on a member of the family of curves.

Then $\tan \phi = r \frac{d\theta}{dr}$ (3)

Let ϕ_1 be the angle between the radius vector and the tangent at any point (r_1, θ_1) on the trajectory.

Then $\tan \phi_1 = r_1 \frac{d\theta_1}{dr_1}$ (4)

At a point of intersection of the given curve and the orthogonal trajectory

$$r_1 = r, \quad \theta_1 = \theta \quad \text{and} \quad \phi_1 = \phi + \frac{\pi}{2}; \quad \text{Hence} \quad \tan \phi_1 = -\cot \phi$$

From (3) and (4),

$$r_1 \frac{d\theta_1}{dr_1} = -\frac{1}{r} \frac{dr}{d\theta} \quad \text{i.e.,} \quad \frac{1}{r} \frac{dr}{d\theta} = -r_1 \frac{d\theta_1}{dr_1}$$

Hence the differential equation of the orthogonal trajectory is obtained by substituting $-r \frac{d\theta}{dr}$ for $\frac{1}{r} \frac{dr}{d\theta}$, i.e., $-r^2 \frac{d\theta}{dr}$ for $\frac{dr}{d\theta}$

$$\text{Therefore the differential equation } F\left(r, \theta, -r^2 \frac{d\theta}{dr}\right) = 0 \quad \dots(5)$$

gives the differential equation of orthogonal trajectory of the family of curves(1)

Solution of (5) is the orthogonal trajectory of the family of curves.

1.18.3 Example

Find the orthogonal trajectory of the family of curves $ay^2 = x^3$, where a is variable parameter.

Solution :

$$\text{Given family of curves is } ay^2 = x^3 \quad \dots(1)$$

$$\text{Differentiating (1) w.r.t. 'x'; } 2ay \frac{dy}{dx} = 3x^2 \quad \dots(2)$$

Eliminating ' a ' from (1) and (2)

$$\begin{aligned} \therefore 2 \left(\frac{x^3}{y^2} \right) \cdot y \frac{dy}{dx} &= 3x^2 \\ \Rightarrow 2x \frac{dy}{dx} &= 3y \quad \dots(3) \end{aligned}$$

is the differential equation of the family (1). Now replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in(3),

gives the differential equation of the orthogonal trajectory to (1) as $-2x \frac{dx}{dy} = 3y$

Integrating both sides $-2 \frac{x^2}{2} = \frac{3y^2}{2} + c$. Therefore

$2x^2 + 3y^2 = c$ is the equation of orthogonal trajectory of (1)

1.18.4 Example

Find the orthogonal trajectory of the family of parabolas $y^2 = 4ax$ where 'a' is the parameter.

Solution :

$$y^2 = 4ax \quad \text{.....(1)}$$

$$\text{differentiating (1) w.r.t. 'x'; } y \frac{dy}{dx} = 2a \quad \text{.....(2)}$$

$$\text{Eliminating 'a' from (1) and (2). } y^2 = 2x \left(y \frac{dy}{dx} \right)$$

$$\therefore y = 2x \frac{dy}{dx} \quad \text{.....(3)}$$

is the differential equation of the family (1)

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in(3)

$$y = 2x \left(-\frac{dx}{dy} \right) \quad \text{.....(4)}$$

is the differential equation of the orthogonal trajectory to (1). Integrating (4)

$$\int y dy = -2 \int x dx + c; 2x^2 + y^2 = c \text{ is the orthogonal trajectory of (1)}$$

1.18.5 Example

Find the orthogonal trajectories of $\frac{x^2}{a^2} + \frac{y^2}{a^2 + \lambda} = 1$ where λ is the parameter.

Solution :

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 + \lambda} = 1 \quad \text{.....(1)}$$

$$\text{Differentiating (1) w.r.t. 'x'; } \frac{2x}{a^2} + \frac{2y}{a^2 + \lambda} \frac{dy}{dx} = 0$$

$$\frac{y^2}{a^2 + \lambda} = -\frac{xy}{a^2} \frac{dy}{dx} \quad \text{.....(2)}$$

Eliminating λ from (1) and (2) $\frac{x^2}{a^2} - \frac{xy}{a^2 \frac{dy}{dx}} = 1$

$$\therefore (x^2 - a^2) \frac{dy}{dx} = xy \quad \dots(3)$$

is the differential equation of the family (1)

Substituting $-\frac{dx}{dy}$ for $\frac{dy}{dx}$ in (3); $(x^2 - a^2) \left(-\frac{dx}{dy}\right) = xy$ i.e.,

$$y \, dy = -\left(\frac{x^2 - a^2}{x}\right) dx$$

is the differential equation of the orthogonal trajectory to (1)

Integrating we get $\frac{y^2}{2} = a^2 \log x - \frac{x^2}{2+c}$

$x^2 + y^2 = 2a^2 \log x + c$ is the orthogonal trajectory of the family of curves(1)

1.18.6 Example

Find the orthogonal trajectories of the family of coaxial circles $x^2 + y^2 + 2gx + c = 0$ where g is the parameter.

Solution :

Given $x^2 + y^2 + 2gx + c = 0 \quad \dots(1)$

Differentiating (1) w.r.t. 'x'; $2x + 2y \frac{dy}{dx} + 2g + 0 = 0 \quad \dots(2)$

Eliminating 'g' from (1) and (2) $x^2 + y^2 - 2x \left(x + y \frac{dy}{dx}\right) + c = 0$

$$y^2 - x^2 - 2xy \frac{dy}{dx} + c = 0 \quad \dots(3)$$

is the differential equation of the family of (1)

Substituting $-\frac{dx}{dy}$ for $\frac{dy}{dx}$ in (3) we get,

$$y^2 - x^2 + 2xy \frac{dx}{dy} + c = 0 \quad \dots(4)$$

Which is the differential equation of orthogonal trajectory. Simplifying

$$2x \frac{dx}{dy} - \frac{1}{y} x^2 = -\frac{c + y^2}{y} \quad \dots(5)$$

is a linear equation of the Bernoulli's form (non – linear differential equation of first order and first degree)

Substituting $x^2 = t$, $2x \frac{dx}{dy} = \frac{dt}{dy}$ in (5)

$$\frac{dt}{dy} - \frac{t}{y} = \frac{(c + y^2)}{y} \quad \dots(6)$$

is linear in t .

$$\therefore \text{I.F} = e^{\int \frac{-1}{y} dy} = \frac{1}{y}$$

General solution of (6) is

$$t \cdot \frac{1}{y} = - \int \frac{c + y^2}{y} \cdot \frac{1}{y} dy + k$$

$$\frac{t}{y} = -y + \frac{c}{y} + k$$

$$x^2 + y - ky - c = 0 \quad (\text{since } t = x^2)$$

is the equation of the orthogonal trajectory of (1)

1.18.7 Example

Find the equation of the system of orthogonal trajectories of the parabolas

$$r = \frac{2a}{1 + \cos \theta}, \text{ where 'a' is the parameter.}$$

Solution :

$$2a = r(1 + \cos \theta) \quad \dots(1)$$

$$2a = 2r \cos^2 \theta/2 \Rightarrow a = r \cos^2 \theta/2$$

$$\log a = \log r + 2 \log \cos \theta/2$$

Differentiating w.r.to ' θ '

$$0 = \frac{1}{r} \frac{dr}{d\theta} + \frac{2}{\cos \theta/2} (-\sin \theta/2) \cdot (1/2) = 0 \quad \text{i.e.,}$$

$$\frac{1}{r} \frac{dr}{d\theta} - \tan \theta/2 = 0 \quad \text{.....(2)}$$

Substituting $-r^2 \frac{d\theta}{dr}$ for $\frac{dr}{d\theta}$ in (2) we get

$$\frac{1}{r} \left(-r^2 \frac{d\theta}{dr} \right) - \tan \theta/2 = 0$$

$$\frac{dr}{r} = -\cot \theta/2 d\theta \quad \text{.....(3)}$$

is the differential equation of the orthogonal trajectory .

Integrating (3)

$$\log r = -2 \log \sin \theta/2 + \log c$$

$$\log \left(\frac{c}{r} \right) = \log \sin^2 \theta/2$$

$$\frac{c}{r} = \sin^2 \theta/2 = \frac{(1 - \cos \theta)}{2} ;$$

$\therefore 2c = r(1 - \cos \theta)$ is the equation of orthogonal trajectory of the family of (1)

1.18.8 Example

Find the orthogonal trajectory of $r^m = a^m \cos m\theta$ where 'a' is the parameter.

Solution :

$$\text{Given } r^m = a^m \cos m\theta \quad \text{.....(1)}$$

$$m \log r = m \log a + \log \cos m\theta$$

$$\text{Differentiating w.r.to '}\theta\text{' ; } \frac{m}{r} \frac{dr}{d\theta} = \frac{1}{\cos m\theta} (-m \sin m\theta)$$

$$\frac{1}{r} \frac{dr}{d\theta} = -\tan(m\theta) \quad \text{.....(2)}$$

is the differential equation of the family of curves (1)

$$\text{Replacing } \frac{dr}{d\theta} \text{ by } -r^2 \frac{d\theta}{dr} \text{ in (2)}$$

$$\frac{1}{r} \cdot \left(-r^2 \frac{d\theta}{dr} \right) = -\tan(m\theta)$$

$$\frac{dr}{r} = \cot(m\theta) d\theta \quad \dots(3)$$

is the differential equation of the orthogonal trajectory .

Integrating (3)

$$\log r = \frac{1}{m} \log \sin(m\theta) + \log c$$

$\log r^m = \log c^m \sin(m\theta)$; $r^m = c^m \sin(m\theta)$ is the orthogonal trajectory of the family (1)

Exercise 4 (s)

- Find the orthogonal trajectories of the following family of curves, where a is a variable parameter.

(i) $y = ax^2$ [Ans: $x^2 + 2y^2 = c$]

(ii) $xy = a^2$ [Ans: $x^2 - y^2 = c$]

(iii) $x^2 - xy + y^2 = a^2$ [Ans: $(x - y) = c(x + y)^2$]

(iv) $2x^2 + y^2 = kx$ [Ans: $x^2 = -y^2 \log\left(\frac{y}{c}\right)$]

(v) $x^2 - y^2 = a^2$ [Ans: $xy = c$]

(vi) $x^{2/3} + y^{2/3} = a^{2/3}$ [Ans: $y^{4/3} - x^{4/3} = c^{4/3}$]

- Show that the system of confocal and coaxial Parabolas $y^2 = 4a(x + a)$ is self orthogonal.
- Find the orthogonal trajectories of the curves $3xy = x^3 - a^3$, a being the parameter.

[Ans $2x^2 = (2y - 1) + ce^{-2y}$]

- Prove that orthogonal trajectories of a system of circles $x^2 + y^2 - ay = 0$ is $x^2 + y^2 - bx = 0$

5. Find the orthogonal trajectories of the following family of curves.

$$(i) \quad r = a\theta \quad [\text{Ans: } r = ce^{-\frac{\theta^2}{2}}]$$

$$(ii) \quad r = e^{a\theta} \quad [\text{Ans: } (\log)^2 + \theta^2 = c]$$

$$(iii) \quad r^n \sin n\theta = a^n \quad [\text{Ans: } r^n \cos n\theta = c^n]$$

$$(iv) \quad r \cos \theta = a \sin^2 \theta \quad [\text{Ans: } r = c(3 + \cos 2\theta)]$$

$$(v) \quad r = a(1 + \cos \theta) \quad [\text{Ans: } r = c(1 - \cos \theta)]$$

$$(vi) \quad r = a \cos \theta \quad [\text{Ans: } r = c \sin \theta]$$

$$(vii) \quad r^2 = a^2 \cos 2\theta \quad [\text{Ans: } r^2 = c^2 \sin 2\theta]$$

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2

Linear Differential Equations with Constant Coefficients and Laplace Transforms

2.1.1 Let

- (i) the differential operator " $\frac{d}{dx}$ " be denoted by 'D'
- (ii) $p_1, p_2, p_3, \dots, p_n$ be either functions of x or constants
- (iii) R be a function of x then the general form of a linear differential equation (L.D.E) of order n is given by

$$D^n y + p_1 D^{n-1} y + p_2 D^{n-2} y + \dots + p_n y = R \quad \dots (1)$$

or simply $(D^n + p_1 D^{n-1} + \dots + p_n)y = 0$, If $p_1, p_2, \dots, p_n$ are constants then (1) is called a L D E with constant coefficients.

Denoting the differential operator $(D^n + p_1 D^{n-1} + \dots + p_n)$ by $f(D)$,

(1) can be written as

$$f(D)y = R \quad \dots (2)$$

2.1.2 Consider the D.E, $\frac{dy}{dx} + Py = 0$

Separating the variables $\frac{dy}{y} = -P(x) dx$

Integration yields $\log y = \int P(x)dx \Rightarrow y = Ce^{-\int P(x)dx} + \log c$ (3)

Solution is $y = Ce^{-\int Pdx}$

Suppose p is a constant $= -m$, say, then solution is $y = Ce^{mx}$

(i.e.) the solution of $(D - m)y = 0$ is $y = Ce^{mx}$ (4)

2.1.3 In this chapter the attention is mostly confined to L.D.E.S with constant coefficients.

If $R = 0$ then equation (2) becomes

$$f(D)y = 0$$

We shall take, here afterwards, $f(m) = 0$ as the auxiliary equation, (A.E).

2.1.4 Consider a second order L.D.E

$$(D^2 + a_1D + a_2)y = 0 \quad \text{..... (6)}$$

A.E. is $m^2 + a_1m + a_2 = 0$.

Let m_1, m_2 be the roots of this equation.

Now four cases arise.

(i) m_1, m_2 are real and distinct

(ii) m_1, m_2 are real and equal

(iii) m_1, m_2 are complex and distinct

(iv) m_1, m_2 are complex and equal

2.1.5 Case i : (6) can be written as

$$[D^2 - (m_1 + m_2)D + m_1m_2]y = 0 \text{ or } (D - m_2)(D - m_1)y = 0 \quad \text{.... (7)}$$

Call $(D - m_1)y = Y$

(7) now becomes $(D - m_2)Y = 0$

From (4) it follows that $Y = Ce^{m_2x}$

From (7) $(D - m_1)y = Ce^{m_2x}$

I.F. $= e^{-m_1x}$

Solution is $y \cdot e^{-m_1x} = C \int e^{(m_2-m_1)x} dx + C_1$ (8)

$$y \cdot e^{-m_1x} = \frac{C}{m_2 - m_1} e^{(m_2-m_1)x} dx + C_1, \text{ Call } \frac{C}{m_2 - m_1} \text{ as } C_2.$$

(i.e.,) $y = C_2 e^{-m_2 x} + C_1 e^{m_1 x}$ where C_1 and C_2 are arbitrary constants

Similarly if $m_1, m_2, \dots, m_n$ are real and distinct roots of $f(m) = 0$ then

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x} \quad \dots (I)$$

(I) is the solution of $f(D)y = 0$ where $C_1, C_2, \dots, C_n$ are arbitrary constants.

2.1.6 (Case ii) : $m_1 = m_2$,

From (8) it follows that the solution of (6) is given by

$$y \cdot e^{-m_1 x} = C \int e^{0 \cdot x} dx + C_1 = Cx + C_1$$

$$\therefore y = (Cx + C_1) e^{m_1 x}$$

Similarly if m_1 is repeated say 'r' times then solution of $(D - m_1)^r y = 0$ is

$$y = (C_1 + C_2 x + C_3 x^2 + \dots + C_r x^{r-1}) e^{m_1 x} \quad \dots (II)$$

(II) is the solution corresponding to a root m_1 repeated r times

2.1.7 Case (iii) : m_1, m_2 are complex, say $(\alpha \pm i\beta)$

Then solution of (6) is given by

$$y = A e^{(\alpha + i\beta)x} + B e^{(\alpha - i\beta)x} \quad \text{where A and B are arbitrary constants}$$

$$\begin{aligned} \text{(i.e.,)} \quad y &= A e^{\alpha x} \cdot e^{i\beta x} + B e^{\alpha x} \cdot e^{-i\beta x} = e^{\alpha x} [A(\cos \beta x + i \sin \beta x) + B(\cos \beta x - i \sin \beta x)] \\ &= e^{\alpha x} [C_1 \cos \beta x + C_2 \sin \beta x] \quad \dots (III) \end{aligned}$$

where $A + B = C_1$ and $i(A - B) = C_2$

(III) gives the solution corresponding to two complex conjugate roots $(\alpha \pm i\beta)$

2.1.8 Case (iv) : If $(\alpha \pm i\beta)$ are repeated say 's' times then the corresponding solution (follows from case (ii) and (iii)) is given by

$$\begin{aligned} y &= e^{\alpha x} [(C_1 + C_2 x + C_3 x^2 + \dots + C_s x^{s-1}) \cos \beta x \\ &\quad + (d_1 + d_2 x + d_3 x^2 + \dots + d_s x^{s-1}) \sin \beta x] \quad \dots (IV) \end{aligned}$$

Example 2.1.9

$$\text{Solve } \frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} + 6y = 0$$

Sol. A.E is $m^2 + 5m + 6 = 0 \Rightarrow (m + 2)(m + 3) = 0$

$$\therefore m = -2, m = -3.$$

$$\text{Solution is } y = A e^{-2x} + B e^{-3x}$$

Example 2.1.10

$$\text{Solve } \frac{d^2y}{dx^2} - 11\frac{dy}{dx} + 30y = 0$$

$$\text{Sol. A.E is } m^2 - 11m + 30 \Rightarrow (m - 5)(m - 6) = 0$$

$$\text{Solution is } y = Ae^{5x} + B e^{6x}$$

Example 2.1.11

$$\text{Solve } 2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 12y = 0$$

$$\text{Sol. A.E is } 2m^2 + 5m - 12 \Rightarrow (2m - 3)(m + 4) = 0$$

$$m = 3/2, m = -4$$

$$\therefore \text{Solution is } y = C_1 e^{\frac{3}{2}x} + c_2 e^{-4x}$$

Example 2.1.12

$$\text{Solve } \frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} - 6y = 0$$

$$\text{Sol. } m^3 + 4m^2 + m - 6 = 0$$

$$(m - 1)(m + 2)(m + 3) = 0$$

$$\therefore \text{Solution is } y = c_1 e^x + c_2 e^{-2x} + C_3 e^{-3x}$$

Exercise - 2(a)

Solve the following differences :

$$1. \quad \frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$$

$$\text{Ans : } y = Ae^x + Be^{-3x}$$

$$2. \quad \frac{d^2y}{dx^2} - 3y = 0$$

$$\text{Ans : } y = Ae^x + Be^{-x}$$

$$3. \quad \frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 6y = 0$$

$$\text{Ans : } y = Ae^{-x} + Be^{2x} + Ce^{-3x}$$

$$4. \quad 9\frac{d^2y}{dx^2} + 18\frac{dy}{dx} - 16y = 0$$

$$\text{Ans : } y = Ae^{\frac{2}{3}x} + Be^{-\frac{8}{3}x}$$

$$5. \quad \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$

$$\text{Ans : } y = Ae^{-x} + Be^{-2x}$$

Solved Examples

Example 2.1.13

$$\text{Solve } \frac{d^2 y}{dx^2} + 6 \frac{dy}{dx} + 9y = 0$$

Sol. A.E is $m^2 + 6m + 9 = 0$

i.e. $(m + 3)^2 = 0$, $m = -3, -3$

$\therefore$ Solution is $y = e^{-3x} (A + Bx)$

Example 2.1.14

$$\text{Solve } \frac{d^3 y}{dx^3} + \frac{d^2 y}{dx^2} = 0$$

Sol. A.E is $m^3 + m^2 = 0 \Rightarrow m^2 (m + 1) = 0$

$\therefore m = 0, 0, -1$

Solution is $y = (A + Bx)e^{0x} + C e^{-x} = (A + Bx) + C e^{-x}$

Example 2.1.15

$$\text{Solve } \frac{d^4 y}{dx^4} + 18 \frac{d^2 y}{dx^2} + 81y = 0$$

Sol. A.E. is $m^4 - 18m^2 + 81 = 0$

$\therefore (m^2 - 9)(m^2 - 9) = 0$

$\therefore m = 3, 3, -3, -3$

Solution is $y = (A + Bx) e^{-3x} + (C + xD)e^{3x} = (C_1 x + C_2)e^{-3x} + (C_3 x + C_4)e^{3x}$

Exercise 2 (b)

1. Solve $\frac{d^2 y}{dx^2} + 10 \frac{dy}{dx} + 25y = 0$ Ans : $y = e^{-5x} (Ax + B)$

2. Solve $\frac{d^3 y}{dx^3} - \frac{d^2 y}{dx^2} + 8 \frac{dy}{dx} - 4y = 0$ Ans : $y = (Ax + B)e^{2x} + C e^x$

3. Solve $\frac{d^4 y}{dx^4} - 2 \frac{d^3 y}{dx^3} + \frac{d^2 y}{dx^2} = 0$ Ans : $y = (Ax + B) + (Cx + D)e^x$

4. Solve $\frac{d^3 y}{dx^3} - 3 \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} - y = 0$ Ans : $y = (Ax^2 + Bx + C) e^x$

Solved Examples

Example 2.1.16

$$\text{Solve } \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 9y = 0$$

Sol. A.E. is $m^2 + 4m + 9 = 0$

$$m = -2 \pm i\sqrt{5}$$

$$\alpha = -2 \text{ and } \beta = \sqrt{5}$$

$$\therefore \text{Solution is } y = e^{-2x} (A \cos x \sqrt{5} + B \sin \sqrt{5} x)$$

Example 2.1.17

$$\text{Solve } \frac{d^3 y}{dx^3} + y = 0$$

Sol. A.E. is $m^3 + 1 = 0$

$$\therefore (m + 1)(m^2 - m + 1) = 0$$

$$\therefore m = -1, m = \frac{1 + \sqrt{3}i}{2}$$

$$\text{Solution is } y = C_1 e^{-x} + e \left(C_2 \cos \frac{\sqrt{3}}{2} + C_3 \sin \frac{\sqrt{3}}{2} x \right)$$

Example 2.1.18

$$\text{Solve } \frac{d^4 y}{dx^4} + 2 \frac{d^3 y}{dx^3} + 3 \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + y = 0$$

Sol: A.E. is $m^4 + 2m^3 + 3m^2 + 2m + 1 = 0$

$$(m^2 + m + 1)^2 = 0$$

$$\therefore m = \frac{-1 \pm \sqrt{3}i}{2}, \frac{-1 \pm \sqrt{3}i}{2}$$

$$\text{Solution is } y = e^{-\frac{1}{2}x} \left[(C_1 + C_2 x) \cos \frac{\sqrt{3}}{2} x + (C_3 + C_4 x) \sin \frac{\sqrt{3}}{2} x \right]$$

Exercise 2 (c)

1. Solve $\frac{d^2 y}{dx^2} + 7y = 0$

Ans : $y = A \cos \sqrt{7}x + B \sin \sqrt{7}x$

2. Solve $2 \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 3y = 0$

Ans : $y = e^{-x} \left(A \cos \frac{x}{\sqrt{2}} + B \sin \frac{x}{\sqrt{2}} \right)$

3. Solve $\frac{d^4 y}{dx^4} + y = 0$

Ans : $y = e^{x/\sqrt{2}} \left(C_1 \cos \frac{x}{\sqrt{2}} + C_2 \sin \frac{x}{\sqrt{2}} \right) + e^{-x/\sqrt{2}} \left(C_3 \cos \frac{x}{\sqrt{2}} + C_4 \sin \frac{x}{\sqrt{2}} \right)$

4. Solve $\frac{d^4 y}{dx^4} - 64y = 0$

Ans : $y = C_1 e^{2x} + C_2 e^{-2x} + e^{-x} (C_3 \cos \sqrt{3}x + C_4 \sin \sqrt{3}x) + e^x (C_5 \cos \sqrt{3}x + C_6 \sin \sqrt{3}x)$

2.2.1 Consider the D.E. $f(D)y = R$ (1)

Operating both sides with $\frac{1}{f(D)}$ (called inverse operator)

we have $\frac{1}{f(D)} [f(D)]y = \frac{1}{f(D)} R$

or $y = \frac{1}{f(D)} R$

This solution is called the particular Integral of (1)

while the solution of $f(D)y = 0$

is called the complementary function (C.F.)

..... (2)

Suppose y_1 is C.F and y_2 is P.I for (1)

Then $f(D)y_1 = 0$ from (2)

and $f(D)y_2 = R$ from (1)

Hence $f(D)[y_1 + y_2] = f(D)y_1 + f(D)y_2$
 $= 0 + R = R$

Which shows that $y_1 + y_2$ is also a solution of (1). $y_1 + y_2$ (i.e.) C.F. + P.I is called the most general solution of (1).

2.2.2 Calculation of $\frac{1}{D-m_1} R$

$\frac{1}{D-m_1} R = y$, say, Operating both sides with $(D-m_1)$, we get $(D-m_1) y = R$

$$(i.e.) \quad \frac{dy}{dx} - m_1 y = R$$

$$I.F. \text{ is } e^{\int -m_1 dx} = e^{-m_1 x}$$

$\therefore$ Solution is given by

$$y \cdot e^{-m_1 x} = \int R e^{-m_1 x} dx + c$$

$$(i.e.) y = C e^{-m_1 x} + e^{m_1 x} \int R e^{-m_1 x} dx = C.F. + P.I$$

$$P.I. = e^{m_1 x} \int R e^{-m_1 x} dx$$

$$\text{If } m_1 = 0, \frac{1}{D} R = \int R dx$$

Thus the operator $\frac{1}{D}$ stands for integration

Example 2.2.3 Find the value of $\frac{1}{D-5} x$

$$\begin{aligned} \text{Sol. } \frac{1}{D-5} x &= e^{5x} \int e^{-5x} x dx = e^{5x} \left[-x \frac{e^{-5x}}{5} + \int \frac{e^{-5x}}{5} dx \right] = e^{5x} \left[-\frac{x e^{-5x}}{5} - \frac{x e^{-5x}}{25} \right] \\ &= -\frac{x}{5} - \frac{1}{25} \end{aligned}$$

Example 2.2.4 Find $\frac{1}{D^2-5D+6} e^x$

$$\begin{aligned} \text{Sol. } \frac{1}{D^2-5D+6} e^x &= \frac{1}{(D-2)(D-3)} \cdot (e^x) \\ &= \frac{1}{(D-2)} \cdot \frac{1}{(D-3)} (e^x) = \frac{1}{(D-2)} e^{3x} \int e^{-3x} \cdot e^x dx = -\frac{1}{(D-2)} \cdot e^{3x} \frac{e^{-2x}}{2} \\ &= -\frac{1}{2} \frac{1}{(D-2)} (e^x) = -\frac{1}{2} e^{2x} \int e^{-2x} \cdot e^x dx = \frac{1}{2} e^{2x} \int e^{-x} dx = \frac{1}{2} e^{2x} \cdot e^{-x} = \frac{e^x}{2} \end{aligned}$$

Example 2.2.5Solve $(D^2 + a^2)y = \tan ax$ **Sol.** A.E is $m^2 + a^2 = 0 \Rightarrow m = \pm ai$ C.F. is $y = C_1 \cos ax + C_2 \sin ax$

$$\begin{aligned} \text{P.I} &= \frac{1}{(D+ai)(D-ai)} \tan ax = \frac{1}{2ai} \left[\frac{1}{D-ai} - \frac{1}{D+ai} \right] \tan ax \\ &= \frac{1}{2ai} \left[e^{iax} \int e^{-iax} \tan ax dx - e^{-iax} \int e^{iax} \tan ax dx \right] \end{aligned}$$

$$\begin{aligned} \int e^{-iax} \tan ax dx &= \int (\cos ax - i \sin ax) \frac{\sin ax}{\cos ax} dx = \int \sin ax - i \frac{(1 - \cos^2 ax)}{\cos ax} dx \\ &= -\frac{\cos ax}{a} - \frac{i}{a} \log(\sec ax + \tan ax) + \frac{i \sin ax}{a} \end{aligned}$$

$$\text{Hence } \int e^{iax} \tan ax dx = -\frac{\cos ax}{a} + \frac{i}{a} \log(\sec ax + \tan ax) - \frac{i \sin ax}{a}$$

$$I_1 = \left(\frac{\cos ax + i \sin ax}{a} \right) [-\cos ax + i \sin ax - i \log(\sec ax + \tan ax)]$$

$$= \frac{1}{a} [-1 - i \cos ax \log(\sec ax + \tan ax) + \sin ax \log(\sec ax + \tan ax)]$$

$$\therefore \text{P.I.} = \frac{1}{2a^2 i} [-2i \cos ax \log(\sec ax + \tan ax)]$$

$$= -\frac{\cos ax}{a} \log(\sec ax + \tan ax)$$

 $\therefore$ Most general solution is

$$y = C_1 \cos ax + C_2 \sin ax - \frac{\cos ax}{a^2} \log(\sec ax + \tan ax)$$

Example 2.2.6Solve $(D^2 - 5D + 6)y = e^x$ **Sol.** A.E. is $m^2 - 5m + 6 = 0$ C.F. is $y = C_1 e^{2x} + C_2 e^{3x}$ P.I. is $y = \frac{1}{D^2 - 5D + 6} (e^x) = \frac{e^x}{2}$ (from 5.2.4) $\therefore$ Complete solution (C.S.) is $y = \text{C.F.} + \text{P.I.}$ (i.e.) $y = C_1 e^{2x} + C_2 e^{3x} + \frac{e^x}{2}$ **Exercise 2 (d)**

1. $\frac{1}{D} \cos x$ 2. $\frac{1}{D^2} (e^{-x})$ 3. $\frac{1}{D} (x^2)$

4. $\frac{1}{D-2} \sin x$ 5. $\frac{1}{D^3 - D^2} x$

Ans : (1) $\sin x$ (2) e^{-x} (3) $\frac{x^3}{3}$ (4) $-\frac{(2 \sin x + \cos x)}{5}$ (5) $-\left(\frac{x^3}{6} + \frac{x^2}{2}\right)$

Solve the following :

6. $\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 3y = e^{2x}$ Ans : $y = C_1 e^{3x} + C_2 e^x + \frac{e^{2x}}{2}$

7. $\frac{d^2 y}{dx^2} + y = 5x + 3$ Ans : $y = C_1 \cos x + C_2 \sin x + (5x + 3)$

8. $\frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} + 2y = e^{-2x}$ Ans : $y = C_1 e^x + C_2 e^{2x} + \frac{e^{-2x}}{2}$

9. $\frac{d^2 y}{dx^2} - \frac{dy}{dx} - 2y = x$ Ans : $y = \left(C_1 e^{-x} + C_2 e^{2x} - \left(\frac{x}{2} - \frac{1}{4} \right) \right)$

10. $\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 4y = x^2 + 1$ Ans : $y = (C_1 x + C_2) e^{2x} + (4x^2 + 8x + 10)$

$$11. \quad (D^2 + a^2)y = \sec ax \quad \text{Ans : } y = C_1 \cos ax + C_2 \sin ax + C_2 \sin ax + \frac{x \sin ax}{a}$$

$$+ \frac{\cos ax}{a^2} \log \cos ax$$

$$12. \quad (D^2 + 9)y = \tan 3x \quad \text{Ans : } y = C_1 \cos 3x + C_2 \sin 3x \\ - \frac{\cos 3x}{9} \log(\sec 3x + \tan 3x)$$

Methods of finding P.I.

We shall now consider the methods for finding P.I. where R is of some special form

2.3.1 Particular integral

Where R is of the form e^{ax}

Case (i) if $f(a) \neq 0$

Since $D e^{ax} = a e^{ax}$; $D^2 e^{ax} = a^2 e^{ax}$ $D^n(e^{ax}) = a^n e^{ax}$

$$f(D) e^{ax} = (D^n + K_1 D^{n-1} + K_2 D^{n-2} + \dots + K_n) e^{ax} \\ = (a^n + K_1 a^{n-1} + \dots + K_n) e^{ax} = f(a) e^{ax}$$

Operating with $\frac{1}{f(D)}$ on both sides

$$\frac{1}{f(D)} [f(D) e^{ax}] = \frac{1}{f(D)} [f(a) e^{ax}]$$

$$(i.e) \quad e^{ax} = \frac{1}{f(D)} f(a) e^{ax} = f(a) \cdot \left[\frac{1}{f(D)} e^{ax} \right]$$

$$(or) \quad \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$$

Note : If K is a constant then $\frac{1}{f(D)}(K) = \frac{1}{f(D)} K \cdot e^{ax} = \frac{1}{f(m)} \cdot K$

$$\text{for example } \frac{1}{D^2 - 2D + 1} e^{3x} \frac{1}{9 - 6 + 1} = e^{3x} = \frac{1}{4} e^{3x}$$

2.3.2 Case ii) If $f(a) = 0$ then it is possible to write $f(D)$ as $\phi(D) (D-a)^r$ where $\phi(a) \neq 0$
 Suppose $r = 1$ then

$$\begin{aligned} \text{P.I.} &= \frac{1}{f(D)} e^{ax} = \frac{1}{D-a} \frac{1}{\phi(D)} e^{ax} \\ &= \frac{1}{D-a} \frac{1}{\phi(a)} e^{ax} = \frac{1}{\phi(a)} \frac{1}{D-a} e^{ax} \\ &= \frac{1}{\phi(a)} e^{ax} \int e^{-ax} \cdot e^{ax} dx = \frac{1}{\phi(a)} e^{ax} dx = \frac{1}{\phi(a)} e^{ax} \cdot x \quad \dots (1) \end{aligned}$$

$$\begin{aligned} \text{if } r = 2 \text{ then } \frac{1}{f(D)} e^{ax} &= \frac{1}{(D-a)^2} \frac{1}{\phi(D)} e^{ax} = \frac{1}{\phi(a)} \frac{1}{(D-a)^2} e^{ax} \\ &= \frac{1}{\phi(a)} \frac{1}{D-a} [xe^{ax}] \end{aligned}$$

$$\text{P.I.} = \frac{1}{f(D)} e^{ax} = \frac{1}{\phi(a)} e^{ax} \cdot \frac{x^2}{2}$$

In general, if $(D-a)$ is repeated 'r' times then

$$\text{P.I.} = \frac{1}{f(D)} e^{ax} = \frac{1}{\phi(a)} e^{ax} \cdot \frac{x^r}{r!}$$

Solved Examples

Example 2.3.3

$$\text{Solve } \frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 5y = 13e^{3x}$$

$$\text{Sol : } (D^2 + 4D + 5) = 13e^{3x}$$

$$\therefore \text{ A.E. is } m^2 + 4m + 5 = 0, \text{ giving } m = -2 \pm i$$

$$\therefore \text{ C.F. is } y = e^{-2x} (A \cos x + B \sin x)$$

$$\text{P.I.} = \frac{1}{D^2 + 4D + 5} (13e^{3x})$$

replace 'D' by '3'

$$= \frac{13e^{3x}}{9 + 12 + 5} = \frac{13e^{3x}}{26} = \frac{e^{3x}}{2}$$

$\therefore$ Complete solution is $y = \text{C.F.} + \text{P.I.} = e^{-2x}(\text{Acos}x + \text{Bsin}x) + \frac{e^{3x}}{2}$

Example 2.3.4

Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 2e^{-2x}$, given that $x = 0, y = 1$ and $\frac{dy}{dx} = -2$

Sol. Given equation is $f(D)y = 2e^{-2x}$ where $f(D) = D^2 + 4D + 5$

$\therefore$ A.E. is $m^2 + 4m + 5 = 0$, giving $m = -2 \pm i$

$\therefore$ C.F. is $y = e^{-2x} (\text{Acos}x + \text{Bsin}x)$

$$\text{P.I.} = \frac{1}{D^2 + 4D + 5} 2e^{-2x} = \frac{2e^{-2x}}{4 - 8 + 5} = 2e^{-2x}$$

$\therefore$ Complete solution is $y = \text{C.F.} + \text{P.I.} = e^{-2x}(\text{Acos}x + \text{Bsin}x) + 2e^{-2x}$ (4)

Now $y(0) = 1$, (given)

Substituting in (4)

$$y(0) = 1(A) + 2; A = -1.$$

$$y^1(x) = e^{-2x}(-\text{Asin}x + \text{Bcos}x) - 2e^{-2x} (\text{Acos}x + \text{Bsin}x) - 4e^{-2x}$$

$$y^1(0) = B - 2A - 4, y^1(0) = -2 \text{ (given)}$$

$$\therefore -2 = B - 2A - 4, \Rightarrow B = 0$$

$$\begin{aligned} \therefore y(x) &= e^{-2x} (-\text{cos}x) + 2e^{-2x} \\ &= 2e^{-2x} - e^{-2x} \text{cos}x \end{aligned}$$

Example 2.3.5

Solve $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x}$

Sol. A.E. is $m^3 + 2m^2 + m = 0$; $m(m+1)^2 = 0$

$\therefore m = 0, -1, -1.$

$$\text{C.F.} = C_1 e^{0x} + (C_2 + C_3 x)e^{-x}$$

$$\text{P.I.} = \frac{1}{D^3 + 2D^2 + D} e^{2x} = \frac{e^{2x}}{18}$$

$\therefore$ Complete solution is $y = C_1 + (C_2 + C_3 x)e^{-x} + \frac{e^{2x}}{18}$

Example 2.3.6.

Solve $\frac{d^2y}{dx^2} - 3 \frac{dy}{dx} + 2y = e^x$

Sol. $f(D) = D^2 - 3D + 2$

A.E. is $m^2 - 3m + 2 = 0 \Rightarrow (m - 2)(m - 1) = 0$; $m = 1, 2$

C.F. is $C_1 e^x + C_2 e^{2x}$ here $f(1) = 0$

$$\therefore \frac{1}{(D-1)} \frac{1}{(D-2)} e^x = \frac{1}{(D-1)} \frac{1}{(1-2)} e^x = \frac{1}{(D-1)} e^x = -e^x \frac{x}{1!} \text{ (using 5.3.2.(3))}$$

$\therefore$ Most general solution is

$$\therefore y = C_1 e^x + C_2 e^{2x} - x e^x$$

2.4.1 R is of the form $\sin ax$ or $\cos ax$

$$D^2 (\sin ax) = (-a^2) \sin ax$$

$$(D^2)^2 (\sin ax) = (-a^2)^2 \sin ax$$

$$(D^2)^3 (\sin ax) = (-a^2)^3 \sin ax$$

.....

.....

.....

$$(D^2)^n (\sin ax) = (-a^2)^n \sin ax$$

Hence $f(D^2) \sin ax = f(-a^2) \sin ax$, provided $f(-a^2) \neq 0$

$$\therefore \frac{1}{f(D^2)} f(D^2) \sin ax = \frac{1}{f(D^2)} f(-a^2) \sin ax$$

$$\text{or } \sin ax = \frac{1}{f(D^2)} f(-a^2) \sin ax$$

$$\frac{1}{f(D^2)} \sin ax = \frac{\sin ax}{f(-a^2)}$$

$$\text{Similarly it can be shown that } \frac{1}{f(-D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax$$

$$\begin{aligned}\text{Thus } \frac{1}{f(D)} [\sin ax \text{ or } \cos ax] &= \frac{1}{\phi_1(D^2) + D\phi_2(D^2)} [\sin ax \text{ or } \cos ax] \\ &= \frac{1}{\phi_1(-a^2) + \phi_2(-a^2)D} [\sin ax \text{ or } \cos ax]\end{aligned}$$

If $f(D)$ vanishes when D^2 is replaced by $-a^2$ then we write

$$\begin{aligned}\frac{1}{f(D)} [\sin ax] &= \frac{1}{f(D)} [\text{Imaginary part of } e^{iax}] \\ &= \text{I.P. of } \frac{1}{f(D)} e^{iax}\end{aligned}$$

$$\text{Similarly } \frac{1}{f(D)} [\cos ax] = \text{R.P. of } \frac{1}{f(D)} e^{iax}$$

Solved Examples

Example 2.4.2

$$\text{Solve } \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin 3x$$

Sol. A.E. is $m^2 - 3m + 2 = 0$; $m = 1, 2$

$$\therefore \text{C.F.} = C_1 e^x + C_2 e^{2x}$$

$$\begin{aligned}\text{P.I. } \frac{1}{D^2 - 3D + 2} (\sin 3x) &= \frac{1}{-9 - 3D + 2} (\sin 3x) = \frac{-1}{(3D + 7)} (\sin 3x) = \frac{(3D - 7)}{9D^2 + 49} (\sin 3x) \\ &= \frac{(3D - 7)}{-81 - 49} (\sin 3x) = -\frac{9\cos 3x + 7\sin 3x}{D^2 - 3D + 2}\end{aligned}$$

Hence complete solution is

$$y = C_1 e^x + C_2 e^{2x} - \frac{1}{130} (9\cos 3x + 7\sin 3x)$$

Example 2.4.3Solve $(D^3 - D)y = \sin x$ **Sol.** A.E. is $m^3 - m = 0$

$$m = 0, m = \pm 1$$

$$\text{C.F. is } C_1 e^{0x} + C_2 e^x + C_3 e^{-x} = C_1 + C_2 e^x + C_3 e^{-x}$$

$$\text{P.I.} = \frac{1 \cdot \sin x}{D^3 - D} = \frac{1}{D^2 - 1} \cdot \frac{1}{D} (\sin x)$$

$$= -\frac{1}{D^2 - 1} \cdot \cos x = \frac{-\cos x}{-1 - 1} = -\frac{x}{2} \sin x$$

The complete solution is

$$y = C_1 + C_2 e^x + C_3 e^{-x} - \frac{x}{2} \sin x$$

Example 2.4.4Solve $(D^2 - D + 1)y = \cos 2x$ **Sol.** A.E. is $m^2 - m + 1 = 0 \Rightarrow m = \frac{1 \pm \sqrt{3}i}{2}$

$$\therefore \text{C.F.} = e^{\frac{1 \pm \sqrt{3}i}{2} x} \left(A \cos \frac{\sqrt{3}}{2} x + B \sin \frac{\sqrt{3}}{2} x \right)$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - D + 1} \cos 2x = \frac{1}{-4 - D + 1} \cos 2x = \frac{1}{-3 - D} \cos 2x = -\frac{(3 - D)}{9 - D^2} \cos 2x \\ &= -\frac{(3 - D) \cos 2x}{13} = -\frac{1}{13} (2 \sin 2x + 3 \cos 2x) \end{aligned}$$

$$\therefore \text{C.F.} + \text{P.I.} = e^{\frac{x}{2}} \left(A \cos \frac{\sqrt{3}}{2} x + B \sin \frac{\sqrt{3}}{2} x \right) - \frac{1}{13} (2 \sin 2x + 3 \cos 2x)$$

Example 2.4.5Solve $D^2(D^2 + 9) = \sin 2x + 5$ **Sol.** A.E. is $m^2(m^2 + 9) = 0 \Rightarrow m = 0, 0, +3i$

$$\text{C.F.} = (C_1 + C_2 x) + C_3 \cos 3x + C_4 \sin 3x$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2(D^2+9)} \sin 2x + \frac{5e^{0x}}{(D^2+9)D^2} \\ &= \frac{\sin 2x}{-4(-4+9z)} + \frac{5}{9} \cdot \frac{1}{D^2}(1) = -\frac{\sin 2x}{20} + \frac{5x^2}{18} \end{aligned}$$

$$\text{Complete solution is } y = (C_1 + C_2x) + C_3\cos_3x + C_4\sin 3x - \frac{\sin 2x}{20} + \frac{5x^2}{18}$$

Example 2.4.6

$$\text{Solve } (D^4 - 2D^3 + 2D^2 - 2D + 1)y = e^x + \sin^2(x/2)$$

$$\text{Sol. A.E. is } f(D) = D^4 - 2D^3 + 2D^2 - 2D + 1$$

$$m^4 - 2m^3 + 2m^2 - 2m + 1 = 0$$

$$= (m-1)^2(m^2+1) = 0$$

$$\therefore m = 1, 1, m = i, -i$$

$$\text{C.F. is } (C_1 + C_2x)e^x + (C_3\cos x + C_4\sin x)$$

$$\text{P.I.} = \frac{1}{f(D)} e^x + \frac{1}{f(D)} \sin^2(x/2)$$

$$\frac{1}{f(D)} e^x = \frac{1}{(D-1)^2(D^2+1)} e^x = \frac{x^2}{2!} \frac{e^x}{2} \text{ [using 2.3.2(3)]}$$

$$\frac{1}{f(D)} \sin^2(x/2) = \frac{1}{(D-1)^2(D^2+1)} \sin^2(x/2) = \frac{1}{(D-1)^2(D^2+1)} \left(\frac{1 - \cos x}{2} \right)$$

$$= \frac{1}{2} \left[\frac{1}{(D-1)^2(D^2+1)} (1 - \cos x) \right] = \frac{1}{2} \left[-1 \frac{1}{(D^2 - 2D + 1)(D^2 + 1)} \cos x \right]$$

$$= \frac{1}{2} - \frac{1}{-4D} \cdot \frac{1}{D^2 + 1} (\cos x)$$

$$= \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{D^2 + 1} \cdot \frac{D}{(D^2)} (\cos x)$$

$$= \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{D^2 + 1} \cdot \frac{D}{(-1)} (-\sin x)$$

$$= \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{D^2 + 1} \cdot \sin x$$

$$\frac{1}{D^2+1} \sin x = \text{I.P. of } \frac{1}{D^2+1} e^{ix} = \text{I.P. of } \frac{1}{(D+i)(D-i)} e^{ix}$$

$$= \text{I.P. of } \frac{1}{2i} x e^{ix} = \frac{x}{2} \text{ I.P. of } -i[\cos x + i \sin x] = -\frac{x}{2} \cos x$$

$$\therefore \text{P.I.} = \frac{1}{2} - \frac{1}{8} x \cos x$$

$\therefore$ Hence the complete solution is

$$y = (C_1 + C_2 x) e^x + C_3 \cos x + C_4 \sin x + \frac{1}{2} \cdot \frac{x^2}{2!} e^x + \frac{1}{2} - \frac{x}{8} \cos x$$

Exercise 2(e)

Solve the following

$$(1) \quad (D^2+4)y = e^x + \sin 2x \quad \text{Ans : } C_1 \cos 2x + C_2 \sin 2x - \frac{x}{2} \cos 2x$$

$$(2) \quad (D^2+D+1)y = \sin 2x$$

$$\text{Ans : } e^{\frac{x}{2}} \left(C_1 \cos \frac{\sqrt{3}}{2} x + C_2 \sin \frac{\sqrt{3}}{2} x - \frac{1}{13} (2 \cos 2x + 3 \sin 2x) \right)$$

$$(3) \quad (D^6+1)y = \sin \frac{3}{2} x \cdot \sin \frac{x}{2}$$

$$\text{Ans. } (C_1 \cos x + C_2 \sin x) + e^{\frac{\sqrt{3}x}{2}} (C_3 \cos \frac{x}{2} + C_4 \sin \frac{x}{2}) + e^{\frac{\sqrt{3}x}{2}} (C_5 \cos \frac{x}{2} + C_6 \sin \frac{x}{2})$$

$$+ \frac{x}{12} \sin x + \frac{1}{126} \cos 2x$$

$$(4) \quad (D^3 + 4D)^2 y = \sin 2x$$

$$\text{Ans. } y = C_1 + C_2 \cos 2x + C_3 \sin 2x - \frac{x \sin x}{8} - \frac{\cos 2x}{16}$$

2.5.1 P.I. is of form $\frac{1}{f(D)} x^m$

$$\frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$$

Now expand $[f(D)]^{-1}$ in ascending powers of D and retain as far as D^m and then operate on x^m

Solved Examples

Example 2.5.2

Solve $(D^2 + 5D + 6)y = x$

Sol. A.E. is $m^2 + 5m + 6 = 0 \Rightarrow m = -2, -3$

C.F. is $C_1 e^{-2x} + C_2 e^{-3x}$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 + 5D + 6} x = \frac{1}{6} \left(1 + \frac{D^2 + 5D}{8}\right)^{-1} x \\ &= \frac{1}{6} \left[1 - \frac{5D}{6}\right] x = \frac{1}{6} \left[x - \frac{5}{6}\right] = \frac{x}{6} - \frac{5}{36} \end{aligned}$$

Solution is therefore $y = C_1 e^{-2x} + C_2 e^{-3x} + \frac{x}{6} - \frac{5}{36}$

Example 2.5.3

Solve $\frac{d^3 y}{dx^3} - 3 \frac{dy}{dx} - 2y = x^2$

Sol. A.E. is $m^3 - 3m - 2 = 0 \Rightarrow (m-2)(m+1)^2 = 0$

C.F. is $C_1 e^{2x} + (C_2 + C_3 x)e^{-x}$

$$\begin{aligned} \text{P.I.} &= \frac{x^2}{D^3 - 3D - 2} = -\frac{1}{2} \left(1 - \left(\frac{D^3 - 3D}{2}\right)\right)^{-1} x^2 \\ &= -\frac{1}{2} \left[1 + \frac{D^3 - 3D}{2} + \left(\frac{D^3 - 3D}{2}\right)^2 + \dots\right] x^2 \\ &= -\frac{1}{2} \left[1 - \frac{3D}{2} + \frac{9}{4} D^2\right] x^2 = -\frac{1}{2} \left[x^2 - \frac{6x}{2} + \frac{9}{2}\right] = \frac{1}{4} [2x^2 - 6x + 9] \end{aligned}$$

Complete Solution is $y = C_1 e^{2x} + (C_2 + C_3 x)e^{-x} - \frac{1}{4} (2x^2 - 6x + 9)$

Exercise 2(f)

1. Solve $(D^3 - 3D^2 + 2)y = x$

$$\text{Ans. } (C_1 + C_2 x)e^x + C_3 e^{-2x} + \frac{1}{2} \left(x + \frac{3}{2}\right)$$

2. Solve $(D^2 + 2D + 3)y = x + x^2$

$$\text{Ans. } y = e^{-x} (C_1 \cos \sqrt{2}x + C_2 \sin \sqrt{2}x) + \frac{14 - 6x}{27}$$

3. Solve $(D^2 + 2D + 1)y = \cos 2x + x^2$

$$\text{Ans. } y = (C_1 x + C_2 e^{-x} - \frac{1}{13}(3 \cos 2x - 2 \sin 2x) + (x^2 - 2x)$$

4. $(D^2 - 4D + 4)y = x^2 + \sin 2x + e^{3x}$

$$\text{Ans. } y = (C_1 + C_2 x)e^{2x} + \frac{2x^2 + x + 3}{8} \frac{\cos 2x}{8} + e^{3x}$$

2.6.1

To find $\frac{1}{f(D)} (e^{ax}V)$ where V is a function of x

$$D(e^{ax}V_1) = e^{ax}D(V_1) + ae^{ax}V_1 = e^{ax}(D+a)V_1$$

$$\text{Similarly } D^2(e^{ax}V_1) = e^{ax}(D+a)^2V_1$$

$$D^3(e^{ax}V_1) = e^{ax}(D+a)^3V_1$$

$$\dots\dots\dots D_n(e^{ax}V_1) = e^{ax}(D+a)^nV_1$$

$$\therefore f(D) (e^{ax}V_1) = e^{ax}f(D+a)V_1$$

$$\text{Calling } f(D+a)V_1 = V, \text{ we get } V_1 = \frac{1}{f(D+a)} V$$

$$\therefore f(D) [e^{ax} \frac{1}{f(D+a)} V] = e^{ax}V \text{ or } e^{ax} \frac{1}{f(D+a)} V = \frac{1}{f(D)} e^{ax}V$$

$$\therefore \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$$

2.6.2 Note

1. While finding $\frac{1}{f(D)} e^{ax}$ when $f(a) = 0$ we can employ the above method for

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(D)} [e^{ax} \cdot e^{0x}] = e^{ax} \cdot \frac{1}{f(D+a)} [e^{0x}]$$

2. Further $\frac{1}{f(D^2)}$ $[\cos ax \text{ or } \sin ax]$ when $f(-a^2) = 0$ we can write

$$\begin{aligned} & \frac{1}{f(D^2)} [\cos ax \text{ or } \sin ax] \\ &= \frac{1}{f(D^2)} [\text{R.P. or I.P. of } e^{iax}] = \text{R.P. or I.P. of } e^{iax} \cdot \frac{1}{f(D+ia)^2} \cdot 1 \end{aligned}$$

Solved Examples

2.6.3 Solve : $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 4y = e^x \cos x$

Sol. A.E. is $m^2 - 2m + 4 = 0$

$$m = 1 \pm \sqrt{3}i$$

C.F. is $e^x (C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x)$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 4} e^x \cos x = e^x \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos x$$

$$= e^x \frac{1}{D^2 + 2D + 1 - 2D - 2 + 4} (\cos x)$$

$$= e^x \frac{1}{D^2 + 3} (\cos x) = e^x \cdot \frac{\cos x}{-1^2 + 3} = \frac{e^x \cos x}{2}$$

$\therefore$ Complete solution is $y = \text{C.F.} + \text{P.I.}$

$$y = e^x (C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x) + \frac{e^x \cos x}{2}$$

Example 2.6.4

Solve $(D^2 - 2)y = e^x \cos x$

Sol. A.E. is $m^2 - 2 = 0$ $m = \pm \sqrt{2}$

$\therefore$ C.F. is $C_1 e^{\sqrt{2}x} + C_2 e^{-\sqrt{2}x}$

$$\text{P.I.} = \frac{1}{(D^2 - 2)} e^x \cos x = e^x \cdot \frac{1}{(D+1)^2 - 2} \cos x = e^x \cdot \frac{1}{D^2 + 2D - 1} \cdot \cos x$$

$$= e^x \cdot \frac{1}{2D - 2} \cdot \cos x$$

$$= e^x \cdot \frac{(2D + 2)}{4D^2 - 4} \cos x = \frac{e^x (-2 \sin x + 2 \cos x)}{-4 - 4} = \frac{e^x}{4} (\sin x - \cos x)$$

$$\therefore y = C_1 e^{\sqrt{2}x} + C_2 e^{-\sqrt{2}x} + \frac{e^x}{4}(\sin x - \cos x)$$

Example 2.6.5

Solve $\frac{d^3 y}{dx^3} - 7 \frac{dy}{dx} - 6y = e^{2x} + xe^{2x}$

Sol. A.E. is $m^3 - 7m - 6 = 0 \therefore m = -1, 3, -2$

$$\therefore \text{C.F.} = C_1 e^{-x} + C_2 e^{3x} + C_3 e^{-2x}$$

$$\begin{aligned} \text{P.I.} &= \frac{(1+x)}{D^3 - 7D - 6} e^{2x} \\ &= e^{2x} \frac{1}{(D+2)^3 - 7(D+2) - 6} (1+x) \\ &= e^{2x} \frac{1}{D^3 - 6D^2 + 5D - 12} \\ &= \frac{-e^{2x}}{12} \left[1 - \left(\frac{D^3 - 6D^2 + 5D}{12} \right) \right]^{-1} (1+x) \\ &= \frac{-e^{2x}}{12} \left(1 + \frac{5D}{12} \right) (1+x) = \frac{-e^{2x}}{12} \left[1 + x + \frac{5}{12} \right] \\ &= \frac{-e^{2x}}{12} \left[x + \frac{17}{12} \right] \end{aligned}$$

$$\therefore y = C_1 e^{-x} + C_2 e^{3x} + C_3 e^{-2x} - \frac{e^{2x}}{12} \left[x + \frac{17}{12} \right]$$

Example 2.6.6

Solve $\frac{d^3 y}{dx^3} - 3 \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} - y = e^x(1+x)$

Sol. A.E. is $m^3 - 3m^2 + 3m - 1 = 0$ or $(m-1)^3 = 0$ or $m = 1, 1, 1$.

$$\text{C.F.} = (C_1 x^2 + C_2 x + C_3) e^x$$

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{(D-1)^3} \cdot (x+1) e^x \\
 &= \frac{e^x \cdot (x+1)}{(D+1-1)^3} = e^x \cdot \frac{1}{D^3} (x+1) = e^x \cdot \frac{1}{D^2} \left(\frac{x^2}{2} + x \right) \\
 &= e^x \frac{1}{D} \left(\frac{x^3}{6} + \frac{x^2}{2} \right) = e^x \left(\frac{x^4}{24} + \frac{x^3}{6} \right)
 \end{aligned}$$

Complete solution is $y = (C_1 x^2 + C_2 x + C_3) e^x + e^x \left(\frac{x^4}{24} + \frac{x^3}{6} \right)$

Example 2.6.7

Solve $\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + y = e^{2x} \sin 2x$

Sol.

A.E. is $m^2 - 4m + 1 = 0$, $m = 2 \pm \sqrt{3}$

C.F. = $C_1 e^{(2+\sqrt{3})x} + C_2 e^{(2-\sqrt{3})x}$

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^2 - 4D + 1} e^{2x} \sin 2x = \frac{e^{2x}}{(D+2)^2 - 4(D+2) + 1} \sin 2x \\
 &= \frac{e^{2x}}{D^2 - 3} \sin 2x = \frac{e^{2x} \sin 2x}{-4 - 3} = -\frac{1}{7} e^{2x} \sin 2x
 \end{aligned}$$

Complete solution is

$$y = \text{C.F.} + \text{P.I.} = C_1 e^{(2+\sqrt{3})x} + C_2 e^{(2-\sqrt{3})x} - \frac{1}{7} e^{2x} \sin 2x$$

Exercise 2(g)

1. Solve $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y = x^2 e^{3x}$

Ans : $y = (C_1 + C_2 x) e^x + \frac{e^{3x}}{8} (2x^2 - 4x + 3)$

2. Solve $(D^3 - 7D - 6)y + e^{2x} + x^2 e^{3x}$

Ans : $y = C_1 e^{-x} + C_2 e^{-2x} + C_3 e^{3x} - e^{2x} \frac{1}{12} \left(x^2 + \frac{5}{6} x + \frac{169}{72} \right)$

3. Solve $(D^2 - 2D + 4)y = e^x \cos^2 x$

Ans : $y = e^x (C_1 \cos 3\sqrt{x} + C_2 \sin 3\sqrt{x}) + \frac{1}{8} e^x - \frac{1}{2} \sin 2x$

4. Solve $\frac{d^2 y}{dx^2} + y = e^x \sin x$

Ans. $y = C_1 \cos x + C_2 \sin x - \frac{1}{5} e^x (2 \cos x - \sin x)$

5. Solve $\frac{d^2 y}{dx^2} + y = e^{2x} \sin x$

Ans. $y = C_1 e^{-x} + e^{x/2} (C_2 \cos \frac{\sqrt{3}}{2} x + C_3 \sin \frac{\sqrt{3}}{2} x) - \frac{1}{130} e^{2x} (11 \cos x - 3 \sin x)$

2.7.1 To find $\frac{1}{f(D)} [xv]$ where v is a function of x

$$D(xv_1) = xDv_1 + v_1$$

$$D^2(xv_1) = D(xDv_1 + v_1) = xD^2v_1 + 2Dv_1 = xD^2v_1 + (D^2)v_1$$

$$\text{Similarly } D^3(xv_1) = xD^3v_1 + (D^3)v_1$$

$$\text{in general } D^n(xv_1) = xD^n v_1 + (D^n)v_1$$

(here 1 indicates differentiation w.r.t 'D')

$$\text{Thus, } f(D)(xv_1) = x(D)v_1 + f^1(D)v_1 \quad \dots(1)$$

$$\text{now let } f(D)v_1 = v$$

$$v_1 = \frac{1}{f(D)} \cdot v$$

$$\text{From (1) } f(D) \left[x \cdot \frac{1}{f(D)} v \right] = xv + f^1(D) \left[\frac{1}{f(D)} v \right]$$

$$\text{Operating both sides with } \frac{1}{f(D)}$$

$$x \frac{1}{f(D)} v = \frac{1}{f(D)} (xv) + \frac{1}{f(D)} \cdot f^1(D) \left[\frac{1}{f(D)} v \right]$$

$$\frac{1}{f(D)} (xv) = x \cdot \frac{1}{f(D)} v - \frac{1}{f(D)} \cdot [f^1(D) \cdot \frac{1}{f(D)} v]$$

$$\frac{1}{f(D)} (xv) = \left[x - \frac{1}{f(D)} f^1(D) \right] \left[\frac{1}{f(D)} v \right]$$

2.7.2. Note : For finding $[x^m \cdot (\sin x \text{ or } \cos x)]$ when $m > 1$ the above method becomes very lengthy. Hence we can take it as $[x^m \cdot (\text{R.P or I.P of } e^{iax})]$ and apply the method (2.6.1).

Solved Examples

Example 2.7.3

Solve $(D^2 + 9)y = x \sin x$

Sol. A.E. is $m^2 + 9 = 0 \Rightarrow m = \pm 3i$

$$\text{P.I.} = \frac{1}{D^2 + 9} (x \sin x) = \left\{ x - \frac{1}{D^2 + 9} \cdot 2D \right\} \cdot \frac{1}{D^2 + 9} \cdot \sin x$$

[using the result of 2.7.1. $v = \sin x$]

$$= \left\{ x - \frac{1}{D^2 + 9} 2D \right\} \cdot \frac{1}{8} \sin x = \frac{x}{8} \sin x - \frac{1}{D^2 + 9} \cdot \frac{2}{8} \cos x$$

$$= \frac{x}{8} \sin x - \frac{2}{8} \cdot \frac{1}{8} \cdot \cos x = \frac{x}{8} \sin x - \frac{1}{32} \cos x$$

Complete solution is $y = C_1 \cos 3x + C_2 \sin 3x + \frac{x}{8} \sin x - \frac{1}{32} \cos x$

Example 2.7.4

Find the solution of $\frac{d^2 y}{dx^2} + 9y = x \cos 2x$

Sol. A.E. is $m^2 + 9 = 0 \Rightarrow m = \pm i\sqrt{3}$

C.F. is $y = C_1 \cos \sqrt{3} x + C_2 \sin \sqrt{3} x$

and $\text{P.I.} = \frac{1}{D^2 + 9} (x \cos 2x)$

$$= \text{Real part of } \frac{1}{D^2 + 9} x (\cos 2x + i \sin 2x)$$

$$= \text{R.P. of } \frac{1}{D^2 + 9} x e^{2ix} = \text{R.P. of } e^{2ix} \frac{1}{(D + 2i)^2 + 9} x = \text{R.P. of } e^{2ix} \frac{1}{D^2 + 4iD + 5} x$$

$$= \text{R.P. of } \frac{e^{2ix}}{5} \left(1 + \frac{D^2 + 4iD}{5} \right)^{-1} x = \text{R.P. of } \frac{e^{2ix}}{5} \left(1 - \frac{D^2 + 4iD}{5} \right) x$$

$$= \text{R.P. of } \frac{e^{2ix}}{5} \left[1 - \frac{4iD}{5} - \frac{D^2}{5} + \dots \right] x$$

$$= \text{R.P. of } \frac{e^{2ix}}{5} \left[x - \frac{4i}{5} \right]$$

$$= \frac{1}{5} [x \cos 2x + \frac{4}{5} \sin 2x]$$

Most general solution is

$$y = C_1 \cos 3x + C_2 \sin 3x + \frac{1}{5} [x \cos 2x + \frac{4}{5} \sin 2x]$$

Example 2.7.5

$$\text{Solve } (D^2 - 4D + 4)y = 8e^{2x} x^2 \sin 2x$$

$$\text{Sol. A.E is } m^2 - 4m + 4 = 0 \Rightarrow m = 2, 2$$

$$\text{C.F. is } (C_1 x + C_2)e^{2x}$$

$$\text{P.I.} = \frac{8}{(D-2)^2} e^{2x} \cdot x^2 \sin 2x = 8e^{2x} \frac{1}{(D+2-2)^2} x^2 \sin 2x = 8e^{2x} \frac{1}{D^2} x^2 \sin 2x$$

$$= 8e^{2x} [\text{I.P. of } \frac{1}{D^2} x^2 e^{2ix}] = 8e^{2x} [\text{I.P. of } e^{2ix} \cdot \frac{1}{(D-2i)^2} x^2]$$

$$= 8e^{2x} [\text{I.P. of } e^{2ix} \frac{1}{-4 + 4iD + D^2} x^2]$$

$$= 8e^{2x} [\text{I.P. of } e^{2ix} \left(\frac{-1}{4} \right) \left[1 - \frac{4iD + D^2}{4} \right]^{-1} x^2]$$

$$= -2e^{2x} [\text{I.P. of } e^{2ix} \left[1 - \frac{4iD + D^2}{4} \right]^{-1} x^2]$$

$$= -2e^{2x} [\text{I.P. of } e^{2ix} \left[1 + \frac{4iD + D^2}{4} + \left(\frac{4iD + D^2}{4} \right)^2 \right] x^2]$$

$$= -2e^{2x} [\text{I.P. of } e^{2ix} \left[x^2 + \frac{8ix + 2}{4} + \frac{16i^2 D^2}{4} (x^2) \right]]$$

$$= -\frac{e^{2x}}{2} [\text{I.P. of } (\cos 2x + i \sin 2x) (4x^2 + 8ix - 30)]$$

$$= \frac{-e^{2x}}{2} \cdot \text{I.P. of } [4x^2 \cos 2x + 8ix \cos 2x - 30 \cos 2x + i4x^2 \sin 2x - 8x \sin 2x - 30i \sin 2x]$$

$$= -e^{2x} [4x \cos 2x + 2x^2 \sin 2x - 15 \sin 2x]$$

General solution is $y = \text{C.F.} + \text{P.I.}$

$$y = (C_1x + C_2)e^{2x} - e^{2x}[2x^2\sin 2x + 4x\cos 2x - 15\sin 2x]$$

Exercise 2(h)

1. Solve $(D^2 - 1)y = x^2\cos 3x$

$$\text{Ans. } y = C_1e^x + C_2e^{-x} - \frac{1}{10} (x^2\cos 3x - \frac{26}{50} \cos 3x - \frac{6}{5} x\sin 3x)$$

2. Solve $(D^2 + 4)y = x\sin^2 x$

$$\text{Ans. } y = C_1\cos 2x + C_2\sin 2x + \frac{x}{4} [\frac{1}{2} + \sin 2x]$$

3. Solve $(D^4 + 2D^2 + 1)y = x^2\cos x$

$$\text{Ans. } y = (C_1x + C_2)\cos x + (C_3x + C_4)\sin x - \frac{1}{48}(x^4 - 9x^2)\cos x + \frac{1}{12}x^3\sin x$$

4. Solve $(D^2 - 2D + 1)y = xe^x\sin x$

$$\text{Ans. } C_1\cos 3x + C_2\sin 3x + \frac{1}{5}(x\cos 2x + \frac{4}{5}\sin 2x)$$

5. Solve $(D^2 - 6D + 13)y = 8e^{3x}\sin 2x$

$$\text{Ans. } y = e^{3x}[C_1\cos 2x + C_2\sin 2x] - 2e^{3x}x\cos 2x$$

6. Solve $(D^4 + D^2 + 1)y = e^{\frac{-x}{2}} \cos(\frac{x\sqrt{3}}{2})$

$$\begin{aligned} \text{Ans. } y = e^{\frac{-x}{2}} [C_1 \cos(\frac{1}{2}x\sqrt{3}) + C_2 \sin(\frac{1}{2}x\sqrt{3})] + e^{\frac{-x}{2}} [C_3 \cos(\frac{1}{2}x\sqrt{3}) \\ + C_4 \sin(\frac{1}{2}x\sqrt{3})] + \frac{e^{\frac{-x}{2}}}{4} [x\cos(\frac{1}{2}x\sqrt{3}) + \frac{1}{\sqrt{3}} x\sin(\frac{1}{2}x\sqrt{3})] \end{aligned}$$

7. Solve $\frac{d^2y}{dx^2} + y = e^x\sin x$

$$\text{Ans. } C_1\cos x + C_2\sin x - \frac{1}{5}e^x(2\cos x - \sin x)$$

2.8.1 Cauchy's Homogenous Linear equation

Most general form of the homogeneous equation is

$$(x^n D^n + K_1 x^{n-1} D^{n-1} + \dots + K_{n-1} x D + = X \quad \dots (1)$$

Where $k_1, k_2, \dots, k_n$ are constants and 'X' is either a constant or a function of x

To solve such equations it is convenient to transform them into linear equations with constant coefficients with the substitution $x = e^z$ i.e. $z = \log x$.

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz} \quad \dots (2)$$

$$\begin{aligned} \frac{d^2 y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{1}{x} \frac{d}{dz} \left(\frac{1}{x} \frac{dy}{dz} \right) = \frac{1}{x} \left(\frac{-1}{x^2} \frac{dx}{dz} \cdot \frac{dy}{dz} + \frac{1}{x} \frac{d^2 y}{dz^2} \right) \\ &= \frac{1}{x} \left[-\frac{1}{x} \frac{dy}{dz} + \frac{1}{x} \frac{d^2 y}{dz^2} \right] \quad \left(\because \frac{dx}{dz} = x \right) \\ &= \frac{1}{x^2} \left(\frac{d^2 y}{dz^2} - \frac{dy}{dz} \right) \quad \dots (3) \end{aligned}$$

$$\text{III } \frac{d^3 y}{dx^3} = \frac{1}{x^3} \left(\frac{d^3 y}{dz^3} - 3 \frac{d^2 y}{dz^2} + 2 \frac{dy}{dz} \right) \quad \dots (4)$$

By taking 'D' to stand for $\frac{d}{dz}$,

$$(2), (3), (4) \Rightarrow x \frac{dy}{dx} = \frac{dy}{dz} = Dy$$

$$x^2 \frac{d^2 y}{dx^2} = D(D-1)y$$

$$x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)y \quad \dots (5)$$

$$x^n \frac{d^n y}{dx^n} = D(D-1)(D-2) \dots (D-n+1)y$$

By substituting (5) in the given equation (1) it transforms into a linear differential equation with constant coefficients.

Solved Examples

2.8.2 Solve $x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + 4y = 0$

Sol. Writing $x = e^z$ (i) reduces to

$$[D(D-1) - D + 4]y = 0, \text{ where } D = \frac{d}{dz}$$

$$(D^2 - 2D + 4)y = 0$$

$$\text{A.E. is } m^2 - 2m + 4 = 0 \Rightarrow m = 1 \pm \sqrt{3} i$$

$$\therefore \text{C.F. is given by } y = (C_1 \cos \sqrt{3}z + C_2 \sin \sqrt{3}z)e^z$$

substituting $z = \log x$,

$$y = C_1 \cos(\sqrt{3} \log x) + C_2 \sin(\sqrt{3} \log x) e^{\log x}$$

$$y = [C_1 \cos(\sqrt{3} \log x) + C_2 \sin(\sqrt{3} \log x)]x$$

2.8.3 Solve $x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} - 4y = 2x^2$ (i)

Sol. Writing $x = e^z$ (i) reduces to

$$[D(D-1) + 3D - 4]y = 2e^{2z},$$

$$(D^2 - D + 3D - 4)y = 2e^{2z}$$

$$\text{A.E. is } m^2 + 2m - 4 = 0 \Rightarrow m = -1 \pm \sqrt{5}$$

$$\therefore \text{C.F. is } y = (C_1 e^{(1+\sqrt{5})z} + C_2 e^{(1-\sqrt{5})z})$$

$$\text{P.I.} = \frac{2}{D^2 + 2D - 4} (e^{2z}) = \frac{2}{4 + 4 - 4} e^{2z} = \frac{2e^{2z}}{4} = \frac{e^{2z}}{2}$$

Most general solution is

$$y = \left(C_1 e^{-(1+\sqrt{5})z} + C_2 e^{-(1-\sqrt{5})z} + \frac{e^{2z}}{2} \right)$$

$$= \left(C_1 e^{-(1+\sqrt{5})\log x} + C_2 e^{-(1-\sqrt{5})\log x} + \frac{e^{2\log x}}{2} \right)$$

$$= C_1 \frac{1}{x^{(1+\sqrt{5})}} + \frac{C_2}{x^{(1-\sqrt{5})}} + \frac{x^2}{2}$$

2.8.4 Solve $x^2 \frac{d^2 y}{dx^2} - 2y = x^2 + \frac{1}{x}$

Sol. Taking $x = e^z$ equation (i) reduces to

$$[D(D-1)-2]y = e^{2z} + e^{-z}$$

$$\text{A.E. is } m^2 - m - 2 = 0 \Rightarrow m = 2, -1$$

$$\therefore \text{C.F. is } y = C_1 e^{2z} + C_2 e^{-z}$$

$$\text{P.I.} = \frac{1}{(D-2)(D+1)} e^{2z} + \frac{1}{(D-2)(D+1)} e^{-z} = \frac{1}{(D-2)(2+1)} e^{2z} + \frac{1}{(-1-2)(D+1)} e^{-z}$$

$$= \frac{1}{3(D-2)} e^{2z} - \frac{1}{3(D+1)} e^{-z}$$

$$= e^{2z} \frac{1}{3(D+2-2)} (1) - e^{-z} \frac{1}{3(D-1+1)} \quad (1)$$

$$= \frac{1}{3} e^{2z} \cdot \frac{1}{D} (1) - \frac{1}{3} e^{-z} \frac{1}{D} (1)$$

$$= \frac{1}{3} z e^{2z} - \frac{1}{3} z e^{-z}$$

Complete solution C.F + P.I.

$$\text{i.e. } y = C_1 e^{2z} + C_2 e^{-z} + \frac{1}{3} z (e^{2z} - e^{-z})$$

$$y = C_1 e^{2\log x} + C_2 e^{-\log x} + \frac{1}{3} (e^{2\log x} - e^{-\log x}) \log x$$

$$y = C_1 x^2 + C_2 \frac{1}{x} + \frac{1}{3} (x^2 - \frac{1}{x}) \log x$$

2.8.5 Solve $x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} + 2y = \left(10x + \frac{10}{x}\right)$

Sol. Taking $x = e^z$ the equation reduces to

$$[D(D-1)(D-2) + 2D(D-1) + 2]y = 10e^z + 10e^{-z}$$

$$(D^3 - D^2 + 2)y = 10(e^z + e^{-z})$$

$$\text{A.E. is } m^3 - m^2 + 2 = 0$$

$$\Rightarrow (m+1)(m^2 - 2m + 2) = 0$$

$$(\text{i.e.}) m = -1, m = 1 \pm i$$

$$\therefore \text{C.F. is } y = C_1 e^{-z} + e^z (C_2 \cos z + C_3 \sin z)$$

$$\begin{aligned} \text{P.I.} &= \frac{10}{D^3 - D^2 + 2} e^z + \frac{10}{D^3 - D^2 + 2} e^{-z} \\ &= \frac{10e^z}{1 - 1 + 2} + \frac{10e^{-z}}{(D+1)(D^2 - 2D + 2)} \\ &= 5e^z + \frac{10}{5(D+1)} e^{-z} = 5e^z + 2e^{-z} \cdot \frac{1}{(D-1)+1} \\ &= 5e^z + 2e^{-z} \cdot \frac{1}{D} (1) = 5e^z + 2e^{-z} \cdot z \end{aligned}$$

Complete solution = C.F. + P.I.

$$\begin{aligned} &= C_1 e^{-z} + e^z (C_2 \cos z + C_3 \sin z) + 5e^z + 2e^{-z} \cdot z \frac{\log x}{x} \\ &= \frac{C_1}{x} + x(C_2 \cos(\log x) + C_3 \sin(\log x)) + 5x + 2 \end{aligned}$$

2.8.6 Solve $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = \log x \sin(\log x)$

Sol. Writing $z = \log x$ the equation reduces to

$$[D(D-1) + D+1]y = z \sin z$$

$$(D^2 + 1)y = z \sin z$$

$$\text{A.E. is } m^2 + 1 = 0 \Rightarrow m = \pm i$$

$$\therefore \text{C.F. is } y = C_1 \cos z + C_2 \sin z$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 + 1} z \sin z \\ &= \text{I.P. of } \frac{1}{D^2 + 1} z e^{iz} = \text{I.P. of } e^{iz} \frac{1}{(D+i)^2 + 1} z \\ &= \text{I.P. of } e^{iz} \frac{1}{D^2 - 1 + 2Di + 1} z \end{aligned}$$

$$\begin{aligned}
&= \text{I.P. of } e^{iz} \frac{1}{D^2 + 2Di} z \\
&= \text{I.P. of } e^{iz} \frac{1}{2Di} \left(1 + \frac{D}{2i}\right)^{-1} z \\
&= \text{I.P. of } e^{iz} \frac{1}{2Di} \left(1 - \frac{D}{2i}\right) z \\
&= \text{I.P. of } e^{iz} \frac{1}{2Di} \left(z - \frac{1}{2i}\right) \\
&= \text{I.P. of } \frac{-e^{iz} i}{2} \left(\frac{z^2}{2} - \frac{1}{2i} z\right) \\
&= \text{Imaginary part of } \left[\frac{-i}{2} (\cos z + i \sin z) \left(\frac{z^2}{2} + \frac{iz}{2}\right) \right] \\
&= -\frac{z^2}{2} \cos z - \frac{1}{4} z \sin z = -\frac{z}{4} (\sin z + 2z \cos z)
\end{aligned}$$

Hence complete solution is

$$\begin{aligned}
y &= C_1 \cos z + C_2 \sin z - \frac{z}{4} (\sin z + 2z \cos z) \\
&= C_1 \cos(\log x) + C_2 \sin(\log x) - \frac{\log x}{4} [\sin(\log x) + 2(\log x) \cos(\log x)]
\end{aligned}$$

Example 2.8.7

Solve $(x^2 D^2 - xD + 4)y = \cos(\log x) + x \sin(\log x)$

Sol. Taking $x = e^z$ or $z = \log x$ the equation reduces to

$$\therefore [D(D-1) - D + 4]y = \cos z + e^z \sin z.$$

$$\text{A.E. is } m^2 - 2m + 4 = 0 \Rightarrow m = 1 \pm \sqrt{3}i$$

$$\therefore \text{C.F.} = e^z (C_1 \cos \sqrt{3}z + C_2 \sin \sqrt{3}z)$$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 4} \cos z + \frac{1}{D^2 - 2D + 4} e^z \sin z$$

$$= \frac{1}{3-2D} \cdot \cos z + \frac{e^z}{(D+1)^2 - 2(D+1) + 4} \sin z$$

$$\frac{(3+2D)}{9+4} \cos z + \frac{e^z \sin z}{3} = \frac{3 \cos z - 2 \sin z}{13} + \frac{e^z \sin z}{3}$$

The complete solution is $y = C.F. + P.I.$

$$y = e^z (C_1 \cos \sqrt{3}z + C_2 \sin \sqrt{3}z) + \frac{1}{13} (3 \cos z - 2 \sin z) + \frac{e^z \sin z}{3}$$

$$= x [C_1 \cos \sqrt{3} (\log x) + C_2 \sin \sqrt{3} (\log x)] + \frac{1}{13} [3 \cos (\log x) - 2 \sin (\log x)] + \frac{x \sin (\log x)}{3}$$

Example 2.8.8

Legendre's Linear equation :

An equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + K_1(ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_n y = X$$

Where K 's are constants and X is either a constant or a function of x is called Legendre's linear equation. To solve such an equation we transform it into linear equation with constant coefficients by the substitution.

$$ax + b = e^z \text{ or } z = \log (ax + b)$$

Now $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{a}{ax+b} \cdot \frac{dy}{dz}$

$$(ax+b)^2 \frac{dy}{dx} = a \frac{dy}{dz} = aDy \quad \text{where } D = \frac{d}{dz}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{a}{ax+b} \frac{dy}{dz} \right) = \frac{a^2}{(ax+b)^2} \cdot \frac{dy}{dz} + \frac{a^2}{(ax+b)^2} \cdot \frac{d^2 y}{dz^2}$$

$$(ax+b)^2 \frac{d^2 y}{dx^2} = a^2 \left(\frac{d^2 y}{dz^2} - \frac{dy}{dz} \right) = a^2 (D(D-1)y)$$

$$\text{Similarly } (ax+b)^3 \frac{d^3 y}{dx^3} = a^3 D(D-1)(D-2)y$$

by substituting these values the given equation reduces to linear equation with constant coefficients.

Solved Examples

Example 2.8.9

$$\text{Solve } (1+x)^2 \frac{d^2 y}{dx^2} + (1+x) \frac{dy}{dx} + y = \sin 2[\log(1+x)] \quad \dots (1)$$

Sol. Write $1+x = e^z \Rightarrow \log(1+x) = z$

$$(1+x) \frac{dy}{dx} = Dy, \quad (1+x)^2 \frac{d^2 y}{dx^2} = D(D-1)y \quad \text{where } D = \frac{d}{dz}$$

Substituting these values (1) reduces to

$$(D^2 - D + D + 1)y = \sin 2z; \text{ (i.e.) } (D^2 + 1)y = \sin 2z$$

$$\text{A.E. is } m^2 + 1 = 0 \Rightarrow m = \pm i$$

$$\text{C.F.} = C_1 \cos z + C_2 \sin z$$

$$\text{P.I.} = \frac{1}{D^2 + 1} \sin 2z = \frac{\sin 2z}{-4 + 1} = \frac{\sin 2z}{-3}$$

$$\text{Complete solution is } = \text{C.F.} + \text{P.I.} = C_1 \cos z + C_2 \sin z - \frac{\sin 2z}{3}$$

$$= C_1 \cos[\log(1+x)] + C_2 \sin[\log(1+x)] - \frac{\sin 2(\log(1+x))}{3}$$

Example 2.8.10

$$\text{Solve } (2x+3)^2 \frac{d^2 y}{dx^2} + 6(2x+3) \frac{dy}{dx} + 6y = \log(2x+3)$$

Sol. Writing $2x+3 = e^z$ or $z = \log(2x+3)$

the given equation reduces to $[2^2.D(D-1) + 6.2D + 6]y = z$

$$\text{i.e. } (4D^2 + 8D + 6)y = z \quad \text{where } D = \frac{d}{dz}$$

$$\text{A.E. is } 4m^2 + 8m + 6 = 0 \Rightarrow 2m^2 + 4m + 3 = 0$$

$$\Rightarrow m = \frac{-2 \pm \sqrt{2}i}{2} = -1 \pm \frac{1}{\sqrt{2}} i$$

$$\text{C.F. is } y = e^{-z} \left[A \cos \frac{1}{\sqrt{2}} z + B \sin \frac{1}{\sqrt{2}} z \right]$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{4D^2 + 8D + 6} z = \frac{1}{6} \frac{1}{1 + \frac{4D}{3} + \frac{2D^2}{3}} \cdot z = \frac{1}{6} \left[1 + \frac{4D}{3} + \frac{2D^2}{3} \right]^{-1} z \\ &= \frac{1}{6} \left[1 - \frac{4D}{3} \right] z = \frac{1}{6} \left[z - \frac{4}{3} \right] = \frac{z}{6} - \frac{2}{9} \end{aligned}$$

Complete solution is $y = \text{C.F.} + \text{P.I.}$

$$\begin{aligned} &= e^{-z} \left[A \cos \frac{1}{\sqrt{2}} z + B \sin \frac{1}{\sqrt{2}} z \right] + \frac{z}{6} - \frac{2}{9} \\ &= \frac{1}{2x+3} \left[A \cos \left(\frac{1}{\sqrt{2}} \log(2x+3) \right) + B \sin \left(\frac{1}{\sqrt{2}} \log(2x+3) \right) \right] + \frac{1}{6} \log(2x+3) - \frac{2}{9} \end{aligned}$$

Exercise 2(h)

Solve the following differential equations

1. $(x^2 D^2 - 3xD + 4)y = 2x^2$

Ans. $(C_1 + C_2 \log x)x^2 + x^2 (\log x)^2$

2. $(x^2 D^2 - 3xD + 5)y = \sin(\log(x))$

Ans. $x^2(C_1 \cos \log x + C_2 \sin \log x) + (\cos \log x + \sin \log x)$

3. $(x^2 D^2 - xD + 2)y = x \log x$

Ans. $x(C_1 \cos \log x + C_2 \sin \log x) + x \log x$

4. $(x^2 D^2 - 3xD + 5)y = x^2 \sin \log x$

Ans. $x^2(C_1 \cos \log x + C_2 \sin \log x) - \frac{1}{2} x^2 \log x \cos \log x$

5. $(x^2 D^2 - 3xD + 1)y = \frac{(\sin \log x) + 1}{x}$

Ans. $x^2 [C_1 \cosh(\sqrt{3} \log x) + C_2 \sinh(\sqrt{3} \log x)] + \frac{1}{6x} + \frac{1}{61x} [5 \sin(\log x) - 6 \cos(\log x)]$

6. $(x^3 D^3 + x^2 D^2)y = x^2 + x + 1$

Ans. $C_1 + (C_2 + C_3 \log x) + x + \frac{(\log x)^2}{2} \cdot x + \frac{x^2}{2}$

7. $(x^4 D^4 + 2x^3 D^3 + x^2 D^2 - xD + 1)y = x + \log x$

Ans. $y = x[C_1 + C_2 \log x + C_3 (\log x)^2 + C_4 (\log x)^3] + \frac{(\log x)^4}{4!} \cdot x^4 + \log x + 4$

8. $(x^2 D^2 + x D + 1)y = (\log x)^2 + \sin(\log x)$

Ans. $C_1 \cos \log x + C_2 \sin \log x + (\log x)^2 - x[2 \cos \log x - \sin \log x]$

9. $(2x-1)^2 \frac{d^2 y}{dx^2} + (2x-1) \frac{dy}{dx} + 2y = 0$

Ans. $y = C_1(2x-1) + \frac{C_2}{\sqrt{2x+1}}$

10. $(5+2x)^2 \frac{d^2 y}{dx^2} - 6(5+2x) \frac{dy}{dx} + 8y = 6x$

Ans. $y = C_1(5+2x)^{2+\sqrt{2}} + C_2(5+2x)^{2-\sqrt{2}} - \frac{3x}{2} - \frac{45}{8}$

11. $(2x+1)^2 \frac{d^2 y}{dx^2} - 6(2x+1) \frac{dy}{dx} + 16y = 8(1+2x)^2$

Ans. $y = C_1 + C_2 \log(2x+1) + [\log(2x+1)^2](2x+1)^2$

12. $(2x+3) \frac{d^2 y}{dx^2} - (2x+3) \frac{dy}{dx} - 12y = 6x$

Ans. $C_1(2x+3)^2 + C_2(2x+3)^{-\frac{1}{2}} - 2x + \frac{3}{2}$

2.9.1 Linear Differential equations of second order

The general form is $\frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Qy = R$

where P, Q, R are functions of x.

Method of variation of Parameters

Let the C.F. of the equation

$$\frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Qy = R$$

be $y = Au + Bv$

where A and B are constants and u, v are two independent solutions of

$$\frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Qy = R \quad \dots (1)$$

$$\text{be } y = Au + Bv \quad \dots (2)$$

where A and B are constants and u, v are two independent solutions of

$$\frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Qy = 0 \quad \dots (3)$$

such that $u_2 + Pu_1 + Qu = 0$ and $v_2 + Pv_1 + Qv = 0$

$$\text{where } u_2 = \frac{d^2 u}{dx^2}, u_1 = \frac{du}{dx} \text{ etc.}$$

In order to obtain the solution of the equation (1) the arbitrary constants A and B are treated as arbitrary functions of x and are chosen in such a way that

$$y = A(x)u + B(x)v \text{ satisfies (1)} \quad \dots (4)$$

Differentiation of (2) gives

$$= Au_1 + uA_1 + Bv_1 + vB_1 \text{ (suffixes indicating the order of the derivative)}$$

$$\text{Now we chose A and B such that } A_1 u + B_1 v = 0 \quad \dots (5)$$

$$\text{So, } \frac{dy}{dx} = Au_1 + Bv_1$$

Again differentiating

$$\frac{d^2 y}{dx^2} = (Au_2 + u_1 A_1) + (Bv_2 + v_1 B_1) \quad \dots (6)$$

substituting (4), (5) and (6) in (1) we get

$$[Au_2 + Bv_2 + A_1 u_1 + B_1 v_1] + P[Au_1 + Bv_1] + Q(Au + Bv) = R$$

$$\text{(or) } A(u_2 + Pu_1 + Qu) + B(v_2 + Pv_1 + Qv) + A_1 u_1 + B_1 v_1 = R$$

$$\therefore A_1 u_1 + B_1 v_1 = R \text{ (} u \text{ and } v \text{ are solutions of (3))} \quad \dots (7)$$

Solving (5) and (7) we get

$$\frac{dA}{dx} = A_1 = \frac{-vR}{uv_1 - u_1 v} \text{ and } \frac{dB}{dx} = B_1 = \frac{uR}{uv_1 - u_1 v}$$

integrating we obtain

$$A(x) = \int \frac{-vR dx}{(uv_1 - u_1 v)} + C_1; B(x) = \int \frac{uR}{uv_1 - u_1 v} + C_2 \quad \dots (8)$$

where C_1 and C_2 are arbitrary constants

Substituting (8) in (4) we get the complete general solution of the equation (1)

Working Rule: First find two independent solutions u and v of (3). Then C.F. is given by $y = Au + Bv$ where A and B are arbitrary constants. Treating A and B as functions of x , we have the solution of (1) as $y = A_1(x)u + B_1(x)v$ where $A(x)$ and $B(x)$ are given by (8)

Solved Examples

Example 2.9.2

Solve by the method of variation of parameters $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

Sol. A.E. is $m^2 - 1 = 0 \Rightarrow m = \pm 1$

C.F. is $y = Ae^x + Be^{-x}$

Assuming A and B as functions of x such that the given equation is satisfied by

$y = Ae^x + Be^{-x}$ we have

$$\frac{dy}{dx} = Ae^x - Be^{-x} + e^x \frac{dA}{dx} + e^{-x} \frac{dB}{dx} = Ae^x - Be^{-x} \quad \dots (1)$$

$$\text{Choosing } A \text{ and } B \text{ so that } e^x \frac{dA}{dx} + e^{-x} \frac{dB}{dx} = 0 \quad \dots (2)$$

$$\text{also } \frac{d^2y}{dx^2} = Ae^x + Be^{-x} + e^x \frac{dA}{dx} + e^{-x} \frac{dB}{dx} = 0 \quad \dots (3)$$

Substituting (1) to (3) in the given equation, we get

$$e^x \frac{dA}{dx} + e^{-x} \frac{dB}{dx} = \frac{2}{1+e^x} \quad \dots (4)$$

$$\text{Solving (2) and (4) we get } \frac{dA}{dx} = \frac{e^{-x}}{e^x + 1} \quad \dots (5)$$

$$\text{and } \frac{dB}{dx} = -\frac{e^x}{e^x + 1} \quad \dots (6)$$

$$\begin{aligned} A &= \int \frac{e^{-x}}{e^x + 1} dx = \int \frac{1}{1 + e^{2x}} dx = \int \frac{e^x + 1 + e^{-x}}{e^x + 1} - 1 dx = \int \frac{e^x}{e^x + 1} + \frac{e^{-x}(e^x + 1)}{e^x + 1} - 1 dx \\ &= \log(e^x + 1) - e^{-x} - x \end{aligned}$$

Integrating (5) and (6) $A = -e^{-x} + \log(1+e^x) - x + C_1$, $B = -\log(1+e^x) + C_2$

Hence the solution is

$$y = [-e^{-x} + \log(e^x + 1) - x + C_1] e^x + [C_2 - \log(1 + e^x)] e^{-x}$$

$$(\text{or}) \quad y = C_1 e^x + C_2 e^{-x} - 1 + e^x [\log(e^x + 1) - x] - e^{-x} \log(e^x + 1)$$

Remarks : If (i) $1 + P + Q = 0$ then $y = e^x$ (ii) $1 - P + Q = 0$ then $y = e^{-x}$

and (iii) $P + Qx = 0$ then $y = x$ are solutions of $D^2y + PDy + Qy = 0$

Example 2.9.3

Solve by the method of variation of parameters

$$\frac{d^3y}{dx^3} + (1 - \cot x) \frac{dy}{dx} - \cot x = \sin^2 x \quad \dots (1)$$

$$\text{Sol:} \quad \frac{d^3y}{dx^3} + (1 - \cot x) \frac{dy}{dx} - \cot x = 0 \quad \dots (2)$$

C.F.P = $1 - \cot x$ $Q = -\cot x$ comparing with original eqn.

$$\therefore 1 - P + Q = 1 - 1 + \cot x - \cot x = 0$$

showing that $v = e^{-x}$ is a solution of (2)

To find another independent solution of (2)

let $u = e^{-x} \cdot w$

By this substitution, the equation (2) reduces to

$$\frac{d^2w}{dx^2} + \frac{dw}{dx} \left(1 - \cot x - \frac{2}{e^{-x}} e^{-x}\right) = 0$$

$$\text{or} \quad \frac{d^2w}{dx^2} = (1 + \cot x) \frac{dw}{dx} \Rightarrow \frac{\frac{d}{dx} \left(\frac{dw}{dx} \right)}{\frac{dw}{dx}} = 1 + \cot x$$

$$\log \frac{dw}{dx} = x + \log \sin x \quad \text{or} \quad \frac{dw}{dx} = e^x \sin x$$

$$\therefore w = \int e^x \sin x \, dx = -\frac{e^x}{2} (\cos x - \sin x)$$

$$u = e^{-x} \left[-\frac{e^x}{2} (\cos x - \sin x) \right] = -\frac{1}{2} (\cos x - \sin x)$$

The second independent solution can be taken as $\cos x - \sin x$

Thus solution of (2) is

$$y = A (\cos x - \sin x) + B e^{-x}$$

For finding the solution of (1) we treat A and B as functions of x

i.e. $y = A(x) (\cos x - \sin x) + B(x) e^{-x}$ is the solution of (1) where A(x) and B(x) are obtained by solving

$$(\cos x - \sin x) A' + e^{-x} \cdot B' = 0 \quad \dots (4)$$

$$\text{and } -(\sin x + \cos x) A' - e^{-x} B' = \sin^2 x \quad \dots (5)$$

$$\text{solving (4) and (5) } \frac{dA}{dx} = -\frac{1}{2} \sin x$$

$$\text{and } \frac{dB}{dx} = e^x \frac{\sin 2x}{4} - e^x \frac{(1 - \cos 2x)}{4}$$

integrating we get

$$A = -\frac{1}{2} \int \sin x dx + C_1 = \frac{\cos x}{2} + C_1 \quad \dots (6)$$

$$\text{and } B = \frac{-e^x}{20} (2\cos 2x - \sin 2x) - \frac{e^x}{4} + \frac{-e^x}{20} (\cos 2x + 2\sin 2x) + C_2 \quad \dots (7)$$

Hence the complete solution is

$$y = C_1 (\cos x - \sin x) + C_2 e^{-x} - \frac{1}{10} (\sin 2x - 2\cos 2x)$$

Example 2.9.4

Solve by the method of variation of parameters

$$(1-x) \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = (1-x)^2$$

The given equation can be written as

$$\frac{d^2 y}{dx^2} + \frac{x}{1-x} \frac{dy}{dx} - \frac{1}{1-x} y = 1-x \quad \dots (1)$$

$$P = \frac{x}{1-x}; Q = \frac{1}{1-x} \text{ and } X = 1-x$$

Clearly $P + Qx = 0$

$$\text{Hence } y = x \text{ is a solution of } \frac{d^2 y}{dx^2} + \frac{x}{1-x} \frac{dy}{dx} - \frac{1}{1-x} y = 0 \quad \dots (2)$$

To obtain the second independent solution of (2)

take $y = vx$

Then (2) reduces to $\frac{d^2v}{dx^2} + \frac{dv}{dx} \left(\frac{x}{1-x} + \frac{2}{x} \right) = 0$

$$\frac{d}{dx} \left[\frac{dv}{dx} \right] + \frac{dv}{dx} \left[\frac{x}{1-x} + \frac{2}{x} \right] = 0$$

$$\frac{\frac{d}{dx} \frac{dv}{dx}}{\frac{dv}{dx}} = - \frac{x}{1-x} - \frac{2}{x} = + \left(\frac{1-x-1}{1-x} \right) - \frac{2}{x} = +1 - \frac{1}{1-x} - \frac{2}{x}$$

$$\log \left(\frac{dv}{dx} \right) = x + \log(x-1) - \log x^2$$

$$\frac{dv}{dx} = \frac{x-1}{x^2} \cdot e^x = e^x \left[\frac{1}{x} + \frac{(-1)}{x^2} \right]$$

$$\therefore v = e^x \cdot \frac{1}{x}$$

The second independent solution is $xv = e^x$

solution of equation (2) is $y = Ae^x + Bx$ (3)

To find the solution of (1) treat A and B as functions of x such that

$$e^x \frac{dA}{dx} + x \frac{dB}{dx} = 0 \quad \text{.... (4)}$$

$$\text{and, } e^x \frac{dA}{dx} + x \frac{dB}{dx} = 1 - x \quad \text{.... (5)}$$

Solving (4) and (5) we get $\frac{dA}{dx} = -xe^{-x}$ and $\frac{dB}{dx} = 1$

integrating, $A = - \int xe^{-x} dx + C_1 = e^{-x}(1+x) + C_1$; $B = x + C_2$

Hence the complete solution is $y = Ae^x + Bx$

$$= [C_1 + e^{-x}(1+x)]e^x + (x+C_2)x = C_1e^x + C_2x + x^2 + (1+x)$$

Exercise (i)

Solve the following by the method of variation of parameters

1. $\frac{d^2y}{dx^2} + a^2y = \sec ax$

Ans. $y = C_1 \cos ax + C_2 \sin ax + \frac{x}{a} \sin ax + \frac{1}{a} \cos ax \log(\cos ax)$

2. $\frac{d^2y}{dx^2} + y = \tan x$

Ans. $y = -[\log(\sec x + \tan x)] \cos x + C_1 \cos x + C_2 \sin x$

3. $\frac{d^2y}{dx^2} + 4y = \operatorname{cosec} 2x$

Ans. $y = C_1 \cos 2x + C_2 \sin 2x + x \cos 2x + \sin 2x \log(\sin 2x)$

4. $x^2 \frac{d^2y}{dx^2} - 2x(1+x) \frac{dy}{dx} + 2(1+x)y = x^3$

Ans. $y = C_1 x e_{2x} + C_2 x - \frac{x^2}{4} - \frac{x}{4}$

3

Mean Value Theorems and Functions of Several Variables

3.1.0 This chapter deals with

- (i) Rolle's theorem
- (ii) Lagrange's mean value theorem also called as first mean value theorem.
- (iii) Cauchy's Mean Value theorem
- (iv) Higher Mean Value theorems.
- (v) Curvature (vi) Centre of curvature .
- (vii) Evolutes (viii) Envelopes

3.1.1 Rolle's Theorem

If $f(x)$ is (i) continuous in $[a, b]$, (ii) differentiable in (a, b)

and (iii) $f(a) = f(b)$,

then there exists $a'c' \in (a, b) \ni f'(c) = 0$

Proof: Suppose

- (i) $f(x)$ is a constant function throughout the interval $[a, b]$,
then $f'(x) = 0 \forall x \in (a, b)$ Hence theorem is proved ...
- (ii) $f(x)$ is not a constant function in $[a, b]$. As $f(x)$ is continuous in $[a, b]$,
there exists a maximum value say, at ' c ' ($a \leq c \leq b$) and a minimum value ,
say at ' d ' ($a \leq d \leq b$) for $f(x)$ in (a, b) .

$f(c) \neq f(d)$ and at least one of them is different from $f(a) = f(b)$

Suppose $f(c) \neq f(a)$ and ' $c+h$ ' be a point in the neighborhood of ' c ',

$$\text{then } \frac{f(c+h) - f(c)}{h} \leq 0 \text{ when } h > 0. \quad \dots(1)$$

$$\text{and } \frac{f(c+h) - f(c)}{h} \geq 0 \text{ when } h < 0. \quad \dots(2)$$

Further $f(x)$ is differentiable in (a, b) .

$$\therefore f'(c) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} \quad \text{As } h \rightarrow 0 \text{ we get from (1) and (2) that}$$

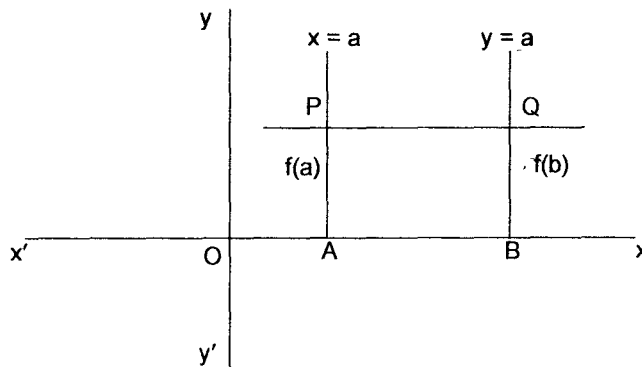
$$\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} \leq 0 \text{ and } \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} \geq 0 \text{ respectively.}$$

$$\text{i.e., } f'(c) \leq 0 \text{ and } f'(c) \geq 0 \text{ simultaneously} \Rightarrow f'(c) = 0$$

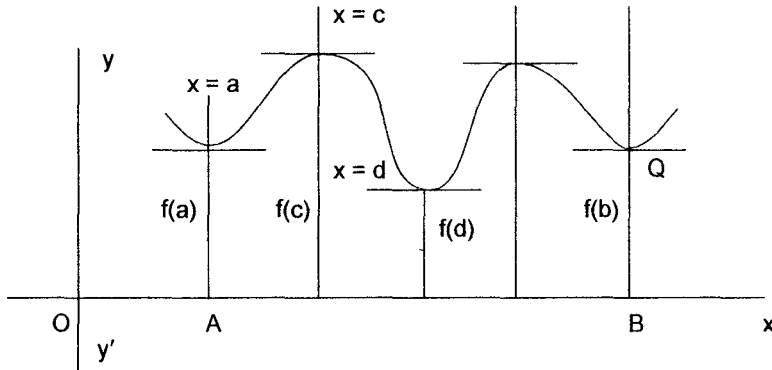
Similarly the theorem can be proved when $f(d) \neq f(a)$

3.1.2 Geometrical interpretation of Rolle's theorem

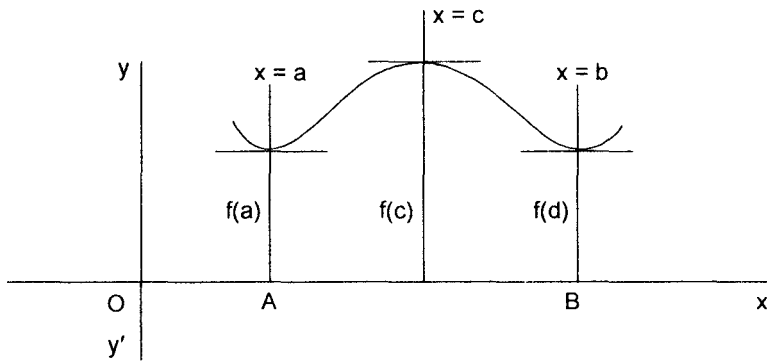
Let P and Q be two points on the curve $y=f(x)$. $AP=BQ$ ordinates $f(a)=f(b)$ and the curve is continuous from P to Q. It can be shown that there is at least one point on the curve $y=f(x)$ between $x=a$ and $x=b$ at which the tangent to the curve is parallel to x-axis.



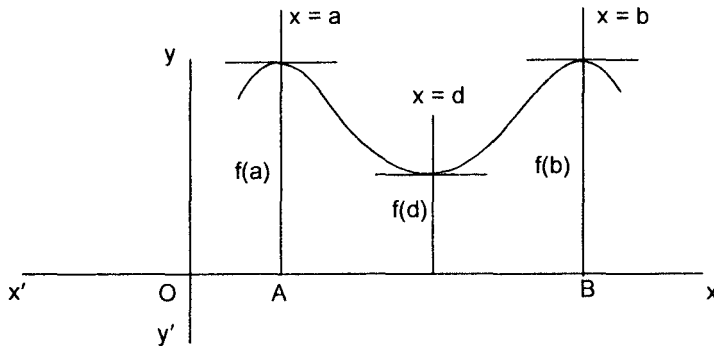
$f(x)$ is a constant function



$f(c)$ and $f(d)$ are both different from $f(a) = f(b)$



$f(c) \neq f(a)$ and $f(d) = f(a) = f(b)$



$f(d) \neq f(a)$ and $f(c) = f(a) = f(b)$

3.1.3 Verify Rolle's theorem for $f(x) = \log \left\{ \frac{x^2 + ab}{x(a+b)} \right\}$ in (a, b)

Solution:

Consider the function $f(x) = \log \left\{ \frac{x^2 + ab}{x(a+b)} \right\}$ in the interval (a, b) .

$$\begin{aligned} \lim_{h \rightarrow 0 \pm} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0 \pm} \frac{1}{h} \left[\log \frac{(x+h)^2 + ab}{(a+b)(x+h)} - \log \frac{x^2 + ab}{(a+b)x} \right] \\ &= \lim_{h \rightarrow 0 \pm} \frac{1}{h} \left[\log \frac{(x^2 + ab) + 2xh + h^2}{(x^2 + ab)} - \log \frac{x+h}{x} \right] \\ &= \lim_{h \rightarrow 0 \pm} \frac{1}{h} \left[\log \left\{ 1 + \frac{2xh + h^2}{x^2 + ab} \right\} - \log \left\{ 1 + \frac{h}{x} \right\} \right] = \lim_{h \rightarrow 0 \pm} \left[\frac{2x}{x^2 + ab} - \frac{1}{x} + 0(h) \right] \end{aligned}$$

where $0(h)$ indicates terms of order h and higher powers of h .

$$f'(x) = \frac{2x}{x^2 + ab} - \frac{1}{x}$$

which indicates that $f(x)$ is differentiable in (a, b) and hence continuous also.

$$\text{Further } f(a) = \log \frac{(a^2 + ab)}{a(a+b)} = \log 1 = 0$$

$$f(b) = \log \frac{(b^2 + ab)}{b(a+b)} = 0. \quad \text{Thus } f(a) = 0 = f(b)$$

All the conditions of Rolle's theorem are satisfied. Hence $\exists c (a < c < b)$ such that $f'(c) = 0$

$$f'(c) = 0 \text{ gives } \frac{2c}{c^2 + ab} - \frac{1}{c} = 0 \Rightarrow c^2 = ab \text{ or } c = \pm ab$$

Clearly $c = +ab$ is the G.M of a and b and so $\in (a, 1)$

The theorem is thus verified.

3.1.4 Verify Rolle's theorem for the function $f(x) = \frac{\sin x}{e^x}$ in $(0, \pi)$

Solution:

$$f(x) = \frac{\sin x}{e^x} \text{ in } (0, \pi)$$

$$\begin{aligned} \lim_{h \rightarrow 0 \pm} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0 \pm} \frac{1}{h} \left[\frac{\sin(x+h)}{e^{x+h}} - \frac{\sin x}{e^x} \right] \\ &= \lim_{h \rightarrow 0 \pm} \frac{e^x}{h} \left[\frac{\sin(x+h) - e^h \sin x}{e^{x+h} \cdot e^x} \right] \\ &= \lim_{h \rightarrow 0 \pm} \frac{1}{h} \left[\frac{\sin(x+h) - \sin x \left\{ 1 + \frac{h}{1!} + \frac{h^2}{2!} + \dots \right\}}{e^{x+h} \cdot e^x} \right] \\ &= \lim_{h \rightarrow 0 \pm} \frac{1}{h} \left[\frac{2 \cos \left(x + \frac{h}{2} \right) \cdot \sin \left(\frac{h}{2} \right) - h \{ \sin x + 0(h) \}}{e^{x+h}} \right] \\ &= \lim_{h \rightarrow 0 \pm} \frac{1}{h} \left[2 \cos \left(x + \frac{h}{2} \right) \cdot \frac{\sin \left(\frac{h}{2} \right)}{\left(\frac{h}{2} \right)} - \{ \sin x + 0(h) \} \right] \end{aligned}$$

$$f'(x) = \frac{\cos x - \sin x}{e^x}$$

Thus $f(x)$ is differentiable in $(0, \pi)$ and hence continuous there.

Further $f(0) = 0 = f(\pi)$. All the conditions of Rolle's theorem are satisfied.

$\therefore \exists c (0 < c < \pi)$ such that $f'(c) = 0$

$$\text{(i.e.,)} \quad \frac{\cos c - \sin c}{e^c} = 0 \Rightarrow c = \frac{\pi}{4} \text{ (principal value)}$$

and clearly $c = \frac{\pi}{4} \in (0, \pi)$. Hence the theorem is verified.

3.1.5 Example

Verify Rolle's theorem for the function $f(x) = |x|$ in $(-1, 1)$.

Solution

Here $f(x) = -x$ for $-1 < x < 0$

$$= 0 \text{ for } x = 0$$

$$= x \text{ for } 0 < x < 1$$

$$f(-1) = 1, f(1) = 1;$$

Hence $f(-1) = f(1)$

$$f'(x) = -1 \quad \text{for} \quad -1 \leq x \leq 0$$

$$f'(x) = 1 \quad \text{for} \quad 0 \leq x \leq 1$$

$\therefore f'(x)$ does not exist at $x = 0$ and hence $f(x)$ is not differentiable in $(-1, 1)$

$\therefore$ Rolle's theorem is not applicable to the function $f(x) = |x|$ in $(-1, 1)$.

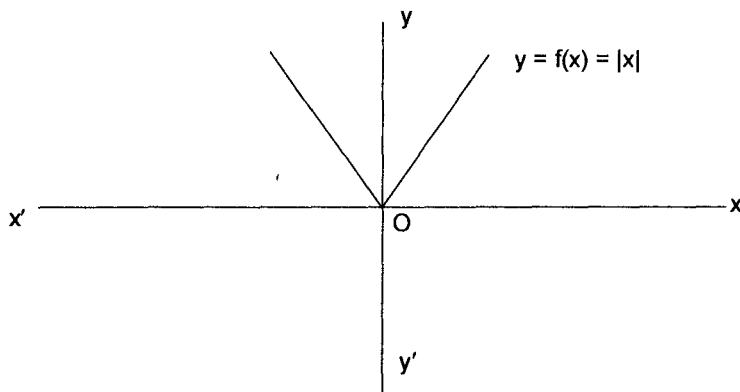


Fig. 3.1 (The curve is not smooth at $x = 0$).

Exercise – 3(A)

I. Verify Rolle's theorem for the following functions :

$$1. f(x) = x(x+3)e^{-x/2} \quad \text{in} \quad (-3, 0) \quad [\text{Ans : } c = -2]$$

$$2. f(x) = \sin x \quad \text{in} \quad (0, \pi) \quad [\text{Ans : } c = \pi/2]$$

$$3. f(x) = e^x (\sin x - \cos x) \quad \text{in} \quad \left(\frac{\pi}{4}, \frac{5\pi}{4} \right) \quad [\text{Ans : } c = \pi]$$

$$4. f(x) = \log \left(\frac{x^2 + 6}{5x} \right) \quad \text{in} \quad (2, 3) \quad [\text{Ans : } c = \sqrt{6}]$$

$$5. f(x) = (x-a)^m (x-b)^n \text{ in } (a, b), m > 0, n > 0 \quad [\text{Ans : } c = \frac{(mb+na)}{(m+n)}]$$

$$6. f(x) = x^2 - 5x + 7 \text{ in } (2, 3) \quad [\text{Ans : } c = 5/2]$$

3.2.1 Lagrange's Mean Value Theorem (First mean Value Theorem):

If $f(x)$ is (i) continuous in $[a, b]$ and (ii) differentiable in (a, b) then $\exists$ at least one

value ' c ' in $(a, b) \ni f'(c) = \frac{f(b) - f(a)}{b - a}$

Proof: Define a new function $\phi x = f(x) + Ax$ (1)

where A is a constant $\ni \phi(a) = \phi(b)$

i.e., $f(a) + A(a) = f(b) + Ab$

i.e., $A = - \left[\frac{f(b) - f(a)}{b - a} \right]$ (2)

$f(x)$ and Ax are continuous in $[a, b]$

Hence $\phi(x)$ is continuous in $[a, b]$ (3)

$f(x)$ and Ax are differential in (a, b) ,

Hence $\phi(x)$ is differential in (a, b) (4)

and $f(a) = \phi(b)$ (5)

(3), (4), (5) show that $\phi(x)$ satisfies all the conditions of Rolle's theorem.

$\therefore \exists$ atleast one value c in $(a, b) \ni \phi' = 0$

$$\phi'(x) = f'(x) + A$$

$$\phi'(c) = f'(c) + A = 0$$

$$\Rightarrow A = -f'(c) \quad \dots(6)$$

From (2) and (6) it follows that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

3.2.2 Geometrical Interpretation of Langrange's Mean Value Theorem

P and Q are two points on the continuous curve $y = f(x)$ corresponding to $x = a$ and $x = b$ respectively.

$\therefore P[a, f(a)], Q[b, f(b)]$ are two points on the curve.

Slope on the line joining the points P and Q is $\frac{f(b) - f(a)}{b - a}$, R is a point on the curve between P and Q corresponding to $x = c$, so that $f'(c)$ is the slope of the tangent line at R $[c, f(c)]$.

$f'(c) = \frac{f(b) - f(a)}{b - a}$ means that the tangent at R is parallel to the chord P Q.

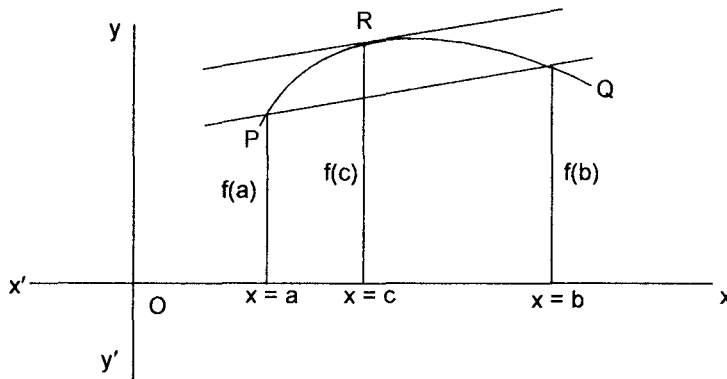


Fig. 3.2

Hence this theorem tells that there is at least one point R on the curve PQ where the tangent to the curve is parallel to the chord PQ.

3.2.3 Example

Verify Lagrange's theorem for the function $f(x) = (x-1)(x-2)(x-3)$ in $(0, 4)$.

Solution

$$f(x) = (x-1)(x-2)(x-3) \text{ and } a = 0, b = 4$$

$f(x) = x^3 - 6x^2 + 11x - 6$ is an algebraic polynomial and $(0, 4)$ is a finite interval.

Hence $f(x)$ is differentiable in $(0, 4)$ and is continuous in $[0, 4]$ showing that the conditions of Lagrange's Mean Value theorem are satisfied.

$\therefore \exists$ atleast one value 'c' in $(0, 4)$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \dots(1)$$

$$f(0) = -6, \quad f(4) = 6$$

$$f'(x) = 3x^2 - 12x + 11$$

$$f'(c) = 3c^2 - 12c + 11$$

$$\text{From (1)} \quad 3c^2 - 12c + 11 = \frac{6 - (-6)}{4 - 0}$$

$$\Rightarrow 3c^2 - 12c + 8 = 0$$

$$\therefore c = \frac{6 \pm 2\sqrt{3}}{3}$$

$$\text{The point } c = \frac{6 \pm 2\sqrt{3}}{3} \text{ which clearly lie in } (0, 4).$$

3.2.4 Example

Verify Lagrange's Mean Value theorem for the function $f(x) = e^x$ in $(0, 1)$:

Solution

$f(x) = e^x$ is differentiable in $(0, 1)$ and continuous in $[0, 1]$

$\therefore \exists$ atleast one value 'c' in $(0, 1)$ such that

$$\exists f'(c) = \frac{f(b) - f(a)}{b - a} \quad \dots(1)$$

$$\exists f'(c) = \frac{f(1) - f(0)}{1 - 0} \quad \dots(1)$$

$$f(0) = e^0 = 1, f(1) = e$$

$$f'(x) = e^x \text{ gives } f'(c) = e^c$$

$$\text{From (1)} \quad e^c = \frac{e-1}{1-0},$$

$$\therefore c = \log(e-1), \text{ this } c \in (0, 1)$$

3.2.5 Example

Verify Langrange's Mean Value theorem for the function $f(x) = 5x^2 + 7x + 6$ in $(3, 4)$.

Solution

$f(x)$ is an algebraic polynomial and the interval $(3, 4)$ is finite, $f(x)$ is differentiable in $(3, 4)$ and continuous in $[3, 4]$.

$\therefore \exists$ atleast one value $c \in (3, 4)$ such that

$$f'(c) = \frac{f(4) - f(3)}{4 - 3} \quad \dots(1)$$

$$\exists f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f(3) = 72, f(4) = 114, f'(x) = 10x + 7$$

$$10c + 7 = \frac{114 - 72}{4 - 3}$$

$$c = 3, 5 \in (3, 4)$$

Exercise – 3(B)

I. Verify Langrange's Mean Value theorem for the following functions :

1. $f(x) = x(x-1)(x-2)$ in $(0, \frac{1}{2})$
2. $f(x) = \log x$ in $(1, e)$
3. $f(x) = x^2 - 3x - 1$ in $(\frac{-11}{4}, \frac{13}{7})$
4. $f(x) = 2x^2 - 7x + 10$ in $(2, 5)$

3.3.1 Cauchy's Mean Value Theorem

If two functions $f(x)$ and $g(x)$ are (i) continuous in $[a, b]$ (ii) differential in (a, b) and (iii) $g'(x) \neq 0$ in (a, b) then $\exists$ atleast one value ' c ' in (a, b) $\exists$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof:

Define a new function $\phi(x) = f(x) + \Lambda g(x)$ (1)

Where Λ is a constant such that $\phi(a) = \phi(b)$ (2)

$$f(a) + \Lambda g(a) = f(b) + \Lambda g(b)$$

$$\Lambda = - \frac{[f(b) - f(a)]}{[g(b) - g(a)]} \quad \text{.....(3)}$$

$f(x)$, $g(x)$ are continuous in $[a, b]$

Hence $\phi(x)$ is continuous in $[a, b]$

$f(x)$, $g(x)$ are differentiable in (a, b) (4)

Hence $\phi(x)$ is differentiable in (a, b) (5)

and $\phi(a) = \phi(b)$ (6)

(4), (5), (6) show that $\phi(x)$ satisfies all the conditions of Rolle's theorem

$\therefore \exists$ atleast one value c in (a, b) $\ni \phi'(c) = 0$

But $\phi'(x) = f'(x) + \Lambda g'(x)$

$\therefore \phi'(c) = f'(c) + \Lambda g'(c) = 0$

$$\Rightarrow \Lambda = - \frac{f'(c)}{g'(c)} \quad \text{.....(7)}$$

From (3), (7) we get $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - f(a)}$

3.3.2 Example

Verify Cauchy's Mean Value theorem for $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x}$ in (a, b)

Solution

$f(x)$, $g(x)$ are differentiable in (a, b) and continuous in $[a, b]$

$\therefore \exists$ atleast one value 'c' in $(a, b) \ni$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - f(a)}$$

Here $f'(x) = \frac{-1}{x^3}$

$$g'(x) = \frac{-1}{x^2}$$

$$\therefore \frac{\frac{-1}{c^3}}{\frac{-1}{c^2}} = \frac{b^{\frac{1}{2}} - \frac{1}{a^2}}{\frac{1}{b} - \frac{1}{a}}$$

$$\frac{2}{c} = \frac{a+b}{ab}$$

$$c = \frac{2ab}{a+b} \text{ which is the Harmonic Mean of 'a' and 'b'}$$

$$c \in (a, b)$$

3.3.3 Example

Verify Cauchy's Mean Value Theorem for $f(x) = e^x$, $g(x) = e^{-x}$ in $(3, 7)$

Solution

$f(x)$, $g(x)$ are differentiable in $(3, 7)$ and continuous in $[3, 7]$

$\therefore \exists$ atleast one value 'c' in $(a, b) \ni$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Here $f'(x) = e^x$
 $g'(x) = e^{-x}$

$$\frac{e^c}{-e^{-c}} = \frac{e^7 - e^3}{e^{-7} - e^{-3}} \Rightarrow +e^{2c} = +e^{10}$$

$$c = 5 \in (3, 7)$$

Exercise – 3(C)

1. Considering the functions $f(x) = x^2$, $g(x) = (x)$ in Cauchy's Mean value theorem for (a, b) prove that 'c' is the arithmetic mean between a and b .
2. Verify Cauchy's Mean Value theorem for $f(x) = \sin x$, $g(x) = \cos x$ in (a, b)
3. Verify Cauchy's Mean Value theorem for $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$ in (a, b)

3.4.1 Higher Mean Value Theorem with Lagrange's form of remainder (Taylor's theorem with Lagrange's form of remainder):

If a function $f(x)$ is such that

- (i) $f(x)$, $f'(x)$, $f''(x)$ $f^{n-1}(x)$ are continuous in $[a, a+h]$
- (ii) $f^n(x)$ exists in $(a, a+h)$, then $\exists$ atleast one number ' θ ' between ' 0 ' and ' 1 '

$$\exists f(a+h) = f(a) + \frac{h}{1!} f'(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^n(a + \theta h)$$

Proof: Define a new function.

$$\begin{aligned} f(x) = f(x) + \frac{(a+h-x)}{1!} f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \dots \\ + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x) + \frac{(a+h-x)^n}{n!} A \end{aligned} \quad \dots(1)$$

where A is a constant $\exists \phi(a) = \phi(a+h)$

$$\begin{aligned} \phi(a) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \dots \\ + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} A \end{aligned} \quad \dots(2)$$

$$\text{and} \quad \phi(a+h) = f(a+h) \quad \dots(3)$$

$$\text{but} \quad \phi(a) = \phi(a+h) \quad \dots(4)$$

$$\text{Thus} \quad f(a+h) = \phi(a+h) = \phi(a)$$

Hence using (2) we get

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} A \quad \dots(5)$$

$f(x)$, $f'(x)$, $f''(x)$, ..., $f^{(n-1)}(x)$ and $(a+h-x)$ etc continuous in $[a, a+h]$ and differentiable in $(a, a+h)$

Hence $\phi(x)$ is continuous in $[a, a+h]$ and differentiable in $(a, a+h)$ (6)

(4) and (6) show that $\phi(x)$ satisfies all the conditions of Rolle's theorem.

$\therefore \exists$ atleast one value 'c' ($a < c < a+h$) $\ni \phi'(c) = 0$

Write $c = a + \theta h$ where $0 < \theta < 1$

$\therefore \exists \theta \in (0, 1) \ni \phi'(a + \theta h) = 0$ (7)

Differentiating (1) with respect to 'x'

$$\begin{aligned} \phi'(x) &= f'(x) + \left[\frac{(a+h-x)}{1!} f''(x) - f'(x) \right] + \left[\frac{(a+h-x)^2}{2!} f'''(x) - (a+h-x)f''(x) \right] + \\ &\dots + \left[\frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n)}(x) - \frac{(n-1)(a+h-x)^{n-2}}{(n-1)!} f^{(n-1)}(x) \right] - \frac{n(a+h-x)^{n-1}}{n!} A \quad \dots(8) \end{aligned}$$

From (7) we get

$$\phi'(a + \theta h) = \frac{(h - \theta h)^{n-1}}{(n-1)!} [f^{(n)}(a + \theta h)] = 0$$

$\therefore A = f^{(n)}(a + \theta h)$ (9)

From (5) and (9) it follows that

$$\begin{aligned} f(a+h) &= f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \dots \\ &\quad + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a + \theta h) \quad (0 < \theta < 1) \end{aligned}$$

The last term $\frac{h^n}{n!} f^n(a + \theta h)$ is called Lagrange's form of remainder.

3.5.1 Higher Mean Value Theorem with Cauchy's form of remainder (Taylor's theorem with Cauchy's form of remainder):

If a function $f(x)$ is such that (i) $f(x)$, $f'(x)$, $f''(x)$ $f^n(x)$ are continuous in $[a, a + h]$ and (ii) $f^n(x)$ exists in $(a, a + h)$ then $\exists$ atleast one number ' θ ' between ' 0 ' and ' 1 ' such that

$$f(a + h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{(n-1)!} (1 - \theta)^{n-1} f^n(a + \theta h)$$

Proof: Define a new function

$$\begin{aligned} \phi(x) = f(x) + \frac{(a + h - x)}{1!} f'(x) + \frac{(a + h - x)^2}{2!} f''(x) + \dots \\ + \frac{(a + h - x)^{n-1}}{(n-1)!} f^{n-1}(x) + (a + h - x) A \end{aligned} \quad \dots(1)$$

$$\text{where } A \text{ is constant } \exists \phi(a + h) = \phi(a) \quad \dots(2)$$

From (1) $\phi(a + h) = f(a + h)$

$$\phi(a) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + h A \quad \dots(4)$$

From (2), (3), (4)

$$\begin{aligned} f(a + h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \dots \\ + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + h A \end{aligned} \quad \dots(5)$$

$f(x)$, $f'(x)$, $f''(x)$ $f^{n-1}(x)$ and $(a + h - x)$, $(a + h - x)^2$ are continuous in $[a, a + h]$ and differentiable in $(a, a + h)$

Hence $\phi(x)$ is continuous in $[a, a + h]$ (6)

$\phi(x)$ is differentiable in $(a, a + h)$ (7)

(2), (6) (7) show that $\phi(x)$ satisfies all the conditions of Rolle's theorem.

$\therefore \exists$ atleast one number ' θ ' in between ' 0 ' and ' 1 ' $\ni \phi'(a + \theta h) = 0$ (8)

Differentiating (1) w.r. to ' x '

$$\begin{aligned} \phi'(x) &= f'(x) + [(a + h - x) f''(x) - f'(x)] \\ &\quad \left[\frac{(a + h - x)^2}{2!} f''(x) - \frac{2(a + h - x)}{2} f''(x) \right] + \dots + \dots + \\ &\quad + \left[\frac{(a + h - x)^{n-1}}{(n-1)!} f^n(x) - (n-1) \frac{(a + h - x)^{n-2}}{(n-1)!} f^{n-1}(x) \right] - A \end{aligned}$$

$$\text{i.e., } \phi'(x) = \frac{(a + h - x)^{n-1}}{(n-1)!} f^n(x) - A \quad \text{.....(9)}$$

From (8) and (9) we get

$$\phi'(a + \theta h) = \frac{(h - \theta h)^{n-1}}{(n-1)!} f^n(a + \theta h) - A = 0$$

$$\therefore A = \frac{h^{n-1}(1 - \theta)^{n-1}}{(n-1)!} f^n(a + \theta h) \quad \text{.....(10)}$$

From (9) and (10)

$$\begin{aligned} f(a + h) &= f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \dots \\ &\quad + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n(1 - \theta)^{n-1}}{(n-1)!} f^n(a + \theta h) \quad (0 < \theta < 1) \end{aligned}$$

The last term $\frac{h^n(1 - \theta)^{n-1}}{(n-1)!} f^n(a + \theta h)$ is called *Cauchy's form of remainder*.

3.5.2 Alternate form of Lagrange's Mean Value theorem

If $f(x)$ is i) continuous in the closed interval $[a, a+h]$ and ii) derivable in the open interval $(a, a+h)$ then there exists at least one number $\theta, 0 < \theta < 1$, such that $f(a+h) = f(a) + f'(a+\theta h)$.

Proof: In (2.1) put $b = a+h$, then from the proof of L . M . V theorem there exists 'c',

$$\text{such that } f'(c) = \frac{f(a+h) - f(a)}{a+h-a} \quad \text{.....(1)}$$

writing $c = a + \theta h$ we have $0 < \theta < 1$ and (1) becomes

$$f'(a+\theta h) = \frac{f(a+h) - f(a)}{a+h-a} \Rightarrow f(a+h) = f(a) + hf'(a+\theta h)$$

3.5.3 Example :

If $f(x+h) = f(x) + hf'(x+\theta h)$, $0 < \theta < 1$, find the value of θ when $f(x)$ is any quadratic expression.

Solution:

Let $f(x) = ax^2 + bx + c$, then

$$f(x+h) = a(x+h)^2 + b(x+h) + c = f(x) + h(2ax + b + ah) \quad \text{.....(1)}$$

It is given that $f(x+h) = f(x) + hf'(x+\theta h)$

.....(2)

From (1) and (2) we get ,

$$f'(x+\theta h) = 2ax + b + ah \quad (\text{i.e.,}) \quad 2a(x+\theta h) + b = 2ax + b + ah$$

(since $f'(x) = 2ax + b$) $\Rightarrow 2\theta = 1$ or $\theta = \frac{1}{2}$

3.5.4 Alternate form of Cauchy's Mean value theorem

If $f(x)$ and $g(x)$ are (i) continuous in the closed interval $[a, a+h]$ (ii) derivable in the open interval $(a, a+h)$ and (iii) $g'(x) \neq 0$ at any point in the open interval $(a, a+h)$, then there exists at least one number $\theta, 0 < \theta < 1$, such that

$$\frac{f'(a+\theta h)}{g'(a+\theta h)} = \frac{f(a+h) - f(a)}{g(a+h) - g(a)}$$

Note: Lagrange's Mean Value theorem can be deduced from C . M . V theorem by replacing $g(x)$ with x in the interval (a, b)

We get $g(b) = b$, $g(a) = a$ and $g'(x) = 1$

$$\frac{f'(c)}{1} = \frac{f(b) - f(a)}{b - a} \text{ which is L. M. V. theorem.}$$

Example: Using Cauchy's Mean Value theorem, show that

$$\sin b - \sin a < b - a \text{ given that } 0 < a < b < \frac{\pi}{2}$$

Solution:

Taking $f(x) = \sin x$ and $g(x) = x$ and applying C.M.V theorem we get that,

$$\exists c \text{ such that } \frac{\sin b - \sin a}{b - a} = \cos c \text{ where } 0 < a < c < b < \frac{\pi}{2}$$

$$\text{Further } \cos c < 1 \text{ in } \left(0, \frac{\pi}{2}\right) \therefore \frac{\sin b - \sin a}{b - a} < 1 \text{ (i.e.,) } \sin b - \sin a < (b - a)$$

3.5.5 Example:

If $f'(x)$ is continuous in $[a, b]$ and $f^{(1)}(x)$ exists in (a, b) then show that

$$f(b) - f(a) = (b - a)f'(a) + \frac{(b - a)^2}{2!} f^{(1)}(a) \text{ where } a < c < b.$$

Solution:

Choose a function $g(x)$ such that $g(x) = h(x) - \left(\frac{b - x}{b - a}\right)^2 h(a)$ where

$$h(x) = f(b) - f(a) - (b - x)f'(x) \quad \text{.....(1)}$$

Now $g(a) = h(a) - h(a) = 0$ and $g(b) = h(b) - 0 = 0$ from (1) and L.M.V theorem.

$$\therefore g(a) = 0 = g(b).$$

It is given that $f'(x)$ is continuous in $[a, b]$ and differentiable in (a, b) .

Hence, so are $h(x)$ and $g(x)$. $\therefore g(x)$ satisfies all the conditions of

Rolle's theorem. Hence $\exists c (a < c < b) \ni g'(c) = 0$ (2)

$$g'(x) = h'(x) + \frac{2(b - x)}{(b - a)^2} h(a) \text{ from (1)}$$

$$= -(b - x)f^{(1)}(x) + \frac{2(b - x)}{(b - a)^2} [f(b) - f(a) - (b - a)f'(a)]$$

Using (2) we get ,

$$(b-c)f''(c) = \frac{2(b-c)}{(b-a)^2} [f(b) - f(a) - (b-a)f'(a)]$$

or
$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''(c) \dots\dots\dots(3)$$

Taking $c = a + \theta h$ where $h = b - a$ and $0 < \theta < 1$

(3) gives $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a+\theta h)$ where $0 < \theta < 1$

3.5.6 Example:

$f(x), g(x)$ and $h(x)$ are three functions derivable in (a, b) , show that there exists c in (a, b) such that

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$$

deduce L . M . V and C . M . V theorems.

Solution:

Consider the function $\phi(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0 \dots\dots\dots(1)$

Clearly $\phi(a) = 0 = \phi(b)$

Also $f(x), g(x)$ and $h(x)$ are all derivable in (a, b) ,

$\therefore$ All the conditions of Rolle's theorem are satisfied . Hence $\exists c \in (a, b)$ such that $\phi'(c) = 0$

(i.e.,)
$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0 \dots\dots\dots(2)$$

Taking $g(x) = x$ and $h(x) = 1$ we have from (2)

$$\begin{vmatrix} f'(c) & 1 & 0 \\ f(a) & a & 1 \\ f(b) & b & 1 \end{vmatrix} = 0 \Rightarrow f'(c)(a-b) - [f(a) - f(b)] = 0$$

$$\text{(i.e.,)} \quad f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{where } c \in (a, b) \quad \dots\dots\dots(3)$$

(3) shows that L . M . V theorem can be deduced from (1) Taking only $h(x) = 1$

$$\begin{aligned} \text{we have,} \quad & \begin{vmatrix} f'(c) & g'(c) & 0 \\ f(a) & g(a) & 1 \\ f(b) & g(b) & 1 \end{vmatrix} = 0 \quad \text{where } c \in (a, b) \\ \Rightarrow & f'(c)[g(a) - g(b)] - g'(c)[f(a) - f(b)] = 0 \\ \Rightarrow & \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \quad \text{where } c \in (a, b) \quad \text{which is C . M . V theorem.} \end{aligned}$$

3.5.7 Example:

Apply Maclaurin's theorem with Lagrange's form of remainder for the function

$$f(x) = e^x \quad \text{and show that } 1 + x + \frac{x^2}{2} \leq e^x \leq 1 + x + \frac{x^2}{2} e^x \quad \text{for every } x \geq 0.$$

Solution:

Maclaurin's theorem with L form of remainder is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \dots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n}{n!} f^n(\theta x)$$

$$\therefore \quad e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} e^{\theta x} \quad \left(\text{since } D^n(e^x) = e^x \right)$$

.....(1)

where $0 < \theta < 1$

Taking $n = 2$ in (1) we get

$$e^x = 1 + x + \frac{x^2}{2} e^{\theta x} \quad \dots\dots\dots(2)$$

For every $x \geq 0$ we have $e^{\theta x} \leq e^x$ where $0 < \theta < 1$

$$\therefore 1+x+\frac{x^2}{2}e^{\theta x} \leq 1+x+\frac{x^2}{2}e^x$$

.....(3)

Further $e^{\theta x} > 1$ whenever $x \geq 0$ and $0 < \theta < 1$

$$\therefore 1+x+\frac{x^2}{2} \leq 1+x+\frac{x^2}{2}e^{\theta x}$$

.....(4)

From (2), (3) and (4) it follows that

$$1+x+\frac{x^2}{2} \leq 1+x+\frac{x^2}{2}e^{\theta x} = e^x \leq 1+x+\frac{x^2}{2}e^x$$

3.5.8 Example

Apply Taylor's theorem to prove that $\log_e(1+x) < 1 - \frac{x^2}{2} + \frac{x^3}{3}$ whenever $x > 0$.

Solution:

According to Taylor's theorem

$$f(a+h) = f(a) + \frac{h}{1} f'(a) + \frac{h^2}{2} f''(a) + \dots + \frac{h^{n-1}}{n-1} f^{(n-1)}(a) + R_n$$

where $R_n = \frac{h^n (1-\theta)^{n-p}}{n!} f^n(a+\theta h)$, $0 < \theta < 1$

R_n with Lagrange's form ($n = p$) is $R_n = \frac{h^n}{n!} f^n(a+\theta h)$

Substituting $a = 1$ and $h = x$ in (1)

$$f(1+x) = f(1) + \frac{x}{1} f'(1) + \frac{x^2}{2} f''(1) + \dots + \frac{x^{n-1}}{n-1} f^{(n-1)}(1) + \frac{x^n}{n!} f^n(1+\theta x)$$

Taking $f(x) = \log x$, $f^n(x) = \frac{(-1)^{n-1} (n-1)!}{x^n}$

$$\therefore \log(1+x) = 0 + x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} \quad 0 < \theta < 1 \text{ (Taking only upto 3 terms)}$$

For any $x > 0$, $(1+\theta x) > 1 \Rightarrow \frac{x^3}{3(1+\theta)^3} < \frac{x^3}{3}$

3.6 Functions of Several Variables

3.6.1

Students are quite familiar with functions of a single independent variable. Functions which depend on more than one independent variable are called *Functions of Several Variables*.

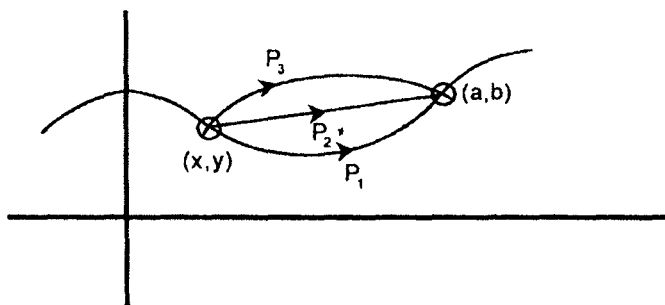
(i) Volume $V = \pi r^2 h$ of a cylinder, r = radius, h = height,
curved surface area $A = 2\pi r h$.

(ii) Area of a triangle $A = \frac{1}{2} xy$, x = base, y = altitude

(iii) Volume of a rectangular parallelopiped $v = xyz$, where x = length, y = breadth, z = height are all examples of functions with more than one variable. (i) and (ii) are examples of functions of two variables and (iii) is a function of three variables.

Let $z = f(x, y)$ is a function of two independent variables. Here x, y are independent variables and z is the dependent variable and let the function $f(x, y)$ be defined in a region R .

Suppose (x, y) be a moving point and (a, b) a fixed point in the region R of the xy -plane. The point (x, y) may approach (a, b) along different paths (see figure) P_1, P_2, P_3 etc.



3.6.2 Concept of a Limit

- Let
- (i) $f(x, y)$ be defined in a region R
 - (ii) (x, y) tend to (a, b) along any path
- and
- (iii) $\varepsilon < 0$ be given.

If $\exists \delta > 0$ such that $|f(x, y) - l| < \epsilon$, $\forall |x - a| < \delta$, $|y - b| < \delta$, then $f(x, y)$ is said to tend to l .

We write
$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) = l$$

or as
$$\lim_{(x, y) \rightarrow (a, b)} f(x, y) = l$$

3.6.3 Example

Consider $\lim_{\substack{x \rightarrow 2 \\ y \rightarrow 1}} \frac{x^2 + y^2}{4xy}$, it can be seen that if $\epsilon = 0.01$ (say) then

$$\left| \frac{x^2 + y^2}{4xy} - \frac{5}{8} \right| < \epsilon \text{ whenever } |x - 2| < 0.02, |y - 1| < 0.02 \text{ thus } \delta = 0.02$$

satisfies

$$\left| \frac{x^2 + y^2}{4xy} - \frac{5}{8} \right| < 0.01. \text{ Hence the desired limit is } \frac{5}{8}.$$

Note : 1

Let $\lim_{(x, y) \rightarrow (a, b)} f(x, y) = l$ and $\lim_{(x, y) \rightarrow (a, b)} g(x, y) = m$

then (i) $\lim_{(x, y) \rightarrow (a, b)} [f \pm g] = l \pm m$

(ii) $\lim_{(x, y) \rightarrow (a, b)} [fg] = lm$

and (iii) $\lim_{(x, y) \rightarrow (a, b)} \frac{f}{g} = \frac{l}{m}$

Note : 2

$\lim_{(x, y) \rightarrow (a, b)}$ exists iff
$$\lim_{(x \rightarrow a)} \left[\lim_{y \rightarrow b} f(x, y) \right] = \lim_{(y \rightarrow b)} \left[\lim_{x \rightarrow a} f(x, y) \right]$$

3.6.4 Example

$$f(x, y) = \frac{x^2 + y}{2x + y^2}, \text{ find } \text{Lt as } (x, y) \rightarrow (2, 1)$$

Solution

$$\text{Lt}_{y \rightarrow 1} \left[\text{Lt}_{x \rightarrow 2} \frac{x^2 + y}{2x + y^2} \right] = \text{Lt}_{y \rightarrow 1} \left[\frac{4 + y}{4 + y^2} \right] = \frac{5}{5} = 1$$

..... (i)

$$\text{Lt}_{x \rightarrow 2} \left[\text{Lt}_{y \rightarrow 1} \frac{x^2 + y}{2x + y^2} \right] = \text{Lt}_{x \rightarrow 2} \left[\frac{x^2 + 1}{2x + 1} \right] = \frac{5}{5} = 1$$

..... (ii)

The two limits (i) and (ii) are equal.

Hence
$$\text{Lt}_{(x, y) \rightarrow (2, 1)} \frac{x^2 + y}{2x + y^2} = 1$$

3.6.5 Example

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2} \text{ find whether the limit exists as } (x, y) \rightarrow (0, 0)$$

Solution

$$\text{Lt}_{x \rightarrow 0} \left[\text{Lt}_{y \rightarrow 0} \frac{x^2 - y^2}{x^2 + y^2} \right] = \text{Lt}_{x \rightarrow 0} \left[\frac{x^2}{x^2} \right] = 1$$

$$\text{Lt}_{y \rightarrow 0} \left[\text{Lt}_{x \rightarrow 0} \frac{x^2 - y^2}{x^2 + y^2} \right] = \text{Lt}_{y \rightarrow 0} \left[\frac{-y^2}{y^2} \right] = -1$$

Thus
$$\text{Lt}_{x \rightarrow 0} \left[\text{Lt}_{y \rightarrow 0} \frac{x^2 - y^2}{x^2 + y^2} \right] \neq \text{Lt}_{y \rightarrow 0} \left[\text{Lt}_{x \rightarrow 0} \frac{x^2 - y^2}{x^2 + y^2} \right]$$

Hence the limit does not exist.

3.6.6 Example

Find
$$\text{Lt}_{(x, y) \rightarrow (0, 0)} \frac{xy}{y^2 - x^2}$$

Solution

Put $y = mx$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{y^2 - x^2} = \lim_{x \rightarrow 0} \frac{mx^2}{m^2x^2 - x^2} = \frac{m}{m^2 - 1}$$

It is clear that limits will be different for different values of m i.e., the limit depends upon the slope of the path along which (x, y) approaches $(0, 0)$. Hence the limit does not exist.

Exercise – 3(D)

I. Examine whether the following limits exist. Find them if they exist.

$$(i) \quad \lim_{(x,y) \rightarrow (2,1)} \frac{x^2 - y^2 + 4}{3xy^2} \quad (ii) \quad \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + 4y^2}{y^2 - 2x^2}$$

$$(iii) \quad \lim_{(x,y) \rightarrow (2,2)} \frac{yx - 2y}{xy - 2x} \quad (iv) \quad \lim_{(x,y) \rightarrow (0,0)} \frac{x^4 - y^4}{x^2 - y^2}$$

$$(v) \quad \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^2 + y^2}$$

3.7.1 Concept of Continuity

Suppose (i) $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$ exists

and (ii) $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$

then $f(x, y)$ is said to be continuous at (a, b) .

Note : 1

If $f(x, y)$ is said to be continuous at every point of a region R , then it is said to be continuous in R .

Note : 2

Let $f(x, y)$ and $g(x, y)$ be continuous at (a, b) then $f \pm g$, fg , and $\frac{f}{g}$ ($g \neq 0$) are all continuous at (a, b) .

3.7.2 Example

Consider the function $f(x, y) = x^2 + y^2 - 2x$ when $(x, y) \neq (0, 0)$ and $f(1, 1) = 0$

$$\begin{aligned} \lim_{(x,y) \rightarrow (1,1)} f(x,y) &= \lim_{x \rightarrow 1} \left[\lim_{y \rightarrow 1} (x^2 + y^2 - 2x) \right] \\ &= \lim_{x \rightarrow 1} [(x^2 + 1 - 2x)] = 0 \\ &= f(1, 1) \end{aligned}$$

Hence the function is continuous at $(1, 1)$

3.7.3 Example

Consider the function $f(x, y) = \frac{x^2 y^2}{x^2 + y^2}$ when $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$

Let $(x, y) \rightarrow (0, 0)$ along the path $y = mx$

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} f(x,y) &= \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^2 + y^2} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 m^2 y^2}{x^2 + m^2 x^2} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{m^2 x^2}{1 + m^2} = 0 = f(0, 0) \end{aligned}$$

Hence $f(x, y)$ is continuous at $(0, 0)$

3.7.4 Example

Consider the function $f(x, y) = \frac{xy}{x^2 - y^2}$ ($x, y \neq 0, 0$), $f(0, 0) = 0$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 - y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{mx^2}{x^2(1 - m^2)}$$

along the path $y = mx$

$$= \frac{m}{1 - m^2} \neq f(0, 0) \text{ except when } m = 0$$

Hence the given function is discontinuous at $(0, 0)$

3.7.5 Example

Consider the function $f(x, y) = \frac{x^2 y^2}{\sqrt{x^2 + y^2}}$ ($x, y) \neq (0, 0)$

$$= 0 \quad \text{at } (x, y) = (0, 0)$$

In this case it is convenient to introduce (polar co-ordinates). Substitutions $x = r \cos \theta$, $y = r \sin \theta$.

$$\begin{aligned} \left| \frac{x^2 y^2}{\sqrt{x^2 + y^2}} \right| &= \left| \frac{r^4 \cos^2 \theta \sin^2 \theta}{\sqrt{r^2 (\cos^2 \theta + \sin^2 \theta)}} \right| \\ &= \left| \frac{r^3}{4} (\sin^2 2\theta) \right| \\ &= \frac{r^3}{4} \quad \sin^2 2\theta \leq 1 \\ &= \frac{(x^2 + y^2)^{3/2}}{4} \end{aligned}$$

Now $\frac{(x^2 + y^2)^{3/2}}{4} \leq \frac{\epsilon}{2}$ provided $|x| < \epsilon^{1/3}$

thus when $|x - 0| < \epsilon^{1/3}$ and $|y - 0| < \epsilon^{1/3}$

$$\left| \frac{x^2 y^2}{\sqrt{x^2 + y^2}} \right| \leq \frac{\epsilon}{\sqrt{2}} < \epsilon \quad \dots (i)$$

Hence $\lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 y^2}{\sqrt{x^2 + y^2}} = 0$ from (i)

$$= f(0, 0)$$

$$f(x, y) = \frac{x^2 y^2}{\sqrt{x^2 + y^2}} \text{ is continuous at } (0, 0)$$

Exercise – 3(E)

I. Investigate the continuity of the following :

$$(i) \quad \left. \begin{aligned} f(x, y) &= 2x + y^2 \\ &= 0 \end{aligned} \right\} \quad \begin{aligned} (x, y) &\neq (2, 3) \\ (x, y) &= (2, 3) \end{aligned} \quad \text{at } (2, 3)$$

Ans: Not continuous

$$(ii) \quad \left. \begin{aligned} f(x, y) &= \frac{xy}{\sqrt{x^2 + y^2}}, \\ &= 0 \end{aligned} \right\} \quad \begin{aligned} (x, y) &\neq (0, 0) \\ (x, y) &= (0, 0) \end{aligned} \quad \text{at } (0, 0)$$

Ans : Not continuous.

$$(iii) \quad \left. \begin{aligned} f(x, y) &= \frac{x - y}{2y + x}, \\ &= 0 \end{aligned} \right\} \quad \begin{aligned} (x, y) &\neq (0, 0) \\ (x, y) &= (0, 0) \end{aligned}$$

Ans : Not continuous

3.8.1 Partial Differentiation

Consider $z = f(x, y)$ where x and y are independent variables and z is the dependent variable.

Keeping one of the two variables x and y as constant and allowing the other to vary we get a partial derivative of 'z' with respect to the variable that is varied.

Keep 'y' constant and allow x to vary, then partial derivative of z w.r. to 'x' is obtained and is denoted by $\frac{\partial z}{\partial x}$ or $f_x(x, y)$ or $D_x f(x, y)$ or $\frac{\partial f}{\partial x}$

$$\text{So} \quad \frac{\partial z}{\partial x} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

$$\text{Similarly} \quad \frac{\partial z}{\partial y} = \lim_{\delta y \rightarrow 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

Note : 1

$z = f(x, y)$ represents a surface in the cartesian co-ordinates (x, y, z) system. The section of the surface $z = f(x, y)$ with the plane $x = k$ (parallel to yz plane) is a curve. (Similarly the sections with planes parallel to xy - plane and zx - planes also will be curves).

Note : 2

$\left(\frac{\partial z}{\partial x}\right)_{x=k}$ gives the slope of the tangent at (x, y, z) to that curve

$\left(\frac{\partial z}{\partial y}\right)_{y=k}$ gives the slope of the tangent at (x, y, z) to the curve obtained as

the section of the surface $z = f(x, y)$ with the plane $y = k$.

3.8.2 Example

$$z = x^3 + x^2y + y^2$$

(i) To find $\frac{\partial z}{\partial y}$, we keep 'y' as constant

partial derivative w.r. to x of $x^3 = 3x^2$

partial derivative w.r. to x of $x^2y = 2xy$

partial derivative w.r. to x of $y^2 = 0$

$$\therefore \frac{\partial z}{\partial x} = 3x^2 + 2xy$$

(ii) To find $\frac{\partial z}{\partial x}$, we keep 'x' as constant

partial derivative w.r., to y of $x^3 = 0$

partial derivative w.r., to y of $x^2y = x^2$

partial derivative w.r., to y of $y^2 = 2y$

$$\therefore \frac{\partial z}{\partial y} = x^2 + 2y$$

3.8.3 Example

$$f = x^2 + y^2 - 2xy$$

$$\frac{\partial f}{\partial x} = 2x - 2y$$

$$\frac{\partial f}{\partial y} = 2y - 2x$$

3.8.4 Example

$$f = (x - y)(x + 2y)$$

This is a product, and the usual product rule applies

$$\frac{\partial f}{\partial x} = (x - y)(1 + 0) + (x + 2y)(1 - 0)$$

$$= x - y + x + 2y = 2x + y$$

$$\frac{\partial f}{\partial y} = (x - y)(0 + 2) + (x + 2y)(0 - 1)$$

$$= 2x - 2y - x - 2y = x - 4y$$

3.8.5 Example

If $f = \frac{x-2y}{x+y}$, Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

Solution

Applying the Quotient rule

$$\frac{\partial f}{\partial x} = \frac{(x+y)(1-0) - (x-2y)(1+0)}{(x+y)^2} = \frac{3y}{(x+y)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(x+y)(-2) - (x-2y)(0+1)}{(x+y)^2} = \frac{-3x}{(x+y)^2}$$

3.8.6 Example

If $f = \sin(ax + by)$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial x} = \cos(ax + by) \times \frac{\partial}{\partial x} (ax + by) = a \cos(ax + by)$$

$$\frac{\partial f}{\partial y} = \cos(ax + by) \times \frac{\partial}{\partial y} (ax + by) = b \cos(ax + by)$$

3.8.7 Example

Consider $f = ax^2 + hxy - by^2$

Then $\frac{\partial f}{\partial x} = 2ax + hy$; $\frac{\partial f}{\partial y} = 2by + hx$

The expression $f_x = 2ax + hy$ is itself a function of x and y . We could therefore find its partial derivative with respect to x or y .

(i) If we differentiate it partially w.r. to x , we get $\frac{\partial}{\partial x} \left[\frac{\partial f}{\partial x} \right]$ and this is

written as $\left[\frac{\partial^2 f}{\partial x^2} \right]$

$$\therefore \left[\frac{\partial^2 f}{\partial x^2} \right] = \frac{\partial}{\partial x} (2ax + hy) = 2a$$

This is called the second partial derivative of ' f ' w.r. to ' x '.

If we differentiate $\frac{\partial f}{\partial x}$ partially w.r. to ' y ' we get $\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right)$ and this is

written as $\frac{\partial^2 f}{\partial y \partial x}$

$$\therefore \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} (2ax + hy) = h$$

Thus $\frac{\partial f}{\partial x} = 2ax + hy, \frac{\partial f}{\partial y} = hx - 2by$

$$\frac{\partial^2 f}{\partial x^2} = 2a, \frac{\partial^2 f}{\partial y^2} = -2b$$

$$\frac{\partial^2 f}{\partial y \partial x} = h$$

Similar steps can be carried out with the expression for $\frac{\partial f}{\partial y}$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = -2b$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial f}{\partial x} \left(\frac{\partial f}{\partial y} \right) = h$$

3.8.8 Example

If $f = ax^3 + hx^2y + by^3$, find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial x \partial y}$

Solution

$$\frac{\partial f}{\partial x} = 3ax^2 + 2hxy; \quad \frac{\partial f}{\partial y} = hx^2 + 3by^2$$

$$\frac{\partial^2 f}{\partial x^2} = 6ax + 2hy; \quad \frac{\partial^2 f}{\partial y^2} = 6by$$

$$\frac{\partial^2 f}{\partial y \partial x} = 2hx; \quad \frac{\partial^2 f}{\partial x \partial y} = 2hx$$

In this example also, we see that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$

3.8.9 Example

If $f = \log(x^2 + y^2)$, prove that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Solution

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2}, \quad \frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{-4xy}{(x^2 + y^2)^2}; \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{-4xy}{(x^2 + y^2)^2}$$

$$\therefore \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

3.8.10 Example

If $u(x + y) = x^2 + y^2$ then prove that $\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \right)^2 = 4 \left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \right)$

Solution

Since $u = \frac{x^2 + y^2}{x + y}$

$$\frac{\partial u}{\partial x} = \frac{x^2 + 2xy - y^2}{(x + y)^2}; \quad \frac{\partial u}{\partial y} = \frac{-x^2 + 2xy + y^2}{(x + y)^2}$$

$$\begin{aligned} \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \right)^2 &= \left[\frac{x^2 + 2xy - y^2 + x^2 - 2xy - y^2}{(x + y)^2} \right]^2 \\ &= 4 \frac{(x - y)^2}{(x + y)^2} \end{aligned} \quad \text{..... (i)}$$

$$\begin{aligned} 4 \left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \right) &= 4 \left[1 - \frac{x^2 + 2xy - y^2}{(x + y)^2} - \frac{-x^2 + 2xy + y^2}{(x + y)^2} \right] \\ &= 4 \frac{(x - y)^2}{(x + y)^2} \end{aligned} \quad \text{..... (ii)}$$

(i) and (ii) prove the result

3.8.11 Example

If $z = 2(ax + by)^2 - (x^2 + y^2)$ and $a^2 + b^2 = 1$, then prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

Solution

$$\frac{\partial z}{\partial x} = 4(ax + by) \cdot a - 2x$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = 4a^2 - 2$$

$$\frac{\partial z}{\partial y} = 4(ax + by) \cdot b - 2y$$

$$\therefore \frac{\partial^2 z}{\partial y^2} = 4b^2 - 2$$

$$\therefore \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 4(a^2 + b^2) - 4 = 0 \quad \because a^2 + b^2 = 1$$

3.8.12 Example

If $z = f\left(\frac{x^2}{y}\right)$ then prove that $x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y} = 0$

$$\text{and } x^2 \frac{\partial^2 z}{\partial x^2} + 3xy \frac{\partial^2 z}{\partial x \partial y} + 2y^2 \frac{\partial^2 z}{\partial y^2} = 0$$

Solution

$$\frac{\partial z}{\partial x} = f'\left(\frac{x^2}{y}\right) \cdot \frac{2x}{y}; \quad \frac{\partial z}{\partial y} = f'\left(\frac{x^2}{y}\right) \left(\frac{-x^2}{y^2}\right)$$

$$\therefore x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y} = f'\left(\frac{x^2}{y}\right) \left[\frac{2x^2}{y} - \frac{2x^2}{y} \right] = 0 \quad \dots (i)$$

Differentiating (i) partially w.r., to 'x'.

$$x\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} + 2y\frac{\partial^2 z}{\partial x \partial y} = 0 \quad \dots (ii)$$

Differentiating (i) w.r., to 'y' partially

$$x\frac{\partial^2 z}{\partial y \partial x} + 2\frac{\partial z}{\partial x} + 2y\frac{\partial^2 z}{\partial y^2} = 0 \quad \dots (iii)$$

Multiplying (ii) by 'x' and (iii) by 'y' and adding

$$x^2 \frac{\partial^2 z}{\partial x^2} + x\frac{\partial z}{\partial x} + 2xy \frac{\partial^2 z}{\partial x \partial y} + xy \frac{\partial^2 z}{\partial y \partial x} + 2y\frac{\partial z}{\partial y} + 2y^2 \frac{\partial^2 z}{\partial y^2} = 0$$

But $x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y} = 0$

$$\therefore x^2 \frac{\partial^2 z}{\partial x^2} + 3xy \frac{\partial^2 z}{\partial x \partial y} + 2y^2 \frac{\partial^2 z}{\partial y^2} = 0$$

3.8.13 Example

If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ then prove that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$

Solution

$$\begin{aligned}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u &= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u \\ &= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) \quad \dots\dots (i)\end{aligned}$$

Now $\frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} (3x^2 - 3yz)$

$$\frac{\partial u}{\partial y} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} (3y^2 - 3xz)$$

$$\frac{\partial u}{\partial z} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} (3z^2 - 3xy)$$

$$\begin{aligned}\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} &= \frac{3x^2 + 3yz - 3y^2 - 3xz + 3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz} \\ &= \frac{3}{x+y+z} \quad (\because (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx) \\ &\quad = x^3 + y^3 + z^3 - 3xyz)\end{aligned}$$

$\therefore$ Hence from (i)

$$\begin{aligned}\text{LHS} &= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \left(\frac{3}{x+y+z}\right) \\ &= 3 \left(-\frac{1}{(x+y+z)^2} - \frac{1}{(x+y+z)^2} - \frac{1}{(x+y+z)^2}\right) \\ &= -\frac{9}{(x+y+z)^2}\end{aligned}$$

3.8.14 Example

If $u = \theta^n e^{\frac{r^2}{4\theta}}$ then find 'n' so that $\frac{\partial u}{\partial \theta} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right)$

Solution

$$\begin{aligned} \frac{\partial u}{\partial \theta} &= n\theta^{n-1} e^{\frac{r^2}{4\theta}} + \theta^n \cdot e^{\frac{r^2}{4\theta}} \cdot \left(\frac{r^2}{4\theta^2} \right) \\ &= \frac{n}{\theta} \cdot u + \frac{r^2}{4\theta^2} \cdot u = \left(\frac{n}{\theta} + \frac{r^2}{4\theta^2} \right) u \end{aligned} \quad \dots (i)$$

also $\frac{\partial u}{\partial r} = \theta^n \cdot e^{\frac{r^2}{4\theta}} \left(-\frac{2r}{4\theta} \right) = -\frac{ru}{2\theta}$

$$\begin{aligned} \therefore \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) &= \frac{\partial}{\partial r} \left(-\frac{r^3 u}{2\theta} \right) = -\frac{ru}{2\theta} \cdot \frac{\partial}{\partial r} (r^3 u) \\ &= \frac{-1}{2\theta} \left[r^3 \frac{\partial u}{\partial r} + 3r^2 u \right] \\ &= \frac{-1}{2\theta} \left[r^4 \frac{u}{2\theta} + 3r^2 u \right] \end{aligned}$$

$$\therefore \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) = \left(\frac{r^2}{4\theta^2} - \frac{3}{2\theta} \right) u \quad \dots (ii)$$

Equating (i) and (ii) we get

$$\left(\frac{n}{\theta} + \frac{r^2}{4\theta} \right) u = \left(\frac{r^2}{4\theta^2} - \frac{3}{2\theta} \right) u$$

$$\therefore \frac{n}{\theta} = \frac{-3}{2\theta}$$

Hence $n = \frac{-3}{2}$

Alternative method :

$$u = \theta^n \cdot e^{\frac{r^2}{4\theta}}$$

Taking logarithms on both sides

$$\log u = n \log \theta + \frac{r^2}{4\theta} \quad \dots (1)$$

Diff. (1) partially w.r.t θ

$$\frac{1}{u} \frac{\partial u}{\partial \theta} = \frac{n}{\theta} + \frac{r^2}{4\theta^2} \quad \therefore \quad \frac{\partial u}{\partial \theta} = \left(n + \frac{r^2}{4\theta} \right) \frac{u}{\theta} \quad \dots (2)$$

Diff.(1) partially w.r.to ' r '

$$\frac{1}{u} \frac{\partial u}{\partial r} = \frac{-r}{2\theta} \Rightarrow \frac{\partial u}{\partial r} = \frac{-u \cdot r}{2\theta}$$

$$r^2 \frac{\partial u}{\partial r} = \frac{-u \cdot r^3}{2\theta}$$

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) = \frac{-1}{2\theta} \left[u \cdot 3r^2 + r^3 \left(\frac{-ur}{2\theta} \right) \right]$$

$$= \frac{ur^2}{\theta} \left[\frac{-3}{4} + \frac{r^2}{4\theta} \right]$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) = \frac{u}{\theta} \left[\frac{-3}{2} + \frac{r^2}{4\theta} \right] \quad \dots (3)$$

From (2) and (3) we get

$$n + \frac{r^2}{4\theta} = \frac{-3}{2} + \frac{r^2}{4\theta}$$

$$\therefore \quad n = \frac{-3}{2}$$

3.8.15 Example

If $x^2 y^y z^z = k$ then prove that at $x = y = z$, $\frac{\partial^2 z}{\partial x \partial y} = [x \log ex]^{-1}$

Solution

Applying logarithms to $x^x y^y z^z = k$, we get $x \log x + y \log y + z \log z = \log k$.

Differentiating this implicit function partially w.r., to 'x'.

$$x \frac{1}{x} + \log x + \left(z \frac{1}{z} + \log z \right) \frac{\partial z}{\partial x} = 0$$

$$\therefore (1 + \log x) + (1 + \log z) \cdot \frac{\partial z}{\partial x} = 0 \quad \dots (i)$$

$$\therefore \frac{\partial z}{\partial x} = - \frac{(1 + \log y)}{1 + \log z}$$

$$\text{Illly} \quad \frac{\partial z}{\partial y} = - \frac{(1 + \log y)}{1 + \log z}$$

Differentiating (i) partially w.r., to 'y'

$$(1 + \log z) \frac{\partial^2 z}{\partial y \partial x} + \frac{\partial z}{\partial x} \left[\frac{1}{z} \cdot \frac{\partial z}{\partial y} \right] = 0$$

Substituting $x = y = z$, we get

$$(1 + \log x) \frac{\partial^2 z}{\partial x \partial y} + \frac{1}{x} = 0$$

$$\begin{aligned} \therefore \frac{\partial^2 z}{\partial x \partial y} &= - \frac{1}{x(1 + \log x)} = - \frac{1}{x(\log e + \log x)} \\ &= - \frac{1}{x \log ex} \quad \text{or} \quad -(x \log ex)^{-1} \end{aligned}$$

Exercise – 3(F)

$$1. \text{ If } f = \frac{x+4y}{4x-y}, \text{ find } \frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y} \quad [\text{Ans : } \frac{-17y}{(4x-y)^2}, \frac{17x}{(4x-y)^2}]$$

$$2. \text{ If } z = ax^2 + bxy + by^2 \text{ find } \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \quad [\text{Ans : } 2ax + by, bx + 2by]$$

3. If $z = (ax + by)(ax - by)$, find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ [Ans : $2xa^2, -2yb^2$]

4. If $z = \tan(ax + by)$, find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ [Ans : $a \sec^2(ax + by), b \sec^2(ax + by)$]

5. If $z = \frac{\sin(ax + by)}{xy}$ find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

[Ans : $\left(\frac{ax \cos(ax + by) - \sin(ax + by)}{x^2 y}, \frac{by \cos(ax + by) - \sin(ax + by)}{x^2 y} \right)$]

6. If $u = \log(\sin y + y \sin x)$, then prove that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

7. If $u = x^2 \tan^{-1} \frac{y}{x} - y^2 \tan^{-1} \left(\frac{x}{y} \right)$, prove that

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{x^2 - y^2}{x^2 + y^2}$$

8. If $u = \tan^{-1} \frac{xy}{\sqrt{x^2 + y^2} + 1}$ then prove that

$$\frac{\partial^2 u}{\partial x \partial y} = \sqrt[3]{1 + x^2 + y^2}$$

9. If $u = \frac{1}{ct^2} e^{\frac{x^2}{4c^2 t}}$ prove that

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

10. If $u = f(x + Ky) + g(x - Ky)$ then show that

$$\frac{\partial^2 u}{\partial y^2} = K^2 \frac{\partial^2 u}{\partial x^2}$$

11. If $u = \frac{y}{z} + \frac{z}{x}$ prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$$

12. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x + y + z}$$

13. If $u = f(r)$ where $r = \sqrt{x^2 + y^2 + z^2}$ prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r} f'(r)$$

14. If $u = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{y+x}$ then prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$$

15. If $x^x y^y z^z = K$, find the value of $\frac{\partial^2 z}{\partial x^2} - 2xy \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2}$ where $x = y = z = 1$

[Ans : -1]

16. If $u = xyf\left(\frac{x}{y}\right)$, prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u$$

17. If $x = r \cos \theta$, $y = r \sin \theta$, then prove that

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

18. If $u = \log r$, $r = x^3 + y^3 - x^2y - xy^2$, prove that

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = -\frac{4}{(x+y)^2}$$

19. If $u = x^2y + y^2z + z^2x$, prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = (x + y + z)^2$$

20. If $u = x^3 + y^3 - 3axy$ then prove that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

21. If $u = e^{ax+by} f(ax-by)$, prove that

$$b \frac{\partial u}{\partial x} + a \frac{\partial u}{\partial y} = 2abu$$

22. If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z}\right)$$

3.9.1 Composite Functions

If u is a function of two variables x and y are themselves functions of an independent variable t .

$$\text{then} \quad \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t} \quad \dots (i)$$

Suppose now that ' u ' is a function of the variables x and y , and x and y are themselves functions of two other variables r and s .

$$\text{then} \quad \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} \quad \dots (A)$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} \quad \dots (B)$$

Similarly if u is a function of r and s , where r and s are themselves functions of x and y then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} \quad \dots (C)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y}$$

The second and higher-order partial derivatives of ' u ' can be obtained by repeated application of the above formulae. Also, the formulae can be extended to functions of three and more variables.

3.9.2 Example

If $z = f(x, y)$, $x = e^u + e^{-v}$, $y = e^{-u} e^v$ then show that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}$$

Solution

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{\partial z}{\partial x} \cdot e^u - \frac{\partial z}{\partial y} \cdot e^{-u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} = -\frac{\partial z}{\partial x} \cdot e^{-v} - \frac{\partial z}{\partial y} \cdot e^v$$

By subtraction

$$\begin{aligned} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) &= (e^u + e^{-v}) \frac{\partial z}{\partial x} - (e^{-u} - e^v) \frac{\partial z}{\partial y} \\ &= x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} \end{aligned}$$

3.9.3 Example

If $u = f(x, y)$, $x = r \cos \theta$, $y = r \sin \theta$ prove that

$$\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 = \left(\frac{\partial u}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial u}{\partial \theta} \right)^2$$

Solution

we know that

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} = \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x} (-r \sin \theta) + \frac{\partial u}{\partial y} (r \cos \theta)$$

$$\left(\frac{\partial u}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial u}{\partial \theta} \right)^2 = \cos^2 \theta \left(\frac{\partial u}{\partial x} \right)^2 + 2 \cos \theta \sin \theta \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} +$$

$$\begin{aligned} & \sin^2 \theta \left(\frac{\partial u}{\partial y} \right)^2 + \sin^2 \theta \left(\frac{\partial u}{\partial x} \right)^2 - 2 \cos \theta \sin \theta \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} + \cos^2 \theta \left(\frac{\partial u}{\partial y} \right)^2 \\ &= \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \end{aligned}$$

3.9.4 Example

If $u = f(x - y, y - z, z - x)$ prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Solution

Let $X = x - y$, $Y = y - z$, $Z = z - x$ then $u = f(X, Y, Z)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y} \cdot \frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z} \cdot \frac{\partial Z}{\partial x} = \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z}$$

$$\text{Similarly} \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial Y} - \frac{\partial u}{\partial X}, \quad \frac{\partial u}{\partial z} = \frac{\partial u}{\partial Z} - \frac{\partial u}{\partial Y}$$

Adding all these gives

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

3.9.5 Example

If $z = f(u, v)$ Where $u = x^2 - y^2$ and $v = 2xy$ prove that

$$(a) \quad x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = 2(x^2 - y^2) \frac{\partial z}{\partial u}$$

$$(b) \quad \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 = 4(x^2 + y^2) \left[\left(\frac{\partial z}{\partial u} \right)^2 + \left(\frac{\partial z}{\partial v} \right)^2 \right]$$

Solution

$$u = x^2 - y^2, \quad v = 2xy$$

$$\therefore \quad \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y, \quad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = 2 \left[x \frac{\partial z}{\partial u} + y \frac{\partial z}{\partial v} \right] \quad \dots (i)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = 2 \left[-y \frac{\partial z}{\partial u} + x \frac{\partial z}{\partial v} \right] \quad \dots (ii)$$

From these, we get

$$x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = 2(x^2 + y^2) \frac{\partial z}{\partial u}$$

From (i) and (ii) we obtain

$$\begin{aligned} \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 &= \left\{ 4 \left[x \frac{\partial z}{\partial u} + y \frac{\partial z}{\partial v} \right]^2 + \left[-y \frac{\partial z}{\partial u} + x \frac{\partial z}{\partial v} \right]^2 \right\} \\ &= 4[x^2 + y^2] \left\{ \left(\frac{\partial z}{\partial u} \right)^2 + \left(\frac{\partial z}{\partial v} \right)^2 \right\} \end{aligned}$$

3.9.6 Example

If $x = r \cos \theta$ and $y = r \sin \theta$, prove the following :

1. $\frac{\partial x}{\partial \theta} = \cos \theta; \quad \frac{\partial x}{\partial \theta} = -r \sin \theta; \quad \frac{\partial y}{\partial r} = \sin \theta; \quad \frac{\partial y}{\partial \theta} = r \cos \theta$
2. $\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}; \quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}; \quad \frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2}, \quad \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2}$
3. $\left(\frac{\partial r}{\partial x} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 = 1$
4. $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}$
5. $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \left(\frac{\partial^2 r}{\partial x \partial y} \right)^2$
6. $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$

Solution

1. We have $x = r \cos \theta$. Differentiating this relation partially w.r.t., r and w.r.t., θ , we get

$$\frac{\partial x}{\partial r} = \cos \theta; \quad \frac{\partial x}{\partial \theta} = -r \sin \theta \quad \dots (i)$$

Similarly, the relation $y = r \sin \theta$ yields

$$\frac{\partial y}{\partial r} = \sin \theta; \quad \frac{\partial y}{\partial \theta} = r \cos \theta \quad \dots (ii)$$

2. From the relations $x = r \cos \theta$, $y = r \sin \theta$, we get

$$x^2 + y^2 = r^2 \quad \dots (iii)$$

$$\frac{y}{x} = \tan \theta \quad \dots (iv)$$

Differentiating (iii) partially w.r.t., ' x ' and w.r.t. ' y ' we get $\frac{\partial r}{\partial x}$; $\frac{\partial r}{\partial y}$

This gives

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}}; \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}} \quad \dots (v)$$

Differentiating again (iv) partially w.r.t., ' x ' and ' y ' we get

$$y \left(-\frac{1}{x^2} \right) = \sec^2 \theta \frac{\partial \theta}{\partial x}, \quad \frac{1}{x} = \sec^2 \theta \frac{\partial \theta}{\partial y}$$

which gives

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2} = \cos^2 \theta = -\frac{y}{r^2} = -\frac{y}{x^2 + y^2} \quad \dots (vi)$$

$$\frac{\partial \theta}{\partial y} = \frac{\cos^2 \theta}{x} = \frac{x}{r^2} = \frac{x}{x^2 + y^2} \quad \dots (vii)$$

3. From (v) we get

$$\left(\frac{\partial r}{\partial x} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{x^2 + y^2}{r^2} = 1 \quad \dots (viii)$$

$$\left(\frac{\partial r}{\partial x} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 = 1$$

4. From (v) we also get

$$\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{x}{r} \right) = \frac{1}{r} + x \left(-\frac{1}{r^2} \right) \frac{\partial r}{\partial x} = \frac{1}{r} - \frac{x^2}{r^3} \quad \dots (ix)$$

$$\frac{\partial^2 r}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{y}{r} \right) = \frac{1}{r} + y \left(-\frac{1}{r^2} \right) \frac{\partial r}{\partial y} = \frac{1}{r} - \frac{y^2}{r^3} \quad \dots (x)$$

Adding (ix) and (x)

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r} \quad \dots (xi)$$

5. From (ix) and (x), we get

$$\left(\frac{\partial^2 r}{\partial x^2} \right) \left(\frac{\partial^2 r}{\partial y^2} \right) = \frac{1}{r^2} + \frac{x^2 y^2}{r^6} - \left[\frac{x^2 + y^2}{r^4} \right] = \frac{x^2 y^2}{r^6} \quad \dots (xii)$$

Differentiating $\frac{\partial r}{\partial y} = \frac{y}{r}$ partially w.r.t., 'x' we get

$$\frac{\partial^2 r}{\partial x \partial y} = \frac{\partial}{\partial x} \left[\frac{y}{r} \right] = y \left[-\frac{1}{r^2} \right] \frac{\partial r}{\partial x} = -\frac{y}{r^2} \cdot \frac{x}{r} = -\frac{xy}{r^3}$$

So that $\left(\frac{\partial r}{\partial x \partial y} \right)^2 = \frac{x^2 y^2}{r^6} \quad \dots (xiii)$

From (xii) and (xiii) we get that

$$\left[\frac{\partial^2 r}{\partial x^2} \right] \left[\frac{\partial^2 r}{\partial y^2} \right] = \left(\frac{\partial^2 r}{\partial x \partial y} \right)^2 \quad \dots (xiv)$$

6. From (vi) we get

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial x} \left[\frac{\partial \theta}{\partial x} \right] = \frac{\partial}{\partial x} \left(\frac{-y}{r^2} \right) = -y \left(-\frac{2}{r^3} \right) \frac{\partial r}{\partial x} = \frac{2y}{r^3} \cdot \frac{x}{r} = \frac{2xy}{r^4} \quad \dots (xv)$$

Similarly from (vii), we get

$$\frac{\partial^2 \theta}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial \theta}{\partial y} \right) = \frac{\partial}{\partial y} \left(\frac{-x}{r^2} \right) = -x \left(-\frac{2}{r^3} \right) \frac{\partial r}{\partial y} = \frac{2x}{r^3} \cdot \frac{y}{r} = \frac{2xy}{r^4} \quad \dots (xvi)$$

Adding (xv) and (xvi) we get

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = \frac{2xy}{r^4} - \frac{2xy}{r^4} = 0$$

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

Note :

1. The variables (r, θ) related to (x, y) through the relations $x = r \cos \theta$, $y = r \sin \theta$ are the polar co-ordinates of the point whose cartesian co-ordinates are (x, y) .
2. In obtaining the results in (i) r and θ are regarded as independent variables and x and y are regarded as dependent variables.
3. In obtaining the results (ii) - (vi), x and y are regarded as independent variables and r and θ are taken as dependent variables.
4. After partial differentiation is carried out, the final expression for the partial derivatives are to be expressed in terms of the independent variables.

5. From the expression for $\frac{\partial r}{\partial x}, \frac{\partial x}{\partial r}, \frac{\partial x}{\partial \theta}, \frac{\partial \theta}{\partial x}$, it may be observed that $\frac{\partial r}{\partial x}$ is not the

reciprocal of $\frac{\partial x}{\partial r}$ and $\frac{\partial x}{\partial \theta}$ is not the reciprocal of $\frac{\partial \theta}{\partial x}$. In fact we note here

$$\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r} \text{ and } \frac{\partial x}{\partial \theta} = r^2 \frac{\partial \theta}{\partial x}. \text{ Similarly } \frac{\partial r}{\partial y} = \frac{\partial y}{\partial r}, \frac{\partial y}{\partial \theta} = r^2 \frac{\partial \theta}{\partial y}.$$

6. The results (i - vi) are useful in calculations involving polar co-ordinates.

Exercise – 3(G)

1. If $u = x + y$, $v = xy$ and $f = f(u, v)$, show that

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = u \frac{\partial f}{\partial u} + 2v \frac{\partial f}{\partial v}$$

2. If $z = f(x, y)$ and $x = e^u + e^{-x}$ and $y = e^{-u} - e^v$ show that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}$$

3. If $z = f(u, v)$ where $u = x^2 - y^2$ and $v = 2xy$ prove that

$$x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = 2(x^2 + y^2) \frac{\partial z}{\partial u}$$

4. If $u = e^x \sin y$, $v = e^x \cos y$ and $z = f(u, v)$ prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (u^2 + v^2) \left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right)$$

5. If $z = \log(u^2 + v)$, $u = e^{x^2 + y^2}$, $v = x^2 + y$, show that

$$\frac{\partial z}{\partial x} = \frac{2x(2u^2 + 1)}{u^2 + v}, \quad \frac{\partial z}{\partial y} = \frac{4yu^2 + 1}{u^2 + v}$$

6. If $w = f(u, v)$ and $u = x^2 - y^2$, $v = 2xy$ prove that

$$\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2} = 4(x^2 + y^2) \left[\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2} \right]$$

7. If $u = \frac{x}{z}$; $v = \frac{y}{z}$ and $w = z$ and $f = f(u, v, w)$ show that

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = w \frac{\partial f}{\partial w}$$

8. If $z = f(x, y)$, $x = u \cosh v$, $y = u \sinh v$ prove that

$$\left(\frac{\partial z}{\partial x} \right)^2 - \left(\frac{\partial z}{\partial y} \right)^2 = \left(\frac{\partial z}{\partial u} \right)^2 - \frac{1}{u^2} \left(\frac{\partial z}{\partial v} \right)^2$$

9. If $z = f(u, v)$, $u = lx + my$, $v = ly - mx$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2) \left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right)$$

10. If $z = f(u, v)$, $u = x^2 - y^2$, $v = y^2 - x^2$ show that

$$x \frac{\partial z}{\partial y} + y \frac{\partial z}{\partial x} = 0$$

11. If $x = e^u \tan v$, $y = e^u \sec v$ find the value of

$$\left(x \frac{\partial u}{\partial x} + y \frac{\partial y}{\partial y} \right) \times \left(x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} \right)$$

[Ans : 0]

$$\left(\text{Hint } u = \frac{1}{2} \log(y^2 - x^2) \text{ and } v = \sin^{-1} \left(\frac{x}{y} \right) \right)$$

12. If $w = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$ prove that

$$\frac{1}{x} \frac{\partial w}{\partial x} + \frac{1}{y} \frac{\partial w}{\partial y} + \frac{1}{z} \frac{\partial w}{\partial z} = 0$$

13. If $w = f(x, y)$, $u = e^x$, $v = e^y$ show that

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial u \partial v} (u v)$$

14. If $x = \sqrt{vw}$, $y = \sqrt{wn}$, $z = \sqrt{uv}$ Prove that

$$x \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y} + z \frac{\partial \phi}{\partial z} = u \frac{\partial \phi}{\partial u} + v \frac{\partial \phi}{\partial v} + w \frac{\partial \phi}{\partial w}$$

where ϕ is a function of x, y, z .

3.10.1 Homogeneous Function

Definition

A function $f(x, y, z, u, \dots)$ is said to be homogeneous in $x, y, z, u, \dots$ if it can be written in the form $x^n g\left(\frac{y}{x}, \frac{z}{x}, \frac{u}{x}, \dots\right)$ and is said to be of degree n .

Ex. 1. $z = \sin^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{y}{x}\right)$ is homogeneous and is of degree '0'. Since

$$z = x^0 \left[\sin^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{y}{x}\right) \right]$$

2. $z = \frac{x^2 + y^2}{x - y}$ is homogeneous and is of degree 1 since

$$z = \frac{x^2 \left[1 + \left(\frac{y}{x}\right)^2 \right]}{x \left[1 - \frac{y}{x} \right]} = x' g\left(\frac{y}{x}\right)$$

$$\begin{aligned}
 3. \quad \phi &= x^3 \sin\left(\frac{y}{x}\right) + xyz \log\left(\frac{x}{y}\right) + xz^2 \tan^{-1}\left(\frac{z}{y}\right) \\
 &= x^3 \left[\sin\left(\frac{y}{x}\right) + \frac{y}{x} \cdot \frac{z}{x} \log\left(\frac{x}{y}\right) + \left(\frac{z}{x}\right)^2 \tan^{-1}\left(\frac{z}{y}\right) \right]
 \end{aligned}$$

Hence ϕ is of degree '3' and homogeneous in x, y, z .

3.10.2 Euler's Theorem

The following theorem, known as the Euler's theorem gives a very useful formula for a specific combination of partial derivatives of homogeneous functions.

Theorem :

If z is a homogeneous function of x and y with degree ' n ' then

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz \quad \text{..... (i)}$$

Proof :

Since z is a homogeneous function of degree ' n ' we can write ' z ' in the form.

$$z = x^n f\left(\frac{y}{x}\right)$$

$$\begin{aligned}
 \text{Now} \quad \frac{\partial z}{\partial x} &= nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right) \left(\frac{-y}{x^2}\right) \\
 &= nx^{n-1} f\left(\frac{y}{x}\right) - x^{n-2} y f'\left(\frac{y}{x}\right)
 \end{aligned}$$

$$\text{and} \quad \frac{\partial z}{\partial y} = x^n f' \cdot \frac{y}{x} = x^{n-1} f' \cdot \frac{y}{x}$$

$$\begin{aligned}
 x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} &= nx^n f\left(\frac{y}{x}\right) - x^{n-1} y f'\left(\frac{y}{x}\right) + x^{n-1} y f'\left(\frac{y}{x}\right) \\
 &= nx^n f\left(\frac{y}{x}\right)
 \end{aligned}$$

$$= nz \quad \left[\because z = x^n f\left(\frac{y}{x}\right) \right]$$

Thus
$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz$$

3.10.3

If z is a homogeneous functions of x and y with degree ' n ' then

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)z \quad \dots (ii)$$

Proof :

Differentiating (i) partially w.r., to x , we obtain

$$x \frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} + y \frac{\partial^2 z}{\partial x \partial y} = n \frac{\partial z}{\partial x}$$

i.e.,
$$x \frac{\partial^2 z}{\partial x^2} + yx \frac{\partial^2 z}{\partial x \partial y} = (n-1) \frac{\partial z}{\partial x} \quad \dots (iii)$$

Differentiating (i) partially w.r., to ' y ', we obtain

$$y \frac{\partial^2 z}{\partial y^2} + x \frac{\partial^2 z}{\partial y \partial x} = (n-1) \frac{\partial z}{\partial y} \quad \dots (iv)$$

Multiplying (iii) by ' x ' and (iv) by ' y ' and adding we get

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = (n-1) \left(x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} \right) = n(n-1)z$$

3.10.3 General Formula

The Euler's theorem may be generalized to homogeneous functions of more than two variables. For a homogeneous function $f(x, y, z)$ of degree ' n ' in three variables, the theorem can be read as :

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = nf \quad \dots (v)$$

In general, for a homogeneous function $f(x_1, \dots, x_k)$ of degree ' n ' in ' k ' independent variables $x_1, \dots, x_k$ the theorem reads

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \dots + x_k \frac{\partial f}{\partial x_k} = nf \quad \dots (vi)$$

The proofs of (v) (vi) are left to the student

3.10.4 Example

If $u = \operatorname{cosec}^{-1} \left[\frac{\sqrt{x} + \sqrt{y}}{\sqrt[3]{x} + \sqrt[3]{y}} \right]$, then prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = -\frac{1}{6} \tan u$

Solution

We note that from the given expression for 'u' we can write

$$\operatorname{cosec} u = \frac{\sqrt{x} + \sqrt{y}}{\sqrt[3]{x} + \sqrt[3]{y}} = f \text{ (say)} \quad \dots (i)$$

Then

$$f = \frac{x^{\frac{1}{2}} \left(1 + \frac{(y)^{\frac{1}{2}}}{x^{\frac{1}{2}}} \right)}{x^{\frac{1}{3}} \left(1 + \frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} \right)}$$

$$= \frac{x^{\frac{1}{6}} \left(1 + \left(\frac{y}{x} \right)^{\frac{1}{2}} \right)}{1 + \left(\frac{y}{x} \right)^{\frac{1}{3}}}$$

$$= x^{\frac{1}{6}} \frac{y}{x}$$

$\therefore f$ is a homogeneous function of degree $\frac{1}{6}$, Hence by Euler's theorem

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = \frac{1}{6} f = \frac{1}{6} \operatorname{cosec} u \quad \dots (ii)$$

i.e.,
$$x \frac{\partial}{\partial x} (\operatorname{cosec} u) + y \frac{\partial}{\partial y} (\operatorname{cosec} u) = \frac{1}{6} \operatorname{cosec} u$$

Carrying out partial differentiation, we get

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{\frac{1}{6} \operatorname{cosec} u}{-\operatorname{cosec} u \cdot \cot u} = -\frac{1}{6} \tan u$$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = -\frac{1}{6} \tan u$$

3.10.5 Example

If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$

Solution

$$u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$$

$$\frac{\partial u}{\partial x} = f' \cdot \left(\frac{1}{y} - \frac{z}{x^2}\right), \quad \frac{\partial u}{\partial y} = f' \cdot \left(\frac{-x}{y^2} + \frac{1}{z}\right)$$

$$\frac{\partial u}{\partial z} = f' \cdot \left(\frac{-y}{z^2} + \frac{1}{x}\right)$$

$$\begin{aligned} \therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} &= f' \cdot \left(\frac{x}{y} - \frac{z}{x} - \frac{x}{y} + \frac{y}{z} - \frac{y}{z} + \frac{z}{x}\right) \\ &= 0 \end{aligned}$$

3.10.6 Example

If $u = \sin^{-1} \frac{x}{y} + \tan^{-1} \frac{y}{x}$ prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$

Solution

$$u = f(x, y) = \sin^{-1} \frac{x}{y} + \tan^{-1} \frac{y}{x}$$

$u = f(x, y)$ is of degree '0'

$\therefore$ By Euler's theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0 \cdot u = 0$$

Alter: $\sin^{-1} \frac{x}{y} = v, \quad \tan^{-1} \frac{y}{x} = w$

$\therefore u = v + w$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}$$

$$\frac{x}{y} = \sin v, \quad \frac{1}{y} = \cos v \frac{\partial v}{\partial x}$$

$$\frac{y}{x} = \tan w, \quad \frac{-y}{x^2} = \sec^2 w \cdot \frac{\partial w}{\partial x}$$

$$\frac{-x}{y^2} = \cos v \frac{\partial v}{\partial y}, \quad \frac{1}{x} = \sec^2 w \cdot \frac{\partial w}{\partial y}$$

$$\begin{aligned} \therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} &= \left(\frac{x}{y \cos v} - \frac{x}{y \cos v} \right) + \left(\frac{-y}{x \sec^2 w} + \frac{y}{x \sec^2 w} \right) \\ &= 0 \end{aligned}$$

3.10.7 Example

If $u = \tan^{-1} \left(\frac{x^3 + y^3}{x + y} \right)$ prove that

(i) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$

(ii) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u$

Solution

Proof:

For the given expression for 'u' we have

$$\tan u = \frac{x^3 + y^3}{x + y} = f, \text{ say}$$

Here ' f ' is a homogeneous functions of degree ' 2 '

$\therefore$ By Euler's theorem

$$x \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = nf = 2f$$

i.e.,
$$x \frac{\partial}{\partial x}(\tan u) + y \frac{\partial}{\partial y}(\tan u) = 2 \tan u$$

Carrying out partial derivatives, we get

$$x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2 \tan u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2 \tan u}{\sec^2 u} = \sin 2u \quad \dots (i)$$

Next differentiating both sides of the result (i) partially w.r., to ' x ' and ' y ' we get

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} = 2 \cos 2u \frac{\partial u}{\partial x} \quad \dots (ii)$$

$$x \frac{\partial^2 u}{\partial y \partial x} + y \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 2 \cos 2u \frac{\partial u}{\partial y} \quad \dots (iii)$$

Multiplying the result (ii) by x , the result (iii), by ' y ' and adding the results we get

$$\begin{aligned} x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} &= (2 \cos 2u - 1) \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) \\ &= (2 \cos 2u - 1) \sin 2u, \quad (\text{from i}) \\ &= \sin 4u - \sin 2u \end{aligned}$$

3.10.8 Example

If $z = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, find the value of $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2}$

Solution

Let
$$v = x^2 \tan^{-1}\left(\frac{y}{x}\right) \quad \text{and} \quad w = y^2 \tan^{-1}\left(\frac{x}{y}\right)$$

Then $z = v + w$

v and w are homogeneous functions of degree '2'. Applying 3.10.3 to ' v ' and ' w ' we get

$$x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = 2v \quad \dots (i)$$

Similarly $x^2 \frac{\partial^2 w}{\partial x^2} + 2xy \frac{\partial^2 w}{\partial x \partial y} + y^2 \frac{\partial^2 w}{\partial y^2} = 2w \quad \dots (ii)$

(i) - (ii) gives $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 2z$

Note :

Let $f(z)$ be a homogeneous function of x and y of degree n ; then from Euler's theorem

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf$$

i.e., $x f' \cdot \frac{\partial z}{\partial x} + y f' \cdot \frac{\partial z}{\partial y} = nf \Rightarrow x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{nf}{f'} \quad \dots (i)$

Differentiating (i) w.r., to ' y ' partially

$$x \frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} + y \frac{\partial^2 z}{\partial x \partial y} = n \left[\frac{f'^2 - f f''}{f'^2} \right] \frac{\partial z}{\partial x} \quad \dots (ii)$$

Differentiating (i) w.r., to ' x ' partially

$$x \frac{\partial^2 z}{\partial x \partial y} + y \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y} = n \left[\frac{f'^2 - f f''}{f'^2} \right] \frac{\partial z}{\partial y} \quad \dots (iii)$$

(ii) x + (iii) y gives

$$\begin{aligned} x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + \left(x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} \right) \\ = n \left[\frac{f'^2 - f f''}{f'^2} \right] \left[x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} \right] \quad \dots (iv) \end{aligned}$$

Using (i) in (iv)

$$\begin{aligned} x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} &= \left[n \left(1 - \frac{ff''}{f'^2} \right) - 1 \right] n \frac{f}{f'} \\ &= n(n-1) \frac{f}{f'} - n^2 \frac{f^2 f''}{f'^3} \quad \dots (v) \end{aligned}$$

Here $f' = \frac{\partial f}{\partial z}, f'' = \frac{\partial^2 f}{\partial z^2}$

3.10.9 Example

If $z = \log(x^3 + y^3 - x^2y - xy^2)$, prove that

$$(i) \quad x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 3 \quad (ii) \quad x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = -3$$

Note : (i) and (iv) can be used as formula in such cases

Hence $f = e^z, f' = e^z, f'' = e^z$ and $n = 3$

From (i) of 3.10.8 note $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 3$

Form (v) of 3.10.8 note $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 3.2 - 9 = -3$

3.10.10 Example

Verify Euler's theorem for the function $u(x, y, z) = \sqrt{\frac{z}{x^2 + y^2}}$

Solution

u is a homogeneous function of degree $\frac{-1}{2}$

We note that $u^2 = u^2 \frac{z}{x^2 + y^2} = z(x^2 + y^2)^{-1} \quad \dots (i)$

So that $2u \frac{\partial u}{\partial x} = -z(x^2 + y^2)^{-2} \cdot 2x$

$$2u \frac{\partial u}{\partial y} = -z(x^2 + y^2)^{-2} \cdot 2y$$

$$2u \frac{\partial u}{\partial z} = (x^2 + y^2)^{-1}$$

gives
$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = \frac{-z}{2u} \left(\frac{1}{x^2 + y^2} \right)$$

$$= -\frac{u^2}{2u} = -\frac{1}{2}u \text{ by (i)}$$

Hence the result

3.10.11 Example

If $f = \cos^{-1} \frac{x^3 + y^3 - z^3}{\sqrt{x^4 + y^4 + z^4}}$ then show that $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = -\cot f$

Solution

$$u = u(x, y, z) = \frac{x^3 + y^3 - z^3}{\sqrt{x^4 + y^4 + z^4}} \quad \dots (i)$$

So that $\cos f = u$

From (i) we see that u is a homogeneous function of degree 1.

$\therefore$ By Euler's theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = u \quad \dots (iii)$$

Form (ii), we find that

$$\frac{\partial u}{\partial x} = -\sin f \cdot \frac{\partial f}{\partial x}, \frac{\partial u}{\partial y} = -\sin f \frac{\partial f}{\partial y}, \frac{\partial u}{\partial z} = -\sin f \cdot \frac{\partial f}{\partial z}$$

Using these and (ii) and (iii), we get

$$-\sin f \left(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} \right) = \cos f$$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = -\cot f$$

3.10.12 Example

Find $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$, if $u = \sin^{-1} \left(\frac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{x^2 + y^2} \right)^{\frac{1}{2}}$

Solution

Taking $z = \left(\frac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{x^2 + y^2} \right)^{\frac{1}{2}}$ then $z = \sin u = f(u)$ say z is a homogeneous function of

$$\text{degree } -\frac{1}{12} = n$$

From Euler's theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)} = -\frac{1}{12} \frac{\sin u}{\cos u} = -\frac{1}{12} \tan u$$

Again $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1) \frac{f}{f'} - \frac{n^2 f^2 f''}{f'^3}$

$$= \left(-\frac{1}{12} \right) \left(-\frac{1}{12} - 1 \right) \tan u - \frac{1}{144} (-\tan u) \cdot \tan^2 u$$

$$(\because f' = \cos u, f'' = -\sin u)$$

$$= \frac{\tan u}{144} (13 + \tan^2 u)$$

3.10.13 Example

If $u = \tan^{-1} \left(\frac{y}{x} \right)$, find $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$

Solution

Let $z = \frac{y}{x}$ degree of z is 0 and $z = \tan u$

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0 \quad \frac{\partial z}{\partial x} = \sec^2 u \frac{\partial u}{\partial x} \quad \frac{\partial z}{\partial y} = \sec^2 u \frac{\partial u}{\partial y}$$

$$\sec^2 u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = 0 \Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$$

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} = 0 \quad \dots (i)$$

$$x \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + y \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots (ii)$$

(i) x + (ii) y gives

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = - \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = 0$$

Exercise – 3(E)

1. Verify Euler's theorem for the following functions :

(a) $z = \frac{xy}{x+y}$

(b) $z = \frac{x^2 + y^2}{\sqrt{x+y}}$

(c) $z = e^{(x^3 + y^3)}$

(d) $z = \sin^{-1} \left(\frac{x^3 - y^3}{x - y} \right)$

(e) $z = \sin^{-1} \left(\frac{x^3 y^3}{x^3 + y^3} \right)$

(f) $z = (x^2 + y^2 + z^2)^{1/2}$

2. If $z = xyf\left(\frac{x}{y}\right)$ then show that $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2z$

3. Show the following by using Euler's theorem :

(a) If $z = \frac{x^3 + y^3}{\sqrt{x+y}}$ then $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{5}{2} z$

(b) If $z = \log \frac{x^3 + y^3}{x+y}$, then $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2$

(c) If $z = \sin^{-1} \left(\frac{x^3 - y^3}{x - y} \right)$ then $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2 \tan z$

4. If $z = \tan^{-1} \left(\frac{x^3 + y^3}{x + y} \right)$, then $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \sin 2u$

5. If $z = \sec^{-1} \left[\frac{x^2 + y^2}{x + y} \right]$, then $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \cot z$

6. If z is a homogeneous function of x and y with degree ' n ' and $z = f(u)$, prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)}$$

7. If $z = x^3 y^2 \sin^{-1} \left(\frac{y}{x} \right)$, show that $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 20z$

8. If $z = \tan^{-1} \sqrt{x^4 + y^4}$, show that

(i) $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \sin 2z$

(ii) $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = \sin 4z - \sin 2z$

9. If $z = \sin^{-1} \left(\frac{x + y}{\sqrt{x} + \sqrt{y}} \right)$, show that

(i) $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{1}{2} \tan z$

(ii) $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = \frac{1}{4} (\tan^3 z - \tan z)$

10. If $z = \tan^{-1} \left(\frac{x}{y} \right) + y \sin^{-1} \left(\frac{x}{y} \right)$ show that

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 0$$

11. If $z = \operatorname{cosec}^{-1} \sqrt{\frac{\frac{1}{x^2} + \frac{1}{y^2}}{\frac{1}{x^3} + \frac{1}{y^3}}}$ then show that

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = \frac{\tan z}{12} \left[\frac{13}{12} + \frac{1}{12} \tan^2 z \right]$$

12. If $z = \tan^{-1} \left(\frac{x^2 + y^2}{x - y} \right)$, then show that

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = -2 \sin^3 z \cos z$$

13. If $z = \log r$ and $r^2 = x^2 + y^2$ then show that $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + 1 = 0$

14. If $z = \sin^{-1} \left[\frac{\frac{1}{x^4} + \frac{1}{y^4}}{\frac{1}{x^5} - \frac{1}{y^5}} \right]$ then show that

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = \frac{1}{400} \tan z (\tan^2 z - 19)$$

15. If $z = \log \left(\frac{\sqrt{x^2 + y^2}}{x + y} \right)$ find the value of $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2}$

3.11.1 Jacobian

If u and v are functions of two independent variables x and y then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \text{ i.e., } \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \text{ is called the Jacobian of } u, v \text{ with respect to } x \text{ and } y$$

and is denoted by

$$\frac{\partial(u, v)}{\partial(x, y)} \quad \text{or} \quad J \left(\frac{u, v}{x, y} \right)$$

If u, v, w are functions of three independent variables x, y, z then the determinant

$$\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

is called the Jacobian of u, v, w with respect to x, y, z and is denoted by

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} \quad \text{or} \quad J\left(\frac{u, v, w}{x, y, z}\right)$$

Similarly

$$\frac{\partial(u_1, u_2, \dots, u_n)}{\partial(x_1, x_2, \dots, x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \dots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \dots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \dots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian of $u_1, u_2, \dots, u_n$ w.r.t $x_1, x_2, \dots, x_n$ and is denoted by

$$\frac{\partial(u_1, u_2, \dots, u_n)}{\partial(x_1, x_2, \dots, x_n)} \quad \text{or} \quad J\left(\frac{u_1, u_2, \dots, u_n}{x_1, x_2, \dots, x_n}\right)$$

SOME PROPERTIES OF JACOBIAN

3.11.2 Property

$$\frac{\partial(u, v)}{\partial(x, y)}, \frac{\partial(x, y)}{\partial(u, v)} = 1 \quad \text{or} \quad JJ' = 1$$

where $J = \frac{\partial(u, v)}{\partial(x, y)}$ and $J' = \frac{\partial(x, y)}{\partial(u, v)}$

Proof :

Let $u = u(x, y)$ and $v = v(x, y)$ so that u and v are functions of x and y . Then differentiating these partially each with respect to u and v , we get

$$1 = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$0 = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$0 = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$1 = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial v}$$

Now

$$\frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

(Interchanging rows and columns in 2nd determinant)

$$= \begin{vmatrix} \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial u} & \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial u} & \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \quad \text{Proved}$$

3.11.3 Property

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)}$$

where u, v are functions of r, s and r, s are functions of x, y .

Proof

By differentiation of function of a functions, partially; we get

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x};$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y}; \quad \dots (i)$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial v}{\partial s} \cdot \frac{\partial s}{\partial x};$$

and

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial v}{\partial s} \cdot \frac{\partial s}{\partial y};$$

$$\begin{aligned} \Rightarrow \frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)} &= \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \times \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix} \\ &= \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \times \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix} \end{aligned}$$

(Interchanging rows and columns of the second determinant)

$$\begin{aligned} &= \begin{vmatrix} \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} & \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} \\ \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial v}{\partial s} \cdot \frac{\partial s}{\partial x} & \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial v}{\partial s} \cdot \frac{\partial s}{\partial y} \end{vmatrix} \\ &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u, v)}{\partial(x, y)} \quad \text{using (i)} \end{aligned}$$

3.11.4 Property

If u, v are functions of two independent variables x and y then u, v are independent if $\frac{\partial(u, v)}{\partial(x, y)} \neq 0$. Otherwise they are dependent. Similarly if u, v, w

(functions of x, y, z) are independent if and only if $\frac{\partial(u, v, w)}{\partial(x, y, z)} \neq 0$

Theorem

If the functions $u_1, u_2, \dots, u_n$ of the variables $x_1, x_2, \dots, x_n$ be defined by the relations $u_1 = f_1(x_1), u_2 = f_2(x_1, x_2), u_3 = f_3(x_1, x_2, x_3), \dots, u_n = f_n(x_1, x_2, \dots, x_n)$.

Then
$$\frac{\partial(u_1, u_2, \dots, u_n)}{\partial(x_1, x_2, \dots, x_n)} = \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdot \frac{\partial u_3}{\partial x_3} \dots \frac{\partial u_n}{\partial x_n}$$

We know that
$$\frac{\partial u_1}{\partial x_2} = 0 = \frac{\partial u_2}{\partial x_4} = \frac{\partial u_3}{\partial x_5} \dots$$

Similarly for $u_3, u_4, \dots, u_n$

Then
$$\frac{\partial(u_1, u_2, \dots, u_n)}{\partial(x_1, x_2, \dots, x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \dots & \frac{\partial u_n}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \dots & \frac{\partial u_n}{\partial x_2} \\ \dots & \dots & \dots & \dots \\ \frac{\partial u_1}{\partial x_n} & \frac{\partial u_2}{\partial x_n} & \dots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \dots & \frac{\partial u_n}{\partial x_1} \\ 0 & \frac{\partial u_2}{\partial x_2} & \dots & \frac{\partial u_n}{\partial x_2} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{\partial u_n}{\partial x_n} \end{vmatrix} = \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \dots \frac{\partial u_n}{\partial x_n}$$

3.11.5 Example

If $x = u^2 - v^2$; $y = 2uv$; find $\frac{\partial(x, y)}{\partial(u, v)}$

Solution

We have $x = u^2 - v^2 \Rightarrow \frac{\partial x}{\partial u} = 2u, \quad \frac{\partial x}{\partial v} = -2v$

$$y = 2uv \Rightarrow \frac{\partial y}{\partial u} = 2v, \frac{\partial y}{\partial v} = 2u$$

$$\therefore \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4(u^2 + v^2)$$

3.11.6 Example

If $x = r \cos \theta$, $y = r \sin \theta$ find $\frac{\partial(x, y)}{\partial(r, \theta)}$ and $\frac{\partial(x, y)}{\partial(x, y)}$

Solution

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$\Rightarrow \frac{\partial x}{\partial r} = \cos \theta, \quad \frac{\partial y}{\partial r} = \sin \theta$$

$$\frac{\partial x}{\partial \theta} = -r \sin \theta, \quad \frac{\partial y}{\partial \theta} = r \cos \theta$$

Then
$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r(\cos^2 \theta + \sin^2 \theta) = r$$

Again $r^2 = x^2 + y^2$; $\theta = \tan^{-1} \frac{y}{x}$

$$\Rightarrow 2r \frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r^2} = \frac{x}{x^2 + y^2}$$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r^2}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r^2}, \quad \frac{\partial \theta}{\partial x} = \frac{-y}{x^2 + y^2} = \frac{-y}{r^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2} = \frac{x}{r^2}$$

$$\Rightarrow \frac{\partial(r, \theta)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{y}{r^2} & \frac{x}{r^2} \end{vmatrix} = \frac{x^2}{r^3} + \frac{y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}$$

3.11.7 Example

If $x = uv$, $y = \frac{u}{v}$ then show that $JJ' = 1$

Solution

We have $\frac{\partial x}{\partial u} = v, \quad \frac{\partial x}{\partial v} = u$

$$\frac{\partial y}{\partial u} = \frac{1}{v}, \quad \frac{\partial y}{\partial v} = -\frac{u}{v^2}$$

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{2u}{v}$$

But $u^2 = xy \quad \therefore \quad 2u \frac{\partial u}{\partial x} = y, \quad 2u \frac{\partial u}{\partial y} = x$

and $v^2 = \frac{x}{y}, \quad 2v \frac{\partial v}{\partial x} = \frac{1}{y}, \quad 2v \frac{\partial v}{\partial y} = -\frac{x}{y^2}$

$$\therefore J' = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{y}{2u} & \frac{x}{2u} \\ \frac{1}{2vy} & -\frac{x}{2vy} \end{vmatrix}$$

$$= -\frac{x}{4uvy} - \frac{x}{4uvy} - \frac{1}{2uv} \cdot \frac{x}{y}$$

$$= -\frac{1}{2uv} v^2 = -\frac{v}{2u}$$

$$JJ' = \left(-\frac{2u}{v} \right) \left(-\frac{v}{2u} \right) = 1$$

3.11.8 Example

Prove that $JJ' = 1$ for $x = e^v \sec u$, $y = e^v \tan u$

Solution

$$\frac{\partial x}{\partial u} = e^v \sec u \tan u, \quad \frac{\partial x}{\partial v} = e^v \sec u$$

and $\frac{\partial y}{\partial u} = e^v \sec^2 u; \quad \frac{\partial y}{\partial v} = e^v \tan u$

$$\therefore J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} e^v \sec u \tan u & e^v \sec u \\ e^v \sec^2 u & e^v \tan u \end{vmatrix} = -e^{2v} \sec u = -xe^v$$

Now $\sec^2 u - \tan^2 u = 1$

$$\therefore x^2 e^{-2v} - y^2 e^{-2v} = 1$$

$$\therefore e^{2v} = x^2 - y^2, \text{ also } \frac{e^v \tan u}{e^v \sec u} = \frac{y}{x}$$

$$\therefore \sin u = \frac{y}{x}$$

$$\therefore u = \sin^{-1} \left(\frac{y}{x} \right) \text{ and } v = \frac{1}{2} \log(x^2 - y^2)$$

$$\therefore \frac{\partial u}{\partial x} = -\frac{y}{x\sqrt{x^2 - y^2}}, \quad \frac{\partial u}{\partial y} = \frac{1}{x\sqrt{x^2 - y^2}}$$

$$\frac{\partial v}{\partial x} = \frac{x}{x^2 - y^2}, \quad \frac{\partial v}{\partial y} = -\frac{y}{x^2 - y^2}$$

$$J' = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} -\frac{y}{x\sqrt{x^2 - y^2}} & \frac{1}{x\sqrt{x^2 - y^2}} \\ \frac{x}{x^2 - y^2} & -\frac{y}{x^2 - y^2} \end{vmatrix}$$

$$= \frac{y^2}{x(x^2 - y^2)^{3/2}} - \frac{x}{(x^2 - y^2)^{3/2}} = \frac{y^2 - x^2}{x(x^2 - y^2)^{3/2}}$$

$$= -\frac{1}{x\sqrt{(x^2 - y^2)}} = \frac{1}{xe^v}$$

$$\therefore JJ' = xe^v \cdot \frac{1}{xe^v} = 1$$

3.11.9 Example

If $x = \frac{vw}{u}$, $y = \frac{wu}{v}$, $z = \frac{uv}{w}$ then show that $\frac{\partial(x, y, z)}{\partial(u, v, w)} = 4$

Solution

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{vw}{u^2} & \frac{w}{u} & \frac{v}{u} \\ \frac{w}{v} & \frac{wu}{v^2} & \frac{u}{v} \\ \frac{v}{w} & \frac{u}{w} & -\frac{uv}{w^2} \end{vmatrix}$$

$$= \begin{vmatrix} -vw & wu & uv \\ vw & -wu & uv \\ vw & uw & -uv \end{vmatrix} \cdot \frac{1}{u^2} \cdot \frac{1}{v^2} \cdot \frac{1}{w^2}$$

$$= \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1(1-1) - 1(-1-1) + 1(1+1) = 4$$

Exercise – 3(I)

1. If $y + x + z = u$, $z + y = uv$, $z = uvw$, find $\frac{\partial(x, y, z)}{\partial(u, v, w)}$

[Ans : $(u^2 v)$]

2. If $u = x + 2y^2 - z^2, v = x^2 yz, w = 2x^2 - xy$ find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ at the point $(1, -1, 0)$

Ans: 19

3. If $x = r \cos \theta, y = r \sin \theta$, find $\frac{\partial(x, y)}{\partial(r, \theta)}$ and $\frac{\partial(r, \theta)}{\partial(x, y)}$ and show that

$$\frac{\partial(x, y)}{\partial(r, \theta)} \cdot \frac{\partial(r, \theta)}{\partial(x, y)} = 1 \quad (\text{Hint : refer 3.6.6})$$

4. If $y_1 = \frac{x_2 x_3}{x_1}, y_2 = \frac{x_1 x_3}{x_2}, y_3 = \frac{x_1 x_2}{x_3}$, show that $\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = 4$

5. If $y_1 = 1 - x_1, y_2 = x_1(1 - x_2), y_3 = x_1 x_3(1 - x_1)$, prove that $\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = (-1)^3 x_1^2 x_3$.

6. If $x = a(\cosh u) \cos v, y = a(\sinh u) \sin v$, then show that

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{a^2}{2} (\cosh 2u - \cosh 2v)$$

7. If $u = f(x_1), v = \phi(x_1, x_2), w = \psi(x_1, x_2, x_3)$ then show that

$$\frac{\partial(u, v, w)}{\partial(x_1, x_2, x_3)} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial v}{\partial x_2} \cdot \frac{\partial w}{\partial x_3}$$

8. If $x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta$ show that $\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin \theta$

Solution:
$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix}$$

$$= \sin \theta \cos \phi [0 + r^2 \sin^2 \theta \cos \phi] + r \cos \theta \cos \phi \cdot r \sin \theta \cos \theta \cos \phi +$$

$$+ (-r \sin \theta \sin \phi) [-r \sin^2 \theta \sin \phi - r \cos^2 \theta \sin \phi]$$

$$= r^2 \sin^3 \theta \cos^2 \phi + r^2 \sin \theta \cos^2 \theta \cos^2 \phi + r^2 \sin^3 \theta \sin^2 \phi + r^2 \cos^2 \theta \sin \theta \sin^2 \phi$$

$$= r^2 \sin^3 \theta + r^2 \sin \theta \cos^2 \theta = r^2 \sin \theta.$$

3.12.0 JACOBIAN OF COMPOSITE FUNCTIONS:

Let (u, v) be functions of x, y where x, y themselves are functions of r, θ . Then u, v are

composite functions of r, θ and $\frac{\partial(u, v)}{\partial(r, \theta)} = \frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(r, \theta)}$ (from 3.6.3)

Note 1: If u, v are functions of x, y we may regard x, y as functions of u, v .

Taking $r = u, \theta = v$ in the above result, we have

$$\frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial(u, v)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \\ \frac{\partial v}{\partial u} & \frac{\partial v}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\therefore \frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(u, v)} = 1 \Rightarrow JJ^1 = 1$$

where $J = \frac{\partial(u, v)}{\partial(x, y)}, J^1 = \frac{\partial(x, y)}{\partial(u, v)}$

Note 2: $\frac{\partial(u, v, w)}{\partial(r, \theta, \phi)} = \frac{\partial(u, v, w)}{\partial(x, y, z)} \cdot \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$ where u, v, w are functions of x, y, z and x, y, z are functions of r, θ, ϕ .

Also $JJ^1 = 1$; where $J = \frac{\partial(u, v, w)}{\partial(x, y, z)}, J^1 = \frac{\partial(x, y, z)}{\partial(u, v, w)}$

3.12.1 Example: If $u = xyz, v = x^2 + y^2 + z^2, w = x + y + z$ find $J = \frac{\partial(x, y, z)}{\partial(u, v, w)}$.

Solution: we first evaluate $J^1 = \frac{\partial(u, v, w)}{\partial(x, y, z)}$, since u, v, w are explicitly given as

functions of x, y, z . $J^1 = \begin{vmatrix} yz & zx & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} = -2(x - y)(y - z)(z - x)$

Hence from the relation that $JJ^1 = 1$, we have

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{-1}{2(x - y)(y - z)(z - x)}$$

Exercise – 3(G)

1. If $u = 2xy, v = x^2 - y^2$, where $x = r \cos \theta, y = r \sin \theta$ show that $\frac{\partial(u, v)}{\partial(r, \theta)} = -4r^3$.
2. If $x = \sqrt{vw}, y = \sqrt{wu}, z = \sqrt{uv}$
where $u = r \sin \theta \cos \phi, v = r \sin \theta \sin \phi, w = r \cos \theta$,
show that $\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \frac{1}{4} r^2 \sin \theta$
3. If $x = uv, y = \frac{u+v}{u-v}$, find $\frac{\partial(u, v)}{\partial(x, y)}$ (Ans: $\frac{(u-v)^2}{4uv}$)
4. If $u = \frac{yz}{x}, v = \frac{zx}{y}, w = \frac{xy}{z}$ prove that $\frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{1}{4}$.
5. If $u = \frac{x+y}{1-xy}, \theta = \tan^{-1}(x) - \tan^{-1}(y)$ show that $\frac{\partial(u, \theta)}{\partial(x, y)} = 0$.
6. If $u = x^2 - 2y, v = x + y + z, w = x - 2y + 3z$ show that $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 10x + 4$

3.12.2 Jacobian of implicit functions

Let u_1, u_2, u_3 , be implicit functions of x_1, x_2, x_3 so that

$$f_r(u_1, u_2, u_3, x_1, x_2, x_3) = 0, \quad r = 1, 2, 3$$

Then $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_1}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_1} + \frac{\partial f_1}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_1} + \frac{\partial f_1}{\partial u_3} \cdot \frac{\partial u_3}{\partial x_1} = 0$ and so on

then $\frac{\partial(f_1, f_2, f_3)}{\partial(u_1, u_2, u_3)} \cdot \frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)}$

$$= \begin{vmatrix} \sum \frac{\partial f_1}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_1} & \sum \frac{\partial f_1}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_2} & \sum \frac{\partial f_1}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_3} \\ \sum \frac{\partial f_2}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_1} & \sum \frac{\partial f_2}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_2} & \sum \frac{\partial f_2}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_3} \\ \sum \frac{\partial f_3}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_1} & \sum \frac{\partial f_3}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_2} & \sum \frac{\partial f_3}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_3} \end{vmatrix}$$

(by the rule of product of determinants)

$$= \begin{vmatrix} -\frac{\partial f_1}{\partial x_1} & -\frac{\partial f_1}{\partial x_2} & -\frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_2}{\partial u_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{vmatrix}$$

$$= (-1)^3 \frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)}$$

Hence
$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = (-1)^3 \frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)} \bigg/ \frac{\partial(f_1, f_2, f_3)}{\partial(u_1, u_2, u_3)}$$

(The result is obvious)

3.12.3 Example

If $u = \frac{x}{\sqrt{1-r^2}}, v = \frac{y}{\sqrt{1-r^2}}, w = \frac{z}{\sqrt{1-r^2}}$ prove that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{1}{(1-r^2)^{3/2}}$$

where $r^2 = x^2 + y^2 + z^2$

Solution

we have $u^2 (1-r^2) = x^2$

$$\Rightarrow f_1(u, v, w, x, y, z) = u^2 (1 - x^2 - y^2 - z^2) - x^2 = 0$$

$$\text{Similarly } f_2 \equiv v^2 (1 - x^2 - y^2 - z^2) - y^2 = 0$$

$$f_3 \equiv w^2 (1 - x^2 - y^2 - z^2) - z^2 = 0$$

$$\therefore \frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} \bigg/ \frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} \quad \dots (i)$$

(see 3.13.2)

$$\therefore \frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = \begin{vmatrix} -2u^2x - 2x & -2u^2y & -2u^2z \\ -2v^2x & -2v^2y - 2y & -2v^2z \\ -2w^2x & -2w^2y & -2w^2z - 2z \end{vmatrix}$$

$$= (-2x)(-2y)(-2z) \begin{vmatrix} u^2 + 1 & u^2 & u^2 \\ v^2 & v^2 + 1 & v^2 \\ w^2 & w^2 & w^2 + 1 \end{vmatrix}$$

$$-8xyz \begin{vmatrix} 1 & 0 & u^2 \\ -1 & 1 & v^2 \\ 0 & -1 & w^2 + 1 \end{vmatrix}$$

by $C_1 - C_2$

$C_2 - C_3$

$$= -8xyz(u^2 + v^2 + w^2 + 1)$$

$$= -8xyz \left[\frac{x^2}{1-r^2} + \frac{y^2}{1-r^2} + \frac{z^2}{1-r^2} + 1 \right]$$

$$= -8xyz \frac{x^2 + y^2 + z^2 + 1 - r^2}{1 - r^2}$$

$$= \frac{-8xyz}{1 - r^2}; \text{ also .}$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(u_1, v_2, w_3)} = \begin{vmatrix} 2u(1-x^2-y^2-z^2) & 0 & 0 \\ 0 & 2u(1-x^2-y^2-z^2) & 0 \\ 0 & 0 & 2u(1-x^2-y^2-z^2) \end{vmatrix}$$

$$= 8uvw(1-x^2-y^2-z^2)^3$$

$$= 8 \frac{x}{\sqrt{1-x^2}} \cdot \frac{y}{\sqrt{1-r^2}} \cdot \frac{z}{\sqrt{1-r^2}} (1-r^2)^3 = 8xyz(1-r^2)^{1/2}$$

Hence from (i), we have $\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \frac{-8xyz}{1-r^2} \bigg/ 8xyz(1-r^2)^{1/2} = \frac{1}{\sqrt{1-r^2}}$

Cor: Given $u = \frac{x}{k}, v = \frac{y}{k}, w = \frac{z}{k}$

where $k = \sqrt{1-x^2-y^2-z^2}$, find $J \frac{(x, y, z)}{(u, v, w)}$ in terms of k. [Ans : k^5]

Exercise - 3(H)

1. If $x+y+z = u$, $y+z = uv$, $z = uvw$, then prove that $\frac{\partial(x, y, z)}{\partial(u, v, w)} = u^2v$
2. If $x^2 + y^2 + u^2 - v^2 = 0$, $uv + xy = 0$, prove that $\frac{\partial(u, v)}{\partial(x, y)} = \frac{x^2 - y^2}{u^2 + v^2}$
3. If $u_1 = x_1 + x_2 + x_3 + x_4$, $u_1u_2 = x_2 + x_3 + x_4$, $u_1u_2u_3 = x_4$ then prove that $\frac{\partial(x_1, x_2, x_3, x_4)}{\partial(u_1, u_2, u_3, u_4)} = u_1^3 u_2^2 u_3$
4. If $u^3 + v^3 + w^3 = x + y + z$, $u^2 + v^2 + w^2 = x^3 + y^3 + z^3$, $u + v + w = x^2 + y^2 + z^2$, then show that $\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$

3.7.4 Use of Jacobians in determining functional dependence and independence of functions.

Let u and v be two functions of x and y connected by the relation $v = f(u)$, then we say that u and v are functionally dependent. We shall prove that the condition for functional dependence is

$$\frac{\partial(u, v)}{\partial(x, y)} = 0$$

Consider $w = v - f(u) = 0$

Now w is a function of u and v where u, v are functions of x and y .

$\therefore$ w is a composite function of x and y

$$\therefore \frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x} \quad \dots (i)$$

$$\therefore \frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x} \quad \dots (ii)$$

But w considered as a function of x, y is identically zero.

$$\text{i.e.,} \quad \frac{\partial w}{\partial x} = 0, \quad \frac{\partial w}{\partial y} = 0$$

∴ From (i) and (ii) we get

$$\frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x} = 0 \quad \dots (iii)$$

and

$$\frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial y} = 0 \quad \dots (iv)$$

Eliminating $\frac{\partial w}{\partial u} \cdot \frac{\partial w}{\partial v}$ from (iii) and (iv) we have

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\Rightarrow \frac{\partial(u, v)}{\partial(x, y)} = 0$$

The concept of functional dependence can be extended to any number of variables. Thus if u, v, w are functions of x, y, z the u, v, w will be functionally dependent (i.e., there will exist a relation between u, v, w) if

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$$

3.14.5 Example

Are $x + y - z, x - y + z, x^2 + y^2 + z^2 - 2yz$ functionally dependent? If so, find a relation between them.

Solution

$$\text{Let } u = x + y - z \quad \dots (i)$$

$$v = x - y + z \quad \dots (ii)$$

$$w = x^2 + y^2 + z^2 - 2yz \quad \dots (iii)$$

$$\begin{aligned} \text{Then } \frac{\partial(u, v, w)}{\partial(x, y, z)} &= \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 2x & 2y-2z & 2z-2y \end{vmatrix} \\ &= - \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 2x & 2y-2z & 2y-2z \end{vmatrix} = 0 \end{aligned}$$

(since 2nd and 3rd columns are identical)

$\therefore u, v, w$ are functionally dependent

To find the relation between u, v, w we notice that, from (i) and (ii)

$$u + v = 2x, \quad u - v = 2(y - z)$$

$\therefore$ From (iii)

$$\begin{aligned} w &= x^2 + (y - z)^2 \\ &= \left(\frac{(u + v)}{2} \right)^2 + \left(\frac{(u - v)}{2} \right)^2 \\ &= \frac{1}{2} (u^2 + v^2) \end{aligned}$$

$$\Rightarrow u^2 + v^2 = 2w, \text{ the required relation}$$

3.14.6 Example

If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = x^3 + y^4 + z^3 - 3xyz$ then prove that u, v, w are not independent and find the relation between them.

Solution

Condition for functional dependence is $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$

$$\begin{aligned} \text{Now } \frac{\partial(u, v, w)}{\partial(x, y, z)} &= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 3x^2 - 3yz & 3y^2 - 3zx & 3z^2 - 3xy \end{vmatrix} \\ &= 6 \begin{vmatrix} 1 & 1 & 1 \\ x & y & 2z \\ x^2 - yz & y^2 - zx & z^2 - 3xy \end{vmatrix} C_1 - C_2, C_2 - C_3 \end{aligned}$$

$$\begin{aligned}
&= 6 \begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ (x-y)x+y+z & y-z(x+y+z) & z^2-xy \end{vmatrix} \\
&= 6(x-y)(y-z) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & z \\ x+y+z & x+y+z & z^2-xy \end{vmatrix} \\
&= 0 \quad (\because C_1 = C_2)
\end{aligned}$$

$\therefore u, v, w$ are functionally dependent. We shall now find the relation between them

we have $w = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$ (A)

$$\begin{aligned}
\text{Now } u^2 - v &= (x + y + z)^2 - (x^2 + y^2 + z^2) \\
&= 2(xy + yz + zx)
\end{aligned}$$

$\therefore$ We write (A) as

$$w = u \left[v - \frac{u-v}{2} \right]$$

$$\text{Thus } 2w = u(3v - u^2)$$

3.14.7 Example

Test whether $u = \frac{x+y}{x-y}, v = \frac{xy}{(x-y)^2}$ are functionally dependent and if so, find

the relation between them.

Solution

Treating u, v as functions of x, y the condition for functional dependence is

$$\frac{\partial(u, v)}{\partial(x, y)} = 0$$

$$u = \frac{x+y}{x-y}, \quad v = \frac{xy}{(x-y)^2}$$

We have $u_x = \frac{(x-y).1 - (x+y).1}{(x-y)^2} = \frac{-2y}{(x-y)^2}$

$$u_y = \frac{(x-y).(1).(x+y).(1)}{(x-y)^2} = \frac{2x}{(x-y)^2}$$

$$\begin{aligned} V_x &= \frac{(x-y)^2 y - 2(x-y).1.xy}{(x-y)^4} \\ &= \frac{((x-y)y - 2xy)}{(x-y)^3} \\ &= \frac{xy - y^2 - 2xy}{(x-y)^3} = \frac{-y(y+x)}{(x-y)^3} \end{aligned}$$

Similarly $v_y = \frac{x(x+y)}{(x-y)^3}$

$$\begin{aligned} \therefore \frac{\partial(u, v)}{\partial(x, y)} &= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \\ &= u_x v_y - v_x u_y \\ &= \frac{-2y}{(x-y)^2} \cdot \frac{x(x+y)}{(x-y)^3} + \frac{y(x+y)}{(x-y)^3} \cdot \frac{2x}{(x-y)^2} = 0 \end{aligned}$$

$\therefore u, v$ are functionally dependent

We shall now find the relation between them

We have $u = \frac{x+y}{x-y}$

$$\therefore \frac{x-y}{x+y} = \frac{1}{u}$$

By componendo and dividendo

$$\frac{x}{y} = \frac{1+u}{1-u}$$

Now
$$v = \frac{xy}{(x-y)^2} = \frac{y^2 \frac{x}{y}}{\left(\frac{x}{y} - 1\right)^2} = \frac{\frac{x}{y}}{\left(\frac{x}{y} - 1\right)^2} = \frac{\frac{1+u}{1-u}}{\left(\frac{(1+u)}{(1-u)} - 1\right)^2} \quad \text{by (A)}$$

$$v = \left(\frac{1-u^2}{4u^2} \right)$$

$$\therefore u^2(4v+1) = 1$$

The required relation of functional dependence between u, v .

3.14.8 Example

Examine for functional dependence of

$$u = \sin^{-1} x + \sin^{-1} y$$

$$v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

and find the relation between them, if it exists.

Solution

We have
$$u_x = \frac{1}{\sqrt{1-x^2}}$$

$$u_y = \frac{1}{\sqrt{1-y^2}}$$

$$\begin{aligned} v_x &= \sqrt{1-y^2} + y \frac{1}{2\sqrt{1-x^2}} (-2x) \\ &= \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} \end{aligned}$$

Similarly
$$v_y = \sqrt{1-x^2} - \frac{xy}{\sqrt{1-y^2}}$$

$$\therefore \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= u_x v_y - u_y v_x$$

$$= \left(1 - \frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}} \right) - \left(1 - \frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}} \right) = 0.$$

$\therefore u, v$ are functionally dependent we shall now determine the relation between u and v

$$\text{Let } A = \sin^{-1} x, B = \sin^{-1} y; \Rightarrow x = \sin A \quad y = \sin B$$

The given function can then be written as $u = A + B$

$$v = \sin A \sqrt{1 - \sin^2 B} + \sin B \sqrt{1 - \sin^2 A} = \sin(A+B) = \sin u$$

Hence the required relation of functional dependence between u and v is $v = \sin u$

3.14.8(a) If $u = e^x \sin y, v = e^x \cos y$, show that u, v are functionally independent

Solution: The Jacobian $\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} e^x \sin y & e^x \cos y \\ e^x \cos y & -e^x \sin y \end{vmatrix} = -e^{2x} \neq 0.$

$\therefore u, v$ are functionally independent.

3.14.9 Verify whether $u = \frac{x-y}{x+z}, v = \frac{x+z}{y+z}$ are functionally dependent, and if so find

the relation.

Solution: Treating u, v as functions of x, y regarding z as an absolute constant, the condition for functional dependent in

$$\frac{\partial(u, v)}{\partial(x, y)} = 0,$$

$$\text{Now } \frac{\partial u}{\partial x} = \frac{y+z}{(x+z)^2}, \frac{\partial u}{\partial y} = -\frac{1}{x+z} \text{ and } \frac{\partial v}{\partial x} = \frac{1}{y+z}, \frac{\partial v}{\partial y} = -\frac{x+z}{(y+z)^2}$$

$$\therefore \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{y+z}{(x+z)^2} & -\frac{1}{x+z} \\ \frac{1}{y+z} & -\frac{x+z}{(y+z)^2} \end{vmatrix} = 0$$

$\therefore$ u, v are functionally dependent. To find the relation between u and v , we eliminate 'x' between the given relation viz:

$$u = \frac{x-y}{x+z} \text{----- (A), } v = \frac{x+z}{y+z} \text{----- (B)}$$

$$\text{From (A) } ux+vy = x-y. \Rightarrow (1-u)x=uz+y \Rightarrow x = \frac{uz+y}{1-u}$$

$$\text{Substituting for 'x' in 'B', } v = \frac{\frac{uz+y}{1-u} + z}{y+z} = \frac{1}{1-u} \cdot \frac{(y+z)}{y+z} = \frac{1}{1-u}$$

Hence the required functional relation between 'u' and 'v' is $v(1-u)=1$.

Exercise – 3(I)

1. Verify whether the following are functionally dependent and if so, find the relation between them.

$$(a) u = \frac{x-y}{x+y}, v = \frac{x+y}{x} \quad (\text{Ans: } u(1+v) = 2)$$

$$(b) u = \frac{x-y}{1-xy}, v = \tan^{-1} x + \tan^{-1} y \quad (\text{Ans: } v = \tan^{-1} u)$$

$$(c) u = \frac{x-y}{x+y}, v = \frac{xy}{(x+y)^2} \quad (\text{Ans: } 4v+u^2 = 1)$$

2. Prove that the following functions are not independent .

$$(a) u = x + 2y + z, v = x - 2y + 3z, w = 2xy - 2z + 4yz - 2z^2$$

$$(b) u = x + y - z, v = x - y + z, w = x^2 + y^2 + z^2 - 2yz$$

$$(c) u = x + y + z, v = xy + yz + zx, w = x^3 + y^3 + z^3 - 3xyz$$

3. If $u = \frac{x+y}{z}, v = \frac{y+z}{x}, w = \frac{y(x+y+z)}{xz}$, show that u, v are connected by a functional relation.

4. **Examine** for functional dependence between $u = \frac{x-y}{1-xy}$, $v = \tan^{-1} x - \tan^{-1} y$. If dependent, find the relation. (Ans: $u = \tan v$)
5. **Prove** that functions $u = y + z$, $v = x + 2z^2$, $w = x - 4yz - 2y^2$ are functionally dependent and find the relation between them. (Ans: $v = w + 2u^2$)

3.15.1 Maxima and Minima

The method of finding the maxima and minima of functions of two and three independent variables is discussed in this topic.

The maximum or minimum values are also called Stationary or extreme or turning values. For finding these values we use Taylor's theorem for functions of two variables.

3.15.2 Taylor's theorem for a function of two variables:

We know that Taylor's theorem for a function $f(x)$ of a single variable 'x' is

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \dots$$

....(1)

Now, consider $f(x,y)$, a function of two independent variables. Keeping 'y' constant and following (1) we get

$$f(x+h, y+k) = f(x, y+k) + h \frac{\partial}{\partial x} f(x, y+k) + \frac{h^2}{2!} \frac{\partial^2}{\partial x^2} f(x, y+k) + \dots \dots (2)$$

Keeping 'x' constant and by applying (1), we get

$$f(x+h, y+k) = f(x, y) + k \frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!} \frac{\partial^2}{\partial y^2} f(x, y) + \dots$$

....(3)

Substituting (3) in (2) we get

$$\begin{aligned} f(x+h, y+k) = & \left[f(x, y) + k \frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!} \frac{\partial^2}{\partial y^2} f(x, y) + \dots \right] \\ & + h \frac{\partial}{\partial x} \left[f(x, y) + k \frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!} \frac{\partial^2}{\partial y^2} f(x, y) + \dots \right] \\ & + \frac{h^2}{2!} \frac{\partial^2}{\partial x^2} \left[f(x, y) + k \frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!} \frac{\partial^2}{\partial y^2} f(x, y) + \dots \right] + \dots \end{aligned}$$

Substituting (iii) in (ii) we get

$$\begin{aligned}
 f(x + hy + k) &= \left[f(x, y) + k \frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!} \frac{\partial^2}{\partial y^2} f(x, y) + \dots \right] \\
 &\quad + h \frac{\partial}{\partial x} \left[f(x, y) + k \frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!} \frac{\partial^2}{\partial y^2} f(x, y) + \dots \right] \\
 &\quad + \frac{h^2}{2} \frac{\partial^2}{\partial x^2} \left[f(x, y) + k \frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2} \frac{\partial^2}{\partial y^2} f(x, y) + \dots \right] \\
 &\quad + \dots \\
 &= \left[f(x, y) + k \frac{\partial f}{\partial y} + \frac{k^2}{2} \frac{\partial^2 f}{\partial y^2} + \dots \right] + \left[h \frac{\partial f}{\partial x} + hk \frac{\partial^2 f}{\partial x \partial y} + \dots \right] \\
 &\quad + \left[\frac{h^2}{2} \frac{\partial^2 f}{\partial x^2} + \dots \right]
 \end{aligned}$$

Thus

$$\begin{aligned}
 f(x + h, y + k) &= f(x, y) + \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) f \\
 &\quad + \frac{1}{2} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^2 f + \dots \quad \dots \text{(iv)}
 \end{aligned}$$

Expression (iv) is known as Taylor's expansion for functions of two variables.

Writing $x = a$, $y = b$ expansion about (a, b) is given by

$$\begin{aligned}
 f(a + h, b + k) &= f(a, b) + \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) f_{(a, b)} \\
 &\quad + \frac{1}{2!} \left[h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right]^2 f_{(a, b)} + \dots
 \end{aligned}$$

Now if we call $h = x - a$, $y - b = k$

we get

$$f(x, y) = f(a, b) + \left((x - a) \frac{\partial}{\partial x} + (y - b) \frac{\partial}{\partial y} \right) f_{(a, b)}$$

$$+ \frac{1}{2!} \left((x-a) \frac{\partial}{\partial x} + (y-b) \frac{\partial}{\partial y} \right)^2 f_{(a,b)} + \dots \quad \dots (v)$$

Writing $a = 0 = b$, we have

$$f(x, y) = f(0, 0) + \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) f_{(0,0)} + \dots \quad \dots (vi)$$

This is Maclaurin's series for two variables.

3.15.3 Example

Expand $e^{x^2+y^2}$ at $(1, 1)$

Solution

$$\text{We have } f(x, y) = e^{x^2+y^2} \Rightarrow f(1,1) = e^2$$

$$f_x(x, y) = 2xe^{x^2+y^2} \Rightarrow f_x(1, 1) = 2e^2$$

$$f_y(x, y) = 2ye^{x^2+y^2} \Rightarrow f_y(1, 1) = 2e^2$$

$$f_{xx}(x, y) = 2e^{x^2+y^2} + 4x^2e^{x^2+y^2} \Rightarrow f_{xx}(1, 1) = 6e^2$$

$$f_{yx}(x, y) = 4xye^{x^2+y^2} \Rightarrow f_{yx}(1, 1) = 4e^2$$

$$f_{yy}(x, y) = 2e^{x^2+y^2} + 4y^2e^{x^2+y^2} \Rightarrow f_{yy}(1, 1) = 6e^2$$

Hence putting these values in second form of Taylor's theorem, we get

$$e^{x^2+y^2} = e^2 [1 + 2(x-1) + 2(y-1)] + \frac{1}{2!} [6(x-1)^2 + 8(x-1)(y-1) + 6(y-1)^2 + \dots]$$

3.15.4 Example

Expand $e^x \sin y$ in powers of x and y as far as terms of third degree

Solution

$$f(x, y) = e^x \sin y \Rightarrow f(0, 0) = 0$$

$$f_x(x, y) = e^x \sin y \Rightarrow f_x(0, 0) = 0$$

$$f_y(x, y) = e^x \cos y \Rightarrow f_y(0, 0) = 1$$

$$\begin{aligned}
f_{xx}(x, y) &= e^x \sin y \Rightarrow f_{xx}(0, 0) = 0 \\
f_{xy}(x, y) &= e^x \cos y \Rightarrow f_{xy}(0, 0) = 1 \\
f_{yy}(x, y) &= e^x \sin y \Rightarrow f_{yy}(0, 0) = 0 \\
f_{xxx}(x, y) &= e^x \sin y \Rightarrow f_{xxx}(0, 0) = 0 \\
f_{xxy}(x, y) &= e^x \sin y \Rightarrow f_{xxy}(0, 0) = 1 \\
f_{xyy}(x, y) &= e^x \sin y \Rightarrow f_{xyy}(0, 0) = 0 \\
f_{yyy}(x, y) &= e^x \cos y \Rightarrow f_{yyy}(0, 0) = -1
\end{aligned}$$

Then by Taylor's theorem we have

$$\begin{aligned}
f(x, y) &= f(0, 0) + [xf_x(0, 0) + yf_y(0, 0)] \\
&+ \frac{1}{2!} [x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] + \dots \\
&= 0 + x(0) + y(1) + \frac{x^2}{2}(0) + xy(1) + \frac{y^2}{2}(0) + \frac{x^2}{6}(0) + \dots \\
&= y + xy + \frac{x^2 y}{2} - \frac{y^3}{6} + \dots
\end{aligned}$$

3.15.5 Example

$f(x, y) = x^2 y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$ by Taylor's theorem

Solution

$$\begin{aligned}
f(x, y) &= f(a, b) + [(x-a)f_x(a, b) + (y-b)f_y(a, b)] \\
&+ \frac{1}{2!} (x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b)f_{xy}(a, b) \\
&+ (y-b)^2 f_{yy}(a, b) + \dots \quad \dots (i)
\end{aligned}$$

Here $a = 1, b = -2$

$$\begin{aligned}
f(x, y) &= x^2 y + 3y - 2 \Rightarrow f(1, -2) = -10 \\
f_x(x, y) &= 2xy \Rightarrow f_x(1, -2) = -4 \\
f_y(x, y) &= x^2 + 3 \Rightarrow f_y(1, -2) = 1 + 3 = 4 \\
f_{xx}(x, y) &= 2y \Rightarrow f_{xx}(1, -2) = 2(-2) = -4 \\
f_{xy}(x, y) &= 2x \Rightarrow f_{xy}(1, -2) = 2(1) = 2 \\
f_{yy}(x, y) &= 0 \Rightarrow f_{yy}(1, -2) = 0
\end{aligned}$$

$$f_{xxx}(x, y) = 0 \Rightarrow f_{xxx}(1, -2) = 0$$

$$f_{xxy}(x, y) = 2 \Rightarrow f_{xxy}(1, -2) = 2$$

$$f_{xyy}(x, y) = 0 \Rightarrow f_{xyy}(1, -2) = 0$$

$$f_{yyy}(x, y) = 0 \Rightarrow f_{yyy}(1, -2) = 0$$

Substituting $a = -1$, $b = 2$ and above values in (i) we have

$$\begin{aligned} x^2y + 3y - 2 &= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 \\ &\quad + 2(x-1)(y+2) + (x-1)^2(y+2) \end{aligned}$$

3.15.6 Example

Evaluate $\log[(1.03)^{\frac{1}{3}} + (0.98)^{\frac{1}{4}} - 1]$, approximately.

Solution

Let $F(x, y) = \log\left(x^{\frac{1}{3}} + y^{\frac{1}{4}} - 1\right)$

Then $F(x + h, y + k) = \log\left((x+h)^{\frac{1}{3}} + (y+k)^{\frac{1}{4}} - 1\right)$

Assuming $x = 1, y = 1, h = 0.03, k = -0.02$

We get $F(x+h, y+k) = \log\left[(1.03)^{\frac{1}{3}} + (0.98)^{\frac{1}{4}} - 1\right]$

$$= F(x, y) + \left(h \frac{\partial F}{\partial x} + k \frac{\partial F}{\partial y}\right) \text{ approximately (A)}$$

$$\frac{\partial F}{\partial x} = \frac{\frac{1}{3} x^{-\frac{2}{3}}}{x^{\frac{1}{3}} + y^{\frac{1}{4}} - 1}; \quad \frac{\partial F}{\partial y} = \frac{\frac{1}{4} y^{-\frac{3}{4}}}{x^{\frac{1}{3}} + y^{\frac{1}{4}} - 1}$$

$$\Rightarrow \log\left[(x+h)^{\frac{1}{3}} + (y+k)^{\frac{1}{4}} - 1\right] = F(x + h, y + k)$$

$$= F(x, y) + h \left(\frac{\partial F}{\partial x}\right) + k \left(\frac{\partial F}{\partial y}\right) \text{ approximately}$$

$$\begin{aligned}
&\Rightarrow \log \left[(1.03)^{\frac{1}{3}} + (0.98)^{\frac{1}{4}} - 1 \right] \\
&= F(1, 1) + 0.03 \left(\frac{\partial F}{\partial x} \right)_{(1,1)} + (-0.02) \left(\frac{\partial F}{\partial y} \right)_{(1,1)} \\
&= 0.03 \left[\frac{\frac{1}{3}}{1+1-1} \right] - 0.02 \left[\frac{\frac{1}{4}}{1+1-1} \right] = 0.005 \text{ approximately}
\end{aligned}$$

$$(\because F(1,1) = 0)$$

Exercise – 3(J)

1. Obtain the expansion using Taylor's theorem of the following :

1. $xy^2 + \cos(xy)$ at $\left(1, \frac{\pi}{4}\right)$

$$[\text{Ans : } \frac{\pi^2}{4} + \left[(x-1) \left(\frac{\pi^2}{4} - \frac{\pi}{2} \right) + \left(y - \frac{\pi}{2} \right) (\pi - 1) \right] + \dots]$$

2. x^y at $(1, 1)$ upto second term

$$[\text{Ans : } x^y = 1 + (x-1) + (x-1) \times (y-1) + \frac{1}{2} (x-1)^2]$$

3. $\sin x \cdot \sin y$ in powers of $\left(x - \frac{\pi}{4}\right)$ and $\left(y - \frac{\pi}{4}\right)$

$$\begin{aligned}
[\text{Ans : } &\frac{1}{2} + \frac{1}{2} \left(x - \frac{\pi}{4} \right) + \frac{1}{2} \left(y - \frac{\pi}{4} \right) - \frac{1}{4} \left(x - \frac{\pi}{4} \right)^2 \\
&+ \frac{1}{2} \left(x - \frac{\pi}{4} \right) \left(y - \frac{\pi}{4} \right) - \frac{1}{4} \left(y - \frac{\pi}{4} \right)^2 + \dots]
\end{aligned}$$

4. $e^{ax} \sin by$ at $(0, 0)$

$$[\text{Ans : } by + xyab + \dots]$$

5. Prove that $\sin(x+y) = x + y - \frac{(x+y)^3}{<3} + \dots$
6. Show that $e^y \log(1+x) = x + xy - \frac{x^2}{2}$ approximately
7. Expand $f(x, y) = \sin xy$ in powers of $(x-1)$ and $\left(y - \frac{\pi}{2}\right)$ upto second degree term.

$$[\text{Ans : } 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)\left(y - \frac{\pi}{2}\right) - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2 + \dots]$$

3.16.7 Maxima and Minima

The function $F(x, y)$ is maximum at (x, y) if for all small positive or negative values of h and k ; we have

$$f(x+h, y+k) - F(x, y) < 0 \quad \dots (i)$$

Similarly, $F(x, y)$ is minimum at (x, y) if for all small positive or negative values of h and k ; we have

$$F(x+h, y+k) - F(x, y) > 0$$

Thus if $F(x, y)$ has a minimum or maximum value then $[F(x+h, y+k) - F(x, y)]$ keeps the same sign for all small positive or negative values of h and k .

Now using Taylor's expansion, we get

$$\begin{aligned} F(x+h, y+k) - F(x, y) &= \left(h \frac{\partial F}{\partial x} + k \frac{\partial F}{\partial y} \right) \\ &+ \frac{1}{2!} \left(h^2 \frac{\partial^2 F}{\partial x^2} + 2hk \frac{\partial^2 F}{\partial x \partial y} + k^2 \frac{\partial^2 F}{\partial y^2} \right) + \dots \dots (iii) \end{aligned}$$

Assuming h, k to be sufficiently small ; the sign of the expression on the left can be made to depend on that of $h \frac{\partial F}{\partial x} + k \frac{\partial F}{\partial y}$

Hence the necessary condition for $f(x, y)$ to be a maximum or minimum is that

$$h \frac{\partial F}{\partial x} + k \frac{\partial F}{\partial y} = 0 \quad \forall h \text{ and } K$$

$$\therefore \quad \frac{\partial F}{\partial x} = 0 \text{ and } \frac{\partial F}{\partial y} = 0$$

$$\text{i.e.,} \quad p = 0 \text{ and } q = 0 \quad \text{where } p = \frac{\partial F}{\partial x}, \quad q = \frac{\partial F}{\partial y}$$

$$\begin{aligned} \text{then (iii)} \Rightarrow f(x+h, y+k) - f(x, y) &= \frac{1}{2!} \left[h^2 \frac{\partial^2 F}{\partial x^2} + 2hk \frac{\partial^2 F}{\partial x \partial y} + K^2 \frac{\partial^2 F}{\partial y^2} \right] + \dots \\ &= \frac{1}{2!} [rh^2 + 2shk + tk^2] \end{aligned}$$

$$\begin{aligned} &\left(\text{where } r = \frac{\partial^2 F}{\partial x^2}, \quad s = \frac{\partial^2 F}{\partial x \partial y}, \quad z = \frac{\partial^2 F}{\partial y^2} \right) \\ &= \frac{r}{2!} \left[h^2 + 2hk \frac{s}{r} + \frac{t}{r} K^2 \right] \\ &= \frac{r}{2!} \left[\left(h + \frac{s}{r} k \right)^2 + \left(\frac{-s^2}{r^2} k^2 + t \frac{k^2}{r^2} \right) \right] \\ &= \frac{r}{2!} \left[\left(h + \frac{s}{r} k \right)^2 + (rt - s^2) \frac{k^2}{r^2} \right] \end{aligned}$$

It has the same sign for all h, k if and only if $(rt - s^2)$ is positive.

When (i) $(rt - s^2)$ positive, $f(x, y)$ is maximum for negative 'r' and minimum for positive 'r'.

(ii) $(rt - s^2)$ is negative, we have neither a maximum nor minimum and such stationary points are called *saddle points*.

Solving $p = 0, q = 0$, we get the extreme points

Extreme value :

$f(a, b)$ is said to be an extreme value of $f(x, y)$ if it is either maximum or minimum.

3.16.8 Example

Find the maximum and minimum values of $x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$.

Solution

We have $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

$$f_x = 3x^2 + 3y^2 - 6x, \quad f_y = 6xy - 6y, \quad f_{yy} = 6x - 6, \quad f_{xy} = 6y$$

We now solve

$$f_x = 0, f_y = 0 \text{ simultaneously}$$

$$\text{i.e., } x^2 + y^2 - 2x = 0 \quad \dots (i)$$

$$\text{and } xy - y = 0 \quad \dots (ii)$$

$$\therefore xy - y = 0 \Rightarrow y = 0 \text{ or } x = 1$$

$$\text{From } x^2 + y^2 - 2x = 0 \text{ we get}$$

$$\text{when } y = 0, x^2 - 2x = 0$$

$$\therefore x = 2 \text{ or } x = 0$$

$$\text{when } x = 1, y^2 - 1 = 0 \text{ i.e., } y = 1 \text{ or } -1$$

Thus the points are (0, 0), (2, 0), (1, 1), (1, -1)

$$(a) \text{ For } x = 0, y = 0, r = f_{xx} = -6, s = 0, t = -6$$

$$\therefore rt - s^2 = 36 - 0 > 0$$

$$\therefore f(x, y) \text{ is stationary at } x = 0, y = 0$$

$$\text{But } r = f_{xx} = -6 < 0$$

$$\therefore f(x, y) \text{ is maximum at } x = 0, y = 0 \text{ and the maximum value } f(0, 0) = 4$$

$$(b) \text{ For } x = 2, y = 0, r = f_{xx} = 6, s = f_{xy} = 0, t = f_{yy} = 6$$

$$\text{Again } rt - s^2 > 0$$

$$\therefore f(x, y) \text{ is stationary at } x = 2, y = 0$$

$$\text{But } r = f_{xx} = 6 > 0$$

$$\therefore f(x, y) \text{ is minimum at } x = 2, y = 0 \text{ and minimum value } f(2, 0) = 0$$

$$(c) \text{ At } x = 1, y = 1 \text{ or at } x = 1, y = -1$$

$rt - s^2 > 0$ and we can reject these points as they are not stationary points.

3.16.9 Example

Discuss the maximum and minimum of

$$f(x, y) = x^2 - y^2 + 6x - 12$$

$$p = 2x + 6, q = -2y, r = 2, s = 0, t = -2$$

$$p = 0 \text{ gives } x = -3, q = 0 \rightarrow y = 0$$

Stationary point is $(-3, 0)$; and here $r = 2 > 0$; and

$$rI - s^2 = 4 - 0 = 4 > 0$$

$x^2 - y^2 + 6x - 12$ is minimum at $(-3, 0)$ the minimum value is

$$f(-3, 0) = -21.$$

Exercise -3(k)

1. Find the maximum and minimum values of :

1. $x^3 + 3xy^2 - 3y^2 + 4$ [Ans : $(0, 0)$ max. its value 4, $(2, 0)$ min . its value '0']

2. $x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$ [Ans : max . value 7 at $(0, 0)$ min.value '3' at $(2, 0)$]

3. $x^2 + y^2 + 6x + 12$ [Ans : min. value '3' at $(-3, 0)$]

4. $x^3 + xy^2 + 21x - 12x^2 - 2y^2$ [Ans : max.value 10 at $(1, 0)$ min. value -98 at $(7, 0)$]

5. $x^4 + xy^2 + y^2$ [Ans: minimum value '0' at $(0, 0)$]

6. $x^4 + y^4 - x^2 - y^2 + 1$

[Ans: max.value 1 at $(0, 0)$ min.value $\frac{1}{2}$ at four points $\left(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}\right)$]

7. $x^2y^2 - x^3y^2 - x^2y^3$ [Ans: max.value $\frac{1}{324}$ at $\left(\frac{1}{2}, \frac{1}{2}\right)$]

8. Discuss the stationary values of $u = x^2y - 3x^2 - 2y^2 - 4y + 3$ [Ans: u is a maximum at $(0, -1)$. The maximum value is 5, u is neither maximum nor minimum at $(\pm 4, 3)$]

9. Find the maximum and minimum values of the following functions.

(a) $x^3y^2(12 - 3x - 4y)$ [Ans: maximum value at $(2, 1)$]

(b) $x^3 + y^3 - 3axy (a > 0)$ [Ans: minimum value is $-a^3$ at (a, a)]

(c) $xy(a - x - y) (a > 0)$ [Ans: maximum value is $\frac{a^3}{27}$ at $(a/3, a/3)$]

10. Determine the values of x, y for which the following functions are maximum or minimum..

(a) $x^3y^2(1 - x - y)$ [Ans: maximum for $x = 1/2$, $y = 1/3$]

(b) $(x^2 + y^2)^2 - 2a^2(x^2 - y^2)$ [Ans: minimum for $x = \pm a, y = 0$]

(c) $\sin x \sin y \sin(x + y)$ [Ans : maximum for $x = y = \pi/3$]

(d) $a[\sin x + \sin y + \sin(x + y)]$ [Ans: maximum for $x = y = \pi/3$]

(e) $xy^2(3x + 6y - 2)$ [Ans : minimum at $x = y = 1/6$]

Lagrange's method of undetermined multipliers :

3.17.0 Theorem: Find the maximum and minimum values of $f(x_1, x_2, x_3, \dots, x_n)$

where $x_1, x_2, x_3, \dots, x_n$ are connected by the following m equations.

$$\phi_1(x_1, x_2, \dots, x_n) = 0$$

$$\phi_2(x_1, x_2, \dots, x_n) = 0$$

$$\dots \dots \dots$$

$$\dots \dots \dots$$

$$\phi_m(x_1, x_2, \dots, x_n) = 0.$$

Proof: Let $u = f(x_1, x_2, \dots, x_n)$. Since n variables are connected by the m given relations, only $n-m$ of the variables are independent.

The maximum and minimum values of u can be found by the method of Lagrange's method of undetermined Multipliers. For u to be max or min of u , $du = 0$

$$\text{i.e. } \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \dots + \frac{\partial u}{\partial x_n} dx_n = 0$$

also from, we have

.....(1)

multiplying the above equations by $1, \lambda_1, \lambda_2, \dots, \lambda_m$ respectively and adding, we get

$$\begin{aligned} & \left(\frac{\partial u}{\partial x_1} + \lambda_1 \frac{\partial \phi_1}{\partial x_1} + \lambda_2 \frac{\partial \phi_2}{\partial x_1} + \dots + \lambda_m \frac{\partial \phi_m}{\partial x_1} \right) dx_1 + \left(\frac{\partial u}{\partial x_2} + \lambda_1 \frac{\partial \phi_1}{\partial x_2} + \lambda_2 \frac{\partial \phi_2}{\partial x_2} + \dots + \lambda_m \frac{\partial \phi_m}{\partial x_2} \right) dx_2 \\ & + \dots \dots \dots \\ & + \left(\frac{\partial u}{\partial x_n} + \lambda_1 \frac{\partial \phi_1}{\partial x_n} + \lambda_2 \frac{\partial \phi_2}{\partial x_n} + \dots + \lambda_m \frac{\partial \phi_m}{\partial x_n} \right) dx_n = 0. \end{aligned} \quad \dots(2)$$

The values $\lambda_1, \lambda_2, \dots, \lambda_m$ are at our choice. We can, therefore choose them so as to satisfy 'm' linear equations.

....(3)

It is immaterial which $n - m$ of the n variables are regarded as independent.

Let these be $x_{m+1}, x_{m+2}, \dots, x_n$. Then since the $n - m$ quantities $dx_{m+1}, dx_{m+2}, \dots, dx_n$ are all independent, their coefficient must be separately zero. This gives additional equations.

The lagrange's multipliers λ_1, λ_2 are choosen such that

$$\frac{\partial \phi}{\partial y} + \lambda_1 \frac{\partial f_1}{\partial y} + \lambda_2 \frac{\partial f_2}{\partial y} = 0 \quad \dots(4)$$

then equation (2) becomes

$$\frac{\partial \phi}{\partial z} + \lambda_1 \frac{\partial f_1}{\partial z} + \lambda_2 \frac{\partial f_2}{\partial z} = 0 \quad \dots(5)$$

These equations (3) ,(4), (5) and given $f_1(x, y, z) = 0$, $f_2(x, y, z) = 0$ are sufficient to find $\lambda_1, \lambda_2, x, y, z$ which give maximum and minimum values of u .

3.17.0(b) Theorem: If $u = \phi(x, y, z)$ and $f(x, y, z) = 0$ find the maximum or minimum values of u .

Proof: For a maximum or a minimum u , we have $du = 0$

$$\text{i.e. } \frac{\partial \phi}{\partial x} .dx + \frac{\partial \phi}{\partial y} .dy + \frac{\partial \phi}{\partial z} .dz = 0 \quad \dots(1)$$

and

from given $f(x, y, z) = 0$.

$$\frac{\partial f}{\partial x} .dx + \frac{\partial f}{\partial y} .dy + \frac{\partial f}{\partial z} .dz = 0 \quad \dots(2)$$

Multiplying (1) by 1 and (2) by λ and adding

$$\left(\frac{\partial \phi}{\partial x} + \lambda \frac{\partial f}{\partial x} \right) dx + \left(\frac{\partial \phi}{\partial y} + \lambda \frac{\partial f}{\partial y} \right) dy + \left(\frac{\partial \phi}{\partial z} + \lambda \frac{\partial f}{\partial z} \right) dz = 0 \quad \dots(3)$$

Lagrange's multiplier λ is chosen such that

$$\frac{\partial \phi}{\partial x} + \lambda \frac{\partial f}{\partial x} = 0$$

....(4)

$$\text{and } \frac{\partial \phi}{\partial y} + \lambda \frac{\partial f}{\partial y} = 0 \quad \dots(5)$$

The equation (3) gives

$$\frac{\partial \phi}{\partial z} + \lambda \frac{\partial f}{\partial z} = 0 \quad \dots(6)$$

The equations (4),(5),(6) and $f(x, y, z) = 0$ enable us to find the values of λ, x, y, z for which u is a maximum or a minimum.

Exercise – 3(I)

3.17.1 Example

Find the point upon the plane $ax + by + cz = p$ at which the function $\phi = x^2 + y^2 + z^2$ has a minimum value and find this minimum ϕ .

Solution

$$ax + by + z = p \Rightarrow z = \frac{p - ax - by}{c} \quad \dots (i)$$

$$\text{Again} \quad \phi = x^2 + y^2 + z^2 = x^2 + y^2 + \left(\frac{p - ax - by}{c} \right)^2$$

Then for min./max., we have

$$\frac{\partial \phi}{\partial x} = 2x - \frac{2a}{c^2} (p - ax - by) = 0 \quad \dots (ii)$$

$$\frac{\partial \phi}{\partial y} = 2y - \frac{2b}{c^2} (p - ax - by) = 0 \quad \dots (iii)$$

$$\text{From (B)} \quad \Rightarrow \frac{x}{a} = \frac{p - ax - by}{c^2}$$

$$\Rightarrow \frac{y}{b} = \frac{p - ax - by}{c^2}$$

$$\Rightarrow \frac{x}{a} = \frac{y}{b} \Rightarrow y = \frac{bx}{a}$$

Now, substituting this value of y in (ii) we obtain

$$x - \frac{a}{c^2} \left(p - ax - \frac{b^2 x}{a} \right) = 0$$

$$\Rightarrow c^2 x - ap + a^2 x + b^2 x = 0$$

$$\Rightarrow x(a^2 + b^2 + c^2) = ap$$

$$\Rightarrow x = \frac{ap}{a^2 + b^2 + c^2} \text{ and } y = \frac{bx}{a} = \frac{bp}{a^2 + b^2 + c^2}$$

$$\text{Then, } r = \frac{\partial^2 \phi}{\partial x^2} = 2 + \frac{2a^2}{c^2}; \frac{\partial^2 \phi}{\partial x \partial y} = \frac{2ab}{c^2}; s; \quad t = \frac{\partial^2 \phi}{\partial y^2} = 2 + \frac{2b^2}{c^2}$$

$$\text{and} \quad \frac{\partial^2 \phi}{\partial x^2} \times \frac{\partial^2 \phi}{\partial y^2} = 4 \left(1 + \frac{a^2}{c^2} + \frac{b^2}{c^2} \right) > 0 \text{ and } r \text{ is positive}$$

Hence $\exists$ a min, when

$$\frac{x}{a} = \frac{y}{b} = \frac{p}{a^2 + b^2 + c^2}$$

$$\begin{aligned} \text{Also } z &= \frac{p - ax - by}{c} = \frac{1}{c} \left[p - \frac{a^2 p}{a^2 + b^2 + c^2} - \frac{b^2 p}{a^2 + b^2 + c^2} \right] \\ &= \frac{cp}{a^2 + b^2 + c^2} \end{aligned}$$

$$\text{i.e., } \frac{z}{c} = \frac{p}{a^2 + b^2 + c^2}$$

Hence $\exists$ a min. at

$$\begin{aligned} \frac{x}{a} = \frac{y}{b} = \frac{z}{c} &= \frac{p}{a^2 + b^2 + c^2} \\ \phi_{\min} = x^2 + y^2 + z^2 &= \frac{a^2 p^2 + b^2 p^2 + c^2 p^2}{(a^2 + b^2 + c^2)^2} = \frac{p^2}{a^2 + b^2 + c^2} \end{aligned}$$

3.17.2 Example

Divide 24 into the three parts such that the continued product of the first, square of the second and cube of third is maximum.

Solution

Let 24 be divided into parts x, y, z then $x + y + z = 24$ (i)

Given $\phi(x, y, z) = x^2 y^2 z$

then $\phi(x, y) = x^2 y^2 (24 - x - y)$ from (i)

or $\phi(x, y) = 24x^3 y^2 - x^4 y^2 - x^3 y^3$

Differentiating, we get

$$p = \frac{\partial \phi}{\partial x} = 72x^2 y^2 - 4x^3 y^2 - 3x^2 y^3$$

$$r = \frac{\partial^2 \phi}{\partial x^2} = 144xy^2 - 12x^2 y^2 - 6xy^3$$

$$q = \frac{\partial \phi}{\partial y} = 48x^3 y - 2x^4 y - 3x^3 y^2$$

$$t = \frac{\partial^2 \phi}{\partial y^2} = 48x^3 - 2x^4 - 6x^3 y$$

and
$$s = \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial y} \right) = 144x^2y^2 - 8x^3y - 9x^2y^2$$

For a maximum value of $\phi(x, y)$ we have

$$\begin{aligned} \frac{\partial \phi}{\partial x} = 0 &\Rightarrow 72x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0 \\ &\Rightarrow x^2y^2(72 - 4x - 3y) = 0 \\ &\Rightarrow 72 - 4x - 3y = 0, \quad x = 0, y = 0 \quad \dots (ii) \end{aligned}$$

and
$$\begin{aligned} \frac{\partial \phi}{\partial y} = 0 &\Rightarrow 48x^3y - 2x^4y - 3x^3y^2 = 0 \\ &\Rightarrow x^3y(48 - 2x - 3y) = 0 \\ &\Rightarrow 48 - 2x - 3y = 0, \quad x = 0, y = 0 \quad \dots (iii) \end{aligned}$$

Solving (ii) and (iii) we get $x = 12$ and $y = 8$ etc

At $(12, 8)$, we have

$$\begin{aligned} r &= 144, 12(8)^2 - 12(12)^2(8) - 6(12)(8)^3 \\ &= 12(8)^2(144 - 144 - 48) = 12(8)^2(-48) \\ r &= 48(12)^3 - 2(12)^4 - 6(12)^3 \cdot 8 \\ &= (12)^3(48 - 24 - 48) \\ s &= 144(12)^2 \cdot 8 - 8(12)^3 \cdot 8 - 9(12)^2(8)^2 \\ &= (12)^2 \cdot 8(144 - 96 - 72) = (12)^2 \cdot 8(-24) \\ \therefore r - s^2 &= 12 \cdot (8)^2(-48) \cdot (12)^3(-24) - [(12)^2 \cdot 8 \cdot (-24)]^2 \\ &= (12)^4 \cdot 8^2 \cdot 24(48 - 24) \\ &= (12)^4 \cdot 8^2 \cdot 24^2 > 0; \end{aligned}$$

Since $r - s^2 > 0$ and $r < 0$, therefore $\phi(x, y)$ is maximum at $(12, 8)$.

Putting $x = 12$, and $y = 8$ in (i), we get $z = 4$

The values of x, y, z are 12, 8, 4 respectively. This is the division of 24 for maximum $\phi(x, y, z)$.

3.17.3 Example

A rectangular box, open at the top, is to have a volume of 32 c.c. Find the dimensions of the box requiring least material for its construction.

Solution

Let l , b and h be the length, breadth and height of the box respectively. Then, we have

$$v = lbh = 32 ; \text{ surface} = 2(l + b)h + lb = S \text{ (say)}$$

$$\therefore S = 2(l + b)h + lb \text{ and } b = \frac{32}{lh} \quad \dots(i)$$

$$\Rightarrow s = 2\left(l + \frac{32}{lh}\right)h + l\left(\frac{32}{lh}\right) = 2lh + \frac{64}{l} + \frac{32}{h}$$

$$\text{Now} \quad \frac{\partial s}{\partial l} = 2h - \frac{64}{l^2}; \quad \frac{\partial s}{\partial h} = 2l - \left(\frac{32}{h^2}\right)$$

$$\Rightarrow \quad \frac{\partial^2 s}{\partial l^2} = \frac{128}{l^3}, \quad \frac{\partial^2 s}{\partial l \partial h} = 2, \quad \frac{\partial^2 s}{\partial h^2} = \frac{64}{h^3}$$

For maximum and minimum of S , we get

$$\frac{\partial s}{\partial l} = 0 = \frac{\partial s}{\partial h}$$

$$\Rightarrow \quad \frac{\partial s}{\partial l} = 0 \Rightarrow 2h - \frac{64}{l^2} = 0, h = \frac{32}{l^2}$$

$$\text{and} \quad \frac{\partial s}{\partial h} = 0 \Rightarrow 2l - \frac{32}{h^2} = 0$$

$$\text{or} \quad l = \frac{16}{h^2}$$

These give $l = 4$, $h = 2$ and $b = 4$

$$\text{Now} \quad \frac{\partial^2 s}{\partial l^2} = \frac{128}{l^3} = \frac{128}{64} = 2, \quad \frac{\partial^2 s}{\partial l \partial h} = 2$$

$$\frac{\partial^2 s}{\partial h^2} = \frac{64}{h^3} = \frac{64}{8} = 8$$

$$\Rightarrow \quad \frac{\partial^2 s}{\partial l^2} \cdot \frac{\partial^2 s}{\partial h^2} - \left(\frac{\partial^2 s}{\partial l \partial h}\right)^2 = 2 \cdot 8 - (2)^2 = 12 = +ve.$$

and $\frac{\partial^2 s}{\partial h^2} = +2$ so 'S' is minimum for $l = 4$, $b = 4$, $h = 2$.

3.17.4 Example: If $u = a^3x^2 + b^3y^2 + c^3z^2$ where $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ show that the

stationary point of u is given by $x = \frac{a+b+c}{a}, y = \frac{a+b+c}{b}, z = \frac{a+b+c}{c}$

Solution: For a stationary value of ' u ', $du = 0$

$$\text{i.e. } a^3 2x dx + b^3 2y dy + c^3 2z dz = 0 \quad \dots(1)$$

Also we have the given relation $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

$$\frac{1}{x^2} dx + \frac{1}{y^2} dy + \frac{1}{z^2} dz = 0$$

.....(2)

Equating the coefficients of dx, dy, dz from (1)+ λ (2) separately to zero

$$\text{We get } a^3x + \frac{\lambda}{x^2} = 0, b^3x + \frac{\lambda}{y^2} = 0, c^3x + \frac{\lambda}{z^2} = 0.$$

$$\therefore a^3x^3 = b^3y^3 = c^3z^3 = (-\lambda)$$

$$\text{or } ax = by = cz = (-\lambda)^{1/3} = k(\text{say})$$

$$x = k/a, y = k/b, z = k/c$$

$$\dots(3)$$

Substituting for x, y, z in $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

$$\frac{a}{k} + \frac{b}{k} + \frac{c}{k} = 1$$

$$\therefore k = a + b + c$$

$$\dots(4)$$

Substituting in (3) we get

$$x = \frac{a+b+c}{a}, y = \frac{a+b+c}{b}, z = \frac{a+b+c}{c} \quad \text{for which } u \text{ is stationary.}$$

Exercise 3(M)

- Find the maximum and minimum distances from the origin to the curve $5x^2 + 6xy + 5y^2 - 8 = 0$.
[Ans : max. 4, min .16,6,3]
- Find the dimensions of the rectangular box, with out a top of maximum capacity whose surface is 108 Sq.inches.
[Ans:6'',6'',3'']

3. If $f = u^2 + v^2 + w^2$ where $u + v + w = 1$, show that f is stationary when $u = v = w = \frac{1}{3}$
4. Use Lagrange's method of multipliers to determine the minimum distance from the origin to the plane $x + 2y + 3z = 14$. [Ans: $\sqrt{14}$]
5. Find the maximum value of $f = xyz$ when $xy + yz + zx = k$. [Ans: $(k/3)^{3/2}$]
6. If the distance from the origin of any point $p(x, y, z)$ on the plane $3x + 2y + z - 12 = 0$ is $p = \sqrt{x^2 + y^2 + z^2}$, find the minimum value of p (use the Lagrange's method).
7. Find the maximum volume of a parallelepiped inscribed in a sphere $x^2 + y^2 + z^2 = r^2$. [Ans: $\frac{8r^2}{3\sqrt{3}}$]
8. Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. [Ans: $\frac{8abc}{3\sqrt{3}}$]
9. Given $x + y + z = k$, find the maximum value of $x^\alpha, y^\beta, z^\gamma$
[Ans: $\left[\frac{\alpha^\alpha \cdot \beta^\beta \cdot \gamma^\gamma \cdot k^{(\alpha+\beta+\gamma)}}{(\alpha + \beta + \gamma)^{\alpha+\beta+\gamma}} \right]$]
10. If $r^2 = x^2 + y^2 + z^2$ and $x+y+z=30$ find the values of x, y, z for which r is a minimum [Ans: $x = y = z = 10$]
11. Find the dimensions of a rectangular box without a top of maximum capacity whose surface area is 108 square inches. [Ans : 6,6,3 inches]
12. Find the dimensions of a rectangular box with open top, so that the total surface area S of the box is a minimum, given that the volume ' V ' of the box is constant.
[Ans: $x = y = 2z = (2v)^{1/3}$]
13. Show that the rectangular solid of maximum volume of that can be inscribed in a sphere is a cube
14. If the total surface area of a closed rectangular box is 108 sq.cm. Find the dimensions of the box having maximum capacity [Ans : $\sqrt{18}, \sqrt{18}, \sqrt{18}$]
15. Perimeter of a triangle is constant, prove that the area of this triangle is maximum when the triangle is equilateral.
16. If $u = \phi(x, y)$ where $f(x, y) = 0$, find the maximum and minimum values of u using Lagrange's multipliers Method

4

Curvature and Curve Tracing

4.1.1 Curvature

The curvature of a curve (bending of a curve) varies from point to point on the curve.

Let P be a point on the curve and Q be a point nearer to P, arc PQ = δs

Let tangents at P and Q make angles, ψ , and $\psi + \delta\psi$ with the X- axis.

$\angle TPT' = \delta\psi$ is the angle through which the tangent at P turns as a point moves along the curve from P to Q through a distance δs along the curve and hence $\delta\psi$ depends on the arc length δs .

$\delta\psi$ is defined as the total bending or total curvature of the arc P Q.

$\frac{\delta\psi}{\delta s}$ is defined as the average curvature of the arc P Q.

Now $\delta s \rightarrow 0$ as $Q \rightarrow P$

$\lim_{\delta s \rightarrow 0} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds}$ is defined as curvature of the curve at the point P and is denoted by K.

Thus $K = \frac{d\psi}{ds}$

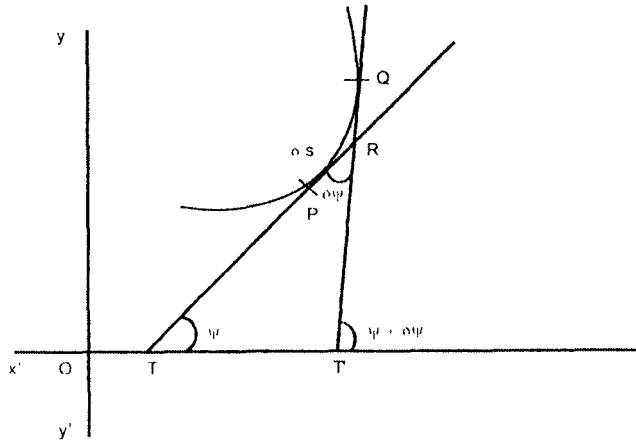


Fig. 4.1

Radius of Curvature

The reciprocal of the curvature of a curve at any point 'P' is called the *radius of curvature* at that point and is denoted by 'ρ'.

Thus $\rho = \frac{ds}{d\psi}$ is called the *radius of curvature* of the curve at the point P.

4.1.2 Theorem

If 'r' is the radius of a circle then the radius of curvature of the circle is same as its radius.

Proof :

Let 'c' be the centre and 'r' be the radius of a circle. P be any point on it and Q be a neighbouring point, arc PQ = ds.

The tangents at P and Q make angles ψ and ψ + δψ respectively.

From the figure

$$\angle T R T' = \delta \psi$$

$$\angle P R Q' = 180 - \delta \psi$$

$$\angle P C Q' = \delta \psi, \text{ from the sector } PCQ.$$

$$\frac{\delta s}{\delta \psi} = r \text{ and as } Q \rightarrow P, \delta s \rightarrow 0$$

$$\therefore \lim_{\delta s \rightarrow 0} \frac{\delta \psi}{\delta s} = \frac{1}{r}$$

$$\text{i.e.,} \quad \frac{1}{\rho} = \frac{1}{r}$$

$$\rho = r$$

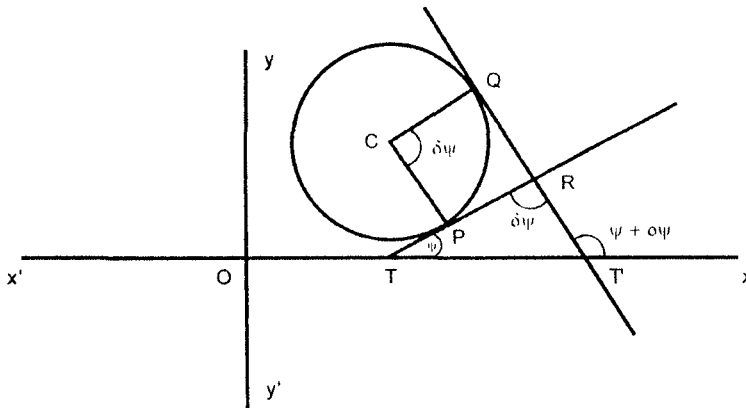


Fig. 4.2

4.1.3 Cartesian form of the Radius of Curvature

Let $y = f(x)$ be the Cartesian equation of a curve and $\tan(\psi)$ be the slope of the tangent to the curve at a point P.

$$\tan \psi = \frac{dy}{dx}$$

Differentiating w.r to 'x'

$$\sec^2 \psi \frac{d\psi}{dx} = \frac{d^2 y}{dx^2}$$

$$\frac{d^2 y}{dx^2} = \sec^2 \psi \frac{d\psi}{ds} \cdot \frac{ds}{dx}$$

We know that

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\frac{d^2y}{dx^2} = (1 + \tan^2 \psi) \frac{1}{\rho} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right]}{\frac{d^2y}{dx^2}}$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\text{i.e., } \rho = \frac{\left[1 + y_1^2\right]^{3/2}}{y_2} \text{ where } y_1 = \frac{dy}{dx}, y_2 = \frac{d^2y}{dx^2}$$

is the radius of curvature of the curve (Cartesian form).

4.1.4 Parametric form of Radius of Curvature

Let $x = f(t)$, $y = g(t)$ be the Parametric form of the equation of a curve.

$$\frac{dx}{dt} = f'(t), \quad \frac{dy}{dt} = g'(t)$$

i.e.,

$$x' = f'(t), \quad y' = g'(t),$$

$$x'' = \frac{d^2x}{dt^2}, \quad y'' = \frac{d^2y}{dt^2}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{y'}{x'}$$

Differentiating w.r. to 'x' on both sides

$$\frac{d^2y}{dx^2} = \frac{x' y'' \cdot \frac{1}{x'} - y' \cdot x'' \cdot \frac{1}{x'}}{(x')^2}$$

$$\frac{d^2y}{dx^2} = \frac{x' y'' - y' \cdot x''}{(x')^3}$$

Substituting $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ values in $\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$

$$\rho = \frac{\left[1 + \left(\frac{y'}{x'}\right)^2\right]^{\frac{3}{2}}}{\frac{x' y'' - y' x''}{(x')^3}}$$

$$\rho = \frac{\left[(x')^2 + (y')^2\right]^{\frac{3}{2}}}{x' y'' - y' x''}$$

where $x' = \frac{dx}{dt}$ $y' = \frac{dy}{dt}$ $x'' = \frac{d^2x}{dt^2}$ $y'' = \frac{d^2y}{dt^2}$

is the parametric form of the radius of curvature.

4.1.5 Polar form of Radius of Curvature

Let P (r, θ) be a point on the curve $r = f(\theta)$. If 'O' is the pole, OX is the initial line.

$$\therefore \angle XOP = \theta$$

The tangent at P makes an angle ψ with the initial line.

ϕ is the angle between the radius vector OP and tangent PT.

From figure

$$\psi = \theta + \phi$$

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds}$$

$$\frac{1}{\rho} = \frac{d\theta}{ds} + \frac{d\phi}{d\theta} \cdot \frac{d\theta}{ds}$$

$$\frac{1}{\rho} = \frac{d\theta}{ds} \left[1 + \frac{d\phi}{d\theta} \right] \quad \dots(1)$$

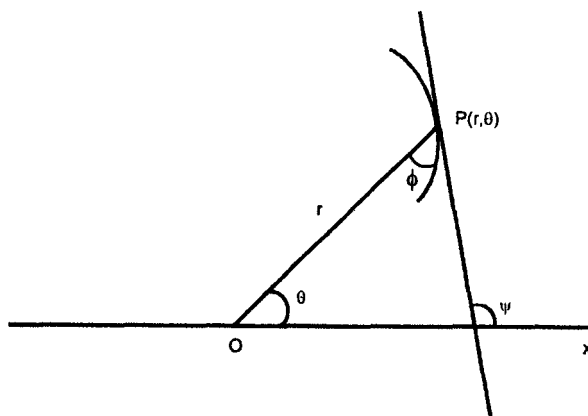


Fig. 4.3

We know

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2} \quad \dots(2)$$

and

$$\tan \phi = r \frac{d\theta}{dr}$$

i.e.,

$$\tan \phi = \frac{r}{\frac{dr}{d\theta}}$$

Differentiating w.r. to 'θ' on both the sides

$$\sec^2 \phi \frac{d\phi}{d\theta} = \frac{\frac{dr}{d\theta} \cdot \frac{dr}{d\theta} - r \frac{d^2 r}{d\theta^2}}{\left(\frac{dr}{d\theta} \right)^2}$$

$$(1 + \tan^2 \phi) \frac{d\phi}{d\theta} = \frac{\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2}$$

$$\therefore \frac{d\phi}{d\theta} = \frac{\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2} \times \frac{1}{1 + \tan^2 \phi}$$

$$\frac{d\phi}{d\theta} = \frac{\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2} \times \frac{1}{1 + \frac{r^2}{\left(\frac{dr}{d\theta}\right)^2}}$$

$$\frac{d\phi}{d\theta} = \frac{\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2r}{d\theta^2}}{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

The value of $1 + \frac{d\phi}{d\theta} = 1 + \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2r}{d\theta^2}}{r^2 + \left(\frac{dr}{d\theta}\right)^2}$

$$1 + \frac{d\phi}{d\theta} = \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2r}{d\theta^2}}{r^2 + \left(\frac{dr}{d\theta}\right)^2} \quad \dots(3)$$

Substituting the value of $\frac{ds}{d\theta}$, $1 + \frac{d\phi}{d\theta}$ from (2), (3) in (1)

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2r}{d\theta^2}}$$

$$\rho = \frac{[r^2 + r_1^2]^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2} \quad \text{where } r_1 = \frac{dr}{d\theta}, r_2 = \frac{d^2r}{d\theta^2}$$

is the polar form of radius of curvature.

4.1.6 Pedal form of Radius of Curvature

Let P be any point on the curve. the tangent P T and P makes an angle ψ with the initial line OA, $\angle P O X = \theta$ (see the figure in 4.1.5)

$$\psi = \theta + \phi \quad \dots(1)$$

Differentiating w.r. to s

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds}$$

$$\frac{1}{\rho} = \frac{1}{r} \cdot r \frac{d\theta}{ds} + \frac{d\phi}{dr} \frac{dr}{ds}$$

$$\frac{1}{\rho} = \frac{1}{r} (\sin\phi) + \frac{d\phi}{dr} \cos\phi,$$

We have $\sin \phi = r \frac{d\theta}{ds}, \frac{1}{\rho} = \frac{1}{r} \left[\sin \phi + r \cos \phi \frac{d\phi}{dr} \right], \cos \phi = \frac{dr}{ds}$

$$\frac{1}{\rho} = \frac{1}{r} \frac{d(r \sin \phi)}{dr} \quad p = r \sin \phi$$

$$\frac{1}{\rho} = \frac{1}{r} \frac{dp}{dr} \quad \therefore \rho = r \frac{dr}{dp}$$

where p is the perpendicular distance from pole to the tangent.

4.1.7 Examples

Prove that the radius of curvature of the curve $y = a \cosh (x/a)$ is $\frac{y^2}{a}$

Solution

$$y = a \cosh (x/a)$$

$$\frac{dy}{dx} = a \sinh(x/a) \frac{1}{a}$$

$$\frac{d^2y}{dx^2} = \cosh(x/a) \frac{1}{a}$$

Substituting the values of $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ in $\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

$$\rho = \frac{\left[1 + \sinh^2 x/a\right]^{3/2}}{\cosh(x/a) \frac{1}{a}}$$

$$\rho = a \cosh^2(x/a)$$

$$\therefore \rho = a \left(\frac{y^2}{a^2}\right), \quad \text{as } y = \cosh(x/a)$$

$$\rho = \left(\frac{y^2}{a}\right)$$

4.1.8 Example

Prove that for the rectangular hyperbola $xy = c^2$ the radius of curvature at any point

is given by $\rho = \frac{r^3}{2c^2}$ where 'r' is the distance of the point from the origin.

Solution

Differentiating $xy = c^2$ w.r. to 'x'

$$x \frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = -y/x$$

$$\frac{d^2y}{dx^2} = - \frac{\left[x \frac{dy}{dx} - y.1 \right]}{x^2} = \frac{2y}{x^2}$$

Substituting for $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ in $\rho = \frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2}}{\frac{d^2y}{dx^2}}$

$$\rho = \frac{\left[1 + \left(-\frac{y}{x} \right)^2 \right]^{3/2}}{\frac{2y}{x^2}}$$

$$\rho = \frac{(x^2 + y^2)^{3/2}}{2xy}$$

$$\rho = \frac{(r^2)^{3/2}}{2c^2} \quad \text{as } r^2 = x^2 + y^2 \text{ and } xy = c^2$$

$$\rho = \frac{r^3}{2c^2}$$

4.1.9 Example

Prove that the square of the radius of curvature at any point on the parabola $y^2 = 4ax$ varies as the cube of the focal distance of the point.

Solution

From $y^2 = 4ax$ (1)

$$\frac{dy}{dx} = \sqrt{\frac{a}{x}}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2} \sqrt{a} x^{-3/2}$$

Hence

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

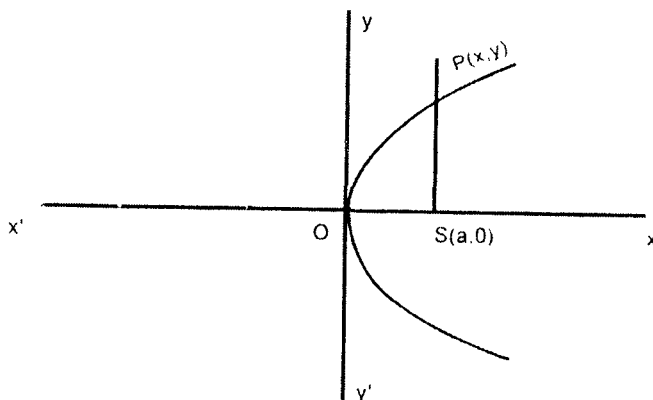


Fig. 4.4

$$\rho = \frac{\left(1 + \frac{a}{x}\right)}{\frac{-1}{2}\sqrt{a}x^{-3/2}}$$

$$\rho = \frac{-2}{\sqrt{a}}(x+a)^{3/2} \quad \text{.....(2)}$$

The focal distance of the point p (x, y)

$$\begin{aligned} SP &= \sqrt{(x-a)^2 + (y-0)^2} \\ &= \sqrt{x^2 + a^2 - 2ax + y^2} \\ &= \sqrt{x^2 + a^2 - 2ax + 4ax} \quad \text{using (1)} \end{aligned}$$

$$\begin{aligned} SP &= \sqrt{(x+a)^2} \\ SP &= \text{focal distance} = (x+a) \quad \text{.....(3)} \end{aligned}$$

From (2) and (3)

$$\rho = \frac{-2}{\sqrt{a}} [\text{Focal distance}]^{3/2}$$

$$\rho^2 = \frac{4}{a} [\text{Focal distance}]^3$$

$\therefore \rho^2 \propto$ cube of the focal distance.

4.1.10 Example

Show that the radius of curvature at an end of the major axis of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is equal to the semi latus-rectum of the ellipse.}$$

Solution

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\text{i.e., } x^2 b^2 + a^2 y^2 = a^2 b^2 \quad \dots(1)$$

Differentiating (1) w.r. to 'x'

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

$$\frac{d^2 y}{dx^2} = -\frac{b^2}{a^2} \left[\frac{y - x \frac{dy}{dx}}{y^2} \right]$$

$$= -\frac{b^2}{a^2} \left[\frac{y - x \left(\frac{-b^2 x}{a^2 y} \right)}{y^2} \right]$$

$$= \frac{-b^4}{a^2 y^3} \left[\frac{x^2}{a^2} + \frac{y^2}{b^2} \right]$$

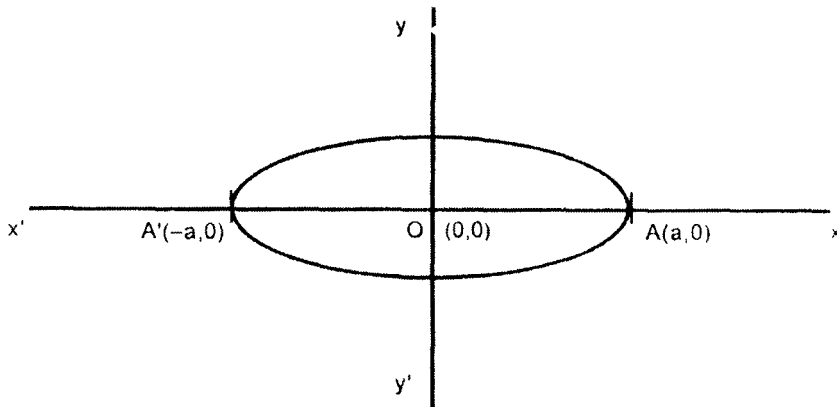


Fig. 4.5

$$\frac{d^2 y}{dx^2} = -\frac{b^2}{a^2 y^3} \quad \left(\text{using } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right)$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{3}{2}}}{\frac{d^2 y}{dx^2}}$$

Hence

$$\rho = \frac{\left[1 + \frac{b^4 x^2}{a^4 y^2} \right]^{\frac{3}{2}}}{\frac{-b^4}{a^2 y^3}} = \frac{(a^4 y^2 + b^4 x^2)^{\frac{3}{2}}}{a^4 b^4} \quad \dots(2)$$

The coordinates of the ends of the major axis are

A (a, 0) and A' (-a, 0)

Taking the end A (a, 0) we have from (2)

$$\rho|_{(a,0)} = \frac{(a^4 \cdot 0 + b^4 \cdot a^2)^{\frac{3}{2}}}{a^4 b^4}$$

$$\rho|_{(a,0)} = \frac{b^2}{a} = \text{Semi latus-rectum of the ellipse.}$$

4.1.11 Example

Prove that ρ at any point of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is three times the length of the perpendicular from origin to the tangent at that point.

Solution

The parametric equations of the curve are

$$x = a \cos^3 \theta, y = a \sin^3 \theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a \sin^2 \theta \cos \theta}{-3a \cos^2 \theta \sin \theta} \frac{d\theta}{dx} = -\tan \theta$$

$$\begin{aligned} \text{and} \quad \frac{d^2 y}{dx^2} &= -\sec^2 \theta \frac{d\theta}{dx} \\ &= -\sec^2 \theta / (-3a \cos^2 \theta \sin \theta) \\ &= \frac{1}{3a \cos^4 \theta \sin \theta} \end{aligned}$$

$$\text{Hence} \quad \rho = \frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2}}{\frac{d^2 y}{dx^2}} \text{ gives}$$

$$\rho = \frac{(1 + \tan^2 \theta)^{3/2}}{1 / (3a \cos^4 \theta \sin \theta)}$$

$$\rho = 3a \cos \theta \sin \theta \quad \dots(1)$$

The perpendicular distance from the origin to the tangent

$$P = \frac{y - x \frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx} \right)^2}} = \frac{a \sin^2 \theta - \cos^2 \theta (-\tan \theta)}{\sqrt{1 + \tan^2 \theta}}$$

$$P = a \sin \theta \cos \theta \quad \dots(2)$$

From (1) & (2) $\rho = 3 p$

4.1.12 Example

Find ρ at any point for the cycloid

$$x = a(\theta - \sin \theta)$$

$$y = a(1 - \cos \theta) \text{ and also find } \rho \text{ at } \theta = 90^\circ$$

Solution

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a(\sin \theta)}{a(1 - \cos \theta)}, \quad \frac{dy}{dx} = \cot \frac{\theta}{2}$$

and $\frac{d^2 y}{dx^2} = \operatorname{cosec}^2 \frac{\theta}{2} \cdot \frac{1}{2} \cdot \frac{d\theta}{dx}$

$$\frac{d^2 y}{dx^2} = \frac{-\frac{1}{2} \operatorname{cosec}^2 \frac{\theta}{2}}{a(1 - \cos \theta)}, \quad \frac{d^2 y}{dx^2} = \frac{-1}{4a \sin^4 \frac{\theta}{2}}$$

$$\therefore \rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2 y}{dx^2}} \quad \rho = \frac{\left(1 + \cot^2 \frac{\theta}{2}\right)^{\frac{3}{2}}}{\frac{-1}{4a \sin^4 \frac{\theta}{2}}}$$

$$\rho = -4a \sin \frac{\theta}{2}$$

at $\theta = 90^\circ$, $\rho = -4a \sin 45^\circ$ i.e., $\rho = -2\sqrt{2} a$

4.1.13 Example

Find ρ at (r, θ) for the curve $r = a(1 + \cos \theta)$ and also find at $\theta = \frac{\pi}{2}$.

Solution

$$r = a(1 + \cos \theta)$$

$$\frac{dr}{d\theta} = r_1 = a(-\sin \theta)$$

$$\frac{d^2 r}{d\theta^2} = r_2 = a \cos \theta$$

$$\begin{aligned}
 \rho &= \frac{(r^2 + r_1^2)^{3/2}}{[r^2 + 2r_1^2 - rr_2]} \\
 &= \frac{[a^2(1 + \cos \theta)^2 + a^2 \sin^2 \theta]^{3/2}}{[a^2(1 + \cos \theta)^2 + 2a^2 \sin^2 \theta + a^2 \cos \theta(1 + \cos \theta)]} \\
 &= \frac{[2a^2(1 + \cos \theta)^{3/2}]}{a^2 + 2a^2 + 3a^2 \cos \theta} = \frac{[2a^2 \cdot 2\cos^2 \theta/2]^{3/2}}{3a^2 \cdot 2\cos^2 \theta/2} = \frac{4}{3} a \cos \theta/2
 \end{aligned}$$

at $\theta = \pi/2$

$$\rho = \frac{4}{3} a \cos 45^\circ = \frac{2}{3} \sqrt{2} a$$

4.1.14 Example

Find ρ for the curve $r^m = a^m \cos m\theta$

Solution

$$r^m = a^m \cos m\theta$$

$$m \log r = m \log a + \log (\cos m\theta)$$

$$m \cdot \frac{1}{r} \frac{dr}{d\theta} = \frac{1}{\cos m\theta} (-m \sin m\theta)$$

$$r_1 = -r \tan m\theta \quad \dots(1)$$

$$r_2 = -r m \sec^2 m\theta - \tan m\theta \cdot \frac{dr}{d\theta}$$

$$r_2 = -rm \sec^2 m\theta + r \tan^2 m\theta \quad \dots(2)$$

From (1)

$$\begin{aligned}
 \rho &= \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2} \\
 &= \frac{(r^2 + r^2 \tan^2 m\theta)^{3/2}}{r^2 + 2r^2 \tan^2 m\theta + mr^2 \sec^2 m\theta - r^2 \tan^2 m\theta}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{r^3 \sec^3 m\theta}{(m+1)r^2 \sec^2 m\theta} \\
 &= \frac{r}{(m+1)\cos m\theta} \\
 &= \frac{r}{(m+1)\frac{r^m}{a^m}} = \frac{a^m}{(m+1)^{m-1}}
 \end{aligned}$$

For the curve $r^m = a^m \cos m\theta$

$$\rho = \frac{a^m}{(m+1)r^{m-1}}$$

4.1.15 Example

Find the radius of curvature of the curve $p^2 = ar$

Solution

$$p^2 = ar \quad \dots(1)$$

Differentiating w.r. to 'r'

$$2p \frac{dp}{dr} = a$$

$$\frac{dp}{dr} = \frac{a}{2p}$$

$$\rho = r \frac{dr}{dp} = r \cdot \frac{2p}{a}$$

From (1)

$$\rho = \frac{2r}{a} \sqrt{ar}$$

$$\rho = 2r \sqrt{\frac{r}{a}}$$

4.1.16 Example

Find the radius of curvature at the point (p, r) on the ellipse $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2 b^2}$

Solution

Differentiating $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2 b^2}$ w.r. to 'r' we get

$$-\frac{2}{p^3} \frac{dp}{dr} = \frac{-2r}{a^2 b^2}$$

$$\Rightarrow r \frac{dr}{dp} = \frac{a^2 b^2}{p^3}$$

$$\therefore \rho = \frac{a^2 b^2}{p^3}$$

4.1.17 Example

Find the radius of curvature ρ of the curve $r^2 \cos 2\theta = a^2$

Solution

$$r^2 = \frac{a^2}{\cos 2\theta} \quad \text{.....(1)}$$

Taking logarithms on both sides we get

$$2 \log r + \log \cos 2\theta = 2 \log a$$

Differentiating w.r.to 'θ'

$$2 \frac{1}{r} \frac{dr}{d\theta} - 2 \tan 2\theta = 0$$

$$\Rightarrow r \frac{d\theta}{dr} = \cot 2\theta$$

$$\therefore \tan \phi = r \frac{d\theta}{dr} = \cot 2\theta$$

$$\tan \phi = \tan \left(\frac{\pi}{2} - 2\theta \right)$$

$$\Rightarrow \phi = \frac{\pi}{2} - 2\theta$$

We have $p = r \sin \phi \Rightarrow p = r \sin \left(\frac{\pi}{2} - 2\theta \right)$

$$p = r \cos 2\theta$$

Substituting $\cos 2\theta = \frac{a^2}{r^2}$ from (1)

$$p = r \left(\frac{a^2}{r^2} \right)$$

$$p = \frac{a^2}{r}$$

$$\frac{dp}{dr} = \frac{a^2}{r^2}$$

$$\therefore \rho = r \frac{dp}{dr} = - \frac{r}{-\frac{a^2}{r^2}}$$

$$\rho = \frac{r^3}{a^2}$$

4.1.18 Curvature at Origin

- (i) When y can be expanded in powers of x by some algebraical or trigonometrical method from the equation of the curve as

$$y = px + \frac{qx^2}{2!} + \dots \text{ then } p = \left. \frac{dy}{dx} \right|_{x=0, y=0} \quad q = \left. \frac{d^2y}{dx^2} \right|_{x=0, y=0}$$

$$\therefore \rho \text{ (at origin)} = \frac{(1+p)^{\frac{3}{2}}}{q}$$

- (ii) *Newtonian Method* : If a curve passes through $(0, 0)$ and the tangent at $(0, 0)$

is X – axis, then we have $x = 0, y = 0$ and $\frac{dy}{dx} = 0$ at origin.

$\therefore$ The expansion of y by Maclaurin's theorem reduces to

$$y = (y_0) + (y_1)_0 + \frac{x^2}{2} (y_2)_0$$

Taking upto terms of $0 (x^2)$

$$y = 0 + 0 \cdot x + q \frac{x^2}{2!} + \dots$$

Dividing each term by x^2 and taking limits

$$q = \lim_{x \rightarrow 0} \frac{2y}{x^2}$$

We have $\rho = (\text{at origin}) = \frac{(1+p)^3}{q}$

as $p = 0$ at $(0, 0)$

and $q = \lim_{x \rightarrow 0} \frac{2y}{x^2}$

and $\rho (\text{at origin}) = \frac{1}{q}$

$$\rho (\text{at origin}) = \lim_{x \rightarrow 0} \left(\frac{x^2}{2y} \right)$$

(iii) Similarly, if a curve passes through $(0, 0)$ and the tangent at $(0, 0)$ is Y - axis

then $\rho (\text{at origin}) = \lim_{x \rightarrow 0} \left(\frac{y^2}{2x} \right)$

(iv) Curvature at pole : If the initial line is the tangent to the curve at the pole

then $\rho = \lim_{x \rightarrow 0} \left(\frac{x^2}{2y} \right) = \lim_{x \rightarrow 0} \frac{r^2 \cos^2 \theta}{2r \sin \theta} \quad \left(\begin{array}{l} \because x = r \cos \theta \\ y = r \sin \theta \end{array} \right)$

$$\rho = \lim_{\theta \rightarrow 0} \frac{r}{2\theta} \cdot \frac{\theta}{\sin \theta} \cdot \cos^2 \theta$$

$$\rho = \lim_{\theta \rightarrow 0} \left(\frac{r}{2\theta} \right) \quad \left(\text{as } \lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} \rightarrow 1, \cos^2 \theta \rightarrow 1 \right)$$

4.1.19 Example

Find ρ at the origin of the curve $y^4 + x^2 + a(x^2 + y^2) - a^2y = 0$ by Newtonian method.

Solution

Equating to zero the terms of the lowest degree in the equation of the curve

$$-a^2y = 0$$

$$\Rightarrow y = 0$$

i.e.,

X - axis is the only tangent at origin

Then

' ρ ' at $(0, 0)$ when X - axis is the tangent is given by

$$\rho = \lim_{\theta \rightarrow 0} \frac{x^2}{2y} \quad \text{.....(1)}$$

Dividing the given equation by y

$$y^3 + x \cdot \frac{x^2}{y} + a \cdot \frac{x^2}{y} + ay = a^2$$

Let

$$x \rightarrow 0$$

So that

$$y \rightarrow 0$$

and

$$\frac{x^2}{y} = 2\rho$$

From (1)

$$0 + 0.2\rho + a2\rho + a.0 = a^2 \Rightarrow 2a\rho = a^2 \Rightarrow \rho = \frac{a}{2}$$

4.1.20 Example

Find ρ at the origin for the curve $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$ by Newtonian method.

Solution

$$x = a(\theta + \sin \theta), \quad y = a(1 - \cos \theta) \text{ is a cycloid}$$

X-axis is the tangent to the curve at $(0, 0)$

$$\begin{aligned} \rho \text{ at } (0, 0) &= \lim_{x \rightarrow 0} \left(\frac{x^2}{2y} \right) \\ &= \lim_{\theta \rightarrow 0} \frac{a^2(\theta + \sin \theta)^2}{2a(1 - \cos \theta)} \\ &= \lim_{\theta \rightarrow 0} \frac{a(\theta + \sin \theta)^2}{2(1 - \cos \theta)}, \quad \left(\text{of the form } \frac{0}{0} \right) \\ &= \lim_{\theta \rightarrow 0} \frac{a(\theta + \sin \theta)^2(1 + \cos \theta)}{2 \sin^2 \theta} \\ &= \lim_{\theta \rightarrow 0} \frac{a}{2} \left(\frac{\theta}{\sin \theta} + 1 \right)^2 (1 + \cos \theta) \\ &= \frac{a}{2} (1 + 1)^2 (1 + 1) \text{ using } \lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} = 1 \end{aligned}$$

$$\rho = \frac{a}{2} [8]$$

$$\rho = 4a$$

4.1.21 Example

Find ρ at $(0, 0)$ for the curve $2x^4 + 3y^3 + 4x^2y + xy - y^2 + 2x = 0$ by Newtonian method.

Solution

Equating to zero the terms of the lowest degree in the given equation of the curve $2x = 0 \Rightarrow x = 0$

$\therefore$ y - axis is the tangent to the curve at $(0, 0)$

$$\therefore \rho \text{ at } (0, 0) = \lim_{x \rightarrow 0} \frac{y^2}{2x}$$

$$\text{So that } \frac{y^2}{x} \Rightarrow 2\rho \quad \dots(1)$$

Dividing the given equation by 'x'

$$2x^3 + 3y^2 \cdot \frac{y^2}{x} + 4xy + y - \frac{y^2}{x} + 2 = 0$$

Taking $x \rightarrow 0, y \rightarrow 0$

$$2.0 + 3.0 \cdot (2\rho) + 4.0 + 0 - 2\rho + 2 = 0; \rho = 1$$

Exercise – 4(A)

I. Find ρ for the following curves :

$$1. \quad y = 4 \sin x - \sin 2x \text{ at } x = \frac{\pi}{2}$$

$$[\text{Ans : } \frac{5\sqrt{5}}{4}]$$

$$2. \quad a^2y = x^3 - a^3$$

$$[\text{Ans : } \frac{(a^4 + ax^4)^{3/2}}{6xa^4}]$$

$$3. \quad x^3 + y^3 = 3axy \text{ at } \left(\frac{3a}{2}, \frac{3a}{2}\right)$$

$$[\text{Ans : } \frac{3\sqrt{2}a}{16}]$$

4. $(x^2 + y^2)^2 (y^2 - x^2) \quad (0, a)$

[Ans : $\left(-\frac{a}{3}\right)$]

5. $\sqrt{x} + \sqrt{y} = \sqrt{a} \text{ at } \frac{a}{4}, \frac{a}{4}$

[Ans : $\frac{a}{\sqrt{2}}$]

6. $x = a(\theta - \sin \theta), y = a(1 - \cos \theta) \text{ at } \theta = \pi$

[Ans : $-4a$]

7. $x = a \sin 2t (1 + \cos 2t)$
 $y = a \cos 2t (1 - \cos 2t) \text{ at any point 't'}$

[Ans : $4a \cos 3t$]

8. $x = a(\cos \theta + \theta \sin \theta)$
 $y = a(\sin \theta - \theta \cos \theta)$

[Ans : $a\theta$]

9. $r^2 = a^2 \cos 2\theta$

[Ans : $\frac{a^2}{3r}$]

10. $r = \frac{a}{2} (1 + \cos \theta) \text{ at } \left(\frac{a}{2}, \frac{\pi}{2}\right)$

[Ans : $\frac{\sqrt{2}}{3}a$]

11. $r^n = a^n \sin n\theta$

[Ans : $\frac{an}{(n+1)r^{n-1}}$]

4.2 Centre of Curvature, Circle of Curvature and Evolute

1. *Definition* : The centre of curvature at a point 'P' of a curves is the point 'C' which lies on the positive direction of the normal at 'P' and is at a distance 'ρ' (in magnitude) from it.
2. *Definition* : The circle of curvature at a point 'P' of a curve is the circle whose centre is at the centre of curvature 'C' and whose radius is 'ρ' in magnitude.

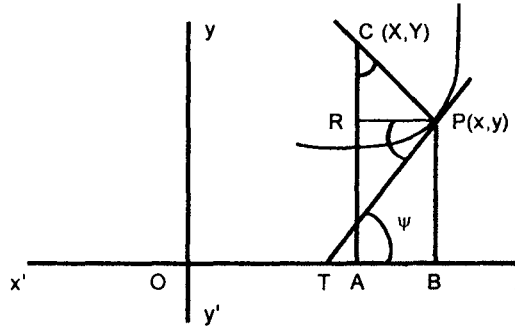


Fig. 4.6

4.2.1 Centre of Curvature

Let $P(x, y)$ be any point on the curve and $C(X, Y)$ be the centre of curvature at 'P' which is on the normal PC at P . Then $PC = \rho =$ radius of curvature at 'P'. The tangent PT at P to the curve makes an angle ψ with X -axis. PB , CA are perpendiculars to X -axis, and PR is perpendicular to CA .

$$\angle PTX = \angle RPT = \angle PCR = \psi$$

$$\cos \psi = \frac{1}{\sec \psi}$$

$$= \frac{1}{(1 + \tan^2 \psi)^{\frac{1}{2}}}$$

$$\cos \psi = \frac{1}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} \quad \text{.....(1)}$$

$$\sin \psi = \tan \psi \cos \psi = \frac{\frac{dy}{dx}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} \quad \text{.....(2)}$$

and we have
$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \quad \dots(3)$$

$$X = OA = OB - AB = OB - PR$$

(i.e.,) $X = x - PC \sin \psi$ (From ΔPRC , $PR = PC \sin \psi$)

$$X = x - \rho \sin \psi$$

From (2) and (3)

$$X = x - \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \cdot \frac{\frac{dy}{dx}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}}$$

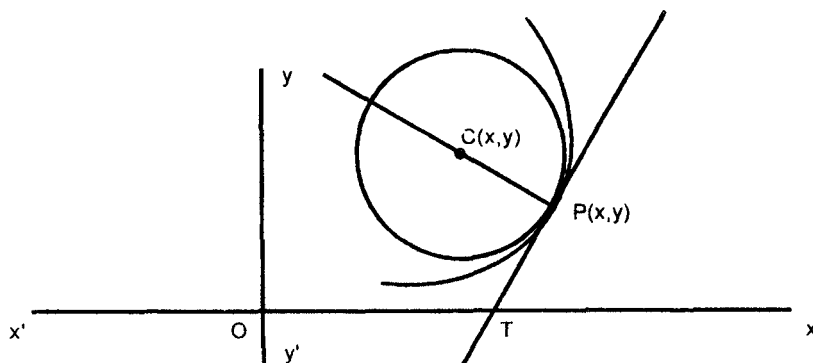
$$\therefore X = x - \frac{\frac{dy}{dx} \left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$$

$$Y = CA = RA + RC = Y + PC \cos \psi$$

From (1) and (3)

$$Y = y + \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \cdot \frac{\frac{dy}{dx}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}}$$

$$Y = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$$



∴ The coordinates of the centre of curvature of a curve

at $P(x, y)$ is $C \left[x - \frac{\left(\frac{dy}{dx}\right) \left[1 + \left(\frac{dy}{dx}\right)^2 \right]}{\frac{d^2y}{dx^2}}, y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} \right]$

Denoting $y_1 = \frac{dy}{dx}$, $y_2 = \frac{d^2y}{dx^2}$ we can write $C \left[x - \frac{y_1(1 + y_1^2)}{y_2}, y + \frac{1 + y_1^2}{y_2} \right]$,

$C(X, Y)$ is the centre and 'ρ' is radius of the circle of curvature

∴ The equation of the circle of curvature is

$$(x - X)^2 + (y - Y)^2 = \rho^2$$

4.2.2 Example

Find the centre of curvature of the curve

$$y = x^3 - 6x^2 + 3x + 1 \text{ at } (1, -1)$$

Solution

$$\frac{dy}{dx} = 3x^2 - 12x + 3$$

$$\frac{dy}{dx}_{(1,-1)} = -6$$

$$\frac{d^2y}{dx^2} = 6x - 12$$

$$\left. \frac{d^2y}{dx^2} \right|_{(1,-1)} = -6$$

$$X = x - \frac{y_1(1+y_1^2)}{y^2} \text{ and } Y = y + \frac{1+y_1^2}{y_2}$$

$$X = 1 - \frac{(-6)(1+36)}{-6} \text{ and } Y = 1 + \frac{1+36}{-6}$$

$$X = -36 \text{ and } Y = \frac{-43}{6}$$

∴ The coordinates of the centre of curvature are $\left(-36, \frac{-43}{6}\right)$

4.2.3 Example

Find the centre of curvature for the cycloid

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta)$$

Solution

$$\frac{dy}{dx} = \frac{a \sin \theta}{a(1 - \cos \theta)} = \cot \frac{\theta}{2}$$

$$\frac{d^2y}{dx^2} = \operatorname{cosec}^2 \frac{\theta}{2} \cdot \frac{1}{2} \cdot \frac{d\theta}{dx} = -\frac{1}{2 \sin^2 \frac{\theta}{2}} \times \frac{1}{a(1 - \cos \theta)}$$

$$\frac{d^2y}{dx^2} = \frac{1}{4a \sin^4 \frac{\theta}{2}}$$

$$X = x - \frac{y_1(1+y_1^2)}{y^2} \text{ and } Y = y + \frac{1+y_1^2}{y_2}$$

$$X = a(\theta - \sin\theta) - \frac{\cot \frac{\theta}{2} \left(1 + \cot^2 \frac{\theta}{2} \right)}{\frac{1}{4a \sin^4 \frac{\theta}{2}}}$$

$$X = a(\theta - \sin\theta) + \frac{4a \sin^4 \frac{\theta}{2} \cot \frac{\theta}{2}}{\sin \frac{\theta}{2}}$$

$$X = a(\theta - \sin\theta) + 2a \sin \theta$$

$$X = a(\theta - \sin\theta)$$

$$Y = y + \frac{1 + y_1^2}{y_2}$$

$$Y = a(1 - \cos\theta) - \frac{\left(1 + \cot^2 \frac{\theta}{2} \right)}{\frac{1}{-4a \sin^4 \frac{\theta}{2}}}$$

$$Y = a(1 - \cos\theta) - 4a \sin^2 \frac{\theta}{2}$$

$$Y = a(1 - \cos\theta) - 2a(1 - \cos\theta)$$

$$Y = -a(1 - \cos\theta)$$

∴ The coordinates of the centre of curvature are

$$[a(\theta + \sin\theta), -a(1 - \cos\theta)]$$

4.2.4 Example

Find the centre of the curvature at any point (x, y) on the ellipse.

Solution

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow b^2 x^2 + a^2 y^2 = a^2 b^2 \quad \dots(1)$$

Diff. (1) w.r. to 'x'

$$2b^2x + 2a^2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{b^2}{a^2} \left(\frac{x}{y} \right) \quad \dots(2)$$

$$\frac{d^2y}{dx^2} = \frac{-b^2}{a^2} \left[\frac{y \cdot 1 - x \frac{dy}{dx}}{y^2} \right]$$

$$\frac{d^2y}{dx^2} = \frac{-b^2}{a^2} \left[\frac{1}{y} - \frac{x}{y^2} \frac{dy}{dx} \right] \quad \dots(3)$$

Using (1) (2) and (3)

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$X = \frac{x^3}{a^4} (a^2 - b^2)$$

$$Y = y + \frac{(1+y^2)}{y^2}$$

$$Y = \frac{-y^3}{a^4} (a^2 - b^2)$$

$\therefore \left[\frac{x^3}{a^4} (a^2 - b^2), \frac{-y^3}{b^4} (a^2 - b^2) \right]$ is the centre of curvature.

4.2.5 Example

Find the circle of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point $\left(\frac{a}{4}, \frac{a}{4} \right)$

Solution

$$\sqrt{x} + \sqrt{y} = \sqrt{a} \quad \dots(1)$$

Diff (1) w.r. to 'x'

$$\frac{1}{2}x^{\frac{1}{2}-1} + \frac{1}{2}y^{\frac{1}{2}-1} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} \quad \text{.....(2)}$$

and

$$\left. \frac{dy}{dx} \right|_{a/4, a/4} = -1 \quad \text{.....(3)}$$

$$\frac{d^2y}{dx^2} = \frac{\left[\sqrt{x} \frac{1}{2} y^{\frac{1}{2}-1} \frac{dy}{dx} - \sqrt{y} \frac{1}{2} x^{\frac{1}{2}-1} \right]}{x}$$

$$\left. \frac{d^2y}{dx^2} \right|_{a/4, a/4} = \frac{4}{a} \quad \text{.....(4)}$$

$$\rho = \frac{[1+y_1^2]^{\frac{3}{2}}}{y_2} = \frac{[1+1]^{\frac{3}{2}}}{\frac{4}{a}} = \frac{a}{\sqrt{2}} \quad \text{.....(5)}$$

is the radius curvature of the curve at $\left(\frac{a}{4}, \frac{a}{4}\right)$

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$X = \frac{a}{4} + \frac{1(1+1)}{\frac{4}{a}} = \frac{3a}{4}$$

$$Y = y + \frac{(1+y_1^2)}{y_2}$$

$$Y = \frac{a}{4} + \frac{(1+1)}{\frac{4}{a}} = \frac{3a}{4}$$

Coordinates of the centre of curvature $\left(\frac{3a}{4}, \frac{3a}{4}\right)$

∴ Equation of the circle of curvature of the curve $(x - X)^2 + (y - Y)^2 = \rho^2$

$$\left(x - \frac{3a}{4}\right)^2 + \left(y - \frac{3a}{4}\right)^2 = \left(\frac{a}{\sqrt{2}}\right)^2$$

Exercise – 4B

1. Find the coordinates of the centre of curvature of the curve $y = x^3$ at the point

$$\left(\frac{1}{2}, \frac{1}{8}\right)$$

[Ans : $\frac{7}{64}, \frac{31}{38}$]

2. Show that the centre of curvature for the curve $y = \frac{x^2 + 9}{x}$ at $(3, 6)$ is $\left(3, \frac{15}{2}\right)$

3. Show that the centre of curvature at the point 't' on the ellipse $x = a \cos t$,

$$y = b \sin t \text{ is } \left[\left(\frac{a^2 - b^2}{a} \right) \cos^3 t, - \left(\frac{a^2 - b^2}{b} \right) \sin^3 t \right]$$

4. Find the centre of curvature at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ of the folium $x^3 + y^3 = 3axy$

[Ans : $\frac{21a}{16}, \frac{21a}{16}$]

5. Find the centre of curvature at the point 'θ' of the curve

$$x = (1 - a\theta) \cos \theta + a \sin \theta$$

$$y = (1 - a\theta) \sin \theta + a \cos \theta$$

[Ans : $a \sin \theta, -a \cos \theta$]

6. Find the coordinates of the centre of curvature of the cycloid $x = a(\theta + \sin \theta)$,

$$y = a(1 - \cos \theta)$$

[Ans : $a(\theta - \sin \theta), a(1 + \cos \theta)$]

7. For the curve $y = mx + \frac{x^2}{a}$ show that the circle of curvature at the origin is

$$x^2 + y^2 = a^2(1 + m^2)(y - mx).$$

4.3

4.3.1 Evolutes

Definition : Corresponding to each point on a curve we can find the curvature of the curve at that point. Drawing the normal at these points we can find centre of curvature corresponding to each of these points. Since the curvature varies from point to point, centres of curvature also differ. The totality of all such centres of curvature of a given curve will define another curve and this curve is called the evolute of the curve.

Thus the locus of centres of curvature of a given curve is called the evolute of that curve.

Notes :

- (i) Elimination of x, y from the coordinates of the centre of curvature (X, Y) gives an equation in terms of X, Y which is called *evolute of the given curve* $y = f(x)$.
- (ii) When the equation of the curve is given in the parametric form (say t) elimination of ' t ' from the coordinates of the centre of curvature (X, Y) gives an equation in terms of X, Y which is called *evolute of the curve* $x = f(t), y = g(t)$.

4.3.2 Example

Find the evolute of the parabola $y^2 = 4ax$

Solution

Differentiating $y^2 = 4ax$ w.r. to ' x '

$$2y \frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{x}} \quad \dots(1)$$

$$\frac{d^2y}{dx^2} = \frac{-\sqrt{a}}{2x^{\frac{3}{2}}} \quad \dots(2)$$

$$X = x - \frac{y_1(1 + y_1^2)}{y_2}$$

$$X = x - \frac{-\frac{\sqrt{a}}{\sqrt{x}} \left(1 + \frac{a}{x}\right)}{\frac{-\sqrt{a}}{2x^{\frac{3}{2}}}}$$

$$X = 3x + 2a \quad \dots(3)$$

$$Y = y + \frac{1+y_1^2}{y_2} = y + \frac{\left(1 + \frac{a}{x}\right)}{\frac{-\sqrt{a}}{3x^2}}$$

$$Y = \frac{-2}{\sqrt{a}} x^3 \quad \dots(4)$$

Squaring

$$Y^2 = \frac{4}{a} x^3 \quad \dots(5)$$

From (3) substituting $x = \frac{X-2a}{3}$ in (5)

$$Y^2 = \frac{4}{a} \left(\frac{X-2a}{3} \right)^3$$

$$\therefore 27 a Y^2 = 4(X-2a)^3$$

$$\therefore 27 a y^2 = 4(x-2a)^3 \text{ is the evolute of the curve.}$$

4.3.3 Example

Show that the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$

Solution

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow b^2 x^2 + a^2 y^2 = a^2 b^2 \quad \dots(1)$$

Differentiating (1) with respect to 'x'

$$2b^2x + 2a^2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-b^2x}{a^2y} \quad \dots(2)$$

$$\begin{aligned}
 \frac{d^2 y}{dx^2} &= \frac{-b^2}{a^2} \left[\frac{y \cdot 1 - x \frac{dy}{dx}}{y^2} \right] \\
 &= \frac{-b^2}{a^2} \left[\frac{y - x \left(\frac{-b^2 x}{a^2 y} \right)}{y^2} \right] \\
 &= \frac{-b^2}{a^2 y^2} \cdot \frac{a^2 b^2}{a^2 y} \\
 \frac{d^2 y}{dx^2} &= \frac{-b^4}{a^2 y^3} \quad \text{.....(3)}
 \end{aligned}$$

$$\begin{aligned}
 X &= x - \frac{y_1(1+y_1^2)}{y_2} \\
 &= x - \frac{\frac{b^2 x}{a^2 y} \left(1 + \frac{b^4 x^2}{a^4 y^2} \right)}{\frac{-b^4}{a^2 y^3}}
 \end{aligned}$$

$$X = x - \frac{x}{a^4 b^2} (a^4 y^2 + b^4 x^2)$$

Substituting $a^2 y^2 = a^2 b^2 - b^2 x^2$ from (1)

$$\begin{aligned}
 X &= x - \frac{x}{a^4 b^2} [a^2 b^2 (a^2 - x^2) + b^4 x^2] \\
 X &= \frac{a^2 - b^2}{a^4} x^3 \quad \text{.....(4)}
 \end{aligned}$$

$$\begin{aligned}
 Y &= y + \frac{1+y_1^2}{y_2} \\
 &= y + \frac{1 + \frac{b^4 x^2}{a^4 y^2}}{\frac{-b^4}{a^2 y^3}}
 \end{aligned}$$

$$Y = y - \frac{y}{a^2 b^4} (a^4 y^2 + b^4 x^2)$$

Substituting $b^2 x^2 = a^2 b^2 - a^2 y^2$ from (1)

$$Y = y - \frac{y}{a^2 b^4} [a^4 y^2 + b^2 a^2 (a^2 - y^2)]$$

$$Y = \frac{a^2 - b^2}{b^4} y^3 \quad \dots(5)$$

Eliminating x, y from (4) (5)

$$x = \left(\frac{a^4 X}{a^2 - b^2} \right)^{\frac{1}{3}} \text{ and } y = \left(\frac{b^4 X}{a^2 - b^2} \right)^{\frac{1}{3}}$$

$$\frac{1}{a^2} \left(\frac{a^4 X}{a^2 - b^2} \right)^{\frac{2}{3}} + \frac{1}{b^2} \left(\frac{b^4 X}{a^2 - b^2} \right)^{\frac{2}{3}} = 1$$

$$(aX)^{\frac{2}{3}} + (bY)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$$

$\therefore (aX)^{\frac{2}{3}} + (bY)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$ is the evolute of the given curve.

4.3.4 Example

Show that the evolute of the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} + a^{\frac{2}{3}} = \text{is } (x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$$

Solution

The parametric equation of the curve is $x = a \cos^3 \theta, y = a \sin^3 \theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a \sin^2 \theta \cos \theta}{-3a \cos^2 \theta \sin \theta}$$

$$\frac{dy}{dx} = -\tan \theta$$

$$\frac{d^2 y}{dx^2} = -\sec^2 \theta \frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = -\sec^2 \theta \times \frac{1}{-3a \cos^2 \theta \sin \theta}$$

$$= \frac{1}{3a \cos^4 \theta \sin \theta}$$

$$X = x - \frac{y_1(1 + y_1^2)}{y^2}$$

$$X = a \cos^3 \theta - \frac{(-\tan \theta)(1 + \tan^2 \theta)}{\frac{1}{3a \cos^4 \theta \sin \theta}}$$

$$X = a \cos^3 \theta + 3 a \sin^2 \theta \cos \theta \quad \dots(1)$$

$$Y = y + \frac{1 + y_1^2}{y_2}$$

$$Y = a \sin^3 \theta + \frac{(1 + \tan^2 \theta)}{\frac{1}{3a \cos^4 \theta \sin \theta}}$$

$$Y = a \sin^3 \theta + 3a \cos^2 \theta \sin \theta \quad \dots(2)$$

From (1) and (2)

$$(X + Y) = a(\cos \theta + \sin \theta)^3 \text{ and } X - Y = a(\cos \theta - \sin \theta)^3$$

$$\Rightarrow (X + Y)^{\frac{2}{3}} + (X - Y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$$

$$(x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = 2a^{\frac{2}{3}} \quad \text{is the evolute of the given curve.}$$

4.3.5 Example

Show that the evolute of the curve

$$x = a(\cos \theta + \sin \theta), \quad y = a(\sin \theta - \theta \cos \theta) \text{ is}$$

$$x^2 + y^2 = a^2$$

Solution

$$x = a(\cos \theta + \sin \theta) \text{ and } y = a(\sin \theta - \theta \cos \theta)$$

Differentiating w.r.t θ

$$\frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta), \quad \frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta)$$

$$\frac{dx}{d\theta} = a\theta\cos\theta, \quad \frac{dy}{d\theta} = a\theta\sin\theta$$

$$\frac{dy}{dx} = \frac{a\theta\sin\theta}{a\theta\cos\theta}$$

$$\frac{dy}{dx} = \tan\theta \quad \text{.....(1)}$$

$$\frac{d^2y}{dx^2} = \sec^2\theta \frac{d\theta}{dx} = \sec^2\theta \cdot \frac{1}{a\theta\cos\theta}$$

$$\frac{d^2y}{dx^2} = \frac{1}{a\theta\cos^3\theta} \quad \text{.....(2)}$$

$$X = x - \frac{y_1(1+y_1^2)}{y_2} = a(\cos\theta + \theta\sin\theta) - \frac{\tan(1+\tan^2\theta)}{\frac{1}{a\theta\cos^3\theta}}$$

$$X = a\cos\theta \quad \text{.....(3)}$$

$$Y = y + \frac{1+y_1^2}{y_2} = a(\sin\theta - \theta\cos\theta) + \frac{(1+\tan^2\theta)}{\frac{1}{a\theta\cos^3\theta}}$$

$$Y = a\sin\theta$$

Eliminating ' θ ' from (3) & (4)

$$X^2 + Y^2 = a^2$$

$\therefore x^2 + y^2 = a^2$ is the evolute of the given curve.

4.3.6 Example

Show that the evolute of the curve $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$, $y = a \sin t$ is

$$y = a \cos h \frac{x}{a}$$

Solution

$$x = a\left(\cos t + \log \tan \frac{t}{2}\right) \quad y = a \sin t$$

$$\frac{dx}{dy} = a \left[-\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \sec^2 \frac{t}{2} \cdot \frac{1}{2} \right], \quad \frac{dy}{dt} = a \cos t$$

$$\frac{dx}{dt} = a \left[-\sin t + \frac{1}{2 \sin \frac{t}{2} \cos \frac{t}{2}} \right] = a \left[-\sin t + \frac{1}{\sin t} \right]$$

$$\frac{dx}{dt} = \frac{a \cos^2 t}{\sin t}, \text{ and } \frac{dy}{dt} = a \cos t,$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a \cos t}{\frac{a \cos^2 t}{\sin t}} = \tan t$$

$$\frac{d^2 y}{dx^2} = \sec^2 t \cdot \frac{dt}{dx} = \frac{\sec^2 t \sin t}{a \cos^2 t} = \frac{\sin t}{a \cos^4 t}$$

$$X = x - \frac{\tan t (1 + \tan^2 t)}{\frac{\sin t}{a \cos^4 t}}$$

$$X = a(\cos t + \log \tan \frac{t}{2}) - a \cos t = a \log \tan \frac{t}{2} \quad \dots(1)$$

$$\frac{1 + y_1^2}{y_2}$$

$$Y = a \sin t + \frac{1 + \tan^2 t}{\frac{\sin t}{a \cos^4 t}}$$

$$Y = \frac{a}{\sin t} \quad \dots(2)$$

From (1) $\tan \frac{t}{2} = e^{X/a}$

From (2) $\sin t = \frac{a}{Y}$

$$(i.e.,) \quad \frac{2 \tan^2 t}{1 + \tan^2 t} = \frac{a}{Y}$$

$$\frac{2e^{\frac{x}{a}}}{1 + e^{\frac{2x}{a}}} = \frac{a}{Y}$$

$$Y = \frac{a}{2} \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right) = a \cosh \left(\frac{x}{a} \right)$$

$\therefore y = a \cosh \left(\frac{x}{a} \right)$ is the evolute of the given curve.

Exercise – 4(C)

1. Find the evolute of the curve $x = ct$ and $y = \frac{c}{t}$

$$[\text{Ans : } (x+y)^2 - (x-y)^2 = (4c)^2]$$

2. Prove that the equation to the evolute of the parabola $x^2 = 4ay$ is $4(y - 2a)^3 = 27ax^2$
3. Show that the evolute of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $(ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}$
4. Show that the evolute of the cycloid $x = a(q - \sin q)$ $y = a(1 - \cos q)$ is another equal cycloid.

4.4

4.4.1 Envelopes

Family of curves :

Let us consider the equation of a straight line $x \cos \alpha + y \sin \alpha = p$. Where α is a parameter. For different values of α , $x \cos \alpha + y \sin \alpha = p$ gives different straight lines but all of them are at a constant distance ' p ' from the origin. The set of all of these straight lines is said to form a family of straight lines and ' α ' is called the parameter of the family. For a given p , different values of ' α ' give different straight lines which touch the circle $x^2 + y^2 = p^2$. The circle $x^2 + y^2 = p^2$ is called the *envelope of the family of straight lines*.

A curve, which touches each member of a given family of curves and each point is the point of contact of some member of the family is called the envelope of the family of curves.

Let $f(x, y, \alpha) = 0$ be the given family of curves and α is the parameter.

Now differentiating with respect to ' α ' partially $\frac{\partial f(x, y, \alpha)}{\partial \alpha} = 0$

Elimination of ' α ' from $f(x, y, \alpha) = 0$ and $\frac{\partial f(x, y, \alpha)}{\partial \alpha} = 0$ given an equation in terms of x, y which is called the *envelope of the family of curves*.

If the given equation of the family of curves $f(x, y, \alpha) = 0$ is the quadratic in ' α ', say of the form $A\alpha^2 + B\alpha + C = 0$ where ' α ' is the parameter and A, B, C are functions of x, y . Then the envelope of the family of curves is $B^2 - 4AC = 0$.

4.4.2 Example

Find the envelope of the family of straight lines $y = mx + \frac{a}{m}$ where ' m ' is the parameter.

Solution

$$y = mx + \frac{a}{m} \quad \text{.....(1)}$$

m is the parameter

Diff (1) w.r. to ' m '

$$0 = x - \frac{a}{m^2}$$

$$m = \sqrt{\frac{a}{x}} \quad \text{.....(2)}$$

Substituting for ' m ' in (1)

$$y = \sqrt{\frac{a}{x}} \cdot x + \frac{a}{\sqrt{\frac{a}{x}}}$$

$$y = 2\sqrt{ax}$$

Squaring we get $y^2 = 4ax$ is the envelope of the family of curves

Aliter :

$$y = m x + \frac{a}{m}$$

$$m^2 x - m y + a = 0$$

is quadratic in parameter ' m ' ($A = x$, $B = -y$, $C = a$)

$B^2 - 4AC = 0$ then gives the required envelop.

$\therefore$ Envelope of the family of curves is $y^2 - 4(a)(x) = 0$

$$y^2 = 4ax \text{ is envelope.}$$

4.4.3 Example

Find the envelope of the family of curves

$$y = m x + \sqrt{a^2 m^2 + b^2} \text{ where 'm' is the parameter.}$$

Solution

$$y = mx + \sqrt{a^2 m^2 + b^2}$$

$$(y - mx)^2 = a^2 m^2 + b^2$$

$$m^2(x^2 - a^2) + m(-2yx) + (y^2 - b^2) = 0$$

is quadratic in parameter ' m '

The envelope of the family of curves

$$(-2x y)^2 - 4(x^2 - a^2)(y^2 - b^2) = 0$$

$$b^2 x^2 + a^2 y^2 = a^2 b^2$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is the envelope of the family of curves.}$$

4.4.4 Example

Find the envelope of the family of curves $x \cos^3 \theta + y \sin^3 \theta = c$ when θ is the parameter.

Solution

$$\text{Given } x \cos^3 \theta + y \sin^3 \theta = c \quad (\theta \text{ is parameter}) \quad \dots(1)$$

Diff. (1) partially w.r. to ' θ '

$$-3x \cos^2 \theta \sin \theta + 3y \sin^2 \theta \cos \theta = 0$$

Hence $\tan \theta = \frac{y}{x}$

$$\therefore \sin \theta = \frac{x}{\sqrt{x^2 + y^2}}, \cos \theta = \frac{y}{\sqrt{x^2 + y^2}}$$

Substituting these values in (1)

$$x \left(\frac{y}{\sqrt{x^2 + y^2}} \right)^3 + y \left(\frac{x}{\sqrt{x^2 + y^2}} \right)^3 = c$$

$x^2 y^2 = c(x^2 + y^2)$ is the envelope of the family curves.

4.4.5 Example

Find the envelope of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where

a, b are parameters connected by the relation $a^2 + b^2 = c^2$

Solution

$$\text{Given} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \dots(1)$$

$$a^2 + b^2 = c^2 \quad \dots(2)$$

Assume that 'a' and 'b' are functions of 't'

Diff. (1) and (2) w.r. to 't'

$$x^2 \left(\frac{-2}{a^3} \right) \frac{da}{dt} + y^2 \left(\frac{-2}{b^3} \right) \frac{db}{dt} = 0$$

$$\frac{x^2}{a^3} \frac{da}{dt} + \frac{y^2}{b^3} \frac{db}{dt} = 0 \quad \dots(3)$$

Diff. $a^2 + b^2 = c^2$ w.r. to 't'

$$2a \frac{da}{dt} + 2b \frac{db}{dt} = 0$$

$$\text{(i.e.,)} \quad a \frac{da}{dt} + b \frac{db}{dt} = 0 \quad \dots(4)$$

Eliminating $\frac{da}{dt}, \frac{db}{dt}$ from (3), (4)

$$\frac{\left(\frac{x^2}{a^3} \right)}{a} = \frac{\left(\frac{y^2}{b^3} \right)}{b} \Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{\frac{x^2}{a^2} + \frac{y^2}{b^2}}{a^2 + b^2}$$

Using (1) and (2) we get

$$\frac{x^2}{a^3} = \frac{y^2}{b^3} = \frac{1}{c^2}$$

(i.e.,) $\frac{x^2}{a^4} = \frac{1}{c^2}$

$\Rightarrow a^4 = x^2 c^2$
 $a^2 = xc$

$$\frac{y^2}{y^4} = \frac{1}{c^2}$$

$\Rightarrow b^2 = yc$

Substituting a^2, b^2 , values in $a^2 + b^2 = c^2$

$\therefore x^2 c^2 + y^2 c^2 = c^2$

$x^2 + y^2 = 1$ is the envelope of the family of curves.

4.4.6 Example

Find the envelope of the family of curves $\frac{x}{a} + \frac{y}{b} = 1$ where a, b are parameters connected by the relation $ab = c^2$.

Solution

Given $\frac{x}{a} + \frac{y}{b} = 1$ (1)

$ab = c^2$ (2)

Assume that a, b are functions of 't'

Diff. (1) and (2) w.r. to 't'

$$x \left(\frac{-1}{a^2} \right) \frac{da}{dt} + y \left(\frac{1}{b^2} \right) \frac{db}{dt} = 0$$

$$\frac{x}{a^2} \frac{da}{dt} + \frac{y}{b^2} \frac{db}{dt} = 0$$
(3)

Diff. $ab = c^2$ w.r. to 't'

$$a \frac{db}{dt} + b \frac{da}{dt} = 0$$
(4)

Eliminating $\frac{da}{dt}$, $\frac{db}{dt}$ from (3), (4)

$$\frac{\frac{x}{a^2}}{b} = \frac{\frac{y}{b^2}}{a} \Rightarrow \frac{\frac{x}{a}}{ab} = \frac{\frac{y}{b}}{ab} = \frac{\frac{x}{a} + \frac{y}{b}}{2ab}$$

Using (1) and (2) we get

$$\frac{\frac{x}{a^2}}{b} = \frac{\frac{y}{b^2}}{a} = \frac{1}{2c^2}$$

$$\frac{\frac{x}{a}}{ab} = \frac{1}{2c^2}$$

$$\frac{x}{a} = \frac{ab}{2c^2}$$

Using (2)

$$\frac{x}{a} = \frac{c^2}{2c^2} \quad \text{.....(5)}$$

$\therefore$

$$a = 2x$$

$$\frac{\frac{y}{b^2}}{a} = \frac{1}{2c^2}$$

$$\frac{y}{b} = \frac{ab}{2c^2}$$

Using (2)

$$\frac{y}{b} = \frac{c^2}{2c^2}$$

$$b = 2y \quad \text{.....(6)}$$

Substituting a, b values from (5), (6) in (2)

$$(2x)(2y) = c^2$$

$4xy = c^2$ is the envelope of the family of curves.

Exercise – 4 (D)

1. Find the envelope of the family of curves :

(a) $y = mx + a\sqrt{1+m^2}$ when m is parameter.

[Ans : $x^2 + y^2 = a^2$]

(b) $y^2 = 2a(x - a)$ where a is the parameter.

[Ans : $x^2 - 2y^2 = 0$]

(c) $x \cos \phi + y \sin \phi = p$ where ϕ is the parameter.

[Ans : $x^2 + y^2 = p^2$]

(d) $\frac{x}{a} + \frac{y}{b} = 1$ when $a + b = c$ (a, b parameters).

[Ans : $\sqrt{x} + \sqrt{y} = \sqrt{c}$]

(e) $\frac{x}{a} + \frac{y}{b} = 1$ when $a^2 + b^2 = c^2$ (a, b parameters).

[Ans : $\sqrt{x} + \sqrt{y} = \sqrt{c}$]

(f) $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$ when $a + b = c$ (a, b parameters).

[Ans : $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$]

(g) Find the envelope of the family of curves $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$ when $a^m b^n = c^{m+n}$

(h) Find the envelope of the family of straight lines $\frac{x}{a} + \frac{y}{b} = 1$ where a & b are related by the equation $a^n + b^n = c^n$, c being a constant.

[Ans : $x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} = c^{\frac{n}{n+1}}$]

4.5 Curve Tracing

Generally a curve is drawn by plotting a number of points and joining them by a smooth line.

If an approximate shape of the curve is sufficient for a given purpose then it is enough to study certain important characteristics. This purpose is served by curve tracing methods.

The points to be observed for tracing of plane algebraic curves are given below.

1. Whether the curve is passing through the origin, if so the equations of the tangents to the curve at the origin.

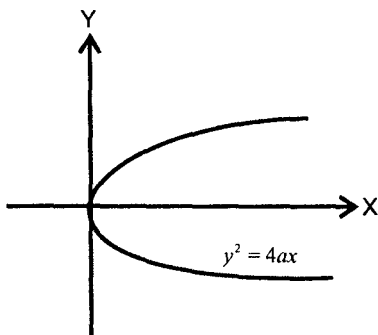
Suppose $F(x,y) = 0$ is the algebraic form of the equation of the curve.

$F(0,0) = 0 \Rightarrow$ the curve is passing through origin (i.e.,) If there is no constant term in $F(x,y)$ then origin lies on the curve.

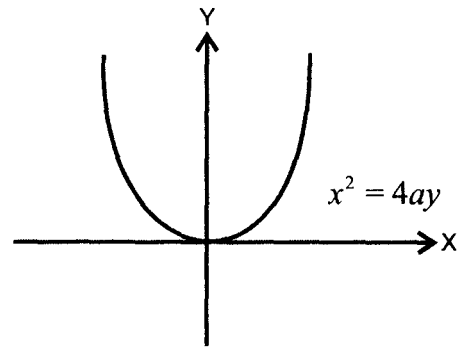
The equations of the tangents to the curve are obtained by equating the lowest degree terms in $F(x,y)$ to zero.

If at $O(0,0)$ the tangents are:

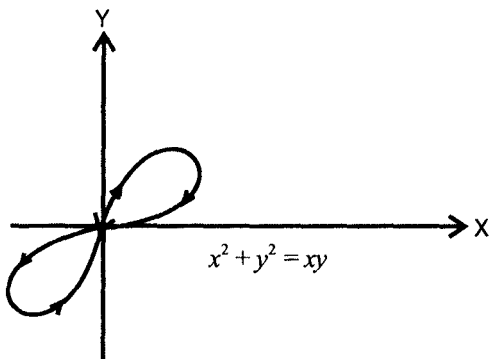
- (i) real and coincident then 'O' is called a cusp
 - (ii) real and different then 'O' is called a node
 - (iii) imaginary then 'O' is called a conjugate point
2. Whether the curve is symmetric about an axis or about other any line. If
 - (i) $F(x, y) = F(x-y) \Rightarrow$ curve is symmetric about x-axis
 - (ii) $F(-x, y) = F(x, y) \Rightarrow$ curve is symmetric about y-axis
 - (iii) $F(-x, -y) = F(x, y) \Rightarrow$ curve is symmetric in opposite quadrants.
 - (iv) $F(y, x) = F(x, y)$ curve is symmetric about $y = x$
 - (v) $F(-y, -x) = F(x, y)$ curve is symmetric about $y = -x$



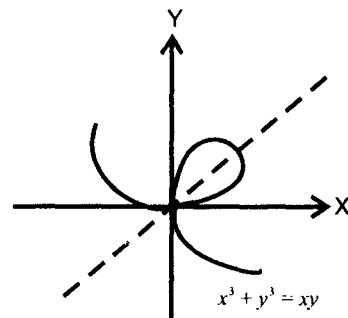
(i)



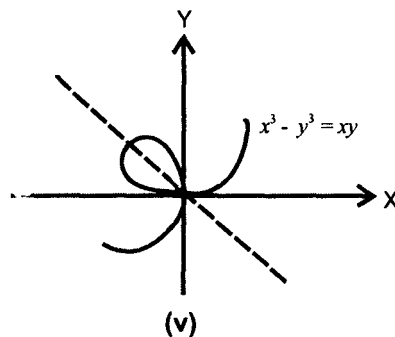
(ii)



(iii)



(iv)



(v)

3. Whether the curve intersects the axes, if so the tangents to the curve at these points.
4. Find the region in which the curve exists (i.e.) the curve is defined. The values of x for which y is defined gives the extent parallel to x -axis and the values of y for which x is defined gives the extent in a direction parallel to y -axis.

If y is imaginary for values of x in a certain region then the curve does not exist in that region.

Similarly with respect to values of y .

5. Finding the asymptotes.

An asymptote is a line that is at a finite distance from $(0, 0)$ and is tangential to the curve at infinity (i.e.) the curve approaches the line at infinity.

- (i) Sum of the coefficients of the highest degree terms in x equated to zero gives the equations of the asymptotes parallel to x – axis.
- (ii) Sum of the coefficients of the highest degree terms in y equated to zero gives the equations of the asymptotes parallel to y -axis.
- (iii) To find the asymptotes that are neither parallel to x -axis nor parallel to y -axis (i.e.) oblique asymptotes, the following method is suggested.

Substitute $y = mx + c$ in $F(x, y) = 0$ and rewrite the equation as a polynomial equation in ' x ' as

$$\Phi_n(m)x^n + \Phi_{n-1}(m)x^{n-1} + \dots + \Phi_0(m) = 0$$

The slopes of the asymptotes are given by $\Phi_n(m) = 0$. Let the slopes be $m_1, m_2, \dots$.

The values of ' c ' can be obtained from $\Phi_{n-1}(m) = 0, \Phi_{n-2}(m) = 0$ (if necessary).

Let the corresponding values of c be $c_1, c_2, \dots$.

Then the asymptotes are $y = m_1x + c_1, y = m_2x + c_2, \dots$.

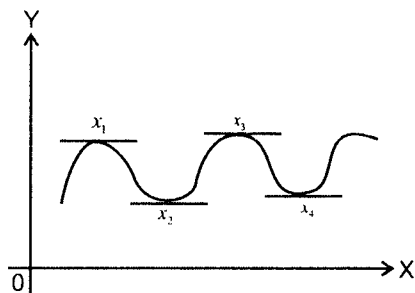
Note: if $\Phi_n(m)$ is a constant then there are no oblique asymptotes to the curve.

6. Obtain $\frac{dy}{dx}$ from $F(x, y) = 0$ by differentiation.

If in the interval for $x, \frac{dy}{dx} > 0$ then the curve is increasing

and $\frac{dy}{dx} < 0$, then the curve is decreasing in that region.

7. The points of maxima and minima are given by $\frac{dy}{dx} = 0$.



$$\frac{dy}{dx} = 0 \text{ gives } x_1, x_2, \dots$$

If in an interval for x

- (i) $\frac{d^2y}{dx^2} > 0$ then the curve is concave upwards ($x_2, x_4, \dots$ in the figure) and has a minima in that interval.
- (ii) $\frac{d^2y}{dx^2} < 0$ the curve is concave downward ($x_1, x_3, \dots$ in the figure) and has a maxima in that interval.
- (iii) If at a point x_2 (say) in the interval $\frac{d^2y}{dx^2} = 0$ then that point is called a point of inflection and at that point the curve changes its concavity to the opposite direction.

4.5.1 Example

Trace the curve $y = \frac{x^2 + 2x}{x + 1}$ (1)

Solution:

Substitution of $x = 0$ in (1) gives $y = 0$. $\therefore$ origin lies on the curve.

$F(x, -y) \neq F(x, y) \neq F(-x, y)$ etc. $\therefore$ The curve is not symmetric about any line

1. can be written as $y(x + 1) - x^2 - 2x = 0$ (2)

(ie) $x^2 - xy - (y - 2x) = 0$ Lowest degree term is $y - 2x = 0$

$\therefore$ Equation of the tangent at $(0, 0)$ is $y = 2x$.

Substituting $y = 0$ in (1) we have $x(x + 2) = 0, x = 0, x = -2$

$\therefore$ The curve crosses x -axis at $(0, 0)$ and $(-2, 0)$

Since $x=0$ in (1) gives $y=0$ only, there is no intercept on y -axis

The curve is not defined at $x = -1$. Hence it is discontinuous.

$\therefore$ The curve is defined in the region $R - \{-1\}$

Coefficient of highest degree term in x (i.e.) of x^2 is 1 which is a constant. Hence there is no asymptote parallel to x -axis.

Coefficient of highest degree term in y is $x+1$. $\therefore x+1=0$ is the asymptote parallel to y -axis.

From (2) $\frac{dy}{dx} = \frac{1}{(x+1)^2}$ which is > 0 always except at $x = -1$

Hence the curve is increasing in the regions.

$$\infty < x < -2, -2 < x < -1, -1 < x < 0 \text{ and } 0 < x < \infty$$

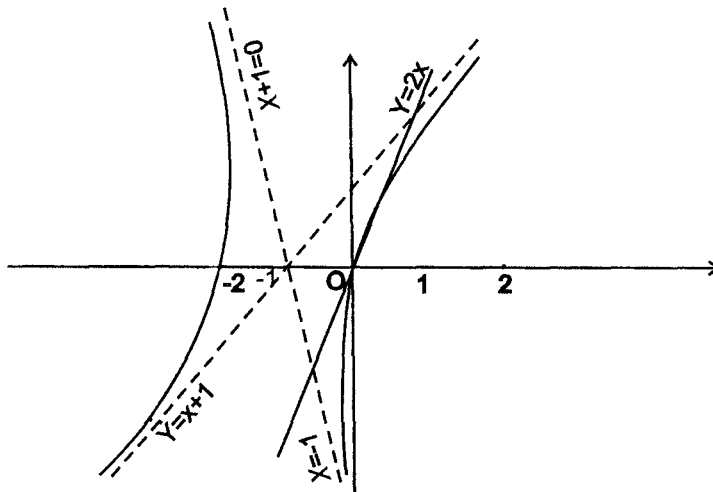


Fig. 4.7

4.5.2 Example

Trace the curve $y = \frac{x^2 + 1}{x^2 - 1}$

Solution :

$$F(x, y) = (x^2 - 1)y - (x^2 + 1) = 0 \quad \dots\dots(1)$$

$x = 0$ in (1) gives $y = -1$ $\therefore$ Hence curve does not pass through $(0,0)$, and y intercept is -1 .

$F(-x, y) = F(x, y)$ curve is symmetric about y -axis

$y = 0$ gives $x^2 + 1 = 0 \Rightarrow x = \pm i$ curve does not cross x -axis.

Coefficient of highest degree term in x is $y-1$ (from(1))

Hence $y = 1$ is the asymptote parallel to x -axis

Coefficient of highest degree term in y is $x^2 - 1$ $\therefore x = \pm 1$ are asymptotes parallel to y -axis

$$\frac{dy}{dx} = \frac{-4x}{(x^2 - 1)^2} \text{ Hence } y' = 0 \text{ gives } x = 0 \text{ and } y''_{x=0} = -\frac{4}{3} < 0 \text{ and } \neq 0$$

$\therefore$ The maximum point on the curve is $(0, -1)$ and there are no points of inflection.

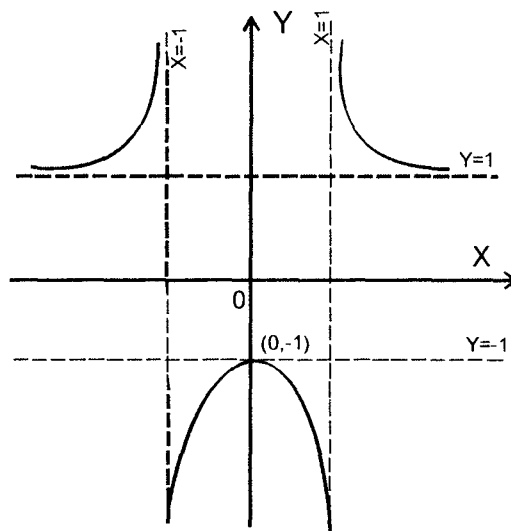


Fig. 4.8

The curve is decreasing in the interval $0 < |x| < 1$ and $1 < x < \infty$ as $\frac{dy}{dx}$ is -ve there and is increasing in $-\infty < x < -1$ as $\frac{dy}{dx}$ is +ve in this interval

4.5.3 Example:

Trace the Folium $x^3 + y^3 = 3axy$

....(1)

Solution:

Clearly $(0,0)$ lies on the curve. No intercepts on y -axis.

$$F(y, x) = y^3 + x^3 - 3axy = 0 = F(x, y) \quad \therefore \text{curve is symmetric about } x\text{-axis}$$

Coefficient of highest degree term in y is 1, a constant, and hence no asymptote parallel to y -axis. Lowest degree term equated to zero gives $xy = 0$, $x = 0$, $y = 0$ (i.e) the axes are the tangents to the curve at the origin.

$y = x$ in (1) gives $2x^3 - 3ax^2 = 0$ $x = 0, \frac{3a}{2}$ (ie) the line $y = x$ meets the curve at $(0,0)$ and $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

$$F(-x, -y) = x^3 - y^3 + 3axy = F(x, y) \quad \therefore \text{curve is not symmetric about } (0,0)$$

To obtain oblique asymptotes : Substitute $y = mx + c$ in (1) and rewriting in terms of ' x ' we get $(1 + m^3)x^3 + (3m^2c - 3am)x^2 + \dots = 0$ (ie) $\phi_3(m) = 1 + m^3$ and $\phi_2(m) = 3m^2c - 3am, \dots$

$\phi_3(m) = 0$ gives $m = -1$, there is only one oblique asymptote with slope -1

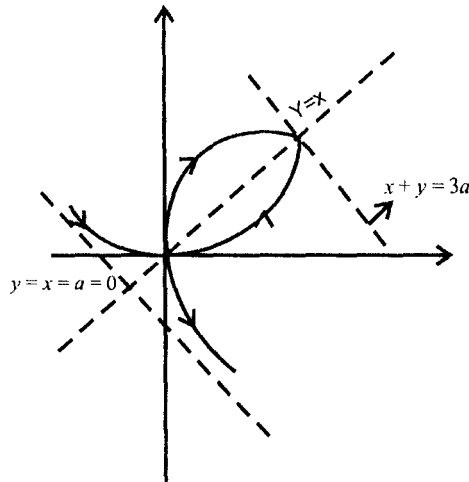


Fig. 4.9

$$\phi_2(m) = 0 \text{ gives } 3m(cm - a) = 0 \Rightarrow cm - a = 0 \quad (\text{e}) \quad c = -a (\because m = -1)$$

$\therefore$ The asymptote is $y = -x - a$ or $y + x + a = 0$ (2)

$$\text{From (1) } \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax} \therefore \left(\frac{dy}{dx} \right)_{\left(\frac{3a}{2}, \frac{3a}{2} \right)} = -1$$

Equation of the tangent to the curve at this point is $y + x = 3a$ which is parallel to (2)

4.5.4 Example:

Trace the curve $a^2 y^2 - x^2 (a^2 - x^2) = 0$ (1)

Solution:

$x = 0$ in (1) gives $y = 0, 0$ (ie) double point on the curve

$F(-x, y) = F(x, -y) = F(x, y)$ curve is symmetric about both the axes .

Lowest degree terms, in (1), equated to zero gives $a^2 (y^2 - x^2) = 0 \Rightarrow y = \pm x$ are the tangents

to the curve at $(0,0)$. $y = 0$ in (1) gives $x = \pm a$

The curve intersects the x-axis at $(a, 0)$ and $(-a, 0)$ From (1) it can be seen that for $|x| > a$

y^2 is negative (ie) the curve does not exist for $|x| > a$

Differentiating(1) w.r.t. 'x' $2a^2 y y_1 = 2x^2 (-x) + 2x(a^2 - x^2)$

$$y_1 = \frac{x(a^2 - 2x^2)}{a^2 y} \therefore y_1 = 0 \text{ gives } x = 0, \pm \frac{a}{\sqrt{2}}, \text{ points of maxima and minima.}$$

Tangents at the points $x = \pm \frac{a}{\sqrt{2}}$ to the curve are parallel to x-axis .Further

$\left(\frac{d^2 y}{dx^2} \right)$ at $x = \frac{a}{\sqrt{2}}$ is negative. $\therefore x = \frac{a}{\sqrt{2}}$ is a maximum point and y is increasing

in $0 < x < \frac{a}{\sqrt{2}}$ while decreasing in $\frac{a}{\sqrt{2}} < x < a$.Thus after tracing the curve in

the first quadrant, the symmetry about both the axes can be utilized for tracing the complete curve.

Substituting $y = mx + c$ in (1) gives $x^4 + a^2 (mx + c)^2 - a^2 x^2 = 0$

$\phi_4(m) = 1$ which is a constant. Hence there are no oblique asymptotes to the curve.

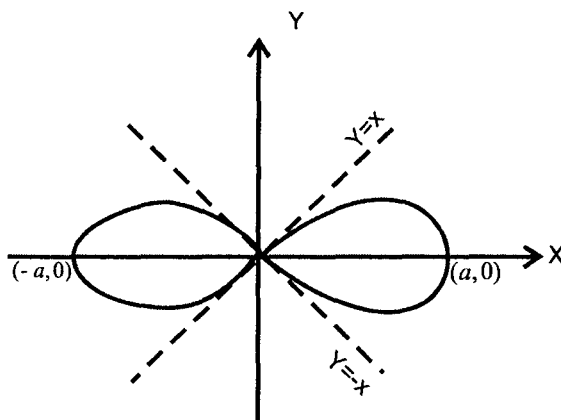


Fig. 4.10

4.5.5 Example:

Trace the curve $y^2(x^2 - a^2) + a^2x^2 = 0$ (1)

Solution:

$x = 0$ in (1) gives $y = 0$ $\therefore (0,0)$ lies on the curve.

$F(x, -y) = F(x, y) = F(x, -y)$ $\therefore$ curve is symmetric about both the axes.

Highest degree terms of y equated to zero gives $(x^2 - a^2)y^2 = 0 \Rightarrow x = \pm a$

$\therefore x = \pm a$ are asymptotes parallel to y -axis. Highest degree terms of x equated to zero gives $x^2(y^2 + a^2) = 0 \Rightarrow x = \pm ia$ $\therefore$ No asymptotes parallel to x -axis.

Equating lowest degree terms to zero we get $a^2(y^2 - x^2) = 0 \Rightarrow y = \pm x$ are tangents at $(0, 0)$ writing $y = mx + c$ in (1) and rearranging in terms of ' x '

$$m^2x^4 + 2mcx^3 + (c^2 - a^2m^2 + a^2)x^2 - 2a^2mcx - c^2a^2 = 0$$

$$\phi_4(m) = 0 \text{ gives } m = 0 \quad \phi_3(m) = 2mc = 0$$

$$\Rightarrow c \text{ is indeterminate since } m = 0$$

$$\phi_2(m) = 0 \text{ gives } c^2 + a^2 = 0 \Rightarrow c = \pm ai$$

oblique asymptotes do not exist. Further there are no intercepts on either of the axes.

(1) can be written as $y^2 = -\frac{a^2 x^2}{x^2 - a^2}$ which indicates that curve does not exist

when $|x| > a$ $\therefore$ The curve exists only in the region $-a < x < a$.

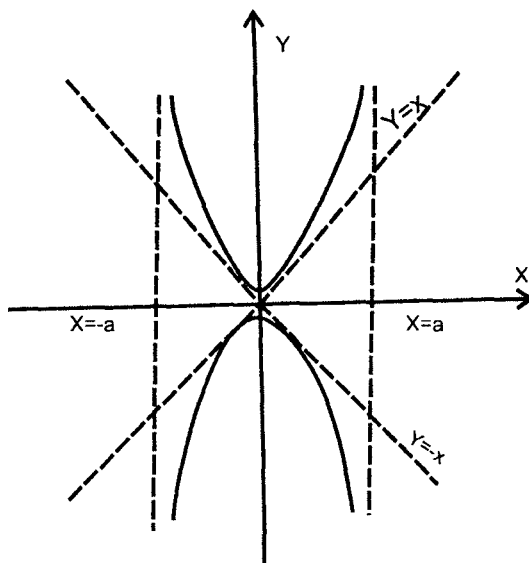


Fig. 4.11

$\frac{dy}{dx} = \frac{a^4 x}{y}$ and $y > 0$ in first quadrant. Hence curve is increasing in $0 < x < a$.

4.5.6 Example:

Trace the curve $y^2(2-x) - x^2(1+x) = 0$ (1)

Solution :

$x = 0$ in (1) gives $y = 0$ $\therefore$ (0,0) lies on the curve $F(x, -y) = F(x, y)$ $\therefore$ curve is symmetric about x-axis. Lowest degree terms equated to zero gives

$2y^2 - x^2 = 0 \Rightarrow y = \pm \frac{x}{\sqrt{2}}$ tangents at (0,0) to the curve $y = 0$ in (1) gives -1, 0

curve passes through (-1,0) in addition to (0,0)

Coefficient of highest degree terms of y equated to zero gives $2 - x = 0$

(ie) $x = 2$ is an asymptote parallel to y-axis

No asymptote parallel to x-axis as coefficient of x^2 is 1, a constant

Substituting $y = mx + c$ in (1) and rearranging

$$x^3(m^2 + 1) + x^2(1 + 2mc - 2m^2) + (c^2 - 4mc)x - 2c^2 = 0$$

$\phi_3(m) = m^2 + 1 \Rightarrow \phi_3(m) = 0$ gives $m = \pm i \therefore$ There are no oblique asymptotes.

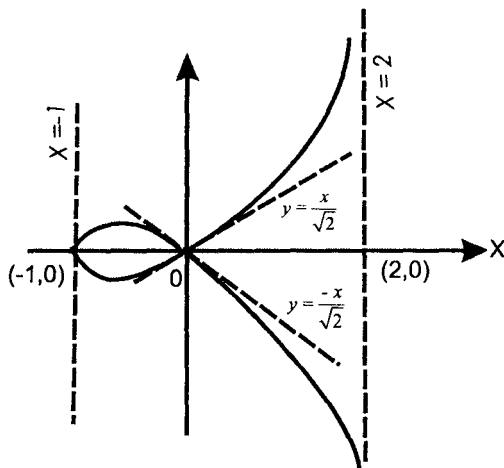


Fig. 4.12

From (1) $y_1 = \frac{y^2 + 2x + 3x^2}{2y(2-x)}$. This shows that $y_1 > 0$ in $0 < x < 2$ where the curve is increasing and $x = 2$ is an asymptote.

$$y_1 = 0 \Rightarrow x = 0, \quad x = \frac{5 \pm \sqrt{57}}{4}.$$

In the second quadrant y is maximum at $x = \frac{5 - \sqrt{57}}{4}$ where the tangent is parallel to x -axis

It can be seen that the curve forms a loop in $[-1, 0]$ due to symmetry about x -axis.

4.5.7 Example

Trace the curve $y^2 - (x-2)(x-4)^2 = 0$ (1)

Solution :

$y = 0$. (1) gives $x = 2, 4$

Curve intersects the x -axis at $(2,0)$ and $(4,0)$

$F(x, -y) = F(x, y) \therefore$ curve is symmetric about x-axis

For $x < 2$, y^2 is -ve $\therefore$ therefore curve does not exist for $x < 2$

From (1) it can be seen that $x = 2$ is a tangent at $(2, 0)$

Coefficient of $x^3 = 1$, a constant $\therefore$ No asymptote parallel to x-axis

Differentiating (1) w.r.t 'x'

$$\frac{dy}{dx} = \frac{(x-4)(3x-8)}{2y}$$

$$\text{At } (4, 0) \quad \frac{dy}{dx} = \lim_{x \rightarrow 4} \pm \frac{(x-4)(3x-8)}{2\sqrt{x-2}(x-4)} = \lim_{x \rightarrow 4} \pm \frac{3x-8}{2\sqrt{x-2}} = \pm\sqrt{2}$$

$\therefore$ Equation of tangents at $(4, 0)$ are $y = \pm\sqrt{2}(x-4)$

Further $\frac{dy}{dx} \rightarrow \infty$ as $x \rightarrow 2 \therefore x = 2$ is a tangent at $(2, 0)$

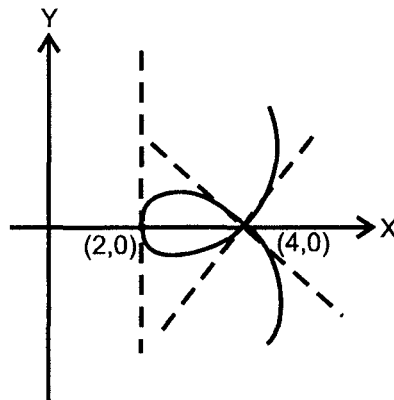


Fig. 4.13

$\frac{dy}{dx} = 0$ gives $x = 8/3$. Thus the tangents at $x = 8/3$ are parallel to x-axis. For the region.

$2 < x < 8/3$, $\frac{dy}{dx} > 0$ above x-axis and $\frac{dy}{dx} < 0$ below x-axis. At $x = 8/3$, y takes

extreme values. For the region $8/3 < x < 4$ $\frac{dy}{dx} < 0$ above x-axis and $\frac{dy}{dx} > 0$ below x-axis.

$\therefore$ the curve forms a loop in $2 \leq x \leq 4$

4.5.8 Example

Trace the curve $y(x^2 + 4a^2) - 8a^3 = 0$ (1)

Solution:

$y \neq 0$ when $x = 0$. $\therefore$ origin does not lie on the curve.

Curve is symmetric about x-axis, since $F(-x, y) = F(x, y)$

y is +ve for all real x $\therefore$ Curve completely lies above x-axis.

Substituting $y = mx + c$ in (1)

$$m^3 + cx^2 + 4a^2mx + 4a^2c - 8a^3 = 0$$

$$\phi_3(m) = 0 \Rightarrow m = 0 \text{ and } \phi_2(m) = 0 \text{ gives } c = 0$$

(ie) $y = 0$ is an asymptote to the curve From (1) $\frac{dy}{dx} = -\frac{16a^3x}{(x^2 + 4a^2)^2}$

In first quadrant $\frac{dy}{dx} < 0$ and hence y is decreasing. $\frac{dy}{dx} = 0$ at $(0, 2a)$. y is maximum at $(0, 2a)$.

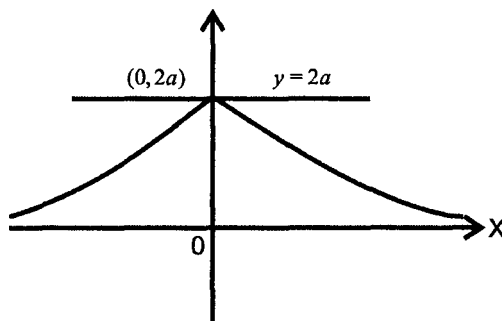


Fig. 4.14

4.5.9 Example:

Trace the curve $y - c \cosh x/c = 0$ (1)

Solution :

$$x = 0 \text{ in (1) gives } y = (c/2)[e^0 + e^{-0}] = c$$

$\therefore$ Curve does not pass through $(0,0)$ and intersects y -axis at $(0, c)$.

$F(-x, y) = y - (c/2)(e^{-x/c} + e^{x/c}) = F(x, y)$ curve is symmetric about y -axis

It can be seen from (1) that $y > 0$ for all real x .

$$\frac{dy}{dx} = \sinh(x/c) = \frac{1}{2}[e^{x/c} - e^{-x/c}]$$

which is > 0 for $x > 0$ and < 0 for $x < 0$.

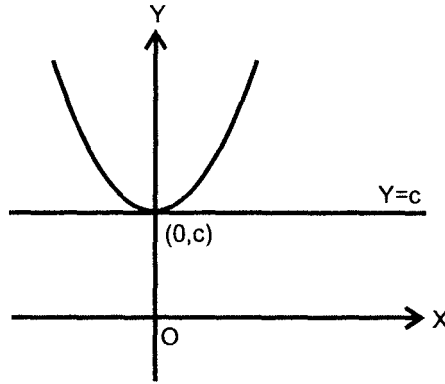


Fig. 4.15

Thus the curve is increasing in $(0, \infty)$, and decreasing in $(-\infty, 0)$.

When $x = 0$ we have $\frac{dy}{dx} = 0$. Further $y = c$ for $x = 0$.

(i.e.) The tangent to the curve at $(0, c)$ is parallel to x -axis.

4.5.10 Example

Trace the curve $(x/a)^{2/3} + (y/a)^{2/3} - 1 = 0$ (1)

Solution:

$y \neq 0$ when $x = 0$ $\therefore (0,0)$ does not lie on the curve.

It can be observed that $F(-x, y) = F(x, y) = F(x, -y)$

The curve is symmetric about both the axes.

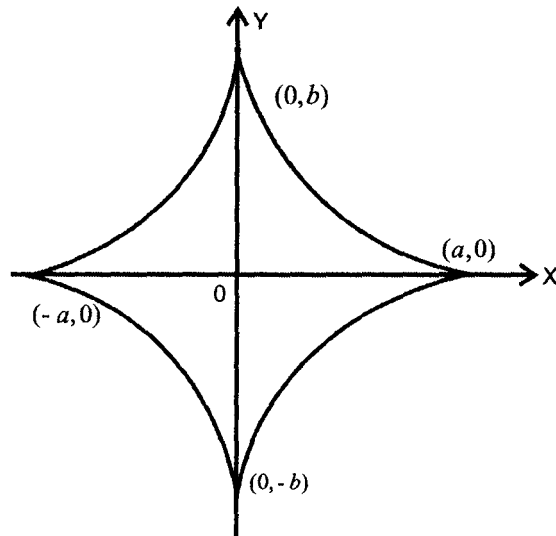


Fig. 4.16

$x = 0$ gives $y^2 = b^2 \Rightarrow y = \pm b$ $y = 0$ gives $x = \pm a$. Thus the curve meets x axis at $(-a, 0), (a, 0)$ and y axis at $(0, b), (0, -b)$.

For $|x| > a$ or $|y| > b$ $(x/a)^{2/3} + (y/b)^{2/3} - 1 > 0 \therefore$ Equation is not satisfied.

$\therefore$ Curve does not exist for $|x| > a$ or $|y| > b$.

$\therefore$ Curve completely lies in the region $|x| \leq a$ and $|y| \leq b$

From (1) $\frac{dy}{dx} = -\frac{b}{a} \left(\frac{ay}{bx} \right)^{1/3}$. In the first quadrant y is decreasing as $\frac{dy}{dx} < 0$ for $0 < x < a$.

4.5.11 Example

Trace the curve $y^2(a-x) - x^3 = 0, a > 0$ (1)

Solution:

Clearly $y = 0$, 0 when $x = 0$. Origin is a double point on the curve.

$F(x, -y) = F(x, y)$. Curve is symmetric about x -axis.

Differentiating (1) w.r.t, 'x' we get $\frac{dy}{dx} = \frac{\sqrt{x} (3a-2x)}{2(a-x)^{3/2}} \therefore \frac{dy}{dx} \Big|_{(0,0)} = 0$.

So x axis is a tangent to the curve at origin.

$x = a$ is an asymptote parallel to y -axis.

As coefficient of x^3 is a constant, there is no asymptote parallel to x -axis.

For $x < 0$ and $x > a$ curve does not exist (as y^2 is negative)

Substituting $y = mx + c$ in (1) and rearranging

$$x^3(m^2 + 1) + x^2(2mc - am^2) + x(c^2 - 2amc) - ac^2 = 0$$

$$\phi_3(m) = m^2 + 1 \Rightarrow m = \pm i.$$

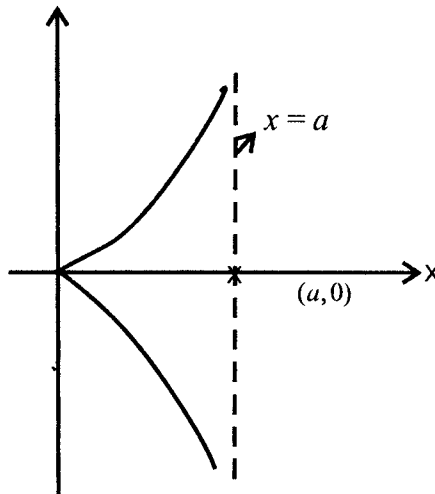


Fig. 4.17

Hence there no oblique asymptotes.

$$\text{From (1) } y_1 = \frac{3x^2 + y^2}{2(a-x)y}$$

For $0 < x < a$, y_1 is positive. Hence y is increasing in $0 < x < a$.

4.5.12 Example

Trace the curve $x^2(a-x) = y^2(b+x)$ (1)

Solution :

Clearly curve passes through origin

$F(x, -y) = F(x, y) \therefore$ Curve is symmetric about x-axis.

Equating lowest degree terms to zero

$$by^2 - ax^2 = 0 \Rightarrow y = \pm \sqrt{\frac{a}{b}}x$$

There are two real, distinct tangents to the curve at $(0, 0)$

$$y = 0 \text{ in (1) gives } x^2(a - x) = 0 \Rightarrow x = 0, a$$

The curve intersects x-axis at $(a, 0)$

There is no asymptote parallel to x-axis as coefficient of x^3 is a con

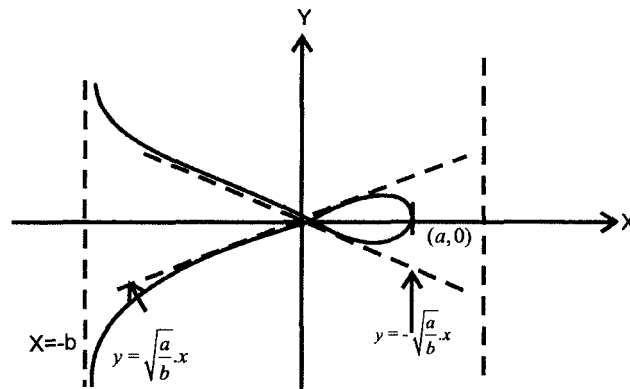


Fig. 4.18

Coefficient of highest degree term in y is $b + x$

$\therefore x + b = 0$ is the asymptotes parallel to y-axis

Substituting $y = mx + c$ in (1) and rearranging

$$x^3(1 + bm^2) + \dots = 0 \text{ i.e., } \phi_3(m) = 0 \text{ gives } m = \pm \frac{i}{\sqrt{b}}$$

Hence no oblique asymptote to the curve.

(1) can be written as $y = \pm x \sqrt{\frac{a-x}{b+x}}$ which shows that y is imaginary in the regions $x > a$ or $x < -b$

Hence the curve completely lies in the region $-b < x \leq a$

From (1) $\frac{dy}{dx} = \pm \frac{(-2x^2 - 3bx + ax + 2ab)}{2(b+x)^{3/2}(a-x)^{1/2}}$ which shows that $\frac{dy}{dx} \rightarrow \infty$ as $x \rightarrow a$

The tangent $x = a$ to the curve at $(a, 0)$ is parallel to y-axis.

Thus the curve forms a loop between $(0, 0)$ and $(a, 0)$.

4.5.13 Example:

Trace the curve $x^2(y^2 - a^2) - b^2y^2 = 0$ (1)

Solution:

$x = 0$ in (1) gives only $y = 0$. Curve passes through origin only and does not intersect either of the axes :

$F(-x, y) = F(x, y) = F(x, -y) \therefore$ Curve is symmetric about both the axes.

Lowest degree terms equated to zero gives $a^2x^2 + b^2y^2 = 0$

showing that there are no tangents to the curve at $(0, 0)$

Coefficient of highest degree terms in y (i.e.) of y^2 is $x^2 - b^2$

$\therefore x = \pm b$ are the asymptotes parallel to y-axis

Coefficient of highest degree terms in x (i.e.) of x^2 is $y^2 - a^2$

$\therefore y = \pm a$ are the asymptotes parallel to x-axis.

Substitution of $y = mx + c$ in (1) gives

$$x^4.m^2 + x^3(2mc) + x^2(c^2 - a^2 - b^2m^2) - 2b^2mcx - b^2c^2 = 0$$

$$\phi_4(m) = m^2 \therefore \phi_4(m) = 0 \text{ gives } m = 0, 0$$

$$\phi_3(m) = 2mc \text{ does not provide value of } c \text{ since } m = 0$$

$$\phi_2(m) = c^2 - a^2 - b^2m^2$$

$$\phi_2(m) = 0 \text{ gives } c^2 = a^2 \Rightarrow c = \pm a$$

The asymptotes are $y = \pm a$ which are parallel to x-axis .

Hence there are no oblique asymptotes.

From (1) $x = \pm \frac{by}{\sqrt{y^2 - a^2}}$ For $|y| < a$, x is imaginary

Similarly for $|x| < b$, y is imaginary. Hence curve does not exist for $|x| < b$, $|y| < a$.

Since the curve is symmetric about both the axes, it is enough to study the curve in the first quadrant.

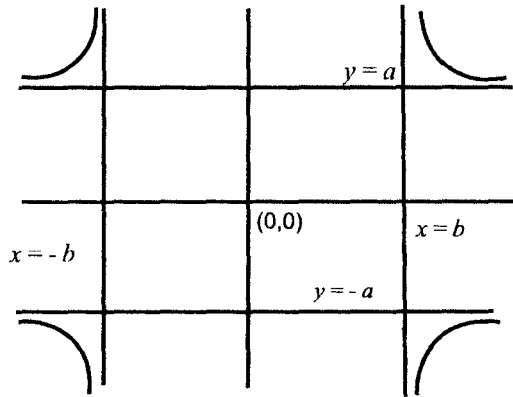


Fig. 4.19

$$\text{From (1) } y = \pm \frac{ax}{\sqrt{x^2 - b^2}} \quad \dots\dots(2)$$

From (2) it is observed that $y \rightarrow \infty$ as $x \rightarrow ba$ and $x \rightarrow \infty$ as $y \rightarrow a$

$$\text{For the first quadrant } y = + \frac{ax}{\sqrt{x^2 - b^2}}$$

$$\therefore y_1 = - \frac{a^2 b^2}{(x^2 - b^2)^{\frac{3}{2}}}. \text{ Thus } y_1 < 0 \text{ for } x > b. \text{ So } y \text{ is decreasing when } x > b \text{ in}$$

first quadrant.

4.5.14 Example

$$\text{Trace the curve } y^2(x-a) - x^2(x+a) = 0 \quad \dots\dots(1)$$

Solution:

$x = 0$ gives $y = 0$ $\therefore (0,0)$ lies on the curve .

Lowest degree terms equated to zero gives $a(x^2 + y^2) = 0 \Rightarrow x^2 + y^2 = 0$

$\therefore$ There are no tangents to the curve at $(0,0)$

coefficient of highest degree terms in y (i.e.) of y^2 is $x - a$

$\therefore x = a$ is the asymptote parallel to y - axis

No asymptotes parallel to x - axis since coefficient of x^3 is constant)

Substituting $y = mx + c$ in (1)

$$x^3(m^2 - 1) + x^2(2mc - am^2 - a) + x(c^2 - 2amc) - ac^2 = 0$$

$$\phi_3(m) = m^2 - 1 = 0 \text{ gives } m = \pm 1$$

$$\phi_2(m) = 0 \text{ gives } c = \frac{a(m^2 + 1)}{2m} = a \text{ when } m = 1 \text{ and } = -a \text{ when } m = -1$$

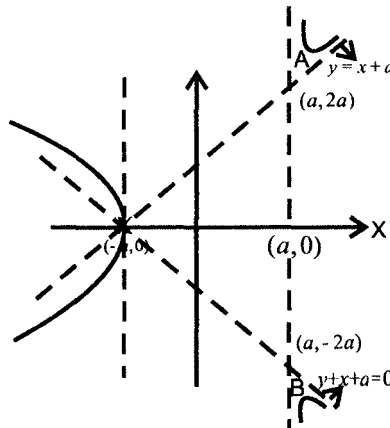


Fig. 4.20

$\therefore$ The oblique asymptotes are $y = x + a$ and $y = x - a = 0$

$$\text{From (1)} \quad \frac{dy}{dx} = \frac{x^2 - ax - a^2}{(x - a)\sqrt{x^2 - a^2}}$$

When $x \rightarrow -a$, $\frac{dy}{dx} \rightarrow \infty$ $\therefore$ Tangent at $(-a, 0)$ is parallel to y - axis

$$\frac{dy}{dx} = 0 \text{ gives } x^2 - ax - a^2 = 0 \Rightarrow x = \frac{1 \pm \sqrt{5}}{2} a$$

Tangents to the curve at $x = \frac{1+\sqrt{5}}{2}a$ (A and B in fig) are parallel to x -axis.

For $0 < x < a$ y is imaginary

$a < x < \infty$ y exists

$-a < x < 0$ y is imaginary

$-\infty < x < -a$ y exists

(i.e.) The curve does not exist in the region $-a < x < a$.

For $x = \frac{1-\sqrt{5}}{2}a$ there is no tangent to the curve (since the curve itself is not defined there).

4.5.15 Example

Trace the curve $y^2(a^2 - x^2) - x^2(a^2 + x^2) = 0$ (1)

Solution :

$x = 0$ in (1) gives $y = 0, 0$ $\therefore$ origin is a double pt on the curve.

The curve is symmetric about both the axes since $F(-x, y) = F(x, y) = F(x, -y)$.

Lowest degree terms equated to zero gives $y = \pm x$ as the tangents at the origin.

(i) can be written as $y^2 = \frac{x^2(a^2 + x^2)}{a^2 - x^2}$. This indicates that $y \rightarrow \pm\infty$ as $x \rightarrow a$.

Hence the curve lies only in $|x| < a$.

$x = \pm a$ are asymptotes parallel to y -axis.

Substitution of $y = mx + c$ in (1) gives $x^4(m^2 + 1) + \dots = 0$

$$\phi_4(m) = m^2 + 1 = 0 \text{ gives } m = \pm i$$

$\therefore$ There are no oblique asymptotes

$$\text{From (1) } \frac{dy}{dx} = \frac{a^4 + 2a^2x^2 - x^4}{(a^2 - x^2)^{3/2}(a^2 + x^2)^{1/2}}$$

In the first quadrant : $\frac{dy}{dx} > 0$ for $0 < x < a$

(i.e.) y is increasing in this region.

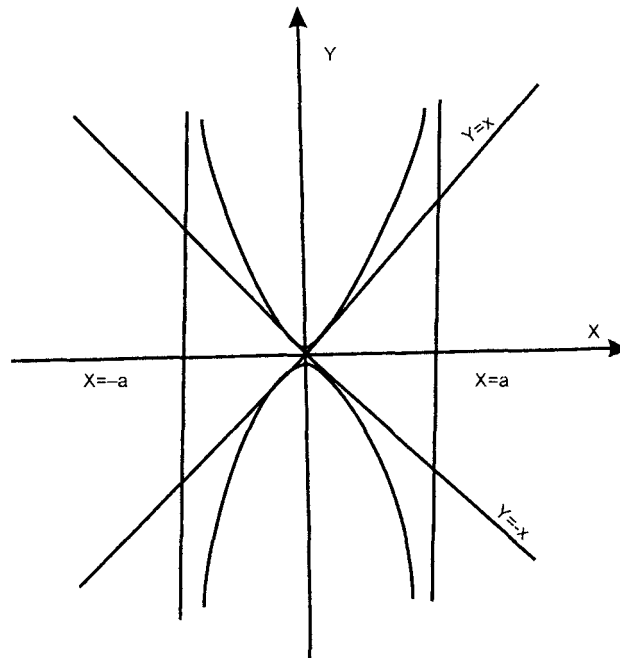


Fig. 4.21

4.1.16 Example

Trace the curve $y^2(a^2 + x^2) - x^2(a^2 - x^2) = 0$ (1)

Solution :

$x = 0$ gives $y = 0, 0$ $\therefore$ origin is a double point on the curve.

$F(-x, y) = F(x, y) = F(x, -y)$ shows that the curve is symmetric about both the axes.

there are no asymptotes parallel to x-axis , since

coefficient of highest degree terms in x ((ie) of x^4) is constant.

The coefficient of highest degree terms in y (ie) of y^2 is $x^2 + a^2$.

$\therefore$ there are no asymptotes parallel to y axis.

From (1) $y = \pm x \sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$ For $|x| > a$, y is imaginary

$\therefore$ The curve entirely lies in the region $-a \leq x \leq a$.

$y = mx + c$ in (1) gives $\phi_a(m) = m^2 + 1 \therefore m = \pm i$

There are no oblique asymptotes $\frac{dy}{dx} = 0$ gives $x = \pm a\sqrt{-1 \pm \sqrt{2}}$

$\therefore$ y values are extreme at $x = \pm a\sqrt{\sqrt{2}-1}$, where the tangents to the curve are parallel to x-axes.

$$\frac{dy}{dx} = \pm \frac{a^4 - 2a^2x^2 - x^4}{(a^2 + x^2)^{3/2} (a^2 - x^2)^{1/2}} \quad \dots(2)$$

$$\frac{dy}{dx} \rightarrow \infty \text{ as } |x| \rightarrow a$$

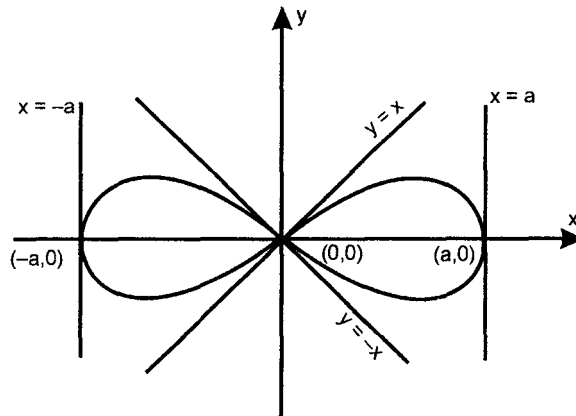


Fig. 4.22

$\therefore x = \pm a$ are tangents to the curve at $(\pm a, 0)$. Further $\left(\frac{dy}{dx}\right)_{(0,0)} = \frac{a^4}{a^4} = 1$ from (2)

Hence $y = \pm x$ are tangents to the curve at $(0, 0)$

The curve thus forms a loop between $(0, 0)$ and $(a, 0)$ and another loop between $(-a, 0)$ and $(0, 0)$.

4.1.17 Example

Trace the curve $ay^2 - 4x^2(a-x) = 0$ (1)

Solution:

$F(x, -y) = F(x, y) \therefore$ curve is symmetric about x-axis

$F(0, 0) = 0$ curve passes through origin .

$y = 0$ in (1) gives $x = 0, a$ (i.e.) the curve cuts the x-axis at $(0,0)$ and $(a,0)$

Equating the terms of lowest degree to zero we get $ay^2 - 4ax^2 = 0 \Rightarrow y = \pm 2x$

Thus $y \pm 2x = 0$ are the tangents to the curve at $(0, 0)$

$$\frac{dy}{dx} = \frac{4x(3a - x^2)}{3ay} \Rightarrow \left(\frac{dy}{dx} \right)_{(a,0)} \rightarrow \infty$$

(ie) the line $x = a$ is a tangent to the curve at $(a, 0)$

substituting $y = mx + c$ in (1) and rearranging the terms

$$4x^3 + ax^2(m^2 - 4) + 2amcx - ac^2 = 0$$

$\phi_3(m) = 4$ constant. Hence there are no asymptotes to the curve.

For $|x| > a$ we find y is imaginary.

Curve does not exist for $|x| > a$.

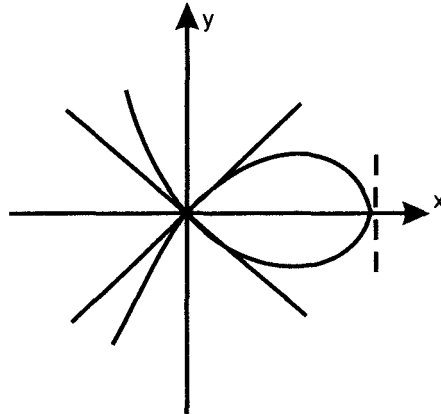


Fig. 4.23

4.1.18 Example:

Trace the curve $a^2 y^2 = x^2 (2a - x)(x - a)$

Solution:

$F(+x, -y) = F$ curve is symmetric about x-axis.

$F(0,0) = 0$ origin lies on the curve (isolated point)

$y = 0$ in (1) gives $x = 0, a, 2a$.

The curve intersects the x-axis at $(a, 0)$ and $(2a, 0)$

Terms of lowest degree are $y^2 + 2x^2$ which shows that there are no tangents to the curve at $(0, 0)$.

$(y^2 + 2x^2 = 0)$ gives $y = \pm\sqrt{2}ix$ (imaginary)

Proceeding as in 3.1.17 it can be seen that the curve has no asymptotes

y is imaginary for $x < a$ and $x > 2a$ (i.e.) the curve exists only in the region $a \leq x \leq 2a$.

Further $\frac{dy}{dx} \rightarrow \infty$ as $x \rightarrow a$ or $x \rightarrow 2a$

$\therefore x = a$ and $x = 2a$ are tangents to the curve parallel to y-axis.

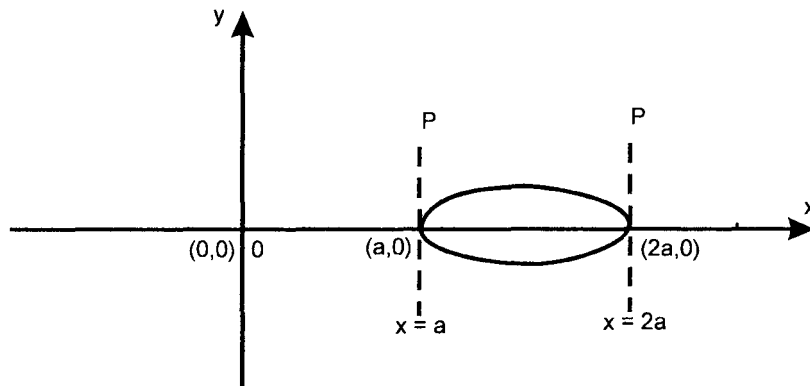


Fig. 4.24

4.1.19 Example

Trace the curve $x(x^2 + y^2) = a(x^2 - y^2)$, $(a > 0)$ (1) (2001 s)

$F(x, -y) = F(x, y)$ curve is symmetric about x-axis

Solution:

$y = 0$ in (1) gives $x = 0, 0, a$. The curve meets the x-axis at $(0, 0)$ and $(a, 0)$.

Equating lowest degree terms to zero we get

$y^2 - x^2 = 0 \Rightarrow y = \pm x$ which are the tangents at $(0, 0)$

$\frac{dy}{dx} \rightarrow \infty$ as $x \rightarrow a$. Thus $x = a$ is a tangent to the curve at $(a, 0)$

For $x > a$ (1) shows that y is imaginary

No part of the curve exists for $x > a$.

$y \rightarrow \infty$ as $x \rightarrow -a \therefore x = -a$ is an asymptote to the curve.

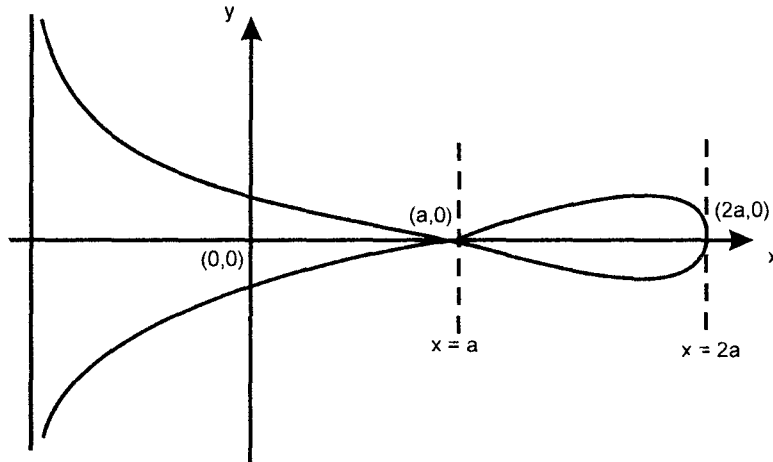


Fig. 4.25

4.1.20 Example

Trace the curve $y^2(a^2 + x^2) = x^2(a^2 - x^2)$ (1)

Solution:

$F(0,0) = 0$ curve passes through origin

$F(-x, y) = F(x, y) = F(x, -y)$ curve is symmetric about both the axes.

Lowest terms equated to zero gives $a^2y^2 - x^2a^2 = 0 \Rightarrow y \pm x$ which are tangents to the curve at $(0,0)$

$\left(\frac{dy}{dx}\right) \rightarrow \infty$ as $x \rightarrow \pm a \therefore x = \pm a$ are the tangents to the curve at $(\pm a, 0)$

y is imaginary when $x < -a$ or $x > a$

$\therefore$ curve completely lies within the region $-a \leq x \leq a$

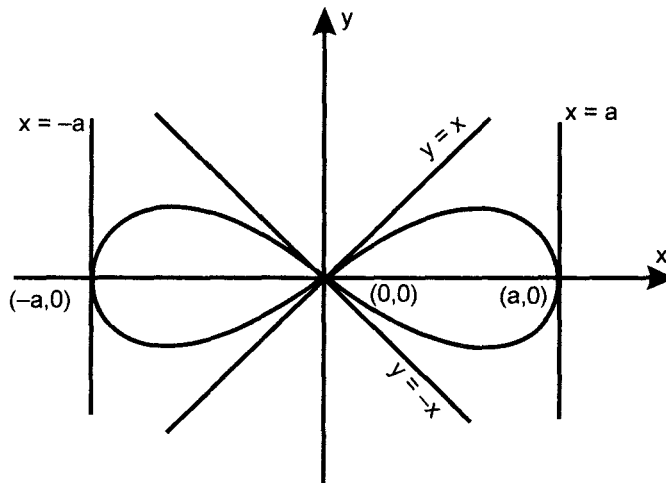


Fig. 4.26

4.1.21 Example

Trace the curve $y(a^2 + x^2) = a^2 x$

Solution:

Let $F(x, y) = y(a^2 + x^2) - a^2 x = 0$

$F(-x, y) \neq F(x, y) \neq F(x, -y)$, Hence the curve is not symmetric about both the axes.

The powers of y are odd. Therefore the curve is symmetric in opposite quadrants.

$F(0, 0) = 0$. So the curve passes through the origin.

The coefficient of highest power of x i.e; of x^2 is y

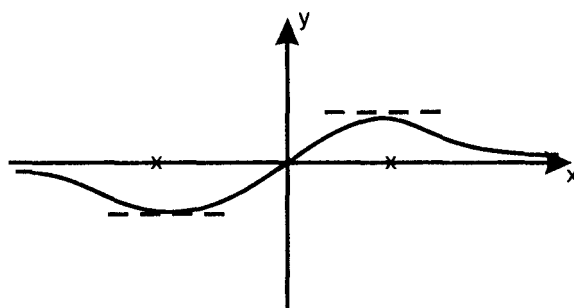
So $y=0$ is an asymptote parallel to x -axis.

The coefficient of highest degree of y is $x^2 + a^2 = 0 \Rightarrow x = \pm ai$

Hence the extreme points are $A(a, a/2)$ $B(-a, -a/2)$

Thus the tangents at A and B are parallel to x -axis.

The shape of the curve is as shown below.



Exercise – 4(E)

Trace the following curves

1. $y = (x^2 + x - 6)(x - 7)$

2. $y^2 x^2 = x^2 - a^2$

3. $y^2 = \frac{x-2}{x^2-1}$

4. $x^3 + y^3 = ax^2$; $x > 0$

5. $9ay^2 = x(x-3a)^2$

6. $(x^2 - a^2)(y^2 - b^2) = a^2 b^2$

7. $y^3 = a^2 x - x^3$

8. $y^2(a-x) = x^3$; $a > 0$

9. $y = \frac{8a^3}{x^2 + 4a^2}$

10. $y^2(a+x) = x^2(b-x)$

11. $y^2 = (x-a)(x-b)(x-c)$; $a, b, c > 0$

12. $y = x^3$

13. $y^2 = \frac{x^2(3a-x)}{x+a}$

14. $xy^2 = a^2(a-x)$

15. $y^2 = (x-2)^2(x-5)$

16. $y^2 = x^3$

17. $xy^2 = 4a^2(2a-x)$

4.2.0 Tracing of curves (polar form)

The equation of the curve in polar coordinates may be in one of the following three forms

$$F(r, \theta) = 0 \quad , \quad r = F(\theta) \quad , \quad \theta = F(r)$$

A study of the following points will simplify the tracing

1. curve is symmetric

(a) About initial line $\theta = 0$ if $F(r, -\theta) = F(r, \theta)$

- (b) About pole 'o' if $F(-r, \theta) = F(r, \theta)$ or
 $F(r, \pi + \theta) = F(r, \theta)$

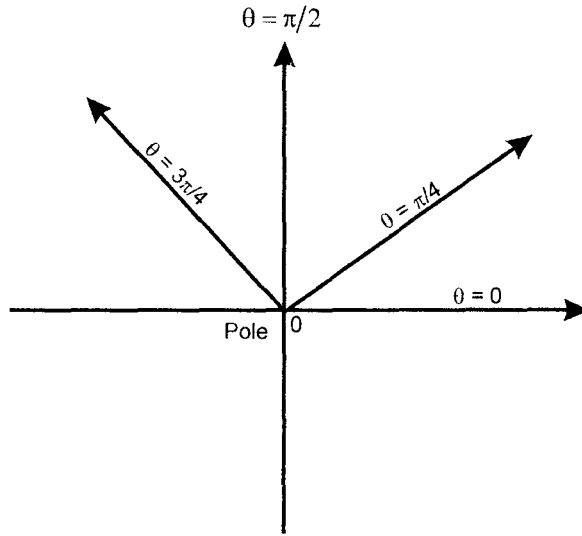


Fig. 4.27(a)

- (c) About $\theta = \frac{\pi}{2}$ if $F(r, \pi - \theta) = F(r, \theta)$
- (d) About $\theta = \frac{\pi}{4}$ if $F(r, \frac{\pi}{2} - \theta) = F(r, \theta)$
- (e) About $\theta = \frac{3\pi}{4}$ if $F(r, \frac{3\pi}{2} - \theta) = F(r, \theta)$
2. (a) Values of θ for which $r = 0$ gives the equation of the tangents to the curve at the pole
- (b) $F(r, 0) = 0$ gives the points of intersection of the curve with the initial line
- (c) $F(r, \frac{\pi}{2}) = 0$ gives the points of intersection of the curve with the line
 $\theta = \frac{\pi}{2}$

3. (a) If r_1 and r_2 are the least and greatest values of r then the curve lies in the annulus between circles of radii r_1 and r_2
- (b) Curve does not exist if r is imaginary for values of θ or if θ is imaginary for values of r
4. In the region in which $\frac{dr}{d\theta} > 0$ ' r ' increases $\frac{dr}{d\theta} < 0$ ' r ' decreases in that region

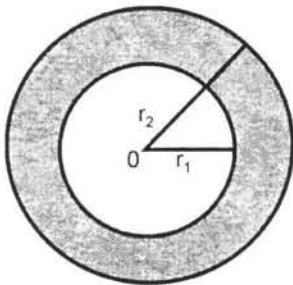


Fig. 4.27(b)

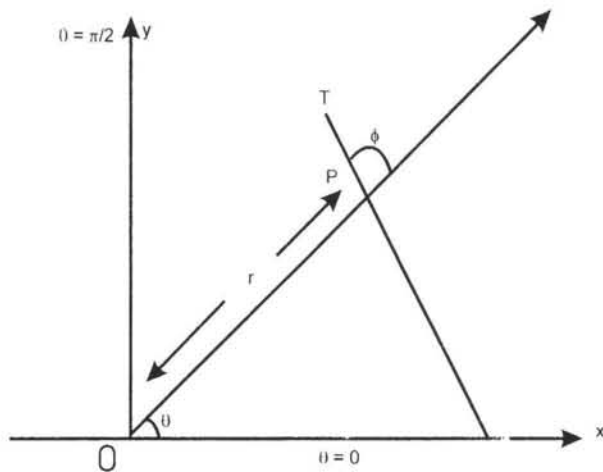
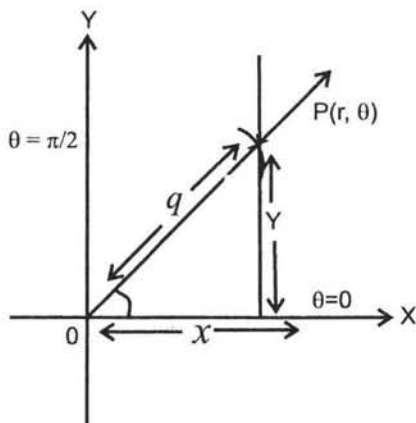


Fig. 4.27(c)

5. Let $p(r, \theta)$ be a point on the curve and P T is the tangent to the curve at P, OR radius vector and ϕ is the angle T P R. Then
- $\tan \phi = r \frac{d\theta}{dr}$ gives equation of PT
 - $\phi = 0$ gives tangent parallel to initial line
 - $\phi = \frac{\pi}{2}$ gives tangent $\perp^{lar}$ to initial line
6. If the curve meets the initial line at two distinct points and curve is symmetric about the initial line then part of the curve forms a loop between these two points.
7. If for any value of θ (say θ_1) such that $\lim_{\theta \rightarrow \infty} r = \infty$ then $\theta = \theta_1$ is an asymptote to the curve

8. Sometimes it may be convenient to change the equation of the curve from polar to Cartesian form by substituting respectively

$$x = r \cos \theta, \quad y = r \sin \theta$$



Few solved examples are given below

4.2.1 Example

Trace the cardioid

$$r - a(1 + \cos \theta) = 0 \text{ --- (1)}$$

Solution:

$F(r, \theta) = r - a(1 + \cos \theta) = F(r, \theta)$ curve is symmetric about the initial line.

When $\theta = \pi$ we get $r = 0$ $\therefore$ curve passes through pole and

$\theta = \pi$ is the tangent at the pole

$$\text{Max } |\cos \theta| = 1 \quad \therefore \text{Max } r = 2a$$

$\therefore$ The curve is entirely within the circle of radius 'a'.

Hence no asymptote to the curve.

θ	0	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	π	$4\pi/3$	$3\pi/2$	$5\pi/2$	2π
r	$2a$	$a(1 + 1/\sqrt{2})$	$3a/2$	a	$a/2$	θ	$a/2$	a	$3a/2$	$2a$

$$r = 2a \text{ when } \theta = 0 \text{ and } r = 0 \text{ when } \theta = \pi$$

$\therefore$ The curve intersects the initial line at $(2a, 0)$ and $(0, \pi)$

The tangents to the curve are given by $\tan \phi = r \frac{d\theta}{dr}$

$$\tan \phi = \frac{a(1 + \cos \theta)}{a(\theta - \sin \theta)} = -\cot \frac{\theta}{2} = \tan \left(\frac{\pi}{2} + \frac{\theta}{2} \right) \dots (2)$$

when $\theta = 0$, (2) gives $\phi = \pi/2$ (ie) The tangents at $(2a, 0)$ is perpendicular to the initial line..

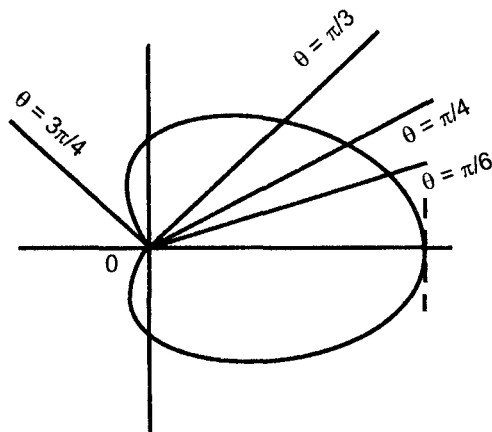


Fig. 4.28

4.2.2 Example

Trace the curve $r = a(1 - \cos \theta)$

Solution:

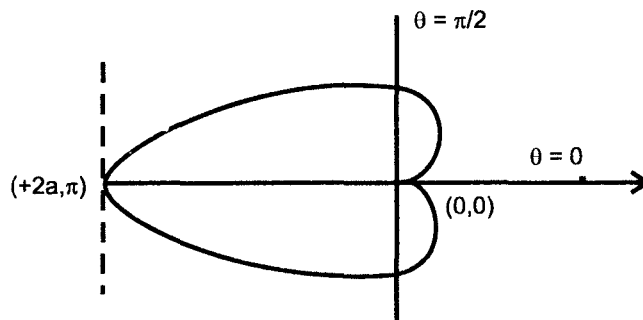


Fig. 4.29

θ	$0, 2\pi$	$\frac{\pi}{4}, \frac{7\pi}{4}$	$\frac{\pi}{3}, \frac{5\pi}{3}$	$\frac{\pi}{2}, \frac{3\pi}{2}$	$\frac{2\pi}{3}, \frac{4\pi}{3}$	π
r	0	$a\left(1 - \frac{1}{\sqrt{2}}\right)$	$\frac{a}{2}$	a	$\frac{3a}{2}$	$2a$

$r = 0$ when $\theta = 0, 2\pi$ and $r = 2a$ when $\theta = \pi$

$\therefore$ The curve intersects the initial line at $(0, 0), (2a, \pi)$

$$\tan \phi = \frac{a(1 - \cos \theta)}{a \sin \theta} = \tan \frac{\theta}{2}$$

$\therefore \tan \phi = 0$ when $\theta = \pi$ (ie) the tangent at $(2a, \pi)$ is perpendicular to the initial line.

(work out remaining points as in 3.2.1)

4.2.3 Example:

Trace the curve $r - a(1 + \sin \theta) = 0$ (1)

Solution: $r = a$ when $\theta = 0, \pi$

$r = 0$ when $\theta = -\pi/2$ (2)

and $r = 2a$ when $\theta = \frac{\pi}{2}$

$\therefore$ The curve intersects the initial line at $(a, 0), (a, \pi)$

and intersects the line $\theta = \pi/2$ at $(2a, \pi/2)$ and $(0, -\pi/2)$

$$F(r, \pi - \theta) = r - a(1 + \sin(\pi - \theta)) = r - a(1 + \sin \theta) = F(r, \theta)$$

$\therefore$ curve is symmetric about $\theta = \pi/2$

The equation of the tangent at the pole is $\theta = -\pi/2$ (from (2))

$\text{Max} |\sin \theta| = 1 \therefore \text{Max } r = 2a$ Hence curve completely lies within the region of the circle $r = 2a$ and there are no asymptotes to the curve.

θ	0	$\pi/6$	$\pi/2$	$5\pi/6$	π	$2\pi/6$	$3\pi/2$	$11\pi/6$	2π
r	a	$3a/2$	$2a$	$3a/2$	a	$a/2$	0	$a/2$	a

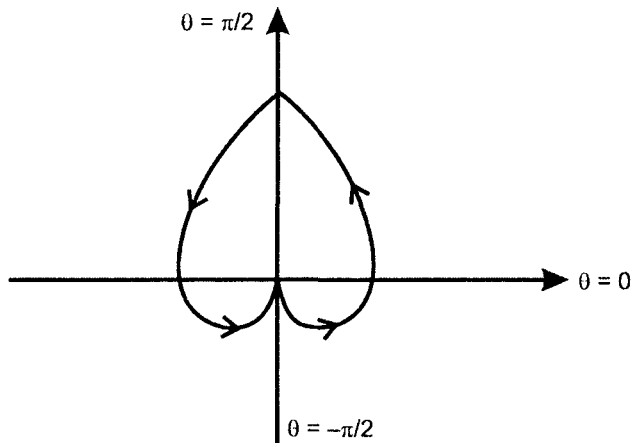


Fig. 4.30

4.2.4 Example:Trace the curve $r - a \sin \theta = 0$ **Solution:**

θ	0	$\pi/12$	$\pi/4$	$\pi/3$	$5\pi/12$	$\pi/2$
r	0	$\frac{a}{2}$	a	$\frac{\sqrt{3}}{2}a$	$\frac{a}{2}$	0

When $\theta = 0$ we have $r = 0$ $\therefore$ pole lies on the curve

$$F(r, \pi + \theta) = r - a \sin(2\pi + 2\theta) = r - a \sin 2\theta = F(r, \theta)$$

 $\therefore$ curve is symmetric about pole.

.....(1)

$$F\left(r, \frac{\pi}{2} - \theta\right) = r - a \sin(\pi - 2\theta) = F(r, \theta)$$

and
$$F\left(r, \frac{3\pi}{2} - \theta\right) = r - a \sin(3\pi - 2\theta) = F(r, \theta)$$

$\therefore$ curve is symmetric about the lines $\theta = \pi/4$ and $\theta = 3\pi/4$ (2)

from (1) and (2) it follows that the curve is symmetric in all the four quadrants.

$r = 0$ gives $\theta = 0, \pi/2, \pi, 3\pi/2$ these four lines are tangents, to the curve, at the pole.

Max $r = \text{Max } |a \sin 2\theta| = a$ $\therefore$ The curve completely lies within the region of the circle $r = a$.

The line $\theta = \pi/4$ meets the curve at $(a, \pi/4)$.

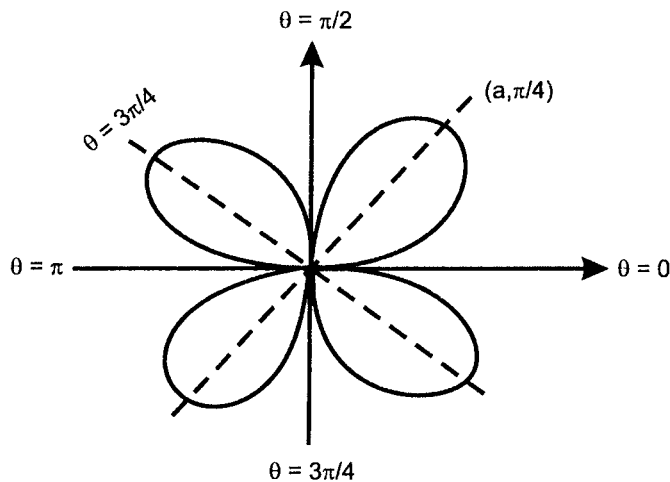


Fig. 4.31

Note : Similarly trace the curve $r = a \cos 2\theta$.

4.2.5 Example:

Trace the curve $r^2 - a^2 \cos 2\theta = 0$ (1)

Solution:

θ	0	$\pi/6$	$\pi/4$	$\pi/3$ to $2\pi/3$	$3\pi/4$	$5\pi/6$	π
r	$\pm a$	$\pm a/\sqrt{2}$	0	imaginary	0	$\pm a/\sqrt{2}$	$\pm a$

$$r = 0 \text{ gives } \cos 2\theta = 0 \Rightarrow \theta = \pm \frac{\pi}{4}$$

curve passes through the pole and $\theta = \pm \frac{\pi}{4}$ are the two tangents at the pole

$$F(r, -\theta) = F(r, \theta) \therefore \text{curve is symmetric about the initial line.}$$

$$F(r, \pi - \theta) = r^2 - a^2 \cos(2\pi - 2\theta) = r^2 - a^2 \cos 2\theta = F(r, \theta)$$

curve is symmetric about $\theta = \pi/2$

$$F(r, \pi + \theta) = r^2 - a^2 \cos(2\pi + 2\theta) = F(r, \theta)$$

curve is symmetric about the pole

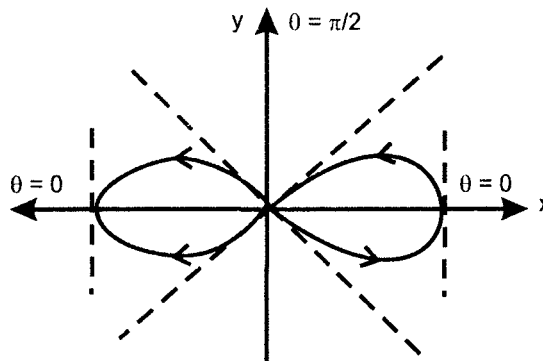


Fig. 4.32

Since $|\cos 2\theta| \leq 1$, Max $r = a$

Curve lies completely within the circle $r = a$

$$\theta = 0 \text{ gives } r^2 = a^2 \Rightarrow r = \pm a$$

curve intersects the initial line at $(a, 0)$ and $(-a, 0)$

$$r^2 \text{ is -ve When } \frac{\pi}{4} < \theta < \frac{3\pi}{4} (\because \cos 2\theta < 0)$$

curve does not exist (above the initial line) in this region.

Differentiating (1) w.r.t ' θ '

$$2r \frac{dr}{d\theta} + 2a^2 \sin 2\theta - 0 \Rightarrow \frac{dr}{d\theta} = \frac{-a^2 \sin 2\theta}{r}$$

$$\tan \phi = r \frac{d\theta}{dr} = r \cdot \frac{r}{-a^2 \sin 2\theta} = -\cot 2\theta = \tan\left(\frac{\pi}{2} + 2\theta\right)$$

$$\therefore \phi = \frac{\pi}{2} + 2\theta \Rightarrow \phi = \frac{\pi}{2} \text{ When } \theta = 0$$

$$\left. \begin{array}{l} \text{Thus when } \theta = 0 \\ \theta = \pi \end{array} \right\} \tan \phi = \infty$$

Hence the tangents at $\theta = 0$ and $\theta = \pi$ are $\perp^{\text{lar}}$ to initial line

$$\text{When } \theta = \frac{\pi}{4}, \tan \phi = \tan\left(\frac{\pi}{2} + \frac{2\pi}{4}\right) = \tan \pi$$

(i.e.,) the radius vector $\theta = \frac{\pi}{4}$ is tangential to the curve at the pole

$$(\because \theta = \frac{\pi}{4} \text{ gives } r = 0)$$

Similarly the radius vector $\theta = \frac{3\pi}{4}$ also is tangential to the curve at the pole.

4.2.6 Example:

Trace the curve $r - a \cos 3\theta = 0$ -----(1)

Solution:

$$F(r, -\theta) = r - a \cos 3\theta = F(r, \theta)$$

Curve is symmetric about initial line

Line $\theta = 0$ intersects the curve at $(a, 0)$

Since $|\cos 3\theta| \leq 1$, from (1), $|r| \leq a$. Curve completely lies

within the region of the circle $r = a$

$$r = 0 \text{ when } 3\theta = 0 \therefore 3\theta = (2n+1)\pi/2, \quad n = 0, 1, 2, \dots$$

(i.e.) curve passes through the pole when $\theta = (2n+1)\pi/6$ ($n = 0$ to 5)

These six lines are tangential to the curve at the pole

The part of the curve from $\theta = 0$ to $\frac{\pi}{6}$ is AA_1O .

This has a reflection AA_2O in the initial line. Thus $AA_1O A_2A$ is a loop.

$$F(r, 4\pi/3 - \theta) = r - a \cos(4\pi - 3\theta) = F(r, \theta)$$

The curve is symmetric about $\theta = 2\pi/3$ OB_1B is the part of the curve

from $\theta = 2\pi/2$ to $2\pi/3$. This has a reflection OB_2B in line $\theta = 2\pi/3$

$$\text{Thus } OB_1BB_2O \text{ forms a loop. } F\left(r, \frac{8\pi}{3} - \theta\right) = r - a \cos(8\pi - 3\theta) = F(r, \theta)$$

Curve is symmetric about $\theta = 4\pi/3$

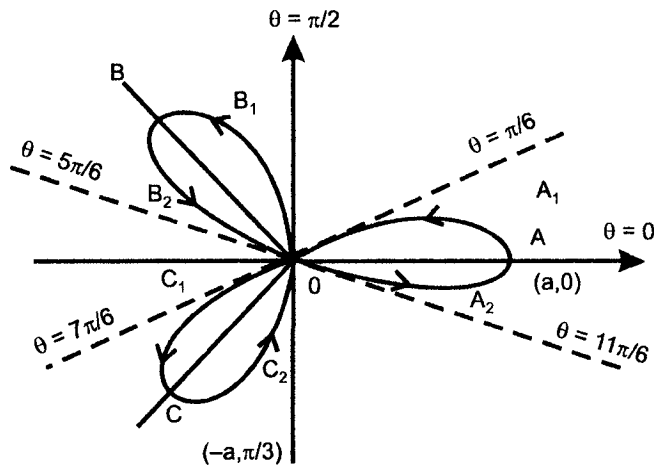


Fig. 4.33

OC_1C is the part of the curve from $\theta = \pi/6$ to $\pi/3$ and

OC_2C is its reflection in $\theta = 4\pi/3$

$\therefore OC_1CC_2O$ forms a loop.

It can be seen that OC_1CC_2O is a reflection of OB_1BB_2O in the line $\theta = \pi$

$$\tan \phi = \frac{a \cos 3\theta}{-3a \sin 3\theta} = -\frac{1}{3} \cot 3\theta$$

$$\phi \rightarrow \infty \text{ when } \theta \rightarrow 0$$

$\therefore$ The tangent at $(a, 0)$ to the curve is $\perp^{lar}$ to the initial line.

4.2.7 Example:

Trace the curve $r^2 - a^2 \sec^2 \theta = 0$

Solution:

The given curve can be written as $F(r, \theta) = r^2 \cos^2 \theta - a^2 = 0$ -----(1)

$\theta = 0$ gives $r = \pm a$. Curve does not pass through the pole.

$F(-r, \theta) = F(r, \theta) \therefore$ Curve is symmetric about the pole

$$F(r, -\theta) = r^2 \cos^2(-\theta) - a^2 = F(r, \theta)$$

Curve is symmetric about the initial line $\theta = 0$

$$F(r, \pi - \theta) = r^2 \cos^2(\pi - \theta) - a^2 = F(r, \theta) \therefore \text{Symmetric about } \theta = \frac{\pi}{2}$$

Since $\cos^2 \theta \leq 1$ (i.e.,) $|\cos \theta| \leq 1$ we get $r^2 \geq a^2$

Curve does not exist in the region $r^2 < a^2$

$$r \rightarrow \infty \text{ as } \theta \rightarrow \frac{\pi}{2} \therefore \theta = \frac{\pi}{2} \text{ is an asymptote to the curve.}$$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	$\pm a$	$\pm \frac{2a}{\sqrt{3}}$	$\pm \sqrt{2}a$	$\pm 2a$	∞

$$r = \frac{dr}{d\theta} = a^2 \sec^2 \theta \tan \theta \quad \frac{d\theta}{dr} = \frac{r}{a^2} \cos^2 \theta \cot \theta$$

$$\tan \phi = r \frac{d\theta}{dr} = \cot \theta = \tan \left(\frac{\pi}{2} - \theta \right) \therefore \phi = \pi/2 \text{ when } \theta = 0 \text{ showing that}$$

the tangent at $(a, 0)$ is $\perp^{lar}$ to the initial line .

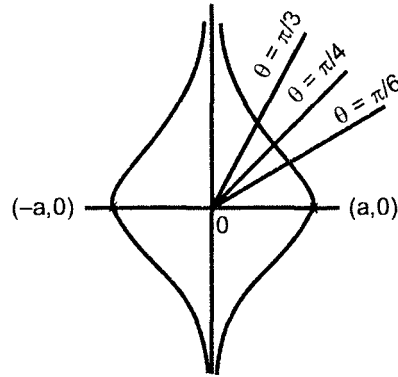


Fig. 4.34

4.2.8 Example

Trace the curve $r^2 - a^2 \cos 2\theta = 0$

Solution:

$\theta = 0$ gives $r = \pm a$ $\therefore$ curve intersects initial line at $(\pm a, 0)$

$r = 0$ gives $\cos 2\theta = 0 \Rightarrow \theta = \pm \pi/4$ pole lies on the curve

and $\theta = \pm \pi/4$ are two tangents to the curve at the pole

$|\cos 2\theta| \leq 1 \Rightarrow |r| \leq a$. Curve completely lies within the region of the circle $r = a$

$$F(r, -\theta) = r^2 - a^2 \cos(-2\theta) = r^2 - a^2 \cos 2\theta = F(r, \theta)$$

Curve is symmetric about the initial line.

$$F(-r, \theta) = (-r)^2 - a^2 \cos 2\theta = F(r, \theta) \text{ symmetric about the pole}$$

$$F(r, \pi - \theta) = r^2 - a^2 \cos(2\pi - 2\theta) = F(r, \theta) \text{ symmetric about } \theta = \pi/2$$

θ	0	$\pm \pi/6$	$\pm \pi/4$	$\pm \pi/3$	$\pm \pi/2$	$\pm 3\pi/4$
r	$\pm a$	$\pm a/\sqrt{2}$	0	imaginary		0

Curve does not exist in the regions $\pi/4 < \theta < 3\pi/4$ and $-3\pi/4 < \theta < -\pi/4$

The curve forms a loop between $(a, 0)$ and $(0, 0)$ and has reflection in $\theta = \pi/2$

$$\begin{aligned} \text{Differentiating (1)} \quad 2r \frac{dr}{d\theta} &= -2a^2 \sin 2\theta \quad \tan \phi = r \frac{d\theta}{dr} = \frac{-r^2}{a^2 \sin 2\theta} = -\cot 2\theta \\ &= \tan (\pi/2 + 2\theta) \end{aligned}$$

$$\theta = 0 \text{ gives } \phi = \pi/2$$

Thus the tangents at $\theta = 0$ (ie) $(a, 0)$ and $(-a, 0)$ are perpendicular to the initial line.

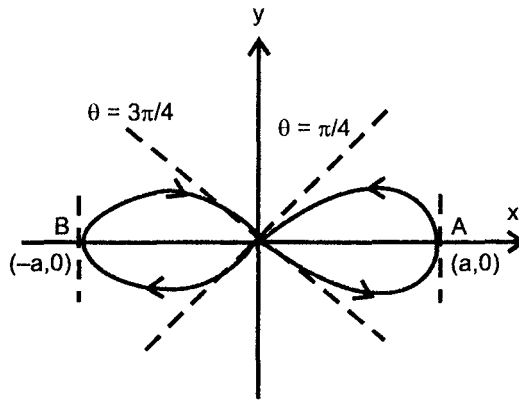


Fig. 4.35

4.2.9 Example:

Trace the curve $r - \{a + b \cos \theta\} = 0$

Solution:

$$F(r, -\theta) = r - \{a + b \cos(-\theta)\} = F(r, \theta)$$

Curve is symmetric about the initial line.

Since $|\cos \theta| \leq 1$ we have $|r| \leq a + b$

The curve completely lies within the region of the circle. $r = a + b$

$\theta = 0$ gives $r = a + b \quad \therefore$ curve meets $\theta = 0$ in $A(a + b, 0)$

$\theta = \pi$ gives $r = a - b$ curve meets $\theta = \pi$ in $B(a - b, 0)$

Thus when $a \neq b$ curve meets the initial line at A and B

When $a = b$ we get $r = 0$ curve passes through the pole .

Curve meets the line $\theta = \pi/2$ at $(a, \pi/2)$.

Curve passes through the pole when $\theta = \cos^{-1}(-a/b)$ (since $r = 0$)

Hence the tangent to the curve at the pole is the line $\theta = \cos^{-1}(-a/b)$ which

- (i) does not exist when $a > b$
- (ii) is $\theta = \cos^{-1}(-1) = \pi$ when $a = b$ and
- (iii) exists when $a < b$

θ	0	$\pi/3$	$\pi/2$	$2\pi/3$	π
r	$a+b$	$a+b/2$	a	$a-b/2$	$a-b$

$$\tan \phi = r \frac{d\theta}{dr} = \frac{a+b \cos \theta}{-b \sin \theta} \text{ showing that } \tan \phi = \infty \text{ when } \theta = 0$$

$\therefore \phi = \pi/2$ (ie) the tangent to the curve at $(a+b, 0)$ is $\perp^{\text{lar}}$ to the initial line.

Further $\phi \neq 0$ for any value of θ and hence no tangent to the curve is parallel to the initial line

When $a = b$

$$\tan \phi = -\frac{1+\cos \theta}{\sin \theta} - \cot \theta/2 = \tan(\pi/2 + \theta/2) .$$

When $\theta = \pi$ we get $\phi = \pi$ (ie) the initial line is a tangent to the curve at $(0, \pi)$

When $a \neq b$ we have $\tan \phi \rightarrow \infty$ as $\theta \rightarrow \pi$ (i.e.) The tangent to the curve at $(a-b, \pi)$ is also perpendicular to the initial line.

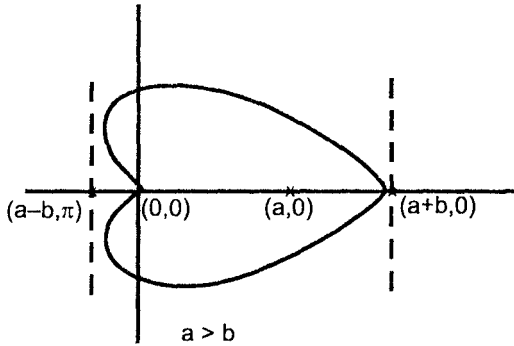


Fig. 4.36 (i)

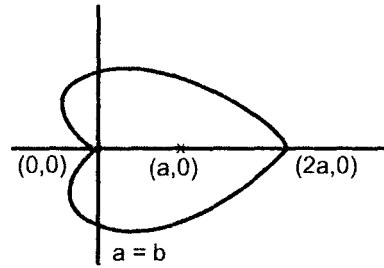


Fig. 4.36 (ii)

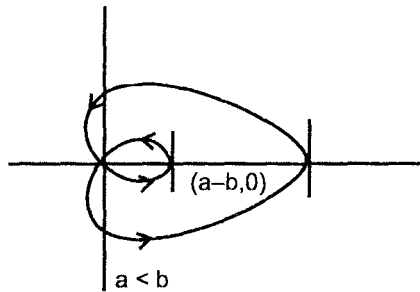


Fig. 4.36 (iii)

4.2.10 Example:

Trace the curve $r = a(\sin \theta + 1/\sin \theta)$ (1)

This example illustrates that changing the coordinate system from polar to cartesian facilitates the tracing.

(1) can be written as $r^2 = a \left(r \sin \theta + \frac{r^2}{r \sin \theta} \right)$ (2)

Substituting $r \cos \theta = x$ and $r \sin \theta = y$ (2) reduces to

$$y(x^2 + y^2) - a(2y^2 + x^2) = 0 \quad \text{....(3)}$$

No constant term in (3) $\therefore$ curve passes through origin

Lowest degree terms equated to zero gives $2y^2 + x^2 = 0 \Rightarrow x \pm y\sqrt{2}i$

The tangents at (0,0) are imaginary.

$x = 0$ in (3) gives $y = 0, 0, 2a$

y -axis cuts the curve at $(0, 2a)$ and $(0, 0)$ is an isolated point

Equating coefficient of the highest degree terms in y to zero gives $y = 0$

(i.e.) x -axis is an asymptotes to the curve.

$F(-x, y) = F(x, y) \therefore$ curve is symmetric about y -axis

Differentiating (3) w.r.t ' x ',

$$\frac{dy}{dx} = \frac{2x(a-y)}{3y^2 + x^2 - 4ay} \therefore \left(\frac{dy}{dx} \right)_{(0, 2a)} = 0$$

The tangent at $(0, 2a)$ is parallel to x -axis

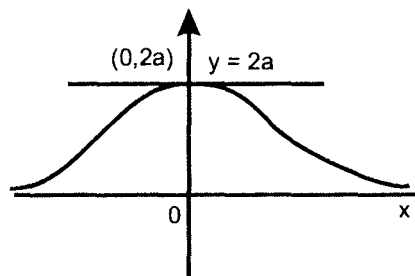


Fig. 4.37

From (3) $x = \pm y \sqrt{\frac{2a-y}{y-a}}$

when $y < a$ or when $y > 2a$, clearly x is imaginary (i.e.,) The curve exists only in the region $a < y \leq 2a$

Exercise 4.2

Trace the following curves

1. $r^2 \cos 2\theta = a^2$
2. $r = 1 + \sqrt{2} \cos \theta$
3. $r = a \sin^2 \theta \sec \theta$
4. $r = 3 + 2 \cos \theta$
5. $r = a \sin 3\theta$
6. $r = a(\cos \theta + \sec \theta)$
7. $r = a \sin 4\theta$
8. $r = a \left(\cos \theta + \frac{1}{\cos \theta} \right)$
9. $r^2 = a \sin 2\theta$
10. $r = a(1 - \sin \theta)$
11. $r^2 = \frac{a^2}{\cos 2\theta}$

4.3.0 Tracing of curves when the equation is given in parametric form:

Suppose the equation of the curve is in the form

$$x = f_1(t) \text{ and } y = f_2(t) \quad \dots(1)$$

where t is the parameter. The study of the following points are useful for tracing the curve

1. If for some value for t , say t_1 , $f_1(t) = 0$ and $f_2(t) = 0$
then the curve passes through the origin.
2. If $f_1(t)$ is an odd function and $f_2(t)$ is an even function
then the curve is symmetric about y -axis.
3. If $f_1(t)$ is even and $f_2(t)$ is odd
then the curve is symmetric about x -axis
4. Intercepts on $X(Y)$ axis are obtained by solving $f_1(t) = 0$ ($f_2(t) = 0$) respectively.
5. Greatest and least value of $f_1(t)$ and $f_2(t)$ give the region in which the curve exists.
6. If $f_1(t)$ or $f_2(t)$ tends to ∞ as $t \rightarrow t_1$ then $t = t_1$ is an asymptote to the curve.
7. $\left(\frac{dy}{dx}\right)_{t=t_1} = 0$ indicates that $t = t_1$ is a tangent to the curve parallel to the x -axis.
8. $\left(\frac{dy}{dx}\right) \rightarrow \infty$ as $t \rightarrow t_1$ then $t = t_1$ is a tangent parallel to the y -axis.
9. If $f_1(t + \alpha) = f_1(t)$ and $f_2(t + \alpha) = f_2(t)$ then the curve is periodic of period α
10. Sometimes it may be convenient to transform the equation of the curve from parametric to cartesian form for tracing the curve

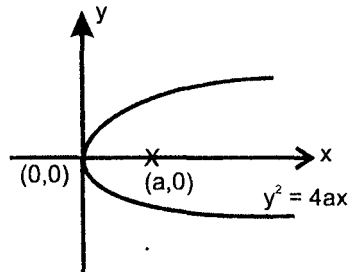
4.3.1 Example:

Trace the curve $x = at^2, y = 2at$

Solution:

Eliminating t from x and y

$$y^2 = 4a^2t^2 = 4a.at^2 = 4ax$$

**Fig. 4.38**

This is a parabola with vertex at origin y axis is a tangent to the curve at the origin.

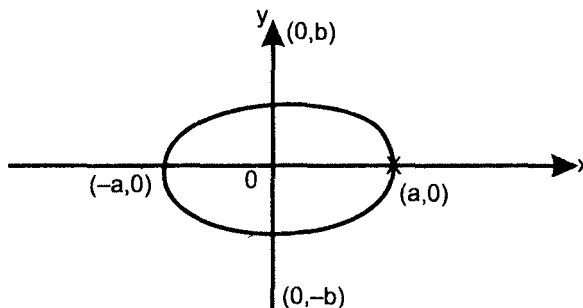
4.3.2 Example:

Trace the curve $x = a \cos t, y = b \sin t$

Solution:

Eliminating t from both x and y

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \text{ which is an ellipse (standard form)}$$

**Fig. 4.39****4.3.3 Example:**

Trace the curve $x = t^2, y = t - \frac{t^3}{3}$

Solution:

Eliminating 't'

$$y^2 = t^2 \left(1 - \frac{t^2}{3}\right)^2 = x \left(1 - \frac{x}{3}\right)^2 = \frac{x}{9} (3-x)^2$$

$$(i.e.,) \quad 9y^2 = x(3-x)^2 \text{ ----- (1)}$$

$$y = 0 \text{ gives } x = 0, x = 3.$$

The curve passes through origin and again intersects x -axis at $(3,0)$.

$$F(x, -y) = 9y^2 - x(3-x)^2 = F(x, y)$$

The curve is symmetric about x -axis and hence it forms a loop between $(0,0)$ and $(3,0)$ $y = \pm \frac{(3-x)\sqrt{x}}{3}$ which shows that the curve does not exist for $x < 0$

(i.e.,) to the left of Y -axis

Lowest degree terms equated to zero gives $x = 0$ (i.e.,) y -axis is a tangent to the curve at origin.

Coefficient of x^3 is 1 (a constant) and coefficient of y^2 is a constant.

Hence, there are no asymptotes parallel to either x or y -axis

Substituting, $y = mx + c$ in (1) and rearranging

$$-x^3 + x^2(9m^2 + 6) + \dots = 0$$

$$\phi_3(m) = -1 \text{ a constant, } \phi_2(m) = 0 \text{ gives } m = \pm \sqrt{\frac{2}{3}}i$$

$\therefore$ No oblique asymptotes exist

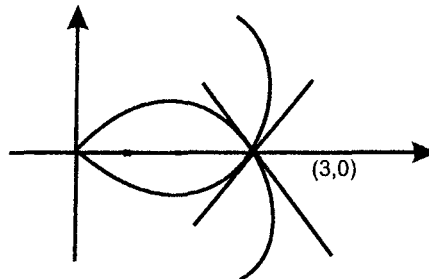


Fig. 4.40

$$\frac{dy}{dx} = \pm \frac{1}{2} \left[\frac{1-x}{\sqrt{x}} \right]$$

$$\left(\frac{dy}{dx} \right)_{x=1} = 0 \therefore \text{The curve is extreme at } x = 1$$

$$\left(\frac{dy}{dx} \right)_{(3,0)} = \pm \frac{1}{\sqrt{3}}$$

$$\therefore \sqrt{3}y = \pm(x-3) \text{ are tangents to the curve at } (3,0)$$

4.3.4 Example:

Trace the curve $x = a(t + \sin t)$, $y = a(1 + \cos t)$

Solution:

For no value of t both x and y simultaneously vanish

$\therefore$ The curve does not pass through the origin.

x is an odd function while y is even

$\therefore$ The curve is symmetric about y -axis

$x(t) = 0$ gives $t = 0$. Correspondingly $x = 0$ and $y = 2a$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{a \sin t}{a(1 + \cos t)} = -\tan \frac{t}{2} = \tan \left(\pi - \frac{t}{2} \right)$$

When $t = 0$ slope $\tan \phi = \tan \pi \Rightarrow \phi = \pi$

$\therefore$ The tangent at $(0, 2a)$ is parallel to x -axis

t	$-\pi$	$-\frac{5\pi}{6}$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π
x	$a\pi$	$-a\left(\frac{1}{2} + \frac{5\pi}{6}\right)$	$-a\left(\frac{\pi}{2} + 1\right)$	0	$a\left(1 + \frac{\pi}{2}\right)$	$a\left(\frac{1}{2} + \frac{5\pi}{6}\right)$	$a\pi$
y	0	$a\left(1 + \frac{\sqrt{3}}{2}\right)$	0	$2a$	a	$a\left(1 + \frac{\sqrt{3}}{2}\right)$	0

$$y(t) = 0 \text{ gives } \cos t = -1 \Rightarrow t = \pm\pi, \pm3\pi, \dots$$

$$\text{(i.e.,)} \quad t = \pm(2n+1)\pi \quad n = 1, 2, 3, \dots$$

corresponding x is $\pm a\pi, \pm 3a\pi, \dots$

Further $x \rightarrow \infty$ as $t \rightarrow \infty$ and $|y| \leq 2a$

$\therefore$ The curve completely lies within the region $0 \leq y \leq 2a$

and is period of period 2π

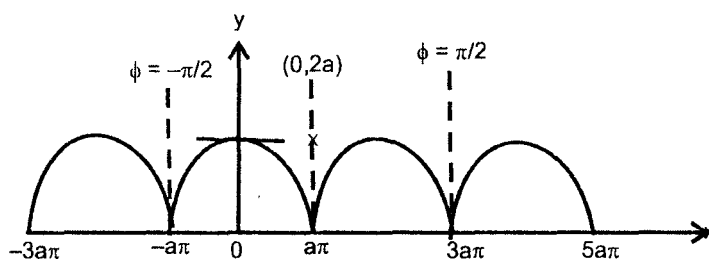


Fig. 4.41

4.3.5 Example:

Trace the curve $x = a(t - \sin t)$; $y = a(1 - \cos t)$

Solution:

$$x(t) = 0 \text{ only when } t = 0$$

$$y(t) = 0 \text{ gives } t = \pm 2n\pi \quad n = 0, 1, 2, \dots$$

$$x(0) = 0 = y(0) \therefore \text{curve passes through origin.}$$

Since $|\cos t| \leq 1$ we have $y \geq 0$

The curve is completely above x -axis and lies within the region $0 \leq y \leq 2a$

$$(y = 2a \text{ when } t = \pm(2n+1)\pi, n = 0, 1, 2, \dots)$$

The curve meets x -axis at $x = \pm 2an\pi, n = 0, 1, 2, \dots$

y is an even function and x is odd

$\therefore$ curve is symmetric about y -axis

$$\lim_{t \rightarrow \infty} x(t) = \infty \text{ but } \lim_{t \rightarrow \infty} y(t) \text{ is finite}$$

$\therefore$ There are no asymptotes to the curve

$$\tan \phi = \frac{dy}{dx} = \cot \frac{t}{2} = \tan \left(\frac{\pi}{2} - \frac{t}{2} \right)$$

$$\Rightarrow \phi = \frac{\pi}{2} - \frac{t}{2}$$

The tangents to the curve at $t = \pm 2n\pi, n = 0, 1, 2, \dots$ are parallel to y -axis

The curve is periodic of period 2π

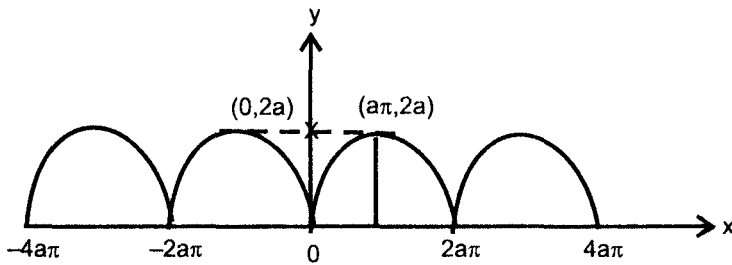


Fig. 4.42

4.3.6 Example:

Trace the curve $x = a \cos^3 \theta, y = b \sin^3 \theta$

Solution:

The functions $\cos^3 \theta$ and $\sin^3 \theta$ are periodic of period 2π .

Hence it is sufficient to trace the curve for one period.

For no value of θ both x and y vanish. $(0, 0)$ does not lie on the curve

$$x(\theta) = 0 \text{ gives } \theta = \pm \frac{\pi}{2}$$

$$y(\theta) = 0 \text{ gives } \theta = 0, \pm \pi$$

∴ The curve meets x -axis at $x = \pm a$ and y -axis at $y = \pm b$

Since $|\cos^3 \theta| \leq 1$ and $|\sin^3 \theta| \leq 1$ we have $|x| \leq a, |y| \leq b$

θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
x	a	$\frac{3\sqrt{3}}{8}a$	$a/8$	0	$-a/8$	$\frac{-2\sqrt{3}a}{8}$	$-a$
y	0	$\frac{1}{8}b$	$\frac{3\sqrt{3}}{8}b$	b	$\frac{3\sqrt{3}}{8}b$	$\frac{b}{8}$	0

$$\tan \phi = \frac{dy}{dx} = -\frac{b}{a} \tan \theta$$

$$\tan \phi = 0 \text{ when } \theta = \pm(2n+1)\pi$$

(i.e.,) The tangent to the curve at these points is parallel to x -axis $\tan \phi \rightarrow \infty$ when

$$\theta \rightarrow \pm\left(\frac{2n+1}{2}\right)\pi$$

∴ The tangent to the curve at these points is parallel to y -axis

It can be observed that for corresponding to x for some θ there is $-x$ for $\pi - \theta$ and corresponding to x for some θ there is $-y$ for $-\theta$.

∴ The curve is symmetric about both the axes

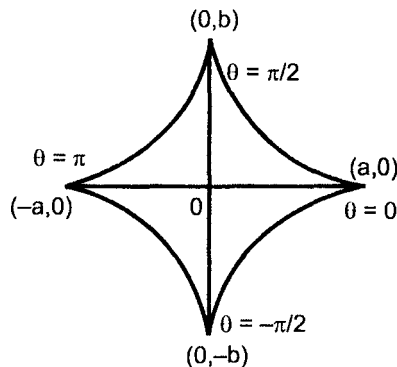


Fig. 4.43

4.3.7 Example:

Trace the curve $x = a \left[\cos \theta + \frac{1}{2} \log \tan^2 \frac{\theta}{2} \right], y = a \sin \theta$

Solution:

$y(0) = 0$ but $x(0) \neq 0$. $(0, 0)$ does not lie on the curve

$$y(\theta) = 0 \text{ gives } \theta = 0 \text{ and } |y|_{\max} = a$$

x is an even function and y is odd. So the curve is symmetric about the x -axis

Corresponding to each $x(\theta)$ there is $-x$ for $\pi - \theta$

$\therefore$ Curve is also symmetric about y -axis

Since $|\sin \theta| \leq 1$ we have $|y| \leq a$. $\therefore$ The curve entirely lies in the region $-a \leq y \leq a$.

$$x\left(\frac{\pi}{2}\right) = a \left[\cos \frac{\pi}{2} + \log \tan \frac{\pi}{4} \right] = a [0 + \log 1] = 0$$

$$\text{Similarly, } x\left[-\frac{\pi}{2}\right] = 0$$

Corresponding y values are $y\left(\frac{\pi}{2}\right) = a$ and $y\left(-\frac{\pi}{2}\right) = -a$

Thus the curve intersects the y -axis at $(0, a)$ and $(0, -a)$

$$\tan \phi = \frac{dy}{dx} = \frac{a \left[-\sin \theta + \frac{1}{\sin \theta} \right]}{a \cos \theta} = \frac{\cos^2 \theta}{\sin \theta \cos \theta} = \cot \theta = \tan \left(\frac{\pi}{2} - \theta \right)$$

$$\therefore \phi = \frac{\pi}{2} - \theta$$

Hence, $\phi = 0$ when $\theta = \frac{\pi}{2}$ and $\phi = \pi$ when $\theta = -\frac{\pi}{2}$

(i.e.,) The tangents to the curve at $\theta = \pm \pi/2$ are parallel to x -axis

Further $\theta \rightarrow \infty$ or $-\infty$ according as $\theta \rightarrow 0$ or π

$\therefore x$ -axis is an asymptote to the curve.

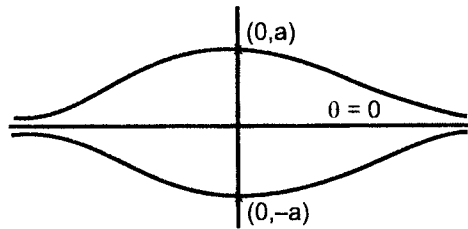


Fig. 4.44

4.3.8 Example:

Trace the curve $x = \frac{3at}{1+t^3}, y = \frac{3at^2}{1+t^3} a > 0$

Solution:

$$x^3 = \left(\frac{3a}{1+t^3} \right)^3 t^3, y^3 = \left(\frac{3a}{1+t^3} \right)^3 t^6$$

$$\therefore x^3 + y^3 = \left(\frac{3a}{1+t^3} \right)^3 t^3 (1+t^3) = \frac{27a^3 t^3}{(1+t^3)^2}$$

$$= 3a \cdot \frac{3at}{1+t^3} \cdot \frac{3at^2}{1+t^3} = 3axy$$

Thus the Cartesian form of the curve is $x^3 + y^3 = 3axy$

For tracing the curve see example 1.3

4.3.9 Example:

Trace the curve $x = \frac{a(1-t^2)}{1+t^2}, y = \frac{2bt}{1+t^2}$

Solution:

$$\frac{x^2}{a^2} = \left(\frac{1-t^2}{1+t^2} \right)^2, \frac{y^2}{b^2} = \frac{4t^2}{(1+t^2)^2}$$

Thus $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (see example 3.3.2)

Exercise 4()

1. $x = a(t + \sin t), y = a(1 + \cos t)$
2. $x = a \sin 2t(1 + \cos 2t), y = a \cos 2t(1 - \cos 2t)$
3. $x = a[\cos \theta - \log(1 + \cos \theta)], y = a \sin \theta$
4. $x = a(t - \sin t), y = a(1 - \cos t)$

Exercise 3.1

1. An asymptote to the curve $y^2(a+x) = x^2(3a-x)$ is

(i) $x-a=0$ (ii) $x+a=0$ (iii) $x-3a=0$ (iv) none of these

[Ans : (ii)]

2. The curve $a^2y^2 = x^2(a^2 - x^2)$ is symmetric about

(i) x-axis (ii) y-axis (iii) both the axes (iv) none of these

[Ans : (iii)]

3. The curve $x^3 + y^3 = 3axy$ is symmetric about the line

[Ans : $y = x$]

4. If the tangents to the curve $y^2 = (x-a)(x-b)^2, (b > a,)$ at the point $x = b$ are inclined at angle θ , $\tan \theta = \dots\dots\dots$

[Ans: $\pm\sqrt{b-a}$]

5. The two tangents to the curve $y^2 = x^3$ at the origin are real and distinct.

[Ans: false]

6. For the curve $r = a \sec \theta \tan \theta$ the origin is a cusp.

[Ans: true]

7. The curve $x = a(t + \sin t), y = a(1 - \cos t)$ is symmetric about x -axis.

[Ans: false]

8. $x - 2a = 0$ is an asymptote to the curve $y^2(2a - x) = x^3$

[Ans:true]

5

Application of Integration to Areas, Lengths, Volumes and Surface areas

5.1 LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

The definite integrals are useful in formulating important physical applications like (i) lengths (ii) areas and (iii) volumes and surface areas of solids of revolution.

- 5.1.1** (a) The area bounded by a curve $y = f(x)$, the axis of x and the two ordinates $x = a$ and $x = b$ is given by the definite integral.

$$\int_a^b y dx = \int_a^b f(x) dx$$

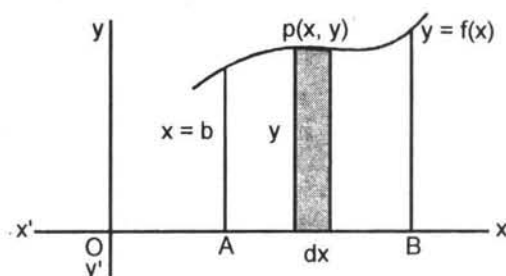


Fig. 5.1.1(a)

Similarly the area bounded by a curve $x = f(y)$ the axis of y and the two abscissae $y = c$ and $y = d$ is given by the definite integral.

$$\int_c^d x dy = \int_c^d f(y) dy$$

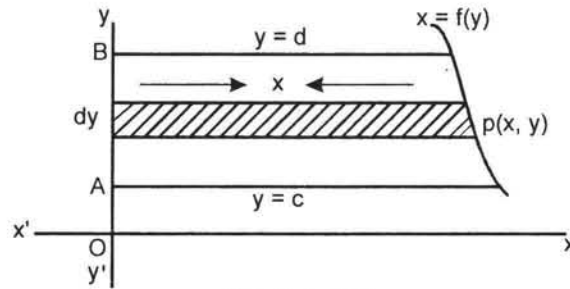


Fig. 5.1.1(b)

- (b) If the equation of the curve is in parametric form, then the area

$$= \int_{t_1}^{t_2} y \frac{dx}{dt} dt \text{ or } \int_{t_1}^{t_2} x \frac{dy}{dt} dt$$

where t_1, t_2 are the limits of integration depending on the boundary of the curve.

- (c) The area enclosed by the polar form of the curve $r = f(\theta)$ and two radii vectors $\theta = \alpha, \theta = \beta$ is

$$= \int_{\alpha}^{\beta} \frac{r^2}{2} d\theta = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta \quad (\text{area of sector OPQ} = \frac{1}{2} r^2 d\theta)$$

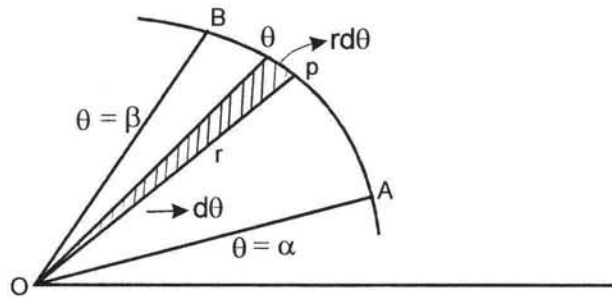


Fig. 5.1.1(c)

5.1.2 Example

Find the total area within the curve $a^2 y^2 = a^2 x^2 - x^4$

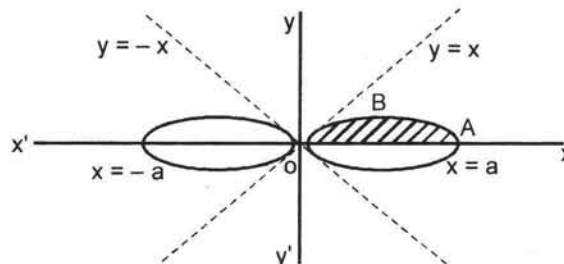


Fig. 5.1.2

Solution:

The curve has two equal loops between the lines $x = a$ and $x = -a$ and is symmetric about x axis. The total area within the curve

$= 4$ (the area of the half of the loop i.e., 'OABO')

$$= 4 \int_0^a y dx$$

$$= 4 \int_0^a \frac{x}{a} \sqrt{a^2 - x^2} dx$$

$$= \frac{4}{a} \times \frac{1}{-2} \int_0^a (a^2 - x^2)^{1/2} (-2x) dx$$

$$= \frac{-2(a^2 - x^2)^{3/2}}{a \cdot \frac{3}{2}} \bigg|_0^a = -\frac{4}{3} (0 - a^2)^{3/2}$$

$$= \frac{4}{3} a^2 \text{ Sq. units}$$

5.1.3 Example:

Find the complete area of the curve $a^2 y^2 = x^3(2a - x)$

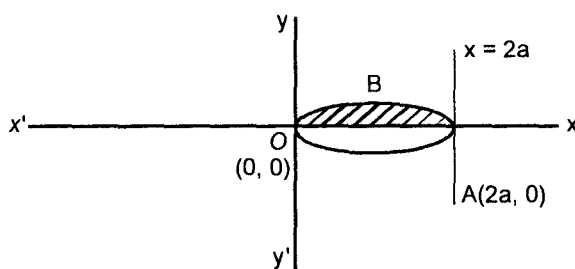


Fig. 5.1.3

Solution:

The curve is symmetrical about x -axis it cuts the x -axis at the points $O(0,0)$, $A(2a,0)$.

The curve consists of a loop lying between $x = 0$, $x = 2a$

The required area

$$= 2\text{OABO}$$

$$= 2 \int_0^{2a} y dx$$

$$= 2 \int_0^{2a} \frac{x^{3/2} \sqrt{2a - x}}{a} dx$$

Substituting $x = 2a \sin^2 \theta$

We have $dx = 4a \sin \theta \cos \theta$

$$x=0 \Rightarrow \theta=0$$

$$x=2a \Rightarrow \theta=\pi/2$$

$$\begin{aligned} \text{Area} &= \frac{2}{a} \int_0^{\pi/2} (2a)^2 \sin^3 \theta \sqrt{2a} \cos \theta \cdot 4a \sin \theta \cos \theta d\theta \\ &= 32a^2 \int_0^{\pi/2} \sin^4 \theta \cos^2 \theta d\theta \\ &= 32a^2 \frac{3.1.1}{6.4.2} \pi/2 = \pi a^2 \text{ Sq. units} \end{aligned}$$

5.1.4 Example:

Find the area of the segment cutoff from the parabola $y^2 = 2x$, by the straight line $y = 4x - 1$.

Solution:

The given curves are $y^2 = 2x$ (i)
 $y = 4x - 1$ (ii)

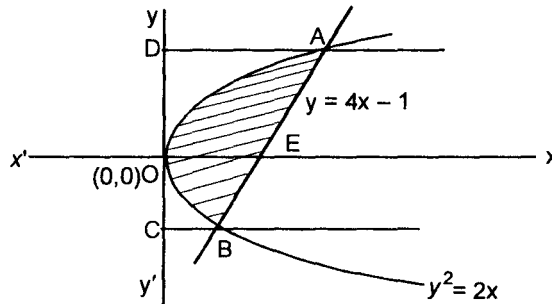


Fig. 5.1.4

Solving (i) and (ii) we get the points of intersection as

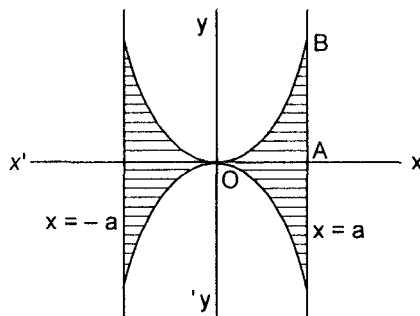
$$A\left(\frac{1}{2}, 1\right), B\left(\frac{1}{8}, -\frac{1}{2}\right)$$

The required area = area AOBA (shaded region)
 = area CBEADOC - area BOADOCB

$$\begin{aligned} &= \int_{-1/2}^1 \frac{1}{4} (y+1) dy - \int_{-1/2}^1 \frac{y^2}{2} dy \text{ (from equations (i) and (ii) respectively),} \\ &= \frac{1}{4} \left[\frac{y^2}{2} + y \right]_{-1/2}^1 - \left[\frac{y^3}{6} \right]_{-1/2}^1 \\ &= \frac{9}{32} \text{ Sq. units} \end{aligned}$$

5.1.5 Example:

Find the area between the curve $x^2y^2 = a^2(y^2 - x^2)$ and its asymptotes.

**Fig. 5.1.5****Solution:**

The given equation can be written as

$$y^2(a^2 - x^2) = a^2x^2 \Rightarrow y^2 = \frac{a^2x^2}{a^2 - x^2}$$

The asymptotes to the curve are given by

$$a^2 - x^2 = 0 \Rightarrow x = \pm a$$

The curve is symmetric about both the axes and passes through the origin.

The shaded portion is the required area.

The complete area = 4(areaOAB in the first quadrant)

$$\begin{aligned} &= 4 \int_0^a y dx \\ &= 4 \int_0^a \frac{ax}{\sqrt{a^2 - x^2}} dx = -2 \int_0^a \frac{-2ax}{(a^2 - x^2)^{1/2}} dx \\ &= -2a \times \frac{(a^2 - x^2)^{1/2}}{1} \Big|_0^a = -4a \left[\sqrt{a^2 - x^2} \right]_0^a \\ &= -4a[0 - a] = 4a^2 \text{ Sq.units.} \end{aligned}$$

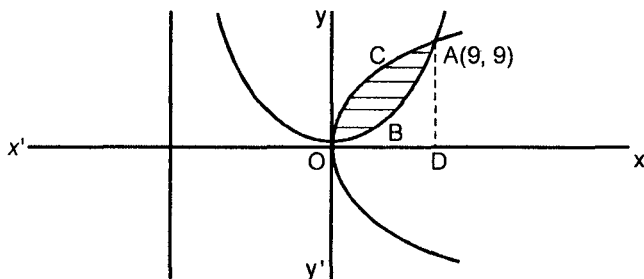
5.1.6 Example:

Find the area bounded by the curves $y^2 = 9x$, $x^2 = 9y$,

Solution:

Solving $y^2 = 9x$ and $x^2 = 9y$

The points of intersection of the two curves are O(0,0) A(9,9)

**Fig. 5.1.6**

The required area = area OBACD

$$= \text{area ODACO} - \text{area OBACO}$$

$$= \int_{x=0}^9 y dx \quad - \int_{x=0}^9 y dx$$

(parabola $y^2 = 9x$) (parabola $x^2 = 9y$)

$$= \int_0^9 3\sqrt{x} dx - \int_0^9 \frac{x^2}{9} dx = 27 \text{ sq. units}$$

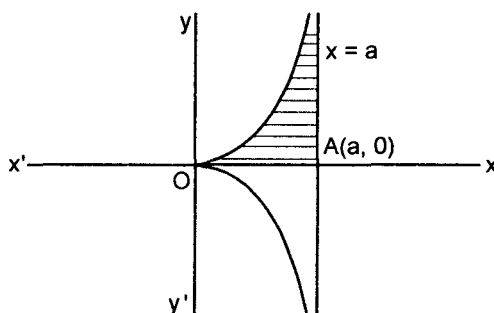
5.1.7 Example:

Find the area bound by the cissoid $x = a \sin^2 t$, $y = \frac{a \sin^3 t}{\cos t}$ and its asymptote.

Solution:

The equation of the curve is

$$x = a \sin^2 t, \quad y = \frac{a \sin^3 t}{\cos t}$$

**Fig. 5.1.7**

$$y^2 = \frac{a^2 \sin^6 t}{\cos^2 t} = \frac{a^2 \left(\frac{x}{a}\right)^3}{\left(1 - \frac{x}{a}\right)} = \frac{x^3}{a - x}$$

The equation of the given curve is transformed into cartesian form. The curve is symmetric about x-axis and $x = a$ is the asymptote.

The required area = 2(shaded area)

$$= 2 \int_0^a y dx = 2 \int_0^a x \left(\frac{x}{a-x} \right)^{1/2} dx$$

Writing

$$x = a \sin^2 \theta, dx = 2a \sin \theta \cos \theta d\theta$$

$$= 4a^2 \int_0^{\pi/2} \sin^4 \theta d\theta = 4a^2 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{3\pi a^2}{4}$$

5.1.8 Example:

Find the area included between the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ and its base.

Solution:

The equation of the curve is

$$x = a(\theta - \sin \theta) \quad y = a(1 - \cos \theta)$$

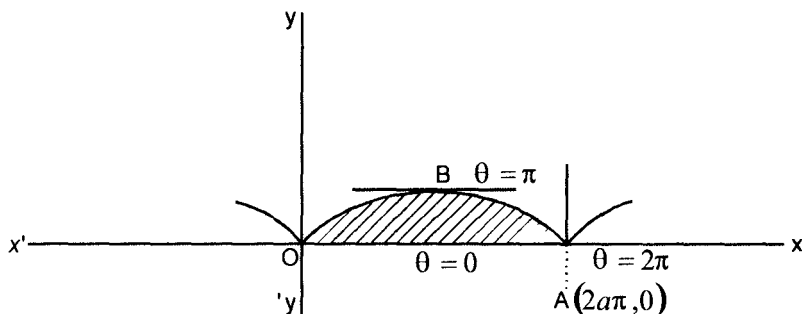


Fig. 5.1.8

The required area = area OABO

$$= \int_0^{2\pi} y dx$$

$$= \int_0^{2\pi} y \frac{dx}{d\theta} d\theta$$

$$= \int_0^{2\pi} a(1 - \cos \theta) a(1 - \cos \theta) d\theta$$

$$= 4a^2 \int_0^{2\pi} \sin^4 \frac{\theta}{2} d\theta = 8a^2 \int_0^{\pi} \sin^4 \frac{\theta}{2} d\theta$$

$$\begin{aligned}
 &= 8a^2 \cdot 2 \int_0^{\pi/2} \sin^4 t \, dt \quad \left(\text{taking } \frac{\theta}{2} = t \right) \\
 &= 16a^2 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \\
 &= 3\pi a^2 \text{ Sq. units}
 \end{aligned}$$

5.1.9 Example:

Find the area between the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ and x-axis.

Solution:

The required area

$$\begin{aligned}
 &= 2 \text{ area OAB} \\
 &= 2 \int_0^{\pi} y \frac{dx}{d\theta} \cdot d\theta = 2 \int_0^{\pi} a(1 - \cos\theta) \cdot a(1 + \cos\theta) d\theta \\
 &= 2a^2 \cdot 2 \int_0^{\pi/2} \sin^2 \theta \, d\theta
 \end{aligned}$$

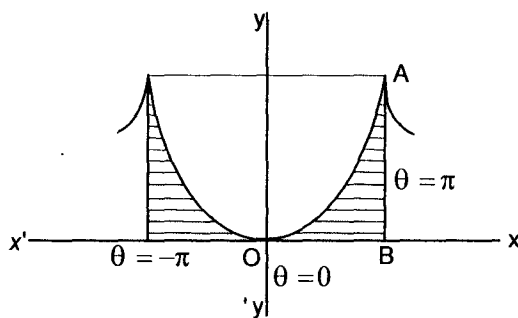


Fig. 5.1.9

$$\begin{aligned}
 &= 4a^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} \\
 &= \pi a^2 \text{ Sq. units}
 \end{aligned}$$

5.1.10 Example:

Find the complete area of the curve given by the equations.

$$x = a \cos^3 \theta, \quad y = b \sin^3 \theta$$

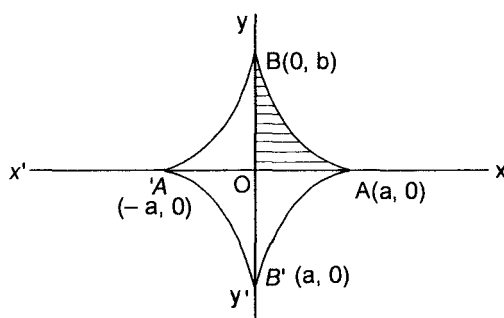


Fig. 5.1.10

Solution:

The required area

$$\begin{aligned}
 &= 4 \text{ area OABO} \\
 &= 4 \int_0^a y dx = 4 \int_{\pi/2}^0 y \frac{dx}{d\theta} d\theta \\
 &= 4 \int_{\pi/2}^0 b \sin^3 \theta (-3\theta \cos^2 \theta \sin \theta) \\
 &= 12ab \int_0^{\pi/2} \sin^4 \theta \cos^2 \theta d\theta \\
 &= 12ab \times \frac{3}{6} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \\
 &= \frac{3}{8} \pi ab \text{ Sq. units.}
 \end{aligned}$$

5.1.11 Example:

Find the area of loop of the curve $r^2 = a^2 \cos 2\theta$.

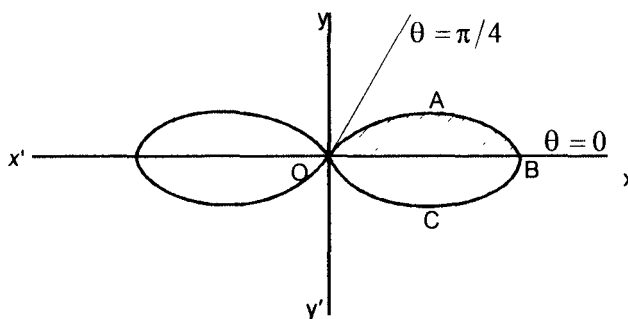


Fig. 5.1.11

Solution:

The required area

$$\begin{aligned}
 &= 4 \text{ area OABO} \\
 &= 2 \int_0^{\pi/4} \frac{r^2}{2} d\theta = \int_0^{\pi/4} r^2 d\theta \\
 &= \int_0^{\pi/4} a^2 \cos 2\theta d\theta = \frac{a^2}{2} [\sin 2\theta]_0^{\pi/4} \\
 &= \frac{a^2}{2}
 \end{aligned}$$

5.1.12 Example:

Find the area of the curve $r = a(1 + \cos\theta)$

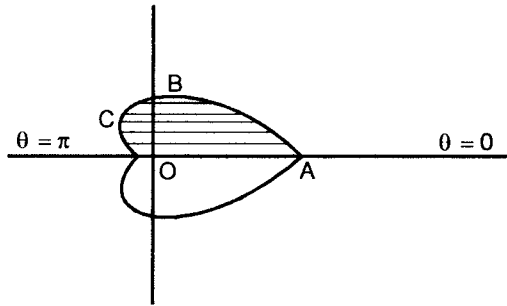
Solution:

Fig. 5.1.12

Required area

$$\begin{aligned}
 &= 2 \cdot (\text{area OABCO}) \\
 &= 2 \int_0^{\pi} \frac{r^2}{2} d\theta \\
 &= \int_0^{\pi} a^2 (1 + \cos\theta)^2 d\theta \\
 &= \int_0^{\pi} a^2 4 \left(\cos^2 \frac{\theta}{2} \right) d\theta \\
 &= 4a^2 \int_0^{\pi} \cos^4 \frac{\theta}{2} d\theta \\
 &= 4a^2 \int_0^{\pi/2} \cos^4 t \cdot 2dt \quad (\text{where } t = \theta/2) \\
 &= 8a^2 \frac{3}{4} \frac{1}{2} \frac{\pi}{2} = \frac{3\pi a^2}{2} \text{ Sq. units}
 \end{aligned}$$

5.1.13 Example:

Find the area of the portion included between the cardioids $r = a(1 + \cos\theta)$ and $r = a(1 - \cos\theta)$.

Solution:

Required area

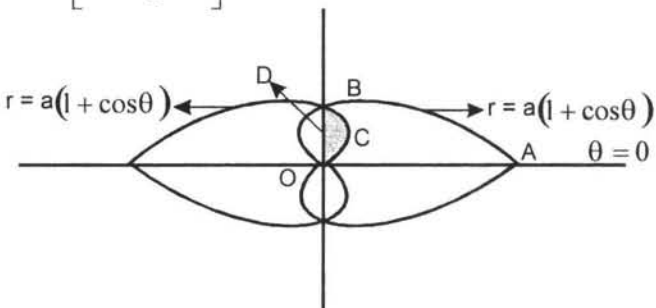
$$\begin{aligned}
 &= 4(\text{OCBDO}) \\
 &= 4 \left[\int_0^{\pi/2} \frac{r^2}{2} d\theta \right]
 \end{aligned}$$


Fig. 5.1.13

$$\begin{aligned}
 &= 2 \int_0^{\pi/2} a^2 (1 - \cos\theta)^2 d\theta \\
 &= 2a^2 \int_0^{\pi/2} a^2 (1 - \cos\theta)^2 d\theta \\
 &= 2a^2 \left\{ \left[\theta - 2\sin\theta \right]_0^{\pi/2} + \frac{1}{2} \cdot \frac{\pi}{2} \right\} = \frac{a^2}{2} (3\pi - 8) \text{ Sq. units}
 \end{aligned}$$

5.1.14 Example:

Find the area of the curve $r^2 = a^2 \sin 2\theta$.

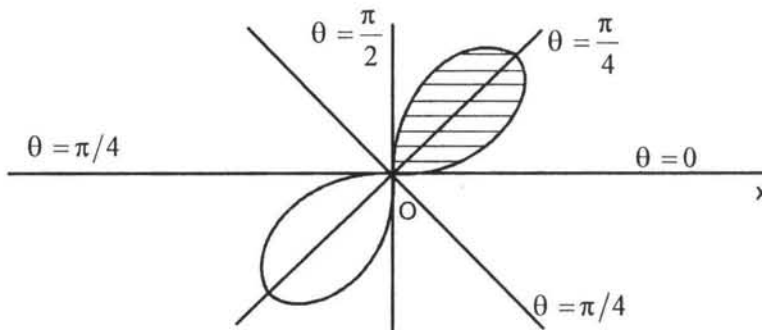


Fig. 5.1.14

Solution:

The total area of the curve = 2 area of one loop of the curve

$$\begin{aligned}
 &= 2 \cdot \frac{1}{2} \int_0^{\pi/2} r^2 d\theta = \int_0^{\pi/2} a^2 \sin 2\theta d\theta \\
 &= \frac{a^2}{2} [-\cos 2\theta]_0^{\pi/2} = \frac{a^2}{2} [1 + 1] = a^2 \text{ Sq. units}
 \end{aligned}$$

Exercise 5 (A)

- Find the whole area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
[Ans: πab]
- Find the whole area of the curve $x^2(x^2 + y^2) = a^2(x^2 - y^2)$
[Ans: $a^2(\pi - 2)$]
- Find the area of the curve $a^2x^2 = y^3(2a - y)$
[Ans: πa^2]
- Find the area bounded by the curve $xy^2 = 4a^2(2a - x)$ and its asymptote.
[Ans: $4\pi a^2$]
- Find the area included between $y^2 = 4ax$ and $y = mx$
[Ans: $\frac{8a^2}{3m^3}$]
- Find the area included between the parabolas $y^2 = 4a(x + a)$ and $y^2 = 4b(b - x)$
[Ans: $\frac{8}{3}(a + b)\sqrt{ab}$]
- Find the area common to the circle $x^2 + y^2 = 9$ and parabola $x^2 = 8y$
[Ans: $\frac{1}{3} \left\{ 2\sqrt{2} + 27 \sin^{-1} \left(\frac{2\sqrt{2}}{3} \right) \right\}$]
- Show that the area of the loop of the curve $a^2y^2 = x^2(2a - x)(x - a)$ is $\frac{3\pi a^2}{8}$
- Find the area of the loop of the curve $r = a \sin 2\theta$.
[Ans: $\frac{\pi a^2}{8}$]
- Find the area common to the circles $r = a\sqrt{2}$, $-2a \cos \theta$
[Ans: $a^2(\pi - 1)$]

5.2.1 Lengths of plane curves:

- (a) The length of the arc of the curve $y = f(x)$ included between the points whose abscissae are 'a' and 'b' is

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \text{or} \quad s = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

- (b) The length of the arc of the curve $x = f(y)$ included between the points whose ordinates are 'c' and 'd' is

$$s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

or

$$s = \int_c^d \sqrt{1 + [f'(y)]^2} dy$$

- (c) The length of the arc of the curve $x = f(t)$, $y = g(t)$ included between two points whose parametric values are ' α ' and ' β ' is

$$s = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

- (d) The length of the arc of the curve $r = f(\theta)$ included between two points whose vectorial angles are θ_1 and θ_2 is

$$s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(r \frac{dr}{d\theta}\right)^2} d\theta$$

- (e) The length of the arc of the curve $\theta = f(r)$ from $r = r_1$ to $r = r_2$ is given by

$$s = \int_{r_1}^{r_2} \sqrt{1 + \left(r \frac{d\theta}{dr}\right)^2} dr$$

5.2.2 Example:

Find the complete length of the curve

$$x^2(a^2 - x^2) = 8a^2y^2$$

Solution:

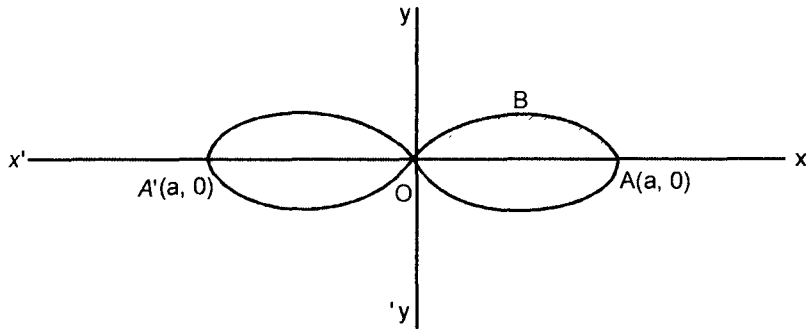


Fig. 5.2.2

The curve is symmetrical about both the axis, one loop is formed in between $x = 0$ and $x = a$ and another loop is formed in between $x = 0$ and $x = -a$.

The total length of the curve = 4(length OABO)

$$\text{Total length} = s = 4 \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \dots(1)$$

The equation of the curve is

$$x^2(a^2 - x^2) = 8a^2y^2$$

Differentiating with respect to 'x'

$$16a^2y \frac{dy}{dx} = 2xa^2 - 4x^3$$

$$\frac{dy}{dx} = \frac{x(a^2 - 2x^2)}{8a^2y}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2(a^2 - 2x^2)^2}{64a^4y^2}$$

Substituting for y^2 from the equation of the curve

$$8a^2y^2 = x^2(a^2 - x^2)$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2(a^2 - 2x^2)^2}{8a^2(a^2 - x^2)}$$

$$= \frac{8a^2(a^2 - x^2) + (a^4 + 4x^4 - 4a^2x^2)}{8a^2(a^2 - x^2)}$$

$$\begin{aligned}
 &= \frac{9a^4 - 12a^2x^2 + 4x^4}{8a^2(a^2 - x^2)} \\
 &= \frac{(3a^2 - 2x^2)^2}{8a^2(a^2 - x^2)} \quad \dots(2)
 \end{aligned}$$

Substituting from (2) in (1)

$$\begin{aligned}
 \text{Total length } s &= 4 \int_0^a \frac{3a^2 - 2x^2}{2\sqrt{2a}\sqrt{a^2 - x^2}} dx \\
 s &= \frac{\sqrt{2}}{a} \int_0^a \frac{a^2 + 2a^2 - 2x^2}{\sqrt{a^2 - x^2}} dx \\
 s &= \frac{\sqrt{2}}{a} \left[\int_0^a \left(\frac{a^2}{\sqrt{a^2 - x^2}} + 2\sqrt{a^2 - x^2} \right) dx \right] \\
 s &= \frac{\sqrt{2}}{a} \left[a^2 \sin^{-1} x/a + 2 \left\{ \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} x/a \right\} \right]_0^a \\
 &= \frac{\sqrt{2}}{a} 2a^2 \sin^{-1}(1) = 2\sqrt{2}a \cdot \frac{\pi}{2} \\
 s &= \sqrt{2}\pi a
 \end{aligned}$$

5.2.3 Example:

Find the length of the loop of the curve

$$3ay^2 = x(x - a)^2$$

Solution:

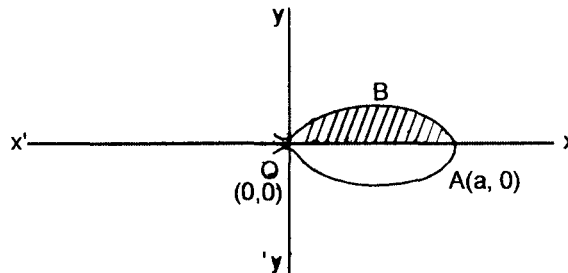


Fig. 5.2.3

The curve is symmetrical about x – axis, the loop is formed between lines $x = 0$ and $x = a$, differentiating $3ay^2 = x(x - a)^2$ w.r., to x .

$$6ay \frac{dy}{dx} = x.2(x - a) + (x - a)^2$$

$$\frac{dy}{dx} = \frac{(x - a)(3x - a)}{6ay}$$

$$1 + \left(\frac{dy}{dx} \right)^2 = 1 + \frac{(x - a)^2(3x - a)^2}{36a^2y^2} \quad \dots(1)$$

Substituting $3ay^2 = x(x - a)^2$ in (1) we get

$$\begin{aligned} 1 + \left(\frac{dy}{dx} \right)^2 &= 1 + \frac{(x - a)^2(3x - a)^2}{12ax(x - a)^2} \\ &= \frac{12ax + 9x^2 + a^2 - 6ax}{12ax} \end{aligned}$$

$$1 + \left(\frac{dy}{dx} \right)^2 = \frac{(3x + a)^2}{12ax} \quad \dots(2)$$

The length of the loop of the curve

$$\begin{aligned} s &= 2 \int_0^a \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx \\ &= 2 \int_0^a \frac{(3x + a)}{2\sqrt{3a}\sqrt{x}} dx \\ s &= \frac{2}{2\sqrt{3a}} \int_0^a 3\sqrt{x} + ax^{-1/2} dx \\ &= \frac{1}{\sqrt{3a}} \left[2x^{3/2} + 2ax^{1/2} \right]_0^a \\ &= \frac{1}{\sqrt{3a}} \left[2a^{3/2} + 2a\sqrt{a} \right] = \frac{4a}{\sqrt{3}} \end{aligned}$$

5.2.4 Example:

Find the perimeter of the loop of the curve $3ay^2 = x^2(a - x)$

Solution:

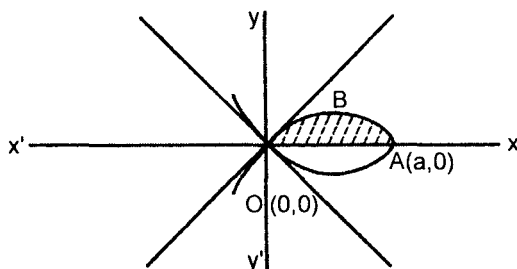


Fig. 5.2.4

The curve is symmetrical about x -axis and the loop of the curve lies between $x = 0$ and $x = a$.

Perimeter of the curve = $2OABO$

$$s = 2 \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$s = 2 \int_0^a \frac{(4a - 3x)}{2\sqrt{3a}\sqrt{a-x}} dx$$

$$s = \frac{1}{a\sqrt{3}} \int_0^a \frac{a + 3(a-x)}{\sqrt{a-x}} dx$$

$$= \frac{1}{a\sqrt{3}} \int_0^a \left[a(a-x)^{-1/2} + 3(a-x)^{-1/2} \right] dx$$

$$= \frac{1}{a\sqrt{3}} \left[-2a(a-x)^{1/2} - 2(a-x)^{3/2} \right]_0^a$$

$$= \frac{1}{a\sqrt{3}} \left[0 + 2a\sqrt{a} + 2a^{3/2} \right] = 4a/\sqrt{3}$$

5.2.5 Example:

Find the length of the arc of the parabola $y^2 = 4ax$ cut off by the line $3y = 8x$.

Solution:

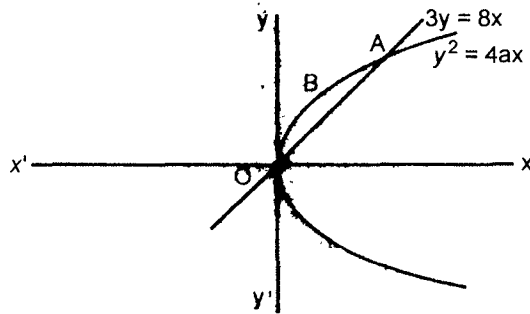


Fig. 5.2.5

Solving the equations of the parabola and the line gives the points of intersections

$$O(0,0) \text{ and } A\left(\frac{9a}{16}, \frac{3a}{2}\right)$$

$$y^2 = 4ax$$

$$2y \frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{2a}{y}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{y^2}{4a^2}$$

The length of the arc OBA of the

$$\text{Parabola} = s = \int_{y=0}^{\frac{3a}{2}} \left[1 + \left(\frac{dx}{dy}\right)^2 \right]^{1/2} dy$$

$$s = \int_{y=0}^{\frac{3a}{2}} \sqrt{\frac{y^2 + 4a^2}{4a^2}} dy = \frac{1}{2a} \int_0^{\frac{3a}{2}} \sqrt{y^2 + 4a^2} dy$$

$$= \frac{1}{2a} \left[\frac{y}{2} \sqrt{y^2 + 4a^2} + \frac{4a^2}{2} \log \left\{ y + \sqrt{y^2 + 4a^2} \right\} \right]_0^{\frac{3a}{2}}$$

$$= a \left[\frac{15}{16} + \log 2 \right]$$

5.2.6 Example:

Find the length of an arc of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

Solution:

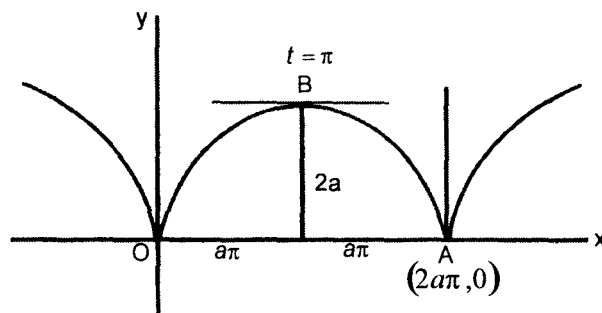


Fig. 5.2.6

$$\frac{dx}{dt} = a(1 - \cos t), \frac{dy}{dt} = a \sin t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \left[a^2(1 - \cos t)^2 + a^2 \sin^2 t\right]^{1/2}$$

$$= 2a \sin t/2$$

The length of an arc of the cycloid

$$= 2 \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= 2 \int_0^\pi 2a \sin t/2 dt$$

$$= 4a [-2 \cos t/2]_0^\pi$$

$$= 8a$$

5.2.7 Example:

Find the total length of the curve

$$\frac{x^{2/3}}{a^{2/3}} + \frac{y^{2/3}}{b^{2/3}} = 1$$

Solution:

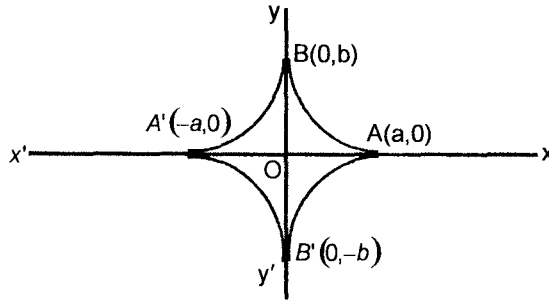


Fig. 5.2.7

The parametric form of the curve is

$$x = a \cos^3 \theta, \quad y = b \sin^3 \theta$$

The total length of the curve = 4. length OABO

$$\begin{aligned} &= 4 \int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \\ s &= 4 \int_0^{\pi/2} \left[(-3a \sin \theta \cos^2 \theta)^2 + (3b \sin^2 \theta \cos \theta)^2 \right]^{1/2} d\theta \\ &= 4 \int_0^{\pi/2} \left\{ a^2 \sin^2 \theta \cos^4 \theta + b^2 \sin^4 \theta \cos^2 \theta \right\}^{1/2} d\theta \\ &= 12 \int_0^{\pi/2} (a^2 \cos^2 \theta + b^2 \sin^2 \theta)^{1/2} \sin \theta \cos \theta d\theta \end{aligned}$$

Substituting

$$a^2 \cos^2 \theta + b^2 \sin^2 \theta = z^2$$

and

$$(b^2 - a^2) 2 \sin \theta \cos \theta d\theta = 2z dz$$

$$\begin{aligned} s &= 12 \int_a^b z \cdot \frac{z}{(b^2 - a^2)} dz = \frac{12}{b^2 - a^2} \cdot \frac{z^3}{3} \Big|_a^b \\ &= \frac{4}{b^2 - a^2} (b^3 - a^3) = \frac{4(a^2 + b^2 + ab)}{a + b} \end{aligned}$$

5.2.8 Example:

Find the length of the arc of the curve given by $x = a \sin 2t(1 + \cos 2t)$, $y = a \cos 2t(1 - \cos 2t)$ measured from the origin to any point.

Solution:

$$x = a \sin 2t(1 + \cos 2t)$$

$$\frac{dx}{dt} = 2a \cos 2t(1 + \cos 2t) + a \sin 2t(-2 \sin 2t)$$

$$\frac{dx}{dt} = 2a(\cos 2t + \cos 4t)$$

$$\frac{dx}{dt} = 4a \cos 3t \cos t \quad \dots(1)$$

and

$$y = a \cos 2t(1 - \cos 2t)$$

$$\frac{dy}{dt} = -2a \sin 2t(1 - \cos 2t) + a \cos 2t(2 \sin 2t)$$

$$\frac{dy}{dt} = 2a(\sin 4t - \sin 2t) = 4a \cos 3t \sin t \quad \dots(2)$$

The length of the curve from origin to any point

$$= \int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Using (1) and (2)

$$s = \int_0^t \sqrt{(4a \cos 3t \cos t)^2 + (4a \cos 3t \sin t)^2} dt$$

$$s = \int_0^t 4a \cos 3t dt = \frac{4a}{3} \sin 3t \Big|_0^t$$

$$s = \frac{4a}{3} \sin 3t$$

5.2.9 Example:

Find the perimeter of the cardioid $r = a(1 + \cos \theta)$.

Solution:

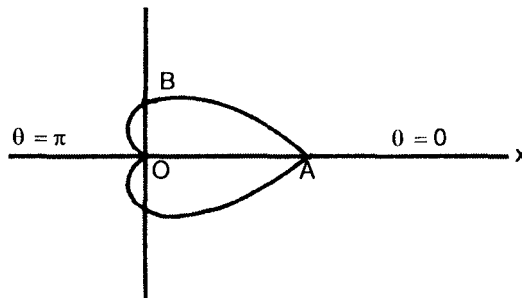


Fig. 5.2.9

$$r = a(1 + \cos \theta) \Rightarrow \frac{dr}{d\theta} = -a \sin \theta$$

The perimeter of the cardioid

$$\begin{aligned} &= 2\text{OABO} \\ &= 2 \int_0^\pi \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \\ &= 2 \int_0^\pi \sqrt{a^2(1 + \cos \theta)^2 + (-a \sin \theta)^2} d\theta \\ &= 2 \int_0^\pi \sqrt{2a^2(1 + \cos \theta)} d\theta \\ &= 2 \int_0^\pi \sqrt{4a^2 \cos^2 \frac{\theta}{2}} d\theta \\ &= 2 \int_0^\pi 2a \cos \frac{\theta}{2} d\theta \\ &= 4a \cdot 2 \sin \frac{\theta}{2} \Big|_0^\pi \\ &= 8a \end{aligned}$$

5.2.10 Example:

Find the perimeter of the curve $r = 2a \cos \theta$.

Solution:

The equation of the curve is $r = 2a \cos \theta$.

It is a circle passing through pole whose centre is on the initial line at a distance $a/2$ from pole.

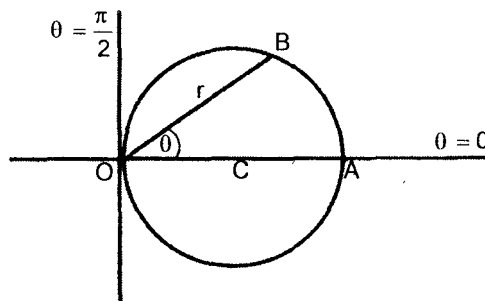


Fig. 5.2.10

The perimeter of the curve

$$= 2OABO$$

$$= 2 \int_0^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

$$= 2 \int_0^{\pi/2} \sqrt{(2a \cos \theta)^2 + (-2a \sin \theta)^2} d\theta$$

$$= 4a \int_0^{\pi/2} d\theta$$

$$= 4a \frac{\pi}{2}$$

$$= 2a\pi$$

5.2.11 Example

Find the length of the arc of the parabola $l/r = 1 + \cos \theta$ cut off by its latus rectum.

Solution:

LSL' = 2l is the latus rectum of the parabola.

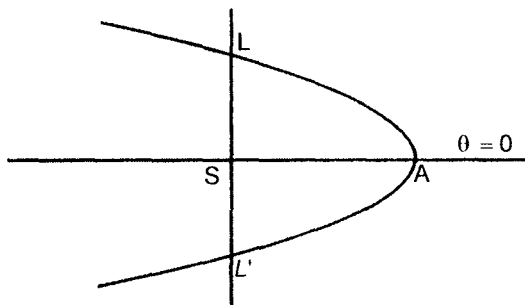


Fig. 5.2.11

The length of the arc L'AL = 2AL

$$= 2 \int_0^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Equations of the curve is $l/r = 1 + \cos \theta$

$$\log r = \log l + \log(1 + \cos \theta)$$

$$\frac{1}{r} \frac{dr}{d\theta} = 0 - \frac{\sin \theta}{1 + \cos \theta}$$

$$\frac{dr}{d\theta} = r \tan \theta/2$$

$$\text{The length of the arc} = 2 \int_0^{\pi/2} \sqrt{r^2 + r^2 \tan^2 \theta/2} d\theta$$

$$= 2 \int_0^{\pi/2} r \sec \theta/2 d\theta$$

$$= 2 \int_0^{\pi/2} \frac{l}{1 + \cos \theta} \cdot \sec \theta/2 d\theta$$

$$= 2 \int_0^{\pi/2} \frac{l}{2 \cos^2 \theta} \cdot \sec \theta/2 d\theta$$

$$= l \int_0^{\pi/2} \sec \frac{\theta}{2} \cdot \sec^2 \theta/2 d\theta$$

$$= l \int_0^{\pi/2} \sqrt{1 + \tan^2 \theta/2} \sec^2 \theta/2 d\theta$$

writing $\tan \theta/2 = t$

$$\sec^2 \theta/2 \cdot \frac{1}{2} d\theta = dt$$

$$= 2l \int_0^1 \sqrt{1 + t^2} dt$$

$$= 2l \left[\frac{t}{2} \sqrt{1 + t^2} + \frac{1}{2} \log(t \sqrt{1 + t^2}) \right]_0^1$$

$$= 2l \left[\frac{1}{2} \sqrt{2} + \frac{1}{2} \log(1 + \sqrt{2}) \right]$$

$$= l[\sqrt{2} + \log(1 + \sqrt{2})]$$

Exercise – 5(B)

- Find the length of the arc of the parabola $y^2 = 4ax$ cut off by its latus rectum.

$$[\text{Ans: } 2a[\sqrt{2} + \log(1 + \sqrt{2})]]$$

- Find the perimeter of the loop of the curve $9ay^2 = (x - 2a)(x - 5a)^2$

$$[\text{Ans: } 4\sqrt{3}a]$$

3. Find the length of the arc of the parabola $x^2 = 4ay$ from vertex to one extremity of the latus rectum.

$$[\text{Ans: } a[\sqrt{2} + \log(1 + \sqrt{2})]]$$

4. Find the length of an arc of the parabola $y = x^2$ measured from the vertex.

$$[\text{Ans: } \frac{x}{2}\sqrt{1+4x^2} + \frac{1}{4}\sin^{-1}(2x)]$$

5. Find that the length of the arc of the curve $y = \log \tanh(x/2)$ from $x = 1$ to $x = 2$

$$[\text{Ans: } \log\left(\frac{e^2+1}{e}\right)]$$

6. Find the length of the arc of the curve $y = x(2-x)$ as x varies from 0 to 2.

$$[\text{Ans: } \frac{1}{2}\log(2+\sqrt{5}) + \sqrt{5}]$$

7. Find the length of the curve $x = a(\theta + \sin\theta)$ $y = a(1 - \cos\theta)$

$$[\text{Ans: } 8a]$$

8. Find the length of the arc of the curve $x = e^\theta \sin\theta$, $y = e^\theta \cos\theta$ from $\theta = 0$ to $\theta = \pi/2$

$$[\text{Ans: } \sqrt{2}(e^{\pi/2} - 1)]$$

9. Prove that the loop of the curve

$$x = t^2, y = t - \frac{t^3}{3} \text{ is of length } 4\sqrt{3}$$

10. Find the perimeter of the cardioid $r = a(1 - \cos\theta)$

$$[\text{Ans: } 8a]$$

11. Show that the arc of the upper half of the curve $r = a(1 - \cos\theta)$ is bisected by $\theta = 2\pi/3$

5.3.1 Volume and Surface of solids of revolution:

- (a) Volume of the solid generated by the revolution of the area bounded by the curve

$y = f(x)$, the x -axis, the ordinates $x = a$ and $x = b$ about the x -axis is $\int_a^b \pi y^2 dx$.

- (b) Volume of the solid obtained by the revolution of the curve $x = f(y)$, the y -axis the

abscissae $y = c$ and $y = d$ about the y -axis $\int_c^d \pi x^2 dy$.

- (c) Volume of the solid obtained by the revolution of the curve $x = f(t)$, $y = \phi(t)$ about

$$x\text{-axis is } \int_{t_1}^{t_2} \pi y^2 \frac{dx}{dt} dt = \int_{t_1}^{t_2} \pi [\phi(t)]^2 \frac{df(t)}{dt} dt$$

- (d) Volume of the solid obtained by the revolution of the curve $x = f(t)$, $y = \phi(t)$ about

$$y\text{-axis is } \int_{t_1}^{t_2} \pi x^2 \frac{dy}{dt} dt = \int_{t_1}^{t_2} \pi [f(t)]^2 \frac{d\phi(t)}{dt} dt$$

- (e) Volume of the solid obtained by the revolution of the curve $r = f(\theta)$ about the initial line is

$$\begin{aligned} &= \int_a^b \pi y^2 dx = \int_{\theta_1}^{\theta_2} \pi (r \sin \theta)^2 \frac{dx}{d\theta} d\theta \\ &= \int_{\theta_1}^{\theta_2} \pi (r \sin \theta)^2 \frac{d(r \cos \theta)}{d\theta} d\theta \end{aligned}$$

- (f) Volume of the solid obtained by the revolution of the curve $r = f(\theta)$ about the line perpendicular to initial line

$$\begin{aligned} &= \int_a^b \pi x^2 dy = \int_{\theta_1}^{\theta_2} \pi (r \cos \theta)^2 \frac{dy}{d\theta} d\theta \\ &= \int_{\theta_1}^{\theta_2} \pi (r \cos \theta)^2 \frac{d(r \sin \theta)}{d\theta} d\theta \end{aligned}$$

- (g) The volume of the solid generated by the revolution about the initial line (x-axis) of the area bounded by the curve $r = f(\theta)$ with the radii vectors $\theta = \alpha$, $\theta = \beta$ is

$$= \int_{\theta=\alpha}^{\theta=\beta} \frac{2\pi}{3} r^3 \sin \theta d\theta$$

- (h) The volume of the solid generated by the revolution of the area about the line $\theta = \pi/2$ (y-axis) of the area bounded by the curve $r = f(\theta)$ with the radii vectors $\theta = \alpha$, $\theta = \beta$ is

$$= \int_{\theta=\alpha}^{\theta=\beta} \frac{2\pi}{3} r^3 \cos \theta d\theta$$

5.3.2 Example:

Find the volume of the solid formed by the revolution of the loop of the curve $y^2(a+x) = x^2(a-x)$ about x-axis.

Solution:

$y = 0$ gives $x = 0, a$

$\therefore$ The loop is formed between $x = 0$ and $x = a$ $x = -a$ is the asymptote to the curve.

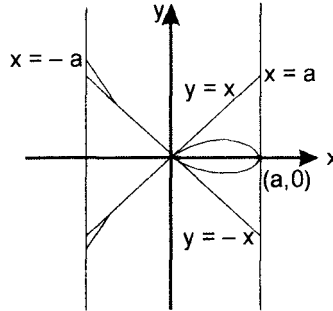


Fig. 5.3.2

$\therefore$ The volume formed by the revolution of the loop about x-axis

$$\begin{aligned}
 &= \int_0^a \pi y^2 dx \\
 &= \pi \int_0^a \frac{x^2(a-x)}{a+x} dx \\
 &= \pi \int_0^a \left[-x^2 + 2ax - 2a^2 + \frac{2a^3}{a+x} \right] dx \\
 &= \pi \left[-\frac{x^3}{3} + ax^2 - 2a^2x + 2a^3 \log(a+x) \right]_0^a \\
 &= 2\pi a^3 \left[\log 2a - \log a - \frac{2}{3} \right] \\
 &= 2\pi a^3 \left[\log 2 - \frac{2}{3} \right] \text{ cubic units.}
 \end{aligned}$$

5.3.3 Example:

Find the volume of the solid generated by revolving the curve $xy^2 = 4(2-x)$ about y-axis.

Solution:

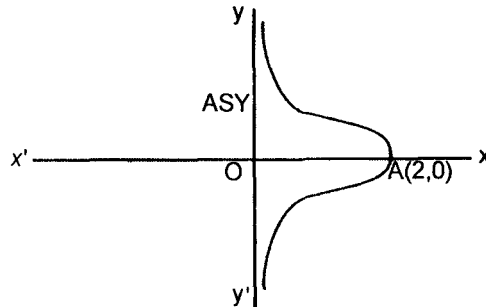


Fig. 5.3.3

The given curve can be written as

$$x(y^2 + 4) = 8 \quad \Rightarrow x = \frac{8}{y^2 + 4}$$

The volume of the solid obtained by revolving the given curve about y-axis

$$\begin{aligned} &= 2 \int_0^{\infty} \pi x^2 dy \\ &= 2\pi \int_0^{\infty} \frac{64}{(y^2 + 4)^2} dy \end{aligned}$$

$$\text{Substituting } y = \tan \theta \Rightarrow \frac{dy}{d\theta} = \sec^2 \theta \, d\theta$$

$$\begin{aligned} &= 128\pi \int_0^{\pi/2} \frac{2 \sec^2 \theta \, d\theta}{[4(\tan^2 \theta + 1)]^2} \\ &= 16\pi \int_0^{\pi/2} \cos^2 \theta \, d\theta \\ &= 8\pi \int_0^{\pi/2} 1 + \cos 2\theta \, d\theta \\ &= 8\pi \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\pi/2} \\ &= 4\pi^2 \text{ cubic units.} \end{aligned}$$

5.3.4 Example:

The part of the parabola $y^2 = 4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the volume of the reel thus generated.

Solution:

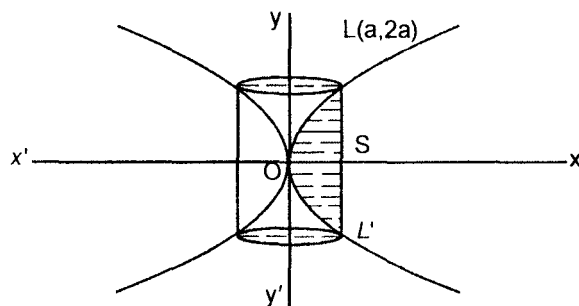


Fig. 5.3.4

Volume of the reel generated

$$\begin{aligned}
 &= 2 \int_0^{2a} \pi x^2 dy \\
 &= 2 \int_0^{2a} \pi \left(\frac{y^2}{4a} \right) dy \\
 &= \frac{\pi}{8a^2} \left[\frac{y^3}{3} \right]_0^{2a} \\
 &= \frac{4\pi a^3}{5} \text{ cubic units}
 \end{aligned}$$

5.3.5 Example:

Find the volume of a sphere of radius 'a'.

Solution:

Equation of the circle of radius 'a' is $x^2 + y^2 = a^2$.

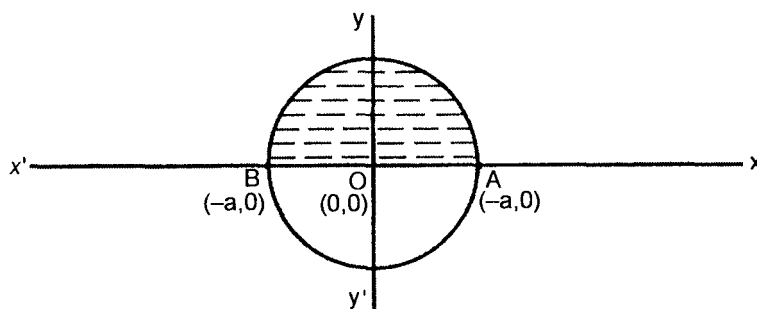


Fig 5.3.5

By revolving this circle about x-axis we get a sphere of radius 'a'.

$$\begin{aligned}
 \text{Volume of the sphere} &= \int_{-a}^a \pi y^2 dx \\
 &= \pi \int_{-a}^a (a^2 - x^2) dx \\
 &= \pi \left[a^2 x - \frac{x^3}{3} \right]_{-a}^a \\
 &= \frac{4}{3} \pi a^3 \text{ cubic units.}
 \end{aligned}$$

5.3.6 Example:

Find the volume of the solid generated by the revolution of the curve $x = a \cos^3 \theta$, $y = b \sin^3 \theta$ about x-axis.

Solution:

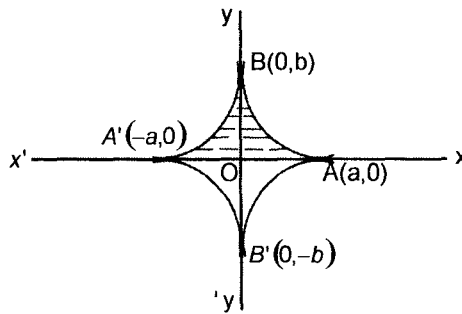


Fig. 5.3.6

The volume of the solid generated by the revolution of the curve $x = a \cos^3 \theta$,

$$\begin{aligned}
 y = b \sin^3 \theta \left(\text{i.e., } \frac{x^{2/3}}{a^{2/3}} + \frac{y^{2/3}}{b^{2/3}} = 1 \right) \text{ is} \\
 &= 2 \int_{\pi/2}^0 \pi y^2 \frac{dx}{d\theta} d\theta \\
 &= 2 \int_{\pi/2}^0 \pi (b \sin^3 \theta)^2 (-3a \cos^2 \theta \sin \theta) d\theta \\
 &= 2 \int_0^{\pi/2} 3\pi ab^2 \sin^7 \theta \cos^2 \theta d\theta \\
 &= 6\pi ab^2 \left[\frac{1}{9} \times \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \right] \\
 &= \frac{32\pi ab^2}{105} \text{ cubic units.}
 \end{aligned}$$

5.3.7 Example:

Find the volume of the solid formed by the revolution of $x = a(\theta - \sin\theta)$,
 $y = a(1 - \cos\theta)$ about its base.

Solution:

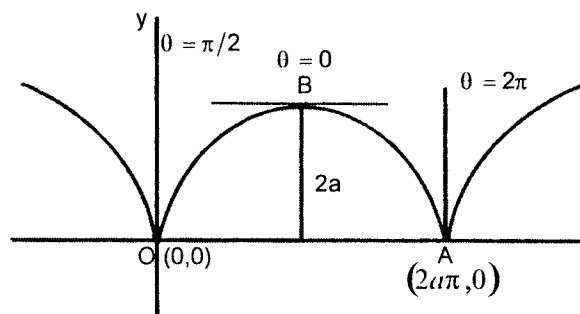


Fig. 5.3.7

Volume obtained by the revolution of the area OABO about the base (x-axis).

$$\begin{aligned}
 &= \int_{x=0}^{x=2a\pi} \pi y^2 dx \\
 &= \pi \int_{\theta=0}^{\theta=2\pi} y^2 \frac{dx}{d\theta} d\theta \\
 &= \pi a^3 \int_{\theta=0}^{\theta=2\pi} (1 - \cos\theta)^3 d\theta \\
 &= \pi a^3 \int_{\theta=0}^{\theta=2\pi} \left[2\sin^2 \theta/2 \right]^3 d\theta \\
 &= \pi a^3 \cdot 8 \int_{\theta=0}^{2\pi} 2\sin^6 \theta/2 d\theta \\
 &= 16\pi a^3 \int_{t=0}^{\pi} \sin^6 t dt \quad (\theta/2 = t \Rightarrow d\theta = 2dt) \\
 &= 16\pi a^3 \cdot 2 \int_0^{\pi/2} \sin^6 t dt = 32a^3 \pi \left(\frac{6-1}{6} \frac{6-3}{6-2} \frac{6-5}{6-4} \cdot \pi/2 \right) \\
 &= 5\pi^2 a^3 \text{ cubic units.}
 \end{aligned}$$

5.3.8 Example:

Find the volume generated by revolving the curve $r = a(1 - \cos\phi)$ about the initial line.

Solution:

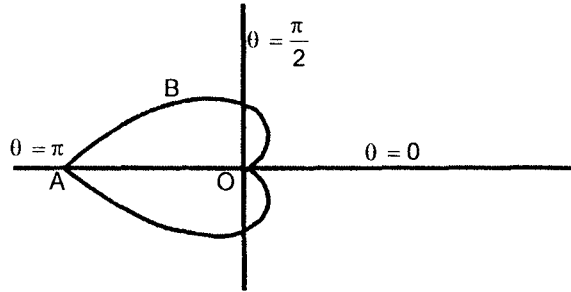


Fig. 5.3.8

The volume generated by the curve

$$\begin{aligned}
 &= \frac{2\pi}{3} \int_0^\pi r^3 \sin \theta \, d\theta \\
 &= \frac{2\pi}{3} \int_0^\pi a^3 (1 - \cos \theta)^3 \sin \theta \, d\theta \\
 &= \frac{2\pi a^3}{3} \int_0^\pi (1 - \cos \theta)^3 \sin \theta \, d\theta \quad \left(\begin{array}{l} 1 - \cos \theta = t \\ \sin \theta \, d\theta = dt \\ \text{and } \theta : 0 \rightarrow 2 \end{array} \right) \\
 &= \frac{2\pi a^3}{3} \int_0^2 t^3 \, dt = \frac{2\pi a^3}{3} \frac{t^4}{4} \Big|_0^2 \\
 &= \frac{8\pi a^3}{3} \text{ cubic units.}
 \end{aligned}$$

5.3.9 Example:

Find the volume of the solid formed by the revolution of the curve $r = a + b \cos \theta$ ($a < b$) about the initial line.

Solution:

$$\begin{aligned}
 \text{Volume} &= \frac{2\pi}{3} \int_0^\pi r^3 \sin \theta \, d\theta \\
 &= \frac{2\pi}{3} \int_0^\pi (a + b \cos \theta)^3 \sin \theta \, d\theta \\
 &\quad \left(a + b \cos \theta = t \Rightarrow \sin \theta \, d\theta = -\frac{dt}{b}, t : a + b \rightarrow a - b \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{2\pi}{3} \int_{a+b}^{a-b} t^3 \left(-\frac{dt}{b} \right) \\
 &= \frac{-2\pi}{3b} \int_{a+b}^{a-b} t^3 dt \\
 &= -\frac{2\pi}{3b} \left[\frac{t^4}{4} \right]_{a+b}^{a-b} \\
 &= \frac{\pi}{6b} \left[(a+b)^4 - (a-b)^4 \right] \\
 &= \frac{4\pi a}{3} (a^2 + b^2) \text{ cubic units.}
 \end{aligned}$$

5.3.10 Example:

Find the volume of the solid generated by revolving the lemniscate $r^2 = a^2 \cos 2\theta$

about the line $\theta = \frac{\pi}{2}$

Solution:

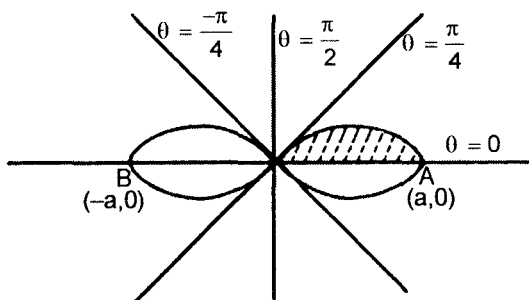


Fig. 5.3.10

The curve is symmetric about $\theta = \pi/2$

$$\begin{aligned}
 \text{Volume} &= 2 \frac{2}{3} \pi \int_0^{\pi/4} r^3 \cos \theta d\theta \\
 &= \frac{4\pi}{3} \int_0^{\pi/4} (a^2 \cos 2\theta)^{3/2} \cos \theta d\theta \\
 &= \frac{4}{3} \pi a^3 \int_0^{\pi/4} (\cos 2\theta)^{3/2} \cos \theta d\theta
 \end{aligned}$$

$$\begin{aligned}
&= \frac{4\pi a^3}{3} \int_0^\pi (1 - 2\sin^2 \theta)^{3/2} \cos \theta \, d\theta \\
&= \frac{2\sqrt{2}\pi a^3}{3} \int_0^{\pi/2} (1 - \sin^2 t)^{3/2} \cos t \, dt \\
&\quad \text{(taking } \sqrt{2} \sin \theta = \sin t \Rightarrow \sqrt{2} \cos \theta d\theta = \cos t dt) \\
&= \frac{2\sqrt{2}\pi a^3}{3} \int_0^{\frac{\pi}{2}} \cos^4 t \, dt \\
&= \frac{2\sqrt{2}\pi a^3}{3} \left[\frac{4-1}{4} \frac{4-3}{4-2} \frac{\pi}{2} \right] \\
&= \frac{\sqrt{2}\pi^2 a^3}{8} \text{ cubic units.}
\end{aligned}$$

Exercise – 5(C)

- Find the volume of the solid generated by revolving the clips $\frac{x^2}{a^2} \times \frac{y^2}{b^2} = 1$
 - About the major axis

[Ans: $\frac{4}{3}\pi ab^2$]
 - About the minor axis

[Ans: $\frac{4}{3}\pi a^2 b$]
- The curve $y^2(a+x) = x^2(3a-x)$ revolves about the x-axis. Find the volume generated by the curve.

[Ans: $\pi a^3(8\log 2 - 3)$]
- Show that the volume of the solid generated by the revolution of the curve $(a-x)y^2 = a^2x$, about its asymptote is $\frac{\pi^2 a^3}{2}$.
- Find the volume of the solid generated by the revolution of the curve $y(a^2 + x^2) = a^3$ about the asymptote

[Ans: $\frac{\pi^2 a^3}{2}$]

5. Find the volume of the solid obtained by revolving the loop of the curve $a^2y^2 = x^2(2a - x)(2a + x)(x - a)$ about x-axis.

$$[\text{Ans: } \frac{123\pi a^3}{60}]$$

6. Find the volume generated by the portion of the arc $y = \sqrt{1+x^2}$ lying between $x = 0$ and $x = 4$ as it revolves about the x-axis.

$$[\text{Ans: } 76\pi/3]$$

7. Show that the volume of the solid generated by the revolution of the cycloid

$$x = a(\theta + \sin\theta), y = a(1 - \cos\theta) \text{ about the y-axis is } \frac{3}{2}\pi^3 a^3 - \frac{8\pi^3}{3}.$$

8. Find the volume of the solid generated by revolving the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ about its base.

$$[\text{Ans: } 5\pi^2 a^3]$$

9. Find the volume of the solid generated by revolving $r^2 = a^2 \cos 2\theta$ about initial line.

$$[\text{Ans: } \frac{\pi a^3 \sqrt{2}}{12} [3 \log(\sqrt{2} + 1) - \sqrt{2}]]$$

10. Find the volume of the solid generated by revolving $r = a \cos 3\theta$ between $\theta = -\pi/6$ and $\theta = \pi/6$ about the initial line.

$$[\text{Ans: } \frac{19\pi a^3}{96\theta}]$$

5.4.1 Area of the Surface of revolution

- (a) Surface area of the solid generated by the revolution about the x-axis of the area bounded by the curves $y = f(x)$, the x-axis the ordinates $x = a$, $x = b$ is

$$\int_{x=a}^b 2\pi y ds = \int_{x=a}^b 2\pi y \frac{ds}{dx} dx \quad \left(\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \right)$$

- (b) Surface area of the solid generated by the revolution about y-axis of the area bounded by the curve $y = f(x)$, the y-axis and the abscissae $y = c$, $y = d$ is

$$\int_{y=c}^d 2\pi x ds = \int_{y=c}^d 2\pi x \frac{ds}{dy} dy \quad \left(\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy} \right)^2} \right)$$

(c) The parametric form is

$$\int_{t=t_1}^{t=t_2} 2\pi y \frac{ds}{dt} dt \qquad \left(\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \right)$$

(d) The polar form is

$$\int_{\theta_1}^{\theta_2} 2\pi r \sin\theta \left(\frac{ds}{d\theta} \right) d\theta \qquad \left(\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \right)$$

5.4.2 Example:

Find the surface area of solid generated by revolving the curve $x^{2/3} + y^{2/3} = a^{2/3}$ about x-axis.

Solution:

The parametric form of the curve is $x = a \cos^3 \theta$, $y = a \sin^3 \theta$.

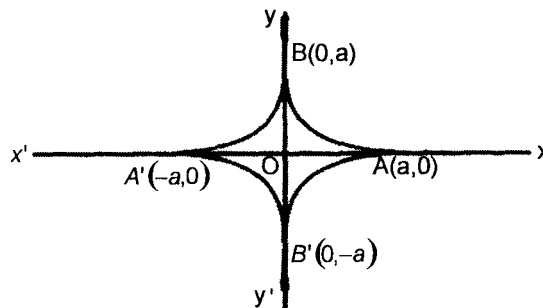


Fig. 5.4.2

The surface area of the solid due to revolution about x-axis

= 2(surface area of the solid generated by revolving an arc in the first quadrant of the astroid is)

$$= 2 \int_0^{\pi/2} 2\pi y \frac{ds}{d\theta} d\theta$$

$$= 4\pi \int_0^{\pi/2} y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

$$= 4\pi \int_0^{\pi/2} a \sin^3 \theta \sqrt{(3a \cos^2 \theta \sin \theta)^2 + (3a \sin^2 \theta \cos \theta)^2} d\theta$$

$$\begin{aligned}
 &= 12\pi a^2 \int_0^{\pi/2} \sin^4 \theta \cos \theta \, d\theta \\
 &= 12\pi a^2 \left[\frac{\sin^5 \theta}{5} \right]_0^{\pi/2} = \frac{12\pi a^2}{5} \text{ Sq. units.}
 \end{aligned}$$

5.4.3 Example:

Find the area of the surface formed by the revolution of the ellipse $x^2 + 4y^2 = 16$ about its:

- (a) major axis (b) minor axis

Solution:

Equation of the ellipse is $\frac{x^2}{16} + \frac{y^2}{4} = 1$

$$\frac{dy}{dx} = \frac{x}{4y}$$

$$\begin{aligned}
 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \sqrt{1 + \left(\frac{x}{4y}\right)^2} \\
 &= \sqrt{\frac{16y^2 + x^2}{16y^2}}
 \end{aligned}$$

- (a) Surface area formed by the revolution about major axis

$$\begin{aligned}
 &= 2 \int_0^4 2\pi y \frac{ds}{dx} dx \\
 &= 4\pi \int_0^4 y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
 &= 4\pi \int_0^4 y \sqrt{1 + \left(\frac{x}{4y}\right)^2} dx \\
 &= 4\pi \int_0^4 \frac{y \sqrt{16y^2 + x^2}}{4y} dx \\
 &= \pi \int_0^4 \sqrt{16\left(\frac{16 - x^2}{4}\right) + x^2} dx
 \end{aligned}$$

$$\begin{aligned}
&= \pi \int_0^4 \sqrt{64 - 3x^2} dx \\
&= \pi\sqrt{3} \int_0^4 \sqrt{\left(\frac{8}{\sqrt{3}}\right)^2 + x^2} dx \\
&= \pi\sqrt{3} \left[\frac{x}{2} \left(\frac{8}{\sqrt{3}}\right)^2 - x^2 + \frac{64}{2.3} \sin^{-1} \left(\frac{x}{\frac{8}{\sqrt{3}}} \right) \right]_0^4 \\
&= 8\pi \left[1 + \frac{4\pi\sqrt{3}}{9} \right] \text{ Sq. units.}
\end{aligned}$$

(b) Surface area formed by the revolution about minor axis

$$\begin{aligned}
&= 2 \int_0^2 2\pi x \frac{ds}{dy} dy = 4\pi \int_0^2 x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \\
&= 4\pi \int_0^2 x \sqrt{1 + \frac{16y^2}{x^2}} dy \\
&= 4\pi \int_0^2 \sqrt{x^2 + 16y^2} dy \\
&= 4\pi \int_0^2 \sqrt{16 - 4y^2 + 16y^2} dy \\
&= 8\sqrt{3}\pi \int_0^2 \sqrt{\left(\frac{2}{\sqrt{3}}\right)^2 + y^2} dy \\
&= 8\sqrt{3}\pi \left[\frac{y}{2} \sqrt{\left(\frac{2}{\sqrt{3}}\right)^2 + y^2} + \frac{\left(\frac{2}{\sqrt{3}}\right)^2}{2} \log \left\{ y + \sqrt{\left(\frac{2}{\sqrt{3}}\right)^2 + y^2} \right\} \right]_0^2 \\
&= 32\pi \left[1 + \frac{1}{6} \sqrt{3} \log(\sqrt{3} + 2) \right] \text{ Sq. units.}
\end{aligned}$$

5.4.4 Example:

Find the surface area formed by revolving cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ about the tangent at the vertex.

Solution:

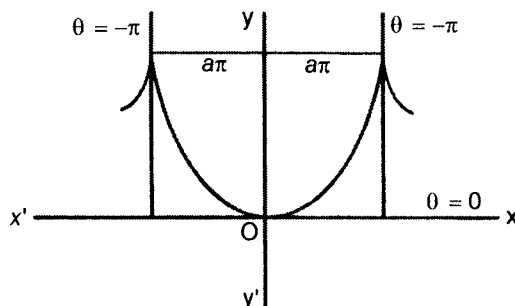


Fig. 5.4.4

Surface area required

= 2(surface area generated by revolving the arc in the first quadrant about OX)

$$\begin{aligned}
 &= 2 \int_0^\pi 2\pi y \frac{ds}{d\theta} d\theta \\
 &= 4 \int_0^\pi \pi y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \\
 &= 4\pi \int_0^\pi a(1 - \cos\theta) \sqrt{[a(1 + \cos\theta)]^2 + a^2 \sin^2\theta} d\theta \\
 &= 16\pi a^2 \cdot \int_0^\pi \sin^2 \frac{\theta}{2} \cos \frac{\theta}{2} d\theta \\
 &\quad \left(\frac{\theta}{2} = t \Rightarrow d\theta = 2dt \right) \\
 &= 32\pi a^2 \int_0^{\pi/2} \sin^2 t \cos t dt \\
 &= \frac{32\pi a^2}{3} \text{ Sq. units.}
 \end{aligned}$$

5.4.5 Example:

Find the area of the surface of reel thus generated by revolution of the part of the parabola $y^2 = 4ax$ bounded by the latus rectum about the tangent at the vertex.

Solution:

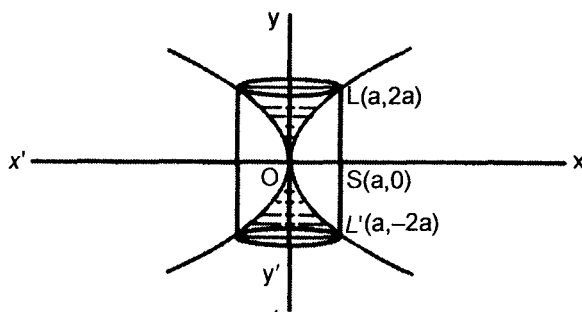


Fig. 5.4.5

$$y^2 = 4ax$$

$$\frac{dy}{dx} = \frac{2a}{y}$$

$$\text{Surface area} = 2 \int_0^a 2\pi x \frac{ds}{dx} dx$$

$$= 2 \int_0^a 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= 4\pi \int_0^a x \sqrt{1 + \left(\frac{2a}{y}\right)^2} dx$$

$$= 4\pi \int_0^a x \sqrt{\frac{y^2 + 4a^2}{y^2}} dx$$

$$= 4\pi \int_0^a x \sqrt{x^2 + ax} dx$$

$$= 4\pi \int_0^a \left[\left(x + \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 \right]^{\frac{1}{2}} dx$$

$$= 4\pi \left\{ \frac{\left(x + \frac{1}{2}\right)}{2} \sqrt{\left(x + \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2} - \frac{\left(\frac{a}{2}\right)^2}{2} \log \left[\left(x + \frac{a}{2}\right)^2 + \sqrt{\left(x + \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2} \right] \right\}_0^a$$

$$= \pi a^2 [3\sqrt{2} - \log(\sqrt{2} + 1)] \text{ Sq. units.}$$

5.4.6 Example:

Find the area of the surface of revolution formed by revolving the curve $r = 2a \cos \theta$ about the initial line.

Solution:

Equation of the circle is

$$r = 2a \cos \theta$$

$$\frac{dr}{d\theta} = -2a \sin \theta$$

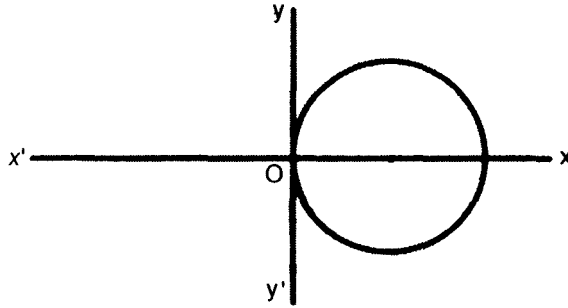


Fig. 5.4.6

$$\text{The surface area} = \int_0^{\pi/2} 2\pi y \frac{ds}{d\theta} d\theta$$

$$= \int_0^{\pi/2} 2\pi r \sin \theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

$$= \int_0^{\pi/2} 2\pi r \sin \theta \sqrt{4a^2 \cos^2 \theta + 4a^2 \sin^2 \theta} d\theta$$

$$= 8\pi a^2 \int_0^{\pi/2} \sin \theta \cos \theta d\theta$$

$$= 4\pi a^2 \text{ Sq. units}$$

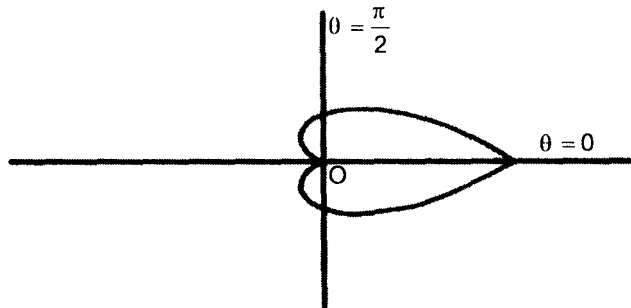
5.4.7 Example:

Find the surface area of the solid formed by the revolution of the cardioid $r = a(1 + \cos\theta)$ about the initial line.

Solution:

Equation of the curve $r = a(1 + \cos\theta)$

$$\frac{dr}{d\theta} = -a \sin \theta$$

**Fig. 5.4.7**

$$\begin{aligned}
 \text{The surface area} &= \int_0^\pi 2\pi y \frac{ds}{d\theta} d\theta \\
 &= 2\pi \int_0^\pi r \sin \theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \\
 &= 2\pi \int_0^\pi r \sin \theta \sqrt{a^2(1 + \cos \theta)^2 + a^2 \sin^2 \theta} d\theta \\
 &= 2\pi \int_0^\pi a(1 + \cos \theta) \sin \theta \cdot \sqrt{a^2(1 + \cos \theta)^2 + a^2 \sin^2 \theta} d\theta \\
 &= 16\pi a^2 \int_0^\pi \cos^3 \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta \\
 &= 16\pi a^2 \int_0^\pi \cos^4 \frac{\theta}{2} \sin \frac{\theta}{2} d\theta \\
 &= 16a^2 \pi \left[-\frac{2}{5} \cos^5 \frac{\theta}{2} \right]_0^\pi \\
 &= \frac{32\pi a^2}{5} \text{ Sq. units}
 \end{aligned}$$

5.4.8 Example:

Find the surface of the solid generated by revolving the lemniscate $r^2 = a^2 \cos 2\theta$ about the initial line.

Solution:

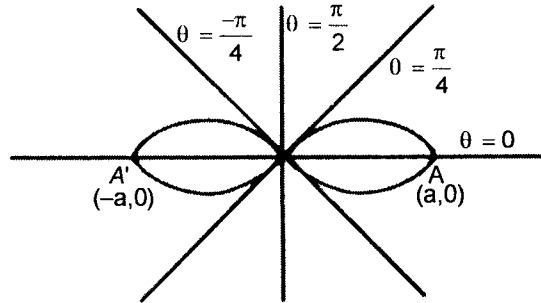


Fig. 5.4.8

Equation of the curve is $r^2 = a^2 \cos 2\theta$

$$\text{Surface area} = 2 \int_0^{\pi/4} 2\pi y \frac{ds}{d\theta} d\theta$$

$$\begin{aligned} & 4\pi \int_0^{\pi/4} r \sin \theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \\ &= 4\pi \int_0^{\pi/4} r \sin \theta \sqrt{r^2 + \frac{a^4}{r^2} \sin^2 2\theta} d\theta \\ &= 4\pi \int_0^{\pi/4} \sin \theta \sqrt{r^4 + a^2 \sin^2 2\theta} d\theta \\ &= 4\pi \int_0^{\pi/4} \sin \theta \sqrt{a^4 \cos^2 2\theta + a^4 \sin^2 2\theta} d\theta \\ &= 4\pi a^2 \left[-\cos \theta \right]_0^{\pi/4} \\ &= 4\pi a^2 \left[1 - \frac{1}{\sqrt{2}} \right] = \frac{4\pi a^2}{\sqrt{2}} (\sqrt{2} - 1) \text{ Sq. units} \end{aligned}$$

Exercise – 5(D)

1. Find the total area of the surface obtained by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its major axis.

$$[\text{Ans: } 2\pi ab \left[\sqrt{1-e^2} + \frac{1}{e} \sin^{-1}(e) \right]]$$

2. Find the surface of a sphere of radius 'a'.

$$[\text{Ans: } 4\pi a^2]$$

3. Find the surface of the solid generated by revolving the arc of the parabola $y^2 = 4ax$ bounded by its latus rectum about x-axis.

$$[\text{Ans: } \frac{8}{3}\pi a^2(2\sqrt{2}-1)]$$

4. Find the area of the surface generated by the revolution of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ about x-axis.

$$[\text{Ans: } \frac{64}{3}\pi a^2]$$

5. Find the area of the surface of the solid formed by the revolution of the cardioid $r = a(1 - \cos \theta)$ about the initial line.

$$[\text{Ans: } \frac{32}{5}\pi a^2]$$

6. The lemniscate $r^2 = a^2 \cos 2\theta$ revolves about a tangent at the pole. Show that the surface area generated is $4\pi a^2$.

5.5.1 Double and triple integrals:

Let $f(x, y)$ be a continuous and single valued function of x and y within a region R bounded by a closed curve 'c' and upon the boundary c . Let the region R be subdivided in any manner into n subregions of areas $\delta R_1, \delta R_2, \dots, \delta R_n$.

Let (x_i, y_i) be any point in the subregion of area δR_i .

Consider the sum

$$\sum_{i=1}^n f(x_i, y_i) \delta R_i$$

The limit of this sum as $n \rightarrow \infty$ ($i = 1, 2, \dots$) is defined as the double integral of $f(x, y)$ over the region R and is written as

$$\iint_R f(x, y) dA$$

$$\therefore \iint_R f(x, y) dx dy = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i, y_i) \delta R_i$$

- (a) Suppose the region R is described by the inequalities
 $c \leq y \leq d$ and $g(y) \leq x \leq h(y)$

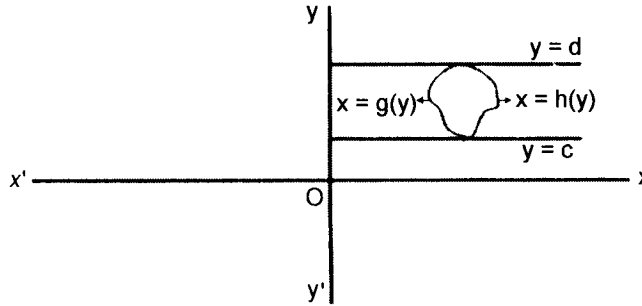


Fig. 5.5.1 (a)

Then

$$\iint_R f(x, y) dy dx = \iint_R f(x, y) dx dy = \int_{y=c}^{y=d} \int_{x=g(y)}^{x=h(y)} f(x, y) dx dy$$

- (b) If the region R is described by inequalities
 $a \leq x \leq b$ and $g_1(x) \leq y \leq h_1(x)$

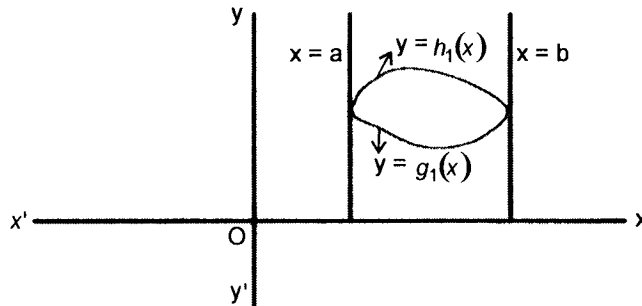


Fig. 5.5.1(b)

Then

$$\iint_R f(x, y) dx dy = \iint_R f(x, y) dy dx = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=h_1(x)} f(x, y) dy dx$$

- (c) If the region R is bounded by the lines $x = a$, $x = b$, $y = c$, $y = d$ (rectangle)

Then

$$\iint_R f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

Note: The order of integration is immaterial for constant limits

5.5.2 Example:

Evaluate $\int_0^1 \int_1^2 (x^2 + y^2) dx dy$

Solution:

$$\begin{aligned}
 \int_0^1 \int_1^2 (x^2 + y^2) dx dy &= \int_0^1 \left[\frac{x^3}{3} + y^2 x \right]_1^2 dy \\
 &= \int_0^1 \left(\frac{8}{3} + 2y^2 - \frac{1}{3} - y^2 \right) dy \\
 &= \int_0^1 \left(y^2 + \frac{7}{3} \right) dy = \frac{y^3}{3} + \frac{7}{3} y \Big|_0^1 \\
 &= \frac{1}{3} + \frac{7}{3} = \frac{8}{3}
 \end{aligned}$$

5.5.3 Example:

Evaluate $\int_1^4 \int_0^{\sqrt{4-x}} xy dy dx$

Solution:

$$\begin{aligned}
 \int_1^4 \int_0^{\sqrt{4-x}} xy dy dx &= \int_{x=1}^4 \int_0^{y=\sqrt{4-x}} [xy dy] dx \\
 &= \int_{x=1}^4 \int_0^{y=\sqrt{4-x}} xy dy . dx \\
 &= \int_{x=1}^4 \left[x \frac{y^2}{2} \right]_{y=0}^{y=\sqrt{4-x}} dx \\
 &= \int_{x=1}^4 \frac{x}{2} (4-x) dx = \frac{4x^2}{4} - \frac{x^3}{6} \Big|_1^4 = \frac{9}{2}
 \end{aligned}$$

5.5.4 Example:

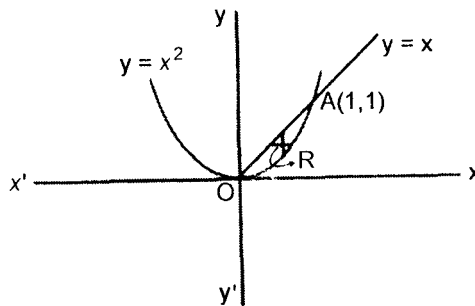
Evaluate
$$\int_0^1 \int_{\sqrt{y}}^{2-y} x^2 dx dy$$

Solution:

$$\begin{aligned} \int_{y=0}^{y=1} \int_{x=\sqrt{y}}^{x=2-y} [x^2] dx dy &= \int_{y=0}^1 \left[\frac{x^3}{3} \right]_{x=\sqrt{y}}^{x=2-y} dy \\ &= \int_{y=0}^1 \frac{1}{3} (2-y)^3 - \frac{1}{3} (\sqrt{y})^3 dy \\ &= \frac{1}{3} \int_{y=0}^1 8 - y^3 - 12y + 6y^2 - y^{\frac{3}{2}} dy \\ &= \frac{1}{3} \left[8y - \frac{y^4}{4} - \frac{12y^2}{2} + \frac{6y^3}{3} - \frac{y^{\frac{3}{2}+1}}{\frac{3}{2}+1} \right]_0^1 \\ &= \frac{1}{3} \left[8 - \frac{1}{4} - 6 + 2 - \frac{2}{5} \right] \\ &= \frac{67}{60} \end{aligned}$$

5.5.5 Example:

Find the value of $\iint_R xy(x+y) dx dy$ taken over the region enclosed by the curves $y = x$ and $y = x^2$.

Solution:**Fig. 5.5.5**

R is the region bounded by the curves $y = x$ and $y = x^2$

$$\begin{aligned}
 I &= \iint_R xy(x+y) dx dy \\
 &= \int_{x=0}^1 \left[\int_{y=x^2}^{y=x} (x^2 y + xy^2) dy \right] dx \\
 &= \int_{x=0}^1 \left[\frac{x^2 y^2}{2} + \frac{xy^3}{3} \right]_{y=x^2}^{y=x} dx \\
 &= \int_{x=0}^1 \frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3} dx \\
 &= \frac{x^5}{10} + \frac{x^5}{15} - \frac{x^7}{14} - \frac{x^8}{24} \Big|_0^1 \\
 &= \frac{1}{10} + \frac{1}{15} - \frac{1}{14} - \frac{1}{24} = \frac{3}{56}
 \end{aligned}$$

5.5.6 Example:

Evaluate $\iint_A \frac{xy}{\sqrt{1-y^2}} dx dy$ where A is the area in the positive quadrant of the circle $x^2 + y^2 = 1$.

Solution:

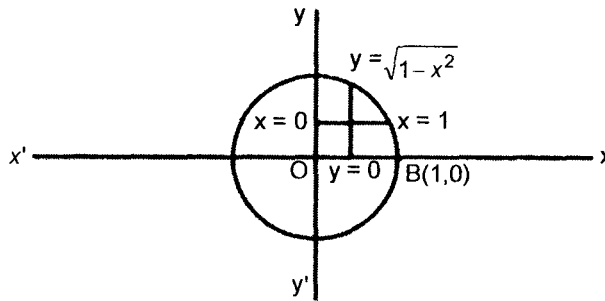


Fig. 5.5.6

$$I = \iint_A \frac{xy}{\sqrt{1-y^2}} dx dy$$

$$\begin{aligned}
 I &= \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} \frac{xy dx dy}{\sqrt{1-y^2}} \\
 I &= \int_{x=0}^1 x \left[\int_{y=0}^{\sqrt{1-x^2}} \frac{(1-y^2)^{-\frac{1}{2}} (-2y) dy}{-2} \right] dx \\
 &= -\frac{1}{2} \int_0^1 x \left[\frac{(1-y^2)^{-\frac{1}{2}}}{\frac{1}{2}} \right]_0^{\sqrt{1-x^2}} dx \\
 &= -\int_0^1 x \left\{ 1 - (1-x^2)^{\frac{1}{2}} \right\} dx \\
 &= -\int_0^1 (x^2 - x) dx \\
 &= \left[\frac{-x^3}{3} + \frac{x^2}{2} \right]_0^1 \\
 &= \frac{-1}{3} + \frac{1}{2} = \frac{1}{6}
 \end{aligned}$$

5.5.7 Example:

Evaluate $\iint e^{ax+by} dx dy$ over the triangle $x = 0, y = 0, ax + by = 1$

Solution:

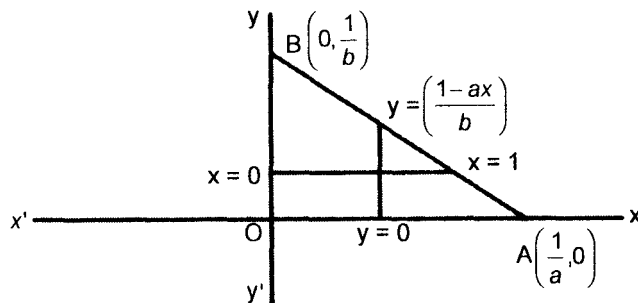


Fig. 5.5.7

$$I = \iint e^{ax+by} dx dy$$

$$I = \int_{x=0}^{1/a} \left[\int_{y=0}^{y=\frac{1-ax}{b}} e^{ax+by} dy \right] dx$$

$$= \int_{x=0}^{1/a} \left[\frac{e^{ax+by}}{b} \right]_0^{\frac{1-ax}{b}} dx$$

$$= \frac{1}{b} \int_0^{\frac{1}{a}} \left[e^{ax+b\left(\frac{1-ax}{b}\right)} - e^{ax+0} \right] dx$$

$$= \frac{1}{b} \int_0^{\frac{1}{a}} (e^1 - e^{ax}) dx$$

$$= \frac{1}{b} \left[ex - \frac{e^{ax}}{a} \right]_0^{\frac{1}{a}}$$

$$= \frac{1}{b} \left[\frac{e}{a} - \frac{e}{a} - 0 + \frac{e^0}{a} \right]$$

$$= \frac{1}{ab}$$

5.5.8 Example:

Evaluate $\iint_R y dx dy$ where R is the region bounded by the parabolas $y^2 = 4x$ and

$$x^2 = 4y.$$

Solution:

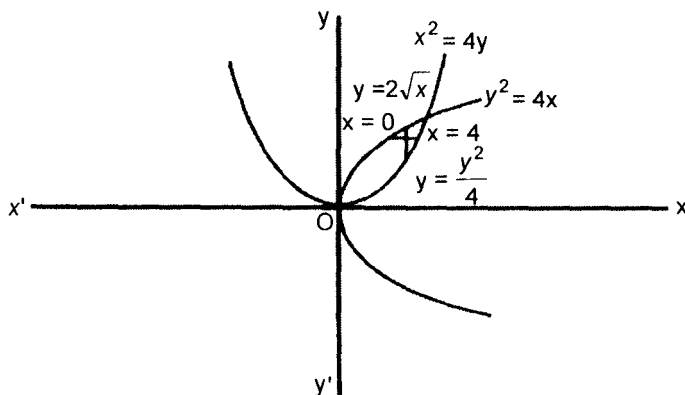


Fig. 5.5.8

Solving the curves $y^2 = 4x$, $x^2 = 4y$

$$\left(\frac{x^2}{4}\right)^2 = 4x$$

$$x = 0, x = 4$$

$\therefore$

$$(0,0), (4,4)$$

$$I = \iint_R y \, dx \, dy$$

$$I = \int_{x=0}^{x=4} \left[\int_{y=\frac{x^2}{4}}^{y=2\sqrt{x}} y \, dy \right] dx$$

$$I = \int_{x=0}^{x=4} \left[\frac{y^2}{2} \right]_{y=\frac{x^2}{4}}^{y=2\sqrt{x}} dx$$

$$I = \int_{x=0}^{x=4} \left(2x - \frac{x^4}{32} \right) dx$$

$$I = \left[x^2 - \frac{x^5}{160} \right]_0^4 = \frac{48}{5}$$

5.5.9 Example:

Evaluate $\iint_R (\sqrt{xy} - y^2) dy dx$ where R is a triangle with vertices at (0, 0), (10, 1), (1, 1).

Solution:

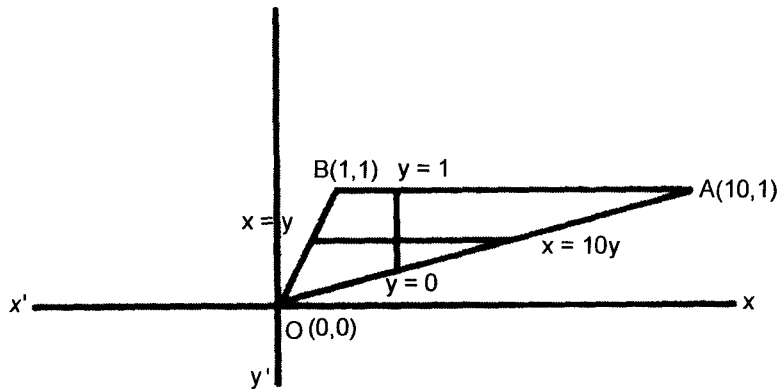


Fig. 5.5.9

The triangle OAB is within the limits $y \leq x \leq 10y$ and $0 \leq y \leq 1$

$$\begin{aligned}
 \iint_R (\sqrt{xy} - y^2) dy dx &= \int_{y=0}^1 \int_{x=y}^{x=10y} (\sqrt{xy} - y^2) dx dy \\
 &= \int_0^1 dy \left[\frac{2}{3} \frac{1}{y} (xy - y^2)^{3/2} \right]_{x=y}^{x=10y} \\
 &= \int_0^1 \frac{2}{3} \frac{1}{y} (9y^2)^{3/2} dy \\
 &= 18 \int_0^1 y^2 dy = 18 \left(\frac{1}{3} \right) = 6
 \end{aligned}$$

Exercise – 5(E)**I. Evaluate the following double integrals:**

$$1. \int_0^2 \int_0^2 (x^2 + y^2) dx dy$$

$$[\text{Ans: } \frac{32}{3}]$$

$$2. \int_{-1}^3 \int_{x^2}^{x+2} dy dx$$

$$[\text{Ans: } \frac{8}{3}]$$

$$3. \int_1^2 \int_3^4 \frac{dx dy}{(x+y)^2}$$

$$[\text{Ans: } \log \frac{25}{24}]$$

$$4. \int_0^a \int_0^{\sqrt{a^2-x^2}} y^3 dy dx$$

$$[\text{Ans: } \frac{25a^5}{15}]$$

$$5. \int_0^a \int_0^b (x^2 + y^2) dx dy$$

$$[\text{Ans: } \frac{ab}{3}(a^2 + b^2)]$$

$$6. \int_0^{2a} \int_0^{\sqrt{2x-x^2}} (x^2 + y^2) dx x$$

$$[\text{Ans: } \frac{3\pi a^4}{4}]$$

$$7. \int_0^1 \int_0^1 \frac{y^2 dx dy}{\sqrt{x^2 + y^2}}$$

$$[\text{Ans: } \frac{1}{3} \log(\sqrt{2} + 1)]$$

$$8. \int_0^a \int_{x^2}^{2a-x} xy dy dx$$

$$[\text{Ans: } \frac{3a^4}{8}]$$

$$9. \text{ Find the value of } \iint xy dx dy \text{ taken over the positive quadrant of the ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$[\text{Ans: } \frac{a^2 b^2}{8}]$$

$$10. \int_0^a \int_0^{\sqrt{a^2 - y^2}} \sqrt{a^2 - x^2 - y^2} dx dy$$

$$[\text{Ans: } \frac{\pi a^3}{6}]$$

$$11. \int_{x=0}^{x=1} \int_{y=0}^{y=x} e^x dx dy$$

$$[\text{Ans: } \frac{e-1}{2}]$$

$$12. \int_{x=0}^1 \int_{y=0}^{\sqrt{1+x^2}} \frac{dx dy}{1+x^2+y^2}$$

$$[\text{Ans: } \frac{\pi}{4} \log(\sqrt{2} + 1)]$$

$$13. \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin(x+y) dx dy$$

$$[\text{Ans: } 1]$$

14. Evaluate $\iint_R xy \, dx \, dy$ where R is the quadrant of the circle $x^2 + y^2 = a^2$ when $x \geq 0, y \geq 0$.

[Ans: $\frac{a^4}{8}$]

15. Evaluate $\iint_R xy \, dx \, dy$ whose R is the region bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.

[Ans: $\frac{a^4}{3}$]

16. Evaluate $\iint_R (x + y) \, dx \, dy$ whose R is the triangular region bounded by $y = 2x, y = \frac{x}{2}$ and $y = 3 - x$.

17. Evaluate $\iint_R xy \, dx \, dy$ where R is the triangular region with vertices A(-6, 2), B(-1, 3) and C(9, -7).

[Ans: $\frac{-1025}{27}$]

18. Evaluate $\iint_R xy \, dx \, dy$ over the region in the positive quadrant for which $x + y \leq 1$

[Ans: $\frac{1}{24}$]

5.5.10 Change of variables:

The variables x, y in $\iint_R f(x, y) \, dx \, dy$ are changed to u, v with the help of the relations

$x = f_1(u, v), y = f_2(u, v)$ then the double integral is transferred into

$$\iint_{R'} \{f_1(u, v) f_2(u, v)\} |J| \, du \, dv$$

where $J = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix}$ and 'R' is the region in the uv plane corresponding to the region

R in the xy plane.

5.5.11 Changing from Cartesian to Polar co-ordinates:

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$J = \begin{vmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} \\ \frac{dy}{dr} & \frac{dy}{d\theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r$$

$$\begin{aligned} \iint_R f(x, y) dx dy &= \iint_{R'} f[r \cos \theta, r \sin \theta] |J| d\theta dr \\ &= \iint_{R'} f[r \cos \theta, r \sin \theta] r d\theta dr \end{aligned}$$

Note: In polar form $dx dy$ is replaced by $r d\theta dr$

5.5.12 Example:

Evaluate $\iint \sqrt{a^2 - x^2 - y^2} dx dy$ over the semi circle $x^2 + y^2 = ax$ in the positive quadrant.

Solution:

Substituting $x = r \cos \theta$,

$$y = r \sin \theta, \text{ in } x^2 + y^2 = ax$$

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = ar \cos \theta \quad r = a \cos \theta$$

which is the equation of the given circle in polar co-ordinates.

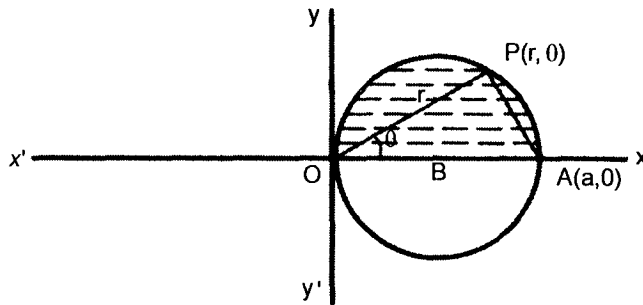


Fig. 5.5.12

From the figure it can be seen that

$$r = OP = a \cos \theta$$

For the upper half of the circle, θ varies from 0 to $\frac{\pi}{2}$ where as for any intermediary value of θ , r varies from 0 to OP i.e., 0 to $a \cos \theta$.

$$\therefore \iint_R \sqrt{a^2 - x^2 - y^2} \, dx \, dy = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a \cos \theta} \sqrt{a^2 - r^2 \cos^2 \theta - r^2 \sin^2 \theta} \, r \, dr \, d\theta$$

where R is the shaded region.

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \int_0^{a \cos \theta} r \sqrt{a^2 - r^2} \, dr \, d\theta &= \int_0^{\frac{\pi}{2}} \left[-\frac{1}{3} (a^2 - r^2)^{\frac{3}{2}} \right]_0^{a \cos \theta} d\theta \\ &= \frac{a^3}{3} \int_0^{\frac{\pi}{2}} (1 - \sin^2 \theta) d\theta \\ &= \frac{a^3}{3} \left\{ \left[\theta \right]_0^{\frac{\pi}{2}} - \frac{2}{3} \right\} \\ &= \frac{a^3}{3} \left(\frac{\pi}{2} - \frac{2}{3} \right) \end{aligned}$$

5.5.13 Example:

Evaluate $\int_0^a \int_0^{\sqrt{a^2 - y^2}} (x^2 + y^2) \, dx \, dy$ by changing into polar coordinates.

Solution:

From the given limit $x = \sqrt{a^2 - y^2}$ it is clear that one of the boundaries is the circle $x^2 + y^2 = a^2$, taking $x = r \cos \theta$, $y = r \sin \theta$. The given region is the first quadrant of the circle. Here r varies from '0' to 'a' and θ varies from '0' to $\frac{\pi}{2}$.

$$\begin{aligned} \int_0^a \int_0^{\sqrt{a^2 - y^2}} (x^2 + y^2) \, dx \, dy &= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^a r^2 \cdot r \, dr \, d\theta \\ &= \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4} \right]_0^a d\theta \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{4} a^4 \Big|_0^{\frac{\pi}{2}} d\theta \\
 &= \frac{a^4}{4} \times \frac{\pi}{2} \\
 &= \frac{\pi a^4}{8}
 \end{aligned}$$

5.5.14 Example:

Transform the integral $\int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy$ to polar coordinates and evaluate it.

Solution:

The limits of x and y are both from 0 to ∞ . Therefore the region is in the first quadrant where r varies from 0 to ∞ and θ varies from 0 to $\frac{\pi}{2}$.

Substituting $x = r \cos \theta$, $y = r \sin \theta$, and $dx dy = r dr d\theta$.

$$\begin{aligned}
 \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy &= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{\infty} e^{-r^2} r dr d\theta \\
 &= \int_0^{\frac{\pi}{2}} \left[\frac{e^{-r^2}}{-2} \right]_0^{\infty} d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta = \frac{1}{2} [\theta]_0^{\frac{\pi}{2}} \\
 &= \frac{\pi}{4}
 \end{aligned}$$

Changing of order of integration:

The double integration can be integrated with respect to y first and then with respect to x or it can be integrated with respect to x first and then with respect to y . In the former case, the limits of integration are determined for the given region by drawing strips parallel to y -axis while in the second case by drawing strips parallel to x -axis.

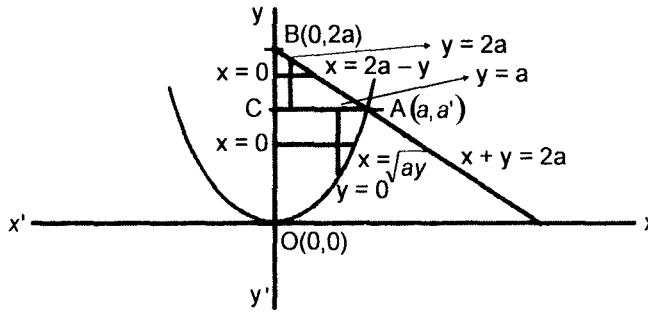
5.5.15 Example:

Evaluate by changing the order of integration $\int_0^a \int_{x^2}^{2a-x} xy dx dy$

Solution:

The given limits are $x: 0 \rightarrow a, y: \frac{x^2}{a} \rightarrow 2a - x$

i.e., the region is bounded by $x = 0, x = a$ and $y = \frac{x^2}{a}, y = 2a - x$ i.e., $x + y = 2a$.

**Fig. 5.5.15**

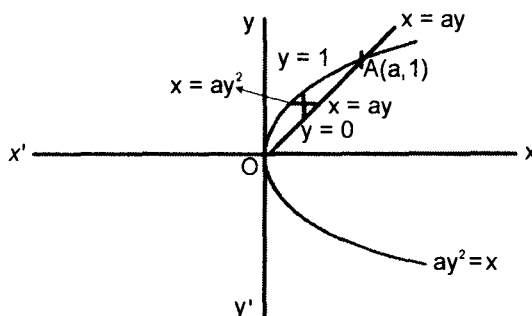
$$\begin{aligned}
 \int_0^a \int_{x^2/a}^{2a-x} xy \, dx \, dy &= \iint_{OAC'O} xy \, dx \, dy + \iint_{C'ABC'} xy \, dx \, dy \\
 &= \int_{y=0}^{y=a} \int_{x=0}^{x=\sqrt{ay}} xy \, dx \, dy + \int_{y=a}^{y=2a} \int_{x=0}^{x=2a-y} xy \, dx \, dy \\
 &= \int_0^a dy \left[\frac{x^2}{2} y \right]_0^{\sqrt{ay}} + \int_a^{2a} dy \left[\frac{x^2}{2} y \right]_0^{2a-y} \\
 &= \frac{a}{2} \int_0^a y^2 \, dy + \frac{1}{2} \int_a^{2a} y(2a-y)^2 \, dy \\
 &= \frac{3a^4}{8}
 \end{aligned}$$

5.5.16 Example:

Change the order of integration and evaluate $\int_0^a \int_{\frac{x}{a}}^{\sqrt{\frac{x}{a}}} (x^2 + y^2) \, dx \, dy$

Solution:

The given limits are $x = 0$, $x = a$ and $y = \frac{x}{a}$ and $y = \sqrt{x}$ i.e., $ay^2 = x$.

**Fig. 5.5.16**

$$\begin{aligned}
 \int_0^a \int_{\frac{x}{a}}^{\sqrt{x}} (x^2 + y^2) dx dy &= \int_{y=0}^{y=1} \left[\int_{x=ya^2}^{x=ay} (x^2 + y^2) dx \right] dy \\
 &= \int_{y=0}^1 dy \left[\frac{x^3}{3} + y^2 x \right]_{x=ya^2}^{x=ay} \\
 &= \int_0^1 \left(\frac{a^3}{3} y^3 - \frac{a^3}{3} y^6 + ay^3 - ay^4 \right) dy \\
 &= \frac{a^3}{28} + \frac{a}{20}
 \end{aligned}$$

5.5.17 Example:

Evaluate $\int_0^a \int_0^{2\sqrt{xa}} x^2 dx dy$ by changing the order of integration.

Solution:

The given limits are $x = 0$, $x = a$ and $y = 0$, $y = 2\sqrt{xa}$ i.e., $y^2 = 4ax$.

$$\int_0^a \int_0^{2\sqrt{xa}} x^2 dx dy = \int_{y=0}^{2a} \left[\int_{x=\frac{y^2}{4a}}^{x=a} x^2 dx \right] dy$$

$$= \int_0^{2a} \left[\frac{x^3}{3} \right]_{\frac{y^2}{4a}}^{a} dy$$

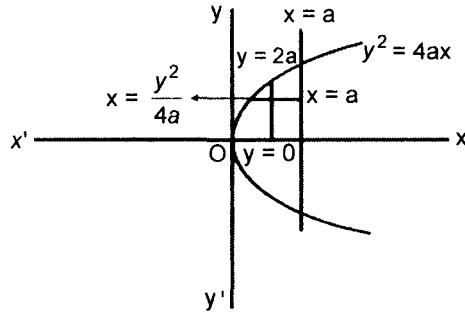


Fig. 5.5.17

$$= \frac{1}{3} \int_0^{2a} \left[a^3 - \left(\frac{y^2}{4a} \right)^3 \right] dy = \frac{1}{3} \left[a^3 y - \frac{1}{64a^3} \times \frac{y^7}{7} \right]_0^{2a}$$

$$= \frac{1}{3} \left[2a^2 - \frac{2a^4}{7} \right] = \frac{4a^4}{7}$$

5.5.18 Example:

Change the order of integration and evaluate $\int_0^4 \int_y^4 \frac{x}{x^2 + y^2} dx dy$.

Solution:

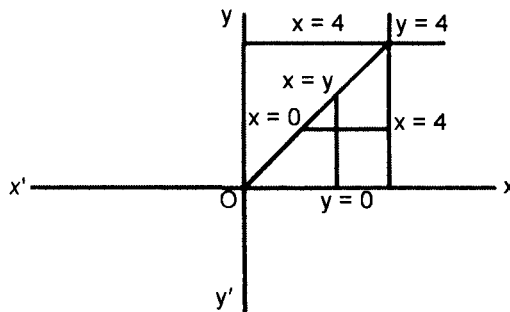


Fig. 5.5.18

The given limits are $y = 0$, $y = 4$ and $x = y$, $x = 4$.

$$\begin{aligned}\int_0^4 \int_0^4 \frac{xdxdy}{x^2 + y^2} &= \int_{x=0}^{x=4} \left[\int_{y=0}^{y=x} \frac{x}{x^2 + y^2} dy \right] dx \\ &= \int_0^4 \left[\tan^{-1} \left(\frac{y}{x} \right) \right]_0^x dx \\ &= \int_0^4 \tan^{-1} dx = \int_0^4 \frac{\pi}{4} dx = \frac{\pi}{4} [x]_0^4 = \pi\end{aligned}$$

5.5.19 Example:

Evaluate by changing the order of integration $\int_0^a \int_0^{\sqrt{a^2 - y^2}} xy dx dy$

Solution:

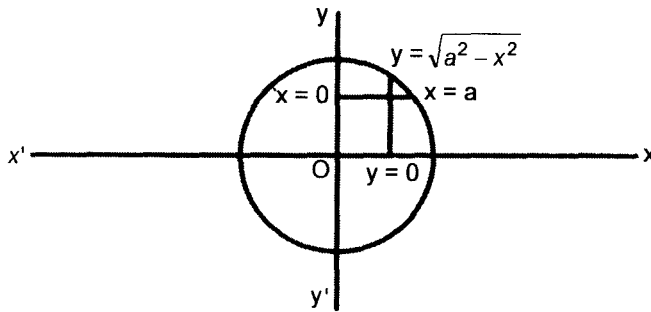


Fig. 5.5.19

The given limit are $y = 0$, $y = a$ and $x = 0$, $x = \sqrt{a^2 - y^2}$ i.e., $x^2 + y^2 = a^2$

$$\begin{aligned}\int_{y=0}^a \int_0^{\sqrt{a^2 - y^2}} xy dx dy &= \int_{x=0}^a \left[\int_{y=0}^{y=\sqrt{a^2 - x^2}} xy dy \right] dx \\ &= \int_0^a \left[x \frac{y^2}{2} \right]_0^{\sqrt{a^2 - x^2}} dx\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} \int_0^a x \left(\sqrt{a^2 - x^2} \right)^2 dx \\
 &= \frac{1}{2} \left[a^2 \frac{x^2}{2} - \frac{x^4}{4} \right]_0^a \\
 &= \frac{1}{2} \left[\frac{a^2}{2} - \frac{a^4}{4} \right] \\
 &= \frac{a^4}{8}
 \end{aligned}$$

Triple integration:

Let $u = f(x, y, z)$ be a single valued function of the independent variables x, y, z defined through the region V . Divide the region V into n subregions $\delta V_1, \delta V_2, \dots, \delta V_n$ and $P(x_i, y_i, z_i)$ be any point inside or on the boundary of the subregion δV_i , then the sum

under the limit $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i, y_i, z_i) \delta V_i$ is defined as triple integral.

$$\therefore \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i, y_i, z_i) \delta V_i = \iiint_V f(x, y, z) dv$$

5.5.20 Example:

Evaluate $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x + y + z) dx dy dz$

Solution:

$$\begin{aligned}
 I &= \int_{-1}^1 dz \int_0^z dx \int_{x-z}^{x+z} (x + y + z) dx dy dz \\
 &= \int_{-1}^1 dz \int_0^z dx \left[xy + \frac{y^2}{2} + zy \right]_{x-z}^{x+z} \\
 &= \int_{-1}^1 dz \int_0^z dx \left[x(x+z) - x(x-z) + \left(\frac{x+z}{2} \right)^2 \left(\frac{x-z}{2} \right)^2 + z(x+z) - z(x-z) \right] \\
 &= \int_{-1}^1 dz \int_0^z [2z(x+z) + 4xz] dx
 \end{aligned}$$

$$\begin{aligned}
 &= \int_{-1}^1 dz \left[2z \frac{x^2}{2} + 2z^2 x + \frac{4x^2}{2} z \right]_0^z \\
 &= \int_{-1}^1 z^3 + 2z^3 + 2z^3 dz \\
 &= 5 \int_{-1}^1 z^3 dz = 5 \left[\frac{z^4}{4} \right]_{-1}^1 = 0
 \end{aligned}$$

5.5.21 Example:

Evaluate $\int_0^2 \int_0^x \int_0^{x+y} e^{x+y+z} dx dy dz$

Solution:

$$\begin{aligned}
 \int_0^2 e^x \left\{ \int_0^x e^y \left(\int_0^{x+y} e^z dz \right) dy \right\} dx &= \int_0^2 e^x \left[\int_0^x e^y (e^z)_0^{x+y} dy \right] dx \\
 &= \int_0^2 e^x \left[\int_0^x e^y (e^{x+y} - 1) dy \right] dx \\
 &= \int_0^2 e^x \left[\int_0^x e^x \cdot \frac{e^{2y}}{2} - e^y \right] dx \\
 &= \int_0^2 e^x \left[\left\{ \frac{e^{3x}}{2} - e^x \right\} - \left\{ e^x \cdot \frac{1}{2} - 1 \right\} \right] dx \\
 &= \int_0^2 \frac{e^{4x}}{2} - \frac{3}{2} e^{2x} + e^x dx \\
 &= \left[\frac{e^{4x}}{8} - \frac{3}{4} e^{2x} + e^x \right]_0^2 \\
 &= \left[\frac{e^8}{8} - \frac{3}{4} e^2 + e \right] - \left[\frac{1}{8} - \frac{3}{4} + 1 \right] \\
 &= \left[\frac{e^8}{8} - \frac{3}{4} e^2 + e - \frac{3}{8} \right]
 \end{aligned}$$

5.5.22 Example:

Evaluate
$$\int_{y=0}^1 \left\{ \int_{x=y^2}^1 \left(\int_{z=0}^{1-x} x \, dz \right) dx \right\} dy$$

$$\begin{aligned}
 I &= \int_{y=0}^1 \int_{x=y^2}^1 \int_{z=0}^{1-x} x \, dz \, dx \, dy \\
 &= \int_{y=0}^1 \left[\int_{x=y^2}^1 x(1-x) \, dx \right] dy \\
 &= \int_{y=0}^1 \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{x=y^2}^{x=1} dy \\
 &= \int_{y=0}^1 \left(\frac{1}{6} - \frac{y^4}{2} + \frac{y^6}{3} \right) dy \\
 &= \left[\frac{y}{6} - \frac{y^5}{10} + \frac{y^7}{21} \right]_0^1 = \frac{4}{35}
 \end{aligned}$$

5.5.23 Example:

Evaluate
$$\int_{z=0}^4 \left[\int_{x=0}^{2\sqrt{z}} \left(\int_{y=0}^{\sqrt{4z-x^2}} dy \right) dx \right] dz$$

Solution:

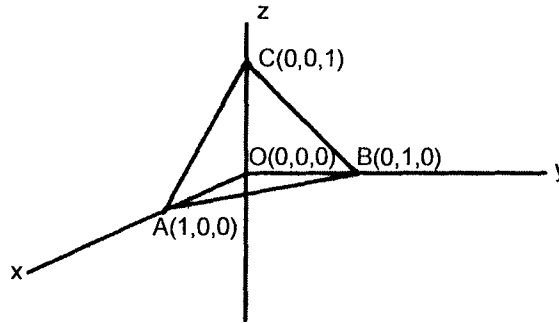
$$\begin{aligned}
 I &= \int_{z=0}^4 \left\{ \int_{x=0}^{2\sqrt{z}} \left[y \right]_{y=0}^{\sqrt{4z-x^2}} dx \right\} dz \\
 &= \int_{z=0}^4 \left\{ \int_{x=0}^{2\sqrt{z}} \sqrt{4z-x^2} \, dx \right\} dz \\
 &= \int_{z=0}^4 \left[\frac{x}{2} \sqrt{4z-x^2} + \frac{4z}{2} \sin^{-1} \left\{ \frac{x}{\sqrt{4z}} \right\} \right]_{x=0}^{2\sqrt{z}} dz \\
 &= \int_0^4 \pi z \, dz = \pi \left[\frac{z^2}{2} \right]_0^4 \\
 &= 8\pi
 \end{aligned}$$

5.5.24 Example:

Evaluate $\iiint_V xyz \, dz \, dy \, dx$ over the volume enclosed by three coordinate planes and the plane $x + y + z = 1$.

Solution:

The plane $x + y + z = 1$ meets the coordinate axes in $A(1,0,0)$, $B(0,1,0)$ and $C(0,0,1)$.

**Fig. 5.5.24**

$$\begin{aligned}
 \iiint_V xyz \, dz \, dy \, dx &= \int_{x=0}^1 \int_{y=0}^{1-x} \left(\int_{z=0}^{1-x-y} xyz \, dz \right) dy \, dx \\
 &= \int_{x=0}^1 \left(\int_{y=0}^{1-x} xy \left[\frac{z^2}{2} \right]_0^{1-x-y} dy \right) dx \\
 &= \int_{x=0}^1 \int_{y=0}^{1-x} \frac{xy}{2} [(1-x-y)^2 - 0] dy \, dx \\
 &= \frac{1}{2} \int_{x=0}^1 \int_{y=0}^{1-x} xy(1+x^2+y^2-2x-2y+2xy) dy \, dx \\
 &= \frac{1}{2} \int_{x=0}^1 \left[xy^2 + \frac{1}{2}x^3y^2 + \frac{1}{4}xy^4 - x^2y^2 - \frac{2}{3}xy^3 + \frac{2}{3}x^2y^3 \right]_0^{1-x} dx \\
 &= \frac{1}{24} \int_0^1 (x + 8x^2 - 30x^3 + 32x^4 - 11x^5) dx \\
 &= \frac{1}{24} \left[\frac{x^2}{2} + \frac{8x^3}{3} - 30\frac{x^4}{4} + 32\frac{x^5}{5} - 11\frac{x^6}{6} \right]_0^1 \\
 &= \frac{7}{21600}
 \end{aligned}$$

5.5.25 Example:

Evaluate $\iiint_V (x^2 + y^2 + z^2) dz dy dx$ where V is the volume of the cube bounded by the coordinate planes and the planes $x = y = z = a$.

Solution:

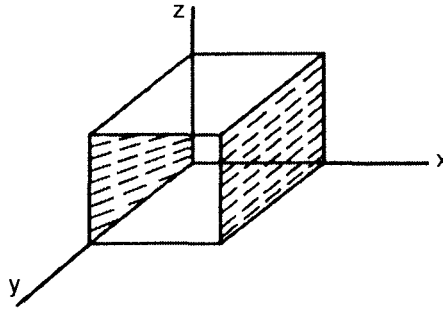


Fig. 5.5.25

$$\begin{aligned}
 \iiint_V (x^2 + y^2 + z^2) dz dy dx &= \int_{x=0}^a \int_{y=0}^a \int_{z=0}^a (x^2 + y^2 + z^2) dz dy dx \\
 &= \int_{x=0}^a \int_{y=0}^a \left[x^2 z + y^2 z + \frac{z^3}{3} \right]_0^a dx dy \\
 &= \int_{x=0}^a \int_{y=0}^a \left(x^2 a + y^2 a + \frac{a^3}{3} \right) dy dx \\
 &= \int_{x=0}^a \left[x^2 a y + \frac{y^3}{3} a + \frac{a^3}{3} y \right]_0^a dx \\
 &= \int_{x=0}^a \left(x^2 a^2 + \frac{a^4}{3} + \frac{a^4}{4} \right) dx \\
 &= \left[\frac{x^3}{3} a^2 + \frac{a^4}{3} x + \frac{a^4}{4} x \right]_0^a \\
 &= a^5
 \end{aligned}$$

5.5.26 Example:

Evaluate by tripple integration the volume of a hemisphere of radius a .

Solution:

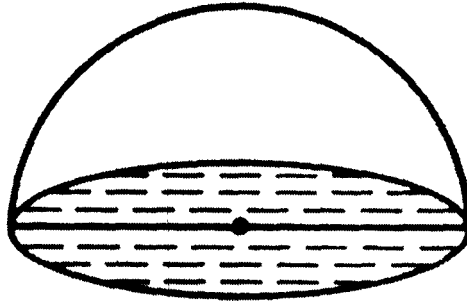


Fig. 5.5.26

The volume of the hemisphere

$$V = 4 \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} dz dy dx$$

$$V = 4 \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2-y^2}} dy dx$$

$$V = 4 \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2} dy dx$$

$$V = 4 \int_{x=0}^a \left[\frac{y}{2} \sqrt{a^2-x^2-y^2} + \frac{1}{2} (a^2-x^2) \sin^{-1} \left(\frac{y}{\sqrt{a^2-x^2}} \right) \right]_{y=0}^{y=\sqrt{a^2-x^2}} dx$$

$$V = 4 \int_{x=0}^a \left[\frac{1}{2} (a^2-x^2) \frac{\pi}{2} \right] dx$$

$$V = \frac{2}{3} \pi a^3$$

Exercise – 5(F)

1. Evaluate the following integrals by changing the order of integration:

$$1. \int_0^3 \int_1^{\sqrt{4-y}} (x+y) dx dy$$

$$[\text{Ans: } \frac{241}{60}]$$

$$2. \int_0^5 \int_0^{x^2} x(x^2 + y^2) dx dy$$

$$[\text{Ans: } \frac{29}{24} \cdot 5^6]$$

$$3. \int_0^{\frac{a}{\sqrt{2}}} \int_0^{\sqrt{a^2-x^2}} y^2 dx dy$$

$$[\text{Ans: } \frac{a^4}{16} + \frac{\pi a^4}{32}]$$

$$4. \int_0^1 \int_x^{2-x} \frac{x}{y} dx dy$$

$$[\text{Ans: } 2 \log 2 - 1]$$

$$5. \int_0^1 \int_y^{\sqrt{y}} xy(x+y) dx dy$$

$$[\text{Ans: } \frac{3}{56}]$$

$$6. \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy dx$$

$$[\text{Ans: } \frac{16a^2}{3}]$$

$$7. \int_1^2 \int_3^4 x^2 + y^2 dx dy$$

$$[\text{Ans: } 10]$$

II. Evaluate the following tripple integrals:

$$1. \int_0^a \int_0^x \int_0^y xyz dx dy dz$$

$$[\text{Ans: } \frac{a^6}{48}]$$

$$2. \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x^2 + y^2 + z^2) dx dy dz$$

$$[\text{Ans: } \frac{1}{20}]$$

$$3. \int_0^1 \int_1^{1-y} \int_0^{1-x^2-y^2} dy dz$$

$$[\text{Ans: } \frac{1}{3}]$$

$$4. \int_0^1 \int_0^2 \int_1^2 x^2 yz dx dy dz$$

$$[\text{Ans: } 1]$$

$$5. \int_0^1 \int_0^1 \int_0^1 (x + y + z) dx dy dz$$

$$[\text{Ans: } \frac{3}{2}]$$

$$6. \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz dx dy dz$$

$$[\text{Ans: } \frac{1}{48}]$$

$$7. \int_1^3 \int_{\frac{1}{x}}^1 \int_0^{\sqrt{xy}} xyz \, dx dy dz$$

$$[\text{Ans: } \left(\frac{13}{9} - \frac{1}{6} \log 3 \right)]$$

$$8. \int_0^1 \int_0^{1-x} \int_0^{x+y} e^z \, dx dy dz$$

$$[\text{Ans: } \frac{1}{2}]$$

$$9. \int_0^4 \int_0^x \int_0^{\sqrt{x+y}} z \, dx dy dz$$

$$[\text{Ans: } 16]$$

$$10. \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^{mx} z^2 \, dx dy dz$$

$$[\text{Ans: } \frac{4}{45} m^2 a^2]$$

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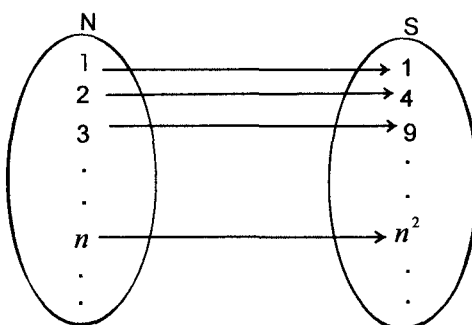
6

Sequences of Series

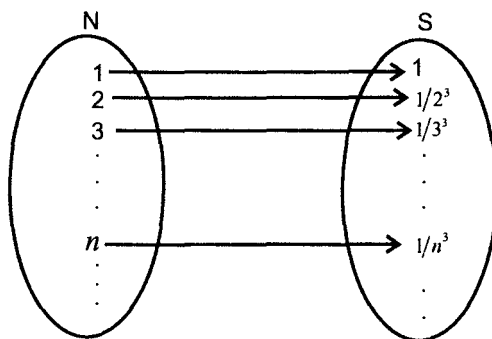
6.0 Sequence

A function $f: \mathbb{N} \rightarrow S$, where S is any nonempty set is called a *Sequence* i.e., for each $n \in \mathbb{N}$, $\exists$ a unique element $f(n) \in S$. The sequence is written as $f(1), f(2), f(3), \dots, f(n), \dots$, and is denoted by $\{f(n)\}$, or $\langle f(n) \rangle$, or $(f(n))$. If $f(n) = a_n$, the sequence is written as $a_1, a_2, \dots, a_n$ and denoted by $\{a_n\}$ or $\langle a_n \rangle$ or (a_n) . Here $f(n)$ or a_n are the n^{th} terms of the Sequence.

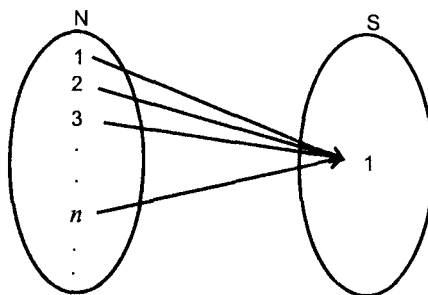
6.1.1 Example : $1, 4, 9, 16, \dots, n^2, \dots$ (or) $\langle n^2 \rangle$



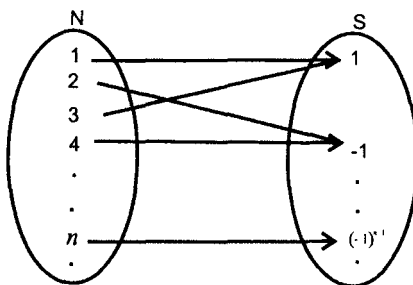
6.1.2 Example : $\frac{1}{1^3}, \frac{1}{2^3}, \frac{1}{3^3}, \dots, \frac{1}{n^3}, \dots$ (or) $\left(\frac{1}{n^3}\right)$



6.1.3 Example : $1, 1, 1, \dots, 1, \dots$ or $\langle 1 \rangle$



6.1.4 Example : $1, -1, 1, -1, \dots$ or $\langle (-1)^{n-1} \rangle$



- Note :**
1. If $S \subseteq \mathbb{R}$ then the sequence is called a *real sequence*.
 2. The range of a sequence is almost a countable set.

6.1.5 Kinds of Sequences

1. *Finite Sequence* : A sequence $\langle a_n \rangle$ in which $a_n = 0 \forall n > m \in \mathbb{N}$ is said to be a finite Sequence. i.e., A finite Sequence has a finite number of terms.
2. *Infinite Sequence* : A sequence, which is not finite is an infinite sequence.

6.1.6 Bounds of a Sequence and Bounded Sequence

1. If $\exists$ a number 'M' $\ni a_n \leq M, \forall n \in \mathbb{N}$, the Sequence $\langle a_n \rangle$ is said to be bounded above or bounded on the right.

Ex: $1, \frac{1}{2}, \frac{1}{3}, \dots$ here $a_n \leq 1 \forall n \in \mathbb{N}$

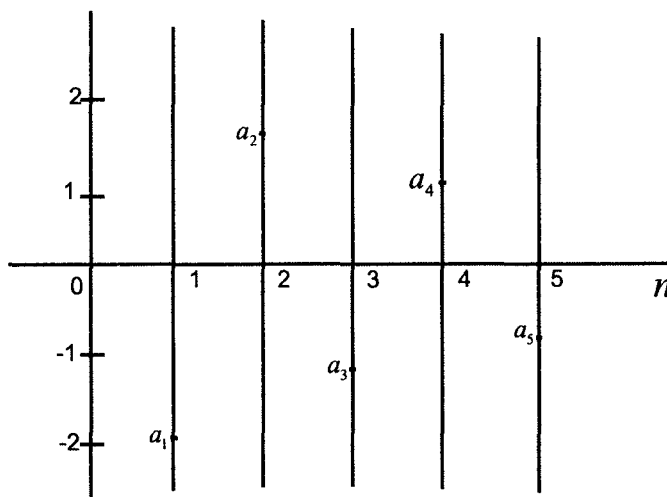
2. If $\exists$ a number 'm' $\ni a_n \geq m, \forall n \in \mathbb{N}$, the sequence $\langle a_n \rangle$ is said to be bounded below or bounded on the left.

Ex: $1, 2, 3, \dots$ here $a_n \geq 1 \forall n \in \mathbb{N}$

3. A sequence which is bounded above and below is said to be bounded.

Ex: Let $a_n = (-1)^n \left(1 + \frac{1}{n} \right)$

n	1	2	3	4	
a_n	-2	3/2	-4/3	5/4	



From the above figure (see also table) it can be seen that $m = -2$ and $M = \frac{3}{2}$.

$\therefore$ The sequence is bounded.

6.1.7 Limits of a Sequence

A Sequence $\langle a_n \rangle$ is said to tend to limit ' l ' when, given any + ve number ' ϵ ', however small, we can always find an integer ' m ' such that $|a_n - l| < \epsilon, \forall n \geq m$, and we write $\lim_{n \rightarrow \infty} a_n = l$ or $\langle a_n \rangle \rightarrow l$

$$\text{Ex : } \quad \text{If } a_n = \frac{n^2 + 1}{2n^2 + 3} \text{ then } \langle a_n \rangle \rightarrow \frac{1}{2}.$$

6.1.8 Convergent, Divergent and Oscillatory Sequences

1. *Convergent Sequence* : A sequence which tends to a finite limit, say ' l ' is called a *Convergent Sequence*. We say that the sequence converges to ' l '
2. *Divergent Sequence* : A sequence which tends to $\pm\infty$ is said to be *Divergent* (or is said to diverge).
3. *Oscillatory Sequence* : A sequence which neither converges nor diverges, is called an *Oscillatory Sequence*.

Examples

1. Consider the sequence $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots$ here $a_n = 1 + \frac{1}{n}$

The sequence $\langle a_n \rangle$ is convergent and has the limit 1

$$a_n - 1 = 1 + \frac{1}{n} - 1 = \frac{1}{n} \text{ and } \frac{1}{n} < \epsilon \text{ whenever } n > \frac{1}{\epsilon}$$

Suppose we choose $\epsilon = .001$, we have $\frac{1}{n} < .001$ when $n > 1000$.

2. If $a_n = 3 + (-1)^n \frac{1}{n}$, $\langle a_n \rangle$ converges to 3.
3. If $a_n = n^2 + (-1)^n \cdot n$, $\langle a_n \rangle$ diverges.
4. If $a_n = \frac{1}{n} + 2(-1)^n$, $\langle a_n \rangle$ oscillates between -2 and 2.

6.2 Infinite Series

6.2.1

If $\langle u_n \rangle$ is a sequence, then the expression $u_1 + u_2 + u_3 + \dots + u_n + \dots$ is called an infinite series. It is denoted by $\sum_{n=1}^{\infty} u_n$ or simply $\sum u_n$

The sum of the first n terms of the series is denoted by s_n

i.e., $s_n = u_1 + u_2 + u_3 + \dots + u_n$; $s_1, s_2, s_3, \dots, s_n$ are called *partial sums*.

6.2.2 Convergent, Divergent and Oscillatory Series

Let $\sum u_n$ be an infinite series. As $n \rightarrow \infty$, there are three possibilities.

- (a) *Convergent series* : As $n \rightarrow \infty, s_n \rightarrow$ a finite limit, say 's' in which case the series is said to be convergent and 's' is called its sum to infinity. Thus $\lim_{n \rightarrow \infty} s_n = s$ (or) simply $Ls_n = s$

This is also written as $u_1 + u_2 + u_3 + \dots + u_n + \dots \text{to } \infty = s$. (or) $\sum_{n=1}^{\infty} u_n = s$

(or) simply $\sum u_n = s$.

- (b) *Divergent series* : If $s_n \rightarrow \infty$ or $-\infty$, the series said to be divergent.
 (c) *Oscillatory Series*: If s_n does not tend to a unique limit either finite or infinite it is said to be an *Oscillatory Series*.

Note : Divergent or Oscillatory series are sometimes called non convergent series.

6.2.3 Geometric Series

The series, $1 + x + x^2 + \dots + x^{n-1} + \dots$ is

- (i) Convergent when $|x| < 1$, and its sum is $\frac{1}{1-x}$
 (ii) Divergent when $x \geq 1$.
 (iii) Oscillates finitely when $x = -1$ and oscillates infinitely when $x < -1$.

Proof:

The given series is a geometric series with common ratio 'x'

$$\therefore s_n = \frac{1-x^n}{1-x} \text{ when } x \neq 1 \text{ [By actual division - verify]}$$

- (i) When $|x| < 1$:

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(\frac{1}{1-x} \right) - \lim_{n \rightarrow \infty} \left(\frac{x^n}{1-x} \right) = \frac{1}{1-x}$$

[since $x^n \rightarrow 0$ as $n \rightarrow \infty$]

$\therefore$ The series converges to $\frac{1}{1-x}$

- (ii) When $x \geq 1$: $s_n = \frac{x^n - 1}{x - 1}$ and $s_n \rightarrow \infty$ as $n \rightarrow \infty$

$\therefore$ The series is divergent.

- (iii) When $x = -1$: when n is even, $s_n \rightarrow 0$ and when n is odd, $s_n \rightarrow 1$

$\therefore$ The series oscillates finitely.

(iv) When $x < -1, s_n \rightarrow \infty$ or $-\infty$ according as n is odd or even.

$\therefore$ The series oscillates infinitely.

6.2.4 Some Elementary Properties of Infinite Series

1. The convergence or divergence of an infinite series is unaltered by an addition or deletion of a finite number of terms from it.
2. If some or all the terms of a convergent series of positive terms change their signs, the series will still be convergent.

3. Let $\sum u_n$ converge to 's'

Let 'k' be a non-zero fixed number. Then $\sum ku_n$ converges to ks .

Also, if $\sum u_n$ diverges or oscillates, so does $\sum ku_n$

4. Let $\sum u_n$ converge to 'l' and $\sum v_n$ converge to 'm'. Then

(i) $\sum (u_n + v_n)$ converges to $(l + m)$ and

(ii) $\sum (u_n - v_n)$ converges to $(l - m)$

6.2.5 Series of Positive Terms

Consider the series in which all terms beginning from a particular term are +ve.

Let the first term from which all terms are +ve be u_1 .

Let $\sum u_n$ be such a convergent series of +ve terms. Then, we observe that the convergence is unaltered by any rearrangement of the terms of the series.

6.2.6 Theorem

If $\sum u_n$ is convergent, then $\lim_{n \rightarrow \infty} u_n = 0$.

Proof:

$$s_n = u_1 + u_2 + \dots + u_n$$

$$s_{n-1} = u_1 + u_2 + \dots + u_{n-1}, \text{ so that, } u_n = s_n - s_{n-1}$$

Suppose $\sum u_n = l$ then $\lim_{n \rightarrow \infty} s_n = l$ and $\lim_{n \rightarrow \infty} s_{n-1} = l$

$$\therefore \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} (s_n - s_{n-1}) ; \lim_{n \rightarrow \infty} s_n - \lim_{n \rightarrow \infty} s_{n-1} = l - l = 0$$

Note : The converse of the above theorem need not be always true. This can be Observed from the following examples.

- (i) Consider the series, $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots$; $u_n = \frac{1}{n}$, $\lim_{n \rightarrow \infty} u_n = 0$

But from p -series test (2.7) it is clear that $\sum \frac{1}{n}$ is divergent.

- (ii) Consider the series, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} + \dots$

$u_n = \frac{1}{n^2}$, $\lim_{n \rightarrow \infty} u_n = 0$, by p series test, clearly $\sum \frac{1}{n^2}$ converges,

Note : If $\lim_{n \rightarrow \infty} u_n \neq 0$ the series is divergent;

Ex: $u_n = \frac{2^n - 1}{2^n}$, here $\lim_{n \rightarrow \infty} u_n = 1$

$\therefore \sum u_n$ is divergent.

Tests for the Convergence of an Infinite Series

In order to study the nature of any given infinite series of +ve terms regarding convergence or otherwise, a few tests are given below.

6.2.7 P-Series Test

The infinite series, $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$, is

- (i) Convergent when $p > 1$, and (ii) Divergent when $p \leq 1$. [JNTU Dec 2002, A 2003]

Proof :

Case (i) Let $p > 1$; $p > 1, 3^p > 2^p$; $\Rightarrow \frac{1}{3^p} < \frac{1}{2^p}$

$\therefore \frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p}$

Similarly, $\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} = \frac{4}{4^p}$

$\frac{1}{8^p} + \frac{1}{9^p} + \dots + \frac{1}{16^p} < \frac{8}{8^p}$, and so on.

Adding we get

$\sum \frac{1}{n^p} < 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \dots$

i.e., $\sum \frac{1}{n^p} < 1 + \frac{1}{2^{(p-1)}} + \frac{1}{2^{2(p-1)}} + \frac{1}{2^{3(p-1)}} + \dots$

The RHS of the above inequality is an infinite geometric series with common ratio $\frac{1}{2^{p-1}} < 1$ (since $p > 1$) The sum of this geometric series is finite.

Hence $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is also finite.

$\therefore$ The given series is convergent.

Case (ii) : Let $p = 1$; $\sum \frac{1}{n^p} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

We have, $\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$
 $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$
 $\frac{1}{9} + \frac{1}{10} + \dots + \frac{1}{16} > \frac{1}{16} + \frac{1}{16} + \dots + \frac{1}{16} = \frac{1}{2}$ and so on

$$\therefore \sum \frac{1}{n^p} = 1 + \left(\frac{1}{2} + \frac{1}{3} \right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} \right) + \dots$$

$$\geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

The sum of RHS series is ∞ $\left(\text{since } s_n = 1 + \frac{n-1}{2} = \frac{n+1}{2} \text{ and } \lim_{n \rightarrow \infty} s_n = \infty \right)$

$\therefore$ The sum of the given series is also ∞ ; $\therefore \sum_{n=1}^{\infty} \frac{1}{n^p}$ ($p = 1$) diverges.

Case (iii) : Let $p < 1$, $\sum \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots$

Since $p < 1$, $\frac{1}{2^p} > \frac{1}{2}$, $\frac{1}{3^p} > \frac{1}{3}$, ... and so on

$$\therefore \sum \frac{1}{n^p} > 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

From the Case (ii), it follows that the series on the RHS of above inequality is divergent.

$$\therefore \sum \frac{1}{n^p} \text{ is divergent, when } P < 1$$

Note : This theorem is often helpful in discussing the nature of a given infinite series.

6.2.8 Comparison Tests

1. Let $\sum u_n$ and $\sum v_n$ be two series of +ve terms and let $\sum v_n$ be convergent.

Then $\sum u_n$ converges,

1. If $u_n \leq v_n, \forall n \in N$
2. or $\frac{u_n}{v_n} \leq k \forall n \in N$ where k is > 0 and finite.
3. or $\frac{u_n}{v_n} \rightarrow$ a finite limit > 0

Proof:

1. Let $\sum v_n = l$ (finite)

Then, $u_1 + u_2 + \dots + u_n + \dots \leq v_1 + v_2 + \dots + v_n + \dots \leq l > 0$

Since l is finite it follows that $\sum u_n$ is convergent

2. $\frac{u_n}{v_n} \leq k \Rightarrow u_n \leq kv_n, \forall n \in N$, since $\sum v_n$ is convergent and $k (> 0)$ is finite, $\sum kv_n$ is convergent $\therefore \sum u_n$ is convergent.

3. Since $\lim_{n \rightarrow \infty} \frac{u_n}{v_n}$ is finite, we can find a +ve constant $k, \exists \frac{u_n}{v_n} < k \forall n \in N$

$\therefore$ from (2), it follows that $\sum u_n$ is convergent

2. Let $\sum u_n$ and $\sum v_n$ be two series of +ve terms and let $\sum v_n$ be divergent.

Then $\sum u_n$ diverges,

- * 1. If $u_n \geq v_n, \forall n \in N$
- or * 2. If $\frac{u_n}{v_n} \geq k, \forall n \in N$ where k is finite and $\neq 0$
- or * 3. If $\lim_{n \rightarrow \infty} \frac{u_n}{v_n}$ is finite and non-zero.

Proof:

1. Let M be a +ve integer however large it may be. Since $\sum v_n$ is divergent, a number m can be found such that

$$v_1 + v_2 + \dots + v_n > M, \forall n > m$$

$$\therefore u_1 + u_2 + \dots + u_n > M, \forall n > m (u_n \geq v_n)$$

$\therefore \sum u_n$ is divergent

2. $u_1 \geq kv_n \forall n$

$\sum v_n$ is divergent $\Rightarrow \sum kv_n$ is divergent

$\therefore \sum u_n$ is divergent

3. Since $\lim_{n \rightarrow \infty} \frac{u_n}{v_n}$ is finite, a + ve constant k can be found such that $\frac{u_n}{v_n} > k, \forall n$

(probably except for a finite number of terms)

$\therefore$ From (2), it follows that $\sum u_n$ is divergent.

Note :

1. In (1) and (2), it is sufficient that the conditions with * hold $\forall n > m \in N$

Alternate form of comparison tests : The above two types of comparison tests 2.8.(1) and 2.8.(2) can be culbed together and stated as follows :

If $\sum u_n$ and $\sum v_n$ are two series of + ve terms such that $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = k$,

where k is non- zero and finite, then $\sum u_n$ and $\sum v_n$ both converge or both diverge.

Note :

1. The above form of comparison tests is mostly used in solving problems.

2. In order to apply the test in problems, we require a certain series $\sum v_n$ whose nature is already known i.e., we must know whether $\sum v_n$ is convergent or divergent. For this reason, we call $\sum v_n$ as an 'auxiliary series'.

3. In problems, the geometric series (2.3.) and the p -series (2.7) can be conveniently used as 'auxiliary series'.

Solved Examples

6.2.9 Example

Test the convergence of the following series:

(a) $\frac{3}{1} + \frac{4}{8} + \frac{5}{27} + \frac{6}{64} + \dots$ (b) $\frac{4}{1} + \frac{5}{4} + \frac{6}{9} + \frac{7}{16} + \dots$

(c) $\sum_{n=1}^{\infty} \left[(n^4 + 1)^{1/4} - n \right]$

Solution(a) *Step 1:*

To find " u_n " the n^{th} term of the given series. The numerators 3, 4, 5, 6.....of the terms, are in AP.

$$n^{\text{th}} \text{ term } t_n = 3 + (n-1) \cdot 1 = n + 2$$

$$\text{Denominators are } 1^3, 2^3, 3^3, 4^3, \dots, n^{\text{th}} \text{ term} = n^3; \therefore u_n = \frac{n+2}{n^3}$$

Step 2:

To choose the auxiliary series $\sum v_n$. In u_n , the highest degree of n in the numerator is 1 and that of denominator is 3.

$$\therefore \text{ we take, } v_n = \frac{1}{n^{3-1}} = \frac{1}{n^2}$$

Step 3 :

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{n+2}{n^3} \times n^2 = \lim_{n \rightarrow \infty} \frac{n+2}{n} = \lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right) = 1,$$

which is non- zero and finite.

*Step 4 :***Conclusion:**

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = 1$$

$\therefore \sum u_n$ and $\sum v_n$ both converge or diverge (by comparison test). But

$\sum v_n = \sum \frac{1}{n^2}$ is convergent by p -series test ($p = 2 > 1$); $\therefore \sum u_n$ is convergent.

$$(b) \quad \frac{4}{1} + \frac{5}{4} + \frac{6}{9} + \frac{7}{16} + \dots$$

$$\text{Step 1 : } 4, 5, 6, 7, \dots \text{ in AP, } t_n = 4 + (n-1)1 = n+3 \quad \therefore u_n = \frac{n+3}{n^2}$$

Step 2 : Let $\sum v_n = \frac{1}{n}$ be the auxiliary series

Step 3 : $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \left(\frac{n+3}{n^2} \right) \times n = \lim_{n \rightarrow \infty} \left(1 + \frac{3}{n} \right) = 1$, which is non-zero and finite.

Step 4 : $\therefore$ By comparison test, both $\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\sum v_n = \sum \frac{1}{n}$ is divergent, by p -series test ($p = 1$); $\therefore \sum u_n$ is divergent.

$$\begin{aligned}
 \text{(c)} \quad \sum_{n=1}^{\infty} \left[(n^4 + 1)^{1/4} - n \right] &= \left\{ n^4 \left(1 + \frac{1}{n^4} \right) \right\}^{1/4} - n = n \left[\left(1 + \frac{1}{n^4} \right)^{1/4} - 1 \right] \\
 &= n \left[1 + \frac{1}{4n^4} + \frac{\frac{1}{4} \left(\frac{1}{4} - 1 \right)}{2!} \cdot \frac{1}{n^8} + \dots - 1 \right] = n \left[\frac{1}{4n^4} - \frac{3}{32n^8} + \dots \right] \\
 &= \frac{1}{4n^3} - \frac{3}{32n^7} + \dots = \frac{1}{n^3} \left[\frac{1}{4} - \frac{3}{32n^4} + \dots \right]
 \end{aligned}$$

Here it will be convenient if we take $v_n = \frac{1}{n^3}$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \left(\frac{1}{4} - \frac{1}{32n^4} + \dots \right) = \frac{1}{4}, \text{ which is non-zero and finite}$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ both converge or both diverge. But

by p -series test $\sum v_n = \sum \frac{1}{n^3}$ is convergent. ($p = 3 > 1$); $\therefore \sum u_n$ is convergent.

6.2.10 Example

If $u_n = \frac{\sqrt[3]{3n^2 + 1}}{\sqrt[4]{2n^3 + 3n + 5}}$ show that $\sum u_n$ is divergent

Solution

As n increases, u_n approximates to

$$\frac{\sqrt[3]{3n^2}}{\sqrt[4]{2n^3}} = \frac{3^{1/3}}{2^{1/4}} \times \frac{n^{2/3}}{n^{3/4}} = \frac{3^{1/3}}{2^{1/4}} \cdot \frac{1}{n^{1/12}}$$

$\therefore$ If we take $v_n = \frac{1}{n^{1/12}}$, $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{3^{1/3}}{2^{1/4}}$ which is finite.

[(or) *Hint*: Take $v_n = \frac{1}{n^{l_1 - l_2}}$, where l_1 and l_2 are indices of 'n' of the largest terms in denominator and nominator respectively of u_n . Here

$$v_n = \frac{1}{n^{\frac{3}{4} - \frac{2}{3}}} = \frac{1}{n^{1/12}}]$$

By comparison test, $\sum v_n$ and $\sum u_n$ converge or diverge together. But

$$\sum v_n = \sum \frac{1}{n^{1/12}} \text{ is divergent by } p\text{-series test (since } p = \frac{1}{12} < 1)$$

$\therefore \sum u_n$ is divergent.

6.2.11 Example

Test for the convergence of the series. $\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{3}} + \sqrt{\frac{3}{4}} + \sqrt{\frac{4}{5}} + \dots$

Solution

Here, $u_n = \sqrt{\frac{n}{n+1}};$

Take $v_n = \frac{1}{n^{\frac{1}{2} - \frac{1}{2}}} = \frac{1}{n^0} = 1$, $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \sqrt{\frac{1}{1 + \frac{1}{n}}} = 1$ (finite)

$\sum v_n$ is divergent by p -series test. ($p = 0 < 1$)

$\therefore$ By comparison test, $\sum u_n$ is divergent, (Students are advised to follow the procedure given in ex. 1.2.9(a) and (b) to find " u_n " of the given series.)

6.2.12 Example

Show that $1 + \frac{1}{\underline{1}} + \frac{1}{\underline{2}} + \dots + \frac{1}{\underline{n}} + \dots$ is convergent.

Solution

$$u_n = \frac{1}{\underline{n}} \text{ (neglecting 1st term)}$$

$$= \frac{1}{1.2.3 \dots n} < \frac{1}{1.2.2.2 \dots n-1 \text{ times}} = \frac{1}{(2^{n-1})}$$

$$\therefore \sum u_n < 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$$

which is an infinite geometric series with common ratio $\frac{1}{2} < 1$

$\therefore \sum \frac{1}{2^{n-1}}$ is convergent. (1.2.3(a)). Hence $\sum u_n$ is convergent.

6.2.13 Example

Test for the convergence of the series, $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \dots$

Solution

$$u_n = \frac{1}{n(n+1)(n+2)};$$

Take $v_n = \frac{1}{n^3}$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{n^3}{n^3 \left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right)} = 1 \text{ (finite)}$$

$\therefore$ By comparison test, $\sum u_n$, and $\sum v_n$ converge or diverge together. But by p -series test, $\sum v_n = \sum \frac{1}{n^3}$ is convergent ($p = 3 > 1$); $\therefore \sum u_n$ is convergent.

6.2.14 Example

If $u_n = \sqrt{n^4 + 1} - \sqrt{n^4 - 1}$, show that $\sum u_n$ is convergent. [JNTU, 200s]

Solution

$$\begin{aligned} u_n &= n^2 \left(1 + \frac{1}{n^4}\right)^{\frac{1}{2}} - n^2 \left(1 - \frac{1}{n^4}\right)^{\frac{1}{2}} \\ &= n^2 \left[\left(1 + \frac{1}{2n^4} - \frac{1}{8n^8} + \frac{1}{16n^{12}} - \dots\right) - \left(1 - \frac{1}{2n^4} - \frac{1}{8n^8} - \frac{1}{16n^{12}} - \dots\right) \right] \\ &= n^2 \left[\frac{1}{n^4} + \frac{1}{8n^{12}} + \dots \right] = \frac{1}{n^2} \left[1 + \frac{1}{8n^{10}} + \dots \right] \end{aligned}$$

Take $v_n = \frac{1}{n^2}$, hence $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = 1$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together. But $\sum v_n = \frac{1}{n^2}$ is convergent by p -series test ($p = 2 > 1$) $\therefore \sum u_n$ is convergent.

6. 2.15 Example

Test the series $\frac{1}{1+x} + \frac{1}{2+x} + \frac{1}{3+x} + \dots$ for convergence.

Solution

$$u_n = \frac{1}{n+x};$$

Take $v_n = \frac{1}{n},$

then $\frac{u_n}{v_n} = \frac{n}{n+x} = \frac{1}{1+\frac{x}{n}}$

$$\lim_{n \rightarrow \infty} \left(\frac{1}{1+\frac{x}{n}} \right) = 1; \sum v_n = \sum \frac{1}{n} \text{ is divergent by } p\text{-series test } (p=1)$$

$\therefore$ By comparison test, $\sum u_n$ is divergent.

6.2.16 Example

Show that $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ is divergent.

Solution

$$u_n = \sin\left(\frac{1}{n}\right); \quad \text{Take } v_n = \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{t \rightarrow 0} \frac{\sin t}{t} \text{ (where } t = \frac{1}{n} \text{)} = 1$$

$\therefore \sum u_n, \sum v_n$ both converge or diverge. But $\sum v_n = \sum \frac{1}{n}$ is divergent

(p -series test, $p=1$); $\therefore \sum u_n$ is divergent.

6.2.17 Example

Test the series $\sum \sin^{-1}\left(\frac{1}{n}\right)$ for convergence.

Solution

$$u_n = \sin^{-1} \frac{1}{n};$$

Take $v_n = \frac{1}{n}$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{\sin^{-1}\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}; = \lim_{\theta \rightarrow 0} \left(\frac{\theta}{\sin \theta} \right) = 1 \left(\text{Taking } \sin^{-1} \frac{1}{n} = \theta \right)$$

But $\sum v_n$ is divergent. Hence $\sum u_n$ is divergent.

6.2.18 Example

Show that the series $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \dots$ is divergent.

Solution

Neglecting the first term, the series is $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \dots$. Therefore

$$\begin{aligned} u_n &= \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)(n+1)^n} = \frac{n^n}{n \left(1 + \frac{1}{n}\right) \cdot n^n \left(1 + \frac{1}{n}\right)^n} \\ &= \frac{1}{n \left(1 + \frac{1}{n}\right) \left(1 + \frac{1}{n}\right)^n}; \quad \text{Take } v_n = \frac{1}{n} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right) \left(1 + \frac{1}{n}\right)^n} = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n \cdot 1} = \frac{1}{e}$$

which is finite and $\sum v_n = \sum \frac{1}{n}$ is divergent by p -series test ($p = 1$)

$\therefore \sum u_n$ is divergent.

6.2.19 Example

Show that the series $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \dots \infty$ is convergent.

Solution

$$\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \dots \infty$$

$$n^{\text{th}} \text{ term} = u_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{1}{n^2} \cdot \frac{\left(2 - \frac{1}{n}\right)}{\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right)}$$

Take $v_n = \frac{1}{n^2}$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{n^2} \frac{\left(2 - \frac{1}{n}\right)}{\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right)} \div \left(\frac{1}{n^2}\right)$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{2-0}{(1+0)(1+0)} = 2 \text{ which is finite and non-zero}$$

$\therefore$ By comparison test $\sum u_n$ and $\sum v_n$ converge or diverge together

But $\sum v_n = \sum \frac{1}{n^2}$ is convergent. $\therefore \sum u_n$ is also convergent.

6.2.20 Example

Test whether the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + \sqrt{n+1}}$ is convergent

Solution

The given series is $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + \sqrt{n+1}}$

$$\begin{aligned} u_n &= \frac{1}{\sqrt{n} + \sqrt{n+1}} \\ &= \frac{\sqrt{n+1} - \sqrt{n}}{(\sqrt{n} + \sqrt{n+1})(\sqrt{n+1} - \sqrt{n})} = \sqrt{n+1} - \sqrt{n} \end{aligned}$$

$$u_n = \sqrt{n} \left\{ \left(1 + \frac{1}{n} \right)^{1/2} - 1 \right\} = \sqrt{n} \left\{ \left(1 + \frac{1}{2n} - \frac{1}{8n^2} + \dots \right) - 1 \right\}$$

$$u_n = \sqrt{n} \left\{ \frac{1}{2n} - \frac{1}{8n^2} + \dots \right\} = \sqrt{n} \left\{ \frac{1}{2} - \frac{1}{8n} + \dots \right\}$$

Take $v_n = \frac{1}{\sqrt{n}}$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left\{ \frac{1}{2} - \frac{1}{8n} + \dots \right\} \div \left(\frac{1}{\sqrt{n}} \right) = \frac{1}{2}$$

which is finite and non-zero

Using comparison test $\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\sum v_n = \sum \frac{1}{\sqrt{n}}$ is divergent (since $p = 1/2$)

$\therefore \sum u_n$ is also divergent.

6.2.21 Example

Test for convergence $\sum_{n=1}^{\infty} [\sqrt[3]{n^3+1} - n]$ [JNTU 1996, 2003, 2003s]

$$n^{\text{th}} \text{ term } u_n = n \left[\left(1 + \frac{1}{n^3} \right)^{1/3} - 1 \right] = n \left[1 + \frac{1}{3n^3} + \frac{1/3(1/3-1)}{1.2} \cdot \frac{1}{n^6} + \dots - 1 \right]$$

$$= \frac{1}{3n^2} - \frac{1}{9n^5} + \dots = \frac{1}{n^2} \left(\frac{1}{3} - \frac{1}{9n^3} + \dots \right); \text{ Let } v_n = \frac{1}{n^2}$$

Then $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \left(\frac{1}{3} - \frac{1}{9n^3} + \dots \right) = \frac{1}{3} \neq 0$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ both converge or diverge.

But $\sum v_n$ is convergent by p -series test (since $p = 2 > 1$) $\therefore \sum u_n$ is convergent.

6.2.22 Example

Show that the series, $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \dots$ is convergent for $p > 2$ and divergent for $p \leq 2$

Solution

$$n^{\text{th}} \text{ term of the given series} = u_n = \frac{n+1}{n^p} = \frac{n\left(1+\frac{1}{n}\right)}{n^p} = \frac{\left(1+\frac{1}{n}\right)}{n^{p-1}}$$

$$\text{Let us take } v_n = \frac{1}{n^{p-1}}; \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = 1 \neq 0;$$

$\therefore \sum u_n$ and $\sum v_n$ both converge or diverge by comparison test.

But $\sum v_n = \sum \frac{1}{n^{p-1}}$ converges when $p-1 > 1$; i.e., $p > 2$ and diverges when $p-1 \leq 1$ i.e. $p \leq 2$; Hence the result.

6.2.23 Example

$$\text{Test for convergence } \sum_{n=1}^{\infty} \left(\frac{2^n + 3}{3^n + 1} \right)^{1/2}$$

Solution

$$u_n = \left[\frac{2^n \left(1 + \frac{3}{2^n}\right)}{3^n \left(1 + \frac{1}{3^n}\right)} \right]^{1/2};$$

$$\text{Take } v_n = \sqrt{\frac{2^n}{3^n}}; \frac{u_n}{v_n} = \left(\frac{1 + \frac{3}{2^n}}{1 + \frac{1}{3^n}} \right)^{1/2}$$

$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = 1 \neq 0$; $\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ behave the same way.

But $\sum v_n = \sum_{n=1}^{\infty} \left(\frac{2}{3} \right)^{n/2} = \sqrt{\frac{2}{3}} + \frac{2}{3} + \left(\frac{2}{3} \right)^{3/2} + \dots$, which is a geometric series with common ratio $\sqrt{\frac{2}{3}} (< 1)$ $\therefore \sum v_n$ is convergent. Hence $\sum u_n$ is convergent.

6.2.24 Example

$$\text{Test for convergence of the series, } \frac{1}{4 \cdot 7 \cdot 10} + \frac{4}{7 \cdot 10 \cdot 13} + \frac{9}{10 \cdot 13 \cdot 16} + \dots$$

Solution

	4, 7, 10,.....is an A . P;	$t_n = 4 + (n-1)3 = 3n+1$
	7, 10, 13,.....is an A . P;	$t_n = 7 + (n-1)3 = 3n+4$
and	10, 13, 16,.....is an A . P;	$t_n = 10 + (n-1)3 = 3n+7$

$$\begin{aligned}\therefore u_n &= \frac{n^2}{(3n+1)(3n+4)(3n+7)} = \frac{n^2}{3n\left(1+\frac{1}{3n}\right) \cdot 3n\left(1+\frac{4}{3n}\right) \cdot 3n\left(1+\frac{7}{3n}\right)} \\ &= \frac{1}{27n\left(1+\frac{1}{3n}\right)\left(1+\frac{4}{3n}\right)\left(1+\frac{7}{3n}\right)};\end{aligned}$$

Taking $v_n = \frac{1}{n}$, we get

$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{1}{27} \neq 0$; $\therefore$ By comparison test, both $\sum u_n$ and $\sum v_n$ behave in the same manner. But by p -series test, $\sum v_n$ is divergent, since $p = 1$. $\therefore \sum u_n$ is divergent.

6.2.25 Example

Test for convergence $\sum \frac{\sqrt{2n^2 - 5n + 1}}{4n^3 - 7n^2 + 2}$

Solution

$$n^{\text{th}} \text{ term of the given series} = u_n = \frac{\sqrt{2n^2 - 5n + 1}}{4n^3 - 7n^2 + 2}$$

$$\text{Let } v_n = \frac{1}{n^2}$$

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{u_n}{v_n} &= \lim_{n \rightarrow \infty} \left[\frac{n\sqrt{2 - \frac{5}{n} + \frac{1}{n^2}}}{n^3\left(4 - \frac{7}{n} + \frac{2}{n^3}\right)} \times \frac{n^2}{1} \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{\sqrt{2 - \frac{5}{n} + \frac{1}{n^2}}}{\left(4 - \frac{7}{n} + \frac{2}{n^3}\right)} \right] = \frac{\sqrt{2}}{4} \neq 0\end{aligned}$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ both converge or diverge. But $\sum v_n$ is convergent. [p series test - $p = 2 > 1$] $\therefore \sum u_n$ is convergent.

6.2.26 Example

Test the series $\sum u_n$, whose n^{th} term is $\frac{1}{(4n^2 - i)}$

Solution

$$u_n = \frac{1}{(4n^2 - i)};$$

$$\text{Let } v_n = \frac{1}{n^2}, \quad \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \left[\frac{n^2}{n^2 \left(4 - \frac{i}{n^2}\right)} \right] = \frac{1}{4} \neq 0$$

$\therefore \sum u_n$ and $\sum v_n$ both converge or diverge by comparison test. But $\sum v_n$ is convergent by p -series test ($p = 2 > 1$); $\therefore \sum u_n$ is convergent.

6.2.27 Example

If $u_n = \left(\frac{1}{n}\right) \cdot \sin\left(\frac{1}{n}\right)$, show that $\sum u_n$ is convergent.

Solution

Let $v_n = \frac{1}{n^2}$, so that $\sum v_n$ is convergent by p -series test.

$$\lim_{n \rightarrow \infty} \left(\frac{u_n}{v_n} \right) = \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \rightarrow \infty} \left(\frac{\sin t}{t} \right)$$

where $t = 1/n$, Thus $\lim_{n \rightarrow \infty} \left(\frac{u_n}{v_n} \right) = 1 \neq 0$

$\therefore$ By comparison test, $\sum u_n$ is convergent.

6.2.28 Example

Test for convergence $\sum \frac{1}{\sqrt{n}} \tan(1/n)$

Take $v_n = \frac{1}{n^{3/2}}$; $\lim_{n \rightarrow \infty} \left[\frac{u_n}{v_n} \right] = 1 \neq 0$ (as in above example)

Hence by comparison test, $\sum u_n$ converges as $\sum v_n$ converges.

6.2.29 Example

Show that $\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)$ is convergent.

Solution

Let $u_n = \sin^2\left(\frac{1}{n}\right)$; Take $v_n = \frac{1}{n^2}$

$$\lim_{n \rightarrow \infty} \left(\frac{u_n}{v_n} \right) = \lim_{n \rightarrow \infty} \left[\frac{\sin^2\left(\frac{1}{n}\right)}{\frac{1}{n^2}} \right] = \lim_{t \rightarrow 0} \left(\frac{\sin t}{t} \right)^2$$

where $t = \frac{1}{n}$; $\lim_{n \rightarrow \infty} \left(\frac{u_n}{v_n} \right) = 1^2 = 1 \neq 0$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ behave the same way.

But $\sum v_n$ is convergent by p-series test, since $p = 2 > 1$; $\therefore \sum u_n$ is convergent.

6.2.30 Example :

Show that $\sum_{n=2}^{\infty} \frac{1}{\log(n^n)}$ is divergent.

Solution

$$u_n = \frac{1}{n \log n}; \log 2 < 1 \Rightarrow 2 \log 2 < 2 \Rightarrow \frac{1}{2 \log 2} > \frac{1}{2};$$

Similarly $\frac{1}{3 \log 3} > \frac{1}{3}, \dots, \frac{1}{n \log n} > \frac{1}{n}, n \in \mathbb{N}$

$\therefore \sum \frac{1}{n \log n} > \sum \frac{1}{n}$; But $\sum \frac{1}{n}$ is divergent by p-series test.

By comparison test, given series is divergent. [If $\sum v_n$ is divergent and $u_n \geq v_n \forall n$ then $\sum u_n$ is divergent.]

(Note : This problem can also be done using Cauchy's integral Test. See ex 1.6.2)

6.2.31 Example

Test the convergence of the series $\sum_{n=1}^{\infty} (c+n)^{-r} (d+n)^{-s}$, where c, d, r, s are all +ve.

Solution

The n^{th} term of the series $= u_n = \frac{1}{(c+n)^r (d+n)^s}$.

$$\text{Let } v_n = \frac{1}{n^{r+s}} \text{ Then } \frac{u_n}{v_n} = \frac{n^{r+s}}{n^r \left(1 + \frac{c}{n}\right)^r \cdot n^s \left(1 + \frac{d}{n}\right)^s} = \frac{1}{\left(1 + \frac{c}{n}\right)^r \left(1 + \frac{d}{n}\right)^s}$$

$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = 1 \neq 0$, $\therefore \sum u_n$ and $\sum v_n$ both converge or diverge, by comparison

test.

But by p -series test, $\sum v_n$ converges if $(r+s) > 1$ and diverges if $(r+s) \leq 1$

$\therefore \sum u_n$ converges if $(r+s) > 1$ and diverges if $(r+s) \leq 1$.

6.2.32 Example

Show that $\sum_{n=1}^{\infty} n^{-(1+\frac{1}{n})}$ is divergent.

Solution

$$u_n = n^{-(1+\frac{1}{n})} = \frac{1}{n \cdot n^{\frac{1}{n}}}$$

Take $v_n = \frac{1}{n}$; $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{n^{\frac{1}{n}}} = 1 \neq 0$

For let $\lim_{n \rightarrow \infty} \frac{1}{n^{\frac{1}{n}}} = y$ say; $\log y = \lim_{n \rightarrow \infty} -\frac{1}{n} \cdot \log n = -\lim_{n \rightarrow \infty} \frac{\log n}{n} = 0$

$\therefore y = e^0 = 1$ $\left(\left(\frac{\infty}{\infty} \right) \text{ using L Hospitals rule} \right)$

By comparison test both $\sum u_n$ and $\sum v_n$ converge or diverge. But p -series test, $\sum v_n$ diverges (since $p=1$); Hence $\sum u_n$ diverges.

6.2.33 Example

Test for convergence the series $\sum_{n=1}^{\infty} \frac{(n+a)^r}{(n+b)^p (n+c)^q}$, a, b, c, p, q, r , being +ve.

Solution

$$\begin{aligned} u_n &= \frac{(n+a)^r}{(n+b)^p (n+c)^q} = \frac{n^r \left(1 + \frac{a}{n}\right)^r}{n^p \left(1 + \frac{b}{n}\right)^p \cdot n^q \left(1 + \frac{c}{n}\right)^q} \\ &= \frac{1}{n^{p+q-r}} \cdot \frac{\left(1 + \frac{a}{n}\right)^r}{\left(1 + \frac{b}{n}\right)^p \left(1 + \frac{c}{n}\right)^q}; \end{aligned}$$

Take $v_n = \frac{1}{n^{p+q-r}}$; $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = 1 \neq 0$;

Applying comparison test both $\sum u_n$ and $\sum v_n$ converge or diverge.

But by p -series test, $\sum v_n$ converges if $(p+q-r) > 1$ and diverges if $(p+q-r) \leq 1$.

Hence $\sum u_n$ converges if $(p+q-r) > 1$ and diverges if $(p+q-r) \leq 1$.

6.2.34 Example

Test the convergence of the following series whose n^{th} terms are :

(a) $\frac{(3n+4)}{(2n+1)(2n+3)(2n+5)}$; (b) $\tan \frac{1}{n}$;

(c) $\left(\frac{1}{n^2}\right)\left(\frac{n+1}{n+3}\right)^n$ (d) $\frac{1}{(3^n + 5^n)}$;

(e) $\frac{1}{n \cdot 3^n}$

Solution

(a) Hint : Take $v_n = \frac{1}{n^2}$; $\sum v_n$ is convergent ; $\lim_{n \rightarrow \infty} \left(\frac{u_n}{v_n}\right) = \frac{3}{8} \neq 0$ (Verify)

Apply comparison test:

$\sum u_n$ is convergent [the student is advised to work out this problem fully]

(b) Proceed as in 1.2.16; $\sum u_n$ is convergent.

(c) Hint : Take $v_n = \frac{1}{n^2}$; $\lim_{n \rightarrow \infty} \left(\frac{u_n}{v_n}\right) = \lim_{n \rightarrow \infty} \frac{(1 + \frac{1}{n})^n}{(1 + \frac{3}{n})^n} = \frac{e}{e^3} = \frac{1}{e^2} \neq 0$

$v_n = \frac{1}{n^2}$ is convergent (work out completely for yourself)

(d) $u_n = \frac{1}{3^n + 5^n} = \frac{1}{5^n} \cdot \frac{1}{1 + \left(\frac{3}{5}\right)^n}$; Take $v_n = \frac{1}{5^n}$; $\lim_{n \rightarrow \infty} \left(\frac{u_n}{v_n}\right) = 1 \neq 0$

$\sum u_n$ and $\sum v_n$ behave the same way. But $\sum v_n$ is convergent since it is a geometric series with common ratio $\frac{1}{5} < 1$

$\therefore \sum u_n$ is convergent by comparison test.

$$(e) \quad \frac{1}{n \cdot 3^n} \leq \frac{1}{3^n}, \forall n \in N, \text{ since } n \cdot 3^n \geq 3^n;$$

$$\therefore \sum \frac{1}{n \cdot 3^n} \leq \sum \frac{1}{3^n} \quad \dots(1)$$

The series on the R.H.S of (1) is convergent since it is geometric series with $r = \frac{1}{3} < 1$.

$\therefore$ By comparison test $\sum \frac{1}{n \cdot 3^n}$ is convergent.

6.2.35 Example

Test the convergence of the following series.

$$(a) \quad 1 + \frac{1+2}{1^2+2^2} + \frac{1+2+3}{1^2+2^2+3^2} + \frac{1+2+3+4}{1^2+2^2+3^2+4^2} + \dots$$

$$(b) \quad 1 + \frac{1^2+2^2}{1^3+2^3} + \frac{1^2+2^2+3^2}{1^3+2^3+3^3} + \frac{1^2+2^2+3^2+4^2}{1^3+2^3+3^3+4^3} + \dots$$

Solution

$$(a) \quad u_n = \frac{1+2+3+\dots+n}{1^2+2^2+3^2+\dots+n^2} = \frac{n \frac{(n+1)}{2}}{n(n+1) \frac{(2n+1)}{6}} = \frac{3}{(2n+1)}$$

$$\text{Take } v_n = \frac{1}{n}; \quad \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \left(\frac{3n}{2n+1} \right) = \frac{3}{2} \neq 0$$

$\sum u_n$ and $\sum v_n$ behave alike by comparison test.

But $\sum v_n$ is diverges by p -series test. Hence $\sum u_n$ is divergent.

$$(b) \quad u_n = \frac{1^2+2^2+\dots+n^2}{1^3+2^3+\dots+n^3} = \frac{n(n+1) \frac{(2n+1)}{6}}{n^2 \frac{(n+1)^2}{4}} = \frac{2(2n+1)}{3n(n+1)}$$

Hint : Take $v_n = \frac{1}{n}$ and proceed as in (a) and show that $\sum u_n$ is divergent.

Exercise 6(a)

1. Test for convergence the infinite series whose n^{th} term are:

- (a) $\frac{1}{n - \sqrt{n}}$ [Ans : divergent]
- (b) $\frac{\sqrt{n+1} - \sqrt{n}}{n}$ [Ans : convergent]
- (c) $\sqrt{n^2 + 1} - n$ [Ans : divergent]
- (d) $\frac{\sqrt{n}}{n^2 - 1}$ [Ans : convergent]
- (e) $\sqrt{n^3 + 1} - \sqrt{n^3}$ [Ans : divergent]
- (f) $\frac{1}{\sqrt{n(n+1)}}$ [Ans : divergent]
- (g) $\frac{\sqrt{n}}{n^2 + 1}$ [Ans : convergent]
- (h) $\frac{2n^3 + 5}{4n^5 + 1}$ [Ans : convergent]

2. Determine whether the following series are convergent or divergent.

- (a) $\frac{1}{1+3^{-1}} + \frac{2}{1+3^{-2}} + \frac{3}{1+3^{-3}} + \dots$ [Ans : divergent]
- (b) $\frac{12}{1^3} + \frac{22}{2^3} + \frac{32}{3^3} + \dots + \frac{2+10n}{n^3} + \dots$ [Ans : convergent]
- (c) $\frac{1}{\sqrt{1} + \sqrt{2}} + \frac{1}{\sqrt{2} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{4}} + \dots$ [Ans : divergent]
- (d) $\frac{2}{3^2} + \frac{3}{4^2} + \frac{4}{5^2} + \dots$ [Ans : divergent]
- (e) $\frac{1}{1^2} + \frac{1}{2^3} + \frac{1}{3^4} + \dots$ [Ans : convergent]

- (f) $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n^2+1}}{\sqrt[4]{4n^2+2n+3}}$ [Ans : divergent]
- (g) $\sum_1^{\infty} (8^{1/n} - 1)$ [Ans : divergent]
- (h) $\sum_1^{\infty} \frac{3n^3+8}{5n^5+9}$ [Ans : convergent]
- (i) $\frac{1}{1.3} + \frac{2}{3.5} + \frac{3}{5.7} + \dots$ [Ans : divergent]

6.3

6.3.1 D' Alembert's Ratio Test

Let (i) $\sum u_n$ be a series of +ve terms and (ii) $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = k (\geq 0)$

Then the series $\sum u_n$ is (i) convergent if $k < 1$ and (ii) divergent if $k > 1$.

Proof: Case (i): $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = k (< 1)$

From the definition of a limit, it follows that

$$\exists m > 0 \text{ and } l (0 < l < 1) \ni \frac{u_{n+1}}{u_n} < l \forall n \geq m$$

$$\text{i.e., } \frac{u_{m+1}}{u_m} < l, \frac{u_{m+2}}{u_{m+1}} < l, \dots$$

$$\begin{aligned} \therefore u_m + u_{m+1} + u_{m+2} + \dots &= u_m \left[1 + \frac{u_{m+1}}{u_m} + \frac{u_{m+2}}{u_m} + \dots \right] \\ &= u_m \left[1 + \frac{u_{m+1}}{u_m} + \frac{u_{m+2}}{u_{m+1}} \cdot \frac{u_{m+1}}{u_m} + \dots \right] \\ &< u_m (1 + l + l^2 + \dots) = u_m \cdot \frac{1}{1-l} (l < 1) \end{aligned}$$

But $u_m \cdot \frac{1}{1-l}$ is a finite quantity $\therefore \sum_{n=m}^{\infty} u_n$ is convergent

By adding a finite number of terms $u_1 + u_2 + \dots + u_{m-1}$, the convergence of the

series is unaltered. $\sum_{n=m}^{\infty} u_n$ is convergent.

Case (ii) $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = k > 1$

There may be some finite number of terms in the beginning which do not satisfy the condition $\frac{u_{n+1}}{u_n} \geq 1$. In such a case we can find a number 'm' $\ni \frac{u_{n+1}}{u_n} \geq 1, \forall n \geq m$

Omitting the first 'm' terms, if we write the series as $u_1 + u_2 + u_3 + \dots$, we have

$$\frac{u_2}{u_1} \geq 1, \frac{u_3}{u_2} \geq 1, \frac{u_4}{u_3} \geq 1 \dots \text{and so on}$$

$$\begin{aligned} \therefore u_1 + u_2 + \dots + u_n &= u_1 \left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2} \cdot \frac{u_2}{u_1} + \dots \right) \quad (\text{to } n \text{ terms}) \\ &\geq u_1 (1 + 1 + 1 + 1 + \dots \text{to } n \text{ terms}) \\ &= nu_1 \end{aligned}$$

$$\lim_{n \rightarrow \infty} \sum_{n=1}^n u_n \geq \lim_{n \rightarrow \infty} nu_1 \text{ which } \rightarrow \infty; \therefore \sum u_n \text{ is divergent}$$

6.3.2

Note : 1

The ratio test fails when $k = 1$. As an example, consider the series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$

Here $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^p = \lim_{n \rightarrow \infty} \left(\frac{1}{1 + \frac{1}{n}} \right)^p = 1$

i.e., $k = 1$ for all values of p ,

But the series is convergent if $p > 1$ and divergent if $p \leq 1$, which shows that when $k = 1$, the series may converge or diverge and hence the test fails.

Note : 2 Ratio test can also be stated as follows :

If $\sum u_n$ is series of +ve terms and if $\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = k$, then $\sum u_n$ is convergent

If $k > 1$ and divergent if $k < 1$ (the test fails when $k = 1$).

Solved Examples

Tests for convergence of Series

6.3.3 Example

(a)
$$\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \dots$$

Solution

$$u_n = \frac{x^n}{n(n+1)}; u_{n+1} = \frac{x^{n+1}}{(n+1)(n+2)},$$

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)(n+2)} \cdot \frac{n(n+1)}{x^n} = \frac{1}{\left(1 + \frac{2}{n}\right)} x.$$

Therefore $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = x$

$\therefore$ By ratio test $\sum u_n$ is convergent When $|x| < 1$ and divergent when $|x| > 1$;

When $x = 1$, $u_n = \frac{1}{n^2(1+1/n)}$; Take $v_n = \frac{1}{n^2}$; $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = 1$

$\therefore$ By comparison test $\sum u_n$ is convergent.

Hence $\sum u_n$ is convergent when $|x| \leq 1$ and divergent when $|x| > 1$

(b)
$$1 + 3x + 5x^2 + 7x^3 + \dots$$

Solution

$$u_n = (2n-1)x^{n-1}; u_{n+1} = (2n+1)x^n$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \left(\frac{2n+1}{2n-1} \right) x = x$$

$\therefore$ By ratio test $\sum u_n$ is convergent when $|x| < 1$ and divergent when $|x| > 1$

When $x = 1$: $u_n = 2n-1$; $\lim_{n \rightarrow \infty} u_n = \infty$; $\therefore \sum u_n$ is divergent.

Hence $\sum u_n$ is convergent when $|x| < 1$ and divergent when $|x| \geq 1$

(c)
$$\sum_{n=1}^{\infty} \frac{x^n}{n^2 + 1} \dots$$

Solution

$$u_n = \frac{x^n}{n^2 + 1} ; u_{n+1} = \frac{x^{n+1}}{(n+1)^2 + 1}$$

Hence
$$\frac{u_{n+1}}{u_n} = \left(\frac{n^2 + 1}{n^2 + 2n + 2} \right) x$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \left[\frac{n^2 \left(1 + \frac{1}{n^2} \right)}{n^2 \left(1 + \frac{2}{n} + \frac{2}{n^2} \right)} \right] (x) = x$$

∴ By ratio test, $\sum u_n$ is convergent when $|x| < 1$ and divergent when $|x| > 1$

When $x = 1 : u_n = \frac{1}{n^2 + 1}$; Take $v_n = \frac{1}{n^2}$

∴ By comparison test, $\sum u_n$ is convergent when $|x| \leq 1$ and divergent when $|x| > 1$

6.3.4 Example

Test the series $\sum_{n \rightarrow \infty} \left(\frac{n^2 - 1}{n^2 + 1} \right) x^n, x > 0$ for convergence.

Solution

$$u_n = \left(\frac{n^2 - 1}{n^2 + 1} \right) x^n ; u_{n+1} = \left[\frac{(n+1)^2 - 1}{(n+1)^2 + 1} \right] x^{n+1}$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \left[\left(\frac{n^2 + 2n}{n^2 + 2n + 2} \right) \left(\frac{n^2 + 1}{n^2 - 1} \right) \right] \cdot x$$

$$= \lim_{n \rightarrow \infty} \left[\frac{n^4 (1 + 2/n)(1 + 1/n)}{n^4 (1 + 2/n + 2/n^2)(1 - 1/n)} \right] = x$$

∴ By ratio test, $\sum u_n$ is convergent when $x < 1$ and divergent when $x > 1$ when $x = 1$,

$$u_n = \frac{n^2 - 1}{n^2 + 1} \quad \text{Take } v_n = \frac{1}{n^0}$$

Applying p -series and comparison test, it can be seen that $\sum u_n$ is divergent when $x = 1$.

$\therefore \sum u_n$ is convergent when $x < 1$ and divergent $x \geq 1$

6.3.5 Example

Show that the series $1 + \frac{2^p}{2} + \frac{3^p}{3} + \frac{4^p}{4} + \dots$, is convergent for all values of p .

Solution

$$u_n = \frac{n^p}{n}; u_{n+1} = \frac{(n+1)^p}{n+1}$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \left[\frac{(n+1)^p}{n+1} \times \frac{n}{n^p} \right] = \lim_{n \rightarrow \infty} \left\{ \frac{1}{(n+1)} \left(\frac{n+1}{n} \right)^p \right\}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{(n+1)} \times \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^p = 0 < 1;$$

$\sum u_n$ is convergent for all ' p '.

6.3.6 Example

Test the convergence of the following series

$$\frac{1}{1^p} + \frac{1}{3^p} + \frac{1}{5^p} + \frac{1}{7^p} + \dots$$

Solution

$$u_n = \frac{1}{(2n-1)^p}; u_{n+1} = \frac{1}{(2n+1)^p}$$

$$\frac{u_{n+1}}{u_n} = \frac{(2n-1)^p}{(2n+1)^p} = \frac{2^p \cdot n^p (1 - 1/2n)^p}{2^p n^p (1 + 1/2n)^p}; \quad \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = 1$$

$\therefore$ Ratio test fails.

$$\text{Take } v_n = \frac{1}{n^p}; \frac{u_n}{v_n} = \frac{n^p}{(2n-1)^p} = \frac{1}{2^p \left(1 - \frac{1}{2n} \right)^p}; \quad \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{1}{2^p},$$

which is non-zero and finite

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ both converge or both diverge.

But by p -series test, $\sum v_n = \sum \frac{1}{n^p}$ converges when $p > 1$ and diverges

when $p \leq 1$

$\therefore \sum u_n$ is convergent if $p > 1$ and divergent if $p \leq 1$.

6.3.7 Example

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^3}, x > 0$

Solution

$$u_n = \frac{(n+1)x^n}{n^3}; u_{n+1} = \frac{(n+2)x^{n+1}}{(n+1)^3}$$

$$\frac{u_{n+1}}{u_n} = \frac{n+2}{(n+1)^3} \cdot x^{n+1} \cdot \frac{n^3}{(n+1)x^n} = \left(\frac{n+2}{n+1}\right) \left(\frac{n}{n+1}\right)^3 \cdot x$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \left(\frac{n+2}{n+1} \right) \left(\frac{n}{n+1} \right)^3 \cdot x = x$$

$\therefore$ By ratio test, $\sum u_n$ converges when $x < 1$ and diverges when $x > 1$.

When $x = 1$, $u_n = \frac{n+1}{n^3}$

Take $v_n = \frac{1}{n^2}$; By comparison test $\sum u_n$ is convergent (give proof)

$\therefore \sum u_n$ is convergent if $x \leq 1$ and divergent if $x > 1$.

6.3.8 Example

Test the convergence of the series

(i) $\sum_{n=1}^{\infty} \left(\frac{n^2}{2^n} + \frac{1}{n^2} \right)$ (ii) $1 + \frac{2.5.8}{1.5.9} + \frac{2.5.8.11}{1.5.9.13} + \dots$ (iii) $\frac{1}{3} + \frac{1.2}{3.5} + \frac{1.2.3}{3.5.7} + \dots$

Solution

(i) $\sum_{n=1}^{\infty} \left(\frac{n^2}{2^n} + \frac{1}{n^2} \right) = \sum_{n=1}^{\infty} \frac{n^2}{2^n} + \sum_{n=1}^{\infty} \frac{1}{n^2}$ Let $u_n = \frac{n^2}{2^n}; v_n = \frac{1}{n^2}$

$$u_{n+1} = \frac{(n+1)^2}{2^{n+1}}; \frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{1}{2} \left(1 + \frac{1}{n}\right)^2 = \frac{1}{2} < 1$$

$\therefore$ By ratio test $\sum u_n$ is convergent. By p -series test, $\sum v_n$ is convergent.

$\therefore$ The given series $\left(\sum u_n + \sum v_n\right)$ is convergent.

(ii) Neglecting the first term, the series can be taken as,

$$\frac{2.5.8}{1.5.9} + \frac{2.5.8.11}{1.5.9.13} +$$

Here, 1st term has 3 fractions, 2nd term has 4 fractions and so on.

$\therefore n^{\text{th}}$ term contains $(n+2)$ fractions

2. 5. 8.....are in A. P.

$\therefore (n+2)^{\text{th}}$ term = $2 + (n+1)3 = 3n+5$;

$\therefore$ 1. 5. 9,.....are in A. P.

$\therefore (n+2)^{\text{th}}$ term = $1 + (n+1)4 = 4n+5$

$$\therefore u_n = \frac{2.5.8.....(3n+5)}{1.5.9.....(4n+5)}$$

$$u_{n+1} = \frac{2.5.8.....(3n+5)(3n+8)}{1.5.9.....(4n+5)(4n+9)}$$

$$\frac{u_{n+1}}{u_n} = \frac{(3n+8)}{(4n+9)};$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{n\left(3 + \frac{8}{n}\right)}{n\left(4 + \frac{9}{n}\right)} = \frac{3}{4} < 1$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

(iii) 1, 2, 3, ...are in A. P. n^{th} term = n ; 3. 5. 7.....are in A.P. n^{th} term = $2n+1$

$$\begin{aligned} \therefore u_n &= \left[\frac{1.2.3.....n}{3.5.7.....(2n+1)} \right] \\ u_{n+1} &= \left[\frac{1.2.3.....n(n+1)}{3.5.7.....(2n+1)(2n+3)} \right] \\ \frac{u_{n+1}}{u_n} &= \left(\frac{n+1}{2n+3} \right) \\ \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} &= \lim_{n \rightarrow \infty} \frac{n \cdot \left(1 + \frac{1}{n} \right)}{n \left(2 + \frac{3}{n} \right)} = \frac{1}{2} < 1 \end{aligned}$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

6.3.9 Exercise

Test for the convergence of the series $\sum \frac{1.2.3.....n}{3.5.7.....(2n+1)}$

Solution

$$\begin{aligned} u_n &= \frac{1.2.3.....n}{3.5.7.....(2n+1)} ; u_{n+1} = \frac{1.2.3.....(n+1)}{3.5.7.....(2n+3)} ; \\ \frac{u_{n+1}}{u_n} &= \frac{n+1}{2n+3} ; \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{n \left(1 + \frac{1}{n} \right)}{n \left(2 + \frac{3}{n} \right)} = \frac{1}{2} < 1 ; \end{aligned}$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

6.3.10 Example

Test for convergence $\sum_{n=1}^{\infty} \frac{1.3.5....(2n-1)}{2.4.6....2n} . x^{n-1} (x > 0)$

Solution

The given series of +ve terms has $u_n = \frac{1.3.5....(2n-1)}{2.4.6....2n} . x^{n-1}$

and $u_{n+1} = \frac{1.3.5....(2n+1)}{2.4.6....(2n+2)} x^n$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \left(\frac{2n+1}{2n+2} \right) x = \lim_{n \rightarrow \infty} \frac{2n \left(1 + \frac{1}{2n} \right)}{2n \left(1 + \frac{2}{2n} \right)} \cdot x = x$$

$\therefore$ By ratio test, $\sum u_n$ is converges when $x < 1$ and diverges when $x > 1$ when $x = 1$, the test fails.

Then
$$u_n = \frac{1.3.5 \dots (2n-1)}{2.4.6 \dots 2n} < 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} u_n \neq 0$$

$\therefore \sum u_n$ is divergent. Hence $\sum u_n$ is convergent when $x < 1$, and divergent when $x \geq 1$

6.3.11 Example

Test for the convergence of $1 + \frac{2}{5}x + \frac{6}{9}x^2 + \dots + \left(\frac{2^n - 2}{2^n + 1} \right) x^{n-1} + \dots (x > 0)$

Solution

Omitting 1st term, $u_n = \left(\frac{2^n - 2}{2^n + 1} \right) x^{n-1}, (n \geq 2)$ and ' u_n ' are all +ve.

$$\begin{aligned} u_{n+1} &= \left(\frac{2^{n+1} - 2}{2^{n+1} + 1} \right) x^n; \quad \lim_{n \rightarrow \infty} \left(\frac{u_{n+1}}{u_n} \right) = \lim_{n \rightarrow \infty} \left(\frac{2^{n+1} - 2}{2^{n+1} + 1} \right) \times \left(\frac{2^n + 1}{2^n - 2} \right) \cdot x \\ &= \lim_{n \rightarrow \infty} \left[\frac{2^{n+1} \left(1 - \frac{1}{2^n} \right)}{2^{n+1} \left(1 + \frac{1}{2^{n+1}} \right)} \cdot \frac{2^n \left(1 + \frac{1}{2^n} \right)}{2^n \left(1 - \frac{2}{2^n} \right)} \cdot x \right] = x; \end{aligned}$$

Hence, by ratio test, $\sum u_n$ converges if $x < 1$ and diverges if $x > 1$.

When $x = 1$, the test fails. Then $u_n = \frac{2^n - 2}{2^n + 1}; \lim_{n \rightarrow \infty} u_n = 1 \neq 0; \therefore \sum u_n$ diverges

Hence $\sum u_n$ is convergent when $x < 1$ and divergent $x > 1$

6.3.12 Example

Using ratio test show that the series $\sum_{n=0}^{\infty} \frac{(3-4i)^n}{n!}$ converges

Solution

$$u_n = \frac{(3-4i)^n}{n!}; \quad u_{n+1} = \frac{(3-4i)^{n+1}}{(n+1)!};$$

$$\lim_{n \rightarrow \infty} \left(\frac{u_{n+1}}{u_n} \right) = \lim_{n \rightarrow \infty} \left(\frac{3-4i}{n+1} \right) = 0 < 1$$

Hence, by ratio test, $\sum u_n$ converges.

6.3.13 Example

Discuss the nature of the series, $\frac{2}{3.4}x + \frac{3}{4.5}x^2 + \frac{4}{5.6}x^3 + \dots \infty (x > 0)$

Solution

Since $x > 0$, the series is of +ve terms ;

$$u_n = \frac{(n+1)}{(n+2)(n+3)} x^n > u_{n+1} = \frac{(n+2)}{(n+3)(n+4)} x^{n+1}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} &= \left[\frac{(n+2)^2 \cdot x}{(n+1)(n+4)} \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{n^2 \left(1 + \frac{2}{n}\right)^2 \cdot x}{n^2 \left(1 + \frac{5}{n} + \frac{4}{n^2}\right)} \right] = x; \end{aligned}$$

Therefore by ratio test, $\sum u_n$ converges if $x < 1$ and diverges if $x > 1$

When $x = 1$, the test fails; Then $u_n = \frac{(n+1)}{(n+2)(n+3)}$;

Taking $v_n = \frac{1}{n}$; $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = 1 \neq 0$

$\therefore$ By comparison test $\sum u_n$ and $\sum v_n$ behave same way. But $\sum v_n$ is divergent

by p -series test. ($p = 1$);

$\therefore \sum u_n$ is diverges when $x = 1$

$\therefore \sum u_n$ is convergent when $x < 1$ and divergent when $x \geq 1$

6.3.14 Example

Discuss the nature of the series $\sum \frac{3.6.9 \dots 3n.5^n}{4.7.10 \dots (3n+1)(3n+2)}$

Solution

$$\begin{aligned}
 \text{Here, } u_n &= \frac{3.6.9.....3n}{4.7.10.....(3n+1)} \frac{5^n}{(3n+2)}; \\
 u_{n+1} &= \frac{3.6.9.....3n(3n+3)5^{n+1}}{4.7.10.....(3n+1)(3n+4)(3n+5)}; \\
 \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} &= \lim_{n \rightarrow \infty} \frac{(3n+2)(3n+3).5}{(3n+4)(3n+5)} \\
 &= \lim_{n \rightarrow \infty} \left[\frac{5.9n^2 \left(1 + \frac{2}{3n}\right) \left(1 + \frac{3}{3n}\right)}{9n^2 \left(1 + \frac{4}{3n}\right) \left(1 + \frac{5}{3n}\right)} \right] = 5 > 1
 \end{aligned}$$

$\therefore$ By ratio test, $\sum u_n$ is divergent.

6.3.15 Example

Test for convergence the series $\sum_{n=1}^{\infty} n^{1-n}$

Solution

$$\begin{aligned}
 u_n &= n^{1-n}; u_{n+1} = (n+1)^{-n}; \\
 \frac{u_{n+1}}{u_n} &= \frac{(n+1)^{-n}}{n^{1-n}} = \frac{n}{n(n+1)^n} = \frac{1}{n} \left(\frac{n}{n+1} \right)^n \\
 \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} &= \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \left(\frac{1}{1 + \frac{1}{n}} \right)^n = 0 \cdot \frac{1}{e} = 0 < 1
 \end{aligned}$$

$\therefore$ By ratio test $\sum u_n$, is convergent

6.3.16 Example

Test the series $\sum_{n=1}^{\infty} \frac{2n^3}{\lfloor n \rfloor}$, for convergence.

Solution

$$u_n = \frac{2n^3}{\lfloor n \rfloor}; u_{n+1} = \frac{2(n+1)^3}{\lfloor n+1 \rfloor}$$

$$\frac{u_{n+1}}{u_n} = \frac{2(n+1)^3}{n+1} \times \frac{\frac{n}{2n^3}}{n^3} = \frac{(n+1)^2}{n^3} = \frac{\left(1 + \frac{1}{n}\right)^2}{n};$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = 0 < 1;$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

6.3.17 Example

Test convergence of the series $\sum \frac{2^n n!}{n^n}$

Solution

$$u_n = \frac{2^n n!}{n^n}; \quad u_{n+1} = \frac{2^{n+1} (n+1)!}{(n+1)^{n+1}};$$

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+1} (n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n n!} = 2 \left(\frac{n}{n+1} \right)^n$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = 2 \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{2}{e} < 1 \quad (\text{since } 2 < e < 3)$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

6.3.20 Example

Test the convergence of the series $\sum u_n$ where u_n is

$$(a) \quad \frac{n^2 + 1}{3^n + 1} \qquad (b) \quad \frac{x^{n-1}}{(2n+1)^a}, (a > 0)$$

$$(c) \quad \left(\frac{1.2.3 \dots n}{4.7.10 \dots 3n+3} \right)^2 \qquad (d) \quad \frac{\sqrt{1+2^n}}{\sqrt{1+3^n}}$$

$$(e) \quad \left(\frac{3n^3 + 7n^2}{5n^9 + 11} \right) x^n$$

Solution

$$(a) \quad \lim_{n \rightarrow \infty} \left(\frac{u_{n+1}}{u_n} \right) = \lim_{n \rightarrow \infty} \left[\frac{(n+1)^2 + 1}{3^{n+1} + 1} \times \frac{3^n + 1}{n^2 + 1} \right]$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \left[\frac{n^2 \left(1 + \frac{2}{n} + \frac{2}{n^2}\right)}{n^2 \left(1 + \frac{1}{n}\right)} \cdot \frac{3^n \left(1 + \frac{1}{3^n}\right)}{3^{n+1} \left(1 + \frac{1}{3^{n+1}}\right)} \right] \\
&= \frac{1}{3} < 1
\end{aligned}$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

$$\begin{aligned}
(b) \quad \lim_{n \rightarrow \infty} \left(\frac{u_{n+1}}{u_n} \right) &= \lim_{n \rightarrow \infty} \left[\frac{x^n}{(2n+3)^a} \times \frac{(2n+1)^a}{x^{n+1}} \right] \\
&= \lim_{n \rightarrow \infty} \left[\frac{2^a n^a \left(1 + \frac{1}{2n}\right)^a}{2^a n^a \left(1 + \frac{3}{2n}\right)^a} \cdot x \right] = x
\end{aligned}$$

By ratio test, $\sum u_n$ convergence if $x < 1$ and diverges if $x > 1$.

When $x = 1$, the test fails; Then, $u_n = \frac{1}{(2n+1)^a}$; Taking $v_n = \frac{1}{n^a}$ we have,

$$\lim_{n \rightarrow \infty} \left(\frac{u_n}{v_n} \right) = \lim_{n \rightarrow \infty} \left(\frac{n}{2n+1} \right)^a = \lim_{n \rightarrow \infty} \frac{1}{\left(2 + \frac{1}{n}\right)^a} = \frac{1}{2^a} \neq 0 \text{ and finite (since } a > 0 \text{)}.$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ have same property

But p -series test, we have

(i) $\sum v_n$ convergent when $a > 1$

and (ii) divergent when $a \leq 1$

$\therefore$ To sum up, (i) $x < 1$, $\sum u_n$ is convergent $\forall a$.

(ii) $x > 1$, $\sum u_n$ is divergent $\forall a$.

(iii) $x = 1$, $a > 1$, $\sum u_n$ is convergent, and

(iv) $x = 1$, $a \leq 1$, $\sum u_n$ is divergent.

$$\begin{aligned}
 \text{(c)} \quad \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} &= \lim_{n \rightarrow \infty} \left[\frac{1.2.3 \dots n(n+1)}{4.7.10 \dots (3n+3)(3n+6)} \times \frac{4.7.10 \dots (3n+3)}{1.2.3 \dots n} \right]^2 \\
 &= \lim_{n \rightarrow \infty} \left[\frac{(n+1)}{3(n+2)} \right]^2 = \frac{1}{9} < 1 \quad ;
 \end{aligned}$$

$\therefore$ By ratio test, $\sum u_n$ is convergent

$$\begin{aligned}
 \text{(d)} \quad \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} &= \lim_{n \rightarrow \infty} \left[\frac{(1+2^{n+1})}{(1+3^{n+1})} \times \frac{(1+3^n)}{(1+2^n)} \right]^{\frac{1}{2}} \\
 &= \lim_{n \rightarrow \infty} \left[\frac{2^{n+1} \left(1 + \frac{1}{2^{n+1}}\right)}{3^{n+1} \left(1 + \frac{1}{3^{n+1}}\right)} \times \frac{3^n \left(1 + \frac{1}{3^n}\right)}{2^n \left(1 + \frac{1}{2^n}\right)} \right]^{\frac{1}{2}} = \left(\frac{2}{3}\right)^{\frac{1}{2}} < 1
 \end{aligned}$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

$$\begin{aligned}
 \text{(e)} \quad \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} &= \lim_{n \rightarrow \infty} \left[\frac{3(n+1)^3 + 7(n+1)^2}{5(n+1)^9 + 11} \times \frac{5n^9 + 11}{3n^3 + 7} \times x \right] \\
 &= \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \left[\frac{3n^3 \left(1 + \frac{1}{n}\right)^3 + 7n^2 \left(1 + \frac{1}{n}\right)^2}{5n^9 \left(1 + \frac{1}{n}\right)^9 + 11} \times \frac{5n^9 \left(1 + \frac{11}{5n^9}\right)}{3n^3 \left(1 + \frac{7}{3n^3}\right)} \times x \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{3n^3 \left\{ \left(1 + \frac{1}{n}\right)^3 + \frac{7}{3n} \left(1 + \frac{1}{n}\right)^2 \right\}}{5n^9 \left\{ \left(1 + \frac{1}{n}\right)^9 + \frac{11}{5n^9} \right\}} \times \frac{5n^9 \left(1 + \frac{11}{5n^9}\right)}{3n^3 \left(1 + \frac{7}{3n^3}\right)} \times x \right] = x
 \end{aligned}$$

$\therefore$ By ratio test, $\sum u_n$ converges when $x < 1$ and diverges when $x > 1$.

When $x = 1$, the test fails,

$$\text{Then } u_n = \frac{3n^3 \left(1 + \frac{7}{3n}\right)}{5n^9 \left(1 + \frac{11}{5n^9}\right)} = \frac{3}{5n^6} \left(1 + \frac{7}{3n}\right)$$

Taking $v_n = \frac{1}{n^6}$, we observe that $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{3}{5} \neq 0$

$\therefore$ By comparison test and p series test, we conclude that $\sum u_n$ is convergent.

$\therefore \sum u_n$ is convergent when $x \leq 1$ and divergent when $x > 1$.

Exercise - 1(b)

1. Test the convergency or divergency of the series whose general term is :

(a) $\frac{x^n}{n}$ [Ans : $|x| < 1$ cgt, $|x| \geq 1$ dgt]

(b) nx^{n-1} [Ans : $|x| < 1$ cgt, $|x| \geq 1$ dgt]

(c) $\left(\frac{2^n - 2}{2^n + 1}\right)x^{n-1}$ [Ans : $|x| < 1$ cgt, $|x| \geq 1$ dgt]

(d) $\left(\frac{n^2 + 1}{n^2 - 1}\right)x^n$ [Ans : $|x| < 1$ cgt, $|x| \geq 1$ dgt]

(e) $\frac{|n|}{n^n}$ [Ans: cgt.]

(f) $\frac{4^n |n|}{n^n}$ [Ans: dgt.]

(g) $\frac{(n^3 + 1)^n}{(3^n + 1)}$ [Ans: cgt.]

2. Determine whether the following series are convergent or divergent :

(a) $\frac{x}{1.2} + \frac{x^2}{3.4} + \frac{x^3}{5.6} + \dots$ [Ans : $|x| \leq 1$ cgt, $|x| > 1$ dgt]

(b) $1 + \frac{x}{2^2} + \frac{x^2}{3^2} + \frac{x^3}{4^2} + \dots$ [Ans : $|x| \leq 1$ cgt, $|x| > 1$ dgt]

(c) $\frac{1}{1.2.3} + \frac{x}{4.5.6} + \frac{x^2}{7.8.9} + \dots$ [Ans : $|x| \leq 1$ cgt, $|x| > 1$ dgt]

(d) $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \dots + \frac{x^n}{n^2 + 1} + \dots$ [Ans : $|x| \leq 1$ cgt, $|x| > 1$ dgt]

(e) $\frac{1.2}{x} + \frac{2.3}{x^2} + \frac{3.4}{x^3} + \dots$ [Ans : $|x| > 1$ cgt, $|x| \leq 1$ dgt]

6.4 Raabe's Test

Let $\sum u_n$ be series of +ve terms and let $\lim_{n \rightarrow \infty} \left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right\} = k$

Then

(i) If $k > 1$, $\sum u_n$ is convergent.

(ii) If $k < 1$, $\sum u_n$ is divergent. (The test fails if $k = 1$)

Proof:

Consider the series $\sum v_n = \sum \frac{1}{n^p}$

$$\begin{aligned} n \left[\frac{v_n}{v_{n+1}} - 1 \right] &= n \left[\left(\frac{n+1}{n} \right)^p - 1 \right] = n \left[\left(1 + \frac{1}{n} \right)^p - 1 \right] \\ &= n \left[\left(1 + \frac{p}{n} + \frac{p(p-1)}{2} \cdot \frac{1}{n^2} + \dots \right) - 1 \right] \\ &= p + \frac{p(p-1)}{2} \cdot \frac{1}{n} + \dots \\ \lim_{n \rightarrow \infty} n \left\{ \frac{v_n}{v_{n+1}} - 1 \right\} &= p \end{aligned}$$

Case (i) :

In this case,

$$\lim_{n \rightarrow \infty} n \left\{ \frac{u_n}{u_{n+1}} - 1 \right\} = k > 1$$

We choose a number ' p ' $\exists k > p > 1$; Comparing the series $\sum u_n$ with $\sum v_n$ which is convergent, we get that $\sum u_n$ will converge if after some fixed number of terms

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} = \left(\frac{n+1}{n} \right)^p$$

i.e. If,

$$n \left(\frac{u_n}{u_{n+1}} - 1 \right) > p + \frac{p(p-1)}{2} \cdot \frac{1}{n} + \dots \text{from (1)}$$

i.e., If

$$\lim_{n \rightarrow \infty} n \left\{ \frac{u_n}{u_{n+1}} - 1 \right\} > p$$

i.e., If $k > p$, which is true. Hence $\sum u_n$ is convergent. The second case also can be proved similarly.

Solved Examples

6.4.1 Example

Test for convergence the series

$$x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4} \cdot \frac{x^5}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{x^7}{7} + \dots$$

Solution

Neglecting the first term, the series can be taken as,

$$\frac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4} \cdot \frac{x^5}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{x^7}{7} + \dots$$

$$1.3.5 \dots \text{are in A.P. } n^{\text{th}} \text{ term} = 1 + (n-1)2 = 2n-1$$

$$2.4.6 \dots \text{are in A.p. } n^{\text{th}} \text{ term} = 2 + (n-1)2 = 2n$$

$$3.5.7 \dots \text{are in A.P } n^{\text{th}} \text{ term} = 3 + (n-1)2 = 2n+1$$

$$\therefore u_n (n^{\text{th}} \text{ term of the series}) =$$

$$= \frac{1.3.5 \dots (2n-1)}{2.4.6 \dots (2n)} \cdot \frac{x^{2n+1}}{2n+1}$$

$$u_{n+1} = \frac{1.3.5 \dots (2n-1)(2n+1)}{2.4.6 \dots (2n)(2n+2)} \cdot \frac{x^{2n+3}}{2n+3}$$

$$\begin{aligned} \frac{u_{n+1}}{u_n} &= \frac{1.3.5 \dots (2n+1)}{2.4.6 \dots (2n+2)} \cdot \frac{x^{2n+3}}{(2n+3)} \cdot \frac{2.4.6 \dots 2n}{1.3.5 \dots (2n-1)} \cdot \frac{(2n+1)}{x^{2n+1}} \\ &= \frac{(2n+1)^2 x^2}{(2n+2)(2n+3)} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{4n^2 \left(1 + \frac{1}{2n}\right)^2}{4n^2 \left(1 + \frac{2}{2n}\right) \left(1 + \frac{3}{2n}\right)} x^2 = x^2$$

$\therefore$ By ratio test, $\sum u_n$ converges if $|x| < 1$ and diverges if $|x| > 1$

If $|x| = 1$ the test fails.

$$\begin{aligned}
 \text{Then } x^2 &= 1 \quad \text{and} \quad \frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2} \\
 \frac{u_n}{u_{n+1}} - 1 &= \frac{(2n+2)(2n+3)}{(2n+1)^2} - 1 = \frac{6n+5}{(2n+1)^2} \\
 \lim_{n \rightarrow \infty} \left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right\} &= \lim_{n \rightarrow \infty} \left(\frac{6n^2 + 5n}{4n^2 + 4n + 1} \right) \\
 &= \lim_{n \rightarrow \infty} \frac{n^2 \left(6 + \frac{5}{n} \right)}{n^2 \left(4 + \frac{4}{n} + \frac{1}{n^2} \right)} = \frac{3}{2} > 1
 \end{aligned}$$

By Raabe's test, $\sum u_n$ converges. Hence the given series is convergent when $|x| \leq 1$ and divergent when $|x| > 1$.

6.4.2 Example

Test for the convergence of the series

$$1 + \frac{3}{7}x + \frac{3.6}{7.10}x^2 + \frac{3.6.9}{7.10.13}x^3 + \dots; x > 0$$

Solution

Neglecting the first term,

$$\begin{aligned}
 u_n &= \frac{3.6.9 \dots 3n}{7.10.13 \dots 3n+4} x^n \\
 u_{n+1} &= \frac{3.6.9 \dots 3n(3n+3)}{7.10.13 \dots (3n+4)(3n+7)} x^{n+1} \\
 \frac{u_{n+1}}{u_n} &= \frac{3n+3}{3n+7} x \quad ; \quad \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = x
 \end{aligned}$$

$\therefore$ By ratio test, $\sum u_n$ is convergent when $x < 1$ and divergent when $x > 1$.

When $x = 1$ The ratio test fails. Then

$$\begin{aligned}
 \frac{u_n}{u_{n+1}} &= \frac{3n+7}{3n+3} ; \frac{u_n}{u_{n+1}} - 1 = \frac{4}{3n+3} \\
 \lim_{n \rightarrow \infty} \left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right\} &= \lim_{n \rightarrow \infty} \left(\frac{4n}{3n+3} \right) = \frac{4}{3} > 1
 \end{aligned}$$

$\therefore$ By Raabe's test, $\sum u_n$ is convergent. Hence the given series converges if $x \leq 1$ and diverges if $x > 1$.

6.4.3 Example

Examine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1^2 \cdot 5^2 \cdot 9^2 \dots (4n-3)^2}{4^2 \cdot 8^2 \cdot 12^2 \dots (4n)^2}$$

Solution

$$u_n = \frac{1^2 \cdot 5^2 \cdot 9^2 \dots (4n-3)^2}{4^2 \cdot 8^2 \cdot 12^2 \dots (4n)^2}$$

$$u_{n+1} = \frac{1^2 \cdot 5^2 \cdot 9^2 \dots (4n-3)^2 (4n+1)^2}{4^2 \cdot 8^2 \cdot 12^2 \dots (4n)^2 (4n+4)^2}$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{(4n+1)^2}{(4n+4)^2} = 1 \quad (\text{verify})$$

$\therefore$ The ratio test fails.

Hence by Raabe's test, $\sum u_n$ is convergent.

6.4.4 Example

Find the nature of the series $\sum \frac{(\lfloor n \rfloor)^2}{\lfloor 2n \rfloor} x^n, (x > 0)$

Solution

$$u_n = \frac{(\lfloor n \rfloor)^2}{\lfloor 2n \rfloor} x^n; \quad u_{n+1} = \frac{(\lfloor n+1 \rfloor)^2}{\lfloor 2n+2 \rfloor} x^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{(2n+1)(2n+2)} x;$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{n^2 \left(1 + \frac{1}{n}\right)^2}{4n^2 \left(1 + \frac{1}{2n}\right) \left(1 + \frac{2}{2n}\right)} x = \frac{x}{4}$$

$\therefore$ By ratio test, $\sum u_n$ converges when $\frac{x}{4} < 1$, i. e. ; $x < 4$; and diverges when $x > 4$; When $x = 4$, the test fails.

$$\begin{aligned}\text{When } x &= 4, \frac{u_n}{u_{n+1}} = \frac{(2n+1)(2n+2)}{4(n+1)^2} \\ \frac{u_n}{u_{n+1}} - 1 &= \frac{-2n-2}{4(n+1)^2} = \frac{-1}{2(n+1)}; \\ \lim_{n \rightarrow \infty} \left[n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right] &= \frac{-1}{2} < 1\end{aligned}$$

$\therefore$ By ratio test, $\sum u_n$ is divergent

Hence $\sum u_n$ is convergent when $x < 4$ and divergent when $x > 4$

6.4.5 Example

Test for convergence of the series $\sum \frac{4.7....(3n+1)}{1.2.3....n} x^n$.

Solution

$$u_n = \frac{4.7....(3n+1)}{1.2.3....n} x^n ; u_{n+1} = \frac{4.7....(3n+1)(3n+4)}{1.2.3....n(n+1)} x^{n+1}$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \left[\frac{(3n+4)}{(n+1)} x \right] = 3x$$

$\therefore$ By ratio test $\sum u_n$ converges if $3x < 1$ i.e., $x < \frac{1}{3}$ and diverges if $x > \frac{1}{3}$;

If $x = \frac{1}{3}$, the test fails

$$\text{When } x = \frac{1}{3}, n \left[\frac{u_n}{u_{n+1}} - 1 \right] = n \left[\frac{(n+1)3}{3n+1} - 1 \right] = n \left[\frac{-1}{3n+4} \right] = -\frac{1}{\left(3 + \frac{4}{n} \right)}$$

$$\lim_{n \rightarrow \infty} n \left[\frac{u_n}{u_{n+1}} - 1 \right] = -\frac{1}{3} < 1$$

$\therefore$ By Raabe's test, $\sum u_n$ is divergent.

$\therefore \sum u_n$ is convergent when $x < \frac{1}{3}$ and divergent when $x \geq \frac{1}{3}$

6.4.6 Example

Test for convergence $2 + \frac{3x}{2} + \frac{4x^2}{3} + \frac{5x^3}{4} + \dots (x > 0)$

Solution

The n^{th} term $u_n = \frac{(n+1)}{n} x^{n-1};$

$$u_{n+1} = \frac{(n+2)}{(n+1)} x^n; \quad \frac{u_{n+1}}{u_n} = \frac{n(n+2)}{(n+1)^2} \cdot x$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{n^2 \left(1 + \frac{2}{n}\right)}{n^2 \left(1 + \frac{1}{n}\right)^2} \cdot x = x$$

$\therefore$ By ratio test, $\sum u_n$ is convergent if $x < 1$ and divergent if $x > 1$

If $x = 1$, the test fails.

$$\begin{aligned} \text{Then } \lim_{n \rightarrow \infty} n \left[\frac{u_n}{u_{n+1}} - 1 \right] &= \lim_{n \rightarrow \infty} n \left[\frac{(n+1)^2}{n(n+2)} - 1 \right] \\ &= \lim_{n \rightarrow \infty} n \left[\frac{1}{n(n+2)} \right] = 0 < 1 \end{aligned}$$

$\therefore$ By Raabe's test $\sum u_n$ is divergent

$\therefore \sum u_n$ is convergent when $x < 1$ and divergent when $x \geq 1$

6.4.7 Example

Find the nature of the series $\frac{3}{4} + \frac{3.6}{4.7} + \frac{3.6.9}{4.7.10} + \dots \infty$

Solution

$$u_n = \frac{3.6.9 \dots 3n}{4.7.10 \dots (3n+1)}; \quad u_{n+1} = \frac{3.6.9 \dots 3n(3n+3)}{4.7.10 \dots (3n+1)(3n+4)}$$

$$\frac{u_{n+1}}{u_n} = \frac{3n+3}{3n+4}; \quad \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{3n \left(1 + \frac{3}{3n}\right)}{3n \left(1 + \frac{4}{3n}\right)} = 1$$

Ratio test fails.

$$\therefore \lim_{n \rightarrow \infty} n \left[\frac{u_n}{u_{n+1}} - 1 \right] = \lim_{n \rightarrow \infty} n \left[\frac{3n+4}{3n+3} - 1 \right]$$

$$= \lim_{n \rightarrow \infty} \frac{n}{3(n+1)} = \lim_{n \rightarrow \infty} \frac{n}{3n(1 + \frac{1}{n})} = \frac{1}{3} < 1$$

$\therefore$ By Raabe's test $\sum u_n$ is divergent.

6.4.8 Example

If $p, q > 0$ and the series

$$1 + \frac{1}{2} \frac{p}{q} + \frac{1.3.p(p+1)}{2.4.q(q+1)} + \frac{1.3.5.p(p+1)(p+2)}{2.4.6.q(q+1)(q+2)} + \dots$$

is convergent, find the relation to be satisfied by p and q .

Solution

$$u_n = \frac{1.3.5 \dots (2n-1) p(p+1) \dots (p+n-1)}{2.4.6 \dots 2n q(q+1) \dots (q+n-1)} \quad [\text{neglecting 1st term}]$$

$$u_{n+1} = \frac{1.3.5 \dots (2n-1)(2n+1) p(p+1) \dots (p+n-1)(p+n)}{2.4.6 \dots 2n(2n+2) q(q+1) \dots (q+n-1)(q+n)}$$

$$\frac{u_{n+1}}{u_n} = \frac{(2n+1)(p+n)}{(2n+2)(q+n)};$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \left[\frac{2n(1 + \frac{1}{2n})}{2n(1 + \frac{1}{2n})} \cdot \frac{n(1 + \frac{p}{n})}{n(1 + \frac{q}{n})} \right] = 1$$

$\therefore$ ratio test fails.

Let us apply Raabe's test

$$\lim_{n \rightarrow \infty} \left[n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right] = \lim_{n \rightarrow \infty} \left[n \left\{ \frac{(q+n)(2n+2)}{(p+n)(2n+1)} - 1 \right\} \right]$$

$$\lim_{n \rightarrow \infty} \left[n \left\{ \frac{2q(n+1) - p(2n+1) + n}{n^2 \left(1 + \frac{p}{n} \right) \left(2 + \frac{1}{n} \right)} \right\} \right]$$

$$\lim_{n \rightarrow \infty} \left[\frac{2q(1 + \frac{1}{n}) - p(2 + \frac{1}{n}) + 1}{2} \right] = \frac{2q - 2p + 1}{2}$$

Since $\sum u_n$ is convergent, by Raabe's test, $\frac{2q-2p+1}{2} > 1$

$\Rightarrow q-p > \frac{1}{2}$, is the required relation.

Exercise 1 (c)

1. Test whether the series $\sum_1^{\infty} u_n$ is convergent or divergent where

$$u_n = \frac{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n-2)^2}{3 \cdot 4 \cdot 5 \cdots (2n-1)(2n)} x^{2n} \quad [\text{Ans : } |x| \leq 1 \text{ cgt, } |x| > 1 \text{ dgt}]$$

2. Test for the convergence the series

$$\sum_1^{\infty} \frac{4 \cdot 7 \cdot 10 \cdots (3n+1)}{n} x^n \quad [\text{Ans : } |x| < \frac{1}{3} \text{ cgt, } |x| \geq \frac{1}{3} \text{ dgt}]$$

3. Test for the convergence the series :

$$(i) \quad \frac{2^2 \cdot 4^2}{3^2 \cdot 3^2} + \frac{2^2 \cdot 4^2 \cdot 5^2 \cdot 7^2}{3^2 \cdot 3^2 \cdot 6^2 \cdot 6^2} + \frac{2^2 \cdot 4^2 \cdot 5^2 \cdot 7^2 \cdot 8^2 \cdot 10^2}{3^2 \cdot 3^2 \cdot 6^2 \cdot 6^2 \cdot 9^2 \cdot 9^2} + \dots$$

[Ans : divergent]

$$(ii) \quad \frac{3 \cdot 4}{1 \cdot 2} x + \frac{4 \cdot 5}{2 \cdot 3} x^2 + \frac{5 \cdot 6}{3 \cdot 4} x^3 + \dots (x > 0)$$

[Ans : cgt if $x \leq 1$ dgt if $x > 1$]

$$(iii) \quad \sum \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \cdot \frac{x^n}{(2n+2)} (x > 0)$$

[Ans : cgt if $x \leq 1$ dgt if $x > 1$]

$$(iv) \quad 1 + \frac{(\underline{1})^2}{\underline{2}} x + \frac{(\underline{2})^2}{\underline{4}} x^2 + \frac{(\underline{3})^2}{\underline{6}} x^3 + \dots (x > 0)$$

[Ans : cgt if $x < 4$ and dgt if $x \geq 4$]

6.5 Cauchy's Root Test

Let $\sum u_n$ be a series of +ve terms and let $\lim_{n \rightarrow \infty} u_n^{1/n} = l$. Then $\sum u_n$ is convergent when $l < 1$ and divergent when $l > 1$

Proof:

$$(i) \quad \lim_{n \rightarrow \infty} u_n^{1/n} = l < 1 \Rightarrow \exists a \text{ +ve number '}\lambda\text{' (}l < \lambda < 1\text{)} \ni u_n^{1/n} < \lambda, \forall n > m$$

$$(\text{or}) u_n < \lambda^n, \forall n > m$$

Since $\lambda < 1$, $\sum \lambda^n$ is a geometric series with common ratio < 1 and therefore convergent.

Hence $\sum u_n$ is convergent.

$$(ii) \quad \lim_{n \rightarrow \infty} u_n^{1/n} = l > 1$$

$\therefore$ By the definition of a limit we can find a number $r \ni u_n^{1/n} > 1 \forall n > r$

i.e., $u_n > \forall n > r$

i.e., after the 1st 'r' terms, each term is > 1 .

$$\lim_{n \rightarrow \infty} \sum u_n = \infty \quad \therefore \sum u_n \text{ is divergent.}$$

Note :

When $\lim_{n \rightarrow \infty} (u_n^{1/n}) = 1$, the root test can't decide the nature of $\sum u_n$. The fact of this statement can be observed by the following two examples.

1. Consider the series $\sum 1/n^3$: $\lim_{n \rightarrow \infty} u_n^{1/n} = \lim_{n \rightarrow \infty} \left(\frac{1}{n^3} \right)^{1/n} = \lim_{n \rightarrow \infty} \left(\frac{1}{n^{1/n}} \right)^3 = 1$
2. Consider the series $\sum 1/n$, in which $\lim_{n \rightarrow \infty} u_n^{1/n} = \lim_{n \rightarrow \infty} \frac{1}{n^{1/n}} = 1$

In both the examples given above, $\lim_{n \rightarrow \infty} u_n^{1/n} = 1$. But series (1) is convergent

(p-series test)

And series (2) is divergent. Hence when the *limit*=1, the test fails.

Solved Examples

6.5.1 Example

Test for convergence the infinite series whose n^{th} terms are:

$$(i) \quad \frac{1}{n^{2n}}$$

$$(ii) \quad \frac{1}{(\log n)^n}$$

$$(iii) \quad \frac{1}{\left[1 + \frac{1}{n}\right]^{n^2}}$$

Solution

$$(i) \quad u_n = \frac{1}{n^{2n}}, u_n^{1/n} = \frac{1}{n^2} \quad ; \quad Lt_{n \rightarrow \infty} u_n^{1/n} = Lt_{n \rightarrow \infty} \frac{1}{n^2} = 0 < 1;$$

By root test $\sum u_n$ is convergent.

$$(ii) \quad u_n = \frac{1}{(\log n)^n}; u_n^{1/n} = \frac{1}{\log n} \quad ; \quad Lt_{n \rightarrow \infty} u_n^{1/n} = Lt_{n \rightarrow \infty} \frac{1}{\log n} = 0 < 1;$$

$\therefore$ By root test, $\sum u_n$ is convergent.

$$(iii) \quad u_n = \frac{1}{\left(1 + \frac{1}{n}\right)^{n^2}}; u_n^{1/n} = \frac{1}{\left(1 + \frac{1}{n}\right)^n}$$

$$Lt_{n \rightarrow \infty} u_n^{1/n} = Lt_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e} < 1;$$

$\therefore$ By root test $\sum u_n$ is convergent.

6.5.2 Example

Find whether the following series are convergent or divergent.

$$(i) \quad \sum_{n=1}^{\infty} \frac{1}{3^n - 1}$$

Solution

$$u_n^{1/n} = \left(\frac{1}{3^n - 1} \right)^{1/n} = \left(\frac{1}{3^n \left(1 - \frac{1}{3^n} \right)} \right)^{1/n}$$

$$Lt_{n \rightarrow \infty} u_n^{1/n} = Lt_{n \rightarrow \infty} \left(\frac{1}{3^n \left(1 - \frac{1}{3^n} \right)} \right)^{1/n} = \frac{1}{3} < 1;$$

By root test, $\sum u_n$ is convergent.

$$(ii) \quad 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \dots$$

Solution

$$u_n = \frac{1}{n^n}; \quad \lim_{n \rightarrow \infty} u_n^{1/n} = \lim_{n \rightarrow \infty} \left(\frac{1}{n^n} \right)^{1/n} = 0 < 1$$

By root test, $\sum u_n$ is convergent.

$$(iii) \quad \sum_{n=1}^{\infty} \frac{[(n+1)x]^n}{n^{n+1}}$$

Solution

$$u_n = \frac{[(n+1)x]^n}{n^{n+1}}$$

$$\lim_{n \rightarrow \infty} u_n^{1/n} = \lim_{n \rightarrow \infty} \left[\frac{\{(n+1)x\}^n}{n^{n+1}} \right]^{1/n}$$

$$\lim_{n \rightarrow \infty} \left[\left\{ \frac{(n+1)x}{n} \right\}^n \cdot \frac{1}{n} \right]^{1/n} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right) x \cdot \frac{1}{n^{1/n}}$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) x \cdot \frac{1}{n^{1/n}} = \lim_{n \rightarrow \infty} x \cdot \frac{1}{n^{1/n}} = x \quad \left(\text{since } \lim_{n \rightarrow \infty} x \cdot \frac{1}{n^{1/n}} = 1 \right)$$

$\therefore \sum u_n$ is convergent if $|x| < 1$ and divergent if $|x| > 1$ and when $|x| = 1$ the test fails.

Then $u_n = \frac{(n+1)^n}{n^{n+1}}; \quad \text{Take } v_n = \frac{1}{n}$

$$\frac{u_n}{v_n} = \frac{(n+1)^n}{n^{n+1}} \cdot n = \frac{(n+1)^n}{n^n} = \left(1 + \frac{1}{n} \right)^n; \quad \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = e > 1$$

$\therefore$ By comparison test, $\sum u_n$ is divergent.

($\sum v_n$ diverges by p -series test)

Hence $\sum u_n$ is convergent if $|x| < 1$ and divergent $|x| \geq 1$

6.5.3 Example

If $u_n = \frac{n^{n^2}}{(n+1)^{n^2}}$, show that $\sum u_n$ is convergent

$$\begin{aligned}
 \lim_{n \rightarrow \infty} u_n^{1/n} &= \lim_{n \rightarrow \infty} \left[\frac{n^{n^2}}{(n+1)^{n^2}} \right]^{1/n} ; = \lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^n} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n \\
 &= \lim_{n \rightarrow \infty} \left(\frac{1}{1 + \frac{1}{n}} \right)^n = \frac{1}{e} < 1 ; \therefore \sum u_n \text{ converges by root test .}
 \end{aligned}$$

6.5.4 Example

Establish the convergence of the series $\frac{1}{3} + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{7}\right)^3 + \dots$

Solution

$$\begin{aligned}
 u_n &= \left(\frac{n}{2n+1} \right)^n \dots\dots(\text{verify}); \\
 \lim_{n \rightarrow \infty} u_n^{1/n} &= \lim_{n \rightarrow \infty} \left(\frac{n}{2n+1} \right) = \frac{1}{2} < 1
 \end{aligned}$$

By root test, $\sum u_n$ is convergent.

6.5.5 Example

Test for the convergence of $\sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}} \cdot x^n$

Solution

$$u_n = \left(\frac{1}{1 + \frac{1}{n}} \right)^{\frac{1}{2}} \cdot x^n ; \lim_{n \rightarrow \infty} u_n^{1/n} = \lim_{n \rightarrow \infty} \left(\frac{1}{1 + \frac{1}{n}} \right)^{\frac{1}{2}} \cdot x = x$$

$\therefore$ By root test, $\sum u_n$ is convergent if $|x| < 1$ and divergent if $|x| > 1$.

When $|x| = 1$: $u_n = \sqrt{\frac{n}{n+1}}$, taking $v_n = \frac{1}{n^0}$ and applying comparison test, it can

be seen that is divergent

$\sum u_n$ is convergent if $|x| < 1$ and divergent if $|x| \geq 1$.

6.5.6 Example

Show that $\sum_{n=1}^{\infty} \left(n^{\frac{1}{n}} - 1 \right)^n$ converges.

Solution

$$u_n = \left(n^{\frac{1}{n}} - 1 \right)^n$$

$$\lim_{n \rightarrow \infty} u_n^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left(n^{\frac{1}{n}} - 1 \right) = 1 - 1 = 0 < 1 \left(\text{since } \lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1 \right);$$

$\therefore \sum u_n$ is convergent by root test.

6.5.7 Example

Examine the convergence of the series whose n^{th} term is $\left(\frac{n+2}{n+3} \right)^n \cdot x^n$

Solution

$$u_n = \left(\frac{n+2}{n+3} \right)^n \cdot x^n; \quad \lim_{n \rightarrow \infty} u_n^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left(\frac{n+2}{n+3} \right) x = x$$

$\therefore$ By root test, $\sum u_n$ converges when $|x| < 1$ and diverges when $|x| > 1$

$$\text{When } |x| = 1: u_n = \left(\frac{n+2}{n+3} \right)^n; \quad \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{2}{n} \right)^n}{\left(1 + \frac{3}{n} \right)^n}$$

$$= \frac{e^2}{e^3} = \frac{1}{e} \neq 0 \quad \text{and the terms are all +ve.}$$

$\therefore \sum u_n$ is divergent. Hence $\sum u_n$ is convergent if $|x| < 1$ and divergent if $|x| \geq 1$.

6.5.8 Example

Show that the series,

$$\left[\frac{2^2}{1^2} - \frac{2}{1} \right]^{-1} + \left[\frac{3^3}{2^3} - \frac{3}{2} \right]^{-2} + \left[\frac{4^4}{3^4} - \frac{4}{3} \right]^{-3} + \dots \text{ is convergent}$$

$$u_n = \left[\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n} \right]^{-n}; \quad = \left(\frac{n+1}{n} \right)^{-n} \left[\left(\frac{n+1}{n} \right)^n - 1 \right]^{-n}$$

$$\left(1 + \frac{1}{n}\right)^{-n} \left[\left(1 + \frac{1}{n}\right)^n - 1 \right]^{-n}; u_n^{1/n} = \left(1 + \frac{1}{n}\right)^{-1} \left[\left(1 + \frac{1}{n}\right)^n - 1 \right]^{-1}$$

$$= \frac{1}{\left(1 + \frac{1}{n}\right) \left\{ \left(1 + \frac{1}{n}\right)^n - 1 \right\}}$$

$$\therefore \lim_{n \rightarrow \infty} u_n^{1/n} = \frac{1}{1} \cdot \frac{1}{e-1} = \frac{1}{e-1} < 1$$

$\therefore$ By root test, $\sum u_n$ is convergent.

6.5.9 Example

Test $\sum_{m=1}^{\infty} u_m$ for convergence when $u_m = \frac{e^{-m}}{\left(1 + \frac{2}{m}\right)^{-m^2}}$

Solution

$$\lim_{m \rightarrow \infty} \left(u_m^{1/m} \right) = \lim_{m \rightarrow \infty} \left[\frac{\left(1 + \frac{2}{m}\right)^{m^2}}{e^m} \right]^{1/m}; \lim_{m \rightarrow \infty} \frac{1}{e} \left(1 + \frac{2}{m}\right)^m = \frac{e^2}{e} = e > 1$$

Hence Cauchy's root tells us that $\sum u_m$ is divergent.

6.5.11 Example :

Test the convergence of the series $\sum \frac{n}{e^{n^2}}$

Solution

$$\lim_{n \rightarrow \infty} u_n^{1/n} = \lim_{n \rightarrow \infty} \frac{n^{1/n}}{e^n} = 0 < 1$$

$\therefore$ By root test, $\sum u_n$ is convergent.

6.5.12 Example

Test the convergence of the series, $\frac{2}{1^2}x + \frac{3^2}{2^3}x^2 + \dots \frac{(n+1)^n \cdot x^n}{n^{n+1}} + \dots, x > 0$

Solution

$$\begin{aligned}
 \lim_{n \rightarrow \infty} u_n^{1/n} &= \lim_{n \rightarrow \infty} \left[\frac{(n+1)^n \cdot x^n}{n^{n+1}} \right]^{1/n} = \lim_{n \rightarrow \infty} \left[\left(\frac{n+1}{n} \right) \cdot \frac{1}{n^{1/n}} \cdot x \right] \\
 &= \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n} \right) \cdot \frac{1}{n^{1/n}} \cdot x \right] = 1 \cdot 1 \cdot x = x \left[\text{since } \lim_{n \rightarrow \infty} n^{1/n} = 1 \right]
 \end{aligned}$$

$\therefore$ By root test, $\sum u_n$ converges if $x < 1$ and diverges when $x > 1$.

When $x = 1$, the test fails.

Then $u_n = \left(1 + \frac{1}{n} \right)^n \cdot \frac{1}{n}$; Take $v_n = \frac{1}{n}$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e \neq 0$$

$\therefore$ By comparison test and p -series test, $\sum u_n$ is divergent.

Hence $\sum u_n$ is convergent when $x < 1$ and divergent when $x \geq 1$.

Exercise 1 (d)

1. Test for convergence the infinite series whose n^{th} terms are :

(a) $\frac{1}{2^n - 1}$ [Ans : convergent]

(b) $\frac{1}{(\log)^{2n}} \cdot (n \neq 1)$ [Ans : convergent]

(c) $\left(\frac{3n+1}{4n+3} \cdot x \right)^n$ [Ans : $|x| < \frac{4}{3}$ cgt, $|x| \geq \frac{4}{3}$ dgt]

(d) $\frac{x^n}{\lfloor n \rfloor}$ [Ans : cgt for all $x \geq 0$]

(e) $\frac{\lfloor n \rfloor}{n^n}$ [Ans : convergent]

(f) $\frac{3^n \cdot \angle n}{n^3}$ [Ans : convergent]

$$(g) \quad \frac{(2n^2 - 1)^n}{(2n)^{2n}} \dots\dots\dots [\text{Ans : convergent}]$$

$$(h) \quad \left(n^{\frac{1}{n}} - 1\right)^{2n} \dots\dots\dots [\text{Ans : convergent}]$$

$$(i) \quad \left(\frac{n-1}{n}\right)^{-n^2} \dots\dots\dots [\text{Ans : divergent}]$$

$$(j) \quad \left(\frac{nx}{n+1}\right)^n, (x > 0) \dots\dots\dots [\text{Ans : } x < 1 \text{ cgt, } x \geq 1 \text{ dgt}]$$

2. Examine the following series for convergence :

$$(a) \quad 1 + \frac{x}{2} + \frac{x^2}{3^2} + \frac{x^3}{4^3} + \dots, x > 0 \dots\dots\dots [\text{Ans : } x \leq 1 \text{ cgt, } x > 1 \text{ dgt}]$$

$$(b) \quad \frac{1}{4} + \left(\frac{2}{7}\right)^2 + \left(\frac{3}{10}\right)^3 + \dots \dots\dots [\text{Ans : convergent}]$$

6.6

6.6.1 Integral Test

+ve term series,

$$\phi(1) + \phi(2) + \dots + \phi(n) + \dots$$

where $\phi(n)$ decreases as n increases is convergent or divergent according as the integral $\int_1^\infty \phi(x) dx$ is finite or infinite.

Proof:

$$\text{Let } S_n = \phi(1) + \phi(2) + \dots + \phi(n)$$

From the above figure, it can be seen that the area under the curve $y = \phi(x)$ between any two ordinates lies between the set of exterior and interior rectangles formed by the ordinates at

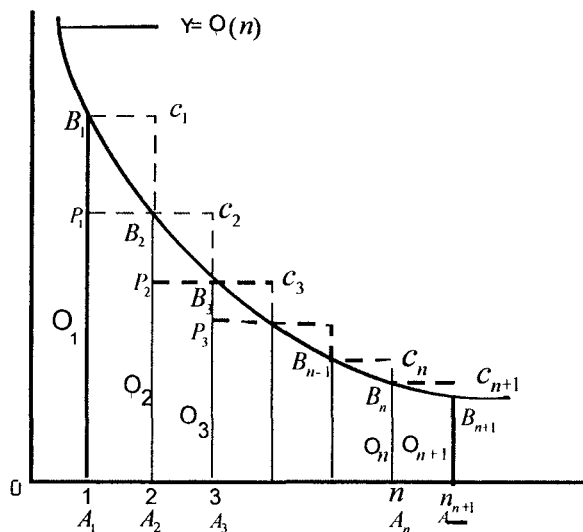
$$n = 1, 2, 3, \dots, n, n+1, \dots$$

Hence the total area under the curve lies between the sum of areas of all interior rectangles and sum of the areas of all the exterior rectangles.

Hence

$$\{\phi(1) + \phi(2) + \dots + \phi(n)\} \geq \int_1^{n+1} \phi(x) dx \geq \{\phi(2) + \phi(3) + \dots + \phi(n+1)\}$$

$$\therefore S_n \geq \int_1^\infty \phi(x) dx \geq S_{n+1} - \phi(1)$$



As $n \rightarrow \infty$, $\text{Lt } S_n$ is finite or infinite according as $\int_1^{\infty} \phi(x) dx$ is finite or infinite.
Hence the theorem.

Solved Examples

6.6.2 Example

Test for convergence the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$

Solution

$$\int_2^{\infty} \frac{1}{x \log x} dx = \text{Lt}_{n \rightarrow \infty} \left[\int_2^n \frac{1}{x \log x} dx \right] = \text{Lt}_{n \rightarrow \infty} [\log \log x]_2^n = \infty$$

$\therefore$ By integral test, the given series is divergent.

6.6.3 Example

Test for convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

Solution

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^p} dx &= \text{Lt}_{n \rightarrow \infty} \left[\int_1^n \frac{1}{x^p} dx \right] = \text{Lt}_{n \rightarrow \infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^n \\ &= \frac{1}{1-p} \text{Lt}_{n \rightarrow \infty} [n^{1-p} - 1] \end{aligned}$$

Case (i)

If $p > 1$, this limit is finite; $\therefore \sum \frac{1}{n^p}$ is convergent.

Case (ii)

If $p < 1$, the limit is in finite; $\therefore \sum \frac{1}{n^p}$ is divergent.

Case (iii)

If $p = 1$, the limit $\lim_{n \rightarrow \infty} \log x \Big|_1^n = \lim_{n \rightarrow \infty} (\log n) = \infty$; $\therefore \sum \frac{1}{n^p}$ is divergent.

Hence $\sum \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$

6.6.4 Example

Test the series $\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$ for convergence.

Solution

$$u_n = \frac{n}{e^{n^2}} = \phi(n) \text{ (say)};$$

$\phi(n)$ is +ve and decreases as n increases. So let us apply the integral test.

$$\begin{aligned} \int_1^{\infty} \phi(x) dx &= \int_1^{\infty} x e^{-x^2} dx = \frac{1}{2} \int_1^{\infty} e^{-t} dt \left\{ t = x^2, dt = 2x dx \right\} \\ &= -\frac{1}{2} e^{-t} \Big|_1^{\infty} = -\frac{1}{2} \left(0 - \frac{1}{e} \right) = \frac{1}{2e}, \text{ which is finite.} \end{aligned}$$

By integral test, $\sum u_n$ is convergent.

6.6.5 Example

Apply integral test to test the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n^2} \sin\left(\frac{\pi}{n}\right)$

Solution

Let $\phi(n) = \frac{1}{n^2} \sin\left(\frac{\pi}{n}\right)$; $\phi(n)$ decreases as n increases and is +ve.

$$\int_2^{\infty} \phi(x) dx = \int_2^{\infty} \frac{1}{x^2} \sin\left(\frac{\pi}{x}\right) dx$$

$$\text{Let } \pi/x = t$$

$$-\frac{1}{\pi} \int_{\pi/2}^0 \sin t dt = \frac{1}{\pi} \cos t \Big|_{\pi/2}^0 = \frac{1}{\pi} \text{ finite, } -\pi/x^2 dx = dt$$

$$1/x^2 dx = -1/\pi dt$$

$\therefore$ By integral test, $\sum u_n$ converges $x = 2 \Rightarrow t = \pi/2$ $x = \infty \Rightarrow t = 0$

6.6.6 Example

Apply integral test and determine the convergence of the following series.

$$(a) \sum_1^{\infty} \frac{3n}{4n^2 + 1} \quad (b) \sum_1^{\infty} \frac{2n^3}{3n^4 + 2} \quad (c) \sum_1^{\infty} \frac{1}{3n+1}$$

Solution

$$(a) \phi(n) = \frac{3n}{4n^2 + 1} \text{ is +ve and decreases as } n \text{ increases}$$

$$\int_1^{\infty} \phi(x) dx = \int_1^{\infty} \frac{3x}{4x^2 + 1} dx \quad \left(\begin{array}{l} 4x^2 + 1 = t \Rightarrow x dx = 1/8 dt \\ x = 1 \Rightarrow t = 5, x = \infty \Rightarrow t = \infty \end{array} \right)$$

$$\int_1^{\infty} \phi(x) dx = \lim_{n \rightarrow \infty} \left[\frac{3}{8} \int_5^t \frac{dt}{t} \right] = \lim_{n \rightarrow \infty} \left[\frac{3}{8} \log t - \log 5 \right] = \infty$$

$\therefore$ By integral test, $\sum u_n$ diverges.

$$(b) \phi(n) = \frac{2n^3}{3n^4 + 2} \text{ decreases as } n \text{ increases and is +ve}$$

$$\begin{aligned} \int_1^{\infty} \phi(x) dx &= \int_1^{\infty} \frac{2x^3}{3x^4 + 2} dx \\ &= \frac{1}{6} \int_5^{\infty} \frac{dt}{t} = \frac{1}{6} [\log t]_5^{\infty} = \infty \quad [\text{where } t = 3x^4 + 2] \end{aligned}$$

By integral test, $\sum u_n$ is divergent.

$$(c) \phi(n) = \frac{1}{3n+1} \text{ is +ve, and decreases as } n \text{ increases.}$$

$$\int_1^{\infty} \phi(x) dx = \int_1^{\infty} \frac{1}{3x+1} dx = \int_4^{\infty} \frac{1}{3} \frac{dt}{t} [t = 3x+1] = \frac{1}{3} \log t \Big|_4^{\infty} = \infty$$

$\therefore$ By integral test, $\sum u_n$ is divergent.

6.7

6.7.1 Alternating Series

A series, $u_1 - u_2 + u_3 - u_4 + - + \dots + (-1)^{n-1} u_n + \dots$, where u_n are all +ve, is an alternating series.

6.7.2 Leibnitz Test

If in an alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$, where u_n are all +ve,

- (i) $u_n > u_{n+1}, \forall n$, and
- (ii) $\lim_{n \rightarrow \infty} u_n = 0$, then the series is convergent.

Proof:

Let $u_1 - u_2 + u_3 - u_4 + \dots$ be an alternating series (' u_n ' are all +ve)

Let $u_1 > u_2 > u_3 > u_4 \dots$, Then the series may be written in each of the following two forms:

$$(u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \dots \quad \dots\dots\dots(\text{A})$$

$$u_1 - (u_2 - u_3) - (u_4 - u_5) - \dots \quad \dots\dots\dots(\text{B})$$

(A) Shows that the sum of any number of terms is +ve and

(B) Shows that the sum of any number of terms is $< u_1$.

Hence the sum of the series is finite. $\therefore$ The series is convergent.

Note:

If $\lim_{n \rightarrow \infty} u_n \neq 0$, then the series is oscillatory.

Solved Examples

6.7.3 Example

Consider the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots$

In this series, each term is numerically less than its preceding term and n^{th} term $\rightarrow 0$ as $n \rightarrow \infty$.

$\therefore$ By Leibnitz's test, the series is convergent.

(Note the sum of the above series is $\log_e 2$)

6.7.4 Example

Test for convergence $\sum \frac{(-1)^{n-1}}{2n-1}$

Solution

The given series is an alternating series $\sum (-1)^{n-1} u_n$, where $u_n = \frac{1}{2n-1}$

We observe that (i) $u_n > 0, \forall n$ (ii) $u_n > u_{n+1}, \forall n$ (iii) $\lim_{n \rightarrow \infty} u_n = 0$

$\therefore$ By Leibnitz's test, the given series is convergent.

6.7.5 Example

Show that the series $S = 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \dots$ converges.

Solution

The given series is $\sum_1 \frac{(-1)^{n-1}}{3^{n-1}} = \sum (-1)^{n-1} u_n$, where $u_n = \frac{1}{3^{n-1}}$ is an alternating series in which 1. $u_n > 0, \forall n$ 2. $u_n > u_{n+1}, \forall n$ and 3. $\lim_{n \rightarrow \infty} u_n = 0$;

Hence by Leibnitz's test, it is convergent.

6.7.6 Example

Test for convergence of the series, $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - + \dots, 0 < x < 1$

Solution

The given series is of the form $\sum \frac{(-1)^{n-1} \cdot x^n}{1+x^n} = \sum (-1)^{n-1} u_n$,

where $u_n = \frac{x^n}{1+x^n}$ Since $0 < x < 1$, $u_n > 0, \forall n$;

$$\begin{aligned} \text{Further, } u_n - u_{n+1} &= \frac{x^n}{1+x^n} - \frac{x^{n+1}}{1+x^{n+1}} \\ &= \frac{x^n - x^{n+1}}{(1+x^n)(1+x^{n+1})} = \frac{x^n(1-x)}{(1+x^n)(1+x^{n+1})} \end{aligned}$$

$0 < x < 1 \Rightarrow$ all terms in numerator and denominator of the above expression are +ve.

$$\therefore u_n > u_{n+1}, \forall n.$$

$$\text{Again, } x^n \rightarrow 0 \text{ as } x^n \rightarrow \infty \text{ since } 0 < x < 1; \therefore \lim_{n \rightarrow \infty} u_n = \frac{0}{1+0} = 0$$

$\therefore$ By Leibnitz's test, the given series is convergent.

6.7.7 Example

Test for convergence $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n(n+1)(n+2)}}$

Solution

The given series is an alternating series $\sum (-1)^{n-1} u_n$

where $u_n = \frac{1}{\sqrt{n(n+1)(n+2)}}; u_n > 0, \forall n;$

Again, $\sqrt{(n+1)(n+2)(n+3)} > \sqrt{n(n+1)(n+2)}$

$\therefore \frac{1}{\sqrt{(n+1)(n+2)(n+3)}} < \frac{1}{\sqrt{n(n+1)(n+2)}}, \forall n$

i.e., $u_{n+1} < u_n, \forall n$

Further, $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n(n+1)(n+2)}} = 0$

$\therefore$ By Leibnitz's test, $\sum_{n=2}^{\infty} (-1)^{n-1} u_n$ is convergent

6.7.8 Example

Test for the convergence of the following series,

$$\frac{1}{6} - \frac{2}{11} + \frac{3}{16} - \frac{4}{21} + \frac{5}{26} - + \dots$$

Solution

Given series, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{5n+1} = \sum (-1)^{n-1} u_n$ is an alternating series

$$u_n = \frac{n}{5n+1} > 0 \forall n;$$

$$\frac{n}{5n+1} - \frac{n+1}{5n+6} = \frac{-1}{(5n+1)(5n+6)} \Rightarrow u_n < u_{n+1}, \forall n$$

Again, $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{n}{5n+1} = \frac{1}{5} \neq 0$

Thus conditions (ii) or (iii) of Leibnitz's test are not satisfied. The given series is not convergent. It is oscillatory.

6.7.9 Example

Test the nature of the following series.

$$(a) \sum_1^{\infty} \frac{(-1)^{n-1}}{\sqrt{n} + \sqrt{n+1}} \quad (b) \sum \frac{(-1)^{n-1}}{n^2 + 1} \quad (c) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\lfloor n+1 \rfloor}$$

Solution

$$(a) \quad u_n = \frac{1}{\sqrt{n} + \sqrt{n+1}} > 0 \forall n ;$$

$$u_n - u_{n+1} = \frac{1}{\sqrt{n} + \sqrt{n+1}} - \frac{1}{\sqrt{n+1} + \sqrt{n+2}}$$

$$= \frac{\sqrt{n+2} - \sqrt{n}}{(\sqrt{n} + \sqrt{n+1})(\sqrt{n+1} + \sqrt{n+2})} = \frac{2}{(\sqrt{n+2} + \sqrt{n})(\sqrt{n} + \sqrt{n+1})(\sqrt{n+1} + \sqrt{n+2})} > 0$$

$\therefore$ By Leibnitz's test the series converges.

$$(b) \quad u_n = \frac{1}{n^2 + 1} > 0, \forall n; \quad \frac{1}{n^2 + 1} > \frac{1}{(n+1)^2 + 1} \Rightarrow u_n > u_{n+1}, \forall n;$$

$$\lim_{n \rightarrow \infty} u_n = 0 \quad \therefore \text{By Leibnitz's test, given series converges.}$$

$$(c) \quad u_n = \frac{1}{\lfloor n+1 \rfloor} > 0, \forall n;$$

$$\lfloor n+2 \rfloor > \lfloor n+1 \rfloor \Rightarrow \frac{1}{\lfloor n+2 \rfloor} < \frac{1}{\lfloor n+1 \rfloor} \Rightarrow u_n > u_{n+1}, \forall n$$

By Leibnitz's test, given series converges.

6.7.10 Example

Test the convergence of the series $\frac{1}{5\sqrt{2}} - \frac{1}{5\sqrt{3}} + \frac{1}{5\sqrt{4}} - + \dots$

Solution

The series can be written as $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{5\sqrt{n+1}}$; $u_n = \frac{1}{5\sqrt{n+1}}$

$$(i) \quad u_n > 0 \forall n$$

$$(ii) \quad 5\sqrt{n+2} > 5\sqrt{n+1} \Rightarrow \frac{1}{5\sqrt{n+2}} < \frac{1}{5\sqrt{n+1}} \Rightarrow u_n > u_{n+1} \forall n$$

$$(iii) \quad \lim_{n \rightarrow \infty} u_n = 0 ;$$

By Leibnitz's test, the given series converges.

6.7.11 Example

Test for convergence the series, $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \dots$

Solution

The given series can be written as $\sum \frac{(-1)^n}{2n}$ (omitting 1st term)

$$\frac{1}{2n} > 0 \forall n; \frac{1}{2n} > \frac{1}{2n+2} \Rightarrow u_n > u_{n+1}, \forall n; \lim_{n \rightarrow \infty} \frac{1}{2n} = 0$$

$\therefore$ By Leibnitz's test, $\sum \frac{(-1)^n}{2n}$ is convergent.

6.7.12 Example

Test for convergence the series, $1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \dots$

Solution

General term of the series is $\frac{(-1)^{n-1}}{(2n-1)!}$

The series is an alternating series; $\frac{1}{(2n-1)!} > 0 \forall n$

$$\frac{1}{(2n-1)!} > \frac{1}{(2n+1)!} \Rightarrow u_n > u_{n+1}, \forall n \in N; \lim_{n \rightarrow \infty} \frac{1}{(2n-1)!} = 0$$

By Leibnitz's test, given series is convergent.

6.8 Absolute convergence

A series $\sum u_n$ is said to be absolutely convergent if the series $\sum |u_n|$ is convergent

6.8.1

Consider the series

$$\sum u_n = 1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \dots$$

$$\sum |u_n| = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

By p -series test, $\sum |u_n|$ is convergent ($p = 3 > 1$)

Hence $\sum u_n$ is absolutely convergent.

Note :

1. If $\sum u_n$ is a series of +ve terms, then $\sum u_n = \sum |u_n|$.

For such a series, there is no difference between convergence and absolute convergence. Thus a series of +ve terms is convergent as well as absolutely convergent.

2. An absolutely convergent series is convergent. But the converse need not be true.

Consider
$$\sum_1^{\infty} (-1)^{n-1} \cdot \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

This series is convergent (1.7.3)

But $\sum_1^{\infty} |(-1)^{n-1} \cdot \frac{1}{n}| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is divergent (p-series test).

Thus $\sum u_n$ is convergent need not imply that $\sum |u_n|$ is convergent (i.e., $\sum u_n$ is not absolutely convergent).

6.9

6.9.1 Conditional Convergence

If the series $\sum |u_n|$ is divergent and $\sum u_n$ is convergent, then $\sum u_n$ is said to be conditionally convergent.

6.9.2 Consider the Series

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ $\sum u_n$ is convergent by Leibnitz's test. (Ex.1.7.3)

But $\sum |u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is divergent by p - series test.

$\therefore \sum u_n$ is conditionally convergent.

6.10

6.10.1 Power Series and Interval of Convergence

A series, $a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$ where ' a_n ' are all constants is a power series in x .

It may converge for some values of x .

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} &= \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \cdot x \quad (1^{\text{st}} \text{ term is omitted.}) \\ &= kx \quad (\text{say}) \quad \text{where} \quad \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = k\end{aligned}$$

Then, by ratio test, the series converges when $|kx| < 1$.

i.e., it converges $\forall x \in \left(-\frac{1}{k}, \frac{1}{k}\right) (k \neq 0)$

The interval $\left(-\frac{1}{k}, \frac{1}{k}\right)$ is known as the interval of convergence of the given power series.

Solved Examples

6.10.2 Example

Find the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{x^n}{n^3}$

Solution

$$u_n = \frac{x^n}{n^3}; u_{n+1} = \frac{x^{n+1}}{(n+1)^3}$$

$$\lim_{n \rightarrow \infty} \left(\frac{u_{n+1}}{u_n} \right) = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^3 \cdot x = \lim_{n \rightarrow \infty} \left(\frac{1}{1 + \frac{1}{n}} \right)^3 \cdot x = x$$

By ratio test, the given series converges when $|x| < 1$, i.e., $x \in (-1, 1)$

When $x = 1$, $\sum u_n = \sum \frac{1}{n^3}$, which, is convergent by p series test.

$\therefore \sum u_n$ is convergent when $x = 1$

Hence, the interval of convergence of the given series is $(-1, 1]$

6.10.3 Example

Test for the convergence of the following series.

(a) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$

(b) $1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \dots$

$$(c) \quad 1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + \dots$$

$$(d) \quad \sum_{n=0}^{\infty} (-1)^n (n+1)x^n, \text{ with } x < \frac{1}{2}$$

Solution

$$(a) \quad \text{The series is of the form } \sum (-1)^{n-1} u_n \text{ where } u_n = \frac{1}{\sqrt{n}}$$

It is an alternating series where (i) $u_n > 0 \forall n$ (ii) $u_n > u_{n+1} \forall n$ and (iii) $\lim_{n \rightarrow \infty} u_n = 0$; $\therefore$ By Leibnitz test, the series is convergent.

Again the series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots$ is divergent, by p -series test.

Hence the given series is conditionally convergent.

$$(b) \quad \sum |u_n| = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ which is convergent by } p\text{-series test.}$$

$\therefore$ The given series is absolutely convergent.

$\therefore$ It is convergent.

(c) The given series is

$$\sum (-1)^{n-1} \cdot \frac{x^{2n-2}}{(2n-2)!} = \sum (-1)^{n-1} u_n; \therefore |u_n| = \frac{x^{2n-2}}{(2n-2)!}$$

$$u_{n+1} = \frac{x^{2n}}{2n!}; \quad \left| \frac{u_{n+1}}{u_n} \right| = \frac{1}{(2n-1)(2n)} \cdot |x^2|; \quad \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = 0 < 1$$

By ratio test, the series $\sum |u_n|$ converges $\forall x$; i.e., $\sum u_n$ is absolutely convergent $\forall x$;

(d) Here, $|u_n| = (n+1)x^n$; $|u_{n+1}| = (n+2)x^{n+1}$ (neglect 1st term)

$$\lim_{n \rightarrow \infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n \rightarrow \infty} \frac{(n+2)}{(n+1)} |x| = \lim_{n \rightarrow \infty} \frac{(1 + \frac{2}{n})}{(1 + \frac{1}{n})} |x| = |x| < 1 \quad (\because x < \frac{1}{2})$$

$\therefore \sum |u_n|$ is convergent $\forall x$, i.e., given series is absolutely convergent and hence convergent.

6.10.4

Show that the series $1 + x + \frac{x^2}{2} + \frac{x^2}{3} + \dots$ converges absolutely $\forall x$

Solution

$$\lim_{n \rightarrow \infty} \frac{|u_{n+1}|}{|u_n|} = \frac{|x|}{n} = 0 < 1 \text{ when } x \neq 0 \text{ [since } |u_n| = \frac{|x^{n-1}|}{(n-1)!}; |u_{n+1}| = \frac{|x^n|}{n!} \text{]}$$

$\therefore$ By ratio test, $\sum |u_n|$ is convergent $\forall x \neq 0$.

When $x = 0$, the series is $(1 + 0 + 0 + \dots)$ and is convergent

$\therefore \sum |u_n|$ converges $\Rightarrow \sum u_n$ is absolutely convergent $\forall x$.

6.10.5 Example

Show that the series, $1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^4} + \dots$ is absolutely convergent.

Solution

$$\sum |u_n| = \sum_{n=1}^{\infty} \frac{1}{3^{n-1}}, \text{ which is a geometric series with common ratio } \frac{1}{3} < 1$$

$\therefore$ It is convergent. Hence given series is absolutely convergent.

6.10.6 Example

Test for convergence, absolute convergence and conditional convergence of the series,

$$1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \dots$$

Solution

The given alternating series is of the form $\sum (-1)^{n-1} u_n$, where, $u_n = \frac{1}{4n-3}$.

$$\text{Hence, } u_n > 0 \forall n \in N; \quad u_{n+1} = \frac{1}{4(n+1)-3} = \frac{1}{4n+1}$$

$$\begin{aligned} u_n - u_{n+1} &= \frac{1}{4n-3} - \frac{1}{4n+1} \\ &= \frac{4n+1-4n+3}{(4n-3)(4n+1)} = \frac{4}{(4n-3)(4n+1)} > 0, \forall n \in N \end{aligned}$$

$$\text{i.e., } u_n > u_{n+1}, \forall n \in N \quad \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{4n-3} = 0;$$

All conditions of Leibnitz's test are satisfied.

Hence $\sum (-1)^{n-1} u_n$ is convergent.

$$|u_n| = \frac{1}{4n-3}; \text{ Take } v_n = \frac{1}{n}; \text{ } \lim_{n \rightarrow \infty} \frac{|u_n|}{v_n} = \lim_{n \rightarrow \infty} \frac{n}{n(4 - \frac{3}{n})} = \frac{1}{4} \neq 0 \text{ and finite.}$$

$\therefore$ By comparison test, $\sum |u_n|$ and $\sum v_n$ behave alike.

But by p -series test, $\sum v_n$ is divergent (since $p = 1$).

$\sum |u_n|$ is divergent and $\therefore$ The given series is conditionally convergent.

6.10.7 Example

Test the series $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{3\sqrt{n}}$, for absolute / conditional convergence.

Solution

The given series is an alternating series of the form $\sum (-1)^{n-1} u_n$.

Here

$$(i) \quad u_n = \frac{1}{3\sqrt{n}}, \forall n \in N$$

$$(ii) \quad 3(n+1) > 3n \Rightarrow 3\sqrt{n+1} > 3\sqrt{n}, \forall n.$$

$$\therefore \frac{1}{3\sqrt{n+1}} < \frac{1}{3\sqrt{n}}, \text{ i.e., } u_{n+1} < u_n, \forall n \in N$$

$$\text{And } \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{3\sqrt{n}} = 0$$

$\therefore$ By Leibnitz's test, the given series is convergent.

But $\sum \left| (-1)^{n-1} \cdot \frac{1}{3\sqrt{n}} \right| = \sum \frac{1}{3\sqrt{n}}$ is divergent by p -series test (since

$$p = \frac{1}{2} < 1)$$

$\therefore$ The given series is conditionally convergent.

6.10.8 Example

Test the following series for absolute / conditional convergence.

$$(a) \quad \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{\sin(n\alpha)}{n^2} \quad (b) \quad \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{n^2}{n^3 + 1}$$

$$(c) \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \quad (d) \quad \sum (-1)^{n-1} \cdot \frac{n\pi^n}{e^{3n+1}}$$

Solution

- (a) $|u_n| = \frac{|\sin n\alpha|}{n^2} < \frac{1}{n^2}$ [since $|\sin n\alpha| < 1$] considering $v_n = \frac{1}{n^2}$ and using comparison and p -series tests, we get that $\sum |u_n|$ is convergent $\sum u_n$ is absolutely convergent.
- (b) By Leibnitz's test, the series converges.
 Taking $v_n = \frac{1}{n}$, by comparison and p -series tests, $\sum \frac{n^2}{n^3+1}$, is seen to be divergent.
 Hence given series is conditionally convergent.
- (c) Take $|u_n| = \frac{1}{2n!}$; $\lim_{n \rightarrow \infty} \frac{|u_{n+1}|}{|u_n|} = 0 < 1$; By ratio test, $\sum |u_n|$ is convergent;
 Hence given series is absolutely convergent.
- (d) $|u_n| = \frac{n\pi^n}{e^{3n+1}}$; By root test, is convergent, $\therefore$ given series is absolutely convergent.
 [In problems (a) to (d) above, hints only are given. Students are advised to do the complete problem themselves]

6.10.9 Example

Find the interval of convergence of the following series.

- (a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n^3}$ (b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n(x+2)^n}{3^n}$
 (c) $\log(1+x)$

Solution

- (a) Let the given series be $\sum u_n$; Then $|u_n| = \frac{|x^n|}{n^3}$; $|u_{n+1}| = \frac{|x^{n+1}|}{(n+1)^3}$

$$\lim_{n \rightarrow \infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^3 \cdot |x| = \lim_{n \rightarrow \infty} \left(\frac{1}{1 + \frac{1}{n}} \right)^3 \cdot |x| = |x|$$

$\therefore$ By ratio test, $\sum |u_n|$ is convergent if $|x| < 1$

i.e., $\sum u_n$ is absolutely convergent if $|x| < 1$;

$\therefore \sum u_n$ is convergent if $|x| < 1$

If $x = 1$, the given series becomes $1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \dots$

which is convergent, since $\sum \frac{1}{n^3}$ is convergent.

Similarly, if $x = -1$, the series becomes $\sum -\frac{1}{n^3} = -\sum \frac{1}{n^3}$ which is also convergent.

Hence the interval of convergence of $\sum u_n$ is $(-1 \leq x \leq 1)$

(b) Proceeding as in (a),

$$\lim_{n \rightarrow \infty} \frac{|u_{n+1}|}{|u_n|} = \frac{|x+2|}{3}$$

$\therefore \sum u_n$ is convergent if $|x+2| < 3$, i.e., if $-3 < x+2 < 3$, i.e., if $-5 < x < 1$.

If $x = -5$, $\sum u_n = \sum (-1)^{2n-1} \cdot n$, and is divergent (in both these cases

If $x = 1$, $\sum u_n = \sum (-1)^{n-1} \cdot n$, and is divergent $\lim_{n \rightarrow \infty} u_n \neq 0$)

Hence the interval of convergence of the series is $(-5 < x < 1)$

(c) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = \sum u_n \quad (\text{say})$$

$$|u_n| = \frac{|x^n|}{n}; \quad |u_{n+1}| = \frac{|x^{n+1}|}{n+1}$$

$$\lim_{n \rightarrow \infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)} |x| = |x|$$

By ratio test, $\sum |u_n|$ is convergent when $|x| < 1$

i.e., $\sum u_n$ is absolutely convergent and hence convergent when $-1 < x < 1$.

When $x = -1$, $\sum u_n = \sum (-1)^{n-1} \cdot \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$,

which is convergent by Leibnitz's test. (give the proof)

When $x = 1$, $\sum u_n = -\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\right)$ which is divergent,

since $\sum \frac{1}{n}$ is divergent by p -series test (prove).

Hence $\sum u_n$ is convergent when $-1 < x \leq 1$

Interval of convergence is $(-1 < x \leq 1)$.

Exercise -1 (e)

1. Use integral test and determine the convergence or divergence of the following series:

1. $\sum \frac{1}{n^2}$ [Ans : convergent]

2. $\sum_2^{\infty} \frac{1}{n(\log n)^2}$ [Ans : convergent]

2. Test for convergence of the following series:

1. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \dots$ [Ans : convergent]

2. $\sum_1^{\infty} \frac{(-1)^{n-1}}{(2n-1)(2n)}$ [Ans : convergent]

3. $\sum_1^{\infty} (-1)^{n-1} n^{-5/2}$ [Ans : convergent]

3. Classify the following series into absolutely convergent and conditionally convergent series :

1. $\sum \frac{(-1)^n}{n^3}$ [Ans : abs.cgt]

2. $\sum \frac{\sin \sqrt{n}}{n^{3/2}}$ [Ans : abs.cgt]

3. $\sum \frac{(-1)^n}{n(\log n)^2}$ [Ans : abs.cgt]

4. Find the interval of convergence of the following series :

1. $\sum \frac{2^n x^n}{n}$ [Ans : $-\infty < x < \infty$]

2. $\sum \frac{x^n}{n^2}$ [Ans : $-1 \leq x \leq 1$]
3. $x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} + \dots$ [Ans : $-1 < x \leq 1$]
5. (a) Show that $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$ is absolutely convergent.
- (b) Show that $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$ is conditionally convergent.

Tests of Convergence – A Summary

- The geometric series $\sum_{n=1}^{\infty} x^{n-1}$ converges if $|x| < 1$, diverges if $x \geq 1$, and oscillates when $x \leq -1$
- If $\sum u_n$ is convergent, $\lim_{n \rightarrow \infty} u_n = 0$ [The convergent need not be necessary]
- p – series test :- $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$
- Comparison test :- The series $\sum u_n$ and $\sum v_n$ are both convergent or both divergent if $\lim_{n \rightarrow \infty} \frac{u_n}{v_n}$ is finite and non – zero.
- D'Alembert's Ratio test :- $\sum u_n$ converges or diverges according as

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} < 1 \text{ or } > 1$$

$\left(\text{Alternately, if } \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} > 1 \text{ or } < 1 \right)$. If the limit = 1, the test fails

- Raabe's test : $\sum u_n$ converges or diverges according as

$$\lim_{n \rightarrow \infty} \left[n \left\{ \frac{u_n}{u_{n+1}} - 1 \right\} \right] > 1 \text{ or } < 1 .$$

- Cauchy's root test : $\sum u_n$ converges or diverges according as $\lim_{n \rightarrow \infty} \left(u_n^{\frac{1}{n}} \right) < 1 \text{ or } > 1$

(If limit = 1, the test fails.)

8. *Integral test* : A series $\sum \phi(n)$ of +ve terms where $\phi(n)$ decreases as n increases is convergent or divergent according as the integral $\int_1^{\infty} \phi(x) dx$ is finite or infinite.
9. *Alternating series – Leibnitz's test*: An alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ convergent if (i) $u_n = u_{n+1}, \forall n$ and (ii) $\lim_{n \rightarrow \infty} u_n = 0$
10. Absolute / conditional convergence :
- $\sum u_n$ is absolutely convergent if $\sum |u_n|$ is convergent.
 - $\sum u_n$ is conditionally convergent if $\sum u_n$ is convergent and $\sum |u_n|$ is divergent.
 - An absolutely convergent series is convergent, but converse need not be true . i.e., a convergent series need not be convergent.

Miscellaneous Exercise - 1 (e)

1. Examine the convergence of the following series:

- $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \dots$ [cgt.]
- $\frac{1^2}{1^3+1} + \frac{2^2}{2^3+1} + \frac{3^2}{3^3+1} + \dots$ [dgt.]
- $\frac{2}{1} + \frac{2^2}{2} + \frac{2^3}{3} + \dots$ [dgt.]
- $\frac{1}{1+2} + \frac{2}{1+2^2} + \frac{3}{1+2^3} + \dots$ [cgt.]
- $\frac{x}{1+x} + \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} + \dots (x > 0)$ [cgt. if $x \leq 1$ dgt. if $x > 1$]
- $2x + \frac{3x^2}{8} + \frac{4x^3}{27} + \dots (x > 0)$ [cgt. if $x \leq 1$ dgt. if $x > 1$]
- $1 + \frac{1}{2} + \frac{1.3}{2.4} + \frac{1.3.5}{2.4.6} + \dots$ [dgt.]
- $\frac{3^2}{6^2} + \frac{3^2.5^2}{6^2.8^2} + \frac{3^2.5^2.7^2}{6^2.8^2.10^2} + \dots$ [cgt.]
- $\frac{3.4}{1.2} + \frac{4.5}{2.3} + \frac{5.6}{3.4} + \dots$ [dig.]

10. $\frac{(1)^2}{2}x + \frac{(2)^2}{4}x^2 + \frac{(3)^2}{6}x^3 + \dots (x > 0)$ [cgt. if $x < 4$, dgt. if $x \geq 4$]
11. $1 + \frac{x}{2^2} + \frac{x^2}{3^2} + \frac{x^3}{4^2} + \dots (x > 0)$ [cgt. if $x \leq 1$, dgt. if $x > 1$]
12. $\frac{3x}{4} + \left(\frac{4}{5}\right)^2 x^2 + \left(\frac{5}{6}\right)^3 + x^3 + \dots (x > 0)$ [cgt if $x < 1$, dgt. if $x \geq 1$]
13. $\sum \left(1 + \frac{1}{n}\right)^{n^2}$ [dgt.]
14. $\sum \frac{2^{3n}}{3^{2n}}$ [cgt.]
15. $\sum \frac{a^n}{1+n^2}, a < 1$ [cgt.]
16. $1 - \frac{1}{2.2} + \frac{1}{3.3} - \frac{1}{4.4} + - + - \dots$ [Abs. cgt.]

2. Examine for absolute and conditional convergence of the following series :

1. $\sum (-1)^n \frac{3^{3n}}{3^{2n}}$ [Abs. cgt.]
2. $\sum \frac{(-1)^n \cdot n}{2^n}$ [Abs. cgt.]
3. $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \dots$ [Cond. cgt]
4. $\sum (-1)^n \frac{(n^2 + 1)}{n^3}$ [Cond.cgt.]

3. Determine the interval of convergence of the following series :

1. $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$ [$-1 \leq x < 1$]
2. $\sum \frac{(x+1)^n}{n \cdot 2^n}$ [$-3 < x < 1$]

Exercise - 1 (g) (Objective type questions)

1. The infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$ is

(i) convergent	(ii) divergent
(iii) oscillatory	(iv) none of these [Ans :(i)]
2. The series $\frac{1+n}{1+n^2}$ is

(i) convergent	(ii) divergent
(iii) oscillatory	(iv) none of these [Ans :(ii)]
3. The series $\frac{1}{1\sqrt{1}} - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \dots$

(i) oscillatory	(ii) absolutely convergent
(iii) conditionally convergent	(iv) none of these [Ans :(ii)]
4. The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ is

(i) oscillatory	(ii) divergent
(iii) convergent	(iv) none of these [Ans :(iii)]
5. The interval of convergence of the series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$, is

(i) $-\infty < x < \infty$	(ii) $-1 < x < 2$
(iii) $-1 < x \leq 1$	(iv) none of these [Ans :(iii)]
6. The series $\frac{1}{1.2} + \frac{2}{3.4} + \frac{3}{5.6} + \dots \infty$ is

(i) convergent	(ii) divergent
(iii) oscillatory	(iv) none of these [Ans :(ii)]
7. The series $\frac{1}{1.3} + \frac{2}{3.5} + \frac{3}{5.7} + \dots \infty$, is

(i) conditionally convergent	(ii) convergent
(iii) divergent	(iv) none of these [Ans :(iii)]
8. The series $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + \dots \infty$ is convergent if

(i) $p < 2$	(ii) $p = 2$
(iii) $p > 2$	(iv) none of these [Ans :(iii)]
9. The series $6 - 10 + 4 + 6 - 10 + 4 + 6 - 10 + 4 + 6 - 10 + \dots \infty$ is

(i) convergent	(ii) oscillatory
(iii) divergent	(iv) none of these [Ans :(ii)]

10. The series $\frac{1}{2.4} + \frac{1}{4.6} + \frac{1}{6.8} + \dots$ is

- (i) convergent
(iii) oscillatory

- (ii) divergent
(iv) none of these

[Ans :(i)]

2. Indicate whether the following statements are true or false :

1. The series $\sum \frac{1}{1+2^{-n}}$ is convergent [False]

2. The series $\sum \frac{n^2+5}{2n^2+7}$ is not convergent [True]

3. The series $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots$ is divergent..... [False]

4. The series $x - \frac{x^3}{3} + \frac{x^5}{5} - + - \dots$, converges when $-1 \leq x \leq 1$.. [True]

5. The series $\sum \frac{(-1)^{n-1}}{n \cdot 5^n}$ is absolutely convergent. [True]

6. The series $x + 2x^2 + 3x^3 + 4x^4 + \dots \infty$ is convergent if $x > 1$ [False]

7. The series $x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \infty$ is divergent if $x \geq 1$ [True]

8. The series $1 + \frac{x}{2} + \frac{2!}{3^2}x^3 + \frac{3!}{4^3}x^3 + \frac{4!}{5^4}x^4 + \dots + \infty$ is convergent
if $x > e$ [True]

9. The series $\sum \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$ is divergent [False]

10. The series $\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \dots$ is convergent
if $x < 1$ [True]

11. The series $1 - 2x + 3x^2 - 4x^3 + \dots \infty (x < 1)$ is divergent..... [False]

12. The series $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \frac{x^4}{1+x^4} + \dots \infty$ is convergent [True]

13. The series $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} + \dots \infty$ converges absolutely..... [True]

14. The series $\sum \frac{(-1)^{n-1}}{\sqrt{n}}$ is conditionally convergent. [True]

15. The series whose n^{th} term is $\frac{3n^2+5}{(n+2)^a}$ is convergent. [False]

3. Fill in the Blanks :

1. The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges if _____. [Ans: $|r| < 1$]

2. If a series of +ve terms $\sum u_n$ is convergent, $\lim_{n \rightarrow \infty} u_n =$ _____. [Ans: 0]

3. $\sum_{n=1}^{\infty} \left\{ \sqrt[3]{n^3+1} - n \right\}$ is _____. [Ans: convergent.]

4. If $\sum_{n=1}^{\infty} \frac{3n^3-4}{(n+5)^p}$ is divergent, value of p is _____. [Ans: ≤ 4]

5. The interval of convergence of $\sum u_n$ where $u_n = \left(\frac{n^2-2}{n^2+2} \right)^{2n} x$, is _____. [Ans: $-1 < x < 1$]

6. $\sum u_n$ is convergent series of +ve series. Then $\lim_{n \rightarrow \infty} (u_n^{1/n})$ is _____. [Ans: < 1]

7. The series $8 - 12 + 4 + 8 - 12 - 4 + \dots$ is _____. [Ans : Oscillatory]

8. If $u_n > 0, \forall n$ and $\sum u_n$ is convergent, then $\lim_{n \rightarrow \infty} \left[n \left\{ \frac{u_n}{u_{n+1}} - 1 \right\} \right]$ is _____. [Ans: > 1]

9. If the series $\sum_{n=1}^{\infty} (-1)^n a_n, (a_n > 0 \forall n)$ is convergent, then for all values of n , $\frac{a_n}{a_{n+1}}$

is _____. [Ans: > 1]

10. If $u_n = \left(1 + \frac{1}{n} \right)^{-n^2}$, $\lim_{n \rightarrow \infty} u_n^{1/n} =$ _____. [Ans: $1/e$]

Tests of Convergence – A Summary

1. The geometric series $\sum_{n=1}^{\infty} x^{n-1}$ converges if $|x| < 1$, diverges if $x \geq 1$ and oscillates when $x \leq -1$.
2. If $\sum u_n$ is convergent, $\lim_{n \rightarrow \infty} u_n = 0$ [The converse need not be necessary]
3. p – series test:- $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$
4. *Comparison test* : The series $\sum u_n$ and $\sum v_n$ are both convergent or both divergent if $\lim_{n \rightarrow \infty} \frac{u_n}{v_n}$ is finite and non – zero.
5. *D’ALEMBERT’S Ratio test* : $\sum u_n$ converges or diverges according as $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} < 1$ or > 1 (Alternately, $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} < 1$ or > 1). If the limit = 1, the test fails.
6. *Raabe’s test*: $\sum u_n$ converges or diverges according as $\lim_{n \rightarrow \infty} \left[n \left\{ \frac{u_n}{u_{n+1}} - 1 \right\} \right] > 1$ or < 1
7. *Cauchy’s root test*: $\sum u_n$ converges or diverges according as $\lim_{n \rightarrow \infty} \left(u_n^{\frac{1}{n}} \right) < 1$ or > 1 ;
(If the Limit = 1, The test fails).
8. *Integral test* : A series $\sum \phi(n)$ of +ve terms where $\phi(n)$ decreases as n increases is convergent or divergent according as integral $\int_1^{\infty} \phi(x) dx$ is finite or infinite.
9. *Alternating series – Leibnitz’s test* : An alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ (where $u_n > 0 \forall n$) is convergent if (i) $u_n > u_{n+1}$, $\forall n$ (ii) $\lim_{n \rightarrow \infty} u_n = 0$
10. *Absolute/ conditional convergence* :
 - (a) $\sum u_n$ is absolutely convergent if $\sum |u_n|$ is convergent .
 - (b) $\sum u_n$ is conditionally convergent if $\sum u_n$ is convergent and $\sum |u_n|$ is divergent.
 - (c) An absolutely convergent series is convergent but converse need not be true. i.e., a convergent series need not be convergent.

Solved University Questions

1. Test the convergence of the series:

$$\frac{1}{1.2.3} + \frac{2}{2.3.4} + \frac{3}{3.4.5} + \dots$$

Solution

Let u_n be the n^{th} term of the series;

$$\text{Then, } u_n = \frac{n}{n(n+1)(n+2)} = \frac{1}{(n+1)(n+2)}$$

$$\begin{aligned} \text{Let } v_n &= \frac{1}{n^2}; \text{ then, } \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)(n+2)} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right)} = 1, \end{aligned}$$

Which is non-zero and finite.

$\therefore$ By comparison test, both $\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\sum v_n$ is convergent by p -series test ($p > 1$) $\therefore \sum u_n$ is convergent.

2. Show that every convergent sequence is bounded

Solution

Let $\langle a_n \rangle$ be a sequence which converges to a limit ' l ' say.

$\lim_{n \rightarrow \infty} a_n = l \Rightarrow$ given any +ve number ϵ , however small,

we can always find an integer ' m ', $\exists$, $|a_n - l| < \epsilon$, $\forall n \geq m$

Taking $\epsilon = 1$, we have, $|a_n - l| < 1$;

i.e., $(l-1) < a_n < (l+1)$, $\forall n \geq m$

Let $\lambda = \min \{a_1, a_2, \dots, a_{m-1}, (l-1)\}$, and $\mu = \max \{a_1, a_2, \dots, a_{m-1}, (l+1)\}$

Then obviously, $\lambda \leq a_n \leq \mu$, $\forall n \in N$;

Hence $\langle a_n \rangle$ is bounded.

3. Show that the series,

$$S = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \dots \text{ converges.}$$

Solution

The given series is an alternating series $\sum (-1)^{n-1} u_n$,

where $u_n = \frac{1}{(2n-1)!}$ We observe that,

(i) $u_n > 0$ and $u_n > u_{n+1}, \forall n$ and

(ii) $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{(2n-1)!} = 0$

$\therefore$ By Leibnitz's test [7.2] the given series converges.

4. Show that the geometric series $\sum_{m=0}^{\infty} q^m = 1 + q + q^2 + \dots$ converges to the sum

$$\frac{1}{1-q} \text{ when } |q| < 1 \text{ and diverges when } |q| \geq 1$$

Solution

See theorem 2.3 (replace 'x' by 'q').

5. Define the convergence of a series. Explain the absolute convergence and conditional convergence of a series. Test the convergence of series

$$\sum \left[1 + \frac{1}{\sqrt{n}} \right]^{-n^2}$$

Solution

For theory part, refer 2.1, 2.2, 8.1, 9.1, and 9.2

$$\begin{aligned} \text{Problem : Let } u_n &= \left(1 + \frac{1}{\sqrt{n}} \right)^{-n^2} ; \lim_{n \rightarrow \infty} \left(u_n^{1/n} \right) = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{\sqrt{n}} \right)^{-n} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\left[1 + \frac{1}{\sqrt{n}} \right]^n} = \frac{1}{e^2} < 1 \end{aligned}$$

By Cauchy's root test, $\sum u_n$ is convergent.

6. Test the convergence of the series, $1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \dots$

Given that $x > 0$.

Solution

Omitting the first term of the series, we have, $\frac{1.3.5 \dots (2n-1)}{2.4.6 \dots 2n} x^n$ and

$$u_n = \frac{1.3.5.(2n-1)}{2.4.6....2n} x^n ; u_{n+1} = \frac{1.3.5.....(2n+1)}{2.4.6.....(2n+2)} x^{n+1} ;$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \left(\frac{2n+1}{2n+2} \right) \cdot x = x$$

By ratio test, $\sum u_n$ is convergent when $x < 1$, and divergent when $x > 1$

The ratio test fails when $x = 1$

$$\text{When } x = 1, \quad \frac{u_n}{u_{n+1}} - 1 = \frac{2n+2}{2n+1} - 1 = \frac{1}{2n+1}$$

$$\lim_{n \rightarrow \infty} \left[n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right] = \lim_{n \rightarrow \infty} \left(\frac{n}{2n+1} \right) = \frac{1}{2} < 1 ;$$

$\therefore$ By Raabe's test, $\sum u_n$ diverges.

$\therefore$ The given series converges when $x < 1$ and diverges when $x \geq 1$.

7. Test the convergence of the series, $\frac{1}{2} + \left(\frac{2}{3}\right)x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \dots x > 0$

Solution

Neglecting the 1st term,

$$u_n = \left[\left(\frac{n+1}{n+2} \right) x \right]^n ;$$

$$u_n^{1/n} = \left(\frac{n+1}{n+2} \right) x = \left(\frac{1 + \frac{1}{n}}{1 + \frac{2}{n}} \right) x$$

$\lim_{n \rightarrow \infty} u_n^{1/n} = x$; By Cauchy's root test, $\sum u_n$ is cgt. when $x < 1$ and dgt. when $x >$

1; when $x = 1$, the test fails.

$$\text{When } x = 1, u_n = \frac{\left(1 + \frac{1}{n}\right)^n}{\left(1 + \frac{2}{n}\right)^n}; \lim_{n \rightarrow \infty} u_n = \frac{e}{e^2} = \frac{1}{e} \neq 0$$

$\therefore \sum u_n$ is divergent.

$\therefore$ is cgt. when $x < 1$ and dgt. when $x \geq 1$.

8. Test the series whose n^{th} term is $(3n-1)/2^n$ for convergence.

Solution

$$u_n = \frac{(3n-1)}{2^n} ; \quad u_{n+1} = \frac{\{3(n+1)-1\}}{2^{n+1}} ;$$

$$\frac{u_{n+1}}{u_n} = \frac{(3n+2)}{2(3n-1)} \quad \text{Lt}_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \frac{1}{2} < 1 ;$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

9. Show by Cauchy's integral test that the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if $p > 1$ and diverges if $0 < p \leq 1$

Solution

Let $\phi(x) = \frac{1}{x(\log x)^p}$; $x \geq 2$; Then $\phi(x)$ decreases as x increases in $[2, \infty]$

$$\int_2^{\infty} \phi(x) dx = \int_2^{\infty} \frac{dx}{x(\log x)^p} = \int_{\log 2}^{\infty} \frac{du}{u^p} = \frac{u^{1-p}}{1-p} \Big|_{\log 2}^{\infty} ;$$

[Taking $\log x = u$, $\frac{1}{x} dx = du$ $x = 2 \Rightarrow u = \log 2$ and $x = \infty \Rightarrow u = \infty$]

Case (i) : $p > 1 \Rightarrow 1-p < 0 \Rightarrow$ Integral is finite, and

Case (ii) : $0 < p \leq 1 \Rightarrow$ Integral is infinite.

Hence, by integral test, the given series converges if $p > 1$ and diverges when $0 < p \leq 1$.

10. Test the convergence of the series $\sum \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

Solution

$$u_n^{1/n} = \left\{ \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}} \right\}^{1/n} = \frac{1}{\left(1 + \frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} ;$$

$$\lim_{n \rightarrow \infty} u_n^{1/n} = \frac{1}{e} < 1 \quad [2 < e < 3].$$

By Cauchy's root test, $\sum u_n$ is convergent.

11. Test the convergence of the series, $\sum_{n=2}^{\infty} \frac{(-1)^n \cdot x^n}{n(n-1)}$, $0 < x < 1$

Solution

The given series is of the form $\sum (-1)^n u_n$, where $u_n = \frac{x^n}{n(n-1)}$.

This is an alternating series in which (i) $u_n > 0$ and $u_n > u_{n+1} \forall n \in N$.

Further $\lim_{n \rightarrow \infty} u_n = 0$. Hence, by Leibnitz test, the series is convergent.

12. Discuss the convergence of the series, $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \dots$

Solution

n^{th} term of the series = $u_n = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$ (omitting 1st term)

$$u_{n+1} = \frac{x^{2n+2}}{(n+3)\sqrt{n+2}}; \quad \frac{u_{n+1}}{u_n} = \frac{\sqrt{n+2}\sqrt{n+1}}{(n+3)} \cdot x^2$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \left[\frac{\sqrt{1+\frac{2}{n}} \cdot \sqrt{1+\frac{1}{n}}}{\left(1+\frac{3}{n}\right)} \cdot x^2 \right] = x^2;$$

$\therefore$ By ratio test, $\sum u_n$ converges if $x^2 < 1$, i.e., if $|x| < 1$, and diverges if $x^2 > 1$, i.e., if $|x| > 1$;

When $x^2 = 1$, $u_n = \frac{1}{(n+2)\sqrt{n+1}}$; taking $v_n = \frac{1}{n^{3/2}}$,

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{n^{3/2}}{n^{3/2} \left(1+\frac{2}{n}\right) \sqrt{1+\frac{1}{n}}} = 1$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ both converge or diverge together;

But $\sum v_n$ is convergent by p -series test.

$\therefore \sum u_n$ is convergent if $|x| \leq 1$ and divergent if $|x| > 1$.

13. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{x^{2n}}{(n+1)\sqrt{n}}$

Solution

$$u_n = \frac{x^{2n}}{(n+1)\sqrt{n}}; \quad u_{n+1} = \frac{x^{2n+2}}{(n+2)\sqrt{n+1}}$$

$$\frac{u_{n+1}}{u_n} = \frac{\sqrt{n}\sqrt{n+1}}{n+2} \cdot x^2 = \frac{\sqrt{1+\frac{1}{n}}}{\left(1+\frac{2}{n}\right)} \cdot x^2; \quad \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = x^2;$$

$\therefore$ By ratio test, $\sum u_n$ converges when $|x| < 1$ and diverges for $|x| > 1$.

When $|x| = 1$, $u_n = \frac{1}{n^{\frac{3}{2}}\left(1+\frac{1}{n}\right)}$ taking $v_n = \frac{1}{n^{\frac{3}{2}}}$ and applying the comparison

test, we observe that $\sum u_n$ is convergent.

Hence $\sum u_n$ converges when $|x| \leq 1$ and diverges when $|x| > 1$.

14. Find the interval of convergence of the series, $\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots \infty$

Solution

For the given series, $u_n = \frac{x^{n+1}}{n+1}; \quad u_{n+1} = \frac{x^{n+2}}{n+2}$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}} \right) x = x$$

By ratio test, $\sum u_n$ converges when $|x| < 1$ i.e., $-1 < x < 1$

When $x = 1$, $u_n = \frac{1}{n+1}$

Taking $u_n = \frac{1}{n}; \quad \frac{u_n}{v_n} = \frac{1}{1+\frac{1}{n}}$

and, $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = 1 \neq 0$ and finite.

$\therefore$ Both $\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\sum v_n$ diverges $\therefore \sum u_n$ also diverges when $x = 1$.

When $x = -1$, the given series is

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots \text{ which is alternating series with}$$

$$u_n > u_{n+1} \forall n \text{ and } u_n \rightarrow 0 \text{ as } n \rightarrow \infty$$

$\therefore$ By Leibnitz's test $\sum u_n$ converges when $x = -1$

$\therefore$ Interval of convergence is $[-1, 1)$ i.e., $-1 \leq x < 1$

15. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1.3.5 \dots (2n+1)}{2.5.8 \dots (3n+2)}$

Solution

$$u_n = \frac{1.3.5 \dots (2n+1)}{2.5.8 \dots (3n+2)}; \quad u_{n+1} = \frac{1.3.5 \dots (2n+3)}{2.5.8 \dots (3n+5)}$$

$$\frac{u_{n+1}}{u_n} = \frac{2n+3}{3n+5}; \quad \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \left[\frac{2 + \binom{3}{n}}{3 + \binom{5}{n}} \right] = \frac{2}{3} < 1$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

16. Prove that the series $\sum \frac{(-1)^n}{n(\log n)^3}$ converges absolutely.

Solution

$$|u_n| = \frac{1}{n(\log n)^3}; \quad \int_2^{\infty} \frac{dx}{x(\log x)^3} = \int_{\log 2}^{\infty} \frac{dt}{t^2}$$

$$(\text{where } t = \log x) = \frac{-1}{t} \Big|_{\log 2}^{\infty} = \frac{1}{\log 2}, \text{ which is finite.}$$

$\therefore$ By integral test $\sum |u_n|$ is convergent.

$\therefore \sum u_n$ converges absolutely.

17. Test the convergence of the series $\sum \frac{(2n+1)}{n^3+1} x^n, x > 0$

Solution

n^{th} term of the given series, $u_n = \frac{2n+1}{n^3+1} x^n$;

$$u_{n+1} = \left[\frac{2(n+1)+1}{(n+1)^3+1} \right] x^{n+1} = \frac{2n+3}{(n+1)^3+1} x^{n+1}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} &= \lim_{n \rightarrow \infty} \frac{(2n+3)x^{n+1}}{\{(n+1)^3+1\}} \times \frac{(n^3+1)}{x^n(2n+1)} \\ \lim_{n \rightarrow \infty} \left[\frac{2n(1+\frac{3}{2n}) \cdot n^3(1+\frac{1}{n^3})}{n^3 \left\{ (1+\frac{1}{n})^3 + \frac{1}{n^3} \right\} \cdot 2n(1+\frac{1}{2n})} \right] x &= x \end{aligned}$$

By ratio test, $\sum u_n$ converges if $x < 1$ and diverges if $x > 1$. If $x = 1$ the test fails.

When $x = 1$, $u_n = \frac{2n+1}{n^3+1}$; Taking $v_n = \frac{1}{n^2}$;

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{2n+1}{n^3+1} \times n^2 = 2 \neq 0 \quad \text{and finite}$$

$\therefore \sum u_n$ and $\sum v_n$ converge or diverge together.

But $\sum v_n$ converges $\therefore \sum u_n$ also converges.

Thus, $\sum u_n$ converges when $x \leq 1$ and diverges when $x > 1$.

18. Test the series $\sum_{n=1}^{\infty} \frac{(-1)^n (\log n)}{n^2}$, for absolute/conditional convergence

Solution

$$u_n = \frac{(-1)^n (\log n)}{n^2}; |u_n| = \frac{(\log n)}{n^2};$$

$$\begin{aligned} \int_2^{\infty} \frac{\log x}{x^2} dx &= \int_{\log 2}^{\infty} t e^{-t} dt \quad [\text{taking } \log x = t, x = e^t, \frac{1}{x} dx = \log t] \\ &= -t e^{-t} + e^{-t} \Big|_{\log 2}^{\infty} = 0 - [1 - \log 2] \cdot e^{-\log 2} = \frac{1}{2} (\log 2 - 1), \end{aligned}$$

which is finite.

$\therefore$ By integral test $\sum |u_n|$ is convergent $\Rightarrow \sum u_n$ converges absolutely.

(Note that $\sum u_n$ is cgt. by Leibnitz's test).

19. Test the convergence of the series $\sum \frac{1}{(\log \log n)^n}$

Solution

$$\text{Given that } u_n = \frac{1}{(\log \log n)^n};$$

$$\lim_{n \rightarrow \infty} u_n^{1/n} = \lim_{n \rightarrow \infty} \left[\frac{1}{\log \log n} \right] = 0 < 1$$

By Cauchy's root test, $\sum u_n$ is convergent.

20. Find the interval of convergence of the series,

$$x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{x^5}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{x^7}{7} + \dots$$

Solution

$$\text{Term of the series, } u_n = \frac{1.3.5 \dots (2n-1)}{2.4.6 \dots 2n} \cdot \frac{x^{2n+1}}{(2n+1)} \quad (\text{neglecting 1st term})$$

$$u_{n+1} = \frac{1.3.5 \dots (2n-1)(2n+1)}{2.4.6 \dots 2n(2n+2)} \cdot \frac{x^{2n+3}}{(2n+3)};$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \left[\frac{(2n+1)^2}{(2n+2)(2n+3)} \cdot x^2 \right] \lim_{n \rightarrow \infty} \left[\frac{n^2 \left(4 + \frac{4}{n} + \frac{1}{n^2} \right)}{n^2 \left(4 + \frac{10}{n} + \frac{6}{n^2} \right)} \cdot x^2 \right] = x^2$$

By ratio test, $\sum u_n$ converges when $x^2 < 1$, i.e., $|x| < 1 \Rightarrow -1 < x < 1$

When $x^2 = 1$, the test fails;

$$\text{Then } \frac{u_n}{u_{n+1}} - 1 = \left(\frac{4n^2 + 10n + 6}{4n^2 + 2n + 1} - 1 \right) = \frac{8n + 5}{4n^2 + 2n + 1}$$

$$\lim_{n \rightarrow \infty} \left[n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right] = \lim_{n \rightarrow \infty} \frac{n^2 \left(8 + \frac{5}{n} \right)}{n^2 \left(4 + \frac{2}{n} + \frac{1}{n^2} \right)} = 2 > 1$$

$\therefore$ By Raabe's test, $\sum u_n$ converges when $x^2 = 1$, i.e., $x = \pm 1$.

$\therefore$ Interval of convergence of $\sum u_n$ is $(-1 \leq x \leq 1)$

7

Vector Differentiation

7.1.1 Vector Point function and vector field

Let P be any point in a region 'D' of space. Let r be the position vector of P. If there exists a vector function F corresponding to each P, then such a function F is called a vector point function and the region D is called a vector field.

Note : In what follows i, j, k are unit vectors along X, Y, Z axes respectively

For example, consider the vector function

$$F = (x - y) i + xyj + yzk \quad \text{.....(1)}$$

Let P be a point whose position vector is

$$r = 2i + j + 3k \text{ in the region D of space.}$$

At P, the value of F is obtained by putting $x = 2, y = 1, z = 3$ in F .

i.e. At P, $F = i + 2j + 3k$

Thus, to each point P of the region D, there corresponds a vector F given by the vector function (1).

Hence F is a vector point function (of scalar variables x, y, z) and the region D is a vector field.

Scalar point function and scalar field.

If there exists a scalar 'f' given by a scalar function 'f' corresponding to each point P (with position vector r) in a region D of space, 'f' is called a scalar point function and D is called a scalar field.

As an example, let P be a point whose position vector is $r = 2i + j + 3k$.

Consider $f = xyz + xy + z$

Then the value of f at P is obtained by putting $x = 2, y = 1, z = 3$

i.e., At P, $f = 2.1.3 + 2.1 + 3 = 11$

Hence the scalar '11' is attached to the point P.

The function 'f' is a scalar point function (of scalar variables x, y, z), and D is a scalar field.

Note : There can be vector and scalar function of one or more scalar variables.

7.1.2 Differentiation of a vector

If $r(u) = r_1(u)i + r_2(u)j + r_3(u)k$, (where r_1, r_2, r_3 , are scalar functions of 'u') be a vector function of 'u', then,

$$\begin{aligned}\frac{dr}{du} &= \lim_{\delta u \rightarrow 0} \frac{\delta r}{\delta u} = \lim_{\delta u \rightarrow 0} \frac{r(u + \delta u) - r(u)}{\delta u} \\ &= \lim_{\delta u \rightarrow 0} \sum \left[\frac{r_1(u + \delta u) - r_1(u)}{\delta u} i \right] \\ &= \sum \frac{dr_1}{du} i = \frac{dr_1}{du} i + \frac{dr_2}{du} j + \frac{dr_3}{du} k\end{aligned}$$

Example

If $r(u) = (3u^2 + 5u + 6)i + 3u^2j - 4uk$, Find $\frac{dr}{du}$, when $u = 1$

$$\frac{dr}{du} = \left\{ \frac{d}{du} (3u^2 + 5u + 6) \right\} i + \left\{ \frac{d}{du} (3u^2) \right\} j + \left\{ \frac{d}{du} (-4u) \right\} k$$

Note : We can apply the above rule of derivative to the case of partial derivatives also

ex : If $A = (x^2yz)i + (xy^2z)j - (3x^3y^2z^2)k$

find $\frac{\partial^2 A}{\partial x \partial y}$ at the point $(1, -1, 2)$

$$\begin{aligned}
 \frac{\partial A}{\partial y} &= \left\{ \frac{\partial}{\partial y} (x^2 y z) \right\} i + \frac{\partial}{\partial y} \{ x y^2 z \} j - \frac{\partial}{\partial y} \{ 3 x^3 y^2 z^2 \} k \\
 &= (x^2 z) i + (2 x y z) j - (6 x^3 y z^2) k \\
 \frac{\partial^2 A}{\partial x \partial y} &= \left\{ \frac{\partial}{\partial x} (x^2 z) \right\} i + \frac{\partial}{\partial x} \{ 2 x y z \} j - \frac{\partial}{\partial x} \{ 6 x^3 y z^2 \} k \\
 &= 2 x z i + 2 y z j - 18 x^2 y z^2 k
 \end{aligned}$$

At the point (1, -1, 2)

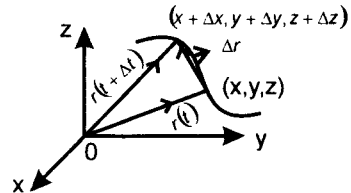
$$\frac{\partial^2 A}{\partial x \partial y} = 4i - 4j + 72k$$

7.1.3 Application to space curves

Let $r(t) = x(t)i + y(t)j + z(t)k$ represent the position vector of a point (x, y, z) on a space curve whose equations are given by $x = x(t)$, $y = y(t)$, $z = z(t)$, where 't' is time.

$$\text{Then } \frac{dr}{dt} = \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k$$

$$\text{and } \frac{d^2 r}{dt^2} = \frac{d^2 x}{dt^2} i + \frac{d^2 y}{dt^2} j + \frac{d^2 z}{dt^2} k$$



- (i) $\frac{dr}{dt}$ represents the velocity vector v (or tangent vector) of the point (x, y, z)
- (ii) $\frac{d^2 r}{dt^2}$ represents the acceleration vector a at the point (x, y, z)

Ex : If a particle moves along a curve $x = e^{-t}$, $y = 2 \cos 2t$, $z = 2 \sin 2t$, where 't' is time.

- 1) find velocity and acceleration at time $t = 0$, and
- 2) find also their magnitudes

Sol : $r = xi + yj + zk$

$$= (e^{-t})i + (2 \cos 2t)j + (2 \sin 2t)k$$

$$v = \frac{dr}{dt} = \frac{d}{dt} (e^{-t})i + \frac{d}{dt} (2\cos 2t)j + \frac{d}{dt} (2\sin 2t)k$$

$$= (-e^{-t})i - (4\sin 2t)j + (4 \cos 2t)k$$

$$a = \frac{d^2r}{dt^2} = \frac{dv}{dt} = \frac{d}{dt} (-e^{-t})i - \frac{d}{dt} (4\sin 2t)j + \frac{d}{dt} (4\cos 2t)k$$

$$= (e^{-t})i - (8\cos 2t)j - (8\sin 2t)k \quad \dots (2)$$

Putting $t = 0$ in (1), velocity as $t = 0$ is $v = -i + 4k$

$$\text{Magnitude} = \sqrt{17}$$

putting $t = 0$ in (2) acceleration at $t = 0$ is $a = i - 8j$

$$\text{Magnitude} = \sqrt{65}$$

7.2 Gradient of Scalar Function

7.2.1 The Vector differential operator 'DEL' or 'NABLA', denoted as ' ∇ ' is defined by

$$\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \quad (i, j, k \text{ are unit vectors in } x, y, z \text{ directions})$$

This operator ' ∇ ' is used in defining the gradient, divergence and curl.

Properties of ' ∇ ' are similar to those of vectors. The operator is applied to both vector and scalar functions.

7.2.2 Gradient

If $\phi(x, y, z)$ is a scalar function, defined at each point (x, y, z) in a certain region of space and is differentiable, the gradient of ϕ (shortly written as $\text{grad } \phi$) is defined as,

$$\begin{aligned} \text{grad } \phi &= \nabla \phi = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \phi \\ &= \left(\frac{\partial \phi}{\partial x} \right) i + \left(\frac{\partial \phi}{\partial y} \right) j + \left(\frac{\partial \phi}{\partial z} \right) k, \end{aligned}$$

(which is a vector function)

$\therefore$ If ϕ defines a scalar field, 'grad' ϕ or $\nabla \phi$ defines a vector field.

7.2.3 Physical significance of 'grad ϕ ' :

If $\phi(x, y, z) = c$ (c being a constant) represents a surface, then 'grad ϕ ' represents the normal vector to the surface at the point (x, y, z)

For, if $\mathbf{r} = xi + yj + zk$, is the position vector of the point (x, y, z) on the surface, we have, $d\mathbf{r} = (dx)i + (dy)j + (dz)k$ which is in the tangent plane to the surface of (x, y, z)

$$\text{Again, } \nabla \phi, d\mathbf{r} = \left[\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k \right] \cdot [dx i + dy j + dz k]$$

$$= \left(\frac{\partial \phi}{\partial x} \right) dx + \left(\frac{\partial \phi}{\partial y} \right) dy + \left(\frac{\partial \phi}{\partial z} \right) dz = d\phi = 0 \quad (\because \phi = c)$$

$\therefore$ The vector ' $\nabla \phi$ ' which is $\perp_{\mathbf{r}}$ to the tangent plane is the normal vector to $\phi = c$ at (x, y, z)

7.2.4 Directional, Derivative

If $\mathbf{a}$ be any vector, $\frac{\nabla \phi \cdot \mathbf{a}}{|\mathbf{a}|}$ which represents the component of $\nabla \phi$ in the direction of $\mathbf{a}$

$\mathbf{a}$ is known as the directional derivative of ' ϕ ' in the direction of $\mathbf{a}$,

(1) Physically the directional derivative is the rate of change of ' ϕ ' in the direction of $\mathbf{a}$.

(2) The directional derivative will be maximum in the direction of $\nabla \phi$ (i.e.,

$\mathbf{a} = \nabla \phi$) and the maximum value of the directional derivative $= \frac{\nabla \phi \cdot \nabla \phi}{|\nabla \phi|} = |\nabla \phi|$.

7.2.5 Some basic properties of the gradient

If ϕ and ψ are two scalar functions,

$$1) \text{ grad } (\phi + \psi) = \text{grad } \phi + \text{grad } \psi \text{ (or) } \nabla(\phi + \psi) = \nabla \phi + \nabla \psi$$

$$\text{Proof: grad } (\phi + \psi) = \nabla(\phi + \psi) = \left\{ \frac{\partial}{\partial x}(\phi + \psi) \right\} i + \left\{ \frac{\partial}{\partial y}(\phi + \psi) \right\} j + \left\{ \frac{\partial}{\partial z}(\phi + \psi) \right\} k$$

$$= \left(\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k \right) + \left(\frac{\partial \psi}{\partial x} i + \frac{\partial \psi}{\partial y} j + \frac{\partial \psi}{\partial z} k \right)$$

$$= \nabla \phi + \nabla \psi$$

$$(2) \text{ grad } (\phi\psi) = \phi (\text{grad } \psi) + \psi (\text{grad } \phi) \text{ (or) } \nabla(\phi\psi) = \phi (\nabla\psi) + (\nabla\phi)\psi$$

$$\text{Proof: } \text{grad } (\phi\psi) = \nabla(\phi\psi)$$

$$= \left\{ \frac{\partial}{\partial x} (\phi\psi) \right\} i + \left\{ \frac{\partial}{\partial y} (\phi\psi) \right\} j + \left\{ \frac{\partial}{\partial z} (\phi\psi) \right\} k$$

$$= \left(\phi \frac{\partial \psi}{\partial x} + \psi \frac{\partial \phi}{\partial x} \right) i + \left(\phi \frac{\partial \psi}{\partial y} + \psi \frac{\partial \phi}{\partial y} \right) j + \left(\phi \frac{\partial \psi}{\partial z} + \psi \frac{\partial \phi}{\partial z} \right) k$$

$$= \phi \left(\frac{\partial \psi}{\partial x} i + \frac{\partial \psi}{\partial y} j + \frac{\partial \psi}{\partial z} k \right) + \psi \left(\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k \right) = \phi(\nabla\psi) + \psi(\nabla\phi)$$

$$= \phi(\text{grad } \psi) + \psi(\text{grad } \phi)$$

$$(3) \text{ If } \psi \neq 0, \text{ grad } \left(\frac{\phi}{\psi} \right) = \frac{\psi(\text{grad } \phi) - \phi(\text{grad } \psi)}{(\psi)^2}$$

(Proof is left to the reader.)

Solved Examples

Ex. 7.2.6 If $f = x^2yz$, find $\text{grad } f$ at the point $(1, -2, 1)$.

Sol: $f = x^2yz$; $\therefore \text{grad } \phi = \nabla \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$,

$$\begin{aligned} \text{grad } f &= \frac{\partial}{\partial x} (x^2yz) i + \frac{\partial}{\partial y} (x^2yz) j + \frac{\partial}{\partial z} (x^2yz) k \\ &= (2xyz)i + (x^2z)j + (x^2y)k \end{aligned}$$

$$\therefore \text{ At the point } (1, -2, 1), \text{ grad } \phi = -4i + j - 2k$$

Ex. 7.2.7 Find the unit normal to the surface $xy + yz + zx = 3$ at the point $(1, 1, 1)$.

Sol: If $\phi = c$ is a surface, $\nabla \phi$ is the normal to it.

Here $f = xy + yz + zx$

$$\therefore \text{normal to } \phi = \nabla \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k.$$

$$= \left\{ \frac{\partial}{\partial x} (xy + yz + zx) \right\} i + \left\{ \frac{\partial}{\partial y} (xy + yz + zx) \right\} j + \left\{ \frac{\partial}{\partial z} (xy + yz + zx) \right\} k$$

$$= (y + z)i + (x + z)j + (y + x)k$$

$$\therefore \text{normal at } (1, 1, 1) = 2i + 2j + 2k$$

$$\therefore \text{Unit normal} = \frac{2i + 2j + 2k}{\sqrt{2^2 + 2^2 + 2^2}} = \frac{i + j + k}{\sqrt{3}}$$

Ex. 7.2.8 (a) Find the directional derivative of $f = 2e^{2x-y+z}$ at $(1, 3, 1)$ in a direction towards the point $(2, 1, 3)$.

Sol :

$$f = 2e^{2x-y+z}; \quad \nabla f = 2e^{2x-y+z} (2i - j + k)$$

$$\nabla f|_{(1,3,1)} = 2e^{2-3+1} (2i - j + k) = 4i - 2j + 2k$$

$$\text{Let } A = (1, 3, 1) \text{ and } B = (2, 1, 3),$$

$$AB = (2-1)i + (1-3)j + (3-1)k = i - 2j + 2k = a \text{ (say)}$$

$$\text{Directional derivative in the direction of } a = \frac{\nabla f \cdot a}{|a|}$$

$$= \frac{(4i - 2j + 2k) \cdot (i - 2j + 2k)}{\sqrt{1 + 4 + 4}} = \frac{(4 + 4 + 4)}{3} = 4$$

(b) In the 'Problem (a)' find the maximum value of the directional derivative

Ans :

$$\text{Maximum value of directional derivative} = |\nabla f|$$

$$= |4i - 2j + 2k| = \sqrt{16 + 4 + 4} = 2\sqrt{6}$$

Ex. 7.2.9 Find the acute angle between the surface $xy^2z = 2$ and $x^2 + y^2 + z^3 = 6$ at the point $(2, 1, 1)$.

Sol :

$$\text{Let } f = xy^2z = 4 \text{ be the surface (1)}$$

Normal vector to (1) at (2, 1, 1) = $\nabla f|_{(2,1,1)} = (y^2z)i + (2xyz)j + (xy^2)k|_{(2,1,1)}$

$$= i + 4j + 2k = a \text{ (say).}$$

Let $g = (x^2y^2 + z^2) = 6$ be the surface (2)

Normal vector to (2) at (2, 1, 1) = $\nabla g|_{(2,1,1)} = (2xi + 2yj + 2zk)|_{(2,1,1)}$

$$= 4i + 2j + 2k = b \text{ (say)}$$

$\therefore$ Angle between the surfaces

= Angle between the normals to them

= Angle between a and b

$$\begin{aligned} &= \left| \cos^{-1} \left(\frac{a \cdot b}{|a||b|} \right) \right| = \cos^{-1} \left| \frac{4 + 8 + 4}{\sqrt{1+16+4} \sqrt{16+4+4}} \right| \\ &= \cos^{-1} \left(\frac{16}{\sqrt{21}\sqrt{24}} \right) = \cos^{-1} \left(\frac{16}{6\sqrt{14}} \right) = \cos^{-1} \left(\frac{8}{3\sqrt{14}} \right) \end{aligned}$$

7.2.10 Find the constants p and q such that the surfaces $px^2 - qyz = (p + 2)x$ and $4x^2y + z^3 = 4$ are orthogonal at the point (1, -1, 2)

Sol : Let $f = px^2 - qyz - (p + 2)x = 0$ be surface (1), and

Let $g = 4x^2y + z^3 = 4$ be surface (2)

Normal to (1) at (1, -1, 2) = $\nabla f|_{(1,-1,2)} = \left(\frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k \right)|_{(1,-1,2)}$

$$= [(2px - p - 2)i - (qz)j - (qy)k]|_{(1,-1,2)} = (p - 2)i - (2q)j - (q)k = a \text{ (say)}$$

Normal to (2) at (1, -1, 2) = $\nabla g|_{(1,-1,2)}$

$$= [(8xy)i + (4x^2)j + (3z^2)k]|_{(1,-1,2)} = -8i + 4j + 12k = b \text{ (say)}$$

Since the surfaces (1) and (2) are orthogonal, $a \cdot b = 0$

$$\therefore -8(p - 2) + 4(-2q) + 12(q) = 0$$

$$\Rightarrow -8p + 16 - 8q + 12q = 0$$

$$\Rightarrow -8p + 4q + 16 = 0$$

$$\Rightarrow 2p - q = 4 \quad \dots (i)$$

Since the point $(1, -1, 2)$ lies on (1), we have,

$$p + 2q - p - 2 = 0 \Rightarrow q = 1$$

from (i) we get, $p = 5/2 \quad \therefore p = 5/2, q = 1$

7.2.11 If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$ show that $\text{grad}(r^3) = 3r \mathbf{r}$

Sol : Let $\phi = r^3 = (x^2 + y^2 + z^2)^{3/2}$

$$\text{Then, } \frac{\partial \phi}{\partial x} = \frac{3}{2} (x^2 + y^2 + z^2)^{3/2-1} \cdot 2x = 3x r$$

$$\text{similarly } \frac{\partial \phi}{\partial y} = 3yr; \text{ and } \frac{\partial \phi}{\partial z} = 3zr$$

$$\text{grad } \phi = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k}$$

$$= (3xr) \mathbf{i} + (3yr) \mathbf{j} + (3zr) \mathbf{k}$$

$$= 3r (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 3r \mathbf{r}$$

Aliter : If $r^2 = x^2 + y^2 + z^2$, we have, $2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

$$\text{Similarly } \frac{\partial r}{\partial y} = \frac{y}{r}; \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\text{grad } \phi = \nabla r^3$$

$$= \frac{\partial}{\partial x} (r^3) \mathbf{i} + \frac{\partial}{\partial y} (r^3) \mathbf{j} + \frac{\partial}{\partial z} (r^3) \mathbf{k} = 3r^2 \frac{\partial r}{\partial x} \mathbf{i} + 3r^2 \frac{\partial r}{\partial y} \mathbf{j} + 3r^2 \frac{\partial r}{\partial z} \mathbf{k}$$

$$= 3r^2 \left[\frac{x}{r} \mathbf{i} + \frac{y}{r} \mathbf{j} + \frac{z}{r} \mathbf{k} \right] = 3r \mathbf{r}$$

7.2.12 Evaluate $\text{grad } r^n$

Sol : Let $\phi = r^n = (x^2 + y^2 + z^2)^{n/2}$

$$\frac{\partial \phi}{\partial x} = \frac{n}{2} (x^2 + y^2 + z^2)^{n/2-1} \cdot 2x = nx (x^2 + y^2 + z^2)^{n/2-1}$$

$$\text{similarly } \frac{\partial \phi}{\partial x} = ny (x^2 + y^2 + z^2)^{n/2-1} \text{ and } \frac{\partial \phi}{\partial z} = nz (x^2 + y^2 + z^2)^{n/2-1}$$

$$\text{grad } r^n = \text{grad } \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

$$= n, (x^2 + y^2 + z^2)^{n/2-1} \cdot (xi + yj + zk) = n r^{n-2} \cdot r$$

Aliter :

$$\text{grad } \phi = \left(\frac{\partial \phi}{\partial x} \right) i + \left(\frac{\partial \phi}{\partial y} \right) j + \left(\frac{\partial \phi}{\partial z} \right) k = \frac{\partial \phi}{\partial r} \cdot \frac{\partial r}{\partial x} i + \frac{\partial \phi}{\partial r} \cdot \frac{\partial r}{\partial y} j + \frac{\partial \phi}{\partial r} \cdot \frac{\partial r}{\partial z} k$$

$$= nr^{n-1} \cdot \frac{x}{r} i + nr^{n-1} \cdot \frac{y}{r} j + nr^{n-1} \cdot \frac{z}{r} k = nr^{n-2} \cdot r$$

Ex.7.2.13 If A is a constant vector prove that $\text{grad } (r \cdot A) = A$

Sol : Let $A = A_1 i + A_2 j + A_3 k$ (A_1, A_2, A_3 being constant functions)

$$r = xi + yj + zk$$

$$r \cdot A = A_1 x + A_2 y + A_3 z = f \text{ (say)}$$

$$\frac{\partial f}{\partial x} = A_1, \frac{\partial f}{\partial y} = A_2, \frac{\partial f}{\partial z} = A_3$$

$$\therefore \text{grad } (r \cdot A) = \text{grad } f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k = A_1 i + A_2 j + A_3 k = A$$

Ex. 7.2.14 Prove $\nabla (\phi(r)) = \frac{\phi'(r) \cdot r}{r}$

Sol : Let $f = \phi(r)$

$$\frac{\partial f}{\partial x} = \phi'(r) \frac{\partial r}{\partial x} = \phi'(r) \cdot \frac{x}{r}$$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial f}{\partial y} = \phi'(r) \frac{\partial r}{\partial y} = \phi'(r) \cdot \frac{y}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial f}{\partial z} = \phi'(r) \frac{\partial r}{\partial z} = \phi'(r) \cdot \frac{z}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r} \text{ (see Aliter of ex. 7.2.11)}$$

$$\therefore \nabla (\phi(r)) = \nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

$$= \frac{\phi'(r)}{(r)} (xi + yj + zk) = \frac{\phi'(r)r}{(r)}$$

Ex. 7.2.15 Find the equations for the tangent plane and normal line to the surface $z = x^2 + y^2$ at the point $(2, 1, 5)$.

Sol : Let $r = xi + yj + zk$ be the position vector of any point $P(x, y, z)$ on the surface.

Let $r_1 = x_1i + y_1j + z_1k$ be the position vector of fixed point $A(x_1, y_1, z_1)$ on the surface.

Then $AP = (x - x_1)i + (y - y_1)j + (z - z_1)k = r - r_1$

Let n be the normal to the surface at A .

Then, since AP is perpendicular to n , we have,

$$(r - r_1) \cdot n = 0 \quad \dots (1)$$

which is the equation to the tangent plane at A .

Here, in the given problem

$$r - r_1 = (x - 2)i + (y - 1)j + (z - 5)k$$

$$\text{and } n = \nabla (x^2 + y^2 - z) \text{ at } A(2, 1, 5)$$

$$= (2xi + 2yj - k) \Big|_{(2,1,5)} = 4i + 2j - k$$

$\therefore$ The tangent plane at $(2, 1, 5)$ is, (from (1)),

$$4(x - 2) + 2(y - 1) - 1(z - 5) = 0 \Rightarrow 4x + 2y - z = 5 \quad \dots (2)$$

From (2), the direction ratios of the normal line at A are $4, 2, -1$

$\therefore$ Equation to the normal line at A are,

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \quad \text{where}$$

$$(x_1, y_1, z_1) = (2, 1, 5) \text{ and } (a, b, c) = (4, 2, -1)$$

$$\therefore \text{The equations of normal line are } \frac{x - 2}{4} = \frac{y - 1}{2} = \frac{z - 5}{-1}$$

From the above example, we have to remember the following :

Let $\phi = c$ be any given surface and (x_1, y_1, z_1) be a point on it; then

(1) Equation to the tangent plane to $\phi = c$ at (x_1, y_1, z_1) is

$$(x - x_1) \frac{\partial \phi}{\partial x} + (y - y_1) \frac{\partial \phi}{\partial y} + (z - z_1) \frac{\partial \phi}{\partial z} = 0$$

(2) Equations to Normal line at (x_1, y_1, z_1) are

$$\frac{x - x_1}{\partial\phi/\partial x} = \frac{y - y_1}{\partial\phi/\partial y} = \frac{z - z_1}{\partial\phi/\partial z}$$

Exercise 7(a)

1. If $\phi = 2xz^3 - 3x^2yz$, find $\nabla \phi$ and $|\nabla \phi|$ at the point $(2, 2, -1)$

$$[\text{Ans : (i) } 22i + 12j - 12k \text{ (ii) } 2\sqrt{193}]$$

2. If $V = 2x i - 3y^2 j + z^3 k$, and $\phi = 2xyz - 3z^2$, find $V \cdot \nabla \phi$ and $V \times \nabla \phi$ at the point $(1, 2, 3)$

$$[\text{Ans : (1) } -426 \text{ (2) } 6i + 352j + 156k]$$

3. If $f = 2xyz$ and $g = x^2y + z$, find $\nabla (f + g)$ and $\nabla (fg)$ at the point $(1, -1, 0)$

$$[\text{Ans : } -2i + j - k; 2k]$$

4. Evaluate $\nabla (3r^2 - 4\sqrt{r} + \frac{6}{\sqrt[3]{r}})$

$$[\text{Ans : } (6 - 2r^{3/2} - 2r^{-\frac{7}{3}}) r]$$

5. If $\phi = r^2 e^{-r}$, show that $\text{grad } \phi = (2-r) e^{-r} r$

6. Find a unit normal vector to the surface $z = x^2 + y^2$ at the point $(1, -2, 5)$

$$[\text{Ans : } \frac{1}{\sqrt{21}}(2i - 4j - k)]$$

7. Find the equations to the tangent plane and the normal line to the surface $xz^2 + x^2y = z - 1$ at the point $(1, -3, 2)$

$$[\text{Ans: (i) } 2x - y - 3z + 1 = 0 \text{ (ii) } \frac{x-1}{-2} = \frac{y+3}{1} = \frac{z-2}{3}]$$

8. Find the equations to the Tangent plane and normal line to the surface $y = x^2 + z^2$ at the point $(1, 5, -2)$

$$[\text{Ans : (i) } 2x - y - 4z = 5 \text{ (ii) } \frac{x-1}{2} = \frac{y-5}{-1} = \frac{z+2}{-4}]$$

9. Find the directional derivative of $U = 4xz^3 - 3x^2y^2z$ at $(2, 1, -2)$ in the direction of $(3i - 2j + 6k)$.

$$[\text{Ans : } \frac{384}{7}]$$

10. Find the directional derivative of $\phi = 4e^{2x-y+z}$ at the point $(1, 1, -1)$ in a direction towards the point $(2, 3, 1)$

$$[\text{Ans : } 8/3]$$

11. Find the values of the constants a, b, c so that the directional derivative of $f = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has a maximum magnitude 64 in a direction parallel to z -axis.

[Hint : Δf at $(1, 2, -1)$ is $\parallel^{lel}$ to z - axis.

$\therefore$ Equate coefficients of i and j to zero and $|\nabla f| = 64$. Thus get 3 equations in a, b, c and solve them]

[Ans: $a = 6, b = 24, c = -8$]

12. Find the acute angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point $(1, -2, 1)$

[Ans : $\cos^{-1} \frac{\sqrt{3}}{7\sqrt{2}}$]

13. Find $\text{grad } \Psi$ if $r = xi + yj + zk, r = |r|$ and

$$(i) \Psi = \text{Log } r \quad (ii) \Psi = \frac{1}{r} \quad (iii) \Psi = r$$

[Ans : (i) r/r^2 (ii) $-r/r^3$ (iii) r/r]

14. Find the directional derivative of $g = x^2y^2 + y^2z^2 + z^2x^2$ at the point $(1, 1, -2)$ in the direction of the tangent to the curve $x = e^{-t}, y = 2 \sin t + 1, z = t - \cos t$ at $t = 0$.

[Hint : Tangent vector to the given curve is

$$\frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k$$

[Ans : $\frac{2}{\sqrt{6}}$]

15. Find the acute angle between the normals to the surface $xy = z^2$ at the points $(1, 9, 3)$ and $(3, 3, -3)$

[Ans : $\cos^{-1} \left(\frac{1}{\sqrt{177}} \right)$]

16. If $r = xi + yj + zk$ and $\phi = x^3 + y^3 + z^3 - 3xyz$, show that $r, \text{grad } \phi = 3\phi$.

7.3 The Divergence of a Vector Function

7.3.1 If $A = A_1i + A_2j + A_3k$ is a vector function, defined and differentiable at each point (x, y, z) in a certain region of space [i.e., A defines a vector field], then the divergence of A (abbreviated as 'Div A ') is defined as,

$$\text{Div } A = \nabla \cdot A$$

$$= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot (A_1i + A_2j + A_3k)$$

$$= \left(\frac{\partial A_1}{\partial x} i + \frac{\partial A_2}{\partial y} j + \frac{\partial A_3}{\partial z} k \right)$$

(since $i \cdot i = j \cdot j = k \cdot k = 1$)

Note : (1) Div A is a scalar field

$$(2) \nabla \cdot A \neq A \cdot \nabla$$

7.3.2 Physical significance of the divergence

If A represents the velocity of fluid in a fluid flow, Div A represents the rate of fluid flow through unit volume. (or) Div A gives the rate at which fluid is originating at a point per unit volume.

Similarly if A represents the Electric flux or heat flux, Div A represents the amount of electric flux or heat flux that diverges per unit volume in unit time.

7.3.3 Some properties of Divergence

If A, B , are vector functions and ' f ' is a scalar function, then, prove that

$$(1) \quad \text{Div } (A + B) = \text{Div } A + \text{Div } B \text{ (i.e.) } \nabla \cdot (A + B) = \nabla \cdot A + \nabla \cdot B$$

Proof: Let $A = A_1i + A_2j + A_3k$

$$B = B_1i + B_2j + B_3k$$

$$A + B = (A_1 + B_1)i + (A_2 + B_2)j + (A_3 + B_3)k$$

$$\text{Div } A = \frac{\partial}{\partial x}(A_1 + B_1) + \frac{\partial}{\partial y}(A_2 + B_2) + \frac{\partial}{\partial z}(A_3 + B_3)$$

$$= \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) + \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} \right)$$

$$= \text{Div } A + \text{Div } B.$$

(2) Prove that, $\text{Div}(f\mathbf{A}) = (\text{grad } f) \cdot \mathbf{A} + f(\text{Div } \mathbf{A})$ i.e. $\nabla \cdot (f\mathbf{A}) = (\nabla f) \cdot \mathbf{A} + f(\nabla \cdot \mathbf{A})$

Proof : Let $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ then $f\mathbf{A} = fA_1\mathbf{i} + fA_2\mathbf{j} + fA_3\mathbf{k}$

$$\begin{aligned}\nabla \cdot (f\mathbf{A}) &= \frac{\partial}{\partial x}(fA_1) + \frac{\partial}{\partial y}(fA_2) + \frac{\partial}{\partial z}(fA_3) \\&= f \frac{\partial A_1}{\partial x} + A_1 \frac{\partial f}{\partial x} + f \frac{\partial A_2}{\partial y} + A_2 \frac{\partial f}{\partial y} + f \frac{\partial A_3}{\partial z} + A_3 \frac{\partial f}{\partial z} \\&= f \left[\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right] + \left[\left(\frac{\partial f}{\partial x} \right) A_1 + \left(\frac{\partial f}{\partial y} \right) A_2 + \left(\frac{\partial f}{\partial z} \right) A_3 \right] \quad \dots (1)\end{aligned}$$

$$\begin{aligned}(\nabla f) \cdot \mathbf{A} &= \left(\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \right) \cdot (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}) \\&= \left(\frac{\partial f}{\partial x} \right) A_1 + \left(\frac{\partial f}{\partial y} \right) A_2 + \left(\frac{\partial f}{\partial z} \right) A_3 \quad \dots (2)\end{aligned}$$

$$f(\nabla \cdot \mathbf{A}) = f \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) \quad \dots (3)$$

$$(1), (2), (3) \Rightarrow \nabla \cdot (f\mathbf{A}) = (\nabla f) \cdot \mathbf{A} + f(\nabla \cdot \mathbf{A})$$

7.3.4 Solenoidal vectors : A vector $\mathbf{A}$ is said to be solenoidal if $\text{Div } \mathbf{A} = 0$

Solved Examples

Ex. 7.3.5 If $\mathbf{A} = (x^2y)\mathbf{i} + (xy^2z)\mathbf{j} + (xyz)\mathbf{k}$, find $\text{div } \mathbf{A}$ at the point $(1, -1, 2)$.

Sol: $\mathbf{A} = (x^2y)\mathbf{i} + (xy^2z)\mathbf{j} + (xyz)\mathbf{k}$

$$\begin{aligned}\text{Div } \mathbf{A} &= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \\&= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial z}(xyz) \\&= 2xy + 2xyz + xy \\&= xy(2z + 3)\end{aligned}$$

$$\therefore \text{At } (1, -1, 2), \text{Div } \mathbf{A} = (1)(-1)[4 + 3] = -7$$

Ex. 7.3.6 If $V = 2xyi + 3x^2yj - 3pyz k$ is solenoidal at $(1, 1, 1)$, find 'p'.

Sol: $\text{Div } V = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(-3pyz)$

$$= 2y + 3x^2 - 3py$$

At $(1, 1, 1)$. $\text{Div } V = 5 - 3p$

Since V is solenoidal, $\text{Div } V = 0 \therefore p = \frac{5}{3}$

Ex. 7.3.7 If $r = xi + yz + zk$, and $r = |r|$, show that $\text{Div } (r^3 r) = 6 r^3$

Sol: $r = \sqrt{x^2 + y^2 + z^2} \therefore r^3 r = (x^2 + y^2 + z^2)^{3/2} (xi + yj + zk)$

$$= A_1 i + A_2 j + A_3 k \text{ (say)}$$

then, $A_1 = x(x^2 + y^2 + z^2)^{3/2}$

$$A_2 = y(x^2 + y^2 + z^2)^{3/2}$$

$$A_3 = z(x^2 + y^2 + z^2)^{3/2}$$

$$\frac{\partial A_1}{\partial x} = x \frac{3}{2} (x^2 + y^2 + z^2)^{\frac{3}{2}-1} 2x + (x^2 + y^2 + z^2)^{3/2} \cdot 1$$

$$= 3x^2 r + r^3$$

similarly $\frac{\partial A_2}{\partial y} = 3y^2 r + r^3; \quad \frac{\partial A_3}{\partial z} = 3z^2 r + r^3$

$$\text{Div } A = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

$$= 3r(x^2 + y^2 + z^2) + 3r^3 = 3r^3 + 3r^3 = 6r^3$$

Aliter: $r^3 r = r^3 xi + r^3 yi + r^3 zk$

$$\frac{\partial}{\partial x}(r^3 x) = r^3 \cdot 1 + x \cdot 3r^2 \frac{\partial r}{\partial x} = r^3 + x \cdot 3r^2 \cdot \frac{x}{r}$$

$$= r^3 + 3x^2 r \quad \left(\because \frac{\partial r}{\partial x} = \frac{x}{r} \right)$$

Similarly $\frac{\partial}{\partial y} (r^3 y) = r^3 + 3y^2 r$ and $\frac{\partial}{\partial z} (r^3 z) = r^3 + 3z^2 r$

$$\text{Div } A = 6r^3$$

Ex. 7.3.8 Evaluate $\text{Div} [r \text{ grad } (r^{-3})]$ or $\nabla \cdot \left\{ r \nabla \left(\frac{1}{r^3} \right) \right\}$

Sol:
$$\text{grad } (r^3) = \left\{ \frac{\partial}{\partial x} (r^3) \right\} i + \left\{ \frac{\partial}{\partial y} (r^3) \right\} j + \left\{ \frac{\partial}{\partial z} (r^3) \right\} k$$

$$\begin{aligned} &= -3r^4 \frac{\partial r}{\partial x} i - 3r^4 \frac{\partial r}{\partial y} j - 3r^4 \frac{\partial r}{\partial z} k = -3r^4 \left(\frac{x}{r} i + \frac{y}{r} j + \frac{z}{r} k \right) \\ &= -3r^5 (xi + yj + zk) \end{aligned}$$

$$r \text{ grad } (r^3) = -3r^4 (xi + yj + zk)$$

$$= A_1 i + A_2 j + A_3 k \quad (\text{say})$$

$$\text{where } A_1 = -3r^4 x, A_2 = -3r^4 y, A_3 = -3r^4 z$$

$$\frac{\partial A_1}{\partial x} = \frac{\partial}{\partial x} [-3r^4 x]$$

$$\begin{aligned} &= -3r^4 \cdot 1 + x \cdot -3 \cdot -4 r^{-5} \cdot \frac{\partial r}{\partial x} = -3r^4 + 12 x r^{-5} \cdot \frac{x}{r} \\ &= -3r^4 + 12 x^2 r^{-6} \end{aligned}$$

$$\text{Similarly, } \frac{\partial A_2}{\partial y} = -3r^4 + 12 y^2 r^{-6}$$

$$\frac{\partial A_3}{\partial z} = -3r^4 + 12 z^2 r^{-6}$$

$$\therefore \text{Div } (r \text{ grad } r^{-3}) = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

$$= -9r^4 + 12 r^{-6} (x^2 + y^2 + z^2) = -9r^4 + 12 r^{-6} \cdot r^2 = 3r^{-4}$$

(Alternate method is left to the reader).

Ex. 7.3.9 Show that $\nabla \cdot (r^n \mathbf{r}) = (n + 3) r^n$. Hence show that $\mathbf{r}/r^3$ is solenoidal.

Sol : $r^n \mathbf{r} = r^n (xi + yj + zk)$

$$\begin{aligned}\nabla \cdot (r^n \mathbf{r}) &= \frac{\partial}{\partial x} (xr^n) + \frac{\partial}{\partial y} (yr^n) + \frac{\partial}{\partial z} (zr^n) \\&= \left[r^n + xnr^{n-1} \cdot \frac{\partial r}{\partial x} \right] + \left[r^n + ynr^{n-1} \frac{\partial r}{\partial y} \right] + \left[r^n + z.n r^{n-1} \frac{\partial r}{\partial z} \right] \\&= 3r^n + n r^{n-1} \left(x \cdot \frac{x}{r} + y \cdot \frac{y}{r} + z \cdot \frac{z}{r} \right) \\&= 3r^n + n \frac{r^{n-1}}{r} (x^2 + y^2 + z^2) \\&= 3r^n + n r^{n-2} \cdot r^2 = (n + 3) r^n \\ \text{If } n = -3, (r^{-3} \mathbf{r}) &= (-3 + 3) r^{-3} = 0\end{aligned}$$

$\therefore \frac{\mathbf{r}}{r^3}$ is solenoidal

Ex. 7.3.10 Prove that $\text{Div} (C_1 \mathbf{A} + C_2 \mathbf{B}) = C_1 \text{Div} \mathbf{A} + C_2 \text{Div} \mathbf{B}$, where C_1, C_2 are constants.

Sol : Let $\mathbf{A} = A_1 \mathbf{i} + A_2 \mathbf{j} + A_3 \mathbf{k}$

$$\mathbf{B} = B_1 \mathbf{i} + B_2 \mathbf{j} + B_3 \mathbf{k}$$

$$C_1 \mathbf{A} + C_2 \mathbf{B} = (C_1 A_1 + C_2 B_1) \mathbf{i} + (C_1 A_2 + C_2 B_2) \mathbf{j} + (C_1 A_3 + C_2 B_3) \mathbf{k}$$

$$\begin{aligned}\text{Div} (C_1 \mathbf{A} + C_2 \mathbf{B}) &= \frac{\partial}{\partial x} (C_1 A_1 + C_2 B_1) + \frac{\partial}{\partial y} (C_1 A_2 + C_2 B_2) \\&\quad + \frac{\partial}{\partial z} (C_1 A_3 + C_2 B_3)\end{aligned}$$

$$= C_1 \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) + C_2 \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} \right)$$

$$= C_1 \text{div} \mathbf{A} + C_2 \text{div} \mathbf{B}$$

Ex. 7.3.11 If $A = 2xi + 3yj + 5zk$ and $f = 2xyz$, find $\text{div}(fA)$ at $(1, 2, 3)$.

Sol : $fA = 2xyz(2xi + 3yj + 5zk)$
 $= (4x^2yz)i + (6xy^2z)j + (10xyz^2)k$

$$\text{Div}(fA) = \frac{\partial}{\partial x}(4x^2yz) + \frac{\partial}{\partial y}(6xy^2z) + \frac{\partial}{\partial z}(10xyz^2)$$

$$= 40xyz$$

$$\therefore \text{At } (1, 2, 3), \text{div}(fA) = 240$$

Aliter : $\text{div}(fA) = D.(fA) = (\nabla f) \cdot A + f(\nabla \cdot A)$

$$\nabla f = 2yzi + 2xzy + 2xyk$$

$$\nabla f \cdot A = 4xyz + 6xyz + 10xyz = 20xyz$$

$$f(\nabla \cdot A) = 2xyz(2 + 3 + 5) = 20xyz$$

$$\text{Hence } \nabla \cdot (fA) = 40xyz \text{ and at } (1, 2, 3) \nabla \cdot (fA) = 240$$

Ex. 7.3.12 If f, g are scalar fields show that $\nabla f \times \nabla g$ is solenoidal

Sol : $\nabla f = \left(\frac{\partial f}{\partial x}\right)i + \left(\frac{\partial f}{\partial y}\right)j + \left(\frac{\partial f}{\partial z}\right)k$

$$\nabla g = \left(\frac{\partial g}{\partial x}\right)i + \left(\frac{\partial g}{\partial y}\right)j + \left(\frac{\partial g}{\partial z}\right)k$$

$$\nabla f \times \nabla g = \begin{vmatrix} i & j & k \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{vmatrix}$$

$$= \Sigma (f_y g_z - f_z g_y)i \quad (\text{Suffixes denote partial derivatives})$$

$$\text{Div}(\nabla f \times \nabla g) = \Sigma \frac{\partial}{\partial x} (f_y g_z - f_z g_y)$$

$$= \Sigma (f_y g_{xz} + g_z f_{xy} - f_z g_{xy} - g_y f_{xz})$$

$$= 0$$

$$\therefore \nabla f \times \nabla g \text{ is solenoidal}$$

Exercise 6(b)

1. If $V = (x^2z)i - (2y^3z^2)j + (xy^2z)k$, find $\text{div } A$ at the point $(1, -1, 1)$
[Ans. -3]
2. If $r = xi + yj + zk$, find $\text{div } r$
[Ans. 3]
3. If $F = (3xyz^2)i + (2xy^3)j - (x^2yz)k$, and $\phi = 3x^2 - yz$,
find (i) $\text{Div } F$ (ii) $\text{Div } (\phi F)$ and (iii) $\text{Div } (\text{grad } \phi)$; at the point $(1, -1, 1)$
[Ans. (i) 4 (ii) 1 (iii) 6]
4. If $V = (3x^2y - z)i + (xz^3 + y^4)j - 2x^3z^2k$, find $\text{grad } (\text{Div } V)$ at the point $(2, -1, 0)$
[Ans. $-6i + 24j - 32k$]
5. Evaluate : (1) $\text{Div } (r^2r)$ (2) $\text{Div } (r \cdot r)$ (3) $\text{grad } \text{Div } (r/r)$ (4) $\text{div } (r/r^3)$
[Ans. (1) $5r^2$ (2) $4r$ (3) $-2r/r^3$ (4) 0]
6. Show that $V = 3y^4z^2i + 4x^3z^2j + 6x^2y^3k$ is solenoidal
7. Show that the vector $F = (2x^2 + 8xy^2z)i + (3x^3y - 3xy)j - (4y^2z^2 + 2x^3z)k$ is not solenoidal, but $G = xyz^2 F$ is solenoidal.
8. If a is a constant vector and $V = a \times r$, where $r = xi + yj + zk$, show that V is a solenoidal vector.
9. Determine the constant 'b' such that the vector, $V = (2x + 3y)i + (by - 3z)j + (6x - 12z)k$ is solenoidal
10. If r_1 and r_2 are vectors joining fixed points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ to a variable point $P(x, y, z)$ prove that $r_1 \times r_2$ is solenoidal.
11. If $r = xi + yj + zk$ and $r = |r|$ show that, $\text{Div } (\text{grad } r^n) = n(n+1)r^{n-2}$.
12. If $g = r^{-2n}$, find $\text{div } (\text{grad } g)$ and find 'n' such that 'g' is solenoidal.

$$[\text{Ans. } \frac{2n(2n-1)}{r^{2n+2}}; n = \frac{1}{2}]$$

7.4 Curl of a vector function

7.4.1 If A is a differential vector function, then $\text{curl } A$ is defined as, $\text{curl } A = \nabla \times A$

$$\text{If } A = A_1i + A_2j + A_3k, \text{ then } \text{Curl } A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$\begin{aligned}
&= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) i + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) j + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) k \\
&= \Sigma \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) i
\end{aligned}$$

Note : The curl of a vector is also a vector

7.4.2 Physical significance of curl

Let $\mathbf{r} = xi + yj + zk$ be the position vector of a point $P(x, y, z)$ of a rigid body rotating about a fixed axis about the origin O with an angular velocity $\omega = \omega_1 i + \omega_2 j + \omega_3 k$. Then the velocity $\mathbf{V}$ of the particle P is given by,

$$\begin{aligned}
\mathbf{V} = \omega \times \mathbf{r} &= \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} \\
&= (\omega_2 z - \omega_3 y)i + (\omega_3 x - \omega_1 z)j + (\omega_1 y - \omega_2 x)k
\end{aligned}$$

$$\begin{aligned}
\text{Curl } \mathbf{V} &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix} \\
&= i \left[\frac{\partial}{\partial y} (\omega_1 y - \omega_2 x) - \frac{\partial}{\partial z} (\omega_3 x - \omega_1 z) \right] + j \left[\frac{\partial}{\partial z} (\omega_2 z - \omega_3 y) - \frac{\partial}{\partial x} (\omega_1 y - \omega_2 x) \right] \\
&\quad + k \left[\frac{\partial}{\partial x} (\omega_3 x - \omega_1 z) - \frac{\partial}{\partial y} (\omega_2 z - \omega_3 y) \right] \\
&= i(\omega_1 + \omega_1) + j(\omega_2 + \omega_2) + k(\omega_3 + \omega_3) = 2\omega
\end{aligned}$$

Thus the curl of velocity vector is twice the angular velocity of rotation.

7.4.3 Irrotational Vector : A vector $\mathbf{V}$ whose curl is zero is said to be an irrotational vector.

7.4.4 Properties : (1) $\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$ (or) $\text{curl}(\mathbf{A} + \mathbf{B}) = \text{curl } \mathbf{A} + \text{curl } \mathbf{B}$

Proof: Let $\mathbf{A} = A_1 i + A_2 j + A_3 k$ and

$\mathbf{B} = B_1 i + B_2 j + B_3 k$ so that

$$\mathbf{A} + \mathbf{B} = (A_1 + B_1)\mathbf{i} + (A_2 + B_2)\mathbf{j} + (A_3 + B_3)\mathbf{k}$$

$$\begin{aligned}\text{Curl } (\mathbf{A} + \mathbf{B}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 + B_1 & A_2 + B_2 & A_3 + B_3 \end{vmatrix} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_1 & B_2 & B_3 \end{vmatrix} \\ &= \text{curl } \mathbf{A} + \text{curl } \mathbf{B}\end{aligned}$$

(2) If ϕ is a scalar function and $\mathbf{A}$ is a vector function

$$\text{Curl } (\phi\mathbf{A}) = \phi(\text{curl } \mathbf{A}) + (\text{grad } \phi) \times \mathbf{A}$$

(or)

$$\nabla \times (\phi\mathbf{A}) = \phi(\nabla \times \mathbf{A}) + (\nabla \phi) \times \mathbf{A}$$

Proof : If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, then $\phi\mathbf{A} = \phi A_1\mathbf{i} + \phi A_2\mathbf{j} + \phi A_3\mathbf{k}$

$$\begin{aligned}\nabla \times (\phi\mathbf{A}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi A_1 & \phi A_2 & \phi A_3 \end{vmatrix} \\ &= \mathbf{i} \left[\frac{\partial}{\partial y} (\phi A_3) - \frac{\partial}{\partial z} (\phi A_2) \right] + \mathbf{j} \left[\frac{\partial}{\partial z} (\phi A_1) - \frac{\partial}{\partial x} (\phi A_3) \right] + \mathbf{k} \left[\frac{\partial}{\partial x} (\phi A_2) - \frac{\partial}{\partial y} (\phi A_1) \right] \\ &= \sum \mathbf{i} \left[\phi \frac{\partial A_3}{\partial y} + A_3 \frac{\partial \phi}{\partial y} - \phi \frac{\partial A_2}{\partial z} - A_2 \frac{\partial \phi}{\partial z} \right] \\ &= \phi \sum \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \mathbf{i} + \sum \left(A_3 \frac{\partial \phi}{\partial y} - A_2 \frac{\partial \phi}{\partial z} \right) \mathbf{i} \\ &= \phi \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} \\ &= \phi(\nabla \times \mathbf{A}) + (\nabla \phi) \times \mathbf{A}\end{aligned}$$

7.4.5 Conservative vector field :

A vector field $\mathbf{F}$, which can be derived from a scalar field ϕ such that $\mathbf{F} = \nabla \phi$, is called a conservative vector field and ϕ is called the scalar potential of $\mathbf{F}$.

Solved Examples

Ex.7.4.6 If $A = (xy)i + (yz)j + (zx)k$, find (a) $\text{curl } A$ and (b) $\text{curl curl } A$ at $(1, 2, -3)$

Sol : $A = xyi + yzj + zxk$

$$\begin{aligned}
 \text{(a)} \quad \text{curl } A &= \nabla \times A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & yz & zx \end{vmatrix} \\
 &= i \left[\frac{\partial}{\partial y}(zx) - \frac{\partial}{\partial z}(yz) \right] + j \left[\frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(zx) \right] + k \left[\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(xy) \right] \\
 &= i(0 - y) + j(0 - z) + k(0 - x) \\
 &= -yi - zj - xk \\
 \therefore \text{curl } A \text{ at } (1, 2, -3) &= -2i + 3j - k
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \text{curl } A &= \nabla \times (\nabla \times A) \\
 &= \nabla \times (-yi - zj - xk) \\
 &= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & -z & -x \end{vmatrix} \\
 &= i \left[\frac{\partial}{\partial y}(-x) - \frac{\partial}{\partial z}(-z) \right] + j \left[\frac{\partial}{\partial z}(-y) - \frac{\partial}{\partial x}(-x) \right] + k \left[\frac{\partial}{\partial x}(-z) - \frac{\partial}{\partial y}(-y) \right] \\
 &= i(0 - (-1)) + j(0 - (-1)) + k(0 - (-1)) \\
 &= i + j + k \\
 \therefore \text{curl } A \text{ at } (1, 2, -3) &= i + j + k
 \end{aligned}$$

Ex. 7.4.7 Show that $V = xi + y^2j + z^3k$ is irrotational

Sol : $\text{Curl } V = \nabla \times V$

$$\begin{aligned}
 &= i \left[\frac{\partial}{\partial y}(z^3) - \frac{\partial}{\partial z}(y^2) \right] + j \left[\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(z^3) \right] + k \left[\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x) \right] \\
 &= 0 \quad \therefore V \text{ is irrotational}
 \end{aligned}$$

Ex. 7.4.8 If $F = (4x + 3y + az)i + (bx - y + z)j + (2x + cy + z)k$ is irrotational, find the constants a, b, c

Sol :
$$\text{Curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (4x + 3y + az) & (bx - y + z) & (2x + cy + z) \end{vmatrix}$$

$$= i \left[\frac{\partial}{\partial y} (2x + cy + z) - \frac{\partial}{\partial z} (bx - y + z) \right] + j \left[\frac{\partial}{\partial z} (4x + 3y + az) - \frac{\partial}{\partial x} (2x + cy + z) \right]$$

$$+ k \left[\frac{\partial}{\partial x} (bx - y + z) - \frac{\partial}{\partial y} (4x + 3y + az) \right]$$

$$= (c - 1)i + (a - 2)j + (b - 3)k$$

Since F is irrotational, $\text{curl } F = 0$

$$\therefore c - 1 = 0, a - 2 = 0, b - 3 = 0$$

$$\text{i.e., } a = 2, b = 3, c = 1$$

Ex. 7.4.9 If $r = xi + yj + zk$, and $r = |r|$, find $\text{curl } (r^n r)$

Sol : $r = xi + yj + zk, r = |r| = \sqrt{x^2 + y^2 + z^2}$

$$r^n r = (x^2 + y^2 + z^2)^{n/2} (xi + yj + zk)$$

$$\text{curl } (r^n r) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x(x^2 + y^2 + z^2)^{n/2} & y(x^2 + y^2 + z^2)^{n/2} & z(x^2 + y^2 + z^2)^{n/2} \end{vmatrix}$$

$$= i \left[z \cdot \frac{n}{2} (x^2 + y^2 + z^2)^{n/2 - 1} \cdot 2y - y \cdot \frac{n}{2} (x^2 + y^2 + z^2)^{n/2 - 1} \cdot 2z \right]$$

$$+ j \left[x \cdot \frac{n}{2} (x^2 + y^2 + z^2)^{n/2 - 1} \cdot 2z - z \cdot \frac{n}{2} (x^2 + y^2 + z^2)^{n/2 - 1} \cdot 2x \right]$$

$$+ k \left[y \cdot \frac{n}{2} (x^2 + y^2 + z^2)^{n/2 - 1} \cdot 2x - x \cdot \frac{n}{2} (x^2 + y^2 + z^2)^{n/2 - 1} \cdot 2y \right]$$

$$= i(0) + j(0) + k(0)$$

$$= 0$$

Aliter: $r^n \mathbf{r} = r^n (xi + yj + zk)$

$$\begin{aligned}\text{curl } (r^n \mathbf{r}) &= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xr^n & yr^n & zr^n \end{vmatrix} \\ &= \sum i \left[\frac{\partial}{\partial y} (zr^n) - \frac{\partial}{\partial z} (yr^n) \right] = \sum i \left[z.nr^{n-1} \frac{\partial r}{\partial y} - y.nr^{n-1} \frac{\partial r}{\partial z} \right] \\ &= \sum i \left[nr^{n-1} \frac{y}{r} - y.nr^{n-1} \frac{z}{r} \right] \\ &= i(o) + j(o) + k(o) = 0\end{aligned}$$

Ex. 7.4.10 Prove that, if $\mathbf{F} = (x + y + 1)i + j - (x + y)k$, $\text{F. curl } \mathbf{F} = 0$

Sol :
$$\text{Curl } \mathbf{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y+1 & 1 & -x-y \end{vmatrix}$$

$$\begin{aligned}&= i[-1-0] + j[0+1] + k[0-1] \\ &= -i + j - k \\ \therefore \text{F curl } \mathbf{F} &= -1(x+y+1) + 1.1 + 1(x+y) = 0\end{aligned}$$

Ex. 7.4.11 $P(x, y, z)$ is a variable point and $Q(x_1, y_1, z_1)$, $R(x_2, y_2, z_2)$ are fixed points.

If $\mathbf{U} = \mathbf{QP}$ and $\mathbf{V} = \mathbf{RP}$; Prove that $\text{curl } (\mathbf{U} \times \mathbf{V})$ is equal to $2(\mathbf{U} - \mathbf{V})$.

Sol : $P(x, y, z)$, $Q = (x_1, y_1, z_1)$, $R = (x_2, y_2, z_2)$

$$\begin{aligned}\therefore \mathbf{QP} &= (x - x_1)i + (y - y_1)j + (z - z_1)k = \mathbf{U} \\ \mathbf{RP} &= (x - x_2)i + (y - y_2)j + (z - z_2)k = \mathbf{V}\end{aligned}$$

$$\mathbf{U} \times \mathbf{V} = \begin{vmatrix} i & j & k \\ (x - x_1) & (y - y_1) & (z - z_1) \\ (x - x_2) & (y - y_2) & (z - z_2) \end{vmatrix}$$

$$\begin{aligned}&= \sum \{(y - y_1)(z - z_2) - (z - z_1)(y - y_2)\} i \\ &= \sum \{y(z_1 - z_2) - z(y_1 - y_2) + (y_1 z_2 - y_2 z_1)\} i\end{aligned}$$

$$\begin{aligned}
 \text{Curl } (U \times V) &= \sum \left[\frac{\partial}{\partial y} \{x(y_1 - y_2) - y(x_1 - x_2) + (x_1 y_2 - x_2 y_1)\} \right. \\
 &\quad \left. - \frac{\partial}{\partial z} \{z(x_1 - x_2) - x(z_1 - z_2) + (z_1 x_2 - z_2 x_1)\} \right] i \\
 &= \Sigma(-(x_1 - x_2) - (x_1 - x_2))i \\
 &= \Sigma -2(x_1 - x_2)i = 2\Sigma(x_2 - x_1)i \\
 &= 2(U - V)
 \end{aligned}$$

Ex. 7.4.12 If F is a conservative vector field show that $\text{curl } F = 0$

Sol. F is a conservative vector field.

$$\therefore \text{There exists a scalar field '}\phi\text{' such that } F = \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$\begin{aligned}
 \therefore \text{Curl } F &= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix} \\
 &= \Sigma i \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \right) = 0
 \end{aligned}$$

Ex. 7.4.13 Show that $F = (6xy + z^3)i + (3x^2 - z)j + (3xz^2 - y)k$ is irrotational. Find ϕ such that $F = \nabla\phi$

$$\begin{aligned}
 \text{Sol : } \text{Curl } F &= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix} \\
 &= i(-1 + 1) + j(3z^2 - 3z^2) + k(6x - 6x) \\
 &= 0
 \end{aligned}$$

$\therefore F$ is irrotational

$$\text{Let } F = \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$\therefore \frac{\partial\phi}{\partial x} = 6xy + z^3 \quad \dots (1)$$

$$\frac{\partial \phi}{\partial y} = 3x^2 - z \quad \dots (2)$$

$$\frac{\partial \phi}{\partial z} = 3xz^2 - y \quad \dots (3)$$

Integrating equation (1) with respect to x we get

$$\phi = 3x^2y + xz^3 + f(y, z) \quad \dots (4)$$

Differentiating (4) partially w.r.t 'y'

$$\frac{\partial \phi}{\partial y} = 3x^2 + \frac{\partial f(y, z)}{\partial y} \quad \dots (5)$$

From (2) and (5) we have

$$\frac{\partial f(y, z)}{\partial y} = -z \quad \dots (6)$$

Integrating (6) w.r.t 'y' we get

$$f(y, z) = -yz + h(z)$$

$$\text{Hence } \phi = 3x^2 + xz^3 - yz + h(z) \quad \dots (7)$$

differentiating (7) w.r.t 'z' we get

$$\frac{\partial \phi}{\partial z} = 3xz^2 - y + h'(z) \quad \dots (8)$$

Comparing (3) and (8) we have

$$h'(z) = 0, \therefore h(z) = \text{constant, 'c' say}$$

$$\text{Hence } \phi = 3x^2y + xz^3 - yz + c \quad (\text{from 7})$$

$$\text{Aliter: } \therefore \frac{\partial \phi}{\partial x} = 6xy + z^3 \quad \dots (1)$$

$$\frac{\partial \phi}{\partial y} = 3x^2 - z \quad \dots (2)$$

$$\frac{\partial \phi}{\partial z} = 3xz^2 - y \quad \dots (3)$$

Integrating the above equations respectively w.r.t x , y , and z , we get

$$\phi = 3x^2y + xz^3 + f(y, z) \quad \dots (4)$$

$$\phi = 3x^2y - yz + g(z, x) \quad \dots (5)$$

$$\phi = xz^3 - yz + h(x, y) \quad \dots (6)$$

ϕ should satisfy all the above three equations simultaneously

(i.e.) (4), (5) & (6)

$\therefore \phi = 3x^2y + xz^3 - yz + c$, where c is a numerical constant.

Note: [Here $f = -yz$, $g = xz^3$, $h = 3x^2y$ will satisfy (4), (5) & (6)]

Ex.6.4.14 If $f(r)$ is differentiable, show that $f(r)r$ is irrotational

Sol : $f(r) r = f(r) (xi + yj + zk)$

$$\text{Curl } (f(r) r) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x f(r) & y f(r) & z f(r) \end{vmatrix}$$

$$= \sum i \left(\frac{\partial}{\partial y} \{z f(r)\} - \frac{\partial}{\partial z} \{y f(r)\} \right)$$

$$= \sum i \left(z \cdot f'(r) \frac{y}{r} - y \cdot f'(r) \frac{\partial r}{\partial z} \right)$$

$$= \sum i \left(z \cdot f'(r) \frac{y}{r} - y \cdot f'(r) \frac{z}{r} \right) = 0$$

$\therefore f(r)r$ is irrotational

Ex. 7.4.15 If U and V are irrotational, prove that $U \times V$ is solenoidal

Sol : Let $U = U_1 i + U_2 j + U_3 k$

$$V = V_1 i + V_2 j + V_3 k$$

U is irrotational $\therefore \text{curl } U = 0$

$$\Rightarrow \sum i \left[\frac{\partial U_3}{\partial y} - \frac{\partial U_2}{\partial z} \right] = 0 \quad \dots (1)$$

$$\text{similarly, } \sum i \left[\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right] = 0 \quad \dots (2)$$

$$\mathbf{U} \times \mathbf{V} = \begin{vmatrix} i & j & k \\ U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \end{vmatrix} = \Sigma i(U_2 V_3 - U_3 V_2)$$

$$\begin{aligned} \text{Div} (\mathbf{U} \times \mathbf{V}) &= \Sigma \frac{\partial}{\partial x} (U_2 V_3 - U_3 V_2) \\ &= \Sigma \left(U_2 \frac{\partial V_3}{\partial x} + V_3 \frac{\partial U_2}{\partial x} - U_3 \frac{\partial V_2}{\partial x} - V_2 \frac{\partial U_3}{\partial x} \right) \end{aligned} \quad \dots (3)$$

$$\text{From (1), we have, } \frac{\partial U_3}{\partial y} = \frac{\partial U_2}{\partial z}; \frac{\partial U_1}{\partial z} = \frac{\partial U_3}{\partial x}; \frac{\partial U_2}{\partial x} = \frac{\partial U_1}{\partial y} \quad \dots (4)$$

$$\text{and from (2), we have, } \frac{\partial V_3}{\partial y} = \frac{\partial V_2}{\partial z}; \frac{\partial V_1}{\partial z} = \frac{\partial V_3}{\partial x}; \frac{\partial V_2}{\partial x} = \frac{\partial V_1}{\partial y} \quad \dots (5)$$

Substituting the six equations of (4) & (5) in (3), we observe that all the 12 terms of (3) will get cancelled. Hence $\text{Div} (\mathbf{U} \times \mathbf{V}) = 0$

$\Rightarrow \mathbf{U} \times \mathbf{V}$ is solenoidal

Exercise - 7(c)

1. If $\mathbf{V} = (2xz^2)\mathbf{i} - (yz)\mathbf{j} + (3xz^3)\mathbf{k}$, and $f = x^2yz$, find the following at the point (1, 1, 1)
(a) $\text{curl } \mathbf{V}$ (b) $\text{curl} (f\mathbf{V})$ (c) $\text{curl} (\text{curl } \mathbf{V})$

[Ans. (a) $i + j$ (b) $5i - 3j - 4k$ (c) $5i + 3k$]

2. If 'g' is a scalar function, show that $\text{curl} (g \text{ grad } g) = 0$
3. Find the value of the constant 'p' for which the vector $\mathbf{V} = (pxy - z^3) + (p - 2)x^2\mathbf{j} + (1 - p)xz^2\mathbf{k}$ is irrotational.

[Ans : 4]

4. Find the constants a, b, c so that the curl of the vector $\mathbf{A} = (x + 2y + az)\mathbf{i} + (bx - 3y - z)\mathbf{j} + (4x + cy + 2z)\mathbf{k}$ is identically equal to zero

[Ans. 4, 2, -1]

5. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $r = |\mathbf{r}|$, show that $\text{curl} \left(\frac{\mathbf{r}}{r^2} \right) = 0$ and find a scalar function 'f' such

that $\frac{\mathbf{r}}{r^2} = -\nabla f$, $f(a) = 0$ where $a > 0$.

[Ans : $f = \log \left(\frac{a}{r} \right)$]

6. If $r = xi + yj + zk$, and p, q are constant vectors, show that (1) $\text{curl} [(r \times p) \times q] = (p \times q)$ and (2) $\text{curl} [(p \cdot q)r] = 0$.

7.5 Laplacian Operator : ∇^2

$$\begin{aligned} 7.5.1 \quad \nabla^2 &= \nabla \cdot \nabla = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \\ &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \text{ is called the Laplacian Operator} \end{aligned}$$

' ∇^2 ' can be applied to both scalar and vector functions as shown below.

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

where ' ϕ ' is scalar function

If $A = A_1 i + A_2 j + A_3 k$, is a vector function, then

$$\begin{aligned} \nabla^2 A &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (A_1 i + A_2 j + A_3 k) \\ &= (\nabla^2 A_1) i + (\nabla^2 A_2) j + (\nabla^2 A_3) k \end{aligned}$$

7.5.2 Vector Identities : we shall give below some vector identities with proofs.

1. If ' ϕ ' is a scalar function $\text{curl grad } \phi = 0$, (or) $\nabla \times \nabla \phi = 0$

$$\text{Proof: } \text{Curl grad } \phi = \nabla \times \left(\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k \right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \sum i \left[\frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial z} \right) - \frac{\partial}{\partial z} \left(\frac{\partial \phi}{\partial y} \right) \right]$$

$$= \sum i \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \right)$$

$= 0$, assuming that ' ϕ ' possesses continuous second order partial derivatives.

(2) If $\mathbf{V}$ is a vector function, $\text{Div} (\text{Curl } \mathbf{A}) = 0$ (or) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

Proof: Let $\mathbf{A} = A_1 \mathbf{i} + A_2 \mathbf{j} + A_3 \mathbf{k}$

$$\text{Curl } \mathbf{A} = \sum i \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right)$$

$$\begin{aligned} \text{Div} (\text{curl } \mathbf{A}) &= \sum \frac{\partial}{\partial x} \left\{ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right\} \\ &= \frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial x \partial z} + \frac{\partial^2 A_4}{\partial y \partial z} - \frac{\partial^2 A_3}{\partial y \partial x} + \frac{\partial^2 A_2}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z \partial y} \\ &= 0 \end{aligned}$$

(3) If $\mathbf{A}$ is a vector function, $\text{curl} (\text{curl } \mathbf{A}) = \text{grad} (\text{Div } \mathbf{A}) - \nabla^2 \mathbf{A}$ (or)

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

Proof: $\text{curl } \mathbf{A} = \sum i \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right)$

$$\begin{aligned} \text{Curl} (\text{curl } \mathbf{A}) &= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) & \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) & \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \end{vmatrix} \\ &= \sum \left[\frac{\partial}{\partial y} \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) - \frac{\partial}{\partial z} \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) \right] i \\ &= \left[\left(-\frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} \right) i + \left(-\frac{\partial^2 A_2}{\partial x^2} - \frac{\partial^2 A_2}{\partial y^2} - \frac{\partial^2 A_2}{\partial z^2} \right) j \right. \\ &\quad \left. + \left(-\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2} - \frac{\partial^2 A_3}{\partial z^2} \right) k \right] + \left[\left(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_2}{\partial y \partial x} + \frac{\partial^2 A_3}{\partial z \partial x} \right) i \right. \\ &\quad \left. + \left(\frac{\partial^2 A_1}{\partial x \partial y} + \frac{\partial^2 A_2}{\partial y^2} + \frac{\partial^2 A_3}{\partial z \partial y} \right) j + \left(\frac{\partial^2 A_1}{\partial x \partial z} + \frac{\partial^2 A_2}{\partial y \partial z} + \frac{\partial^2 A_3}{\partial z^2} \right) k \right] \end{aligned}$$

(adding and subtracting $\frac{\partial^2 A_1}{\partial x^2} i$, $\frac{\partial^2 A_2}{\partial y^2} j$ and $\frac{\partial^2 A_3}{\partial z^2} k$).

$$\begin{aligned}
 &= -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(A_1 i + A_2 j + A_3 k) \\
 &\quad + i \frac{\partial}{\partial x} \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + j \frac{\partial}{\partial y} \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + k \frac{\partial}{\partial z} \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) \\
 &= -\nabla^2 A + \nabla \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) \\
 &= -\nabla^2 A + \nabla(\nabla \cdot A)
 \end{aligned}$$

- (4) $\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$, (A, B are vector functions) (or) $\text{Div}(A \times B) = B \cdot \text{curl } A - A \cdot \text{curl } B$.

Proof: Let $A = A_1 i + A_2 j + A_3 k$ and $B = B_1 i + B_2 j + B_3 k$

$$A \times B = \begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= \sum i(A_2 B_3 - A_3 B_2)$$

$$\begin{aligned}
 \text{Div}(A \times B) &= \sum \frac{\partial}{\partial x} (A_2 B_3 - A_3 B_2) \\
 &= \sum \left(A_2 \frac{\partial B_3}{\partial x} + B_3 \frac{\partial A_2}{\partial x} + A_3 \frac{\partial B_2}{\partial x} + B_2 \frac{\partial A_3}{\partial x} \right) \quad \dots(1)
 \end{aligned}$$

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix} = \sum \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) i$$

$$B \cdot (\nabla \times A) = \sum B_1 \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \quad \dots (2)$$

Similarly,

$$A. (\nabla \times B) = \sum A_i \left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z} \right) \dots (3)$$

Expanding the summations of (1), (2) & (3), we observe that,

$$\text{Div} (A \times B) = B. (\nabla \times A) - A. (\nabla \times B)$$

7.5.3 Operation of ∇ on product of two functions

Suppose ϕ and Ψ , are two scalar or vector point functions. When ∇ is operated on the product of ϕ and ψ , the following rule is useful.

$\nabla (\phi\Psi) = \nabla (\phi_0\Psi) + \nabla (\phi\Psi_0)$ wherein the suffix '0' indicates that the function is not to be varied, that is, ∇ is not to be operated on that function. After the completion of operation of ∇ the suffixes are dropped.

While proving the identities the following are useful.

- (a) $a.b = b.a$
- (b) $a \times b = -b \times a$
- (c) $a \times a = 0$
- (d) $a. (b \times c) = (a \times b).c$
- (e) $a \times (b \times c) = (a.c)b - (a.b)c$
- (f) $(a.c)b = a \times (b \times c) + (a.b)c$
- (g) $a.a = a^2$
- (h) $\times$ operation is always between vectors only

vector identities using ∇ operation:

$$(i) \nabla.(FG) = \nabla.(FG_0) + \nabla.(F_0G)$$

$$= \nabla F.G_0 + F_0 \nabla.G$$

$$\therefore \text{div} (fG) = G. \text{grad} F + F \text{div} G$$

$$(ii) \nabla \times (FG) = \nabla \times (FG_0) + \nabla \times (F_0G)$$

$$\nabla \times (FG_0) = \nabla F \times G_0$$

due to (h)

$$\nabla \times (F_0G) = (\nabla \times G) F_0 = F_0(\nabla \times G)$$

$$\therefore \nabla \times (FG) = \nabla F \times G + F(\nabla \times G)$$

$$(i.e.) \text{curl} (FG) = (\text{grad} F) \times G + F \text{curl} G$$

$$(iii) \nabla \cdot (F \times G) = \nabla \cdot (F \times G_0) + \nabla \cdot (F_0 \times G)$$

$$\nabla \cdot (F \times G_0) = (\nabla \times F) \cdot G_0 \quad \text{due to (d)}$$

$$= G_0 \cdot (\nabla \times F) \quad \text{due to (a)}$$

$$\nabla \cdot (F_0 \times G) = -\nabla \cdot (G \times F_0) \quad \text{due to (b)}$$

$$= -\nabla \times (G \cdot F_0) \quad \text{due to (d)}$$

$$= -(\nabla \times G) \cdot F_0$$

$$= -F_0 \cdot (\nabla \times G) \quad \text{due to (a)}$$

$$\therefore \nabla \cdot (F \times G) = G \cdot (\nabla \times F) - F \cdot (\nabla \times G)$$

$$\text{Thus div } (F \times G) = G \cdot \text{curl } F - F \cdot \text{curl } G$$

$$(iv) \nabla \times (F \times G) = \nabla \times (F \times G_0) + \nabla \times (F_0 \times G)$$

$$\nabla \times (F \times G_0) = (\nabla \cdot G_0) F - (\nabla \cdot F) G_0 \quad \text{due to (e)}$$

$$= (G_0 \cdot \nabla) F - G_0 (\nabla \cdot F) \quad \text{due to (a)}$$

$$\nabla \times (F_0 \times G) = (\nabla \cdot G) F_0 - (\nabla \cdot F_0) G \quad \text{due to (e)}$$

$$= F_0 (\nabla \cdot G) - (F_0 \cdot \nabla) G \quad \text{due to (a)}$$

$$\therefore \nabla \times (F \times G) = (G \cdot \nabla) F - G(\nabla \cdot F) + F(\nabla \cdot G) - (F \cdot \nabla) G$$

$$\text{Thus curl } (F \times G) = (G \cdot \nabla) F - (F \cdot \nabla) G + F(\nabla \cdot G) - G(\nabla \cdot F)$$

$$(v) \nabla (F \cdot G) = \nabla (F_0 \cdot G) + \nabla (F \cdot G_0)$$

$$\nabla (F_0 \cdot G) = F_0 \times (\nabla \times G) + (F_0 \cdot \nabla) G \quad \text{due to (f)}$$

$$\nabla (F \cdot G_0) = \nabla (G_0 \cdot F) = G_0 \times (\nabla \times F) + (G_0 \cdot \nabla) F \quad \text{due to (f)}$$

$$\therefore \nabla (F \cdot G) = F \times (\nabla \times G) + G \times (\nabla \times F) + (F \cdot \nabla) G + (G \cdot \nabla) F$$

$$\text{Thus grad } (F \cdot G) = F \times \text{curl } G + G \times \text{curl } F + (F \cdot \nabla) G + (G \cdot \nabla) F$$

$$(vi) \text{curl } (\text{grad } F) = \nabla \times (\nabla F) = (\nabla \times \nabla) F = 0 \quad \text{due to (c)}$$

$$(vii) \text{div } (\text{curl } F) = \nabla \cdot (\nabla \times F) = \nabla \times (\nabla \cdot F) \quad \text{due to (d)}$$

$$= (\nabla \times \nabla) \cdot F = 0 \quad \text{due to (c)}$$

$$(viii) \text{curl } (\text{curl } F) = \nabla \times (\nabla \times F)$$

$$= (\nabla \cdot F) \nabla - (\nabla \cdot \nabla) F \quad \text{due to (e)}$$

$$= \nabla (\nabla \cdot F) - \nabla^2 F \quad (\text{since } \nabla \cdot \text{ must not appear at the end})$$

$$= \text{grad } (\text{div } F) - \nabla^2 F$$

Solved examples

Ex. 7.5.4 If $f = x^2y^3z^2$, find $\nabla^2 f$ at $(1, 2, 1)$

Sol:
$$\begin{aligned}\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\ &= 2y^3z^2 + 6x^2yz^2 + 2x^2y^3 \\ \therefore \text{ at } (1, 2, 1), \nabla^2 f &= 16 + 12 + 16 = 44.\end{aligned}$$

Ex. 7.5.5 Show that, if $r = xi + yj + zk$, $r = |r|$, then $\nabla^2 r^n = n(n+1)r^{n-2}$

Sol: $r^n = (x^2 + y^2 + z^2)^{n/2}$, ($\because r = \sqrt{x^2 + y^2 + z^2}$)

$$\begin{aligned}\frac{\partial}{\partial x}(r^n) &= \frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n-2}{2}-1} \cdot 2x = nx(x^2 + y^2 + z^2)^{\frac{n-2}{2}} \\ \frac{\partial^2}{\partial x^2}(r^n) &= n \left[x \cdot \frac{n-2}{2}(x^2 + y^2 + z^2)^{\frac{n-2}{2}-1} \cdot 2x + (x^2 + y^2 + z^2)^{\frac{n-2}{2}} \right] \\ &= n \left[(n-2)x^2(x^2 + y^2 + z^2)^{\frac{n-4}{2}} + (x^2 + y^2 + z^2)^{\frac{n-2}{2}} \right] \\ &= n(n-2)x^2r^{n-4} + nr^{n-2} \quad \dots (1)\end{aligned}$$

Similarly, $\frac{\partial^2}{\partial y^2}(r^n) = n(n-2)y^2r^{n-4} + nr^{n-2} \quad \dots (2)$

and $\frac{\partial^2}{\partial z^2}(r^n) = n(n-2)z^2r^{n-4} + nr^{n-2} \quad \dots (3)$

$$\begin{aligned}\nabla^2(r^n) &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) r^n \\ &= 3nr^{n-2} + n(n-2)r^{n-4}(x^2 + y^2 + z^2) \quad \text{adding (1), (2) \& (3)} \\ &= 3nr^{n-2} + n(n-2)r^{n-2} \quad (\because x^2 + y^2 + z^2 = r^2) \\ &= n(n+1)r^{n-2}\end{aligned}$$

Aliter:
$$\frac{\partial}{\partial x}(r^n) = nr^{n-1} \cdot \frac{\partial r}{\partial x} = nr^{n-1} \cdot \frac{x}{r}$$

$$= nr^{n-2} \cdot x \quad \left(\because \frac{\partial r}{\partial x} = \frac{x}{r} \right)$$

$$\frac{\partial^2}{\partial x^2}(r^n) = n \left(r^{n-2} \cdot 1 + x(n-2)r^{n-3} \cdot \frac{\partial r}{\partial x} \right)$$

$$= n \left(r^{n-2} + (n-2)r^{n-3} \cdot \frac{x^2}{r} \right)$$

$$= nr^{n-2} + n(n-2)r^{n-4} \cdot x^2 \quad \dots (1)$$

Similarly,
$$\frac{\partial^2}{\partial y^2}(r^n) = nr^{n-2} + n(n-2)r^{n-4} \cdot y^2 \quad \dots (2)$$

and
$$\frac{\partial^2}{\partial z^2}(r^n) = nr^{n-2} + n(n-2)r^{n-4} \cdot z^2 \quad \dots (3)$$

Adding (1), (2) & (3) we get,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) r^n = 3nr^{n-2} + n(n-2)r^{n-4}(x^2 + y^2 + z^2)$$

$$\Rightarrow \nabla^2 r^n = 3nr^{n-2} + n(n-2)r^{n-4} \cdot r^2$$

$$= r^{n-2}[3n + n^2 - 2n]$$

$$= n(n+1)r^{n-2}$$

Note: If $n = -1$, we have $\nabla^2 \left(\frac{1}{r} \right) = 0$, which means that $\left(\frac{1}{r} \right)$ satisfies the Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \text{ or } \nabla^2 \phi = 0$$

Ex.7.5.6 If $\nabla \cdot U = 0$, $\nabla \cdot V = 0$, $\nabla \times U = -\frac{\partial V}{\partial t}$, $\nabla \times V = \frac{\partial U}{\partial t}$, show that U and V satisfy

$$\text{the wave equation } \nabla^2 U = \frac{\partial^2 U}{\partial t^2}$$

Sol:
$$\nabla \times (\nabla \times U) = \nabla \times \left(-\frac{\partial V}{\partial t} \right) = -\frac{\partial}{\partial t} (\nabla \times V)$$

$$= -\frac{\partial}{\partial t} \left(\frac{\partial U}{\partial t} \right) = -\frac{\partial^2 U}{\partial t^2} \quad \dots (1)$$

$$\begin{aligned}\text{But } \nabla \times (\nabla \times U) &= \nabla (\nabla \cdot U) - \nabla^2 U \\ &= -\nabla^2 U \quad (\because \nabla \cdot U = 0)\end{aligned}\quad \dots(2)$$

from (1) & (2), $\frac{\partial^2 U}{\partial t^2} = \nabla^2 U$, which shows that U satisfies the wave equation.

Similarly,

$$\nabla \times (\nabla \times V) = \nabla \times \frac{\partial U}{\partial t} = \frac{\partial}{\partial t} (\nabla \times U) = \frac{-\partial^2 V}{\partial t^2} \quad \dots(3)$$

$$\text{and } \nabla \times (\nabla \times V) = \nabla (\nabla \cdot V) - \nabla^2 V = -\nabla^2 V \quad (\because \nabla \cdot V = 0) \quad \dots(4)$$

(3) and (4) $\Rightarrow V$ satisfies the wave equation.

Ex. 7.5.7 If $\nabla \cdot V = 0$, show that $\nabla \times [\nabla \times \{\nabla \times (\nabla \times V)\}] = \nabla^4 V$ (or)

$$\text{curl} [\text{curl} \{\text{curl} (\text{curl } V)\}] = \nabla^4 V$$

Sol: We know that

$$\begin{aligned}\nabla \times (\nabla \times V) &= \nabla (\nabla \cdot V) - \nabla^2 V && [\text{from 7.5.2 (3)}] \\ &= -\nabla^2 V \quad (\because \nabla \cdot V = 0) \\ &= -U \quad (\text{say})\end{aligned}$$

Then the given expression

$$\begin{aligned}&= -\nabla \times (\nabla \times U) = \nabla^2 U - \nabla (\nabla \cdot U) && [\text{from 7.5.2 (3)}] \\ &= \nabla^2 (\nabla^2 V) - \nabla (\nabla \cdot U) && (\because \nabla^2 V = U) \\ &= \nabla^4 V - \nabla (\nabla \cdot U)\end{aligned}$$

$$\begin{aligned}\nabla \cdot U &= \nabla \cdot (\nabla^2 V) \\ &= \left\{ \sum i \frac{\partial}{\partial x} \right\} \cdot \{\nabla^2 V\} \\ &= \sum \left\{ i \frac{\partial}{\partial x} \right\} \cdot (\nabla^2 (V_1 i + V_2 j + V_3 k)) \\ &= \nabla^2 \left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \right) \\ &= \nabla^2 (\text{Div } V) = \nabla^2 (0) = 0 && (\because \text{Div } V = \nabla \cdot V = 0)\end{aligned}$$

Hence the problem

Exercise 7(d)

(1) Show that $\nabla^2(\log r) = \frac{1}{r^2}$

(2) Prove that $\nabla^2(fg) = f(\nabla^2 g) + 2(\nabla f) \cdot (\nabla g) + g(\nabla^2 f)$

(3) Show that $\nabla^2\left(\nabla \cdot \frac{\mathbf{r}}{r^2}\right) = \frac{2}{r^4}$

(4) Show that $\nabla^2\phi(r) = \frac{d^2\phi}{dr^2} + \frac{2}{r} \frac{d\phi}{dr}$

show also that, if $\nabla^2\phi = 0$, then $\phi = C_1 + \frac{C_2}{r}$ where C_1, C_2 are constants

(5) If $\mathbf{r} = xi + yj + zk$, prove that, $\text{curl}(\mathbf{k} \times \text{grad} \frac{1}{r}) + \text{grad}(\mathbf{k} \cdot \text{grad} \frac{1}{r}) = 0$

7.6 VECTOR INTEGRATION**7.6.1 Ordinary integration of vectors**

- (1) If $\mathbf{A}(u) = A_1(u)\mathbf{i} + A_2(u)\mathbf{j} + A_3(u)\mathbf{k}$ be a vector function of a scalar variable 'u' ($A_1(u), A_2(u), A_3(u)$ assumed to be continuous in any given interval), the indefinite integral of $\mathbf{A}(u)$ is given by.

$$\int \mathbf{A}(u) du = \mathbf{i} \int A_1(u) du + \mathbf{j} \int A_2(u) du + \mathbf{k} \int A_3(u) du$$

- (2) If there exists a vector $\mathbf{B}(u)$ such that $\mathbf{A}(u) = \frac{d}{du}(\mathbf{B}(u))$, we can write

$$\int \mathbf{A}(u) du = \int \frac{d}{du}(\mathbf{B}(u)) du = \mathbf{B}(u) + c$$

where c is a constant of integration independent of u .

- (3) The definite integral of $\mathbf{A}(u)$ between the limits 'a' and 'b' in the above, is written as,

$$\int_a^b \mathbf{A}(u) du = \int_a^b \frac{d}{du}(\mathbf{B}(u)) du = \mathbf{B}(u) + c \Big|_a^b = \mathbf{B}(b) - \mathbf{B}(a)$$

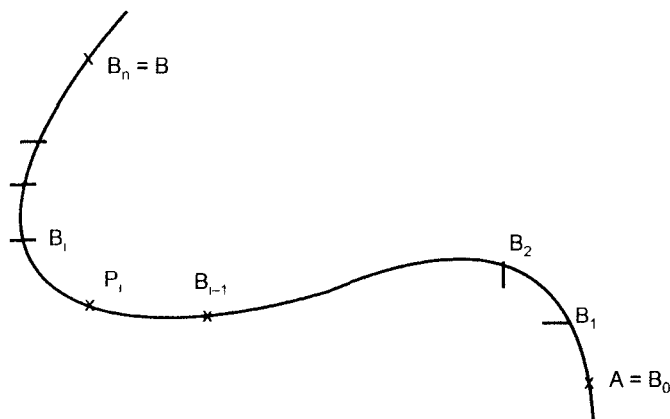
7.6.2 Line Integrals:

Let $\mathbf{A}(x, y, z)$ be a continuous vector function defined in the entire region of space.

Let c be any curve in the region. Divide c into n intervals by taking points

$\mathbf{A} = \mathbf{B}_0, \mathbf{B}_1, \mathbf{B}_2 \dots \mathbf{B}_n (= \mathbf{B})$.

Let P_i be any point in the interval $\mathbf{B}_{i-1} \mathbf{B}_i$



Let $r_0, r_1, \dots, r_n$ be the position vectors of points $B_0, B_1, B_2, \dots, B_n$ respectively. Let us consider the sum,

$$\sum A(P_i) \cdot \delta r_i$$

The limit of this sum as $n \rightarrow \infty$ and $|\delta r_i| \rightarrow 0$ is defined as the line integral of A along the curve c and is denoted symbolically by

$$\int_c A \cdot dr \quad \text{or} \quad \int_c A \cdot \frac{dr}{dt} dt; \quad \text{which is a scalar.}$$

If c is a closed curve, the integral is written as $\oint A \cdot dr$

Cartesian form of line integral:

If $A = A_1 i + A_2 j + A_3 k$

$$dr = (dx)i + (dy)j + (dz)k,$$

$$\int_c A \cdot dr = \int_c A_1 dx + A_2 dy + A_3 dz$$

Note: $\int_c \phi dr$, and $\int_c A \times dr$ are also examples of line integrals.

7.6.3 Physical applications:

(1) Work done by a force

(1) If A represents a force and dr is an element of the path of the particle along a curve c , then the line integral

$$\int_P^Q A \cdot dr \quad (P, Q \text{ are 2 points on } c)$$

represents the work done by force A in moving the particle from P to Q .

(2) flow or circulation:

(2) If A is the electric field strength, the line integral given above i.e., $\int_c A \cdot dr$, is called the flow of A along c . If c is a closed curve it is often referred to as circulation of A around c .

In general, the line integral $\int_P^Q A \cdot dr$, will depend on the path from P to Q

7.6.4 Theorem: Prove that the necessary and sufficient condition for the integral $\int_c A \cdot dr$, to be independent of the path c joining any two points is that A is a conservative vector field, (or) there exists a scalar field ϕ such that $A = \nabla\phi$ (or) $\text{curl } A = 0$, [i.e., the work done by the force A in moving the particle from one point to another is independent of the path if $A = \nabla\phi$]

Proof: Let $P = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$ be any two given points on the curve c . Let $A = \nabla\phi$ where ϕ is single-valued and has continuous derivatives.

$$\begin{aligned}
 (1) \quad \text{Work done} &= \int_P^Q A \cdot dr \\
 &= \int_P^Q \nabla\phi \cdot dr = \int_P^Q \left(i \frac{\partial\phi}{\partial x} + j \frac{\partial\phi}{\partial y} + k \frac{\partial\phi}{\partial z} \right) (dx i + dy j + dz k) \\
 &= \int_P^Q \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz = \int_P^Q d\phi = \phi(Q) - \phi(P) \\
 &= \phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1)
 \end{aligned}$$

i.e., the integral depends upon the two points only but not on the path joining them. (This is true only if ϕ is single-valued at all points P and Q).

(2) The integral is independent of the path. Then,

$$\phi(x, y, z) = \int_{(x_1, y_1, z_1)}^{(x, y, z)} A \cdot dr = \int_{(x_1, y_1, z_1)}^{(x, y, z)} A \cdot \frac{dr}{ds} ds$$

Differentiation with respect to 's' gives

$$\frac{d\phi}{ds} = A \cdot \frac{dr}{ds} \quad \dots\dots (1)$$

But $\frac{d\phi}{ds} = \nabla\phi \frac{dr}{ds}$ (2)

(1) – (2) $\Rightarrow (A - \nabla\phi) \cdot \frac{dr}{ds} = 0$, which is true irrespective of $\frac{dr}{ds}$

$\therefore A = \nabla\phi$

Solved Examples

Ex. 7.6.5: $F(t) = (3t^2 - t)i + (2 - 6t)j - 4tk$, find (a) $\int F(t)dt$ (b) $\int_2^4 F(t)dt$

Sol: (a) $\int F(t)dt = i \int (3t^2 - t)dt + j \int (2 - 6t)dt - k \int 4t dt$

$$= \left(t^3 - \frac{t^2}{2} \right) i + (2t - 3t^2)j - 2t^2 k + c$$

(b) $\int_2^4 F(t)dt = \left(t^3 - \frac{t^2}{2} \right) i + (2t - 3t^2)j - 2t^2 k + c \Big|_2^4 = 50i - 32j - 24k$

Ex. 7.6.6 Evaluate: $\int_0^{\pi/2} [(3\sin\theta)i + (2\cos\theta)j] d\theta$

Sol: Given integral

$$= i \int_0^{\pi/2} (3\sin\theta) d\theta + j \int_0^{\pi/2} (2\cos\theta) d\theta = -3\cos\theta \Big|_0^{\pi/2} + 2\sin\theta \Big|_0^{\pi/2} = 3i + 2j$$

Ex. 7.6.7 The acceleration a of a particle at any time ' t ' ≥ 0 is given by,

$$a = e^{-t}i - 6(t+1)j + (3\sin t)k$$

Find the velocity V and displacement r at any time ' t ' given that $V = 0$ when $t = 0$ and $r = 0$ when $t = 0$.

Sol: $a = e^{-t}i - 6(t+1)j + (3\sin t)k = \frac{d^2r}{dt^2}$

$$\therefore V = \frac{dr}{dt} = \int a dt = -e^{-t}i - 6\left(\frac{t^2}{2} + t\right)j - (3\cos t)k + C_1 \quad \text{.....(1)}$$

(C_1 is constant of integration)

But $V = 0$ when $t = 0$

$$\therefore -i - 3k + C_1 = 0 \Rightarrow C_1 = i + 3k$$

Substituting in (1), we get

$$\text{Velocity } V = (1 - e^{-t})i - (3t^2 + 6t)j + (3 - 3\cos t)k$$

Integrating,

$$r = (1 + e^{-t})i - (t^3 + 3t^2)j + (3t - 3\sin t)k + C_2 \quad \dots (2)$$

(C_2 being constant of integration)

But $r = 0$ when $t = 0$.

$$\therefore (2) \Rightarrow +i + C_2 = 0 \therefore C_2 = -i$$

$$\text{From (2), } r = (t - 1 + e^{-t})i - (t^3 + 3t^2)j + (3t - 3\sin t)k$$

Ex. 7.6.8 Show that

$$\int F \times \frac{d^2 F}{dt^2} = F \times \frac{dF}{dt} + c$$

Sol: We know that

$$\begin{aligned} \frac{d}{dt} \left(F \times \frac{dF}{dt} \right) &= F \times \frac{d}{dt} \left(\frac{dF}{dt} \right) + \frac{dF}{dt} \times \frac{dF}{dt} \\ &= F \times \frac{d^2 F}{dt^2} \quad \left(\because \frac{dF}{dt} \times \frac{dF}{dt} = 0 \right) \end{aligned}$$

Hence the result.

Ex. 7.6.9 If $A = 2ti + 3t^2j - (4t + 1)k$, and $B = ti + 2j + t^2k$, find

$$(i) \int_0^2 (A \cdot B) dt \quad (ii) \int_0^2 (A \times B) dt$$

Sol: (i) $A \cdot B = (2t)t + (3t^2)(2) - t^2(4t + 1) = 2t^2 + 6t^2 - 4t^3 - t^2 = 7t^2 - 4t^3$

$$\int_0^2 (A \cdot B) dt = \left[\frac{7t^3}{3} - t^4 \right]_0^2 = \frac{8}{3}$$

$$(ii) A \times B = \begin{vmatrix} i & j & k \\ 2t & 3t^2 & -(4t+1) \\ t & 2 & t^2 \end{vmatrix}$$

$$= (3t^4 + 8t + 2)i + (-4t^2 - t - 2t^3)j + (4t - 3t^3)k$$

$$\int_0^2 (A \times B) dt = \left[\left(\frac{3t^5}{5} + 4t^2 + 2t \right) i + \left(-\frac{4t^3}{3} - \frac{t^2}{2} - \frac{2t^4}{4} \right) j + \left(2t^2 - \frac{3t^4}{4} \right) k \right]_0^2$$

$$= \frac{196}{5}i - \frac{62}{3}j - 4k$$

Ex.7.6.10 If $F(2) = i + 2j - 2k$ and $F(3) = 6i - 2j + 3k$,

evaluate $\int_2^3 F \cdot \frac{dF}{du} du$

Sol: From vector differentiation,

We get, $\frac{d}{du}(F \cdot F) = F \cdot \frac{dF}{du} + \frac{dF}{du} \cdot F = 2 \left(F \cdot \frac{dF}{du} \right)$, so that

$$F \cdot \frac{dF}{du} = \frac{1}{2} \frac{d}{du}(F \cdot F) = \frac{1}{2} \frac{d}{du} |F|^2$$

$$\therefore \int_2^3 F \cdot \frac{dF}{du} du = \frac{1}{2} \int_2^3 \left\{ \frac{d}{du} |F|^2 \right\} du = \left[\frac{1}{2} |F|^2 \right]_2^3$$

But $F(2) = i + 2j - 2k \Rightarrow |F|^2 = 1 + 4 + 4 = 9$, when $u = 2$

and $F(3) = 6i + 2j - 3k \Rightarrow |F|^2 = 36 + 4 + 9 = 49$, when $u = 3$

$$\therefore \int_2^3 F \cdot \frac{dF}{du} du = \frac{1}{2} [49 - 9] = 20$$

Ex. 7.6.11 If $F = (x^2 - 2y)i - 6yzj + 8xz^2k$, evaluate $\int_c F \cdot dr$ from the point $(0,0,0)$ to the

point $(1,1,1)$ along the following paths

(1) $x = t, y = t^2, z = t^3$.

(2) the straight line from $(0,0,0)$ to $(1,0,0)$, then to $(1,1,0)$ and then to $(1,1,1)$ and

(3) the straight line joining $(0,0,0)$ to $(1,1,1)$.

Sol.

(1) $x = t; \quad y = t^2; \quad z = t^3;$

$dx = dt; \quad dy = 2t dt; \quad dz = 3t^2 dt;$

when $t = 0$, the point is $(0,0,0)$, when $t = 1$, the point is $(1,1,1)$

$$\int_c F \cdot dr = \int_c (x^2 - 2y)dx - 6yz dy + 8xz^2 dz$$

$$= \int_{t=0}^{t=1} (t^2 - 2t^2)dt - 6.t^2.t^3(2t)dt + 8t(t^3)^2(3t^2)dt = \int_0^1 -t^2 dt - 12t^6 dt + 24t^9 dt$$

$$\begin{aligned}
 &= \int_0^1 (24t^9 - 12t^6 - t^2) dt = \frac{24t^{10}}{10} - \frac{12t^7}{7} - \frac{t^3}{3} \Big|_0^1 \\
 &= \frac{12}{5} - \frac{12}{7} - \frac{1}{3} = \frac{252 - 180 - 35}{105} = \frac{37}{105}
 \end{aligned}$$

Aliter: Along C, $F = (t^2 - 2t^2)i - 6.t^2 t^3 j + 8.t.t^6 k = -t^2 i - 6t^5 j + 8t^7 k$
 $dr = (dx)i + (dy)j + (dz)k = dt i + (2t dt)j + (3t^2 dt)k$

$$\begin{aligned}
 \int_C F \cdot dr &= \int_{t=0}^1 (-t^2 - 12t^6 + 24t^9) dt \quad (\text{Taking dot product of } F \text{ and } dr) \\
 &= \frac{37}{105}
 \end{aligned}$$

(2) Let $O = (0,0,0)$, $P = (1,0,0)$, $Q = (1,1,0)$, $R = (1,1,1)$.

Then along OP, $y = 0$, $z = 0$, $dy = 0$, $dz = 0$ and x varies from 0 to 1.

$$\therefore \int_{OP} F \cdot dr = \int_{x=0}^1 (x^2 - 2.0) dx - 6(0)(0)(0) + 8x(0)^2(0) = \int_0^1 x^2 dx = \frac{1}{3} \quad \dots (1)$$

Along PQ, $x = 1$, $z = 0$, $dx = 0$, $dz = 0$ and y varies from 0 to 1.

$$\therefore \int_{PQ} F \cdot dr = \int_{y=0}^1 (1^2 - 2y) dy - 6y(0)(0) + 8.1.(0)^2(0) = \int_0^1 0 dy = 0 \quad \dots (2)$$

Along QR, $x = 1$, $y = 1$, $dx = 0$, $dy = 0$ and z varies from 0 to 1.

$$\therefore \int_{QR} F \cdot dr = \int_{z=0}^1 (1^2 - 2.1)(0) - 6.1.z(0) + 8.1.z^2 dz = \int_0^1 8z^2 dz = \frac{8}{3} \quad \dots (3)$$

$$\text{Adding (1), (2), (3), } \int F \cdot dr = \frac{1}{3} + 0 + \frac{8}{3} = 3$$

(3) The equation of the straight line joining $(0,0,0)$ to $(1,1,1)$ is $\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t$ (say),
 so that $x = t$, $y = t$, $z = t$, $dx = dy = dz = dt$ 't' takes values from 0 to 1.

$$\begin{aligned}
 \therefore \int_C F \cdot dr &= \int_0^1 [(t^2 2t) - 6.1.t + 8t.t^2] dt = \int_0^1 (8t^3 - 5t^2 - 2t) dt \\
 &= \left[2t^4 - \frac{5t^3}{3} - t^2 \right]_0^1 = 2 - \frac{5}{3} - 1 = \frac{-2}{3}
 \end{aligned}$$

Ex.7.6.12 Find the total work done by a force $F = 2xyi - 4zj + 5xk$ along the curve $x = t^2$, $y = 2t + 1$, $z = t^3$, from the points $t = 1$ to $t = 2$.

Sol: $x = t^2$, $dx = 2t dt$; $y = 2t + 1$, $dy = 2dt$; $z = t^3$, $dz = 3t^2 dt$

$$\text{Total work done} = \int_c F \cdot dr$$

$$= \int_c (2xyi - 4zj + 5xk) \cdot \{(dx)i + (dy)j + (dz)k\} = \int_c 2xy dx - 4z dy + 5x dz$$

$$= \int_{t=1}^2 [2t^2(2t+1)]2t dt - (4t^3)2dt + (5t^2)3t^2 dt = \int_1^2 (8t^4 + 4t^3 - 8t^3 + 15t^4) dt$$

$$= \int_1^2 (23t^4 - 4t^3) dt = \left[23 \frac{t^5}{5} - t^4 \right]_1^2 = \frac{23}{5} (32 - 1) - (16 - 1) = \frac{713}{5} - 15 = \frac{638}{5}$$

Ex.7.6.13 If c is the curve $y = 3x^2$ in the xy -plane and $F = (x + 2y)i - xyj$,

evaluate $\int F \cdot dr$, from the point $(0,0)$ to $(1,3)$.

Sol: Since c is a curve in xy -plane, we take $r = xi + yj$, so that

$$F \cdot dr = \{(x + 2y)i - xyj\} \cdot \{(dx)i + (dy)j\} = (x + 2y) dx - xy dy.$$

1st Method: By taking the curve in parametric coordinates as,

$x = t$, $y = 3t^2$, $dx = dt$, $dy = 6t dt$, so that t varies from 0 to 1 to get the points $(0,0)$ & $(1,3)$: We have

$$\int_{(0,0)}^{(1,3)} F \cdot dr = \int_{t=0}^1 (t + 6t^2) dt - t(3t^2)(6t dt) = \int_0^1 (t + 6t^2 - 18t^4) dt$$

$$= \left[\frac{t^2}{2} + \frac{6t^3}{3} - \frac{18t^5}{5} \right]_0^1 = \frac{1}{2} + 2 - \frac{18}{5} = \frac{5}{2} - \frac{18}{5} = -\frac{11}{10}$$

2nd Method: $y = 3x^2$, $dy = 6x dx$ and x varies from 0 to 1.

$$\therefore \int_c F \cdot dr = \int_0^1 (x + 6x^2) dx - x \cdot 3x^2 \cdot 6x dx$$

$$= \int_0^1 (x + 6x^2 - 18x^4) dx = -\frac{11}{10}$$

Ex.7.6.14 Find the work done by the force F in moving a particle once around the circle 'C' in xy -plane, if the centre of the circle is origin and radius is '2' and

$$F = (x + y + z)i + (2x + y)j + (2x - y + z)k.$$

Sol: xy -plane is $z = 0$,

$$\therefore F = (x + y)i + (2x + y)j + (2x - y)k \text{ and } r = xi + yj \Rightarrow dr = (dr)i + (dy)j$$

$$F \cdot dr = (x + y)dx + (2x + y)dy.$$

The equation of the circle is $x = 2\cos\theta$, $y = 2\sin\theta$

$$\therefore dx = -2\sin\theta d\theta, dy = 2\cos\theta d\theta$$

θ varies from 0 to 2π

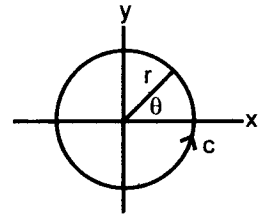
$$\text{Work done} = \int F \cdot dr$$

$$= \int_0^{2\pi} (2\cos\theta + 2\sin\theta)(-2\sin\theta)d\theta + (2 \cdot 2\cos\theta + 2\sin\theta)(2\cos\theta)d\theta$$

$$= \int_0^{2\pi} (-2\sin 2\theta - 4\sin^2\theta + 8\cos^2\theta + 2\sin 2\theta)d\theta = \int_0^{2\pi} (8\cos^2\theta - 4\sin^2\theta)d\theta$$

$$= \int_0^{2\pi} [4(1 + \cos 2\theta) - 2(1 - \cos 2\theta)]d\theta = \int_0^{2\pi} (2 + \cos 2\theta)d\theta = [2\theta + \frac{1}{2}\sin 2\theta]_0^{2\pi}$$

$$= 4\pi$$



Ex.7.6.15 Show that the necessary and sufficient condition for a vector field V to be conservative is $\text{curl } V = 0$

Sol: a) Necessary condition: If V is conservative, $\exists$ a ' ϕ ' $\ni V = \nabla \phi$.

$$\text{curl } V = \text{curl } (\nabla \phi) = 0 \quad (\text{see 7.5.2 (1)})$$

b) Sufficient condition: Let $V = V_1i + V_2j + V_3k$

$$\text{Curl } V = \nabla \times V = 0 \Rightarrow \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_1 & V_2 & V_3 \end{vmatrix} = 0$$

$$\Rightarrow \sum \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right) i = 0$$

$$\Rightarrow \frac{\partial V_3}{\partial y} = \frac{\partial V_2}{\partial z}; \quad \frac{\partial V_1}{\partial z} = \frac{\partial V_3}{\partial x}; \quad \frac{\partial V_2}{\partial x} = \frac{\partial V_1}{\partial y} \quad \dots (1)$$

The work done by the force field V in moving a particle from (x_1, y_1, z_1) to (x, y, z)

$$\text{is } \int_c V \cdot dr$$

$$= \int_c V_1(x, y, z)dx + V_2(x, y, z)dy + V_3(x, y, z)dz$$

where V is a path joining (x_1, y_1, z_1) to (x, y, z)

Let us choose a particular path consisting straight line segments from (x_1, y_1, z_1) to (x, y_1, z_1) to (x, y, z_1) to (x, y, z) and denote the work done along this path by a scalar function $\phi(x, y, z)$;

$$\therefore \phi(x, y, z) = \int_{x_1}^x V_1(x, y_1, z_1)dx + \int_{y_1}^y V_2(x, y, z_1)dy + \int_{z_1}^z V_3(x, y, z)dz \quad \dots (2)$$

From (2), it can be seen that,

$$\frac{\partial \phi}{\partial z} = V_3(x, y, z) \quad \dots (3)$$

$$\frac{\partial \phi}{\partial y} = V_2(x, y, z_1) + \int_{z_1}^z \frac{\partial V_3}{\partial y}(x, y, z)dz = V_2(x, y, z_1) + \int_{z_1}^z \frac{\partial V_2}{\partial z}(x, y, z)dz \quad [\text{from (1)}]$$

$$\begin{aligned} &= V_2(x, y, z_1) + V_2(x, y, z) \Big|_{z_1}^z = V_2(x, y, z_1) - V_2(x, y, z) - V_2(x, y, z_1) \\ &= V_2(x, y, z) \end{aligned} \quad \dots (4)$$

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= V_1(x, y_1, z_1) + \int_{y_1}^y \frac{\partial V_2}{\partial x}(x, y, z_1)dy + \int_{z_1}^z \frac{\partial V_3}{\partial x}(x, y, z_1)dz \\ &= V_1(x, y_1, z_1) + \int_{y_1}^y \frac{\partial V_1}{\partial y}(x, y, z_1)dy + \int_{z_1}^z \frac{\partial V_1}{\partial z}(x, y, z)dz \quad \text{from} \quad (1) \end{aligned}$$

$$\begin{aligned} &= V_1(x, y_1, z_1) + V_1(x, y, z_1) \Big|_{y_1}^y + V_1(x, y, z) \Big|_{z_1}^z \\ &= V_1(x, y_1, z_1) + V_1(x, y, z_1) - V_1(x, y_1, z_1) + V_1(x, y, z) - V_1(x, y_1, z_1) \\ &= V_1(x, y, z) \end{aligned} \quad \dots (5)$$

$$(3), (4), (5) \Rightarrow \frac{\partial \phi}{\partial x}i + \frac{\partial \phi}{\partial y}j + \frac{\partial \phi}{\partial z}k = V_1(x, y, z)i + V_2(x, y, z)j + V_3(x, y, z)k = V$$

$$\Rightarrow V = \nabla \phi$$

Hence the proof.

Ex. 7.6.16a) Show that $F = y^2i + (2xy + z^2)j + 2yzk$ is a conservative force field.

b) Find its scalar potential.

c) Find the work done in moving an object in this field from (1,2,1) to (3,1,4)

Sol:

$$\begin{aligned} \text{a) } \nabla \times F &= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & 2xy + z^2 & 2yz \end{vmatrix} \\ &= i(2z - 2z) + j(0 - 0) + k(2y - 2y) = 0 \end{aligned}$$

$\therefore F$ is a conservative force field.

b) Let ' ϕ ' be the scalar potential of F .

1st method:

$$F = \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$\therefore \frac{\partial\phi}{\partial x} = y^2 \quad \dots (1) \quad \frac{\partial\phi}{\partial y} = 2xy + z^2 \quad \dots (2) \quad \frac{\partial\phi}{\partial z} = 2yz \quad \dots (3)$$

Integrating (1) w.r.t x , (2) w.r.t y , & (3) w.r.t z , respectively, we get,

$$\phi = xy^2 + f(y,z), \quad \phi = xy^2 + yz^2 + g(z,x), \quad \text{and} \quad \phi = yz^2 + h(x,y).$$

These equations will be consistent if f, g, h are taken as

$$f(y,z) = yz^2, \quad g(z,x) = 0, \quad h(x,y) = xy^2.$$

Hence $\phi = xy^2 + yz^2 + \text{constant}$

2nd Method: Since F is conservative, $\int_c F \cdot dr$ is independent of path joining

(x_1, y_1, z_1) to (x, y, z)

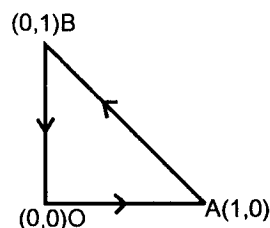
using method of problem 7.6.15(b),

$$\begin{aligned} \phi &= \int_{x_1}^x (y_1^2) dx + \int_{y_1}^y (2xy_1 + z_1^2) dy + \int_{z_1}^z 2yz dz = xy_1^2 \Big|_{x_1}^x + (xy_1^2 + z_1^2 y) \Big|_{y_1}^y + yz^2 \Big|_{z_1}^z \\ &= xy^2 - x_1 y_1^2 + xy_1^2 + z_1^2 y - x_1 y_1^2 - z_1^2 y_1 + yz^2 - yz_1^2 \\ &= xy^2 + yz^2 - x_1 y_1^2 - z_1^2 y_1 = xy^2 + yz^2 + \text{constant}. \end{aligned}$$

3rd Method: Since $F \cdot dr = \nabla\phi \cdot dr = \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz = d\phi$,

$$\begin{aligned} d\phi &= (y^2)dx + (2xy + z^2)dy + (2yz)dz = (y^2dx + 2xydy) + (z^2dy + 2yz)dz \\ &= d(xy^2) + d(yz^2) = d(xy^2 + yz^2) \\ \Rightarrow \phi &= xy^2 + yz^2 + \text{constant} \end{aligned}$$

$$\begin{aligned} \text{c) work done} &= \int_c F \cdot dr = \int_{P_1=(1,2,1)}^{P_2=(3,1,4)} d\phi \\ &= \phi(P_2) - \phi(P_1) = xy^2 + yz^2 \Big|_{(1,2,1)}^{(3,1,4)} \\ &= (3 \cdot 1^2 + 3 \cdot 4^2) - (1 \cdot 2^2 + 1 \cdot 1^2) = 51 - 5 = 46. \end{aligned}$$



Ex. 7.6.17 Evaluate $\int_c (x^3 dy + y^2 dx)$ where c is the boundary of the triangle whose vertices

are $(0,0)$, $(1,0)$, $(0,1)$.

Sol: Let $I = \int_c (x^3 dy + y^2 dx)$

$$I = I_1 + I_2 + I_3, \text{ where } I_1 = \int_{OA}, I_2 = \int_{AB}, I_3 = \int_{BO}$$

(i) Along OA, $y = 0 \Rightarrow dy = 0$

$$\therefore I_1 = \int x^3 \cdot 0 + 0^2 \cdot dx = 0$$

(ii) Along AB, $x + y = 1$, $y = 1 - x \Rightarrow dy = -dx$, x varies from 1 to 0.

$$\therefore I_2 = \int_1^0 x^3 (-dx) + (1-x)^2 dx$$

$$= \int_0^1 (x^2 - x^3 + 1 - 2x) dx = -\frac{x^3}{3} - \frac{x^4}{4} \Big|_1^0 = 0 - \left(-\frac{1}{3} - \frac{1}{4} \right) = -\frac{1}{12}$$

(iii) Along BO, $x = 0 \Rightarrow dx = 0 \therefore I_3 = \int 0 \cdot dy + y^2 \cdot 0 = 0$

$$\therefore I = I_1 + I_2 + I_3 = -\frac{1}{12}$$

Ex. 7.6.18: If $f = xy^2z^2$, evaluate $\int_c f dr$ where the curve ' c ' is given by

$$x = t, y = t^2, z = t^3 \text{ from } t = 0 \text{ to } 1.$$

Sol: $f = xy^2z^2 = t(t^2)^2 \cdot (t^3)^2 = t^{11}$

$$dr = dx \, i + dy \, j + dz \, k$$

$$dr = (dt) i + (2tdt) j + (3t^2 dt) k = (i + 2tj + 3t^2 k) dt$$

$$\begin{aligned} \int_c f dr &= \int_{t=0}^1 t^{11} (i + 2tj + 3t^2 k) dt = i \int_0^1 t^{11} dt + j \int_0^1 2t^{12} dt + k \int_0^1 3t^{13} dt \\ &= i \frac{t^{12}}{12} \Big|_0^1 + j \frac{2t^{13}}{13} \Big|_0^1 + k \frac{3t^{14}}{14} \Big|_0^1 = \frac{1}{12} i + \frac{2}{13} j + \frac{3}{14} k \end{aligned}$$

Ex.7.6,18: If $A = 3zi - 2xj + yk$, and c is the curve given by

$$x = \cos t, y = \sin t, z = 2\cos t,$$

evaluate $\int_c A \cdot dr$ from $t = 0$ to $t = \frac{\pi}{2}$.

Sol:
$$A \cdot dr = \begin{vmatrix} i & j & k \\ 3z & -2x & y \\ dx & dy & dz \end{vmatrix}$$

$$= (-2x \, dz - y \, dy) i + (y \, dx - 3z \, dz) j + (3z \, dy + 2x \, dx) k$$

$$x = \cos t, y = \sin t, z = 2\cos t$$

$$\Rightarrow dx = (-\sin t) dt, dy = (\cos t) dt, dz = (-2\sin t) dt$$

$$\therefore (1) \Rightarrow A \cdot dr = i[(-2\cos t)(-2\sin t) - \sin t \cdot \cos t] dt + j[(\sin t)(-2\sin t) - 3(2\cos t)(-2\sin t)] dt + k[3(2\cos t)(\cos t) + (2\cos t)(-2\sin t)] dt.$$

$$= i(3\sin t \cos t) dt + j[(12 \sin t \cos t) - \sin^2 t] dt + k(6\cos 2t - 2\sin t \cos t) dt.$$

$$\begin{aligned} \therefore \int_c A \cdot dr &= i \int_0^{\pi/2} \frac{3}{2} \sin 2t dt + j \int_0^{\pi/2} \left[6 \sin 2t - \frac{1}{2}(1 - \cos 2t) \right] dt \\ &\quad + k \int_0^{\pi/2} [3(1 + \cos 2t) - \sin 2t] dt \end{aligned}$$

$$= i \frac{3}{2} \left(\frac{-\cos 2t}{2} \right) \Big|_0^{\pi/2} + j \left[-3 \cos 2t - \frac{t}{2} + \frac{\sin 2t}{4} \right] \Big|_0^{\pi/2} + k \left[3t + \frac{3}{2} \sin 2t + \frac{\cos 2t}{2} \right] \Big|_0^{\pi/2}$$

$$= i \left(\frac{-3}{4} \right) (-1 - 1) + j \left[-3(-1 - 1) - \frac{\pi}{4} + 0 \right] + k \left[\frac{3\pi}{2} + 0 + \frac{1}{2}(-1 - 1) \right]$$

$$= \frac{3}{2} i + \left(6 - \frac{\pi}{4} \right) j + \left(\frac{3\pi}{2} - 1 \right) k$$

Exercise – 7(e)

1) If $U(t) = (2t^2 - 1)i + 3tj + (2 - t)k$. Find (a) $\int U(t)dt$, (b) $\int_2^4 U(t)dt$

[Ans: (a) $\left(\frac{2t^3}{3} - t\right)i + \frac{3t^2}{2}j + \left(2t - \frac{t^2}{2}\right)k$, (b) $\frac{106}{3}i + 18j - 2k$]

2) Evaluate: $\int_0^{\pi/2} (6\sin u)i - (3\cos u)j + uk$

[Ans: $6i - 3j + \frac{\pi^2}{8}k$]

3) If $A(s) = si - s^2j + (s + 1)k$, $B(s) = 2si + 6sk$ find (a) $\int_0^2 A \cdot B ds$, (b) $\int_0^2 A \times B ds$.

[Ans: (a) $\frac{100}{3}$, (b) $-48i - 12j + 16k$]

4) If $A = ti - j + 2tk$, $B = t^2i - tj + 2k$, $C = i - 2j + 2k$, evaluate

(a) $\int_0^1 \{A \cdot (B \times C)\} dt$, (b) $\int_0^1 \{A \times (B \times C)\} dt$

[Ans: (a) $-\frac{1}{3}$, (b) $-\frac{5}{2}i + \frac{37}{6}j + \frac{8}{3}k$]

5) The acceleration of a particle a at any time $t \geq 0$ is given by $a = e^t i + (2\cos 2t)j + (2\sin 2t)k$. If the velocity V and displacement r are both zero at $t = 0$, find V and r at any time t .

[Ans: $V = (e^t - 1)i + (\sin 2t)j + (1 - \cos 2t)k$,

$r = (e^t - t - 1)i + \frac{1}{2}(1 - \cos 2t)j + \left(t - \frac{1}{2}\sin 2t\right)k$]

6) Evaluate $\int_2^3 A \cdot \frac{dA}{du} du$, given that, $A(2) = 4i - 2j + 3k$ and $A(3) = 2i + j + 2k$

[Ans: -10]

Exercise 7(f)

1. If $\phi = xyz$, evaluate $\int_c \phi dr$, where c is the curve $x = t^3, y = t^2, z = t$, from $t = 0$ to 1

$$[\text{Ans: } \frac{1}{3}i + \frac{1}{4}j + \frac{1}{7}k]$$

2. If $F = xi - yzj + z^2k$, and c is the curve given by $x = t, y = t^3, z = t^2$, evaluate

(i) $\int_c F.dr$ and (ii) $\int_c F.dr$, from $t = 0$ to $t = 1$.

$$[\text{Ans: (i) } -\frac{5}{7}i - \frac{7}{15}j + \frac{11}{12}k, \text{ (ii) } \frac{23}{24}]$$

3. If $A = (2x + 3)i + xyj + (zx - y)k$, evaluate $\int_c A.dr$ along c where c is

(a) the curve $x = t^3, y = 2t^2, z = t$ from $t = 0$ to 1 .

(b) The straight lines from $(0,0,0)$ to $(1,0,0)$, then to $(1,0,1)$ and then to $(1,2,1)$

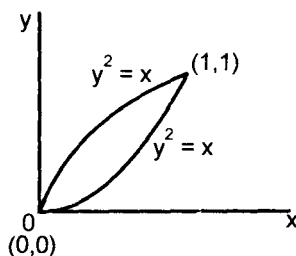
(c) The straight line joining $(0,0,0)$ and $(1,2,1)$.

$$[\text{Ans: (a) } \frac{491}{105}, \text{ (b) } \frac{13}{2}, \text{ (c) } \frac{14}{3}]$$

4. If $A = (3xy - 2y^2)i + (x - y)j$, evaluate $\int_c A.dr$ along the curve c in xy -plane given by $y = x^3$ from the point $(0,0)$ to $(2,8)$.

$$[\text{Ans: } \frac{1308}{35}]$$

5. If $F = (2x - y)i + (x - 2y)j$, evaluate $\int_c A.dr$ where c is the closed curve shown in the figure below.



$$[\text{Ans: } (\frac{1}{3})]$$

6. If $A = (3x + 2y)i + (x + y)j$, and c is the boundary of the triangle whose vertices are $(0,0)$, $(1,0)$, $(0,1)$, evaluate $\int_c A \cdot dr$.

[Ans: $\frac{3}{2}$]

7. If $A = (2x + y)i + (3x - 2y)j$, compute the circulation of A about the circle C : $x^2 + y^2 = 4$, traversed in the positive direction.

[Ans: 8π]

8. Find the work done in moving a particle in the force field.

$F = 2x^2i + (2yz - x)j + yk$, along (a) the straight line from $(0,0,0)$ to $(3,1,2)$

(b) the space curve $x = 3t^2$, $y = t$, $z = 3t^2 - t$ from $t = 0$ to $t = 1$

[Ans: (a) $\frac{113}{6}$, (b) $\frac{58}{3}$]

9. (a) Prove that $V = (2x \sin y - 3)i + (x^2 \cos y + z^2)j + 2(yz + 1)k$ is a conservative force field. (b) Find the scalar potential of V . (c) Find the work done in moving an object in this field from $(1,0,-1)$ to $(2, \frac{\pi}{2}, 1)$.

[Ans: (b) $x^2 \sin y + yz^2 - 3x + 2z + \text{constant}$, (c) $\frac{\pi}{2} + 5$]

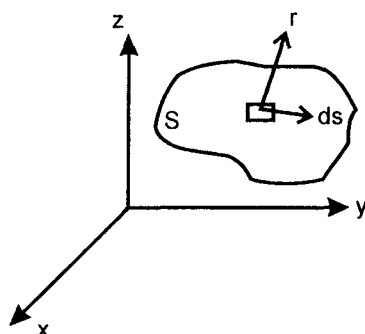
10. If $A = (9x^2y - 2xz^3)i + 3x^3j - 3x^2z^2k$, (a) prove that $\int_c A \cdot dr$ is independent of the curve 'C' joining two given points. (b) show that there is a differentiable function ϕ such that $A = \nabla\phi$ and find it.

[Ans: $\phi = 3x^3y - x^2z^3 + c$]

7.7 SURFACE INTEGRALS

- 7.7.1** Let S be a two-sided surface. Let one side be taken as the positive side. If S is a closed surface, the outer side is considered as the positive side. Let A be a vector function. Consider an element of area 'ds' in the surface. Let n be the unit normal vector to ds in the positive direction. It can be seen that $A \cdot n = A \cos\theta$. (where ' θ ' is the angle between A and n and $A = |A|$) is the normal component of A . Let ds be a vector whose magnitude is ds and whose direction is that of n .

$\therefore ds = n \, ds$.



Then the integral,

$$\iint_S A \cdot ds = \iint_S A \cdot n \cdot ds \quad \dots (1)$$

is an example of a surface integral which is also called as flux of A over s.

If 'f' is a scalar function,

$$\iint_S \phi \cdot ds \quad \dots (2)$$

$$\iint_S \phi \cdot n \cdot ds \quad \dots (3)$$

$$\iint_S A \times ds = \iint_S A \times n \cdot ds \quad \dots (4)$$

are some other examples of surface integrals.

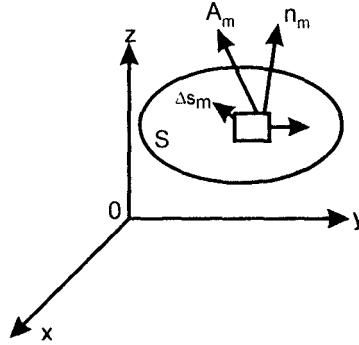
Note (1) The surface integrals can also be defined in terms of limits of sums. (see 7.7.2)

(2) The notation $\oint_S$ is also used to denote a surface integral over the closed surface s.

(3) Sometimes the notation $\oint_S$ may also be used for surface integrals.

(4) Surface integrals can be conveniently evaluated by expressing them as double integrals over the projected area of s on one of the coordinate planes (see 7.7.3)

7.7.2 Definition of surface integral as the limit of a sum:



The area S is divided into 'L' elements of area ΔS_m , $m = 1, 2, \dots, L$,

let $P_m = (x_m, y_m, z_m)$ be any point in ΔS_m . Let $A(x_m, y_m, z_m) = A_m$

Let n_m be the positive unit normal to ΔS_m at P_m . Then $(A_m \cdot n_m)$ is the normal component of A_m at P_m . Consider the sum,

$$\sum_{m=1}^L A_m \cdot n_m \Delta S_m \quad \dots (1)$$

The limit of the sum (1) as $L \rightarrow \infty$ such that the largest dimension of each $\Delta S_m \rightarrow 0$ (if the limit exists) is known as the surface integral of the normal component of A

over S and denoted by $\iint_S A \cdot n \, ds$

7.7.3 Evaluation of a surface integral

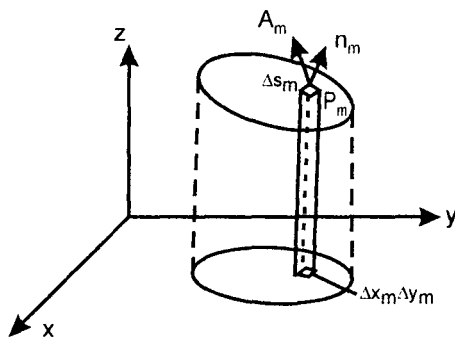
To evaluate surface integrals, it is convenient to express them as double integrals taken over the projected area of the surface S on one of the coordinate planes (xy , yz , or zx planes).

If R is the projection of S on the xy – plane, it can be shown that

$$\iint_S A \cdot n \, ds = \iint_R A \cdot n \frac{dx dy}{|n \cdot k|}$$

From 7.7.2, the surface integral is the limit of the sum

$$\sum_{m=1}^L A_m \cdot n_m \Delta S_m \quad \dots (1)$$



The projection of ΔS_m on the xy -plane is $|n_m \Delta S_m \cdot k|$ (or) $|n_m \cdot k| \Delta S_m$ which is equal to $\Delta x_m \Delta y_m$

$$\therefore \Delta S_m = \frac{\Delta x_m \Delta y_m}{|n_m \cdot k|}$$

$\therefore$ The sum (1) becomes

$$\sum_{m=1}^L A_m \cdot n_m \frac{\Delta x_m \Delta y_m}{|n_m \cdot k|}$$

By the fundamental theorem of integral calculus the limit of this sum as $L \rightarrow \infty$ in such a manner that the largest Δx_m and Δy_m approach zero is

$$\iint_R A \cdot n \frac{dx dy}{|n \cdot k|} \text{ which is the required result.}$$

Note: Similarly if R is the projection of S on yz and zx planes respectively, it can be seen as

$$\iint_S A \cdot n ds = \iint_R A \cdot n \frac{dy dz}{|n \cdot i|}$$

and

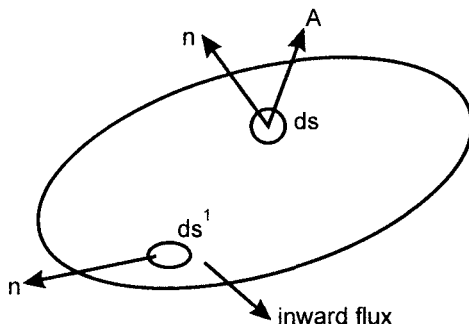
$$\iint_S A \cdot n ds = \iint_R A \cdot n \frac{dz dx}{|n \cdot j|}$$

7.7.4 Physical interpretation of surface integrals:

Let A denote the velocity of a moving fluid. Let S be a fixed surface in the fluid. Let ds be an element of surface. Then $A \cdot n ds = A \cdot dS$ represents the amount of fluid that passes normally through dS in unit time at any point. If the direction of n is outward or positive, the amount of fluid flow is positive. Similarly if dS' is another element for which n is in the negative direction, the fluid flow is negative at that point.

to $\int_S A \cdot n \, ds$ and it is known as the total flux of A through the entire surface S .

A can be a vector denoting physical quantities such as electric force, magnetic



force, flux of heat or gravitational force etc. In all these cases, $\int_S A \cdot n \, ds$ denotes total flux of A through S .

Solved Examples

Ex. 7.7.5 Evaluate $\iint_S A \cdot n \, ds$ where $A = (x + y^2)i - 2xj + 2yzk$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant.

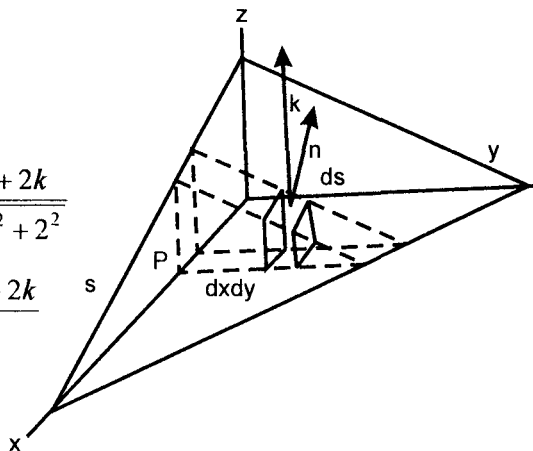
Sol: $A = (x + y^2)i - 2xj + 2yzk$

Let $\phi = 2x + y + 2z - 6$

$$\nabla \phi = \frac{\partial \phi}{\partial x}i + \frac{\partial \phi}{\partial y}j + \frac{\partial \phi}{\partial z}k = 2i + j + 2k$$

$$\begin{aligned} \text{Unit normal } n \text{ to } S &= \frac{\nabla \phi}{|\nabla \phi|} = \frac{2i + j + 2k}{\sqrt{2^2 + 1^2 + 2^2}} \\ &= \frac{2i + j + 2k}{3} \end{aligned}$$

$$A \cdot n = \frac{1}{3} [2(x + y^2) - 2x + 4yz]$$



$$= \frac{1}{3} [2x + 2y^2 - 2x + 2y(6 - 2x - y)]$$

[Substituting $2z = 6 - 2x - y$ ($\because 2x + y + 2z = 6$)]

$$= \frac{1}{3} (12y - 4xy)$$

If R be the projection of S on the xy - plane.

$$|n.k| = \frac{2}{3} \Rightarrow ds = \frac{dxdy}{|n.k|} = \frac{3}{2} dxdy$$

$$\begin{aligned} \therefore \iint_R A.nds &= \iint_R (A.n) \frac{dxdy}{|n.k|} \\ &= \iint_R \frac{1}{3} (12y - 4xy) \frac{3}{2} dxdy \\ &= \iint_R (6y - 2xy) dxdy \quad \dots (1) \end{aligned}$$

To evaluate this double integral over R,

- (i) Keep x fixed and integrate w.r.t. y from $y = 0$ to $(6 - 2x)$, ii) and then integrate w.r.t. x from $x = 0$ to $x = 3$.

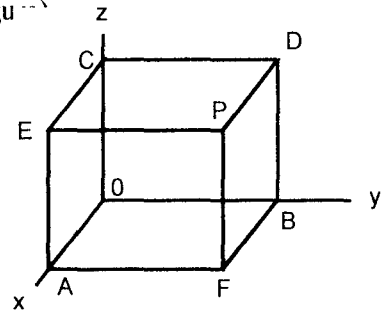
$\therefore$ Given integral

$$\begin{aligned} &\int_{x=0}^3 \int_{y=0}^{6-2x} (6y - 2xy) dy dx \\ &= \int_{x=0}^3 [3y^2 - xy^2]_{y=0}^{6-2x} dx = \int_{x=0}^3 (3-x)(6-2x)^2 dx = \int_{x=0}^3 (108 - 108x + 36x^2 - 4x^3) dx \\ &= [108x - 54x^2 + 12x^3 - x^4]_0^3 = 324 - 486 + 324 - 81 = 81 \end{aligned}$$

Ex. 7.7.6 If $F = 4xzi - y^2j + yzk$, evaluate $\iint_S F.nds$ where S is the surface of the cube

bounded by $x = 0, x = 1, y = 0, y = 1, z = 0$ and $z = 1$.

Sol. The surface S can be divided into 6 faces (see figure)



(i) S_1 : Face EPFA

(ii) S_2 : Face OBDC

(iii) S_3 : Face PFBD

(iv) S_4 : Face OCEA

(v) S_5 : Face PDCE

(vi) S_6 : Face OBFA

$$\iint_S F \cdot n \, ds = \iint_{S_1} F \cdot n \, ds + \iint_{S_2} F \cdot n \, ds + \iint_{S_3} F \cdot n \, ds + \iint_{S_4} F \cdot n \, ds + \iint_{S_5} F \cdot n \, ds + \iint_{S_6} F \cdot n \, ds$$

On S_1 : $n = i$, $x = 1$

$$\iint_{S_1} F \cdot n \, ds = \int_0^1 \int_0^1 (4zi - y^2 j + yzk) \cdot i \, dy \, dz = \int_0^1 \int_0^1 4z \, dy \, dz = \int_0^1 2z^2 \Big|_0^1 dy = \int_0^1 2 \, dy = 2$$

On S_2 : $n = -i$, $x = 0$

$$\iint_{S_2} F \cdot n \, ds = \int_0^1 \int_0^1 (-y^2 j + yzk) \cdot (-i) \, dy \, dz = \int_0^1 \int_0^1 (0) \, dy \, dz = 0$$

On S_3 : $n = j$, $y = 1$

$$\iint_{S_3} F \cdot n \, ds = \int_0^1 \int_0^1 (4xzi - j + zk) \cdot j \, dx \, dz = \int_0^1 \int_0^1 (-1) \, dx \, dz = \int_0^1 -z \Big|_0^1 dz = \int_0^1 -1 \, dz = -1$$

On S_4 : $n = -j$, $y = 0$

$$\therefore \iint_{S_4} F \cdot n \, ds = \int_0^1 \int_0^1 (4xzi) \cdot (-j) \, dx \, dz = \int_0^1 \int_0^1 (0) \, dx \, dz = 0$$

On S_5 : $n = k$, $z = 1$

$$\therefore \iint_{S_5} F \cdot n \, ds = \int_0^1 \int_0^1 (4xi - y^2 j + yk) \cdot k \, dx \, dy = \int_0^1 \int_0^1 y \, dx \, dy = \int_0^1 \frac{y^2}{2} \Big|_0^1 dy = \int_0^1 \frac{1}{2} \, dy = \frac{1}{2}$$

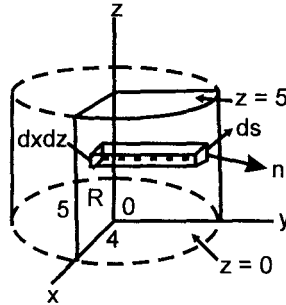
On S_6 : $n = -k$, $z = 0$

$$\iint_{S_6} F \cdot n \, ds = \int_0^1 \int_0^1 (-y^2 j) \cdot (-k) \, dx \, dy = 0$$

$$\therefore \iint_S F \cdot n \, ds = 2 + 0 - 1 + 0 + \frac{1}{2} + 0 = \frac{3}{2}$$

Ex. 7.7.7 If $A = z^2i + x^2j - y^2zk$, and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$, evaluate $\iint_S A \cdot n \, ds$

Sol:



Project s on xz plane and let the projection be R . (See figure)

$$\iint_S A \cdot n \, ds = \iint_R A \cdot n \frac{dxdy}{|n \cdot j|} \quad \dots (1)$$

The normal to $x^2 + y^2 = 16$ is

$$\begin{aligned} \nabla(x^2 + y^2) &= \frac{\partial}{\partial x}(x^2 + y^2)i + \frac{\partial}{\partial y}(x^2 + y^2)j \\ &= 2xi + 2yj \end{aligned}$$

$$\text{Unit normal } n = \frac{2xi + 2yj}{\sqrt{(2x)^2 + (2y)^2}} = \frac{2(xi + yj)}{2\sqrt{x^2 + y^2}} = \frac{xi + yj}{4} \quad (\because x^2 + y^2 = 16)$$

$$A \cdot n = \frac{z^2x + x^2y}{4}$$

$$n \cdot j = \frac{y}{4}$$

$$\begin{aligned} \therefore \text{From (1)} \quad \iint_S A \cdot n \, ds &= \iint_R \frac{z^2x + x^2y}{4} \frac{dxdz}{y/4} \\ &= \iint_R \frac{(z^2x + x^2y)}{y} dxdz = \int_{z=0}^5 \left[\int_{x=0}^4 \left(\frac{xz^2}{\sqrt{16-x^2}} + x^2 \right) dx \right] dz \\ &= \int_{z=0}^5 \left(4z^2 \frac{64}{3} \right) dz \quad \left[\because \int_0^4 \frac{x}{\sqrt{16-x^2}} dx = 4, \quad \int_0^4 x^2 dx = \frac{64}{3} \right] \\ &= \frac{4z^3}{3} + \frac{64z^5}{3} \bigg|_0^5 = \frac{820}{3} \end{aligned}$$

Ex. 7.7.8 Evaluate $\iint_S \phi n ds$ where S is the surface of problem 7.7.7 above and $\phi = \frac{xyz}{8}$

Sol: We have $\iint_S \phi n ds = \iint_R \phi n \frac{dx dz}{|n \cdot j|}$

using $n = \frac{xi + yj}{4}$, $n \cdot j = \frac{y}{4}$, the integral on R.H.S. becomes,

$$\iint_R \frac{1}{8} xyz \frac{(xi + yj)}{4} \cdot \frac{dx dz}{y/4}$$

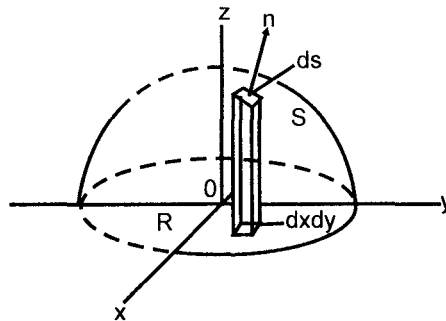
$$\frac{1}{8} \int_{z=0}^5 \int_{x=0}^4 xz(xi + yj) dx dz = \frac{1}{8} \int_{z=0}^5 \left[\int_0^4 (x^2 zi + xz\sqrt{16-x^2}) j dx \right] dz$$

$$\frac{1}{8} \int_0^5 \left(\frac{64z}{3} i + \frac{64z}{3} j \right) dz \quad \left[\because \int_0^4 x\sqrt{16-x^2} dx = \frac{64}{3} \int_0^4 x^2 dx = \frac{64}{3} \right]$$

$$\frac{1}{8} \cdot \frac{64}{3} \left(\frac{z^2}{2} i + \frac{z^2}{2} j \right) \Big|_0^5 = \frac{8}{3} \cdot \frac{25}{2} (i + j) = \frac{100}{3} (i + j)$$

Ex. 7.7.9 If $A = yi + (x - 2xz)j - xyk$, evaluate $\iint_S (\text{curl } A) n ds$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 4$ above the xy plane

Sol:



$$\text{Curl } A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x - 2xz & -xy \end{vmatrix}$$

$$= i\left\{\frac{\partial}{\partial y}(-xy) - \frac{\partial}{\partial z}(x - 2xz)\right\} + j\left\{\frac{\partial}{\partial z}(y) - \frac{\partial}{\partial x}(-xy)\right\} + k\left\{\frac{\partial}{\partial x}(x - 2xz) - \frac{\partial}{\partial y}(y)\right\}$$

$$= xi + yj - 2zk$$

The normal to the surface is $\nabla (x^2 + y^2 + z^2) = 2xi + 2yj + 2zk$

$$\text{Unit normal } n = \frac{2xi + 2yj + 2zk}{\sqrt{(2x)^2 + (2y)^2 + (2z)^2}}$$

$$= \frac{xi + yj + zk}{2} \quad (\because x^2 + y^2 + z^2 = 4)$$

$$(\text{Curl } A) \cdot n = \frac{x^2 y^2 - 2z^2}{2}$$

The projection of S on xy plane is the circle $x^2 + y^2 = 4$, $z = 0$ (see figure)

$$\therefore \iint_S (\text{curl } A \cdot n) ds = \iint_R (\text{curl } A \cdot n) \frac{dx dy}{|n \cdot k|}$$

$$= \iint_R \frac{(x^2 + y^2 - 2z^2)}{2} \frac{dx dy}{z/2}$$

$$= \int_{x=-2}^2 \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{x^2 + y^2 - 2(4 - x^2 - y^2)}{\sqrt{4 - x^2 - y^2}} dx dy \quad (\because z^2 = 4 - x^2 - y^2)$$

$$= \int_{x=-2}^2 \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{3(x^2 + y^2) - 8}{\sqrt{4 - x^2 - y^2}} dx dy$$

Changing into polar coordinates by taking $x = r \cos \theta$, $y = r \sin \theta$, $dx dy = r dr d\theta$, the integral becomes

$$\int_{\theta=0}^{2\pi} \int_{r=0}^2 \left(\frac{3(r^2 - 4) + 4}{\sqrt{4 - r^2}} \right) r dr d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\int_{r=0}^2 \left(-3r\sqrt{4 - r^2} + \frac{4r}{\sqrt{4 - r^2}} \right) dr \right] d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\left(4 - r^2 \right)^{3/2} - 4\sqrt{4 - r^2} \right]_{r=0}^2 d\theta = \int_{\theta=0}^{2\pi} (8 - 8) d\theta = 0$$

Ex.7.7.10 Evaluate $\iint_S A \cdot n \, ds$ where $A = yzi + xzj + xyk$ and S is the part of the sphere $x^2 + y^2 + z^2 = 9$ which lies in the first octant.

Sol: Unit normal n to $S = \frac{\nabla(x^2 + y^2 + z^2)}{|\nabla(x^2 + y^2 + z^2)|}$

$$= \frac{2xi + 2yj + 2zk}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{xi + yj + zk}{3} \quad (\because x^2 + y^2 + z^2 = 9)$$

$$A \cdot n = 3xyz; \quad n \cdot k = \frac{z}{3}$$

If R is the projection of S on xy -plane, we have,

$$\begin{aligned} \iint_S A \cdot n \, ds &= \iint_R A \cdot n \frac{dx dy}{|n \cdot k|} \\ &= \iint_R 3xyz \frac{dx dy}{z/3} = 9 \iint_R xy \, dx dy \end{aligned}$$

The region R is bounded by x -axis, y -axis and the circle $x^2 + y^2 = 9$; $z = 0$ changing to polar coordinates, the last integral becomes

$$\begin{aligned} 9 \int_{\theta=0}^{\pi/2} \int_{r=0}^3 (r \cos \theta)(r \sin \theta) r \, dr \, d\theta &= 9 \int_0^{\pi/2} \int_0^3 r^3 \cos \theta \sin \theta \, dr \, d\theta \\ &= 9 \int_0^{\pi/2} \left(\frac{r^4}{4} \right)_0^3 \cos \theta \sin \theta \, d\theta = 9 \cdot \frac{81}{4} \int_0^{\pi/2} \cos \theta \sin \theta \, d\theta \\ &= \frac{729}{4} \cdot \frac{1}{2} = \frac{729}{8} \quad \left(\because \int_0^{\pi/2} \cos \theta \sin \theta \, d\theta = \frac{1}{2} \right) \end{aligned}$$

Exercise – 7(g)

1. If $F = 18zi - 12xj + 3yk$, evaluate $\iint_S F \cdot n \, ds$ where S is that part of the plane

$2x + 3y + 6z = 12$ which is located in the first octant

(Hint: Take projection of S on the xy plane and $ds = \frac{dx dy}{|n \cdot k|}$)

[Ans: 24]

2. Evaluate $\iint_S V \cdot n \, ds$, where $V = yi + 2xj - zk$ and s is the surface of the plane

$2x + y = 6$ in the first octant cut off by the plane $z = 4$

[Ans: 108]

3. If $r = xi + yj + zk$ find the value of the integral $\iint_S r \cdot n \, ds$ over

a) the surface S of the unit cube bounded by the coordinate planes and the planes $x = y = z = 1$ and

b) the surface of the sphere of radius 'a' with centre at the origin

[Ans: a) 3, b) $4\pi a^3$]

4. Evaluate $\iint_S F \cdot ds$ over the entire surface of the region above the xy plane bounded

by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$, if $F = 4xzi + xy^2zj + 3zk$

[Ans: 320π]

5. If S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$, evaluate $\iint_S F \cdot n \, ds$, where $F = 2yi - zj + x^2k$

[Hint: $\iint_S F \cdot n \, ds = \iint_R F \cdot n \frac{dydz}{|n \cdot i|}$, where R is the projection of S on the yz plane]

[Ans: 132]

6. Evaluate $\iint_S V \cdot n \, ds$ over the entire surface S of region bounded by the cylinder

$x^2 + z^2 = 9$, $x = 0$, $y = 0$, $z = 0$ and $y = 8$ if $V = 6zi + (2x + y)j - xk$

[Ans: 18π]

7. Evaluate $\iint_S A \cdot n \, ds$ where $A = zi + xj - 3y^2zk$ and S is the surface of the cylinder

$x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$

[Ans: 90]

8. If $V = yzi + xzj + xyk$ and S is that part of the sphere $x^2 + y^2 + z^2 = 1$ which lies in the first octant, find the value of $\iint_S V \cdot n \, ds$.

[Ans: $\frac{3}{8}$]

9. Evaluate $\iint_S \{(x^3 - yz^2)i - (2x^2y)j + 2k\} . ds$ over a cube with edges of length 'r' parallel to the coordinate axes

[Ans: $\frac{1}{2}$]

10. If $V = xi + yj + zk$, and S is the triangle with vertices at (1,0,0), (0,1,0) and (0,0,1), find the value of $\iiint_S V . ds$.

[Ans: $\frac{1}{2}$]

11. If $A = xi - yj + (z^2 - 1)k$, find the value of $\iint_S A . nds$, where S is the closed surface bounded by the planes $z = 0$, $z = 1$ and the cylinder $x^2 + y^2 = 1$

[Ans: π]

7.8 VOLUME INTEGRALS

- 7.8.1** Consider a closed surface in space enclosing a volume V. Then, integrals of the form $\iiint_V A dv$ and $\iiint_V \phi dv$, [A is a vector function, ϕ is a scalar function] are examples of volume integrals.

7.8.2 Expression of volume integral as the limit of a sum:

Let A be a continuous vector function. Let S be a surface enclosing the region D. Divide this region D into a finite number of subregions $D_1, \dots, = 1, 2, \dots, n$.

Let Δv_i be the volume of the subregion D_i enclosing any point whose position vector is r_i .

Consider the sum

$$V = \sum_{i=1}^n A(r_i) \Delta v_i$$

The limit of this sum as $n \rightarrow \infty$ such that $\Delta v_i \rightarrow 0$, is called the volume integral of

A over D and denoted by $\iiint_D A dv$

If $A = A_1(x, y, z)i + A_2(x, y, z)j + A_3(x, y, z)k$,

so that $dv = dx dy dz$

$$\iiint_D A dv = i \iiint_D A_1(x, y, z) dx dy dz + j \iiint_D A_2(x, y, z) dx dy dz + k \iiint_D A_3(x, y, z) dx dy dz$$

Solved Examples

Ex. 7.8.3 If $F = (2x^2 - 3z)i - 2xyj - 4xk$, evaluate $\iiint_V \nabla \cdot F dv$ where V is the closed region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$.

Sol: $\nabla \cdot F = \frac{\partial}{\partial x}(2x^2 - 3z) - \frac{\partial}{\partial y}(2xy) - \frac{\partial}{\partial z}(4x) = 4x - 2x = 2x$

$$\begin{aligned} \therefore \iiint_V \nabla \cdot F dv &= \int_{x=0}^2 \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} 2x dz dy dx = \int_{x=0}^2 \int_{y=0}^{2-x} [2xz]_{z=0}^{4-2x-2y} dy dx \\ &= \int_{x=0}^2 \int_{y=0}^{2-x} 2xz dy dx = \int_{x=0}^2 \int_{y=0}^{2-x} 2x(4-2x-2y) dy dx = \int_{x=0}^2 [8x - 4x^2 - 4xy]_{y=0}^{2-x} dx \\ &= \int_{x=0}^2 (8xy - 4x^2y - 2xy^2)_{y=0}^{2-x} dx = \int_{x=0}^2 [8x(2-x) - 4x^2(2-x) - 2x(2-x)^2] dx \\ &= \int_{x=0}^2 (8x - 8x^2 + 2x^3) dx = (4x^2 - \frac{8x^3}{3} + \frac{2x^4}{4}) \Big|_0^2 = 16 - \frac{64}{3} + 8 = \frac{8}{3} \end{aligned}$$

Ex. 7.8.4 Evaluate $\iiint_V (\nabla \cdot A) dv$ over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$,

where $A = 4xi - 2y^2j + z^2k$.

Sol: $\nabla \cdot A = \frac{\partial}{\partial x}(4x) - \frac{\partial}{\partial y}(2y^2) + \frac{\partial}{\partial z}(z^2) = 4 - 4y + 2z$

$$\therefore \iiint_V (\nabla \cdot A) dv = \iiint_V (4 - 4y + 2z) dv = \int_{x=-2}^2 \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^3 [4 - 4y + 2z] dz dy dx$$

$$\begin{aligned}
&= \int_{x=-2}^2 \int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} \left[(4z - 4yz + z^2) \right]_0^3 dy dx = \int_{x=-2}^2 \left[\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} (21 - 12y) dy \right] dx \\
&= \int_{x=-2}^2 (21y - 6y^2) \Big|_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} dx = \int_{x=-2}^2 42\sqrt{4-x^2} dx = 84 \int_0^2 \sqrt{4-x^2} dx \\
&= 84 \left[\frac{x}{2} \sqrt{4-x^2} + 2 \sin^{-1} \left(\frac{x}{2} \right) \right]_0^2 = 84 \left[0 + 2 \left(\frac{\pi}{2} \right) - 0 \right] = 84\pi
\end{aligned}$$

Ex. 7.8.5 Evaluate $\iiint_V \phi dv$ taken over the rectangular parallelepiped $0 \leq x < a$, $0 \leq y < b$,

$$0 \leq z < c \text{ and } \phi = 2(x + y + z)$$

Sol:

$$\begin{aligned}
\iiint_V \phi dv &= \iiint_V 2(x + y + z) dv = \int_{x=0}^a \int_{y=0}^b \left[\int_{z=0}^c 2(x + y + z) dz \right] dy dx \\
&= \int_{x=0}^a \int_{y=0}^b \left[2xz + 2yz + z^2 \right]_{z=0}^c dy dx = \int_{x=0}^a \int_{y=0}^b (2cx + 2cy + c^2) dy dx \\
&= \int_{x=0}^a \left[2cxy + cy^2 + c^2 y \right]_0^b dx = \int_{x=0}^a (2bcx + cb^2 + c^2 b) dx = bcx^2 + (b^2c + bc^2)x \Big|_0^a \\
&= a^2bc + a(b^2c + bc^2) = abc(a + b + c)
\end{aligned}$$

Ex. 7.8.6 If $\phi = 4y + 2xz$, evaluate $\iiint_V \phi dv$ over the region in the first octant bounded by

$$x^2 + y^2 = 9, z = 0, z = 2.$$

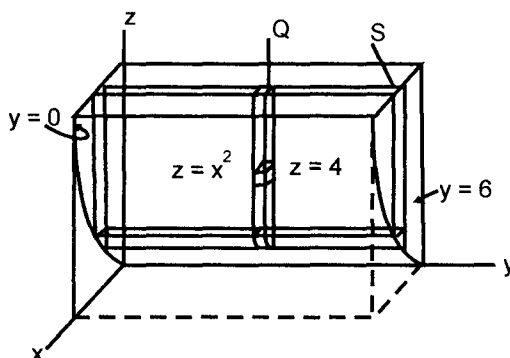
Sol:

$$\begin{aligned}
\iiint_V \phi dv &= \iiint_V (4y + 2xz) dv \\
&= \int_{x=0}^3 \int_{y=0}^{y=\sqrt{9-x^2}} \left[\int_{z=0}^{z=2} (2xz + 4y) dz \right] dy dx = \int_0^3 \int_0^{\sqrt{9-x^2}} (4yz + xz^2) \Big|_0^2 dy dx \\
&= \int_0^3 \left[\int_0^{\sqrt{9-x^2}} (8y + 4x) dy \right] dx = \int_0^3 (4y^2 + 4xy) \Big|_0^{\sqrt{9-x^2}} dx \\
&= \int_0^3 [4(9 - x^2) + 4x\sqrt{9 - x^2}] dx = 108
\end{aligned}$$

Ex. 7.8.7 Evaluate $\iiint_V F dv$ where $F = xzi - 2xj + 2y^2k$ and V is the region bounded by

the surfaces $x = 0$, $y = 0$, $y = 6$, $z = x^2$, and $z = 4$

Sol:



The region V is covered by (see the figure) (a) keeping x and y fixed and integrating from $z = x^2$ to $z = 4$ (base top of column PQ) (b) then by keeping x fixed and integrating from $y = 0$ to $y = 6$ (R to S in the slab) and (c) finally integrating from $x = 0$ to $x = 2$ (where $z = x^2$ meets $z = 4$).

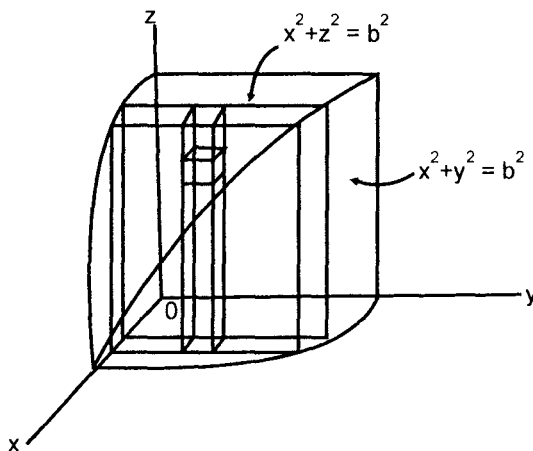
$\therefore$ The required integral is

$$\begin{aligned}
 & \int_{x=0}^2 \int_{y=0}^6 \left[\int_{z=x^2}^4 (xzi - 2xj + 2y^2k) dz \right] dy dx \\
 &= i \int_{x=0}^2 \int_{y=0}^6 \left[\int_{z=x^2}^4 xz dz \right] dy dx + j \int_{x=0}^2 \int_{y=0}^6 \left[\int_{z=x^2}^4 -2xz dz \right] dy dx + k \int_{x=0}^2 \int_{y=0}^6 \left[\int_{z=x^2}^4 2y^2 dz \right] dy dx \\
 &= i \int_{x=0}^2 \int_{y=0}^6 \left. \frac{xz^2}{2} \right|_{z=x^2}^4 dy dx + j \int_{x=0}^2 \int_{y=0}^6 \left. -2xz \right|_{z=x^2}^4 dy dx + k \int_{x=0}^2 \int_{y=0}^6 \left. 2y^2 z \right|_{z=x^2}^4 dy dx
 \end{aligned}$$

$$\begin{aligned}
&= i \int_{x=0}^2 \left[\int_{y=0}^6 \left(8x - \frac{x^5}{2} \right) dy \right] dx + j \int_{x=0}^2 \left[\int_{y=0}^6 (2x^3 - 8x) dy \right] dx + k \int_{x=0}^2 \left[\int_{y=0}^6 (8y^2 - 2x^2 y) dy \right] dx \\
&= i \int_{x=0}^2 \left. 8xy - \frac{x^5 y}{2} \right|_{y=0}^6 dx + j \int_{x=0}^2 \left. 2x^3 y - 8xy \right|_{y=0}^6 dx + k \int_{x=0}^2 \left. \frac{8y^3}{3} - \frac{2x^2 y^2}{2} \right|_{y=0}^6 dx \\
&= i \int_0^2 (48x - 3x^5) dx + j \int_0^2 (12x^3 - 48x) dx + k \int_0^2 (576 - 144x^2) dx \\
&= i \left(24x^2 - \frac{1}{2}x^6 \right) \Big|_{x=0}^2 + j \left(3x^4 - 24x^2 \right) \Big|_{x=0}^2 + k \left(576x - 48x^3 \right) \Big|_{x=0}^2 \\
&= 64i - 48j + 768k
\end{aligned}$$

Ex. 7.8.8 Find the volume of the region common to the intersecting cylinders $x^2 + y^2 = b^2$ and $x^2 + z^2 = b^2$.

Sol:



Required volume is equal to 8 times the volume of the region shown in the figure (as the axes cut the volume into 8 equal parts one in each octant)

$$\begin{aligned}
 \text{Volume} &= 8 \int_{x=0}^b \int_{y=0}^{\sqrt{b^2-x^2}} \left(\int_{z=0}^{\sqrt{b^2-x^2}} dz \right) dy dx = 8 \int_{x=0}^b \left(\int_{y=0}^{\sqrt{b^2-x^2}} \sqrt{b^2-x^2} dy \right) dx \\
 &= 8 \int_{x=0}^b (b^2 - x^2) dx = 8 \left[\left(b^2 x - \frac{x^3}{3} \right) \right]_0^b = 16 \frac{b^2}{3}.
 \end{aligned}$$

Exercise – 7(h)

1. Evaluate $\iiint_V (2x + y) dv$ where V is the closed region bounded by the cylinder

$$z = 4 - x^2 \text{ and the planes } x = 0, y = 2 \text{ and } z = 0. \quad [\text{Ans: } \frac{80}{3}]$$

2. If $\mathbf{A} = (2x^2 - 3z)\mathbf{i} - 2xy\mathbf{j} - 4x\mathbf{k}$, and V is the closed region bounded by the planes

$$x=0, y=0, z=0 \text{ and } 2x + 2y + z = 4, \text{ find the value of } \iiint_V (\text{curl } \mathbf{A}) dv$$

$$[\text{Ans: } \frac{8}{3}(j - k)]$$

3. Evaluate $\iiint_V f dv$ where $f = 45x^2y$ and v is the closed region bounded by the planes

$$4x + 2y + z = 8, x = 0, y = 0 \text{ and } z = 0 \quad [\text{Ans: } 128]$$

4. If $\mathbf{A} = 2xzi - xj + y^2k$ and v is the region bounded by the surfaces

$$x = 0, y = 0, y = 6, z = x^2 \text{ and } z = 4, \text{ find the value of } \iiint_V \mathbf{A} dv \quad \text{Ans: } 128i - 24j + 384k$$

5. Evaluate $\iiint_V (\text{Div } \mathbf{A}) dv$ taken over the rectangular parallelepiped,

$$0 \leq x \leq 1, 0 \leq y \leq 2,$$

$$0 \leq z \leq 3, \text{ where } \mathbf{A} = (x^2 - yz)\mathbf{i} + (y^2 - zx)\mathbf{j} + (z^2 - xy)\mathbf{k} \quad [\text{Ans: } 36]$$

6. Evaluate $\iiint_V (\text{Div } \mathbf{F}) dv$ for the volume of a cube with edges of length unity

$$\text{parallel to the coordinate axes where } \mathbf{F} = (x^3 - yz^2)\mathbf{i} - (2x^2y)\mathbf{j} + 2k$$

$$[\text{Ans: } 1/3]$$

7.9 CURVILINEAR COORDINATES

7.9.0 The students are already familiar with the coordinate systems such as i) cartesian
ii) Polar and iii) parametric.

A study of yet another coordinate system called “curvilinear coordinate system” will be now taken up.

7.9.1 Transformation of coordinates

Suppose the rectangular coordinates (x, y, z) of any point be expressed as functions of variables v_1, v_2, v_3 as

$$x = x(v_1, v_2, v_3), y = y(v_1, v_2, v_3), z = z(v_1, v_2, v_3) \quad \text{.....(1)}$$

Let the above system of equations (i) be solved for v_1, v_2, v_3 and another system of equations

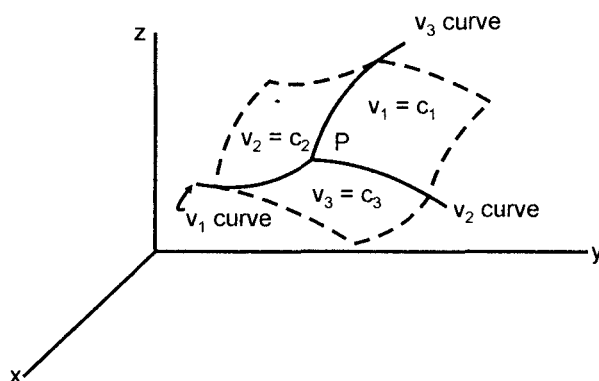
$$v_1 = v_1(x, y, z), v_2 = v_2(x, y, z), v_3 = v_3(x, y, z), \text{ be obtained} \quad \text{..... (2)}$$

If the functions in (1) and (2) are assumed to be single valued and to have continuous derivatives, the correspondence between the variables x, y, z and v_1, v_2, v_3 is unique.

Thus, to a given point P with cartesian coordinates (x, y, z) , we can associate from (2) a unique set of coordinates (v_1, v_2, v_3) which are called curvilinear coordinates.

The systems of equations (1) and (2) give the “transformation of coordinates”.

7.9.2 Orthogonal Curvilinear coordinates



The surfaces $v_1 = c_1, v_2 = c_2, v_3 = c_3$ (c_1, c_2, c_3 being constants) are called ‘coordinate surfaces’. (see figure).

Each pair of the above surfaces intersect in curves called ‘coordinate curves or lines’. i.e., $v_1 = c_1$ and $v_2 = c_2$ intersect in v_3 curve, $v_2 = c_2, v_3 = c_3, v_1 = c_1$ in v_2 curve.

If the coordinate surfaces taken in pairs intersect at right angles the coordinate system is called 'an orthogonal curvilinear coordinate system'.

Note: The coordinate surfaces described above are similar to the coordinate planes of the rectangular system and the coordinate curves are similar to the coordinate axes. Infact, the rectangular coordinate system is also an example of curvilinear system with $v_1 = x$, $v_2 = y$, $v_3 = z$; $c_1 = c_2 = c_3 = 0$

7.9.3 Unit vectors of a curvilinear system

Consider the position vector $r = xi + yj + zk$, of a point P. using (1), it can be written as $r = r(v_1, v_2, v_3)$

- (i) A tangent vector to the v_1 curve at P (for which v_2, v_3 are constants) is $\frac{\partial r}{\partial v_1}$. If e_1 is

the unit tangent vector we have $e_1 = \frac{\partial r / \partial v_1}{\left| \frac{\partial r}{\partial v_1} \right|}$

If $\left| \frac{\partial r}{\partial v_1} \right| = \lambda_1$, we have, $\frac{\partial r}{\partial v_1} = \lambda_1 e_1$

|||y if $\frac{\partial r}{\partial v_2}, \frac{\partial r}{\partial v_3}$ are tangent vectors to the v_2 and v_3 curves at P respectively and $e_2,$

e_3 are unit vectors in these directions,

we can write, $\frac{\partial r}{\partial v_2} = \lambda_2 e_2, \frac{\partial r}{\partial v_3} = \lambda_3 e_3$

where $\lambda_2 = \left| \frac{\partial r}{\partial v_2} \right|$ and $\lambda_3 = \left| \frac{\partial r}{\partial v_3} \right|$

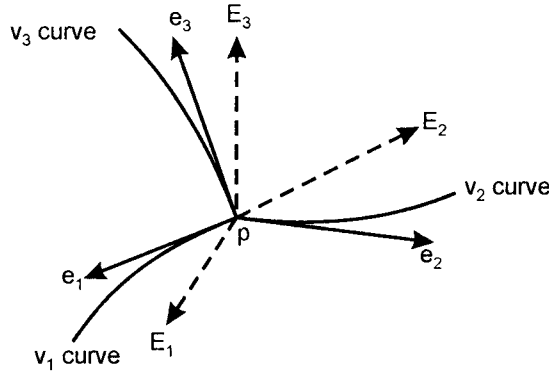
The unit vectors e_1, e_2, e_3 are in the directions of increasing v_1, v_2, v_3 respectively. The quantities $\lambda_1, \lambda_2, \lambda_3$ are called 'scale factors'.

- (ii) ' ∇v_1 ' is a vector at P normal to the surface $v_1 = c_1$. If E_1 is a unit vector in this direction,

$$E_1 = \frac{\nabla v_1}{|\nabla v_1|}$$

|||y the unit vectors, $E_2 = \frac{\nabla v_2}{|\nabla v_2|}, E_3 = \frac{\nabla v_3}{|\nabla v_3|}$ at P are respectively normal the surfaces

$v_2 = c_2$, and $v_3 = c_3$.



Thus, in general, at a given point P there exist two sets of unit vectors,

- (i) e_1, e_2, e_3 tangential to the coordinate curves and
- (ii) E_1, E_2, E_3 are normal to the coordinate surfaces (see figure)

7.9.4 Representations of a vector

- (a) A vector A can be represented in terms of the unit base vectors e_1, e_2, e_3 or E_1, E_2, E_3 in the following forms:

$$(i) \quad A = a_1 e_1 + a_2 e_2 + a_3 e_3$$

$$(or) \quad (ii) \quad A = A_1 E_1 + A_2 E_2 + A_3 E_3,$$

Here a_1, a_2, a_3 or A_1, A_2, A_3 are the components of A in the two systems

respectively (b) The vectors $\frac{\partial r}{\partial v_1}, \frac{\partial r}{\partial v_2}, \frac{\partial r}{\partial v_3}$ and $\nabla v_1, \nabla v_2, \nabla v_3$, are called

‘Unitary base vectors (need not be unit vectors in general). We can also represent A in one of the following forms; as

$$A = c_1 \frac{\partial r}{\partial v_1} + c_2 \frac{\partial r}{\partial v_2} + c_3 \frac{\partial r}{\partial v_3} = c_1 \alpha_1 + c_2 \alpha_2 + c_3 \alpha_3$$

$$(or) \quad A = C_1 \nabla v_1 + C_2 \nabla v_2 + C_3 \nabla v_3 = C_1 \beta_1 + C_2 \beta_2 + C_3 \beta_3$$

Here

c_1, c_2, c_3 are called the ‘contravariant components of A and C_1, C_2, C_3 are called the ‘Covariant components of A

7.9.5 Arc length, area (surface) and volume elements

(i) Let $r = r(v_1, v_2, v_3)$

$$\therefore dr = \frac{\partial r}{\partial v_1} dv_1 + \frac{\partial r}{\partial v_2} dv_2 + \frac{\partial r}{\partial v_3} dv_3 = \lambda_1 (dv_1) e_1 + \lambda_2 (dv_2) e_2 + \lambda_3 (dv_3) e_3 \quad \dots (3)$$

Since $ds^2 = dr \cdot dr$, the differential ds of arc length is given by,

$$ds^2 = \lambda_1^2 dv_1^2 + \lambda_2^2 dv_2^2 + \lambda_3^2 dv_3^2 \text{ for an orthogonal coordinate system}$$

$$(\because e_1 \cdot e_2 = e_2 \cdot e_3 = e_3 \cdot e_1 = 0)$$

(ii) Let dv_1, dv_2 , be vectors specifying arc-elements along v_1 and v_2 curves respectively at point P.

i.e., $dv_1 = dr$ along v_1 curve and $dv_2 = dr$ along v_2 curve.

Since v_2, v_3 are constants along v_1 curve and v_1, v_3 are constants along v_2 curve, we have, from (3),

$$dv_1 = (\lambda_1 dv_1) e_1; dv_2 = (\lambda_2 dv_2) e_2 \quad \dots (4)$$

The area of the parallelogram formed with the vectors of (4) as adjacent sides is the area element or surface element ' ds_3 ' on the surface $v_3 = c_3$ (see fig) and it is given by

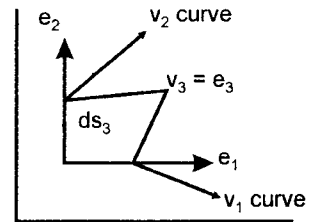
$$ds_3 = |dv_1 \times dv_2| = |\lambda_1 \lambda_2 dv_1 dv_2 e_3| = \lambda_1 \lambda_2 dv_1 dv_2$$

$$||^{\text{ly}} ds_1 = \lambda_2 \lambda_3 dv_2 dv_3$$

$$ds_2 = \lambda_3 \lambda_1 dv_3 dv_1$$

are surface elements on surfaces

$v_1 = c_1$ and $v_2 = c_2$ respectively.



(iii) The volume element ' dv ' for an orthogonal curvilinear coordinate system is equal to the volume of the parallelepiped whose coterminous edges are dv_1, dv_2, dv_3

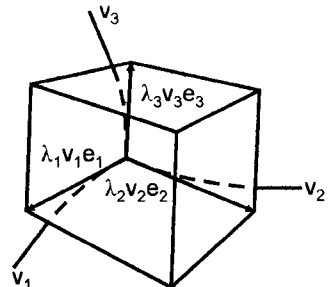
$$\text{i.e., } \lambda_1 dv_1 e_1, \lambda_2 dv_2 e_2, \lambda_3 dv_3 e_3$$

(see figure)

$$\therefore dv = |(\lambda_1 dv_1 e_1) \cdot (\lambda_2 dv_2 e_2) \times (\lambda_3 dv_3 e_3)|$$

$$= \lambda_1 \lambda_2 \lambda_3 dv_1 dv_2 dv_3 |e_1 \cdot e_2 \times e_3|$$

$$= \lambda_1 \lambda_2 \lambda_3 dv_1 dv_2 dv_3 \quad (\because |e_1 \cdot e_2 \times e_3| = 1)$$



7.9.6 EXPRESSIONS FOR GRADIENT, DIVERGENCE AND CURL IN AN ORTHOGONAL CURVILINEAR COORDINATE SYSTEM:

7.9.6(1) Gradient of a scalar function ϕ :

Prove that $\nabla\phi$ (i.e. grad ϕ) = $\left[\frac{1}{\lambda_1} \frac{\partial\phi}{\partial v_1} e_1 + \frac{1}{\lambda_2} \frac{\partial\phi}{\partial v_2} e_2 + \frac{1}{\lambda_3} \frac{\partial\phi}{\partial v_3} e_3 \right]$

Proof: Let $\nabla\phi = g_1 e_1 + g_2 e_2 + g_3 e_3$, where g_1, g_2, g_3 are to be determined

$$\begin{aligned} d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial v_1} dv_1 + \frac{\partial \mathbf{r}}{\partial v_2} dv_2 + \frac{\partial \mathbf{r}}{\partial v_3} dv_3 \\ &= (\lambda_1 e_1) dv_1 + (\lambda_2 e_2) dv_2 + (\lambda_3 e_3) dv_3 \end{aligned}$$

$$\therefore d\phi = \nabla\phi \cdot d\mathbf{r},$$

$$d\phi = \lambda_1 g_1 dv_1 + \lambda_2 g_2 dv_2 + \lambda_3 g_3 dv_3 \quad \dots (A)$$

$$\text{Again, } d\phi = \frac{\partial\phi}{\partial v_1} dv_1 + \frac{\partial\phi}{\partial v_2} dv_2 + \frac{\partial\phi}{\partial v_3} dv_3 \quad \dots (B)$$

$$(A) \text{ and } (B) \Rightarrow \lambda_1 g_1 = \frac{\partial\phi}{\partial v_1}, \quad \lambda_2 g_2 = \frac{\partial\phi}{\partial v_2}, \quad \lambda_3 g_3 = \frac{\partial\phi}{\partial v_3}$$

$$\Rightarrow g_1 = \frac{1}{\lambda_1} \frac{\partial\phi}{\partial v_1}, \quad g_2 = \frac{1}{\lambda_2} \frac{\partial\phi}{\partial v_2}; \quad g_3 = \frac{1}{\lambda_3} \frac{\partial\phi}{\partial v_3}$$

$$\therefore \text{Grad } \phi = \nabla\phi = \frac{e_1}{\lambda_1} \frac{\partial\phi}{\partial v_1} + \frac{e_2}{\lambda_2} \frac{\partial\phi}{\partial v_2} + \frac{e_3}{\lambda_3} \frac{\partial\phi}{\partial v_3}$$

Note: The above expression indicates that $\nabla = \left[\frac{e_1}{\lambda_1} \frac{\partial}{\partial v_1} + \frac{e_2}{\lambda_2} \frac{\partial}{\partial v_2} + \frac{e_3}{\lambda_3} \frac{\partial}{\partial v_3} \right]$

7.9.6(2) If v_1, v_2, v_3 are orthogonal coordinates, prove that

$$(a) |\nabla v_l| = \lambda_l^{-1}, \quad (l = 1, 2, 3)$$

$$(b) e_l = E_l \quad (l = 1, 2, 3)$$

We know that, (from 7.9.6 (1))

$$\nabla\phi = \frac{e_1}{\lambda_1} \frac{\partial\phi}{\partial v_1} + \frac{e_2}{\lambda_2} \frac{\partial\phi}{\partial v_2} + \frac{e_3}{\lambda_3} \frac{\partial\phi}{\partial v_3}$$

$$\text{Put } \phi = v_1$$

$$\therefore \nabla v_1 = \frac{e_1}{\lambda_1}; \text{ |||ly we get } \nabla v_2 = \frac{e_2}{\lambda_2}, \quad \nabla v_3 = \frac{e_3}{\lambda_3}$$

Hence $|\nabla v_1| = \left| \frac{e_1}{\lambda_1} \right| = \lambda_1^{-1}$

Similarly $|\nabla v_2| = \left| \frac{e_2}{\lambda_2} \right| = \lambda_2^{-1}$, $|\nabla v_3| = \left| \frac{e_3}{\lambda_3} \right| = \lambda_3^{-1}$, which proves (a)

(b) $E_1 = \frac{\nabla v_1}{|\nabla v_1|}$ (by definition, see 7.9.3(ii))

$$= \frac{\frac{e_1}{\lambda_1}}{\frac{1}{\lambda_1}} = e_1, \quad l = 1, 2, 3. \quad [\text{from (a)}]$$

7.9.6. (3) Prove that, in orthogonal curvilinear coordinates,

$$\nabla \cdot (A_1 e_1) = \frac{1}{\lambda_1 \lambda_2 \lambda_3} \frac{\partial}{\partial v_1} (A_1 \lambda_2 \lambda_3) \text{ and hence show that,}$$

$$\text{div } A = \frac{1}{\lambda_1 \lambda_2 \lambda_3} \left[\frac{\partial}{\partial v_1} (A_1 \lambda_2 \lambda_3) + \frac{\partial}{\partial v_2} (A_2 \lambda_3 \lambda_1) + \frac{\partial}{\partial v_3} (A_3 \lambda_1 \lambda_2) \right]$$

Proof: we have, $\nabla v_1 = \frac{e_1}{\lambda_1}$, $\nabla v_2 = \frac{e_2}{\lambda_2}$, $\nabla v_3 = \frac{e_3}{\lambda_3}$

$$\therefore \nabla v_2 \times \nabla v_3 = \frac{1}{\lambda_2 \lambda_3} (e_2 \times e_3) = \frac{e_1}{\lambda_2 \lambda_3} \quad \dots (C)$$

$$e_1 = \lambda_2 \lambda_3 (\nabla v_2 \times \nabla v_3)$$

Similarly $e_2 = \lambda_3 \lambda_1 (\nabla v_3 \times \nabla v_1)$ and $e_3 = \lambda_1 \lambda_2 (\nabla v_1 \times \nabla v_2)$

$$\begin{aligned} \therefore \nabla \cdot (A_1 e_1) &= \nabla \cdot A_1 \lambda_2 \lambda_3 (\nabla v_2 \times \nabla v_3) \\ &= \nabla (A_1 \lambda_2 \lambda_3) (\nabla v_2 \times \nabla v_3) + A_1 \lambda_2 \lambda_3 \{ \nabla \cdot (\nabla v_2 \times \nabla v_3) \} \\ &= \nabla (A_1 \lambda_2 \lambda_3) \cdot \frac{e_1}{\lambda_2 \lambda_3} + A_1 \lambda_2 \lambda_3 \left\{ \nabla \cdot \frac{e_1}{\lambda_2 \lambda_3} \right\} \end{aligned} \quad \dots (D)$$

(Using (C))

$$\text{Now, } \nabla \cdot \frac{e_1}{\lambda_2 \lambda_3} = \left[\frac{e_1}{\lambda_1} \frac{\partial}{\partial v_1} + \frac{e_2}{\lambda_2} \frac{\partial}{\partial v_2} + \frac{e_3}{\lambda_3} \frac{\partial}{\partial v_3} \right] \cdot \frac{e_1}{\lambda_2 \lambda_3}$$

(Writing the expression for ∇)

$$= \frac{1}{\lambda_1} \frac{\partial}{\partial v_1} \left(\frac{1}{\lambda_2 \lambda_3} \right) + 0 + 0 \quad [\because e_1 \cdot e_1 = 1, e_1 \cdot e_2 = 0, e_1 \cdot e_3 = 0]$$

$$= 0 \quad \left[\frac{1}{\lambda_2 \lambda_3} \text{ is constant} \right]$$

$$\begin{aligned} \therefore (D) \Rightarrow \nabla \cdot (A_1 e_1) &= \nabla (A_1 \lambda_2 \lambda_3) \cdot \frac{e_1}{\lambda_2 \lambda_3} \\ &= \frac{1}{\lambda_1 \lambda_2 \lambda_3} \frac{\partial}{\partial v_1} (A_1 \lambda_2 \lambda_3) \end{aligned} \quad \dots (E)$$

(Operating ∇)

$$||| \text{ly, we get, } \nabla \cdot A_2 e_2 = \frac{1}{\lambda_1 \lambda_2 \lambda_3} \frac{\partial}{\partial v_2} (A_2 \lambda_3 \lambda_1) \quad \dots (F)$$

$$\text{and } \nabla \cdot A_3 e_3 = \frac{1}{\lambda_1 \lambda_2 \lambda_3} \frac{\partial}{\partial v_3} (A_3 \lambda_1 \lambda_2) \quad \dots (G)$$

Hence, if $A = A_1 e_1 + A_2 e_2 + A_3 e_3$,

$$\begin{aligned} \text{Div } A &= \nabla \cdot (A_1 e_1 + A_2 e_2 + A_3 e_3) \\ &= \nabla \cdot (A_1 e_1) + \nabla \cdot (A_2 e_2) + \nabla \cdot (A_3 e_3) \\ &= \frac{1}{\lambda_1 \lambda_2 \lambda_3} \left[\frac{\partial}{\partial v_1} (A_1 \lambda_2 \lambda_3) + \frac{\partial}{\partial v_2} (A_2 \lambda_3 \lambda_1) + \frac{\partial}{\partial v_3} (A_3 \lambda_1 \lambda_2) \right] \quad [\text{adding R.H.S of (E), (F), (G)}] \end{aligned}$$

7.9.6 (4) Prove that in orthogonal curvilinear coordinates,

$$\nabla \times (A_1 e_1) = \frac{e_2}{\lambda_3 \lambda_1} \frac{\partial}{\partial v_3} (A_1 \lambda_1) - \frac{e_3}{\lambda_1 \lambda_2} \frac{\partial}{\partial v_2} (A_1 \lambda_1) \text{ and hence,}$$

$$\text{Curl } A = \nabla \times A = \frac{1}{\lambda_1 \lambda_2 \lambda_3} \begin{vmatrix} \lambda_1 e_1 & \lambda_2 e_2 & \lambda_3 e_3 \\ \partial/\partial v_1 & \partial/\partial v_2 & \partial/\partial v_3 \\ A_1 \lambda_1 & A_2 \lambda_2 & A_3 \lambda_3 \end{vmatrix}$$

Proof: Let $A = A_1 e_1 + A_2 e_2 + A_3 e_3$,

$\therefore \nabla v_1 = \frac{e_1}{\lambda_1}$, we have,

$$\begin{aligned}
 \nabla \times (A_1 e_1) &= \nabla \times (A_1 \lambda_1 \nabla v_1) \\
 &= \nabla (A_1 \lambda_1) \times \nabla v_1 + (A_1 \lambda_1) (\nabla \times \nabla v_1) \\
 &= \nabla (A_1 \lambda_1) \times \frac{e_1}{\lambda_1} + 0 \quad (\text{from 7.5.2 (i) curl grad} = \nabla v_1 = 0) \\
 &= \left[\frac{e_1}{\lambda_1} \frac{\partial}{\partial v_1} (A_1 \lambda_1) + \frac{e_2}{\lambda_2} \frac{\partial}{\partial v_2} (A_1 \lambda_1) + \frac{e_3}{\lambda_3} \frac{\partial}{\partial v_3} (A_1 \lambda_1) \right] \times \frac{e_1}{\lambda_1} \\
 &\quad (\text{substituting for } \nabla) \\
 &= \frac{e_2}{\lambda_3 \lambda_1} \frac{\partial}{\partial v_3} (A_1 \lambda_1) - \frac{e_3}{\lambda_1 \lambda_2} \frac{\partial}{\partial v_2} (A_1 \lambda_1) \quad [\because e_1 \times e_1 = 0, e_2 \times e_1 = -e_3, e_3 \times e_1 = e_2] \\
 &\quad \dots (i)
 \end{aligned}$$

Similarly, it can be shown that

$$\nabla \times (A_2 e_2) = \frac{e_3}{\lambda_1 \lambda_2} \frac{\partial}{\partial v_1} (A_2 \lambda_2) - \frac{e_1}{\lambda_2 \lambda_3} \frac{\partial}{\partial v_3} (A_2 \lambda_2) \quad \dots (ii)$$

and

$$\nabla \times (A_3 e_3) = \frac{e_1}{\lambda_2 \lambda_3} \frac{\partial}{\partial v_2} (A_3 \lambda_3) - \frac{e_2}{\lambda_3 \lambda_1} \frac{\partial}{\partial v_1} (A_3 \lambda_3) \quad \dots (iii)$$

$$\therefore \text{curl } A = \nabla \times A = \nabla \times (A_1 e_1 + A_2 e_2 + A_3 e_3)$$

$$\begin{aligned}
 &= \nabla \times (A_1 e_1) + \nabla \times (A_2 e_2) + \nabla \times (A_3 e_3) \\
 &= \frac{e_1}{\lambda_2 \lambda_3} \left\{ \frac{\partial}{\partial v_2} (A_3 \lambda_3) - \frac{\partial}{\partial v_3} (A_2 \lambda_2) \right\} + \frac{e_2}{\lambda_3 \lambda_1} \left\{ \frac{\partial}{\partial v_3} (A_1 \lambda_1) - \frac{\partial}{\partial v_1} (A_3 \lambda_3) \right\} \\
 &\quad + \frac{e_3}{\lambda_1 \lambda_2} \left\{ \frac{\partial}{\partial v_1} (A_2 \lambda_2) - \frac{\partial}{\partial v_2} (A_1 \lambda_1) \right\} \quad (\text{Adding R.H.S. of (i), (ii) and (iii)})
 \end{aligned}$$

$$= \frac{1}{\lambda_1 \lambda_2 \lambda_3} \begin{vmatrix} \lambda_1 e_1 & \lambda_2 e_2 & \lambda_3 e_3 \\ \partial/\partial v_1 & \partial/\partial v_2 & \partial/\partial v_3 \\ A_1 \lambda_1 & A_2 \lambda_2 & A_3 \lambda_3 \end{vmatrix}$$

Ex. 7.9.6. (5) Derive an expression for $\nabla^2\phi$ in orthogonal curvilinear coordinates.

Sol: $\nabla\phi = \frac{e_1}{\lambda_1} \frac{\partial\phi}{\partial v_1} + \frac{e_2}{\lambda_2} \frac{\partial\phi}{\partial v_2} + \frac{e_3}{\lambda_3} \frac{\partial\phi}{\partial v_3}$ (from 7.9.6(1))

Let $A = \nabla\phi = A_1 e_1 + A_2 e_2 + A_3 e_3$, so that,

$$A_1 = \frac{1}{\lambda_1} \frac{\partial\phi}{\partial v_1}, \quad A_2 = \frac{1}{\lambda_2} \frac{\partial\phi}{\partial v_2}, \quad A_3 = \frac{1}{\lambda_3} \frac{\partial\phi}{\partial v_3}$$

$$\therefore \nabla^2\phi = \nabla \cdot \nabla\phi = \nabla \cdot A$$

$$= \frac{1}{\lambda_1 \lambda_2 \lambda_3} \left[\frac{\partial}{\partial v_1} (A_1 \lambda_2 \lambda_3) + \frac{\partial}{\partial v_2} (A_2 \lambda_3 \lambda_1) + \frac{\partial}{\partial v_3} (A_3 \lambda_1 \lambda_2) \right]$$

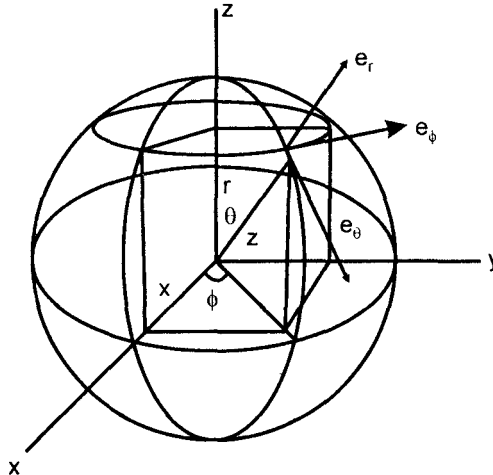
(from 7.9.6(3))

$$= \frac{1}{\lambda_1 \lambda_2 \lambda_3} \left[\frac{\partial}{\partial v_1} \left(\frac{\lambda_2 \lambda_3}{\lambda_1} \frac{\partial\phi}{\partial v_1} \right) + \frac{\partial}{\partial v_2} \left(\frac{\lambda_3 \lambda_1}{\lambda_2} \frac{\partial\phi}{\partial v_2} \right) + \frac{\partial}{\partial v_3} \left(\frac{\lambda_1 \lambda_2}{\lambda_3} \frac{\partial\phi}{\partial v_3} \right) \right]$$

Note: In the expressions for ∇ , $\text{grad}\phi$, $\text{Div } A$, $\text{curl } A$, and $\nabla^2\phi$ derived in 7.9.6(1) to 7.9.6(5), if we put $v_1 = x$, $v_2 = y$, $v_3 = z$; $\lambda_1 = \lambda_2 = \lambda_3 = 1$, and $e_1 = i$, $e_2 = j$, $e_3 = k$; we obtain the corresponding expressions in (rectangular) Cartesian coordinate system. [The reader is advised to verify]

7.9.7 Spherical Polar coordinates (Spherical Coordinates)

7.9.7(1) If we take $v_1 = r$, $v_2 = \theta$, $v_3 = \phi$, the curvilinear coordinate system becomes a “spherical polar coordinate system”.



The equations of transformation are,

$$\left. \begin{aligned} x &= r \sin \theta \cos \phi, \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \theta \end{aligned} \right\} \quad \dots (i)$$

where $r \geq 0$, $0 \leq \phi < 2\pi$, and $0 \leq \theta \leq \pi$.

7.9.7(2) Coordinate surfaces and coordinate curves.

(i) The coordinate surfaces (level surfaces) in a spherical coordinate system are
 $r = c_1$ which are spheres with centre at origin [or origin itself if $c_1 = 0$]

$\theta = c_2$ which are cones with vertex at origin

(lines if $c_2 = 0$ or π and xy -plane if $c_2 = \pi/2$)

$\phi = c_3$ which are planes through the z -axis

(ii) The coordinates curves are:

(a) Intersection of $\theta = c_2$ and $\phi = c_3$ is the r -curve and it is a line

(b) Intersection of $r = c_1$ and $\phi = c_3$ is the θ -curve. It is semi circle if $c_1 \neq 0$

(c) Intersection of $r = c_1$ and $\theta = c_2$ is the ϕ -curve. It is a circle (or point).

7.9.7(3) Scale factors:

$$\mathbf{r} = xi + yj + zk$$

$$= (r \sin \theta \cos \phi)i + (r \sin \theta \sin \phi)j + (r \cos \theta)k$$

$$\frac{\partial \mathbf{r}}{\partial v_1} = \frac{\partial \mathbf{r}}{\partial r} = (\sin \theta \cos \phi)i + (\sin \theta \sin \phi)j + (\cos \theta)k$$

$$\begin{aligned} \lambda_1 &= \left| \frac{\partial \mathbf{r}}{\partial r} \right| = \sqrt{(\sin \theta \cos \phi)^2 + (\sin \theta \sin \phi)^2 + (\cos \theta)^2} \\ &= 1 \end{aligned}$$

$$\frac{\partial \mathbf{r}}{\partial v_2} = \frac{\partial \mathbf{r}}{\partial \theta} = (r \cos \theta \cos \phi)i + (r \cos \theta \sin \phi)j - (r \sin \theta)k$$

$$\begin{aligned} \therefore \lambda_2 &= \left| \frac{\partial \mathbf{r}}{\partial \theta} \right| = \sqrt{(r \cos \theta \cos \phi)^2 + (r \cos \theta \sin \phi)^2 + (-r \sin \theta)^2} \\ &= r \end{aligned}$$

$$\frac{\partial r}{\partial v_3} = \frac{\partial r}{\partial \phi} = (-r \sin \theta \sin \phi)i + (r \sin \theta \cos \phi)j$$

$$\begin{aligned}\lambda_3 &= \left| \frac{\partial r}{\partial \phi} \right| = \sqrt{(-r \sin \theta \sin \phi)^2 + (r \sin \theta \cos \phi)^2} \\ &= r \sin \theta\end{aligned}$$

∴ The scale factors of spherical coordinate system are

$$\lambda_1 = 1, \quad \lambda_2 = r, \quad \lambda_3 = r \sin \theta.$$

7.9.7(4) Base vectors: The base vectors e_1, e_2, e_3 are taken for the sake of convenience as e_r, e_θ, e_ϕ respectively. They are,

$$e_r = e_1 = \frac{1}{\lambda_1} \frac{\partial r}{\partial v_1} = \frac{1}{\lambda_1} \frac{\partial r}{\partial r} = (\sin \theta \cos \phi)i + (\sin \theta \sin \phi)j + (\cos \theta)k$$

$$\begin{aligned}e_\theta = e_2 &= \frac{1}{\lambda_2} \frac{\partial r}{\partial \theta} = \frac{1}{r} [(r \cos \theta \cos \phi)i + (r \cos \theta \sin \phi)j - (r \sin \theta)k] \\ &= (\cos \theta \cos \phi)i + (\cos \theta \sin \phi)j - (\sin \theta)k\end{aligned}$$

$$\begin{aligned}\text{and } e_\phi = e_3 &= \frac{1}{\lambda_3} \frac{\partial r}{\partial \phi} = \frac{1}{r \sin \theta} (-r \sin \theta \sin \phi)i + (r \sin \theta \cos \phi)j \\ &= (-\sin \phi)i + (\cos \phi)j\end{aligned}$$

From the above expressions, it can be observed that

$$e_r \cdot e_\theta = e_\theta \cdot e_\phi = e_r \cdot e_\phi = 0$$

$$\text{and } e_r \times e_\theta = e_\phi; \quad e_\theta \times e_\phi = e_r; \quad e_\phi \times e_r = e_\theta;$$

[students are advised to verify the above equations]

which show that “the spherical polar coordinate system is a right-handed orthogonal coordinate system”.

7.9.7(5) Arc length, Area element, and volume element in spherical coordinates:

$$\lambda_1 = 1, \quad \lambda_2 = r, \quad \lambda_3 = r \sin \theta$$

$$\text{i) } ds^2 = \lambda_1^2 dv_1^2 + \lambda_2^2 dv_2^2 + \lambda_3^2 dv_3^2 \quad (\text{from 7.9.5(1)})$$

$$= 1^2(dr)^2 + r^2(d\theta)^2 + (r \sin \theta)^2(d\phi)^2$$

gives the arc length ‘ds’.

ii) From 7.9.5(2), $ds_1 = \lambda_2 \lambda_3 dv_2 dv_3 \Rightarrow ds_r = r^2 \sin \theta d\theta d\phi$

$$ds_2 = \lambda_3 \lambda_1 dv_3 dv_1 \Rightarrow ds_\theta = r \sin \theta dr d\phi$$

and $ds_3 = \lambda_1 \lambda_2 dv_1 dv_2 \Rightarrow ds_\phi = r dr d\theta$

ds_r, ds_θ, ds_ϕ are the area elements on the respective surfaces

$$r = c_1, \quad \theta = c_2, \quad \text{and} \quad \phi = c_3.$$

iii) From 7.9.5(3), $dv = \lambda_1 \lambda_2 \lambda_3 dv_1 dv_2 dv_3$
 $= 1.r.(r \sin \theta) dr d\theta d\phi$

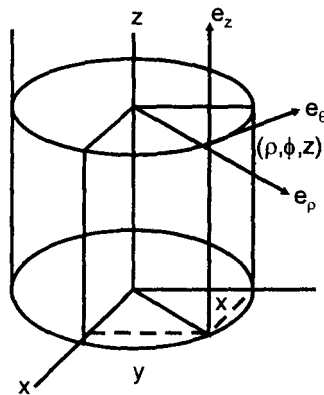
i.e., $dv = r^2 \sin \theta dr d\theta d\phi$ is the volume element.

7.9.8 Cylindrical polar coordinates (cylindrical coordinates)

7.9.8(1) In this case, we take,

$$v_1 = \rho, v_2 = \theta, v_3 = z.$$

Then the curvilinear coordinate system becomes a 'cylindrical coordinate system'.



The equations of transformation are

$$x = \rho \cos \theta, \quad y = \rho \sin \theta, \quad z = z.$$

where $\rho \geq 0, \quad 0 \leq \theta < 2\pi, \quad -\infty < z < \infty.$

7.9.8(2) Scale factors:

$$\begin{aligned} \mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ &= (\rho \cos \theta)\mathbf{i} + (\rho \sin \theta)\mathbf{j} + z\mathbf{k} \end{aligned}$$

$$\frac{\partial \mathbf{r}}{\partial v_1} = \frac{\partial \mathbf{r}}{\partial \rho} = (\cos \theta) \mathbf{i} + (\sin \theta) \mathbf{j}$$

$$\lambda_1 = \left| \frac{\partial \mathbf{r}}{\partial \rho} \right| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1$$

$$\frac{\partial \mathbf{r}}{\partial v_2} = \frac{\partial \mathbf{r}}{\partial \theta} = (-\rho \sin \theta) \mathbf{i} + (\rho \cos \theta) \mathbf{j}$$

$$\lambda_2 = \left| \frac{\partial \mathbf{r}}{\partial \theta} \right| = \sqrt{(-\rho \sin \theta)^2 + (\rho \cos \theta)^2} = \rho$$

$$\frac{\partial \mathbf{r}}{\partial v_3} = \frac{\partial \mathbf{r}}{\partial z} = \mathbf{k}; \quad \lambda_3 = \left| \frac{\partial \mathbf{r}}{\partial z} \right| = 1$$

$\therefore$ scale factors are 1, ρ , 1.

7.9.8(3) Coordinate surfaces and coordinate curves:

(i) The coordinate surfaces (level surfaces) in a cylindrical coordinate system are.

$\rho = c_1$, which are cylinders coaxial with z -axis (or z -axis itself if $c_1 = 0$)

$\theta = c_2$, which are planes through z -axis

and $z = c_3$, which are planes $\perp^r$ to z -axis

(ii) Coordinate curves are:

(a) Intersection of $\theta = c_2$ and $z = c_3$ is the ρ -curve which is a straight line.

(b) Intersection of $\rho = c_1$ and $z = c_3$ is the θ -curve which is a circle (or Point)

(c) Intersection of $\rho = c_1$ and $\theta = c_2$ is the z -curve which is a straight line.

7.9.8(4) Base vectors: $(\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_z)$

$$\mathbf{e}_\rho = \frac{1}{\lambda_1} \frac{\partial \mathbf{r}}{\partial \rho} = (\cos \theta) \mathbf{i} + (\sin \theta) \mathbf{j}$$

$$\begin{aligned} \mathbf{e}_\theta &= \frac{1}{\lambda_2} \frac{\partial \mathbf{r}}{\partial \theta} = \frac{1}{\rho} [(-\rho \sin \theta) \mathbf{i} + (\rho \cos \theta) \mathbf{j}] \\ &= (-\sin \theta) \mathbf{i} + (\cos \theta) \mathbf{j} \end{aligned}$$

$$e_z = \frac{1}{\lambda_3} \frac{\partial \mathbf{r}}{\partial z} = \mathbf{k};$$

observe that, $e_\rho \times e_\theta = e_\theta \cdot e_z = e_z \cdot e_\rho = 0$

$$\text{and } e_\rho \times e_\theta = e_z, \quad e_\theta \times e_z = e_\rho; \quad e_z \times e_\rho = e_\theta$$

which shows that cylindrical coordinate system is a righthanded orthogonal coordinate system.

7.9.8(5) Arc length, Area element and volume element in cylindrical coordinate system:

$$\begin{aligned} \text{(i)} \quad ds^2 &= \lambda_1^2 dv_1^2 + \lambda_2^2 dv_2^2 + \lambda_3^2 dv_3^2 \\ &= 1^2 (d\rho)^2 + \rho^2 (d\theta)^2 + 1^2 (dz)^2 \\ &= (d\rho)^2 + \rho^2 (d\theta)^2 + (dz)^2 \end{aligned}$$

gives arc length 'ds'.

$$\text{(ii)} \quad ds_1 = \lambda_2 \lambda_3 dv_2 dv_3 \Rightarrow ds_\rho = r^2 \sin \theta d\theta \cdot dz$$

$$ds_2 = \lambda_3 \lambda_1 dv_3 dv_1 \Rightarrow ds_\theta = d\rho \cdot dz$$

$$\text{and } ds_3 = \lambda_1 \lambda_2 dv_1 dv_2 \Rightarrow ds_z = \rho d\rho d\theta$$

ds_ρ , ds_θ , ds_z represent the area elements on the surfaces $\rho = c_1$, $\theta = c_2$, and $z = c_3$ respectively.

(iii) Volume element 'dv':

$$\begin{aligned} dv &= \lambda_1 \lambda_2 \lambda_3 dv_1 dv_2 dv_3 \\ &= 1 \cdot \rho \cdot d\rho \cdot d\theta \cdot dz = \rho d\rho \cdot d\theta \cdot dz. \end{aligned}$$

7.9.9: EXPRESSIONS FOR GRADIENT, DIVERGENCE, CURL, LAPLACIAN AND JACOBIAN IN SPHERICAL AND CYLINDRICAL COORDINATES:

7.9.9(1) Gradient of a scalar function 'f'

$$\text{Grad } f = \nabla f = \frac{1}{\lambda_1} \frac{\partial f}{\partial v_1} e_1 + \frac{1}{\lambda_2} \frac{\partial f}{\partial v_2} e_2 + \frac{1}{\lambda_3} \frac{\partial f}{\partial v_3} e_3 \quad (\text{from 7.9.6(1)})$$

a) *spherical coordinates*:

$$\lambda_1 = 1, \quad \lambda_2 = r, \quad \lambda_3 = r \sin \theta$$

$$e_1 = e_r = (\sin \theta \cos \phi)i + (\sin \theta \sin \phi)j + (\cos \theta)k$$

$$e_2 = e_\theta = (\cos \theta \cos \phi)i + (\cos \theta \sin \phi)j - (\sin \theta)k$$

$$e_3 = e_\phi = (-\sin \phi)i + (\cos \phi)j$$

$$\therefore \nabla f = \frac{\partial f}{\partial r}e_r + \frac{1}{r} \frac{\partial f}{\partial \theta}e_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi}e_\phi$$

b) *cylindrical coordinates*:

$$\lambda_1 = 1, \quad \lambda_2 = \rho, \quad \lambda_3 = 1$$

$$e_1 = e_\rho = (\cos \theta)i + (\sin \theta)j$$

$$e_2 = e_\theta = (-\sin \theta)i + (\cos \theta)j$$

$$e_3 = e_z = k$$

$$\therefore \nabla f = \frac{\partial f}{\partial \rho}e_\rho + \frac{\partial f}{\partial \theta}e_\theta + \frac{\partial f}{\partial z}e_z$$

7.9.9(2) Divergence of a vector function

If $A = A_1e_1 + A_2e_2 + A_3e_3$, the expression for Div A is given by

$$\text{Div } A = \frac{1}{\lambda_1\lambda_2\lambda_3} \left[\frac{\partial}{\partial v_1}(A_1\lambda_2\lambda_3) + \frac{\partial}{\partial v_2}(A_2\lambda_3\lambda_1) + \frac{\partial}{\partial v_3}(A_3\lambda_1\lambda_2) \right] \quad (\text{from 7.9.6(3)})$$

(a) *In spherical coordinates*

$$\lambda_1 = 1, \quad \lambda_2 = r, \quad \lambda_3 = r \sin \theta, \quad v_1 = r, \quad v_2 = \theta, \quad v_3 = \phi$$

$$\therefore \text{Div } A = \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r}(r^2 \sin \theta A_1) + \frac{\partial}{\partial \theta}(r \sin \theta A_2) + \frac{\partial}{\partial \phi}(r A_3) \right]$$

(b) *In cylindrical coordinates*

$$\lambda_1 = 1, \quad \lambda_2 = \rho, \quad \lambda_3 = 1, \quad v_1 = \rho, \quad v_2 = \theta, \quad v_3 = z$$

$$\therefore \text{Div } A = \frac{1}{\rho} \left[\frac{\partial}{\partial \rho}(\rho A_1) + \frac{\partial}{\partial \theta}(A_2) + \frac{\partial}{\partial z}(\rho A_3) \right]$$

7.9.9(3) Laplacian (∇^2): Let f be a scalar function. Then the expression for $\nabla^2 f$ in curvilinear coordinates is given by

$$\nabla^2 f = \frac{1}{\lambda_1 \lambda_2 \lambda_3} \left[\frac{\partial}{\partial v_1} \left(\frac{\lambda_2 \lambda_3}{\lambda_1} \frac{\partial f}{\partial v_1} \right) + \frac{\partial}{\partial v_2} \left(\frac{\lambda_3 \lambda_1}{\lambda_2} \frac{\partial f}{\partial v_2} \right) + \frac{\partial}{\partial v_3} \left(\frac{\lambda_1 \lambda_2}{\lambda_3} \frac{\partial f}{\partial v_3} \right) \right]$$

[from 7.9.6(5)]

a) *In spherical coordinates:*

$$\lambda_1 = 1, \quad \lambda_2 = r, \quad \lambda_3 = r \sin \theta, \quad v_1 = r, \quad v_2 = \theta, \quad v_3 = \phi$$

$$\begin{aligned} \therefore \nabla^2 f &= \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} \left(r^2 \sin \theta \frac{\partial f}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{\partial}{\partial \phi} \left(\frac{1}{\sin \theta} \frac{\partial f}{\partial \phi} \right) \right] \\ &= \frac{\partial^2 f}{\partial r^2} + \frac{2}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial f}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} \end{aligned}$$

(after differentiation and simplification)

b) *In cylindrical coordinates:*

$$\lambda_1 = 1, \quad \lambda_2 = \rho, \quad \lambda_3 = 1, \quad v_1 = \rho, \quad v_2 = \theta, \quad v_3 = z$$

$$\begin{aligned} \therefore \nabla^2 f &= \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{\partial}{\partial \theta} \left(\frac{1}{\rho} \frac{\partial f}{\partial \theta} \right) + \frac{\partial}{\partial z} \left(\rho \frac{\partial f}{\partial z} \right) \right] \\ &= \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \end{aligned}$$

(after differentiation and simplification)

7.9.9(4) Curl of a vector function $\mathbf{A}$.

$$\text{If } \mathbf{A} = A_1 \mathbf{e}_1 + A_2 \mathbf{e}_2 + A_3 \mathbf{e}_3,$$

$$\text{curl } \mathbf{A} = \frac{1}{\lambda_1 \lambda_2 \lambda_3} \begin{vmatrix} \lambda_1 \mathbf{e}_1 & \lambda_2 \mathbf{e}_2 & \lambda_3 \mathbf{e}_3 \\ \partial/\partial v_1 & \partial/\partial v_2 & \partial/\partial v_3 \\ A_1 \lambda_1 & A_2 \lambda_2 & A_3 \lambda_3 \end{vmatrix} \quad (\text{from 7.9.6(4)})$$

a) *In spherical coordinates:*

$$\lambda_1 = 1, \quad \lambda_2 = r, \quad \lambda_3 = r \sin \theta, \quad v_1 = r, \quad v_2 = \theta, \quad v_3 = \phi$$

$$\therefore \text{curl } \mathbf{A} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} e_r & r e_\theta & (r \sin \theta) e_\phi \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial \phi \\ A_1 & r A_2 & (r \sin \theta) A_3 \end{vmatrix}$$

b) *In cylindrical coordinates:*

$$\lambda_1 = 1, \quad \lambda_2 = \rho, \quad \lambda_3 = 1, \quad v_1 = \rho, \quad v_2 = \theta, \quad v_3 = z$$

$$\therefore \text{curl } \mathbf{A} = \frac{1}{\rho} \begin{vmatrix} e_\rho & \rho e_\theta & e_z \\ \partial/\partial \rho & \partial/\partial \theta & \partial/\partial z \\ A_1 & \rho A_2 & A_3 \end{vmatrix}$$

7.9.9(5) Jacobian:

(i) *Orthogonal curvilinear coordinates:*

Let $\mathbf{x} = \mathbf{x}(v_1, v_2, v_3)$; $\mathbf{y} = \mathbf{y}(v_1, v_2, v_3)$; $\mathbf{z} = \mathbf{z}(v_1, v_2, v_3)$ represent an orthogonal curvilinear coordinate system

$$\text{Jacobian} = J \left(\frac{\mathbf{x}, \mathbf{y}, \mathbf{z}}{v_1, v_2, v_3} \right) = \begin{vmatrix} \partial \mathbf{x} / \partial v_1 & \partial \mathbf{y} / \partial v_1 & \partial \mathbf{z} / \partial v_1 \\ \partial \mathbf{x} / \partial v_2 & \partial \mathbf{y} / \partial v_2 & \partial \mathbf{z} / \partial v_2 \\ \partial \mathbf{x} / \partial v_3 & \partial \mathbf{y} / \partial v_3 & \partial \mathbf{z} / \partial v_3 \end{vmatrix}$$

since $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\frac{\partial \mathbf{r}}{\partial v_1} = \frac{\partial x}{\partial v_1} \mathbf{i} + \frac{\partial y}{\partial v_1} \mathbf{j} + \frac{\partial z}{\partial v_1} \mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial v_2} = \frac{\partial x}{\partial v_2} \mathbf{i} + \frac{\partial y}{\partial v_2} \mathbf{j} + \frac{\partial z}{\partial v_2} \mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial v_3} = \frac{\partial x}{\partial v_3} \mathbf{i} + \frac{\partial y}{\partial v_3} \mathbf{j} + \frac{\partial z}{\partial v_3} \mathbf{k}$$

$$\therefore J \left(\frac{\mathbf{x}, \mathbf{y}, \mathbf{z}}{v_1, v_2, v_3} \right) = \left[\frac{\partial \mathbf{r}}{\partial v_1}, \frac{\partial \mathbf{r}}{\partial v_2}, \frac{\partial \mathbf{r}}{\partial v_3} \right] \quad [\text{Scalar Triple Product}]$$

$$= [\lambda_1 e_1, \lambda_2 e_2, \lambda_3 e_3] = \lambda_1 \lambda_2 \lambda_3 [e_1 e_2 e_3] = \lambda_1 \lambda_2 \lambda_3 \quad \{ \because [e_1 e_2 e_3] = 1 \}$$

(ii) *Spherical coordinates:*

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$z = r \cos \theta$, are the equations of transformation of spherical coordinate system.

$$\text{Jacobian} = J \left(\frac{x, y, z}{r, \theta, \phi} \right)$$

$$= \begin{vmatrix} \partial x / \partial r & \partial y / \partial r & \partial z / \partial r \\ \partial x / \partial \theta & \partial y / \partial \theta & \partial z / \partial \theta \\ \partial x / \partial \phi & \partial y / \partial \phi & \partial z / \partial \phi \end{vmatrix}$$

$$= \begin{vmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ r \cos \theta \cos \phi & r \cos \theta \sin \phi & -r \sin \theta \\ -r \sin \theta \sin \phi & r \sin \theta \cos \phi & 0 \end{vmatrix}$$

$$= r^2 \sin \theta \quad (\text{or}) [J = \lambda_1 \lambda_2 \lambda_3 = 1 \cdot r \cdot r \sin \theta = r^2 \sin \theta]$$

(iii) *Cylindrical coordinates:*

$x = \rho \cos \theta, y = \rho \sin \theta, z = z$, are the equations of transformation.

$$\text{Jacobian} = J \left(\frac{x, y, z}{\rho, \theta, z} \right) = \begin{vmatrix} \partial x / \partial \rho & \partial y / \partial \rho & \partial z / \partial \rho \\ \partial x / \partial \theta & \partial y / \partial \theta & \partial z / \partial \theta \\ \partial x / \partial z & \partial y / \partial z & \partial z / \partial z \end{vmatrix}$$

$$= \begin{vmatrix} \cos \theta & \sin \theta & 0 \\ -\rho \sin \theta & \rho \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho$$

$$[(\text{or}) J = \lambda_1 \lambda_2 \lambda_3 = 1 \cdot \rho \cdot 1 = \rho]$$

The summary of important results of this chapter is given in a table in the next page for the convenience of the student.

	Cartesian Coordinates (x, y, z)	Spherical Coordinates (r, θ, ϕ)	Cylindrical Coordinates (ρ, θ, z)
1) Equations of transformation of coordinates	$x = x$ $y = y$ $z = z$	$x = r \sin\theta \cos\phi$ $y = r \sin\theta \sin\phi$ $z = r \cos\theta$	$x = \rho \cos\theta$ $y = \rho \sin\theta$ $z = z$
2) Scale factors	$\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$	$\lambda_1 = 1, \lambda_2 = r, \lambda_3 = r \sin\theta$	$\lambda_1 = 1, \lambda_2 = \rho, \lambda_3 = 1$
3) Base vectors	i, j, k	$\mathbf{e}_r = (\sin\theta \cos\phi)i + (\sin\theta \sin\phi)j + (\cos\theta)k$ $\mathbf{e}_\theta = (\cos\theta \cos\phi)i + (\cos\theta \sin\phi)j - (\sin\theta)k$ $\mathbf{e}_\phi = (-\sin\phi)i + (\cos\phi)j$	$\mathbf{e}_\rho = (\cos\theta)i + (\sin\theta)j$ $\mathbf{e}_\theta = (-\sin\theta)i + (\cos\theta)j$ $\mathbf{e}_z = k$
4) Jacobian (J)	$\frac{\partial(x, y, z)}{\partial(x, y, z)} = 1$	$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta$	$\frac{\partial(x, y, z)}{\partial(\rho, \theta, z)} = \rho$
5) (Arc Length) ²	$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$	$(ds)^2 = (dr)^2 + r^2(d\theta)^2 + r^2 \sin^2\theta(d\phi)^2$	$ds^2 = (d\rho)^2 + \rho^2(d\theta)^2 + (dz)^2$
6) Area elements on the coordinate surfaces	$ds_1 = dy dz$ $ds_2 = dz dx$ $ds_3 = dx dy$	$ds_r = r^2 \sin\theta d\theta d\phi$ $ds_\theta = r \sin\theta d\phi dr$ $ds_\phi = r dr d\theta$	$ds_\rho = \rho d\theta dz$ $ds_\theta = dz d\rho$ $ds_z = \rho d\rho d\theta$
7) Volume element (dv)	$dv = dx dy dz$	$dv = r^2 \sin\theta dr d\theta d\phi$	$dv = \rho d\rho d\theta dz$

	Cartesian Coordinates (x, y, z)	Spherical Coordinates (r, θ, ϕ)	Cylindrical Coordinates (ρ, θ, z)
8) Grad f	$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$	$\nabla f = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi$	$\nabla f = \frac{\partial f}{\partial \rho} \mathbf{e}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta + \frac{\partial f}{\partial z} \mathbf{e}_z$
9) Div A	$\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$	$\text{Div A} = \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} (r^2 \sin \theta A_1) + \frac{\partial}{\partial \theta} (r \sin \theta A_2) + \frac{\partial}{\partial \phi} (r A_3) \right]$	$\frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho A_1) + \frac{\partial}{\partial \theta} (A_2) + \frac{\partial}{\partial z} (\rho A_3) \right]$
10) Laplacian ($\nabla^2 f$)	$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{\partial^2 f}{\partial r^2} + \frac{2}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial f}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}$	$\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$
11) Curl A	$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix}$	$\frac{1}{r^2 \sin \theta} \begin{vmatrix} \mathbf{e}_r & r \mathbf{e}_\theta & (r \sin \theta) \mathbf{e}_\phi \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial \phi \\ A_1 & r A_2 & (r \sin \theta) A_3 \end{vmatrix}$	$\frac{1}{\rho} \begin{vmatrix} \mathbf{e}_\rho & \rho \mathbf{e}_\theta & \mathbf{e}_z \\ \partial/\partial \rho & \partial/\partial \theta & \partial/\partial z \\ A_1 & \rho A_2 & A_3 \end{vmatrix}$

Solved examples

Ex. 7.9.10 Obtain the equations of transformation from cylindrical to cartesian coordinates.

Sol: Equations of transformation from cartesian to cylindrical coordinates are

$$x = \rho \cos \theta \quad \dots (1), \quad y = \rho \sin \theta \quad \dots (2) \quad z = z \quad \dots (3)$$

from (1) and (2), we get, $\rho^2 = x^2 + y^2$

$$\Rightarrow \rho = \sqrt{x^2 + y^2}, \quad \rho > 0$$

from (1) and (2), we also get $\frac{y}{x} = \tan \theta$

$$\Rightarrow \theta = \tan^{-1} \left(\frac{y}{x} \right)$$

$\therefore$ The required equations are

$$\rho = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} \left(\frac{y}{x} \right), \quad z = z$$

Note: If a point lies on z-axis, $x = 0, y = 0 \Rightarrow \theta$ is indeterminate. These points on z-axis are known as singular points of the transformation.

Ex. 7.9.11 Represent the vector $A = 2xi - yj + z^2k$ in cylindrical polar coordinates.

Sol: The base vectors in cylindrical coordinates are

$$e_\rho = (\cos \theta)i + (\sin \theta)j \quad \dots (1)$$

$$e_\theta = (-\sin \theta)i + (\cos \theta)j \quad \dots (2)$$

$$e_z = k \quad \dots (3)$$

Solving (1) and (2), we get

$$i = (\cos \theta)e_\rho - (\sin \theta)e_\theta$$

$$j = (\sin \theta)e_\rho + (\cos \theta)e_\theta$$

$$\text{Then } A = 2[(\cos \theta)e_\rho - (\sin \theta)e_\theta]x - [(\sin \theta)e_\rho + (\cos \theta)e_\theta]y + z^2e_z.$$

$$\begin{aligned} A &= (2x \cos \theta - y \sin \theta)e_\rho - [2x \sin \theta + y \cos \theta]e_\theta + z^2e_z \\ &= (2\rho \cos^2 \theta - \rho \sin^2 \theta)e_\rho - [2\rho \sin \theta \cos \theta + \rho \sin \theta \cos \theta]e_\theta + z^2e_z \end{aligned}$$

(substituting $x = \rho \cos \theta, y = \rho \sin \theta$)

$$= \rho(2 \cos^2 \theta - \sin^2 \theta)e_\rho - (3\rho \sin \theta \cos \theta)e_\theta + z^2e_z$$

Aliter: Since cylindrical coordinates form an orthogonal coordinate system, we can write A

$$\text{as } A = a_1 e_\rho + a_2 e_\theta + a_3 e_z \quad \dots (1) \quad \text{where } a_1, a_2, a_3 \text{ are to be determined.}$$

From (1) we find,

$$A \cdot e_\rho = a_1, \quad A \cdot e_\theta = a_2; \quad A \cdot e_z = a_3 \quad \dots (2)$$

For cylindrical coordinates, we have,

$$x = \rho \cos \theta, \quad y = \rho \sin \theta, \quad z = z$$

$\therefore$ given A becomes,

$$A = (2\rho \cos \theta)i - (\rho \sin \theta)j + z^2 k \quad \dots (3)$$

$$\text{Also, } \left. \begin{aligned} e_\rho &= (\cos \theta)i + (\sin \theta)j \\ e_\theta &= (-\sin \theta)i + (\cos \theta)j \\ e_z &= k \end{aligned} \right\} \quad \dots (4)$$

using (3) and (4) in (2), we get

$$a_1 = 2\rho \cos^2 \theta - \rho \sin^2 \theta;$$

$$a_2 = -2\rho \sin \theta \cos \theta - \rho \sin \theta \cos \theta = -3\rho \sin \theta \cos \theta; \quad a_3 = z^2$$

$$\begin{aligned} \therefore A &= a_1 e_\rho + a_2 e_\theta + a_3 e_z \\ &= (2 \cos^2 \theta - \sin^2 \theta) \rho e_\rho - (3\rho \sin \theta \cos \theta) e_\theta + z^2 e_z \end{aligned}$$

Ex. 7.9.12 Represent the vector $A = xyi - zj + xzk$ in the spherical coordinate system.

Sol: Since the base vectors e_r, e_θ, e_ϕ are mutually orthogonal, we can write

$$A = a_1 e_r + a_2 e_\theta + a_3 e_\phi \quad \dots (1)$$

where a_1, a_2, a_3 are to be found.

$$\text{From (1) we get, } A \cdot e_r = a_1, \quad A \cdot e_\theta = a_2, \quad A \cdot e_\phi = a_3 \quad \dots (2)$$

For spherical coordinates, we have,

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta \quad \dots (3)$$

$$\therefore A = (r^2 \sin^2 \theta \sin \phi \cos \phi)i - (r \cos \theta)j + (r^2 \sin \theta \cos \theta \cos \phi)k \quad \dots (4)$$

Also we have,

$$\begin{aligned} e_r &= (\sin \theta \cos \phi)i + (\sin \theta \sin \phi)j + (\cos \theta)k \\ e_\theta &= (\cos \theta \cos \phi)i + (\cos \theta \sin \phi)j - (\sin \theta)k \\ e_\phi &= (-\sin \phi)i + (\cos \phi)j \end{aligned} \quad \dots (5)$$

Using (4) and (5) in (2), we get,

$$a_1 = r^2 \sin^3 \theta \sin \phi \cos^2 \phi - r \sin \theta \cos \theta \sin \phi + r^2 \sin \theta \cos^2 \theta \cos \phi.$$

$$a_2 = r^2 \sin^2 \theta \cos \theta \sin \phi \cos^2 \phi - r \cos^2 \theta \sin \phi - r^2 \sin^2 \theta \cos \theta \cos \phi$$

$$a_3 = -r^2 \sin^2 \theta \sin^2 \phi \cos \phi - r \cos \theta \cos \phi$$

Hence

$$\begin{aligned} A &= a_1 e_r + a_2 e_\theta + a_3 e_\phi \\ &= (r^2 \sin^3 \theta \sin \phi \cos^2 \phi - r \sin \theta \cos \theta \sin \phi + r^2 \sin \theta \cos^2 \theta \cos \phi) e_r \\ &\quad + (r^2 \sin^2 \theta \cos \theta \sin \phi \cos^2 \phi - r \cos^2 \theta \sin \phi - r^2 \sin^2 \theta \cos \theta \cos \phi) e_\theta \\ &\quad + (-r^2 \sin^2 \theta \sin^2 \phi \cos \phi - r \cos \theta \cos \phi) e_\phi \end{aligned}$$

Ex. 7.9.13 Prove the following:

$$(i) \dot{e}_r = \dot{\theta} e_\theta + (\sin \theta) \dot{\phi} e_\phi$$

$$(ii) \dot{e}_\theta = -\dot{\theta} e_r + (\cos \theta) \dot{\phi} e_\phi$$

$$(iii) \dot{e}_\phi = (-\sin \theta) \dot{\phi} e_r - (\cos \theta) \dot{\phi} e_\theta$$

where ‘ $\dot{\bullet}$ ’ denotes differentiation w.r.t. time ‘ t ’.

Sol: (i) We know that

$$e_r = (\sin \theta \cos \phi) i + (\sin \theta \sin \phi) j + (\cos \theta) k$$

$$\begin{aligned} \dot{e}_r &= \left\{ \sin \theta (-\dot{\sin \phi}) + \cos \theta \cos \phi \dot{\theta} \right\} i + \left\{ \cos \theta \sin \phi \dot{\theta} + \sin \theta \cos \phi \dot{\phi} \right\} j - (\sin \theta) \dot{\theta} k \\ &= \dot{\theta} \{ (\cos \theta \cos \phi) i + (\cos \theta \sin \phi) j - (\sin \theta) k \} + \dot{\phi} \{ (-\sin \theta \sin \phi) i + (\sin \theta \cos \phi) j \} \\ &= \dot{\theta} e_\theta + \sin \theta \dot{\phi} e_\phi \end{aligned}$$

$$(ii) e_\theta = (\cos \theta \cos \phi) i + (\cos \theta \sin \phi) j - (\sin \theta) k$$

$$\begin{aligned} \dot{e}_\theta &= \left(-\sin \theta \dot{\theta} \cos \phi - \cos \theta \sin \phi \dot{\theta} \right) i + \left(-\sin \theta \dot{\theta} \sin \phi + \cos \theta \cos \phi \dot{\phi} \right) j - \left(\cos \theta \dot{\theta} \right) k \\ &= -\dot{\theta} \{ (\sin \theta \cos \phi) i + (\sin \theta \sin \phi) j + (\cos \theta) k \} + \cos \theta \dot{\phi} \{ (-\sin \phi) i + (\cos \phi) j \} \\ &= -\dot{\theta} e_r + \cos \theta \dot{\phi} e_\phi \end{aligned}$$

$$(iii) e_\phi = (-\sin \phi) i + (\cos \phi) j$$

$$\dot{e}_\phi = -\cos \phi \dot{\phi} i - \sin \phi \dot{\phi} j \quad \dots (i)$$

$$(-\sin \theta) \dot{\phi} e_r - \cos \theta \dot{\phi} e_\theta$$

$$\begin{aligned} &= \dot{\phi} [-\sin \theta \{ \sin \theta \cos \phi i + \sin \theta \sin \phi j + \cos \theta k \} \\ &\quad - \cos \theta \{ \cos \theta \cos \phi i + \cos \theta \sin \phi j - \sin \theta k \}] \end{aligned}$$

$$\begin{aligned}
&= \dot{\phi} \left[-\cos \phi (\sin^2 \theta + \cos^2 \theta) i - \sin \phi (\sin^2 \theta + \cos^2 \theta) j \right] \\
&= -\cos \phi \cdot \dot{\phi} i - \sin \phi \cdot \dot{\phi} j \\
&= e_\phi \text{ from (i)}
\end{aligned}$$

Ex. 7.9.14 If e_ρ , e_θ are base vectors in cylindrical coordinates, show that

$$(i) \left(\dot{e}_\rho \right) = \dot{\theta} e_\theta \text{ and } (ii) \left(\dot{e}_\theta \right) = -\dot{\theta} e_\rho,$$

where ‘.’ denotes differentiation w.r.t. ‘t’

Sol: $e_\rho = (\cos \theta) i + (\sin \theta) j$

$$\begin{aligned}
\dot{e}_\rho &= -\sin \theta \cdot \dot{\theta} i + \cos \theta \cdot \dot{\theta} j \\
&= \dot{\theta} \{ (-\sin \theta) i + (\cos \theta) j \} \\
&= \dot{\theta} e_\theta
\end{aligned}$$

$$e_\theta = -\sin \theta i + \cos \theta j, \quad \dot{e}_\theta = -\dot{\theta} (\cos \theta) i - \dot{\theta} (\sin \theta) j = -\dot{\theta} e_\rho$$

Ex. 7.9.15 Express the velocity V and acceleration a of a particle in cylindrical coordinates.

Sol: In cartesian coordinates, the position vector $r = xi + yj + zk$.

$$\text{Velocity } V = \frac{dr}{dt} = \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k$$

$$\text{Acceleration } a = \frac{d^2 r}{dt^2} = \frac{d^2 x}{dt^2} i + \frac{d^2 y}{dt^2} j + \frac{d^2 z}{dt^2} k$$

In cylindrical coordinates, $x = \rho \cos \theta$, $y = \rho \sin \theta$, $z = z$ and

$$e_\rho = (\cos \theta) i + (\sin \theta) j, \quad e_\theta = (-\sin \theta) i + (\cos \theta) j, \quad e_z = k$$

$$\therefore i = \cos \theta e_\rho - \sin \theta e_\theta; \quad j = (\sin \theta) e_\rho + (\cos \theta) e_\theta \text{ (Solving above equations for } i \text{ and } j)$$

$$\therefore r = (\rho \cos \theta)(\cos \theta e_\rho - \sin \theta e_\theta) + \rho \sin \theta (\sin \theta e_\rho + \cos \theta e_\theta) + z e_z = \rho e_\rho + z e_z$$

$$\therefore V = \frac{dr}{dt} = \frac{d\rho}{dt} e_\rho + \rho \frac{de_\rho}{dt} + \frac{dz}{dt} e_z \quad \left[\because \frac{de_z}{dt} = \frac{d}{dt}(k) = 0 \right]$$

$$= \dot{\rho} e_\rho + \rho \left(\dot{\theta} e_\theta \right) + \dot{z} e_z \quad (\text{from Ex. 7.9.14})$$

$$\begin{aligned}
 \mathbf{a} &= \frac{dV}{dt} = \frac{d}{dt} [\dot{\rho} \mathbf{e}_\rho + \rho \dot{\theta} \mathbf{e}_\theta + \dot{z} \mathbf{e}_z] = \dot{\rho} \dot{\mathbf{e}}_\rho + \ddot{\rho} \mathbf{e}_\rho + \dot{\rho} \dot{\theta} \mathbf{e}_\theta + \rho \ddot{\theta} \mathbf{e}_\theta + \rho \dot{\theta} \dot{\mathbf{e}}_\theta + \dot{z} \dot{\mathbf{e}}_z \\
 &\quad (\text{since } \dot{\mathbf{e}}_z = 0) \\
 &= \dot{\rho} \dot{\mathbf{e}}_\theta + \dot{\rho} \dot{\mathbf{e}}_\rho + \dot{\rho} \dot{\theta} \mathbf{e}_\theta + \rho \ddot{\theta} \mathbf{e}_\theta + \rho \dot{\theta} (-\dot{\theta} \mathbf{e}_\rho) + \dot{z} \mathbf{e}_z, \text{ (using results of (ex 7.9.14))} \\
 &\quad [\ddot{\rho} - \rho \dot{\theta}^2] \mathbf{e}_\rho + [\rho \ddot{\theta} + 2\dot{\rho} \dot{\theta}] \mathbf{e}_\theta + \dot{z} \mathbf{e}_z
 \end{aligned}$$

Ex. 7.9.16 If $f = xyz$, find 'gradf' in (a) Cylindrical coordinates (b) Spherical coordinates

Solution:

In cylindrical coordinates (ρ, θ, z) , we have, $x = \rho \cos \theta, y = \rho \sin \theta, z = z$

$$\text{and} \quad \nabla f = \left(\frac{\partial f}{\partial \rho} \right) \mathbf{e}_\rho + \frac{1}{\rho} \left(\frac{\partial f}{\partial \theta} \right) \mathbf{e}_\theta + \left(\frac{\partial f}{\partial z} \right) \mathbf{e}_z \quad \dots(1)$$

$$f = xyz = \rho^2 z \sin \theta \cos \theta = \frac{\rho^2 z}{2} \sin 2\theta$$

$$\frac{\partial f}{\partial \rho} = \rho z \sin 2\theta; \frac{\partial f}{\partial \theta} = \rho^2 z \cos 2\theta; \frac{\partial f}{\partial z} = \frac{\rho^2}{2} \sin 2\theta$$

$$\therefore (1) \text{ gives, } \text{grad}f = \nabla f = (\rho z \sin 2\theta) \mathbf{e}_\rho + (\rho z \cos 2\theta) \mathbf{e}_\theta + \left(\frac{\rho^2}{2} \sin 2\theta \right) \mathbf{e}_z$$

(b) In spherical coordinates (r, θ, ϕ) , we have,

$x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta$, and

$$\nabla f = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi \quad \dots(2)$$

Here $f = xyz = 3r^2 \sin^2 \theta \cos \theta \cos \phi \sin \phi$

$$\therefore \frac{\partial f}{\partial \theta} = \frac{r^3}{2} \sin 2\phi \{-\sin^3 \theta + \cos \theta 2 \sin \theta \cos \theta\}$$

$$= \frac{r^3}{2} \sin 2\phi \{-\sin^3 \theta + 2 \sin \theta \cos^2 \theta\}$$

$$\frac{\partial f}{\partial \phi} = r^3 \sin^2 \theta \cos \theta \cos 2\phi$$

$\therefore$ from (2)

$$\begin{aligned} \nabla f = & \left(3r^2 \sin^2 \theta \cos \theta \sin \phi \cos \phi \right) e_r + \frac{r^2}{2} \sin 2\phi \{-\sin^3 \theta + 2 \sin \theta \cos^2 \theta\} e_\theta \\ & + \left(r^2 \sin \theta \cos \theta \cos 2\phi \right) e_\phi. \end{aligned}$$

E. 7.9.17 If $f = \rho z \cos \theta$ (in cylindrical coordinates) find ∇f .

Sol: $f = \rho z \cos \theta$.

$$\frac{\partial f}{\partial \rho} = z \cos \theta, \quad \frac{\partial f}{\partial \theta} = -\rho z \sin \theta, \quad \frac{\partial f}{\partial z} = \rho \cos \theta$$

$$\begin{aligned} \text{Hence } \nabla f &= \frac{\partial f}{\partial \rho} e_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \theta} e_\theta + \frac{\partial f}{\partial z} e_z \\ &= (z \cos \theta) e_\rho - (z \sin \theta) e_\theta + (\rho \cos \theta) e_z \end{aligned}$$

Ex. 7.9.18 If $f = r^2 \sin 2\theta \sin \phi$ in spherical coordinates, find ∇f .

Sol: $\frac{\partial f}{\partial r} = 2r \sin 2\theta \sin \phi$

$$\frac{\partial f}{\partial \theta} = 2r^2 \cos 2\theta \sin \phi$$

$$\frac{\partial f}{\partial \phi} = r^2 \sin 2\theta \cos \phi$$

$$\begin{aligned} \therefore \nabla f &= \frac{\partial f}{\partial r} e_r + \frac{1}{r} \frac{\partial f}{\partial \theta} e_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} e_\phi \\ &= (2r \sin 2\theta \sin \phi) e_r + (2r \cos 2\theta \sin \phi) e_\theta + (2r \cos \theta \cos \phi) e_\phi \end{aligned}$$

Ex. 7.9.19 Show that the vector field $A = z\{(\sin \theta) e_\rho + (\cos \theta) e_\theta\} - (\rho \cos \theta) e_z$, in cylindrical polar coordinates is solenoidal.

Sol: If $A = A_1 e_\rho + A_2 e_\theta + A_3 e_z$ then, $A_1 = z \sin \theta$; $A_2 = z \cos \theta$; $A_3 = -\rho \cos \theta$

$$\text{In cylindrical coordinates, } \text{Div } A = \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho A_1) + \frac{\partial}{\partial \theta} (A_2) + \frac{\partial}{\partial z} (\rho A_3) \right]$$

$$\begin{aligned}
 &= \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho z \sin \theta) + \frac{\partial}{\partial \theta} (z \cos \theta) + \frac{\partial}{\partial z} (-\rho^2 \cos \theta) \right] \\
 &= \frac{1}{\rho} [z \sin \theta - z \sin \theta + 0] = 0
 \end{aligned}$$

$\therefore A$ is solenoidal

Ex. 7.9.20 If $A = \left(r^2 \cos \theta e_r + \frac{1}{r} e_\theta + \frac{1}{r \sin \theta} e_\phi \right)$, in spherical coordinates, find $\text{Div } A$.

Sol: Here $A_1 = r^2 \cos \theta$, $A_2 = \frac{1}{r}$, $A_3 = \frac{1}{r \sin \theta}$

$$\begin{aligned}
 \text{Div } A &= \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} (r^2 \sin \theta A_1) + \frac{\partial}{\partial \theta} (r \sin \theta A_2) + \frac{\partial}{\partial \phi} (r A_3) \right] \\
 &= \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} \left(\frac{r^4}{2} \sin 2\theta \right) + \frac{\partial}{\partial \theta} (\sin \theta) + \frac{\partial}{\partial \phi} (\text{cosec } \theta) \right] \\
 &= \frac{1}{r^2 \sin \theta} \left[\frac{4r^3}{2} \sin 2\theta + \cos \theta + 0 \right] \\
 &= \frac{1}{r^2 \sin \theta} [4r^3 \sin \theta \cos \theta + \cos \theta] \\
 &= 4r \cos \theta + \frac{\cot \theta}{r^2}
 \end{aligned}$$

Ex. 7.9.21 If $f = \rho^2 z^2 \cos 2\theta$, show that $\nabla^2 f = 2\rho^2 \cos 2\theta$

Sol: In cylindrical coordinates,

$$\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \quad \dots (1)$$

$$f = \rho^2 z^2 \cos 2\theta$$

$$\frac{\partial f}{\partial \rho} = 2\rho z^2 \cos 2\theta; \quad \frac{\partial^2 f}{\partial \rho^2} = 2z^2 \cos 2\theta; \quad \frac{\partial f}{\partial \theta} = -2\rho^2 z^2 \sin 2\theta;$$

$$\frac{\partial^2 f}{\partial \theta^2} = -4\rho^2 z^2 \cos 2\theta;$$

$$\frac{\partial f}{\partial z} = 2\rho^2 z \cos 2\theta; \quad \frac{\partial^2 f}{\partial z^2} = 2\rho^2 \cos 2\theta$$

∴ From (1) we get,

$$\nabla^2 f = 2z^2 \cos 2\theta + 2z^2 \cos 2\theta - 4z^2 \cos 2\theta + 2\rho^2 \cos 2\theta = 2\rho^2 \cos 2\theta.$$

Ex. 7.9.22 Using spherical coordinates, show that $\nabla^2 r^n = n(n+1)r^{n-2}$ when n is a constant and $r \neq 0$ if $n < 2$

Sol: In spherical coordinates,

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial f}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} \quad \dots (1)$$

Here, $f = r^n$

$$\therefore \frac{\partial f}{\partial r} = nr^{n-1}; \quad \frac{\partial^2 f}{\partial r^2} = n(n-1)r^{n-2}$$

$$\frac{\partial f}{\partial \theta} = 0, \quad \frac{\partial^2 f}{\partial \theta^2} = 0, \quad \frac{\partial f}{\partial \phi} = 0, \quad \frac{\partial^2 f}{\partial \phi^2} = 0$$

$$\begin{aligned} \therefore (1) \Rightarrow \nabla^2 f &= \nabla^2 r^n = n(n-1)r^{n-2} + \frac{2}{r} nr^{n-1} \\ &= n(n-1)r^{n-2} + 2nr^{n-2} = n(n+1)r^{n-2}. \end{aligned}$$

Ex. 7.9.23 If $f = r^2 \sin 2\theta + \cos^2 \phi$, find $\nabla^2 f$.

Sol: $\frac{\partial f}{\partial r} = 2r \sin 2\theta; \quad \frac{\partial^2 f}{\partial r^2} = 2 \sin 2\theta$

$$\frac{\partial f}{\partial \theta} = 2r^2 \cos 2\theta, \quad \frac{\partial^2 f}{\partial \theta^2} = -4r^2 \sin 2\theta$$

$$\frac{\partial f}{\partial \phi} = -\sin 2\phi, \quad \frac{\partial^2 f}{\partial \phi^2} = -2 \cos 2\phi$$

$$\begin{aligned} \therefore \nabla^2 f &= \frac{\partial^2 f}{\partial r^2} + \frac{2}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial f}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} \\ &\quad + \left[\frac{\cot \theta}{r^2} \times 2r^2 \cos 2\theta + \frac{1}{r^2 \sin^2 \theta} \times (-2 \cos 2\phi) \right] \\ &= 2 \sin 2\theta + \frac{2}{r} (2r \sin 2\theta) + \frac{1}{r^2} (-4r^2 \sin 2\theta) + \frac{\cot \theta}{r^2} \times -2 \cos 2\phi \\ &= 2 \sin 2\theta + 4 \sin 2\theta - 4 \sin 2\theta + 2 \cot \theta \cos 2\theta - \frac{2}{r^2} \operatorname{cosec}^2 \theta \cos 2\phi \\ &= 2 \sin 2\theta + 2 \cot \theta \cos 2\theta - \frac{2}{r^2} \operatorname{cosec}^2 \theta \cos 2\phi \end{aligned}$$

Ex.7.9.24 If A is a vector field, find curl A where,

$$(1) A = (r^2 \sin \theta) e_r + \left(\frac{1}{r} \cos \theta \right) e_\theta + \frac{1}{r^2} e_\phi$$

$$(2) A = (z \cos \theta) e_\rho - (z \sin \theta) e_\theta$$

Sol: If $A = A_1 e_r + A_2 e_\theta + A_3 e_\phi$, We have, in spherical coordinates,

$$\text{Curl } A = \frac{1}{r^2 \sin \theta} \begin{vmatrix} e_r & r e_\theta & (r \sin \theta) e_\phi \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial \phi \\ A_1 & r A_2 & (r \sin \theta) A_3 \end{vmatrix}$$

$$\text{Here, } A_1 = r^2 \sin \theta, A_2 = \frac{1}{r} \cos \theta, A_3 = \frac{1}{r^2}$$

$$\begin{aligned} \therefore \text{Curl } A &= \frac{1}{r^2 \sin \theta} \begin{vmatrix} e_r & r e_\theta & (r \sin \theta) e_\phi \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial \phi \\ r^2 \sin \theta & \cos \theta & (1/r) \sin \theta \end{vmatrix} \\ &= \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial \theta} \left\{ \left(\frac{1}{r} \sin \theta \right) - \frac{\partial}{\partial \phi} (\cos \theta) \right\} e_r + \left\{ \frac{\partial}{\partial \phi} (r^2 \sin \theta) - \frac{\partial}{\partial r} \left(\frac{\sin \theta}{r} \right) \right\} r e_\theta \right. \\ &\quad \left. + \left\{ \frac{\partial}{\partial r} (\cos \theta) - \frac{\partial}{\partial \theta} (r^2 \sin \theta) \right\} r \sin \theta e_\phi \right] \\ &= \frac{1}{r^2 \sin \theta} \left[\frac{1}{r} \cos \theta e_r + \frac{1}{r^2} \sin \theta r e_\theta - r^2 \cos \theta r \sin \theta e_\phi \right] \\ &= \frac{1}{r^2 \sin \theta} \left[\frac{1}{r} \cos \theta e_r + \frac{\sin \theta}{r} e_\theta - r^3 \sin \theta \cos \theta e_\phi \right] \\ &= \frac{\cot \theta}{r^3} e_r + \frac{1}{r^3} e_\theta - r \cos \theta e_\phi \end{aligned}$$

(2) In cylindrical coordinates, if $A = A_1 e_\rho + A_2 e_\theta + A_3 e_z$,

$$\text{we have, } \text{Curl } A = \frac{1}{\rho} \begin{vmatrix} e_\rho & \rho e_\theta & e_z \\ \partial/\partial \rho & \partial/\partial \theta & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix}$$

Here, $A_1 = z \cos \theta$, $A_2 = -z \sin \theta$, $A_3 = 0$.

$$\begin{aligned}
 \therefore \text{Curl } \mathbf{A} &= \frac{1}{\rho} \begin{vmatrix} e_\rho & \rho e_\theta & e_z \\ \partial/\partial \rho & \partial/\partial \theta & \partial/\partial z \\ z \cos \theta & -\rho z \sin \theta & 0 \end{vmatrix} \\
 &= \frac{1}{\rho} \left[\left\{ \frac{\partial}{\partial \theta}(0) - \frac{\partial}{\partial z}(-\rho z \sin \theta) \right\} e_\rho + \left\{ \frac{\partial}{\partial z}(z \cos \theta) - \frac{\partial}{\partial \rho}(0) \right\} \rho e_\theta \right. \\
 &\quad \left. + \left\{ \frac{\partial}{\partial \rho}(-\rho z \sin \theta) - \frac{\partial}{\partial \theta}(z \cos \theta) \right\} e_z \right] \\
 &= \frac{1}{\rho} [\rho \sin \theta e_\rho + \rho \cos \theta e_\theta + (-z \sin \theta + z \sin \theta) e_z] \\
 &= (\sin \theta) e_\rho + (\cos \theta) e_\theta
 \end{aligned}$$

Ex. 7.9.25 Show that the vector field $\mathbf{A} = (\rho z \sin 2\theta) e_\rho + (\rho z \cos 2\theta) e_\theta + \left(\frac{1}{2} \rho^2 \sin 2\theta\right) e_z$ is irrotational.

Sol: Here, $A_1 = \rho z \sin 2\theta$, $A_2 = \rho z \cos 2\theta$, $A_3 = \frac{1}{2} \rho^2 \sin 2\theta$

$$\begin{aligned}
 \text{Curl } \mathbf{A} &= \frac{1}{\rho} \begin{vmatrix} e_\rho & \rho e_\theta & e_z \\ \partial/\partial \rho & \partial/\partial \theta & \partial/\partial z \\ A_1 & \rho A_2 & A_3 \end{vmatrix} \\
 &= \frac{1}{\rho} \begin{vmatrix} e_\rho & \rho e_\theta & e_z \\ \partial/\partial \rho & \partial/\partial \theta & \partial/\partial z \\ \rho z \sin 2\theta & \rho^2 z \cos 2\theta & \frac{1}{2} \rho^2 \sin 2\theta \end{vmatrix} \\
 &= \frac{1}{\rho} \left[\left\{ \frac{\partial}{\partial \theta} \left(\frac{1}{2} \rho^2 \sin 2\theta \right) - \frac{\partial}{\partial z} (\rho^2 z \cos 2\theta) \right\} e_\rho \right. \\
 &\quad \left. + \left\{ \frac{\partial}{\partial z} (\rho z \sin 2\theta) - \frac{\partial}{\partial \rho} \left(\frac{1}{2} \rho^2 \sin 2\theta \right) \right\} \rho e_\theta \right]
 \end{aligned}$$

$$\begin{aligned}
& + \left\{ \frac{\partial}{\partial \rho} (\rho^2 z \cos 2\theta) - \frac{\partial}{\partial \theta} (\rho z \sin 2\theta) \right\} e_z \Big] \\
& = \frac{1}{\rho} (\rho^2 \cos 2\theta - \rho^2 \cos 2\theta) e_\rho + (\rho \sin 2\theta - (\rho \sin 2\theta) \rho) e_\theta + (2\rho z \cos 2\theta) e_z \\
& = \frac{1}{\rho} [(0) e_\rho + (0) e_\theta + (0) e_z] = 0 \quad \therefore A \text{ is irrotational.}
\end{aligned}$$

Ex.7.9.26 Show that $\text{div}(\text{curl } A) = 0$, where A is a vector function, in orthogonal curvilinear coordinates.

Sol: Let $A = A_1 e_1 + A_2 e_2 + A_3 e_3$

Then

$$\begin{aligned}
\text{curl } A &= \frac{1}{\lambda_1 \lambda_2 \lambda_3} \begin{vmatrix} \lambda_1 e_1 & \lambda_2 e_2 & \lambda_3 e_3 \\ \partial/\partial v_1 & \partial/\partial v_2 & \partial/\partial v_3 \\ \lambda_1 A_1 & \lambda_2 A_2 & \lambda_3 A_3 \end{vmatrix} \\
&= \sum \frac{1}{\lambda_1 \lambda_2 \lambda_3} \left[\left\{ \frac{\partial}{\partial v_2} (\lambda_3 A_3) - \frac{\partial}{\partial v_3} (\lambda_2 A_2) \right\} \lambda_1 e_1 \right] \\
&= \sum \left(\frac{1}{\lambda_2} \cdot \frac{\partial A_3}{\partial v_2} - \frac{1}{\lambda_3} \cdot \frac{\partial A_2}{\partial v_3} \right) e_1 \\
\text{Div } f &= \sum \frac{1}{\lambda_1 \lambda_2 \lambda_3} \left[\frac{\partial}{\partial v_1} (f_1 \lambda_2 \lambda_3) \right] \\
\text{Div curl } A &= \sum \frac{1 \lambda_2 \lambda_3}{\lambda_1 \lambda_2 \lambda_3} \left[\frac{\partial}{\partial v_1} \left\{ \frac{1}{\lambda_2} \frac{\partial A_3}{\partial v_2} - \frac{1}{\lambda_3} \frac{\partial A_2}{\partial v_3} \right\} \right] \\
&= \left[\frac{1}{\lambda_1 \lambda_2} \frac{\partial^2 A_3}{\partial v_1 \partial v_2} - \frac{1}{\lambda_1 \lambda_3} \frac{\partial^2 A_2}{\partial v_1 \partial v_3} + \frac{1}{\lambda_2 \lambda_3} \frac{\partial^2 A_1}{\partial v_2 \partial v_3} - \frac{1}{\lambda_1 \lambda_2} \frac{\partial^2 A_3}{\partial v_2 \partial v_1} \right. \\
&\quad \left. + \frac{1}{\lambda_3 \lambda_1} \frac{\partial^2 A_2}{\partial v_3 \partial v_1} - \frac{1}{\lambda_3 \lambda_2} \frac{\partial^2 A_1}{\partial v_3 \partial v_2} \right] \\
&= 0
\end{aligned}$$

Exercise – 7(i)

1. Show that the spherical coordinate system is an orthogonal system.
2. Show that the cylindrical coordinate system is an orthogonal system.

[Hint for 1 and 2: Show that $e_1.e_2 = e_2.e_3 = e_3.e_1 = 0$ etc]

3. Express the vector, $F = 2xi - 3y^2j + zrk$, in cylindrical polar coordinates.

$$[\text{Ans: } F = (2\rho \cos^2 \theta - 3\rho^2 \sin^2 \theta)e_\rho - (\rho \sin 2\theta + 3\rho^2 \sin^2 \theta)e_\theta + (\rho z \cos \theta)e_z]$$

4. Represent the vector, $A = xyi + yzj + zrk$, in spherical polar coordinates.

$$[\text{Ans: } A = r^2 \sin \theta \{ \sin^2 \theta \cos^2 \phi \sin \phi + \sin \theta \cos \theta \sin^2 \phi + \cos^2 \theta \cos \phi \} e_r \\ + \{ \sin \theta \cos \theta \cos^2 \phi \sin \phi + \cos^2 \theta \sin^2 \phi - \sin \theta \cos \theta \cos \phi \} e_\theta \\ + \{ \cos \theta \sin \phi \cos \phi - \sin \theta \sin^2 \phi \cos \phi \} e_\phi]$$

5. Express the vectors a) $A = yi + xj$ and b) $F = zi - 2xj + yk$, in cylindrical coordinates.

$$[\text{Ans: a) } A = \rho e_\theta \text{ b) } F = (z \cos \theta - \rho \sin 2\theta)e_\rho - (z \sin \theta + 2\rho \cos^2 \theta)e_\theta + (\rho \sin \theta)e_z]$$

6. Express the vectors a) $A = -yi + xj$, b) $F = xi + 2yj + yzk$, in spherical polar coordinates.

$$[\text{Ans: a) } A = r \sin \theta e_\phi \text{ b) } r \sin \theta [\{ \sin \theta (1 + \sin^2 \phi) + r \cos^2 \phi \sin \phi \} e_r \\ + \{ \cos \theta (1 + \sin^2 \phi) - r \sin \theta \cos \theta \sin \phi \} e_\theta + \sin \phi \cos \phi e_\phi]$$

7. Find the gradients of the following function

$$\text{i) } f = \rho z \sin \theta \quad \text{ii) } f = \frac{\sin \theta}{r^2} \quad \text{iii) } f = r^2 \sin^2 \theta + \cos^3 \phi$$

$$[\text{Ans: i) } \nabla f = (z \sin \theta)e_\rho + (z \cos \theta)e_\theta + (\rho \sin \theta)e_z]$$

$$\text{ii) } \nabla f = \frac{1}{r^3} [(-2 \sin \theta)e_r + (\cos \theta)e_\theta]$$

$$\text{iii) } \nabla f = (2r \sin^2 \phi)e_r - (r \sin 2\theta)e_\theta - \left(\frac{3}{r} \operatorname{cosec} \theta \cos^2 \phi \sin \phi \right) e_\phi]$$

8. Find the divergence of the following vector functions.

$$\text{i) } A = (\cos \theta + \sin \theta) e_\rho + (\cos \theta - \sin \theta) e_\theta + e_z$$

$$\text{ii) } F = (r^2 \sin \theta) e_r + \left(\frac{1}{r} \cos \theta \right) e_\theta + \frac{1}{r^2} e_\phi, \text{ iii) } V = (\rho \cos \theta) e_r + (\rho \sin \theta) e_\phi$$

$$[\text{Ans: (i) } 0 \quad (\text{ii}) 4r \sin \theta + \frac{1}{r^2} \frac{\cos 2\theta}{\sin \theta} \quad (\text{iii}) 3 \cos \theta]$$

9. Show that the following vector fields are solenoidal.

$$(\text{i}) \mathbf{A} = (z \cos \theta) \mathbf{e}_\rho - (z \sin \theta) \mathbf{e}_\theta \quad (\text{ii}) \mathbf{F} = \left(\frac{1}{r^3} \cot \theta \right) \mathbf{e}_r - \frac{1}{r^3} \mathbf{e}_\theta + (r \cos \theta) \mathbf{e}_\phi$$

10. Find the curl of the following vector fields :

$$(\text{i}) \mathbf{A} = (z \sin \theta) \mathbf{e}_\rho + (z \cos \theta) \mathbf{e}_\theta - (\rho \cos \theta) \mathbf{e}_z$$

$$(\text{ii}) \mathbf{F} = (r \sin \theta) \mathbf{e}_r + \left(\frac{1}{r} \cos \theta \right) \mathbf{e}_\theta + \left(\frac{1}{r} \right) \mathbf{e}_\phi$$

$$(\text{iii}) \mathbf{V} = \frac{1}{r} \tan \frac{\theta}{2} \mathbf{e}_\phi$$

$$[\text{Ans: (i)} (\sin \theta - \cos \theta) \mathbf{e}_\rho + (\sin \phi + \cos \phi) \mathbf{e}_\theta \quad (\text{ii}) \left(\frac{1}{r^2} \cot \theta \right) \mathbf{e}_r - (\cos \theta) \mathbf{e}_\phi \quad (\text{iii}) \frac{1}{r^2} \mathbf{e}_\phi]$$

11. If 'f' is a scalar function in orthogonal curvilinear coordinates v_1, v_2, v_3 prove that ' ∇f ' is irrotational.

7.10 GREEN'S THEOREM IN THE PLANE

7.10.1 Green's Theorem:

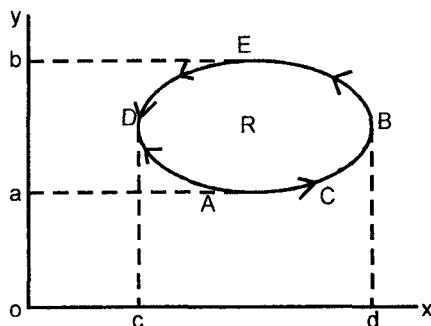
Let

- (i) R be a closed region of the xy plane bounded by a simple closed curve C
- (ii) P(x,y) and Q(x,y) be continuous functions of x and y having continuous

$$\text{derivatives in R. Then } \int_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

where C is in the positive direction.

Proof:



Let the equations of the curves DAB and DEB (see the figure) be respectively $y = f_1(x)$ and $y = f_2(x)$. Let R be the region bounded by the curve C. Then,

$$\begin{aligned}\iint_R \frac{\partial P}{\partial y} dx dy &= \int_{x=c}^d \left[\int_{y=f_1(x)}^{f_2(x)} \frac{\partial P}{\partial y} dy \right] dx \\&= \int_{x=c}^d P(x, y) \Big|_{y=f_1(x)}^{y=f_2(x)} dx = \int_{x=c}^d [P(x, f_2) - P(x, f_1)] dx = - \int_c^d P(x, f_1) dx - \int_d^c P(x, f_2) dx \\&= - \oint_c P(x, y) dx\end{aligned}$$

$$\text{i.e., } \oint_c P(x, y) dx = - \iint_R \frac{\partial P}{\partial y} dx dy \quad \dots (1)$$

|||^y if the equations of the curves ADE be and ABE be taken respectively as $x = g_1(y)$ and $x = g_2(y)$, we have,

$$\begin{aligned}\iint_R \frac{\partial Q}{\partial x} dx dy &= \int_{y=a}^b \left[\int_{x=g_1(y)}^{g_2(y)} \frac{\partial Q}{\partial x} dx \right] dy \\&= \int_a^b [Q(g_2, y) - Q(g_1, y)] dy = \int_b^a Q(g_1, y) dy + \int_a^b Q(g_2, y) dy \\&= \oint_c Q(x, y) dy\end{aligned}$$

$$\text{Thus } \oint_c Q(x, y) dy = \iint_R \frac{\partial Q}{\partial x} dx dy \quad \dots (2)$$

Adding (1) and (2),

$$\oint_c P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

7.10.2 Vector notation of Green's theorem

Green's Theorem in the plane can be put in vector notation in the following way .

Let $F = P(x, y)i + Q(x, y)j$

and $r = xi + yj$, so that

$$dr = (dx)i + (dy)j$$

$$\therefore F \cdot dr = P dx + Q dy$$

$$\begin{aligned} \text{Again, } \nabla \times F &= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix} \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) k, \text{ so that} \end{aligned}$$

$$(\nabla \times F) \cdot k = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Taking $dR = dxdy$, Green's theorem in the plane can be stated in the vector form as,

$$\oint_C F \cdot dr = \iint_R (\nabla \times F) \cdot k dR$$

..... (A)

7.10.3 Physical interpretation of Green's theorem

1. Let F denote the force field acting on a particle.

Then $\oint_C F \cdot dr$ represents the work done in moving the particle around a closed curve C .

$\therefore$ From (A) it follows that the work done is determined by $\text{curl } F = \nabla \times F$.

2. In particular, if $\nabla \times F = 0$ i.e., if F is conservative (or $F = \nabla f$)

Then $\oint_C F \cdot dr = 0$. i.e., the work done is independent of the path.

3. Conversely, if the integral is independent of the path, i.e., if

$$\oint_C F \cdot dr = 0, \text{ then } \nabla \times F = 0$$

In the plane, $\nabla \times F = 0$ is equivalent to saying that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ where $F = Pi + Qj$.

7.10.4 Application of Green's theorem to the evaluation of area of a simple closed curve.

The area bounded by a simple closed curve $C = \frac{1}{2} \oint_C xdy - ydx$

Proof: By Green's theorem, we have,

$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

if we put $P = -y$, and $Q = x$,

$$\text{i.e., } \frac{\partial P}{\partial y} = -1, \quad \frac{\partial Q}{\partial x} = 1, \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2,$$

$\therefore$ We get,

$$\oint_C xdy - ydx = \iint_R 2dxdy = 2A, \text{ where } A \text{ is the required area.}$$

$$\text{i.e., } A = \frac{1}{2} \oint_C xdy - ydx.$$

Solved Examples

Ex. 7.10.5 Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol: Parametric equations of the ellipse are,

$$x = a \cos \theta, \quad y = b \sin \theta$$

$$dx = -a \sin \theta d\theta, \quad dy = b \cos \theta d\theta.$$

$\therefore$ By Green's theorem,

$$\text{Area of the ellipse} = \frac{1}{2} \oint_C xdy - ydx$$

$$= \frac{1}{2} \int_0^{2\pi} (a \cos \theta b \cos \theta + b \sin \theta a \sin \theta) d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} ab d\theta = \pi ab$$

Ex. 7.10.6 Evaluate $\oint_c (y - \sin x)dx + (\cos x)dy$, a) directly and b) using Green's theorem,

where c is the boundary of the triangle in xy -plane whose vertices are $(0,0)$,

$\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, 1\right)$ traversed in the positive direction.

Sol: Let $I = \oint_c (y - \sin x)dx + (\cos x)dy$

a) along OA: $y = 0, \quad dy = 0$

$$\therefore \int_{OA} (y - \sin x)dx + (\cos x)dy = \int_0^{\pi/2} -\sin x dx = \cos x \Big|_0^{\pi/2} = -1$$

along AB: $x = \frac{\pi}{2}, \quad dx = 0$

$$\therefore \int_{AB} (y - \sin x)dx + (\cos x)dy = \int_0^1 0 dy = 0$$

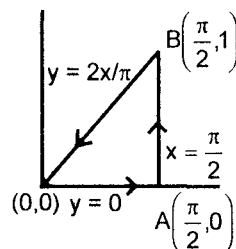
along BO: Equation of OB is $y = \frac{2x}{\pi}$, $dy = \frac{2}{\pi} dx$,

$$\therefore \int_{BO} (y - \sin x)dx + \cos x dy = \int_{\pi/2}^0 \left[\left(\frac{2x}{\pi} - \sin x \right) + \frac{2}{\pi} \cos x \right] dx$$

$$= \frac{x^2}{\pi} + \cos x + \frac{2}{\pi} \sin x \Big|_{\pi/2}^0 = (0 + 1 + 0) - \left(\frac{\pi}{4} + 0 + \frac{2}{\pi} \right) = 1 - \frac{\pi}{4} - \frac{2}{\pi}.$$

$\therefore I = \text{Sum of the integrals along OA, AB and BO}$

$$= -1 + 0 + 1 - \frac{\pi}{4} - \frac{2}{\pi} = -\frac{\pi}{4} - \frac{2}{\pi}$$



(b) By Green's theorem,

$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy, \quad \text{Here } P = y - \sin x, \quad Q = \cos x$$

$$\frac{\partial Q}{\partial x} = -\sin x, \quad \frac{\partial P}{\partial y} = 1$$

$$\begin{aligned} \therefore I &= \iint_R (-\sin x - 1) dx dy = \int_{x=0}^{\pi} \left[\int_{y=0}^{2x} (-\sin x - 1) dy \right] dx = \int_{x=0}^{\pi} -y \sin x - y \Big|_0^{2x} dx \\ &= \int_0^{\pi} \left(-\frac{2x}{\pi} \sin x - \frac{2x}{\pi} \right) dx = -\frac{2}{\pi} \int_0^{\pi} x \sin x dx - \frac{2}{\pi} \int_0^{\pi} x dx \\ &= -\frac{2}{\pi} \left[-x \cos x + \sin x + \frac{x^2}{2} \right]_0^{\pi} - \frac{2}{\pi} \left[0 + 1 + \frac{\pi^2}{8} \right] = -\frac{2}{\pi} - \frac{\pi}{4} \end{aligned}$$

Ex.6.10.7 Evaluate $\oint_C (3x + 4y)dx + (2x - 3y)dy$ where C is the circle in xy plane with

centre at origin and radius 2 units.

Sol: By Green's theorem ,

$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy,$$

$$\text{Here } P = 3x + 4y, \quad Q = 2x - 3y ; \quad \frac{\partial P}{\partial y} = 4, \quad \frac{\partial Q}{\partial x} = 2$$

$$\begin{aligned} \therefore \text{The given integral} &= \iint_R (2 - 4) dx dy \\ &= -2 \iint_R dx dy = -2A, \end{aligned}$$

where A is the area of the circle C.

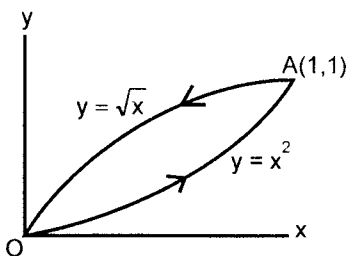
$$= -2 \times \pi \times 2^2 = -8\pi$$

7.10.8 Verify Green's theorem in the plane for the integral $\oint_C (3x^2 - 8y^2)dx + 4(4y - 6xy)dy$

where C is the boundary of the region given by

$$(1) \quad y = \sqrt{x}, y = x^2 \quad (2) \quad x = 0, y = 0, x + y = 1$$

(1) The given region is shown in the figure below.



$$\text{let } I = \oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy.$$

The points of intersection of $y = x^2$ and $y = \sqrt{x}$ are $(0, 0)$ and $(1, 1)$.

We have to integrate I along

(1) $y = x^2$ from O to A .

(2) along $y = \sqrt{x}$ from A to O , and add the two values.

Along $y = x^2$, $dy = 2xdx$

$$\therefore I = \int_{AO} (3x^2 - 8x^4)dx + (4x^2 - 6x^3)2xdx = \int_{x=0}^{x=1} (3x^2 - 20x^4 + 8x^3)dx = x^3 - 4x^5 + 2x^4 \Big|_0^1 = -1$$

Along $y = \sqrt{x}$; $x = y^2$, $dx = 2ydy$

$$\therefore I = \int_{OA} (3y^4 - 8y^2)dy + (4y - 6x^3)2ydy = \int_{y=1}^{y=0} (6y^5 - 22y^3 + 4y)(2y)dy = y^6 - \frac{11}{2}y^4 + 2y^2 \Big|_1^0 = \frac{5}{2}$$

$$\therefore I = -1 + \frac{5}{2} = \frac{3}{2}$$

By Green's Theorem,

$$\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

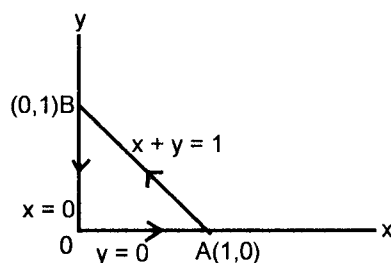
Here, $P = 3x^2 - 8y^2$; $Q = 4y - 6xy$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -6y - (-16y) = 10y$$

$$\therefore I = \int_R \int 10y dx dy = \int_{x=0}^1 \left[\int_{y=x^2}^{\sqrt{x}} 10y dy \right] dx = \int_{x=0}^1 5y^2 \Big|_{x^2}^{\sqrt{x}} dx = \int_{x=0}^1 (5x - 5x^4) dx = \frac{3}{2}$$

Green's theorem is verified.

2. The given region is shown in the figure below.



Along OA, $y = 0 \Rightarrow dy = 0$

$$\therefore \text{Given integral} = \int_{x=0}^1 3x^2 dx = x^3 \Big|_0^1 = 1$$

Along AB, $y = 1 - x \Rightarrow dy = -dx$

$\therefore$ Given integral

$$= \int_{x=1}^0 \{3x^2 - 8(1 - x^2)\} dx + \{4(1 - x) - 6x(1 - x)(-dx)\} = \int_{x=1}^0 (-11x^2 + 26x - 12) dx$$

$$= -\frac{11}{3}x^3 + 13x^2 - 12x \Big|_1^0 = \frac{11}{3} - 13 + 12 = \frac{8}{3}$$

Along BO, $x = 0 \Rightarrow dx = 0$

$$\text{Given integral} = \int_{y=1}^{y=0} 4y dy = 2y^2 \Big|_1^0 = -2$$

$$\therefore \text{The given integral} = 1 + \frac{8}{3} - 2 = \frac{5}{3}$$

By green's theorem,

$$\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

Here, $P = 3x^2 - 8y^2$, $Q = 4y - 6xy$

$$\frac{\partial Q}{\partial x} \cdot \frac{\partial P}{\partial y} = -6y - (-16y) = 10y$$

$$\begin{aligned} \therefore \text{The given integral} &= \iint_R 10y \, dx \, dy = \int_{x=0}^1 \left[\int_{y=0}^{1-x} 10y \, dy \right] dx = \int_{x=0}^1 5y^2 \Big|_0^{1-x} dx = \int_0^1 5(1-x)^2 dx \\ &= \frac{5(1-x)^3}{3x-1} \Big|_0^1 = -\frac{5}{3}(0-1) = \frac{5}{3} \end{aligned}$$

Hence the theorem is verified.

Ex. 7.10.9 Evaluate $\int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3 \, dx - 3x^2y^2 \, dy)$ along the path $x^4 - 6xy^3 - 4y^2 = 0$

Sol: $P = 10x^4 - 2xy^3$, $Q = 3x^2y^2$

$$\frac{\partial P}{\partial y} = -6xy^2 = \frac{\partial Q}{\partial x}$$

The integral is independent of the path. Hence we can use any path. For example, if we use the path from points $(0, 0)$ to $(2, 0)$ and then from $(2, 0)$ to $(2, 1)$; we can evaluate the integral.

(i) From $(0, 0)$ to $(2, 0)$; $y = 0$, $dy = 0$

$$\therefore \text{The integral} = \int_0^2 10x^4 \, dx - 2x^5 \Big|_0^2 = 64$$

(ii) From $(2, 0)$ to $(2, 1)$; $x = 2$, $dx = 0$

$$\therefore \text{The integral} = \int_0^2 10x^4 \, dx = 2x^5 \Big|_0^2 = 64 \quad \int_0^2 12y^2 \, dy = 4y^3 \Big|_0^2 = -4$$

$$\therefore \text{The value of the integral} = 64 - 4 = 60$$

Aliter : Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, we know that $\{10x^4 - 2xy^3\}dx - 3x^2y^2 \, dy\}$ is an exact

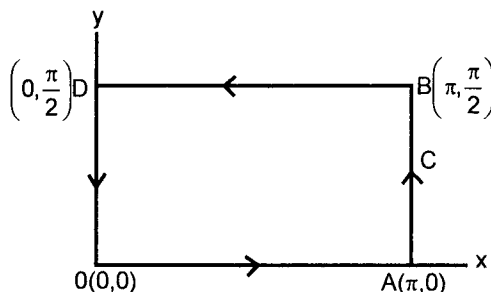
differential of $(2x^5 - x^2y^3)$.

$$\text{The given integral} = \int_{(0,0)}^{(2,1)} d(2x^5 - x^2y^3) = 2x^5 - x^2y^3 \Big|_{(0,0)}^{(2,1)} = 2 \cdot 2^5 - 2^2 \cdot 1^3 = 60$$

Ex. 7.10.10 Verify Green's theorem for $\int_C (e^{-x} \sin y) dx + (e^{-x} \cos y) dy$ where C is the

boundary of the rectangle whose vertices are $(0, 0)$ $(\pi, 0)$ $(\pi, \frac{\pi}{2})$ and $(0, \frac{\pi}{2})$ traversed in the +ve direction.

Sol:



By Green's theorem,

$$\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

Here, $P = e^{-x} \sin y$, $Q = e^{-x} \cos y$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -e^{-x} \cos y - e^{-x} \cos y = -2e^{-x} \cos y$$

$$\therefore \oint_C P dx + Q dy = \iint_R -2e^{-x} \cos y dx dy$$

$$= -2 \int_{x=0}^{\pi} \left[\int_{y=0}^{\pi/2} e^{-x} \cos y dy \right] dx = -2 \int_{x=0}^{\pi} e^{-x} \sin y \Big|_0^{\pi/2} dx$$

$$= -2 \int_C e^{-x} dx = 2e^{-x} \Big|_0^{\pi} = 2(e^{-\pi} - 1) \quad \dots(i)$$

$$\text{Again, } \oint_C P dx + Q dy = \int_C \int_{OA} + \int_{AB} + \int_{BD} + \int_{DO}$$

Along OA; $y = 0$, $dy = 0$

$$\therefore \int_{OA} P dx + Q dy = 0$$

Along AB: $x = \pi$, $dx = 0$

$$\int_{AB} P dx + Q dy = \int_0^{\pi/2} e^{-\pi} \cos y dy = e^{-\pi} \sin y \Big|_0^{\pi/2} = e^{-\pi}$$

Along BD ; $y = \frac{\pi}{2}, \quad dy = 0$

$$\int_{BD} P dx + Q dy = \int_{\pi}^0 e^{-x} dx = \frac{e^{-x}}{-1} \Big|_{\pi}^0 = -1 + e^{-\pi}$$

Along DO ; $x = 0, \quad dx = 0$

$$\therefore \int_{DO} P dx + Q dy = \int_{\pi/2}^0 \cos y dy = \sin y \Big|_{\pi/2}^0 = -1$$

$$\therefore \int_c P dx + Q dy = 0 + e^{-\pi} - 1 + e^{-\pi} - 1 = 2(e^{-\pi} - 1) \quad \dots (ii)$$

$\therefore$ From (i) and (ii) it is proved that

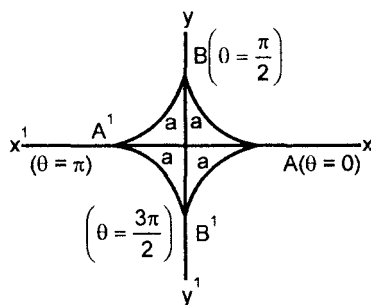
$$\int_c P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

Hence the theorem is verified.

Ex. 7.10.11 Apply Green's theorem to obtain the area bounded by the curve

$$x^{2/3} + y^{2/3} = a^{2/3}, \quad a > 0.$$

Sol: The parametric equations of the curve are $x = a \cos^3 \theta, y = a \sin^3 \theta$. A rough sketch of the curve is given below :



$$x = a \cos^3 \theta, \quad y = a \sin^3 \theta$$

$$dx = -3 a \cos^2 \theta \sin \theta d\theta, \quad dy = 3 a \sin^2 \theta \cos \theta d\theta$$

By Green's theorem, the area bounded by a simple closed curve C is given by

$$\frac{1}{2} \int_c (x dy - y dx)$$

∴ The area bounded by the given curve C

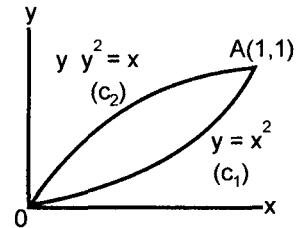
$$\begin{aligned}
 &= \frac{1}{2} \int_0^{2\pi} \{a \cos^3 \theta \cdot 3a \sin^2 \theta \cos \theta + a \sin^3 \theta \cdot 3a \cos^2 \theta \sin \theta\} d\theta \\
 &= \frac{3a^2}{2} \int_0^{2\pi} \cos^2 \theta \sin^2 \theta (\cos^2 \theta + \sin^2 \theta) d\theta \\
 &= \frac{3a^2}{2} \int_0^{2\pi} \cos^2 \theta \sin^2 \theta d\theta = \frac{3a^2}{8} \int_0^{2\pi} \sin^2 2\theta d\theta = \frac{3a^2}{16} \int_0^{2\pi} (1 - \cos 4\theta) d\theta = \frac{3\pi a^2}{8}
 \end{aligned}$$

Ex. 7.10.12 Verify Green's theorem for $\int_C (2xy - x^2)dx + (x + y^2)dy$, where C is the closed curve in xy-plane bounded by the curves $y = x^2$ and $y^2 = x$.

Sol: By Green's theorem,

$$\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

Here, $P = 2xy - x^2$, $Q = x + y^2$, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 2x$



$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \int_{x=0}^1 \left[\int_{y=x^2}^{y=\sqrt{x}} (1 - 2x) dy \right] dx = \int_0^1 y - 2xy \Big|_{y=x^2}^{y=\sqrt{x}} dx$$

$$= \int_0^1 (\sqrt{x} - 2x\sqrt{x} - x^2 + 2x^3) dx = \frac{2}{3} x^{\frac{3}{2}} - 2 \cdot \frac{2}{5} x^{\frac{5}{2}} - \frac{x^3}{3} + 2 \cdot \frac{x^4}{4} \Big|_0^1$$

$$= \frac{1}{2} - \frac{4}{5} - \frac{1}{3} + \frac{1}{2} = \frac{1}{30}$$

..... (i)

Again $\int_C Pdx + Qdy = \int_{C_1} (Pdx + Qdy) + \int_{C_2} (Pdx + Qdy)$

Along C_1 , $y = x^2$, $dy = 2xdx$, "x" varies from 0 to 1

$$\therefore \int_{C_1} Pdx + Qdy = \int_{x=0}^1 (2x^3 - x^2)dx + (x + x^4)2xdx = \int_0^1 (2x^3 + x^2 + 2x^5)dx$$

$$= \frac{x^4}{2} + \frac{x^3}{3} + \frac{x^6}{3} \Big|_0^1 = \frac{7}{6}$$

Along C_2 , $x = y^2$, $dx = 2ydy$, 'y' varies from 1 to 0

$$\begin{aligned}\therefore \int_{C_2} Pdx + Qdy &= \int_{y=1}^0 (2y^3 - y^4)2ydy + (y^2 + y^2)dy = \int_1^0 (4y^4 - 2y^5 + 2y^2)dy \\ &= \frac{4}{5}y^5 - \frac{2}{6}y^6 + \frac{2}{3}y^3 \Big|_1^0 = \frac{4}{5} + \frac{1}{3} - \frac{2}{3} = -\frac{17}{15}\end{aligned}$$

$$\therefore \int_c Pdx + Qdy = \frac{7}{6} - \frac{17}{15} = \frac{1}{30} \quad \dots (ii)$$

From (i) and (ii), the theorem is verified.

Exercise – 7(j)

- Evaluate $\oint (x^2 + y^2)dx + 3xy^2dy$, (a) directly (b) by Green's theorem, where c is the circle $x^2 + y^2 = 4$, traversed in the +ve direction.
(Ans : 12π)
- Evaluate $\oint (x^2 + 2xy)dx + (x^2y + 3)dy$ around the boundary C of the region given by $y^2 = 8x$ and $x = 2$, (a) directly and (b) by Green's theorem.
(Ans : 128)
- Verify Green's theorem for the integral $\oint (3x^2 + 2y)dx - (x + 3\cos y)dy$ where C is the boundary of the parallelogram with vertices at $(0, 0)$, $(2, 0)$, $(3, 1)$ and $(1, 1)$
(Ans : -6)
- Evaluate $\int_{(0,0)}^{(\pi,2)} (6xy - y^2)dx + (3x^2 - 2xy)dy$ along the cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$. Verify the result by using Green's theorem.
(Ans : $6\pi^2 - 4\pi$)

5. Using Green's theorem find the area bounded by one arch of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$, $a > 0$, and x-axis.

[Ans: $3\pi a^2$]

6. Evaluate $\oint_C (2x^2 - y^2)dx + (x^2 + y^2)dy$ by Green's theorem where C is the boundary of the surface in the xy plane enclosed by x axis and the semi-circle $y = \sqrt{1 - x^2}$

[Ans: $4/3$]

7. Evaluate $\oint_C (\cos x \sin y - xy)dx + \sin x \cos y dy$, using Green's theorem where c is the circle $x^2 + y^2 = 1$

[Ans: 0]

8. Verify Green's Theorem in the plane for $\oint_C (x^2 - xy^3)dx + (y^2 - 2xy)dy$ where C is the square with vertices at (0,0), (2,0) and (0,2)

[Ans: 8]

9. Evaluate $\int_{(0,0)}^{(2,1)} (12x^3 - 2xy^3)dx - 3x^2y^2dy$ along the path $x^3 - y^3 + y - 4xy = 0$

[Hint : Proceed as in aliter of 7.10.9]

[Ans: $3x^4 - x^2y^3 \Big|_{(0,0)}^{(2,1)} = 44$]

7.11 Gauss Divergence Theorem

7.11.1 Gauss' Divergence Theorem

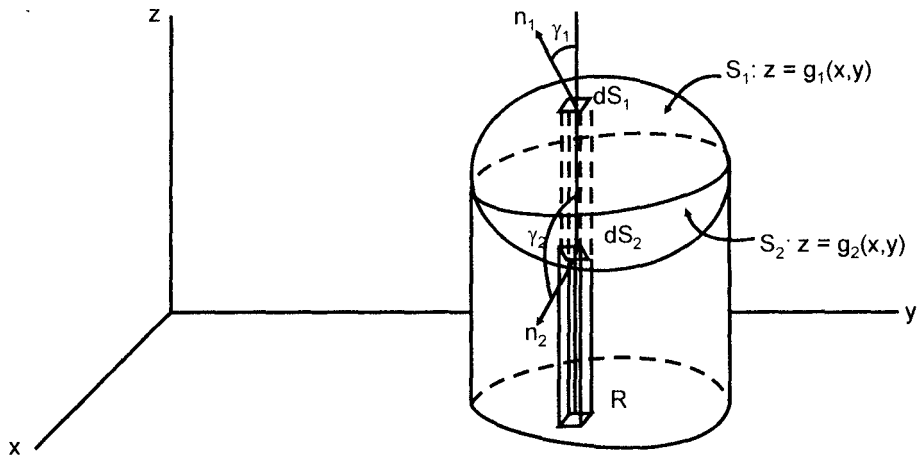
Let (1) V be the volume bounded by a closed surface S

(2) A be a vector function of position with continuous derivatives.

$$\text{Then, } \iiint_V \nabla \cdot A \, dv = \iint_S A \cdot n \, ds = \oint_S A \cdot ds$$

where n is positive unit normal (outside drawn normal) to S .

Proof:



S is a closed surface

Let any line $\parallel$ to the coordinate axes cut S in at most two points.

Let $z = g_1(x, y)$ and $z = g_2(x, y)$ be the equations of the upper portion S_1 and lower portion S_2 respectively.

Let R be the projection of S on the xy -plane.

If $A = A_1 i + A_2 j + A_3 k$

$$\nabla \cdot A = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

Then,

$$\begin{aligned}
 \iiint_V \frac{\partial A_3}{\partial z} dv &= \iiint_V \frac{\partial A_3}{\partial z} dz dy dx \\
 &= \int_R \int_{z=g_2(x,y)}^{g_1(x,y)} \left[\frac{\partial A_3}{\partial z} dz \right] dy dx = \int_R \left[A_3(x, y, z) \right]_{z=g_2}^{z=g_1} dy dx \\
 &= \int_R [A_3(x, y, g_1) - A_3(x, y, g_2)] dy dx
 \end{aligned}$$

For the upper portion S_1 , $dy dx = (\cos \gamma_1) dS_1 = k \cdot n_1 dS_1$

since the normal n_1 to S_1 makes an acute angle γ_1 with k .

For the lower portion S_2 , $dy dx = (-\cos \gamma_2) dS_2 = -k \cdot n_2 dS_2$,

since the normal n_2 to S_2 makes an obtuse angle γ_2 with k .

$$\therefore \int_R \int A_3(x, y, g_1) dy dx = \int_{S_1} \int A_3 k \cdot n_1 dS_1 \text{ and}$$

$$\int_R \int A_3(x, y, g_2) dy dx = \int_{S_2} \int A_3 k \cdot n_2 dS_2,$$

$$\begin{aligned}
 \text{So that, } \int_R \int [A_3(x, y, g_1) - A_3(x, y, g_2)] dy dx &= \int_{S_1} \int A_3 k \cdot n_1 dS_1 + \int_{S_2} \int A_3 k \cdot n_2 dS_2 \\
 &= \int_S \int A_3 k \cdot n dS \quad \dots (1)
 \end{aligned}$$

Similarly, by projecting S on the other coordinate planes, we get,

$$\iiint_V \frac{\partial A_1}{\partial x} dv = \int_S \int A_1 i \cdot n ds \quad \dots (2)$$

$$\text{and } \iiint_V \frac{\partial A_2}{\partial y} dv = \int_S \int A_2 j \cdot n ds \quad \dots (3)$$

$$(1) + (2) + (3) \Rightarrow$$

$$\iiint_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) dv = \int_S \int (A_1 i + A_2 j + A_3 k) \cdot n dS$$

$$\text{i.e. } \iiint_V \nabla \cdot A dv = \int_S \int A \cdot n dS \text{ which proves the theorem.}$$

7.11.2 (a) Express Gauss' Theorem in words and (b) obtain its Cartesian form .

(a) Gauss' Theorem states that "The surface integral of the normal component of vector $\mathbf{A}$ taken over a closed surface is equal to the integral of the divergence of $\mathbf{A}$ taken over the volume enclosed by the surface.

(b) Let $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$,

$$\text{Then } \text{div } \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

Let the unit normal $\mathbf{n}$ to S make angles α, β, γ with the +ve coordinate axes so that, $\cos \alpha = \mathbf{n} \cdot \mathbf{i}$, $\cos \beta = \mathbf{n} \cdot \mathbf{j}$, $\cos \gamma = \mathbf{n} \cdot \mathbf{k}$ and $\cos \alpha$, $\cos \beta$, $\cos \gamma$ are the direction cosines of $\mathbf{n}$

Then if $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$

$$\mathbf{n} = (\cos \alpha)\mathbf{i} + (\cos \beta)\mathbf{j} + (\cos \gamma)\mathbf{k} \quad [\mathbf{n} \cdot \mathbf{i} = n_1 = \cos \alpha \text{ etc.}]$$

$$\therefore \mathbf{A} \cdot \mathbf{n} = A_1 \cos \alpha + A_2 \cos \beta + A_3 \cos \gamma$$

Hence the divergence theorem can be written in Cartesian form as,

$$\begin{aligned} \iiint_v \left(\frac{\partial A}{\partial x} + \frac{\partial A}{\partial y} + \frac{\partial A}{\partial z} \right) dx dy dz &= \int_S (A_1 \cos \alpha + A_2 \cos \beta + A_3 \cos \gamma) ds \\ &= \int_S A_1 dy dz + A_2 dz dx + A_3 dx dy \end{aligned}$$

Solved Examples

Ex. 7.11.3: Evaluate using the divergence theorem $= \int_S \mathbf{F} \cdot \mathbf{n} \, ds$ where

$\mathbf{F} = 2xy\mathbf{i} + yz^2\mathbf{j} + xz\mathbf{k}$ and S is the surface of the parallelepiped bounded by $x=0$, $y=0$, $z=0$, $x=2$, $y=1$ and $z=3$.

Sol: By the divergence theorem ; $\int_S \mathbf{F} \cdot \mathbf{n} \, ds = \iiint_v \nabla \cdot \mathbf{F} \, dv$

$$\begin{aligned} \text{Here, } \nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(xz) \\ &= (2y + z^2 + x) \end{aligned}$$

$$\therefore \int_S \mathbf{F} \cdot \mathbf{n} \, ds = \iiint_v (2y + z^2 + x) \, dv$$

$$\begin{aligned}
&= \int_{x=0}^2 \int_{y=0}^1 \int_{z=0}^3 (2y + z^2 + x) dx dy dz = \int_{x=0}^2 \int_{y=0}^1 \left[\int_{z=0}^3 (2y + z^2 + x) dz \right] dx dy \\
&= \int_0^2 \int_0^1 \left(2yz + \frac{z^3}{3} + xz \right) \Big|_0^3 dx dy = \int_0^2 \int_0^1 (6y + 9 + 3x) dx dy \\
&= \int_{x=0}^2 \left[\int_{y=0}^1 (6y + 9 + 3x) dy \right] dx = \int_0^2 \left(3y^2 + 9y + 3xy \right) \Big|_0^1 dx = \int_0^2 (12 + 3x) dx = 12x + \frac{3x^2}{2} \Big|_0^2 \\
&= 24 + 6 = 30
\end{aligned}$$

Ex.7.11.4 Verify the divergence theorem for $F = (4xy)i - (y^2)j + (xz)k$, over the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0$ and $z = 1$.

Sol : By the divergence theorem

$$\iiint_V F \cdot n \, dS = \iiint_V \nabla \cdot F \, dv$$

$$\text{Here } F = (4xy)i - (y^2)j + (xz)k$$

$$\nabla \cdot F = \frac{\partial}{\partial x}(4xy) - \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(xz) = 4y - 2y + x = x + 2y$$

$$\begin{aligned}
\iiint_V \nabla \cdot F \, dv &= \int_{x=0}^1 \int_{y=0}^1 \int_{z=0}^1 (x + 2y) dz dy dx = \int_{x=0}^1 \int_{y=0}^1 \left[\int_{z=0}^1 (x + 2y) dz \right] dy dx \\
&= \int_{x=0}^1 \int_{y=0}^1 [xz + 2yz]_0^1 dy dx
\end{aligned}$$

$$= \int_{x=0}^1 \left[\int_{y=0}^1 (x + 2y) dy \right] dx = \int_{x=0}^1 \left(xy + y^2 \right) \Big|_0^1 dx = \int_{x=0}^1 (x + 1) dx = \frac{x^2}{2} + x \Big|_0^1 = \frac{3}{2}$$

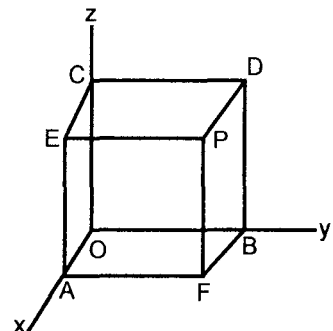
To evaluate $\iiint_S F \cdot n \, dS$

The surface S contains 6 faces (see figure)

S_1 - Face EPFA, S_2 - Face OBDC

S_3 - Face PFBD, S_4 - Face OCEA

S_5 - Face PDCE, S_6 - Face OBFA



The surface integral $\iint_S \mathbf{F} \cdot \mathbf{n} dS$ is equal to the sum of the surface integrals on the above 6 faces.

on S_1 , $\mathbf{n} = \mathbf{i}$, $x = 1$, $\mathbf{F} \cdot \mathbf{n} = 4y$; $dS = dy dz$

$$\therefore \iint_S \mathbf{F} \cdot \mathbf{n} dS = \int_{y=0}^1 \int_{z=0}^1 4y dy dz = \int_{y=0}^1 \left[\int_{z=0}^1 4y dz \right] dy = \int_{y=0}^1 4yz \Big|_0^1 dy = \int_0^1 4y dy = 2y^2 \Big|_0^1 = 2$$

on S_2 , $\mathbf{n} = -\mathbf{i}$, $x = 0$, $\mathbf{F} \cdot \mathbf{n} = 0$

$$\therefore \iint_{S_2} \mathbf{F} \cdot \mathbf{n} dS = 0$$

on S_3 , $\mathbf{n} = \mathbf{j}$, $y = 1$, $\mathbf{F} \cdot \mathbf{n} = -1$, $dS = dx dz$.

$$\therefore \iint_{S_3} \mathbf{F} \cdot \mathbf{n} dS = \int_{x=0}^1 \int_{z=0}^1 [-1 dz] dx = \int_{x=0}^1 \left[-z \Big|_0^1 \right] dx = \int_{x=0}^1 -1 dx = -x \Big|_0^1 = -1$$

on S_4 , $\mathbf{n} = -\mathbf{j}$, $y = 0$, $\mathbf{F} \cdot \mathbf{n} = 0$

$$\therefore \iint_{S_4} \mathbf{F} \cdot \mathbf{n} dS = 0$$

on S_5 , $\mathbf{n} = \mathbf{k}$, $z = 1$, $\mathbf{F} \cdot \mathbf{n} = x$; $dS = dx dy$

$$\therefore \iint_{S_5} \mathbf{F} \cdot \mathbf{n} dS = \int_{x=0}^1 \int_{y=0}^1 x dx dy = \int_{x=0}^1 \left[\int_{y=0}^1 x dy \right] dx = \int_{x=0}^1 xy \Big|_0^1 dx = \int_{x=0}^1 x dx = \frac{x^2}{2} \Big|_0^1 = \frac{1}{2}$$

on S_6 , $\mathbf{n} = \mathbf{k}$, $z = 0$, $\mathbf{F} \cdot \mathbf{n} = 0$

$$\therefore \iint_{S_6} \mathbf{F} \cdot \mathbf{n} dS = 0$$

$$\text{Hence } \therefore \iint_S \mathbf{F} \cdot \mathbf{n} dS = 2 - 1 + \frac{1}{2} = \frac{3}{2}$$

The divergence theorem is verified

Ex.6.11.5: If $\mathbf{r} = xi + yj + zk$, evaluate $\therefore \iint_S \mathbf{r} \cdot \mathbf{n} dS$ where S is any closed surface.

Sol: By the divergence theorem

$$\therefore \iint_S \mathbf{r} \cdot \mathbf{n} dS = \iiint_V \nabla \cdot \mathbf{r} dv, \text{ where } V \text{ is the volume enclosing } S.$$

$$\nabla.r = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

$$\therefore \int \int_s r.n \, ds = \int \int \int_v 3 \, dv = 3v$$

Ex.7.11.6 : If ' ψ ' is a scalar function, prove that $\int \int \int_v \nabla \psi \, dv = \int \int_s \Psi n \, ds$

Sol : By divergence theorem

$$\int \int \int_v \nabla.A \, dv = \int \int_s A.n \, ds \quad \dots (1)$$

Take $A = \psi c$, $C = c_1 i + c_2 j + c_3 k$ is any constant vector.

$$\therefore (1) \Rightarrow \int \int \int_v \nabla.(\Psi c) \, dv = \int \int_s (\Psi c).n \, ds \quad \dots (2)$$

But,

$$\nabla.(\psi c) = (\nabla \psi).c + \psi(\nabla.C)$$

$$= (\nabla \psi).c$$

$$[\because \nabla.c = \frac{\partial}{\partial x}(c_1) + \frac{\partial}{\partial y}(c_2) + \frac{\partial}{\partial z}(c_3) = 0,$$

$$= c. \nabla \psi$$

c_1, c_2, c_3 , being constant]

$$\text{and } (\psi c).n = c.(\psi n)$$

$$[\because a.b = b.a]$$

so that (2) becomes,

$$\int \int \int_v (c. \nabla \Psi) \, dv = \int \int_s \{c.(\Psi n)\} \, ds$$

$$\Rightarrow c. \int \int \int_v \nabla \Psi \, dv = c. \int \int_s \Psi n \, ds$$

$$\Rightarrow \int \int \int_v \nabla \Psi \, dv = \int \int_s \Psi n \, ds \quad (\because c \text{ is arbitrary})$$

Ex.7.11.7 : If s is a closed surface enclosing a volume V and if $F = (lx)i + (my)j + (nz)k$,

$$l, m, n \text{ being constants, evaluate } \int \int_s F.n \, ds$$

Sol : By Gauss' theorem

$$\iint_s F \cdot n \, ds = \iiint_v \nabla \cdot F \, dv$$

$$\nabla \cdot F = \frac{\partial}{\partial x}(lx) + \frac{\partial}{\partial y}(my) + \frac{\partial}{\partial z}(nz) = l + m + n$$

$$\begin{aligned} \therefore \iint_s F \cdot n \, ds &= \iiint_v (l + m + n) \, dv \\ &= (l + m + n) V \end{aligned}$$

Ex.7.11.8 : If $E = \text{curl } A$, evaluate $\iint_s E \cdot n \, ds$ where s is any closed surface

Sol : By Gauss' theorem,

$$\iint_s E \cdot n \, ds = \iiint_v (\text{div } E) \, dv \text{ where } V \text{ is the volume enclosed by } s$$

$$\text{But } \text{Div } E = \text{div } (\text{curl } A) = 0$$

[See 7.5.2 (2)]

$$\therefore \iint_s E \cdot n \, ds = 0$$

Ex.7.11.9 : If s any closed surface and n is unit +ve normal to s , show that $\iint_s n \, ds = 0$

Sol : Consider a constant vector

$$a = a_1 i + a_2 j + a_3 k \quad (\text{i.e. } a_1, a_2, a_3 \text{ are constants})$$

$$\text{Then, } a \cdot \left[\iint_s n \, ds \right] = \iint_s (a \cdot n) \, ds$$

$$= \iiint_v (\nabla \cdot a) \, dv \quad (\text{by divergence theorem})$$

$$= 0 \quad \left[\because \nabla \cdot a = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} = 0 \right]$$

Thus $a \cdot \left[\iint_s n \, ds \right] = 0$ is true where a is any arbitrary vector.

$$\therefore \iint_s n \, ds = 0$$

Ex.7.11.10 : Evaluate $\iint_s (F \cdot n) \, ds$, where s is the region bounded by $y^2 = 8x$, $x = 2$, $z = 0$,

$z = 3$, and $F = 2xi + 3yj + \frac{1}{3}z^3k$, using Gauss' theorem

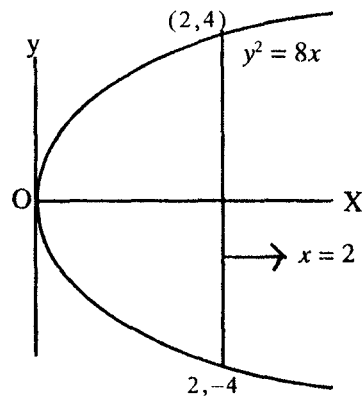
Sol: By the divergence theorem ,

$$\iint_S (\mathbf{F} \cdot \mathbf{n}) ds = \iiint_V (\nabla \cdot \mathbf{F}) dv \quad \dots(1)$$

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}\left(\frac{z^3}{3}\right) = 5 + z^2$$

$$\text{From (1), } \iint_S (\mathbf{F} \cdot \mathbf{n}) ds = \iiint_V (5 + z^2) dv ,$$

$$\begin{aligned} &= \int_{x=0}^2 \int_{y=2\sqrt{2x}}^{2\sqrt{2x}} \left[\int_{z=0}^3 (5 + z^2) dz \right] dy dx \\ &= \int_{x=0}^2 \int_{y=2\sqrt{2x}}^{2\sqrt{2x}} \left[5z + \frac{z^3}{3} \right]_0^3 dy dx \\ &= \int_{x=0}^2 \left[\int_{y=2\sqrt{2x}}^{2\sqrt{2x}} 24 dy \right] \left[\int_{x=0}^2 24 y \Big|_{-2\sqrt{2x}}^{2\sqrt{2x}} dx \right] \\ &= \int_{x=0}^2 96\sqrt{2x} dx = 96\sqrt{2} \left[\frac{x^2}{2} \right]_0^2 = 192\sqrt{2} \end{aligned}$$



Ex.7.11.11 Evaluate $\iint_S (2xi - 3y^2j + z^2k) \cdot \mathbf{n} ds$ over the surface bounded by

$$x^2 + y^2 = 1, z = 0, z = 2, \text{ using Gauss' theorem.}$$

Sol: By the divergence theorem ,

$$\iint_S (\mathbf{F} \cdot \mathbf{n}) ds = \iiint_V (\nabla \cdot \mathbf{F}) dv ,$$

$$\text{Here, } \mathbf{F} = 2xi - 3y^2j + z^2k$$

$$\begin{aligned} \text{Div } \mathbf{F} &= \frac{\partial}{\partial x}(2x) - \frac{\partial}{\partial y}(3y^2) + \frac{\partial}{\partial z}(z^2) \\ &= 2 - 6y + 2z \end{aligned}$$

$$\therefore \iint_S (\mathbf{F} \cdot \mathbf{n}) ds = \int_{x=1}^{+1} \int_{y=\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^2 (2 - 6y + 2z) dz dy dx$$

$$\begin{aligned}
&= \int_{x=-1}^{+1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[\int_{z=0}^2 (2-6y+2z) dz \right] dy dx = \int_{x=-1}^{+1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2z-6yz+z^2) \Big|_{-1}^1 dy dx \\
&= \int_{x=-1}^{+1} \left[\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (8-12y) dy \right] dx = \int_{x=-1}^1 (8y-6y^2) \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = \int_{-1}^1 16\sqrt{1-x^2} dx \\
&= 16 \left[\frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x \right]_{-1}^1 = 8\pi
\end{aligned}$$

Ex.7.11.12: Evaluate using the divergence theorem $\iiint_V (F.n) dS$ where S is the surface of the sphere $x^2 + y^2 + z^2 = b^2$ in the first octant and $F = yi + zj + xk$

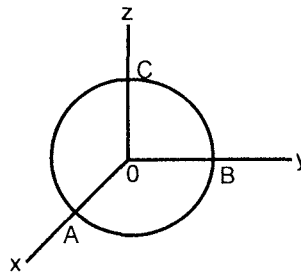
Sol: By divergence theorem,

$$\therefore \iiint_V F.n ds = \iiint_V \nabla.F dv \quad \dots (i)$$

$$F = yi + zj + xk$$

$$\nabla.F = 0$$

$$\therefore \iiint_V (\nabla.F) dv = 0 \quad \dots (ii)$$



Let us evaluate the surface integrals over the faces OAB, OBC and OCA.

$$\int_{OAB} F.n ds = - \int_{x=0}^b \int_{y=0}^{\sqrt{b^2-x^2}} x dx dy \quad (\because n = -k)$$

$$= - \int_0^b xy \Big|_0^{\sqrt{b^2-x^2}} dx = - \int_0^b x \sqrt{b^2-x^2} dx = -\pi \frac{b^3}{3}$$

$$\begin{aligned}
 \text{Similarly } \iint_{OAB} \mathbf{F} \cdot \mathbf{n} ds &= \iint_{OAB} \mathbf{F} \cdot \mathbf{n} ds + \iint_{ABC} \mathbf{F} \cdot \mathbf{n} ds + \iint_{OCA} \mathbf{F} \cdot \mathbf{n} ds + \iint_{ABC} \mathbf{F} \cdot \mathbf{n} ds \\
 &= -\pi b^3 + \iint_{ABC} \mathbf{F} \cdot \mathbf{n} ds \quad \dots(\text{iii})
 \end{aligned}$$

From (i), (ii), and (iii), we get

$$\begin{aligned}
 0 &= -\pi b^3 + \iint_{ABC} \mathbf{F} \cdot \mathbf{n} ds \\
 \iint_{ABC} \mathbf{F} \cdot \mathbf{n} ds &= \pi b^3
 \end{aligned}$$

Ex. 7.11.13: Verify divergence theorem for $\mathbf{F} = 4xi - 2y^2j + z^2k$ taken over the region bounded by

$$x^2 + y^2 = 4, \quad z = 0 \quad \text{and} \quad z = 3.$$

Sol: By the divergence theorem, we have

$$\iiint_v \text{Div} \vec{F} dv = \iint_s \vec{F} \cdot \vec{n} ds \quad \dots(1)$$

$$(1) \text{ Div} \vec{F} = \frac{\partial}{\partial x}(4x) - \frac{\partial}{\partial y}(2y^2) + \frac{\partial}{\partial z}(z^2) = 4 - 4y + 2z$$

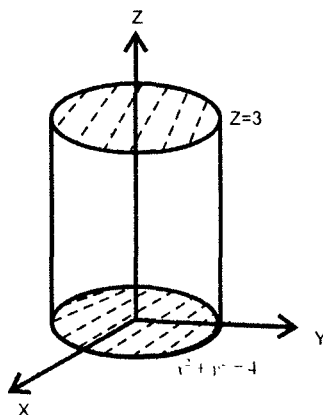
$$\therefore \text{ L.H.S. of (1) } = \int_{x=-2}^2 \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[\int_{z=0}^3 (4 - 4y + 2z) dz \right] dy dx$$

$$= \int_{x=-2}^2 \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z - 4yz + z^2 \right]_{z=0}^3 dy dx = \int_{x=-2}^2 \left[\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y) dy \right] dx$$

$$= 42 \int_{x=-2}^2 \sqrt{4-x^2} dx = 84\pi$$

[Do the integration w.r.t.x yourself, taking $x = 2 \sin \theta$]

$$(2) \text{ Evaluate of surface integral } \iint_s \vec{F} \cdot \vec{n} ds$$



The given surface of the cylinder can be divided into 3 parts, namely

- (a) S_1 : the circular surface $z = 0$
- (b) S_2 : the surface $z = 3$ (circular) and
- (c) S_3 : the cylindrical portion of $S : x^2 + y^2 = 4, z = 0, z = 3$

we now find $\iint \vec{F} \cdot \vec{n} \, ds$ over S_1, S_2, S_3 . If we add them, we get R.H.S of (1).

(a) on $S_1 : z = 0$; $\vec{n} = -\vec{k}$; $\vec{F} \cdot \vec{n} = -(4xi - 2y^2j) \cdot \vec{k} = 0$; $\therefore \iint_{S_1} \vec{F} \cdot \vec{n} \, ds = 0$.

(b) on $S_2 : z = 3$; $\vec{n} = \vec{k}$; $\vec{F} \cdot \vec{n} = (4xi - 2y^2j + 9k) \cdot \vec{k} = 9$; $ds = \frac{dxdy}{|\vec{n} \cdot \vec{k}|} = dxdy$

$$\therefore \iint_{S_2} \vec{F} \cdot \vec{n} \, ds = \iint_{S_2} 9 \, dxdy = 9A, \text{ where } A \text{ is the area of the circle}$$

$$x^2 + y^2 = 4, = 9\pi(2^2) = 36\pi$$

(c) on S_3 : Let $\phi = x^2 + y^2 - 4 = 0$;

$$\vec{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{2(xi + yi)}{2\sqrt{x^2 + y^2}} = \frac{xi + yi}{2} \text{ (since } x^2 + y^2 = 4 \text{)}$$

$$\vec{F} \cdot \vec{n} = \frac{4x^2 - 2y^3}{2} = 2x^2 - y^3 ;$$

To evaluate $\iint_{S_3} \vec{F} \cdot \vec{n} \, ds$, take $x = 2 \cos \theta, y = 2 \sin \theta$,

and $ds = 2d\theta dz$; limits of z are 0 to 3 and those of θ are 0 to 2π .

$$\begin{aligned}\text{Hence } \iint_{S_3} \vec{F} \cdot \vec{n} \, ds &= \int_{\theta=0}^{2\pi} \int_{z=0}^3 (8\cos^2 \theta - 8\sin^3 \theta) 2d\theta \, dz \\ &= 16 \int_{\theta=0}^{2\pi} \left[(\cos^2 \theta - \sin^3 \theta) z \right]_{z=0}^3 d\theta = 48 \int_{\theta=0}^{2\pi} (\cos^2 \theta - \sin^3 \theta) d\theta = 48\pi\end{aligned}$$

[Do the integration w.r.t. θ yourself]

$\therefore$ R.H.S of (1) = $0 + 36\pi + 48\pi = 84\pi$; $\therefore$ L.H.S = R.H.S

Hence the theorem is verified.

Exercise – 7K

1. Verify Gauss's divergence theorem for $\mathbf{A} = (x^2 - yz)\mathbf{i} + (y^2 - zx)\mathbf{j} + (z^2 - xy)\mathbf{k}$ taken over the rectangular parallalopiped $0 \leq x \leq 2$, $0 \leq y \leq 3$, $0 \leq z \leq 1$. [Ans:36]
2. Use the divergence theorem to find $\iiint_S \mathbf{F} \cdot \mathbf{n} \, dS$, where $\mathbf{f} = (3x + 2z^2)\mathbf{i} - (z^2 - 2y)\mathbf{j} + (y^3 - 2z)\mathbf{k}$ and S is the surface of the sphere with centre at $(2, -1, 3)$ and radius 2 units. [Ans:32 π]
3. Verify the divergence theorem for the vector function, $\mathbf{A} = (4xz)\mathbf{i} - (y^2)\mathbf{j} + (yz)\mathbf{k}$, taken over the unit cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = 1$. [Ans:3/2]
4. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and S is the surface of the rectangular parallelepiped bounded by planes $x = 0, y = 0, z = 0, x = a, y = b$ and $z = c$, find the value of $\iiint_S \mathbf{r} \cdot \mathbf{n} \, dS$ using Gauss's theorem. Verify your answer by direct evaluation of the integral. [Ans:3abc]
5. Use the divergence theorem to evaluate $\iiint_S \mathbf{A} \cdot \mathbf{n} \, ds$ for $\mathbf{A} = (2x)\mathbf{i} - (2y)\mathbf{j} + (3z)\mathbf{k}$ where s is the sphere given by $(x-1)^2 + y^2 + z^2 = 1$ [Ans:4 π]
6. If $\mathbf{V} = (lx)\mathbf{i} + (my)\mathbf{j} + (nz)\mathbf{k}$ and l, m, n being constants show that $\iiint_S \mathbf{V} \cdot \mathbf{ds} = \frac{32\pi}{3}(l+m+n)$. where S is the surface the sphere $(x-3)^2 + (y-2)^2 + (z-1)^2 = 4$.

7.12 Stoke's Theorem

7.12.1 Stoke's Theorem

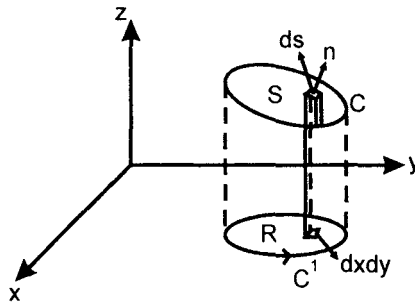
Let (1) S be an open, two-sided surface bounded by a simple closed curve C .

(2) A be a vector function having continuous derivatives

$$\text{Then, } \oint_C A \cdot dr = \iint_S (\nabla \times A) \cdot n ds = \iint_S (\nabla \times A) \cdot ds$$

where C travels in the +ve direction and n is the unit +ve (outward drawn) normal to S .

Proof :



Let S be the surface. Let the projections of S on the coordinate planes be regions bounded by simple closed curves.

Let ' R ' the projection of S on xy plane be bounded by C^1 . (see the figure above).

Let the equation of S be $z = \phi_1(x, y)$ where ϕ_1 is a single valued, continuous and differentiable function.

Let $A = A_1 i + A_2 j + A_3 k$

$$\text{Then } \nabla \times (A_1 i) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & 0 & 0 \end{vmatrix} = \frac{\partial A_1}{\partial z} j - \frac{\partial A_1}{\partial y} k$$

$$\{\nabla \times (A_1 i)\} \cdot n \, ds = \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_1}{\partial y} \right) \cdot n \, ds = \left\{ \frac{\partial A_1}{\partial z} (n \cdot j) - \frac{\partial A_1}{\partial y} (n \cdot k) \right\} ds \quad \dots (i)$$

The position vector r of any point on S can be taken as

$$\begin{aligned} r &= xi + yj + zk \\ &= xi + yj + \phi_1(x, y)k \end{aligned}$$

$$\text{and } \frac{\partial r}{\partial y} = j + \frac{\partial \phi_1}{\partial y} k$$

But $\frac{\partial r}{\partial y}$ being the vector tangent to S , it is $\perp$ to n .

$$\therefore \frac{\partial r}{\partial y} \cdot n = 0 \Rightarrow n \cdot j = -\frac{\partial \phi_1}{\partial y} (n \cdot k)$$

$$\begin{aligned} \therefore (i) \Rightarrow \{ \nabla \times A_1 i \} \cdot n \, ds &= \left(-\frac{\partial A_1}{\partial z} \frac{\partial \phi_1}{\partial y} n \cdot k - \frac{\partial A_1}{\partial y} n \cdot k \right) ds \\ &= -\left(\frac{\partial A_1}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial A_1}{\partial y} \right) (n \cdot k) ds \quad (\because z = \phi_1) \end{aligned} \quad \dots (ii)$$

on S , $A_1(x, y, z) = A_1(x, y, \phi_1(y)) = G(x, y)$ (say)

$$\frac{\partial A_1}{\partial y} + \frac{\partial A_1}{\partial z} \cdot \frac{\partial z}{\partial y} = \frac{\partial G}{\partial y}$$

$\therefore$ (ii) becomes,

$$\{ \nabla \times (A_1 i) \} \cdot n \, ds = -\frac{\partial G}{\partial y} (n \cdot k) \, ds = -\frac{\partial G}{\partial y} \, dx \, dy \quad [\because n \cdot k \, ds = dx \, dy]$$

$$\iint_S \{ \nabla \times A_1 i \} \cdot n \, ds = \iint_R -\frac{\partial G}{\partial y} \, dx \, dy = \oint_C G \, dx, \text{ by Green's theorem in the plane. Now}$$

at each point (x, y) of C , the value of G is the same as the value of A_1 at each point (x, y, z) of C , and since dx is same for both curves, we have

$$\oint_C G \, dx = \oint_C A_1 \, dx$$

Hence,
$$\iint_S \{\nabla \times (A_1 i)\} \cdot n \, ds = \oint_C A_1 \, dx \quad \dots (iii)$$

|||ly by projecting S on yz and zx planes, it can be shown that,

$$\iint_S \{\nabla \times (A_2 j)\} \cdot n \, ds = \oint_C A_2 \, dy \quad \dots (iv)$$

$$\iint_S \{\nabla \times (A_3 k)\} \cdot n \, ds = \oint_C A_3 \, dz \quad \dots (v)$$

Adding (iii), (iv) and (v),
$$\iint_S \nabla \times A \cdot n \, ds = \oint_C A \cdot dr$$

(since $A \cdot dr = A_1 \, dx + A_2 \, dy + A_3 \, dz$)

Hence the theorem is proved.

Ex.7.12.1 : (a) Express Stoke's theorem in words and (b) obtain its cartesian form.

Sol : (a) The line integral of the tangential component of a vector A taken around a simple closed curve C is equal to the surface integral of the normal component of curl A taken over a surface S having C as its boundary.

(b) As in 7.11.2(b)

$$A = A_1 i + A_2 j + A_3 k$$

$$n = (\cos \alpha) i + (\cos \beta) j + (\cos \gamma) k$$

$$\text{Then, } \nabla \times A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) i + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) j + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) k$$

$$(\nabla \times A) \cdot n = \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \cos \alpha + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) \cos \beta + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \cos \gamma$$

$$A \cdot dr = (A_1 i + A_2 j + A_3 k) \cdot (dx i + dy j + dz k) = A_1 dx + A_2 dy + A_3 dz$$

Hence the cartesian form of the Stoke's theorem can be stated as,

$$\begin{aligned} & \iint_S \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \cos \alpha + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) \cos \beta + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \cos \gamma \right] ds \\ &= \oint_C A_1 dx + A_2 dy + A_3 dz \end{aligned}$$

Solved Examples

Ex.7.12.3 : Verify Stoke's Theorem for $A = (x - 2y)i + yz^2j + y^2zk$, where S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

Sol : The boundary of the projection of S in the xy -plane is a circle with centre at origin and unit radius. Its parametric equations are $x = \cos\theta$, $y = \sin\theta$, $z = 0$, $0 \leq \theta < 2\pi$

$$\therefore dx = (-\sin\theta)d\theta, dy = (\cos\theta)d\theta.$$

$$\therefore \oint_C A \cdot dr = \oint_C (x - 2y)dx + yz^2 dy + y^2 z dz$$

$$= \int_{\theta=0}^{2\pi} [\cos\theta - 2\sin\theta] (-\sin\theta) d\theta \quad (\because z = 0)$$

$$= \int_0^{2\pi} \left[\frac{-\sin 2\theta}{2} + 1 - \cos 2\theta \right] d\theta = \frac{\cos 2\theta}{4} + \theta - \frac{\sin 2\theta}{2} \Bigg|_0^{2\pi} = 2\pi$$

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x - 2y) & yz^2 & y^2 z \end{vmatrix} = 2k$$

$$\iint_S (\nabla \times A) \cdot n ds = \iint_S 2(n \cdot k) ds = 2 \iint_R dx dy$$

($n \cdot k ds = dx dy$ and R is the projection of S on the xy plane)

$$\iint_S (\nabla \times A) \cdot n ds = 2 \int_{x=-1}^{+1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} dy dx = 4 \int_{-1}^{+1} \sqrt{1-x^2} dx = 8 \int_0^1 \sqrt{1-x^2} dx = 2\pi$$

Hence Stoke's theorem is verified.

Ex.7.12.4 : Prove that a necessary and sufficient condition that $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for every closed curve C is that $\nabla \times \mathbf{F} = 0$.

Proof: (a) The condition is necessary : Let $\nabla \mathbf{F} = 0$;

Then by Stoke's theorem , $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, ds = 0$

(b) The condition is sufficient:

Suppose that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ around every closed path c.

Assume that $\nabla \times \mathbf{F} \neq 0$ at some point P. then , assuming that $\nabla \mathbf{F}$ is continuous , there exists a region with P as its interior point where $\nabla \mathbf{F} = 0$. Let S be surface contained in this region and let the normal n to S at each point has the same direction as $\nabla \mathbf{F}$.

Then $\nabla \mathbf{F} = a \mathbf{n}$, (a being a +ve constant); Let C be the boundary of S.

Then by Stokes theorem , $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \mathbf{F}) \cdot \mathbf{n} \, ds = a \iint_S \mathbf{n} \cdot \mathbf{n} \, ds > 0$

Which is a contradiction to the hypothesis that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$; $\therefore \nabla \mathbf{F} = 0$

Note: It follows that $\nabla \mathbf{F} = 0$ is also a necessary and sufficient condition for the line integral $\int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r}$ to be independent of path joining the points P_1 and P_2 . (see 7.6.4)

Ex.7.12.5 If $\mathbf{r} = xi + yj + zk$, show that $\oint_C \mathbf{r} \cdot d\mathbf{r} = 0$

By Stoke's theorem , $\oint_C \mathbf{r} \cdot d\mathbf{r} = \iint_C (\text{Curl } \mathbf{r}) \cdot \mathbf{n} \, ds$ (1)

$$\text{But curl } \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = 0$$

$$\therefore (1) \Rightarrow \int_C \mathbf{r} \cdot d\mathbf{r} = 0$$

Ex.7.12.6 : If ' f ' and ' g ' are scalar functions, show that

$$\int_C f(\text{grad } g) \cdot d\mathbf{r} = - \int_C g(\text{grad } f) \cdot d\mathbf{r}$$

Sol : By stoke's theorem,

$$\int_C \{\text{grad } (fg)\} \cdot d\mathbf{r} = \iint_S [\text{curl}\{\text{grad}(fg)\}] \cdot \mathbf{n} \, ds = 0, \text{ since } \text{curl grad } (fg) = 0$$

$$\therefore \int_C \{\text{grad } (fg)\} \cdot d\mathbf{r} = 0 \quad \dots (1)$$

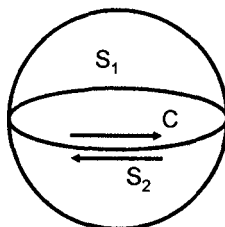
$$\text{But } \text{grad}(fg) = f(\text{grad } g) + g(\text{grad } f) \quad \dots (2)$$

Hence the result [from (1) and (2)]

Ex.7.12.7 : If $\mathbf{A}$ is any vector function, prove by stoke's theorem that $\text{div curl } \mathbf{A} = 0$.

Sol : Let V be any volume enclosed by a closed surface S . Then by Gauss' divergence theorem. We get,

$$\iiint_V \nabla \cdot (\text{curl } \mathbf{A}) \, dv = \iint_S (\text{curl } \mathbf{A}) \cdot \mathbf{n} \, ds \quad \dots (1)$$



Divide the surface S into two portions S_1 and S_2 by a closed curve C .

$$\text{Then } \iint_S (\text{curl } \mathbf{A}) \cdot \mathbf{n} \, ds$$

$$= \iint_{S_1} (\text{curl } \mathbf{A}) \cdot \mathbf{n} \, ds_1 + \iint_{S_2} (\text{curl } \mathbf{A}) \cdot \mathbf{n} \, ds_2$$

$$= \int_C \mathbf{A} \cdot d\mathbf{r} - \int_C \mathbf{A} \cdot d\mathbf{r}$$

= 0, by Stoke's theorem, since the +ve directions along the boundaries of S_1 and S_2 are opposite.

$$\therefore (1) \Rightarrow \iiint_V (\nabla \cdot (\text{curl } A)) dv = 0$$

Since this is true for all volume elements V , we have,

$$\nabla \cdot \text{curl } A = 0 \Rightarrow \text{div}(\text{curl } A) = 0$$

Ex.7.12.8 : Use Stoke's theorem and prove that $\text{curl grad } f = 0$, where 'f' is a scalar function.

Sol : If S is a surface enclosed by a simple closed curve C , we have, by Stoke's theorem,

$$\iint_S \{\text{curl}(\text{grad } f)\} \cdot n \, ds = \int_C (\text{grad } f) \cdot dr \quad \dots (1)$$

Now, $\text{grad } f \cdot dr$

$$= \left(\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k \right) \cdot (dx i + dy j + dz k)$$

$$= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = df$$

$$\therefore \int_C (\text{grad } f) \cdot dr = \int_A^A df$$

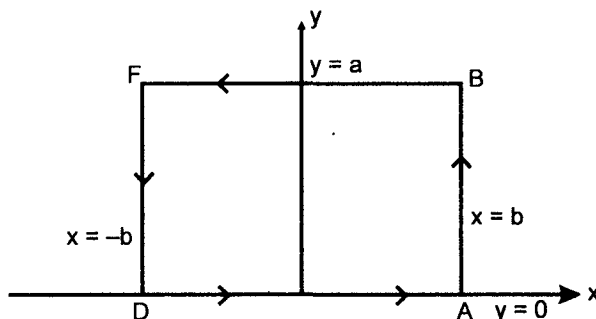
$$= f(A) - f(A) = 0, \text{ where } A \text{ is any point on } C$$

$$\therefore (1) \Rightarrow \iint_S \{\text{curl}(\text{grad } f)\} \cdot n \, ds = 0$$

Since this equation is true for all surface elements S ,

we have, $\text{curl}(\text{grad } f) = 0$

Ex.7.12.9 : Verify stoke's theorem for $A = y^2 j - 2xy j$ taken round the rectangle bounded by $x = \pm b, y = 0, y = a$



Sol : $\text{curl } A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & -2xy & 0 \end{vmatrix} = -4yk$

For the given surface S, $n = k$

$$\therefore (\text{curl } A) \cdot n = -4y$$

Hence $\iiint_S (\text{curl } A) \cdot n \, ds$

$$\begin{aligned} &= \iint_S -4y \, dx \, dy = \int_{y=0}^a \left[\int_{x=-b}^b -4y \, dx \right] dy = \int_0^a -4xy \Big|_{-b}^b \, dy = \int_0^a -8by \, dy \\ &= -4by^2 \Big|_0^a = -4a^2b \end{aligned} \quad \dots (1)$$

$$\int_C A \cdot dr = \int_{DA} + \int_{AB} + \int_{BF} + \int_{FD}$$

$$\int_C A \cdot dr = y^2 dx - 2xy dy$$

Along DA, $y = 0, dy = 0, \Rightarrow \int_{DA} A \cdot dr = 0 \quad (\because A \cdot dr = 0)$

Along AB, $x = b, dx = 0$

$$\therefore \int_{AB} A \cdot dr = \int_{y=0}^a -2by \, dy = -by^2 \Big|_0^a = -a^2b$$

Along BF, $y = a, dy = 0$

$$\therefore \int_{BF} A \cdot dr = \int_b^{-b} a^2 \, dx = -2a^2b$$

Along FD, $x = -b, dx = 0$

$$\therefore \int_{FD} A \cdot dr = \int_a^0 2by \, dy = -by^2 \Big|_a^0 = -a^2b$$

$$\therefore \int_C A \cdot dr = 0 - a^2b - 2a^2b - a^2b = -4a^2b \quad \dots (2)$$

From (1) and (2), $\int_C A \cdot dr = \iiint_S \text{curl}(A) \cdot n \, ds$

Hence the theorem is verified

Ex.7.12.10: Use stoke's theorem to evaluate the integral $\int_C A \cdot dr$ where $A = 2y^2i + 3x^2j - (2x+z)k$, and C is the boundary of the triangle whose vertices are $(0,0,0)$, $(2,0,0)$, $(2,2,0)$.

Sol:
$$\text{Curl } A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y^2 & 3x^2 & -2x-z \end{vmatrix}$$

$$= 2j + (6x - 4y)k$$

Since the z -coordinate of each vertex of the triangle is zero, the triangle lies in the xy -plane.

$$\therefore n = k.$$

$$\therefore (\text{curl } A) \cdot n = 6x - 4y$$

consider the triangle in xy -plane.

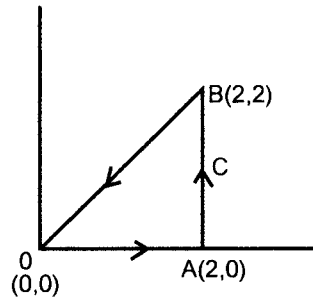
Equation of the straight line OB is $y = x$

By Stoke's theorem,

$$\int_C A \cdot dr = \iint_S (\text{Curl } A) \cdot n \, ds$$

$$= \int_{x=0}^2 \int_{y=0}^{y=x} (6x - 4y) \, dx \, dy = \int_{x=0}^2 \left[\int_{y=0}^x (6x - 4y) \, dy \right] dx$$

$$= \int_{x=0}^2 \left[6xy - 2y^2 \right]_0^x dx = \int_0^2 (6x^2 - 2x^2) dx = 4 \left[\frac{x^3}{3} \right]_0^2 = \frac{32}{3}.$$



Ex. 7.12.11 Use Stoke's theorem to evaluate $\iint_S (\text{curl } A) \cdot n \, ds$, where $A = 2yi + (x - 2zx)j + xyk$, and S is the surface of the sphere $x^2 + y^2 + z^2 = b^2$ above the xy -plane.

Sol: The boundary C of the surface S is the circle $x^2 + y^2 + z^2 = b^2$, $z = 0$.

The parametric equations of C are $x = b \cos \theta$, $y = b \sin \theta$, $z = 0$, $0 \leq \theta < 2\pi$

By Stoke's theorem, we have,

$$\iint_S (\text{Curl } A) \cdot n \, ds = \int_C A \cdot dr$$

$$\begin{aligned}
&= \int_C 2y dx + (x - 2zx) dy + xy dz = \int_C 2y dx + x dy \quad (\because z = 0, dz = 0 \text{ on } C) \\
&= \int_0^{2\pi} (2b \sin \theta)(-b \sin \theta) d\theta + b \cos \theta \cdot b \cos \theta d\theta \quad [\because x = b \cos \theta \Rightarrow dx = -b \sin \theta d\theta \\
&\quad \text{and } y = b \sin \theta \Rightarrow dy = b \cos \theta d\theta] \\
&= b^2 \int_0^{2\pi} (\cos^2 \theta - 2 \sin^2 \theta) d\theta = b^2 \int_0^{2\pi} \left[\frac{1 + \cos 2\theta}{2} - (1 - \cos 2\theta) \right] d\theta \\
&= \frac{b^2}{2} \int_0^{2\pi} (-1 + 3 \cos 2\theta) d\theta = \frac{b^2}{2} \left[-\theta + \frac{3 \sin 2\theta}{2} \right]_0^{2\pi} = -\frac{b^2}{2} \cdot 2\pi = -\pi b^2
\end{aligned}$$

Ex. 7.12.12 Apply Stoke's theorem to evaluate $\int A \cdot dr$, where $A = (x - y)i + (2y + z)j +$

$(y - z)k$ and C is the boundary of the triangle whose vertices are $\left(\frac{1}{6}, 0, 0\right)$, $\left(0, \frac{1}{3}, 0\right)$

and $\left(0, 0, \frac{1}{2}\right)$.

Sol: Let $A = \left(\frac{1}{6}, 0, 0\right)$, $B = \left(0, \frac{1}{3}, 0\right)$, $C = \left(0, 0, \frac{1}{2}\right)$.

The equation of the plane ABC is (by intercept form),

$$\frac{x}{1/6} + \frac{y}{1/3} + \frac{z}{1/2} = 1 \Rightarrow 6x + 3y + 2z = 1 \quad \dots (1)$$

The direction ratios of the normal to (1) are 6, 3, 2

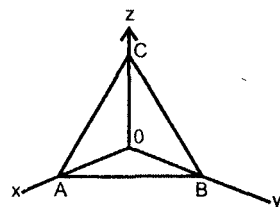
$$\therefore \text{Direction cosines are } \frac{6}{7}, \frac{3}{7}, \frac{2}{7} \quad \left(\because \sqrt{6^2 + 3^2 + 2^2} = 7\right)$$

If n is the unit normal to the plane, $n = \frac{6}{7}i + \frac{3}{7}j + \frac{2}{7}k$

$$A = (x - y)i + (2y + z)j + (y - z)k$$

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x - y & 2y + z & y - z \end{vmatrix} = k$$

$$\therefore (\nabla \times A) \cdot n = \frac{2}{7}$$



∴ By Stoke's theorem,

$$\int_C \mathbf{A} \cdot d\mathbf{r} = \iint_S (\nabla \mathbf{F}) \cdot \mathbf{n} \, ds = \frac{2}{7} \iint_S ds = \frac{2}{7} \text{ (Area of triangle ABC)} \quad \dots(2)$$

To find the area of triangle ABC :

$$AB = \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{\sqrt{5}}{6},$$

$$AC = \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{10}}{6}$$

Direction ratios of AB are $\frac{-1}{6}, \frac{1}{3}, 0$.

Direction ratios of AC are $\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0$

Direction ratios of BC are $\frac{-1}{\sqrt{10}}, 0, \frac{3}{\sqrt{10}}$

$$\cos \hat{CAB} = \left(\frac{-1}{\sqrt{5}}\right)\left(\frac{-1}{\sqrt{10}}\right) + 0 + 0 = \frac{1}{\sqrt{50}}$$

$$\sin \hat{CAB} = \frac{7}{\sqrt{50}}$$

$$\therefore \text{Area of triangle } ABC = \frac{1}{2} AB \cdot AC \sin \hat{CAB} = \frac{1}{2} \cdot \frac{\sqrt{5}}{6} \cdot \frac{\sqrt{10}}{6} \cdot \frac{7}{\sqrt{50}} = \frac{7}{72}$$

$$\int_C \mathbf{A} \cdot d\mathbf{r} = \frac{2}{7} \times \frac{7}{72} = \frac{1}{36} \text{ [from (2)]}$$

Ex.7.12.13 Evaluate $\iint_S (\text{curl } \mathbf{A}) \cdot \mathbf{n} \, ds$ taken over the portion s of the surface

$x^2 + y^2 + z^2 - 2fx + fz = 0$ above the xy plane $z = 0$, if

$\mathbf{A} = \sum (x^2 + y^2 - z^2)\mathbf{i}$ and verify Stoke's theorem.

Solution: Let 'S' denote the portion of the surface, $x^2 + y^2 + z^2 - 2fx + fz = 0$ above the xy -plane $z = 0$.

The surface S meets the xy -plane in the circle 'C', whose equations are

$$x^2 + y^2 - 2fx = 0, z = 0.$$

$$\Rightarrow (x-f)^2 + y^2 = f^2, z=0$$

$\therefore$ The parametric equations of 'C' can be taken as

$$x = f + f \cos \theta, \quad y = f \sin \theta, \quad z = 0 \quad (0 \leq \theta < 2\pi)$$

Let S_1 denote the plane region bounded by C. If S^1 is the surface consisting of S and S_1 , S^1 is a closed surface.

$\therefore$ From example 7.11.8 on divergence theorem, we have

$$= 0$$

$$\text{i.e., } \iint_S (\text{Curl } A) \cdot 2 \, ds + \iint_{S_1} (\text{Curl } A) \cdot n \, ds = 0 \quad [\because S^1 \text{ consists of S and } S_1]$$

$$\text{i.e., } \iint_S (\text{Curl } A) \cdot n \, ds - \iint_{S_1} (\text{curl } A) \cdot k \, ds = 0 \quad [\because n = -k \text{ on } S_1]$$

$$\therefore \iint_S (\text{Curl } A) \cdot n \, ds = \iint_{S_1} (\text{curl } A) \cdot k \, ds \quad \dots (1)$$

$$\text{Now, curl } A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 + y^2 - z^2 & y^2 + z^2 - x^2 & z^2 + x^2 - y^2 \end{vmatrix}$$

$$= i(-2y - 2z) + j(-2z - 2x) + k(-2x - 2y)$$

$$\therefore (\text{curl } A) \cdot k = -2(x + y)$$

$$\therefore \text{From (1), } \iint_S (\text{curl } A) \cdot n \, ds = -2 \iint_{S_1} (x + y) \, ds$$

Polar equation of S_1 is $r = 2f \cos \theta$.

$\therefore$ changing to polar coordinates,

$$\begin{aligned} \iint_{S_1} (\text{curl } A) \cdot n \, ds &= -2 \int_{\theta=0}^{\pi} \int_{r=0}^{2f \cos \theta} (r \cos \theta + r \sin \theta) r \, d\theta \, dr \\ &= -2 \int_{\theta=0}^{\pi} \left[\int_{r=0}^{2f \cos \theta} (\cos \theta + \sin \theta) r^2 \, dr \right] d\theta = -2 \int_{\theta=0}^{\pi} (\cos \theta + \sin \theta) \frac{r^3}{3} \Big|_0^{2f \cos \theta} d\theta \end{aligned}$$

$$\begin{aligned}
&= -2 \times \frac{8f^3}{3} \int_0^\pi (\cos \theta + \sin \theta) \cos^3 \theta d\theta = -\frac{16f^3}{3} \int_0^\pi \cos^4 \theta d\theta \quad [\because \int_0^\pi \cos^3 \theta \sin \theta d\theta = 0] \\
&= \frac{16f^3}{3} \times 2 \int_0^{\pi/2} \cos^4 \theta d\theta = -\frac{32f^3}{3} \cdot \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = -2\pi f^3 \quad \dots (2)
\end{aligned}$$

Again, $\int_C A \cdot dr$

$$\begin{aligned}
&= \int_C (x^2 + y^2 - z^2) dx + (y^2 + z^2 - x^2) dy + (z^2 + x^2 - y^2) dz \\
&= \int_C (x^2 + y^2) dx + (y^2 - x^2) dy \quad [\because \text{on } C \quad z = 0, \quad dz = 0] \\
&= \int_0^{2\pi} [(f + f \cos \theta)^2 + f^2 \sin^2 \theta] (-f \sin \theta) d\theta \\
&\quad + \int_0^{2\pi} [f^2 \sin^2 \theta - (f + f \cos \theta)^2] (f \cos \theta) d\theta \\
&= f^3 \left[\int_0^{2\pi} \{ (1 + \cos \theta)^2 + \sin^2 \theta \} (-\sin \theta) + \{ \sin^2 \theta - (1 + \cos \theta)^2 \} \cos \theta d\theta \right] \\
&= f^3 \left[\int_0^{2\pi} -2 \sin \theta d\theta + \int_0^{2\pi} -2 \sin 2\theta d\theta + \int_0^{2\pi} -2 \cos \theta d\theta + \int_0^{2\pi} -\cos^3 \theta d\theta \right. \\
&\quad \left. - \int_0^{2\pi} (1 + \cos 2\theta) d\theta + \int_0^{2\pi} \sin^2 \theta \cos \theta d\theta \right] \quad (\text{on simplification}) \\
&= f^3 (-2\pi) = -2\pi f^3 \quad (\text{all other integrals vanish}) \quad \dots (3)
\end{aligned}$$

from (2) and (3), we have

$$\int_S (\text{Curl } A) \cdot nds = \int_C A \cdot dr$$

which verifies Stoke's theorem.

Exercise-7(I)

1. $f F = (xe^x)i + (3y^2)j - (z)k$, and C is the $x^2 + y^2 = 9, z = 2$, evaluate $\oint_C F \cdot dr$ using Stoke's theorem.
[Ans: 0]
2. Apply Stoke's theorem to obtain the value of the integral $\oint_C V \cdot dr$, where $V = (3y^2)i + (2x^2)j - (x + 2z)k$ and C is the boundary of the triangle whose vertices are $(0,0,0), (1,0,0)$ and $(1,1,0)$
[Ans: 1]
3. Verify Stoke's theorem for $F = (2x - y)i - (yz^2)j - (y^2z)k$ if S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is the boundary.
[Ans: π]
4. If $F = (y - z + 2)i + (yz + 4)j - (xz)k$ and S represents the surface of the cube $x = 0, y = 0, z = 0, x = 2, y = 2, z = 2$ above the xy plane, verify that $\iint_S (\text{Curl } F) \cdot ds = \oint_C F \cdot dr$, C being the boundary of S traversed in the +ve direction.
[Ans: Each integral = -4]
5. Find the value of the integral $\oint_C (yz)dx + (zx)dy + (xy)dz$, using Stoke's theorem where C is the Curve $x^2 + y^2 = 4, z = y^2$
[Ans: 0]
6. Verify Stoke's theorem for the function $V = (3x^2)i + (2xy)j$, integrated along the square $x = 0, y = 0, x = 1, y = 1$ in the xy -plane..
[Ans: 0]
7. Evaluate by Stoke's theorem the integral $\oint_C A \cdot dr$ where $A = (2 \sin z)i - (3 \cos x)j + (\sin y)k$; where C is the boundary of the rectangle $0 \leq x \leq \pi, 0 \leq y \leq 1, z = 0$
[Ans: 6]

Exercise – 7(m)

1. If $f(x, y, z) = x^l y^m z^n - 1$, find the directional derivative of f at $(1, 1, 1)$ in the direction of $(i + 2j + 2k)$

$$[\text{Ans: } \frac{1}{3}(l + 2m + 2n)]$$

2. Find the acute angle between the surfaces

$$x^2 + y^2 + z^2 = 6 \text{ and } 3xyz + y^2z - xy + 3 = 0 \text{ at } (1, -1, 2)$$

$$[\text{Ans: } \cos^{-1}\left(\frac{\sqrt{5}}{3}\right)]$$

3. If $r = xi + yj + zk$, and p, q are constant vectors, show that $\text{Div}\{(r \times p) \times q\} = -2(p, q)$

4. If $F = (x^2y)i - (y^2z)j + (z^2x)k$, find $\text{curl } F$ at $(1, -2, 3)$

5. If $f = xyz(x + y + z)$, prove that $\text{curl grad } f = 0$

6. A fluid motion is given by $V = (z^3)i - (y^2)j + (3xz^2)k$. Show that it is irrotational. Find its velocity potential ϕ such that $V = \nabla \phi$

$$[\text{Ans: } \phi = xz^3 - \frac{y^3}{3} + c]$$

7. Evaluate $\int_{(0,0)}^{(2,1)} (4x^3 - 12x^2y^2)dx - (8x^3y)dy$ along the path $x^3 - 3xy^2 = 2y^3$

$$[\text{Ans: } 16]$$

8. If $V = (xy)i - (yz)j + (zx^2)k$ and S is the surface of the cube bounded by $x = 0, x = 2, y = 0, y = 2, z = 0, z = 2$, evaluate $\int_S V \cdot n \, ds$

$$[\text{Ans: } 32/3]$$

9. If $f = 4x + yz$, evaluate $\iiint_V f \, dv$ over the region in the first octant bounded by

$$x^2 + y^2 = 1, \quad z = 0, \quad z = 3$$

$$[\text{Ans: } 11/2]$$

10. Express $F = xi - y^2j + zk$ in (a) cylindrical polar coordinates (b) spherical polar coordinates.

$$(\text{Ans: (a) } \rho(\cos^2 \theta - \rho \sin^3 \theta).e_\rho - \rho \sin \theta \cos \theta(1 + \rho \sin \theta).e_\theta + ze_z$$

$$(b) (r \sin^2 \theta \cos^2 \phi - r^2 \sin^3 \theta \sin^3 \phi + r \cos^2 \theta)e_r$$

$$+ (r \sin \theta \cos \theta \cos^2 \phi - r^2 \sin^2 \theta \cos \theta \sin^3 \phi - r \sin \theta \cos \theta)e_\theta$$

$$+ (-r \sin \theta \sin \phi \cos \phi - r^2 \sin^2 \theta \sin^2 \phi \cos \phi)e_\phi]$$

11. If $f = \rho z \sin 2\theta$, find $\text{grad } f$ in cylindrical coordinates

$$[(z \sin 2\theta)e_\rho + (2z \cos 2\theta)e_\theta + (\rho \sin 2\theta)e_z]$$

12. If $\mathbf{A} = (r \cos \theta)e_r - \left(\frac{1}{r} \sin \theta\right)e_\theta + re_\phi$, find the $\text{curl } \mathbf{A}$ in spherical coordinates

$$[\text{Ans: } (\cot \theta)e_r + 2e_\theta - (2 \sin \theta)e_\phi]$$

13. Verify Green's theorem in the plane for $\int_C (x^3 - y^2)dx + (x^2 - 2xy)dy$, where C is

the boundary of the square bounded by $0 \leq x \leq 1$, $0 \leq y \leq 1$ [Ans: 1]

14. Using Gauss divergence theorem, prove that $\iint_S \mathbf{F} \cdot \mathbf{n} \, ds = \pi r^2 l^2$, where

$\mathbf{f} = (y^2z)\mathbf{i} + (xz)\mathbf{j} + (z^2)\mathbf{k}$, and S is the surface bounded by $x^2 + y^2 = r^2$, $z = 0$, and $z = l$

15. Show that the Stoke's theorem, when restricted to the xy -plane, is Green's theorem in the plane.

(Hint : In Stoke's theorem, take $\mathbf{A} = P\mathbf{i} + Q\mathbf{j}$; $n = k$; and $ds = dxdy$)

Exercise-7(n)**I. Choose the correct answer in the following questions**

1. The tangent vector at the point $t=1$ on the curve $x = t^2 + 1, y = 4t - 3, z = t^3$ is
(a) $2i-4j+3k$ (b) $2i+4j+3k$ (c) $2i-4j-3k$ (d) $2i+4j-3k$ [b]
2. The magnitude of acceleration at $\theta = 0$ on the curve $x = 2 \cos 3\theta, y = 2 \sin 3\theta, z = 3\theta$ is
(a) 6 (b) 9 (c) 18 (d) 3 [c]
3. if $f = xyz$, the value of $|\text{grad } f|$ at the point $(1, 2, -1)$ is
(a) 0 (b) 1 (c) 2 (d) 3 [d]
4. The maximum rate of change of $f = xy^2 + yz + zx^2$ at the point $(1, 1, 1)$ is
(a) $\sqrt{11}$ (b) 0 (c) 3 (d) none [d]
5. The angle between the normals to the sphere $x^2 + y^2 + z^2 = 9$ at the points $(1, 2, 2)$ and $(2, 1, 2)$ is
(a) $\cos^{-1}(8/9)$ (b) $\frac{\pi}{2}$ (c) $\cos^{-1}(3/4)$ (d) $\frac{\pi}{4}$ [a]
6. If $\mathbf{a}$ is a constant vector and $\mathbf{r} = xi + yj + zk$, then $\nabla(\mathbf{a} \cdot \mathbf{r})$ is
(a) 0 (b) $\mathbf{a}$ (c) $\mathbf{r}$ (d) r [b]
7. If $(x+3y)i + (2-3y)j + (x+az)k$ is a solenoidal vector, the value of a is
(a) 0 (b) 1 (c) 2 (d) 3 [c]
8. If $\mathbf{r} = xi + yj + zk$, and $\mathbf{a} = \frac{1}{3}\mathbf{r}$, $\text{div } \vec{a} =$
(a) 0 (b) 1 (c) -1 (d) 2 [b]

Exercise - 7(O)

1. If $f(x,y,z) = x^l y^m z^n$, find the directional derivative of f at $(1,1,1)$ in the direction of $(i+2j+2k)$ [Ans: $\frac{1}{3} (1 + 2m + 2n)$]

2. Find the acute angle between the surfaces $x^2 + y^2 + z^2 = 6$ and

$$3xyz + y^2z - xy + 3 = 0 \text{ at } (1, -1, 2) \quad [\text{Ans: } \cos^{-1}\left(\frac{\sqrt{5}}{3}\right)]$$

3. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and $\mathbf{p}$, $\mathbf{q}$, are constant vectors show that

$$\text{Div } \{\mathbf{r} \times \mathbf{p}\} \times \mathbf{q} = -2(\mathbf{p} \cdot \mathbf{q})$$

4. If $\mathbf{F} = (x^2y)\mathbf{i} - (y^2z)\mathbf{j} + (z^2x)\mathbf{k}$, find $\text{curl curl } \mathbf{F}$ at $(1, -2, 3)$

$$[\text{Ans: } 6\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}]$$

5. If $f = xyz(x + y + z)$, prove that $\text{curl grad } f = 0$

6. A fluid motion is given by $\mathbf{V} = (z^3)\mathbf{i} - (y^2)\mathbf{j} + (3xz^2)\mathbf{k}$. Show that it is irrotational. Find its velocity potential ϕ such that $\mathbf{V} = \nabla\phi$

$$[\text{Ans: } \phi = xz^3 - \frac{y^3}{3} + c]$$

7. Evaluate $\int_{(0,0)}^{(2,1)} (4x^3 - 12x^2y^2)dx - (8x^3y)dy$ along the path $x^3 - 3xy^2 = 2y^3$

$$[\text{Ans: } 16]$$

8. If $\mathbf{V} = (xy)\mathbf{i} - (yz)\mathbf{j} + (zx^2)\mathbf{k}$ and S is the surface of the cube bounded by $x = 0$, $x = 2$, $y = 0$, $y = 2$, $z = 0$ and $z = 2$, evaluate $\int_S \mathbf{V} \cdot \mathbf{n} ds$

$$[\text{Ans: } 32/3]$$

9. If $f = 4x + yz$, evaluate $\iiint_V f dv$ over the region in the first octant bounded by $x^2 + y^2 = 1$, $z = 0$, $z = 3$

$$[\text{Ans: } 11/2]$$

10. Express $F = xi - y^2j + zk$ in

a) cylindrical polar coordinates

b) spherical polar coordinates.

$$[\text{Ans: a) } \rho(\cos^2\theta - \rho\sin^3\theta)e_\rho - \rho\sin\theta\cos\theta(1 + \rho\sin\theta).e_\theta + ze_z$$

$$\text{b) } (r\sin^2\theta\cos^2\phi - r^2\sin^3\theta\sin^3\phi + r\cos^2\theta)e_r + (r\sin\theta\cos\theta\cos^2\phi - r^2\sin^2\theta\cos\theta\sin^3\phi - r\sin\theta\cos\theta)e_\theta + (-r\sin\theta\sin\phi\cos\phi - r^2\sin^2\theta\sin^2\phi\cos\phi)e_\phi]$$

11. If $f = \rho z \sin 2\theta$, find $\text{grad } f$ in cylindrical coordinates

$$[\text{Ans: } (z\sin 2\theta)e_\rho + (2x\cos 2\theta)e_\theta + (\rho\sin 2\theta)e_z]$$

12. If $A = (r\cos\theta)e_r - \left(\frac{1}{r}\sin\theta\right)e_\theta + re_\phi$, find $\text{curl } A$ in spherical coordinates

$$[\text{Ans: } (\cot\theta)e_r + 2e_\theta - (\sin\theta)e_\phi]$$

13. Verify Green's theorem in the plane for $\int_C (x^3 - y^2)dx + (x^2 - 2xy)dy$, where C is the boundary of the square bounded by $0 \leq x \leq 1$, $0 \leq y \leq 1$

$$[\text{Ans: } 1]$$

14. Using Gauss' divergence theorem, prove that $\iint_S F \cdot nds = \pi r^2 l^2$, where

$$F = (y^2z)i + (xz)j + (z^2)k, \text{ and } S \text{ is the surface bounded by } x^2 + y^2 = r^2, z=0, \text{ and } z=l$$

15. Show that the Stoke's theorem, when restricted to the xy plane, is Green's theorem in the plane

$$[\text{Hint: In Stoke's theorem, take } A = Pi + Qj; n = k; \text{ and } ds = dx dy]$$

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8

Laplace Transforms

8.1 Laplace Transforms

Introduction

The theory of Laplace transform is an essential part of the mathematical background required by engineers, physicists and mathematicians. It gives an easy and effective means for solving certain types of differential and integral equations. It is the foundation of the modern form of operational calculus, which was originated in an attempt to justify certain operational methods used by an electrical engineer Oliver Heaviside, in the latter part of the 19th century for solving equations in electromagnetic theory.

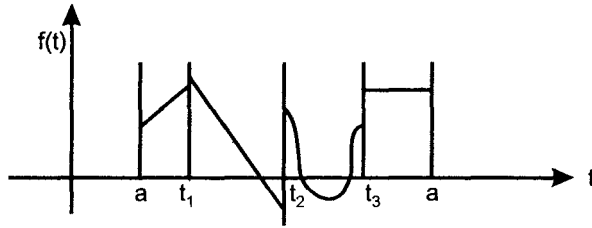
The Laplace transform reduces the problem of solving a differential equation to an algebraic problem. It is particularly useful for solving problems where the mechanical or electrical driving force has discontinuities, acts for a short time only or is periodic but not merely a sine or cosine function.

- 8.1.1** A class of linear transformations of particular importance is that of the integral transforms. The Laplace transformation (L.T) is a special case of an integral transformation and is defined by

$$F(s) = L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$$

Def. Piecewise continuity of a function :

A function of $f(t)$ is sectionally continuous (Piecewise continuous) in an interval $0 \leq t \leq b$ if $f(t)$ is continuous over every finite interval $0 \leq t \leq b$ except possibly at a finite number of points, where there are jump discontinuities and at which the function approaches different limits from left and right.



Sectionally continuous function

Def. Functions of exponential Order :-

A function $f(t)$ is said to be of exponential order σ , as $t \rightarrow \infty$ if $\exists M > 0$

$$\exists |f(t)| < M e^{\sigma t}, \forall t \geq 0 \quad \dots (1)$$

or equivalently, $|e^{-\sigma t} f(t)| \leq M, \forall t \geq 0$.

Remark : $e^{\sigma t}$ and $\sin bt$ are of exponential order, while e^{t^2} is not of exponential order since

$$|e^{-\sigma t} e^{t^2}| = e^{t^2 - \sigma t}$$

which can be made larger than any given constant by increasing 't' indefinitely.

8.1.2 Conditions for L.T to exist

By using above remarks we can prove the conditions that are sufficient for the existence of Laplace transforms.

Theorem : If $f(t)$ is piecewise continuous in every finite interval $[0, N]$ for $N > 0$, and of exponential order σ , that is $|f(t)| \leq M e^{\sigma t}, \forall t \geq 0$ and $M > 0$. (from 1)

then the Laplace transform of $F(t)$ exists for all $s > \sigma$

Proof: Let $I = |F(s)| = \left| \int_0^{\infty} e^{-st} f(t) dt \right| \leq \int_0^{\infty} |f(t)| e^{-st} dt \leq \int_0^{\infty} M e^{\sigma t} e^{-st} dt = \frac{M}{s - \sigma}$

Here, $s > \sigma$ since I is non-negative everywhere. This shows that above conditions are sufficient for the existence of $L[f(t)]$.

Properties of Laplace Transforms

8.1.3 *Linearity property:* Let $f_1(t)$ and $f_2(t)$ be two functions on $[0, \infty]$ such that the Laplace transforms $L[f_1(t)]$ and $L[f_2(t)]$ exist. If k_1 and k_2 are two constants, then

$$L[k_1 f_1(t) + k_2 f_2(t)] = k_1 L[f_1(t)] + k_2 L[f_2(t)] \quad \text{.....(i)}$$

This property is valid since

$$\int_0^{\infty} [k_1 f_1(t) + k_2 f_2(t)] e^{-st} dt = \int_0^{\infty} k_1 f_1(t) e^{-st} dt + \int_0^{\infty} k_2 f_2(t) e^{-st} dt = k_1 L[f_1(t)] + k_2 L[f_2(t)]$$

This result can be extended to the linear combinations of more than two functions.

8.1.4 If $f(t)$ is piecewise continuous on $[0, N]$ for each $N > 0$, and $f(t)$ is of exponential order σ , that is by 8.10.1(1), $|f(t)| \leq M e^{\sigma t}$, $\forall t \geq 0$ and $M > 0$ and if $L[f(t)] = F(s)$ then we have

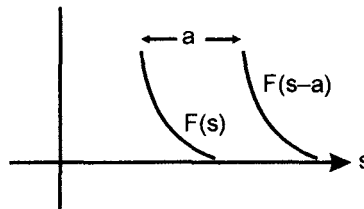
$$(1) L[e^{at} f(t)] = F(s-a) \quad \forall s > \sigma + a \quad \forall a > 0 \quad \text{.....(i)}$$

$$(2) L[f_1(t)] = e^{-as} F(s)$$

$$\text{where } f_1(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases} \quad (a > 0)$$

$$\text{and } (3) L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right) \quad (\text{for any } a > 0)$$

Proof (1)



From definition, we have

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt = F(s)$$

so that

$$L[e^{at} f(t)] = \int_0^{\infty} e^{-st} [e^{at} f(t)] dt = \int_0^{\infty} e^{-(s-a)t} f(t) dt = F(s-a)$$

This property shows shifting on the s-axis and is called the *first shifting property*.

This means that replacing s by $s - a$ in the Laplace transform corresponds to multiplying the original function by e^{at}

$$\begin{aligned}
 (2) \quad L(f_1(t)) &= \int_0^{\infty} e^{-st} f_1(t) dt = \int_0^a e^{-st} f_1(t) dt + \int_a^{\infty} e^{-st} f_1(t) dt \\
 &= \int_0^a e^{-st} \cdot 0 dt + \int_a^{\infty} e^{-st} f(t-a) dt && \text{from definition of } f_1(t) \\
 &= \int_0^{\infty} e^{-st} f(t-a) dt = \int_0^{\infty} e^{-s(p+a)} f(p) dp && (\text{where } t = p+a) \\
 &= e^{-as} \int_0^{\infty} e^{-ps} f(p) dp = e^{-as} F(s)
 \end{aligned}$$

This property indicates shifting on the t-axis and is called the second shifting property

$$\begin{aligned}
 (3) \quad L[f(at)] &= \int_0^{\infty} e^{-st} f(at) dt = \int_0^{\infty} e^{-s(p/a)} f(p) \frac{dp}{a} && [\text{taking } t = p/a] \\
 &= \frac{1}{a} \int_0^{\infty} e^{-sp/a} f(p) dp = \frac{1}{a} F\left(\frac{s}{a}\right)
 \end{aligned}$$

This property is known as change of scale property.

8.1.5 Laplace Transforms of standard functions

$$(a) \quad L(e^{-at}) = \frac{1}{s+a} \quad (s > a) \quad \dots\dots(1)$$

Proof: we have $L(e^{-at}) = \int_0^{\infty} e^{-st} \cdot e^{-at} dt = \int_0^{\infty} e^{-(s+a)t} dt = \frac{-1}{s+a} [e^{-(s+a)t}]_0^{\infty} = \frac{1}{s+a}$

if $a = 0$ the $L(1) = \frac{1}{s}$ (s > 0) $\dots\dots(2)$

Also changing the sign of 'a' we get

$$L(e^{at}) = \frac{1}{s-a} \quad \dots\dots(3)$$

$$(b) \quad L(\sin at) = \frac{a}{s^2 + a^2} \quad \text{and} \quad L(\cos at) = \frac{s}{s^2 + a^2} \quad (s > 0)$$

Proof: $L(\cos at + i \sin at) = L(e^{iat}) = \frac{1}{s - ia} = \frac{s + ia}{s^2 + a^2}$ from (a)

Equating real and imaginary parts we get

$$L(\sin at) = \frac{a}{s^2 + a^2}, L(\cos at) = \frac{s}{s^2 + a^2} \quad \dots(4)$$

$$(c) \quad L(\sinh at) = \frac{a}{s^2 - a^2} \text{ and } L(\cosh at) = \frac{s}{s^2 - a^2} \quad (s > |a|)$$

Proof:
$$L(\sinh at) = L\left(\frac{e^{at} - e^{-at}}{2}\right) = \frac{1}{2}L(e^{at}) - \frac{1}{2}L(e^{-at})$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] \quad \text{from (a)}$$

$$\text{Thus } L(\sinh at) = \frac{a}{s^2 - a^2}$$

$$\text{similarly we can show that } L(\cosh at) = \frac{s}{s^2 - a^2} \quad \dots(5)$$

$$(d) \quad L(t^n) = \frac{n!}{s^{n+1}} \quad [(n+1) > 0 \text{ and } s > 0]$$

Proof:
$$L(t^n) = \int_0^\infty e^{-st} t^n dt = \int_0^\infty e^{-p} \cdot \frac{p^n}{s^n} \cdot \frac{dp}{s} \quad \text{where } p = st$$

$$= \int_0^\infty \frac{e^{-p} \cdot p^n}{s^{n+1}} dp = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$\text{Since } \Gamma(n+1) = \int_0^\infty e^{-x} x^n dx \quad \text{and} \quad \Gamma(n+1) = n! \quad \text{if } n \text{ is a +ve integer}$$

$$\therefore L(t^n) = \frac{n!}{s^{n+1}} \quad \dots(6)$$

From (6) we have

$$L(1) = \frac{1}{s}, \quad L(t) = \frac{1}{s^2}, \quad L(t^2) = \frac{2}{s^3}, \quad L(t^3) = \frac{6}{s^4}$$

Solved Examples

8.1.6 Find the Laplace transform of

1. $2t^3 + \cos 4t + e^{-2t}$

4. $3t^2 + e^{-t} + \sin^3 2t$

2. $e^{2t} + 4t^3 - \sin 2t \cos 3t$

5. $\cos^3 2t$

3. $\cos(\omega t + \beta)$

Sol: 1. $L(2t^3 + \cos 4t + e^{-2t}) = 2 \times \frac{6}{s^4} + \frac{s}{s^2 + 16} + \frac{1}{s + 2} = \frac{12}{s^4} + \frac{s}{s^2 + 16} + \frac{1}{s + 2}$

2. $L(e^{2t}) + 4L(t^3) - L(\sin 2t \cos 3t)$

$$\frac{1}{s - 2} + 4 \times \frac{6}{s^4} - \left(\frac{5}{s^2 + 25} - \frac{1}{s^2 + 1} \right)$$

$$[\because \sin 2t \cos 3t = \frac{1}{2} (\sin 5t - \sin t)]$$

3. $L \cos(wt + \beta) = L[\cos wt \cdot \cos \beta - \sin wt \sin \beta]$

$$= \cos \beta L(\cos wt) - \sin \beta L(\sin wt) = \cos \beta \frac{s}{s^2 + w^2} - \sin \beta \frac{w}{s^2 + w^2}$$

4. $L[3t^2 + e^{-t} + \sin^3 2t] = 3L(t^2) + L(e^{-t}) + L(\sin^3 2t)$

$$= 3 \cdot \frac{2}{s^3} + \frac{1}{s + 1} + \frac{3}{2(s^2 + 4)} - \frac{3}{2} \frac{1}{s^2 + 36}$$

$$= \frac{2}{s^3} + \frac{1}{s + 1} + \frac{3}{2} \frac{1}{s^2 + 4} - \frac{3}{2} \frac{1}{s^2 + 36}$$

$$(\because L(\sin^3 2t) = \frac{1}{4} L[3\sin 2t - \sin 6t])$$

5. $L(\cos^3 2t) = L \left[\frac{\cos 6t + 3\cos 2t}{4} \right]$

$$= \frac{1}{4} \left(\frac{s}{s^2 + 36} + \frac{3s}{s^2 + 4} \right)$$

Exercise 8(A)

Find the Laplace transform of

1. $4t^2 + \sin 3t + e^{-2t}$ Ans. $\frac{8}{s^3} + \frac{3}{s^2 + 9} + \frac{1}{s + 2}$

2. $(\sin 2t - \cos 2t)^2$ Ans. $\frac{1}{s} - \frac{4}{s^2 + 16}$

3. $(\cosh^2 4t)$ Ans. $\frac{1}{2} \left(\frac{1}{s} + \frac{s}{s^2 - 64} \right)$

4. $t^2 - e^{-2t} + \cosh^2 4t$ Ans. $\frac{2}{s^3} - \frac{1}{s + 2} + \frac{1}{2} \left(\frac{1}{s} + \frac{s}{s^2 - 64} \right)$

5. $t^4 + 2at^2 + a^2$ Ans. $\frac{a^2 s^4 + 4as^2 + 24}{s^5}$

6. $\cos^2(2bt)$ Ans. $\frac{1}{2} \left(\frac{1}{s} + \frac{s}{s^2 + 16b^2} \right)$
7. $\sinh^3 3t$ Ans. $\frac{162}{(s^2 - 81)(s^2 - 9)}$
8. $\sinh at - \sin bt$ Ans. $\frac{a}{s^2 - a^2} - \frac{b}{s^2 + b^2}$
9. $\cosh^3 2t$ Ans. $\frac{s(s^2 - 28)}{(s^2 - 36)(s^2 - 4)}$
10. $\cosh at - \cos bt$ Ans. $\frac{s}{s^2 - a^2} - \frac{s}{s^2 + b^2}$

8.1.7 Examples on properties 8.1.4

Find the Laplace transform of

1. $e^{-bt} \cos at$ 2. $t^3 e^{-4t}$ 3. $e^{2t} \sin 3t$
 4. $e^{-3t} \sin 3t$ 5. $e^{-2t} [2 \cos 3t - 3 \sin 3t]$

Sol.

1. $L(\cos at) = \frac{s}{s^2 + a^2}$

$\therefore L(e^{-bt} \cos at) = \frac{s + b}{(s + b)^2 + a^2}$ (from 8.1.4(1) first shifting theorem)

2. $L(t^3) = \frac{6}{s^4}$

By shifting theorem 8.1.4(1)

$L(e^{-4t} \cdot t^3) = \frac{6}{(s + 4)^4}$

3. $L(\sin 3t) = \frac{3}{s^2 + 9}$

by 8.1.4(1),

$L(e^{2t} \sin 3t) = \frac{3}{(s - 2)^2 + 9}$

4. $e^{-3t} \sin 3t$

$$\begin{aligned} L(\sin 3t) &= L\left[\frac{1}{4}(3 \sin t - \sin 3t)\right] = \frac{3}{4}L(\sin t) - \frac{1}{4}L(\sin 3t) \\ &= \frac{3}{4} \frac{1}{s^2 + 1} - \frac{1}{4} \frac{3}{s^2 + 9} \end{aligned}$$

By 8.1.4(1)

$$L(e^{-3t} \sin 3t) = \frac{3}{4} \cdot \frac{1}{(s+3)^2 + 1} - \frac{1}{4} \cdot \frac{1}{(s+3)^2 + 9}$$

5. $e^{-2t}(2 \cos 3t - 3 \sin 3t)$

$$L(2 \cos 3t) = 2 \cdot \frac{s}{s^2 + 9}$$

$$L(3 \sin 3t) = \frac{9}{s^2 + 9}$$

By shifting theorem 8.1.4(1)

$$\begin{aligned} &2L(e^{-2t} \cos 3t) - 3L(e^{-2t} \sin 3t) \\ &= \frac{2(s+2)}{(s+2)^2 + 9} - \frac{9}{(s+2)^2 + 9} \end{aligned}$$

8.1.8 If $L[f(t)] = \frac{3s}{s^2 + 9}$, find $L[f(3t)]$

Sol. $L[f(t)] = \frac{3s}{s^2 + 9}$

$$\text{By 8.1.4 (3), } L[f(3t)] = \frac{1}{3} \frac{3 \cdot \frac{s}{3}}{\left(\frac{s}{3}\right)^2 + 9} = \frac{3s}{s^2 + 27}$$

Exercise 8(B)

Find the Laplace Transform of

1. $4e^{3t} \sin(2t + \alpha)$

Ans. $4 \cos \alpha \frac{2}{(s-3)^2 + 16} + \sin \frac{s-3}{(s-3)^2 + 16}$

2. $e^{-4t} \sin 3t$

Ans. $\frac{3}{4} \frac{1}{(s+4)^2 + 1} - \frac{1}{4} \frac{3}{(s+4)^2 + 9}$

3. $\cosh 2t \cdot \cos 2t$

Ans. $\frac{1}{2} \left[\frac{s-2}{(s-2)^2 + 4} + \frac{s+2}{(s+2)^2 + 4} \right]$

4. $\cosh at \sin at$ Ans. $\frac{a(s^2 + 2a^2)}{s^4 + 2a^4}$
5. $\cos 3t \cdot \cosh 4t$ Ans. $\frac{s(s^2 - 7)}{s^4 + 625 - 14s^2}$
6. $e^{-2t}(2\cos 3t - 3\sin 3t)$ Ans. $\frac{2s - 5}{s^2 + 4s + 13}$
7. $\cosh at \cos at$ Ans. $\frac{s^3}{s^4 + 4a^4}$
8. $e^t \sin 4t \cdot \cos 2t$ Ans. $\frac{3}{(s-1)^2 + 36}$
9. $t^2 - 3t + 5 + e^{2t} \cdot t^2$ Ans. $\frac{5s^2 - 3s + 2}{s^3}$
10. $t^n e^{-at} + \sin^3 t$ Ans. $\frac{n+1}{(s+a)^{n+1}} + \frac{6}{(s^2+1)(s^2+a)}$

8.1.9 Property (4) : Effect of multiplication by t

If $L[f(t)] = F(s)$ then $L[tf(t)] = -F'(s)$

Proof: We know that $F(s) = \int_0^{\infty} e^{-st} f(t) dt$.

differentiating with respect to s

$$F(s) = \int_0^{\infty} \frac{\partial}{\partial s} [e^{-st} f(t)] dt = \int_0^{\infty} e^{-st} f(t) dt$$

$$\therefore -F'(s) = \int_0^{\infty} e^{-st} [tf(t)] dt = L[tf(t)]$$

8.1.10 Examples

1. $L(e^{-2t}) = \frac{3}{s+2}, L(te^{-2t}) = -\frac{d}{ds} \frac{1}{s+2} = \frac{1}{(s+2)^2}$
2. $L(t \cos at) = (-1) \frac{d}{ds} [L(\cos at)] = -\frac{d}{ds} \left(\frac{s}{s^2 + a^2} \right) = \frac{2s^2}{(s^2 + a^2)^2}$
3. $L(t \sin t) = -\frac{d}{ds} \left(\frac{1}{s^2 + 1} \right) = \frac{2s}{(s^2 + 1)^2}$

4. $L(\text{atsinat} - \text{cosat}) = aL(\text{tsinat}) - L(\text{cosat}) = a \left[-\frac{d}{ds} \left(\frac{-a}{s^2 + a^2} \right) \right] - \frac{s}{s^2 + a^2}$
 $= a \left[\frac{2as}{(s^2 + a^2)^2} \right] - \frac{s}{(s^2 + a^2)} = \frac{2a^2s}{(s^2 + a^2)^2} - \frac{s}{(s^2 + a^2)}$
5. $L(1+t^2e^t)^3 = L[1+3t^2e^t+3t^2e^{2t}+t^3e^{3t}]$
 $= L(1) + 3L(te^t) + 3L(t^2e^{2t}) + L(t^3e^{3t})$
 $= \frac{1}{s} - 3 \frac{d}{ds} \frac{1}{(s-1)} + 3 \frac{d^2}{ds^2} \frac{1}{(s-2)} + \frac{d^3}{ds^3} \left(\frac{1}{s-2} \right) = \frac{1}{2} + \frac{3}{(s-1)^2} + \frac{6}{(s-2)^3} + \frac{6}{(s-3)^4}$
6. $L(t^2 \cos t) = \frac{d^2}{ds^2} [L(\cos t)] = \frac{d^2}{ds^2} \left(\frac{s}{s^2 + 1} \right) = \frac{d}{ds} \left[-\frac{s^2}{(s^2 + 1)^2} \right]$
 $= \left[\frac{(s+1)^2 \cdot 2s - 2(s^2 + 1) \cdot 2s \cdot s^2}{(s^2 + 1)^4} \right] = \frac{2s(s^2 - 1)}{(s^2 + 1)^3}$
7. $L(te^{4t} \sin 2t) = \frac{d}{ds} L[e^{4t} \sin 2t] = -\frac{d}{ds} \left[\frac{2}{(s-4)^2 + 4} \right]$ by shifting property
 $= -\frac{d}{ds} \left[\frac{2}{s^2 - 8s + 20} \right] = \frac{4s - 16}{(s^2 - 8s + 20)^2}$
8. $L(t^2 e^t \cos 4t) = \frac{d^2}{ds^2} L(e^t \cos 4t) = \frac{d^2}{ds^2} \left[\frac{s-1}{(s-1)^2 + 16} \right]$ by shifting theorem
 $= -\frac{d}{ds} \left[\frac{s^2 - 2s - 15}{(s^2 - 2s + 17)^2} \right] = -\frac{(2s^3 - 6s^2 - 90s + 94)}{(s^2 - 2s + 17)^3}$

Exercise 8(C)

Find the Laplace transform of the following:

1. $te^{-3t} \sin 2t$ Ans. $\frac{1}{2} \left[\frac{1}{(s-3)^2} - \frac{(s^2 - s + 5)}{(s^2 - 6s + 13)^2} \right]$
2. $te^{-3t} \sin 3t$ Ans. $\frac{6s - 18}{(s^2 - 6s + 18)^2}$
3. $t e^{-2t} \sin t$ Ans. $\frac{2s + 4}{(s^2 + 4s + 5)^2}$

$$4. \quad 2t \sin 3t - 3t \cos 3t \quad \text{Ans.} \quad \frac{9 + 12s - 3s^2}{(s^2 + 9)^2}$$

$$5. \quad te^t \sin 2t \cos t \quad \text{Ans.} \quad \frac{3s - 3}{(s^2 - 2s + 10)^2}$$

8.1.11 Property (5) If $L[f(t)] = f(s)$, then $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty f(s) ds$

Proof: $f(s) = \int_s^\infty e^{-st} f(t) dt$

Integrating both sides w.r.t 's' between the limits s, ∞ and changing the order of integration on right hand side (R.H.S),

$$\begin{aligned} \int_s^\infty f(s) ds &= \int_0^\infty \left[\int_s^\infty e^{-st} f(t) ds \right] dt = \int_0^\infty \left[\frac{e^{-st}}{t} f(t) \right]_0^\infty dt = \int_0^\infty e^{-st} \frac{f(t)}{t} dt \\ &= \int_0^\infty e^{-st} \frac{f(t)}{t} dt = \left[\frac{1}{t} f(t) \right] \end{aligned}$$

8.1.12 Examples

Find the Laplace Transforms of the following

$$1. \quad \frac{1}{t}(1 - \cos at) \quad 2. \quad \frac{1}{t}(e^{-t} \sin at) \quad 3. \quad \frac{1}{t}(1 - e^{2t})$$

$$4. \quad \frac{1}{t}(e^{-3t} \sin 2t) \quad 5. \quad \frac{1}{t}(e^{at} - e^{bt})$$

Sol. 1. $L(1 - \cos at) = L(1) - L(\cos at) = \frac{1}{s} - \frac{s}{s^2 + a^2}$

$$L\left(\frac{1}{t}(1 - \cos at)\right) = \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2 + a^2} \right] ds = \left[\log s - \frac{1}{2} \log(s^2 + a^2) \right]_0^\infty = \frac{1}{2} \log \left(\frac{s^2 + a^2}{s^2} \right)$$

2. $L(e^{-t} \sin at) = \frac{a}{(s+1)^2 + a^2}$

$$L\left[\frac{1}{t}(e^{-t} \sin at)\right] = \int_s^\infty \frac{a}{(s+1)^2 + a^2} ds = \left[\tan^{-1} \frac{(s+1)}{a} \right]_s^\infty = \frac{\pi}{2} - \tan^{-1} \frac{s+1}{a} = \cot^{-1} \left(\frac{s+1}{a} \right)$$

3. $L(1 - e^{2t}) = L(1) - L(e^{2t}) = \frac{1}{s} - \frac{1}{s-2}$
- $$L\left[\frac{1}{t}(1 - e^{2t})\right] = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-2}\right) ds = [\log s - \log(s-2)]_s^\infty = \log \frac{(s-2)}{s}$$
4. $L(e^{-3t} \sin 2t) = \frac{2}{(s+3)^2 + 4}$
- $$L\left(\frac{1}{t} e^{-3t} \sin 2t\right) = \int_s^\infty \frac{2}{(s+3)^2 + 4} ds = \tan^{-1}\left(\frac{s+3}{2}\right)_s^\infty$$
- $$= \frac{\pi}{2} = \tan^{-1}\left(\frac{s+3}{2}\right) = \cot^{-1}\left(\frac{s+3}{2}\right)$$
5. $L(e^{at}) = \frac{1}{s-a}$, $L(e^{bt}) = \frac{1}{s-b}$
- $$L\left[\frac{1}{t}(e^{at} - e^{bt})\right] = \int_s^\infty \left(\frac{1}{s-a} - \frac{1}{s-b}\right) ds = [\log(s-a) - \log(s-b)]_s^\infty$$
- $$= \left[\log\left(\frac{s-a}{s-b}\right)\right]_s^\infty = -\log\left(\frac{s-a}{s-b}\right) = \log\left(\frac{s-b}{s-a}\right)$$

Exercise 8(D)

Find the Laplace transform of the following:

1. $\frac{1}{t}(1 - \cos t)$ Ans. $\frac{1}{2} \log\left(\frac{s^2+1}{s}\right)$
2. $\frac{1}{t}e^{-t} \sin t$ Ans. $\cot^{-1}(s+1)$
3. $\frac{1}{t}(e^{-at} - e^{-bt})$ Ans. $\log\left(\frac{s+b}{s+a}\right)$
4. $\frac{1}{t}(e^{-3t} \sin 2t)$ Ans. $\cot^{-1}\left(\frac{s-3}{2}\right)$
5. $\frac{1}{t}(\sin 2t)$ Ans. $\frac{1}{4} \log\left(\frac{s^2+4}{s^2}\right)$

$$6. \quad \frac{1}{t} (\cos at - \cos bt) \quad \text{Ans.} \quad \frac{1}{2} \log \left(\frac{s^2 + b^2}{s^2 + a^2} \right)$$

$$7. \quad \frac{1}{t} (1 - \cos 3t) \quad \text{Ans.} \quad \frac{1}{2} \log \left(\frac{s^2 + 9}{s^2} \right)$$

$$8. \quad \frac{1}{t} (\sin 3t) \quad \text{Ans.} \quad \frac{1}{4} \left[3 \cot^{-1} s - \cot^{-1} \frac{s}{3} \right]$$

8.2.1 Theorem: If $f(t)$ is continuous $\forall t \geq 0$ and of exponential order, say σ and has a derivative $f'(t)$ which is piecewise continuous on every finite interval $[0, N]$ for each $N > 0$, then the Laplace transform of the derivative $f'(t)$ exists for $s > \sigma$ and

$$L\{f'(t)\} = sLf(t) - f(0)$$

Proof:
$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$$

$$L[f'(t)] = \int_0^{\infty} e^{-st} f'(t) dt$$

Integration by parts gives

$$L[f'(t)] = \left[e^{-st} f(t) \right]_0^{\infty} + s \int_0^{\infty} e^{-st} f(t) dt$$

Since $f(t)$ is of an exponential order, the integrand in first integral on the R.H.S. is zero at the upper limit when $s > \sigma$ and if $f(0)$ at the lower limit. Thus, we have

$$L[f'(t)] = 0 - f(0) + s \int_0^{\infty} e^{-st} f(t) dt = sLf(t) - f(0)$$

If $\lim_{t \rightarrow 0} t f(t) = f(0+)$ exists, but is not equal to $f(0)$, which may or may not exist then

$$L[f'(t)] = sLf(t) - f(0+).$$

Note: If $f(t)$ and its derivatives $f'(t), f''(t), \dots, f^{(n-1)}(t)$ are continuous $\forall t \geq 0$ and are functions of exponential order for some $M > 0$ and σ , and the derivative $f^{(n)}(t)$ is Piecewise continuous on every finite interval in the range $t \geq 0$, then the Laplace transform of $f^{(n)}(t)$ exists when $s > \sigma$ and is given by

$$L[f^{(n)}(t)] = s^n [Lf(t) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)]$$

If $n = 2$, $L[f''(t)] = s^2[Lf(t)] - sf(0) - f'(0)$ and so on.

8.2.2 Example

Prove that $L(\cos at) = \frac{s}{s^2 + a^2}$

Sol. We can show this with the help of above theorem

$$\text{Let } f(t) = \cos at \Rightarrow f(0) = 1$$

$$\text{then } f'(t) = -a \sin t \Rightarrow f'(0) = 0$$

$$f''(t) = -a^2 \cos at$$

Hence, we have

$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$$

$$\therefore L[-a^2 \cos at] = s^2 L(\cos at) - s - 0 = s^2 L(\cos at) - s$$

$$s^2 L(\cos at) + a^2 L(\cos at) = s$$

$$\therefore L(\cos at) = \frac{s}{s^2 + a^2}$$

8.2.3 Theorem: If f is Piecewise continuous on $[0, N]$ for each $N > 0$ and is of exponential

order σ , then $L \int_0^t f(u) du = \frac{1}{s} L[f(t)] \quad \forall \quad s > \sigma,$

Proof. Let $\phi(t) = \int_0^t f(u) du$. Then we have $\phi'(t) = f(t)$ and $\phi(0) = 0$;

$$L[\phi'(t)] = s L[\phi(t)] - \phi(0) = s[L(\phi(t))] = sL \left[\int_0^t f(u) du \right]$$

$$\therefore L \left[\int_0^t f(u) du \right] = \frac{1}{s} L[\phi'(t)] = \frac{1}{s} [L(f(t))]$$

8.2.4 Examples

Find the Laplace transforms of

$$1. \int_0^t \sin 2p dp \quad 2. \int_0^t p \cosh p dp \quad 3. \int_0^t \frac{\sin p}{p} dp \quad 4. \int_0^t e^p \cdot \frac{\sin p}{p} dp$$

Sol. 1. $L(\sin 2t) = \frac{2}{s^2 + 4} = f(s)$, say

$$\int_0^t \sin 2p dp = \frac{1}{s} f(s) = \frac{2}{s(s^2 + 4)}$$

$$2. \quad L(\cosh t) = \frac{s}{s^2 - a^2} \quad L(t \cosh t) = -\frac{(s^2 + a^2)}{(s^2 - a^2)} = f(s)$$

$$\text{then } L \int_0^t p \cosh p dp = \frac{1}{s} f(s) = -\frac{(s^2 + a^2)}{(s^2 - a^2)s}$$

$$3. \quad L(\sin t) = \frac{1}{s^2 + 1}; \quad L\left(\frac{1}{t} \sin t\right) = \int_s^\infty \frac{1}{s^2 + 1} ds$$

$$= \left[\tan^{-1} s \right]_s^\infty = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s = f(s) \text{ say}$$

$$L \int_0^t \frac{\sin p}{p} dp = \frac{1}{s} f(s) = \frac{1}{s} \cot^{-1} s$$

$$4. \quad L(\sin t) = \frac{1}{s^2 + 1} \quad \therefore L(e^t \sin t) = \frac{1}{(s-1)^2 + 1}$$

$$L\left[\frac{1}{t}(e^t \sin t)\right] = \int_s^\infty \frac{ds}{(s-1)^2 + 1}$$

$$= \left[\tan^{-1}(s-1) \right]_s^\infty = \frac{\pi}{2} - \tan^{-1}(s-1) = \cot^{-1}(s-1) = f(s) \text{ say}$$

$$\therefore L \int_0^t e^p \frac{\sin p}{p} dp = \frac{1}{s} f(s) = \frac{1}{s} \cot^{-1}(s-1)$$

Exercise 8(E)

Find the Laplace Transform of

$$1. \quad \int_0^t \cos 2p dp$$

$$2. \quad \int_0^t p e^{-3} \sin 4p dp$$

$$3. \quad \int_0^t \frac{1}{p} \sin 3p dp$$

$$\text{Ans. } 1. \quad \frac{s}{s(s^2 + 4)}$$

$$2. \quad \frac{8(s+3)}{s(s^2 + 6s + 25)^2}$$

$$3. \quad \frac{1}{s} \cot^{-1} s$$

Laplace Transforms of Periodic Functions

8.2.5 A function $f(t)$ is said to be periodic with period $p > 0$ if $f(t+p) = f(t)$ for all $t > 0$

Since periodic appear in many practical problems and in most cases are more complicated than single sine or cosine functions, we find it useful to establish the following result:

$$L[f(t)] = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t) dt$$

Proof: Let $f(t)$ be a periodic function of period 'p' so that $f(t) = f(t+p) = f(t+2p) = \dots$

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt = \int_0^p e^{-st} f(t) dt + \int_p^{2p} e^{-st} f(t) dt + \dots$$

writing $t = u+p$

$$\text{now } \int_p^{2p} e^{-st} f(t) dt = \int_0^p e^{-s(u+p)} f(u+p) du$$

$$= e^{-ps} \int_0^p e^{-st} f(t+p) dt \quad (\text{change } u \text{ to } t)$$

$$= e^{-ps} \int_0^p e^{-st} f(t) dt \quad [\because f(t+p) = f(t)]$$

$$\text{similarly, } \int_{2p}^{3p} e^{-st} f(t) dt = e^{-2ps} \int_0^p e^{-st} f(t) dt \text{ and so on}$$

$$L[f(t)] = (1 + e^{-ps} + e^{-2ps} + \dots \infty) \int_0^p e^{-st} f(t) dt$$

The expression within parentheses is a G.P. with c.r. $= e^{-ps}$

$$\therefore L[f(t)] = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t) dt$$

Solved Problems

8.2.6 Find the Laplace Transform of

1. $f(t) = k \frac{t}{p}$ for $0 < t < p$ and $f(t) = f(t+p)$
2. $f(t) = 1$ for $0 < t < a$ and $f(t) = -1$, $a < t < 2a$ and $f(t)$ is periodic with period $2a$

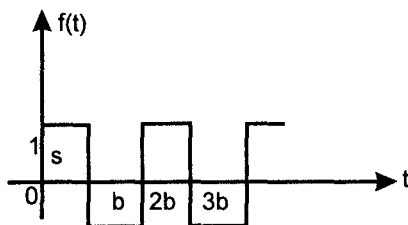
Sol. Since $f(t)$ is periodic with period p .

$$\begin{aligned}
 L[f(t)] &= \frac{1}{1-e^{-sp}} \int_0^p e^{-st} f(t) dt = \frac{1}{1-e^{-sp}} \int_0^p e^{-st} \cdot \frac{kt}{p} dt \\
 &= \frac{1}{1-e^{-sp}} \frac{k}{p} \int_0^p e^{-st} t dt = \frac{1}{1-e^{-sp}} \frac{k}{p} \left[-t \frac{e^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^p \\
 &= \frac{1}{1-e^{-sp}} \frac{k}{p} \left[-p \frac{e^{-sp}}{s} - \frac{e^{-sp}}{s^2} + \frac{1}{s^2} \right] = \frac{1}{1-e^{-sp}} \frac{k}{p} \left[\frac{1}{s^2} (1 - e^{-sp}) - \frac{pe^{-sp}}{s} \right]
 \end{aligned}$$

2. $f(t)$ is peirodic with peirod $2a$

$$\begin{aligned}
 L[f(t)] &= \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t) dt = \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} \cdot 1 dt + \int_a^{2a} e^{-st} (-1) dt \right] \\
 &= \frac{1}{1-e^{-2as}} \left[\left\{ \frac{-e^{-st}}{s} \right\}_0^a - \left\{ \frac{-e^{-st}}{s} \right\}_a^{2a} \right] = \frac{1}{s} \frac{1}{1-e^{-2as}} \cdot (1 - e^{-as})^2 \\
 &= \frac{1}{s} \cdot \left[\frac{1 - e^{-2as}}{1 + e^{-2as}} \right] = \frac{1}{s} \cdot \left[\frac{e^{\frac{as}{2}} - e^{-\frac{as}{2}}}{e^{\frac{as}{2}} + e^{-\frac{as}{2}}} \right] = \frac{1}{s} \tanh \left[\frac{as}{2} \right]
 \end{aligned}$$

8.2.7 Find the Laplace transform of the rectangular wave shown in the figure:



Rectangular wave

Sol: The period of the given function is '2b'.

$$\begin{aligned}
 \text{Hence } L[f(t)] &= \frac{1}{1-e^{-2bs}} \int_0^{2b} e^{-st} f(t) dt = \frac{1}{1-e^{-2bs}} \left[\int_0^b e^{-st} f(t) dt + \int_b^{2b} (-1)e^{-st} dt \right] \\
 &= \frac{1}{1-e^{-2bs}} \left[\left\{ \frac{-e^{-st}}{-s} \right\}_0^b - \left\{ \frac{-e^{-st}}{-s} \right\}_b^{2b} \right] = \frac{1}{1-e^{-2bs}} \left(\frac{1-2e^{-bs}+e^{-2bs}}{s} \right) = \frac{(1-e^{-bs})^2}{s(1-e^{-bs})(1+e^{-bs})} \\
 &= \frac{1}{s} \left(\frac{1-e^{-bs}}{1+e^{-bs}} \right) = \frac{e^{\frac{bs}{2}} - e^{-\frac{bs}{2}}}{s \left(e^{\frac{bs}{2}} + e^{-\frac{bs}{2}} \right)} \\
 &= \frac{1}{s} \tanh \frac{bs}{2}
 \end{aligned}$$

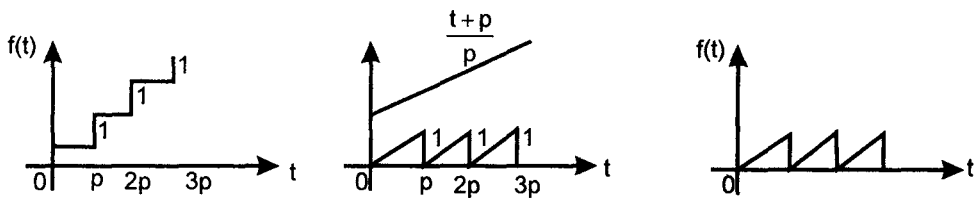
8.2.8 What is the Laplace transform of the staircase function (fig(a)) given by

$$f(t) = n+1, np < t < (n+1)p \text{ for } n = 0, 1, 2, \dots$$

Sol. The easiest way of getting the Laplace transform of the staircase function is to consider it as the difference between the two functions shown in Fig(b) the

transform of the linear function $\frac{(t+p)}{p}$ can easily be found by using first shifting, second shifting and change of scale properties.

Thus, we have



(a) Staircase function (b) solution of the staircase function (c) Saw-tooth function.

$$L\left(\frac{t+p}{p}\right) = \frac{1}{p} \left(\frac{1}{s^2} + \frac{p}{s} \right)$$

The function $f_1(t) = \frac{t}{p}$ is called a saw-tooth function (fig(c)) and its transform is obtained as shown below.

$$\begin{aligned} L[f_1(t)] &= \frac{1}{1-e^{-sp}} \int_0^p e^{-st} \frac{t}{p} dt = \frac{1}{1-e^{-sp}} \cdot \frac{1}{p} \left[\frac{e^{-st}}{s^2} (-st-1) \right]_0^p \\ &= \frac{1-(1+ps)e^{-ps}}{ps^2(1-e^{-sp})} \end{aligned}$$

This can be simplified to

$$L[f_1(t)] = \frac{1}{p} \left[\frac{1+ps}{s^2} - \frac{p}{s(1-e^{-ps})} \right] \quad \dots(2)$$

Thus from (1) and (2), we get

$$\begin{aligned} L[f(t)] &= \frac{1}{p} \left(\frac{1}{s^2} + \frac{p}{s} \right) - \frac{1}{p} \left(\frac{1+ps}{s^2} - \frac{p}{s(1-e^{-ps})} \right) \\ &= \frac{1}{s(1-e^{-ps})} \end{aligned}$$

Exercise 8(F)

1. If $Lf(t) = \phi(s)$, Prove that
 $L[f''(t)] = s^3 \phi(s) - s^2 \phi(s) - s\phi'(s) - \phi''(s)$
2. If $L[f''(t)] = \tan^{-1}\left(\frac{1}{s}\right)$, $f(0) = 2$ and $f'(0) = -1$ find $L[f(t)]$

$$(\text{Ans. } \frac{2s-1+\tan^{-1}\left(\frac{1}{s}\right)}{s^3})$$

3. A periodic function $f(t)$ of period $2a$ is defined by

$$\begin{aligned} f(t) &= t, & \text{for } 0 \leq t \leq a, \\ &= 2a - t, & \text{for } a < t < 2a \end{aligned}$$

$$\text{show that } L[f(t)] = \frac{1}{s^2} \tanh\left(\frac{as}{2}\right)$$

4. If $f(t) = t^2$, $0 < t < 2$ and $f(t+2) = f(t)$
find $L[f(t)]$.

$$\text{Ans. } \frac{2 - 2^{-2s} - 4se^{-2s} - 4s^2e^{-2s}}{s^3(1 - e^{-2s})}$$

5. If $f(s) = L[f(t)]$, Prove that

$$L\left[\int_0^t \int_0^t f(u) du\right] = \frac{F(s)}{s^2} \text{ and hence generalise the result.}$$

8.2.9 Laplace transform of the unit step function

The unit step function or the Heaviside function is defined as

$$H(t-a) = 0 \quad \text{if } t < a,$$

$$H(t-a) = 1 \text{ if } t > a$$

where a is a non-negative constant. In particular, when $a = 0$, we have

$$H(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t > 0 \end{cases}$$

The unit step function is the basic building block of certain functions whose knowledge greatly increases the usefulness of the Laplace transformation.

It is easy to show that the Laplace transform of $H(t-a)$ is given by

$$\begin{aligned} L(H(t-a)) &= \frac{e^{-as}}{s} \\ L[H(t-a)] &= \int_0^\infty e^{-st} H(t-a) dt = \int_0^a e^{-st} 0 dt + \int_a^\infty e^{-st} 1 dt \\ &= \left. \frac{-e^{-st}}{s} \right|_a^\infty = \frac{e^{-as}}{s} \end{aligned}$$

In Particular, when $a=0$, we get $L(H(t)) = \frac{1}{s}$

8.2.10 Heaviside shift theorem: If $Lf(t) = f(s)$, then $L[f(t-a)H(t-a)] = e^{-as} f(s)$.

Proof: By definition, we have

$$\begin{aligned} L[f(t-a)H(t-a)] &= \int_0^\infty e^{-st} f(t-a)H(t-a) dt = \int_0^a e^{-st} f(t-a) 0 dt + \int_a^\infty e^{-st} f(t-a) 1 dt \\ &= \int_0^\infty e^{-s(p+a)} f(p) dp = e^{-sa} \int_0^\infty e^{-sp} f(p) dp \quad (\text{consider } p = t-a) \end{aligned}$$

8.2.1 Examples

Find the L.T. of the function

$$1. \quad f(t) = \begin{cases} 1 & \text{if } 0 < t < \pi \\ 0 & \text{if } \pi < t < 2\pi \\ \sin t & \text{if } t > 2\pi \end{cases}$$

Sol. Let us represent $f(t)$ in terms of unit step functions as $f(t) = H(t) - H(t - \pi) + H(t - 2\pi) \sin t$

$$L[f(t)] = L[H(t)] - L[H(t - \pi)] + L[H(t - 2\pi) \sin t] = \frac{1}{s} - \frac{e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s}$$

2. Find L.T. of $t^2 H(t - 3)$

Let us express $\phi(t) = t^2$ as a function of $(t - 3)$ by using Taylor's series, which states that

$$f(x) = f(a) + (x - a) f'(a) + \frac{(x - a)^2}{2!} f''(a) + \dots + \dots$$

$$\phi(t) = 9 + (t - 3) \cdot 6 + \frac{(t - 3)^2}{2!} \cdot 2$$

Now, by the above result

$$L[f(t) \cdot H(t - a)] = L[9 + 6(t - 3) + (t - 3)^2 H(t - 3)]$$

$$= L[f(t - 3) \cdot H(t - 3)] = e^{-3s} Lf(t) \quad \text{where } f(t) = 9 + 6t + t^2$$

$$L[t^2 H(t - a)] = e^{-3s} \left[\frac{9}{s} + \frac{6}{s^2} + \frac{2}{s^3} + \dots \right]$$

3. Express $F(t) = (t - 2)^2$ when $t > 2$ and $f(t) = 0$ when $0 < t < 2$ in terms of Heaviside unit step function and find its Laplace transform

Sol. It is easy to see that

$$F(t) = (t - 2)^2 = (t - 2)^2 \cdot H(t - 2) \quad (\text{since for } t < 2, H(t - 2) = 0 \text{ and for } t \geq 2, H(t - 2) = 1)$$

$$L[f(t) \cdot H(t - 2)] = L[(t - 2)^2 H(t - 2)] = e^{-2s} \cdot Lf(t) \quad \text{where } f(t) = t^2$$

$$= e^{-2s} \cdot \frac{2!}{s^3}$$

Exercise 8(G)

1. Find the Laplace transform of $f(t)$ using the unit step function, where

$$f(t) = \begin{cases} 1 & \text{if } 0 < t < 1 \\ 2 & \text{if } 1 < t < 3 \\ 4 & \text{if } 3 < t < 4 \\ -2 & \text{if } 4 < t \end{cases}$$

$$\text{Ans. } \left[\frac{1}{s} + \frac{e^{-s}}{s} + \frac{2e^{-3s}}{s} - \frac{6e^{-4s}}{s} \right]$$

2. Determine the Laplace transform of the following function:

$$(i) \quad f(t) = 6 - 2H(t - \pi) - 4H(t - 2\pi) \quad \text{Ans. } \frac{6 - 2e^{-\pi s} - 4e^{-2\pi s}}{s}$$

$$(ii) \quad f(t) = e^{2t}H(t - \log 5) \quad \text{Ans. } \frac{5^{2-s}}{5-s}$$

$$(iii) \quad f(t) = (t-a)H(t-a) \quad \text{Ans. } \frac{e^{-as}}{s^2}$$

$$(iv) \quad f(t) = t^2H(t-2) \quad \text{Ans. } \frac{e^{-2s}(2+4s+4s^2)}{s^3}$$

$$(v) \quad f(t) = H(t-e^4) \quad \text{Ans. } \frac{1}{s}$$

$$(vi) \quad f(t) = tH(t-2) \quad \text{Ans. } e^{-2s} \left[\frac{2}{s} + \frac{1}{s^2} \right]$$

$$(vii) \quad f(t) = (1+2t-3t^2+4t^3)H(t-2) \quad \text{Ans. } e^{-2s} \left[\frac{25}{s} + \frac{38}{s^2} + \frac{42}{s^3} + \frac{24}{s^4} \right]$$

$$(viii) \quad f(t) = \sin t \cdot H(t-\pi) \quad \text{Ans. } e^{-\pi s} \frac{1}{s^2+1}$$

8.3.1 Inverse Laplace Transforms

If $L[f(t)] = F(s)$ then $f(t)$ is called the inverse Laplace transform of $F(s)$ and is denoted by $L^{-1}[F(s)] = f(t)$

if f_1 and f_2 are continuous functions on $[0, \infty]$ such that

$$L[f_1(t)] = L[f_2(t)] \text{ or } F_1(s) = F_2(s) \ (\forall s > 0) \text{ then } f_1(t) = f_2(t) \ (\forall t \geq 0)$$

Some of the important properties of the inverse laplace transforms are analogous to the corresponding properties of Laplace transforms.

1. *Linearity Property:* If k_1, k_2 are any constants and $F_1(s)$ and $F_2(s)$ are the Laplace transforms of $f_1(t)$ and $f_2(t)$ respectively, then

$$L^{-1}[k_1 F_1(s) + k_2 F_2(s)] = k_1 f_1(t) + k_2 f_2(t).$$

2. *First shifting property:* If $L^{-1}[F(s)] = f(t)$, then

$$L^{-1}[e^{-as} F(s)] = \begin{cases} f(t-a) & \text{for } t > a \\ 0 & \text{for } t < a \end{cases}$$

3. *Change of scale property:* if $L^{-1}[F(s)] = f(t)$, then

$$L^{-1}[F(as)] = \frac{1}{a} f\left(\frac{t}{a}\right)$$

4. *Inverse Laplace transforms of derivatives:* If $L^{-1}[F(s)] = f(t)$ then

$$L^{-1}[F^n(s)] = L^{-1}\left[\frac{d^n}{ds^n} F(s)\right] = (-1)^n t^n f(t)$$

5. *Inverse Laplace transform of integrals:* If $L^{-1}[F(s)] = f(t)$ then

$$L^{-1}\left\{\int_0^s F(u) du\right\} = \frac{f(t)}{t}$$

6. *Multiplication by 's':* If $L^{-1}[F(s)] = f(t)$ and $f(0) = 0$, then

$$L^{-1}[sF(s)] = f'(t)$$

7. *Division by 's':* If $L^{-1}[F(s)] = f(t)$ then

$$L^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(u) du.$$

This result can be extended to $L^{-1}\left(\frac{F(s)}{s^n}\right)$

$$= \int_0^t \int_0^t \dots \int_0^t f(u) du^n.$$

(the proof is left to the student)

8.3.2 Table of inverse transforms

$L[f(t)]$	$L^{-1}[F(s)]$
$L(1) = \frac{1}{s}$	$L^{-1}\left(\frac{1}{s}\right) = 1$
$L(e^{-at}) = \frac{1}{s+a}$	$L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$
$L(t^n) = \frac{n!}{s^{n+1}}$	$L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{(n+1)!}$
if 'a' is a positive integer $L(\sin at) = \frac{a}{s^2 + a^2}$	$L^{-1}\left(\frac{1}{s^2 + a^2}\right) = \frac{1}{a} \sin at$
$L(\cos at) = \frac{s}{s^2 + a^2}$	$L^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos at$
$L(\sinh at) = \frac{a}{s^2 - a^2}$	$L^{-1}\left(\frac{1}{s^2 - a^2}\right) = \frac{1}{a} \sinh at$
$L(\cosh at) = \frac{s}{s^2 - a^2}$	$L^{-1}\left(\frac{s}{s^2 - a^2}\right) = \cosh at$

8.3.3 Using the above results inverse Laplace transforms of some simple functions can be obtained as below.

Find the inverse Laplace transform of:

1. $\frac{3+2s+s^2}{s^3}$

2. $\frac{2s+3}{s^2+9}$

3. $\frac{3s+5}{9s^2-25}$

4. $\frac{1}{4s+5}$

5. $\frac{1}{4s-5}$

6. $\frac{3}{s+4}$

7. $\frac{2s+1}{(s+1)}$

8. $\frac{s+3}{s^2+4}$

Sol. From the above table, we get

$$1. \quad L^{-1} \frac{3}{s^3} + L^{-1} \left(\frac{2}{s^2} \right) + L^{-1} \left(\frac{1}{s} \right) = 3L^{-1} \frac{1}{s^3} + 2L^{-1} \frac{1}{s^2} + L^{-1} \left(\frac{1}{s} \right) = 3 \frac{t^2}{2} + 2t + 1$$

$$2. \quad L^{-1} \left(\frac{2s+3}{s^2+9} \right) = L^{-1} \left(\frac{2s}{s^2+9} \right) + L^{-1} \left(\frac{3}{s^2+9} \right) = 2L^{-1} \left(\frac{s}{s^2+9} \right) + L^{-1} \left(\frac{3}{s^2+9} \right) \\ = 2\cos 3t + \sin 3t$$

$$3. \quad L^{-1} \left(\frac{3s+5}{9s^2-25} \right) = L^{-1} \left(\frac{3s}{9s^2-25} \right) + L^{-1} \left(\frac{5}{9s^2-25} \right) = \frac{1}{3} \cosh \left(\frac{5t}{3} \right) + \frac{1}{3} \sinh \left(\frac{5t}{3} \right)$$

$$4. \quad L^{-1} \left(\frac{1}{4s+5} \right) = L^{-1} \frac{1}{4 \left(s + \frac{5}{4} \right)} = \frac{1}{4} e^{-\left(\frac{5}{4}\right)t}$$

$$5. \quad L^{-1} \left(\frac{1}{4s-5} \right) = L^{-1} \frac{1}{4 \left(s - \frac{5}{4} \right)} = \frac{1}{4} e^{\left(\frac{5}{4}\right)t}$$

$$6. \quad L^{-1} \left(\frac{3}{s+4} \right) = 3L^{-1} \frac{1}{s+4} = 3e^{-4t}$$

$$7. \quad L^{-1} \frac{2s+1}{(s^2+1)} = 2L^{-1} \left(\frac{s}{s^2+1} \right) + L^{-1} \left(\frac{1}{s^2+1} \right) = 2\cos t + \sin t$$

$$8. \quad L^{-1} \left(\frac{s+3}{s^2+4} \right) = L^{-1} \left(\frac{s}{s^2+4} \right) + L^{-1} \left(\frac{3}{s^2+4} \right) = \cos 2t + \frac{3}{2} \sin 2t$$

Exercise 8(H)

Find the invere Laplace transform of:

$$1. \quad \frac{2}{s} + \frac{1}{s^3} + \frac{1}{s+4} \quad 2. \quad \frac{3s+4}{s^2+16} \quad 3. \quad \frac{2s+3}{s^2-9} \quad 4. \quad \frac{(s^2-1)^2}{s^5} \quad 5. \quad \frac{4s+15}{16s^2-25}$$

Ans. 1. $2 + \frac{t^2}{2} + e^{-4t}$ 2. $3\cos 4t + \sin 4t$ 3. $2\cosh 3t + \sinh 3t$

4. $1 + t^2 + \frac{t^4}{24}$ 5. $\frac{1}{4}\cosh\left(\frac{5t}{4}\right) = \frac{3}{4}\sinh\frac{5}{4}t$

Examples 8.3.4

Find the inverse Laplace transform of:

1. $\frac{6s-4}{s^2-4s+20}$ 2. $\frac{1}{(s-3)^3}$ 3. $\frac{1}{(s-2)^4}$ 4. $\frac{1}{(s-1)^2} + \frac{s-2}{s^2-4s+5}$

Sol. 1. $L^{-1}\left(\frac{6s-4}{s^2-4s+20}\right) = L^{-1}\frac{6s-12+8}{(s-2)^2+16} = L^{-1}\frac{6s-12}{(s-2)^2+4^2} + L^{-1}\frac{8}{(s-2)^2+4^2}$
 $= 6L^{-1}\frac{(s-2)}{(s-2)^2+4^2} + 2L^{-1}\frac{4}{(s-2)^2+4^2} = 6e^{2t}\cos 4t + 2e^{2t}\sin 4t$

2. $L^{-1}\left(\frac{1}{(s-3)^3}\right) = \frac{1}{2}e^{3t}t^2$

3. $L^{-1}\left(\frac{1}{(s-2)^4}\right) = \frac{1}{2}e^{2t}t^3$

4. $L^{-1}\left(\frac{1}{(s-2)^4}\right) + L^{-1}\frac{(s-2)}{(s^2-4s+5)} = L^{-1}\left(\frac{1}{(s-2)^2}\right) + L^{-1}\left(\frac{(s-2)}{s^2-4s+4+1}\right)$
 $= L^{-1}\left(\frac{1}{(s-2)^2}\right) + L^{-1}\left(\frac{s-2}{(s-2)^2+1^2}\right) = e^t.t + e^{2t}.\cos t$

Exercise 8(I)

Find the inverse Laplace transform of:

(i) $\frac{s+4}{s^2-4s+13}$

(ii) $\frac{s+6}{s^2-8s+25}$

(iii) $\frac{4s+12}{s^2+8s+16}$

Ans. (i) $e^{2t}\cos 3t + 2e^{2t}\sin 3t$ (ii) $e^{4t}\cos 3t + e^{4t}\sin 3t$ (iii) $4e^{-4t}.1 - 4e^{-4t}$ or $4e^{-4t}(1-t)$

8.3.5 Method of partial fractions:

Whenever possible we express the given function $F(s)$ into the sum of linear or quadratic partial fractions as

$$F(s) = \frac{A}{(s+a)^r} + \frac{Bs+C}{(s^2+a^2)^r} \text{ and then use the standard results to find the inverse L.T.}$$

Examples (1) Find $L^{-1}\left[\frac{s+4}{s^3+s}\right]$ for all $s>0$,

Sol. By partial fraction expansion, we have

$$\frac{s+1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$s+1 = A(s^2+1) + (Bs+C)s = As^2 + A + Bs^2 + Cs$$

comparing coefficients of different powers of s from both sides

$$0 = A + B$$

$$1 = C$$

$$\therefore 1 = A, B = -1$$

$$\therefore \frac{s+1}{s(s^2+1)} = \frac{1}{s} + \frac{-s+1}{s^2+1}$$

By the linearity property of L^{-1} , we have

$$L^{-1}\left(\frac{s+1}{s^3+s}\right) = L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s^2+1}\right) = 1 - \cos t + \sin t$$

2. If $L[f(t)] = \frac{2s^2 - 6s + 5}{s^3 + 6s^2 + 11s - 6}$ find $[f(t)]$

Sol. By partial fractions we have

$$f(s) = \frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6} = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$$

$$\therefore 2s^2 - 6s + 5 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

put $s = 1$ we get

$$1 = 2A$$

$$A = 1/2$$

put $s = 2$

$$B = -1$$

Similarly $s = 3$ gives

$$C = 5/2$$

$$\therefore L^{-1}[f(s)] = \frac{1}{2}L^{-1}\frac{1}{(s-1)} - L^{-1}\left(\frac{1}{s-2}\right) + \frac{5}{2}L^{-1}\left(\frac{A}{s-3}\right) = \frac{1}{2}e^t - e^{2t} + \frac{5}{2}e^{3t}$$

3. If $Lf(t) = \frac{3s+1}{(s-1)(s^2+1)}$, find $f(t)$

Sol. By partial fractions we have

$$f(s) = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

$\therefore$ Solving for A, B C

we have $A = 2$, $B = -2$, $C = -1$

$$\begin{aligned} L^{-1}[f(s)] &= 2L^{-1}\frac{1}{s-1} - 2L^{-1}\frac{s}{s^2+1} + L^{-1}\frac{1}{s^2+1} \\ &= 2e^t - 2\cos t + \sin t \end{aligned}$$

4. If $L[f(t)] = \frac{s+2}{(s+3)(s+1)^3}$ find $f(t)$

Sol.
$$f(s) = \frac{s+2}{(s+3)(s+1)^3} = \frac{1}{8}\frac{1}{s+3} - \frac{1}{8}\frac{1}{s+1} + \frac{1}{4}\frac{1}{(s+1)^2} + \frac{1}{2}\frac{1}{(s+1)^3}$$

(after expressing the function in partial fractions)

$$\begin{aligned} \therefore L^{-1}[f(s)] &= \frac{1}{8}L^{-1}\frac{1}{s+3} - \frac{1}{8}L^{-1}\frac{1}{s+1} + \frac{1}{4}L^{-1}\frac{1}{(s+1)^2} + \frac{1}{2}L^{-1}\frac{1}{(s+1)^3} \\ &= \frac{1}{8}e^{-3t} - \frac{1}{8}e^{-t} + \frac{1}{4}te^{-t} + t^2e^{-t} = \frac{1}{8}[(2t^2 + 2t - 1)e^{-t} + e^{-3t}] \end{aligned}$$

5. If $L[f(t)] = \frac{(s^2+2s-42)}{(s^2+2s+5)(s^2+2s+2)}$ find $f(t)$

Sol: Let $f(s) = \frac{as+b}{s^2+2s+5} + \frac{cs+d}{s^2+2s+2}$

$$\therefore s^2 + 2s - 42 = (as+b)(s^2+2s+2) + (cs+d)(s^2+2s+5)$$

Equating the coefficients of like powers of s we get $a = 0$, $b = 3$, $c = 0$, $d = -2$.

$$\therefore f(s) = \frac{3}{s^2 + 2s + 5} - \frac{2}{s^2 + 2s + 2} = \frac{3}{(s+1)^2 + 2^2} - \frac{2}{(s+1)^2 + 1^2}$$

$$\therefore L^{-1}[f(s)] = 3L^{-1}\left[\frac{1}{(s+1)^2 + 2^2}\right] - 2L^{-1}\left[\frac{1}{(s+1)^2 + 1^2}\right]$$

$$= \frac{3}{2} e^{-t} \sin 2t - 2e^{-t} \sin t.$$

6. If $L[f(t)] = \frac{2s}{4s^2 + 16}$, find $f(t)$

Sol. By change of scale property we have

$$L^{-1}\left[\frac{2s}{4s^2 + 16}\right] = \frac{1}{2} \cos 2t \quad \text{since } L^{-1}\left[\frac{s}{s^2 + 16}\right] = \cos 4t$$

7. If $L[f(t)] = \frac{e^{-5s}}{(s-2)^4}$ find $f(t)$

Sol. We know $L^{-1}\left[\frac{1}{(s-2)^4}\right] = e^{2t} L^{-1}\left[\frac{1}{s^4}\right] = e^{2t} \cdot \frac{t^3}{3!} = \frac{1}{6} t^3 e^{2t}$

Thus by second shifting property

$$L^{-1}\left[\frac{e^{-5s}}{(s-2)^4}\right] = \begin{cases} \frac{1}{6} (t-5)^3 e^{2(t-5)} & \text{if } t > 5 \\ 0 & \text{if } t < 5 \end{cases}$$

Exercise 8(J)

Determine the inverse Laplace transform of each of the following functions.

(i) $\frac{3}{s+2}$ (ii) $\frac{s}{s^2 + 2s + 6}$ (iii) $\frac{2s+1}{s(s+1)}$ (iv) $\log\left(\frac{s+3}{s+2}\right)$

(v) $\frac{s-5}{s^2 + 6s + 13}$ (vi) $\frac{e^{-2s}}{s(s+1)}$ (vii) $\frac{s^2}{(s^2 + 9)(s^2 + 16)}$

$$(viii) \frac{2s^2 + s - 10}{(s-4)(s^2 + 2s + 2)}$$

$$(ix) \log \left(1 + \frac{1}{s^2} \right) \quad (x) \frac{1}{(s^4 + 64)}$$

$$(xi) \frac{1}{s(s+1)(s+2)(s+3)}$$

$$(xii) \frac{s}{(s^4 + s^2 + 1)}$$

$$(xiii) \frac{s^2 + 10s + 13}{s - 10(s^2 - 5s + 6)}$$

$$(xiv) \frac{s}{(s^2 + 1)(s^2 + 4)}$$

$$(xv) \frac{s + 29}{(s+4)(s^2 + 9)}$$

$$(xvi) \frac{5s + 3}{(s-1)(s^2 + 2s + 5)}$$

$$(xvii) \frac{4s + 4}{(s-1)^2(s+2)}$$

$$(xviii) \frac{s^2 + 1}{s^3 + 3s^2 + 2s}$$

Ans. (i) $3e^{-2t}$ (ii) $e^{-1} \cos \sqrt{5}t - \frac{1}{\sqrt{5}} e^{-1} \sin \sqrt{5}t$ (iii) $1 + e^{-t}$ (iv) $\frac{1}{t}(e^{-2t} - e^{-3t})$

(v) $e^{-3t} \cos 2t - 4e^{-3t} \sin 2t$ (vi) $4(1 - e^{-2t})(t - 2)$ (vii) $\left(\frac{4}{7} \sin 4t - \frac{3}{7} \sin 3t \right)$

(viii) $e^{4t} - e^{-t}(\cos 2t + 2 \sin t)$ (ix) $\frac{2}{t}(1 - \cos t)$ (x) $\frac{1}{8} \sin 2t \sinh 2t$

(xi) $\frac{1}{6} \cdot 1 - \frac{1}{2} e^{-t} + \frac{1}{2} e^{-2t} - \frac{1}{6} e^{-3t}$ (xii) $\frac{2}{\sqrt{3}} \left(\sin \frac{\sqrt{5}}{2} t \cdot \sinh \frac{t}{2} \right)$

(xiii) $12e^t - 37e^{2t} + 25e^{3t}$ (xiv) $\frac{1}{3} (\cos t - \cos 2t)$ (xv) $e^{-4t} - \cos 3t + \frac{5}{3} \sin 3t$

(xvi) $e^t - \frac{1}{2} e^{-t} \cos 2t + \frac{3}{2} e^{-t} \sin 2t$ (xvii) $\frac{4}{9} (e^t - e^{-2t} + 6te^t)$

(xviii) $\frac{1}{2} - 2^{-et} + \frac{5}{2} e^{-2t}$

8.4 Convolution Theorem

If $L[f(t)] = F(s)$ and $L[g(t)] = G(s)$ the inverse transform of $F(s)$. $G(s)$ denoted by $H(t)$, is called the convolution of f and g is defined as given below.

$$H(t) = (f * g)(t) = \int_0^t (t-u)g(u)du, \forall t \geq 0 \quad \dots(1)$$

The convolution operation $*$ has the following properties.

$$(f * g)(t) = (g * f)(t), \quad \forall t \geq 0 \text{ (commutative)}$$

$$[f * (g * h)](t) = [(f * g) * h](t) \quad \forall t \geq 0 \text{ (associative)}$$

$$[f * (g + h)](t) = (f * g)(t) + (f * h)(t) \quad \forall t \geq 0 \text{ (distributive)}$$

8.4.1 Theorem: Let $f(t)$ and $g(t)$ be the inverse transforms of $F(s)$ and $G(s)$ respectively, and satisfy the hypothesis of the existence theorem (8.10.2) then the the inverse transform $h(t)$ of the product $F(s) G(s)$ is the convolution of $f(t)$ and $g(t)$.

$$h(t) = (f * g)(t) = \int_0^t f(t-u)g(u)du$$

$$\text{i.e. } L^{-1}[F(s)G(s)] = \int_0^t f(t-u)g(u)du \quad \dots(i)$$

Proof: It is sufficient if we show that

$$F(s) G(s) = L \left[\int_0^t f(t-u)g(u)du \right]$$

$$\begin{aligned} \text{Now } F(s)G(s) &= \left[\int_0^\infty e^{-sv} f(v)dv \right] \left[\int_0^\infty e^{-su} g(u)du \right] \\ &= \int_0^\infty \int_0^\infty e^{-(u+v)s} f(v)g(u)dvdu = \int_0^\infty \int_0^\infty e^{-s(v+u)} f(v)dv \Big] g(u)du. \end{aligned}$$

where the integration is extended over the first quadrant ($u \geq 0, v \geq 0$) in the uv -plane. We now introduce a new variable 't' in the inner integral of the last expression by taking $v = t-u$ so that $u + v = t$ and $dv = dt$ (u being fixed during this integration). Thus

$$F(s)G(s) = \int_0^\infty \left[\int_u^\infty e^{-st} f(t-u)dt \right] g(u)du = \int_0^\infty \int_u^\infty e^{-st} f(t-u)g(u)dtdu$$

$$\begin{aligned}
 F(s)G(s) &= \int_0^\alpha \int_0^1 [e^{-st} f(t-u)g(u)du]dt = \int_0^\alpha e^{-st} \int_0^1 [f(t-u)g(u)du]dt \\
 &= L \left[\int_0^1 f(t-u)g(u)du \right]
 \end{aligned}$$

8.4.2 Examples

(1) Evaluate $L^{-1} \left[\frac{1}{s^2(s+1)^2} \right]$, using the convolution theorem

Sol. we have $L^{-1} \left(\frac{1}{s^2} \right) = t$ and $L^{-1} \left(\frac{1}{(s+1)^2} \right) = te^{-t}$

By the convolution theorem

$$\begin{aligned}
 &= L^{-1} \left[\frac{1}{s^2(s+1)^2} \right] = \int_0^t (ue^{-u} - (t-u))du = \int_0^t (ut - u^2)e^{-u}du \\
 &= [(ut - u^2)e^{-u} - (t-2u)e^{-u} + (-2)(-e^{-u})] = t - e^{-t} + 2e^{-t} + t - 2
 \end{aligned}$$

(2) Evaluate $L^{-1} \left[\frac{s}{(s^2 + 2^2)^2} \right]$ using the convolution theorem.

Sol. Let $F(s) = \frac{s}{s^2 + 4}$, $G(s) = \frac{s}{s^2 + 4}$ and Let $\phi(s) = F(s) G(s)$

$$f(t) = L^{-1} [F(s)] = L^{-1} \left(\frac{s}{s^2 + 4} \right) = \cos 2t$$

$$g(t) = L^{-1} [G(s)] = L^{-1} \left(\frac{1}{s^2 + 4} \right) = \frac{1}{2} \sin 2t$$

$$f(u) = \cos 2u \text{ and } g(t-u) = \frac{1}{2} \sin 2(t-u)$$

$$\begin{aligned}
 L^{-1}[\varphi(s)] &= \int_0^t \cos 2u \frac{1}{2} \sin 2(t-u) du \\
 &= \frac{1}{4} \int_0^t \sin 2t + \sin 2(t-2u) du \\
 &= \frac{1}{4} \left[\sin 2t \cdot u + \frac{1}{2} \cos 2(t-2u) \right]_0^t \frac{1}{4} t \sin 2u
 \end{aligned}$$

(3) Evaluate $L^{-1}\left[\frac{1}{s\sqrt{s+4}}\right]$ using convolution theorem.

Sol. Let $F(s) = \frac{1}{\sqrt{s+4}}$ $G(s) = \frac{1}{s}$ and choose

$$\phi(s) = F(s) \cdot G(s)$$

$$\therefore f(t) = L^{-1}[F(s)] = L^{-1} \frac{1}{(s+4)^{1/2}} = e^{-4t} L^{-1} \frac{1}{s^{1/2}} = e^{-4t} \frac{t^{1/2}}{\Gamma \frac{1}{2}} = \frac{e^{-4t}}{\sqrt{\pi t}}$$

$$g(t) = L^{-1}[G(s)] = L^{-1} \left(\frac{1}{s} \right) = 1$$

$$f(u) = \frac{e^{-4u}}{\sqrt{\pi u}} \text{ and } g(t-u) = 1$$

By convolution theorem

$$L^{-1}[\phi(s)] = \int_0^t \frac{e^{-4u}}{\sqrt{\pi u}} du = \frac{1}{\sqrt{\pi}} \int_0^{2\sqrt{t}} e^{-x^2} dx$$

$$\text{where } 4u = x^2 \quad 2du = xdx \quad \sqrt{u} = \frac{x}{2}$$

$$L^{-1}[\phi(s)] = \frac{1}{2} \operatorname{erf}(2\sqrt{t}) \quad (\text{where } \operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx)$$

(erft stands for error function)

Exercise 8(K)

Using the convolution theorem find the inverse Laplace transforms of

$$(i) \frac{1}{(s-2)(s+2)^2} \quad (ii) \frac{s^2}{(s^2+1)(s^2+4)} \quad (iii) \frac{s^2}{(s^2+1)^2} \quad (iv) \frac{1}{s} \log \frac{(s+2)}{s+1}$$

$$(v) \log \left(\frac{s+2}{s+1} \right) \quad (vi) \frac{1}{(s+1)(s^2+1)}$$

$$\text{Ans. (i)} \quad \frac{1}{16} (e^{2t} - e^{-2t} - 4te^{-2t}) \quad (ii) \quad \frac{1}{3} (2\sin 2t - \sin t) \quad (iii) \quad \frac{1}{3} t \sin t$$

$$(iv) \int_0^{\infty} \frac{e^{-u} - e^{-2u}}{u} du \quad (vi) \quad \frac{1}{2} [e^{-4} - \cos t - \sin t]$$

8.4.3 Applications to differential equations

Suppose we wish to find the particular solution of the differential equation

$$\alpha y'' + \beta y' + \gamma y = f(t) \quad \dots (1)$$

which satisfies the initial conditions $y(0) = y_0$ and $y'(0) = y_0^1$

Taking the Laplace transform on both sides of (1) and using its linearity, it follows that

$$\alpha L(y'') + \beta L(y') + \gamma L(y) = L[f(t)]$$

$$\text{but } L(y'') = s^2 L(y) - sy(0) - y'(0)$$

$$L(y') = sL(y) - y(0)$$

$$\alpha[s^2 L(y) - sy(0) - y'(0)] + \beta[sL(y) - y(0)] + \gamma L(y) = L[f(t)]$$

and hence

$$(\alpha s^2 + \beta s + \gamma)L(y) - L[f(t)] + (\alpha s + \beta)y_0 + \alpha y_0^1$$

$$\Rightarrow L(y) = \frac{L[f(t)] + (\alpha s + \beta)y_0 + \alpha y_0^1}{\alpha s^2 + \beta s + \gamma}$$

The function $L[f(t)]$ is a specific function of s since $f(t)$ is known; and since $\alpha, \beta, \gamma, y_0$ and y_0^1 are constants, $L(y)$ is completely known as a function of s . The inverse Laplace transform of $L(y)$ will be the solution of the given diff. eqn.

(Here it is assumed that the functions $f(t), y, y'$ and y'' must have Laplace transforms).

Solved Examples

8.4.4

Find the solution of $\frac{d^2 y}{dt^2} + \frac{dy}{dt} - 2y = \sin t$ which satisfies the initial conditions $y = 0$.

$y' = 0$ when $t = 0$

Sol. Applying the Laplace transforms to both sides of the given equation and by linearity, we have

$$L(y'') + L(y') - 2L(y) = L(\sin t)$$

As we know that $L(\sin t) = \frac{1}{s^2 + 1}; s > 0$

$$[s^2 L(y) - sy(0) - y'(0)] + [sL(y) - y(0)] - 2L(y) = \frac{1}{s^2 + 1}$$

Given conditions $y(0) = 0, y'(0) = 0$

$$\Rightarrow s^2 L(y) + sL(y) - 2L(y) = \frac{1}{s^2 + 1}$$

$$L(y) = \frac{1}{(s^2 + s - 2)(s^2 + 1)} = \frac{1}{(s - 1)(s + 2)(s^2 + 1)}$$

we have the partial fraction expansion

$$\frac{1}{(s - 1)(s + 2)(s^2 + 1)} = \frac{A}{s - 1} + \frac{B}{s + 2} + \frac{Cs + D}{s^2 + 1}$$

where $A = \frac{1}{6}, B = \frac{-1}{15}, C = \frac{-1}{10}$ and $D = \frac{-3}{10}$

Therefore we get

$$\begin{aligned} y(t) &= L^{-1} \left[\frac{1}{(s - 1)(s + 2)(s^2 + 1)} \right] \\ &= \frac{1}{6} L^{-1} \left[\frac{1}{s - 1} \right] - \frac{1}{15} L^{-1} \left[\frac{1}{s + 2} \right] - \frac{1}{10} L^{-1} \left[\frac{s}{s^2 + 1} \right] - \frac{3}{10} L^{-1} \left[\frac{1}{s^2 + 1} \right] \\ &= \frac{1}{6} e^t - \frac{1}{15} e^{-2t} - \frac{1}{10} \cos t - \frac{3}{10} \sin t \end{aligned}$$

as the solution to the given initial value problem.

8.4.5 Example

Solve the initial value problem

$$y'' - 3y' + 2y = e^{3t}, y = 1, y' = 0 \text{ where } t = 0$$

Sol. Applying Laplace transform to both sides of the differential equation, we get

$$[s^2 Ly(t) - sy(0) - y'(0)] - 3[sLy(t) - y(0)] + 2L(t) = L(e^{3t}) = \frac{1}{s-3}$$

Applying the given conditions, we get

$$(s^2 - 3s + 2) Ly(t) = s - 3 + \frac{1}{s-3}$$

$$\therefore L[y(t)] = \frac{s-3}{(s^2-3s+2)} + \frac{1}{(s-3)(s^2-3s+2)}$$

$$\therefore L[y(t)] = \frac{1}{(s-3)(s^2-3s+2)} + \frac{s-3}{(s^2-3s+2)}$$

$$= \frac{1}{(s-1)(s-2)(s-3)} + \frac{s-3}{(s-1)(s-2)} \quad \dots(1)$$

To find $y(t)$, we expand each term on the right hand side of (1) in partial functions, Thus

$$\frac{1}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} - \frac{B}{s-2} + \frac{C}{s-3}$$

$$1 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

substituting $s = 1$ gives $A = \frac{1}{2}$; similarly substituting $s = 2$ gets $B = -1$ and substituting

$$s = 3 \text{ gives } C = \frac{1}{2} \quad \text{hence}$$

$$\frac{1}{(s-1)(s-2)(s-3)} = \frac{1}{2(s-1)} - \frac{1}{s-2} + \frac{1}{2(s-3)}$$

Similarly we can also write

$$\frac{s-3}{(s-1)(s-2)} = \frac{D}{s-1} + \frac{E}{s-2}$$

$$s-3 = D(s-2)E(s-1)$$

Taking $s = 2$, we get $E = -1$

similarly $s = 1$ we get $D = 2$

$$\begin{aligned}\text{Hence } L[f(t)] &= \frac{1}{2(s-1)} - \frac{1}{s-2} + \frac{1}{2(s-3)} + \frac{1}{s-1} - \frac{1}{s-2} \\ &= \frac{2}{2(s-1)} - \frac{2}{s-2} + \frac{1}{2(s-3)}\end{aligned}$$

$$(\text{or}) \quad L[y(t)] = L\left[\frac{5}{2}e^t - 2e^{2t} + \frac{1}{2}e^{3t}\right]$$

$$y(t) = \frac{5}{2}e^t - 2e^{2t} + \frac{1}{2}e^{3t}$$

Example 8.4.6

Solve $y'' + 3y' + 2y = 2t^3 + 2t + 2$ with $y(0) = 2, y'(0)$

Sol. Applying Laplace transform to both sides of the given equation we get

$$L(y'') + 3L(y') + 2L(y) = 2L(t^2) + 2L(t) + 2$$

$$[s^2Ly(t) - sy(0) - y'(0)] + 3[sLy(t) - y(0)] + 2Ly(t) = \frac{4}{s^3} + \frac{2}{s^2} + \frac{2}{s} \quad \dots(1)$$

using the given conditions (1) gives

$$(s^2+3s+2)Ly(t) = \frac{2(2+s+s^2)}{s^3} + 2s + 6 = \frac{2(s^4+3s^3+s^2+s+2)}{s^3}$$

$$\therefore L[y(t)] = \frac{2(s^4+3s^3+s^2+s+2)}{s^3(s^2+s+2)}$$

By partial fractions,

$$y(t) = 3L^{-1}\left(\frac{1}{s}\right) - 2L^{-1}\left(\frac{1}{s^2}\right) + 2L^{-1}\left(\frac{1}{s^3}\right) - L^{-1}\left(\frac{1}{s+2}\right)$$

$$y(t) = 3 - 2t + t^2 - e^{-2t}$$

Exercise 8(I)

Using Laplace transform method solve the following differential equations with the given conditions.

1. $y'' + 2y' - 3y = 0$ at $y = 0$, $y' = 4$ when $x = 0$
2. $y'' + 4y = 4t$ at $y = 1$, $y' = 0$ when $t = 0$
3. $y'' - 2y' + y = e^t$ at $y = 2$, $y' = -1$ when $t = 0$
4. $y'' + k^2y = \cos kt$ at $y = 0$, $y' = \frac{1}{2}$ when $t = 0$
5. $y'' - y = 1$ at $y = -1$, $y' = 1$ when $t = 0$
6. $y'' + 2y' + 3y = 3e^{-t} \sin t$; $y(0) = y'(0) = 0$
7. $x'' + 4x' = -8t$ given $x(0) = 0$, $x'(0) = 0$
8. $y'' + 9y = 18t$ given $y(0) = 0$, $y'(0) = 0$
9. $y'' + y = \sin t$ given $y(0) = -1$, $y'(0) = \frac{1}{2}$

Ans. 1. $y = e^x - e^{-3x}$

2. $y = \frac{1}{4}t - \frac{1}{8}\sin t + \cos t$ 3. $y = 2e^t - te^t + \frac{1}{2}t^2e^t$

4. $y = \frac{1}{2k}\sin kt + \frac{1}{2}t\sin t$ 5. $y = -1 + e^t - \cos ht$ 6. $y = 3e^{-t}\sin t$

7. $x = \frac{-1}{8} + \frac{1}{2}t - t^2 + \frac{1}{8}e^{-4t}$ 8. $y = 2t - \frac{2}{3}\sin 3t$ 9. $y = \left(1 - \frac{1}{2}\right)\cos t$