Geometry of the Generalized Geodesic Flow and Inverse Spectral Problems

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Second Edition

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Contents

Preface

This monograph is devoted to the analysis of some inverse problems concerning the spectrum of the Laplace operator in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and of the scattering length spectrum (SLS) (the set of sojourn times of reflecting rays) of the scattering kernel associated with scattering in the exterior Ω of a bounded obstacle $K \subset \mathbb{R}^n$, $n \geq 2$. In both cases our aim is to obtain some geometric information about Ω (resp. K) from spectral (resp. scattering) data. We treat both inverse problems by using similar techniques based on properties of the generalized geodesic flow in Ω and on microlocal analysis of the corresponding mixed problems.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a closed bounded domain with C^∞ smooth boundary $\partial\Omega$, and let A be the self-adjoint operator in $L^2(\Omega)$ related to the *Laplacian*

$$-\Delta = -\sum_{j=1}^{n} \partial_{x_j}^2$$

in Ω with Dirichlet boundary condition on $\partial\Omega$. The *spectrum* of A is given by a sequence

$$0 \leq \lambda_1^2 \leq \lambda_2^2 \leq \cdots \leq \lambda_m^2 \leq \cdots \tag{0.1}$$

of eigenvalues λ_j^2 for which the problem

$$\begin{cases} -\Delta\varphi_j = \lambda_j^2 \varphi_j & \text{in } \Omega, \\ \varphi_j = 0 & \text{on } \partial\Omega \end{cases}$$

has a non-trivial solution $\varphi_j \in C^\infty(\Omega)$. The *counting function*

$$N(\lambda) = \#\{j : \lambda_j^2 \leq \lambda^2\},$$

where every eigenvalue is counted with its multiplicity, admits a polynomial bound

$$N(\lambda) \leq C\lambda^n, \quad \lambda \to +\infty. \tag{0.2}$$

Moreover, it is known (see [Se], [H4], [SaV]) that $N(\lambda)$ has a Weyl type asymptotic

$$N(\lambda) = \frac{(4\pi)^{-n/2}}{\Gamma(n/2+1)} \mathrm{Vol}_n(\Omega)\lambda^n + \mathcal{O}(\lambda^{n-1}) \tag{0.3}$$

as $\lambda \to \infty$. Thus, from the spectrum (0.1) we can recover the volume of Ω. In 1911, Weyl [W] conjectured that for every bounded domain Ω in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ we have

$$N(\lambda) = \frac{(4\pi)^{-n/2}}{\Gamma(n/2+1)}\mathrm{Vol}_n(\Omega)\lambda^n - \frac{(4\pi)^{-(n-1)/2}}{4\Gamma(n-1/2+1)}\mathrm{Vol}_{n-1}(\partial\Omega) + o(\lambda^{n-1}) \tag{0.4}$$

as $\lambda \to \infty$. Ivrii [Ivl] proved that if the points $(x, v) \in \partial\Omega \times \mathbb{S}^{n-1}$ for which there exists a periodic billiard trajectory in Ω issued from x in direction v form a subset of Lebesgue measure zero in the space $\partial\Omega \times \mathbb{S}^{n-1}$, then the asymptotic (0.4) holds. Therefore, for such domain $\mathrm{Vol}_{n-1}(\partial\Omega)$ becomes another spectral invariant. It is not known so far if the assumption in Ivrii's result is always satisfied.

To obtain more information from the knowledge of the spectrum $\{\lambda_j^2\}$, it is convenient to examine some distributions determined by the sequence (0.1). The distribution

$$\tau(t) = \sum_j e^{-\lambda_j^2 t} \in \mathcal{D}'(\bar{\mathbb{R}}_+)$$

has the asymptotic

$$\tau(t) \sim \sum_{j=1}^{\infty} c_j t^{-(n/2)+j/2} \text{ as } t \searrow 0, \tag{0.5}$$

and the constants c_j are spectral invariants. Moreover, one can recover $\mathrm{Vol}_n(\Omega)$ and $\mathrm{Vol}_{n-1}(\partial\Omega)$ from c_0 and c_1.

In his classical work Kac [Kac] posed the problem of recovering the shape of a strictly convex domain $\Omega \subset \mathbb{R}^2$ from the spectrum (0.1). This article has had a big influence on the investigations of various inverse spectral problems for manifolds with and without boundary as well as on the analysis of the so-called isospectral manifolds, that is manifolds for which the spectra of the corresponding Laplace–Beltrami operators coincide.

To determine a strictly convex planar domain Ω, modulo Euclidean transformations, it suffices to know the curvature $\mathcal{K}(x)$ of $\partial\Omega$ at each point $x \in \partial\Omega$. In general, the spectral data $\{c_j\}_{j=0}^{\infty}$, given by (0.5), is not sufficient to determine the function $\mathcal{K}(x)$. Let us mention that the distribution $\tau(t)$ is singular only at $t = 0$. A distribution related to $\{\lambda_j^2\}$ having a larger *singular set* is

$$\sigma(t) = \sum_{j=1}^{\infty} \cos(\lambda_j t) \in \mathcal{S}'(\mathbb{R}). \tag{0.6}$$

This distribution is singular at 0 and

$$\sigma(t) \sim \sum_{j=0}^{\infty} d_j t^{-n+j}$$

(see [Me3], [Iv2]). The constants d_j provide other spectral invariants, and the first two determine again $\mathrm{Vol}_n(\Omega)$ and $\mathrm{Vol}_{n-1}(\partial\Omega)$.

It turns out that the set of singularities of $\sigma(t)$ is related to the so-called *length spectrum* L_Ω of Ω. By definition, L_Ω is the set of periods (lengths) of all *periodic generalized geodesics* in Ω. Let us mention that the generalized geodesics are the projections in Ω of the generalized bicharacteristics of the wave operator $\Box = \partial_t^2 - \Delta_x$ in $T^*(\mathbb{R} \times \Omega)$ defined by Melrose and Sjöstrand ([MS1], [MS2]). We refer to Chapter 1 for the precise definitions. The so-called *Poisson relation for manifolds with boundary* has the form

$$\text{sing supp } \sigma(t) \subset \{0\} \cup \{T \in \mathbb{R} : \ |T| \in L_\Omega\}. \tag{0.7}$$

For strictly convex (concave) domains this relation has been established by Anderson and Melrose [AM]. Its proof for general domains is based on the results in [MS2] on the propagation of C^∞ singularities. A relation similar to (0.6) was first established for Riemannian manifolds without boundary. This was achieved independently by Chazarain [Ch2] and Duistermaat and Guillemin [DG]. Moreover, under certain assumptions on the periodic geodesics with period T, the leading singularity at T was examined in [DG].

It is natural to investigate the inverse inclusion in (0.7), however in the general case, very little is known so far. For certain strictly convex planar domains Ω Marvizi and Melrose [MM] found a sequence of closed billiard trajectories in Ω whose lengths belong to sing supp $\sigma(t)$. It was expected ([Cl], [GM3]) that for generic strictly convex domains in $\mathbb{R}^2$ the inclusion (0.7) could become an equality. Such a result was established in [PS2] (see also [PSl]) for all generic domains (not necessarily convex). Its analogue in the case $n > 2$ is proved only for strictly convex domains [S3]. The results, just mentioned, form one of the main topics in this book.

If the equality

$$\text{sing supp } \sigma(t) = \{0\} \cup \{T : \ |T| \in L_\Omega\} \tag{0.8}$$

holds for some domain Ω, then the lengths of the periodic geodesics in Ω can be considered as spectral invariants. From them one can determine various spectral invariants. The reader may consult [MM], [Cl], [Pol], [Po2], [Po3], [PoT], [HeZ] and [Z] for more information and further results in this direction.

Let $\mathcal{L}_\Omega$ be the set of all periodic geodesics in Ω. For $\gamma \in \mathcal{L}_\Omega$ we denote by T_γ the *period (length)* of γ. There are three types of elements of $\mathcal{L}_\Omega$: periodic reflecting rays (i.e. closed billiard trajectories in Ω), closed geodesics on $\partial\Omega$ and *periodic geodesics of mixed type*, containing both linear segments in Ω and geodesic segments on $\partial\Omega$. Amongst the periodic reflecting rays we will distinguish those without segments tangent to the boundary $\partial\Omega$; such rays will be called *ordinary*. Similarly to the case of closed geodesics on $\partial\Omega$, for each ordinary periodic reflecting ray γ one can naturally define a *Poincaré map* $\mathcal{P}_\gamma$ such that the spectrum $\text{spec}\,(P_\gamma)$ of the linearization P_γ of $\mathcal{P}_\gamma$ contains certain information about the behaviour of billiard flow along γ. Such a ray γ will be called *non-degenerate* if $1 \notin \text{spec}\, P_\gamma$. Poincaré maps for periodic reflecting rays are defined in Chapter 2.

Given a smooth submanifold X of $\mathbb{R}^n$, we denote by $C^\infty(X, \mathbb{R}^n)$ the *space of all smooth maps* $f : X \to \mathbb{R}^n$, endowed with the Whitney C^∞ topology (see Chapter 1). Let $\mathbf{C}(X) = C^\infty_{emb}(X, \mathbb{R}^n)$ be its subspace consisting of all smooth embedding of X into $\mathbb{R}^n$. Being open in $C^\infty(X, \mathbb{R}^n), \mathbf{C}(X)$ is a Baire space, so every residual (countable intersection of open dense subsets) subset of $\mathbf{C}(X)$ is dense in it.

Throughout the book we will consider very often the situation when Ω is a compact domain with smooth boundary $\partial\Omega$ and $X = \partial\Omega$. Then for every $f \in \mathbf{C}(X)$ there exists a unique compact domain Ω_f in $\mathbb{R}^n$ with boundary $\partial\Omega_f = f(X) = f(\partial\Omega)$. Let us note that if Ω is strictly convex, the set $\mathcal{O}(\Omega)$ of those $f \in \mathbf{C}(X)$ such that Ω_f is strictly convex, is open in $\mathbf{C}(X)$, and so it is a Baire topological space, too. If Ω is a domain in $\mathbb{R}^n$ with bounded complement, for $f \in \mathbf{C}(X)$ we denote by Ω_f the unbounded domain in $\mathbb{R}^n$ with $\partial\Omega_f = f(X)$. In the following we sometimes say that a property is generically satisfied (briefly a *generic property*) in some classes of objects, say for the compact domains in $\mathbb{R}^n$ with smooth boundaries. By this we mean a property S such that for every bounded domain with smooth boundary $X = \partial\Omega$ there exists a residual subset R of $\mathbf{C}(X)$ such that Ω_f has the property S for every $f \in R$. In the same way considering residual subsets of $\mathcal{O}(\Omega)$, one can talk about generic properties of the strictly convex domains, etc.

Let us note that in the whole book 'smooth' means C^∞ (although many separate arguments work replacing C^∞ by C^k for some $k \geq 1$). By a domain we always mean a domain with smooth boundary.

Exploiting the Multijet Transversality Theorem (see Section 1.1), we establish that the following properties of the compact domains in $\mathbb{R}^n$ are generic:

(I) $T_\gamma / T_\delta \notin \mathbb{Q}$ for all periodic ordinary reflecting rays γ and δ such that neither of them is a multiple of the other.

(II) Every periodic reflecting ray in Ω is ordinary and non-generate.

As a consequence of this, it is established that the asymptotic (0.4) holds for generic domains $\Omega \subset \mathbb{R}^n$. Using (i) and (ii), we prove (0.8) for generic strictly convex domains in the plane. In fact, if Ω has the properties (i) and (ii), then each periodic reflecting ray in Ω has a period T_γ which is an isolated point in L_Ω. The kernel $\mathcal{E}(t, x, y)$ of the operator $\cos(t\sqrt{A})$ satisfies the equality

$$\sigma(t) = \int_\Omega \mathcal{E}(t, x, x) dx.$$

One can compute the leading singularity of $\sigma(t)$ for t close to T_γ by the Poisson summation formula discussed in Chapter 4. This leads to (0.8), since by (i) the singularities, related to different periodic rays, cannot be cancelled.

In general, a domain $\Omega \subset \mathbb{R}^2$ might admit periodic geodesics of mixed type. The analysis of the singularities of $\sigma(t)$, related to the periods of such geodesics, leads to some rather difficult problems. We overcome this difficulty by showing that the following property is generic for domains $\Omega \subset \mathbb{R}^2$:

(III) There are no periodic geodesics of mixed type in Ω.

The analysis of the generic properties, such as (i)–(iii), is the second main topic of this book. To establish (0.8) for generic convex domains in $\mathbb{R}^n$, $n \geq 3$, in Chapter 7 we prove an analogue of the classical bumpy metric theorem of Abraham–Klingenberg–Takens–Anosov, considering Riemannian metrics on $X \subset \mathbb{R}^n$, induced by smooth embeddings of X into $\mathbb{R}^n$.

Our third topic concerns the kernel $s(t - t', \theta, \omega)$ of the operator

$$S - Id: \ L^2(\mathbb{R} \times \mathbb{S}^{n-1}) \to L^2(\mathbb{R} \times \mathbb{S}^{n-1}).$$

Here $\theta, \omega \in \mathbb{S}^{n-1}, t, t' \in \mathbb{R}$, and S is the scattering operator related to the Dirichlet problem for the wave operator $\Box = \partial_t^2 - \Delta_x$ in the exterior of a bounded obstacle K with smooth boundary $\partial\Omega = \partial K$ (see [LP1]). For fixed $\theta, \omega \in \mathbb{S}^{n-1}$ the *scattering kernel* $s(t, \theta, \omega)$ is a tempered distribution in $\mathcal{S}'(\mathbb{R})$. The Fourier transform $\mathcal{F}_{t\to\lambda} s(t, \theta, \omega)$ with respect to t yields the *scattering amplitude*

$$\overline{a(\lambda, \theta, \omega)} = \left(\frac{2\pi}{i\lambda}\right)^{(n-1)/2} \mathcal{F}_{t\to\lambda} s(t, \theta, \omega).$$

It is well known that the scattering amplitude $a(\lambda, \theta, \omega)$ determines uniquely the obstacle K (see for instance [LP1]). On the other hand, in the applications for given directions ω, θ is difficult to measure $a(\lambda, \theta, \omega)$ for all $\lambda \in \mathbb{R}$ and we can measure only the singularities of $s(t, \theta, \omega)$. It turns out that these singularities are related to *sojourn times* of generalized (ω, θ)-rays in Ω. These rays are generalized geodesics in Ω, incoming with direction ω and outgoing with direction θ. For such a ray γ the sojourn time was defined by Guillemin [G1] as an analogue of the notion of a period of a periodic geodesic; this notion appears also in the physical literature.

The sojourn time measures the time which a point, moving along γ with a unit speed, spends near the obstacle K. For strictly convex obstacles K and fixed $\theta \neq \omega$ one has

$$\text{sing supp}_t \ s(t, \theta, \omega) = \{-T_\gamma\},$$

γ being the unique (ω, θ)-ordinary reflecting ray in Ω (see [Ma2]). In general, the set $\mathcal{L}_{(\omega,\theta)}(\Omega)$ of *all* (ω, θ)*-generalized rays* in Ω could contain more than one element. Assuming that for $(\omega, \theta) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ every (ω, θ) ray γ in Ω is the projection of a uniquely extendible generalized bicharacteristic $\tilde{\gamma}$ of $\Box$, we prove the inclusion

$$\text{sing supp}_t \ s(t, \theta, \omega) \subset \{-T_\gamma : \ \gamma \in \mathcal{L}_{(\omega,\theta)}(\Omega)\}, \tag{0.9}$$

which is called the *Poisson relation for the scattering kernel*. The above assumption for the (ω, θ) rays is fulfilled for generic obstacles as well as for generic directions, that is for (ω, θ) in a subset $\mathcal{R}$ of $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ whose complement has Lebesgue measure zero. We prove that the relation (0.9) becomes an equality for $(\theta, \omega) \in \mathcal{R}$ and also for generic obstacles in $\mathbb{R}^3$ and all directions $\theta \neq \omega$. For this purpose we study generic properties of (ω, θ)-rays, similar to (i)–(iii). Here the analogue of a periodic reflecting ray is an ordinary reflecting (ω, θ)-ray and that of Poincaré map is the so-called differential cross section dJ_γ of an ordinary reflecting (ω, θ)-ray.

The non-degeneracy of such a ray γ means that $\det dJ_\gamma \neq 0$. The analogue of (iii) says that, given $(\theta, \omega) \in \mathcal{R} \subset \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$, there are no (ω, θ)-rays of mixed type in Ω. For an ordinary reflecting non-degenerate (ω, θ)-ray γ whose sojourn time T_γ is an isolated point in $\mathcal{L}_{(\omega,\theta)}(\Omega)$, we find the leading singularity of $s(t, \theta, \omega)$ for t sufficiently close to $-T_\gamma$. To do this, as in the analysis of the singularities of $\sigma(t)$ for t close to a period T_γ, we construct a global parametrix for the mixed problem by a global Fourier integral operator and we obtain a precise information about the principal symbol of this operator after multiple reflections. In this way the calculation of the singularity is reduced to the asymptotic of an oscillatory integral for which we apply the stationary phase argument. It turns out that the leading singularity of $\sigma(t)$, as well as that of $s(t, \theta, \omega)$, is given by some global geometric characteristics. This is the third main topic of this book.

Similar to the length spectrum for bounded domains, the right-hand side of (0.9) contains certain information about the geometry of the obstacle K; we call it the *scattering length spectrum* (SLS) with respect to ω, θ. The sojourn times of the (ω, θ)-rays are easy to be observed and they form scattering data for the inverse scattering problems. The fourth main topic in this book concerns *inverse scattering results*. First, in Chapter 10 we study inverse scattering problems for obstacles K that are finite disjoint unions of several strictly convex domains. Under a geometric condition (H), introduced by M. Ikawa, a hyperbolic property of the billiard trajectories in the exterior Ω of the obstacles is established. This allows us to show that all periodic reflecting rays in Ω can be approximated by (ω, θ)-rays for appropriately fixed directions ω and θ and that their periods can be determined from the sojourn times of these rays. Also we find the asymptotic of the coefficients in front of the leading singularities of the scattering kernel, corresponding to the sojourn times of the approximating (ω, θ)-rays.

A more general approach to the inverse problem of recovering information about an obstacle from the SLS is discussed in Chapter 13. It turns out that if two obstacles K and L have (almost) the same scattering length spectra, then the generalized geodesic flows in their exteriors are naturally conjugated on the non-trapping parts of their phase spaces via a time-preserving conjugacy. We use this result to show that certain properties of obstacles are recoverable from the SLS and also that some classes of obstacles can be uniquely recovered from their SLS.

In this book we assume some knowledge of differential geometry, including basic facts in symplectic geometry, as well as some knowledge of differential topology. The analysis of the generalized bicharacteristics is based on several deep and important results from microlocal analysis and the calculus of global Fourier integral operators. We present a summary of known results in this area proving for convenience some of them in Chapter 1. On the other hand, in Chapter 11 we present detailed proofs of some new properties of the generalized bicharacteristics that are essentially used in Chapters 12 and 13. The main references for these results are the monographs of Hörmander [Hl]–[H4]. The reader might read these results informally, omitting their proofs, and then proceed to Chapters 2, 7–10.

The first edition of this monograph was published in 1992 (see [PS7]). The present (second) edition is an improved version of the first. Various misprints and arguments have been corrected and several details added to the exposition. Apart from that, in the present edition Chapters 11–13 are entirely new. These chapters contain several results established after 1992 which could be also of independent interest.

Most of the publications cited in the References concern inverse spectral results for manifolds with boundary and inverse scattering results related to the singularities of the scattering kernel. It was not possible and we have not even attempted, to cover the immense range of works devoted to inverse spectral and inverse scattering results.

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1

Preliminaries from differential topology and microlocal analysis

Here we collect some facts concerning manifolds of jets, spaces of smooth maps and transversality, as well as some material from microlocal analysis. A special emphasis is given to the definition and main properties of the generalized bicharacteristics of the wave operator and the corresponding generalized geodesics.

1.1 Spaces of jets and transversality theorems

We begin with the notion of transversality, manifolds of jets and spaces of smooth maps. The reader is referred to Golubitsky and Guillemin [GG] or Hirsch [Hir] for a detailed presentation of this material.

In this book **smooth** means C^∞.

Let X and Y be smooth manifolds and let $f : X \longrightarrow Y$ be a smooth map. Given $x \in X$, we will denote by $T_x f$ the *tangent map* of f at x. Sometimes we will use the notation $d_x f = T_x f$. If $\mathrm{rank}(T_x f) = \dim(X) \leq \dim(Y)$ (resp. $\mathrm{rank}(T_x f) = \dim(Y) \leq \dim(X)$), then f is called an *immersion* (resp. a *submersion*) at x. Let W be a smooth submanifold of Y. We will say that f is *transversal* to W at $x \in X$, and will denote this by $f \pitchfork_x W$, if either $f(x) \notin W$ or $f(x) \in W$ and $\mathrm{Im}(T_x f) + T_{f(x)} W = T_{f(x)} Y$. Here for every $y \in W$ we identify $T_y W$ with its image under the map $T_y i : T_y W \longrightarrow T_y W$, where $i : W \longrightarrow Y$ is the inclusion. Clearly, if f is a submersion at x, then $f \pitchfork W$ for every submanifold W of Y. If $Z \subset X$ and $f \pitchfork W$

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for every $x \in Z$, we will say that f is transversal to W on Z. Finally, if f is transversal to W on the whole X, we will say that f is transversal to W and write $f \pitchfork W$.

The next proposition contains a basic property of transversality that will be used several times throughout.

Proposition 1.1.1: *Let $f : X \longrightarrow Y$ be a smooth map, and let W be a smooth submanifold of Y such that $f \pitchfork W$. Then $f^{-1}(W)$ is a smooth submanifold of X with*

$$\text{codim}\,(f^{-1}(W)) = \text{codim}\,(W). \tag{1.1}$$

In particular:

(a) if $\dim(X) < \text{codim}\,(W)$, *then* $f^{-1}(W) = \emptyset$, *that is* $f(X) \cap W = \emptyset$.

(b) if $\dim(X) = \text{codim}\,(W)$, *then* $f^{-1}(W)$ *consists of isolated points in* X.

Consequently, if f is a submersion, then for every submanifold W of Y, $f^{-1}(W)$ is a submanifold of X with (1.1). Thus, in this case, $f^{-1}(y)$ is a submanifold of X of codimension equal to $\dim(Y)$ for every $y \in Y$.

Let again X and Y be smooth manifolds and let $x \in X$. Given two smooth maps $f, g : X \longrightarrow Y$, we will write $f \sim_x g$ if $d_x f = d_x g$. For an integer $k \geq 2$, we will write $f \sim_x^k g$ if for the smooth maps $df, dg : TX \longrightarrow TY$, we have $df \sim_\xi^{k-1} dg$ for every $\xi \in T_x X$. In this way by induction one defines the relation $f \sim_x^k g$ for all integers $k \geq 1$. Fix for a moment $x \in X$ and $y \in Y$. Denote by $J_k(X,Y)_{x,y}$ the family of all equivalence classes of smooth maps $f : X \longrightarrow Y$ with $f(x) = y$ with respect to the equivalence relation $\sim_x^k$. Define the *space of k-jets* by

$$J^k(X,Y) = \bigcup_{(x,y) \in X \times Y} J^k(X,Y)_{x,y}.$$

So, for each k-jet $\sigma \in J^k(X,Y)$, there exist $x \in X$ and $y \in Y$ with $\sigma \in J^k(X,Y)_{x,y}$. We set $\alpha(\sigma) = x$ and $\beta(\sigma) = y$, thus obtaining maps

$$\alpha : J^k(X,Y) \longrightarrow X, \quad \beta : J^k(X,Y) \longrightarrow Y, \tag{1.2}$$

called the *source* and the *target* map, respectively. Given an arbitrary smooth $f : X \longrightarrow Y$, let

$$j^k f : X \longrightarrow J^k(X,Y) \tag{1.3}$$

be the map assigning to every $x \in X$ the equivalence class $j^k f(x)$ of f in $J^k(X,Y)_{x,f(x)}$.

There is a natural structure of a smooth manifold on $J^k(X,Y)$ for every k. We refer the reader to [GG] or [Hir] for its description and main properties. Let us only mention that with respect to this structure for every smooth map f the maps (1.2) and (1.3) are also smooth.

For a non-empty set A and an integer $s \geq 1$, define

$$A^{(s)} = \{(a_1, \ldots, a_s) \in A^s : a_i \neq a_j, 1 \leq i < j \leq s\}.$$

Note that if A is a topological space, then $A^{(s)}$ is an open (dense) subset of the product space A^s. If $f : A \longrightarrow B$ is an arbitrary map, define $f^s : A^s \longrightarrow B^s$ by

$$f^s(a_1, \ldots, a_s) = (f(a_1), \ldots, f(a_s)),$$

Let X and Y be smooth manifolds, let s and k be natural numbers and let $\alpha^s : (J^k(X,Y))^s \longrightarrow X^s$. The open submanifold

$$J^k_s(X,Y) = (\alpha^s)^{-1}(X^{(s)})$$

of $(J^k(X,Y))^s$ is called an *s-fold k-jet bundle*. For a smooth $f : X \longrightarrow Y$, define the smooth map

$$j^k_s f : X^{(s)} \longrightarrow J^k_s(X,Y)$$

by

$$j^k_s f(x_1, \ldots, x_s) = (j^k f(x_1), \ldots, j^k f(x_s)).$$

We will now define the Whitney C^k topology on *the space* $C^\infty(X,Y)$ *of all smooth maps* from X into Y. Let $k \geq 0$ be an integer and let U be an open subset of $J^k(X,Y)$. Set

$$M(U) = \{f \in C^\infty(X,Y) : j^k f(X) \subset U\}.$$

The family $\{M(U)\}_U$, where U runs over the open subsets of $J^k(X,Y)$, is the basis for a topology on $C^\infty(X,Y)$, called *the Whitney C^k topology*. The supremum of all Whitney C^k topologies for $k \geq 0$ is called *the Whitney C^∞ topology*. It follows from this definition that $f_n \to f$ as $n \to \infty$ in the C^∞ topology if $f_n \to f$ in the C^k topology for all $k \geq 0$. Note that if X is not compact (and $\dim(Y) > 0$), then any of the C^k topologies (including the case $k = \infty$) does not satisfy the first axiom of countability, and therefore is not metrizable. On the other hand, if X is compact, then all C^k topologies on $C^\infty(X,Y)$ are metrizable with complete metrics.

In this book we always consider $C^\infty(X,Y)$ with the Whitney C^∞ topology. An important fact about these spaces, which will be often used in what follows, is that whenever X and Y are smooth manifolds, the space $C^\infty(X,Y)$ is a Baire topological space. Recall that a subset R of a topological space Z is called *residual* in Z if R contains a countable intersection of open dense subsets of Z. If every residual subset of Z is dense in it, then Z is called a *Baire space*.

In some of the next chapters we will consider spaces of the form $C^\infty(X, \mathbb{R}^n)$, X being a smooth submanifold of $\mathbb{R}^n$ for some $n \geq 2$. Let us note that these spaces have a natural structure of Frechet spaces. Moreover, if X is compact, then $C^\infty(X, \mathbb{R}^n)$ has a natural structure of a Banach space. Denote by

$$\mathbf{C}(X) = C^\infty_{emb}(X, \mathbb{R}^n)$$

the *subset of* $C^\infty(X, \mathbb{R}^n)$ *consisting of all smooth embeddings* $X \longrightarrow \mathbb{R}^n$. Then $\mathbf{C}(X)$ is open in $C^\infty(X, \mathbb{R}^n)$ (cf. Chapter II in [Hir]), and therefore it is a Baire topological space with respect to the topology induced by $C^\infty(X, \mathbb{R}^n)$. Finally, notice that for compact X the space $\mathbf{C}(X)$ has a natural structure of a Banach manifold. We refer the reader to [Lang] for the definition of Banach manifolds and their main properties.

The following theorem is known as the *multijet transversality theorem* and will be used many times later in this book.

Theorem 1.1.2: *Let X and Y be smooth manifolds, let k and s be natural numbers and let W be a smooth submanifold of $J_s^k(X, Y)$. Then*

$$T_W = \{F \in C^\infty(X, Y) : j_s^k F \pitchfork W\}$$

is a residual subset of $C^\infty(X, Y)$. Moreover, if W is compact, then T_W is open in $C^\infty(X, Y)$.

For $s = 1$, this theorem coincides with Thom's transversality theorem.

We conclude this section with a special case of the Abraham transversality theorem which will be used in Chapter 6. Now by a smooth manifold we mean a smooth Banach manifold of finite or infinite dimension (cf. [Lang]).

Let $\mathcal{A}$, X and Y be smooth manifolds, and let

$$\rho : \mathcal{A} \longrightarrow C^\infty(X, Y) \tag{1.4}$$

be a map, $\mathcal{A} \ni a \mapsto \rho_a$. Define

$$\mathrm{ev}_\rho : \mathcal{A} \times X \longrightarrow Y \tag{1.5}$$

by $\mathrm{ev}_\rho(a, x) = \rho + a(x)$.

The next theorem is a special case of Abraham's transversality theorem (see [AbR]).

Theorem 1.1.3: *Let ρ have the form (1.4) and let W be a smooth submanifold of Y.*

(a) If the map (1.5) is C^1 and K is a compact subset of X, then

$$\mathcal{A}_{K,W} = \{a \in \mathcal{A} : \rho_a \pitchfork_x W, x \in K\}$$

is an open subset of $\mathcal{A}$.

(b) Let $\dim(X) = n < \infty$, codim $(W) = q < \infty$ *and let r be a natural number with $r > n - q$. Suppose that the manifolds $\mathcal{A}$, X and Y satisfy the second axiom of countability, the map (1.5) is C^r and* $\mathrm{ev}_\rho \pitchfork W$. *Then*

$$\mathcal{A}_W = \{a \in \mathcal{A} : \rho_a \pitchfork W\}$$

is a residual subset of $\mathcal{A}$.

1.2 Generalized bicharacteristics

Our aim in this section is to define the generalized bicharacteristics of the *wave operator*

$$\Box = \partial_t^2 - \Delta_x$$

and to present their main properties which will be used throughout the book. Here we use the notation from Section 24 in [H3]. In what follows Ω is a closed domain in $\mathbb{R}^{n+1}$ with a smooth boundary $\partial\Omega$.

Given a point on $\partial\Omega$, we choose local normal coordinates

$$x = (x_1, \dots, x_{n+1}), \quad \xi = (\xi_1, \dots, \xi_{n+1})$$

in $T^*(\mathbb{R}^{n+1})$ about it such that the boundary $\partial\Omega$ is given by $x_1 = 0$ and Ω is locally defined by $x_1 \geq 0$. We assume that the coordinates ξ_i are those dual to x_i. The coordinates x, ξ can be chosen so that the *principal symbol* of $\Box$ has the form

$$p(x, \xi) = \xi_1^2 - r(x, \xi'),$$

where

$$x' = (x_2, \dots, x_{n+1}), \quad \xi' = (\xi_2, \dots, \xi_{n+1}),$$

and $r(x, \xi')$ is homogeneous of order 2 in ξ'. Introduce the sets

$$\begin{aligned}
\Sigma &= \{(x, \xi) \in T^*\mathbb{R}^{n+1} \setminus \{0\} : p(x, \xi) = 0\}, \\
\Sigma_0 &= \{(x, \xi) \in T^*\mathbb{R}^{n+1} : x_1 > 0\}, \\
H &= \{(x, \xi) \in \Sigma : x_1 = 0, r(0, x', \xi') > 0\}, \\
G &= \{(x, \xi) \in \Sigma : x_1 = 0, r(0, x', \xi') = 0\}.
\end{aligned}$$

The sets Σ, H and G are called the *characteristic set*, the *hyperbolic set* and the *glancing set*, respectively. Let

$$r_0(x', \xi') = r(0, x', \xi'), \quad r_1(x', \xi') = \frac{\partial r}{\partial x_1}(0, x', \xi').$$

The *diffractive* and the *gliding* sets are defined by

$$\begin{aligned}
G_d &= \{(x, \xi) \in G : r_1(x', \xi') > 0\}, \\
G_g &= \{(x, \xi) \in G : r_1(x', \xi') < 0\},
\end{aligned}$$

respectively.

Next, consider the Hamiltonian vector fields

$$H_p = \sum_{j=1}^{n+1} \left(\frac{\partial p}{\partial \xi_j} \cdot \frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j} \cdot \frac{\partial}{\partial \xi_j} \right),$$

$$H_{r_0} = \sum_{j=1}^{n+1} \left(\frac{\partial r_0}{\partial \xi_j} \cdot \frac{\partial}{\partial x_j} - \frac{\partial r_0}{\partial x_j} \cdot \frac{\partial}{\partial \xi_j} \right).$$

Notice that $d_\xi p(x, \xi) \neq 0$ on Σ and $d_{\xi'} r_0(x, \xi') \neq 0$ on G, so H_p and H_{r_0} are not radial on Σ and G, respectively. Next, introduce the sets

$$G^k = \{(x, \xi) \in G : r_1 = H_{r_0}(r_1) = \ldots = H_{r_0}^{k-3}(r_1) = 0\}, \quad k \geq 3,$$

$$G^\infty = \bigcap_{k=3}^{\infty} G^k.$$

The above definitions are independent of the choice of local coordinates. Let us mention that if $\partial\Omega$ is given locally by $\varphi = 0$ and Ω by $\varphi > 0$, φ being a smooth function, then

$$H = \{(x, \xi) \in T^*(\mathbb{R} \times \Omega) : p(x, \xi) = 0,\ H_p\varphi(x, \xi) \neq 0\},$$

$$G = \{(x, \xi) \in T^*(\mathbb{R} \times \Omega) : p(x, \xi) = 0,\ H_p\varphi(x, \xi) = 0\},$$

$$G_d = \{(x, \xi) \in G : H_p^2\varphi(x, \xi) > 0\},$$

$$G_g = \{(x, \xi) \in G : H_p^2\varphi(x, \xi) < 0\},$$

$$G^k = \{(x, \xi) \in G : H_p^j\varphi(x, \xi) = 0,\ 0 \leq j < k\}.$$

We define the generalized bicharacteristics of $\Box$ using the special coordinates (x, ξ) chosen above.

Definition 1.2.1: Let I be an open interval in $\mathbb{R}$. A curve

$$\gamma : I \longrightarrow \Sigma \tag{1.6}$$

is called a *generalized bicharacteristic* of $\Box$ if there exists a discrete subset B of I such that the following conditions hold:

(i) If $\gamma(t_0) \in \Sigma_0 \cup G_d$ for some $t_0 \in I \setminus B$, then γ is differentiable at t_0 and

$$\frac{d}{dt}\gamma(t_0) = H_p(\gamma(t_0)).$$

(ii) If $\gamma(t_0) \in G \setminus G_d$ for some $t_0 \in I \setminus B$, then $\gamma(t) = (x_1(t), x'(t), \xi_1(t), \xi'(t))$ is differentiable at t_0 and

$$\frac{dx_1}{dt}(t_0) = \frac{d\xi_1}{dt}(t_0) = 0, \quad \frac{d}{dt}(x'(t), \xi'(t))_{|t=t_0} = H_{r_0}(\gamma(t_0)).$$

(iii) If $t_0 \in B$, then $\gamma(t_0) \in \Sigma_0$ for all $t \neq t_0$, $t \in I$, with $|t - t_0|$ sufficiently small. Moreover, for $\xi_1^{\pm}(x', \xi') = \pm\sqrt{r_0(x', \xi')}$, we have

$$\lim_{t \to t_0, \pm(t-t_0)>0} \gamma(t) = (0, x'(t_0), \xi_1^{\pm}(x'(t_0)), \xi'(t_0)) \in H.$$

This definition does not depend on the choice of the local coordinates. Note that when $\partial\Omega$ is given by $\varphi = 0$ and Ω by $\varphi > 0$, then the condition (ii) means that if $\gamma(t_0) \in G \setminus G_d$, then

$$\frac{d\gamma}{dt}(\gamma(t_0)) = H_p^G(\gamma(t_0)),$$

where

$$H_p^G = H_p + \frac{H_p^2 \varphi}{H_\varphi^2 p} \, H_\varphi$$

is the so-called *glancing vector field* on G.

It follows from the above definition that if (1.6) is a generalized bicharacteristic, the functions $x(t)$, $\xi'(t)$, $|\xi_1(t)|$ are continuous on I, while $\xi_1(t)$ has jump discontinuities at any $t \in B$. The functions $x'(t)$ and $\xi'(t)$ are continuously differentiable on I and

$$\frac{dx'}{dt} = -\frac{\partial r}{\partial \xi'}, \quad \frac{d\xi'}{dt} = \frac{\partial r}{\partial x'}. \tag{1.7}$$

Moreover, for $t \in B$, $x_1(t)$ admits left and right derivatives

$$\frac{d^{\pm} x_1}{dt}(t) = \lim_{\epsilon \to +0} \pm \frac{x_1(t \pm \epsilon) - x_1(t)}{\epsilon} = 2\xi_1(t \pm 0). \tag{1.8}$$

The function $\xi_1(t)$ also has a left derivative and a right derivative. For $\gamma(t) \notin G_g$, we have

$$\frac{d^{\pm}\xi_1}{dt}(t) = \lim_{\epsilon \to +0} \pm \frac{\xi_1(t \pm \epsilon) - \xi_1(t)}{\epsilon} = \frac{\partial r}{\partial x_1}(x(t), \xi'(t)), \tag{1.9}$$

while $\frac{d^{\pm}\xi_1}{dt}(t) = 0$ for $\gamma(t) \in G_g$. Thus, if $\gamma(t)$ remains in a compact set, then the functions $x(t), \xi'(t), \xi_1^2(t)$ and $x_1(t)\xi_1(t)$ satisfy a uniform Lipschitz condition. For the left and right derivatives of $|\xi_1(t)|$, one gets

$$\left| \frac{d^{\pm}|\xi_1(t)|}{dt} \right| \leq \left| \frac{\partial r}{\partial x_1}(x(t), \xi'(t)) \right|. \tag{1.10}$$

Melrose and Sjöstrand [MS2] (see also Section 24 in [H3]) showed that for each $z_0 \in \Sigma$, there exists a generalized bicharacteristic (1.6) of $\Box$ with $\gamma(t_0) = z_0$ for some $t_0 \in I$. Since the vector fields H_p and H_p^G are not radial on Σ and G, respectively, such a bicharacteristic γ can be extended for all $t \in \mathbb{R}$. However, in general, γ is not unique. We refer the reader to [Tay] or [H3] for examples demonstrating this.

For $\rho \in \Sigma$, denote by $C_t(\rho)$ the set of those $\mu \in \Sigma$ such that there exists a generalized bicharacteristic (1.6) with $0, t \in I$, $\gamma(0) = \rho$ and $\gamma(t) = \mu$. In many cases $C_t(\rho)$ is related to a uniquely determined bicharacteristic γ. In the general case it is convenient to introduce the following.

Definition 1.2.2: A generalized bicharacteristic $\gamma : \mathbb{R} \longrightarrow \Sigma$ of $\Box$ is called *uniquely extendible* if for each $t \in \mathbb{R}$, the only generalized bicharacteristics (up to a change of parameter) passing through $\gamma(t)$ is γ. That is, for $\rho = \gamma(0)$, we have $C_t(\rho) = \{\gamma(t)\}$ for all $t \in \mathbb{R}$.

It was proved by Melrose and Sjöstrand [MS1] that if $\mathrm{Im}(\gamma) \subset \Sigma \setminus G^\infty$, then γ is uniquely extendible. If $z_0 = \gamma(t_0) \in H$ for some $t_0 \in B$, then $\gamma(t)$ meets $\partial\Omega$ transversally at $x(t_0)$ and (iii) holds. For $z_0 \in \Sigma_0 \cup G_d$ we have $\gamma(t) \in \Sigma_0$ for $|t - t_0|$ small enough, while in the case $z_0 \in G_g$ for small $|t - t_0|$, $\gamma(t)$ coincides with the gliding ray

$$\gamma_0(t) = (0, x'(t), 0, \xi'(t)), \tag{1.11}$$

where $(x'(t), \xi(t))$ is a null bicharacterstic of the Hamiltonian vector field H_{r_0}.

To discuss the local uniqueness of generalized bicharacteristics, let $\gamma(t) = (x(t), \xi(t))$ be such a bicharacteristic and let $y'(t), \eta'(t))$ be the solution of the problem

$$\begin{cases} \dfrac{dy'}{dt}(t) = \dfrac{\partial r_0}{\partial \xi'}(y'(t), \eta'(t)), \\ \dfrac{d\eta'}{dt}(t) = -\dfrac{\partial r_0}{\partial x'}(y'(t), \eta'(t)), \\ y'(0) = x'(0), \;\; \eta'(0) = \xi'(0). \end{cases} \tag{1.12}$$

Then setting $e(t) = r_1(y'(t), \eta'(t))$, we have the following local description of γ.

Proposition 1.2.3: *Let $\gamma(0) \in G^3$. If $e(t)$ increases for small $t > 0$, then for such t the bicharateristic $\gamma(t)$ is a trajectory of H_p. If $e(t)$ decreases for $0 \le t \le T$, then for such t, $\gamma(t)$ is a gliding ray of the form (1.11).*

A proof of this proposition and some other properties of generalized bicharacteristics can be found in Section 24.3 in [H3].

It should be mentioned that for $k \ge 3$ and $\gamma(0) \in G^k \setminus G^{k+1}$, we have

$$e(t) = \frac{1}{2(k-2)!} H_p^k \varphi(\gamma(0))\; t^{k-2} + \mathrm{O}(t^{k-1}),$$

therefore the sign of $H_p^k \varphi(\gamma(0))$ determines the local behaviour of $e(t)$.

Corollary 1.2.4: *In each of the following cases, every generalized bicharacteristic of $\Box$ is uniquely extendible:*

(a) the boundary $\partial\Omega$ is a real analytic manifold;

(b) there are no points $y \in \partial\Omega$ at which the normal curvature of $\partial\Omega$ vanished of infinite order in some direction $\xi \in T_y(\partial\Omega)$;

(c) $\partial\Omega$ is given locally by $\varphi = 0$ and

$$H_p^2\varphi(z) \leq 0 \tag{1.13}$$

for every $z \in G$. If $\partial\Omega$ is locally convex in the domain of φ, then (1.13) holds.

Proof: In the case (a) the symbols $r_0(x', \xi')$ and $r_1(x', \xi')$ are real analytic, so the solution $(y'(t), \eta'(t))$ of (1.12) is analytic in t. Consequently, the function $e(t)$ is analytic and we can use its Taylor expansion in order to apply Proposition 1.2.3.

In the case (c), using the special coordinates x, ξ, and combining (1.13) with (1.9), we get $\frac{d^{\pm}\xi_1}{dt}(t) \geq 0$. On the other hand, if $\xi_1(t)$ has a jump at $\gamma(t) \in H$, then this jump is equal to $2r_0(x'(t), \xi'(t)) > 0$. Thus, the function $\xi_1(t)$ is increasing. If $e(t) = 0$ for $0 \leq t \leq t_0$, we get $x_1(t) = \xi_1(t) = 0$ for such t, so $\{\gamma(t) : 0 \leq t \leq t_1\}$ is a gliding ray. Assume that there exists a sequence $t_k \searrow 0$ such that $e(t_k) \neq 0$ for all $k \geq 1$. Then $\xi_1(t) > 0$ for all sufficiently small $t > 0$. Now (1.8) shows that $x_1(t)$ is increasing for such t, therefore there is $t' > 0$ such that $\{\gamma(t) : 0 \leq t \leq t'\}$ coincides with a trajectory of H_p.

Let $p = \sum_{j=1}^n \xi_j^2 - \xi_{n+1}^2$ and let φ depend on $x_1, \ldots, x_n$ only. Then

$$(H_p^2\varphi)(x, \xi) = 4\sum_{i,j=1}^n \frac{\partial^2\varphi}{\partial x_i\, \partial x_j}(x)\xi_i\xi_j,$$

and if the boundary $\partial\Omega$ is locally convex, we obtain (1.13).

Finally, in the case (b), for each $x \in \partial\Omega$ there exists a multi-index α, depending on x, such that $(\partial^\alpha\varphi)(x) \neq 0$. This implies $G^\infty = \emptyset$, which completes the proof. ■

According to Lemma 6.1.2, in the generic case discussed in Chapter 6 the assumption (b) is always satisfied.

Let $Q = \Omega \times \mathbb{R}$. We will again use the coordinates $x = (x_1, \ldots, x_{n+1})$, this time denoting the last coordinate by t, that is $t = x_{n+1}$. For $x \in \partial Q = \partial\Omega \times \mathbb{R}$, let $N_x(\partial Q)$ be the space of covectors $\xi \in T_x^*Q$ vanishing on $T_x(\partial Q)$. Define the equivalence relation $\sim$ on T^*Q by $(x, \xi) \sim (y, \eta)$ if and only if either $x = y \in Q \setminus \partial Q$ and $\xi = \eta$, or $x = y \in \partial Q$ and $\xi - \eta \in N_x(\partial Q)$. Then $T^*Q/\sim$ can be naturally identified over ∂Q with $T^*(\partial Q)$. Consider the map

$$\sim : T^*Q \ni (x, \xi) \mapsto (x, \xi_{|T_x(\partial Q)}) \in T^*(\partial Q),$$

defined as the identity on $T^*(Q \setminus \partial Q)$. Then $\widetilde{\Sigma} = \Sigma_b$ is called the *compressed characteristic set*, while the image $\tilde{\gamma}$ of a bicharacteristic γ under $\sim$ is called a *compressed generalized bicharacteristic*. Clearly $\tilde{\gamma}$ is a continuous curve in Σ_b.

Given $\rho = (x, \xi)$, $\mu = (y, \eta) \in T^*Q$, denote by $d(\rho, \mu)$ the standard Euclidean distance between ρ and μ. For $\rho, \mu \in \Sigma$ define

$$D(\rho, \mu) = \inf_{\nu', \nu'' \in \Sigma, \nu \sim \nu''} (\min\{d(\rho, \mu), d(\rho, \nu') + d(\nu'', \mu)\}).$$

Clearly, $D(\rho, \mu) = 0$ if and only if $\rho \sim \mu$, and $D(\rho, \mu) = D(\rho', \mu')$ provided $\rho \sim \rho'$ and $\mu \sim \mu'$. It is easy to check that D is symmetric and satisfies the triangle inequality. Thus, D is a pseudo-metric on Σ, which induces a metric on Σ_b.

For the next lemma we assume that I is a closed non-trivial interval in $\mathbb{R}$, $(y_0, \eta_0) \in \Sigma$ and Γ is a neighbourhood of (y_0, η_0) in Q.

Lemma 1.2.5: *There exists a constant $C_0 > 0$ depending only on Γ and I such that for every generalized bicharacteristic $\gamma : I \longrightarrow \Sigma \cap \gamma$ we have*

$$D(\gamma(t), \gamma(s)) \leq C_0 |t - s|$$

for all $t, s \in I$.

Proof: It is enough to consider the case when $|t - s|$ is small. Then we can use the local coordinates introduced earlier. From (1.7), (1.8) and (1.10), we get

$$|x(t) - x(s)| + |\xi'(t) - \xi'(s)| \leq C_1 |t - s|, \quad ||\xi_1(t)| - |\xi_1(s)|| \leq C_1 |t - s|,$$

where $C_1 > 0$ is a constant independent of t and s. Thus, if $\xi_1(t) = 0$ or $\xi_1(s) = 0$ we get $|\xi_1(t) - \xi_1(s)| \leq C_1 |t - s|$. The latter holds also in the case when $\gamma(t') \notin \partial\Omega$ for all $t' \in (t, s)$. Consequently, $D(\gamma(t), \gamma(s)) \leq C_2 |t - s|$ whenever either $\xi_1(t)\xi_1(s) = 0$ or $\gamma(t') \in \partial\Omega$ only for finitely many $t' \in (t, s)$.

Assume that there are infinitely many $t' \in (t, s)$ such that $\gamma(t')$ is a reflection point of γ. Then there exists $u \in [t, s]$ with $\gamma(u) \in G$. Hence,

$$D(\gamma(t), \gamma(u)) \leq C_2 |t - u|, \quad D(\gamma(u), \gamma(s)) \leq C_2 |u - s|,$$

and using the triangle inequality for D, we complete the proof of the assertion. ∎

The next lemma shows that any sequence of generalized bicharacteristics has a subsequence that is convergent on a given compact interval.

Lemma 1.2.6: *Let $I = [a, b]$ be a compact interval in $\mathbb{R}$, let K be a compact subset of Σ and let $\gamma^{(k)}(t) = (x^{(k)}(t), \xi^{(k)}(t)) : I \longrightarrow K \subset \Sigma$ be a generalized bicharacteristic of $\Box$ for every natural number k. Then there exists an infinite sequence $k_1 < k_2 < \dots$ of natural numbers and a generalized bicharacteristic $\gamma(t) = (x(t), \xi(t)) : I \longrightarrow \Sigma$ such that*

$$\lim_{m \to \infty} D(\gamma^{(k_m)}(t), \gamma(t)) = 0 \tag{1.14}$$

for all $t \in I$.

Proof: Using local coordinates, we see that the derivatives of $(x^{(k)})'(t)$ and $(\xi^{(k)})'(t)$ and the left and right derivatives of $x_1^{(k)}(t)$ and $\xi_1^{(k)}(t)$ are uniformly bounded for $t \in I$ and $k \geq 1$. Hence the maps $x^{(k)}(t)$, $(\xi^{(k)})'(t)$, $x_1^{(k)}(t)\xi_1^{(k)}(t)$ and $(\xi_1^{(k)}(t))^2$ are uniformly Lipschitz, which implies that there exists an infinite sequence $k_1 < k_2 < \ldots$ of natural numbers such that the sequences $x^{(k_m)}(t)$, $(\xi^{(k_m)})'(t)$, $(\xi_1^{(k_m)}(t))^2$ and $x_1^{(k_m)}(t), \xi_1^{(k_m)}(t)$ are uniformly convergent for $t \in I$. It now follows from Proposition 24.3.12 in [H3] that there exists a generalized bicharacteristic $\gamma(t) : I \to \Sigma$ of $\Box$ such hat

$$\lim_{m \longrightarrow \infty} \gamma^{(k_m)}(t) = \gamma(t) \tag{1.15}$$

for all $t \in I$ with $\gamma(t) \notin H$.

Let $t' \in I$ be such that $\gamma(t')$ is a reflection point of γ. Then there exists a sequence $t_j \to t'$ with $\gamma(t_j) \in \Sigma_0 \cup G$ for all j. Thus,

$$\begin{aligned} &D(\gamma^{(k_m)}(t'), \gamma(t')) \\ &\leq D(\gamma^{(k_m)}(t'), \gamma^{(k_m)}(t_j)) + D(\gamma(t_j), \gamma(t')) + D(\gamma^{(k_m)}(t_j), \gamma(t_j)). \end{aligned}$$

By Lemma 1.2.5, the first two terms in the right-hand side can be estimated uniformly with respect to m, while for the third term we can use (1.15). Taking j and m sufficiently large, we obtain (1.14), which proves the lemma. ■

In what follows we will use local coordinates $(t, x) \in \mathbb{R} \times \Omega$ and the corresponding local coordinates $(t, x; \tau, \xi) \in T^*(\mathbb{R} \times \Omega)$. In these coordinates the principal symbol p of $\Box$ has the form

$$p(x, \tau, \xi) = \xi_1^2 - q_2(x, \xi') - \tau^2,$$

where $\xi' = (\xi_2, \ldots, \xi_n)$ and $q_2(x, \xi')$ is homogeneous of order 2 in ξ'. Consequently, the vector fields H_p and H_p^G do not involve derivatives with respect to τ, so by Definition 1.2.1, the variable τ remains constant along each generalized bicharacteristic. This makes it possible to parametrize every generalized bicharacteristic by the time t.

Given $(y, \eta) \in T^*(\Omega) \setminus \{0\}$, consider the points

$$\mu_\pm = (0, y, \mp|\eta|, \eta) \in \Sigma.$$

Assume that locally $\partial\Omega$ is given by $x_1 = 0$ and Ω by $x_1 \geq 0$. Let μ_+be a hyperbolic point and let $\xi_1^\pm(y', \eta)$ be the different real roots of the equation

$$p(0, y', |\eta|, z, \eta') = 0$$

with respect to z. Denote by γ the generalized bicharacteristic parameterized by a parameter s such that

$$\lim_{s \searrow 0} \gamma(s) = \mu_+.$$

Then $\tau = -|\eta| < 0$ along γ and the time t increases when s increases. Such a bicharacteristic will be called *forward*. For the right derivative of $x_1(t)$ we get

$$\frac{d^+x_1}{dt} = \frac{d^+x_1/ds}{dt/ds} = \frac{\xi_1(+0)}{-\tau} > 0,$$

since for small $t > 0$, $\gamma(t)$ enters the interior of Ω and $x_1(t) > 0$. Therefore, setting

$$\xi_1^{\pm}(y', \eta) = \pm\sqrt{|\eta|^2 + q_2(0, y', \eta')},$$

we find

$$\lim_{s \searrow 0} \xi_1(s) = \xi_1^+(y', \eta).$$

In the case $\mu_+ \in G$ it may happen that there exist several forward bicharacteristic passing through μ_+. Denote by C_+ the set of those

$$(t, x, y; \tau, \xi, \eta) \in T^*(\mathbb{R} \times \Omega \times \Omega) \setminus \{0\}$$

such that $\tau = -|\xi| = -|\eta|$ and (t, x, τ, ξ) and $(0, y, \tau, \eta)$ lie on forward generalized bicharacteristics of $\Box$. In a similar way we define C_- using a *backward bicharacteristic*, determined as the forward ones replacing μ_+ by μ_-. The set $C = C_+ \cup C_-$ is called the *bicharacteristic relation* of $\Box$. If $\mu = (0, y, \tau, \eta) \in H$ and $\tau < 0$ (resp. $\tau > 0$), we will say that μ is a reflection point of a forward (resp. backward) bicharacteristic. Similarly, if $\rho = (t, x, \tau, \xi) \in H$, then ρ is a reflection point of a generalized bicharacteristic passing through $(0, y, \tau, \eta)$, and, working in local coordinates as before, the sign of τ determines uniquely $\xi_1(t + 0)$. The sets $C_\pm$ and C are homogeneous with respect to (τ, ξ, η), that is $(t, x, y, \tau, \xi, \eta) \in C_\pm$ implies $(t, x, y, s\tau, s\xi, s\eta) \in C_\pm$ for all $s \in \mathbb{R}^+$.

Lemma 1.2.7: *The sets $C_\pm$ are closed in $T^*(\mathbb{R} \times \Omega \times \Omega) \setminus \{0\}$.*

Proof: Since C_+ is homogeneous, it is sufficient to show that if

$$C_+ \ni z_k = (t_k, x_k, y_k, -1, \xi_k, \eta_k), \quad |\xi_k| = |\eta_k| = 1$$

for all $k \geq 1$ and there exists

$$\lim_{k \to \infty} z_k = z_0 = (t_0, x_0, y_0, -1, \xi_0, \eta_0),$$

then $z_0 \in C_+$. Let $\gamma^{(k)}(t)$ be a generalized bicharacteristic of $\Box$ such that $(t_k, x_k, -1, \xi_k)$ and $(0, y_k, -1, \eta_k)$ lie on $\mathrm{Im}(\gamma^{(k)})$. If one of these points belongs to H, we consider it as a reflection point of $\gamma^{(k)}$, according to the above-mentioned convention by suitably choosing $\xi_1^{(k)}(t)$. Assume $|t_k| \leq T$. Then there exists a compact set $K \subset \Sigma$ such that $\gamma^{(k)}(t) \in K$ for all $|t| \leq T$, so we can apply the argument in the proof of Lemma 1.2.6. Consequently, there exists an infinite sequence $k_1 < k_2 < \dots$ of natural numbers and a generalized bicharacteristic γ satisfying (1.14) and (1.15). Then for the Euclidean distance d we find

$$d(\gamma^{(k_m)}(t_{k_m}), \gamma(t_0)) \le d(\gamma^{(k_m)}(t_{k_m}), \gamma^{k_m)}(t_0)) + d(\gamma^{(k_m)}(t_0), \gamma(t_0)).$$

If $\gamma(t_0) \in \Sigma_0 \cup G$, according to (1.15) and the continuity of $x(t)$, $\xi'(t)$ and $|\xi_1(t)|$, we get

$$d(\gamma^{(k_m)}(t_{k_m}), \gamma(t_0)) \to 0 \tag{1.16}$$

as $m \to \infty$, which shows that $z_0 \in C_+$. If $\gamma(t_0) \in H$, then by our convention, $\xi_1(t+0)$ and $\xi_1^{(k_m)}(t+0)$ have the same sign for large m, which implies $z_0 \in C_+$.

Therefore, C_+ is closed. In the same way one proves that C_- is closed as well. ■

Using C_+ we now define the so-called *generalized Hamiltonian flow* $\mathcal{F}_t$ of $\Box$; it is sometimes called the *broken Hamiltonian flow*. Given $(y, \eta) \in T^*\Omega \setminus \{0\}$, set

$$\mathcal{F}_t(y, \eta) = \{(x, \xi) \in T^*\Omega \setminus \{0\} : (t, x, y, -|\eta|, \xi, \eta) \in C_+\}.$$

In general, $\mathcal{F}_t(y, \eta)$ is not a one-point set. Nevertheless, setting

$$\mathcal{F}_t(V) = \{\mathcal{F}_t(y, \eta) : (y, \eta) \in V\}$$

for $V \subset T^*\Omega \setminus \{0\}$, we have the group property

$$\mathcal{F}_{t+s}(y, \eta) = \mathcal{F}_t(\mathcal{F}_s(y, \eta)).$$

The flow generated by C_- is $\mathcal{F}_t(y, -\eta)$.

Let $\partial\Omega$ be locally given by $x_1 = 0$ and let

$$p(x, \tau, \xi) = \xi_1^2 - q_2(x, \xi') - \tau^2$$

be the principal symbol of $\Box$. A point

$$\sigma = (t, x', \tau, \xi') \in T^*(\mathbb{R} \times \partial\Omega) \setminus \{0\}$$

is called *hyperbolic* (resp. *glancing*) for $\Box$ if the equation

$$p(0, x', \tau, \xi_1, \xi') = 0 \tag{1.17}$$

with respect to ξ_1 has two different real roots (resp. a double real root). These definitions are invariant with respect to the choice of the local coordinates. If (1.17) has no real roots, then σ is called an *elliptic* point. Clearly, the set of hyperbolic points is open in $T^*(\mathbb{R} \times \partial\Omega)$, while that of the glancing points is closed.

Let $\pi : T^*(\mathbb{R} \times \Omega) \longrightarrow \Omega$ be the *natural projection*, $\pi(t, x, \tau, \xi) = x$.

Definition 1.2.8: A continuous curve $g : [a, b] \longrightarrow \Omega$ is called a *generalized geodesic* in Ω if there exists a generalized bicharacteristic $\gamma : [a, b] \longrightarrow \Sigma$ such that

$$g(t) = \pi(\gamma(t)), \quad t \in [a, b]. \tag{1.18}$$

Notice that, in general, a generalized geodesic is not uniquely determined by a point on it and the corresponding direction. If the generalized bicharacteristic γ with (1.18) satisfies

$$\gamma(t) \in \Sigma_0 \cup H, \quad t \in [a, b],$$

we will say that g (or $\mathrm{Im}(g)$) is a *reflecting ray* in Ω. Two special kinds of such rays will be studied in detail in Chapter 2. One of them is defined as follows.

Definition 1.2.9: A point $(x, \xi) \in T^*\Omega \setminus \{0\}$ is called *periodic* with *period* $T \neq 0$ if

$$(T, x, x, \pm|\xi|, \xi, \xi) \in C.$$

A generalized bicharacteristic $\gamma(t) = (t, x(t), \tau, \xi(t)) \in \Sigma$, $t \in \mathbb{R}$, will be called *periodic* with *period* $T \neq 0$ if for each $t \in \mathbb{R}$ the point $(x(t), \xi(t))$ is periodic with period T. The projections on Ω of the periodic generalized bicharacteristics of $\Box$ are called *periodic generalized geodesics*.

Notice that if $(T, x, x, -|\xi|, \xi, \xi) \in C_+$, then $(T, x, x, |\xi|, -\xi, -\xi) \in C_-$, since we can change the orientation on the bicharacteristic passing through $(0, x, -|\xi|, \xi)$. A uniquely extendible bicharacteristic γ is periodic provided $\mathrm{Im}(\gamma)$ contains a periodic point. If T is the period of a generalized geodesic g, then $|T|$ coincides with the standard length of the curve $\mathrm{Im}(g)$.

Let $\mathcal{L}_\Omega$ be the *set of all periodic generalized geodesics* in Ω. For $g \in \mathcal{L}_\Omega$ we denote by T_g the *length* of $\mathrm{Im}(g)$. We call *length spectrum* the following set

$$L_\Omega = \{T_g : g \in \mathcal{L}_\Omega\}.$$

Lemma 1.2.10: *The set L_Ω is closed in $\mathbb{R}$ and $0 \notin L_\Omega$.*

Proof: Consider a convergent sequence $\{T_k\}$ of elements of L_Ω converging to some $T_0 \in \mathbb{R}$ as $k \to \infty$. Then for every $k \geq 1$ there exists a generalized bicharacteristic $\gamma^{(k)}$ of $\Box$ with period T_k passing through a point of the form $(0, x_k, -1, \xi_k)$. If $T_0 \neq 0$, choosing a subsequence as in the proof of Lemma 1.2.7, we obtain $T_0 \in L_\Omega$.

It remains to show that the case $T_0 = 0$ is impossible. Assume $T_0 = 0$. Passing to an appropriate subsequence, we may assume that there exists $\lim_{k\to\infty}(x_k, \xi_k) = (x_0, \xi_0)$ and for every t there exists

$$\lim_{k\to\infty} \gamma^{(k)}(t) = \lim_{k\to\infty} (t, x^{(k)}(t), -1, \xi^{(k)}(t)) = \gamma_0(t) = (t, x_0(t), -1, \xi_0(t)),$$

provided $\gamma_0(t) \notin H$ and $|t| \leq T$. If x_0 is in the interior of Ω, then x_k is also in the interior of Ω for large k. Then for such k, $x^{(k)}(t)$ is in the interior of Ω for sufficiently small $t > 0$, which is a contradiction. If there exists t' with $|t'| \leq T$ and $x_0(t')$ in the interior of Ω, then we get a contradiction by the same argument.

It remains to consider the case when $\gamma_0(t) \in G$ for all $t \in [-T, T]$. Then for such t, $\gamma_0(t) = (x_0(t), \xi_0(t))$ is an integral curve of the glancing vector field H_p^G. Since the latter is not radial, $\gamma_0(t)$ has no stationary points for $t \in [-T, T]$. Given a small neighbourhood U of x_0 in $\partial\Omega$, there exist δ_0, δ_1 such that $0 < \delta_0 < \delta_1 \leq T$ and $x_0(t) \notin U$ for $\delta_0 \leq |t| \leq \delta_1$. Since $x^{(k)}(t) \to x_0(t)$ as $k \to \infty$ uniformly for $|t| \leq T$, for sufficiently large k there exists a natural number m_k with

$$\delta_0 \leq m_k T_k \leq \delta_1, \quad x^{(k)}(T_k) = x^{(k)}(m_k T_k).$$

Then $x_0 = \lim_{k\to\infty} x_k = \lim_{k\to\infty} x^{(k)}(T_k) \notin U$, which is a contradiction. This proves that $T_0 \neq 0$ and this completes the proof of the proposition. ■

1.3 Wave front sets of distributions

In this section we collect some basic facts concerning wave fronts of distributions. For more details, we refer the reader to the books of Hörmander [Hl], [H3].

Let X be an open subset of $\mathbb{R}^n$ and let $\mathcal{D}'(X)$ be the *space of all distributions* on X. The *singular support* $\operatorname{sing\,supp}(u)$ of $u \in \mathcal{D}'(X)$ is a closed subset of X such that if $x_0 \notin \operatorname{sing\,supp}(u)$ there exists an open neighbourhood U of x_0 in X and a smooth function $f \in C^\infty(U)$ such that

$$\langle u, \varphi \rangle = \int f(x)\varphi(x)\, dx, \quad \varphi \in C_0^\infty(U).$$

For a more precise analysis of $\operatorname{sing\,supp}(u)$, it is useful to consider the directions $\xi \in \mathbb{R}^n \setminus \{0\}$ along which the *Fourier transform* $\widehat{\varphi u}(\xi)$ of the distribution $\varphi u \in \mathcal{E}'(X)$ is not rapidly decreasing, provided $\varphi \in C_0^\infty(X)$ and $\operatorname{supp}(\varphi) \cap \operatorname{sing\,supp}(u) \neq \emptyset$.

Definition 1.3.1: Let $u \in \mathcal{D}'(X)$ and let $\mathcal{O}$ be the set of all $(x_0, \xi_0) \in X \times \mathbb{R}^n \setminus \{0\}$ for which there exists an open neighbourhood U of x_0 in X and an open conic neighbourhood V of ξ_0 in $\mathbb{R}^n$ so that for $\varphi \in C_0^\infty(U)$ and $\xi \in V$ we have

$$|\widehat{\varphi u}(\xi)| \leq C_m(1 + |\xi|)^{-m}, \quad m \in \mathbb{N}.$$

The closed subset

$$WF(u) = (X \times \mathbb{R}^n) \setminus \{0\}$$

of $X \times \mathbb{R}^n \setminus \{0\}$ is called the *wave front set* of u.

It is easy to see that $WF(u)$ is a conic subset of $X \times \mathbb{R}^n \setminus \{0\}$ with the property

$$\pi(WF(u)) = \text{sing supp}(u),$$

where $\pi : X \times \mathbb{R}^n \longrightarrow X$ is the natural projection.

For our aims in Chapter 3 we will describe the wave front sets of distributions given by oscillatory integrals. Such integrals have the form

$$\int e^{\mathbf{i}\varphi(x,\theta)} a(x,\theta)\, d\theta. \tag{1.19}$$

Here the *phase* $\varphi(x, \theta)$ is a C^∞ real-valued function, defined for $(x, \theta) \in \Gamma \subset X \times (\mathbb{R}^N \setminus \{0\})$, and Γ is an open conic set, i.e. $(x, \theta) \in \Gamma$ implies $(x, t\theta) \in \Gamma$ for all $t > 0$. We assume that φ has the properties:

$$\varphi(x, t\theta) = t\, \varphi(x, \theta), \quad (x, \theta) \in \Gamma, t > 0,$$

$$d_{x,\theta}\varphi(x, \theta) \neq 0, \quad (x, \theta) \in \Gamma.$$

The *amplitude* $a(x, \theta)$ belongs to the class of symbols $S^m(X \times \mathbb{R}^N)$, formed by C^∞ functions on $X \times \mathbb{R}^N$ such that for each compact $K \subset X$ and all multi-indices α, β, we have

$$|\partial^\alpha \partial^\beta a(x, \theta)| \leq C_{\alpha,\beta,K}\ (1 + |\theta|)^{m-|\beta|}, \quad x \in K, \quad \theta \in \mathbb{R}^N. \tag{1.20}$$

We endow $S^m(X \times \mathbb{R}^N)$ with the topology defined by the semi-norms

$$p_{\alpha,\beta,j}(a) = \sup_{x \in K_j, \theta \in \mathbb{R}^N} (1 + |\theta|)^{-m+|\beta|}\ |\partial^\alpha \partial^\beta a(x, \theta)|,$$

where $\{K_j\}$ is an increasing sequence of compact sets with $\cup_{j=1}^\infty K_j = X$.

Let $F \subset \Gamma \cup (X \times \{0\})$ be a closed cone and let $\text{supp}\,(a) \subset F$. For $\psi \in C_0^\infty(X)$ we will now define the integral

$$\int e^{\mathbf{i}\varphi(x,\theta)} a(x, \theta)\ \psi(x)\ dx\ d\theta$$

to obtain a distribution in $\mathcal{D}'(X)$. To do this, we need a regularization, since the integral in θ is not convergent for $m > -N$.

Choose a function $\chi \in C_0^\infty(\mathbb{R}^N)$ such that $\chi(\theta) = 1$ for $|\theta| \leq 1$ and $\chi(\theta) = 0$ for $|\theta| \geq 2$. For $0 < \epsilon \leq 1$, the functions $\chi(\epsilon\theta)$ form a bounded set in $S^0(X \times \mathbb{R}^N)$. Then the functions $a_\epsilon = a(x, \theta)\chi(\epsilon\theta)$ also form a bounded set in $S^0(X \times \mathbb{R}^N)$ and

$$a_\epsilon \to a \in S^{m'}(X \times \mathbb{R}^N)$$

as $\epsilon \to 0$ for each $m' > m$.

Consider the operator

$$L = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{N} b_j \frac{\partial}{\partial \theta_j} + \chi$$

with

$$a_j = -\mathbf{i}(1-\chi)\kappa^{-1}\varphi_{x_j}, \quad b_j = -\mathbf{i}(1-\chi)\kappa^{-1}|\theta|^2\varphi_{\theta_j},$$

and $\kappa = |\varphi_x|^2 + |\theta|^2|\varphi_\theta|^2$. For each compact set $K \subset X$ we have

$$\kappa(x,\theta) \geq \delta_K|\theta|^2, \quad x \in K, \quad (x,\theta) \in \Gamma,$$

where $\delta_K > 0$ depends on K only. Clearly

$$L(e^{\mathbf{i}\varphi}) = e^{\mathbf{i}\varphi},$$

and the operator ${}^tL$ formally adjoint to L has the form

$${}^tL = -\sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j} - \sum_{j=1}^{N} b_j \frac{\partial}{\partial \theta_j} + c$$

with

$$a_j \in S^{-1}(X \times \mathbb{R}^N), \;\; b_j \in S^0(X \times \mathbb{R}^N), \;\; c \in S^{-1}(X \times \mathbb{R}^N).$$

The operator ${}^t(L)^k$ is a continuous map of S^m onto S^{m-k}. Define the linear map $I_{\varphi,a} : C_0^\infty(X) \to \mathbb{R}$ by

$$\begin{aligned} I_{\varphi,a}(\psi) &= \lim_{\epsilon\to 0} \int\int e^{\mathbf{i}\varphi(x,\theta)} a(x,\theta)\chi(\epsilon\theta)\psi(x)\; dx\; d\theta \\ &= \lim_{\epsilon\to 0} \int\int e^{\mathbf{i}\varphi(x,\theta)} ({}^tL)^k \;[a(x,\theta)\chi(\epsilon\theta)\psi(x)]\; dx\; d\theta. \qquad (1.21) \end{aligned}$$

For $m - k < -N$ the integral on the right-hand side of (1.21) is absolutely convergent, and it is easy to see that $I_{\varphi,a}$ becomes a distribution in $\mathcal{D}'(X)$. Thus, we obtain the following.

Proposition 1.3.2: *Let $\varphi(x,\theta)$ and $a(x,\theta)$ be as above. Then the oscillatory integral (1.19) defines a distribution $I_{\varphi,a}$ given by (1.21).*

We are now going to describe the set $WF(I_{\varphi,a})$.

Theorem 1.3.3: *We have*

$$WF(I_{\varphi,a}) \subset \{(x, \varphi_x(x,\theta)) : (x,\theta) \in F, \;\; \varphi_\theta(x,\theta) = 0\}. \qquad (1.22)$$

Proof: Let $f \in C_0^\infty(X)$. Then the Fourier transform

$$\widehat{fI_{\varphi,a}}(\xi) = \int\int e^{\mathbf{i}(\varphi(x,\theta)-\langle x,\xi\rangle)}a(x,\theta)f(x)\,dx\,d\theta$$

is expressed by an oscillatory integral. Let V be a closed cone in $\mathbb{R}^N$ such that

$$V \cap \{\varphi_x(x,\theta) : (x,\theta) \in F,\ x \in \text{supp}\,(f),\ \varphi_\theta(x,\theta) = 0\} = \emptyset.$$

By compactness, there exists $\delta > 0$ such that

$$\mu = |\xi - \varphi_x(x,\theta)|^2 + |\theta|^2|\varphi_\theta(x,\theta)|^2 \geq \delta(|\theta| + |\xi|)^2 \tag{1.23}$$

for $(x,\theta) \in F$, $x \in \text{supp}\,(f)$ and $\xi \in V$. To obtain (1.23) it suffices to observe that if the latter conditions are satisfied, then the left-hand side of (1.23) is positive and then use the homogeneity with respect to (θ,ξ). As above, consider the operator

$$L = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{N} b_j \frac{\partial}{\partial \theta_j} + \chi$$

with

$$a_j = -\frac{\mathbf{i}(1-\chi)}{\mu}(\varphi_{x_j} - \xi_j), \quad b_j = -\frac{\mathbf{i}(1-\chi)}{\mu}|\theta|^2\varphi_{\theta_j}.$$

Then

$$\widehat{fI_{\varphi,a}}(\xi) = \lim_{\epsilon\to 0}\int\int e^{\mathbf{i}(\varphi(x,\theta)-\langle x,\xi\rangle)}({}^tL)^k[a(x,\theta)\chi(\epsilon\theta)f(x)]\,dx\,d\theta,$$

and applying (1.23), we conclude that

$$|\widehat{fI_{\varphi,a}}(\xi)| \leq C_N(1+|\xi|)^{-N}, \quad \xi \in V.$$

This implies (1.22). ■

For asymptotics of oscillatory integrals depending on a parameter $\lambda \in \mathbb{R}$ we have the following.

Lemma 1.3.4: *Let $u \in \mathcal{D}'(X)$, $f \in C_0^\infty(X)$ and let $\varphi \in C_0^\infty(X)$ be a real-valued function. Assume*

$$WF(u) \cap \{(x,\varphi_x) : x \in \text{supp}\,(f)\} = \emptyset.$$

Then for each $m \in \mathbb{N}$ we have

$$|\langle u, f(x)e^{\mathbf{i}\lambda\varphi(x)}\rangle| \leq C_m(1+|\lambda|)^{-m}, \quad \lambda \in \mathbb{R}.$$

Proof: Choosing a finite partition of unity, we can restrict our attention to the case $u \in \mathcal{E}'(X)$. Set

$$\Sigma_f = \{\xi \in \mathbb{R}^n \setminus \{0\} : \exists x \in \text{supp}\,(f) \text{ with } (x, \xi) \in WF(u)\}.$$

Then

$$\begin{aligned}\langle u, f(x)e^{\mathbf{i}\lambda\varphi(x)}\rangle &= (2\pi)^{-n} \int\int e^{\mathbf{i}(\langle x,\xi\rangle - \lambda\varphi(x))} f(x)\hat{u}(\xi)\, dx\, d\xi \\ &= \int_X \int_W + \int_X \int_{\mathbb{R}^n \setminus W} = I_1(\lambda) + I_2(\lambda).\end{aligned}$$

Here W is a closed conic set such that $\Sigma_f \subset W$,

$$W \cap \{\varphi_x(x) : x \in \text{supp}\,(f)\} = \emptyset,$$

and $I_1(\lambda)$ is interpreted as an oscillatory integral. For $x \in \text{supp}\,(f)$ and $\xi \in W$ we have

$$|\xi - \lambda\varphi_x(x)| \geq \delta(|\xi| + |\lambda|), \quad \lambda \in \mathbb{R},$$

with $\delta > 0$. Using the same argument as in the proof of Theorem 1.3.3, we see that $I_1(\lambda) = O(|\lambda|^{-m})$ for all $m \in \mathbb{N}$. For $I_2(\lambda)$ we use the fact that if $\xi \in \mathbb{R}^n \setminus W$ and $\text{supp}\,(u) \cap \text{supp}\,(f) \neq \emptyset$, then $\hat{u}(\xi)$ is rapidly decreasing. This proves the assertion. ■

Now let $\Gamma \subset X \times \mathbb{R}^n \setminus \{0\}$ be a closed conic set. Set

$$\mathcal{D}'_\Gamma(X) = \{u \in \mathcal{D}'(X) : WF(u) \subset \Gamma\}.$$

Using an argument similar to that in the proof of Lemma 1.3.4, it is easy to see that $u \in \mathcal{D}'_\Gamma(X)$ if and only if for each $\varphi \in C_0^\infty(X)$ and each closed cone $V \subset \mathbb{R}^n$ with

$$(\text{supp}\,(\varphi) \times V) \cap \Gamma = \emptyset \tag{1.24}$$

we have

$$\sup_{\xi \in V} |\xi|^m |\widehat{\varphi u}(\xi)| < \infty, \quad m \in \mathbb{N}.$$

This makes it possible to introduce the following.

Definition 1.3.5: Let $\{u_j\}_j \subset \mathcal{D}'_\Gamma(X)$ and let $u \in \mathcal{D}'_\Gamma(X)$. We will say that the sequence $\{u_j\}$ converges to u in $\mathcal{D}'_\Gamma(X)$ if:

(a) $u_j \to u$ weakly in $\mathcal{D}'(X)$,

(b) $\sup_{j \in \mathbb{N}} \sup_{\xi \in V} |\xi|^m |\widehat{\varphi u_j}(\xi)| < \infty$ for every $m \in \mathbb{N}$, every $\varphi \in C_0^\infty(X)$ and every closed cone V satisfying (1.24).

For every $u \in \mathcal{D}'_\gamma(X)$ there exists a sequence $\{u_j\} \subset C_0^\infty(X)$ converging to u in $\mathcal{D}'_\Gamma(X)$. To prove this, consider two sequences $\chi_j, \varphi_j \in C_0^\infty(X)$ such that $\chi_j = 1$ on K_j, $\varphi_j \geq 0$, $\int \varphi_j(x)\, dx = 1$ and $\text{supp}\,(\chi_j) + \text{supp}\,(\varphi_j) \subset X$. Then

$$u_j = \varphi_j * \chi_j u \in C_0^\infty(X)$$

and $u_j \to u$ in $\mathcal{D}'(X)$. Moreover, the condition (b) also holds, so $u_j \to u$ in $\mathcal{D}'_\Gamma(X)$.

For our aims in Chapter 3 we need to justify some operations on distributions (see [Hl] for more details). For convenience of the reader we list these properties, including only one proof of these – namely that of the existence of the pull-back f^*. We use the notation from [Hl].

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be open sets and let $f : X \longrightarrow Y$ be a smooth map. Consider a closed cone $\Gamma \subset Y \times \mathbb{R}^m \setminus \{0\}$ and set

$$N_f = \{(f(x), \eta) \in Y \times \mathbb{R}^n : {}^t f'(x)\eta = 0\},$$
$$f^*(\Gamma) = \{(x, {}^t f'(x)\eta : (f(x), \eta) \in \Gamma\}.$$

For $u \in C_0^\infty(Y)$, consider the map

$$(f^*u)(x) = u(f(x)).$$

Theorem 1.3.6: *Let $N_f \cap \Gamma = \emptyset$. Then the map f^*u can be extended uniquely on the space $\mathcal{D}'_\Gamma(Y)$ such that*

$$WF(f^*u) \subset f^*\Gamma. \tag{1.25}$$

Proof: Using a partition of unity, we may consider only the case when X and Y are small open neighbourhoods of $x_0 \in X$ and $y_0 \in Y$, respectively. Set

$$\Gamma_y = \{\eta : (y, \eta) \in \Gamma\}.$$

Choose a small compact neighbourhood X_0 of x_0 and a closed conic neighbourhood V of Γ_{y_0} so that

$${}^t f'(x)\eta \neq 0 \;\text{ for } x \in X_0, \eta \in V.$$

Next, choose a small compact neighbourhood Y_0 of y_0 with $\Gamma_y \subset V$ for all $y \in Y_0$.

Now let $\chi \in C_0^\infty(X_0)$ and let $\{u_j\}_j \subset C_0^\infty(Y)$ be a sequence such that $u_j \to u$ in $\mathcal{D}'_\Gamma(Y)$. Choosing $\varphi \in C_0^\infty(Y_0)$ with $\varphi = 1$ on $f(X_0)$, we have

$$\langle f^*u_j, \chi\rangle = \langle f^*(\varphi u_j), \chi\rangle = (2\pi)^{-m} \int \widehat{\varphi u_j}(\eta) I_\chi \, d\eta = \int_V + \int_{\mathbb{R}^m \setminus V} = I_1 + I_2,$$

where

$$I_\chi(\eta) = \int e^{\mathbf{i}\langle f(x), \eta\rangle} \chi(x)\, dx.$$

For $x \in \text{supp}\,(\chi)$ and $\eta \in V$ we obtain

$$|\nabla_x \langle f(x), \eta\rangle| \geq \delta|\eta|, \quad \delta > 0.$$

Using the operator

$$L = \frac{-\mathbf{i}}{|\nabla_x \langle f(x), \eta\rangle|^2} \sum_{j=1}^{n} \partial_{x_j}(\langle f(x), \eta\rangle) \frac{\partial}{\partial x_j},$$

we integrate by parts in $I_\chi(\eta)$ and get

$$|I_\chi(\eta)| \le C_p(1+|\eta|)^{-p}, \quad \eta \in V,$$

for all $p \in \mathbb{N}$. On the other hand, there exists $M > 0$ such that

$$|\widehat{\varphi u_j}(\eta)| \le C(1+|\eta|)^{-M},\ j \in \mathbb{N}.$$

Thus, I_1 is absolutely convergent, and we can consider the limit as $j \to \infty$. To deal with I_2, notice that $(\operatorname{supp}\varphi \setminus V) \cap \Gamma = \emptyset$. For $\eta \notin V$, (b) yields the estimates

$$|\widehat{\varphi u_j}(\eta)| \le C'_p(1+|\eta|)^{-p},\ p \in \mathbb{N}, \tag{1.26}$$

uniformly with respect to j. Thus, we can let $j \to \infty$ in I_2.

To establish (1.25), replace $\chi(x)$ by $\chi(x)e^{-\mathbf{i}\langle x,\xi\rangle}$ and write

$$I_\chi(\eta, \epsilon) = (2\pi)^{-n} \int e^{\mathbf{i}\langle f(x),\eta\rangle - \mathbf{i}\langle x,\xi\rangle} \chi(x)\, dx.$$

Choose a small open conic neighbourhood W of the set

$$\{\xi = {}^t f'(x_0)\eta : (f(x_0), \eta) \in \Gamma\}$$

so that $x \in X_0$ and $\eta \in V$ imply ${}^t f'(x)\eta \in W$. As above, for $x \in X_0$, $\eta \in V$ and $\xi \notin W$ we deduce the estimate

$$|\xi - {}^t f'(x)\eta| \ge \delta(|\xi| + |\eta|), \quad \delta > 0.$$

For such ξ and η we integrate by parts in $I_\chi(\eta, \epsilon)$ and obtain

$$|I_\chi(\eta, \epsilon)| \le C''_p(1 + |\xi| + |\eta|)^{-p}, \quad p \in \mathbb{N}.$$

For $\eta \notin V$, $\xi \notin W$ we choose a function $\psi(\xi) \in C_0^\infty(\mathbb{R})$ with $\psi(\xi) = 1$ for $|\xi| \le 1$, and consider the operator

$$L = -\mathbf{i}(1 - \psi(\xi))|\xi|^{-2} \left\langle \xi, \frac{\partial}{\partial x} \right\rangle + \psi(x).$$

Then $L(e^{\mathbf{i}\langle x,\xi\rangle}) = e^{\mathbf{i}\langle x,\xi\rangle}$, and, as in the previous case, for $\eta \notin V$and $\xi \notin W$, we get the estimates

$$|I_\chi(\eta, \epsilon)| \le C_p(1+|\eta|)^p(1+|\xi|)^{-p}, \quad p \in \mathbb{N}.$$

Combining these estimates with (1.26), we obtain

$$|\chi(\widehat{f^* u_j})(\xi)| \le C_N(1+|\xi|)^{-N}$$

for $\xi \notin W$, where the constant C_N does not depend on j. Letting $j \to \infty$ proves (1.25). ∎

By an easy modification of the above-mentioned argument, one proves the following modification of Theorem 1.3.6 for distributions depending on a parameter.

Corollary 1.3.7: *Let Z be a compact subset of $\mathbb{R}^p$ and let*

$$Z \ni z \mapsto (u, \cdot, z) \in \mathcal{D}'_{\Gamma}(Y)$$

be a continuous map. Under the assumptions of Theorem 1.3.6, the map

$$Z \ni z \mapsto f^*(u, \cdot, z) \in \mathcal{D}'_{f^*\Gamma}(X)$$

is continuous.

Next, consider a linear continuous map

$$\mathcal{K} : C_0^\infty(Y) \longrightarrow \mathcal{D}'(X).$$

By Schwartz's theorem (cf. Theorem 5.2.1 in [Hl]), there exists a distribution $K \in \mathcal{D}'(X \times Y)$, called the *kernel* of $\mathcal{K}$, such that

$$\langle K, \varphi(x) \otimes \psi(y) \rangle = \langle (\mathcal{K}\psi)(x), \varphi(x) \rangle$$

for all $\varphi \in C_0^\infty(X)$ and $\psi \in C_0^\infty(Y)$. $WF(K)$ will be called the *wave front set* of $\mathcal{K}$. Set

$$WF'(K) = \{(x, y, \xi, \eta) : (x, y, \xi, -\eta) \in WF(K)\},$$
$$WF(K)_X = \{(x, \xi) : (x, y, \xi, 0) \in WF(K) \text{ for some } y \in Y\},$$
$$WF'(K)_Y = \{(y, \eta) : (x, y, 0, \eta) \in WF'(K) \text{ for some } x \in X\},$$

and consider the composition

$$WF'(K) \circ WF(u) = \{(x, \xi) : \exists (y, \eta) \in WF(u) \text{ with } (x, y, \xi, \eta) \in WF'(K)\}.$$

The following two results will also be necessary for Chapter 3. Their proofs can be found in Section 8.2 of [Hl].

Theorem 1.3.8: *For $\psi \in C_0^\infty(Y)$ we have*

$$WF(\mathcal{K}\psi) \subset \{(x, \xi) : (x, y, \xi, 0) \in WF(K) \text{ for some } y \in \text{supp}\,(\psi)\}.$$

Theorem 1.3.9: *There exists a unique extension of $\mathcal{K}$ on the set*

$$\{u \in \mathcal{E}'(Y) : WF(u) \cap WF'(K)_Y = \emptyset\}$$

such that for each compact $M \subset Y$ and each closed conic set Γ with $\Gamma \cap WF'(K)_Y = \emptyset$ the map

$$\mathcal{E}'(M) \cap \mathcal{D}'_\Gamma(Y) \ni u \mapsto \mathcal{K}u \in \mathcal{D}'(X)$$

is continuous. Moreover, the inclusion

$$WF(\mathcal{K}u) \subset WF(K)_X \cup WF'(K) \circ WF(u)$$

holds.

The wave front of $u \in \mathcal{D}'(X)$ can be described by means of the characteristic set of pseudo-differential operators on X. Denote by $L^m(X)$ the *class of all pseudo-differential operators (PDO) in X of order m.* If $x(x,\xi) \in S^m(X \times \mathbb{R}^n)$ is the symbol of $A \in L^m(X)$, then the oscillatory integral

$$K_A(x,\eta) = (2\pi)^{-n} \int e^{\mathbf{i}\langle x-y,\xi\rangle} a(x,\xi)\, d\xi$$

determines the kernel of A and $WF(A) = WF(K_A)$. The operator $A \in L^m(X)$ is called *properly supported* if for each compact $K \subset X$ there exists another compact $K' \subset X$ so that $\mathrm{supp}\,(u) \subset K$ implies $\mathrm{supp}\,(Au) \subset K'$, and if $u = 0$ on K', then $Au = 0$ on K. A point $(x_0,\xi_0) \in T^*X \setminus \{0\}$ is called *non-characteristic* for a properly supported PDO $A \in L^m(X)$ if there exists a properly supported PDO $B \in L^{-m}(X)$ so that

$$(x_0,\xi_0) \notin WF(AB - Id) \cup WF(BA - Id).$$

In this case A is called *elliptic* at (x_0,ξ_0).

Proposition 1.3.10: *If there exists a properly supported PDO $A \in L^m(X)$, elliptic at (x_0,ξ_0), such that $Au \in C^\infty(X)$, then $(x_0,\xi_0) \notin WF(u)$.*

The reader may consult Section 18 in [Hl] for the main properties of PDOs and for a proof of the above-mentioned proposition.

1.4 Boundary problems for the wave operator

Let $\Omega \subset \mathbb{R}^n$ be a domain in $\mathbb{R}^n, n \geq 2$ with C^∞ smooth compact boundary $\partial\Omega$. Consider the problem

$$\begin{cases} (\partial_t^2 - \Delta_x)u = f \text{ in } \mathbb{R} \times \Omega^\circ, \\ u = u_0 \text{ on } \mathbb{R} \times \partial\Omega, \\ u|_{t<t_0} = 0. \end{cases} \tag{1.27}$$

Here the trace $u|_{(t,x)\in\mathbb{R}\times\partial\Omega}$ exists, since the boundary $\mathbb{R} \times \partial\Omega$ is not characteristic for the operator $\Box = \partial_t^2 - \Delta_x$. For the existence of a solution of (1.27) we refer to

[H3], Section 24. In particular, we have the following result proved in [H3], Theorem 24.1.1.

Theorem 1.4.1: *Let* $f \in H_s^{loc}(\mathbb{R} \times \Omega^\circ), u_0 \in H_{s+1}^{loc}(\mathbb{R} \times \partial\Omega)$ *with* $s \geq 0$. *Assume that* f *and* u_0 *vanish for* $t < t_0$. *Then there exists an unique solution* $u \in H_{s+1}^{loc}(\Omega^\circ)$ *of (1.27).*

We may apply the above theorem when Ω is a bounded domain in $\mathbb{R}^n$, as well as in the case when $\Omega = \mathbb{R}^n \setminus \bar{K}$, K being a bounded non-empty open obstacle with smooth boundary.

To study the singularities of the solution of the Dirichlet problem

$$\begin{cases} (\partial_t^2 - \Delta_x)u = f \text{ in } \mathbb{R} \times \Omega^\circ, \\ u = u_0 \text{ on } \mathbb{R} \times \partial\Omega, \end{cases} \tag{1.28}$$

we need to introduce the wave front set $WF_b(u)$. Let $Q = \mathbb{R} \times \Omega$. Consider the space $\widetilde{T}^*(Q) = T^*(Q^\circ) \cup T^*(\partial Q)$ of equivalence classes in T^*Q with respect to the equivalence relation $\sim$ defined in Section 1.2. It will be called the *compressed cotangent bundle* of Q. For the solution u of (1.27) we can define the *generalized wave front set* $WF_b(u) \subset \widetilde{T}^*Q \setminus \{0\}$ in such a way that

$$WF_b(u)|_{T^*(Q^\circ)} = WF(u_{|Q^\circ}),$$

and

$$WF_b(u)|_{T^*(\partial Q)} \subset \Sigma_b.$$

(See Section 1.2 for the definition of Σ_b.) For this purpose, as in Section 1.2, introduce local coordinates (x_1, x'), $x' = (x_2, \dots, x_n, x_{n+1})$, $x_{n+1} = t$ in Q so that ∂Q is locally given by $x_1 = 0$. Let (ξ_1, ξ') be the dual coordinates to (x_1, x').

Now define $WF_b(u)|_{T^*(\partial Q)}$ as the subset of $T^*(\partial Q) \setminus \{0\}$, the complement of which consists of all $(x_0', \xi_0') \in T^*(\partial Q) \setminus \{0\}$ such that there exists a PDO $B(x, D')$, depending smoothly on x_1, elliptic at $(0, x_0, \xi_0')$, and such that $B(x, D_{x'})u \in C^\infty(Q)$. This definition does not depend on the choice of the local coordinates.

In Q° the set $WF(u) \setminus WF(f)$ is contained in the characteristic set Σ and it is propagating along the bicharacteristics of $\Box$ which are rays. For simplicity assume that $f \in C^\infty(Q^\circ)$. The singularities of the solution $u|_Q$ of (1.28) can be described by means of $WF_b(u)$. The simplest case is when $(0, x', \xi') \in H$ is a hyperbolic point. Then if $(0, x', \xi') \in (WF_b(u) \cap H) \setminus WF(u_0)$, the outgoing and incoming bicharacteristics issued from this point are included in $WF_b(u)$ over a small neighbourhood of $(0, x_0', \xi_0')$. If $(0, x_0', \xi_0') \in G$ is a gliding point, the situation is more complicated and we must consider the generalized compressed bicharacteristics of $\Box$ issued from this point. The following result was proved by Melrose and Sjöstrand [MS2] (see also Section 24 and Theorem 24.5.3 in [H3]).

Theorem 1.4.2: *Let $u \in \mathcal{D}'(Q)$ be a solution of problem (1.28) with $f \in C^\infty(Q)$ and $u_0 \in \mathcal{D}'(\partial\Omega)$ and let*

$$\hat{z} \in (WF_b(u) \setminus WF(u_0)) \cap \{(x,\xi) \in \widetilde{T}^*Q : x_{n+1} = t > t_0\}.$$

Then $\hat{z}$ is either a characteristic point in Σ_0 or a point in $T^*(\partial\Omega) \cap \Sigma = H \cup G$, and there exists a maximal compressed generalized bicharacteristics $\tilde{\gamma}(\sigma) = (x(\sigma), \xi(\sigma))$ *of $\Box$, passing through $\hat{z}$ and staying in $WF_b(u)$ as long as $t(\sigma) = x_{n+1}(\sigma) > t_0$.*

One can also describe the singularities of a boundary problem with non-homogeneous boundary condition

$$\begin{cases} (\partial_t^2 - \Delta_x)u = f \text{ in } \mathbb{R} \times \Omega^\circ, \\ u = g \text{ on } \mathbb{R} \times \partial\Omega, \end{cases} \tag{1.29}$$

with $f = 0, g = 0$ for $t < t_0$. In this situation we have the following result established in [MS2] (see Theorem 6.14).

Theorem 1.4.3: *Let u be a solution of (1.29) and let $f \in C^\infty$. Then $WF_b(u)$ is a complete union of the generalized half-bicharacteristics issued from $WF(g)$.*

Here half-bicharacteristics means that we consider these bicharacteristics γ for which the time increases when we move along γ.

The same results hold for the boundary problem

$$\begin{cases} (\partial_t^2 - \Delta_x)u = f \text{ in } \mathbb{R} \times \Omega^\circ, \\ (\partial_\nu + \alpha(x))u = u_0 \text{ on } \mathbb{R} \times \partial\Omega, \end{cases} \tag{1.30}$$

where ∂_ν is the derivative with respect to a normal field of $\partial\Omega$ and $\alpha(x)$ is a C^∞ function on $\partial\Omega$. For $\alpha(x) = 0$ we have the Neumann problem, while for $\alpha(x) \neq 0$ we obtain the Robin problem.

1.5 Notes

The results in Section 1.1 can be found with detailed proofs in [GG] and [Hir]. In Section 1.2 we follow [MS1], [MS2] and [H3]. Lemma 1.2.5 is proved in [MS1], while Lemmas 1.2.6, 1.2.7 and 1.2.10 can be found in [H3]. The results in Section 1.3 concerning wave front sets of distributions and operators are due to Hörmander [Hl], [H3]. The definition of generalized wave front set $WF_b(u)$ was introduced by Melrose and Sjöstrand [MS1]. Theorem 1.3.11 was established in [MS1], [MS2]. We refer the reader to Section 24 in [H3] for more details concerning the generalized bicharacteristics and the propagation of singularities for the Dirichlet problem.

2

Reflecting rays

In this chapter we begin with some elementary properties of periodic reflecting rays in a domain Ω, relating these rays to critical points of certain length functions. In a similar way in Section 2.4 we deal with scattering rays. For both kinds of rays the special case is considered when the complement of Ω is a finite disjoint union of strictly convex bounded domains. The results obtained in this case will be useful for our considerations in subsequent chapters.

The linear Poincaré map P_γ of a periodic reflecting ray γ is frequently used in this book. It is defined in Section 2.3, where a useful matrix representation of it is also described. The analogue of a Poincaré map for a reflecting scattering ray γ, the so-called differential cross section dJ_γ, is defined and studied in Section 2.4. This section contains also the main definitions concerning scattering rays which will be frequently used in the next chapters.

2.1 Billiard ball map

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, with a smooth boundary $X = \partial\Omega$. The *billiard flow* in Ω is the dynamical system generated by the motion of a material point in Ω. The point is moving with constant velocity in the interior of Ω making reflections at $\partial\Omega$ according to the usual law of geometrical optics 'the angle of incidence is equal to the angle of reflection'. The successive positions of the point at $\partial\Omega$ are described by the so-called *billiard ball map*, which will be denoted hereafter by B. This map is defined on a subset M' of the set

$$M = \{(x, v) \in X \times \mathbb{S}^{n-1} : \langle \nu(x), v \rangle \geq 0\},$$

$\nu(x)$ being the *unit normal* to $\partial\Omega$ at x pointing into the interior of Ω, as follows. Let $q = (x, v) \in M$ be such that the straight-line ray γ issued from x in direction

Geometry of the Generalized Geodesic Flow and Inverse Spectral Problems, Second Edition.
Vesselin M. Petkov and Luchezar N. Stoyanov.

v has a common point with X and let y be the first such point, that is $y \in X$ and the open segment (x, y) does not contain any points of X. By definition, the subset M' consists of those q so that γ intersects X transversally at y. For such q define $w = v - 2\langle v, \nu(y)\rangle\, \nu(y)$ and set $B(q) = (y, w)$. Thus, one obtains a map $B : M' \longrightarrow M$. Clearly M' and M are open subsets of $X \times \mathbb{S}^{n-1}$ and so they have natural structures of smooth manifolds. It is a standard exercise to show that B is a smooth map. From a dynamical point of view it is more convenient to consider B as a map $B : M_0 \longrightarrow M_0$, where

$$M_0 = \cap_{n=-\infty}^{\infty} B^n(M')$$

and $B^n = B \circ B \circ \cdots \circ B$ (n times). The points $q \in M_0$ with $B^k(q) = q$ for some integer $k \geq 2$ will be called *periodic points* of *period* k of B. Clearly B has no fixed points in M', that is points q with $B(q) = q$.

Notice that in general M' is a proper subset of M. For unbounded domains Ω this is clear, while for bounded but non-convex Ω this is due to the existence of rays tangent to $\partial\Omega$. In the latter case it is more convenient to deal with the generalized geodesic flow on $T^*\Omega$ introduced in Chapter 1. Using the standard identification of $T\Omega$ with $T^*\Omega$, one can consider the billiard flow as a subsystem of the generalized geodesic flow, and respectively, the billiard ball map B as a map on some subset of the *cosphere bundle* $S^*(\partial\Omega)$ (see Section 4.2).

Example 2.1.1: Let Ω be a strictly convex-bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Clearly in this case, $M_0 = M = M'$. Moreover, B can be naturally extended to a diffeomorphism $B : \overline{M} \longrightarrow \overline{M}$ by $B(q) = q$ for every $q \in \overline{M} \setminus M$, where

$$\overline{M} = \{(x, v) \in X \times \mathbb{S}^{n-1} : \langle \nu(x), v\rangle \geq 0\}$$

is the closure of M in $X \times \mathbb{S}^{n-1}$. In fact, $\overline{M}$ is a manifold with boundary $\partial\overline{M} = \overline{M} \setminus M$. We refer the reader to [Ko] for a proof of the fact that B is smooth on $\overline{M}$.

It can be easily shown that for any integer $s \geq 2$, there exists a periodic point $q \in M$ of period s. Indeed, fix an arbitrary s and consider the function

$$F = F_s : \Omega^s \longrightarrow \mathbb{R}$$

given by

$$F(x_1, \ldots, x_s) = \sum_{i=1}^{s} \|x_i - x_{i+1}\| , \tag{2.1}$$

where $x_{s+1} = x_1$ by definition. Since Ω^s is compact and F is continuous, there exists $x = (x_1, \ldots, x_s) \in \Omega^s$ such that F has a maximum at x. A trivial argument shows that $x_i \in \partial\Omega$ and $x_i \neq x_{i+1}$ for every $i = 1, \ldots, s$. Then the restriction G of F to $(\partial\Omega)^s$ has a maximum at x. Since G is smooth on $(\partial\Omega)^s$, x is a critical point of G. It then follows that $x_1, x_2, \ldots, x_s$ are the successive reflection points of a periodic billiard trajectory (see Proposition 2.1.3 for a rigorous proof), that is

$B(x_i, v_i) = (x_{i+1}, v_{i+1})$, where $v_i = (x_{i+1} - x_i)/ \|x_{i+1} - x_i\|$. Therefore, $B^s(x_i, v_i) = (x_i, v_i)$, which shows that B has at least s distinct periodic points of period s.

In the case $n = 2$, that is $\Omega \subset \mathbb{R}^2$, one can modify the above argument to prove the existence of periodic points of B of arbitrary period s and a given rotation number $k \leq s/2$. Given $x = (x_1, \ldots , x_s) \in \Omega^s$, define the *rotation (winding) number* $r(x)$ as follows. Set $x_{s+1} = x_1$, and for any $i = 1, \ldots , s$, denote by ℓ_i the length of the segment $[x_i, x_{i+1}]$ on $\partial\Omega$ with respect to the positive (counter-clockwise) orientation of $\partial\Omega$, and by ℓ'_i its length with respect to the negative orientation of $\partial\Omega$. Denote by L the *length* of $\partial\Omega$. The integer part of $\frac{1}{L}\sum_{i=1}^{s} \ell_i$ will be denoted by $r_+(x)$ and that of $\frac{1}{L}\sum_{i=1}^{s} \ell'_i$ by $r_-(x)$. Since $\ell_i + \ell'_i = L$, we have $r_+(x) + r_-(x) = s$, therefore

$$r(x) = \min\{r_+(x),\, r_-(x)\} \leq \frac{s}{2}.$$

The number $r(x)$ is called the *rotation number* of $x = (x_1, \ldots , x_s)$.

Given two integers $s \geq 2$ and k with $1 \leq k \leq s/2$, applying the above argument to the function $F = F_s$ on the set of those $x \in \Omega^s$ with $r(x) = k$, one gets that there exists $x \in \Omega^s$ with $r(x) = k$ such that x is generated by a periodic point (x_1, v_1) of B of period s.

For our needs in subsequent chapters it will be convenient to put the notion of a periodic orbit of the billiard in a more general setting. This will allow us to consider such orbits in arbitrary domains Ω with smooth boundaries $\partial\Omega$.

Let X be a smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$, $n \geq 2$. Given two linear segments ℓ_1 and ℓ_2 with a common end $x \in X$, we will say that these segments *satisfy the law of reflection* at x with respect to X if ℓ_1 and ℓ_2 make equal acute angles with one of the unit normals $\nu(x)$ to X at x , and ℓ_1, ℓ_2 and $\nu(x)$ lie in a common two-dimensional plane.

Definition 2.1.2: Let X be as above and let γ be a curve in $\mathbb{R}^n$ of the form

$$\gamma = \cup_{i=1}^{k} \ell_i,$$

where $\ell_i = [x_i, x_{i+1}]$ is a straight-line segment, $x_i \in X$ for each $i = 1, \ldots , k$, $k \geq 2$, and we set for convenience $x_{k+1} = x_1$ and $\ell_{k+1} = \ell_1$. We will say that γ is a *periodic reflecting ray* for X if the following conditions are satisfied:

(a) for each $i = 1, \ldots , k$ the open segments ℓ_i° do not intersect transversally X (but may have common tangent points with X);

(b) for each $i = 1, \ldots , k$ the segments ℓ_i and ℓ_{i+1} satisfy the law of reflection at x_{i+1} with respect to X.

The points $x_1, \ldots , x_s$ will be called *reflection points* of γ, while

$$\ell_\gamma = \sum_{i=1}^{k} \|x_i - x_{i+1}\|$$

will be called the *length* of γ.

Notice that if Ω is a domain with boundary X, then a periodic reflecting ray for X may not be entirely in Ω (see Figure 2.1). If γ contains a segment orthogonal to X at some of its end points, then γ will be called *symmetric*, otherwise it will be called *non-symmetric*. In general a periodic reflecting ray γ may have segments tangent to X at some of its interior points (see Figure 2.2). If γ has no such segments, we will say that γ is *ordinary*. Let us mention that in what follows, with the exception of Section 2.2, points like z_1 in Figure 2.2 are not considered as reflection points. In general a periodic reflecting ray can pass two or more times through some of its reflection points, and two different periodic reflecting rays could have some common reflection points. Given a periodic reflecting ray γ and an integer $k \geq 2$, one defines naturally the k-multiple δ of γ. Clearly, as a subset of $\mathbb{R}^n$, δ coincides with γ; however, the number of reflection points of δ is ks, where s is the number of reflection points of γ. We will say that γ is *primitive* if γ is not a multiple of any periodic reflecting ray (Figures 2.3, 2.4).

We conclude this section with an elementary fact, which however will be important later on. Namely, we will show that there is a natural one-to-one correspondence between the periodic reflecting rays with s reflection points for a given submanifold X and some kind of critical points of the restriction of the map $F = F_s$ defined by (2.1) on X^s. Notice that F is well defined and continuous on $(\mathbb{R}^n)^s$ and F is smooth on the set U_s of those $y = (y_1, \ldots, y_s) \in (\mathbb{R}^n)^s$ such that $y_i \neq y_{i+1}$ for all $i = 1, \ldots, s$ (as before $y_{s+1} = y_1$ by definition).

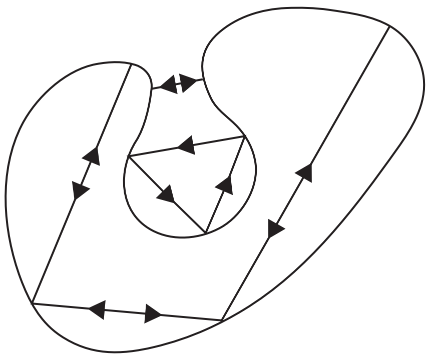

Figure 2.1 Periodic reflecting rays.

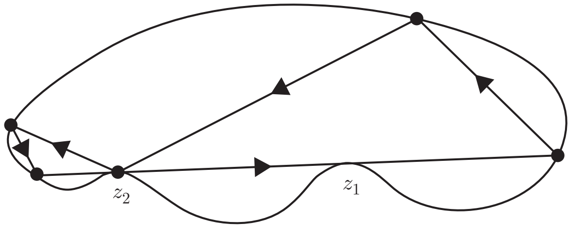

Figure 2.2 Tangent rays.

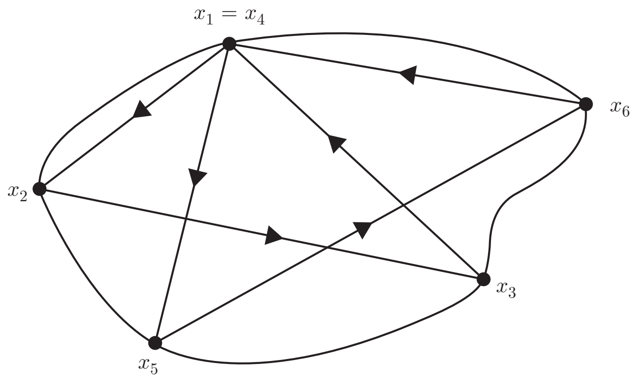

Figure 2.3 Multiple reflections through a point.

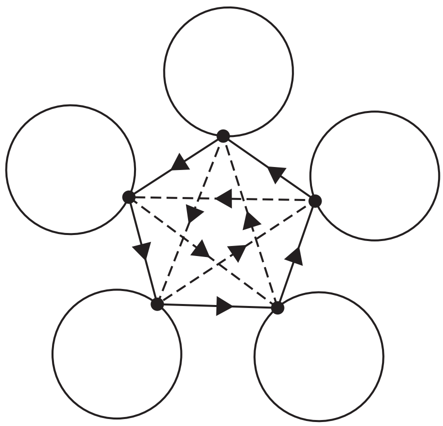

Figure 2.4 Periodic rays with common reflection points.

Proposition 2.1.3: *Let γ be a curve in $\mathbb{R}^n$ of the form $\gamma = \cup_{i=1}^{s} \ell_i$, where $\ell_i = [x_i, x_{i+1}]$ is a straight-line segment, $x_i \in X$ for each $i = 1, \dots, s$, $s \geq 2$, $x_{s+1} = x_1$, and such that the open segments ℓ_i° do not intersect X transversally. Then γ is a periodic reflecting ray for X if and only if $x = (x_1, \dots, x_s)$ is a critical point of the map $F_{|(X^s \cap U_s)}$.*

Proof: Take arbitrary smooth charts

$$\varphi_j : \mathbb{R}^{n-1} \longrightarrow U_j \subset X, \quad j = 1, \dots, s,$$

such that $\varphi_j(0) = x_j$. Then $\{\frac{\partial \varphi_j}{\partial u_j^{(t)}}(0)\}_{t=0}^{n-1}$ is a basis of the tangent space to X at x_j. Here we use the notation $u_j = (u_j^{(1)}, \dots, u_j^{(n-1)}) \in \mathbb{R}^{n-1}$. Consider the function

$$G : (\mathbb{R}^{n-1})^s \longrightarrow \mathbb{R}$$

defined by

$$G(u_1, \dots, u_s) = F(\varphi_1(u_1), \dots, \varphi_s(u_s)).$$

Let $\varphi_j = (\varphi_j^{(1)}, \dots, \varphi_j^{(n-1)})$. Clearly for any $u \in (\mathbb{R}^{n-1})^s$ sufficiently close to 0, G is differentiable at u and

$$\frac{\partial G}{\partial u_j^{(t)}}(u) = \left\langle \frac{\varphi_j(u_j) - \varphi_{j-1}(u_{j-1})}{\|\varphi_j(u_j) - \varphi_{j-1}(u_{j-1})\|} + \frac{\varphi_j(u_j) - \varphi_{j+1}(u_{j+1})}{\|\varphi_j(u_j) - \varphi_{j+1}(u_{j+1})\|}, \frac{\partial \varphi_j}{\partial u_j^{(t)}}(u_j) \right\rangle .$$

Setting

$$v_{i,j} = \frac{x_i - x_j}{\|x_i - x_j\|},$$

we get

$$\frac{\partial G}{\partial u_j^{(t)}}(0) = \left\langle v_{j,j-1} + v_{j,j+1}, \frac{\partial \varphi_j}{\partial u_j^{(t)}}(0) \right\rangle .$$

Notice that the segments ℓ_j and ℓ_{j+1} satisfy the law of reflection at x_{j+1} with respect to X if and only if the vector $v_{j,j-1} + v_{j,j+1}$ is orthogonal to X at x_{j+1}. According to the above argument, this is equivalent to the fact that $\frac{\partial G}{\partial u_j^{(t)}}(0) = 0$ for all $t = 1, \dots, n-1$. Hence, γ is a periodic reflecting ray for X if and only if 0 is a critical point of G. This proves the proposition. ∎

2.2 Periodic rays for several convex bodies

In this section we study periodic reflecting rays in a domain Ω in $\mathbb{R}^n$ such that the complement $K = \mathbb{R}^n \setminus \Omega$ has the form

$$K = K_1 \cup \cdots \cup K_s, \tag{2.2}$$

$s \geq 3$. Here each K_i is a compact convex domain in $\mathbb{R}^n$ with C^2-smooth boundary $\Gamma_i = \partial K_i$ and $K_i \cap K_j = \emptyset$ whenever $i \neq j$.

Our aim in what follows is to provide a coding for the periodic reflecting rays in Ω. Namely, we associate with any periodic reflecting ray γ a finite sequence

$$\alpha_\gamma = (i_1, \dots, i_k) \in \{1, \dots, s\}^k,$$

where k is the number of reflections of γ such that the jth successive reflection point belongs to K_{i_j} for any $j = 1, \dots, k$. Clearly, for such a sequence we have $i_j \neq i_{j+1}$ for $j = 1, \dots, k-1$and $i_k \neq i_1$. Every

$$\alpha = (i_1, \dots, i_k) \in \{1, \dots, s\}^k \tag{2.3}$$

with the latter property will be called a *configuration* of length $|\alpha| = k$. Denote by $\mathcal{A}_k$ the *set of all configurations of length* k. We will show that if all K_i are strictly convex, then the correspondence $\gamma \longrightarrow \alpha_\gamma \in \mathcal{A}_k$ is invertible, and under some additional assumption, it is moreover bijective. It is easy to construct examples showing that in general this map is not surjective.

Clearly

$$\Gamma = \partial\Omega = \Gamma_1 \cup \cdots \cup \Gamma_s.$$

For $q \in \Gamma$, the *unit normal vector* to Γ at q pointing into the interior of Ω will be denoted by $\nu(q)$. The second fundamental form of Γ is non-positive definite at any $q \in \Gamma$ with respect to this choice of the normal field.

Definition 2.2.1: We will say that K satisfies the **condition (H)** if for any $i \neq j$ the convex hull of $K_i \cup K_j$ has no common points with K_r for any $r \notin \{i, j\}$.

In what follows the reflecting rays for Γ contained in $\overline{\Omega}$ will be called briefly *reflecting rays* in $\bar{\Omega}$.

Proposition 2.2.2: *Let K satisfy the condition (H). Then for every integer $k \geq 2$ and every $\alpha \in \mathcal{A}_k$ there exists a periodic reflecting ray γ in $\bar{\Omega}$ such that $\alpha_\gamma = \alpha$.*

Proof: Fix an arbitrary α of the form (2.3) and consider the function

$$F = F_\alpha : K_\alpha = K_{i_1} \times \cdots \times K_{i_k} \longrightarrow \mathbb{R}, \tag{2.4}$$

defined by

$$F(q_1, \ldots, q_k) = \sum_{j=1}^{k} \| q_j - q_{j+1} \| , \tag{2.5}$$

where we use the notation $q_{k+1} = q_1$. Since F is continuous and K_α is compact, there exists $q = (q_, \ldots, q_k) \in K_\alpha$ such that F has an absolute minimum at q. A simple geometric argument shows that $q_j \in \Gamma_{i_j}$ for all j. It follows from the condition (H) that each of the open segments (q_j, q_{j+1}) has no common points with K. Now applying Proposition 2.1.3 one gets that $q_1, \ldots, q_k$ are the successive reflection points of a periodic reflecting ray for Γ. Clearly $\gamma \subset \bar{\Omega}$ and $\alpha_\gamma = \alpha$. This proves the assertion. ■

In what follows up to the end of this section we deal with the general case, that is we do not assume the condition (H) to be satisfied. Now it is more convenient to consider the points of tangency of periodic reflecting rays as reflection points (we will do the same in Section 2.4). To this end we need an extension of the billiard ball map B similar to that in Example 2.1.1.

Let $L_q\Gamma$ be the tangent hyperplane to Γ at $q \in \Gamma$. An element $x = (q, v) \in \Gamma \times \mathbb{S}^{n-1}$ will be called *regular* if either $\langle \nu(q), v \rangle > 0$ or $\langle \nu(q), v \rangle = 0$ and there exists a neighbourhood U of q in Γ such that $U \cap L_q\Gamma = \{q\}$. If all K_i are strictly convex, then each point x with $\langle \nu(q), v \rangle \geq 0$ is regular; however in the general case this condition is not enough.

Denote by M'' the set of those regular elements x such that the straight-line ray δ starting at q with direction v has a common point with Γ and if p is the first common point (i.e. the open segment (q, p) has no common points with Γ), then $y = (p, w)$ is a regular element of $\Gamma \times \mathbb{S}^{n-1}$, where $w = v - 2\langle \nu(p), v \rangle \nu(p)$. We set $B(x) = y$, thus extending B to a map $B : M'' \longrightarrow \Gamma \times \mathbb{S}^{n-1}$. We will be interested in the restriction

$$B : M_1 \longrightarrow M_1$$

of B, where

$$M_1 = \cap_{m=0}^{\infty} B^{-m}(M'').$$

More specifically, we will study the periodic points $x \in M_1$ of B, that is the points for which there exists $k \in \mathbb{N}$ with $B^k(x) = x$.

Let $\pi : \Gamma \times \mathbb{S}^{n-1} \longrightarrow \Gamma$ be the *natural projection* and let $\alpha \in \mathcal{A}_k$. A point $x = (q, v) \in M_1$ will be called a periodic point of type α for B if $B^k(x) = x$ and

$$q_j = \pi \circ B^{j-1}(x) \in \Gamma_{i_j} \tag{2.6}$$

for all $j = 1, \ldots, k$. If the segment $[q_j, q_{j+1}]$ is tangent to Γ at q_j, we will say that q_j is a *tangential reflection point* of the corresponding periodic billiard trajectory $\gamma(x)$; otherwise q_j will be called a *proper reflection point*.

In general there could exist distinct periodic points of B having the same type α. One can construct examples involving periodic reflecting rays with parallel corresponding segments arranging suitably obstacles with flat parts of their boundaries (see Figure 2.5). As we see from the next theorem these are in fact the only possibilities to construct such examples.

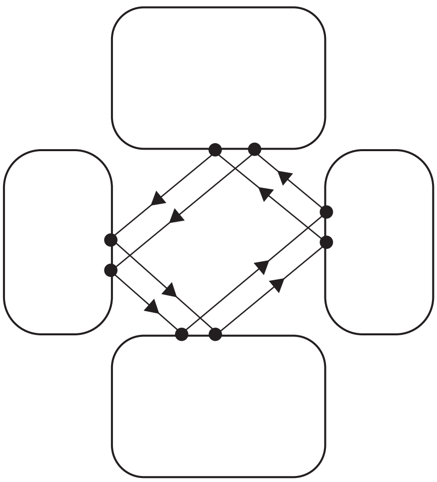

Figure 2.5 Parallel periodic rays.

Theorem 2.2.3: *Let $\alpha \in \mathcal{A}_k$, $k \geq 2$, and suppose there exist two distinct periodic points (q, v) and (p, w) of type α for B and let $q_j = \pi \circ B^{j-1}(q, v)$ and $p_j = \pi \circ B^{j-1}(p, w)$ for all $j = 1, 2, \ldots$. Then $v = w$ and for any $j \geq 1$ the segments $[q_j, q_{j+1}]$ and $[p_j, p_{j+1}]$ are parallel. If q_j is a proper reflection point, then*

$$tq_j + (1-t)p_j \in \Gamma_{i_j} \tag{2.7}$$

for all $t \in [0, 1]$. If all q_j are proper reflection points, then for $t \in (0, 1)$ sufficiently close to 1 the points $(tq + (1-t)p, v)$ are periodic reflection points of type α for B, generating periodic billiard trajectories in $\bar{\Omega}$ with equal lengths and parallel corresponding segments.

In other words for any $\alpha \in \mathcal{A}_k$, there are three possibilities: (i) there are no periodic points of type α for B; (ii) there is exactly one periodic point of type α for B; (iii) the periodic points of type α generate a (continuous or discrete) family of periodic billiard trajectories in $\bar{\Omega}$ of equal lengths and with parallel corresponding segments.

Before proceeding with the proof of the above theorem, we consider some consequences of it.

Corollary 2.2.4: *Let α have the form (2.3) and let Γ_{i_j} be strictly convex for all $j = 1, \ldots, k$. Then there exists at most one periodic point of type α for B.*

For $k \geq 2$ set $a_k = \sharp \mathcal{A}_k$. Clearly

$$a_2 = s(s-1), \quad a_3 = s(s-1)(s-2).$$

On the other hand, it is easy to show that for $k \geq 4$ we have

$$a_k = (s-2)a_{k-1} + (s-1)a_{k-2}.$$

Thus,

$$a_k = (s-1)^k + (-1)^k(s-1)$$

for any $k \geq 2$. Combining the latter with Corollary 2.2.4 and Proposition 2.2.2, we deduce the following.

Corollary 2.2.5: *Let K_i be strictly convex for all $i = 1, \ldots, s$ and let P_k be the number of periodic points of B of period k. Then*

$$P_k \leq a_k = (s-1)^k + (-1)^k(s-1), \tag{2.8}$$

and therefore

$$\limsup_{k \to \infty} \frac{\log P_k}{k} \leq \log(s-1). \tag{2.9}$$

If an addition Ω satisfies the condition (H), then there are equalities in both (2.8) and (2.9).

Here $\log = \log_c$ with an arbitrary constant $c > 1$.

The rest of this section is devoted to the proof of Theorem 2.2.3.

Fix an α of the form (2.3) and consider the function (2.4) defined by (2.5). Set

$$\Gamma_\alpha = \Gamma_{i_1} \times \cdots \times \Gamma_{i_k}.$$

Clearly, K_α is a compact convex subset of $(\mathbb{R}^n)^k$; however, Γ_α is not its boundary. In fact, Γ_α is a 'very thin' subset of ∂K_α.

Lemma 2.2.6: *Let $x = (q, v)$ be a periodic point of type α of B and let the points q_j be defined by (2.6) for all $j = 1, \dots, k$. Then:*

(a) The map $F : K_\alpha \longrightarrow \mathbb{R}$ has a local minimum at $\tilde{q} = (q_1, \dots, q_k)$;

(b) If there exists at least one j such that Γ is strictly convex at q_j and q_{j+1} is a proper reflection point, then F has a strict local minimum at $\tilde{q}$.

Proof: Since the case $k = 2$ is trivial, we assume $k \geq 3$.

There exist C^2-smooth charts

$$\varphi_j : \mathbb{R}^{n-1} \longrightarrow U_j \subset \Gamma_{i_j}$$

such that $\varphi_j(0) = q_j$. Consider the function

$$G : (\mathbb{R}^{n-1})^k \longrightarrow \mathbb{R}$$

defined by

$$G(u_1, \dots, u_k) = F(\varphi_1(u_1), \dots, \varphi_k(u_k)).$$

First we will show that G has a local minimum at 0; this would imply that $F_{|\Gamma_\alpha}$ has a local minimum at $\tilde{q}$.

Let $\varphi_j = (\varphi_j^{(1)}, \dots, \varphi_j^{(n)})$ and let $u = (u_1, \dots, u_k) \in (\mathbb{R}^{n-1})^k$. In what follows we also use the following notation:

$$I_j = \{j-1, j+1\}, \quad u_j = (u_j^{(1)}, \dots, u_j^{(n-1)}) \in \mathbb{R}^{n-1},$$
$$a_{ij} = 1/\|q_i - q_j\|, \quad v_{ij} = a_{ij}(q_i - q_j).$$

Clearly, $a_{ij} = a_{ji} > 0$ and $v_{ji} = -v_{ij} \in \mathbb{S}^{n-1}$.

For $j = 1, \dots, k$, $t = 1, \dots, n-1$ and u sufficiently close to 0 we have

$$\frac{\partial G}{\partial u_j^{(t)}}(u) = \sum_{i \in I_j} \left\langle \frac{\varphi_j(u_j) - \varphi_i(u_i)}{\|\varphi_j(u_j) - \varphi_i(u_i)\|}, \frac{\partial \varphi_j}{\partial u_j^{(t)}}(u_j) \right\rangle. \tag{2.10}$$

By Proposition 2.1.3, 0 is a critical point of G. We will prove that the second fundamental form of G at 0 is non-negative defined. We now need the derivatives

$$\frac{\partial^2 G}{\partial u_j^{(t)} \partial u_i^{(m)}}(0) \tag{2.11}$$

for $i, j = 1, \ldots, k$ and $t, m = 1, \ldots, n-1$. Having fixed j, there are three possibilities for i.

Case 1. $i \notin I_j \cup \{j\}$. Then clearly the derivative (2.11) is 0.

Case 2. $i \in I_j$. In this case (2.10) implies

$$\begin{aligned}\frac{\partial^2 G}{\partial u_j^{(t)} \partial u_i^{(m)}}(0) = &-a_{ij} \left\langle \frac{\partial \varphi_i}{\partial u_j^{(t)}}(0), \frac{\partial \varphi_i}{\partial u_i^{(m)}}(0) \right\rangle \\ &+a_{ji} \left\langle \frac{\partial \varphi_j}{\partial u_j^{(t)}}(0), v_{ji} \right\rangle \left\langle \frac{\partial \varphi_i}{\partial u_i^{(m)}}(0), v_{ji} \right\rangle .\end{aligned}$$

Case 3. $i = j$. Then

$$\begin{aligned}\frac{\partial^2 G}{\partial u_j^{(t)} \partial u_j^{(m)}}(0) = &\sum_{i \in I_j} \left\langle v_{ji}, \frac{\partial^2 \varphi_j}{\partial u_j^{(t)} \partial u_j^{(m)}}(0) \right\rangle \\ &+\sum_{i \in I_j} a_{ji} \left\langle \frac{\partial \varphi_j}{\partial u_j^{(t)}}(0), \frac{\partial \varphi_j}{\partial u_j^{(m)}}(0) \right\rangle \\ &-\sum_{i \in I_j} a_{ji} \left\langle \frac{\partial \varphi_j}{\partial u_j^{(t)}}(0), v_{ji} \right\rangle \left\langle \frac{\partial \varphi_j}{\partial u_j^{(m)}}(0), v_{ji} \right\rangle .\end{aligned}$$

Fix an arbitrary vector $\xi = (\xi_j^{(t)})_{1 \le j \le k, 1 \le t \le n-1} \in (\mathbb{R}^{n-1})^k$. We have to establish that

$$\sigma = \sum_{i,j=1}^{k} \sum_{t,m=1}^{n-1} \frac{\partial^2 G}{\partial u_j^{(t)} \partial u_i^{(m)}}(0)\, \xi_j^{(t)} \xi_i^{(m)} \ge 0.$$

Set

$$\xi_j = (\xi_j^{(1)}, \ldots, \xi_j^{(n-1)}) \in \mathbb{R}^{n-1}$$

and

$$z_j = \sum_{t=1}^{n-1} \xi_j^{(t)} \frac{\partial \varphi_j}{\partial u_j^{(t)}}(0).$$

Notice that for $\nu_j = \nu(q_j)$, there exists $\lambda_j > 0$ such that

$$v_{jj-1} + v_{jj+1} = -\lambda_j v_j.$$

Since the hypersurface $U_j = \varphi_j(\mathbb{R}^{n-1}) \subset \Gamma$ is convex at q_j, the choice of the normal field ν shows that the second fundamental form B_j of U_j at q_j is a non-positive definite. That is,

$$B_j(\xi_j, \xi_j) = \sum_{t,m=1}^{n-1} \left\langle \nu_j, \frac{\partial^2 \varphi_j}{\partial u_j^{(t)} \partial u_j^{(m)}}(0) \right\rangle \xi_j^{(t)} \xi_j^{(m)} \leq 0$$

for all $\xi_j \in \mathbb{R}^{n-1}$.

Using the expressions for the second derivatives of G at 0 in the three possible cases, we get

$$\begin{aligned}
\sigma &= \sum_{j=1}^{k} \sum_{t,m=1}^{n-1} \frac{\partial^2 G}{\partial u_j^{(t)} \partial u_j^{(m)}}(0)\, \xi_j^{(t)} \xi_j^{(m)} \\
&\quad + \sum_{j=1}^{k} \sum_{i \in I_j} \sum_{t,m=1}^{n-1} \frac{\partial^2 G}{\partial u_j^{(t)} \partial u_i^{(m)}}(0)\, \xi_j^{(t)} \xi_i^{(m)} \\
&= \left(-\sum_{j=1}^{k} \lambda_j \sum_{t,m=1}^{n-1} \left\langle \nu_j, \frac{\partial^2 \varphi_j}{\partial u_j^{(t)} \partial u_i^{(m)}}(0) \right\rangle \xi_j^{(t)} \xi_i^{(m)} \right. \\
&\quad + \sum_{j=1}^{k} \sum_{i \in I_j} \sum_{t,m=1}^{n-1} a_{ji} \left\langle \frac{\partial \varphi_j}{\partial u_j^{(t)}}(0), \frac{\partial \varphi_j}{\partial u_j^{(m)}}(0) \right\rangle \xi_j^{(t)} \xi_j^{(m)} \\
&\quad \left. - \sum_{j=1}^{k} \sum_{i \in I_j} \sum_{t,m=1}^{n-1} a_{ji} \left\langle \frac{\partial \varphi_j}{\partial u_j^{(t)}}(0), v_{ji} \right\rangle \left\langle \frac{\partial \varphi_j}{\partial u_j^{(m)}}(0), v_{ji} \right\rangle \xi_j^{(t)} \xi_j^{(m)} \right) \\
&\quad + \left(-\sum_{j=1}^{k} \sum_{i \in I_j} \sum_{t,m=1}^{n-1} a_{ji} \left\langle \frac{\partial \varphi_j}{\partial u_j^{(t)}}(0), \frac{\partial \varphi_i}{\partial u_i^{(m)}}(0) \right\rangle \xi_j^{(t)} \xi_i^{(m)} \right. \\
&\quad \left. + \sum_{j=1}^{k} \sum_{i \in I_j} \sum_{t,m=1}^{n-1} a_{ji} \left\langle \frac{\partial \varphi_j}{\partial u_j^{(t)}}(0), v_{ji} \right\rangle \left\langle \frac{\partial \varphi_i}{\partial u_i^{(m)}}(0), v_{ji} \right\rangle \xi_j^{(t)} \xi_i^{(m)} \right) \\
&= -\sum_{j=1}^{k} \lambda_j B_j(\xi_j, \xi_j) + \sum_{j=1}^{k} \sum_{i \in I_j} a_{ji} \langle z_j, z_j \rangle - \sum_{j=1}^{k} \sum_{i \in I_j} a_{ji} \langle z_j, v_{ji} \rangle^2 \\
&\quad - \sum_{j=1}^{k} \sum_{i \in I_j} a_{ji} \langle z_j, z_i \rangle + \sum_{j=1}^{k} \sum_{i \in I_j} a_{ji} \langle z_j, v_{ji} \rangle \langle z_i, v_{ji} \rangle.
\end{aligned}$$

Since $i \in I_j$ is equivalent to $j \in I_i$, $a_{ji} = a_{ij}$ and $v_{ji} = -v_{ij}$, it now follows that

$$\begin{aligned}\sigma = &-\sum_{j=1}^{k} \lambda_j B_j(\xi_j, \xi_j) + \sum_{j=1}^{k} a_{jj+1} \big(\| z_j \|^2 - \langle z_j, v_{jj+1}\rangle^2 \\ &-\langle z_j, z_{j+1}\rangle + \langle z_j, v_{jj+1}\rangle\langle z_{j+1}, v_{jj+1}\rangle + \| z_{j+1}\|^2 \\ &-\langle z_{j+1}, v_{j+1j}\rangle^2 - \langle z_{j+1}, z_j\rangle + \langle z_{j+1}, v_{j+1j}\rangle\langle z_j, v_{j+1j}\rangle\big) \\ = &-\sum_{j=1}^{k} \lambda_j B_j(\xi_j, \xi_j) + \sum_{j=1}^{k} a_{jj+1}(\| z_j - z_{j+1}\|^2 - \langle z_j - z_{j+1}, v_{jj+1}\rangle^2).\end{aligned}$$

Since $\| v_{jj+1} \|= 1$, we have

$$\langle z_j - z_{j+1}, v_{jj+1}\rangle^2 \leq \|z_j - z_{j+1}\|^2,$$

which yields $\sigma \geq 0$.

Next, assume that $\xi \neq 0$ and $\sigma = 0$, and let there exist j such that Γ is strictly convex at q_j and q_{j+1} is a proper reflection point. It follows from $\sigma = 0$ that $B_j(\xi_j, \xi_j) = 0$ and the vector $z_j - z_{j+1}$ is parallel to v_{jj+1}, that is to the segment $[q_j, q_{j+1}]$. Since B_j is definite, one gets $\xi_j = 0$, that is $z_j = 0$. On the other hand, the vector z_{j+1} lies in the tangent hyperplane to Γ at q_{j+1}; therefore, $[q_j, q_{j+1}]$ is tangent to Γ at q_{j+1}, a contradiction with the assumption that q_{j+1} is a proper reflection point. Thus, the assumptions in (b) imply $\sigma > 0$ for any choice of $\xi \neq 0$.

In this way we have established that G has a local minimum at 0, hence $F_{|\Gamma_\alpha}$ has a local minimum at $\tilde{q}$. Moreover, if the assumptions in (b) are satisfied, then $F_{|\Gamma_\alpha}$ has a strict local minimum at $\tilde{q}$.

Next, for every j, fix a neighbourhood V_j of q_j in K_{i_j} such that $F(\tilde{q}) \leq F(\tilde{p})$ whenever $\tilde{p} \in V \cap \Gamma_\alpha$, where $V = V_1 \times \cdots \times V_k$. Since the points $B^{j-1}(q, v)$ are regular, we may assume that neighbourhoods V_j are chosen so that for each $\tilde{p} \in V$ and each $j = 1, \ldots, k$ the straight line determined by the segment $[p_j, p_{j+1}]$ intersects Γ_{i_j} and $\Gamma_{i_{j+1}}$ at points in V_j and V_{j+1}, respectively. Indeed, if q_j is a tangential reflection point, we may define V_j by

$$V_j = \{p_j \in K_{i_j} : \langle p_j - q_j, \nu(q_j)\rangle > -\epsilon_j\}$$

for some sufficiently small $\epsilon_j > 0$. If q_j is a proper reflection point, consider an open ball D_j in $\mathbb{R}^n$ centred at q_j and having a sufficiently small radius $\epsilon_j > 0$ and set $V_j = K_{i_j} \cap D_j$.

Let $\tilde{p} = (p_1, \ldots, p_k) \in V$. Denote by p'_1 the intersection point of Γ_{i_1} with the segment $[p_1, p_2]$. Then $p'_1 \in V_1$ and the triangle inequality implies

$$F(p_1, \ldots, p_k) \geq F(p'_1, p_2, \ldots, p_k).$$

Next, for the intersection point p'_2 of Γ_{i_2} with the segment $[p'_1, p_2]$ we get

$$F(p'_1, p_2, p_3, \ldots, p_k) \geq F(p'_1, p'_2, p_3, \ldots, p_k),$$

etc. Continuing in this way, for any j we find a point $p'_j \in \Gamma_{i_j} \cap V_j$ such that $F(\tilde{p}) \geq F(\tilde{p}')$ holds for $\tilde{p}' = (p'_1, p'_2, \ldots, p'_k) \in \Gamma_\alpha \cap V$. It now follows from the choice of V that $F(\tilde{p}') \geq F(\tilde{q})$ and therefore $F(\tilde{p}) \geq F(\tilde{q})$. This concludes the proof of part (a).

The proof of (b) follows easily from the above arguments. We leave the details to the reader. ■

Proof of Theorem 2.2.3: Fix α of the form (2.3) and let

$$F : K_\alpha \longrightarrow \mathbb{R}$$

be defined as above. Clearly F is a convex function, that is

$$F(\tilde{q} + (1-t)\tilde{p}) \leq tF(\tilde{q}) + (1-t)F(\tilde{p})$$

for all $t \in [0,1]$ and all $\tilde{q}, \tilde{p} \in K_\alpha$.

Assume that there exist two different periodic points (q, v) and (p, w) of type α for B. Set

$$\tilde{q} = (q_1, \ldots, q_k), \quad \tilde{p} = (p_1, \ldots, p_k).$$

Then $\tilde{q}, \tilde{p} \in K_\alpha$ and Lemma 2.2.6 implies that F has local minima at both these points. For $t \in [0,1]$ set

$$q_j^{(t)} = tq_j + (1-t)p_j, \quad \tilde{q}^{(t)} = (q_1^{(t)}, \ldots, q_k^{(t)}).$$

Clearly $\tilde{q}^{(t)} = t\tilde{q} + (1-t)\tilde{p} \in K_\alpha$.

We will show that $F(\tilde{q}) = F(\tilde{p})$. Suppose that $F(\tilde{q}) > F(\tilde{p})$. Then for every $t \in (0,1)$ we have

$$F(\tilde{q}^{(t)}) = F(t\tilde{q} + (1-t)\tilde{p}) \leq tF(\tilde{q}) + (1-t)F(\tilde{p}) < F(\tilde{q}). \tag{2.12}$$

Since $\tilde{q}^{(t)} \longrightarrow \tilde{q}$ as $t \longrightarrow 1$, we get a contradiction with the fact that F has a local minimum at $\tilde{q}$. Therefore, $F(\tilde{q}) \leq F(\tilde{p})$. Similarly, we obtain $F(\tilde{q}) \geq F(\tilde{p})$, so $F(\tilde{q}) = F(\tilde{p})$.

Combining the latter with (2.12), we get more, namely that

$$F(\tilde{q}^{(t)}) = F(\tilde{q}) = F(\tilde{p}) \tag{2.13}$$

for all t sufficiently close to 0 or 1. Now the convexity of F implies that (2.13) holds for all $t \in [0,1]$.

Let us recall that for $p \neq p'$, $q \neq q'$ and $t \in (0,1)$ the equality

$$\| (tq + (1-t)p) - (tq' + (1-t)p') \| = t \| q - q' \| + (1-t) \| p - p' \|$$

holds if and only if the segments $[p, p']$ and $[q, q']$ are parallel. In our situation this yields that the segments $[q_j, q_{j+1}]$ and $[p_j, p_{j+1}]$ are parallel for every j. In particular, $v = w$.

Choose the neighbourhoods V_j of the points q_j as in the proof of Lemma 2.2.5. There exists $t_0 \in (0,1)$ such that $q_j^{(t)} \in V_j$ for any $t \in (t_0, 1]$. Clearly F has a minimum at $q_j^{(t)}$ in $V = V_1 \times \cdots \times V_k$ for every $t \in (t_0, 1]$. Let q_j be a proper reflection point for some j and assume that $q_j^{(t)} \notin \Gamma_{i_j}$ for some $t \in (t_0, 1)$. Set

$$\tilde{r} = (q_1^{(t)}, \ldots, q_{j-1}^{(t)}, q_j', q_{j+1}^{(t)}, \ldots, q_k^{(t)}),$$

where q_j' is the intersection point of Γ_{i_j} with the segment $[q_j^{(t)}, q_{j+1}^{(t)}]$. Since q_j is a proper reflection point, it follows from the above remark that

$$\| q_{j-1}^{(t)} - q_j^{(t)} \| + \| q_j^{(t)} - q_{j+1}^{(t)} \| > \| q_{j-1}^{(t)} - q_j' \| + \| q_j' - q_{j+1}^{(t)} \| ,$$

therefore $F(\tilde{q}^{(t)}) > F(\tilde{r})$. This is a contradiction with the minimality of $F(\tilde{q}^{(t)})$. Thus, $q_j^{(t)} \in \gamma_{i_j}$ for all $t \in (t_0, 1]$ sufficiently close to 1.

Finally, if all q_j are proper reflection points, the last argument shows that for all $t \in (0,1)$ sufficiently close to 1 the points $(tq + (1-t)p, v)$ are periodic points of type α of B. Clearly, these generate periodic billiard trajectories in Ω with lengths $F(\tilde{q}) = F(\tilde{p})$ and parallel corresponding segments. ■

2.3 The Poincaré map

Throughout this section Ω will be a closed domain in $\mathbb{R}^n$ with smooth boundary $X = \partial\Omega$ and γ will be an ordinary periodic reflecting ray in Ω with successive reflection points $q_1, q_2, \ldots, q_m, q_{m+1} = q_1$ and period (length) $T > 0$. Here we define the linear Poincaré map P_γ of γ and present a useful representation of it. There are different ways to define this map, however all of them are equivalent in the sense that the *spectrum* $\mathrm{spec}(P_\gamma)$ is the same.

Let $\tilde{\gamma}$ be a generalized bicharacteristic of $\Box$ in $T^*(\mathbb{R} \times \Omega)$ such that $\pi_x(\tilde{\gamma}) = \gamma$, where

$$\pi_x : T^*(\mathbb{R} \times \Omega) \longrightarrow \Omega$$

is the composition of the natural projections

$$T^*(\mathbb{R} \times \Omega) \longrightarrow \mathbb{R} \times \Omega \longrightarrow \Omega.$$

Given $\rho \in \tilde{\gamma}$ with $\pi_x(\rho) \neq q_i$ for all $i = 1, \ldots, m$, there exists a small conic neighbourhood V of ρ in $T^*(\Omega^\circ)$ such that for every $(y, \eta) \in V$ the generalized bicharacteristic of $\Box$, parameterized by the time and issued from (y, η) has exactly m reflections at $\partial\Omega$ for all $t \in [0, T]$. The Hamiltonian flow $\mathcal{F}_T$, introduced in Section 1.2, maps V into a conic neighbourhood W of ρ, and we can define the map

$$(d\mathcal{F}_T)(\rho) : T_\rho(T^*(\Omega)) \longrightarrow T_\rho(T^*(\Omega)).$$

Clearly the tangent vectors e to $\tilde{\gamma}$ at ρ and the direction f of the cone axis at ρ are invariant with respect to $(d\mathcal{F}_T)(\rho)$. Let E_ρ be the two-dimensional space generated by e and f, and let

$$\Sigma_\rho = T_\rho(T^*(\Omega))/E_\rho$$

be the corresponding quotient space. The linear map

$$P_\gamma(\rho) = d\mathcal{F}_T(\rho)_{|\Sigma_\rho}$$

will be called the *(linear) Poincaré map* of γ at ρ. Clearly P_γ preserves the natural symplectic structure of Σ_ρ (cf. e.g. [AbM]).

If ρ and μ are two different points on $\tilde{\gamma}$ such that $\pi_x(\rho) \notin \partial\Omega$ and $\pi_x(\mu) \notin \partial\Omega$, then for some $\tau \in \mathbb{R}$ we have $\Phi^\tau(\rho) = \mu$ and therefore

$$\mathcal{F}_\tau \circ \mathcal{F}_T(\sigma) = \mathcal{F}_T \circ \mathcal{F}_\tau(\sigma), \quad \sigma \in V.$$

Thus,

$$d\mathcal{F}_\tau(\rho) \circ d\mathcal{F}_T(\mu) = d\mathcal{F}_T(\mu) \circ d\mathcal{F}_\tau(\rho),$$

so the Poincaré map $P_\gamma(\rho)$ is conjugated to the Poincaré map $P_\gamma(\mu)$. Therefore, the *spectrum* $\mathrm{spec}(P_\gamma)$ of $P_\gamma(\rho)$ is independent of the choice of ρ. We will say that γ is *non-degenerate* if $1 \notin \mathrm{spec}(P_\gamma)$.

Denote by Π_i the hyperplane in $\mathbb{R}^n$ passing through the point q_i and orthogonal to the line $q_i q_{i+1}$ and by ω_i the unit vector determined by the vector $\overrightarrow{q_i, q_{i+1}}$. In what follows we assume for convenience that for $j \equiv i$ (mod m) we have $\Pi_j = \Pi_i$, $q_j = q_i$, etc. We also assume that for each i the hyperplane Π_i is endowed with a linear basis such that $q_i = 0$.

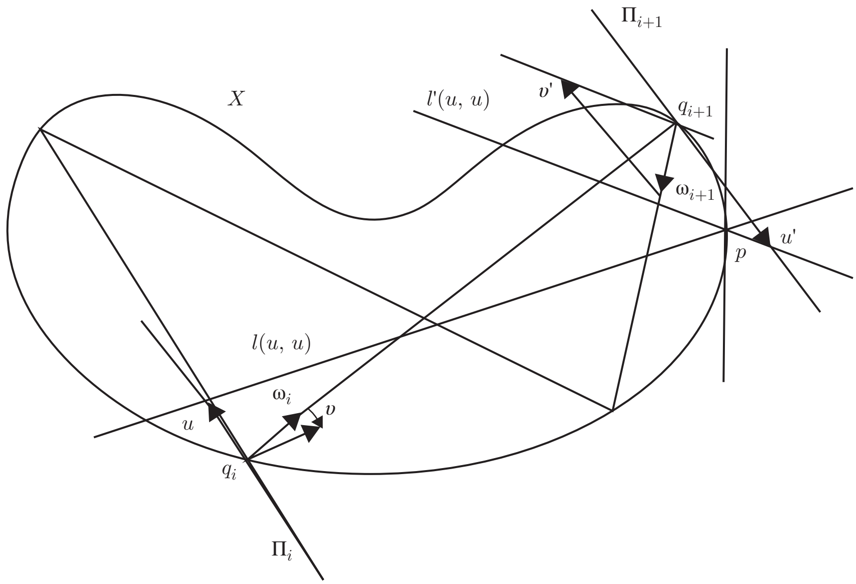

Figure 2.6 The billiard ball map.

For a pair $(u, v) \in \Pi_i \times \Pi_i$ sufficiently close to $(0, 0)$ let $\ell(u, v)$ be the oriented line passing through u and having direction $\omega_i + v$. Here we identify the point v with the vector v. If (u, v) is sufficiently close to $(0, 0)$, then $\ell(u, v)$ intersects transversally X at some point $p = p(u, v)$ close to q_{i+1}. Let $\ell'(u, v)$ be the oriented line symmetric to $\ell(u, v)$ with respect to the tangent hyperplane to X at p, and let u' be the intersection point of $\ell'(u, v)$ with Π_{i+1} (clearly such a point exists for (u, v) close to $(0, 0)$). There is a unique $v' \in \Pi_{i+1}$ such that $\omega_{i+1} + v'$ has the direction of the line $\ell'(u, v)$ (see Figure 2.6). Thus, we obtain a map

$$\Phi_{i+1} : \Pi_i \times \Pi_i \ni (u, v) \mapsto (u', v') \in \Pi_{i+1} \times \Pi_{i+1},$$

defined for (u, v) in a small neighbourhood of $(0, 0)$. The smoothness of this map follows from the smoothness of the billiard ball map. Consider the composition

$$\mathcal{P}_\gamma = \Phi_m \circ \cdots \circ \Phi_1 : \Pi_m \times \Pi_m \longrightarrow \Pi_m \times \Pi_m,$$

and the linear map

$$d\mathcal{P}_\gamma(0, 0) : \Pi_m \times \Pi_m \longrightarrow \Pi_m \times \Pi_m.$$

Let $\rho \in \tilde{\gamma}$ be a fixed point such that $z = \pi_x(\rho)$ lies on the open segment (q_m, q_1). Consider the hyperplane Π_z passing through z and orthogonal to $\overrightarrow{q_m q_1}$. Then we can identify $\Pi_z \times \Pi_z$ with the space $T_\rho(T^*(\Omega))/E_\rho$. Given $(u, v) \in \Pi_m \times \Pi_m$ sufficiently close to $(0, 0)$, consider the oriented line $\ell(u, v)$ passing through u with direction $\omega_m + v$. Let $\ell(u, v)$ intersect Π_z at $\tilde{u}$. Write the unit vector $\omega(u, v)$ with the direction of $\ell(u, v)$ in the form $\omega(u, v) = \omega_m + \tilde{v}$, where $\tilde{v} \in \Pi_z$. Thus, we obtain a map

$$\Phi_z : \Pi_m \times \Pi_m \ni (u, v) \mapsto (\tilde{u}, \tilde{v}) \in \Pi_z \times \Pi_z,$$

defined for (u, v) sufficiently close to $(0, 0)$ such that $\Phi_z(0, 0) = (0, 0)$.

For $(\tilde{u}, \tilde{v}) \in \Pi_z \times \Pi_z$ let $t(\tilde{u}, \tilde{v})$ be the minimal positive number such that

$$\mathcal{F}_{t(\tilde{u}, \tilde{v})}(\tilde{u}, \omega_m + \tilde{v}) = (p, \omega_m + q)$$

with $(p, q) \in \Pi_z \times \Pi_z$. Setting

$$Q_z(\tilde{u}, \tilde{v}) = (p, q) \in \Pi_z \times \Pi_z,$$

we obtain a map defined in a small neighbourhood of $(0, 0)$ in $\Pi_z \times \Pi_z$. Clearly, $t(0, 0) = T$, $Q_z(0, 0) = (0, 0)$, and for (u, v) close to $(0, 0)$ we have

$$(\Phi_z \circ \mathcal{P}_\gamma)(u, v) = (Q_z \circ \Phi_z)(u, v).$$

By using the local smoothness of $\mathcal{F}_t(\sigma)$ with respect to t and σ we get

$$d\Phi_z(0, 0) \circ d\mathcal{P}_\gamma(0, 0) = (d\mathcal{F}_T)_{|\Pi_z \times \Pi_z}(0, 0) \circ d\Phi_z(0, 0).$$

Therefore, $d\mathcal{P}_\gamma(0, 0)$ is conjugated to the Poincaré map P_γ of γ.

In what follows very often we will use the notation P_γ for $d\mathcal{P}_\gamma(0,0)$ and call it the *Poincaré map* of γ.

Next, we proceed to describe a useful representation of $P_\gamma = d\mathcal{P}_\gamma(0,0)$. We need some additional notation. Set

$$\lambda_i = \| q_{i-1} - q_i \| \ , \tag{2.14}$$

and denote by α_i the *tangent hyperplane* to X at q_i, by σ_i the *symmetry* with respect to α_i, and by Π'_i the hyperplane passing through $q_i$and orthogonal to ω_{i-1}. Clearly Π'_i is parallel to Π_{i-1}; see Figure 2.6. Moreover,

$$\sigma_i(\omega_{i-1}) = \omega_i, \sigma_i(\Pi'_i) = \Pi_i.$$

Choose a continuous *unit normal field* $\nu_i(q)$ to X for $q \in X$ near q_i such that

$$\langle \nu_i(q_i), \omega_i \rangle > 0.$$

For $u \in \Pi'_i$ close to 0 (i.e. to q_i) denote by $\ell(u,0)$ the line through u orthogonal to Π'_i. The intersection points of $\ell(u,0)$ with X and α_i will be denoted by $f_i(u)$ and $\pi_i(u)$, respectively (we choose $f_i(u)$ close to q_i). Then

$$\pi_i : \Pi'_i \longrightarrow \alpha_i$$

is the *natural projection* along the vector ω_{i-1}, while f_i is a local diffeomorphism

$$f_i : (\Pi'_i, q_i) \longrightarrow (X, q_i),$$

which can be considered as a parameterization of X about q_i.

The *second fundamental form* $S(\xi,\eta)$ of X at q_i is defined for $\xi, \eta \in \alpha_i$ by

$$S(\xi,\eta) = \langle G_i(\xi), \eta \rangle,$$

where

$$G_i = d\nu_i(q_i) : \alpha_i \longrightarrow \alpha_i.$$

Since S is symmetric and bilinear (cf. e.g. [GKM]), there exists a unique symmetric linear map

$$\tilde{\psi}_i : \Pi_i \longrightarrow \Pi_i \tag{2.15}$$

such that

$$\langle \tilde{\psi}_i \sigma_i(\xi), \sigma_i(\eta) \rangle = -2\langle \omega_{i-1}, \nu_i(q_i) \rangle \ \langle G_i(\pi_i(\xi)), \pi_i(\eta) \rangle \tag{2.16}$$

for all $\xi, \eta \in \Pi'_i$ (Figure 2.7).

Since $\sigma_i(\Pi'_i) = \Pi_i$ with respect to the linear basis fixed in Π_i and that in Π'_i obtained by identifying the latter with Π_{i-1} using the translation along the line $q_{i-1}q_i$ we may regard $(\sigma_i)_{|\Pi'_i}$ as a real symmetric $(n-1) \times (n-1)$ matrix. For the sake of brevity we will denote this matrix again by σ_i. Next, we write the linear maps

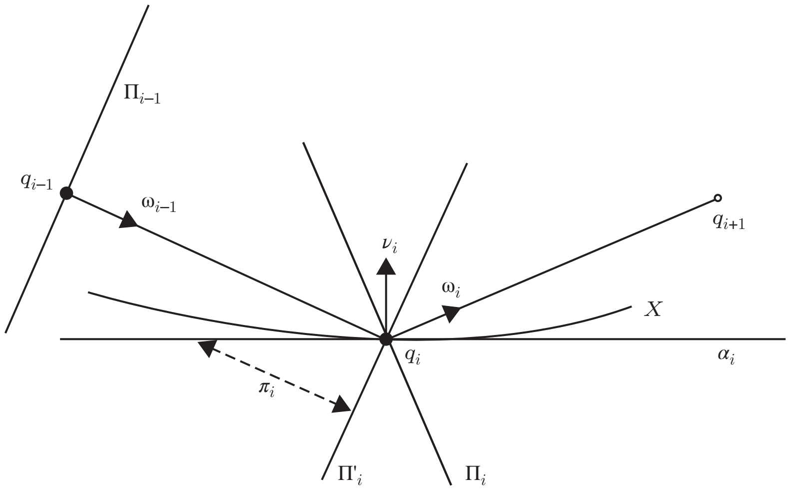

Figure 2.7 The reflection operator.

between products of the type $\Pi_j \times \Pi_j$ as $2(n-1) \times 2(n-1)$ block matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, D are real $(n-1) \times (n-1)$ matrices. By I we denote the *identity matrix* on any Π_j.

After these preparations, fix an arbitrary $i = 1, \dots, m$ and write the map Φ_i in the form

$$\Phi_i = \Phi_i^{(r)} \circ \Phi_i^{(t)},$$

where

$$\Phi_i^{(t)} : \Pi_{i-1} \times \Pi_{i-1} \longrightarrow \Pi'_i \times \Pi'_i, \quad \Phi_i^{(r)} : \Pi'_i \times \Pi'_i \longrightarrow \Pi_i \times \Pi_i$$

are defined in small neighbourhoods of $(0, 0)$ as follows. For $u, v \in \Pi_{i-1}$ let $\ell(u, v)$ be the oriented line defined as above. Denote by u' the intersection point of $\ell(u, v)$ with Π'_i and set $\Phi_i^{(t)}(u, v) = (u', v')$, where $v' \in \Pi'_i$ is such that the vector $v' + \omega_{i-1}$ has the direction of $\ell(u, v)$. Finally, set

$$\Phi_i^{(r)} = \Phi_i \circ (\Phi_i^{(t)})^{-1}.$$

Clearly $\Phi_i^{(t)}$ is a linear map and

$$\Phi_i^{(t)} = \begin{pmatrix} I & \lambda_i I \\ 0 & I \end{pmatrix}. \tag{2.17}$$

Next, write the linear map $R_i = d\Phi_i^{(r)}(0,0)$ in the form

$$R_i = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We will now determine the matrices A, B, C, D in terms of λ_i, σ_i and $\tilde{\psi}_i$.

Take $u = 0$ and $v \in \Pi'_i$ close to 0. Then clearly $\Phi_i^{(r)}(0, v) = (0, \sigma_i(v))$, which yields

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ \sigma_i(v) \end{pmatrix},$$

and therefore $B = 0$, $D = \sigma_i$.

Now take $u \in \Pi'_i$ close to 0 and $v = 0$ and set

$$(u', v') = \Phi_i^{(r)}(u, 0) \in \Pi_i \times \Pi_i.$$

It follows from the definition of $\Phi_i^{(r)}$ that

$$u' = f_i(u) + t\omega, \quad v' = \omega' - \langle \omega', \omega_i \rangle \omega_i \tag{2.18}$$

for some $t \in \mathbb{R}$, where ω' is the vector symmetric to ω_{i-1} with respect to the tangent hyperplane to X at $f_i(u)$, that is

$$\omega' = \omega_{i-1} - 2\langle \omega_{i-1}, \nu_i(f_i(u)) \rangle \; \nu_i(f_i(u)).$$

Setting $\nu_i = \nu_i(q_i)$, we have

$$\nu_i(f_i(u)) = \nu_i + G_i(\pi_i(u)) + O(\|u\|^2),$$

and, taking into account that

$$\sigma_i(\omega_{i-1}) = \omega_i = \omega_{i-1} - 2\langle \omega_{i-1}, \nu_i \rangle \; \nu_i,$$

we find

$$\omega' = \omega_i - 2\langle \omega_{i-1}, G_i(\pi_i(u)) \rangle \; \nu_i - 2\langle \omega_{i-1}, \nu_i \rangle \; G_i(\pi_i(u)) + O(\|u\|^2). \tag{2.19}$$

Since $\langle \omega_{i-1}, \nu_i \rangle = \langle \omega_i, \nu_i \rangle$ and $G_i(\pi_i(u)) \in \alpha_i$, (2.19) implies

$$\begin{aligned} \langle \omega', \omega_i \rangle &= 1 - 2\langle \omega_{i-1}, G_i(\pi_i(u)) \rangle \; \langle \nu_i, \omega_i \rangle - 2\langle \omega_{i-1}, \nu_i \rangle \, \langle G_i(\pi_i(u)), \omega_i \rangle + O(\|u\|^2) \\ &= 1 + O(\|u\|^2). \end{aligned}$$

It now follows from (2.18) and (2.19) that

$$v' = -2\langle \omega_i, G_i(\pi_i(u)) \rangle \; \nu_i + 2\langle \omega_i, \nu_i \rangle \; G_i(\pi_i(u)) + O(\|u\|^2). \tag{2.20}$$

To compute u', first combine (2.18)–(2.20) to get

$$u' = f_i(u) + t(\omega_i - v') + O(\|u\|^2) = \pi_i(u) + t(\omega_i - v') + O(\|u\|^2). \tag{2.21}$$

Since $\langle u', \omega_i\rangle = \langle v', \omega_i\rangle = 0$, (2.21) yields

$$t = -\langle \pi_i(u), \omega_i\rangle + O(\|u\|^2),$$

so

$$\begin{aligned} u' &= \pi_i(u) - \langle \pi_i(u), \omega_i\rangle \; \omega_i + O(\|u\|^2) \\ &= \sigma_i(\pi_i(u) - \langle \pi_i(u), \omega_i\rangle \; \omega_{i-1}) + O(\|u\|^2). \end{aligned}$$

On the other hand, it is easily seen that

$$u = \pi_i(u) - \langle \pi_i(u), \omega_i\rangle \omega_{i-1}.$$

Combining this with the latter expression for u' gives

$$u' = \sigma_i(u) + O(\|u\|^2). \tag{2.22}$$

Now it follows from (2.22) and (2.20) that the components A and C of the matrix $R = \Phi_i^{(r)}(0,0)$ have the form $A = \sigma_i$ and

$$Cu = -2\langle \omega_i, G_i(\pi_i(u))\rangle \; \nu_i + 2\langle \omega_i, \nu_i\rangle \; G_i(\pi_i(u)). \tag{2.23}$$

We will show now that $C = \tilde{\psi}_i\sigma_i$. Indeed, since

$$\sigma_i(v) = \pi_i(v) - \langle \pi_i(v), \omega_i\rangle \; \omega_i$$

for $v \in \Pi_i'$ and $\langle Cu, \omega_i\rangle = 0$, it follows from (2.23) that

$$\begin{aligned} \langle Cu, \sigma_i(v)\rangle &= \langle Cu, \pi_i(v)\rangle = 2\langle \omega_i, \nu_i\rangle \; \langle G_i(\pi_i(u)), \pi_i(v)\rangle \\ &= -2\langle \omega_{i-1}, \nu_i\rangle \; \langle G_i(\pi_i(u)), \pi_i(v)\rangle. \end{aligned}$$

Combining the latter with (2.16), one gets

$$\langle (C - \tilde{\psi}_i\sigma_i)(u), \sigma_i(v)\rangle = 0$$

for all $u, v \in \Pi_i'$. Therefore, $C = \tilde{\psi}_i\sigma_i$ which shows that

$$\Phi_i^{(r)}(0,0) = \begin{pmatrix} \sigma_i & 0 \\ \tilde{\psi}_i\sigma_i & \sigma_i \end{pmatrix}.$$

Finally, the latter and (2.17) imply

$$\begin{aligned} d\Phi_i(0,0) &= \begin{pmatrix} \sigma_i & 0 \\ \tilde{\psi}_i\sigma_i & \sigma_i \end{pmatrix} \begin{pmatrix} I & \lambda_i I \\ 0 & I \end{pmatrix} = \begin{pmatrix} \sigma_i & \lambda_i\sigma_i \\ \tilde{\psi}_i\sigma_i & (I + \lambda_i\tilde{\psi}_i)\sigma_i \end{pmatrix} \\ &= \begin{pmatrix} I & \lambda_i I \\ \tilde{\psi}_i & I + \lambda_i\tilde{\psi}_i \end{pmatrix} \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}. \end{aligned}$$

Next, we identify the hyperplanes Π_{i-1} and Π'_i using the translation along the line $q_{i-1}q_i$ (which is orthogonal to both hyperplanes). Then we can write $\sigma_i(\Pi_{i-1}) = \Pi_i$ and for the composition

$$s_i = \sigma_i \circ \sigma_{i-1} \circ \cdots \circ \sigma_1 \tag{2.24}$$

one has $s_i(\Pi_m) = \Pi_i$. Consider the symmetric linear map

$$\psi_i = s_i^{-1}\tilde{\psi}_i s_i : \Pi_m \longrightarrow \Pi_m. \tag{2.25}$$

Now we can view the matrix

$$\begin{pmatrix} I & \lambda_i I \\ \psi_i & I + \lambda_i \psi_i \end{pmatrix} = \begin{pmatrix} s_i^{-1} & 0 \\ 0 & s_i^{-1} \end{pmatrix} \begin{pmatrix} I & \lambda_i I \\ \tilde{\psi}_i & I + \lambda_i \tilde{\psi}_i \end{pmatrix} \begin{pmatrix} s_i & 0 \\ 0 & s_i \end{pmatrix}$$

as a linear map $\Pi_m \times \Pi_m \longrightarrow \Pi_m \times \Pi_m$. Combining this with the representation of $d\Phi_i(0,0)$, we obtain the following.

Theorem 2.3.1: *Under the assumptions and conventions above, the Poincaré map*

$$P_\gamma : \Pi_m \times \Pi_m \longrightarrow \Pi_m \times \Pi_m$$

is a linear symplectic map that has the following matrix representation

$$P_\gamma = \begin{pmatrix} s_m & 0 \\ 0 & s_m \end{pmatrix} \begin{pmatrix} I & \lambda_m I \\ \psi_m & I + \lambda_m \psi_m \end{pmatrix} \cdots \begin{pmatrix} I & \lambda_1 I \\ \psi_1 & I + \lambda_1 \psi_1 \end{pmatrix}, \tag{2.26}$$

where s_m, ψ_j and λ_j are given by (2.24), (2.25), (2.16) and (2.14).

Let us recall that $X = \partial\Omega$ is called *strictly convex* (convex) at q_i with respect to the unit normal field $\nu_i(q)$ provided the linear operator $G_i(q_i)$, defined above, is positive definite (resp. non-negatively semi-definite). It follows from (2.16) that this condition is equivalent to the fact that the linear symmetric map $\tilde{\psi}_i$ is positive definite (resp. non-negatively semi-definite).

Proposition 2.3.2: *Let Ω and γ be as in the beginning of this section, and assume that $X = \partial\Omega$ is strictly convex at q_i with respect to the normal field $\nu_i(q)$ for every $i = 1, \ldots, m$. Then the Poincaré map P_γ is hyperbolic, that is* $\text{spec}(P_\gamma)$ *has no common points with the unit circle.*

Proof: For any $k = 1, \ldots, m$ denote by $\mathcal{M}_k$ the space of all linear symmetric maps $M : \Pi_k \longrightarrow \Pi_k$. Recall that $\Pi_0 = \Pi_m$ according to our notation above.

Let $M_0 \in \mathcal{M}_0$ be non-negatively semi-definite (we will denote this by $M_0 \geq 0$). Consider the linear subspace

$$L_0 = \{(u, M_0 u) : u \in \Pi_0\}$$

of $\Pi_0 \times \Pi_0$. Then L_0 is a Lagrangian subspace of $\Pi_0 \times \Pi_0$ with respect to the natural symplectic structure of that space, and $d\Phi_1(0,0)(L_0)$ coincides with the linear subspace

$$L_1 = \{\sigma_1(I + \lambda_1 M_0)u, \sigma_1((I + \lambda_1\psi_1)M_0 + \psi_1)u : u \in \Pi_0\}$$

of $\Pi_1 \times \Pi_1$. It is convenient to introduce the operators

$$\mathcal{A}_i : \mathcal{M}_{i-1} \longrightarrow \mathcal{M}_i \tag{2.27}$$

defined by

$$\mathcal{A}_i(M) = \sigma_i M(I + \lambda_i M)^{-1} + \tilde{\psi}_i. \tag{2.28}$$

Then for $M_1 = \mathcal{A}_1(M_0)$ we have

$$L_1 = \{(v, M_1 v) : v \in \Pi_1\}.$$

Define inductively

$$M_k = \mathcal{A}_k(M_{k-1}), \quad L_k = \{(u, M_k u) : u \in \Pi_k\}.$$

Then $M_k \in \mathcal{M}_k$ and L_k is a linear subspace of $\Pi_k \times \Pi_k$.

Choose an arbitrary $\epsilon > 0$ such that $\tilde{\psi}_i \geq \epsilon I$ for every $i = 1, \ldots, m$ and denote by $\mathcal{M}_k(\epsilon)$ the subspace of $\mathcal{M}_k$ consisting of all $M \in \mathcal{M}_k$ such that $M \geq \epsilon I$. Notice that $\mathcal{A}_k(\mathcal{M}_{k-1}(\epsilon)) \subset \mathcal{M}_k(\epsilon)$ and for any $A, B \in \mathcal{M}_{k-1}(\epsilon)$ we have

$$\mathcal{A}_k(A) - \mathcal{A}_k(B) = \sigma_k((I + \lambda_k A)^{-1}(A - B)(I + \lambda_k B)^{-1})\sigma_k.$$

Therefore,

$$\| \mathcal{A}_k(A) - \mathcal{A}_k(B) \| \leq (1 + \epsilon\lambda_k)^{-2} \| A - B \| \leq \frac{\| A - B \|}{(1 + \epsilon\lambda)^2},$$

where $\lambda = \min \lambda_k$. This shows that for any k the map $\mathcal{A}_k$ is a contraction from $\mathcal{M}_{k-1}(\epsilon)$ to $\mathcal{M}_k(\epsilon)$. Then the map

$$\mathcal{A} = \mathcal{A}_m \circ \mathcal{A}_{m-1} \circ \cdots \circ \mathcal{A}_1$$

is a contraction from $\mathcal{M}_0(\epsilon)$ into $\mathcal{M}_0(\epsilon)$. Consequently, there exists a (unique) fixed point $M_0 \in \mathcal{M}_0(\epsilon)$ of $\mathcal{A}$. Now taking into account (2.26) and (2.27), we see that for any $u \in \Pi_m = \Pi_0$ we have

$$P_\gamma \begin{pmatrix} u \\ M_0 u \end{pmatrix} = \begin{pmatrix} Su \\ M_0 Su \end{pmatrix},$$

where the linear map $S : \Pi_m \longrightarrow \Pi_m$ is defined by

$$S = \sigma_m(I + \lambda_m \mathcal{A}'_{m-1}(M_0)) \circ \sigma_{m-1}(I + \lambda_{m-1}\mathcal{A}'_{m-2}(M_0)) \circ \cdots$$
$$\circ \sigma_2(I + \lambda_2 \mathcal{A}'_1(M_0)) \circ \sigma_1(I + \lambda_1 M_0),$$

and $\mathcal{A}'_k = \mathcal{A}_k \circ \mathcal{A}_{k-1} \circ \cdots \circ \mathcal{A}_1$. Moreover, it follows from the expression for S that

$$\|Sx\| \geq \prod_{i=1}^{m} (1 + \lambda_i \epsilon) \, \|x\|$$

for any $x \in \Pi_m$. Consequently,

$$\text{spec}(S) \subset \{z \in \mathbb{C} : |z| > 1\}.$$

The eigenvalues of S are clearly eigenvalues of P_γ. Hence P_γ has $n-1$ eigenvalues z_j with $|z_j| > 1$. Since P_γ is symplectic, $1/z_j$ are eigenvalues, too. This proves the proposition. ■

Corollary 2.3.3: *Let $\Omega = \mathbb{R}^n \setminus (K_1 \cup K_2)$, where K_1 and K_2 are compact disjoint strictly convex domains in $\mathbb{R}^n$ with smooth boundaries ∂K_1 and ∂K_2. Let γ be the unique periodic reflecting ray in Ω with two reflection points $q_1 \in K_1$ and $q_2 \in K_2$. Then*

$$\text{spec}(P_\gamma) \subset (0, 1) \cup (1, \infty).$$

Proof: We use the argument from the proof of Proposition 2.3.2. In the present case, we have $\lambda_1 = \lambda_2 = \lambda$, and moreover Π_1 and Π_2 are parallel and can be identified. Thus,

$$S = \sigma_2(I + \lambda \mathcal{A}'_1(M_0)) \circ \sigma_1(I + \lambda M_0),$$

and setting $M_2 = M_0(I + \lambda M_0)^{-1} + \psi_1$, we obtain

$$(I + \lambda M_2)^{-1/2} S (I + \lambda M_2)^{1/2} = (I + \lambda M_2)^{-1/2}(I + \lambda M_0)(I + \lambda M_2)^{1/2}.$$

Therefore, the eigenvalues of S are real and greater than 1 which proves the assertion. ■

2.4 Scattering rays

Let Ω be a closed connected domain in $\mathbb{R}^n$, $n \geq 2$, with smooth boundary $X = \partial\Omega$ and bounded complement. Set

$$K = \overline{\mathbb{R}^n \setminus \Omega}. \tag{2.29}$$

Clearly K is a compact domain with smooth boundary $\partial K = X$ and the complement of Ω is $K \setminus X$. Let ω and θ be two fixed unit vectors in $\mathbb{R}^n$.

Definition 2.4.1: Let γ be a curve in Ω of the form $\gamma = \cup_{j=0}^{k} \ell_i$ for some $k \geq 1$, where $\ell_i = [x_i, x_{i+1}]$ are finite segments for $i = 1, \ldots, k-1$, $x_i \in X$ for all i, and ℓ_0 (resp. ℓ_k) is the infinite segment starting at x_0 (resp. at x_k) and having direction $-\omega$ (resp. θ). The curve γ is called a reflecting (ω, θ)-ray in Ω if for every

$i = 0, 1, \dots, k-1$ the segments ℓ_i and ℓ_{i+1} satisfy the law of reflection at x_{i+1} with respect to X. For such γ the points $x_1, \dots, x_k$ will be called reflection points of γ.

Clearly, for every generalized (ω, θ)-geodesic $c : \mathbb{R} \longrightarrow \Omega$, which has no gliding segments on $\partial\Omega$ and only finitely many reflection points, the curve $\gamma = \mathrm{Image}\,(c)$ is a reflecting (ω, θ)-ray in Ω (cf. Section 1.2). It is easy to construct examples showing that the converse is not true.

By a *scattering ray* in Ω we mean a reflecting (ω, θ)-ray for some unit vectors ω and θ. Such a ray γ will be called *symmetric* if some segment of γ is orthogonal to X at some of its end points; otherwise γ will be called *non-symmetric*. Clearly, a symmetric reflecting (ω, θ)-ray may exist only if $\theta = -\omega$, and for such a ray γ we have $k = 2m + 1$ and $\ell_{m-i} = \ell_{m+i-1}$ for all $i = 0, 1, \dots, m$. A scattering ray without segments tangent to X will be called *ordinary*.

Next, we define two important notions related to a scattering ray. Fix an arbitrary open ball U_0 with radius $a > 0$ containing K. For any $\xi \in \mathbb{S}^{n-1}$ denote by Z_ξ the hyperplane orthogonal to ξ and tangent to U_0 such that ξ is pointing into the interior of the open half-space H_ξ with boundary Z_ξ and containing U_0.

Let $\pi_\xi : \mathbb{R}^n \longrightarrow Z_\xi$ be the orthogonal projection. For a reflecting (ω, θ)-ray γ in Ω with successive reflection points $x_1, \dots, x_k$ the *sojourn time* T_γ of γ is defined by

$$T_\gamma = \| \pi_\omega(x_1) - x_1 \| + \sum_{i=1}^{k-1} \| x_i - x_{i+1} \| + \| x_k - \pi_{-\theta}(x_k) \| - 2a. \tag{2.30}$$

Clearly, $T_\gamma + 2a$ coincides with the length of the part of γ that lies in $H_\omega \cap H_{-\theta}$ (see Figure 2.8).

It is easy to see that T_γ does not depend on the choice of the ball U_0. Indeed, we have

$$\| \pi_\omega(x_1) - x_1 \| = a + \langle \omega, x_1 \rangle, \quad \| x_k - \pi_{-\theta}(x_k) \| = a - \langle \theta, x_k \rangle,$$

and (2.30) implies

$$T_\gamma = \langle \omega, x_1 \rangle + \sum_{i=1}^{k-1} \| x_i - x_{i+1} \| - \langle \theta, x_k \rangle. \tag{2.31}$$

This proves that T_γ does not depend on the choice of U_0.

Let γ be a reflecting (ω, θ)-ray as above. Set $u_\gamma = \pi_\omega(x_i)$ and assume that γ is ordinary, that is it has no segments tangent to $X = \partial\Omega$. Then there exists a neighbourhood $W = W_\gamma$ of u_γ in Z_ω such that for every $u \in W$ there are unique $\theta(u) \in \mathbb{S}^{n-1}$ and points $x_1(u), \dots, x_k(u) \in X$ which are the successive reflection points of a reflecting $(\omega, \theta(u))$-ray in Ω with $\pi_\omega(x_1(u)) = u$. We set $J_\gamma(u) = \theta(u)$, thus obtaining a map

$$J_\gamma : W_\gamma \longrightarrow \mathbb{S}^{n-1}.$$

It follows immediately from the smoothness of the billiard ball map related to an appropriately chosen domain $\Omega' \subset \Omega \cap H_\omega \cap H_{-\theta}$, that the map J_γ is smooth, too. It is an easy exercise to check the latter fact directly.

A scattering ray γ will be called *non-degenerate* if $\mathrm{rank}(dJ_\gamma) = n - 1$.

Next, applying Theorem 2.3.1, we will obtain a matrix representation for $dJ_\gamma(u_\gamma)$. Set $m = k + 2$, $q_i = x_i$ for $i = 1, \dots, k$, $q_0 = \pi_\omega(q_1)$, $q_{k+1} = \pi_{-\theta}(q_k)$,

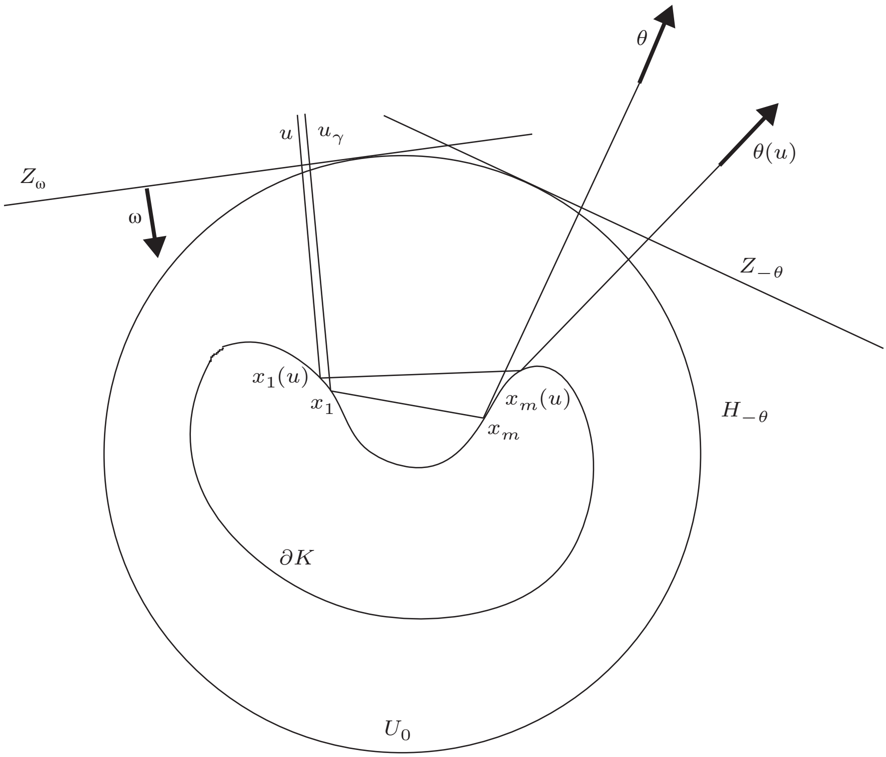

Figure 2.8 The map J_γ.

$\lambda_i = \| q_{i-1} - q_i \|$, $\Pi_0 = Z_\omega$, $\Pi_{k+1} = Z_{-\theta}$. For $i = 1, \ldots, k$ define Π_i, σ_i, $\tilde{\psi}_i$ as in Section 2.3. We assume again that in every Π_i a linear basis is fixed with $q_i = 0$. Define the maps

$$\Phi_{i+1} : \Pi_i \times \Pi_i \longrightarrow \Pi_{i+1} \times \Pi_{i+1},$$

$i = 0, 1, \ldots, k$, as in Section 2.3. Then by the same argument one gets

$$d\Phi_i(0,0) = \begin{pmatrix} \sigma_i & \lambda_i \sigma_i \\ \tilde{\psi}_i \sigma_i & \sigma_i + \lambda_i \tilde{\psi}_i \sigma_i \end{pmatrix}, \quad i = 0, 1, \ldots, k+1.$$

On the other hand,

$$dJ_\gamma(q_0)u = \mathrm{pr}_2 \left(d\Phi_{k+1}(0,0) \circ \cdots \circ d\Phi_1(0,0) \begin{pmatrix} u \\ 0 \end{pmatrix} \right)$$

for $u \in \Pi_0$, where $\mathrm{pr}_2 \begin{pmatrix} u \\ v \end{pmatrix} = v$. Therefore,

$$dJ_\gamma(q_0)u = \mathrm{pr}_2 \begin{pmatrix} \sigma_k & \lambda_k \sigma_k \\ \sigma_k \tilde{\psi}_k & \sigma_k + \lambda_k \sigma_k \tilde{\psi}_k \end{pmatrix} \times \cdots \times \begin{pmatrix} \sigma_1 & \lambda_1 \sigma_1 \\ \tilde{\psi}_1 \sigma_1 & \sigma_1 + \lambda_1 \tilde{\psi}_1 \sigma_1 \end{pmatrix} \begin{pmatrix} u \\ 0 \end{pmatrix}. \tag{2.32}$$

The next proposition treats a special case of scattering rays.

Proposition 2.4.2: *Under the notation and conventions above, assume in addition that for every $i = 1, \dots, k$, $X = \partial K$ is (strictly) convex at $q_i = x_i$ with respect to the unit normal ν_i pointing into the interior of Ω. Then for every $u \in \Pi_0 = Z_\omega$ we have*

$$dJ_\gamma(q_0)u = M_k\sigma_k(I + \lambda_k M_{k-1})\sigma_{k-1}(I + \lambda_{k-1}M_{k-2})\cdots\sigma_2(I + \lambda_2 M_1)\sigma_1 u, \tag{2.33}$$

where

$$M_i : \Pi_i \longrightarrow \Pi_i, \quad i = 1, \dots, k, \tag{2.34}$$

are non-negative semi-definite (resp. positive definite) symmetric linear maps defined inductively by

$$M_1 = \tilde{\psi}_1, \quad M_i = \sigma_i M_{i-1}(I + \lambda_i M_{i-1})^{-1}\sigma_i + \tilde{\psi}_i, \tag{2.35}$$

$i = 2, 3, \dots, k + 1$. *In particular,* det $dJ_\gamma(q_0) \neq 0$.

Proof: Since ∂K is (strictly) convex at q_i, it follows from the definitions of $\tilde{\psi}_i$ (cf. Section 2.3) that they are non-negatively semi-definite (resp. positive definite) symmetric linear maps. Now we can use (2.34) to define the maps M_i inductively; the definition is correct and $M_i \geq 0$ (resp. $M_i > 0$) for all $i = 1, \dots, k$. Set

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix} = d\Phi_i(0,0) \circ \cdots \circ d\Phi_1(0,0) \begin{pmatrix} u \\ 0 \end{pmatrix}.$$

Clearly, $u_1 = \sigma_1 u$, $v_1 = \tilde{\psi}_1\sigma_1 u$. We will prove by induction that

$$u_i = \sigma_i(I + \lambda_i M_{i-1})u_{i-1}, \;\; v_i = M_i u_i \tag{2.36}$$

for every $i = 2, 3, \dots, k + 1$. From this the equality (2.33) follows immediately. Assume that $v_{i-1} = M_{i-1}u_{i-1}$ for some $i > 1$. Then

$$\begin{aligned}
\begin{pmatrix} u_i \\ v_i \end{pmatrix} &= d\Phi_i(0,0)\begin{pmatrix} u_{i-1} \\ v_{i-1} \end{pmatrix} = \begin{pmatrix} I & \lambda_i I \\ \tilde{\psi}_i & I + \lambda_i\tilde{\psi}_i \end{pmatrix} = \begin{pmatrix} \sigma_i u_{i-1} \\ \sigma_i M_{i-1}u_{i-1} \end{pmatrix} \\
&= \begin{pmatrix} \sigma_i(I + \lambda_i M_{i-1})u_{i-1} \\ (\tilde{\psi}_i\sigma_i + \sigma_i M_{i-1} + \lambda_i\tilde{\psi}_i\sigma_i M_{i-1})u_{i-1} \end{pmatrix}.
\end{aligned}$$

Thus, $u_i = \sigma_i(I + \lambda_i M_{i-1})u_{i-1}$ and

$$\begin{aligned}
v_i &= (\sigma_i M_{i-1} + \tilde{\psi}_i\sigma_i(I + \lambda_i M_{i-1}))u_{i-1} \\
&= (\sigma_i M_{i-1}(I + \lambda_i M_{i-1})^{-1}\sigma_i + \tilde{\psi}_i)\sigma_i(I + \lambda_i M_{i-1})u_{i-1} = M_i u_i.
\end{aligned}$$

Therefore, (2.36) holds which proves the assertion. ∎

From now on until the end of this section we will assume that K has the form (2.2), where $s \geq 2$ and K_i are disjoint strictly convex compact domains in $\mathbb{R}^n$ with smooth boundaries $\partial K_i = \Gamma_i$. As before ω and θ will be two fixed unit vectors and U_0 an open ball containing K. We will also use the notation Z_ξ, H_ξ and π_ξ introduced above.

For any $u \in Z_\omega$ consider the *billiard semi-trajectory*

$$\gamma(u) = \{S_t(u) : t \geq 0\}$$

such that $S_0(u) = u$ and $N_0(u) = \omega$, where $N_t(u)$ is the velocity vector of the trajectory $S_t(u)$ at time t, that is the unit vector determining the direction of the trajectory at that point. For $y = S_t(u) \in X$ we set

$$N_{-t}(u) = \lim_{\epsilon \to 0} N_{t-\epsilon}(u), \quad \epsilon > 0,$$

and $N_{+t}(u) = \sigma_y(N_{-t}(u))$, where σ_y is the *symmetry* with respect to the tangent hyperplane to X at y. By $x_1(u), x_2(u), \dots$ we denote the successive reflection points of $\gamma(u)$ and by $t_1(u), t_2(u), \dots$ the corresponding times (moments) of reflection. Notice that the points $x_j(u)$ include not only the proper (transversal) reflection points of $\gamma(u)$ but its tangent (reflection) points as well. For convenience set

$$x_0(u) = u, \quad t_0(u) = 0,$$

and denote by $r(u)$ the *number of reflection points* of $\gamma(u)$. Thus, $r(u)$ is a non-negative integer or ∞.

Denote by $\mathcal{A}'_k$ the set of all symbols of the form (2.3) such that $i_j \neq i_{j+1}$ for all $j = 1, \dots, k-1$. Here we do not assume that $i_1 \neq i_k$, so for $k > 1$ and $s > 1$ the set $\mathcal{A}_k$ introduced in Section 2.2 is a proper subset of $\mathcal{A}'_k$. The elements of the latter set will be called *configurations* of length k.

Let $\alpha \in \mathcal{A}'_k$ be a fixed configuration of the form (2.3). Set

$$F_\alpha = \{u \in Z_\omega : r(u) \geq k\ , \quad x_j(u) \in K_{i_j} \text{ for all } j = 1, \dots, k\}, \tag{2.37}$$

and denote by U_α the set of those $u \in F_\alpha$ such that $x_j(u)$ is a proper reflection point of $\gamma(u)$ for all $j = 1, \dots, k$. Clearly, F_α is a bounded subset of Z_ω (it is contained in $\pi_\omega(K)$) consisting of those $u \in Z_\omega$ for which the trajectory $\gamma(u)$ has at least k reflection points and the first k reflections 'follow' the configuration α. In general F_α is not a closed subset of Z_ω. However, it is easy to see that U_α is open in Z_ω.

For every $u \in F_\alpha$ set

$$J_\alpha(u) = N_{t+}(u) \in \mathbb{S}^{n-1}, \quad t = t_k(u).$$

Thus, we obtain a continuous map

$$J_\alpha : F_\alpha \longrightarrow \mathbb{S}^{n-1}. \tag{2.38}$$

Clearly J_α is smooth on U_α.

The rest of this section is devoted to the study of the map J_α. First, we consider some immediate consequences of the definition of this map.

Lemma 2.4.3:

(a) For every $u \in \overline{F_\alpha}$, there exists $\beta = (i'_1, \dots, i'_p) \in \mathcal{A}'_p$ for some $p \geq k$ such that $u \in F_\beta$ and there exists a sequence $p_1 < p_2 < \dots < p_k = p$ with $i_{p_j} = i_j$ for any $j = 1, \dots, k$. Moreover, if $x_r(u)$ is a proper reflection point of $\gamma(u)$ for some $r = 1, \dots, k$, then $i_r = i_{p_j}$ for some $j = 1, \dots, k$.

(b) J_α can be extended to a continuous map $J_\alpha : \overline{F_\alpha} \longrightarrow \mathbb{S}^{n-1}$.

Before continuing let us introduce some additional notation. Set

$$L_\alpha = \{u \in \overline{F_\alpha} : N_t(u) = \omega, \quad t \geq 0\}, \tag{2.39}$$

$$M_\alpha = \overline{F_\alpha} \setminus L_\alpha, \quad E_\alpha = J_\alpha(\overline{F_\alpha}). \tag{2.40}$$

Note that L_α is a compact subset of Z_ω contained in the boundary (in Z_ω) of the convex subset $\pi_\omega(K_{i_1})$. Hence L_α has Lebesgue measure zero and empty interior in Z_ω. In fact, it is a smooth compact $(n-2)$-dimensional submanifold. It is clear from the above definition that L_α is non-empty only for special K and special configurations α. Since $\overline{F_\alpha}$ is compact, E_α is a compact subset of $\mathbb{S}^{n-1}$. Finally, since $J_\alpha(u) = \omega$ for any $u \in L_\alpha$, the set $J_\alpha(M_\alpha)$ coincides either with E_α (in most cases) or with $E_\alpha \setminus \{\omega\}$.

We will use the main result in Section 2.2 to prove the following property of the map J_α.

Proposition 2.4.4: *For every configuration α the map*

$$J_\alpha : M_\alpha \longrightarrow J_\alpha(M_\alpha) \tag{2.41}$$

is a homeomorphism.

Proof: It is enough to prove that if $u \in M_\alpha$, $v \in \overline{F_\alpha}$ and $J_\alpha(u) = J_\alpha(v)$, then $u = v$. Indeed, assume this is true. Then (2.41) is a continuous bijection and we have to show that its inverse is continuous. Let $\{u_k\} \subset M_\alpha$ and $u \in M_\alpha$ be such that $J_\alpha(u_k) \longrightarrow J_\alpha(u)$ as $k \longrightarrow \infty$, and let v be an arbitrary limit point of the sequence $\{u_k\}$. Then $v \in \overline{F_\alpha}$ and clearly $J_\alpha(u) = J_\alpha(v)$. Then we must have $u = v$. The compactness of $\overline{F_\alpha}$ now shows that $u_k \longrightarrow u$ as $k \longrightarrow \infty$. Thus, the map (2.41) is a homeomorphism.

Let α have the form (2.3) and let $u \in M_\alpha$ and $v \in \overline{F_\alpha}$, $u \neq v$. It follows from the definition of M_α that $\gamma(u)$ has at least one proper reflection point. Let $j_1 < \dots < j_m$ ($m \leq k$) be all natural numbers not greater than k such that $y_\ell = x_{i_{j_\ell}}(u)$ are proper reflection points of $\gamma(u)$. It follows from Lemma 2.4.3(a) that there exists a configuration β with the properties listed in the lemma such that $v \in F_\beta$. Moreover, there exist reflection points $z_\ell \in K_{i_{j_\ell}}$ ($\ell = 1, \dots, m$) of $\gamma(v)$ such that the successive proper reflection points of $\gamma(v)$ form a subsequence of $z_1, \dots, z_m$.

Assume that $J_\alpha(u) = J_\alpha(v) = \eta \in \mathbb{S}^{n-1}$. Consider arbitrary convex domains K_0 and K_{s+1} with smooth boundaries in $\mathbb{R}^n$ such that $K_0 \subset \mathbb{R}^n \setminus H_\omega$, $K_{s+1} \subset H_\eta$, $\pi_\omega(K) \subset \partial K_0$ and $\pi_{-\eta}(K) \subset \partial K_{s+1}$.

Now we are in a position to apply Lemma 2.2.5 to

$$K' = K_0 \cup K_1 \cup \cdots \cup K_s \cup K_{s+1}$$

and the configuration

$$\lambda = (0, i_{j_1}, \ldots, i_{j_m}, s+1, i_{j_m}, \ldots, i_{j_1}, 0).$$

Consider the convex domain

$$K_\lambda = K_0 \times K_{i_{j_1}} \times \cdots \times K_{i_{j_m}} \times K_{s+1} \times K_{i_{j_m}} \times \cdots \times K_{i_{j_1}} \times K_0$$

in $(\mathbb{R}^n)^{2m+3}$ and the corresponding length function

$$F = F_\lambda : K_\lambda \longrightarrow \mathbb{R}$$

(see (2.4) and (2.5)). Then by Lemma 2.2.6(b), F has a strict local minimum at the point

$$\tilde{q} = (q_0, q_1, \ldots, q_m, q_{m+1}, q_m, \ldots, q_1, q_0),$$

where $q_i = x_i(u)$ for $i = 0, 1, \ldots, m$ and $q_{m+1} = \pi_\eta(q_m)$. It then follows from the convexity of F that it has no other local minima in K_λ. On the other hand, by Lemma 2.2.6(a), F has also a local minimum at the point

$$\tilde{p} = (p_0, p_1, \ldots, p_m, p_{m+1}, p_m, \ldots, p_1, p_0),$$

where $p_i = x_i(v)$ for $i = 0, 1, \ldots, m$ and $p_{m+1} = \pi_\eta(p_m)$. Therefore, $\tilde{q} = \tilde{p}$ which implies $u = q_0 = p_0 = v$, in contradiction with $u \neq v$.

This shows that $J_\alpha(u) \neq J_\alpha(v)$ and thus completes the proof of the proposition. ∎

Combining Propositions 2.4.2 and 2.4.4, we get the following.

Corollary 2.4.5: *For every configuration α the map*

$$J_\alpha : U_\alpha \longrightarrow J_\alpha(U_\alpha)$$

is a diffeomorphism.

Let α be a configuration of the form (2.3) and let γ be a reflecting (ω, θ)-ray in Ω with successive reflection points $x_1, \ldots, x_k$. We will say that γ *is of type α if $x_j \in K_{i_j}$ for all $j = 1, \ldots, k$.*

The following is another direct consequence of Proposition 2.4.4.

Corollary 2.4.6: *If $\omega \neq \theta$, then for every configuration α there is at most one reflecting (ω, θ)-ray of type α in Ω.*

Under some conditions on α (resp. K), ω and θ, it is shown in Section 10.3 that there exists a (unique) reflecting (ω, θ)-ray of type α in Ω.

2.5 Notes

The material discussed in Section 2.1 was already known to Birkhoff [Bir]. The reader is referred to [KozT] for more details on this and related topics.

The results in Section 2.2 are taken from [S2], where the more general case of semi-dispersing billiards is considered. In the particular case when the condition (H) is satisfied the statement of Corollary 2.2.4 was proved by Ikawa [I4] using a different technique.

The representation of the linear Poincaré map P_γ described in Theorem 2.3.1 was obtained in [PV]. Proposition 2.3.2 generalizes a result from [BGR] concerning two disjoint strictly convex domains. Most of the material in Section 2.4 is taken from [PS5], however the proof of Proposition 2.4.4 is different from that in [PS5].

A general definition of a billiard on a Riemann manifold with boundary can be found in [CFS] (see also [Sin2]). A more general type of dynamical systems is studied in [KS]. Ergodic properties of billiards related to certain problems in statistical mechanics have been studied very intensively in the 1970s, 1980s and 1990s by Sinai, Bunimovich, Chernov and many others (see [Sinl], [BunS], [BCS], [Cher1], [KSS], [Sim] and the references there). One should mention in particular the solutions of the Boltzmann–Sinai Ergodic Hypothesis by N. Simányi [Sim].

For dispersing billiards in the plane the results in [BCS] imply an exponential estimate from below for the number P_k of periodic points of period *k* (*cf.* also Corollary 2.2.5), and the set of periodic points is dense in the phase space of such billiards. Further results concerning periodic points of billiards are contained in [Cher2]. As mentioned in Section 3.4 in [Cher2], according to some arguments in [BCS] the estimate (2.8) in Corollary 2.2.5 does not hold for dispersing billiards on the flat torus; for the latter it might even happen that $P_k = \infty$ for some k. It follows from [Cher2] that for dispersing billiards

$$\liminf \frac{\log P_k}{k}$$

can be estimated from below by the metric entropy of the billiard ball map (see e.g. [CFS] or [Wa] for the definition of entropy). Let us mention that for the geodesic flow φ_t on a compact manifold with constant negative curvature Margulis [Marg] proved the same formula.

There has been a significant activity in the last 20 years or so in studying dynamical systems involving billiards of various kinds. Apart from what was mentioned above, we refer the reader to the monographs [ChM], [KozT], [Pla], [Tab] and the articles [BCST], [ChD], [Gut], [Katl], [Kat2], [Kat3], [KP], [Mor], [PlaR], [S5] and the references there for general information. We should note however that our list of references covers only a small percentage of the large number of works in this area.

3

Poisson relation for manifolds with boundary

This chapter is devoted to the analysis of the singularities of the distribution

$$\sigma(t) = \sum_j \cos \lambda_j t,$$

where $\{\lambda_j^2\}_{j=1}^{\infty}$ are the eigenvalues of the Laplacian in a bounded domain Ω with Dirichlet boundary condition on $\partial\Omega$. The main purpose is to establish the so-called *Poisson relation for manifolds with boundary*, namely that

$$\text{sing supp}\,\sigma(t) \subset \{0\} \cup \{\pm T_\gamma : \gamma \in \mathcal{L}_\Omega\},$$

where $\mathcal{L}_\Omega$ is the set of all generalized periodic bicharacteristics of $\Box$ in Ω and T_γ denotes the period of T_γ. In Section 3.1, the fundamental solutions $e_0(t, x-y)$ and $h_0(t, x-y)$ of $\Box$ and $\Box^2$, respectively, are studied. We describe the singularities of e_0 and h_0 on the diagonal of $\partial\Omega \times \partial\Omega$. In Section 3.2 we introduce the distribution $\sigma(t)$ and show that it coincides with the trace of the fundamental solution $E(t, x, y)$ of the Dirichlet problem for $\Box$.

In Section 3.3 the proof of the Poisson relation is reduced to the analysis of the trace of a distribution $B(t, x, y)$, defined by (3.13), on the manifold without boundary $\partial\Omega$. For convex domains, we examine the singularities of $B(t, x, y)$ and those of the trace $B(t, x, x), x \in \partial\Omega$.

The analysis of the singularities of $B(t, x, y)$ for non-convex Ω leads to some difficulties. For this reason for general domains we study in Section 3.4 separately the singularities of $E(t, x, y)$ for $x, y \in \Omega^\circ$ and those for x, y close to $\partial\Omega$, provided $t \notin \{\pm T_\gamma : \gamma \in \mathcal{L}_\Omega\}$. For this purpose we apply a localization that will be exploited in the next chapter. The advantage of the proof in Section 3.4 is that it works without

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any change for Neumann and Robin boundary conditions, according to the results on propagation of singularities in [MS1] and [MS2].

3.1 Traces of the fundamental solutions of $\Box$ and $\Box^2$

Let $e_0(t, x-y)$ be the solution of the problem

$$\begin{cases}(\partial_t^2 - \Delta_x)e_0(t,x-y) = \delta(t)\delta(x-y),\\ \text{supp } e_0(t,x-y) \subset \{(t,x,y) \in \mathbb{R}_t \times \mathbb{R}^{2n} :\ t \geq 0\}.\end{cases}$$

The second condition implies that the Fourier transform $\hat{e}_0(\tau, x-y)$ of $e_0(t, x-y)$ with respect to t admits an analytic continuation in $\{\tau \in \mathbb{C} :\ \text{Im}\,\tau < 0\}$. This implies

$$\hat{e}_0(\tau, x-y) = (2\pi)^{-n} \int e^{\mathbf{i}\langle x-y,\xi\rangle}[\xi^2 - (\tau - \mathbf{i}0)^2]^{-1} d\xi,$$

with $\xi^2 = \langle \xi, \xi \rangle$ and

$$e_0(t,x-y) = (2\pi)^{-n-1} \lim_{\epsilon \to +0} \int\int e^{\mathbf{i}\tau|\xi|}[1 - (\tau - \mathbf{i}\epsilon)^2]^{-1} e^{\mathbf{i}\langle x-y,\xi\rangle}|\xi|^{-1} d\tau\, d\xi.$$

For $\epsilon > 0$ the theorem of residues yields

$$\int_{-\infty}^{\infty} e^{\mathbf{i}\tau|\xi|}[1 - (\tau - \mathbf{i}\epsilon)^2]^{-1} d\tau = -\mathbf{i}\pi(e^{\mathbf{i}t|\xi|(1+\mathbf{i}\epsilon)} - e^{\mathbf{i}t|\xi|(\mathbf{i}\epsilon-1)}).$$

Letting $\epsilon \to +0$, for $t \geq 0$ we obtain

$$e_0(t,x-y) = \frac{(2\pi)^{-n}}{2\mathbf{i}} \int (e^{\mathbf{i}(t|\xi| + \langle x-y,\xi\rangle)} - e^{\mathbf{i}(-t|\xi|+\langle x-y,\xi\rangle)}) \frac{d\xi}{|\xi|}.$$

The integral for $|\xi| \geq 1$ can be considered as a sum of oscillatory integrals with phase functions

$$\varphi_{\pm}(t,x,y,\xi) = \pm t|\xi| + \langle x-y, \xi\rangle$$

(see Proposition 1.3.2). Moreover, by using Theorem 1.3.3, it is easy to find the wave front set

$$\begin{aligned} WF(e_0(t,x-y)) \subset \{(t,x,y,\tau,\xi,\eta) \in T^*(\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_y) \setminus \{0\} :\\ t > 0, x = y \mp t\frac{\xi}{|\xi|}, \tau = \pm|\xi|,\ \xi + \eta = 0\} \\ \cup\{(t,x,y,\tau,\xi,\eta)\} \in T^*(\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_y) \setminus \{0\} : t = 0, x = y, \xi + \eta = 0\}. \quad (3.1)\end{aligned}$$

Now let $\Omega \subset \mathbb{R}^n$ be a bounded closed domain with smooth boundary $\partial\Omega$. We wish to define the *trace*

$$f_0(t,x-y) = e_0(t,x-y)|_{\mathbb{R}_t \times \partial\Omega \times \partial\Omega},$$

and to describe $WF(f_0(t, x-y))$. To do this, consider the inclusion map

$$j : \mathbb{R}_t \times \partial\Omega \times \partial\Omega \to \mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_y,$$

$j(t, x, y) = (t, x, y)$. The *normal set* N_j of j has the form

$$N_j = \{(t, x, y, \tau, \xi, \eta) \in T^*(\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_y) : x \in \partial\Omega, \ y \in \partial\Omega,$$
$$\tau = 0, \xi|_{T_x(\partial\Omega)} = \eta|_{T_y(\partial\Omega)} = 0\}.$$

Obviously, for $\mathbb{R}^+_t = \{t \in \mathbb{R} : \ t > 0\}$ we have

$$WF\left(e_0(t, x-y)|_{\mathbb{R}^+_t \times \partial\Omega \times \partial\Omega}\right) \cap N_j = \emptyset.$$

According to Theorem 1.3.6, we can define $f_0(t, x-y)$ for $t > 0$ by setting

$$f_0(t, x-y) = j^* e_0(t, x-y),$$

j^* being the *pull-back* of j. The same procedure works for $t = 0, x \neq y$.

The definition of $f_0(t, x-y)$ for $t = 0, x = y, x \in \partial\Omega$ is more difficult, since the set N_j contains some points lying over the set

$$\{(0, x, x) \in \mathbb{R}_t \times \partial\Omega \times \partial\Omega\}.$$

To cover this case we make a more precise analysis of f_0 for (t, x, y) close to $(0, x^0, x^0), x^0 \in \partial\Omega$. For $x \in \Omega^\circ, y \in \partial\Omega$, we have

$$\hat{e}_0(\tau, x-y) = (2\pi)^{-n} \int e^{\mathbf{i}\langle x-y, \xi\rangle} (\xi^2 - \tau^2)^{-1} d\xi, \ \operatorname{Im} \tau < 0.$$

To find the limit of the right-hand side as $x \to x^0, \operatorname{Im} \tau \to -0$, we introduce near $x^0 \in \partial\Omega$ local coordinates $(x_1, x'), x' = (x_2, \dots, x_n)$ so that in a neighbourhood of x^0, the domain Ω is given by $x_1 \geq g(x')$, while $\partial\Omega$ has the form $x_1 = g(x')$, g being a smooth function. Let $H(x', y') \in C^\infty(\mathbb{R}^{n-1}, \mathbb{R}^{n-1})$ be a smooth vector-valued function such that

$$g(x') - g(y') = \langle x' - y', H(x', y')\rangle$$

with $H(x', x') = dg(x')$. Here and below $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^{n-1}$. The metric on $T(\partial\Omega)$, inherited from the standard Euclidean one in $\mathbb{R}^n$, has the form

$$m^2(x', y'; \nu') = \langle \nu', \nu' \rangle - \frac{\langle \nu', H(x', y')\rangle^2}{1 + |H(x', y')|^2}.$$

In the local coordinates (x_1, x'), for $x \in \Omega^\circ, y \in \partial\Omega$, we get

$$\hat{e}_0(\tau, x, y') = (2\pi)^{-n} \int e^{\mathbf{i}\langle x' - y', \xi'\rangle} e^{\mathbf{i}\xi_1(x_1 - g(x'))} \left(1 + |dg(y')|^2\right)^{1/2} (\xi^2 - \tau^2)^{-1} d\xi. \quad (3.2)$$

Here the factor $(1 + |dg(y')|^2)^{1/2}$ appears, since in the coordinates (z_1, y') with $z_1 = y_1 - g(y')$ we have

$$\delta_{\partial\Omega} = (1 + |dg(y')|^2)^{1/2} \otimes \delta(z_1).$$

Next, change the variables

$$\begin{cases} \zeta_1 = \xi_1, \\ \zeta' = \xi' + \xi_1 H(x', y'). \end{cases}$$

Then (3.2) becomes

$$\hat{e}_0(\tau, x, y') = (2\pi)^{-n} \int e^{\mathbf{i}\langle x'-y', \xi'\rangle + \rho\xi_1} (1 + |dg(y')|)^{1/2} (\xi^2 - \tau^2) d\zeta_1 d\zeta',$$

with $\rho = x_1 - g(y')$ and

$$\xi^2 - \tau^2 = (\zeta')^2 - 2\zeta_1 \langle \zeta', H\rangle + (1 + |H|^2)\zeta_1^2 - \tau^2.$$

The roots $z_\pm$ of the equation $\xi^2 - \tau^2 = 0$ with respect to ζ_1 have the form

$$z_\pm = \frac{\langle \zeta', H\rangle}{1 + |H|^2} \pm \mathbf{i} \left(\frac{m^2 - \tau^2}{1 + |H|^2} \right)^{1/2}.$$

Here we choose the square root so that $\mathrm{Re}\,(m^2 - \tau^2)^{1/2} > 0$. Hence $\mathrm{Im}\, z_+ > 0$, and by the theorem of residues we get

$$\int_{-\infty}^{\infty} e^{\mathbf{i}\rho\zeta_1} \Big((1 + |H|^2)(\zeta_1 - z_+)(\zeta_1 - z_-) \Big)^{-1} d\zeta_1$$

$$= \pi e^{\mathbf{i}\rho\zeta_+} \Big((1 + |H|^2)(m^2 - \tau^2) \Big)^{-1/2}.$$

Letting $\rho \to +0$, for $x' \in \partial\Omega$, $y' \in \partial\Omega$, we obtain

$$\hat{e}_0(\tau, x', y') = \frac{(2\pi)^{-n-1}}{2} \int \frac{e^{\mathbf{i}\langle x'-y', \zeta'\rangle} (1 + |dg(y')|^2)^{1/2}}{[(1 + |H(x', y')|^2)(m^2(x', y'; \zeta') - \tau^2)]^{1/2}} d\zeta'.$$

The form $m^2(x', y'; \zeta')$ is positively definite, so there exists a positively definite symmetric matrix $Q(x', y')$ such that

$$m^2(x', y'; \zeta') = |Q(x', y')^{-1}\zeta'|^2.$$

Changing the variables $\zeta' \to Q(x', y')\xi'$, we deduce

$$\hat{e}_0(\tau, x', y') = F(x', y') \int e^{\mathbf{i}\langle Q(x', y')(x'-y'), \xi'\rangle} (\xi'^2 - (\tau - \mathbf{i}0)^{-1/2} d\xi'$$

with $F(x', y') \in C^\infty$.

Taking the inverse Fourier transform in τ, we have to examine the integrals

$$I_1 + I_2 = \int\int_{|\tau|\leq|\xi'|+1} e^{\mathbf{i}t\tau+\langle Q(x',y')(x'-y'),\xi'\rangle}[\xi' - (\tau - \mathbf{i}0)^2]^{-1/2} d\tau\, d\xi'$$
$$+ \int\int_{|\tau|>|\xi'|+1} e^{\mathbf{i}t\tau+\langle Q(x',y')(x'-y'),\xi'\rangle}[\xi' - (\tau - \mathbf{i}0)^2]^{-1/2} d\tau\, d\xi'.$$

Setting $\tau = \mu|\xi'|$, we may consider I_1, modulo smooth terms, as an oscillatory integral with a phase function

$$\Psi_1 = t\mu|\xi'| + \langle Q(x',y')(x'-y'),\xi'\rangle,$$

since $\Psi_{1,x'} \neq 0$ for $x' = y', |\xi'| \geq 1$. On the other hand, we can treat I_2 as an oscillatory integral with a phase function

$$\Psi_2 = t\tau + \langle Q(x',y')(x'-y'),\xi'\rangle,$$

since $|\Psi_{2,t}| + |\Psi_{2,x'}| \neq 0$ for $x' = y'$. Consequently, we may define $e_0(t, x - y)|_{\mathbb{R}_t\times\partial\Omega\times\partial\Omega}$ for $t = 0$, and the analysis of the wave front sets of the integrals I_1, I_2 yields

$$WF(e_0(t,x-y)|_{\mathbb{R}_t\times\partial\Omega\times\partial\Omega}) \cap \{t = 0\} \subset \{(t, x', y', \tau, \xi', \xi')$$
$$\in T^*(\mathbb{R}_t \times \partial\Omega \times \partial\Omega) \setminus \{0\} : \ t = 0, x' = y', \xi' + \eta' = 0\}. \tag{3.3}$$

In a similar way one examines the distribution $h_0(t, x - y)$ determined as the solution of the problem

$$\begin{cases} (\partial_t^2 - \Delta_x)^2 h_0(t, x-y) = \delta(t)\delta(x-y), \\ \text{supp } h_0(t, x-y) \subset \{(t,x,y) \in \mathbb{R}_t \times \mathbb{R}^{2n} : \ t \geq 0\}. \end{cases}$$

Write the Fourier transform $\hat{h}_0(\tau, x - y)$ as

$$\hat{h}_0(\tau, x-y) = (2\pi)^{-n} \int e^{\mathbf{i}\langle x-y,\xi\rangle}[\xi^2 - (\tau - \mathbf{i}0)^2]^{-2} d\xi.$$

Then for $t \geq 0$ we obtain the oscillatory integral

$$h_0(t, x-y) = \frac{\mathbf{i}(2\pi)^{-n}}{2} \int [e^{\mathbf{i}\varphi_+}(\mathbf{i}t|\xi| - 1) + e^{\mathbf{i}\varphi_-}(\mathbf{i}t|\xi| + 1)]|\xi|^{-3} d\xi.$$

This implies a relation completely analogous to (3.1). Thus for $t > 0$ and for $t = 0, x \neq y$, we can define

$$F_1(t, x-y) = j^* h_0(t, x-y).$$

For the trace on $t = 0$ and $x = y$, we repeat the above procedure. For $\operatorname{Im} \tau < 0$ and $x', y' \in \partial\Omega$, one obtains

$$\lim_{\rho \to +0} \int_{-\infty}^{\infty} e^{\mathbf{i}\rho\xi_1} (\xi^2 - \tau^2)^{-2} d\xi_1 = \frac{\pi}{2} (1 + |H|^2)^{-1/2} (m^2 - \tau^2)^{-3/2}$$

and we deduce

$$\hat{f}_1(\tau, x', y') = F_1(x', y') \int e^{\mathbf{i}\langle Q(x',y')(x'-y'), \xi' \rangle} (\xi'^2 - (\tau - \mathbf{i}0)^2)^{-3/2} d\xi'$$

with $F_1(x', y') \in C^\infty$. It remains to study two oscillatory integrals, similar to I_1 and I_2. The analysis is completely analogous to the previous one and we leave the details to the reader. Finally, combining the action of the pull-back j^* with (3.3), we obtain the following.

Theorem 3.1.1: *The wave front sets of the distributions* $f_k(t, x - y) \in \mathcal{D}'(\mathbb{R}_t \times \partial\Omega \times \partial\Omega)$, $k = 0, 1$, *are contained in the set of points*

$$(t, x', y', \tau, \tilde{\xi}, \tilde{\eta}) \in T^*(\mathbb{R}_t \times \partial\Omega \times \partial\Omega) \setminus \{0\}$$

satisfying the following conditions:

(i) $t > 0$ *and there exists* $\xi \in \mathbb{R}^n \setminus \{0\}$ *such that* $x' = y' \mp t\xi/|\xi|, \tau = \pm|\xi|, \tilde{\xi} = p_x(\xi), \tilde{\xi} + \tilde{\eta} = 0$*;*

(ii) $t = 0, x' = y', \tilde{\xi} + \tilde{\eta} = 0.$

Here $p_x : T^*_x(\mathbb{R}^n) \longrightarrow T^*_x(\partial\Omega)$ is the *canonical projection.*

3.2 The distribution $\sigma(t)$

Let $\Omega \subset \mathbb{R}^n, n \geq 2$, be a bounded closed domain with C^∞ smooth boundary $\partial\Omega$ and $\Omega^\circ \neq \emptyset$. Let $H^1_0(\Omega)$ be the closure of the space $C^\infty_0(\Omega)$ with respect to the norm

$$\|u\|_1^2 = \sum_{|\alpha| \leq 1} \|\partial^\alpha u\|^2,$$

$\|.\|$ being the norm in $L^2(\Omega)$.

Introduce the operator A by $Au = -\Delta u$ for $u \in C^\infty_0(\Omega)$, and extend it to the domain

$$D_A = \{u \in H^1_0(\Omega) : \Delta u \in L^2(\Omega)\},$$

where Δu is interpreted in the sense of distributions. Let $(\cdot, \cdot)$ be the inner product in $L^2(\Omega)$. Clearly

$$(Au, u) = (u, Au) \text{ for } u, v \in D_A.$$

Thus, A is symmetric and closed operator in $L^2(\Omega)$. Moreover,

$$((A+1)u, u) = \|u\|^2, \; u \in D_A,$$

and the form $\mathcal{A}(u, u) = ((A+1)u, u)$ is continuous and coercive in $H_0^1(\Omega)$. Consequently, by Lax–Milgram theorem for each $f \in L^2(\Omega)$, there exists an unique solution $u \in H_0^1(\Omega)$ of the problem

$$-\Delta u + u = f \text{ in } \mathcal{D}'(\Omega).$$

This implies that $A+1$ is a bijection from D_A into $L^2(\Omega)$ and

$$\|(A+1)^{-1} f\|_{H^1(\Omega)} \leq C \, \|f\|, \; \forall f \in L^2(\Omega). \tag{3.4}$$

Hence, -1 is not in the spectrum of A, and, by the well-known criteria for self-adjointness, A is a self-adjoint operator in $L^2(\Omega)$, related to the *Laplacian* $-\Delta$ *with Dirichlet boundary condition on* $\partial\Omega$.

The estimate (3.4) shows that the resolvent $(A+1)^{-1}$ is compact in $L^2(\Omega)$ and we deduce that the spectrum of A is formed by an infinite number of eigenvalues

$$0 < \lambda_1^2 \leq \lambda_2^2 \leq \, \dots \, \leq \lambda_m^2 \leq \, \dots$$

with finite multiplicities. Let $\{\varphi_j(x)\}_{j=1}^{\infty}$ be an orthonormal set of eigenfunctions of A so that

$$\begin{cases} -\Delta \varphi_j(x) = \lambda_j^2 \varphi_j(x), \; x \in \Omega, \\ \varphi_j(x) = 0, \; x \in \partial\Omega. \end{cases}$$

Therefore, $\varphi_j(x) \in C^\infty(\Omega)$, and we can introduce the *spectral function*

$$e(x, y, \lambda) = \sum_{\lambda_j^2 \leq \lambda^2} \varphi_j(x) \varphi_j(y),$$

which is the kernel of the *spectral projection* E_λ of A. Moreover, we have the estimate

$$\sup\{|e(x, y, \lambda)| : \; x, y \in \Omega\} \leq C\lambda^n, \; \lambda \geq 1 \tag{3.5}$$

with a constant $C > 0$ independent of λ. We refer to Hörmander [[H3], Section 17.5] for the proof of (3.5).

Let $N(\lambda) = \#\{j : \; \lambda_j^2 \leq \lambda^2\}$ be the *counting function* of eigenvalues. Clearly,

$$N(\lambda) = \int_\Omega e(x, x, \lambda^2) dx,$$

and (3.5) yields

$$N(\lambda) \leq C_1 \lambda^n, \; \lambda \longrightarrow +\infty.$$

Introduce the tempered distribution

$$\sigma(t) = \sum_{j=1}^{\infty} \cos \lambda_j t = \mathrm{Re} \sum_{j=1}^{\infty} \exp\ (\lambda_j t) \in \mathcal{S}'(\mathbb{R}).$$

Since A is a non-negative self-adjoint operator, by the spectral calculus we may define the operator $\cos(A^{1/2}t)$. Let

$$\mathcal{E}(t, x, y) \in \mathcal{D}'(\mathbb{R} \times \Omega \times \Omega)$$

be the *kernel* of $\cos(tA^{1/2})$. Therefore,

$$(\cos(tA^{1/2})f, f) = \int_0^{\infty} \cos(\lambda t) \int_{\Omega} \int_{\Omega} e(x, y, \lambda^2) f(x) \overline{f(y)} dx\, dy,$$

and a simple calculus implies

$$\mathcal{E}(t, x, y) = \sum_{j=1}^{\infty} (\cos \lambda_j t) \varphi_j(x) \varphi(y).$$

Hence

$$\sigma(t) = \int_{\Omega} \mathcal{E}(t, x, y) dx \tag{3.6}$$

and $\mathcal{E}(t, x, y)$ is a solution of the problem

$$\begin{cases} (\partial_t^2 - \Delta_x)\mathcal{E}(t, x, y) = 0,\ t \in \mathbb{R},\ x \in \Omega^{\circ},\ y \in \Omega^{\circ}, \\ \mathcal{E}(t, x, y)|_{x \in \partial\Omega} = \mathcal{E}(t, x, y)|_{y \in \partial\Omega} = 0, \\ \mathcal{E}(0, x, y) = \delta(x - y),\ \partial_t \mathcal{E}(0, x, y) = 0. \end{cases} \tag{3.7}$$

Following the results for the propagation of singularities in [[MS2], [H3]] (see Theorem 1.4.2), we can describe the singularities of $\mathcal{E}(t, x, y)$. Since the trace defining $\sigma(t)$ is taken over a manifold with boundary, Theorems 1.3.8 and 1.3.9, concerning the calculus with wave front sets, cannot be applied immediately. To describe the singularities of $\sigma(t)$, in the next section we find another representation of $\sigma(t)$ involving integration over $\partial\Omega$.

3.3 Poisson relation for convex domains

As in the previous section, Ω is a compact domain in $\mathbb{R}^n, n \geq 2$. Our first argument concerns the general case and we do not assume that Ω is convex.

Let

$$E(t, x, y) = \sum_{j=1}^{\infty} \lambda_j^{-1} \sin(\lambda_j t) \varphi_j(x) \varphi_j(y)$$

be the *kernel of the operator* $A^{-1/2}\sin(tA^{1/2})$. Clearly,

$$\mathcal{E}(t,x,y) = \partial_t E(t,x,y)$$

and $E(t,x,y)$ is the solution of the problem

$$\begin{cases}(\partial_t^2 - \Delta_x)E(t,x,y) = 0,\ t \in \mathbb{R},\ x \in \Omega^\circ,\ y \in \Omega^\circ,\\ E(t,x,y)|_{x\in\partial\Omega} = 0,\\ E(0,x,y) = 0,\ \partial_t E(0,x,y) = \delta(x-y).\end{cases}$$

Next, we examine the trace

$$\int E(t,x,x)dx.$$

For $y \in \partial\Omega$ introduce the distribution $K(t,x,y)$ as the solution of the problem

$$\begin{cases}(\partial_t^2 - \Delta_x)K(t,x,y) = 0,\ t \in \mathbb{R},\ x \in \Omega^\circ,\ y \in \Omega^\circ,\\ K(t,x,y) = \delta(t)\otimes\delta(x-y),\ x \in \partial\Omega,\\ \operatorname{supp} K(t,x,y) \subset \{t,x,y) \in \mathbb{R}_t \times \Omega \times \partial\Omega :\ t \geq 0\}.\end{cases} \tag{3.8}$$

Notice that $K(t-s,x,y)$ is the kernel of a continuous operator

$$\mathcal{K} : C_0^\infty(\mathbb{R}\times\partial\Omega) \ni f(s,y) \to (\mathcal{K}f)(t,x) \in \bar{\mathcal{D}}'(\mathbb{R}\times\Omega).$$

Here $(\mathcal{K}f)(t,x)$ is the solution of the problem

$$\begin{cases}(\partial_t^2 - \Delta_x)\mathcal{K}f = 0 \text{ in } \mathbb{R}\times\Omega^\circ,\\ \mathcal{K}f - f = 0, \text{ on } \mathbb{R}\times\partial\Omega,\\ \mathcal{K}f|_{t<0} = 0,\end{cases}$$

where $\bar{\mathcal{D}}'(\mathbb{R}\times\Omega)$ is the space of all distributions u on $\mathbb{R}\times\Omega^\circ$ for which there exists an open neighbourhood $\tilde{\Omega}$ of Ω such that u can be extended to a distribution in $\mathcal{D}'(\mathbb{R}\times\tilde{\Omega})$. Now we wish to extend $\mathcal{K}$ as a continuous operator on $\mathcal{E}'(\mathbb{R}\times\partial\Omega)$. To do this, given $f(s,y) \in \mathcal{E}'(\mathbb{R}\times\partial\Omega)$, write it in the form

$$f(s,y) = \sum_{j=1}^{N} L_j(s,y,D_s,D_y)f_j(s,y)$$

with $f_j(s,y) \in \mathcal{E}'(\mathbb{R}\times\partial\Omega)\cap H_s(\mathbb{R}\times\partial\Omega), s \geq 1$ and $L_j(s,y,D_s,D_y)$ differential operators, involving only derivatives along directions tangential to $\mathbb{R}\times\partial\Omega$. By Theorem 1.4.1, it follows that we can define $\mathcal{K}g$ for $g \in H_s(\mathbb{R}\times\partial\Omega), s \geq 1$. Applying this, we find $(\mathcal{K}f_j)(t,x)$ for $j = 1,\dots,N$, and we define

$$(\mathcal{K}f)(t,x) = \sum_{j=1}^{N} L_j(\mathcal{K}f_j)(t,x).$$

Let $e_0(t, x-y)$ be the fundamental solution of $\Box$, defined in Section 3.1. Notice that it coincides with the solution of the Cauchy problem:

$$\begin{cases}(\partial_t^2 - \Delta_x)e_0(t, x-y) = 0,\ t \in \mathbb{R}_t^+,\ x, y \in \mathbb{R}^n,\\ e_0(t, x-y) = 0,\ \partial_t e_0(0, x-y) = \delta(x-y),\end{cases}$$

extended as 0 for $t < 0$. Next, introduce the distribution

$$\begin{aligned}\widetilde{E}(t, x, y) &= \int_\infty^\infty \int_{\partial\Omega} K(s, x, z)e_0(t-s, z-y)ds\,dz\\ &= \int_\infty^\infty \int_{\partial\Omega} K(t-s, x, z)e_0(s, z-y)ds\,dz,\end{aligned} \tag{3.9}$$

where the integral is interpreted as the action of the distribution $e_0(t-s, z-y)|_{(z,y)\in\partial\Omega\times\Omega}$ on $K(s, x, z)$. The integration in (3.9) is over compact set with respect to (s, z) and to justify this action, we apply Theorem 1.3.9 and the fact that

$$\begin{aligned}WF(e_0(t-s, z-y)|_{(z,y)\in\partial\Omega\times\Omega}) \cap \{(t, s, y, z, 0, -\sigma, 0, -\zeta)\\ \in T^*(\mathbb{R}_t \times \mathbb{R}_s \times \Omega \times \partial\Omega) \setminus \{0\}\} = \emptyset.\end{aligned} \tag{3.10}$$

To verify (3.10), observe that

$$(\partial_t + \partial_s)e_0(t-s, .) = 0$$

leads to $\tau + \sigma = 0$ on $WF(e_0(t-s, .))$, while $\eta + \zeta = 0$ on $WF(e_0(t-s, .))$ follows from Theorem 3.1.1.

The hyperplane $t = 0$ is not characteristic for $\Box$, so the traces $\widetilde{E}(0, x, y)$ and $(\partial_t\widetilde{E})(0, x, y)$ exist and

$$\widetilde{E}(0, x, y) = \int_\infty^\infty \int_{\partial\Omega} K(-s, x, z)e_0(s, z-y)ds\,dz = 0$$

since $e_0(0, z-y) = 0$ for $t \leq 0$. Similarly, we find $(\partial_t\widetilde{E})(0, x, y) = 0$. In this way we see that $\widetilde{E}(t, x, y)$ is a solution of the problem

$$\begin{cases}(\partial_r^2 - \Delta_x)\widetilde{E}(t, x, y) = 0, t \in \mathbb{R}, x \in \Omega^\circ,\ y \in \Omega^\circ,\\ \widetilde{E}(t, x, y) - e_0(t, x-y) = 0,\ (t, x) \in \mathbb{R} \times \partial\Omega,\\ \widetilde{E}(0, x-y) = \partial_t\widetilde{E}(0, x-y) = 0.\end{cases}$$

Consequently,

$$E(t, x, y) = e_0(t, x-y) - \widetilde{E}(t, x, y).$$

In what follows we investigate the singularities of $\sigma(t)$ for $t > 0$. First, notice that

$$\text{sing supp } e_0(t, x-y) \subset \{(t, x, y) \in \mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_y^n :\ |x-y| = t\}. \tag{3.11}$$

This follows from the propagation of singularities of the solution of Cauchy problem satisfied by $e_0(t, x - y)$. Consequently, for $t > 0$ we have

$$\int_{\Omega} e_0(t, 0)dx \in C^{\infty}(\mathbb{R}^+).$$

Thus, for $t > 0$, modulo C^{∞} terms, we obtain

$$\int_{\Omega} E(t,x,x)dx = -\int_{\Omega} \widetilde{E}(t,x,x)dx = -\int_{\infty}^{\infty}\int_{\partial\Omega} K(t-s,x,z)e_0(s,z-x)ds\,dz$$
$$= -\int_0^t \int_{\partial\Omega} K(t-s,x,z)(\partial_s^2 - \Delta_x)h_0(s,z-x)ds\,dz,$$

where $h_0(t, x-y)$ is the fundamental solution of $\Box^2$, introduced in Section 3.1 and extended as 0 for $t < 0$. Integrating by parts with respect to s and x, we find

$$\int_{\Omega} \widetilde{E}(t,x,x)dx = \int_0^t ds \int_{\partial\Omega} dz \int_{\partial\Omega} [\partial_{\nu_x} K(t-s,x,z)h_0(s,z-x)$$
$$-K(t-s,x,z)\partial_{\nu_x} h_0(s,z-x)]dx. \tag{3.12}$$

Here, ∂_{ν_x} denotes the derivative with respect to the exterior normal ν_x at $x \in \partial\Omega$. On the other hand, sing supp $h_0(t, x-y)$ satisfies the relation (3.11) since $h_0(t, x-y)$ is a solution of the Cauchy problem

$$\begin{cases} (\partial_t^2 - \Delta_x)^2 h_0(t, x-y) = 0, \ t \in \mathbb{R}_t^+, \ x, y \in \mathbb{R}^n, \\ \partial_t^j h_0(0, x-y) = 0, \ j = 0,1,2, \\ \partial_t^3 h_0(0, x-y) = \delta(x-y), \end{cases}$$

where $h_0(t, x-y)$ is extended as 0 for $t < 0$. Therefore, the second term in the right-hand side of (3.12) becomes

$$\int_0^t ds \int_{\partial\Omega} dz \int_{\partial\Omega} (\partial_{\nu_x} h_0(t,0)dx \in C^{\infty}(\mathbb{R}_t^+).$$

The boundary $\partial\Omega$ is not characteristic for the operator $\Box$ and $K(t,x,y)$ satisfies (3.8). We can apply the partial hypoellipticity of $K(t,x,y)$ with respect to the normal direction ν_x to introduce the trace

$$k(t,x,y) = \partial_{\nu_x} K(t,x,z)|_{(x,z)\in\partial\Omega\times\partial\Omega}.$$

Applying Theorem 3.1.1, the wave front

$$WF(h_0(t-s,z-y)|_{(y,z)\in\partial\Omega\times\partial\Omega}$$

satisfies a relation analogous to (3.10) and we can define the distribution

$$B(t,x,y) = -\int_{\infty}^{\infty}\int_{\partial\Omega} k(s,x,z)h_0(t-s,z-y)ds\,dz, \tag{3.13}$$

interpreted as the action of $h_0(t-s, z-y)$ on $k(s,x,z) \in \bar{\mathcal{D}}'(\mathbb{R}_s \times \partial\Omega \times \partial\Omega)$.

Consider $B(t,x,y)$ for $t > 0, x, y \in \partial\Omega, |x-y| < t$. A finite speed of propagation argument yields

$$\text{sing supp } k(t,x,z) \subset \{(t,x,z) \in \mathbb{R} \times \partial\Omega \times \partial\Omega : |x-z| \leq t\}.$$

Combining this with the information for sing supp $h_0(t-s, z-y)$, we conclude that the integrand in (3.13) is singular only if (t,s,x,y,z) satisfies

$$t > |x-y| \geq |z-y| - |x-z| \geq t - 2s.$$

To describe $WF(B(t,x,y)|_{t>0,\ |x-y|<t})$, we need to know only the singularities of $k(t,x,y)$ for $t > 0$.

Recall that the relation C, introduced in Section 1.2, is the set of points

$$(t,x,y,\tau,\xi,\eta) \in T^*(\mathbb{R} \times \Omega \times \Omega) \setminus \{0\},$$

such that $\tau^2 = |\xi|^2$, and (t,x,τ,ξ) and $(0,y,\tau,\eta)$ lie on a generalized bicharacteristic of $\Box$. According to Lemma 1.2.7, the relation C is closed. Denote by C_b the set of those

$$(t,x,y,\tau,\tilde{\xi},\tilde{\eta}) \in T^*(\mathbb{R} \times \partial\Omega \times \partial\Omega) \setminus \{0\}$$

such that there exist $\xi \in T_x^*(\Omega), \eta \in T_y^*(\Omega)$ with $\tilde{\xi} = p_x(\xi), \tilde{\eta} = p_y(\eta)$ and $(t,x,y,\tau,\xi,\eta) \in C$. Here, p_x is the projection, introduced at the end of Section 3.1. Repeating the proof of Lemma 1.2.7, we conclude that C_b is closed.

Proposition 3.3.1: *We have*

$$WF'\left(k(t,x,y)|_{t>0}\right) \subset C_b. \tag{3.14}$$

Proof: It is convenient to introduce the set $\tilde{C}_b$ of those

$$(t,s,x,y,\tau,\sigma,\tilde{\xi},\tilde{\eta}) \in T^*(\mathbb{R}_t \times \mathbb{R}_s \times \partial\Omega \times \partial\Omega) \setminus \{0\}$$

such that $\tau = \sigma$, and there exist $\xi \in T_x^*(\Omega), \eta \in T_y^*(\Omega)$ with $p_x(\xi) = \tilde{\xi}, p_y(\eta) = \tilde{\eta}, \tau^2 = |\xi|^2 = |\eta|^2$ such that (t,x,τ,ξ) and (s,y,τ,η) lie on a generalized bicharacteristic of $\Box$. We will prove that

$$WF'\left(k(t-s,x,y)|_{t>s}\right) \subset \tilde{C}_b \tag{3.15}$$

and (3.14) follows from (3.15) taking $s = 0$.

Consider an arbitrary $\rho_0 = (t_0, s_0, x_0, y_0, \tau_0, \sigma_0, \tilde{\xi}_0, \tilde{\eta}) \notin \tilde{C}_b$ with $t_0 > s_0$. Since $\tau = \sigma \neq 0$ on $WF'(k(t-s,\cdot,\cdot))$, we may assume $\tau_0 = \sigma_0 \neq 0$. The relation $\tilde{C}_b$ is closed, so there exist open conic neighbourhoods Γ_1 of $(s_0, y_0, \tau_0, \tilde{\xi}_0)$

and Γ_2 of $(t_0, x_0, \tau_0, \tilde{\xi}_0)$ so that $(s, y, \sigma, \tilde{\eta}) \in \Gamma_1$ and $(t, x, \tau, \tilde{\xi}) \in \Gamma_2$ imply $(t, s, x, y, \tau, \sigma, \tilde{\xi}, \tilde{\eta}) \notin \tilde{C}_b$. Let

$$\pi : T^*(\mathbb{R} \times \Omega) \longrightarrow \mathbb{R}$$

be the natural projection. We may choose Γ_1 and Γ_2 in a such a way that

$$\pi(\Gamma_1) = (\alpha, \beta), \pi(\Gamma_2) = (\gamma, \delta),\ \beta < \gamma.$$

Let $A_1 \in L^0(\mathbb{R} \times \partial\Omega)$ be a pseudo-differential operator on $\mathbb{R} \times \partial\Omega$ with full symbol equal to 1 in a small conic neighbourhood of $(s_0, y_0, \tau_0, \tilde{\eta}_0)$, and with wave front set $WF(A_1) \subset \Gamma_1$. We choose A_1 so that the kernel of A_1 has a compact support, contained in $(\alpha, \beta) \times \partial\Omega \times (\alpha, \beta) \times \partial\Omega$. In a similar way we choose a pseudo-differential operator $A_2 \in L^0(\mathbb{R} \times \partial\Omega)$ with full symbol equal to 1 in a small conic neighbourhood of $(t_0, x_0, \tau_0, \tilde{\xi}_0)$ such that $WF(A_2) \subset \Gamma_2$ and the kernel of A_2 has a compact support, contained in $\mathbb{R} \times \partial\Omega \times \mathbb{R} \times \partial\Omega$.

Fix $p \geq 2$. Then for each $f \in H_p^{loc}(\mathbb{R} \times \partial\Omega)$ we have

$$A_1 f \in H_p(\mathbb{R} \times \partial\Omega),\ \text{supp}\ A_1 f \subset (\alpha, \beta) \times \partial\Omega.$$

By Theorem 1.4.1, there exists a solution $(SA_1 f)(t, x)$ of the problem

$$\begin{cases} (\partial_t^2 - \Delta_x) SA_1 f = 0 \text{ in } \mathbb{R} \times \Omega^\circ, \\ SA_1 f - A_1 f = 0 \text{ on } \mathbb{R} \times \partial\Omega, \\ (SA_1 f)|_{t < \alpha} = 0. \end{cases} \tag{3.16}$$

Thus, we obtain a linear map

$$H_p^{loc}(\mathbb{R} \times \partial\Omega) \ni f \to SA_1 f \in \bar{H}_p^{loc}(\mathbb{R} \times \Omega^\circ).$$

Define

$$Rf = \frac{\partial}{\partial \nu_x} (SA_1 f)|_{x \in \partial\Omega},$$

and note that Rf can be obtained by the action of the distribution $k(t - s, x, z)$ to $(A_1 f)(s, z)$. For the solution $SA_1 f$ of (3.16) we are in a position to apply the results for propagation of singularities in [MS2] (see Theorem 1.4.3). Consequently, for $t \in (\gamma, \delta)$, the singularities of $(SA_1 f)(t, x)$ are described by the generalized bicharacteristics of $\Box$ issued from the points $(s, y, \sigma, \eta) \in T^*(\mathbb{R} \times \Omega)$ such that

$$\sigma^2 = |\eta|^2,\ (s, y, \sigma, p_y(\eta)) \in WF(A_1 f) \subset \Gamma_1.$$

Our assumptions yield $WF(Rf) \cap \Gamma_2 = \emptyset$, hence we obtain a linear map

$$H_p^{loc}(\mathbb{R} \times \partial\Omega) \ni f \longrightarrow A_2 Rf \in C^\infty(\mathbb{R} \times \partial\Omega).$$

The energy estimates for (3.16) (see Section 24.1 in [H3]) show that A_2R is a closed map, defined on $H_p^{loc}(\mathbb{R} \times \partial\Omega)$, and by the closed graph theorem one concludes that A_2R is continuous. Therefore, the kernel of A_2R is C^∞ smooth and

$$A_2(t, x, D_t, D_x)A_1^*(s, y, D_s, D_y)k(t - s, x, y) \in C^\infty,$$

A_1^* being the operator formally adjoint to A_1. Choose a zero order pseudo-differential operator

$$C(t, x, x, y, D_t, D_x, D_s, D_y) \in L^0$$

which is elliptic at ρ_0. Then $CA_2A_1^*$ is an operator in L^0, which is elliptic at ρ. Now $CA_2A_1^*k \in C^\infty$ implies $\rho_0 \notin WF'(k(t - s, x, y))$, and this completes the proof of the proposition. ∎

From now on we assume that Ω is convex. We wish to find the composition

$$WF'(k(s, x, z)) \circ WF'\left(h_0(t - s, z - y)|_{(y,z)\in\partial\Omega\times\partial\Omega}\right).$$

By Theorem 3.1.1, if

$$(t, s, z, y, \tau, \sigma, \tilde{\xi}, \tilde{\eta}) \in WF'\left(h_0(t - s, z - y)|_{(y,z)\in\partial\Omega\times\partial\Omega}\right),$$

we have

$$z = y \pm (t - s)\frac{\eta}{|\eta|},\ \tilde{\xi} = \tilde{\eta},\ z \in \partial\Omega,$$

with $\eta \in T_y^*(\Omega),\ |\eta|^2 = \tau^2,\ p_y(\eta) = \tilde{\eta}$. Moreover, the intersection

$$\left\{z = y + \mu\frac{\eta}{|\eta|} :\ \mu \in \mathbb{R}\right\} \cap \Omega$$

is convex. Therefore, we may consider the generalized bicharacteristic of $\Box$, issued from a point $(z, \zeta) \in T^*(\Omega)$ with $p_z(\zeta) = \tilde{\zeta} = \tilde{\eta}$, $|\zeta|^2 = \tau^2$, as a part of a generalized bicharacteristic issued from (y, η). Then, taking into account (3.13) and applying Theorem 1.3.9, we obtain

$$WF'\left(B(t, x, y)|_{t>0,\ |x-y|<t}\right) \subset C_b. \tag{3.17}$$

The advantage of the equality

$$\int_\Omega E(t, x, x)dx = \int_{\partial\Omega} B(t, x, x)dx \mod C^\infty(\mathbb{R}_t^+)$$

is that $\partial\Omega$ is a manifold without boundary and we may exploit (3.17) to describe the singularities of the right-hand side integral. Consider the map

$$\kappa :\ \mathbb{R} \times \partial\Omega \ni (t, x) \to (t, x, x) \in \mathbb{R} \times \partial\Omega \times \partial\Omega.$$

The normal set N_κ of κ satisfies $C_b \cap N_\kappa = \emptyset$. By using the pull-back κ^* of κ, an application of Theorem 1.3.6 yields

$$\begin{aligned}&WF(B(t,x,x)|_{t>0,\, x\in\partial\Omega})\\&\subset \{(t,x,\tau,\tau,\tilde{\xi}-\tilde{\eta}) \in T^*(\mathbb{R}\times\partial\Omega)\setminus\{0\} :\ (t,x,x,\tau,\tilde{\xi},\tilde{\eta}) \in C_b\}.\end{aligned}$$

Finally, we apply Theorem 1.3.8 for the integral over small open sets ω_j covering $\partial\Omega$ and deduce that

$$WF\left(\int_{\partial\Omega} B(t,x,x)dx|_{t>0}\right)$$

is contained in the set

$$\{(t,\tau) \in T^*(\mathbb{R}_t^+)\setminus\{0\} :\ \exists(x,\tilde{\xi}) \in T^*(\partial\Omega) \text{ with } (t,x,x,\tau,\tilde{\xi},\tilde{\xi}) \in C_b\}.$$

On the other hand, as we have mentioned in Section 1.2, $(T,x,x,\tau,\xi,\xi) \in C$ means that there exists a periodic generalized bicharacteristic of $\Box$ with period T passing through (x,ξ).

Thus, we obtain the following.

Theorem 3.3.2: *For every compact convex domain Ω in $\mathbb{R}^n$, $n \geq 2$, with C^∞ smooth boundary, we have*

$$\text{sing supp } \sigma(t) \subset \{0\} \cup \{\pm T_\gamma : \gamma \in \mathcal{L}_\Omega\}, \tag{3.18}$$

where $\mathcal{L}_\Omega$ is the set of all periodic generalized bicharacteristics of $\Box$ in Ω.

3.4 Poisson relation for arbitrary domains

We consider the same problem as in the previous section treating arbitrary domains Ω. If we follow the proof in the convex case, we must establish the inclusion (3.17). Since the intersection of straight lines, issued from y and Ω, could be non-convex, we are going to consider the generalized bicharacteristics passing through all points $(s,z) \in \mathbb{R}\times\partial\Omega$ such that

$$(t-s,z,y) \in \text{sing supp } h_0(t-s,z-y)|_{y\in\partial\Omega, z\in\partial\Omega}.$$

This leads to some singularities that are not described by the relation C_b.

Throughout this section we apply the notation from the previous ones. The following assertion can be established repeating the proof of Proposition 3.3.1. We leave the details to the reader.

Proposition 3.4.1: *The intersection $WF'(K(t,x,z) \cap T^*(\mathbb{R}_t^+ \times \Omega^\circ \times \partial\Omega)$ is contained in the set of all points*

$$(t,x,z,\tau,\xi,\tilde{\zeta}) \in T^*(\mathbb{R}_t^+ \times \Omega^\circ \times \partial\Omega)\setminus\{0\}$$

such that there exists $\zeta \in T_z^*(\Omega)$ *with* $p_z(\zeta) = \tilde{\zeta}, \tau^2 = |\xi|^2 = |\zeta|^2$ *and* (t, x, τ, ξ) *and* $(0, z, \tau, \zeta)$ *lie on a common generalized bicharacteristic of* $\Box$.

In the following we examine $E(t, x, y)$ for $t > 0$ and $|x - y| < t$. First, we deal with the case $x, y \in \Omega^\circ$.

Proposition 3.4.2: *We have the inclusion*

$$WF'(E(t, x, y)|_{t>0, |x-y|<t}) \cap \{(t, x, y, \tau, \xi, \eta) \in T^*(\mathbb{R} \times \Omega^\circ \times \Omega^\circ)\} \subset C. \tag{3.19}$$

Proof: Assume that $q_0 = (t_0, x_0, y_0, \tau_0, \xi_0, \eta_0) \notin C$ for some

$$t_0 > 0, x_0, y_0 \in \Omega^\circ, \ |x_0 - y_0| < t_0.$$

Notice that for $t > 0, x, y \in \Omega^\circ$ we have

$$(\partial_t^2 - \Delta_x) E(t, x, y) = 0, \ (\partial_t^2 - \Delta_y) E(t, x, y) = 0,$$

consequently $\tau^2 = |\xi|^2 = |\eta|^2$. In particular, $\tau_0^2 = |\xi_0|^2 = |\eta_0|^2$ if $q_0 \in WF'$ $(E(t, x, y))$. We assume that $\tau_0 < 0$, the case $\tau_0 > 0$ is treated similarly. Choose small conic neighbourhoods $\Gamma_1 = V_1 \times \Sigma_1$ of (y_0, η_0) and $\Gamma_2 = V_2 \times \Sigma_2$ of (x_0, ξ_0) and $\delta_0 > 0$ such that $(y, \eta) \in \Gamma_1, (x, \xi) \in \Gamma_2, |t - t_0| < \delta_0$ imply $(t, x, y, \tau, \xi, \eta) \notin C$ for $\tau^2 = |\xi|^2 = |\eta|^2$. We may take V_i so that $\bar{V}_i \subset \Omega^\circ, i = 1.2$.

Let $A_1(y, D_y) \in L^0(\Omega^\circ)$ be a pseudo-differential operator, the full symbol $a_1(y, \eta)$ of which equals to 1 in some conic neighbourhood of (y_0, η_0), the kernel of A_1 has a compact support in $\Omega^\circ \times \Omega^\circ$ and $WF(A_1) \subset \gamma_1$. Consider the straight line

$$L_0 = \{z \in \mathbb{R}^n : \ z = y_0 + \sigma \eta_0, \sigma \in \mathbb{R}\},$$

and let $L_0 \cap \Omega = \cup_{j \in P} l_j$, where $l_j = [x_j, x_{j+1}], x_j \in \partial\Omega, l_j \cap l_k = \emptyset$ for $k \neq j$ and $P \subset \mathbb{Z} \cap [-p, q], p, q$ being non-negative integers or ∞. We may assume that $y_0 \in l_0$. Notice that in general $l_0 \cap \partial\Omega$ could contain non-trivial segments lying on $\partial\Omega$.

Let $j \in P$ and $x_j = y_0 + \sigma_j \eta_0 / |\eta_0|$. Assume that $-2, 2 \in P$. Then

$$0 \leq \sigma_1 < \sigma_2, \sigma_{-1} < \sigma_0 \leq 0.$$

Choose $\epsilon' < (\sigma_2 - \sigma_1)/2, \epsilon' < (\sigma_0 - \sigma_{-1})/2$ and two functions $\kappa(t), \chi(t) \in C^\infty(\mathbb{R})$ such that

$$\kappa(t) = \begin{cases} 1 \text{ for } t \geq \sigma_0 - \epsilon', \\ 0 \text{ for } t \leq \sigma_{-1} + \epsilon', \end{cases}$$

$$\chi(t) = \begin{cases} 1 \text{ for } t \leq \sigma_1 + \epsilon', \\ 0 \text{ for } t \geq \sigma_2 - \epsilon'. \end{cases}$$

Now set

$$e_1(t,x,y) = A_1(y,D_y)e_0(t,x-y),$$
$$e_2(t,x,y) = e_0(t,x,y) - e_1(t,x,y),$$
$$\tilde{e}_1(t,x,y) = \kappa(t)\chi(t)A_1(y,D_y)e_0(t,x-y).$$

Notice that $(\partial_t^2 - \Delta_x)(e_1 - \tilde{e}_1)(t,x,y) \neq 0$ implies $\sigma_1 + \epsilon' \leq t \leq \sigma_2 - \epsilon'$ or $\sigma_{-1} + \epsilon' \leq t \leq \sigma_0 - \epsilon'$. For sufficiently small Γ_1 and ϵ' we obtain

$$(\partial_t^2 - \Delta_x)(\tilde{e}_1 + e_2) \in C^\infty(\mathbb{R}\times\Omega\times\Omega),$$
$$(\tilde{e}_1 + e_2)(0,x,y) = 0,$$
$$\partial_t(\tilde{e}_1 + e_2)(0,x,y) - \delta(x-y) \in C^\infty(\Omega\times\Omega).$$

This implies

$$E(t,x,y) - (\tilde{e}_1 + e_2)(t,x,y) + \int_\infty^\infty \int_{\partial\Omega} K(s,x,z)(\tilde{e}_1 + e_2) \times(t-s,z,y)ds\,dz \in C^\infty(\mathbb{R}\times\Omega\times\Omega). \tag{3.20}$$

For $|x-y| < t$ the sum $(\tilde{e}_1 + e_2)(t,x,y)$ is C^∞ smooth, so we have to examine the integral

$$\widetilde{E}_1(t,x,y) = \int_\infty^\infty \int_{\partial\Omega} K(s,x,z)\tilde{e}_1(t-s,z,y)ds\,dz \tag{3.21}$$

and a similar one with e_2 instead of $\tilde{e}_1$. Applying Theorem 1.3.9, we deduce

$$WF'(\widetilde{E}_1(t,x,y)) \cap \{(t,x,y,\tau,\xi,\eta) \in T^*(\mathbb{R}^+\times V_2\times V_1) :\ |x-y| < t\}$$
$$\subset WF'\left(K(s,x,z)|_{x\in V_2, z\in\partial\Omega}\right) \circ WF'\left(\tilde{e}_1(t-s,z,y)|_{y\in V_1, z\in\partial\Omega}\right), \tag{3.22}$$

where the composition of wave front sets is taken with respect to s and z. Now, it is easy to see that $q_0 \notin WF'(\widetilde{E}_1(t,x,y))$. Indeed, assume that for some $(\hat{s},\hat{z},\hat{\sigma},\hat{\zeta}) \in T^*(\mathbb{R}\times\partial\Omega)$, we have

$$(t_0,\hat{s},\hat{z},y_0,\tau_0,\hat{\sigma},\hat{\zeta},\eta_0) \in WF'(\tilde{e}_1(t-s,z,y)),$$
$$(\hat{s},x_0,\hat{z},\hat{\sigma},\xi_0,\hat{\zeta}) \in WF'(K(s,x,z)).$$

First, assume that $t - \hat{s} > 0$. Then

$$\hat{\sigma} = \tau_0,\ \ \hat{z} = y_0 + (t_0 - \hat{s})\frac{\eta_0}{|\eta_0|},\ \ \hat{\zeta} = p_{\hat{z}}(\eta_0)$$

and the construction of $\tilde{e}_1$ yields $\hat{z} \in l_0$. Thus, the generalized bicharacteristic issued from $(t_0 - \hat{s},\hat{z},\tau_0,\zeta)$ with $p_{\hat{z}}(\zeta) = \hat{\zeta}, |\zeta|^2 = \tau_0^2$ is a part of a generalized bicharacteristic issued from $(0,y_0,\tau_0,\eta_0)$ and passing through (t_0,x_0,τ_0,ξ_0). This is a contradiction with the choice of q_0. The case $t_0 = \hat{s}$ is handled by a similar argument.

The analysis of the integral involving e_2 is trivial and we conclude that

$$q_0 \notin WF'(E(t,x,y)),$$

which completes the proof of the proposition. ∎

Now we pass to the analysis of the singularities of $E(t,x,y)$ for x and y close to some point $z_0 \in \partial\Omega$. In an open neighbourhood U of z_0, we choose local coordinates $x = (x_1, \ldots, x_{n-1}, x_n) = (x_1, x'), x_1 = \text{dist}(x, \partial\Omega)$ such that $\partial\Omega \cap U$ is given by $x_1 = 0$ and $\Omega \cap U$ lies in the half-space $x_1 \geq 0$. As in Section 1.2, in these local coordinates the principal symbol of $\partial_t^2 - \Delta_x$ has the form $q(x,\tau,\xi) = \xi_1^2 - q_2(x_1, x', \xi') - \tau^2$, where $\xi' = (\xi_2, \ldots, \xi_n)$ and $-q_2(x,\xi')$ is a positively definite quadratic form in ξ'. For $t > 0$ we have $(\partial_t^2 - \Delta_x)E(t,x,y) = 0$ and the boundary $\partial\Omega$ is non-characteristic for $\partial_t^2 - \Delta_x$. Therefore, the partial hypoellipticity of $E(t,x,y)$ implies that $E(t,x,y)|_{t>0}$ is a C^∞ smooth function of x_1 with values in the space of distributions $\mathcal{D}'(\mathbb{R}_t^+ \times \mathbb{R}_{x'}^{n-1} \times \Omega)$. Since the same argument works for the variables $y = (y_1, y')$, for sufficiently small $\epsilon > 0$ we obtain a C^∞ function

$$H : [0,\epsilon] \times [0,\epsilon] \longrightarrow \mathcal{D}'(\mathbb{R}^+ \times U' \times U'), H(x_1, y_1) = E(t, x_1, x', y_1, y'),$$

where $U' \subset \mathbb{R}^{n-1}$ is a small neighbourhood of 0. For this purpose we exploit the smoothness of E with respect to y_1 and we apply Theorem B.2.9 in [H3], considering y_1 as a parameter. The proof of the later theorem can be trivially extended to cover a smooth dependence on a parameter and we deduce the above assertion.

Let L_Ω be *the set of periods of all periodic generalized bicharacteristics of* $\Box$ *in* Ω. According to Lemma 1.2.10, the set L_Ω is closed in $\mathbb{R}$.

Proposition 3.4.3: *Let* $t_0 > 0, t_0 \notin L_\Omega$. *Then for sufficiently small* $\epsilon > 0$ *we have*

$$(t_0, x', x', \tau, \xi', \xi') \notin WF'(H(x_1, y_1)),$$

whenever $(x_1, y_1) \in [0,\epsilon] \times [0,\epsilon], \tau \in \mathbb{R}$ *and* $(x', \xi') \in U \times \mathbb{R}^{n-1}$.

The proof of this proposition is long, and we begin with some preparations. The main difficulty is to provide uniformity with respect to x_1 and y_1. We follow the idea of the proof of Proposition 3.4.2, by studying the singularities of a distribution similar to $\widetilde{E}_1(t,x,y)$. If

$$(t, x', y', \tau, \xi', \eta') \in WF'(H(x_1, y_1)),\ t > 0,$$

then the point (t, x', τ, ξ') belongs to the compressed characteristic set Σ_b of $\partial_t^2 - \Delta_x$, with respect to the surface $x_1 = const$, introduced in Section 1.2. This means that the equation $q(x,\tau,\xi) = \xi_1^2 - q_2(x_1, x', \xi') - \tau^2 = 0$ with respect to ξ_1 has only real roots and this implies $-q_2(x_1, x', \xi') \leq \tau^2$. Therefore, $\tau = 0$ leads to $\xi' = 0$ and $\eta' = 0$. Hence we may assume $\tau \neq 0$.

Consider a point $\rho_0 = (t_0, x_0', x_0', \tau_0, \xi_0', \xi_0')$ with

$$\tau_0 < 0, -q_2(0, x_0', \xi_0') \leq \tau_0^2, (x_0', \xi_0') \in U' \times \mathbb{R}^{n-1}.$$

The case $\tau_0 > 0$ can be treated by a similar argument.

Below we choose $0 < \epsilon < t_0/4$ and obtain

$$(t_0, x_1, x_0', y_1, y_0') \notin \text{sing supp } e_0(t, x - y)$$

for $x_1, y_1 \in [0, \epsilon]$. For technical reasons it is more convenient to study the distribution

$$\Xi(t - t', x, y) = E(t - t', x, y)Y(t - t'),$$

where

$$Y(t) = \begin{cases} 1, \ t \geq 0, \\ 0, \ t < 0, \end{cases}$$

is the *Heaviside function*, and to take the trace $t' = 0$. Let

$$A_1(t', y, D_{t'}, D_{y'}) \in L^0(\mathbb{R}_{t'} \times \mathbb{R}^{n-1}_{y'})$$

be a zero order pseudo-differential operator depending smoothly on $y_1 \in [0, \epsilon]$ such that the full symbol $a_1(t', y, \tau', \eta')$ of A_1 is equal to 1 in some conic neighbourhood $\tilde{\Gamma}_1$ of $(0, x_0', \tau_0, \xi_0')$ for $y_1 \in [0, \epsilon]$. Let $\Gamma_1 \supset \tilde{\Gamma}_1$ be another conic neighbourhood of the same point and let $WF(A_1(\cdot, y_1, \cdot)) \subset \Gamma_1$. Moreover, we assume that the kernel of $A_1(\cdot, y_1, \cdot)$ has a compact support in $\mathcal{T} \times U' \times \mathcal{T} \times U', \mathcal{T}$ being a small neighbourhood of 0 in $\mathbb{R}$.

Consider the equation

$$q(0, x_0', \tau_0, \xi_1, \xi_0') = 0 \tag{3.23}$$

with respect to ξ_1. First, we treat the case when this equation has a double real root ξ_1^0. Set $y_0 = (0, x_0'), \eta_0 = (\xi_1^0, \xi_0')$ and consider the line L_0 and the linear segments l_j introduced in the proof of Proposition 3.4.2. Next, define

$$\begin{aligned} e_1(t - t', x, y) &= A_1(t', y, D_{t'}, D_{y'})e_0(t - t', x, t), \\ e_2(t - t', x, y) &= e_0(t - t', x, y) - e_1(t - t', x, y), \\ \tilde{e}_1(t - t', x, y) &= \kappa(t - t')\chi(t - t')e_0(t - t', x, y), \end{aligned}$$

where $\kappa(t)$ and $\chi(t)$ are the functions introduced in the proof of Proposition 3.4.2. Thus, for $\Gamma_1, \mathcal{T}$ and ϵ sufficiently small we are going to study

$$\tilde{\Xi}_1(t - t', x, y) = \int_{\infty}^{\infty} \int_{\partial\Omega} K(s, z, z)\tilde{e}_1(t - t' - s, z, y) ds\, dz.$$

For the distribution $\tilde{\Xi}_1(t-t', x, y)$ we need an inclusion similar to that in Proposition 3.4.1 for x close to $\partial\Omega$. For this purpose by using the fact that $\partial\Omega$ is not characteristic for $\Box$, for $t > 0$ and for small $\epsilon > 0$, one introduces the C^∞ function

$$\mathcal{K} : [0, \epsilon] \longrightarrow \mathcal{D}'(\mathbb{R}_t^+ \times U' \times \partial\Omega), \ \mathcal{K}(x_1) = K(t, x_1, x', z).$$

We have the following.

Proposition 3.4.4: *For sufficiency small $\epsilon > 0$ and $x_1 \in [0, \epsilon], WF'(\mathcal{K}(x_1))$ is contained in the set of all points $(t, x', z, \tau, \xi', \tilde{\zeta}) \in T^*(\mathbb{R}_t^+ \times U' \times \partial\Omega) \setminus \{0\}$ such that there exist $\zeta \in T_z^*(\Omega)$ and $\xi_1 \in \mathbb{R}$ with $p_z(\zeta) = \tilde{\zeta}, \tau^2 = |\zeta|^2, q(x_1, x', \tau, \xi_1, \xi') = 0$, and $(0, z, \tau, \zeta)$ and $(t, x_1, x', \tau, \xi_1, \xi')$ lie on a common generalized bicharacteristic of $\Box$.*

Proof: For $\hat{x}_1 > 0$ the result follows from Proposition 3.4.1 taking the trace on $x_1 = \hat{x}_1$. For $\hat{x}_1 = 0$ the proof is a modification of that of Proposition 3.3.1. By using the notation of this proof, it suffices to apply the results in [MS2] or [H3], Section 24, concerning the propagation of the generalized wave front set $WF_b(SAf)$ introduced in Section 1.4. Then for some pseudo-differential operator $B_0(t, x_1, x', D_{t'}, D_{x'}) \in L^0(\mathbb{R}_t \times \mathbb{R}_{x'}^{n-1})$, depending smoothly on x_1, we obtain a linear map

$$H_p^{loc}(\mathbb{R} \times \partial\Omega) \ni f \longrightarrow B_0 SAf \in C^\infty(\mathbb{R}_t \times \mathbb{R}^n),$$

and we apply the argument of the proof of Proposition 3.3.1. ∎

Consider the inclusion

$$\begin{aligned} &WF'\left(\tilde{\Xi}_1(t-t, x_1, x', y_1, y')\right) \cap \ T^*(\mathbb{R}_t^+ \times \mathbb{R}_{t'} \times U' \times U') \\ &\quad \subset WF'\left(K(s, x', x_n, z)|_{x' \in U', z \in \partial\Omega}\right) \circ WF'\left(\tilde{e}_1(t-t'-s, z, y_1, y')|_{y' \in U', z \in \partial\Omega}\right), \end{aligned} \tag{3.24}$$

where x_1, y_1 are parameters. Observe that for $\Gamma_1, \mathcal{T}$ and ϵ sufficiently small, the wave front set $WF'(\tilde{e}_1(t-t'-s, z, y_1, y'))$ has a projection on $T^*(\mathbb{R}_t \times \mathbb{R}^n)$, which is related to the straight lines issued from y with direction η, (y, η) being close to (y_0, η_0). The precise choice of ϵ will be discussed below.

Introduce

$$l_0(\sigma) = \left\{ \left(\sigma, y_0 + \sigma \frac{\eta_0}{|\eta_0|}\right) : \ 0 \leq \sigma \leq \sigma_1 \right\}.$$

Recall that the set $C_t(\mu)$ is formed by points $\nu \in T^*(\mathbb{R} \times \Omega)$ such that there exists a generalized bicharacteristic $\Gamma(t)$ of $\Box$ with $\gamma(0) = \mu, \gamma(t) = \nu$. Consider the metric $D(\rho, \mu)$ defined in Section 1.2 and recall that $D(\rho, \mu) = 0$ implies $\rho = \mu$ or $\rho = (x, \xi), \mu = (x, \eta)$ with $x \in \partial\Omega, p_x(\xi) = p_x(\eta)$. We denote by $\gamma(t; \mu)$ one of the generalized bicharacteristics of $\Box$ parameterized by the time t and passing through μ for $t = 0$. Thus, we have $C_t(\mu) = \cup\gamma(t;\mu)$, where the union is taken over all bicharacteristics issued from μ. Set $\nu_0 = (0, 0, x_0', \tau_0, \xi_1^0, \xi_0') = (0, y_0, \tau_0, \eta_0)$.

Lemma 3.4.5: *For each $\delta > 0$, there exists $\epsilon(\delta) > 0$ such that if $D(\mu, l_0(\sigma)) < \epsilon(\delta)$ for some $\sigma \in [0, \sigma_1]$, then for each $\nu \in C_{t_0-\sigma}(\mu)$ we have*

$$D(\nu, C_{t_0}(\nu_0)) = \inf_{\rho\in C_{t_0}(\nu_0)} D(\nu, \rho) < \delta.$$

Remark 3.4.6: This lemma says that if $\bar{\gamma}(\sigma;\nu)$ is a curve that coincides with a linear segment, passing through ν for some $0 \leq \sigma \leq \hat{\sigma}$ and with the generalized bicharacteristic issued from $\mu = \bar{\gamma}(\hat{\sigma}, \nu)$ for $\hat{\sigma} \leq \sigma \leq t_0$, then $\bar{\gamma}(t_0;\nu)$ is close to $C_{t_0}(\nu_0)$, provided ν is close to ν_0. We need this property because in general $\bar{\gamma}(\sigma;\nu), 0 \leq \sigma \leq t_0$, is not a generalized bicharacteristic of $\Box$.

Proof of Lemma 3.4.5: Assume that there exist sequences $\{\sigma_k\}, 0 \leq \sigma_k \leq \sigma_1$, $\{\mu_k\}$ and $\nu_k = \gamma(t_0 - \sigma_k; \mu_k)$ so that

$$D(\mu_k, l_0(\sigma_k)) < \frac{1}{k},\ \ D(\nu_k, C_{t_0}(\nu_0)) \geq \delta,\ \ \forall k \in \mathbb{N}.$$

Taking subsequences, we can suppose

$$\sigma_k \to \hat{\sigma},\ \ D(\mu_k, l_0(\hat{\sigma})) \to 0 \text{ as } \quad k \to +\infty.$$

A simple argument shows that there exists a subsequence $\{\mu_{k_m}\}$ converging to $v_0 = l_0(\hat{\sigma})$ or to v_1 such that $D(v_0, v_1) = 0$. The latter is possible only if $\hat{\sigma} = \sigma_1$ and if l_0 hits transversally $\partial\Omega$.

Next, we suppose that $\mu_k \to v$ in the usual sense, where $v = v_0$ or $v = v_1$. Consider the sequence of generalized bicharacteristics

$$\{\gamma(t;\mu_k)\},\ \ 0 \leq t \leq t_0.$$

According to Lemma 1.2.6, there exists a subsequence $\{\mu_{k_m}\}$ and a generalized bicharacteristic $\tilde{\gamma}(t;v)$ of $\Box$ so that for all $t \in [0, t_0]$, we have

$$D(\gamma(t;\mu_{k_m}), \tilde{\gamma}(t;v)) \to 0 \text{ as } m \to +\infty.$$

Without loss of generality, we may assume that $k_m = m$ for each m. Applying the triangle inequality for the metric D, we get

$$\begin{aligned} D(\nu_k, \tilde{\gamma}(t_0 - \hat{\sigma}; v)) &\leq D(\gamma(t_0 - \sigma_k; \mu_k), \gamma(t_0 - \hat{\sigma}; \mu_k) \\ &\quad + D(\gamma(t_0 - \hat{\sigma}; \mu_k), \tilde{\gamma}(t_0 - \hat{\sigma}; v)). \end{aligned}$$

Following Lemma 1.2.5, for the first term on the right-hand side we apply the estimate

$$D(\gamma(t';\mu),\gamma(t'';\mu)) \leq C_0|t'-t''|$$

for $t', t'' \in [0, t_0]$ uniformly on μ in some small neighbourhood of v. Thus, for k_0 large enough one obtains

$$D(\nu_k, \tilde{\gamma}(t_0-\hat{\sigma};v)) \leq \delta,\ k \geq k_0.$$

Define the generalized bicharacteristic

$$\gamma_0(\sigma;\nu_0) = \begin{cases} l_0(\sigma),\ 0 \leq \sigma \leq \hat{\sigma}, \\ \tilde{\gamma}(\sigma-\hat{\sigma};v),\ \hat{\sigma} \leq \sigma. \end{cases}$$

Then $\tilde{\gamma}(t_0-\hat{\sigma};v) \in C_{t_0}(\nu_0)$ and we obtain a contradiction. This completes the proof. ■

Proof of Proposition 3.4.3: Consider again the inclusion (3.24). As we have already observed, for $0 \leq t_0 - s \leq \sigma_1$, the set $WF'(\tilde{e}_1(t_0-s, z, y_1, y'))$ has a projection on Ω close to $l_0(\sigma)$. Choosing $\Gamma_1, \mathcal{T}$ and ϵ sufficiently small, we apply Lemma 3.4.5 and Proposition 3.4.4 and conclude that for $t = t_0, t' = 0$, the projection of the wave front set $WF'(\tilde{\Xi}_1)$ on $T^*(\mathbb{R}\times\Omega)$, is sufficiency close to $C_{t_0}(\nu_0)$. On the other hand, $t_0 \notin L_\Omega$ implies $(t_0, y_0, \tau_0, \eta_0) \notin C_{t_0}(\nu_0)$. Now we take the trace $t' = 0$ and exploit the fact that $C_{t_0}(\nu_0)$ is closed as a consequence of Lemma 1.2.7. Thus, for small Γ_1 and ϵ we obtain

$$\rho_0 \notin WF'(H(x_1,y_1)) \text{ for } (x_1,y_1) \in [0,\epsilon]\times[0,\epsilon]. \tag{3.25}$$

Here the choice of ϵ depends on that of Γ_1. This completes the analysis of the case when (3.23) has a double root.

Next, we pass to the case when the point $(0, x_0', \tau_0, \xi_0')$ is hyperbolic for $\Box$, that is equation (3.23) with respect to ξ_1 has two distinct real roots $\xi_\pm^0$. Let Γ_1 be a small conic neighbourhood of $(0, x_0', \tau_0, \xi_0')$ such that the points $(t, x', \tau, \xi') \in \Gamma_1$ are hyperbolic (see Section 1.2). Let $A_1(t', y, D_{t'}, D_{y'})$ be a pseudo-differential operator as mentioned above with $WF(A_1) \subset \Gamma_1$. The singularities of the distribution

$$e_1(t-t',x,y) = A_1(t',y,D_{t'},D_{y'})e_0(t-t',x-y)$$

for small $t - t' > 0$ are propagating along the outgoing and incoming bicharacteristics entering the exterior or interior of Ω, respectively. For $0 < 2\delta < \epsilon$ introduce a function $\kappa_\delta(t) \in C^\infty(\mathbb{R})$ such that

$$\kappa_\delta(t) = \begin{cases} 1,\ t \leq \delta, \\ 0,\ t \geq 2\delta. \end{cases}$$

For sufficiently small δ consider the distributions

$$e_2(t-t',x,y)(1-A)e_0(t-t',x-y),$$
$$\tilde{e}_1(t-t',x,y) = e_1(t-t',x,y)\kappa_\delta(t-t')\mathbf{1}_\Omega(x),$$

$\mathbf{1}_\Omega(x)$ being the characteristic function of Ω. For $(t,x) \in \mathbb{R} \times \Omega^\circ$ we have

$$(\partial_t^2 - \Delta_x)\tilde{e}_1(t-t',x,y) = f_\delta(t-t',x,y),$$

and for sufficiently small $\delta > 0$, we deduce

$$\text{sing supp } f_\delta(t-t',x,y) \subset \{(t,x,t',y) :\ \delta \le t-t' \le 2\delta, x \in \mathcal{O}_\delta\}, \tag{3.26}$$

where $\bar{\mathcal{O}}_\delta \subset \Omega^\circ$. Hence the singularities of f_δ for t' small enough are bounded away from the boundary $\partial\Omega$. We extend $E(t,x,y)$ as 0 for $t < 0$, and for $t \ge 0$ we write

$$\begin{aligned}\Xi(t-t',x,y) = (\tilde{e}_1 + e_2)(t-t',x,y) \\ &- \int_\infty^\infty \int_{\partial\Omega} K(s,x,z)(\tilde{e}_1+e_2)(t-t'-s,z,y)ds\,dz \\ &- \int_\infty^\infty \int_{\partial\Omega} E(t-t'-s,x,z)f_\delta(s,z,y)ds\,dz.\end{aligned}$$

The integrals are interpreted in the sense of distributions. The smoothness of $e_0(t,x-y)$ with respect to $t \in \mathbb{R}^+$ and (3.26) make this possible.

The analysis of the term involving $\tilde{e}_1$ is easy since the singularities of $\tilde{e}_1$ are concentrated around the bicharacteristics entering Ω°. For the term involving f_δ we apply a similar argument. To do this, we prove an assertion similar to Proposition 3.4.4 with $\partial\Omega$ replaced by an open domain $\mathcal{O}, \mathcal{O} \subset \Omega^\circ$. After this, we establish an analogue of (3.19) for

$$WF'(E(t,x_1,x',y)) \cap T^*(\mathbb{R}_t^+ \times U' \times \mathcal{O})$$

uniformly with respect to $x_1 \in [0,\epsilon]$. This can be done applying the arguments from the proofs of Propositions 3.4.2 and 3.4.4 with slight modifications. We leave the details to the reader. Combining these results and taking the trace $t' = 0$, we obtain (3.25).

In the above argument the choice of ϵ has been related to that of the conic neighbourhood Γ_1. To obtain uniformity with respect to U', consider a covering

$$\mathcal{T} \times U' \times (\mathbb{R}^n \setminus \{0\}) \subset \cup_{j=1}^M \Gamma_j$$

with conic neighbourhoods Γ_j for which our argument works with $\epsilon = \epsilon_j > 0$. Taking $\epsilon = \min \epsilon_j$, we obtain $\epsilon > 0$ that depends on U' and T_0, only. This completes the proof of Proposition 3.4.3, since the relation (3.25) holds for $\epsilon > 0$ chosen above. ■

Now it is easy to establish the inclusion (3.18). Let

$$\Omega \subset \cup_{k=1}^{m} U_k \tag{3.27}$$

be a covering with open sets $U_k \subset \mathbb{R}^n$. Assume that for $k = 1, \dots, m_0$, we have $U_k \subset \Omega^\circ$, while $U_k \cap \partial\Omega \neq \emptyset$ for $k = m_0 + 1, \dots, m$. Let $t_0 \neq L_\Omega, t_0 > 0$.

Fix a neighbourhood $U_k \subset \Omega^\circ$. Applying (3.19), as at the end of Section 3.3, we deduce that

$$WF\left(\int_{U_k} E(t, x, x) dx|_{t>0}\right)$$

is contained in the set of all $(t, \tau) \in T^*(\mathbb{R}) \setminus \{0\}$ such that there exists $(x, \xi) \in T^*(U_k)$ with $(t, x, x, \tau, \xi, \xi) \in C$.

We pass to the case $U_k \cap \partial\Omega \neq \emptyset$. Introduce in U_k local coordinates (x_1, x') so that $U_k \cap \partial\Omega$ has the form $x_1 = 0$, while $U_k \cap \Omega$ is given by $x' \in U', 0 \leq x_1 \leq \epsilon$. For sufficiently small $\delta_0 > 0$ we have $t_0 - \delta_0 > 0$ and

$$(t_0 - \delta_0, t_0 + \delta_0) \cap L_\Omega = \emptyset.$$

Set $\Delta_0 = (t_0 - \delta_0, t_0 + \delta_0)$. Taking δ_0 and ϵ small enough, Proposition 3.4.3 implies

$$(t, x', x', \tau, \xi', \xi') \notin WF(H(x_1, y_1)) \tag{3.28}$$

for $t \in \Delta_0, (x', \xi') \in U' \times \mathbb{R}^{n-1}, (x_1, y_1) \in [0, \epsilon] \times [0, \epsilon]$. We consider t, x_1, y_1 as parameters and apply the argument at the end of Section 3.3 for the trace $x' = y'$ and the integral over U'. Therefore, the relation (3.28) yields

$$\int_{U'} E(t, x_1, x', x_1, x') dx'|_{t \in \Delta_0, x_1 \in [0, \epsilon]} \in C^\infty,$$

and integrating with respect to x_1, we get

$$\int_0^\epsilon \int_{U'} E(t, x_1, x', x_1, x') dx_1 \; dx' \in C^\infty(\Delta_0).$$

Consequently, we can find a sufficiently fine covering (3.27) such that

$$t_0 \notin \text{sing supp} \sum_{k=1}^{m} \int_{U_k \cap \Omega} E(t, x, x) dx$$

and we obtain the inclusion (3.18) for an arbitrary domain Ω with smooth boundary $\partial\Omega$.

The above argument, with some trivial modifications, can be applied to the analysis of the distribution

$$\sigma_N(t) = \sum_{j=1}^{\infty} \cos \lambda_j t \in \mathcal{S}'(\mathbb{R}),$$

where $0 \leq \lambda_1^2 \leq \lambda_2^2 \leq \cdots \leq \lambda_m^2 \leq \cdots$ are the eigenvalues of the self-adjoint operator A_N in $L^2(\Omega)$ related to the Laplacian in Ω with Neumann or Robin boundary conditions on $\partial\Omega$. The corresponding eigenfunctions $\{\varphi_j(x)\}_{j=1}^{\infty}$ satisfy

$$\begin{cases} -\Delta\varphi_j(x) = \lambda_j^2\varphi(x) \text{ in } \Omega, \\ (\partial_\nu + \alpha(x))\varphi_j(x) = 0 \text{ on } \partial\Omega. \end{cases}$$

Here ∂_ν denotes the derivative with respect to a continuous normal field $\nu(x)$ to $\partial\Omega$ and $\alpha(x) \in C^\infty(\partial\Omega)$. We define $\mathcal{E}(t, x, y)$ and $E(t, x, y)$ in the same way as in Section 3.2.

The proof of (3.18) for arbitrary domains goes without any change, by using the results for propagation of singularities in [MS2] concerning the Neumann and Robin boundary problems. For example, we define $K(t, x, y)$ as the solution of the problem

$$\begin{cases} (\partial_t^2 - \Delta_x)K(t, x, y) = 0 \text{ in } \mathbb{R} \times \Omega^\circ, \\ (\partial_\nu + \alpha(x))K(t, x, y) = \delta(t) \otimes \delta(x - y) \text{ for } x \in \partial\Omega, \\ \operatorname{supp} K(t, x, y) \subset \{(t, x, y) \in \mathbb{R}_t \times \Omega \times \partial\Omega : \ t \geq 0\}. \end{cases}$$

The assertions of Propositions 3.4.1 and 3.4.4 are true for K. Summing up the above results, we get the following.

Theorem 3.4.7: *Let Ω be a compact domain in $\mathbb{R}^n, n \geq 2$, with C^∞ smooth boundary $\partial\Omega$. Then*

$$\operatorname{sing\,supp}\ \sigma(t) \subset \{0\} \cup \{\pm T_\gamma : \ \gamma \in \mathcal{L}_\Omega\}. \tag{3.29}$$

The same is true for sing supp $\sigma_N(t)$*, where $\sigma_N(t)$ is related to Neumann or Robin boundary problem for the Laplacian.*

3.5 Notes

The analysis of the fundamental solution in Section 3.1 is taken from [BLR]. The idea of the proof in Section 3.3 of the Poisson relation for convex domains was proposed in [BLR]. For strictly convex (concave) domains this relation was obtained previously by Anderson and Melrose [AM]. The argument of [AM] can be generalized for general domains by using the results of propagation of singularities established in [MS1] and [MS2]. We followed the approach in [AM] exploiting the continuity properties of the generalized bicharacteristics described by Lemmas 1.2.6 and 3.4.5. A detailed investigation of $N(\lambda), e(x, y, \lambda)$ and the distribution $E(t, x, y)$ is contained in [H3] and [H4]. In particular, in [H4], Proposition 29.3.2, it was proved that $WF'(E(t, x, y)|_{\mathbb{R}_t \times \Omega^\circ \times \Omega^\circ}) \subset C$.

4

Poisson summation formula for manifolds with boundary

In this chapter the leading singularity of $\sigma(t) = \sum_j \cos \lambda_j t$ near the period T_γ of a periodic ordinary reflecting bicharacteristic γ of $\Box$ is examined. We assume that if δ is another period bicharacteristic of $\Box$ in Ω with the same period T_γ, then the projections of γ and δ on Ω coincide. Moreover, we suppose that the Poincaré map P_γ of γ has no eigenvalues equal to 1.

In Section 4.1 a global parametrix for the mixed problem characterizing $\mathcal{E}(t, x, y)$ is constructed. For this purpose we apply global Fourier integral distributions to express the successive reflections. The principal symbol of the parametrix is investigated in Section 4.2. The singularity of $\sigma(t)$ is studied in Section 4.3 and a Poisson summation formula for manifolds with boundary is obtained in Theorem 4.3.1.

4.1 Global parametrix for mixed problems

In this section we use the notation of Chapter 3. Our aim is to construct a global parametrix for the operator $\mathcal{E}_B = \cos(tA)B(y, D_y)$, where $B(y, D_y)$ is a zero order pseudo-differential operator with $WF(B) \subset \Gamma, \Gamma = U \times V$ being a small conic neighbourhood of a fixed point $(y_0, \eta_0) \in T^*(\Omega^\circ) \setminus \{0\}$. First, we assume that $\bar{U} \subset \Omega^\circ$ and that the kernel of B has compact support in $U \times U$.

Let $T_0 > 0$ be fixed. We consider the generalized bicharacteristics $\gamma(t; \mu_\pm)$ of $\Box$ issued from $\mu_\pm = (0, y, \mp|\eta|, \eta)$ with $(y, \eta) \in \Gamma$ and parameterized by the time t. In this section we treat the case when $\gamma(t; \mu_\pm)$ is reflecting at $\partial\Omega$ and without tangent segments for $|t| \leq T_0$. We will treat $\gamma_+(t; \mu) = \gamma(t; \mu_+)$ and for simplicity of the notation we will omit the sign $+$ in μ_+.

Geometry of the Generalized Geodesic Flow and Inverse Spectral Problems, Second Edition.
Vesselin M. Petkov and Luchezar N. Stoyanov.

Let $F_B(t,x,y)$ be the kernel of $\mathcal{E}_B$. Our purpose is to construct a global Fourier integral distribution $\hat{F}_B(t,x,y)$ so that

$$(F_B(t,x,y) - \hat{F}_B(t,x,y))|_{[0,T_0]\times\Omega\times U} \in C^\infty.$$

The distribution $\hat{F}_B$ will be obtained as a sum of global Fourier integral distributions related to the reflections of $\gamma_+(t;\mu)$.

For reader's convenience, we recall some basic facts concerning global Fourier integral distributions and we refer to Sections 21 and 25 in [H4] for more details. Let $W = X \times (\mathbb{R}^N \setminus \{0\}), \dim X = n$, be an open conic set and let $\varphi(x,\theta) \in C^\infty(W)$ be a real-valued phase function homogeneous of order 1 in θ. The phase φ is non-degenerate if $d\varphi \neq 0$ and $d(\frac{\partial\varphi}{\partial\theta_j}), j = 1,\dots,N$, are linearly independent on

$$C_\varphi = \{(x,\theta) \in W : d_\theta\varphi = 0\}$$

which is a smooth manifold of dimension N. Consider the immersion

$$i_\varphi : C_\varphi \ni (x,\theta) \to \{(x, d_x\varphi) \in T^*(X) \setminus \{0\}\} = \Lambda_\varphi.$$

Then Λ_φ is Lagrangian manifold in $T^*(X) \setminus \{0\}$. Consider the Fourier integral distribution

$$I(x) = \int_{\mathbb{R}^N} e^{\mathbf{i}\varphi(x,\theta)} a(x,\theta) d\theta \in I^m(X, \Lambda_\varphi),$$

where $a \in S^{m+\frac{n-2N}{4}}(X \times \mathbb{R}^N)$ is a classical symbol. Following Hörmander (Section 25, [H4]), it is convenient to consider $I(x)$ as a distribution in $\mathcal{D}'(X, \Omega_X^{1/2})$ with values the half-density bundle $\Omega_X^{1/2}$ of X. The corresponding class of Fourier distributions is denoted by $I^m(X, \Lambda_\varphi; \Omega_X^{1/2})$. A Fourier integral operator

$$V : C_0^\infty(Y, \Omega_Y^{1/2}) \to \mathcal{D}'(X, \Omega_X^{1/2})$$

has a kernel given by a Fourier integral distribution

$$I(x,y) = \int_{\mathbb{R}^N} e^{\mathbf{i}\varphi(x,y,\theta)} b(x,y,\theta) d\theta \in I^m(X \times Y, C'; \Omega_{X\times Y}^{1/2}).$$

Here

$$C = \{(x, d_x\varphi, y, d_y\varphi) \in T^*(X) \setminus \{0\} \times T^*(Y) \setminus \{0\} : \ \varphi_\theta(x,y,\theta) = 0\}$$

is called a *canonical relation* of V or $I(x,y)$ and

$$\Lambda = C' = \{(x, d_x\varphi, y, -d_y\varphi) \in T^*(X) \setminus \{0\} \times T^*(Y) \setminus \{0\} : \ \varphi_\theta(x,y,\theta) = 0\}$$

is a *Lagrangian manifold* with respect to the form $\sigma_X + \sigma_Y$, where σ_X and σ_Y are the symplectic forms on $T^*(X)$ and $T^*(Y)$ trivially lifted to $T^*(X) \times T^*(Y)$.

The principal symbol v_0 of V belongs to the class

$$S^{m+(\dim\ X+\dim\ Y-2N)/4}\left(X \times Y, M_\Lambda \otimes \Omega_\Lambda^{1/2}\right),$$

where M_Λ is the Maslov bundle of Λ, while $\Omega_\Lambda^{1/2}$ is the bundle of half-densities on Λ. We refer to [H4], Section 25, for the precise definitions of $M_\Lambda, \Omega_\Lambda^{1/2}$ and $S^m(X \times Y, M_\Lambda \otimes \Omega_\Lambda^{1/2})$. The Maslov bundle M_Λ is locally trivial and the bundle $\Omega_\Lambda^{1/2}$ of half-densities can be trivialized by choosing a canonical half-density on Λ.

The situation is more simple if X and Y are n-dimensional manifolds, and $C = \mathrm{graph}\, r$ is the graph of a homogeneous canonical transformation

$$r : T^*(Y) \setminus \{0\} \to T^*(X) \setminus \{0\}.$$

Then v_0 is a product of a half-density on C and a Maslov factor. Since C is diffeomorphic to $T^*(Y) \setminus \{0\}$, we may consider v_0 as an object on $T^*(Y) \setminus \{0\}$ parameterized by the *symplectic coordinates* $(y, \eta) \in T^*(Y)$. Next on $T^*(Y) \setminus \{0\}$ there exists a *canonical half-density* $d_{can} = |dy \wedge d\eta|^{1/2}$ related to the coordinates (y, η), and the symbol v_0, modulo Maslov factors, becomes $f_0 d_{can}$ with a classical symbol f_0. Moreover, in this case the operator $V \in I^m(X \times Y, C')$ defines a continuous map from the Sobolev spaces $H_{(s)}^{comp}(Y, \Omega_Y^{1/2})$ to $H_{(s-m)}^{loc}(X, \Omega_X^{1/2})$ for every $s \in \mathbb{R}$ (see Section 25.3 in [H4]). In particular, we may define VG for distribution with compact support $G \in \mathcal{E}'(Y)$.

We start by the fundamental solution

$$\begin{aligned}\mathcal{R}_0(t, x-y) &= \frac{(2\pi)^{-n}}{2}\left(\int_{\mathbb{R}^n} e^{\mathbf{i}t|\eta|+\mathbf{i}\langle x-y,\eta\rangle} d\eta + \int_{\mathbb{R}^n} e^{-\mathbf{i}t|\eta|+\mathbf{i}\langle x-y,\eta\rangle} d\eta\right)\\ &= \frac{1}{2}(\mathcal{R}_0^- + \mathcal{R}_0^+),\end{aligned}$$

satisfying

$$\begin{cases}(\partial_t^2 - \Delta_x)\mathcal{R}_0 = 0,\\ \mathcal{R}_0(0, x-y) = \delta(x-y), \partial_t \mathcal{R}_0(0, x-y) = 0.\end{cases}$$

We consider $\mathcal{R}_0^\pm$ as a Fourier integral distribution

$$\mathcal{R}_0^\pm \in I^{-1/4}(\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_y^n, (C_0^\pm)'),$$

where

$$C_0^\pm = \left\{(t, x, y, \tau, \xi, \eta) \in T^*(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n) \setminus \{0\};\ x = y \pm t\frac{\eta}{|\eta|},\right.$$
$$\left.\xi = \eta, \tau = \mp|\eta|\right\}$$

is parameterized by $(y,\eta)\in\Gamma\subset T^*(\mathbb{R}^n)\setminus\{0\}$ and

$$C_0' = \{(t,x,y,\tau,\xi,-\eta):\ (t,x,y,\tau,\xi,\eta)\in C_0\}.$$

In the following we deal with $\mathcal{R}_0^+$ related to C_0^+ and given by the Fourier integral with phase function $\varphi_- = -t|\eta| + \langle x-y,\eta\rangle$. Denote by $t_1(y,\eta)$ the time of the first reflection of $\gamma_+(t;\mu)$ and set

$$t_1 := \inf_{(y,\eta)\in\Gamma} t_1(y,\eta), T_1 := \sup_{(y,\eta)\in\Gamma} t_1(y,\eta), \mathcal{I}_1 = [t_1, T_1].$$

We need to examine the trace on $\partial\Omega$ of the distribution

$$\mathcal{R}_B^+(t,x,y) = (2\pi)^{-n}\int e^{\mathbf{i}\varphi_-}\beta(y,\eta)d\eta$$

for $t\in\mathcal{I}_1$, provided Γ is sufficiently small. Here $\beta(y,\eta)\in S^0(\mathbb{R}^n\times\mathbb{R}^n)$ is the symbol of a zero order pseudo-differential operator $B(y,D_y)$ with $WF(B)\subset\Gamma$.

Consider the inclusion map

$$i:\mathbb{R}\times\partial\Omega\ni(t,x)\longrightarrow(t,x)\in\Omega$$

and denote by i^* the operator of the trace on $\mathbb{R}\times\partial\Omega$. The kernel $\mathcal{K}$ of i^* is a Fourier integral distribution in $I^{-1/4}(\mathbb{R}\times\partial\Omega\times\mathbb{R}\times\Omega,\mathcal{N}')$, where the canonical relation $\mathcal{N}$ has the form

$$\mathcal{N}=\Big\{(t,x,\tau,\tilde{\xi},t,x,\tau,\xi)\in T^*(\mathbb{R}\times\partial\Omega)\times T^*|_{\mathbb{R}\times\partial\Omega}(\mathbb{R}\times\Omega);\ x\in\partial\Omega,\tilde{\xi}=\xi|_{T_x(\partial\Omega)}\Big\}.$$

If $\partial\Omega$ is given by $x_1=0$ in normal geodesic coordinates (x_1,x'), then

$$\mathcal{K}(t,x',t,y) = (2\pi)^{-n}\int e^{\mathbf{i}\langle x'-y',\xi'\rangle-\mathbf{i}y_1\xi_1}d\xi$$

and this explains the order $-1/4$ of $\mathcal{K}$. We wish to prove that for any fixed $t\in\mathcal{I}_1$ the trace $i^*\mathcal{R}_B^+$ is a Fourier integral distribution, and for this purpose it is necessary to compose the canonical relation $\mathcal{N}$ and

$$C_\Gamma = \{(t,x,y,\tau,\xi,\eta)\in C_0^+,(y,\eta)\in\Gamma\}.$$

Introduce

$$Z = T^*(\mathbb{R}\times\partial\Omega)\times\Delta(T^*(\mathbb{R}\times\Omega)\setminus\{0\})\times T^*(U)\setminus\{0\},$$

where $\Delta(\mathcal{U}) = \{(m,m):\ m\in\mathcal{U}\}$ denotes the diagonal of $\mathcal{U}$.

Let $\gamma_+(t;\mu)$ hit (transversally) $\partial\Omega$ at $x^1(y,\eta)$ for $t=t_1(y,\eta)$. Assume that $x^1(y,\eta)\in\omega_1\subset\partial\Omega$ for $(y,\eta)\in\Gamma$, and let ω_1 have in local normal coordinates (x_1,x') the form $x_1=0$. Let the principal symbol q of $\Box$ have the form

$$q(x,\tau,\xi) = \xi_1^2 + m(x,\xi') - \tau^2$$

in these local coordinates. A simple calculus shows that $\operatorname{codim}(\mathcal{N} \times C_\Gamma) = 4n+2$, $\operatorname{codim} Z = 2n+2$. On the other hand, the transversality of $\gamma_+(t;\mu)$ to $\partial\Omega$ yields

$$\frac{\partial q}{\partial \xi_1} = 2\xi_1 = \sqrt{\tau^2 - m(0, x', \xi')} \neq 0 \tag{4.1}$$

for $x' = x'(y,\eta), \xi' = \xi'(y,\eta)$. This implies easily that $\operatorname{codim}((\mathcal{N} \times C_\Gamma) \cap Z) = 6n+3$. The manifold $\mathcal{N} \times C_\Gamma$ intersects Z cleanly, which means that for each $u \in (\mathcal{N} \times C_\Gamma) \cap Z$ we have

$$T_u((\mathcal{N} \times C_\Gamma) \cap Z) = T_u(\mathcal{N} \times C_\Gamma) \cap T_u(Z).$$

The excess e of this clean intersection is 1 and the composition of the canonical relations $\rho_0 = \mathcal{N} \circ C_\Gamma$ will be a canonical relation (see [H3], Section 21). Since the projection

$$\pi_0 : (t_1(y,\eta), x^1(y,\eta), y, -|\eta|, \tilde{\xi}, \eta) \in \rho \longrightarrow (y,\eta) \in \Gamma$$

is a diffeomorphism, the relation ρ_0 is locally the graph of a homogeneous canonical transformation

$$r_0 : \Gamma \to T^*(\mathbb{R} \times \partial\Omega).$$

By the calculus for Fourier integral operators with clean intersection (see [H4], Section 25), we deduce that $i^*\mathcal{R}_B^+$ is a Fourier integral distribution

$$i^*\mathcal{R}_B^+(t,x,y) \in I^0(\mathbb{R} \times \partial\Omega \times U, \rho_0'). \tag{4.2}$$

Notice that we obtain a Fourier integral distribution of order $-1/4 - 1/4 + e/2 = 0$. Next, we modify $\mathcal{R}_B^+(t,x,y)$ for $t > T_1 + \epsilon_1$ and small $\epsilon_1 > 0$, so that

$$\mathcal{R}_B^+ \in C^\infty, \text{ for } t > T_1 + \epsilon_1$$

and for $t \geq 0$ the trace $i^*\mathcal{R}_B^+$, modulo smooth functions, coincides with (4.2). This continuation is possible since the singularities of $\mathcal{R}_B^+(t,x,y)$ are propagating along the bicharacteristics of $\Box$ lying in the exterior of Ω for $T_1 + \epsilon \leq t \leq T_1 + 2\epsilon$, provided ϵ small enough.

To construct the parametrix for $t > t_1$, we need to satisfy the boundary condition on $\partial\Omega$. Let $\nu(x)$ be the exterior unit normal at $x \in \partial\Omega$. Consider the set

$$\Sigma_1 = \Big\{ (t_1(y,\eta), z, \tau, \tilde{\xi}) \in T^*(\mathbb{R} \times \partial\Omega) \setminus \{0\} : \ z = y + t_1(t,\eta)\frac{\eta}{|\eta|} ,$$

$$\tau = -|\eta|, \eta|_{T_z(\partial\Omega)} = \tilde{\xi}, \langle \nu(z), \eta \rangle > 0, (y,\eta) \in \Gamma \Big\}.$$

which depends on Γ but for the brevity of the notation we will omit this later. Introduce the map

$$\Phi(t,\rho) : \Sigma_1 \ni \rho = (s, z, \tau, \tilde{\xi}) \to (\exp tH_q)(s, z, \tau, \xi),$$

where $\xi = \eta - 2\langle \nu(z), \eta\rangle \nu(z)$ is the *reflected direction* determined from the *incoming direction* η. Notice also that $|\xi| = |\eta|$ and the principal symbol q of $\Box$ vanishes on $\Phi(t,\rho)$. Moreover, τ remains constant along the action of Φ, and $\Phi(t,\cdot)$ is one-parameter group along the trajectories of the Hamiltonian field H_q of the principal symbol q. Consider the map

$$l_1 : \Sigma_1 \ni (s, z, \tau, \tilde{\xi}) \to (s, z, \tau, \xi) \in l_1(\Sigma_1) = \mathcal{U}_1 \subset H,$$

where ξ is the reflected direction introduced above and H is the set of hyperbolic points introduced in Section 1.2. Thus,

$$\Sigma_1 \xrightarrow{\;l_1\;} \mathcal{U}_1 \xrightarrow{\exp(tH_q)} T^*(\mathbb{R} \times \Omega).$$

It is easy to see that $\Phi(t,\rho)$ is an immersion. Indeed, for $t = 0$, $\Phi(0,\rho)$ maps Σ_1 diffeomorphically into $T^*(\mathbb{R} \times \Omega)$ since ξ is uniquely determined from $\eta|_{T_z(\partial\Omega)}$. Moreover, $d\Phi|_{t=0}$ maps $\frac{\partial}{\partial t}$ into the Hamiltonian field H_q. The condition (4.1) implies that H_q is transversal to $\partial\Omega$, and this shows that $d\Phi|_{t=0}$ is injective. Finally, the group property $\Phi(t_1 + t_2, \cdot) = \Phi(t_1, \Phi(t_2, .))$ implies that $d\Phi(t,\rho)$ is injective for all t and $\Phi(t,\rho)$ is an immersion.

Introduce

$$C_1 = \{(\Phi(t,\rho), \rho) \in T^*(\mathbb{R} \times \Omega) \setminus \{0\} \times T^*(\mathbb{R} \times \partial\Omega) \setminus \{0\} :\ t \geq 0, \rho \in \Sigma_1\}.$$

To prove that C_1 is a canonical relation on $T^*(\mathbb{R} \times \Omega) \setminus \{0\} \times T^*(\mathbb{R} \times \partial\Omega) \setminus \{0\}$, consider the symplectic forms $d\alpha$ and $d\alpha_b$ on $T^*(\mathbb{R} \times \Omega)$ and $T^*(\mathbb{R} \times \partial\Omega)$, respectively, where α and α_b are canonical one-forms. For example, if $w \in T_{(s,\sigma)}(T^*(\mathbb{R} \times \partial\Omega))$ with $s \in \mathbb{R} \times \partial\Omega, \sigma \in T^*_s(\mathbb{R} \times \partial\Omega)$, we have

$$\langle \alpha_b, w\rangle = \langle \sigma, w'\rangle,$$

$w' \in T_s(\mathbb{R} \times \partial\Omega)$ being the projection of w on $T_s(\mathbb{R} \times \partial\Omega)$.

By using a trivial lifting, consider α and α_b as one-forms on $W = T^*(\mathbb{R} \times \Omega) \times T^*(\mathbb{R} \times \partial\Omega)$ and denote them by the same notation. Our purpose is to show that $\alpha - \alpha_b$ vanishes on the image of $d\Phi$. This will imply that C_1 is a Lagrangian submanifold with respect to the symplectic structure defined by $\alpha - \alpha_b$. First consider the case $t = 0$ and set $\Lambda_0 = \{(\Phi(0,\rho), \rho) :\ \rho \in \Sigma_1\}$. Since $(d\Phi|_{t=0})(\frac{\partial}{\partial t}) = (H_q, 0)$, we get

$$\left\langle \alpha - \alpha_b, (d\Phi|_{t=0}) \left(\frac{\partial}{\partial t}\right)\right\rangle = \langle \alpha, H_q\rangle(\Phi(0,\rho)) = q(\Phi(0,\rho)) = 0,$$

because q vanishes on $\Phi(0,\rho)$. Now consider $w \in T_{(s,\sigma)}(T^*(\mathbb{R} \times \partial\Omega))$ and let

$$d\Phi(0,\rho)w = v \in T_{(s,\zeta)}(T^*(\mathbb{R} \times \Omega)).$$

Then $\zeta|_{T_s(\mathbb{R}\times\Omega)} = \sigma$ and the projection v' of v on $T_s(\mathbb{R} \times \partial\Omega)$ coincides with w' defined above. Consequently,

$$\langle \alpha, d\Phi(0,\rho)w\rangle = \langle \zeta, w'\rangle = \langle \sigma, w'\rangle$$

and

$$\langle \alpha - \alpha_b, (d\Phi(0,\rho)w, w)\rangle = 0.$$

Thus, Λ_0 is the embedded Lagrangian submanifold. Next

$$\Lambda_t = \{(\Phi(t,\rho),\rho) :\ t > 0,\ \rho \in \Sigma_1\}$$

is also an embedded Lagrangian submanifold with respect to $d\alpha - d\alpha_b$ because $\Phi(t,.)$ is the Hamiltonian flow of q. Thus, we have proved the following.

Lemma 4.1.1: *C_1 is a canonical relation. For every fixed $t > 0$ the map $\mathcal{U}_1 \ni u \to \Phi(t,u) \in H$ preserves the 2-form on H induced by the canonical symplectic form in $T^*(\mathbb{R} \times \Omega)$.*

The second statement follows from the fact that for fixed t, the set $\{\Phi(t, l_1(\rho)) : \rho \in \Sigma_1\}$ is diffeomorphic to $\mathcal{U}_1$. The fact that for fixed time the shift along the billiard trajectories issued from Γ and reflecting on $\mathbb{R} \times \partial\Omega$ preserves the fundamental one-form. $\langle \eta, dy\rangle$ has been proved by a direct calculus in Proposition 2.3.5 in [SaV]. Clearly, a transformation preserving $\langle \eta, dy\rangle$ is a canonical one.

Now, let $\omega_1 = \pi(\Sigma_1)$, where $\pi : (t,x,\tau,\tilde{\xi}) \to x$ is the projection on $\partial\Omega$. To arrange the Dirichlet boundary condition on ω_1, we introduce a Fourier integral operator

$$R_1^+ \in I^{-1/4}(\mathbb{R} \times \partial\Omega \times \mathbb{R} \times \partial\Omega, C_1')$$

satisfying the conditions

$$\begin{cases} (\partial_t^2 - \Delta_x)R_1^+ \equiv 0, \\ i_{\omega_1}^* R_1^+ f \equiv f, \end{cases} \tag{4.3}$$

for every distribution $f \in \mathcal{E}'(\mathbb{R} \times \partial\Omega)$ with $WF(f) \subset \Sigma_1$. Here $\equiv$ means an equality modulo operators with smooth kernels or equality modulo C^∞ terms, while $i_{\omega_1}^*$ is the trace on ω_1. Notice that $i_{\omega_1}^*$ does not coincide with the identity on $\partial\Omega$ and $i_{\omega_1}^*$ is a Fourier integral operator. Moreover, for sufficiently small ω_1, hence for small enough Γ, we have $\Phi(t,\rho)|_{\partial\Omega} \in \mathcal{U}_1$ only for $t = 0$. For the composition $i_{\omega_1}^* R_1^+$, we apply the above argument based on a clean intersection and $i_{\omega_1}^* R_1^+$ is related to the canonical relation $\{(\rho,\rho) :\ \rho \in \Sigma_1\}$, hence it is a zero order pseudo-differential operator.

To construct R_1^+, we must solve the transport equations for the principal and lower-order symbols of R_1^+. To do this, recall the notion of the principal symbol discussed in the beginning of this section. For brevity of the notations, put

$$Y = \mathbb{R} \times \Omega \times \partial\Omega,\ \ \Lambda = C_1'.$$

Then the kernel of R_1^+ is given by a distribution

$$\mathcal{R}_1^+ \in I^{-1/4}\Big(Y, \Lambda; \Omega_Y^{1/2}\Big)$$

and its principal symbol a_0 belongs to the class $S^0\big(Y, M_\Lambda \otimes \Omega_\Lambda^{1/2}\big)$. Denote by q the principal symbol of $\Box$ and consider H_q as a vector field on Y by using a trivial lifting. Since the subprincipal symbol of $\Box$ vanishes, we obtain the transport equation

$$\mathcal{L}_{H_q} a_0 = 0, \tag{4.4}$$

where $\mathcal{L}_{H_q}$ denotes the Lie derivative along the Hamiltonian field H_q (see Theorem 25.2.4 in [H4]). Since Maslov factors are locally constant, it is sufficient to solve equation (4.4) for the half-density $f_0 d_\Lambda$ part of a_0, where d_Λ is a half-density of Λ. In the next section we prove that we can choose d_Λ invariant with the action of H_q and the problem is reduced to the equation $H_q f_0 = 0$. Since H_q is transversal to $\partial\Omega$, we solve the last equation with initial condition $f_0|_{\omega_1} = 1$, so that the principal symbol of $i^*_{\omega_1} R^*_1$, modulo Maslov factors, is d_Λ. Then we get

$$(\partial_t^2 - \Delta_x) R_1^+ \in I^{-1/4+1}\big(Y, \Lambda; \Omega_Y^{1/2}\big).$$

Next for the lower-order symbols $a_j = f_j d_\Lambda$ of R_1^+, we obtain the equations

$$H_q f_j = g_j(f_0, \dots, f_{j-1}), \;\; j \geq 1$$

which we solve with zero initial conditions on ω_1. Thus, the full symbol of $(\partial_t^2 - \Delta_x) R_1^+$ is 0, that of $i^*_{\omega_1} R_1^+$ is equal to 1 and we arrange (4.3).

Since Φ is an immersion, it follows easily that the set $C_1 \times (\mathcal{N} \circ C_\Gamma)$ intersects

$$T^*(\mathbb{R} \times \Omega) \setminus \{0\} \times \Delta(T^*(\mathbb{R} \times \partial\Omega) \setminus \{0\}) \times T^*(U) \setminus \{0\}$$

transversally. Then the composition $\Gamma_+^1 = C_1 \circ (\mathcal{N} \circ C_\Gamma)$ is a canonical relation, and by the calculus of Fourier integral operators we deduce

$$R_1^+ i^* \mathcal{R}_B^+ \in I^{-1/4}\left(\mathbb{R} \times \Omega \times U, (\Gamma_+^1)'\right).$$

Notice that $\Gamma_+^1 \subset C_+$, where C_+ is the relation introduced in Section 1.2 and

$$\Gamma_+^1 = \{(t, -|\eta|, \mathcal{F}_{t-t_1(y,\eta)}(x^1(y,\eta), \xi^1(y,\eta)), y, \eta) \in T^*(\mathbb{R} \times \Omega \times \partial\Omega) \setminus \{0\}\}$$

is parameterized by (y, η). Here, the reflected direction $\xi_1(y, \eta)$ is determined as in the definition of Σ_1, and $\mathcal{F}_t$ denotes the generalized Hamiltonian flow.

Let $t_k(y, \eta)$ be the time of the kth reflection of $\gamma_+(t; \mu)$. Let $\gamma_+(t, \mu)$ hit $T^*(\partial\Omega)$ at points $\lambda_1(y, \eta), \dots, \lambda_k(y, \eta), \dots$ with $\lambda_k(y, \eta) = (x^k(y, \eta), \eta^k(y, \eta)) \in T^*(\partial\Omega)$ and let $\widehat{\lambda_k(y, \eta)} \in T^*(\partial\Omega)$ be obtained from $\lambda_k(y, \eta)$ by changing the incoming direction $\eta^k(y, \eta)$ with the reflecting one $\xi^k(y, \eta)$ so that

$$\langle \eta^k, \nu(x^k) \rangle (y, \eta) = -\langle \xi^k, \nu(x^k) \rangle (y, \eta).$$

Set

$$t_k = \inf_{(y,\eta) \in \Gamma} t_k(y, \eta), T_k = \sup_{(y,\eta) \in \Gamma} t_k(y, \eta).$$

Our analysis in Section 3.1 shows that $WF(i^*\mathcal{R}_B^+) \subset \Sigma_1$. Setting $V_0^+ = \mathcal{R}_B^+$, we get

$$i^*_{\omega_1} R_1^+ i^* V_0^+ \equiv i^* V_0^+$$

and we define

$$V_1^+ = R_1^+ i^* V_0^+. \tag{4.5}$$

Then for $0 < t < t_2$ the trace of $W_1^+ = -V_1^+ + V_0^+$ on $\partial\Omega$ vanishes modulo smooth functions. This completes the construction related to the first reflection.

Now we repeat the above procedure for other reflections. For $t \in [t_2, T_2]$, the generalized bicharacteristics $\gamma_+(t;\mu)$ hit transversally $\partial\Omega$. To prove that

$$\rho_1 = \mathcal{N} \circ C_1 \subset T^*(\mathbb{R} \times \partial\Omega) \setminus \{0\} \times T^*(\mathbb{R} \times \partial\Omega) \setminus \{0\}$$

is a canonical relation, we apply the above argument showing that $\mathcal{N} \times C_1$ intersects cleanly

$$T^*(\mathbb{R} \times \partial\Omega) \times \Delta(T^*(\mathbb{R} \times \Omega)) \setminus \{0\}) \times T^*(\mathbb{R} \times \partial\Omega).$$

Moreover, the map

$$\mathcal{U}_1 \ni u \to \Phi(t_2(y,\eta), u) \in \mathcal{U}_2 \subset H$$

is a diffeomorphism. Taking the projection on Σ_1, we conclude that ρ_1 is locally the graph of a homogeneous canonical transformation

$$r_1 : \Sigma_1 \to \Sigma_2 = r_1(\Sigma_1) \subset T^*(\mathbb{R} \times \partial\Omega).$$

Next, we modify R_1^+ for $t \notin [t_1 - \epsilon_2, T_2 + \epsilon_2]$ and small $\epsilon_2 > 0$, so that $R_1^+ Qf \in C^\infty$ if $t \notin [t_1 - \epsilon_2, T_2 + \epsilon_2]$ and Q is a pseudo-differential operators with $WF(Q) \subset \Sigma_1$. The singularities of the kernel $\mathcal{R}_1^+$ of R_1^+ and the form of C_1 make this possible. Set $\Sigma_2 = r_1(\Sigma_1), \omega_2 = \pi(\Sigma_2) \subset \partial\Omega$ (see the diagram), and consider the Fourier integral operator $i^*_{\omega_2} R_1^+ \in I^0(\mathbb{R} \times \partial\Omega \times \mathbb{R} \times \partial\Omega, \rho_1')$.

$$\begin{array}{ccc}
\mathcal{U}_1 & \xrightarrow{\Phi(t_2(y,\eta),.)} & \mathcal{U}_2 \\
l_1 \big\uparrow & & l_2 \big\uparrow \\
\Sigma_1 & \xrightarrow{r_1} & \Sigma_2 \\
\pi \big\downarrow & & \pi \big\downarrow \\
\omega_1 & & \omega_2
\end{array}$$

Following this procedure, define

$$\Sigma_k = r_{k-1}(\Sigma_{k-1}),\ \pi(\Sigma_k) = \omega_k,\ l_k(\Sigma_k) = \mathcal{U}_k \subset H,$$

the canonical relation

$$C_k = \{(\Phi(t, l_k(\rho)) \in T^*(\mathbb{R} \times \Omega \times \mathbb{R} \times \partial\Omega) \setminus \{0\} : t \geq 0, \rho \in \Sigma_k\}$$

related to the kth reflection of $\gamma_+(t;\mu)$. For small Γ the sets $\mathcal{U}_k$ belong to H and the inverse map l_k^{-1} is diffeomorphism. Applying Lemma 4.1.1, we conclude that

$$r_k : \Sigma_k \to \Sigma_{k+1} \subset T^*(\mathbb{R} \times \partial\Omega),$$

is a homogeneous canonical transformation . Denote by $i^*_{\omega_k}$ the trace on ω_k. Repeating the construction of R_1^+, we can find a Fourier integral operator

$$R_k^+ \in I^{-1/4}(\mathbb{R} \times \Omega \times \mathbb{R} \times \partial\Omega, C_k')$$

satisfying the conditions

$$\begin{cases} (\partial_t^2 - \Delta_x) R_k^+ \equiv 0, \\ i^*_{\omega_k} R_k^+ f \equiv f, \end{cases}$$

for each $f \in \mathcal{E}'(\mathbb{R} \times \partial\Omega)$ with $WF(f) \subset \Sigma_k$. As above, we modify the kernel $\mathcal{R}_k^+$ of R_k^+ for $t \notin [t_k - \epsilon_k, T_{k+1} + \epsilon_k], \epsilon_k > 0$ being sufficiently small. Our construction shows that the trace $i^* R_k^+$, modulo smoothing operators, coincides with the sum of the traces $i^*_{\omega_k} R_k^+$ and $i^*_{\omega_{k+1}} R_k^+$. To satisfy the boundary conditions, introduce a zero order pseudo-differential operator $M_k \in L^0(\mathbb{R} \times \Omega)$ such that

$$WF(M_k) \cap \mathcal{U}_{k+1} = \emptyset,\ WF(Id - M_k) \cap \mathcal{U}_k = \emptyset,\ k \geq 1.$$

This is possible since $\mathcal{U}_k \cap \mathcal{U}_{k+1} = \emptyset$, provided Γ small enough. Consequently,

$$i^*(Id - M_{k-1})R_{k-1}^+ = i^*_{\omega_{k-1}}(Id - M_{k-1})R_{k-1}^+ + i^*_{\omega_k}(I - M_{k-1})R_{k-1}^+ \equiv i^*_{\omega_k} R_{k-1}^+$$

and $i^*_{\omega_k} R_k^+ i^*(Id - M_{k-1})R_{k-1}^+ \equiv i^*(Id - M_{k-1})R_{k-1}^+$.

After this preparation define

$$V_k^+ = R_k^+ i^*(Id - M_{k-1})V_{k-1}^+, k \geq 2$$

and set

$$W_p^+ = \sum_{k=0}^{p} (-1)^k V_k^+.$$

Here the coefficients $(-1)^k$ are added to arrange the boundary conditions on $\partial\Omega$ for time $0 \leq t < t_{k+1}$. Notice that $V_k^+ \in I^{-1/4}(\mathbb{R} \times \Omega \times U, (\Gamma_+^k)')$, where the canonical relation Γ_+^k has the form

$$\Gamma_+^k = \Big\{ \Big(t, -|\eta|, \mathcal{F}_{t - t_k(y,\eta)} \widehat{\lambda_k(y,\eta)}, y, \eta\Big) \in T^*(\mathbb{R} \times \Omega \times U) \Big\} \subset C_+. \tag{4.6}$$

Therefore, for $0 \le t < t_{p+1}$ we have

$$\begin{cases} (\partial_t^2 - \Delta_x) W_p^+ \in C^\infty, \\ i^* W_p^+ \in C^\infty. \end{cases}$$

Notice that the construction of W_p^+ works if $\gamma_+(t;\mu)$ have at least p reflections for $0 \le t \le T_0$.

In the same way we treat the relation C_0^- and the generalized bicharacteristics $\gamma_-(t;\mu)$ of $\Box$ issued from $\nu(0, y, |\eta|, \eta)$, $(y, \eta) \in \Gamma$. Let

$$W_p^- = \sum_{k=0}^{p} (-1)^k V_k^-, V_0 = \mathcal{R}_B^- = (2\pi)^{-n} \int e^{\mathrm{i}\varphi_+} \beta(y,\eta) d\eta$$

be the corresponding distribution. Here $V_k^- \in I^{-1/4}(\mathbb{R} \times \Omega \times U, (\Gamma_-^k)')$, where

$$\Gamma_-^k = \Big\{ \Big(t, |\eta|, \mathcal{F}_{-t-t_k(y,\eta)} \widehat{\lambda_{k,-}(y,\eta)}, y, \eta\Big) \in T^*(\mathbb{R} \times \Omega \times U) \setminus \{0\} \Big\} \subset C_-$$

with $\lambda_{k,-}(y,\eta)$ related to $\gamma_-(y,\mu)$ and $\widehat{\lambda_{k,-}(y,\eta)}$ obtained from $\lambda_{k,-}(y,\eta)$ changing the incoming direction by the reflecting one. Therefore,

$$W_p = \frac{1}{2} \sum_{k=0}^{p} (-1)^k (V_k^+ + V_k^-)$$

will be the solution of the problem

$$\begin{cases} (\partial_t^2 - \Delta_x) W_p \in C^\infty, \\ i^* W_p \in C^\infty, \\ W_p|_{t=0} = B^*(y, D_y)\delta(x-y), \partial_t W_p|_{t=0} = 0 \end{cases}$$

for $0 \le t \le \hat{t}_{p+1}$, where $\hat{t}_{p+1}$ depends on the time of the $(p+1)$th reflection of $\gamma_\pm(y;\mu)$ and $B^*(y, D_y)$ is the operator adjoint to $B(y, D_y)$. For large p we obtain the distribution $\hat{F}_B(t,x,y)$ for $0 \le t \le T_0$.

Now let γ be a periodic ordinary reflecting bicharacteristic of $\Box$ with period $T > 0$ passing through (y_0, η_0). Let m_γ be the number of reflections of γ. The operator V_k^+ is related to the relation Γ_k^+. Let U and $\epsilon > 0$ be sufficiently small and let $\mathcal{I} = (T-\epsilon, T+\epsilon)$. Then for $t \in \mathcal{I}$ modulo smooth terms, we obtain

$$W_{m_\gamma}^+ \equiv (-1)^{m_\gamma} V_{m_\gamma}^+ \in I^{-1/4}(\mathcal{I} \times U \times U, (\Gamma_+^{m_\gamma})').$$

Finally, we take

$$\hat{F}_B(t,x,y)|_{\mathcal{I}\times U \times U} = \frac{1}{2}(-1)^{m_\gamma}(V_{m_\gamma}^+ + V_{m_\gamma}^-) = \frac{1}{2}(\hat{F}_B^+ + \hat{F}_B^-). \tag{4.7}$$

This proves the following.

Proposition 4.1.2: *For* $0 \le t \le T_0$ *we have*

$$(F_B(t,x,y) - \hat{F}_B(t,x,y))|_{\Omega\times U} \in C^\infty.$$

Now we will discuss briefly the case when U is a small open neighbourhood of a point $y_0 \in \partial\Omega$. Let $(y_1, y'), y_1 = \mathrm{dist}(x, \partial\Omega)$ be local normal coordinates. Then $\partial\Omega \cap U$ has the form $y_1 = 0$ and let

$$\Omega \cap U = \{(y_1, y') : \ y' \in U', 0 \le y_1 \le \alpha\}.$$

Let

$$q(y,\tau,\eta) = \eta_1^2 - \tau^2 + m(y,\eta'), m(y,\eta') \ge c_0|\eta'|^2, c_0 > 0,$$

be the principal symbol of $\Box$ in the local coordinates and let $\mu_0 = (0, y_0', \tau_0, \eta_0') \in T^*(\mathbb{R}^n) \setminus \{0\}$ be a hyperbolic point of $\Box$. This means that the equation $q(0, y_0', \tau_0, \eta_1, \eta_0') = 0$ with respect to η_1 has two distinct real roots. Let $\mathcal{O} = \mathcal{J} \times U'$, $\mathcal{J}$ being a small neighbourhood of 0, and let

$$\Gamma = \mathcal{O} \times V \subset T^*(\mathbb{R}_{t'} \times \mathbb{R}^{n-1}_{y'}) \setminus \{0\}$$

be an open conic neighbourhood of μ_0 such that

$$\{(t', y_1, y', \tau', \eta') : \ (t', y', \tau', \eta') \in \Gamma, 0 \le y_1 \le \alpha\}$$

are hyperbolic points for $\Box$. The latter means that ${\tau'}^2 - m(y,\eta') > 0$. Consider a zero order pseudo-differential operator $B(t', y, D_{t'}, D_{y'})$, depending smoothly on y_1, so that $WF(B(\cdot, y_1, \cdot)) \subset \Gamma$ and the kernel of $B(\cdot, y_1, \cdot)$ has a compact support in $\mathcal{O} \times \mathcal{O}$. As in Section 3.4, it is more convenient to study

$$\mathcal{E}_B = \cos((t-t')A)Y(t-t')B(t', y, D_{t'}, D_{y'}),$$

where $0 \le y_1 \le \alpha$ is considered as a parameter and $\alpha > 0$ is sufficiently small.

Let $\eta_1^\pm(y_1, y', \tau, \eta')$ be the real roots of $q(y_1, y', \tau', \eta_1, \eta') = 0$ with respect to η_1. Assume that the bicharacteristics $\gamma_\pm(t;\mu)$ issued from $\mu = (t', y_1, y', \tau', \eta_1^\pm, \eta')$ are reflecting for $|t| \le T_0$. We treat the case with $\eta_1 = \eta_1^+$. Similarly to the situation examined above, introduce

$$\Sigma_1' = \{(t, z, \tau, \tilde{\xi}) \in T^*(\mathbb{R} \times \partial\Omega) \setminus \{0\} : \ t = t_1(\gamma_+(t;\mu)), z = y + t_1(\gamma_+(t;\mu))\frac{\eta}{|\eta|},$$
$$\tau = -\tau', \eta|_{T_z(\partial\Omega)} = \tilde{\xi}, \langle \nu(z), \eta\rangle > 0\}.$$

Here $t_1(\gamma_+(t;\mu))$ is the time of the first reflection of $\gamma_+(t;\mu)$. Clearly, for μ with $y_1 = 0$ we have $t_1(\gamma(t;\mu)) = 0$ and $z = y \in \partial\Omega$. On the other hand, we have $(\partial_t + \partial_{t'})\mathcal{E}_B = 0$ and on the wave front set of $\mathcal{E}_B$ one has $\tau + \tau' = 0$. We consider y_1 as a parameter, and for fixed y_1, the set Σ_1' is parameterized by the symplectic coordinates (t', y', τ', η'). Next, we define the flow $\Phi(t, l_1(\rho)), \rho \in \Sigma_1'$ and repeat the construction of V_k^+.

As it was mentioned in Section 3.4, the distribution $\mathcal{E}(t - t', x_1, x', y_1, y')$ for $0 \leq x_1 \leq \alpha, 0 \leq y_1 \leq \alpha$, depends smoothly on x_1, y_1. For small $\alpha > 0$ a periodic bicharacteristic issued from $y_1 = \hat{y}_1$ and transversal to this hyperplane after reflections will return to this hyperplane. Thus, we construct a Fourier integral distribution

$$W_{m_\gamma}(t - t', x, y) = \frac{1}{2}\left(\mathcal{F}_B^+(t - t', x, y) + \mathcal{F}_B^-(t - t', x, y)\right)$$

such that

$$(\mathcal{E}_B(t, t', x_1, x', y_1, y) - W_{m_\gamma}(t - t', x_1, x', y_1, y'))|_{\mathcal{I} \times \mathcal{J} \times U \times U} \in C^\infty. \tag{4.8}$$

Here $\mathcal{J} \subset \mathbb{R}$ is a small neighbourhood of $0, \mathcal{I} = (T - \epsilon, T + \epsilon), \epsilon > 0, \mathcal{E}_B(t, t', x, y)$ is the kernel of $\mathcal{E}_B$ and $0 \leq x_1, y_1 \leq \alpha$. The Fourier integral distributions $\mathcal{F}_B^\pm(t - t', x, y)$ are related to the canonical relations

$$\begin{aligned}\mathcal{M}_{\Gamma,\pm} = \{&(t, x, \tau, \xi, t', y, \tau', \eta) \in T^*(\mathcal{I} \times U \times \mathcal{J} \times U) \setminus \{0\} : \ \tau = \tau', \\ &\eta_1 = \eta_1^\pm(y, \tau, \eta'), (t, x, \tau, \xi) \ \text{ and } \ (t', y, \tau, \eta) \\ &\text{lie on a generalized bicharacteristic of } \Box \text{ and } (t', y', \tau', \eta') \in \Gamma\}.\end{aligned}$$

4.2 Principal symbol of $\hat{F}_B$

In this section we shall study the principal symbol of the distribution $\hat{F}_B(t, x, y)$ given by (4.7). To do this, we must examine the principal symbol of $V_{m_\gamma}^\pm$ following the rules for composition of Fourier integral operators (see [H4]).

Consider the kernel $\mathcal{R}_1^+$ of the operator R_1^+. The principal symbol a_0 of $\mathcal{R}_1^+$ satisfies equation (4.5). Setting $\Lambda = C_1'$, the symbol a_0 is a section of the bundle $M_\Lambda \otimes \Omega_\Lambda^{1/2}$, where M_Λ is the Maslov bundle over Λ. Ignoring the Maslov factors in M_Λ, which are locally constant, consider the half-density part of a_0. It is convenient to trivialize the bundle of half-densities $\Omega_\Lambda^{1/2}$ by using a half-density d_Λ on Λ , which is invariant with respect to the action of H_q. We choose a canonical half-density $d_{can} = (|dy| \wedge |d\eta|)^{1/2}$ on $T^*(U)$ related to the symplectic coordinates (y, η). As it was proved in the previous section, Σ_1 can be parameterized by the symplectic coordinates $(y, \eta) \in \Gamma$ and we can define a half-density

$$((r_0)^{-1})^*(|dy| \wedge |d\eta|)^{1/2} = d_{can}$$

on Σ_1 by using the projection π_0 and the canonical transformation r_0. In the same way we choose a half-density $\delta_{can} = (|dt| \wedge |dy| \wedge |d\eta|)^{1/2}$ on $\mathbb{R} \times l_1(\Sigma_1)$. Therefore, since C_1 is parameterized by $(t, \rho) \in \mathbb{R} \times \Sigma_1$,

$$\left((\Phi(t, .)^{-1}\right)^* \delta_{can} = \delta_{can}$$

is a half-density on Λ and $(\Gamma_+^1)'$. Here we used the fact that for every fixed t_0, the map $\Phi(t_0, .)$ is a canonical transformation from $l_1(\Sigma_1)$ to $\{\Phi(t_0, u) : u \in l_1(\Sigma_1)\}$, hence $(\Phi(t_0, .)^{-1})^*$ leaves d_{can} invariant. Now it is clear that δ_{can} is invariant under the action of $\Phi(t, \cdot)^*$ and this implies

$$\mathcal{L}_{H_q}(\delta_{can}) = \frac{\partial}{\partial t}(\Phi^*(t, .)\delta_{can})|_{t=0} = 0.$$

The half-density part of a_0 has the form $a_0 = f_0\delta_{can}$ with a function f_0 homogeneous of order 0. The transport equation $\mathcal{L}_{H_q} a_0 = 0$ yields $H_q f_0 = 0$. Thus, f_0 is constant along the orbits of H_q and the initial condition on ω_1 for a_0 implies $f_0 = 1$.

Next, consider the operator $(Id - M_1)i^*R_1^+$ related to the canonical relation $\rho_1 = \operatorname{graph} r_1$. The condition $WF(M_1) \cap \Sigma_2 = \emptyset$ shows that the symbol of M_1 vanishes on Σ_2. As we mentioned in the previous section, we can parameterize Σ_2 by (y, η) exploiting the homogeneous canonical transformation r_1 . Therefore, on ρ_1, we have a canonical half-density $((r_1)^{-1})^* d_{can} = d_{can}$. Repeating this procedure for the operator $(Id - M_k)i^*R_k^+$, we are in a position to apply the rule for the computation of the principal symbol of a product of Fourier integral operators associated with homogeneous canonical transformations (see [H4], Section 25). Setting

$$\sigma_k = r_k \circ \cdots \circ r_1 : \Sigma_1 \to \Sigma_{k+1}, k \geq 2,$$

introduce a half-density d_{can} on $\rho_k = \operatorname{graph} \sigma_k$ and a half-density δ_{can} on $(\Gamma_+^k)'$ parameterized by (t, y, η).

Then the principal symbol of the operator

$$(-1)^k R_k^+ i^*(Id - M_{k-1})i^*R_{k-1}^+ \ldots (Id - M_1)i^*R_1^+,$$

modulo Maslov factor, is $(-1)^k\delta_{can}$. To obtain a local representation of $\hat{F}_B^+$, consider a small conic neighbourhood $Z_0 \subset (\Gamma_+^{m_\gamma})'$ of $\nu_0 = (T, y_0, y_0, -|\eta_0|, \eta_0, \eta_o)$ $\in (\Gamma_+^{m_\gamma})'$ related to a periodic reflecting ray $\gamma = \gamma_+(t; \mu_0)$ with period T issued from $\mu_0 = (y_0, \eta_0)$. For fixed $t \in \mathcal{I}$ the projection

$$Z_0 \ni (t, x, y, \tau, \xi, \eta) \to (y, \eta) \in U \times V$$

is a diffeomorphism. Let $\mathcal{F}_t(y, \eta)$ be the generalized Hamiltonian flow of $\Box$ introduced in Section 1.2. Then for fixed t, the generalized bicharacteristics hit the surface $\{\mathcal{F}_t(y, \eta);\ (y, \eta) \in \Gamma\}$ transversally and the map $(x, \xi) = \mathcal{F}_t(y, \eta)$ is a homogeneous canonical transformation according to the argument in the previous section. Therefore, it is possible to find a phase function $\varphi(t, x, \eta)$, determined in a small conic neighbourhood $Y = \tilde{\mathcal{I}} \times \tilde{U} \times \tilde{V}$ of (T, y_0, η_0) and homogeneous of order 1 in η, so that (see Proposition 25.3.3 in [H4])

$$(\mathcal{F}_t(y, \eta), (y, \eta)) = (x, \varphi_\eta, \varphi_x, \eta), \det \varphi_{x,\eta}(t, x, \eta) \neq 0, (t, x, \eta) \in Y.$$

This implies

$$Z_0 = \{(t, x, \varphi_\eta, -|\eta|, \varphi_x, \eta) :\ (t, x, \eta) \in Y\}.$$

For $\mathcal{I}, U, V$ small enough we arrange $\mathcal{I} \times U \times V \subset Y$ and for $t \in \mathcal{I}, x, y \in U$, modulo C^∞ terms, we have the representation

$$\hat{F}_B^+(t,x,y) = (2\pi)^{-n} \int_{|\eta|\geq 1} e^{\mathbf{i}\varphi(t,x,\eta) - \mathbf{i}\langle y,\eta\rangle} b(t,x,y,\eta) d\eta. \tag{4.9}$$

Here t is a parameter,

$$b(t,x,y,\eta) \sim \sum_{j=0}^{\infty} b_j(t,x,y,\eta)$$

and $b_j(t,x,y,\eta)$ are homogeneous of order $(-j)$ with respect to η. For fixed $t \in \mathcal{I}$ we may consider (x,η) as local coordinates on Z_0 and $(|dx| \wedge |d\eta|)^{1/2}$ becomes a half-density on Z_0. With respect to this localization, the principal symbol of (4.9), modulo Maslov factors, has the form $b_0(t,x,\varphi_x,\eta)(|dx| \wedge |d\eta|)^{1/2}$.

On the other hand, for fixed t_0 it is possible to express the principal symbol of $\hat{F}_B^+(t_0,x,y)$ by using the symplectic coordinates (y,η) on $T^*(U)$ and the half-density d_{can} on

$$Z_0|_{(\Phi^{t_0}(y,\eta),(y,\eta))}$$

related to (y,η). The principal symbol of $(-1)^{m_\gamma} V_{m_\gamma}^+$ is $(-1)^{m_\gamma}\delta_{can}$, and we must take the restriction on $t = t_0$. Following the rules of composition of half-densities, it is easy to see that the half-density part of the restriction on $t = t_0$ is $(-1)^{m_\gamma} d_{can}$. We will explain a similar argument below in a more difficult case when we take the trace of δ_{can} on $\partial\Omega \times \partial\Omega$.

Since $y = \varphi_x(x,\eta)$ on Z_0, we obtain

$$|dy|^{1/2} \wedge |d\eta|^{1/2} = |\det \varphi_{x\eta}(t,x,\eta)|^{1/2} |dx|^{1/2} \wedge |d\eta|^{1/2}.$$

Comparing this with the form of this symbol in the coordinates (x,η), we deduce

$$b_0(t,x,\varphi_x,\eta) = (-1)^{m_\gamma} e^{\mathbf{i}\frac{\pi}{2}\sigma} |\det \varphi_{x\eta}(t,x,\eta)|^{1/2}. \tag{4.10}$$

Here $\sigma \in \mathbb{N}$ and $e^{\mathbf{i}\frac{\pi}{2}\sigma}$ is a Maslov factor.

The integer σ depends on γ, only. To see this, we will express σ by the signatures of the matrices $d^2_{\eta\eta}\varphi_j$, where φ_j are phase functions parameterizing C_+ in small neighbourhoods Z_j along γ. Let $\gamma = \gamma(t)$ be parameterized by the time, and let $0 = t_0 \leq t_1 \leq \cdots \leq t_l = T$ be a sequence of times along γ with $t_j = t_{j+1}$ only if $\gamma(t_j)$ is a reflection point of γ.

Assume $t_j < t_{j+1}$ and let $\gamma(t)$ have no reflections for $t_j < t < t_{j+1}$. Suppose $\gamma(t_j) \in Z_j \cap Z_{j+1}$ and let Z_k for $k = j, j+1$ be expressed by φ_k as earlier Z_0 has been expressed by φ. Then passing from the representation of $\hat{F}_B^+$ by φ_j to that related to φ_{j+1}, we must add the Maslov factor

$$\mathbf{i}^{1/2(\text{sign } d^2_{\eta\eta}\varphi_j - \text{sign } d^2_{\eta\eta}\varphi_{j+1})}. \tag{4.11}$$

Now suppose $\gamma(t_j) = (t_j, x_0, \tau_0, \xi_o)$ is a reflection point of γ. Let $x_0 \in \omega_k \subset \partial\Omega$ and let ω_k have locally the form $x_1 = 0, x_0 = (0, x_0')$. Denote as above by $q(x, \tau, \xi)$ the principal symbol of $\Box$ in these local coordinates. For (x', τ, ξ') close to (x_0', τ_0, ξ_0'), denote by $\xi_1^{\pm}(x, \tau, \xi')$ the roots of the equation $q(x, \tau, \xi_1, \xi') = 0$ with respect to ξ_1. For t close to t_j, the distribution $\hat{F}_B^+$ has the form

$$\hat{F}_B^+ = (-1)^k (R_k^+ (Id - M_{k-1}) i^* L_{k-1} - L_{k-1}) i^* \mathcal{R}_B^+.$$

The operator $i^*_{\omega_k} L_{k-1}$ has a canonical relation that is the graph of σ_k. Let $\chi(t, x', \tau, \xi')$ be the generating function of σ_k, that is

$$\text{graph } \sigma_k^{-1} = \left(\begin{pmatrix} t & x' \\ \chi_t & \chi_{x'} \end{pmatrix}, \begin{pmatrix} \chi_\tau & \chi_{\xi'} \\ \tau & \xi' \end{pmatrix} \right).$$

By convention, assume that $(\xi_1^{\pm}, \xi')$ is close to the direction of $\gamma(t)$ for $\mp(t - t_j) > 0$ and $|t - t_j|$ sufficiently small. In other words, ξ_1^- (resp. ξ_1^+) corresponds to incoming (resp. outgoing) segments reflecting on ω_k.

Introduce the phase functions $\varphi^{\mp}(t, x, \tau, \xi')$ as solutions of the Cauchy problems

$$\begin{cases} \frac{\partial \varphi^{\mp}}{\partial x_1} = \xi_1^{\mp}(x, \varphi_t^{\mp}, \varphi_{x'}^{\mp}), \\ \varphi^{\mp}|_{x_1=0} = \chi(t, x', \tau, \xi'). \end{cases}$$

Then the kernel of L_{k-1} for t close to t_j admits the representation

$$\int e^{\mathbf{i}\varphi^-(t,x,\tau,\xi') - \mathbf{i}t'\tau - \mathbf{i}\langle x', \xi' \rangle} b^{(k)}(t, x, \tau, \xi') d\tau \, d\xi', \tag{4.12}$$

while the kernel of $R_k^+(Id - M_{k-1}) i^* L_{k-1}$ has a similar representation by $\varphi^+(t, x, \tau, \xi')$. Hence putting $\zeta = (\tau, \xi')$, by the initial conditions for $\varphi^{\mp}$, we deduce

$$\text{sign } d^2_{\zeta\zeta} \varphi^- = \text{sign } d^2_{\zeta\zeta} \varphi^+. \tag{4.13}$$

Thus, the reflection at $\gamma(t_j)$ does not involve a Maslov factor.

Denote by $\mathcal{M}$ the set of $j \in \mathbb{R}$ such that $t_j = t_{j+1}$. Then, according to (4.11) and (4.13), the Maslov factor σ in (4.10) has the form

$$\sigma_\gamma = \frac{1}{2} \sum_{j=0, j \notin \mathcal{M}}^{L-1} \left(\text{sign } d^2_{\eta\eta} \varphi_j - \text{sign } d^2_{\eta\eta} \varphi_{j+1} \right). \tag{4.14}$$

Clearly, σ_γ depends on γ, only, since the choice of $\gamma(0)$ is not important for the sum in (4.14).

For $\hat{F}_B^-$ we follow a completely similar argument. For $t \in \mathcal{I}, x, y \in U$ we get a representation

$$\hat{F}_B^-(t, x, y) = (2\pi)^{-n} \int_{|\eta| \geq 1} e^{\mathbf{i}\psi(t,x,\eta) + \mathbf{i}\langle y, \eta \rangle} c(t, x, \eta) d\eta \tag{4.15}$$

with phase function $\psi(t,s,\eta)$ representing locally $(\Gamma_-^{m_\gamma})'$ and

$$c(t,x,\eta) \sim \sum_j c_j(t,x,\eta).$$

The principal symbol has the form

$$c_0(t,x,\eta) = (-1)^{m_\gamma} e^{-\mathbf{i}\frac{\pi}{2}\sigma_\gamma} |\det \psi_{x\eta}(t,x,\eta)|^{1/2}. \tag{4.16}$$

In fact, we repeat the construction of $\hat{F}_B^+$ following a covering of $\gamma(t)$, where we change the orientation because $\tau > 0$ on $(\Gamma_-^{m_\gamma})'$ and the time t decreases when we move along $\gamma(t)$.

Now we pass to the case when U is a neighbourhood of a point in $\partial\Omega$, and we use the notation of the previous section. The purpose is to describe the principal symbol of $\mathcal{E}_B|_{t'=0}$ for t close to a period T of a periodic bicharacteristic. We introduce normal coordinates (y_1, y', η_1, η') and (x_1, x', ξ_1, ξ') so that the boundary $\partial\Omega$ is given locally, respectively, by $y_1 = 0$ or $x_1 = 0$. Let the principal symbol q of $\Box$ have the forms $\eta_1^2 - \tau^2 + m(y,\eta')$ and $\xi_1^2 - \tau^2 + m_1(x,\xi')$, respectively, and assume that on Ω° locally we have $y_1 > 0, x_1 > 0$, respectively. Let $i_{\partial\Omega}$ be the restriction on $\partial\Omega$ and let

$$i_{\partial\Omega} \times i_{\partial\Omega} : \mathbb{R} \times \partial\Omega \times \partial\Omega \to \mathbb{R} \times \Omega \times \Omega$$

be the inclusion map. Here we use $i_{\partial\Omega}$ for the restriction on $\partial\Omega$, while in Section 4.1, the operator i was the restriction on $\mathbb{R} \times \partial\Omega$.

Clearly,

$$i_{\partial\Omega} \times i_{\partial\Omega} \in I^{1/4}(\mathbb{R} \times \partial\Omega \times \partial\Omega \times \mathbb{R} \times \Omega \times \Omega, \mathcal{N}'_{\partial\Omega\times\partial\Omega})$$

is a Fourier integral operator with canonical relation

$$\begin{aligned}\mathcal{N}_{\partial\Omega\times\partial\Omega} = \{&(t,x,y,\tau,\tilde{\xi},\tilde{\eta},t,x,y,\tau,\xi,\eta) \in T^*(\mathbb{R}\times\partial\Omega\times\partial\Omega)\times T^*(\mathbb{R}\times\Omega\times\Omega) :\\ &x\in\partial\Omega, y\in\partial\Omega, \tilde{\xi} = \xi|_{T_x(\partial\Omega)}, \tilde{\eta} = \eta|_{T_y(\partial\Omega)}\}.\end{aligned}$$

As in the previous section, it is easy to check that $\mathcal{N}_{\partial\Omega\times\partial\Omega} \times \Gamma_+^k$ intersects transversally

$$T^*(\mathbb{R}\times\partial\Omega)\setminus\{0\} \times \Delta(T^*(\mathbb{R}\times\Omega)\setminus\{0\}) \times T^*(\mathbb{R}\times\Omega)\setminus\{0\}$$

and this explains the order $1/4$ of the operator $i_{\partial\Omega}\times i_{\partial\Omega}$. Therefore, following [H4] for the composition of canonical relations with transversal intersection, we deduce that

$$\Gamma_{\partial,+}^k = \mathcal{N}_{\partial\Omega\times\partial\Omega} \circ \Gamma_+^k \subset T^*(\mathbb{R}\times\partial\Omega\times\partial\Omega)\setminus\{0\}$$

is a canonical relation.

We need to introduce a suitable parameterization of $\Gamma_{\partial,+}^k$. For this purpose, we introduce the billiard map. Consider the *unit ball bundle*

$$B^*(\partial\Omega) = \{(y,\eta) \in T^*(\partial\Omega) : \; h_0(y,\eta) < 1\},$$

where $h_0(y,\eta)$ is the induced Riemannian metric on $\partial\Omega$. In local coordinates, given above, a point $(0,y',\frac{\eta'}{\tau})$ is in $B^*(\partial\Omega)$ if $m(0,y',\eta')<\tau^2$, since $m(0,y',\eta')$ is the induced Riemannian metric on $\partial\Omega$ in local coordinates. This is just the condition that $(0,y',\tau,\eta')$ is a hyperbolic point for the operator $\Box$ (see Section 1.2). Let

$$S^*(\partial\Omega)=\{(y,\eta)\in T^*(\Omega): y\in\partial\Omega, |\eta|=1\}$$

be the cosphere bundle and let

$$\Sigma^{\pm}=\{(y,\eta)\in S^*(\partial\Omega):\ \pm\langle\eta,\nu(y)\rangle>0\}.$$

Consider the projection

$$\pi_{\Sigma^+}:\Sigma^+\ni(y,\eta)\to(y,\eta|_{T(\partial\Omega)})\in B^*(\partial\Omega).$$

Then the *billiard map* has the form

$$\beta=\pi_{\Sigma^+}\circ B\circ\pi_{\Sigma^+}^{-1}:B^*(\partial\Omega)\to B^*(\partial\Omega),$$

where

$$B:\Sigma^+\longrightarrow\Sigma^+$$

is the billiard ball map defined in Section 2.1. Thus, β is defined in a neighbourhood of a hyperbolic point if the bicharacteristic issued from this point hits transversally $T^*(\partial\Omega)$.

Consider the principal symbol $g=|\eta|$ of $\sqrt{-\Delta}$. In local coordinates, $g=(|\eta_1|^2+m(y,\eta')|)^{1/2}$. Let $W_1=\{(t,1,x,\xi')\in T^*(\mathbb{R}\times\partial\Omega)\cap H\}$, H being the hyperbolic set introduced in Section 1.2. Then W_1 is a submanifold of codimension 1 of the set

$$\{(t,\tau,x,\xi')\in T^*(\mathbb{R}\times\partial\Omega)\cap H\}$$

and $\mathcal{F}_t$ induces a map φ on W_1. Let ω_{W_1} be the restriction on W_1 of the canonical symplectic form of $T^*(\mathbb{R}\times\Omega)$ and consider the projection $\pi: W_1\ni(t,1,x,\xi')\to(x,\xi')\in\Sigma^+$. Therefore, there exists a unique two-form ω_S on Σ^+ of maximal rank such that $(\pi)^*\omega_S=\omega_{W_1}$ and Σ^+ has a symplectic structure with canonical symplectic form ω_S. Now consider the diagram

$$\begin{array}{ccc}
W_1 & \xrightarrow{\varphi} & W_1\\
\pi\downarrow & & \pi\downarrow\\
\Sigma^+ & \xrightarrow{B} & \Sigma^+\\
\pi_{\Sigma^+}\downarrow & & \pi_{\Sigma^+}\downarrow\\
B^*(\partial\Omega) & \xrightarrow{\beta} & B^*(\partial\Omega)
\end{array}$$

Here φ is the restriction of the generalized flow on W_1. It is clear that φ preserves W_1 and the fibration $W_1 \longrightarrow \Sigma^+$, and we can define the billiard map B as $B = \pi \circ \varphi \circ \pi^{-1}$. On the other hand,

$$(\pi^{-1} \circ \pi_{\Sigma^+}^{-1}) B^*(\partial\Omega) \subset W_1 \cap H.$$

Applying Lemma 4.1.1, we deduce that φ preserves the form ω_{W_1} on $W_1 \cap H$, and this implies that B preserves ω_S on $\pi_{\Sigma^+}^{-1}(B^*(\partial\Omega))$. Next, we define a symplectic form $\omega_B = (\pi_{\Sigma^+}^{-1})^* \omega_S$ on $B^*(\partial\Omega)$ and β becomes a symplectic map preserving ω_B. In local coordinates, we have $\omega_B = \sum_{j=1}^{n-1} dy_j \wedge d\eta_j$. Notice that $\pi_{\Sigma^+}^{-1}(B^*(\partial\Omega))$ does not contain directions in $S^*(\partial\Omega)$, which are tangent to $\partial\Omega$ and π_{Σ^+} is a diffeomorphism on $\pi(W_1)$.

By using the map β^k, we introduce a parameterization of $\Gamma^k_{\partial,+}$ given by

$$\Gamma \ni (0, y', \tau, \eta') \to \left\{ \left(T_k\left((\pi_{\Sigma^+})^{-1} \left(0, y', \frac{\eta'}{\tau} \right) \right), \tau, \tau\beta^k \left(0, y', \frac{\eta'}{\tau} \right), y', \eta' \right) \right\} \in \Gamma^k_{\partial,+}. \tag{4.17}$$

Here $T_k((\pi_{\Sigma^+})^{-1}(0, y', \frac{\eta'}{\tau}))$ is the length of the reflecting bicharacteristic issued from $\pi_{\Sigma^+}^{-1}(0, y', \frac{\eta'}{\tau}')$ and hitting the boundary after k reflections at $\beta^k(0, y', \frac{\eta'}{\tau})$, while $\tau(x, \xi) = (x, \tau\xi)$. Obviously, on $\Gamma^k_{\partial,+}$, we have a half-density $(|dy'| \wedge |d\eta'| \wedge |d\tau|)^{1/2}$.

Recall that the principal symbol of $\hat{F}_B(t, x, y)$ over $\Omega \times \Omega$ for t close to T has the form

$$(-1)^{m_\gamma} e^{\mathbf{i}\frac{\pi}{\sigma_\gamma}} \delta_{can}.$$

We ignore the coefficient in front of δ_{can}, and for simplicity we write k instead of m_γ. After the application of $i^*_{\partial\Omega\times\partial\Omega}$, we must find the composition of the half-density $\delta_{can} = (|dt| \wedge |dy| \wedge |d\eta|)^{1/2}$ on Γ^k_+ with the half-density on $\mathcal{N}_{\partial\Omega\times\partial\Omega}$. According to the rules of composition of densities in the case of transversal intersection (see [H4] and [DG]), first we take the exterior tensor product of the half-density

$$|dt \wedge d\tau \wedge dx' \wedge d\xi_1 \wedge d\xi' \wedge dy' \wedge d\eta_1 \wedge d\eta'|^{1/2}$$

on $\mathcal{N}_{\partial\Omega\times\partial\Omega}$ and the half-density

$$|ds \wedge dz \wedge d\zeta|^{1/2}$$

on Γ^k_+ at the points of the fiber product F, where the components in $T^*(\mathbb{R}) \times T^*(\Omega) \times T^*(\Omega)$ of these half-densities are equal (see the following diagram).

$$\begin{array}{ccccc} \Gamma^k_+ & \xleftarrow{\ \pi\ } & F & \xrightarrow{\ \alpha\ } & \mathcal{N}_{\partial\Omega\times\partial\Omega} \circ \Gamma^k_+ = \Gamma^k_{\partial,+} \\ i\big\downarrow & & \pi\big\downarrow & & \\ T^*(\mathbb{R}) \times T^*(\Omega) \times T^*(\Omega) & \xleftarrow{\ \pi_\Omega\ } & \mathcal{N}_{\partial\Omega\times\partial\Omega} & & \end{array}$$

Second, we divide the tensor product by the canonical half-density

$$|ds \wedge d\tau \wedge dx_1 \wedge dx' \wedge d\xi_1 \wedge d\xi' \wedge dz \wedge d\zeta|^{1/2},$$

on the common $T^*(\mathbb{R}) \times T^*(\Omega) \times T^*(\Omega)$ component. Thus, we obtain the half-density

$$\frac{|dt \wedge dy' \wedge d\eta_1 \wedge d\eta'|^{1/2}}{|dx_1|^{1/2}}$$

on $\Gamma^k_{\partial,+}$.

The numerator above is a half-density on $\mathbb{R} \times T^*_{\partial\Omega}(\mathbb{R}^n)$. The problem is to compare this half-density on $\Gamma^k_{\partial,+}$ with the half-density $|dy' \wedge d\eta' \wedge d\tau|^{1/2}$ in the parameterization (4.17).

Consider a parameterization

$$\Phi_+ : \mathbb{R} \times T^*_{\partial\Omega}(\mathbb{R}^n) \to T^*(\mathbb{R}) \times T^*(\Omega) \times T^*_{\partial\Omega}(\Omega),$$

of the graph of the billiard flow given by

$$\begin{aligned}&\Phi_+(t, 0, y', \eta_1, \eta')\\ &\quad= (t, -|\eta|, \exp\ (tH_g)(0, y', \eta_1, \eta'), 0, y', \eta_1, \eta') \in T^*(\mathbb{R}) \times T^*(\Omega) \times T^*_{\partial\Omega}(\Omega)\end{aligned}$$

and denote by $\omega_{T^*(\mathbb{R}^n)}$ the canonical $2n$ symplectic form on $T^*(\mathbb{R}^n)$. This form can be considered as a $2n$ form on $T^*(\mathbb{R}) \times T^*(\Omega) \times T^*_{\partial\Omega}(\Omega)$ by a trivial lifting. Therefore, $|\Phi_+^* \omega_{T^*(\mathbb{R}^n)}|^{1/2}$ is a half-density on $\mathbb{R} \times T^*_{\partial\Omega}(\mathbb{R}^n)$. Set

$$\gamma_1(y', \eta', \tau) = \sqrt{1 - \frac{m(0, y', \eta')}{\tau^2}}.$$

Then have the following.

Lemma 4.2.1: *We have the equality*

$$|dt \wedge dy' \wedge d\eta_1 \wedge d\eta'|^{1/2} = \left| \frac{\eta_1}{\sqrt{\eta_1^2 + m(y, \eta')}} \right|^{-1/2} |\Phi_+^* \omega_{T^*(\mathbb{R}^n)}|^{1/2}$$

as half-densities on $\mathbb{R} \times T^*_{\partial\Omega}(\mathbb{R}^n)$. *On the set* $\Gamma^k_{\partial,+}|_{y_1=0,t=0}$, *the factor on the right-hand side becomes*

$$\gamma_1(y', \eta', \tau)^{-1/2}.$$

Proof: We have $d\Phi_+ \frac{\partial}{\partial t} = \frac{\partial}{\partial t} \Phi_+(0, y', \eta_1, \eta') = H_g$, where H_g is the Hamiltonian vector field of $g = (\eta_1^2 + m(y, \eta'))^{1/2}$. Thus,

$$\begin{aligned}\frac{\Phi_+^*\omega_{T^*(\mathbb{R}^n)}}{dt\wedge dy'\wedge d\eta_1\wedge d\eta'} &= \omega_{T^*(\mathbb{R}^n)}\left(\frac{\partial}{\partial t}\Phi_+(0,y',\eta_1,\eta'), d\Phi_+\frac{\partial}{\partial y'}, d\Phi_+\frac{\partial}{\partial \eta_1}, d\Phi_+\frac{\partial}{\partial \eta'}\right)\\ &= \omega_{T^*(\mathbb{R}^n)}\left(H_g, d\Phi_+\frac{\partial}{\partial y'}, d\Phi_+\frac{\partial}{\partial \eta_1}, d\Phi_+\frac{\partial}{\partial \eta'}\right)\\ &= \omega_{T^*(\mathbb{R}^n)}\left(H_g, \frac{\partial}{\partial y'}, \frac{\partial}{\partial \eta_1}, \frac{\partial}{\partial \eta'}\right).\end{aligned}$$

Here we have used the fact that $d\Phi_+|_{T(\Omega)}$ is a symplectic diffeomorphism. On the other hand, $H_g = \frac{\eta_1}{\sqrt{\eta_1^2+m(y,\eta')}}\frac{\partial}{\partial y_1} + \cdots$, where $\cdots$ denotes vector fields spanned by $\frac{\partial}{\partial y'}, \frac{\partial}{\partial \eta_1}, \frac{\partial}{\partial \eta'}$. Evaluating the form $\omega_{T^*(\mathbb{R}^n)}$ by using the coordinates (y_1, y', η_1, η'), we have

$$\omega_{T^*(\mathbb{R}^n)}\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial \eta_1}, \frac{\partial}{\partial \eta'}\right) = 1.$$

Now consider the restriction of Φ_+ on

$$\left\{(0,y',\eta_1,\eta') :\ \eta_1 = \sqrt{\tau_2 - m(0,y,\eta')}, (0,y',\tau,\eta')\in\Gamma\right\},$$

and the parameterization (4.17) of $\Gamma^k_{\partial,+}$ over the domain points. Then $\eta_1^2 + m(0,y',\eta') = \tau^2$, and the factor $(\frac{\eta_1}{\sqrt{\eta_1^2+m(0,y',\eta')}})^{-1/2}$ becomes $\gamma_1^{-1/2}$. ∎

Next we are going to consider the points $\Gamma^k_{\partial,+}|_{x_1=0,t=T^k}$ in the image of Φ_+.

Lemma 4.2.2: *On the set* $\Gamma^k_{\partial,+}|_{x_1=0,t=T^k} \subset \Phi_+(\mathbb{R}\times T^*_{\partial\Omega}(\mathbb{R}^n))$ *we have*

$$\frac{|\Phi_+^*\omega_{T^*(\mathbb{R}^n)}|^{1/2}}{|dx_1|^{1/2}} = \gamma_1^{-1/2}\left(\tau\beta^k\left(y', \frac{\eta'}{\tau}\right), \tau\right)|d\tau\wedge dy'\wedge d\eta'|^{1/2}.$$

Proof: We are going to examine the half-density $|\Phi_+^*\omega_{T^*(\mathbb{R}^n)}|^{1/2}|dx_1|^{-1/2}$. Notice that $\Phi_+^* d\tau = d\tau$ and $\Phi_+(y_1) = x_1$ yields $\Phi_+^*(dy_1) = dx_1$. Thus,

$$|\Phi_+^*\omega_{T^*(\mathbb{R}^n)}|^{1/2}|dx_1|^{-1/2} = \left|\Phi_+^*\frac{\omega_{T^*(\mathbb{R}^n)}}{|dy_1|}\right|^{1/2} = |\Phi_+^*(dy'\wedge d\eta'\wedge d\eta_1)|^{1/2}.$$

Next we have $\eta_1 = \sqrt{\tau^2 - m(0,y',\eta')}$, hence

$$\begin{aligned}|\Phi_+^*(dy'\wedge d\eta'\wedge d\eta_1)| &= |\Phi_+^*(dy'\wedge d\eta')\wedge\Phi_+^*(d\eta_1)|\\ &= \left|(dy'\wedge d\eta')\wedge\Phi_+^* d\left(\sqrt{\tau^2 - m(0,y',\eta')}\right)\right|\end{aligned}$$

and

$$\left|\Phi_+^* d\left(\sqrt{\tau^2 - m(0,y,\eta')}\right)\right| = \left|\Phi_+^* \frac{\tau d\tau}{\sqrt{\tau^2 - m(0,y',\eta')}}\right| \wedge \cdots$$
$$= \left(1 - \frac{m\left(0, \tau(\beta^k)^*(y', \frac{\eta'}{\tau})\right)}{\tau^2}\right)^{-1/2} |d\tau| \wedge \cdots,$$

where ... denotes forms which multiplied by $(dy' \wedge d\eta')$ vanish. Here we have used the equality

$$\tau\Phi_+^*\left(dy' \wedge d\frac{\eta'}{\tau}\right) = \tau\beta^{*k}\left(dy' \wedge d\frac{\eta'}{\tau}\right)$$
$$= (dy' \wedge d\eta')|_{\tau\beta^k(y',\eta'/\tau)}$$

which follows from the fact that β preserves the symplectic form on $B^*(\partial\Omega)$. This completes the proof. ∎

Combining these two lemmas, we obtain the following.

Proposition 4.2.3: *The principal symbol of* $(i_{\partial\Omega} \times i_{\partial\Omega})^* V_+^{m_\gamma}$ *in the local coordinates* (4.17)*, modulo Maslov factors, has the form*

$$(-1)^{m_\gamma}\left(\gamma_1(\tau, y', \eta')\gamma_1\left(\tau, \tau\beta^k(y', \eta'/\tau)\right)\right)^{-1/2} |d\tau \wedge dy' \wedge d\eta'|^{-1/2}. \qquad (4.18)$$

Remark 4.2.4: The same argument works if we consider the symbol

$$(i_{x_1=\hat{x}_1} \times i_{y_1=\hat{y}_1})^* V_+^{m_\gamma},$$

where the restriction is on the surfaces $y_1 = \hat{y}_1$ and $x_1 = \hat{x}_1$. Then the coefficients γ_1 is defined by $m(y_1, y', \eta')$.

4.3 Poisson summation formula

In this section we use the notation of the previous sections. Let γ be a periodic ordinary reflecting bicharacteristic of $\Box$ in Ω with period $T = T_\gamma > 0$. Let P_γ be the Poincaré map of γ introduced in Section 2.3. Denote by $\pi : T^*(\mathbb{R} \times \Omega) \longrightarrow \Omega$ the usual projection. Then $\pi(\gamma) = \tilde{\gamma}$ will be a (generalized) periodic geodesic in Ω. Throughout this section we make the following assumptions:

(i) if δ is a periodic bicharacteristic of $\Box$ in Ω with period T, then $\pi(\delta) = \pi(\gamma)$,

(ii) $\det(P_\gamma - Id) \neq 0$.

Since by Lemma 1.2.10 the set L_Ω of periods of periodic bicharacteristics in Ω is closed, T is an isolated point in sing supp $\sigma(t)$. Indeed, if there exist bicharacteristics with periods $T_k \to T$, passing over $(x_k, \xi_k) \in T^*(\Omega)$, we can find subsequences $x_k \to x_0$,
$\xi_k \to \xi_0$, and a bicharacteristic δ with period T passing over (x_0, ξ_0). If γ does not pass through (x_0, ξ_0), we obtain a contradiction with (i). If γ passes through (x_0, ξ_0), we obtain a contradiction with (ii).

Choose $\epsilon > 0$ and $\mathcal{I} = (T - \epsilon, T + \epsilon) \subset \mathbb{R}^+$ so that

$$\text{sing supp}\, \sigma(t) \cap \mathcal{I} = \{T\}.$$

Let $\mathcal{O}_\gamma$ be a sufficiently small open neighbourhood of $\tilde{\gamma}$. Then (i) and the choice of $\mathcal{I}$ yield

$$\{(t, x, x, \tau, \xi, \xi) \in C : \ t \in \mathcal{I}, x \in \Omega^\circ \setminus \mathcal{O}_\gamma\} = \emptyset.$$

Applying the argument at the end of Section 3.4, with $\mathcal{I}$ instead of Δ_0, for $W \subset \Omega^\circ \setminus \mathcal{O}_\gamma$ we obtain

$$\int_W \mathcal{E}(t, x, x)dx \in C^\infty(\mathcal{I}).$$

For $W \cap \partial\Omega \neq \emptyset$ and $W \cap \mathcal{O}_\gamma = \emptyset$ we obtain the same result by using Proposition 3.4.3.

To study the leading singularity of $\sigma(t)$ for t close to T, we must examine the traces

$$\sum_{j=1}^{M} \int_{\Omega \cap U_j} \mathcal{E}(t, x, x)ds,$$

where

$$\mathcal{O}_\gamma \subset \bigcup_{j=1}^{M} U_j \tag{4.19}$$

is a covering and $U_j \subset \Omega^\circ$ for $j = 1, \dots, M_0, U_j \cap \partial\Omega \neq \emptyset$ for $M_0 + 1 \leq j \leq M$.

First we study the trace on $U_j \subset \Omega^\circ$, and for simplicity we write U instead of U_j. To microlocalize the problem, introduced a covering

$$T^*(U) \setminus \{0\} \subset \bigcup_{k=1}^{N} (U \times V_k),$$

V_k being small conic neighbourhoods. As in Section 4.1, suppose that γ pass over $(y_0, \eta_0) \in U \times V_{k_0}$. Choose the above covering sufficiently fine to arrange $\eta_0 \notin V_k$, whereas $k \neq j_0$,

$$\{(t, x, x, \tau, \xi, \xi) \in C : \ t \in \mathcal{I}, (x, \xi) \in U \times V_k\} = \emptyset. \tag{4.20}$$

Choose pseudo-differential operators $\tilde{B}_j \in L^0(\Omega^\circ)$ with principal symbols $\tilde{b}_j(y,\eta)$ and full symbols $\tilde{\beta}_j(y,\eta)$ so that

$$\sum_{j=1}^{N} \tilde{b}_j = 1 \text{ on } T^*(U) \setminus \{0\},$$

$$\text{conesupp}\, \tilde{\beta}_j(y,\eta) \subset U \times V_j.$$

The operator $\sum_j \tilde{B}_j$ is elliptic on U, and there exists a pseudo-differential operator $P \in L^0(\Omega^\circ)$ such that

$$\sum_j P\tilde{B}_j = Id + R(y, D_y),$$

$R(y,D_y)$ being an operator with C^∞ kernel. Setting $B_j = P\tilde{B}_j$, we obtain a partition of unity on $T^*(U) \setminus \{0\}$ given by $\sum_j B_j = Id + R$.

The kernel of the operator $\cos(A^{1/2}t)B_k$ has the form $B_k^*(y,D_y)\mathcal{E}(t,x,y)$, where B_k^* is the operator adjoint to B_k. By using Proposition 3.4.2, it is easy to see that

$$WF(B_k^*(y,D_y)\mathcal{E}(t,x,y))|_{\mathcal{I}\times U\times U} \subset$$
$$\{(t,x,y,\tau,\xi,\eta) \in C: \ (y,\eta) \in WF(B_k)\}.$$

As in the proof of Proposition 3.3.1, we can choose a pseudo-differential operator C_k, elliptic at $\rho = (t,x,\hat{y},\tau,\xi,\hat{\eta}) \in C$ and a pseudo-differential operator A_k with $(\hat{y},\hat{\eta}) \notin WF(A_k)$, so that $C_k A_k^* B_k^* \mathcal{E}(t,x,y) \in C^\infty$. Thus,

$$\rho \notin WF(B_k^*(y,D_y)\mathcal{E}(t,x,y)|_{\mathcal{I}\times U\times U}).$$

In the case $\rho \notin C$ the result follows immediately and (4.20) implies for $k \neq k_0$ that

$$\int_U (B_k^*(y,D_y)\mathcal{E})(t.x,x)dx \in C^\infty(\mathcal{I}).$$

We are going to study $\exp(\mp itA)B_{k_0}$, and we omit the index k_0 writing B and V instead of B_{k_0} and V_{k_0}. Set $\Gamma = U \times V$ and apply the construction of Section 4.1 for B with $WF(B) \subset \Gamma$. If $F_\pm(t,x,y)$ are the kernels of $\exp\,(\mp \mathbf{i}tA)$, we have

$$B^*(y,D_y)F_\pm(t,x,y) - \hat{F}_B^\pm(t,x,y) \in C^\infty(\mathcal{I}\times U\times U).$$

For the distribution $\hat{F}_B^+ = (-1)^{m_\gamma} V_{m_\gamma}^+$, given by (4.9), we obtain modulo terms in $C^\infty(\mathcal{I})$ the representation

$$\int_U \hat{F}_B^+(t,x,x)dx = (2\pi)^{-n}\int_U dx \int_1^\infty dr \int_{V\times\mathbb{S}^{n-1}} e^{\mathbf{i}r(\varphi(t,x,\omega)-\langle x,\omega\rangle)}$$
$$\times \sum_j b_j(t,x,\omega) r^{n-1-j} d\omega,$$

where the integral is interpreted in the sense of distributions. We can assume that the coordinates are chosen so that

$$U = \{(x_1, x') \in \mathbb{R}^n : \; x' \in U', 0 < \alpha < x_1 < \beta\},$$
$$\eta_0 = (1, 0, \cdots, 0), \eta_1 > 0 \text{ on } V,$$

where $\{(x_1, 0) : \; \alpha \leq x_1 \leq \beta\} \subset \gamma$. Here $U' \subset \mathbb{R}^{n-1}$ is an open neighbourhood of $y_0' = 0$ and $\alpha < y_{0,1} < \beta$. Introduce the integral

$$I_{\alpha,\beta} = (2\pi)^{-n} \int_\alpha^\beta dx_1 \int_1^\infty dr \int_{U'} \int_{V \cap \mathbb{S}^{n-1}} e^{\mathbf{i}r(\varphi(t,x,\omega) - \langle x, \omega \rangle)} \times \sum_j b_j(t, x, \omega) r^{n-1-j} dx' \, d\omega. \tag{4.21}$$

Our aim is to apply a stationary phase argument for the integral with respect to x' and ω, considering x_1 as a parameter.

The critical points satisfy the equalities

$$\varphi_{x'} = \omega', \varphi_\omega = x.$$

The representation of C_+ by the phase φ, given in Section 4.2, implies

$$\varphi_t(t, x, \omega) = -|\varphi_x(t, x, \omega)| = -1.$$

Moreover, $\omega_1 > 0$ and $\varphi_{x_1} > 0$ on $V \cap \mathbb{S}^{n-1}$. Thus, at the critical points $(\hat{x}', \hat{\omega}')$, we get $\varphi_t = -1, \varphi_x = \omega$ and the form of C_+ yields

$$(x_1, \hat{x}', \hat{\omega}) = \Phi^t(x_1, \hat{x}', \hat{\omega}).$$

This equality is possible for $t = T, \hat{x}' = 0, \hat{\omega} = \eta_0$, only. Next, introduce local coordinates

$$\mathbb{R}^{n-1} \supset W \ni \eta' \to \left(\sqrt{1 - |\eta'|^2}, \eta'\right) \in V \cap \mathbb{S}^{n-1}$$

and write the phase in the form

$$\varphi\left(t, x_1, x', \sqrt{1 - |\eta'|^2}, \eta'\right) - x_1\sqrt{1 - |\eta'|^2} - \langle x', \eta' \rangle.$$

To apply a stationary phase argument, we need to examine the matrix

$$\Delta(\rho(x_1)) = \begin{pmatrix} \varphi_{x'x'} & \varphi_{x'\eta'} - I_{n-1} \\ \varphi_{\eta'x'} - I_{n-1} & \varphi_{\eta'\eta'} \end{pmatrix} (\rho(x_1)),$$

where $\rho(x_1) = (T, x_1, 0, \eta_0)$ and I_m denotes the $(m \times m)$ identity matrix. Here we have used the equality

$$\varphi_{\omega_1}(\rho(x_1)) = x_1 \text{ for } \alpha < x_1 < \beta.$$

Consider the generalized Hamiltonian flow $\mathcal{F}_t : (\varphi_\eta, \eta) \to (x, \varphi_x)$ defined in Section 1.2. For the differential $d\mathcal{F}_T$, we get

$$(d\mathcal{F}_T)\begin{pmatrix}\varphi_{\eta x}\delta x + \varphi_{\eta\eta}\delta\eta \\ \delta\eta\end{pmatrix} = \begin{pmatrix}\delta x \\ \varphi_{xx}\delta x + \varphi_{x\eta}\delta\eta\end{pmatrix},$$

hence

$$\begin{aligned} d\mathcal{F}_T - I_n = Q_\varphi &= \begin{pmatrix}\varphi_{\eta x}^{-1} - I_n & -\varphi_{\eta x}^{-1}\varphi_{\eta\eta} \\ \varphi_{xx}\varphi_{\eta x}^{-1} & -\varphi_{xx}\varphi_{\eta x}^{-1}\varphi_{\eta\eta} + \varphi_{x\eta} - I_n\end{pmatrix} \\ &= \begin{pmatrix}0 & -I_n \\ I_n & 0\end{pmatrix}\begin{pmatrix}\varphi_{xx} & \varphi_{x\eta} - I_n \\ \varphi_{\eta x} - I_n & \varphi_{\eta\eta}\end{pmatrix}\begin{pmatrix}\varphi_{\eta x}^{-1} & -\varphi_{\eta x}^{-1}\varphi_{\eta\eta} \\ 0 & I_n\end{pmatrix}. \end{aligned}$$

Clearly,

$$\begin{gathered}\varphi_{x\eta_1}(\rho(x_1)) = \eta_0,\ \varphi_{\eta\eta_1}(\rho(x_1)) = 0, \\ \det\varphi_{x\eta}(\rho(x_1)) = \det\varphi_{x'\eta'}(\rho(x_1)).\end{gathered}$$

According to the definition of P_γ in Section 2.3, the Poincaré map P_γ, modulo conjugations, is the restriction of $d\Phi^T$ on the linear space

$$\mathcal{L} = \{(0, \delta x', 0, \delta\eta') :\ \delta x' \in \mathbb{R}^{n-1}, \delta\eta' \in \mathbb{R}^{n-1}\}.$$

Let Q'_φ be the $(n-1)\times(n-1)$ matrix obtained from Q_φ replacing I_n by I_{n-1} and making only the derivatives of φ with respect to x' and η'. Therefore, for each $l' = (0, \delta x', 0, \delta\eta') \in \mathcal{L}$, we have

$$\langle (d\mathcal{F}_T - I_n)l', l'\rangle = \left\langle Q'_\varphi\begin{pmatrix}\delta x' \\ \delta\eta'\end{pmatrix}, \begin{pmatrix}\delta x', \\ \delta\eta'\end{pmatrix}\right\rangle.$$

This implies

$$(\det\varphi_{x'\eta'})^{-1}(\det\Delta)(\rho(x_1)) = \det(P_\gamma - I).$$

We may apply the representation (4.10) at the points $\rho(x_1)$ and we get

$$b_0(\rho(x_1)) = (-1)^{m_\gamma}\exp\left(\mathbf{i}\frac{\pi}{2}\sigma_\gamma\right)|\det\varphi_{x'\eta'}|^{1/2}(\rho(x_1)),$$

$\sigma_\gamma \in \mathbb{N}$ being the Maslov index of γ. By the Euler equality for the phase φ, we deduce

$$\varphi(t, x_1, 0, \eta_0) = T - t + x_1.$$

Applying the stationary phase method, we get

$$(2\pi)^{-n}\int_{U'}\int_{V\cap\,\mathbb{S}^{n-1}} e^{\mathbf{i}r(\varphi(t,x,\omega)-\langle x,\omega\rangle)}\sum_j b_j(t,x,\omega)r^{n-1-j}dx'\,d\omega$$
$$= e^{ir(T-t)}\left(\sum_{k=0}^{N-1} c_k r^{-k} + \mathcal{O}(r^{-N}\right),$$

where

$$c_0 = \frac{\mathbf{i}}{2\pi}\exp\left(\mathbf{i}\frac{\pi}{2}\beta_\gamma\right)|\det(P_\gamma - I)|^{-1/2}$$

and

$$\beta_\gamma = 2m_\gamma + \sigma_\gamma + \frac{\text{sign }\Delta}{2} - 1. \tag{4.22}$$

The number β_γ is locally constant since for small $\beta-\alpha$, we have sign $\Delta(\rho(x_1)) =$ const and the integers $m_\gamma \in \mathbb{N}, \sigma \in \mathbb{N}$ depend on γ, only. Thus, c_0 is independent of x_1 and r.

To complete the computation of the leading term of $I_{\alpha,\beta}$, take the Fourier transform of the Heaviside function $Y(r)$. The integral in r can be interpreted in the sense of distributions, hence

$$\int_1^\infty e^{\mathbf{i}(T-t)}dr == -\mathbf{i}(t-T-\mathbf{i}0)^{-1} + L^1_{loc}(\mathbb{R}).$$

Finally, for $t\in\mathcal{I}$ we have

$$I_{\alpha,\beta} = \frac{\beta-\alpha}{2\pi}\exp\left(\mathbf{i}\frac{\pi}{2}\beta_\gamma\right)|\det(P_\gamma - I)|^{-1/2}(t-T-\mathbf{i}0)^{-1} + L^1_{loc}(\mathbb{R}). \tag{4.23}$$

A similar argument with trivial modifications works for $\mathcal{R}^-_B(t,x,y)$. On the other hand, we may use the equality

$$F_-(t,x,y) = \overline{F_+(t,x,y)}.$$

Then, modulo $C^\infty(\mathcal{I})$, we get

$$\int_U \hat{F}^-_B(t,x,x)dx = \int_U F^-(t,x,x)dx$$
$$= \int_U \overline{F^+(t,x,x)}dx = \int_U \overline{\hat{F}^+_B(t,x,x)}dx = \overline{I_{\alpha,\beta}} + L^1_{loc}(\mathbb{R}).$$

Thus,

$$\int_U \mathcal{E}(t,x,x)dx = \text{Re}\int_U \hat{F}^+(t,x,x)dx = \frac{\beta-\alpha}{2\pi}|\det(P_\gamma - I)|^{-1/2}$$
$$\times\text{Re}\left[\exp\left(\mathbf{i}\frac{\pi}{2}\beta_\gamma\right)(t-T-\mathbf{i}0)^{-1}\right] + L^1_{loc}(\mathbb{R}). \tag{4.24}$$

Now we turn to the case $\partial\Omega \cap U \neq \emptyset$. Let

$$\pi(\gamma) = \tilde{\gamma} = \bigcup_{j=1}^{p_\gamma} l_j, l_j = [q_j, q_{j+1}]$$

with $q_j \in \partial\Omega, q_{p_\gamma+1} = q_1$ and $m_\gamma = mp_\gamma$. Let $q_{j+1} \in U$ and let $(y_1, y'), y' \in U' \subset \mathbb{R}^{n+1}$ be local coordinates chosen at the end of Section 4.1. Choose a small neighbourhood $\mathcal{J}$ of $0 \in \mathbb{R}$ and set $\mathcal{O} = \mathcal{J} \times U'$, $\mathcal{J}, U'$ being the same as in Section 4.1. Introduce a covering

$$T^*(\mathcal{O}) \setminus \{0\} \subset \bigcup_{k=1}^{N} (\mathcal{O} \times V_k),$$

where V_k are small conic neighbourhoods in $\mathbb{R}_t \setminus \{0\} \times \mathbb{R}^{n-1}_{y'} \setminus \{0\}$. Choose zero order pseudo-differential operators $B(t', y, D_{t'}, D_{y'})$ depending smoothly on $0 \leq y_1 \leq \alpha$ and satisfying $WF((B_k(\cdot, y_1, \cdot)) \subset \mathcal{O} \times V_k$. As earlier, construct a microlocal partition of unity on $T^*(\mathcal{O}) \setminus \{0\}$ given by

$$\sum_k B_k(t', y, D_{t'}, D_{y'}) = Id + R',$$

R' being an operator with C^∞ smooth kernel.

Recall that $\Omega \cap U = \{(y_1, y') : \ y' \in U', \ 0 \leq y_1 \leq \alpha\}$ and denote by

$$q(y, \tau, \eta) = \eta_1^2 - \tau^2 + m(y, \eta')$$

the principal symbol of $\Box$ in the local coordinates, where $m(y, \eta') \geq c_0|\eta'|^2, \ c_0 > 0$. Let $q_{j+1} = (0, y_0')$, and let $\eta_1^\pm$ be the roots of the equation $q(y, \tau, \eta_1, \eta') = 0$ with respect to η_1. Assume that γ passes over $(0, y_0', \tau_0, \eta_0')$ and set

$$\mu_\pm = (0, y_0', \pm\tau_0, \eta_0'), \ \eta_0^\pm = \eta_1^\pm(0, y_0', \tau_0, \eta_0').$$

Since $\tau_0^2 - m(y_0, \eta_0') > 0$, we have $\tau_0 \neq 0$ and without loss of generality we suppose that $\tau_0 < 0$. By convention , choose $\eta_1^\pm$ so that

$$\pm\frac{\partial q}{\partial \eta_1}(0, y_0', \tau_0, \eta_0^\pm, \eta_0') > 0.$$

Then (η_0^+, η_0') (resp. (η_0^-, η_0')) is collinear with the direction of l_{j+1} (resp. l_j) at q_{j+1}. Suppose $\Gamma_\pm = \mathcal{O} \times V_\pm$ are small open conic neighbourhoods of $\mu_\pm$. By the assumptions (i), (ii), we may choose $\Gamma_\pm$ small enough, so that for $\mu \notin \Gamma_+ \cup \Gamma_-$ there are no periodic bicharacteristics of $\Box$ in Ω passing over μ and having periods in $\mathcal{I} = (T - \epsilon, T + \epsilon)$, where $\epsilon > 0$ is small enough. Taking a suitable partition of unity on $T^*(\mathcal{O}) \setminus \{0\}$, assume that for $k \neq k_0$ and $0 \leq y_1 \leq \alpha$ we have

$$WF(B_k(\cdot, y_1, \cdot)) \cap \Gamma_\pm = \emptyset.$$

Let $\mathcal{F}^{\pm}_{B_k}(t-t',x,y)$ be the kernels of the operators

$$\exp(\mp\mathbf{i}(t-t')A)Y(t-t')B_k(t',y,D_{t'},D_{y'}).$$

According to the results of Section 3.4 for the kernels of $\exp(\mp\mathbf{i}(t-t')A)Y(t-t')$, for $t\in\mathcal{I}, t'\in\mathcal{J}$ we get

$$(t,t',x',x',\tau,\tau,\xi',\xi')\notin WF(\mathcal{F}^{\pm}_{B_k}(t-t',x_1,x',y_1,y'),$$

whenever $(x_1,y_1)\in[0,\alpha]\times[0,\alpha], \tau\in\mathbb{R}, (t',y',\tau,\eta')\in WF(B_k(\cdot,y_1,\cdot))$. Then for $k\neq k_{\pm}$ we have

$$\int_{\omega\cap U}\mathcal{F}^{\pm}_{B_k}(t-t',x,x)dx\in C^{\infty}(\mathcal{I}\times\mathcal{J}).$$

In the following we treat $\mathcal{F}^{\pm}_{B_{k_{\pm}}}$ and we omit $k_{\pm}$ in the notation of $\mathcal{F}^{\pm}_{B}$ and B. Let $\Gamma=\mathcal{O}\times V$ be open conic neighbourhood of $\mu_+=\mu$ and suppose $WF(B(\cdot,y_1,\cdot))\subset\Gamma$. As we have mentioned in Section 4.1, we can construct Fourier integral distribution $\mathcal{F}^{+}_{B}(t-t',x,y)$ related to the canonical relation $\mathcal{M}_+=\mathcal{M}_{\Gamma,+}$ defined at the end of Section 4.1.

In the following we consider $(x_1,y_1)\in[0,\alpha]\times[0,\alpha]$ as parameters. For fixed x_1,y_1 and for $t\in\mathcal{I}, t'\in\mathcal{J}$, the projection

$$\mathcal{M}_+\ni(t,t',x,y,\tau,\tau,\xi,\eta)\to(t',y',\tau,\eta')\in\mathcal{J}\times U'\times V$$

is locally a diffeomorphism. In fact for fixed $\hat{x}_1,\hat{y}_1$, the generalized flow induces a homogeneous canonical transformation from

$$\{(t',\hat{y}_1,y',\tau,\eta'):\ (t',y',\tau,\eta')\in\mathcal{J}\times U'\times V\}$$

into

$$\{t,\hat{x}_1,x',\tau,\eta'):\ (t,x',\tau,\xi')\in\mathcal{I}\times U'\times V\}$$

Then there exists a generating function $\varphi(t,x_1,x',y_1,\tau,\eta')$, depending smoothly on x_1,y_1 and homogeneous of order 1 in (τ,η'), such that $\mathcal{M}_+$ has locally the form

$$\{(t,\varphi_\tau,x_1,x',y_1,\varphi_{\eta'},\varphi_t,\tau,\varphi_x,\eta_1^+,\eta'):\ (t,x',\tau,\eta')\in\mathcal{I}\times U'\times V\}.$$

Here $\eta_1^+(t,x_1,x',y_1,\tau,\eta')$ is determined from the equation $q(y_1,\varphi_{\eta'},\tau,\eta_1,\eta')=0$ with respect to η_1 so that $\eta_1^+(T,0,y_0',0,\tau_0,\eta_0')=\eta_0^+$. Moreover,

$$\det\begin{pmatrix}\varphi_{t,\tau} & \varphi_{t,\eta'}\\ \varphi_{x',\tau} & \varphi_{x',\eta'}\end{pmatrix}\neq 0.$$

Now we write

$$\mathcal{F}_B^+(t-t',x,y) = (2\pi)^{-n}\int e^{\mathbf{i}\varphi(t,x_1,x',y_1,\tau,\eta')-\mathbf{i}t'\tau-\mathbf{i}\langle y',\eta'\rangle}$$
$$\times\sum_{j=0}^{\infty}\tilde{b}_j(t,t',x,y,\tau,\eta')d\tau\ d\eta'$$

with $\tilde{b}_j$ homogeneous of order $(-j)$ with respect to (τ,η'). Notice that $\tau<0$ and $|\eta'|\leq C_0|\tau|$ on $\mathcal{M}_+$ with $C_0>0$ independent of η'. We may suppose that b_j vanish for $|\eta'|\geq C_0|\tau|$. Thus, taking $t'=0$ and setting $\tau=-r,\ r>0$, modulo C^∞ terms, we get

$$I_\alpha = \int_{\Omega\cap U}\mathcal{F}_B^+(t,x,x)dx = (2\pi)^{-n}\int_0^\alpha dx_1\int_1^\infty dr$$
$$\times\int_{U'}\int_{|\eta'|\leq C_0} e^{\mathbf{i}r\Psi^+(t,x,\eta')}\sum_j b_j(t,x,x_1,-1,\eta')r^{n-1-j}dx'\ d\eta'$$

with

$$\Psi^+(t,x,\eta') = \varphi(t,x_1,x',x_1,-1,\eta') - \langle x',\eta'\rangle,$$
$$b_j(t,x,x_1,-1,\eta') = \tilde{b}_j(t,0,x_1,x',x_1,x',-1,\eta').$$

For the integral with respect to x' and η', we wish to apply a stationary phase argument. The critical points $\hat{x}',\hat{\eta}'$ satisfy the equalities

$$\varphi_{x'} = \hat{\eta}',\varphi_{\eta'} = \hat{x}',$$
$$q(x_1,\hat{x}',-1,\varphi_{x_1}(t,x_1,\hat{x}',x_1,-1,\hat{\eta}'),\hat{\eta}') = q(x_1,\hat{x}',-1,\eta_1^+,\hat{\eta}') = 0.$$

Since $\varphi_{x_1}(T,0,y_0',0,-1,\eta_0') = \eta_0^+(0,y_0',-1,\eta_0')$, we deduce $\varphi_{x_1}=\eta_1^+$ at the critical points and

$$(x_1,\hat{x}',\eta_1^+(\cdots),\hat{\eta}') = \Phi^{t-\varphi_\tau}(x_1,\hat{x}',\eta_1^+(\cdots),\hat{\eta}').$$

This is possible only for

$$\hat{x}'=y_0',\ \hat{\eta}'=\eta_0',\ \varphi_\tau(T,x_1,y_0',x_1,-1,\eta_0')=0,\ t=T.$$

Since on $\mathcal{M}_+$ we have $\varphi_t(\dots)=\tau$, we get $\varphi_t(\dots)=-1,\varphi_{tt}(\dots)=0$ and deduce

$$\Psi^+(t,x_1,x',\eta') = \Psi^+(T,x_1,x',\eta') + (T-t).$$

Therefore, we reduce the analysis of the integral with respect to x',η' to that with phase function $\Psi^+(T,x_1,x',\eta')$.

Set

$$\Delta_+(T, x_1, x', \eta') = \begin{pmatrix} \Psi^+_{x'x'} & \Psi^+_{x'\eta'} \\ \Psi^+_{\eta'x'} & \Psi^+_{\eta'\eta'} \end{pmatrix} (T, x_1, x', \eta')$$

$$= \begin{pmatrix} \varphi_{x'x'} & \varphi_{x'\eta'} - I_{n-1} \\ \varphi_{\eta'x'} - I_{n-1} & \varphi_{\eta'\eta'} \end{pmatrix} (T, x_1, x', x_1, -1, \eta').$$

Since $\det \varphi_{x'\eta'} \neq 0$, as in the previous case of a neighbourhood included in Ω°, we conclude by using the Poincaré map P_γ and the transversality of l_{j+1} to $\partial\Omega$ that

$$\det \Delta_+(T, 0, y'_0, \eta'_0) = \det \varphi_{x'\eta'}(T, 0, y'_0, 0, -1, \eta'_0) \det(P_\gamma - I) \neq 0.$$

By the implicit function theorem, there exist functions $x'(x_1), \eta'(x_1)$ determined for $0 \leq x_1 \leq \epsilon$ so that

$$\mathrm{grad}_{x'\eta'} \Psi^+(T, x_1, x'(x_1), \eta'(x_1)) = 0,$$

$x'(0) = y'_0, \eta'(0) = \eta'_0$. Moreover, by the Euler equality for the function $\Psi^+(T, x_1, x', \eta')$ homogeneous with respect to η', we deduce $\Psi^+(T, x_1, x_1(x'), \eta'(x_1)) = 0$. Assuming $\alpha \leq \epsilon$ and Γ^+ small enough, we apply the stationary phase argument with parameter x_1 for the integral with respect to (x', η') (see, for example Theorem 7.7.6 in [Hl]) and obtain

$$(2\pi)^{-n} \int_{U'} \int_{\eta'|\leq C_0} e^{\mathbf{i}r\Phi^+(T,x_1,x',\eta')} \sum_j b_j(t, x_1, x', x_1, \tau, \eta') r^{n-1-j} dx' \, d\eta'$$

$$= e^{\mathbf{i}\pi s/4} |\Delta^+(x_1)|^{-1/2} b_0(t, x_1, x'(x_1), x_1, -1, \eta'(x_1)) + \mathcal{O}(r^{-1}),$$

where

$$\Delta_+(x_1) = \begin{pmatrix} \Psi^+_{x'x'} & \Psi_{x'\eta'} \\ \Psi^+_{\eta'x'} & \Psi^+_{\eta'\eta'} \end{pmatrix} (T, x_1, x'(x_1), \eta'(x_1))$$

and $s = \mathrm{sign}\, \Delta^+(0)$. The term $\mathcal{O}(r^{-1})$ yields lower order singularity after the integration with respect to r. On the other hand, writing

$$b_0(t, x_1, x'(x_1), x_1, -1, \eta'(x_1))$$
$$= b_0(T, x_1, x'(x_1), x_1, -1, \eta'(x_1)) + (t - T)c_0(t, x_1),$$

we conclude that the term with coefficient $(t - T)$ yields lower order singularity since the Fourier transform of the Heaviside function $\mathcal{F}_{r\to t}Y(r)$ is the distribution $\pi\delta(t) - \mathbf{i}v.p.\frac{1}{t}$ and

$$\frac{1}{2\pi} \int_1^\infty e^{-\mathbf{i}r(t-T)}(t - T)dr = -\frac{\mathbf{i}}{2\pi}(t - T)v.p.\frac{1}{t - T} \in L^1_{loc}(\mathbb{R}).$$

Now we apply Proposition 4.2.3 for the principal symbol b_0 at

$$\rho(x_1) = (T, x_1, x'(x_1), x_1, -1, \eta'(x_1)).$$

The point $(x_1, x'(x_1), \eta'(x_1))$ is periodic and exploiting Proposition 4.2.3 and the remark at the end of the previous section, the factors in (4.18) for $\tau = 1$ become

$$(1 - m\,(x_1, x'(x_1), \eta'(x_1)))^{-1/2} = \beta_0(x_1).$$

Consequently, since $\tau = -1$, the symbol $b_0(\rho(x_1))$, modulo the half-density $|d\tau \wedge dx' \wedge d\eta'|^{1/2}$, has the form

$$b_0(\rho(x_1)) = (-1)^{m_\gamma} e^{\mathbf{i}\frac{\pi}{2}\sigma_\gamma} |\det \varphi_{x'\eta'}(\rho(x_1))|^{1/2} \beta_0(x_1).$$

For the integral with respect to x_1, we get

$$\int_0^\alpha |\Delta_+(x_1)|^{-1/2} b_0(\rho(x_1)) dx_1 = (-1)^{m_\gamma} e^{\mathbf{i}\frac{\pi}{2}\sigma_\gamma} |\det(P_\gamma - I)|^{-1/2} \int_0^\alpha \beta_0(x_1) dx_1.$$

On the other hand, $\int_0^\alpha \beta_0(x_1)dx_1 = l_{j,0}$, where $l_{j,0}$ is the length of the segment l_j lying in $U \cap \Omega$. In fact, $l_{j,0}$ is parameterized by $(x_1, x'(x_1)), 0 \leq x_1 \leq \alpha$, and

$$1 + m\left(x_1, x'(x_1), \frac{dx'(x_1)}{dx_1}\right) = 1 + m(x_1, x'(x_1), \eta'(x_1))\beta_0^2(x_1) = \beta_0^2(x_1).$$

Thus, modulo smooth terms, we get

$$\int_{\Omega\cap U} \mathcal{F}^+(t,x,x)dx = l_{j,0} \exp\left[\mathbf{i}\frac{\pi}{2}\left(2m_\gamma + \sigma_\gamma + \frac{\text{sign}\,\Delta_+(0)}{2} - 1\right)\right]$$
$$\times |\det(P_\gamma - I)|^{-1/2}(t - T - \mathbf{i}0)^{-1} + L^1_{loc}(\mathbb{R}). \quad (4.25)$$

To find the trace of $\mathcal{F}_{B_{k_-}}(t-t',x,y)$, we use once more the argument based on the equality

$$\mathcal{F}^-(t-t',x,y) = \overline{\mathcal{F}^+(t-t',x,y)},$$

$\mathcal{F}^-(t-t',x,y)$ being the kernel of the operator $\exp\,(\mathbf{i}(t-t')A)$. Then for $t \in \mathcal{I}$, $t' \in \mathcal{J}$, modulo C^∞ terms, we have

$$\int_{\Omega\cap U} \mathcal{F}^-(t-t',x,x)dx = \int_{\Omega\cap U} \mathcal{F}^-_{B_{k_-}}(t-t',x,x)dx$$
$$= \int_{\Omega\cap U} \overline{\mathcal{F}^+(t-t',x,x)}dx = \int_{\Omega\cap U} \overline{\mathcal{F}^+_{B_k+}(t-t',x,x)}dx$$

and taking $t' = 0$, we deduce

$$\int_{\Omega\cap U} \mathcal{F}^-_{B_{k_-}}(t,x,x)dx = l_{j,0} \exp\left[-\mathbf{i}\frac{\pi}{2}\left(2m_\gamma + \sigma_\gamma + \frac{\text{sign}\,\Delta_+(0)}{2} - 1\right)\right]$$
$$\times |\det(P_\gamma - I)|^{-1/2}(t - T + \mathbf{i}0)^{-1} + L^1_{loc}(\mathbb{R}). \quad (4.26)$$

On the other hand, we may exploit a local representation of the canonical relation $\mathcal{M}_-$ by a phase function $\psi(t, x_1, x', y_1, \tau, \eta')$. Setting

$$\Psi^-(t, x, \eta') = \psi(t, x_1, x', x_1, 1, \eta') - \langle x', \eta' \rangle,$$

$$\Delta_-(T, x_1, x', \eta') = \begin{pmatrix} \Psi^-_{x'x'} & \Psi^-_{x'\eta'} \\ \Psi^-_{\eta'x'} & \Psi^-_{\eta'\eta'} \end{pmatrix} (T, x_1, x', \eta'),$$

we repeat the above argument to compute the trace of $\mathcal{F}^-_{B_{k_-}}$ over U. Comparing the arguments of the leading terms, and taking into account that for the leading term of $\mathcal{F}^-_{B_{k_-}}$ we have Maslov index $-\sigma_\gamma$, we get

$$-\frac{\text{sign}\,\Delta_+(0)}{2} = \frac{\text{sign}\,\Delta_-(0)}{2} \pmod 4. \tag{4.27}$$

Below we consider β_γ, given by (4.21), as an element of $\mathbb{Z}_4$. Our aim is to show that β_γ modulo 4 depends only on γ. If for some segment l_j of γ we have

$$\Omega^\circ \supset U_k \cap l_j \neq \emptyset,\ \Omega^\circ \supset U_m \cap l_j \neq \emptyset,$$

covering l_j by a chain of neighbourhoods connecting U_k and U_m and using the leading terms in (4.23), we conclude that β_γ does not depend on the choice of U_k. To prove that β_γ does not depend on the choice of a segment l_j, take two sufficiently small neighbourhoods $U_k \subset \Omega^\circ$, $k = j, j+1$, so that

$$U_j \cap l_j \neq \emptyset, U_{j+1} \cap l_{j+1} \neq \emptyset,$$
$$U_k \subset \{(x_1, x'), x' \in U', 0 < \epsilon \leq x_1 \leq \alpha\}, k = j, j+1.$$

For the analysis of the contribution of the outgoing segment l_{j+1} we study the trace of $\mathcal{F}^+_B$ over U_{j+1}, and from (4.23) and (4.25) for the number $\beta_{\gamma,j+1}$ related to l_{j+1} we deduce

$$\beta_{\gamma,j+1} = 2m_\gamma + \sigma_\gamma + \frac{\text{sign}\,\Delta_+^{(j+1)}}{2} - 1.$$

Here $\Delta_+^{(j+1)}$ is related to a phase function φ_{j+1} parameterizing $\mathcal{M}_+$. Next, for the analysis of the contribution coming from the incoming segment l_j we must study the trace of $\mathcal{F}^+_B = \overline{\mathcal{F}^-_B}$ over U_j, and for the corresponding number $\beta_{\gamma,j}$ we get

$$\beta_{\gamma,j} = 2m_\gamma + \sigma_\gamma - \frac{\text{sign}\,\Delta_-^{(j)}}{2} - 1,$$

where $\Delta_-^{(j)}$ is related to a phase function ψ_j parameterizing $\mathcal{M}_-$. In the Fourier integral operator related to $\mathcal{F}^+_B$, we take the phase ψ_j with sign – and this explains the sign – in the expression of $\beta_{\gamma,j}$. Finally, by (4.27), we have

$$\beta_{\gamma,j+1} = \beta_{\gamma,j} \pmod 4,$$

so $\beta_\gamma \in \mathbb{Z}_4$ depends only on γ.

Summing up the contributions of l_j in the leading terms in (4.24), (4.25), we obtain the following.

Theorem 4.3.1: *Let γ be a periodic ordinary reflecting bicharacteristic of $\Box$ in Ω with period $T_\gamma > 0$ and primitive period $T_\gamma^\#$. Assume the conditions* (i) *and* (ii) *fulfilled. Then the distribution $\sigma(t)$ near T_γ has the form*

$$\sigma(t) = \frac{T_\gamma^\#}{2\pi}\mathrm{Re}\,[\exp\,(\mathbf{i}\frac{\pi}{2}\beta_\gamma)(t - T_\gamma - \mathbf{i}0)^{-1}]|\det(P_\gamma - I)|^{-1/2} + L^1_{loc}(\mathbb{R}). \quad (4.28)$$

Remark 4.3.2: For the proof of (4.28), we need a weaker result concerning the behaviour of term (4.25). In fact, it suffices to show that

$$\int_{\Omega\cap U} \mathcal{F}^+(t,x,x)dx = \mathcal{O}(\alpha)|\det(P_\gamma - I)|^{-1/2}(t - T - \mathbf{i}0)^{-1} + L^1_{loc}(\mathbb{R})$$

with $\mathcal{O}(\alpha) \to 0$ as $\alpha \to 0$. To obtain this, it is not necessary to know the precise form of $b_0(T, x_1, x'(x_1), x_1, -1, \eta'(x_1))$.

Now we discuss briefly the Neumann and Robin boundary problems. For these problems we repeat the construction from Section 4.1. Let V_k^+ be Fourier integral operators related to the same canonical relations as in the case of Dirichlet problem. We consider

$$W_p = \sum_{k=0}^{p} V_k^+. \quad (4.29)$$

We must satisfy for $0 \le t \le \hat{t}_p$ the boundary conditions

$$i^*\left(\frac{\partial}{\partial\nu} + \alpha(x)\right)(V_k^+ + V_{k-1}^+) \in C^\infty, \quad k = 1, \cdots, p. \quad (4.30)$$

Here $\nu(x)$ is the unit normal to $\partial\Omega$, pointing into the exterior of Ω, while $\alpha(x) \in C^\infty(\partial\Omega)$.

By using the notations of Section 4.2, consider the term

$$i_{\omega_k}\frac{\partial}{\partial\nu}(R_k^+(I - M_{k-1})i^*L_{k-1} + L_{k-1})i^*\mathcal{R}_B^+. \quad (4.31)$$

Introduce local normal coordinates (y_1, y'), where $y_1 = \mathrm{dist}(x, \partial\Omega)$ and let $\partial\Omega$ have locally the form $y_1 = 0$. Let $q(y,\tau,\eta) = \eta_1^2 + m(x,\eta') - \tau^2$ be the principal symbol of $\Box$ in the local coordinates (y_1, y', η_1, η'), where $m(y,\eta')$ is homogeneous of order 2 in η' and $m(y,\eta') \ge c_0|\eta'|^2, c_0 > 0$. The derivative with respect to the normal vector field in the new coordinates is transformed into $\frac{\partial}{\partial y_1}$. The roots $\xi_1^\pm$ of the equation $q(x,\tau,\xi_1,\xi') = 0$ with respect to ξ_1 become

$$\xi_1^\pm = \mp\sqrt{\tau^2 - m(x,\xi')} = \mp\sqrt{\mu(x,\tau,\xi')}.$$

Now let $\varphi^{\pm}(t, x, \tau, \xi')$ be the phase functions introduced in Section 4.2. The kernel of L_{k-1} has the form (4.12), and the principal symbol of $i_{\omega_k} \frac{\partial}{\partial \nu} L_{k-1}$ becomes

$$\mathbf{i}\xi_1^-(x, \varphi_t, \varphi_{x'}|_{x_1=0}) = \mathbf{i}\sqrt{\mu(x, \varphi_t^-, \varphi_{x'}^-)}|_{x_1=0}.$$

Similarly, the principal symbol of

$$i_{\omega_k} \frac{\partial}{\partial \nu}(R_k^+(I - M_{k-1})i^* L_{k-1})$$

is equal to $-\mathbf{i}\sqrt{\mu(x, \varphi_t^-, \varphi_{x'}^-)}|_{x_1=0}$. On the other hand, on $x_1 = 0$, we have $\varphi^+ = \varphi^-$, hence the principal symbol of (4.31) vanishes. Choosing suitably the lower order symbols of R_k^+, we arrange (4.30). To find the leading singularity of $\sigma(t)$ neat T_γ, we need only to know the principal symbol of $V_{m_\gamma}^+$ in $\Omega^\circ \times \Omega^\circ$, which differs from that in the case of Dirichlet boundary conditions by the absence of the factor $(-1)^{m_\gamma}$. For the analysis close to the boundary according to the Remark 4.3.2, we can obtain a $\mathcal{O}(\alpha)$ term. Thus, repeating the argument of this section, we get the following.

Theorem 4.3.3: *Under the assumptions of Theorem 4.3.1, the distributions $\sigma(t)$, related to the eigenvalues of Neumann and Robin boundary problems in Ω, near T_γ has the form*

$$\sigma(t) = \frac{T_\gamma^\#}{2\pi} \mathrm{Re}\,[\exp(\mathbf{i}\frac{\pi}{2}\beta_\gamma)(t - T_\gamma - \mathbf{i}0)^{-1}]|\det(P_\gamma - I)|^{-1/2} + L^1_{loc}(\mathbb{R}), \quad (4.32)$$

where $\delta_\gamma = \sigma_\gamma + \frac{\mathrm{sign}\,\Delta}{2} - 1 \in \mathbb{N}$ depends only on γ.

In the special case when $\beta_\gamma = -1$ or $\delta_\gamma = -1$, the formulae (4.28) and (4.32) can be simplified. Indeed, notice that

$$2\mathrm{Re}\,(-\mathbf{i}(t - T_\gamma - \mathbf{i}0)^{-1}) = \mathbf{i}(t - T_\gamma + \mathbf{i}0)^{-1} - \mathbf{i}(t - T_\gamma - \mathbf{i}0)^{-1} = 2\pi\delta(t - T_\gamma).$$

hence we have the following.

Corollary 4.3.4: *Under the assumptions of Theorems 4.3.1, let $\beta_\gamma = 2m_\gamma - 1$ (resp. $\delta_\gamma = -1$ for Robin boundary problem). Then for t near T_γ, we have*

$$\sigma(t) = \frac{1}{2}(-1)^{m_\gamma} T_\gamma^\# |\det(P_\gamma - I)|^{-1/2}\delta(t - T_\gamma) + L^1_{loc}(\mathbb{R}), \quad (4.33)$$

where for Robin problem the factor $(-1)^{m_\gamma}$ is omitted.

The analysis of the singularity of $\sigma(t)$ can be applied if we study a periodic reflecting non-generated ray γ in an unbounded domain. Therefore, such a ray lies in a compact set and we may repeat the above argument. In the case when the unbounded domain is the exterior of a finite union of strictly convex disjoint obstacles in $\mathbb{R}^3$, the Maslov index σ_γ is zero. In fact, we may construct a global phase functions (see [I4], [Bu1]) related to every segment l_j of γ, hence σ_γ is zero along l_j. Since σ_γ does not change after reflections, we get $\sigma_\gamma = 0$. Moreover, $\operatorname{sign} \Delta = 0$ (see [16]) and we may apply Corollary 4.3.4.

Finally, notice that the classical Poisson summation formula has the form

$$\sum_{k \in \mathbb{Z}} e^{-\mathbf{i}kt} = 2\pi \sum_{k \in \mathbb{Z}} \delta(t - 2\pi k), \tag{4.34}$$

where the equality is interpreted in the sense of distributions in $\mathcal{D}'(\mathbb{R})$. This means that for every function $\varphi \in C_0^\infty(\mathbb{R})$, we have

$$\sum_{k \in \mathbb{Z}} \hat{\varphi}(k) = 2\pi \sum_{k \in \mathbb{Z}} \varphi(2\pi k).$$

The Laplace–Beltrami operator $-\frac{d^2}{ds^2}$ on $\mathbb{S}^1$ has eigenvalues $\lambda_k^2 = k^2, k \in \mathbb{N}$, and all periodic geodesics on $\mathbb{S}^1$ have primitive period $T_\gamma^\# = 2\pi$. Then (4.33) can be considered as a Poisson summation formula for $-\frac{d^2}{ds^2}$.

4.4 Notes

The construction of the global parametrix in Section 4.1 follows the work of Guillemin and Melrose [GM1] (see also [Chl], [Ch2] and [HeZ] for similar constructions). The analysis of the principal symbol of $\hat{F}_B$ in the interior of Ω is based on [GM1] and [DG]. The form of the principal symbol on the boundary was established in [HeZ] (see also [PoT] and [SaV] for similar investigations). Lemmas 4.2.1, 4.2.2 and Proposition 4.2.3 are due to [HeZ]. Theorems 4.3.1 and 4.3.3 have been proved in [GM1]. In Section 4.3 we present a more detailed proof concerning the leading term in (4.27) and (4.31). An application of these results is considered in [GM2], where a simple inverse problem for the ellipse is studied. For more fine inverse spectral results we refer to [PoT], [Z] and [HeZ].

For manifolds without boundary the singularity of $\sigma(t)$ related to the set of periodic bicharacteristics has been examined in [Ch3], [C2] and [DG]. For the tools related to symplectic geometry, calculus with half-densities and global theory of Fourier integral operators the reader may find more details in [H3], [H4], [GS].

5

Poisson relation for the scattering kernel

This chapter is devoted to the proof of a relation analogous to that obtained in Section 3.4. We introduce the scattering kernel $s(t, \theta, \omega)$, which is also given by the Fourier transform of the scattering amplitude well known in the physical literature. In Section 5.2, the contributions of the rays incoming with direction ω are localized. The Poisson relation for the scattering kernel has the form

$$\text{sing supp } s(t, \theta, \omega) \subset \{ -T_\gamma : \gamma \in \mathcal{L}_{(\omega,\theta)}(\Omega)\}.$$

Here $\mathcal{L}_{(\omega,\theta)}(\Omega)$ is the set of all (ω, θ)-rays in Ω and T_γ is the sojourn time of γ. The above relation is established in Section 5.3 under the assumption that each (ω, θ)-ray is the projection of a uniquely extendible generalized bicharacteristic. The results for propagation of singularities in [MS2] are not sufficient to eliminate all contributions, which must be cancelled from physical point of view in order to obtain singularities related only to the sojourn times of (ω, θ)-rays. To overcome this difficulty, we apply an argument based on the $(\mathbf{i}\lambda)$-solutions of the reduced wave equation $(\Delta + \lambda^2)u = 0$.

5.1 Representation of the scattering kernel

Let K be a compact subset of $\mathbb{R}^n, n \geq 2$, with non-empty interior and C^∞ smooth boundary ∂K. Assume that $\Omega = \overline{\mathbb{R}^n \setminus K}$ is connected. Clearly, $\partial\Omega = \partial K$. In this section we introduce and study the scattering kernel related to the scattering operator

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for the wave operator $\Box = \partial_t^2 - \Delta_x$ with Dirichlet boundary condition in the exterior of K.

Fix $\rho_0 > 0$ so that $K \subset \{x \in \mathbb{R}^n : |x| \leq \rho_0\}$. First we will treat the case when n is odd, and at the end of this section we will discuss the case n even. Consider the Dirichlet problem

$$\begin{cases} (\partial_t^2 - \Delta_x)u = 0 \text{ in } \mathbb{R} \times \Omega^\circ, \\ u = 0 \text{ on } \mathbb{R} \times \partial\Omega, \\ u|_{t=0} = f_1, u_t|_{t=0} = f_2. \end{cases} \tag{5.1}$$

Denote by $H_D(\Omega)$ the completion of the space $C_0^\infty(\Omega^\circ)$ with respect to the norm

$$\|\varphi\|_D = \left(\int_\Omega \|\nabla_x \varphi\|^2 dx\right)^{1/2}$$

and introduce the *energy space* $H = H_D(\Omega) \oplus L^2(\Omega)$. There exists a unitary group $U(t) = e^{\mathbf{i}tG}$ of H with generator $\mathbf{i}G$ such that for $f = (f_1, f_2) \in H$ we have

$$U(t)f = (u(t, \cdot), u_t(t, \cdot)),$$

$u(t, x)$ being the solution of (5.1) in the sense of distributions. Here the operator G has form

$$G = -\mathbf{i}\begin{pmatrix} 0 & 1 \\ \Delta_D & 0 \end{pmatrix}$$

and domain

$$D(G) = \{(u, v) : u \in H_0^1(\Omega) \cap H^2(\Omega),\ v \in H_D(\Omega)\} \subset H.$$

Moreover, G is self-adjoint in H and Δ_D is the Dirichlet Laplacian in $L^2(\Omega)$ with domain $D(\Delta_D) = H_0^1(\Omega) \cap H^2(\Omega)$.

Similarly, define the space $H_0(\mathbb{R}^n)$ as above, replacing Ω by $\mathbb{R}^n$, and consider the energy space $H_0 = H_D(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$. For the solution $u_0(t, x)$ of the Cauchy problem for $\Box$ in $\mathbb{R}_t \times \mathbb{R}_x^n$ with initial date $f = (f_1, f_2) \in H_0$ we have

$$U_0(t)f = (u_0(t, x), \partial_t u_0(t, x)),$$

where $U_0(t) = e^{\mathbf{i}tG_0}$ is a unitary group in H_0 with generator $\mathbf{i}G_0$, where

$$G_0 = -\mathbf{i}\begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$$

is self-adjoint in H_0, Δ is the Laplacian in $L^2(\mathbb{R}^n)$ with domain $D(\Delta) = H^2(\mathbb{R}^n)$ and

$$D(G_0) = \{(u, v) :\ u \in H^2(\mathbb{R}^n),\ \ v \in H^1(\mathbb{R}^n)\} \subset H_0.$$

The space H can be considered as a subspace of H_0, extending $f \in H$ as 0 in K. Let

$$J :\ H_0 \to H$$

be the orthogonal projection. Consider the wave operators

$$W_{\pm} f = \lim_{t \to \mp\infty} U(t) J U_0(-t) f, \ \ f \in H_0.$$

These operators exist for each $f \in H_0$ and, moreover, they are isometrics from H_0 onto H_0 (see [LP1]). The operator W_+ (resp. W_-) is related to the evolution when the time $t \to +\infty$ (resp. $t \to -\infty$).

The operators $W_{\pm}$ are complete, that is Image $W_+ =$ Image W_-, and this makes possible to define the *scattering operator*

$$S = (W_+)^{-1} \circ W_-$$

as a unitary operator from H_0 onto H_0. We refer to [LP1] for the existence of $W_{\pm}$ and the main properties of S. Notice that for all $t \in \mathbb{R}$ we have

$$SU_0(t) = U_0(t) S.$$

By using the Radon transform, we can construct an isometric isomorphism (see [LP1])

$$\mathcal{R} : H_0 \longrightarrow L^2(\mathbb{R} \times \mathbb{S}^{n-1})$$

so that

$$\mathcal{R} U_0(t) = T_t \mathcal{R},$$

where T_t is the *translation operator* in $L^2(\mathbb{R} \times \mathbb{S}^{n-1})$ having the form

$$T_t f(\sigma, \omega) = f(\sigma - t, \omega), \ f \in L^2(\mathbb{R} \times \mathbb{S}^{n-1}).$$

Therefore,

$$\tilde{S} = \mathcal{R} \circ S \circ \mathcal{R}^{-1} : \ L^2(\mathbb{R}_t \times \mathbb{S}^{n-1}) \longrightarrow L^2(\mathbb{R}_t \times \mathbb{S}^{n-1})$$

becomes a unitary operator commuting with the translations in t and $\tilde{S} - Id$ is a linear continuous map from $C_0^\infty(\mathbb{R}_t \times \mathbb{S}^{n-1})$ into $\mathcal{D}'(\mathbb{R}_t \times \mathbb{S}^{n-1})$. By the Schwartz theorem, the operator $\tilde{S} - Id$ has a kernel

$$s(t - t', \theta, \omega) \in \mathcal{D}'(\mathbb{R}_t \times \mathbb{S}^{n-1} \times R_{t'} \times \mathbb{S}^{n-1}),$$

where $\theta, \omega \in \mathbb{S}^{n-1}$. Since for fixed ω and θ, the kernel s depends only on $t - t'$, we can consider the distribution

$$s(t, \theta, \omega) \in \mathcal{D}'(\mathbb{R} \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}),$$

called the *scattering kernel*.

To obtain a representation of $s(t, \theta, \omega)$, introduce the solution $w(t, x; \omega)$ of the problem

$$\begin{cases} (\partial_t^2 - \Delta_x) w(t, x; \omega) = 0 \text{ in } \mathbb{R} \times \Omega^\circ, \\ w = 0 \text{ on } \mathbb{R} \times \partial\Omega, \\ w|_{t < -\rho_0} = \delta(t - \langle x, \omega \rangle). \end{cases} \tag{5.2}$$

Then we have the following representation

$$s(\sigma, \theta, \omega) = C_n \int_{\partial\Omega} \partial_t^{n-2} \partial_\nu w(\langle x, \theta \rangle - \sigma, x; \omega) dS_x, \tag{5.3}$$

where $\nu(x)$ is the unit normal to $x \in \partial\Omega$ pointing into Ω, dS_x is the Lebesgue measure on $\partial\Omega$ and

$$C_n = (-1)^{(n+1)/2} 2^{-n} \pi^{1-n}.$$

This representation was established by Majda [Ma2] (see also [LP1] and Chapter 8 in [P5] for related results). The integral (5.3) is interpreted in the sense of distributions, and for $\rho(t) \in C_0^\infty(\mathbb{R})$ we have

$$\langle s(t, \theta, \omega), \rho(t) \rangle = C_n \int_{-\infty}^{\infty} \int_{\partial\Omega} \partial_\nu w(t, x; \omega) \frac{d^{n-2}\rho}{dt^{n-2}}(\langle x, \theta \rangle - t) dt\, dS_x. \tag{5.4}$$

Here $\frac{d^{n-2}\rho}{dt^{n-2}}(\langle x, \theta \rangle - t)$ has compact support in $\mathbb{R}_t \times \partial\Omega$. Moreover, it follows from (5.4) that $s(t, \theta, \omega)$ depends smoothly on (θ, ω) and takes values in the space of tempered distributions in t. To justify the existence of a solution of (5.2), set $w^+(t, x; \omega) = w(t, x; \omega) - \delta(t - \langle x, \omega \rangle)$ and introduce the function

$$H_2(t) = \begin{cases} t^2/2, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Then $w^+ = (d^3/dt^3) w_2$, where w_2 is the solution of the problem

$$\begin{cases} \Box w_2 = 0 \text{ in } \mathbb{R} \times \Omega^\circ, \\ w_2 + H_2(t - \langle x, \omega \rangle) = 0 \text{ on } \mathbb{R} \times \partial\Omega, \\ w_2|_{t < -\rho_0} = 0. \end{cases}$$

As in the proof of Proposition 3.3.1 in Section 3.3, applying Theorem 24.1.1 in [H3], we obtain the existence of the solution w_2 and this implies the existence of w.

In the physical literature the function $a(\lambda, \theta, \omega)$, given by

$$\overline{a(\lambda, \theta, \omega)} = \left(\frac{2\pi}{\mathbf{i}\lambda} \right)^{(n-1)/2} \mathcal{F}_{t \to \lambda} s(t, \theta, \omega), \;\; \lambda \in \mathbb{R}$$

is called the *scattering amplitude*. Here $\mathcal{F}_{t \to \lambda}$ is the Fourier transform with respect to t. In the following exposition we assume that $\lambda \in \mathbb{R}$. It is easy to see that

$$\mathcal{F}_{t \to \lambda}((\partial_t^{n-2} \partial_\nu) w(\langle x, \theta \rangle - t, x; \omega)) = (-\mathbf{i}\lambda)^{n-2} e^{-\mathbf{i}\lambda \langle x, \theta \rangle} \partial_\nu \Big(e^{\mathbf{i}\lambda \langle x, \omega \rangle} + \overline{v_{sc}(\lambda, x; \omega)} \Big),$$

where $v_{sc}(\lambda, x; \omega) = \mathcal{F}_{t \to \lambda}(w^+(t, x; \omega))$. Consequently,

$$a(\lambda, \theta, \omega) = -\frac{(\mathbf{i}\lambda)^{(n-3)/2}}{2(2\pi)^{(n-1)/2}} \int_{\partial\Omega} \Big[e^{\mathbf{i}\lambda \langle x, \theta \rangle} \partial_\nu v_{sc}(\lambda, x; \omega) - e^{\mathbf{i}\lambda \langle x, \theta - \omega \rangle} \mathbf{i}\lambda \langle \nu, \omega \rangle \Big] dS_x. \tag{5.5}$$

The function $v_{sc}(\lambda, x; \omega)$ is a solution of the problem

$$\begin{cases} (\Delta_x + \lambda^2) v_{sc}(\lambda, x; \omega) = 0 \text{ in } \Omega^\circ, \\ v_{sc}(\lambda, x; \omega) + e^{-\mathbf{i}\lambda\langle x, \omega\rangle} = 0 \text{ on } \partial\Omega. \end{cases} \tag{5.6}$$

Clearly, $w^+(t, x; \omega)$ is outgoing since it vanishes for $t < -\rho_0$. To define the outgoing solutions of the reduced wave equation $(\Delta + \lambda^2)u = g$, we must introduce the *outgoing resolvent* $R_0(\lambda) = (-\Delta - \lambda^2)^{-1}$ of the self-adjoint Laplacian $-\Delta$ in $L^2(\mathbb{R}^n)$ defined above. Let $\mathrm{Im}\ \lambda < 0$. Then $\lambda^2 \notin [0, +\infty]$ and our choice of outgoing resolvent is such that $R_0(\lambda)$ becomes a bounded operator from $L^2(\mathbb{R}^n)$ to $H^2(\mathbb{R}^n)$. Let $W_0(t)$ be the operator satisfying

$$(\partial_t^2 - \Delta_x) W_0(t) = 0,\ \ W_0(0) = 0,\ \ \partial_t(W_0(t)) = Id.$$

This operator has the form

$$W_0(t) = \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}$$

and the kernel $w_0(t, x, y)$ of $W_0(t)$ is the distribution $w_0(t, x - y)$ determined as the solution of the Cauchy problem

$$\begin{cases} (\partial_t^2 - \Delta_x) w_0(t, x - y) = 0, \\ w_0(0, x - y) = 0,\ \ \partial_t w_0(0, x - y) = \delta(x - y). \end{cases}$$

If we extend $w_0(t, x - y)$ as 0 for $t < 0$, we obtain a solution of the problem

$$\begin{cases} (\partial_t^2 - \Delta_x) e_0(t, x - y) = \delta(t)\delta(x - y), \\ \text{supp } e_0(t, x - t) \subset \{(t, x, y) :\ t \geq 0\}. \end{cases}$$

Thus $w_0(t, x - y) = H(t) e_0(t, x - y)$, where $H(t)$ is the function $\mathbf{1}_{[0,\infty]}(t)$ and $e_0(t, x - y)$ is the distribution studied in Section 3.1.

Notice that by the strong Huygens principle for n odd, we have $w_0(t, x - y) = 0$ for $|x - y| \neq t, t > 0$. For $\mathrm{Im}\ \lambda < 0$ the outgoing resolvent is given by the integral

$$R_0(\lambda) = \int_0^\infty e^{-\mathbf{i}\lambda t} W_0(t) dt.$$

Passing to the kernels, we have

$$R_0(\lambda, x - y) = \int_0^\infty e^{-\mathbf{i}\lambda t} w_0(t, x - y) dt.$$

By the Huygens principle, $w_0(t, x - y)$ vanishes if $|x - y| > t$, so for $|x - y| < A$ we have

$$R_0(\lambda, x - y) = \int_0^{A+1} e^{-\mathbf{i}\lambda t} w_0(t, x - y) dt$$

and $R_0(\lambda)$ admits an analytic continuation in $\mathbb{C}$ from $L^2_{comp}(\mathbb{R}^n)$ to $H^2_{loc}(\mathbb{R}^n)$.

Definition 5.1.1: A function $u(\lambda, x)$ is called $(\mathbf{i}\lambda)$-outgoing if there exist $R_1 > 0$ and $f(x) \in L^2_{comp}(\mathbb{R}^n)$ such that

$$u(\lambda, x) = R_0(\lambda) f(x) \text{ for } |x| \geq R_1.$$

The Fourier transform of w^+ will be $(\mathbf{i}\lambda)$-outgoing solution of the reduced wave equation

$$(\Delta + \lambda^2) u = 0$$

(see [LP1] and [P5]) and it is sufficient to prove that the Fourier transform of w_2 is $(\mathbf{i}\lambda)$-outgoing. To see this, let $\chi(x) \in C_0^\infty(\mathbb{R}^n)$ be a function such that $\chi(x) = 0$ for $|x| \leq \rho_0 + 1$, $\chi(x) = 1$ for $|x| \geq \rho_0 + 2$. Therefore,

$$\Box(\chi w_2) = 2\langle \nabla\chi, \nabla w_2 \rangle - (\Delta\chi) w_2 = h_2(t, x),$$

and for each $t \in \mathbb{R}$ we have $h(t, \cdot) = (0, h_2(t, \cdot)) \in H_0$. Moreover, $h(t, x)$ has a compact support with respect to x. Since χw_2 is an outgoing solution of the wave equation vanishing for $t < -\rho_0$, we can write

$$\chi w_2(t, x; \omega) = \int_{-\infty}^{t} (U_0(t - \tau) h((\tau, x))_1 d\tau = \int_0^\infty (U_0(\sigma) h(t - \sigma, x))_1 d\sigma.$$

Here $(g)_1$ denotes the first component of $g = (g_1, g_2) \in H_0$.

Let $v_2(\lambda, x; \omega)$ be the Fourier transform of $\chi w_2(t, x; \omega)$ with respect to t. For each $\varphi(\lambda) \in \mathcal{S}(\mathbb{R})$ we have

$$\begin{aligned}
\langle v_2(\lambda, x; \omega), \varphi(\lambda) \rangle &= \left\langle \chi w_2(t, x; \omega), \int e^{-\mathbf{i}\lambda t} \varphi(\lambda) d\lambda \right\rangle \\
&= \int_{-\infty}^{\infty} \int_0^\infty (U_0(\sigma)(0, h_2(t - \sigma, x)))_1 d\sigma \left(\int e^{-\mathbf{i}\lambda} \varphi(\lambda) d\lambda \right) dt \\
&= \left\langle \int_0^\infty e^{-\mathbf{i}\lambda\sigma} (U_0(\sigma)(0, g_2(\lambda, x)))_1 d\sigma,\ \varphi(\lambda) \right\rangle,
\end{aligned}$$

where $g_2(\lambda, x) = \mathcal{F}_{t \to \lambda} h_2(t, x)$. Thus,

$$v_2(\lambda, x; \omega) = \int e^{-\mathbf{i}\lambda\sigma} (U_0(\sigma)(0, g_2(\lambda, x)))_1 d\sigma.$$

On the other hand,

$$\mathbf{i}(G_0 - \lambda)^{-1} = \int_0^\infty e^{-\mathbf{i}\lambda\sigma} U_0(\sigma) d\sigma, \text{ Im } \lambda < 0$$

and

$$(G_0 - \lambda)^{-1} = \begin{pmatrix} \lambda R_0(\lambda) & -\mathbf{i} R_0(\lambda) \\ -\mathbf{i}\Delta R_0(\lambda) & \lambda R_0(\lambda) \end{pmatrix}.$$

Using the analytic continuation of $R_0(\lambda)$, this yields

$$\mathbf{i}((G_0 - \lambda)(0, g_2))_1 = R_0(\lambda)g_2, \ \ \lambda \in R.$$

Consequently, $v_2(\lambda, x; \omega)$ is $(\mathbf{i}\lambda)$-outgoing and the same is true for the Fourier transform of $w_2(t, x; \omega)$. This implies that $v_{sc}(\lambda, x; \omega)$ is $(\mathbf{i}\lambda)$-outgoing.

The kernel $G^+_{\mathbf{i}\lambda}(x-y)$ of $-R_0(\lambda) = (\Delta + \lambda^2)^{-1}$ is called *outgoing Green function*. A precise formula for $G^+_{\mathbf{i}\lambda}$ is given in [LP1] (see also Chapter 2 in [P5]). It is possible to find $G^+_{\mathbf{i}\lambda}$ from the Fourier transform with respect to λ of the fundamental solution of the wave operator $e_0(t, x-y)$. The following representation of $e_0(t, x)$ is obtained in Section 6.2 in [Hl]. For a function $\psi(x) \in C_0^\infty(\mathbb{R}^n)$ we have

$$\langle e_0(t, x), \psi(x)\rangle = \frac{1}{4\pi^{(n-1)/2}} \left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{(n-3)/2} t^{n-2} \int_{\mathbb{S}^{n-1}} \psi(t\omega) d\omega, \ t > 0.$$

Setting $t = r = |x|, x = r\omega$, we get

$$\langle G^+_{\mathbf{i}\lambda}(x), \psi(x)\rangle = -\frac{1}{4\pi^{(n-1)/2}} \int_0^\infty \int_{\mathbb{S}^{n-1}} e^{-\mathbf{i}\lambda r} \left(\frac{1}{2r}\frac{\partial}{\partial r}\right)^{(n-3)/2} r^{n-2}\psi(r\omega) dr\, d\omega.$$

Since $r^{n-1}dr\ d\omega = dx$, after an integration by parts, we deduce

$$G^+_{\mathbf{i}\lambda}(x) = -\frac{(-1)^{(n-3)/2}}{2(2\pi)^{(n-1)/2}} \left(\frac{1}{r}\partial_r\right)^{(n-3)/2} \left(\frac{e^{-\mathbf{i}\lambda r}}{r}\right).$$

Notice that this implies easily the following asymptotic

$$G^+_{\mathbf{i}\lambda}(x) = -\frac{(\mathbf{i}\lambda)^{(n-3)/2}}{2(2\pi)^{(n-1)/2}} \frac{e^{-\mathbf{i}\lambda r}}{r^{(n-1)/2}} + \mathcal{O}\left(\frac{e^{-\mathbf{i}\lambda r}}{r^{(n+1)/2}}\right), \ r \to +\infty. \tag{5.7}$$

It is easy to see that a $(\mathbf{i}\lambda)$-outgoing solution $u(\lambda, x)$ satisfies for $|x| \to \infty$ the condition

$$\begin{cases} u(\lambda, x) = \dfrac{e^{-\mathbf{i}\lambda r}}{r^{(n-1)/2}} b\left(\lambda, \dfrac{x}{|x|}\right) + \mathcal{O}\left(\dfrac{e^{-\mathbf{i}\lambda r}}{r^{(n+1)/2}}\right), \\ \dfrac{\partial u}{\partial r}(\lambda, x) + \mathbf{i}\lambda u(\lambda, x) = \mathcal{O}\left(\dfrac{e^{-\mathbf{i}\lambda r}}{r^{(n+1)/2}}\right), \end{cases} \tag{5.8}$$

called $(\mathbf{i}\lambda)$-outgoing Sommerfeld radiation condition (see [LP1]).

Applying (5.8) to $v_{sc}(\lambda, x; \omega)$, we get

$$v_{sc}(\lambda, x; \omega) = \frac{e^{-\mathbf{i}\lambda r}}{r^{(n-1)/2}} \tilde{a}(\lambda, \theta, \omega) + \mathcal{O}\left(\frac{e^{-\mathbf{i}\lambda r}}{r^{(n+1)/2}}\right), \ x = r\theta, \ \ r \to \infty. \tag{5.9}$$

On the other hand, exploiting (5.8) and applying the Green formula, it is easy to see that

$$v_{sc}(\lambda, x; \omega) = \int_{\partial\Omega} \left[G^+_{\mathbf{i}\lambda}(x-y)\frac{\partial v_{sc}}{\partial \nu}(\lambda, y; \omega) - \frac{\partial G^+_{\mathbf{i}\lambda}}{\partial \nu}(x-y) v_{sc}(\lambda, y; \omega)\right] dS_y.$$

Multiplying both sides of this equality by $e^{\mathbf{i}\lambda r}r^{(n-1)/2}$ and setting $x = r\theta$, $r = |x|$, we find

$$\begin{aligned}\tilde{a}(\lambda, \theta, \omega) &= \lim_{r\to\infty} e^{\mathbf{i}\lambda r}r^{(n-1)/2}v_{sc}(\lambda, r\theta; \omega)\\ &= -\frac{(\mathbf{i}\lambda)^{(n-3)/2}}{2(2\pi)^{(n-1)/2}}\int_{\partial\Omega} e^{\mathbf{i}\lambda\langle x,\theta\rangle}\left[\frac{\partial v_{sc}}{\partial\nu}(\lambda, x; \omega) - \mathbf{i}\lambda\langle\nu, \theta\rangle v_{sc}(\lambda, x; \omega)\right] dS_x\\ &= -\frac{(\mathbf{i}\lambda)^{(n-3)/2}}{2(2\pi)^{(n-1)/2}}\int_{\partial\Omega}\left(e^{\mathbf{i}\lambda\langle x,\theta\rangle}\frac{\partial v_{sc}}{\partial\nu}(\lambda, x; \omega) - \mathbf{i}\lambda\langle\nu, \omega\rangle e^{\mathbf{i}\lambda\langle x,\theta-\omega\rangle}\right) dS_x.\end{aligned}$$

Here we have used the equality

$$\int_{\partial\Omega} e^{\mathbf{i}\lambda\langle x,\theta-\omega\rangle}\langle\nu, \theta+\omega\rangle dS_x = \int_K \frac{\partial}{\partial(\theta+\omega)}(e^{\mathbf{i}\langle y,\theta-\omega\rangle})dy = 0.$$

Thus, $a(\lambda, \theta, \omega) = \tilde{a}(\lambda, \theta, \omega)$ and the scattering amplitude can be considered as the asymptotic profile of the outgoing solution $v_{sc}(\lambda, x; \omega)$ of the problem (5.6).

The scattering amplitude determines uniquely the obstacle K (see for instance [LP1]). On the other hand, in the application it is possible to measure only the singularities of $s(t, \theta, \omega)$ and their leading terms. Hence, in general, we cannot measure the Fourier transform of $s(t, \theta, \omega)$, and for this reason it is more important to investigate the inverse scattering problems related to the singularities of $s(t, \theta, \omega)$.

The above analysis can be applied to other boundary problems for the wave equation in $\mathbb{R} \times \Omega$. For example, to study the Neumann problem

$$\begin{cases}(\partial_t^2 - \Delta_x)u = 0 \text{ in } \mathbb{R}\times\Omega^\circ,\\ \partial_\nu u = 0 \text{ on } \mathbb{R}\times\partial\Omega,\\ u|_{t=0} = f_1,\ \ u_t|_{t=0} = f_2,\end{cases} \tag{5.10}$$

we introduce the energy space $H_N = H^1(\Omega) \oplus L^2(\Omega)$ and the unitary group $U_N(t) = e^{\mathbf{i}G_N t}$. The operator G_N has the form

$$G_N = -\mathbf{i}\begin{pmatrix} 0 & 1\\ \Delta_N & 0\end{pmatrix},$$

where D_N is the Laplace operator in $L^2(\Omega)$ with Neumann boundary condition with domain

$$D(D_N) = \{f \in H^2(\Omega) :\ \partial_\nu f|_{\partial\Omega} = 0\}.$$

We can define the wave operators $W_\pm$ and the scattering operator S, and for the scattering kernel $s_N(t, \theta, \omega)$ related to Neumann problem we obtain the representation

$$s_N(\sigma, \theta, \omega) = C_n\int_{\partial\Omega}\langle\nu, \theta\rangle\partial_t^{n-2}w_N(\langle x, \theta\rangle - \sigma, x; \omega)dS_x, \tag{5.11}$$

where $w_N(t, x; \omega)$ is the solution of the problem

$$\begin{cases} (\partial_t^2 - \Delta_x) w_N(t, x; \omega) = 0 \text{ in } \mathbb{R} \times \Omega^\circ, \\ \partial_\nu w_N = 0 \text{ on } \mathbb{R} \times \partial\Omega, \\ w_N|_{t < -\rho_0} = \delta(t - \langle x, \omega \rangle). \end{cases} \tag{5.12}$$

For the proof of this representation and for other boundary problems, we refer to Section 8.6 in [P5], where problems for domains with time-dependent boundaries are considered.

To examine the case of even dimensions $n \geq 2$, notice that the strong Huygens principle is not valid for n even. The wave operators $W_\pm$ and the scattering operators S are defined as above, but following [LP2] (see also Chapter 2 in [P5]) we have an outgoing and an incoming translation representations of $U_0(t)$ denoted, respectively, by $\mathcal{R}^+$ and $\mathcal{R}^-$. They satisfy the relation $\mathcal{R}^+ = K\mathcal{R}^-$, where

$$\mathcal{K}\varphi(s, \omega) = \mathcal{F}^{-1}_{\sigma \to s} (\text{sing}\, \sigma) \mathcal{F}_{s \to \sigma} \varphi(s, \omega)$$

is the Hilbert transform. The scattering operator in $L^2(\mathbb{R} \times \mathbb{S}^{n-1})$ is defined by

$$\tilde{S} = \mathcal{R}^+ \circ S \circ (\mathcal{R}^-)^{-1}.$$

The kernel $s(t - t', \theta, \omega)$ of the operator $\tilde{S} - K$ is called scattering kernel and for $s(t, \theta, \omega)$ we obtain again the representation (5.3). We define the scattering amplitude in the same way and $v_{sc}(t, x; \omega)$ is $(\mathbf{i}\lambda)$-outgoing solution of the problem (5.6). Here for the definition of the $(\mathbf{i}\lambda)$-outgoing solutions, we use the outgoing resolvent $R_0(\lambda)$ whose kernel for $n \geq 2$ has the form

$$R_0(\lambda, x - y) = -\frac{\mathbf{i}}{4} \left(\frac{\lambda}{2\pi |x - y|} \right)^{(n-2)/2} \left(H^{(2)}_{\frac{n-2}{2}}(u) \right) \Big|_{u = \lambda |x - y|}, \tag{5.13}$$

where $H^{(2)}_{\frac{n-2}{2}}(u)$ is the Hankel function of second kind. Moreover, we can deduce (see [Va]) that

$$R_0(\lambda) = \begin{cases} E(\lambda) + \lambda^{n-2} E_1(\lambda), & n \text{ odd}, \\ F(\lambda) + \lambda^{n-2} \log \lambda F_1(\lambda), & n \text{ even}, \end{cases}$$

where $E(\lambda), E_1(\lambda), F(\lambda), F_1(\lambda)$ are entire operator-valued functions from $L^2_{comp}(\mathbb{R}^n)$ to $H^2_{loc}(\mathbb{R}^n)$. For dimension $n = 3$ we have

$$H^{(2)}_{1/2}(r) = \mathbf{i} \left(\frac{2}{\pi r} \right)^{1/2} e^{-\mathbf{i} r}$$

and we obtain

$$R_0(\lambda, x) = \frac{e^{-\mathbf{i}\lambda |x|}}{4\pi |x|}.$$

For $n = 2$ we get

$$R_0(\lambda, x) = -\frac{\mathbf{i}}{4} H^{(2)}_0(\lambda |x|).$$

On the other hand, for $\mu \geq 0$ and $|z| = r$ we have the asymptotic

$$H_\mu^{(2)}(z) = \left(\frac{2}{\pi r}\right)^{1/2} e^{-\mathbf{i}(z - \frac{\mu\pi}{2} - \frac{\pi}{4})} + \mathcal{O}(r^{-3/2}),\ -2\pi < \arg z < \pi,\ r \to +\infty \tag{5.14}$$

and for all $n \geq 2$ we obtain (5.7) for $G^+_{\mathbf{i}\lambda}(x) = -R_0(\lambda, x)$. Repeating the above argument for the $(\mathbf{i}\lambda)$-outgoing solution of the problem (5.6), we deduce for n even the representation (5.5) for the scattering amplitude $a(\lambda, \theta, \omega)$.

Remark 5.1.2: In our exposition we choose the outgoing resolvent $R_0(\lambda)$ and the outgoing Green function $G^+_{\mathbf{i}\lambda}(x) = -R_0(\lambda, x)$ so that they are analytic for $\mathrm{Im}\ \lambda < 0$. The choice of $G^+_{\mathbf{i}\lambda}$ is the same as in [LP1]. In other papers and books, the outgoing resolvent and outgoing Green function are defined to be analytic for $\mathrm{Im}\ \lambda > 0$. Under this condition, the outgoing resolvent has the form (5.13) with $H^{(1)}_{\frac{n-2}{2}}(u)$ instead of $H^{(2)}_{\frac{n-2}{2}}(u)$. In particular for $n = 3$ one has

$$R_0(\lambda, x) = \frac{e^{\mathbf{i}\lambda|x|}}{4\pi|x|}.$$

5.2 Location of the singularities of $s(t, \theta, \omega)$

In this section we begin by the analysis of the singularities of $s(t, \theta, \omega)$. Let $\theta \neq \omega$ be fixed. Recall that $\mathcal{L}_{\omega,\theta}(\Omega)$ denotes the set of all (ω, θ)-rays in Ω. As usual, π is the natural projection $T^*(\mathbb{R} \times \Omega) \to \Omega$. We fix $\rho_0 > 0$ so that $\Omega \subset \{x \in \mathbb{R}^n : |x| \leq \rho_0\}$. Recall that γ is an (ω, θ)-ray if $\gamma = \pi \circ \tilde{\gamma}$, where

$$\tilde{\gamma}(t) = (t, x(t), \pm 1, \xi(t)) \in T^*(\mathbb{R} \times \Omega)$$

is a generalized bicharacteristic of $\Box$ in Ω such that there exist real numbers $t_1 > t_2$ with

$$\xi(t) = -\omega \text{ for } t \leq t_1,\ \ \xi(t) = -\theta \text{ for } t \geq t_2. \tag{5.15}$$

This means that the curve $x(t)$ has direction ω for $t \leq t_1$ and direction θ for $t \geq t_2$. We assume that the time t increases when we move along $\tilde{\gamma}(t)$. Denote by T_γ the *sojourn time* of γ introduced in Section 2.4.

In the following we consider a fixed t_0 such that

$$-t_0 \notin \{-T_\gamma : \gamma \in \mathcal{L}_{\omega,\theta}(\Omega)\}.$$

Take $T > 0$ with $|t_0| < T$ and consider the set

$$\Gamma_T = \{T_\gamma : |T_\gamma| \leq T,\ \gamma \in \mathcal{L}_{\omega,\theta}(\Omega)\}.$$

To check that this set is closed, take a sequence $\{\gamma_k\} \subset \mathcal{L}_{\omega,\theta}(\Omega)$ with $T_{\gamma_k} \to T_0$. There exists a compact set M such that $\gamma_k(t) \in M$ for $|t| \leq T + 3\rho_0$ and all k. Let

$\tilde{\gamma}_k$ be a generalized bicharacteristic of $\Box$ in Ω such that $\gamma_k = \pi \circ \tilde{\gamma}_k$. There exist $t_1 < t_2$ such that $|t_1| < T + 3\rho_0, i = 1, 2$, and for each k (5.15) holds replacing $\xi(t)$ by $\xi_k(t)$. It follows by Lemma 1.2.6 that there exists a generalized bicharacteristic $\tilde{\gamma}$ of $\Box$ such that $\pi \circ \tilde{\gamma}$ is an (ω, θ)-ray with sojourn time T_0. Hence Γ_T is closed.

Choose $\epsilon_0 > 0$ so that

$$T_\gamma \notin [t_0 - \epsilon_0, t_0 + \epsilon_0],\ \gamma \in \mathcal{L}_{\omega,\theta}(\Omega). \tag{5.16}$$

Let $\rho(t) \in C_0^\infty(\mathbb{R}), \rho(t) = 1$ for $|t| \leq \frac{1}{2}, \rho(t) = 0$ for $|t| \geq 1$. Set $\rho_\delta(t) = \rho(t/\delta)$ for $0 < \delta \leq \epsilon_0/2$ and consider the integral

$$\begin{aligned} J(\lambda) &= \langle s(t,\theta,\omega), \rho_\delta(t+t_0)e^{-\mathbf{i}\lambda t}\rangle \\ &= \sum_{k=0}^{n-2} c_k(-\mathbf{i}\lambda)^{n-2-k}\int_{\mathbb{R}}\int_{\partial\Omega} e^{\mathbf{i}\lambda(t-\langle x,\theta\rangle)}\rho_\delta^{(k)}(\langle x,\theta\rangle - t + t_0)\frac{\partial w}{\partial\nu}(t,x;\omega)dt\, dS_x. \end{aligned}$$

Here $w(t,x;\omega)$ is the solution of (5.2), $c_k = \text{const}$, $c_0 = C_n$ and $\rho_\delta^{(k)} = \frac{d^k\rho_\delta}{dt^k}$.

Our aim is to show that for sufficiently small δ, the integral $J(\lambda)$ is rapidly decreasing with respect to λ. In the following we study the term with $k = 0$. The analysis of the other terms is completely analogous.

Without loss of the generality, we may assume that $\omega = (0, \ldots, 0, 1)$. Consider the hyperplane

$$Z(\tau) = \{x \in \mathbb{R}^n :\ x_n = \tau\},$$

where $\tau < -\rho_0$ will be fixed below. Let

$$\mathbb{R}_\tau^+ = \{t \in \mathbb{R} :\ t > \tau\}$$

and let

$$\varphi_j(x') \in C_0^\infty(\mathbb{R}^{n-1}), x' = (x_1, \ldots, x_{n-1}).$$

Consider the problems

$$\begin{cases} \Box v_j = 0 \text{ in } \mathbb{R}_\tau^+ \times \mathbb{R}_x^n, \\ v_j(\tau,x) = \varphi_j(x')\delta(\tau - x_n), \\ \dfrac{\partial v_j}{\partial t}(\tau,x) = \varphi_j(x')\delta'(t - x_n), \end{cases} \tag{5.17}$$

$$\begin{cases} \Box W_j = 0 \text{ in } \mathbb{R} \times \Omega^\circ, \\ W_j = \text{on } \mathbb{R} \times \partial\Omega, \\ W_j(\tau,x) = \varphi_j(x')\delta(\tau - x_n), \\ \dfrac{\partial W_j}{\partial t}(\tau,x) = \varphi_j(x')\delta'(\tau - x_n). \end{cases} \tag{5.18}$$

There exists a compact set $F_0' \subset \mathbb{R}^{n-1}$ such that if $\operatorname{supp} \varphi_j \cap F_0' = \emptyset$, then the straight lines issued from (x', τ), $x' \in \operatorname{supp} \varphi_j$, with direction ω do not meet $\partial\Omega$. Hence for such j we obtain

$$WF\left(\left(\frac{\partial W_j}{\partial \nu}\right)\Big|_{\mathbb{R}\times\partial\Omega}\right) \cap \left\{(t, x, 1 - \theta|_{T_x(\partial\Omega)}) : \ |t| \leq T + \rho_0 + 1, \ x \in \partial\Omega\right\} = \emptyset. \tag{5.19}$$

Covering $\partial\Omega$ by small open neighbourhoods ω_k, we can apply Theorem 1.3.4 to the integrals over $\mathbb{R} \times \omega_k$. Thus, we obtain

$$\int_{\mathbb{R}} \int_{\partial\Omega} e^{\mathrm{i}\lambda(t - \langle x, \theta\rangle)} \rho_\delta(\langle x, \theta\rangle - y + t_0) \frac{\partial W_j}{\partial \nu} dt \, dS_x = \mathcal{O}(|\lambda|^{-m}), \ \forall m \in \mathbb{N}. \tag{5.20}$$

Set

$$F_0 = \{x \in \mathbb{R}^n : x \in F_0', x_n = \tau\}$$

and denote by $l(u_0)$ the straight line passing through $u_0 \in F_0$ with direction ω. First, consider the case

$$\emptyset \neq l(u_0) \cap \bar{K} \subset \partial\Omega,$$

that is $l(u_0)$ could meet $\partial\Omega$ only at points, where it is tangent to $\partial\Omega$. Let $\gamma_0(t)$ be a generalized bicharacteristic of $\Box$ in Ω with Im $(\pi \circ \gamma_0) = l(u_0)$. Then γ_0 is uniquely extendible in the sense of Definition 1.2.2. To prove this, assume that $\partial\Omega$ is locally given by $\varphi(x) = 0$ and Ω by $\varphi(x) \geq 0$. If $\hat{x} \in l(u_0) \cap \partial\Omega$, the above-mentioned geometric assumption implies $\varphi_{x_n, x_n}(\hat{x}) \geq 0, \varphi_{x_n}(\hat{x}) = 0$. Now we can apply the argument of the proof of Corollary 1.2.4 based on condition (c) to conclude that γ_0 is uniquely extendible.

Next, we use the sets $C_t(\mu)$ and the metric $D(\rho, \mu)$ introduced in Section 1.2. Set $\mu_u = (\tau, u, 1 - \omega)$. Then we have $C_t(\mu_{u_0}) = \gamma_0(t)$. Applying the argument of the proof of Lemma 1.2.6 for fixed $\epsilon > 0$, we find a small neighbourhood $\mathcal{O}(u_0) \subset F_0$ of u_0 such that for $|t| \leq T + \rho_0 + 1$ and $u \in \mathcal{O}(u_0)$ we have

$$D(C_t(\mu_u), \gamma_0(t)) = \inf_{\nu \in C_t(\mu_u)} D(\nu, \gamma_0(t)) < \epsilon.$$

Now, take φ_j with $\operatorname{supp} \varphi_j \subset \mathcal{O}(u_0)$ and consider the solution W_j of (5.18). The singularities of W_j are contained in the set

$$\{C_t(\mu_u) : u \in F_0 \cap \operatorname{supp} \ \varphi_j\}.$$

By the above inequality with ϵ small enough, we arrange (5.19) and hence (5.20) holds.

Secondly, consider the case when $l(u_0)$ has common points with the interior of K. Choose $x_1(u_0) \in l(u_0)$ such that the linear segment $[u_0, x_1(u_0)]$ is the maximal one that has no common points with the interior of K. There are two possibilities:

(i) $l(u_0)$ meets transversally $\partial\Omega$ at $x_1(u_0)$;

(ii) $l(u_0)$ is tangent to $\partial\Omega$ at $x_1(u_0)$ and ω is an asymptotic direction for $\partial\Omega$ at $x_1(u_0)$.

Notice that in the case (ii) for each neighbourhood V of $x_1(u_0)$, we have $l(u_0) \cap V \cap K^\circ \neq \emptyset$.

Set $t_1(u_0) = |u_0 - x_1(u_0)|$. As in Section 3.1, it is easy to write the solution v_j of (5.17) as an oscillatory integral and to find that $WF(v_j)$ is contained in the set of all points $(t, x, \pm\sigma, \mp\sigma\omega) \in T^*(\mathbb{R}^{n=1}) \setminus \{0\}$ such that $\sigma > 0$ and there exist $\hat{x} \in Z(1), \hat{x}' \in \text{supp}\, \varphi_j, s \geq 0$ with $t = \tau \pm s, x = \hat{x} \pm s\omega$. In the case (i) we modify v_j to $\tilde{v}_j$ on the intersection of the interior of K with a small neighbourhood of $x_1(u_0)$ so that $\tilde{v}_j = v_j$ for $t < T_1 + \epsilon, \tilde{v}_j = 0$ for $t > t_1 + 2\epsilon$. Here $t_1 = \max\{t_1(u) : u \in \mathcal{O}(u_0)\}, \mathcal{O}(u_0)$ and $\epsilon > 0$ are chosen sufficiently small and supp $\varphi_j \subset \mathcal{O}(u_0)$. Thus, we preserve the condition

$$\Box\tilde{v}_j = 0 \text{ in } \mathbb{R}^+_\tau \times \Omega. \tag{5.21}$$

In the case (ii) we repeat the same procedure, modifying v_j in the interior of K so that $\tilde{v}_j = 0$ for $t > t_1 + 2\epsilon$. For this choose $z \in l(u_0) \cap K^\circ$ sufficiently close to $x_1(u_0)$ and modify v_j in a small neighbourhood $W \subset K^\circ$ of z so that (5.21) remains valid.

Set $h_j = v_j|_{\mathbb{R}^+_\tau \times \partial\Omega}$ and notice that $h_j = 0$ for t sufficiently close to τ. Extending h_j as 0 for $t < \tau$, consider the solution w_j of the problem

$$\begin{cases} \Box w_j = 0 \text{ in } \mathbb{R} \times \Omega^\circ, \\ w_j + h_j = 0 \text{ on } \mathbb{R} \times \partial\Omega, \\ w_j|_{t<\tau} = 0. \end{cases} \tag{5.22}$$

Since $\frac{\partial}{\partial t}(w_j + \tilde{v}_j)|_{\mathbb{R}^+_\tau \times \partial\Omega} = 0$, we are going to study the integrals

$$I_{j,\delta}(\lambda) = \int_{\mathbb{R}} \int_{\partial\Omega} e^{\mathbf{i}\lambda(t - \langle x, \theta \rangle)} \rho_\delta(\langle x, \theta \rangle - t + t_0) \left(\frac{\partial}{\partial \nu} - \langle \nu, \theta \rangle \frac{\partial}{\partial t} \right) w_j dt\, dS_x,$$

$$I_{j,\delta}(\lambda) = \int_{\mathbb{R}} \int_{\partial\Omega} e^{\mathbf{i}\lambda(t - \langle x, \theta \rangle)} \rho_\delta(\langle x, \theta \rangle - t + t_0) \left(\frac{\partial}{\partial \nu} - \langle \nu, \theta \rangle \frac{\partial}{\partial t} \right) \tilde{v}_j dt\, dS_x.$$

We will examine these integrals in the next section.

5.3 Poisson relation for the scattering kernel

In this section we use the notation of the previous ones. We begin by the analysis of the integral

$$I(\lambda) = \int_{\mathbb{R}} \int_{\partial\Omega} e^{\mathbf{i}\lambda(t - \langle x, \theta \rangle)} \rho_\delta(\langle x, \theta \rangle - t + t_0) \left(\frac{\partial}{\partial \nu} - \langle \nu, \theta \rangle \frac{\partial}{\partial t} \right) v\; dt\, dS_x.$$

Here $v \in \bar{\mathcal{D}}'(\mathbb{R} \times \Omega))$ is the solution of the problem

$$\begin{cases} \Box v = F \text{ in } \mathbb{R} \times \Omega^\circ, \\ v = h \text{ on } \mathbb{R} \times \partial\Omega, \\ v|_{t<\tau} = 0, \end{cases} \tag{5.23}$$

where $\tau < -\rho_0$ is fixed, $F \in \mathcal{E}'(\mathbb{R} \times \Omega^\circ), h \in H^s_{loc}(\mathbb{R} \times \partial\Omega)$ and $F = 0, h = 0$ for $t < \tau$. By $\bar{\mathcal{D}}'(\mathbb{R} \times \Omega)$ we denote the space of distributions in $\mathbb{R}_t \times \Omega^\circ$ admitting extensions as distributions on $\mathbb{R}_t \times \mathbb{R}^n_x$. Since $\mathbb{R} \times \partial\Omega$ is not characteristic for $\Box$, the traces

$$\left(\frac{\partial^j v}{\partial \nu^j}\right)\Big|_{\mathbb{R}\times\partial\Omega} \in \mathcal{D}'(\mathbb{R} \times \partial\Omega),$$

$j = 0, 1$, exist (see [H3]) and we can interpret $I(\lambda)$ in the sense of distributions. Set $T_1 = \rho_0 + |t_0| + 1$.

Next, we will use the generalized wave front set

$$WF_b(v) \subset T^*(\mathbb{R} \times \Omega^\circ) \cup T^*(\mathbb{R} \times \partial\Omega) = \tilde{T}^*(\mathbb{R} \times \Omega)$$

and the map $\sim$ introduced in Sections 1.3 and 1.4. Recall that for $x \in \partial\Omega$, we have

$$\sim\colon\ T^*(\mathbb{R} \times \Omega) \ni (t, x, \tau, \xi) \to (t, x, \tau, \xi|_{T_x(\partial\Omega)}) \in T^*(\mathbb{R} \times \partial\Omega).$$

To examine $I_{j,\delta}(\lambda)$ and $J_{j,\delta}(\lambda)$, we need the following.

Proposition 5.3.1: *Assume that there exists $\eta > 0$ such that*

$$WF_b(v) \cap \{\mu \in \tilde{T}^*(\mathbb{R} \times \Omega) :\ \mu = (t, \widetilde{x, 1 - \theta}),\ T_1 + \eta \le t \le T_1 + 2\eta\} = \emptyset, \tag{5.24}$$

$$WF(F) \cap \{\mu \in T^*(\mathbb{R} \times \Omega^\circ) :\ \mu = (t, x, 1 - \theta)\} = \emptyset. \tag{5.25}$$

Then $I(\lambda) = \mathcal{O}(|\lambda|^{-m})$ for all $m \in \mathbb{N}$.

Proof: Suppose that $F = 0$ for $|x| \ge R$ and choose two functions $\alpha(t) \in C_0^\infty(\mathbb{R}), \beta(x) \in C_0^\infty(\mathbb{R}^n)$ such that

$$\alpha(t) = \begin{cases} 1 \text{ for } t \le T_1 + \eta, \\ 0 \text{ for } t \ge T_1 + 2\eta, \end{cases}$$

$$\beta(x) = \begin{cases} 1 \text{ for } |x| \le \tau_1 + R + T_1 + 2\eta, \\ 0 \text{ for } |x| \ge \tau_1 + R + T_1 + 3\eta \end{cases}$$

with $\tau_1 = \rho_0 - \tau$. Setting $\tilde{v}(t, x) = \alpha(t)\beta(x)v(t, x)$, we get

$$\begin{cases} \Box\tilde{v} = \tilde{F} \text{ in } \mathbb{R} \times \Omega^\circ, \\ \tilde{v} = \alpha\beta h \text{ on } \mathbb{R} \times \partial\Omega, \\ \tilde{v}|_{t<\tau} = 0 \end{cases}$$

with

$$\tilde{F} = 2\alpha_t\beta v_t + \alpha_{tt}\beta v - 2\alpha\langle\nabla\beta, \nabla v\rangle - \alpha(\Delta\beta)v + \alpha\beta F.$$

A finite speed of propagation argument implies $v \in C^\infty$ for $|t| \leq T_1 + 2\eta, |x| \geq \tau_1 + R + T_1 + 2\eta$. Consequently, the terms involving derivatives of β are smooth and (5.24) and (5.25) yield

$$WF_b(\tilde{F}) \cap \{\mu \in \tilde{T}^*(\mathbb{R} \times \Omega) : \ \mu = \widetilde{(t, x, 1 - \theta)}\} = \emptyset. \tag{5.26}$$

Since v and $\tilde{F}$ have compact supports, we can take the partial Fourier transform with respect to t. Setting

$$V(\lambda, x) = \langle v(t, x), e^{-\mathbf{i}\lambda t}\rangle, \ f(\lambda, t) = \langle \tilde{F}(t, x), \ e^{-\mathbf{i}\lambda t}\rangle,$$
$$g(\lambda, x) = \langle \alpha\beta h(t, x), \ e^{-\mathbf{i}\lambda t}\rangle,$$

we obtain

$$\begin{cases} (\Delta + \lambda^2)V(\lambda, x) = -f(\lambda, x) \text{ in } \Omega^\circ, \\ V = g \text{ on } \partial\Omega, \\ V \text{ is } (\mathbf{i}\lambda)\text{–outgoing}. \end{cases}.$$

The $(\mathbf{i}\lambda)$-outgoing condition follows from $v|_{t<\tau} = 0$ by using the argument from Section 5.1. Thus for $|x| \to \infty$ we have the representation

$$V(\lambda, x) = \int_{\partial\Omega} \left[\frac{\partial V}{\partial \nu}(\lambda, x) G^+_{\mathbf{i}\lambda}(x - y) - V(\lambda, y)\frac{\partial G^+_{\mathbf{i}\lambda}}{\partial \nu}(x - y)\right] dS_y$$
$$- \int_\Omega G^+_{\mathbf{i}\lambda}(x - y) f(\lambda, y) du,$$

$G^+_{\mathbf{i}\lambda}(x)$ being the $(\mathbf{i}\lambda)$-outgoing Green function given by (5.7). Let $x = r\theta, r = |x|$ and multiply the above equality by $e^{\mathbf{i}\lambda r} r^{(n-1)/2}$. As in Section 5.1, letting $r \to \infty$, we deduce

$$\int_{\partial\Omega} e^{\mathbf{i}\lambda\langle x,\theta\rangle} \left[\frac{\partial V}{\partial \nu}(\lambda, y) - \mathbf{i}\lambda\langle \nu, \theta\rangle V(\lambda, y)\right] dS_y = \int_{\mathbb{R}} \int_\Omega e^{-\mathbf{i}\lambda(t - \langle x,\theta\rangle)} \tilde{F}(t, y) dt \, dy, \tag{5.27}$$

the integrals being interpreted in the sense of distributions.

Let $\{U_k\}_{k=1}^N$ be a covering of $\{x \in \Omega : \ |x| \leq \tau_1 + R + T_1 + 3\eta\}$ consisting of small open sets $U_k \subset \mathbb{R}^n$. For $U_k \subset \Omega^\circ$ we apply Theorem 1.3.4 and (5.26) to get

$$\int_{\mathbb{R}} \int U_k e^{-\mathbf{i}\lambda(t - \langle x,\theta\rangle)} \tilde{F}(t, x) dt \, dx = \mathcal{O}(|\lambda|^{-m}), \ \forall m \in \mathbb{N}.$$

In the case $U_k \cap \partial\Omega \neq \emptyset$, let $\partial\Omega$ be locally given by $x_n = \psi(x'), x' = (x_1, \ldots, x_{n-1})$ and let $d\psi(\hat{x}') = 0$. Changing the variables, we may assume

$$\Omega \cap U_k = \{x \in \mathbb{R}^n : \ x' \in U', \ 0 \leq x_n \leq c\}.$$

Since $\mathbb{R} \times \partial\Omega$ is not characteristic for $\Box$, the distribution v depends smoothly on $x_n \in [0, c]$. The integral over $\mathbb{R} \times (\Omega \cap U_k)$ becomes

$$\int_{\mathbb{R}} \int_0^\infty dx_n e^{\mathbf{i}\lambda x_n \theta_n} \int_{U'} e^{-\mathbf{i}\lambda(t - \psi(x')\theta_n - \langle x',\theta'\rangle)} F_k(t, x', x_n) dt \, dx',$$

where $F_k(\cdot, x_n) \in \mathcal{E}'(\mathbb{R}_t \times \mathbb{R}^{n-1}_{x'})$ depends smoothly on the parameter $x_n \in [0, c]$. For U' and c small enough by (5.26), we obtain

$$(t, x', 1 - d\psi(x')\theta_n - \theta') \notin WF(F_k(\cdot, x_n)).$$

Thus, we may apply Theorem 1.3.4 for the integral with respect to t, x'. Consequently, the right-hand side of (5.27) can be estimated by $\mathcal{O}(|\lambda|^{-m})$ for all $m \in \mathbb{N}$. Next, we have

$$\begin{aligned}(2\pi)^{-1} &\int_{\mathbb{R}} d\lambda e^{\mathbf{i}\lambda t} \int_{\partial\Omega} e^{\mathbf{i}\lambda\langle x,\theta\rangle} \left[\frac{\partial V}{\partial \nu}(\lambda, y) - \mathbf{i}\lambda\langle\nu, \theta\rangle V(\lambda, y)\right] dS_y \\ &= \int_{\partial\Omega} \left(\frac{\partial \tilde{v}}{\partial \nu} - \langle \nu, \theta\rangle \frac{\partial \tilde{v}}{\partial t}\right)(t + \langle x, \theta\rangle, y) dS_y \in C_0^\infty(\mathbb{R})\end{aligned}$$

and

$$\begin{aligned}I(\lambda) &= \int_{\mathbb{R}} \left(\int_{\partial\Omega} \left(\frac{\partial \tilde{v}}{\partial \nu} - \langle \nu, \theta\rangle \frac{\partial \tilde{v}}{\partial t}\right)(t + \langle y, \theta\rangle, y) dS_y\right) e^{\mathbf{i}\lambda t} \rho_\delta(-t + t_0) dt \\ &= \mathcal{O}(|\lambda|^{-m}), \ \forall m \in \mathbb{N}.\end{aligned}$$

This proves the proposition. ■

Now consider the integral $J_{j,\delta}(\lambda)$ defined in Section 5.2. Notice that $\tilde{v}_j = v_j$ for $t < \tau$. Choose a function $\alpha_1(t) \in C_0^\infty(\mathbb{R})$ such that

$$\alpha_1(t) = \begin{cases} 0 \text{ for } t \leq \tau - T_1 - 2\epsilon, \\ 1 \text{ for } t \geq \tau - T_1 - \epsilon \end{cases}$$

with $\epsilon > 0$. Then $\theta \neq \omega$ implies

$$WF(\tilde{v}_j) \cap \{(t, x, 1 - \theta) \in T^*(\mathbb{R}^{n+1}) : \ t < \tau\} = \emptyset.$$

Setting $v(t, x) = \alpha_1(t)\tilde{v}_1(t, x)$, we can apply Proposition 5.3.1 with η large enough. The integral $J_{j,\delta}(\lambda)$ does not change when we replace $\tilde{v}_j$ by v, hence

$$J_{j,\delta}(\lambda) = \mathcal{O}(|\lambda|^{-m}), \ \forall m \in \mathbb{N}. \tag{5.28}$$

Now we turn to the analysis of $I_{j,\delta}(\lambda)$. For our analysis we make an additional assumption.

($U_{(\omega,\theta)}$) Each (ω, θ)-ray γ in Ω is the projection of a uniquely extendible generalized bicharacteristic $\tilde{\gamma}$ of $\Box$.

Notice that ($U_{(\omega,\theta)}$) concerns only the (ω, θ)-rays. In particular, there could exist generalized bicharacteristics of $\Box$ that are not uniquely extendible, the projections on Ω of which are not (ω, θ)-rays.

For each $u_0 \in F$ satisfying the conditions (i), (ii) from Section 5.2, choose a sufficiently small neighbourhood $\mathcal{O}(u_0) \subset Z(\tau)$ and take φ_j with $\operatorname{supp} \varphi_j \subset \mathcal{O}(u_0)$. The singularities of the solution w_j of (5.22) are contained in the set of generalized bicharacteristics of $\Box$ issued from $\mathcal{O}(u_0)$ with direction ω. For brevity of the notation, set $C_t(u) = C_t(\mu_u)$ with $\mu_u = (\tau, u, 1, -\omega)$. There are two cases.

Case A. For all $\sigma > T_1$ we have

$$C_\sigma(u_0) \cap \{(\sigma, x, 1 - \theta) \in T^*(\mathbb{R} \times \Omega) : \rho_0 \leq |x| \leq \tau_1 + \sigma + 1\} = \emptyset. \tag{5.29}$$

Then for all $\tau \geq t$ we have

$$C_t(u_0) \cap \{(t, x, 1, -\theta) \in T^*(\mathbb{R} \times \Omega) : \rho_0 \leq |x|\} = \emptyset.$$

Indeed, suppose that for some $\hat{t} \in [\tau, T_1]$, there exists a generalized bicharacteristic $\gamma(t)$ such that $\gamma(t) \in C_t(u_0)$ and $(\hat{t}, \hat{x}, 1, -\theta) \in \gamma(\hat{t})$ with $|\hat{x}| \geq \rho_0$. Then $\gamma(\sigma)$ would have direction θ for all $\sigma \geq \hat{t}$, which is a contradiction with (5.29).

Exploiting Lemma 1.2.6, we can find a small neighbourhood $\mathcal{O}(u_0)$ such that for all $u \in \mathcal{O}(u_0)$ and all $t \in [\tau, T_1]$ we have

$$C_t(u) \cap \{(t, x, 1, -\theta) \in T^*(\mathbb{R} \times \Omega) :\ \rho_0 \leq |x| \leq \rho_0 + 1\} = \emptyset.$$

Now choose a function $\beta \in C_0^\infty(\mathbb{R})$ such that

$$\beta(x) = \begin{cases} 1 \text{ for } |x| \leq \rho_0, \\ 0 \text{ for } |x| \geq \rho_0 + 1. \end{cases}$$

Take φ_j with supp $\varphi_j \subset \mathcal{O}(u_0)$ and consider the solution w_j of (5.22). Let

$$\Box(\beta w_j) = -2\langle \nabla\beta, \nabla w_j\rangle - (\Delta\beta) w_j = F_j.$$

The above analysis implies

$$WF(F_j) \cap \{\mu \in \tilde{T}^*(\mathbb{R} \times \Omega) :\ \mu = (t, \widetilde{x, 1} - \theta),\ t \leq T_1\} = \emptyset. \tag{5.30}$$

Let $u_j(\lambda, x)$ be the solution of the problem

$$\begin{cases} (\Delta + \lambda^2) u_j = 0 \text{ in } \Omega^\circ, \\ u_j + \mathcal{F}_{t\to\lambda}(h_j) = 0 \text{ on } \partial\Omega, \\ u_j \text{ is } (\mathbf{i}\lambda)\text{–outgoing}, \end{cases}$$

$\mathcal{F}_{t\to\lambda}$ being the Fourier transform with respect to t. It is easy to check that $u_j(\lambda, x)$ is a tempered distribution with respect to λ for $|x| \leq R$, hence the Fourier transform

$$\tilde{w}_j(\lambda, x) = \mathcal{F})_{t\to\lambda}(\beta w_j) = \beta u_j$$

exists. Setting $\tilde{F}_j(\lambda, x) = \mathcal{F}_{t\to\lambda}(F_j)$, as in the proof of Proposition 5.3.1, we obtain

$$\int_{\partial\Omega} e^{\mathbf{i}\lambda\langle x,\theta\rangle} \left(\frac{\partial \tilde{w}_j}{\partial\nu}(\lambda, x) - \mathbf{i}\lambda\langle\nu, \theta\rangle \tilde{w}_j(\lambda, x) \right) dS_x$$
$$= \int_\Omega e^{\mathbf{i}\lambda\langle x,\theta\rangle} \tilde{F}_j(\lambda, x) dx.$$

Taking the inverse Fourier transform, we have

$$\int_{\partial\Omega} \left(\frac{\partial \tilde{w}_j}{\partial \nu} - \langle \nu, \theta \rangle \frac{\partial w_j}{\partial t} \right) (t + \langle x, \theta \rangle, x) dS_x = \int_{\Omega} F_j(t + \langle x, \theta \rangle, x) dx,$$

and, applying (5.30), we deduce

$$\begin{aligned} I_{j,\delta}(\lambda) &= \int_{\mathbb{R}} \int_{\Omega} e^{\mathbf{i}\lambda t} \rho_\delta(-t + t_0) F_j(t + \langle x, \theta \rangle, x) dt\, dx \\ &= \int_{\mathbb{R}} \int_{\Omega} e^{\mathbf{i}\lambda(t - \langle x, \theta \rangle)} \rho_\delta(\langle x, \theta \rangle - t + t_0) F_j(t, x) dt\, dx \\ &= \mathcal{O}(|\lambda|^{-m}), \forall m \in \mathbb{N}. \end{aligned} \tag{5.31}$$

Case B. There exists $\sigma > T_1$ with

$$C_\sigma(u_0) \cap \{(\sigma, x, 1, -\theta) \in T^*(\mathbb{R} \times \Omega) : \ \rho_0 \le |x| \le \tau_1 + \sigma + 1\} \neq \emptyset.$$

Then there exists a generalized bicharacteristic γ of $\Box$ issued from μ_{u_0} and passing for $t = \sigma$ over some point $y, |y| \ge \rho_0$ with direction θ. This means that $\pi \circ \gamma$ is an (ω, θ)-ray. The assumption $(U_{(\omega,\theta)})$ implies that γ is uniquely extendible, hence $C_t(u_0) = \gamma(t)$. Let T_γ be the sojourn time of γ and let

$$\operatorname{Im} \gamma = \{(t, x(t), 1, -\xi(t)) \in T^*(\mathbb{R} \times \Omega) : \ |\xi(t)| = 1, t \ge \tau\},$$

where for $\gamma(t) \in H$ instead of $\xi(t)$, we determine $\xi(t + 0)$ and $\xi(t - 0)$. Recall that H is the set of hyperbolic points introduced in Section 1.2.

Introduce the numbers

$$\begin{aligned} T_2 &= \inf \{\sigma : \ \sigma \ge \tau, \ \xi(t) = \theta \text{ for } t > \sigma\}, \\ T_3 &= \inf \{\sigma : \sigma \ge \tau, \ x(t) \notin \partial\Omega \text{ for } t > \sigma\}. \end{aligned}$$

Clearly, $T_2 \le T_3$. It is easy to see that

$$t - \langle x(t), \theta \rangle = T_\gamma \text{ for } T_2 \le t \le T_3.$$

For $t = T_3$ this follows from the definition of T_γ and the parameterization of $\gamma(t)$ by t. For $T_2 \le t \le T_3$ we use the equality $\langle x(T_3) - x(t), \theta \rangle = T_3 - t$. Taking into account (5.16), we obtain

$$|\langle x(t), \theta \rangle - t + t_0| \ge \epsilon_0, \ \ T_2 \le t \le T_3.$$

Now choose $\mathcal{O}(u_0)$ small enough and assume $\operatorname{supp} \varphi_j \subset \mathcal{O}(u_0)$. Then for $\tau \le t \le T_3 + 1$, the singularities of w_j will be contained in a small neighbourhood of $\gamma(t)$. This makes it possible to choose $s < T_2$ sufficiently close to T_2 and to arrange

$$\gamma(s) \notin H, \xi(s) \neq \theta, \tag{5.32}$$

$$|\langle x, \theta \rangle - t + t_0| \geq \frac{\epsilon_0}{2} \tag{5.33}$$

for $t \geq s$ and

$$(t, x) \in \text{sing supp } (\omega_j|_{\mathbb{R}\times\partial\Omega}) \cup \text{sing supp } \left(\left(\frac{\partial w_j}{\partial \nu} \right) \Big|_{\mathbb{R}\times\partial\Omega} \right).$$

It is necessary to satisfy (5.33) for $s \leq t \leq T_3 + \epsilon_1$, where $\epsilon_1 > 0$ is chosen so that the singularities of w_j for $t \geq T_3 + \epsilon_1$ lie in the interior of Ω. Moreover, if $\gamma(t)$ is a glancing point, (5.32) yields

$$\xi(s) \neq \theta|_{T_{x(s)}(\partial\Omega)} \tag{5.34}$$

because $|\xi(s)| = |\theta| = 1$. For $\mathcal{O}(u_0)$ small enough (5.32) and (5.34) imply

$$WF_b(w_j) \cap \{\mu \in \tilde{T}^*(\mathbb{R} \times \Omega) : \ \mu = (s, \widetilde{x, 1 - \theta})\} = \emptyset.$$

Since $WF_b(w_j)$ is closed, for small $\epsilon > 0$ we have

$$WF_b(w_j) \cap \{\mu \in \tilde{T}^*(\mathbb{R} \times \Omega) : \ \mu = (t, \widetilde{x, 1 - \theta}), \ s \leq t \leq s + \epsilon\} = \emptyset. \tag{5.35}$$

Choose a function $\alpha_2(t) \in C_0^\infty(\mathbb{R})$ such that

$$\alpha_2(t) = \begin{cases} 1 \text{ for } t \leq s, \\ 0 \text{ for } t \geq s + \epsilon. \end{cases}$$

Write

$$I_{j,\delta}(\lambda) = I'_{j,\delta}(\lambda) + I''_{j,\delta}(\lambda),$$

where $I'_{j,\delta}(\lambda)$ (resp. $I''_{j,\delta}(\lambda)$) is obtained from $I_{j,\delta}(\lambda)$ replacing w_j by $\alpha_2 w_j$ (resp. by $(1 - \alpha_2)(w_j)$). We may assume that the inequality (5.33) holds for (t, x) in some neighbourhood of

$$\text{sing supp } \left(\left(\frac{\partial}{\partial \nu} - \langle \nu, \theta \rangle \frac{\partial}{\partial t} \right) (1 - \alpha_2) \omega_j|_{\mathbb{R}\times\partial\Omega} \right).$$

Then $\delta \leq \epsilon_0/2$ implies $\rho_\delta(\langle x, \theta \rangle - t + t_0) = 0$ for such (t, x) and

$$I''_{j,\delta}(\lambda) = \mathcal{O}(|\lambda|^{-m}), \ \forall m \in \mathbb{N}.$$

To examine $I'_{j,\delta}(\lambda)$, set $F_j = \Box(\alpha_2 w_j)$. It follows from (5.35) that

$$WF_b(F_j) \cap \{\mu \in \tilde{T}^*(\mathbb{R} \times \Omega) : \ \mu = (t, \widetilde{x, 1, -\theta})\} = \emptyset.$$

Then for $I'_{j,\delta}(\lambda)$, we can apply the argument from the proof of Proposition 5.3.1. Therefore, (5.31) holds for $I'_{j,\delta}(\lambda)$ in the case B.

In both cases A and B we have chosen a neighbourhood $\mathcal{O}(u_0)$ of each $u_0 \in F_0$. There exists a finite set $\{u_0^{(j)} : 1 \leq j \leq M\} \subset F_0$ such that

$$F_0 \subset \cup_{j=1}^M \mathcal{O}(u_0^{(j)}).$$

Suppose that the points $u_0^{(j)}, j \leq N, N \leq M$, satisfy condition (i) or (ii) of Section 5.2. Choose a partition of unity $\{\varphi_j(x')\}_{j=1}^{\infty}$ so that $\operatorname{supp} \varphi_j \subset \mathcal{O}(u_0^{(j)})$, $j = 1, \dots, N, \operatorname{supp} \varphi_j \cap F_0' = \emptyset$ for $j > M$. Set

$$\tilde{w} = \sum_{j=1}^{N} (w_j + \tilde{v}_j) + \sum_{j>N} W_j,$$

where $w_j, W_j, \tilde{v}_j$ are introduced in the previous section. Clearly

$$\begin{cases} \Box \tilde{w} = 0 \text{ in } \mathbb{R}_\tau^+ \times \Omega^\circ, \\ \tilde{w} = 0 \text{ on } \mathbb{R}_\tau^+ \times \partial\Omega, \\ \tilde{w}|_{t=\tau} = \delta(\tau - x_n), \ \dfrac{\partial \tilde{w}}{\partial t}|_{t=\tau} = \delta'(\tau - x_n). \end{cases}$$

This implies $w = \tilde{w}$ in $\mathbb{R}_\tau^+ \times \Omega$. Choosing $\tau < -T_1$, we can replace w by $\tilde{w}$ in $J(\lambda)$. Finally, (5.20), (5.28) and (5.31) show that $J(\lambda)$ is rapidly decreasing, and we conclude that $-t_0 \notin \operatorname{sing\,supp} s(t, \theta, \omega)$.

Thus, we have proved the following.

Theorem 5.3.2: *Let $\theta \neq \omega$ be fixed and let the condition $(U_{(\omega,\theta)})$ be fulfilled. Then*

$$\operatorname{sing\,supp} s(t, \theta, \omega) \subset \{ -T_\gamma : \gamma \in \mathcal{L}_{(\omega,\theta)}(\Omega)\}. \tag{5.36}$$

The inclusion (5.36) is called the Poisson relation for the scattering kernel by analogy with the relation (3.29) in Chapter 3.

The condition $(U_{(\omega,\theta)})$ has been used only for the analysis of $C_t(u)$ in the case B. If $C_t(u)$ contains many generalized bicharacteristics, scattering with different directions, a localization of $C_t(u)$ might be done to eliminate the contributions related to the rays having outgoing directions $\eta \neq \theta$.

5.4 Notes

The representation (5.4) of $s(t, \theta, \omega)$ was obtained in [Ma2] (see also [LP1] and Chapter 8 in [P5] for related results). The outgoing solutions of the reduced wave equation and the outgoing Green function are examined in more detail in [LP1]. The Poisson relation (5.36) for non-convex domains K has been studied in [Pl] under some geometric restrictions concerning the rays incoming with directions $\pm\omega$. For several convex obstacles, (5.36) has been proved in [PS5] (see also [Nal], [Na2] and [NS] for partial results). Theorem 5.3.2 was obtained in [CPS] (see also [Me4] for similar result). For the description of the uniquely extendible bicharacteristics we may apply Corollary 1.2.4. For generic domains Ω the condition $(U_{(\omega,\theta)})$ is satisfied for all $\omega, \theta \in \mathbb{S}^{n-1}$. Moreover, according to the results in Chapter 11, $(U_{(\omega,\theta)})$ is satisfied for all $(\omega, \theta) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \setminus \mathcal{R}$, where the residual set $\mathcal{R}$ has measure zero. For the scattering theory for the wave equation the reader may find more references in [LP1], [LP2], [P5], [Me4], [Me2], [PP].

6

Generic properties of reflecting rays

In this chapter we establish several properties of periodic reflecting rays in bounded domains and scattering rays in domains with bounded complements. These properties will be used in Chapters 7 and 11 to investigate certain inverse spectral problems for generic domains. In Section 6.1 we prove a general theorem that provides existence of residual sets of smooth embeddings of a given submanifold of $\mathbb{R}^n$ satisfying certain conditions. As a consequence of it, some elementary generic properties are derived for the two kinds of reflecting rays under consideration. In particular, we show that Herman Weyl's conjecture is true for generic bounded domains. The main result in Section 6.1, as well as the scheme of its proof, will be used several times in this book. Following this scheme, we prove in the present chapter that for generic domains the reflecting rays under consideration are ordinary and non-degenerate.

6.1 Generic properties of smooth embeddings

Let X be a smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$, $n \geq 2$. In this section we prove a general theorem establishing the existence of a residual set of smooth embeddings f of X into $\mathbb{R}^n$ satisfying some particular properties. This theorem will be applied several times in the present chapter.

Theorem 6.1.1: *Let $n \geq 2$, $s \geq 2$, p and q be natural numbers, let U be an open subset of $(\mathbb{R}^n)^{(s)}$ and let X be a smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$. Let*

$$H = (H_1, \ldots, H_p) : U \longrightarrow \mathbb{R}^p$$

Geometry of the Generalized Geodesic Flow and Inverse Spectral Problems, Second Edition.
Vesselin M. Petkov and Luchezar N. Stoyanov.

be a smooth map such that for every $y \in U$ and every $i = 1, \dots, s$ there exists an integer $r_i = 1, \dots, p$ with

$$\mathrm{grad}_{y_i} H_{r_i}(y) \neq 0, \tag{6.1}$$

where $y = (y_1, \dots, y_s)$.

(A) *Let $p = 1$ and let T_1 be the set of those $f \in \mathbf{C}(X)$ for which the critical points x of $H \circ f^s$ with $f^s(x) \in U$ form a discrete subset of $X^{(s)}$. Then T_1 contains a residual subset of $\mathbf{C}(X)$.*

(B) *Let p be arbitrary and let*

$$L = (L_1, \dots, L_q) : U \longrightarrow \mathbb{R}^q$$

be a smooth map such that $dL(y) \neq 0$ for any $y \in U$ with $L(y) = 0$. Let T_2 be the set of all $f \in \mathbf{C}(X)$ such that if x is a critical point of $H \circ f^s$ with $f^s(x) \in U$, then $L(f^s(x)) \neq 0$. Then T_2 contains a residual subset of $\mathbf{C}(X)$.

Here we use the notation

$$\mathrm{grad}_{y_i} H_{r_i}(y) = \left(\frac{\partial H_{r_i}}{\partial y_i^{(j)}}(y) \right)_{j=1}^{n} \in \mathbb{R}^n.$$

For the proof of the theorem we need some preparation.

First, notice that it is enough to consider the case $q = 1$ in part (B). Indeed, assume that the assertion in (B) is true for $q = 1$. Let $q > 1$. For $m = 1, \dots, q$ set

$$U_m = \{y \in U : dL_m(y) \neq 0\}.$$

Then $U_1, \dots, U_q$ are open subsets of U and $U = \cup_{m=1}^{q} U_m$. Moreover, we have

$$\cap_{m=1}^{q} T_2^{(m)} \subset T_2,$$

where $T_2^{(m)}$ is the set of all $f \in \mathbf{C}(X)$ such that if x is a critical point of $H \circ f^s$ with $f^s(x) \in U_m$, then $L_m(f^s(x)) \neq 0$. It follows from our assumption that $T_2^{(m)}$ contains a residual subset of $\mathbf{C}(X)$ for every $m = 1, \dots, q$. Hence T_2 has the same property.

Thus, assume $q = 1$. We may assume in addition that for every $i = 1, \dots, s$ there exists $r_i = 1, \dots, p$ such that (6.1) holds whenever $y \in U$. Indeed, for any $\mathbf{r} = (r_1, \dots, r_s)$, let $U_{\mathbf{r}}$ be the set of those $y \in U$ such that (6.1) holds for $i = 1, \dots, s$. Then $U = \cup U_{\mathbf{r}}$, where $\mathbf{r}$ runs over all s-tuples of the considered type (there are only finitely many of them), and therefore $\cap_{\mathbf{r}} T_2^{(\mathbf{r})} \subset T_2$, where $T_2^{(\mathbf{r})}$ is the set of those $f \in \mathbf{C}(X)$ such that if x is a critical point of $H \circ f^s$ with $f^s(x) \in U_{\mathbf{r}}$, then $L(f^s(x)) \neq 0$. Thus, if every $T_2^{(\mathbf{r})}$ contains a residual subset of $\mathbf{C}(X)$, then T_2 has the same property.

From now on we assume that $q = 1$ and for every $i = 1, \dots, s$ there exists $r_i = 1, \dots, p$ such that (6.1) holds whenever $y \in U$. We will consider (A) and (B) simultaneously.

Let $J^1_s(X, \mathbb{R}^n)$ be the s-fold bundle of 1-jets, and let α and β be the corresponding source and target maps (cf. Section 1.1). Set

$$M = (\alpha^s)^{-1}(X^{(s)}) \cap (\beta^s)^{-1}(U). \tag{6.2}$$

Clearly, M is an open subset (and therefore a submanifold) of $J^1_s(X, \mathbb{R}^n)$. Let Σ_1 be the set of those

$$\sigma = (j^1 f_1(x_1), \dots, j^1 f_s(x_s)) \in M \tag{6.3}$$

such that $x = (x_1, \dots, x_s)$ is a critical point of the map $H \circ (f_1 \times \dots \times f_s)$. For a given $f \in \mathbf{C}(X)$ set

$$A_f = \{x \in X^{(s)} : j^1_s f(x) \in \Sigma_1\}.$$

It then follows that

$$T_1 = \{f \in \mathbf{C}(X) : A_f \text{ is a discrete subset of } X^{(s)}\}. \tag{6.4}$$

To describe T_2 in a similar way, consider the set Σ_2 of those $\sigma \in \Sigma_1$ of the form (6.3) such that $L \circ (f_1 \times \dots \times f_s) = 0$. Then

$$T_2 = \{f \in \mathbf{C}(X) : j^1_s f(X^{(s)}) \cap \Sigma_2 = \emptyset\}. \tag{6.5}$$

The central point in the proof of Theorem 6.1.1 is in obtaining relevant information about the sets Σ_1 and Σ_2. Our aim is to show that each of these sets can be covered by a countable family of smooth submanifolds of M of sufficiently large codimension.

Lemma 6.1.2: *When $p = 1$, Σ_1 is a smooth submanifold of M with*

$$\text{codim } (\Sigma_1) = (n-1)s. \tag{6.6}$$

For every $p \in \mathbb{N}$, there exists a finite or countable family $\{W_m\}$ of smooth submanifolds of M with

$$\text{codim } (W_m) = (n-1)s + 1 \tag{6.7}$$

for all m such that

$$\Sigma_2 \subset \cup_m W_m. \tag{6.8}$$

Proof: Consider an arbitrary $\sigma_0 \in M$. We will construct a chart on M defined on a neighbourhood of σ_0. There exist coordinate neighbourhoods $V_1, \dots, V_s$ of elements of X such that $V_i \cap V_j = \emptyset$ for $i \neq j$ and

$$\sigma_0 \in \prod_{i=1}^{s} J^1(V_i, \mathbb{R}^n) \subset J^1_s(X, \mathbb{R}^n).$$

Set

$$D = M \cap \left(\prod_{i=1}^{s} J^1(V_i, \mathbb{R}^n) \right). \tag{6.9}$$

Clearly, D is an open neighbourhood of σ_0 in M (see Section 1.1). Consider arbitrary charts $\varphi_i : V_i \longrightarrow \mathbb{R}^{n-1}$ and define the chart

$$\varphi : D \longrightarrow (\mathbb{R}^{n-1})^{(s)} \times (\mathbb{R}^n)^{(s)} \times \mathbb{R}^{n(n-1)s} \tag{6.10}$$

by

$$\varphi(\sigma) = (u; v; a), \tag{6.11}$$

where σ has the form (6.3) and

$$u = (u_1, \dots, u_s), \quad v = (v_1, \dots, v_s), \quad a = (a_{ij}^{(t)})_{1 \le i \le s, 1 \le j \le n-1, 1 \le t \le n}, \tag{6.12}$$

$$u_i = \varphi_i(x_i), \quad v_i = f_i(x_i), \tag{6.13}$$

$$a_{ij}^{(t)} = \frac{\partial \left(f_i^{(t)} \circ \varphi_i^{-1} \right)}{\partial u_i^{(j)}} (u_i) \tag{6.14}$$

for $i = 1, \dots, s$, $j = 1, \dots, n-1$ and $t = 1, \dots, n$. Here we use the notation

$$f_i = (f_i^{(1)}, \dots, f_i^{(n)}), \;\; u_i = (u_i^{(1)}, \dots, u_i^{(n-1)}) \in \mathbb{R}^{n-1},$$
$$v_i = (v_i^{(1)}, \dots, v_i^{(n)}) \in \mathbb{R}^n.$$

Let us mention that if $F : U \longrightarrow \mathbb{R}$ is a smooth function, then

$$\frac{\partial \left(F \circ \left((f_1 \circ \varphi_1^{-1}) \times \cdots \times (f_s \circ \varphi_s^{-1}) \right) \right)}{\partial u_i^{(j)}} (u) = \sum_{t=1}^{n} \frac{\partial F}{\partial y_i^{(t)}} (v) a_{ij}^{(t)}, \tag{6.15}$$

where $y_i = (y_i^{(1)}, \dots, y_i^{(n)})$.

Since each of the sets Σ_1 and Σ_2 can be covered by a countable number of coordinate neighbourhoods, it is sufficient to prove that if D is an arbitrary coordinate neighbourhood of the form described earlier, then for $p = 1$, $\varphi(D \cap \Sigma_1)$ is a smooth submanifold of $\varphi(D)$ of codimension $(n-1)s$, while for arbitrary $p \in \mathbb{N}$, $\varphi(D \cap \Sigma_2)$ is contained in a smooth submanifold of $\varphi(D)$ of codimension $(n-1)s + 1$.

We will write the elements ξ of $\varphi(D)$ of the form

$$\xi = (u; v; a) \in (\mathbb{R}^{n-1})^{(s)} \times (\mathbb{R}^n)^{(s)} \times \mathbb{R}^{n(n-1)s},$$

where u, v and a are determined by (6.12)–(6.14). It follows from our assumptions that for every $i = 1, \dots, s$ there exists $r_i = 1, \dots, p$ such that (6.1) holds for all $y \in U$. For $i = 1, \dots, s$ and $j = 1, \dots, n-1$ set

$$b_{ij}(\xi) = \sum_{i=1}^{n} \frac{\partial H_{r_i}}{\partial y_i^{t)}} (v) a_{ij}^{(t)}. \tag{6.16}$$

Define the maps

$$R_1 : \varphi(D) \longrightarrow \mathbb{R}^{(n-1)s}, \quad R_2 : \varphi(D) \longrightarrow \mathbb{R}^{(n-1)s} \times \mathbb{R}$$

by

$$R_1(\xi) = (b_{ij}(\xi))_{1\le i\le s, 1\le j\le n-1}, \quad R_2(\xi) = (R_1(\xi), \tilde{L}(\xi)),$$

where $\tilde{L}(\xi) = L(v)$ by definition.

Note that for $p = 1$ we have $r_i = 1$ for any i, so

$$\varphi(D \cap \Sigma_1) = R_1^{-1}(0). \tag{6.17}$$

For an arbitrary $p \in \mathbb{N}$ the definitions of Σ_2 and R_2 yield

$$\varphi(D \cap \Sigma_2) \subset R_2^{-1}(0). \tag{6.18}$$

We will now show that R_2 is a submersion on $R_2^{-1}(0)$. Let $\xi = (u; v; a) \in R_2^{-1}(0)$. Assume that

$$\sum_{i=1}^{s}\sum_{j=1}^{n-1} B_{ij} \operatorname{grad} b_{ij}(\xi) + C \operatorname{grad} \tilde{L}(\xi) = 0 \tag{6.19}$$

for some constants C and B_{ij} $(1 \le i \le s,\ 1 \le j \le n-1)$. Here we consider $\operatorname{grad} b_{ij}(\xi)$ and $\operatorname{grad} \tilde{L}(\xi)$ as vectors in

$$(\mathbb{R}^{n-1})^{(s)} \times (\mathbb{R}^{n})^{(s)} \times \mathbb{R}^{n(n-1)s}.$$

It follows from (6.11)–(6.14), $D \subset M$ and $\xi \in \varphi(D)$ that $v \in U$. Fix arbitrary $i = 1, \dots, s$ and $j = 1, \dots, n-1$. There exists $t = 1, \dots, n$ such that

$$\frac{\partial H_{r_i}}{\partial y_i^{(t)}}(v) \ne 0.$$

Then, according to (6.16), we get

$$\frac{\partial b_{ij}}{\partial a_{ij}^{(t)}}(\xi) = \frac{\partial H_{r_i}}{\partial y_i^{(t)}}(v) \ne 0, \quad \frac{\partial b_{i'j'}}{\partial a_{ij}^{(t)}}(\xi) = 0 \text{ for } i' \ne i, j' \ne j.$$

Moreover, $\frac{\partial \tilde{L}}{\partial a_{ij}^{(t)}}(\xi) = 0$. Considering the derivatives with respect to $a_{ij}^{(t)}$ in (6.19), we deduce

$$B_{ij} \frac{\partial H_{r_i}}{\partial y_i^{(t)}}(v) = 0,$$

hence $B_{ij} = 0$. The latter holds for all $i = 1, \dots, s$ and $j = 1, \dots, n-1$. Now (6.19) becomes $C \operatorname{grad} \tilde{L}(\xi) = 0$. Since $\xi \in R_2^{-1}(0)$, we have $R_2(\xi) = 0$, so in particular

$L(v) = \tilde{L}(\xi) = 0$. The assumption on L and $v \in U$ now implies $dL(v) \neq 0$. Thus, $C = 0$, and therefore, R_2 is a submersion at ξ.

Hence R_2 is a submersion on $R_2^{-1}(0)$, and therefore (see Section 1.1) $R_2^{-1}(0)$ is a smooth submanifold of $\varphi(D)$ with

$$\text{codim } (R_2^{-1}(0)) = \dim(\mathbb{R}^{(n-1)s} \times \mathbb{R}) = (n-1)s + 1.$$

Applying the above argument in the case $p = 1$, we derive that R_1 is a submersion on $R_1^{-1}(0)$. Therefore, $R_1^{-1}(0)$ is a submanifold of $\varphi(D)$ with

$$\text{codim } (R_1^{-1}(0)) = \dim(\mathbb{R}^{n-1)s}) = (n-1)s.$$

Now the assertion follows from (6.17) and (6.18). ■

Proof of Theorem 6.1.1: (A) Let $p = 1$. According to the above lemma, Σ_1 is a smooth submanifold of M with (6.6). Since M is open in $J_s^1(X, \mathbb{R}^n)$, Σ_1 is a smooth submanifold of $J_s^1(X, \mathbb{R}^n)$ of the same codimension. Applying the Multijet Transversality Theorem (see Section 1.1), we derive that the set

$$T_1' = \{f \in C^\infty(X, \mathbb{R}^n) : j_s^1 f \pitchfork \Sigma_1\}$$

is residual in $C^\infty(X, \mathbb{R}^n)$. Since $\mathbf{C}(X)$ is open in $C^\infty(X, \mathbb{R}^n)$, the set $T_1'' = T_1' \cap \mathbf{C}(X)$ is residual in $\mathbf{C}(X)$.

Now to prove (A) it is enough to establish $T_1'' \subset T_1$. Let $f \in T_1''$; then $j_s^1 f \pitchfork \Sigma_1$. Since

$$j_s^1 f : X^{(s)} \longrightarrow J_s^1(X, \mathbb{R}^n)$$

and

$$\dim(X^{(s)}) = (n-1)s = \text{codim } (\Sigma_1),$$

it follows that A_f is a discrete subset of Σ_1, that is $f \in T_1$. Thus, $T_1'' \subset T_1$, which concludes the proof of part (A).

(B) It follows from Lemma 6.1.2 that there exists a finite or countable family $\{W_m\}$ of smooth submanifolds of M (and therefore of $J_s^1(X, \mathbb{R}^n)$) satisfying (6.7) and (6.8). By the Multijet Transversality Theorem, for any m the set

$$S_m = \{f \in \mathbf{C}(X) : j_s^1 f \pitchfork W_m\}$$

is residual in $\mathbf{C}(X)$. It remains to check that

$$\cap_m S_m \subset T_2. \tag{6.20}$$

Let $f \in \cap_m S_m$. Then for every m we have $f \in S_m$, so $j_s^1 f \pitchfork W_m$. By (6.16), we have

$$\dim(X^{(s)}) = (n-1)s < \text{codim } W_m,$$

so $j_s^1 f(X^{(s)}) \cap W_m = \emptyset$. This holds for all m, so by (6.8), $j_s^1(X^{(s)}) \cap \Sigma_2 = \emptyset$. This and (6.5) imply $f \in T_2$ which proves (6.20). Hence, T_2 contains a residual subset of $\mathbf{C}(X)$. ■

6.2 Elementary generic properties of reflecting rays

In this section we apply Theorem 6.1.1 to obtain some properties of generic domains Ω in $\mathbb{R}^n$ concerning the behaviour of periodic reflecting rays and scattering rays in Ω. In particular, we establish that for generic Ω the following properties are satisfied:

(a) The lengths of any two distinct primitive periodic reflecting rays are independent over the rationals;

(b) For every $x \in \partial\Omega$ there exists at most one direction $\xi \in \mathbb{S}^{n-1}$ (up to the symmetry with respect to the normal to $\partial\Omega$ at x) such that (x, ξ) generates a periodic reflecting ray in Ω.

Similar properties are considered for scattering rays.

Clearly in the general case neither (a) nor (b) are satisfied. A simple example is a disk Ω in the plane. Let us mention that property (b) is equivalent to the following: any two distinct primitive periodic reflecting rays in Ω have no common reflection point, and any primitive periodic reflecting ray passes only once through each of its reflection points. More precisely, the latter is true only for non-symmetric rays (see Section 2.1); for symmetric rays this property can be formulated in a similar way.

We begin with a simple combinatorial classification of the periodic reflecting rays.

Let $k \geq s \geq 2$ be natural numbers, and let

$$\alpha : \{1, \ldots, k\} \longrightarrow \{1, \ldots, s\} \tag{6.21}$$

be a map with

$$\alpha(i) \neq \alpha(i+1) \tag{6.22}$$

for all $i = 1, \ldots, k$. Here for convenience we set $\alpha(m) = \alpha(i)$ for $m = i + pk$, $1 \leq i \leq k$, p being an integer. If

$$\{\alpha(i), \alpha(i+1)\} \neq \{\alpha(j), \alpha(j+1)\} \tag{6.23}$$

holds for all $1 \leq i < j \leq k$, we will say that α is a *non-symmetric map*. If $k = 2m$ and there is $i_0 = 1, \ldots, k$ such that (6.22) holds for $i_0 \leq i < j \leq i_0 + m$ and

$$\alpha(i_0 + m + j) = \alpha(i_0 + m - j), \quad j = 1, \ldots, m-1, \tag{6.24}$$

then we will say that α is a *symmetric map*. By an *admissible map* we mean a map (6.21) that is either a non-symmetric or a symmetric map.

Next, assume that an admissible map of the form (6.21) is fixed. Consider the sets

$$I_i = I_i(\alpha) = \{j : \exists t = 1, \ldots, k \text{ with } \{i, j\} = \{\alpha(t), \alpha(t+1)\}\} \tag{6.25}$$

for $i = 1, \ldots, s$. Denote by U_α the set of all $y = (y_1, \ldots, y_s) \in (\mathbb{R}^n)^{(s)}$ such that

$$y_i \notin \text{conv}\{y_j : j \in I_i\}, \quad i = 1, \ldots, s,$$

and for any $i = 1, \ldots, s$, if q, j, r and t are distinct elements of I_i, then at least one of the triples y_i, y_q, y_j and y_i, y_r, y_t consists of non-collinear points. It is easily seen that U_α is open in $(\mathbb{R}^n)^{(s)}$ and the function

$$F = F_\alpha : U_\alpha \longrightarrow \mathbb{R}, \tag{6.26}$$

given by

$$F(y) = \sum_{i=1}^{k} \| y_{\alpha(i)} - y_{\alpha(i+1)} \| , \tag{6.27}$$

is smooth.

Notice that if γ is a periodic reflecting ray with reflection points $y_1, \ldots, y_s$ (their ordering does not matter in this case), there exist $k \geq s$ and an admissible map (6.21) such that

$$y_{\alpha(1)}, \ldots, y_{\alpha(k)} \tag{6.28}$$

are the successive reflection points of γ. In such a case we will say that γ *is of type* α. Then we have $y = (y_1, \ldots, y_s) \in U_\alpha$ and $F(y)$ is equal to the length T_γ of γ. Moreover, for any $i = 1, \ldots, s$ the relation $j \in I_i$ is equivalent to the fact that there exists a segment of γ connecting y_i and y_j. Clearly, the type α of a periodic reflecting ray γ is not uniquely determined.

In what follows X will be an arbitrary fixed smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$.

The following proposition is a consequence of Proposition 2.1.3.

Proposition 6.2.1: *Let α be an admissible map of the form (6.21) and let $x_1, \ldots, x_s$ be all distinct reflection points of a periodic reflecting ray γ of type α for X, that is $x_{\alpha(1)}, \ldots, x_{\alpha(s)}$ are the successive reflection points of γ. Then $x = (x_1, \ldots, x_s)$ is a critical point of the map $(F_\alpha)_{|X^{(s)}}$.*

Proof: Define the maps

$$g : U_\alpha \longrightarrow (\mathbb{R}^n)^k, \quad G : (\mathbb{R}^n)^k \longrightarrow \mathbb{R}$$

by

$$g(y) = (y_{\alpha(1)}, \ldots, y_{\alpha(s)}),$$

and

$$G(z_1, \ldots, z_k) = \sum_{i=1}^{k} \| z_i - z_{i+1} \| .$$

Then $F_\alpha = G \circ g$. Moreover, $g(X^{(s)}) \subset X^k$. It follows from our assumptions and Proposition 2.1.3 that $g(x)$ is a critical point of $G_{|X^k}$. Therefore, x is a critical point of $(F_\alpha)_{|X^{(s)}}$. ■

In order to apply the above proposition we need another property of the map $F = F_\alpha$.

Lemma 6.2.2: *Let α be an admissible map of the form (6.21). For every $i = 1, \ldots, s$ and every $y \in U_\alpha$ there exists $j = 1, \ldots, n$ such that*

$$\frac{\partial F}{\partial y_i^{(j)}}(y) \neq 0. \tag{6.29}$$

Proof: Fix arbitrary $i = 1, \ldots, s$ and $y \in U_\alpha$. For any $j = 1, \ldots, n$ we have

$$\frac{\partial F}{\partial y_i^{(j)}}(y) = a \sum_{t \in I_i} \frac{y_i^{(j)} - y_t^{(j)}}{\| y_i - y_t \|}, \tag{6.30}$$

where $a = 1$ for a non-symmetric map α and $a = 2$ if α is symmetric.

Assume that the derivative (6.30) is 0 for all j. Then

$$\sum_{t \in I_i} \frac{y_i^{(j)} - y_t^{(j)}}{\| y_i - y_t \|} = 0,$$

which can be re-written as

$$y_i = \sum_{t \in I_i} a_t y_t,$$

where

$$a_t = \frac{1}{\| y_i - y_t \| \left(\sum_{j \in I_i} 1 / \| y_i - y_j \| \right)}.$$

Since $\sum_{t \in I_i} a_t = 1$, we get

$$y_i \in \operatorname{conv}\{y_t : t \in I_i\},$$

which is a contradiction with $y \in U_\alpha$. ■

Denote by $\mathcal{R}$ the set of those $f \in \mathbf{C}(X)$ such that every two primitive periodic reflecting rays for $f(X)$ have rationally independent lengths. Consider also the set $\mathcal{A}$ of those $f \in \mathbf{C}(X)$ such that for every $y \in f(X)$ there exists at most one direction $\xi \in \mathbb{S}^{n-1}$ (up to symmetry with respect to the normal to $f(X)$ at y) such that (y, ξ) generates a periodic reflecting ray for $f(X)$.

Theorem 6.2.3: *Each of the sets $\mathcal{R}$ and $\mathcal{A}$ contains a residual subset of* $\mathbf{C}(X)$.

Proof: Fix an arbitrary surjective non-symmetric map (6.21) and suppose that $k > s$. Without loss of generality we will assume that

$$\alpha(1) = 1, \quad |\alpha^{-1}(1)| > 1.$$

Denote by $\mathcal{A}_\alpha$ the set of those $f \in \mathbf{C}(X)$ such that there is no periodic reflecting ray of type α for $f(X)$. We will show that $\mathcal{A}_\alpha$ contains a residual subset of $\mathbf{C}(X)$. To do this we are going to apply Theorem 6.1.1(B) with $U = U_\alpha$, $p = 1$ and $H = F_\alpha$.

Choose arbitrary distinct elements j_1, j_2 of $\alpha^{-1}(1)$. Then

$$q = \alpha(j_1 - 1), \quad j = \alpha(j_1 + 1), \quad r = \alpha(j_2 - 1), \quad t = \alpha(j_2 + 1)$$

are distinct elements of $I_1 = I_1(\alpha)$. Set

$$L_u(y) = \left\langle \frac{y_q - y_1}{\|y_q - y_1\|} + (-1)^u \frac{y_j - y_1}{\|y_j - y_1\|}, \frac{y_r - y_1}{\|y_r - y_1\|} - (-1)^u \frac{y_t - y_1}{\|y_t - y_1\|} \right\rangle$$

for $u = 1, 2$, $y \in U = U_\alpha$, and define

$$L : U_\alpha \longrightarrow \mathbb{R}^2$$

by $L(y) = (L_1(y), L_2(y))$. We will check that if $L(y) = 0$ for some $y \in U_\alpha$, then $dL(y) \neq 0$. Let $y \in U_\alpha$ be such that $L(y) = 0$. If

$$\frac{\partial L_1}{\partial y_m^{(\ell)}}(y) = 0, \quad \ell = 1, \dots, n,$$

then a simple calculation implies that $y_q - y_1$ is collinear with the vector

$$v = \frac{y_r - y_1}{\|y_r - y_1\|} + \frac{y_t - y_1}{\|y_t - y_1\|}.$$

Notice that $y \in U$ implies $v \neq 0$. Since $L_1(y) = 0$ and $(y_q - y_1)/\|y_q - y_1\|$ and $(y_j - y_1)/\|y_j - y_1\|$ are unit vectors, we deduce that $y_j - y_1$ is collinear with v, too. Therefore, y_1, y_q, y_j are collinear.

Next, assume that

$$\frac{\partial L_2}{\partial y_m^{(\ell)}}(y) = 0, \quad \ell = 1, \dots, n.$$

In the same way one obtains that y_1, y_r, y_t are collinear, which is a contradiction with $y = U_\alpha$ and the definition of U_α. Hence $dL(y) \neq 0$.

Finally, note that if $y_1, \dots, y_s$ are the reflection points of a periodic reflecting ray of type α, then for $y = (y_1, \dots, y_s)$ we have $y \in U_\alpha$ and $L(y) = 0$. Now applying Theorem 6.1.1(B) we obtain that $\mathcal{A}_\alpha$ contains a residual subset of $\mathbf{C}(X)$.

Next, we consider the case when α is a surjective symmetric map of the form (6.21). Let $k = 2m$ and let $y_1, \ldots, y_s$ be all distinct reflection points of a periodic reflecting ray γ for X such that (6.28) are the successive reflection points of γ (i.e. γ is of type α). Then y is a critical point of the map $F_{|X^{(s)}}$. Moreover,

$$F(y) = 2\sum_{i=1}^{m} \|y_{\alpha(i)} - y_{\alpha(i+1)}\| .$$

Assume that $k > 2s - 2$ and define $\mathcal{A}_\alpha$ as in the non-symmetric case. Using a slight modification of the above argument, replacing F by G and applying Theorem 6.1.1(B) again, we deduce that $\mathcal{A}_\alpha$ contains a residual subset of $\mathbf{C}(X)$.

Hence the set

$$\mathcal{A}_1 = \cap_\alpha \mathcal{A}_\alpha,$$

where α runs over the surjective maps (6.21) with $k > s$ for non-symmetric α and with $k > 2s - 2$ for symmetric α, contains a residual subset of $\mathbf{C}(X)$.

Let $Z = f(X)$ for some $f \in \mathcal{A}_1$. Suppose that there exist two different primitive periodic reflecting rays γ and δ for Z which have a common reflection point. We may assume that $z_1, \ldots, z_s$ are the successive reflection points of γ, $u_1, \ldots, u_t$ those of δ and $z_1 = u_1$. For $k \in \mathbb{N}$ set

$$\alpha_k = \mathrm{id} : \{1, \ldots, k\} \longrightarrow \{1, \ldots, k\}$$

and

$$F_k = F_{\alpha_k} : U_k = U_{\alpha_k} \longrightarrow \mathbb{R}.$$

Let $z_i = f(x_i)$, $u_j = f(y_j)$ for $i = 1, \ldots, s$ and $j = 1, \ldots, t$. Then for $x = (x_1, \ldots, x_s)$ and $y = (y_1, \ldots, y_t)$ we have $z = f^s(x) \in U_s$, $u = f^t(y) \in U_t$, x is a critical point of $F_s \circ f^s$ and y is a critical point of $F_t \circ f^t$. Hence $(x, y) \in U_s \times U_t$ and (x, y) is a critical point of the map $H \circ f^{s+t}$, where

$$H : U_s \times U_t \longrightarrow \mathbb{R}^2 \tag{6.31}$$

is defined by

$$H(z, u) = (F_s(z), F_t(u)). \tag{6.32}$$

The fact that $z_1 = u_1$ can be expressed by

$$L(f^s(x), f^t(y)) = 0,$$

where $L : U_s \times U_t \longrightarrow \mathbb{R}^n$ is given by $L(z, u) = z_1 - u_1$.

It is easily seen that $U = U_s \times U_t$, H and L satisfy the assumptions of Theorem 6.1.1(B). Consequently, there exists a residual subset $\mathcal{A}_2(s, t)$ of $\mathbf{C}(X)$ such that for any $f \in \mathcal{A}_2(s, t)$, we have $L(f^s(x), f^t(y)) \neq 0$ for any critical point

(x, y) of H with $(f^s(x), f^t(y)) \in U_s \times U_t$. Then

$$\mathcal{A}_2 = \cap\{\mathcal{A}_2(s,t) : s,\ t \geq 2\}$$

is also a residual subset of $\mathbf{C}(X)$.

It follows from the definitions of $\mathcal{A}_1$ and $\mathcal{A}_2$ that if $f \in \mathcal{A}_1 \cap \mathcal{A}_2$, then any two different primitive periodic reflecting rays for $f(X)$ have no common reflection point. Therefore, $\mathcal{A}_1 \cap \mathcal{A}_2 \subset \mathcal{A}$, so $\mathcal{A}$ contains a residual subset of $\mathbf{C}(X)$.

Next, consider the subset $\mathcal{R}$ of $\mathbf{C}(X)$. Let $Z = f(X)$ for some $f \in \mathcal{A}$. Suppose that there exist two different primitive periodic reflecting rays γ and δ for Z such that $pT_\gamma = qT_\delta$ for some $p, q \in \mathbb{N}$. Let $z_1, \dots, z_s$ and $u_1, \dots, u_t$ be the successive reflection points of γ and δ, respectively. Set $x_i = f^{-1}(z_i)$, $y_j = f^{-1}(u_j)$,

$$x = (x_1, \dots, x_s), \quad y = (y_1, \dots, y_t).$$

Consider the map H given by (6.31) and (6.32). As above one shows that (x, y) is a critical point of $H \circ f^{s+t}$. Moreover, for the map

$$K : U_s \times U_t \longrightarrow \mathbb{R}^n,$$

defined by $K(z, u) = pF_s(z) - qF_t(u)$, we have $K \circ f^{s+t}(x, y) = 0$. It follows from Lemma 6.2.2 that $dK(z, u) \neq 0$ for all $(z, u) \in U_s \times U_t$. Now applying Theorem 6.1.1(B) we deduce that there exists a residual subset $\mathcal{R}(p, q, s, t)$ of $\mathbf{C}(X)$ such that if $f \in \mathcal{R}(p, q, s, t)$, then whenever $(x, y) \in X^s \times X^t$, $(f^s(x), f^t(y)) \in U_s \times U_t$ and (x, y) is a critical point of $H \circ f^{s+t}$, we have $K(f^s(x), f^t(y)) \neq 0$. This yields the inclusion

$$\cap\{\mathcal{R}(p, q, s, t) : p, q, s, t \in \mathbb{N}, \quad s, t \geq 2\} \cap \mathcal{A} \subset \mathcal{R},$$

which shows that $\mathcal{R}$ contains a residual subset of $\mathbf{C}(X)$. This completes the proof of the theorem. ■

Let us mention that the property (b) from the beginning of this section implies that the measure of the set of periodic points in the phase space of the billiard system related to Ω is zero. More precisely, let Σ be the set of those $(x, \xi) \in X \times \mathbb{S}^{n-1}$ such that there exists a periodic reflecting ray for X passing through x with direction ξ. If μ denotes the standard Lebesgue measure on $X \times \mathbb{S}^{n-1}$, then for generic X we have $\mu(\Sigma) = 0$. In Section 6.4 we will establish stronger results, namely we will show that for generic X in $\mathbb{R}^n$ there exist at most countably many periodic reflecting rays for X. However, the weaker property proved in this section is already sufficient to make an application.

As a direct consequence of the result of Ivrii in [Ivl] concerning the asymptotic (0.4) for the counting function $N(\lambda)$ related to the point spectrum of the Laplacian, and Theorem 6.2.3 we obtain the following.

Corollary 6.2.4: *For every bounded domain Ω in $\mathbb{R}^n$, $n \geq 2$, with smooth boundary $X = \partial\Omega$ there exists a residual subset T of $\mathbf{C}(X)$ such that for every $f \in T$ the asymptotic (0.4) holds replacing Ω by Ω_f.*

In other words the Herman Weyl conjecture (see the Preface) is true for generic domains in $\mathbb{R}^n$.

The second part of this section is devoted to some properties of generic compact domains K in $\mathbb{R}^n$ with smooth boundaries $X = \partial K$ concerning reflecting (ω, θ)-rays in $\Omega = \overline{\mathbb{R}^n \setminus K}$. These properties are analogous to the properties of periodic reflecting rays considered in the first part of this section, and we will use the same technique in their proofs. That is why we will omit some of the details in the following considerations.

In what follows X will be a fixed compact smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$, $n \geq 2$, and ω and θ will be fixed unit vectors in $\mathbb{R}^n$.

It is convenient to introduce a notion slightly more general than that of an (ω, θ)-ray. Let γ be a curve of the form

$$\gamma = \cup_{i=0}^{k} \ell_i \subset \mathbb{R}^n$$

such that $\ell_i = [x_i, x_{i+1}]$ are finite straight-line segments for $i = 1, \dots, k-1$, $x_i \in X$ for all i, and ℓ_0 (resp. ℓ_k) is the infinite straight-line segment starting at x_1 (resp. x_k) and having direction $-\omega$ (resp. θ). Then γ will be called an *(ω, θ)-trajectory for X* if for any $i = 0, 1, \dots, k-1$ the segments ℓ_i and ℓ_{i+1} satisfy the law of reflection at x_{i+1} with respect to X. The points $x_1, \dots, x_k$ will be called reflection points of γ and the sojourn time T_γ of γ is defined by (2.31). As for reflecting (ω, θ)-rays (see Section 2.4), we distinguish two types of (ω, θ)-trajectories–symmetric and non-symmetric ones. The definitions are the same.

We should note that in general the segments of an (ω, θ)-trajectory for X may intersect transversally X. According to Definition 2.4.1 every reflecting (ω, θ)-ray is an (ω, θ)-trajectory but the converse is not true in general.

Define the subsets $\mathcal{B}$, $\mathcal{P}$, $\mathcal{S}$ of $\mathbf{C}(X)$ as the sets of those $f \in \mathbf{C}(X)$ such that:

(a) $\mathcal{B}$: for every $x \in f(X)$ there exists at most one direction $\xi \in \mathbb{S}^{n-1}$ (up to the symmetry with respect to the normal $\nu(x)$ to $f(X)$) so that there exists an (ω, θ)-trajectory for $f(X)$ passing through x with direction ξ;

(b) $\mathcal{P}$: there is no (ω, θ)-trajectory for $f(X)$ having different parallel segments;

(c) $\mathcal{S}$: $T_\gamma \neq T_\delta$ for any two different (ω, θ)-trajectories for $f(X)$.

Before going on, let us make the following remark. Let T be a subset of $\mathbf{C}(X)$, and let $\{U_k\}$ be a sequence of open subset of $\mathbb{R}^n$ such that $\cup_k U_k = \mathbb{R}^n$ and $X \subset U_k$ for any k. Then T is a residual subset of $\mathbf{C}(X)$ if and only if $T \cap C^\infty_{emb}(X, U_k)$ is residual in $C^\infty_{emb}(X, U_k)$ for each k. This follows easily from the fact that $C^\infty_{emb}(X, U_k)$ is open in $\mathbf{C}(X)$ for all k and

$$\mathbf{C}(X) = \cup_k C^\infty_{emb}(X, U_k).$$

Theorem 6.2.5: *Each of the sets $\mathcal{B}$, $\mathcal{P}$ and $\mathcal{S}$ defined above is a residual subset of* $\mathbf{C}(X)$.

Proof: Fix an arbitrary open ball U_0 with radius a in $\mathbb{R}^n$ such that $X \subset U_0$. For the sake of brevity, set

$$Z_1 = Z_\omega, \quad Z_2 = Z_{-\theta}, \quad \pi_1 = \pi_\omega, \quad \pi_2 = \pi_{-\theta},$$

where we use the notation Z_ξ and π_ξ from Section 2.4. According to the above remark, it is enough to establish that the intersection of each of the sets $\mathcal{B}$, $\mathcal{P}$ and $\mathcal{S}$ with

$$\mathbf{C}(X, U_0) = C^\infty_{emb}(X, U_0)$$

is a residual subset of $\mathbf{C}(X, U_0)$.

We need a modification of the notion of an admissible map introduced in the present section.

Let $k \geq s \geq 2$ be natural numbers, and let α be a map of the form (6.21) such that (6.22) holds for $i = 1, \ldots, k-1$. We will say that α is a *weakly non-symmetric map* if (6.22) holds for $1 \leq i < j \leq k-1$. If $k = 2m+1$, $\alpha(m-i+1) = \alpha(m+i+1)$ for $i = 0, 1, \ldots, m$ and (6.22) holds for $1 \leq i < j \leq m$, we will say that α is a *weakly symmetric map*. A map of the form (6.21) that is either weakly non-symmetric or weakly symmetric will be called a *weakly admissible map*.

Next, consider a fixed weakly admissible map (6.21) and set

$$\alpha(0) = 0, \quad \alpha(k+1) = s+1.$$

Thus, we regard α as a map

$$\alpha : \{0, 1, \ldots, k, k+1\} \longrightarrow \{0, 1, \ldots, s, s+1\}.$$

Define the sets $I_i = I_i(\alpha)$ by (6.25) for $i = 1, \ldots, s$. In this proof we will also use the notation $y_0 = \pi_1(y_1)$ and $y_{s+1} = \pi_2(y_{\alpha(k)})$ for any $y = (y_1, \ldots, y_s) \in (\mathbb{R}^n)^{(s)}$. Set $U^*_\alpha = U_\alpha \cap U_0^{(s)}$, where U_α is as in the beginning of this section. Then U^*_α is an open subset of $U_0^{(s)}$ and the function

$$F^* = F^*_\alpha : U^*_\alpha \longrightarrow \mathbb{R}, \tag{6.33}$$

given by

$$F^*(y) = \sum_{i=0}^{k} \| y_{(\alpha_i)} - y_{\alpha(i+1)} \| , \tag{6.34}$$

is smooth. If $y_1, \ldots, y_s$ are the distinct reflection points of an (ω, θ)-trajectory γ for X such that (6.28) are the successive reflection points of γ, we will say that γ *is of type* α. In this case we have $y = (y_1, \ldots, y_s) \in U^*_\alpha$ and $F^*(y) = T_\gamma$. Moreover, y is a critical point of the map

$$(F^*)_{|X^s} : X^s \longrightarrow \mathbb{R}.$$

It is also clear that for every (ω, θ)-trajectory γ there exists a surjective weakly admissible map α such that γ is of type α.

The following lemma can be proved in the same way as Lemma 6.2.2. We leave the details to the reader. ■

Lemma 6.2.6: *For every $i = 1, \ldots, s$ and every $y \in U^*_\alpha$ there exists $j = 1, \ldots, n$ such that $\frac{\partial F^*_\alpha}{\partial y_i^{(j)}}(y) \neq 0$.*

Next, assume that (6.21) is a surjective weakly non-symmetric map such that $k > s$. Denote by $\mathcal{B}_\alpha$ the set of those $f \in \mathrm{C}(X, U_0)$ such that there does not exist an (ω, θ)-trajectory of type α for $f(X)$. To show that $\mathcal{B}_\alpha$ contains a residual subset of $\mathbf{C}(X, U_0)$ we will use again Theorem 6.1.1(B), this time with $U = U^*_\alpha$, $p = 1$ and $H = F^*$.

Since $k > s$, there exists $i = 1, \ldots, s$ such that $\alpha^{-1}(i)$ contains more than one element. Take two arbitrary distinct $j_1, j_2 \in \alpha^{-1}(i)$. Then $q = \alpha(j_1 - 1)$, $j = \alpha(j_1 + 1)$, $r = \alpha(j_2 - 1)$, $t = \alpha(j_2 + 1)$ are distinct elements of I_i. Since $\{q, j\} \neq \{0, s+1\}$, either q or j is not in $\{0, s+1\}$. We may assume that $q \notin \{0, s+1\}$; otherwise we will simply change our notation by setting $q = \alpha(j_1 + 1)$ and $j = \alpha(j_1 - 1)$. Similarly, we may assume that $r \notin \{0, s+1\}$. For $u = 1, 2$ and $y \in U^*_\alpha$ set

$$L_u(y) = \left\langle \frac{y_q - y_i}{\|y_q - y_i\|} + (-1)^u \frac{y_j - y_i}{\|y_j - y_i\|}, \frac{y_r - y_i}{\|y_r - y_i\|} - (-1)^u \frac{y_t - y_i}{\|y_t - y_i\|} \right\rangle,$$

and define

$$L : U^*_\alpha \longrightarrow \mathbb{R}^2$$

by $L(y) = (L_1(y), L_2(y))$. Now repeating the corresponding part in the proof of Theorem 6.2.3 we deduce that $\mathcal{B}_\alpha$ contains a residual subset of $\mathbf{C}(X, U_0)$.

Next, suppose that $\theta = -\omega$ and α is a surjective weakly symmetric map of the form (6.21) with $k > 2s - 1$. Let $k = 2m + 1$ and let $y_1, \ldots, y_s$ be the distinct reflection points of an (ω, θ)-trajectory γ for X of type α, that is (6.28) are the successive reflection points of γ. Then one gets easily that $y' = (y_1, \ldots, y_m, y_{m+1})$ is a critical point of the map $G^*_{|X^{(m+1)}}$, where

$$G^* : (\mathbb{R}^n)^{(m+1)} \longrightarrow \mathbb{R}$$

is defined by

$$G^*(y) = \sum_{i=0}^{m} \|y_{\alpha(i)} - y_{\alpha(i+1)}\| .$$

Define $\mathcal{B}_\alpha$ as in the previous case. Now, using an argument similar to the above, replacing F^* by G^* and applying again Theorem 6.1.1(B), we see that $\mathcal{B}_\alpha$ contains a residual subset of $\mathbf{C}(X, U_0)$.

In this way we have established that the set $\mathcal{B}_1 = \cap_\alpha \mathcal{B}_\alpha$, where α runs over the set of surjective weakly admissible maps (6.21) with $k > s$ in the non-symmetric case and with $k > 2s - 1$ in the symmetric case, containing a residual subset of $\mathbf{C}(X, U_0)$.

Next, we have to prove that the set $\mathcal{B}_2$ of those $f \in \mathbf{C}(X, U_0)$ such that any two distinct (ω, θ)-trajectories for $f(X)$ have no common reflection points contains a residual subset of $\mathbf{C}(X, U_0)$. This can be established using a modification of the corresponding argument in the proof of Theorem 6.2.3. We leave the details to the reader.

Since $\mathcal{B}_1 \cap \mathcal{B}_2 \subset \mathcal{B}$, it follows that $\mathcal{B}$ contains a residual subset of $\mathbf{C}(X, U_0)$.

Next, we proceed with the set $\mathcal{S}$. Let

$$f \in \mathcal{B} \cap \mathbf{C}(X, U_0),$$

and let γ and δ be distinct non-symmetric (ω, θ)-trajectories for $Y = f(X)$ such that $T_\gamma = T_\delta$. Let $y_i = f(x_i)$ be all distinct reflection points of γ and δ taken together. Since $f \in \mathcal{B}$, we may assume that $y_1, \ldots, y_r$ are the successive reflection points of γ for some $r < s$, while $y_{r+1}, \ldots, y_s$ are those of δ. Consider the functions

$$F^{**}, G^{**} : U \longrightarrow \mathbb{R}$$

defined by

$$F^{**}(z) = \|\pi_1(z_1) - z_1\| + \sum_{i=1}^{r-1} \|z_i - z_{i+1}\| + \|z_r - \pi_2(z_r)\| , \tag{6.35}$$

$$G^{**}(z) = \|\pi_1(z_{r+1}) - z_{r+1}\| + \sum_{i=r+1}^{s-1} \|z_i - z_{i+1}\| + \|z_s - \pi_2(z_s)\| . \tag{6.36}$$

Here U is the set of all $z \in (\mathbb{R}^n)^{(s)}$ such that $z_i \notin [z_{i-1}, z_{i+1}]$ for all $i = 2, 3, \ldots, s-1$ with $i \neq r$, $z_1 \notin [\pi_1(z_1), z_2]$, $z_r \notin [z_{r-1}, \pi_2(z_r)]$, $z_{r+1} \notin [\pi_1(z_{r+1}), z_{r+2}]$, $z_s \notin [z_{s-1}, \pi_2(z_s)]$. Clearly, $F^{**}(y) = G^{**}(y)$. Applying Theorem 6.1.1(B) with $H = (F^{**}, G^{**})$ and $L : U \longrightarrow \mathbb{R}$, defined by $L(z) = F^{**}(z) - G^{**}(z)$, we derive that the set

$$\mathcal{S}_1(r, s) = \{f \in \mathbf{C}(X, U_0) : \mathrm{grad}_x(H \circ f^s)(x) = 0 \ \Rightarrow \ L(f^s(x)) \neq 0\}$$

contains a residual subset of $\mathbf{C}(X, U_0)$. Hence

$$\mathcal{S}_1 = \cap_{r<s} \mathcal{S}_1(r, s) \cap \mathcal{B}$$

has the same property. Moreover, it follows from our considerations above that for $f \in \mathcal{S}_1$ we have $T_\gamma \neq T_\delta$ for any two different non-symmetric (ω, θ)-trajectories γ and δ for $f(X)$.

In a similar way one constructs $\mathcal{S}_2$ and $\mathcal{S}_3$ containing residual subsets of $\mathbf{C}(X, U_0)$ such that for $f \in \mathcal{S}_2$ (resp. $f \in \mathcal{S}_3$) we have $T_\gamma \neq T_\delta$ for any two different symmetric (ω, θ)-trajectories γ and δ (resp. for any symmetric γ and non-symmetric δ). Since

$$\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_3 \subset \mathcal{S},$$

it follows that $\mathcal{S}$ contains a residual subset of $\mathbf{C}(X, U_0)$.

Finally, we have to deal with $\mathcal{P}$. For this we use an argument similar to these above. One can express analytically the fact that two segments $[y_i, y_{i+1}]$ and $[y_j, y_{j+1}]$ of an (ω, θ)-trajectory are parallel by using a map $L : U \longrightarrow \mathbb{R}$ of the form

$$L(y) = \frac{y_i - y_{i+1}}{\|y_i - y_{i+1}\|} + \epsilon\, \frac{y_j - y_{j+1}}{\|y_j - y_{j+1}\|}.$$

Here $\epsilon = \pm 1$ and the set $U \subset (\mathbb{R}^n)^{(s)}$ are defined appropriately. A standard application of Theorem 6.1.1(B) shows that $\mathcal{P}$ contains a residual subset of $\mathbf{C}(X, U_0)$. The details are left to the reader.

6.3 Absence of tangent segments

In this section X will again be a fixed smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$, $n \geq 2$. Recall that a reflecting ray for X is called *ordinary* if it has no segments tangent to X.

Theorem 6.3.1: *Let $\mathcal{T}$ be the set of those $f \in \mathbf{C}(X)$ such that every periodic reflecting ray for $f(X)$ is ordinary. Then $\mathcal{T}$ contains a residual subset of $\mathbf{C}(X)$.*

To prove this theorem we need the following technical lemma.

Lemma 6.3.2: *Let $p = 1, \ldots, n$ be fixed and let Q_p be the set of those $v = (v_0, v_1, v_2) \in (\mathbb{R}^n)^{(3)}$ such that $v_1^{(p)} \neq v_0^{(p)}$. For any $m = 1, \ldots, n$ let*

$$d^{(m)} : Q_p \longrightarrow \mathbb{R}$$

be defined by

$$d^{(m)}(v) = \frac{v_1^{(m)} - v_0^{(m)}}{\|v_1 - v_0\|} + \frac{v_2^{(m)} - v_0^{(m)}}{\|v_2 - v_0\|}.$$

Then the vectors $\operatorname{grad}(d^{(m)})(v)$, *$m = 1, \ldots, n$, $m \neq p$, are linearly independent over $\mathbb{R}$ for every $v \in Q_p$.*

Proof: Fix an arbitrary $v \in Q_p$ and assume that there exist real constants $D^{(m)}$ such that

$$\sum_{1 \leq m \leq n, m \neq p} D^{(m)} \operatorname{grad}(d^{(m)})(v) = 0. \tag{6.37}$$

Set $D^{(p)} = 0$ and $D = (D^{(1)}, \dots, D^{(n)}) \in \mathbb{R}^n$. For convenience set

$$\omega_i = \frac{v_i - v_0}{\| v_i - v_0 \|}, \quad z_i = \frac{1}{\| v_i - v_0 \|}$$

for $i = 1, 2$. By a direct calculation we find

$$\frac{\partial d^{(m)}}{\partial v_0^{(t)}}(v) = z_1 \omega_1^{(m)} \omega_1^{(t)} + z_2 \omega_2^{(m)} \omega_2^{(t)}$$

for all $t = 1, \dots, n$, $t \neq m$, and

$$\frac{\partial d^{(t)}}{\partial v_0^{(t)}}(v) = -(z_1 + z_2) + z_1 (\omega_1^{(t)})^2 + z_2 (\omega_2^{(t)})^2$$

for all $t = 1, \dots, n$. It then follows from (6.37) that for any t we have

$$\begin{aligned} 0 &= \sum_{m=1}^{n} D^{(m)} \frac{\partial d^{(t)}}{\partial v_0^{(t)}}(v) \\ &= \sum_{m=1}^{n} D^{(m)} (z_1 \omega_1^{(m)} \omega_1^{(t)} + z_2 \omega_2^{(m)} \omega_2^{(t)}) - D_t (z_1 + z_2), \end{aligned}$$

which is equivalent to

$$(z_1 + z_2) D^{(t)} = z_1 \langle D, \omega_1 \rangle \omega_1^{(t)} + z_2 \langle D, \omega_2 \rangle \omega_2^{(t)}.$$

Since the latter holds for any $t = 1, \dots, n$, we get

$$(z_1 + z_2) D = z_1 \langle D, \omega_1 \rangle \omega_1 + z_2 \langle D, \omega_2 \rangle \omega_2. \tag{6.38}$$

Considering the inner product of (6.38) with ω_1, we find

$$\langle D, \omega_1 \rangle = \langle D, \omega_2 \rangle \ \langle \omega_1, \omega_2 \rangle.$$

In the same way we see that

$$\langle D, \omega_2 \rangle = \langle D, \omega_1 \rangle \ \langle \omega_1, \omega_2 \rangle.$$

Combining the last two equalities implies

$$\langle D, \omega_1 \rangle (1 - \langle \omega_1, \omega_2 \rangle^2) = 0 = \langle D, \omega_2 \rangle (1 - \langle \omega_1, \omega_2 \rangle^2). \tag{6.39}$$

First, assume that $\langle \omega_1, \omega_2 \rangle^2 \neq 1$. Then (6.39) implies $\langle D, \omega_1 \rangle = \langle D, \omega_2 \rangle = 0$, and by (6.38) we must have $D = 0$. Let $\langle \omega_1, \omega_2 \rangle^2 = 1$. Then $\omega_2 = \epsilon \omega_1$ for some $\epsilon = \pm 1$ and (6.38) becomes

$$(z_1 + z_2) D = (z_1 + z_2) \langle D, \omega_1 \rangle \ \omega_1.$$

Hence $D = \langle D, \omega_1 \rangle\, \omega_1$, and comparing the p-components of these vectors, we find $0 = D^{(p)} = \langle D, \omega_1 \rangle\, \omega_1^{(p)}$. Since $v \in Q_p$, we have $\omega_1^{(p)} \neq 0$, therefore $\langle D, \omega_1 \rangle = 0$. Using the above, this gives $D = 0$. ■

Proof of Theorem 6.3.1: Let $\mathcal{A}$ be the subset of $\mathbf{C}(X)$ defined in Section 6.3.2 (cf. the text before Theorem 6.2.3). Let $f \in \mathcal{A}$ and let $s \geq 2$ be a natural number. Suppose that there exists a periodic reflecting ray γ for $f(X)$ with s reflection points such that some segment of γ is tangent to $f(X)$. We may assume that $y_1 = f(x_1), \ldots, y_s = f(x_s)$ are the successive reflection points of γ and the segment $[y_1, y_2]$ is tangent to $f(X)$ at some point $y_0 = f(x_0)$. The latter condition is equivalent to

$$\frac{y_1 - y_0}{\|y_1 - y_0\|} + \frac{y_2 - y_0}{\|y_2 - y_0\|} = 0$$

and $\langle y_1 - y_2, \nu(y_0) \rangle = 0$, where $\nu(y_0)$ denotes (one of) the unit normal vectors to $f(X)$ at y_0.

Fix $s \geq 2$ and define U_s and $F = F_s$ as in the proof of Theorem 6.2.3. Denote by $\mathcal{T}_s$ the set of those $f \in \mathbf{C}(X)$ such that there does not exist a critical point

$$x = (x_0, x_1, \ldots, x_s) \in X^{(s+1)}$$

for which $x' = (x_1, \ldots, x_s)$ is a critical point of the map $F \circ f^s$ on $X^{(s)}$, $f^s(x') \in U_s$, and the segment $[f(x_1), f(x_2)]$ is tangent to $f(X)$ at $f(x_0)$. We will prove that $\mathcal{T}_s$ contains a residual subset of $\mathbf{C}(X)$. To do this we will use an argument similar to that in the proof of Theorem 6.1.1.

Set

$$V = \{j^1 f(x) \in J^1(X, \mathbb{R}^n) : \operatorname{rank}(df(x)) = n - 1\}.$$

Denote by M the set of those

$$\sigma = (j^1 f_0(x_0), j^1 f_1(x_1), \ldots, j^1 f_x(x_s)) \in V^{s+1} \tag{6.40}$$

such that

$$x = (x_0, x_1, \ldots, x_s) \in X^{s+1}, \quad x' = (x_1, \ldots, x_s) \in X^{(s)}, \quad f^s(x') \in U_s.$$

Clearly M is an open subset of $J^1_{s+1}(X, \mathbb{R}^n)$.

Let Σ be the set of all elements σ of M of the form (6.40) such that x' is a critical point of $F \circ (f_1 \times \cdots \times f_s)$ and the segment $[f_1(x_1), f_2(x_2)]$ is tangent to $f_0(X)$ at $f_0(x_0)$. Clearly we can write

$$\mathcal{T}_s = \{f \in \mathbf{C}(X) : j^1_{s+1}(X^{(s+1)}) \cap \Sigma = \emptyset\}. \tag{6.41}$$

We will establish that Σ can be covered by a countable family of smooth submanifolds of M of codimension greater than $\dim(X^{(s+1)}) = (s+1)(n-1)$.

Consider a coordinate neighbourhood D of an element of Σ. We may take D of the same form as in the proof of Lemma 6.1.2. Namely, fix arbitrary charts

$$\varphi_i : V_i \longrightarrow \mathbb{R}^{n-1}, \quad i = 0, 1, \ldots, s,$$

where V_i are open subsets of X with $V_i \cap V_j = \emptyset$ for all $1 \le i, j \le s$, $i \ne j$. Then we define D by (6.9), (6.11)–(6.14), the only difference being that i runs from 0 to s (instead of $1 \le i \le s$). Notice that the vector

$$N_i = (N_i^{(1)}, \ldots, N_i^{(n)})$$

given by

$$N_i^{(t)} = (-1)^t \det \begin{pmatrix} a_{i1}^{(1)} & \cdots & a_{i1}^{(t-1)} & a_{i1}^{(t+1)} & \cdots & a_{i1}^{(n)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{in-1}^{(1)} & \cdots & a_{in-1}^{(t-1)} & a_{in-1}^{(t+1)} & \cdots & a_{in-1}^{(n)} \end{pmatrix} \tag{6.42}$$

is normal to $f_i(X)$ at $f_i(x_i)$.

We will show that $\varphi(D \cap \Sigma)$ is contained in a finite union of smooth submanifolds of $\varphi(D)$ of codimension $(s+1)(n-1)+1$. The elements ξ of $\varphi(D)$ will be written in the form

$$\xi = (u; v; a) \in (\mathbb{R}^{n-1})^{(s+1)} \times (\mathbb{R}^n)^{(s+1)} \times \mathbb{R}^{n(n-1)(s+1)},$$

where u, v and a are determined as in the proof of Lemma 6.1.2. For $p = 1, \ldots, n$ consider the open subset

$$G_p = \{\xi \in \varphi(D) : v_1^{(p)} \ne v_0^{(p)}\}$$

of $\varphi(D)$. Since $\varphi(D) = \cup_{p=1}^n G_p$, it is sufficient to show that $G_p \cap \varphi(D \cap \Sigma)$ is contained in a smooth submanifold of $\varphi(D)$ of codimension $(s+1)(n-1)+1$ for every p.

Fix an arbitrary p. For $\xi \in \varphi(D)$ set

$$N(\xi) = (N_0^{(1)}(\xi), \ldots, N_0^{(n)}(\xi)),$$

where the components $N_0^{(t)}(\xi)$ are given by (6.42). Since $D \subset M \subset V^{s+1}$, we have $N(\xi) \ne 0$ for any $\xi \in \varphi(D)$. Set

$$c_{ij}(\xi) = \sum_{t=1}^{n} \frac{\partial F}{\partial y_i^{(t)}}(v)\, a_{ij}^{(t)}$$

for $\xi \in \varphi(D)$, $1 \le i \le s$, $1 \le j \le n-1$. Notice that if $\xi = \varphi(\sigma)$ and σ has the form (6.40), then $x' = (x_1, \ldots, x_s)$ is a critical point of the map $F \circ (f_1 \times \cdots \times f_s)$ if and only if $c_{ij}(\xi) = 0$ for all $i = 1, \ldots, s$ and all $j = 1, \ldots, n-1$. Consider the map

$$K : G_p \longrightarrow \mathbb{R}^{s(n-1)} \times \mathbb{R}^{n-1} \times \mathbb{R},$$

defined by

$$K(\xi) = ((c_{ij}(\xi))_{1\leq i\leq s, 1\leq j\leq n-1}, \quad (d^{(m)})_{1\leq m\leq n, m\neq p}, L(\xi)),$$

where

$$d^{(m)}(\xi) = \frac{v_1^{(m)} - v_0^{(m)}}{\|v_1 - v_0\|} + \frac{v_2^{(m)} - v_0^{(m)}}{\|v_2 - v_0\|},$$

and

$$L(\xi) = \langle v_1 - v_2, N(\xi)\rangle.$$

It is straightforward to check that

$$G_p \cap \varphi(D \cap \Sigma) \subset K^{-1}(0).$$

Now it is sufficient to show that K is a submersion at any point of G_p. This will imply that $K^{-1}(0)$ is a submanifold of G_p of codimension $(s+1)(n-1)+1$.

Fix an arbitrary $\xi \in G_p$ and assume that there exist real constants C_{ij}, D_m and E such that

$$\sum_{i=1}^{s}\sum_{j=1}^{n-1} C_{ij} \text{ grad}(c_{ij})(\xi) + \sum_{\substack{m=1\\ m\neq p}}^{n} D_m \text{ grad}(d^{(m)})(\xi) + E \text{ grad } L(\xi) = 0. \quad (6.43)$$

It follows from Lemma 6.2.2 that for any $i = 1, \ldots, s$ there exists $t = 1, \ldots, n$ such that (6.29) holds with $y = v$. Considering the derivatives with respect to $a_{ij}^{(t)}$ in (6.43), we find $C_{ij} = 0$ for all i and j. Consequently, the first double sum in (6.43) is zero. Since $(\partial L/\partial v_0^{(t)})(\xi) = 0$ for any $t = 1, \ldots, n$, we can use (6.43) to apply Lemma 6.3.2. It then follows that $D_m = 0$ for every $m \neq p$. Finally, consider an arbitrary $t = 1, \ldots, n$ with $N^{(t)}(\xi) \neq 0$. Then by (6.43) and

$$\frac{\partial L}{\partial v_1^{(t)}}(\xi) = N^{(t)}(\xi) \neq 0$$

it follows that $E = 0$. Hence K is a submersion at ξ.

In this way we have shown that $K^{-1}(0)$ is a smooth submanifold of G_p of codimension $(s+1)(n-1)+1$. As we have already mentioned above, this implies that $D \cap \Sigma$ is contained in a finite union of smooth submanifolds W_m of M with

$$\text{codim } (W_m) = (s+1)(n-1)+1.$$

Then (6.41) yields

$$\cap_m T_s^{(m)} \subset \mathcal{T}_s, \quad (6.44)$$

where

$$T_s^{(m)} = \{f \in \mathbf{C}(X) : j^1_{s+1} f(X^{s+1}) \cap W_m = \emptyset\}.$$

Since M is open in $J^1_{s+1}(X, R^n)$, W_m is a smooth submanifold of $J^1_{s+1}(X, \mathbb{R}^n)$ with

$$\text{codim } (W_m) = (s+1)(n-1) + 1 > \dim(X^{s+1}).$$

On the other hand,

$$j^1_{s+1} f : X^{s+1} \longrightarrow J^1_{s+1}(X, \mathbb{R}^n),$$

therefore

$$j^1_{s+1} f(X^{s+1}) \cap W_m = \emptyset \Longleftrightarrow j^1_{s+1} f \pitchfork W_m.$$

It then follows from the Multijet Transversality Theorem (see Section 1.1) that $T_s^{(m)}$ is a residual subset of $\mathbf{C}(X)$ for every m, and by (6.44), $\mathcal{T}_s$ contains a residual subset of $\mathbf{C}(X)$.

Since $\cap_{s=2}^{\infty} \mathcal{T}_s \cap \mathcal{A} \subset \mathcal{T}$ and, according to Theorem 6.2.3, $\mathcal{A}$ contains a residual subset of $\mathbf{C}(X)$, it follows that $\mathcal{T}$ has the same property. ■

The next theorem can be proved applying the previous argument with some small modifications. We leave the details to the reader.

Theorem 6.3.3: *Let ω and θ be two fixed unit vectors in $\mathbb{R}^n$, and let X be a compact smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$. Let $\mathcal{T}(\omega, \theta)$ be the set of those $f \in \mathbf{C}(X)$ such that every reflecting (ω, θ)-ray for $f(X)$ is ordinary. Then $\mathcal{T}(\omega, \theta)$ contains a residual subset of $\mathbf{C}(X)$.*

6.4 Non-degeneracy of reflecting rays

In this section it is proved that for generic Ω in $\mathbb{R}^n$ all periodic reflecting rays are non-degenerate. It is also established that, given two fixed unit vectors ω and θ, for generic Ω with bounded complements, every reflecting (ω, θ)-ray γ in Ω is non-degenerate, that is $\det(J_\gamma) \neq 0$.

In what follows X will be a fixed compact smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$, $n \geq 2$.

Theorem 6.4.1: *Let Λ be an arbitrary countable set of complex numbers and let T_Λ be the set of those $f \in \mathbf{C}(X)$ such that every periodic reflecting ray γ for $f(X)$ is ordinary and $\text{spec}(P_\gamma)$ does not contain elements of Λ. Then T_Λ contains a residual subset of $\mathbf{C}(X)$.*

Since the definition of the Poincaré map involves second derivatives, there is no way to prove the above theorem applying somehow Theorem 6.1.1. However, as in Section 6.3, an appropriate modification of the scheme of proof of Theorem 6.1.1 will be useful again.

We begin with some preliminary considerations.

Let γ be an ordinary primitive non-symmetric reflecting ray for X with successive reflection points $q_1, \dots, q_m, q_{m=1} = q_1$. We will assume that the points $q_1, \dots, q_m$ are all different. Recall the notation Π_i, σ_i, λ_i, ν_i, G_i $\tilde{\psi}_i$, ψ_i, s_i from Section 2.3. Then P_γ has the representation (2.26). For brevity set

$$A_j = \begin{pmatrix} I & \lambda_j I \\ \psi_j & I + \lambda_j \psi_j \end{pmatrix} \tag{6.45}$$

for $j = 1, \dots, m$. Since P_γ is a symplectic matrix, we have $0 \notin \mathrm{spec}(P_\gamma)$. Let $1/\mu \in \mathbb{C}$ be an eigenvalue of P_γ. Then by (2.26) we get

$$\begin{aligned} 0 &= \det \left(\frac{1}{\mu} I - \begin{pmatrix} s_m & 0 \\ 0 & s_m \end{pmatrix} A_m \cdots A_1 \right) \\ &= \det \left(I - A_m \cdots A_1 \begin{pmatrix} \mu s_m & 0 \\ 0 & \mu s_m \end{pmatrix} \right). \end{aligned}$$

Set

$$E = \det \left(I - A_m \cdots A_1 \begin{pmatrix} \mu s_m & 0 \\ 0 & \mu s_m \end{pmatrix} \right), \tag{6.46}$$

and denote by $d_{ij}^{(t)}$ the elements of the matrix ψ_t, $t = 1, \dots, m$, $i, j = 1, \dots, n-1$. Then clearly E can be expressed as a polynomial of the elements $d_{ij}^{(t)}$. The terms in E involving only products of elements $d_{ij}^{(1)}$ $(i, j = 1, \dots, n-1)$ are contained in the determinant

$$\begin{aligned} D &= \det \left(I - \begin{pmatrix} I & \lambda_m I \\ 0 & I \end{pmatrix} \cdots \begin{pmatrix} I & \lambda_2 I \\ 0 & I \end{pmatrix} A_1 \begin{pmatrix} \mu s_m & 0 \\ 0 & \mu s_m \end{pmatrix} \right) \\ &= \det \left(A_1^{-1} - \begin{pmatrix} \mu s_m & \mu(\sum_{i=2}^m \lambda_i) s_m \\ 0 & \mu s_m \end{pmatrix} \right). \end{aligned}$$

Since

$$A_1^{-1} = \begin{pmatrix} I + \lambda_1 \psi_1 & -\lambda_1 I \\ -\psi_1 & I \end{pmatrix},$$

we find

$$\begin{aligned} D &= \det \begin{pmatrix} I + \lambda_1 \psi_1 - \mu s_m & -\mu(\sum_{i=2}^m \lambda_i) s_m - \lambda_1 I \\ -\psi_1 & I - \mu s_m \end{pmatrix} \\ &= \det \begin{pmatrix} I - \mu s_m & -\mu(\sum_{i=1}^m \lambda_i) s_m \\ -\psi_1 & I - \mu s_m \end{pmatrix}. \end{aligned}$$

It is now clear that the product $d_{11}^{(1)} d_{22}^{(1)} \cdots d_{n-1n-1}^{(1)}$ has a non-zero coefficient

$$\epsilon \mu \left(\sum_{i=1}^m \lambda_i \right) \det(s_m),$$

where $\epsilon = \pm 1$. Consequently, E is a non-trivial polynomial of $d_{ij}^{(t)}$ $(1 \leq t \leq m$, $1 \leq i, j \leq n-1)$ with coefficients depending on μ, λ_i and σ_i.

Consider the multiindices

$$\tau = ((i_1, j_1, t_1), \ldots, (i_\ell, j_\ell, t_\ell)), \tag{6.47}$$

consisting of triples (i_s, j_s, t_s) of integers such that

$$1 \leq i_s, j_s \leq n-1, \quad 1 \leq t_s \leq m, \quad t_1 \geq t_2 \geq \cdots \geq t_\ell \geq 1. \tag{6.48}$$

For $i \leq \ell$ denote by $p_i(\tau)$ the number of those triples (i_s, j_s, t_s) in τ such that $t_s = t_i$. Set

$$|\tau| = \sum_{i=1}^{\ell} t_i, \quad d^\tau = d_{i_1 j_1}^{(t_1)} d_{i_2 j_2}^{(t_2)} \cdots d_{i_\ell j_\ell}^{(t_\ell)},$$

and define the function $\partial^\tau E$ by

$$\partial^\tau E = \frac{\partial^{|\tau|} E}{\partial d_{i_1 j_1}^{(t_1)} \partial d_{i_2 j_2}^{(t_2)} \cdots \partial d_{i_\ell j_\ell}^{(t_\ell)}}.$$

It follows from our arguments above that

$$E = \sum_{\tau} c_\tau d^\tau,$$

where τ runs over the set of the multiindices (6.47) satisfying (6.48) such that $|\tau| \leq m(n-1)$ and $p_i(\tau) \leq n-1$. Here

$$c_\tau = c_\tau(\mu, \lambda_1, \ldots, \lambda_m, \sigma_1, \ldots, \sigma_m)$$

are real coefficients.

Next, we will define an open subset M of the m-fold bundle of 2-jets $J_m^2(X, \mathbb{R}^n)$. First, consider the open subset

$$V = \{j^2 f(x) \in J^2(X, \mathbb{R}^n)\colon \operatorname{rank}(df(x)) = n-1\}$$

of $J^2(X, \mathbb{R}^n)$. Denote by U_m the set of those $y = (y_1, \ldots, y_m) \in (\mathbb{R}^n)^{(m)}$ such that for every $i = 1, \ldots, m$, the point y_i does not belong to the segment $[y_i, y_{i+1}]$. As before we set for convenience $y_0 = y_m$ and $y_{m+1} = y_1$. We will also need the function

$$F = F_m : U_m \longrightarrow \mathbb{R},$$

given by

$$F(y) = \sum_{i=1}^{m} \|y_i - y_{i+1}\|.$$

Finally, set

$$M = (\alpha^m)^{-1}(X^{(m)}) \cap (\beta^m)^{-1}(U_m) \cap V^m,$$

where α and β are the source and the target maps, respectively, defined in Section 1.1.

An atlas for M can be described as in Sections 6.1 and 6.3. Namely, consider arbitrary coordinate neighbourhoods $V_1, \dots, V_m$ of distinct elements of X such that $V_i \cap V_j = \emptyset$ for $i \neq j$, and let

$$\varphi_j : V_i \longrightarrow \mathbb{R}^{n-1}$$

be arbitrary smooth charts. Set

$$D = M \cap \prod_{i=1}^{m} J^2(V_i, \mathbb{R}^n), \tag{6.49}$$

and define the chart

$$\varphi : D \longrightarrow (\mathbb{R}^{n-1})^{(s)} \times (\mathbb{R}^n)^{(s)} \times \mathbb{R}^{s(n-1)n} \times \mathbb{R}^{s(n-1)(n-2)n/2}$$

by

$$\varphi(\sigma) = (u; v; a; b)$$

for every

$$\sigma = (j^2 f_1(x_1), \dots, j^2 f_m(x_m)) \in D, \tag{6.50}$$

where u, v, a are defined by (6.12) (replacing s by m), (6.13) and (6.14), while

$$b = (b^{(t)}_{ij\ell})_{1\le i\le m, 1\le j,\ell\le n-1, 1\le t\le n} \tag{6.51}$$

is given by

$$b^{(t)}_{ij\ell} = \frac{\partial^2 (f_i^{(t)} \circ \varphi_i^{-1})}{\partial u_i^{(j)} \partial u_i^{(\ell)}}(u_i). \tag{6.52}$$

As mentioned in Section 6.3, the vector

$$N_i = (N_i^{(1)}, \dots, N_i^{(n)}), \tag{6.53}$$

determined by (6.42), is orthogonal to $f_i(X)$ at the point $f_i(x_i)$.

Let $\sigma \in D$ have the form (6.50) and let $q_i = f_i(x_i)$, $i = 1, \dots, m$. Now we define the numbers λ_i and the maps σ_i as in the case of a periodic reflecting ray for a given submanifold $f(X)$. Namely, we first define λ_i by (2.14) and N_i by (6.53) and (6.42). Next, denote by Π_i the hyperplane passing through $f_i(x_i)$ and orthogonal to $\overrightarrow{f_i(x_i) f_{i+1}(x_{i+1})}$ and by α_i the hyperplane passing through $f_i(x_i)$ and orthogonal to

N_i. Now the symmetry σ_i is defined as in Section 2.4. Till now we have only used the data $j^1 f_1(x_1), \ldots, j^1 f_m(x_m)$.

Next, define the differential of the Gauss map

$$G_i : \alpha_i \longrightarrow \alpha_i$$

by means of $j^2 f_i(x_i)$ and determine ψ_i and $\tilde{\psi}_i$ by (2.16), (2.24) and (2.25). Finally, define the matrices $A_i = A_i(\sigma)$ by (6.45) and the function $E = E(\sigma)$ by (6.46), where μ is a fixed complex number. Clearly, E is completely determined by σ, therefore E can be viewed as a function

$$E : D \longrightarrow \mathbb{R}.$$

Denote by Σ the set of all elements σ of M of the form (6.50) such that $x = (x_1, \ldots, x_m)$ is a critical point of $F_m \circ f^m$, $f^m(x) \in U_m$ and $E(\sigma) = 0$.

Lemma 6.4.2: *Σ is contained in the union of a countable family of smooth submanifolds of M of codimension $s(n-1)+1$.*

Proof: Consider a coordinate neighbourhood D of the form (6.49) of an element of Σ, and let the chart φ on D be defined as earlier. To prove the lemma, it is enough to show that $\varphi(D \cap \Sigma)$ is contained in a finite union of smooth submanifolds of $\varphi(D)$ of codimension $s(n-1)+1$.

The elements ξ of $\varphi(D)$ have the form

$$\xi = (u; v; a; b),$$

where u, v, a, b are given by (6.12) with s replaced by m, (6.13), (6.14), (6.51) and (6.52). Set $G(\xi) = E(\varphi^{-1}(\xi))$ and

$$c_{ij}(\xi) = \sum_{t=1}^{n} \frac{\partial F_m}{\partial y_i^{(t)}}(v) a_{ij}^{(t)}$$

for $1 \le i \le m$, $1 \le j \le n-1$. Consider the multiindices

$$\delta_p = ((1,1,1), (2,2,1), \ldots, (p,p,1)).$$

For any $p = 0, 1, \ldots, n-2$ denote by M_p the set of those $\xi \in \varphi(D)$ such that $c_{ij}(\xi) = 0$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n-1$, and

$$\partial^{\delta_p} G(\xi) = 0, \quad \partial^{\delta_{p+1}} G(\xi) \neq 0.$$

Here we set for convenience $\partial^{\delta_0} G = G$.

Given two multiindices τ and τ' of the form (6.47) with (6.48), we will write $\tau < \tau'$ if $|\tau| < |\tau'|$ and τ' contains all triples in τ. Denote by $\mathcal{M}$ the set of all multiindices τ of the form (6.47) with (6.48) such that $|\tau| \le m(n-1)$, $\tau > \delta_{n-1}$ and τ contains exactly $n-1$ triples (i,j,t) with $t=1$. Let $\mathcal{M}_2$ be the set of all pairs $(\tau,\tau')\in\mathcal{M}_2$ such that $\tau<\tau'$ and $|\tau'|=|\tau|+1$. Given $(\tau,\tau')\in\mathcal{M}_2$, set

$$M(\tau,\tau')=\{\xi\in\varphi(D): c_{ij}(\xi)=0\;\forall i=1,\dots,m,\\ \forall j=1,\dots,n-1, \partial^{\tau}G(\xi)=0\;, \partial^{\tau'}G(\xi)\neq 0\}.$$

It then follows from above that

$$\varphi(D\cap\Sigma)\subset\cup_{p=0}^{n-2}M_p\cup\cup_{(\tau,\tau')\in\mathcal{M}_2}M(\tau,\tau').$$

Therefore, the lemma will be proved if we establish that each of the sets M_p and $M(\tau,\tau')$ is a smooth submanifold of $\varphi(D)$ of codimension $s(n-1)+1$.

Before going on let us make the following remark. Let $A=(A_{ij})$ and $B=(B_{ij})$ be $n\times n$ symmetric matrices such that

$$UAV=B \tag{6.54}$$

for some invertible matrices U and V. Let $\psi(B_{11},B_{12},\dots,B_{nn})$ be a smooth function of the elements B_{ij} of the matrix B. If the elements of the matrices U and V do not depend on (A_{ij}) and (B_{ij}), then $\partial\psi/\partial B_{ij}=0$ for all $i,j=1,\dots,n$ is equivalent to $\partial\psi/\partial A_{ij}=0$ for all $i,j=1,\dots,n$. Here ψ is considered as a function of (A_{ij}) replacing each B_{ij} by the corresponding function of (A_{ij}) according to (6.54).

For every $k=1,\dots,m$ we have

$$\psi_k=\sigma_1\cdots\sigma_k\pi_k^*G_k\pi_k\sigma_k\cdots\sigma_1,$$

where the projection π_k is defined as in Section 2.4. Let $G_k=(g_{ij}^{(k)})_{i,j=1}^{n-1}$. It then follows from the above remark that if

$$\frac{\partial}{\partial d_{ij}^{(k)}}(\partial^{\tau}G)(\xi)\neq 0 \tag{6.55}$$

for some (i,j,k), then there exist i' and j' such that

$$\frac{\partial}{\partial g_{i'j'}^{(k)}}(\partial^{\tau}G)(\xi)\neq 0.$$

Notice that

$$g_{j\ell}^{(k)}=-\sum_{t=1}^{n}b_{kj\ell}^{(t)}N_k^{(t)},$$

where $b_{kj\ell}^{(t)}$ are given by (6.52) and $N_k^{(t)}$ by (6.42). Thus, $G(\xi)$ is a polynomial of $b_{kj\ell}^{(t)}$ $(1\leq k\leq m,\ 1\leq j,\ell\leq n-1,\ 1\leq t\leq n)$. Therefore, if

$$\frac{\partial}{\partial g_{j\ell}^{(k)}}(\partial^{\tau}G)(\xi)\neq 0$$

for some (k, j, ℓ), then there exists $t = 1, \dots, n$ such that

$$\frac{\partial}{\partial b^{(t)}_{kj\ell}} (\partial^\tau G)(\xi) \neq 0. \tag{6.56}$$

Fix an arbitrary $(\tau, \tau') \in \mathcal{M}_2$ and denote by $\mathcal{O}_{\tau'}$ the set of those $\xi \in \varphi(D)$ such that $\partial^{\tau'} G(\xi) \neq 0$. Define the map

$$L : \mathcal{O}_{\tau'} \longrightarrow \mathbb{R}^{s(n-1)+1}$$

by

$$L(\xi) = ((c_{ij}(\xi))_{1 \le i \le m, 1 \le j \le n-1} \; ; \; \partial^\tau G(\xi)).$$

Clearly,

$$M(\tau, \tau') = L^{-1}(0) \subset \mathcal{O}_{\tau'}.$$

We are going to show that L is a submersion on $\mathcal{O}_{\tau'}$; this will imply that $M(\tau, \tau')$ is a smooth submanifold of $\varphi(D)$ of codimension $s(n-1)+1$.

Let $\xi \in \mathcal{O}_{\tau'}$ and assume that

$$\sum_{i=1}^{m} \sum_{j=1}^{n-1} C_{ij} \operatorname{grad}(c_{ij})(\xi) + A \operatorname{grad}(\partial^\tau G)(\xi) = 0 \tag{6.57}$$

for some constants C_{ij} and A. Since $\xi \in M(\tau, \tau')$, there exists (i_0, j_0, k_0) such that (6.55) holds with $i = i_0, j = j_0$ and $k = k_0$. It then follows from our above reasoning that there exist i, j and t such that (6.56) holds with $k = k_0$. Since the functions $c_{ij}(\xi)$ do not depend on the variables $b^{(t)}_{kj\ell}$, we get $A = 0$. Next, fix arbitrary i and j and consider the derivatives with respect to $a^{(t)}_{ij}$ in (6.57). Since $v \in U_m$, according to Lemma 6.2.2 and using the same idea as that in the proof of Lemma 6.1.2, we find $C_{ij} = 0$. Therefore, L is a submersion at ξ.

This shows that $M(\tau, \tau')$ is a smooth submanifold of $\varphi(D)$ of codimension $s(n-1)+1$. Applying the same argument with a slight modification, we also see that M_p has the same property for any $p = 0, 1, \dots, n-2$. This completes the proof of the lemma. ■

Proof of Theorem 6.4.1: Given a complex number μ and an integer $m \ge 2$, set

$$T'(\mu, m) = \{f \in \mathbf{C}(X) : j^2_m f(X^{(m)}) \cap \Sigma = \emptyset\}.$$

Then $T'(\mu, m)$ contains a residual subset of $\mathbf{C}(X)$. This follows easily from Lemma 6.4.2, applying the Multijet Transversality Theorem and an argument similar to that in the proofs of Theorems 6.1.1 and 6.3.1.

Let $f \in T'(\mu, m)$. If $x = (x_1, \dots, x_m) \in X^{(m)}$ is a critical point of the map $F_m \circ f^m$ with $f^m(x) \in U_m$, then we have $E(\sigma) \neq 0$, where σ is defined by (6.50) with

$f_1 = \cdots = f_m = f$. According to our considerations at the beginning of this section, this implies that for any ordinary primitive non-symmetric periodic reflecting ray γ for $f(X)$, we have $\mu \notin \mathrm{spec}(P_\gamma)$.

Set

$$\widetilde{\Lambda} = \{z \in \mathbb{C} : \exists k \in \mathbb{N} \text{ with } z^k \in \Lambda\}.$$

Since Λ is countable, $\widetilde{\Lambda}$ is also countable. Let $\mathcal{A}$ and $\mathcal{T}$ be the subsets of $\mathbf{C}(X)$ from Theorems 6.2.3 and 6.3.1, respectively. It follows from the considerations above and the theorem just cited that

$$T' = \cap_{\mu \in \tilde{\Lambda}, m \geq 2} T'(\mu, m)$$

contains a residual subset of $\mathbf{C}(X)$. Moreover, the properties of the sets $T'(\mu, m)$ mentioned earlier yield that for every $f \in T'$ and every ordinary non-symmetric periodic reflecting ray γ for $f(X)$, $\mathrm{spec}(P_\gamma)$ does not contain elements of Λ.

In a similar way one constructs $T'' \subset \mathbf{C}(X)$ containing a residual subset of $\mathbf{C}(X)$ such that for every $f \in T''$ and every ordinary symmetric periodic reflecting ray γ for $f(X)$ we have $\mathrm{spec}(P_\gamma) \cap \Lambda = \emptyset$. To do this one can repeat most of the considerations in this section with minor modifications. Let us mention that if $q_i = f(x_i)$, $i = 1, \ldots, m$, are the successive reflection points of a symmetric primitive periodic reflecting ray for $f(X)$ with $f \in \mathcal{A} \cap \mathcal{T}$, then we can always assume that the segment $[q_1, q_2]$ is orthogonal to $f(X)$ at q_1. Moreover, $f \in \mathcal{A} \cap \mathcal{T}$ implies that γ is ordinary and the points $q_1, \ldots, q_k$ with $k = (m-2)/2$ are all different. Finally, $x = (x_1, \ldots, x_k)$ is a critical point of the function $G_k \circ f^k$, where

$$G_k : U'_k \longrightarrow \mathbb{R}$$

is given by

$$G_k(y) = \sum_{i=1}^{k-1} \| y_i - y_{i+1} \|,$$

and $U'_k \subset (\mathbb{R}^n)^{(k)}$ is defined appropriately. After these remarks, one can define the function E in the same way as above and repeat the argument from the non-symmetric case. The necessary modifications are very easy and we leave them to the reader.

Since $T' \cap T'' \subset T_\Lambda$, it follows that T_Λ contains a residual subset of $\mathbf{C}(X)$. ■

Combining Theorems 6.2.3, 6.3.1 and 6.4.1, we deduce that the set

$$\mathcal{F} = \mathcal{A} \cap \mathcal{T} \cap T_{\mathbb{Q}/\mathbb{Z}}$$

contains a residual subset of $\mathbf{C}(X)$. Here

$$\mathbb{Q}/\mathbb{Z} = \{z \in \mathbb{C} : z^k = 1 \text{ for some } k \in \mathbb{N}\},$$

and, as before, X is a compact $(n-1)$-dimensional submanifold of $\mathbb{R}^n$, $n \geq 2$.

The next theorem shows that for generic domains Ω in $\mathbb{R}^n$ there exist at most countably many periodic reflecting rays in Ω.

Theorem 6.4.3: *For every $f \in \mathcal{F}$ and every integer $s \geq 2$ there exist only finitely many periodic reflecting rays for $f(X)$ with exactly s reflection points.*

Proof: Fix arbitrary $f \in \mathcal{F}$ and $s \geq 2$. Without loss of generality we may assume that $f = \mathrm{id}$; otherwise we will replace X by $f(X)$. Denote by K_s the set of all $x = (x_1, \dots, x_s) \in X^s$ such that $x_1, \dots, x_s$ are the successive reflection points of a periodic reflecting ray for X. We will prove that K_s is finite.

Assume that K_s is infinite. It then follows by the compactness of X that there exists a sequence

$$\{(x_{1,m}, \dots, x_{s,m})\}_{m=1}^{\infty}$$

of different elements of K_s such that $x_i = \lim_{m\to\infty} x_{i,m}$ exists for every $i = 1, \dots, s$. As before, we set for convenience $x_{s+1,m} = x_{1,m}$ and $x_{s+1} = x_1$.

Lemma 6.4.4: *There exist $i \neq j$ with $x_i \neq x_j$.*

Proof of Lemma 6.4.4: Set

$$e_{i,m} = \frac{x_{i+1,m} - x_{i,m}}{\| x_{i+1,m} - x_{i,m} \|}, \quad a_{i,m} = \frac{\| x_{i+1,m} - x_{i,m} \|}{\sum_{j=1}^{s} \| x_{j+1,m} - x_{j,m} \|}.$$

Then $\| e_{i,m} \| = 1$ and $\sum_{i=1}^{s} a_{i,m} = 1$. Without loss of generality we may assume that there exist

$$\lim_{m\to\infty} e_{i,m} = e_i, \quad \lim_{m\to\infty} a_{i,m} = a_i.$$

Then $\| e_i \| = 1$ for all $i = 1, \dots, s$ and $\sum_{i=1}^{s} a_i = 1$.

Assume that $x_1 = \dots = x_s$. Then using

$$\lim_{m\to\infty} x_{1,m} = \lim_{m\to\infty} x_{2,m} = \lim_{m\to\infty} x_{3,m},$$

we find that $e_2 = e_1$. In the same way we get $e_{i+1} = e_i$ for all i, so $e_1 = e_2 = \dots = e_s$. Then

$$\sum_{i=1}^{s} (x_{i+1,m} - x_{i,m}) = 0$$

implies $\sum_{i=1}^{s} a_{i,m} e_{i,m} = 0$. Letting $m \to \infty$, gives $\sum_{i=1}^{s} a_i e_i = 0$. However,

$$\sum_{i=1}^{s} a_i e_i = \left(\sum_{i=1}^{s} a_i \right) e_1 = e_1,$$

so we must have $e_1 = 0$, a contradiction. This proves the lemma. ■

We now continue with the proof of Theorem 6.4.3. According to the above lemma, without loss of generality we may assume that $x_1 \neq x_2$. There exists a uniquely determined sequence

$$i_1 = 1 < i_2 < \cdots < i_k \leq s, \quad i_{k+1} = s + 1$$

of integers such that for every $j = 2, \ldots, k$, i_j is the maximal integer $i > i_{j-1}$ so that the points

$$x_{i_{j-1}}, x_{i_{j-1}+1}, \ldots, x_i$$

are collinear. It then follows that the points

$$x_{i_1}, x_{i_2}, \ldots, x_{i_k}$$

are the successive reflection points of a periodic reflecting ray γ for X.

Lemma 6.4.5: *We have $k = s$ and $i_j = j$ for every $j = 1, \ldots, s$.*

Proof of Lemma 6.4.5: Suppose that $i_2 > 2$; then $i_2 \geq 3$.

Case 1. There exists i with $1 < i < i_2$ and $x_i \neq x_{i_2}$. In this case the segment $[x_1, x_{i_2}]$ of γ is tangent to X at the point x_i, which is a contradiction with $\mathrm{id} \in \mathcal{F} \subset \mathcal{T}$ (γ must be ordinary).

Case 2. $x_2 = x_3 = \cdots = x_{i_2}$. Denote by θ_m the measure of the angle between the vector $x_{3,m} - x_{2,m}$ and the tangent hyperplane to X at $x_{2,m}$. Since

$$\lim_{m\to\infty} x_{3,m} = \lim_{m\to\infty} x_{2,m} = x_2,$$

we have $\lim_{m\to\infty}\theta_m = 0$. On the other hand, θ_m coincides with the measure of the angle between the vector $x_{1,m} - x_{2,m}$ and the tangent hyperplane to X at $x_{2,m}$. Thus, the vector

$$x_1 - x_2 = \lim_{m\to\infty} (x_{1,m} - x_{2,m})$$

must be tangent to X at $x_2 = x_{i_2}$. The latter implies that x_{i_2} is contained in the segment $[x_1, x_{i_3}]$, which is a contradiction with the choice of i_2.

In this way we have shown that $i_2 = 2$. Proceeding in the same way, we prove that $i_3 = 3, \ldots, i_s = s$. In particular, $k = s$. ■

It follows from the last two lemmas that $x_i \neq x_{i+1}$ for all $i = 1, \ldots, s$. Moreover, it is clear that $x_1, \ldots, x_s$ are the successive reflection points of a periodic reflecting ray γ for X. Let us remark that in general some of the reflection points of γ could coincide even though $x_{1,m}, \ldots, x_{s,m}$ are different for all m. For example,

γ could be a symmetric periodic reflecting ray with $1 + s/2$ different reflection points.

Denote by γ_m the periodic reflecting ray for X with reflection points $x_{1,m}, \dots, x_{s,m}$. Set

$$\eta_i = \frac{x_{i+1} - x_i}{\|x_{i+1} - x_i\|}, \quad \eta_{i,m} = \frac{x_{i+1,m} - x_{i,m}}{\|x_{i+1,m} - x_{i,m}\|}.$$

For every i we can define the billiard ball map B in a neighbourhood of (x_i, η_i) in $X \times \mathbb{S}^{n-1}$ so that it takes values in a neighbourhood of (x_{i+1}, η_{i+1}). Then $B^s(x_1, \eta_1) = (x_1, \eta_1)$ and $B^s(x_{1,m}, \eta_{1,m}) = (x_{1,m}, \eta_{1,m})$ for every $m \geq 1$. On the other hand, it is easy to see that the linear map $L = dB^s(x_1, \eta_1)$ is conjugate to the linear Poincaré map P_γ. Now $\mathrm{id} \in \mathcal{F} \subset T_{\mathbb{Q}/\mathbb{Z}}$ implies that $1 \notin \mathrm{spec}(L)$. This is a contradiction with the fact that B^s has an infinite sequence of fixed points $(x_{1,m}, \eta_{1,m})$ approaching (x_1, η_1).

Thus, the set K_s is finite which concludes the proof of the theorem. ■

Next, we consider (ω, θ)-trajectories for X, where ω and θ are two fixed unit vectors in $\mathbb{R}^n$.

Theorem 6.4.6: *Let* $T(\omega, \theta)$ *be the set of those* $f \in \mathbf{C}(X)$ *such that every reflecting* (ω, θ)*-trajectory for* $f(X)$ *is ordinary and* $\det(dJ_\gamma) \neq 0$. *Then* $T(\omega, \theta)$ *contains a residual subset of* $\mathbf{C}(X)$.

Proof of Theorem 6.4.6: Fix an open ball U_0 containing X and set

$$Z_1 = Z_\omega, \quad Z_2 = Z_{-\theta}, \quad \pi_1 = \pi_\omega, \quad \pi_2 = \pi_{-\theta},$$

where Z_ξ and π_ξ are defined in Section 2.4. According to the remark before Theorem 6.2.5, it is sufficient to establish that

$$T(\omega, \theta) \cap \mathbf{C}(X, U_0)$$

contains a residual subset of $\mathbf{C}(X, U_0)$.

Let $f \in \mathcal{B} \cap \mathcal{T}(\omega, \theta) \cap \mathbf{C}(X, U_0)$, where $\mathcal{B}$ and $\mathcal{T}(\omega, \theta)$ are the sets from Theorems 6.2.6 and 6.3.3, respectively. Let γ be a reflecting (ω, θ)-trajectory for $f(X)$ with successive reflection points $q_1, \dots, q_k$. Set $u_\gamma = q_0 = \pi_1(q_1)$ and $q_{k+1} = \pi_2(q_k)$ and define Π_i, λ_i, σ_i, ψ_i, etc., as in Section 2.4. Then for $dJ_\gamma(u_\gamma)$, we have the representation (2.32).

Next, consider the function $E = \det(dJ_\gamma)(u_\gamma)$. Clearly, E is a polynomial of the elements $\tilde{d}_{ij}^{(t)}$ $(1 \leq i, j \leq n-1)$ of the matrices $\tilde{\psi}_t$, $t = 1, \dots, k$.

First, assume that γ is non-symmetric. Since $f \in \mathcal{B} \cap \mathcal{T}(\omega, \theta)$, γ is ordinary and the reflection points $q_1, \dots, q_k$ of γ are all different. We will now show that the coefficient in front of

$$\tilde{d}_{11}^{(1)} \tilde{d}_{22}^{(1)} \cdots \tilde{d}_{n-1n-1}^{(1)} \tag{6.58}$$

in E is non-zero. Let s_i be given by (2.24). Then we have

$$\begin{aligned}&\begin{pmatrix}\sigma_k & \lambda_k\sigma_k\\ 0 & \sigma_k\end{pmatrix}\cdots\begin{pmatrix}\sigma_2 & \lambda_2\sigma_2\\ 0 & \sigma_2\end{pmatrix}\begin{pmatrix}\sigma_1 & \lambda_1\sigma_1\\ \tilde{\psi}_1\sigma_1 & \sigma_1+\lambda_1\tilde{\psi}_1\sigma_1\end{pmatrix}\\ &= \begin{pmatrix}s_k & 0\\ 0 & s_k\end{pmatrix}\begin{pmatrix}I & \sum_{i=2}^k \lambda_i I\\ 0 & I\end{pmatrix}\begin{pmatrix}I & \lambda_1 I\\ \psi_1 & I+\lambda_1\psi_1\end{pmatrix}.\end{aligned}$$

Therefore

$$\mathrm{p}r_2\begin{pmatrix}s_k & 0\\ 0 & s_k\end{pmatrix}\begin{pmatrix}I & \sum_{i=2}^k \lambda_i I\\ 0 & I\end{pmatrix}\begin{pmatrix}I & \lambda_1 I\\ \psi_1 & I+\lambda_1\psi_1\end{pmatrix}\begin{pmatrix}u\\ 0\end{pmatrix} = s_k\psi_1(u),$$

where

$$\mathrm{p}r_2\begin{pmatrix}u\\ v\end{pmatrix} = v.$$

This clearly implies that the coefficient in front of the product (6.58) in E is 1. Thus, E is a non-trivial polynomial of the variables $\tilde{d}_{ij}^{(t)}$ with coefficients depending on λ_i, σ_i $(i = 1,\ldots,k)$.

Next, repeating most of the considerations in the proof of Theorem 6.4.1 and replacing F by the function F^* determined by (6.33) and (6.34), we prove that there exists a residual subset R' of $\mathbf{C}(X)$ with $R' \subset \mathcal{B}\cap\mathcal{T}(\omega,\theta)$ such that $f \in R'$ yields $dJ_\gamma(u_\gamma) \neq 0$ for any non-symmetric (ω,θ)-trajectory γ for $f(X)$.

To deal with the symmetric case, assume $\theta = -\omega$, and let again $f \in \mathcal{B}\cap\mathcal{T}(\omega,\theta)\cap\mathbf{C}(X,U_0)$. Let $q_1,\ldots,q_k$ be the successive reflection points of a symmetric (ω,θ)-trajectory γ for $f(X)$. Then γ is ordinary, $k = 2m+1$ and the points $q_1,\ldots,q_m$ are different. Clearly $q_{m+i+1} = q_{m-i+1}$ for $i = 0,1,\ldots,m$. We have

$$\begin{aligned}M &= \begin{pmatrix}\sigma_1 & \lambda_1\sigma_1\\ 0 & \sigma_1\end{pmatrix}\cdots\begin{pmatrix}\sigma_{m-1} & \lambda_{m-1}\sigma_{m-1}\\ 0 & \sigma_{m-1}\end{pmatrix}\begin{pmatrix}\sigma_m & \lambda_m\sigma_m\\ \tilde{\psi}_m\sigma_m & \sigma_m+\lambda_m\tilde{\psi}_m\sigma_m\end{pmatrix}\\ &\quad\times\begin{pmatrix}\sigma_{m-1} & \lambda_{m-1}\sigma_{m-1}\\ 0 & \sigma_{m-1}\end{pmatrix}\cdots\begin{pmatrix}\sigma_1 & \lambda_1\sigma_1\\ 0 & \sigma_1\end{pmatrix}\\ &= \begin{pmatrix}s_{m-1}^{-1} & 0\\ 0 & s_{m-1}^{-1}\end{pmatrix}\begin{pmatrix}I & \sum_{i=1}^{m-1}\lambda_i I\\ 0 & I\end{pmatrix}\begin{pmatrix}I & \lambda_m I\\ \psi_m & I+\lambda_m\psi_m\end{pmatrix}\\ &\quad\times\begin{pmatrix}s_m & 0\\ 0 & s_m\end{pmatrix}\begin{pmatrix}I & \sum_{i=1}^{m-1}\lambda_i I\\ 0 & I\end{pmatrix}.\end{aligned}$$

Therefore

$$\mathrm{p}r_2\left(M\begin{pmatrix}u\\ 0\end{pmatrix}\right) = s_m^{-1}\psi_m s_m(u),$$

which shows that the coefficient in front of the product $\tilde{d}_{11}^{(m)}\tilde{d}_{22}^{(m)}\cdots\tilde{d}_{n-1n-1}^{(m)}$ in E is non-zero. Thus, E is a non-trivial polynomial, and we can apply the argument from the proof of Theorem 6.4.1, replacing F by the map

$$G^* : U_m^* \longrightarrow \mathbb{R},$$

defined by

$$G^*(y) = \| y_1 - \pi_1(y_1) \| + \sum_{i=1}^{m-1} \| y_i - y_{i+1} \|,$$

where U_m^* is an appropriately defined open subset of $(\mathbb{R}^n)^{(m)}$. In this way we prove that there exists a residual subset R'' of $\mathbf{C}(X)$ with $R'' \subset \mathcal{B} \cap \mathcal{T}(\omega, \theta)$ such that $f \in R''$ implies $\det(dJ_\gamma)(u_\gamma) \neq 0$ for any symmetric (ω, θ)-trajectory γ for $f(X)$.

Finally, notice that $R' \cap R'' \subset T(\omega, \theta)$. Since each of the sets R' and R'' contains a residual subset of $\mathbf{C}(X)$, the same is true for $T(\omega, \theta)$. This completes the proof of the theorem. ■

Consider the subset

$$\mathcal{F}(\omega, \theta) = \mathcal{B} \cap \mathcal{T}(\omega, \theta) \cap T(\omega, \theta)$$

of $\mathbf{C}(X)$, where $\mathcal{B}$ and $\mathcal{T}(\omega, \theta)$ are the sets from Theorems 6.2.6 and 6.3.3, respectively. It follows from Theorems 6.2.6, 6.3.3 and 6.4.6 that $\mathcal{F}(\omega, \theta)$ contains a residual subset of $\mathbf{C}(X)$. Applying the argument from the proof of Theorem 6.4.3, we get the following.

Corollary 6.4.7: *For every $f \in \mathcal{F}(\omega, \theta)$ and every integer $m \geq 1$ there exist only finitely many (ω, θ)-trajectories for $f(X)$ with m reflection points.*

6.5 Notes

Theorem 6.1.1 was proved in a slightly different form in [PS2] (see also [PSl]). The material in Section 6.2 is a modification of parts of [PS2], [S1] and [CPS], while that of Section 6.3 is taken from [PS4]. The case $n = 2$ of Theorem 6.4.1 was proved in [PS2] (see also [PSl]). In its present form this theorem as well as Theorem 6.4.6 was established in [PS3]. Finally, Theorem 6.4.3 and Corollary 6.4.7 are taken from [PS4].

Generic properties of reflecting rays were first considered by Lazutkin [L2], [Ll] who proved an analogue of the Kupka–Smale theorem for billiards in strictly convex planar domains. It should be noted that Theorem 6.4.1 can be considered as a first part of a Kupka–Smale theorem for general billiards.

7

Bumpy surfaces

In this chapter we study some particular properties of generic compact submanifolds M of $\mathbb{R}^n$ with $1 \leq \dim(M) < n$. These concern the behaviour of the geodesic flow on M determined by the standard Riemannian metric on M inherited from the Euclidean structure of $\mathbb{R}^n$. Combined with the generic properties concerning periodic reflecting rays, established in Chapter 6, these properties will be essentially used when we study the Poisson relation for generic domains in $\mathbb{R}^n$ in Chapter 8.

Our aim in the present chapter is to prove the existence of a residual set of smooth embeddings F of M into $\mathbb{R}^n$ such that the standard Riemannian metric on $M' = F(M)$ is a *bumpy metric*, that is all closed geodesics on M' are non-degenerate. As a consequence we obtain the classical Bumpy Metric Theorem of Abraham–Klingenberg–Takens–Anosov: for every compact smooth manifold M there exists a residual set in the space of all smooth Riemannian metrics on M consisting of bumpy metrics.

7.1 Poincaré maps for closed geodesics

In this section we recall some standard facts from the theory of ordinary differential equations which will be used in subsequent sections.

Let $\Delta = [0, a] \subset \mathbb{R}$ for some $a > 0$, and let

$$X = (X^{(1)}, \dots, X^{(k)}) : \Delta \times U \longrightarrow \mathbb{R}^k$$

Geometry of the Generalized Geodesic Flow and Inverse Spectral Problems, Second Edition.
Vesselin M. Petkov and Luchezar N. Stoyanov.

be a C^1 map, where U is an open neighbourhood of 0 in $\mathbb{R}^k$. For $u \in U$ close to 0 let $x(t;u)$ be the solution to the differential equation

$$\dot{x}(t;u) = X(t;x(t;u)). \tag{7.1}$$

Here $\dot{x}$ denotes the derivative with respect to t. We will assume that $x(t;u)$ exists for all $t \in \Delta$ whenever u belongs to a small neighbourhood V of 0 in $\mathbb{R}^k$ with $V \subset U$. Define the map

$$\mathcal{P}_t : V \longrightarrow \mathbb{R}^k$$

by $\mathcal{P}_t(u) = x(t,u)$.

The proof of the following proposition can be found in standard texts on ordinary differential equations (see e.g. [Pon]).

Proposition 7.1.1: *Under the above assumptions, the map $\mathcal{P}_t$ is differentiable at 0, and for any $t \in \Delta$ the matrix $P_t = d\mathcal{P}_t(0)$ is a solution to the problem*

$$\begin{cases} \dot{P}_t = d_x X(t;x(t;0)) \cdot P_t, \quad t \in \Delta, \\ P_0 = I, \end{cases} \tag{7.2}$$

where

$$d_x X = \begin{pmatrix} \dfrac{\partial X^{(1)}}{\partial x_1} & \cdots & \dfrac{\partial X^{(1)}}{\partial x_k} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial X^{(k)}}{\partial x_1} & \cdots & \dfrac{\partial X^{(k)}}{\partial x_k} \end{pmatrix},$$

and I is the identity matrix.

Corollary 7.1.2: *Under the assumptions of Proposition 7.1.1, assume also that $Y : \Delta \times U \longrightarrow \mathbb{R}^k$ is another C^1 map. Consider the differential equation*

$$\dot{y}(t;u) = Y(t;y(t;u)), \tag{7.3}$$

and assume that $x(t;0)$, $t \in \Delta$, is a solution to both (7.1) and (7.3), and

$$\frac{\partial X}{\partial x_i}(t;x(t;0)) = \frac{\partial Y}{\partial x_i}(t;x(t;0))$$

for all $i = 1, \ldots, k$ and all $t \in \Delta$. Define the map $\mathcal{Q}_t : V \longrightarrow \mathbb{R}^k$ (possibly with a smaller neighbourhood V of 0 in $\mathbb{R}^k$) by $\mathcal{Q}_t(u) = y(t;u)$. Then $d\mathcal{P}_t(0) = d\mathcal{Q}_t(0)$ for all $t \in \Delta$.

Proof: Since $d_x X(t;x(t;0)) = d_x Y(t;x(t;0))$ for all $t \in \Delta$, it follows that $d\mathcal{P}_t(0)$ and $d\mathcal{Q}_t(0)$ are both solutions to (7.2). Thus, they must coincide. ∎

Let M be a smooth manifold and let

$$\pi : T^*M \longrightarrow M$$

be the *natural projection* on the *cotangent bundle* T^*M of M. Let $m = \dim(M) - 1$, then

$$\dim(M) = m + 1.$$

We assume that $m \geq 1$.

Let ω be the *canonical symplectic form* on T^*M (see e.g. [AbM]), and let g be a fixed smooth Riemannian metric on M. Define the *energy function*

$$H = H_g : T^*M \longrightarrow \mathbb{R}$$

by

$$H(q) = \frac{1}{2}\langle q, q\rangle_g,$$

where $\langle \cdot, \cdot \rangle$ is the *inner product* on T^*M determined by the Riemannian metric g. The *Hamiltonian vector field* determined by H (and so by g) is the unique vector field $X = X_g$ on T^*M such that

$$\omega(X, Y) = dH \cdot Y$$

for any smooth vector field Y on T^*M. The flow on T^*M determined by X is called the *geodesic flow*. A curve c on T^*M is an integral curve of X if and only if the curve $\pi \circ c$ in M is a geodesic with respect to the metric g. We refer the reader to Chapter 3 in [AbM] for the basic facts concerning Hamiltonian dynamics.

Next, consider a closed integral curve

$$c : [0, \theta] \longrightarrow T^*M, \quad \theta > 0, \tag{7.4}$$

of X, that is a curve with $c(0) = c(\theta)$ and $\dot{c}(t) = X(c(t))$ for all $t \in [0, \theta]$. Then θ is called the *period* of c. If $c(t) \neq c(0)$ for all $t \in (0, \theta)$, we will say that θ is the *minimal period* of c and that c is *primitive*. Similar terminology will be used for the closed geodesic

$$\gamma : [0, \theta] \longrightarrow M, \quad \gamma = \pi \circ c, \tag{7.5}$$

on (M, g).

To define the Poincaré map of c, set $q = c(0)$, $p = \pi(q)$ and consider a smooth m-dimensional submanifold Σ^* of M containing p such that $\dot{c}(0) = X(q)$is transversal to the $(2m + 1)$-dimensional submanifold $\Sigma = \pi^{-1}(\Sigma^*)$ of T^*M at q. For $q' \in \Sigma$ close to q the integral curve of X passing through q' at $t = 0$, after time t close to θ, intersects Σ at some point $q'' \in \Sigma$. In this way we obtain a map

$$\mathcal{P} = \mathcal{P}^{(g)} : \Sigma \ni q' \mapsto q'' \in \Sigma,$$

defined in a small neighbourhood of q in Σ. This map leaves the $2m$-dimensional submanifold

$$\widetilde{\Sigma} = \{q' \in \Sigma : H(q') = H(q)\}$$

invariant and preserves the natural symplectic form on $\widetilde{\Sigma}$ induced by ω. In this way $\mathcal{P}$ induces a local symplectic diffeomorphism

$$\mathcal{P} : (\widetilde{\Sigma}, q) \longrightarrow (\widetilde{\Sigma}, q).$$

The linear map

$$P = P^{(g)} = d\mathcal{P}^{(g)}(q) : T_q\widetilde{\Sigma} \longrightarrow T_q\widetilde{\Sigma}$$

is called the linear *Poincaré map* of the integral curve c (or the closed geodesic γ). The reader may refer Chapters 7 and 8 in [AbM] for more details concerning the definition of the Poincaré map, as well as the proof of the fact that, up to conjugacy, it does not depend on the choice of the initial point q and the submanifold Σ^*. The latter shows that the *spectrum* $\mathrm{spec}(P)$ of P does not depend on the choice of q and Σ^*. We will say that c (resp. γ) is *non-degenerate* as an integral curve (resp. closed geodesic) of period θ if $1 \notin \mathrm{spec}(P)$. If all closed geodesics on (M, g) are non-degenerate, then g is called a *bumpy metric* on M.

In what follows we assume that g is a fixed smooth Riemannian metric on M and (7.1) is a fixed closed integral curve of X with **minimal period** $\theta > 0$. Define γ by (7.2). There exist *Fermi coordinates* in a neighbourhood of $\Im(\gamma)$ in M (see e.g. Section 1.12 in [Kl]). This means that there exist an open neighbourhood U of $\Im(\gamma)$ in M and a local diffeomorphism

$$r : V = (-\alpha, \theta + \alpha) \times B_\alpha(0) \longrightarrow U, \tag{7.6}$$

for some constant $\alpha > 0$, where

$$B_\alpha(0) = \{x \in \mathbb{R}^m : \|x\| < \alpha\},$$

such that the following conditions are satisfied:

(i) $\gamma(t) = r(t, 0, \dots, 0)$ for every $t \in [0, \theta]$;

(ii) the 1-jet of g_{00} coincides with the 1-jet of the constant 1 at all points of

$$\gamma_0 = \{(t, 0, \dots, 0) \in \mathbb{R}^{m+1} : 0 \le t \le \theta\};$$

(iii) $g_{0i} = 0$ on γ_0 for all $i = 1, \dots, m$.

Here g_{ij} $(i, j = 0, 1, \dots, m)$ are the components of the metric g with respect to the coordinates $x_0, x_1, \dots, x_m$ provided by r. To be more precise, $x_i : U \longrightarrow \mathbb{R}$ are smooth functions such that

$$r(x_0(\xi), x_1(\xi), \dots, x_m(\xi)) = \xi, \quad \xi \in U.$$

Since r is only a local diffeomorphism, x_i provide coordinates only locally, that is every point in U has an open neighbourhood $W \subset U$ such that the restriction of r to $r^{-1}(W)$ is a diffeomorphism between $r^{-1}(W)$ and W. In what follows this is already sufficient in order to treat the x_i's in the same way as if they were coordinates on the whole of U.

Let $y_0, y_1, \ldots, y_m$ be the coordinates dual to $x_0, x_1, \ldots, x_m$; then

$$x_0, x_1, \ldots, x_m;\ y_0, \quad y_1, \ldots, y_m$$

are (local) coordinates in T^*M in a neighbourhood of $\Im(c)$. With respect to these coordinates we have

$$\omega = \sum_{i=0}^{m} dx_i \wedge dy_i$$

and

$$H(x_0, x_1, \ldots, x_m; y_0, y_1, \ldots, y_m) = \frac{1}{2} \sum_{i,j=0}^{m} g_{ij}(x_0, \ldots, x_m)\, y_i y_j$$

(see e.g. Chapter 3 in [AbM]).

For convenience we will use the following abbreviations:

$$x = (x_0; x'), \quad x' = (x_1, \ldots, x_m), \quad y = (y_0; y'), \quad y' = (y_1, \ldots, y_m).$$

For $0 \leq t \leq \theta$ define

$$\Sigma(t) = \{(x; y) : x_0 = t\}, \quad \widetilde{\Sigma}(t) = \{(x; y) : x_0 = t, y_0 = 1\}.$$

Given $t \geq 0$, let

$$\mathcal{P}_t : \Sigma(0) \longrightarrow \Sigma(t)$$

be the map defined in a small neighbourhood of $q = c(0)$ which assigns to each $q' \in \Sigma(0)$ the first intersection point of the positive integral curve of X through q' with $\Sigma(t)$.

Next, consider a perturbation

$$\tilde{g} = g + g'$$

of g with a small g' which is another smooth Riemannian metric on M and g' satisfies the following conditions:

(ii′) the 1-jet of g'_{00} is zero on γ_0;

(iii′) $g'_{0i} = 0$ on γ_0 for all $i = 1, \ldots, m$.

Then $\widetilde{X} = X_{\tilde{g}}$ can be written in the form $\widetilde{X} = X + X'$, where X' is the Hamiltonian vector field on T^*M (defined only locally near $\Im(c)$) determined by the Hamiltonian function

$$H'(x;y) = \frac{1}{2} \sum_{i,j=0}^{m} g'_{ij}(x)\, y_i y_j.$$

Notice that $c(t)$ is an integral curve for not only X but $\widetilde{X}$ as well, that is $\gamma(t)$ is a geodesic on $(M, \tilde{g})$. This follows immediately from the conditions (ii′) and (iii′), writing down the corresponding Hamiltonian system of differential equations for an integral curve of X.

Define the maps

$$\mathcal{P}'_t : \Sigma(0) \longrightarrow \Sigma(t)$$

in the same way as $\mathcal{P}_t$ using the vector field $\widetilde{X}$ instead of X.

In a neighbourhood of $\Im(c)$ the vector fields X and X' have the form

$$X = \sum_{i=0}^{m} \left(\frac{\partial H}{\partial y_i} \cdot \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i} \cdot \frac{\partial}{\partial y_i} \right),$$

$$X' = \sum_{i=0}^{m} \left(\frac{\partial H'}{\partial y_i} \cdot \frac{\partial}{\partial x_i} - \frac{\partial H'}{\partial x_i} \cdot \frac{\partial}{\partial y_i} \right).$$

Now define the time-dependent vector fields X_t and X'_t on $\Sigma(0)$ by

$$X_t(\zeta) = \sum_{i=1}^{m} \left(\frac{\partial H}{\partial y_i}(t;\zeta) \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}(t;\zeta) \frac{\partial}{\partial y_i} \right),$$

$$X' = \sum_{i=1}^{m} \left(\frac{\partial H'}{\partial y_i}(t;\zeta) \frac{\partial}{\partial x_i} - \frac{\partial H'}{\partial x_i}(t;\zeta) \frac{\partial}{\partial y_i} \right)$$

for $t \in [0, \theta]$ and

$$\zeta = (x_1, \ldots, x_m; y_0, y_1, \ldots, y_m) \in \mathbb{R}^{2m+1}$$

close to 0. The corresponding local diffeomorphisms

$$\widetilde{\mathcal{P}}_t, \tilde{\mathcal{P}}'_t : (\Sigma(0), c(0)) \longrightarrow (\Sigma(t), c(t))$$

are determined in the same way as $\mathcal{P}_t$ and $\mathcal{P}'_t$, replacing X by X_t and X' by X'_t. More precisely, given $\zeta \in \Sigma(0)$ close to $c(0)$, $\widetilde{\mathcal{P}}_t(\zeta)$ is the first intersection point of the positive integral curve of the time-dependent vector field X_t starting at ζ with $\Sigma(t)$. The definition of $\tilde{\mathcal{P}}'_t$ is similar.

Lemma 7.1.3: *For every $t \in [0, \theta]$ we have*

$$d\widetilde{\mathcal{P}}_t(0) = d\mathcal{P}_t(0), \quad d\tilde{\mathcal{P}}'_t(0) = d\mathcal{P}_t(0).$$

Proof: We will prove the first of the above equalities; the proof of the second is very similar.

Notice that the zero component of X corresponding to the coordinate x_0 has the form

$$X^{(0)}(x;y) = \frac{\partial H}{\partial y_0}(x;y) = \sum_{i=0}^{m} g_{i0}(x) y_i.$$

Therefore,

$$\frac{\partial X^{(0)}}{\partial x_k}(c(t)) = \frac{g_{00}}{\partial x_k}(t;0) = 0$$

and

$$\frac{\partial X^{(0)}}{\partial y_k}(c(t)) = g_{k0}(t;0) = 0$$

for every $k = 0, 1, \ldots, m$. Here we used the fact that the metric g satisfies the conditions (i), (ii) and (iii).

Similarly, for the $(m+1)$st component $X^{(m+1)}$ of X corresponding to the coordinate y_0, we find

$$X^{(m+1)}(x;y) = -\frac{\partial H}{\partial x_0}(x;y) = -\frac{1}{2}\sum_{i,j=0}^{m} \frac{\partial g_{ij}(x)}{\partial x_0}\, y_i y_j.$$

Therefore,

$$\frac{\partial X^{(m+1)}}{\partial x_k}(c(t)) = -\frac{1}{2}\frac{\partial^2 g_{00}}{\partial x_k \partial x_0}(t;0) = 0,$$

and

$$\frac{\partial X^{(m+1)}}{\partial y_k}(c(t)) = -\frac{\partial g_{k0}}{\partial x_0}(t;0) = 0$$

for every $k = 0, 1, \ldots, m$.

Now the assertion follows from Corollary 7.1.2. ■

The definition of X_t implies that for every integral curve $\xi(t)$ of X_t we have $y_0(t) = \text{const}$. The same is true for the integral curves of X'_t, so

$$\widetilde{\mathcal{P}}_t(\widetilde{\Sigma}(0)) \subset \widetilde{\Sigma}(t), \quad \widetilde{\mathcal{P}}'_t(\widetilde{\Sigma}(0)) \subset \widetilde{\Sigma}(t) \tag{7.7}$$

for every $t \in [0, \theta]$. Notice that similar inclusions are in general not satisfied for $\mathcal{P}_t$ and $\mathcal{P}'_t$. However, according to Lemma 7.1.3 and (7.7), we have

$$d\mathcal{P}_t(c(0))(T_{c(0)}\widetilde{\Sigma}(0)) \subset T_{c(t)}\widetilde{\Sigma}(t), \quad d\mathcal{P}'_t(c(0))(T_{c(0)}\widetilde{\Sigma}(0)) \subset T_{c(t)}\widetilde{\Sigma}(t)$$

for every $t \in [0, \theta]$.

Let P_t and P'_t be the *matrices* of the restrictions of the linear maps $d\mathcal{P}_t(c(0))$ and $d\mathcal{P}'_t(c(0))$, respectively, on $T_{c(0)}\widetilde{\Sigma}(0)$. Here using the coordinates $x_1, \dots, x_m$, $y_1, \dots, y_m$, we identify $\widetilde{\Sigma}(t)$ for all t with an open neighbourhood of 0 in $\mathbb{R}^{2m}$. Correspondingly, the tangent spaces $T_{c(t)}\widetilde{\Sigma}(t)$ are identified with $\mathbb{R}^{2m} \times \mathbb{R}^{2m}$, so P_t and P'_t are $2m \times 2m$ symplectic matrices smoothly depending on t. In particular,

$$R_t = P_t^{-1} P'_t \tag{7.8}$$

is also a symplectic matrix smoothly depending on t.

We are going to show that the matrix function R_t is the solution of a certain matrix differential equation. To do this we need the following simple fact.

Lemma 7.1.4: *Let Δ be an interval in $\mathbb{R}$ and let Q_t, Q'_t, Y_t, Y'_t ($t \in \Delta$) be $k \times k$ real matrices, differentiable with respect to t in Δ, such that Q_t is invertible for all $t \in \Delta$. If*

$$\dot{Q}_t = Y_t Q_t, \quad \dot{Q}'_t = (Y_t + Y'_t) Q'_t, \quad t \in \Delta,$$

then for $S_t = Q_t^{-1} Q'_t$ we have

$$\dot{S}_t = (Q_t^{-1} Y'_t Q_t) S_t, \quad t \in \Delta.$$

Proof: It follows from $Q_t^{-1} Q_t = I$ that

$$(Q_t^{-1})^{\bullet} Q_t + Q_t^{-1} \dot{Q}_t = 0,$$

and therefore

$$(Q_t^{-1})^{\bullet} = -Q_t^{-1} \dot{Q}_t Q_t^{-1}$$

for all $t \in \Delta$. Then

$$\begin{aligned} \dot{S}_t &= (Q_t^{-1})^{\bullet} Q'_t + Q_t^{-1} \dot{Q}'_t = -Q_t^{-1} \dot{Q}_t Q_t^{-1} Q'_t + Q_t^{-1} (Y_t + Y'_t) Q'_t \\ &= -Q_t^{-1} Y_t Q'_t + Q_t^{-1} Y_t Q'_t + (Q_t^{-1} Y'_t Q_t) Q_t^{-1} Q'_t = (Q_t^{-1} Y'_t Q_t) S_t \end{aligned}$$

for every $t \in \Delta$. ■

Next, consider $\widetilde{X}_t(\xi)$ as a time-dependent vector field defined for

$$\xi = (x'; y') = (x_1, \dots, x_m; y_1, \dots, y_m) \in \widetilde{\Sigma}(0).$$

Then

$$d_\xi \widetilde{X}_t(\xi) = J \begin{pmatrix} d^2_{x'x'} H(t;\xi) & d^2_{x'y'} H(t;\xi) \\ d^2_{x'y'} H(t;\xi) & d^2_{y'y'} H(t;\xi) \end{pmatrix},$$

where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

is the *canonical* $2m \times 2m$ *symplectic matrix*, and

$$d^2_{x'x'} H(t;\xi) = \left(\frac{\partial^2 H}{\partial x_i \partial x_j}(t;\xi) \right)_{i,j=1}^{m}, \quad d^2_{x'y'} H(t;\xi) = \left(\frac{\partial^2 H}{\partial x_i \partial y_j}(t;\xi) \right)_{i,j=1}^{m},$$

etc. In the same way one obtains

$$d_\xi \widetilde{X}'_t(\xi) = J \begin{pmatrix} d^2_{x'x'} H'(t;\xi) & d^2_{x'y'} H'(t;\xi) \\ d^2_{x'y'} H'(t;\xi) & d^2_{y'y'} H'(t;\xi) \end{pmatrix}. \tag{7.9}$$

Now applying Proposition 7.1.1 and Lemma 7.1.3, we deduce that P_t and P'_t are solutions of the following problems:

$$\begin{cases} \dot{P}_t = d_\xi \widetilde{X}_t(0) P_t, & t \in [0,\theta], \\ P_0 = I, \end{cases}$$

$$\begin{cases} \dot{P}'_t = d_\xi (\widetilde{X}_t(0) + d_\xi \widetilde{X}'_t(0)) P_t, & t \in [0,\theta], \\ P'_0 = I. \end{cases}$$

Now it follows from Lemma 7.1.4 that the matrix function R_t determined by (7.8) is a solution of the problem

$$\begin{cases} \dot{R}_t = (P_t^{-1} d_\xi \widetilde{X}'_t(0) P_t) R_t, & t \in [0,\theta], \\ R_0 = I. \end{cases} \tag{7.10}$$

Next, observe that

$$\frac{\partial H'}{\partial x_i} = \frac{1}{2} \sum_{j,k=0}^{m} \frac{\partial g'_{kj}}{\partial x_i}(x)\, y_k y_j, \quad \frac{\partial H'}{\partial y_i} = \sum_{j=0}^{m} g'_{ij}(x)\, y_j.$$

Since $x_0 = t$, $y_0 = 1$ and $x' = y' = 0$ on $c(t)$, we get

$$\frac{\partial^2 H'}{\partial x_i \partial x_j}(c(t)) = \frac{1}{2} \frac{\partial^2 g'_{00}}{\partial x_i \partial x_j}(t;0), \quad \frac{\partial^2 H'}{\partial x_i \partial y_j}(c(t)) = \frac{\partial g'_{0j}}{\partial x_i}(t;0),$$
$$\frac{\partial^2 H'}{\partial y_i \partial y_j}(c(t)) = g'_{ij}(t;0).$$

Consider the *homogeneous polynomial*

$$\widetilde{H}_t(x';y') = \sum_{i,j=1}^{m} \left(\frac{1}{2} a_{ij}(t) x_i x_j + b_{ij}(t) x_i y_j + \frac{1}{2} c_{ij}(t) y_i y_j \right), \tag{7.11}$$

where

$$a_{ij}(t) = \frac{\partial^2 g'_{00}}{\partial x_i x_j}(t;0), \quad b_{ij}(t) = \frac{\partial g'_{0j}}{\partial x_i}(t;0), \quad c_{ij}(t) = \frac{1}{2} g'_{ij}(t;0). \tag{7.12}$$

Clearly, $\widetilde{H}_t(x';y')$ is the sum of those terms in the Taylor series of the function $H'_{|\widetilde{\Sigma}(t)}$ that involve second derivatives.

It follows from (7.9) and the expression for the second derivatives of H' along $c(t)$ that $d_\xi \widetilde{X}'_t(0) = J \cdot D(t)$, where

$$D(t) = \begin{pmatrix} A(t) & B(t) \\ B(t)^T & C(t) \end{pmatrix}, \tag{7.13}$$

and

$$A(t) = (a_{ij}(t)), \quad B(t) = (b_{ij}(t)), \quad C(t) = (c_{ij}(t)). \tag{7.14}$$

Clearly, $J \cdot D(t)$ belongs to the *Lie algebra* $\mathrm{sp}(2m)$ of the *symplectic Lie group* Sp$(2m)$ for all t (see e.g. [AbM]). Finally, from (7.10) and the last expression for $d_\xi \widetilde{X}'_t(0)$, we derive the following.

Proposition 7.1.5: *The matrix function $R_t = P_t^{-1} P_t$ is a solution of the problem*

$$\begin{cases} \dot{R}_t = P_t^{-1} J D(t) P_t R_t, & t \in [0, \theta], \\ R_0 = I, \end{cases} \tag{7.15}$$

where $D(t)$ is given by (7.13), (7.14) and (7.12).

7.2 Local perturbations of smooth surfaces

Let M be a smooth submanifold of $\mathbb{R}^n$, $n \geq 3$. As before, set

$$\dim(M) = m + 1$$

and assume that $1 \leq m \leq n-2$, that is $\dim(M) \leq n-1$. On M we consider the *standard Riemannian metric* g inherited from the Euclidean structure of $\mathbb{R}^n$. Set $H = H_g$ and $X = X_g$ (cf. the notation in Section 7.1). Given $F \in \mathbf{C}(M)$, denote by g_F the Riemannian metric on M such that

$$F : (M, g_F) \longrightarrow (F(M), g)$$

is an isometry.

In what follows we assume that (7.4) is a fixed integral curve of X with minimal period $\theta > 0$ and (7.5) is a corresponding closed geodesic on M.

Theorem 7.2.1: *Let Λ be an arbitrary countable set of complex numbers. Under the above assumptions, there exists $t_0 \in (0, \theta)$ such that for every neighbourhood $\mathcal{U}$ of 0* *in* $C^\infty(M,\mathbb{R}^n)$ *and every neighbourhood* $\mathcal{W}$ *of* $\gamma(t_0)$ *in* M *there exists* $f \in \mathcal{U}$ *with* $\operatorname{supp}(f) \subset \mathcal{W}$ *such that* $F = \mathrm{id} + f \in \mathbf{C}(M)$, γ *is a geodesic on* (M, g_F) *and the spectrum of the Poincaré map related to* γ *with respect to the matrix* g_F *does not contain elements of the set* Λ.

The proof of this theorem is rather lengthy and is broken into several lemmas.

As in Section 7.1, consider a local diffeomorphism (7.6) satisfying the conditions (i), (ii) and (iii). We will use again the coordinates

$$x_0, x_1, \ldots, x_m;\ y_0, y_1, \ldots, y_m$$

in a neighbourhood of $\Im(c)$ in T^*M. Notice that the components g_{ij} of the metric g have the form

$$g_{ij}(x) = \left\langle \frac{\partial r}{\partial x_i}(x), \quad \frac{\partial r}{\partial x_j}(x) \right\rangle,\ x \in V,\ i, j = 0, 1, \ldots, m,$$

where r is considered as a map $r : V \longrightarrow \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$.

Given an embedding $F \in \mathbf{C}(M)$, we can write it in the form $F = \mathrm{id} + f$ for some $f \in C^\infty(M, \mathbb{R}^n)$. The corresponding perturbed metric on M has the form

$$\tilde{g} = g_F = g + g'$$

for some smooth two-form g'. Clearly, in the special case under consideration, we have

$$g'_{ij}(x) = \left\langle \frac{\partial r}{\partial x_i}(x), \frac{\partial (f \circ r)}{\partial x_i}(x) \right\rangle + \left\langle \frac{\partial r}{\partial x_j}(x), \frac{\partial (f \circ r)}{\partial x_i}(x) \right\rangle + \left\langle \frac{\partial (f \circ r)}{\partial x_i}(x), \frac{\partial (f \circ r)}{\partial x_j}(x) \right\rangle \tag{7.16}$$

for all $x \in V$ and all $i, j = 0, 1, \ldots, m$.

Next, we will use the symplectic matrices P_t, P'_t and R_t and the homogeneous polynomial $\widetilde{H}_t$ from Section 7.1. We will only consider smooth two-forms g' with supports in a small neighbourhood of $\gamma(t)$ for some $t \in (0, \theta)$, so that the matrix functions $D(t)$, defined by (7.12)–(7.14), have compact supports in $(0, \theta)$.

Now the problem is to find a perturbation $F = \mathrm{id} + f$ with a small f such that if R_t is the solution of the problem (7.15) for the corresponding matrix function $D(t)$, then the spectrum of the matrix $P'_\theta = P_\theta R_\theta$ does not contain elements of the set Λ.

Since γ is a closed curve, the vector $\partial r/\partial x_0$ is not constant along γ_0, so there exists $t_0 \in (0, \theta)$ such that $(\partial^2 r/\partial^2 x_0)(t; 0) \neq 0$. Fix t_0 with this property and an arbitrary small neighbourhood $\mathcal{W}$ of $\gamma(t_0)$ in M. There exist real numbers a and b with

$$\begin{cases} 0 < a < t_0 < b < \theta_0, \\ \dfrac{\partial^2 r}{\partial^2 x_0}(t; 0) \neq 0 \quad \forall t \in [a, b], \\ r(t; 0) \in \mathcal{W} \quad \forall t \in [a, b]. \end{cases} \tag{7.17}$$

Choose an arbitrary β with

$$0 < \beta < \min\{a, \theta - b, \alpha\},$$

and consider an arbitrary smooth function

$$\rho : \mathbb{R}^{m+1} \longrightarrow [0,1]$$

such that

$$\mathrm{supp}\,(\rho) \subset (0,\theta) \times (-\beta,\beta)^m$$

and $\rho(x) = 1$ for all $x \in [a,b] \times [-\beta/2, \beta/2]^m$.

In what follows the numbers t_0, a, b, β and the function ρ with the above properties will be fixed.

Define the map $h : V \longrightarrow \mathbb{R}^n$ by

$$h(x) = \frac{1}{2} \sum_{i,j=1}^{m} v_{ij}(x_0) x_i x_j, \tag{7.18}$$

where $v_{ij} : [0,\theta] \longrightarrow \mathbb{R}^n$ are smooth maps with

$$v_{ji} = v_{ij}, \quad \mathrm{supp}\,(v_{ij}) \subset [a,b] \quad (i,j = 1,\dots,m) \tag{7.19}$$

which will be constructed later. Set

$$f(\xi) = \begin{cases} \rho(r^{-1}(\xi)) h(r^{-1}(\xi)), & \xi \in U, \\ 0, & \xi \in M \setminus U. \end{cases} \tag{7.20}$$

Then $f : M \longrightarrow \mathbb{R}^n$ is a smooth map with

$$\mathrm{supp}\,(f) \subset r([a,b] \times [-\beta,\beta]^m).$$

Clearly, $\mathrm{supp}\,(f) \subset \mathcal{W}$, provided β is chosen sufficiently small. Moreover, if the maps $v_{ij} \in C^\infty([0,\theta], \mathbb{R}^n)$ are sufficiently close to 0 in the C^∞ topology, then $f \in \mathcal{U}$ and $F = \mathrm{id} + f \in \mathbf{C}(M)$.

Set $g' = g - g_F$ as above. Notice that for x close to γ_0 we have $f(r(x)) = h(x)$, so (7.16) implies

$$g'_{ij}(x) = \left\langle \frac{\partial r}{\partial x_i}(x), \frac{\partial h}{\partial x_j}(x) \right\rangle + \left\langle \frac{\partial r}{\partial x_j}(x), \frac{\partial h}{\partial x_i}(x) \right\rangle$$
$$+ \left\langle \frac{\partial h}{\partial x_i}(x), \frac{\partial h}{\partial x_j}(x) \right\rangle. \tag{7.21}$$

Using the form (7.18) of h, by a direct calculation one checks that g' satisfies the conditions (ii′) and (iii′) in Section 7.1; therefore, γ is a geodesic in (M, g_F).

We will now show that choosing the maps v_{ij} with (7.19) in a special way, we obtain all perturbations g' of g of a certain kind.

Lemma 7.2.2: *Let* $a_{ij}, b_{ij} : [0,\theta] \longrightarrow \mathbb{R}^n$ *be smooth maps such that*

$$\begin{cases} a_{ij} = a_{ji}, \quad b_{ij} = b_{ji}, \\ \text{supp}\,(a_{ij}) \subset [a,b], \quad \text{supp}\,(b_{ij}) \subset [a,b], \end{cases} \tag{7.22}$$

for all $i, j = 1, \ldots, m$*. Then there exists a neighbourhood* $\mathcal{V}$ *of 0 in* $C^\infty([0,\theta], \mathbb{R}^n)$ *such that if all* a_{ij}, b_{ij} *are in* $\mathcal{V}$*, then there exists a smooth map* h *of the form (7.18) with (7.19) for which the map* f*, defined by (7.20), belongs to* $\mathcal{U}$*, and for* $g' = g - g_F$*,* $F = \text{id} + f$*, the corresponding polynomial* $\widetilde{H}_t$ *has the form*

$$\widetilde{H}_t(x'; y') = \sum_{i,j=1}^{m} \left(\frac{1}{2} a_{ij}(t) x_i x_j + b_{ij}(t) x_i y_j \right). \tag{7.23}$$

Proof: We have to choose the maps v_{ij} so that

$$\frac{\partial^2 g'_{00}}{\partial x_i \partial x_j}(t;0) = a_{ij}(t), \quad \frac{\partial g'_0}{\partial x_j}(t;0) = b_{ij}(t), \quad g'_{ij}(t;0) = 0, \tag{7.24}$$

for all $i, j = 1, \ldots, n$ and all $t \in [0, \theta]$.

Let $v_{ij} : [0,\theta] \longrightarrow \mathbb{R}^n$ be arbitrary smooth maps satisfying (7.19). Then for $i = 1, \ldots, n$ and $t \in [0,\theta]$ we have $\frac{\partial h}{\partial x_i}(t;0) = 0$, so $g'_{ij} = 0$ by (7.21). Then for $x \in V$ close to γ_0 and any $i, j \geq 1$ we find

$$\frac{\partial^2 h}{\partial x_0 \partial x_i}(x) = \sum_{j=1}^{n} \frac{\partial v_{ij}}{\partial x_0}(x_0) x_j, \quad \frac{\partial^3 h}{\partial x_0 \partial x_i \partial x_j}(x) = \frac{\partial v_{ij}}{\partial x_0}(x_0).$$

Next, differentiating (7.21) we get

$$\frac{\partial^2 g'_{00}}{\partial x_i \partial x_j}(t;0) = 2 \left\langle \frac{\partial r}{\partial x_0}(t;0), v'_{ij}(t) \right\rangle, \quad \frac{\partial g'_{0i}}{\partial x_j}(t;0) = \left\langle \frac{\partial r}{\partial x_0}(t;0), v_{ij}(t) \right\rangle$$

for all $i, j \geq 1$ and $t \in [0, \theta]$.

Set $w(t) = \frac{\partial r}{\partial x_0}(t;0)$. Then by (7.17),

$$\|w(t)\| = \left\| \frac{\partial^2 r}{\partial^2 x_0}(t;0) \right\| > 0$$

whenever $t \in [a, b]$.

Fix arbitrary $i, j = 1, \ldots, n$. According to (7.24), we have to choose the maps v_{ij} in such a way that

$$\begin{cases} \langle w(t), v_{ij}(t) \rangle = b_{ij}(t) \\ 2\langle w(t), \dot{v}_{ij}(t) \rangle = a_{ij}(t) \end{cases} \tag{7.25}$$

for all $t \in [0,\theta]$. Define $v_{ij} : [0,\theta] \longrightarrow \mathbb{R}^n$ by $v_{ij}(t) = 0$ for $t \notin [a,b]$ and

$$v_{ij}(t) = b_{ij}(t) w(t) + \frac{\dot{b}_{ij}(t) - \frac{1}{2} a_{ij}(t)}{\| w(t) \|^2} w(t)$$

for $t \in [a, b]$. It follows from above that v_{ij} are well-defined smooth maps with supp $(v_{ij}) \subset [a, b]$. A straightforward verification shows that (7.25) and (7.19) hold. Moreover, if all a_{ij} and b_{ij} are taken in a small neighbourhood $\mathcal{V}$ of 0 in $C^\infty([0, \theta], \mathbb{R}^n)$, then h is C^∞ close to 0 and the map f, defined by (7.20), belongs to $\mathcal{U}$. This proves the assertion. ■

Our interest to Hamiltonians of the form (7.23) with $c_{ij} = 0$ for all i, j and symmetric matrices $A = (a_{ij})$ and $B = (b_{ij})$ leads naturally to the consideration of a special subset of $\mathrm{sp}(2m)$. Let $\mathbf{a}$ be the *linear subspace* of $\mathrm{sp}(2m)$ consisting of all matrices of the form

$$\begin{pmatrix} B & 0 \\ A & -B \end{pmatrix},$$

where A and B are symmetric $m \times m$ real matrices. The following lemma shows that $\mathbf{a}$ is sufficiently large for our aims.

Lemma 7.2.3: *Let*

$$P = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$$

be an arbitrary $2m \times 2m$ *real matrix. For every* $\epsilon > 0$ *there exists* $N \in \mathbf{a}$ *such that* $\|N\| < \epsilon$ *and* $\det(P - \exp N) \neq 0$.

Proof: Fix an arbitrary $\epsilon > 0$. Take $\delta > 0$ so small that if $N_i \in \mathbf{a}$, $\|N_i\| < \delta$ for $i = 1, 2$ and $\exp N = (\exp N_1)(\exp N_2)$, then $\|N\| < \epsilon$.

Denote by $\mathbf{a}'$ the set of all $2m \times 2m$ matrices of the form

$$\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix},$$

where A is a symmetric $m \times m$ real matrix. It is easy to see that $\mathbf{a}'$ is a Lie subalgebra of $\mathrm{sp}(2m)$, $\mathbf{a}' \subset \mathbf{a}$, and the commutator $[\mathbf{a}, \mathbf{a}']$ is contained in $\mathbf{a}'$. Therefore, for every $N_1 \in \mathbf{a}$ the set

$$\mathbf{a}(N_1) = \{tN_1 + X : t \in \mathbb{R}, \quad X \in \mathbf{a}'\}$$

is a Lie subalgebra of $\mathrm{sp}(2m)$. Hence if $N_1 \in \mathbf{a}$ and $N_2 \in \mathbf{a}'$ are sufficiently close to 0 and $\exp N = (\exp N_1)(\exp N_2)$, then $N \in \mathbf{a}(N_1)$ and in particular $N \in \mathbf{a}$.

Next, consider matrices N_1, N_2 of the form

$$N_1 = \begin{pmatrix} -B & 0 \\ 0 & B \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix},$$

where A and B are symmetric $m \times m$ real matrices. Then $N_i \in \mathbf{a}$ and there exists $\delta' > 0$ such that if $\|A\| < \delta'$ and $\|B\| < \delta'$, then $\|N_i\| < \delta$ for $i = 1, 2$. Define N by $\exp\ N = (\exp\ N_1)(\exp\ N_2)$. Then $N \in \mathbf{a}$, and setting $D = \exp B$, we obtain

$$\exp N = \begin{pmatrix} D^{-1} & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ A & I \end{pmatrix} = \begin{pmatrix} D^{-1} & 0 \\ DA & D \end{pmatrix}.$$

Choose $\tau > 0$ such that if D is a symmetric positive definite $m \times m$ matrix with $\|D - I\| < \tau$, then $D = \exp B$ for some symmetric $m \times m$ matrix B with $\|B\| < \delta'$.

Assume that

$$\det(P - \exp N) = 0 \tag{7.26}$$

for every choice of the symmetric matrices A and B with $\|A\| < \delta'$ and $\|B\| < \delta'$. Consider an arbitrary symmetric positive definite matrix D with $\|D - I\| < \tau$. We can write D in the form $D = E^{-1} D_1 E$, where E is an orthogonal matrix and

$$D_1 = \begin{pmatrix} y_1 & 0 & \cdots & 0 \\ 0 & y_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \cdots & y_m \end{pmatrix}, \quad |y_i - 1| < \tau, \quad i = 1, \ldots, m.$$

Fix E, A with $\|A\| < \delta'$ and $y_2, \ldots, y_m$ with $|y_i - 1| < \tau$ for $i = 2, \ldots, m$. Then (7.26) implies that

$$\det \begin{pmatrix} X - E^{-1} D_1^{-1} E & Y \\ Z - E^{-1} D_1 E A & T - E^{-1} D_1 E \end{pmatrix} = 0 \tag{7.27}$$

holds for every $y_1 \in \mathbb{R}$ with $|y_1 - 1| < \tau$. The left-hand side of (7.27) is a rational function of y_1 determined for all $y_1 \neq 0$. Since it vanishes for infinitely many values of y_1, it must be zero for all $y_1 \neq 0$. Therefore for fixed $E, A, y_2, \ldots, y_m$, (7.27) holds for all $y_1 \neq 0$. Using the same argument, we get by induction that for fixed E and A, (7.27) holds for all $y_1 \neq 0, \ldots, y_m \neq 0$. This is true for any choice of the orthogonal matrix E, hence

$$\begin{pmatrix} X - D^{-1} & Y \\ Z - DA & T - D \end{pmatrix} = 0 \tag{7.28}$$

holds for every non-singular symmetric matrix D. Next, for any non-singular symmetric matrix D, we can apply an argument similar to the previous one to show that (7.28) holds for all symmetric matrices A.

In this way we have established that (7.28) holds for all symmetric matrices A and D with $\det(D) \neq 0$. On the other hand, it is well known from linear algebra that any square matrix Z can be written in the form $Z = D_0 A_0$, where A_0 and D_0 are symmetric matrices and $\det(D_0) \neq 0$. Let A_0 and D_0 be such matrices. Set $D = y D_0$, $A = \frac{1}{y} A_0$ for an arbitrary real $y \neq 0$. Then $Z = DA$ and (7.28) implies

$$\det(X - y D_0^{-1}) \det\left(T - \frac{1}{y} D_0\right) = \det \begin{pmatrix} X - D^{-1} & Y \\ 0 & T - D \end{pmatrix} = 0.$$

Since each of the equalities $\det(X - y D_0^{-1}) = 0$ and $\det(T - \frac{1}{y} D_0) = 0$ is satisfied only for finitely many $y \in \mathbb{R}$, we get a contradiction. This proves the lemma. ■

The next lemma concerns analytical dependence on parameters of solutions of a special kind of systems of ordinary differential equations.

Lemma 7.2.4: *Let L be a smooth map from $[0, \theta]$ into the space of $k \times k$ real matrices such that $\|L(t)\| < c\,\psi(t)$ for every $t \in [0, \theta]$, where $c \in (0, 1)$ is a constant and*

$$\psi : [0, \theta] \longrightarrow [0, N]$$

for some $N > 0$ is an integrable function with

$$\int_0^\theta \psi(t)\, dt \leq 1.$$

Let $Y_\epsilon(t)$ ($\epsilon \in \mathbb{R}$) be the $k \times k$ matrix function which is the solution of the problem:

$$\begin{cases} \dot{Y}_\epsilon(t) = \epsilon L(t) Y_\epsilon(t), \quad t \in [0, \theta]\ , \\ Y_\epsilon(0) = I. \end{cases} \tag{7.29}$$

Then there exist smooth matrix functions $B_p(t)$, $p = 0, 1, \ldots,$ such that

$$\|B_p(t)\| \leq c^p, \quad \|\dot{B}_p(t)\| \leq N c^p \quad (p = 0, 1, \ldots; \quad t \in [0, \theta]), \tag{7.30}$$

and

$$Y_\epsilon(t) = \sum_{p=0}^{\infty} \epsilon^p B_p(t) \tag{7.31}$$

for all $t \in [0, \theta]$ and $|e| < \frac{1}{c}$.

Proof: Set

$$Z^{(p)}(t, \epsilon) = \frac{d^p}{d\epsilon^p} Y_\epsilon(t).$$

Writing down the variational equation of (7.29), by induction we get

$$\begin{cases} \dot{Z}^{(p)}(t, \epsilon) = \epsilon L(t) Z^{(p)}(t, \epsilon) + p L(t) Z^{(p-1)}(t, \epsilon), \\ Z^{(p)}(0, \epsilon) = 0, \end{cases}$$

for all $t \in [0, \theta]$, $\epsilon \in \mathbb{R}$ and $p = 1, 2, \ldots$. Then for $\epsilon = 0$ one finds

$$\dot{Z}^{(p)}(t, 0) = p L(t) Z^{(p-1)}(t, 0), \quad Z^{(p)}(0, 0) = 0$$

for all $t \in [0, \theta]$, $p = 1, 2, \ldots$. Clearly $\|Z^{(0)}(t, 0)\| = \|Y_0(t)\| = 1$. Suppose that for some integer $p > 0$ we have

$$\|Z^{(p-1)}(t, 0)\| \leq (p-1)! c^{p-1}$$

for all $t \in [0, \theta]$. It then follows from above that

$$\|Z^{(p)}(t,0)\| = \left\| p \int_0^t L(s) Z^{(p-1)}(s,0)\ ds \right\| \leq p! c^{p-1} \int_0^t \|L(s)\|\ \ ds \leq p! c^p.$$

Moreover, $\|Z^{(0)}(t,0)\| = 0$ and for $p \geq 1$ we get

$$\|\dot{Z}^{(p)}(t,0)\| \leq p\ \|L(t)\|\ \|Z^{(p-1)}(t,0)\| \leq pcn(p-1)! c^{p-1} = Np! c^p.$$

Therefore, the maps

$$B_p(t) = \frac{1}{p!} Z^{(p)}(t,0) = \frac{1}{p!} \frac{d^p}{d\epsilon^p} Y_\epsilon(t)_{|\epsilon=0}$$

satisfy the conditions (7.30). Hence the series

$$G(t,\epsilon) = \sum_{p=0}^{\infty} \epsilon^p B_p(t)$$

is uniformly and absolutely convergent for $t \in [0, \theta]$, provided $|\epsilon| \leq \text{const} < \frac{1}{c}$. Then for ϵ satisfying the latter condition we have

$$\begin{aligned}
\dot{G}(t,\epsilon) &= \sum_{p=0}^{\infty} \epsilon^p \dot{B}_p(t) = \sum_{p=0}^{\infty} \frac{\epsilon^p}{p!} \dot{Z}(t,0) \\
&= \epsilon L(t) \sum_{p=1}^{\infty} \frac{\epsilon^{p-1}}{(p-1)!} Z^{(p-1)}(t,0) \\
&= \epsilon L(t) \sum_{p=1}^{\infty} \epsilon^{p-1} B_{p-1}(t) = \epsilon L(t) G(t,\epsilon).
\end{aligned}$$

Moreover,

$$G(0,\epsilon) = B_0(0) = Y_\epsilon(0) = I.$$

Therefore, $G(t,\epsilon)$ coincides with the solution $Y_\epsilon(t)$ of (7.29). This is true for every ϵ with $|\epsilon| < 1/c$, hence (7.31) is satisfied. ■

We are now ready to prove the main result of this section.

Proof of Theorem 7.2.1: We will use the map r, the coordinates x_i, y_i and the numbers a, b and t_0 satisfying (7.17). Take an arbitrary $\mu > 0$ such that

$$c = \mu \max\{\ \|P_t\|^2 : t \in [0,\theta]\} < 1. \tag{7.32}$$

In what follows the numbers a, b, t_0, μ and c will be fixed.

Fix an arbitrary neighbourhood $\mathcal{W}$ of $\gamma(t_0)$ in M and an arbitrary neighbourhood $\mathcal{U}$ of 0 in $C^\infty(M, \mathbb{R}^n)$. Given a neighbourhood $\mathcal{O}$ of 0 in $C^\infty([0,\theta], \mathbf{a})$, denote by $\mathcal{A}$ the set of all $N \in \mathcal{O}$ with supp $(N) \subset [a,b]$. We will consider $\mathcal{A}$ with the topology induced by $\mathcal{O} \subset C^\infty([0,\theta], \mathbf{a})$. Then, being an open subset of a closed linear subspace of $C^\infty([0,\theta], \mathbf{a})$, $\mathcal{A}$ is a Baire space. For $N \in \mathcal{A}$ we have

$$N(t) = \begin{pmatrix} (b_{ij}(t)) & 0 \\ (a_{ij}(t)) & -(b_{ij}(t)) \end{pmatrix}, \tag{7.33}$$

where the functions $a_{ij}(t)$ and $b_{ij}(t)$ satisfy (7.22). Let $\mathcal{V}$ be a neighbourhood of 0 in $C^\infty([0,\theta], \mathbb{R}^n)$ as in Lemma 7.2.2. Fix $\mathcal{O}$ in such a way that $a_{ij}, b_{ij} \in \mathcal{V}$ for all $i,j = 1,\ldots,m$. Then for every $N \in \mathcal{A}$, we construct $h = h_N$ and $f = f_N$ as in Lemma 7.2.2 and set $F_N = \mathrm{id} + f_N$. Denote by $R_N(t)$ the smooth $2m \times 2m$ matrix function which is the solution of the problem

$$\begin{cases} \dot{R}_N(t) = P_t^{-1} N(t) P_t R_N(t), \quad t \in [0,\theta], \\ R_N(0) = I. \end{cases} \tag{7.34}$$

For a given $\lambda \in \mathbb{C}$ set

$$\mathcal{A}_\lambda = \{N \in \mathcal{A} : \det(P_\theta R_N(\theta) - \lambda) \neq 0\}.$$

It is easy to see that $\mathcal{A}_\lambda$ is open in $\mathcal{A}$; we leave this as an exercise for the reader.

We will now show that $\mathcal{A}_\lambda$ is dense in $\mathcal{A}$. It is sufficient to show that $\mathcal{A}_\lambda$ contains elements arbitrarily close to 0. Indeed, for $N \in \mathcal{A}$ we can apply this to the submanifold $M' = F_N(M)$ to show that there exist elements of $\mathcal{A}_\lambda$ arbitrarily close to N.

By Lemma 7.2.3 there exists $A \in \mathbf{a}$ such that $\|A\| < \mu$ and

$$\det(\exp A - \lambda P_{t_0} P_\theta^{-1} P_{t_0}^{-1}) \neq 0.$$

Fix a matrix A with these properties and take an arbitrary smooth function

$$\varphi : \mathbb{R} \longrightarrow [0,1]$$

such that supp $(\varphi) \subset [-1,1]$ and $\int_{\mathbb{R}} \varphi(t)\, dt = 1$. For $\delta > 0$ set

$$\varphi_\delta(t) = \frac{1}{\delta}\, \varphi\left(\frac{t - t_0}{\delta}\right).$$

Then supp $(\varphi_\delta) \subset [t_0 - \delta, t_0 + \delta]$, $0 \leq \varphi_\delta \leq 1/\delta$ and $\int_{\mathbb{R}} \varphi_\delta(t)\, dt = 1$. Let $R_\delta(t)$ be the solution of (7.34) for $N(t) = \varphi_\delta(t) A$. (Notice that $\varphi_\delta A \notin \mathcal{A}$ for sufficiently small $\delta > 0$.) Then for $t > t_0$ we have

$$R_\delta(t) \to \exp(P_{t_0}^{-1} A P_{t_0}) = P_{t_0}^{-1} (\exp A) P_{t_0}$$

as $\delta \longrightarrow 0$. The choice of A now implies the existence of $\delta > 0$ with

$$\det\bigl(R_\delta(\theta) - \lambda P_\theta^{-1}\bigr) \neq 0. \tag{7.35}$$

Fix an arbitrary $\delta > 0$ with this property and set

$$L(t) = \varphi_\delta(t)\ P_t^{-1} A P_t, \quad t \in [0, \theta].$$

Then

$$\|L(t)\| = \varphi_\delta(t)\ \ \|J P_t^T J A P_t\| \leq \mu\ \varphi_\delta(t)\ \max_s \|P_s\|^2 = c\ \varphi_\delta(t),$$

which yields that the assumptions of Lemma 7.2.4 are satisfied for $L(t)$, c, $N = 1/\delta$ and $\psi = \varphi_\delta$. By the lemma, the solution $Y_\epsilon(t)$ of (7.29) satisfies (7.31) and (7.30). In particular,

$$\chi_\lambda(\epsilon) = \det(Y_\epsilon(\theta) - \lambda P_\theta^{-1}) \tag{7.36}$$

is an analytic function of $\epsilon \in \mathbb{R}$ for $|\epsilon| < 1/c$. Notice that for $\epsilon = 1$ the solution of (7.29) coincides with the solution of (7.34) for $N = \varphi_\delta A$. Hence $Y_1(t) = R_\delta(t)$ for every t, and (7.36) and (7.35) imply

$$\chi_\lambda(1) = \det(R_\delta(\theta) - \lambda P_\theta^{-1}) \neq 0.$$

On the other hand, $c < 1$ by (7.33), so $1 < 1/c$. This shows that $\chi_\lambda(\epsilon)$ is an analytic function for $|\epsilon| < 1/c$ which is not trivially zero. In particular, 0 is not a limit point of the set

$$E_\lambda = \{\epsilon \in (0, 1/c) : \chi_\lambda(\epsilon) = 0\}.$$

Hence for all sufficiently small $\epsilon > 0$ the map N_ϵ, defined by $N_\epsilon(t) = \epsilon\varphi(t)A$, belongs to $\mathcal{A}$ and $\chi_\lambda(\epsilon) \neq 0$, i.e. $N_\epsilon \in \mathcal{A}_\lambda$. Since $N_\epsilon \to 0$ as $\epsilon \to 0$, it follows that $\mathcal{A}_\lambda$ contains elements arbitrarily close to 0.

In this way we have established that $\mathcal{A}_\lambda$ is open and dense in $\mathcal{A}$ for each complex number λ. Since Λ is countable,

$$\mathcal{A}_\Lambda = \cap_{\lambda \in \Lambda} \mathcal{A}_\lambda$$

is a residual subset of $\mathcal{A}$. In particular, there exists $N \in \mathcal{A}_\Lambda$ such that $f = f_N \in \mathcal{U}$ and $F = F_N$ satisfy the requirements of the theorem. ∎

7.3 Non-degeneracy and transversality

Our aim in this section is to describe the notion of non-degeneracy of a closed geodesic by means of some transversality condition. This will allow us to apply the Transversality Theorem of Abraham in the next section and establish the existence of a residual set of embeddings inducing bumpy metrics on M.

Throughout M will be a compact smooth $(m+1)$-dimensional submanifold of $\mathbb{R}^n$, $1 \leq m \leq n-2$. Let

$$\sigma : S^* M \longrightarrow M$$

be the *co-sphere bundle* of M. That is,

$$S^*M = \cup_{x \in M} S^*_x M,$$

where for every $x \in M$, $S^*_x M$ is the space of straight lines through 0 in $T^*_x M$ endowed with the quotient topology, and $\sigma(v) = x$ for every $v \in S^*_x M$. Let

$$p : T^*M \longrightarrow S^*M$$

be the *canonical projection*. Consider the map

$$\mathcal{H} : T^*M \times \mathbb{R}^+ \times \mathbf{C}(M) \longrightarrow T^*M,$$

which assigns to $(X, t, F) \in T^*M \times \mathbb{R}^+ \times \mathbf{C}(M)$ the shift of X after time t under the action of the geodesic flow on T^*M generated by the metric g_F. There exists a unique map

$$\mathcal{F} : S^*M \times \mathbb{R}^+ \times \mathbf{C}(M) \longrightarrow S^*M$$

such that $\mathcal{F} \circ (p \times \mathrm{id} \times \mathrm{id}) = p \circ \mathcal{H}$.

In what follows we will use not only the topological structure of $\mathbf{C}(M)$ but its structure of a Banach manifold as well. The latter is inherited naturally from the Banach space structure of $C^\infty(M, \mathbb{R}^n)$. We will now describe it briefly. As a general reference on infinite dimensional manifolds the reader may consult [Lang].

Let $2 \leq s \leq \infty$. Fix a finite number of charts

$$\varphi_i : B_r \longrightarrow U_i, \quad i = 1, \ldots, k,$$

where U_i are open subsets of M, B_r is the open ball with centre 0 and radius $r > 0$ in $\mathbb{R}^{m+1}$ and $\cup_{i=1}^k K_i = M$, where $K_i = \varphi_i(\overline{B_{r/2}})$. For $f \in C^s(M, \mathbb{R}^n)$ define

$$\|f\|_s = \sum_{j=0}^{s} \frac{1}{j!} \max_{1 \leq i \leq k} \sup_{x \in K_i} \|D^j (f \circ \varphi_i^{-1})(x)\| .$$

Then $\| \cdot \|_s$ is a norm in $C^s(M, \mathbb{R}^n)$ with respect to which the latter is a Banach space. This norm generates the Whitney C^s topology (see Section 2.1 in [Hir]). Denote by $\mathbf{C}^s(M)$ the subset of $C^s(M, \mathbb{R}^n)$ consisting of all C^s embeddings of M into $\mathbb{R}^n$. Then $\mathbf{C}^s(M)$ is an open subset of $C^s(M, \mathbb{R}^n)$ (see e.g. Section 2.1 in [Hir]) and therefore has a natural structure of a Banach manifold with a model space $C^\infty(M, \mathbb{R}^n)$. In what follows we consider $\mathbf{C}^s(M)$ with this structure. As before, the space $\mathbf{C}^\infty(M)$ will be denoted by $\mathbf{C}(M)$ for brevity.

The following lemma is a consequence of Chapter 5 in [AbR], and in fact can be derived easily from some standard facts about smooth dependence of solutions to systems of differential equations on parameters. We omit the proof.

Lemma 7.3.1: *The maps $\mathcal{F}$ and $\mathcal{H}$ are smooth.*

The concept of non-degeneracy of a closed geodesic introduced in the previous section has a local character and at a first glance seems to be inconvenient when we try to perturb the whole manifold M in order to make all geodesics on M non-degenerate. However, this concept can be described in terms of some global characteristics of the corresponding geodesic flow.

Given a fixed smooth Riemannian metric g on M, for $t \in \mathbb{R}$ and $Y \in T^*M$ denote by $\varphi_t(Y)$ the shift of Y after time t under the action of the geodesic flow on T^*M determined by g. Let ψ_t be the projection of φ_t on S^*M, that is

$$\psi_t : S^*M \longrightarrow S^*M$$

such that $\psi_t \circ p = p \circ \varphi_t$ for all $t \in \mathbb{R}$. Let $Y \in T^*M \setminus \{0\}$ generate a periodic trajectory with period $\theta > 0$, that is $\varphi_\theta(Y) = Y$. Consider the linear map

$$T\varphi_\theta : T_Y(T^*M) \longrightarrow T_Y(T^*M).$$

The following proposition is a special case of a general fact in the theory of dynamical systems (see e.g. Sections 7.1 and 8.1 in [AbM]). We omit the proof.

Proposition 7.3.2: *Under the above assumptions, let P be the linear Poincaré map related to the closed integral curve*

$$c = \{\varphi_t(Y) : t \in [0, \theta]\}$$

determined by Y. Then

$$\operatorname{spec}(T\varphi_\theta) = \{1\} \cup \operatorname{spec}(P),$$

and the multiplicity of 1 in $\operatorname{spec}(T\varphi_\theta)$ *is $k + 2$, where k is the multiplicity of 1 in* $\operatorname{spec}(P)$.

According to the definition in Section 7.1, the curve c (and therefore the corresponding closed geodesic on M) is non-degenerate as a curve of period θ if $k = 0$ in the above proposition. This is equivalent to the fact that 1 occurs with multiplicity exactly 2 in $\operatorname{spec}(T\varphi_\theta)$. Setting $X = p(Y)$, we see that c is non-degenerate if and only if 1 occurs with multiplicity exactly 1 in $\operatorname{spec}(T\psi_\theta)$.

Denote by $\mathbb{D}$ the *diagonal* of $S^*M \times S^*M$, that is

$$\mathbb{D} = \{(X, X) : X \in S^*M\}.$$

The following lemma is the central point in this section.

Lemma 7.3.3: *Let*

$$(Y_0, \theta, F_0) \in (T^*M \setminus \{0\}) \times \mathbb{R}^+ \times \mathbf{C}(M)$$

be such that

$$c : [0, \theta] \longrightarrow T^*M, \quad c(t) = \mathcal{H}(Y_0, t, F_0),$$

is a closed geodesic with respect to the metric g_{F_0} of period θ, and let $X_0 = p(Y_0)$.

(a) If c is non-degenerate as a curve of period θ, then the map

$$\mathcal{F}_1 : S^*M \times \mathbb{R}^+ \longrightarrow S^*M \times S^*M,$$

defined by $\mathcal{F}_1(X, t) = (X, \mathcal{F}(X, t, F_0))$, is transversal to $\mathbb{D}$ at (X_0, θ).

(b) If θ is the minimal period of c, then the map

$$\mathcal{F}_2 : \mathbb{R}^+ \times \mathbf{C}(M) \longrightarrow S^*M \times S^*M,$$

defined by $\mathcal{F}_2(t, F) = (X_0, \mathcal{F}(X_0, t, F))$, is transversal to $\mathbb{D}$ at (θ, F_0).

Proof: It is sufficient to consider the case $F_0 = \mathrm{id}$; the general case follows by replacing M by $F_0(M)$ and g by g_{F_0}.

So, assume $F_0 = \mathrm{id}$. Consider coordinates $x_0, \ldots, x_m; y_0, \ldots, y_m$ in a neighbourhood of $\Im(c)$ in T^*M by means of a map (7.6) as in Section 7.1 such that

$$Y_0 = (0, \ldots, 0; 1, 0, \ldots, 0).$$

Using these coordinates, we will identify $T_{Y_0}(T^*M)$ with

$$\mathbb{R}^{2m+2} = (\mathbb{R} \times \mathbb{R}^m) \times (\mathbb{R} \times \mathbb{R}^m).$$

Notice that we can use $x_0, \ldots, x_m; y_1, \ldots, y_m$ as coordinates in a neighbourhood of $p \circ c$ in S^*M.

(a) Let c be non-degenerate as an integral curve of period θ. Define

$$\mathcal{H}_1 : T^*M \times \mathbb{R}^+ \longrightarrow T^*M \times T^*M$$

by

$$\mathcal{H}_1(Y, t) = (Y, \mathcal{H}(Y, t, \mathrm{id})).$$

For $s, t \in \mathbb{R}^+$ and $Y = (s, 0; 1, 0)$ we have

$$\mathcal{H}_1(Y, t) = (Y, (s + t, 0; 1, 0)).$$

Therefore,

$$\Im(T\mathcal{H}_1(Y_0, \theta)) \supset (\mathbb{R} \times \{0\}) \times (\mathbb{R} \times \{0\}). \tag{7.37}$$

Fix $t = \theta$ and consider $Y \in T^*M$ of the form $Y = (0, x; 0, y)$, where $x, y \in \mathbb{R}^{2m}$. Then $Y \in \Sigma(0)$ (see the notation in Section 7.1) and $\mathcal{H}_1(Y, \theta) = (Y, Y')$, where $Y' = (0, x'; 0, y') \in \Sigma(\theta)$ and $(x', y') = \mathcal{P}_\theta(x, y)$. Since $1 \notin \mathrm{spec}(d\mathcal{P}_\theta(0, 0))$, the map

$$(x, y) \mapsto ((x, y), \mathcal{P}_\theta(x, y))$$

of $\mathbb{R}^{2m}$ into $\mathbb{R}^{2m}$ is transversal to the diagonal at $(0, 0)$. Letting $\mathbb{D}'$ denote the diagonal in

$$\mathbb{R}^{2m+2} \times \mathbb{R}^{2m+2} = (T_{Y_0}(T^*M))^2,$$

we get

$$\mathbb{D}' + \Im(T\mathcal{H}_1(Y_0, \theta)) \supset (\{0\} \times \mathbb{R}^m)^4.$$

Combining this with (7.37), we find

$$\mathbb{D}' + \Im(T\mathcal{H}_1(Y_0, \theta)) \supset (\mathbb{R}^{m+1} \times (\{0\} \times \mathbb{R}^m))^2,$$

which proves part (a).

(b) Let θ be the minimal period of c. Fix the numbers a, b, t_0 with (7.17) and a smooth map $\rho : \mathbb{R}^{m+1} \longrightarrow [0, 1]$ as in Section 7.2. Let $\varphi : \mathbb{R} \longrightarrow [0, 1]$ be a smooth function with

$$\text{supp}\,(\varphi) \subset [-1, 1], \qquad \int_{\mathbb{R}} \varphi(t)\; dt = 1.$$

For $\epsilon, \delta > 0$ set

$$\varphi_\delta(t) = \frac{1}{\delta}\, \varphi\left(\frac{t - t_0}{\delta}\right), \qquad \chi(t) = \epsilon\; \varphi_\delta(t).$$

Later we will determine how small ϵ and δ should be. Clearly, if δ is sufficiently small, then $\text{supp}\,(\chi) \subset [a, b]$.

It follows from (7.17) that for $\lambda(t) = \frac{\partial r}{\partial x_0}(t; 0)$ we have $\dot{\lambda}(t) \neq 0$ whenever $t \in [a, b]$. As in the proof of Lemma 7.2.2 we construct smooth maps $e, d : [0, \theta] \longrightarrow \mathbb{R}^n$ with supports in $[a, b]$ such that

$$\begin{cases} \langle \lambda(t), e(t) \rangle = \chi(t), \quad \langle \lambda(t), \dot{e}(t) \rangle = 0, \\ \langle \lambda(t), d(t) \rangle = 0, \quad 2\langle \lambda(t), \dot{e}(t) \rangle = -\chi(t) \end{cases} \tag{7.38}$$

for all $t \in [0, \theta]$. Namely, we set

$$e(t) = \chi(t)\; \lambda(t) + \frac{\dot{\chi}(t)}{\|\dot{\lambda}(t)\|^2}\; \dot{\lambda}(t),$$

and

$$d(t) = \frac{\chi(t)}{2\; \|\dot{\lambda}(t)\|^2}\; \dot{\lambda}(t)$$

for all $t \in [0, \theta]$. Next, define

$$W = \{\omega = (u_1, \ldots, u_m; v_1, \ldots, v_m) \in \mathbb{R}^{2m} : \|\omega\| < 1\},$$

and consider the map

$$\Omega : W \longrightarrow \mathbf{C}(M), \quad \Omega(\omega) = F = \mathrm{id} + f,$$

where f is determined by (7.20) by means of the map h given by

$$h(x) = \left(\sum_{i=1}^{m} u_i x_i \right) e(x_0) + \left(\sum_{i=1}^{m} v_i x_i \right) d(x_0) \tag{7.39}$$

for $x = (x_0, x_1, \dots, x_m) \in V$. If e and d are sufficiently close to 0 in the C^∞ topology (and so χ is C^∞ close to 0), then Ω is well defined. In fact, Ω coincides with the restriction of a linear map $\mathbb{R}^{2m} \longrightarrow C^\infty(M, \mathbb{R}^n)$.

To prove part (b), it is sufficient to show that the map

$$(t, F) \mapsto \mathcal{F}(X_0, t, F)$$

from $\mathbb{R}^+ \times \mathbf{C}(M)$ into S^*M is a submersion at (θ, id). Consider the map

$$\mathcal{H}_2 : \mathbb{R}^+ \times W \longrightarrow T^*M$$

defined by

$$\mathcal{H}_2(t, w) = \mathcal{H}(Y_0, t, \Omega(w)).$$

The assertion will be proved if we show that

$$\Im(T\mathcal{H}_2(\theta, 0)) = (\mathbb{R} \times \mathbb{R}^m) \times (\{0\} \times \mathbb{R}^m). \tag{7.40}$$

Let $\omega = 0$; then $\mathcal{H}_2(t, 0) = (t, 0; 1, 0)$ and therefore

$$\Im(T\mathcal{H}_2(\theta, 0)) \supset (\mathbb{R} \times \{0\}) \times (\{0\} \times \{0\}). \tag{7.41}$$

Let $\omega = (u; v) \in W$. Define h by (7.39) and set $\tilde{g} = g_F$, $F = \Omega(\omega)$, $g' = \tilde{g} - g$. Then $\tilde{g}_{ij} = g_{ij} + g'_{ij}$, where g'_{ij} are determined by (7.21). In a small neighbourhood of $\Im(c)$ in T^*M the Hamiltonian function corresponding to $\tilde{g}$ has the form

$$\tilde{H}(x; y) = \frac{1}{2} \sum_{i,j=0}^{m} \tilde{g}_{ij}(x) y_i y_j,$$

where $x = (x_0, \dots, x_m)$, $y = (y_0, \dots, y_m)$. Therefore, the coordinate functions $\tilde{x}(t; \omega)$ and $\tilde{y}(t; \omega)$ of $\mathcal{H}(Y_0, t, \Omega(\omega))$ with $t \in [0, \theta]$ satisfy the Hamiltonian equations

$$\dot{\tilde{x}}_k = \frac{\partial \widetilde{H}}{\partial y_k}(\tilde{x}; \tilde{y}; \omega), \quad \dot{\tilde{y}}_k = -\frac{\partial \widetilde{H}}{\partial x_k}(\tilde{x}; \tilde{y}; \omega). \tag{7.42}$$

Set $x(t) = \tilde{x}(t; 0)$, $y(t) = \tilde{y}(t; 0)$; then $x(t) = (t, 0, \dots, 0)$ and $y(t) = (1, 0, \dots, 0)$ are the coordinates of $c(t)$.

Next, consider $\tilde{x}(t;\omega)$ and $\tilde{y}(t;\omega)$ as column vectors. For $q = 1, \ldots, m$ and $t \in [0, \theta]$ set

$$\xi_q(t) = \left(\frac{d}{du_q}\right)_{|\omega=0} \begin{pmatrix} \tilde{x}(t;\omega) \\ \tilde{y}(t;\omega) \end{pmatrix}, \quad \eta_q(t) = \left(\frac{d}{dv_q}\right)_{|\omega=0} \begin{pmatrix} \tilde{x}(t;\omega) \\ \tilde{y}(t;\omega) \end{pmatrix}.$$

Writing the variational equations of (7.42), we find $\xi_q(0) = 0$ and

$$\dot{\xi}_q(t) = S(t)\xi_q(t) + R_q(t), \quad t \in [0, \theta], \tag{7.43}$$

where

$$S(t) = \begin{pmatrix} S_1(t) & S_2(t) \\ S_3(t) & S_4(t) \end{pmatrix},$$

$$S_1(t) = \left(\frac{\partial^2 \widetilde{H}}{\partial y_k \partial x_i}(x(t); y(t); 0)\right), \quad S_2(t) = \left(\frac{\partial^2 \widetilde{H}}{\partial y_k \partial y_i}(x(t); y(t); 0)\right),$$

$$S_3(t) = \left(-\frac{\partial^2 \widetilde{H}}{\partial x_k \partial x_i}(x(t); y(t); 0)\right), \quad S_4(t) = \left(-\frac{\partial^2 \widetilde{H}}{\partial x_k \partial y_i}(x(t); y(t); 0)\right),$$

and

$$R_q(t) = \left(\left(-\frac{\partial^2 \widetilde{H}}{\partial y_k \partial u_q}(x(t); y(t); 0)\right)_k; \ \left(-\frac{\partial^2 \widetilde{H}}{\partial x_k \partial u_q}(x(t); y(t); 0)\right)_k\right)^T.$$

Since $y(t) = (1, 0, \ldots, 0)$, it follows that

$$\frac{\partial^2 \widetilde{H}}{\partial y_k \partial x_i}(x(t); y(t); 0) = \sum_{j=0}^{m} \frac{\partial \tilde{g}_{kj}}{\partial x_i}(x(t); 0) y_j(t) = \frac{\partial \tilde{g}_{k0}}{\partial x_i}(x(t); 0).$$

On the other hand, (7.21) and (7.39) imply

$$\tilde{g}_{k0} = g_{k0} + \left\langle \frac{\partial r}{\partial x_0}, \frac{\partial h}{\partial x_k} \right\rangle + \left\langle \frac{\partial r}{\partial x_k}, \frac{\partial h}{\partial x_0} \right\rangle + O(\| \omega\|^2).$$

Differentiating the latter equality with respect to x_i and evaluating at $x_0 = t$, $x_1 = \cdots = x_m = 0$, we find

$$\frac{\partial \tilde{g}_{k0}}{\partial x_i}(x(t); 0) = \frac{\partial g_{k0}}{\partial x_i}(t; 0).$$

Therefore,

$$S_1(t) = \left(\frac{\partial g_{k0}}{\partial x_i}(t; 0)\right)_{k,i}.$$

By similar calculations one obtains $S_2(t) = (g_{ki}(t;0))_{k,i}$,

$$S_3(t) = \left(-\frac{1}{2}\frac{\partial^2 g_{00}}{\partial x_k \partial x_i}(t;0)\right)_{k,i}, \quad S_4(t) = \left(-\frac{\partial g_{i0}}{\partial x_k}(t;0)\right)_{k,i}.$$

Moreover, (7.38) yields

$$\frac{\partial^2 \widetilde{H}}{\partial y_k \partial u_q}(x(t);y(t);0) = \frac{\partial \tilde{g}_{k0}}{\partial u_q}(x(t);0) = \begin{cases} 0, & k \neq q, \\ \langle \lambda(t), e(t) \rangle = \chi(t), & k = q, \end{cases}$$

and

$$\frac{\partial^2 \widetilde{H}}{\partial x_k \partial u_q}(x(t);y(t);0) = \frac{\partial^2 \tilde{g}_{00}}{\partial x_k \partial u_q}(x(t);0) = 0$$

for $k = 0, 1, \dots, m$ and $q = 1, \dots, m$. Thus, the column vector $R_q(t)$ has the form

$$R_q(t) = (0, \dots, 0, \chi(t), 0, \dots, 0; 0, \dots, 0)^T,$$

where $\chi(t)$ is the qth component. Since $g_{00}(t;0) = 1$,

$$\frac{\partial g_{00}}{\partial x_i}(t;0) = \frac{\partial^2 g_{00}}{\partial x_0 \partial x_i}(t;0) = \frac{\partial g_{i0}}{\partial x_0}(t;0) = 0$$

for $i = 0, 1, \dots, m$ and $g_{i0}(t;0) = 0$ for $i = 1, \dots, m$, it follows that the 0th row of $S(t)$ has the form $(0, \dots, 0; 1, 0, \dots, 0)$, while the $(m+1)$st one consists of zeros only. Combining this with (7.43), we obtain

$$\dot{\xi}_q^{(0)}(t) = \xi_q^{(m+1)}(t), \quad \dot{\xi}_q^{(m+1)}(t) = 0, \quad t \in [0, \theta].$$

Here, $\xi_q^{(i)}$ is the ith component of the vector ξ_q. Consequently, we get $\xi_q^{(0)}(t) = \xi_q^{(m+1)}(t) = 0$ for all $t \in [0, \theta]$ and all $q = 1, \dots, m$. In particular,

$$\xi_q(\theta) \in (\{0\} \times \mathbb{R}^m) \times (\{0\} \times \mathbb{R}^m), \quad q = 1, \dots, m.$$

In the same way one gets similar inclusions for the vectors $\eta_q(\theta)$, $q = 1, \dots, m$.

Next, notice that the subspace $T\mathcal{H}_2(\theta, 0)(\{0\} \times T_0 W)$ is generated by the vectors

$$\xi_1(\theta), \dots, \xi_m(\theta), \quad \eta_1(\theta), \dots, \eta_m(\theta) \tag{7.44}$$

(see the beginning of the proof of part (b)). Therefore,

$$T\mathcal{H}_2(\theta, 0)(\{0\} \times T_0 W) \subset (\{0\} \times \mathbb{R}^m) \times (\{0\} \times \mathbb{R}^m). \tag{7.45}$$

Now we will show that if δ and ϵ are sufficiently small, then (7.45) becomes an equality. In fact, it is sufficient to choose δ so small that the vectors (7.44) are linearly independent. To see that this is possible, consider the fundamental solution $Z(t)$of (7.43). That is, $Z(t)$ is the $(2m+2) \times (2m+2)$ smooth matrix function with $Z(0) = I$

and $\dot{Z}(t) = S(t)Z(t)$ for $t \in \mathbb{R}$. Since $S(t) \in \mathrm{sp}(2m)$, we have $Z(t) \in \mathrm{Sp}(2m)$ for every t. Now (7.43) implies

$$\xi_q(t) = Z(t) \int_0^t Z(s)^{-1} R_q(s)\, ds, \quad t \in \mathbb{R}. \tag{7.46}$$

Let $\lambda_0(t), \ldots, \lambda_m(t), \mu_0(t), \ldots, \mu_m(t)$ be the successive column vectors of $Z(t)^{-1}$. Then

$$\int_0^\theta \varphi_\delta(s)\lambda_q(s)\, ds \to \lambda_q(t_0)$$

and

$$\int_0^\theta \varphi_\delta(s)\mu_q(s)\, ds \to \mu_q(t_0)$$

as $\delta \to 0$. Hence if δ is sufficiently small, the vectors

$$\lambda_1, \ldots, \lambda_m, \quad \mu_1, \ldots, \mu_m,$$

given by

$$\lambda_q = \int_0^\theta \varphi_\delta(s)\lambda_q(s)\, ds, \quad \mu_q = \int_0^\theta \varphi_\delta(s)\mu_q(s)\, ds,$$

are linearly independent. Fix such a $\delta > 0$ and choose $\epsilon > 0$ so small that $\Omega(\omega) \in \mathbf{C}(M)$ for every $\omega \in W$ (see (7.38) and (7.39)). Using (7.46), we find

$$\xi_q(\theta) = Z(\theta) \int_0^\theta Z(s)^{-1} R_q(s)\, ds = \epsilon Z(\theta) \int_0^\theta \varphi_\delta(s)\lambda_q(s)\, ds = \epsilon Z(\theta)\lambda_q.$$

In the same way one gets $\eta_q(\theta) = \epsilon Z(\theta)\mu_q$. Thus, the vectors (7.44) are linearly independent, and therefore (7.45) becomes an equality with this choice of δ. This and (7.41) imply (7.40), which concludes the proof of the lemma. ■

7.4 Global perturbations of smooth surfaces

Throughout we use the notation from the previous section.

Our aim in this section is to prove the following

Theorem 7.4.1: *Let M be a smooth compact submanifold of $\mathbb{R}^n$, $n \geq 3$, with $\dim(M) < n$. There exists a residual subset $\mathcal{B}$ of $\mathbf{C}(M)$ such that for every $F \in \mathcal{B}$ the standard metric on $F(M)$ is a bumpy metric.*

Proof: Given $F \in \mathbf{C}(M)$, $X \in S^*M$ and $t \in \mathbb{R}$, set

$$\psi_t^F(X) = \mathcal{F}(X, t, F).$$

For $0 < a \le b$ denote by $\mathbf{C}(a,b)$ the set of those $F \in \mathbf{C}(M)$ such that if

$$\tilde{c} = \{\psi_t^F(v) : 0 \le t \le \omega\}$$

is a periodic trajectory of the flow ψ_t^F of period $\omega \le b$ and having minimal period $\theta \le a$, then $\tilde{c}$ is non-degenerate as a curve of period ω. Notice that $\mathbf{C}(a',b') \subset \mathbf{C}(a,b)$ if $a \le a'$, $b \le b'$ and $a' \le b'$. Consider the set

$$\mathcal{B} = \cap_{k=1}^{\infty} \mathbf{C}(k,k).$$

Then clearly for every $F \in \mathcal{B}$ the standard metric on $F(M)$ is a bumpy metric. Thus, the theorem will be proved if we show that $\mathbf{C}(k,k)$ is an open and dense subset of $\mathbf{C}(M)$ for every $k = 1, 2 \dots$.

Next, notice that for every $F \in \mathbf{C}(M)$ there exist a neighbourhood $\mathcal{U}$ of F in $\mathbf{C}(M)$ and $\alpha > 0$ such that for every $G \in \mathcal{U}$ the flow ψ_t^G has no periodic trajectories with period $\le \alpha$. Indeed, suppose this is not true. Then there exist sequences $\{F_k\} \subset \mathbf{C}(M)$, $\{t_k\} \subset \mathbb{R}^+$, $\{v_k\} \subset S^*M$ such that $F_k \to F$, $t_k \to 0$ and $\psi_{t_k}^{F_k} v_k = v_k$ for every $k \ge 1$. Due to the compactness of S^*M we may assume that $v_k \to v \in S^*M$. Fix an arbitrary $t > 0$. For each k write $t = m_k t_k + s_k$ with $m_k \in \mathbb{N}$ and $0 \le s_k < t_k$. Then $s_k \to 0$, so we have

$$\psi_t^{F_k} v_k = \psi_{s_k}^{F_k} v_k \to v$$

as $k \to \infty$. On the other hand, clearly $\psi_t^{F_k} v_k \to \psi_t^F v$. Thus, $\psi_t^F v = v$ for all $t > 0$, which is a contradiction with the well-known fact that the geodesic flow ψ_t^F has no fixed points.

Now let $0 < a \le b$ be fixed numbers. We will show that $\mathbf{C}(a,b)$ is open in $\mathbf{C}(M)$. Assume that $F_k \to F$ with $F_k \notin \mathbf{C}(a,b)$ for all $k \ge 1$. Then for every $k \ge 1$ there exist $v_k \in S^*M$, $t_k \in (0,a]$ and $\ell_k \in \mathbb{N}$ such that $\psi_{t_k}^{F_k} v_k = v_k$, $\ell_k t_k \le b$ and $\operatorname{spec}(T\psi_{\ell_k t_k}^{F_k}(v_k))$ contains 1 with multiplicity at least 2. According to the above argument, there exist a neighbourhood $\mathcal{U}$ of F in $\mathbf{C}(M)$ and $\alpha > 0$ such that for every $G \in \mathcal{U}$ the flow ψ_t^G has no periodic trajectories of period $\le \alpha$. Without loss of generality we may assume that $F_k \in \mathcal{U}$ for all k and $t_k \to t$, $v_k \to v \in S^*M$ as $k \to \infty$. Now we see that the sequence $\{\ell_k\}$ must be bounded, so we may assume $\ell_k = \ell$ for all k. It then follows that $\psi_t^F(v) = v$, $\ell t \le b$ and $\operatorname{spec}(T\psi_{\ell t}^F(v))$ contains 1 with multiplicity at least 2. Thus, $F \notin \mathbf{C}(a,b)$, which proves that $\mathbf{C}(a,b)$ is open.

Given $a > 0$, consider the map

$$\mathcal{F}^a : S^*M \times (0, 2a) \times \mathbf{C}(a, 2a) \longrightarrow S^*M \times S^*M,$$

defined by $\mathcal{F}^a(X,t,F) = (X, \mathcal{F}(X,t,F))$. It follows from Lemma 7.3.3 that $\mathcal{F}^a \pitchfork \mathbb{D}$. Indeed, let $\mathcal{F}^a(X_0, \theta, F_0) \in \mathbb{D}$. Then $\theta < 2a$ and $\tilde{c} = \{\psi_t^{F_0}(X) : 0 \le t \le \theta\}$ is a periodic trajectory of period θ. If θ is the minimal period, then $\mathcal{F}^a(X_0,t,F) = \mathcal{F}_2(t,F)$ and Lemma 7.3.3(b) imply that $\mathcal{F}^a \pitchfork \mathbb{D}$ at (X_0, θ, F_0). Next, assume that the minimal period ω of $\tilde{c}$ is $< \theta$; then clearly $\omega \le \theta/2 \le a$. Since $F_0 \in \mathbf{C}(a, 2a)$, $\tilde{c}$ is non-degenerate as a trajectory with period θ. By Lemma 7.3.3(a) and $\mathcal{F}^a(X,t,F_0) = \mathcal{F}_1(X,t)$, we see that $\mathcal{F}^a \pitchfork \mathbb{D}$ at (X_0,θ,F_0) again. Hence $\mathcal{F}^a \pitchfork \mathbb{D}$.

Next, for $F \in \mathbf{C}(M)$ define

$$\mathcal{F}^a_F : S^*M \times (0,2a) \longrightarrow S^*M \times S^*M$$

by $\mathcal{F}^a_F(X,t) = \mathcal{F}^a(X,t,F)$. Set

$$U = \{F \in \mathbf{C}(a,2a) : \mathcal{F}^a_F \pitchfork \mathbb{D}\}.$$

It follows from Abraham's Transversality Theorem (see Section 1.1) that U is open and dense in $\mathbf{C}(a,2a)$. On the other hand, $U \subset \mathbf{C}(3a/2,3a/2)$. Indeed, if $\mathcal{F}^a \pitchfork \mathbb{D}$, then every periodic trajectory of ψ^F_t of period $\theta < 2a$ is non-degenerate as a trajectory of period θ, so $F \in \mathbf{C}(3a/2,3a/2)$. Thus, $\mathbf{C}(3\mathbf{a}/2,3\mathbf{a}/2) \cap \mathbf{C}(\mathbf{a},2\mathbf{a})$ is an open and dense subset of $\mathbf{C}(a,2a)$.

Next, fix an arbitrary $a > 0$. We will now show that $\mathbf{C}(a,2a)$ is dense in $\mathbf{C}(a,a)$. Consider an arbitrary $F_0 \in \mathbf{C}(a,a)$ and an arbitrary open neighbourhood $\mathcal{U}$ of F_0 in $\mathbf{C}(a,a)$. Notice that if $X \in S^*M$ generates a periodic trajectory $\tilde{c}$ of $\psi^{F_0}_t$ with period not greater than a, then $\tilde{c}$ is non-degenerate as a trajectory with period a. Therefore, there exists a neighbourhood V of X in S^*M such that there is no $Y \in V \setminus \{X\}$ generating a periodic trajectory of period $\le a$. Now the compactness of S^*M shows that there exist only finitely many periodic trajectories

$$\tilde{c}_i = \{\psi^{F_0}_t X_i : 0 \le t \le \theta_i\}, \quad i = 1,\dots,k,$$

of the flow $\psi^{F_0}_t$ with minimal periods $\theta_i \le a$. A standard result from the theory of differential equations implies that there exist a neighbourhood $\mathcal{V} \subset \mathcal{U}$ of F_0 and continuous maps

$$Z_i : \mathcal{V} \longrightarrow S^*M, \quad \omega_i : \mathcal{V} \longrightarrow \mathbb{R}^+, \quad i = 1,\dots,k,$$

with $Z_i(F_0) = X_i$, $\omega_i(F_0) = \theta_i$ for all i and such that for every $F \in \mathcal{V}$

$$\tilde{c}_i(F) = \{\psi^F_t Z_i(F) : 0 \le t \le \omega_i(F)\}, \quad i = 1,\dots,k, \tag{7.47}$$

are periodic trajectories of the flow ψ^F_t with minimal periods $\omega_i(F)$, respectively. Moreover, using the non-degeneracy of the trajectories $\tilde{c}_i$ and a compactness argument very similar to one of those already applied in this proof, we see that if $\mathcal{V}$ is sufficiently small, then for every $F \in \mathcal{V}$ if $\tilde{c} = \{\psi^F_t(X)\}$ is a periodic trajectory of period $\le a$, then it coincides with some of the trajectories (7.47). Then by Theorem 7.2.1 there exists $F \in \mathcal{V}$ such that the trajectories (7.47) are non-degenerate as trajectories with periods $k\omega_i(F)$, respectively, for every integer $k \ge 1$. However, as we have already mentioned, the trajectories (7.47) are the only periodic trajectories of ψ^F_t having minimal periods $\le a$. Therefore, $F \in \mathbf{C}(a,2a)$.

In this way we have proved that $\mathbf{C}(a,2a)$ is dense in $\mathbf{C}(a,a)$. Hence $\mathbf{C}(3a/2,3a/2)$ is dense in $\mathbf{C}(a,a)$. The latter implies that $\mathbf{C}((3a/2)^k,(3a/2)^k)$ is dense in $\mathbf{C}(a,a)$ for every integer $k \geq 1$, and therefore $\mathbf{C}(b,b)$ is dense in $\mathbf{C}(a,a)$ for every $b \geq a$.

Let again $a > 0$ be a fixed number. To show that $\mathbf{C}(a,a)$ is dense in $\mathbf{C}(M)$, fix an arbitrary $F \in \mathbf{C}(M)$ and an arbitrary neighbourhood $\mathcal{U}$ of F in $\mathbf{C}(M)$. We may assume that $\mathcal{U}$ is so small that there exists $\alpha \in (0,a)$ such that for every $G \in \mathcal{U}$ the flow ψ_t^G has no periodic trajectories with periods $\leq \alpha$. Then obviously $\mathcal{U} \subset \mathbf{C}(\alpha,\alpha)$. Now the density of $\mathbf{C}(a,a)$ in $\mathbf{C}(\alpha,\alpha)$ yields that $\mathcal{U} \cap \mathbf{C}(a,a)$ is non-empty. Thus, $\mathbf{C}(a,a)$ is dense in $\mathbf{C}(M)$.

The above shows that each of the sets $\mathbf{C}(k,k)$ is open and dense in $\mathbf{C}(M)$. Thus, $\mathcal{B}$ is a residual subset of $\mathbf{C}(M)$, which proves the theorem. ■

The Classical Bumpy Metric Theorem concerns the *space $\mathcal{G}M$ of all smooth Riemannian metrics* on a given smooth compact manifold M. More precisely, $\mathcal{G}M$ is the set of all smooth symmetric and positive definite tensors $g \in \mathcal{T}_2^0(M)$ (see e.g. Sections 1.7 and 2.5 in [AbM]), endowed with the Whitney C^∞ topology. Using the well-known fact that for every $g \in \mathcal{G}M$, the Riemannian manifold (M,g) can be isometrically embedded in some $\mathbb{R}^n$ (with the standard metric), and applying Theorems 7.2.1 and 7.4.1 we deduce the following.

Corollary 7.4.2 (Bumpy Metric Theorem): *There exists a residual subset of $\mathcal{G}M$ consisting of bumpy metrics on M.*

7.5 Notes

The Classical Bumpy Metric Theorem (Corollary 7.4.2) was announced by Abraham [Ab] giving an idea of a proof. More general results were established by Klingenberg and Takens [KT]. Given a smooth manifold M with $\dim(M) = m+1$, an integer $k > 0$ and open, dense and invariant subset Q of the Lie group of k-jets of smooth local symplectic maps

$$(\mathbb{R}^{2m}, 0) \longrightarrow (\mathbb{R}^{2m}, 0),$$

it was proved in [KT] that there exists a residual subset R of $\mathcal{G}M$ such that for $g \in R$ the k-jet of the Poincaré map of every closed geodesic on (M,g) belongs to Q. This global result was derived as a consequence of the following local result in [KT]: if $g \in \mathcal{G}M$ and γ is a closed geodesic on (M,g), then there exists $g' \in \mathcal{G}M$ arbitrarily close to g such that γ is a geodesic on (M,g') and the k-jet of the Poincaré map of γ with respect to g' belongs to Q. Different proofs of these results for $k=1$ and $k=3$ were given in [K2]. In fact, both [KT] and [K2] do not contain detailed proofs of the global theorem in the case $k=1$ which includes the Bumpy Metric Theorem. The latter theorem was proved in details by Anosov [An] using the local theorem in [KT].

The main results in this chapter, which are analogous to special cases of the results in [KT], are taken from [S3]. The material in Section 7.1 is a mixture of parts of [KT] and [S3], presented in a different form here. Sections 7.2 and 7.3 follow [S3] with some modifications, while the globalization argument in Section 7.4 is a modification of Section 4 in [An]. Lemma 7.3.3 is an analogue of the main Lemma 1 in [An].

Complete analogues of the results in [KT] for hypersurfaces in $\mathbb{R}^n$ endowed with the standard metric were established in [ST].

Generic properties of more general Hamiltonian systems were studied by Robinson [R], Meyer and Palmore [MeyP], Takens [T], Rademacher [Ra], Contreras [Con] and many others (see also [AbM] and the historical remarks and the references in the articles just cited). It is worth mentioning that the analogue of the Bumpy Metric Theorem for general Hamiltonian systems is not true, as an example in [MeyP] shows.

8

Inverse spectral results for generic bounded domains

In this chapter we study the Poisson relation for generic bounded domains Ω in $\mathbb{R}^n$, $n \geq 2$, with smooth boundaries $\partial\Omega$. We begin with the simplest case $n = 2$ and show that for generic $\Omega \subset \mathbb{R}^2$, without any conditions concerning convexity, the Poisson relation becomes an equality. A similar result for $n \geq 3$ is only known in the case of strictly convex domains. The main tools used in the proof of this are the ones developed in Chapters 3 and 4. Here we also have to deal with the lengths T_γ of the closed geodesics γ on $\partial\Omega$. To do this, one needs to know that non-degenerate closed geodesics on $\partial\Omega$ can be approximated arbitrarily well by periodic reflecting rays in Ω. A proof of this approximation result is given in Section 8.3 using Melrose' interpolating Hamiltonians studied in Section 8.2. Throughout this chapter $\Im(c)$ denotes Image (c).

8.1 Planar domains

In this section we establish that for generic bounded domains Ω in $\mathbb{R}^2$ with smooth boundaries $X = \partial\Omega$ the Poisson relation becomes an equality, that is we have

$$\text{sing supp } \sigma_\Omega = \{0\} \cup \{\pm T_\gamma : \gamma \in \mathcal{L}_\Omega\}. \tag{8.1}$$

In the proof of this we use results from Chapters 3, 4 and 6. The main point here is to show that for generic Ω, $\mathcal{L}_\Omega$ consists only of periodic reflecting rays and, possibly, multiples of the boundary $\partial\Omega$.

Geometry of the Generalized Geodesic Flow and Inverse Spectral Problems, Second Edition.
Vesselin M. Petkov and Luchezar N. Stoyanov.

Let X be an arbitrary fixed smooth curve in $\mathbb{R}^2$. A curve γ in $\mathbb{R}^2$ of the form

$$\gamma = \cup_{i=1}^{k-1} \ell_i$$

will be called a *degenerate broken ray* for X if the following conditions are satisfied:

(i) there exist points $x_1, \dots, x_k \in X$ such that $\ell_i = [x_i, x_{i+1}]$ and the interior of the segment ℓ_i does not intersect X transversally for every $i = 1, \dots, k-1$;

(ii) ℓ_i and ℓ_{i+1} satisfy the law of reflection at x_{i+1} with respect to X for every $i = 1, \dots, k-2$;

(iii) the curvature of X vanishes at x_1 and x_k, and ℓ_1 and ℓ_{k-1} are tangent to X at x_1 and x_k, respectively.

The points $x_1, \dots, x_k$ will be called *vertices* of γ.

Clearly when $k = 2$ the condition (ii) can be dropped. Examples of degenerate broken rays are given in Figures 8.1 and 8.2. If γ contains a segment orthogonal to X at some of its end points, we will say that γ is *symmetric* (see Figure 8.2); otherwise γ will be called *non-symmetric*. Notice that $\ell_1 = \ell_{k-1}$ for symmetric γ.

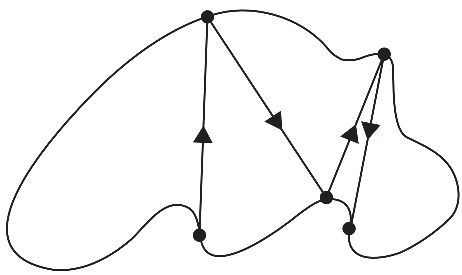

Figure 8.1 A degenerate trajectory.

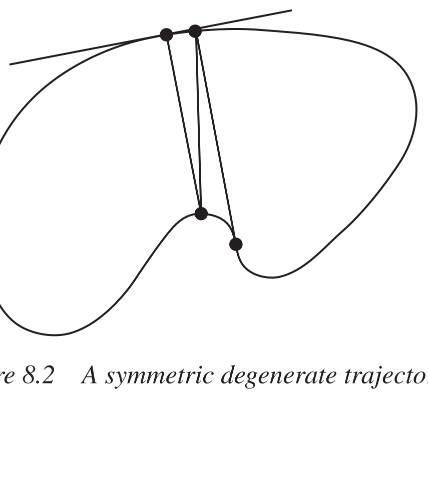

Figure 8.2 A symmetric degenerate trajectory.

Proposition 8.1.1: *There exists a residual subset $\mathcal{D}$ of $\mathbf{C}(X)$ such that for every $f \in \mathcal{D}$ there are no degenerate broken rays for $f(X)$.*

Proof of Proposition 8.1.1: We will use the technique from Sections 6.2–6.4, slightly changing the definitions of the main objects.

Fix arbitrary integers $k \geq s \geq 2$ and consider a non-symmetric map

$$\alpha : \{1, \dots, k\} \longrightarrow \{1, \dots, s\} \tag{8.2}$$

(see the definition in Section 6.2) such that

$$\alpha(1) = 1, \quad \alpha(2) = 2, \quad \alpha(k) = s. \tag{8.3}$$

We will use the notation $I_i = I_i(\alpha)$ given by (6.25). As in Chapter 6, denote by U_α the set of all $y = (y_1, \dots, y_s) \in (\mathbb{R}^2)^{(s)}$ such that

$$y_i \notin \text{conv}\{y_j : j \in I_i\}$$

for all $i = 1, \dots, s-1$. Define $H = H_\alpha : U_\alpha \longrightarrow \mathbb{R}$ by

$$H(y) = \sum_{i=1}^{s-1} \|y_{\alpha(i)} - y_{\alpha(i+1)}\|.$$

Notice that, given a non-symmetric degenerate broken ray γ for $Y = f(X)$ with $f \in \mathbf{C}(X)$, there exist integers $k \geq s \geq 2$, a non-symmetric map (8.2) with (8.3) and distinct points $y_1 = f(x_1), \dots, y_s = f(x_s) \in Y$ such that $y_{\alpha(1)}, \dots, y_{\alpha(k)}$ are the successive vertices of γ. Then we will say that γ has *type* α. In this case we have $x = (x_1, \dots, x_s) \in X^{(s)}$, $f^s(x) \in U_\alpha$ and

$$\text{grad}_{\,x'}(H \circ f^s)(x) = 0, \tag{8.4}$$

where

$$x' = (x_2, \dots, x_{s-1}) \in X^{(s-2)}.$$

The proof of (8.4) is almost the same as that of Proposition 6.2.1, and we leave it as an exercise to the reader. Notice that the curvature of $Y = f(X)$ vanishes at the points $f(x_1)$ and $f(x_s)$ and the segment $[f(x_1), f(x_2)]$ is tangent to $f(X)$ at $f(x_2)$. Similar conditions are satisfied at $f(x_s)$; however, we do not need them here.

Next, consider the open subset

$$V = \{j^2 f(x) \in J^2(X, \mathbb{R}^2) \colon \text{rank}\ df(x) = 1\}$$

of $J^2(X, \mathbb{R}^2)$. Denote by M the set of all

$$\sigma = (j^2 f_1(x_1), \dots, j^2 f_s(x_s)) \tag{8.5}$$

such that $\sigma \in V^s$, $x = (x_1, \dots, x_s) \in X^{(s)}$, $f^s(x) \in U_\alpha$. Clearly M is an open subset of the smooth manifold $J^2_s(X, \mathbb{R}^2)$. Let Σ be the set of those elements (8.5) of M such that the curvature of $f_i(X)$ vanishes at $f_i(x_i)$ for $i = 1$ and $i = s$,

$$\text{grad}_{\,x'}(H \circ (f_1 \times \cdots \times f_s))(x) = 0,$$

and

$$\langle f_2(x_2) - f_1(x_1), N_1 \rangle = 0,$$

N_1 being a non-zero normal vector to $f_1(X)$ at $f_1(x_1)$. It then follows from the above that for any f in the set

$$\mathcal{D}_\alpha = \{ f \in \mathbf{C}(X) : j^2_s f(X^{(s)}) \cap \Sigma = \emptyset \}, \tag{8.6}$$

there are no degenerate broken rays of type α for $f(X)$.

We will now show that $\mathcal{D}_\alpha$ is a residual subset of $\mathbf{C}(X)$. To do this, we will first prove that Σ is a smooth submanifold of M of codimension $s + 1$.

Consider a coordinate neighbourhood D of an element of Σ in M. We may assume that

$$D = M \cap \prod_{i=1}^{s} J^2(V_i, \mathbb{R}^2),$$

where $V_1, \dots, V_s$ are coordinate neighbourhoods of different elements of X such that $V_i \cap V_j = \emptyset$ whenever $i \neq j$. Fix arbitrary smooth charts $\varphi_i : V_i \longrightarrow \mathbb{R}$ and define the chart

$$\varphi : D \longrightarrow \mathbb{R}^{(s)} \times (\mathbb{R}^2)^{(s)} \times \mathbb{R}^{2s} \times \mathbb{R}^{2s}$$

by $\varphi(\sigma) = (u; v; a; b)$ for every $\sigma \in D$ of the form (8.5), where u and v are determined by (6.12) and (6.13),

$$a = (a_i^{(t)})_{1 \le i \le s, 1 \le t \le 2}, \quad b = (b_i^{(t)})_{1 \le i \le s, 1 \le t \le 2},$$
$$a_i^{(t)} = \frac{\partial (f_i^{(t)} \circ \varphi_i^{-1})}{\partial u_i}(u_i), \quad b_i^{(t)} = \frac{\partial^2 (f_i^{(t)} \circ \varphi_i^{-1})}{\partial u_i^2}(u_i) \tag{8.7}$$

for all $i = 1, \dots, s$ and $t = 1, 2$. Recall that the vector $N_i = (N_i^{(1)}, N_i^{(2)})$ determined by (6.42) for $n = 2$ is orthogonal to $f_i(X)$ at the point $f_i(x_i)$. It is easy to see that the curvature of $f_i(X)$ vanishes at $f_i(x_i)$ if and only if $a_i^{(1)} b_i^{(2)} - a_i^{(2)} b_i^{(1)} = 0$.

To see that Σ is a smooth submanifold of M of codimension $s + 1$, it is sufficient to establish that $\varphi(D \cap \Sigma)$ is a smooth submanifold of $\varphi(D)$ of the same codimension. We will use the procedure applied several times in Chapter 6.

Define the map $R : \varphi(D) \longrightarrow \mathbb{R}^{s+1}$ by

$$R(\xi) = (\, (c_i(\xi))_{2 \le i \le s-1}\, ;\, L_1(\xi)\, ;\, L_s(\xi)\, ;\, K(\xi)\,),$$

where the elements ξ of $\varphi(D)$ are written in the form $\xi = (u; v; a; b)$ with u, v, a, b given by (6.12), (6.13) and (8.7), and the functions c_i, L_1, L_s and K are defined as follows:

$$c_i(\xi) = \frac{\partial H}{\partial y_i^{(1)}}(v)a_i^{(1)} + \frac{\partial H}{\partial y_i^{(2)}}(v)a_i^{(2)}, \quad i = 2, \dots, s-1,$$
$$L_j(\xi) = a_j^{(1)} b_j^{(2)} - a_j^{(2)} b_j^{(1)}, \quad j = 1, s,$$
$$K(\xi) = (v_2^{(1)} - v_1^{(1)})a_1^{(2)} - (v_2^{(2)} - v_1^{(2)})a_1^{(1)}.$$

It is now clear that

$$\varphi(D \cap \Sigma) = R^{-1}(0),$$

so showing that R is a submersion on $\varphi(D)$ will prove that $\varphi(D \cap \Sigma)$ is a smooth submanifold of $\varphi(D)$ of codimension $s+1$.

Let $\xi \in \varphi(D)$ and assume that

$$\sum_{i=2}^{s-1} C_i \text{grad } c_i(\xi) + A_1 \text{grad } L_1(\xi) + A_s \text{grad } L_s(\xi) + B \text{grad } K(\xi) = 0$$

for some constants C_i, A_j and B. For a given $j = 1, s$ we have either $a_j^{(1)} \neq 0$ or $a_j^{(2)} \neq 0$. Thus, considering the derivatives with respect to $b_j^{(t)}$ in the above equality, we get $A_1 = A_s = 0$. In a similar way, using the fact that the functions $c_i(\xi)$ do not depend on $a_1^{(t)}$ and $v_1 = (v_1^{(1)}, v_1^{(2)}) \neq 0$, one gets $B = 0$. Finally, using an argument very similar to that in the proof of Lemma 6.1.2 and an obvious analogue of Lemma 6.2.2 for the present function H, we obtain $C_i = 0$ for all $i = 2, \dots, s-1$. Hence, R is a submersion on $\varphi(D)$, so $\varphi(D \cap \Sigma)$ is a smooth submanifold of $\varphi(D)$ of codimension $s+1$.

In this way we have established that Σ is a smooth submanifold of M of codimension $s+1$. By the definition $\mathcal{D}_\alpha$ we have

$$\mathcal{D}_\alpha = \{f \in \mathbf{C}(X) : j_s^2 f \pitchfork \Sigma\}$$

(see e.g. the end of the proof of Theorem 6.1.1), and now Theorem 1.1.2 implies that $\mathcal{D}_\alpha$ is a residual subset of $\mathbf{C}(X)$.

Set $\mathcal{D}' = \cap_\alpha \mathcal{D}_\alpha$, where α runs over the set of all non-symmetric maps (8.2) with (8.3). Then $\mathcal{D}'$ is a residual subset of $\mathbf{C}(X)$. It follows from the above that for any $f \in \mathcal{D}'$, there are no non-symmetric degenerate broken rays for $f(X)$.

As in Chapter 6, the treatment of the symmetric case is very similar to the non-symmetric case. We leave it to the reader to prove in details that there exists a residual subset $\mathcal{D}''$ of $\mathbf{C}(X)$ so that for any $f \in \mathcal{D}''$ there are no symmetric degenerate broken rays for $f(X)$. Then the set $\mathcal{D} = \mathcal{D}' \cap \mathcal{D}''$ has the required properties. ■

Using the above proposition, we intend to show that for generic Ω in $\mathbb{R}^2$ every generalized periodic geodesic in Ω, which is not contained in $\partial\Omega$, is a periodic reflecting ray. To do so, we will apply results of Melrose and Sjöstrand on properties of generalized geodesics; however, we need to know that the curvature of $\partial\Omega$ does not vanish of infinite order. The following proposition shows that a much stronger property is satisfied by generic domains. We prove it in a more general form, having in mind another application in Chapter 11.

Proposition 8.1.2: *Let X be a smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$, $n \geq 2$, and let $\mathcal{K}$ be the set of those $f \in \mathbf{C}(X)$ for which there are no points $y \in f(X)$ and directions $v \in T_y F(X) \setminus \{0\}$ such that the curvature of $f(X)$ at y vanishes of order $2n-3$ in the direction of v. Then $\mathcal{K}$ is a residual subset of $\mathbf{C}(X)$.*

In the proof of this proposition we will use the following lemma that can be established using the argument from the proof of Thom's Transversality Theorem (see [GG]). We omit the details.

Lemma 8.1.3: *Let Σ, X and Y be smooth manifolds, let $k \geq 1$ be an integer and let $g : \Sigma \longrightarrow J^k(X, Y)$ be a smooth map. Then*

$$R = \{f \in C^\infty(X, Y) : g \pitchfork j^k f\}$$

is a residual subset of $C^\infty(X, Y)$.

Proof of Proposition 8.1.2: Set $k = 2n - 1$ and denote by $\mathbb{P}R^n$ the *projective space* of all lines through 0 in $\mathbb{R}^n$. Let M be the set of all

$$(w, j^k f(x)) \in \mathbb{P}\mathbb{R}^n \times J^k(X, \mathbb{R}^n)$$

such that $\mathrm{rank}(df(x)) = n - 1$ and w is tangent to $f(X)$ at $f(x)$. It is easily seen that M is a smooth submanifold of $\mathbb{P}R^n \times J^k(X, \mathbb{R}^n)$. Let Σ be the set of those $(w, j^k f(x)) \in M$ such that the curvature of $f(X)$ vanishes of order $2n - 3$ in the direction of w. In the following we will describe this last condition analytically.

We claim that Σ is a smooth submanifold of M of codimension $2n - 2$. To prove this, consider an open covering $\{\mathcal{O}_j\}_{j=1}^n$ of $\mathbb{P}\mathbb{R}^n$, where $\mathcal{O}_j$ is determined by all vectors $w \in \mathbb{R}^n$ with $w^{(j)} \neq 0$. It is sufficient to show that for every j,

$$\Sigma_j = \{(w, j^k f(x)) \in \Sigma : w \in \mathcal{O}_j\}$$

is a submanifold of M of codimension $2n - 2$. We will do this for $j = 1$; the other cases are similar.

Consider an arbitrary smooth chart $\varphi : U \longrightarrow V \subset X$, where U is an open subset of $\mathbb{R}^{n-1}$ and V is an open neighbourhood of some element of X. Define the chart

$$\psi : W = M \cap (\mathcal{O}_1 \times J^k(X, \mathbb{R}^n)) \longrightarrow \mathbb{R}^{n-2} \times \mathbb{R}^{n-1} \times \mathbb{R}^n \times \mathbb{R}^n$$

by

$$\psi(w, j^k f(x)) = (\lambda; u; v; a).$$

Here $\lambda = (\lambda_2, \dots, \lambda_{n-1}) \in \mathbb{R}^{n-2} \setminus \{0\}$, w is determined by the vector

$$\frac{\partial\varphi}{\partial u_1}(u) + \sum_{j=2}^{n-1} \lambda_j \frac{\partial\varphi}{\partial u_j}(u), \quad \varphi(u) = x,$$

$v = f(x)$ and $a = (a_{\mathbf{i}}^{(t)})$ is given by

$$a_{\mathbf{i}}^{(t)} = \frac{\partial^{\mathbf{i}}(f^{(t)} \circ \varphi)}{\partial u^{\mathbf{i}}}(u)$$

for every $t = 1, \dots, n$ and every multiindex $\mathbf{i} = (i_1, \dots, i_{n-1})$ with

$$|\mathbf{i}| = i_1 + \dots + i_k \leq k.$$

As before, we use the notation $f = (f^{(1)}, \dots, f^{(n)})$.

The assertion will be proved if we show that $\psi(\Sigma \cap W)$ is a smooth submanifold of $\psi(W)$ of codimension $2n - 2$. For each $u \in U$ choose a unit normal vector $\nu(u)$ to $f(X)$ at $f(\varphi(u))$ so that the map $u \mapsto \nu(u)$ is continuous. Then the coefficients $b_{ij}(u)$ of the second fundamental form of $f(X)$ at $f(x)$, $x = \varphi(u)$, are determined by

$$b_{ij}(u) = \left\langle \frac{\partial^2\varphi}{\partial u_i \partial u_j}(u) \, , \nu(u) \right\rangle,$$

for $i, j = 1, \dots, n-1$. Now set $\lambda_1 = 1$, and for $j = 0, 1, 2, \dots,$ consider the functions

$$K_j(\lambda; u) = \left(\sum_{i=1}^{n-1} \lambda_i \frac{\partial}{\partial u_i} \right)^j \left(\sum_{p,q=1}^{n-1} \lambda_p \lambda_q b_{pq}(u) \right).$$

The curvature of $f(X)$ at $f(x)$ vanishes of order m in the direction of

$$w(\lambda) = \sum_{i=1}^{n-1} \lambda_i \frac{\partial\varphi}{\partial u_i}(u)$$

if and only if $K_j(\lambda; u) = 0$ for all $j = 0, 1, \dots, m$ (see e.g. [GKM]).

Clearly, there exist smooth functions

$$\tilde{\nu}, \tilde{K}_j : \psi(W) \longrightarrow \mathbb{R}$$

such that $\tilde{\nu}(\xi) = \nu(u)$ and $\tilde{K}_j(\xi) = K_j(\lambda; u)$ whenever $\xi = (\lambda; u; v; a) = \psi(w; j^k f(x))$. Define the map

$$K : \psi(W) \longrightarrow \mathbb{R}^{2n-2}$$

by $K(\xi) = (\tilde{K}_j(\xi))_{j=0}^{2n-3}$. Then $\psi(W \cap \Sigma) = K^{-1}(0)$, and it is sufficient to show that K is a submersion at any point of $\psi(W \cap \Sigma)$.

Let $\xi = (\lambda; u; v; a) \in \psi(W \cap \Sigma)$, and let

$$\sum_{j=0}^{2n-3} A_j \text{grad } \tilde{K}_j(\xi) = 0 \tag{8.8}$$

for some real constants A_j. Since $W \subset M$ and $\xi \in \psi(W)$, there exists $t = 1, \ldots, n$ with $\tilde{\nu}^{(t)}(\xi) \neq 0$. Fix such a t and set $\mathbf{i} = (2n-1, 0, \ldots, 0)$. Next, considering the derivatives with respect to $a_{\mathbf{i}}^{(t)}$ in (8.7), we get $A_{2n-3} = 0$. In the same way one obtains $A_{2n-4} = 0$, etc. Thus, $A_j = 0$ for all j, so K is a submersion. This shows that Σ is a smooth submanifold of M of codimension $2n-2$.

Now we can apply Lemma 8.1.3 for Σ, X, $Y = \mathbb{R}^n$ and $g = \pi \circ i$, where $i : \Sigma \longrightarrow M$ is the inclusion and $\pi : M \longrightarrow J^k(X, \mathbb{R}^n)$ is the natural projection. Since

$$\text{codim } (\Im(T_\sigma \pi)) \geq (2n-2) - (n-2) = n$$

and $\dim\,(X) = n - 1 < n$, the condition $j^k f \,\overline{\pitchfork}\, g$ is equivalent to the relation $j^k f(X) \cap \pi(\Sigma) = \emptyset$. On the other hand, $\mathcal{K}$ coincides with the set of those $f \in \mathbf{C}(X)$ satisfying the latter relation. Therefore, by Lemma 8.1.3, $\mathcal{K}$ is a residual subset of $\mathbf{C}(X)$. ■

In the rest of this section we consider the case $X \subset \mathbb{R}^2$, that is $n = 2$. We will also assume that X is **compact and connected**; then $X = \partial\Omega$ for some bounded domain Ω in $\mathbb{R}^2$. Given $f \in \mathbf{C}(X)$, we will denote by Ω_f the bounded domain in $\mathbb{R}^2$ with boundary $f(X)$ and by L_f the length of the curve $f(X)$. Using the notation in Propositions 8.1.1 and 8.1.2, we have the following.

Corollary 8.1.4: *Let $f \in \mathcal{D} \cap \mathcal{K}$. Then every periodic generalized geodesic in Ω_f which is not contained in $f(X)$ is a periodic reflecting ray in Ω_f.*

Proof: Consider an arbitrary periodic generalized geodesic γ in Ω_f which is not entirely contained in $f(X)$. Since $f \in \mathcal{K}$, the curvature of $\partial\Omega_f = f(X)$ can only simply vanish. Therefore, γ is the projection of a uniquely extendible bicharacteristic of $\Box$, so γ consists of a finite number of linear segments and segments (arcs) of $\partial\Omega_f$ (see Section 1.2). Clearly, γ has at least one linear segment with points in the interior of Ω_f. If γ contains at least one non-trivial segment (arc) of $\partial\Omega_f$, then it would contain a whole degenerate broken ray for $f(X)$, which is a contradiction with $f \in \mathcal{D}$. Thus, γ consists of linear segments only, so it is a periodic reflecting ray in Ω_f. ■

The next lemma will be useful when we consider generic strictly convex domains Ω in $\mathbb{R}^2$ and try to prove that the length of $\partial\Omega$ is contained in $\text{sing supp}\sigma_\Omega$.

Lemma 8.1.5: *There exists a residual subset $\mathcal{W}$ of $\mathbf{C}(X)$ such that every $f \in \mathcal{W}$ has the following properties:*

(a) every periodic reflecting ray in Ω_f is ordinary and non-degenerate;

(b) $T_\gamma/T_\delta \notin \mathbb{Q}$ for every two distinct primitive periodic reflecting rays γ and δ in Ω_f;

(c) for every integer $s \geq 2$ there are only finitely many periodic reflecting rays in Ω_f with s reflection points;

(d) $T_\gamma/L_f \notin \mathbb{Q}$ for every periodic reflecting ray γ in Ω_f.

Proof of Lemma 8.1.5: It follows from Theorems 6.2.3, 6.3.1, 6.4.1 and 6.4.3 that there exists a residual subset $\mathcal{W}'$ such that every $f \in \mathcal{W}'$ has the properties (a)–(c).

Next, fix arbitrary $p, q, s \in \mathbb{N}$, $s \geq 2$, and denote by $\mathcal{W}(p,q,s)$ the set of those $f \in \mathcal{W}'$ such that $pT_\gamma \neq qL_f$ for every periodic reflecting ray γ in Ω_f having s reflection points. We will show that $\mathcal{W}(p,q,s)$ is open and dense in $\mathcal{W}'$.

To establish the density, we may assume that $\mathrm{id} \in \mathcal{W}'$. Then we have to prove that $\mathcal{W}(p,q,s)$ contains elements arbitrarily close to id in the C^∞ topology. Let $\gamma_1, \ldots, \gamma_m$ be all the periodic reflecting rays in $\Omega = \Omega_{\mathrm{id}}$ with s reflection points; since $\mathrm{id} \in \mathcal{W}'$, there are only finitely many of them. There exists a non-trivial closed segment (arc) Δ of X which does not contain reflection points of γ_i for all $i = 1, \ldots, m$. We claim that for every $f \in \mathcal{W}'$ which is sufficiently close to id and $f(x) = x$ for all $x \in X \setminus \Delta$, the only periodic reflecting rays in Ω_f with s reflection points are $\gamma_1, \ldots, \gamma_m$. If not, there would exist a sequence $\{f_k\} \subset \mathcal{W}'$ converging to id such that for every k there exists a periodic reflecting ray δ_k in Ω_{f_k} with s reflection points $y_{1,k}, \ldots, y_{m,k}$ and $\delta_k \neq \gamma_i$ for all $i = 1, \ldots, m$. The latter implies that at least one reflection point of δ_k is in $f_k(\Delta)$. We may assume that $y_{1,k} \in f_k(\Delta)$ for all k. Using the compactness of X, we may assume that there exists $\lim_{k\to\infty} y_{i,k} = y_i$ for all $i = 1, \ldots, m$. Then $y_1 \in \Delta$, and a simple continuity argument shows that $y_1, \ldots, y_m$ are the reflection points of a periodic reflection ray γ for X. Now $y_1 \in \Delta$ shows that $\gamma \neq \gamma_i$ for all i, which is a contradiction. Thus, for every $f \in \mathcal{W}'$ sufficiently close to id and $f = \mathrm{id}$ on $X \setminus \Delta$, $\gamma_1, \ldots, \gamma_m$ are the only periodic reflecting rays in Ω_f with s reflection points. Clearly we can choose such $f \in \mathcal{W}'$ so that $pT_{\gamma_i} \neq qL_f$ for all $i = 1, \ldots, m$; then we will have $f \in \mathcal{W}(p,q,s)$. This shows that $\mathcal{W}(p,q,s)$ is dense in $\mathcal{W}'$.

To prove that $\mathcal{W}(p,q,s)$ is open in $\mathcal{W}'$, consider an arbitrary sequence $\{f_k\} \subset \mathcal{W}' \setminus \mathcal{W}(p,q,s)$ with $f_k \to f \in \mathcal{W}'$. We have to show that $f \notin \mathcal{W}(p,q,s)$. Without loss of generality, we may assume that $f = \mathrm{id}$. By the assumptions, for every k there exists a periodic reflecting ray δ_k in Ω_{f_k} with s reflection points $y_{1,k}, \ldots, y_{s,k}$ such that $pT_{\delta_k} = qL_{f_k}$. Using again the compactness of X, we may assume that there exist $\lim_{k\to\infty} y_{i,k} = y_i$ for all $i = 1, \ldots, s$. Then $y_1, \ldots, y_s$ are the reflection points of a periodic reflecting ray δ in Ω with $pT_\delta = qL_f$, which implies $f = \mathrm{id} \notin \mathcal{W}(p,q,s)$. This proves that $\mathcal{W}(p,q,s)$ is open in $\mathcal{W}'$.

Setting

$$\mathcal{W} = \bigcap_{p,q,s\in\mathbb{N},\, s\geq 2} \mathcal{W}(p,q,s),$$

we obtain a residual subset of $\mathbf{C}(X)$ having the required properties. ■

What follows now is the main point in this section. Denote by Ξ the *family of all bounded domains* Ω *in* $\mathbb{R}^2$ *with smooth connected boundaries* $\partial\Omega$ *with the following properties*:

(ND) every periodic reflecting ray in Ω is ordinary and non-degenerate;

(R) $T_\gamma/T_\delta \notin \mathbb{Q}$ for every two distinct primitive trajectories γ and δ in $\mathcal{L}_\Omega$;

(K) the curvature of $\partial\Omega$ does not vanish of order $k \geq 1$, and there are no degenerate broken rays in Ω.

Notice that for strictly convex Ω the condition (K) is trivially satisfied. If $X = \partial\Omega = \Im(\gamma)$ is (smoothly) parameterized by a parameter $t \in \mathbb{R}$ and $\kappa(t)$ is the curvature of X at $\gamma(t)$, then (K) implies that whenever $\kappa(t) = 0$ for some t, then $\kappa'(t) \neq 0$.

As in Chapters 3 and 4, given a domain Ω, we denote by $\{\lambda_j^2\}$ the spectrum of the Laplace operator in Ω and by $\sigma(t) = \sigma_\Omega(t)$ the corresponding distribution defined in Section 3.2.

Theorem 8.1.6:
(a) For every bounded domain Ω *in* $\mathbb{R}^2$ *with smooth connected boundary* $X = \partial\Omega$*, there exists a residual subset* $\mathcal{W}$ *of* $\mathbf{C}(X)$ *such that* $\Omega_f \in \Xi$ *for all* $f \in \mathcal{W}$.

(b) The equality (8.1) holds for every $\Omega \in \Xi$*. Moreover, if* $\Omega \in \Xi$*, then for every periodic reflecting ray* γ *in* Ω*,* $\mathrm{spec}(P_\gamma)$ *is uniquely determined from the spectrum* $\{\lambda_j^2\}$.

Before proceeding with the proof of the theorem, let us mention that the second part in (b) means the following: If for $\Omega_1, \Omega_2 \in \Xi$, the corresponding spectra $\{\lambda_j^2\}$ of the Dirichlet problems for the Laplacians are the same, then there exists a bijection $\mu : \mathcal{L}_{\Omega_1} \longrightarrow \mathcal{L}_{\Omega_2}$, $\mu(\gamma) = \gamma'$, such that $T_\gamma = T_{\gamma'}$ and $\mathrm{spec}(P_\gamma) = \mathrm{spec}(P_{\gamma'})$ for every $\gamma \in \mathcal{L}_{\Omega_1}$.

Proof of Theorem 8.1.6:
(a) Given Ω with $X = \partial\Omega$, consider the residual set $\mathcal{W}$ from Lemma 8.1.5. Then each $f \in \mathcal{W}$ has the properties (a)–(d) from Lemma 8.1.5. Hence, $\Omega_f \in \Xi$.

(b) Let $\Omega \in \Xi$. Then for every periodic reflecting ray γ in Ω, the conditions (ND) and (R) are satisfied. Thus, for each $\gamma \in \mathcal{L}_\Omega$ the assumptions of Theorem 4.3.1 are fulfilled, so $T_\gamma \in \mathrm{sing\,supp}\sigma_\Omega$.

First, suppose that Ω is strictly convex. Parameterizing $\partial\Omega$ by arc length, we obtain a primitive generalized geodesic δ. It follows from Example 2.1.1 that there exists a sequence $\{\gamma_k\}$ of periodic reflecting rays in Ω such that $T_{\gamma_k} \to T_\delta$ as $k \to \infty$. Since $\mathrm{sing\,supp}\sigma_\Omega$ is a closed subset of $\mathbb{R}$, it follows that $T_\delta \in \mathrm{sing\,supp}\sigma_\Omega$. Now Theorems 3.4.7 and 4.3.1 show that (8.1) holds.

Next, assume that $\Omega \in \Xi$ is not strictly convex. We will show that all elements of $\mathcal{L}_\Omega$ are periodic reflecting rays. Assume that there exists $\delta \in \mathcal{L}_\Omega$ which is not a periodic reflecting ray. Then the condition (K) and the connectedness of $X = \partial\Omega$ yield that $\Im(\delta) = X$. On the other hand, since Ω is not strictly convex, there exists a point $x \in X$ at which the curvature of X vanishes. Let $\delta(t_0) = x$, and let $\kappa(t)$ be the curvature of X at $\delta(t)$. Since $\kappa(t_0) = 0$, the condition (K) implies $\kappa'(t_0) \neq 0$, so $\kappa(t)$ changes its sign at $t = t_0$. Then by the properties of generalized bicharacteristics discussed in Proposition 1.2.3, it follows that there exists $\epsilon > 0$ such that either $\{\delta(t) : t_0 - \epsilon < t \leq t_0\}$ or $\{\delta(t) : t_0 \leq t < t_0 + \epsilon\}$ is a linear segment parallel to $\dot{\delta}(t_0)$ and having an end x. Now $\Im(\delta) = X$ implies that X contains a non-trivial linear segment which is a contradiction with (K). Thus, in this case $\mathcal{L}_\Omega$ contains only periodic reflecting rays, and as above we conclude that the equality (8.1) holds.

Finally, let $\Omega \in \Xi$ and let γ be a periodic reflecting ray in Ω. We have to show that $\{\lambda_j^2\}$ determines $\mathrm{spec}(P_\gamma)$. Since γ is non-degenerate, we have $\mathrm{spec}(P_\gamma) = \{a, 1/a\}$ for some $a \neq \pm 1$. It follows from Theorem 4.3.1 that for t close to T_γ we have

$$(2\pi)\sigma_\Omega(t) = \mathrm{Re}(cT_\gamma^\sharp |\det(I - P_\gamma)|^{-1/2}(t - T_\gamma - \mathbf{i}0)^{-1}) + L^1_{loc},$$

where $c \in \{\pm 1, \pm\mathbf{i}\}$ is a constant. Since

$$(t - T_\gamma - \mathbf{i}0)^{-1} - (t - T_\gamma + \mathbf{i}0)^{-1} = 2\pi\mathbf{i}\ \delta(t - T_\gamma),$$

from $\sigma_\Omega(t)$, we determine

$$T_\gamma^\sharp |\det(I - P_\gamma)|^{-1/2}.$$

Therefore, $T_\gamma^\sharp |\det(I - P_\gamma)|^{-1/2}$ can be determined from $\{\lambda_j^2\}$. On the other hand, since $\Omega \in \Xi$, $T_\gamma^\sharp$ can be determined by T_γ. In fact, $T_\gamma^\sharp$ is the smallest positive number $u \in \text{sing supp}\sigma$ such that T_γ/u is an integer. Hence we can determine the number

$$d = |\det(I - P_\gamma)| = 2 - (a + 1/a).$$

Since the elements of $\mathrm{spec}(P_\gamma)$ are the roots of the equation $x^2 - (2 - d)x + 1 = 0$, this completes the proof of the assertion. ■

8.2 Interpolating Hamiltonians

The present and the next sections are devoted to the study of the billiard ball map B in a small neighbourhood of a closed geodesic on the boundary of a strictly convex compact domain in $\mathbb{R}^n$. A very useful tool here will be the so-called interpolating Hamiltonians. Their existence was established by Melrose [Me1] in a more general situation which we are now going to present briefly.

Let (S, ω) be a smooth symplectic manifold with boundary ∂S and $\dim\ (S) = 2m + 2$, and let F and G be two hypersurfaces of S, that is F and G are smooth submanifolds of codimension 1 of S. Let $s_0 \in F \cap G$. There exists an open neighbourhood V of s_0 in S and C^∞ functions $f, g : V \longrightarrow \mathbb{R}^+$ such that

$$F \cap V = f^{-1}(0), \quad G \cap V = g^{-1}(0), \quad df \neq 0, \quad dg \neq 0 \text{ on } V.$$

Then f and g are called *defining functions* in V for F and G, respectively. The considerations that follow are only local near s_0, so we will assume that $V = S$, that is f and g are globally defined on S.

Assume that F and G have a transversal intersection at s_0, that is $df(s_0)$ and $dg(s_0)$ are linearly independent. Shrinking $S = V$ if necessary, we may assume that F and G have a transversal intersection at any point of

$$J = F \cap G.$$

Then J is a smooth submanifold of S of codimension 2. A point $s \in J$ will be called a *glancing point* of F and G if the Poisson bracket $\{f, g\}$ of f and g (see e.g. [AbM] or [H3]) vanishes at s. Thus,

$$K = \{s \in J : \{f, g\}(s) = 0\}$$

is the set of all glancing points of F and G. If $s \in K$ and

$$\{f, \{f, g\}\}(s) \neq 0, \quad \{g, \{f, g\}\}(s) \neq 0, \tag{8.9}$$

then s will be called a *non-degenerate glancing point*.

From now on we will assume that (8.9) holds for every $s \in K$. Then K is a smooth submanifold of S of codimension 3. To prove this, first recall that

$$\{f, g\} = X_f g,$$

where X_f is the *Hamiltonian vector field on S determined by the function f*, and that for every $s \in J$ the condition $\{f, g\}(s) = 0$ is equivalent to the fact that the integral curve of X_f (which is contained in F) is tangent to G at s. Given $s \in K$, we have $X_f f(s) = X_f g(s) = 0$, and if $d\{f, g\}(s)$ is a linear combination of $df(s)$ and $dg(s)$, then

$$\{f, \{f, g\}\}(s) = (X_f\{f, g\})(s) = 0,$$

in contradiction with (8.9). Thus, $df(s)$, $dg(s)$ and $d\{f, g\}(s)$ are linearly independent for every $s \in K$, which shows that K is a smooth submanifold of S of codimension 3.

Next, fix an arbitrary smooth submanifold M_F of F of codimension 1 intersecting transversally ∂S at $s_0 \in \partial S \cap F$. For $s \in F$ denote by $\pi_F(s)$ the intersection point of the integral curve of X_f through s with M_F. Assuming that $S = V$ is sufficiently small, the map

$$\pi_F : F \longrightarrow M_F$$

is a well-defined smooth submersion. In fact, M_F can be identified with the quotient space $F/\sim$, where $s_1 \sim s_2$ if s_1 and s_2 lie on the same integral curve of X_f in F. Then

$$\pi_F : F \longrightarrow F/\sim$$

is just the canonical projection.

In a similar way one defines $\pi_G : G \longrightarrow M_G$. Finally, set

$$J_F = \pi_F(J) \subset M_F, \quad J_G = \pi_G(J) \subset M_G.$$

We can now state the result of Melrose [Me1] which is crucial for the approximation theorem in the next section.

Theorem 8.2.1: *Under the assumptions and notation above, we have the following:*

(a) J_F has a natural structure of a smooth symplectic manifold inherited from (S, ω) such that $\pi_F : J \longrightarrow J_F$ is a smooth symplectic map and $\partial J_F = \pi_F(K)$. There exist two uniquely determined continuous maps

$$\alpha_\pm : J_F \longrightarrow J$$

such that

$$\pi_F \circ \alpha_\pm = \mathrm{id}, \quad \Im(\alpha_+) \cup \Im(\alpha_-) = J,$$

and $\alpha_\pm$ are smooth on $J_F \setminus \partial J_F$. A similar statement holds for J_G. Let

$$\beta_\pm : J_G \longrightarrow J$$

be the corresponding continuous inverses of π_G. Then the maps

$$\delta_\pm : J_F \longrightarrow J_F$$

defined by

$$\delta_\pm = \pi_F \circ \beta_\pm \circ \pi_G \circ \alpha_\pm,$$

are continuous on J_F and smooth and symplectic on $J_F \setminus \partial J_F$, $\delta_\pm(\partial J_F) \subset \partial J_F$, and locally $\delta_\pm \circ \delta_\mp = \mathrm{id}$.

(b) There exist smooth symplectic coordinates

$$(x;\xi) = (x_0, x_1, \ldots, x_m; \xi_0, \xi_1, \ldots, \xi_m)$$

in a neighbourhood of s_0 in S such that $s_0 = (0,0)$,

$$F = \{(x;\xi) \in S : x_0 = 0\}, \quad G = \{(x,\xi) \in S : \xi_0^2 - x_0 - \xi_1 = 0\}.$$

These coordinates induce canonical coordinates $(x_1, \dots, x_m; \xi_1, \dots, \xi_m)$ *in* J_F *such that* $\xi_1 \geq 0$ *in* J_F, $\partial J_F = \{\xi_1 = 0\}$, *with respect to which the maps* $\delta_\pm$ *have the form*

$$\delta_\pm(x_1, \dots, x_m; \xi_1, \dots, \xi_m) = (x_1 \pm 2\sqrt{\xi_1}, x_2, \dots, x_m; \xi_1, \dots, \xi_m).$$

For a proof of this theorem we refer the reader to [Me1] (see also Section 21.4 in [H3]).

As an immediate consequence of the above one gets the following.

Corollary 8.2.2: *Let* $s_0 \in \partial S$ *be a non-degenerate glancing point of* F *and* G. *There exists a neighbourhood* V *of* s_0 *in* S *and a smooth function* $h : V \longrightarrow \mathbb{R}^+$, *which is a defining function for* ∂S *in* V, *such that* $\delta_\pm$ *have the following form in* V*:*

$$\delta_\pm(s) = \exp(\pm\sqrt{h}\ X_h)(s). \tag{8.10}$$

If $h_1 : V_1 \longrightarrow \mathbb{R}^+$ *is another smooth function with these properties, then* $h - h_1$ *vanishes of infinite order on* $\partial S \cap V \cap V_1$.

A smooth function $h : V \longrightarrow \mathbb{R}^+$, which is a defining function for ∂S in V and satisfies (8.10), will be called a local *interpolating Hamiltonian* for the maps $\delta_\pm$.

The following elementary lemma concerns the canonical form of the maps $\delta_\pm$ from Theorem 8.2.1(b) and will be useful later.

Lemma 8.2.3: *Let* $a < b$, $\omega > 0$ *and* $\epsilon > 0$ *be real numbers such that* $\omega \leq \epsilon^2$. *Define the map*

$$u : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R} \times \mathbb{R}^+$$

by $u(x, \xi) = (x + \sqrt{\xi}, \xi)$. *Then for*

$$V = [a, a + \epsilon] \times (0, \omega],$$

we have

$$[a, \infty) \times (0, \omega] \subset \cup_{j=0}^{\infty} u^j(V).$$

Proof of Lemma 8.2.3: Fix arbitrary $y > a$ and $\eta \in (0, \omega]$. Notice that

$$u^j([a, a + \epsilon] \times \{\eta\}) = [a + j\sqrt{\eta}, a + \epsilon + j\sqrt{\eta}] \times \{\eta\}.$$

On the other hand, $a + \epsilon + j\sqrt{\eta} \geq a + (j + 1)\sqrt{\eta}$, since $\epsilon \geq \sqrt{\omega} \geq \sqrt{\eta}$. Therefore,

$$\cup_{j=0}^{\infty} u^j([a, a + \epsilon] \times \{\eta\}) = [a, \infty) \times \{\eta\},$$

which proves the assertion. ∎

Consider again a non-degenerate glancing point s_0 of F and G, and let U' be a neighbourhood of s_0 in S on which there exist smooth symplectic coordinates $x_1, \dots, x_m; \xi_1, \dots, \xi_m$ with the properties listed in Theorem 8.2.1(b). Notice that if for some $a < b$,

$$c : [a, b] \longrightarrow U = \pi_F(U') \subset J_F$$

is the projection under π_F of an integral curve of X_g through s_0, then c has the form

$$c(t) = (A + t, 0, \dots, 0; 0, \dots, 0)$$

for some real constant $A > 0$. Let ϵ and ω be such that $0 < \epsilon < b - a$, $0 < 4\omega < \epsilon^2$, and let

$$U(a, b, \omega) = \{(x; \xi) : x_1 \in [a, b], 0 < \xi_1 \leq \omega, |x_i| \leq \omega,$$
$$|\xi_i| \leq \omega, i = 2, \dots, m\} \subset U.$$

For every integer $j \geq 0$ set

$$U_j(a, a + \epsilon, \omega) = U(a, a + \epsilon, \omega) \cap \cap_{i=0}^{j} \delta_+^{-i}(U).$$

It follows from Lemma 8.2.3 and the form of the map δ_+ that

$$U(a, b, \omega) \subset \cup_{j=0}^{\infty} \delta_+^j(U_j(a, a + \epsilon, \omega)).$$

In particular,

$$\overline{\cup_{j=0}^{\infty} \delta_+^j(U_j(a, a + \epsilon, \omega))}$$

contains an open neighbourhood of $c((a, b))$ in J_F.

We will now consider the case when $\Im(c)$ is no longer contained in a small coordinate neighbourhood of a point.

For $\lambda > 0$ denote by V_λ the set of all $x_1, \dots, x_m; \xi_1, \dots, \xi_m \in \mathbb{R}^{2m}$ such that $x_1 \in (a - \lambda, b + \lambda)$, $\xi_1 \in [0, \lambda)$, $|x_i| < \lambda$ and $|\xi_i| < \lambda$ for all $i = 2, \dots, m$. Then V_λ is a smooth manifold with boundary $\partial V_\lambda = \{\xi_1 = 0\}$.

Lemma 8.2.4: *Let $a < b$ be real numbers and let*

$$\tilde{c} : [a, b] \longrightarrow K \subset G$$

be an integral curve of the Hamiltonian vector field X_g such that all points of $\Im(\tilde{c})$ are non-degenerate glancing points of F and G. Let $c = \pi_F \circ \tilde{c}$ be its projection in J_F, and suppose that δ_+ is a well-defined and continuous invertible map in a neighbourhood U of $\Im(c)$ having the properties listed in Theorem 8.2.1(b). Assume that for some $\lambda > 0$

$$\Psi : V_\lambda \longrightarrow U$$

is a smooth map that is a local diffeomorphism and

$$\Phi(t,0,\ldots,0;0,\ldots,0)=c(t),\quad t\in(a-\lambda,b+\lambda).$$

Then there exist $\mu\in(0,\lambda)$*, a continuous invertible map*

$$\tilde{\delta}_+ : V_\mu \longrightarrow V_\lambda \tag{8.11}$$

with

$$\Phi\circ\tilde{\delta}_+(v)=\delta_+\circ\Phi(v),\quad v\in V_\mu, \tag{8.12}$$

and a smooth function $\tilde{h}:V_\mu\longrightarrow\mathbb{R}^+$ *which is a defining function of* ∂V_μ *and*

$$\tilde{\delta}_+(v)=\left(\exp\left(\sqrt{\tilde{h}}\,X_{\tilde{h}}\right)\right)(v) \tag{8.13}$$

holds for all $v\in V_\mu$*, where* V_μ *is considered as a symplectic manifold with respect to the pull-back by* Φ *of the symplectic structure of* J_F *inherited from* S*.*

Proof of Lemma 8.2.4: Clearly $\Im(\tilde{c})$ can be covered by a finite family of coordinate neighbourhoods $U_1',\ldots,U_k'$, each of them possessing smooth symplecting coordinates with the properties listed in Theorem 8.2.1(b). We will consider only the case $k=2$; the general case follows in the same way using a simple induction on k.

So, assume that $\Im(\tilde{c})\subset U_1'\cup U_2'$. Then there exist open subsets U_1,U_2 of J_F such that

$$\Im(c)\subset U_1\cup U_2,\quad \overline{U_i}\subset\pi_F(U_i')\subset U\quad (i=1,2).$$

We may assume that $c(a)\in U_1$ and $c(b)\in U_2$. Set

$$V=U_1\cap U_2,\quad V_j=V\cap\cap_{i=0}^{j}\delta_+^{-i}(U),\quad j=0,1,2,\ \ldots\ .$$

According to the choice of the neighbourhoods U_i', there exist smooth symplectic coordinates $x_1,\ldots,x_m;\xi_1,\ldots,\xi_m$ in U_2 with the properties listed in Theorem 8.2.1(b). There exists $a'\in(a,b)$ such that

$$c([a,a'])\subset U_1,\quad c([a',b])\subset U_2.$$

With respect to the coordinates in U_2 we have

$$c(t)=(A+t,0,\ldots,0;0,\ldots,0)$$

for some constant A. Shifting the coordinate x_1, we may assume that $A=0$.

Next, fix an arbitrary $\epsilon>0$ such that $c([a',a'+\epsilon])\subset U_1\cap U_2$ and take $\omega>0$ with $4\omega<\epsilon^2$. We take ϵ and ω so small that

$$\begin{aligned}W=\{(x;\xi)\in U_2 : a'-\omega\le x_1\le b+\omega, 0\le\xi_1\le\omega,\\ |x_i|\le\omega,|\xi_i|\le\omega, i=2,\ldots,m\}\end{aligned}$$

is contained in $U_2 \cap \delta_+^{-1}(U_2)$, and

$$V = \{(x;\xi) \in W : a' \leq x_1 \leq a' + \epsilon\} \subset U_1 \cap U_2.$$

Using the reasoning just after Lemma 8.2.3, it follows that

$$W \setminus \partial J_F \subset \cup_{j=0}^{\infty} \delta_+^j(V), \quad W \subset \overline{\cup_{j=0}^{\infty} \delta_+^j(V)}. \tag{8.14}$$

We now begin the construction of μ, $\tilde{\delta}_+$ and $\tilde{h}$. First, choose $\mu \in (0, \lambda)$ so that every $(x;\xi) = \Phi(v)$ with $v \in V_\mu$ and $x_1 \in [a', b]$ is contained in W. Since Φ is a local diffeomorphism, there is a unique continuous map (8.11) with (8.12). To define $\tilde{h}$, fix an arbitrary local interpolating Hamiltonian h for δ_+ in U_1; its existence follows from the choice of U_1. Set

$$U_\mu = V_\mu \cap \Phi^{-1}(U_1),$$

and define

$$\tilde{h}(v) = h \circ \Phi(v), \quad v \in U_\mu. \tag{8.15}$$

Then (8.12) and the fact that h is an interpolating Hamiltonian for δ_+ in U_1 imply that (8.13) holds for $v \in U_\mu$. We extend $\tilde{h}$ as follows. For $v \in \partial V_\mu$ we simply set $\tilde{h}(v) = 0$; then (8.13) is trivially satisfied on ∂V_μ. Next, assume $v \in V_\mu \setminus \partial(V_\mu \cup U_\mu)$. Then $\Phi(v) \in W \setminus \partial J_F$, hence (8.14) implies $\Phi(v) = \delta_+^j(s)$ for some $j \geq 0$ and $s \in V$. Set $\tilde{h}(v) = h(s)$. To check the correctness of this definition, assume that $\Phi(v) = \delta_+^\ell(s')$ for some $\ell \geq 0$ and $s' \in V$. Let $j \geq \ell$; the other case is similar. Then $\delta_+^{j-\ell}(s) = s' \in V \subset U_1$, and since h is constant along the orbits of δ_+, one gets $h(s) = h(s')$. This shows that the definition of $\tilde{h}$ is correct. Moreover, for $\Phi(v) = \delta_+^j(s)$, we can find a small neighbourhood Q of s in $V \setminus J_F$ such that $\delta_+^j(Q)$ is a neighbourhood of $\Phi(v)$ in $U_2 \setminus J_F$, and then we can define $\tilde{h}$ on $\Phi^{-1}(\delta_+^j(Q))$ by $\tilde{h}(\Phi^{-1}(\delta_+^j(s'))) = h(s')$ for all $s' \in Q$. This shows that $\tilde{h}$ is smooth in a neighbourhood of v and (8.13) holds.

Thus, $\tilde{h}$ is smooth on $V_\mu \setminus \partial V_\mu$ and (8.13) holds for all $v \in V_\mu$. It remains to show that $\tilde{h}$ is smooth on ∂V_μ.

Since the function $(x;\xi) \mapsto \xi_1$ is a local interpolating Hamiltonian for δ_+ in U_2, it follows from the second part of Corollary 8.2.2 that the function $h - \xi_1$ vanishes of infinite order on $V \cap \partial J_F$. This implies that for every integer $p > 0$ there exists a constant $C_p > 0$ such that

$$|h(x_1, \ldots, x_m; \xi_1, \ldots, \xi_m) - \xi_1| \leq C_p \xi_1^p \tag{8.16}$$

for all $(x;\xi) \in V$. Let $v \in \partial V_\mu \setminus U_\mu$. Then

$$s = \Phi(v) = (x_1, \ldots, x_m; 0, \xi_2, \ldots, \xi_m) \in W.$$

For every $\xi_1 \in (0, \omega)$ we have

$$s' = (x_1, \ldots, x_m; \xi_1, \xi_2, \ldots, \xi_m) \in W \subset U_2,$$

and there exist $y_1 \in [a', a' + \epsilon]$ and $j \in \mathbb{N}$ such that $y_1 + j\sqrt{\xi_1} = x_1$, that is $\delta_+^j(s'') = s'$, where

$$s'' = (y_1, x_2, \ldots, x_m; \xi_1, \xi_2, \ldots, \xi_m) \in V.$$

Defining $h_1(s') = h(s'')$ and $h_1 = 0$ on ∂J_F, one gets a function h_1 on a neighbourhood of s in U_2 such that $\tilde{h}(v') = h_1 \circ \Phi(v')$ for v' in a small neighbourhood of v in V_μ. Although the choice of j and y_1 above depends on s', according to (8.16) we always have

$$|h_1(s') - \xi_1| = |h(s'') - \xi_1| \le C_p \xi_1^p$$

for all $p \ge 0$. This shows that h_1 is smooth in a neighbourhood of s in U_2; in fact, all derivatives of h_1 on ∂J_F coincide with the corresponding derivatives of the function ξ_1. Now the smoothness of Φ implies that $\tilde{h}$ is smooth in a neighbourhood of v in V_μ. ■

8.3 Approximations of closed geodesics by periodic reflecting rays

Let Ω be a compact strictly convex domain in $\mathbb{R}^n$, $n \ge 2$, with smooth boundary $\partial\Omega$, and let $\gamma : [0, L] \longrightarrow \partial\Omega$ be a closed geodesic on $\partial\Omega$. Consider the corresponding integral curve $c : [0, L] \longrightarrow T^*(\partial\Omega)$ of the Hamiltonian vector field determined by the standard Riemannian metric on $\partial\Omega$ (see the beginning of Section 7.1). We will assume that $L > 0$ is the primitive period of γ (and c) and that γ (resp. c) is non-degenerate as a curve of period mL for every $m = 1, 2, \ldots$. The latter means that $\mathrm{spec}(P_\gamma)$ does not contain any roots of unity.

There is a natural way to define a winding number for any finite sequence of points lying in a small neighbourhood U of $\Im(c)$ in $T^*\Omega$. We take U so that

$$\Im(c) \subset \cup_{s \in \Im(c)} N_s(\epsilon)$$

for some small $\epsilon > 0$, where $N_s(\epsilon)$ is the ϵ-neighbourhood of s in the orthogonal complement of $T_s^*(\Im(c))$ in $T_s^*\Omega$. If $\epsilon > 0$ is sufficiently small, then the projection $\mu : U \longrightarrow \Im(c)$, defined by $\mu(s') = s$ for $s' \in N_s(\epsilon)$, is a well-defined smooth submersion. As a closed oriented curve without self-intersections with the direction determined by $\dot{c}(0)$, $\Im(c)$ is homeomorphic to the unit circle $\mathbb{S}^1$ with the counterclockwise orientation. Now for every sequence $s_1, \ldots, s_k$ in U define the *winding number* $\mathrm{wn}(\{s_i\})$ to be the winding number of the sequence $\mu(s_1), \ldots, \mu(s_k)$ in $\Im(c)$ (see Section 2.1). We define the *winding number of a closed billiard trajectory* as the winding number of the sequence of its successive reflection points.

Let N **be an arbitrary positive integer**. It will stay fixed until the end of this section. Our aim is to prove the following theorem.

Theorem 8.3.1: *Every neighbourhood of $\Im(\gamma)$ in Ω contains a closed billiard trajectory in Ω with winding number N.*

For the proof of this theorem we need to consider multiples of the closed curves c and γ. It is convenient to extend c and γ periodically with period L, that is so that $c(t+L) = c(t)$ for all $t \in \mathbb{R}$ and similarly for γ. In fact, we will need these extensions only on a sufficiently large compact interval I containing $[0, NL]$.

Next, we are going to use the previous section to study the billiard ball map in a neighbourhood of $\Im(\gamma)$.

Consider the symplectic manifold $S = T^*\mathbb{R}^n$ endowed with the canonical symplectic form

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i,$$

where $p_1, \ldots, p_n; q_1, \ldots, q_n$ are the standard coordinates in $T^*\mathbb{R}^n$. Consider the hypersurfaces

$$F = T^*_{\partial\Omega}\mathbb{R}^n = \{(p,q) \in T^*\mathbb{R}^n : p \in \partial\Omega\},$$
$$G = S^*\mathbb{R}^n = \{(p,q) \in T^*\mathbb{R}^n : |q| = 1\}.$$

Clearly $g : T^*\mathbb{R}^n \longrightarrow \mathbb{R}$, $g(p,q) = |q|^2 - 1$ is a defining function for G. To get a similar function for F, fix an arbitrary function φ, defined and smooth on a neighbourhood of $\partial\Omega$ in $\mathbb{R}^n$ such that $\partial\Omega = \varphi^{-1}(0)$ and $d\varphi \neq 0$ on $\partial\Omega$. Then $f(p,q) = \varphi(p)$ provides a defining function f for F. Since f depends on p only and g on q, it is clear that F and G intersect transversally at any point of

$$J = F \cap G = S^*_{\partial\Omega}\mathbb{R}^n.$$

To describe the set K of glancing points of F and G, notice that

$$\begin{aligned}\{f,g\}(p,q) &= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i}(p,q) - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i}(p,q) \right) \\ &= 2 \sum_{i=1}^{n} q_i \frac{\partial \varphi}{\partial p_i}(p) = 2\langle q, \nabla\varphi(p)\rangle,\end{aligned}$$

where $\nabla\varphi(p)$ is the *gradient* of φ at p. Since $\nabla\varphi(p)$ is parallel to the *unit normal vector* $\nu(p)$ to $\partial\Omega$, pointing into the interior of Ω, the condition $\{f, g, \}(p,q) = 0$ is equivalent to $\langle q, \nu(p)\rangle = 0$. Therefore, the set K of glancing points coincides with $S^*(\partial\Omega)$.

To apply Theorem 8.2.1, we need to know that the points of K are non-degenerate. For $(p,q) \in K$ we have

$$\{f, \{f,g\}\}(p,q) = 2 \sum_{i=1}^{n} \left(\frac{\partial \varphi}{\partial p_i}(p) \right)^2 \neq 0,$$

since $d\varphi(p) \neq 0$. On the other hand, the strict convexity of $\partial\Omega$ at p implies

$$\{g, \{f, g\}\}(p, q) = 4 \sum_{i,j=1}^{n} q_i q_j \frac{\partial^2 \varphi}{\partial p_i \partial p_j}(p) \neq 0$$

whenever $q \neq 0$. Therefore, every point of K is non-degenerate, so Theorem 8.2.1 is applicable in the present situation.

To describe the maps $\delta_\pm$, first we give a geometric interpretation of the spaces M_F and M_G. Notice that

$$X_f(p, q) = (0; -\nabla\varphi(p)), \quad X_g(p, q) = 2(q; 0).$$

This shows that every integral curve of X_f has the form

$$p(t) = p(0), \quad q(t) = q(0) - t\nabla\varphi(p(0)).$$

To such a trajectory we assign the point $(p(0), q'(0))$, where $q'(0)$ is the orthogonal projection of $q(0)$ onto $T^*_{p(0)}(\partial\Omega)$. In other words, we will identify M_F with $T^*(\partial\Omega)$. Then $\pi_F : F \longrightarrow M_F = T^*(\partial\Omega)$ is the projection just defined. Now we have

$$J_F = \pi_F(J) = B^*(\partial\Omega) = \{(p, q) \in T^*(\partial\Omega) : |q| \leq 1\},$$

and clearly $\partial J_F = S^*(\partial\Omega) = K$. Here $B^*(\partial\Omega)$ is the closure of the set $B^*(\partial\Omega)$ introduced in Section 4.2. In fact, $\pi_F = \mathrm{id}$ on K. The maps

$$\alpha_\pm : B^*(\partial\Omega) \longrightarrow J = S^*_{\partial\Omega}\mathbb{R}^n$$

are now defined as follows. For $(p, q) \in B^*(\partial\Omega)$, let $q^\pm \in \mathbb{S}^{n-1}$ be such that $q^\pm = q \pm \lambda\nu(p)$ for some $\lambda \geq 0$. Then $\langle q^\pm, \nu(p)\rangle = \pm\lambda$. Set $\alpha_\pm(p, q) = (p, q^\pm)$. It is easy to see that these are exactly the maps from Theorem 8.2.1(a).

To deal with G, notice that the integral curves of X_g have the form

$$p(t) = p(0) + 2tq(0), \quad q(t) = q(0).$$

Thus, M_G can be naturally identified with the space of all oriented lines in $\mathbb{R}^n$: to the integral curve $(p(t), q(t))$ we assign the line trough $p(0)$ with direction $q(0)$. The projection

$$\pi_G : G = S^*\mathbb{R}^n \longrightarrow M_G$$

is similarly defined: $\pi_G(p, q)$ is the line through p in the direction of q. In this setting, clearly J_G is the subset of M_G consisting of those lines that have a common point with $\partial\Omega$. Then $\partial J_G = \pi_G(K)$ is the set of all oriented lines tangent to $\partial\Omega$, and π_G induces a diffeomorphism between $K = S^*(\partial\Omega)$ and ∂J_G. It is now easy to describe the inverses

$$\beta_\pm : J_G \longrightarrow J = S^*_{\partial\Omega}\mathbb{R}^n.$$

Given an oriented line $\ell \in J_G$ with direction $q \in \mathbb{S}^{n-1}$, denote by $p^{\pm}$ the intersection points of ℓ with $\partial\Omega$ (which may coincide) in such a way that $p^+ = p^- + \lambda q$, $\lambda \geq 0$ (see Figure 8.3(a)). Set $\beta_{\pm}(\ell) = (p^{\pm}, q)$. These maps clearly have the properties described in Theorem 8.2.1(a).

We can now describe the map

$$\delta_+ = \pi_F \circ \beta_+ \circ \pi_G \circ \alpha_+ : B^*(\partial\Omega) \longrightarrow B^*(\partial\Omega).$$

Given $(p, q) \in B^*(\partial\Omega)$, we have $(p, q^+) \in S^*_{\partial\Omega}\mathbb{R}^n$. Then $\ell = \pi_G(p, q^+)$ is the line through p with direction q^+. Thus, $\beta_+(\ell) = (p^+, q^+)$, where p^+ is the other intersection point of ℓ and $\partial\Omega$ (clearly we will have $p^+ = p$ if $q^+ = q$). Finally, $\pi_F(p^+, q^+) = (p^+, r) \in B^*(\partial\Omega)$, where r is the orthogonal projection of q^+ on $T^*_{p^+}(\partial\Omega)$, and then $\delta_+(p, q) = (p^+, r)$ (see Figure 8.3(b)). Thus, δ_+ is globally defined and is naturally equivalent to the billiard ball map defined in Section 2.1. In this section we will simply set $B = \delta_+$ and call it the *billiard ball map*. This map is the extension of the billiard map β defined in Section 4.2. Thus,

$$B = \pi_F \circ \beta_+ \circ \pi_G \circ \alpha_+ : B^*(\partial\Omega) \longrightarrow B^*(\partial\Omega),$$

and $B^{-1} = \delta_-$.

Setting $m = n - 1$, we have $\dim\ (\partial\Omega) = m$. Consider the closed interval

$$I = [-L, NL + L] \subset \mathbb{R}.$$

As in Section 7.1, there exist an open neighbourhood $\mathcal{O}$ of $\Im(\gamma)$ in $\partial\Omega$ and a local diffeomorphism

$$r : \mathcal{O}_u = (-u - L, u + NL + L) \times B_u(0) \longrightarrow \mathcal{O},$$

where $u > 0$ and $B_u(0)$ is the open u-ball about 0 in $\mathbb{R}^{m-1}$ with the following properties:

(i) $\gamma(t) = r(t, 0, \ldots, 0)$ for every $t \in (-u - L, u + NL + L)$;

(ii) the 1-jet of g_{11} coincides with the 1-jet of the constant 1 at all points of

$$\gamma_0 = \{(t, 0, \ldots, 0) \in \mathbb{R}^m : t \in (-u - L, u + NL + L)\};$$

(iii) $g_{1i} = 0$ on γ_0 for all $i = 2, \ldots, m$.

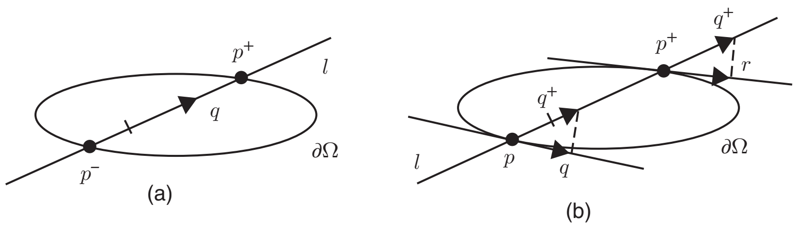

Figure 8.3 The billiard ball map $B = \delta_+$.

Here g_{ij} $(i, j = 1, \ldots, m)$ are the components of the standard metric on $\partial\Omega$ in the coordinates $x_1, \ldots, x_m$ provided by r. Let $\xi_1, \ldots, \xi_m$ be the corresponding dual coordinates in $T^*(\partial\Omega)$. Then, as before, in a sufficiently small neighbourhood of any point of $\Im(c)$,

$$x_1, \ldots, x_m;\ \xi_1, \ldots, \xi_m$$

can be used as coordinates. In such coordinates

$$\omega = \sum_{i=1}^{m} dx_i \wedge d\xi_i$$

is the canonical symplectic form on $T^*(\partial\Omega)$, while

$$H(x;\xi) = \frac{1}{2} \sum_{i,j=1}^{m} g_{ij}(x)\xi_i\xi_j$$

is the function whose Hamiltonian vector field X_H determines the geodesic flow on $T^*(\partial\Omega)$. That is, the geodesics on $\partial\Omega$ are exactly the projections of the integral curves of X_H on $T^*(\partial\Omega)$.

In what follows it will be convenient to make the correspondence between the coordinates $(x;\xi)$ and points in $T^*(\partial\Omega)$ more clear. For $u > \lambda > 0$ set

$$\widetilde{W}_\lambda = \{(x;\xi) : x \in \mathcal{O}_u, 1 - \lambda < 2H(x;\xi) \leq 1\}.$$

Clearly $\widetilde{W}_\lambda$ is a submanifold of $\mathcal{O}_u$ with boundary

$$\partial\widetilde{W}_\lambda = \{(x;\xi) \in \mathcal{O}_u : 2H(x;\xi) = 1\}.$$

Let Ψ be the map that assigns to any $(x;\xi) \in \widetilde{W}_\lambda$ the unique point in $T^*(\partial\Omega)$ with coordinates $(x;\xi)$ as explained above. If $\lambda > 0$ is sufficiently small, this map is well defined. Notice that

$$W_\lambda = \Psi(\widetilde{W}_\lambda)$$

is an open subset of $B^*(\partial\Omega)$ and

$$\partial W_\lambda = W_\lambda \cap S^*(\partial\Omega) = \Psi(\partial\widetilde{W}_\lambda).$$

Moreover,

$$\Psi : \widetilde{W}_\lambda \longrightarrow W_\lambda$$

is a local diffeomorphism.

In order to apply Lemma 8.2.4, first we will slightly change the map Ψ to get a map

$$\Phi : V_\lambda \longrightarrow T^*(\partial\Omega)$$

satisfying the corresponding requirements. Denote the points in V_λ (see the text just before Lemma 8.2.4 for this notation) by

$$(y;\eta) = (y_1,\dots,y_m;\eta_1,\dots,\eta_m).$$

Define the map

$$\psi : \widetilde{W}_\lambda \longrightarrow V_\lambda$$

by $\psi(x;\xi) = (y;\eta)$, where $y = x$, $\eta_i = \xi_i$ for $i = 2,\dots,m$ and

$$\eta_1 = 1 - 2H(x;\xi) = 1 - \sum_{i,j=1}^{m} g_{ij}(x)\xi_i\xi_j.$$

Since $\xi_1 = 1$, $\xi_2 = \cdots = \xi_m = 0$ along $\Im(c)$, we have

$$\frac{\partial\eta_1}{\partial\xi_1} = -2\sum_{i=1}^{m} g_{1i}(x_1,0,\dots,0)\xi_i = -2$$

at all points of $\Im(c)$. Therefore, if $\lambda > 0$ is sufficiently small, the map ψ is a diffeomorphism.

Assume now that $\lambda > 0$ is fixed small enough so that it satisfies the above requirements. Then $\Phi = \Psi \circ \psi^{-1}$ is a local diffeomorphism between V_λ and $\Phi(V_\lambda) = W_\lambda \subset T^*(\partial\Omega)$. To get the situation in Lemma 8.2.4, we endow V_λ with the symplectic structure induced by the canonical symplectic form ω on $\widetilde{W}_\lambda$ via the diffeomorphism ψ. Then Φ becomes a smooth symplectic map. Applying Lemma 8.2.4 and replacing the pair (Φ, V_λ) by $(\Psi, \widetilde{W}_\lambda)$ using the diffeomorphism ψ, we get the following.

Lemma 8.3.2: *There exist $\mu \in (0,\lambda)$, a continuous invertible map*

$$\tilde{B} : \widetilde{W}_\lambda \longrightarrow \widetilde{W}_\lambda \tag{8.17}$$

with

$$\Psi \circ \tilde{B}(v) = B \circ \Psi(v), \quad v \in \widetilde{W}_\lambda, \tag{8.18}$$

and a smooth function $\tilde{h} : \widetilde{W}_\lambda \longrightarrow \mathbb{R}_+$, which is a defining function for $\partial\widetilde{W}_\lambda$ and

$$\tilde{B}(v) = \left(\exp\sqrt{\tilde{h}}\, X_{\tilde{h}}\right)(v) \tag{8.19}$$

for all $v \in \widetilde{W}_\lambda$.

In other words, $\tilde{B}$ is a covering map for the billiard ball map B by means of the covering map Ψ, and $\tilde{h}$ is an interpolating Hamiltonian for $\tilde{B}$ on the whole domain $\widetilde{W}_\lambda$.

Using the canonical local normal fibration of $B^*(\partial\Omega)$ in the neighbourhood W_λ of $\Im(c)$, we get a smooth family $\Sigma(t)$, $t \in \mathbb{R}$, of smooth submanifolds of W_λ of codimension one such that for every t, $\Sigma(t)$ contains $c(t)$ and is transversal (in fact, orthogonal) to $\Im(c)$ at $c(t)$. Each $\Sigma(t)$ is a manifold with boundary

$$\partial\Sigma(t) = \Sigma(t) \cap S^*(\partial\Omega),$$

and

$$\Sigma(t + L) = \Sigma(t) \tag{8.20}$$

for all t. Notice that if $(p, q) \in \Sigma(t)$, with p_i, q_i being the standard coordinates in $T^*\mathbb{R}^n$, then $p = c(t)$ and $(p, q/|q|) \in \partial\Sigma(t)$.

Next, consider the curve

$$d(t) = (t, 0, \ldots, 0; 0, \ldots, 0) \in \widetilde{W}_\lambda,$$

which is the preimage of the integral curve c with respect to Ψ. For each $t \in (-u - L, u + NL + L)$ there exists a smooth submanifold $\tilde{\Sigma}(t)$ of $\widetilde{W}_\lambda$, passing through $d(t)$ and transversal to $\Im(d)$ at $d(t)$, such that

$$\Psi(\tilde{\Sigma}(t)) = \Sigma(t).$$

Thus, $\tilde{\Sigma}(t)$ is the smooth fibration of $\widetilde{W}_\lambda$ corresponding to $\Sigma(t)$ in W_λ. For $t \in I$ define the map

$$\tilde{\mathcal{P}}_t : \tilde{\Sigma}(t) \longrightarrow \tilde{\Sigma}(t + NL)$$

locally around $d(t)$ as follows. Given $v \in \tilde{\Sigma}(t)$, consider the integral curve $v(t)$ of the Hamiltonian vector field $X_{\tilde{h}}$ through v with $v(0) = v$, and denote by $\tilde{\mathcal{P}}_t(v)$ the intersection point of this curve with $\tilde{\Sigma}(t + NL)$. Then $\tilde{\mathcal{P}}_t(v) = v(T)$ for some $T = T_v$. Setting $\mathcal{P}_t(\Psi(v)) = \Psi(v(T))$, we get another map

$$\mathcal{P}_t : \Sigma(t) \longrightarrow \Sigma(t) = \Sigma(t + NL).$$

Clearly, $\tilde{\mathcal{P}}_t$ is a smooth local symplectic map of a small neighbourhood of $d(t)$ in $\tilde{\Sigma}(t)$ onto a neighbourhood of $d(t + NL)$ in $\tilde{\Sigma}(t + NL)$. $\mathcal{P}_t$ has similar properties and moreover

$$\mathcal{P}_t \circ \Psi = \Psi \circ \tilde{\mathcal{P}}_t$$

locally around $d(t)$ in $\tilde{\Sigma}(t)$ for each $t \in I$. Notice that the notation $\mathcal{P}_t$, $\tilde{\mathcal{P}}_t$, $\Sigma(t)$ and $\tilde{\Sigma}(t)$ differs from the corresponding one in Section 7.1. However, the restriction of $\mathcal{P}_t$ on $\partial\Sigma(t) = \Sigma(t) \cap S^*(\partial\Omega)$ is exactly the Poincaré map of the integral curve $c(t)$ ($t \in [0, NL]$) of X_H, that is defined by means of the geodesic flow on $S^*(\partial\Omega)$.

Consider the map

$$\tau : W_\lambda \longrightarrow \partial W_\lambda = W_\lambda \cap S^*(\partial\Omega) \tag{8.21}$$

which is the restriction of the orthogonal projection of $B^*(\partial\Omega) \setminus \{0\}$ on $S^*(\partial\Omega)$ along the normal fibres. More precisely, using again the standard coordinates p_i, q_i

in $T^*\mathbb{R}^n$ from the beginning of this section, we have

$$\tau(p,q) = (p, q/|q|).$$

We assume that $\lambda > 0$ is taken so small that $q \neq 0$ for all $(p,q) \in W_\lambda$. Then (8.21) is a well-defined smooth submersion. Notice that the corresponding map

$$\tilde{\tau} : \widetilde{W}_\lambda \longrightarrow \partial\widetilde{W}_\lambda,$$

for which $\Phi \circ \tilde{\tau} = \tau \circ \Phi$, has the form

$$\tilde{\tau}(x;\xi) = (x; \xi/\sqrt{2H(x;\xi)}),$$

and is also a smooth submersion.

Next, we are going to exploit the non-degeneracy of the curves c and γ.

Lemma 8.3.3: *For all sufficiently small $\epsilon \in (0,\mu)$ there exists a unique family of smooth maps*

$$c_t : [1-\epsilon, 1] \longrightarrow \Sigma(t), \quad t \in I,$$

such that

$$\tau \circ \mathcal{P}_t \circ c_t(s) = \tau \circ c_t(s), \quad s \in [1-\epsilon, 1], \tag{8.22}$$

and the map

$$I \times (1-\epsilon, 1] \longrightarrow W_\lambda, \quad (t,s) \mapsto c_t(s),$$

is smooth.

The uniqueness means the following. If

$$\ell_t : [1-\delta, 1] \longrightarrow \Sigma(t), \quad t \in I,$$

is another smooth family with the same properties, then $c_t(s) = \ell_t(s)$ for all $t \in I$ and all $s \in [1 - \min\{\epsilon, \delta\}, 1]$.

Proof of Lemma 8.3.3: For all sufficiently small $\delta > 0$, Ψ induces a diffeomorphism between

$$\tilde{U}_t = \{(x;\xi) \in \widetilde{W}_\delta : t-\delta < x_1 < t+\delta, |x_i| < \delta, |\xi_i| < \delta, i = 2,\dots,m\} \subset \widetilde{W}_\delta$$

and $U_t = \Psi(\tilde{U}_t) \subset W_\delta$ for all $t \in I$. Then we can use $(x;\xi)$ as coordinates in each U_t by means of the chart $(\Psi, \tilde{U})$.

Fix an arbitrary $t_0 \in I$ and introduce new coordinates $(x;\eta)$ in $U = U_{t_0}$ setting

$$\eta_1 = \xi_1, \quad \eta' = \xi'/\sqrt{2H(x;\xi)}.$$

Here we use the notation

$$\eta' = (\eta_2, \dots, \eta_m), \quad \xi' = (\xi_2, \dots, \xi_m).$$

Clearly, $(x;\xi) = (y;\eta)$ on $U \cap S^*(\partial\Omega)$, and in the new coordinates the map τ has the form

$$\tau(x;\eta_1,\eta') = (x;\hat{\eta}_1,\eta').$$

Fix an arbitrary $t \in \Delta = (t_0 - \delta, t_0 + \delta)$. Then the points in $\Sigma(t)$ have the form $(t, x'; s, \eta')$, where $s \in (1-\delta, 1]$ and in these coordinates the map

$$\mathcal{P}_t : \Sigma(t) \longrightarrow \Sigma(t) \subset U$$

takes the form

$$\mathcal{P}_t(t, x'; s, \eta') = (t, z'; \chi, \zeta') \tag{8.23}$$

for some $\chi \in \mathbb{R}$, $z', \zeta' \in \mathbb{R}^{m-1}$.

Fix $s \in [1-\delta, 1]$ and consider the map

$$Q_s : \mathbb{R}^{2m-2} \longrightarrow \mathbb{R}^{2m-2}$$

defined in a small neighbourhood of 0 by

$$Q_s(x';\eta') = (z';\zeta'),$$

where z' and ζ' are determined by (8.23). As we have already mentioned above, for $s = 1$, $\mathcal{P}_t$ coincides with the Poincaré map of c; therefore, the non-degeneracy of c implies that $\mathrm{id} - dQ_1$ is invertible at 0. Since dQ_s depends smoothly on s, this yields that $\mathrm{id} - dQ_s(0)$ is an invertible map for all $s \leq 1$ sufficiently close to 1. Applying the Implicit Function Theorem to the system

$$\begin{cases} x' = z'(x'; s, \eta') \\ \eta' = \zeta'(x'; s, \eta'), \end{cases}$$

we find $0 < \epsilon < \mu < \lambda$ and maps

$$s \mapsto x'(s), \quad s \mapsto \eta'(s),$$

defined and smooth for $s \in [1-\epsilon, 1]$ such that

$$\mathcal{P}_t(t, x'(s); s, \eta'(s)) = (t, x'(s); \chi(s), \eta'(s)), \quad s \in [1-\epsilon, 1]. \tag{8.24}$$

Now we define the curve

$$c_t : [1-\epsilon, 1] \longrightarrow \Sigma(t)$$

by

$$c_t(s) = (t, x'(s); s, \eta'(s)). \tag{8.25}$$

The smoothness and the uniqueness of the map $(t,s) \mapsto c_t(s)$ for $t \in \Delta$, $s \in [1-\epsilon, 1]$, follow from the Implicit Function Theorem, while (8.22) for $t \in \Delta$ is a consequence of the definition of $c_t(s)$, (8.24) and the form of the map τ in the coordinates $(x;\eta)$.

Finally, take $t_1, \ldots, t_k \in I$ such that the intervals $\Delta_i = (t_i - \delta, t_i + \delta)$, $i = 1, \ldots, k$, cover I and choose $\epsilon > 0$ so small that the maps $c_t^{(i)}(s)$ determined as above replacing Δ by Δ_i are all defined for $s \in [1-\epsilon, 1]$. By the uniqueness of these functions, it follows that $c_t^{(i)}(s) = c_t^{(j)}(s)$ for $s \in \Delta_i \cap \Delta_j$. Therefore, setting $c_t(s) = c_t^{(i)}(s)$ whenever $t \in \Delta_i$ and $s \in [1-\epsilon, 1]$, we obtain a smooth family of maps having the desired properties. ■

For $t \in I$ there exists an unique smooth map

$$\tilde{c}_t : [1-\epsilon, 1] \longrightarrow \tilde{\Sigma}(t) \subset \tilde{U}$$

such that

$$c_t = \Psi \circ \tilde{c}_t.$$

Denote by $T_t(s)$ the time required for the points $\tilde{c}_t(s) \in \tilde{\Sigma}(t)$ to flow along the corresponding integral curve of the Hamiltonian $X_{\tilde{h}}$ to the point $\tilde{\mathcal{P}}_t(\tilde{c}_t(s)) \in \tilde{\Sigma}(t + NL)$. Then, assuming that $\epsilon > 0$ is sufficiently small, $T_t(s)$ is a continuous bounded function of (t,s) with

$$T_t(1) = NL > 0, \quad t \in I.$$

Lemma 8.3.4: *For all sufficiently large $\epsilon \in (0,\mu)$ there exists an integer $k_0 > 0$ such that if $0 < \epsilon \le \epsilon_0$, then for every $k \ge k_0$ there is a unique smooth function*

$$s_k : I \longrightarrow [1-\epsilon, 1]$$

with

$$k\sqrt{\tilde{h}(\tilde{c}_t(s_k(t)))} = T_t(s_k(t)) \tag{8.26}$$

for all $t \in I$.

Proof of Lemma 8.3.4: Recall that $\tilde{h}$ is a defining function for

$$\partial\widetilde{W}_\mu = \widetilde{W}_\mu \cap S^*(\partial\Omega)$$

in $\widetilde{W}_\mu$ (see Lemma 8.3.2). On the other hand, from the proof of Lemma 8.3.3, the curve $c_t(s)$ has the form (8.25) in the coordinates $(x;\eta)$. We may choose $\epsilon \in (0,\mu)$ with the properties in Lemma 8.3.3 such that

$$\frac{\partial\tilde{h}}{\partial\eta_1}(x;\eta) \neq 0$$

for all $(x;\eta) \in W_\mu$ with $\eta \in [1-\epsilon, 1]$.

Next, notice that all derivatives of $\tilde{h}$, except that with respect to η_1, are zero for $\eta_1 = 1$. Thus, the absolute values of these derivatives will be arbitrarily small for all $\eta_1 \in [1-\epsilon, 1]$, provided ϵ is chosen sufficiently small. By (8.25) for such ϵ we have

$$\frac{d}{ds}\tilde{h}(\tilde{c}_t(s)) \neq 0, \; s \in [1-\epsilon, 1].$$

This implies that for any sufficiently large integer $k > 0$ the function

$$g_t(s) = k\sqrt{\tilde{h}(\tilde{c}_t(s))} - T_t(s)$$

is strictly monotone in $[1-\epsilon, 1]$ and takes values with distinct signs at $1-\epsilon$ and 1. Therefore, there exists a unique $s_k(t) \in [1-\epsilon, 1]$ such that $g_t(s_k(t)) = 0$, that is (8.26) holds. Now the Implicit Function Theorem implies that $s_k(t)$ depends smoothly on t. ■

For later use, let us just mention that if $\epsilon > 0$ is chosen sufficiently small as in the proof above, then there exist constants $C_2 > C_1 > 0$ such that

$$C_1(1-\eta_1) \leq |\tilde{h}(x;\eta)| \leq C_2(1-\eta_1), \quad \eta_1 \in [1-\epsilon, 1]. \tag{8.27}$$

From now on we assume that $\epsilon \in (0, \mu)$ is fixed so small that the corresponding requirements of Lemmas 8.3.3 and 8.3.4 are satisfied and (8.27) holds whenever $(x;\eta) \in \widetilde{W}_\mu$.

For $k \geq k_0$ setting

$$\rho_k(t) = c_t(s_k(t)), \quad t \in I,$$

we obtain smooth functions

$$\rho_k : I \longrightarrow W_\mu \subset B^*(\partial\Omega)$$

such that

$$\rho_k(t) \in \Sigma(t), \quad t \in I.$$

Using (8.18) and (8.19) for the covering map (8.17) of B, we will now show that

$$B^k(\rho_k(t)) \in \Sigma(t) \tag{8.28}$$

for all $t \in I$. Indeed, for $v = \tilde{c}_t(s_k(t))$, (8.19) implies

$$\tilde{B}^k(v) = \left(\exp\left(k\sqrt{\tilde{h}}X_{\tilde{h}}\right)\right)(v).$$

Combining this with (8.26) and the definition of $T_t(s)$, we find

$$\tilde{B}^k(v) = \tilde{\mathcal{P}}_t(v).$$

Using the map Ψ and taking (8.18) into account, we get

$$B^k(\Psi(v)) = \mathcal{P}_t(\Psi(v)).$$

Finally, $c_t = \Psi \circ \tilde{c}_t$ and the definition of $\rho_k(t)$ yield (8.28).

Next, considering the functions $\rho_k(t - L)$ on an appropriate subinterval of I and using the uniqueness from Lemmas 8.3.3 and 8.3.4 (and decreasing ϵ once again if necessary), we see that $\rho_k(t)$ is periodic in t with period L, that is

$$\rho_k(t + L) = \rho_k(t) \tag{8.29}$$

for all $t \in I$ with $t + L \in I$ and all $k \geq k_0$. Thus, each ρ_k can be considered as a smooth map $\rho_k : \mathbb{S}^1 \longrightarrow B^*(\partial\Omega)$.

Lemma 8.3.5: *For every $k \geq k_0$ and every $t \in [0, L]$ the sequence $\{B^j(\rho_k(t))\}_{j=0}^{k-1}$ has winding number N and $\rho_k(t) \to c(t)$ as $k \to \infty$ uniformly for $t \in [0, L]$.*

Proof of Lemma 8.3.5: The first assertion follows immediately from the construction of $\rho_k(t)$. To prove the second assertion, it is sufficient to show that $s_k(t) \to 1$ as $k \to \infty$ uniformly for $t \in [0, L]$, which will follow trivially from the inequalities

$$\frac{K_1^2}{k^2 C_2} < 1 - s_k(t) < \frac{K_2^2}{k^2 C_1} \tag{8.30}$$

for $t \in [0, L]$, $k \geq k_0$. Here C_1, C_2 are the constants from (8.27), while $K_i > 0$ can be chosen so that

$$K_1 \leq T_t(s) \leq K_2 \tag{8.31}$$

for all $t \in I$, $s \in (1 - \epsilon, 1]$.

To prove (8.30), first combine (8.26) and (8.31) to get

$$\frac{K_1^2}{k^2} \leq \tilde{h}(\tilde{c}_t(s_k(t))) \leq \frac{K_2^2}{k^2}. \tag{8.32}$$

Recall that using the coordinates $(x; \eta)$ from Lemma 8.3.2, the η_1 component of $\tilde{c}_t(s)$ is exactly s. Thus, applying (8.27), we find

$$C_1(1 - s_k(t)) \leq \tilde{h}(\tilde{c}_t(s_k(t))) \leq C_2(1 - s_k(t)). \tag{8.33}$$

Combining (8.32) and (8.33), one gets (8.30) immediately, which proves the lemma. ■

We now turn to the final step in the proof of the main theorem.

Proof of Theorem 8.3.1: According to Lemma 8.3.5, the theorem will follow if we show that for every integer $k \geq k_0$ there exists $t \in [0, L]$ with

$$B^k(\rho_k(t)) = \rho_k(t). \tag{8.34}$$

Fix an arbitrary $k \geq k_0$. Let

$$\pi : T^*\mathbb{R}^n \longrightarrow \mathbb{R}^n$$

be the natural projection on the first component, that is $\pi(x;\xi) = x$, and let $\|\cdot\|$ be the standard norm in $\mathbb{R}^n$. Define the function

$$F_k : B^*(\partial\Omega) \longrightarrow \mathbb{R}$$

by

$$F_k(v) = \sum_{j=0}^{k-1} \|\pi(B^j(v)) - \pi(B^{j+1}(v))\|.$$

Clearly F_k is continuous. It is convenient to write it as a composition $F_k = R \circ Q$, where

$$Q : B^*(\partial\Omega) \longrightarrow (\partial\Omega)^k, \quad R : (\partial\Omega)^k \longrightarrow \mathbb{R}$$

are given by

$$Q(v) = (\pi(v), \pi(B(v)), \dots, \pi(B^{k-1}(v)))$$

and

$$R(x_1, \dots, x_k) = \sum_{j=1}^{k} \|x_j - x_{j+1}\|.$$

These maps are continuous, Q is smooth for $v \in B^*(\partial\Omega) \setminus S^*(\partial\Omega)$ with $dQ(v) \neq 0$ and R is smooth on $(\partial\Omega)^k$.

Since the map $\rho_k(t)$ is periodic, the function $F_k \circ \rho_k(t)$ has a minimum and a maximum on $[0, L]$. Therefore, there exists at least one $t \in [0, L]$ with

$$\frac{d(F_k \circ \rho_k)}{dt}(t) = 0. \tag{8.35}$$

Fix an arbitrary t with this property. We will now prove that (8.34) holds for this choice of t.

It follows from $\rho_k(t) \in \Sigma(t)$ that $\rho_k(t) = (\gamma(t);\xi)$, γ being the initial geodesic on $\partial\Omega$ (see the beginning of this section). Then by (8.28), we have $B^k(\rho_k(t)) = (\gamma(t);\zeta) \in \Sigma(t)$, which implies

$$\zeta = C\xi, \quad C > 0.$$

It is now sufficient to show that $C = 1$; this would clearly imply (8.34).

Set $Q(\rho_k(t)) = (x_1, \dots, x_k)$ and

$$v_1 = \frac{x_2 - x_1}{\|x_2 - x_1\|}, \quad v_k = \frac{x_1 - x_k}{\|x_1 - x_k\|}.$$

Then $x_1 = \gamma(t)$ and, identifying $T_x(\partial\Omega)$ and $T_x^*(\partial\Omega)$ via the natural duality, we have that the orthogonal projections of v_1 and v_k on $T_{\gamma(t)}(\partial\Omega)$ coincide with ξ and ζ, respectively. As in the proof of Proposition 2.1.3, take arbitrary smooth charts

$$\varphi_j : \mathbb{R}^{n-1} \longrightarrow U_j \subset \partial\Omega, \quad j = 1, \dots, k,$$

with $\varphi_j(0) = x_j$. Consider the function

$$G : (\mathbb{R}^{n-1})^k \longrightarrow \mathbb{R}$$

defined by

$$G(u_1, \dots, u_k) = R(\varphi_1(u_1), \dots, \varphi_k(u_k)).$$

Since the segments $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$ satisfy the law of reflection at x_j with respect to $\partial\Omega$ for every $j = 2, 3, \dots, k$, using the calculations from the proof of Proposition 2.1.3, we find

$$\frac{\partial G}{\partial u_j^{(i)}}(0) = 0$$

for all $j = 2, \dots, k$, $i = 1, \dots, n-1$. Moreover,

$$\begin{aligned} \frac{\partial G}{\partial u_1^{(i)}}(0) &= \left\langle v_1 + v_k, \frac{\partial \varphi_1}{\partial u_1^{(i)}}(0) \right\rangle = \left\langle \zeta - \xi, \frac{\partial \varphi_1}{\partial u_1^{(i)}}(0) \right\rangle \\ &= (C-1) \left\langle \xi, \frac{\partial \varphi_1}{\partial u_1^{(i)}}(0) \right\rangle, \end{aligned} \tag{8.36}$$

according to the above remark about v_1 and v_k and taking into account the fact that $\frac{\partial \varphi_1}{\partial u_1^{(i)}}(0)$ is tangent to $\partial\Omega$ at x_1. There exists a smooth map

$$\chi = (\chi_1, \dots, \chi_k) : \Delta \longrightarrow (\mathbb{R}^{n-1})^k$$

defined on a small open interval Δ around t in $\mathbb{R}$ such that $\chi(0) = 0$ and $G(\chi(s)) = F_k(\rho_k(s))$. It now follows from (8.35) and (8.36) that

$$\begin{aligned} 0 = d(G \circ \chi)(t) &= \sum_{j=1}^{k} \sum_{i=1}^{n-1} \frac{\partial G}{\partial u_j^{(i)}}(\chi(t))\, d\chi_j^{(i)}(t) = \sum_{i=1}^{n-1} \frac{\partial G}{\partial u_1^{(i)}}(\chi(t))\, d\chi_1^{(i)}(t) \\ &= (C-1) \left\langle \xi, \sum_{i=1}^{n-1} \frac{\partial \varphi_1}{\partial u_1^{(i)}}(\chi(t))\, d\chi_1^{(i)}(t) \right\rangle = (C-1)\, \langle \xi, d(\varphi_1 \circ \chi_1)(t) \rangle. \end{aligned}$$

Therefore to prove $C = 1$, it is sufficient to show that $\langle \xi, d(\varphi_1 \circ \chi_1)(t) \rangle \neq 0$. Notice that $\varphi_1 \circ \chi_1(s)$ coincides with the first component of $Q \circ \rho_k(s)$, which is $\gamma(s)$. Thus, identifying $\dot{\gamma}(t)$ with $d\gamma(t)$, using the natural identification of $T(\partial\Omega)$ with $T^*(\partial\Omega)$, we have

$$\langle \xi, d(\varphi_1 \circ \chi_1)(t) \rangle = \langle \xi, \dot{\gamma}(t) \rangle,$$

which is non-zero for sufficiently large k, since

$$\rho_k(t) \to c(t) = (\gamma(t), b\dot{\gamma}(t))$$

for some $b > 0$. This completes the proof of the theorem. ■

8.4 The Poisson relation for generic strictly convex domains

Here we prove that the equality (8.1) holds for generic strictly convex domains Ω in $\mathbb{R}^n$, $n \geq 3$, with smooth boundaries $\partial\Omega$.

As we have already mentioned, for a strictly convex $\Omega \subset \mathbb{R}^n$ every $\gamma \in \mathcal{L}_\Omega$ is either a closed geodesic on $\partial\Omega$ or a periodic reflecting ray in Ω. If γ is not a multiple of another element δ of $\mathcal{L}_\Omega$, then γ is called *primitive*. We will say that γ is a *non-degenerate* element of $\mathcal{L}_\Omega$ if the Poincaré map P_γ of γ has no eigenvalues that are roots of unity.

Denote by $\Xi = \Xi(n)$ the family of all strictly convex compact domains Ω in $\mathbb{R}^n$ with C^∞ boundaries $\partial\Omega$ satisfying the following two conditions:

(R) $T_\gamma/T_\delta \notin \mathbb{Q}$ for every two different primitive elements γ and δ of $\mathcal{L}_\Omega$;

(ND) every element of $\mathcal{L}_\Omega$ is non-degenerate.

Given a strictly convex domain Ω (with a smooth boundary, which will be assumed throughout this section), denote by $\mathcal{O}_\Omega$ the set of all $F \in \mathbf{C}(\partial\Omega)$ such that Ω_F is strictly convex. Recall that Ω_F is the domain with boundary $F(\partial\Omega)$. Clearly, $\mathcal{O}_\Omega$ is an open subset of $\mathbf{C}(\partial\Omega)$ containing id. In particular, $\mathcal{O}_\Omega$ is a Baire topological space with respect to the topology inherited from $\mathbf{C}(\partial\Omega)$, therefore every residual subset of $\mathcal{O}_\Omega$ is dense in it.

The first main result in this section shows that the family Ξ is 'very large' in some topological sense. In particular, for every strictly convex domain Ω there exist smooth perturbations of Ω arbitrarily close to id with respect to the C^∞ topology such that the perturbed domain is in Ξ. Moreover, almost all perturbations in $\mathcal{O}_\Omega$ have this property.

Theorem 8.4.1: *Let Ω be an arbitrarily strictly convex domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. There exists a residual subset $R(\Omega)$ of $\mathcal{O}_\Omega$ such that $\Omega_F \in \Xi$ for every $F \in R(\Omega)$.*

The proof of this theorem is postponed to the end of this section. We now proceed to prove that the Poisson relation becomes an equality for all $\Omega \in \Xi$.

Define the *length spectrum* L_Ω of Ω by

$$L_\Omega = \{T_\gamma : \gamma \in \mathcal{L}_\Omega\}.$$

Applying the approximation theorem from the previous section, we obtain the following characterization of the isolated points in L_Ω for $\Omega \in \Xi$.

Proposition 8.4.2: *Let $\Omega \in \Xi$ and let $\gamma \in \mathcal{L}_\Omega$. Then γ is a periodic reflecting ray in Ω if and only if T_γ is an isolated point in L_Ω.*

Proof of Proposition 8.4.2: Let γ be a closed geodesic on $\partial\Omega$. Since Ω is strictly convex and γ is non-degenerate by (ND), it follows from Theorem 8.3.1 that γ can be approximated by periodic reflecting rays in Ω. In particular, there exists a sequence $\{\gamma_k\}$ of periodic reflecting rays in Ω with $T_{\gamma_k} \to T_\gamma$ as $k \to \infty$. Condition (R) implies $T_{\gamma_k} \neq T_\gamma$ for all k. Thus, T_γ is not isolated in L_Ω.

Suppose now that γ is a periodic reflecting ray in Ω. We will show that T_γ is isolated in L_Ω. Assume that there exists a sequence $\{\gamma_k\} \subset \mathcal{L}_\Omega$ with $T_{\gamma_k} \to T_\gamma$ and $T_{\gamma_k} \neq T_\gamma$ for all k. For any k choose an arbitrary point $x_k \in \gamma_k \cap \partial\Omega$ and denote by $v_k \in \mathbb{S}^{n-1}$ the outgoing direction of γ_k at x_k. We may assume that there exist the limits $x_k \to x \in \partial\Omega$ and $v_k \to v \in \mathbb{S}^{n-1}$. Using the continuity of the generalized geodesic flow (see Section 1.2), there exists $\delta \in \mathcal{L}_\Omega$ passing through x with direction v such that $T_{\gamma_k} \to T_\delta$. Thus, $T_\gamma = T_\delta$ and (R) implies $\delta = \gamma$, that is δ is a periodic reflecting ray in Ω. In particular, v is transversal to $\partial\Omega$ at x, and therefore v_k is transversal to $\partial\Omega$ at x_k for sufficiently large k. Thus, γ_k is a periodic reflecting ray with the same number of reflection points as $\delta = \gamma$ for large k. This now implies $1 \in \mathrm{spec}(P_\gamma)$, which is a contradiction with the non-degeneracy of γ. Hence T_γ is an isolated point in L_Ω. ■

The central moment in this section is the following.

Theorem 8.4.3: *For every $\Omega \in \Xi$, we have*

$$\mathrm{sing\ supp}\sigma_\Omega(t) = \{0\} \cup \{\pm T_\gamma : \gamma \in \mathcal{L}_\Omega\}. \tag{8.37}$$

Proof of Theorem 8.4.3: The inclusion $\subset$ follows from Theorem 3.3.2. To check the converse inclusion, consider an arbitrary periodic reflecting ray γ in Ω. Since (R) and (ND) hold, it follows from Theorem 4.3.1 that $T_\gamma \in \mathrm{sing\ supp}\sigma_\Omega$. Finally, let γ be a closed geodesic on $\partial\Omega$. Since there exists a sequence $\{\gamma_k\}$ of periodic reflecting rays in Ω with $T_{\gamma_k} \to T_\gamma$ as $k \to \infty$ (see the first part of the proof of Proposition 8.4.2) and $T_{\gamma_k} \in \mathrm{sing\ supp}\sigma_\Omega$ for all k, it follows that $T_\gamma \in \mathrm{sing\ supp}\sigma_\Omega$. This proves (8.37). ■

The above theorem shows that knowing the point spectrum of the Laplacian for a domain $\Omega \in \Xi$, we can recover the length spectrum of Ω, which clearly contains some geometric information about Ω. It turns out that one can also recover some part of $\mathrm{spec}(P_\gamma)$ for every $\gamma \in \mathcal{L}_\Omega$. To see this, the following result of Stark will be useful.

Lemma 8.4.4: *Let P and Q be $2n \times 2n$ matrices such that*

$$|\det(I - P^k)| = |\det(I - Q^k)|$$

for all $k = 1, 2, \ldots$. Then

$$\mathrm{spec}(P) \setminus \mathbb{S}^1 = \mathrm{spec}(Q) \setminus \mathbb{S}^1,$$

and there exists an integer $N = N(P, Q) > 0$ such that $\mathrm{spec}(P^N) = \mathrm{spec}(Q^N)$.

For a proof of this lemma we refer the reader to the Appendix in [DG]. Combining it with Theorem 8.4.3 and the main result of Chapter 4, we obtain the following.

Corollary 8.4.5: *Let $\Omega_1, \Omega_2 \in \Xi$ be such that the Dirichlet problem for the Laplacian has the same spectrum for Ω_1 and Ω_2. Then there exists a bijection*

$$\mathcal{L}_{\Omega_1} \to \mathcal{L}_{\Omega_2}, \quad \gamma \mapsto \gamma',$$

such that for every $\gamma \in \mathcal{L}_{\Omega_1}$ we have $T_{\gamma'} = T_\gamma$; γ is a periodic reflecting ray in Ω_1 if and only if γ' is a periodic reflecting ray in Ω_2;

$$\mathrm{spec}(P_{\gamma'}) \setminus \mathbb{S}^1 = \mathrm{spec}(P_\gamma) \setminus \mathbb{S}^1$$

and there exists an integer $N = N(\gamma) > 0$ with $\mathrm{spec}(P_{\gamma'}^N) = \mathrm{spec}(P_\gamma^N)$.

Proof: It follows from our assumptions and (8.37) for $\Omega = \Omega_1$ and $\Omega = \Omega_2$ that

$$\{T_\gamma : \gamma \in \mathcal{L}_{\Omega_1}\} = \{T_{\gamma'} : \gamma' \in \mathcal{L}_{\Omega_2}\}.$$

Since (R) holds for both Ω_1 and Ω_2, for every $\gamma \in \mathcal{L}_{\Omega_1}$ there exists a unique $\gamma' \in \Omega_2$ with $T_{\gamma'} = T_\gamma$. Moreover, the map $\gamma \mapsto \gamma'$ is a bijection.

Let γ be a periodic reflecting ray in Ω_1. By Proposition 8.4.2, T_γ is an isolated point in $\mathcal{L}_{\Omega_1}$. Thus, $T_{\gamma'}$ is an isolated point in $\mathcal{L}_{\Omega_2}$, so γ' is a periodic reflecting ray. Applying Theorem 4.3.1 to the k-multiples of γ and γ', we get

$$|\det(I - P_\gamma^k)| = |\det(I - P_{\gamma'}^k)|$$

for all $k = 1, 2, \ldots$. Lemma 8.4.4 now implies that the third property listed in the Corollary is fulfilled. The same property in the case of a closed geodesic γ on $\partial\Omega_1$ follows from the classical result of Duistermaat and Guillemin [DG] concerning the Laplace–Beltrami operator on manifolds without boundary, the condition (ND) and Lemma 8.4.4. ■

The following lemma will be used in the proof of Theorem 8.4.1 below.

Lemma 8.4.6: *Let M be a compact smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$, let γ be a primitive closed geodesic on M and let K be a finite subset of $\Im(\gamma)$.*

There exists a point $q_0 \in \Im(\gamma) \setminus K$ *such that for every neighbourhood* U *of* q_0 *in* M*, there exist* $\lambda > 0$ *and a continuous family* $F_\mu = \mathrm{id} + f_\mu$*,* $\mu \in (-\lambda, \lambda)$*, of elements of* $\mathbf{C}(M)$ *such that* $f_0 = 0$*, and for every* $\mu \in (-\lambda, \lambda)$ *we have*

$$\mathrm{supp}\,(f_\mu) \subset U, \tag{8.38}$$

the curve $\gamma_\mu = F_\mu \circ \gamma$ *is a closed geodesic on* $F_\mu(M)$ *with period* $\theta(\mu)$ *depending smoothly on* μ *and* $\dot{\theta}(0) \neq 0$*.*

Proof of Lemma 8.4.6: Let $\gamma : [0, \theta] \longrightarrow M$, where $\theta > 0$ is the minimal period of γ. Clearly there exists $t_0 \in (0, \theta)$ such that $\gamma''(t_0) \neq 0$ and $q_0 = \gamma(t_0) \notin K$. Fix t_0 and q_0 with these properties and consider an arbitrary neighbourhood U of q_0 in M with $U \cap K = \emptyset$. We may assume that U is so small that there exist semi-geodesic coordinates along $\Im(\gamma)$ in U. Namely, U is the image of a chart

$$r : V = (t_0 - a, t_0 + a) \times B_a(0) \longrightarrow U \subset M,$$

where $a > 0$ and $B_a(0)$ is the open ball of radius a and centre 0 in $\mathbb{R}^m$, $m = n - 2$, and in the coordinates $x_0, x_1, \ldots, x_m$ in U provided by r, for all $y = (y_1, \ldots, y_m) \in B_a(0)$ the curves $\{(t; y) : t \in (t_0 - a, t_0 + a)\}$ are geodesic lines orthogonal to any surface

$$\{(s; y) : y \in B_a(0)\}, \quad s \in (t_0 - a, t_0 + a).$$

Then for $x \in V$ we have $g_{00}(x) = 1$, $g_{0i}(x) = 0$ for all $i \geq 1$. Here g is the standard metric on M. We will assume that $a > 0$ is chosen so small that

$$w(t) = \frac{\partial r}{\partial x_0}(t; 0) \neq 0$$

for all $t \in [t_0 - a, t_0 + a]$.

Fix arbitrary smooth functions

$$\rho : \mathbb{R}^m \longrightarrow [0, 1]$$

with compact $\mathrm{supp}\,(\rho) \subset B_a(0)$, $\rho(y) = 1$ for all $y \in B_{a/2}(0)$, and

$$\varphi : \mathbb{R} \longrightarrow [0, 1]$$

such that $\varphi(t_0) > 0$ and $\mathrm{supp}\,(\varphi) \subset [t_0 - a/2, t_0 + a/2]$. Define the map $v : V \longrightarrow \mathbb{R}^n$ by

$$v(t; y) = -\frac{\varphi(t)\ \rho(y)}{\|\dot{w}(t)\|^2}\ \dot{w}(t).$$

Then v is smooth with compact $\mathrm{supp}\,(v) \subset [t_0 - a/2, t_0 + a/2] \times B_a(0)$ and

$$\left\langle w(t), \frac{\partial v}{\partial x_0}(t; 0) \right\rangle = \varphi(t)$$

for all $t \in (t_0 - a, t_0 + a)$. For μ close to 0 define $f = f_\mu : M \longrightarrow \mathbb{R}^n$ by

$$f(z) = \begin{cases} z, & z \notin U, \\ r(x) + \mu\, v(x), & z = r(x) \in U. \end{cases}$$

Clearly, there exists a sufficiently small $\lambda > 0$ such that for all $\mu \in (-\lambda, \lambda)$ we have $F_\mu = \mathrm{id} + f_\mu \in \mathbf{C}(M)$.

Fix an arbitrary $\mu \in (-\lambda, \lambda)$. The map

$$\tilde{r} = r + \mu\, v : V \longrightarrow \mathbb{R}^n$$

provides coordinates $x_0, x_1, \ldots, x_m$ near $\Im(\gamma_\mu)$ on $\tilde{M} = F_\mu(M)$, where $\gamma_\mu = F_\mu \circ \gamma$. Let $\tilde{g}$ be the standard metric on $\tilde{M}$. Then for $s \in (t_0 - a, t_0 + a)$ and $y \in B_{a/2}(0)$ we have

$$\tilde{g}_{00}(s; y) = 1 + 2\mu\, \varphi(s) + O(\mu^2), \tag{8.39}$$

where the last term depends only on s. Moreover, $\tilde{g}_{0i}(s; y) = 0$ for all $i > 0$. It is now clear that the curve $\gamma_\mu(t) = \tilde{r}(t; 0)$, $t \in [0, \theta]$, is a closed geodesic on $\tilde{M}$ (although t is not a natural parameter for it). Let $\theta(\mu)$ be the minimal period of γ_μ. It follows from the construction of F_μ that $\theta(\mu)$ depends smoothly on μ.

It remains to show that $\dot{\theta}(0) \neq 0$. It follows from (8.39) that

$$\theta(\mu) = \int_0^\theta \sqrt{\tilde{g}_{00}(t; 0)}\; dt = \int_0^\theta \sqrt{1 + 2\mu\, \varphi(t) + O(\mu^2)}\; dt.$$

Differentiating this equality with respect to μ and evaluating at $\mu = 0$, gives

$$\theta'(0) = \int_0^\theta \varphi(t)\; dt > 0,$$

which completes the proof of the lemma. ∎

Proof of Theorem 8.4.1: Set $M = \partial\Omega$ and fix an arbitrary integer $q > 0$. Denote by $\mathcal{L}_\Omega(q)$ the set of all $\gamma \in \mathcal{L}_\Omega$ such that $T_\gamma \leq q$, and if γ is a periodic reflecting ray, then it has not more than q reflection points. It follows from the results in Chapters 6 and 7 that there exists a residual subset R'_q of $\mathcal{O}_\Omega$ such that every $F \in R'_q$ has the following properties: every element of $\mathcal{L}_{\Omega_F}(q)$ is non-degenerate; $T_\gamma \neq T_\delta$ for any two different periodic reflecting rays $\gamma, \delta \in \mathcal{L}_{\Omega_F}(q)$. Notice that $\mathcal{L}_{\Omega_F}(q)$ is finite whenever $F \in R'_q$ (see Theorem 6.4.3).

Let R_q be the set of those $F \in R'_q$ such that $T_\gamma \neq T_\delta$ for any two different elements γ and δ of $\mathcal{L}_{\Omega_F}(q)$. We will show that R_q is open and dense in R'_q.

To prove the openness, we will show that $R'_q \setminus R_q$ is closed in R'_q. Let

$$\{F_k\} \subset R'_q \setminus R_q, \quad F_k \longrightarrow F \in R'_q,$$

in the C^∞ topology. We have to check that $F \notin R_q$. Without loss of generality, we may assume $F = \mathrm{id}$. Since $F_k \notin R_q$, we can find two elements $\gamma_k \neq \delta_k$ of $\mathcal{L}_{\Omega_k}(q)$, where $\Omega_k = \Omega_{F_k}$, with $T_{\gamma_k} = T_{\delta_k}$, γ_k being a closed geodesic passing through some point $F_k(x_k)$, $x_k \in M$, with direction $dF_k(u_k)$, $u_k \in S_{x_k}M$. Considering an appropriate subsequence, we may assume that either δ_k is a periodic reflecting ray with at most q reflection points for all k, or δ_k is a closed geodesic passing through some point $F_k(y_k)$, $y_k \in M$, with direction $dF_k(v_k)$, $v_k \in S_{y_k}M$, for all k. In the first case, using a standard continuity argument and taking subsequences, one finds a closed geodesic γ on M and a periodic reflecting ray δ in Ω with at most q reflection points such that $T_\gamma = T_\delta \leq q$, which implies id $\notin R_q$. We leave the details to the reader.

Consider the second case when all δ_k are closed geodesics. We may assume $x_k \to x$ and $y_k \to y$ in M, $u_k \to u$ and $v_k \to v$ in $\mathbb{S}^{n-1}$, and also $T_{\gamma_k} = T_{\delta_k} \to T$ as $k \to \infty$. Then there are closed geodesics γ and δ on M determined by (x, u) and (y, v), respectively, such that $T_\gamma = T_\delta = T$. We claim that $\gamma \neq \delta$. Assume that $\gamma = \delta$. Let g be the standard metric on M and let g_k be the Riemannian metric on M so that $F_k : (M, g_k) \longrightarrow F_k(M)$ is an isometry. If $t' \in [0, T)$ is the time for which the geodesic flow on (M, g) shifts (y, v) to (x, u), then for the same time t' the geodesic flow on (M, g_k) shifts (y_k, v_k) along δ_k to $(y'_k, v'_k) \neq (x_k, u_k)$ such that $(y'_k, v'_k) \to (x, u)$. Therefore, without loss of generality, we may assume $y = x$ and $v = u$.

Denote by Σ the hyperplane in $\mathbb{R}^n$ passing through x and orthogonal to u. Shifting x_k and y_k along the geodesic γ_k, we may also assume $x_k, y_k \in \Sigma$ for all sufficiently large k. Using the natural identification of T^*M with TM, we may consider the Poincaré map $\mathcal{P}_\gamma$ as a local symplectic map

$$\mathcal{P}_\gamma : S_{\Sigma \cap M}M \longrightarrow S_{\Sigma \cap M}M$$

defined in a small neighbourhood U of (x, u). For $(z, w) \in U$ consider the geodesic on (M, g_k) passing through z in direction w. After some time close to T this curve intersects Σ at some point z' with direction $w' \in \mathbb{S}^{n-1}$. Define $\mathcal{P}_k(z, w) = (z', w')$. Since $g_k \to g$, there exists a neighbourhood V of (x, u) in U and $k_0 > 0$ such that for $k \geq k_0$ the map $\mathcal{P}_k$ is well defined and smooth on V and $(x_k, v_k) \in V$, $(y_k, v_k) \in V$. So, $\mathcal{P}_k$ can be viewed as a Poincaré map for γ_k in the neighbourhood V of both (x_k, u_k) and (y_k, v_k). Since $(x_k, u_k) \neq (y_k, v_k)$ are fixed points for $\mathcal{P}_k$, for large k there exists $(z_k, w_k) \in V$ such that $(d\mathcal{P}_k - \mathrm{id})(z_k, w_k)$ is not invertible. Now $\mathcal{P}_k \to \mathcal{P}_\gamma$, $z_k \to x$ and $w_k \to u$ imply that $(d\mathcal{P}_\gamma - \mathrm{id})(x, u)$ is not invertible. Thus, $1 \in \mathrm{spec}(P_\gamma)$ in contradiction with the non-degeneracy of γ. This proves that $\gamma \neq \delta$ which implies id $\notin R_q$. Hence R_q is open in R'_q.

To establish the density, it is enough to assume id $\in R'_q$ and prove that there are elements of R_q arbitrarily close to id. So, assume id $\in R'_q$ and let $\mathcal{W}$ be an arbitrary neighbourhood of id in $\mathbf{C}(M)$. Since id $\in R'_q$, there are only finitely many closed geodesics $\gamma_1, \ldots, \gamma_s$ on M with periods $\leq q$, and finitely many periodic reflecting rays $\delta_1, \ldots, \delta_k$ with periods $\leq q$ and not more than q reflection points. Applying Lemma 8.4.6 to $\gamma = \gamma_1$ and

$$K = \Im(\gamma_1) \cap (\cup_{i=2}\Im(\gamma_i) \cup \cup_{j=1}^k \Im(\delta_j)),$$

we find $F = \mathrm{id} + f \in \mathcal{W}$ such that $\mathrm{supp}\,(f)$ is contained in a small neighbourhood U of a point in $\Im(\gamma_1)$ with

$$U \cap (\cup_{i=2}\Im(\gamma_i) \cup \cup_{j=1}^{k}\Im(\delta_j)) = \emptyset,$$

for which $\tilde{\gamma}_1 = F \circ \gamma_1$ is a closed geodesic on $F(M)$ and

$$T_{\tilde{\gamma}_1} \notin \mathbb{Q}\boldsymbol{\cdot}\{T_{\delta_1}, \dots, T_{\delta_k}\}.$$

We choose F in such a way that $T_{\tilde{\gamma}_1} < q$ if $T_{\gamma_1} < q$ and $T_{\tilde{\gamma}_1} > q$ if $T_{\gamma_1} = q$. Notice that if U is sufficiently small and f is sufficiently close to 0 in the C^∞ topology, then $\delta_1, \dots, \delta_k$ are the only periodic reflecting rays in Ω_F with periods $\leq q$ and not more than q reflection points. Moreover, the closed geodesics on $F(M)$ with periods $\leq q$ are $\gamma_2, \dots, \gamma_s$ and possibly $\tilde{\gamma}_1$. Thus, choosing U small enough and f sufficiently close to 0, we have $F \in R'_q$.

Repeating this procedure $s - 1$ times, we find $G \in \mathcal{W} \cap R_q$ which proves the density of R_q in R'_q. Therefore, R_q is a residual subset of $\mathcal{O}_\Omega$. Finally, setting

$$R(\Omega) = \cap_{q=1}^{\infty} R_q,$$

we obtain a residual subset of $\mathcal{O}_\Omega$ which has all the desired properties. This proves the theorem. ■

8.5 Notes

A slightly different version of Proposition 8.1.2 was proved by Landis [Lan]. Lemma 8.1.3 is also contained in [Lan]. The rest of Section 8.1 is taken from [PS2] (see also [PSl]). Theorem 8.2.1 and Corollary 8.2.2 are due to Melrose [Me1]. The other material in Sections 8.2 and 8.3 is a modification of a part of Magnuson's thesis [Mag]. The results in Section 8.4 are taken from [S3] and, as one can see, rely heavily on previous results of Anderson and Melrose [AM], Duistermaat and Guillemin [DG], Guillemin and Melrose [GM1], Magnuson [Mag], Melrose [Me1], Petkov and Stoyanov [PS2], [PS3], [PS4], Stoyanov [S1], [S3] and others. For other inverse spectral results see [Ber], [Prot], [MSi], [Vi].

9

Singularities of the scattering kernel

In this chapter a formula is proved for the leading singularity of the scattering kernel at $-T_\gamma$, where T_γ is the sojourn time of an ordinary non-degenerate reflecting (ω, θ)-ray satisfying some additional assumptions. As a consequence, applying the results from Chapter 5, certain information on the singular set of the scattering kernel is obtained. A special emphasis is given to three-dimensional generic domains. We prove that for such domains, the (ω, θ)-rays of mixed type disappear and any singularity of the scattering kernel has the form $-T_\gamma$ for some reflecting (ω, θ)-ray.

9.1 Singularity of the scattering kernel for a non-degenerate (ω, θ)-ray

Let $\Omega = \overline{\mathbb{R}^n \setminus K}$ be a connected closed domain in $\mathbb{R}^n$, $n \geq 2$, with bounded complement and smooth boundary $\partial\Omega$, introduced in Chapter 5. Let $\omega \neq \theta$ be fixed unit vectors. Throughout this chapter we use the notation from Chapter 5 (see also Section 2.4).

In this section γ will be a fixed ordinary reflecting non-degenerate (ω, θ)-ray with sojourn time T_γ satisfying the following assumption:

(I) $T_\delta \neq T_\gamma$ for every $\delta \in \mathcal{L}_{(\omega,\theta)}(\Omega) \setminus \{\gamma\}$.

As in Section 5.3, it is easy to check that for sufficiently small $\epsilon > 0$ we have

$$(T_\gamma - \epsilon, T_\gamma + \epsilon) \cap \text{sing supp } s(t, \theta, \omega) = \{T_\gamma\}.$$

Geometry of the Generalized Geodesic Flow and Inverse Spectral Problems, Second Edition.
Vesselin M. Petkov and Luchezar N. Stoyanov.

Fix a such $\epsilon > 0$ and let $\rho_\delta(t) \in C_0^\infty(\mathbb{R})$, $0 < \delta \le \epsilon$, be the function defined in Section 5.2. Let $q_1, \ldots, q_m$ be the successive reflection points of γ, and let $\tilde{\gamma}$ be the generalized bicharacteristic of $\Box$ such that $\pi(\tilde{\gamma}) = \gamma$. We suppose that $\tilde{\gamma}$ is issued from x_0, $|x_0| > \rho_0$, with (incoming) direction ω. Following the localization argument in Section 5.2, introduce the distribution $u(t, x; \omega)$ as the solution of the problem:

$$\begin{cases} (\partial_t^2 - \Delta_x)u = 0 \text{ in } \mathbb{R} \times \Omega^\circ, \\ u + \varphi(x)\delta(t - \langle x, \omega\rangle) = 0 \text{ on } \mathbb{R} \times \partial\Omega, \\ u|_{t < -\rho_0} = 0. \end{cases} \tag{9.1}$$

Here $\varphi \in C_0^\infty(\mathbb{R})$ is such a way that $\varphi(q_1) = 1$ and $\varphi(x) = 0$ outside a small neighbourhood $\mathcal{O}_1$ of q_1. Let $u_0 = -\varphi(x)\delta(t - \langle x, \omega\rangle)|_{\mathbb{R}\times\partial\Omega}$.

In what follows $\delta \in (0, \epsilon]$ will be fixed, and for sake of brevity we set $\rho = \rho_\delta$. Our aim is to examine the integral

$$I(\lambda) = \int_{\mathbb{R}} \int_{\partial\Omega} e^{\mathrm{i}\lambda(t - \langle x, \theta\rangle)} \rho(\langle x, \theta\rangle - t + T_\gamma) \left(\frac{\partial}{\partial \nu} - \langle \nu, \theta\rangle \frac{\partial}{\partial t}\right) u \, dt \, dS_x, \tag{9.2}$$

interpreted in the sense of distributions.

Next, with some modifications we apply the construction of Section 4.1 to the mixed problem (9.1) and we use the notations R_k^+, V_k^+, M_k introduced in this section. The difference is that we must study a problem with non-homogeneous boundary condition. Consider local coordinates (y_1, y') in $\mathcal{O}_1$ and let $\omega_1 = \mathcal{O}_1 \cap \partial\Omega$ be given by $y_1 = g(y')$. Assume that $q_1 = (0, y_0')$ and let $y_1 > g(x')$ in $\mathcal{O}_1 \cap \Omega^\circ$. For $\eta \in \mathbb{R}^n$ consider the hyperplane $Z_{\eta,a} \subset \mathbb{R}^n$ with normal η such that distance of q_1 to $Z_{\eta,a}$ is $a > 0$. Let $\pi_\eta : \partial\Omega \to Z_{\eta,a}$ be the projection introduced in Section 2.4 along the direction η.

Applying the construction of Section 4.1, consider the Fourier integral operators $R_1^+, R_2^+, \ldots, R_m^+$ associated with the canonical relations $C_1, \ldots, C_m$. We modify the operator $R_1^+ = V_1^+$ to satisfy the inhomogeneous boundary condition. The symbol of R_1^+, modulo Maslov factors, is $f|dt \wedge dy \wedge d\eta|^{1/2}$ with $f \sim \sum_{j=0}^\infty f_j$. In Section 4.1 we have choose $f_0 = 1$ and the principal symbol of $i_{\omega_1}^* R_1^+$, modulo Maslov factors, in local coordinates (y', η') becomes $\gamma_1(y', \eta', 1)^{-1/2} \alpha_1(y', \eta')|d\tau \wedge dy' \wedge d\eta'|^{1/2}$, where γ_1 is given by Lemma 4.2.1 and $\alpha_1(y', \eta') > 0$ is related to the expression of the half-density $|d\tau \wedge dy' \wedge d\eta'|^{1/2}$ in normal geodesic coordinates used in Lemma 4.2.1. Notice that $\alpha_1(y_0', \eta') = 1$.

To obtain the identity operator in local coordinates (y', η'), we choose $\gamma_1^{1/2}\alpha_1^{-1}$ as initial condition for f_0 on ω_1 and next we determine f_0 from the transport equation $H_q f_0 = 0$ taking f_0 constant along the orbits of H_q. Therefore, the principal symbol of R_1^+ will be $\gamma_1^{1/2}\alpha_1^{-1}|dt \wedge dy \wedge d\eta|^{-1/2}$. With this choice, we guarantee that

$$i_{\omega_1}^* V_1^+ u_0 \equiv u_0,$$

provided that $WF(u_0) \subset \Sigma_1$.

To satisfy the last condition, we must define Σ_1 suitably. First, choose a small conic neighbourhood V of ω. Next for $\eta \in V$ consider

$$U_{\varphi,\eta,a} = \{\pi_\eta(z) \in Z_{\eta,a} : z \in \text{supp } \varphi \cap \partial\Omega\}.$$

The vector ω is transversal to $\partial\Omega$ for z close to q_1, hence for small V and supp φ the map π_η is a diffeomorphism. We introduce $U = \{x \in U_{\varphi,\eta,a} : |a - a_0| \leq \epsilon, \eta \in W\}$, where $a_0 > 0$ is fixed and $\epsilon > 0$ is small enough. With this choice, we define $\Gamma = U \times V$ and the set Σ_1 as in Section 4.1. It is easy to see that $WF(u_0) \subset \Sigma_1$.

By using the notation of Section 4.1, introduce

$$V_k^+ = R_k^+(Id - M_{k-1})i^* V_{k-1}^+, \ k \geq 2,$$

and consider

$$U_m = \sum_{k=1}^{m} (-1)^{k-1} V_k^+ u_0.$$

Notice that the factors $(-1)^k$ are chosen to obtain vanishing boundary condition on $\omega_k, k \geq 2$. Then $u - U_m \in C^\infty(\mathbb{R} \times \Omega)$ for $t < t_{m+1}$, where t_{m+1} is related to the mth reflection of the rays issued from Σ_1. Consequently, we may study (9.2) replacing u by U_m. To do this, we need a representation of

$$L_m = (-1)^{m-1} i^*_{\omega_m} \left(\frac{\partial}{\partial \nu} - \langle \nu, \theta \rangle \frac{\partial}{\partial t} \right) R_m^+(Id - M_{m-1}) i^* V_{m-1} u_0.$$

Consider the operator $\mathcal{R}_m = R_m^+ i^* V_{m-1}^+$ and neglect the term involving M_{m-1}, which yields a smooth term when we take the trace on ω_m. This operator is related to the canonical relation

$$\mathcal{G}_+^m = \{t + s, \tau, \Phi^{t+s-T_m(s,\tau,y',\eta')} \widehat{\lambda_{m-1}}(s, \tau, y', \eta'), s, \tau, y', \eta') \\ \in T^*(\mathbb{R} \times \Omega \times \mathbb{R} \times \omega_1)\}.$$

Similar to the canonical relation (4.6), here $\lambda_{m-1}(s, \tau, y', \eta')$ is the point on $T^*_{\partial\Omega}(\mathbb{R} \times \Omega)$ obtained after $(m-1)$ reflections of the generalized bicharacteristics issued from $(s, \tau, g(y'), y', \eta_1, \eta'), \eta_1 = \sqrt{\tau^2 - |\eta'|^2}$, while $\widehat{\lambda_{m-1}}(\dots)$ denotes the point obtained from $\lambda_{m-1}(\dots)$ after mth reflection and $T_m(s, \tau, y', \eta')$ is the length of the projection of these bicharacteristics on Ω.

Let $\mathcal{O}_m \subset \mathbb{R}^n$ be a small neighbourhood of q_m and let in local coordinates $(h(x'), x')$, $\omega_m = \mathcal{O}_m \cap \partial\Omega$ be given by $x_1 = h(x')$ with $q_m = (0, x_0')$ and $x_1 > h(x')$ in $\mathcal{O}_m \cap \Omega^\circ$. In local coordinates (s, y', τ, η') and (t, x', σ, ξ') in $T^*(\mathbb{R} \times \omega_1)$ and $T^*(\mathbb{R} \times \omega_m)$, respectively, the operator $i^*_{\omega_m} R_m^+ i^*_{\omega_m} V_{m-1}^+$ is related to the graph of a canonical transformation σ_{m-1} and it can be expressed by a homogeneous of order 1 with respect to (τ, η') generating function $\chi(t, x', \tau, \eta')$ (see Section 4.1) such that

$$\det \begin{pmatrix} \chi_{t\tau} & \chi_{t\eta'} \\ \chi_{x'\tau} & \chi_{x'\eta'} \end{pmatrix} \neq 0,$$

and

$$\sigma_{m-1} : (\chi_\tau, \chi_{\eta'}, \tau, \eta') \to (t, x', \chi_t, \chi_{x'}).$$

Clearly, τ is constant with respect to the action of σ_{m-1}, and we get

$$\chi_{t\tau} = 1, \ \ \chi_{tt} = \chi_{t\eta'} = \chi_{tx'} = 0. \tag{9.3}$$

This yields $\det \chi_{x'\eta'} \neq 0$.

We have the representation

$$\begin{aligned} J_m &= (-1)^{m-1} i^*_{\omega_m} R^+_m i^*_{\omega_m} V^+_{m-1} \equiv (-1)^{m-1} i^*_{\omega_m} V^+_{m-1} \\ &= (2\pi)^{-n} (-1)^{m-1} \int e^{\mathbf{i}\chi(t,x',\tau,\eta') - \mathbf{i}s\tau - \mathbf{i}\langle y',\eta'\rangle} b(t, x', s, y', \tau, \eta') ds\, dy'\, d\tau\, d\eta', \end{aligned}$$

where $b \sim \sum_{k=0}^{\infty} b_k(t, x', s, y', \tau, \eta')$ with b_k homogeneous of order $-k$ with respect to (τ, η').

Applying J_m to u_0, one obtains

$$\begin{aligned} J_m u_0 = (2\pi)^{-n} (-1)^m \int & e^{\mathbf{i}\chi(t,x',\tau,\eta') - \mathbf{i}(\langle y',\omega'\rangle + g(y')\omega_1)\tau - \mathbf{i}\langle y',\eta'\rangle} \\ & \times b(t, x', \langle y', \omega'\rangle + g(y')\omega_1, y', \tau, \eta')(1+|dg(y')|^2)^{1/2} \varphi(g(y'), y') dy'\, d\tau\, d\eta'. \end{aligned}$$

Now we pass to L_m. Let $p_m(x, \tau, \xi)$ be the principal symbol of $\Box$ in local coordinates (x_1, x') and let $\xi_1^+(x, \tau, \xi')$ be the outgoing root of the equation $p_m(x, \tau, \xi_1, \xi') = 0$ with respect to ξ_1 satisfying the condition $\frac{-\xi_1^+}{\tau} > 0$. Consider the phase function $\varphi^+(t, x, \tau, \xi')$ determined as the solution of the problem

$$\begin{cases} \dfrac{\partial \varphi^+}{\partial x_1} = \xi_1^+(x, \varphi_t^+, \varphi_{x'}^+), \\ \varphi^+|_{x_1 = h(x')} = t\tau + \langle x', \xi'\rangle. \end{cases}$$

Therefore, for x close to ω_m we have

$$(R^+_m g)(t, x) = (2\pi)^{-n} \int e^{\mathbf{i}\varphi^+(t,x,\tau,\xi') - \mathbf{i}s\tau - \mathbf{i}\langle y',\xi'\rangle} a(t, x, \tau, \xi') g(s, y') ds\, dy'\, d\tau\, d\xi'$$

with $a \sim \sum_{k=0}^{\infty} a_k(t, x, \tau, \xi')$, a_k being homogeneous of order $-k$ with respect to (τ, ξ'). Moreover,

$$a_0|_{x_1 = h(x')} = 1, \ a_k|_{x_1 = h(x')} = 0, \ \ k \geq 1.$$

The singularities of $R^+_m g$ are propagating along the bicharacteristics of $\Box$ entering in Ω. Therefore,

$$i^*_{\omega_m} \left(\frac{\partial}{\partial \nu} - \langle \nu, \theta \rangle \frac{\partial}{\partial t} \right) R^+_m g = B_m i^*_\omega R^+_m g,$$

where B_m is a first order pseudo-differential operator. The principal symbol of B_m has the form

$$\beta_1(x',\tau,\xi') = \mathbf{i}\Big[(1+|dh(x')|^2)^{-1/2}\left(\xi_1^+(h(x'),x',\tau,\xi') - \langle \xi', h_{x'}\rangle\right) \\ -\langle \nu(h(x'),x'),\theta\rangle\tau\Big], \tag{9.4}$$

and modulo smooth terms, we have $L_m = B_m J_m u_0$. On the other hand, $B_m J_m u_0$ is Fourier integral operator with the same phase as $J_m u_0$ and principal symbol

$$\tilde{b}_1(t,x',y',\tau,\eta') = \beta_1(x',\chi_t,\chi_{x'})b_0(t,x',\langle y',\omega'\rangle + g(y')\omega_1, y',\tau,\eta').$$

We set $\psi = t - \langle x,\theta\rangle$ and later we study the asymptotic of $(B_m J_m u_0, \rho e^{\mathbf{i}\lambda\psi})$. The leading term is the integral

$$I_0(\lambda) = (-1)^m\left(\frac{\lambda}{2\pi}\right)^n \int e^{\mathbf{i}\lambda\Phi(t,x',y',\tau,\eta')}\tilde{b}_1(t,x',y',\tau,\eta') \\ \times \rho(\langle x',\theta'\rangle - t + T_\gamma)(1+|dg(y')|^2)^{1/2} dt\ dx'\ dy'\ d\tau\ d\eta' \\ + \text{lower order terms.}$$

Here the phase function Φ has the form

$$\Phi(t,x',y',\tau,\eta') = t - h(x')\theta_1 - \langle x',\theta'\rangle + \chi(t,x',\tau,\eta') \\ -(\langle y',\omega'\rangle + g(y')\omega_1)\tau - \langle y',\eta'\rangle.$$

The critical points of Φ are determined from the equations:

$$\begin{cases} \chi_t(t,x',\tau,\eta') = -1, \\ \chi_{x'}(t,x',\tau,\eta') = \theta' + dh(x')\theta_1, \\ \eta' = -(\omega' + dg(y')\omega_1)\tau, \\ \chi_\tau(t,x',\tau,\eta') = \langle y',\omega'\rangle + g(y')\omega_1, \\ \chi_{\eta'}(t,x',\tau,\eta') = y'. \end{cases}$$

Let $(\hat{t},\hat{x}',\hat{y}',\hat{\tau},\hat{\eta}')$ be a critical point of Φ. Since τ is constant along the generalized bicharacteristics of $\Box$, we get $\hat{\tau} = 1$ and we may parameterize the bicharacteristics by the time t. This implies

$$-\hat{\eta}' = \omega' + dg(\hat{y}')\omega_1.$$

Set $\hat{y} = (g(\hat{y}'),\hat{y}')$ and let $p(y,\tau,\eta)$ be the principal symbol of $\Box$. Denote by $\hat{\eta}_1^-(y',\tau,\eta')$ the incoming root of the equation $p(g(y'),y',1,\eta_1,\eta') = 0$ with respect to η_1, that is $\langle \hat{\eta}_1^-, \nu\rangle < 0$. Then for $\hat{y}'$ close to y_0', we conclude that $-\hat{\eta} = -(\hat{\eta}_1^-(\hat{y}',1,\hat{\eta}'),\hat{\eta}')$ will be close to the reflected direction of γ at q_1 having the form

$$\omega - 2\langle \nu(q_1),\omega\rangle\nu(q_1).$$

Let $\hat{T}$ be the length of the generalized geodesics $\hat{l}$ issued from $\hat{y}$ with direction $-\hat{\eta}$ and joining $\hat{y}$ and $\hat{x} = (h(\hat{x}'), \hat{x}')$. Then

$$\langle \hat{y}, \omega \rangle = \chi_\tau(\hat{t}, \hat{x}', 1, \hat{\eta}') = \hat{t} - \hat{T},$$

since $\chi_\tau = s$ and $\hat{t} = \hat{s} + \hat{T}$. The reflecting direction of $\hat{l}$ is close to θ and $\langle \hat{x}, \theta \rangle - \hat{t} + T_\gamma \in \text{supp } \rho_\delta$ yields $\hat{t} = \langle \hat{x}, \theta \rangle + T_\gamma + \mathcal{O}(\delta)$. Thus if $\text{supp } \varphi$ and δ are small enough, the sojourn time of the ray issued from $(\hat{y}, -\hat{\eta})$ is close to T_γ. Exploiting the condition (I) and the non-degeneracy of γ, we get

$$\hat{x}' = x_0',\ \hat{y}' = y_0',\ \hat{t} - \langle \hat{x}_0, \theta \rangle = T_\gamma,\ \hat{\eta}' = -\omega'.$$

Moreover, on the critical points of Φ we have

$$\Phi_{tt} = \Phi_{tx'} = \Phi_{ty'} = \Phi_{t\eta'} = 0, \Phi_{t\tau} = 1.$$

Put

$$G = \langle \nu(y_0), \omega \rangle g_{y'y'}(y_0'), H = \langle \nu(x_0), \theta \rangle h_{x'x'}(x_0'),$$

and consider the matrix

$$\Delta = \begin{pmatrix} G & 0 & I \\ 0 & -H - \chi_{x'x'} & \chi_{x'\eta'} \\ I & -\chi_{\eta'x'} & -\chi_{\eta'\eta'} \end{pmatrix}.$$

We have

$$\det \Delta = (\det \chi^2_{x'\eta'}) \det \begin{pmatrix} G & 0 & I \\ 0 & \chi_{x'\eta'}(\chi_{x'x'} + H)\chi_{\eta'x'} & I \\ -I & I & \chi_{\eta'\eta'} \end{pmatrix}.$$

To find $\det \Delta$, we apply the following.

Lemma 9.1.1: *Let F, M, L be $n \times n$ matrices. Then*

$$\det \begin{pmatrix} F & 0 & I \\ 0 & M & I \\ -I & I & L \end{pmatrix} = \det (FLM + M - F). \tag{9.5}$$

Proof: First assume that F is invertible. Then

$$\det \begin{pmatrix} F & 0 & I \\ 0 & M & I \\ -I & I & L \end{pmatrix} = (\det F) \det \begin{pmatrix} M & I \\ I & L + F^{-1} \end{pmatrix}.$$

It is easy to see that

$$\det \begin{pmatrix} M & I \\ I & L + F^{-1} \end{pmatrix} = \det ((L + F^{-1})M - I),$$

from which (9.5) follows directly. In the general case replace F by $F_\epsilon = F + \epsilon I$, where $\epsilon > 0$ is taken small so that F_ϵ is invertible. Applying (9.6) for F_ϵ and letting $\epsilon \to 0$, we complete the proof. ■

The symmetry of the matrices $\chi_{y'y'}, \chi_{\eta'\eta'}, H, G$ and Lemma 9.1.1 yield

$$\det\, \chi_{x'\eta'}^{-1} \det \Delta = \det\, \Big((H + \chi_{x'x'})\chi_{x'\eta'}^{-1}(I + \chi_{\eta'\eta'}G) - \chi_{\eta'\eta'}G \Big). \tag{9.6}$$

By exploiting the non-degeneracy of γ, we will show that the right-hand side of (9.6) is not zero. To do this, we are going to use the hyperplane Z_ω, orthogonal to ω and the orthogonal projection π_ω onto Z_ω (cf. Section 2.4 for the notation). Consider the diagram

$$\begin{array}{ccccccccccc}
Z_\omega & \xleftarrow{\pi_\omega} & \partial\Omega & \xrightarrow{\mu_1} & B^*(\partial\Omega) & \xrightarrow{r} & B^*(\partial\Omega) & \xrightarrow{\mu_2} & B_x^*(\partial\Omega) & \xrightarrow{p_x} & \mathbb{S}^{n-1}\\
 & & j_1\uparrow & & {j_1}^*\downarrow & & {j_2}^*\uparrow & & j_2\downarrow & & \\
 & & \mathbb{R}^{n-1} & \xrightarrow{\tilde\mu_1} & T^*(\mathbb{R}^{n-1}) & \xrightarrow{\tilde r} & T^*(\mathbb{R}^{n-1}) & \xrightarrow{\tilde\mu_2} & \mathbb{R}^{n-1} & &
\end{array}$$

the maps in it being defined as follows. First, μ_1 has the form

$$\mu_1(y) = (y, \langle \nu(y), \omega\rangle \nu(y) - \omega) \in B^*(\partial\Omega).$$

Here $\eta(\omega) = -\omega + 2\langle \nu(y), \omega\rangle \nu(y) \in \mathbb{S}^{n-1}$ is the reflected direction associated with $-\omega$. To define r, we use the canonical transformation σ_{m-1}. Applying (9.3), from the equality $(t, x, 1, \tilde\xi) = \sigma_{m-1}(s, y, 1, \tilde\eta)$ we determine two functions $x(y, \tilde\eta), \tilde\xi(y, \tilde\eta)$ and define

$$r : B^*(\partial\Omega) \ni (y, \tilde\eta) \longrightarrow (x(y, \tilde\eta), \tilde\xi(y, \tilde\eta)) \in B^*(\partial\Omega).$$

The map μ_2 is related to the reflection on $\partial\Omega$ in a neighbourhood of q_m and has the form

$$\mu_2(x, \tilde\xi) = -\xi + \langle \nu(x), \xi\rangle \nu(x) \in T_x^*(\partial\Omega),$$

where $\xi \in T_x^*(\Omega)$ is the unique vector with $\langle \nu(x), \xi\rangle < 0$ the orthogonal projection of which on $T_x(\partial\Omega)$ coincides with $\tilde\xi$. The map $p_x : B_x^*(\partial\Omega) \longrightarrow \mathbb{S}^{n-1}$ is defined by

$$p_x : B_x^*(\partial\Omega) \ni \zeta - \langle \nu(x), \zeta\rangle \nu(x) \to \zeta \in \mathbb{S}^{n-1}.$$

The maps j_1, j_2, j_1^*, j_2^* are diffeomorphisms associated with the local coordinates x', y' chosen earlier. For example,

$$j_1(y') = (g(y'), y'), j_2(\langle dh(x'), \xi'\rangle, \xi') = \xi'.$$

Finally, the maps $\tilde\mu_1, \tilde r, \tilde\mu_2$ are defined in such a way that the considered diagram is commutative.

It is clear that $dj_1(y_0') = I, dj_2(0, \theta') = I$ and

$$d(\mu_2 \circ r \circ \mu_1)(q_1) = d(\tilde{\mu}_2 \circ \tilde{r} \circ \tilde{\mu}_1)(y_0').$$

In local coordinates y' we have

$$\tilde{\mu}_1(y') = (y', -(1 + |dg(y')|^2)^{-1}(\omega_1 - \langle dg(y'), \omega' \rangle)(-1, dg(y')) - \omega),$$

hence

$$d\tilde{\mu}_1(y_0') = \begin{pmatrix} I \\ -G \end{pmatrix}.$$

Similarly, $d\tilde{\mu}_2(0, x_0'') = (H, I)$. To obtain $d\tilde{r}$, observe that the map $\tilde{r}$ is given by the generating function $\chi(\hat{t}, x', 1, \eta')$, that is

$$\tilde{r} : \begin{pmatrix} \chi_{\eta'} \\ \eta' \end{pmatrix} \to \begin{pmatrix} x' \\ \chi_{x'} \end{pmatrix}.$$

A simple calculation shows that

$$d\tilde{r}(\chi_{\eta'}, \eta') = \begin{pmatrix} \chi_{\eta' x'}^{-1} & -\chi_{\eta' x'}^{-1} \chi_{\eta' \eta'} \\ \chi_{x' x'} \chi_{\eta' x'}^{-1} & \chi_{x' \eta'} - \chi_{x' x'} \chi_{\eta' x'}^{-1} \chi_{\eta' \eta'} \end{pmatrix}.$$

Therefore, by using the form of $d\tilde{\mu}_1(y_0')$ and $d\tilde{\mu}_2(x_0', \theta')$, we find

$$d(\tilde{\mu}_2 \circ \tilde{r} \circ \tilde{\mu}_1)(y_0') = -(H + \chi_{x'x'} \chi_{\eta' x'}^{-1}(I + \chi_{\eta' \eta'} G) + \chi_{x' \eta'} G.$$

In the same way consider the local inverse $\pi_\omega^{-1} : Z_\omega \to \partial\Omega$ around q_1. It is easy to see that $\det\, d(\pi_\omega^{-1})(u_\gamma) = -\langle \nu(q_1), \omega \rangle^{-1}, \det\, d(p_x)(\theta') = \langle \nu(q_m), \theta \rangle^{-1}$.

Since γ is non-degenerate, there exists a neighbourhood W_γ of $u_\gamma = \pi_\omega(q_1)$ in Z_ω such that the map

$$J_\gamma : W_\gamma \to \mathbb{S}^{n-1}$$

is well defined and smooth (see Section 2.4). Moreover, we have

$$J_\gamma = p_x \circ \mu_2 \circ r \circ \mu_1 \circ \pi_\omega^{-1}$$

and $\det\, dJ_\gamma(u_\gamma) \neq 0$. On the other hand, (9.6) implies

$$\begin{aligned} |\det\, dJ_\gamma(u_\gamma)| &= \Big(|\langle \nu(q_1), \omega \rangle| |\langle \nu(q_m), \theta \rangle| \Big)^{-1} \\ &\quad \times \Big| \det(H + \chi_{x'x'}) \chi_{\eta' x'}^{-1}(I + \chi_{\eta' \eta'} G) - \chi_{x' \eta'} G \Big| \\ &= |\langle \nu(q_1), \omega \rangle|^{-1} |\langle \nu(q_m), \theta \rangle|^{-1} |(\det\, \chi_{x' \eta'})^{-1}| |\det\, \Delta|. \end{aligned}$$

Therefore, $\det\, \Delta \neq 0$ and we are in position to apply the stationary phase argument to the integral $I_0(\lambda)$. It follows from the Euler equality for $\chi(t, x', \tau, \eta')$ that

$$\Phi(\langle x_0, \theta \rangle + T_\gamma, x_0', y_0', 1 - \omega') = T_\gamma.$$

On the other hand, by (9.4) for $\hat{t} = \langle x_0, \theta\rangle + T_\gamma$, we have

$$\begin{aligned} d_1 &= \beta_1(x_0', \chi_t(\hat{t}, x_0', 1, \eta_0'), \chi_{x'}(\hat{t}, x_0', 1, \eta_0')) \\ &= \mathbf{i}(\xi_1^+(0, x_0', 1, \theta') + \langle \nu(q_m), \theta\rangle) = 2\mathbf{i}\langle \nu(q_m), \theta\rangle, \end{aligned}$$

since the outgoing condition implies

$$\xi_1^+(0, x_0', 1, \theta') = \sqrt{1 - |\theta'|^2} = \theta_1.$$

It remains to find the form of

$$d_0 = b_0(\hat{t}, x_0', \langle \chi_{\eta'}(\hat{t}, x_0', 1, \eta_0'), \omega\rangle, \chi_{\eta'}(\hat{t}, x_0', 1, \eta_0'), 1, \eta_0').$$

Recall that this symbol is attached to the half-density $|d\tau \wedge dx' \wedge d\eta'|^{1/2}$. As in Section 4.2, we have

$$|d\tau \wedge dy' \wedge d\eta'|^{1/2} = |\det\, \chi_{x'\eta'}|^{1/2} |d\tau \wedge dx' \wedge d\eta'|^{1/2}.$$

Next the principal symbol $b_0(\hat{t}, x', \hat{s}, y', \tau, \eta')$ of J_m with respect to the half-density $|d\tau \wedge dy' \wedge d\eta'|^{1/2}$ according to Lemma 4.2.2 and our choice of V_1^+, modulo Maslov factor, is equal to

$$\gamma_1^{1/2}(y', \eta', 1)\gamma_1^{-1/2}(\beta^m(y', \eta'), 1)\alpha_1^{-1}(y', \eta')\alpha_m(x', \xi').$$

Here the coefficient $\gamma_1(y', \eta', 1)$ is related to the choice of initial conditions, the coefficient $\gamma_1^{-1/2}(\beta^m(y', \eta'), 1)$ comes from Lemma 4.2.2, while a_1 and a_m are related to the change of coordinates passing to normal geodesic coordinates used in Section 4.2 to local coordinates introduced in the beginning of this section. It is easy to see that $\alpha_1 = \alpha_m = 1$ at the points (y_0', η_0') and (x_0', ξ_0'), respectively, since the unit normal vectors at $(0, y_0')$ and $(0, x_0')$ have the form $(1, 0, \ldots 0)$. Therefore,

$$d_0 = \left(\sqrt{1 - |\theta'|^2}\right)^{-1/2} \left(\sqrt{1 - |\omega'|^2}\right)^{1/2} = \langle \nu(q_m), \theta\rangle^{-1/2} |\langle \nu(q_1), \omega\rangle|^{1/2},$$

and we obtain

$$\begin{aligned} &\tilde{b}_1(\hat{t}, x_0', \langle \chi_{\eta'}(\hat{t}, x_0', 1, \eta_0'), \omega\rangle, \chi_{\eta'}(\hat{t}, x_0', 1, \eta_0'), 1, \eta_0') \\ &\quad = e^{\mathbf{i}\frac{\pi}{2}\sigma_\gamma} d_1 d_0 |\det\, \chi_{x'\eta'}|^{1/2} = 2\mathbf{i}e^{\mathbf{i}\frac{\pi}{2}\sigma_\gamma} (\langle \nu(q_m), \theta\rangle)^{1/2} |\langle \nu(q_1), \omega\rangle|^{1/2} |\det\, \chi_{x'\eta'}|^{1/2}, \end{aligned}$$

σ_γ being the Maslov index.

By the stationary phase argument, we get

$$\begin{aligned} I_0(\lambda) = 2\mathbf{i}\left(\frac{2\pi}{\lambda}\right)^{(n-1)/2} &(-1)^m \exp\left(\mathbf{i}\frac{\pi}{2}\beta_\gamma\right) e^{\mathbf{i}\lambda T_\gamma} \\ &\times |\det\, dJ_\gamma(u_\gamma)|^{-1/2} + \mathcal{O}(\lambda^{-(n+1)/2})), \end{aligned}$$

where

$$\beta_\gamma = \sigma_\gamma + \frac{1}{2} \text{ sign } \Delta \in \mathbb{N}.$$

Recall that

$$(s(t,\theta,\omega), \rho(t+T_\gamma)e^{-\mathbf{i}\lambda t}) = \sum_{k=0}^{n-2} c_k(-\mathbf{i}\lambda)^{n-2-k}$$
$$\times \int_{\mathbb{R}}\int_{\partial\Omega} e^{\mathbf{i}\lambda(t-\langle x,\theta\rangle)}\rho^{(k)}(\langle x,\theta\rangle - t + T_\gamma)\left(\frac{\partial}{\partial\nu} - \langle\nu,\theta\rangle\frac{\partial}{\partial t}\right) u\,dt\,dS_x$$

with $c_0 = \frac{1}{2}(-1)^{(n-1)/2}(2\pi)^{1-n}$. Thus, we obtain the following.

Theorem 9.1.2: *Let γ be an ordinary non-degenerate reflecting (ω,θ)-rays with m_γ reflection points satisfying assumption (I). Then*

$$-T_\gamma \in \text{sing supp } s(t,\theta,\omega),$$

and for t sufficiently close to $-T_\gamma$ the scattering kernel has the form

$$s(t,\theta,\omega) = (2\pi\mathbf{i})^{(1-n)/2}(-1)^{m_\gamma-1}\exp\left(\mathbf{i}\frac{\pi}{2}\beta_\gamma\right)$$
$$\times |\text{det } dJ_\gamma(u_\gamma)|^{-1/2}\delta^{(n-1)/2}(t+T_\gamma) + \textit{lower order singularities}. \tag{9.7}$$

In the particular case when $m_\gamma = 1$, the integral $I_0(\lambda)$ can be written with a phase function

$$\psi(x') = \langle x', \omega' - \theta'\rangle + h(x')(\omega_1 - \theta_1).$$

Then

$$\text{det } \Delta(x'_0) = \langle\nu(q_1), \omega - \theta\rangle^{n-1}\text{det } h_{x'x'}(x'_0),$$

and $\langle\nu(q_1), \omega+\theta\rangle = 0$. Thus,

$$|\text{det } dJ_\gamma(u_\gamma)|^{-1/2} = 2^{(1-n)/2}|\langle\nu(q_1),\omega\rangle|^{(3-n)/2}|\text{det } h_{x'x'}(x'_0)|^{-1/2}.$$

Since $\nu(q_1) = (\theta-\omega)/|\theta-\omega|$, we get

$$\langle\nu(q_1),\omega\rangle = \frac{\langle\theta,\omega\rangle - 1}{|\theta-\omega|} = -\frac{1}{2}|\theta-\omega|,$$

and therefore

$$|\text{det } dJ_\gamma(u_\gamma)|^{-1/2} = \frac{1}{2}|\theta-\omega|^{(3-n)/2}|\mathcal{K}(q_1)|^{-1/2},$$

where $\mathcal{K}(q_1)$ is the Gauss curvature of $\partial\Omega$ at q_1. In this way we have established the following.

Corollary 9.1.3: *Let γ be as in Theorem 9.1.2 and assume in addition that γ has exactly one reflection point q_1. Then for t sufficiently close to $-\mathbb{T}_\gamma$ we have*

$$s(t,\theta,\omega) = \frac{1}{2}(2\pi\mathbf{i})^{(1-n)/2} \exp\left(\mathbf{i}\frac{\pi}{4}\text{sign}\,\Delta(q_1)\right)|\theta-\omega|^{(3-n)/2} \\ \times|\mathcal{K}(q_1)|^{-1/2}\delta^{(n-1)/2}(t+\langle\omega-\theta,q_1\rangle) + \textit{lower order singularities.}$$

Moreover, if $\partial\Omega$ is strictly convex at q_1 with respect to the normal field ν, then sign $\Delta(q_1) = (n-1)$ *and*

$$s(t,\theta,\omega) = \frac{1}{2}(2\pi)^{(1-n)/2}|\theta-\omega|^{(3-n)/2} \\ \times|\mathcal{K}(q_1)|^{-1/2}\delta^{(n-1)/2}(t+\langle\omega-\theta,q_1\rangle) + \textit{lower order singularities.}$$

These results imply an asymptotic of the *scattering amplitude* $a(\lambda,\theta,\omega)$ given in Section 5.1 by

$$\overline{a(\lambda,\theta,\omega)} = \left(\frac{2\pi}{\mathbf{i}\lambda}\right)^{(n-1)/2}\mathcal{F}_{t\to\lambda}s(t,\theta,\omega),$$

where $\mathcal{F}_{t\to\lambda}$ is the Fourier transform with respect to t.

9.2 Singularities of the scattering kernel for generic domains

Let Ω,ω,θ be as in the previous section. In this section we consider an application of Theorem 9.1.2 exploiting some results in Chapter 6.

Denote by $\mathcal{L}^m_{\omega,\theta}(\Omega)$ the set of all (ω,θ)-rays of mixed type in Ω. Then

$$L_{\omega,\theta} = \mathcal{L}_{\omega,\theta}(\Omega)\setminus\mathcal{L}^m_{\omega,\theta}(\Omega)$$

is exactly the set of all reflecting (ω,θ)-rays in Ω. Set

$$\mathcal{G}(\Omega) = \{-T_\gamma : \gamma\in\mathcal{L}^m_{\omega,\theta}(\Omega)\}.$$

Theorem 9.2.1: *There exists a residual subset $\mathcal{R}$* of $\mathbf{C}(X)$ *such that for every $f\in\mathcal{R}$ we have*

$$\{-T_\gamma : \gamma\in L_{\omega,\theta}(\Omega_f)\}\setminus\mathcal{G}(\Omega_f)\subset \text{sing supp }\, s_{\Omega_f}(t,\theta,\omega), \tag{9.8}$$

and for t close to $-T_\gamma$, (9.7) holds with Ω replaced by Ω_f, provided $-T_\gamma$ belongs to the left-hand side of (9.8).

Proof: It follows by Theorems 6.2.3, 6.3.3, 6.4.6 and 8.1.2 that there exists a residual subset $\mathcal{R}$ of $\mathbf{C}(X)$ such that every f in it has the following properties:

(i) every reflecting (ω, θ)-ray in Ω_f is ordinary and non-degenerate;

(ii) $T_\gamma \neq T_\delta$ for every two different reflecting (ω, θ)-rays γ and δ in Ω_f;

(iii) the normal curvature of $f(X)$ does not vanish of infinite order.

To check that $\mathcal{R}$ has the desired properties, fix an arbitrary $f \in \mathcal{R}$. We claim that $\mathcal{G}(\Omega_f)$ is closed in $\mathbb{R}$. Indeed, let $\{\gamma_k\} \subset \mathcal{L}^m_{\omega,\theta}$ and let $T_{\gamma_k} \stackrel{k\to\infty}{\longrightarrow} T$. We may assume that γ_k hits $\partial\Omega$ for first time at x_k and choosing a subsequence, which we denote again by $\{x_k\}$, we have $x_k \to x \in \partial\Omega$ for $k \to \infty$. By using Lemma 1.2.6, we may assume that $\{\gamma_k\}$ converges to some $\gamma \in \mathcal{L}_{\omega,\theta}(\Omega_g)$ and $T_\gamma = T$. If γ is a reflecting (ω, θ)-ray, then it would be ordinary by $f \in \mathcal{R}$, and therefore for all sufficiently large k, γ_k would be a reflecting (ω, θ)-ray, which is a contradiction with the non-degeneracy of γ (cf. (i)). Thus, γ is an (ω, θ)-ray of mixed type, which shows that $\mathcal{G}(\Omega_f)$ is closed in $\mathbb{R}$.

Now the desired properties of f follows from Theorem 9.1.2 and conditions (i) and (ii). ■

As an immediate consequence of the above-mentioned theorem, one gets the following.

Corollary 9.2.2: *Assume in addition that $K = \overline{\mathbb{R} \setminus \Omega}$ is a finite disjoint union of strictly convex domains K_i. Let $\mathcal{O}$ be the set of those $f \in \mathbf{C}(X)$ such that $f(K_i)$ is strictly convex for every i. Then there exists a residual subset $\mathcal{S}$ of $\mathcal{O}$ such that for every $f \in \mathcal{S}$ the relation (5.34) becomes an equality with Ω replaced by Ω_f and for each $\gamma \in \mathcal{L}_{\omega,\theta}(\Omega_f)$ and t close to $-T_\gamma$ we have (9.7).*

9.3 Glancing ω-rays

From now till the end of this chapter we consider domains Ω in $\mathbb{R}^3$. As in Section 9.2, Ω will be a connected close domain with smooth boundary $\partial\Omega$ and bounded complement.

Let Ω be fixed and let Z_ω and π_ω be as in Section 2.4. A curve $\gamma = \cup_{i=0}^k l_k$ in Ω consisting of linear segments $l_i = [x_i, x_{i+1}]$, $x_i \in \partial\Omega$, $i = 1, \ldots, k$, and an infinite ray l_0 starting from x_i with direction $-\omega$, will be called a glancing ω-ray in Ω if it has the following properties:

(i) l_i and l_{i+1} satisfy the law of reflection at x_{i+1} with respect to $\partial\Omega$ for every $i = 0, 1, \ldots, k-1$;

(ii) l_k is tangent to $\partial\Omega$ and the normal curvature of $\partial\Omega$ at x_k vanishes in direction l_k.

The points $x_1, \ldots, x_k$ will be called vertices of γ.

Our aim in this section is to show that for generic domains Ω, for any $k \geq 1$ the glancing ω-rays with k vertices in Ω form a discrete subset of a certain manifold. This fact will be applied in the next section.

Let γ be a glancing ω-ray in Ω and let $x_1, \dots, x_s$ be all different vertices of it such that x_s is the last one. Then there exists a surjective ns-map (cf. Section 6.2)

$$\alpha: \{1, \dots, k\} \to \{1, \dots, s\} \tag{9.9}$$

such that

$$\alpha(k) = s, \tag{9.10}$$

and $x_{\alpha(1)}, \dots, x_{\alpha(k)}$ are the successive vertices of γ. If $s > 1$, we may assume that

$$\alpha(k-1) = s - 1. \tag{9.11}$$

Fix arbitrary integers $k \geq s \geq 1$ and a surjective ns-map (9.9) with (9.10) and (9.11), the latter provided $s > 1$. Set $X = \partial\Omega$. Recall that for $f \in \mathbf{C}(X)$ by Ω_f we denote the unbounded domain in $\mathbb{R}^3$ with boundary $f(X)$. Denote by $G(\omega, \alpha)$ the set of those $f \in \mathbf{C}(X)$ such that there does not exist $y = (y_1, \dots, y_s) \in f(X)^{(s)}$ such that $y_{\alpha(1)}, \dots, y_{\alpha(k)}$ are successive vertices of a glancing ω-ray in Ω_f.

Lemma 9.3.1: *Let α be non-invertible, that is $k > s$. Then $G(\omega, \alpha)$ contains a residual subset of* $\mathbf{C}(X)$.

Proof: We assume $s > 1$. The case $s = 1$ can be considered using some part of the following reasonings.

Given $i - 1, \dots, s - 1$, we determine $I_i(\alpha)$ by (6.25). Denote by U_α the set of those $y = (y_1, \dots, y_s) \in (\mathbb{R}^3)^{(s)}$ such that y_i does not belong to the convex hull of the set $\{y_j : j \in I_i(\alpha)\}$ for every $i = 1, \dots, s-1$. It is convenient to set

$$y_0 = \pi_\omega(y_1). \tag{9.12}$$

Define $F : U_\alpha \to \mathbb{R}$ by

$$F(y) = \sum_{i=0}^{s-1} \left\| y_{\alpha(i)} - y_{\alpha(i+1)} \right\| .$$

Let γ be a glancing ω-ray of type α in $\Omega_f, f \in \mathbf{C}(X)$, that is there exists an ordering $y_1, \dots, y_s$ of different vertices of γ such that $y_{\alpha(1)}, \dots y_{\alpha(k)}$ are the successive vertices of γ. Then $t = (y_1, \dots, y_s) \in U_\alpha$. Moreover, for $x = (x_1, \dots, x_s), f(x_i) = y_i, x' = (x_1, \dots, x_{s-1})$, we have $\text{grad}_{x'} F(f^s(x)) = 0$ and $\langle y_s - y_{s-1}, \nu(y_s) \rangle = 0$ (the fact that the normal curvature of $f(X)$ at y_i vanishes in direction $y_s - y_{s-1}$ is not needed here).

Assuming $k > s$, we use almost the same argument as in the proof of Theorem 6.2.3. After the above preparation, the details are rather standard and we leave them to the reader. ■

In view of Lemma 9.3.1, we can restrict our attention to glancing ω-rays that pass only once through each of their vertices.

Fix $k \in \mathbb{N}$ and denote by $D(\omega, k)$ the set of those $f \in \mathbf{C}(X)$ such that the elements $y = (y_1, \ldots, y_k)$ of $f(X)^{(k)}$ for which $y_1, \ldots, y_k$ are the successive vertices of a glancing ω-ray in Ω_f form a discrete subset of $f(X)^{(k)}$.

Lemma 9.3.2: *$D(\omega, k)$ contains a residual subset of $\mathbf{C}(X)$.*

Proof: We assume again $k > 1$, the case $k = 1$ can be proved by using a part of the following argument.

Let U_k be the set of those $y = (y_1, \ldots, y_k) \in (\mathbb{R}^3)^{(k)}$ such that $y_i \notin [y_{i-1}, y_{i+1}]$ for every $i = 1, \ldots, s-1$. Define $H : U_k \to \mathbb{R}$ by

$$H(y) = \sum_{i=0}^{k-1} \|y_i - y_{i+1}\|.$$

As earlier, if $y_1, \ldots, y_k$ are the successive vertices of a glancing ω-ray in Ω_f, then $y = (y_1, \ldots, y_k) \in U_k$, and for $x = (x_1, \ldots, x_k), f(x_i) = y_i, x' = (x_1, \ldots, x_{k-1})$ we have $\text{grad}_{x'} H(f(^k(x)) = 0, \langle y_k - y_{k-1}, \nu(y_k)\rangle = 0$, and the normal curvature of $Y = f(X)$ at y_k vanishes in direction $\omega = (y_k - y_{k-1})/\|y_k - y_{k-1}\|$. The later condition can be expressed analytically as follows. Let $r : V \to Y$ be a smooth chart, V being an open neighbourhood of 0 in $\mathbb{R}^2$ and $r(V)$ an open neighbourhood of y_k in Y, $r(0) = y_k$. Writing the standard coordinates in V by $v = (v_1, v_2)$, we have

$$w = \lambda \frac{\partial r}{\partial v_1}(0) + \mu \frac{\partial r}{\partial v_2}(0)$$

for real λ, μ. Recall the coefficients of the second fundamental form Y at y_k:

$$L = \left\langle \frac{\partial^2 r}{\partial v_1^2}(0), \nu(y_k) \right\rangle, M = \left\langle \frac{\partial^2 r}{\partial v_1 \partial v_1}(0), \nu(y_k) \right\rangle, N = \left\langle \frac{\partial^2 r}{\partial v_1^2}(0), \nu(y_k) \right\rangle.$$

Then the fact that the normal curvature of Y at y_k is zero in direction w is equivalent to

$$L\lambda^2 + 2M\lambda\mu + N\mu^2 = 0. \tag{9.13}$$

Indeed,

$$\lambda = \left\langle w, \left(G \frac{\partial r}{\partial v_1} - F \frac{\partial r}{\partial v_2} \right) \Big/ (EG - F^2) \right\rangle,$$

$$\mu = \left\langle w, \left(E \frac{\partial r}{\partial v_2} - F \frac{\partial r}{\partial v_1} \right) \Big/ (EG - F^2) \right\rangle,$$

where $E = \|\partial r/\partial v_1\|^2, F = \langle \partial r/\partial v_1, \partial r/\partial v_2\rangle, G = \|\partial r/\partial v_2\|^2$ are the coefficients of the first fundamental form. Therefore, (9.14) is equivalent to

$$L\left\langle w, \left(G\frac{\partial r}{\partial v_1} - F\frac{\partial r}{\partial v_2}\right)\right\rangle + 2M\left\langle w, \left(G\frac{\partial r}{\partial v_1} - F\frac{\partial r}{\partial v_2}\right)\right\rangle$$
$$\times\left\langle w, \left(E\frac{\partial r}{\partial v_2} - F\frac{\partial r}{\partial v_1}\right)\right\rangle + N\left\langle w, \left(E\frac{\partial r}{\partial v_2} - F\frac{\partial r}{\partial v_1}\right)\right\rangle^2 = 0.$$

Next, we proceed as in the proofs of Theorems 6.3.1 and 6.4.1. To do this, we need the bundle $J^2(X, \mathbb{R}^3)$ of 2-jets. Let $\mathcal{M}$ be the set of those

$$\tau = (j^2 f_1(x_1), \ldots, j^2 f_k(x_k)) \in J_k^2(X, \mathbb{R}^3)$$

such that $(x_1, \ldots, x_k) \in X^{(k)}, (f_1(x_1), \ldots, f_k(x_k)) \in U_k$, rank $df_i(x_i) = 2$ for all $i = 1, \ldots, k$. Clearly, $\mathcal{M}$ is an open submanifold of $J_k^2(X, \mathbb{R}^3)$. The singular set Σ is now defined as the set of all $\tau \in \mathcal{M}$ such that

$$\text{grad}_{x'} F \circ (f_1 \times \cdots \times f_k)(x) = 0,\ \langle f_k(x_k) - f_{k_1}(x_{k-1}), \nu\rangle = 0,$$

where $x = (x_1, \ldots, x_k), \nu$ is a non-zero normal vector to $f_k(X)$ at $f_k(x_k)$, and the normal curvature of $f_k(X)$ at $f_K(x_k)$ vanishes in direction $w = f_k(x_k) - f_{k-1}(x_{k-1})$.

We shall show that Σ is a smooth manifold of $\mathcal{M}$ of codimension $2k$. Fix coordinate neighbourhoods $V_i, i = 1, \ldots, k$ with $V_i \cap V_j = \emptyset$ whenever $i \neq j$ and set

$$V = M \cap \left(\prod_{i=1}^{k} J^2(V_i, \mathbb{R}^3)\right).$$

Consider arbitrary charts $\varphi_i : W_i \to V_i, W_i \subset \mathbb{R}^2$ and define the chart

$$\varphi : V \to (\mathbb{R}^2)^{(k)} \times (\mathbb{R}^2)^{(k)} \times (\mathbb{R}^2)^{(2k)} \times (\mathbb{R}^2)^{(4k)}$$

by $\varphi(j^2 f_1(x_1), \ldots, j^2 f_k(x_k)) = (u; v; a; b)$, where

$$u = (u_1, \ldots, u_2), v = (v_1, \ldots, v_k),$$
$$a = (a_{ij}^{(t)}), b = (b_{kjl}^{(t)}), \varphi_i(u_i) = x_i,\ v_i = f(x_i),$$
$$a_{ij}^{(t)} = \frac{\partial(f_i^{(t)} \circ \varphi_i)}{\partial u_i^{(j)}}(u_i), b_{kjl}^{(t)} = \frac{\partial^2(f_i^{(t)} \circ \varphi_i)}{\partial u_i^{(j)} \partial u_i^{(l)}}(u_i) \tag{9.14}$$

for $i = 1, \ldots, k, j, l = 1, 2, t = 1, 2, 3$. As before, we use the notation $u_i = (u_i^{(1)}, u_i^{(2)}) \in \mathbb{R}^2, v_i = (v_1^{(1)}, v_i^{(2)}, v_i^{(3)}) \in \mathbb{R}^3, f_i = (f_i^{(1)}, f_i^{(2)}, f_i^{(3)})$.

In what follows we write the elements of $\varphi(v)$ in the form $\xi = (u; v; a; b)$, where u, v, a, b are determined by (9.14). For such a ξ set

$$\nu(\xi) = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ a_{k1}^{(1)} & a_{k1}^{(2)} & a_{k1}^{(3)} \\ a_{k2}^{(1)} & a_{k2}^{(2)} & a_{k2}^{(3)} \end{pmatrix},$$

where $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$. Then $\nu(\xi) \neq 0$ is a normal vector to $f_k(X)$ at $f_k(x_k)$.

Our aim is to show that $\varphi(V \cap \Sigma)$ is a smooth manifold of $\varphi(V)$ of codimension $2k$. For $\xi \in \varphi(V)$ define $E(\xi) = \|a_{k1}\|^2, F(\xi) = \langle a_{k1}, a_{k2}\rangle, G(\xi) = \|a_{k2}\|^2, L(\xi) = \langle b_{k11}, \nu(\xi)\rangle, M(\xi) = \langle b_{k12}, \nu(\xi)\rangle, N(\xi) = \langle b_{k22}, \nu(\xi)\rangle, \lambda(\xi) = \langle v_k - v_{k-1}, G(\xi)a_{k1} - F(\xi)a_{k2}\rangle, \mu(\xi) = \langle v_k - v_{k-1}, E(\xi)a_{k2} - F(\xi)a_{k1}\rangle$, where a_{kj} and b_{ijl} are the vectors in $\mathbb{R}^3$ with components $a_{ij}^{(t)}$ and $b_{ijl}^{(t)}$, respectively.

For $m = 1, 2, 3$ set

$$\mathcal{O}_m = \{\xi \in \varphi(V) :\ \nu^{(m)}(\xi) \neq 0\}.$$

Clearly, $\mathcal{O}_m$ is an open subset of $\varphi(V)$ that covers $\varphi(V)$. Thus, it is sufficient to show that $\mathcal{O}_m \cap \varphi(V \cap \Sigma)$ is a smooth submanifold of $\mathcal{O}_m$ of codimension $2k$. We shall do this for $m = k - 1$, the other cases are the same.

Consider the map $K :\ \mathcal{O}_1 \to \mathbb{R}^{2k}$, defined by

$$K(\xi) = ((d_{p,q})(\xi))_{p=1,\dots,k-1;q=1,2}; (K_i(\xi))_{i=1,2}),$$

where

$$d_{pq}(\xi) = \sum_{t=1}^{3} \frac{\partial H}{\partial y_p^{(l)}}(v) a_{pq}^{(l)}, K(\xi) = \langle v_k - v_{k-1}, \nu(\xi)\rangle,$$

$$K_2(\xi) = L(\xi)\lambda(\xi)^2 + 2M(\xi)\lambda(\xi)\mu(\xi) + N(\xi)\mu(\xi)^2.$$

The map K is so defined that $\mathcal{O}_1 \cap \varphi(V \cap \Sigma) = K^{-1}(0)$. Therefore, it is sufficient to establish that K is a submersion on $\mathcal{O}_1 \cap \varphi(V \cap \Sigma)$.

Fix an arbitrary $\xi \in \mathcal{O}_1 \cap \varphi(V \cap \Sigma)$ and assume that

$$\sum_{p=1}^{k-1} \sum_{q=1}^{2} D_{pq} \text{grad}\, d_{pq}(\xi) + \sum_{i=1}^{2} A_i \text{grad}\, K(\xi) = 0 \tag{9.15}$$

for some real constants D_{pq} and A_i. Since $\xi = (u; v; a; b) \in \varphi(V)$ and $V \subset \mathcal{M}$, we have $v_k - v_{k-1} \neq 0$. It follows by the definitions of $\nu(\xi), \lambda(\xi), \mu(\xi)$ and $K_1(\xi) = 0$ that

$$v_k - v_{k-1} = C(\lambda(\xi)a_{k1} + \mu(\xi)a_{k2})$$

with some coefficient $C \neq 0$. Consequently, either $\lambda(\xi) \neq 0$ or $\mu(\xi) \neq 0$. Let $\lambda(\xi) \neq 0$, the other case is similar. Consider in (9.16) the derivatives with respect to $b^{(1)}_{kl1}$. The only non-zero derivative is

$$\frac{\partial K_2}{\partial b^{(1)}_{kl1}} = \lambda(\xi)^2 \nu^{(1)}(\xi) \neq 0$$

because $\xi \in \mathcal{O}_1$. Hence $A_2 = 0$. Next, considering the derivatives with respect to $a^{(t)}_{pq}$, as in the proofs of Theorems 6.3.1 and 6.4.1, we obtain $D_{pq} = 0$ for all p, q. Now $A_1 = 0$ follows trivially and K is a submersion at ξ.

In this way we established that Σ is a smooth submanifold of $\mathcal{M}$ of codimension $2k$. Then for $f \in G(\omega, \alpha)$ the condition $j^2_k f \pitchfork \Sigma$ is equivalent to the fact that $\{x \in X^{(k)} : j^2_k f(x) \in \Sigma\}$ is a discrete subset of $X^{(k)}$, that is $f \in D(\omega, k)$. According to the Multijet Transversality Theorem, we see that $D(\omega, k)$ contains a residual subset of $\mathbf{C}(X)$. ■

9.4 Generic domains in $\mathbb{R}^3$

Let Ω be an arbitrary domain with smooth compact boundary $X = \partial\Omega$ and bounded complement in $\mathbb{R}^3$, and let $\omega \neq \theta$ be two unit vectors in $\mathbb{R}^3$. Our aim in this section is to establish the following result.

Theorem 9.4.1: *There exists a residual subset $\mathcal{R}$ of $\mathbf{C}(X)$ such that for every $f \in \mathcal{R}$ there are no (ω, θ)-rays of mixed type in Ω_f.*

As an immediate consequence of this theorem and the results in the previous sections, we obtain the following.

Corollary 9.4.2: *The generic connected domains Ω in $\mathbb{R}^3$ with smooth boundaries and bounded complements have the following properties:*

(a) every (ω, θ)-ray in Ω is an ordinary non-degenerate reflecting (ω, θ)-ray;
(b) $T_\gamma \neq T_\delta$ for every two different $\gamma, \delta \in \mathcal{L}_{\omega,\theta}(\Omega)$;
(c) the relation (5.30) becomes an equality;
(d) for every $\gamma \in \mathcal{L}_{\omega,\theta}(\Omega)$ and every t close to $-T_\gamma$, we have (9.7).

The rest of this section is devoted to the proof of Theorem 9.4.1. We begin with a technical lemma.

Lemma 9.4.3: *Let X be a smooth surface in $\mathbb{R}^3$ and let $c : [a, b] \longrightarrow X, b > a$, be a geodesic on X. Let $t_0 \in (a, b)$ be such that $c(t_0)$ is not a point of self-intersection of c. For every sufficiently small interval (α, β) containing t_0 and every sufficiently small open neighbourhood U of $c(t_0)$ in X with*

$$c(\alpha, \beta) \subset U, U \cap \operatorname{Im} c = c([\alpha, \beta]), \tag{9.16}$$

there exists $f \in \mathbf{C}(X)$, arbitrarily close to id with respect to the C^∞ topology such that $f(x) = x$ for all $x \in X \setminus U$, and if $\tilde{c}: [a,b] \longrightarrow \tilde{X} = f(X)$ is the geodesic on $\tilde{X}$ with $\tilde{c}(t) = c(t)$ for $t \in [a, \alpha]$, then

$$\tilde{c}((\alpha,\beta]) \cap c((\alpha,\beta]) = \emptyset. \tag{9.17}$$

Proof: We take U and (α,β) so small that there exist local coordinates x_0, x_1 in U, determined by a chart

$$r:\ V = (\alpha,\beta) \times (-\delta,\delta) \to U \subset X$$

with $\delta > 0$, such that the components g_{ij} of the standard metric on X have the form

$$g_{00}(x) = 1,\ \ g_{01}(x) = 0,\ \ g_{11}(x) = G(x) > 0$$

for all $x = (x_0, x_1) \in V$. Moreover, we may assume that

$$G(x) < 1 \text{ for every } x \in V. \tag{9.18}$$

Otherwise we can replace r by another chart $\tilde{r}:\ V \to X$, given by $\tilde{r}(x_0,x_1) = r(x_0, \epsilon x_1)$. If $\epsilon > 0$ is chosen sufficiently small, then

$$\tilde{g}_{11}(x_0,x_1) = \epsilon^2 g_{11}(x_0,x_1) < 1$$

for all $x \in V$. Clearly, (9.18) holds for sufficiently small intervals (α,β) and $\delta > 0$. Next, we assume that α, β and δ are fixed with these properties. Finally, mention that $r(t,0) = c(t)$ for all $t \in [\alpha,\beta]$.

Fix two smooth functions $\lambda, \mu:\ \mathbb{R} \to [0,1]$ such that

$$\text{supp } \lambda = [\alpha,\beta],\ p = p(0) > 0,\ \ q = \mu'(0) > 0. \tag{9.19}$$

For $\epsilon > 0$ define $f_\epsilon(y) = y$ for $y \in X \setminus U$, and

$$f_\epsilon(y) = r(x) + \epsilon\lambda(x_0)\mu(x_1)\frac{\partial r}{\partial x_0}(x)$$

for $y = r(x) \in U, x = (x_0,x_1) \in V$. It is easy to see that for all sufficiently small ϵ we have $f_\epsilon \in \mathbf{C}(X)$, therefore $X_\epsilon = f_\epsilon(X)$ is a smooth surface. Moreover,

$$\psi(x) = r(x) + \epsilon\lambda(x_0)\mu(x_1)\frac{\partial r}{\partial x_0}(x)$$

determines a chart $\psi:\ V \to \psi(V) \subset X_\epsilon$. Denote by $g_{ij}(\epsilon;x)$ the components of the standard metric on $\psi(V) \subset X_\epsilon$. We have

$$g_{00}(\epsilon;x) = 1 + 2\epsilon\lambda'(x_0)\mu(x_1) + \mathcal{O}(\epsilon^2),$$
$$g_{01}(\epsilon;x) = \epsilon\lambda(x_0)\mu'(x_1) + \mathcal{O}(\epsilon^2)$$
$$g_{11}(\epsilon;x) = G(x_0,x_1) + 2\epsilon\lambda(x_0)\mu(x_1)\left\langle \frac{\partial r}{\partial x_1}(x), \frac{\partial^2 r}{\partial x_0 \partial x_1}(x)\right\rangle + \mathcal{O}(\epsilon^2),$$

as $\epsilon \to 0$.

Using the coordinates x_0, x_1, introduce the canonical coordinates x_0, x_1, y_0, y_1 in $T^*(X_\epsilon)$, and consider the Hamiltonian vector field generated by the Hamiltonian

$$H(\epsilon; x, y) = \frac{1}{2} g_{00}(\epsilon; x) y_0^2 + g_{01}(\epsilon; x) y_0 y_1 + \frac{1}{2} g_{11}(\epsilon; x) y_1^2,$$

where $x = x_0, x_1, y = (y_0, y_1)$. Denote by $c(\epsilon; x)$ the geodesic on X_ϵ such that $c(\epsilon; x) = c(t)$ for $t \in [a, \alpha]$, and let $(x(\epsilon; t), y(\epsilon; t)$ be the corresponding integral curve on $T^*(X_\epsilon)$. Writing the Hamiltonian equations for this curve and then the corresponding variational equations for

$$X_i(t) = \frac{d}{dt} x_i(\epsilon; x)|_{\epsilon=0}, \; Y_i(t) = \frac{d}{dt} y_i(\epsilon; t)|_{\epsilon=0},$$

according to (9.19), we obtain

$$\begin{cases} X_0'(t) = Y_0(t) + 2p\lambda'(t), \\ X_1'(t) = G(t,0)Y_1(t) + q\lambda(t), \\ Y_0'(t) = -p\lambda''(t), \\ Y_1'(t) = -q\lambda'(t), \\ X_0(\alpha) = X_1(\alpha) = Y_0(\alpha) = Y_1(\alpha) = 0, \end{cases}$$

for $t \in [\alpha, \beta]$. Consequently, $Y_1(t) = -q\lambda(t)$, and so $X_1'(t) = q\lambda(t)(1 - G(t, 0))$. This and (9.18) imply $X_1'(t) > 0$ for all $t \in (\alpha, \beta)$. Therefore, for all sufficiently small $\epsilon > 0$ we have

$$\frac{d}{dt} x_1(\epsilon; x) > 0, \;\; t \in (\alpha, \beta).$$

Fix such an ϵ. Then $x_1(\epsilon; t) > 0$ for all $t \in (\alpha, \beta)$, and $f = f_\epsilon$ has the desired properties. This proves the assertion. ■

It follows by Lemma 8.1.2 that there exists a residual subset $\mathcal{K}$ of $\mathbf{C}(X)$ such that for $f \in \mathcal{K}$, the normal curvature of $f(X)$ can vanish only of finite order. Then by the properties of the generalized bicharacteristics of $\Box$ (cf. Section 1.2), we have that if $f \in \mathcal{K}$ and $\gamma : \mathbb{R} \to \Omega_f$ is an (ω, θ)-ray in Ω_f, then $\mathrm{Im}\,\gamma$ is a finite union of linear segments (two of them are infinite) and geodesic segments on $\partial\Omega$. The ends of these (linear or geodesic) segments will be called vertices of γ (or of $\mathrm{Im}\,\gamma$).

Proof of Theorem 9.4.1: We are going to construct by induction a sequence

$$\mathcal{V}_1 \supset \mathcal{V}_2 \supset \ldots \supset \mathcal{V}_k \supset \ldots$$

of residual subsets of $\mathcal{K}$ such that for every k and every $f \in \mathcal{V}_k$ there are no (ω, θ)-rays of mixed type in Ω_f with not more than $k + 1$ different (linear or geodesic) segments.

Set $\mathcal{V}_1 = \mathcal{K}$. Clearly, for every $f \in \mathcal{V}_1$ there are no (ω, θ)-rays of mixed type in Ω_f having exactly two segments and therefore only one vertex.

Let $k > 1$ and assume that $\mathcal{V}_1 \supset \cdots \supset \mathcal{V}_{k-1}$ are already constructed and have the desired properties. To construct $\mathcal{V}_k$, we need some technical preparation. A map of the form

$$\kappa : \{1, 2, \ldots, k\} \to \{0, 1\}$$

will be called k-design, if κ is not identically zero, $\kappa(k) = 0$ and $\kappa(i)\kappa(i+1) = 0$ for all $i = 1, \ldots, k-2$. An (ω, θ)-ray γ will be called (ω, θ)-ray with design κ if Im γ has $k+1$ segments $l_0, l_1, \ldots, l_k$ and for every $i = 1, \ldots, k-1$, l_i is a linear segment if and only if $\kappa(i) = 0$.

Fix an arbitrary k-design κ and set

$$q = \max\{i : 1 \le i \le k-1, \kappa(i) = 1\},$$
$$p = \min\{i : 1 \le i \le k-1, \kappa(i) = 1\}.$$

It follows from Lemma 9.3.2 that for every $\eta \in \mathbb{S}^2$ and every $m \in \mathbb{N}$ there exists a residual subset $D(\eta, m)$ of $\mathbf{C}(X)$ such that for $f \in D(\eta, m)$ the elements $y = (y_1, \ldots, y_m)$ of $f(X)^{(m)}$ such that $y_1, \ldots, y_m$ are the successive vertices of a glancing η-ray in Ω_f form a discrete subset of $f(X)^{(m)}$. On the other hand, Theorem 6.3.3 and Corollary 6.4.7 imply the existence of a residual subset $\mathcal{T}_k$ of $\mathbf{C}(X)$ such that for every $f \in \mathcal{T}_k$ there are only finitely many reflecting (ω, θ)-rays in Ω_f with not more than k reflection points, and all of them are ordinary. We set

$$\mathcal{W} = \mathcal{V}_{k-1} \cap D(\omega, p) \cap D(-\theta, q) \cap \mathcal{T}_k.$$

Then $\mathcal{W}$ is a residual subset of $\mathbf{C}(X)$ which is clearly contained in $\mathcal{K}$.

Denote by $\mathcal{V}(k, \kappa)$ the set of those $f \in \mathcal{W}$ such that there are no (ω, θ)-rays of mixed type with design κ in Ω_f.

We will first show that $\mathcal{V}(k, \kappa)$ is dense in $\mathcal{W}$. To this end, we may assume that id $\in \mathcal{W}$ and then prove that there are $f \in \mathcal{V}(k, \kappa)$ arbitrarily close to id. Suppose id $\in \mathcal{W}$. We claim that there exist only finitely many (ω, θ)-rays of mixed type with design κ in Ω. Assume the contrary, and let $\gamma_1, \ldots, \gamma_m, \ldots$ be an infinite sequence of different (ω, θ)-rays of mixed type with design κ in Ω. Denote by $x_1^{(m)}, \ldots, x_k^{(m)}$ the successive vertices of Im γ_m. We may assume that for all m, one has $\gamma_m(0) = x_1^{(m)}$ and $\dot{\gamma}_m(t) = \omega$ for $t < 0$. Moreover, considering an appropriate subsequence, we may assume that there exists

$$\lim_{m \to \infty} x_i^{(m)} = x_i \in \partial\Omega$$

for all $i = 1, \ldots, k$. It then follows by the continuity of the broken Hamiltonian flow that for every $t \in \mathbb{R}$ there exists

$$\lim_{m \to \infty} \gamma_m(t) = \gamma(t) \in \Omega,$$

and γ is an (ω, θ)-ray in Ω. Roughly speaking, the ith finite segment of Im γ has end points x_i and x_{i+1}. However, this is not precise, since in general some of these segments can vanish. Denote by $l_0^{(m)},\ldots, l_k^{(m)}$ the successive segments of Im γ_m. Set

$$l_i = \lim_{m\to\infty} l_i^{(m)}, i = 0, 1,\ldots, k.$$

Then each l_i is either the one-point set $\{x_i\}$, or a linear segment, or a geodesic segment on $\partial\Omega$. Clearly, the first segments l_0 is the infinite ray starting from x_1 and having direction $-\omega$.

Let $s + 1$ be the number of different segments of γ, then $s \leq k$. There are two cases.

Case 1. $s = k$. Then each l_i is a non-degenerate linear or geodesic segment, and clearly γ is an (ω, θ)-ray of mixed type with design κ in Ω. Consequently, $x_1,\ldots, x_p$ are the successive vertices of a glancing ω-ray in Ω. Moreover, for every $m \in \mathbb{N}, x_1^{(m)},\ldots, x_p^{(m)}$ are the successive vertices of a glancing ω-ray in Ω. Since $\lim_{m\to\infty} x_i^{(m)} = x_i$ for every i, this is a contradiction with id $\in \mathcal{W} \subset D(\omega, p)$.

Case 2. $s < k$. Then $s \leq k - 1$, and id $\in \mathcal{W} \subset \mathcal{V}_{k-1}$ implies that γ cannot be an (ω, θ)-ray of mixed type. Therefore, l_i is one point for all i with $\kappa(i) = 1$, and γ is a reflecting (ω, θ)-ray with s reflection points. Since l_i vanishes for at least one i, some segment of γ is tangent to $\partial\Omega$, which is a contradiction with id $\in \mathcal{W} \subset \mathcal{T}_k$.

In both possible cases we got a contradiction. This shows that there are only finitely many (ω, θ)-rays of mixed type with design κ in Ω. Let $\gamma, \gamma_1,\ldots, \gamma_s$ be all of them. Denote by $x_1,\ldots, x_k$ the successive vertices of Im γ and by $l_0, l_1,\ldots, l_k$ its successive segments. Since id $\in D(-\theta, q)$, there exists a neighbourhood V of x_k in X such that $V \cap \text{Im}\,\gamma_i = \emptyset$ for all $i = 1,\ldots, s$, and if δ is a glancing (ω, θ)-ray in Ω with $k - q$ vertices and the first y_1 of them belongs to V, then $y_1 = x_k$ (and therefore δ is a part of Im γ). Let $0 = t_1 < \cdots, t_k$ be the times with $\gamma(t_i) = x_i$. Since l_q is a geodesic segment on X, we can find $t_0 \in (\alpha, \beta) \subset (t_q, t_{q+1})$ and a small coordinate neighbourhood U of $\gamma(t_0)$ in X such that

$$U \cap \left((\cup_{i=1}^s \text{Im}\,\gamma_i) \cup \bar{V} \cup \cup_{j=0, j\neq q}^k l_j\right) = \emptyset,$$

and (9.16) holds for $c(t) = \gamma(t), a = t_q, b = t_{q+q}, \alpha, \beta$. By Lemma 9.4.1, there exists $f \in \mathbf{C}(X)$, arbitrary close to id, such that $f = \text{id}$ on $X \setminus U$ and (9.18) is satisfied for the geodesic $\tilde{c}$ on $\tilde{X} = f(X)$ with $\tilde{c}(t) = c(t)$ for $t \in [t_q, \alpha]$. We claim that if f is chosen in this way and it is sufficiently close to id, then the only (ω, θ)-rays of mixed type with design κ in Ω_f are $\gamma_1,\ldots, \gamma_s$. Indeed, assume this is not true. Then we can find a sequence $f_m \longrightarrow_m$ id of such f_m so that for every m there exists an (ω, θ)-ray of mixed type δ_m with design κ in Ω_{f_m}, different from $\gamma_1,\ldots, \gamma_s$. Since id $\in D(\omega, p)$, for large m, the first vertex of Im δ_m is necessarily x_1. We may assume $\delta_m(0) = x_1$, and then we obtain $\delta_m(t) = \gamma(t)$ for all $t \leq \alpha$. Now the construction of f_m shows that the last vertex $y_k^{(m)}$ of δ_m cannot be x_k. Otherwise, we would have $\delta_m(\beta) = \gamma(\beta)$, which would imply $\tilde{c}(\beta) = c(\beta)$ in contradiction with (9.17). So $y_k^{(m)} \neq x_k$, and the choice of V yields $t_k^{(m)} \notin V$.

Considering appropriate convergence subsequences, we may assume that there exists $\delta(t) = \lim_{m\to\infty} \delta_m(t)$ for every t. Then δ is clearly an (ω, θ)-ray in Ω with first reflection point of δ x_1, therefore δ coincides with γ. On the other hand, the last reflection point of δ is $y_k = \lim_m y_k^{(m)} \notin V$, so δ cannot coincide with γ. This contradiction shows that $\gamma_1, \ldots, \gamma_s$ are the only (ω, θ)-rays of mixed type with design κ in Ω_f, provided f is constructed as above and is sufficiently close to id. Moreover, if f is sufficiently close to id, then $f \in \mathcal{W}$. Repeating this procedure s times, we find $g \in \mathcal{W}$, arbitrarily close to id, such that there are no (ω, θ)-rays of mixed type with design κ in Ω_g. Then $g \in \mathcal{V}(k, \kappa)$, which shows that $\mathcal{V}(k, \kappa)$ is dense in $\mathcal{W}$.

To establish that $\mathcal{V}(k, \kappa)$ is open in $\mathcal{W}$, we may assume that $f_m \to_m \text{id} \in \mathcal{W}$ for some sequence $\{f_m\} \subset \mathcal{W} \setminus \mathcal{V}(k, \kappa)$. Then we have to prove that there exists an (ω, θ)-ray of mixed type with design κ in Ω. For every m there exists an (ω, θ)-ray of mixed type δ_m with design κ in Ω_{f_m}. Choosing again appropriate subsequence and repeating a part of the above argument, we see that there exists an (ω, θ)-ray of mixed type in Ω. Thus, $\mathcal{V}(k, \kappa)$ is open in $\mathcal{W}$.

Set $\mathcal{V}_k = \cap_k \mathcal{V}(k, \kappa)$, where κ runs over the set of all k-designs. Since the later is finite, $\mathcal{V}_k$ is open and dense in $\mathcal{W}$, so it is residual in $\mathbf{C}(X)$. Clearly, $\mathcal{V}_k$ has all desired properties. This completes the construction of the sequence $\{\mathcal{V}_k\}$.

Finally, define $\mathcal{V} = \cap_{k=1}^{\infty} \mathcal{V}_k$. Then $\mathcal{V}$ is a residual subset of $\mathbf{C}(X)$, and for every $f \in \mathcal{V}$ there are no (ω, θ)-rays of mixed type in Ω_f. This concludes the proof of the theorem. ■

9.5 Notes

Under more restrictive assumptions on γ and Ω, Theorem 9.1.2 was proved in [[Pl]]. The case of strictly convex obstacles (cf. Corollary 9.1.3) was studied sometime before by Majda [[Mal]] (see also [[Ma2], [MaT], [So2], [Y]]). In Section 9.1, we follow the analysis in [[Pl]] based on the construction of a global parametrix in [[GM1]]. Lemmas 4.2.1 and 4.2.2 make possible to precise the form of the coefficient in (9.7). The results of Section 9.2 were proved in [[CPS]]. The material in Sections 9.3 and 9.4 is taken from [[S4]]. Note that to prove a result similar to Corollary 9.4.2 for domains in $\mathbb{R}^n, n > 3$, it is desirable to have an analogue of Theorem 9.4.1 for such domains. Except Lemma 9.3.2, all arguments in Sections 9.3 and 9.4 can be modified to cover the general case. We do not know whether their analogues are true for $n > 3$ for either Lemma 9.3.2 or Theorem 9.4.1.

10

Scattering invariants for several strictly convex domains

In this chapter we study scattering rays and related singularities of the scattering kernel in the exterior Ω of an obstacle K which is a disjoint union of a finite number of strictly convex compact domains in $\mathbb{R}^n$. It is shown first that if $\omega \in \mathbb{S}^{n-1}$ is fixed, then for most of the vectors $\theta \in \mathbb{S}^{n-1}$ all (ω, θ)-rays γ are ordinary with distinct sojourn times providing singularities of the scattering kernel $s(t, \theta, \omega)$.

Starting from the second section, we assume that the convex hull of every two connected components of K does not contain points of any other connected component of K. Under this condition, we prove that for fixed configurations α of connected components of K, one can choose appropriate ω and θ so that for every integer $q \leq 1$ there exists a unique (ω, θ)-ray γ_q with q reflection points following the given configurations in a certain way. It turns out that the reflection points of these rays approximate the corresponding reflection points of the unique periodic reflecting ray γ_α in Ω having type α. Moreover, the period of γ_α is an invariant which can be determined from the asymptotic of the sojourn times T_γ as $q \to \infty$. Another geometric invariant, related to the Poincaré map of γ_α, can be determined by the asymptotic of the coefficients c_q in front of the main singularity of $s(t, \theta, \omega)$ for $t \sim -T_{\gamma_q}$.

10.1 Singularities of the scattering kernel for generic θ

Let $K \subset \mathbb{R}^n, n \geq 2$, have the form

$$K = K_1 \cup \cdots \cup K_s, \tag{10.1}$$

Geometry of the Generalized Geodesic Flow and Inverse Spectral Problems, Second Edition.
Vesselin M. Petkov and Luchezar N. Stoyanov.

where K_i are strictly convex compact domains in $\mathbb{R}^n$ with smooth boundaries $\Gamma_i = \partial K_i$. As in Chapters 5 and 9, we shall use the notation $\Omega = \overline{\mathbb{R}^n \setminus K}$.

Fix arbitrary vector, $\omega \in \mathbb{S}^{n-1}$. Our aim in this section is to prove the following.

Theorem 10.1.1: *There exists a residual subset $R(\omega)$ of $\mathbb{S}^{n-1}$ such that for every $\theta \in R(\omega)$ one has the following properties:*

(a) each (ω, θ)-ray in Ω is ordinary and every two different (ω, θ)-rays in Ω have distinct sojourn times;

(b) if n is odd we have sing $\operatorname{supp} s_\Omega(t, \theta, \omega) = \{ -T_\gamma : \gamma \in \mathcal{L}_{\omega,\theta}(\Omega)\}$, *and for every $\gamma \in \mathcal{L}_{\omega,\theta}(\Omega)$ and t close to $-T_\gamma$, we have* (9.7).

Notice that property (b) follows immediately from (a) and Theorems 5.3.2 and 9.1.2. Thus, we have to construct $R(\omega)$ in a such way that (a) is satisfied.

Let us recall some notation from Section 2.4, which will be used in the whole chapter. Given $u \in Z_\omega$, denote by $S_t(u)$ the shift of u after time t along the billiard semi-trajectory $\gamma(u)$ in Ω, starting at u in direction ω. Then

$$\gamma(u) = \{S_t(u) : t \geq 0\}.$$

Let $N_t(u)$ be the velocity vector (i.e. the direction) of $\gamma(u)$ at $S_t(u)$. If t is such that $x = S_t(u) \in \partial\Omega$, we shall use the notation

$$N_{-t}(u) = \lim_{\epsilon \searrow 0} N_{t-\epsilon}(u),$$

and $N_{+t}(u) = \sigma_x(N_{-t}(u))$, σ_x being the symmetry with respect to the tangent hyperplane $T_x(\partial\Omega)$. Note that $S_0(u) = u$ and $N_0(u) = \omega$. By $x_1(u), x_2(u), \ldots$, we denote the *successive reflections points* of γ if any, and by $t_1(u), t_2(u), \ldots$ the corresponding times (moments) of reflection. It is convenient to set $x_0(u) = u$ and $t_0(u) = 0$. Finally, denote by $r(u)$ the number of reflections of $\gamma(u), 0 \leq r(u) \leq \infty$.

Furthermore, let $Z_\omega^{(0)}$ be the set of those $u \in Z_\omega$ such that $r(u) < \infty$. Define the map $T : Z_\omega^{(0)} \to \mathbb{R}$ as follows. Given $u \in Z_\omega^{(0)}$, the vector $N_t(u)$ is constant for sufficiently large t, so there exists $\eta = \lim_{t \to +\infty} N_t(u)$. Clearly, there is exactly one $t > 0$ such that $S_t(u) \in Z_{-\eta}$. We set $T(u) = t$. Let us mention that under the latter notation, if γ is the unique (ω, θ)-ray in Ω, passing through u, then $T_\gamma = T(u) + 2a$.

Recall the notion of a configuration from Section 2.4. Given an integer $m \geq 1$, by a *configuration of length m* we mean a sequence

$$\alpha = (i_1, \ldots, i_m) \in \{1, 2, \ldots, s\}^m \tag{10.2}$$

such that $i_j \neq i_{j+1}$ for $j = 1, \ldots, m-1$. For such an α we set $|\alpha| = m$ and define the sets $U_\alpha \subset F_\alpha \subset Z_\omega$ and the map

$$J_\alpha : \bar{F}_\alpha \to \mathbb{R}$$

as in Section 2.4.

Lemma 10.1.2:
(a) There exists a sequence

$$\mathcal{L}_1 \supset \mathcal{L}_2 \supset \cdots \supset \mathcal{L}_m \supset \cdots$$

of open dense subsets of $\mathbb{S}^{n-1}$ *such that for every* $m \in \mathbb{N}$, *if* α *is a configuration of length* $m, u \in F_\alpha$ *and* $J_\alpha(u) \in \mathcal{L}_m$, *then* $u \in U_\alpha$;
(b) For every two configurations α *and* β *there exists a residual subset* $\mathcal{L}(\alpha, \beta)$ *of* $\mathbb{S}^{n-1}$ *such that for* $\theta \in \mathcal{L}(\alpha, \beta)$ *the conditions*

$$u \in U_\alpha, v \in U_\beta,\ r(u) = |\alpha|,\ r(v) = |\beta|,\ J_\alpha(u) = J_\beta(v) = \theta, \tag{10.3}$$

imply $T(u) \neq T(v)$.

For the proof of this lemma we need a general fact concerning the propagation of convex wave fronts in Ω.

Let X be a smooth hypersurface lying in the interior of Ω and let $e(x), x \in X$, be a continuous field of unit normal vectors to X. We assume that X is **convex** at every $x \in X$ with respect to the normal field $e(x)$, that is the corresponding second fundamental form of X is non-negative semi-definite everywhere in X. Consider an open coordinate neighbourhood V of some point x_0 in X and a smooth chart

$$U \ni u \to x(u) \in V \subset X$$

of an open neighbourhood U of 0 in $\mathbb{R}^{n-1}$ onto V with $x_0 = x(0)$. Set $e(u) = e(x(u))$. With respect to this parameterization, the *second fundamental* form $\mathrm{II}_u^{(X)}$ of X at u (i.e. at $x(u)$) is given by

$$\mathrm{II}_u^{(X)}(\xi, \eta) = \sum_{i,j=1}^{n-1} b_{ij}^{(X)}(u)\xi_i\eta_j,$$

where

$$b_{ij}^{(X)}(u) = \left\langle e(u), \frac{\partial^2 x}{\partial u_i \partial u_j}(u) \right\rangle,\ \ i, j = 1,\dots,n-1$$

are the coefficients of $\mathrm{II}_u^{(X)}$ at u. The *first fundamental form* $\mathrm{I}_u^{(X)}$ of X at u is given by

$$\mathrm{I}_u^{(X)}(\xi, \eta) = \sum_{i,j=1}^{n-1} g_{ij}^{(X)}(u)\xi_i\xi_j,$$

with

$$g_{ij}^{(X)}(u) = \left\langle \frac{\partial x}{\partial u_i}(u), \frac{\partial x}{\partial u_j}(u) \right\rangle, \quad i, j = 1,\dots,n-1.$$

The *sectional normal curvature* of X at $x(u)$ in direction ξ (more precisely, in direction $\sum_{i=1}^{n-1} \xi_i \frac{\partial x}{\partial u_i}(u) \in T_{x(u)}X$) is defined by

$$\kappa_u^{(X)}(\xi) = -\frac{\mathrm{II}_u^{(X)}(\xi,\xi)}{\mathrm{I}_u^{(X)}(\xi,\xi)}.$$

For $x \in X$ and $t \in \mathbb{R}$ let $Q_t(x)$ be the shift of x along the billiard semi-trajectory $\chi(x)$ in Ω starting at x in direction $e(x)$ and let $M_t(x)$ be the direction of the trajectory at $Q_t(x)$. Then

$$\chi(x) = \{Q_t(x) :\ t \geq 0\}.$$

Assume that the trajectory $\chi(x_0)$ hits $\partial\Omega = \partial K$ at some point y_0, reflecting transversally on ∂K at y_0. Then for the angle φ_0 between $\nu(y_0)$ and the direction $N_{t_0}(x_0)$ at $\gamma(x_0)$ at y_0, we have $\cos\varphi_0 > 0$. Here $t_0 = \|x_0 - y_0\| > 0$.

Furthermore, take $T > t_0$ such that $Q_t(x_0)$ makes only one reflection when t runs from 0 to T. Restricting our considerations to a small neighbourhood of x_0 in X, we may assume that for every $x \in X$ the trajectory $Q_t(x)$ makes exactly one transversal reflection on ∂K when t runs from 0 to T. Then

$$Z = Z^{(T)} = \{Q_T(x) :\ x \in X\}$$

is a smooth hypersurface in $\mathbb{R}^n$. This follows easily, for example, from the next considerations and the implicit function theorem. Set $z_0 = Q_T(x(0))$, and denote by $\kappa(T)$ the minimum of the sectional curvatures of Z at z_0 and by κ_0 the same for ∂K at y_0. Finally, set

$$\kappa_+(t_0) = \lim_{T \searrow t_0} \kappa(T).$$

Before going on, let us mention that for $0 < t < t_0$, we have

$$\kappa_u^{(X)}(\xi) = \frac{\kappa_u^{(X)}(\xi)}{1 + t\kappa_u^{(X)}(\xi)}.$$

This formula can be easily derived from a part of the argument in the proof of the following lemma. The latter shows that the wave front $Z = Z^{(T)}$ is always strictly convex and gives an estimate for its minimal normal curvature. Actually, we need only the strict convexity, the estimate of the curvature will not be used in our next analysis.

Lemma 10.1.3: *Under the above assumptions, the hypersurface Z is strictly convex at z_0 and*

$$\kappa(T) \geq \frac{2\kappa_0 \cos\varphi_0}{1 + 2(T - t_0)\kappa_0 \cos\varphi_0}. \tag{10.4}$$

In particular,

$$\kappa_+(t_0) \geq 2\kappa_0 \cos\varphi_0. \tag{10.5}$$

Proof: Note that (10.4) follows from (10.5) and the above result. Thus, it is sufficient to prove only (10.5).

For $u \in U$ denote by $y(u)$ the reflection point of $\chi(x(u))$ on ∂K, and set $Y = \partial K$,

$$z(u) = z^{(T)}(u) = S_T(x(u)), \;\; f(u) = \frac{z(u) - y(u)}{\|z(u) - y(u)\|}.$$

We shall see later that $f(u)$ is the unit normal to Z at $z(u)$. It follows by our assumptions that $u \mapsto y(u), u \mapsto z(u)$ provide smooth parameterization for ∂K around y_0 and for Z around z_0. Setting $t(u) = \|y(u) - x(u)\|$, we have

$$y(u) = x(u) + t(u)e(u) \tag{10.6}$$

and

$$z(u) = y(u) + (T - t(u))f(u). \tag{10.7}$$

Differentiating (10.6) and (10.7) with respect to $u_i, i = 1,\dots, n-1$, one finds

$$\frac{\partial y}{\partial u_i} = \frac{\partial x}{\partial u_i} + \frac{\partial t}{\partial u_i}e + t\frac{\partial e}{\partial u_i}, \tag{10.8}$$

$$\frac{\partial z}{\partial u_i} = \frac{\partial y}{\partial u_i} - \frac{\partial t}{\partial u_i}f + (T - t(u))\frac{\partial f}{\partial u_i}. \tag{10.9}$$

Taking the inner product of (10.8) with $e(u)$, we get

$$\frac{\partial t}{\partial u_i} = \left\langle e(u), \frac{\partial y}{\partial u_i}(u) \right\rangle.$$

Then (10.9) implies

$$\frac{\partial z}{\partial u_i}(u) = \frac{\partial y}{\partial u_i}(u) - \left\langle \frac{\partial y}{\partial u_i}(u), e(u) \right\rangle f(u) + \mathcal{O}(T - t_0), \tag{10.10}$$

where $\mathcal{O}(T - t_0)$ denotes a term that tends to 0 as $T \searrow t_0$. Notice that $f(u)$ does not depend on T, provided $T - t_0 > 0$ is sufficiently small. By (10.10), we first obtain

$$\left\langle \frac{\partial z}{\partial u_i}(u), f(u) \right\rangle = \left\langle \frac{\partial y}{\partial u_i}(u), f(u) \right\rangle - \left\langle \frac{\partial y}{\partial u_i}(u), e(u) \right\rangle = 0,$$

since $e(u)$ and $f(u)$ are symmetric with respect to the tangent hyperplane to ∂K at $y(u)$. This shows that $f(u)$ is the unit normal to Z at $z(u)$.

Next, (10.10) implies

$$g_{ij}^{(Z)}(u) = g_{ij}^{(Y)}(u) - \left\langle e(u), \frac{\partial y}{\partial u_i}(u) \right\rangle \left\langle e(u), \frac{\partial y}{\partial u_j}(u) \right\rangle.$$

Consider an arbitrary $\xi = (\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{n-1}$ and set

$$\eta = \sum_{i=1}^{n-1} \xi_i \frac{\partial y}{\partial u_i}(u) \in T_{y(u)}(\partial K).$$

Then

$$\begin{aligned} \mathrm{I}_u^{(Z)}(\xi,\xi) = \sum_{i,j=1}^{n-1} \xi_i \xi_j g_{ij}^{(Z)}(u) &= \mathrm{I}_u^{(Y)}(\xi,\xi) - \langle e(u), \eta \rangle^2 + \mathcal{O}(T - y_0) \\ &\le \mathrm{I}_u^{(Y)}(\xi,\xi) + \mathcal{O}(T - t_0). \end{aligned} \tag{10.11}$$

We are going to deduce similar relations between the second fundamental forms. Differentiating (10.8) and (10.9) with respect to u_j, we get

$$\frac{\partial^2 y}{\partial u_i \partial u_j} = \frac{\partial^2 x}{\partial u_i \partial u_j} + \frac{\partial t}{\partial u_i \partial u_j} e + \frac{\partial t}{\partial u_i} \frac{\partial e}{\partial u_j} + t \frac{\partial^2 e}{\partial u_i \partial u_j}, \tag{10.12}$$

$$\frac{\partial^2 z}{\partial u_i \partial u_j} = \frac{\partial^2 y}{\partial u_i \partial u_j} - \frac{\partial^2 t}{\partial u_i \partial u_j} f - \frac{\partial t}{\partial u_i} \frac{\partial f}{\partial u_j} - \frac{\partial t}{\partial u_j} \frac{\partial f}{\partial u_i} + (T - t(u)) \frac{\partial^2 f}{\partial u_i \partial u_j}. \tag{10.13}$$

To find the term $\partial^2 t / \partial u_i \partial u_j$, we take the inner product of (10.10) with $e(u)$ and get

$$\frac{\partial^2 t}{\partial u_i \partial u_j} = \left\langle \frac{\partial^2 y}{\partial u_i \partial u_j}, e \right\rangle - \left\langle \frac{\partial^2 x}{\partial u_i \partial u_j}, e \right\rangle - t \left\langle \frac{\partial^2 e}{\partial u_i \partial u_j}, e \right\rangle.$$

Replacing this term in (10.13), we obtain

$$\begin{aligned} \left\langle \frac{\partial^2 z}{\partial u_i \partial u_j}(0), e(0) \right\rangle &= \left\langle \frac{\partial^2 y}{\partial u_i \partial u_j}(0), f(0) \right\rangle - \frac{\partial^2 t}{\partial u_i \partial u_j}(0) + \mathcal{O}(T - t_0) \\ &= \left\langle \frac{\partial^2 y}{\partial u_i \partial u_j}(0), f(0) - e(0) \right\rangle + \left\langle \frac{\partial^2 x}{\partial u_i \partial u_j}(0), e(0) \right\rangle \\ &\quad + t_0 \left\langle \frac{\partial^2 e}{\partial u_i \partial u_j}(0), e(0) \right\rangle + \mathcal{O}(T - t_0) \\ &= 2 \cos \varphi_0 \left\langle \frac{\partial^2 y}{\partial u_i \partial u_j}(0), \nu(0) \right\rangle + \left\langle \frac{\partial^2 x}{\partial u_i \partial u_j}(0), e(0) \right\rangle \\ &\quad - t_0 \left\langle \frac{\partial e}{\partial u_i}(0), \frac{\partial e}{\partial u_j}(0) \right\rangle + \mathcal{O}(T - t_0). \end{aligned} \tag{10.14}$$

Here we have taken into account that $f(0) - e(0) = 2(\cos \varphi_0) \nu(0)$ and

$$\left\langle \frac{\partial^2}{\partial u_i \partial u_j}, e \right\rangle = - \left\langle \frac{\partial e}{\partial u_i}, \frac{\partial e}{\partial u_j} \right\rangle.$$

It follows immediately from (10.14) that for $u = 0$ and $\zeta = \sum_{i=1}^{n-1} \xi_i \frac{\partial e}{\partial u_i}(0)$ we have

$$\mathrm{II}_u^{(Z)}(\xi,\xi) = 2\cos\varphi_0 \mathrm{II}_u^{(Y)}(\xi,\xi) + \mathrm{II}_u^{(Y)}(\xi,\xi) - t_0\|\zeta\|^2 + \mathcal{O}(T - t_0).$$

The convexity of X implies $\mathrm{II}_u^{(X)} \leq 0$, so

$$\mathrm{II}_u^{(Z)}(\xi,\xi) \leq 2\cos\varphi_0 \mathrm{II}_u^{(Y)}(\xi,\xi) + \mathcal{O}(T - t_0).$$

Finally, combining the latter with (10.11), one gets

$$-\frac{\mathrm{II}_0^{(Z)}(\xi,\xi)}{\mathrm{I}_0^{(Z)}(\xi,\xi)} \geq -\frac{2\cos\varphi_0 \mathrm{II}_u^{(Y)}(\xi,\xi) + \mathcal{O}(T - t_0)}{\mathrm{I}_0^{(Y)}(\xi,\xi)}$$

$$\geq 2\cos\varphi_0 + \mathcal{O}(T - t_0),$$

which clearly yields (10.5). ■

Proof of Lemma 10.1.2: (a) Set $\mathcal{L}_1 = \mathbb{S}^{n-1} \setminus \{\omega\}$. Assume that the sets $\mathcal{L}_1 \supset \cdots \supset \mathcal{L}_m$ are already constructed and have the desired properties. We are going to construct $\mathcal{L}_{m+1}$.

Fix an arbitrary configuration $\alpha = (i_1, \ldots, i_m, i_{m+1})$ of length $m + 1$, and for $k \leq m$ set $\alpha_k = (i_1, \ldots, i_k)$. Since $\mathcal{L}_m$ is open and dense in $\mathbb{S}^{n-1}$,

$$F = \mathcal{S} \setminus \mathcal{L}_m$$

is a compact subset of $\mathbb{S}^{n-1}$ with empty interior. Fix for a moment an arbitrary $k \leq m$ and set $\beta = \alpha_k$. Next, we use the notations (2.39) and (2.40) from Section 2.4, as well as Proposition 2.4.4. First consider the map $J_\beta : \bar{F}_\beta \to E_\beta$. Since $F \cap E_\beta$ is a compact subset of E_β with empty interior, and

$$J_\beta^{-1}(F) = J_\beta^{-1}(F \cap E_\beta) \subset (J_\beta^{-1}(F \cap E_\beta) \cap M_\alpha) \cup L_\alpha,$$

it follows by Proposition 2.4.4 that $J_\beta^{-1}(F)$ is a compact subset of Z_ω with empty interior. By the definition of β, we have $F_\alpha \subset F_\beta$ and $L_\alpha \subset L_\beta$. On the other hand, $\omega \notin \mathcal{L}_1 \supset \mathcal{L}_m$, and the definitions of J_β and F imply $J_\beta^{-1}(F) \cap L_\beta = \emptyset$. Consequently, $J_\beta^{-1}(F) \cap L_\alpha = \emptyset$, which shows that $J_\beta^{-1}(F)\bar{F}_\alpha$ is contained in M_α. Applying again Proposition 2.4.4, we deduce that $J_\alpha(J_\beta^{-1}(F) \cap \bar{F}_\alpha)$ is a compact subset of $\mathbb{S}^{n-1}$ with empty interior. In this way, we have established that

$$V = \mathcal{L}_m \setminus \cup_{k=1}^m J_\alpha(F \cap J_{\alpha_k}^{-1}(F))$$

is an open dense subset of $\mathbb{S}^{n-1}$. Note that if $u \in F_\alpha$ and $J_\alpha(u) \in V$, then the first m reflection points of $\gamma(u)$ are proper, that is transversal ones.

Denote by $\mathcal{L}_{m+1}^{(\alpha)}$ the set of those $\theta \in V$ such that if $J_\alpha = \theta$ for some $u \in F_\alpha$, then the first $m + 1$ reflection points of $\gamma(u)$ are proper ones. We claim that $\mathcal{L}_{m+1}^{(\alpha)}$ is open and dense in V. The openness is clear. To establish the density, fix an arbitrary

$\theta \in V \setminus \mathcal{L}^{(\alpha)}_{m+1}$. Then $\theta = J_\alpha(u_0)$ for some $u_0 \in F_\alpha$ such that the first m reflection points $x_1(u_0), \ldots, x_m(u_0)$ of $\gamma(u_0)$ are proper ones and $\gamma(u_0)$ is tangent to ∂K at $x_{m+1}(U_0)$. Take an arbitrary t with $t_m(u_0) < t < t_{m+1}(u_0)$, and choose an open $(n-2)$-dimensional ball U in Z_ω with centre u_0 so small that for every $u \in U$ the trajectory $\{S_s(u) : 0 \leq s \leq t\}$ has exactly m reflections, all of them being proper ones. Then, according to Lemma 10.1.3, $Y = S_t(U)$ is a smooth strictly convex hypersurface in $\mathbb{R}^n$ (Figure 10.1). Note that $\{N_t(u) : u \in U\}$ is a normal field for Y. It is clear now that there exists $v \in U$ such that the ray, starting at $S_t(v)$ in direction $N_t(v)$, intersects transversally $\Gamma_{i_{m+1}}$, which means that $v \in \mathcal{L}^{(\alpha)}_{m+1}$. Therefore, $\mathcal{L}^{(\alpha)}_{m+1}$ is dense in V. Since V is open and dense in $\mathbb{S}^{n-1}$, we deduce that $\mathcal{L}^{(\alpha)}_{m+1}$ is also open and dense in $\mathbb{S}^{n-1}$.

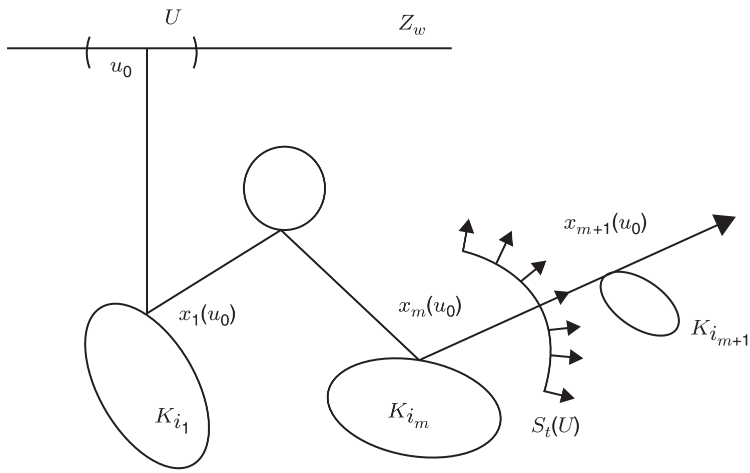

Figure 10.1 Strictly convex hypersurface $S_i(U)$.

Set $\mathcal{L}_{m+1} = \cap_\alpha \mathcal{L}^{(\alpha)}_{m+1}$, where α runs over the finite set of all configurations of length $m+1$. Then $\mathcal{L}_{m+1}$ is open and dense in $\mathbb{S}^{n-1}$ and has the desired properties. This concludes the proof of (a).

(b) Let $\{\mathcal{L}_m\}$ be the sequence constructed in (a), and set

$$R'(\Omega) = \cap_{m=1}^\infty \mathcal{L}_m.$$

Then $R'(\omega)$ is a residual subset of $\mathbb{S}^{n-1}$ such that for every configuration α we have

$$J_\alpha^{-1}(R'(\omega)) \cap F_\alpha \subset U_\alpha.$$

Fix two arbitrary configurations α and β, and set $m = \max\{|\alpha|, |\beta|\}$. Denote by $\mathcal{L}(\alpha, \beta)$ the set of all $\theta \in R'(\omega)$ such that (10.3) implies $T(u) \neq T(v)$. We are going to show that $\mathcal{L}(\alpha, \beta)$ is open and dense in $R'(\omega)$.

First, we prove that $R'(\omega) \setminus \mathcal{L}(\alpha, \beta)$ is closed in $R'(\omega)$. Consider an arbitrary sequence

$$R'(\omega) \setminus \mathcal{L}(\alpha, \beta) \ni \theta_k \to \theta \in R'(\omega).$$

Then for every k there exist $u_k \in U_\alpha \cap J_\alpha^{-1}(\theta_k)$ and $v_k \in U_\beta \cap J_\beta^{-1}(\theta_k)$ with $r_k(u_k) = |\alpha|, r(v_k) = |\beta|, T(u_k) = T(v_k)$. Taking appropriate subsequences, we may assume that $u_k \to u \in \bar{F}_\alpha, v_k \to v \in \bar{F}_\beta$. Assume $u \notin F_\alpha$. Then $u \in F_{\alpha'}$ for some α', and $\gamma(u)$ is tangent to ∂K at some of its points. On the other hand,

$$J_{\alpha'}(u) = J_\alpha(u) = \lim_k J_\alpha(u_k) = \lim_k \theta_k = \theta \in R'(\omega),$$

which is a contradiction with the properties of $R'(\omega)$. Hence $u \in F_\alpha$, and now $J_\alpha(u) = \theta \in R'(\omega)$ implies $u \in U_\alpha$. In the same way, one finds $J_\beta(v) = \theta$ and $v \in U_\beta$. Moreover, we have $r(u) = |\alpha|, r(v) = |\beta|$. Thus, $\theta \in R'(\omega) \setminus \mathcal{L}(\alpha, \beta)$, which shows that $\mathcal{L}(\alpha, \beta)$ is open in $R'(\omega)$.

To establish the density, fix an arbitrary $\theta \in R'(\omega) \setminus \mathcal{L}(\alpha, \beta)$ and set $\eta = -\theta$. Then there exist u and v having the properties (10.3) with $T(u) = T(v)$. Applying again Lemma 10.1.3, we find open $(n-2)$-dimensional balls $U' \subset U_\alpha$ and $V' \subset U_\beta$ centred at u and v, respectively, such that for $t = T(u) = T(v)$, $S_t(U')$ and $S_t(V')$ are smooth strictly convex hypersurfaces in $\mathbb{R}^n$ (Figure 10.2). Note that these hypersurfaces are tangent to Z_η at $S_t(u)$ and $S_y(v)$, respectively, so these are contained in the closed half-space, determined by Z_η and η. It follows by the strict convexity that

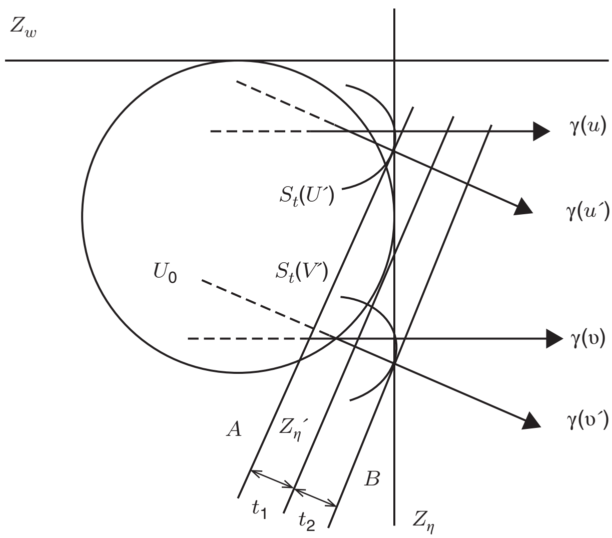

Figure 10.2 Strictly convex hypersurfaces $S_t(U)$ and $S_t(V)$.

for every $\theta' \in \mathbb{S}^{n-1}$ close to θ, there are unique $u' \in U'$ and $v' \in V'$ with $J_\alpha(u') = J_\beta(v') = \theta'$ and $S_t(U')$ and $S_t(V')$ have a common tangent hyperplane at $S_t(u')$ and

$S_t(v')$, respectively. It is easy to see that D has empty interior in $\mathbb{S}^{n-1}$; in fact, it is contained in a smooth submanifold of $\mathbb{S}^{n-1}$ with positive codimension. Since $R'(\omega)$ is dense in $\mathbb{S}^{n-1}$, there exists $\theta' \in R'(\omega) \setminus D$ arbitrary close to θ. Take such a θ' and denote by A (resp. B) the hyperplane tangent to $S_t(U')$ at $S_t(u')$ (resp. tangent to $S_t(V')$ at $S_t(v')$). Then for $\eta' = -\theta'$, the three hyperplanes A, B and $Z_{\eta'}$ are parallel. Setting $t_1 = T(u') - t, t_2 = T(v') - t$, we see that $|t_1|$ is the distance between A and $Z_{\eta'}$, while $|t_2|$ is the distance between B and $Z_{\eta'}$. Since $A \neq B$, we have $t_1 \neq t_2$; therefore,

$$T(u') = t + t_1 \neq t + t_2 = T(v').$$

Moreover, it follows by the properties of the maps J_α and J_β that (10.3) holds if we replace u by u', v by v' and θ by θ'. Thus, $\theta' \in \mathcal{L}(\alpha, \beta)$. This proves the density of $\mathcal{L}(\alpha, \beta)$ in $R'(\omega)$.

Since $R'(\omega)$ is residual in $\mathbb{S}^{n-1}$ and $\mathcal{L}(\alpha, \beta)$ is open and dense in $R'(\omega)$, we get that $\mathcal{L}(\alpha, \beta)$ is residual in $\mathbb{S}^{n-1}$, too. This concludes the proof of (b). ∎

Proof of Theorem 10.1.1: Let $\mathcal{L}(\alpha, \beta)$ be the residual subsets of $\mathbb{S}^{n-1}$ from Lemma 10.1.2, chosen as subsets of $R'(\omega)$, the latter being defined in the proof of Lemma 10.1.2 (b). Set

$$R(\omega) = \cap_{\alpha,\beta} \mathcal{L}(\alpha, \beta),$$

where α and β run over the set of all configurations. Then $R(\omega)$ is residual in $\mathbb{S}^{n-1}$ and has the property (a) from Theorem 10.1.1. As we have already mentioned, the property (b) follows immediately from (a) and Theorems 8.3.2 and 9.1.2. ∎

10.2 Hyperbolicity of scattering trajectories

Let Ω and K be as in the previous section. From now on, we assume that K satisfies the following condition:

$$\text{(H)} \begin{cases} \text{for any three distinct connected components,} \\ L, M, N \text{ of } K \text{ the convex hull of} \\ L \cup M \text{ does not contain points of } N. \end{cases}$$

In what follows, we use the notation from the beginning of the previous section. Recall that $\kappa_0 > 0$ denotes the minimum of the normal curvatures of ∂K.

Let $m \geq 2$ be an arbitrary integer, and let

$$L', L_1, L_2, \ldots, L_m, L'' \tag{10.15}$$

be an arbitrary sequence of connected components of K such that $L_i \neq L_{i+1}$ for all $i = 1, \ldots, m-1$, $L' \neq L_1$, $L_m \neq L''$. Note that some of these connected components may coincide. This is clearly the case when m is larger than the number of connected components of K.

Before going on, let us mention that the condition (H) implies the existence of $\varphi_0 \in (0, \pi/2)$ with the following property: if x, y, z belong to the boundaries of connected components L, M and N of K, $L \neq M, M \neq N$, the open segments (x, y) and (y, z) have no common points with K and $[x, y]$ and $[y, z]$ satisfy the law of reflection at y with respect to ∂K, then $\varphi < \varphi_0$, where $\varphi \in (0, \pi/2]$ is the angle between $[y, z]$ and the normal $\nu(y)$ to ∂K at y.

Furthermore, it is easy to see that there exists $\psi_0 \in (0, \pi/2)$ with the following property: for every two distinct connected components M and L of K, there is a tangent hyperplane H to M such that M and L are contained in one and a same half-space with respect to H, and the minimal angle between H and a straight line, having common point with both M and L, is greater than or equal to φ_0. Clearly, ψ_0 depends only on K. Let Z' be a hyperplane, tangent to L' and having the properties of H with respect to the pair $M = L', L = L_1$, and Z'' be a hyperplane tangent to L'' and having similar properties with respect to the pair $M = L'', L = L_m$. Denote by V the set of those $x \in Z'$ such that if $y \in \partial L_1, z \in \partial L_2$, the open segments (x, y) and (y, z) have no common points with $K \setminus L_1$ and $[x, y]$ and $[y, w]$ satisfy the law of reflection at y with respect to ∂K, then for the angle $\varphi \in (0, \pi/2]$ between $[y, w]$ and the normal $\nu(y)$ we have $\varphi < \varphi_0$. Clearly, V is an open subset of Z'. In a similar way we define an open subset W of Z'', this time the corresponding property concerns points $x \in L_{m_1}, y \in L_m, z \in W$.

The central moment in this section is the following.

Lemma 10.2.1: *There exist constants $C > 0$ and $\delta \in (0, 1)$, depending only on K, with the following property: if*

$$y_0' \in V,\ \ y_1' \in \partial L_1, \ldots, y_m' \in \partial L_m,\ \ y_{m+1}' \in W,$$

and

$$y_0'' \in V,\ \ y_1'' \in \partial L_1, \ldots, y_m'' \in \partial L_m,\ \ y_{m+1}'' \in W$$

are two sequences of points such that for every $j = 1, \ldots, m$ the segments $[y_{j-1}', y_j']$ and $[y_j', y_{j+1}']$ satisfy the law of reflection at y_j' with respect to ∂L_j and the segments $[y_{j-1}'', y_j'']$ and $[y_j'', y_{j+1}'']$ satisfy the law of reflection at y_j'' with respect to ∂L_j, then

$$\|y_i' - y_i''\| \leq C(\delta^i + \delta^{m-i}) \tag{10.16}$$

for all $i = 1, \ldots, m$.

To prove this lemma, we need some preparation. Define the function

$$F : V \times \partial L_1 \times \cdots \times \partial L_m \times W \to \mathbb{R}$$

by

$$F(v; y_1, \ldots, y_m; w) = \|v - y_1\| + \sum_{i=1}^{m-1} \|y_i - y_{i+1}\| + \|y_m - w\|.$$

Applying a standard argument and using the choice of V and W, we see that for every $v \in V$ and every $w \in W$ there exist $y_1(v,w) \in \partial L_1,\dots, y_m(v,w) \in \partial L_m$ such that

$$F(v; y_1(v,w),\dots, y_m(v,w); w) = \min\{F(v; y_1,\dots, y_m; w) : (y_1,\dots, y_m) \in \partial L_1 \times \dots \times \partial L_m\}$$

(cf., e.g. the proof of Proposition 10.3.2). Moreover, by an argument, very similar to that in the proof of Proposition 2.4.4, one gets that $y_i(v,w)$ are unique with this property. In fact, $y_i(v,w)$ are successive reflection points of a billiard trajectory in Ω connecting v and w. This shows that the maps $y_i(v,w)$ depend smoothly on (v,w). The latter can be derived also by the implicit function theorem.

Next, we identify every tangent hyperplane $T_x(\partial K), x \in \partial K$ (including Z' and Z'') with the $(n-1)$-dimensional linear subspace of $\mathbb{R}^n$ parallel to it, and we measure the lengths of the vectors $\xi \in T_x(\partial K)$ using the standard norm in $\mathbb{R}^n$. In this way we define also the norm of a linear operator between two tangent spaces. That is, we use the standard Riemannian metric on ∂K. We also assume that some basis in Z' is fixed and denote the elements of Z' by $v = (v^{(1)},\dots, v^{(n-1)})$. In the same way, we fix a basis in Z'' and set $w = (w^{(1)},\dots, w^{(n-1)}) \in Z''$. For fixed w, let

$$\partial_v y_i(v,w) : Z' \to T_{y_i}(v,w)\partial L_i$$

be the tangential map of the map $V \ni v \to y_i(v,w) \in \partial L_i$ at v. In the same way we define the tangential map

$$\partial_w y_i(v,w) : Z'' \to T_{y_i(v,w)}\partial L_i$$

of the map $W \ni w \mapsto y_i(v,w) \in \partial L_i$. We are going to estimate the norms of the linear operators $\partial_v y_i(v,w)$ and $\partial_w y_i(v,w)$.

Lemma 10.2.2: *For all $v \in V, w \in W$ and every $j = 1,\dots, m$ we have*

$$\|\partial_v y_j(v,w)\| \le C' e^{-j\epsilon} \tag{10.17}$$

and

$$\|\partial_w y_j(v,w)\| \le C' e^{-(m-j)\epsilon}, \tag{10.18}$$

where $C' > 0$ and $\epsilon > 0$ are constants depending only on K.

Proof: Fix arbitrary $v_0 \in V$ and $w_0 \in W$. For every $j = 1,\dots, m$ take a smooth chart

$$\varphi_j : \mathbb{R}^{n-1} \to U_j \subset \partial L_j$$

such that $\varphi_j(0) = y_j(v_0, w_0)$ and $\{\frac{\partial \varphi_j}{\partial u_j^{(p)}}(0)\}_{p=1}^{n-1}$ is an orthonormal basis in $T_{y_j(v_0,w_0)}\partial L_j$. As in the proof of Lemma 2.2.6, define the function

$$G : V \times (\mathbb{R}^{n-1})^m \times W \to \mathbb{R}$$

by

$$G(v; u_1, \dots, u_m; w) = F(v; \varphi_1(u_1), \dots, \varphi_m(u_m); w).$$

Given $i, j = 1, \dots, m$, consider the $(n-1) \times (n-1)$ symmetric matrix

$$G_{ij}(v, w) = \left(\frac{\partial^2 G}{\partial u_i^{(p)} \partial u_j^{(q)}} (u(v, w)) \right)_{p,q=1}^{n-1}.$$

Then

$$G_{uu}(v, w) = \begin{pmatrix} G_{11} & G_{12} & \cdots & G_{1m} \\ G_{21} & G_{22} & \cdots & G_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ G_{m1} & G_{m2} & \cdots & G_{mm} \end{pmatrix}$$

is a symmetric $m \times m$ block matrix. According to our computations in the proof of Lemma 2.2.6, we have

$$G_{ij}(v, w) = 0 \quad \text{whenever} \quad |i - j| > 1. \tag{10.19}$$

Consequently, there exists a constant $c_0 > 0$, depending only on K such that

$$\|G_{ij}(v, w)\| \leq c_0$$

for all $i, j = 0, 1, \dots, m-1$. It then follows by (10.19) that for $C_0 = 6c_0$ we have

$$\|G_{uu}(v, w)\| \leq C_0. \tag{10.20}$$

Moreover, using again the computation in the proof of Lemma 2.2.6, we see that $G_{uu}(v, w)$ is positive definite and

$$2\kappa_0 \cos \varphi_0 I \leq G_{uu}(v, w) \leq C_0 I, \tag{10.21}$$

I being the *identity matrix*.

In what follows denote the matrices $G_{ij}(v_0, w_0), G_{uu}(v_0, w_0)$, etc., briefly by G_{ij}, G_{uu}, etc. We should note that G_{ij} is considered as the matrix of a symmetric linear operator $T_{y_i(v_0,w_0)} \partial L_i \to T_{y_j(v_0,w_0)} \partial L_j$, so G_{uu} is the matrix of a symmetric positive definite linear operator

$$G_{uu} : \mathcal{T} = \prod_{i=1}^{m} T_{y_i(v_0,w_0)} \partial L_i \to \prod_{i=1}^{m} T_{y_i(v_0,w_0)} \partial L_i.$$

For $(v, w) \in V \times W$ close to (v_0, w_0) and $j = 1, \dots, m$ there is a uniquely determined smooth map

$$(v, w) \to u_j(v, w) \in \mathbb{R}^{n-1}$$

such that $y_j(v, w) = \varphi_j(u_j(v, w))$. Next, consider the map

$$w \to u_j(v_0, w).$$

For $l = 1,\ldots, n-1$ and $i = 1,\ldots, m$ define the column-vectors

$$\partial_l u_i(w) = \left(\frac{\partial u_i^{(p)}}{\partial w^{(l)}}(v_0, w)\right)_{p=1}^{n-1}, \quad G_i(u_i) = \left(\frac{\partial G}{\partial u_i^{(p)}}(u_i)\right)_{p=1}^{n-1}.$$

For the sake of brevity, we set $\partial_l u_i = \partial_l u_i(v_0, w_0)$ and $G_i = G_i(0)$. We shall also consider the elements ξ of $\mathcal{T}$ as column-vectors consisting of m blocks so that the ith block of ξ corresponds to a column-vectors from $T_{y_i(v_0,w_0)}\partial L_i, i = 1,\ldots, m$.

Consider the linear operator of $\mathcal{T}$ into itself with diagonal block matrix

$$D = \begin{pmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & D_m \end{pmatrix},$$

the ith block D_i being a diagonal $(n-1) \times (n-1)$ matrix

$$D_i = \begin{pmatrix} d_i & 0 & \cdots & 0 \\ 0 & d_i & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & d_i \end{pmatrix},$$

with $d_i = e^{(m-i)\epsilon}$. The constant $\epsilon > 0$ will be chosen in a special way later. Note that $d_m = 1$, so D_m is the identity $(n-1) \times (n-1)$ matrix. For $c = e^\epsilon - 1 > 0$ we have

$$\left|\frac{d_i}{d_{i+1}} - 1\right| \le c, \quad \left|\frac{d_{i+1}}{d_i} - 1\right| \le c$$

for all $i = 1,\ldots, m-1$. Using these inequalities and according to the choice of C_0 and c_0, by a direct computation we get

$$\|G_{uu} - DG_{uu}D^{-1}\| \le 2cC_0.$$

Therefore, for $\xi \in \mathcal{T}$ we have

$$\|G_{uu}\xi\| \le \|DG_{uu}D^{-1}\xi\| + 2cC_0\|\xi\|.$$

On the other hand, (10.21) implies

$$2\kappa_0 \cos\varphi_0 \|\xi\| \le \|G_{uu}\xi\|,$$

which combined with the previous inequality gives

$$\|\xi\| \le \frac{1}{2\kappa_0 \cos\varphi_0}(\|DG_{uu}D^{-1}\xi\| + 2cC_0\|\xi\|).$$

We now choose $\epsilon > 0$ such that

$$c = e^\epsilon - 1 = \frac{\kappa_0 \cos\varphi_0}{2C_0}, \tag{10.22}$$

then $cC_0/\kappa_0 \cos\varphi_0 = \frac{1}{2}$, and therefore

$$\|\xi\| \leq \frac{1}{\kappa_0 \cos\varphi_0} \|DG_{uu}D^{-1}\xi\|.$$

Setting

$$b_0 = \frac{1}{\kappa_0 \cos\varphi_0} > 0$$

and $\eta = D^{-1}\xi \in \mathcal{T}$ in the latter inequality, we find

$$\|\eta\| \leq b_0 \|DG_{uu}\eta\|, \;\; \eta \in \mathcal{T}. \tag{10.23}$$

Fix an arbitrary $i = 1,\dots,m$. Clearly, for $w \in W$ close to w_0 we have $G_i(u_i(v_0, w)) = 0$. Differentiating this equality with respect to $w^{(l)}$ and evaluating at $w = w_0$, one gets

$$\sum_{j=1}^{m} G_{ij}\partial_i u_j + \partial_i G_i = 0. \tag{10.24}$$

Here

$$\partial_i G_i = \left(\frac{\partial^2 G}{\partial u_i^{(p)} \partial v^{(l)}}(0) \right)_{p=1}^{n-1}$$

is a column-vector in $T_{y_i(v_0,w_0)}\partial L_i$. Again according to the computations in the proof of Lemma 2.2.6, we observe that $\partial_i G_i = 0$ for all $i < m$, while

$$\|\partial_l G_m\| \leq c_0',$$

with a constant $c_0' > 0$, depending only on K. Exchanging c_0, we may assume that $c_0' = c_0$. Let $\partial_i G \in \mathcal{T}$ be the vector, consisting of m blocks each of them having $n-1$ entries, such that the first $m-1$ blocks are zero, while the mth block coincides with the column-vector $\partial_i G_m$. Define $\xi \in \mathcal{T}$ so that its ith block coincides with the column-vector $\partial_i u_i$. Using (10.24) for all $i = 1,\dots,m$, we obtain

$$G_{uu}\eta + \partial_i G = 0,$$

and therefore

$$DG_{uu}\eta + D\partial_l G = 0.$$

Since $D_m = I$ and all blocks of $\partial_l G$, expect the last one, are zero, we deduce that $DG_{uu}\eta = -\partial_l G$, hence

$$\|DG_{uu}\eta\| \leq c_0.$$

This and (10.23) imply $\|D\eta\| \leq b_0 c_0$. The ith block of the vector $D\eta \in \mathcal{T}$ has the form $d_i \partial_l i_i$, and according to $d_i = e^{(m-i)\epsilon}$, we obtain

$$\|\partial_l u_i\| \leq b_0 c_0 e^{-(m-i)\epsilon}.$$

This is true for all $i = 1,\ldots,m$ and all $l = 1,\ldots,n-1$. Consequently, for the map $\partial_w u_i(v_0, w_0)$, we have

$$\|\partial_w u_i(v_0, w_0)\| \leq b_0 c_0 \sqrt{n-1} e^{-(m-1)\epsilon}.$$

Since $y_i(v,w) = \varphi_i(u_i(v,w))$ and φ_i was chosen in such a way that $\partial_{u_i}\varphi_i(0) = I$, the same estimate holds for $\partial_w y_i(v_0, w_0)$. Therefore, (10.18) is satisfied with

$$C' = b_0 c_0 \sqrt{n-1},\ \ \epsilon = \log\left(1 + \frac{\kappa_0 \cos\varphi_0}{2C_0}\right),$$

which clearly depends only on K.

To establish (10.17), we proceed in the same way, considering the maps $v \mapsto y_i(v, w_0)$ and determining the matrix D by $d_j = e^{j\epsilon}$, where ϵ is defined as above. This concludes the proof of the assertion. ■

Proof of Lemma 10.2.1: Let $\epsilon > 0$ be defined as above and let $\delta = e^{-\epsilon}$. Then $0 < \delta < 1$ and δ depends only on K. According to our assumptions, we have

$$y_i(v', w') = y_i',\ \ y_i(v', w'') = y_i'',$$

for all $i = 1,\ldots,m$, where $v' = y_0'$, $v'' = y_0'' \in V$ and $w' = y_{m+1}'$, $w'' = y_{m+1}'' \in W$. For $s \in [0,1]$ set $w_s = s\omega' + (1-s)\omega''$. Given $i = 1,\ldots,m$, consider the smooth curve $c(s) = y_i(v', w_s)$ on ∂L_i. Since $\operatorname{diam} W \leq C_1$, we have $\|w_0 - w_1\| \leq C_1$ with some constant $C_1 > 0$ depending only on K. Combining this with (10.18), we obtain

$$\|\dot{c}(s)\| \leq C' C_1 \delta^{m-i}.$$

Integrating this inequality for s from 0 to 1, we see that the length of the curve c is not greater than $C\delta^{m-1}$, where $C = C'C_1 > 0$. Hence there exists a curve with length $\leq C\delta^{m-i}$ on ∂L_i, joining $y_i(y', w')$ and $y_i(v', w'')$. Consequently,

$$\|y_i(v', w') - y_i(v', w'')\| \leq C\delta^{m-i}. \tag{10.25}$$

Now set $v_s = sv' + (1-s)v''$ and consider the curve $d(s) = y_i(v_s, w'')$ on ∂L_i. Applying the same argument and according to (10.17), we get

$$\|y_i(v', w'') - y_i(v'', w'')\| \leq C\delta^{m-i}.$$

Combining this with (10.25), one gets

$$\|y_i(v', w') - y_i(v'', w'')\| \leq C(\delta^i + \delta^{m-i}),$$

which proves (10.16). ■

We conclude this section with another lemma, the proof of which is very similar to that of Lemma 10.2.1.

Fix an arbitrary $w \in \mathbb{S}^{n-1}$ and set $Z = Z_\omega$. Let

$$M_1, M_2, \ldots, M_r, \ \ r \geq 2, \tag{10.26}$$

be a sequence of connected components of K such that $M_i \neq M_{i+1}$ for all $i = 1, \ldots, r-1$. We assume that

$$\pi_\omega(M_1) \cap \pi_\omega(M_2) = \emptyset. \tag{10.27}$$

Using this assumption, we may choose φ_0 from the beginning of the section in such a way that if $x \in Z, y \in M_1, z \in M_2$, the open segment (x, y) has direction ω, (x, y) and (y, z) have no common points with K and $[x, y]$ and $[y, z]$ satisfy the law of reflection at y with respect to ∂K, then $\varphi < \varphi_0$, where $\varphi \in [0, \pi/2]$ is the angle between $[y, z]$ and the normal $\nu(y)$ to ∂K at y. Hereafter, we assume that φ_0 is chosen in this way. In this case φ_0 depends on K and the choice of Z (i.e. on the choice of ω).

We are going to study trajectories $\gamma(u)$ issued from $u \in Z$ with direction ω having $r(u) \geq r$ reflection points $x_1(u), x_2(u), \ldots, x_{r(u)}(u)$ and $x_i(u) \in \Gamma_i$ for every $i = 1, \ldots, r$. For $j \leq r(u)$ denote by $\varphi_j(u) \in [0, \pi/2]$ the angle between the normal $\nu(x_j(u))$ and the vector $N_{+t_j(u)}(u)$, where $t_j(u)$ is the time of the jth reflection of $\gamma(u)$. Then it follows from the choice of φ_0 that

$$\varphi_j(u) < \varphi_0, \ \ j < r(u), \tag{10.28}$$

for every $u \in Z$. Denote by $\mathcal{U}_r$ the set of all $u \in Z$ with these properties.

Lemma 10.2.3: *For all $u', u'' \in \mathcal{U}_r$ and every $i = 0, 1, \ldots, r$ we have*

$$\|x_i(u') - x_i(u'')\| \leq B\delta^{r-i}, \tag{10.29}$$

where $x_0(u) = u, B > 0$ and $\delta \in (0, 1)$ are constants depending only on K and Z.

Proof: We use almost the same argument as in the proof of Lemma 10.2.1. Set $Z' = Z$ and choose Z'' as above, considering the pair $M = M_r, L = M_{r-1}$. Define V, W and F in the same way. This time is more convenient to denote the elements of V by $u_0 = (u_0^{(1)}, \ldots, u_0^{(n-1)})$. Given $w \in W$, there exist uniquely determined $z_0(w) \in Z, z_1(w) \in \partial M_1, \ldots, z_{r-1}(w) \in \partial M_{r-1}$ such that

$$F(z_0(w), z_1(w), \ldots, z_{r-1}(w); w) = \min\{F(z_0, z_1, \ldots, z_{r-1}; w) :$$
$$(z_0, z_1, \ldots, z_{r-1}) \in Z \times \partial M_1 \times \ \ldots \ \times \partial M_{r-1}\}.$$

Then for $u = z_0(w)$ we have $z_i(w) = x_i(u), i = 1, \ldots, r-1$ and w is a point on the ray starting at $x_{r-1}(u)$ in the direction $N_{+t_{r-1}(u)}(u)$.

We claim that for every $w \in W$ and every $j = 0, 1, \ldots, r-1$ we have

$$\|\partial_w z_j(w)\| \leq C' e^{-(r-j-1)\epsilon}, \tag{10.30}$$

where $C' > 0$ and $\epsilon > 0$ are the same constants as in Lemma 10.2.2.

Fix $w_0 \in W$ and take a smooth chart

$$\varphi_j : \mathbb{R}^{n-1} \to U_j \subset \partial M_j,\ \ j = 1,\ldots, r-1,$$

such that $\varphi_j(0) = z_j(w_0)$ and $\{\frac{\partial \varphi_j}{\partial u_j^{(p)}}(0)\}_{p=1}^{n-1}$ is an orthonormal basis in $T_{z_j(w_0)}\partial M_j$. Take $\varphi_0(u_0) = u_0 + z_0(w_0)$. Define the function

$$G:\ V \times (\mathbb{R}^{n-1})^{r-1} \times W \to \mathbb{R}$$

by

$$G(u_0, u_1,\ldots, u_{r-1}; w) = F(\varphi_0(u_0), \varphi_1(u_1),\ldots, \varphi_{r-1}(u_{r-1}); w).$$

Next, we repeat the argument from the proof of (10.18) by slightly exchanging the notation. After the change of φ_0, the constants c_0, C_0, ϵ, etc., are the same. In this way we establish the inequalities (10.30).

Let $u', u'' \in \mathcal{U}_r$, then $u' = z_0(w')$ and $u'' = z_0(w'')$ for some $w', w'' \in W$. By using an integration, as in the proof of Lemma 10.2.1, we see that for every $i = 0, 1,\ldots, r-1$ we have

$$\|z_(w') - z_i(w'')\| \leq B\delta^{r-i},$$

where $\delta = e^{-\epsilon}$ is the same as above, and $B = C\epsilon = C'C_1 e$. Since $x_i(u') = z_i(w'), x_i(u'') = z_i(w'')$, this implies (10.29). ∎

10.3 Existence of scattering rays and asymptotic of their sojourn times

Let the obstacle K have the form (10.1) and let Ω be the closure of its complements. We assume again the condition (H) is satisfied. In this section we show that for every configuration α, under a special choice of $\omega, \theta \in \mathbb{S}^{n-1}$, there exists an infinite sequence γ_q of (ω, θ)-rays in Ω, following α in a certain way, and we study the asymptotics of the sojourn times T_{γ_q} as $q \to \infty$.

In what follows, we use the notation from Section 10.1. Given $x \in \mathbb{R}^n, \eta \in \mathbb{S}^{n-1}$, denote by $l(x, \eta)$ the linear ray starting at x with direction η. Sometimes it will be convenient to use the notation $\mathrm{dist}(x, y) = \|x - y\|$.

Fix an arbitrary configuration α of the form (10.2). We say that the pair (ω, θ) of element of $\mathbb{S}^{n-1}$ is α-*admissible* if the following conditions are satisfied:

(i) every (ω, θ)-ray in Ω is ordinary and any two different (ω, θ)-rays in Ω have distinct sojourn times;

(ii) for every $x \in \Gamma_{i_1}$, the ray $l(x, -\omega)$(resp. $l(x, \omega)$) has no common points with $K \setminus K_{i1}$ (resp. with K_{i_2});

(iii) for every $x \in \Gamma_{i_m}$, the ray $l(x, \theta)$ (resp. $l(x, -\theta)$) has no common points with $K \setminus K_{i_m}$ (resp. $K_{i_{m-1}}$).

Lemma 10.3.1: *For every configuration α there exist unit vectors $\omega \neq \theta$ such that (ω, θ) is α-admissible.*

Proof: Set $D_1 = K_1, D_2 = K_{i_2}$. Take an arbitrary hyperplane A separating D_1 and D_2 such that A is tangent to D_1 at some point x and to D_2 at another point y. Set $\omega' = (x - y)/\|x - y\|$ and consider the convex cone

$$C = \{y + t(u - y) : \ u \in D_1, \ t \geq 0\}.$$

It is easy to check that the orthogonal projection of D_1 on the hyperplane $Z_{\omega'}$ is contained in C. We claim that

$$l(u, \omega') \cap (K \setminus D_1) = \emptyset, \ u \in D_1. \tag{10.31}$$

To prove this, assume that there exist $u \in D_1, t > 0, j \neq i_1$ such that $v = u - t\omega' \in K_j$. Then u and v have a common orthogonal projection u' on $Z_{\omega'}$. Since $u, u' \in C$, we have $v \in C$. On the other hand, the definition of C implies that the segment $[y, v]$ contains points of D_1 which is a contradiction of the condition (H). Thus, (10.31) holds. Then by the compactness of $K \setminus D_1$, there exists $\epsilon > 0$ such that

$$l(u, -\omega) \cap (K \setminus D_1) = \emptyset, \ u \in D_1,$$

holds, provided $\omega \in \mathbb{S}^{n-1}$ and $\|\omega - \omega'\| < \epsilon$. Take an arbitrary $\omega \in \mathbb{S}^{n-1}$, satisfying the latter inequality and such that $\langle -\omega, \nu(x) \rangle > 0$. Then clearly the condition (ii) is satisfied.

Fix an ω with property (ii) and denote by $R(\omega)$ the residual subset of $\mathbb{S}^{n-1}$ from Theorem 10.1.1. Using the density of this set, and applying the above argument for $D_1 = K_{i_k}$ and $D_2 = K_{i_{k-1}}$, we find $\theta \in R(\omega)$ satisfying (iii). Now $\theta \in R(\omega)$ implies that (i) is also satisfied. This proves the assertion. ■

From now on till the end of the chapter, $\alpha = (i_1, \ldots, i_k)$ will be a *fixed configuration* with $k \geq 2$ and $i_1 \neq i_k$ and l will be a *a fixed integer* with $1 \leq l \leq k$. For every integer $q \geq 0$ set

$$\alpha_{q,l} = (i_1, \ldots, i_k; \ \ldots \ ; i_1, \ldots, i_k; i_1, \ldots, i_l), \tag{10.32}$$

where the block $(i_1, \ldots, i_k)$ is repeated q times. Clearly, $\alpha_{q,l}$ is a configuration of length $qk + l$.

Now we fix two arbitrary unit vectors $\omega \neq \theta$ such that the pair (ω, θ) is $\alpha_{1,l}$-admissible. The existence of such vectors is guaranteed by Lemma 10.3.1. The pair (ω, θ) will also be fixed till the end of the chapter. As earlier, we shall use the notation $Z = Z_\omega$.

Proposition 10.3.2: *For every integer $q \geq 0$ there exists a unique (ω, θ)-ray γ_q of type $\alpha_{q,l}$ in Ω.*

Proof: Set for convenience $\eta = -\theta$. Fix an arbitrary integer $q \geq 0$ and set $m = qk + l$ and

$$D = Z \times \Gamma_{i_1} \times \cdots \times \Gamma_{i_m} \times Z_\eta,$$

where i_j are the successive components of $\alpha_{q,l}$. Define the function $F : D \to \mathbb{R}$ by

$$F(\zeta) = \|z_1 - x_1\| + \sum_{j=1}^{m-1} \|x_j - x_{j+1}\| + \|x_m - z_2\|$$

for every $\zeta = (z_1; x_1, \ldots, x_m; z_2) \in D$. Clearly, F is continuous and, considering its restriction on an appropriate compact subset of D, we see that there exists $\zeta' = (z_1'; x_1', \ldots, x_m'; z_2') \in D$ with $F(\zeta') = \min F$. Clearly, z_1' is the orthogonal projection of x_1' on Z, while z_2' is the projection of x_m' on Z_η. Since (ω, θ) satisfies the condition (ii), the segment $[z_1', x_2']$ has no common points with K_{i_1}. For $c > 0$ consider the rotative ellipsoid

$$E_c = \{x \in \mathbb{R}^n : \ \|z_1' - x\| + \|x - x_2'\| \leq c\}.$$

Let $c > 0$ be the minimal number with $E_c \cap K_{i_1} \neq \emptyset$. Then E_c is tangential to K_{i_1} at some of its points y_1. It is now clear that $y_1 = x_1'$, since F has total minimum at ζ'. Therefore, the segments $[z_1', x_1']$ and $[x_1', x_2']$ satisfy the law of reflection at x_1' with respect to Γ_{i_1}.

Repeating this argument several times and using the condition (H), we see that $x_1', \ldots, x_m'$ are the successive reflection points of a (ω, θ)-ray of type of $\alpha_{q,l}$ in Ω. The uniqueness of this ray follows from Corollary 2.4.6. ■

For every integer $q \geq 0$ set $U_q = U_{\alpha_{q,l}}$. Then U_q is an open subset of $Z = Z_\omega$, and the above proposition implies $U_q \neq \emptyset$. More precisely, there exists a unique $u_q \in U_q$ such that $\gamma_q = \gamma(u_q)$ is an (ω, θ)-ray in Ω of type $\alpha_{q,l}$.

As in Section 10.2, the condition (H) and the choice of Z imply the existence of $\varphi_0 \in (0, \pi/2)$ and using the notation of this section, we have

$$\varphi_j(u) < \varphi_0, j = 1, \ldots, r(u), \tag{10.33}$$

for every $u \in Z$. Then as an immediate consequence of Lemma 10.2.3, we get the following.

Lemma 10.3.3: *There exist constants $C > 0$ and $\delta > 0$, depending only on K and Z, such that*

$$\text{dist}\,(x_i(u), x_i(v)) \leq C\delta^{qk+l-i}, \ \ i = 0, 1, \ldots, qk + l \tag{10.34}$$

for every integer $q \geq 0$ and all $u, v \in U_q$.

We can also apply Lemma 10.2.3, considering the hyperplane $Z_{-\theta}$ instead of $Z = Z_\omega$. Note that γ_q is a (ω, θ)-ray; therefore, taking its reflection points in the opposite order, we get a sequence

$$x_{ql+k}(u_q), x_{qk+l-1}(u_q), \ldots, x_2(u_q), x_1(u_q),$$

which is the sequence of the successive reflection points of a $(-\theta, -\omega)$-ray in Ω. In a similar way we can consider also the ray γ_{q+1}. Now, eventually exchanging the constants C and δ (making them depending on K, Z_ω and $Z_{-\theta}$) to get an analogue of Lemma 10.3.3 with Z replaced by $Z_{-\theta}$, we obtain the inequalities

$$\operatorname{dist}(x_{qk+l-i}(u_q), x_{(q+1)k+l-i}(u_{q+1})) \leq C\delta^{qk+l-i-1}$$

for all $i = 0, 1, \ldots, qk + l$. Setting $j = qk + l - i$, the latter implies

$$\operatorname{dist}(x_j(u_q), x_{j+k}(u_{q+1})) \leq C\delta^{j-1}, \ \ j = 0, 1, \ldots, qk + l. \tag{10.35}$$

Next, we assume that C and δ are fixed having the above properties.

Using (10.34) for $i = 0$, we see that $\operatorname{diam} \bar{U}_q \leq C\delta^{qk+l}$ for every q. Since

$$U_1 \supset \cdots \supset U_q \supset \cdots \ ,$$

we deduce that the intersection of the closures of these sets consists of exactly one point u^∞, that is

$$\cap_{q=0}^{\infty} \bar{U}_q = \{u^\infty\}.$$

It is clear that the trajectory $\gamma^\infty = \gamma(u^\infty)$ has infinitely many reflection points. Moreover, for $x_i^\infty = x_i(u^\infty)$ we have

$$x_{qk+l}^\infty \in \Gamma_{i_j}, \ \ q \geq 0, \ \ 1 \leq j \leq k.$$

Fix for a moment $q \geq 0, r \geq 0$ and $j = 1, \ldots, k$. Take an arbitrary integer $p > q$ and apply Lemma 10.2.1 for $m = pk + j - 2$, the connected components

$$L' = K_1, \ \ L_1 = K_{i_2}, \ldots, L_m = K_{i_{j-1}}, \ \ L'' = K_{i_j}$$

of K and the sequences of points

$$x_1^\infty, x_2^\infty, \ldots, x_{pk+j-1}^\infty, x_{pk+j}^\infty$$

and

$$x_{rk+1}^\infty, x_{rk+2}^\infty, \ldots, x_{(p+r)k+j-1}^\infty, x_{(p+r)k+j}^\infty.$$

Then, we obtain

$$\operatorname{dist}(x_{qk+j}^\infty, x_{(q+r)k+j}^\infty) \leq C(\delta^{qk+j} + \delta^{(p-q)k-2}). \tag{10.36}$$

Here we have used the inequality (10.16) for $i = qk + j$. Since (10.36) holds for all $p \geq q$, letting $p \to q$, we get

$$\operatorname{dist}(x_{qk+j}^\infty, x_{(q+r)k+j}^\infty) \leq C\delta^{qk+j} \tag{10.37}$$

for all q and r, which shows that the sequence $\{x^{\infty}_{qk+j}\}_q$ is convergent. Denote z_j by its limit. Then $z_j \in \Gamma_{i_j}$ and (10.37) implies

$$\text{dist}\,(x^{\infty}_{qk+j}, z_j) \leq C\delta^{qk+j},\;\; q \geq 0,\;\; 1 \leq j \leq k. \tag{10.38}$$

Moreover, it follows from

$$z_{j+k} = \lim_q\, x^{\infty}_{(q+1)k+j} = \lim_q\, x^{\infty}_{qk+j} = z_j$$

that $z_1, z_2, \ldots, z_k$ are the successive reflection points of a periodic reflecting ray γ_α of type α in Ω. Note that by Corollary 2.2.4, there exists only one such ray.

Remark 10.3.4: One can avoid the application of Lemma 10.2.1 in the above argument, according to the inequalities (10.35), which follows in fact from Lemma 10.2.3. In other words, for our considerations in this section and the following one as well, only Lemma 10.2.3 from Section 10.2 is necessary. However, Lemma 10.2.1 presents an important property of the billiard trajectories, and its proof is almost the same as that of Lemma 10.2.3, we include it in this book.

Set

$$d_j = \sum_{p=1}^{j} \|z_p - z_{p=1}\|, \quad 1 \leq j \leq k,$$

$d_\alpha = d_k$, and

$$L^{\infty}_m = \langle x^{\infty}_1, \omega\rangle + \sum_{p=1}^{m} \|x^{\infty}_p - x^{\infty}_{p+1}\|.$$

Clearly, d_α is the period (length) of γ_α. Using (10.38) for $q \geq 0$ and $r \geq 0$, we find

$$\begin{aligned}
&|(L^{\infty}_{q+r)k+j} - (q+r)d_\alpha - d_j) - (L^{\infty}_{qk+j} - qd_\alpha - d_j)| \\
&\qquad \leq 2C \sum_{p=1}^{rk+1} \delta^{qk+t+j} \leq C_1\delta^q,
\end{aligned}$$

where the constant $C_1 > 0$ is determined by C and δ. Hence for every $j \leq k$, there exists

$$L_j = L_{\alpha,\omega,j} = \lim_q\, (L^{\infty}_{qk+j} - qd_\alpha - d_j).$$

Moreover, we have the asymptotic

$$L^{\infty}_{qk+j} = qd_\alpha + d_j + L_j + \mathcal{O}(\delta^q) \text{ as } q \to \infty. \tag{10.39}$$

In the same way as above for $u^\infty \in Z = Z_\omega$, we find a unique $v^\infty \in Z_\eta$ such that the billiard trajectory $\tilde{\gamma}^\infty$, starting from η^∞ in direction $\eta = -\theta$ has infinitely many reflection points y^{∞}_i such that $y^{\infty}_{qk+r} \in \Gamma_{j_r}$, for all $q \geq 0, 1 \leq r \leq k$, where

$$(j_1, \ldots, j_k) = (i_l, i_{l-1}, \ldots, i_1; i_k, i_{k-1}, \ldots, i_{l+2}, i_{l+1}).$$

Now for

$$G_m^\infty = -\langle y_1^\infty, \theta\rangle + \sum_{p=1}^{m} \|y_p^\infty - y_{p+1}^\infty\|$$

we get the asymptotic

$$G_{qk}^\infty = qd_\alpha + L_{\alpha,\theta} + \mathcal{O}(\delta^q) \text{ as } q \to \infty. \tag{10.40}$$

where $L_{\alpha,\theta}$ is a constant, depending only on K, α and θ. This can be proved by the same argument as above, and we omit the details.

Set $T_q = T_{\gamma_q}$,

$$L_p^{(q)} = \langle x_1(u_q), \omega\rangle + \sum_{r=1}^{p} \|x_r(u_q) - x_{r+1}(u_q)\|,$$

and

$$G_p^{(q)} = -\langle x_{qk+l(u_q)}, \theta\rangle + \sum_{r=p+1}^{qk+l-1} \|x_r(u_q) - x_{r+1}(u_q)\|.$$

Given $q \geq 0$, we define $p = p(q)$ by $p = k\left[\frac{q}{2}\right] + l - 1$. Using the choice of the constants C and δ (cf. Lemma 10.3.3), we get the following:

$$|L_p^{(q)} - L_p^\infty| + |G_p^{(q)} - G_{(q-[q/2])k}|$$
$$\leq 2\sum_{r=1}^{p+1} \|x_r(u_q) - x_r^\infty\| + 2\sum_{r=1}^{(q-[q/2])k+1} \|x_{qk+l-r+1}(u_q) - y_r^\infty\| \leq C_2\delta^q,$$

where $C_1 > 0$ is a constant, depending only on K, α, ω and θ. Combining this with the asymptotics (10.39) and (10.40), and using the fact that for every q we have $T_q = L_p^{(q)} + G_p^{(q)}$, one obtains the following.

Theorem 10.3.5: *The sojourn times T_q have the asymptotic*

$$T_q = qd_\alpha + L_{\alpha,\omega,\theta} + \mathcal{O}(\delta^q) \text{ as } q \to \infty, \tag{10.41}$$

where

$$L_{\alpha,\omega,\theta} = L_{l-1} + L_{\alpha,\theta} + d_{l-1},$$

and the constants d_j, L_j and $L_{\alpha,\theta}, d_0 = 0$, are determined as above.

In particular, from the sojourn times T_q one can recover the period d_α of the periodic reflecting ray γ_α.

Corollary 10.3.6: *Let $s = 2$, that is $K = K_1 \cup K_2$, and let d be the distance between K_1 and K_2. Let (ω, θ) be α-admissible for $\alpha = (1, 2)$. Then for every*

$i = 1, 2$ and every integer $m \geq 1$ there exists a unique (ω, θ)-ray in Ω with m reflection points, the first of which belongs to K_i. Let $T_m^{(i)}$ be the sojourn time of this (ω, θ)-ray. There exists a constant $\delta \in (0, 1)$ and for $i = 1, 2, j = 0, 1$ a constant $L_{\omega,\theta}^{(i,j)}$, such that

$$T_{2q+l}^{(i)} = 2qd + L_{\omega,\theta}^{(i,j)} + \mathcal{O}(\delta^q) \text{ as } q \to \infty.$$

Example 10.3.7: Under the notation in Corollary 10.3.6, assume in addition that $n = 2$ and K_1 and K_2 are discs in $\mathbb{R}^2 = Oxy$, having one and the same radius r and centers $(-a, 0)$ and $(a, 0)$, respectively. We suppose that $a > r > 0$, so K_1 and K_2 are disjoint. Set

$$\omega = (0, -1), \;\; \theta = (\cos\theta_0, \sin\theta_0),$$

where $\theta_0 \in (0, \pi/2)$ is taken close to $\pi/2$. By Lemma 10.3.1 we can choose θ_0 in such a way that the pair (ω, θ) is α-admissible for $\alpha = (1, 2)$. In fact, under certain assumptions on r and α, it can be shown that this is true for all θ in a small neighbourhood of $-\omega$ in $\mathbb{S}^{n-1}$ (cf., e.g. [NS]). Using the notation $T_m^{(i)}$ from the above corollary, we then have $T_m^{(i)} \neq T_n^{(j)}$ whenever $(m, i) \neq (n, j)$.

Next, consider the two-dimensional torus K' in $\mathbb{R}^3 = Oxyz$, obtained by rotating K (or K_1 only) about the axis Oy. Let Ω' be the closure of the complement of K' in $\mathbb{R}^3$. We shall consider ω and θ as vectors in $\mathbb{R}^3$ having third component 0. Then it is easy to see that the scattering length spectrum of Ω' coincides with

$$\{T_m^{(i)} : \; m \in \mathbb{N}, \;\; i = 1.2\}.$$

Moreover, for $m \in \mathbb{N}$ and $i = 1, 2$ if γ is (ω, θ)-ray in Ω' with sojourn time $T_m^{(i)}$, then close to $-T_\gamma = T_m^{(i)}$ we have (9.7).

10.4 Asymptotic of the coefficients of the main singularity

We continue with the notation and assumptions from the previous section. For the sake of brevity, set $\kappa = \cos\varphi_0$.

Given $q \geq 0$, denote by c_q the *coefficient in front of the main singularity* of the scattering kernel $s_\Omega(t, \theta, \omega)$ for t close to $-T_q$. In other words, c_q is the coefficient in front of $\delta^{(n-1)/2}(t + T_q)$ in the formula (9.7) for the (ω, θ)-ray $\gamma = \gamma_q$ and t close to $-T_q$.

Our aim in this section is to find the asymptotic of $|c_q|$ as $q \to \infty$. Set

$$D = \text{diam}\, K, \;\; d' = \min_{i \neq j} \text{dist}\,(K_i, K_j), \;\; d'' = \frac{1}{d'}.$$

Since the domains K_i are compact and strictly convex, there exist constants $\mu_2 > \mu_1 > 0$ such that

$$\mu_1(v, v) \leq \langle G_x v, v\rangle \leq \mu_2(v, v), \;\; v \in T_x(\partial K),$$

$G_x : T_x(\partial K) \to T_x(\partial K)$ being the differential of the Gauss map of ∂K at x.

Let $x \in \Gamma_i, y \in \Gamma_j, i \neq j$, and assume that the segment $[x, y]$ is contained in Ω and is transversal to both Γ_i and Γ_j. Denote by Π the hyperplan passing through x and orthogonal to $[x, y]$. Set $e = (y - x)/\|y - x\|$, and denote by π the projection $\Pi \to T_x(\partial K)$ along the vector e. As in Section 2.3, define the symmetric linear map $\tilde{\psi} : \Pi \to \Pi$ by

$$\left\langle \tilde{\psi}(u), u \right\rangle = 2 \left\langle e, \nu(x) \right\rangle \left\langle G_x(\pi(u)), \pi(u) \right\rangle, \ \ u \in \Pi. \tag{10.42}$$

We shall say that $\tilde{\psi}$ is the operator determined by the segment $[x, y]$. A standard exercise shows that

$$\operatorname{spec} \tilde{\psi} \subset [2\mu_1 \langle \nu(x), e\rangle, 2\mu_2 \langle \nu(x), e\rangle^{-1}]. \tag{10.43}$$

We shall prove two technical lemmas which will be used several times later.

Lemma 10.4.1: *Let $x, x' \in \Gamma_i, y, y' \in \Gamma_j, i \neq j$, and let $e > 0$ be such that dist $(x, x') < \epsilon$ and dist $(y, y') < \epsilon$. Introduce the vectors*

$$e = \frac{y - x}{\|y - x\|}, \quad e' = \frac{y' - x'}{\|y' - x'\|},$$

and assume that $\langle e, \nu(x)\rangle \geq \kappa, \langle e', \nu(x')\rangle \geq \kappa$. Let

$$\tilde{\psi} : \Pi \to \Pi, \ \ \tilde{\psi}' : \Pi' \to \Pi'$$

be the operators determined by the segments $[x, y]$ and $[x', y']$, respectively. Then there exist a linear isometry $A : \mathbb{R}^n \to \mathbb{R}^n$ and a constant $C' > 0$, depending only on K and κ, such that $A(\Pi') = \Pi$, $\|A - I\| < C'\epsilon$ and $\|\tilde{\psi} - A\tilde{\psi}' A^{-1}\| < C'\epsilon$.

Proof: Let $A_1 : \mathbb{R}^n \to \mathbb{R}^n$ be the translation determined by the vectors $x' - x$. Set $\nu'' = A_1(\nu(x'))$ and denote by A_2 the rotation around a line in $\mathbb{R}^n$ in a angle $\varphi = \arccos\langle \nu(x), \nu(x')\rangle$, for which $A_2(\nu'') = \nu(x)$ and such that $A_2 =$ id on $\{\nu(x), \nu''\}^{\perp}$. Finally, let $e'' = A_2 \circ A_1(e')$ and let A_3 be the rotation with $A_3(e'') = e$, which is identical on $\{e, e''\}^{\perp}$. We set $A = A_3 \circ A_2 \circ A_1$. Clearly, A is a linear isometry.

It is easy to check that $\|A_i - I\| < \text{const}\, \epsilon$ for every $i = 1, 2, 3$. For example,

$$\|A_2 - I\| = \left\| \begin{pmatrix} 1 - \cos\varphi & \sin\varphi \\ -\sin\varphi & 1 - \cos\varphi \end{pmatrix} \right\| = \sqrt{2(1 - \cos\varphi)} = \|\nu(x) - \nu(x')\|,$$

and the smoothness of G_x and the compactness of K imply

$$\|\nu(x) - \nu(x')\| < \text{const}\, \|x - x'\| < \text{const}\, \epsilon.$$

Similar simple estimates can be written for A_1 and A_3.

Thus, $\|A - I\| <$ const ϵ. Next, set $\tilde{\chi} = A\tilde{\psi}' A^{-1}, G = G_x, G' = G_{x'}$. Take an arbitrary $u \in \Pi$ with $\|u\| = 1$ and set $u' = A^{-1}u$. Then $u' \in \Pi'$ and $\|u'\| = 1$. Let

$$\pi : \Pi \to T_x(\partial K),\ \ \pi' = \Pi' \to T_{x'}(\partial K)$$

be the projections along the vectors e and e', respectively, and let $v = \pi(u)$, $v' = \pi'(u')$ (see the diagram).

$$\begin{array}{ccccccc}
\Pi & \xleftarrow{\ \tilde{\psi}\ } & \Pi & \xleftarrow{\ A\ } & \Pi' & \xrightarrow{\ \tilde{\psi}'\ } & \Pi' \\
 & & \pi \big\downarrow & & \pi' \big\downarrow & & \\
 & & T_x(\partial K) & & T_{x'}(\partial K) & &
\end{array}$$

We have

$$\begin{aligned}
|\langle\tilde{\psi}(u), u\rangle - \langle\tilde{\chi}(u), u\rangle| &= |\langle\tilde{\psi}(u), u\rangle - \langle\tilde{\psi}'(u'), u'\rangle| \\
= |2\langle e, \nu(x)\rangle\langle G(\pi(u)), \pi(u)\rangle &- 2\langle e', \nu(x')\rangle\langle G'(\pi'(u')), \pi'(u')\rangle| \\
\leq 2|\langle e, \nu(x)\rangle - \langle e', \nu(x')\rangle||\langle G(v), v\rangle &+ 2\langle e', \nu(x')\rangle|\langle G(v), v\rangle - \langle G'(v'), v'\rangle| \\
< \text{const } \epsilon &+ 2|\langle G(v), v\rangle - \langle G'(v'), v'\rangle|.
\end{aligned}$$

It follows by the smoothness of the Riemannian metric on ∂K that

$$|\langle G(v), v\rangle - \langle G'(v'), v'\rangle| < \text{const } \|v - v'\|.$$

Now taking into account that $\|\pi\| \leq 1/\kappa$, $\|\pi'\| \leq 1/\kappa$, we find

$$\begin{aligned}
\|v - v'\| &= \|\pi(u) - \pi'(A^{-1}(u))\| \\
\leq \|\pi - \pi' \circ A^{-1}\| &\leq (\|\pi\| + \|\pi'\|)\|(A - I)\| < \text{const } \epsilon.
\end{aligned}$$

Therefore, $|\langle(\tilde{\psi} - \tilde{\chi})(u), u\rangle| <$ const ϵ, which implies $\|\tilde{\psi} - \tilde{\chi}\| <$ const ϵ. Thus proves the assertion. ■

Furthermore, we are going to apply the above lemma for two sequences of points. Let $x_1, \ldots, x_p$ and $x'_1, \ldots, x'_p$ be points of K such that for every $j = 1, \ldots, p$, the points x_j and x'_j belong to Γ_i for one and the same $i = i(j)$. Assume that

$$\text{dist }(x_j, x'_j) \leq Da^j, \quad j = 1, \ldots, p \tag{10.44}$$

for some constants $D > 0$ and $a > 0$. Introduce the unit vectors

$$e_j = \frac{x_{j+1} - x_j}{\|x_{j+1} - x_j\|}, \quad e'_j = \frac{x'_{j+1} - x'_j}{\|x'_{j+1} - x'_j\|},$$

and assume that

$$\langle e_j, \nu(x_j)\rangle \geq \kappa, \quad \langle e'_j, \nu(x'_j)\rangle \geq \kappa, \quad j = 1, \ldots, p.$$

It follows from the above lemma that for every $j \le p$ there exists a linear isometry A_j in $\mathbb{R}^n$ such that $A_j(\Pi)' = \Pi_j$ and

$$\|A_j - I\| < C'D(1+a)a^j, \quad \|\tilde{\psi}_j - A_j\tilde{\psi}'_j A_j^{-1}\| < C'D(1+a)a^j. \tag{10.45}$$

Here $\tilde{\psi}_j : \Pi_j \to \Pi_j$ and $\tilde{\psi}' : \Pi'_j \to \Pi'_j$ are the operators determined by the segments $[x_j, x_{j+1}]$ and $[x'_j, x'_{j+1}]$, respectively. Let $M_1 : \Pi_1 \to \Pi_1$ and let $M'_1 : \Pi'_1 \to \Pi'_1$ be arbitrary symmetric non-negative definite linear operators. Define recursively

$$M_i = \sigma_i M_{i-1}(I + \lambda_i M_{i-1})^{-1}\sigma_i + \tilde{\psi}_i, \quad i = 2,\ldots,p, \tag{10.46}$$

where $\lambda_i = \text{dist}\,(x_{i-1}, x_i)$ and σ_i is the symmetry with respect to Π_i. We define the maps $M'_i, i = 2,\ldots,p$, in the same way, replacing $\tilde{\psi}_i, \sigma_i, \lambda_i$ and x_i by $\tilde{\psi}'_i, \sigma'_i, \lambda'_i$ and x'_i, respectively. Finally, set

$$b = (1 + 2\mu_1\kappa d')^{-1}, \quad a_1 = \begin{cases} a \text{ if } a \ge 1, \\ \max\{a, b\} \text{ if } a < 1. \end{cases} \tag{10.47}$$

Next, we use the notation $\log = \log_e$.

Lemma 10.4.2: *Under the above assumptions, there exist constants $E > 0, E' > 0$, depending only on K, κ and a, such that*

$$\|M_j - A_j M'_j A_j^{-1}\| < DEa_1^j + b^{2(j-r)}\|M_r - A_r M'_r A_r^{-1}\| \tag{10.48}$$

and

$$\begin{aligned} &|\log\det((I + \lambda_{i+1}M_j)(I + \lambda'_{j+1}M'_{j+1})^{-1}| \\ &< DE'a_1^j + (n-1)db^{2(j-r)+1}\|M_r - A_r M'_r A_r^{-1}\| \end{aligned} \tag{10.49}$$

for all $1 \le r \le j \le p$.

Proof: First, note that $|\lambda_i - \lambda'_i| < D(1+a)a^i$. Moreover, for every symmetric non-negative definite linear operator M we have

$$\|(I + \lambda M)^{-1}\| \le (1 + \lambda\sigma)^{-1}, \quad \|(M(1 + \lambda M)^{-1}\| \le \frac{1}{\lambda},$$

where $\sigma = \min\,(\text{spec}\, M)$. It follows by (10.46) that

$$\min\,(\text{spec}\, M_{i-1}) \ge \min\,(\text{spec}\, \tilde{\psi}'_{i-1}), \quad i \ge 2,$$

therefore

$$\|(I + \lambda_i M_{i-1})^{-1}\| \le b, \ \|M_{i-1}(I + \lambda_i M_{i-1})^{-1}\| \le \frac{1}{\lambda_i} \le d''.$$

Introduce the operator

$$L_i = A_i M_i' A_i^{-1} : \Pi_i \to \Pi_i.$$

Since $\sigma_i' = A_I^{-1}\sigma_i A_i$, we find

$$L_i = \sigma_i B_i L_{i-1}(I + \lambda_i' L_{i-1})^{-1} B_i^{-1}\sigma_i + A_i \tilde{\psi}_i' A_i^{-1}$$

for $B_i = A_i \circ A_{i-1}^{-1}$. Using (10.46) and the trivial inequality

$$\|X - B_i Y B_i^{-1}\| \leq 2\|X\|\|I - B_i\| + \|X - Y\|,$$

we have

$$\begin{aligned}\|M_i - L_i\| \leq \|M_{i-1}(I + \lambda_i M_{i-1})^{-1} - B_i L_{i-1}(I + \lambda_i' L_{i-1})^{-1} B_i^{-1}\| \\ +\|\tilde{\psi}_i' - A_i \tilde{\psi}_i' A_i^{-1}\| < CD(1+a)a^i + 2C'd''(1+a)^2 a^{i-1} \\ +\|M_{i-1}(I + \lambda_i M_{i-1})^{-1} - L_{i-1}(I + \lambda_i' L_{i-1})^{-1}\|.\end{aligned}$$

The last term can be estimated as follows:

$$\begin{aligned}\|M_{i-1}(I + \lambda_i M_{i-1})^{-1} - L_{i-1}(I + \lambda_i' L_{i-1})^{-1}\| \\ \leq |\lambda_i - \lambda_i'|\|(I + \lambda_i M_{i-1})^{-1}\|\|L_{i-1}(I + \lambda_i' L_{i-1})^{-1}\| \\ +b^2\|M_{i-1} - L_{i-1}\| < Dd''^2(1+a)a^i + b^2\|M_{i-1} - L_{i-1}\|.\end{aligned}$$

Therefore,

$$\|M_i - L_i\| < DE''a^i + b^2\|M_{i-1} - L_{i-1}\|, \ i = 2, \ldots, p,$$

where $E'' = (1+a)(C' + 2d''(1+a)a^{-1}C' + d''^2) > 0$.

Repeating this procedure $j - r$ times, one gets

$$\|M_j - L_j\| < DE'' \sum_{t=0}^{j-r-1} a^{j-i} b^{2t} + b^{2(j-r)}\|M_r - L_r\|, \tag{10.50}$$

for all $1 \leq r \leq j \leq p$. There are two cases.

Case 1. $a \geq 1$. Then $a > b^2$, and (10.50) implies

$$\begin{aligned}\|M_j - L_j\| < DE''a^j \sum_{t=0}^{j-r-1} (b^2/a)^t + b^{2(j-r)}\|M_r - L_r\| \\ < DE''a^j \left(1 - \frac{b^2}{a}\right)^{-1} + b^{2(j-r)}\|M_r - L_r\|.\end{aligned}$$

In this case, we set $E = E''a(a - b^2)^{-1}$.

Case 2. $0 < a < 1$. Then $a_1 = \max\{a, b\} < 1$, and (10.50) implies

$$\|M_j - L_j\| < DE'' \sum_{t=0}^{j-r-1} a_1^{j+t} + b^{2(j-r)}\|M_r - L_r\|$$
$$< DE''(1 - a_1)^{-1}a_1^j + b^{2(j-r)}\|M_r - L_r\|.$$

Now set $E = E''(1 - a_1)^{-1}$.

It follows by the choice of E in both cases that (10.48) holds for $1 \le r \le j \le p$.

Before going on, let us note that if A is an arbitrary linear operator in $\mathbb{R}^k$, then

$$|\det A| \le (1 + \|A - I\|)^k.$$

Using this, we find the following estimates:

$$\det((I + \lambda_{i+1}M_i)(I + \lambda'_i M'_i)^{-1})$$
$$\le \Big(1 + \|I - (I + \lambda_{i+1}M_i)(I + \lambda'_{i+1}L_i)^{-1}\|\Big)^{n-1}$$
$$= \Big(1 + \|\lambda'_{i+1}L_i - \lambda_{i+1}M_i)(I + \lambda'_{i+1}L_i)^{-1}\|\Big)^{n-1}$$
$$\le \left(1 + \frac{|\lambda'_{i+1} - \lambda_{i+1}|}{\lambda'_{i+1}} + b\lambda_{i+1}\|M_i - L_i\|\right)^{n-1}$$
$$< (1 + D(1 + a)d''a^{i+1} + bd\|M_i - L_i\|)^{n-1}.$$

In the same way one gets a similar inequality for

$$\det((I + \lambda_{i+1}M_i)^{-1}(I + \lambda'_{i+1}M'_i)),$$

therefore

$$|\log\det(I + \lambda_{i+1}M_i) - \log\det(I + \lambda'_{i+1}M'_i)|$$
$$< (n - 1)\log(1 + Dd''(1 + a)a^{i+1} + bd\|M_i - L_i\|)$$
$$< (n - 1)(Dd''(1 + a)a^{i+1} + bd\|M_i - L_i\|).$$

Finally, applying (10.48) for $j = i$, we get (10.49) with $E' = (n - 1)(a_1^2 + a_1)d'' + bdE)$ for all $1 \le r \le j \le p$. This completes the proof of the lemma. ■

For the reflection points z_i of the unique periodic reflecting ray of type α in Ω (cf. Section 10.3), we define $z_m = z_j$, whenever m has the form $m = qk + j, 1 \le j \le k$. Let $\tilde{\psi}''_j : \Pi''_j \to \Pi''_j$ be the operator determined by $[z_j, z_{j+1}]$, and let $\mathcal{M}(\Pi''_j)$ be the *space of all symplectic positively definite linear maps* $M : \Pi''_j \to \Pi''_j$. Define $\mathcal{F}_j : \mathcal{M}(\Pi''_j) \to \mathcal{M}(\Pi''_j)$ by

$$\mathcal{F}_j(M) = \sigma''_{j+1}M(I + \lambda''_{j+1}M)^{-1}\sigma''_{j+1} + \tilde{\psi}''_{j+1},$$

where $\lambda_j'' = \text{dist}\,(z_j, z_{j-1})$ and σ_j'' is the symmetry with respect to the tangent hyperplane to ∂K at z_j. Using the argument from the proof of Proposition 2.3.2, we deduce that the map

$$\mathcal{F}_k \circ \mathcal{F}_{k-1} \circ \ldots \circ \mathcal{F}_1 : \mathcal{M}(\Pi_1'') \to \mathcal{M}(\Pi_1'')$$

has a unique fixed point M_1''. Then $M_2'' = \mathcal{F}_1(\mathcal{M}_1'')$ is the unique fixed point of $\mathcal{F}_k \circ \mathcal{F}_{k-1} \circ \ldots \circ \mathcal{F}_2$, etc.

Let $q \geq 0$ be an arbitrary integer. Consider the configuration $\alpha_{q,l}$, and set

$$J_q = J_{\alpha_q,l} : F_{\alpha_q,l} \to \mathbb{S}^{n-1}.$$

Recall that $u_q \in U \subset Z$ is the unique point in Z for which there exists a (ω, θ)-ray of type $\alpha_{q,l}$ intersecting Z at u_q. We denote this (ω, θ)-ray by γ_q.

Set $m = qk + l$ and $x_i = x_i(u_q)$ for $i = 1,\ldots,m$ and $x_i' = x_i(u_{q+1})$ for $i = 1,\ldots,m+k$. Define the operators $\tilde{\psi}_i'$ and $\tilde{\psi}'$ as in the text before Lemma 10.4.2, and set $M_1 = \tilde{\psi}_1, M_1' = \tilde{\psi}_1'$. Next, define M, recursively by (10.46), and M_i' in a similar way. Finally, set

$$p = \left[\frac{m}{2}\right],\ t = \left[\frac{p}{2}\right]. \tag{10.51}$$

Clearly, $2p \leq m < 2p+1, 2t \leq p < 2t+1$, which implies $4t \leq m < 4p+3$. Applying the inequalities (10.34), we get

$$\text{dist}\,(x_i, x_i') < C\delta^{p-i}, \quad i = 1,\ldots,t, \tag{10.52}$$

$$\text{dist}\,(x_i, x_i') < C\delta^{i}, \quad i = t+1,\ldots,p. \tag{10.53}$$

Next, (10.35) implies

$$\text{dist}\,(x_{p+i}, x_{p+k+i}') < C\delta^{p-i-1}, \quad i = 1,\ldots,t,$$

$$\text{dist}\,(x_{p+i}, x_{p+k+i}') < C\delta', \quad i = t+1,\ldots,m-p. \tag{10.54}$$

Finally, apply Lemma 10.2.1 to the sequences $x_1', x_2', \ldots, x_m'$ and $z_1, z_2, \ldots, z_m$. Instead of referring to Lemma 10.2.1, one can use the inequalities (10.35) only (cf. Remark 10.3.4). Then we find

$$\text{dist}\,(x_{rk+j}', z_j) \leq C(\delta^{rk+j} + \delta^{m+k-(rk+j)})$$

for $rk + j \leq p + k$. Since $m \geq 2p$, this implies

$$\text{dist}\,(x_{rk+j}', z_j) < C''\delta^{rk+j},\ rk+j \leq p+k,$$

where $C'' = C(1+\delta^{-k}) > C$. Set $D = C''\delta^p, a = 1/\delta > 1$, then (10.44) takes the form

$$\text{dist}\,(x_i, x_i') < Da^i,\ i = 1,\ldots,t.$$

Now we can apply Lemma 10.4.2 to the sequences $x_1,\ldots,x_t$ and $x_1',\ldots,x_t'$. Since

$$\|M_1 - A_1 M_1' A_1^{-1}\| = \|\tilde{\psi}_1 - A_1 \tilde{\psi}_1' A_1^{-1}\| < C'D(1+a)a < 2CC'\delta^{p-2},$$

it follows by (10.49) for $r = 1$ and $a_1 = 1/\delta$ that

$$\begin{aligned} &|\log\det(I + \lambda_{i+1}M_i) - \log\det(I + \lambda'_{i+1}M'_i)| \\ &< DE\delta^{-i} + 2(n-1)db^{2i-1}C'C\delta^{p-2} \\ &< CE'\delta_1^{p-i} + 2(n-1)dCC'\delta_1^{p+2i-3} < F_1\delta_1^{p-i} \end{aligned}$$

for all $i = 1,\ldots,t$, where $F_1 = CE' + 2(n-1)dCC'$ and

$$\delta_1 = \max\{b, \delta\} \in (0, 1).$$

Furthermore, observe that (10.52) implies

$$\text{dist}\,(x_i, x'_i) < C\delta^i, \quad i = 1,\ldots,t.$$

This and (10.53) show that Lemma 10.4.2 is applicable with $D = C, a = \delta = a_1$, $r = 1$. Then we get

$$|\log\det(I + \lambda_{i+1}M_i) - \log\det(I + \lambda'_{i+1}M'_i)| < F_1\delta_1^i, i = 1,\ldots,p. \tag{10.55}$$

Next, we apply Lemma 10.4.2 to the sequences $z_1,\ldots,z_{p+k}$ and $x'_1,\ldots,x'_{p+k}$. It follows by (10.43) that

$$\|M'_1\| = \|\tilde{\psi}'_1\| \leq \frac{2\mu_2}{\kappa}.$$

Therefore, (10.49) implies

$$|\log\det(I + \lambda_{p+j+1}M_{p+j}) - \log\det(I + \lambda''_{p+j+1}M''_{p+j})| < F_2\delta_1^{p+j}, \quad j = 1,\ldots,\kappa, \tag{10.56}$$

where $F_2 > 0$ is a constant.

Consider the sequences $x_1,\ldots,x_p$ and $x'_{k+1},\ldots,x'_{k+p}$, and the corresponding isometries $A'_j : \Pi'_{j+k} \to \Pi_j$ (cf. Lemma 10.4.1). Applying again Lemma 10.4.2 and the inequalities (10.35), we find a constant $F_3 > 0$ such that

$$\|M_p - A'_pM'_{p+k}A'^{-1}_p\| < F_3\delta_1^p. \tag{10.57}$$

Now we turn to the sequences $x_{p+1},\ldots,x_{p+t}$ and $x_{p+k+1},\ldots,x_{r+k+t}$. It follows by (10.54), (10.57) and Lemma 10.4.2 for $D = c\delta^p, a = 1/\delta = a_1, r = p$, that

$$\begin{aligned} &|\log\det(I + \lambda_{p+j+1}M_{p+j}) - \log\det(I + \lambda''_{p+k+j+1}M''_{p+k+j})| \\ &< F_4\delta_1^{p-j}, j = 1,\ldots,t, \end{aligned} \tag{10.58}$$

where $F_4 > 0$ is a constant.

Finally, applying Lemma 10.4.2 twice more, we find constants $F_5 > 0, F_6 > 0$ such that

$$|\log\det(I + \lambda_{p+j+1}M_{p+j}) - \log\det(I + \lambda''_{p+k+j+1}M''_{p+k+j})| \\ < F_5\delta_1^j, \;\; j = t+1,\dots, m-p, \tag{10.59}$$

$$|\log|\det\; M_m| - \log|\det\; M'_{m+k}|| < F_6\delta_1^p. \tag{10.60}$$

Set $F = \max\{F_1,\dots, F_6\}$ and

$$\tilde{c} = -\sum_{j=1}^{k}\log\det(I + \lambda''_{j+1}M''_j) < 0. \tag{10.61}$$

Using the matrix representations of $\mathrm{d}J_q(u_q)$ and $\mathrm{d}J_{q+1}(u_{q+1})$ from Proposition 2.4.2, one finds

$$\log|\det\; dJ_{q+1}(u_q)| = \log|\det\; dJ_q(u_q)| - \tilde{c} + \epsilon_{q,l}, \tag{10.62}$$

where

$$\epsilon_{q,l} = \sum_{i=1}^{p}(\log\det(I + \lambda'_{i+1}M'_i) - \log\det(I + \lambda_{i+1}M_i)) \\ + \sum_{i=1}^{k}\Big(\log\det(I + \lambda'_{p+i+1}M'_{p+1}) - \log\det(I + \lambda''_{p+i+1}M''_{p+i})\Big) \\ + \sum_{j=1}^{m-p-1}\Big(\log\det(I + \lambda''_{p+k+j+1}M''_{p+k+j}) - \log\det(I + \lambda_{p+j+1}M_{p+j})\Big) \\ + \big(\log|\det\; M'_{m+k}| - \log|\det\; M_m|\big)\,.$$

Now combining (10.55), (10.56), (10.58 –10.60), we obtain

$$|\epsilon_{q,l}| < F\left(2\sum_{i=1}^{t}\delta_1^{p-i} + \sum_{i=t+1}^{p}\delta_1^i + \sum_{i=1}^{k}\delta_1^{p+i} + \sum_{j=t+1}^{m-p-1}\delta_1^j + \delta_1^p\right) \\ < 6F(1-\delta_1)^{-1}\delta_1^t < F'_0\delta_0^{kq}, \tag{10.63}$$

with $\delta_0 = \delta_1^{1/4}$ and $F'_0 = 6F(1-\delta_1)^{-1}\delta_1^{-3/4}$.

Recall from the previous section that x_i^∞ are the reflection points of the trajectory $\gamma(u^\infty)$ and $r(u^\infty) = \infty$. Define the operators M_i^∞ in the same way as M_i, replacing the points x_i by x_i^∞. Set $\lambda_i^\infty = \mathrm{dist}\,(x_{i-1}^\infty, x_i^\infty)$ and

$$c_l = c_l(\omega,\theta) = \log|\det\; M_l^\infty| + \sum_{i=1}^{l-1}\log\det(I + \lambda_{i+1}^\infty M_i^\infty) + \sum_{j=1}^{\infty}\epsilon_{j,l}.$$

Then, applying (10.62) q times, we get

$$\log|\det\; dJ_q(u_q)| = -q\tilde{c} + c_1 + \delta_{q,l},$$

where

$$\delta_{q,l} = -\sum_{j=q}^{\infty} \epsilon_{j,l} + (\log|\det\ dJ_\beta(u_q)| - \log|\det\ dJ_\beta(u^\infty)|)$$

and $\beta = (i_1, \ldots, i_l)$. The expression in the parentheses on the right-hand side of the latter equality can be estimated from above with $F_0''\delta_0^{kq}$ for some constant $F_0'' > 0$. Therefore, by (10.63),

$$|\delta_{q,l}| < F_0'\delta_0^{kq}(1-\delta_0^k)^{-1} + F_0''\delta_0^{kq} = F_0\delta_0^{kq},$$

where $F_0 = F_0'(1-\delta_0^k)^{-1} + F_0'' > 0$.

Finally, set $c_\alpha = \frac{\tilde{c}}{2}$. As a consequence of the considerations in this section, we obtain the following.

Theorem 10.4.3: *There exist constant Q_α and $\delta_0, 0 < \delta_0 < 1$, depending only on K, α, ω and θ, such that*

$$\log|c_q| = qc_\alpha + Q_\alpha + \mathcal{O}(\delta_0^q) \text{ as } q \to \infty. \tag{10.64}$$

Let us note that the constant c_α has a certain geometrical meaning. Namely, we have

$$c_\alpha = -\frac{1}{2}\sum_{j=1}^{n-1}\log|\mu_j| = -\frac{1}{2}\log\left(\prod_{j=1}^{n-1}\mu_j\right), \tag{10.65}$$

$\mu_1, \ldots, \mu_{n-1}$ being the eigenvalues of the linear Poincaré map P_{γ_α} with modulus greater than 1. Indeed, the latter eigenvalues are precisely the eigenvalues of the operator S from the proof of Proposition 2.3.2. Using the representation of S found there, we get

$$\log\det\ S = \log\prod_{i=1}^{k}\det(I + \lambda_{i+1}''M_i'') = -\tilde{c} = -2c_\alpha,$$

which proves (10.65).

Notice that in the case of two strictly convex disjoint obstacles $K_1, K_2 \subset \mathbb{R}^3$, the constants $d = \text{dist}\ (K_1, K_2)$ and $c_0 = -\frac{1}{2}\log(\mu_1\mu_2)$ determine a sequences of resonances of the scattering matrix (see [I3], [Ger]) given by

$$z_m = \frac{m\pi}{d} + \mathbf{i}\frac{c_0}{2d}, m \in \mathbb{Z}.$$

10.5 Notes

Theorem 10.1.1 and Lemma 10.1.2, as well as the whole of Sections 10.3 and 10.4, are taken from [PS5]. Some results related to Theorem 10.3.5 are proved by Nakamura and Soga (cf. [NaI], [Na2], [NS]). Lemma 10.3.4 is well known in the theory of dispersing billiards. It seems that it was first proved for curves in the plane by

Sinai [Sinl], see also [Sin2]. The material in Section 10.2 is an adaptation of a part of Appendix 9(b) in [Sjo], perhaps it might be derived also from Section 3 in [I4]. As we have already mentioned in Remark 10.3.4, Lemma 10.2.3 is sufficient for our aims in this chapter. Namely, one can slightly exchange the arguments in Sections 10.3 and 10.4 to prove the same results using only Lemma 10.2.3 from Section 10.2. For two strictly convex obstacles, it is proved in [S9] a more precise result than Corollary 10.3.5. Namely, it was shown that we have an asymptotic with remainder $\mathcal{O}(\Lambda^q)$, where Λ is one of the Lyapunov exponent of the billiard ball map related to the shortest periodic ray. For the scattering poles for several strictly convex disjoint obstacles and the behaviour of the dynamical zeta function see [P3], [I3], [I4], [S10].

11

Poisson relation for the scattering kernel for generic directions

In this chapter we prove that the Poisson relation for the scattering kernel $s(t, \omega, \theta)$ becomes an equality for almost all pairs of unit vectors (ω, θ). Apart from the main results in Chapters 5 and 9, this requires a certain regularity property of the related generalized Hamiltonian flow. In Section 11.3 we prove that for each $T > 0$ the phase space of the generalized Hamiltonian flow can be represented as a countable union of compact subsets S_i such that on each S_i, $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ coincides with the restriction of a one-parameter family $\mathcal{G}_t^{(i)}$ of uniformly Lipschitz maps defined in a neighbourhood of S_i such that for all but finitely many t, $\mathcal{G}_t^{(i)}$ is smooth and its restriction to transversal sections is symplectic. For 'Sard's theorem-type' applications, this regularity property is as good as smoothness, and in particular implies that the generalized Hamiltonian flow preserves the Hausdorff dimension $\dim_H$ of Borel subsets of its phase space.

11.1 The Poisson relation for the scattering kernel

Let K be a compact subset of $\mathbb{R}^n, n \geq 2$, with C^∞ boundary ∂K such that $\Omega_K = \overline{\mathbb{R}^n \setminus K}$ is connected. As before, such a set K is called an *obstacle* in $\mathbb{R}^n$. Denote by $\mathcal{K}$ the *class of obstacles* K such that for $(x, \xi) \in T^*(\partial\Omega_K)$, if the curvature of ∂K at x vanishes of infinite order in direction ξ, then ∂K is convex at x

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in direction ξ. It should be mentioned that $\mathcal{K}$ is of second Baire category in the space of all obstacles with smooth boundaries endowed with the Whitney C^∞ topology.

The main result in this chapter is the following.

Theorem 11.1.1: *Let $K \in \mathcal{K}$. There exists a subset $\mathcal{R}$ of full Lebesgue measure in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ such that*

$$\text{sing supp}\, s_K(t, \theta, \omega) = \{ -T_\gamma : \gamma \in \mathcal{L}_{\omega,\theta}\}$$

holds for all $(\omega, \theta) \in \mathcal{R}$.

It can be seen from the proof that the set $\mathcal{R}$ is also of second Baire category in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$.

This theorem is related to some problems in inverse scattering theory. It shows that for most obstacles K the singularities of the scattering kernel are completely determined by some geometric objects – the scattering length spectrum of K. In general these objects are not enough to recover the obstacle. One may conjecture that for obstacles K satisfying a natural accessibility condition the scattering length spectrum completely determines K. This is indeed so for some special classes of obstacles (see Chapter 13).

To prove Theorem 11.1.1 we need some preparation.

Consider again an obstacle K in $\mathbb{R}^n$ and let $\Omega = \Omega_K$. It follows from results of Melrose and Sjöstrand [MS2] (see also [[H3], Theorem 24.3.9]) that every (ω, θ)-ray γ in Ω_K that does not contain gliding segments is a reflecting (ω, θ)-ray, that is it consists of finitely many straight line segments in Ω (two of them are in fact infinite rays). For such a ray γ we have (cf. Section 2.4 and [G1])

$$T_\gamma = \langle \omega, x_1 \rangle + \sum_{i=1}^{s-1} \|x_i - x_{i+1}\| - \langle \theta, x_s \rangle,$$

where $x_1, \dots, x_s$ are the successive reflection points of γ and $<, >$ denotes the standard inner product in $\mathbb{R}^n$.

Recall from Chapter 2 that a reflecting (ω, θ)-ray γ in Ω is called *ordinary* if γ has no tangencies to $\partial\Omega$, and an ordinary (ω, θ)-ray γ is called *non-degenerate* if $\det(dJ_\gamma) \neq 0$.

Proposition 11.1.2:

(a) There exists a set $\mathcal{R} \subset \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ the complement of which is a countable union of compact subsets of measure zero in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ such that for every pair $(\omega, \theta) \in \mathcal{R}$ all (ω, θ)-trajectories for $X = \partial K$ are ordinary.

(b) For every $\omega \in \mathbb{S}^{n-1}$ there exists a set $\mathcal{S}(\omega) \subset \mathbb{S}^{n-1}$ the complement of which is a countable union of compact subsets of $\mathbb{S}^{n-1}$ of measure zero such that if $\theta \in \mathcal{S}(\omega)$, then any ordinary reflecting (ω, θ)-ray in Ω is non-degenerate and any two different ordinary reflecting (ω, θ)-rays in Ω have distinct sojourn times.

(c) There exists a set $\mathcal{S} \subset \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ the complement of which is a countable union of compact subsets of measure zero in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ such that for every pair $(\omega, \theta) \in \mathcal{S}$ all (ω, θ)-trajectories for $X = \partial K$ are ordinary and non-degenerate, and any two different ordinary reflecting (ω, θ)-rays in Ω have distinct sojourn times.

We prove this proposition in Section 11.5.

Fix a hyperplane Z in $\mathbf{R}^n$ such that K is contained in one of the open half-spaces determined by Z.

As a simple corollary of our argument in Section 11.5 we get also the following result, which is in fact a consequence of a result of Melrose and Sjöstrand [MS2] (see also Chapter 24 in [H3]).

Proposition 11.1.3: *There exists $\mathcal{T} \subset Z \times \mathbb{S}^{n-1}$, the complement of which is a countable union of compact subsets of measure zero in $Z \times \mathbb{S}^{n-1}$, such that for every $(x, \omega) \in \mathcal{T}$ the trajectory of the generalized geodesic flow in Ω starting at x in direction ω has no tangencies to ∂K.*

For convenience of the reader we will now briefly recall some definitions from previous chapters that will be used here.

Let Ω be a domain in $\mathbb{R}^n$. Consider the symplectic manifold $\hat{S} = T^*(\Omega \times \mathbb{R})$ and the smooth function $\hat{p}(x, t; \xi, \tau) = \sum_{i=1}^{n} \xi_i^2 - \tau^2$. Since both vector fields $H_{\hat{p}}$ and $H_{\hat{p}}^G$ do not depend on t, we have $\tau = \text{const}$ along each generalized integral curve of $\hat{p}$. The change of τ can only affect the parameterization along a generalized integral curve which is not important for our aim. Thus, we may assume that $\tau = \pm 1$. There is a natural correspondence between the generalized integral curves of $\hat{p}$ with this property and the generalized integral curves of the Hamiltonian function

$$p(x, \xi) = \sum_{i=1}^{n} \xi_i^2 - 1 \tag{11.1}$$

on the symplectic manifold $S = T^*(\Omega)$; the correspondence being given by

$$(x(t), \xi(t)) \mapsto (x(t), t; \xi(t), \pm 1).$$

Let $\mathcal{F}_t$ be the generalized Hamiltonian flow on $T^*(\Omega) \setminus \{0\}$ generated by the function (11.1). Notice that the submanifold $\Sigma = p^{-1}(0)$ of S coincides with the *cosphere bundle* $S^*(\Omega)$. The restriction of $\mathcal{F}_t$ to $S^*(\Omega)$ will be called the *generalized geodesic flow*

$$\mathcal{F}_t = \mathcal{F}_t^{(K)} : S^*(\Omega) \longrightarrow S^*(\Omega), \quad t \in \mathbb{R}. \tag{11.2}$$

Let $\text{pr}_1 : T^*(\Omega) \longrightarrow \Omega$ and $\text{pr}_2 : T^*(\Omega) \longrightarrow \mathbb{R}^n$ be the natural projections. A curve γ in Ω is called a *generalized geodesic* in Ω if there exist an interval I and

$\rho \in T^*(\Omega)$ such that $\gamma = \{\mathrm{pr}_1(F_t(\rho)) : t \in I\}$. We will say that γ is a *gliding segment* on $\partial\Omega$ (resp. *reflecting ray* in Ω) if $\{F_t(\rho) : t \in I\}$ is a gliding trajectory (resp. reflecting trajectory) of p.

Next, assume that $\Omega = \Omega_K$ for some obstacle K in $\mathbb{R}^n$ and $\omega, \theta \in \mathbb{S}^{n-1}$. Let $\gamma = \{\mathrm{pr}_1(\Gamma(t)) : t \in \mathbb{R}\}$, where $\Gamma : \mathbb{R} \longrightarrow S^*(\Omega)$ is a generalized integral curve of p. The curve γ is called an (ω, θ)-ray in Ω if there exist real numbers $a < b$ such that $\mathrm{pr}_2(\Gamma(t)) = \omega$ for $t \leq a$ and $\mathrm{pr}_2(\Gamma(t)) = \theta$ for $t \geq b$. If γ is a reflecting ray, that is it does not contain gliding segments on $\partial\Omega$ and has only finitely many reflection points, it is called a *reflecting* (ω, θ)*-ray* in Ω. By $\mathcal{L}_{\omega,\theta}(K)$ we denote the *set of all* (ω, θ)*-rays in* Ω_K.

Fix an open ball $\mathcal{O}$ that contains K. Given $\xi \in \mathbb{S}^{n-1}$ denote by Z_ξ *the hyperplane in* $\mathbb{R}^n$ *orthogonal to* ξ *and tangent to* $\mathcal{O}$ such that $\mathcal{O}$ is contained in the open half-space R_ξ determined by Z_ξ and having ξ as an inner normal. Given an (ω, θ)-ray γ in Ω, *the sojourn time* T_γ of γ is defined by $T_\gamma = T'_\gamma - 2a$, where T'_γ is the length of that part of γ which is contained in $R_\omega \cap R_{-\theta}$ and a is the radius of the ball $\mathcal{O}$. It is known (cf. [G1] and Section 2.4) that this definition does not depend on the choice of the ball $\mathcal{O}$.

Given $T > 0$, denote by $\mathcal{T}_T$ *the set of those* $\rho \in \Sigma$ *such that*

$$\{\mathcal{F}_t(\rho) : 0 \leq t \leq T\} \bigcap G_g \neq \emptyset,$$

that is the trajectory $\{\mathcal{F}_t(\rho) : 0 \leq t \leq T\}$ contains a non-trivial gliding part on ∂S.

Recall that if (S, ω) is a symplectic manifold with a two-form ω, a submanifold L of S is called *isotropic* if $T_x L$ is isotropic for every $x \in L$, that is $\omega_x(u, v) = 0$ for all $u, v \in T_x L$. If L is isotropic and has maximal possible dimension, that is $\dim(L) = \frac{1}{2}\dim(S)$, then L is called *Lagrangian* (see e.g. Chapter 21 in [H3]).

For any metric space X denote by $\dim_H(X)$ the *Hausdorff dimension* of X (see e.g. [E]).

The following proposition is a special case of Theorem 11.4.1 proved below.

Proposition 11.1.4: *Let* $S = T^*(\Omega)$ *and let* $\mathcal{L}_0$ *be an isotropic submanifold of* $S \setminus \partial S$ *of dimension* $n - 1$ *such that* $H_p(\rho)$ *is not tangent to* $\mathcal{L}_0$ *at each* $\rho \in \mathcal{L}_0$. *Then for every* $T > 0$ *we have* $\dim_H \mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0) \leq n - 2$. *Moreover, if for a given* T *we have* $\mathcal{F}_T(\mathcal{L}_0) \subset S \setminus \partial S$, *then there exists a countable family* $\{\mathcal{I}_m\}$ *of smooth* $(n - 2)$*-dimensional isotropic submanifolds of* S *such that* $\mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0) \subset \bigcup_m \mathcal{I}_m$.

Proof of Theorem 11.1.1: Let K be an obstacle in $\mathbb{R}^n$ of the class $\mathcal{K}$. We are going to show that there exists a subset $\mathcal{R}$ of full Lebesgue measure in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ such that for each $(\omega, \theta) \in \mathcal{R}$ the only (ω, θ)-rays in Ω_K are reflecting (ω, θ)-rays.

Consider the domain $\Omega = \Omega_K$, the symplectic manifold $S = T^*(\Omega)$ and the corresponding generalized geodesic flow (11.2) of the function (11.1). As above, denote by $\mathcal{O}$ an open ball in $\mathbb{R}^n$ containing the obstacle K and by C the boundary sphere of $\mathcal{O}$. Fix $\omega \in \mathbb{S}^{n-1}$, $x_0 \in C$ and consider the generalized geodesic $(x(t), \xi(t)) = \mathcal{F}_t(x_0, \omega)$. Let $T > 0$ be such that $x(T) \in C$. Set $\mathcal{T} = \mathcal{T}_T$ and

$$S'_0 = \{(x, \xi) \in S : x \in C, \xi \text{ is transversal to } C\}.$$

Since $\Sigma = p^{-1}(0) = S^*(\Omega)$, using the notation

$$S^*_C(\Omega) = \{(x,\xi) \in S^*(\Omega) : x \in C\},$$

we have

$$S_0 = S_0' \cap \Sigma = \{(x,\xi) \in S^*_C(\Omega) : \xi \text{ is transversal to } C\}.$$

Then S_0 is a symplectic submanifold of S. Let $\mathcal{P} : S_0' \longrightarrow S_0'$ be the local map defined in a neighbourhood of (x_0, ω) using the shift along the flow $\mathcal{F}_t$; then $\mathcal{P}(S_0) \subset S_0$. Consider the Lagrangian submanifold

$$\mathcal{L}_0 = \{(x,\xi) \in S_0 : \xi = \omega\}$$

of S_0. Then $\mathcal{L}_0$ is an isotropic submanifold of S.

Applying Proposition 11.1.4 to $\mathcal{L}_0$ gives that $\mathcal{F}_T(\mathcal{L}_0 \cap \mathcal{T})$ is contained in a countable union of isotropic $(n-2)$-dimensional submanifolds of S. Since locally near (x_0, ω) the map $\mathcal{F}_T : S_0' \longrightarrow \mathcal{F}_T(S_0')$ is smooth, $\mathcal{F}_T(S_0')$ is a $(2n-1)$-dimensional submanifold of S transversal to the flow $\mathcal{F}_t$ at $\mathcal{F}_T(x_0, \omega)$. Consequently, locally near $\mathcal{F}_T(x_0, \omega) \in \mathcal{F}_T(S_0') \cap S_0'$, the shift $\mathcal{Q}$ along $\mathcal{F}_t$ from $\mathcal{F}_T(S_0')$ to S_0' is a smooth map. Moreover, $\mathcal{Q}$ maps $\mathcal{F}_T(S_0)$ into S_0 (since $p^{-1}(0)$ is invariant under the flow $\mathcal{F}_t$), the restriction $\mathcal{Q} : \mathcal{F}_T(S_0) \longrightarrow S_0$ is a local symplectic map, and $\mathcal{P} = \mathcal{Q} \circ \mathcal{F}_T$. Hence the set $\mathcal{P}(\mathcal{L}_0 \cap \mathcal{T}) = \mathcal{Q}(\mathcal{F}_T(\mathcal{L}_0 \cap \mathcal{T}))$ is contained in a countable union of isotropic $(n-2)$-dimensional submanifolds of S. The projection $j : S_0 \longrightarrow \mathbb{S}^{n-1}$, $j(x,\xi) = \xi$, is smooth, so Sard's theorem now gives that the set $j(\mathcal{P}(\mathcal{L}_0 \cap \mathcal{T}))$ has Lebesgue measure zero in $\mathbb{S}^{n-1}$. Hence there exists a neighbourhood U of x_0 in C and a subset

$$\mathcal{R}_\omega(U) = \mathbb{S}^{n-1} \setminus j(\mathcal{P}(\mathcal{L} \cap \mathcal{T}))$$

of full Lebesgue measure in $\mathbb{S}^{n-1}$ such that for $x \in U$ every generalized (ω,θ)-ray in Ω passing through x with $\theta \in \mathcal{R}_\omega(U)$ is a reflecting (ω,θ)-ray. Covering C by a countable family of neighbourhoods U_i, we find a subset $\mathcal{R}_\omega = \cap_{i=1}^\infty \mathcal{R}_\omega(U_i)$ of full Lebesgue measure in $\mathbb{S}^{n-1}$ such that every (ω,θ)-ray in Ω with $\theta \in \mathcal{R}_\omega$ is a reflecting (ω,θ)-ray. It now follows from Fubini's theorem that

$$\widetilde{\mathcal{R}} = \{(\omega,\theta) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} : \theta \in \mathcal{R}_\omega\}$$

is a subset of full Lebesgue measure in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$. Moreover, it is clear that for $(\omega,\theta) \in \widetilde{\mathcal{R}}$ all (ω,θ)-rays in Ω are reflecting ones.

According to Proposition 11.1.2, there exists a subset $\mathcal{R}$ of $\mathbb{S}^{n-1}$ of full Lebesgue measure such that for $(\omega,\theta) \in \mathcal{R}$ every reflecting (ω,θ)-ray in Ω_K is ordinary and non-degenerate and $T_\gamma \neq T_\delta$ whenever γ and δ are different reflecting (ω,θ)-rays in Ω_K. Intersecting $\mathcal{R}$ with $\widetilde{\mathcal{R}}$, we may simply assume that $\mathcal{R} \subset \tilde{\mathcal{R}}$. Then, given $(\omega,\theta) \in \mathcal{R}$, it follows from Theorem 9.1.2 that

$$-T_\gamma \in \text{sing supp } s_K(t,\theta,\omega)$$

for all $\gamma \in \mathcal{L}_{\omega,\theta}(\Omega_K)$. Combining this with Theorem 5.3.2 proves the theorem. ■

11.2 Generalized Hamiltonian flow

We begin with the definition of Melrose and Sjöstrand [MS1], [MS2] of the generalized Hamiltonian flow (GHF) in the symplectic invariant form given by Hörmander [H3] (see Section 24.3). This is rather similar to what we did in Section 1.2 in the case $S = T^*(\Omega)$, where $\Omega \subset \mathbb{R}^n$ for some $n \geq 2$. Here we deal with the general case of an arbitrary symplectic manifold.

Let S be a symplectic manifold with boundary ∂S, $\dim S = 2n$, $n \geq 2$, and let $p : S \longrightarrow \mathbb{R}$ be a smooth (C^∞) function. Let $\varphi \in C^\infty(S)$ be a *defining function* of ∂S, that is $\varphi > 0$ in $S \setminus \partial S$, $\varphi = 0$ on ∂S and $d\varphi \neq 0$ on ∂S (φ might be only locally defined near a compact part of ∂S). The first assumption that we make about S and p is the following.

A1. $dp_{|\partial S} \neq 0$ and $\{\varphi, \{\varphi, p\}\}(\rho) \neq 0$ whenever $\rho \in \partial S$ and $\{\varphi, p\}(\rho) = 0$.

Here $\{f, g\}$ denotes the *Poisson bracket* of f and g. Denote by H_p the *Hamiltonian vector field* determined by the function p and consider the following subsets of S:

$G = \{\sigma \in S : \varphi(\sigma) = H_p\varphi(\sigma) = 0\}$ (glancing set),

$G_d = \{\sigma \in G : H_p^2\varphi(\sigma) > 0\}$ (diffractive set),

$G_g = \{\sigma \in G : H_p^2\varphi(\sigma) < 0\}$ (gliding set),

$G^k = \{\sigma \in G : H_p^j\varphi(\sigma) = 0,\ j = 0, 1, \dots, k-1\}$ and

$G^\infty = \cap_{k=2}^\infty G^k$.

The *gliding vector field* H_p^G on G is defined by

$$H_p^G = H_p + \frac{H_p^2\varphi}{H_\varphi^2 p} H_\varphi.$$

In fact, H_p^G is a well-defined smooth vector field in a neighbourhood of G in S.

In order to properly define the GHF, one should be able to define a 'reflected trajectory' at a point $\rho \in \partial S$ where the flow of H_p hits transversally ∂S. This requires some sort of hyperbolic structure of H_p near such points $\rho \in \partial S$.

In what follows we make the following assumption about the symplectic manifold S and the function p:

A2. For every point $\rho_0 \in \partial S \cap p^{-1}(0)$ there exists an open neighbourhood $\mathcal{O}$ of ρ_0 in S and symplectic coordinates $(x, \xi) = (x_1, \dots, x_n; \xi_1, \dots, \xi_n)$ in $\mathcal{O}$ such that $\varphi(x, \xi) = x_1$ in $\mathcal{O}$, that is

$$S \cap \mathcal{O} = \{(x, \xi) : x_1 \geq 0\}, \quad \partial S \cap \mathcal{O} = \{(x, \xi) : x_1 = 0\},$$

and such that

$$p(x, \xi) = g(x, \xi)[\xi_1^2 - r(x, \xi')], \quad (x, \xi) \in \mathcal{O}, \tag{11.3}$$

for some smooth functions $g(x, \xi)$ and $r(x, \xi')$ with $|g(x, \xi)| \geq a > 0$ in $\mathcal{O}$ for some constant $a > 0$.

Here we use the notation $x' = (x_2, \dots, x_n)$, $\xi' = (\xi_2, \dots, \xi_n)$.

In all cases known to the authors where the generalized Hamiltonian flow has been involved (e.g. propagation of singularities for second-order linear differential operators; cf. [MS1], [MS2], [H3], [Tay] and the references there), the condition **A2** has been satisfied. Notice also that when $\rho_0 \in G$, then **A2** follows from **A1** and the Malgrange preparation theorem (see [MS1] or Section 24.3 in [H3]). In this case we may also assume that the coordinates (x, ξ) are centred at ρ_0, that is $\rho_0 = (0, 0)$.

The following definition is due to Melrose and Sjöstrand [MS1], [MS2]. Here we consider it in the form given by Hörmander (cf. Definition 24.3.6 in [H3]).

Definition 11.2.1: Let $I \subset \mathbb{R}$ be an interval. A curve $\Gamma : I \longrightarrow S$ is called a generalized integral curve of p if there exists a discrete subset B of I such that:

(i) if $t \in I \setminus B$ and $\Gamma(t) \in (S \setminus \partial S) \cup G_d$, then there exists $\Gamma'(t) = H_p(\Gamma(t))$;

(ii) if $t \in I \setminus B$ and $\Gamma(t) \in G \setminus G_d$, then there exists $\Gamma'(t) = H_p^G(\Gamma(t))$;

(iii) for each $t \in B$, $\Gamma(t + s) \in S \setminus \partial S$ for all small $s \neq 0$ and there exist the limits $\Gamma(t - 0) \neq \Gamma(t + 0)$, which are points of the same integral curve of φ on ∂S.

We will only consider integral curves on the *zero bicharacteristic set* $\Sigma = p^{-1}(0)$. For $k = 2, 3, \dots$ denote

$$G_-^k = \{\sigma \in G^k : H_p^k(\sigma) < 0\}, \quad G_+^k = \{\sigma \in G^k : H_p^k(\sigma) > 0\}.$$

The third assumption that we make about S and p is the following:

A3. $\Sigma \bigcap G^\infty \bigcap \overline{\bigcup_{k=2}^{\infty} G_-^k} = \emptyset.$

In this case one can define a (local) flow

$$\mathcal{F}_t = \mathcal{F}_t^{(K)} : \Sigma \longrightarrow \Sigma, \quad t \in \mathbb{R},$$

such that $\{\mathcal{F}_t(\sigma) : t \in \mathbb{R}\}$ is an integral curve of p for each $\sigma \in \Sigma$. This flow is called the *generalized Hamiltonian flow* (GHF) generated by p.

Let $\tilde{S} = S/\sim$ be the quotient space with respect to the following equivalence relation on S: $\rho \sim \sigma$ iff either $\rho = \sigma$ or $\rho \in S \cap \partial S$, $\sigma \in S \cap \partial S$ and ρ and σ lie on the same integral curve of φ on ∂S. $\tilde{S}$ carries a natural structure of a manifold with boundary. Using the natural map $\pi : S \longrightarrow \tilde{S}$, the flow $\mathcal{F}_t$ gives rise to another flow $\mathcal{F}_t : \tilde{S} \longrightarrow \tilde{S}$, called the *compressed Hamiltonian flow*. We will consider this flow on the invariant set $\tilde{\Sigma} = \pi(\Sigma)$. It follows from Theorem 3.22 in [MS2] that $\mathcal{F}_t$ is continuous on $\tilde{\Sigma}$.

Let $\Gamma : I \longrightarrow S$ be a generalized integral curve of p. We say that Γ is *gliding* on ∂S if the set of those $t \in I$ such that $\Gamma(t) \in G_g$ is dense in I. In this case the trajectory $\{\Gamma(t) : t \in I\}$ is called a *gliding trajectory* of p on ∂S. If $\Gamma(I) \cap G_g = \emptyset$, then Γ is called a *reflecting integral curve* of p and $\{\Gamma(t) : t \in I\}$ a *reflecting trajectory*.

Remark 11.2.2: The maps $\mathcal{F}_t$ depend on φ and in general φ is only locally defined near ∂S. However the integral curves of $\mathcal{F}_t$ are globally defined and do not depend on the choice of φ. Since the behaviour of $\mathcal{F}_t$ away from ∂S is trivial (a smooth Hamiltonian flow on a symplectic manifold without boundary), the emphasis here is on the study of $\mathcal{F}_t$ near ∂S.

The main result in this section is the following.

Theorem 11.2.3: *The generalized Hamiltonian flow $\mathcal{F}_t$ preserves the Hausdorff dimension of Borel subsets of the phase space Σ.*

We derive this from a technical result whose statement involves Lipschitz maps on subsets of Σ.

Without loss of generality, we may assume that S is part of a *symplectic manifold $\mathcal{V}$ of the same dimension and without boundary* and that p is a smooth function on $\mathcal{V}$. Denote by H_p the *Hamiltonian vector field* on $\mathcal{V}$ determined by p and by Φ_t the corresponding *smooth Hamiltonian flow* on $\mathcal{V}$. Clearly, if $\rho \in S \setminus \partial S$ and $\mathcal{F}_t(\rho) \in S \setminus \partial S$ for all $t \in I = (a, b)$, then $\mathcal{F}_t(\rho) = \Phi_t(\rho)$ for all $t \in I$. The apparent difference between $\mathcal{F}_t$ and Φ_t is that the latter is smooth and has no reflections at ∂S; in fact the trajectories of Φ_t can cross ∂S and enter $\mathcal{V} \setminus S$. Set[1] $\tilde{\mathcal{V}} = \mathcal{V}/\sim$, where $\sim$ is the same equivalence relation by means of which we defined $\tilde{S}$.

For every point $\rho \in \partial S$ there is a symplectic chart $\mathcal{O}$ in $\mathcal{V}$ with the properties described in **A2**. (In fact, $\mathcal{V}$ can be constructed by gluing such charts around ∂S.) There exists a *metric* d_0 on $\mathcal{V}$ that is equivalent to the standard metric $\|x - y\| + \|\xi - \eta\|$ on each chart $\mathcal{O}$. In what follows d_0 will denote a fixed metric on $\mathcal{V}$ with this property.

There exists a *pseudometric* d on $\mathcal{V}$ such that

$$\begin{aligned} c \min\{d_0(\rho', \sigma') : \pi(\rho') = \pi(\rho), \pi(\sigma') = \pi(\sigma)\} &\leq d(\rho, \sigma) \\ \leq C \min\{d_0(\rho', \sigma') : \pi(\rho') = \pi(\rho), \pi(\sigma') &= \pi(\sigma)\} \end{aligned} \tag{11.4}$$

for all $\rho, \sigma \in \mathcal{V}$, where $C > c > 0$ are constants. Given a coordinate open subset $\mathcal{O}$ of S defined by a Darboux chart as in **A2** with d_0 equivalent to

$$d_0'((x, \xi), (y, \eta)) = \|x - y\| + \|\xi - \eta\|$$

[1] This notation is introduced just for convenience; the set $\tilde{\mathcal{V}}$ does not have any geometric importance and will not be used significantly later.

on $\mathcal{O}$, we can define a *pseudometric* d' on $\mathcal{O}$, say by

$$d'((x,\xi),(y,\eta)) = \|x-y\| + \|x_1\xi - y_1\eta\| + \min\{\|\xi-\eta\|, \|\xi-\overline{\eta}\|\},$$

where $\overline{\eta} = (-\eta_1, \eta_2, \dots, \eta_n)$. Being the sum of three pseudometrics on $\mathcal{O}$, d' is a pseudometric, too. Moreover, (11.4) holds with d_0 and d replaced by d'_0 and d', respectively. Gluing appropriately the above locally defined pseudometrics, one gets a globally defined pseudometric d on $\mathcal{V}$ satisfying the condition (11.4). Then *the projection* $\tilde{d}$ of d to $\tilde{\mathcal{V}}$ is a metric. Since the projection of $\mathcal{F}_t$ to $\tilde{\Sigma}$ is continuous with respect to the metric $\tilde{d}$ ([MS2]), the flow $\mathcal{F}_t$ on S is continuous with respect to the pseudometric d.

Remark 11.2.4: One can easily see that for any Borel subset B of Σ, $\dim_H(B)$ calculated with respect to the metric d_0 is the same as $\dim_H(B)$ calculated with respect to the pseudometric d. To check this, it is enough to consider separately three cases: $B \subset S \setminus \partial S$ (trivial since d_0 is equivalent to d locally in $S \setminus \partial S$), $B \subset G$ (trivial since d_0 is equivalent to d on G) and $B \subset \partial S \setminus G$. Consider the last case. Then $B = B_- \cup B_+$, where $B_- = \{\sigma \in B : \sigma = \lim_{t\nearrow 0}\mathcal{F}_t(\sigma)\}$ and B_+ is defined similarly. Then $B_\pm$ are Borel subsets of Σ and it is enough to show that $\dim_H(B_\pm)$ is the same with respect to d_0 and d. However this follows trivially since d is a metric on each of the sets $B_\pm$ equivalent to d_0. From the last case one also obtains that for any Borel subset B of Σ we have $\dim_H(B) = \dim_H(\pi(B))$.

Given $\sigma \in S$, denote

$$\ell(\sigma) = \{\mathcal{F}_t(\sigma) : 0 \le t \le T\}.$$

Theorem 11.2.5: *Let $T > 0$. There exists a representation of Σ as a countable union $\Sigma = \cup_{i\in I} S_i$ of Borel subsets S_i such that for each $i \in I$ there exist an open neighbourhood V_i of S_i in $\mathcal{V}$ and a family of maps*

$$\mathcal{G}_t^{(i)} : V_i \longrightarrow \mathcal{V}, \quad 0 \le t \le T,$$

with the following properties:

(a) $\mathcal{G}_t^{(i)}(\sigma) = \mathcal{F}_t(\sigma)$ for all $\sigma \in S_i$ and all $t \in [0,T]$;

(b) For every $\sigma \in S_i$ and every $t \in (0,T]$, there exists an open neighbourhood $W = W(\sigma,i,t)$ of σ in V_i such that $\mathcal{G}_t^{(i)} : (W, d_1) \longrightarrow (\mathcal{V}, d_2)$ is Lipschitz, where $d_1 = d_0$ if $\sigma \in (S\setminus\partial S)\cup G$ and $d_1 = d$ if $\sigma \in \partial S \setminus G$, and similarly $d_2 = d_0$ if $\mathcal{F}_t(\sigma) \in (S\setminus\partial S)\cup G$ and $d_2 = d$ if $\mathcal{F}_t(\sigma) \in \partial S\setminus G$;

(c) If $\sigma \in S_i \cap [(S\setminus\partial S)\cup G]$ and $t \in (0,T]$ is such that $\mathcal{F}_t(\sigma) \in (S\setminus\partial S)\cup G$, then there exists an open neighbourhood $W = W(\sigma,i,t)$ of σ in V_i such that the map $\mathcal{G}_t^{(i)} : W \longrightarrow \mathcal{V}$ is smooth. If moreover both σ and $\mathcal{F}_t(\sigma)$ are not ends of gliding segments of $\{\mathcal{F}_s(\sigma) : s \in [-\epsilon, T+\epsilon]\}$ for any small $\epsilon > 0$, then W can be chosen in such a way that the restriction of $\mathcal{G}_t^{(i)}$ to any smooth cross section in W at σ is a contact transformation.

The latter means that if $\mathcal{M}$ is a smooth local submanifolds of W of codimension 1 containing σ and transversal to $\ell(\sigma)$, then $\mathcal{G}_t^{(i)}{}_{|\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{G}_t^{(i)}(\mathcal{M})$ is a contact (canonical) transformation with respect to the standard contact structures on $\mathcal{M}$ and $\mathcal{G}_t^{(i)}(\mathcal{M})$ inherited from the symplectic structure of $\mathcal{V}$ (cf. e.g. Section 5.2 and Proposition 8.1.3 in [AbM]). In particular, $\mathcal{M} \cap \Sigma$ and $\mathcal{G}_t^{(i)}(\mathcal{M} \cap \Sigma)$ are symplectic submanifolds of $\mathcal{V}$ of codimension 2 and the restriction

$$\mathcal{G}_t^{(i)} : \mathcal{M} \cap \Sigma \longrightarrow \mathcal{G}_t^{(i)}(\mathcal{M} \cap \Sigma)$$

is a symplectic map. However, in general, Σ is not invariant under $\mathcal{G}_t^{(i)}$, that is $\mathcal{G}_t^{(i)}(V_i \cap \Sigma)$ is not necessarily a subset of Σ.

It follows from [MS2] that for any given $\sigma \in \Sigma$ the trajectory $\ell(\sigma)$ has only finitely many transversal reflection points and finitely many gliding segments, so part (c) in the above theorem concerns all but finitely many $t \in [0, T]$. It can be seen from the proof of Theorem 11.2.5 that $\mathcal{G}_t^{(i)}$ is actually a 'finite combination' of local Hamiltonian flows in $\mathcal{V}$.

Clearly Theorem 11.2.3 would have been trivial (and Theorem 11.2.5 would have been unnecessary for its proof) if the maps $\mathcal{F}_t$ were Lipschitz. However, it is well known and easy to see that this is not the case. Locally near a point $\rho \in \tilde{S}$ the map $\mathcal{F}_t$ is Lipschitz on a neighbourhood of ρ for small $|t|$ when $\rho \notin \partial S$ or ρ is a transversal reflection point. Whenever $\rho \in G$, the map $\mathcal{F}_t$ is not Lipschitz (cf. [MS1] or [H3]). For example, in the simplest case of a diffractive tangent point $\rho \in G_d$, the map $\mathcal{F}_t$ has a singularity of 'square root type' at ρ, so it is clearly not Lipschitz.

Theorem 11.2.5 is proved in Section 11.3.

Proof of Theorem 11.2.3: It is enough to show that for each t the map $\mathcal{F}_t : \Sigma \longrightarrow \Sigma$ does not increase the Hausdorff dimension of Borel subsets; then using the same property for $\mathcal{F}_{-t}$, one concludes that $\mathcal{F}_t$ actually preserves $\dim_H$. For a similar reason it is enough to consider the case $t > 0$.

Let B be a Borel subset of Σ and let $t > 0$ be a fixed number. We have to show that $\dim_H(\mathcal{F}_t(B)) \leq \dim_H(B)$. From the properties of Hausdorff dimension (cf. e.g. [E]) we have $\dim_H(B) = \max_{1 \leq i \leq 3} \dim_H(B_i)$, where $B = B_1 \cup B_2 \cup B_3$ with $B_1 = B \setminus \partial S$, $B_2 = B \cap G$, $B_3 = (B \cap \partial S) \setminus G$. So, it is enough to prove that $\dim_H(\mathcal{F}_t(B_i)) \leq \dim_H(B_i)$ for $i = 1, 2, 3$. This essentially means that we have to consider separately three cases: $B \subset S \setminus \partial S$, $B \subset G$ and $B \subset \partial S \setminus G$.

First, assume that $B \subset S \setminus \partial S$. Take an arbitrary $T > t$ and let $\Sigma = \cup_{i \in I} S_i$ be a representation of Σ as a countable union of Borel subsets S_i of S with the properties listed in Theorem 11.2.5. To prove $\dim_H(\mathcal{F}_t(B)) \leq \dim_H(B)$, it is enough to show that $\dim_H(\mathcal{F}_t(B \cap S_i)) \leq \dim_H(B \cap S_i)$ for each i, for which in turn it is enough for each $\sigma \in B \cap S_i$ to find an open neighbourhood U of σ in S such that

$$\dim_H(\mathcal{F}_t(B \cap S_i \cap U)) \leq \dim_H(B \cap S_i \cap U).$$

Fix for a moment i and $\sigma \in B \cap S_i$. Then by Theorem 11.2.5(b) there exists a neighbourhood W of σ in V_i such that $\mathcal{G}_t^{(i)} : (W, d_0) \longrightarrow (\mathcal{V}, d_2)$ is Lipschitz, where $d_2 =$

d or d_0 depending on where $\mathcal{F}_t(\sigma)$ belongs. Since $d \leq \text{const}\ d_0$, it follows that $\mathcal{G}_t^{(i)} : (W, d_0) \longrightarrow (\mathcal{V}, d)$ is Lipschitz. Moreover, we can take W such that $\overline{W}$ is compact and has no common points with ∂S. Then d_0 is equivalent to d on W and so $\mathcal{G}_t^{(i)} : (W, d) \longrightarrow (\mathcal{V}, d)$ is Lipschitz. Using this and the fact that $\mathcal{G}_t^{(i)} = \mathcal{F}_t$ on S_i (see condition (a) in Theorem 11.2.5), we get

$$\dim_H(\mathcal{F}_t(B \cap S_i \cap W)) = \dim_H(\mathcal{G}_t^{(i)}(B \cap S_i \cap W)) \leq \dim_H(B \cap S_i \cap W).$$

Set $U = W \cap \Sigma$; then U is a neighbourhood of σ in Σ having the desired property. This completes the proof in the case $B \subset S \setminus \partial S$.

The case $B \subset G$ is very similar to the first one – since d_0 is equivalent to d on G, one can use Theorem 11.2.5(b) again as above.

Finally, consider the case $B \subset \partial S \setminus G$. It is enough for each $\sigma \in B \cap S_i$ to find an open neighbourhood U of σ in Σ such that $\dim_H(\mathcal{F}_t(B \cap S_i \cap U)) \leq \dim_H(B \cap S_i \cap U)$. By Theorem 11.2.5(b), there exists an open neighbourhood W of σ in V_i (the domain of $\mathcal{G}_t^{(i)}$) such that $\mathcal{G}_t^{(i)} : (W, d) \longrightarrow (\mathcal{V}, d_2)$ is Lipschitz, where again $d_2 = d_0$ or $d_2 = d$. As in the first case, one concludes that $\mathcal{G}_t^{(i)} : (W, d) \longrightarrow (\mathcal{V}, d)$ is Lipschitz and that

$$\dim_H(\mathcal{F}_t(B \cap S_i \cap U)) \leq \dim_H(B \cap S_i \cap U),$$

where $U = W \cap \Sigma$. This completes the proof of $\dim_H(\mathcal{F}_t(B)) \leq \dim_H(B)$. ■

Using Theorem 11.2.3, we will now show that most rays incoming from infinity are not trapped by the obstacle K.

Denote by $\Omega_{\hat{K}}$ the closure of the complement of the *convex hull* $\hat{K}$ of K. In the following it is important that we consider points $(x, \xi) \in S^*(\Omega_{\hat{K}})$. In general, it is not clear at all whether the points $(x, \xi) \in S^*(\Omega_K)$ (with $x \in \hat{K}$) that generate bounded trajectories form a set of Lebesgue measure zero in $S^*(\Omega_K)$. The example of M. Livshitz (see Chapter 13) shows that for some obstacles K, there is a non-trivial open set of elements (x, ξ) in $S^*(\Omega_K)$ that generate bounded (trapped) trajectories.

Proposition 11.2.6: *If $K \in \mathcal{K}$, then the set of those $(x, \xi) \in S^*(\Omega_{\hat{K}})$ such that the trajectory $\{\mathcal{F}_t(x, \xi) : t \geq 0\}$ is bounded has Lebesgue measure zero in $S^*(\Omega_{\hat{K}})$.*

Proof of Proposition 11.2.6: Let $K \in \mathcal{K}$, let $\mathcal{O}$ be an open ball containing K and C be the boundary sphere of $\mathcal{O}$. Set $\Omega = \Omega_K$. For $(x, \omega) \in S_C^*(\Omega)$, let $\delta(x, \omega)$ be the generalized geodesic in Ω_K issued from x in direction ω. Assume that there exists a subset W of positive Lebesgue measure in $S_C^*(\Omega)$ such that $\delta(x, \omega) \subset \mathcal{O}$ for all $(x, \omega) \in W$. According to Theorem 11.2.5 and Proposition 11.1.2, we may assume that for all $(x, \omega) \in W$, the generalized geodesic $\delta(x, \omega)$ does not contain gliding segments on $\partial\Omega$ and has only transversal reflections at ∂K. Given $(x, \omega) \in W$, denote by x' the first common point of $\delta(x, \omega)$ with ∂K and by ω' the reflected direction of $\delta(x, \omega)$ at x', that is

$$\omega' = \omega - 2\langle \omega, N_K(x') \rangle N_K(x').$$

Then the set $W' = \{(x', \omega') : (x, \omega) \in W\}$ is a subset of positive Lebesgue measure in $S^*_{\partial K}(\Omega)$.

Denote by M the set of those $(y, \eta) \in S^*_{\partial K}(\Omega)$ for which the billiard ball map B is well defined (see the general definition of a billiard in Section 4.2; see also Ch. 6 of [CFS]). Moreover, B preserves the so-called Liouville's measure μ on M which is equivalent to the Lebesgue measure.

Next, we use the argument from the proof of the Poincaré recurrence theorem (see [CFS]). It follows from the definition of W' that $B^k(W') \subset M$ and $\mu(B^k(W')) = \mu(W') > 0$ for all $k = 0, 1, 2, \ldots$. On the other hand, in the situation under consideration, we clearly have $\mu(\cup_{k=0}^{\infty} B^k(W')) < \infty$. Therefore, there exist non-negative integers $k < m$ with $B^k(W') \cap B^m(W') \neq \emptyset$. Since B is invertible, this means that there exists $(x', \omega') \in W' \cap B^{m-k}(W')$. Then $(x', \omega') = B(y, \eta)$ for some $(y, \eta) \in B^{m-k-1}(W') \subset M$. Now the choice of W and the definition of W' show that W' has no common points with $B(M)$. This is a contradiction that proves the proposition. ■

11.3 Invariance of the Hausdorff dimension

This section is devoted to the proof of Theorem 11.2.5. Throughout this section S will be a $2n$-dimensional symplectic manifold and p a smooth function of S satisfying the conditions **A1**, **A2** and **A3**, and $T > 0$ will be a fixed real number.

Let $\rho_0 \in \partial S \setminus G$. There exist a neighbourhood U of ρ_0 in S and $T' > 0$ such that for every $\rho \in U$ the trajectory $\{\mathcal{F}_t(\rho) : 0 \leq t \leq T'\}$ has at most one common point with ∂S that is a transversal reflection point. In fact, taking $T' > 0$ and U sufficiently small, for every $\rho \in U$ there exists a unique real number $t(\rho)$ with $|t(\rho)| < T'$ such that $\mathcal{F}_{t(\rho)}(\rho) \in \partial S$.

Lemma 11.3.1: *Under the assumptions above, if the neighbourhood U is taken sufficiently small, then the family of maps*

$$\mathcal{F}_t : U \longrightarrow S, \quad 0 \leq t \leq T',$$

is uniformly Lipschitz with respect to the pseudometric d on S. That is, there exists a constant $C > 0$ such that $d(\mathcal{F}_t(\rho), \mathcal{F}_t(\sigma)) \leq C d(\rho, \sigma)$ for all $\rho, \sigma \in U$ and all $t \in [0, T']$.

Proof: It is enough to show that the map $U \ni \rho \mapsto t(\rho)$ is uniformly Lipschitz with respect to the pseudometric d. The rest follows from the smoothness of the Hamiltonian flow of H_p, its transversality to ∂S at ρ_0, and the fact that the pseudometric d is equivalent to the metric d_0 on any subset W of S such that $\sigma = \lim_{t \searrow 0} \mathcal{F}_t(\sigma)$ for any $\sigma \in W$ (or $\sigma = \lim_{t \nearrow 0} \mathcal{F}_t(\sigma)$ for any $\sigma \in W$).

Let $\mathcal{O}$ be a coordinate neighbourhood of ρ_0 of the type described in **A2**. Then for $\rho_0 = (x^{(0)}, \xi^{(0)})$ we have $\xi_1^{(0)} \neq 0$. Take U so small that $|\xi_1| > \dfrac{2|\xi_1^{(0)}|}{3}$ for every $\rho = (x, \xi) \in U$. Notice that since $|\xi_1|$ is uniformly bounded from below, we have

$|t(\rho)| \leq \text{const } x_1$ for all $\rho = (x, \xi) \in U$, where const means a positive constant that does not depend on ρ (and σ later on).

Given $\rho = (x, \xi) \in U$ and $\sigma = (y, \eta) \in U$, we have to show that

$$|t(\rho) - t(\sigma)| \leq \text{const } d(\rho, \sigma). \tag{11.5}$$

If both $t(\rho)$ and $t(\sigma)$ are non-negative or non-positive, this follows again from the smoothness of the flow of H_p. Assume $t(\rho) > 0$ and $t(\sigma) < 0$ (the other remaining case is similar). Then $\xi_1 < 0$ and $\eta_1 > 0$. It follows from the main property of d that $|x_1\xi_1 - y_1\eta_1| \leq \text{const } d(\rho, \sigma)$. Hence $x_1 + y_1 \leq \text{const } d(\rho, \sigma)$, so $x_1 \leq \text{const } d(\rho, \sigma)$ and $y_1 \leq \text{const } d(\rho, \sigma)$. This implies

$$|t(\rho)| \leq \text{const } x_1 \leq \text{const } d(\rho, \sigma),$$

and similarly $|t(\sigma)| \leq \text{const } \tilde{d}(\rho, \sigma)$. Thus, (11.5) holds in all possible cases for $t(\rho)$ and $t(\sigma)$. ■

To every $\rho \in \Sigma$, we will now associate a string

$$\alpha = (k_0, k_1, \ldots, k_m, k_{m+1}; l_0, l_1, \ldots, l_m, l_{m+1}; q_0, q_1, \ldots, q_m; q), \tag{11.6}$$

of integers that roughly describes the geometry of the trajectory $\ell(\rho)$. For example, m will be the number of different gliding segments contained in the interior of $\ell(\rho)$, k_i and l_i will be the orders of tangency of $\ell(\rho)$ to ∂S at the initial and terminal point of the ith gliding segment, and q_i will be the number of transversal reflections of $\ell(\rho)$ between the ith and the $(i+1)$st gliding segments. The numbers k_0, l_0, k_{m+1}, l_{m+1} will describe the combinatorial type of $\ell(\rho)$ at its initial and terminal points. For example, if $\rho \notin \partial S$, we will have $k_0 = l_0 = -1$; if $\rho \in \partial S \setminus G$, then $k_0 = l_0 = 0$; if $\ell(\rho)$ begins with a gliding segment, then k_0 and l_0 will be the orders of tangency of this segment to ∂S at its initial and terminal points, etc. The pair k_{m+1}, l_{m+1} will play a similar role at the end of the trajectory $\ell(\rho)$. Finally, $1/q$ will be (roughly speaking) a lower bound of the distance to the set G at any transversal reflection of $\ell(\rho)$ at ∂S.

For the precise definition it is better to start with a given α and define the set of points $\rho \in \Sigma$ whose type is represented by α.

Notice that a point $\rho \in S$ *belongs to a gliding segment* if there exist $a < b$ such that $0 \in [a, b]$ and $\{\mathcal{F}_t(\rho) : a \leq t \leq b\}$ is a gliding segment on ∂S (cf. Section 11.2). Then $\rho \in G$ but in general we do not necessarily have $\rho \in G_g$. However, according to condition **A3**, we do have $\rho \notin G^\infty$, hence $\rho \in G^k \setminus G^{k+1}$ for some $k \geq 2$.

Let (11.6) be a string of integers, where $m = m(\alpha) \geq 0$, $k_i \geq 3$, $l_i \geq 3$ $(1 \leq i \leq m)$, $k_0, l_0, k_{m+1}, l_{m+1} \geq -1$, $q_i \geq 0$ $(0 \leq i \leq m)$, and $q \geq 1$. We will say that α is *admissible* if whenever $k_0 \leq 1$ (resp. $l_{m+1} \leq 1$) we have $l_0 = k_0$ (resp. $k_{m+1} = l_{m+1}$) and when $k_0 \geq 2$ (resp. $l_{m+1} \geq 2$) we have $l_0 \geq 2$ (resp. $k_{m+1} \geq 2$).

Definition 11.3.2: Let α be an admissible string of the form (11.6). Denote by S_α the set of those $\rho \in \Sigma$ for which there exist a sequence of real numbers

$$0 = t_0(\rho) \leq s_0(\rho) < t_1(\rho) < s_1(\rho) < \ldots < t_m(\rho) < s_m(\rho) < t_{m+1}(\rho) \leq s_{m+1}(\rho) = T \tag{11.7}$$

with the following properties:

(i) For every $i = 0, 1, \dots, m$ the curve $\{\mathcal{F}_t(\rho) : t \in [s_i(\rho), t_{i+1}(\rho)]\}$ has exactly q_i transversal reflections at ∂S and no common points with G_g;

(ii) For all $i = 0, 1, \dots, m, m+1$, $\{\mathcal{F}_t(\rho) : t \in [t_i(\rho), s_i(\rho)]\}$ is an integral curve of the vector field H_p^G on G and $\mathcal{F}_t(\rho) \in G_g$ for almost all $t \in [t_i(\rho), s_i(\rho)]$;

(iii) For every $i = 1, \dots, m$ we have $\mathcal{F}_{t_i(\rho)}(\rho) \in G^{k_i} \setminus G^{k_i+1}$ and $\mathcal{F}_{s_i(\rho)}(\rho) \in G^{l_i} \setminus G^{l_i+1}$;

(iv) If $k_0 \le 1$, then $t_0(\rho) = s_0(\rho) = 0$ and: $\rho \notin \partial S$ for $k_0 = -1$, $\rho \in \partial S \setminus G$ for $k_0 = 0$, $\rho \in G$ but ρ does not belong to a gliding segment for $k_0 = 1$. If $k_0 \ge 2$, then ρ belongs to a gliding segment, $\rho \in G^{k_0} \setminus G^{k_0+1}$ and $\mathcal{F}_{s_0(\rho)}(\rho) \in G^{l_0} \setminus G^{l_0+1}$;

(v) If $l_{m+1} \le 1$, then $t_{m+1}(\rho) = s_{m+1}(\rho) = T$ and: $\mathcal{F}_T(\rho) \notin \partial S$ for $l_{m+1} = -1$, $\mathcal{F}_T(\rho) \in \partial S \setminus G$ for $l_{m+1} = 0$, $\mathcal{F}_T(\rho) \in G$ but $\mathcal{F}_T(\rho)$ does not belong to a gliding segment for $l_{m+1} = 1$. If $l_{m+1} \ge 2$, then $\mathcal{F}_T(\rho)$ belongs to a gliding segment, $\mathcal{F}_{t_{m+1}(\rho)}(\rho) \in G^{k_{m+1}} \setminus G^{k_{m+1}+1}$ and $\mathcal{F}_T(\rho) \in G^{l_{m+1}} \setminus G^{l_{m+1}+1}$.

(vi) For every $t \in [0, T]$ such that $\mathcal{F}_t(\rho) \in \partial S \setminus G$ we have $d(\mathcal{F}_t(\rho), G) \ge \frac{1}{q}$.

One can check that each S_α is a Borel subset of Σ (this can be derived using arguments from the proof of Lemma 11.3.5). Notice that some of the sets S_α may be empty and any $\rho \in \Sigma$ belongs to many (in fact, infinitely many) S_α.

Remark 11.3.3: Notice that condition (i) **does not** exclude the possibility that $\{\mathcal{F}_t(\rho) : t \in (s_i(\rho), t_{i+1}(\rho))\}$ has some other common points with ∂S apart from the q_i transversal reflections. In general $\partial S \cap \{\mathcal{F}_t(\rho) : t \in (s_i(\rho), t_{i+1}(\rho))\}$ may be a very complicated set (e.g. a Cantor set). Most of the points in this set (in fact all except the q_i transversal reflections) will be points from the set $G^\infty \cup \cup_{k=2}^\infty G_+^k$, which according to condition **A3** is far from G_g. Because of this possibility the construction of the maps $\mathcal{G}_t^{(i)}$ is a bit more complicated than perhaps anticipated.

Lemma 11.3.4: *We have $\Sigma = \cup_\alpha S_\alpha$, where α runs over all admissible strings.*

Proof: Let $\rho \in \Sigma$. It follows from [MS1] that $\{\mathcal{F}_t(\rho) : 0 \le t \le T\}$ has only finitely many transversal reflections at ∂S and finitely many gliding segments on ∂S. Take a small $\epsilon > 0$ and let

$$E = \overline{\{t \in [-\epsilon, T+\epsilon] : \mathcal{F}_t(\rho) \in G_g\}} \cap [0, T].$$

Then E is a finite disjoint union of closed subintervals of the interval $[0, T]$. If $0 \notin E$, set $s_0(\rho) = 0$ and $k_0 = l_0 = -1$ if $\rho \notin \partial S$; $k_0 = l_0 = 0$ if $\rho \in \partial S \setminus G$; $k_0 = l_0 = 1$ if $\rho \in G$.

If $0 \in E$, then $[0, s_0(\rho)]$ is a connected component of E for some $s_0(\rho) \ge 0$ (i.e. ρ belongs to a gliding segment). Consequently, there exist $k_0 \ge 2$ and $l_0 \ge 2$ such that

$\rho \in G^{k_0} \setminus G^{k_0+1}$ and $\mathcal{F}_{s_0(\rho)}(\rho) \in G^{l_0} \setminus G^{l_0+1}$. Notice that $s_0(\rho) > 0$ implies $l_0 \geq 3$ (cf. Section 24.3 in [H3]), while $0 = s_0(\rho)$ yields $k_0 = l_0 \geq 3$. This defines completely the pair of integers k_0, l_0. In a similar way one defines the pair k_{m+1}, l_{m+1}.

Since $\ell(\rho)$ has only finitely many transversal reflections, there exists an integer $q \geq 1$ such that $d(\mathcal{F}_t(\rho), G) \geq \frac{1}{q}$ whenever $\mathcal{F}_t(\rho)$ is a transversal reflection point $(0 \leq t \leq T)$. If every connected component of E contains either 0 or T, set $m = 0$ and $\alpha = (k_0, k_1; l_0, l_1; q_0; q)$, with k_0, l_0 and k_1, l_1 already defined and q_0 being the number of reflections of $\ell(\rho)$.

Assume that the union of connected components of E that do not contain 0 or T is not empty; then it has the form $\cup_{i=1}^m [t_i(\rho), s_i(\rho)]$. For each $i = 1, \ldots, m$, according to assumption **A3** again, there exist integers $l_i \geq 3$, $k_i \geq 3$ such that condition (iii) in Definition 11.3.2 holds. Finally, denote by q_i the number of transversal reflections of $\{\mathcal{F}_t(\rho) : s_i(\rho) < t < t_{i+1}(\rho)\}$ at ∂S and define α by (11.6). Then $\rho \in S_\alpha$ which proves the assertion. ■

Theorem 11.2.5 follows immediately from the following.

Lemma 11.3.5: *Let α be an admissible string of the form (11.6). For every $\rho \in S_\alpha$ there exist an open neighbourhood $V(\alpha, \rho)$ of ρ in $\mathcal{V}$ and a family of maps $\mathcal{G}_t^{(\alpha,V)} : V(\alpha, \rho) \longrightarrow \mathcal{V}$, $0 \leq t \leq T$, such that:*

(a) $\mathcal{G}_t^{(\alpha,V)}(\sigma) = \mathcal{F}_t(\sigma)$ for all $\sigma \in S_\alpha \cap V(\alpha, \rho)$ and all $t \in [0, T]$;

(b) For every $\sigma \in S_\alpha \cap V(\alpha, \rho)$ and every $t \in (0, T]$ there exists an open neighbourhood $W = W(\sigma, \alpha, t)$ of σ in $V(\alpha, \rho)$ such that $\mathcal{G}_t^{(\alpha,V)} : (W, d_1) \longrightarrow (\mathcal{V}, d_2)$ is Lipschitz, where d_1 and d_2 are as in Theorem 11.2.5(b);

(c) If $\sigma \in S_\alpha \cap V(\alpha, \rho) \cap [(S \setminus \partial S) \cup G]$ and $t \in (0, T]$ are such that $\mathcal{F}_t(\sigma) \in (S \setminus \partial S) \cup G$, then there exists an open neighbourhood $W = W(\sigma, \alpha, t)$ of σ in $V(\alpha, \rho)$ such that the map $\mathcal{G}_t^{(\alpha,V)} : W \longrightarrow \mathcal{V}$ is smooth. If moreover both σ and $\mathcal{F}_t(\sigma)$ are not ends of gliding segments of $\{\mathcal{F}_s(\sigma) : s \in [-\epsilon, T + \epsilon]\}$ for any $\epsilon > 0$, then W can be chosen in such a way that the restriction of $\mathcal{G}_t^{(\alpha,V)}$ to any smooth local cross section at σ in W is a contact transformation.

Proof of Theorem 11.2.5: For each $\rho \in S_\alpha$ fix a neighbourhood $V(\alpha, \rho)$ and a family of maps $\mathcal{G}^{(\alpha,V)}$ as in the above lemma. For each α there exists a countable open cover of S_α consisting of sets of the form $V(\alpha, \rho_j(\alpha))$, $j = 1, 2, \ldots$. Then the set I of pairs $i = (\alpha, j)$ with α an admissible string of the form (11.6) and j a positive integer is countable. For each $i = (\alpha, j)$ set $V_i = V(\alpha, \rho_j(\alpha))$, $S_i = S_\alpha \cap V_i$ and $\mathcal{G}_t^{(i)} = \mathcal{G}_t^{(\alpha,\rho_j(\alpha))}$ $(0 \leq t \leq T)$. According to Lemma 11.3.5, these objects have all the properties required in Theorem 11.2.5. ■

The rest of this section is devoted to the proof of Lemma 11.3.5.

Before we go on, let us briefly describe the idea of the construction of the maps $\mathcal{G}_t^{(\alpha,V)}$.

Recall the gliding vector field H_p^G from Section 11.2. We will slightly change it to make a Hamiltonian vector field in $\mathcal{V}$. The function $\dfrac{H_p^2\varphi}{H_\varphi^2 p}$ is well defined and smooth near ∂S. Fix an arbitrary smooth extension f of $\dfrac{H_p^2\varphi}{H_\varphi^2 p}$ to $\mathcal{V}$ and denote

$$\hat{p} = p + f\varphi,$$

thus obtaining another smooth function on $\mathcal{V}$. Notice that $\hat{p} = p$ on ∂S, so $p^{-1}(0) \cap \partial S = \hat{p}^{-1}(0) \cap \partial S$. Moreover, $H_{\hat{p}} = H_p + fH_\varphi = H_p^G$ on ∂S.

Denote by Ψ_t the *flow of the Hamiltonian vector field* $H_{\hat{p}}$ on $\mathcal{V}$. Since the flows Φ_t and Ψ_t are smooth on $\mathcal{V}$, the families of maps $\{\Phi_t\}_{0\leq t\leq T}$ and $\{\Psi_t\}_{0\leq t\leq T}$ are uniformly Lipschitz on any subset U' of $\mathcal{V}$ with $\overline{U'}$ compact.

Idea of the construction of the maps $\mathcal{G}_t^{(\alpha,V)}$. For simplicity consider the case $\alpha = (0, k_1, 0; 0, l_1, 0; 1, 0; q)$. Given $\rho \in S_\alpha$, we have $0 = s_0(\rho) < t_1(\rho) < s_1(\rho) < t_2(\rho) = T$, and $\ell(\rho)$ has exactly one transversal reflection at time $a \in (0, t_1(\rho))$. Take b and c very close to $t_1(\rho)$ such that $0 < b < a < c < t_1(\rho)$ and $\mathcal{F}_t(\rho) \in S \setminus \partial S$ for all $t \in [b, a)$ and $t \in (a, c]$. Consider arbitrary smooth local cross sections $\mathcal{B}$ and $\mathcal{C}$ in S to $\mathcal{F}_t$ containing the points $\mathcal{F}_b(\rho)$ and $\mathcal{F}_c(\rho)$, respectively. Choosing appropriately small neighbourhoods U_1 and W_1 of $\mathcal{F}_{t_1(\rho)}(\rho)$ and $\mathcal{F}_{s_1(\rho)}(\rho)$ in $\mathcal{V}$, set

$$\mathcal{M}_1 = \{\sigma \in U_1 : H_p^{k_1-1}\varphi(\sigma) = 0\}, \quad \mathcal{N}_1 = \{\sigma \in W_1 : H_p^{l_1-1}\varphi(\sigma) = 0\};$$

these are then smooth local cross sections to $\mathcal{F}_t$ at $\mathcal{F}_{t_1(\rho)}(\rho)$ and $\mathcal{F}_{s_1(\rho)}(\rho)$, respectively. On a small neighbourhood $V = V(\alpha, \rho)$ of ρ in $\mathcal{V}$ 'the flow' $\mathcal{G}_t = \mathcal{G}_t^{(\alpha,V)}$ is defined as follows: it carries $\sigma \in V$ along the trajectory $\Phi_t(\sigma)$ until it hits the hypersurface $\mathcal{B}$; between the hypersurfaces $\mathcal{B}$ and $\mathcal{C}$ 'the flow' $\mathcal{G}_t$ coincides with $\mathcal{F}_t$; between $\mathcal{C}$ and $\mathcal{M}_1$, $\mathcal{G}_t$ acts as Φ_t again; between $\mathcal{M}_1$ and $\mathcal{N}_1$, $\mathcal{G}_t$ coincides with the flow Ψ_t of $H_{\hat{p}}$; and finally from $\mathcal{N}_1$ 'onwards' $\mathcal{G}_t$ coincides with Φ_t. As one can see, the idea is quite simple – the action of $\mathcal{G}_t$ between any two consecutive distinguished cross-sections ($\mathcal{B}$, $\mathcal{C}$, $\mathcal{M}_1$, $\mathcal{N}_1$) coincides with the action of one of the flows Φ_t, Ψ_t and $\mathcal{F}_t$ (the first two being smooth Hamiltonian flows in $\mathcal{V}$). Notice that $\Phi_t = \mathcal{F}_t$ near $\mathcal{B}$ and $\mathcal{C}$, so there is no loss of smoothness there. The places where we can (and actually do) loose smoothness are the transversal reflections and the cross-sections at the ends of gliding segments ($\mathcal{M}_1$ and $\mathcal{N}_1$ in our example). One can easily observe that if V is chosen sufficiently small, then $\mathcal{G}_t(\sigma) = \mathcal{F}_t(\sigma)$ whenever $\sigma \in S_\alpha \cap V$.

We will now give a detailed proof of Lemma 11.3.5.

Fix $\rho \in \Sigma$ **and an admissible string** α **of the form (11.6) such that** $\rho \in S_\alpha$. We are going to define the neighbourhood $V = V(\alpha, \rho)$ and the family of maps $\mathcal{G}_t = \mathcal{G}_t^{(\alpha,V)}$ required in Lemma 11.3.5.

There are several possible cases for the pairs k_0, l_0 and k_{m+1}, l_{m+1} described in (iv) and (v) in Definition 11.3.2. We will consider in details one of these; the others can be dealt with in the same way with minor modifications at the ends of the trajectory $\ell(\rho)$ (see the end of this section for some details).

We will assume that

$$k_0 = -1, \quad l_{m+1} \geq 2. \tag{11.8}$$

Let $t_i = t_i(\rho)$, $s_i = s_i(\rho)$ **be the corresponding numbers** from (11.7). The assumption (11.8) implies that (see (iv) and (v) in Definition 11.3.2) $0 = t_0(\rho) = s_0(\rho)$, $\rho \notin \partial S$ and $\mathcal{F}_T(\rho)$ belongs to a gliding segment. Thus, $\{\mathcal{F}_t(\rho) : t_{m+1} \leq t \leq T\}$ is a gliding segment on ∂S if $t_{m+1} < s_{m+1} = T$, and $\{\mathcal{F}_t(\rho) : T \leq t \leq T + \epsilon\}$ is a gliding segment on ∂S for some $\epsilon > 0$ if $t_{m+1} = s_{m+1} = T$.

For every $i = 1, 2, \ldots, m+1$, $\rho \in S_\alpha$ gives $\mathcal{F}_{t_i}(\rho) \in G^{k_i} \setminus G^{k_i+1}$, thus $H_p^{k_i}\varphi(\mathcal{F}_{t_i}(\rho)) \neq 0$. From Definition 11.3.2 it also follows that $\mathcal{F}_t(\rho) \in S \setminus \partial S$ for $t < t_i$ sufficiently close to t_i. This is only possible if $H_p^{k_i}\varphi(\mathcal{F}_{t_i}(\rho)) < 0$ (cf. Section 24.3 in [H3]). In the same way one gets $H_p^{l_i}\varphi(\mathcal{F}_{s_i}(\rho)) > 0$.

Fix small open neighbourhoods U_i of $\mathcal{F}_{t_i}(\rho)$ $(1 \leq i \leq m+1)$ and W_i of $\mathcal{F}_{s_i}(\rho)$ $(1 \leq i \leq m)$ in $\mathcal{V}$ such that $H_p^{k_i}\varphi(\sigma) < 0$ for $\sigma \in U_i$ $(1 \leq i \leq m+1)$ and $H_p^{l_i}\varphi(\sigma) > 0$ for $\sigma \in W_i$ $(1 \leq i \leq m)$ and $H_p^{l_{m+1}}\varphi(\sigma) < 0$ for $\sigma \in W_{m+1}$. Define

$$\mathcal{M}_i = \{\rho \in U_i : H_p^{k_i - 1}\varphi(\rho) = 0\}, \quad \mathcal{N}_i = \{\rho \in W_i : H_p^{l_i - 1}\varphi(\rho) = 0\}$$

for $1 \leq i \leq m+1$ and $1 \leq i \leq m$, respectively. Since

$$\{p, H_p^{k_i - 1}\varphi\}(\mathcal{F}_{t_i}(\rho)) = H_p^{k_i}(\mathcal{F}_{t_i}(\rho)) \neq 0,$$

shrinking the neighbourhood U_i if necessary, we have that $\mathcal{M}_i$ is a smooth $(2n-1)$-dimensional submanifold of $\mathcal{V}$ containing $\mathcal{F}_{t_i}(\rho_0)$ and transversal to the flow $\mathcal{F}_t$ at this point. Similarly, $\mathcal{N}_i$ is a smooth $(2n-1)$-dimensional submanifold of $\mathcal{V}$ containing $\mathcal{F}_{s_i}(\rho_0)$ and transversal to $\mathcal{F}_t$ at $\mathcal{F}_{s_i}(\rho_0)$.

It follows from the definition of the numbers t_i, s_i that the part $\{\mathcal{F}_t(\rho) : s_i \leq t \leq t_{i+1}\}$ of the trajectory of ρ does not contain gliding segments to ∂S and has exactly q_i transversal reflections at ∂S. However, it may have some other common points with ∂S (cf. Remark 11.3.3). Our plan is to isolate the times of transversal reflections in small open intervals; then on the rest of $[s_i, t_{i+1}]$, which we denote by $I_i(\rho)$, the trajectory of ρ will be an integral curve of H_p in S and therefore of Φ_t in $\mathcal{V}$. The latter is smooth and we can use it to define the orbit of $\mathcal{G}_t$ over $I_i(\rho)$ for any point $\sigma \in \mathcal{V}$ sufficiently close to ρ.

Let $i = 0, 1, \ldots, m$ be such that $q_i > 0$ and let $a_i^{(1)} < \ldots < a_i^{(q_i)}$ be the *times of the transversal reflections* of $\{\mathcal{F}_t(\rho) : s_i \leq t \leq t_{i+1}\}$. For each $j = 1, \ldots, q_i$ fix arbitrary numbers $b_i^{(j)}$ and $c_i^{(j)}$ close to $a_i^{(j)}$ such that

$$t_i < b_i^{(1)} < a_i^{(1)} < c_i^{(1)} < b_i^{(2)} < a_i^{(2)} < c_i^{(2)} < \ldots < b_i^{(q_i)} < a_i^{(q_i)} < c_i^{(q_i)} < s_{i+1},$$

and $\mathcal{F}_t(\rho) \in S \setminus \partial S$ for $t \in [b_i^{(j)}, a_i^{(j)}) \cup (a_i^{(j)}, c_i^{(j)}]$, $j = 1, 2, \ldots, q_i$.

Next, choose arbitrary smooth local $(2n-1)$-dimensional submanifolds $\mathcal{B}_i^{(j)}$ and $\mathcal{C}_i^{(j)}$ of S so that $\mathcal{B}_i^{(j)}$ (resp. $\mathcal{C}_i^{(j)}$) contains $\mathcal{F}_{b_i^{(j)}}(\rho)$ (resp. $\mathcal{F}_{c_i^{(j)}}(\rho)$) and is transversal to H_p at $\mathcal{F}_{b_i^{(j)}}(\rho)$ (resp. at $\mathcal{F}_{c_i^{(j)}}(\rho)$). We take these submanifolds in such a way that $\overline{\mathcal{B}_i^{(j)}} \cap \partial S = \overline{\mathcal{C}_i^{(j)}} \cap \partial S = \emptyset$. Using the continuity of the flows $\mathcal{F}_t$, Φ_t and Ψ_t and a simple (backward) induction, we may assume that these local cross-sections are such that:

(C) the shift along the flow Φ_t maps $\mathcal{C}_i^{(q_{i-1})}$ to $\mathcal{M}_i$ ($1 \le i \le m$ with $q_i > 0$);
(M) the shift along the flow Ψ_t maps $\mathcal{M}_i$ to $\mathcal{N}_i$ ($1 \le i \le m$);
(N) the shift along the flow Φ_t maps $\mathcal{N}_i$ to $\mathcal{B}_i^{(1)}$ if $q_i > 0$ and to $\mathcal{M}_{i+1}$ if $q_i = 0$ ($1 \le i \le m$);
(B) the shift along the flow $\mathcal{F}_t$ maps $\mathcal{B}_i^{(j)}$ to $\mathcal{C}_i^{(j)}$ ($0 \le i \le m$ with $q_i > 0$, $1 \le j \le q_i$).

Finally, using again the continuity of $\mathcal{F}_t$, choose an *open neighbourhood* $V = V(\alpha, \rho)$ of ρ in S (hence in $\mathcal{V}$) such that the shift along the flow Φ_t maps V onto $\mathcal{B}_0^{(1)}$ if $q_0 > 0$. If $q_0 = 0$, we choose V so small that $\Phi_{t_1(\sigma)}(\sigma) \in \mathcal{M}_1$ for all $\sigma \in V$.

Definition 11.3.6: Given $\sigma \in V$, consider the curve $\{\mathcal{G}_t(\sigma) : 0 \le t \le T\}$ in $\mathcal{V}$ with the following properties:

(i) There exist numbers $t_i(\sigma)$ ($1 \le i \le m+1$) and $s_i(\sigma)$ ($1 \le i \le m$) with

$$0 < t_1(\sigma) < s_1(\sigma) < \ \ldots \ < t_m(\sigma) < s_m(\sigma) < \ \ldots \ < t_{m+1}(\sigma)$$

such that $\mathcal{G}_{t_i(\sigma)}(\sigma) \in \mathcal{M}_i$ for $i = 1, \ldots, m, m+1$, and $\mathcal{G}_{s_i(\sigma)}(\sigma) \in \mathcal{N}_i$ for $i = 1, \ldots, m$.

(ii) For every $i = 0, 1, \ldots, m$ with $q_i > 0$ and every $j = 1, \ldots, q_i$ there exist numbers $a_i^{(j)}(\sigma)$, $b_i^{(j)}(\sigma)$ and $c_i^{(j)}(\sigma)$ such that

$$\begin{aligned} s_i(\sigma) < b_i^{(1)}(\sigma) < a_i^{(1)}(\sigma) < c_i^{(1)}(\sigma) < \ \ldots \ < b_i^{(q_i)}(\sigma) \\ < a_i^{(q_i)}(\sigma) < c_i^{(q_i)}(\sigma) < t_{i+1}(\sigma), \end{aligned}$$

$\mathcal{G}_{b_i^{(j)}(\sigma)}(\sigma) \in \mathcal{B}_i^{(j)}$, $\mathcal{G}_{c_i^{(j)}(\sigma)}(\sigma) \in \mathcal{C}_i^{(j)}$ and $\mathcal{G}_{a_i^{(j)}(\sigma)}(\sigma) \in \partial S$;

(iii) $\{\mathcal{G}_t(\sigma) : t \in I\}$ is a trajectory of the following:

- Φ_t for any interval I contained in

$$I_i(\sigma) = [s_i(\sigma), t_{i+1}(\sigma)] \setminus \cup_{j=1}^{q_i} (b_i^{(j)}(\sigma), c_i^{(j)}(\sigma))$$

for some $i = 0, 1, \ldots, m$;

- Ψ_t for $I = [t_i(\sigma), s_i(\sigma)]$, $i = 1, \ldots, m, m+1$;
- $\mathcal{F}_t$ for $I = [b_i^{(j)}(\sigma), a_i^{(j)}(\sigma))$ or $I = (a_i^{(j)}(\sigma), c_i^{(j)}(\sigma)]$ for any $i = 0, 1, \ldots, m$ with $q_i > 0$ and $j = 1, \ldots, q_i$.

Clearly the definition of $\mathcal{G}_t$ can be carried out step by step – first on the interval $[0, b_0^{(1)}(\sigma)]$ (assuming $q_0 > 0$), then on $[b_0^{(1)}(\sigma), c_0^{(1)}(\sigma)]$, $[c_0^{(1)}(\sigma), b_0^{(2)}(\sigma)]$, etc. The numbers $b_0^{(j)}(\sigma)$, $a_0^{(j)}(\sigma)$, $c_0^{(j)}(\sigma)$, $t_1(\sigma)$, $s_1(\sigma)$, etc., are also defined step by step following the inductive construction. From this procedure, which is effectively described by Definition 11.3.6, one can see that if the neighbourhood $V = V(\alpha, \rho)$ of ρ in $\mathcal{V}$ is taken sufficiently small, then the curve $\{\mathcal{G}_t(\sigma) : 0 \leq t \leq T\}$ is well defined for all $\sigma \in V$. Moreover, we can choose V in such a way that

$$d(\mathcal{G}_{a_i^{(j)}(\sigma)}(\sigma), G) \geq \frac{1}{2q}, \ \ \sigma \in V$$

for all $i = 0, 1, \ldots, m$ with $q_i > 0$ and $j = 1, \ldots, q_i$. Notice that in (i) it may happen that $t_{m+1}(\sigma) > T$. For such σ in the corresponding parts in (iii) the last interval involved will be $[s_m(\sigma), T]$ if $q_m = 0$ and $[c_m^{(q_m)}(\sigma), T]$ if $q_m > 0$.

In what follows we will use the notation $I_i^{(j)}(\sigma) = \left(b_i^{(j)}(\sigma), c_i^{(j)}(\sigma)\right)$. Clearly, it makes sense only when $q_i > 0$.

Proof of Lemma 11.3.5: We will show that $V = V(\alpha, \rho)$ and the maps $\mathcal{G}_t = \mathcal{G}^{(\alpha, V)}$ have the properties listed in Lemma 11.3.5. We are still considering the case (11.8). As promised earlier, at the end of the proof we will say how to deal with the other possible cases.

Step 1. We will show that the real-valued functions $t_i(\sigma)$, $s_i(\sigma)$, $a_i^{(j)}(\sigma)$, $b_i^{(j)}(\sigma)$, $c_i^{(j)}(\sigma)$ $(i \leq m)$ and the corresponding points $\mathcal{G}_{t_i(\sigma)}(\sigma)$, $\mathcal{G}_{s_i(\sigma)}(\sigma)$, $\mathcal{G}_{a_i^{(j)}(\sigma)}(\sigma)$, $\mathcal{G}_{b_i^{(j)}(\sigma)}(\sigma)$, $\mathcal{G}_{c_i^{(j)}(\sigma)}(\sigma)$ depend smoothly on $\sigma \in V$. If $q_0 = 0$, then $t_1(\sigma)$ is just the (first) time when the trajectory $\{\Phi_t(\sigma) : 0 \leq t \leq T\}$ hits the cross-section $\mathcal{M}_1$. Since Φ_t is a smooth (Hamiltonian) flow in $\mathcal{V}$, it follows that both $t_1(\sigma)$ and $\mathcal{G}_{t_1(\sigma)}(\sigma)$ depend smoothly on $\sigma \in V$. If $q_0 > 0$, the first number we have to define is $b_0^{(1)}(\sigma)$. This is the time when $\{\Phi_t(\sigma) : 0 \leq t \leq T\}$ hits the cross-section $\mathcal{B}_0^{(1)}$, so for the same reason as above, $b_0^{(1)}(\sigma)$ and $\mathcal{G}_{b_0^{(1)}(\sigma)}(\sigma)$ depend smoothly on σ. From $\mathcal{B}_0^{(1)}$ to ∂S our trajectory follows $\mathcal{F}_t$ which in $S \setminus \partial S$ is a smooth Hamiltonian flow ($= \Phi_t$ in $S \setminus \partial S$) transversal to ∂S at $\mathcal{G}_{a_0^{(1)}(\sigma)}(\sigma)$. Hence $a_0^{(1)}(\sigma) - b_0^{(1)}(\sigma)$ and therefore $a_0^{(1)}(\sigma)$ depend smoothly on σ. This also implies that $\mathcal{G}_{a_0^{(1)}(\sigma)}(\sigma)$ is smooth. The corresponding statement for $c_0^{(1)}(\sigma)$ follows similarly.

Next, suppose we have shown that $t_1(\sigma)$ and $\mathcal{G}_{t_1(\sigma)}(\sigma) \in \mathcal{M}_1$ depend smoothly on $\sigma \in V$. From the cross-section $\mathcal{M}_1$ to the cross-section $\mathcal{N}_1$, $\mathcal{G}_t$ acts as the smooth Hamiltonian flow Ψ_t in $\mathcal{V}$. Thus, $s_1(\sigma) - t_1(\sigma)$ (and therefore $s_1(\sigma)$) and $\mathcal{G}_{s_1(\sigma)}(\sigma)$ depend smoothly on σ. Proceeding in this way inductively, one completes Step 1. By the same procedure, it follows that $t_{m+1}(\sigma)$ is a smooth function of $\sigma \in V(\alpha, \rho)$. However, as mentioned earlier if $t_{m+1}(\rho) = T$, then we may have $t_{m+1}(\sigma) > T$ for some $\sigma \in V(\alpha, \rho)$ arbitrarily close to ρ.

Step 2. We are going to show that $\mathcal{G}_t = \mathcal{F}_t$ on $S_\alpha \cap V$; this will prove part (a) of Lemma 11.3.5. Let $\sigma \in S_\alpha \cap V$. It follows from the choice of the neighbourhood V and the definition of the numbers $t_i(\sigma)$ and $s_i(\sigma)$ that

$\mathcal{F}_{t_i(\sigma)}(\sigma) \in \mathcal{M}_i$, $\mathcal{F}_{s_i(\sigma)}(\sigma) \in \mathcal{N}_i$. Moreover, the definition of $\mathcal{G}_t$ gives that on each interval contained in $I_0(\sigma)$, $\mathcal{G}_t$ acts as the flow Φ_t. However, $\sigma \in S_\alpha$ implies that $\mathcal{F}_t(\sigma)$ has no transversal reflections or gliding segments on $I_0(\sigma)$; so on any time interval contained in $I_0(\sigma)$ the action of the flow $\mathcal{F}$ is the same as that of Φ. On the intervals $I_0^{(j)}(\sigma)$ containing the times of transversal reflections $\mathcal{G}_t$ acts as $\mathcal{F}_t$ by definition. Therefore, $\mathcal{G}_t(\sigma) = \mathcal{F}_t(\sigma)$ for all $t \in [0, t_1(\sigma)]$.

Next, $\sigma \in S_\alpha$ implies that $\{\mathcal{F}_t(\sigma) : t \in [t_1(\sigma), s_1(\sigma)]\}$ is an integral curve of the vector field H_p^G contained in $G \subset \partial S$. Since $H_{\hat{p}} = H_p^G$ on ∂S, it follows that $\{\mathcal{F}_t(\sigma) : t \in [t_1(\sigma), s_1(\sigma)]\}$ is an integral curve of the vector field $H_{\hat{p}}$, too. This agrees with the definition of $\mathcal{G}_t$, so $\mathcal{F}_t(\sigma) = \mathcal{G}_t(\sigma)$ for all $t \in [0, s_1(\sigma)]$. This and the definition of $\mathcal{G}_t$ yield $\mathcal{F}_t(\sigma) = \mathcal{G}_t(\sigma)$ for $t \in [0, s_2(\sigma)]$, etc. Applying the above procedure inductively, we get $\mathcal{F}_t(\sigma) = \mathcal{G}_t(\sigma)$ for all $t \in [0, T]$.

Step 3. Next, we check condition (c) of Lemma 11.3.5. Let

$$\sigma \in S_\alpha \cap V \cap [(S \setminus \partial S) \cup G]$$

and $t \in (0, T]$ be such that $\mathcal{F}_t(\sigma) \in (S \setminus \partial S) \cup G$, and let $\mathcal{M}$ be an arbitrary smooth cross-section to the flow $\mathcal{F}_t$ with $\sigma \in \mathcal{M} \subset V$. First, consider the case $0 < t < t_1(\sigma)$. If $q_0 = 0$, then $I_0(\sigma) = [0, t_1(\sigma)]$, and so $\mathcal{G}_s(\sigma) = \Phi_s(\sigma)$ for all $s \in [0, t_1(\sigma)]$. Moreover for σ' in a small open neighbourhood W of σ in V, we have $\mathcal{G}_s(\sigma') = \Phi_s(\sigma')$ for all $s \in [0, t]$. Since Φ_s is a smooth Hamiltonian flow in $\mathcal{V}$, it follows that $\mathcal{G}_t : W \longrightarrow \mathcal{V}$ is smooth and $\mathcal{G}_t : \mathcal{M} \cap W \longrightarrow \mathcal{G}_t(\mathcal{M} \cap W)$ is a contact transformation. Let $q_0 \geq 1$. Then $\{\mathcal{F}_s(\sigma) : 0 \leq s \leq t_1(\sigma)\}$ has transversal reflections for $s = a_0^{(j)}$ $(j = 1, \dots, q_0)$ and possibly some other common points with ∂S. For $s \in [0, b_0^{(1)}(\sigma)]$, we have $\mathcal{G}_s(\sigma) = \Phi_s(\sigma)$, so for $t \leq b_0^{(1)}(\sigma)$ the map $\mathcal{G}_t : V \longrightarrow \mathcal{V}$ is smooth and $\mathcal{G}_t : \mathcal{M} \longrightarrow \mathcal{G}_t(\mathcal{M})$ is a contact transformation. Also notice that

$$\mathcal{G}_{b_0^{(1)}(\sigma')}(\sigma') = \Phi_{b_0^{(1)}(\sigma')}(\sigma') \in \mathcal{B}_0^{(1)}$$

depends smoothly on $\sigma' \in V$ and defines a contact transformation from $\mathcal{M}$ to $\mathcal{B}_0^{(1)}$.

Let $b_0^{(1)}(\sigma) < t \leq c_0^{(1)}(\sigma)$; then the definition of $\mathcal{G}$ implies

$$\mathcal{G}_t(\sigma') = \mathcal{F}_{t - b_0^{(1)}(\sigma')} \circ \Phi_{b_0^{(1)}(\sigma')}(\sigma')$$

on a small neighbourhood W of σ in V. The only $s \in [b_0^{(1)}(\sigma'), c_0^{(1)}(\sigma')]$ with $\mathcal{F}_s(\sigma') = \mathcal{G}_s(\sigma') \in \partial S$ is $s = a_0^{(1)}(\sigma')$, the time of the corresponding transversal reflection. Assuming $t \neq a_0^{(1)}(\sigma)$, we can take the neighbourhood W so small that $\mathcal{G}_t(W) \cap \partial S = \emptyset$; then $\mathcal{G}_t$ is smooth on W and $\mathcal{G}_t : \mathcal{M} \cap W \longrightarrow \mathcal{G}_t(\mathcal{M} \cap W)$ is a contact transformation. Moreover,

$$\mathcal{G}_{c_0^{(1)}(\sigma')}(\sigma') = \mathcal{F}_{c_0^{(1)}(\sigma') - b_0^{(1)}(\sigma')} \circ \Phi_{b_0^{(1)}(\sigma')}(\sigma')$$

is smooth on the whole V and defines a contact transformation between $\mathcal{M}$ and $\mathcal{C}_0^{(1)}$. Continuing in this way by induction, one checks that condition (ii) in Lemma 11.3.5

holds for $t < t_1(\sigma)$. Apart from that, we get that the map $V \ni \sigma' \mapsto \mathcal{G}_{t_1(\sigma')}(\sigma') \in \mathcal{M}_1$ (which is smooth by Step 1) defines a contact transformation between $\mathcal{M}$ and $\mathcal{M}_1$.

Next, consider the case $t_1(\sigma) < t < s_1(\sigma)$. On this time interval $\mathcal{G}_t$ acts as the smooth Hamiltonian flow Ψ_t, so condition (ii) is again trivially satisfied. More precisely, we have $\mathcal{G}_t(\sigma') = \Psi_{t-t_1(\sigma')} \circ \mathcal{G}_{t_1(\sigma')}(\sigma')$ on a small neighbourhood W of σ in V. Moreover, $V \ni \sigma' \mapsto \mathcal{G}_{s_1(\sigma')}(\sigma') \in \mathcal{N}_1$ is smooth and its restriction to $\mathcal{M}$ defines a contact transformation.

Proceeding in this way, we show that for every t which (in the case under consideration) is different from $t_i(\sigma)$ $(i = 1, \dots, m+1)$, $s_i(\sigma)$ $(i = 1, \dots, m)$ and $a_i^{(j)}(\sigma)$ $(i = 0, 1, \dots, m$ with $q_i > 0$ and $j = 1, \dots, q_i)$, there exists an open neighbourhood W of σ in V such that $\mathcal{G}_t : W \longrightarrow \mathcal{V}$ is smooth and $\mathcal{G}_t : \mathcal{M} \cap W \longrightarrow \mathcal{G}_t(\mathcal{M} \cap W)$ is a contact transformation.

Step 4. Let us now prove Lemma 11.3.5(b). Let $\sigma \in S_\alpha \cap V$ and $t \in (0, T]$. If t is different from all $t_i(\sigma)$, $s_i(\sigma)$ and $a_i^{(j)}(\sigma)$, then it follows from the previous step that $\mathcal{G}_t : W \longrightarrow \mathcal{V}$ is smooth for some neighbourhood W of σ in V, thus (possibly shrinking W so that $\overline{W}$ is contained in the domain of smoothness of $\mathcal{G}_t$), $\mathcal{G}_t : (W, d_0) \longrightarrow (\mathcal{V}, d_0)$ is Lipschitz.

Next, assume that $t = a_i^{(j)}(\sigma)$ for some $i = 0, 1, \dots, m$ with $q_i > 0$ and some $j = 1, \dots, q_i$. Then by Definition 11.3.6,

$$\mathcal{G}_t(\sigma') = \mathcal{F}_{t - b_i^{(j)}(\sigma')} \circ \mathcal{G}_{b_i^{(j)}(\sigma')}(\sigma')$$

for all $\sigma' \in V$. Since

$$V \ni \sigma' \mapsto \mathcal{G}_{b_i^{(j)}(\sigma')}(\sigma') \in \mathcal{B}_i^{(j)}$$

is smooth, it is Lipschitz with respect to the metric d_0 on every neighbourhood W of σ in V with $\overline{W}$ compact and contained in V. On the other hand, $\overline{\mathcal{B}_i^{(j)}} \cap \partial S = \emptyset$ shows that d and d_0 are equivalent on $\mathcal{B}_i^{(j)}$. Thus, taking W sufficiently small, Lemma 11.3.1 gives that $\mathcal{G}_t : (W, d_0) \longrightarrow (\mathcal{V}, d)$ is Lipschitz.

Finally, assume that $t = t_i(\sigma)$ for some $i = 1, \dots, m, m+1$ (the case $t = s_i(\sigma)$ is almost identical). Take some $\tau < t$ close to t so that $\mathcal{G}_\tau(\sigma) \in S \setminus \partial S$ (such τ exists according to Proposition 24.3.8 in [H3]). Then by Step 3, $\mathcal{G}_\tau : W \longrightarrow \mathcal{V}$ is smooth for some small neighbourhood W of σ in V. For $\sigma' \in W$, we have

$$\mathcal{G}_t(\sigma') = \Phi_{t-\tau} \circ \mathcal{G}_\tau(\sigma')$$

if $t_i(\sigma') \geq t$ and $\mathcal{G}_t(\sigma') = \Psi_{t - t_i(\sigma')} \circ \Phi_{t_i(\sigma') - \tau} \circ G_\tau(\sigma')$ if $t_i(\sigma') < t$. From this it follows easily that $\mathcal{G}_t : (W, d_0) \longrightarrow (\mathcal{V}, d_0)$ is Lipschitz.

With this the proof of Lemma 11.3.5 in the case $k_0 = -1$ and $l_{m+1} \geq 2$ is complete.

Step 5. Let us now explain how to deal with the other possible cases for k_0 and l_{m+1}.

Case 2. $k_0 = 0, l_{m+1} \geq 2$. Given $\rho \in S_\alpha$, we have that $\rho \in \partial S \setminus G$. Take an arbitrary $c_0 > 0$ close to 0 such that $\{\mathcal{F}_t(\rho) : 0 < t \leq c_0\} \subset S \setminus \partial S$ and a smooth local cross section $\mathcal{C}_0$ at $\mathcal{F}_{c_0}(\rho)$. We take $\mathcal{C}_0$ such that $\overline{\mathcal{C}_0} \cap \partial S = \emptyset$. Then d_0 is equivalent to d on $\mathcal{C}_0$. Taking a sufficiently small neighbourhood V of ρ in $\mathcal{V}$, we now define $\mathcal{G}_t$ slightly changing Definition 11.3.6 in the following way: $\mathcal{G}_t(\sigma) = \mathcal{F}_t(\sigma)$ for $t \leq c_0(\sigma)$, where $\mathcal{F}_{c_0(\sigma)}(\sigma) \in \mathcal{C}_0$. From the cross-section $\mathcal{C}_0$ 'onwards', we define the action of $\mathcal{G}_t$ as in Definition 11.3.6. One proves (a) of Lemma 11.3.5 as in Step 1. To prove (b), consider arbitrary $\sigma \in V$ and $t \in (0, T]$. Using Lemma 11.3.1 as in Step 4, one shows that if $t \leq c_0(\sigma)$, then $\mathcal{G}_t : (W, d) \longrightarrow (\mathcal{V}, d)$ is Lipschitz. Let $t > c_0(\sigma)$. Then for $t'(\sigma') = t - c_0(\sigma')$ we have $\mathcal{G}_t(\sigma') = \mathcal{G}_{t'(\sigma')} \circ \mathcal{G}_{c_0(\sigma')}(\sigma')$. For the map $\sigma' \mapsto \mathcal{G}_{t'(\sigma')}$ we can apply the arguments in the previous steps. Since d is equivalent to d_0 on $\mathcal{C}_0$, condition (b) of Lemma 11.3.5 follows. Condition (c) does not apply to the case under consideration.

Case 3. $k_0 \geq 2$, $l_{m+1} \geq 2$. Then $\rho \in S_\alpha$ implies that ρ belongs to a gliding segment. Taking a small open neighbourhood V of ρ in $\mathcal{V}$, set $\mathcal{M}_0 = \{\rho' \in V : H_p^{l_0-1}\varphi(\rho') = 0\}$. Change Definition 11.3.6 in the following way: for $\sigma \in V$ there exists $s_0(\sigma)$ (which may be negative if $s_0(\rho) = 0$) such that $\mathcal{F}_{s_0(\sigma)}(\sigma) \in \mathcal{M}_0$; if $s_0(\sigma) > 0$, then $\mathcal{G}_t(\sigma) = \Psi_t(\sigma)$ for $0 \leq t \leq s_0(\sigma)$, while for $t > s_0(\sigma)$ the orbit $\mathcal{G}_t(\sigma)$ is defined as in Definition 11.3.6; if $s_0(\sigma) \leq 0$, then the orbit $\mathcal{G}_t(\sigma)$ is defined as in Definition 11.3.6. One proves (a) and (b) as in the first case with minor modifications. The only difference comes when one deals with condition (b) of Lemma 11.3.5. Now W has to be considered with the metric d_0. The rest is the same.

Case 4. $k_0 = -1$, $l_{m+1} = -1$. This is in fact the easiest case to deal with. Now $\rho \in S_\alpha$ implies that both ρ and $\mathcal{F}_T(\rho)$ are in $S \setminus \partial S$, and we can take $V = V(\alpha, \rho)$ in such a way that $\overline{V} \cap \partial S = \emptyset$ and $\mathcal{F}_T(\overline{V}) \cap \partial S = \emptyset$. The rest is the same.

Case 5. $k_0 = -1$, $l_{m+1} = 0$. Similar to Case 2, take a smooth local cross section $\mathcal{B}_{m+1}$ at some point $\mathcal{F}_{b_{m+1}(\rho)}(\rho)$, where $b_{m+1}(\rho)$ is less than but very close to T. We take $\mathcal{B}_{m+1}$ such that $\overline{\mathcal{B}_{m+1}} \cap \partial S = \emptyset$, then d_0 is equivalent to d on $\mathcal{B}_{m+1}$. We change Definition 11.3.6 so that for any $\sigma \in V$, $\mathcal{G}_t$ acts as $\mathcal{F}_t$ on the interval $[b_{m+1}(\sigma), T]$, where $\mathcal{G}_{b_{m+1}(\sigma)}(\sigma) \in \mathcal{B}_{m+1}$. Given $\sigma \in V$ and $t \in (0, b_{m+1}(\sigma)]$, the corresponding statements in Lemma 11.3.5 follow immediately from the first case considered. For $t > b_{m+1}(\sigma)$ we have

$$\mathcal{G}_t(\sigma') = \mathcal{F}_{t-b_{m+1}(\sigma')} \circ \mathcal{G}_{b_{m+1}(\sigma')}(\sigma')$$

on a sufficiently small neighbourhood W of σ in V. Moreover, as in the first case, one shows that $\mathcal{G}_{b_{m+1}(\sigma')}(\sigma')$ is smooth on V (provided the latter is small enough). Combining this with Lemma 11.3.1, one derives that $\mathcal{G}_t : (W, d_0) \longrightarrow (S, d)$ is Lipschitz for a sufficiently small neighbourhood W of σ. If $\sigma \in S_\alpha \cap V$ and $t > b_{m+1}(\sigma)$ are such that $\mathcal{F}_t(\sigma) = \mathcal{G}_t(\sigma) \in (S \setminus \partial S) \cup G$, then as in the first case we derive that $\mathcal{G}_t$ is smooth on a small neighbourhood W.

Cases 6–9. The remaining cases can be easily dealt with combining arguments from the previous cases considered. We leave the details to the reader. ■

11.4 Further regularity of the generalized Hamiltonian flow

Given $T > 0$, denote by $\mathcal{T}_T$ *the set of those* $\rho \in \Sigma$ *such that*

$$\{\mathcal{F}_t(\rho) : 0 \leq t \leq T\} \bigcap G_g \neq \emptyset,$$

that is the trajectory $\{\mathcal{F}_t(\rho) : 0 \leq t \leq T\}$ contains a non-trivial gliding segment on ∂S.

In this section we prove the following additional regularity property of the generalized Hamiltonian flow, which implies Proposition 11.1.4 immediately.

Theorem 11.4.1: *Let $\mathcal{L}_0$ be an isotropic submanifold of $\Sigma \setminus \partial S$ of dimension $n-1$ such that $H_p(\rho)$ is not tangent to $\mathcal{L}_0$ at any $\rho \in \mathcal{L}_0$. Then for every $T > 0$ we have*

$$\dim_H(\mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0)) \leq n - 2.$$

Moreover, if for a given T we have $\mathcal{F}_T(\mathcal{L}_0) \subset S \setminus \partial S$, then there exists a countable family $\{\mathcal{I}_m\}$ of smooth $(n-2)$-dimensional isotropic submanifolds of S such that $\mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0) \subset \bigcup_m \mathcal{I}_m$.

Remark 11.4.2: The above statement is not true if we replace $\mathcal{T}_T$ by the set $\tilde{\mathcal{T}}_T$ of those $\rho \in \Sigma$ such that $\{\mathcal{F}_t(\rho) : 0 \leq t \leq T\} \bigcap G \neq \emptyset$. Using simple caustics in the plane, one can easily construct examples when $\dim_H(\mathcal{F}_T(\tilde{\mathcal{T}}_T \cap \mathcal{L}_0)) = n - 1$.

Let $\mathcal{L}_0$ be an isotropic submanifold of $\Sigma \setminus \partial S = p^{-1}(0) \setminus \partial S$ of dimension $n-1$ such that $H_p(\rho)$ is not tangent to $\mathcal{L}_0$ at each $\rho \in \mathcal{L}_0$ and let $T > 0$. It is sufficient to consider the case when $\mathcal{L}_0$ is contained in a small open neighbourhood of some of its points. That is why we may assume that there exists a $(2n-1)$-dimensional submanifold S_0 of S which is transversal to H_p and such that $S_0' = S_0 \cap p^{-1}(0)$ is a $(2n-2)$-dimensional symplectic submanifold of S containing $\mathcal{L}_0$. The main point is to prove the following local version of Theorem 11.4.1.

Lemma 11.4.3: *For every admissible string α of the form (11.6) and every $\rho \in \mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha$, there exists an open neighbourhood $W = W(\alpha, \rho)$ of ρ in S such that*

$$\dim_H(\mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha \cap W)) \leq n - 2.$$

Moreover, if $\mathcal{F}_T(\rho) \notin \partial S$, then W can be chosen in such a way that

$$\mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha \cap W)$$

is contained in an $(n-2)$-dimensional isotropic submanifold of S.

We will now use Lemma 11.4.3 to prove Theorem 11.4.1.

Proof of Theorem 11.4.1: Assume that for each string α of the form (11.6) and each $\rho \in \mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha$ there exists a neighbourhood $W(\alpha, \rho)$ as stated in Lemma 11.4.3. Since $\mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha$ is a separable metric space, there exists a sequence $\rho_1(\alpha), \dots, \rho_m(\alpha), \dots$ of elements of $\mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha$ such that

$$\mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha \subset \cup_{m=1}^{\infty} W(\alpha, \rho_m(\alpha)).$$

Thus, we have

$$\mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha) \subset \cup_{m=1}^{\infty} \mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha \cap W(\alpha, \rho_m(\alpha))),$$

which implies $\dim_H \mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha) \leq n-2$. Since

$$\mathcal{T}_T \cap \mathcal{L}_0 \subset \cup_\alpha (\mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha),$$

where α runs over the countable set of all configurations of the form (11.6), it now follows that $\dim_H(\mathcal{T}_T \cap \mathcal{L}_0) \leq n-2$.

In the case $\mathcal{F}_T(\mathcal{L}_0) \subset S \setminus \partial S$, we may assume (according to Lemma 11.4.3) that each

$$\mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha \cap W(\alpha, \rho_m(\alpha)))$$

is contained in an $(n-2)$-dimensional isotropic submanifold of S. Then $\mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha)$ is contained in a countable union of $(n-2)$-dimensional isotropic submanifolds of S, and so $\mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0)$ has the same property. ■

For the proof of Lemma 11.4.3 we need the following fact.

Proposition 11.4.4: *Let $\mathcal{N}$ be a symplectic manifold without boundary with $\dim \mathcal{N} = 2k$, $k \geq 2$, and let $\mathcal{E}$ be a symplectic submanifold of $\mathcal{N}$ with $\dim \mathcal{E} = 2k-2$. For every Lagrangian submanifold $\mathcal{L}$ of $\mathcal{N}$ and every $\rho_0 \in \mathcal{L} \cap \mathcal{E}$ there exist an open neighbourhood $\mathcal{U}$ of ρ_0 in $\mathcal{N}$ and a Lagrangian submanifold $\mathcal{L}'$ of $\mathcal{E}$ such that $\rho_0 \in \mathcal{L} \cap \mathcal{E} \cap \mathcal{U} \subset \mathcal{L}'$.*

Proof of Proposition 11.4.4: Since the statement is of a local nature, we may assume $\mathcal{N} = \mathbb{R}^k \times \mathbb{R}^k$ with the standard symplectic form ω and $\rho_0 = 0$. Given $\mathcal{L}$ with $0 \in \mathcal{L} \cap \mathcal{E}$, denote $E = T_0\mathcal{E}$ and $L = T_0\mathcal{L}$. Then E is a symplectic linear subspace of $N = T_0\mathcal{N} = \mathbb{R}^k \times \mathbb{R}^k$ with $\dim E = 2k-2$, while L is a Lagrangian subspace of N. For $A \subset N$ set

$$A^\perp = \{v \in N : \omega(v, u) = 0 \ \forall\, u \in A\}.$$

Now the assumptions on L and E imply $L = L^\perp$ and $E \cap E^\perp = \{0\}$. It then follows that $E^\perp$ is not contained in L. Indeed, if $E^\perp \subset L$, then $L = L^\perp \subset E$ and therefore $E^\perp \subset L \subset E$ which is a contradiction since $\dim E^\perp = 2$ and $E \cap E^\perp = \{0\}$. Hence the linear subspace $E^\perp \cap L$ is either zero or one dimensional.

Case 1. $\dim(E^\perp \cap L) = 0$. Then $N = E + L$, so $\mathcal{E}$ and $\mathcal{L}$ are transversal at 0. Hence there exists a neighbourhood $\mathcal{U}$ of 0 in $\mathcal{N}$ such that $\mathcal{L}'' = \mathcal{E} \cap \mathcal{L} \cap \mathcal{U}$ is a smooth submanifold of $\mathcal{L}$ with codimension 2 in $\mathcal{L}$, that is $\dim \mathcal{L}'' = k - 2$. Being a submanifold of $\mathcal{L}$, $\mathcal{L}''$ is isotropic, so (possibly shrinking $\mathcal{U}$) it is contained in a Lagrangian submanifold $\mathcal{L}'$ of $\mathcal{E}$.

Case 2. $\dim(E^\perp \cap L) = 1$. Locally near 0, we may assume that $\mathcal{E} = f^{-1}(0) \cap g^{-1}(0)$, where f and g are smooth functions such that $df(\rho)$ and $dg(\rho)$ are linearly independent and $\{f, g\}(\rho) = 1$ for each ρ in an open ball $\mathcal{U}$ with centre 0 in $\mathcal{N}$. Then $E^\perp = \text{span}\,\{X_f(0), X_g(0)\}$ and therefore there exist $a, b \in \mathbb{R}$, $(a, b) \neq (0, 0)$, with $aX_f(0) + bX_g(0) \in L$. We may assume that $X_g(0) \in L$; otherwise one can replace g by an appropriate linear combination of f and g. With this assumption, we have $X_g(0) \in E^\perp \cap L$.

Since $(E + L)^\perp = E^\perp \cap L$ is one dimensional, $\dim(E + L) = 2k - 1$ and therefore $\dim(E \cap L) = k - 1$. Fix an arbitrary basis $v_2, \ldots, v_k$ in $E \cap L$ and set $v_1 = X_g(0)$. Then $v_1 \in E^\perp \cap L$, and $E \cap E^\perp = \{0\}$ implies that $v_1, v_2, \ldots, v_k$ is a basis in L.

Set $u_1 = X_f(0)$. Then $u_1 \in E^\perp$ gives $\omega(u_1, v_i) = 0$ for all $i = 2, \ldots, k$. Moreover,

$$\omega(u_1, v_1) = \omega(X_f(0),\ X_g(0)) = \{f, g\}(0) = 1.$$

There exists $u_2, \ldots, u_k \in E$ such that $u_2, \ldots, u_k, v_2, \ldots, v_k$ form a symplectic basis in E. From $u_1, v_1 \in E^\perp$ we get $\omega(u_1, u_i) = \omega(v_1, u_i) = 0$ for all $i = 2, \ldots, k$ which shows that $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k$ is a symplectic basis in N. Then shrinking $\mathcal{U}$ again if necessary, there exist symplectic coordinates $x_1 = g, x_2, \ldots, x_k$, $\xi_1 = f, \xi_2, \ldots, \xi_k$ such that

$$u_i = X_{x_i}(0) = -\frac{\partial}{\partial \xi_i}, \quad v_i = X_{\xi_i}(0) = \frac{\partial}{\partial x_i}$$

for all $i = 1, \ldots, k$. In these coordinates we have

$$\mathcal{E} \cap \mathcal{U} = \{\rho = (x, \xi) \in \mathcal{U} : x_1 = \xi_1 = 0\}.$$

Moreover,

$$L = \text{span}\{v_1, \ldots, v_k\} = \text{span}\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\right\} = \mathbb{R}^k \times \{0\}.$$

Therefore, taking $\mathcal{U}$ small enough, the Lagrangian submanifold $\mathcal{L} \cap \mathcal{U}$ can be written as a graph of a smooth map:

$$\mathcal{L} \cap \mathcal{U} = \{(x, h(x)) : x \in W\},$$

where W is a neighbourhood of 0 in $\mathbb{R}^k$ and $h(x) = (h_1(x), \ldots, h_k(x))$ is smooth in W. Then

$$\frac{\partial h_i}{\partial x_j}(x) = \frac{\partial h_j}{\partial x_i}(x), \quad i, j = 1, \ldots, k, x \in W. \tag{11.9}$$

This follows, for example, from the fact that $\mathcal{L} \cap \mathcal{U}$ is the graph of the 1-form

$$\beta(x) = \sum_{i=1}^{k} h_i(x) dx_i.$$

It is known (cf. e.g. [AbM]) that in such a case, $\mathcal{L} \cap \mathcal{U}$ is Lagrangian iff β is closed, that is $d\beta = 0$ on $\mathcal{U}$. Since

$$d\beta = \sum_{i=1}^{k} dh_i \wedge dx_i = \sum_{i=1}^{k} \sum_{j=1}^{n} \frac{\partial h_i}{\partial x_j} dx_j \wedge dx_i = \sum_{j<i} \left(\frac{\partial h_i}{\partial x_j} - \frac{\partial h_j}{\partial x_i} \right) dx_j \wedge dx_i,$$

it is clear that $d\beta = 0$ on $\mathcal{U}$ is equivalent to (11.9).

Locally near 0 we have

$$\mathcal{L} \cap \mathcal{E} = \{(0, x'; 0, h_2(0, x'), \dots, h_k(0, x')) : h_1(0, x') = 0\},$$

where $x' = (x_2, \dots, x_k) \in \mathbb{R}^{k-1}$. Consider the local submanifold

$$\mathcal{L}' = \{(0, x', 0, h_2(0, x'), \dots, h_k(0, x')) : x' \in \mathbb{R}^{k-1}\}.$$

It follows from the above argument and (11.9) that $\mathcal{L}'$ is a Lagrangian submanifold of

$$\mathbb{R}^{k-1} \times \mathbb{R}^{k-1} = (\{0\} \times \mathbb{R}^{k-1}) \times (\{0\} \times \mathbb{R}^{k-1}) \subset \mathcal{N}.$$

Since $\mathcal{L} \cap \mathcal{E} \subset \mathcal{L}'$ locally near 0, this proves the assertion. ■

Proof of Lemma 11.4.3: Let α be a string of the form (11.6) and let $\rho \in \mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha$. We have to find a neighbourhood $W = W(\alpha, \rho)$ of ρ in S with

$$\dim_H(\mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0 \cap W \cap S_\alpha)) \leq n - 2$$

and in the case $\mathcal{F}_T(\rho) \notin \partial S$ such that $\mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0 \cap W \cap S_\alpha)$ is contained in a smooth $(n-2)$-dimensional isotropic submanifold of S. Using Lemma 11.3.5, there exists an open neighbourhood $V = V(\alpha, \rho)$ of ρ in $\mathcal{V}$ and a family of maps

$$\mathcal{G}_t^{(\alpha, V)} : V(\alpha, \rho) \longrightarrow \mathcal{V}, \quad 0 \leq t \leq T,$$

with the properties listed in Lemma 11.3.5. Since $\rho \in S \setminus \partial S$, we can take $V \subset \overline{V} \subset S \setminus \partial S$. Moreover, we have $k_0 = -1$.

From $\rho \in \mathcal{T}_T$ we get $\mathcal{F}_t(\rho) \in G_g$ for some $t \in [0, T]$. Therefore, either the number $m = m(\alpha)$ in (11.6) is positive or $m = 0$ and $\mathcal{F}_T(\rho)$ belongs to a gliding segment, so we must have $t_1(\rho) < s_1(\rho) = T$. (It is impossible to have $t_1(\rho) = T$, since $\mathcal{F}_t(\rho) \in G_g$ for some $t \leq T$.) Here $t_i = t_i(\rho)$ and $s_i = s_i(\rho)$ are the numbers given by (11.7). In both cases there exists c with $t_1 < c < s_1$ and $\mathcal{F}_c(\rho) \in G_g$. Then we can take an open neighbourhood W of ρ in V so small that for all $\sigma \in W \cap S_\alpha$ we have $t_1(\sigma) < c < s_1(\sigma)$ and $\mathcal{F}_c(\sigma) \in G_g$ (the latter is possible, since G_g is an open subset of G). Using Lemma 11.3.5(c) with $\sigma = \rho$ and $t = c$, we can take the neighbourhood W of ρ in such a way that $\mathcal{G}_c : W \longrightarrow \mathcal{V}$ is smooth and $\mathcal{G}_c : S_0 \cap W \longrightarrow \mathcal{G}_c(S_0 \cap W)$ is a contact transformation.

For $\rho' = \mathcal{G}_c(\rho) = \mathcal{F}_c(\rho)$ there exists a symplectic submanifold $\mathcal{N}'$ of S of dimension $2n-2$ such that $\mathcal{N}' \subset p^{-1}(0)$ and $\mathcal{E} = \mathcal{N}' \cap G$ is a symplectic submanifold of S of dimension $2n-4$. Indeed, take local symplectic coordinates x, ξ in a neighbourhood $\mathcal{O}$ of ρ' in $\mathcal{V}$ as in condition **A2** in Section 2. Then $G = \{(x,\xi) : x_1 = \xi_1 = 0\}$. Since $\{x_1, p\} = \{\xi_1, p\} = 0$ on G, the Darboux lemma implies that there exists a smooth function g in $\mathcal{O}$ (possibly shrinking $\mathcal{O}$) such that $\{x_1, g\} = \{\xi_1, g\} = 0$, $dg \neq 0$ and $\{p, g\} = 1$. Consequently

$$\mathcal{N}' = \{(x,\xi) \in \mathcal{O} : p(x,\xi) = g(x,\xi) = 0\}$$

and $\mathcal{E} = \mathcal{N}' \cap G$ are symplectic submanifolds of $\mathcal{V}$ of dimension $2n-2$ and $2n-4$, respectively. We will also need the submanifold $\mathcal{N} = g^{-1}(0)$. Clearly, this is a $(2n-1)$-dimensional submanifold of $\mathcal{V}$ containing the point $\mathcal{F}_c(\rho)$ and transversal to $H_p(\rho)$. Notice that

$$\mathcal{N}' = \mathcal{N} \cap p^{-1}(0) = \mathcal{N} \cap \Sigma.$$

Assuming W is small enough, for each $\sigma \in W$ the curve $\{\mathcal{G}_t(\sigma) : t \in [0,T]\}$ intersects transversally $\mathcal{N}$ at some point $\mathcal{P}'_\alpha(\sigma)$. As in the proof of Lemma 11.3.5 one shows that the resulting map $\mathcal{P}'_\alpha : W \longrightarrow \mathcal{N}$ is smooth and its restriction $\mathcal{P}'_\alpha : W \cap S_0 \longrightarrow \mathcal{P}'_\alpha(W \cap S_0)$ is a contact transformation. Next, define the map $\mathcal{P}''_\alpha : \mathcal{N} \longrightarrow \mathcal{V}$ by $\mathcal{P}''_\alpha(\mathcal{P}'_\alpha(\sigma)) = \mathcal{G}_T(\sigma)$. Using again an argument from the proof of Lemma 11.3.5, it follows that $\mathcal{P}''_\alpha : (\mathcal{N}, d_0) \longrightarrow (\mathcal{V}, d)$ is a Lipschitz map. Now define $\mathcal{P}' : W \cap S_0 \longrightarrow \mathcal{N}$ like $\mathcal{P}'_\alpha$ using the flow $\mathcal{F}_t$ instead of $\mathcal{G}_t$, and set $\mathcal{P}''(\mathcal{P}'(\sigma)) = \mathcal{F}_T(\rho)$ for $\sigma \in W$.

It follows from Lemma 11.3.5(a) that $\mathcal{G}_t = \mathcal{F}_t$ on $V \cap S_\alpha$ for all $t \in [0,T]$. Consequently $\mathcal{P}'_\alpha = \mathcal{P}'$ on $W \cap S_\alpha$ and $\mathcal{P}''_\alpha = \mathcal{P}''$ on $\mathcal{P}'(W \cap S_\alpha)$. On the other hand, it follows from the choice of W and the definition of $\mathcal{P}'$ that $\mathcal{P}'(W \cap S_\alpha) \subset G$. Since $p^{-1}(0)$ is invariant under the flow $\mathcal{F}_t$ and $\mathcal{N}' = \mathcal{N} \cap p^{-1}(0)$, we have $\mathcal{P}'(S'_0 \cap W \cap S_\alpha) \subset \mathcal{N}' \cap G = \mathcal{E}$.

Recall from above that $\mathcal{P}'_\alpha : W \cap S_0 \longrightarrow \mathcal{P}'_\alpha(W \cap S_0)$ is a contact transformation. Hence $\mathcal{L} = \mathcal{P}'_\alpha(W \cap \mathcal{L}_0)$ is a Lagrangian submanifold of $\mathcal{P}'_\alpha(W \cap S'_0)$ and

$$\mathcal{P}'(\mathcal{L}_0 \cap W \cap S_\alpha) = \mathcal{P}'_\alpha(\mathcal{L}_0 \cap W \cap S_\alpha) \subset \mathcal{L} \cap \mathcal{E}.$$

Now Proposition 11.4.4 implies that if the neighbourhood $\mathcal{O}$ of $\rho' = \mathcal{F}_c(\rho)$ is sufficiently small, then $\mathcal{O} \cap \mathcal{L} \cap \mathcal{E}$ is contained in a Lagrangian submanifold $\mathcal{L}'$ of $\mathcal{E}$. Without loss of generality we may assume that $\mathcal{P}'_\alpha(W) \subset \mathcal{O}$; then $\mathcal{P}'_\alpha(\mathcal{L}_0 \cap W \cap S_\alpha) \subset \mathcal{L}'$. Since $\dim \mathcal{L}' = n-2$, we have $\dim_H(\mathcal{L}') = n-2$. As we observed above, $\mathcal{P}''_\alpha : (\mathcal{N}, d_0) \longrightarrow (\mathcal{V}, d)$ is a Lipschitz map. Moreover, (as we mentioned in Section 11.2), for Borel subsets of $\Sigma = p^{-1}(0)$, $\dim_H$ calculated with respect to d_0 or d is the same. Hence $\dim_H(\mathcal{P}''_\alpha(\mathcal{L}')) \leq n-2$. This and

$$\mathcal{F}_T(\mathcal{L}_0 \cap W \cap S_\alpha) = \mathcal{P}''_\alpha \circ \mathcal{P}'_\alpha(\mathcal{L}_0 \cap W \cap S_\alpha) \subset \mathcal{P}''_\alpha(\mathcal{L}')$$

yield $\dim_H(\mathcal{F}_T(\mathcal{L}_0 \cap W \cap S_\alpha)) \leq n-2$.

Next, assume that $\mathcal{F}_T(\rho) \in S \setminus \partial S$. Shrinking W if necessary, we may assume that $\mathcal{F}_T(\overline{W}) \subset S \setminus \partial S$. In this case, as for $\mathcal{P}'_\alpha$, one shows that $\mathcal{P}''_\alpha$ is smooth and its restriction $\mathcal{P}''_\alpha : \mathcal{N} \longrightarrow \mathcal{P}''_\alpha(\mathcal{N})$ is contact. Consequently, $S_T = \mathcal{P}''_\alpha(\mathcal{N}')$ is a symplectic submanifold of S of dimension $2n-2$ and $\mathcal{P}''_\alpha : \mathcal{N}' \longrightarrow S_T$ is a local symplectic map. Clearly $\mathcal{L}'$ (being a Lagrangian submanifold of $\mathcal{E}$) is an $(n-2)$-dimensional isotropic submanifold of $\mathcal{N}$, so $\mathcal{I} = \mathcal{P}''_\alpha(\mathcal{L}')$ is an $(n-2)$-dimensional isotropic submanifold of S_T. Moreover, it follows from above that $\mathcal{F}_T(\mathcal{L}_0 \cap W \cap S_\alpha) \subset \mathcal{I}$. This proves the lemma. ■

11.5 Proof of Proposition 11.1.2

Let K and $\Omega = \Omega_K$ be as in Section 11.1, and let $X = \partial K$. The arguments considered here actually work for any compact smooth $(n-1)$-dimensional submanifold X of $\mathbb{R}^n$, $n \geq 2$. As in previous chapters, we will use the notation

$$X^{(s)} = \{(x_1, \ldots, x_s) \in X^s \ : \ x_i \neq x_j \ , i \neq j \}$$

for any integer $s \geq 1$.

Fix a large number $R > 0$ so that the open ball $B = B_R$ in $\mathbb{R}^n$ with centre 0 and radius R contains X. Given a vector $\omega \in \mathbb{S}^{n-1}$, we will denote by Z_ω the hyperplane tangent to B and orthogonal to ω such that the half-space determined by Z_ω and having ω as an inner normal contains X. For a string of the form $x = (x_1, \ldots, x_s)$ we will denote by $x_0 = x_0(x)$ (resp. $x_{s+1} = x_{s+1}(x)$) the orthogonal projection of x_1 on Z_ω (resp. of x_s on $Z_{-\theta}$).

Fix two integers k and s with $s \geq 1$ and $0 \leq k \leq s$. Set

$$\Xi_s = \mathbb{S}^{n-1} \times X^{(s)} \times X \times \mathbb{S}^{n-1}.$$

Consider the *set* $\Xi(s,k)$ *of those* $\xi = (\omega; x; y; \theta)$ with $\omega, \theta \in \mathbb{S}^{n-1}$, $x = (x_1, \ldots, x_s) \in X^{(s)}$ and $y \in X$ for which there exists an (ω, θ)-trajectory[2] for X with successive (transversal) reflection points $x_1, \ldots, x_s$ such that the segment $[x_k, x_{k+1}]$ is tangent to X at the point $y \in (x_k, x_{k+1})$.

Lemma 11.5.1: $\Xi(s,k)$ *is a smooth submanifold of* Ξ_s *of dimension* $2n-3$.

Proof of Lemma 11.5.1: We will use arguments similar to some of those used in Chapter 6 – see for example the proof of Theorem 6.3.1.

[2] See Section 6.2 for the definition of an (ω, θ)-trajectory.

Consider the open subsets

$$V_r(s,k) = \{(\omega; x; y; \theta) \in \Xi_s : x_k^{(r)} \neq x_{k+1}^{(r)}\}, \quad r = 1, \dots, n$$

of Ξ_s. Clearly they cover Ξ_s, that is $\Xi_s = \cup_{r=1}^n V_r(s,k)$. To prove the lemma it is therefore enough to show that for each $r = 1, \dots, n$ the set

$$\Xi_r(s,k) = \Xi(s,k) \cap V_r(s,k)$$

is a smooth submanifold of Ξ_s of dimension $2n-3$. We will deal in details with the case $r = n$; the other cases are similar.

Consider an arbitrary

$$\eta = (\omega^{(0)}; x^{(0)}; y^{(0)}; \theta^{(0)}) \in \Xi_n(s,k).$$

Choose smooth charts $\varphi_i : U_i \longrightarrow X$ of X around $x_i^{(0)}$ and $\psi : V \longrightarrow X$ of X around $y^{(0)}$ such that $\varphi_i(U_i) \cap \varphi_{i+1}(U_{i+1}) = \emptyset$, $i = 1, \dots, s-1$, $\varphi_k(U_k) \cap \psi(V) = \emptyset$, $\varphi_{k+1}(U_{k+1}) \cap \psi(V) = \emptyset$ (for $k = 0$ or s the corresponding condition is to be deleted). Fix an integer p with $\omega_p^{(0)} \neq 0$. Then on a small neighbourhood of $\omega^{(0)}$, we may parameterize $\mathbb{S}^{n-1}$ by

$$D_1 \ni \omega' = (\omega_1, \dots, \omega_{p-1}, \omega_{p+1}, \dots, \omega_n) \mapsto \omega(\omega') \in \mathbb{S}^{n-1},$$

where $\omega_p = \epsilon(1 - |\omega'|^2)^{1/2}$ for some constant $\epsilon = \pm 1$ and D_1 is an open subset of $\mathbb{R}^{n-1}$. Similarly, we may assume that $\mathbb{S}^{n-1}$ is parameterized around $\theta^{(0)}$ by

$$\theta'' = (\theta_1, \dots, \theta_{q-1}, \theta_{q+1}, \dots, \theta_n) \in D_2$$

for some $q = 1, \dots, n$ and some open subset D_2 of $\mathbb{R}^{n-1}$. In this way we get a chart

$$\chi : U = D_1 \times U_1 \times \dots \times U_s \times V \times D_2 \longrightarrow D \subset \Xi_s,$$

defined by

$$\chi(\xi) = (\omega; \varphi_1(u_1), \dots, \varphi_s(u_s); \psi(v); \theta)$$

for $\xi = (\omega'; u; v; \theta'') \in U$. Here $\omega = \omega(\omega')$ and $\theta = \theta(\theta'')$, $u_i = (u_i^{(1)}, \dots, u_i^{(n-1)}) \in U_i$.

As we have done several times in Chapter 6, we will now use the length function $F : U \longrightarrow \mathbb{R}$ defined by

$$F(\xi) = \sum_{i=1}^{s-1} \|\varphi_i(u_i) - \varphi_{i+1}(u_{i+1})\|.$$

First, consider the case $0 < k < s$. Let $\xi = (\omega'; u; v; \theta'') \in U$ be such that $\chi(\xi) \in \Xi_n(s,k)$. Then we have

$$\operatorname{grad}_{u_i} F(\xi) = 0, \ \ i = 2, \dots, s-1,$$

$$\left\langle \frac{\varphi_2(u_2) - \varphi_1(u_1)}{\|\varphi_2(u_2) - \varphi_1(u_1)\|} - \omega, \frac{\partial \varphi_1}{\partial u_1^{(j)}}(u_1) \right\rangle = 0, \ \ j = 1, \dots, n-1,$$

$$\left\langle \frac{\varphi_s(u_s) - \varphi_{s-1}(u_{s-1})}{\|\varphi_s(u_s) - \varphi_{s-1}(u_{s-1})\|} - \theta, \frac{\partial \varphi_s}{\partial u_s^{(j)}}(u_s) \right\rangle = 0, \ \ j = 1, \dots, n-1,$$

$$\frac{\psi(v) - \varphi_k(u_k)}{\|\psi(v) - \varphi_k(u_k)\|} + \frac{\psi(v) - \varphi_{k+1}(u_{k+1})}{\|\psi(v) - \varphi_{k+1}(u_{k+1})\|} = 0,$$

and

$$\langle \varphi_{k+1}(u_{k+1}) - \varphi_k(u_k), \ N(\xi) \rangle = 0,$$

where

$$N(\xi) = \det \begin{pmatrix} f_1 & f_2 & \dots & f_n \\ \frac{\partial \psi^{(1)}}{\partial v^{(1)}}(v) & \frac{\partial \psi^{(2)}}{\partial v^{(1)}}(v) & \dots & \frac{\partial \psi^{(n)}}{\partial v^{(1)}}(v) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \psi^{(1)}}{\partial v^{(n-1)}}(v) & \frac{\partial \psi^{(2)}}{\partial v^{(n-1)}}(v) & \dots & \frac{\partial \psi^{(n)}}{\partial v^{(n-1)}} \end{pmatrix}$$

is a normal vector to X at $\psi(v)$. Here $f_1, \dots, f_n$ are the standard basis vectors in $\mathbb{R}^n$.

To use the above conditions, introduce the functions

$$K_i^{(j)}(\xi) = \frac{\partial F}{\partial u_i^{(j)}}(\xi), \quad i = 2, \dots, s-1, j = 1, \dots, n-1,$$

$$L_j(\xi) = \left\langle \frac{\varphi_2(u_2) - \varphi_1(u_1)}{\|\varphi_2(u_2) - \varphi_1(u_1)\|} - \omega, \frac{\partial \varphi_1}{\partial u_1^{(j)}}(u_1) \right\rangle, \ \ j = 1, \dots, n-1,$$

$$M_j(\xi) = \left\langle \frac{\varphi_s(u_s) - \varphi_{s-1}(u_{s-1})}{\|\varphi_s(u_s) - \varphi_{s-1}(u_{s-1})\|} - \theta, \frac{\partial \varphi_s}{\partial u_s^{(j)}}(u_s) \right\rangle, \ \ j = 1, \dots, n-1,$$

$$P_j(\xi) = \frac{\psi^{(j)}(v) - \varphi_k^{(j)}(u_k)}{\|\psi(v) - \varphi_k(u_k)\|} + \frac{\psi^{(j)}(v) - \varphi_{k+1}^{(j)}(u_{k+1})}{\|\psi(v) - \varphi_{k+1}(u_{k+1})\|}, \ \ j = 1, \dots, n-1,$$

and

$$Q(\xi) = \langle \varphi_{k+1}(u_{k+1}) - \varphi_k(u_k), N(\xi) \rangle.$$

Finally, define the map

$$G : U \longrightarrow (\mathbb{R}^{n-1})^{s-2} \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}$$

by

$$G(\xi) = ((K_i^{(j)}(\xi))_{2\le i\le s-1}^{1\le j\le n-1}; (L_j(\xi))_{1\le j\le n-1}; (M_j(\xi))_{1\le j\le n-1}; (P_j(\xi))_{1\le j\le n-1}; Q(\xi)).$$

Clearly G is smooth and it follows from the above that

$$\chi^{-1}(D \cap \Xi_n(s,k)) = G^{-1}(0).$$

To prove the lemma, we have to establish that $G^{-1}(0)$ is a smooth submanifold of U of dimension $2n-3$, and, as we have done several times in Chapter 6, to do this it is sufficient to show that G is submersion at any point of $G^{-1}(0)$. This would then imply that $G^{-1}(0)$ is a smooth submanifold with

$$\dim G^{-1}(0) = (s+3)(n-1) - [(s+1)(n-1)+1] = 2n-3.$$

Fix $\xi \in G^{-1}(0)$ and assume that

$$\sum_{i=2}^{s-1}\sum_{j=1}^{n-1} A_i^{(j)} \text{ grad } K_i^{(j)}(\xi) + \sum_{j=1}^{n-1} B_j \text{ grad } L_j(\xi)$$
$$+\sum_{j=1}^{n-1} C_j \text{ grad } M_j(\xi) + \sum_{j=1}^{n-1} p_j \text{ grad } P_j(\xi) + q \text{ grad } Q(\xi) = 0 \quad (11.10)$$

for some real coefficients $A_i^{(j)}, B_j, C_j, p_j, q$. We will prove that these constants are zero.

Considering in (11.10) the derivatives with respect to ω_r, $r \neq p$, and using

$$\omega_p = \epsilon(1-\omega_1^2 - \ldots - \omega_{p-1}^2 - \omega_{p+1}^2 - \ldots - \omega_n^2)^{1/2}$$

for some constant $\epsilon = \pm 1$, we get

$$\sum_{j=1}^{n-1} B_j \left(-\frac{\partial \varphi_1^{(r)}}{\partial u_1^{(j)}}(u_1) + \frac{\omega_r}{\omega_p} \cdot \frac{\partial \varphi_1^{(p)}}{\partial u_1^{(j)}}(u_1) \right) = 0 \quad (11.11)$$

for $r \neq p$. Clearly (11.11) holds for $r = p$, as well. Setting

$$c = \frac{1}{\omega_p} \sum_{j=1}^{n-1} B_j \frac{\partial \varphi_1^{(p)}}{\partial u_1^{(j)}}(u_1),$$

(11.11) gives

$$c\omega_r = \sum_{j=1}^{n-1} B_j \frac{\partial \varphi_1^{(r)}}{\partial u_1^{(j)}}(u_1), \quad r = 1, \ldots, n,$$

which implies

$$c\omega = \sum_{j=1}^{n-1} B_j \frac{\partial \varphi_1}{\partial u_1^{(j)}}(u_1). \tag{11.12}$$

So, $c\omega$ is a tangent vector to X at $\varphi_1(u_1)$. On the other hand, $\xi = (\omega; u; v; \theta) \in G^{-1}(0)$ implies $\chi(\xi) \in \Xi_n(s,k) \subset \Xi(s,k)$, and so $\varphi_1(u_1)$ is the first (transversal) reflection point of an (ω, θ)-trajectory for X. Hence ω is not tangent to X at $\varphi_1(u_1)$. Thus, $c = 0$ and now (11.12) gives $B_1 = \ldots = B_{n-1} = 0$.

Similarly, using the derivatives with respect to $\theta_2, \ldots, \theta_n$ in (11.10), we find $C_1 = \ldots = C_{n-1} = 0$.

To prove $A_m^{(1)} = \ldots = A_m^{(n-1)} = 0$ for all $m = 2, \ldots, k$, we will assume $k \geq 2$; the case $k = 1$ is trivial. Now $k \geq 2$ implies that the functions P_j, Q and $K_i^{(j)}$ for $i \geq 3$ do not depend on the variables $u_1^{(r)}$. On the other hand,

$$K_2^{(j)}(\xi) = \frac{\partial F}{\partial u_2^{(j)}}(\xi) = \left\langle \frac{\varphi_2(u_2) - \varphi_1(u_1)}{\|\varphi_2(u_2) - \varphi_1(u_1)\|} + \frac{\varphi_2(u_2) - \varphi_3(u_3)}{\|\varphi_2(u_2) - \varphi_3(u_3)\|}, \frac{\partial \varphi_2}{\partial u_2^{(j)}}(u_2) \right\rangle,$$

so

$$\frac{\partial K_2^{(j)}}{\partial u_1^{(r)}}(\xi) = -a_1 \left[\left\langle \frac{\partial \varphi_1}{\partial u_1^{(r)}}(u_1), \frac{\partial \varphi_2}{\partial u_2^{(j)}}(u_2) \right\rangle - \left\langle e_1, \frac{\partial \varphi_1}{\partial u_1^{(r)}}(u_1) \right\rangle \left\langle e_1, \frac{\partial \varphi_2}{\partial u_2^{(j)}}(u_2) \right\rangle \right],$$

where

$$a_i = \frac{1}{\|\varphi_i(u_i) - \varphi_{i+1}(u_{i+1})\|}, \quad e_i = \frac{\varphi_i(u_i) - \varphi_{i+1}(u_{i+1})}{\|\varphi_i(u_i) - \varphi_{i+1}(u_{i+1})\|}.$$

Considering the derivatives with respect to $u_1^{(r)}$ in (11.10), we obtain

$$\sum_{j=1}^{n-1} A_2^{(j)} \frac{\partial K_2^{(j)}}{\partial u_1^{(r)}}(\xi) = 0. \tag{11.13}$$

Set

$$w = \sum_{j=1}^{n-1} A_2^{(j)} \frac{\partial \varphi_2}{\partial u_2^{(j)}}(u_2) \in T_{\varphi_2(u_2)} X. \tag{11.14}$$

The expression for $\frac{\partial K_2^{(j)}}{\partial u_1^{(r)}}$ found above implies that (11.13) can be written in the form

$$\left\langle \frac{\partial \varphi_1}{\partial u_1^{(r)}}(u_1), w \right\rangle - \langle e_1, w \rangle \left\langle e_1, \frac{\partial \varphi_1}{\partial u_1^{(r)}}(u_1) \right\rangle = 0.$$

Thus,

$$\left\langle \frac{\partial \varphi_1}{\partial u_1^{(r)}}(u_1), w - \langle e_1, w \rangle e_1 \right\rangle = 0.$$

This is true for all $r = 1, \dots, n-1$, so

$$w - \langle e_1, w\rangle e_1 = \lambda N_1 \tag{11.15}$$

for some $\lambda \in \mathbb{R}$, where N_1 is an unit normal vector to X at $\varphi_1(u_1)$. Note that $\langle e_1, N_1\rangle \neq 0$. Taking inner product of (11.15) with e_1 gives $0 = \lambda\langle N_1, e_1\rangle$, and so $\lambda = 0$. Using (11.15) again, $w = \langle e_1, w\rangle e_1$. Since $\xi \in G^{-1}(0)$, we cannot have $e_1 \in T_{\varphi_2(u_2)}X$, therefore $\langle e_1, w\rangle = 0$ and so $w = 0$. Now (11.14) gives $A_2^{(1)} = \dots = A_2^{(n-1)} = 0$.

Using the above procedure several times, by induction one obtains $A_m^{(j)} = 0$ for all $m = 2, \dots, k$ and $j = 1, \dots, n-1$.

The case $k < s-1$ is dealt with similarly – repeating the above argument, consider the derivatives with respect to $u_s^{(r)}$ and show that $A_{s-1}^{(j)} = 0$ for $j = 1, \dots, n-1$. In a similar way, by induction one gets $A_m^{(j)} = 0$ for all $m = s-1, s-2, \dots, k+1$ and $j = 1, \dots, n-1$. Therefore, all coefficients $A_m^{(j)}$ in (11.10) are zero.

We can now write (11.10) in the simpler form

$$\sum_{j=1}^{n-1} p_j \operatorname{grad} P_j(\xi) + q \operatorname{grad} Q(\xi) = 0. \tag{11.16}$$

Notice that

$$\frac{\psi(v) - \varphi_k(u_k)}{\|\psi(v) - \varphi_k(u_k)\|} = -\frac{\psi(v) - \varphi_{k+1}(u_{k+1})}{\|\psi(v) - \varphi_{k+1}(u_{k+1})\|} = e_k.$$

Setting

$$b_1 = \frac{1}{\|\psi(v) - \varphi_k(u_k)\|}, \quad b_2 = \frac{1}{\|\psi(v) - \varphi_{k+1}(u_{k+1})\|},$$

we calculate

$$\frac{\partial P_j}{\partial u_k^{(r)}}(\xi) = -b_1 \left[\frac{\partial \varphi_k^{(j)}}{\partial u_k^{(r)}}(u_k) - \left\langle e_k, \frac{\partial \varphi_k}{\partial u_k^{(r)}}(u_k)\right\rangle e_k^{(j)}\right]$$

and

$$\frac{\partial P_j}{\partial u_{k+1}^{(r)}}(\xi) = -b_2 \left[\frac{\partial \varphi_{k+1}^{(j)}}{\partial u_{k+1}^{(r)}}(u_{k+1}) - \left\langle e_k, \frac{\partial \varphi_{k+1}}{\partial u_{k+1}^{(r)}}(u_{k+1})\right\rangle e_{k+1}^{(j)}\right].$$

Next, set $p_n = 0$ and $p = (p_1, \dots, p_n) \in \mathbb{R}^n$. Considering the derivatives with respect to $u_k^{(r)}$ in (11.16), we get

$$-b_1 \sum_{j=1}^{n-1} p_j \left[\frac{\partial \varphi_k^{(j)}}{\partial u_k^{(r)}}(u_k) - \left\langle e_k, \frac{\partial \varphi_k}{\partial u_k^{(r)}}(u_k)\right\rangle e_k^{(j)}\right] - q\left\langle \frac{\partial \varphi_k}{\partial u_k^{(r)}}(u_k), N(\xi)\right\rangle = 0.$$

In vector form this is

$$-b_1\left\langle p, \frac{\partial\varphi_k}{\partial u_k^{(r)}}(u_k)\right\rangle + b_1\langle e_k, p\rangle\left\langle e_k, \frac{\partial\varphi_k}{\partial u_k^{(r)}}(u_k)\right\rangle - q\left\langle N(\xi), \frac{\partial\varphi_k}{\partial u_k^{(r)}}(u_k)\right\rangle = 0,$$

which can be written as

$$\left\langle b_1 p - \langle e_k, b_1 p\rangle e_k + qN(\xi), \frac{\partial\varphi_k}{\partial u_k^{(r)}}(u_k)\right\rangle = 0.$$

This is true for all $r = 1, \dots, n-1$, so

$$b_1 p - \langle e_k, b_1 p\rangle e_k + qN(\xi) = \mu N_k \tag{11.17}$$

for some $\mu \in \mathbb{R}$, where N_k is an unit normal vector to X at $\varphi_k(u_k)$. Taking the inner product of (11.17) with e_k, gives $\mu\langle N_k, e_k\rangle = q\langle N(\xi), e_k\rangle = 0$, since the segment $[\varphi_k(u_k), \varphi_{k+1}(u_{k+1})]$ is tangent to X at $\psi(v)$ and $N(\xi)$ is a normal vector to X at $\psi(v)$. Now $\langle N_k, e_k\rangle \neq 0$ implies $\mu = 0$. Using this back in (11.17), gives

$$b_1 p - \langle e_k, b_1 p\rangle e_k + qN(\xi) = 0. \tag{11.18}$$

Similarly, using the derivatives with respect to $u_{k+1}^{(r)}$ in (11.16), we get

$$b_2 p - \langle e_k, b_2 p\rangle e_k - qN(\xi) = 0. \tag{11.19}$$

Set $b = b_1 + b_2 > 0$ and combine (11.18) and (11.19) to get

$$p = \langle e_k, p\rangle e_k. \tag{11.20}$$

So, p is parallel to e_k which combined with (11.18) gives $q = 0$. Since $\xi \in G^{-1}(0)$, we have $\chi(\xi) \in \Xi_n(s,k)$, so $\varphi_k^{(n)}(u_k) \neq \varphi_{k+1}^{(n)}(u_{k+1})$ and therefore $e_k^{(n)} \neq 0$. On the other hand, $p_n = 0$ by definition, so (11.20) implies $\langle e_k, p\rangle = 0$. Now (11.20) shows that $p = 0$. Hence all coefficients in (11.10) are zero.

This completes the proof in the case $0 < k < s$. The cases $k = 0$ and $k = s$ can be dealt with in a similar way. We leave the details as an exercise to the reader. ∎

Proof of Proposition 11.1.2(a): Given integers $0 \leq k \leq s$, the projection

$$\pi_s : \Xi_s = \mathbb{S}^{n-1} \times X^{(s)} \times X \times \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1} \times \mathbb{S}^{n-1},$$

$\pi_s(\omega; x; y; \theta) = (\omega, \theta)$, is clearly smooth. Since $\Xi_r(s,k)$ is a smooth submanifold of dimension $2n-3 < \dim(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$, by Sard's theorem, the set $\Lambda_r(s,k) = \pi_s(\Xi_r(s,k)) \subset \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ has measure zero. Clearly, $\Xi_r(s,k)$ can be written as an union of a finite or countable number of compact sets, so $\Lambda_r(s,k)$ is a finite or countable union of compact subsets of $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ of measure zero. Setting $\Lambda = \cup_{0 \leq k \leq s} \cup_{r=1}^{n} \Lambda_r(s,k)$, it follows immediately that $\mathcal{R} = (\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}) \setminus \Lambda$ has the desired properties. ∎

Proof of Proposition 11.1.3: Consider the projections

$$\tilde{\pi}_s : \Xi_s \longrightarrow \mathbb{S}^{n-1} \times X,$$

defined by $\tilde{\pi}_s(\omega; x_1, \ldots, x_s; y; \theta) = (\omega, x_1)$. Using the argument from the proof of Proposition 11.1.2(a), one shows that

$$\mathcal{T} = (\mathbb{S}^{n-1} \times X) \setminus \cup_{0 \le k \le s} \cup_{r=1}^{n} \tilde{\pi}_s(\Xi_r(s,k))$$

has the desired properties. ■

We now turn to the proof of Proposition 11.1.2(b). We begin with some additional notation and technical preparation.

For any $x \in X$ we will denote by $N(y)$ the *unit normal to* X at x pointing into Ω.

Fix an arbitrary $\omega \in \mathbb{S}^{n-1}$ and let $Z = Z_\omega$. For a given integer $k \ge 1$ denote by U_k the open subset of Z consisting of the points $x \in Z$ generating a trajectory $\gamma(u)$ of the generalized geodesic flow in Ω which is an ordinary reflecting ray with exactly k reflection points. Let $J_k(u) \in \mathbb{S}^{n-1}$ be the direction of $\gamma(u)$ after the last reflection.

Let $k \ge 1$ and $s \ge 1$ be two arbitrary integers. For $u \in U_k$ denote by $f_k(u)$ the sojourn time of the scattering ray determined by the generalized geodesic $\gamma(u)$. The definition of the set U_k shows that $f_k : U_k \longrightarrow \mathbb{R}$ is a smooth function.

Given $u \in U_k$, denote by $x_1(u), \ldots, x_k(u) \in X$ the successive reflection points of $\gamma(u)$. Clearly,

$$J_k(u) = \frac{x_k(u) - x_{k-1}(u)}{\|x_k(u) - x_{k-1}(u)\|} - 2\left\langle \frac{x_k(u) - x_{k-1}(u)}{\|x_k(u) - x_{k-1}(u)\|}, N(x_k(u)) \right\rangle N(x_k(u))$$

and

$$f_k(u) = \sum_{i=0}^{k-1} \|x_{i+1}(u) - x_i(u)\| + t - 2R,$$

where $x_0(u) = u$ and $x_{k+1}(u)$ denotes the orthogonal projection of $x_k(u))$ on $Z_{-\theta}$ with $\theta = J_k(u)$. Set $t = \|x_k(u) - x_{k+1}(u)\|$. Since $x_k - (R-t)\theta \perp \theta$, we have $\langle \theta, x_k + t\theta - R\theta \rangle = 0$. Therefore $t = R - \langle \theta, x_k \rangle$, and so

$$f_k(u) = \sum_{i=0}^{k-1} \|x_{i+1}(u) - x_i(u)\| - \langle x_k(u), J_k(u) \rangle - R.$$

Given $v \in U_s$, we denote the successive reflection points of $\gamma(v)$ by $y_1(v)$ $, \ldots, y_s(v)$; then set $y_0(v) = v$ and denote by $y_{s+1}(v)$ the orthogonal projection of $y_s(v)$ on $Z_{-\theta'}$ with $\theta' = J_s(v)$.

Let $\Delta(k,s)$ be the set of those $(u,v) \in U_k \times U_s$ such that $J_k(u) = J_s(v)$, $f_k(u) = f_s(v)$ and $\operatorname{rank} \mathrm{d}J_k(u) = \operatorname{rank} \mathrm{d}J_s(v) = n-1$.

Lemma 11.5.2: $\Delta(k, s)$ *is a smooth* $(n-2)$*-dimensional submanifold of* $U_k \times U_s$.

Proof: The argument here is very similar to that in the proof of Lemma 11.5.1.

Fix an arbitrary $w^{(0)} = (u^{(0)}, v^{(0)})$ in $\Delta(k, s)$. Since

$$\text{rank}\,(dJ_k(u^{(0)})) = \text{rank}\,(dJ_s(v^{(0)})) = n-1,$$

there exists a neighbourhood V of $w^{(0)}$ in $U_k \times U_s$ such that we have

$$\text{rank}\,(dJ_k(u)) = \text{rank}\,(dJ_s(v)) = n-1, \ (u, v) \in V.$$

The map needed here is

$$H : U \longrightarrow \mathbb{R}^n$$

defined by

$$H(u, v) = (\lambda(u, v); (\chi^{(j)}(u, v))_{1 \le j \le n-1}),$$

where

$$\lambda(u, v) = f_k(u) - f_s(v), \quad \chi(u, v) = J_k(u) - J_s(v).$$

We then have $\Delta(k, s) \cap U \subset H^{-1}(0)$, so we have to show that H is submersion at any point of $H^{-1}(0)$.

To keep the notation simple, we will just show that H is submersion at $w^{(0)}$; for all other points in $H^{-1}(0)$ the argument is the same.

Set $\theta = J_k(u_0)$. Without loss of generality we may assume $\theta^{(n)} \neq 0$. Let

$$\sum_{j=1}^{n-1} A_j \text{grad}\, \chi^{(j)}(w_0) + C \text{grad}\, \lambda(w_0) = 0 \tag{11.21}$$

for some constants A_j, C. Set $A_n = 0$, $A = (A_1, \dots, A_n) \in \mathbb{R}^n$ and

$$e_i = \frac{x_{i+1}(u_0) - x_i(u_0)}{\|x_{i+1}(u_0) - x_i(u_0)\|},$$

for $i = 1, \dots, k-1$. Then for all $p = 1, \dots, n-1$ and $i = 1, \dots, k-1$ we have

$$\begin{aligned} \frac{\partial}{\partial u_p} \|x_{i+1} - x_i\|(u_0) &= \frac{1}{\|x_{i+1} - x_i\|} \left\langle x_{i+1} - x_i, \frac{\partial x_{i+1}}{\partial u_p}(u_0) - \frac{\partial x_i}{\partial u_p}(u_0) \right\rangle \\ &= \left\langle e_i, \frac{\partial x_{i+1}}{\partial u_p}(u_0) - \frac{\partial x_i}{\partial u_p}(u_0) \right\rangle. \end{aligned}$$

Since $\dfrac{\partial x_i}{\partial u_p}(u_0)$ is tangent to X at $x_i(u_0)$, we have

$$\left\langle e_{i-1}, \frac{\partial x_i}{\partial u_p}(u_0)\right\rangle = \left\langle e_i, \frac{\partial x_i}{\partial u_p}(u_0)\right\rangle,$$

so

$$\frac{\partial f}{\partial u_p}(u_0) = \sum_{i=0}^{k-1}\left\langle e_i, \frac{\partial x_{i+1}}{\partial u_p}(u_0) - \frac{\partial x_i}{\partial u_p}(u_0)\right\rangle - \left\langle \frac{\partial x_k}{\partial u_p}(u_0), J_k(u_0)\right\rangle - \left\langle x_k, \frac{\partial J_k}{\partial u_p}(u_0)\right\rangle.$$

This and

$$\left\langle \frac{\partial x_k}{\partial u_p}(u_0), J_k(u_0)\right\rangle = \left\langle e_k, \frac{\partial x_k}{\partial u_p}(u_0)\right\rangle$$

imply

$$\frac{\partial f}{\partial u_p}(u_0) = \sum_{i=1}^{k}\left\langle e_{i-1}, \frac{\partial x_i}{\partial u_p}(u_0)\right\rangle - \sum_{i=0}^{k-1}\left\langle e_i, \frac{\partial x_i}{\partial u_p}(u_0)\right\rangle - \left\langle e_k, \frac{\partial x_k}{\partial u_p}(u_0)\right\rangle$$
$$- \left\langle x_k(u_0), \frac{\partial J_k}{\partial u_p}(u_0)\right\rangle,$$

and using $e_0 = \omega$, we get

$$\frac{\partial f}{\partial u_p}(u_0) = -\left\langle e_0, \frac{\partial x_0}{\partial u_p}(u_0)\right\rangle - \left\langle x_k(u_0), \frac{\partial J_k}{\partial u_p}(u_0)\right\rangle = -\left\langle x_k(u_0), \frac{\partial J_k}{\partial u_p}(u_0)\right\rangle.$$

Since $\langle x_k(u_0), \theta\rangle = 0$, it follows that

$$\frac{\partial \lambda}{\partial u_p}(u_0) = \frac{\partial f}{\partial u_p}(u_0) = -\left\langle x_k(u_0), \frac{\partial J_k}{\partial u_p}(u_0)\right\rangle = 0.$$

On the other hand, clearly $\frac{\partial \chi^{(j)}}{\partial u_p}(u_0) = \frac{\partial J_k^{(j)}}{\partial u_p}(u_0)$. Considering the derivatives with respect to u_p in (11.21) now gives

$$0 = \sum_{j=1}^{n-1} A_j \frac{\partial J_k^{(j)}}{\partial u_p}(u_0) - C\left\langle x_k(u_0), \frac{\partial J_k^{(j)}}{\partial u_p}(u_0)\right\rangle$$
$$= \left\langle A, \frac{\partial J_k}{\partial u_p}(u_0)\right\rangle - C\left\langle x_k(u_0), \frac{\partial J_k^{(j)}}{\partial u_p}(u_0)\right\rangle = \left\langle A - Cx_k(u_0), \frac{\partial J_k^{(j)}}{\partial u_p}(u_0)\right\rangle$$

for all $p = 1, \ldots, n-1$. Since rank $\mathrm{d}J_k(u_0) = n-1$, this yields

$$A - Cx_k(u_0) = a\theta \tag{11.22}$$

for some $a \in \mathbb{R}$. Similarly, considering the derivatives with respect to v_p in (11.21), one derives

$$A - Cy_s(v_0) = b\theta \tag{11.23}$$

for some $b \in \mathbf{R}$. Combining (11.22) and (11.23), we get $C(x_k(u_0) - y_s(v_0)) = 0$. On the other hand, $x_k(u_0) \neq y_s(v_0)$, so we must have $C = 0$, and (11.2) becomes $A = a\theta$. However $A_n = 0$ by definition, so $a\theta^{(n)} = 0$ and the assumption $\theta^{(n)} \neq 0$ implies $a = 0$. Thus $A = a\theta = 0$, that is $A_1 = \ldots = A_{n-1} = 0$. This proves that H is submersion at $w^{(0)}$. ■

Proof of Proposition 11.1.2(b): Let again $\omega \in \mathbb{S}^{n-1}$ be a fixed vector and let $Z = Z_\omega$. We will use the sets $U_k \subset Z$, the functions f_k and J_k, and Lemma 11.5.2.

Given integers k and s, the map

$$h_{k,s} : U_k \times U_s \longrightarrow \mathbb{S}^{n-1}, \quad h_{k,s}(u, v) = J_k(u)$$

is smooth and by Lemma 11.5.2, $\Delta(k, s)$ is a smooth submanifold of $U_k \times U_s$ with $\dim \Delta(s, k) = n - 2$. By Sard's theorem, $h_{k,s}(\Delta(s, k))$ is a countable union of compact subsets of $\mathbb{S}^{n-1}$ of measure zero. For any $k \geq 1$ consider the closed subset

$$F_k = \{u \in U_k\text{: rank } (dJ_k(u)) \leq n - 2\}$$

of U_k. Clearly, F_k can be represented as a countable union of compact subsets, $F_k = \cup_i F_{k,i}$. Using Sard's theorem again, $J_k(F_k)$ is a countable union of compact sets of measure zero in $\mathbb{S}^{n-1}$. Thus,

$$\mathcal{S}'(\omega) = \mathbb{S}^{n-1} \setminus (\cup_k J_k(F_k) \cup \cup_{k,s}\, h_{k,s}(\Delta(k, s))),$$

is a subset of $\mathbb{S}^{n-1}$ the complement of which is a countable union of compact subsets of $\mathbb{S}^{n-1}$ of measure zero. So, $\mathcal{S}'(\omega)$ has full measure in $\mathbb{S}^{n-1}$.

Denote by G_k the set of regular values of the map $J_k : U_k \longrightarrow \mathbb{S}^{n-1}$. By Sard's theorem again, G_k has full measure in $\mathbb{S}^{n-1}$, so $G(\omega) = \cap_{k=0}^{\infty} G_k$ has full measure in $\mathbb{S}^{n-1}$, as well. Hence $\mathcal{S}(\omega) = G(\omega) \cap \mathcal{S}'(\Omega)$ has full measure in $\mathbb{S}^{n-1}$.

We claim that $\mathcal{S}(\omega)$ has the desired properties. First, for any $\theta \in \mathcal{S}(\omega)$, we have $\theta \in G(\omega)$. Given an ordinary (ω, θ)-ray γ in Ω, it is incoming through a point $u \in U_k$ for some k, and then $\theta = J_k(u) \in G_k$ shows that $\text{rank}(dJ_k(u)) = n - 1$, that is γ is non-degenerate.

Next, for any $\theta \in \mathcal{S}'(\omega)$ any two distinct (ω, θ)-rays in Ω have different sojourn times. Indeed, let $\theta \in \mathcal{S}'(\omega)$. Assume that there exist $u \neq v$ in Z which determines ordinary reflecting (ω, θ)-rays in Ω with coinciding sojourn times. Then we have $u \in U_k$ and $v \in U_s$ for some k and s. Since $\theta = J_k(u) \notin J_k(F_k)$, we must have $\text{rank}\,(dJ_k(u)) = n - 1$. Similarly, $\text{rank}\,(dJ_s(v)) = n - 1$, so $(u, v) \in \Delta(k, s)$. Thus, $\theta = J_k(u) = h_{k,s}(u, v)$ which is again a contradiction with $\theta \in \mathcal{S}'(\omega)$. Hence any two different ordinary reflecting (ω, θ)-rays in Ω have distinct sojourn times. ■

Proof of Proposition 11.1.2(c): Define

$$\mathcal{S} = \{(\omega, \theta) \in \mathcal{R} : \theta \in \mathcal{S}(\omega)\}.$$

It follows by the properties of the sets $\mathcal{R}$ and $\mathcal{S}(\omega)$ that the complement of $\mathcal{S}$ in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ has measure zero and for any $(\omega, \theta) \in \mathcal{S}$ all (ω, θ)-trajectories for X are ordinary and non-degenerate and any two different ordinary reflecting (ω, θ)-rays in Ω have distinct sojourn times. ■

11.6 Notes

The exposition of Sections 11.1–11.4 follows mostly [S6]. The main Theorem 11.1.1 was proved in [S6] using results from [Pl] (see Theorems 5.3.2 and 9.1.2). In the special case when K is a finite disjoint union of convex bodies Theorem 11.1.1 was proved in [PS7]. Proposition 11.2.6 first appeared in [LP1]. Here we give a different (and more rigorous) proof. Proposition 11.1.2 was proved in [PS5].

12

Scattering kernel for trapping obstacles

Let Ω be a closed domain in $\mathbb{R}^n, n \geq 2$, with bounded complement K and smooth boundary $\partial\Omega$. In this chapter we show that if the obstacle $K = \mathbb{R}^n \setminus \Omega^\circ$ is trapping and $K \in \mathcal{K}$, then there exists a sequence of ordinary reflecting non-degenerate (ω_m, θ_m)-rays γ_m in Ω with sojourn times $T_{\gamma_m} \to \infty$ such that

$$-T_{\gamma_m} \in \text{sing supp } s(t, \theta_m, \omega_m).$$

Here $\mathcal{K}$ is the class of obstacles introduced in the beginning of Section 11.1. We obtain a representation of the scattering amplitude $a(\lambda, \theta, \omega)$, introduced in Section 5.1 by the cut-off outgoing resolvent of the Dirichlet Laplacian. From this we deduce a meromorphic continuation of $a(\lambda, \theta, \omega)$ in $\mathbb{C}$ for n odd and in the logarithmic covering of $\mathbb{C}$ for n even. Finally, we introduce weakly non-degenerate trapping rays and examine the estimate of the scattering amplitude if there exists at least one such ray.

12.1 Scattering rays with sojourn times tending to infinity

Let K be a compact subset of $\mathbb{R}^n, n \geq 2$, with smooth boundary and $\Omega = \overline{R^n \setminus K}$. Set $Q = \mathbb{R} \times \Omega$ and denote by (τ, ξ) the variables dual to (t, x) in $T^*(Q)$. The *characteristic set* of the wave operator $\Box$ has the form

$$\Sigma = \{(t, x, \tau, \xi) \in T^*(Q) \setminus \{0\} : \tau^2 =| \xi|^2\}.$$

Geometry of the Generalized Geodesic Flow and Inverse Spectral Problems, Second Edition.
Vesselin M. Petkov and Luchezar N. Stoyanov.

Consider the generalized bicharacteristics of the operator $\Box$ introduced in Section 1.2. In general, the generalized flow

$$\mathcal{F}_t : (0, x_0, 1, \xi_0) \to (x(t), \xi(t))$$

is not smooth, and in some cases there may exist two different integral curves issued from the same point in the phase space (see [T] for an example). To avoid this situation throughout this chapter, recall that an obstacle K belongs to the class $\mathcal{K}$ introduced in Section 11.1, if at any point $(x, \xi) \in T^*(\partial K)$, where the curvature of ∂K vanishes of infinite order in direction ξ, the boundary ∂K is convex in direction ξ. Then every generalized bicharacteristic of $\Box$ is uniquely extendible and every generalized ray with finite length can be uniformly approximated by ordinary reflecting ones (see Section 7 in [MS2]).

Consider the *compressed cotangent bundle* $\tilde{T}^*(Q)$ introduced in Section 1.4 and the map

$$\sim : T^*(Q) \ni (t, x, \tau, \xi) \to (t, x, \tau, \xi|_{T_x(\partial\Omega)}) \in T^*(\partial Q)$$

defined as identity on $T^*(Q \setminus \partial Q)$ (see Section 1.2). Recall that $\tilde{\Sigma} = \Sigma_b$ is the *compressed characteristic set*, and the image $\tilde{\gamma} = \sim (\gamma)$ of a generalized bicharacteristic γ of $\Box$ is a *compressed generalized bicharacteristic*.

Let $\rho_0 > 0$ be fixed so that $K \subset B_0 = \{x \in \mathbb{R}^n : \|x\| \leq \rho_0\}$. Given a point $\nu = (0, x, 1, \xi) \in \Sigma_b, (x, \xi) \in T^*(\partial\Omega)$, consider the compressed generalized bicharacteristic $\gamma_\nu(t) = (t, x(t), 1, \xi(t)) \in T^*(Q)$ of $\Box$, parameterized by the time t and passing through ν for $t = 0$. Denote by $T(\nu) \in \mathbb{R}^+ \cup \infty$ the maximal $T > 0$ such that $x(t) \in B_0$ for $0 \leq t \leq T(\nu)$, and denote by Σ_∞ the set of those $\nu = (0, x, 1, \xi) \in \Sigma_b, (x, \xi) \in T^*(\partial\Omega)$ such that $T(\nu) = \infty$. By using the continuity of the generalized Hamiltonian flow of $\Box$ (see Section 1.2 and Theorem 3.22 in [MS2]), it is easy to see that Σ_∞ is closed in Σ_b. On the other hand, $\Sigma_\infty \neq \Sigma_b$. Indeed, take a hyperplane Π tangent to $\partial\Omega$ such that K is contained in a half-space determined by Π. Consider an arbitrary $\nu_0 = (0, x_0, 1, \xi_0) \in \Sigma_b$ with $(x_0, \xi_0) \in T^*(\partial\Omega)$, $x_0 \in \partial\Omega \cap \Pi$ and ξ_0 tangent to $\partial\Omega$. Then we have $T(\nu_0) < \infty$, since $\gamma_{\nu_0}(t)$ leaves B_0 for $t > 2\rho_0$.

Definition 12.1.1: *The obstacle K is trapping if $\Sigma_\infty \neq \emptyset$.*

For trapping obstacles the boundary $\partial\Sigma_\infty$ of Σ_∞ in Σ_b is not empty. Choose an arbitrary $\hat{\nu} \in \partial\Sigma_\infty$. Since $\Sigma_b \setminus \Sigma_\infty \neq \emptyset$, there exists a sequence of points $\nu_m = (0, x_m, 1, \xi_m) \in \Sigma_b$ with $(x_m, \xi_m) \in T^*(\partial\Omega)$ such that $\nu_m \notin \Sigma_\infty$ for all m and $\nu_m \to \hat{\nu} \in \Sigma_\infty$. Consider the compressed generalized bicharacteristics

$$\gamma_{\nu_m}(t) = (t, x_m(t), 1, \xi_m(t))$$

passing through ν_m for $t = 0$ and such that $T(\nu_m) < \infty$. If the sequence $\{T(\nu_m)\}$ is bounded, this would imply $T(\hat{\nu}) < \infty$ in contradiction with $\hat{\nu} \in \Sigma_\infty$. Therefore, $\{T(\nu_m)\}$ is unbounded and we may assume $T(\nu_m) \longrightarrow_{m \longrightarrow \infty} +\infty$. Set

$$y_m = x_m(T(\nu_m)) \in \partial B_0, \ \omega_m = \xi_m(T(\nu_m)) \in \mathbb{S}^{n-1}.$$

Passing to a subsequence, we may assume that $y_m \to \hat{z} \in \partial B_0$ and $\omega_m \to \hat{\omega} \in \mathbb{S}^{n-1}$. Consider the generalized bicharacteristic $\gamma_\mu(t) = (t, y(t), 1, \xi(t))$ of $\Box$ issued from $\mu = (0, \hat{z}, 1, \hat{\omega})$. Then by continuity one obtains $T(\gamma_\mu) = \infty$ and $y(t) \in B_0$ for $t \geq 0$.

As in Section 2.4, let $Z_{\hat{\omega}}$ be the hyperplane passing through $\hat{z}$ and orthogonal to $\hat{\omega}$. Denote by Z_∞ the set of those points $y \in Z_{\hat{\omega}}$ such that the generalized bicharacteristic γ_{μ_y} passing through $\mu_y = (0, y, 1, \hat{\omega})$ has the property $T(\mu_y) = \infty$. A simple argument shows that Z_∞ is closed in $Z_{\hat{\omega}}$ and clearly $Z_\infty \neq Z_{\hat{\omega}}$. Consequently, there exists a sequence $z_m \to y_0$ with $z_m \in Z_{\hat{\omega}} \setminus Z_\infty$ for all m such that $T(\gamma_{z_m}) < \infty$ for all m and $T(\gamma_{z_m}) \to \infty$. In general the bicharacteristic $\gamma_{\mu_{z_m}}$ could contain gliding or glancing segments. According to the results in [MS2] mentioned above and the fact that $K \in \mathcal{K}$, every generalized bicharacteristic $\gamma_{\mu_{z_m}}$ can be approximated by multiple ordinary reflecting rays δ_m with $T_{\delta_m} \to +\infty$. Moreover, applying Proposition 11.2.6, we may assume that δ_m are unbounded in both directions, that is δ_m is a $(\hat{\omega}, \theta_m)$-ray for some $\theta \in \mathbb{S}^{n-1}$. Thus, taking (z'_m, ω'_m) sufficiently close to $(z_m, \hat{\omega})$, we obtain the following result.

Proposition 12.1.2: *Let the obstacle $K \in \mathcal{K}$ be trapping. Then there exists a sequence of ordinary reflecting (ω'_m, θ'_m)-rays γ_m such that $T_{\gamma_m} \longrightarrow \infty$.*

Now consider an obstacle K having the form

$$K = \cup_{j=1}^{N} K_j, \;\; K_i \cap K_j = \emptyset \text{ for } i \neq j, \tag{12.1}$$

where K_j is convex for all $j = 1, \ldots, N$. Obviously, $K \in \mathcal{K}$ and $\Sigma_\infty \neq \emptyset$. Thus, we obtain the following corollary.

Corollary 12.1.3: *Let the obstacle K have the form* (12.1). *Then there exists a sequence of ordinary reflecting (ω'_m, θ'_m)-rays γ_m such that $T_{\gamma_m} \longrightarrow \infty$.*

Notice that Theorem 11.1.1 cannot be applied directly since it is not known whether the sequence $\{\omega'_m, \theta'_m\}$ is in the subset $\mathcal{R} \subset \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$. Recall that, following our analysis in Section 9.1, to obtain the singularity of $s(t, \theta, \omega)$, produced by the sojourn time of an ordinary reflection (ω, θ)-ray γ, we need to know that

$$\gamma \text{ is a non} - \text{degenerate ordinary reflecting ray} \tag{12.2}$$

and

$$T_\gamma \neq T_\delta \;\text{ for every }\; \delta \in \mathcal{L}_{(\omega,\theta)}(\Omega) \setminus \{\gamma\}. \tag{12.3}$$

First we will examine the case when K has the form (12.1). For $z \in \partial K$ denote by $\mathcal{K}(z)$ the Gauss curvature of ∂K at z.

Lemma 12.1.4: *Let K have the form* (12.1) *and let γ be an ordinary reflecting (ω, θ)-ray with reflection points $x_1, \ldots, x_k$. Suppose that there exists $1 \leq j \leq k$ such that $\mathcal{K}(x_j) > 0$. Then the ray γ is non-degenerate.*

Proof: Using the notation of Section 2.4, for the map $J_\gamma(u_\gamma)$ one has the representation

$$dJ_\gamma(u_\gamma)u = M_k\sigma_k(I+\lambda_k M_{k-1})\sigma_{k-1}(I+\lambda_{k-1}M_{k-2})\cdots\sigma_2(I+\lambda_2 M_1)\sigma_1 u$$

given in Proposition 2.4.2. Recall that $\lambda_i = \|x_{i-1}-x_i\|, i=1,\dots,k, x_0 = u_\gamma$, σ_i is a linear map associated with the symmetry with respect to the tangent plane to ∂K at x_i and M_i are symmetric linear maps having the form $M_1 = \tilde{\psi}_1, M_i = \sigma_i M_{i-1}(I+\lambda_i M_{i-1})^{-1}\sigma_i + \tilde{\psi}_i, i = 2,\dots,k$, where $\tilde{\psi}_i$ is a linear symmetric map depending on the second fundamental form of ∂K at x_i. The obstacles K_j are convex, so the maps $\tilde{\psi}_i \geq 0$ are definitive non-negative for every $i=1,\dots,k$. By induction this implies that $M_i \geq 0$ for $i=1,\dots,k$. By assumption we have $\mathcal{K}(x_j) > 0$, hence $\tilde{\psi}_j > 0$. One deduces that $M_i > 0$ for $i = j, j+1,\dots,k$. Therefore, if $dJ_\gamma(u_\gamma)u = 0$, we get $u = 0$ and the map $dJ_\gamma(u_\gamma)$ is invertible. ■

Next, we consider a fixed ordinary reflecting (ω_m,θ_m)-ray γ_m which is non-degenerate. We wish to replace (ω_m,θ_m) by a pair (ω'_m,θ'_m) sufficiently close to (ω_m,θ_m) for which there exist ordinary reflecting non-degenerate (ω'_m,θ'_m)-rays δ_m such that T_{δ_m} satisfy (12.3). By Proposition 11.1.2, we know that we may find an approximation by ordinary reflecting rays (ω'_m,θ'_m)-rays for which (12.3) holds. We wish to show that the property of an ordinary ray to be non-degenerate is locally preserved.

To do this, we use a corollary of the inverse mapping theorem (cf. [Hl], Theorem 1.1.7). Let U and V be open subsets of $\mathbb{R}^m$ and let $F : U \ni x \mapsto f(x) \in V$ be a C^∞ map. Suppose that $x_0 \in U$ is such that $\det df(x_0) \neq 0$. Then

$$\alpha = 1/\|df(x_0)^{-1}\| > 0,$$

$\|\cdot\|$ being the standard norm in the space of linear maps $\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)$. Set $y_0 = f(x_0)$ and choose $\delta > 0$ small enough so that $U_\delta = \{x \in \mathbb{R}^m : \|x - x_0\| < \delta\} \subset U$, $\|df(x) - df(x_0)\| \leq \frac{\alpha}{2}$ for $x \in U_\delta$, $V_\delta = \{y \in \mathbb{R}^m : \|y - y_0\| < \frac{\delta\alpha}{2}\} \subset V$. Then it follows by the inverse mapping theorem that the map f is injective on U_δ and surjective on V_δ.

Now let $x_0 = z_m$ and $\omega_0 = \omega_m$. Consider the hyperplane $Z = Z_{\omega_0}$. For ω sufficiently close to ω_0, the (ω,θ)-rays issued from $y \in Z_\omega$ in direction ω can be considered as suitable (ω,θ)-rays issued from a point $x \in Z$, provided y is close enough to x_0. Thus, we obtain a C^∞ map

$$U = \mathcal{O}\times\Gamma \ni (x,\omega) \mapsto f(x,\omega) \in \mathbb{S}^{n-1},$$

where $\mathcal{O} \subset Z$ is a small neighbourhood of x_0, $\Gamma \subset \mathbb{S}^{n-1}$ is a small neighbourhood of ω_0 and $f(x,\omega)$ is the outgoing direction of the ray issued from x in direction ω. Since γ_m is non-degenerate by assumption, we have $\det df_x(x_0,\omega_0) \neq 0$. We may assume that U is chosen so small that $\det df_x(x,\omega) \neq 0$ holds for all $(x,\omega) \in \overline{U}$. Let

$$\max_{(x,\omega)\in U} \|(df_x(x,\omega))^{-1}\| = \frac{1}{\alpha}.$$

Then there exists $\delta > 0$ such that for $(x, \omega) \in U$ with $\|x - x_0\| < \delta, \|\omega - \omega_0\| < \delta$, we have $\|df_x(x, \omega) - df_x(x_0, \omega_0)\| \leq \frac{1}{4}\alpha$. We may assume that δ is so small that

$$\mathcal{O}_\delta = \{x \in Z : \|x - x_0\| < \delta\} \subset \mathcal{O},$$
$$\Gamma_\delta = \{\omega \in \mathbb{S}^{n-1} : \|\omega - \omega_0\| < \delta\} \subset \Gamma.$$

Clearly, for $\omega \in \Gamma_\delta$ fixed the map $\mathcal{O}_\delta \ni x \mapsto f(x, \omega) \in \mathbb{S}^{n-1}$ is injective. Denote $\theta_0 = f(x_0, \omega_0)$ and consider the set $W_\delta = \{\theta \in \mathbb{S}^{n-1} : \|\theta - \theta_0\| < \frac{\delta\alpha}{4}\}$. Choose $\delta' \in (0, \delta)$ so small that $\|f(x_0, \omega) - \theta_0\| < \frac{\delta\alpha}{4}$ for $\omega \in \Gamma_{\delta'}$. Then for $\omega \in \Gamma_{\delta'}$ and $\theta \in W_\delta$, we deduce $\|\theta - f(x_0, \omega)\| < \frac{\delta\alpha}{2}$. Consequently, applying the version of inverse mapping theorem mentioned above, for each fixed $\omega \in \Gamma_{\delta'}$ and each fixed $\theta \in W_\delta$ we can find $x_{(\omega,\theta)} \in \mathcal{O}_\delta$ with $f(x_{(\omega,\theta)}, \omega) = \theta$. Thus, locally we obtain non-degenerate (ω, θ) ordinary reflecting rays. Combining this with the properties (a) and (b) of Proposition 11.1.2, we obtain the following.

Theorem 12.1.5: *Let K have the form* (12.1). *Assume that there are no points $z \in \partial K$ such that the Gauss curvature $\mathcal{K}(u)$ of ∂K vanishes for every u in some neighbourhood U_z of z in ∂K. Then there exists a sequence of ordinary reflecting non-degenerate (ω_m, θ_m)-rays in $\bar{\Omega}$ with sojourn times $T_m \to \infty$ such that for t near $-T_m$, we have*

$$s(t, \theta_m, \omega_m) = A_m \delta^{(n-1)/2}(t + T_m) + \text{lower order singularities}$$

with $A_m \neq 0$.

In the case $n = 3$ the assumption of Theorem 12.1.5 means that there are no points $z \in \partial K$ such that the standard metric on ∂K is locally flat around z.

To satisfy the condition (12.2), we need to construct a sequence of ordinary reflecting non-degenerate rays. If we have a degenerate ray γ_m, we must replace it by non-degenerate one with sojourn time sufficiently close to T_γ. Let $C = \{x \in \mathbb{R}^n : \|x\| = \rho_0\}$ be the boundary of B_0. We may study the rays issued from a small neighbourhood $W \subset C \times \mathbb{S}^{n-1}$ of the point $(\hat{z}, \hat{\omega}) \in C \times \mathbb{S}^{n-1}$ introduced in proof of Proposition 12.1.2. Let $\mathcal{O}(W)$ be the set of all pairs of directions $(\omega, \theta) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ for which there exists an ordinary reflecting (ω, θ)-ray issued from $(x, \omega) \in W$ with *outgoing* direction $\theta \in \mathbb{S}^{n-1}$. To establish an approximation with (ω, θ)-rays issued from W, it is useful to know that $\mathcal{O}(W)$ has a positive measure in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ for all sufficiently small neighbourhoods $W \subset C \times \mathbb{S}^{n-1}$ of $(\hat{z}, \hat{\omega})$. For this purpose, one introduces the following.

Definition 12.1.6: A generalized bicharacteristic γ issued from $(y, \eta) \in C \times \mathbb{S}^{n-1}$ is called weakly non-degenerate if for every neighbourhood $W \subset C \times \mathbb{S}^{n-1}$ of (y, η) the set $\mathcal{O}(W)$ has a positive measure in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$.

This definition generalizes that of non-degenerate ordinary reflecting ray given in Section 2.4. This follows from the above argument based on the implicit function theorem.

Remark 12.1.7: In general a weakly non-degenerate ordinary reflecting ray does not need to be non-degenerate. In fact, the set of points $(y, \eta) \in C \times \mathbb{S}^{n-1}$ that generate weakly non-degenerate bicharacteristics is closed in $C \times \mathbb{S}^{n-1}$. As an example, consider the special case when K is convex with vanishing Gauss curvature at some point $x_0 \in \partial K$ and strictly positive Gauss curvature at any other point of ∂K. Consider a reflecting ray γ in $\mathbb{R}^n$ with a single reflection point at x_0. Therefore, it is easy to see that γ is degenerate following the definition in Section 2.4. However, arbitrarily close to γ, we can find an ordinary reflecting ray δ_m with a single reflection point $x_m \neq x_0$. Then δ_m is non-degenerate and hence it is weakly non-degenerate. Thus, γ can be approximated arbitrarily well with weakly non-degenerate rays, and since the set of points generating weekly non-degenerate rays is close, the ray γ itself is weakly non-degenerate.

For obstacles with weekly non-degenerate generalized bicharacteristic we have the following.

Theorem 12.1.8: *Let the obstacle $K \in \mathcal{K}$ have at least one trapping weakly non-degenerate bicharacteristic δ issued from $(y, \eta) \in C \times \mathbb{S}^{n-1}$. Then there exists a sequence of ordinary reflecting non-degenerate (ω_m, θ_m)-rays γ_m with sojourn times $T_{\gamma_m} \longrightarrow \infty$ so that*

$$-T_{\gamma_m} \in \text{sing supp } s(t, \theta_m, \omega_m), \quad \forall m \in \mathbb{N}. \tag{12.4}$$

Proof: Let $W_m \subset C \times \mathbb{S}^{n-1}$ be a neighbourhood of (y, η) such that for every $z \in W_m$, the generalized bicharacteristic γ_z issued from z satisfies the condition $T(\gamma_z) > m$. For every $m \in N$ the continuity of the compressed generalized flow guarantees the existence of W_m and one has $W_{m+1} \subset W_m$. Consider the open subset F_m of $C \times \mathbb{S}^{n-1} \times C \times \mathbb{S}^{n-1}$ consisting of the points (x, ω, z, θ) such that $(x, \omega) \in W_m$, and there exists an ordinary reflecting (ω, θ)-ray issued from $(x, \omega) \in W_m$ and passing through z with direction θ.

The projection $F_m \ni (x, \omega, z, \theta) \longrightarrow (\omega, \theta)$ is smooth and Sard's theorem implies the existence of a set $\mathcal{D}_m \subset \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ with measure zero so that if $(\omega, \theta) \notin \mathcal{D}_m$ the corresponding (ω, θ)-ray issued from $(x, \omega) \in W_m$ is non-degenerate. Then the set $\mathcal{O}(W_m) \setminus \mathcal{D}_m$ has a positive measure and taking $(\omega_m, \theta_m) \in \mathcal{O}(W_m) \setminus \mathcal{D}_m$, we obtain an ordinary reflecting non-degenerate (ω_m, θ_m)-ray δ_m with sojourn time T_m issued from $z_m \in W_m$. Next we choose

$$q(m) > \max\{m+1, T_m\}, \; q(m) \in \mathbb{N}$$

and repeat the same argument for $W_{q(m)}$ and $F_{q(m)}$. This proves the existence of a sequence of ordinary reflecting non-degenerate (ω_m, θ_m)-rays γ_m with sojourn times $T_{\gamma_m} \to \infty$.

Now let γ_m be an ordinary reflecting non-degenerate (ω_m, θ_m)-ray issued from $C \times \mathbb{S}^{n-1}$. Applying (a) and (b) of Proposition 11.1.2, we deduce that for almost all directions $(\omega, \theta) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$, the sojourn times of ordinary reflecting rays

(ω, θ)-ray are different. Hence we can approximate (ω_m, θ_m) by directions (ω'_m, θ'_m) so that the sojourn times of the corresponding ordinary reflecting non-degenerate (ω'_m, θ'_m)-rays are different. Next, since γ_m is non-degenerate, by using the inverse mapping theorem, it is possible to find ordinary reflecting non-degenerate (ω''_m, θ''_m)-rays γ'' with sojourn time T''_γ sufficiently close to T_{γ_m} so that (12.2) and (12.3) hold for γ''_m. This completes the proof. ■

12.2 Scattering amplitude and the cut-off resolvent

We start with the representation (5.3) of the scattering kernel given in Section 5.1. Consider the solution of the problem

$$\begin{cases} (\partial_t^2 - \Delta_x) w(t, x; \omega) = 0 \text{ in } \mathbb{R} \times \Omega^\circ, \\ w = 0 \text{ on } \mathbb{R} \times \partial\Omega, \\ w|_{t<-\rho_0} = \delta(t - \langle x, \omega \rangle). \end{cases} \tag{12.5}$$

A finite speed of propagation argument implies that

$$\text{supp } w(t, x; \omega)|_{x \in \partial\Omega} \subset \{(t, x) \in \mathbb{R} \times \partial\Omega : \; t \geq \langle x, \omega \rangle\}.$$

Thus,

$$\text{supp } w(\langle x, \theta \rangle - s)|_{x \in \partial\Omega} \subset \{(s, x) \in \mathbb{R} \times \partial\Omega : \; s \leq \langle x, \theta - \omega \rangle\}$$

and we deduce

$$\text{supp}_t \, s(t, \theta, \omega) \subset \{t \in \mathbb{R} : \; t \leq \max_{x \in \partial\Omega} \langle x, \theta - \omega \rangle\}. \tag{12.6}$$

For $\theta = -\omega$, the above-mentioned relation yields

$$\text{supp}_t (t, -\omega, \omega) \leq -2\rho(\omega), \tag{12.7}$$

where $\rho(\omega) = \min_{x \in \partial\Omega} \langle x, \omega \rangle$ is the *support function* in direction ω. For strictly convex obstacles one has

$$\langle x_+, \theta - \omega \rangle = -T_{\gamma_+},$$

where T_{γ_+} is the sojourn time of the ordinary reflecting (ω, θ)-ray reflecting at $x_+ \in \partial\Omega$ and for $\theta \neq \omega$ we have

$$\text{sing supp}_t \, s(t, \theta, \omega) = \{ -T_{\gamma_+} \}.$$

Consequently, for strictly convex obstacles, the singularities of the scattering kernel determine the obstacle. For arbitrary obstacles we may determine the convex hull of the obstacle from (12.7) and the maximal singularity $-2\rho(\omega)$ of $s(t, -\omega, \omega)$, where ω belongs to an open dense set in $\mathbb{S}^{n-1}$ (see [Ma2] for more details).

Now consider the Fourier transform

$$\hat{s}(\lambda, \theta, \omega) = \mathcal{F}_{t \longrightarrow \lambda}(s(t, \theta, \omega)) = \int e^{-\mathbf{i}\lambda t} s(t, \theta, \omega) dt,$$

where the integral is interpreted in the sense of tempered distributions. Therefore,

$$\overline{\hat{s}(\lambda, \theta, \omega)} = \int e^{\mathbf{i}\lambda t} \overline{s(t, \theta, \omega)} dt$$

admits an analytic continuation for $\operatorname{Im} \lambda < 0$ since $\operatorname{supp}_t s(t, \theta, \omega) \subset (-\infty, 2\rho_0]$. The same is true for the scattering amplitude $a(\lambda, \theta, \omega)$ defined in Section 5.1.

Recall the representation (5.5) of $a(\lambda, \theta, \omega)$. Ignoring the factor $(\mathbf{i}\lambda)^{(n-3)/2}$, which for n even has a singularity at 0, we will study the integral

$$\int_{\partial\Omega} \left[e^{\mathbf{i}\lambda\langle x, \theta \rangle} \partial_\nu v_{sc}(x, \lambda; \omega) - \mathbf{i}\lambda e^{\mathbf{i}\lambda\langle x, \theta - \omega \rangle} \langle \nu, \omega \rangle \right] dS_x, \tag{12.8}$$

where $v_{\text{sc}}(x, \lambda; \omega)$ is a $(\mathbf{i}\lambda)$-outgoing solution of the problem (5.6).

Consider the problem

$$\begin{cases} (\Delta + \lambda^2) u(x, \lambda) = f \text{ in } \Omega^\circ, \\ u(x, \lambda) = 0 \text{ on } \partial\Omega, \\ u(x, \lambda) \text{ is } (\mathbf{i}\lambda) - \text{outgoing}. \end{cases} \tag{12.9}$$

For $\operatorname{Im} \lambda \leq 0$ and $f \in L^2(\Omega)$ this problem has an unique solution

$$u(x, \lambda) = (-\Delta_D - \lambda^2)^{-1} f$$

and $R(\lambda) = (-\Delta_D - \lambda^2)^{-1}$ is the $(\mathbf{i}\lambda)$-outgoing resolvent of the Dirichlet Laplacian. Moreover,

$$R(\lambda) = (-\Delta_D - \lambda^2)^{-1} : L^2_{comp}(\Omega) \ni f \longrightarrow u(x, \lambda) \in H^2_{loc}(\Omega), \ \operatorname{Im} \lambda \leq 0$$

has a meromorphic extension in $\mathbb{C}$ for n odd and in the logarithmic covering of $\mathbb{C}$ for n even (see for instance [LP1], [LP2], [Z1]) with poles λ_j, $\operatorname{Im} \lambda_j > 0$.

We will express $v_{sc}(x, \lambda; \omega)|_{\partial\Omega}$ by using the operator $R(\lambda)$. Consider a function $\varphi_1 \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi_1 = 1$ on a neighbourhood of K. We get

$$\begin{aligned} v_{sc}(x, \lambda; \omega) + \varphi_1(x) e^{-\mathbf{i}\lambda\langle x, \omega \rangle} &= -R(\lambda) \left((\Delta + \lambda^2)(\varphi_1 e^{-\mathbf{i}\lambda\langle x, \omega \rangle}) \right) \\ &= -R(\lambda)([\Delta, \varphi_1] e^{-\mathbf{i}\lambda\langle x, \omega \rangle}). \end{aligned}$$

Next, choose a function $\varphi_2(x) \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi_2(x) = 1$ on a neighbourhood of K and $\varphi_1(x) = 1$ on $\operatorname{supp} \varphi_2$. Then the normal derivative becomes

$$\left. \frac{\partial v_{sc}}{\partial \nu} \right|_{\partial\Omega} = \mathbf{i}\lambda e^{-\mathbf{i}\lambda\langle x, \omega \rangle} \langle \nu, \omega \rangle|_{\partial\Omega} - \frac{\partial}{\partial \nu} \left(\varphi_2 R(\lambda)([\Delta, \varphi_1] e^{-\mathbf{i}\lambda\langle x, \omega \rangle} \right) |_{\partial\Omega}$$

and the term $\mathbf{i}\lambda e^{\mathbf{i}\lambda\langle x,\theta-\omega\rangle}\langle\nu,\omega\rangle|_{\partial\Omega}$ cancels the same term with sign $(-)$ in (12.8). On the other hand, by using Green's formula, we obtain

$$\int_{\Omega} e^{\mathbf{i}\lambda\langle x,\theta\rangle}(\Delta+\lambda^2)\Big(\varphi_2 R(\lambda)([\Delta,\varphi_1]e^{-\mathbf{i}\lambda\langle x,\omega\rangle})\Big)dx$$
$$= -\int_{\partial\Omega} e^{\mathbf{i}\lambda\langle x,\theta\rangle}\partial_\nu\Big(\varphi_2 R(\lambda)([\Delta,\varphi_1]e^{-\mathbf{i}\lambda\langle x,\omega\rangle})\Big)dS_x$$
$$= \int_{\mathbb{R}^n} e^{\mathbf{i}\lambda\langle x,\theta\rangle}[\Delta,\varphi_2]R(\lambda)([\Delta,\varphi_1]e^{-\mathbf{i}\lambda\langle x,\omega\rangle})dx.$$

To see that the last integral is independent of the choice of φ_2, assume that $\varphi_1 \in C_0^\infty(\mathbb{R}^n)$ is equal to 1 on the support of $\chi_i \in C_0^\infty(\mathbb{R}^n), i = 1, 2$, while χ_i are equal to 1 on K. Let $v_1 = e^{\mathbf{i}\lambda\langle x,\theta\rangle}$, $v_2 = e^{\mathbf{i}\lambda\langle x,\omega\rangle}$. Then the last integral has the form

$$\Big(R(\lambda)[\Delta,\chi_1-\chi_2]v_1, [\Delta,\varphi_1]v_2\Big)_{L^2(\Omega)}$$

and

$$\Big(R(\lambda)((\Delta+\lambda^2)(\chi_1-\chi_2)-(\chi_1-\chi_2)(\Delta+\lambda^2))v_1, [\Delta,\varphi_1]v_2\Big)_{L^2(\Omega)}$$
$$= \Big(-(\chi_1-\chi_2)v_1, [\Delta,\varphi_1]v_2\Big)_{L^2(\Omega)} = 0.$$

In the same way we can assume that $\varphi_2 = 1$ on the support of φ_1 so we may switch the conditions on φ_2 and φ_1. Indeed, let $\varphi_1 = 1$ on the support $\tilde{\varphi}_1$, $\tilde{\varphi}_1 = 1$ on a neighbourhood of K and let $\varphi_2 = 1$ on the supports of φ_1 and $\tilde{\varphi}_1$. Then by the same argument, we get

$$v_1[\Delta,\varphi_2-\tilde{\varphi}_1]R(\lambda)[\Delta,\varphi_1]\bar{v_2} = 0$$

since $[\Delta,\tilde{\varphi}_1]\varphi_1 = 0$ and $\varphi_2[\Delta,\varphi_1] = 0$. Thus, we obtain the following.

Proposition 12.2.1: *Let $\varphi_i \in C_0^\infty(\mathbb{R}^n), i = 1, 2$, be such that $\varphi_i = 1$ on $\bar{K}$ and let $\varphi_1 = 1$ on the support of φ_2 or $\varphi_2 = 1$ on the support of φ_1. Then the scattering amplitude has the representation*

$$a(\lambda,\theta,\omega) = -\frac{(\mathbf{i}\lambda)^{(n-3)/2}}{2(2\pi)^{(n-1)/2}}\int_{\Omega} e^{\mathbf{i}\lambda\langle x,\theta\rangle}[\Delta,\varphi_1]R(\lambda)[\Delta,\varphi_2]e^{-\mathbf{i}\lambda\langle x,\omega\rangle}dx,\ \lambda\in\mathbb{R}, \tag{12.10}$$

and this representation is independent of the choice of φ_1 and φ_2.

Remark 12.2.2: It is easy to see that if we replace in (12.10) the resolvent of the Dirichlet Laplacian $R(\lambda)$ by the resolvent $R_0(\lambda) = (-\Delta-\lambda^2)^{-1}$ of the Laplacian in $\mathbb{R}^n$, then the integral (12.10) vanishes. This follows from the fact that $R_0(\lambda)$ and Δ commute.

Let $\psi(x) \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function such that $\psi(x) = 1$ on supp $\varphi_i, i = 1, 2$. Then we obtain a representation of $a(\lambda,\theta,\omega)$ by the cut-off resolvent

$R_\psi(\lambda) = \psi(x)R(\lambda)\psi(x)$. Since the cut-off resolvent has a meromorphic continuation in $\mathbb{C}$ for n odd and in the logarithmic covering of $\mathbb{C}$ for n even, we obtain a meromorphic continuation of the scattering amplitude $a(\lambda, \theta, \omega)$ with poles in $\text{Im } z > 0$.

For $\omega \in \mathbb{S}^{n-1}$ introduce the operators

$$L^2_{comp}(\mathbb{R}^n) \ni f \to (\mathbf{E}_\pm(z,\omega)f)(\omega) = \int_{\mathbb{R}^n} e^{\pm \mathbf{i} z\langle x,\omega\rangle} f(x)dx \in L^2(\mathbb{S}^{n-1}),$$

$$L^2(\mathbb{S}^{n-1}) \ni g \to ({}^t\mathbf{E}_\pm(z,\omega)g)(x) = \int_{\mathbb{S}^{n-1}} e^{\pm \mathbf{i} z\langle x,\omega\rangle} g(\omega)d\omega \in L^2_{loc}(\mathbb{R}^n).$$

The operator

$$(K(\lambda)f)(\theta) = \int_{\mathbb{S}^{n-1}} a(\lambda,\theta,\omega)f(\omega)d\omega, \ \lambda \in \mathbb{R}$$

is called *far-field* operator. Choosing $\chi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on the supports of $\varphi_i, i = 1, 2$, we get the representation

$$(K(\lambda)f)(\theta) = c_n(\lambda^{(n-3)/2})\,{}^t\mathbf{E}_+(\lambda,\theta)[\Delta,\varphi_1]R(\lambda)[\Delta,\varphi_2]\mathbf{E}_-(\lambda,\omega)f. \quad (12.11)$$

The operator

$$S(\lambda) = Id + K(\lambda), \lambda \in \mathbb{R}$$

is called *scattering matrix* (see [LP1] for odd dimensions and [PZ], [Z1] for even dimension). For odd dimensions the operator $S(\lambda)$ is analytic in $\{z \in \mathbb{C} : \ \text{Im } z \leq 0\}$ and has a meromorphic continuation for $\text{Im } z > 0$. For even dimensions one obtains a meromorphic continuation on the logarithmic covering of $\mathbb{C}$.

It is possible to show that the scattering matrix $S(\lambda)$ has the same poles as the operator

$$R(\lambda) : L^2_{comp}(\Omega) \to H^2_{loc}(\Omega)$$

and the multiplicities of these poles coincide (see [LP1] for n odd and [Z1], [PZ] for n even). In the following we discuss only the generic case of simple poles and refer to [PZ], [Z1] for multiple poles and their multiplicities.

Let $z_0 \in \mathbb{C}$ be a simple pole of $R(\lambda)$ in a neighbourhood $\mathcal{U} \subset \mathbb{C}$ of z_0 and let

$$R(\lambda) = \frac{A}{z - z_0} + B(z),$$

where A is a rank one operator and the operator-valued function $B(z)$ is analytic in $\mathcal{U}$. Therefore, there exist $\Phi, \Psi \in L^2_{loc}(\Omega)$ so that

$$Af = (f, \Psi)_{L^2(\Omega)}\Phi, \ \ \forall f \in L^2_{comp}(\Omega).$$

It is clear that $(\Delta + z_0^2)A = 0$ and choosing f so that $C = (f, \Psi)_{L^2(\Omega)} \neq 0$, we get $(\Delta + z_0^2)\Phi = 0$. On the other hand, applying Cauchy integral formula, we have

$$\Phi = \frac{Af}{C} = \frac{1}{2\pi \mathbf{i} C}\int_{|z-z_0|=\epsilon} R(z)f \ dz,$$

provided $\epsilon > 0$ small enough. To prove that Φ is z_0-outgoing, we take $R > \rho_0$ large enough and consider $\mathbb{1}_{|x|\geq R}R(z)f$. Choosing a function $\chi \in C^\infty(\Omega)$ such that $\chi = 0$ for $|x| \leq \rho_0, \chi = 0$ on supp f and $\chi = 1$ for $|x| \geq R > \rho_0$, we deduce

$$\mathbb{1}_{|x|\geq R}R(z)f = \mathbb{1}_{|x|\geq R}\chi R(z)f = -\mathbb{1}_{|x|>R}R_0(z)[\Delta, \chi]R(z)f,$$

provided $\operatorname{Im} z < 0$. By analytic continuation, the above equality will be true for $|z - z_0| = \epsilon$ and this yields

$$\mathbb{1}_{|x|\geq R}\Phi = \mathbb{1}_{|x|\geq R}R_0(z_0)g(x)$$

with $g(x) = -\frac{1}{2\pi \mathbf{i} C}[\Delta, \chi]\int_{|z-z_0|=\epsilon} R(z)f\ dz = [\Delta, \chi]g_1(x) \in L^2_{comp}(\Omega)$.

Next, if z is not pole of $R(z)$, the operator $R(z)$ is symmetric with respect to the bilinear form $(v, \bar{w})_{L^2(\Omega)}$, so A must be symmetric, too. This implies $\Psi = \bar{\Phi}$ and we have $Au = (u, \bar{\Phi})_{L^2(\Omega)}\Phi$, hence

$$R(z) = \frac{\Phi \otimes \Phi}{\lambda - z_0} + B(z),\ z \in \mathcal{U}.$$

Let

$$g \in L^2_{comp}(\Omega),\ \operatorname{supp} g \subset \{x \in \mathbb{R}^n :\ \rho_0 < a \leq \|x\| \leq b\}$$

and let $\chi \in C_0^\infty(\Omega)$, $\chi = 1$ on supp g, Then

$$\mathbf{E}_\pm(z, \omega)g = \mathbf{E}_\pm(z, \omega)[\Delta, \chi]R_0(z)g.$$

In fact, for every $f \in L^2(\mathbb{S}^{n-1})$ by integration by parts, we get

$$([\Delta, \chi]R_0(z)g, {}^t\mathbf{E}_\pm(z, \omega)f)_{L^2(\Omega)} = ((\Delta + z^2)\chi R_0(z)g, {}^t\mathbf{E}_\pm(z, \omega)f)_{L^2(\Omega)}$$
$$+(g, {}^t\mathbf{E}_\pm(z, \omega)f)_{L^2(\Omega)} = (\mathbf{E}_\pm(z, \omega)g, f)_{L^2(\mathbb{S}^{n-1})}.$$

Therefore, $K(z)$ has kernel

$$c_n z^{(n-3)/2}\frac{(\mathbf{E}_+(z_0, \theta)g)(\mathbf{E}_-(z_0, \omega)g)}{z - z_0} + B_1(z, \theta, \omega), c_n \neq 0$$

with $B_1(z, \theta, \omega)$ analytic in $\mathcal{U}$, and for a dense set of directions $\theta, \omega \in \mathbb{S}^{n-1}$ we have

$$\mathbf{E}_-(z_0, \omega)g \neq 0,\ \mathbf{E}_+(z_0, \theta)g \neq 0.$$

This implies that $K(z)$ as well as $S(z)$ have simple pole at z_0.

12.3 Estimates for the scattering amplitude

Consider the cut-off resolvent $R_\psi(\lambda)$ with

$$\operatorname{supp}\ \psi \subset \mathcal{C}_{a,b} = \{x \in \mathbb{R}^n :\ 0 < a \leq \|x\| < b\}.$$

For $\lambda \in \mathbb{R}$ and sufficiently large a N. Burq [Bu3] and Cardoso–Vodev [CV] established the estimate

$$\|R_\psi(\lambda)\|_{L^2(\Omega)\longrightarrow L^2(\Omega)} \leq \frac{C_2}{1+|\lambda|},\ \lambda \in \mathbb{R} \tag{12.12}$$

without any geometrical restriction of K. Clearly, this implies

$$|a(\lambda,\theta,\omega)| \leq C_0(1+|\lambda|)^{(n-3)/2},\ \forall(\theta,\omega) \in \mathbb{S}^{n-1}\times\mathbb{S}^{n-1},\ \lambda \in \mathbb{R}. \tag{12.13}$$

On the other hand, if we have poles λ_j of $R_\psi(\lambda)$ converging sufficiently fast to the real axis, the norm $\|\chi R(\lambda)\chi\|_{L^2(\Omega)\longrightarrow L^2(\Omega)}$ with an *arbitrary cut-off function* $\chi \in C_0^\infty(\mathbb{R}^n), \chi = 1$ on K may increase like $\mathcal{O}(e^{\mathbb{C}|\lambda|})$ for $\lambda \in \mathbb{R}, |\lambda| \longrightarrow \infty$. (see [Bu2]). This happens if there exists an elliptic periodic ordinary reflecting ray with Poincaré map satisfying some technical conditions (see [Po3] for the existence of quasi-modes and [SV] for the existence of poles). Consequently, the existence of trapping rays influences the estimates of $R_\chi(\lambda) = \chi R(\lambda)\chi$ with $\chi(x)$ equal to 1 on a neighbourhood of the obstacle, and the behaviours of the scattering amplitude $a(\lambda,\theta,\omega)$ and $R_\chi(\lambda)$ with arbitrary χ are rather different for $\lambda \in \mathbb{R}$ if we have trapping rays.

By using the notation of Section 12.1, an obstacle $K \in \mathcal{K}$ is called *non-trapping* if $\Sigma_\infty = \emptyset$. From the results on propagation of singularities given in [MS1], [MS2], it follows that if $K \in \mathcal{K}$ is non-trapping, there exist $\epsilon > 0$ and $d > 0$ so that $R_\chi(\lambda)$ has no poles in the domain

$$U_{\epsilon,d} = \{\lambda \in \mathbb{C} :\ 0 \leq \operatorname{Im}\lambda \leq \epsilon \log\ (1+|\lambda|) - d\}.$$

Moreover, for non-trapping obstacles we have the estimate (see [Va])

$$\|R_\chi(\lambda)\|_{L^2(\Omega)\longrightarrow L^2(\Omega)} \leq \frac{C}{1+|\lambda|}e^{C|\operatorname{Im}\lambda|},\ \forall\lambda \in U_{\epsilon,d}.$$

We conjecture that the existence of singularities $t_m \longrightarrow -\infty$ of the scattering kernel $s(t,\theta_m,\omega_m)$ for suitable directions $(\theta_m,\omega_m) \in \mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$ implies that for every $\epsilon > 0$ and $d > 0$ we have poles of the cut-off resolvent in $U_{\epsilon,d}$. In general, without any information of the geometry of trapping rays, this is a difficult problem.

Here we prove a weaker result assuming an estimate of the scattering amplitude.

Theorem 12.3.1: *Let $K \in \mathcal{K}$ and let n be odd. Suppose that there exist $m \in \mathbb{N}$, $\alpha \geq 0, \epsilon > 0, d > 0$ and $C > 0$ so that $a(\lambda,\theta,\omega)$ is analytic in $U_{\epsilon,d}$ and*

$$|a(\lambda,\theta,\omega)| \leq C(1+|\lambda|)^m e^{\alpha|\operatorname{Im}\lambda|},\ \forall(\omega,\theta) \in \mathbb{S}^{n-1}\times\mathbb{S}^{n-1},\ \forall\lambda \in U_{\epsilon,d}. \tag{12.14}$$

Then there are no trapping weakly non-degenerate bicharacteristics in $T^*(\bar{\Omega})$.

For the proof we need the following.

Lemma 12.3.2: *Let* $u \in \mathcal{S}'(\mathbf{R})$ *be a distribution with* $\text{supp } u \subset \{t \in \mathbb{R} : t \leq \tau\}$. *Assume that the Fourier transform*

$$\hat{u}(\lambda) = \langle u, e^{\mathrm{i}t\lambda} \rangle_{\mathcal{D}'(\mathbb{R})},\ \operatorname{Im} \lambda < 0$$

admits an analytic continuation in the domain $U_{\epsilon,d}$ *such that for all* $\zeta \in U_{\epsilon,d}$ *we have*

$$| \hat{u}(\zeta) | \leq C(1+ | \zeta |)^N e^{\alpha|\operatorname{Im}\zeta|},\ \alpha \geq 0. \tag{12.15}$$

Then for each $q \in \mathbb{N}$ *there exist* $t_q < \tau$ *and* $v_q \in C^q(\mathbb{R})$ *such that* $u = v_q$ *for* $t \leq t_q$.

Proof: Choose a function $\varphi \in C_0^\infty(\mathbb{R})$ such that

$$\text{supp } \varphi \subset (-1, 1), \int_{\mathbf{R}} \varphi(t)dt = 1.$$

Set $\varphi_\delta(t) = \frac{1}{\delta}\varphi(\frac{t}{\delta})$, $0 < \delta \leq 1$, and consider $u * \varphi_\delta$. Introduce the path

$$\Gamma_\epsilon : \mathbb{R} \setminus [-\gamma, \gamma] \ni \xi \mapsto \zeta = \xi + i(d - \epsilon \log (1+ | \xi |) \in U_{\epsilon,d},$$

where $\gamma = \exp\ (\frac{d}{\epsilon}) - 1$ and d is given in the definition of $U_{\epsilon,d}$. Clearly, (12.15) implies

$$| \hat{u}(\zeta)\hat{\varphi}(\delta\zeta) | \leq C_M(1+ | \zeta |)^N(1+ | \delta\zeta |)^{-M} e^{(a+\delta)|\operatorname{Im} \zeta|},\ \zeta \in U_{\epsilon,d}.$$

By using the analyticity of $\hat{u}(\zeta)$ in $U_{\epsilon,d}$ combined with the estimate obtained above for fixed δ, we write the integral

$$(u * \varphi_\delta)(t) = (2\pi)^{-1} \int_{\mathbb{R}} e^{it\xi} \hat{u}(\xi)\hat{\varphi}(\delta\xi)d\xi$$

as a sum of two integrals

$$(2\pi)^{-1} \int_{\Gamma_\epsilon} e^{it\zeta}\hat{u}(\zeta)\hat{\varphi}(\delta\zeta)d\zeta + (2\pi)^{-1} \int_{|\xi| \leq \gamma} e^{it\zeta}\hat{u}(\xi)\hat{\varphi}(\delta\xi)d\xi.$$

Since the second integral is over a compact interval, passing to a limit as $\delta \to 0$, we get a C^∞ function.

Next, for $\zeta \in \Gamma_\epsilon$, $0 < \delta \leq 1$ we have the estimate

$$| \hat{u}(\zeta)\hat{\varphi}(\delta\zeta)e^{it\zeta} | \leq C(1+ | \zeta |)^N e^{-(a+1+t)\operatorname{Im} \zeta} \leq C'(1+ | \operatorname{Re} \zeta |)^{N+\epsilon(a+1+t)},$$

provided $a + 1 + t < 0$. Given $q \in \mathbb{N}$, choose

$$t_q = \frac{1}{\epsilon}(-N - n - 1 - q - \epsilon(a + 1)).$$

Then $t \leq t_q$ implies

$$\epsilon(a + 1 + t) \leq -N - n - 1 - q$$

and, since $d\zeta = F(\xi)d\xi \to d\xi$ as $|\ \xi\ | \to \infty$, for $t \leq t_q$ the integral on Γ_ϵ is uniformly convergent for $0 \leq \delta \leq 1$. The same is true if we take the derivatives with respect to t up to order q. Taking the limit $\delta \to 0$ and exploiting the fact that $\hat{\varphi}(\delta\zeta) \to 1$, by Lebesgue theorem we deduce that for $t \leq t_q$ we have $u * \varphi_\delta \longrightarrow_{\delta\to 0} f$, f being a C^q function. This completes the proof of the lemma. ■

Proof of Theorem 12.3.1: If $K \in \mathcal{K}$ is trapping and has a weakly non-degenerate trapping bicharacteristic, we can apply Theorem 12.1.8. Then we have delta-type singularities $t_m \to -\infty$ of $u_m(t) = \overline{s(t, \theta_m, \omega_m)}$. From (12.15), we obtain a uniform estimate with respect to (θ_m, ω_m) for the Fourier transform of $u_m(t)$. For $q = 0$ we choose $t_0 = \frac{1}{\epsilon}(-N - n - 1 - \epsilon(a + 1))$. Next, we take m large enough so that $t_m < t_0$ and fix m. Applying Lemma 12.3.2, we obtain a contradiction. ■

Since $S(\lambda) - Id$ is a Hilbert–Schmidt operator with kernel $a(\lambda, \theta, \omega)$, a suitable estimate for the scattering amplitude in a domain $\mathcal{U}$ in $\mathbb{C}$, where we have no poles of the cut-off resolvent, will imply an estimate for the scattering matrix for $\lambda \in \mathcal{U}$. On the other hand, for the estimate of the scattering amplitude we need an estimate for the norm

$$\|\mathbb{1}_{\mathcal{C}_{a,b}} R(\lambda) \mathbb{1}_{\mathcal{C}_{a,b}}\|_{L^2(\Omega) \longrightarrow L^2(\Omega)},\ \lambda \in \mathcal{U},$$

provided $a > \rho_0$ large enough (see [BP2] for the estimates of the above norm).

If an obstacle K is trapping and there are no sequences of poles λ_j with $\lim_{j\to\infty} \mathrm{Im}\ \lambda_j = 0$, then there exists $\delta > 0$ such that in

$$V_\delta = \{z \in \mathbb{C} :\ 0 \leq \mathrm{Im}\ z \leq \delta : |z| \geq a_0 > 0\}$$

there are no poles. One expects that under this condition the scattering amplitude admits an exponential estimate, that is for $0 < \mu < \delta$ there exists constants $C_\mu > 0, C \geq 0$ so that

$$|a(\lambda, \theta, \omega)| \leq C_\mu e^{C|\lambda|},\ \ \forall(\theta, \omega) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1},\ \ \forall\lambda \in V_\mu. \tag{12.16}$$

For dimension $n = 2$ the estimate (12.16) has been established in [BP1] without any condition of trapping rays. For dimensions $n \geq 3$ this is an interesting open problem.

12.4 Notes

The results in Section 12.1 have been obtained in [PS6], [PS8], [PS9] and [PS10], and the notion of weakly non-degenerate trapping bicharacteristic has been introduced in [PS9]. The results in Section 12.2 concerning the maximal singularity of $s(t, -\omega, \omega)$ are obtained by [Ma2] (see [P5] for a similar result). The Proposition 12.2.1 has been proved in [PZ]. The link between the poles of the cut-off resolvent and those of the operator $S(\lambda)$ has been studied in many papers. We refer to the classical book [LP1] for odd dimension and to [Z2] for an analysis covering all dimensions $n \geq 2$. Lemma 12.3.2 was proved in [PS8] and our argument follows the proof of Theorem 7.3.8 in [Hl]. Theorem 12.3.1 was established in [PS10].

13

Inverse scattering by obstacles

In this chapter we discuss the inverse problem of recovering information about an obstacle from the singularities of the scattering kernel. As we already know, this scattering data is closely related to the set of sojourn times of scattering rays in the exterior of the obstacle, the so-called scattering length spectrum (SLS). We will in fact try to recover information about the obstacle from its SLS. The first observation that we make, and it is rather important, is that if two obstacles K and L have (almost) the same scattering length spectra, then the generalized geodesic flows in their exteriors are naturally conjugated on the non-trapping parts of their phase spaces via a time-preserving conjugacy. This is explained in Section 13.1 and the proof of the main result is given in Section 13.2. In subsequent sections we use this result to show that certain properties of obstacles are recoverable from the SLS, and also that some classes of obstacles can be uniquely recovered from their SLS.

13.1 The scattering length spectrum and the generalized geodesic flow

Let K be an obstacle in $\mathbb{R}^n$, $n \geq 2,$ that is a compact subset of $\mathbb{R}^n$ with C^∞ boundary ∂K such that $\Omega_K = \overline{\mathbb{R}^n \setminus K}$ is connected. Given $(\omega, \theta) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$, let $\mathcal{S}L_K(\omega, \theta)$ be the set of sojourn times T_γ of all (ω, θ)-rays γ in Ω_K. The *SLS* of K is by definition the map

$$\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \ni (\omega, \theta) \mapsto \mathcal{S}L_K(\omega, \theta) \subset [0, \infty).$$

As we know from Chapter 11, for 'most' obstacles K we have

$$\mathcal{S}L_K(\omega, \theta) = \text{sing supp } s_K(t, \theta, \omega)$$

Geometry of the Generalized Geodesic Flow and Inverse Spectral Problems, Second Edition.
Vesselin M. Petkov and Luchezar N. Stoyanov.

for almost all (ω, θ), where s_K is the scattering kernel related to the scattering operator for the wave equation in $\mathbb{R} \times \Omega_K$ with Dirichlet boundary condition on $\mathbb{R} \times \partial\Omega_K$.

It is a rather important problem in inverse scattering by obstacles to get information about the geometry of the obstacle K from its SLS. It is well known and easy to see that the *convex hull* $\widehat{K}$ of K can be recovered from $\mathcal{S}L_K$; this has been noted by various people –Majda [Ma2] (see also Majda and Ralston [MaR]) and Lax and Phillips [LP2]. Indeed, one can recover all supporting hyperplanes to K by using back-scattering only, that is pairs (ω, θ) with $\theta = -\omega$ (see our comments in Section 12.2). In fact, since we have to work with generic $(\omega, \theta) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$, we need to use pairs (ω, θ) with θ very close to $-\omega$. An obvious consequence from this is that convex obstacles K are completely determined by its SLS. Similarly, connected obstacles with real analytic boundaries are also uniquely determined by their SLS (in the class of obstacles with real analytic boundaries). However, as the next example shows, in general $\mathcal{S}L_K$ does not determine K uniquely.

Example 13.1.1: (Livshits' Example, adapted from Chapter 5 of [Me4])
Consider the obstacle K in $\mathbb{R}^2$ bounded by the closed curve in Figure 13.1. Here the curve E is half an ellipse with end points A and B, while F_1 and F_2 are the foci of the ellipse. According to a well-known property of the ellipse, if a ray enters the area inside the ellipse between the foci F_1 and F_2, after reflection at E, it will go out again between the foci. Thus, no scattering ray 'coming from infinity' has a common point with the bold lines from A to F_1 and from B to F_2.

As one can see in Livshits' example, there are points in Ω_K that cannot be reached by scattering rays 'incoming from infinity' and 'outgoing to infinity' after a finite amount of time spent 'near the obstacle'. In fact, there is a whole open subset of $S^*(\Omega_K)$ consisting of points $\sigma = (x, \xi)$ which do not belong to an (ω, θ)-ray for any (ω, θ). Such points are called trapped points.

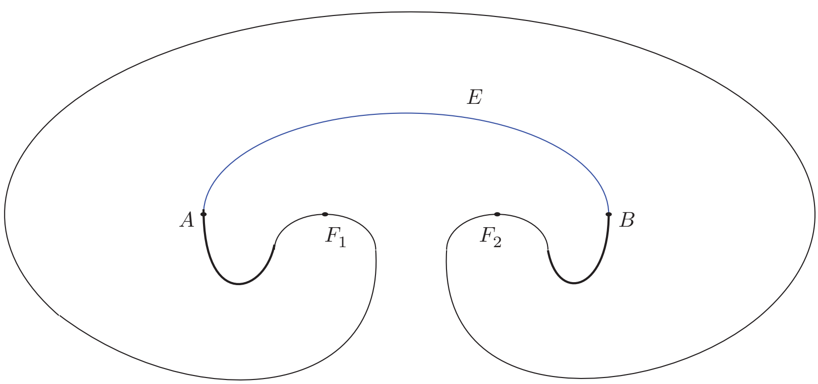

Figure 13.1 Livshits' example.

More precisely, considering again an arbitrary obstacle K, a point $\sigma = (x, \xi) \in S^*(\Omega_K)$ is called *non-trapped* if both curves

$$\gamma_K^+(\sigma) = \{\mathrm{pr}_1(\mathcal{F}_t^{(K)}(\sigma)) : t \leq 0\}$$

and

$$\gamma_K^-(\sigma) = \{\mathrm{pr}_1(\mathcal{F}_t^{(K)}(\sigma)) : t \geq 0\}$$

in Ω_K are unbounded in $\mathbb{R}^n$. Otherwise σ is called a *trapped point* and σ belongs to the set Σ_∞ introduced in Section 12.1. As before, here we use the notation

$$\mathrm{pr}_1(y, \eta) = y, \quad \mathrm{pr}_2(y, \eta) = \eta.$$

Denote by $\mathrm{Trap}(\Omega_K)$ the *set of all trapped points*. As we have already mentioned, Livshits' example shows that in general $\mathrm{Trap}(\Omega_K)$ may have positive Lebesgue measure and a non-empty interior in $S^*(\Omega_K)$. K is called a *non-trapping obstacle* if $\mathrm{Trap}(\Omega_K) = \emptyset$. For example, as we will see in Section 13.3, all star-shaped obstacles the curvature of whose boundaries does not vanish of infinite order are non-trapping.

For any $\sigma = (x, \xi) \in S^*(\Omega_K)$ we will also use the notation

$$\gamma_K(\sigma) = \{\mathrm{pr}_1(\mathcal{F}_t^{(K)}(\sigma)) : t \in \mathbb{R}\} = \gamma_K^+(\sigma) \cup \gamma_K^-(\sigma).$$

Here

$$\mathcal{F}_t^{(K)} : \dot{T}^*(\Omega_K) \longrightarrow \dot{T}^*(\Omega_K)$$

is the *generalized geodesic flow* in Ω_K (see Section 1.2), where

$$\dot{T}(\Omega) = \{(x, \xi) \in T^*(\Omega) : \xi \neq 0\}.$$

Fix a large open ball $\mathcal{O}$ containing K in its interior and set

$$\Omega_0 = \overline{\mathbb{R}^n \setminus \mathcal{O}}.$$

In what follows, for convenience, all obstacles K considered will be contained in $\mathcal{O}$, therefore we will always have $\Omega_0 \subset \Omega_K$.

Denote by $\mathcal{K}$ the *class of obstacles* K in $\mathbb{R}^n$ such that for each $(x, \xi) \in S^*(\partial K)$ if the curvature of ∂K at x vanishes of infinite order in direction ξ, then all points (y, η) sufficiently close to (x, ξ) are diffractive points (see Section 1.2). It follows from a result in [MS2] that for $K \in \mathcal{K}$, the flow $\mathcal{F}_t^{(K)}$ is well defined and continuous. Let $\mathcal{K}_0$ the *class of all obstacles* $K \in \mathcal{K}$ satisfying the following non-degeneracy conditions: $\gamma_K(\sigma)$ is a non-degenerate ordinary reflecting ray for almost all $\sigma \in S^*(\Omega_0)$ such that $\gamma_K(\sigma) \cap \partial K \neq \emptyset$, and ∂K does not contain non-trivial open flat subsets. Using arguments from Chapter 6, it is not difficult to show that obstacles in the class $\mathcal{K}_0$ are 'generic'. This means that for every obstacle K in $\mathbb{R}^n$, applying suitable arbitrarily small C^∞ deformations to ∂K, one gets obstacles from the class $\mathcal{K}_0$. 'Most' deformations in topological sense have this property.

We will say that two obstacles K and L have *almost the same SLS* if there exists a subset $\mathcal{R}$ of full Lebesgue measure in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ such that

$$\mathcal{S}L_K(\omega, \theta) = \mathcal{S}L_L(\omega, \theta) \tag{13.1}$$

for all $(\omega, \theta) \in \mathcal{R}$.

We will now state the main result in this section – briefly, it says that if two obstacles $K, L \in \mathcal{K}_0$ have almost the same SLS, then their generalized geodesic flows are conjugate with a time-preserving conjugacy on the non-trapping parts of their phase spaces. This will be used several times in subsequent sections.

Theorem 13.1.2: *Assume that two obstacles $K, L \in \mathcal{K}_0$ have almost the same SLS. Then there exists a homeomorphism*

$$\Phi : \dot{T}^*(\Omega_K) \setminus \mathrm{Trap}(\Omega_K) \longrightarrow \dot{T}^*(\Omega_L) \setminus \mathrm{Trap}(\Omega_L),$$

which has the following properties:

(i) Φ *defines a symplectic map on an open dense subset of* $\dot{T}^*(\Omega_K) \setminus \mathrm{Trap}(\Omega_K)$*,*

(ii) Φ *maps* $S^*(\Omega_K) \setminus \mathrm{Trap}(\Omega_K)$ *onto* $S^*(\Omega_L) \setminus \mathrm{Trap}(\Omega_L)$*,*

(iii) $\mathcal{F}_t^{(L)} \circ \Phi = \Phi \circ \mathcal{F}_t^{(K)}$ *for all* $t \in \mathbb{R}$*,*

(iv) $\Phi = id$ *on* $\dot{T}^*(\Omega_0) \setminus \mathrm{Trap}(\Omega_K) = \dot{T}^*(\Omega_0) \setminus \mathrm{Trap}(\Omega_L)$.

Conversely, if $K, L \in \mathcal{K}_0$ are two obstacles for which there exists a homeomorphism $\Phi : S^*(\Omega_K) \setminus \mathrm{Trap}(\Omega_K) \longrightarrow S^*(\Omega_L) \setminus \mathrm{Trap}(\Omega_L)$ such that $\mathcal{F}_t^{(L)} \circ \Phi = \Phi \circ \mathcal{F}_t^{(K)}$ for all $t \in \mathbb{R}$ and $\Phi = \mathrm{id}$ on $S^*(\Omega^0) \setminus \mathrm{Trap}(\Omega_K)$, then K and L have the same SLS.

We prove Theorem 13.1.2 in Section 13.2, and, as an easy consequence of it, there we also derive the following.

Corollary 13.1.3: *If two obstacles $K, L \in \mathcal{K}_0$ have almost the same SLS and the sets of trapped points of both K and L have Lebesgue measure zero, then* $\mathrm{Vol}(K) = \mathrm{Vol}(L)$.

It is clear from Livshits' example that the above conclusion is not true without any assumption about the sets of trapped points. Recall that, according to Proposition 11.2.6, for any obstacle K the set $S^*(\Omega_0) \cap \mathrm{Trap}(\Omega_K)$ is relatively small, that is it has zero measure. On the other hand, Livshits' example demonstrates that in some cases, we may have $\dim_H(\mathrm{Trap}(\Omega_K)) = 2n - 1 = \dim_H(S^*(\Omega_K))$. So, from the SLS, we cannot 'see' whether the trapped set $\mathrm{Trap}(\Omega_K)$ is significant or not – the trapped set that we 'see' far from the obstacle is always small.

Apart from this, the local version of the above problem appears to be even less informative about the obstacle. The following example shows that in the corresponding local problem there is no uniqueness. It concerns obstacles containing flat spots on their boundaries, so it is natural to ask whether similar examples exist where the Gauss curvatures of ∂K and ∂L are non-zero. At present we do not know the answer to this question.

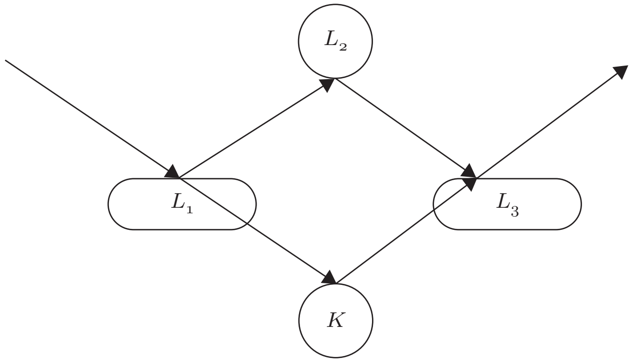

Figure 13.2 Local non-uniqueness.

Example 13.1.4: Let K and $L = L_1 \cup L_2 \cup L_3$ be two obstacles in $\mathbb{R}^n$ (n can be any integer ≥ 2), as in Figure 13.2. Here K and L_2 are strictly convex domains in $\mathbb{R}^n$, which are symmetric with respect to a hyperplane H containing the 'top parts' of ∂L_1 and ∂L_3. L_1 and L_3 are convex domains in $\mathbb{R}^n$ contained in the 'lower' half-space with respect to H. Both contain non-trivial open subsets of H. The rays on the figure are generated by some σ_0 in $S^*(\Omega_0)$. Clearly, for σ close to σ_0, we have $\mathcal{F}_t^{(K)}(\sigma) = \mathcal{F}_t^{(L)}(\sigma)$ for $t \gg 0$. Moreover, both trajectories $\gamma_K(\sigma)$ and $\gamma_L(\sigma)$ have common points with the corresponding obstacles and are non-degenerate. At the same time, the obstacles K and L are completely different, both globally and locally. In fact, $K \cap L = \emptyset$.

We conclude this section with two easy consequences of Theorem 13.1.2.

Given an obstacle K, recall that for any $x \in \partial K$, $\nu_K(x)$ denotes the *exterior unit normal* to ∂K at x. Points of the form $(x, \nu_K(x)) \in S^*(\Omega_K)$ which are non-trapped clearly define back-scattering rays. Indeed, if for some $t > 0$ we have $\mathcal{F}_t^{(K)}(x, \nu_K(x)) = (y, \eta) \in S^*(\Omega_0)$, then the scattering ray γ issued from y in direction $-\eta$ will hit ∂K orthogonally at x after time t, and so after time $2t$ it will return to its 'initial' position y. Denote by $\mathrm{Trap}^{(n)}(\partial K)$ the set of those $x \in \partial K$ such that $(x, \nu_K(x)) \in \mathrm{Trap}(\Omega_K)$.

Proposition 13.1.5: *Assume that two obstacles K and L from the class $\mathcal{K}_0$ have almost the same SLS. Then there exists a homeomorphism*

$$\varphi : \partial K \setminus \mathrm{Trap}^{(n)}(\partial K) \longrightarrow \partial L \setminus \mathrm{Trap}^{(n)}(\partial L)$$

such that for all $x \in \partial K \setminus \mathrm{Trap}^{(n)}(\partial K)$, setting $y = \varphi(x)$, we have $\Phi(x, \nu_K(x)) = (y, \nu_L(y))$.

Proof: Let $K, L \in \mathcal{K}_0$ have almost the same SLS, and let Φ be the conjugacy from Theorem 13.1.2. Given $x \in \partial K \setminus \mathrm{Trap}^{(n)}(\partial K)$, there exists $t > 0$ so large that $(z, \zeta) = \mathcal{F}_t^{(K)}(x, \nu_K(x)) \in S^*(\Omega_0)$. This simply means that

$\mathcal{F}_t^{(K)}(z,-\zeta) = (x, \nu_K(x))$ and $\mathcal{F}^{(K)}{}_{2t}(z,-\zeta) = (z,\zeta)$. Since $\Phi(z,\zeta) = (z,\zeta)$, it follows that $\mathcal{F}_{2t}^{(L)}(z,-\zeta) = (z,\zeta)$. Setting $(y,\eta) = \mathcal{F}_t^{(L)}(z,-\zeta)$, we must have $y \in \partial L$ and $\eta \perp \partial L$ at y. Using the identification of directions at y, we may assume $\eta = \nu_L(y)$. So, $\Phi(x, \nu_K(x)) = (y, \nu_L(y))$ for some $y \in \partial L \setminus \mathrm{Trap}^{(n)}(\partial L)$. ■

If under the assumptions of Proposition 13.1.5 we have the additional information that the trapped sets $\mathrm{Trap}^{(n)}(\partial K)$ and $\mathrm{Trap}^{(n)}(\partial L)$ are small, then we can conclude that K and L have the same number of connected components.

Corollary 13.1.6: *Assume that two obstacles $K, L \in \mathcal{K}_0$ in $\mathbb{R}^n$, $n \geq 3$, have almost the same SLS, and moreover $\dim_H(\mathrm{Trap}^{(n)}(\partial K)) < n-2$ and $\dim_H (\mathrm{Trap}^{(n)}(\partial L)) < n-2$. Then K and L have the same number of connected components.*

Proof: It is known that for compact subsets of $\mathbb{R}^n$ the *topological dimension* $\dim$ does not exceed the Hausdorff dimension $\dim_H$ (see e.g. [E]). So, we have $\dim(\mathrm{Trap}^{(n)}(\partial K)) < n-2$ and $\dim(\mathrm{Trap}^{(n)}(\partial L)) < n-2$.

Let $K_1, \dots, K_p$ be the connected components of K. Since $\dim(\mathrm{Trap}^{(n)}(\partial K)) < n-2$, it follows that $\partial K_i \setminus \mathrm{Trap}^{(n)}(\partial K)$ is a connected open subset of ∂K_i for any $i = 1, \dots, p$ (see e.g. Theorem IV.4 in [HW]). Therefore, the sets $\partial K_i \setminus \mathrm{Trap}^{(n)}(\partial K)$ are the open connected components of $\partial K \setminus \mathrm{Trap}^{(n)}(\partial K)$. The homeomorphism φ from Proposition 13.1.5 has to map these into the open connected components of $\partial L \setminus \mathrm{Trap}^{(n)}(\partial L)$. Thus, the number of connected components of $\partial L \setminus \mathrm{Trap}^{(n)}(\partial L)$ is the same as the number of connected components of $\partial K \setminus \mathrm{Trap}^{(n)}(\partial K)$. We know already that the latter is the same as the number of connected components of K. A similar argument applies to L. ■

13.2 Proof of Theorem 13.1.2

Throughout we assume that $K, L \in \mathcal{K}_0$ are two obstacles in $\mathbb{R}^n$, $n \geq 2$, with almost the same SLS. As before we will denote by $\mathcal{O}$ a large open ball in $\mathbb{R}^n$ that contains K and L, and Ω_0 will be the closure of the complement of $\mathcal{O}$ in $\mathbb{R}^n$. As in Chapter 11, given $\xi \in \mathbb{S}^{n-1}$, we will denote by Z_ξ *the hyperplane in* $\mathbb{R}^n$ *orthogonal to* ξ *and tangent to* $\mathcal{O}$ such that $\mathcal{O}$ is contained in the open half-space determined by Z_ξ and having ξ as an inner normal.

The following lemma is the main step in the proof of Theorem 13.1.2.

Lemma 13.2.1: *If two obstacles $K, L \in \mathcal{K}_0$ have almost the same SLS, then for every $\sigma \in S^*(\Omega_0)$ and every $t \in \mathbb{R}$ with $\mathcal{F}_t^{(K)}(\sigma) \in S^*(\Omega_0)$ we have $\mathcal{F}_t^{(K)}(\sigma) = \mathcal{F}_t^{(L)}(\sigma)$.*

Proof of Lemma 13.2.1: Assume that $K, L \in \mathcal{K}_0$ have almost the same SLS, and let $\mathcal{R}$ be a subset of $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ of full Lebesgue measure in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ such that

$$\mathcal{SL}_K(\omega,\theta) = \mathcal{SL}_L(\omega,\theta), \quad (\omega,\theta) \in \mathcal{R}. \tag{13.2}$$

Using Proposition 11.1.2, we may assume that $\mathcal{R}$ has the following additional properties:

(i) for $(\omega, \theta) \in \mathcal{R}$ all (ω, θ)-rays in Ω_K and Ω_L are non-degenerate simply reflecting (ω, θ)-rays;

(ii) if $(\omega, \theta) \in \mathcal{R}$ and γ and δ are (ω, θ)-rays in Ω_K (or Ω_L), then $T_\gamma \neq T_\delta$.

Moreover, shrinking $\mathcal{R}$ a bit if necessary, we may assume that $(\omega, \omega) \notin \mathcal{R}$ for any $\omega \in \mathbb{S}^{n-1}$. Then for $(\omega, \theta) \in \mathcal{R}$ any (ω, θ)-ray in Ω_K or Ω_L must have at least one reflection point.

We now need the following technical lemma. ■

Lemma 13.2.2: *Assume that γ is a non-degenerate simply reflecting (ω_0, θ_0)-ray in Ω_K for some $(\omega_0, \theta_0) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ with successive reflection points $x_1, \ldots, x_k$ ($k \geq 1$). Then there exist:*

(a) a neighbourhood U of (ω_0, θ_0) in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$,

(b) for each $i = 1, \ldots, k$ a neighbourhood U_i of x_i in ∂K,

(c) for every $(\omega, \theta) \in U$ unique $x_1(\omega, \theta) \in U_1, \ldots, x_k(\omega, \theta) \in U_k$ which are the successive reflection points of an (ω, θ)-ray in Ω_K,

(d) a sufficiently large $T_0 > 0$,

such that $x_i(\omega, \theta)$ depends smoothly on $(\omega, \theta) \in U$ for each $i = 1, \ldots, k$, and for every $T \geq T_0$ there exists an open neighbourhood W of (x_1, ω_0) in $\partial K \times \mathbb{S}^{n-1}$ such that the map $H(y, \omega) = (\omega, \mathrm{pr}_2(\mathcal{F}_t^{(K)}(y, \omega)))$ is a diffeomorphism $H : W \longrightarrow U$.

Proof of Lemma 13.2.2: There exist $\sigma_0 = (x_0, \xi_0) \in S^*(\Omega_0)$ and $T_0 > 0$ such that $\gamma = \gamma_K(\sigma_0)$ and $\mathcal{F}_t^{(K)}(\sigma_0) \in S^*(\Omega_0)$ for all $t \in (-\infty, 0] \cup [T_0, \infty)$. Fix such σ_0 and T_0. Then each $x_i = \mathrm{pr}_1(\mathcal{F}^{(K)}{}_{t_i}(\sigma_0))$ for some $t_i > 0$.

Let Π and Π' be the hyperplanes in $\mathbb{R}^n$ through x_0 and y_0, respectively, perpendicular to the trajectory $\gamma_K(\sigma_0)$ at these two points. The cross-sectional map $\mathcal{P}_K : S^*(\Pi) \longrightarrow S^*(\Pi')$ defined by the shift along the flow $\mathcal{F}_t^{(K)}$ is smooth on a small open neighbourhood V of σ_0 in $S^*(\Pi)$. Choose V so small that for all points $\sigma \in V$ the trajectory $\gamma_K(\sigma)$ is simply reflecting and has exactly k reflection points $x_1(\sigma), \ldots, x_k(\sigma)$. Then $x_j(\sigma)$ are smooth maps of $\sigma \in V$ into ∂K.

Another smooth map is given by

$$V \ni (x, \xi) \mapsto J(x, \xi) = \mathrm{pr}_2(\mathcal{P}_K(x, \xi)) \in \mathbb{S}^{n-1}.$$

For ξ close to ξ_0 consider the map $J_\xi = J(., \xi)$. Clearly, $\det dJ_\xi(x)$ depends smoothly on $(x, \xi) \in V$, and moreover $\det dJ_{\xi_0}(x_0) \neq 0$ since γ is non-degenerate. Taking V small enough, we may assume that $\det dJ_\xi(x) \neq 0$ for all $(x, \xi) \in V$.

We will now use the Inverse Function Theorem for the map

$$F : V \longrightarrow \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$$

defined by

$$F(x,\xi) = (\xi, \mathrm{pr}_2(\mathcal{P}_K(x,\xi))) = (\xi, J(x,\xi)).$$

The Jacobian matrix of F at $(x,\xi) \in V$ has the form

$$dF(x,\xi) = \begin{pmatrix} 0 & I \\ dJ_\xi(x) & * \end{pmatrix},$$

where I is the identity $(n-1)\times(n-1)$ matrix. Hence $dF(x,\xi)$ is non-singular and by the Inverse Function Theorem, F has a local inverse which is a smooth map as well. Assuming again that V is taken small enough, F is a diffeomorphism between V and an open neighbourhood U of (ω_0,θ_0) in $\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$. Now for $G = F^{-1}$ we have $G(\omega,\theta) = (x(\omega,\theta),\xi)$ for $(\omega,\theta)\in U$, and the smoothness of G implies that $x_j(\omega,\theta) = x_j(x(\omega,\theta),\omega)$ depends smoothly on $(\omega,\theta)\in U$ for all $j = 1,\dots,k$. Moreover, the definition of F shows that $x_1(\omega,\theta),\dots,x_k(\omega,\theta)$ are the successive reflection points of a reflecting (ω,θ)-ray in Ω_K, and if $y_1,\dots,y_k$ are the successive reflection points of an (ω,θ)-ray in Ω_K and for each j, y_j is sufficiently close to $x_j(\omega_0,\theta_0)$, then $y_j = x_j(\omega,\theta)$ for any $j=1,\dots,k$.

Next, given $(x,\xi)\in V$, let $x_1(x,\xi)\in\partial K$ be the first reflection point of $\gamma_K(x,\xi)$. Setting $L(x,\xi) = (x_1(x,\xi),\xi)$, we get a diffeomorphism between V and an open neighbourhood W of (x_1,ξ) in $\partial K\times\mathbb{S}^{n-1}$. It is easy to see now that the diffeomorphism

$$H = F\circ L^{-1} : W \longrightarrow U$$

has the required properties. Indeed, if $T\geq T_0$, the definitions of F and L yield

$$H(y,\omega) = (\omega, \mathrm{pr}_2(\mathcal{F}_t^{(K)}(y,\omega)))$$

for all $(y,\omega)\in W$. ∎

We now continue with the proof of Lemma 13.2.1.

Fix an arbitrary $\sigma_0 = (q_0,u_0)\in S^*(\Omega_0)$ and $t_0>0$ such that $\mathcal{F}^{(K)}{}_{t_0}(\sigma_0)\in S^*(\Omega_0)$. We have to prove that $\mathcal{F}_{t_0}^{(K)}(\sigma_0) = \mathcal{F}_{t_0}^{(L)}(\sigma_0)$. It follows from Proposition 11.2.6 that $S^*(\Omega_0)\cap\mathrm{Trap}(\Omega_K)$ has measure zero in $S^*(\Omega_0)$. Similarly for $S^*(\Omega_0)\cap\mathrm{Trap}(\Omega_L)$, so $S^*(\Omega_0)\setminus(\mathrm{Trap}(\Omega_K)\cup\mathrm{Trap}(\Omega_L))$ is dense in $S^*(\Omega_0)$. Since both $\mathcal{F}_t^{(K)}$ and $\mathcal{F}_t^{(L)}$ are continuous flows, it is enough to consider the case $\sigma_0\in S^*(\Omega_0)\setminus(\mathrm{Trap}(\Omega_K)\cup\mathrm{Trap}(\Omega_L))$.

So, we assume from now on that

$$\sigma_0\in S^*(\Omega_0)\setminus(\mathrm{Trap}(\Omega_K)\cup\mathrm{Trap}(\Omega_L)).$$

Then $\gamma_K(\sigma_0)$ is an (ω_0,θ_0)-ray for some $(\omega_0,\theta_0)\in\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$. Reversing directions along the trajectory $\gamma_K(\sigma_0)$, we may assume that $u_0=\omega_0$. Thus, if $\gamma_K(\sigma_0)\cap\partial K\neq\emptyset$, then this intersection happens forwards, that is $\gamma_K^-(\sigma_0)\cap\partial K=\emptyset$ and $\gamma_K^+(\sigma_0)\cap\partial K\neq\emptyset$.

Case 1. $\gamma_K(\sigma_0) \cap \partial K \neq \emptyset$. Then, according to the above, the forward trajectory $\gamma_K^+(\sigma_0)$ has a common point with ∂K. So, there exists $s_0 > 0$ with $\mathcal{F}_{s_0}^{(K)}(\sigma_0) = (x_0, \xi_0)$ and $x_0 \in \partial K$. Take the minimal $s_0 > 0$ with this property; then $\text{pr}_1(\mathcal{F}_s^{(K)}(\sigma_0)) \notin \partial K$ for $s \in [0, s_0)$. It follows from Proposition 11.1.3 that there exist points $\sigma_0' \in S^*(\Omega_0)$ arbitrarily close to σ_0 such that $\gamma_K(\sigma_0')$ and $\gamma_L(\sigma_0')$ are simply reflecting rays. Moreover, using the assumption that $K \in \mathcal{K}_0$ and $\gamma_K(\sigma_0) \cap \partial K \neq \emptyset$, we may take σ_0' so that $\gamma_K(\sigma_0')$ is a non-degenerate (ω_0', θ_0')-ray for some (ω_0', θ_0') close to (ω_0, θ_0).

Using again the continuity of the flows $\mathcal{F}_t^{(K)}$ and $\mathcal{F}_t^{(L)}$, we may simple assume this for the trajectories generated by σ_0. Thus, we will assume from now on that $\gamma_K(\sigma_0)$ is a simply reflecting non-degenerate ray. Moreover, since $\mathcal{R}$ is dense in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$, using the diffeomorphism H from Lemma 13.2.2 for $\gamma = \gamma_k(\sigma_0)$ and perturbing slightly $(x_0, \xi_0) \in \partial K \times \mathbb{S}^{n-1}$ if necessary, we may assume that $(\omega_0, \theta_0) \in \mathcal{R}$.

Let $x_1, \ldots, x_k \in \partial K$ $(k \geq 1)$ be the successive reflection points of $\delta = \gamma_K(\sigma_0)$. It follows from (13.2) and $(\omega_0, \theta_0) \in \mathcal{R}$ that there exists (at least one) (ω_0, θ_0)-ray δ' in Ω_L with

$$T_\delta = T_{\delta'}. \tag{13.3}$$

Fix one δ' with this property. Since $(\omega_0, \theta_0) \in \mathcal{R}$, δ' is a simply reflecting non-degenerate (ω_0, θ_0)-ray in Ω_L. Let $y_1, \ldots, y_m \in \partial L$ $(m \geq 1)$ be the reflection points of δ'. Using Lemma 13.2.2, choose a neighbourhood U of (ω_0, θ_0) in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ such that for each $(\omega, \theta) \in U$ there are a unique reflecting (ω, θ)-ray $\delta(\omega, \theta)$ in Ω_K with reflection points $x_1(\omega, \theta), \ldots, x_k(\omega, \theta)$ close to $x_1, \ldots, x_k$ and a unique reflecting (ω, θ)-ray $\delta'(\omega, \theta)$ in Ω_L with reflection points $y_1(\omega, \theta), \ldots, y_m(\omega, \theta)$ close to $y_1, \ldots, y_m$.

We now need the following lemma.

Lemma 13.2.3: *Assume that $T_{\delta(\omega,\theta)} = T_{\delta'(\omega,\theta)}$ for all $(\omega, \theta) \in U$. Then for every $(\omega, \theta) \in U$ the vector $y_1(\omega, \theta) - x_1(\omega, \theta)$ is parallel to ω (unless it is the zero vector), and similarly $(y_m(\omega, \theta) - x_k(\omega, \theta))$ is parallel to θ.*

Proof of Lemma 13.2.3: Choose an arbitrary smooth parameterization of U:

$$\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \supset W \ni (u, v) \mapsto (\omega(u), \theta(v)).$$

Set

$$x_j(u, v) = x_j(\omega(u), \theta(v)), \quad y_j(u, v) = y_j(\omega(u), \theta(v)).$$

The assumptions of the lemma show that the time functions

$$T(u, v) = \langle \omega(u), x_1(u, v) \rangle + \sum_{i=1}^{k-1} \|x_i(u, v) - x_{i+1}(u, v)\| - \langle x_k(u, v), \theta(v) \rangle,$$

$$S(u,v) = \langle \omega(u), y_1(u,v)\rangle + \sum_{i=1}^{m-1} \|y_i(u,v) - y_{i+1}(u,v)\| - \langle y_m(u,v), \theta(v)\rangle$$

coincide, that is $T(u,v) = S(u,v)$ for all $(u,v) \in W$.

Calculating the derivatives of these functions (as we have done before; see e.g. Chapter 2), we get

$$\frac{\partial T}{\partial u_j}(u,v) = \left\langle \frac{\partial \omega}{\partial u_j}, x_1 \right\rangle + \left\langle \omega, \frac{\partial x_1}{\partial u_j} \right\rangle + \sum_{i=1}^{k-1} \left\langle \frac{x_{i+1} - x_i}{\|x_{i+1} - x_i\|}, \frac{\partial x_{i+1}}{\partial u_j} - \frac{\partial x_i}{\partial u_j} \right\rangle - \left\langle \frac{\partial x_k}{\partial u_j}, \theta \right\rangle.$$

Using the vectors $p_i = \frac{x_{i+1}-x_i}{\|x_{i+1}-x_i\|}$, the reflection law implies

$$\frac{\partial T}{\partial u_j}(u,v) = \left\langle \frac{\partial \omega}{\partial u_j}, x_1 \right\rangle + \left\langle \omega - p_1, \frac{\partial x_1}{\partial u_j} \right\rangle + \left\langle p_1 - p_2, \frac{\partial x_2}{\partial u_j} \right\rangle + \cdots + \left\langle p_{k-2} - p_{k-1}, \frac{\partial x_{k-1}}{\partial u_j} \right\rangle + \left\langle p_{k-1} - \theta, \frac{\partial x_k}{\partial u_j} \right\rangle = \left\langle \frac{\partial \omega}{\partial u_j}, x_1 \right\rangle.$$

Similarly,

$$\frac{\partial S}{\partial u_j}(u,v) = \left\langle \frac{\partial \omega}{\partial u_j}, y_1 \right\rangle.$$

Since $T(u,v) = S(u,v)$ on W, we must have $\frac{\partial T}{\partial u_j}(u,v) = \frac{\partial S}{\partial u_j}(u,v)$ for all $w = (u,v) \in W$, so

$$\left\langle \frac{\partial \omega}{\partial u_j}, x_1 \right\rangle = \left\langle \frac{\partial \omega}{\partial u_j}, y_1 \right\rangle$$

for all $j = 1, \ldots, n-1$. That is,

$$\left\langle \frac{\partial \omega}{\partial u_j}, y_1 - x_1 \right\rangle = 0$$

for all j. Since the vectors $\frac{\partial \omega}{\partial u_j}$, $j = 1, \ldots, n-1$, span a plane perpendicular to ω, it follows now that $(y_1 - x_1) \parallel \omega$.

Similarly, we derive $(y_m - x_k) \parallel \theta$. ■

Next, we continue with the proof of Lemma 13.2.1. We will use the notation introduced just before the statement of Lemma 13.2.3.

We will now use again the assumption (13.2). For every $(\omega,\theta) \in \mathcal{R} \cap U$ there exists a unique reflecting (ω,θ)-ray $\delta''(\omega,\theta)$ in Ω_L with

$$T_{\delta''(\omega,\theta)} = T_{\delta(\omega,\theta)}. \tag{13.4}$$

The uniqueness follows from the choice of the set $\mathcal{R}$. Moreover, choosing the neighbourhood U of (ω_0, θ_0) in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ sufficiently small, we have $\delta''(\omega, \theta) = \delta'(\omega, \theta)$ for all $(\omega, \theta) \in \mathcal{R} \cap U$. Indeed, if this is not true, then there exists a sequence $\{(\omega_p, \theta_p)\}_{p=1}^{\infty} \subset \mathcal{R} \cap U$ with $(\omega_p, \theta_p) \to (\omega_0, \theta_0)$ as $p \to \infty$ such that

$$\delta''(\omega_p, \theta_p) \neq \delta'(\omega_p, \theta_p), \quad p \geq 1.$$

Consider the hyperplane $Z = Z_{\omega_0}$, and for each $p \geq 1$ let y_p be the (incoming) intersection point of $\delta''(\omega_p, \theta_p)$ with Z, so that $\delta''(\omega_p, \theta_p) = \gamma_L(y_p, \omega_p)$. Choosing a subsequence, we may assume that $y_p \to y \in Z$ as $p \to \infty$. Since $\omega_p \to \omega_0$, it follows that $\delta'' = \gamma_L(y, \omega_0)$ is an (ω_0, θ_0)-ray in Ω_L with

$$T_{\delta''} = \lim_{p \to \infty} T_{\delta''(\omega_p, \theta_p)} = \lim_{p \to \infty} T_{\delta(\omega_p, \theta_p)} = T_{\delta}.$$

Here we used (13.4). Now (13.3) implies $T_{\delta''} = T_{\delta} = T_{\delta'}$. From this, $(\omega_0, \theta_0) \in \mathcal{R}$ and the choice of $\mathcal{R}$, it follows that $\delta'' = \delta'$. Hence u belongs to $\delta' = \delta'(\omega_0, \theta_0)$; therefore, for large p the ray $\delta''(\omega_p, \theta_p)$ has m reflection points belonging to the neighbourhoods U'_j, respectively. From the choice of U and the uniqueness of the (ω, θ)-rays $\delta'(\omega, \theta)$ for $(\omega, \theta) \in U$, it now follows that $\delta''(\omega_p, \theta_p) = \delta'(\omega_p, \theta_p)$. This is a contradiction which proves that $\delta''(\omega, \theta) = \delta'(\omega, \theta)$ for all $(\omega, \theta) \in \mathcal{R} \cap U$.

Hence

$$T_{\delta'(\omega, \theta)} = T_{\delta(\omega, \theta)} \tag{13.5}$$

for all $(\omega, \theta) \in \mathcal{R} \cap U$, and the density of $\mathcal{R} \cap U$ in U and the continuity of $T_{\delta(\omega,\theta)}$ and $T_{\delta'(\omega,\theta)}$with respect to (ω, θ) imply that (13.5) holds for all $(\omega, \theta) \in U$. Now Lemma 13.2.3 gives $(y_1(\omega, \theta) - x_1(\omega, \theta)) \parallel \omega$ and $(y_m(\omega, \theta) - x_k(\omega, \theta)) \parallel \theta_0$. So,

$$y_1(\omega, \theta) = x_1(\omega, \theta) + \lambda(\omega, \theta)\, \omega, \quad y_m(\omega, \theta) = x_k(\omega, \theta) + \mu(\omega, \theta)\, \omega \tag{13.6}$$

for some real numbers $\lambda(\omega, \theta), \mu(\omega, \theta)$.

By assumption we have $(y, \eta) = \mathcal{F}_{t_0}^{(K)}(\sigma_0) \in S^*(\Omega_0)$. Thus, either $\eta = \omega_0$ and $y = x_1 + s\omega_0$ for some s, or $\eta = \theta_0$ and $y = x_k + s\theta_0$ for some $s > 0$. The same holds for $\mathcal{F}_{t_0}^{(L)}(\sigma_0) = (y', \eta')$. Now (13.5) and (13.6) imply $(y, \eta) = (y', \eta')$. Indeed, in the case $\eta = \omega_0$ and $y = x_1 + s\omega_0$ for some s this is trivial. Assume that $y = x_k + s\theta_0$ for some $s > 0$. Then $\eta = \theta_0$ and (13.6) shows that $\eta' = \theta_0$, too. It remains to see that $y = y'$. Recall the hyperplane Z_{ω_0} tangent to the ball $\mathcal{O}$ and perpendicular to ω_0 and so that ω_0 points into the half-space determined by Z_{ω_0} and containing $\mathcal{O}$. Shifting points along the trajectory $\gamma_K(\sigma_0)$, it is enough to consider the case when $q_0 \in Z_{\omega_0}$ and $y \in Z_{-\theta_0}$. If R is the radius of the ball $\mathcal{O}$, we then have $T_{\delta} = t_0 - 2R$ (see [G1]). Now (13.5) gives $T_{\delta'} = t_0 - 2R$ which implies that the point y' must lie in the hyperplane $Z_{-\theta_0}$. This and (13.6) yield $y' = y$. Thus, $\mathcal{F}_{t_0}^{(K)}(\sigma_0) = \mathcal{F}_{t_0}^{(L)}(\sigma_0)$.

This completes the proof in Case 1.

Case 2.$\gamma_K(\sigma_0) \cap \partial K = \emptyset$. Then we must have $\gamma_L(\sigma_0) \cap \partial L = \emptyset$. Otherwise the above argument (swapping the roles of K and L) implies $\gamma_K(\sigma_0) \cap \partial K \neq \emptyset$, which is a contradiction. So, both $\gamma_K(\sigma_0)$ and $\gamma_L(\sigma_0)$ are free rays in $\mathbb{R}^n$ and therefore $\mathcal{F}_t^{(K)}(\sigma_0) = \mathcal{F}_t^{(L)}(\sigma_0)$ for all $t \in \mathbb{R}$. ■

Proof of Theorem 13.1.2: For any $\sigma \in \dot{T}^*(\Omega) \setminus \mathrm{Trap}(\Omega_K)$ take $t = t(\sigma) \in \mathbb{R}$ so large that $\mathrm{pr}_1(\mathcal{F}_t^{(K)}(\sigma)) \in S^*(\Omega_0)$ and set $\Phi(\sigma) = \mathcal{F}_{-t}^{(L)} \circ \mathcal{F}_t^{(K)}(\sigma)$. The definition of Φ is correct by Lemma 13.2.1, and clearly $\mathcal{F}_t^{(L)} \circ \Phi = \Phi \circ \mathcal{F}_t^{(K)}$ for all $t \in \mathbb{R}$. Apart from that $\Phi(\sigma) = \sigma$ for $\sigma \in \dot{T}^*(\Omega_0) \setminus \mathrm{Trap}(\Omega_K)$. The general properties of the generalized geodesic flows $\mathcal{F}_t^{(K)}$ and $\mathcal{F}_t^{(L)}$ (see [MS2] or Sect. 24.3 in [H3]) imply that Φ is a homeomorphism and it is a symplectic map on an open dense subset of $\dot{T}^*(\Omega_0) \setminus \mathrm{Trap}(\Omega_K) = \dot{T}^*(\Omega_0) \setminus \mathrm{Trap}(\Omega_K)$.

Conversely, assume that for two obstacles $K, L \in \mathcal{K}_0$ there exists a homeomorphism

$$\Phi : S^*(\Omega_K) \setminus \mathrm{Trap}(\Omega_K) \longrightarrow S^*(\Omega_L) \setminus \mathrm{Trap}(\Omega_L)$$

with $\mathcal{F}_t^{(L)} \circ \Phi = \Phi \circ \mathcal{F}_t^{(K)}$ for all $t \in \mathbb{R}$ and such that $\Phi = \mathrm{id}$ on $S^*(\Omega_0) \setminus \mathrm{Trap}(\Omega_K)$.

Consider an arbitrary $(\omega, \theta) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$. Given an (ω, θ)-ray γ in Ω_K, there exist $\sigma \in S^*(\Omega_0)$ and $T > 0$ with $\gamma = \gamma_K(\sigma)$ and $\mathcal{F}_t^{(K)}(\sigma) \in S^*(\Omega_0)$ for all $t \leq 0$ and $t \geq T$. Thus, for all $t \in (-\infty, 0] \cup [T, \infty)$ we have

$$\mathcal{F}_t^{(L)}(\sigma) = \mathcal{F}_t^{(L)} \circ \Phi(\sigma) = \Phi \circ \mathcal{F}_t^{(K)}(\sigma) = \mathcal{F}_t^{(K)}(\sigma),$$

so $\gamma_L(\sigma)$ is an (ω, θ)-ray in Ω_L having the same sojourn time as $\gamma_K(\sigma)$. In particular, $\mathcal{SL}_K(\omega, \theta) \subset \mathcal{SL}_L(\omega, \theta)$ for all (ω, θ). By symmetry, we must have $\mathcal{SL}_K(\omega, \theta) = \mathcal{SL}_L(\omega, \theta)$ for all (ω, θ). ■

Proof of Corollary 13.1.3: By the assumptions, $\Phi : S^*(\mathcal{O} \cap \Omega_K) \longrightarrow S^*(\mathcal{O} \cap \Omega_L)$ is defined almost everywhere and is measure-preserving with respect to the Liouville measures μ_K on $S^*(\Omega_K)$ and μ_L on $S^*(\Omega_L)$. So, we must have

$$\mu_K(S^*(\mathcal{O} \cap \Omega_K)) = \mu_L(S_b^*(\mathcal{O} \cap \Omega_L)).$$

By Fubini's theorem,

$$\begin{aligned}\mu_K(S^*(\mathcal{O} \cap \Omega_K)) &= \mathrm{Vol}_{n-1}(\mathbb{S}^{n-1})\mathrm{Vol}_n(\mathcal{O} \cap \Omega_K) \\ &= \mathrm{Vol}_{n-1}(\mathbb{S}^{n-1})[\mathrm{Vol}_n(\mathcal{O}) - \mathrm{Vol}_n(K)],\end{aligned}$$

and similarly

$$\begin{aligned}\mu_L(S^*(\mathcal{O} \cap \Omega_L)) &= \mathrm{Vol}_{n-1}(\mathbb{S}^{n-1})\mathrm{Vol}_n(\mathcal{O} \cap \Omega_L) \\ &= \mathrm{Vol}_{n-1}(\mathbb{S}^{n-1})[\mathrm{Vol}_n(\mathcal{O}) - \mathrm{Vol}_n(L)].\end{aligned}$$

Hence, $\mathrm{Vol}_n(K) = \mathrm{Vol}_n(L)$. ■

13.3 An example: star-shaped obstacles

An obstacle K in $\mathbb{R}^n$ is called *star-shaped* if there exists a point x_0 in the interior of K such that the segment $[x_0, x]$ is entirely in K for any $x \in \partial K$.

Proposition 13.3.1: *If K is a star-shaped obstacle and the curvature of K does not vanish of infinite order, then K is non-trapping, that is* $\mathrm{Trap}(\Omega_K) = \emptyset$.

Proof: Assume that K is star shaped with respect to 0, that is for every $x \in \partial K$ the segment $[0, x]$ lies in K.

Consider the function $f : S^*(\Omega_K) \longrightarrow \mathbb{R}$ given by $f(x, \xi) = \langle x, \xi \rangle$. We will show that f is increasing along reflecting rays in Ω_K from one reflection point to the next.

Let $x \in \partial K$ and $\xi \in \mathbb{S}^{n-1}$ be such that $\langle \nu_K(x), \xi \rangle > 0$. Let $y = x + t\xi$ for some $t > 0$ be the first intersection point of the straight-line ray issued from x in direction ξ with ∂K, and let η be the reflected direction of that ray, that is

$$\eta = \xi - 2\langle \nu_K(y), \xi \rangle \; \nu_K(y).$$

Then

$$\begin{aligned} f(y, \eta) = \langle y, \eta \rangle &= \langle x + t\xi, \xi - 2\langle \nu_K(y), \xi \rangle \; \nu_K(y) \rangle \\ &= \langle x, \xi \rangle + t - 2\langle \nu_K(y), \xi \rangle \; \langle \nu_K(y), y \rangle \\ &= \langle x, \xi \rangle + t + 2\langle \nu_K(y), \eta \rangle \; \langle \nu_K(y), y \rangle \geq f(x, \xi) + t. \end{aligned}$$

Let R be the radius of a large ball $\mathcal{O}$ with centre 0 containing K, and let $T = 2R$. Assume that $\mathrm{Trap}(\Omega_K) \neq \emptyset$; then there exists $\sigma = (x, \xi) \in S^*_{\partial K}(\Omega_K)$ such that $\|\mathrm{pr}_1(\mathcal{F}_t^{(K)}(\sigma))\| \leq R$ for all $t \geq 0$. Since the curvature of K does not vanish of infinite order, σ can be approximated arbitrarily well with points σ' such that the trajectory $\{\mathcal{F}_t^{(K)}(\sigma') : t \in [0, T]\}$ has only simple (transversal) reflections (see Theorem 7.16 in [MS2] or [H3]), so without loss of generality we will assume that

$$\gamma = \{\mathcal{F}_t^{(K)}(\sigma) : t \in [0, T]\}$$

has only simple (transversal) reflections and

$$\|\mathrm{pr}_1(\mathcal{F}_t^{(K)}(\sigma))\| \leq R, \quad 0 \leq t \leq T.$$

Let (x_i, ξ_i), $i = 1, 2, \ldots$, be the successive reflection points (with reflected directions) of γ. Set $(x_0, \xi_0) = (x, \xi)$ and let $(x_{i+1}, \xi_{i+1}) = \mathcal{F}_{t_i}^{(K)}(x_i, \xi_i)$ for all $i \geq 0$. Then for any $m \geq 1$ we have

$$\begin{aligned} \|x_m\| \geq \langle x_m, \xi_m \rangle = f(x_m, \xi_m) &\geq f(x_{m-1}, \xi_{m-1}) + t_{m-1} \\ &\geq t_{m-1} + t_{m-2} + \cdots + t_1 + \langle x_0, \xi_0 \rangle \geq t_{m-1} + t_{m-2} + \cdots + t_1. \end{aligned}$$

So, for any time $t \in [0, T]$ we have $\|\mathrm{pr}_1(\mathcal{F}_t^{(K)}(\sigma))\| \geq t$. In particular, $\|\mathrm{pr}_1(\mathcal{F}_t^{(K)}(\sigma))\| \geq T > R$, which is impossible. Hence $\mathrm{Trap}(\Omega_K) = \emptyset$. ■

Remark 13.3.2: Notice that in the calculation above involving (x, ξ) and $y = x + t\,\xi$ the inequality $\|y\| < \|x\|$ is only possible if $f(x, \xi) = \langle x, \xi \rangle < 0$. Indeed,

$$\|y\|^2 = \|x + t\xi\|^2 = \|x\|^2 + t^2 + 2t\,\langle x, \xi \rangle,$$

so clearly $\|y\|^2 < \|x\|^2$ implies $\langle x, \xi \rangle < 0$.

As an application of Theorem 13.1.2 we will derive the following.

Proposition 13.3.3: *Let $K, L \in \mathcal{K}_0$ have almost the same SLS. If K is star shaped, then $\partial K \subset \partial L$. If moreover L is non-trapped or ∂L is connected, then $K = L$.*

Proof: Given $x \in \partial K$, as before we will denote by $\nu_K(x)$ the outward unit normal to K at x. In the following for brevity we will use the notation $\gamma_K(x) = \gamma_K(x, \nu_K(x))$.

Assume $K, L \in \mathcal{K}_0$ have almost the same SLS and K is star shaped. Then the conclusion of Theorem 13.1.2 holds for the generalized geodesic flows in Ω_K and Ω_L. We will assume that in the above definition of star-shaped $x_0 = 0$, that is for every $x \in \partial K$ the segment $[0, x]$ lies in K.

Set

$$W_r = \{y \in \mathbb{R}^n : \|y\| > r\}$$

for $r > 0$, and let

$$a = \inf\{R > 0 : W_r \cap \partial K \subset \partial L\; \forall r \geq R\}.$$

Obviously, $W_a \cap \partial K \subset \partial L$. We will prove that $a = 0$.

We claim that

$$W_a \cap \partial K = W_a \cap \partial L. \tag{13.7}$$

Indeed, if $W_a \cap \partial K$ is a proper subset of $W_a \cap \partial L$, then there exists $x \in \partial L \setminus \partial K$ with $\|x\| > a$, so

$$b = \sup\{\|x\| : x \in \partial L \setminus \partial K\} \in (a, \infty)$$

is well defined. By the definition of b we can find a sequence $\{x_m\}$ of points in $\partial L \setminus \partial K$ such that $\|x_m\| \nearrow b$ as $m \to \infty$. Since ∂L is compact, taking a subsequence, we may assume that $x_m \to x \in \partial L$. Then $\|x\| = b > a$, so $x \in \partial K$ is impossible; otherwise by the choice of a we would have $\partial K = \partial L$ near x, a contradiction. Thus, $x \in \partial L \setminus \partial K$.

This shows that b is not just a supremum, it is a maximum, that is

$$b = \max\{\|x\| : x \in \partial L \setminus \partial K\}.$$

Take $x \in \partial L \setminus \partial K$ such that $\|x\| = b$. Then the normal to ∂L at x must be perpendicular to ∂L, that is $\nu_L(x) = x/\|x\|$. Moreover, the choice of b shows that for $r \geq b$ we have $W_r \cap \partial K \subset \partial L$. Thus, the ray $\ell = \{x + t\nu_L(x) : t > 0\}$ has no common points with $\partial L \setminus \partial K$, so $\ell \cap \partial L = \ell \cap \partial K$. If there exists a point $y \in \ell \cap \partial K$, since K is star shaped, we must have $[0, y] \subset K$. However, clearly $x \in [0, y]$, so we must have $x \in K$, which is a contradiction. Thus, $\ell \cap \partial L = \ell \cap \partial K = \emptyset$. Thus, $\gamma_L(x)$ defines a back-scattering ray in Ω_L, that is a scattering ray with just one reflection point x where the ray is perpendicular to ∂L. Now using the conjugacy Φ from Theorem 13.1.2, it follows immediately that $\gamma_L(x)$ coincides with a corresponding ray in Ω_K, that is we must have $x \in \partial K$. This is a contradiction that proves (13.7).

We are now ready to prove that $a = 0$. Assume $a > 0$. Then there exists $x \in \partial K$ with $\|x\| = a$. Fix for a moment an arbitrary x with this property. If $\{x + t\nu_k(x) : t > 0\} \cap \partial K = \emptyset$, then there exists an open neighbourhood $\mathcal{O}_x$ of x in $\mathbb{R}^n$ such that $\{y + t\nu_k(y) : t > 0\} \cap \partial K = \emptyset$ for all $y \in \mathcal{O}_x \cap \partial K$. In this case it follows easily, using the conjugacy Φ from Theorem 13.1.2, that $\partial K = \partial L$ in a neighbourhood of x, and we may take $\mathcal{O}_x$ so that $\mathcal{O}_x \cap \partial K = \mathcal{O}_x \cap \partial L$.

If $\{x + t\nu_k(x) : t > 0\} \cap \partial K \neq \emptyset$, then there exist $\epsilon > 0$ and an open neighbourhood $\mathcal{O}_x$ of x in $\mathbb{R}^n$ such that $\{y + t\nu_k(y) : t > 0\} \cap \partial K \neq \emptyset$ and

$$t_y = \min\{t > 0 : y + t\nu_K(y) \in \partial K\} > \epsilon$$

for all $y \in \mathcal{O}_x \cap \partial K$. Shrinking $\mathcal{O}_x$ if necessary, we may assume that $\|y\| + t_y > a$ for all $y \in \mathcal{O}_x$. Given $y \in \mathcal{O}_x$, consider the trajectory $\delta(y) = \gamma_K^+(y, \nu_K(y))$. Let f be the function from the proof of Proposition 13.3.1. Since $f(y, \nu_K(y)) > 0$, it follows from the argument in the proof of Proposition 13.3.1 that for any reflection point (z, ζ) of $\delta(y)$ we have $f(z, \zeta) > f(y, \nu_K(y)) > 0$. Combining this with Remark 13.3.2, shows that $\|z\| > \|y\|$. Thus, whenever $y \in \mathcal{O}_x$, we have that all reflection points of $\delta(y)$ belong to W_a. Now (13.7) implies $(y, \nu_K(y)) = (y, \nu_L(y)$ for any $y \in \mathcal{O}_x$, so in particular $\mathcal{O}_x \cap \partial K = \mathcal{O}_x \cap \partial L$.

Thus, for any $x \in \Gamma_a = \{y \in \partial K : \|y\| = a\}$ we can choose an open neighbourhood $\mathcal{O}_x$ of x in $\mathbb{R}^n$ with $\mathcal{O}_x \cap \partial K = \mathcal{O}_x \cap \partial L$. Since Γ_a is compact, there exist $x_1, \ldots, x_p \in \Gamma_a$ with $\Gamma_a \subset \cup_{i=1}^p \mathcal{O}_{x_i}$. The latter is then an open neighbourhood of Γ_a in $\mathbb{R}^n$, so for $r < a$, r sufficiently close to a, we will have $\Gamma_r \subset \cup_{i=1}^p \mathcal{O}_{x_i}$. This immediately implies $W_r \cap \partial K \subset \partial L$ for such r, which is a contradiction with the choice of a. Thus, we must have $a = 0$.

This proves that $\partial K \subset \partial L$. The second part of the statement in Proposition 13.3.3 follows trivially from the first. ■

13.4 Tangential singularities of scattering rays I

As one may expect the behaviour of the generalized geodesic flow near a scattering ray $\gamma_K(\sigma)$ tells us whether the ray has some kind of tangency to ∂K. A simple way to look at this is by using Poincaré maps between appropriate cross-sections of $\gamma_K(\sigma)$.

As before, throughout $\mathcal{O}$ denotes a large open ball in $\mathbb{R}^n$ and $\Omega_0 = \overline{\mathbb{R}^n \setminus \mathcal{O}}$.

Let K be an obstacle in $\mathbb{R}^n$ and let $\sigma_0 = (x_0, \xi_0) \in S^*(\Omega_0)$ generate a scattering ray, i.e. there exists $T > 0$ such that $\mathcal{F}_t^{(K)}(\sigma_0) \in S^*(\Omega_0)$ for all $t \leq 0$ and all $t \geq T$. Consider arbitrary smooth $(n-1)$-dimensional local submanifolds (e.g. hyperplanes) X and Y of $\mathbb{R}^n$ with $x_0 \in X$, $y_0 = \mathrm{pr}_1(\mathcal{F}_t^{(K)}(\sigma_0)) \in Y$ and such that X and Y are transversal to $\gamma_K(\sigma_0)$ at x_0 and y_0, respectively. Define the *Poincaré map* (or cross-sectional map)

$$\mathcal{P}_K : S^*X \longrightarrow S^*Y$$

by using the shift along the flow $\mathcal{F}_t^{(K)}$ from S^*X near σ_0 to S^*Y near $\rho_0 = \mathcal{F}_{t_0}^{(K)}(\sigma_0)$. If L is another obstacle contained in $\mathcal{O}$, $K, L \in \mathcal{K}_0$ and K and L have almost the same SLS, then according to Theorem 13.1.2 we have $\mathcal{P}_K = \mathcal{P}_L$. So whatever singularities we observe for $\mathcal{P}_K$, exactly the same we have for $\mathcal{P}_L$.

We discuss simple tangencies of scattering rays in this section. In typical situations we can also distinguish between diffractive and gliding behaviour, however the proof of this is a bit more involved and will be done in the next section.

First, let us consider the local cross-sectional map near a tangent point of a generalized geodesic.

Let $\rho_0 = (y, \xi) \in S^*(\partial K)$ be a diffractive point (i.e. $\rho_0 \in G_d$; see Section 1.2). Take $\epsilon > 0$ so small that

$$\ell = \{\mathrm{pr}_1(\mathcal{F}_t^{(K)}(\rho_0)) : -\epsilon \leq t \leq \epsilon\}$$

is a straight-line segment in Ω_K. Denote by Π and Π' the hyperplanes in $\mathbb{R}^n$ through $x = \mathrm{pr}_1(\mathcal{F}_{-\epsilon}^{(K)}(\rho_0))$ and $z = \mathrm{pr}_1(\mathcal{F}_{\epsilon}^{(K)}(\rho_0))$, respectively, and perpendicular to ℓ. Let $\sigma = (x, \xi)$ and let

$$\mathcal{P} : S^*\Pi \longrightarrow S^*\Pi'$$

be the (local) cross-sectional map defined by the shift along the flow $\mathcal{F}_t^{(K)}$ near the trajectory ℓ. Consider the Gauss map

$$G = d\nu(y) : T_y(\partial K) \longrightarrow T_y(\partial K)$$

of ∂K at y. Then $\kappa = \langle G(\xi), \xi \rangle > 0$, since (y, ξ) is a diffractive point. As we have done in Sections 2.3 and 2.4, for $\rho \in S^*(\Pi)$ close to σ one can identify the tangent space $T_\rho(S^*\Pi)$ with $\Pi \times \Pi$ in a natural way.

The following lemma is an easy consequence of Proposition 2.4.2.

Lemma 13.4.1: *Assume that $\{\sigma_m\}_{m=1}^{\infty} \subset S^*(\Pi)$ is a sequence converging to σ such that for each m the generalized geodesic*

$$\ell_m = \{\mathrm{pr}_1(\mathcal{F}_t^{(K)}(\sigma_m)) : -\epsilon \leq t \leq \epsilon\}$$

has a transversal reflection point y_m near y. (This is then its only reflection point.) If $\rho_0 = (y, \xi)$ is a diffractive point, that is $\rho_0 \in G_d$, then for each $m \geq 1$ there exists

$u_m \in \Pi$ *with* $\|u_m\| = 1$ *such that*

$$\|d\mathcal{P}(\sigma_m)(u_m, 0)\| \geq \frac{2\kappa}{\langle \xi_m, \nu(y_m) \rangle},$$

where ξ_m *is the reflected direction of* ℓ_m *at* y_m*. In particular,* $\|d\mathcal{P}(\sigma_m)\| \to \infty$ *as* $m \to \infty$. ■

In what follows and also in the next few sections we will work with obstacles K so that the normal curvature of K does not vanish of infinite order. Denote by $\mathcal{K}^{(\text{fin})}$ the *class of obstacles* with this property. Clearly $\mathcal{K}^{(\text{fin})} \subset \mathcal{K}$. Set

$$\mathcal{K}_0^{(\text{fin})} = \mathcal{K}^{(\text{fin})} \cap \mathcal{K}_0.$$

As shown in the following proposition, the points in $S^*(\Omega_0)$ generating scattering rays having multiple tangencies can be well approximated by points generating rays with just one tangency to ∂K.

Proposition 13.4.2: *For any obstacle* $K \in \mathcal{K}^{(\text{fin})}$ *and any point*

$$\sigma = (y, \eta) \in S^*(\partial K) \setminus \text{Trap}(\Omega_K)$$

such that the Gauss curvature of ∂K *at* y *is non-zero, there exists* $\sigma' = (y', \eta') \in S^*(\partial K)$ *arbitrarily close to* σ *such that* y' *is the only tangent point of the scattering ray* $\gamma_K(\sigma')$ *to* ∂K*.*

We prove this proposition in Section 13.8. Now we will use to derive an important consequence.

Proposition 13.4.3: *Assume that two obstacles* $K, L \in \mathcal{K}_0^{(\text{fin})}$ *have almost the same SLS. For any point* $\sigma_0 = (x_0, \xi_0) \in S^*(\Omega_0) \setminus \text{Trap}(\Omega_K)$ *the scattering ray* $\gamma_K(\sigma_0)$ *contains a point of tangency to* ∂K *if and only if* $\gamma_L(\sigma_0)$ *contains a point of tangency to* ∂L*.*

Proof of Proposition 13.4.3: Replacing ξ_0 by $-\xi_0$ if necessary, we may assume that there exists $T > 0$ such that $\mathcal{F}_t^{(K)}(\sigma_0) \in S^*(\Omega_0)$ for all $t \leq 0$ and all $t \geq T$. Consider the hyperplanes X and Y of $\mathbb{R}^n$ with $x_0 \in X$, $y_0 = \text{pr}_1(\mathcal{F}_t^{(K)}(\sigma_0)) \in Y$ and such that X and Y are perpendicular to $\gamma_K(\sigma_0)$ at x_0 and y_0, respectively. Define the cross-sectional maps $\mathcal{P}_K, \mathcal{P}_L : S^*X \longrightarrow S^*Y$ as before. Then $\mathcal{P}_K = \mathcal{P}_L$.

Assume $\gamma_K(\sigma_0) \cap S^*(\partial K) \neq \emptyset$, that is $\gamma_K(\sigma_0)$ contains a tangent point to ∂K. If for some $t_0 > 0$ the point $\rho_0 = \mathcal{F}_{t_0}^{(K)}(\sigma_0) \in S^*(\partial K)$ is a single tangent point, that is it does not belong to a gliding segment on ∂K, then there exists a diffractive point $\rho \in S^*(\partial K)$ arbitrarily close to ρ_0 (see e.g. Section 24.3 in [H3]). By Proposition 13.4.2, we can take ρ so that it is the only tangent point of $\gamma_K(\rho)$ to ∂K. Take such a point ρ and let $\sigma \in S^*X$ be so that $\mathcal{F}_t^{(K)}(\sigma) = \rho$ for some $t > 0$ close to t_0. Then $\gamma_K(\sigma) = \gamma_K(\rho)$ has only one tangent point to ∂K which is the projection of a diffractive point in $S^*(\partial K)$. Now Lemma 13.4.1 and $\mathcal{P}_K = \mathcal{P}_L$ imply that the map $\mathcal{P}_L$ has a singularity at σ, so $\gamma_L(\sigma)$ must contain a tangent point to ∂L. Letting $\rho \to \rho_0$, we have $\sigma \to \sigma_0$ and therefore $\gamma_L(\sigma_0)$ contains a tangent point to ∂L.

Next, consider the case when $\gamma_K(\sigma_0)$ contains a gliding segment on ∂K, however it has no simple tangencies to ∂K. Let $\rho_0 = \mathcal{F}_{t_0}^{(K)}(\sigma_0) \in S^*(\partial K)$ be one end of such a gliding segment; then $\rho_0 \in G^3$ (see Section 1.2), and nearby ρ_0 there are points from the set G_g on the trajectory $\{\mathcal{F}_t^{(K)}(\sigma_0) : t \in \mathbb{R}\}$. Since $K \in \mathcal{K}^{(\mathrm{fin})}$, setting $\rho_0 = (x_0, \xi_0)$, the curvature of ∂K in the direction of ξ_0 must change sign at x_0, so there exist diffractive points $\rho \in S^*(\partial K)$ arbitrarily close to ρ_0 (see Section 1.2). Now repeating the argument from the previous case, we derive again that the trajectory $\gamma_L(\rho)$ must have a tangency to ∂L. Letting $\rho \to \rho_0$ shows that $\gamma_L(\sigma_0)$ contains a tangent point to ∂L. ■

13.5 Tangential singularities of scattering rays II

This section deals with a more comprehensive approach in studying tangential singularities of scattering rays. It is rather technical and might be skipped at first reading. It will be used in subsequent sections, and the reader may then need to look into this section.

It turns out that the behaviour of the sojourn time function near σ tells us, for example, whether $\gamma_K(\sigma)$ has a simple tangency at ∂K or it contains a whole gliding segment on ∂K. Some other information can be obtained as well.

We begin with some local considerations.

Let again K be an obstacle in $\mathbb{R}^n$, $n \geq 2$, such that the normal curvature of K does not vanish of infinite order. In this section for brevity we will use the notation

$$\mathcal{F}_t = \mathcal{F}_t^{(K)}, \quad \Omega = \Omega_K.$$

Let φ be a defining function for ∂K in a small neighbourhood V_0 of ∂K. That is, $\varphi : V_0 \longrightarrow \mathbb{R}$ is smooth, $d\varphi \neq 0$ on ∂K, and $\varphi^{-1}(0) = \partial K$. Consider the Hamiltonian function

$$p : T^*(\mathbb{R}^n) \longrightarrow \mathbb{R}, \quad p(x, \xi) = \frac{1}{2}(|\xi|^2 - 1).$$

The corresponding Hamiltonian vector field is

$$H_p = (\xi_1, \ldots, \xi_n; 0, \ldots, 0) = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i} .$$

As before, $\mathcal{O}$ will denote a *fixed open ball* in $\mathbb{R}^n$ containing K and $\Omega_0 = \overline{\mathbb{R}^n \setminus \mathcal{O}}$. Set $S_0 = \partial\mathcal{O}$.

In this section we will first study the sets

$$\mathcal{T}_k = \{\sigma \in S^*(V_0) : H_p^j \varphi(\sigma) = 0 \quad \text{for} \quad j = 0, 1, \ldots, k \quad \text{and} \quad H_p^{k+1}\varphi(\sigma) \neq 0 \},$$

where k is a positive integer. In general these sets are not manifolds, however locally each of them is contained in a submanifold of codimension 2 in $S^*(V_0)$. What is more important, it turns out that for $k \geq 2$ the set $\mathcal{T}_k$ is locally contained in a submanifold of codimension 3 (see Proposition 13.5.1). An important consequence of this is that the set of those $\sigma \in S^*(\Omega_0)$ that generate trajectories containing gliding segments on ∂K can be covered by a countable family of submanifolds of codimension 2 in $S^*(\Omega_0)$, so topologically it does not divide $S^*(\Omega_0)$ (see e.g. [F]).

Proposition 13.5.1: *For each $k \geq 1$ and each $\sigma \in \mathcal{T}_k$ there exists an open neighbourhood $V(\sigma)$ of σ in $T^*(V_0)$ and a smooth submanifold $\Gamma(\sigma)$ of $V(\sigma)$ such that $\mathcal{T}_k \cap V(\sigma) \subset \Gamma(\sigma) \subset S^*(V_0)$ and the codimension of $\Gamma(\sigma)$ in $T^*(V_0)$ is 3 for $k = 1$ and 4 for $k \geq 2$. Consequently, as a submanifold of $S^*(V_0)$, the codimension of $\Gamma(\sigma)$ is 2 for $k = 1$ and 3 for $k \geq 2$.*

Proof of Proposition 13.5.1: Denote by V' the set of those $\rho \in T^*(V_0)$ such that $\xi \neq 0$ and $H_p^{k+1}\varphi(\rho) \neq 0$, and define $g : V' \longrightarrow \mathbb{R}$ by

$$g(\rho) = H_p\varphi(\rho) = \sum_{i=1}^{n} \xi_i \frac{\partial \varphi}{\partial x_i}(x),$$

where $\rho = (x, \xi)$.

First, consider the case $k = 1$. We claim that

$$\mathcal{T}_1 = \{\rho \in V' : p(\rho) = \varphi(\rho) = g(\rho) = 0\}$$

is a submanifold of $V(\sigma)$. For this of course one has to show that $dp(\rho)$, $d\varphi(\rho)$ and $dg(\rho)$ are linearly independent on $\mathcal{T}_1$. Let $\rho = (x, \xi) \in \mathcal{T}_1$ and assume that

$$u\, dp(\rho) + a\, d\varphi(\rho) + b\, dg(\rho) = 0 \tag{13.8}$$

for some $u, a, b \in \mathbb{R}$. Here $d = d_{(x,\xi)}$. Considering derivatives with respect to x_j, (13.8) implies

$$a\, \frac{\partial \varphi}{\partial x_j}(x) + b\, \sum_{i=1}^{n} \xi_i \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(x) = 0.$$

Multiplying the latter by ξ_j and summing up, gives $aH_p\varphi(\rho) + bH_p^2\varphi(\rho) = 0$. Since $H_p\varphi(\rho) = 0$ and $H_p^2\varphi(\rho) \neq 0$, the above implies $b = 0$. This and (13.8) yield $u = a = 0$. Hence $\mathcal{T}_1$ is a submanifold of $V(\sigma)$ of codimension 3.

Next, consider the case $k \geq 2$. Define another function f on V' by

$$f(\rho) = H_p^k\varphi(\rho) = \sum_{i_1, i_2, \ldots, i_k = 1}^{n} \xi_{i_1}\xi_{i_2} \cdots \xi_{i_k} \frac{\partial^k \varphi}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}}(x)\, .$$

Set

$$\Gamma = \{\rho \in V' : p(\rho) = \varphi(\rho) = g(\rho) = f(\rho) = 0\}.$$

Clearly, $\mathcal{T}_k \cap V' \subset \Gamma \subset S^*(V_0)$. We will show that dp, $d\varphi$, dg and df are linearly independent at any point $\rho \in \mathcal{T}_k \cap V'$ (in particular at σ). Let $\rho = (x, \xi) \in \mathcal{T}_k \cap V'$ and let

$$u\ dp(\rho) + a\ d\varphi(\rho) + b\ dg(\rho) + c\ df(\rho) = 0 \tag{13.9}$$

for some $u, a, b, c \in \mathbb{R}$. Considering derivatives with respect to x_m, (13.9) implies

$$a\ \frac{\partial \varphi}{\partial x_m}(x) + b\ \sum_{i=1}^{n} \xi_i \frac{\partial^2 \varphi}{\partial x_i \partial x_m}(x)$$
$$+c \sum_{i_1, i_2, \dots, i_k = 1}^{n} \xi_{i_1}\xi_{i_2}\cdots\xi_{i_k} \frac{\partial^{k+1}\varphi}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_k}\partial x_m}(x) = 0$$

for all $m = 1, \dots, n$. Multiplying the latter by ξ_m and summing up, we get

$$aH_p\varphi(\rho) + bH_p^2\varphi(\rho) + cH_p^{k+1}\varphi(\rho) = 0.$$

Since $\rho \in \mathcal{T}_k$ and $k \geq 2$, we have $H_p\varphi(\rho) = H_p^2\varphi(\rho) = 0$ and $H_p^{k+1}\varphi(\rho) \neq 0$. Hence $c = 0$. Next, considering the terms in (13.9) corresponding to derivatives with respect to ξ_i, we get

$$u\ \xi_i + b\ \frac{\partial \varphi}{\partial x_i}(x) = 0$$

for all i. Multiplying this by ξ_i and summing up, gives $0 = u\ |\xi|^2 + b\ H_p\varphi(\rho) = u$, so $u = 0$. Returning to the previous equality and using the fact that $d\varphi(\rho) \neq 0$, one gets $b = 0$. Then (13.9) yields $a = 0$ as well. This shows that dp, $d\varphi$, dg and df are linearly independent at σ. By continuity, there exists an open neighbourhood $V(\sigma)$ of σ in V' such that dp, $d\varphi$, dg and df are linearly independent on $V(\sigma)$. Then $\Gamma \cap V(\sigma)$ is a submanifold of codimension 4 in $V(\sigma)$ and $\mathcal{T}_k \cap V(\sigma) \subset \Gamma \cap V(\sigma)$. ■

Next, we are going to show that for $k \geq 2$ the set $\mathcal{T}_k$ can be covered by a countable family of codimension 2 submanifolds of $S^*(S_0)$.

Fix for a moment $k \geq 2$ and $m \geq 0$. Denote by $\mathcal{T}_k^{(m)}$ the *set of those* $\sigma \in \mathcal{T}_k$ such that there exists $t > 0$ with $\mathcal{F}_t(\sigma) \in S^*(S_0)$ and the trajectory $\{\mathcal{F}_s(\sigma) : 0 < s \leq t\}$ has no common points with $\cup_{r=2}^{\infty}\mathcal{T}_r$ and has exactly m transversal reflection points at ∂K (and possibly some tangent points that belong to $\mathcal{T}_1$). For $\sigma \in \mathcal{T}_k^{(m)}$ denote by $s(\sigma)$ the *minimal number* $t > 0$ with $\mathcal{F}_t^{(K)}(\sigma) \in S^*(S_0)$.

Lemma 13.5.2: *For every $\sigma_0 \in \mathcal{T}_k^{(m)}$ there exists an open neighbourhood $U_k^{(m)}(\sigma_0)$ of σ_0 in $S^*(V_0)$ such that the set*

$$N_k^{(m)}(\sigma_0) = \{\mathcal{F}_{s(\sigma)}(\sigma) : \sigma \in \mathcal{T}_k^{(m)} \cap U_k^{(m)}(\sigma)\}$$

is contained in a smooth codimension 2 submanifold of $S^*(S_0)$.

Proof: We will use the above proposition and an argument from Section 11.3.

Fix an arbitrary $\sigma_0 \in \mathcal{T}_k^{(m)}$ and let $s_1 < s_2 < \cdots < s_m$ be the times of the transversal reflections of $\{\mathcal{F}_s(\sigma_0) : 0 < s \leq s(\sigma_0)\}$. Clearly $0 < s_1$ and $s_m < s(\sigma)$. For each $j = 1, \ldots, m$ fix two numbers a_j and b_j close to s_j and such that

$$b_0 = 0 < a_1 < s_1 < b_1 < a_2 < s_2 < b_2 < \cdots < a_m < s_m < b_m < a_{m+1} = s(\sigma_0).$$

For each $j = 1, \ldots, m$ choose arbitrary smooth cross sections (i.e. submanifolds of $S^*(\Omega)$ of codimension 1 transversal to the flow $\mathcal{F}_t$) $\mathcal{A}_j$ and $\mathcal{B}_j$ to the trajectory $\{\mathcal{F}_s(\sigma_0) : 0 < s \leq s(\sigma_0)\}$ such that $\mathcal{F}_{a_j}(\sigma_0) \in \mathcal{A}_j$ and $\mathcal{F}_{b_j}(\sigma_0) \in \mathcal{B}_j$. We assume that $\mathcal{A}_j$ and $\mathcal{B}_j$ are so small and so close to the reflection point $\mathcal{F}_{s_j}(\sigma_0)$ that for any $\rho \in \mathcal{A}_j$ the trajectory of ρ under $\mathcal{F}_t$ makes exactly one (transversal) reflection at ∂K before intersecting transversally $\mathcal{B}_j$.

Let $V(\sigma_0)$ be an open neighbourhood of σ_0 in $S^*(V_0)$ with the properties described in Proposition 13.5.1. For ρ in a small neighbourhood W of σ_0 in $V(\sigma_0)$ we denote by $\psi_t(\rho)$ the unique curve in $S^*(\mathbb{R}^n)$ for which there exists a sequence of numbers

$$b_0(\rho) = 0 < a_1(\rho) < s_1 < b_1(\rho) < a_2(\rho) < \cdots < a_m(\rho) < s_m < b_m(\rho) < a_{m+1}(\rho)$$

with the following properties:

(i) $\psi_{a_j(\rho)}(\rho) \in \mathcal{A}_j$ and $\psi_{b_j(\rho)}(\rho) \in \mathcal{B}_j$ for all $j = 1, \ldots, m$, $\psi_{a_{m+1}(\rho)}(\rho) \in S^*(S_0)$, and $\psi_s(\rho) \in S^*(\mathcal{O})$ for $s \in (0, a_{m+1}(\rho))$;

(ii) for each $j = 0, 1, \ldots, m$ the curve $\{\psi_t(\rho) : b_j(\rho) \leq t \leq a_{j+1}(\rho)\}$ is a trajectory of the vector field H_p in $\mathbb{R}^n$ (this curve could have common points with the interior of K);

(iii) for each $j = 1, \ldots, m$ the curve $\{\psi_t(\rho) : a_j(\rho) \leq t \leq b_j(\rho)\}$ is a trajectory of the GHF $\mathcal{F}_t$ in Ω.

It is clear that if the neighbourhood W of σ_0 in $V(\sigma_0)$ is sufficiently small, then the curve $\psi_t(\rho)$ is well defined for all $\rho \in W$. Set

$$\Lambda(\sigma_0) = \{\rho \in W : H_p^k \varphi(\rho) = 0\}.$$

We assume that $H_p^{k+1}\varphi \neq 0$ on $V(\sigma_0)$ (which follows from the construction of $V(\sigma_0)$ in the proof of Proposition 13.5.1). Then $\Lambda(\sigma_0)$ is a codimension 1 submanifold of W transversal to the vector field H_p. Consequently, the map

$$\Lambda(\sigma_0) \ni \rho \mapsto \psi_{a_1(\rho)}(\rho) \in \mathcal{A}_1$$

is smooth and a local bijection, so it defines a local diffeomorphism. Dealing in the same way with the shift along the curve $\psi_t(\rho)$ between successive cross sections, one derives that the map $\Psi_k^{(m)}(\rho) = \psi_{a_{m+1}(\rho)}(\rho)$ from $\Lambda(\sigma_0)$ to $S^*(S_0)$ is a local diffeomorphism.

Set $U_k^{(m)}(\sigma_0) = W$. Let $\Gamma(\sigma)$ be as in Proposition 13.5.1. Then $\Gamma(\sigma_0) \cap U_k^{(m)}$ is a codimension 2 submanifold of $\Lambda(\sigma_0)$. Hence $\Psi_k^{(m)}(\Gamma(\sigma_0) \cap U_k^{(m)}(\sigma_0))$ is a codimension 2 submanifold of $S^*(S_0)$. It remains to show that

$$N_k^{(m)}(\sigma_0) = \Psi_k^{(m)}(\Gamma(\sigma_0) \cap U_k^{(m)}(\sigma_0)).$$

To check this, observe that for any $\rho \in U_k^{(m)}(\sigma_0) \cap \mathcal{T}_k^{(m)}$ we have $\psi_s(\rho) = \mathcal{F}_s(\rho)$ for all $s \in [0, a_{m+1}(\rho)]$. Indeed, for such ρ the trajectory $\{\mathcal{F}_t(\rho) : t \geq 0\}$ has exactly m transversal reflection points $\mathcal{F}_{s_i(\rho)}(\rho)$, $i = 1, \dots, m$, where $s_i(\rho)$ is close to s_i for each i. There exist real numbers $a_i(\rho)$ close to a_i and $b_i(\rho)$ close to b_i such that $\mathcal{F}_{a_i(\rho)}(\rho) \in \mathcal{A}_i$ and $\mathcal{F}_{b_i(\rho)}(\rho) \in \mathcal{B}_i$ for all $i = 1, \dots, m$ and $\mathcal{F}_{a_{m+1}(\rho)}(\rho) \in S^*(S_0)$. Hence the curve

$$\psi_s(\rho) = \mathcal{F}_s(\rho), s \in [0, a_{m+1}(\rho)],$$

satisfies the conditions (i)–(iii). In particular, $s(\sigma) = a_{m+1}(\rho)$ and therefore $N_k^{(m)}(\sigma_0) = \Psi_k^{(m)}(\Gamma(\sigma_0) \cap U_k^{(m)}(\sigma_0))$. This proves the lemma. ■

Denote by $\mathcal{G}_K$ the set of those $\sigma \in S^*(S_0)$ such that $\mathcal{F}_t(\sigma) \in \mathcal{T}_k$ for some $t \in \mathbb{R}$ and some $k \geq 2$. Then $\mathcal{G}_K$ contains any $\sigma \in S^*(S_0)$ that generates a trajectory containing a gliding segment on ∂K (see Section 1.2).

As an immediate consequence of the above lemma we get the following.

Proposition 13.5.3: *There exists a countable family $\{N_i\}$ of codimension 2 submanifolds of $S^*(S_0)$ such that $\mathcal{G}_K \subset \cup_i N_i$.*

Proof: Cover the set $\mathcal{T}_k^{(m)}$ with a countable family $U_k^{(m)}(\sigma_j^{(k,m)})$ of open subsets of $S^*(V_0)$ with the properties listed in Lemma 13.5.2. Then

$$\mathcal{G}_K \subset \cup_{j=1}^{\infty} \cup_{m=0}^{\infty} \cup_{k=1}^{\infty} N_k^{(m)}(\sigma_j^{(k,m)}),$$

so the statement follows from Lemma 13.5.2. ■

Finally, it remains to deal with the case $k = 1$.

Lemma 13.5.4: *For every $\sigma_0 \in \mathcal{T}_1$ there exists an open neighbourhood $U(\sigma_0)$ of σ_0 such that the set*

$$N(\sigma_0) = \{\mathcal{F}_{s(\sigma)}(\sigma) : \sigma \in \mathcal{T}_1 \cap U(\sigma_0)\}$$

is contained in a smooth codimension 1 submanifold of $S^*(S_0)$.

Proof: This is essentially a repetition of the proof of Lemma 13.5.2 with minor modifications. It is actually simpler since there are no further tangencies that have to be avoided. We leave the details to the reader. ■

An important consequence of the last lemma is the following.

Proposition 13.5.5: *There exists a countable family $\{M_i\}$ of codimension 1 submanifolds of $S^*(S_0) \setminus (\text{Trap}(\Omega_K) \cup \mathcal{G}_K)$ such that every*

$$\sigma \in S^*(S_0) \setminus (\text{Trap}(\Omega_K) \cup \mathcal{G}_K \cup \cup_i M_i)$$

generates a simply reflecting trajectory in Ω. Moreover, the family $\{M_i\}$ is locally finite in $S^*(S_0) \setminus (\text{Trap}(\Omega_K) \cup \mathcal{G}_K)$, that is any compact subset of $S^*(S_0) \setminus (\text{Trap}(\Omega_K) \cup \mathcal{G}_K)$ has common points with only finitely many of the submanifolds M_i.

Proof of Proposition 13.5.5: For every $\sigma \in \mathcal{T}_1 \setminus \text{Trap}(\Omega_K)$ choose an open neighbourhood $U(\sigma)$ as in Lemma 13.5.4. Shrinking $U(\sigma)$ if necessary, we may assume that $H_p^2\varphi \neq 0$ on the closure $\overline{U(\sigma)}$ of $U(\sigma)$, and therefore it does not have common points with $\mathcal{T}_k$ for any $k \geq 2$. Also, we choose $U(\sigma)$ such that $\overline{U(\sigma)} \cap \text{Trap}(\Omega_K) = \emptyset$. Consequently, for the set $N(\sigma)$ from Lemma 13.5.4 we have $\text{dist}(N(\sigma), \text{Trap}(\Omega_K) \cup \mathcal{G}_K) > 0$, and so there exists a smooth codimension 1 submanifold $M(\sigma)$ of $S^*(S_0)$ such that $N(\sigma) \subset M(\sigma)$ and $\overline{M(\sigma)}$ has no common points with $\text{Trap}(\Omega_K) \cup \mathcal{G}_K$.

Choose a countable set of elements $\sigma_i \in \mathcal{T}_1 \setminus \text{Trap}(\Omega_K)$ such that $\mathcal{T}_1 \setminus \text{Trap}(\Omega_K) \subset \cup_i U(\sigma_i)$ and $\{U(\sigma_i) \cap \mathcal{T}_1\}$ is a locally finite family in $\mathcal{T}_1 \setminus \text{Trap}(\Omega_K)$. Denote $M_i = M(\sigma_i)$. It is now clear that any

$$\sigma \in S^*(S_0) \setminus (\text{Trap}(\Omega_K) \cup \mathcal{G}_K \cup \cup_i M_i)$$

generates a simply reflecting trajectory in Ω. It remains to show that $\{M_i\}$ is locally finite in $S^*(S_0) \setminus (\text{Trap}(\Omega_K) \cup \mathcal{G}_K)$. Let L be a compact subset of $S^*(S_0) \setminus (\text{Trap}(\Omega_K) \cup \mathcal{G}_K)$. Then the sojourn (travelling) times of trajectories generated by elements of L are uniformly bounded. Assume that there exists $\rho_m \in M_{i_m} \cap L$ for infinitely many i_m. Choosing a subsequence, we may assume that $\rho_m \to \rho \in L$ as $m \to \infty$. Since $\rho_m \in M_{i_m}$, there exists $t_m \in \mathbb{R}$ with $\mathcal{F}_{t_m}(\rho_m) \in U(\sigma_{i_m}) \cap \mathcal{T}_1$. Since the sequence $\{t_m\}$ is bounded, we may assume that $t_m \to t \in \mathbb{R}$ as $t \to \infty$. Then $\mathcal{F}_{t_m}(\rho_m) \to \mathcal{F}_t(\rho)$, so for $\sigma = \mathcal{F}_t(\rho)$ we must have $H_p\varphi(\sigma) = 0$. On the other hand, $\rho \in L$ shows that $\rho \notin \text{Trap}(\Omega_K) \cup \mathcal{G}_K$, so the trajectory generated by ρ is not trapped and does not have common points with $\mathcal{T}_k$ for any $k \geq 2$. Hence $\sigma \in \mathcal{T}_1 \setminus \text{Trap}(\Omega_K)$, and therefore there exists a compact neighbourhood W of σ in $S^*(\mathbb{R}^n)$ such that $W \cap \mathcal{T}_1 \cap \text{Trap}(\Omega_K) = \emptyset$. Then $\mathcal{F}_{t_m}(\rho_m) \to \sigma$ implies that W has common points with $U(\sigma_{i_m}) \cap \mathcal{T}_1$ for all sufficiently large m, a contradiction with the local finiteness of the family $\{U(\sigma_i) \cap \mathcal{T}_1\}$ in $\mathcal{T}_1 \setminus \text{Trap}(\Omega_K)$. Hence L can have common points with only finitely many M_i's. ■

Remark 13.5.6: In general different submanifolds M_i and M_j may have common points ρ and they are not necessarily transversal at such points. Clearly for such a point ρ the trajectory $\gamma_K(\rho)$ will have more than one tangent point to ∂K. However, it follows from the construction in the proof of Proposition 13.5.5 that if $M_i \neq M_j$ and $\sigma \in M_i \cap M_j$, then there exist points in $M_i \setminus M_j$ arbitrarily close to σ. Moreover, the construction shows that $\cup_i M_i$ is a closed subset of $S^*(S_0) \setminus (\mathrm{Trap}(\Omega_K) \cup \cup_q N_q)$. Indeed, by the local finiteness of the family $\{M_i\}$, if $\sigma \in \overline{M_i} \setminus (\mathrm{Trap}(\Omega_K) \cup \cup_q N_q)$ for some i, then $\sigma \in M_j$ for some j.

13.6 Reflection points of scattering rays and winding numbers

Let again $K \in \mathcal{K}$ be an obstacle in $\mathbb{R}^n$. We already know (see Proposition 11.1.3) that the set of points $\sigma \in S^*(\Omega_K)$ generating rays tangent to ∂K has Lebesgue measure zero in $S^*(\Omega_K)$. The question that we will discuss at some stage here is how to distinguish[1] between rays containing simple tangencies only and rays containing non-trivial segments on ∂K. It turns out that this is possible – near points generating rays with gliding segments there is a large variety of points generating rays with simple tangencies. To demonstrate this we will use winding numbers. We did use winding numbers in Section 8.3, however here we will proceed in a different way using an idea of Melrose and Sjöstrand [MS2].

As in the previous section, consider a defining function $\varphi \in C^\infty(\mathbb{R}^n)$ of ∂K. We choose φ with the following properties:

(i) $\varphi \geq 0$ in Ω_K and $\varphi^{-1}(0) = \partial K$;

(ii) $\|\nabla\varphi\| = 1$ in a neighbourhood V of ∂K;

(iii) $\varphi = 1$ on $\Omega_K \setminus V_0$, where V_0 is a neighbourhood of ∂K with $\overline{V} \subset V_0$.

Then $\nabla\varphi(x) = \nu_K(x)$ for $x \in \partial K$. Consider the continuous function

$$\lambda : S^*(\Omega_K) \times \mathbb{R} \longrightarrow \mathbb{C}$$

defined by

$$\lambda(\sigma, t) = [\varphi(x(t)) + \mathbf{i}\langle \xi(t), \nabla\varphi(x(t))\rangle]^2,$$

where $\mathcal{F}_t^{(K)}(\sigma) = (x(t), \xi(t))$, $t \in \mathbb{R}$. Now observe that when $x(t) \in \Omega_K \setminus V_0$, then $\varphi(x(t)) = 1$ and $\nabla\varphi(x(t)) = 0$, so $\lambda(\sigma, t) = 1$. So, in particular, if σ is a non-trapped point, then it will generate an (ω, θ)-ray for some $(\omega, \theta) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$, and then most of the points on the incoming ray and those on the outgoing ray will be outside V_0. Thus, for such σ, $t \mapsto \lambda(\sigma, t)$ defines a closed curve in $\mathbb{C}$ beginning and ending at 1.

Also, $\lambda(\sigma, t) = 0$ is equivalent to $\varphi(x(t)) = 0$ and $\langle \xi(t), \nabla\varphi(x(t))\rangle = 0$. This means that the ray $\gamma_K(\sigma)$ is tangent to ∂K at $x(t)$.

[1] For example, by using cross-sectional maps, as in Section 13.4.

So, if σ is a non-trapped point such that $\gamma(\sigma)$ is a simply reflecting ray, then $t \mapsto \lambda(\sigma, t)$ is a closed continuous curve in $\mathbb{C} \setminus \{0\}$, so it has a well-defined *winding number* $\mathcal{WN}(\sigma)$ with respect to 0. This is the degree of the map

$$\mathbb{R} \ni t \mapsto \frac{\lambda(\sigma, t)}{|\lambda(\sigma, t)|} \in \mathbb{S}^1$$

(see e.g. [Hir]).

We can now prove the following simple but rather important fact.

Proposition 13.6.1: *Assume that $K \in \mathcal{K}$ and $\sigma \in S^*(\Omega_K) \setminus \mathrm{Trap}(\Omega_K)$ generates a simply reflecting ray $\gamma_K(\sigma)$ in Ω_K. Then the number of reflection points of $\gamma_K(\sigma)$ equals $\mathcal{WN}(\sigma)$.*

Proof: Notice that for the number k of reflections of $\gamma(\sigma)$ we have

$$k = \sharp\{t : \lambda(\sigma, t) \in \mathbb{R}_-\}.$$

Indeed,

$$\lambda(\sigma, t) = \varphi(x(t))^2 - \langle \xi(t), \nabla\varphi(x(t))\rangle^2 + 2\mathbf{i}\varphi(x(t))\langle \xi(t), \nabla\varphi(x(t))\rangle \in \mathbb{R}_-$$

if and only if $\varphi(x(t))\langle \xi(t), \nabla\varphi(x(t))\rangle = 0$ and $\varphi(x(t))^2 - \langle \xi(t), \nabla\varphi(x(t))\rangle^2 < 0$. These two relations can happen only when $\varphi(x(t)) = 0$ and $\langle \xi(t), \nabla\varphi(x(t))\rangle \neq 0$, that is when $\gamma_K(\sigma)$ has a simple (transversal) reflection at $x(t)$.

Moreover, if $\gamma_K(\sigma)$ has a reflection at $x(t_0)$ for some $t_0 \in \mathbb{R}$, then for t near t_0 we have $\mathrm{Im}(\lambda(\sigma, t)) = \varphi(x(t))\langle \xi(t), \nabla\varphi(x(t))\rangle < 0$ for $t < t_0$ and $\mathrm{Im}(\lambda(\sigma, t)) > 0$ for $t > t_0$. So, the curve $\lambda(\sigma, t)$ intersects $\mathbb{R}_-$ at $t = t_0$ from below to above.[2] Thus,

$$\mathcal{WN}(\sigma) = \sharp\{t : \lambda(\sigma, t) \in \mathbb{R}_-\},$$

so $\mathcal{WN}(\sigma) = k$. ∎

Corollary 13.6.2: *Let $K \in \mathcal{K}$ and let $\sigma(s)$, $s \in [a, b]$, be a continuous curve in $S^*(\Omega_K)$ consisting of non-trapped points such that $\gamma_K(\sigma(s))$ is simply reflecting for all $s \in [a, b]$. Then the number of reflection points of $\gamma_K(\sigma(s))$ is the same for all $s \in [a, b]$.*

Proof: It is well known that winding numbers are homotopy invariant (see e.g. [Hir]). For any $a \leq s_1 < s_2 \leq b$ the map $\Lambda : \mathbb{R} \times [s_1, s_2] \longrightarrow \mathbb{C} \setminus \{0\}$, defined by $\Lambda(s, t) = \lambda(\sigma(s), t)$, is a continuous homotopy between the curves $\lambda(\sigma(s_1), t)$ and $\lambda(\sigma(s_2), t)$. Now the assertion follows from Proposition 13.6.1. ∎

[2] That is, for t near t_0, when t increases, the point $\lambda(\sigma, t)$ moves from the lower half of the complex plane to the upper one.

We will now characterize the non-trapped points σ generating rays $\gamma_K(s)$ containing a gliding segment on ∂K.

Proposition 13.6.3: *Assume that $K \in \mathcal{K}$ and $\sigma \in S^*(\Omega_0) \setminus \mathrm{Trap}(\Omega_K)$.*

(a) If $\gamma_K(\sigma)$ contains a gliding segment on ∂K, then for every continuous curve $\sigma(s)$ ($0 \leq s \leq a$ for some $a > 0$) in $S^*(\Omega_0)$ with $\sigma(0) = \sigma$ there exist infinitely many $s \in (0, a]$ for which the trajectory $\gamma_K(\sigma(s))$ is tangent to ∂K at some of its points.

(b) Suppose $K \in \mathcal{K}^{(\mathrm{fin})}$ and $\gamma_K(\sigma)$ does not contain a gliding segment on ∂K. Then there exists a continuous curve $\sigma(s)$ ($0 \leq s \leq a$ for some $a > 0$) in $S^*(\Omega_0)$ with $\sigma(0) = \sigma$ such that $\gamma_K(\sigma(s))$ is a simply reflecting ray for all $s \in (0, a]$.

Proof:

(a) Let $\sigma(s)$, $0 \leq s \leq a$, be a continuous curve in $S^*(\Omega_0)$ with $\sigma(0) = \sigma$. Assume that for some $\epsilon \in (0, a]$ the trajectory $\gamma_K(\sigma(s))$ is simply reflecting for all $s \in (0, \epsilon]$. Then by Corollary 13.6.2, the number $k(s)$ of reflection points of $\gamma_K(\sigma(s))$ is constant: $k(s) = k$ for all $s \in (0, \epsilon]$. However it follows from results in [MS2] and the assumption that $\gamma_K(\sigma)$ contains a gliding segment on ∂K that we must have $k(s) \to \infty$ as $s \searrow 0$, a contradiction. This proves the assertion.

(b) In this case $\gamma_K(\sigma)$ has only finitely many common points with $S^*(\partial K)$; some of them might be tangent points but these will be simple tangencies. Since $\sigma \notin \mathrm{Trap}(\Omega_K)$ and $\gamma_K(\sigma)$ do not contain a gliding segment, it follows from Propositions 13.5.3 and 13.5.5 that there exist an open neighbourhood V of σ in $S^*(\Omega_0)$ and finitely many smooth submanifolds $S_1, \ldots, S_k$ of V of positive codimension such that $\sigma \in \cup_{i=1}^k S_i$ and any point $\rho \in V \setminus \cup_{i=1}^k S_i$ generates a simply reflecting scattering ray in Ω_K. As one can easily observe, we can find a continuous curve $\sigma(s)$, $0 \leq s \leq a$, so that $\sigma(s) \notin \cup_{i=1}^k S_i$ for all $s \in (0, a]$. Then $\gamma_K(\sigma(s))$ will be a simply reflecting ray in Ω_K for all $s \in (0, a]$. ■

We can apply the developments so far in this section to get a consequence concerning recovering information about an obstacle from its SLS.

Corollary 13.6.4: *Assume that $K, L \in \mathcal{K}_0^{(\mathrm{fin})}$ have almost the same SLS and that for some $\sigma \in S^*(\Omega_0)$, $\gamma_K(\sigma)$ is a scattering ray in Ω_K containing a gliding segment on ∂K. Then $\gamma_L(\sigma)$ is a scattering ray in Ω_L containing a gliding segment on ∂L.*

Proof: Let $\sigma = (x, \xi)$. Replacing ξ_0 by $-\xi_0$ if necessary, we may assume that there exists $T > 0$ such that $\mathcal{F}_t^{(K)}(\sigma) \in S^*(\Omega_0)$ for all $t \leq 0$ and all $t \geq T$. Consider the hyperplanes X and Y of $\mathbb{R}^n$ with $x \in X$, $y = \mathrm{pr}_1(\mathcal{F}_t^{(K)}(\sigma)) \in Y$ and such that X and Y are perpendicular to $\gamma_K(\sigma)$ at x and y, respectively. Define the cross-sectional maps $\mathcal{P}_K, \mathcal{P}_L : S^*X \longrightarrow S^*Y$ as before. Then, by Theorem 13.1.2, $\mathcal{P}_K = \mathcal{P}_L$.

It follows from Proposition 13.4.1 that $\gamma_L(\sigma)$ contains a tangent point to ∂L. If $\gamma_L(\sigma)$ does not contain a gliding ray on ∂L, it follows from Proposition 13.6.3(b) that we can find a continuous curve $\sigma(s)$, $0 \leq s \leq a$, in $S^*(\Omega_0)$ with $\sigma(0) = \sigma$ such that $\gamma_L(\sigma(s))$ is a simply reflecting ray for all $s \in (0, a]$. Since $\mathcal{P}_K = \mathcal{P}_L$, it follows that

$\gamma_K(\sigma(s))$ is also a simply reflecting ray for all $s \in (0, a]$. This contradicts Proposition 13.6.3(a). Thus, $\gamma_L(\sigma)$ must contain a gliding segment on ∂L. ■

For an obstacle K in $\mathbb{R}^n$ denote by $U^{(K)}$ the *set of these non-trapped points* $\sigma \in S^*(\Omega_0)$ *such that* $\gamma_K(\sigma)$ *is a simply reflecting ray*. We know from Propositions 11.1.3 and 11.2.6 that $U^{(K)}$ is open and dense and has full Lebesgue measure in $S^*(\Omega_0)$. Moreover, by Proposition 13.4.3, if $K, L \in \mathcal{K}_0^{(\text{fin})}$ have almost the same SLS, then $U^{(K)} = U^{(L)}$.

Since $S^*(S_0) \setminus \text{Trap}(\Omega_K)$ is an open subset of $S^*(S_0)$, it is a union of (possibly infinitely many) disjoint connected open subsets of $S^*(S_0)$. Clearly some of these connected components contain points $\rho \in S^*(S_0)$ generating $\gamma_K(\rho)$ in Ω_K, that is rays with no common points with ∂K.

Definition 13.6.5: A point $\sigma \in S^*(S_0)$ will be called *accessible* if it belongs to a connected component of $S^*(S_0) \setminus \text{Trap}(\Omega_K)$ containing a point that generates a free ray. Similarly, a connected component of $S^*(S_0) \setminus \text{Trap}(\Omega_K)$ will be called *accessible* if it contains a point generating a free ray.

Denote by $A^{(K)}$ the *set of all accessible points* of $S^*(S_0) \setminus \text{Trap}(\Omega_K)$. Clearly, $A^{(K)}$ is a union of (open) connected components of $S^*(S_0) \setminus \text{Trap}(\Omega_K)$, so this is an open subset of $S^*(\mathbb{S}_0)$.

Denote by $U_0^{(K)}$ the *set of all accessible points* $\rho \in U^{(K)}$, that is

$$U_0^{(K)} = A^{(K)} \cap U^{(K)}.$$

Clearly $U_0^{(K)}$ is dense in $A^{(K)}$.

The central point in this section is the following.

Theorem 13.6.6: *Assume that two obstacles* $K, L \in \mathcal{K}_0^{(\text{fin})}$ *have almost the same SLS.*

(a) For every connected component W *of* $S^*(S_0) \setminus \text{Trap}(\Omega_K)$ *there exists an integer* $m = m(K, L, W)$ *such that*

$$\#(\gamma_K(\sigma) \cap \partial K) = \#(\gamma_L(\sigma) \cap \partial L) + m \tag{13.10}$$

for all $\sigma \in W \cap U^{(K)}$.

(b) If W *is an accessible connected component of* $S^*(S_0) \setminus \text{Trap}(\Omega_K)$*, then* $m(K, L, W) = 0$*, that is*

$$\#(\gamma_K(\sigma) \cap \partial K) = \#(\gamma_L(\sigma) \cap \partial L) \tag{13.11}$$

for all $\sigma \in W \cap U^{(K)}$.

The first step in the proof of the above theorem uses the submanifolds M_i of $S^*(S_0) \setminus (\text{Trap}(\Omega_K) \cup \mathcal{G}_K)$ constructed in the proof of Proposition 13.5.5.

Lemma 13.6.7: *Let $\sigma_0 \in M_{i_0} \setminus \cup_{i \neq i_0} M_i$ for some i_0 be such that $\rho_0 = \mathcal{F}_{t_0}^{(K)}(\sigma_0) \in G_d$ for some (unique) $t_0 > 0$. Then for every smooth (C^1) curve $\sigma(s)$, $|s| < \epsilon_0$, in $S^*(S_0)$ transversal to M_{i_0} at $\sigma(0) = \sigma_0$ there exists $\epsilon \in (0, \epsilon_0]$ such that*

$$\# \left[\gamma_K(\sigma(s')) \cap \partial K\right] \neq \# \left[\gamma_K(\sigma(s'')) \cap \partial K\right]$$

for all $-\epsilon < s' < 0 < s'' < \epsilon$.

Proof of Lemma 13.6.7: Assume that $\sigma(s)$, $|s| < \epsilon_0$, is a smooth curve in $S^*(S_0)$ transversal to M_{i_0} at $\sigma_0 = \sigma(0)$. By assumption $\gamma_K(\sigma_0)$ has only one tangent point ρ_0 to ∂K, that is $\{\mathcal{F}_t^{(K)}(\sigma_0) : t \in \mathbb{R}\} \cap S^*(\partial K)$ consists of one single point ρ_0. There exists an open neighbourhood V of ρ_0 in $S^*(\mathbb{R}^n)$ such that $H_p^2 \varphi > 0$ on V. Let $k \geq 0$ be the number of transversal reflections of $\gamma_K(\sigma_0)$ at ∂K. Taking the neighbourhood V sufficiently small, we may assume that ρ_0 is the only point of $\{\mathcal{F}_t^{(K)}(\sigma_0) : t \in \mathbb{R}\} \cap S^*(\partial K)$ contained in V. Now take $\epsilon \in (0, \epsilon_0]$ so small that

$$\{\sigma(s) : 0 < |s| < \epsilon\} \cap (\cup_i M_i) = \emptyset.$$

Then for any $0 < |s| < \epsilon$, the ray $\{\mathcal{F}_t^{(K)}(\sigma(s)) : t \in \mathbb{R}\}$ has only transversal reflection points (no tangencies) and their number is either k or $k+1$, depending on whether the ray has a common point with ∂K near ρ_0 or not.

Consider the subsets

$$G^2 \cap V \subset \Gamma = \{\rho \in V : H_p \varphi(\rho) = 0\}$$

of V. It is easy to see that both are submanifolds of V, Γ has codimension 1 in V, while $G^2 \cap V$ has codimension 1 in Γ. Moreover, Γ is transversal to the flow $\mathcal{F}_t^{(K)}$ at ρ_0. Let V' be an open neighbourhood of σ_0 in $S^*(S_0)$ such that $\mathcal{F}_{t_0}^{(K)}(V') \subset V$. Shrinking V' if necessary and taking $\delta > 0$ sufficiently small, for every $\sigma \in V'$ we can find a real number $t(\sigma) \in (0, t_0 - \delta]$ close to $t_0 - \delta$ with $\mathcal{F}_{t(\sigma)}^{(K)}(\sigma) \in V \cap S^*(\mathbb{R}^n \setminus K)$. Given $\sigma \in V'$, the forward trajectory of H_p in $S^*(\mathbb{R}^n)$ starting at $\mathcal{F}_{t(\sigma)}^{(K)}(\sigma)$ (its projection in $\mathbb{R}^n$ is just a straight line) intersects Γ transversally at some point $\psi(\sigma)$. This defines a smooth map $\psi : V \longrightarrow \Gamma$, and so $\rho(s) = \psi(\sigma(s))$ is a smooth curve in Γ.

We now need to recall the construction of the submanifolds M_i from the proof of Proposition 13.5.5. In particular, since $\sigma_0 \in M_{i_0}$ and $\rho_0 = \psi(\sigma_0) \in G^2 \cap V$, the definition of the submanifold M_{i_0} shows that $M_{i_0} = \psi^{-1}(G^2 \cap V)$ locally near σ_0. In particular, the curve $\rho(s)$ in Γ is transversal to $G^2 \cap V$ at ρ_0. Recalling the definition of the set G^2 from Section 1.2, we have

$$G^2 \cap V = \{\rho \in \Gamma : \varphi(\rho) = 0\}.$$

Since $H_p \varphi$ is a defining function $H_p \varphi$ for Γ and the differentials of $H_p \varphi$ and φ are linearly independent of V, we have

$$\frac{d}{ds} \varphi(\rho(s))_{|s=0} = \langle \nabla \varphi(\rho_0), \dot{\rho}(0) \rangle \neq 0,$$

so $\varphi(\rho(s))$ has a different sign for $s < 0$ and for $s > 0$ (s close to 0). Then for s in one of the intervals $[-\delta, 0)$ and $(0, \delta]$, the trajectory $\{\mathcal{F}_t^{(K)}(\sigma(s)) : t \in \mathbb{R}\}$ has a common point with $S^*_{\partial K}(\mathbb{R}^n)$ near ρ_0, while for s in the other interval, it does not. This proves the assertion. ■

Proof of Theorem 13.6.6: Let $\{N_i^{(K)}\}$ and $\{M_i^{(K)}\}$ be the submanifolds of $S^*(S_0)$ constructed in Propositions 13.5.3 and 13.5.5 for the obstacle K, and let $\{N_i^{(L)}\}$ and $\{M_i^{(L)}\}$ be corresponding submanifolds of $S^*(S_0)$ for the obstacle L. We will also use the particular construction of the submanifolds $M_i^{(K)}$ and $M_i^{(L)}$ in the proof of Proposition 13.5.5.

Consider an arbitrary connected component W of $S^*(S_0) \setminus \mathrm{Trap}(\Omega_K) = S^*(S_0) \setminus \mathrm{Trap}(\Omega_L)$, an arbitrary point $\sigma_0 \in W \cap U^{(K)} = W \cap U^{(L)}$ and set

$$m = \#(\gamma_K(\sigma_0) \cap \partial K) - \#(\gamma_L(\sigma_0) \cap \partial L).$$

Fix an arbitrary $\sigma_1 \in W \cap U$. We will show that (13.10) holds for $\sigma = \sigma_1$ and the number m defined above.

Since $\sigma_0 \in W \subset S^*(S_0) \setminus \mathrm{Trap}(\Omega_K)$, we can take a smooth curve $\sigma(s)$, $0 \le s \le a$, in $S^*(S_0) \setminus \mathrm{Trap}(\Omega_K)$ such that $\sigma(0) = \sigma_0$, $\sigma(a) = \sigma_1$. Then take a compact neighbourhood W_0 of the curve $\lambda = \{\sigma(s) : s \in [0, a]\}$ in $S^*(S_0) \setminus \mathrm{Trap}(\Omega_K)$. Slightly perturbing λ in the interior of W_0, we can make it transversal to any of the submanifolds $M_i^{(K)}$, $M_i^{(L)}$, $N_i^{(K)}$ and $N_i^{(L)}$ (see e.g. [Hir]). However, $N_i^{(K)}$ and $N_i^{(L)}$ have codimension 2 in $S^*(S_0)$, so the transversality to these submanifolds simply means that $\lambda \cap N_i^{(K)} = \lambda \cap N_i^{(L)} = \emptyset$ for all i. Recalling Proposition 13.5.3, we now obtain $\lambda \cap \mathcal{G}_K = \lambda \cap \mathcal{G}_L = \emptyset$. On the other hand, $\mathcal{G}_K \cap W_0$ and $\mathcal{G}_L \cap W_0$ are closed in W_0 since W_0 is a compact set without common points with $\mathrm{Trap}(\Omega_K) = \mathrm{Trap}(\Omega_L)$. So, shrinking W_0 if necessary, we may assume $W_0 \cap \mathcal{G}_K = W_0 \cap \mathcal{G}_L = \emptyset$.

Next, since the family $\{M_i^{(K)}\}$ is locally finite in $S^*(S_0) \setminus (\mathrm{Trap}(\Omega_K) \cup \mathcal{G}_K)$, we have that $\lambda \cap M_i^{(K)} \ne \emptyset$ only for finitely many i. So, there are only finitely many $s_1, \ldots, s_p \in [0, a]$ such that $\gamma_K(\sigma(s_j))$ contains a tangent point to ∂K. According to Proposition 13.4.3, $\gamma_L(\sigma(s))$ has a tangency to ∂L for exactly the same values of s, that is for $s = s_1, \ldots, s_p$. Now by Proposition 13.4.2 (or rather, its proof in Section 13.8; see also Remark 13.5.6), we can perturb the curve $\sigma(s)$ near each s_j, so that each of the rays $\gamma_K(\sigma(s_i))$ and $\gamma_L(\sigma(s_i))$ has a single point of tangency to ∂K and ∂L, respectively, for any $i = 1, \ldots, p$.

Fix for a moment $s \in [0, a]$. Proceeding for example as in the proof of Proposition 13.4.3, we may assume that there exists $T > 0$ such that $\mathcal{F}_t^{(K)}(\sigma(s)) \in S^*(\Omega_0)$ for all $t \le 0$ and all $t \ge T$. Consider the hyperplanes $X = X(s)$ and $Y = Y(s)$ of $\mathbb{R}^n$ with $x = x(s) \in X$, $y = y(s) = \mathrm{pr}_1(\mathcal{F}_t^{(K)}(\sigma(s))) \in Y$ and such that X and Y are perpendicular to $\gamma_K(\sigma(s))$ at x and y, respectively. Define the cross-sectional map $\mathcal{P}_K(s) : S^*X \longrightarrow S^*Y$ as before, and observe (as before) that $\mathcal{P}_K(s) = \mathcal{P}_L(s)$ for all s.

Set $s_0 = 0$, $s_{p+1} = a$. It follows from Corollary 13.6.2 that for every $j = 0, 1, \dots, p$ there exists a constant k_j (resp. ℓ_j) such that for every $s \in (s_j, s_{j+1})$, the number of reflection points of $\gamma_K(\sigma(s))$ (resp. $\gamma_L(\sigma(s))$) is equal to k_j (resp. ℓ_j). We claim that $k_j = \ell_j + m$ for all j. We will prove this by induction. By the definition of m we have $k_0 = \ell_0 + m$. Assume $k_j = \ell_j + m$ for some $j \leq p$. By construction, $\sigma(s_j) \in M_i$ for some i and $\gamma_K(\sigma(s_j))$ has a single tangency to ∂K which is the projection of a point in $G^2 \setminus G^3$. Moreover, the curve $\sigma(s)$ is transversal to M_i at $\sigma(s_j)$. Applying Lemma 13.6.7, it follows now that either $k_{j+1} = k_j + 1$ or $k_{j+1} = k_j - 1$. By the same argument, $l_{j+1} = l_j \pm 1$. The difference between the cases $k_{j+1} = k_j + 1$ and $k_{j+1} = k_j - 1$ is that in the former we have $\limsup_{s \nearrow s_j} \|d\mathcal{P}_K(s)\| < \infty$, while in the latter $\limsup_{s \nearrow s_j} \|d\mathcal{P}_K(s)\| = \infty$. Since $\mathcal{P}_K(s) = \mathcal{P}_L(s)$, we now see that in the former case we must have $\ell_{j+1} = \ell_j + 1$, while in the latter, $\ell_{j+1} = \ell_j - 1$. Hence $k_{j+1} = \ell_{j+1} + m$. By induction, $k_j = \ell_j + m$ for all j. This proves the relationship (13.10) for $\sigma = \sigma_1$, and thus completes the proof of part (a).

(b) This follows trivially from part (a), since if W is accessible, we can choose σ_0 so that it generates a free ray in Ω_K. By Theorem 13.1.1, $\gamma_L(\sigma_0)$ is also a free ray, so we must have $m = 0$ in (13.10). That is, (13.11) holds. ■

13.7 Recovering the accessible part of an obstacle

It is natural to expect that the accessible points on the boundary ∂K of an obstacle K will be easier to recover using the SLS of the generalized geodesic flow in Ω_K. In this section we will demonstrate that to a big extend this is indeed so (see Definition 13.6.5 for the definition of an accessible point).

Throughout $K \in \mathcal{K}^{(\mathrm{fin})}$ will be a fixed obstacle in $\mathbb{R}^n$, $n \geq 2$. As before S_0 will denote the boundary sphere of a large ball $\mathcal{O}$ in $\mathbb{R}^n$ containing the obstacle K. Fix a countable set $\{M_i\} = \{M_i^{(K)}\}$ of submanifolds of $S^*(S_0)$ with the properties described in Proposition 13.5.5 (see also Remark 13.5.6). We will also use the set $U^{(K)}$ and $U_0^{(K)}$ from Section 13.5.

Definition 13.7.1: Let $\sigma(s), 0 \leq s \leq a$, be a smooth curve in $S^*(S_0)$ for some $a > 0$, and let the family $\{M_i\}$ be as above. The curve $\sigma(s)$ will be called *regular* if it satisfies the following conditions:

(i) $\gamma_K(\sigma(0))$ is a free ray in Ω_K,

(ii) $\sigma(a) \notin \cup_i M_i$;

(iii) $\sigma(s) \notin \mathrm{Trap}(\Omega_K) \cup \mathcal{G}_K$ for all $s \in [0, a]$;

(iv) $\sigma \pitchfork M_i$ for all i and $\sigma(s) \notin M_i \cap M_j$ for any $i \neq j$ and any $s \in [0, a]$.

Clearly, by (ii) and (iii), $\gamma_K(\sigma(a))$ is a simply reflecting ray, while (iii) and (iv) show that all trajectories $\gamma_K(\sigma(s))$ are scattering rays with at most one tangent point to ∂K. Notice that if $\sigma(s)$ is a regular curve as above and $\rho(s)$ $(0 \leq s \leq a)$ is uniformly close to $\sigma(s)$ in the C^1 topology, then $\rho(s)$ is also regular.

The following is easy to derive using the argument in the first part of the proof of Theorem 13.6.6. We leave the details to the reader.

Lemma 13.7.2: *For every $\sigma_0 \in U_0^{(K)}$ there exists a regular curve $\sigma(s)$, $0 \leq s \leq a$, with $\sigma(a) = \sigma_0$.*

We will now define a sequence of subsets of the boundary ∂K of K.

Definition 13.7.3: The sequence

$$\partial K^{(1)} \subset \partial K^{(2)} \subset \cdots \subset \partial K^{(m)} \subset \cdots \subset \partial K$$

is defined recursively as follows:

(a) Set $\partial K^{(0)} = \emptyset$.

(b) Let $\partial K^{(1)}$ be the set of those $x \in \partial K$ such that there exists a regular curve $\sigma(s)$ $(0 \leq s \leq a)$ in $S^*(S_0)$ with $\gamma_K(\sigma(s))$ having at most one common point with ∂K for all $s \in [0, a]$ and $x \in \gamma_K(\sigma(a))$.

(c) Assume that the subsets $\partial K^{(1)} \subset \cdots \subset \partial K^{(m)}$ of ∂K have been constructed already for some $m \geq 1$. Denote by $\partial K^{(m+1)}$ the set of those $x \in \partial K$ such that there exists a regular curve $\sigma(s)$ $(0 \leq s \leq a)$ in $S^*(S_0)$ with $\gamma_K(\sigma(s))$ having at most one common point with ∂K that is not in $\partial K^{(m)}$ for all $s \in [0, a]$ and $x \in \gamma_K(\sigma(a))$.

Clearly, each of the sets $\partial K^{(m)}$ is open in ∂K.

The *strongly accessible part* of ∂K is by definition

$$\partial K^{(\infty)} = \overline{\cup_{m=1}^{\infty} \partial K^{(m)}}.$$

The obstacle K will be called *strongly accessible* if $\partial K^{(\infty)} = \partial K$.

Simple examples show that in many cases the strongly accessible part of the obstacle is significant, in fact even the set $\partial K^{(1)}$ is already substantial. For example, it is easy to see that the star-shaped obstacles from Section 13.3 are strongly accessible.

It seems natural to conjecture that all non-trapping obstacles are strongly accessible, however no proof of this is known to the authors.

As is natural to expect, for ‘relatively regular’ obstacles, the strongly accessible part of the boundary can be completely recovered from the SLS of the obstacle. This is the main result in this section.

Theorem 13.7.4: *Let $K, L \in \mathcal{K}_0^{(\text{fin})}$ be two obstacles in $\mathbb{R}^n$, $n \geq 2$, with almost the same SLS.*

(a) We have $\partial K^{(m)} = \partial L^{(m)}$ for all $m \geq 0$, and therefore $\partial K^{(\infty)} = \partial L^{(\infty)}$.

(b) If in addition it is known that K is strongly accessible, then $L = K \cup L'$ for some connected component L' of L with $L' \cap K = \emptyset$.

(c) If in addition it is known that K is strongly accessible and any connected component of L can be reached by a trajectory $\gamma_L(\rho)$ generated by an accessible point $\rho \in S^*(S_0) \setminus \mathrm{Trap}(\Omega_L)$, then $K = L$.

Proof: (a) We will use induction on m. Assume $\partial K^{(m-1)} = \partial L^{(m-1)}$ for some $m \geq 1$. We have to prove $\partial K^{(m)} = \partial L^{(m)}$. By symmetry, it is enough to show that $\partial K^{(m)} \subset \partial L^{(m)}$.

Given $x \in \partial K^{(m)}$, there exist a regular (with respect to the submanifolds $M_i^{(K)}$ of $S^*(S_0)$) curve $\sigma(s)$, $0 \leq s \leq a$, with $\gamma_K(\sigma(s))$ having at most one common point with ∂K that is not in $\partial K^{(m-1)}$ for all $s \in [0, a]$ and $x \in \gamma_K(\sigma(a))$. Thus, $\sigma(a) = \mathcal{F}_t^{(K)}(x, \xi)$ for some $t > 0$ and $\xi \in \mathbb{S}^{n-1}$ with $\langle \xi, \nu_K(x) \rangle > 0$. Applying an arbitrarily small in the C^1 Whitney topology (see e.g. [Hir]) perturbation of the curve $\sigma(s)$ in $S^*(S_0)$, we may assume that the curve σ is transversal to each of the submanifolds $N_i^{(L)}$, $M_i^{(L)}$, it has the same end point $\sigma(a)$, and that $\sigma(s)$ belongs to at most one of the submanifolds $M_i^{(L)}$ for all $s \in [0, a]$ (see the proof of Theorem 13.6.6). As before, since the codimension of each $N_i^{(L)}$ in $S^*(S_0)$ is 2, one gets $\sigma(s) \notin \cup_i N_i^{(L)}$, and therefore $\sigma(s) \notin \mathcal{G}_L$ for all $s \in [0, a]$. Finally, Proposition 13.4.3 and $\sigma(a) \notin \mathcal{G}_K \cup \cup_i M_i^{(K)}$ imply $\sigma(a) \notin \mathcal{G}_L \cup \cup_i M_i^{(L)}$. All this shows that the perturbed curve $\sigma(s)$ ($0 \leq s \leq a$) is a regular curve with respect to the family $\{M_i^{(L)}\}$.

Lemma 13.7.5: *We have $\partial K = \partial L$ in a neighbourhood of the point x and*

$$\gamma_K(\sigma(s)) = \gamma_L(\sigma(s)) \tag{13.12}$$

for all $s \in [0, a]$.

Proof of Lemma 13.7.5: Since $\gamma_K(\sigma(0))$ is a free ray in $\mathbb{R}^n$, and so $\gamma_K(\sigma(0)) = \gamma_L(\sigma(0))$, we have

$$s_0 = \sup\{s \in [0, a] : \gamma_K(\sigma(t)) = \gamma_L(\sigma(t)), \quad 0 \leq t \leq s\} > 0.$$

We will prove that $s_0 = a$. Assume $s_0 < a$. By continuity, $\rho_0 = \sigma(s_0) \in S^*(S_0)$ satisfies

$$\gamma_K(\rho_0) = \gamma_L(\rho_0). \tag{13.13}$$

For brevity set

$$U_0 = U_0^{(K)} = U_0^{(L)} \subset U = U^{(K)} = U^{(L)}.$$

By Theorem 13.6.6(b),

$$\#(\gamma_K(\rho) \cap \partial K) = \#(\gamma_L(\rho) \cap \partial L) \tag{13.14}$$

for all $\rho \in U_0$.

Denote by $x_1, \ldots, x_k$ the successive common points of $\gamma_K(\rho_0)$ with ∂K. Since $\rho_0 = \sigma(s_0)$ and σ is a regular curve, there is at most one $i_0 = 1, \ldots, k$ such that

$x_{i_0} \notin \partial K^{(m-1)}$. If there is no such i_0, there is nothing to prove. Assume that $x = x_{i_0} \notin \partial K^{(m-1)}$ for some i_0, and so $x_i \in \partial K^{(m-1)} = \partial L^{(m-1)}$ for all $i \neq i_0$.

There are two possible cases to consider.

Case 1. x_i is a transversal reflection point of $\gamma_K(\rho_0)$ for all $i = 1, \dots, k$. Then by Proposition 13.4.3, $\gamma_L(\rho_0)$ also has only transversal reflections at ∂L, that is no tangencies to ∂L. By (13.13), the reflection points of $\gamma_L(\rho_0)$ coincide with those of $\gamma_K(\rho_0)$, that is they are $x_1, \dots, x_k$. Clearly this implies that every $\rho \in S^*(S_0)$ sufficiently close to ρ_0 generates a scattering ray $\gamma_K(\rho)$ with exactly k common points $x'_1, \dots, x'_k$ with ∂K and also a scattering ray $\gamma_L(\rho)$ with exactly k common points $x''_1, \dots, x''_k$ with ∂L, where for all i the points x'_i and x''_i are close to x_i.

By assumption $\partial K^{(m-1)} = \partial L^{(m-1)}$. Since this is open in both ∂K and ∂L, assuming ρ is sufficiently close to ρ_0, we have $x'_i, x''_i \in \partial K^{(m-1)} = \partial L^{(m-1)}$ for all $i \neq i_0$. By Theorem 13.1.2, $\mathcal{F}_t^{(K)}(\rho) = \mathcal{F}_t^{(L)}(\rho)$ for $|t| \gg 0$, so we must have $x'_i = x''_i$ for $i < i_0$ and also for $i > i_0$. The only other reflection points of $\gamma_K(\rho)$ and $\gamma_L(\rho)$ are x'_{i_0} and x''_{i_0}, respectively, and now a simple geometric argument implies that we must have $x'_{i_0} = x''_{i_0}$. That is, $\gamma_K(\rho) = \gamma_L(\rho)$ for all $\rho \in S^*(S_0)$ sufficiently close to $\rho_0 = \sigma(s_0)$. This is a contradiction with the choice of s_0 which proves that we must have $s_0 = a$. Thus, (13.12) holds for all $s \in [0, a]$. Moreover, the above argument, replacing ρ_0 by $\sigma(a)$, shows that $\partial K = \partial L$ in a neighbourhood of the point x.

Case 2. $\gamma_K(\rho_0)$ has a tangency to ∂K at some point $x_q \in \partial K$. This tangent point is then unique by the definition of a regular curve. Clearly, there are $s \in (0, a]$ arbitrarily close to a for which $\sigma(s) \in U_0^{(K)}$, so (13.14) holds with $\rho = \sigma(s)$ for such s. Hence (13.14) holds for $\rho = \rho_0$ as well.

Take an open connected neighbourhood W of ρ_0 in $S^*(S_0) \setminus (\mathrm{Trap}(\Omega_K) \cup \mathcal{G}_K)$ so small that both $\gamma_K(\rho)$ and $\gamma_L(\rho)$ have transversal reflections near x_i for any $i \neq q$ and any $\rho \in W$. Using the fact that $\rho_0 \notin \mathcal{G}_K$, we may assume W is taken so small that for any $\rho \in W$ the trajectory $\gamma_K(\rho)$ has at most k common points with ∂K. Indeed, suppose this is not the case. Since $W \cap \mathrm{Trap}(\Omega_K) = \emptyset$, we can find a sequence $\{\rho_p\} \subset S^*(S_0)$ such that $\rho_p \to \rho_0$ as $p \to \infty$ and for every p there exist real numbers $t'_p \neq t''_p$ with $\mathrm{pr}_1(\mathcal{F}_{t'_p}^{(K)}(\rho_p)) \in \partial K$ and $\mathrm{pr}_1(\mathcal{F}_{t''_p}^{(K)}(\rho_p)) \in \partial K$ for all p. Choosing an appropriate subsequence, we may assume $t'_p \to t$ and $t''_p \to t$ as $p \to \infty$ for some $t \in \mathbb{R}$. This gives $\mathcal{F}_t^{(K)}(\rho_0) \in S^*(\partial K)$ and also it follows from known facts (cf. e.g. Section 24.3 in [H3]) that $\mathcal{F}_t^{(K)}(\rho_0) \in G^3$. This is a contradiction with $\rho_0 \notin \mathcal{G}_K$. Thus, shrinking the open neighbourhood W of ρ_0, we may assume that for any $\rho \in W$, the trajectory $\gamma_K(\rho)$ has at most k common points with ∂K. Fix a neighbourhood W with this property.

We now claim that

$$\mathcal{F}_t^{(K)}(\rho) = \mathcal{F}_t^{(L)}(\rho), \quad \rho \in W, t \in \mathbb{R}. \tag{13.15}$$

Let $x_i = \mathrm{pr}_1(\mathcal{F}_{t_i}^{(K)}(\rho_0))$ for some $0 < t_1 < \cdots < t_k$, and let $\mathcal{F}_t^{(K)}(\rho_0) \in S^*(S_0)$ for some $T > t_k$. Then $\mathrm{pr}_1(\mathcal{F}_t^{(K)}(\rho_0))$ will be the last common point of $\gamma_K(\rho_0)$ with $\overline{\mathcal{O}}$ before it 'escapes to ∞'. Taking a sufficiently small $\epsilon > 0$ with $t_{i_0-1} + 2\epsilon < t_{i_0}$, $t_{i_0} + 2\epsilon < t_{i_0+1}$ and $t_k + 2\epsilon < T$, and shrinking W if necessary, we may assume that $\mathcal{F}_t^{(K)}(\rho) = \mathcal{F}_t^{(L)}(\rho)$ for all $\rho \in W$ and all $t < t_{i_0} - \epsilon$. Set $t_0 = t_{i_0}$, $t' = t_0 - \epsilon$ and $t'' = t_0 + \epsilon$. By Theorem 13.1.2, $\mathcal{F}_t^{(K)}(\rho) = \mathcal{F}_t^{(L)}(\rho)$ for $\rho \in W$ and all $t \ll 0$ and $t \gg T$. Since $\partial K = \partial L$ near x_i for all $i \neq i_0$, it then follows that

$$\mathcal{F}_t^{(K)}(\rho) = \mathcal{F}_t^{(L)}(\rho), \quad \rho \in W, t \leq t', t \geq t''. \tag{13.16}$$

Moreover, the 'short trajectory' $\delta_K = \{\mathrm{pr}_1(\mathcal{F}_t^{(K)}(\rho)) : t \in [t', t'']\}$ is either:

(i) a straight-line segment (this can only happen if $q = i_0$), which may have a common point with ∂K near x_{i_0}, or

(ii) δ_K has just one transversal reflection at some point $x'(\rho) \in \partial K$ near x_{i_0} (this will necessarily happen if $q \neq i_0$ and it may also happen if $q = i_0$).

Similarly, the 'short trajectory' $\delta_L = \{\mathrm{pr}_1(\mathcal{F}_t^{(L)}(\rho)) : t \in [t', t'']\}$ has either one or zero common points with ∂L. It is now easy to see that in case (i), δ_L must also be a straight-line segment, and moreover (13.16) implies $\delta_L = \delta_K$. In the case (ii), a similar argument shows that $\delta_L = \delta_K$ again. This proves (13.15).

Let $\rho_0' = (y_0, \eta_0) = \mathcal{F}_{t_0}^{(K)}(\rho_0)$; then $x_{i_0} = y_0$. Using the continuity of the flow $\mathcal{F}_t^{(K)}$, there exists an open neighbourhood V of ρ_0' in $S^*(\Omega_K)$ such that $\mathcal{F}_{-t(\omega)}^{(K)}(\omega) \in W$ for all $\omega \in V$, where $t(\omega)$ is a number close to $t_0 = t_{i_0}$. Then there exists an open neighbourhood V' of y_0 in ∂K such that for every $y \in V'$, we can find a vector $\eta \in S_y(\partial K)$ close to η_0 such that $\omega = (y, \eta) \in V$. Setting $t = t(\omega)$ and using $\rho = \mathcal{F}_{-t}^{(K)}(\omega) \in W$ and (13.15), we get

$$y = \mathrm{pr}_1(\mathcal{F}_t^{(K)}(\mathcal{F}_{-t}^{(K)}(\omega)) = \mathrm{pr}_1(\mathcal{F}_t^{(K)}(\rho)) = \mathrm{pr}_1(\mathcal{F}_t^{(L)}(\rho)) \in \partial L.$$

Thus, $V' \subset \partial L$ which shows that $\partial K = \partial L$ near x_{i_0}.

So, by (13.15) we have $\gamma_K(\rho) = \gamma_L(\rho)$ for all $\rho \in S^*(S_0)$ sufficiently close to $\rho_0 = \sigma(s_0)$. This a contradiction with the choice of s_0 which proves $s_0 = a$. Finally, as in Case 1, repeating the latest argument above, replacing ρ_0 by $\sigma(a)$, shows that $\partial K = \partial L$ in a neighbourhood of the point x. This completes the proof in Case 2, thus proving the lemma. ■

We now continue with the proof of the theorem.

Lemma 13.7.5 gives $x \in \partial L^{(m)}$, and so this proves $\partial K^{(m)} \subset \partial L^{(m)}$. By symmetry $\partial K^{(m)} = \partial L^{(m)}$. Hence by induction $\partial K^{(m)} = \partial L^{(m)}$ for all $m \geq 0$, and therefore $\partial K^{(\infty)} = \partial L^{(\infty)}$.

(b) Assume that K is strongly accessible, that is $\partial K^{(\infty)} = \partial K$. Then the above relation implies $\partial K \subset \partial L$. This is only possible when $L = K \cup L'$ for some (obstacle) L' with $L' \cap K = \emptyset$.

(c) Assume that K is strongly accessible and any connected component of L can be reached by a scattering ray $\gamma_L(\rho)$ for some accessible $\rho \in S^*(S_0) \setminus \mathrm{Trap}(\Omega_L)$. By part (b), $K = L \cup L'$ for some obstacle L'. Assume $L' \neq \emptyset$, otherwise there is nothing to prove. Let L'' be a connected component of L'. By assumption, there exists an accessible $\rho \in S^*(S_0) \setminus \mathrm{Trap}(\Omega_L)$ such that $\gamma_L(\rho) \cap \partial L'' \neq \emptyset$.

We can choose ρ so that $\rho \notin \cup_i N_i^{(L)} \cup \cup_i M_i^{(L)}$. Repeating an argument from the proof of Theorem 13.6.6, we find a smooth curve $\sigma(s)$, $0 \leq s \leq 1$, in $S^*(S_0) \setminus (\text{Trap}(\Omega_L) \cup \mathcal{G}_L)$ transversal to each of the submanifolds $M_i^{(L)}$ such that $\sigma(0)$ generates a free ray in Ω_L and $\sigma(1) = \rho$.

Set $s' = \min\{s \in [0,1] : \gamma_L(\sigma(s)) \cap \partial L' \neq \emptyset\}$ and $\rho' = \sigma(s')$. Then the trajectory $\gamma_L(\rho')$ must have a tangency to $\partial L'$ at some point $y \in \partial L'$; otherwise we get a contradiction with the choice of s'. It now follows from the choice of the curve σ that $\gamma_L(\rho')$ has no other tangencies to L, and the choice of s' shows that $\gamma_L(\rho')$ has no other common points with L'. Thus, all other common points of $\gamma_L(\rho')$ and L belong to $L \setminus L' = K$, so $\gamma_K(\rho') = \gamma_L(\rho')$. However this is a contradiction with Proposition 13.4.3 since $\gamma_K(\rho')$ has no tangency to ∂K, while $\gamma_L(\rho')$ has a tangent point to ∂L. Hence $L' = \emptyset$ and therefore $K = L$. ■

13.8 Proof of Proposition 13.4.2

Most of this section is taken by the proofs of two technical lemmas. These proofs are relatively elementary however rather lengthy.

Lemma 13.8.1: *Let X be a C^∞ smooth submanifold of codimension 1 in $\mathbb{R}^n$, and let $x_0 \in X$ and $\xi_0 \in T_x X$, $\|\xi_0\| = 1$, be such that the normal curvature of X at x_0 in the direction ξ_0 is non-zero. Then for every $\epsilon > 0$ there exist an open neighbourhood V of x_0 in X, a smooth map*

$$V \ni x \mapsto \xi(x) \in T_x X$$

and a smooth positive function $t(x) \in [\delta, \epsilon]$ on V for some $\delta \in (0, \epsilon)$ such that

$$Y = \{y(x) = x + t(x)\xi(x) : x \in V\}$$

is a smooth strictly convex surface with unit normal field $\mu(y(x)) = \xi(x)$, $x \in V$. That is, the normal field of Y consists of vectors tangent to X at the corresponding points of V.

Proof of Lemma 13.8.1: Considering X with the Riemannian metric induced by $\mathbb{R}^n$, there exists a local smooth codimension 1 submanifold X' of X containing x and perpendicular to ξ_0 at x_0 such that the second fundamental form of X' in X with respect to the normal ξ_0 is negative definite at x_0. For example, we can take an appropriate strictly convex $(n-1)$-dimensional submanifold Z of $\mathbb{R}^n$ containing x_0 and having 'outward' unit normal ξ_0 at x_0, and set $X' = X \cap Z$. Parameterize X' by $h(u')$, $u' = (u_2, \ldots, u_{n-1})$, $h(0) = x_0$, and let $\xi(u')$ be a continuous unit normal field to X' with $\xi(u') \in T_{h(u')}X$ for all u' and $\xi(0) = \xi_0$. For any u' let $c(t; u')$ be the geodesic on X parameterized by arc length t such that $c(0; u') = h(u')$ and $\dot{c}(0; u') = \xi(u')$. Define

$$r(u_1, u_2, \ldots, u_{n-1}) = c(u_1; u').$$

It then follows that for $|u_1|$ and $\|u'\|$ small enough $r(u)$ is a smooth parameterization of an open neighbourhood V of x_0 in X such that $x_0 = r(0)$, $\xi_0 = \frac{\partial r}{\partial u_1}(0)$ and

$$\left\| \frac{\partial r}{\partial u_1}(u) \right\| = 1, \quad \left\langle \frac{\partial r}{\partial u_1}(u), \frac{\partial r}{\partial u_i}(u) \right\rangle = 0, \quad i > 1, \;\; u \in U. \tag{13.17}$$

Shrinking the neighbourhood V of x_0 if necessary, we may assume that u runs over some open ball U in $\mathbb{R}^{n-1}$. Notice that the second fundamental form of X' in X at x_0 has the form

$$II'(v') = \sum_{i,j=2}^{n-1} v_i v_j \left\langle \xi_0, \frac{\partial^2 r}{\partial u_i \partial u_j}(0) \right\rangle,$$

where $v' = (v_2, \ldots, v_{n-1}) \in \mathbb{R}^{n-2}$. So the choice of X' implies $II'(v') < 0$ whenever $v' \neq 0$.

Fix a small $\epsilon > 0$ and set

$$y(u) = r(u) + (\epsilon - u_1)\frac{\partial r}{\partial u_1}(u), \quad u \in U.$$

Shrinking the ball U if necessary, we may assume that $|u_1| < \epsilon/2$ for all $u \in U$. It will become clear later how small ϵ should be.

We claim that $Y = \{y(u) : u \in U\}$ is a smooth submanifold of $\mathbb{R}^n$ (provided $\epsilon > 0$ and U are small enough), $y(u)$ is a smooth parameterization of Y and $\mu(u) = \frac{\partial r}{\partial u_1}(u)$ is a normal vector to Y at $y(u)$.

First, notice that differentiating (13.17) with respect to u_j, implies

$$\left\langle \frac{\partial r}{\partial u_1}(u), \frac{\partial^2 r}{\partial u_1 \partial u_j}(u) \right\rangle = 0, \quad 1 \leq j \leq n-1, \tag{13.18}$$

$$\left\langle \frac{\partial^2 r}{\partial u_1 \partial u_j}(u), \frac{\partial r}{\partial u_i}(u) \right\rangle + \left\langle \frac{\partial r}{\partial u_1}(u), \frac{\partial^2 r}{\partial u_i \partial u_j}(u) \right\rangle = 0,$$

$$2 \leq i \leq n-1, \;\; 1 \leq j \leq n-1. \tag{13.19}$$

From these two equalities it follows in particular that $\frac{\partial^2 r}{\partial u_1^2}(u) \perp \frac{\partial r}{\partial u_i}(u)$ for any $i = 1, \ldots, n-1$, and therefore $\frac{\partial^2 r}{\partial u_1^2}(u)$is a normal vector to X at $r(u)$. On the other hand, the assumption that the normal curvature of X at $x_0 = r(0)$ in the direction of $\xi_0 = \frac{\partial r}{\partial u_1}(0)$ is non-zero implies that $\frac{\partial^2 r}{\partial u_1^2}(0) \neq 0$. In particular, the vectors

$$\frac{\partial^2 r}{\partial u_1^2}(0), \frac{\partial r}{\partial u_2}(0), \ldots, \frac{\partial r}{\partial u_{n-1}}(0)$$

in $\mathbb{R}^n$ are linearly independent. Without loss of generality, we may assume that the matrix formed by the first $n-1$ coordinates of these vectors has a non-zero determinant.

Since

$$\frac{\partial y}{\partial u_1}(u) = (\epsilon - u_1)\frac{\partial^2 r}{\partial u_1^2}(u), \quad \frac{\partial y}{\partial u_i}(u) = \frac{\partial r}{\partial u_i}(u) + (\epsilon - u_1)\frac{\partial^2 r}{\partial u_1 \partial u_i}(u), \quad i > 1, \tag{13.20}$$

(13.17) and (13.18) imply that $\mu(u) = \frac{\partial r}{\partial u_1}(u)$ is a normal vector to Y at $y(u)$, provided Y is a smooth submanifold with parameterization $y(u)$.

To prove that Y is locally a smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$, it is enough to show that if $\epsilon > 0$ is small enough, then the vectors $\frac{\partial y}{\partial u_i}(0)$ $(1 \leq i \leq n-1)$ are linearly independent. Assume the contrary, that is there exists $a > 0$ such that for any $\epsilon \in (0, a]$ the corresponding vectors $\frac{\partial y}{\partial u_i}(0)$ are linearly dependent. Let $y(u) = (y_1(u), \ldots, y_n(u))$ and $r(u) = (r_1(u), \ldots, r_n(u))$. Define $z(u) = (y_1(u), \ldots, y_{n-1}(u))$ and $h(u) = (r_1(u), \ldots, r_{n-1}(u))$. Then the vectors $\frac{\partial z}{\partial u_i}(0)$ $(1 \leq i \leq n-1)$ are also linearly dependent, so we must have

$$0 = \det \frac{\partial z}{\partial u}(0) = \det \begin{pmatrix} \frac{\partial z}{\partial u_1}(0) \\ \frac{\partial z}{\partial u_2}(0) \\ \cdots \\ \frac{\partial z}{\partial u_{n-1}}(0) \end{pmatrix} = \det \begin{pmatrix} \epsilon \frac{\partial^2 h}{\partial u_1^2}(0) \\ \frac{\partial h}{\partial u_2}(0) + \epsilon \frac{\partial^2 h}{\partial u_1 \partial u_2}(0) \\ \cdots \\ \frac{\partial h}{\partial u_{n-1}}(0) + \epsilon \frac{\partial^2 h}{\partial u_1 \partial u_{n-1}}(0) \end{pmatrix},$$

where the rows in the matrices above are vectors in $\mathbb{R}^{n-1}$. Dividing by ϵ the first row of the determinant in the right-hand side and then letting $\epsilon \to 0$, we obtain that the matrix formed by the first $n-1$ coordinates of the vectors

$$\frac{\partial^2 r}{\partial u_1^2}(0), \frac{\partial r}{\partial u_2}(0), \ldots, \frac{\partial r}{\partial u_{n-1}}(0)$$

has a zero determinant – contradiction with our assumption above.

Thus, there exists arbitrarily small $\epsilon > 0$ such that the vectors $\frac{\partial y}{\partial u_i}(0)$ $(1 \leq i \leq n-1)$ are linearly independent. Given such an ϵ, shrinking U if necessary, we may assume that $\frac{\partial y}{\partial u_i}(u)$ $(1 \leq i \leq n-1)$ are linearly independent of any $u \in U$. Then Y is a smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ and $y(u)$ is a smooth parameterization of Y. As we observed above, $\mu(u) = \frac{\partial r}{\partial u_1}(u)$ is then a unit normal to Y at $y(u)$. Moreover, by the definition of $y(u)$, the segment $[r(u), y(u)]$ is tangent to X at $r(u)$.

It remains to show that the normal curvature of Y with respect to the normal field $\mu(u)$ is negative. For this we need the second derivatives of $y(u)$ which we get from (13.20):

$$\frac{\partial^2 y}{\partial u_1^2}(0) = -\frac{\partial^2 r}{\partial u_1^2}(0) + \epsilon \frac{\partial^3 r}{\partial u_1^3}(0), \quad \frac{\partial^2 y}{\partial u_1 \partial u_i}(0) = \epsilon \frac{\partial^3 r}{\partial u_1^2 \partial u_i}(0), \quad 2 \leq i \leq n-1,$$

$$\frac{\partial^2 y}{\partial u_i \partial u_j}(0) = \frac{\partial^2 r}{\partial u_i \partial u_j}(0) + \epsilon \frac{\partial^3 r}{\partial u_1 \partial u_i \partial u_j}(0), \quad 2 \leq i, j \leq n-1.$$

Hence the coefficients

$$c_{ij} = \left\langle \mu(0), \frac{\partial^2 y}{\partial u_i \partial u_j}(0) \right\rangle$$

of the second fundamental form of Y at $y(0)$ have the form:

$$c_{1j} = \epsilon \left\langle \frac{\partial r}{\partial u_1}(0), \frac{\partial^3 r}{\partial u_1 \partial u_1^2 \partial u_j}(0) \right\rangle, \quad 2 \leq j \leq n-1 \, ,$$

$$c_{ij} = \left\langle \frac{\partial r}{\partial u_1}(0), \frac{\partial^2 r}{\partial u_i \partial u_j}(0) \right\rangle + \epsilon \left\langle \frac{\partial r}{\partial u_1}(0), \frac{\partial^3 r}{\partial u_1 \partial u_i \partial u_j}(0) \right\rangle, \quad 2 \leq i, j \leq n-1.$$

On the other hand, differentiating (13.18) with respect to u_i, one gets

$$\left\langle \frac{\partial r}{\partial u_1}(0), \frac{\partial^3 r}{\partial u_1 \partial u_i \partial u_j}(0) \right\rangle = -\left\langle \frac{\partial^2 r}{\partial u_1 \partial u_i}(0), \frac{\partial^2 r}{\partial u_1 \partial u_j}(0) \right\rangle \, .$$

Thus, the second fundamental form $II(v)$, $v = (v_1, \ldots, v_{n-1}) \in \mathbb{R}^{n-1}$, of Y at $y(0)$ has the form

$$\begin{aligned} II(v) = \epsilon \sum_{i,j=1}^{n-1} v_i v_j c_{ij} &= \sum_{i,j=1}^{n-1} v_i v_j \left\langle \frac{\partial r}{\partial u_1}(0), \frac{\partial^3 r}{\partial u_1 \partial u_i \partial u_j}(0) \right\rangle \\ &+ \sum_{i,j=2}^{n-1} v_i v_j \left\langle \frac{\partial r}{\partial u_1}(0), \frac{\partial^2 r}{\partial u_i \partial u_j}(0) \right\rangle \\ &= -\epsilon \sum_{i,j=1}^{n-1} v_i v_j \left\langle \frac{\partial^2 r}{\partial u_1 \partial u_i}(0), \frac{\partial^2 r}{\partial u_1 \partial u_j}(0) \right\rangle + II'(v') \\ &= -\epsilon \, \|w\|^2 + II'(v'), \end{aligned}$$

where $v' = (v_2, \ldots, v_{n-1})$ and

$$w = \sum_{i=1}^{n-1} v_i \frac{\partial^2 r}{\partial u_1 \partial u_i}(0).$$

Since by the choice of the submanifold X' we have $II'(v') < 0$ for $v' \neq 0$, it now follows that $II(v) < 0$ whenever $v \neq 0$. Thus, Y is strictly convex at $y(0)$ with respect to the normal $\mu(0) = \frac{\partial r}{\partial u_1}(0)$, and therefore the same conclusion holds for any $y(u)$ close enough to $y(0)$. So, shrinking U (and therefore V), we get that the whole submanifold Y is strictly convex in $\mathbb{R}^n$. ■

Let again X be a smooth bounded $(n-1)$-dimensional submanifold of $\mathbf{R}^n$, $n \geq 2$. Recall the definition of an (ω, θ)-trajectory for X from Section 6.2.

Let $\mathcal{O}$ be an open ball containing X. Given $\omega \in \mathbb{S}^{n-1}$, define the hyperplane Z_ω as in Section 13.2. For an integer $p \geq 1$ consider the smooth manifolds

$$X^{(p)} = \{(x_1, \ldots, x_p) \in X^p \, : \, x_i \neq x_j \, , i \neq j \, \}, \quad M_p = \mathbb{S}^{n-1} \times X^{(p)} \times \mathbb{S}^{n-1} \, .$$

Fix integers k, m and $s \geq 1$ and $0 \leq k < m \leq s$. Denote by $M(s, k, m)$ the set of those $\eta = (\omega; x; y; z; \theta) \in M_{s+2}$ with $x = (x_1, \dots, x_s)$, $y, z \in X$, such that there exists an (ω, θ)-trajectory for X with successive (transversal) reflection points $x_1, \dots, x_s$, the segment $[x_k, x_{k+1}]$ of which is tangent to X at the point $y \in (x_k, x_{k+1})$, the segment $[x_m, x_{m+1}]$ is tangent to X at $z \in (x_m, x_{m+1})$ and the Gauss curvature of X either at y or at z is non-zero. Here by x_0 (resp. x_{s+1}) we denote the orthogonal projection of x_1 on Z_ω (resp. of x_s on $Z_{-\theta}$).

Lemma 13.8.2: *$M(s, k, m)$ is a smooth submanifold of M_{s+2} of dimension $2n - 4$.*

Proof of Lemma 13.8.2: We will use an argument similar to that in the proof of Lemma 11.5.1.

Assume $0 < k$ and $m < s$; the cases $k = 0$ and/or $m = s$ are similar.

Given $\hat{\eta} = (\hat{\omega}; \hat{x}; \hat{y}; \hat{z}; \hat{\theta}) \in M(s, k, m)$, choose smooth charts $\varphi_i : U_i \longrightarrow X$ of X around $\hat{x}_i$, $\psi : V \longrightarrow X$ of X around $\hat{y}$ and $\chi : W \longrightarrow X$ of X around $\hat{z}$ such that $\varphi_i(U_i) \cap \varphi_{i+1}(U_{i+1}) = \emptyset$, $i = 1, \dots, s-1$, $\varphi_k(U_k) \cap \psi(V) = \emptyset$, $\varphi_{k+1}(U_{k+1}) \cap \psi(V) = \emptyset$, $\varphi_m(U_m) \cap \chi(W) = \emptyset$ and $\varphi_{m+1}(U_{m+1}) \cap \chi(W) = \emptyset$. Let $\omega(\omega')$, $\omega' \in D_1 \subset \mathbb{R}^{n-1}$, be a smooth parameterization of $\mathbb{S}^{n-1}$ near $\hat{\omega}$ (say, ω' is an appropriate choice of $n-1$ coordinates of ω), and let $\theta(\theta')$, $\theta' \in D_2 \subset \mathbb{R}^{n-1}$ be a similar parameterization of $\mathbb{S}^{n-1}$ near $\hat{\theta}$. Consider the chart

$$\Phi : U = D_1 \times U_1 \times \cdots \times U_s \times V \times W \times D_2 \longrightarrow D \subset M_{s+2} ,$$

defined by

$$\Phi(\xi) = (\omega(\omega'); \varphi_1(u_1), \dots, \varphi_s(u_s); \psi(v); \chi(w); \theta(\theta'))$$

for $\xi = (\omega'; u; v; w; \theta') \in U$.

Because of the symmetry of the roles of k and m, we may assume that the Gauss curvature of X at $\hat{y}$ is non-zero. Let $\hat{\xi} = (\hat{\omega}'; \hat{u}; \hat{v}; \hat{w}; \hat{\theta}') \in U$ be such that $\Phi(\hat{\xi}) = \hat{\eta}$. Notice that $\hat{\eta} \in M(s, k, m)$ implies $\nu_1(\hat{v}) \perp \hat{x}_{k+1} - \hat{x}_k$, where $\nu_1(\hat{v})$ is the naturally defined normal to X at $\psi(\hat{v})$ (see below). Choosing an appropriate coordinate system in $\mathbb{R}^n$, we may assume that

$$\hat{x}_{k+1} - \hat{x}_k = (0, \dots, 0, a), \quad \nu_1(\hat{v}) = (1, 0, \dots, 0) \tag{13.21}$$

for some $a > 0$. Shrinking U_m and U_{m+1} if necessary, we can find $i_0 = 1, \dots, n$ so that

$$\varphi_{m+1}^{(i_0)}(u_{m+1}) - \varphi_m^{(i_0)}(u_m) \neq 0, \quad u_m \in U_m, \ u_{m+1} \in U_{m+1}.$$

Fix i_0 with this property and set $B = \{1, \dots, n\} \setminus \{i_0\}$.

Define $F : U \longrightarrow \mathbf{R}$ by

$$F(\xi) = \sum_{i=1}^{s-1} \|\varphi_i(u_i) - \varphi_{i+1}(u_{i+1})\|.$$

Here $u_i = (u_i^{(1)}, \dots, u_i^{(n-1)}) \in U_i$. Let

$$f_1 = (1, 0, \dots, 0), \dots, f_n = (0, \dots, 0, 1).$$

As in the proof of Lemma 11.5.1, to express the condition

$$\xi = (\omega'; u; v; w; \theta'') \in \Phi^{-1}(M(s,k,m)),$$

we will use the naturally defined normals

$$\nu_1(v) = \det \begin{pmatrix} f_1 & \cdots & f_n \\ \frac{\partial \psi^{(1)}}{\partial v^{(1)}}(v) & \cdots & \frac{\partial \psi^{(n)}}{\partial v^{(1)}}(v) \\ \vdots & \vdots & \vdots \\ \frac{\partial \psi^{(1)}}{\partial v^{(n-1)}}(v) & \cdots & \frac{\partial \psi^{(n)}}{\partial v^{(n-1)}} \end{pmatrix}$$

and

$$\nu_2(w) = \det \begin{pmatrix} f_1 & \cdots & f_n \\ \frac{\partial \chi^{(1)}}{\partial w^{(1)}}(w) & \cdots & \frac{\partial \chi^{(n)}}{\partial w^{(1)}}(w) \\ \vdots & \vdots & \vdots \\ \frac{\partial \chi^{(1)}}{\partial w^{(n-1)}}(w) & \cdots & \frac{\partial \chi^{(n)}}{\partial w^{(n-1)}}(w) \end{pmatrix}$$

to X at $\psi(v)$ and $\chi(w)$, respectively, and the functions

$$K_i^{(j)}(\xi) = \frac{\partial F}{\partial u_i^{(j)}}(\xi), \quad i = 2, \dots, s-1, \quad j = 1, \dots, n-1,$$

$$L_j(\xi) = \left\langle \frac{\varphi_2(u_2) - \varphi_1(u_1)}{\|\varphi_2(u_2) - \varphi_1(u_1)\|} - \omega(\omega'), \frac{\partial \varphi_1}{\partial u_1^{(j)}}(u_1) \right\rangle, \quad j = 1, \dots, n-1,$$

$$M_j(\xi) = \left\langle \frac{\varphi_s(u_s) - \varphi_{s-1}(u_{s-1})}{\|\varphi_s(u_s) - \varphi_{s-1}(u_{s-1})\|} - \theta(\theta'), \frac{\partial \varphi_s}{\partial u_s^{(j)}}(u_s) \right\rangle, \quad j = 1, \dots, n-1,$$

$$P_j(\xi) = \frac{\psi^{(j)}(v) - \varphi_k^{(j)}(u_k)}{\|\psi(v) - \varphi_k(u_k)\|} + \frac{\psi^{(j)}(v) - \varphi_{k+1}^{(j)}(u_{k+1})}{\|\psi(v) - \varphi_{k+1}(u_{k+1})\|}, \quad j = 1, \dots, n-1,$$

$$Q_j(\xi) = \frac{\chi^{(j)}(w) - \varphi_m^{(j)}(u_m)}{\|\chi(w) - \varphi_m(u_m)\|} + \frac{\chi^{(j)}(w) - \varphi_{m+1}^{(j)}(u_{m+1})}{\|\chi(w) - \varphi_{m+1}(u_{m+1})\|}, \quad j \in B,$$

$R(\xi) = \langle \varphi_{k+1}(u_{k+1}) - \varphi_k(u_k), \nu_1(v) \rangle$, $T(\xi) = \langle \varphi_{m+1}(u_{m+1}) - \varphi_m(u_m), \nu_2(w) \rangle$.

Consider the map

$$G : U \longrightarrow (\mathbb{R}^{n-1})^{s-2} \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$$

defined by

$$G(\xi) = (\ (K_i^{(j)}(\xi))_{2\le i\le s-1}^{1\le j\le n-1}; (L_j(\xi))_{1\le j\le n-1}; (M_j(\xi))_{1\le j\le n-1}; \\ (P_j(\xi))_{1\le j\le n-1}; (Q_j(\xi))_{j\in B}; R(\xi); T(\xi)\ ).$$

Then G is smooth and we have $\Phi^{-1}(D \cap M(s,k,m)) = G^{-1}(0)$. We will show that G is submersion at $\hat{\xi}$. Then shrinking U (and therefore D), G is a submersion on the whole U, so $G^{-1}(0)$ is a smooth submanifold of U with

$$\dim(G^{-1}(0)) = (s+4)(n-1) - [(s+2)(n-1)+2] = 2n-4\ .$$

Assume that

$$\sum_{i=2}^{s-1}\sum_{j=1}^{n-1} A_i^{(j)} \nabla K_i^{(j)}(\hat{\xi}) + \sum_{j=1}^{n-1} B_j \nabla L_j(\hat{\xi}) + \sum_{j=1}^{n-1} C_j \nabla M_j(\hat{\xi}) \\ + \sum_{j=1}^{n-1} p_j \nabla P_j(\hat{\xi}) + \sum_{j\in B} q_j \nabla Q_j(\hat{\xi}) + r\nabla R(\hat{\xi}) + t\nabla T(\hat{\xi}) = 0 \qquad (13.22)$$

for some real numbers $A_i^{(j)}, B_j, C_j, p_j, q_j, r, t$. We will show that all these are zero. Here ∇ means ∇_ξ, the gradient with respect to $\xi = (\omega'; u; v; w; \theta'')$.

First, exactly as in the proof of Lemma 11.5.1, considering the derivatives with respect to ω' and θ' in (13.22), one shows that

$$B_1 = \cdots = B_{n-1} = C_1 = \cdots = C_{n-1} = 0.$$

Then, repeating again the corresponding argument from the proof of Lemma 11.5.1, it follows that $A_i^{(j)} = 0$ for all $j = 1, \dots, n-1$, $1 \le i \le k$ and $m \le i \le s$. Now (13.22) gets the form

$$\sum_{i=k+1}^{m}\sum_{j=1}^{n-1} A_i^{(j)} \nabla K_i^{(j)}(\hat{\xi}) \ + \ \sum_{j=1}^{n-1} p_j\ \nabla\ P_j(\hat{\xi}) + \sum_{j\in B} q_j\ \nabla\ Q_j(\hat{\xi}) \\ +\ r\ \nabla\ R(\hat{\xi}) + t\nabla\ T(\hat{\xi}) = 0. \qquad (13.23)$$

Set for convenience

$$b_1 = \frac{1}{\|\psi(\hat{v}) - \varphi_k(\hat{u}_k)\|}, \quad b_2 = \frac{1}{\|\psi(\hat{v}) - \varphi_{k+1}(\hat{u}_{k+1})\|}, \\ b = b_1 + b_2, \quad e = \frac{\hat{x}_{k+1} - \hat{x}_k}{\|\hat{x}_{k+1} - \hat{x}_k\|}\ .$$

Since $\hat{\xi} \in M(s,k,m)$, we have $\psi(\hat{v}) \in [\hat{x}_k, \hat{x}_{k+1}]$. This and (13.21) imply

$$\frac{\psi(\hat{v}) - \varphi_k(\hat{u}_k)}{\|\psi(\hat{v}) - \varphi_k(\hat{u}_k)\|} = -\frac{\psi(\hat{v}) - \varphi_{k+1}(\hat{u}_{k+1})}{\|\psi(\hat{v}) - \varphi_{k+1}(\hat{u}_{k+1})\|} = e = (0,\dots,0,1)\ .$$

Setting $p_n = 0$ and $p = (p_1, \dots, p_n) \in \mathbf{R}^n$, as in the proof of Lemma 11.5.1, one derives from (13.23) that

$$b_1 p - \langle e, b_1 p\rangle e + r\nu_1(\hat{v}) = 0 \ .$$

On the other hand, $e = (0, \dots, 0, 1)$ and the definition of p give $p \perp e$, so $b_1 p + r\nu_1(\hat{v}) = 0$. That is,

$$p = -\frac{r}{b_1}\nu_1(\hat{v}). \tag{13.24}$$

Next, we have

$$\frac{\partial R}{\partial v_i}(\xi) = \left\langle \varphi_{k+1}(u_{k+1}) - \varphi_k(u_k), \frac{\partial \nu_1}{\partial v_i}(v) \right\rangle ,$$

and

$$\begin{aligned}\frac{\partial P_j}{\partial v_i}(\xi) &= b_1 \frac{\partial \psi^{(j)}}{\partial v_i}(v) - \frac{\psi^{(j)}(v) - \varphi_k^{(j)}(u)}{\|\psi(v) - \varphi_k(u)\|^3} \left\langle \psi(v) - \varphi_k(u), \frac{\partial \psi}{\partial v_i}(v) \right\rangle \\ &\quad + b_2 \frac{\partial \psi^{(j)}}{\partial v_i}(v) - \frac{\psi^{(j)}(v) - \varphi_{k+1}^{(j)}(u)}{\|\psi(v) - \varphi_{k+1}(u)\|^3} \left\langle \psi(v) - \varphi_{k+1}(u), \frac{\partial \psi}{\partial v_i}(v) \right\rangle \\ &= b \frac{\partial \psi^{(j)}}{\partial v_i}(v) - b_1 e^{(j)} \left\langle e, \frac{\partial \psi}{\partial v_i}(v) \right\rangle - b_2 e^{(j)} \left\langle e, \frac{\partial \psi}{\partial v_i}(v) \right\rangle \\ &= b \frac{\partial \psi^{(j)}}{\partial v_i}(v) - b e^{(j)} \left\langle e, \frac{\partial \psi}{\partial v_i}(v) \right\rangle .\end{aligned}$$

Considering the derivatives with respect to v_i in (13.23), we get

$$\begin{aligned} 0 &= \sum_{j=1}^{n} b p_j \left[\frac{\partial \psi^{(j)}}{\partial v_i}(\hat{v}) - e^{(j)} \left\langle e, \frac{\partial \psi}{\partial v_i}(\hat{v}) \right\rangle \right] + r \left\langle \hat{x}_{k+1} - \hat{x}_k, \frac{\partial \nu_1}{\partial v_i}(\hat{v}) \right\rangle \\ &= b \left\langle p, \frac{\partial \psi}{\partial v_i}(\hat{v}) \right\rangle - b\langle p\, , e\rangle \left\langle e\, , \frac{\partial \psi}{\partial v_i}(\hat{v}) \right\rangle + ra \left\langle e\, , \frac{\partial \nu_1}{\partial v_i}(\hat{v}) \right\rangle \end{aligned}$$

Using $e \perp p$, $\nu_1(\hat{v}) \perp \frac{\partial \psi}{\partial v_i}(\hat{v})$ for all $i = 1, \dots, n-1$ and (13.24), this implies $0 = \left\langle e\, , \frac{\partial \nu_1}{\partial v_i}(\hat{v}) \right\rangle$ for all $i = 1, \dots, n-1$. Since the Gauss curvature of X at $\psi(\hat{v})$ is non-zero and $a \neq 0$, it follows that $r = 0$. Now (13.24) implies $p = 0$, so (13.23) takes the form

$$\sum_{i=k+1}^{m} \sum_{j=1}^{n-1} A_i^{(j)} \nabla K_i^{(j)}(\hat{\xi}) + \sum_{j \in B} q_j \ \nabla\, Q_j(\hat{\xi}) + t \nabla\, T(\hat{\xi}) = 0 \ . \tag{13.25}$$

Then, applying again an argument from the proof of Lemma 11.5.1, one gets $A_i^{(j)} = 0$ for all i and j, $q_j = 0$ for all $j \in B$ and $t = 0$. Thus, G is a submersion at $\hat{\xi}$. ∎

We will also need the following lemma.

Lemma 13.8.3: *Let X and Z be smooth local $(n-1)$-dimensional submanifolds of $\mathbb{R}^n$ $(n \geq 2)$ with $X \cap Z = \emptyset$, and let $(x_0, \xi_0) \in S^*(X)$ be such that $(z_0, \xi_0) \in S^*(Z)$, where $z_0 = x_0 + t_0\xi_0$ for some $t_0 > 0$. If the curvature of X in the direction of ξ_0 is non-zero, then there exists $(x, \xi) \in S^*(X)$ arbitrarily close to (x_0, ξ_0) such that $(x + t\xi, \xi) \notin S^*(Z)$ for any $t > 0$.*

Proof of Lemma 13.8.3: It follows from Lemma 13.8.1 that there exist $\epsilon > 0$, an open neighbourhood V of x_0 in X, a smooth positive function $t(x)$, $x \in V$, with $t(x) \in [\delta, \epsilon]$ for all $x \in V$, and a smooth map $V \ni x \mapsto \xi(x) \in T_x(X)$ such that

$$Y = \{y(x) = x + t(x)\xi(x) : x \in V\}$$

is a smooth strictly convex surface with unit normal field $\xi(x)$. Given $z' \in Z$ with $z' = x + t\xi(x)$ for some $t > 0$ and a sufficiently small open neighbourhood W of z' in $\mathbb{R}^n$, the orthogonal projection $\varphi : W \longrightarrow Y$ is well defined and smooth. Thus, $\varphi : W \cap Z \longrightarrow Y$ is a smooth map. If $z \in W \cap Z$ is such that the ray $\{y(x) + t\xi(x) : t > 0\}$ is tangent to Z at z, then z is a critical point and x is a critical value of φ. By Sard's theorem (see e.g. [Hir]), the set of critical values of φ has Lebesgue measure zero in Y. Covering Z by a finite or countable family of neighbourhoods W, one shows that the set of those $y(x) \in Y$ such that $\{y(x) + t\xi(x) : t > 0\}$ is tangent to Z has measure zero in Y. ■

Proof of Proposition 13.4.2: Let $\sigma = (y, \eta) \in S^*(\partial K)$ be such that $\gamma_K(\sigma)$ is a scattering ray in Ω_K. Since the curvature of ∂K does not vanish of infinite order, $\gamma_K(\sigma)$ contains only finitely many gliding segments (if any) and finitely many diffractive tangent points to ∂K [MS1]. Using Proposition 13.5.1 and perturbing slightly σ in $S^*(\partial K)$ if necessary, we may assume that $\gamma_K(\sigma)$ does not contain gliding segments on ∂K. Then $\gamma_K(\sigma)$ has only finitely many tangent points to ∂K, all of them being diffractive tangent points. We will assume that γ_K has $\ell \geq 2$ different tangent points to ∂K; otherwise there is nothing to prove. One of these is y; denote one of the others (if there are more than two) by z.

The case when $\gamma_K(\sigma)$ has no transversal reflections follows immediately from Lemma 13.8.3. Assume that $\gamma_K(\sigma)$ has $s \geq 1$ transversal reflection points $x_1, \ldots, x_s$ at ∂K. As we have done before, denote by x_0 (resp. x_{s+1}) the orthogonal projection of x_1 on the hyperplane Z_ω (resp. of x_s on $Z_{-\theta}$).

We claim that there exists $\sigma' \in S^*(\partial K)$ arbitrarily close to σ such that $\gamma_K(\sigma')$ has at most $\ell - 1$ tangent points to ∂K. To prove this, consider some small open neighbourhood U_i of x_i $(i = 1, \ldots, s)$, V of y and W of z in ∂K such that $V \cap U_i = W \cap U_i = \emptyset$ for all i. We take these so small that

$$X = V \cup W \cup (\cup_{i=1}^s U_i)$$

does not contain any other tangent points of $\gamma_K(\sigma)$ to ∂K. We will now apply Lemma 13.8.2 to the smooth submanifold X of $\mathbb{R}^n$.

Let $k, m = 0, 1, \dots, s$ be such that y and z belong to the segments $[x_k, x_{k+1}]$ and $[x_m, x_{m+1}]$, respectively. We may assume that $k \leq m$; otherwise, we can replace σ by $(y, -\eta)$. If $k = m$, then the claim stated above follows immediately from Lemma 13.8.3. Assume $k < m$. Then, setting $x = (x_1, \dots, x_s)$ and denoting by ω and θ the incoming and outgoing directions of $\gamma_K(\sigma)$, we have

$$(\omega; x, y, z; \theta) \in M(s, k, m) \subset M_{s+2}.$$

The natural projection $p : M_{s+2} \longrightarrow M_{s+1}$ defined by

$$p(\tilde{\omega}; \tilde{x}; \tilde{y}; \tilde{z}; \tilde{\theta}) = (\tilde{\omega}; \tilde{x}; \tilde{y}; \tilde{\theta})$$

determines a smooth map $p : M(s, k, m) \longrightarrow M(s, k)$. It follows from Lemmas 11.5.1 and 13.8.2 that

$$\dim(M(s, k)) = 2n - 3 > 2n - 4 = \dim(M(s, k, m)).$$

Now Sard's theorem gives that $M(s, k) \setminus p(M(s, k, m))$ contains points $(\omega'; x'; y'; \theta')$ arbitrarily close to $(\omega; x, y, \theta)$. Setting

$$\eta' = \frac{x'_{k+1} - x'_k}{\|x'_{k+1} - x'_k\|},$$

we get points $\sigma' = (y', \eta') \in S^*(\partial K)$ arbitrarily close to σ such that $\partial K(\sigma')$ does not have a tangency to ∂K in W, that is $\gamma_K(\sigma')$ has at most $\ell - 1$ tangent points to ∂K.

Proceeding by induction, one finds $\sigma' \in S^*(\partial K)$ arbitrarily close to σ such that $\gamma_K(\sigma')$ has only one tangency to ∂K. ■

13.9 Notes

The inverse scattering problem discussed in this chapter resembles the problem concerning the so-called lens equivalence of geodesic flows on Riemannian manifolds [Cr]. Various other problems related to recovering information from the length spectrum of certain types of geodesics have been considered in Riemannian geometry – see for example [SU], [SUV], [CULV] for general information and many references.

The main results in this chapter were proved in [S8] and [S7]. More precisely, the main Theorem 13.1.2 and its consequences in Sections 13.1 and 13.3 were established in [S8]. The main result Proposition 13.4.3 of Section 13.4 was also obtained in [S8], however the main tool for it, Proposition 13.4.2, was proved in [S7]. The exposition of Section 13.5 also follows [S7]. The results of Sections 13.6 and 13.7 were established in [S8]. Section 13.8 devoted to the proof of Proposition 13.4.2 follows [S7].

There are some very recent results (not covered in this book) concerning problems related to recovering information about obstacles from scattering rays in their exteriors – see [NS1] and [NS2]. For example, it was proved in [NS1] that if K and L are two obstacles in $\mathbb{R}^n$, $n \geq 2$, such that each of them is a finite disjoint union of strictly convex domains with smooth boundaries and K and L have almost the same SLS, then $K = L$. For obstacles with real analytic boundaries this was established earlier in [S7]. A higher-dimensional version of Livshits' example was constructed recently in [NS3].

Various other problems related to billiards in the exterior of certain subsets of Euclidean spaces have been considered as well – see for example [Pla] and[PlaR] for some information and references in this direction.

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Symbol Index

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