Guochang Xu · Jia Xu

Orbits

2^{nd} Order Singularity-free Solutions

Second Edition

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Guochang Xu
GeoForschungsZentrum Potsdam
Potsdam
Germany

Chinese Academy of Space Technology
Beijing
China

Jia Xu
Brettnacher Straße 19
Berlin
Germany

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To
Liping, Yuxi and Pan

Preface to the Second Edition

As mentioned in the preface of the first edition of this book, the symbolic mathematical operation software design is one of the key components for deriving formulas and developing theory. During the time of this revision, Mr. Jia Xu has taken over most of the work concerning program optimization and design, results computation and validation. Due to his elegant software design, the solution of solar radiation disturbance on the satellite orbit, which was not successfully derived in the first edition, has been obtained. Through optimization and homogeneous truncation, the solution series of geopotential disturbance is now derived up to 8×8 degrees and orders which are sufficient for application of MEO and GEO satellite orbit determination. Because of the extreme importance of Jia Xu's contribution, Springer has agreed to include him as co-author of this new edition, and this is certainly well deserved.

After the first edition of this book was published at the end of 2008, the first author was very happy to put aside the hard work of writing and concentrate – with a small team – on the practical application of developed analytical satellite orbit theory. Experiences from the practice and implementation of the theory into software gave the author a strong feeling that the theory must be further developed. At the beginning of 2010, the first author started to revise and refine the theory and wrote a series of papers for books and journals for review and publication (Xu 2010, Xu et al. 2010a, b, 2011, Xu and Xu 2012, 2013c). Due to the difference in their scientific working areas, the first author very much regrets that the contribution of the second author on software design and strategic discussion is not properly acknowledged. Through the reviews of and communications with some of the top experts in celestial mechanics, the purpose of this new edition is becoming clearer: to revise and derive complete singularity-free solutions for satellite orbits disturbed by all perturbation forces (Chaps. 5, 6, 7, and 10) and to make the theory applicable (Chaps. 8 and 9). The highlight of this edition is that the singularity problem in orbital mechanics is solved for the first time cf. Chap. 10 or Xu and Xu 2012). Orbital problems in the solar gravity field are also discussed (Chap. 11). We further concentrate on the theory of second-order solutions instead of attempting to derive those of the third order. The papers published in the top journal MNRAS are invaluable for revising and supplementing the first edition, for modifying parts of the key contents and reporting on new theoretical progress and knowledge and for laying down a solid theoretical basis for future applications. With the coming of the satellite era, analytical satellite orbit theory will surely be important for applications such as orbit determination without global coverage of the monitoring station network, long-term orbit forecasting, on-board autonomous orbit determination and state controlling and monitoring, solar radiation effects monitoring as well as space mission orbit design. The authors are grateful to the readers of the first edition, whose interest led Springer to request for this second edition as early as 2010.

Compared to the book *GPS – Theory, Algorithms and Applications* which took more than 2 years to write the first edition (Xu 2003) and 9 months to revise for the second edition (Xu 2007), only 6 months of extreme hard work were needed for the first edition of this book *Orbits* (Xu 2008); however, more than 2 years have already been expended in preparing this second edition. The authors hope that this very extensively revised and supplemented edition will be complete in theory yet easy to use in applications. Worth mentioning is that there is a review of the first edition of this book which is published in MathSciNet (MR2494776) of the American Mathematical Society written by Pierre Rochus, professor and associate director of Spatial Centre of the University of Liege. He stated to the author through an email on 11 August 2009: “In my modest opinion, I think that your book is very original. I read a lot of books on Celestial Mechanics (I gave lectures on Celestial Mechanics at the University of Liege for 15 years) and your book brings new ideas”. In this revised version, complete solutions on all possible disturbances are given in detail with applicable and comprehensive methods.

The analytical solution of the C_{20} disturbance discussed in Chap. 5 is greatly revised by taking into account the geopotential coefficient with a special truncation order of eccentricity. The analytical solutions of the geopotential disturbance up to 8×8 degrees and orders are derived and given in Chap. 6 and Appendix 2. In Chap. 7, one of the key contents of this book concerning extraterrestrial disturbances is most intensively revised by taking into account the theoretical parts of the papers by Xu et al. (2010a, b) and Xu and Xu (2012), by using so-called basic Lagrangian and Gaussian equations of motion and different truncations, so that simpler and more precise formulas are obtained. The solutions of all kinds of disturbances are derived in a similar but independent way. Solutions of extraterrestrial disturbances such as solar radiation pressure, atmospheric drag and disturbances by the sun, moon and planets are given. Application of orbital analytical solutions is discussed in more detail. An additional chapter (Chap. 11) is presented for orbit mechanics in the solar gravity field, including solar oblateness disturbance (Xu et al. 2011) for planetary perihelion precession and solar radiation disturbance (Xu and Xu 2013c) for modification of Kepler’s third law through comparison with traditional mass correction results. A singularity-free theory is also developed (Chap. 10) based on the research described in the first edition of this book and a review of theoretical orbit theory (Xu 2010) as well as an intensive study of Xu and Xu (2012). A summary is given for the so-called Lagrange-Xu and

Gauss-Xu equations of motion by using basic Lagrangian and Gaussian equations. How to form the singularity-free solutions based on solutions of the basic equations and how to solve the critical inclination problem are described in detail with examples and new defined criteria (Chap. 10).

The extended contents have been reviewed by individual and anonymous journal reviewers. We thank the reviewers for their thought-provoking comments and valuable suggestions: Prof. Rudolf Dvorak of Vienna University, Prof. Ludwig Comblinck of Hertro Astronomical Observatory, Prof. Rolf Romer and Dr. Svetozar Petrovic of GFZ and Prof. Zhiping Lü of the Institute of Surveying and Mapping in Zhengzhou. We also thank Prof. Pierre Rochus of Liege University in Belgium for reviewing the first edition of this book (Rochus 2009).

The first author wishes to thank sincerely Prof. Wu Chen of the Polytech University of Hong Kong, Prof. Ta-Kang Yeh of the Taipeh University of Taiwan, Prof. Yuanxi Yang of the Institute of Surveying and Mapping (ISM) in Xi'an, Prof. Qin Zhang of ChangAn University (CAU), and his team members, scientist Dr. Tianhe Xu of GFZ and ISM, PhD candidate Xin Shen of GFZ and Wuhan University, Keifei He of GFZ and Technical University of Berlin, Guanwen Huang of GFZ and CAU, Xiao Cao of GFZ and Chinese Academy of Space Technology and Yan Xu of CAU and GFZ for their cooperation and discussion as well as co-authorship in several scientific journal papers. Theoretical progress of the paper by Xu et al. (2010a) has been stimulated by an application report of Dr. Tianhe Xu, which was one of the key contributions during his research time in GFZ. The second author, Jia Xu, is especially thanked for constant support and cooperation over the years. We co-authored the significant paper "On the singularity problem in orbital mechanics" Xu and Xu (2012) which solves for the first time the singularity problem in orbital mechanics. We are now inseparable cooperators on a completely new and original book *N-Body – Solar System Analytical Solution* from Springer (contracted), in which we will present for the first time the analytical solutions of the N-body problem in our solar system, and a further software companion book *GPS – Software Companion* from Springer (contracted) for the third edition of *GPS – Theory, Algorithms and Applications* (contracted).

The first author wishes to thank sincerely his former directors Prof. Dr. Christoph Reigber and Prof. Dr. Markus Rothacher and new director Prof. Dr. Harald Schuh for their support and trust during their time at the GFZ and for granting him great freedom of research, without which some of the scientific activities would never have been possible to realise. He is also grateful to the acting heads of the sections of GFZ, Dr. Christoph Förste, Dr. Frank Flechtner and Dr. Jens Wickert, for their support. He also thanks the Chinese government for honouring him with the title of National Distinguished Expert, owing to which he has been fortunate enough to have had the involvement of the Chinese Academy of Space Technology for research in GNSS navigation and orbital mechanics as well as in space missions. He thanks the Southwest Jiaotong University for honouring him with the Honorary Professor title. He is also grateful to his many close friends for their friendship and support. His special thanks go to Prof. Jörg Reinking of Jade University of Applied Sciences, Dr. Ludger Timmen of University of Hannover, Prof. Ludwig Comblinck of Hertro Astronomical Observatory, Prof. Pierre Rochus of Liege University, Prof. Yuanxi Yang of ISM in Xi'an, Prof. Qin Zhang of CAU, Prof. Zhilin Li of the Polytech University of Hong Kong, Prof. Zhiping Lü and Prof. Shulong Zhu of the Institute of Surveying and Mapping in Zhengzhou, Dr. Gim Der of DerAstrodynamics, Prof. Yunzhong Shen of Tongji University and Dr. Hong He of Helmholtz's representative office in Beijing. He is also grateful for the valuable feedback from readers and from students through his professorships at ChangAn University and the NTSC CAS as well as Neubrandenburg University of Applied Sciences.

May 2012 Guochang Xu and Jia Xu

Preface to the First Edition

The purpose of this reference book is to describe and to derive the analytical solutions of the equations of satellite motion perturbed by extraterrestrial and geopotential disturbances of the second order. The equations of satellite motion perturbed by extraterrestrial disturbances are solved by means of discretization and approximated potential function as well as Gaussian equations. The equations perturbed by geopotential disturbances are solved by symbolic mathematical operations. The traditional problem of singularity in the solutions is solved by so-called singularity-free orbit theory. Simplified disturbed equations of motion are proposed to simplify the solutions. Applications of the theory for analytical orbit determination are also discussed. Indeed, this is the first book since the satellite era, which describes systematically the orbit theory with analytical solutions, with respect to all of extraterrestrial and geopotential disturbances of the second order, and the solutions are free of singularity. Based on such a theory, the algorithms of orbit determination can be renewed; deeper insight into the physics of disturbances becomes possible; the way to a variety of new applications and refinements is opened.

My primary knowledge of the orbit theory came from my education of mathematics while studying physics and theoretical mechanics (1981). My first practical experience with orbits came from the research activity at the Technical University (TU) Berlin on orbit corrections of satellite altimetry data (1988–1992). The extensive experience on orbit came from the GPS/Galileo software development for orbit determination and geopotential mapping at the GFZ (2001–2004). The traditional adjustment model of the solar radiation used in numerical orbit determination is investigated and considered not reasonable physically; and a new adjustment model is proposed in the user manual of the Multi-Functional GPS/Galileo software (MFGsoft) (Xu 2004), which is also reported in the 2nd edition of the book "*GPS – Theory, Algorithms and Applications*" (Xu 2007). Indeed, one of the ways to obtain the solutions of the extraterrestrial disturbances of the satellite motion is found during that investigation. However, it has not been realised until two scientists, Dr. Xiaochun Lu and Dr. Xiaohui Li of the National Time Service Center (NTSC) in Xi'an, came to visit and to cooperate with me at GFZ. We discussed the virtual navigation system and tried to solve the stability problem of the 3-D positioning of the system. By considering what is significant in theory and, what is more important than our numerical study, the idea of solving the disturbed equations of motion was obtained, and the solutions of the extraterrestrial disturbances of the equation of satellite motion were found. Because of the importance of the geopotential disturbances, great efforts were then made to derive the related solutions. Thereafter, alternative solutions of the extraterrestrial disturbances were found by using different means (besides the discretization, also approximated potential function and Gaussian disturbed equations). To simplify the solutions, the simplified disturbed equations were proposed. To solve the problem of singularity, the singularity-free theory was also developed.

After publishing my book, "*GPS – Theory, Algorithms and Applications*", in 2003, I did not want to ever write another scientific book because this process took more than 2 years of extremely hard work. However, I must finish this book because some of the scientists have contributed their lifetime to the theoretical solutions of the geopotential disturbances of the equation of satellite motion and now the results are here. The solutions of the extraterrestrial disturbances of the orbit motion are of extreme importance for practice, but they are rarely investigated because they are highly complex. From the theory, a special confusion related to the solar radiation from the pure numerical orbit determination has been cleared. Many interesting applications will follow soon. To make the process of writing easy, a small portion of the basic contents of my GPS book is partly modified and imported or rearranged and used.

The book includes ten chapters. After a brief introduction, the coordinate and time systems are described in the second chapter. The third chapter is dedicated to the Keplerian satellite orbits – the orbits of the satellite under the attraction of the central force of the Earth.

The fourth chapter deals with perturbations of the orbit. Perturbed equations of satellite motion are derived. Perturbation forces of the satellite motion are discussed in detail, including the perturbations of the Earth's gravitational field, Earth tide and ocean tide, the sun, the moon and planets, solar radiation pressure, and atmospheric drag, as well as coordinate perturbation.

The fifth chapter covers the analytical solution of $\bar{C}_{20}$ perturbation, including the complete formulas of the long term, and long and short periodic terms. The derivation also gives the algorithm and model of orbit correction. The solutions of other geopotential disturbances of higher order and degree are described in the sixth chapter. As examples, solutions of disturbances of $\bar{C}_{30}$, D_{21} and D_{22} are given. General solutions of disturbance of D_{lm} are derived. Symbolic operation software for deriving solutions of geopotential disturbances of any order and degrees are designed and used.

The seventh chapter covers the solutions of extraterrestrial disturbances such as solar radiation pressure, atmospheric drag and the disturbances of the sun, the moon and planets. The principle and strategy that lead to the solution are described. The solutions are derived via discretization and approximated potential function as well as Gaussian perturbed equations of motion. Simplified disturbed equations are proposed and used partly. The ephemeris of the sun, the moon and planets are given for practical use.

The eighth chapter is dedicated to numerical orbit determination, including its principle, the algebraic solutions of the variation equations, and the numerical integration and interpolation algorithms, as well as the related derivatives.

The ninth chapter describes the principle of analytical orbit determination based on the proposed new solutions. Real time ability and properties of the analytical orbit solutions are discussed.

The final chapter includes algorithms that lead to singularity-free orbit theory and the equations of motion in non-inertial frame as well as discussions concerning the further development of the orbit theory and its applications as well as comments on some remaining problems.

The book has been subjected to an individual review of chapters and sections and a general review. I am grateful to reviewers Prof. Markus Rothacher of GFZ, Prof. Dieter Lelgemann of TU Berlin, Prof. Yuanxi Yang of the Institute of Surveying and Mapping (ISM) in Xi'an, Dr. Jianfeng Guo of Information Engineering University (IEU) in Zhengzhou, Prof. Xuhai Yang of NTSC in Xi'an, Dr. Junping Chen of GFZ. A grammatical check of technical English writing has been performed by Springer Heidelberg.

I wish to thank sincerely Prof. Markus Rothacher for his support and trust during my research activities at GFZ. Dr. Jürgen Kusche is thanked for his encouragement and help. Prof. Dr. Christoph Reigber is thanked for granting me special freedom of research. My grateful thanks go to Dr. Xiaochun Lu and Dr. Xiaohui Li of NTSC in Xi'an. Their visit to and cooperation at the GFZ have led to the derivations of the key contents of this book. Dr. Jiangfeng Guo of IEU in Zhengzhou followed a part of my derivation and checked it for correctness. Volker Grund of GFZ helped me greatly by assisting in the application of software tools, which is another key to the solution of geopotential disturbances. Qianxin Wang of GFZ helped to check some of the formula typing. Dr. Jinghui Liu of the educational department of the Chinese Embassy in Berlin, Prof. Yuanxi Yang of ISM in Xi'an, Prof. Heping Sun of the Institute of Geodesy and Geophysics (IGG) in Wuhan and Prof. Qin Zhang of ChangAn University in Xi'an are thanked for their friendly support during my scientific activities in China. The Chinese Academy of Sciences is thanked for the Outstanding Overseas Chinese Scholars Fund, which supported greatly many valuable scientific activities even outside China.

During this work, many valuable discussions have been held with many scientists and friends. My special thanks go to Dr. Luisa Bastos of the Astronomical Observatory of University Porto, Dr. Rene Forsberg of Danish National Space Center, Prof. Jörg Reinking of Oldenburg University of Applied Sciences, Prof. Jikun Ou and Prof. Yunbin Yuan of IGG in Wuhan, Prof. Wu Chen of Hong Kong Polytechnic University, Prof. Yunzhong Shen of Tongji University in Shanghai, Dr. Yanxiong Liu of the First Oceanic Institute in Qingdao, Prof. Jiancheng Li of Wuhan University, Prof. Ta-Kang Yeh of the ChingYun University of Taiwan, Dr. Jürgen Neumeyer, Dr. Franz Barthelmes, and Dr. Svetozar Petrovic of GFZ,

Dr. Uwe Meyer of GeoZentrum Hannover, Dr. Ludger Timmen of University Hannover, Dr. Xiong Li of Hugro Inc. Houston, Dr. Daniela Morujao of Lisbon University, Prof. Klaus Hehl of Technical University of Applied Sciences Berlin, etc.

I also wish to thank sincerely Angelika Svarovsky and Hartmut Pflug of GFZ for their kind help. I am also grateful to Dr. Chris Bendall of Springer Heidelberg for his valuable advice.

My wife Liping, son Jia, daughters Yuxi, Pan and Yan are thanked for their constant support and understanding, as well as for their help.

October 2007 Guochang Xu

Abbreviations

AU	Astronomical Units
BDT	Barycentric Dynamic Time
CAS	Chinese Academy of Sciences
CAST	Chinese Academy of Space Technology
CHAMP	Challenging Mini-satellite Payload
CIO	Conventional International Origin
CRF	Conventional Reference Frame
CTS	Conventional Terrestrial System
DGK	Deutsche Geodätische Kommission
DGPS	Differential GPS
ECEF	Earth-Centred-Earth-Fixed (system)
ECI	Earth-Centred Inertial (system)
ECSF	Earth-Centred-Space-Fixed (system)
ESA	European Space Agency
EU	European Union
Galileo	Global Navigation Satellite System of EU
GAST	Greenwich Apparent Sidereal Time
GEO	Geostationary Earth Orbit (satellite)
GFZ	GeoForschungsZentrum Potsdam
GIS	Geographic Information System
GLONASS	Global Navigation Satellite System of Russia
GLOT	GLONASS Time
GMST	Greenwich Mean Sidereal Time
GNSS	Global Navigation Satellite System
GPS	Global Positioning System
GPST	GPS Time
GRACE	Gravity Recovery and Climate Experiment
GRS	Geodetic Reference System
GST	Galileo System Time
IAG	International Association of Geodesy
IAT	International Atomic Time

IAU	International Astronomical Union
IERS	International Earth Rotation Service
IGG	Institute of Geodesy and Geophysics (CAS)
IGS	International GPS Geodynamics Service
INS	Inertial Navigation System
ION	Institute of Navigation
ITRF	IERS Terrestrial Reference Frame
IUGG	International Union for Geodesy and Geophysics
JD	Julian Date
JPL	Jet Propulsion Laboratory
KMS	National Survey and Cadastre (Denmark)
KSGsoft	Kinematic/Static GPS Software
LEO	Low Earth Orbit (satellite)
LS	Least Squares (adjustment)
mas	micro arc second
MEO	Medium Earth Orbit (satellite)
MFGsoft	Multi-Functional GPS/Galileo Software
MIT	Massachusetts Institute of Technology
MJD	Modified Julian Date
MNRAS	Monthly Note of Royal Astronomical Society
NASA	National Aeronautics and Space Administration
NAVSTAR	Navigation System with Time and Ranging
NGS	National Geodetic Survey
NTSC	National Time Service Center (CAS)
OD	Orbit Determination
PC	Personal Computer
PZ-90	Parameters of the Earth Year 1990
RMS	Root Mean Square
SLR	Satellite Laser Ranging
SNR	Signal-to-Noise Ratio
SST	Satellite-Satellite Tracking
SV	Space Vehicle
TAI	International Atomic Time
TDB	Barycentric Dynamic Time
TDT	Terrestrial Dynamic Time
TJD	Time of Julian Date
TOPEX	(Ocean) Topography Experiment
TOW	Time of Week
TRANSIT	Time Ranging and Sequential
TT	Terrestrial Time
UT	Universal Time
UTC	Universal Time Coordinated
UTC_{SU}	Moscow Time UTC
WGS	World Geodetic System
ZfV	Zeitschrift für Vermessungswesen

Contents

Constants

Symbol	Value	Unit	Explanation
a_e	6378137	m	Semi-major axis of WGS-84
f_e	1/298.2572236		Flattening factor of WGS-84
a_p	6378136	m	Semi-major axis of PZ-90
f_p	1/298.2578393		Flattening factor of PZ-90
a_{eI}	6378136.54	m	Semi-major axis of ITRF-96
f_{eI}	1/298.25645		Flattening factor of ITRF-96
ε	84381.″412		Obliquity of the ecliptic at J2000.0
JDGPS	2444244.5	JD	Julian Date of GPS standard epoch (1980 Jan. 6, 0 h)
JD2000.0	2451545.0	JD	Julian Date of 2000 January 1, 12 h
G	6.67259e–11	$m^3s^{-2}\ kg^{-1}$	Constant of gravitation
μ_e	3.986004418e14	m^3s^{-2}	Geocentric gravitational constant
ω_e	7.292115e–5	$rads^{-1}$	Nominal mean angular velocity of the Earth
C	299792458	ms^{-1}	Speed of light
μ_s	1.327124e20	m^3s^{-2}	Heliocentric gravitational constant
μ_m	$\mu_e(M_m/M_e)$	m^3s^{-2}	Gravitational constant of the moon
M_m/M_e	0.0123000345		Moon-Earth mass ratio
h_2, h_3	0.6078, 0.292		Love number
l_2	0.0847		Shida number
P_s	4.5605e–6	Nm^{-1}	Luminosity of the sun
a_s	1.0000002 AU	m	Semi-major axis of the orbit of the sun
AU	149597870691	m	Astronomical units
a_m	384401000	m	Semi-major axis of the orbit of the moon

Chapter 1
Introduction

The desire to understand the orbits of the planets has a history as long as that of mankind. How and why the planets orbit around the sun are questions in two categories. One focuses on geometry and the other on physics. However, without knowing the answer to why, the how may not be answered theoretically, with the exception made by astronomical genius Kepler. After Newton's second law, all three Kepler laws may be derived theoretically.

Without any doubt, the milestones of orbit theory were crossed by Nicolaus Copernicus (1473–1543) with his heliocentric cosmology in "De revolutionibus orbium coelestium" (1543), Johannes Kepler (1571–1630) with his laws of planetary motion in "Astronomia nova" (1609) and "Harmonics Mundi" (1619) and Isaac Newton (1643–1727) with his universal gravitation and laws of motion in "Principia mathematica" (1687). The Keplerian orbit describes satellite (or planet) motion under the attraction of the central force of the Earth (or sun). After the first satellite was launched in 1957, William Kaula (1926–2000) crossed the milestones with the first-order solution of the equation of satellite motion disturbed by geopotential perturbations in "Theory of Satellite Geodesy" (1966). Thereafter, many scientists devoted themselves to the second-order orbit solution of geopotential disturbances. The complexity of the theory is such that only a few people understand it, and the theory, in turn, is rarely applied in practice. Numerical orbit determination is developed directly to meet the needs of satellite missions and to overcome the problem caused by the lack of analysis solutions for the equations of satellite motion.

Apparently most studies of orbit theory are focused on the solution of geopotential disturbances. Therefore, there exists a void in the literature on the solution of extraterrestrial disturbances. Meanwhile, it appears that the numerical algorithms are very robust and are not affected much by the obvious unphysical models and by the singularity caused by parameterisation of the problem.

Descriptions of Keplerian motion of satellites under the influence of the central force of the Earth are perfect and exact, and have a mathematical beauty (see Chap. 3). As soon as it is found that a satellite is moving in an orbital plane, the equations of motion are re-represented in the orbital plane and the Keplerian

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motion is then derived completely. Even in the case of a central force field, without the coordinate transformation step, it would be nearly impossible to derive the solution. This indicates the extreme importance of the selection of the coordinate system.

Recall the Kaula's solution to satellite motion under the influence of the geopotential field. The equations of satellite motion are represented in an inertial coordinate system according to Newton's law. However, the geopotential function is represented in the Earth-fixed system. To transform the geopotential function from the Earth-fixed system to the inertial one, a so-called Kaula's function is created, which is extremely complicated and leads to an extremely complicated solution. Some expressions of the solution are implicit. It is very difficult even to try to get the explicit expressions of the $\overline{C}_{20}$ solutions from the Kaula's solution. After Kaula's theory, studies on orbit theory are partly based on alternative variables, i.e. alternative coordinate systems.

The use of alternative coordinate systems is the first key to the solution of extraterrestrial disturbances of the equation of motion. The expansion (approximation) of models of the disturbance forces is the second. Two adjustment models of solar radiation and atmospheric drag used in numerical orbit determination are proposed by Xu (2004), in which alternative coordinate systems are suggested and approximation methods to simplify the force (adjustment) models are given. The path to solutions of extraterrestrial disturbances is then open; however, this was realised first at the beginning of 2007. The solutions are then derived and given via discretization, expanded potential function and Gaussian equations.

The extraterrestrial disturbances are second order ones. To derive complete solutions of second order, the solutions of geopotential disturbances are searched intensively. A method to derive solutions of geopotential disturbance of l order and m degree is described generally and used to derive 8×8 order and degree solutions. In GNSS orbit determination only a few lower order and degrees of the geopotential disturbances need to be considered. Therefore, the examples given will be enough for analytical orbit determination of satellites at higher altitudes with small eccentricities. The higher the order and degrees of the geopotential disturbances are, the more complex the analytical solutions will be. Therefore, application of the complete solution of geopotential disturbances will be a challenge for the future.

To describe a complete theory of satellite orbit, coordinate and time systems, the Keplerian orbit of the satellite must be discussed (Chaps. 2 and 3). Perturbation forces such as gravitational field, tide, the sun, the moon, planets, solar radiation and atmospheric drag, etc. and the disturbed equations of the satellite motion also have to be discussed (Chap. 4). Then the solutions of $\overline{C}_{20}$ disturbance and 8×8 order and degree geopotential disturbances can be derived (Chaps. 5 and 6). The solution of extraterrestrial disturbances such as solar radiation pressure, atmospheric drag and the disturbance of the sun, the moon and planets, are then given (Chap. 7). Numerical orbit determination is dealt with (Chap. 8) before discussing analytical orbit determination and application of orbit theory (Chap. 9).

Singularity-free orbit theory and discussions are given (Chap. 10). The orbit in solar gravity field is discussed in the last chapter (Chap. 11).

This book covers satellite orbit theory from theoretical and numerical aspects, with an emphasis on analytical solutions and applications. The analytical solutions of extraterrestrial disturbances and geopotential disturbances and the singularity-free theory are newly derived. The theory has opened and will further open very interesting research areas concerning satellite orbits. Part of the contents is refined theory, obtained from extensive research on individual problems. Because of the strong research and application background, the theories are conformably described with complexity. A brief summary of the contents is given in the Preface.

Some literature books is recommended for further reading: Diacu 1992, 1996; Dvorak and Lhotka 2013; Goldstein 1980; Hargihara 1970; Bate et al. 1971; Battin 1999; Boccaletti and Pucacco 2001; Brumberg 1995; Brouwer and Clemence 1961; Kaula 1966/2001; Chobotov 1991; Collins 2004; Cui 1990, 1997; Herrick 1972; Meeus 1992; Montenbruck and Gill 2000; Vallado 2007; Van Kamp 1967; Xu 2003, 2007, 2008.

Chapter 2
Coordinate and Time Systems

Satellites orbit around the Earth or travel in the planetary system of the sun. They are generally observed from the Earth. To describe the orbits of satellites (positions and velocities), suitable coordinate and time systems have to be defined.

2.1 Geocentric Earth-Fixed Coordinate Systems

It is convenient to use the Earth-Centred Earth-Fixed (ECEF) coordinate system to describe the location of a station on the Earth's surface. The ECEF coordinate system is a right-handed Cartesian system (x, y, z). Its origin and the Earth's centre of mass coincide, while its z-axis and the mean rotational axis of the Earth coincide; the x-axis points to the mean Greenwich meridian, while the y-axis is directed to complete a right-handed system (Fig. 2.1). In other words, the z-axis points to a mean pole of the Earth's rotation. Such a mean pole, defined by international convention, is called the Conventional International Origin (CIO). The xy-plane is called the mean equatorial plane, and the xz-plane is called the mean zero-meridian.

The ECEF coordinate system is also known as the Conventional Terrestrial System (CTS). The mean rotational axis and mean zero-meridian used here are necessary. The true rotational axis of the Earth changes its direction all the time with respect to the Earth's body. If such a pole is used to define a coordinate system, then the coordinates of the station would also change all the time. Because the survey is made in our true world, it is obvious that the polar motion has to be taken into account and will be discussed later.

The ECEF coordinate system can, of course, be represented by a spherical coordinate system (r, ϕ, λ), where r is the radius of the point (x, y, z), and ϕ and λ are the geocentric latitude and longitude, respectively (Fig. 2.2). λ is counted eastward from the zero-meridian. The relationship between (x, y, z) and (r, ϕ, λ) is obvious:

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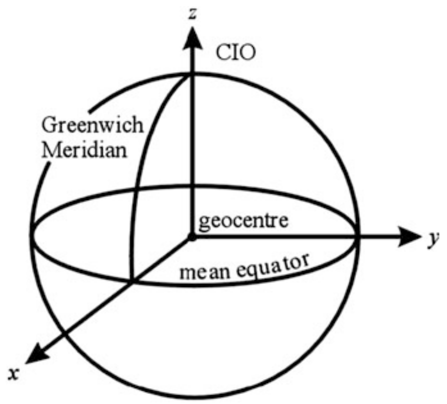


Fig. 2.1 Earth-Centred Earth-Fixed coordinates

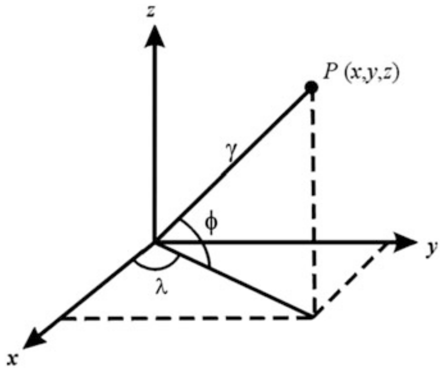
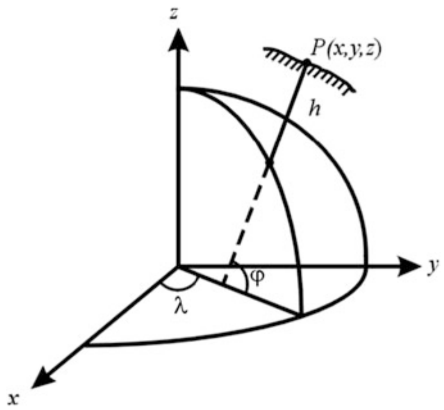


Fig. 2.2 Cartesian and spherical coordinates

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\phi\cos\lambda \\ r\cos\phi\sin\lambda \\ r\sin\phi \end{pmatrix}, \text{ or } \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \tan\lambda = y/x \\ \tan\phi = z/\sqrt{x^2 + y^2} \end{cases} \tag{2.1}$$

An ellipsoidal coordinate system (φ, λ, h) may also be defined on the basis of the ECEF coordinates; however, geometrically, two additional parameters are needed to define the shape of the ellipsoid (Fig. 2.3). φ, λ and h are geodetic latitude, longitude and height, respectively. The ellipsoidal surface is a rotational ellipse. The ellipsoidal system is also called the geodetic coordinate system. Geocentric longitude and geodetic longitude are identical. The two geometric parameters could be the semi-major radius (denoted by a) and the semi-minor radius (denoted by b) of the rotating ellipse, or the semi-major radius and the flattening (denoted by f) of the ellipsoid. They are equivalent sets of parameters. The relationship between (x, y, z) and (φ, λ, h) is (see, e.g. Torge 1991):

Fig. 2.3 Ellipsoidal coordinate system



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (N+h)\cos\varphi\cos\lambda \\ (N+h)\cos\varphi\sin\lambda \\ (N(1-e^2)+h)\sin\varphi \end{pmatrix}, \quad (2.2)$$

or

$$\begin{cases} \tan\varphi = \dfrac{Z}{\sqrt{x^2+y^2}}\left(1-e^2\dfrac{N}{N+h}\right)^{-1} \\ \tan\lambda = \dfrac{y}{x} \\ h = \dfrac{\sqrt{x^2+y^2}}{\cos\varphi} - N \end{cases}, \quad (2.3)$$

where

$$N = \frac{a}{\sqrt{1-e^2\sin^2\varphi}}. \quad (2.4)$$

N is the radius of curvature in the prime vertical and e is the first eccentricity. The geometric meaning of N is shown in Fig. 2.4. In (2.3) the φ and h have to be solved by iteration; however, the iteration process converges quickly, since $h << N$. The flattening and the first eccentricity are defined as

$$f = \frac{a-b}{a}, \text{ and } e = \frac{\sqrt{a^2-b^2}}{a}. \quad (2.5)$$

In cases where $\varphi = \pm 90°$ or h is very large, the iteration formulas of (2.3) could be unstable. Alternatively, using

$$\operatorname{ctan}\varphi = \frac{\sqrt{x^2+y^2}}{z+\Delta z} \text{ and } \Delta z = e^2 N\sin\varphi = \frac{ae^2\sin\varphi}{\sqrt{1-e^2\sin^2\varphi}}$$

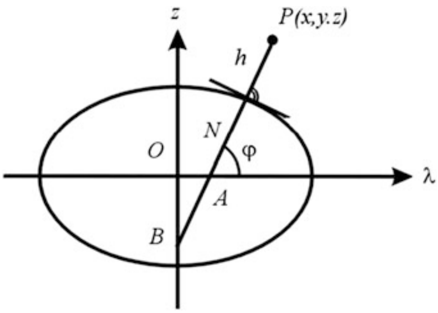


Fig. 2.4 Radius of curvature in the prime vertical

may lead to a stably iterated result of φ (see Lelgemann 2002). Δz and e^2N are the lengths of $\bar{O}\bar{B}$ and $\bar{A}\bar{B}$ (see Fig. 2.4), respectively. The geodetic height h can be obtained using Δz, i.e.

$$h = \sqrt{x^2 + y^2 + (z + \Delta z)^2} - N.$$

The two geometric parameters used in the World Geodetic System 1984 (WGS-84) are $a = 6378137$ m, $f = 1/298.2572236$. In International Terrestrial Reference Frame 1996 (ITRF-96) the two parameters are $a = 6378136.49$ m, $f = 1/298.25645$. ITRF uses the International Earth Rotation Service (IERS) Conventions (see McCarthy 1996). In the PZ-90 (Parameters of the Earth Year 1990) coordinate system of GLONASS, the two parameters are $a = 6378136$ m, $f = 1/298.2578393$.

The relation between the geocentric and geodetic latitude ϕ and φ (see (2.1) and (2.3)) may be given by

$$\tan\phi = \left(1 - e^2 \frac{N}{N + h}\right) \tan\varphi. \tag{2.6}$$

2.2 Coordinate System Transformations

Any Cartesian coordinate system can be transformed to another Cartesian coordinate system through three successive rotations if their origins are the same and if they are both right-handed or left-handed coordinate systems. These three rotational matrices are

$$R_1(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix},$$
$$R_2(\alpha) = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}, \tag{2.7}$$
$$R_3(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where α is the rotating angle, which has a positive sign for a counter-clockwise rotation as viewed from the positive axis to the origin. R_1, R_2 and R_3 are called the rotating matrix around the x, y and z-axis, respectively. For any rotational matrix R there are properties of $R^{-1}(\alpha) = R^T(\alpha)$ and $R^{-1}(\alpha) = R(-\alpha)$; that is, the rotational matrix is an orthogonal one, where R^{-1} and R^T are the inverse and transpose of the matrix R.

For two Cartesian coordinate systems with different origins and different length units, the general transformation can be given in vector (matrix) form as

$$X_n = X_0 + \mu R X_{\text{old}}, \tag{2.8}$$

or

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \mu R \begin{pmatrix} x_{\text{old}} \\ y_{\text{old}} \\ z_{\text{old}} \end{pmatrix},$$

where μ is the scale factor (or the ratio of the two length units) and R is a transformation matrix that can be formed by three suitably successive rotations. x_{n} and x_{old} denote the new and old coordinates, respectively; x_0 denotes the translation vector and is the coordinate vector of the origin of the old coordinate system in the new one.

If rotational angle α is very small, then one has $\sin\alpha \approx \alpha$ and $\cos\alpha \approx 1$. In such a case, the rotational matrix can be simplified. If the three rotational angles α_1, α_2, α_3 in R of (2.8) are very small, then R can be written as

$$R = \begin{pmatrix} 1 & \alpha_3 & -\alpha_2 \\ -\alpha_3 & 1 & \alpha_1 \\ \alpha_2 & -\alpha_1 & 1 \end{pmatrix}, \tag{2.9}$$

where α_1, α_2, α_3 are small rotating angles around the x, y and z-axis, respectively (see, e.g. Lelgemann and Xu 1991). Using the simplified R, the transformation (2.8) is called the Helmert transformation.

As an example, the transformation from WGS-84 to ITRF-90 (McCarthy 1996) is given by

$$\begin{pmatrix} x_{\mathrm{ITRF-90}} \\ y_{\mathrm{ITRF-90}} \\ z_{\mathrm{ITRF-90}} \end{pmatrix} = \begin{pmatrix} 0.060 \\ -0.517 \\ -0.223 \end{pmatrix} + \mu \begin{pmatrix} 1 & -0.0070'' & -0.0003'' \\ 0.0070'' & 1 & -0.0183'' \\ 0.0003'' & 0.0183'' & 1 \end{pmatrix} \begin{pmatrix} x_{\mathrm{WGS-84}} \\ y_{\mathrm{WGS-84}} \\ z_{\mathrm{WGS-84}} \end{pmatrix},$$

where $\mu = 0.999999989$ and the translation vector has the unit of meter.

The transformation between two coordinate systems can be generally represented by (2.8), where the scale factor $\mu = 1$ (i.e. the units of length used nowadays are the same). A formula of velocity transformation between different coordinate systems can be obtained by differentiating (2.8) with respect to time.

2.3 Local Coordinate System

The local left-handed Cartesian coordinate system (x', y', z') can be defined by placing the origin to the local point $P_1(x_1, y_1, z_1)$, whose z'-axis is pointed to the vertical, x'-axis is directed to the north and y' is pointed to the east (see Fig. 2.5). The $x'y'$-plane is called the horizontal plane; the vertical is defined perpendicular to the ellipsoid. Such a coordinate system is also called a local horizontal coordinate system. For any point P_2, whose coordinates in the global and local coordinate system are (x_2, y_2, z_2) and (x', y', z'), respectively, one has relations of

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = d \begin{pmatrix} \cos A \sin Z \\ \sin A \sin Z \\ \cos Z \end{pmatrix}, \text{ and } \begin{pmatrix} d = \sqrt{x'^2 + y'^2 + z'^2} \\ \tan A = y'/x' \\ \cos Z = z'/d \end{pmatrix}, \tag{2.10}$$

where A is the azimuth, Z is the zenith distance and d is the radius of the P_2 in the local system. A is measured from the north clockwise; Z is the angle between the vertical and the radius d.

The local coordinate system (x', y', z') can indeed be obtained by two successive rotations of the global coordinate system (x, y, z) by $R_2(90°–\varphi)R_3(\lambda)$ and then by changing the x-axis to a right-handed system. In other words, the global system has to be rotated around the z-axis by angle λ, then around the y-axis by angle $90°–\varphi$, and then change the sign of the x-axis. The total transformation matrix R is then

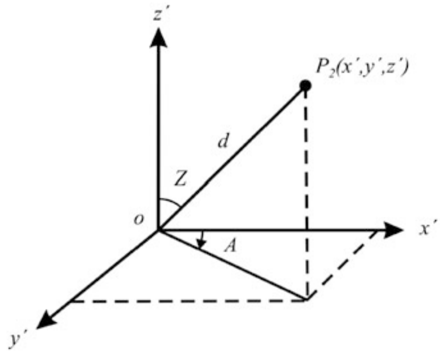


Fig. 2.5 Astronomical coordinate system

$$R = \begin{pmatrix} -\sin\varphi\cos\lambda & -\sin\varphi\sin\lambda & \cos\varphi \\ -\sin\lambda & \cos\lambda & 0 \\ \cos\varphi\cos\lambda & \cos\varphi\sin\lambda & \sin\varphi \end{pmatrix}, \tag{2.11}$$

and there are

$$X_{\text{local}} = RX_{\text{global}} \quad \text{and} \quad X_{\text{global}} = R^T X_{\text{local}}, \tag{2.12}$$

where X_{local} and X_{global} are the same vector represented in local and global coordinate systems. (φ,λ) are the geodetic latitude and longitude of the local point.

If the vertical direction is defined as the plumb line of the gravitational field at the local point, then such a local coordinate system is called an astronomic horizontal system (its x'-axis is pointed to the north, left-handed system). The plumb line of gravity g and the vertical line of the ellipsoid at the point p do not generally coincide with each other; however the difference is very small. The difference is omitted in GPS practice.

Combining (2.10) and (2.12), the zenith angle and azimuth of a point P_2 (satellite) related to the station P_1 can be directly computed by using the global coordinates of the two points by

$$\cos Z = \frac{z'}{d} \quad \text{and} \quad \tan A = \frac{y'}{x'}, \tag{2.13}$$

where

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}, \\ x' &= -(x_2 - x_1)\sin\varphi\cos\lambda - (y_2 - y_1)\sin\varphi\sin\lambda + (z_2 - z_1)\cos\varphi, \\ y' &= -(x_2 - x_1)\sin\lambda + (y_2 - y_1)\cos\lambda \quad \text{and} \\ z' &= (x_2 - x_1)\cos\varphi\cos\lambda + (y_2 - y_1)\cos\varphi\sin\lambda + (z_2 - z_1)\sin\varphi. \end{aligned}$$

2.4 Earth-Centred Inertial Coordinate System

To describe the motion of the GPS satellites, an inertial coordinate system has to be defined. The motion of the satellites follows Newtonian mechanics, and Newtonian mechanics is valid and expressed in an inertial coordinate system. For various reasons, the Conventional Celestial Reference Frame (CRF) is suitable for our purpose. The *xy*-plane of the CRF is the plane of the Earth's equator; the coordinates are celestial longitude, measured eastward along the equator from the vernal equinox, and celestial latitude. The vernal equinox is a crossover point of the ecliptic and the equator. So the right-handed Earth-centred inertial (ECI) system uses the Earth centre as the origin, CIO (Conventional International Origin) as the *z*-axis, and its *x*-axis is directed to the equinox of J2000.0 (Julian Date of 12 h 1st January 2000). Such a coordinate system is also called equatorial coordinates of date. Because of the motion (acceleration) of the Earth's centre, ECI is indeed a quasi-inertial system, and the general relativistic effects have to be taken into account in this system. The system moves around the sun, however, without rotating with respect to the CIO. This system is also called the Earth-centred space-fixed (ECSF) coordinate system.

An excellent figure has been given by Torge (1991) to illustrate the motion of the Earth's pole with respect to the ecliptic pole (see Fig. 2.6). The Earth's flattening, combined with the obliquity of the ecliptic, results in a slow turning of the equator on the ecliptic due to the differential gravitational effect of the moon and the sun. The slow circular motion with a period of about 26,000 years is called precession, and the other quicker motion with periods ranging from 14 days to 18.6 years is called nutation. Taking the precession and nutation into account, the Earth's mean pole (related to the mean equator) is transformed to the Earth's true pole (related to the true equator). The *x*-axis of the ECI is pointed to the vernal equinox of date.

The angle of the Earth's rotation from the equinox of date to the Greenwich meridian is called Greenwich Apparent Sidereal Time (GAST). Taking GAST into account (called the Earth's rotation), the ECI of date is transformed to the true equatorial coordinate system. The difference between the true equatorial system and the ECEF system is polar motion. So we have transformed the ECI system in a geometric way to the ECEF system. Such a transformation process can be written as

$$X_{\mathrm{ECEF}} = R_{\mathrm{M}} R_{\mathrm{S}} R_{\mathrm{N}} R_{\mathrm{P}} X_{\mathrm{ECI}}, \tag{2.14}$$

where R_{P} is the precession matrix, R_{N} is the nutation matrix, R_{S} is the Earth rotation matrix, R_{M} is the polar motion matrix, X is the coordinate vector, and indices ECEF and ECI denote the related coordinate systems.

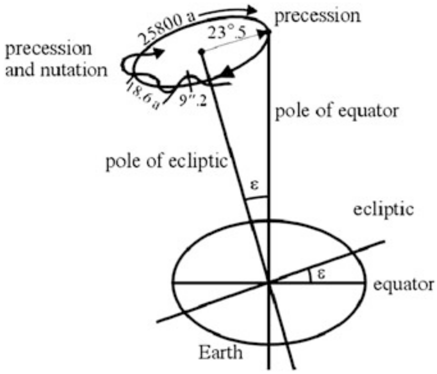


Fig. 2.6 Precession and nutation

2.4.1 Precession

The precession matrix consists of three successive rotational matrices, i.e. (see, e.g. Hofmann-Wellenhof et al. 1997/2001; Leick 1995/2004; McCarthy 1996)

$$R_P = R_3(-z)R_2(\theta)R_3(-\zeta)$$
$$= \begin{pmatrix} \cos z\cos\theta\cos\zeta - \sin z\sin\zeta & -\cos z\cos\theta\sin\zeta - \sin z\cos\zeta & -\cos z\sin\theta \\ \sin z\cos\theta\cos\zeta + \cos z\sin\zeta & -\sin z\cos\theta\sin\zeta + \cos z\cos\zeta & -\sin z\sin\theta \\ \sin\theta\cos\zeta & -\sin\theta\sin\zeta & \cos\theta \end{pmatrix}, \tag{2.15}$$

where z, θ, ζ are precession parameters and

$$\begin{aligned} z &= 2\,306.''2181T + 1.''09468T^2 + 0.''018203T^3, \\ \theta &= 2\,004.''3109T - 0.''42665T^2 - 0.''041833T^3 \text{ and} \\ \zeta &= 2\,306.''2181T + 0.''30188T^2 + 0.''017998T^3, \end{aligned} \tag{2.16}$$

where T is the measuring time in Julian centuries (36,525 days) counted from J2000.0 (see Sect. 2.8 time systems).

2.4.2 Nutation

The nutation matrix consists of three successive rotational matrices, i.e. (see, e.g. Hofmann-Wellenhof et al. 1997/2001; Leick 1995/2004; McCarthy 1996)

$$R_{\mathrm{N}} = R_1(-\varepsilon - \Delta\varepsilon)R_3(-\Delta\psi)R_1(\varepsilon)$$
$$= \begin{pmatrix} \cos\Delta\psi & -\sin\Delta\psi\cos\varepsilon & -\sin\Delta\psi\sin\varepsilon \\ \sin\Delta\psi\cos\varepsilon_t & \cos\Delta\psi\cos\varepsilon_t\cos\varepsilon + \sin\varepsilon_t\sin\varepsilon & \cos\Delta\psi\cos\varepsilon_t\sin\varepsilon - \sin\varepsilon_t\cos\varepsilon \\ \sin\Delta\psi\sin\varepsilon_t & \cos\Delta\psi\sin\varepsilon_t\cos\varepsilon - \cos\varepsilon_t\sin\varepsilon & \cos\Delta\psi\sin\varepsilon_t\sin\varepsilon + \cos\varepsilon_t\cos\varepsilon \end{pmatrix}$$
$$\approx \begin{pmatrix} 1 & -\Delta\psi\cos\varepsilon & -\Delta\psi\sin\varepsilon \\ \Delta\psi\cos\varepsilon_t & 1 & -\Delta\varepsilon \\ \Delta\psi\sin\varepsilon_t & \Delta\varepsilon & 1 \end{pmatrix}, \quad (2.17)$$

where ε is the mean obliquity of the ecliptic angle of date, $\Delta\psi$ and $\Delta\varepsilon$ are nutation angles in longitude and obliquity, $\varepsilon_t = \varepsilon + \Delta\varepsilon$, and

$$\varepsilon = 84\,381.''448 - 46.''8150T - 0.''00059T^2 + 0.''001813T^3. \quad (2.18)$$

The approximation is made by letting $\cos\Delta\psi = 1$ and $\sin\Delta\psi = \Delta\psi$ for very small $\Delta\psi$. For precise purposes, the exact rotation matrix shall be used. The nutation parameters $\Delta\psi$ and $\Delta\varepsilon$ can be computed using International Astronomical Union (IAU) theory or IERS theory:

$$\Delta\psi = \sum_{i=1}^{106} \left(A_i + A_i'T\right)\sin\beta,$$
$$\Delta\varepsilon = \sum_{i=1}^{106} \left(B_i + B_i'T\right)\cos\beta$$

or

$$\Delta\psi = \sum_{i=1}^{263} (A_i + A_i'T)\sin\beta + A''_i\cos\beta,$$

$$\Delta\varepsilon = \sum_{i=1}^{263} (B_i + B'_iT)\cos\beta + B''_i\cos\beta,$$

where argument

$$\beta = N_{1i}l + N_{2i}l' + N_{3i}F + N_{4i}D + N_{5i}\Omega,$$

where l is the mean anomaly of the moon, l' is the mean anomaly of the sun, $F = L - \Omega$, D is the mean elongation of the moon from the sun, Ω is the mean longitude of the ascending node of the moon and L is the mean longitude of the moon. The formulas of l, l', F, D and Ω, are given in Sect. 7.7. The coefficient values of $N_{1i}, N_{2i}, N_{3i}, N_{4i}, N_{5i}, A_i, B_i$, A'_i, B'_i A''_i and B''_i can be found in, e.g. McCarthy (1996). The updated formulas and tables can be found in updated IERS conventions. For convenience, the coefficients of the IAU 1980 nutation model are given in Appendix 1.

2.4.3 Earth Rotation

The Earth rotation matrix can be represented as

$$R_S = R_3(\text{GAST}), \tag{2.19}$$

where GAST is Greenwich Apparent Sidereal Time and

$$\text{GAST} = \text{GMST} + \Delta\psi \cos\varepsilon + 0.''00264 \sin\Omega + 0.''000063 \sin 2\Omega, \tag{2.20}$$

where GMST is Greenwich Mean Sidereal Time. Ω is the mean longitude of the ascending node of the moon; the second term on the right-hand side is the nutation of the equinox. Furthermore,

$$\text{GMST} = \text{GMST}_0 + \alpha\text{UT1}, \tag{2.21}$$

$$\text{GMST}_0 = 6 \times 3\,600.''0 + 41 \times 60.''0 + 50.''54841 \\ + 8\,640\,184\,.''812866\; T_0 + 0.''093104\; T_0^2 - 6.''2 \times 10^{-6} T_0^3,$$

$$\alpha = 1.0027379093\;50795 + 5.9006 \times 10^{-11} T_0 - 5.9 \times 10^{-15} T_0^2,$$

where GMST_0 is Greenwich Mean Sidereal Time at midnight on the day of interest. α is the rate of change. UT1 is the polar motion corrected Universal Time (see Sect. 2.8). T_0 is the measuring time in Julian centuries (36,525 days) counted from J2000.0 to 0 h UT1 of the measuring day. By computing GMST, UT1 is used (see Sect. 2.8).

2.4.4 Polar Motion

As shown in Fig. 2.7, polar motion is defined as the angles between the pole of date and the CIO pole. The polar motion coordinate system is defined by *xy*-plane coordinates, whose *x*-axis points to the south and coincides with the mean Greenwich meridian, and whose *y*-axis points to the west. x_p and y_p are the angles of the pole of date, so the rotation matrix of polar motion can be represented as

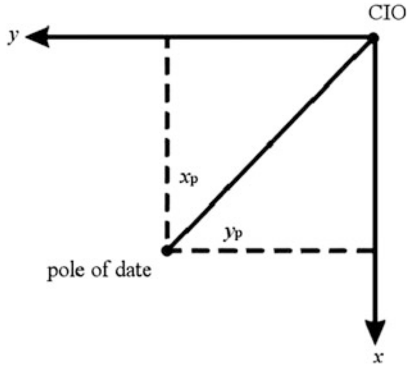


Fig. 2.7 Polar motion

$$R_M = R_2(-x_p)R_1(-y_p) = \begin{pmatrix} \cos x_p & \sin x_p \sin y_p & \sin x_p \cos y_p \\ 0 & \cos y_p & -\sin y_p \\ -\sin x_p & \cos x_p \sin y_p & \cos x_p \cos y_p \end{pmatrix}.$$
$$\approx \begin{pmatrix} 1 & 0 & x_p \\ 0 & 1 & -y_p \\ -x_p & y_p & 1 \end{pmatrix} \quad (2.22)$$

The IERS determined x_p and y_p can be obtained from the home pages of IERS.

2.5 IAU 2000 Framework

At its 2000 General Assembly, the International Astronomical Union (IAU) adopted a set of resolutions that provide a consistent framework for defining barycentric and geocentric celestial reference systems (Petit 2002). The consequence of the resolution is that coordinate transformation from celestial reference system (CRS, i.e. the ECI system) to the terrestrial reference system (TRS, i.e. the ECEF system) has the form

$$X_{ECEF} = R_M R_S R_{NP} X_{ECI}, \quad (2.23)$$

where R_{NP} is the precession-nutation matrix, R_S is the Earth rotation matrix, R_M is the polar motion matrix, X is the coordinate vector, and indices ECEF and ECI denote the related coordinate systems. The rotation matrices are functions of time T which is defined (see McCarthy and Petit 2003) by

$$T = (\mathrm{TT} - 2000\,\mathrm{January\ 1d\ 12h\ TT})\ \mathrm{in\ days}/36525, \quad (2.24)$$

where TT is the Terrestrial Time (for details see Sect. 2.8) and

$$R_{\mathrm{M}} = R_2(-x_{\mathrm{p}})R_1(-y_{\mathrm{p}})R_3(s'),$$
$$R_{\mathrm{S}} = R_3(\vartheta) \quad \text{and}$$
$$R_{\mathrm{NP}} = R_3(-s)R_3(-E)R_2(d)R_3(E), \tag{2.25}$$

where x_{p} and y_{p} are the angles of the pole of date (or polar coordinates of the Celestial Intermediate Pole (CIP) in TRS), and s' is a function of x_{p} and y_{p}:

$$s' = \frac{1}{2}\int_{T0}^{T} (x_{\mathrm{p}}\dot{y}_{\mathrm{p}} - \dot{x}_{\mathrm{p}}y_{\mathrm{p}})dt \quad \text{or}$$

approximately (see McCarthy and Capitaine 2002)

$$s' = (-47\mu as)\,T, \tag{2.26}$$

where T is time in Julian Century counted from J2000.0 and

$$\vartheta = 2\pi(0.7790572732\ 640 + 1.002737811\ 91135448\ T_u), \tag{2.27}$$

where T_u = (Julian UT1 date – 2451545.0) and UT1 = UTC + (UT1-UTC). (UT1-UTC) is published by the IERS.

E and d are such that the coordinates of the CIP in the CRS are

$$X = \sin d \cos E$$
$$Y = \sin d \sin E.$$
$$Z = \cos d \tag{2.28}$$

Equivalently, R_{NP} can be given by

$$R_{\mathrm{NP}} = R_3(-s) \cdot \begin{pmatrix} 1 - aX^2 & -aXY & X \\ -aXY & 1 - aY^2 & Y \\ -X & -Y & 1 - a(X^2 + Y^2) \end{pmatrix}^{-1} \tag{2.29}$$

where

$$a = \frac{1}{1 + \cos d} \approx \frac{1}{2} + \frac{1}{8}(X^2 + Y^2). \tag{2.30}$$

The developments of X and Y can be found on the website of the IERS Conventions and have the following form (in mas: microarcsecond) (Captinaine 2002):

$$
\begin{aligned}
X =& - 16616.99'' + 2004191742.88''T - 427219.05''T^2 \\
& - 198620.54''T^3 - 46.05''T^4 + 5.98''T^5 \\
& + \sum\nolimits_i [(a_{s,0})_i \sin\beta + (a_{c,0})_i \cos\beta] \\
& + \sum\nolimits_i [(a_{s,1})_i T \sin\beta + (a_{c,1})_i T \cos\beta] \\
& + \sum\nolimits_i [(a_{s,2})_i T^2 \sin\beta + (a_{c,2})_i T^2 \cos\beta] + \ldots
\end{aligned} \tag{2.31}
$$

$$
\begin{aligned}
Y =& - 6950.78'' - 25381.99''T - 22407250.99''T^2 \\
& + 1842.28''T^3 - 1113.06''T^4 + 0.99''T^5 \\
& + \sum\nolimits_i [(b_{s,0})_i \sin\beta + (b_{c,0})_i \cos\beta] \\
& + \sum\nolimits_i [(b_{s,1})_i T \sin\beta + (b_{c,1})_i T \cos\beta] \\
& + \sum\nolimits_i [(b_{s,2})_i T^2 \sin\beta + (b_{c,2})_i T^2 \cos\beta] + \ldots
\end{aligned} \tag{2.32}
$$

s in (2.29) is the accumulated rotation, between the reference epoch and the date T, of CEO on the true equator due to the celestial motion of CIP, and can be expressed as

$$
s(T) = -\frac{1}{2}[X(T)Y(T) - X(T_0)Y(T_0)] + \int_{T_0}^{T} \dot{X}Y dt - \left(\sigma_0 N_0 - \sum\nolimits_0 N_0\right)
$$

where σ_0 and $\sum_0$ are the positions of CEO at J2000.0 and the x-origin of CRS, respectively and N_0 is the ascending node at J2000.0 in the equator of CRS. In above equation, term $s(T) + \frac{1}{2}[X(T)Y(T)]$ can be expressed as (in mas)

$$
\begin{aligned}
s + XY/2 =& \, 94.0 + 3808.35T - 119.94T^2 \\
& -72574.09T^3 + 27.70T^4 + 15.61T^5 \\
& + \sum\nolimits_i [(c_{s,0})_i \sin\beta + (c_{c,0})_i \cos\beta] \\
& + \sum\nolimits_i [(c_{s,1})_i T \sin\beta + (c_{c,1})_i T \cos\beta] \\
& + \sum\nolimits_i [(c_{s,2})_i T^2 \sin\beta + (c_{c,2})_i T^2 \cos\beta] + \ldots
\end{aligned} \tag{2.33}
$$

In (2.31), (2.32), and (2.33), coefficients $(a_{s,j})_i$, $(a_{c,j})_i$, $(b_{s,j})_i$, $(b_{c,j})_i$ and $(c_{s,j})_i$, $(c_{c,j})_i$ can be extracted from Tables 5.2a, 5.2b and 5.2c (available at ftp://tai.bipm.org/iers/conv2003/chapter5/). β is the combination of the fundamental arguments of nutation theory:

$$
\beta = \sum_{j=1}^{14} N_j F_j \tag{2.34}
$$

The first five F_j are the Delaunary variables l, l', F, D, Ω (given in Sect. 7.7); the amplitudes of sines and cosines β can be derived from the amplitudes of the precession and nutation series (see McCarthy and Petit 2003); F_6 to F_{13} are the mean longitudes of the planets (Mercury to Neptune), including the Earth; F_{14} is the general precession in longitude. They are given in radians and T in Julian Centuries of TDB (see Sect. 2.8). The coefficients N_j are functions of index i and can be found in the IERS website:

$$\begin{aligned}
F_6 &= l_{Me} = 4.402608842 + 2608.7903141574T \\
F_7 &= l_{Ve} = 3.176146697 + 1021.3285546211T \\
F_8 &= l_E = 1.753470314 + 628.3075849991T \\
F_9 &= l_{Ma} = 6.203480913 + 334.0612426700T \\
F_{10} &= l_{Ju} = 0.599546497 + 52.9690962641T \\
F_{11} &= l_{Sa} = 0.874016757 + 21.3299104960T \\
F_{12} &= l_{Ur} = 5.481293872 + 7.4781598567T \\
F_{13} &= l_{Ne} = 5.311886287 + 3.8133035638T \\
F_{14} &= P_a = 0.024381750T + 0.00000538691T^2
\end{aligned} \tag{2.35}$$

Using the new paradigm, the complete procedure of transforming GCRS to ITRS, which is compatible with the IAU2000 precession-nutation, is based on the expressions of (2.31), (2.32), and (2.33).

An equivalent way to realise the transformation between TRS and CRS under the definition of IAU 2000 can be implemented in a classical way by adding IAU2000 corrections to the corresponding rotating angles. This is done by using the transformation formula (2.14), where the three precession rotating angles (see McCarthy and Petit 2003) are

$$\begin{aligned}
z &= -2.5976176'' + 2\,306.0803226''T + 1.0947790''T^2 \\
&\quad + 0.0182273''T^3 + 0.0000470''T^4 - 0.0000003''T^5, \\
\theta &= 2\,004.1917476''T - 0.4269353''T^2 - 0.0418251''T^3 \\
&\quad - 0.0000601''T^4 - 0.0000001''T^5 \quad \text{and} \\
\zeta &= 2.5976176'' + 2\,306.0809506''T + 0.3019015''T^2 \\
&\quad + 0.0179663''T^3 - 0.0000327''T^4 - 0.0000002''T^5.
\end{aligned} \tag{2.36}$$

The IAU 2000 nutation model is given by two series for nutation in longitude $\Delta\psi$ and obliquity $\Delta\varepsilon$, referred to the mean equator and equinox of date, with T measured in Julian centuries from epoch J2000.0:

$$\Delta\psi = \sum_{i=1}^{N} \left(A_i + A_i'T\right) \ \cos\beta + \left(A_i'' + \mathrm{A}_i'''T\right) \ \cos\beta, \tag{2.37}$$

$$\Delta\varepsilon = \sum_{i=1}^{N} (B_i + B_i'T) \ \cos\beta + (B_i'' + B_i''') \ \cos\beta,$$

where argument β can be found on the IERS website. For these two formulas, rate and bias corrections are necessary because of the new definition of the Celestial Intermediate Pole and the Celestial and Terrestrial Ephemeris Origin:

$$\begin{aligned} d\Delta\psi &= (-0.0166170 \pm 0.0000100)'' + (-0.29965 \pm 0.00040)''T, \\ d\Delta\varepsilon &= (-0.0068192 \pm 0.0000100)'' + (-0.02524 \pm 0.00010)''T. \end{aligned} \tag{2.38}$$

The Earth rotation angle (i.e. the apparent Greenwich Sidereal Time GST or GAST) can be computed by adding a correction EO to the GMST in (2.27) (in mas):

$$\begin{aligned} EO = {} & 14506 + 4612157399.66T + 1396677.21T^2 - 93.44T^3 + 18.82T^4 \\ & + \Delta\psi \cos\varepsilon + \sum\nolimits_i \left[(d_{s,0})_i \sin\beta + (d_{c,0})_i \cos\beta\right] \\ & + \sum\nolimits_i \left[(d_{s,1})_i T \sin\beta + (d_{c,1})_i T \cos\beta\right] + \ldots \end{aligned} \tag{2.39}$$

where coefficients $(d_{s,j})_i, (d_{c,j})_i$ can be extracted from Table 5.4 (available at ftp://tai.bipm.org/iers/conv2003/chapter5/). $\Delta\psi$ is defined in (2.37) and ε is defined in (2.18).

Similarly, the rotation matrix of polar motion shall be represented as the first formula of (2.25) and (2.26).

2.6 Geocentric Ecliptic Inertial Coordinate System

As discussed above, ECI uses the CIO pole in the space as the z-axis (through consideration of the polar motion, nutation and precession). If the ecliptic pole is used as the z-axis, then an ecliptic coordinate system is defined which may be called the Earth Centred Ecliptic Inertial (ECEI) coordinate system. ECEI places the origin at the mass centre of the Earth, its z-axis is directed to the ecliptic pole (or, the xy-plane is the mean ecliptic), and its x-axis is pointed to the vernal equinox of date. The coordinate transformation between the ECI and ECEI systems can be represented as

$$X_{\mathrm{ECEI}} = R_1(-\varepsilon)X_{\mathrm{ECI}}, \tag{2.40}$$

where ε is the ecliptic angle (mean obliquity) of the ecliptic plane related to the equatorial plane. The formula for ε is given in Sect. 2.4. Usually, coordinates of the sun and the moon, as well as planets, are given in the ECEI system.

2.7 Satellite Fixed Coordinate System

The orbit data, which describes the position of the satellite, are usually refers to the mass centre of the satellite. However, the orbit determination is usually measured through an instrument which is not exactly at the mass centre of the satellite. Therefore a satellite fixed coordinate system is necessary to be defined for describing the position of the instrument (e.g. antenna or reflector). Such antenna centre correction (also called mass centre correction) has to be applied to the satellite coordinates in precise applications.

A satellite fixed coordinate system shall be set up for describing the antenna phase centre offset to the mass centre of the satellite. As shown in Fig. 2.8, the origin of the frame coincides with the mass centre of the satellite, the z-axis is parallel to the antenna pointing direction, the y-axis is parallel to the solar-panel axis and the x-axis is selected to complete the right-handed frame. A solar vector is a vector from the satellite mass centre pointed to the sun. During the motion of the satellite, the z-axis is always pointing to the Earth, and the y-axis (solar-panel axis) shall be kept perpendicular to the solar vector. In other words, the y-axis is always perpendicular to the plane, which is formed by the sun, the Earth and satellite. The solar-panel can be rotated around its axis to keep the solar-panel perpendicular to the ray of the sun for optimally collecting the solar energy. The solar angle β is defined as the angle between the z-axis and the solar unit vector $\vec{n}_{\text{sun}}$ (see Fig. 2.9). Denoting the unit vector of the satellite fixed frame as $(\vec{e}_x\,\vec{e}_y\,\vec{e}_z)$, the solar unit vector can be represented as

$$\vec{n}_{\text{sun}} = (\sin\beta \quad 0 \quad \cos\beta). \tag{2.41}$$

β is needed for computation of the solar radiation pressure in orbit determination.

Using $\vec{r}$ for the geocentric satellite vector and $\vec{r}_{\text{s}}$ for the geocentric solar vector (Fig. 2.10),

$$\vec{r} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \qquad \vec{r}_S = \begin{pmatrix} X_{\text{sun}} \\ Y_{\text{sun}} \\ Z_{\text{sun}} \end{pmatrix}, \tag{2.42}$$

then in a geocentric coordinate system one has

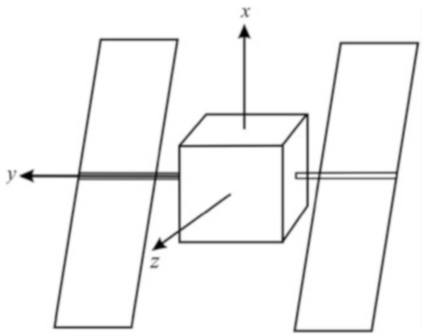


Fig. 2.8 Satellite fixed coordinate system

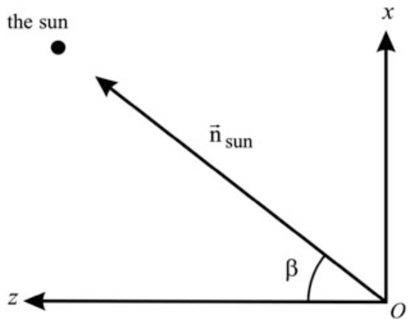


Fig. 2.9 The sun vector in satellite fixed frame

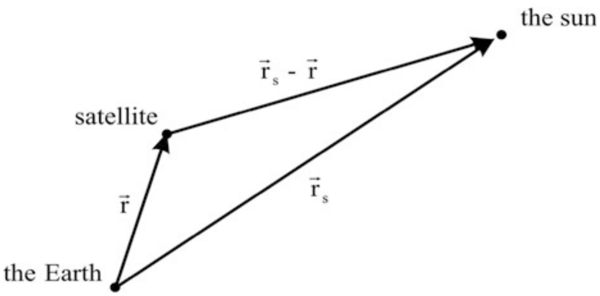


Fig. 2.10 The Earth-sun-satellite vectors

$$\vec{e}_z = -\frac{\vec{r}}{|\vec{r}|}, \tag{2.43}$$

$$\vec{e}_y = \frac{\vec{e}_z \times \vec{n}_{\text{sun}}}{|\vec{e}_z \times \vec{n}_{\text{sun}}|},$$

$$\vec{e}_x = \vec{e}_y \times \vec{e}_z, \tag{2.44}$$

$$\vec{n}_{\text{sun}} = \frac{\vec{r}_{\text{s}} - \vec{r}}{|\vec{r}_{\text{s}} - \vec{r}|} \quad \text{and} \tag{2.45}$$

Table 2.1 GPS satellite antenna phase centre offset

Satellite	x	y	z
Block I	0.2100	0.0	0.8540
Block II/IIA	0.2794	0.0	1.0259
Block IIR	0.0000	0.0	1.2053

$$\cos\beta = \vec{n}_{\text{sun}} \cdot \vec{e}_z, \tag{2.46}$$

or

$$\vec{e}_z = \frac{-1}{r}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \qquad r = \sqrt{X^2 + Y^2 + Z^2}, \tag{2.47}$$

$$\vec{n}_{\text{sun}} = \frac{1}{R}\begin{pmatrix} X_{\text{sun}} - X \\ Y_{\text{sun}} - Y \\ Z_{\text{sun}} - Z \end{pmatrix}, \tag{2.48}$$

$$\vec{e}_y = \frac{-1}{S}\begin{pmatrix} YZ_{\text{sun}} - Y_{\text{sun}}Z \\ ZX_{\text{sun}} - Z_{\text{sun}}X \\ XY_{\text{sun}} - X_{\text{sun}}Y \end{pmatrix} \quad \text{and} \tag{2.49}$$

$$\vec{e}_x = \frac{1}{S \cdot r}\begin{pmatrix} (ZX_{\text{sun}} - Z_{\text{sun}}X)Z - (XY_{\text{sun}} - X_{\text{sun}}Y)Y \\ (XY_{\text{sun}} - X_{\text{sun}}Y)X - (YZ_{\text{sun}} - Y_{\text{sun}}Z)Z \\ (YZ_{\text{sun}} - Y_{\text{sun}}Z)Y - (ZX_{\text{sun}} - Z_{\text{sun}}X)X \end{pmatrix} \tag{2.50}$$

where

$$R = \sqrt{(X_{\text{sun}} - X)^2 + (Y_{\text{sun}} - Y)^2 + (Z_{\text{sun}} - Z)^2} \quad \text{and} \tag{2.51}$$

$$S = \sqrt{(YZ_{\text{sun}} - Y_{\text{sun}}Z)^2 + (ZX_{\text{sun}} - Z_{\text{sun}}X)^2 + (XY_{\text{sun}} - X_{\text{sun}}Y)^2}. \tag{2.52}$$

Suppose the satellite antenna phase centre in the satellite fixed frame is (x,y,z); then the offset vector in the geocentric frame can be obtained by substituting (2.47), (2.49), and (2.50) into the following formula:

$$\vec{d} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z, \tag{2.53}$$

which may be added to the vector $\vec{r}$.

GPS satellite antenna phase centre offsets in the satellite fixed frame are given in Table 2.1.

The dependence of the phase centre on the signal direction and frequencies is not considered for the satellite here. A misorientation of the $\vec{e}_y$ ($\vec{e}_x$ too) of the satellite with respect to the sun may cause errors in the geometrical phase centre correction. In the Earth's shadow (for up to 55 min), the misorientation becomes worse. The geometrical misorientation may be modelled and estimated.

2.8 Time Systems

The three time systems used in satellite surveying are sidereal time, dynamic time and atomic time (see, e.g. Hofmann-Wellenhof et al. 1997/2001; Leick 1995/2004; McCarthy 1996; King et al. 1987).

Sidereal time is a measure of the Earth's rotation and is defined as the hour angle of the vernal equinox. If the measure is counted from the Greenwich meridian, the sidereal time is called Greenwich Sidereal Time. Universal Time (UT) is the Greenwich hour angle of the apparent sun, which is orbiting uniformly in the equatorial plane. Because the angular velocity of the Earth's rotation is not constant, sidereal time is not a uniformly-scaled time. The oscillation of UT is also partly caused by the polar motion of the Earth. The universal time corrected for the polar motion is denoted by UT1.

Dynamic time is a uniformly-scaled time used to describe the motion of bodies in a gravitational field. Barycentric Dynamic Time (TDB) is applied in an inertial coordinate system (its origin is located at the centre-of-mass (Barycentre)). Terrestrial Dynamic Time (TDT) is used in a quasi-inertial coordinate system (such as ECI). Because of the motion of the Earth around the sun (or, say, in the sun's gravitational field), TDT will vary with respect to TDB. However, both the satellite and the Earth are subject to almost the same gravitational perturbations. TDT may be used for describing the satellite motion without taking into account the influence of the gravitational field of the sun. TDT is also called Terrestrial Time (TT).

Atomic Time is a time system kept by atomic clocks such as International Atomic Time (TAI). It is a uniformly-scaled time used in the ECEF coordinate system. TDT is realised by TAI in practice with a constant offset (32.184 s). Because of the slowing down of the Earth's rotation with respect to the sun, Coordinated Universal Time (UTC) is introduced to keep the synchronisation of TAI to the solar day (by inserting leap seconds). GPS Time (GPST) is also atomic time.

The relationships between different time systems are as follows:

$$\begin{aligned} \mathrm{TAI} &= \mathrm{GPST} + 19.0 \ \mathrm{sec} \\ \mathrm{TAI} &= \mathrm{TDT} - 32.184 \ \mathrm{sec}, \\ \mathrm{TAI} &= \mathrm{UTC} + n \ \mathrm{sec} \\ \mathrm{UT1} &= \mathrm{UTC} + \mathrm{dUT1} \end{aligned} \tag{2.54}$$

where dUT1 can be obtained by IERS (dUT1 < 0.7 s, see Zhu et al. 1996; dUT1 is also broadcasted with the navigation data), n is the number of leap seconds of date and is inserted into UTC on the 1st of January and 1st of July of the relevant years. The actual n can be found in the IERS report.

Time argument T (Julian centuries) is used in the formulas given in Sect. 2.4. For convenience, T is denoted by TJD, and TJD can be computed from the civil date (Year, Month, Day, and Hour) as follows:

$$\mathrm{JD} = \mathrm{INT}(365.25Y) + \mathrm{INT}(30.6001\,(M+1)) + \mathrm{Day} + \mathrm{Hour}/24 + 1\,720\,981.5 \text{ and}$$

$$\mathrm{TJD} = \mathrm{JD}/36\,525, \tag{2.55}$$

where

$$\begin{array}{lll} Y = \mathrm{Year} - 1, & M = \mathrm{Month} + 12, & \text{if Month} \le 2, \\ Y = \mathrm{Year}, & M = \mathrm{Month}, & \text{if Month} > 2, \end{array}$$

where JD is the Julian Date, Hour is the time of UT and INT denotes the integer part of a real number. The Julian Date counted from JD2000.0 is then JD2000 = JD–JD2000.0, where JD2000.0 is the Julian Date of 2000 January 1st 12 h and has the value of 2451545.0 days. One Julian century is 36,525 days.

Inversely, the civil date (Year, Month, Day and Hour) can be computed from the Julian Date (JD) as follows:

$$\begin{aligned} b &= \mathrm{INT}(\mathrm{JD} + 0.5) + 1537, \\ c &= \mathrm{INT}\left(\frac{\mathrm{b} - 122.1}{365.25}\right), \\ d &= \mathrm{INT}(365.25c), \\ e &= \mathrm{INT}\left(\frac{b-d}{30.6001}\right), \\ \mathrm{Hour} &= \mathrm{JD} + 0.5 - \mathrm{INT}(\mathrm{JD} + 0.5), \\ \mathrm{Day} &= b - d - \mathrm{INT}(30.6001\,e), \\ \mathrm{Month} &= e - 1 - 12\,\mathrm{INT}\left(\frac{e}{14}\right) \text{ and} \\ \mathrm{Year} &= c - 4\,715 - \mathrm{INT}\left(\frac{7 + \mathrm{Month}}{10}\right), \end{aligned} \tag{2.56}$$

where b, c, d and e are auxiliary numbers.

Because the GPS standard epoch is defined as JD = 2444244.5 (1980 January 6, 0 h), GPS week and the day of week (denoted by Week and N) can be computed by

$$N = \text{modulo}(\text{INT}(\text{JD} + 1.5), 7) \text{ and}$$
$$\text{Week} = \text{INT}\left(\frac{\text{JD} - 2\,444\,244.5}{7}\right), \tag{2.57}$$

where N is the day of week ($N = 0$ for Monday, $N = 1$ for Tuesday, and so on).

For saving digits and counting the date from midnight instead of noon, the Modified Julian Date (MJD) is defined as

$$\text{MJD} = (\text{JD} - 2400000.5) \tag{2.58}$$

GLONASS time (GLOT) is defined by Moscow time UTC_{SU}, which equals UTC plus 3 h (corresponding to the offset of Moscow time to Greenwich time), theoretically. GLOT is permanently monitored and adjusted by the GLONASS Central Synchroniser (see Roßbach 2006). UTC and GLOT then have the simple relation

$$\text{UTC} = \text{GLOT} + \tau_c - 3 \text{ h},$$

where τ_c is the system time correction with respect to UTC_{SU}, which is broadcasted by the GLONASS ephemeris and is less than 1 μs. Therefore, approximately,

$$\text{GPST} = \text{GLOT} + m - 3 \text{ h},$$

where m is the number of "leap seconds" between GPS and GLONASS (UTC) time and is given in the GLONASS ephemeris. m is indeed the leap seconds since GPS standard epoch (1980 January 6, 0 h).

Galileo system time (GST) will be maintained by a number of UTC laboratory clocks. GST and GPST are time systems of various UTC laboratories. After the offset of GST and GPST is made available to the user, interoperability will be assured.

Chapter 3
Keplerian Orbits

3.1 Keplerian Motion

The simplified satellite orbiting is called Keplerian motion, and the problem is called the two-bodies problem. The satellite is supposed to move in a central force field. The equation of satellite motion is described by Newton's second law of motion by

$$\vec{f} = m \cdot a = m \cdot \ddot{\vec{r}}, \tag{3.1}$$

where $\vec{f}$ is the attracting force, m is the mass of the satellite, a, or alternatively $\ddot{\vec{r}}$, is the acceleration of the motion (second order differentiation of vector $\vec{r}$ with respect to the time), and according to Newton's law

$$\vec{f} = -\frac{GMm}{r^2}\frac{\vec{r}}{r}, \tag{3.2}$$

where G is the universal gravitational constant, M is the mass of the Earth and r is the distance between the mass centre of the Earth and the mass centre of the satellite. The equation of satellite motion is then

$$\ddot{\vec{r}} = -\frac{\mu}{r^2}\frac{\vec{r}}{r}, \tag{3.3}$$

where μ (=GM) is called Earth's gravitational constant.

Satellite motion can be considered a motion of the satellite under the central force field of the Earth and the disturbed motion caused by other perturbation forces. Therefore, the Keplerian orbits are important in orbit theory and will be discussed in this chapter.

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Equation 3.3 of satellite motion is valid only in an inertial coordinate system, so the ECSF coordinate system discussed in Sect. 3.4 will be used for describing the orbit of the satellite. The vector form of the equation of motion can be rewritten through the three components x, y and z ($\vec{r} = (x, y, z)$) as

$$\begin{aligned}\ddot{x} &= -\frac{\mu}{r^3}x \\ \ddot{y} &= -\frac{\mu}{r^3}y. \\ \ddot{z} &= -\frac{\mu}{r^3}z\end{aligned} \tag{3.4}$$

Multiplying y, z by the first equation of (3.4), x, z by the second and x, y by the third, and then forming the differences of them, one gets

$$\begin{aligned}y\ddot{z} - z\ddot{y} &= 0 \\ z\ddot{x} - x\ddot{z} &= 0 \\ x\ddot{y} - y\ddot{x} &= 0,\end{aligned} \tag{3.5}$$

or in vector form

$$\vec{r} \times \ddot{\vec{r}} = 0. \tag{3.6}$$

Equations (3.5) and (3.6) are equivalent to

$$\begin{aligned}\frac{\mathrm{d}(y\dot{z} - z\dot{y})}{\mathrm{d}t} &= 0 \\ \frac{\mathrm{d}(z\dot{x} - x\dot{z})}{\mathrm{d}t} &= 0, \\ \frac{\mathrm{d}(x\dot{y} - y\dot{x})}{\mathrm{d}t} &= 0\end{aligned} \tag{3.7}$$

$$\frac{\mathrm{d}(\vec{r} \times \dot{\vec{r}})}{\mathrm{d}t} = 0. \tag{3.8}$$

Integrating (3.7) and (3.8) leads to

$$\begin{aligned}y\dot{z} - z\dot{y} &= A \\ z\dot{x} - x\dot{z} &= B\ , \\ x\dot{y} - y\dot{x} &= C\end{aligned} \tag{3.9}$$

$$\vec{r} \times \dot{\vec{r}} = \vec{h} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}, \tag{3.10}$$

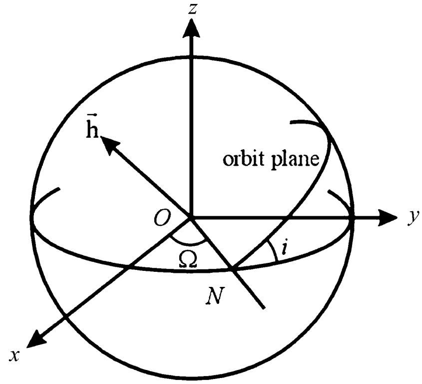


Fig. 3.1 Orbital plane

where A, B, C are integration constants; they form the integration constant vector $\vec{h}$. That is

$$h = \sqrt{A^2 + B^2 + C^2} = \left|\vec{r} \times \dot{\vec{r}}\right|. \qquad (3.11)$$

The constant h is twice the area that the radius vector sweeps during unit time. This is indeed Kepler's second law. Then $h/2$ is called the area velocity of the radius of the satellite.

Multiplying x, y and z by the three equations of (3.9) and adding them together, one has

$$Ax + By + Cz = 0. \qquad (3.12)$$

That is, the satellite motion fulfils the equation of a plane, and the origin of the coordinate system is in the plane. In other words, the satellite moves in a plane in the central force field of the Earth. The plane is called the orbital plane of the satellite.

The angle between the orbital plane and the equatorial plane is called inclination of the satellite (denoted by i, see Fig. 3.1). Alternatively, the inclination i is the angle between the vector $\vec{z} = (0, 0, 1)$ and $\vec{h} = (A, B, C)$, i.e.

$$\cos i = \frac{\vec{z} \cdot \vec{h}}{|\vec{z}| \cdot \left|\vec{h}\right|} = \frac{C}{h}. \qquad (3.13)$$

The orbital plane cuts the equator at two points. They are called ascending node N and descending node (see the next section for details). Vector $\vec{s}$ denotes the vector from the Earth centre pointed to the ascending point. The angle between the ascending node and the x-axis (vernal equinox) is called the right ascension of the ascending node (denoted by Ω). Thus

$$\vec{s} = \vec{z} \times \vec{h},$$

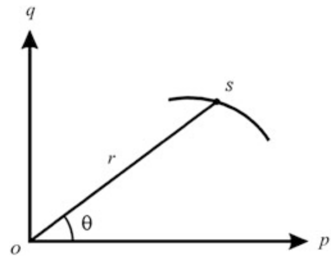


Fig. 3.2 Polar coordinates in the orbital plane

and

$$\cos \Omega = \frac{\vec{s} \cdot \vec{x}}{|\vec{s}| \cdot |\vec{x}|} = \frac{-B}{\sqrt{A^2 + B^2}},$$

$$\sin \Omega = \frac{\vec{s} \cdot \vec{y}}{|\vec{s}| \cdot |\vec{y}|} = \frac{A}{\sqrt{A^2 + B^2}}. \quad (3.14)$$

Parameters i and Ω uniquely define the place of the orbital plane and are therefore called orbital plane parameters. Ω, i and h are then selected as integration constants, which have significant geometric meanings in the satellite orbits.

3.2 Satellite Motion in the Orbital Plane

In the orbital plane, a two-dimensional rectangular coordinate system is given in Fig. 3.2. The coordinates can be represented in polar coordinate r and ϑ as

$$\begin{aligned} p &= r \cos \vartheta \\ q &= r \sin \vartheta \end{aligned}. \quad (3.15)$$

The equation of motion in pq-coordinates is similar to (3.4) as

$$\begin{aligned} \ddot{p} &= -\frac{\mu}{r^3} p \\ \ddot{q} &= -\frac{\mu}{r^3} q \end{aligned}. \quad (3.16)$$

From (3.15), one has

$$\begin{aligned} \dot{p} &= \dot{r} \cos \vartheta - r\dot{\vartheta} \sin \vartheta \\ \dot{q} &= \dot{r} \sin \vartheta + r\dot{\vartheta} \cos \vartheta \\ \ddot{p} &= \left(\ddot{r} - r\dot{\vartheta}^2\right) \cos \vartheta - \left(r\ddot{\vartheta} + 2\dot{r}\dot{\vartheta}\right) \sin \vartheta. \\ \ddot{q} &= \left(\ddot{r} - r\dot{\vartheta}^2\right) \sin \vartheta + \left(r\ddot{\vartheta} + 2\dot{r}\dot{\vartheta}\right) \cos \vartheta \end{aligned} \quad (3.17)$$

Substituting (3.17) and (3.15) into (3.16), one gets

$$\begin{aligned} \left(\ddot{r} - r\dot{\vartheta}^2\right)\cos\vartheta - \left(r\ddot{\vartheta} + 2\dot{r}\dot{\vartheta}\right)\sin\vartheta &= -\frac{\mu}{r^2}\cos\vartheta \\ \left(\ddot{r} - r\dot{\vartheta}^2\right)\sin\vartheta + \left(r\ddot{\vartheta} + 2\dot{r}\dot{\vartheta}\right)\cos\vartheta &= -\frac{\mu}{r^2}\sin\vartheta \end{aligned}. \tag{3.18}$$

The point from which the polar angle ϑ is measured is arbitrary. So setting ϑ as zero, the equation of motion is then

$$\begin{aligned} \ddot{r} - r\dot{\vartheta}^2 &= -\frac{\mu}{r^2} \\ r\ddot{\vartheta} + 2\dot{r}\dot{\vartheta} &= 0 \end{aligned}. \tag{3.19}$$

Multiplying r by the second equation of (3.19) gives

$$\frac{\mathrm{d}(r^2\dot{\vartheta})}{\mathrm{d}t} = 0. \tag{3.20}$$

Because $r\dot{\vartheta}$ is the tangential velocity, $r^2\dot{\vartheta}$ is twice the area velocity of the radius of the satellite. Integrating (3.20) and comparing it with the discussion in Sect. 3.1, one has

$$r^2\dot{\vartheta} = h. \tag{3.21}$$

$h/2$ is the area velocity of the radius of the satellite.

For solving the first differential equation of (3.19), the equation has to be transformed into a differential equation of r with respect to variable f. Let

$$u = \frac{1}{r}, \tag{3.22}$$

then from (3.21) one gets

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}t} = hu^2 \tag{3.23}$$

and

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}t} &= \frac{\mathrm{d}r}{\mathrm{d}\vartheta}\frac{\mathrm{d}\vartheta}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}\vartheta}\left(\frac{1}{u}\right)hu^2 = -h\frac{\mathrm{d}u}{\mathrm{d}\vartheta} \\ \frac{\mathrm{d}^2r}{\mathrm{d}t^2} &= -h\frac{\mathrm{d}^2u}{\mathrm{d}\vartheta^2}\frac{\mathrm{d}\vartheta}{\mathrm{d}t} = -h^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\vartheta^2} \end{aligned}. \tag{3.24}$$

Substituting (3.22) and (3.24) into the first equation of (3.19), the equation of motion is then

$$\frac{d^2u}{d\vartheta^2} + u = \frac{\mu}{h^2}, \tag{3.25}$$

and its solution is

$$u = d_1 \cos\vartheta + d_2 \sin\vartheta + \frac{\mu}{h^2},$$

where d_1 and d_2 are constants of integration. The above equation may be simplified as

$$u = \frac{\mu}{h^2}(1 + e\cos(\vartheta - \omega)), \tag{3.26}$$

where

$$d_1 = \frac{\mu}{h^2} e\cos\omega, \quad d_2 = \frac{\mu}{h^2} e\sin\omega.$$

Thus the moving equation of satellite in the orbital plane is

$$r = \frac{h^2/\mu}{1 + e\cos(\vartheta - \omega)}. \tag{3.27}$$

Comparing (3.27) with a standard polar equation of conic,

$$r = \frac{a(1-e^2)}{1 - e\cos\varphi}, \tag{3.28}$$

orbit (3.27) is obviously a polar equation of conic section with the origin at one of the foci. Where parameter e is the eccentricity, for $e = 0$, $e < 1$, $e = 1$ and $e > 1$, the conic is a circle, an ellipse, a parabola and a hyperbola, respectively. For the satellite orbiting around the Earth, generally, $e < 1$. Thus the satellite orbit is an ellipse, and this is indeed Kepler's first law. Parameter a is the semi-major axis of the ellipse, and

$$\frac{h^2}{\mu} = a\left(1 - e^2\right). \tag{3.29}$$

It is obvious that parameter a has more significant geometric sense than that of h, so a is preferable to use. Parameters a and e define the size and shape of the ellipse and are called ellipse parameters. The ellipse cuts the equator at the ascending and

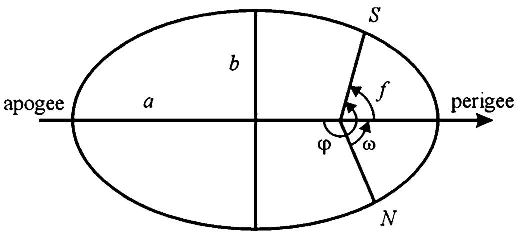


Fig. 3.3 Ellipse of satellite motion

descending nodes. Polar angle φ is counted from the apogee of the ellipse. This can be seen by letting $\varphi = 0$; then $r = a(1 + e)$. φ has a 180° difference with the angle $\vartheta - \omega$. Letting $f = \vartheta - \omega$, where f is called the true anomaly of the satellite counted from the perigee, then the orbit (3.27) can be written as

$$r = \frac{a(1 - e^2)}{1 + e\cos f}. \tag{3.30}$$

In the case of $f = 0$, i.e. the satellite is at the point of perigee, $\omega = \vartheta$, ϑ is the polar angle of the perigee counted from the p-axis. Supposing the p-axis is an axis in the equatorial plane and points to the ascending node N; then ω is the angle of perigee counted from the ascending node (see Fig. 3.3) and is called the argument of perigee. The argument of perigee defines the axis direction of the ellipse related to the equatorial plane.

3.3 The Keplerian Equation

Up to now, five integration constants have been derived. They are inclination angle i, right ascension of ascending node Ω, semi-major axis a, eccentricity e of the ellipse and argument of perigee ω. Parameters i and Ω decide the place of the orbital plane, a and e decide the size and shape of the ellipse and ω decides the direction of the ellipse (see Fig. 3.4). To describe the satellite position in the ellipse, the velocity of motion has to be discussed.

The period T of satellite motion is the area of the ellipse divided by area velocity:

$$T = \frac{\pi ab}{\frac{1}{2}h} = \frac{2\pi ab}{\sqrt{\mu a(1 - e^2)}} = 2\pi a^{3/2}\mu^{-1/2}. \tag{3.31}$$

The average angular velocity n is then

$$n = \frac{2\pi}{T} = a^{-3/2}\mu^{1/2}. \tag{3.32}$$

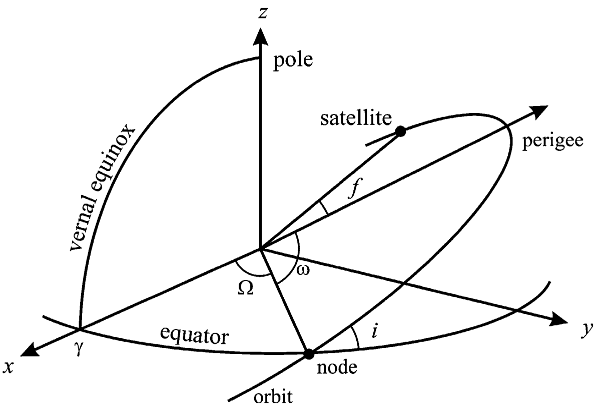


Fig. 3.4 Orbital geometry

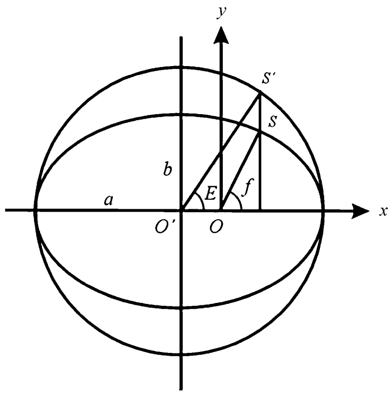


Fig. 3.5 Mean anomaly of satellite

Equation (3.32) is Kepler's third law. It is obvious that it is easier to describe the angular motion of the satellite under the average angular velocity n in the geometric centre of the ellipse (rather than in the geocentre). For simplifying the problem, an angle called the eccentric anomaly is defined (denoted by E, see Fig. 3.5). S' is the vertical projection of the satellite S on the circle with a radius of a (semi-major axis of the ellipse). The distance between the geometric centre O' of the ellipse and the geocentre O is ae. Thus,

$$
\begin{aligned}
x &= r\cos f = a\cos E - ae \\
y &= r\sin f = b\sin E = a\sqrt{1-e^2}\sin E,
\end{aligned}
\tag{3.33}
$$

where the second equation can be obtained by substituting the first into the standard ellipse equation ($x^2/a^2 + y^2/b^2 = 1$) and omitting the small terms that contain e (for the satellite, generally, $e << 1$), where b is the semi-minor axis of the ellipse. The orbit equation can then be represented by variable E as

$$r = a(1 - e\cos E). \tag{3.34}$$

The relation between true and eccentric anomalies can be derived by using (3.33) and (3.34):

$$\tan\frac{f}{2} = \frac{\sin f}{1+\cos f} = \frac{\sin E}{1+\cos E}\frac{\sqrt{1-e^2}}{1-e} = \frac{\sqrt{1+e}}{\sqrt{1-e}}\tan\frac{E}{2}. \tag{3.35}$$

If the *xyz*-coordinates are rotated so that the *xy*-plane coincides with the orbital plane, then the area velocity formulas of (3.9) and (3.10) have only one component along the *z*-axis, i.e.

$$x\dot{y} - y\dot{x} = h = \sqrt{\mu a(1-e^2)}. \tag{3.36}$$

From (3.33), one has

$$\begin{aligned}\dot{x} &= -a\sin E\frac{\mathrm{d}E}{\mathrm{d}t}\\ \dot{y} &= a\sqrt{1-e^2}\cos E\frac{\mathrm{d}E}{\mathrm{d}t}.\end{aligned} \tag{3.37}$$

Substituting (3.33) and (3.37) into (3.36) and taking (3.32) into account, a relation between E and t is obtained:

$$(1 - e\cos E)\mathrm{d}E = \sqrt{\mu}a^{-3/2}\mathrm{d}t = n\mathrm{d}t. \tag{3.38}$$

Suppose that at time t_p the satellite is at the point perigee, i.e. $E(t_\mathrm{p}) = 0$, and at any time t, $E(t) = E$, then integration of (3.38) from 0 to E, namely from t_p to t is

$$E - e\sin E = M, \tag{3.39}$$

where

$$M = n\left(t - t_p\right). \tag{3.40}$$

Equation (3.39) is the Keplerian equation. E is given as a function of M, namely t. Because of (3.34), the Keplerian equation indirectly assigns r as a function of t. M is called the mean anomaly and describes the satellite as orbiting the Earth with a mean angular velocity n. t_p is called the perigee passage and is the sixth integration constant of the equation of satellite motion in a centre-force field.

Knowing M to compute E, the Keplerian equation (3.39) may be solved iteratively. Because of the small e, the convergence can be achieved very quickly.

Three anomalies (true anomaly f, eccentric anomaly E and mean anomaly M) are equivalent through the relations of (3.35) and (3.39). They are functions of time t (including the perigee passage t_p) and they describe the position changes of the satellite with time in the ECSF coordinates.

3.4 State Vector of the Satellite

Consider the orbital right-handed coordinate system: if the xy-plane is the orbital plane, the x-axis is pointing to the perigee, the z-axis is in the direction of vector $\vec{h}$, and the origin is in the geocentre, then the position vector $\vec{q}$ of the satellite is (see (3.33))

$$\vec{q} = \begin{pmatrix} a(\cos E - e) \\ a\sqrt{1 - e^2}\sin E \\ 0 \end{pmatrix} = \begin{pmatrix} r\cos f \\ r\sin f \\ 0 \end{pmatrix}. \tag{3.41}$$

Differentiating (3.41) with respect to time t and taking (3.38) into account, the velocity vector of the satellite is then

$$\dot{\vec{q}} = \begin{pmatrix} -\sin E \\ \sqrt{1 - e^2}\cos E \\ 0 \end{pmatrix} \frac{na}{1 - e\cos E} = \begin{pmatrix} -\sin f \\ e + \cos f \\ 0 \end{pmatrix} \frac{na}{\sqrt{1 - e^2}}. \tag{3.42}$$

The second part of the above equation can be derived from the relation between E and f. The state vector of the satellite in the orbital coordinate system can be rotated to the ECSF coordinate system by three successive rotations. First, a clockwise rotation around the 3rd-axis from the perigee to the node is given by (see Fig. 3.4)

$$R_3(-\omega).$$

Next, a clockwise rotation around the 1st-axis with the angle of inclination i is given by

$$R_1(-i).$$

Finally, a clockwise rotation around the 3rd-axis from the node to the vernal equinox is given by

$$R_3(-\Omega).$$

So the state vector of the satellite in the ECSF coordinate system is

$$\begin{pmatrix} \vec{r} \\ \dot{\vec{r}} \end{pmatrix} = R_3(-\Omega)R_1(-i)R_3(-\omega)\begin{pmatrix} \vec{q} \\ \dot{\vec{q}} \end{pmatrix}, \tag{3.43}$$

where

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \dot{\vec{r}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}.$$

For given six Keplerian elements (Ω, i, ω, a, e, M_0) of t_0, where $M_0 = n(t_0 – t_\mathrm{p})$, the satellite state vector of time t can be computed, e.g. as follows:

1. Using (3.32) to compute the mean angular velocity n
2. Using (3.40), (3.39), (3.33) and (3.30) to compute the three anomalies M, E, f and r
3. Using (3.41) and (3.42) to compute the state vector $\vec{q}$ and $\dot{\vec{q}}$ in orbital coordinates
4. Using (3.43) to rotate state vector $\vec{q}$ and $\dot{\vec{q}}$ to the ECSF coordinates

Keplerian elements can be given in practice at any time. For example, with t_0, where only f is a function of t_0, other parameters are constants. In this case, the related E and M can be computed by (3.35) and (3.39), and thus t_p can be computed by (3.40).

From (3.42), one has

$$v^2 = \frac{a^2n^2}{(1 - e\cos E)^2}\left[\sin^2 E + \left(1 - e^2\right)\cos^2 E\right] = \frac{a^2n^2(1 + e\cos E)}{1 - e\cos E}. \tag{3.44}$$

Taking (3.32) and (3.34) into account leads to

$$v^2 = \frac{\mu(1 + e\cos E)}{r} = \frac{\mu(2 - r/a)}{r} = \mu\left(\frac{2}{r} - \frac{1}{a}\right), \tag{3.45}$$

where $v^2/2$ is the kinetic energy scaled by mass, μ/r is the potential energy and a is the semi-major axis of the ellipse. This is the total energy conservative law of mechanics.

Rotate the vector $\vec{q}$ and $\dot{\vec{q}}$ in (3.41) and (3.42) by $R_3(-\omega)$ and denote by $\vec{p}$ and $\dot{\vec{p}}$, i.e.

$$\vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = R_3(-\omega)\begin{pmatrix} r\cos f \\ r\sin f \\ 0 \end{pmatrix} = \begin{pmatrix} r\cos(\omega + f) \\ r\sin(\omega + f) \\ 0 \end{pmatrix}, \tag{3.46}$$

and

$$\dot{\vec{p}} = \begin{pmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \end{pmatrix} = R_3(-\omega) \begin{pmatrix} -\sin f \\ e + \cos f \\ 0 \end{pmatrix} \frac{na}{\sqrt{1-e^2}}$$
$$= \begin{pmatrix} -\sin(\omega+f) - e\sin\omega \\ \cos(\omega+f) + e\cos\omega \\ 0 \end{pmatrix} \frac{na}{\sqrt{1-e^2}}. \quad (3.47)$$

The reverse problem of (3.43), i.e. for given rectangular satellite state vector $(\vec{r}, \dot{\vec{r}})^T$ to compute the Keplerian elements, can be carried out as follows. $\omega + f$ is called argument of latitude and is denoted by u:

1. Using the given state vector to compute the modulus r and v ($r = |\vec{r}|$, $v = |\dot{\vec{r}}|$)
2. Using (3.10) and (3.11) to compute vector $\vec{h}$ and its modulus h
3. Using (3.13) and (3.14) to compute inclination i and the right ascension of ascending node Ω
4. Using (3.45), (3.29), and (3.32) to compute semi-major axis a, eccentricity e and average angular velocity n
5. Rotating $\vec{r}$ by $\vec{p} = R_1(i)R_3(\Omega)\vec{r}$ and then using (3.46) to compute $\omega + f$
6. Rotating $\dot{\vec{r}}$ by $\dot{\vec{p}} = R_1(i)R_3(\Omega)\,\dot{\vec{r}}$ and then using (3.47) to compute ω and f
7. Using (3.33), (3.39) and (3.40) to compute E, M and t_p

To transform the GPS state vector from the ECSF coordinate system to other coordinate systems, the formulas discussed in Chap. 2 can be used.

Chapter 4
Perturbations on the Orbits

Satellites are attracted not only by the central force of the Earth but also by the non-central force of the Earth, the attracting forces of the sun and the moon, and the drag force of the atmosphere. They are also affected by solar radiation pressure, Earth and ocean tides, general relativity effects and coordinate perturbations. Equations of satellite motion have to be represented by perturbed equations. In this chapter, after discussions of the perturbed equations of motion, emphasis is given to the attracting forces and the order estimation of the disturbances.

4.1 Perturbed Equation of Satellite Motion

The perturbed equation of motion of the satellite is described by Newton's second law in an inertial Cartesian coordinate system as

$$m\ddot{\vec{r}} = \vec{f}, \tag{4.1}$$

where $\vec{f}$ is the summated force vector acting on the satellite, and $\vec{r}$ is the radius vector of the satellite with mass m. $\ddot{\vec{r}}$ is the acceleration. Equation 4.1 is a second-order differential equation. For convenience, it can be written as two first-order differential equations as follows:

$$\begin{aligned} \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} &= \dot{\vec{r}} \\ \frac{\mathrm{d}\dot{\vec{r}}}{\mathrm{d}t} &= \frac{1}{m}\vec{f} \end{aligned}. \tag{4.2}$$

G. Xu and J. Xu, *Orbits*, DOI 10.1007/978-3-642-32793-3_4,

Denoting the state vector of the satellite as

$$\vec{X} = \begin{pmatrix} \vec{r} \\ \dot{\vec{r}} \end{pmatrix}, \tag{4.3}$$

(4.2) can be written as

$$\dot{\vec{X}} = \vec{F}, \tag{4.4}$$

where

$$\vec{F} = \begin{pmatrix} \dot{\vec{r}} \\ \vec{f}/m \end{pmatrix}. \tag{4.5}$$

Equation (4.4) is called the state equation of the satellite motion. Integrating (4.4) from t_0 to t, one has

$$\vec{X}(t) = \vec{X}(t_0) + \int_{t_0}^{t} \vec{F}\mathrm{d}t, \tag{4.6}$$

where $\vec{X}(t)$ is the instantaneous state vector of the satellite, $\vec{X}(t_0)$ is the initial state vector at time t_0 and $\vec{F}$ is a function of the state vector $\vec{X}(t)$ and time t. Denoting the initial state vector as $\vec{X}_0$, then the perturbed satellite orbit problem turns out to be a problem of solving the differential state equation under the initial condition as

$$\begin{cases} \dot{\vec{X}}(t) = \vec{F} \\ \vec{X}(t_0) = \vec{X}_0 \end{cases}. \tag{4.7}$$

4.1.1 Lagrangian Perturbed Equation of Satellite Motion

If the force $\vec{f}$ includes only the conservative forces, then there is a potential function V so that

$$\frac{\vec{f}}{m} = \mathrm{grad}V = \begin{pmatrix} \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{pmatrix}, \tag{4.8}$$

where (x, y, z) are Cartesian coordinates. Denoting R as the disturbance potential and V_0 as the potential of the centred force $\vec{f}_0$, then

$$R = V - V_0, \qquad \frac{\vec{f} - \vec{f}_0}{m} = \mathrm{grad}R. \tag{4.9}$$

The perturbed equation of satellite motion (4.2) in Cartesian coordinates is then

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \dot{x} \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= \dot{y} \\
\frac{\mathrm{d}z}{\mathrm{d}t} &= \dot{z} \\
\frac{\mathrm{d}\dot{x}}{\mathrm{d}t} &= -\frac{\mu}{r^3}x + \frac{\partial R}{\partial x} \\
\frac{\mathrm{d}\dot{y}}{\mathrm{d}t} &= -\frac{\mu}{r^3}y + \frac{\partial R}{\partial y} \\
\frac{\mathrm{d}\dot{z}}{\mathrm{d}t} &= -\frac{\mu}{r^3}z + \frac{\partial R}{\partial z},
\end{aligned}
\tag{4.10}
$$

where μ is the gravitational constant of the Earth. The state vector ($\vec{r}$, $\dot{\vec{r}}$) of the satellite corresponds to an instantaneous Keplerian ellipse (a, e, ω, i, Ω, M). Using the relationships between the two sets of parameters (see Chap. 3), the perturbed equation of motion (4.10) can be transformed into a so-called Lagrangian perturbed equation system (see for example Kaula 1966/2001):

$$
\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2}{na}\frac{\partial R}{\partial M} \\
\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{1-e^2}{na^2e}\frac{\partial R}{\partial M} - \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R}{\partial \omega} \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R}{\partial e} - \frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i} \\
\frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\left(\cos i\frac{\partial R}{\partial \omega} - \frac{\partial R}{\partial \Omega}\right) \\
\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i} \\
\frac{\mathrm{d}M}{\mathrm{d}t} &= n - \frac{2}{na}\frac{\partial R}{\partial a} - \frac{1-e^2}{na^2e}\frac{\partial R}{\partial e}.
\end{aligned}
\tag{4.11}
$$

On the basis of this equation system, Kaula derived the first-order perturbed analysis solution (see Kaula 1966/2001). In the case of a small e ($e \ll 1$), the orbit is nearly circular, so that the perigee and the related Keplerian elements f and ω are not defined (this is not to be confused with the force vector $\vec{f}$ and true anomaly f). To overcome this problem, let $u = f + \omega$, and a parameter set of (a, i, Ω, ξ, η, λ) is used to describe the motion of the satellite, where

$$\begin{aligned}\xi &= e\cos\omega\\ \eta &= -e\sin\omega\\ \lambda &= M+\omega.\end{aligned} \tag{4.12}$$

Thus, one has

$$\begin{aligned}\frac{\mathrm{d}\xi}{\mathrm{d}t} &= \frac{\xi}{e}\frac{\mathrm{d}e}{\mathrm{d}t}+\eta\frac{\mathrm{d}\omega}{\mathrm{d}t}\\ \frac{\mathrm{d}\eta}{\mathrm{d}t} &= \frac{\eta}{e}\frac{\mathrm{d}e}{\mathrm{d}t}-\xi\frac{\mathrm{d}\omega}{\mathrm{d}t}\\ \frac{\mathrm{d}\lambda}{\mathrm{d}t} &= \frac{\mathrm{d}M}{\mathrm{d}t}+\frac{\mathrm{d}\omega}{\mathrm{d}t}\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}\frac{\partial R}{\partial\omega} &= \frac{\partial R}{\partial(\xi,\ \eta,\ \lambda)}\frac{\partial(\xi,\ \eta,\ \lambda)}{\partial\omega}=\frac{\partial R}{\partial(\xi,\ \eta,\ \lambda)}(\eta,-\xi,\ 1)^T=\eta\frac{\partial R}{\partial\xi}-\xi\frac{\partial R}{\partial\eta}+\frac{\partial R}{\partial\lambda}\\ \frac{\partial R}{\partial e} &= \frac{\partial R}{\partial(\xi,\ \eta,\ \lambda)}\frac{\partial(\xi,\ \eta,\ \lambda)}{\partial e}=\frac{\partial R}{\partial(\xi,\ \eta,\ \lambda)}\left(\frac{\xi}{e},\ \frac{\eta}{e},\ 0\right)^T=\frac{\xi}{e}\frac{\partial R}{\partial\xi}+\frac{\eta}{e}\frac{\partial R}{\partial\eta}\\ \frac{\partial R}{\partial M} &= \frac{\partial R}{\partial(\xi,\ \eta,\ \lambda)}\frac{\partial(\xi,\ \eta,\ \lambda)}{\partial M}=\frac{\partial R}{\partial(\xi,\ \eta,\ \lambda)}(0,\ 0,\ 1)^T=\frac{\partial R}{\partial\lambda}.\end{aligned} \tag{4.14}$$

Substituting (4.14) into (4.11), and then substituting the second, third and sixth equations of (4.11) into (4.13), one has

$$\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2}{na}\frac{\partial R}{\partial\lambda}\\ \frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\left[\cos i\left(\eta\frac{\partial R}{\partial\xi}-\xi\frac{\partial R}{\partial\eta}+\frac{\partial R}{\partial\lambda}\right)-\frac{\partial R}{\partial\Omega}\right]\\ \frac{\mathrm{d}\Omega}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i}\\ \frac{\mathrm{d}\xi}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{na^2}\frac{\partial R}{\partial\eta}-\eta\frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i}+\xi\frac{1-e^2-\sqrt{1-e^2}}{na^2e^2}\frac{\partial R}{\partial\lambda}\\ \frac{\mathrm{d}\eta}{\mathrm{d}t} &= -\frac{\sqrt{1-e^2}}{na^2}\frac{\partial R}{\partial\xi}+\xi\frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i}+\eta\frac{1-e^2-\sqrt{1-e^2}}{na^2e^2}\frac{\partial R}{\partial\lambda}\\ \frac{\mathrm{d}\lambda}{\mathrm{d}t} &= n-\frac{2}{na}\frac{\partial R}{\partial a}-\frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i}-\frac{1-e^2-\sqrt{1-e^2}}{na^2e^2}\left(\xi\frac{\partial R}{\partial\xi}+\eta\frac{\partial R}{\partial\eta}\right).\end{aligned} \tag{4.15}$$

The new variables of (4.12) do not have clear geometric meanings. An alternative is to use the Hill variables (see for example Cui 1990).

4.1.2 Gaussian Perturbed Equation of Satellite Motion

Considering the non-conservative disturbance forces such as solar radiation and air drag, no potential functions exist for use; therefore, the Lagrangian perturbed equation of motion cannot be used directly in such a case. The equation of motion perturbed by non-conservative disturbance force has to be derived.

Considering any force vector $\vec{f} = (f_x \; f_y \; f_z)^T$ in ECSF coordinate system, one has

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = R_3(-\Omega)R_1(-i)R_3(-u)\begin{pmatrix} f_r \\ f_\alpha \\ f_h \end{pmatrix}, \tag{4.16}$$

where $(f_r \; f_\alpha \; f_h)^T$ is a force vector with three orthogonal components in an orbital plane coordinate system, the first two components are in the orbital plane, f_r is the radial force component, f_α is the force component perpendicular to f_r and points in the direction of satellite motion and f_h completes a right-handed system. For convenience, the force vector may also be represented by tangential, central components in the orbital plane (f_t, f_c) as well as f_h (see Fig. 4.1). It is obvious that

$$\begin{pmatrix} f_r \\ f_\alpha \\ f_h \end{pmatrix} = R_3(-\beta)\begin{pmatrix} f_t \\ f_c \\ f_h \end{pmatrix}, \tag{4.17}$$

where

$$\tan\beta = r\frac{\mathrm{d}f}{\mathrm{d}r} = \frac{a(1-e^2)}{1+e\cos f}\frac{\mathrm{d}f}{\dfrac{a(1-e^2)}{(1+e\cos f)^2}e\sin f\mathrm{d}f} = \frac{1+e\cos f}{e\sin f} \tag{4.18}$$

or

$$\begin{aligned} \sin\beta &= \frac{1+e\cos f}{\sqrt{1+2e\cos f+e^2}} \\ \cos\beta &= \frac{e\sin f}{\sqrt{1+2e\cos f+e^2}}. \end{aligned} \tag{4.19}$$

To replace the partial derivatives $\partial R/\partial\sigma$ by force components, the relationships between them have to be derived, where σ is a symbol for all Keplerian elements. Using the regulation of partial derivatives, one has

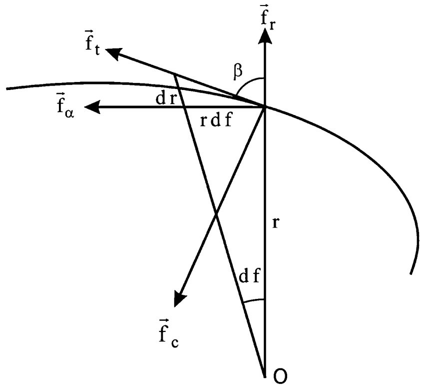


Fig. 4.1 Relation of radial and tangential forces

$$\frac{\partial R}{\partial \sigma} = \frac{\partial R}{\partial \vec{r}} \cdot \frac{\partial \vec{r}}{\partial \sigma} = \vec{f} \cdot \left(\frac{\partial r}{\partial \sigma} \vec{e}_r + r \frac{\partial \vec{e}_r}{\partial \sigma} \right)$$
$$= R_3(-\Omega) R_1(-i) R_3(-u) \begin{pmatrix} f_r \\ f_\alpha \\ f_h \end{pmatrix} \cdot \left(\frac{\partial r}{\partial \sigma} \vec{e}_r + r \frac{\partial \vec{e}_r}{\partial \sigma} \right), \quad (4.20)$$

where $\vec{e}_r$ is the radial unit vector of the satellite, the dot is the vector dot product, and

$$\vec{e}_r = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix} = R_3(-\Omega) R_1(-i) R_3(-u) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\Omega \cos u - \sin\Omega \cos i \sin u \\ \sin\Omega \cos u + \cos\Omega \cos i \sin u \\ \sin i \sin u \end{pmatrix}$$
$$\frac{\partial \vec{e}_r}{\partial \sigma} = \begin{pmatrix} \sin\Omega \sin i \sin u \frac{\partial i}{\partial \sigma} - \varepsilon_2 \frac{\partial \Omega}{\partial \sigma} - (\cos\Omega \sin u + \sin\Omega \cos i \cos u) \frac{\partial u}{\partial \sigma} \\ -\cos\Omega \sin i \sin u \frac{\partial i}{\partial \sigma} + \varepsilon_1 \frac{\partial \Omega}{\partial \sigma} - (\sin\Omega \sin u - \cos\Omega \cos i \cos u) \frac{\partial u}{\partial \sigma} \\ \cos i \sin u \frac{\partial i}{\partial \sigma} + \sin i \cos u \frac{\partial u}{\partial \sigma} \end{pmatrix}. \quad (4.21)$$

Substituting (4.21) into (4.20) and simplifying it, one has

$$\frac{\partial R}{\partial \sigma} = \frac{\partial r}{\partial \sigma} f_r + r \left(\cos i \frac{\partial \Omega}{\partial \sigma} + \frac{\partial u}{\partial \sigma} \right) f_\alpha + r \left(\sin u \frac{\partial i}{\partial \sigma} - \sin i \cos u \frac{\partial \Omega}{\partial \sigma} \right) f_h. \quad (4.22)$$

For deriving the partial derivatives of r and u $(= f + \omega)$ with respect to the six Keplerian elements, the following basic relations (see Chap. 3) are used:

$$\begin{aligned}
&r = \frac{a(1-e^2)}{1+e\cos f} = a(1-e\cos E) \\
&r\cos f = a(\cos E - e) \\
&r\sin f = a\sqrt{1-e^2}\,\sin E \\
&\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \\
&E - e\sin E = M,
\end{aligned} \tag{4.23}$$

where E is a function of (e, M), f is a function of (e, E), i.e. (e, M), r is a function of (a, e, M) and u is a function of (ω, f), i.e. (ω, e, M). Thus

$$\begin{aligned}
&\frac{\partial E}{\partial(e,M)} = \left(\frac{a}{r}\sin E,\ \frac{a}{r}\right) \\
&\frac{\partial f}{\partial(e,M)} = \left(\frac{2+e\cos f}{1-e^2}\sin f,\quad \left(\frac{a}{r}\right)^2\sqrt{1-e^2}\right) \\
&\frac{\partial r}{\partial M} = ae\sin E\frac{\partial E}{\partial M} = \frac{a^2e}{r}\sin E = \frac{ae}{\sqrt{1-e^2}}\sin f \\
&\frac{\partial r}{\partial(a,e,i,\Omega,\omega)} = \left(\frac{r}{a}, -a\cos f,\quad 0,\ 0,\ 0\right) \\
&\frac{\partial u}{\partial e} = \frac{\partial u}{\partial f}\frac{\partial f}{\partial e} = \frac{2+e\cos f}{1-e^2}\sin f \\
&\frac{\partial u}{\partial M} = \frac{\partial u}{\partial f}\frac{\partial f}{\partial M} = \left(\frac{a}{r}\right)^2\sqrt{1-e^2} \\
&\frac{\partial u}{\partial(a,i,\Omega,\omega)} = (0,\ 0,\ 0,\ 1).
\end{aligned} \tag{4.24}$$

Substituting (4.24) into (4.22), one has

$$\begin{aligned}
&\frac{\partial R}{\partial a} = \frac{r}{a}f_r \\
&\frac{\partial R}{\partial e} = -a\cos f\cdot f_r + \frac{r\sin f}{1-e^2}(2+e\cos f)\cdot f_\alpha \\
&\frac{\partial R}{\partial i} = r\sin u\cdot f_h \\
&\frac{\partial R}{\partial \Omega} = i\cos i\cdot f_\alpha - r\sin i\cos u\cdot f_h \\
&\frac{\partial R}{\partial \omega} = r\cdot f_\alpha \\
&\frac{\partial R}{\partial M} = \frac{ae}{\sqrt{1-e^2}}\sin f\cdot f_r + \frac{a(1+e\cos f)}{\sqrt{1-e^2}}\cdot f_\alpha.
\end{aligned} \tag{4.25}$$

Putting (4.25) into Lagrangian perturbed equations of motion (4.11), the so-called Gaussian perturbed equations of motion are then

$$
\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2}{n\sqrt{1-e^2}}[e\sin f\cdot f_r + (1+e\cos f)\cdot f_\alpha] \\
\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{na}[\sin f\cdot f_r + (\cos E+\cos f)\cdot f_\alpha] \\
\frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{(1-e\cos E)\cos u}{na\sqrt{1-e^2}}\cdot f_h \\
\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= \frac{(1-e\cos E)\sin u}{na\sqrt{1-e^2}\sin i}\cdot f_h \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{nae}\left[-\cos f\cdot f_r + \frac{2+e\cos f}{1+e\cos f}\sin f\cdot f_\alpha\right] - \cos i\frac{\mathrm{d}\Omega}{\mathrm{d}t} \\
\frac{\mathrm{d}M}{\mathrm{d}t} &= n - \frac{1-e^2}{nae}\left[-\left(\cos f - \frac{2e}{1+e\cos f}\right)\cdot f_r + \frac{2+e\cos f}{1+e\cos f}\sin f\cdot f_\alpha\right].
\end{aligned} \tag{4.26}
$$

The force components of (f_r, f_α, f_h) are used. Using (4.17), the Gaussian perturbed equations of motion can be represented by a disturbed force vector of (f_t, f_c, f_h).

4.2 Perturbation Forces of Satellite Motion

Perturbation forces of satellite motion will be discussed in this section. They are the gravitational forces of the Earth, the attracting forces of the sun, the moon and the planets, the drag force of the atmosphere, solar radiation pressure, Earth and ocean tides and coordinate perturbations.

4.2.1 Perturbation of the Earth's Gravitational Field

After a brief review of the Earth's gravitational field, the perturbation force of the Earth will be outlined here.

4.2.1.1 The Earth's Gravitational Field

The complete real solution of the Laplace equation is called potential function V of the Earth. In spherical coordinates, V can be expressed by (Moritz 1980; Sigl 1989)

$$
V = \sum_{lmi}\frac{1}{r^{l+1}}V_{lmi} = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{1}{r^{l+1}}P_{lm}(\sin\varphi)[C_{lm}\cos m\lambda + S_{lm}\sin m\lambda], \tag{4.27}
$$

where r is the radius, φ is the latitude and λ is the longitude measured eastward (counter-clockwise looking toward the origin from the positive end of the z-axis). One can, of course, use the co-latitude ϑ (or polar distance) instead of the latitude φ ($\sin\varphi = \cos\vartheta$). The subscript i in the first term denotes the $\cos m\lambda$ or $\sin m\lambda$ term. $P_{lm}(\sin\varphi)$ is the so-called associated Legendre function, V_{lmi} denotes surface spherical harmonics, C_{lm}, S_{lm} are coefficients of the spherical functions, and

$$P_{lm}(\sin\varphi) = \cos^m\varphi \sum_{t=0}^{k} T_{lmt} \sin^{l-m-2t}\varphi, \tag{4.28}$$

where k is the integer part of $(l–m)/2$, and

$$T_{lmt} = \frac{(-1)^t(2l-2t)!}{2^l t!(l-t)!(l-m-2t)!}. \tag{4.29}$$

An important property of surface spherical harmonics V_{lmi} is that they are orthogonal ones. For the integration over the surface of a sphere there is (Heiskanen and Moritz 1967; Kaula 1966/2001)

$$\int_{\text{sphere}} V_{LMI}V_{lmi}\mathrm{d}\sigma = 0, \quad \text{if} \quad L \neq l \quad \text{or} \quad M \neq m \quad \text{or} \quad I \neq i. \tag{4.30}$$

The integral of the square of V_{lmi} for $C_{lm} = 1$ or $S_{lm} = 1$ is

$$\int_{\text{sphere}} V^2{}_{lmi}\mathrm{d}\sigma = \left[\frac{(l+m)!}{(l-m)!(2l+1)(2-\delta_{0m})}\right]4\pi, \tag{4.31}$$

where the Kronecker delta δ_{0m} is equal to 1 for $m = 0$ and 0 for $m \neq 0$.

The normalised Legendre functions can be defined and denoted by

$$\bar{P}_{lm}(x) = P_{lm}(x)\left[\frac{(l-m)!(2l+1)(2-\delta_{0m})}{(l+m)!}\right]^{1/2}, \tag{4.32}$$

where $x = \sin\varphi = \cos\vartheta$. Recurrence formulas can be easily derived (Wenzel 1985):

$$\bar{P}_{(l+1)(l+1)}(x) = \bar{P}_{ll}(x)\left[\frac{(2l+3)}{(l+1)(2-\delta_{0l})}\right]^{1/2}(1-x^2)^{1/2},$$

$$\bar{P}_{(l+1)l}(x) = \bar{P}_{ll}(x)[2l+3]^{1/2}x \qquad l \geq 1,$$

$$\bar{P}_{(l+1)m}(x) = \bar{P}_{lm}(x)\left[\frac{(2l+1)(2l+3)}{(l+m+1)(l-m+1)}\right]^{1/2} x$$
$$- \bar{P}_{(l-1)m}(x)\left[\frac{(l+m)(l-m)(2l+3)}{(l+m+1)(l-m+1)(2l-1)}\right]^{1/2}$$

and

$$\bar{P}_{00}(x) = 1, \qquad \bar{P}_{10}(x) = \sqrt{3}\,x, \qquad \bar{P}_{11}(x) = \sqrt{3(1-x^2)}. \tag{4.33}$$

Since the first term of V (i.e. $l = 0$) is represented by GM/r, the fully normalised geopotential function is taken as follows (Torge 1989; Rapp 1986):

$$V(r,\varphi,\lambda) = \frac{GM}{r}\left[1 + \sum_{l=2}^{\infty}\sum_{m=0}^{l}\left(\frac{a}{r}\right)^{l}\bar{P}_{lm}(\sin\varphi)[\bar{C}_{lm}\cos m\lambda + \bar{S}_{lm}\sin m\lambda]\right], \tag{4.34}$$

where GM is the geocentric gravitational constant, $\bar{C}_{lm}$, $\bar{S}_{lm}$ are normalised coefficients and a is the mean equatorial radius of the Earth. The first term of V is the potential of the central force of the Earth. The perturbation potential of the Earth is then (denoting $GM=\mu$)

$$R_{\text{geo}}(r,\varphi,\lambda) = \frac{\mu}{r}\sum_{l=2}^{\infty}\sum_{m=0}^{l}\left(\frac{a}{r}\right)^{l}\bar{P}_{lm}(\sin\varphi)[\bar{C}_{lm}\cos m\lambda + \bar{S}_{lm}\sin m\lambda]. \tag{4.35}$$

For any initial external potential of the Earth

$$U(r,\varphi,\lambda) = \frac{\mu}{r}\left[1 + \sum_{l=2}^{L}\sum_{m=0}^{l}\left(\frac{a}{r}\right)^{l}\bar{P}_{lm}(\sin\varphi)\left[\bar{C}^{N}_{lm}\cos m\lambda + \bar{S}^{N}_{lm}\sin m\lambda\right]\right], \tag{4.36}$$

the disturbing potential T is then

$$T = V - U = \frac{\mu}{r}\left[\sum_{l=2}^{\infty}\sum_{m=0}^{l}\left(\frac{a}{r}\right)^{l}\bar{P}_{lm}(\sin\varphi)[\Delta\bar{C}_{lm}\cos m\lambda + \Delta\bar{S}_{lm}\sin m\lambda]\right], \tag{4.37}$$

where $\bar{C}^{N}_{lm}$, $\bar{S}^{N}_{lm}$ are known normalised coefficients of the disturbing potential and

$$\bar{C}_{lm} = \Delta\bar{C}_{lm} - \bar{C}^{N}_{lm}, \qquad \bar{S}_{lm} = \Delta\bar{S}_{lm} - \bar{S}^{N}_{lm}, \qquad (l \le L). \tag{4.38}$$

4.2.1.2 Perturbation Force of the Earth's Gravitational Field

Denoting (x', y', z') as three orthogonal Cartesian coordinates in the ECEF system, then the force vector is

$$\vec{f}_{\mathrm{ECEF}} = \begin{pmatrix} \dfrac{\partial V}{\partial x'} \\ \dfrac{\partial V}{\partial y'} \\ \dfrac{\partial V}{\partial z'} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial V}{\partial(r,\varphi,\lambda)} \dfrac{\partial(r,\varphi,\lambda)}{\partial x'} \\ \dfrac{\partial V}{\partial(r,\varphi,\lambda)} \dfrac{\partial(r,\varphi,\lambda)}{\partial y'} \\ \dfrac{\partial V}{\partial(r,\varphi,\lambda)} \dfrac{\partial(r,\varphi,\lambda)}{\partial z'} \end{pmatrix} = \left(\frac{\partial V}{\partial(r,\varphi,\lambda)} \frac{\partial(r,\varphi,\lambda)}{\partial(x',y',z')} \right)^T. \tag{4.39}$$

From the relation between the Cartesian and spherical coordinates

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} r\cos\varphi\cos\lambda \\ r\cos\varphi\sin\lambda \\ r\sin\varphi \end{pmatrix}, \qquad \begin{pmatrix} r = \sqrt{x'^2 + y'^2 + z'^2} \\ \phi = \tan^{-1}\frac{z'}{\sqrt{x'^2+y'^2}} \\ \lambda = \tan^{-1}\frac{y'}{x'} \end{pmatrix}, \tag{4.40}$$

one has

$$\frac{\partial(r,\varphi,\lambda)}{\partial(x',y',z')} = \begin{pmatrix} \cos\varphi\cos\lambda & \cos\varphi\sin\lambda & \sin\varphi \\ -\dfrac{1}{r}\sin\varphi\cos\lambda & -\dfrac{1}{r}\sin\varphi\sin\lambda & \dfrac{1}{r}\cos\varphi \\ -\dfrac{1}{r\cos\varphi}\sin\lambda & \dfrac{1}{r\cos\varphi}\cos\lambda & 0 \end{pmatrix}. \tag{4.41}$$

For differentiations of the associated Legendre function, from (4.33) one has similar recurrence formulas:

$$\frac{\mathrm{d}\bar{P}_{00}(\sin\varphi)}{\mathrm{d}\varphi} = 0, \tag{4.42}$$

$$\frac{\mathrm{d}\bar{P}_{10}(\sin\varphi)}{\mathrm{d}\varphi} = \sqrt{3}\cos\varphi,$$

$$\frac{\mathrm{d}\bar{P}_{11}(\sin\varphi)}{\mathrm{d}\varphi} = -\sqrt{3}\sin\varphi,$$

$$\frac{\mathrm{d}\bar{P}_{(l+1)(l+1)}(\sin\varphi)}{\mathrm{d}\varphi} = -q\sin\varphi\,\bar{P}_{ll}(\sin\varphi) + q\cos\varphi\frac{\mathrm{d}\bar{P}_{ll}(\sin\varphi)}{\mathrm{d}\varphi},$$

$$q = \sqrt{\frac{2l+3}{2l+2}}, \quad l \ge 1,$$

$$\frac{\mathrm{d}\bar{P}_{(l+1)l}(\sin\varphi)}{\mathrm{d}\varphi} = g\cos\varphi\bar{P}_{ll}(\sin\varphi) + g\sin\varphi\frac{\mathrm{d}\bar{P}_{ll}(\sin\varphi)}{\mathrm{d}\varphi}, \quad l \ge 1,$$

$$g = \sqrt{2l+3},$$

$$\frac{\mathrm{d}\bar{P}_{(l+1)m}(\sin\varphi)}{\mathrm{d}\varphi} = h\cos\varphi\bar{P}_{lm}(\sin\varphi) + h\sin\varphi\frac{\mathrm{d}\bar{P}_{lm}(\sin\varphi)}{\mathrm{d}\varphi} - k\frac{\mathrm{d}\bar{P}_{(l-1)m}(\sin\varphi)}{\mathrm{d}\varphi},$$

$$h = \sqrt{\frac{(2l+1)(2l+3)}{(l+m+1)(l-m+1)}},$$

$$k = \sqrt{\frac{(l+m)(l-m)(2l+3)}{(l+m+1)(l-m+1)(2l-1)}}$$

and

$$\frac{\mathrm{d}^2\bar{P}_{00}(\sin\varphi)}{\mathrm{d}\varphi^2} = 0, \tag{4.43}$$

$$\frac{\mathrm{d}^2\bar{P}_{10}(\sin\varphi)}{\mathrm{d}\varphi^2} = -\sqrt{3}\sin\varphi,$$

$$\frac{\mathrm{d}^2\bar{P}_{11}(\sin\varphi)}{\mathrm{d}\varphi^2} = -\sqrt{3}\cos\varphi,$$

$$\frac{\mathrm{d}^2\bar{P}_{(l+1)(l+1)}(\sin\varphi)}{\mathrm{d}\varphi^2} = -q\cos\varphi\bar{P}_{ll}(\sin\varphi) - 2q\sin\varphi\frac{\mathrm{d}\bar{P}_{ll}(\sin\varphi)}{\mathrm{d}\varphi} + q\cos\varphi\frac{\mathrm{d}^2\bar{P}_{ll}(\sin\varphi)}{\mathrm{d}\varphi^2},$$

$$\frac{\mathrm{d}^2\bar{P}_{(l+1)l}(\sin\varphi)}{\mathrm{d}\varphi^2} = -g\sin\varphi\bar{P}_{ll}(\sin\varphi) + 2g\cos\varphi\frac{\mathrm{d}\bar{P}_{ll}(\sin\varphi)}{\mathrm{d}\varphi} + g\sin\varphi\frac{\mathrm{d}^2\bar{P}_{ll}(\sin\varphi)}{\mathrm{d}\varphi^2},$$

$$l \ge 1,$$

$$\begin{aligned}\frac{\mathrm{d}^2\bar{P}_{(l+1)m}(\sin\varphi)}{\mathrm{d}\varphi^2} &= -h\sin\varphi\bar{P}_{lm}(\sin\varphi) + 2h\cos\varphi\frac{\mathrm{d}\bar{P}_{lm}(\sin\varphi)}{\mathrm{d}\varphi} \\ &\quad + h\sin\varphi\frac{\mathrm{d}^2\bar{P}_{lm}(\sin\varphi)}{\mathrm{d}\varphi^2} - k\frac{\mathrm{d}^2\bar{P}_{(l-1)m}(\sin\varphi)}{\mathrm{d}\varphi^2}.\end{aligned}$$

The partial derivatives of the potential function with respect to the spherical coordinates are

$$\frac{\partial V}{\partial r} = -\frac{\mu}{r^2}\left[1 + \sum_{l=2}^{\infty}\sum_{m=0}^{l}(l+1)\left(\frac{a}{r}\right)^l \bar{P}_{lm}(\sin\varphi)[\bar{C}_{lm}\cos m\lambda + \bar{S}_{lm}\sin m\lambda]\right]$$
$$\frac{\partial V}{\partial \varphi} = \frac{\mu}{r}\sum_{l=2}^{\infty}\sum_{m=0}^{l}\left(\frac{a}{r}\right)^l \frac{\mathrm{d}\bar{P}_{lm}(\sin\varphi)}{\mathrm{d}\varphi}[\bar{C}_{lm}\cos m\lambda + \bar{S}_{lm}\sin m\lambda]$$
$$\frac{\partial V}{\partial \lambda} = \frac{\mu}{r}\sum_{l=2}^{\infty}\sum_{m=0}^{l} m\left(\frac{a}{r}\right)^l \bar{P}_{lm}(\sin\varphi)[-\bar{C}_{lm}\sin m\lambda + \bar{S}_{lm}\cos m\lambda]. \quad (4.44)$$

Using the transformation formula of (2.14), the perturbation force of the Earth's gravitational field in the ECSF system is then

$$\vec{f}_{\mathrm{ECSF}} = R_{\mathrm{P}}^{-1}R_{\mathrm{N}}^{-1}R_{\mathrm{S}}^{-1}R_{\mathrm{M}}^{-1}\vec{f}_{\mathrm{ECEF}}. \quad (4.45)$$

The computation process of disturbance force of the Earth's gravitational field in the ECSF coordinate system may be carried out by:

1. Using (2.14) to transform the satellite coordinates in the ECSF system to the ECEF system.
2. Using (4.40) to compute the spherical coordinates of the satellite in the ECEF system.
3. Using (4.39) to compute the force vector in the ECEF system.
4. Using (4.45) to transform the force vector to the ECSF system.

4.2.2 Perturbation of the Sun and the Moon as well as Planets

The equations of motion of two point-masses M and m under their mutual action can be given by

$$M\ddot{\vec{r}}_M = GMm\frac{\vec{r}_{Mm}}{r_{Mm}^3} \quad \text{and} \quad m\ddot{\vec{r}}_m = GMm\frac{\vec{r}_{mM}}{r_{mM}^3}, \quad (4.46)$$

where r is the length of the vector $\vec{r}$, index Mm means the vector is pointing from point-mass M to m and single index M or m means the vector is pointing to point-mass M or m. Introducing additional point-masses $m(j)$, $j = 1,2, \ldots,$ the attractions of $m(j)$ on M and m can be given as equations similar to (4.46), and the total attractions may be obtained by summations

$$\begin{aligned} M\ddot{\vec{r}}_M &= GMm\frac{\vec{r}_{Mm}}{r_{Mm}^3} + \sum_j GMm(j)\frac{\vec{r}_{Mm(j)}}{r_{Mm(j)}^3} \\ m\ddot{\vec{r}}_m &= GMm\frac{\vec{r}_{mM}}{r_{mM}^3} + \sum_j Gmm(j)\frac{\vec{r}_{mm(j)}}{r_{mm(j)}^3}. \end{aligned} \quad (4.47)$$

By dividing these two equations with –M and m, respectively, then adding them together, one has

$$\ddot{\vec{r}}_m - \ddot{\vec{r}}_M = -G(M+m)\frac{\vec{r}_{Mm}}{r_{mM}^3} + \sum_j Gm(j)\left[\frac{\vec{r}_{mm(j)}}{r_{mm(j)}^3} - \frac{\vec{r}_{Mm(j)}}{r_{Mm(j)}^3}\right]. \quad (4.48)$$

Letting $\vec{r} = \vec{r}_m - \vec{r}_M$, i.e. using the point-mass M as the origin, substituting $\vec{r}_{mm(j)} = -(\vec{r}_m - \vec{r}_{m(j)})$ in the right-hand side of (4.48) and omitting the mass m (mass of satellite), one has

$$\ddot{\vec{r}} = -GM\frac{\vec{r}}{r^3} - \sum_j Gm(j)\left[\frac{\vec{r} - \vec{r}_{m(j)}}{\left|\vec{r} - \vec{r}_{m(j)}\right|^3} + \frac{\vec{r}_{m(j)}}{r_{m(j)}^3}\right]. \quad (4.49)$$

It is obvious that the first term on the right-hand side is the central force of the Earth; the disturbance forces of multiple point-masses acting on the satellite are then

$$\vec{f}_m = -m\sum_j Gm(j)\left[\frac{\vec{r} - \vec{r}_{m(j)}}{\left|\vec{r} - \vec{r}_{m(j)}\right|^3} + \frac{\vec{r}_{m(j)}}{r_{m(j)}^3}\right], \quad (4.50)$$

where $Gm(j)$ are the gravitational constants of the sun and the moon as well as the planets. For the term j of the force (4.50) the potential function is then

$$V_j = -m_l\mu_j\left(\frac{-1}{\left|\vec{r}_l - \vec{r}_j\right|} + \frac{r_l\cos\alpha}{r_j^2}\right).$$

The correctness of the above potential function can be verified directly by making gradient operation on it with respect to the coordinates of planet l and comparing the results with the force vector (4.50).

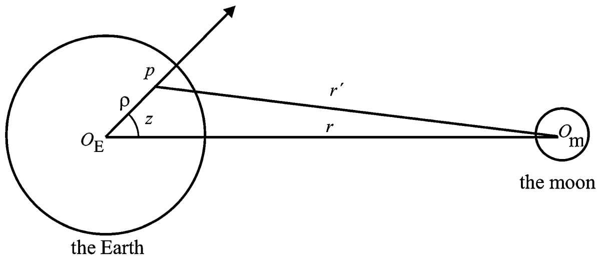
4.2.3 Earth Tide and Ocean Tide Perturbations

The tidal potential generated by the moon and the sun can be written as

$$W_P = \sum_{j=1}^{2}\mu_j\sum_{n=2}^{\infty}\frac{\rho^n}{r_j^{n+1}}P_n(\cos z_j)$$

or

Fig. 4.2 The Earth-moon system



$$W_P = \sum_{j=1}^{2} \mu_j \sum_{n=2}^{\infty} \frac{\rho^n}{r_j^{n+1}} \left[\begin{array}{l} P_n(\sin\varphi)P_n(\sin\delta_j) \\ + 2\sum_{k=1}^{n} \frac{(n-k)!}{(n+k)!} P_{nk}(\sin\varphi)P_{nk}(\sin\delta_j)\cos kh_j \end{array} \right], \tag{4.51}$$

where j is the index of the moon ($j = 1$) and the sun ($j = 2$), μ_j is the gravitational constant of body j, ρ is the geocentric distance of the Earth's surface (set as a_e), r_j is the geocentric distance of the body j, $P_n(x)$ and $P_{nk}(x)$ are the Legendre function and associated Legendre function, z_j is the zenith distance of the body j, δ_j and h_j are the declination and local hour angle of body j, $h_j = H_j - \lambda$ and H_j is the hour angle of j (see Fig. 4.2). The tidal deformation of the Earth caused by the tidal potential can be considered a tidal deformation potential acting on the satellite by Dirichlet's theorem (Melchior 1978; Dow 1988):

$$\delta V = \sum_{j=1}^{2} \mu_j \sum_{n=2}^{\infty} k_n \left(\frac{\rho}{r}\right)^{n+1} \frac{\rho^n}{r_j^{n+1}} P_n(\cos z_j)$$

or

$$\delta V = \sum_{j=1}^{2} \mu_j \sum_{n=2}^{N} k_n \frac{a_e^{2n+1}}{r^{n+1} r_j^{n+1}} \left[\begin{array}{l} P_n(\sin\varphi)P_n(\sin\delta_j) \\ + 2\sum_{k=1}^{n} \frac{(n-k)!}{(n+k)!} P_{nk}(\sin\varphi)P_{nk}(\sin\delta_j)\cos kh_j \end{array} \right], \tag{4.52}$$

where k_n is the Love number, (r, φ, λ) is the spherical coordinate of the satellite in the ECEF system, and N is the truncating number. The recurrence formulas of the Legendre function are (see for example Xu 1992)

$$\begin{aligned} &(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \\ &(1-x^2)\frac{\mathrm{d}P_n(x)}{\mathrm{d}x} = nP_{n-1}(x) - nxP_n(x) \\ &P_0(x) = 1 \quad P_1(x) = x. \end{aligned} \tag{4.53}$$

The disturbing force vector of the tidal potential in the ECEF coordinate system is then

$$\vec{f}_{\mathrm{ECEF}} = \begin{pmatrix} \frac{\partial \delta V}{\partial x'} \\ \frac{\partial \delta V}{\partial y'} \\ \frac{\partial \delta V}{\partial z'} \end{pmatrix} = \begin{pmatrix} \frac{\partial \delta V}{\partial (r,\varphi,\lambda)} \frac{\partial (r,\varphi,\lambda)}{\partial x'} \\ \frac{\partial \delta V}{\partial (r,\varphi,\lambda)} \frac{\partial (r,\varphi,\lambda)}{\partial y'} \\ \frac{\partial \delta V}{\partial (r,\varphi,\lambda)} \frac{\partial (r,\varphi,\lambda)}{\partial z'} \end{pmatrix} = \left(\frac{\partial \delta V}{\partial (r,\varphi,\lambda)} \frac{\partial (r,\varphi,\lambda)}{\partial (x',y',z')} \right)^T, \quad (4.54)$$

where

$$\frac{\partial \delta V}{\partial r} = \sum_{j=1}^{2} \mu_j \sum_{n=2}^{N} -k_n \frac{(n+1)a_{\mathrm{e}}^{2n+1}}{r^{n+2} r_j^{n+1}} \times \left[\begin{array}{l} P_n(\sin\varphi) P_n(\sin\delta_j) \\ +2\sum_{k=1}^{n} \frac{(n-k)!}{(n+k)!} P_{nk}(\sin\varphi) P_{nk}(\sin\delta_j) \cos kh_j \end{array} \right],$$

$$\frac{\partial \delta V}{\partial \varphi} = \sum_{j=1}^{2} \mu_j \sum_{n=2}^{N} k_n \frac{a_{\mathrm{e}}^{2n+1}}{r^{n+1} r_j^{n+1}} \left[\begin{array}{l} \frac{n}{\cos\varphi} (P_{n-1}(\sin\varphi) - \sin\varphi P_n(\sin\varphi)) P_n(\sin\delta_j) \\ +2\sum_{k=1}^{n} \frac{(n-k)!}{(n+k)!} (P_{n(k+1)}(\sin\varphi) - k\tan\varphi P_{nk}(\sin\varphi)) \\ \qquad \cdot P_{nk}(\sin\delta_j) \cos kh_j \end{array} \right]$$

and

$$\frac{\partial \delta V}{\partial \lambda} = \sum_{j=1}^{2} \mu_j \sum_{n=2}^{N} k_n \frac{a_{\mathrm{e}}^{2n+1}}{r^{n+1} r_j^{n+1}} \left[2\sum_{k=1}^{n} \frac{(n-k)!}{(n+k)!} k P_{nk}(\sin\varphi) P_{nk}(\sin\delta_j) \sin kh_j \right]. \quad (4.55)$$

Other partial derivatives in (4.54) have been given in Sect. 4.2.1. The transformation of the force vector from the ECEF to the ECSF coordinate system can be made by (4.45).

The ocean tidal potential generated by tide element $\sigma H \mathrm{d}s$ can be written as

$$\frac{G\sigma H \mathrm{d}s}{r'} \quad or \quad G\sigma H \mathrm{d}s \sum_{n=0}^{\infty} \frac{a_{\mathrm{e}}^{n}}{r^{n+1}} P_n(\cos z), \tag{4.56}$$

where H is the ocean tide height of the area ds, G is the gravitational constant, σ is the water density, r' is the distance between the satellite and the water element ds, r is the geocentric distance of the satellite, z is the zenith distance of the ds and a_{e} is the radius of the Earth. Using the spherical triangle

$$\cos z = \sin\varphi \sin\varphi_s + \cos\varphi \cos\varphi_s \cos(\lambda_s - \lambda),$$

where (φ_s, λ_s) is the spherical coordinate of ds and (r, φ, λ) is the spherical coordinate of satellite in the ECEF system, (4.56) turns out to be (denoted by Q)

$$Q = G\sigma H \mathrm{d}s \sum_{n=0}^{\infty} \frac{a_{\mathrm{e}}^{n}}{r^{n+1}} \left[\begin{array}{l} P_n(\sin\varphi) P_n(\sin\varphi_s) + (2 - \delta_{0n}) \\ \cdot \displaystyle\sum_{k=0}^{n} \frac{(n-k)!}{(n+k)!} P_{nk}(\sin\varphi) P_{nk}(\sin\varphi_s) \cos k(\lambda_s - \lambda) \end{array} \right]. \tag{4.57}$$

The direct ocean tide potential is then the integration of $Q/\mathrm{d}s$ over the ocean (denoted by O), including the potential of the deformation of the ocean loading. The ocean tide potential is then

$$\begin{aligned} \delta V_1 = & \oiint_O G\sigma H \sum_{n=0}^{\infty} (1 + k_n') \frac{a_{\mathrm{e}}^{n}}{r^{n+1}} \\ & \times \left[\begin{array}{l} P_n(\sin\varphi) P_n(\sin\varphi_s) + (2 - \delta_{0n}) \\ \cdot \displaystyle\sum_{k=0}^{n} \frac{(n-k)!}{(n+k)!} P_{nk}(\sin\varphi) P_{nk}(\sin\varphi_s) \cos k(\lambda_s - \lambda) \end{array} \right] \mathrm{d}s\,, \end{aligned} \tag{4.58}$$

where k_n' is the ocean loading Love number. Equation 4.58 does not include the potential changing because of the loading deformation over the continents, which may give a non-negligible contribution to the orbit motion of the satellite (see Knudsen et al. 1999). The loading deformation generated by the ocean tide can be represented as

$$u_r(\varphi, \lambda) = \oiint_{\text{ocean}} \sigma H u(z) \mathrm{d}s \qquad \text{and}$$

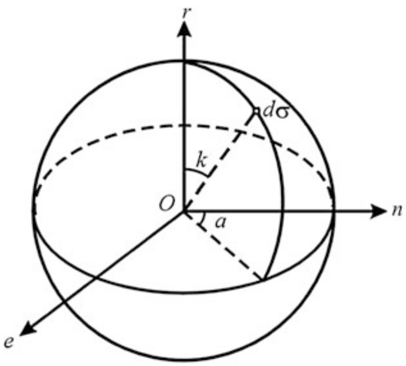


Fig. 4.3 Ocean tide and loading

$$u(z) = \frac{a_e h'_\infty}{2M\sin(z/2)} + \frac{a_e}{M}\sum_{n=0}^{N}(h'_n - h'_\infty)P_n(\cos z), \tag{4.59}$$

where a_e is the radius of the Earth, M is the mass of the Earth, z is the geocentric zenith distance of the loading point (related to the computing point; see Fig. 4.3), $P_n(\cos z)$ is the Legendre function, $u(z)$ is the radial loading displacement Green function, h'_n is the loading Love number of order n and u_r is the radial loading deformation. Substituting u_r for H in (4.57) and integrating $Q/\mathrm{d}s$ over the continents (denoted by C), the potential of the loading deformation is then

$$\delta V_2 = \oiint_C G\sigma_e u_r \sum_{n=0}^{\infty}\frac{a_e^n}{r^{n+1}}\left[\begin{array}{l} P_n(\sin\varphi)P_n(\sin\varphi_s) + (2-\delta_{0n}) \\ \cdot\sum_{k=0}^{n}\frac{(n-k)!}{(n+k)!}P_{nk}(\sin\varphi)P_{nk}(\sin\varphi_s)\cos k(\lambda_s-\lambda)\end{array}\right]\mathrm{d}s, \tag{4.60}$$

where σ_e is the density of the mass $u_r\mathrm{d}s$ on the Earth's surface. The total ocean tide potential disturbance is the summation of (4.58) and (4.60). Similar to (4.54), the disturbing force can be derived and transformed to the ECSF system. There are

$$\vec{f}_{\mathrm{ECEF}} = \begin{pmatrix} \frac{\partial(\delta V_1+\delta V_2)}{\partial x'} \\ \frac{\partial(\delta V_1+\delta V_2)}{\partial y'} \\ \frac{\partial(\delta V_1+\delta V_2)}{\partial z'} \end{pmatrix} = \left(\frac{\partial(\delta V_1+\delta V_2)}{\partial(r,\varphi,\lambda)}\frac{\partial(r,\varphi,\lambda)}{\partial(x',y',z')}\right)^T, \tag{4.61}$$

where

$$\frac{\partial \delta V_1}{\partial r} = \oiint_O G\sigma H \sum_{n=0}^{\infty} (1+k'_n) \frac{-(n+1)a_e^n}{r^{n+2}} \times \left[\begin{array}{l} P_n(\sin\varphi)P_n(\sin\varphi_s) + (2-\delta_{0n}) \\ \times \sum_{k=0}^{n} \frac{(n-k)!}{(n+k)!} P_{nk}(\sin\varphi)P_{nk}(\sin\varphi_s)\cos k(\lambda_s - \lambda) \end{array} \right] \mathrm{d}s,$$

$$\frac{\partial \delta V_1}{\partial \varphi} = \oiint_O G\sigma H \sum_{n=0}^{\infty} (1+k'_n) \frac{a_e^n}{r^{n+1}} \times \left[\begin{array}{l} \frac{\mathrm{d}P_n(\sin\varphi)}{\mathrm{d}\varphi} P_n(\sin\varphi_s) + (2-\delta_{0n}) \\ \times \sum_{k=0}^{n} \frac{(n-k)!}{(n+k)!} \frac{\mathrm{d}P_{nk}(\sin\varphi)}{\mathrm{d}\varphi} P_{nk}(\sin\varphi_s)\cos k(\lambda_s - \lambda) \end{array} \right] \mathrm{d}s,$$

$$\frac{\partial \delta V_1}{\partial \lambda} = \oiint_O G\sigma H \sum_{n=0}^{\infty} (1+k'_n) \frac{a_e^n}{r^{n+1}} \times \left[\begin{array}{l} (2-\delta_{0n}) \\ \times \sum_{k=0}^{n} \frac{(n-k)!}{(n+k)!} P_{nk}(\sin\varphi)P_{nk}(\sin\varphi_s) k \sin k(\lambda_s - \lambda) \end{array} \right] \mathrm{d}s,$$

$$\frac{\partial \delta V_2}{\partial r} = \oiint_C G\sigma_e u_r \sum_{n=0}^{\infty} \frac{-(n+1)a_e^n}{r^{n+2}} \times \left[\begin{array}{l} P_n(\sin\varphi)P_n(\sin\varphi_s) + (2-\delta_{0n}) \\ \times \sum_{k=0}^{n} \frac{(n-k)!}{(n+k)!} P_{nk}(\sin\varphi)P_{nk}(\sin\varphi_s)\cos k(\lambda_s - \lambda) \end{array} \right] \mathrm{d}s,$$

$$\frac{\partial \delta V_2}{\partial \varphi} = \oiint_C G\sigma_e u_r \sum_{n=0}^{\infty} \frac{a_e^n}{r^{n+1}} \times \left[\begin{array}{l} \frac{\mathrm{d}P_n(\sin\varphi)}{\mathrm{d}\varphi} P_n(\sin\varphi_s) + (2-\delta_{0n}) \\ \times \sum_{k=0}^{n} \frac{(n-k)!}{(n+k)!} \frac{\mathrm{d}P_{nk}(\sin\varphi)}{\mathrm{d}\varphi} P_{nk}(\sin\varphi_s)\cos k(\lambda_s - \lambda) \end{array} \right] \mathrm{d}s$$

and

$$\frac{\partial \delta V_2}{\partial \lambda} = \oiint\limits_C G\sigma_e u_r \sum_{n=0}^{\infty} \frac{a_e^n}{r^{n+1}} \times \begin{bmatrix} (2-\delta_{0n}) \\ \times \sum_{k=0}^{n} \frac{(n-k)!}{(n+k)!} P_{nk}(\sin\varphi) P_{nk}(\sin\varphi_s) k \sin k(\lambda_s - \lambda) \end{bmatrix} \mathrm{d}s. \tag{4.62}$$

4.2.4 Solar Radiation Pressure

Solar radiation pressure is sunlight-exerted force acting on the satellite's surface. The radiation force (see for example Seeber 1989/1993) can be represented as

$$\vec{f}_{\text{solar}} = m\gamma P_s C_r r_{\text{sun}}^2 \frac{S}{m} \frac{\vec{r} - \vec{r}_{\text{sun}}}{|\vec{r} - \vec{r}_{\text{sun}}|^3}, \tag{4.63}$$

where γ is the shadow factor, P_s is the luminosity of the sun, C_r is the surface reflectivity, r_{sun} is the geocentric distance of the sun, (S/m) is the surface to mass ratio of the satellite and $\vec{r}$ and $\vec{r}_{\text{sun}}$ are the geocentric vectors of the satellite and the sun. Usually P_s has the value of 4.5605×10^{-6} Nm^{-1} (Newton/meter), C_r has values from 1 to 2, 1 is for the complete absorption of the sunlight, and for aluminium, $C_r = 1.95$.

The shadow factor is defined as

$$\gamma = 1 - \frac{A_{ss}}{A_s}, \tag{4.64}$$

where A_s is the sight surface of the sun viewed from the satellite, and A_{ss} is the shadowed sight surface of the sun. The sunlight may be shadowed by the Earth and the moon. For convenience, we will discuss both parameters that are only in the satellite-Earth-sun system (see Fig. 4.4). It is obvious that the half sight angles of the Earth and the moon, as well as the sun, viewed from the satellite are

$$\alpha_e = \sin^{-1}\left(\frac{a_e}{|\vec{r}|}\right)$$
$$\alpha_m = \sin^{-1}\left(\frac{a_m}{|\vec{r}_m - \vec{r}|}\right)$$
$$\alpha_s = \sin^{-1}\left(\frac{a_s}{|\vec{r}_s - \vec{r}|}\right), \tag{4.65}$$

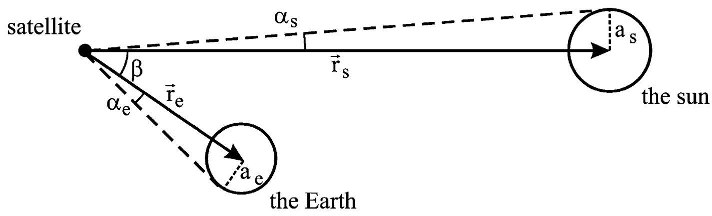


Fig. 4.4 Satellite–Earth–sun system

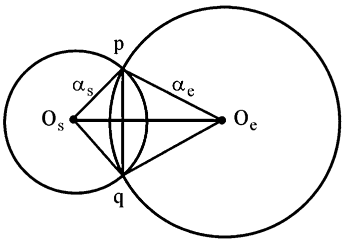


Fig. 4.5 Shadowed surface area

where a_e, a_s and a_m are semi-major radii of the Earth, sun and moon, respectively; $a_m = 0.272493a_e$ and $a_s = 959.63\pi/(3{,}600 \times 180)$ (AU). For the GPS satellite, $\alpha_s < 0.3°, \alpha_e \approx 16.5°$ and $\alpha_m \approx \alpha_s \pm 0.03°$. Furthermore, $A_s = \alpha_s^2\pi$ and $A_m = \alpha_m^2\pi$. The angles between the centre of the Earth and the sun, as well as the centre of the moon and the sun, are

$$\beta_{es} = \cos^{-1}\left(\frac{-\vec{r}\cdot(\vec{r}_s - \vec{r})}{r|\vec{r}_s - \vec{r}|}\right)$$
$$\beta_{ms} = \cos^{-1}\left(\frac{(\vec{r}_m - \vec{r})\cdot(\vec{r}_s - \vec{r})}{|\vec{r}_m - \vec{r}|\cdot|\vec{r}_s - \vec{r}|}\right), \tag{4.66}$$

where the vectors with indices s and m are the geocentric vectors of the sun and moon, respectively. The vector without an index is the geocentric vector of the satellite, and $r = |\vec{r}|$. If $\beta_{es} \geq \alpha_e + \alpha_s$, then the satellite is not in the shadow of the Earth (i.e. $A_{ss} = 0$). If $\beta_{es} \geq \alpha_e - \alpha_s$, then the sun is not in view of the satellite (i.e. $A_{ss} = A_s$). If $\alpha_e - \alpha_s < \beta_{es} < \alpha_e + \alpha_s$, then the sunlight is partly shadowed by the Earth. The formula of the shadowed surface can be derived as follows (see Fig. 4.5). The two circles with radius α_e and α_s cut each other at points p and q, line $\overline{pq}$ is called a chord (denoted by 2a), the chord-related central angle at origin o_s is denoted by ϕ_1 and the surface area between the chord and the arc of the circle α_s on the right-hand side of the chord is denoted by A_1. Line $\overline{pq}$ cuts $\overline{O_sO_e}$ at point g while $\overline{O_sg}$ and $\overline{gO_e}$ are denoted by b and b_1. Then one has

$$a^2 = \alpha_s^2 - b^2, \qquad b_1 = \frac{\alpha_e^2 + \beta_{es}^2 - \alpha_s^2}{2\beta_{es}}$$
$$b = \begin{cases} \beta_{es} - b_1 & \text{if} \quad b_1 \le \alpha_e \\ b_1 - \beta_{es} & \text{if} \quad b_1 > \alpha_e \end{cases}$$
$$\phi_1 = \begin{cases} 2\cos^{-1}\left(\frac{b}{\alpha_s}\right) & \text{if} \quad b_1 \le \alpha_e \\ 2\pi - 2\cos^{-1}\left(\frac{b}{\alpha_s}\right) & \text{if} \quad b_1 > \alpha_e \end{cases}$$
$$A_1 = \begin{cases} \frac{1}{2}\phi_1\alpha_s^2 - ab & \text{if} \quad b_1 \le \alpha_e \\ \frac{1}{2}\phi_1\alpha_s^2 + ab & \text{if} \quad b_1 > \alpha_e \end{cases}. \tag{4.67}$$

Similarly, the chord-related central angle at origin o_e is denoted by ϕ_2, while the surface area between the chord and the arc of the circle α_e on the left side of chord is denoted by A_2. Then one has

$$\phi_2 = 2\cos^{-1}\left(\frac{b_1}{\alpha_e}\right), \qquad A_2 = \frac{1}{2}\phi_2\alpha_e^2 - ab_1 \tag{4.68}$$

and

$$\gamma = 1 - \frac{A_1 + A_2}{\alpha_s^2\pi}. \tag{4.69}$$

A similar discussion applies for the moon. If $\beta_{ms} \ge \alpha_m + \alpha_s$, then the satellite is not in the shadow of the moon, i.e. $A_{ss} = 0$. If $\beta_{ms} \ge \alpha_m - \alpha_s$, then the full shadow has occurred, i.e. $A_{ss} = \min(A_s, A_m)$. If $|\alpha_m - \alpha_s| < \beta_{ms} < \alpha_m + \alpha_s$, then the sunlight is partially shadowed by the moon. The formula of the shadowed surface can be similarly derived by changing the index e to m in (4.67) and (4.68). Because of the small sight angle of the moon viewed from the satellite, the shadowed time will be very short if it happens. By GPS satellite dynamic orbit determination (e.g. in IGS orbit determination), only the data that have the γ value of 0 or 1 are used.

Because of the complex shape of the satellite and the use of constant reflectivity and homogenous luminosity of the sun, as well as the existence of indirect solar radiation (reflected from the Earth's surface), the model of (4.63) discussed earlier is not accurate enough and will be used as a first-order approximation. A further model for the adjustment to fit solar radiation effects is needed.

The force vector points from the sun to the satellite. The satellite fixed coordinate system is introduced in Sect. 2.7 (see that section for details). The solar radiation force vector in the ECSF system is then

$$\vec{f}_{\text{solar}} = m\gamma P_s C_r \frac{S}{m} \frac{r_{\text{sun}}^2}{|\vec{r} - \vec{r}_{\text{sun}}|^2} \vec{n}_{\text{sun}}$$
$$= m\gamma P_s C_r \frac{S}{m} \frac{r_{\text{sun}}^2}{|\vec{r} - \vec{r}_{\text{sun}}|^2} (\sin\beta \cdot \vec{e}_x + \cos\beta \cdot \vec{e}_z), \tag{4.70}$$

where

$$\vec{e}_z = -\frac{\vec{r}}{|\vec{r}|}, \quad \vec{e}_y = \frac{\vec{e}_z \times \vec{n}_{\text{sun}}}{|\vec{e}_z \times \vec{n}_{\text{sun}}|}, \quad \vec{e}_x = \vec{e}_y \times \vec{e}_z \quad \text{and}$$
$$\vec{n}_{\text{sun}} = \frac{\vec{r} - \vec{r}_{\text{sun}}}{|\vec{r} - \vec{r}_{\text{sun}}|}. \tag{4.71}$$

Further formulas of (4.71) can be found in Sect. 2.7. Taking the remaining error of the radiation pressure into account, the solar radiation force model can be represented as (see Fliegel et al. 1992; Beutler et al. 1994)

$$\vec{f}_{\text{solar-force}} = \vec{f}_{\text{solar}} + \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 \\ \cos u \\ \sin u \end{pmatrix}. \tag{4.72}$$

That is, nine parameters are used to model the solar radiation force error for every satellite.

An alternative adjustment model of solar radiation is given by introducing a so-called disturbance coordinate system and will be outlined in the next section (see Xu 2004).

4.2.4.1 Disturbance Coordinate System and Radiation Error Model

The solar radiation force vector points from the sun to the satellite. If the shadow factor is computed exactly, the luminosity of the sun is a constant, and the surface reflectivity of the satellite is a constant; then the length of the solar force vector can also be considered a constant, because

$$\frac{r_{sun}^2}{(r_{sun} + r)^2} \le \frac{r_{\text{sun}}^2}{|\vec{r} - \vec{r}_{\text{sun}}|^2} \le \frac{r_{sun}^2}{(r_{sun} - r)^2}, \tag{4.73}$$

and for GPS satellites

$$\frac{r_{sun}^2}{(r_{sun} \pm r)^2} = \left(\frac{r_{sun}}{r_{sun} \pm r}\right)^2 \approx \left(1 \mp \frac{r}{r_{sun}} \pm \ldots\right)^2 \approx 1 \mp \frac{2r}{r_{sun}} \approx 1 \mp 3.5 \times 10^{-4}.$$

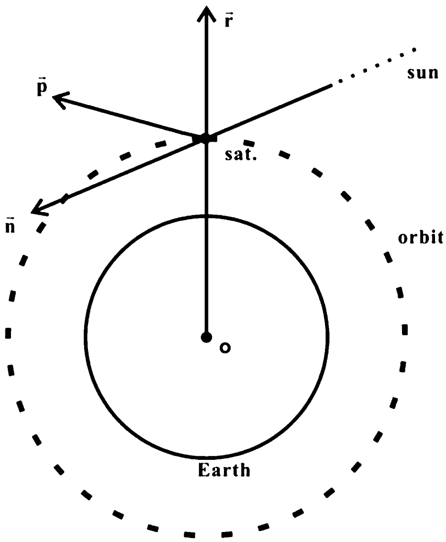


Fig. 4.6 Disturbance coordinate system

Any bias error in P_s, C_r and (S/m) may cause a model error of $\alpha\vec{f}_{solar}$, where α is a parameter. So the $\alpha\vec{f}_{solar}$ can be considered a main error model of the solar radiation. Because the ratio of the geocentric distances of the satellite and the sun is so small, the direction and distance changes of the sun-satellite vector are negligible. With the motion of the sun, the solar radiation force vector changes its direction with the time in the ECSF (Earth-Centred-Space-Fixed) coordinate system ca. 1° per day. Such an effect can only be considered a small drift, not a periodical change for orbit determination. To model such an effect in the ECSF system one needs three bias parameters in three coordinate axes and three drift terms instead of a few periodical parameters. It is obvious that to model such an effect in the direction of $\vec{n}$, just one parameter α is needed. Therefore, it is very advantageous to define a so-called disturbance coordinate system as follows cf. Fig. 4.6): the origin is the mass-centre of the satellite, and the three axes are defined by $\vec{r}$ (radial vector of the satellite), $\vec{n}$ (the sun-satellite unit vector) and $\vec{p}$ (the atmospheric drag unit vector). These three axes are always in the main disturbance directions of the indirect solar radiation (reflected from the Earth's surface), direct solar radiation and atmospheric drag, respectively. This coordinate system is not a Cartesian one and the axes are not orthogonal to each other. The parameters in individual axes are mainly used to model the related disturbance effects, and meanwhile to absorb the remained error of other un-modelled effects.

In the so-called disturbance coordinate system, the solar radiation pressure error model can be represented alternatively by (see Xu 2004)

$$\alpha\vec{f}_{solar} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} \begin{pmatrix} 1 \\ t \end{pmatrix}, \tag{4.74}$$

where b-terms are very small.

4.2.5 Atmospheric Drag

Atmospheric drag is the disturbance force acting on the satellite's surface caused by the air. Air drag force can be represented as (see for example Seeber 1993; Liu and Zhao 1979)

$$\vec{f}_{\text{drag}} = -m\frac{1}{2}\left(\frac{C_{\text{d}}S}{m}\right)\sigma\left|\dot{\vec{r}} - \dot{\vec{r}}_{\text{air}}\right|\left(\dot{\vec{r}} - \dot{\vec{r}}_{\text{air}}\right), \tag{4.75}$$

where S is the cross section (or effective area) of the satellite, C_{d} is the drag factor, m is the mass of the satellite, $\dot{\vec{r}}$ and $\dot{\vec{r}}_{\text{air}}$ are the geocentric velocity vectors of the satellite and the atmosphere and σ is the density of the atmosphere. Usually S has a value of 1/4 of the outer surface area of the satellite, and C_{d} has laboratory values of 2.2 ± 0.2. The velocity vector of the atmosphere can be modelled by

$$\dot{\vec{r}}_{\text{air}} = k\vec{\omega} \times \vec{r} = k\omega\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}, \tag{4.76}$$

where $\vec{\omega}$ is the angular velocity vector of the Earth's rotation, and $\omega=|\vec{\omega}|$, k is the atmospheric rotation factor. For the lower layer of the atmosphere, $k = 1$, i.e. the lower layer of the atmosphere is considered to be rotating with the Earth. For the higher layer, $k = 1.2$, because the higher ionosphere is accelerated by the Earth's magnetic field.

The gravity-balanced atmospheric-density model has the exponential form of (see Liu and Zhao 1979)

$$\sigma = \sigma_0(1 + q)\exp\left(-\frac{r - \rho}{H}\right), \tag{4.77}$$

where σ_0 is the atmospheric density at the reference point ρ, q is the daily change factor of the density, r is the geocentric distance of the satellite and H is the density-height scale factor. For the spherical and rotating ellipsoidal layer atmospheric models, one has

$$\rho = a_{\text{e}} + h_i \tag{4.78}$$

and

$$\rho = (a_{\text{e}} + h_i)\sqrt{1 - e^2}\sqrt{\frac{1 + \tan^2\varphi}{1 + \tan^2\varphi - e^2}}, \tag{4.79}$$

Table 4.1 Reference of atmospheric densities

h_i (km)	$\sigma_0(i)$ (g km^{-3})	h_i (km)	$\sigma_0(i)$ (g km^{-3})
100	497,400	600	0.08–0.64
200	255–316	700	0.02–0.22
300	17–35	800	0.07–0.01
400	2.2–7.5	900	0.003–0.04
500	0.4–2.0	1,000	0.001–0.02

respectively, where a_e is the semi-major radius of the Earth, h_i ($i = 1,2, \ldots$) is a set of numbers, φ is the geocentric latitude of the satellite and e is the eccentricity of the ellipsoid. Equations (4.78) and (4.79) represent a sphere with radius $a_e + h_i$ and rotating ellipsoid with semi-major axis $a_e + h_i$. Equation (4.79) can be derived from the relation of tanφ and the ellipsoid equation

$$z^2 = (x^2 + y^2)\tan^2\varphi$$
$$x^2 + y^2 + z^2 \frac{1}{1-e^2} = (a_e + h_i)^2.$$

A reference of atmospheric densities can be read from Table 4.1, which is given by Cappellari et al. (1976) (see Seeber 1989/1993).

The density–height scale H between every two layers can be computed from these values. It is notable that the air density may change its value up to a factor of 10 because of the radiation of the sun. The density of the atmosphere at a defined point reaches its maximum value at 14 h local time and its minimum at 3.5 h. The most significant period of change is the daily change and is represented by the daily changing factor as

$$q = \frac{f-1}{f+1}\cos\psi, \tag{4.80}$$

where f is the ratio of the maximum density and the minimum density and ψ is the angle between the satellite vector $\vec{r}$ and the daily maximum density direction $\vec{r}_m$. The f may have the value of 3 and

$$\cos\psi = \frac{\vec{r}\cdot\vec{r}_m}{|\vec{r}|\cdot|\vec{r}_m|}, \tag{4.81}$$

where

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\text{sun}} = \begin{pmatrix} r\cos\delta\cos\alpha \\ r\cos\delta\sin\alpha \\ r\sin\delta \end{pmatrix}, \quad \begin{pmatrix} r = \sqrt{x^2+y^2+z^2} \\ \delta = \tan^{-1}\frac{z}{\sqrt{x^2+y^2}} \\ \alpha = \tan^{-1}\frac{y}{x} \end{pmatrix}$$
$$\vec{r}_m = \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\text{m}} = \begin{pmatrix} r\cos\delta\cos(\alpha+\pi/6) \\ r\cos\delta\sin(\alpha+\pi/6) \\ r\sin\delta \end{pmatrix}, \tag{4.82}$$

where (α,δ) are the coordinates (right ascension, latitude) of the sun in the ECSF coordinate system.

Taking the remaining error of the atmospheric drag into account, the air drag force model can be represented as

$$\vec{f}_{\text{air-drag}} = \vec{f}_{\text{drag}} + (1+q)\Delta\vec{f}_{\text{drag}}, \tag{4.83}$$

where the force error vector is denoted by $\Delta\vec{f}_{\text{drag}}$ and the time variation part of atmospheric density is considered in parameter q.

4.2.5.1 Error Model in Disturbance Coordinate System

In the atmospheric drag model (4.75), the velocity vector of the atmosphere is always perpendicular to the z-axis of the ECSF coordinates and the satellite velocity vector is always in the tangential direction of the orbit. The variation of the term $\left|\dot{\vec{r}} - \dot{\vec{r}}_{\text{air}}\right|$ (denoted by g) is dominated by the direction changes of the velocity vectors of the satellite and the atmosphere. Any bias error in S (effective area of the satellite), C_{d} (drag factor) and σ (density of the atmosphere) may cause a model error of $\mu\vec{f}_{drag}$, where μ is a parameter. So the $\mu\vec{f}_{drag}$ can be considered a main error model of the unmodelled atmospheric drag. To simplify our discussion, we consider the velocities of the satellite and atmosphere are constants, and call the satellite positions with max(z) and −max(z), the highest and lowest points, respectively. With the satellite at the lowest point, the two velocity vectors are in the same direction and therefore g reaches the minimum. At the ascending node, the two vectors have the maximum angle of inclination i and g reaches the maximum. Then g reaches the minimum again at the highest point and reaches the maximum again at the descending node, and at the end reaches the minimum at the lowest point. It is obvious that, besides the constant part, g has a dominant periodical component of cos2f and sin2f, where f is the true anomaly of the satellite.

In the so-called disturbance coordinate system the atmospheric drag error model can be represented alternatively by (see Xu 2004)

$$\mu\vec{f}_{drag} = [a + b\varphi(2\omega)\cos(2f) + c\varphi(3\omega)\cos(3f) + d\varphi(\omega)\cos f]\,\vec{p}, \tag{4.84}$$

where

$$\varphi(k\omega) = \begin{cases} \sin k\omega & if\ \cos k\omega = 0 \\ \frac{1}{\cos k\omega} & if\ \cos k\omega \neq 0 \end{cases}, \quad k = 1,\ 2,\ 3 \tag{4.85}$$

where ω is the angle of perigee and f is the true anomaly of the satellite; a, b, c and d are model parameters to be determined. According to the simulation, a-term and

b-term are the most significant terms. The amount of d is just about 1 % of that of c, and the amount of c is about 1 % that of b.

4.2.6 Additional Perturbations

As mentioned earlier, the disturbed equation of motion of the satellite is valid only in an inertial coordinate system, or ECSF system. Therefore, the state vector and force vectors as well as the disturbing potential function have also to be represented in the ECSF system. As seen earlier, for some reason the state vector and the force vectors as well as the disturbing potential function R, are sometimes given in the ECEF system and then transformed to the ECSF system by (see Sect. 4.2.4)

$$\begin{aligned} \vec{X}_{\text{ECSF}} &= R_{\text{t}} \cdot \vec{X}_{\text{ECEF}} \\ \vec{f}_{\text{ECSF}} &= R_{\text{t}} \cdot \vec{f}_{\text{ECEF}} \\ R_{\text{ECSF}} &= R(R_{\text{t}}^{-1} X_{\text{ECSF}}) \quad \text{for} \quad R(X_{\text{ECEF}}), \end{aligned} \tag{4.86}$$

where R_{t} is the transformation matrix in general. Variable transformation is further denoted by $X_{\text{ECSF}} = R_{\text{t}} X_{\text{ECEF}}$. We have also seen that sometimes the state vectors (of the satellite, the sun, the moon) in the ECSF system have to be transformed to the ECEF system for use, and then the result vectors will be transformed back to the ECSF system again. However, due to the complication of transformation R_{t}^{-1}, quite often a simplified R_{s}^{-1} is used (in later discussions, for example, to represent the disturbing potential function using Keplerian elements, only the Earth rotation is considered). Thus

$$R_{\text{ECSF}} = \{R(R_{\text{t}}^{-1} X_{\text{ECSF}}) - R(R_{\text{s}}^{-1} X_{\text{ECSF}})\} + R(R_{\text{s}}^{-1} X_{\text{ECSF}}), \tag{4.87}$$

where the first term on the right-hand side is the correction because of the approximation using the second term. The transformations of (4.86) and (4.87) are exact operations, and their differentiation with respect to time t and the partial derivatives with respect to variable X_{ECSF} are then

$$\begin{aligned} \frac{\text{d}\vec{X}_{\text{ECSF}}}{\text{d}t} &= \frac{\text{d}R_{\text{t}}}{\text{d}t} \vec{X}_{\text{ECEF}} + R_{\text{t}} \frac{\text{d}\vec{X}_{\text{ECEF}}}{\text{d}t} \\ \frac{\text{d}\vec{f}_{\text{ECSF}}}{\text{d}t} &= \frac{\text{d}R_{\text{t}}}{\text{d}t} \vec{f}_{\text{ECEF}} + R_{\text{t}} \frac{\text{d}\vec{f}_{\text{ECEF}}}{\text{d}t} \\ \frac{\partial R_{\text{ECSF}}}{\partial X_{\text{ECSF}}} &= \frac{\partial \left[R(R_{\text{t}}^{-1} X_{\text{ECSF}}) - R(R_{\text{s}}^{-1} X_{\text{ECSF}})\right]}{\partial X_{\text{ECSF}}} + \frac{\partial R(R_{\text{s}}^{-1} X_{\text{ECSF}})}{\partial X_{\text{ECSF}}}. \end{aligned} \tag{4.88}$$

That is, the time differentiations of the state vector and force vectors cannot be transformed directly as in (4.86). In other words, if the state vector and force vectors are not directly given in the ECSF system, they are not allowed to be differentiated as usual afterward. An approximated and transformed perturbing potential function will introduce an error. The first term on the right-hand side of (4.88) signifies additional perturbations (i.e. coordinate perturbations). The order of such perturbations can be estimated by the first term on the right-hand side. If the relationship between two coordinate systems changes with time or the transformation has not been made exact, such perturbations will occur. Recalling

$$R = R_{\mathrm{P}}^{-1} R_{\mathrm{N}}^{-1} R_{\mathrm{S}}^{-1} R_{\mathrm{M}}^{-1}$$

and their definitions (see Chap. 2), one has

$$\begin{aligned}\frac{\mathrm{d}R}{\mathrm{d}t} &= R_{\mathrm{P}}^{-1} R_{\mathrm{N}}^{-1} R_{\mathrm{S}}^{-1} \frac{\mathrm{d}R_{\mathrm{M}}^{-1}}{\mathrm{d}t} + R_{\mathrm{P}}^{-1} R_{\mathrm{N}}^{-1} \frac{\mathrm{d}R_{\mathrm{S}}^{-1}}{\mathrm{d}t} R_{\mathrm{M}}^{-1} \\ &\quad + R_{\mathrm{P}}^{-1} \frac{\mathrm{d}R_{\mathrm{N}}^{-1}}{\mathrm{d}t} R_{\mathrm{S}}^{-1} R_{\mathrm{M}}^{-1} + \frac{\mathrm{d}R_{\mathrm{P}}^{-1}}{\mathrm{d}t} R_{\mathrm{N}}^{-1} R_{\mathrm{S}}^{-1} R_{\mathrm{M}}^{-1},\end{aligned} \tag{4.89}$$

where

$$\begin{aligned}\frac{\mathrm{d}R_{\mathrm{M}}^{-1}}{\mathrm{d}t} &= \begin{pmatrix} 0 & 0 & -\dot{x}_p \\ 0 & 0 & \dot{y}_p \\ \dot{x}_p & -\dot{y}_p & 0 \end{pmatrix}, \qquad \frac{\mathrm{d}R_{\mathrm{S}}^{-1}}{\mathrm{d}t} = \frac{\mathrm{d}R_3(\mathrm{GAST})}{\mathrm{d}t} \\ \frac{\mathrm{d}R_{\mathrm{N}}^{-1}}{\mathrm{d}t} &= \frac{\mathrm{d}R_1(-\varepsilon)}{\mathrm{d}t} R_3(\Delta\psi) R_1(\varepsilon+\Delta\varepsilon) + R_1(-\varepsilon) \frac{\mathrm{d}R_3(\Delta\psi)}{\mathrm{d}t} R_1(\varepsilon+\Delta\varepsilon) \\ &\quad + R_1(-\varepsilon) R_3(\Delta\psi) \frac{\mathrm{d}R_1(\varepsilon+\Delta\varepsilon)}{\mathrm{d}t} \\ \frac{\mathrm{d}R_{\mathrm{P}}^{-1}}{\mathrm{d}t} &= \frac{\mathrm{d}R_3(\zeta)}{\mathrm{d}t} R_2(-\theta) R_3(z) + R_3(\zeta) \frac{\mathrm{d}R_2(-\theta)}{\mathrm{d}t} R_3(z) + R_3(\zeta) R_2(-\theta) \frac{\mathrm{d}R_3(z)}{\mathrm{d}t},\end{aligned} \tag{4.90}$$

where all elements are defined and given in Chap. 2, $(\dot{x}_p, \dot{y}_p)$ is the polar motion rate of time, and

$$\begin{aligned}\frac{\mathrm{d}R_1(\alpha)}{\mathrm{d}t} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin\alpha & \cos\alpha \\ 0 & -\cos\alpha & -\sin\alpha \end{pmatrix} \frac{\mathrm{d}\alpha}{\mathrm{d}t} \\ \frac{\mathrm{d}R_2(\alpha)}{\mathrm{d}t} &= \begin{pmatrix} -\sin\alpha & 0 & -\cos\alpha \\ 0 & 0 & 0 \\ \cos\alpha & 0 & -\sin\alpha \end{pmatrix} \frac{\mathrm{d}\alpha}{\mathrm{d}t} \\ \frac{\mathrm{d}R_3(\alpha)}{\mathrm{d}t} &= \begin{pmatrix} -\sin\alpha & \cos\alpha & 0 \\ -\cos\alpha & -\sin\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\mathrm{d}\alpha}{\mathrm{d}t}.\end{aligned} \tag{4.91}$$

Further formulas may be easily derived.

Table 4.2 Accelerations ($m\ s^{-2}$) caused by forces (see Seeber 1993; Kang 1998)

Central force acceleration	0.56
Gravitational C_2 acceleration	5×10^{-5}
Other gravitational acceleration	3×10^{-7}
The moon's central force acceleration	5×10^{-6}
The sun's central force acceleration	2×10^{-6}
Planets' central force acceleration	3×10^{-10}
The Earth's tidal acceleration	2×10^{-9}
Ocean's tidal acceleration	5×10^{-10}
Solar pressure acceleration	1×10^{-7}
Atmosphere drag acceleration (Topex)	4×10^{-10}
General relativity acceleration	3×10^{-10}

4.2.7 Order Estimations of Perturbations

Perturbation forces that are scaled by the mass of the satellite are the accelerations. The accelerations caused by the discussed forces have been estimated for the GPS satellite by several authors and are summarised in Table 4.2.

If the coordinate system is used without taking precession and nutation into account, additional perturbation acceleration can reach up to 3×10^{-10} ($m\ s^{-2}$). Additional acceleration of gravitational potential can reach up to 1×10^{-9} (see Liu and Zhao 1979).

Chapter 5
Solutions of $\overline{C}_{20}$ Perturbation

Satellites are attracted not only by the central force of the Earth, but also by the non-central force of the Earth, the attracting forces of the sun and the moon as well as planets, and the drag force of the atmosphere, solar radiation pressure, Earth and ocean tides, and coordinate perturbations (see Chap. 4). Equations of satellite motion have to be represented by perturbed equations. In this chapter, emphasis is given to the analytical solution of the $\overline{C}_{20}$ perturbation. Orbit correction is discussed based on the solution.

5.1 $\overline{C}_{20}$ Perturbed Equations of Motion

The geopotential term $\overline{C}_{20}$ is a zonal term. Compared with other geopotential terms, $\overline{C}_{20}$ has a value that is at least 100 times larger. According to the order estimation discussed in Sect. 4.2.7, $\overline{C}_{20}$ perturbation is one of the most significant disturbing factors and is a perturbation of the first order. The analytic solution of the $\overline{C}_{20}$ perturbation will give a clear insight into orbit disturbance. The related perturbing potential is (see the section *The Earth's Gravitational Field* in Chap. 4)

$$R_2 = \frac{\mu a_e^2}{r^3}\overline{C}_{20}\overline{P}_{20}(\sin\varphi)$$

or

$$R_2 = \frac{b}{r^3}(3\sin^2\varphi - 1), \tag{5.1}$$

G. Xu and J. Xu, *Orbits*, DOI 10.1007/978-3-642-32793-3_5,

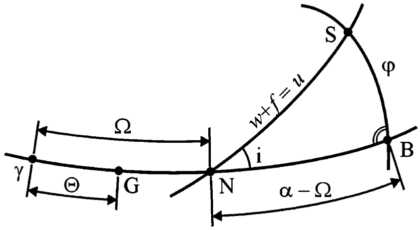


Fig. 5.1 Orbit-equator-meridian triangle

where

$$b = \frac{\sqrt{5}\mu a_e^2}{2}\overline{C}_{20}.$$

The variables (r, φ, λ) of the geopotential disturbance function in the ECEF system are transformed into orbital elements in the ECSF system by using the following relations (Fig. 5.1; see Kaula 1966/2001):

$$\begin{aligned} \sin\varphi &= \sin i \sin u \\ \lambda &= \alpha - \Theta = \Omega - \Theta + (\alpha - \Omega) \\ \cos(\alpha - \Omega) &= \frac{\cos u}{\cos\varphi} \\ \sin(\alpha - \Omega) &= \frac{\sin u \cos i}{\cos\varphi}. \end{aligned} \tag{5.2}$$

where α is the right ascension of the satellite, $u = \omega + f$, Θ is the Greenwich Sidereal Time and other parameters are Keplerian elements. It is obvious that such a coordinate transformation only takes the Earth's rotation into account; this will cause a coordinate perturbation (see Sect. 4.2.6). However, such an effect can be neglected by the first order solution. Substituting the first formula of (5.2) into (5.1) and taking the trigonometric formula (for reducing the order) into account, one has

$$R_2 = \frac{b}{r^3}\left[\frac{3}{2}\sin^2 i(1 - \cos 2u) - 1\right], \tag{5.3}$$

where

$$r = \frac{a(1 - e^2)}{1 + e\cos f}, \tag{5.4}$$

where Ω does not appear in the zonal disturbance. Taking into account the partial derivatives of f with respect to (M, e) and r with respect to (a, M, e) (see Sect. 4.1), the derivatives of R_2 with respect to Keplerian elements are then

$$\frac{\partial R_2}{\partial a} = \frac{\partial R_2}{\partial r}\frac{\partial r}{\partial a} = \frac{-3}{a}R_2, \qquad \frac{\partial R_2}{\partial \Omega} = 0,$$

$$\frac{\partial R_2}{\partial i} = \frac{b}{r^3}\left[\frac{3}{2}\sin 2i(1-\cos 2u)\right],$$

$$\frac{\partial R_2}{\partial \omega} = \frac{b}{r^3}\left[3\sin^2 i \sin 2u \frac{\partial u}{\partial \omega}\right] = \frac{3b}{r^3}\sin^2 i \sin 2u,$$

$$\begin{aligned}\frac{\partial R_2}{\partial e} &= \frac{-3R_2}{r}\frac{\partial r}{\partial e} + \frac{b}{r^3}\left[3\sin^2 i \sin 2u \frac{\partial u}{\partial e}\right] \\ &= \frac{3a\cos f}{r}R_2 + \frac{b}{r^3}\left[3\sin^2 i \sin 2u \frac{2+e\cos f}{1-e^2}\sin f\right]\end{aligned}$$

and

$$\begin{aligned}\frac{\partial R_2}{\partial M} &= \frac{-3R_2}{r}\frac{\partial r}{\partial M} + \frac{b}{r^3}\left[3\sin^2 i \sin 2u \frac{\partial u}{\partial M}\right] \\ &= \frac{-3ae\sin f}{r\sqrt{1-e^2}}R_2 + \frac{b}{r^3}\left[3\sin^2 i \sin 2u \left(\frac{a}{r}\right)^2\sqrt{1-e^2}\right].\end{aligned} \tag{5.5}$$

Substituting these derivatives and R_2 into the equation of motion (4.11), one has the $\overline{C}_{20}$ perturbed equations of motion:

$$\frac{da}{dt} = \frac{6b\sqrt{1-e^2}}{na^4}\left\{\frac{-e}{(1-e^2)}\frac{a^4}{r^4}\sin f\left[\frac{3}{2}\sin^2 i(1-\cos 2u)-1\right] + \frac{a^5}{r^5}\left[\sin^2 i \sin 2u\right]\right\},$$

$$\begin{aligned}\frac{de}{dt} =& \frac{3b(1-e^2)^{3/2}}{na^5 e}\left\{\frac{-e}{(1-e^2)}\frac{a^4}{r^4}\sin f\left[\frac{3}{2}\sin^2 i(1-\cos 2u)-1\right] + \frac{a^5}{r^5}\left[\sin^2 i \sin 2u\right]\right\} \\ &- \frac{3b\sqrt{1-e^2}}{na^5 e}\frac{a^3}{r^3}\sin^2 i \sin 2u,\end{aligned}$$

$$\begin{aligned}\frac{d\omega}{dt} =& \frac{3b\sqrt{1-e^2}}{na^5 e}\left\{\frac{a^4}{r^4}\cos f\left[\frac{3}{2}\sin^2 i(1-\cos 2u)-1\right] + \frac{a^3}{r^3}\left[\sin^2 i \sin 2u \frac{2+e\cos f}{1-e^2}\sin f\right]\right\} \\ &- \frac{3b}{na^5\sqrt{1-e^2}}\frac{a^3}{r^3}\left[\cos^2 i(1-\cos 2u)\right],\end{aligned}$$

$$\frac{di}{dt} = \frac{3b}{2na^5\sqrt{1-e^2}}\frac{a^3}{r^3}\sin 2i \sin 2u,$$

$$\frac{d\Omega}{dt} = \frac{3b}{na^5\sqrt{1-e^2}}\frac{a^3}{r^3}[\cos i(1-\cos 2u)]$$

and

$$\begin{aligned}\frac{dM}{dt} = n + \frac{6b}{na^5}\frac{a^3}{r^3}\left[\frac{3}{2}\sin^2 i(1-\cos 2u) - 1\right] \\ - \frac{3b(1-e^2)}{na^5 e}\left\{\begin{array}{l}\frac{a^4}{r^4}\cos f\left[\frac{3}{2}\sin^2 i(1-\cos 2u) - 1\right] \\ +\frac{a^3}{r^3}\left[\sin^2 i \sin 2u \frac{2+e\cos f}{1-e^2}\sin f\right]\end{array}\right\}.\end{aligned} \tag{5.6}$$

5.2 Solutions of $\overline{C}_{20}$ Perturbed Orbit

For convenience the right-hand side of (5.6) will be separated into three parts:

$$\frac{d\sigma_j}{dt} = \left(\frac{d\sigma_j}{dt}\right)_0 + \left(\frac{d\sigma_j}{dt}\right)_\omega + \left(\frac{d\sigma_j}{dt}\right)_f \tag{5.7}$$

or

$$\frac{d\sigma_j}{dt} = \dot{\sigma}_{j0} + \left(\frac{d\sigma_j}{dt}\right)_\omega + \left(\frac{d\sigma_j}{dt} - \dot{\sigma}_{j0} - \dot{\sigma}_{j\omega}\right), \tag{5.8}$$

where the first term (denoted by $\dot{\sigma}_{j0}$) on the right-hand side includes all terms that are only functions of (a, i, e), the second term includes all terms of ω (without f) (denoted by $\dot{\sigma}_{j\omega}$), and the third term includes all terms of f. They are denoted by the sub-index of 0, ω and f, respectively. Equation 5.8 is needed for later integral variable transformation. The second terms on the right-hand side of the above two equations are the same. It is worth noting that r is a function of f. The solution of the R_2 perturbed orbit is the integration of these equations between initial epoch t_0 and any instantaneous epoch t. The three terms on the right-hand side can be integrated with the integral variable of t, ω and M, respectively.

All terms of ω are represented in terms of sin2u and cos2u. Omitting terms of sin2u and cos2u in (5.6), the remaining terms of f are included in the following functions:

$$\left(\frac{a}{r}\right)^3, \quad \left(\frac{a}{r}\right)^4 \sin f \quad \text{and} \quad \left(\frac{a}{r}\right)^4 \cos f, \tag{5.9}$$

where

$$\frac{a}{r} = \frac{1 + e\cos f}{1 - e^2}, \quad \left(\frac{a}{r}\right)^2 = \frac{1 + 0.5e^2 + 2e\cos f + 0.5e^2\cos 2f}{(1 - e^2)^2},$$

$$\left(\frac{a}{r}\right)^3 = \frac{1 + 1.5e^2 + (3e + 0.75e^3)\cos f + 1.5e^2\cos 2f + 0.25e^3\cos 3f}{(1 - e^2)^3},$$

$$\left(\frac{a}{r}\right)^4 = \frac{1}{(1 - e^2)^4}\begin{bmatrix} \left(1 + 3e^2 + \frac{3}{8}e^4\right) + (4e + 3e^3)\cos f \\ +(3e^2 + 0.5e^4)\cos 2f + e^3\cos 3f + \frac{1}{8}e^4\cos 4f \end{bmatrix},$$

$$\left(\frac{a}{r}\right)^4 \sin f = \frac{1}{(1 - e^2)^4}\begin{bmatrix} \left(1 + 1.5e^2 + \frac{1}{8}e^4\right)\sin f + (2e + e^3)\sin 2f \\ +\left(1.5e^2 + \frac{3}{16}e^4\right)\sin 3f + 0.5e^3\sin 4f + \frac{1}{16}e^4\sin 5f \end{bmatrix},$$

$$\left(\frac{a}{r}\right)^4 \cos f = \frac{1}{(1 - e^2)^4}\begin{bmatrix} (2e + 1.5e^3) + \left(1 + 4.5e^2 + \frac{5}{8}e^4\right)\cos f \\ +(2e + 2e^3)\cos 2f + \left(1.5e^2 + \frac{5}{16}e^4\right)\cos 3f \\ +0.5e^3\cos 4f + \frac{1}{16}e^4\cos 5f \end{bmatrix}, \tag{5.10}$$

and

$$\begin{aligned} \sin jf \sin mf &= -0.5[\cos(j+m)f - \cos(j-m)f] \\ \cos jf \cos mf &= 0.5[\cos(j+m)f + \cos(j-m)f] \\ \sin jf \cos mf &= 0.5[\sin(j+m)f + \sin(j-m)f]. \end{aligned} \tag{5.11}$$

Then the first term (equation of long term perturbation) in (5.8) is

$$
\begin{aligned}
&\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)_0 = \left(\frac{\mathrm{d}e}{\mathrm{d}t}\right)_0 = \left(\frac{\mathrm{d}i}{\mathrm{d}t}\right)_0 = 0 \\
&\left(\frac{\mathrm{d}\omega}{\mathrm{d}t}\right)_0 = \frac{3b}{na^5(1-e^2)^{3.5}}\left(4\sin^2 i - 3 + \frac{15}{4}e^2\sin^2 i - 3e^2\right) \\
&\left(\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right)_0 = \frac{3b}{2na^5}\cos i \frac{(2+3e^2)}{(1-e^2)^{3.5}} \\
&\left(\frac{\mathrm{d}M}{\mathrm{d}t}\right)_0 = n + \frac{9b}{2na^5}\left(\frac{3}{2}\sin^2 i - 1\right)\frac{e^2}{(1-e^2)^3}.
\end{aligned}
\tag{5.12}
$$

The solutions of the long term perturbation are then the integration of these equations from t_0 to t. t is the instantaneous time of interesting and t_0 is the initial time and is set to 0 for convenience. It is obvious that the long term perturbations are then

$$
\Delta\sigma_j(t)_0 = \left(\frac{\mathrm{d}\sigma_j}{\mathrm{d}t}\right)_0 t, \quad j = 1, 2, \ldots, 6. \tag{5.13}
$$

The integral variable transformation between t and ω, Ω and M can be approximated by

$$
\mathrm{d}t = \left(\frac{\mathrm{d}\omega}{\mathrm{d}t}\right)_0^{-1}\mathrm{d}\omega,\ \mathrm{d}t = \left(\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right)_0^{-1}\mathrm{d}\Omega \ \text{ and } \ \mathrm{d}t = \left(\frac{\mathrm{d}M}{\mathrm{d}t}\right)_0^{-1}\mathrm{d}M \tag{5.14}
$$

The second term (long periodic perturbation) in (5.8) exists only in sin2u and cos2u related terms. All sin2u and cos2u terms are factorised by the following functions:

$$
\left(\frac{a}{r}\right)^3,\ \left(\frac{a}{r}\right)^5,\ \left(\frac{a}{r}\right)^4\sin f,\ \left(\frac{a}{r}\right)^4\cos f \ \text{ and } \ \left(\frac{a}{\mathrm{r}}\right)^3\frac{2+e\cos f}{1-e^2}\sin f, \tag{5.15}
$$

where

$$
\left(\frac{a}{r}\right)^5 = \frac{1}{(1-e^2)^5}\left[\begin{array}{l}\left(1+5e^2+1\frac{7}{8}e^4\right)+\left(5e+7.5e^3+\frac{5}{8}e^5\right)\cos f \\ +(5e^2+2.5e^4)\cos 2f+\left(2.5e^3+\frac{5}{16}e^5\right)\cos 3f \\ +\frac{5}{8}e^4\cos 4f+\frac{1}{16}e^5\cos 5f\end{array}\right] \quad \text{and}
$$

$$\left(\frac{a}{r}\right)^3 \frac{2+e\cos f}{1-e^2}\sin f = \frac{1}{(1-e^2)^4}\begin{bmatrix} \left(2+2.25e^2+\frac{1}{8}e^4\right)\sin f \\ +(3.5e+0.25e^3)\sin 2f \\ +\left(2.5e^2+\frac{3}{16}e^4\right)\sin 3f \\ +\frac{5}{8}e^3\sin 4f+\frac{1}{16}e^4\sin 5f \end{bmatrix}. \tag{5.16}$$

From properties of (5.12) and

$$\begin{aligned} \sin 2u &= \sin 2\omega\cos 2f + \cos 2\omega\sin 2f \\ \cos 2u &= \cos 2\omega\cos 2f - \sin 2\omega\sin 2f, \end{aligned} \tag{5.17}$$

it is obvious that all ω terms (without f) may be created only by multiplying sin2u and cos2u by sin2f and cos2f in (5.15). In other words, only $\sin^2 2f$ and $\cos^2 2f$ will lead to a constant of 0.5. Therefore, when seeking the ω terms (without f), just sin2f and cos2f related terms in (5.15) have to be taken into account. Thus, the second term (long periodic term perturbation) of (5.8) is

$$\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)_\omega = \frac{3be^2(2+e^2)}{na^4(1-e^2)^{4.5}}\sin^2 i\sin 2\omega,$$

$$\left(\frac{\mathrm{d}e}{\mathrm{d}t}\right)_\omega = \frac{3be(1+5e^2)}{4na^5(1-e^2)^{3.5}}\sin^2 i\sin 2\omega,$$

$$\left(\frac{\mathrm{d}\omega}{\mathrm{d}t}\right)_\omega = \frac{3b}{8na^5(1-e^2)^{3.5}}\left(6e^2+(2-17e^2)\sin^2 i\right)\cos 2\omega,$$

$$\left(\frac{\mathrm{d}i}{\mathrm{d}t}\right)_\omega = \frac{9be^2}{8na^5(1-e^2)^{3.5}}\sin 2i\sin 2\omega,$$

$$\left(\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right)_\omega = \frac{-9be^2}{4na^5(1-e^2)^{3.5}}\cos i\cos 2\omega$$

and

$$\left(\frac{\mathrm{d}M}{\mathrm{d}t}\right)_\omega = -\frac{3b(2+11e^2)}{8na^5(1-e^2)^3}\sin^2 i\cos 2\omega. \tag{5.18}$$

The solutions of the long periodic term perturbation are then the integration of the above equations from ω_0 to ω. $\omega_0 = \omega(t_0)$ and $\omega = \omega(t)$. It is obvious that the solutions of the long periodic perturbations are then

$$\left(\Delta\sigma_j(t)\right)_\omega - \left(\Delta\sigma_j(t_0)\right)_\omega = \left(\Delta\sigma_j(\omega)\right)_\omega - \left(\Delta\sigma_j(\omega_0)\right)_\omega, \tag{5.19}$$

i.e.

$$(\Delta a(\omega))_\omega = -\left(\frac{\mathrm{d}\omega}{\mathrm{dt}}\right)_0^{-1} \frac{3be^2(2+e^2)}{2na^4(1-e^2)^{4.5}} \sin^2 i \cos 2\omega,$$

$$(\Delta e(\omega))_\omega = -\left(\frac{\mathrm{d}\omega}{\mathrm{dt}}\right)_0^{-1} \frac{3be(1+5e^2)}{8na^5(1-e^2)^{3.5}} \sin^2 i \cos 2\omega,$$

$$(\Delta \omega(\omega))_\omega = \left(\frac{\mathrm{d}\omega}{\mathrm{dt}}\right)_0^{-1} \frac{3b}{16na^5(1-e^2)^{3.5}} \left(6e^2 + (2-17e^2)\sin^2 i\right) \sin 2\omega,$$

$$(\Delta i(\omega))_\omega = -\left(\frac{\mathrm{d}\omega}{\mathrm{dt}}\right)_0^{-1} \frac{9be^2}{16na^5(1-e^2)^{3.5}} \sin 2i \cos 2\omega,$$

$$(\Delta \Omega(\omega))_\omega = \left(\frac{\mathrm{d}\omega}{\mathrm{dt}}\right)_0^{-1} \frac{-9be^2}{8na^5(1-e^2)^{3.5}} \cos i \sin 2\omega$$

and

$$(\Delta M(\omega))_\omega = -\left(\frac{\mathrm{d}\omega}{\mathrm{dt}}\right)_0^{-1} \frac{3b(2+11e^2)}{16na^5(1-e^2)^{3}} \sin^2 i \sin 2\omega. \tag{5.20}$$

The second term on the right-hand side of (5.19) can be obtained by replacing the ω in the first term by ω_0.

The third term of (5.8) includes all terms of f and can be denoted and represented by

$$\left(\frac{\mathrm{d}\sigma_j}{\mathrm{d}t}\right)_f = \left(\frac{\mathrm{d}\sigma_j}{\mathrm{d}t} - \dot{\sigma}_{j0} - \dot{\sigma}_{j\omega}\right). \tag{5.21}$$

This equation can be obtained by withdrawing (5.12) and (5.18) from (5.6). All terms in (5.21) are periodic functions of f. They can be transformed to functions of M by using the relations (see Liu and Zhao 1979)

$$\sin f = \sin M + e\sin 2M + \frac{1}{8}e^2(9\sin 3M - 7\sin M) + \frac{1}{6}e^3(8\sin 4M - 7\sin 2M),$$

$$\cos f = \cos M + e(\cos 2M - 1) + \frac{9}{8}e^2(\cos 3M - \cos M) + \frac{4}{3}e^3(\cos 4M - \cos 2M). \tag{5.22}$$

These two relations have precision of order $O(e^4)$. For more precision (see Liu and Zhao 1979):

$$\sin f = 2\sqrt{1-e^2}\sum_{n=1}^{\infty}\frac{1}{n}\frac{d}{de}[J_n(ne)]\sin nM$$
$$= 2\sqrt{1-e^2}\sum_{n=1}^{\infty}\frac{1}{2}[J_{n-1}(ne) - J_{n+1}(ne)]\sin nM$$

$$\cos f = -e + \frac{2}{e}(1-e^2)\sum_{n=1}^{\infty}J_n(ne)\cos nM$$

$$J_n(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(n+k)!k!}\left(\frac{x}{2}\right)^{n+2k}$$

where $J_n(x)$ is the Bessel function. After the transformation, the index f in (5.21) can be changed to M:

$$\left(\frac{\mathrm{d}\sigma_j}{\mathrm{d}t}\right)_M = \left(\frac{\mathrm{d}\sigma_j}{\mathrm{d}t} - \dot{\sigma}_{j0} - \dot{\sigma}_{j\omega}\right)_M \tag{5.23}$$

and the short periodic disturbances (functions of M) can be obtained by

$$(\Delta\sigma_j(t))_M = \int_{M_0}^{M}\left(\frac{d\sigma_j}{dt}\right)_M\left(\frac{dM}{dt}\right)_0^{-1}dM. \tag{5.24}$$

For convenience, denote these results of integration by

$$(\Delta\sigma_j(t))_M = (\Delta\sigma_j(M))_M - (\Delta\sigma_j(M_0))_M. \tag{5.25}$$

The first term on the right-hand side can be obtained by indefinite integration of (5.24) and will be given below. (The process of an alternative and software based derivation will be outlined in detail in the next chapter by deriving other perturbations of geopotential disturbances.) The second term on the right-hand side can be obtained by replacing the M in the first term by M_0. The constant factor in (5.23) is not taken into account in the following solutions, for various reasons. In the application of the following formulas, this factor should be multiplied. Define

$$(\Delta\sigma_j(M))_M = b_j\left(c_jM + d_j(\omega)M + \sum_{k=1}^{10}A_{jk}\cos kM + \sum_{k=1}^{10}B_{jk}\sin kM\right), \tag{5.26}$$

where j is the index of Keplerian elements. Then there are commonly

$$b_0 = \frac{3\sqrt{5}\mu a_e \overline{C}_{20}}{128a^5 e(1-e^2)^{9/2} n}, \quad b_1 = 2aeb_0, \quad b_2 = (1-e^2)b_0,$$
$$b_3 = b_2, \quad b_4 = 8e(1-e^2)b_0, \quad b_5 = b_4, \quad b_6 = -(1-e^2)^{3/2} b_0. \tag{5.27}$$

For $j = 1$, there are

$$c_j = 0,$$
$$d_j = -64e^2 \sin 2\omega \sin^2 i, \tag{5.28}$$

$$A_{j1} = e(-64 + (96 + 16\cos 2\omega)\sin^2 i),$$
$$A_{j2} = -96e^2 + (144e^2 - (32 - 224e^2)\cos 2\omega)\sin^2 i$$
$$A_{j3} = -112e\cos 2\omega \sin^2 i$$
$$A_{j4} = -272e^2 \cos 2\omega \sin^2 i \tag{5.29}$$

$$B_{j1} = -16e \sin 2\omega \sin^2 i$$
$$B_{j2} = (32 - 224e^2)\sin 2\omega \sin^2 i$$
$$B_{j3} = 112e \sin 2\omega \sin^2 i$$
$$B_{j4} = 272e^2 \sin 2\omega \sin^2 i \tag{5.30}$$

For $j = 2$, there are

$$c_j = 0$$
$$d_j = -16e^2 \sin 2\omega \sin^2 i \tag{5.31}$$

$$A_{j1} = e(-64 + (96 - 16\cos 2\omega)\sin^2 i)$$
$$A_{j2} = e^2(-96 + (144 + 16\cos 2\omega)\sin^2 i)$$
$$A_{j3} = -\frac{112}{3} e\cos 2\omega \sin^2 i$$
$$A_{j4} = -136e^2 \cos 2\omega \sin^2 i \tag{5.32}$$

$$B_{j1} = 16e\sin 2\omega \sin^2 i$$

$$B_{j2} = -16e^2 \sin 2\omega \sin^2 i$$

$$B_{j3} = \frac{112}{3} e \sin 2\omega \sin^2 i$$

$$B_{j4} = 136e^2 \sin 2\omega \sin^2 i \tag{5.33}$$

For $j = 3$, there are

$$c_j = 64e - 96e\sin^2 i \tag{5.34}$$

$$d_j = -16e\sin^2 i \cos 2\omega$$

$$A_{j1} = -32e^2 \sin 2\omega + (16 - 36e^2) \sin 2\omega \sin^2 i$$

$$A_{j2} = 32e \sin 2\omega + 48e \sin 2\omega \sin^2 i$$

$$A_{j3} = e^2 \left(\frac{224}{3}\right) \sin 2\omega - \left(\frac{112}{3} - \frac{962}{3} e^2\right) \sin 2\omega \sin^2 i$$

$$A_{j4} = -136e \sin 2\omega \sin^2 i$$

$$A_{j5} = -338e^2 \sin 2\omega \sin^2 i \tag{5.35}$$

$$\begin{aligned} B_{j1} = &-64 - 152e^2 - 32e^2 \cos 2\omega \\ &+ (96 + 132e^2 + \cos 2\omega(16 - 40e^2))\sin^2 i \end{aligned}$$

$$B_{j2} = -96e + 32e \cos 2\omega + (144e + 48e \cos 2\omega)\sin^2 i$$

$$\begin{aligned} B_{j3} = &-\frac{424}{3} e^2 + \frac{224}{3} e^2 \cos 2\omega \\ &+ \left(212e^2 + \left(-\frac{112}{3} + \frac{962}{3} e^2\right) \cos 2\omega\right)\sin^2 i \end{aligned}$$

$$B_{j4} = -136e \cos 2\omega \sin^2 i$$

$$B_{j5} = -338e^2 \cos 2\omega \sin^2 i \tag{5.36}$$

For $j = 4$, there are

$$c_j = 0$$

$$d_j = -3e^2 \sin 2i \sin 2\omega \tag{5.37}$$

$$A_{j1} = 2e \cos 2\omega \sin 2i$$

$$A_{j2} = (-2 + 11e^2) \cos 2\omega \sin 2i$$

$$A_{j3} = -\frac{14}{3} e \cos 2\omega \sin 2i$$

$$A_{j4} = -\frac{17}{2} e^2 \cos 2\omega \sin 2i \tag{5.38}$$

$$B_{j1} = -2e \sin 2\omega \sin 2i$$

$$B_{j2} = (2 - 11e^2) \sin 2\omega \sin 2i$$

$$B_{j3} = \frac{14}{3} e \sin 2\omega \sin 2i$$

$$B_{j4} = \frac{17}{2} e^2 \sin 2\omega \sin 2i \tag{5.39}$$

For $j = 5$, there are

$$c_j = -24e^2 \cos i$$

$$d_j = 6e^2 \cos i \cos 2\omega \tag{5.40}$$

$$A_{j1} = 4e \sin 2\omega \cos i$$

$$A_{j2} = (-4 + 22e^2) \sin 2\omega \cos i$$

$$A_{j3} = -\frac{28}{3} e \sin 2\omega \cos i$$

$$A_{j4} = -17e^2 \sin 2\omega \cos i \tag{5.41}$$

$$B_{j1} = (24e + 4e \cos 2\omega) \cos i$$

$$B_{j2} = (18e^2 + (-4 + 22e^2)\cos 2\omega)\cos i$$

$$B_{j3} = -\frac{28}{3} e \cos 2\omega \cos i$$

$$B_{j4} = -17e^2 \cos 2\omega \cos i \tag{5.42}$$

For $j = 6$, there are

$$c_j = 64e - 96e\sin^2 i$$

$$d_j = -16e\sin^2 i \cos 2\omega \tag{5.43}$$

$$A_{j1} = (16 - 164e^2)\sin 2\omega \sin^2 i$$

$$A_{j2} = 176e \sin 2\omega \sin^2 i$$

$$A_{j3} = \left(-\frac{112}{3} + \frac{1858}{3} e^2\right) \sin 2\omega \sin^2 i$$

$$A_{j4} = -136e \sin 2\omega \sin^2 i$$

$$A_{j5} = -338e^2 \sin 2\omega \sin^2 i \tag{5.44}$$

$$B_{j1} = (-64 + 424e^2) + (96 - 636e^2 + (16 - 168e^2)\cos 2\omega)\sin^2 i$$

$$B_{j2} = -96e + (144e + 176e \cos 2\omega)\sin^2 i$$

$$B_{j3} = -\frac{424}{3} e^2 + \left(212e^2 + \left(-\frac{112}{3} + \frac{1858}{3} e^2\right) \cos 2\omega\right) \sin^2 i$$

$$B_{j4} = -136e \cos 2\omega \sin^2 i$$

$$B_{j5} = -338e^2 \cos 2\omega \sin^2 i \tag{5.45}$$

These are the solutions of the $\overline{C}_{20}$ perturbations on satellite orbits. The original long formulas given in Xu (2008) is truncated and given above. The complete solutions of the $\overline{C}_{20}$ perturbations on satellite orbits will be given homogenously in the next chapter. Discussions and comments will be given in the following section.

It is notable that the solutions of the $\overline{C}_{20}$ perturbations on satellite orbits are solved by using Lagrangian equations of motion.

5.3 Properties of the Solutions of $\overline{C}_{20}$ Perturbations

These derived solutions of $\overline{C}_{20}$ perturbations are mathematically rigorous except for the series truncation of the transformation from f to M. It is obvious that the series of the solutions may also be truncated according to the precision requirements. The total perturbations on the orbit disturbed by $\overline{C}_{20}$ can be represented as (see (5.7), (5.19), and (5.25))

$$\Delta\sigma_j = \Delta\sigma_j(t)_0 + (\Delta\sigma_j(\omega) - \Delta\sigma_j(\omega_0))_\omega + (\Delta\sigma_j(M) - \Delta\sigma_j(M_0))_M \\ j = 1, 2, \ldots, 6. \tag{5.46}$$

Adding the $\overline{C}_{20}$ perturbations (5.46) on the Keplerian orbit (i.e. the satellite orbit under the acting of the central force of the Earth), one obtains the mathematical expressions of satellite orbit under the central force field and $\overline{C}_{20}$ disturbance.

The disturbances of the $\overline{C}_{20}$ on the different Keplerian elements are inhomogeneous. There are no long term disturbances on the (a, e, i). The order ratios of the long term disturbances on the (ω, Ω, M) are (1, e, e^2), respectively. These indicate that the disturbance on the orientation of the ellipse is stronger than that on the orbital plane. In addition to the constant motion of M, there is a small (e^2) long term change. All Keplerian elements are subjected to long periodic disturbances and they can be grouped in orders with a, (ω, M), e, and (i, Ω). The orders of the short periodic perturbations are dependent on the coefficients given in (5.26) and can be grouped with a, (ω, e, i, Ω) and M. Long and short periodic disturbances tend to change the semi-axis of the orbits very strongly, to change the orientation of the ellipse and motion of the satellite to a great extent and to change the orbital plane and shape to a lesser extent. The order ratios of the long periodic perturbations of the zonal term of $\overline{C}_{20}$ are (ae^2, e, 1, e^2, e^2, 1), respectively. The order ratios of the short periodic perturbations of zonal term $\overline{C}_{20}$ are (ae, e, e, e, e, 1), respectively. These are useful for considering the truncation of the short term solutions. Three different truncating orders should be selected for the six series, depending on the kind of the satellite orbit (or the numbers of a, e and the coefficients of e).

It may be noted from the coefficients (5.27) that there is a divisor e in the second, third and sixth equations of (5.26). By carefully checking on the formulas given in Sect. 5.2, it is obvious that there exists singularity in case of circular orbits in the short term solutions of the third and sixth equations (i.e. in (5.35, 5.36) and (5.44, 5.45)).

It is interesting that in the solutions for the short periodic disturbance there exist long term and long periodic terms, respectively. The short periodic term is related to the variable f. The transformation from f to M given in (5.22) is not a transformation between two periodic functions. There is a small term of e which leads to the terms represented in (5.26) by the first two terms (Note that the transformation is just one of the reasons.) Using the relation (5.14) the dM (i.e. M) can be transformed to dt (i.e. t). In addition

$$\left(\frac{\mathrm{d}M}{\mathrm{d}t}\right)_0^{-1}\mathrm{d}M=\mathrm{d}t=\left(\frac{\mathrm{d}\omega}{\mathrm{d}t}\right)_0^{-1}\mathrm{d}\omega \tag{5.47}$$

can be used to transform easily the $d_j(\omega)M$ terms to functions of ω and the formulas are left to interested readers.

For consistency with the solutions of other order and degree geopotential disturbances given in the next chapter, the complete solution (the indefinite integration of the disturbed equation of motion, to be exact) of $\overline{C}_{20}$ disturbance may be rewritten as

$$\Delta\sigma_j=b_j\left(c'_jM+d'_j(\omega)M+\sum_{k=1}^{10}A_{jk}\cos kM+\sum_{k=1}^{10}B_{jk}\sin kM\right). \tag{5.48}$$

It is obvious that this solution is the summation of the solution given in (5.12), (5.18), and (5.26). Most coefficients are the same as that of (5.26) except the c'_j and d'_j, which can be easily obtained. The relations of the integral variable transformation can be obtained as

$$\left(\frac{\mathrm{d}\sigma_j}{\mathrm{d}t}\right)_0=c'_j,\ \text{and}\ \left(\frac{\mathrm{d}\sigma_j}{\mathrm{d}t}\right)_\omega=d'_j. \tag{5.49}$$

5.4 Orbit Correction

When the orbit errors of GPS satellites become non-negligible for special GPS applications, a process of orbit correction is the first option. Generally, orbit correction is applied to the regional or very long baseline of GPS precise positioning. Even IGS precise GPS orbits are not homogenously precise because they are dependent on the distribution of the IGS reference stations and the length and quality of the data used. The orbit correction is an adjustment or filtering process in which, as well as the station position, the orbit errors are modelled, determined and corrected, based on a known orbit.

Keplerian elements also describe the orbit geometry for instantaneous time. Orbit errors can be considered geometric element errors of the orbit in general. Recalling earlier discussions on the $\overline{C}_{20}$ perturbed orbit solution, a general orbit model can be written as

$$\begin{aligned}\sigma_j(t)=\sigma_{j\mathrm{c}}(t)+\dot\sigma_{j0}(t-t_0)+A_{j\omega}\cos 2\omega+B_{j\omega}\sin 2\omega\\+\sum_{m=1}^{m(j)}\left[A_{jm}\cos mM+B_{jm}\sin mM\right],\end{aligned} \tag{5.50}$$

where $\sigma_j(t), \sigma_{jc}(t), \dot{\sigma}_{j0}$ are true orbit element at time t, computed element at t, element rate with respect to the initial epoch t_0, $A_{j\omega}$, $B_{j\omega}$, A_{jm}, B_{jm} are the coefficients of the long and short periodic perturbations, respectively, and $m(j)$ is the maximum integer of index m related to the jth Keplerian element. ω and M are Keplerian elements. Generally speaking, the coefficients of $A_{j\omega}$, $B_{j\omega}$, A_{jm}, B_{jm} are also functions of ω and i which can be considered in the short periodic term as constants.

For a general model, the order of the polynomial term can be raised to 2, further terms of ω (and Ω) may also be added and $m(j)$ is selectable. The selection of the number of the order depends on the need and situation of orbit errors.

In the GPS observation equations (see Xu 2003, 2007), the orbit state vector is represented in the range or range rate functions. It depends on the use of the GPS observables. We generally denote both the range and range rate together as ρ; their partial derivatives with respect to the orbit state vector are given in Sect. 8.3 and have the forms

$$\frac{\partial \rho}{\partial \vec{r}} \quad \text{and} \quad \frac{\partial \rho}{\partial \dot{\vec{r}}}, \tag{5.51}$$

where the satellite state vector is $(\vec{r}, \dot{\vec{r}})$. The relations between $(\vec{r}, \dot{\vec{r}})$ and Keplerian elements σ_j are discussed in Sect. 3.4. Also, the relations between σ_j and the parameters of the orbit correction model are given in (5.50). Therefore, the orbit correction parts in the GPS observation equations are

$$\frac{\partial \rho}{\partial \vec{r}} \frac{\partial \vec{r}}{\partial \vec{\sigma}} \frac{\partial \vec{\sigma}}{\partial \vec{y}} \Delta \vec{y}^T + \frac{\partial \rho}{\partial \dot{\vec{r}}} \frac{\partial \dot{\vec{r}}}{\partial \vec{\sigma}} \frac{\partial \vec{\sigma}}{\partial \vec{y}} \Delta \vec{y}^T, \tag{5.52}$$

where $\vec{y}$ and $\Delta\vec{y}$ are the parameter vectors in model (5.50) and the parameter correction vector of the model, and $\vec{\sigma}$ is the vector of Keplerian elements. If the initial parameter vector is selected as zero, then $\vec{y} = \Delta\vec{y}$. It is obvious that

$$\vec{y} = \left(\dot{\sigma}_{j0}, A_{j\omega}, B_{j\omega}, A_{jm}, B_{jm}\right) \tag{5.53}$$

and

$$\frac{\partial \sigma_j}{\partial(\dot{\sigma}_{j0}, A_{j\omega}, B_{j\omega}, A_{jm}, B_{jm})} = ((t - t_0), \cos 2\omega, \sin 2\omega, \cos mM, \sin mM). \tag{5.54}$$

Here parameters A_{jm}, B_{jm} represent symbolically the unknowns of all m. For the convenience of representing the partial derivatives of the state vector with respect to the Keplerian elements, the Keplerian element vector is reordered as

$$\vec{\sigma} = (\Omega, i, \omega, a, e, M). \tag{5.55}$$

This does not affect (5.54), because the right-hand side of the equation has nothing to do with index j. According to the formulas in Sect. 3.4 ((3.41), (3.42), and (3.43))

$$\begin{pmatrix} \vec{r} \\ \dot{\vec{r}} \end{pmatrix} = R_3(-\Omega)R_1(-i)R_3(-\omega)\begin{pmatrix} \vec{q} \\ \dot{\vec{q}} \end{pmatrix}, \tag{5.56}$$

where

$$\vec{q} = \begin{pmatrix} a(\cos E - e) \\ a\sqrt{1-e^2}\sin E \\ 0 \end{pmatrix} = \begin{pmatrix} r\cos f \\ r\sin f \\ 0 \end{pmatrix} \text{ and} \tag{5.57}$$

$$\dot{\vec{q}} = \begin{pmatrix} -\sin E \\ \sqrt{1-e^2}\cos E \\ 0 \end{pmatrix} \frac{na}{1-e\cos E} = \begin{pmatrix} -\sin f \\ e+\cos f \\ 0 \end{pmatrix} \frac{na}{\sqrt{1-e^2}}, \tag{5.58}$$

one has

$$\frac{\partial \vec{r}}{\partial(\Omega,\, i,\, \omega)} = \frac{\partial R}{\partial(\Omega,\, i,\, \omega)}\vec{q} \quad \text{and} \quad \frac{\partial \dot{\vec{r}}}{\partial(\Omega,\, i,\, \omega)} = \frac{\partial R}{\partial(\Omega,\, i,\, \omega)}\dot{\vec{q}}, \tag{5.59}$$

where $(\vec{q},\, \dot{\vec{q}})$ are position and velocity vectors of the satellite in the orbital plane coordinate system, and

$$R = R_3(-\Omega)R_1(-i)R_3(-\omega) \text{ and} \tag{5.60}$$

$$\frac{\partial R}{\partial(\Omega,i,\omega)} = \left(\frac{\partial R_3(-\Omega)}{\partial \Omega}R_1(-i)R_3(-\omega), R_3(-\Omega)\frac{\partial R_1(-i)}{\partial i}R_3(-\omega), R_3(-\Omega)R_1(-i)\frac{\partial R_3(-\omega)}{\partial \omega}\right),$$

where

$$\frac{\partial R_1(-i)}{\partial i} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin i & -\cos i \\ 0 & \cos i & -\sin i \end{pmatrix},$$

$$\frac{\partial R_3(-\Omega)}{\partial \Omega} = \begin{pmatrix} -\sin\Omega & -\cos\Omega & 0 \\ \cos\Omega & -\sin\Omega & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\frac{\partial R_3(-\omega)}{\partial \omega} = \begin{pmatrix} -\sin\omega & -\cos\omega & 0 \\ \cos\omega & -\sin\omega & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For the Keplerian elements in the orbital plane (a, e, M), one has

$$\frac{\partial \vec{r}}{\partial(a,\ e,\ M)} = R\frac{\partial \vec{q}}{\partial(a,\ e,\ M)} \quad \text{and} \quad \frac{\partial \dot{\vec{r}}}{\partial(a,\ e,\ M)} = R\frac{\partial \dot{\vec{q}}}{\partial(a,\ e,\ M)}, \tag{5.61}$$

where

$$\frac{\partial \vec{q}}{\partial(a,e,M)} = \begin{pmatrix} \cos E - e & \dfrac{-a\sin^2 E}{1-e\cos E} - a & \dfrac{-a\sin E}{1-e\cos E} \\ \sqrt{1-e^2}\sin E & a\sqrt{1-e^2}\left(\dfrac{\sin 2E}{2(1-e\cos E)} - \dfrac{e\sin E}{1-e^2}\right) & \dfrac{a\sqrt{1-e^2}\cos E}{1-e\cos E} \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\frac{\partial \dot{\vec{q}}}{\partial(a,e,M)} = \begin{pmatrix} \dfrac{n\sin E}{2(1-e\cos E)} & \dfrac{na\sin E(e-2\cos E+e\cos^2 E)}{(1-e\cos E)^3} & \dfrac{na(e-\cos E)}{(1-e\cos E)^3} \\ \dfrac{-n\sqrt{1-e^2}\cos E}{2(1-e\cos E)} & \dfrac{na\left[1+e^2-2e\cos E+\sin^2 E(e\cos E-2)\right]}{\sqrt{1-e^2}(1-e\cos E)^3} & \dfrac{-na\sqrt{1-e^2}\sin E}{(1-e\cos E)^3} \\ 0 & 0 & 0 \end{pmatrix}.$$

The partial derivative formulas given in Sect. 4.1 and the relation in (3.32) between n and a (mean angular velocity and semi–major axis of the satellite) given in Chap. 3 are used, i.e.

$$\frac{\partial E}{\partial(e,M)} = \left(\frac{a}{r}\sin E,\ \frac{a}{r}\right)$$

and

$$n^2 = \mu/a^3. \tag{5.62}$$

Chapter 6
Solutions of Geopotential Perturbations

After an introduction, the principle of the derivations of the geopotential perturbations on an Earth satellite will be discussed first. Then the disturbed basic Lagrangian equations of satellite motion and the simplifying algorithms are outlined. The orbit solutions disturbed by the geopotential are derived up to 8×8 orders and degrees, which are sufficient and necessary for orbit determination of satellites similar to that of GPS.

6.1 Introduction

The geopotential disturbance force is amongst the largest acting on an Earth satellite orbit. Therefore, the satellite orbit disturbed by geopotential is one of the most studied topics in celestial mechanics (Brouwer and Clemence 1961; Van Kamp 1967; Wnuk 1990a, b; Chobotov 1991; Boccaletti and Pucacco 2001; Eberle et al. 2008; Gooding 1990; King-Hele and Walker 1989; Petrovskaya and Vershkov 2007; Kudryavtsev 2007). After Kaula's (1966) first order solution of Lagrangian equations of motion disturbed by geopotential, many scientists have contributed to the derivation of the second order solution using alternative variables and equations; however, the analytical orbit theory, due to its complexities, is rarely applied and the numerical algorithm dominates in the orbit determination practice (Herrick 1972; Meeus 1992; Seeber 2003; Vallado 2007). An analytical solution gives the theoretical integrals and shows the physical effects with very clear spectral properties (Cui 1990; Schneider and Cui 2005; Licandro et al. 2008; Lynden-Bell 2009; Pal 2009), and may give a direct insight into the physical phenomenon of disturbance (Torge 1991; Mysen 2009; Desmars et al. 2009). Xu (2008) introduced a method to solve the equations of satellite motion disturbed by the geopotential force; however only the solutions for a few geopotential terms D_{20}, D_{21}, D_{22} and D_{30} were given. The algorithms used will be further developed, (simplified, generalized) and described here. The solution will be derived for higher

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orders and degrees of 8 × 8 geopotential functions so that applications for such as GPS satellite or onboard orbit determination could be possible.

6.2 Principle of the Derivations

From the solution process of the equation of satellite motion perturbed by the geopotential term $\bar{C}_{20}$ given in Chap. 4, one notices that the derivation is very complicated, even if the potential function of the perturbation is relatively simple. An alternative method is to use symbolic mathematical operation software such as Mathematica, Maple, etc. However, the principle and strategy of the derivation still have to be carefully created.

For simplification, geopotential disturbance function of order l and degree m can be written as (see (4.35))

$$R_{lm} = \frac{\mu}{r}\left(\frac{a_e}{r}\right)^l \bar{P}_{lm}(\sin\phi)[\bar{C}_{lm}\cos m\lambda + \bar{S}_{lm}\sin m\lambda]. \tag{6.1}$$

Here μ is the geocentric gravitational constant, r is the radius, φ is the latitude and λ is the longitude. $\bar{P}_{lm}(\sin\varphi)$ is the so-called associated Legendre function, $\bar{C}_{lm}$, $\bar{S}_{lm}$ are normalised coefficients of the spherical functions and a_e is the mean equatorial radius of the Earth, l and m are called order and degree respectively and $m \le l$, $l \ge 2$. (For $l = m = 0$ R_{00} is the potential of the central force of the Earth.) Let

$$\begin{aligned} \bar{C}_{lm} &= D_{lm}\cos m\lambda_{lm} \\ \bar{S}_{lm} &= D_{lm}\sin m\lambda_{lm}, \\ \bar{\lambda} &= \lambda - \lambda_{lm} \end{aligned} \tag{6.2}$$

where

$$\begin{aligned} D_{lm} &= \sqrt{\bar{C}_{lm}^2 + \bar{S}_{lm}^2} \\ \cos m\lambda_{lm} &= \frac{\bar{C}_{lm}}{D_{lm}}, \\ \sin m\lambda_{lm} &= \frac{\bar{S}_{lm}}{D_{lm}} \end{aligned} \tag{6.3}$$

and note that for zonal terms $m = 0$, $\bar{S}_{lm} = 0$, $\lambda_{lm} = 0$, so that $D_{lm} = \bar{C}_{lm}$, the expression (6.1) turns out to be

$$R_{lm} = \frac{b_{lm}}{r^{l+1}}\bar{P}_{lm}(\sin\phi)\cos(m\bar{\lambda}), \tag{6.4}$$

$$b_{lm} = \mu a_e^l D_{lm}$$

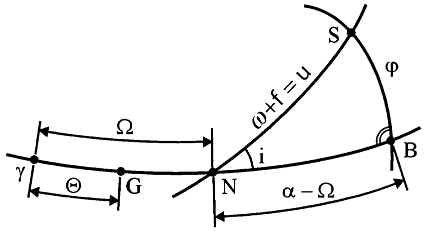

Fig. 6.1 Orbit-equator-meridian triangle

To transform the geographic coordinates into the Keplerian variables, the following relations are needed (see (5.2) or Kaula 2001, cf. Fig. 6.1):

$$\begin{aligned}
&\sin\phi = \sin i \sin u \\
&\bar{\lambda} = \alpha - \Theta - \lambda_{lm} = (\Omega - \Theta - \lambda_{lm}) + (\alpha - \Omega). \\
&\cos(\alpha - \Omega) = \frac{\cos u}{\cos\phi} \\
&\sin(\alpha - \Omega) = \frac{\sin u \cos i}{\cos\phi}
\end{aligned} \tag{6.5}$$

In (6.5) and Fig. 6.1, γ is the vernal equinox, G is the Greenwich, N is the ascending node, S is the satellite, α is the right ascension of the satellite, $u = \omega + f$, Θ is the Greenwich Sidereal Time, and other Keplerian elements (a, e, ω, i, Ω, M, f) are the semi-major axis, the eccentricity of the ellipse, the argument of perigee, the inclination angle, the right ascension of ascending node, the mean anomaly and the true anomaly, respectively. We rewrite the well-known mathematical expansions (Bronstein and Semendjajew 1987; Wang et al. 1979) as follows.

Because (see Wang et al. 1979)

$$\cos(my) = \sum_{j=0}^{[m/2]} (-1)^j \binom{m}{2j} (\cos y)^{m-2j} (\sin y)^{2j},$$

$$\sin(my) = \sum_{j=0}^{[(m-1)/2]} (-1)^j \binom{m}{2j+1} (\cos y)^{m-2j-1} (\sin y)^{2j+1}. \tag{6.6}$$

here $[z]$ represents the integer part of z and the binomial form has the well-known expression of

$$b(m,j) = \binom{m}{k} = \frac{m!}{k!(m-k)!}. \tag{6.7}$$

Let

$$\begin{aligned}\bar{\Omega} &= \Omega - \Theta - \bar{\lambda}_{lm} \\ y &= \alpha - \Omega,\end{aligned} \tag{6.8}$$

then it follows

$$\begin{aligned}\cos m\bar{\lambda} &= \cos(m\bar{\Omega} + my) = \cos m\bar{\Omega}\cos my - \sin m\bar{\Omega}\sin my \\ &= \frac{1}{\cos^m\varphi}\sum_{j=0}^{[m/2]}(-1)^j\cos m\bar{\Omega}\binom{m}{2j}(\cos u)^{m-2j}(\sin u\cos i)^{2j} \\ &\quad - \frac{1}{\cos^m\varphi}\sum_{j=0}^{[(m-1)/2]}(-1)^j\sin m\bar{\Omega}\binom{m}{2j+1}(\cos u)^{m-2j-1}(\sin u\cos i)^{2j+1}\end{aligned} \tag{6.9}$$

Note that in the definition of (4.28) there is a factor of $\cos^{\mathrm{m}}\varphi$ in the expression of $\bar{P}_{lm}(\sin\phi)$; therefore, let

$$\begin{aligned}q_m(\Omega, u, i) &= \sum_{j=0}^{[m/2]}(-1)^j\cos m\bar{\Omega}\binom{m}{2j}(\cos u)^{m-2j}(\sin u\cos i)^{2j} \\ &\quad - \sum_{j=0}^{[(m-1)/2]}(-1)^j\sin m\bar{\Omega}\binom{m}{2j+1}(\cos u)^{m-2j-1}(\sin u\cos i)^{2j+1}\end{aligned} \tag{6.10}$$

$$Q_{lm}(x) = \bar{P}_{lm}(x)/(1-x^2)^{m/2} = N_{lm}\sum_{k=0}^{K}T_{lmk}x^{l-m-2k} \tag{6.11}$$

$$N_{lm} = \sqrt{\frac{(l-m)!(2l+1)(2-\delta_{0m})}{(l+m)!}},$$

$$T_{lmk} = \frac{(-1)^k(2l-2k)!}{2^l k!(l-k)!(l-m-2k)!}. \tag{6.12}$$

Here K is the integer part of $(l–m)/2$, or $K = [(l–m)/2]$ and δ_{0m} is the Kronecker Delta (if $m = 0$, $\delta_{0m} = 1$, else $\delta_{0m} = 0$). The expression (6.1) turns out to be

$$R_{lm} = \frac{b_{lm}}{r^{l+1}}Q_{lm}(x)q_m(\Omega, u, i), \tag{6.13}$$

and then it follows that

$$
\begin{aligned}
\frac{\partial R_{lm}}{\partial a} &= \frac{\partial R_{lm}}{\partial r}\,\frac{\partial r}{\partial a} = \frac{-(l+1)}{a}\,\frac{b_{lm}}{r^{l+1}} Q_{lm}(x) q_m(\Omega, u, i) \\
\frac{\partial R_{lm}}{\partial \Omega} &= \frac{b_{lm}}{r^{l+1}} Q_{lm}(x) \frac{\partial q_m(\Omega, u, i)}{\partial \Omega} \\
\frac{\partial R_{lm}}{\partial i} &= \frac{b_{lm}}{r^{l+1}}\,\frac{\partial Q_{lm}(x)}{\partial x}\,\frac{\partial x}{\partial i} q_m(\Omega, u, i) + \frac{b_{lm}}{r^{l+1}} Q_{lm}(x) \frac{\partial q_m(\Omega, u, i)}{\partial i}, \\
\frac{\partial R_{lm}}{\partial \omega} &= \frac{b_{lm}}{r^{l+1}}\,\frac{\partial Q_{lm}(x)}{\partial x}\,\frac{\partial x}{\partial u} q_m(\Omega, u, i) + \frac{b_{lm}}{r^{l+1}} Q_{lm}(x) \frac{\partial q_m(\Omega, u, i)}{\partial u} \\
\frac{\partial R_{lm}}{\partial e} &= \frac{b_{lm}(-l-1)}{r^{l+2}}\,\frac{\partial r}{\partial e} Q_{lm}(x) q_m(\Omega, u, i) \\
&\quad + \frac{b_{lm}}{r^{l+1}}\,\frac{\partial Q_{lm}(x)}{\partial x}\,\frac{\partial x}{\partial u}\,\frac{\partial u}{\partial e} q_m(\Omega, u, i) + \frac{b_{lm}}{r^{l+1}} Q_{lm}(x) \frac{\partial q_m(\Omega, u, i)}{\partial u}\,\frac{\partial u}{\partial e}
\end{aligned}
\tag{6.14}
$$

and

$$
\begin{aligned}
\frac{\partial R_{lm}}{\partial M} = &\frac{b_{lm}(-l-1)}{r^{l+2}}\,\frac{\partial r}{\partial M} Q_{lm}(x) q_m(\Omega, u, i) \\
&+ \frac{b_{lm}}{r^{l+1}}\,\frac{\partial Q_{lm}(x)}{\partial x}\,\frac{\partial x}{\partial u}\,\frac{\partial u}{\partial M} q_m(\Omega, u, i) + \frac{b_{lm}}{r^{l+1}} Q_{lm}(x) \frac{\partial q_m(\Omega, u, i)}{\partial u}\,\frac{\partial u}{\partial M}
\end{aligned}
$$

Here

$$
\begin{aligned}
\frac{\partial q_m(\Omega, u, i)}{\partial \Omega} = &\, m \sum_{j=0}^{[m/2]} (-1)^{j+1} b(m, 2j) \sin\, m\bar{\Omega}\,(\cos\, u)^{m-2j} (\sin\, u\, \cos\, i)^{2j} \\
&- m \sum_{j=0}^{[(m/2)/2]} (-1)^{j}\, b(m, 2j+1) \cos\, m\bar{\Omega}\,(\cos\, u)^{m-2j-1} (\sin\, u\, \cos\, i)^{2j+1}
\end{aligned}
\tag{6.15}
$$

$$
\begin{aligned}
\frac{\partial q_m(\Omega, u, i)}{\partial u} = &\sum_{j=0}^{[m/2]} (-1)^{j} b(m, 2j) b_1 \cos\, m\bar{\Omega}\,(\cos\, u)^{m-2j-1} (\sin\, u)^{2j-1}\,(\cos\, i)^{2j} \\
&- m \sum_{j=0}^{[(m/2)/2]} (-1)^{j}\, b(m, 2j+1) b^2\, \sin\, m\bar{\Omega}\,(\cos\, u)^{m-2j-2} \\
&\times (\sin\, u)^{2j} (\cos\, i)^{2j+1}
\end{aligned}
$$

$$
\begin{aligned}
b_1 &= (2j - m \sin^2 u) \\
b_2 &= (2j + 1 - m \sin^2 u)
\end{aligned}
\tag{6.16}
$$

$$\frac{\partial q_m(\Omega,u,i)}{\partial i}=$$
$$\sum_{j=0}^{[m/2]}2j(-1)^{j+1}b(m,2j)\cos m\bar{\Omega}\,(\cos u)^{m-2j}(\sin u)^{2j}\,(\cos i)^{2j-1}\sin i$$
$$+\sum_{j=0}^{[(m/1)/2]}(-1)^j b_3(m,j)\,\sin m\bar{\Omega}\,(\cos u)^{m-2j-1}(\sin u)^{2j+1}\,(\cos i)^{2j}\,\sin i$$

$$b_3(m,j)=(2j+1)b(m,2j+1) \tag{6.17}$$

$$\frac{\partial Q_{lm}(x)}{\partial x}=N_{lm}\sum_{k=0}^{K}W_{lmk}x^{l-m-2k-1}$$
$$W_{lmk}=T_{lmk}(l-m-2k) \tag{6.18}$$

$$x=\sin\varphi=\sin u\,\sin i \tag{6.19}$$

$$\frac{\partial x}{\partial u}=\cos u\sin i$$
$$\frac{\partial x}{\partial u}=\sin u\,cos\,i$$
$$\sin u=\sin(f+\omega)=\sin f\,\cos\omega+\cos f\,\sin\omega$$
$$\cos u=\cos(f+\omega)=\cos f\,\cos\omega+\sin f\,\sin\omega \tag{6.20}$$

Furthermore, the following relations are needed (see (4.24) or Kaula 2001):

$$\frac{\partial f}{\partial(e,M)}=\left(\frac{2+e\cos f}{1-e^2}\sin f,\left(\frac{a}{r}\right)^2\sqrt{1-e^2}\right)$$
$$\frac{\partial r}{\partial(a,e,\omega,i,\Omega,M)}=\left(\frac{r}{a},-a\,\cos f,\,0,0,0,\frac{ae}{\sqrt{1-e^2}}\sin f\right). \tag{6.21}$$

These derivations lead to simplified formulae for the perturbation function and are necessary and sufficient to transform the differential equations of motion into functions of Keplerian variables. They are used to derive the solutions of perturbations of geopotential function in orders and degrees of 8 × 8 and are the basis for deriving the general solution of the perturbation of order l and degree m.

6.3 Singularity-Free Basic Lagrangian Equations of Motion

For convenience, we rewrite the Lagrangian equations of satellite motion as follows (see (4.11), or e.g. Battin 1999; Montenbruck and Gill 2000; Kaula 1966/2001):

$$
\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2}{na}\frac{\partial V}{\partial M} \\
\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{1-e^2}{na^2 e}\frac{\partial V}{\partial M} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial V}{\partial \omega}, \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial V}{\partial e} - \frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial V}{\partial i} \\
\frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\left(\cos i\frac{\partial V}{\partial \omega} - \frac{\partial V}{\partial \Omega}\right) \\
\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial V}{\partial i}. \\
\frac{\mathrm{d}M}{\mathrm{d}t} &= n - \frac{2}{na}\frac{\partial V}{\partial a} - \frac{1-e^2}{na^2 e}\frac{\partial V}{\partial e}
\end{aligned}
\tag{6.22}
$$

Here n is the mean angular velocity and the term "n" in the last equation of (6.22) will be omitted later on, V is the disturbance potential function and in our case is R_{lm} given in (6.13).

The Lagrangian equations can be written as (cf. Chap. 10)

$$
\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2}{na}\frac{\mathrm{d}a_1}{\mathrm{d}t} = h_1\frac{\mathrm{d}a_1}{\mathrm{d}t} \\
\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{1-e^2}{na^2 e}\frac{\mathrm{d}a_1}{\mathrm{d}t} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\mathrm{d}e_1}{\mathrm{d}t} = h_2\frac{\mathrm{d}a_1}{\mathrm{d}t} - h_3\frac{\mathrm{d}e_1}{\mathrm{d}t} \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{na^2 e}\frac{\mathrm{d}\omega_1}{\mathrm{d}t} - \cos i\frac{\mathrm{d}\Omega}{\mathrm{d}t} = h_3\frac{\mathrm{d}\omega_1}{\mathrm{d}t} - h_5(\cos i/\sin i)\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} \\
\frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\left(\cos i\frac{de_1}{dt} - \frac{\mathrm{d}i_1}{\mathrm{d}t}\right) = h_4\frac{1}{\sin i}\left(\cos i\frac{de_1}{dt} - \frac{\mathrm{d}i_1}{\mathrm{d}t}\right) \\
\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} = h_5\frac{1}{\sin i}\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} \\
\frac{\mathrm{d}M}{\mathrm{d}t} &= n - \frac{2}{na}\frac{\mathrm{d}M_1}{\mathrm{d}t} - \frac{1-e^2}{na^2 e}\frac{\mathrm{d}\omega_1}{\mathrm{d}t} = n - h_1\frac{\mathrm{d}M_1}{\mathrm{d}t} - h_2\frac{\mathrm{d}\omega_1}{\mathrm{d}t}
\end{aligned}
\tag{6.23}
$$

where coefficients (h_1, h_2, h_3, h_4, $h_5 = h_4$) are defined, and

$$\frac{\mathrm{d}a_1}{\mathrm{d}t}=\frac{\partial R}{\partial M},\quad \frac{\mathrm{d}e_1}{\mathrm{d}t}=\frac{\partial R}{\partial \omega},\quad \frac{\mathrm{d}\omega_1}{\mathrm{d}t}=\frac{\partial R}{\partial e},$$
$$\frac{\mathrm{d}i_1}{\mathrm{d}t}=\frac{\partial R}{\partial \Omega},\quad \frac{\mathrm{d}\Omega_1}{\mathrm{d}t}=\frac{\partial R}{\partial i},\quad \frac{\mathrm{d}M_1}{\mathrm{d}t}=\frac{\partial R}{\partial a}. \tag{6.24}$$

Equation 6.24 are called basic Lagrangian equations of motion. By integrating (6.23) the mean value theorem for integration is used where the functions of variables (a, e, i) are considered constants. For solving the Lagrangian equations of motion (6.22), i.e. (6.23), the simplest way is to solve the basic Lagrangian equations (6.24). Substituting solutions of (6.24) into (6.23) the solutions of (6.22) can be obtained. Therefore, the potential force disturbing orbit problem turns out to be the problem of solving the basic Lagrangian equations of motion (6.24). It is notable that the basic Lagrangian equations of motion are singularity-free.

6.4 Simplifying Algorithms

According to mathematic expansion formulas (Bronstein and Semendjajew 1987; Wang et al. 1979)

$$(1+y)^N=\sum_{n=0}^{N}\binom{N}{n}y^n=1+Ny+\frac{N(N-1)}{2!}y^2+\frac{N(N-1)(N-2)}{3!}y^3+\ldots$$
$$(1-y)^{-N}=1+Ny+\frac{N(N+1)}{2!}y^2+\frac{N(N+1)(N+2)}{3!}y^3+\ldots |y|<1 \tag{6.25}$$

it follows that (in case of truncation to e^3)

$$\begin{aligned}\frac{1}{r^N}&=\frac{(1+e\cos f)^N}{a^N(1-e^2)^N}\\ &\approx\frac{1}{a^N}\left(1+Ne\cos f+\frac{N(N-1)}{2}e^2\cos^2 f+\frac{N(N-1)(N-2)}{6}e^3\cos^3 f\right)(1+Ne^2).\\ &\approx\frac{1}{a^N}\left(1+Ne\cos f+\frac{N(N-1)}{2}e^2\cos^2 f+\frac{N(N-1)(N-2)}{6}e^3\cos^3 f+Ne^2+N^2e^3\cos f\right)\end{aligned} \tag{6.26}$$

Here N is an integer. Approximation (6.26) means that the order l in the first factor of the geopotential function (6.13) can be considered as a general parameter and the truncation can be made with respect to e depending on the accuracy required. This is important for simplifying our algorithms.

For the second factor of the geopotential function (6.11), it follows that (for $m=0$, $l=2, 3, 4, \ldots$)

$$
\begin{aligned}
&Q_{20}(x) = N_{20}(T_{200}x^2 + T_{201}) \\
&Q_{30}(x) = N_{30}(T_{300}x^3 + T_{301}x) \\
&Q_{40}(x) = N_{40}(T_{400}x^4 + T_{401}x^2 + T_{402}) \\
&Q_{50}(x) = N_{50}(T_{500}x^5 + T_{501}x^3 + T_{502}x) \\
&Q_{60}(x) = N_{60}(T_{600}x^6 + T_{601}x^4 + T_{602}x^2 + T_{603}) \\
&\ldots\ldots \\
&Q_{l0}(x) = N_{l0}(T_{l00}x^l + T_{l01}x^{l-2} + \ldots + T_{60K}x^{l-2K})
\end{aligned}
\tag{6.27}
$$

From (6.27), it is notable that, first, Q_{l0} is an l-order polynomial of x; second, the polynomial has $[l/2] + 1$ terms in total; third, the power of x has a step size of 2; fourth, for even l there is a constant term in the polynomial. Instead of deriving the influence of Q_{l0}, one may first derive the effects of x^l, and then multiply them by the respective coefficients and add them together. In this way the effects of x^l will be computed only once. Except for the effects of x^0 and x^1, for any l the effects of Q_{l0} can be obtained through the effects of x^l combined with other effects already derived. In this way the working load is greatly reduced and the long and complicated formulas are reduced to shorter and simpler ones. This is possible because the order l has the property of a general parameter as discussed at the beginning of this section.

Furthermore, e.g. if disturbance of x^l is derived, then (for any even l) the constant part of disturbance in Q_{l0} (see (6.27)) is obtained. Similarly, if disturbance of x^l is derived, then for any $L > l$ the x^l part of disturbance in Q_{L0} is obtained. This is an effective way for deriving the solution.

The function (6.11) is a weighted summation of the disturbances of

$$\{x^l\} \text{ or } \{\sin^l \varphi\} \text{ i.e. } \{\sin^{l-2k} u, \ (k = 0, \ldots, K)\}. \tag{6.28}$$

Because of

$$
\begin{aligned}
\sin^{2n}\varphi &= \frac{1}{2^{2n-1}}\left(\sum_{j=0}^{n-1}(-1)^{n+j}\binom{2n}{j}\cos(2n-2j)\varphi + \frac{1}{2}\binom{2n}{n}\right) \\
\sin^{2n+1}\varphi &= \frac{1}{2^{2n}}\left(\sum_{j=0}^{n}(-1)^{n+j}\binom{2n+1}{j}\sin(2n-2j+1)\varphi\right),
\end{aligned}
\tag{6.29}
$$

it follows that for any even l, x^l is a linear combination of function basis $\{1, \cos 2\varphi, \cos 4\varphi, \ldots, \cos l\varphi\}$, and for any odd l, x^l is a linear combination of function basis $\{\sin\varphi, \sin 3\varphi, \ldots, \sin l\varphi\}$. This means that the order l has the meaning of frequency and x^l includes all frequencies that are equal or lower than l.

For deriving the effects of the disturbances of (6.28), the following formulas are needed:

$$(y+z)^L = \sum_{j=0}^{L} \binom{L}{j} y^{L-j} z^j, \tag{6.30}$$

$$\begin{aligned} \sin^L u &= (\cos\,\omega \sin f + \sin\,\omega \cos f)^L \\ &= \sum_{j=0}^{L} \binom{L}{j} \cos^{L-j}\,\omega \sin^j\,\omega \sin^{L-j} f \cos^j f, \end{aligned} \tag{6.31}$$

$$\begin{aligned} \cos^L u &= (\cos\,\omega \cos f + \sin\,\omega \sin f)^L \\ &= \sum_{j=0}^{L} \binom{L}{j} (-1)^j \cos^{L-j}\,\omega \sin^j\,\omega \cos^{L-j} f \sin^j f \end{aligned} \tag{6.32}$$

For $L = 1, 2, 3, 4, \ldots$, $\sin^L u$ can be represented by sets of functional basis:

$$\begin{aligned} &\sin^1\, u \in \{\sin f, \cos f\} \\ &\sin^2 u \in \{\sin^2 f, \sin f\, \cos f,\, \cos^2 f\} \\ &\sin^3 u \in \{\sin^3 f, \sin^2 f\, \cos f,\, \sin f \cos^2 f,\, \cos^3\} \\ &\ldots\ldots \\ &\sin^L u \in \{\sin^L f, \sin^{L-1} f\, \cos f, \ldots\ldots, \sin f \cos^{L-1} f,\, \cos^L f\} \end{aligned} \tag{6.33}$$

The Q_{L0} term turns out to be a series of the following functional set:

$$\sin^L u \in \{\sin^k f \cos^m f,\, k = 0, \ldots, L,\, m = L-k\}. \tag{6.34}$$

The disturbance of Q_{l0} is a weighted summation of effects of functions given in (6.34).

To transform the trigonometric functions of the true anomaly f to the mean anomaly M, the following relations are used (see, e.g. (5.22), truncated to e^2):

$$\begin{aligned} \sin f &= \left(1 - \frac{7}{8}e^2\right) \sin M + e \sin 2M + \frac{9}{8}e^2 \sin 3M, \\ \cos f + e &= \left(1 - \frac{9}{8}e^2\right) \cos M + e \cos 2M + \frac{9}{8}e^2 \cos 3M. \end{aligned} \tag{6.35}$$

It follows that (see also (6.31), truncated to e^2)

$$\begin{aligned}\sin^k f &= \sum_{j=0}^{k} \binom{k}{j} \sin^{k-j} M \left(e \sin 2M + \frac{e^2}{8}(9\sin 3M - 7\sin M) \right)^j \\ &\approx \sin^k M + k \sin^{k-1} M \left(e \sin 2M + \frac{e^2}{8}(9\sin 3M - 7\sin M) \right) \\ &\quad + \frac{k(k-1)}{2} e^2 \sin^{k-2} M \sin^2 2M \end{aligned} \tag{6.36}$$

$$\begin{aligned}\cos^m f &= \sum_{j=0}^{m} \binom{m}{j} \cos^{m-j} M \left(e(\cos 2M - 1) + \frac{9e^2}{8}(\cos 3M - \cos M) \right)^j \\ &\approx \cos^m M + m \cos^{m-1} M \left(e(\cos 2M - 1) + \frac{9e^2}{8}(\cos 3M - \cos M) \right) \\ &\quad + \frac{m(m-1)}{2} e^2 \cos^{m-2} M (\cos 2M - 1)^2 \end{aligned} \tag{6.37}$$

For more precise formulas (5.22) can be used:

$$\sin f = \sin M + e \sin 2M + \frac{1}{8} e^2 (9\sin 3M - 7\sin M) + \frac{1}{6} e^3 (8\sin 4M - 7\sin 2M),$$

$$\cos f = \cos M + e(\cos 2M - 1) + \frac{9}{8} e^2 (\cos 3M - \cos M) + \frac{4}{3} e^3 (\cos 4M - \cos 2M).$$

The related (6.36) and (6.37) can be similarly derived.

The above discussions lead to simplifications of the two factors of the potential function (6.13). Substituting all the formulas into (6.22) or (6.24) the respective equations can be obtained.

6.5 Solutions of Geopotential Disturbance

The basic Lagrangian equations of satellite motion disturbed by the individual geopotential term have the form

$$\frac{d\sigma_{j1}}{dt} = d_j + \sum_{k=1} (b_{jk} \cos kM + c_{jk} \sin kM). \tag{6.38}$$

Here σ_{j1} is the jth Keplerian element with index 1 (cf. (6.24)), d, b, c are functions of $(a, e, \omega, i, \Omega)$. All terms of M are short-periodic perturbations and all d-terms include long-periodic and linear perturbations. Sometimes the lengths of the formulas of b and c are very long.

Using mean value theorem for integration it holds that (Wang et al. 1979; Bronstein and Semendjajew 1987)

$$\int_0^T h(y(t))g(M(t))dt = h(y(\xi))\int_0^T g(M(t))dt, \quad 0 \le \xi \le T. \tag{6.39}$$

Here functions h (i.e. b and c in (6.38)) and g (i.e. sinkM and coskM in (6.38)) can be integrated. Transforming the time interval [0, T] into the mean motion angular intervals of $\{[2(j-1)\pi/k, 2j\pi/k], j = 1, \ldots J\}$ and $[2J\pi/k, nT]$ (for any k, the integer J can be obtained by relation $nT - 2J\pi/k < 2\pi/k$), the integrals of the short periodic terms g are zero, except over the rest (non-full cycle) intervals $[2J\pi/k, nT]$. Therefore, it holds that

$$\int_0^T h(y(t))g(M(t))dt = h(y(\xi))\int_{2J\pi/k}^{nT} g(M)\frac{1}{n}dM. \tag{6.40}$$

Here, y(t) are slowly changing Keplerian elements (except M) and can be considered as constants over $[2J\pi/k, nT]$ by selecting suitable T.

The short periodic terms in (6.38) can be easily integrated with respect to M by using the relation $M = nt$ (and considering other Keplerian elements as constants and/or using the mean value theorem for integration) and have the form

$$\Delta\sigma_{j1} = \Delta d_j + \sum_{k=1}\frac{1}{kn}(b_{jk}\sin kM - c_{jk}\cos kM), \tag{6.41}$$

where Δd_j denote symbolically the integrals of the long periodic and linear terms. The indefinite integrals (symbolically given in (6.41)) of the short periodic terms are valid over [0, T]. For longer interval [0, kT] the integrations can be made stepwise over [0, T], [T, $2T$], … and the integrals can be then accumulated.

Omitting the terms with M on the right-hand side of (6.38), (6.38) can be further reduced to

$$\frac{d\sigma_{j1}}{dt} = d_j = D_j + \sum_{k,m}(g_{jkm}\cos(k\Omega + m\omega) + q_{jkm}\sin(k\Omega + m\omega)). \tag{6.42}$$

Here coefficients D, g and q are functions of (a, e, i). D-terms are secular. The long-periodic terms of ω and Ω in (6.42) can also be easily integrated with respect to $k\Omega + m\omega$ by using the relation $\Omega = n_\Omega t$, $\omega = n_\omega t$ (n_Ω and n_ω are the mean motions of node and perigee (secular disturbances of Ω and ω) and will be given later) and have the form

Table 6.1 Truncation table

Truncation	Disturbance of geopotential D_{lm}
e^3	D_{20}
e^2	D_{21}, D_{22}
	$D_{30}, D_{31}, D_{32}, D_{33}$
	$D_{40}, D_{41}, D_{42}, D_{43}, D_{44}$
e	$D_{50}, D_{51}, D_{52}, D_{53}, D_{54}, D_{55}$
	$D_{60}, D_{61}, D_{62}, D_{63}, D6_4, D_{65}, D_{66}$
	$D_{70}, D_{71}, D_{72}, D_{73}, D_{74}, D_{75}, D_{76}, D_{77}$
	$D_{80}, D_{81}, D_{82}, D_{83}, D_{84}, D_{85}, D_{86}, D_{87}, D_{88}$

$$\Delta\sigma_{j1} = \Delta d_j = \Delta D_j + \sum_{k,m} K_m^k (g_{jkm} \sin(k\Omega + m\omega) - q_{jkm} \cos(k\Omega + m\omega))$$
$$K_m^k = \frac{1}{kn_\Omega + mn_\omega}. \tag{6.43}$$

Here ΔD_j denotes symbolically the integrals of the linear terms.

The integrals of long-periodic terms (6.43) are obtained by using the mean value theorem for integration. The integrations have only to be done over the rest non-full cycle intervals and the elements a, e, i could be considered as constants during the rest non-full cycle time-span. In case the Fourier coefficients of long-periodic terms cannot be considered as constants, the integration time interval should be selected so that the assumption will be valid and the integrations can be done stepwise and then accumulated.

For convenience, we gave the solutions (6.41) in forms of the following elements:

$$\frac{\mathrm{d}a_1}{\mathrm{d}t}, \frac{\mathrm{d}e_1}{\mathrm{d}t}, \frac{\mathrm{d}\omega_1}{\mathrm{d}t}, \frac{\mathrm{d}i_1}{\mathrm{d}t}, \frac{\mathrm{d}\Omega_1}{\mathrm{d}t}, \frac{\mathrm{d}M_1}{\mathrm{d}t},$$

and

$$d_1, d_2, d_3, d_4, d_5, d_6. \tag{6.44}$$

Only non-zero elements will be listed. For all solutions of disturbance of D_{lm} there is a factor omitted which has to be noticed as follows:

$$\frac{b_{lm}}{a^{l+1}}, \quad l = 2, 3, 4, 5, 6, 7, 8, \quad m = 0, 1, 2, 3, 4, 5, 6, 7, 8. \tag{6.45}$$

Totals given are solutions of 8×8 orders and degrees of the geopotentials. Truncations are made in three ways as listed in Table 6.1. Formulas are given with or without terms of M as listed in Table 6.2.

It is notable that the solutions of the geopotential perturbations on satellite orbits are solved by using basic Lagrangian equations of motion. All the original formulas with M in pdf format are available upon request by authors. (xu@gfz-potsdam.de, j.xu@arcor.de).

Table 6.2 Disturbance formula table

Disturbances given with M terms	Disturbances given without M terms
D_{20}, D_{21}, D_{22}	
$D_{30}, D_{31}, D_{32}, D_{33}$	
$D_{40}, D_{41}, D_{42}, D_{43}, D_{44}$	
D_{50}	$D_{51}, D_{52}, D_{53}, D_{54}, D_{55}$
D_{60}	$D_{61}, D_{62}, D_{63}, D64_{4}, D_{65}, D_{66}$
D_{70}	$D_{71}, D_{72}, D_{73}, D_{74}, D_{75}, D_{76}, D_{77}$
D_{80}	$D_{81}, D_{82}, D_{83}, D_{84}, D_{85}, D_{86}, D_{87}, D_{88}$

6.5.1 Solutions of D_{20} *Perturbation*

Denote

$$cw = \cos\omega, \quad sw = \sin\omega, \quad ci = \cos i, \quad si = \sin i. \tag{6.46}$$

The solutions of D_{20} perturbation can be obtained according to the discussion of Sect. 6.5 using following basic Lagrangian equations of motion (truncation to e^3).

$$\begin{aligned}\frac{\mathrm{d}a_1}{\mathrm{d}t} = &\frac{3}{64}(-16cw\,e\,si^2sw + 12cw\,e^3si^2sw)\cos M + (64cw\,si^2sw - 160cw\,e^2si^2sw)\\ &\times\cos 2M + (336cw\,e\,si^2sw - 578cw\,e^3si^2sw)\cos 3M + 1088cw\,e^2si^2sw\\ &\times\cos 4M + 2646cw\,e^3si^2sw\cos 5M + (e^3(36 + si^2(-58cw^2 - 50sw^2))\\ &+ e(32 + si^2(-56cw^2 - 40sw^2)))\sin M + (si^2(32cw^2 - 32sw^2)\\ &+ e^2(96 + si^2(-224cw^2 - 64sw^2)))\sin 2M + (e\,si^2(168cw^2 - 168sw^2)\\ &+ e^3(212 + si^2(-607cw^2 - 29sw^2)))\sin 3M + e^2si^2(544cw^2 - 544sw^2)\sin 4M\\ &+ e^3si^2(1323cw^2 - 1323sw^2)\sin 5M)\end{aligned} \tag{6.47}$$

$$\begin{aligned}\frac{\mathrm{d}e_1}{\mathrm{d}t} = &\frac{3si^2}{32}((-16cw\ e\ sw + 8cw\ e^3sw)\cos M + (32cw\ sw - 80cw\ e^2sw)\cos 2M\\ &+ (112cw\ e\ sw - 166cw\ e^3sw)\cos 3M + 272cw\ e^2sw\cos 4M\\ &+ 478cw\ e^3sw\cos 5M + (e^3(-2cw^2 + 2sw^2) + e(-8cw^2 + 8sw^2))\sin M\\ &+ (16cw^2 - 16sw^2 + e^2(-40cw^2 + 40sw^2))\sin 2M + (e(56cw^2 - 56sw^2)\\ &+ e^3(-83cw^2 + 83sw^2))\sin 3M + e^2(136cw^2 - 136sw^2)\sin 4M\\ &+ e^3(239cw^2 - 239sw^2)\sin 5M)\end{aligned}$$

$$\frac{\mathrm{d}\omega_1}{\mathrm{d}t} = -\frac{3}{64}(e^3(80 + si^2(-176cw^2 - 184sw^2)) + e(32 + si^2(-48cw^2 - 48sw^2))$$
$$+ (32 + si^2(-56cw^2 - 40sw^2) + e^2(108 + si^2(-158cw^2 - 166sw^2)))$$
$$\times \cos M + (e^3(192 + si^2(-244cw^2 - 332sw^2)) + e(96 + si^2(-224cw^2$$
$$- 64sw^2))) \cos 2M + (si^2(56cw^2 - 56sw^2) + e^2\ (212$$
$$+ si^2(-687cw^2 + 51sw^2))) \cos 3M + (e\ si^2(272cw^2 - 272sw^2)$$
$$+ e^3(368 + si^2(-1488cw^2 + 504sw^2))) \cos 4M + e^2si^2(845cw^2 - 845sw^2)$$
$$\cos 5M + e^3si^2(1908cw^2 - 1908sw^2) \cos 6M + (16cw\ si^2sw - 4cw\ e^2si^2sw)$$
$$\sin M + (160cw\ e\ si^2sw - 40cw\ e^3si^2sw) \sin 2M + (-112cw\ si^2sw$$
$$+ 738cw\ e^2si^2sw) \sin 3M + (-544cw\ e\ si^2sw + 1992cw\ e^3si^2sw) \sin 4M$$
$$- 1690cw\ e^2si^2sw \sin 5M - 3816cw\ e^3si^2sw \sin 6M)$$

$$\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} = -\frac{3\,ci\,si}{32}(-16 + e^2(-24) + (e^3(-90cw^2 - 98sw^2) + e(-56cw^2 - 40sw^2))\cos M$$
$$+ (16cw^2 - 16sw^2 + e^2(-112cw^2 - 32sw^2))\cos 2M + (e(56cw^2 - 56sw^2)$$
$$+ e^3(-149cw^2 + 17sw^2))\cos 3M + e^2(136cw^2 - 136sw^2)\cos 4M$$
$$+ e^3(239cw^2 - 239sw^2)\cos 5M + (16cw\,e\,sw + 4cw\,e^3\,sw)\sin M$$
$$+ (-32cw\,sw + 80cw\,e^2 sw)\sin 2M + (-112cw\,e\,sw + 166cw\,e^3 sw)\sin 3M$$
$$- 272cw\,e^2 sw\,\sin 4M - 478cw\,e^3 sw\,\sin 5M)$$

$$\frac{\mathrm{d}M_1}{\mathrm{d}t} = \frac{3}{64\,a}(32 + si^2(-48) + e^2(48 + si^2(-72cw^2 - 72sw^2))$$
$$+ (e^3(108 + si^2(-270cw^2 - 294sw^2)) + e(96 + si^2(-168cw^2 - 120sw^2)))$$
$$\cos M + (si^2(48cw^2 - 48sw^2) + e^2(144 + si^2(-336cw^2 - 96sw^2))) \cos 2M$$
$$+ (e\ si^2(168cw^2 - 168sw^2) + e^3(212 + si^2(-447cw^2 + 51sw^2))) \cos 3M$$
$$+ e^2si^2(408cw^2 - 408sw^2) \cos 4M + e^3si^2(717cw^2 - 717sw^2) \cos 5M$$
$$+ (48cw\ e\ si^2sw + 12cw\ e^3si^2sw) \sin M + (-96cw\ si^2sw + 240cw\ e^2si^2sw)$$
$$\sin 2M + (-336cw\ e\ si^2sw + 498cw\ e3\ si^2sw) \sin 3M - 816cw\ e^2si^2sw$$
$$\sin 4M - 1434cw\ e^3si^2sw\ \ \sin 5M)$$

The long periodic terms and long terms included in the above formulas are as follows:

$$d_3 = -\frac{3\,e}{16}(8 + 20e^2 + (-12 - 45e^2)si^2 + e^2 si^2 \cos 2w) \tag{6.48}$$

$$d_5 = \frac{3\,ci}{4}(2 + 3e^2)si$$

$$d_6 = -\frac{3}{8a}(2 + 3e^2)(-2 + 3si^2)$$

6.5.2 Solutions of D_{21} Perturbation

Denote

$$cw = \cos\omega, \quad sw = \sin\omega, \quad ci = \cos i, \quad si = \sin i, \quad co = \cos\Omega, \quad so = \sin\Omega. \tag{6.49}$$

The solutions of D_{21} perturbation can be obtained according to the discussion of Sect. 6.5 using the following basic Lagrangian equations of motion (truncation to e^2):

$$\begin{aligned}
\frac{\mathrm{d}a_1}{\mathrm{d}t} = \frac{3\,si}{8}&(co(4) + e^2(6ci\ cw\ so\ sw + co(15cw^2 + 21sw^2))\\
&+ e(8ci\ cw\ so\ sw + co(16cw^2 + 24sw^2))\cos M\\
&+ (-8ci\ cw\ so\ sw + co(4cw^2 - 4\ sw^2)\\
&+ e^2(44ci\ cw\ so\ sw + co(18cw^2 + 62sw^2)))\cos 2M\\
&+ e(-48ci\ cw\ so\ sw + co(24cw^2 - 24sw^2))\cos 3M\\
&+ e^2(-166ci\ cw\ so\ sw + co(83cw^2 - 83sw^2))\cos 4M\\
&+ e(8co\ cw\ sw + ci\ so(16cw^2 + 8sw^2))\sin M\\
&+ (-8co\ cw\ sw + ci\ so(-4cw^2 + 4\ sw^2)\\
&+ e^2(44co\ cw\ sw + ci\ so(58cw^2 + 14sw^2)))\sin 2M\\
&+ e(-48co\ cw\ sw + ci\ so(-24cw^2 + 24\ sw^2))\sin 3M\\
&+ e^2(-166co\ cw\ sw + ci\ so(-83cw^2 + 83sw^2))\sin 4M)
\end{aligned}$$

$$\frac{\mathrm{d}e_1}{\mathrm{d}t} = \frac{3si}{2}(e(2ci\ cw\ so\ sw + co(-cw^2 + sw^2))\cos M$$
$$+ (-4ci\ cw\ so\ sw + co(2cw^2 - 2sw^2)$$
$$+ e^2(10ci\ cw\ so\ sw + co(-5cw^2 + 5sw^2)))\cos 2M$$
$$+ e(-14ci\ cw\ so\ sw + co(7cw^2 - 7sw^2))\cos 3M$$
$$+ e^2(-34ci\ cw\ so\ sw + co(17cw^2 - 17sw^2))\cos 4M$$
$$+ e(2co\ cw\ sw + ci\ so(cw^2 - sw^2))\sin M$$
$$+ (-4co\ cw\ sw + ci\ so(-2cw^2 + 2sw^2)$$
$$+ e^2(10co\ cw\ sw + ci\ so(5cw^2 - 5sw^2)))\sin 2M$$
$$+ e(-14co\ cw\ sw + ci\ so(-7cw^2 + 7sw^2))\sin 3M$$
$$+ e^2(-34co\ cw\ sw + ci\ so(-17cw^2 + 17sw^2))\sin 4M)$$

$$\frac{\mathrm{d}\omega_1}{\mathrm{d}t} = \frac{3si}{32}(e(24co\ cw\ sw + ci\ so(-36cw^2 - 60sw^2))$$
$$+ (16co\ cw\ sw + ci\ so(-40cw^2 - 56sw^2)$$
$$+ e^2(88co\ cw\ sw + ci\ so(-118cw^2 - 206sw^2)))\cos M$$
$$+ e(-32co\ cw\ sw + ci\ so(-160cw^2 - 128sw^2))\cos 2M$$
$$+ (80co\ cw\ sw + ci\ so(40cw^2 - 40sw^2)$$
$$+ e^2(-354co\ cw\ sw + ci\ so(-495cw^2 - 141sw^2)))\cos 3M$$
$$+ e(392co\ cw\ sw + ci\ so(196cw^2 - 196sw^2))\cos 4M$$
$$+ e^2(1226co\ cw\ sw + ci\ so(613cw^2 - 613sw^2))\cos 5M$$
$$+ (-16ci\ cw\ so\ sw + co(40cw^2 + 24sw^2)$$
$$+ e^2(-20ci\ cw\ so\ sw + co(34cw^2 + 14sw^2)))\sin M$$
$$+ e(32ci\ cw\ sosw + co(72cw^2 + 104sw^2))\sin 2M$$
$$+ (-80ci\ cw\ so\ sw + co(40cw^2 - 40sw^2)$$
$$+ e^2(354ci\ cw\ so\ sw + co(7cw^2 + 361sw^2)))\sin 3M$$
$$+ e(-392ci\ cw\ so\ sw + co(196cw^2 - 196sw^2))\sin 4M$$
$$+ e^2(-1226ci\ cw\ so\ sw + co(613cw^2 - 613sw^2))\sin 5M)$$

$$\begin{aligned}\frac{\mathrm{d}i_1}{\mathrm{d}t} = &\frac{3si}{4}(ci\ co\ e^2(-3) + ci\ co(-2) + e(2cw\ so\ sw + ci\ co(-7cw^2 - 5sw^2))\cos M\\
&+ (-4cw\ so\ sw + ci\ co(2cw^2 - 2sw^2) + e^2(10cw\ so\ sw\\
&+ ci\ co(-14cw^2 - 4sw^2)))\cos 2M + e(-14cw\ so\ sw\\
&+ ci\ co(7cw^2 - 7sw^2))\cos 3M + e^2(-34cw\ so\ sw\\
&+ ci\ co(17cw^2 - 17sw^2))\cos 4M + e(2ci\ co\ cw\ sw + so(cw^2 - sw^2))\sin M\\
&+ (-4ci\ co\ cw\ sw + so(-2cw^2 + 2sw^2)\\
&+ e^2(10ci\ co\ cw\ sw + so(5cw^2 - 5sw^2)))\sin 2M\\
&+ e(-14ci\ co\ cw\ sw + so(-7cw^2 + 7sw^2))\sin 3M\\
&+ e^2(-34ci\ co\ cw\ sw + so(-17cw^2 + 17sw^2))\sin 4M)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} = &\frac{3}{4}(ci^2so(-2) + si^2so(2) + e^2(ci^2so(-3) + si^2so(3))\\
&+ e(-2ci\ co\ cw\ sw + ci^2so(-7cw^2 - 5sw^2) + si^2so(7cw^2 + 5sw^2))\cos M\\
&+ (4ci\ co\ cw\ sw + ci^2so(2cw^2 - 2sw^2) + si^2so(-2cw^2 + 2sw^2)\\
&+ e^2(-10ci\ co\ cw\ sw + ci^2so(-14cw^2 - 4sw^2)\\
&+ si^2so(14cw^2 + 4sw^2)))\cos 2M + e(14ci\ co\ cw\ sw + ci^2so(7cw^2 - 7sw^2)\\
&+ si^2so(-7cw^2 + 7sw^2))\cos 3M + e^2(34ci\ co\ cw\ sw + ci^2so(17cw^2 - 17sw^2)\\
&+ si^2so(-17cw^2 + 17sw^2))\cos 4M + e\ (2ci^2cw\ so\ sw - 2cw\ si^2so\ sw\\
&+ ci\ co(-cw^2 + sw^2))\sin M + (-4ci^2cw\ so\ sw + 4cw\ si^2so\ sw\\
&+ ci\ co(2cw^2 - 2sw^2) + e^2(10ci^2cw\ so\ sw - 10cw\ si^2so\ sw\\
&+ ci\ co(-5cw^2 + 5sw^2)))\sin 2M + e(-14ci^2cw\ so\ sw + 14cw\ si^2so\ sw\\
&+ ci\ co(7cw^2 - 7sw^2))\sin 3M + e^2(-34ci^2cw\ so\ sw + 34cw\ si^2so\ sw\\
&+ ci\ co(17cw^2 - 17sw^2))\sin 4M)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}M_1}{\mathrm{d}t} = &-\frac{9si}{4\,a}(ci\ e^2so(-3) + ci\ so(-2)\\
&+ e(-2co\ cw\ sw + ci\ so(-7cw^2 - 5sw^2))\cos M\\
&+ (4co\ cw\ sw + ci\ so(2cw^2 - 2sw^2)\\
&+ e^2(-10co\ cw\ sw + ci\ so(-14cw^2 - 4sw^2)))\cos 2M\\
&+ e(14co\ cw\ sw + ci\ so(7cw^2 - 7sw^2))\cos 3M\\
&+ e^2(34co\ cw\ sw + ci\ so(17cw^2 - 17sw^2))\cos 4M\\
&+ e(2ci\ cw\ so\ sw + co(-cw^2 + sw^2))\sin M\\
&+ (-4ci\ cw\ so\ sw + co(2cw^2 - 2sw^2)\\
&+ e^2(10ci\ cw\ so\ sw + co(-5cw^2 + 5sw^2)))\sin 2M\\
&+ e(-14ci\ cw\ so\ sw + co(7cw^2 - 7sw^2))\sin 3M\\
&+ e^2(-34ci\ cw\ so\ sw + co(17cw^2 - 17sw^2))\sin 4M)\end{aligned} \tag{6.50}$$

The long periodic terms (there are no long terms) included in above formulas are given as follows:

$$d_3 = \frac{9}{16}(-8ci\ \sin(o) + (-1+ci)\sin(o-2w) + (1+ci)\sin(o+2w))e\ si \tag{6.51}$$

$$d_4 = -\frac{3}{4}\cos(o)\ ci\ si\ (2+3e^2)$$

$$d_5 = -\frac{3}{4}\sin(o)(2+3e^2)(ci-si)(ci+si)$$

$$d_6 = \frac{9}{4a}\sin(o)\ ci\ si\ (2+3e^2)$$

where $o = \Omega, \quad w = \omega$.

6.5.3 Solutions of $\mathbf{D}_{22}$ Perturbation

$$\begin{aligned}
\frac{\mathrm{d}a_1}{\mathrm{d}t} = {} & 9\sin(\frac{M}{2})\cos(\frac{M}{2})e\, so^2 + co^2 + ci^2(-co^2+so^2) \\
& + e(-8ci\ co\ cw\ so\ sw + so^2(-7cw^2-5sw^2) + co^2(7cw^2+5sw^2) \\
& + ci^2(co^2(-5cw^2-7sw^2) + so^2(5cw^2+7sw^2)))\cos M \\
& + (-8\ ci\ co\ cw\ so\ sw + co^2(cw^2-sw^2) + so^2(-cw^2+sw^2) \\
& + ci^2(co^2(cw^2-sw^2) + so^2(-cw^2+sw^2)))\cos 2M \\
& + e(-40ci\ co\ cw\ so\ sw + co^2(5cw^2-5sw^2) + so^2(-5cw^2+5sw^2) \\
& + ci^2(co^2(5cw^2-5sw^2) + so^2(-5cw^2\ +5sw^2)))\cos 3M \\
& + e(-2co^2cw\ sw + 2cw\ so^2sw + ci^2(-2co^2cw\ sw + 2cw\ so^2sw) \\
& + ci\ co\ so(-4cw^2+4sw^2))\sin M + (-2co^2cw\ sw + 2cw\ so^2sw \\
& + ci^2(-2co^2cw\ sw + 2cw\ so^2sw) + ci\ co\ so\ (-4cw^2+4sw^2))\sin 2M \\
& + e(-10co^2cw\ sw + 10cw\ so^2sw + ci^2(-10co^2cw\ sw + 10cw\ so^2sw) \\
& + ci\ co\ so(-20cw^2+20sw^2))\sin 3M
\end{aligned} \tag{6.52}$$

$$
\begin{aligned}
\frac{\mathrm{d}e_1}{\mathrm{d}t} = &-\frac{3}{2}(e(-2co^2cw\,sw + 2cw\,so^2sw + ci^2(-2co^2cw\,sw + 2cw\,so^2sw) \\
&+ ci\,co\,so(24 - 28cw^2 - 20sw^2))\cos M + (4co^2cw\,sw - 4cw\,so^2sw \\
&+ ci^2(4co^2cw\,sw - 4cw\,so^2sw) + ci\,co\,so(8cw^2 - 8sw^2) \\
&+ e^2(-10co^2cw\,sw + 10cw\,so^2sw + ci^2(-10co^2cw\,sw + 10cw\,so^2sw) \\
&+ ci\,co\,so(36 - 56cw^2 - 16sw^2)))\cos 2M + e(14co^2cw\,sw - 14cw\,so^2sw \\
&+ ci^2(14co^2cw\,sw - 14cw\,so^2sw) + ci\,co\,so(28cw^2 - 28sw^2))\cos 3M \\
&+ e^2(34co^2cw\,sw - 34cw\,so^2sw + ci^2(34co^2cw\,sw - 34cw\,so^2sw) \\
&+ ci\,co\,so(68cw^2 - 68sw^2))\cos 4M + e(8ci\,co\,cw\,so\,sw \\
&+ so^2(cw^2 - sw^2) + co^2(-cw^2 + sw^2) + ci^2(so^2(cw^2 - sw^2) \\
&+ co^2(-cw^2 + sw^2)))\sin M + (-16ci\,co\,cw\,so\,sw + co^2(2cw^2 - 2sw^2) \\
&+ so^2(-2cw^2 + 2sw^2) + ci^2(co^2(2cw^2 - 2sw^2) + so^2(-2cw^2 + 2sw^2)) \\
&+ e^2(40ci\,co\,cw\,so\,sw + so^2(5cw^2 - 5sw^2) + co^2(-5cw^2 + 5sw^2) \\
&+ ci^2(so^2(5cw^2 - 5sw^2) + co^2(-5cw^2 + 5sw^2))))\sin 2M \\
&+ e(-56ci\,co\,cw\,so\,sw + co^2(7cw^2 - 7sw^2) + so^2(-7cw^2 + 7sw^2) \\
&+ ci^2(co^2(7cw^2 - 7sw^2) + so^2(-7cw^2 + 7sw^2)))\sin 3M \\
&+ e^2(-136ci\,co\,cw\,so\,sw + co^2(17cw^2 - 17sw^2) + so^2(-17cw^2 + 17sw^2) \\
&+ ci^2(co^2(17cw^2 - 17sw^2) + so^2(-17cw^2 + 17sw^2)))\sin 4M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\omega_1}{\mathrm{d}t} = &\frac{9}{32}(e(-64\,ci\,co\,cw\,so\,sw + so^2(-24cw^2 - 8sw^2) + co^2(24cw^2 + 8sw^2) \\
&+ ci^2(co^2(-8cw^2 - 24sw^2) + so^2(8cw^2 + 24sw^2))) \\
&+ (-64ci\,co\,cw\,so\,sw + so^2(-24cw^2 - 8sw^2) + co^2(24cw^2 + 8sw^2) \\
&+ ci^2(co^2(-8cw^2 - 24sw^2) + so^2(8cw^2 + 24sw^2)) \\
&+ e^2(-224ci\,co\,cw\,so\,sw + so^2(-82cw^2 - 26sw^2) + co^2(82cw^2 + 26sw^2) \\
&+ ci^2(co^2(-26cw^2 - 82sw^2) + so^2(26cw^2 + 82sw^2))))\cos M \\
&+ e(-128ci\,co\,cw\,so\,sw + so^2(-64cw^2 - 32sw^2) + co^2(64cw^2 + 32sw^2) \\
&+ ci^2(co^2(-32cw^2 - 64sw^2) + so^2(32cw^2 + 64\,sw^2)))\cos 2M \\
&+ (-64ci\,co\,cw\,so\,sw + co^2(8cw^2 - 8sw^2) + so^2(-8cw^2 + 8sw^2) \\
&+ ci^2(co^2(8cw^2 - 8sw^2) + so^2(-8cw^2 + 8sw^2)) \\
&+ e^2(-40ci\,co\,cw\,so\,sw + so^2(-111cw^2 - 101sw^2) \\
&+ co^2(111cw^2 + 101sw^2) + ci^2(co^2(-101cw^2 - 111sw^2) \\
&+ so^2(101cw^2 + 111sw^2))))\cos 3M
\end{aligned}
$$

$$
\begin{aligned}
&+e(-320ci\ co\ cwso\ sw+co^2(40cw^2-40sw^2)+so^2(-40cw^2+40sw^2)\\
&+ci^2(co^2(40\ cw^2-40sw^2)+so^2(-40cw^2+40sw^2)))\cos 4M\\
&+e^2(-1016ci\ co\ cw\ so\ sw+co^2(127cw^2-127sw^2)\\
&+so^2\ (-127cw^2+127sw^2)+ci^2(co^2(127cw^2-127sw^2)\\
&+so^2(-127cw^2+127\ sw^2)))\cos 5M+(-16co^2cw\ sw+16cw\ so^2sw\\
&+ci^2(-16co^2cw\ sw+16\ cw\ so^2sw)+ci\ co\ so(-32cw^2+32sw^2)\\
&+e^2(-12co^2\ cw\ sw+12cw\ so^2sw+ci^2(-12co^2cw\ sw+12cw\ so^2sw)\\
&+ci\ co\ so(-24cw^2+24sw^2)))\sin M+e(-32co^2cw\ sw+32cw\ so^2sw\\
&+ci^2(-32co^2cw\ sw+32cw\ so^2sw)+ci\ co\ so(-64cw^2+64sw^2))\sin 2M\\
&+(-16co^2cw\ sw+16cw\ so^2sw+ci^2(-16co^2cw\ sw+16cw\ so^2sw)\\
&+ci\ co\ so(-32cw^2+32sw^2)+e^2(-10co^2cw\ sw+10cw\ so^2sw\\
&+ci^2(-10co^2cw\ sw+10cw\ so^2sw)+ci\ co\ so(-20cw^2+20\ sw^2)))\sin 3M\\
&+e(-80co^2cw\ sw+80cw\ so^2sw+ci^2(-80co^2cw\ sw+80cw\ so^2sw)\\
&+ci\ co\ so(-160cw^2+160sw^2))\sin 4M+e^2(-254co^2cw\ sw\\
&+254cw\ so^2sw+ci^2(-254co^2cw\ sw+254cw\ so^2sw)\\
&+ci\ co\ so(-508cw^2+508sw^2))\sin 5M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}i_1}{\mathrm{d}t}=&-3(ci^2co\ so(-2+2co\ so+e^2(-3ci^2\ co\ so+3co\ so\\
&+e(ci(-2co^2cw\ sw+2cw\ so^2sw)+ci^2co\ so(-7cw^2-5sw^2)\\
&+co\ so(5cw^2+7sw^2))\cos M+(ci(4co^2cw\ sw-4cw\ so^2sw)\\
&+co\ so(2cw^2-2sw^2)+ci^2co\ so(2cw^2-2sw^2)+e^2(ci(-10co^2cw\ sw\\
&+10cw\ so^2sw)+ci^2co\ so(-14cw^2-4sw^2)+co\ so(4cw^2+14sw^2)))\cos 2M\\
&+e(ci(14\ co^2\ cw\ sw-14cw\ so^2sw)+co\ so(7cw^2-7sw^2)\\
&+ci^2co\ so(7cw^2-7sw^2))\cos 3M+e^2(ci(34co^2cw\ sw-34cw\ so^2sw)\\
&+co\ so(17cw^2-17sw^2)+ci^2co\ so(17cw^2-17sw^2))\cos 4M\\
&+e(2co\ cw\ so\ sw+2ci^2\ co\ cw\ so\ sw+ci(so^2\ (cw^2-sw^2)\\
&+co^2(-cw^2+sw^2)))\sin M+(-4co\ cw\ so\ sw-4ci^2co\ cw\ so\ sw\\
&+ci(co^2(2cw^2-2sw^2)+so^2(-2cw^2+2sw^2))+e^2(10co\ cw\ so\ sw\\
&+10ci^2co\ cw\ so\ sw+ci(so^2(5cw^2-5sw^2)+co^2(-5cw^2+5\ sw^2))))\sin 2M\\
&+e(-14co\ cw\ so\ sw-14ci^2co\ cw\ so\ sw+ci(co^2(7cw^2-7sw^2)\\
&+so^2(-7cw^2\ +7sw^2)))\sin 3M+e^2(-34co\ cw\ so\ sw-34ci^2co\ cw\ so\ sw\\
&+ci(co^2(17cw^2-17sw^2)+so^2(-17cw^2+17sw^2)))\sin 4M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} = & -\frac{3si}{2}(ci(-2co^2+2so^2)+ci\,e^2(-3co^2+3so^2)+e(4co\,cw\,so\,sw \\
& +ci(co^2(-7cw^2-5sw^2)+so^2(7cw^2+5sw^2)))\cos M+(-8co\,cw\,so\,sw \\
& +ci(co^2(2cw^2-2sw^2)+so^2(-2cw^2+2sw^2))+e^2(20co\,cw\,so\,sw \\
& +ci(co^2(-14cw^2-4sw^2)+so^2(14cw^2+4sw^2))))\cos 2M \\
& +e(-28co\,cw\,so\,sw+ci(co^2(7cw^2-7sw^2)+so^2(-7cw^2+7sw^2)))\cos 3M \\
& +e^2(-68co\,cw\,so\,sw+ci(co^2(17cw^2-17sw^2) \\
& +so^2(-17cw^2+17sw^2)))\cos 4M+e(ci(2co^2cw\,sw-2cw\,so^2sw) \\
& +co\,so(2cw^2-2sw^2))\sin M+(ci(-4co^2cw\,sw+4cw\,so^2sw) \\
& +co\,so(-4cw^2+4sw^2)+e^2(ci(10co^2cw\,sw-10cw\,so^2sw) \\
& +co\,so(10cw^2-10sw^2)))\sin 2M+e(ci(-14co^2cw\,sw+14cw\,so^2sw) \\
& +co\,so(-14cw^2+14sw^2))\sin 3M+e^2(ci(-34co^2cw\,sw+34cw\,so^2sw) \\
& +co\,so(-34cw^2+34sw^2))\sin 4M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}M_1}{\mathrm{d}t} = & -\frac{9}{4a}(-2so^2+2co^2+ci^2(-2co^2+2so^2)+e^2(-3so^2+3co^2 \\
& +ci^2(-3co^2+3so^2))+e(8ci\,co\,cw\,so\,sw+so^2(-5cw^2-7sw^2) \\
& +co^2(5cw^2+7sw^2)+ci^2(co^2(-7cw^2-5sw^2)+so^2(7cw^2+5sw^2)))\cos M \\
& +(-16ci\,co\,cw\,so\,sw+co^2(2cw^2-2sw^2)+so^2(-2cw^2+2sw^2) \\
& +ci^2(co^2(2cw^2-2sw^2)+so^2(-2cw^2+2sw^2))+e^2(40ci\,co\,cw\,so\,sw \\
& +so^2(-4cw^2-14sw^2)+co^2(4cw^2+14sw^2)+ci^2(co^2(-14cw^2-4sw^2) \\
& +so^2(14cw^2+4sw^2))))\cos 2M+e(-56ci\,co\,cw\,so\,sw+co^2(7cw^2-7sw^2) \\
& +so^2(-7cw^2+7sw^2)+ci^2(co^2(7cw^2-7sw^2)+so^2(-7cw^2+7sw^2)))\cos 3M \\
& +e^2(-136ci\,co\,cw\,so\,sw+co^2(17cw^2-17sw^2)+so^2(-17cw^2+17sw^2) \\
& +ci^2(co^2(17cw^2-17sw^2)+so^2(-17cw^2+17sw^2)))\cos 4M+e^2(co^2cw\,sw \\
& -2cw\,so^2sw+ci^2(2co^2cw\,sw-2cw\,so^2sw)+ci\,co\,so(4cw^2-4sw^2))\sin M \\
& +(-4co^2cw\,sw+4cw\,so^2sw+ci^2(-4co^2cw\,sw+4cw\,so^2sw) \\
& +ci\,co\,so(-8cw^2+8sw^2)+e^2(10co^2cw\,sw-10cw\,so^2sw \\
& +ci^2(10co^2cw\,sw-10\,cw\,so^2sw)+ci\,co\,so(20cw^2-20sw^2)))\sin 2M \\
& +e(-14co^2cw\,sw+14cw\,so^2sw+ci^2(-14\,co^2cw\,sw+14cw\,so^2sw) \\
& +ci\,co\,so(-28\,cw^2+28sw^2))\sin 3M+e^2(-34co^2cw\,sw+34cw\,so^2sw \\
& +ci^2(-34co^2cw\,sw+34cw\,so^2sw)+ci\,co\,so(-68cw^2+68sw^2))\sin 4M)
\end{aligned}
$$

The long periodic terms (there are no long terms) included in above formulas are given as follows:

$$d_3 = \frac{8e}{9}((4-4ci^2)\cos(2o) + (1-2ci+ci^2)\cos(2o-2w) + (1+2ci+ci^2)\cos(2o+2w)) \tag{6.53}$$

$$d_4 = 3\sin(o)\cos(o)(-1+ci)(1+ci)(2+3e^2)$$

$$d_5 = \left(3ci + \frac{9ci\,e^2}{2}\right) si\ \cos(2o)$$

$$d_6 = \left(\frac{9(-2+2ci^2)}{4a} + \frac{9(-3+3ci^2)e^2}{4a}\right)\cos(2o)$$

6.5.4 Solutions of D_{30} Perturbation

$$\begin{aligned}\frac{\mathrm{d}a_1}{\mathrm{d}t} = -\frac{si}{64}&((96cw + si^2(-120cw) + e^2(60cw + si^2(-70cw^3 - 90cw\ sw^2)))\cos M \\ &+ e(576cw + si^2(-800cw^3 - 480cw\ sw^2))\cos 2M \\ &+ (si^2(120cw^3 - 360cw\ sw^2) + e^2(1908cw + si^2(-3105cw^3 \\ &- 225cw\ sw^2)))\cos 3M + e\ si^2(800cw^3 - 2400cw\ sw^2)\cos 4M \\ &+ e^2 si^2(3175cw^3 - 9525cw\ sw^2)\cos 5M + (-96sw + si^2(120sw) \\ &+ e^2(-324sw + si^2(390cw^2sw + 410sw^3)))\sin M + e(-576sw \\ &+ si^2(960cw^2\ sw + 640sw^3))\sin 2M + (si^2(-360cw^2sw + 120sw^3) \\ &+ e^2(-1908sw + si^2(4545cw^2sw + 1665sw^3)))\sin 3M + e\ si^2(-2400cw^2sw \\ &+ 800sw^3)\sin 4M + e^2si^2(-9525cw^2sw + 3175sw^3)\sin 5M)\end{aligned} \tag{6.54}$$

$$\begin{aligned}\frac{\mathrm{d}e_1}{\mathrm{d}t} = -\frac{3si}{64}&(e(32cw + si^2(-40cw)) + (32cw + si^2(-40cw) \\ &+ e^2(108cw + si^2(-130cw^3 - 150cw\ sw^2)))\cos M \\ &+ e(96cw - 160cw^3si^2)\cos 2M + (si^2(40cw^3 - 120cw\ sw^2) \\ &+ e^2(212cw + si^2(-505cw^3 + 455cw\ sw^2)))\cos 3M \\ &+ e\ si^2(200cw^3 - 600cw\ sw^2)\cos 4M + e^2si^2(635cw^3 - 1905cw\ sw^2)\cos 5M \\ &+ (-32sw + si^2(40sw) + e^2(-20sw + si^2(10cw^2sw + 30sw^3)))\sin M \\ &+ e(-96sw + si^2(240cw^2sw + 80sw^3))\sin 2M + (si^2(-120cw^2sw + 40sw^3) \\ &+ e^2(-212sw + si^2(985cw^2sw + 25sw^3)))\sin 3M \\ &+ e\ si^2(-600cw^2sw + 200sw^3)\sin 4M \\ &+ e^2si^2\ (-1905cw^2sw + 635sw^3)\sin 5M)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}\omega_1}{\mathrm{d}t} =& \frac{si}{32}(-48sw + si^2(60sw) + e^2(-360sw + si^2(450sw)) + e(-324sw\\
&+ si^2(390cw^2sw + 410sw^3))\cos M + (-144sw + si^2(240cw^2sw + 160sw^3)\\
&+ e^2(-672sw + si^2(615cw^2sw + 915sw^3)))\cos 2M + e(-636sw\\
&+ si^2(1515cw^2sw + 555sw^3))\cos 3M + (si^2(-300cw^2sw + 100sw^3)\\
&+ e^2(-1848sw + si^2(6270cw^2sw + 990sw^3)))\cos 4M\\
&+ e\,si^2(-1905cw^2sw + 635sw^3)\cos 5M + e^2si^2(-7335cw^2sw\\
&+ 2445sw^3)\cos 6M + e(-60cw + si^2(70cw^3 + 90cw\,sw^2))\sin M\\
&+ (-144cw + si^2(200cw^3 + 120cw\,sw^2) + e^2(-120cw\\
&+ si^2(75cw^3 + 375cw\,sw^2)))\sin 2M + e(-636cw + si^2(1035cw^3\\
&+ 75cw\,sw^2))\sin 3M + (si^2(-100cw^3 + 300cw\,sw^2) + e^2(-1848cw\\
&+ si^2(3630cw^3 - 1650cw\,sw^2)))\sin 4M + e\,si^2(-635cw^3\\
&+ 1905cw\,sw^2)\sin 5M + e^2si^2(-2445cw^3 + 7335cw\,sw^2)\sin 6M)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} =& \frac{3ci}{64}(e(-32sw + si^2(120sw)) + (-32sw + si^2(120sw) + e^2(-108sw\\
&+ si^2(390cw^2sw + 410sw^3)))\cos M + e(-96sw + si^2(480cw^2sw\\
&+ 320sw^3))\cos 2M + (si^2(-120cw^2sw + 40sw^3) + e^2(-212sw\\
&+ si^2(1515cw^2sw + 555sw^3)))\cos 3M + e\,si^2(-600cw^2sw\\
&+ 200sw^3)\cos 4M + e^2\,si^2(-1905cw^2sw + 635sw^3)\cos 5M\\
&+ (-32cw + si^2(120cw) + e^2(-20cw + si^2(70cw^3 + 90cw\,sw^2)))\sin M\\
&+ e(-96cw + si^2(400cw^3 + 240cw\,sw^2))\sin 2M + (si^2(-40cw^3\\
&+ 120cw\,sw^2) + e^2(-212cw + si^2(1035cw^3 + 75cw\,sw^2)))\sin 3M\\
&+ e\,si^2(-200cw^3 + 600cw\,sw^2)\sin 4M\\
&+ e^2si^2(-635cw^3 + 1905cw\,sw^2)\sin 5M)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}M_1}{\mathrm{d}t} =& -\frac{si}{16a}(e(-96sw + si^2(120sw)) + (-96sw + si^2(120sw) + e^2(-324sw\\
&+ si^2(390cw^2sw + 410sw^3)))\cos M + e(-288sw + si^2(480\,cw^2sw\\
&+ 320sw^3))\cos 2M + (si^2(-120cw^2sw + 40sw^3) + e^2(-636sw\\
&+ si^2(1515cw^2sw + 555sw^3)))\cos 3M + e\,si^2(-600cw^2sw + 200sw^3)\cos 4M\\
&+ e^2si^2(-1905cw^2sw + 635sw^3)\cos 5M + (-96cw + si^2(120cw)\\
&+ e^2(-60cw + si^2(70cw^3 + 90cw\,sw^2)))\sin M + e(-288cw + si^2(400cw^3\\
&+ 240cw\,sw^2))\sin 2M + (si^2(-40cw^3 + 120cw\,sw^2) + e^2(-636cw\\
&+ si^2(1035cw^3 + 75cw\,sw^2)))\sin 3M + e\,si^2(-200cw^3\\
&+ 600cw\,sw^2)\sin 4M + e^2si^2(-635cw^3 + 1905cw\,sw^2)\sin 5M)\end{aligned}$$

The long periodic terms (there are no long terms) included in above formulas are given as follows:

$$d_2 = \frac{3}{8}\cos(w)\ e\ si(-4+5si^2) \tag{6.55}$$

$$d_3 = \frac{3}{16}\sin(w) si(-4+5si^2)(2+15e^2)$$

$$d_5 = \frac{3}{8}\sin(w)\ ci\ e(-4+15si^2)$$

$$d_6 = -\frac{3}{2a}\sin(w)\ e\ si(-4+5si^2)$$

6.5.5 Solutions of D_{31} Perturbation

$$\begin{aligned}
\frac{\mathrm{d}a_1}{\mathrm{d}t} =& \frac{3}{32} e(-32ci\ cw\ so - 32co\ sw + si^2(ci\ so(40cw) + co(120sw))) \\
&+ (si^2(ci\ so\ (-40cw) + co(40sw)) + e^2(-144ci\ cw\ so - 144\mathrm{co}\ sw \\
&+ si^2(ci\ so(150cw^3 + 170cw\ sw^2) + co(550cw^2sw + 570sw^3)))) \cos M \\
&+ e(32ci\ cw\ so + 32co\ sw + si^2(160co\ sw^3 + ci\ so(-320cw^3 \\
&- 160cw\ sw^2))) \cos 2M + (si^2(ci\ so(40cw^3 - 120cw\ sw^2) + co(120cw^2sw \\
&- 40sw^3)) + e^2(144ci\ cw\ so + 144co\ sw + si^2(ci\ so(-1295cw^3 - 15cw\ sw^2) \\
&+ co(-705cw^2sw + 575sw^3)))) \cos 3M + e\ si^2(ci\ so(280cw^3 - 840cw\,sw^2) \\
&+ co(840cw^2sw - 280sw^3)) \cos 4M + e^2si^2(ci\ so(1145cw^3 - 3435cw\ sw^2) \\
&+ co(3435cw^2sw - 1145sw^3)) \cos 5M + (si^2(co(40cw) + ci\ so(40sw)) \\
&+ e^2(16co\ cw - 16ci\ so\ sw + si^2(co(70cw^3 + 90cw\ sw^2) + ci\ so(170cw^2sw \\
&+ 150sw^3)))) \sin M + e(32co\ cw - 32ci\ so\ sw + si^2(co(80cw^3 + 240cw\ sw^2) \\
&+ ci\ so(400cw^2sw + 240sw^3))) \sin 2M + (si^2(co(40cw^3 - 120cw\ sw^2) \\
&+ ci\ so(-120cw^2sw + 40sw^3)) + e^2(144co\ cw - 144ci\ so\ sw \\
&+ si^2(co(-65cw^3 + 1215cw\ sw^2) + ci\ so(1935cw^2sw + 655sw^3)))) \sin 3M \\
&+ e\ si^2(co(280cw^3 - 840cw\ sw^2) + ci\ so(-840cw^2sw + 280sw^3)) \sin 4M \\
&+ e^2si^2(co(1145cw^3 - 3435cw\ sw^2) + ci\ so(-3435cw^2sw \\
&+ 1145sw^3)) \sin 5M)
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}e_1}{\mathrm{d}t} =& \frac{3}{64} e(32ci\ cw\ so + 32co\ sw + si^2(ci\ so(-120cw) + co(-40sw)))\\
&+ (32ci\ cw\ so + 32co\ sw + si^2(ci\ so(-120cw) + co(-40sw))\\
&+ e^2(108ci\ cw\ so + 108co\ sw + si^2(ci\ so(-390cw^3 - 450cw\ sw^2)\\
&+ co(-90cw^2 sw - 150sw3)))) \cos M + e(96ci\ cw\ so + 96co\ sw\\
&+ si^2(-480ci\ cw^3 so - 480co\ cw^2 sw)) \cos 2M\\
&+ (si^2(ci\ so(120cw^3 - 360cw\ sw^2) + co(360cw^2 sw - 120sw^3))\\
&+ e^2(212ci\ cw\ so + 212co\ sw + si^2(ci\ so(-1515cw^3 + 1365cw\ sw^2)\\
&+ co(-2425cw^2 sw + 455sw^3)))) \cos 3M + e\ si^2(ci\ so(600cw^3\\
&- 1800cw\ sw^2) + co(1800cw^2 sw - 600sw^3)) \cos 4M\\
&+ e^2 si^2(ci\ so(1905cw^3 - 5715cw\ sw^2) + co(5715cw^2 sw - 1905sw^3)) \cos 5M\\
&+ (32co\ cw - 32ci\ so\ sw + si^2(co(-40cw) + ci\ so(120sw))\\
&+ e^2(20co\ cw - 20ci\ so\ sw + si^2(co(-10cw^3 - 70cw\ sw^2)\\
&+ ci\ so(30cw^2 sw + 90sw^3)))) \sin M\\
&+ e(96co\ cw - 96ci\ so\ sw + si^2(co(-240cw^3 + 240cw\ sw^2)\\
&+ ci\ so(720cw^2 sw + 240sw^3))) \sin 2M\\
&+ (si^2(co(120cw^3 - 360cw\ sw^2) + ci\ so(-360cw^2 sw + 120sw^3))\\
&+ e^2(212co\ cw - 212ci\ so\ sw + si^2(co(-985cw^3 + 1895cw\ sw^2)\\
&+ ci\ so(2955cw^2 sw + 75sw^3)))) \sin 3M\\
&+ e\ si^2(co(600cw^3 - 1800cw\ sw^2) + ci\ so(-1800cw^2 sw\\
&+ 600sw^3)) \sin 4M + e^2 si^2(co(1905cw^3 - 5715cw\ sw^2)\\
&+ ci\ so(-5715cw^2 sw + 1905sw^3)) \sin 5M)
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}\omega_1}{\mathrm{d}t} =& -\frac{3}{16}(16co\ cw - 16ci\ so\ sw + si^2(co(-40cw) + ci\ so(40sw))\\
&+ e^2(92co\ cw - 92ci\ so\ sw + si^2(co(-190cw^3 - 210cw\ sw^2)\\
&+ ci\ so(250cw^2 sw + 270sw^3))) + e(88co\ cw - 88ci\ so\ sw\\
&+ si^2(co(-190cw^3 - 210cw\ sw^2) + ci\ so(230cw^2 sw + 250sw^3))) \cos M\\
&+ (16co\ cw - 16ci\ so\ sw + ci\ si^2 so(80cw^2 sw + 80sw^3)\\
&+ e^2(176co\ cw - 176ci\ so\ sw + si^2(co(-345cw^3 - 485cw\ sw^2)\\
&+ ci\ so(395cw^2 sw + 535sw^3)))) \cos 2M + e(72co\ cw - 72ci\ so\ sw\\
&+ si^2(co(-65cw^3 + 175cw\ sw^2) + ci\ so(535cw^2 sw + 295sw^3))) \cos 3M\\
&+ (si^2(co(40cw^3 - 120cw\ sw^2) + ci\ so(-120cw^2 sw + 40sw^3))\\
&+ e^2(212co\ cw - 212ci\ so\ sw + si^2(co(-450cw^3 + 1250cw\ sw^2)
\end{aligned}$$

$$
\begin{aligned}
&+ ci\ so(2310cw^2sw + 610sw^3)))) \cos 4M + e\ si^2(co(255cw^3 \\
&- 765cw\ sw^2) + ci\ so(-765cw^2sw + 255sw^3)) \cos 5M \\
&+ e^2si^2(co(985cw^3 - 2955cw\ sw^2) + ci\ so(-2955cw^2sw \\
&+ 985sw^3)) \cos 6M + e(-8ci\ cw\ so - 8co\ sw + si^2(ci\ so(30cw^3 \\
&+ 50cw\ sw^2) + co(-10cw^2sw + 10sw^3))) \sin M \\
&+ (-16ci\ cw\ so - 16co\ sw + ci\ si^2so(80cw^3 + 80cw\ sw^2) \\
&+ e^2(-16ci\ cw\ so - 16co\ sw + si^2(ci\ so(35cw^3 + 175cw\ sw^2) \\
&+ co(-95cw^2sw + 45\ sw^3)))) \sin 2M + e(-72ci\ cw\ so \\
&- 72co\ sw + si^2(ci\ so(415cw^3 + 175cw\ sw^2) \\
&+ co(185cw^2sw - 55sw^3))) \sin 3M + (si^2(ci\ so(-40cw^3 \\
&+ 120cw\ sw^2) + co(-120cw^2sw + 40sw^3)) + e^2(-212ci\ cw\ so \\
&- 212co\ sw + si^2(ci\ so(1460cw^3 - 240cw\ sw^2) \\
&+ co(1300cw^2sw - 400sw^3)))) \sin 4M + e\ si^2(ci\ so(-255cw^3 \\
&+ 765cw\ sw^2) + co(-765cw^2sw + 255sw^3)) \sin 5M \\
&+ e^2si^2(ci\ so(-985cw^3 + 2955cw\ sw^2) \\
&+co(-2955cw^2sw + 985sw^3)) \sin 6M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}i_1}{\mathrm{d}t} = \frac{3}{64}&(e(32cw\ so + 32ci\ co\ sw + si^2(so(-40cw) + ci\ co(-120sw))) \\
&+ (32cw\ so + 32ci\ co\ sw + si^2(so(-40cw) + ci\ co(-120sw)) \\
&+ e^2(108cw\ so + 108ci\ co\ sw + si^2(so(-130cw^3 - 150cw\ sw^2) \\
&+ ci\ co(-390cw^2sw - 410sw^3)))) \cos(M) + e(96cw\ so + 96ci\ co\ sw \\
&+ si^2(-160cw^3so + ci\ co(-480cw^2\ sw - 320sw^3))) \cos 2M \\
&+ (si^2(so(40cw^3 - 120cw\ sw^2) + ci\ co(120cw^2sw - 40sw^3)) \\
&+ e^2(212cw\ so + 212ci\ co\ sw + si^2(so(-505cw^3 + 455cw\ sw^2) \\
&+ ci\ co(-1515cw^2sw - 555sw^3)))) \cos 3M + e\ si^2(so(200cw^3 \\
&- 600cw\ sw^2) + ci\ co(600cw^2sw - 200sw^3)) \cos 4M \\
&+ e^2si^2(so(635cw^3 - 1905cw\ sw^2) + ci\ co(1905cw^2sw - 635sw^3)) \cos 5M \\
&+ (32ci\,co\ cw - 32so\ sw + si^2(ci\ co(-120cw^3 - 120cw\ sw^2) \\
&+ so(40cw^2sw + 40sw^3)) + e^2(20ci\ co\ cw - 20so\ sw \\
&+ si^2(ci\ co(-70cw^3 - 90cw\ sw^2) + so(10cw^2sw + 30sw^3)))) \sin M \\
&+ e(96ci\ co\ cw - 96so\ sw + si^2(ci\ co(-400cw^3 - 240cw\ sw^2) \\
&+ so(240cw^2sw + 80sw^3)))\sin 2M + (si^2(ci\ co(40cw^3 - 120cw\ sw^2)
\end{aligned}
$$

$$+\,so - 120cw^2sw + 40\ sw^3)) + e^2(212ci\ co\ cw - 212so\ sw$$
$$+\,si^2(ci\ co(-1035cw^3 - 75cw\ sw^2) + so(985cw^2sw$$
$$+\,25sw^3))))\sin 3M + e\ si^2(ci\ co(200cw^3 - 600cw\ sw^2)$$
$$+\,so(-600cw^2sw + 200sw^3))\sin 4M + e^2si^2(ci\ co(635cw^3$$
$$-1905cw\ sw^2) + so(-1905cw^2sw + 635sw^3))\sin 5M)$$

$$\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} = \frac{3si}{64}(e(-32so\ sw + ci\ co(80cw) + ci^2so(-240sw) + si^2so(120sw))$$
$$+\,(-32so\ sw + ci\ co(80cw) + ci^2so(-240sw) + si^2so(120sw)$$
$$+\,e^2(-108so\ sw + ci\ co(260cw^3 + 300cw\ sw^2)$$
$$+\,ci^2so(-780cw^2sw - 820sw^3) + si^2so(390cw^2sw + 410sw^3)))\cos M$$
$$+\,e(320ci\ co\ cw^3 - 96so\ sw + ci^2so(-960cw^2sw - 640sw^3)$$
$$+\,si^2so(480cw^2sw + 320sw^3))\cos 2M + (ci\ co(-80cw^3 + 240cw\ sw^2)$$
$$+\,ci^2so(240cw^2sw - 80sw^3) + si^2so(-120cw^2sw + 40sw^3)$$
$$+\,e^2(-212so\ sw + ci\ co(1010cw^3 - 910cw\ sw^2) + ci^2so(-3030cw^2sw$$
$$-\,1110sw^3) + si^2so(1515cw^2sw + 555sw^3)))\cos 3M + e(ci\ co(-400cw^3$$
$$+\,1200cw\ sw^2) + ci^2so(1200cw^2sw - 400sw^3) + si^2so(-600cw^2sw$$
$$+\,200sw^3))\cos 4M + e^2(ci\ co(-1270cw^3 + 3810cw\ sw^2)$$
$$+\,ci^2so(3810cw^2sw - 1270sw^3) + si^2so(-1905cw^2sw$$
$$+\,635sw^3))\cos 5M + (-32cw\ so + ci^2so(-240cw) + si^2so(120cw)$$
$$+\,ci\ co(-80sw) + e^2(-20cw\ so + ci^2so(-140cw^3 - 180cw\ sw^2)$$
$$+\,si^2so(70cw^3 + 90cw\ sw^2) + ci\ co(-20cw^2sw - 60sw^3)))\sin M$$
$$+\,e(-96cw\ so + ci^2so(-800cw^3 - 480cw\ sw^2) + si^2so(400cw^3$$
$$+\,240cw\ sw^2) + ci\ co(-480cw^2sw - 160sw^3))\sin 2M + (ci^2so(80cw^3$$
$$-\,240cw\ sw^2) + si^2so(-40cw^3 + 120cw\ sw^2) + ci\ co(240cw^2sw - 80sw^3)$$
$$+\,e^2(-212cw\ so + ci^2so(-2070cw^3 - 150cw\ sw^2) + si^2so(1035cw^3$$
$$+\,75cw\ sw^2) + ci\ co(-1970cw^2sw - 50sw^3)))\sin 3M + e(ci^2so(400cw^3$$
$$-\,1200cw\ sw^2) + si^2so(-200cw^3 + 600cw\ sw^2) + ci\ co(1200cw^2sw$$
$$-\,400sw^3))\sin 4M + e^2(ci^2so(1270cw^3 - 3810cw\ sw^2)$$
$$+\,si^2so(-635cw^3 + 1905cw\ sw^2) + ci\ co(3810cw^2sw$$
$$-\,1270sw^3))\sin 5M)$$

$$
\begin{aligned}
\frac{\mathrm{d}M_1}{\mathrm{d}t} = & \frac{3}{16a}(e(32co\ cw - 32ci\ so\ sw + si^2(co(-40cw) + ci\ so(120sw))) \\
& + (32co\ cw - 32ci\ so\ sw + si^2(co(-40cw) + ci\ so(120sw)) \\
& + e^2(108co\ cw - 108ci\ so\ sw + si^2(co(-130cw^3 - 150cw\ sw^2) \\
& + ci\ so(390cw^2 sw + 410sw^3)))) \cos M + e(96co\ cw - 96ci\ so\ sw \\
& + si^2(-160co\ cw^3 + ci\ so(480cw^2 sw + 320sw^3))) \cos 2M \\
& + (si^2(co(40cw^3 - 120cw\ sw^2) + ci\ so(-120cw^2 sw + 40sw^3)) \\
& + e^2(212co\ cw - 212ci\ so\ sw + si^2(co(-505cw^3 + 455cw\ sw^2) \\
& + ci\ so(1515cw^2 sw + 555sw^3)))) \cos 3M + e\ si^2(co(200cw^3 - 600cw\ sw^2) \\
& + ci\ so(-600cw^2 sw + 200sw^3)) \cos 4M + e^2 si^2(co(635cw^3 - 1905cw\ sw^2) \\
& + ci\ so(-1905cw^2 sw + 635sw^3)) \cos 5M + (-32ci\ cw\ so - 32co\ sw \\
& + si^2(ci\ so(120cw^3 + 120cw\ sw^2) + co(40cw^2 sw + 40sw^3)) \\
& + \mathrm{e}^2(\text{-20ci cw so-20co sw} + \mathrm{si}^2(\mathrm{ci\ so}(70\mathrm{cw}^3 + 90\mathrm{cw\ sw}^2) \\
& + \mathrm{co}(10\mathrm{cw}^2\mathrm{sw} + 30\mathrm{sw}^3)))) \sin M + \mathrm{e}(-96\mathrm{ci\ cw\ so} - 96\mathrm{co\ sw} \\
& + si^2(ci\ so(400cw^3 + 240cw\ sw^2) + co(240cw^2 sw + 80sw^3))) \sin 2M \\
& + (si^2(ci\ so(-40cw^3 + 120cw\ sw^2) + co(-120cw^2 sw + 40sw^3)) \\
& + e^2(-212ci\ cw\ so - 212co\ sw + si^2(ci\ so(1035cw^3 + 75cw\ sw^2) \\
& + co(985cw^2 sw + 25sw^3)))) \sin 3M + e\ si^2(ci\ so(-200cw^3 + 600cw\ sw^2) \\
& + co(-600cw^2 sw + 200sw^3)) \sin 4M + e^2 si^2(ci\ so(-635cw^3 \\
& + 1905cw\ sw^2) + co(-1905cw^2 sw + 635sw^3)) \sin 5M)
\end{aligned}
\tag{6.56}
$$

The long periodic terms (there are no long terms) included in above formulas are given as follows:

$$
\begin{aligned}
d_2 = & \frac{3}{16}e((4 - 4ci + (-5 + 15ci)si^2) \sin(o - w) \\
& + (-4 - 4ci + (5 + 15ci)si^2) \sin(o + w))
\end{aligned}
\tag{6.57}
$$

$$
\begin{aligned}
d_3 = & -\frac{3}{32}((5 - 5ci)e^2 si^2 \cos(o - 3w) + (16 - 16ci + (92 - 92ci)e^2 \\
& + (-40 + 40ci + (-195 + 265ci)e^2)si^2) \cos(o - w) + (16 + 16ci \\
& + (92 + 92ci)e^2 + (-40 - 40ci + (-195 - 265ci)e^2)si^2) \cos(o + w) \\
& + (5 + 5ci)e^2 si^2 \cos(o + 3w))
\end{aligned}
$$

$$
\begin{aligned}
d_5 = & -\frac{3}{16}e\ si((4 - 10ci + 30ci^2 - 15si^2) \cos(o - w) \\
& + (-4 - 10ci - 30ci^2 + 15si^2) \cos(o + w))
\end{aligned}
$$

$$d_6 = -\frac{3}{4a}e((-4+4ci+(5-15ci)si^2)\cos(o-w) + (-4-4ci+(5+15ci)si^2)\cos(o+w))$$

6.5.6 Solutions of $\mathbf{D}_{32}$ Perturbation

$$\begin{aligned}\frac{\mathrm{d}a_1}{\mathrm{d}t} = \frac{15}{32}si(&e(so^2(-32cw)+co^2(32cw)+ci\ co\ so(-128sw)+ci^2(so^2(-32cw)\\
&+co^2(32cw)))+(so^2(-24cw)+co^2(24cw)+ci\ co\ so(-32sw)\\
&+ci^2(co^2(-8cw)+so^2(8cw))+e^2(so^2(-154cw^3-174cw\ sw^2)\\
&+co^2(154cw^3+174cw\ sw^2)+ci\ co\ so(-536cw^2\ sw-616sw^3)\\
&+ci^2(so^2(-134cw^3-154cw\ sw^2)+co^2(134cw^3+154cw\ sw^2))))\cos M\\
&+e(so^2(-96cw^3-160cwsw^2)+co^2(96cw^3+160cw\ sw^2)\\
&+ci\ co\ so(128cw^2sw-128sw^3)+ci^2(co^2(-96cw^3-32cw\ sw^2)\\
&+so^2(96cw^3+32cw\ sw^2)))\cos 2M+(co^2(8cw^3-24cw\ sw^2)\\
&+so^2(-8cw^3+24cw\ sw^2)+ci\ co\ so(-96cw^2\ sw+32sw^3)\\
&+ci^2(co^2(8cw^3-24cw\ sw^2)+so^2(-8cw^3+24cw\ sw^2))\\
&+e^2(so^2(-221cw^3-669cw\ sw^2)+co^2(221cw^3+669cw\ sw^2)\\
&+ci\ co\ so(1284cw^2sw-508sw^3)+ci^2(so^2\ (415cw^3-33cw\ sw^2)\\
&+co^2(-415cw^3+33cw\ sw^2))))\cos 3M+e(co^2(64cw^3-192cw\ sw^2)\\
&+so^2(-64cw^3+192cw\ sw^2)+ci\ co\ so(-768cw^2sw+256sw^3)\\
&+ci^2(co^2(64cw^3-192cw\ sw^2)+so^2(-64cw^3+192cw\ sw^2)))\cos 4M\\
&+e^2(co^2(281cw^3-843cw\ sw^2)+so^2(-281cw^3+843cw\ sw^2)\\
&+ci\ co\ so(-3372cw^2sw+1124sw^3)+ci^2(co^2(281cw^3-843cw\ sw^2)\\
&+so^2(-281cw^3+843cw\ sw^2)))\cos 5M+(ci\ co\ so(-32cw)\\
&+co^2(-24sw)+so^2(24sw)+ci^2(so^2(-8sw)+co^2(8sw))\\
&+e^2(ci\ co\ so(-24cw^3-104cw\ sw^2)+co^2(-50cw^2sw-70sw^3)\\
&+so^2(50cw^2sw+70sw^3)+ci^2(so^2(-58cw^2sw-38sw^3)\\
&+co^2(58cw^2sw+38sw^3))))\sin M+e(-256ci\ co\ cw\ so\ sw^2\\
&+co^2(-64cw^2sw-128sw^3)+so^2(64cw^2sw+128sw^3)\\
&+ci^2(so^2(-128cw^2sw-64sw^3)+co^2(128cw^2sw+64sw^3)))\sin 2M\end{aligned}$$

$$+ (ci\ co\ so(-32cw^3 + 96cw\ sw^2) + so^2\ (24cw^2 sw - 8sw^3)$$
$$+ co^2(-24cw^2 sw + 8sw^3) + ci^2(so^2(24cw^2 sw - 8sw^3)$$
$$+ co^2(-24cw^2 sw + 8sw^3)) + e^2(ci\ co\ so(388cw^3 - 1404cw\ sw^2)$$
$$+ co^2(3cw^2 sw - 445sw^3) + so^2(-3cw^2 sw + 445sw^3)$$
$$+ ci^2(so^2(-639cw^2 sw - 191sw^3) + co^2(639cw^2 sw + 191sw^3)))) \sin 3M$$
$$+ e(ci\ co\ so(-256cw^3 + 768cw\ sw^2) + so^2(192cw^2 sw - 64sw^3)$$
$$+ co^2(-192cw^2 sw + 64sw^3) + ci^2(so^2(192cw^2 sw - 64sw^3)$$
$$+ co^2(-192cw^2 sw + 64sw^3))) \sin 4M + e^2(ci\ co\ so(-1124cw^3$$
$$+ 3372cw\ sw^2) + so^2(843cw^2 sw - 281sw^3) + co^2(-843cw^2 sw + 281sw^3)$$
$$+ ci^2(so^2(843cw^2 sw - 281sw^3) + co^2(-843cw^2 sw + 281sw^3))) \sin 5M)$$

$$\frac{\mathrm{d}e_1}{\mathrm{d}t} = \frac{15}{32} si(e(so^2(-8cw) + co^2(8cw) + ci\ co\ so((-128 + 160cw^2)sw + 160sw^3)$$
$$+ ci^2(co^2(-24cw) + so^2(24cw))) + (so^2(-8cw) + co^2(8cw)$$
$$+ ci\ co\ so((-128 + 160cw^2)sw + 160sw^3) + ci^2(co^2(-24cw) + so^2(24cw))$$
$$+ e^2(so^2(-30cw^3 - 18cw\ sw^2) + co^2(30cw^3 + 18cw\ sw^2)$$
$$+ ci\ co\ so((-432 + 504cw^2)sw + 552sw^3) + ci^2(co^2(-78cw^3 - 90cw\ sw^2)$$
$$+ so^2(78cw^3 + 90cw\ sw^2)))) \cos M + e(ci^2(-96co^2 cw^3 + 96cw^3 so^2)$$
$$+ 96co^2 cw\ sw^2 - 96cw\ so^2 sw^2 + ci\ co\ so((-384 + 768cw^2)sw + 384sw^3))$$
$$\cos 2M + (co^2(24cw^3 - 72cw\ sw^2) + so^2(-24cw^3 + 72cw\ sw^2)$$
$$+ ci\ co\ so(-288\ cw^2 sw + 96sw^3) + ci^2(co^2(24cw^3 - 72cw\ sw^2)$$
$$+ so^2(-24cw^3 + 72cw\ sw^2)) + e^2(so^2(91cw^3 - 485cw\ sw^2)$$
$$+ co^2(-91cw^3 + 485cw\ sw^2) + ci\ co\ so((-848 + 2788cw^2)sw + 484sw^3)$$
$$+ ci^2(so^2(303cw^3 - 273cw\ sw^2) + co^2(-303cw^3 + 273cw\ sw^2)))) \cos 3M$$
$$+ e(co^2(120cw^3 - 360cw\ sw^2) + so^2(-120cw^3 + 360cw\ sw^2)$$
$$+ ci\ co\ so(-1440cw^2 sw + 480sw^3) + ci^2(co^2(120cw^3 - 360cw\ sw^2)$$
$$+ so^2(-120cw^3 + 360cw\ sw^2))) \cos 4M + e^2(co^2(381cw^3 - 1143cw\ sw^2)$$
$$+ so^2(-381cw^3 + 1143cw\ sw^2) + ci\ co\ so(-4572cw^2 sw + 1524sw^3)$$
$$+ ci^2(co^2(381cw^3 - 1143cw\ sw^2) + so^2(-381cw^3 + 1143cw\ sw^2))) \cos 5M$$
$$+ (ci\ co\ so(-128cw + 160cw^3 + 160cw\ sw^2) + co^2(-8sw) + so^2(8sw)$$
$$+ ci^2(so^2(-24sw) + co^2(24sw)) + e^2(ci\ co\ so(-80cw + 88cw^3$$
$$+ 136cw\ sw^2) + co^2(-14cw^2 sw - 2sw^3) + so^2(14cw^2 sw + 2sw^3)$$
$$+ ci^2(so^2(-6cw^2 sw - 18sw^3) + co^2(6cw^2 sw + 18sw^3)))) \sin M$$

$$
\begin{aligned}
&+ e(ci\ co\ so(-384cw + 576cw^3 + 192cw\ sw^2) + co^2(48cw^2sw - 48sw^3)\\
&+ so^2(-48cw^2sw + 48sw^3) + ci^2(so^2(-144cw^2sw - 48sw^3)\\
&+ co^2(144cw^2sw + 48sw^3)))\sin(2M) + (ci\ co\ so(-96cw^3 + 288cw\ sw^2)\\
&+ so^2(72cw^2sw - 24sw^3) + co^2(-72cw^2sw + 24sw^3) + ci^2(so^2(72cw^2sw\\
&- 24sw^3) + co^2(-72cw^2sw + 24sw^3)) + e^2(ci\ co\ so(-848cw + 1636cw^3\\
&- 668cw\ sw^2) + co^2(379cw^2sw - 197sw^3) + so^2(-379cw^2sw + 197sw^3)\\
&+ ci^2(so^2(-591cw^2sw - 15sw^3) + co^2(591cw^2sw + 15sw^3))))\sin 3M\\
&+ e(ci\ co\ so(-480cw^3 + 1440cw\ sw^2) + so^2(360cw^2sw - 120sw^3)\\
&+ co^2(-360cw^2sw + 120sw^3) + ci^2(so^2(360cw^2sw - 120sw^3)\\
&+ co^2(-360cw^2sw + 120\ sw^3)))\sin 4M + e^2(ci\ co\ so(-1524cw^3\\
&+ 4572cw\ sw^2) + so^2(1143cw^2sw - 381sw^3) + co^2(-1143cw^2sw\\
&+ 381sw^3) + ci^2(so^2(1143cw^2sw - 381sw^3)\\
&+ co^2(-1143cw^2sw + 381sw^3)))\sin 5M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\omega_1}{\mathrm{d}t} = &\frac{15}{16} si(ci\ co\ so(-48cw) + co^2(-4sw) + so^2(4sw) + ci^2(co^2(-20sw)\\
&+ so^2(20sw)) + e^2(ci\ co\ so(-232cw^3 - 296cw\ sw^2)\\
&+ co^2(10cw^2sw - 6sw^3) + so^2(-10cw^2sw + 6sw^3)\\
&+ ci^2(co^2(-110cw^2sw - 126sw^3) + so^2(110cw^2sw + 126sw^3)))\\
&+ e(ci\ co\ so(-232cw^3 - 280cw\ sw^2) + co^2(2cw^2sw - 10sw^3)\\
&+ so^2(-2cw^2sw + 10sw^3) + ci^2(co^2(-106cw^2sw - 118sw^3)\\
&+ so^2(106cw^2sw + 118sw^3)))\cos M + (-64ci\ co\ cw\ so\ sw^2\\
&+ so^2(-32cw^2sw - 16sw^3) + co^2(32cw^2sw + 16sw^3)\\
&+ ci^2(co^2(-16cw^2sw - 32sw^3) + so^2(16cw^2sw + 32sw^3))\\
&+ e^2(ci\ co\ so(-428cw^3 - 636cw\ sw^2) + co^2(31cw^2sw - 21sw^3)\\
&+ so^2(-31cw^2sw + 21sw^3) + ci^2(co^2(-193cw^2sw - 245sw^3)\\
&+ so^2(193cw^2sw + 245sw^3))))\cos 2M + e(ci\ co\ so(-76cw^3 - 76cw\ sw^2)\\
&+ so^2(-87cw^2sw - 87sw^3) + co^2(87cw^2sw + 87sw^3)\\
&+ ci^2(co^2(-125cw^2sw - 125sw^3) + so^2(125cw^2sw + 125sw^3)))\cos 3M\\
&+ (ci\ co\ so(48cw^3 - 144cw\ sw^2) + co^2(36cw^2sw - 12sw^3)\\
&+ so^2(-36cw^2sw + 12sw^3) + ci^2(co^2(36cw^2sw - 12sw^3)\\
&+ so^2(-36cw^2sw + 12sw^3)) + e^2(ci\ co\ so(-536cw^3 + 680cw\ sw^2)\\
&+ so^2(-22cw^2sw - 326sw^3) + co^2(22cw^2sw + 326sw^3)
\end{aligned}
$$

$$
\begin{aligned}
&+ ci^2(co^2(-594cw^2sw - 290sw^3) + so^2(594cw^2sw + 290sw^3)))) \cos 4M \\
&+ e(ci\ co\ so(308cw^3 - 924cw\ sw^2) + co^2(231cw^2sw - 77sw^3) \\
&+ so^2(-231cw^2sw + 77sw^3) + ci^2(co^2(231cw^2sw - 77sw^3) \\
&+ so^2(-231cw^2sw + 77sw^3))) \cos 5M + e^2(ci\ co\ so(1196cw^3 \\
&- 3588cw\ sw^2) + co^2(897cw^2sw - 299sw^3) + so^2(-897cw^2sw + 299sw^3) \\
&+ ci^2(co^2(897cw^2sw - 299sw^3) + so^2(-897cw^2sw + 299sw^3))) \cos 6M \\
&+ e(co^2(10cw^3 - 2cw\ sw^2) + so^2(-10cw^3 + 2cw\ sw^2) \\
&+ ci\ co\ so(-24cw^2sw + 24sw^3) + ci^2(co^2(-10cw^3 - 22cw\ sw^2) \\
&+ so^2(10cw^3 + 22cw\ sw^2))) \sin M + (so^2(-24cw^3 - 8cw\ sw^2) \\
&+ co^2(24cw^3 + 8cw\ sw^2) + ci\ co\ so(-32cw^2sw + 32sw^3) \\
&+ ci^2\ (co^2(-24cw^3 - 40cw\ sw^2) + so^2(24cw^3 + 40cw\ sw^2)) \\
&+ e^2(co^2(27cw^3 - 25cw\ sw^2) + so^2(-27cw^3 + 25cw\ sw^2) \\
&+ ci\ co\ so(-132cw^2sw + 76sw^3) + ci^2(co^2(-13cw^3 - 65cw\ sw^2) \\
&+ so^2(13cw^3 + 65cw\ sw^2)))) \sin 2M + e(so^2(-87cw^3 - 87cw\ sw^2) \\
&+ co^2(87cw^3 + 87cw\ sw^2) + ci\ co\ so(76cw^2sw + 76sw^3) \\
&+ ci^2(co^2(-125cw) + so^2(125cw))) \sin 3M + (co^2(12cw^3 - 36cw\ sw^2) \\
&+ so^2(-12cw^3 + 36cw\ sw^2) + ci\ co\ so(-144cw^2sw + 48sw^3) \\
&+ ci^2(co^2\ (12cw^3 - 36cw\ sw^2) + so^2(-12cw^3 + 36cw\ sw^2)) \\
&+ e^2(so^2(-174cw^3 - 478cw\ sw^2) + co^2(174cw^3 + 478cw\ sw^2) \\
&+ ci\ co\ so(1144cw^2sw - 72sw^3) + ci^2(co^2(-442cw^3 - 138cw\ sw^2) \\
&+ so^2(442cw^3 + 138cw\ sw^2)))) \sin 4M + e(co^2(77cw^3 - 231cw\ sw^2) \\
&+ so^2(-77cw^3 + 231cw\ sw^2) + ci\ co\ so(-924cw^2sw + 308sw^3) \\
&+ ci^2(co^2(77cw^3 - 231cw\ sw^2) + so^2(-77cw^3 + 231cw\ sw^2))) \sin 5M \\
&+ e^2(co^2(299cw^3 - 897cw\ sw^2) + so^2(-299cw^3 + 897cw\ sw^2) \\
&+ ci\ co\ so(-3588cw^2sw + 1196sw^3) + ci^2(co^2(299cw^3 - 897cw\ sw^2) \\
&+ so^2(-299cw^3 + 897cw\ sw^2))) \sin 6M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}i_1}{\mathrm{d}t} = -\frac{15}{8} si(&e(ci^2co\ so(-24sw) + co\ so(8sw) + ci(so^2(-8cw) + co^2(8cw))) \\
&+ (ci^2co\ so(-24sw) + co\ so(8sw) + ci(so^2(-8cw) + co^2(8cw)) \\
&+ e^2(ci^2co\ so(-78cw^2sw - 82sw^3) + co\ so(30cw^2sw + 26\ sw^3) \\
&+ ci(so^2(-26cw^3 - 30cw\ sw^2) + co^2(26cw^3 + 30cw\ sw^2)))) \cos M \\
&+ e(ci(32co^2cw^3 - 32cw^3\ so^2) + 32co\ so\ sw^3
\end{aligned}
$$

$$+ ci^2 co\, so(-96cw^2sw - 64sw^3)) \cos 2M + (co\, so(24cw^2sw - 8sw^3)$$
$$+ ci^2 co\, so(24cw^2sw - 8sw^3) + ci(so^2(8cw^3 - 24cw\, sw^2)$$
$$+ co^2(-8cw^3 + 24cw\, sw^2)) + e^2(ci^2 co\, so(-303cw^2sw - 111sw^3)$$
$$+ co\, so(-91cw^2sw + 101sw^3) + ci(co^2(101cw^3 - 91cw\, sw^2)$$
$$+ so^2(-101cw^3 + 91cw\, sw^2)))) \cos 3M + e(co\, so(120cw^2sw - 40sw^3)$$
$$+ ci^2 co\, so(120cw^2sw - 40sw^3) + ci(so^2(40cw^3 - 120cw\, sw^2)$$
$$+ co^2(-40cw^3 + 120cw\, sw^2))) \cos 4M + e^2(co\, so(381cw^2sw - 127sw^3)$$
$$+ ci^2 co\, so(381cw^2sw - 127sw^3) + ci(so^2(127cw^3 - 381cw\, sw^2)$$
$$+ co^2(-127cw^3 + 381cw\, sw^2))) \cos 5M + (ci^2 co\, so(-24cw) + co\, so(8cw)$$
$$+ ci(co^2(-8sw) + so^2(8sw)) + e^2(ci^2 co\, so\, (-14cw^3 - 18cw\, sw^2)$$
$$+ co\, so(6cw^3 + 2cw\, sw^2) + ci(co^2(-2cw^2sw - 6sw^3)$$
$$+ so^2(2cw^2sw + 6sw^3)))) \sin M + e(ci^2 co\, so(-80cw)$$
$$+ co\, so(16cw^3 + 48cw\, sw^2) + ci(co^2(-48cw^2sw - 16sw^3)$$
$$+ so^2(48cw^2sw + 16sw^3))) \sin 2M + (co\, so(8cw^3 - 24cw\, sw^2)$$
$$+ ci^2 co\, so(8cw^3 - 24cw\, sw^2) + ci(co^2(24cw^2sw - 8sw^3)$$
$$+ so^2(-24cw^2sw + 8sw^3)) + e^2(ci^2 co\, so(-207cw^3 - 15cw\, sw^2)$$
$$+ co\, so(5cw^3 + 197cw\, sw^2) + ci(co^2(-197cw^2sw - 5sw^3)$$
$$+ so^2(197cw^2\, sw + 5sw^3)))) \sin 3M + e(co\, so(40cw^3 - 120cw\, sw^2)$$
$$+ ci^2 co\, so(40cw^3 - 120cw\, sw^2) + ci(co^2(120cw^2sw - 40sw^3)$$
$$+ so^2(-120cw^2sw + 40sw^3))) \sin 4M + e^2(co\, so(127cw^3 - 381cw\, sw^2)$$
$$+ ci^2 co\, so(127cw^3 - 381cw\, sw^2) + ci(co^2(381cw^2sw - 127sw^3)$$
$$+ so^2(-381cw^2sw + 127sw^3))) \sin 5M)$$

$$\frac{d\Omega_1}{dt} = \frac{15}{32}(e(ci^2 co\, so(-32cw) + ci(so^2(-8w) + co^2(8sw)) + ci^3(co^2(-24\, sw)$$
$$+ so^2(24sw)) + si^2(co\, so(32cw) + ci\, (so^2(-48sw) + co^2(48sw))))$$
$$+ (ci^2 co\, so(-32cw) + ci\, (so^2(-8sw) + co^2(8sw))$$
$$+ ci^3(co^2(-24sw) + so^2(24sw)) + si^2(co\, so(32cw)$$
$$+ ci(so^2(-48sw) + co^2(48sw))) + e^2(ci^2 co\, so(-104cw^3 - 120cw\, sw^2)$$
$$+ ci(so^2(-30cw^2sw - 26sw^3) + co^2(30cw^2sw + 26sw^3))$$
$$+ ci^3(co^2(-78cw^2sw - 82sw^3) + so^2(78cw^2sw + 82sw^3))$$
$$+ si^2(co\, so(104cw^3 + 120cw\, sw^2) + ci(so^2(-156cw^2sw - 164sw^3)$$
$$+ co^2(156cw^2sw + 164sw^3))))) \cos M + e(-128ci^2 co\, cw^3 so$$

$$
\begin{aligned}
&+ ci\ (32co^2sw^3 - 32so^2sw^3) + ci^3(co^2(-96cw^2sw - 64sw^3)\\
&+ so^2(96cw^2sw + 64sw^3)) + si^2(128co\ cw^3so\\
&+ ci(so^2(-192cw^2sw - 128\ sw^3) + co^2(192cw^2sw + 128sw^3)))) \cos 2M\\
&+ (ci^2co\ so(32cw^3 - 96cw\ sw^2) + ci(co^2(24cw^2sw - 8sw^3)\\
&+ so^2(-24cw^2sw + 8sw^3)) + ci^3(co^2(24cw^2sw - 8sw^3)\\
&+ so^2(-24cw^2sw + 8sw^3)) + si^2(co\ so(\text{-}32cw^3 + 96\ cw\ sw^2)\\
&+ ci(so^2(48cw^2sw - 16sw^3) + co^2(-48cw^2sw + 16sw^3)))\\
&+ e^2(ci^2co\ so(-404cw^3 + 364cw\ sw^2) + ci(so^2(91cw^2sw - 101sw^3)\\
&+ co^2(-91cw^2sw + 101sw^3)) + ci^3(co^2(-303cw^2sw - 111sw^3)\\
&+ so^2(303cw^2sw + 111sw^3)) + si^2(co\ so(404cw^3 - 364cw\ sw^2)\\
&+ ci(so^2(-606cw^2sw - 222sw^3) + co^2(606cw^2sw + 222sw^3))))) \cos 3M\\
&+ e(ci^2co\ so(160cw^3 - 480cw\ sw^2) + ci(co^2(120cw^2sw - 40sw^3)\\
&+ so^2(-120cw^2sw + 40sw^3)) + ci^3(co^2(120cw^2sw - 40sw^3)\\
&+ so^2(-120cw^2sw + 40sw^3)) + si^2(co\ so(-160cw^3 + 480cw\ sw^2)\\
&+ ci(so^2(240cw^2sw - 80sw^3) + co^2(-240cw^2sw + 80sw^3)))) \cos 4M\\
&+ e^2(ci^2co\ so(508cw^3 - 1524cw\ sw^2) + ci(co^2(381cw^2sw\text{-}127sw^3)\\
&+ so^2(-381cw^2sw + 127sw^3)) + ci^3(co^2(381cw^2sw - 127sw^3)\\
&+ so^2(-381cw^2sw + 127sw^3)) + si^2(co\ so(-508cw^3 + 1524cw\ sw^2)\\
&+ ci(so^2(762cw^2sw - 254sw^3) + co^2(-762cw^2sw + 254sw^3)))) \cos 5M\\
&+ (ci^2\ co\ so(32sw) + ci(so^2(-8cw) + co^2(8cw)) + ci^3(co^2(-24cw)\\
&+ so^2(24cw)) + si^2(co\ so(-32sw) + ci(so^2(-48cw) + co^2(48cw)))\\
&+ e^2(ci^2co\ so(8cw^2sw + 24sw^3) + ci(so^2(-6cw^3 - 2cw\ sw^2)\\
&+ co^2(6\ cw^3 + 2cw\ sw^2)) + ci^3(co^2(-14cw^3 - 18cw\ sw^2)\\
&+ so^2(14\ cw^3 + 18cw\ sw^2)) + si^2(co\ so(-8cw^2sw - 24sw^3)\\
&+ ci(so^2(-28cw^3 - 36cw\ sw^2) + co^2(28\ cw^3 + 36cw\ sw^2))))) \sin M\\
&+ e(ci^2co\ so(192cw^2sw + 64\ sw^3) + ci\ (so^2(-16cw^3 - 48cw\ sw^2)\\
&+ co^2(16cw^3 + 48cw\ sw^2)) + ci^3(co^2(-80cw^3 - 48cw\ sw^2)\\
&+ so^2(80cw^3 + 48cw\ sw^2)) + si^2(co\ so(-192cw^2sw - 64\ sw^3)\\
&+ ci(so^2(-160cw^3 - 96cw\ sw^2) + co^2(160cw^3 + 96cw\ sw^2)))) \sin 2M\\
&+ (ci^2co\ so(-96cw^2sw + 32sw^3) + ci(co^2(8cw^3 - 24cw\ sw^2)\\
&+ so^2(-8cw^3 + 24cw\ sw^2)) + ci^3(co^2(8cw^3 - 24cw\ sw^2)\\
&+ so^2(-8cw^3 + 24cw\ sw^2)) + si^2(co\ so(96cw^2sw - 32sw^3)\\
&+ ci(so^2(16cw^3 - 48cw\ sw^2) + co^2(-16cw^3 + 48cw\ sw^2)))
\end{aligned}
$$

$$+ e^2(ci^2\, co\, so(788cw^2sw + 20sw^3) + ci^3(co^2(-207cw^3 - 15cw\, sw^2)$$
$$+ so^2(207cw^3 + 15cw\, sw^2)) + ci(so^2(-5cw^3 - 197cw\, sw^2)$$
$$+ co^2(5cw^3 + 197cw\, sw^2)) + si^2(co\, so(-788cw^2sw - 20sw^3)$$
$$+ ci(so^2(-414cw^3 - 30cw\, sw^2) + co^2(414cw^3 + 30cw\, sw^2))))) \sin 3M$$
$$+ e(ci^2\, co\, so(-480cw^2sw + 160sw^3) + ci(co^2(40cw^3 - 120cw\, sw^2)$$
$$+ so^2(-40cw^3 + 120cw\, sw^2)) + ci^3(co^2(40cw^3 - 120cw\, sw^2)$$
$$+ so^2\,(-40cw^3 + 120cw\, sw^2)) + si^2(co\, so(480cw^2sw - 160sw^3)$$
$$+ ci(so^2(80cw^3 - 240cw\, sw^2) + co^2(-80cw^3 + 240cw\, sw^2)))) \sin 4M$$
$$+ e^2(ci^2co\, so(-1524cw^2sw + 508sw^3) + ci(co^2(127cw^3 - 381cw\, sw^2)$$
$$+ so^2(-127cw^3 + 381cw\, sw^2)) + ci^3(co^2(127cw^3 - 381cw\, sw^2)$$
$$+ so^2(-127cw^3 + 381cw\, sw^2)) + si^2(co\, so(1524cw^2sw - 508sw^3)$$
$$+ ci(so^2(254cw^3 - 762cw\, sw^2) + \; co^2(-254cw^3 + 762cw\, sw^2)))) \sin 5M)$$

$$\frac{\mathrm{d}M_1}{\mathrm{d}t} = -\frac{15si}{8a}(e(ci\, co\, so(-32cw) + so^2(-8sw) + co^2(8sw) + ci^2(co^2(-24sw)$$
$$+ so^2(24sw))) + (ci\, co\, so\,(-32cw) + so^2\,(-8sw) + co^2(8sw)$$
$$+ ci^2(co^2(-24sw) + so^2(24sw)) + e^2(ci\, co\, so(-104cw^3 - 120cw\, sw^2)$$
$$+ so^2(-30cw^2sw - 26sw^3) + co^2(30cw^2sw + 26sw^3)$$
$$+ ci^2(co^2(-78cw^2sw - 82sw^3) + so^2(78cw^2sw + 82sw^3)))) \cos M$$
$$+ e(-128ci\, co\, cw^3so + 32co^2sw^3 - 32so^2sw^3 + ci^2(co^2(-96cw^2sw$$
$$- 64sw^3) + so^2(96cw^2sw + 64sw^3))) \cos 2M + (ci\, co\, so(32cw^3$$
$$- 96cw\, sw^2) + co^2(24cw^2sw - 8sw^3) + so^2(-24cw^2sw + 8sw^3)$$
$$+ ci^2(co^2(24cw^2sw - 8sw^3) + so^2(-24cw^2sw + 8sw^3))$$
$$+ e^2(ci\, co\, so(-404cw^3 + 364cw\, sw^2) + so^2(91cw^2sw - 101sw^3)$$
$$+ co^2(-91cw^2sw + 101sw^3) + ci^2(co^2(-303cw^2sw - 111sw^3)$$
$$+ so^2(303cw^2sw + 111sw^3)))) \cos 3M + e(ci\, co\, so(160cw^3 - 480cw\, sw^2)$$
$$+ co^2(120cw^2sw - 40sw^3) + so^2(-120cw^2sw + 40sw^3)$$
$$+ ci^2(co^2(120cw^2sw - 40sw^3) + so^2(-120cw^2sw + 40sw^3))) \cos 4M$$
$$+ e^2(ci\, co\, so(508cw^3 - 1524cw\, sw^2) + co^2(381cw^2sw - 127sw^3)$$
$$+ so^2(-381cw^2sw + 127sw^3) + ci^2(co^2(381cw^2sw - 127sw^3)$$
$$+ so^2(-381cw^2sw + 127sw^3))) \cos 5M + (so^2(-8cw) + co^2(8cw)$$
$$+ ci\, co\, so(32sw) + ci^2\,(co^2(-24cw) + so^2(24cw))$$
$$+ e^2(so^2(-6cw^3 - 2cw\, sw^2) + co^2(6cw^3 + 2cw\, sw^2) + ci\, co\, so(8cw^2sw$$

$$
\begin{aligned}
&+24sw^3)+ci^2(co^2(-14cw^3-18cw\,sw^2)+so^2(14cw^3+18cw\,sw^2))))\\
&\sin M+e(so^2(-16cw^3-48cw\,sw^2)+co^2(16cw^3+48cw\,sw^2)\\
&+ci\,co\,so(192cw^2sw+64sw^3)+ci^2(co^2(-80cw^3-48cw\,sw^2)\\
&+so^2(80cw^3+48cw\,sw^2)))\sin 2M+(co^2(8cw^3-24cw\,sw^2)\\
&+so^2(-8cw^3+24cw\,sw^2)+ci\,co\,so(-96cw^2sw+32sw^3)\\
&+ci^2(co^2(8cw^3-24cw\,sw^2)+so^2(-8cw^3+24cw\,sw^2))\\
&+e^2(so^2(-5cw^3-197cw\,sw^2)+co^2(5cw^3+197cw\,sw^2)\\
&+ci\,co\,so(788cw^2sw+20sw^3)+ci^2(co^2(-207cw^3-15cw\,sw^2)\\
&+so^2(207cw^3+15cw\,sw^2))))\sin 3M+e(co^2(40cw^3-120cw\,sw^2)\\
&+so^2(-40cw^3+120cw\,sw^2)+ci\,co\,so(-480cw^2sw+160sw^3)\\
&+ci^2(co^2(40cw^3-120cw\,sw^2)+so^2(-40cw^3+120cw\,sw^2)))\sin 4M\\
&+e^2\,(co^2(127cw^3-381cw\,sw^2)+so^2(-127cw^3+381cw\,sw^2)\\
&+ci\,co\,so(-1524cw^2sw+508sw^3)+ci^2(co^2(127cw^3-381cw\,sw^2)\\
&+so^2(-127cw^3+381cw\,sw^2)))\sin 5M)
\end{aligned}
\tag{6.58}
$$

The long periodic terms included in above formulas are given as follows:

$$
d_2=-\frac{15}{8}e\;si((-1-2ci+3ci^2)\cos(2o-w)+(-1+2ci+3ci^2)\cos(2o+w))
\tag{6.59}
$$

$$
\begin{aligned}
d_3=&\frac{15}{16}si((-2+4ci-2ci^2)e^2\sin(2o-3w)\\
&+(2-12ci+10ci^2+(1-62ci+61ci^2)e^2)\sin(2o-w)\\
&+(-2-12ci-10ci^2+(-1-62ci-61ci^2)\;e^2)\sin(2o+w)\\
&+(2+4ci+2ci^2)e^2\sin(2o+3w))
\end{aligned}
$$

$$
d_4=-\frac{15}{4}e\;si((1+2ci-3ci^2)\cos(2o-w)+(-1+2ci+3ci^2)\cos(2o+w))
$$

$$
\begin{aligned}
d_5=&-\frac{15}{8}e((ci+2ci^2-3ci^3+(-2+6ci)si^2)\sin(2o-w)+(-ci+2ci^2+3ci^3\\
&+(-2-6ci)si^2)\sin(2o+w))
\end{aligned}
$$

$$
d_6=\frac{15}{2a}e\;si((1+2ci-3ci^2)\sin(2o-w)+(-1+2ci+3ci^2)\sin(2o+w))
$$

6.5.7 Solutions of D_{33} Perturbation

$$
\begin{aligned}
\frac{\mathrm{d}a_1}{\mathrm{d}t} =& -15\sin(\frac{M}{2})\cos(\frac{M}{2})e(e(co\,so^2(-45cw)+co^3(15cw)+ci^2(co^3(-15cw)\\
&+co\,so^2(45cw))+ci(co^2so(-45sw)+so^3\,(15sw))+ci^3(so^3(-15sw)\\
&+co^2so(45sw)))+(co\,so^2(-18cw)+co^3(6cw)+ci^2(co^3(-6cw)\\
&+co\,so^2(18cw))+ci(co^2so(-18sw)+so^3(6sw))+ci^3(so^3(-6sw)\\
&+co^2so(18sw)))\cos M+e(co\,so^2(-84cw^3-72cw\,sw^2)\\
&+co^3(28cw^3+24cw\,sw^2)+ci^2(co^3(-24cw^3-36cw\,sw^2)\\
&+co\,so^2(72cw^3+108cw\,sw^2))+ci(co^2so(-108cw^2sw-72sw^3)\\
&+so^3(36cw^2sw+24sw^3))+ci^3(so^3(-24cw^2sw-28sw^3)\\
&+co^2so(72cw^2sw+84sw^3)))\cos 2M+(co^3(2cw^3-6cw\,sw^2)\\
&+co\,so^2(-6cw^3+18cw\,sw^2)+ci^2(co^3(6cw^3-18cw\,sw^2)\\
&+co\,so^2(-18cw^3+54cw\,sw^2))+ci^3(so^3(6cw^2sw-2sw^3)\\
&+co^2so(-18cw^2sw+6sw^3))+ci(so^3(18cw^2sw-6sw^3)\\
&+co^2so(-54cw^2sw+18sw^3)))\cos 3M+e(co^3(13cw^3-39cw\,sw^2)\\
&+co\,so^2(-39cw^3+117cw\,sw^2)+ci^2(co^3(39cw^3-117cw\,sw^2)\\
&+co\,so^2\,(-117cw^3+351cw\,sw^2))+ci^3(so^3(39cw^2sw-13sw^3)\\
&+co^2so(-117cw^2sw+39sw^3))+ci(so^3(117cw^2sw-39sw^3)\\
&+co^2so(-351cw^2sw+117sw^3)))\cos 4M+(co^3(-6sw)+co\,so^2(18sw)\\
&+ci(co^2so(-18cw)+so^3(6cw))+ci^3(so^3(-6cw)+co^2so(18cw))\\
&+ci^2(co\,so^2(-18sw)+co^3(6sw)))\sin M+e(co^3(-30cw^2sw-26sw^3)\\
&+co\,so^2(90cw^2sw+78sw^3)+ci(co^2so(-90cw^3-54cw\,sw^2)\\
&+so^3(30cw^3+18cw\,sw^2))+ci^3(so^3(-26cw^3-30cw\,sw^2)\\
&+co^2so(78cw^3+90cw\,sw^2))+ci^2(co\,so^2(-54cw^2sw-90sw^3)\\
&+co^3(18cw^2sw+30sw^3)))\sin 2M+(co\,so^2(18cw^2sw-6sw^3)\\
&+co^3(-6cw^2sw+2sw^3)+ci^3(so^3(2cw^3-6cw\,sw^2)\\
&+co^2so(-6cw^3+18cw\,sw^2))+ci(so^3(6cw^3-18cw\,sw^2)\\
&+co^2so(-18cw^3+54cw\,sw^2))+ci^2(co\,so^2(54cw^2sw-18sw^3)\\
&+co^3(-18cw^2sw+6sw^3)))\sin 3M+e(co\,so^2(117cw^2sw-39sw^3)\\
&+co^3(-39cw^2sw+13sw^3)+ci^3(so^3(13cw^3-39cw\,sw^2)\\
&+co^2so(-39cw^3+117cw\,sw^2))+ci(so^3(39cw^3-117cw\,sw^2)\\
&+co^2so(-117cw^3+351cw\,sw^2))+ci^2(co\,so^2(351cw^2sw-117sw^3)\\
&+co^3(-117cw^2sw+39sw^3)))\sin 4M)
\end{aligned}
\tag{6.60}
$$

$$
\begin{aligned}
\frac{\mathrm{d}e_1}{\mathrm{d}t} = &-\frac{45}{32}(e(co\,so^2(-24sw)+co^3(8sw)+ci^3(co^2so(-24cw)+so^3(8cw))\\
&+ci(co^2so(96cw-72cw^3-72cw\,sw^2)+so^3(-32cw+24cw^3\\
&+24cw\,sw^2))+ci^2(co^3((64-72cw^2)sw-72sw^3)\\
&+co\,so^2((-192+216cw^2)sw+216sw^3)))+(co\,so^2(-24sw)\\
&+co^3(8sw)+ci^3(co^2so(-24cw)+so^3(8cw))+ci(co^2so(96cw-72cw^3\\
&-72cw\,sw^2)+so^3(-32cw+24cw^3+2cw\,sw^2))\\
&+ci^2(co^3((64-72cw^2)sw-72sw^3)+co\,so^2((-192+216cw^2)sw\\
&+216sw^3))+e^2(co\,so^2(-90cw^2sw-78sw^3)+co^3(30cw^2sw+26sw^3)\\
&+ci^3(co^2so(-78cw^3-90cw\,sw^2)+so^3(26cw^3+30cw\,sw^2))\\
&+ci(co^2so(324cw-234cw^3-270cw\,sw^2)+so^3(-108cw+78cw^3\\
&+90cw\,sw^2))+ci^2(co^3((216-234cw^2)sw-246sw^3)\\
&+co\,so^2((-648+702cw^2)sw+738sw^3))))\cos M+e(ci^3(-96co^2cw^3so\\
&+32cw^3so^3)+ci(co^2(288cw-288cw^3)so+(-96cw+96cw^3)so^3)\\
&+32co^3sw^3-96co\,so^2sw^3+ci^2(co^3((192-288cw^2)sw-192sw^3)\\
&+co\,so^2((-576+864cw^2)sw+576sw^3)))\cos 2M\\
&+(co^3(24cw^2sw-8sw^3)+co\,so^2(-72cw^2sw+24sw^3)\\
&+ci^3(co^2so(24cw^3-72cw\,sw^2)+so^3(-8cw^3+24cw\,sw^2))\\
&+ci(co^2so(72cw^3-216cw\,sw^2)+so^3(-24cw^3+72cw\,sw^2))\\
&+ci^2(co^3(72cw^2sw-24sw^3)+co\,so^2(-216cw^2sw+72sw^3))\\
&+e^2(co\,so^2(273cw^2sw-303sw^3)+co^3(-91cw^2sw+101sw^3)\\
&+ci^3(so^3(101cw^3-91cw\,sw^2)+co^2so(-303cw^3+273cw\,sw^2))\\
&+ci(so^3(-212cw+303cw^3-273cw\,sw^2)+co^2so(636cw-909cw^3\\
&+819cw\,sw^2))+ci^2(co^3((424-909cw^2)sw-333sw^3)\\
&+co\,so^2((-1272+2727cw^2)sw+999sw^3))))\cos 3M\\
&+e(co^3(120cw^2sw-40sw^3)+co\,so^2(-360cw^2sw+120sw^3)\\
&+ci^3(co^2so(120cw^3-360cw\,sw^2)+so^3(-40cw^3+120cw\,sw^2))\\
&+ci(co^2so(360cw^3-1080cw\,sw^2)+so^3(-120cw^3+360cw\,sw^2))\\
&+ci^2(co^3(360cw^2sw-120sw^3)+co\,so^2(-1080cw^2sw+360sw^3)))\cos 4M\\
&+e^2(co^3(381cw^2sw-127sw^3)+co\,so^2(-1143cw^2sw+381sw^3)\\
&+ci^3(co^2so(381cw^3-1143cw\,sw^2)+so^3(-127cw^3+381cw\,sw^2))\\
&+ci(co^2so(1143cw^3-3429cw\,sw^2)+so^3(-381cw^3+1143cw\,sw^2))\\
&+ci^2(co^3(1143cw^2sw-381sw^3)+co\,so^2(-3429cw^2sw
\end{aligned}
$$

$$
\begin{aligned}
&+1143sw^3)))\cos 5M + (co\,so^2(-24cw) + co^3(8cw) \\
&+ci^2(co^3(64cw - 72cw^3 - 72cw\,sw^2) + co\,so^2(-192cw + 216cw^3 \\
&+216cw\,sw^2)) + ci^3(so^3(-8sw) + co^2so(24sw)) \\
&+ci(so^3((32 - 24cw^2)sw - 24sw^3) + co^2so((-96 + 72cw^2)sw + 72sw^3)) \\
&+e^2(co\,so^2(-18cw^3 - 6cw\,sw^2) + co^3(6cw^3 + 2cw\,sw^2) + ci^2(co^3(40cw \\
&-42cw^3 - 54cw\,sw^2) + co\,so^2(-120cw + 126cw^3 + 162cw\,sw^2)) \\
&+ci^3(so^3(-2cw^2sw - 6\,sw^3) + co^2so(6\,cw^2sw + 18\,sw^3)) \\
&+ci(so^3((20 - 6cw^2)sw - 18sw^3) + co^2so((-60 + 18cw^2)sw \\
&+54sw^3))))\sin M + e(co\,so^2(-48cw^3 - 144cw\,sw^2) + co^3(16cw^3 \\
&+48cw\,sw^2) + ci^2(co^3(192cw - 240cw^3 - 144cw\,sw^2) + co\,so^2(-576cw \\
&+720cw^3 + 432cw\,sw^2)) + ci^3(so^3(-48cw^2sw - 16sw^3) \\
&+co^2so(144cw^2sw + 48\,sw^3)) + ci(so^3((96 - 144cw^2)sw - 48sw^3) \\
&+\ co^2so((-288 + 432cw^2)sw + 144sw^3)))\sin 2M \\
&+(co^3(8cw^3 - 24cw\,sw^2) + co\,so^2(-24cw^3 + 72cw\,sw^2) \\
&+ci^2(co^3(24cw^3 - 72cw\,sw^2) + co\,so^2(-72cw^3 + 216cw\,sw^2)) \\
&+ci^3(so^3(24cw^2sw - 8sw^3) + co^2so(-72cw^2sw + 24sw^3)) \\
&+ci(so^3(72cw^2sw - 24sw^3) + co^2so(-216cw^2sw + 72sw^3)) \\
&+e^2(co\,so^2(-15cw^3 - 591cw\,sw^2) + co^3(5cw^3 + 197cw\,sw^2) \\
&+ci^2(co^3(424cw - 621cw^3 - 45cw\,sw^2) + co\,so^2(-1272cw + 1863cw^3 \\
&+135cw\,sw^2)) + ci^3\,(so^3\,(-197cw^2\,sw - 5sw^3) + co^2\,so(591\,cw^2\,sw \\
&+15sw^3)) + ci(so^3\,((212 - 591cw^2)sw - 15sw^3) + co^2\,so\,((-636 \\
&+1773cw^2)sw + 45sw^3))))\sin 3M + e(co^3(40cw^3 - 120cw\,sw^2) \\
&+co\,so^2(-120cw^3 + 360cw\,sw^2) + ci^2(co^3(120cw^3 - 360cw\,sw^2) \\
&+co\,so^2(-360cw^3 + 1080cw\,sw^2)) + ci^3(so^3(120cw^2sw - 40sw^3) \\
&+co^2so(-360cw^2sw + 120sw^3)) + ci(so^3(360cw^2sw - 120sw^3) \\
&+co^2so(-1080cw^2sw + 360sw^3)))\sin 4M + e^2(co^3(127cw^3 \\
&-381cw\,sw^2) + co\,so^2(-381cw^3 + 1143cw\,sw^2) \\
&+ci^2(co^3(381cw^3 - 1143cw\,sw^2) + co\,so^2(-1143cw^3 + 3429cw\,sw^2)) \\
&+ci^3(so^3(381cw^2sw - 127sw^3) + co^2so(-1143cw^2sw + 381sw^3)) \\
&+ci(so^3(1143cw^2sw - 381sw^3) + co^2so(-3429cw^2sw \\
&+1143sw^3)))\sin 5M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\omega_1}{\mathrm{d}t} = \frac{15}{4} & (co\,so^2(-18cw) + co^3(6cw) + ci^2(co^3(-6cw + co\,so^2(18cw)) \\
& + ci(co^2so(-18sw) + so^3(6sw)) + ci^3(so^3(-6sw) + co^2so(18sw)) \\
& + e^2(co\,so^2(-108cw^3 - 90cw\,sw^2) + co^3(36cw^3 + 30cw\,sw^2) \\
& + ci^2(co^3(-30cw^3 - 48cw\,sw^2) + co\,so^2(90cw^3 + 144cw\,sw^2)) \\
& + ci(co^2so(-144cw^2sw - 90sw^3) + so^3(48cw^2sw + 30sw^3)) \\
& + ci^3(so^3(-30cw^2sw - 36sw^3) + co^2so(90cw^2sw + 108sw^3))) \\
& + e(co\,so^2(-102cw^3 - 90cw\,sw^2) + co^3(34cw^3 + 30cw\,sw^2) \\
& + ci^2(co^3(-30cw^3 - 42cw\,sw^2) + co\,so^2(90cw^3 + 126cw\,sw^2)) \\
& + ci(co^2so(-126cw^2sw - 90sw^3) + so^3(42cw^2sw + 30sw^3)) \\
& + ci^3(so^3(-30cw^2sw - 34sw^3) + co^2so(90cw^2sw + 102sw^3)))\cos M \\
& + (8co^3cw^3 - 24co\,cw^3so^2 + ci(-72co^2cw^2so\,sw + 24cw^2so^3sw) \\
& + ci^2(-24co^3cw\,sw^2 + 72co\,cw\,so^2sw^2) + ci^3(24co^2so\,sw^3 - 8so^3sw^3) \\
& + e^2(co\,so^2(-207cw^3 - 171cw\,sw^2) + co^3(69cw^3 + 57cw\,sw^2) \\
& + ci^2(co^3(-57cw^3 - 93cw\,sw^2) + co\,so^2(171cw^3 + 279cw\,sw^2)) \\
& + ci(co^2so(-279cw^2sw - 171sw^3) + so^3(93cw^2sw + 57sw^3)) \\
& + ci^3(so^3(-57cw^2sw - 69sw^3) + co^2so(171cw^2sw + 207sw^3))))\cos 2M \\
& + e(co\,so^2(-99cw^3 - 27cw\,sw^2)\ ?co^3(33cw^3 + 9cw\,sw^2) \\
& + ci^2(co^3(-9cw^3 - 81cw\,sw^2) + co\,so^2(27cw^3 + 243cw\,sw^2)) \\
& + ci(co^2so(-243cw^2sw - 27sw^3) + so^3(81cw^2sw + 9sw^3)) \\
& + ci^3(so^3(-9cw^2sw - 33sw^3) + co^2so(27cw^2sw + 99sw^3)))\cos 3M \\
& + (co^3(2cw^3 - 6cw\,sw^2) + co\,so^2(-6cw^3 + 18cw\,sw^2) \\
& + ci^2(co^3(6cw^3 - 18cw\,sw^2) + co\,so^2(-18cw^3 + 54cw\,sw^2)) \\
& + ci^3(so^3(6cw^2sw - 2sw^3) + co^2so(-18cw^2sw + 6sw^3)) \\
& + ci(so^3(18cw^2sw - 6sw^3) + co^2so(-54cw^2sw + 18sw^3)) \\
& + e^2(co\,so^2(-252cw^3 - 198cw\,sw^2) + co^3(84cw^3 + 66cw\,sw^2) \\
& + ci^2(co^3(-66cw^3 - 120cw\,sw^2) + co\,so^2(198cw^3 + 360cw\,sw^2)) \\
& + ci(co^2so(-360cw^2sw - 198sw^3) + so^3(120cw^2sw + 66sw^3)) \\
& + ci^3(so^3(-66cw^2sw - 84sw^3) + co^2so(198cw^2sw + 252sw^3))))\cos 4M \\
& + e(co^3(13cw^3 - 39cw\,sw^2) + co\,so^2(-39cw^3 + 117cw\,sw^2) \\
& + ci^2(co^3(39cw^3 - 117cw\,sw^2) + co\,so^2(-117cw^3 + 351cw\,sw^2)) \\
& + ci^3(so^3(39cw^2sw - 13sw^3) + co^2so(-117cw^2sw + 39sw^3)) \\
& + ci(so^3(117cw^2sw - 39sw^3) + co^2so(-351cw^2sw + 117sw^3)))\cos 5M
\end{aligned}
$$

$$
\begin{aligned}
&+e^2(co^3(51cw^3-153cw\,sw^2)+co\,so^2(-153cw^3+459cw\,sw^2)\\
&+ci^2(co^3(153cw^3-459cw\,sw^2)+co\,so^2(-459cw^3+1377cw\,sw^2))\\
&+ci^3(so^3(153cw^2sw-51sw^3)+co^2so(-459cw^2sw+153sw^3))\\
&+ci(so^3(459cw^2sw-153sw^3)+co^2so(-1377cw^2sw+459sw^3)))\cos 6M\\
&+e(co^3(-6cw^2sw-2sw^3)+co\,so^2(18cw^2sw+6sw^3)\\
&+ci(so^3(6cw^3-6cw\,sw^2)+co^2so(-18cw^3+18cw\,sw^2))\\
&+ci^3(so^3(-2cw^3-6cw\,sw^2)+co^2\,so(6cw^3+18cw\,sw^2))\\
&+ci^2(co\,so^2(18cw^2sw-18sw^3)+co^3(-6cw^2sw+6sw^3)))\sin M\\
&+(co^3(-12cw^2sw-4sw^3)+co\,so^2(36cw^2sw+12sw^3)\\
&+ci(so^3(12cw^3-12cw\,sw^2)+co^2so(-36cw^3+36cw\,sw^2))\\
&+ci^3(so^3(-4cw^3-12cw\,sw^2)+co^2so(12cw^3+36cw\,sw^2))\\
&+ci^2(co\,so^2(36cw^2sw-36sw^3)+co^3(-12cw^2sw+12sw^3))\\
&+e^2(co^3(-15cw^2sw-3sw^3)+co\,so^2(45cw^2sw+9sw^3)\\
&+\ ci^3(so^3(-3cw^3-15cw\,sw^2)\\
&+\ e^2(co^3(-15cw^2sw-3sw^3)+co\,so^2(45cw^2sw+9sw^3)\\
&+ci^3(so^3(-3cw^3-15cw\,sw^2)+co^2so(9cw^3+45cw\,sw^2))\\
&+ci(so^3(15cw^3-21cw\,sw^2)+co^2so(-45cw^3+63cw\,sw^2))\\
&+ci^2(co\,so^2(63cw^2sw-45sw^3)+co^3(-21cw^2sw+15sw^3))))\sin 2M\\
&+e(co^3(-45cw^2sw-21sw^3)+co\,so^2(135cw^2sw+63sw^3)\\
&+ci(so^3(45cw^3-27cw\,sw^2)+co^2so(-135cw^3+81cw\,sw^2))\\
&+ci^3(so^3(-21cw^3-45cw\,sw^2)+co^2so(63cw^3+135cw\,sw^2))\\
&+ci^2(co\,so^2(81cw^2sw-135sw^3)+co^3(-27cw^2sw+45sw^3)))\sin 3M\\
&+(co\,so^2(18cw^2sw-6sw^3)+co^3(-6cw^2sw+2sw^3)\\
&+ci^3(so^3(2cw^3-6cw\,sw^2)+co^2so(-6cw^3+18cw\,sw^2))\\
&+ci(so^3(6cw^3-18cw\,sw^2)+co^2so(-18cw^3+54cw\,sw^2))\\
&+ci^2(co\,so^2(54cw^2sw-18sw^3)+co^3(-18cw^2sw+6sw^3))\\
&+e^2(co^3(-93cw^2sw-75sw^3)+co\,so^2(279cw^2sw+225sw^3)\\
&+ci(co^2so(-279cw^3-117cw\,sw^2)+so^3(93cw^3+39cw\,sw^2))\\
&+ci^3(so^3(-75cw^3-93cw\,sw^2)+co^2so(225cw^3+279cw\,sw^2))\\
&+ci^2(co\,so^2(-117cw^2sw-279sw^3)+co^3(39cw^2sw+93sw^3))))\sin 4M\\
&+e(co\,so^2(117cw^2sw-39sw^3)+co^3(-39cw^2sw+13sw^3)\\
&+ci^3(so^3(13cw^3-39cw\,sw^2)+co^2so(-39cw^3+117cw\,sw^2))
\end{aligned}
$$

$$+ci(so^3(39cw^3-117cw\,sw^2)+co^2so(-117cw^3+351cw\,sw^2))$$
$$+ci^2(co\,so^2(351cw^2sw-117sw^3)+co^3(-117cw^2sw+39sw^3)))\sin 5M$$
$$+e^2(co\,so^2(459cw^2sw-153sw^3)+co^3(-153cw^2sw+51sw^3)$$
$$+ci^3(so^3(51cw^3-153cw\,sw^2)+co^2so(-153cw^3+459cw\,sw^2))$$
$$+ci(so^3(153cw^3-459cw\,sw^2)+co^2so(-459cw^3+1377cw\,sw^2))$$
$$+ci^2(co\,so^2(1377cw^2sw-459sw^3)+co^3(-459cw^2sw+153sw^3)))\sin 6M)$$

$$\frac{\mathrm{d}i_1}{\mathrm{d}t}=-\frac{45}{32}(e(so^3(-24cw)+co^2so(72cw)+ci^2(co^2so(-72cw)$$
$$+so^3(24cw))+ci(co\,so^2(-72sw)+co^3(24sw))+ci^3(co^3(-24w)$$
$$+co\,so^2(72sw)))+(so^3(-24cw)+co^2so(72cw)+ci^2(co^2so(-72cw)$$
$$+so^3(24cw))+ci(co\,so^2(-72sw)+co^3(24sw))+ci^3(co^3(-24sw)$$
$$+co\,so^2(72sw))+e^2(so^3(-82cw^3-78cw\,sw^2)+co^2so(246cw^3$$
$$+234cw\,sw^2)+ci^2(co^2so(-234cw^3-270cw\,sw^2)+so^3(78cw^3$$
$$+90cw\,sw^2))+ci(co\,so^2(-270cw^2sw-234sw^3)+co^3(90cw^2sw$$
$$+78sw^3))+ci^3(co^3(-78cw^2sw-82sw^3)+co\,so^2(234cw^2sw$$
$$+246sw^3))))\cos M+e(ci^2(-288co^2cw^3so+96cw^3so^3)$$
$$+so^3(-64cw^3-96cw\,sw^2)+co^2so(192cw^3+288cw\,sw^2)$$
$$+ci(96co^3sw^3-288\mathrm{co}\,so^2sw^3)+ci^3(co^3(-96cw^2sw-64sw^3)$$
$$+co\,so^2(288cw^2sw+192sw^3)))\cos 2M+(co^2so(24cw^3-72cw\,sw^2)$$
$$+so^3(-8cw^3+24cw\,sw^2)+ci^2(co^2so(72cw^3-216cw\,sw^2)$$
$$+so^3(-24cw^3+72cw\,sw^2))+ci^3(co^3(24cw^2sw-8sw^3)$$
$$+co\,so^2(-72cw^2sw+24sw^3))+ci(co^3(72cw^2sw-24sw^3)$$
$$+co\,so^2(-216cw^2sw+72sw^3))+e^2(so^3(-111cw^3-303cw\,sw^2)$$
$$+co^2so(333cw^3+909cw\,sw^2)+ci^2(so^3(303cw^3-273cw\,sw^2)$$
$$+co^2so(-909cw^3+819cw\,sw^2))+ci(co\,so^2(819cw^2sw-909sw^3)$$
$$+co^3(-273cw^2sw+303sw^3))+ci^3(co^3(-303cw^2sw-111sw^3)$$
$$+co\,so^2(909cw^2sw+333sw^3))))\cos 3M+e(co^2so(120cw^3-360cw\,sw^2)$$
$$+so^3(-40cw^3+120cw\,sw^2)+ci^2(co^2so(360cw^3-1080cw\,sw^2)$$
$$+so^3(-120cw^3+360cw\,sw^2))+ci^3(co^3(120cw^2sw-40sw^3)$$
$$+co\,so^2(-360cw^2sw+120sw^3))+ci(co^3(360cw^2sw-120sw^3)$$
$$+co\,so^2(-1080cw^2sw+360sw^3)))\cos 4M+e^2(co^2so(381cw^3$$

$$
\begin{aligned}
&- 1143cw\,sw^2) + so^3(-127cw^3 + 381cw\ sw^2) + ci^2(co^2so(1143cw^3\\
&- 3429cw\,sw^2) + so^3(-381cw^3 + 1143cw\ sw^2)) + ci^3(co^3(381cw^2sw\\
&- 127sw^3) + co\ so^2(-1143cw^2sw + 381sw^3)) + ci(co^3(1143cw^2sw\\
&- 381sw^3) + co\ so^2(-3429cw^2sw + 1143sw^3)))\cos 5M\\
&+ (co^2so(-72sw) + so^3(24sw) + ci(co\ so^2(-72cw) + co^3(24cw))\\
&+ ci^3(co^3(-24cw) + co\ so^2(72cw)) + ci^2(so^3(-24sw) + co^2so(72sw))\\
&+ e^2(co^2so(-54cw^2sw - 42sw^3) + so^3(18cw^2sw + 14sw^3)\\
&+ ci(co\ so^2(-54cw^3 - 18cw\ sw^2) + co^3(18cw^3 + 6cw\ sw^2))\\
&+ ci^3(co^3(-14cw^3 - 18cw\ sw^2) + co\ so^2(42cw^3 + 54cw\ sw^2))\\
&+ ci^2(so^3(-6cw^2sw - 18sw^3) + co^2so(18cw^2sw + 54sw^3))))\sin M\\
&+ e(co^2so(-144cw^2sw - 240sw^3) + so^3(48cw^2sw + 80sw^3)\\
&+ ci(co\ so^2(-144cw^3 - 432cw\ sw^2) + co^3(48cw^3 + 144cw\ sw^2))\\
&+ ci^3(co^3(-80cw^3 - 48cw\ sw^2) + co\ so^2(240cw^3 + 144cw\ sw^2))\\
&+ ci^2(so^3(-144cw^2sw - 48sw^3) + co^2so(432cw^2sw + 144sw^3)))\sin 2M\\
&+ (so^3(24cw^2sw - 8sw^3) + co^2so(-72cw^2sw + 24sw^3)\\
&+ ci^3(co^3(8cw^3 - 24cw\ sw^2) + co\ so^2(-24cw^3 + 72cw\ sw^2))\\
&+ ci(co^3(24cw^3 - 72cw\ sw^2) + co\ so^2(-72cw^3 + 216cw\ sw^2))\\
&+ ci^2(so^3(72cw^2sw - 24sw^3) + co^2so(-216cw^2sw + 72sw^3))\\
&+ e^2(co^2so(-45cw^2sw - 621sw^3) + so^3(15cw^2sw + 207sw^3)\\
&+ ci^3(co^3(-207cw^3 - 15cw\ sw^2) + co\ so^2(621cw^3 + 45cw\ sw^2))\\
&+ ci(co\ so^2(-45cw^3 - 1773cw\ sw^2) + co^3(15cw^3 + 591cw\ sw^2))\\
&+ ci^2(so^3(-591cw^2sw - 15sw^3) + co^2so(1773cw^2sw + 45sw^3))))\sin 3M\\
&+ e(so^3(120cw^2sw - 40sw^3) + co^2so(-360cw^2sw + 120sw^3)?\\
&ci^3(co^3(40cw^3 - 120cw\ sw^2) + co\ so^2(-120cw^3 + 360cw\ sw^2))\\
&+ ci(co^3(120cw^3 - 360cw\ sw^2) + co\ so^2(-360cw^3 + 1080cw\ sw^2))\\
&+ ci^2(so^3(360cw^2sw - 120sw^3) + co^2so(-1080cw^2sw + 360sw^3)))\sin 4M\\
&+ e^2(so^3(381cw^2sw - 127sw^3) + co^2so(-1143cw^2sw + 381sw^3)\\
&+ ci^3(co^3(127cw^3 - 381cw\ sw^2) + co\ so^2(-381cw^3 + 1143cw\ sw^2))\\
&+ ci(co^3(381cw^3 - 1143cw\ sw^2) + co\ so^2(-1143cw^3 + 3429cw\ sw^2))\\
&+\ ci^2(so^3(1143cw^2sw - 381sw^3) + co^2so(-3429cw^2sw\\
&+ 1143sw^3)))\sin 5M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} =& \frac{45}{32} si(e(so^3(-8sw) + co^2 so(24sw) + ci(co\ so^2(-48cw) + co^3(16cw)) \\
&+ ci^2(co^2 so(-72sw) + so^3(24sw))) + (so^3(-8sw) + co^2 so(24sw) \\
&+ ci(co\ so^2(-48cw) + co^3(16cw)) + ci^2(co^2 so(-72sw) + so^3(24sw)) \\
&+ e^2(so^3(-30cw^2 sw - 26sw^3) + co^2 so(90cw^2 sw + 78sw^3) \\
&+ ci(co\ so^2(-156cw^3 - 180cw\ sw^2) + co^3(52cw^3 + 60cw\ sw^2)) \\
&+ ci^2(co^2 so(-234cw^2 sw - 246sw^3) + so^3(78cw^2 sw + 82sw^3)))) \cos M \\
&+ e(ci(64co^3 cw^3 - 192co\, cw^3 so^2) + 96co^2 so\ sw^3 - 32so^3 sw^3 \\
&+ ci^2(co^2 so(-288cw^2 sw - 192sw^3) + so^3(96cw^2 sw + 64sw^3))) \cos 2M \\
&+ (co^2 so(72cw^2 sw - 24sw^3) + so^3(-24cw^2 sw + 8sw^3) \\
&+ ci(co\ so^2(48cw^3 - 144cw\ sw^2) + co^3(-16cw^3 + 48cw\ sw^2)) \\
&+ ci^2(co^2 so(72cw^2 sw - 24sw^3) + so^3(-24cw^2 sw + 8sw^3)) \\
&+ e^2(so^3(91cw^2 sw - 101sw^3) + co^2 so(-273cw^2 sw + 303sw^3) \\
&+ ci(co^3(202cw^3 - 182cw\ sw^2) + co\ so^2(-606cw^3 + 546cw\ sw^2)) \\
&+ ci^2(co^2 so(-909cw^2 sw - 333sw^3) + so^3(303cw^2 sw + 111sw^3)))) \cos 3M \\
&+ e(co^2 so(360cw^2 sw - 120sw^3) + so^3(-120cw^2 sw + 40sw^3) \\
&+ ci(co\ so^2(240cw^3 - 720cw\ sw^2) + co^3(-80cw^3 + 240cw\ sw^2)) \\
&+ ci^2(co^2 so(360cw^2 sw - 120sw^3) + so^3(-120cw^2 sw + 40sw^3))) \cos 4M \\
&+ e^2(co^2 so(1143cw^2 sw - 381sw^3) + so^3(-381cw^2 sw + 127sw^3) \\
&+ ci(co\ so^2(762cw^3 - 2286cw\ sw^2) + co^3(-254cw^3 + 762cw\ sw^2)) \\
&+ ci^2(co^2 so(1143cw^2 sw - 381sw^3) + so^3(-381cw^2 sw + 127sw^3))) \cos 5M \\
&+ (so^3(-8cw) + co^2 so(24cw) + ci^2(co^2 so(-72cw) + so^3(24cw)) \\
&+ ci(co^3(-16sw) + co\ so^2(48sw)) + e^2(so^3(-6cw^3 - 2cw\ sw^2) \\
&+ co^2 so(18cw^3 + 6cw\ sw^2) + ci^2(co^2 so(-42cw^3 - 54cw\ sw^2) \\
&+ so^3(14cw^3 + 18cw\ sw^2)) + ci(co^3(-4cw^2 sw - 12sw^3) \\
&+ co\ so^2(12cw^2 sw + 36sw^3)))) \sin M + e(so^3(-16cw^3 - 48cw\ sw^2) \\
&+ co^2 so(48cw^3 + 144cw\ sw^2) + ci^2(co^2 so(-240cw^3 - 144cw\ sw^2) \\
&+ so^3(80\ cw^3 + 48cw\ sw^2)) + ci(co^3(-96cw^2 sw - 32sw^3) \\
&+ co\ so^2(288cw^2 sw + 96sw^3))) \sin 2M + (co^2 so(24cw^3 - 72cw\ sw^2) \\
&+ so^3(-8cw^3 + 24cw\ sw^2) + ci^2(co^2 so(24cw^3 - 72cw\ sw^2) \\
&+ so^3(-8cw^3 + 24cw\ sw^2)) + ci(co^3(48cw^2 sw - 16sw^3) \\
&+ co\ so^2(-144cw^2 sw + 48sw^3)) + e^2(so^3(-5cw^3 - 197cw\ sw^2)
\end{aligned}
$$

$$+co^2so(15cw^3+591cw\,sw^2)+ci^2(co^2so(-621cw^3-45cw\,sw^2)$$
$$+so^3(207cw^3+15cw\,sw^2))+ci(co^3(-394cw^2sw-10sw^3)$$
$$+co\,so^2(1182cw^2sw+30sw^3))))\sin 3M+e(co^2so(120cw^3-360cw\,sw^2)$$
$$+so^3(-40cw^3+120cw\,sw^2)+ci^2(co^2so(120cw^3-360cw\,sw^2)$$
$$+so^3(-40cw^3+120cw\,sw^2))+ci(co^3(240cw^2sw-80sw^3)$$
$$+co\,so^2(-720cw^2sw+240sw^3)))\sin 4M+e^2(co^2so(381cw^3$$
$$-1143cw\,sw^2)+so^3(-127cw^3+381cw\,sw^2)+ci^2(co^2so(381cw^3$$
$$-1143cw\,sw^2)+so^3(-127cw^3+381cw\,sw^2))$$
$$+\ ci(co^3(762cw^2sw-254sw^3)+co\,so^2(-2286cw^2sw+762sw^3)))\sin 5M)$$

$$\frac{\mathrm{d}M_1}{\mathrm{d}t}=\frac{-15}{8a}(e(co\,so^2(-72cw)+co^3(24cw)+ci^2(co^3(-24cw)+co\,so^2(72cw))$$
$$+ci(co^2so(-72sw)+so^3(24sw))+ci^3(so^3(-24sw)+co^2so(72sw)))$$
$$+(co\,so^2(-72cw)+co^3(24cw)+ci^2(co^3(-24cw)+co\,so^2(72cw)$$
$$+ci(co^2so(-72sw)+so^3(24sw))+ci^3(so^3(-24sw)+co^2so(72sw))$$
$$+e^2(co\,so^2(-246cw^3-234cw\,sw^2)+co^3(82cw^3+78cw\,sw^2)$$
$$+ci^2(co^3(-78cw^3-90cw\,sw^2)+co\,so^2(234cw^3+270cw\,sw^2))$$
$$+ci(co^2so(-270cw^2sw-234sw^3)+so^3(90cw^2sw+78sw^3))$$
$$+ci^3(so^3(-78cw^2sw-82sw^3)+co^2so(234cw^2sw+246sw^3))))\cos M$$
$$+e(ci^2(-96co^3cw^3+288co\,cw^3so^2)+co\,so^2(-192cw^3-288cw\,sw^2)$$
$$+co^3(64cw^3+96cw\,sw^2)+ci(-288co^2so\,sw^3+96so^3sw^3)$$
$$+ci^3(so^3(-96cw^2sw-64sw^3)+co^2so(288cw^2sw+192sw^3)))\cos 2M$$
$$+(co^3(8cw^3-24cw\,sw^2)+co\,so^2(-24cw^3+72cw\,sw^2)$$
$$+ci^2(co^3(24cw^3-72cw\,sw^2)+co\,so^2(-72cw^3+216cw\,sw^2))$$
$$+ci^3(so^3(24cw^2sw-8sw^3)+co^2so(-72cw^2sw+24sw^3))$$
$$+ci(so^3(72cw^2sw-24sw^3)+co^2so(-216cw^2sw+72sw^3))$$
$$+e^2(co\,so^2(-333cw^3-909cw\,sw^2)+co^3(111cw^3+303cw\,sw^2)$$
$$+ci^2(co\,so^2(909cw^3-819cw\,sw^2)+co^3(-303cw^3+273cw\,sw^2))$$
$$+ci(co^2so(819cw^2sw-909sw^3)+so^3(-273cw^2sw+303sw^3))$$
$$+ci^3(so^3(-303cw^2sw-111sw^3)+co^2so(909cw^2sw+333sw^3))))\cos 3M$$
$$+e(co^3(40cw^3-120cw\,sw^2)+co\,so^2(-120cw^3+360cw\,sw^2)$$
$$+ci^2(co^3(120cw^3-360cw\,sw^2)+co\,so^2(-360cw^3+1080cw\,sw^2))$$
$$+ci^3(so^3(120cw^2sw-40sw^3)+co^2so(-360cw^2sw+120sw^3))$$

$$
\begin{aligned}
&+ci(so^3(360cw^2sw-120sw^3)+co^2so(-1080cw^2sw+360sw^3)))\cos 4M\\
&+e^2(co^3(127cw^3-381cw\,sw^2)+co\,so^2(-381cw^3+1143cw\,sw^2)\\
&+ci^2(co^3(381cw^3-1143cw\,sw^2)+co\,so^2(-1143cw^3+3429cw\,sw^2))\\
&+ci^3(so^3(381cw^2sw-127sw^3)+co^2so(-1143cw^2sw+381sw^3))\\
&+ci(so^3(1143cw^2sw-381sw^3)+co^2so(-3429cw^2sw\\
&+1143sw^3)))\cos 5M+(co^3(-24sw)+co\,so^2(72sw)\\
&+ci(co^2so(-72cw)+so^3(24cw))+ci^3(so^3(-24cw)+co^2so(72cw))\\
&+ci^2(co\,so^2(-72sw)+co^3(24sw))+e^2(co^3(-18cw^2sw-14sw^3)\\
&+co\,so^2(54cw^2sw+42sw^3)+ci(co^2so(-54cw^3-18cw\,sw^2)\\
&+so^3(18cw^3+6cw\,sw^2))+ci^3(so^3(-14cw^3-18cw\,sw^2)\\
&+co^2so(42cw^3+54cw\,sw^2))+ci^2(co\,so^2(-18cw^2sw-54sw^3)\\
&+co^3(6cw^2sw+18sw^3))))\sin M+e(co^3(-48cw^2sw-80sw^3)\\
&+co\,so^2(144cw^2sw+240sw^3)+ci(co^2so(-144cw^3-432cw\,sw^2)\\
&+so^3(48cw^3+144cw\,sw^2))+ci^3(so^3(-80cw^3-48cw\,sw^2)\\
&+co^2so(240cw^3+144cw\,sw^2))+ci^2(co\,so^2(-432cw^2sw-144sw^3)\\
&+co^3(144cw^2sw+48sw^3)))\sin M+(co\,so^2(72cw^2sw-24sw^3)\\
&+co^3(-24cw^2sw+8sw^3)+ci^3(so^3(8cw^3-24cw\,sw^2)+co^2so(-24cw^3\\
&+72cw\,sw^2))+ci(so^3(24cw^3-72cw\,sw^2)+co^2so(-72cw^3\\
&+216cw\,sw^2))+ci^2(co\,so^2(216cw^2sw-72sw^3)\\
&+co^3(-72cw^2sw+24sw^3))+e^2(co^3(-15cw^2sw-207sw^3)\\
&+co\,so^2(45cw^2sw+621sw^3)+ci^3(so^3(-207cw^3-15cw\,sw^2)\\
&+co^2so(621cw^3+45cw\,sw^2))+ci(co^2so(-45cw^3-1773cw\,sw^2)\\
&+so^3(15cw^3+591cw\,sw^2))+ci^2(co\,so^2(-1773cw^2sw-45sw^3)\\
&+co^3(591cw^2sw+15sw^3))))\sin 3M+e(co\,so^2(360cw^2sw-120sw^3)\\
&+co^3(-120cw^2sw+40sw^3)+ci^3(so^3(40cw^3-120cw\,sw^2)\\
&+co^2so(-120cw^3+360cw\,sw^2))+ci(so^3(120cw^3-360cw\,sw^2)\\
&+co^2so(-360cw^3+1080cw\,sw^2))+ci^2(co\,so^2(1080cw^2sw-360sw^3)\\
&+co^3(-360cw^2sw+120sw^3)))\sin 4M+e^2(co\,so^2(1143cw^2sw-381sw^3)\\
&+co^3(-381cw^2sw+127sw^3)+ci^3(so^3(127cw^3-381cw\,sw^2)\\
&+co^2so(-381cw^3+1143cw\,sw^2))+ci(so^3(381cw^3-1143cw\,sw^2)\\
&+co^2so(-1143cw^3+3429cw\,sw^2))+ci^2(co\,so^2(3429cw^2sw-1143sw^3)\\
&+\ co^3(-1143cw^2sw+381sw^3)))\sin 5M)
\end{aligned}
$$

The long periodic terms included in above formulas are given as follows:

$$
\begin{aligned}
d_2 = & \frac{45}{8} e(-1+ci)(1+ci)((-1+ci)\sin(3o-w) \\
& + (1+ci)\sin(3o+w))
\end{aligned} \tag{6.61}
$$

$$
\begin{aligned}
d_3 = & \frac{45}{16}((1-3ci+3ci^2-ci^3)e^2\cos(3o-3w)+(4-4ci-4ci^2 \\
& +4ci^3+(23-23ci-23ci^2+23ci^3)e^2)\cos(3o-w) \\
& +(4+4ci-4ci^2-4ci^3+(23+23ci-23ci^2-23ci^3)e^2) \\
& \cos(3o+w)+(1+3ci+3ci^2+ci^3)e^2\cos(3o+3w))
\end{aligned}
$$

$$
d_4 = \frac{135}{8} e(-1+ci)(1+ci)((1-ci)\sin(3o-w)+(1+ci)\sin(3o+w))
$$

$$
d_5 = \frac{45}{8} e\ si((1+2ci-3ci^2)\cos(3o-w)+(-1+2ci+3ci^2)\cos(3o+w))
$$

$$
d_6 = \frac{45}{2a} e(-1+ci)(1+ci)((1-ci)\cos(3o-w)+(1+ci)\cos(3o+w))
$$

6.5.8 Solutions of D_{40} *Perturbation*

$$
\begin{aligned}
\frac{\mathrm{d}a_1}{\mathrm{d}t} = & -\frac{5}{128}(e(48cw\ si^2sw+si^4(-56cw\,sw))\cos M+(192cw\ si^2sw \\
& +si^4(-224cw\,sw)+e^2(192cw\ si^2sw+si^4(-168cw^3sw \\
& -280cw\ sw^3)))\cos 2M+e(1296cw\ si^2sw+si^4(-1764cw^3sw \\
& -1260cw\ sw^3))\cos 3M+(si^4(224cw^3sw-224cw\ sw^3) \\
& +e^2(5088cw\,si^2sw+si^4(-8400cw^3sw-3472cw\ sw^3)))\cos 4M \\
& +e\ si^4(1820cw^3sw-1820cw\ sw^3)\cos 5M+e^2si^4(8568cw^3sw \\
& -8568cw\ sw^3)\cos 6M+e(48+si^2(-216cw^2-264sw^2) \\
& +si^4(182cw^4+420cw^2sw^2+238sw^4))\sin M+(si^2(96cw^2-96sw^2) \\
& +si^4(-112cw^4+112sw^4)+e^2(192+si^2(-864cw^2-1056sw^2) \\
& +si^4(742cw^4+1596cw^2sw^2+966sw^4)))\sin 2M \\
& +e(si^2(648cw^2-648sw^2)+si^4(-819cw^4+378cw^2sw^2
\end{aligned}
$$

$$+ 693sw^4)) \sin 3M + (si^4(56cw^4 - 336cw^2sw^2 + 56sw^4)$$
$$+ e^2(si^2(2544cw^2 - 2544sw^2) + si^4(-3584cw^4$$
$$+ 3696cw^2sw^2 + 2352sw^4))) \sin 4M$$
$$+ e\ si^4(455cw^4 - 2730cw^2sw^2 + 455sw^4) \sin 5M$$
$$+ e^2\ si^4(2142cw^4 - 12852cw^2sw^2 + 2142sw^4) \sin 6M)$$

$$\begin{aligned}
\frac{de_1}{dt} = & -\frac{5}{32}si^2(e^2(36cw\ sw + si^2(-42cw^3sw - 42cw\ sw^3)) + e(24cw\ sw \\
& + si^2(-28cw^3sw - 28cw\ sw^3))\cos M + (48cw\ sw + si^2(-56cw^3sw \\
& - 56cw\ sw^3) + e^2(48cw\ sw + si^2(-28cw^3sw - 84cw\ sw^3)))\cos 2M \\
& + e(216cw\ sw + si^2(-336cw^3sw - 168cw\ sw^3))\cos 3M \\
& + (si^2(56cw^3sw - 56cw\ sw^3) + e^2(636cw\ sw + si^2(-1358cw^3sw \\
& - 126cw\ sw^3)))\cos 4M + e\ si^2(364cw^3sw - 364cw\ sw^3)\cos 5M \\
& + e^2si^2(1428cw^3sw - 1428cw\ sw^3)\cos 6M + e(12cw^2 - 12sw^2 \\
& + si^2(-14cw^4 + 14sw^4))\sin M + (24\mathrm{cw}^2 - 24\mathrm{sw}^2 + \mathrm{si}^2(-28\mathrm{cw}^4 \\
& + 28sw^4) + e^2(24cw^2 - 24sw^2 + si^2(-21cw^4 - 42cw^2sw^2 + 35sw^4)))\sin 2M \\
& + e(108cw^2 - 108sw^2 + si^2(-147cw^4 + 126cw^2sw^2 + 105sw^4))\sin 3M \\
& + (si^2(14cw^4 - 84cw^2sw^2 + 14sw^4) + e^2(318cw^2 - 318sw^2 \\
& + si^2(-525cw^4 + 924cw^2sw^2 + 217sw^4)))\sin 4M + \mathrm{e\ si}^2(91\mathrm{cw}^4 - 546\mathrm{cw}^2\mathrm{sw}^2 \\
& + 91sw^4)\sin 5M + \mathrm{e}^2\mathrm{si}^2(357\mathrm{cw}^4 - 2142\mathrm{cw}^2\mathrm{sw}^2 + 357\mathrm{sw}^4)\sin 6M)
\end{aligned} \tag{6.62}$$

$$\begin{aligned}
\frac{d\omega_1}{dt} = & \frac{5}{1024}(e(768 + si^2(-3264cw^2 - 4416sw^2) + si^4(2688cw^4 + 6720cw^2sw^2 \\
& + 4032sw^4)) + (384 + si^2(-1728cw^2 - 2112sw^2) + si^4(1456cw^4 \\
& + 3360cw^2sw^2 + 1904sw^4) + e^2(3888 + si^2(-15888cw^2 - 22992sw^2) \\
& + si^4(12859cw^4 + 34062cw^2sw^2 + 21147sw^4)))\cos M + e(1536 \\
& + si^2(-6912cw^2 - 8448sw^2) + si^4(5936cw^4 + 12768cw^2sw^2 \\
& + 7728sw^4))\cos 2M + (si^2(1728cw^2 - 1728sw^2) + si^4(-2184cw^4 \\
& + 1008cw^2sw^2 + 1848sw^4) + e^2(4176 + si^2(-21096cw^2 - 20664sw^2) \\
& + si^4(20097cw^4 + 27090cw^2sw^2 + 19593sw^4)))\cos 3M + e(si^2(10176cw^2 \\
& - 10176sw^2) + si^4(-14336cw^4 + 14784cw^2sw^2 + 9408sw^4))\cos 4M \\
& + (si^4(728cw^4 - 4368cw^2sw^2 + 728sw^4) + e^2(si^2(36984cw^2 - 36984sw^2) \\
& + si^4(-59213cw^4 + 96390cw^2sw^2 + 27083sw^4)))\cos 5M + e\ si^4(5712cw^4
\end{aligned}$$

$$- 34272cw^2sw^2 + 5712sw^4)\cos 6M + e^2si^4(26257cw^4 - 157542cw^2sw^2$$
$$+ 26257sw^4)\cos 7M + (-384cw\ si^2sw + si^4(448cw^3sw + 448cw\ sw^3)$$
$$+ e^2(-2400cw\ si^2sw + si^4(2828cw^3sw + 2772cw\ sw^3)))\sin M$$
$$+ e(-1536cw\ si^2sw + si^4(1344cw^3sw + 2240cw\ sw^3))\sin 2M$$
$$+ (-3456cw\ si^2sw + si^4(4704cw^3sw + 3360cw\ sw^3) + e^2(432cw\ si^2sw$$
$$+ si^4(-6804cw^3sw + 5796cw\ sw^3)))\sin 3M + e(-20352cw\ si^2sw$$
$$+ si^4(33600cw^3sw + 13888cw\ sw^3))\sin 4M + (si^4(-2912cw^3sw$$
$$+ 2912cw\ sw^3) + e^2(-73968cw\ si^2sw + si^4(150556cw^3sw$$
$$+ 22036cw\ sw^3)))\sin 5M + e\ si^4(-22848cw^3sw + 22848cw\ sw^3)\sin 6M$$
$$+ e^2si^4(-105028cw^3sw + 105028cw\ sw^3)\sin 7M)$$

$$\frac{d\Omega_1}{dt} = \frac{5}{32}ci\ si(-24 + 42si^2 + e^2(-102cw^2 - 138sw^2 + si^2(168cw^4 + 420cw^2sw^2$$
$$+ 252sw^4)) + e(-108cw^2 - 132sw^2 + si^2(182cw^4 + 420cw^2sw^2$$
$$+ 238sw^4))\cos M + (24cw^2 - 24sw^2 + si^2(-56cw^4 + 56sw^4)$$
$$+ e^2(-216cw^2 - 264sw^2 + si^2(371cw^4 + 798cw^2sw^2 + 483sw^4)))\cos 2M$$
$$+ e(108cw^2 - 108sw^2 + si^2(-273cw^4 + 126cw^2sw^2 + 231sw^4))\cos 3M$$
$$+ (si^2(14cw^4 - 84cw^2sw^2 + 14sw^4) + e^2(318cw^2 - 318sw^2$$
$$+ si^2(-896cw^4 + 924cw^2sw^2 + 588sw^4)))\cos 4M + e\ si^2(91cw^4$$
$$- 546cw^2sw^2 + 91sw^4)\cos 5M + e^2si^2(357cw^4 - 2142cw^2sw^2$$
$$+ 357sw^4)\cos 6M + e(-24cw\ sw + si^2(56cw\ sw))\sin M + (-48cw\ sw$$
$$+ si^2(112cw\,sw) + e^2(-48cw\ sw + si^2(84cw^3sw + 140cw\ sw^3)))\sin 2M$$
$$+ e(-216cw\ sw + si^2(588cw^3sw + 420cw\ sw^3))\sin 3M$$
$$+ (si^2(-56cw^3sw + 56cw\ sw^3) + e^2(-636cw\ sw + si^2(2100cw^3sw$$
$$+ 868cw\ sw^3)))\sin 4M + e\ si^2(-364cw^3sw + 364cw\ sw^3)\sin 5M$$
$$+ e^2si^2(-1428cw^3sw + 1428cw\ sw^3)\sin 6M)$$

$$\frac{dM_1}{dt} = -\frac{5}{128a}(48 + si^2(-240) + si^4\ (210) + e^2(240 + si^2(-1020cw^2 - 1380sw^2)$$
$$+ si^4(840cw^4 + 2100cw^2sw^2 + 1260sw^4)) + e(240 + si^2(-1080cw^2$$
$$- 1320sw^2) + si^4(910cw^4 + 2100cw^2sw^2 + 1190sw^4))\cos M + (si^2(240cw^2$$
$$- 240sw^2) + si^4\ (-280cw^4 + 280sw^4) + e^2(480 + si^2(-2160cw^2$$
$$- 2640sw^2) + si^4(1855cw^4 + 3990cw^2sw^2 + 2415sw^4)))\cos 2M$$

$$+e(si^2(1080cw^2-1080sw^2)+si^4(-1365cw^4+630cw^2sw^2$$
$$+1155sw^4))\cos 3M+(si^4(70cw^4-420cw^2sw^2+70sw^4)+e^2(si^2(3180cw^2$$
$$-3180sw^2)+si^4(-4480cw^4+4620cw^2sw^2+2940sw^4)))\cos 4M$$
$$+e\,si^4(455cw^4-2730cw^2sw^2+455sw^4)\cos 5M+e^2si^4(1785cw^4$$
$$-10710cw^2sw^2+1785sw^4)\cos 6M+e(-240cw\,si^2sw+si^4(280cw^3sw$$
$$+280cw\,sw^3))\sin M+(-480cw\,si^2sw+si^4(560cw^3sw+560cw\,sw^3)$$
$$+e^2(-480cw\,si^2sw+si^4(420cw^3sw+700cw\,sw^3)))\sin 2M$$
$$+e(-2160cw\,si^2sw+si^4(2940cw^3sw+2100cw\,sw^3))\sin 3M$$
$$+(si^4(-280cw^3sw+280cw\,sw^3)+e^2(-6360cw\,si^2sw$$
$$+si^4(10500cw^3sw+4340cw\,sw^3)))\sin 4M+e\,si^4(-1820cw^3sw$$
$$+1820cw\,sw^3)\sin 5M+e^2si^4(-7140cw^3sw+7140cw\,sw^3)\sin 6M)$$

The long periodic terms and long terms included in above formulas are given as follows:

$$d_2=\frac{15}{16}\sin(w)\cos(w)\;e^2si^2(-6+7si^2) \tag{6.63}$$

$$d_3=-\frac{15}{32}e(-8+40si^2-35si^4+(-6si^2+7si^4)\cos 2w)$$

$$d_5=-\frac{15}{16}ci\;si(4+20e^2+(-7-35e^2)si^2+(-3e^2+7e^2si^2)\cos 2w)$$

$$d_6=\frac{15}{64a}(-8-40e^2+(40+200e^2)si^2+(-35-175e^2)si^4$$
$$+(-30e^2si^2+35e^2si^4)\cos 2w)$$

6.5.9 Solutions of D_{41} Perturbation

$$\frac{da_1}{dt}=-\frac{5}{64}si(co(48)-84co\,si^2+e^2(240ci\,cw\,so\,sw+co(360cw^2+600sw^2)$$
$$+si^2(ci\,so(-280cwsw)+co(-560cw^4-1680cw^2sw^2-1120sw^4)))$$
$$+e(96ci\,cw\,so\,sw+co(288cw^2+384sw^2)+si^2(ci\,so(-56cwsw)$$
$$+co(-448cw^4-1176cw^2sw^2-728sw^4)))\cos M+(-96ci\,cw\,so\,sw$$

$$
\begin{aligned}
&+co(48cw^2-48sw^2)+ci\,si^2so(336cwsw)+e^2(384ci\,cw\,so\,sw\\
&+co(648cw^2+1032sw^2)+si^2(ci\,so(-308cw^3sw-140cw\,sw^3)\\
&+co(-889cw^4-3066cw^2sw^2-2009sw^4))))\cos 2M\\
&+e(-768ci\,cw\,so\,sw+co(384cw^2-384sw^2)+si^2(ci\,so(2856cw^3sw\\
&+1960cw\,sw^3)+co(-252cw^4+672cw^2sw^2+28sw^4)))\cos 3M\\
&+(si^2(ci\,so(-336cw^3sw+336cw\,sw^3)+co(84cw^4-504cw^2sw^2\\
&+84sw^4))+e^2(-3264ci\,cw\,so\,sw+co(1632cw^2-1632sw^2)\\
&+si^2(ci\,so(14000cw^3sw+5488cw\,sw^3)+co(-1904cw^4+6384cw^2sw^2\\
&-224sw^4))))\cos 4M+e\,si^2(ci\,so(-2800cw^3sw+2800cw\,sw^3)\\
&+co(700cw^4-4200cw^2sw^2+700sw^4))\cos 5M+e^2si^2(ci\,so(-13412cw^3sw\\
&+13412cw\,sw^3)+co(3353cw^4-20118cw^2sw^2+3353sw^4))\cos 6M\\
&+e(96co\,cw\,sw+ci\,so(288cw^2+192sw^2)+si^2(co(-280cwsw)\\
&+ci\,so(-448cw^4-840cw^2sw^2-392sw^4)))\sin M+(-96co\,cw\,sw\\
&+ci\,so(-48cw^2+48sw^2)+ci\,si^2so(168cw^4-168sw^4)+e^2(384co\,cw\,sw\\
&+ci\,so(1152cw^2+768sw^2)+si^2(co(-1204cw^3sw-1036cw\,sw^3)\\
&+ci\,so(-1813cw^4-3234cw^2sw^2-1589sw^4))))\sin 2M+e(-768co\,cw\,sw\\
&+ci\,so(-384cw^2+384sw^2)+si^2(co(728cw^3sw-168cw\,sw^3)\\
&+ci\,so(1316cw^4-672cw^2sw^2-1092sw^4)))\sin 3M+(si^2(co(-336cw^3sw\\
&+336cw\,sw^3)+ci\,so(-84cw^4+504cw^2sw^2-84sw^4))\\
&+e^2(-3264co\,cw\,sw+ci\,so(-1632cw^2+1632sw^2)+si^2(co(5936cw^3sw\\
&-2576cw\,sw^3)+ci\,so(5936cw^4-6384cw^2sw^2-3808sw^4))))\sin 4M\\
&+e\,si^2(co(-2800cw^3sw+2800cw\,sw^3)+ci\,so(-700cw^4+4200cw^2sw^2\\
&-700sw^4))\sin 5M+e^2si^2(co(-13412cw^3sw+13412cw\,sw^3)\\
&+ci\,so(-3353cw^4+20118cw^2sw^2-3353sw^4))\sin 6M)
\end{aligned}
\tag{6.64}
$$

$$
\begin{aligned}
\frac{\mathrm{d}e_1}{\mathrm{d}t}=&-\frac{5}{16}si(e^2(-36ci\,cw\,so\,sw+co(18cw^2-18sw^2)+si^2(ci\,so(84cwsw)\\
&+co(-21cw^4+21sw^4)))+e(-24ci\,cw\,so\,sw+co(12cw^2-12sw^2)\\
&+si^2(ci\,so(56cwsw)+co(-14cw^4+14sw^4)))\cos M+(-48ci\,cw\,so\,sw\\
&+co(24cw^2-24sw^2)+si^2(ci\,so(112cwsw)+co(-28cw^4+28\,sw^4))\\
&+e^2(-48ci\,cw\,so\,sw+co(24cw^2-24sw^2)+si^2(ci\,so(56cw^3sw\\
&+168cw\,sw^3)+co(-14cw^4-84cw^2sw^2+42sw^4))))\cos 2M
\end{aligned}
$$

$$
\begin{aligned}
&+e(-216ci\ cw\ so\ sw+co(108cw^2-108sw^2)+si^2(ci\ so(672cw^3sw\\
&+336cw\ sw^3)+co(-168cw^4+252cw^2sw^2+84sw^4)))\cos 3M\\
&+(si^2(ci\ so(-112cw^3sw+112cw\ sw^3)+co(28cw^4-168cw^2sw^2\\
&+28sw^4))+e^2(-636ci\ cw\ so\ sw+co(318cw^2-318sw^2)\\
&+si^2(ci\ so(2716cw^3sw+252cw\ sw^3)+co(-679cw^4+1848cw^2sw^2\\
&+63sw^4))))\cos 4M+e\ si^2(ci\ so(-728cw^3sw+728cw\ sw^3)+co(182cw^4\\
&-1092cw^2sw^2+182sw^4))\cos 5M+e^2si^2(ci\ so(-2856cw^3sw\\
&+2856cw\ sw^3)+co(714cw^4-4284cw^2sw^2+714sw^4))\cos 6M\\
&+e(-24co\ cw\ sw+ci\ so(-12cw^2+12sw^2)+si^2(co(28cwsw)\\
&+ci\ so(28cw^4-28sw^4)))sin(M)+(-48co\ cw\ sw+ci\ so(-24cw^2\\
&+24sw^2)+si^2(co(56cwsw)+ci\ so(56cw^4-56sw^4))+e^2(-48co\ cw\ sw\\
&+ci\ so(-24cw^2+24sw^2)+si^2(112co\ cw\ sw^3+ci\ so(42cw^4+84cw^2sw^2\\
&-70sw^4))))\sin 2M+e(-216co\ cw\ sw+ci\ so(-108cw^2+108sw^2)\\
&+si^2(co(420cw^3sw+84cw\ sw^3)+ci\ so(294cw^4-252cw^2sw^2\\
&-210sw^4)))\sin 3M+(si^2(co(-112cw^3sw+112cw\ sw^3)\\
&+ci\ so(-28cw^4+168cw^2sw^2-28sw^4))+e^2(-636co\ cw\ sw\\
&+ci\ so(-318cw^2+318sw^2)+si^2(co(1974cw^3sw-490cw\ sw^3)\\
&+ci\ so(1050cw^4-1848cw^2sw^2-434sw^4))))\sin 4M\\
&+e\ si^2(co(-728cw^3sw+728cw\ sw^3)+ci\ so(-182cw^4+1092cw^2sw^2\\
&-182sw^4))\sin 5M+e^2si^2(co(-2856cw^3sw+2856cw\ sw^3)\\
&+\ ci\ so(-714cw^4+4284cw^2sw^2-714sw^4))\sin 6M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\omega_1}{\mathrm{d}t}=&-\frac{5}{256}si(e(1248co\ cw\ sw+ci\ so(-1296cw^2-2544sw^2)\\
&+si^2(co(-2240cwsw)+ci\ so(2296cw^4+6720cw^2sw^2+4424sw^4)))\\
&+(576co\ cw\ sw+ci\ so(-672cw^2-1248sw^2)+si^2(co(-1120cwsw)\\
&+ci\ so(1232cw^4+3360cw^2sw^2+2128sw^4))+e^2(6816co\ cw\ sw\\
&+ci\ so(-6312cw^2-13128sw^2)+si^2(co(-11676cw^3sw-11844cw\ sw^3)\\
&+ci\ so(10983cw^4+33894cw^2sw^2+23079sw^4))))\cos M+e(2304co\ cw\ sw\\
&+ci\ so(-2688cw^2-4992sw^2)+si^2(co(-4144cw^3sw-4816cw\ sw^3)\\
&+ci\ so(5012cw^4+12936cw^2sw^2+8596sw^4)))\cos 2M\\
&+(1344co\ cw\ sw+ci\ so(672cw^2-672sw^2)+si^2(co(-1344cw^3sw
\end{aligned}
$$

$$
\begin{aligned}
&- 896 cw\,sw^3) + ci\,so(-1848 cw^4 + 336 cw^2 sw^2 + 1736 sw^4))\\
&+ e^2(4392 co\,cw\,sw + ci\,so(-8244 cw^2 - 12636 sw^2) + si^2(co(-5880 cw^3 sw\\
&- 15120 cw\,sw^3) + ci\,so(16989 cw^4 + 29610 cw^2 sw^2 + 21861 sw^4))))\cos 3M\\
&+ e(7968 co\,cw\,sw + ci\,so(3984 cw^2 - 3984 sw^2) + si^2(co(-12992 cw^3 sw\\
&- 448 cw\,sw^3) + ci\,so(-12152 cw^4 + 9408 cw^2 sw^2 + 9016 sw^4)))\cos 4M\\
&+ (si^2(co(2464 cw^3 sw - 2464 cw\,sw^3) + ci\,so(616 cw^4 - 3696 cw^2 sw^2\\
&+ 616 sw^4)) + e^2(29112 co\,cw\,sw + ci\,so(14556 cw^2 - 14556 sw^2)\\
&+ si^2(co(-71680 cw^3 sw + 22120 cw\,sw^3) + ci\,so(-50281 cw^4\\
&+ 70350 cw^2 sw^2 + 26831 sw^4))))\cos 5M + e\,si^2(co(19376 cw^3 sw\\
&- 19376 cw\,sw^3) + ci\,so(4844 cw^4 - 29064 cw^2 sw^2 + 4844 sw^4))\cos 6M\\
&+ e^2 si^2(co(89236 cw^3 sw - 89236 cw\,sw^3) + ci\,so(22309 cw^4\\
&- 133854 cw^2 sw^2 + 22309 sw^4))\cos 7M + (-576 ci\,cw\,so\,sw\\
&+ co(672 cw^2 + 96 sw^2) + si^2(ci\,so(896 cw^3 sw + 896 cw\,sw^3)\\
&+ co(-1232 cw^4 - 1344 cw^2 sw^2 - 112 sw^4)) + e^2(-2160 ci\,cw\,so\,sw\\
&+ co(2136 cw^2 - 24 sw^2) + si^2(ci\,so(3836 cw^3 sw + 4004 cw\,sw^3)\\
&+ co(-3647 cw^4 - 3822 cw^2 sw^2 - 7 sw^4))))\sin M + e(-2304 ci\,cw\,so\,sw\\
&+ co(2592 cw^2 + 288 sw^2) + si^2(ci\,so(3248 cw^3 sw + 3920 cw\,sw^3)\\
&+ co(-4676 cw^4 - 5544 cw^2 sw^2 - 196 sw^4)))\sin 2M + (-1344 ci\,cw\,so\,sw\\
&+ co(672 cw^2 - 672 sw^2) + si^2(ci\,so(3808 cw^3 sw + 3360 cw\,sw^3)\\
&+ co(-616 cw^4 + 336 cw^2 sw^2 + 504 sw^4)) + e^2(-4392 ci\,cw\,so\,sw\\
&+ co(5940 cw^2 + 1548 sw^2) + si^2(ci\,so(252 cw^3 sw + 9492 cw\,sw^3)\\
&+ co(-10647 cw^4 - 20034 cw^2 sw^2 - 147 sw^4))))\sin 3M\\
&+ e(-7968 ci\,cw\,so\,sw + co(3984 cw^2 - 3984 sw^2) + si^2(ci\,so(27440 cw^3 sw\\
&+ 14896 cw\,sw^3) + co(-4928 cw^4 + 9408 cw^2 sw^2 + 1792 sw^4)))\sin 4M\\
&+ (si^2(ci\,so(-2464 cw^3 sw + 2464 cw\,sw^3) + co(616 cw^4 - 3696 cw^2 sw^2\\
&+ 616 sw^4)) + e^2(-29112 ci\,cw\,so\,sw + co(14556 cw^2 - 14556 sw^2)\\
&+ si^2(ci\,so(124012 cw^3 sw + 30212 cw\,sw^3) + co(-24115 cw^4\\
&+ 70350 cw^2 sw^2 + 665 sw^4))))\sin 5M + e\,si^2(ci\,so(-19376 cw^3 sw\\
&+ 19376 cw\,sw^3) + co(4844 cw^4 - 29064 cw^2 sw^2 + 4844 sw^4))\sin 6M\\
&+ e^2 si^2(ci\,so(-89236 cw^3 sw + 89236 cw\,sw^3)\\
&+ co(22309 cw^4 - 133854 cw^2 sw^2 + 22309 sw^4))\sin 7M)
\end{aligned}
$$

$$\begin{aligned}
\frac{\mathrm{d}i_1}{\mathrm{d}t} = &-\frac{5}{32}si(ci\ co(-24) + ci\ co\ si^2(42) + e^2(-36cw\ so\ sw \\
&+ ci\ co(-102cw^2 - 138sw^2) + si^2(so(42cwsw) + ci\ co(168cw^4 \\
&+ 420cw^2sw^2 + 252sw^4))) + e(-24cw\ so\ sw + ci\ co(-108cw^2 \\
&- 132sw^2) + si^2(so(28cwsw) + ci\ co(182cw^4 + 420cw^2sw^2 \\
&+ 238sw^4)))\cos M + (-48cw\ so\ sw + ci\ co(24cw^2 - 24sw^2) \\
&+ si^2(so(56cw^3sw + 56cw\ sw^3) + ci\ co(-56cw^4 + 56sw^4)) \\
&+ e^2(-48cw\ so\ sw + ci\ co(-216cw^2 - 264sw^2) + si^2(so(28cw^3sw \\
&+ 84cw\ sw^3) + ci\ co(371cw^4 + 798cw^2sw^2 + 483sw^4))))\cos 2M \\
&+ e(-216cw\ so\ sw + ci\ co(108cw^2 - 108sw^2) + si^2(so(336cw^3sw \\
&+ 168cw\ sw^3) + ci\ co(-273cw^4 + 126cw^2sw^2 + 231sw^4)))\cos 3M \\
&+ (si^2(so(-56cw^3sw + 56cw\ sw^3) + ci\ co(14cw^4 - 84cw^2sw^2 + 14sw^4)) \\
&+ e^2(-636cw\ so\ sw + ci\ co(318cw^2 - 318sw^2) + si^2(so(1358cw^3sw \\
&+ 126cw\ sw^3) + ci\ co(-896cw^4 + 924cw^2sw^2 + 588sw^4))))\cos 4M \\
&+ e\ si^2(so(-364cw^3sw + 364cw\ sw^3) + ci\ co(91cw^4 - 546cw^2sw^2 \\
&+ 91sw^4))\cos 5M + e^2si^2(so(-1428cw^3sw + 1428cw\ sw^3) \\
&+ ci\ co(357cw^4 - 2142cw^2sw^2 + 357sw^4))\cos 6M + e(-24ci\ co\ cw\ sw \\
&+ so(-12cw^2 + 12sw^2) + si^2(ci\ co(56cwsw) + so(14cw^4 - 14sw^4)))\sin M \\
&+ (-48ci\ co\ cw\ sw + so(-24cw^2 + 24\ sw^2) + si^2(ci\ co(112cwsw) \\
&+ so(28cw^4 - 28sw^4)) + e^2(-48ci\ co\ cw\ sw + so(-24cw^2 + 24sw^2) \\
&+ si^2(ci\ co(84cw^3sw + 140cw\ sw^3) + so(21cw^4 + 42cw^2sw^2 \\
&- 35sw^4))))\sin 2M + e(-216ci\ co\ cw\ sw + so(-108cw^2 + 108sw^2) \\
&+ si^2(ci\ co(588cw^3sw + 420cw\ sw^3) + so(147cw^4 - 126cw^2sw^2 \\
&- 105sw^4)))\sin 3M + (si^2(ci\ co(-56cw^3sw + 56cw\ sw^3) + so(-14cw^4 \\
&+ 84cw^2sw^2 - 14sw^4)) + e^2(-636ci\ co\ cw\ sw + so(-318cw^2 + 318sw^2) \\
&+ si^2(ci\ co(2100cw^3sw + 868cw\ sw^3) + so(525cw^4 - 924cw^2sw^2 \\
&- 217sw^4))))\sin 4M + e\ si^2(ci\ co(-364cw^3sw + 364cw\ sw^3) \\
&+ so(-91cw^4 + 546cw^2sw^2 - 91sw^4))\sin 5M + e^2si^2(ci\ co(-1428cw^3sw \\
&+ 1428cw\ sw^3) + so(-357cw^4 + 2142cw^2sw^2 - 357sw^4))\sin 6M)
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} = &-\frac{5}{32}(ci^2so(-24) + si^4so(-42) + si^2(so(24) + ci^2so(126)) \\
&+ e^2(36ci\ co\ cw\ sw + ci^2so(-102cw^2 - 138sw^2) + si^4so(-168cw^4 \\
&- 420cw^2sw^2 - 252sw^4) + si^2(so(102cw^2 + 138sw^2) + ci\ co(-126cw^3sw
\end{aligned}$$

$$
\begin{aligned}
&- 126cw\,sw^3) + ci^2so(504cw^4 + 1260cw^2sw^2 + 756sw^4)))\\
&+ e(24ci\,co\,cw\,sw + ci^2so(-108cw^2 - 132sw^2) + si^4so(-182cw^4\\
&- 420cw^2sw^2 - 238sw^4) + si^2(so(108cw^2 + 132sw^2)\\
&+ ci\,co(-84cw^3sw - 84cw\,sw^3) + ci^2so(546cw^4 + 1260cw^2sw^2\\
&+ 714sw^4)))\cos M + (48ci\,co\,cw\,sw + ci^2so(24cw^2 - 24sw^2)\\
&+ si^4so(56cw^4 - 56sw^4) + si^2(so(-24cw^2 + 24sw^2) + ci\,co(-168cwsw)\\
&+ ci^2\,so(-168cw^4 + 168sw^4)) + e^2(48ci\,co\,cw\,sw + ci^2so(-216cw^2\\
&- 264sw^2) + si^4so(-371cw^4 - 798cw^2sw^2 - 483sw^4) + si^2(so(216cw^2\\
&+ 264sw^2) + ci\,co(-84cw^3sw - 252cw\,sw^3) + ci^2so(1113cw^4\\
&+ 2394cw^2sw^2 + 1449sw^4))))\cos 2M + e(216ci\,co\,cw\,sw + ci^2so(108cw^2\\
&- 108sw^2) + si^4so(273cw^4 - 126cw^2sw^2 - 231sw^4) + si^2(so(-108cw^2\\
&+ 108sw^2) + ci\,co(-1008cw^3sw - 504cw\,sw^3) + ci^2so(-819cw^4\\
&+ 378cw^2sw^2 + 693sw^4)))\cos 3M + (si^4so(-14cw^4 + 84cw^2sw^2\\
&- 14sw^4) + si^2(ci\,co(168cw^3sw - 168cw\,sw^3)\\
&+ ci^2so(42cw^4 - 252cw^2sw^2 + 42sw^4)) + e^2(636ci\,co\,cw\,sw\\
&+ ci^2so(318cw^2 - 318sw^2) + si^4so(896cw^4 - 924cw^2sw^2 - 588sw^4)\\
&+ si^2(so(-318cw^2 + 318sw^2) + ci\,co(-4074cw^3sw - 378cw\,sw^3)\\
&+ ci^2so(-2688cw^4 + 2772cw^2sw^2 + 1764sw^4))))\cos 4M + e(si^4so(-91cw^4\\
&+ 546cw^2sw^2 - 91sw^4) + si^2(ci\,co(1092cw^3sw - 1092cw\,sw^3)\\
&+ ci^2so(273cw^4 - 1638cw^2sw^2 + 273sw^4)))\cos 5M + e^2(si^4so(-357cw^4\\
&+ 2142cw^2sw^2 - 357sw^4) + si^2(ci\,co(4284cw^3sw - 4284cw\,sw^3)\\
&+ ci^2so(1071cw^4 - 6426cw^2sw^2 + 1071sw^4)))\cos 6M + e(-24ci^2cw\,so\,sw\\
&+ ci\,co(12cw^2 - 12sw^2) + si^4so(-56cwsw) + si^2(24cw\,so\,sw\\
&+ ci^2so(168cwsw) + ci\,co(-42cw^4 + 42sw^4)))\sin M + (-48ci^2cw\,so\,sw\\
&+ ci\,co(24cw^2 - 24sw^2) + si^4so(-112cwsw) + si^2(48cw\,so\,sw\\
&+ ci^2so(336cwsw) + ci\,co(-84cw^4 + 84sw^4)) + e^2(-48ci^2cw\,so\,sw\\
&+ ci\,co(24cw^2 - 24sw^2) + si^4so(-84cw^3sw - 140cw\,sw^3)\\
&+ si^2(48cw\,sosw + ci^2so(252cw^3sw + 420cw\,sw^3) + ci\,co(-63cw^4\\
&- 126cw^2sw^2 + 105sw^4))))\sin 2M + e(-216ci^2cw\,so\,sw + ci\,co(108cw^2\\
&- 108sw^2) + si^4so(-588cw^3sw - 420cw\,sw^3) + si^2(216cw\,so\,sw\\
&+ ci^2so(1764cw^3sw + 1260cw\,sw^3) + ci\,co(-441cw^4 + 378cw^2sw^2\\
&+ 315sw^4)))\sin 3M + (si^4so(56cw^3sw - 56cw\,sw^3) + si^2(ci^2so(-168cw^3sw
\end{aligned}
$$

$$
\begin{aligned}
&+168cw\,sw^3)+ci\,co(42cw^4-252cw^2sw^2+42sw^4))\\
&+e^2(-636ci^2cw\,so\,sw+ci\,co(318cw^2-318sw^2)\\
&+si^4so(-2100cw^3sw-868cw\,sw^3)+si^2(636cw\,so\,sw\\
&+ci^2so(6300cw^3sw+2604cw\,sw^3)+ci\,co(-1575cw^4+2772cw^2sw^2\\
&+651sw^4))))\sin 4M+e(si^4so(364cw^3sw-364cw\,sw^3)\\
&+si^2(ci^2so(-1092cw^3sw+1092cw\,sw^3)+ci\,co(273cw^4\\
&-1638cw^2sw^2+273sw^4)))\sin 5M+e^2(si^4so(1428cw^3sw\\
&-1428cw\,sw^3)+si^2(ci^2so(-4284cw^3sw+4284cw\,sw^3)\\
&+\ ci\,co(1071cw^4-6426cw^2sw^2+1071sw^4)))\sin 6M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dM_1}{dt}=\frac{25}{32a}&si(ci\,so(-24)+ci\,si^2so(42)+e^2(36co\,cw\,sw+ci\,so(-102cw^2\\
&-138sw^2)+si^2(co(-42cwsw)+ci\,so(168cw^4+420cw^2sw^2+252sw^4)))\\
&+e(24co\,cw\,sw+ci\,so(-108cw^2-132sw^2)+si^2(co(-28cwsw)\\
&+ci\,so(182cw^4+420cw^2sw^2+238sw^4)))\cos M+(48co\,cw\,sw\\
&+ci\,so(24cw^2-24sw^2)+si^2(co(-56cwsw)+ci\,so(-56cw^4+56sw^4))\\
&+e^2(48co\,cw\,sw+ci\,so(-216cw^2-264sw^2)+si^2(co(-28cw^3sw\\
&-84cw\,sw^3)+ci\,so(371cw^4+798cw^2sw^2+483sw^4))))\cos 2M\\
&+e(216co\,cw\,sw+ci\,so(108cw^2-108sw^2)+si^2(co(-336cw^3sw\\
&-168cw\,sw^3)+ci\,so(-273cw^4+126cw^2sw^2+231sw^4)))\cos 3M\\
&+(si^2(co(56cw^3sw-56cw\,sw^3)+ci\,so(14cw^4-84cw^2sw^2+14sw^4))\\
&+e^2(636co\,cw\,sw+ci\,so(318cw^2-318sw^2)+si^2(co(-1358cw^3sw\\
&-126cw\,sw^3)+ci\,so(-896cw^4+924cw^2sw^2+588sw^4))))\cos 4M\\
&+e\,si^2(co(364cw^3sw-364cw\,sw^3)+ci\,so(91cw^4-546cw^2sw^2\\
&+91sw^4))\cos(5M)+e^2si^2(co(1428cw^3sw-1428cw\,sw^3)\\
&+ci\,so(357cw^4-2142cw^2sw^2+357sw^4))\cos(6M)+e(-24ci\,cw\,so\,sw\\
&+co(12cw^2-12sw^2)+si^2(ci\,so(56cwsw)+co(-14cw^4+14sw^4)))\sin M\\
&+(-48ci\,cw\,so\,sw+co(24cw^2-24sw^2)+si^2(ci\,so(112cwsw)\\
&+co(-28cw^4+28sw^4))+e^2(-48ci\,cw\,so\,sw+co(24cw^2-24\,sw^2)\\
&+si^2(ci\,so(84cw^3sw+140cw\,sw^3)+co(-21cw^4-42cw^2sw^2\\
&+35sw^4))))\sin 2M+e(-216ci\,cw\,so\,sw+co(108cw^2-108sw^2)\\
&+si^2(ci\,so(588cw^3sw+420cw\,sw^3)+co(-147cw^4+126cw^2sw^2
\end{aligned}
$$

$$+105sw^4)))\sin 3M + (si^2(ci\ so(-56cw^3sw + 56cw\ sw^3) + co(14cw^4 - 84cw^2sw^2 + 14sw^4)) + e^2(-636ci\ cw\ so\ sw + co(318cw^2 - 318sw^2) + si^2(ci\ so(2100cw^3sw + 868cw\ sw^3) + co(-525cw^4 + 924cw^2sw^2 + 217sw^4))))\sin 4M + e\ si^2(ci\ so(-364cw^3sw + 364cw\ sw^3) + co(91cw^4 - 546cw^2sw^2 + 91sw^4))\sin 5M + e^2si^2(ci\ so(-1428cw^3sw + 1428cw\ sw^3) + co(357cw^4 - 2142cw^2sw^2 + 357sw^4))\sin 6M)$$

The long periodic terms included in above formulas are given as follows:

$$\begin{aligned} d_2 =& \frac{15}{32}e^2si((-6+6ci+(7-14ci)si^2)\cos(o-2w) \\ &+(-6-6ci+(7+14ci)si^2)\cos(o+2w)) \end{aligned} \tag{6.65}$$

$$\begin{aligned} d_3 =& \frac{5}{64}e\ si((480ci-840ci\ si^2)\sin(o) \\ &+(78-78ci+(-140+133ci)si^2)\sin(o-2w) \\ &+(-78-78ci+(140+133ci)si^2)\sin(o+2w)) \end{aligned}$$

$$\begin{aligned} d_4 =& \frac{15}{64}si((16ci+80ci\ e^2+(-28ci-140ci\ e^2)si^2)\cos(o) \\ &+((6-6ci)e^2+(-7+14ci)e^2si^2)\cos(o-2w) \\ &+((-6-6ci)e^2+(7+14ci)e^2si^2)\cos(o+2w)) \end{aligned}$$

$$\begin{aligned} d_5 =& \frac{15}{64}((16ci^2+80ci^2e^2+(-16-84ci^2+(-80-420ci^2)e^2)si^2 \\ &+(28+140e^2)si^4)\sin(o)+((6ci-6ci^2)e^2 \\ &+(6-21ci+42ci^2)e^2si^2-14e^2si^4)\sin(o-2w) \\ &+((-6ci-6ci^2)e^2+(6+21ci+42ci^2)e^2si^2-14e^2si^4)\sin(o+2w)) \end{aligned}$$

$$\begin{aligned} d_6 =& -\frac{75}{64a}si((16ci+80ci\ e^2+(-28ci-140ci\ e^2)si^2)\sin(o) \\ &+((6-6ci)e^2+(-7+14ci)e^2si^2)\sin(o-2w)+((-6-6ci)e^2 \\ &+(7+14ci)e^2si^2)\sin(o+2w)) \end{aligned}$$

6.5.10 Solutions of D_{42} Perturbation

$$
\begin{aligned}
\frac{\mathrm{d}a_1}{\mathrm{d}t} = & \frac{15}{32}(ci\,co\,si^2 so(-112cw^4 - 224cw^2sw^2 - 112sw^4) \\
& + e^2(-80co^2cw\,sw + 80cw\,so^2sw + ci^2(-80co^2cw\,sw + 80cw\,so^2sw) \\
& + ci\,co\,so(-160cw^2 + 160sw^2) + si^2(so^2(-280cwsw) + co^2(280cwsw) \\
& + ci\,co\,so(-560cw^4 - 2240cw^2sw^2 - 1680sw^4) \\
& + ci^2(so^2(-280cwsw) + co^2(280cwsw)))) \\
& + e(-40co^2cw\,sw + 40cw\,so^2sw + ci^2(-40co^2cw\,sw + 40cw\,so^2sw) \\
& + ci\,co\,so(-80cw^2 + 80sw^2) + si^2(so^2(-168cwsw) + co^2(168cwsw) \\
& + ci\,co\,so(-504cw^4 - 1568cw^2sw^2 - 1064sw^4) + ci^2(so^2(-112cwsw) \\
& + co^2(112cwsw))))\cos M + (si^2(so^2(-112cwsw) + co^2(112cwsw) \\
& + ci^2(co^2(-112cwsw) + so^2(112cwsw))) + e^2(-160co^2cw\,sw + 160cw\,so^2sw \\
& + ci^2(-160co^2cw\,sw + 160cw\,so^2sw) + ci\,co\,so(-320cw^2 + 320sw^2) \\
& + si^2(so^2(-700cw^3sw - 644cw\,sw^3) + co^2(700cw^3sw + 644cw\,sw^3) \\
& + ci\,co\,so(-812cw^4 - 4088cw^2sw^2 - 3052sw^4) \\
& + ci^2(so^2(-476cw^3sw - 420cw\,sw^3) + co^2(476cw^3sw + 420cw\,sw^3)))))\cos 2M \\
& + e(40co^2cw\,sw - 40cw\,so^2sw + ci^2(40co^2cw\,sw - 40cw\,so^2sw) \\
& + ci\,co\,so(80cw^2 - 80sw^2) + si^2(so^2(-420cw^3sw - 812cw\,sw^3) \\
& + co^2(420cw^3sw + 812cw\,sw^3) + ci\,co\,so(-476cw^4 + 1176cw^2sw^2 + 84sw^4) \\
& + ci^2(co^2(-1092cw^3sw - 700cw\,sw^3) + so^2(1092cw^3sw + 700cw\,sw^3))))\cos 3M \\
& + (si^2(co^2(112cw^3sw - 112cw\,sw^3) + so^2(-112cw^3sw + 112cw\,sw^3) \\
& + ci\,co\,so(112cw^4 - 672cw^2sw^2 + 112sw^4) + e^2(240co^2\,cw\,sw - 240cw\,so^2sw \\
& + ci^2(240co^2cw\,sw - 240cw\,so^2sw) + ci\,co\,so(480cw^2 - 480sw^2) \\
& + si^2(so^2(-336cw^3sw - 3920cw\,sw^3) + co^2(336cw^3sw + 3920cw\,sw^3) \\
& + ci\,co\,so(-3472cw^4 + 10752cw^2sw^2 - 112sw^4) + ci^2(co^2(-5600cw^3sw \\
& - 2016cw\,sw^3) + so^2(5600cw^3sw + 2016cw\,sw^3)))))\cos 4M \\
& + e\,si^2(co^2(980cw^3sw - 980cwsw^3) + so^2(-980cw^3sw + 980\,cwsw^3) \\
& + ci\,co\,so(980cw^4 - 5880cw^2sw^2 + 980sw^4) + \mathrm{ci}^2(\mathrm{co}^2(980\mathrm{cw}^3\mathrm{sw} - 980\mathrm{cw\,sw}^3) \\
& + \mathrm{so}^2(\text{-}980\mathrm{cw}^3\mathrm{sw} + 980\mathrm{cw\,sw}^3)))\cos 5M + \mathrm{e}^2\mathrm{si}^2(\mathrm{co}^2(4844\mathrm{cw}^3\mathrm{sw} - 4844\mathrm{cw\,sw}^3) \\
& + so^2(-4844cw^3sw + 4844cw\,sw^3) + ci\,co\,so(4844cw^4 - 29064cw^2sw^2 \\
& + 4844sw^4) + ci^2(co^2(4844cw^3sw - 4844cwsw^3) + so^2(-4844cw^3sw \\
& + 4844cwsw^3)))\cos 6M + e(160ci\,co\,cw\,so\,sw + so^2(-20cw^2 - 60sw^2) \\
& + co^2(20cw^2 + 60sw^2) + ci^2(co^2(-60cw^2 - 20sw^2) + so^2(60cw^2 + 20sw^2))
\end{aligned}
$$

$$+si^2(ci\,co\,so(-560cw^3sw-560cw\,sw^3)+co^2(14cw^4-140cw^2sw^2-154sw^4)$$
$$+so^2(-14cw^4+140cw^2sw^2+154sw^4)+ci^2(so^2(-266cw^4-420cw^2sw^2$$
$$-154sw^4)+co^2(266cw^4+420cw^2sw^2+154sw^4))))\sin M+(si^2(co^2(56cw^4$$
$$-56sw^4)+so^2(-56cw^4+56sw^4)+ci^2(so^2(56cw^4-56sw^4)+co^2(-56cw^4$$
$$+56sw^4)))+e^2(640ci\,co\,cw\,so\,sw+so^2(-80cw^2-240sw^2)+co^2(80cw^2$$
$$+240sw^2)+ci^2(co^2(-240cw^2-80sw^2)+so^2(240cw^2+80sw^2))$$
$$+si^2(ci\,co\,so(-2352cw^3sw-2128cw\,sw^3)+co^2(63cw^4-602cw^2sw^2$$
$$-609sw^4)+so^2(-63cw^4+602cw^2sw^2+609sw^4)+ci^2(so^2(-1071cw^4$$
$$-1638cw^2sw^2-623sw^4)+co^2(1071cw^4+1638cw^2sw^2+623sw^4)))))\sin 2M$$
$$+e(-160ci\,co\,cw\,so\,sw+co^2(20cw^2-20sw^2)+so^2(-20cw^2+20sw^2)$$
$$+ci^2(co^2(20cw^2-20sw^2)+so^2(-20cw^2+20sw^2))$$
$$+si^2(ci\,co\,so(1344cw^3sw-224cw\,sw^3)+co^2(259cw^4+294cw^2sw^2-357sw^4)$$
$$+so^2(-259cw^4-294cw^2sw^2+357sw^4)+ci^2(so^2(497cw^4-294cw^2sw^2$$
$$-399sw^4)+co^2(-497cw^4+294cw^2sw^2+399sw^4))))\sin 3M$$
$$+(si^2(ci\,co\,so(-448cw^3sw+448cw\,sw^3)+so^2(-28cw^4+168cw^2sw^2-28sw^4)$$
$$+co^2(28cw^4-168cw^2sw^2+28sw^4)+ci^2(so^2(-28cw^4+168cw^2sw^2-28sw^4)$$
$$+co^2(28cw^4-168cw^2sw^2+28sw^4)))+e^2(-960ci\,co\,cw\,so\,sw+co^2(120cw^2$$
$$-120sw^2)+so^2(-120cw^2+120sw^2)+ci^2(co^2(120cw^2-120sw^2)$$
$$+so^2(-120cw^2+120sw^2))+si^2(ci\,co\,so(10528cw^3sw-3808cw\,sw^3)$$
$$+co^2(616cw^4+2688cw^2sw^2-1512sw^4)+so^2(-616cw^4-2688cw^2sw^2$$
$$+1512sw^4)+ci^2(so^2(2352cw^4-2688cw^2sw^2-1456sw^4)+co^2(-2352cw^4$$
$$+2688cw^2sw^2+1456sw^4)))))\sin 4M+e\,si^2(ci\,co\,so(-3920cw^3sw$$
$$+3920cw\,sw^3)+so^2(-245cw^4+1470cw^2sw^2-245sw^4)+co^2(245cw^4$$
$$-1470cw^2sw^2+245sw^4)+ci^2(so^2(-245cw^4+1470cw^2sw^2-245sw^4)$$
$$+co^2(245cw^4-1470cw^2sw^2+245sw^4)))\sin 5M+e^2si^2(ci\,co\,so(-19376cw^3sw$$
$$+19376cw\,sw^3)+so^2(-1211cw^4+7266cw^2sw^2-1211sw^4)+co^2(1211cw^4$$
$$-7266cw^2sw^2+1211sw^4)+ci^2(so^2(-1211cw^4+7266cw^2sw^2-1211sw^4)$$
$$+co^2(1211cw^4-7266cw^2sw^2+1211sw^4)))\sin 6M)$$

$$\frac{\mathrm{d}e_1}{\mathrm{d}t}=\frac{15}{8}(ci\,co\,si^2so(-56cw^2+56cw^4+(-56+112cw^2)sw^2+56sw^4)$$
$$+e^2(6co^2cw\,sw-6cw\,so^2sw+ci^2(6co^2cw\,sw-cw\,so^2sw)$$
$$+ci\,co\,so(80-68cw^2-92sw^2)+si^2(ci\,co\,so(-238cw^2+196cw^4$$
$$+(-322+560cw^2)sw^2+364sw^4)+ci^2(co^2(-42cw^3sw-42cw\,sw^3)$$

$$
\begin{aligned}
&+ so^2(42cw^3sw + 42cw\, sw^3)))) + e(4co^2cw\, sw - 4cw\, so^2sw \\
&+ ci^2(4co^2cw\, sw - 4cw\, so^2sw) + ci\, co\, so(80 - 72cw^2 - 88sw^2) \\
&+ si^2(ci\, co\, so(-252cw^2 + 224cw^4 + (-308 + 560cw^2)sw^2 + 336sw^4) \\
&+ ci^2(co^2(-28cwsw) + so^2(28cwsw))))\cos M + (8co^2cw\, sw - 8cw\, so^2sw \\
&+ ci^2(8co^2cw\, sw - 8cw\, so^2sw) + ci\, co\, so(16cw^2 - 16sw^2) \\
&+ si^2(ci\, co\, so(56cw^2 - 112cw^4 - 56sw^2 + 112sw^4) \\
&+ ci^2(co^2(-56cwsw) + so^2(56cwsw))) + e^2(8co^2cw\, sw - 8cw\, so^2sw \\
&+ ci^2(8co^2cw\, sw - 8cw\, so^2sw) + ci\, co\, so(160 - 144cw^2 - 176sw^2) \\
&+ si^2(co^2(28cw^3sw - 28cw\, sw^3) + so^2(-28cw^3sw + 28cw\, sw^3) \\
&+ ci\, co\, so(-504cw^2 + 476cw^4 + (-616 + 952cw^2)sw^2 + 700sw^4) \\
&+ ci^2(co^2(-28cw^3sw - 84cw\, sw^3) + so^2(28cw^3sw + 84cw\, sw^3))))))\cos 2M \\
&+ e(36co^2cw\, sw - 36cw\, so^2sw + ci^2(36co^2cw\, sw - 36cw\, so^2sw) \\
&+ ci\, co\, so(72cw^2 - 72sw^2) + si^2(so^2(84cw^3sw - 84cw\, sw^3) \\
&+ co^2(-84cw^3sw + 84cw\, sw^3) + ci\, co\, so(252cw^2 - 588cw^4 \\
&+ (-252 + 504cw^2)sw^2 + 420sw^4) + ci^2(co^2(-336cw^3sw - 168cw\, sw^3) \\
&+ so^2(336cw^3sw + 168cw\, sw^3))))\cos 3M + (si^2(co^2(56cw^3sw - 56cw\, sw^3) \\
&+ so^2(-56cw^3sw + 56cw\, sw^3) + ci\, co\, so(56cw^4 - 336cw^2sw^2 + 56sw^4) \\
&+ ci^2(co^2(56cw^3sw - 56cw\, sw^3) + so^2(-56cw^3sw + 56cw\, sw^3))) \\
&+ e^2(106co^2cw\, sw - 106cw\, so^2sw + ci^2(106co^2cw\, sw - 106cw\, so^2sw) \\
&+ ci\, co\, so(212cw^2 - 212sw^2) + si^2(so^2(616cw^3sw - 616cw\, sw^3) \\
&+ co^2(-616cw^3sw + 616cw\, sw^3) + ci\, co\, so(742cw^2 - 2100cw^4 \\
&+ (-742 + 3696cw^2)sw^2 + 868sw^4) + ci^2(co^2(-1358cw^3sw \\
&- 126cw\, sw^3) + so^2(1358cw^3sw + 126cw\, sw^3))))))\cos 4M \\
&+ e\, si^2(co^2(364cw^3sw - 364cw\, sw^3) + so^2(-364cw^3sw + 364cw\, sw^3) \\
&+ ci\, co\, so(364cw^4 - 2184cw^2sw^2 + 364sw^4) + ci^2(co^2(364cw^3sw \\
&- 364cw\, sw^3) + so^2(-364cw^3sw + 364cw\, sw^3)))\cos 5M \\
&+ e^2si^2(co^2(1428cw^3sw - 1428cw\, sw^3) + so^2(-1428cw^3sw \\
&+ 1428cw\, sw^3) + ci\, co\, so(1428cw^4 - 8568cw^2sw^2 + 1428sw^4) \\
&+ ci^2(co^2(1428cw^3sw - 1428cw\, sw^3) + so^2(-1428cw^3sw \\
&+ 1428cwsw^3)))\cos 6M + e(-16ci\, co\, cw\, so\, sw + co^2(2cw^2 - 2sw^2) \\
&+ so^2(-2cw^2 + 2sw^2) + ci^2(co^2(2cw^2 - 2sw^2) + so^2(-2cw^2 + 2sw^2)) \\
&+ si^2(ci\, co\, so((-56cw + 112cw^3)sw + 112cw\, sw^3) + ci^2(so^2(14cw^4 \\
&- 14sw^4) + co^2(-14cw^4 + 14\, sw^4))))\sin M + (-32ci\, co\, cw\, so\, sw
\end{aligned}
$$

$$
\begin{aligned}
&+co^2(4cw^2-4sw^2)+so^2(-4cw^2+4sw^2)+ci^2(co^2(4cw^2-4sw^2)\\
&+so^2(-4cw^2+4sw^2))+si^2(ci\,co\,so((-112cw+224cw^3)sw\\
&+224cw\,sw^3)+ci^2(so^2(28cw^4-28sw^4)+co^2(-28cw^4+28sw^4)))\\
&+e^2(-32ci\,co\,cw\,so\,sw+co^2(4cw^2-4sw^2)+so^2(-4cw^2+4sw^2)\\
&+ci^2(co^2(4cw^2-4sw^2)+so^2(-4cw^2+4sw^2))+si^2(ci\,co\,so((-112cw\\
&+112cw^3)sw+336cw\,sw^3)+so^2(-7cw^4+42cw^2sw^2-7sw^4)\\
&+co^2(7cw^4-42cw^2sw^2+7sw^4)+ci^2(so^2(21cw^4+42cw^2sw^2-35sw^4)\\
&+co^2(-21cw^4-42cw^2sw^2+35sw^4))))))\sin 2M+e(-144ci\,co\,cw\,so\,sw\\
&+co^2(18cw^2-18sw^2)+so^2(-18cw^2+18sw^2)+ci^2(co^2(18cw^2-18sw^2)\\
&+so^2(-18cw^2+18sw^2))+si^2(ci\,co\,so((-504cw+1344cw^3)sw\\
&+672cw\,sw^3)+co^2(-21cw^4+126cw^2sw^2-21sw^4)+so^2(21cw^4\\
&-126cw^2sw^2+21sw^4)+ci^2(so^2(147cw^4-126cw^2sw^2-105sw^4)\\
&+co^2(-147cw^4+126cw^2sw^2+105sw^4)))))\sin 3M\\
&+(si^2(ci\,co\,so(-224cw^3sw+224cw\,sw^3)+so^2(-14cw^4+84cw^2sw^2\\
&-14sw^4)+co^2(14cw^4-84cw^2sw^2+14sw^4)+ci^2(so^2(-14cw^4\\
&+84cw^2sw^2-14sw^4)+co^2(14cw^4-84cw^2sw^2+14sw^4)))\\
&+e^2(-424ci\,co\,cw\,so\,sw+co^2(53cw^2-53sw^2)+so^2(-53cw^2+53sw^2)\\
&+ci^2(co^2(53cw^2-53sw^2)+so^2(-53cw^2+53sw^2))\\
&+si^2(ci\,co\,so((-1484cw+5432cw^3)sw+504cw\,sw^3)\\
&+ci^2(so^2(525cw^4-924cw^2sw^2-217sw^4)\\
&+co^2(-525cw^4+924cw^2sw^2+217sw^4))))))\sin 4M\\
&+e\,si^2(ci\,co\,so(-1456cw^3sw+1456cw\,sw^3)+so^2(-91cw^4\\
&+546cw^2sw^2-91sw^4)+co^2(91cw^4-546cw^2sw^2+91sw^4)\\
&+ci^2(so^2(-91cw^4+546cw^2sw^2-91sw^4)+co^2(91cw^4-546cw^2sw^2\\
&+91sw^4)))\sin 5M+e^2si^2(ci\,co\,so(-5712cw^3sw+5712cw\,sw^3)\\
&+so^2(-357cw^4+2142cw^2sw^2-357sw^4)+co^2(357cw^4-2142cw^2sw^2\\
&+357sw^4+ci^2(so^2(-357cw^4+2142cw^2sw^2-357sw^4)\\
&+co^2(357cw^4-2142cw^2sw^2+357sw^4)))\sin 6M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\omega_1}{\mathrm{d}t}=-\frac{15}{256}&(e(-2560ci\,co\,cw\,so\,sw+so^2(-960cw^2-320sw^2)\\
&+co^2(960cw^2+320sw^2)+ci^2(co^2(-320cw^2-960sw^2)\\
&+so^2(320cw^2+960sw^2))+si^2(ci\,co\,so(8960cw^3sw+8960cw\,sw^3)
\end{aligned}
$$

$$
\begin{aligned}
&+co^2(-1904cw^4-2240cw^2sw^2-336sw^4)+so^2(1904cw^4\\
&+2240cw^2sw^2+336sw^4)+ci^2(so^2(-1904cw^4-6720cw^2sw^2-4816sw^4)\\
&+co^2(1904cw^4+6720cw^2sw^2+4816sw^4))))\\
&+(-1280ci\ co\ cw\ so\ sw+so^2(-480cw^2-160sw^2)\\
&+co^2(480cw^2+160sw^2)+ci^2(co^2(-160cw^2-480sw^2)\\
&+so^2(160cw^2+480sw^2))+si^2(ci\ co\ so(4480cwsw)+co^2(-1008cw^4\\
&-1120cw^2sw^2-112sw^4)+so^2(1008cw^4+1120cw^2sw^2+112sw^4)\\
&+ci^2(so^2(-1008cw^4-3360cw^2sw^2-2352sw^4)+co^2(1008cw^4\\
&+3360cw^2sw^2+2352sw^4)))+e^2(-13440ci\ co\ cw\ so\ sw\\
&+so^2(-4920cw^2-1560sw^2)+co^2(4920cw^2+1560sw^2)\\
&+ci^2(co^2(-1560cw^2-4920sw^2)+so^2(1560cw^2+4920sw^2))\\
&+si^2(ci\ co\ so(46256cw^3sw+47824cw\ sw^3)+co^2(-9429cw^4\\
&-11634cw^2sw^2-1813sw^4)+so^2(9429cw^4+11634cw^2sw^2+1813sw^4)\\
&+ci^2(so^2(-9107cw^4-33726cw^2sw^2-25011sw^4)+co^2(9107cw^4\\
&+33726cw^2sw^2+25011sw^4)))))\cos M+e(-5120ci\ co\ cw\ so\ sw\\
&+so^2(-1920cw^2-640sw^2)+co^2(1920cw^2+640sw^2)\\
&+ci^2(co^2(-640cw^2-1920sw^2)+so^2(640cw^2+1920sw^2))\\
&+si^2(ci\ co\ so(17024cw^3sw+18816cw\ sw^3)+co^2(-3976cw^4\\
&-4816cw^2sw^2-392sw^4)+so^2(3976cw^4+4816cw^2sw^2+392sw^4)\\
&+ci^2(so^2(-4088cw^4-13104cw^2sw^2-9464sw^4)+co^2(4088cw^4\\
&+13104cw^2sw^2+9464sw^4))))\cos 2M+(-1280ci\ co\ cw\ so\ sw\\
&+co^2(160cw^2-160sw^2)+so^2(-160cw^2+160sw^2)+ci^2(co^2(160cw^2\\
&-160sw^2)+so^2(-160cw^2+160sw^2))+si^2(ci\ co\ so(3584cw^3sw\\
&+5376cw\ sw^3)+co^2(504cw^4-336cw^2sw^2-392sw^4)+so^2(-504cw^4\\
&+336cw^2sw^2+392sw^4)+ci^2(so^2(1512cw^4+336cw^2sw^2-1624sw^4)\\
&+co^2(-1512cw^4-336cw^2sw^2+1624sw^4)))+e^2(-12000ci\ co\ cw\ so\ sw\\
&+so^2(-4980cw^2-1980sw^2)+co^2(4980cw^2+1980sw^2)\\
&+ci^2(co^2(-1980cw^2-4980sw^2)+so^2(1980cw^2+4980sw^2))\\
&+si^2(ci\ co\ so(30240cw^3sw+53760cw\ sw^3)+so^2(10731cw^4\\
&+16590cw^2sw^2-21sw^4)+co^2(-10731cw^4-16590cw^2sw^2+21sw^4)\\
&+ci^2(so^2(-13881cw^4-32130cw^2sw^2-24129sw^4)+co^2(13881cw^4\\
&+32130cw^2sw^2+24129sw^4)))))\cos(3M)+e(-7680ci\ co\ cw\ so\ sw
\end{aligned}
$$

$$
\begin{aligned}
&+co^2(960cw^2-960sw^2)+so^2(-960cw^2+960sw^2)+ci^2(co^2(960cw^2\\
&-960sw^2)+so^2(-960cw^2+960sw^2))+si^2(ci\,co\,so(37632cw^3sw\\
&+16128cw\,sw^3)+co^2(1904cw^4+4032cw^2sw^2-3248sw^4)\\
&+so^2(-1904cw^4-4032cw^2sw^2+3248sw^4)+ci^2(so^2(9968cw^4\\
&-4032cw^2sw^2-8624sw^4)+co^2(-9968cw^4+4032cw^2sw^2\\
&+8624sw^4))))\cos 4M+(si^2(ci\,co\,so(-8064cw^3sw+8064cw\,sw^3)\\
&+so^2(-504cw^4+3024cw^2sw^2-504sw^4)+co^2(504cw^4-3024cw^2sw^2\\
&+504sw^4)ci^2(so^2(-504cw^4+3024cw^2sw^2-504sw^4)+co^2(504cw^4\\
&-3024cw^2sw^2+504sw^4)))+e^2(-28320ci\,co\,cw\,so\,sw+co^2(3540cw^2\\
&-3540sw^2)+so^2(-3540cw^2+3540sw^2)+ci^2(co^2(3540cw^2-3540sw^2)\\
&+so^2(-3540cw^2+3540sw^2))+si^2(ci\,co\,so(217280cw^3sw\\
&-19040cw\,sw^3)+co^2(1799cw^4+44310cw^2sw^2-16569sw^4)\\
&+so^2(-1799cw^4-44310cw^2sw^2+16569sw^4)+ci^2(so^2(41349cw^4\\
&-44310cw^2sw^2-26579sw^4)+co^2(-41349cw^4+44310cw^2sw^2\\
&+26579sw^4)))))\cos 5M+e\,si^2(ci\,co\,so(-63616cw^3sw+63616cw\,sw^3)\\
&+so^2(-3976cw^4+23856cw^2sw^2-3976sw^4)+co^2(3976cw^4\\
&-23856cw^2sw^2+3976sw^4)+ci^2(so^2(-3976cw^4+23856cw^2sw^2\\
&-3976sw^4)+co^2(3976cw^4-23856cw^2sw^2+3976sw^4)))\cos 6M\\
&+e^2si^2(ci\,co\,so(-293776cw^3sw+293776cw\,sw^3)+so^2(-18361cw^4\\
&+110166cw^2sw^2-18361sw^4)+co^2(18361cw^4-110166cw^2sw^2\\
&+18361sw^4)+ci^2(so^2(-18361cw^4+110166cw^2sw^2-18361sw^4)\\
&+co^2(18361cw^4-110166cw^2sw^2+18361sw^4)))\cos 7M+(-320co^2cw\,sw\\
&+320cw\,so^2sw+ci^2(-320co^2cw\,sw+320cw\,so^2sw)+ci\,co\,so(-640cw^2\\
&+640sw^2)+si^2(so^2(-896cwsw)+co^2(896cwsw)+ci\,co\,so(4032cw^4\\
&+3584cw^2sw^2-448sw^4)+ci^2(so^2(-1344cwsw)+co^2(1344cwsw)))\\
&+e^2(-1040co^2cw\,sw+1040cw\,so^2sw+ci^2(-1040co^2cw\,sw\\
&+1040cw\,so^2sw)+ci\,co\,so(-2080cw^2+2080sw^2)\\
&+si^2(so^2(-2044cw^3sw-2436cw\,sw^3)+co^2(2044cw^3sw+2436cw\,sw^3)\\
&+ci\,co\,so(12012cw^4+11032cw^2sw^2-2548sw^4)+ci^2(so^2(-4844cw^3sw\\
&-5236cw\,sw^3)+co^2(4844cw^3sw+5236cw\,sw^3)))))\sin M\\
&+e(-1280co^2cw\,sw+1280cw\,so^2sw+ci^2(-1280co^2cw\,sw\\
&+1280cw\,so^2sw)+ci\,co\,so(-2560cw^2+2560sw^2)\\
&+si^2(so^2(-3360cw^3sw-3808cw\,sw^3)+co^2(3360cw^3sw+3808cw\,sw^3)
\end{aligned}
$$

$$
\begin{aligned}
&+ci\,co\,so(15456cw^4+14784cw^2sw^2-2464sw^4)\\
&+ci^2(so^2(-5152cw^3sw-5600cw\,sw^3)+co^2(5152cw^3sw\\
&+5600cw\,sw^3))))\sin 2M+(-320co^2cw\,sw+320cw\,so^2sw\\
&+ci^2(-320co^2cw\,sw+320cw\,so^2sw)+ci\,co\,so(-640cw^2+640sw^2)\\
&+si^2(co^2(-1120cw^3sw-672cw\,sw^3)+so^2(1120cw^3sw+672cw\,sw^3)\\
&+ci\,co\,so(2016cw^4+1344cw^2sw^2-2464sw^4)+ci^2(so^2(-2912cw^3sw\\
&-3360cw\,sw^3)+co^2(2912cw^3sw+3360cw\,sw^3)))+e^2(-3000co^2cw\,sw\\
&+3000cw\,so^2sw+ci^2(-3000co^2cw\,sw+3000cw\,so^2sw)\\
&+ci\,co\,so(-6000cw^2+6000sw^2)+si^2(so^2(-7812cw^3sw-13692cw\,sw^3)\\
&+co^2(7812cw^3sw+13692cw\,sw^3)+ci\,co\,so(35532cw^4+52584cw^2sw^2\\
&-6468sw^4)+ci^2(so^2(-7308cw^3sw-13188cw\,sw^3)+co^2(7308cw^3sw\\
&+13188cw\,sw^3)))))\sin 3M+e(-1920co^2cw\,sw+1920cw\,so^2sw\\
&+ci^2(-1920co^2cw\,sw+1920cw\,so^2sw)+ci\,co\,so(-3840cw^2\\
&+3840sw^2)+si^2(co^2(-2464cw^3sw-7840cw\,sw^3)+so^2(2464cw^3sw\\
&+7840cw\,sw^3)+ci\,co\,so(16128cw^4-16128cw^2sw^2-10752sw^4)\\
&+ci^2(so^2(-21280cw^3sw-15904cw\,sw^3)+co^2(21280cw^3sw\\
&+15904cw\,sw^3))))\sin 4M+(si^2(so^2(2016cw^3sw-2016cw\,sw^3)\\
&+co^2(-2016cw^3sw+2016cw\,sw^3)+ci\,co\,so(-2016cw^4+12096cw^2sw^2\\
&-2016sw^4)+ci^2(so^2(2016cw^3sw-2016cw\,sw^3)+co^2(-2016cw^3sw\\
&+2016cw\,sw^3)))+e^2(-7080co^2cw\,sw+7080cw\,so^2sw\\
&+ci^2(-7080co^2cw\,sw+7080cw\,so^2sw)+ci\,co\,so(-14160cw^2\\
&+14160sw^2)+si^2(co^2(11172cw^3sw-47908cw\,sw^3)+so^2(-11172cw^3sw\\
&+47908cw\,sw^3)+ci\,co\,so(79100cw^4-177240cw^2sw^2-20020sw^4)\\
&+ci^2(so^2(-97468cw^3sw-38388cw\,sw^3)+co^2(97468cw^3sw\\
&+38388cw\,sw^3)))))\sin 5M+e\,si^2(so^2(15904cw^3sw-15904cw\,sw^3)\\
&+co^2(-15904cw^3sw+15904cw\,sw^3)+ci\,co\,so(-15904cw^4\\
&+95424cw^2sw^2-15904sw^4)+ci^2(so^2(15904cw^3sw-15904cw\,sw^3)\\
&+co^2(-15904cw^3sw+15904cw\,sw^3)))\sin 6M+e^2si^2(so^2(73444cw^3sw\\
&-73444cw\,sw^3)+co^2(-73444cw^3sw+73444cw\,sw^3)\\
&+ci\,co\,so(-73444cw^4+440664cw^2sw^2-73444sw^4)\\
&+ci^2(so^2(73444cw^3sw-73444cw\,sw^3)+co^2(-73444cw^3sw\\
&+73444cw\,sw^3)))\sin 7M)
\end{aligned}
$$

$$\begin{aligned}
\frac{\mathrm{d}i_1}{\mathrm{d}t} =& \frac{15}{8}(ci^2co\,so(-8) + co\,so(8) + si^2(co\,so(-14) + ci^2co\,so(42))\\
&+ e^2(ci(12co^2cw\,sw - 12cw\,so^2sw) + ci^2co\,so(-34cw^2 - 46sw^2)\\
&+ co\,so(46cw^2 + 34sw^2) + si^2(co\,so(-70) + ci^2co\,so(168cw^4\\
&+ 420cw^2sw^2 + 252sw^4) + ci(co^2(-42cw\,sw) + so^2(42cw\,sw))))\\
&+ e(ci(8co^2cw\,sw - 8cw\,so^2sw) + ci^2co\,so(-36cw^2 - 44sw^2)\\
&+ co\,so(44cw^2 + 36sw^2) + si^2(co\,so(-70) + ci^2co\,so(182cw^4\\
&+ 420cw^2sw^2 + 238sw^4) + ci(co^2(-28cw\,sw) + so^2(28cw\,sw))))\cos M\\
&+ (ci(16co^2cw\,sw - 16cw\,so^2sw) + co\,so(8cw^2 - 8sw^2)\\
&+ ci^2co\,so(8cw^2 - 8sw^2) + si^2(ci^2co\,so(-56cw^4 + 56sw^4)\\
&+ ci(co^2(-56cw\,sw) + so^2(56cw\,sw))) + e^2(ci(16co^2cw\,sw - 16cw\,so^2sw)\\
&+ ci^2co\,so(-72cw^2 - 88sw^2) + co\,so(88cw^2 + 72sw^2)\\
&+ si^2(co\,so(-133cw^4 - 322cw^2sw^2 - 133sw^4) + ci^2co\,so(371cw^4\\
&+ 798cw^2sw^2 + 483sw^4) + ci(co^2(-28cw^3sw - 84cw\,sw^3)\\
&+ so^2(28cw^3sw + 84cw\,sw^3)))))\cos 2M + e(ci(72co^2cw\,sw - 72cw\,so^2sw)\\
&+ co\,so(36cw^2 - 36sw^2) + ci^2co\,so(36cw^2 - 36sw^2)\\
&+ si^2(co\,so(-21cw^4 + 126cw^2sw^2 - 21sw^4) + ci^2co\,so(-273cw^4\\
&+ 126cw^2sw^2 + 231sw^4) + ci(co^2(-336cw^3sw - 168cw\,sw^3)\\
&+ so^2(336cw^3sw + 168cw\,sw^3))))\cos 3M + ci(co^2(56cw^3sw - 56cw\,sw^3)\\
&+ so^2(-56cw^3sw + 56cw\,sw^3))) + e^2(ci(212co^2cw\,sw - 212cw\,so^2sw)\\
&+ co\,so(106cw^2 - 106sw^2) + ci^2co\,so(106cw^2 - 106sw^2)\\
&+ si^2(co\,so(-154cw^4 + 924cw^2sw^2 - 154sw^4) + ci^2co\,so(-896cw^4\\
&+ 924cw^2sw^2 + 588sw^4) + ci(co^2(-1358cw^3sw - 126cw\,sw^3)\\
&+ so^2(1358cw^3sw + 126cw\,sw^3)))))\cos 4M + e\,si^2(co\,so(91cw^4\\
&- 546cw^2sw^2 + 91sw^4) + ci^2co\,so(91cw^4 - 546cw^2sw^2 + 91sw^4)\\
&+ ci(co^2(364cw^3sw - 364cw\,sw^3) + so^2(-364cw^3sw + 364cw\,sw^3)))\\
&\cos 5M + e^2si^2(co\,so(357cw^4 - 2142cw^2sw^2 + 357sw^4) + ci^2co\,so(357cw^4\\
&- 2142cw^2sw^2 + 357sw^4) + ci(co^2(1428cw^3sw - 1428cw\,sw^3)\\
&+ so^2(-1428cw^3sw + 1428cw\,sw^3)))\cos 6M + e(-8co\,cw\,so\,sw\\
&- 8ci^2co\,cw\,so\,sw + ci(co^2(4cw^2 - 4sw^2) + so^2(-4cw^2 + 4sw^2))\\
&+ si^2(ci^2co\,so(56cw\,sw) + ci(so^2(14cw^4 - 14sw^4) + co^2(-14cw^4\\
&+ 14sw^4))))\sin M + (-16co\,cw\,so\,sw - 16ci^2co\,cw\,so\,sw
\end{aligned}$$

$$
\begin{aligned}
&+ ci(co^2(8cw^2 - 8sw^2) + so^2(-8cw^2 + 8sw^2)) + si^2(ci^2 co\ so(112cw\ sw) \\
&+ ci(so^2(28cw^4 - 28sw^4) + co^2(-28cw^4 + 28sw^4))) \\
&+ e^2(-16co\ cw\ so\ sw - 16ci^2 co\ cw\ so\ sw + ci(co^2(8cw^2 - 8sw^2) \\
&+ so^2(-8cw^2 + 8sw^2)) + si^2(co\ so(-28cw^3 sw + 28cw\ sw^3) \\
&+ ci^2 co\ so(84cw^3 sw + 140cw\ sw^3) + ci(so^2(21cw^4 + 42cw^2 sw^2 - 35sw^4) \\
&+ co^2(-21cw^4 - 42cw^2 sw^2 + 35sw^4)))))\sin 2M + e(-72co\ cw\ so\, sw \\
&- 72ci^2 co\ cw\ so\ sw + ci(co^2(36cw^2 - 36\ sw^2) + so^2(-36cw^2 + 36sw^2)) \\
&+ si^2(co\ so(84cw^3 sw - 84cw\ sw^3) + ci^2 co\ so(588cw^3 sw + 420cw\ sw^3) \\
&+ ci(so^2(147cw^4 - 126cw^2 sw^2 - 105sw^4) + co^2(-147cw^4 + 126cw^2 sw^2 \\
&+ 105sw^4))))\sin 3M + (si^2(co\ so(-56cw^3 sw + 56cw\ sw^3) \\
&+ ci^2 co\ so(-56cw^3 sw + 56cw\ sw^3) + ci(so^2(-14cw^4 + 84cw^2 sw^2 - 14sw^4) \\
&+ co^2(14cw^4 - 84cw^2 sw^2 + 14sw^4))) + e^2(-212co\ cw\ so\ sw \\
&- 212ci^2 co\ cw\ so\ sw + ci(co^2(106cw^2 - 106sw^2) + so^2(-106cw^2 \\
&+ 106\ sw^2)) + si^2(co\ so(616cw^3 sw - 616cw\ sw^3) + ci^2 co\ so(2100cw^3 sw \\
&+ 868cw\ sw^3) + ci(so^2(525cw^4 - 924cw^2 sw^2 - 217sw^4) + co^2(-525cw^4 \\
&+ 924cw^2 sw^2 + 217sw^4)))))\sin 4M + e\ si^2(co\ so(-364cw^3 sw + 364cw\ sw^3) \\
&+ ci^2 co\ so(-364cw^3 sw + 364cw\ sw^3) + ci(so^2(-91cw^4 + 546cw^2 sw^2 \\
&- 91sw^4) + co^2(91cw^4 - 546cw^2 sw^2 + 91sw^4)))\sin 5M \\
&+ e^2 si^2(co\ so(-1428cw^3 sw + 1428cw\ sw^3) + ci^2 co\ so(-1428cw^3 sw \\
&+ 1428cw\ sw^3) + ci(so^2(-357cw^4 + 2142cw^2 sw^2 - 357sw^4) \\
&+ co^2(357cw^4 - 2142cw^2 sw^2 + 357sw^4)))\sin 6M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} = &-\frac{15}{16} si(ci(co^2(8cw^2 - 14cw^4 + (8 - 28cw^2)sw^2 - 14sw^4) \\
&+ so^2(-8cw^2 + 14cw^4 + (-8 + 28cw^2)sw^2 + 14sw^4)) \\
&+ ci^3(so^2(-42) + co^2(42)) + ci\ si^2(co^2(-42) + so^2(42)) \\
&+ e^2(24co\ cw\ so\ sw + ci^2 co\ so(168cw\ sw) + ci(co^2(34cw^2 - 70cw^4 \\
&+ (46 - 140cw^2)sw^2 - 70sw^4) + so^2(-34cw^2 + 70cw^4 \\
&+ (-46 + 140cw^2)sw^2 + 70sw^4)) + ci^3(so^2(-168cw^4 - 420cw^2 sw^2 \\
&- 252sw^4) + co^2(168cw^4 + 420cw^2 sw^2 + 252sw^4)) \\
&+ si^2(co\ so(-84cw\ sw) + ci(co^2(-168cw^4 - 420cw^2 sw^2 - 252sw^4) \\
&+ so^2(168cw^4 + 420cw^2 sw^2 + 252sw^4)))) + e(16co\ cw\ so\ sw \\
&+ ci^2 co\ so(112cw\ sw) + ci(co^2(36cw^2 - 70cw^4 + (44 - 140cw^2)sw^2
\end{aligned}
$$

$$
\begin{aligned}
&-70sw^4)+so^2(-36cw^2+70cw^4+(-44+140cw^2)sw^2+70sw^4))\\
&+ci^3(so^2(-182cw^4-420cw^2sw^2-238\,sw^4)+co^2(182cw^4+420cw^2sw^2\\
&+238sw^4))+si^2(co\,so(-56cw\,sw)+ci(co^2(-182cw^4-420cw^2sw^2\\
&-238sw^4)+so^2(182cw^4+420cw^2sw^2+238sw^4))))\cos M\\
&+(32co\,cw\,so\,sw+ci^2co\,so(224cw\,sw)+ci(so^2(8cw^2-8sw^2)\\
&+co^2(-8cw^2+8sw^2))+ci^3(so^2(56cw^4-56sw^4)+co^2(-56\,cw^4\\
&+56\,sw^4))+si^2(co\,so(-112cw\,sw)+ci(co^2(56cw^4-56sw^4)\\
&+so^2(-56cw^4+56sw^4)))+e^2(32co\,cw\,so\,sw+ci^2co\,so(112cw^3sw\\
&+336cw\,sw^3)+ci(co^2(72cw^2-133cw^4+(88-322cw^2)sw^2-133sw^4)\\
&+so^2(-72cw^2+133cw^4+(-88+322cw^2)sw^2+133sw^4))\\
&+ci^3(so^2(-371cw^4-798cw^2sw^2-483sw^4)+co^2(371cw^4+798cw^2sw^2\\
&+483sw^4))+si^2(co\,so(-56cw^3sw-168cw\,sw^3)+ci(co^2(-371cw^4\\
&-798cw^2sw^2-483sw^4)+so^2(371cw^4+798cw^2sw^2+483sw^4)))))\cos 2M\\
&+e(144co\,cw\,so\,sw+ci^2co\,so(1344cw^3sw+672cw\,sw^3)\\
&+ci(co^2(-36\,cw^2-21cw^4+(36+126\,cw^2)sw^2-21sw^4)+so^2(36cw^2\\
&+21cw^4+(-36-126cw^2)sw^2+21sw^4))+ci^3(so^2(273cw^4\\
&-126cw^2sw^2-231sw^4)+co^2(-273cw^4+126cw^2sw^2+231sw^4))\\
&+si^2(co\,so(-672cw^3sw-336cw\,sw^3)+ci(co^2(273cw^4-126cw^2sw^2\\
&-231sw^4)+so^2(-273cw^4+126cw^2sw^2+231sw^4))))\cos 3M\\
&+(ci^2co\,so(-224cw^3sw+224cw\,sw^3)+ci(so^2(-14cw^4+84cw^2sw^2\\
&-11sw^4)+co^2(14cw^4-84cw^2sw^2+14sw^4))+ci^3(so^2(-14cw^4\\
&+84cw^2sw^2-14sw^4)+co^2(14cw^4-84cw^2sw^2+14sw^4))\\
&+si^2(co\,so(112cw^3sw-112cw\,sw^3)+ci(co^2(-14cw^4+84cw^2sw^2\\
&-14sw^4)+so^2(14cw^4-84cw^2sw^2+14sw^4)))+e^2(424co\,cw\,so\,sw\\
&+ci^2co\,so(5432cw^3sw+504cw\,sw^3)+ci(co^2(-106cw^2-154cw^4\\
&+(106+924cw^2)sw^2-154sw^4)+so^2(106cw^2+154cw^4+(-106\\
&-924cw^2)sw^2+154sw^4))+ci^3(so^2(896cw^4-924cw^2sw^2-588sw^4)\\
&+co^2(-896cw^4+924cw^2sw^2+588sw^4))+si^2(co\,so(-2716cw^3sw\\
&-252cw\,sw^3)+ci(co^2(896cw^4-924cw^2sw^2-588sw^4)+so^2(-896cw^4\\
&+924cw^2sw^2+588sw^4)))))\cos 4M+e(ci^2co\,so(-1456cw^3sw\\
&+1456cw\,sw^3)+ci(so^2(-91cw^4+546cw^2sw^2-91sw^4)+co^2(91cw^4\\
&-546cw^2sw^2+91sw^4))+ci^3(so^2(-91cw^4+546cw^2sw^2-91sw^4)\\
&+co^2(91cw^4-546cw^2sw^2+91sw^4))+si^2(co\,so(728cw^3sw-728cw\,sw^3)
\end{aligned}
$$

$$+ci(co^2(-91cw^4+546cw^2sw^2-91sw^4)+so^2(91cw^4-546cw^2sw^2$$
$$+91sw^4))))\cos 5M+e^2(ci^2co\,so(-5712cw^3sw+5712cw\,sw^3)$$
$$+ci(so^2(-357cw^4+2142cw^2sw^2-357sw^4)+co^2(357cw^4-2142cw^2sw^2$$
$$+357sw^4))+ci^3(so^2(-357cw^4+2142cw^2sw^2-357sw^4)+co^2(357cw^4$$
$$-2142cw^2sw^2+357sw^4))+si^2(co\,so(2856cw^3sw-2856cw\,sw^3)$$
$$+ci(co^2(-357cw^4+2142cw^2sw^2-357sw^4)+so^2(357cw^4-2142cw^2sw^2$$
$$+357sw^4))))\cos 6M+e(ci(8co^2cw\,sw-8cw\,so^2sw)$$
$$+co\,so(8cw^2-8sw^2)+ci^2co\,so(56cw^4-56sw^4)+ci^3(so^2(-56cw\,sw)$$
$$+co^2(56cw\,sw))+si^2(co\,so(-28cw^4+28sw^4)+ci(co^2(-56cw^3sw)$$
$$+so^2(56cw\,sw))))\sin M+(ci(16co^2cw\,sw-16cw\,so^2sw)+co\,so(16cw^2$$
$$-16sw^2)+ci^2co\,so(112cw^4-112sw^4)+ci^3(so^2(-112cw\,sw)$$
$$+co^2\,(112cw\,sw))+si^2(co\,so(-56cw^4+56\,sw^4)+ci(co^2(-112cw\,sw)$$
$$+so^2(112cw\,sw)))+e^2(co\,so(16cw^2-16sw^2)+ci^2co\,so(84cw^4$$
$$+168cw^2sw^2-140sw^4)+ci(so^2((-16cw+28cw^3)sw-28cw\,sw^3)$$
$$+co^2((16cw-28cw^3)sw+28cw\,sw^3))+ci^3(so^2(-84cw^3sw$$
$$-140cw\,sw^3)+co^2(84cw^3sw+140cw\,sw^3))+si^2(co\,so(-42cw^4$$
$$-84cw^2sw^2+70sw^4)+ci(co^2(-84cw^3sw-140cw\,sw^3)+so^2(84cw^3sw$$
$$+140cw\,sw^3)))))\sin 2M+e(co\,so(72cw^2-72sw^2)+ci^2co\,so(588cw^4$$
$$-504cw^2sw^2-420sw^4)+ci(co^2((72cw+84cw^3)sw-84cw\,sw^3)$$
$$+so^2((-72cw-84cw^3)sw+84cw\,sw^3))+ci^3(so^2(-588cw^3sw$$
$$-420cw\,sw^3)+co^2(588cw^3sw+420cw\,sw^3))+si^2(co\,so(-294cw^4$$
$$+252cw^2sw^2+210sw^4)+ci(co^2(-588cw^3sw-420cw\,sw^3)$$
$$+so^2(588cw^3sw+420cw\,sw^3))))\sin 3M+(ci^2co\,so(-56cw^4+336cw^2sw^2$$
$$-56sw^4)+ci(so^2(56cw^3sw-56cw\,sw^3)+co^2(-56cw^3sw+56cw\,sw^3))$$
$$+ci^3(so^2(56cw^3sw-56cw\,sw^3)+co^2(-56cw^3sw+56cw\,sw^3))$$
$$+si^2(co\,so(28cw^4-168cw^2sw^2+28sw^4)+ci(co^2(56cw^3sw-56cw\,sw^3)$$
$$+so^2(-56cw^3sw+56cw\,sw^3)))+e^2(co\,so(212cw^2-212sw^2)$$
$$+ci^2co\,so(2100cw^4-3696cw^2sw^2-868sw^4)+ci(co^2((212cw$$
$$+616cw^3)sw-616cw\,sw^3)+so^2((-212cw-616cw^3)sw$$
$$+616cw\,sw^3))+ci^3(so^2(-2100cw^3sw-868cw\,sw^3)+co^2(2100cw^3sw$$
$$+868cw\,sw^3))+si^2(co\,so(-1050cw^4+1848cw^2sw^2+434sw^4)$$
$$+ci(co^2(-2100cw^3sw-868cw\,sw^3)+so^2(2100cw^3sw$$
$$+868cw\,sw^3)))))\sin 4M+e(ci^2co\,so(-364cw^4+2184cw^2sw^2-364sw^4)$$

$$
\begin{aligned}
&+ci(so^2(364cw^3sw-364cw\,sw^3)+co^2(-364cw^3sw+364cw\,sw^3))\\
&+ci^3(so^2(364cw^3sw-364cw\,sw^3)+co^2(-364cw^3sw+364cw\,sw^3))\\
&+si^2(co\,so(182cw^4-1092cw^2sw^2+182sw^4)+ci(co^2(364cw^3sw\\
&-364cw\,sw^3)+so^2(-364cw^3sw+364cw\,sw^3))))\sin 5M\\
&+e^2(ci^2co\,so(-1428cw^4+8568cw^2sw^2-1428sw^4)\\
&+ci(so^2(1428cw^3sw-1428cw\,sw^3)+co^2(-1428cw^3sw+1428cw\,sw^3))\\
&+ci^3(so^2(1428cw^3sw-1428cw\,sw^3)+co^2(-1428cw^3sw+1428cw\,sw^3))\\
&+si^2(co\,so(714cw^4-4284cw^2sw^2+714sw^4)+ci(co^2(1428cw^3sw\\
&-1428cw\,sw^3)+so^2(-1428cw^3sw+1428cw\,sw^3))))\sin 6M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}M_1}{\mathrm{d}t}=&\frac{75}{32a}(so^2(-8)+co^2(8)+ci^2(co^2(-8)+so^2(8))+si^2(co^2(-14)\\
&+so^2(14)+ci^2(so^2(-42)+co^2(42)))+e^2(-48ci\,co\,cw\,so\,sw\\
&+so^2(-46cw^2-34sw^2)+co^2(46cw^2+34sw^2)+ci^2(co^2(-34cw^2\\
&-46sw^2)+so^2(34cw^2+46sw^2))+si^2(ci\,co\,so(168cw\,sw)+co^2(-70)\\
&+so^2(70)+ci^2(so^2(-168cw^4-420cw^2sw^2-252sw^4)+co^2(168cw^4\\
&+420cw^2sw^2+252sw^4))))+e(-32ci\,co\,cw\,so\,sw+so^2(-44cw^2\\
&-36sw^2)+co^2(44cw^2+36sw^2)+ci^2(co^2(-36cw^2-44sw^2)\\
&+so^2(36cw^2+44sw^2))+si^2(ci\,co\,so(112cw\,sw)+co^2(-70)+so^2(70)\\
&+ci^2(so^2(-182cw^4-420cw^2sw^2-238sw^4)+co^2(182cw^4+420cw^2sw^2\\
&+238sw^4))))\cos M+(-64ci\,co\,cw\,so\,sw+co^2(8cw^2-8sw^2)\\
&+so^2(-8cw^2+8sw^2)+ci^2(co^2(8cw^2-8sw^2)+so^2(-8cw^2+8sw^2))\\
&+si^2(ci\,co\,so(224cw\,sw)+ci^2(so^2(56cw^4-56sw^4)+co^2(-56cw^4\\
&+56sw^4)))+e^2(-64ci\,co\,cw\,so\,sw+so^2(-88cw^2-72sw^2)\\
&+co^2(88cw^2+72sw^2)+ci^2(co^2(-72cw^2-88sw^2)+so^2(72cw^2\\
&+88\,sw^2))+si^2(ci\,co\,so(112cw^3sw+336cw\,sw^3)+co^2(-133cw^4\\
&-322cw^2sw^2-133sw^4)+so^2(133cw^4+322cw^2sw^2+133sw^4)\\
&+ci^2(so^2(-371cw^4-798cw^2sw^2-483\,sw^4)+co^2(371cw^4+798cw^2sw^2\\
&+483sw^4)))))\cos 2M+e(-288ci\,co\,cw\,so\,sw+co^2(36cw^2-36sw^2)\\
&+so^2(-36cw^2+36sw^2)+ci^2(co^2(36cw^2-36sw^2)+so^2(-36cw^2\\
&+36sw^2))+si^2(ci\,co\,so(1344cw^3sw+672cw\,sw^3)+co^2(-21cw^4\\
&+126cw^2sw^2-21sw^4)+so^2(21cw^4-126cw^2sw^2+21sw^4)\\
&+ci^2(so^2(273cw^4-126cw^2sw^2-231sw^4)+co^2(-273cw^4+126cw^2sw^2
\end{aligned}
$$

$$
\begin{aligned}
&+231sw^4))))\cos 3M + (si^2(ci\ co\ so(-224cw^3 sw + 224cw\ sw^3)\\
&+so^2(-14cw^4+84cw^2sw^2-14sw^4)+co^2(14cw^4-84cw^2sw^2+14sw^4)\\
&+ci^2(so^2(-14cw^4+84cw^2sw^2-14sw^4)+co^2(14cw^4-84cw^2sw^2\\
&+14sw^4)))+e^2(-848ci\ co\ cw\ so\ sw+co^2(106cw^2-106sw^2)\\
&+so^2(-106cw^2+106sw^2)+ci^2(co^2(106cw^2-106sw^2)+so^2(-106\ cw^2\\
&+106sw^2))+si^2(ci\ co\ so(5432cw^3sw+504cw\ sw^3)+co^2(-154cw^4\\
&+924cw^2sw^2-154sw^4)+so^2(154cw^4-924cw^2sw^2+154sw^4)\\
&+ci^2(so^2(896cw^4-924cw^2sw^2-588sw^4)+co^2(-896cw^4+924cw^2sw^2\\
&+588sw^4)))))\cos 4M+e\ si^2(ci\ co\ so(-1456cw^3sw+1456cw\ sw^3)\\
&+so^2(-91cw^4+546cw^2sw^2-91sw^4)+co^2(91cw^4-546cw^2sw^2\\
&+91sw^4)+ci^2(so^2(-91cw^4+546cw^2sw^2-91sw^4)+co^2(91cw^4\\
&-546cw^2sw^2+91sw^4)))\cos 5M+e^2si^2(ci\ co\ so(-5712cw^3sw\\
&+5712cw\ sw^3)+so^2(-357cw^4+2142cw^2sw^2-357sw^4)+co^2(357cw^4\\
&-2142cw^2sw^2+357sw^4)+ci^2(so^2(-357cw^4+2142cw^2sw^2-357sw^4)\\
&+co^2(357cw^4-2142cw^2sw^2+357sw^4)))\cos 6M+e(-8co^2cw\ sw\\
&+8cw\ so^2sw+ci^2(-8co^2cw\ sw+8cw\ so^2sw)+ci\ co\ so(-16cw^2\\
&+16sw^2)+si^2(ci\ co\ so(56cw^4-56sw^4)+ci^2(so^2(-56cw\ sw)\\
&+co^2(56cw\ sw))))\sin(M)+(-16co^2cw\ sw+16cw\ so^2sw\\
&+ci^2(-16co^2cw\ sw+16cw\ so^2sw)+ci\ co\ so(-32cw^2+32sw^2)\\
&+si^2(ci\ co\ so(112cw^4-112sw^4)+ci^2(so^2(-112cw\ sw)+co^2(112cw\ sw)))\\
&+e^2(-16co^2cw\ sw+16cw\ so^2sw+ci^2(-16co^2cw\ sw+16cw\ so^2sw)\\
&+ci\ co\ so(-32cw^2+32sw^2)+si^2(so^2(28cw^3sw-28cw\ sw^3)\\
&+co^2(-28cw^3sw+28cw\ sw^3)+ci\ co\ so(84cw^4+168cw^2sw^2-140sw^4)\\
&+ci^2(so^2(-84cw^3sw-140cw\ sw^3)+co^2(84cw^3sw+140cw\ sw^3)))))\sin 2M\\
&+e(-72co^2cw\ sw+72cw\ so^2sw+ci^2(-72co^2cw\ sw+72cw\ so^2sw)\\
&+ci\ co\ so(-144cw^2+144sw^2)+si^2(co^2(84cw^3sw-84cw\ sw^3)\\
&+so^2(-84cw^3sw+84cw\ sw^3)+ci\ co\ so(588cw^4-504cw^2sw^2-420sw^4)\\
&+ci^2(so^2(-588cw^3sw-420cw\ sw^3)+co^2(588cw^3sw+420cw\ sw^3))))\\
&\sin 3M+(si^2(so^2(56cw^3sw-56cw\ sw^3)+co^2(-56cw^3sw+56cw\ sw^3)\\
&+ci\ co\ so(-56cw^4+336cw^2sw^2-56sw^4)+ci^2(so^2(56cw^3sw-56cw\ sw^3)\\
&+co^2(-56cw^3sw+56cw\ sw^3)))+e^2(-212co^2cw\ sw+212cw\ so^2sw\\
&+ci^2(-212co^2cw\ sw+212cw\ so^2sw)+ci\ co\ so(-424cw^2+424sw^2)\\
&+si^2(co^2(616cw^3sw-616cw\ sw^3)+so^2(-616cw^3sw+616cw\ sw^3)
\end{aligned}
$$

$$+ ci\,co\,so(2100cw^4 - 3696cw^2sw^2 - 868sw^4) + ci^2(so^2(-2100cw^3sw - 868cw\,sw^3) + co^2(2100cw^3sw + 868cw\,sw^3)))))\sin 4M$$
$$+ e\,si^2(so^2(364cw^3sw - 364cw\,sw^3) + co^2(-364cw^3sw + 364cw\,sw^3) + ci\,co\,so(-364\,cw^4 + 2184cw^2sw^2 - 364sw^4) + ci^2(so^2(364cw^3sw - 364cw\,sw^3) + co^2(-364cw^3sw + 364cw\,sw^3)))\sin 5M$$
$$+ e^2si^2(so^2(1428cw^3sw - 1428cw\,sw^3) + co^2(-1428cw^3sw + 1428cw\,sw^3) + ci\,co\,so(-1428cw^4 + 8568cw^2sw^2 - 1428sw^4) + ci^2(so^2(1428cw^3sw - 1428cw\,sw^3) + co^2(-1428\,cw^3sw + 1428cw\,sw^3)))\sin 6M)$$

The long periodic terms included in above formulas are given as follows:

$$d_2 = -\frac{45}{32}e^2((2 - 4ci + 2ci^2 + (14ci - 7ci^2)si^2)\sin(2o - 2w) + 14ci^2si^2\sin 2w + (-2 - 4ci - 2ci^2 + (14ci + 7ci^2)si^2)\sin(2o + 2w)) \tag{6.67}$$

$$d_3 = \frac{15e}{32}((-80 + 80\,ci^2 + (140 - 420ci^2)si^2)\cos(2o) + (-20 + 40ci - 20ci^2 + (49 - 140ci + 91ci^2)si^2)\cos(2o - 2w) + (-20 - 40ci - 20ci^2 + (49 + 140ci + 91ci^2)si^2)\cos(2o + 2w))$$

$$d_4 = -\frac{15}{16}((-8 + 8ci^2 + (-40 + 40ci^2)e^2 + (14 - 42ci^2 + (70 - 210ci^2)e^2)si^2)\sin 2o + ((-3 + 6ci - 3ci^2)e^2 + (-21ci + 21ci^2)e^2si^2)\sin(2o - 2w) + ((-3 - 6ci - 3ci^2)e^2 + (21ci + 21ci^2)e^2si^2)\sin(2o + 2w))$$

$$d_5 = \frac{45}{32}si((4ci - 28ci^3 + (20ci - 140ci^3)e^2 + (28ci + 140ci\,e^2)si^2)\cos(2o) + ((-2 + 2ci - 14ci^2 + 14ci^3)e^2 + (7 - 14ci)e^2si^2)\cos(2o - 2w) + ((2 + 2ci + 14ci^2 + 14ci^3)e^2 + (-7 - 14ci)e^2si^2)\cos(2o + 2w))$$

$$d_6 = -\frac{75}{32a}((-8 + 8ci^2 + (-40 + 40ci^2)e^2 + (14 - 42ci^2 + (70 - 210ci^2)e^2)si^2)\cos 2o + ((-3 + 6ci - 3ci^2)e^2 + (-21ci + 21ci^2)e^2si^2)\cos(2o - 2w) + ((-3 - 6ci - 3ci^2)e^2 + (21ci + 21ci^2)e^2si^2)\cos(2o + 2w))$$

6.5.11 Solutions of $\mathrm{D}_{43} \sim \mathrm{D}_{88}$ *Perturbations*

Due the extensive formulas the solutions are given in Appendix 2. Furthermore, the electronic files of the formulas are available upon request by authors (xu@gfz-potsdam.de, j.xu@arcor.de).

6.6 Properties of the Solutions of Geopotential Perturbations

The properties of the solutions of geopotential disturbances of lower orders and degrees may be summarised from the solutions given in Sects. 6.5.2 ~ 6.5.10 and Appendix 2.

6.6.1 Long Term Perturbations

Only the disturbances of the even order zonal terms D_{l0} (l = even) have long term components and only the Keplerian elements of (ω, Ω, M) are long term perturbed. Only the (ω, M) are disturbed by the even order zonal terms D_{l0} (l = even) long periodically. That is to say, to study the long term effects of geopotential disturbances only the even order zonal terms D_{l0} (l = even) have to be taken into account. Because inclination i is not disturbed secularly, $\sin i$ and $\cos i$ are non-periodic functions. Long terms are functions of (a, e, i), which are considered constants here. Because of slow changes of (ω, Ω), long periodic terms are periodic functions of (ω, Ω). Short periodic terms are functions of M.

6.6.2 Long and Short Periodic Perturbations

All Keplerian elements (a, e, ω, i, Ω, M) are disturbed by geopotential long periodically and short periodically. All functions of $\sin\omega$, $\cos\omega$, $\sin\Omega$, $\cos\Omega$ should be reduced to sine and cosine functions of $n\omega + m\Omega$.

Chapter 7
Solutions of Extraterrestrial Disturbances

Solutions of extraterrestrial disturbances of solar radiation pressure, drag force of the atmosphere, and attracting forces of the sun, moon, and planets are given in this chapter. For convenience, the ephemeris of the sun and moon, as well as planets, are described.

7.1 Analytical Solution of Satellite Orbit Disturbed by Solar Radiation Pressure

The disturbing force vector is first transformed and rotated to the orbital frame so that it can be used in the Gaussian equations of satellite motion. Then the force vector is expanded to trigonometric functions of the Keplerian angular elements and the disturbances are separated into three parts: short periodic terms with functions of M, long periodic terms with functions of (ω, Ω) and secular terms (non-periodic functions of (a, e, i)) using software for mathematic symbolic operations. The integrations are then carried out with respect to M, (ω, Ω) and t, respectively, to obtain the analytical solutions of disturbed satellite orbits. Solutions are derived and given at two precision levels. All Keplerian elements are disturbed by the solar radiation short- and long-periodically. The semi-major axis a and the eccentricity e of the orbital ellipse as well as inclination angle i are not disturbed linearly. The argument of perigee ω and the mean anomaly M are subjected to linear perturbations in two opposite directions (perigee precession/regression and mean motion delaying/advancing) depending on a so-called inclination factor. The right ascension of ascending node Ω is disturbed linearly by a function of $\cos i$.

7.1.1 Introduction

Earth satellite orbits are disturbed in a more complicated fashion by solar radiation pressure because of the non-conservative property of this force (Boccaletti and Pucacco 2001; Brouwer and Clemence 1961; Chobotov 1991; Krivov et al. 1996).

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The effects of the solar radiation pressure on a satellite orbit are discussed in numerous items in the literature (Bate et al. 1971; Van Kamp 1967; Vallado 2007; Herrick 1972; Meeus 1992; Kudak et al. 2010; Levin 1968; Musen 1960; Parkinson et al. 1960; Vilhena De Moraes 1981; Saad et al. 2010; Vokrouhlicky and Milani 2000; Vokrouhlicky et al. 1993, 1994; Zizka and Vokrouhlicky 2011); short reviews on the research can be found, e.g. in Boulton (1983); Hughes (1977); Harwood and Swinerd (1995); however, without providing commonly acknowledged and applicable analytical solution. An analytical solution gives the theoretical integrals and shows the physical effects with clear spectral properties (Licandro et al. 2008; Cui 1990; Schneider and Cui 2005; Lynden-Bell 2009; Pal 2009), and may give a direct insight into the physical disturbing phenomenon (Touma et al. 2009; Desmars et al. 2009). An analytical solution has been given in Xu (2008); however, approximations are made directly with respect to the force vector. Therefore, the solving method is not extendable for a more precise solution of the third order (see discussion in the next section). A further problem is that the truncation of the formula is not well done, so that the solution seems to be extremely bloated. Here, quite a similar but still independent way to solve the dilemma is developed and the solution of solar radiation disturbance is derived.

The force vector is given in Sect. 7.1.2. The basic Gaussian equations are given and mathematical formulas are used to expand the right-hand sides of the equations into linear trigonometric functions of the Keplerian angular elements in Sects. 7.1.3 and 7.1.4. The solutions with explicit short periodic terms are derived in Sect. 7.1.5. Explicit long-periodic and secular solutions are discussed in Sect. 7.1.6. More precise solutions are given in Sect. 7.1.7, followed by a concluding summary.

7.1.2 Disturbing Force of Solar Radiation Pressure

Solar radiation pressure, caused by solar light, is a disturbing force (scaled by mass) acting on the satellite's surface and can be represented as (Kezerashvili and Vazquez-Poritz 2009; Hughes 1977; Kubo-oka and Sengoku 1999; Montenbruck and Gill 2000; Harwood and Swinerd 1995; Seeber 2003; Xu 2008)

$$\vec{f}_{\text{solar}} = \gamma P_s C_r \frac{S}{m} \frac{r_s^2}{|\vec{r} - \vec{r}_s|^2} \vec{n}_s, \quad \vec{n}_s = \frac{\vec{r} - \vec{r}_s}{|\vec{r} - \vec{r}_s|}, \tag{7.1}$$

where γ is the shadow factor, P_{s} is the luminosity of the sun, C_{r} is the surface reflectivity, r_{s} is the geocentric distance of the sun, (S/m) is the area to mass ratio of the satellite, and $\vec{r}$ and $\vec{r}_{\mathrm{s}}$ are the geocentric vectors of the satellite and the sun, $\vec{n}_s$ represents the unit vector from the sun to the satellite. Using ξ to represent the coefficient part of (7.1) yields

$$\vec{f}_{\text{solar}} = \xi \frac{r_s^2}{|\vec{r} - \vec{r}_s|^2} \vec{n}_s = \xi \frac{r_s^2}{|\vec{r} - \vec{r}_s|^3} (\vec{r} - \vec{r}_s). \tag{7.2}$$

ξ could be slightly different in the various literature because of different scale and/ or units (Kezerashvili and Vazquez-Poritz 2009). Two approximations are made in Xu (2008) to derive the solution. First, the ratio of the geocentric distance of the sun and the distance of the sun to the satellite is considered as 1 (with precision of 3.5×10^{-4} for a GPS satellite); second, the sun-satellite unit vector is approximated by the sun-Earth one. Instead of approximations, the expansion method will be used here so that the solution can be derived as precisely as required.

The disturbance of the solar radiation pressure on the satellite can be solved by using Gaussian equations of motion. The force vector has to be rotated from the ECSF (Earth-Centred Space-Fixed) frame to the orbital coordinate frame by cf. (4.16) or, e.g. Battin 1999; Kaula 1966/2001; Montenbruck and Gill 2000)

$$\begin{pmatrix} f_r \\ f_\alpha \\ f_h \end{pmatrix} = R_3(f)R_3(\omega)R_1(i)R_3(\Omega) \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}, \tag{7.3}$$

Here $(f_r, f_\alpha, f_h)^T$ is a force vector with three orthogonal components in an orbital plane coordinate system, the first two components are in the orbital plane, f_r is the radial force component, f_α is the force component perpendicular to f_r and pointed in the direction of the satellite motion (see Fig. 7.1) and f_h completes a right-handed system. $\vec{f} = (f_x, f_y, f_z)^T$ is the same force vector in the ECSF system. R_k is the rotational matrix around the axis k. Keplerian elements $(a, e, \omega, i, \Omega, M, f)$ are (the semi-major axis, the eccentricity of the ellipse, the argument of perigee, the inclination angle, the right ascension of the ascending node, the mean anomaly and the true anomaly), respectively. Satellite position vector in orbital frame is (see (3.41), or Battin 1999; Kaula 1966/2001; Montenbruck and Gill 2000)

$$\vec{q} = \begin{pmatrix} a(\cos E - e) \\ a\sqrt{1-e^2}\sin E \\ 0 \end{pmatrix} = \begin{pmatrix} r\cos f \\ r\sin f \\ 0 \end{pmatrix}. \tag{7.4}$$

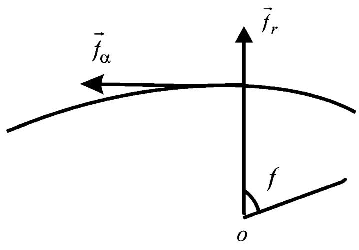
Here E is the eccentric anomaly. The vector can be rotated from the orbital frame to the ECSF frame by (see (3.43), or Battin 1999; Kaula 1966/2001; Montenbruck and Gill 2000)

$$\vec{r} = R_3(-\Omega)R_1(-i)R_3(-\omega)\vec{q}. \tag{7.5}$$

Similarly, the position vector of the sun can be represented by:

$$\vec{r}_s = R_1(\varepsilon)R_3(-\Omega_s)R_1(-i_s)R_3(-\omega_s)\vec{q}_s, \tag{7.6}$$

Fig. 7.1 Radial force vector and its perpendicular one in the orbital plane



where index s is used to denote the sun related Keplerian elements and vector, ε is the angle of mean obliquity of the ecliptic plane related to the equatorial plane and can be regarded as a constant. The solar position vector ((7.4) with index s) is represented in the Earth Centred Ecliptic Inertial (ECEI) frame.

Vectors (7.5) and (7.6) can both be rotated to the orbital frame by using (7.3). Denoting the following matrix as R and its elements as R_{km}:

$$R = R_3(\omega)R_1(i)R_3(\Omega)R_1(\varepsilon)R_3(-\Omega_s)R_1(-i_s)R_3(-\omega_s), \tag{7.7}$$

and

$$B = R_3(\omega)R_1(i)R_3(\Omega) = \begin{pmatrix} \cos\omega\cos\Omega - \sin\omega\cos i\sin\Omega & \cos\omega\sin\Omega + \sin\omega\cos i\ \cos\Omega & \sin\omega\sin i \\ -\sin\omega\cos\Omega - \cos\omega\cos i\sin\Omega & -\sin\omega\sin\Omega + \cos\omega\cos i\cos\Omega & \cos\omega\sin i \\ \sin i\sin\Omega & -\sin i\cos\Omega & \cos i \end{pmatrix}$$
$$A = R_1(\varepsilon)R_3(-\Omega_s)R_1(-i_s)\,R_3(-\omega_s) \tag{7.8}$$

$$\vec{p} = R\vec{q}_s = B\vec{r}_s = Br_s\vec{n}_s = BA\vec{q}_s = r_s\begin{pmatrix} R_{11}\cos f_s + R_{12}\sin f_s \\ R_{21}\cos f_s + R_{22}\sin f_s \\ R_{31}\cos f_s + R_{32}\sin f_s \end{pmatrix} \tag{7.9}$$

and further noting that the three rotational angles of Keplerian elements with index s can be considered as constants, one has the solar radiation pressure force vector in orbital frame

$$\begin{pmatrix} f_r \\ f_\alpha \\ f_h \end{pmatrix} = \xi\frac{r_s^2}{|\vec{r} - \vec{r}_s|^3} = R_3(f)(\vec{q} - \vec{p}). \tag{7.10}$$

The force vector (7.10) is represented completely in Keplerian elements and therefore can be used in Gaussian equations to solve the problem. B and $\vec{p}$ given in (7.8) and (7.9) are functions of satellite Keplerian elements (ω, i, Ω). Components of $\vec{p}$ are later denoted by p with indices. Elements of the matrix A

are functions of the solar Keplerian elements (except for the f_s) and are therefore considered as constants (the elements of A are denoted by indices). The unit vector of (7.6) can be written as

$$\vec{n}_s = \frac{\vec{r}_s}{r_s} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} A_{11}\cos f_s + A_{12}\sin f_s \\ A_{21}\cos f_s + A_{22}\sin f_s \\ A_{31}\cos f_s + A_{32}\sin f_s \end{pmatrix}. \tag{7.11}$$

For a more precise solution, the solar Keplerian element f_s shall be considered as a variable. However, this will make our derivation much more complicated. Using the daily average f_{sa} to approximate the changing f_s will cause an angle error of at most 0.017/2 rad/day, which corresponds to $\mathrm{d}f = f_s - f_{sa} <= 0.0085$. Using the Taylor expansions

$$\begin{aligned} \sin(f_s) &\approx \sin(f_{sa}) + \cos(f_{sa})df - \sin(f_{sa})df^2, \\ \cos(f_s) &\approx \cos(f_{sa}) + \sin(f_{sa})df - \cos(f_{sa})df^2 \end{aligned} \tag{7.12}$$

the unit vector (7.11) can be considered as a daily constant. Because of the small e_s (≈ 0.0167) (the same as for the Earth) and

$$\begin{aligned} r_s &= \frac{a_s(1-e_s^2)}{1+e_s\cos f_s} \approx a_s(1 - e_s\cos f_s + \cdots), \\ r_s^2 &= \frac{a_s^2(1-e_s^2)^2}{(1+e_s\cos f_s)^2} \approx a_s^2(1 - 2e_s\cos f_s + \cdots), \end{aligned} \tag{7.13}$$

the solar radius r_s and the components of the solar vector $\vec{r}_s$ can also be regarded as daily constants. These assumptions make our further derivation a daily valid one; however, the daily results can be added together to get the solutions valid for desired time-span.

7.1.3 Basic Gaussian Equations of Motion

The Gaussian perturbed equations of motion are (Kaula 1966/2001)

$$\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2}{n\sqrt{1-e^2}}[e\sin f\cdot f_r + (1+e\cos f)\cdot f_\alpha] \\ \frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{na}[\sin f\cdot f_r + (\cos E + \cos f)\cdot f_\alpha] \end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{nae}\left[-\cos f\cdot f_r + \frac{2+e\cos f}{1+e\cos f}\sin f\cdot f_\alpha\right] - \cos i\frac{\mathrm{d}\Omega}{\mathrm{d}t}\\
\frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{(1-e\cos E)\cos u}{na\sqrt{1-e^2}}\cdot f_h\\
\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= \frac{(1-e\cos E)\sin u}{na\sqrt{1-e^2}\sin i}\cdot f_h\\
\frac{\mathrm{d}M}{\mathrm{d}t} &= n - \frac{2}{na}\left(\frac{1-e^2}{1+e\cos f}\right)\cdot f_r - \frac{1-e^2}{nae}\left[-\cos f\cdot f_r + \frac{2+e\cos f}{1+e\cos f}\sin f\cdot f_\alpha\right]
\end{aligned} \tag{7.14}$$

and

$$u = \omega + f, \quad \cos E = \frac{e+\cos f}{1+e\cos f}. \tag{7.15}$$

Here $(f_r, f_\alpha, f_h)^T$ is a force vector in an orbital plane coordinate system, the first two components are in the orbital plane, f_r is the radial component, f_α is the component perpendicular to f_r and pointed in the direction of the satellite motion (see Fig. 7.1, where o is the focus of the orbital ellipse and f is the true anomaly counted from the perigee of the orbital ellipse), and f_h completes a right-handed system. E is the eccentric anomaly. The relation between eccentric anomaly E and true anomaly f is given in Fig. 7.2 (where S' is the vertical projection of the satellite S on the circle with a radius of a (semi-major axis of the ellipse), b is the semi-minor axis of the ellipse, O' is the centre of the circle and O is the focus of the ellipse).

The Gaussian equations can be rewritten as

$$\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2}{n\sqrt{1-e^2}}\frac{\mathrm{d}a_1}{\mathrm{d}t} = h_1\frac{\mathrm{d}a_1}{\mathrm{d}t}\\
\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{na}\frac{\mathrm{d}e_1}{\mathrm{d}t} = h_2\frac{\mathrm{d}e_1}{\mathrm{d}t}\\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{nae}\frac{\mathrm{d}\omega_1}{\mathrm{d}t} - \cos i\frac{\mathrm{d}\Omega}{\mathrm{d}t} = h_3\frac{\mathrm{d}\omega_1}{\mathrm{d}t} - \cos i\frac{\mathrm{d}\Omega}{\mathrm{d}t} = h_3\frac{\mathrm{d}\omega_1}{\mathrm{d}t} - h_7\frac{\mathrm{d}\Omega_1}{\mathrm{d}t}\\
\frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{1}{na\sqrt{1-e^2}}\frac{\mathrm{d}i_1}{\mathrm{d}t} = h_4\frac{\mathrm{d}i_1}{\mathrm{d}t}\\
\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= \frac{1}{na\sqrt{1-e^2}\sin i}\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} = h_5\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} = h_4\frac{1}{\sin i}\frac{\mathrm{d}\Omega_1}{\mathrm{d}t}\\
\frac{\mathrm{d}M}{\mathrm{d}t} &= n - 2\left(\frac{1-e^2}{na}\right)\frac{\mathrm{d}M_1}{\mathrm{d}t} - \frac{1-e^2}{nae}\frac{\mathrm{d}\omega_1}{\mathrm{d}t} = n - 2h_6 e\frac{\mathrm{d}M_1}{\mathrm{d}t} - h_6\frac{\mathrm{d}\omega_1}{\mathrm{d}t}
\end{aligned} \tag{7.16}$$

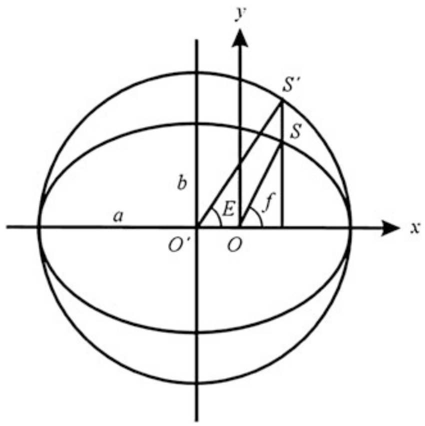


Fig. 7.2 The eccentric and true anomalies (f and E) of a satellite

where coefficients of (h_1, h_2, h_3, h_4, h_5, h_6, $h_7 = h_5\cos i = h_4\cos i/\sin i$) are defined, and

$$
\begin{aligned}
\frac{\mathrm{d}a_1}{\mathrm{d}t} &= e\sin f\cdot f_r + (1+e\cos f)\cdot f_\alpha \\
\frac{\mathrm{d}e_1}{\mathrm{d}t} &= \sin f\cdot f_r + (\cos E+\cos f)\cdot f_\alpha \\
\frac{\mathrm{d}\omega_1}{\mathrm{d}t} &= -\cos f\cdot f_r + \frac{2+e\cos f}{1+e\cos f}\sin f\cdot f_\alpha \\
\frac{\mathrm{d}i_1}{\mathrm{d}t} &= (1-e\cos E)\cos u\cdot f_h \\
\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} &= (1-e\cos E)\sin u\cdot f_h \\
\frac{\mathrm{d}M_1}{\mathrm{d}t} &= \frac{1}{1+e\cos f}\cdot f_r
\end{aligned}
\tag{7.17}
$$

Equation 7.17 are called basic Gaussian equations of motion. By integrating equations (7.17) the mean value theorem for integration is used where the functions of variables (a, e, i) are considered constants. For solving the Gaussian equations of motion (7.16), the simplest way is to solve the basic Gaussian equations (7.17). Substituting solutions of (7.17) into (7.16) the solutions of (7.16) can be obtained. Therefore, without exception, the non-conservative force disturbing orbit problem turns out to be the problem of solving the basic Gaussian equations of motion (7.17). It is notable that the basic Gaussian equations of motion are singularity-free.

7.1.4 Expansions of Solar Radiation Pressure Force

The divisor (1 + *e*cos*f*) of (cf. (7.15))

$$\cos E = \frac{e + \cos f}{1 + e\cos f}, \tag{7.18}$$

can be expanded by using (Bronstein and Semendjajew 1987; Wang et al. 1979)

$$(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + \ldots |x| < 1. \tag{7.19}$$

The vector part of (7.10) can be further simplified as

$$R_3(f)(\vec{q} - \vec{p}) = \begin{pmatrix} r - p_1 \cos f - p_2 \sin f \\ p_1 \sin f - p_2 \cos f \\ -p_3 \end{pmatrix}. \tag{7.20}$$

According to the geometric relations of the vectors it follows (see Fig. 7.3) that

$$\begin{aligned} \vec{r} &= \vec{r}_s + (\vec{r} - \vec{r}_s) \\ |\vec{r} - \vec{r}_s|^2 &= r_s^2 + r^2 - 2rr_s\cos\alpha, \\ \cos\alpha &= \vec{n}_r.\vec{n}_s \end{aligned} \tag{7.21}$$

where α is the angle between the geocentric vectors of the satellite and the sun, and $\cos\alpha$ is the inner product of the two related unit vectors.

Using the expansion formula (Bronstein and Semendjajew 1987; Wang et al. 1979)

$$\begin{aligned} \sqrt[q]{(1+x)^p} =& 1 + \frac{p}{q}x + \frac{p(p-q)}{q \cdot 2q}x^2 + \frac{p(p-q)(p-2q)}{q \cdot 2q \cdot 3q}x^3 + \ldots \\ & |x| \le 1, p > 0 \, or \, q > 0 \end{aligned} \tag{7.22}$$

and taking (7.21) into account, the ratio factor in (7.10) can be expanded by

$$\begin{aligned} \frac{r_s^2}{|\vec{r} - \vec{r}_s|^3} &= \frac{r_s^2}{r_s^3(1 + \frac{r^2}{r_s^2} - \frac{2r}{r_s}\cos\alpha)^{3/2}} \\ &\approx \frac{1}{r_s}\left(1 + 3\frac{r}{r_s}\cos\alpha - \frac{r^2}{r_s^2}(\frac{3}{2} - \frac{15}{8}\cos^2\alpha)\right), \end{aligned} \tag{7.23}$$

where

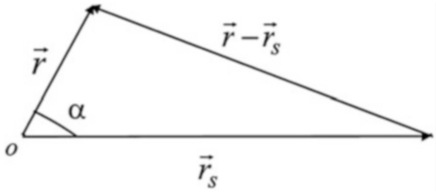


Fig. 7.3 Geometric relationship between the geocentric vectors of the satellite and the sun

$$r = \frac{a(1-e^2)}{1+e\cos f} \approx a(1-e^2)(1-e\cos f + e^2\cos^2 f),$$
$$r^2 = \frac{a^2(1-e^2)}{(1+e\cos f)^2} \approx a^2(1-2e^2)(1-2e\cos f + 3e^2\cos^2 f). \qquad (7.24)$$

The terms on the right-hand side of (7.23) are in orders of (r/r_s). To derive the second or third order solutions of solar radiation pressure disturbance, taking the first two or three terms into account will be precise enough. (see also discussions in Sect. 7.1.6). Furthermore, it holds that

$$\cos\alpha = \frac{\vec{r}}{r}\cdot\frac{\vec{r}_s}{r_s} = B^T\begin{pmatrix}\cos f\\ \sin f\\ 0\end{pmatrix}\cdot\begin{pmatrix}n_1\\ n_2\\ n_3\end{pmatrix}$$
$$= n_1(B^T_{11}\cos f + B^T_{12}\sin f) + n_2(B^T_{21}\cos f + B^T_{22}\sin f)$$
$$+ n_3(B^T_{31}\cos f + B^T_{32}\sin f) \qquad (7.25)$$

where B^{T} is the transpose of matrix B in (7.8) and the two indices denote the related element. The n with indices are the components of the solar unit vector given in (7.11).

Substituting the derived force vector into basic Gaussian equations of motion (7.17), the related solution can be derived.

7.1.5 Solutions of the Disturbed Equations

For convenience for the integration with respect to time t, the following functions are used to transform the trigonometric function of true anomaly f to mean anomaly M (see, e.g. (5.22), or Battin 1999; Kaula 1966/2001; Montenbruck and Gill 2000):

$$\sin f = (1-\frac{7}{8}e^2)\sin M + e(1-\frac{7}{6}e^2)\sin 2M + \frac{9}{8}e^2\sin 3M,$$
$$\cos f + e = (1-\frac{9}{8}e^2)\cos M + e(1-\frac{4}{3}e^2)\cos 2M + \frac{9}{8}e^2\cos 3M. \qquad (7.26)$$

Here terms smaller than e^2 are neglected. To demonstrate the principle of the method in a simple way, only the first term in (7.23) is taken into account for the moment. Substituting (7.23) and (7.18) into (7.10) allows the force vector to be represented completely by Keplerian elements. The basic Gaussian equations (7.17) can be transformed in terms of $\cos^m f \sin^k f$ and (using (7.24)) reduced to a functional series of $\{\cos mM, \sin kM\}$, which have the forms of

$$\frac{\mathrm{d}\sigma_{1j}}{\mathrm{d}t} = D_j + \sum_{k=1}^{3} (b_{jk} \cos kM + c_{jk} \sin kM), \tag{7.27}$$

where σ_1 with index j denotes the jth Keplerian element in (7.17); D, b, c are functions of (a, e, ω, i, Ω). All terms of M are short periodic perturbations and D-terms long periodic and linear ones.

Using mean value theorem for integration it holds that (Bronstein and Semendjajew 1987; Wang et al. 1979)

$$\int_0^T h(y(t))g(M(t))dt = h(y(\varsigma)) \int_0^T g(M(t))dt, 0 \le \varsigma \le T. \tag{7.28}$$

Here h (functions of b and c in (7.27)) and g (i.e. sinkM and coskM in (7.27), $k = 1$, 2, 3) can be integrated. The integration time interval [0, T] can be transformed into mean motion angular intervals of $\{[2(j-1)\pi/k, 2j\pi/k], j = 1, \ldots j\}$ and $[2J\pi/k, nT]$ (for any k the integer J can be obtained by $nT - 2J\pi/k < 2\pi/k)$); the integrals of the short periodic terms g are zero, except over the rest (non-full cycle) intervals $[2J\pi/k, nT]$. Therefore it holds that

$$\int_0^T h(y(t))g(M(t))dt = h(y(\varsigma)) \int_{2J\pi/k}^{nT} g(M\frac{1}{n}dM \tag{7.29}$$

where y(t) are slow changing Keplerian elements (except M). Selecting T so that h can be considered as constants over $[2J\pi/k, nT]$ (i.e. (7.29) is for $\varsigma = T$ valid) then the integrals (7.29) are is generally valid. For integration over time interval longer than T the integration can be made step-wise and then accumulated. In case of non-solar-synchronous satellite orbit, the integration should be done piece-wise over all the non-shadowing time intervals.

The solutions (i.e. indefinite integrals) of the differential equations (7.27) are then ($M = nt$ and mean value theorem for integration are used)

$$\Delta\sigma_{1j} = \Delta D_j + \sum_{k=1}^{3} \frac{1}{kn} (b_{jk} \sin kM - c_{jk} \cos kM), \tag{7.30}$$

where ΔD_j denotes symbolically the integrals of the long periodic and linear terms and will be discussed in the next section. These transformations and integrations are carried out using software for mathematic symbolic operations.

The explicit solutions (i.e. the short periodic terms in (7.30)) are (truncated to the terms of the order of e and $(1/r_s)$, for example)

$$\begin{aligned}\Delta a_1 &= (-B_{21}n_1 - B_{22}n_2 - B_{23}n_3)(\sin M + \frac{e}{2}\sin 2M)\\ &\quad + (B_{11}n_1 + B_{12}n_2 + B_{13}n_3)(-\cos M - \frac{e}{2}\cos 2M) - \frac{ae}{r_s}\cos M,\end{aligned} \tag{7.31}$$

$$\begin{aligned}\Delta e_1 &= (B_{21}n_1 + B_{22}n_2 + B_{23}n_3)(\frac{3}{4} - e\sin M - \frac{1}{4}\sin 2M - \frac{e}{4}\sin 3M)\\ &\quad + \frac{a}{r_s}(-\cos M - \frac{e}{4}\cos 2M) + (B_{11}n_1 + B_{12}n_2 + B_{13}n_3)\\ &\quad \times (\frac{e}{4}\cos M - \frac{1}{4}\cos 2M - \frac{e}{4}\cos 3M),\end{aligned} \tag{7.32}$$

$$\begin{aligned}\Delta\omega_1 &= (B_{11}n_1 + B_{12}n_2 + B_{13}n_3)(\frac{3e}{4}\sin M - \frac{1}{4}\sin 2M - \frac{e}{4}\sin 3M)\\ &\quad + \frac{a}{r_s}(-\sin M - \frac{e}{4}\sin 2M) + (B_{21}n_1 + B_{22}n_2 + B_{23}n_3)\\ &\quad \times (-\frac{5e}{4}\cos M + \frac{1}{4}\cos 2M + \cos 3M),\end{aligned} \tag{7.33}$$

$$\begin{aligned}\Delta i_1 &= (B_{31}n_1 + B_{32}n_2 + B_{33}n_3)\cos\omega(-\sin M - \frac{e}{4}\sin 2M)\\ &\quad + (B_{31}n_1 + B_{32}n_2 + B_{33}n_3)\sin\omega(-\cos M - \frac{e}{4}\cos 2M),\end{aligned} \tag{7.34}$$

$$\begin{aligned}\Delta\Omega_1 &= (B_{31}n_1 + B_{32}n_2 + B_{33}n_3)\cos\omega(-\sin M - \frac{e}{4}\sin 2M)\\ &\quad - (B_{31}n_1 + B_{32}n_2 + B_{33}n_3)\sin\omega(-\cos M - \frac{e}{4}\cos 2M),\end{aligned} \tag{7.35}$$

$$\begin{aligned}\Delta M_1 &= \frac{a}{r_s}(-2e\sin M) + (B_{11}n_1 + B_{12}n_2 + B_{13}n_3)(-\sin M - \frac{e}{4}\sin 2M)\\ &\quad + (B_{21}n_1 + B_{22}n_2 + B_{23}n_3)(\cos M - \frac{e}{4}\cos 2M)\end{aligned} \tag{7.36}$$

Formulas (7.31), (7.32), (7.33), (7.34), (7.35), and (7.36) show that all Keplerian elements are short-periodically disturbed by the solar radiation. Comparing (7.31), (7.32), (7.33), (7.34), (7.35), and (7.36) with (7.30), the coefficients of b and c can be easily obtained; therefore the equations in (7.27) are implicitly given to avoid too many formulas. It is notable that the solutions of the solar radiation perturbations on satellite orbits given above are solved by using basic Gaussian equations of motion (7.17).

7.1.6 Long Periodic and Secular Solutions

Usually the trigonometric functions of Keplerian angular element i are considered as long periodic ones. However, this is true only if the related element is linearly perturbed at least partly and dominantly. This can also be easily understood; e.g. if

$i = i_0 + \alpha \sin z, \alpha < \pi$ (z is any suitable variable), the definition interval of i is less than 2π and therefore sin$i$ and cos$i$ are not periodic functions.

Omitting the terms with M on the right-hand side of (7.27), (7.27) includes now only the long-periodic and secular terms and can be further reduced to have the forms

$$\frac{d\sigma_{1j}}{dt} = D_j = G_j + \sum_{k,m} (g_{jkm} \cos(k\Omega + m\omega) + q_{jkm} \sin(k\Omega + m\omega). \tag{7.37}$$

Here all terms of Ω and ω are long periodic disturbances. G are functions of (a, e, i) and are secular disturbances.

Using mean value theorem for integration the long-periodic terms of (7.37) can be similarly integrated. Due to the daily approximation of the solar unit vector (7.11), (7.12), and (7.13), the equations at (7.37) are daily valid. Selecting T so that g and q in (7.37) can be considered as constants over the interval [0, T], then the integrals of (7.37) can be obtained. For integration over a time interval longer than T, the integration can be made step-wise and then accumulated. It holds that

$$\begin{aligned} \Delta\sigma_{1j} &= \Delta G_j + \sum_{k,m} K_m^k (g_{jkm} \sin(k\Omega + m\omega) - q_{jkm} \cos(k\Omega + m\omega)), \\ K_m^k &= \frac{1}{kn_\Omega + mn_\omega} \end{aligned} \tag{7.38}$$

where ΔG_j are the integrals of the linear terms and n_Ω and n_ω are mean motion velocities of the ascending node and the argument of perigee. Or explicitly

$$\begin{aligned} \Delta a_1 &= 0, \\ \Delta e_1 &= \frac{3}{4} K_{-1}^1 (1 - \cos i)(n_2 \sin(\Omega - \omega) + n_1 \cos(\Omega - \omega)) \\ &\quad - \frac{3}{2} K_1^0 n_3 \sin i \sin\omega - \frac{3}{4} K_1^1 (1 + \cos i)(n_2 \sin(\Omega + \omega) + n_1 \cos(\Omega + \omega)), \\ \Delta\omega_1 &= \frac{3ae}{2r_s} t + \frac{3}{4} K_{-1}^1 (1 - \cos i)(n_1 \sin(\Omega - \omega) + n_2 \cos(\Omega - \omega)) \\ &\quad + \frac{3}{4} K_1^1 (1 + \cos i)(n_1 \sin(\Omega + \omega) - n_2 \cos(\Omega + \omega)) - \frac{3}{2} K_1^0 n_3 \sin i \cos\omega, \\ \Delta i_1 &= -\frac{3e}{4} K_{-1}^1 \sin i (n_2 \sin(\Omega - \omega) + n_1 \cos(\Omega - \omega)) \\ &\quad - \frac{3}{4} K_1^1 \sin i (n_2 \sin(\Omega + \omega) + n_1 \cos(\Omega + \omega)) - \frac{3e}{2} K_1^0 n_3 \cos i \sin\omega, \\ \Delta\Omega_1 &= \frac{3e}{4} K_{-1}^1 \sin i (n_1 \sin(\Omega - \omega) - n_1 \cos(\Omega - \omega)) \\ &\quad - \frac{3e}{4} K_1^1 \sin i (n_1 \sin(\Omega + \omega) + n_2 \cos(\Omega + \omega)) - \frac{3e}{2} K_1^0 n_3 \cos i \cos\omega, \end{aligned}$$

$$\Delta M_1 = \frac{a}{r_s}t - \frac{3}{2}K_1^0 n_3 \sin i \cos\omega + \frac{3}{4}K_{-1}^1(1-\cos i)(n_1\sin(\Omega-\omega) - n_2\cos(\Omega-\omega)) + \frac{3}{4}K_1^1(1+\cos i)(n_1\sin(\Omega+\omega) + n_2\cos(\Omega+\omega)), \quad (7.39)$$

where all effects are long-periodic ones except for the following two secular influences:

$$\Delta\omega_1(\text{secular}) = \frac{3ae}{2r_s}t, \quad \Delta M_1(\text{secular}) = \frac{a}{r_s}t \quad (7.40)$$

or equivalently

$$\mathrm{d}\omega = n_\omega \mathrm{d}t, \quad n_\omega = h_3\frac{3ae}{2r_s}, \quad \mathrm{d}M = (n+n_M)\mathrm{d}t, \quad n_M = -h_6\frac{7ae}{2r_s}. \quad (7.41)$$

Equation 7.41 represent contributions of the solar radiation to the mean motion of Keplerian elements of a satellite. Without knowing the mean motion, the integration is difficult. This shows the importance of our algorithms to separate the disturbances into secular, long- and short-periodic parts.

Comparing (7.37) with (7.39), the coefficients of g and q can be obtained (i.e. the equations at (7.37) are indirectly given). Again, the solutions of the solar radiation perturbations on satellite orbits given above are solved by using basic Gaussian equations of motion (7.17).

7.1.7 More Precise Solutions

The solutions given above are approximated to the order of e and/or $(1/r_s)$. More precise solutions, approximated to the order of e^2 and (a/r_s^2), can be similarly derived by using a slightly modified software. For simplicity, we give here the solutions of basic Gaussian equations. The solutions can be derived without problems using the above-mentioned algorithms. It follows that (factor ξ is omitted, $rs = r_s$, $cw = \cos\omega$, $sw = \sin\omega$)

$$
\begin{aligned}
\frac{\mathrm{d}a_1}{\mathrm{d}t} = & -\frac{1}{8rs^2}(a^2e(-12B_{21}n_1-12B_{22}n_2-12B_{23}n_3)+(a^2e^2(-12B_{21}n_1-12B_{22}n_2\\
& -12B_{23}n_3)+a\,e(B_{11}(-36B_{21}n_1^2-36B_{22}n_1n_2-36B_{23}n_1n_3)\\
& +B_{12}(-36B_{21}n_1n_2-36B_{22}n_2^2-36B_{23}n_2n_3)+B_{13}(-36B_{21}n_1n_3-36B_{22}n_2n_3\\
& -36B_{23}n_3^2))rs+(8B_{21}n_1+8B_{22}n_2+8B_{23}n_3+e^2(-9B_{21}n_1-9B_{22}n_2\\
& -9B_{23}n_3))rs^2)\cos M+(a^2e(12B_{21}n_1+12B_{22}n_2+12B_{23}n_3)+a(B_{11}(24B_{21}n_1^2\\
& +24B_{22}n_1n_2+24B_{23}n_1n_3)+B_{12}(24B_{21}n_1n_2+24B_{22}n_2^2+24B_{23}n_2n_3)\\
& +B_{13}(24B_{21}n_1n_3+24B_{22}n_2n_3+24B_{23}n_3^2)+e^2(B_{11}(-72B_{21}n_1^2-72B_{22}n_1n_2\\
& -72B_{23}n_1n_3)+B_{12}(-72B_{21}n_1n_2-72B_{22}n_2^2-72B_{23}n_2n_3)+B_{13}(-72B_{21}n_1n_3\\
& -72B_{22}n_2n_3-72B_{23}n_3^2)))rs+e(8B_{21}n_1+8B_{22}n_2+8B_{23}n_3)rs^2)\cos 2M\\
& +(a^2e^2(12B_{21}n_1+12B_{22}n_2+12B_{23}n_3)+ae(B_{11}(36B_{21}n_1^2+36B_{22}n_1n_2\\
& +36B_{23}n_1n_3)+B_{12}(36B_{21}n_1n_2+36B_{22}n_2^2+36B_{23}n_2n_3)+B_{13}(36B_{21}n_1n_3\\
& +36B_{22}n_2n_3+36B_{23}n_3^2))\,rs+e^2(9B_{21}n_1+9B_{22}n_2+9B_{23}n_3)rs^2)\cos 3M\\
& +a\,e^2(B_{11}(48B_{21}n_1^2+48B_{22}n_1n_2+48B_{23}n_1n_3)+B_{12}(48B_{21}n_1n_2+48B_{22}n_2^2\\
& +48B_{23}n_2n_3)+B_{13}(48B_{21}n_1n_3+48B_{22}n_2n_3+48B_{23}n_3^2))rs\cos 4M\\
& +(a^2e^2(36B_{11}n_1+36B_{12}n_2+36B_{13}n_3)+a\,e(-8+30B_{11}^2n_1^2-6B_{21}^2n_1^2\\
& +30B_{12}^2n_2^2-6B_{22}^2n_2^2+60B_{12}B_{13}n_2n_3-12B_{22}B_{23}n_2n_3+30B_{13}^2n_3^2-6B_{23}^2n_3^2\\
& +B_{11}(60B_{12}n_1n_2+60B_{13}n_1n_3)+B_{21}(-12B_{22}n_1n_2-12B_{23}n_1n_3))rs\\
& +(-8B_{11}n_1-8B_{12}n_2-8B_{13}n_3+e^2(7B_{11}n_1+7B_{12}n_2+7B_{13}n_3))rs^2)\sin M\\
& +(a^2e(-12B_{11}n_1-12B_{12}n_2-12B_{13}n_3)+a(-12B_{11}^2n_1^2+12B_{21}^2n_1^2\\
& -12B_{12}^2n_2^2+12B_{22}^2n_2^2-24B_{12}B_{13}n_2n_3+24B_{22}B_{23}n_2n_3-12B_{13}^2n_3^2+12B_{23}^2n_3^2\\
& +B_{11}(-24B_{12}n_1n_2-24B_{13}n_1n_3)+B_{21}(24B_{22}n_1n_2+24B_{23}n_1n_3)\\
& +e^2(-4+42B_{11}^2n_1^2-30B_{21}^2n_1^2+42B_{12}^2n_2^2-30B_{22}^2n_2^2+84B_{12}B_{13}n_2n_3\\
& -60B_{22}B_{23}n_2n_3+42B_{13}^2n_3^2-30B_{23}^2n_3^2+B_{11}(84B_{12}n_1n_2+84B_{13}n_1n_3)\\
& +B_{21}(-60B_{22}n_1n_2-60B_{23}n_1n_3)))rs+e(-8B_{11}n_1-8B_{12}n_2\\
& -8B_{13}n_3)rs^2)\sin 2M+(a^2e^2(-12B_{11}n_1-12B_{12}n_2-12B_{13}n_3)\\
& +a\,e(-18B_{11}^2n_1^2+18B_{21}^2n_1^2-18B_{12}^2n_2^2+18B_{22}^2n_2^2-36B_{12}B_{13}n_2n_3\\
& +36B_{22}B_{23}n_2n_3-18B_{13}^2n_3^2+18B_{23}^2n_3^2+B_{11}(-36B_{12}n_1n_2-36B_{13}n_1n_3)\\
& +B_{21}(36B_{22}n_1n_2+36B_{23}n_1n_3))rs+e^2(-9B_{11}n_1-9B_{12}n_2-9B_{13}n_3)rs^2)\sin 3M\\
& +a\,e^2(-24B_{11}^2n_1^2+24B_{21}^2n_1^2-24B_{12}^2n_2^2+24B_{22}^2n_2^2-48B_{12}B_{13}n_2n_3\\
& +48B_{22}B_{23}n_2n_3-24B_{13}^2n_3^2+24B_{23}^2n_3^2+B_{11}(-48B_{12}n_1n_2-48B_{13}n_1n_3)\\
& +B_{21}(48B_{22}n_1n_2+48B_{23}n_1n_3))rs\sin 4M)
\end{aligned}
\tag{7.42}
$$

$$
\begin{aligned}
\frac{\mathrm{d}e_1}{\mathrm{d}t} = &-\frac{1}{32rs^2}(a^2(-48B_{21}n_1 - 48B_{22}n_2 - 48B_{23}n_3 + e^2(48B_{21}n_1 + 48B_{22}n_2 \\
&+ 48B_{23}n_3)) + a\ e(B_{11}(-240B_{21}n_1^2 - 240B_{22}n_1n_2 - 240B_{23}n_1n_3) \\
&+ B_{12}(-240B_{21}n_1n_2 - 240B_{22}n_2^2 - 240B_{23}n_2n_3) + B_{13}(-240B_{21}n_1n_3 \\
&- 240B_{22}n_2n_3 - 240B_{23}n_3^2))rs + (48B_{21}n_1 + 48B_{22}n_2 + 48B_{23}n_3)rs^2 \\
&+ (a^2e(-48B_{21}n_1 - 48B_{22}n_2 - 48B_{23}n_3) + a(B_{11}(144B_{21}n_1^2 + 144B_{22}n_1n_2 \\
&+ 144B_{23}n_1n_3) + B_{12}(144B_{21}n_1n_2 + 144B_{22}n_2^2 + 144B_{23}n_2n_3) \\
&+ B_{13}(144B_{21}n_1n_3 + 144B_{22}n_2n_3 + 144B_{23}n_3^2) + e^2(B_{11}(36B_{21}n_1^2 \\
&+ 36B_{22}n_1n_2 + 36B_{23}n_1n_3) + B_{12}(36B_{21}n_1n_2 + 36B_{22}n_2^2 + 36B_{23}n_2n_3) \\
&+ B_{13}(36B_{21}n_1n_3 + 36B_{22}n_2n_3 + 36B_{23}n_3^2)))rs + e(-24B_{21}n_1 - 24B_{22}n_2 \\
&- 24B_{23}n_3)rs^2)\cos M + (a^2(48B_{21}n_1 + 48B_{22}n_2 + 48B_{23}n_3 + e^2(-96B_{21}n_1 \\
&- 96B_{22}n_2 - 96B_{23}n_3)) + a\ e(B_{11}(-48B_{21}n_1^2 - 48B_{22}n_1n_2 - 48B_{23}n_1n_3) \\
&+ B_{12}(-48B_{21}n_1n_2 - 48B_{22}n_2^2 - 48B_{23}n_2n_3) + B_{13}(-48B_{21}n_1n_3 \\
&- 48B_{22}n_2n_3 - 48B_{23}n_3^2))rs + (16B_{21}n_1 + 16B_{22}n_2 + 16B_{23}n_3 \\
&+ e^2(-32B_{21}n_1 - 32B_{22}n_2 - 32B_{23}n_3))rs^2)\cos 2M + (a^2e(48B_{21}n_1 \\
&+ 48B_{22}n_2 + 48B_{23}n_3) + a(B_{11}(48B_{21}n_1^2 + 48B_{22}n_1n_2 + 48B_{23}n_1n_3) \\
&+ B_{12}(48B_{21}n_1n_2 + 48B_{22}n_2^2 + 48B_{23}n_2n_3) + B_{13}(48B_{21}n_1n_3 + 48B_{22}n_2n_3 \\
&+ 48B_{23}n_3^2) + e^2(B_{11}(-186B_{21}n_1^2 - 186B_{22}n_1n_2 - 186B_{23}n_1n_3) \\
&+ B_{12}(-186B_{21}n_1n_2 - 186B_{22}n_2^2 - 186B_{23}n_2n_3) + B_{13}(-186B_{21}n_1n_3 \\
&- 186B_{22}n_2n_3 - 186B_{23}n_3^2)))rs + e(24B_{21}n_1 + 24B_{22}n_2 \\
&+ 24B_{23}n_3)rs^2)\cos 3M + (a^2e^2(48B_{21}n_1 + 48B_{22}n_2 + 48B_{23}n_3) \\
&+ a\ e(B_{11}(96B_{21}n_1^2 + 96B_{22}n_1n_2 + 96B_{23}n_1n_3) + B_{12}(96B_{21}n_1n_2 + 96B_{22}n_2^2 \\
&+ 96B_{23}n_2n_3) + B_{13}(96B_{21}n_1n_3 + 96B_{22}n_2n_3 + 96B_{23}n_3^2))rs + e^2(32B_{21}n_1 \\
&+ 32B_{22}n_2 + 32B_{23}n_3)rs^2)\cos 4M + a\ e^2(B_{11}(150B_{21}n_1^2 + 150B_{22}n_1n_2 \\
&+ 150B_{23}n_1n_3) + B_{12}(150B_{21}n_1n_2 + 150B_{22}n_2^2 + 150B_{23}n_2n_3) \\
&+ B_{13}(150B_{21}n_1n_3 + 150B_{22}n_2n_3 + 150B_{23}n_3^2))rs\ \cos 5M + (a^2e(144B_{11}n_1 \\
&+ 144B_{12}n_2 + 144B_{13}n_3) + a(-32 - 24B_{11}^2n_1^2 + 120B_{21}^2n_1^2 - 24B_{12}^2n_2^2 \\
&+ 120B_{22}^2n_2^2 - 48B_{12}B_{13}n_2n_3 + 240B_{22}B_{23}n_2n_3 - 24B_{13}^2n_3^2 + 120B_{23}^2n_3^2 \\
&+ B_{11}(-48B_{12}n_1n_2 - 48B_{13}n_1n_3) + B_{21}(240B_{22}n_1n_2 + 240B_{23}n_1n_3) \\
&+ e^2(20 - 6B_{11}^2n_1^2 - 54B_{21}^2n_1^2 - 6B_{12}^2n_2^2 - 54B_{22}^2n_2^2 - 12B_{12}B_{13}n_2n_3 \\
&- 108B_{22}B_{23}n_2n_3 - 6B_{13}^2n_3^2 - 54B_{23}^2n_3^2 + B_{11}(-12B_{12}n_1n_2 - 12B_{13}n_1n_3) \\
&+ B_{21}(-108B_{22}n_1n_2 - 108B_{23}n_1n_3)))rs + e(8B_{11}n_1 + 8B_{12}n_2 \\
&+ 8B_{13}n_3)rs^2)\sin M + (a^2(-48B_{11}n_1 - 48B_{12}n_2 - 48B_{13}n_3 + e^2(120B_{11}n_1
\end{aligned}
$$

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\begin{aligned}
&+120B_{12}n_2+120B_{13}n_3))+a\,e(-16+48B_{11}^2n_1^2+48B_{12}^2n_2^2+96B_{12}B_{13}n_2n_3\\
&+48B_{13}^2n_3^2+B_{11}(96B_{12}n_1n_2+96B_{13}n_1n_3))rs+(-16B_{11}n_1-16B_{12}n_2\\
&-16B_{13}n_3+e^2(24B_{11}n_1+24B_{12}n_2+24B_{13}n_3))rs^2)\sin 2M+(a^2e(-48B_{11}n_1\\
&-48B_{12}n_2-48B_{13}n_3)+a(-24B_{11}^2n_1^2+24B_{21}^2n_1^2-24B_{12}^2n_2^2+24B_{22}^2n_2^2\\
&-48B_{12}B_{13}n_2n_3+48B_{22}B_{23}n_2n_3-24B_{13}^2n_3^2+24B_{23}^2n_3^2+B_{11}(-48B_{12}n_1n_2\\
&-48B_{13}n_1n_3)+B_{21}(48B_{22}n_1n_2+48B_{23}n_1n_3)+e^2(-12+111B_{11}^2n_1^2\\
&-75B_{21}^2n_1^2+111B_{12}^2n_2^2-75B_{22}^2n_2^2+222B_{12}B_{13}n_2n_3-150B_{22}B_{23}n_2n_3\\
&+111B_{13}^2n_3^2-75B_{23}^2n_3^2+B_{11}(222B_{12}n_1n_2+222B_{13}n_1n_3)+B_{21}(-150B_{22}n_1\,n_2\\
&-150B_{23}n_1n_3)))rs+e(-24B_{11}n_1-24B_{12}n_2-24B_{13}n_3)rs^2)\sin 3M\\
&+(a^2e^2(-48B_{11}n_1-48B_{12}n_2-48B_{13}n_3)+a\,e(-48B_{11}^2n_1^2+48B_{21}^2n_1^2\\
&-48B_{12}^2n_2^2+48B_{22}^2n_2^2-96B_{12}B_{13}n_2n_3+96B_{22}B_{23}n_2n_3-48B_{13}^2n_3^2+48B_{23}^2n_3^2\\
&+B_{11}(-96B_{12}n_1n_2-96B_{13}n_1n_3)+B_{21}(96B_{22}n_1n_2+96B_{23}n_1n_3))rs\\
&+a\,e^2(-32B_{11}n_1-32B_{12}n_2-32B_{13}n_3)rs^2)\sin 4M+a\,e^2(-75B_{11}^2n_1^2\\
&+75B_{21}^2n_1^2-75B_{12}^2n_2^2+75B_{22}^2n_2^2-150B_{12}B_{13}n_2n_3+150B_{22}B_{23}n_2n_3\\
&-75B_{13}^2n_3^2+75B_{23}^2n_3^2+B_{11}(-150B_{12}n_1n_2-150B_{13}n_1n_3)\\
&+B_{21}(150B_{22}n_1n_2+150B_{23}n_1n_3))rs\sin 5M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\omega_1}{\mathrm{d}t}=-\frac{1}{32rs^2}&(a^2(48B_{11}n_1+48B_{12}n_2+48B_{13}n_3+e^2(192B_{11}n_1+192B_{12}n_2\\
&+192B_{13}n_3))+a\,e(-48+192B_{11}^2n_1^2-48B_{21}^2n_1^2+192B_{12}^2n_2^2-48B_{22}^2n_2^2\\
&+384B_{12}B_{13}n_2n_3-96B_{22}B_{23}n_2n_3+192B_{13}^2n_3^2-48B_{23}^2n_3^2\\
&+B_{11}(384B_{12}n_1n_2+384B_{13}n_1n_3)+B_{21}(-96B_{22}n_1n_2-96B_{23}n_1n_3))rs\\
&+(-48B_{11}n_1-48B_{12}n_2-48B_{13}n_3)rs^2+(a^2e(-240B_{11}n_1-240B_{12}n_2\\
&-240B_{13}n_3)+a(32-120B_{11}^2n_1^2+24B_{21}^2n_1^2-120B_{12}^2n_2^2+24B_{22}^2n_2^2\\
&-240B_{12}B_{13}n_2n_3+48B_{22}B_{23}n_2n_3-120B_{13}^2n_3^2+24B_{23}^2n_3^2\\
&+B_{11}(-240B_{12}n_1n_2-240B_{13}n_1n_3)+B_{21}(48B_{22}n_1n_2+48B_{23}n_1n_3)\\
&+e^2(-12+114B_{11}^2n_1^2-78B_{21}^2n_1^2+114B_{12}^2n_2^2-78B_{22}^2n_2^2\\
&+228B_{12}B_{13}n_2n_3-156B_{22}B_{23}n_2n_3+114B_{13}^2n_3^2-78B_{23}^2n_3^2\\
&+B_{11}(228B_{12}n_1n_2+228B_{13}n_1n_3)+B_{21}(-156B_{22}n_1n_2-156B_{23}n_1n_3)))rs\\
&+e(-24B_{11}n_1-24B_{12}n_2-24B_{13}n_3)rs^2)\cos M+(a^2(48B_{11}n_1\\
&+48B_{12}n_2+48B_{13}n_3+e^2(-144B_{11}n_1-144B_{12}n_2-144B_{13}n_3))\\
&+a\,e(16-144B_{11}^2n_1^2+96B_{21}^2n_1^2-144B_{12}^2n_2^2+96B_{22}^2n_2^2-288B_{12}B_{13}n_2n_3\\
&+192B_{22}B_{23}n_2n_3-144B_{13}^2n_3^2+96B_{23}^2n_3^2+B_{11}(-288B_{12}n_1n_2\\
&-288B_{13}n_1n_3)+B_{21}(192B_{22}n_1n_2+192B_{23}n_1n_3))rs+(16B_{11}n_1+16B_{12}n_2\\
&+16B_{13}n_3+e^2(-32B_{11}n_1-32B_{12}n_2-32B_{13}n_3))rs^2)\cos 2M
\end{aligned}
$$

$$
\begin{aligned}
&+(a^2e(48B_{11}n_1+48B_{12}n_2+48B_{13}n_3)+a(24B_{11}^2n_1^2-24B_{21}^2n_1^2+24B_{12}^2n_2^2\\
&-24B_{22}^2n_2^2+48B_{12}B_{13}n_2n_3-48B_{22}B_{23}n_2n_3+24B_{13}^2n_3^2-24B_{23}^2n_3^2\\
&+B_{11}(48B_{12}n_1n_2+48B_{13}n_1n_3)+B_{21}(-48B_{22}n_1n_2-48B_{23}n_1n_3)\\
&+e^2(12-189B_{11}^2n_1^2+153B_{21}^2n_1^2-189B_{12}^2n_2^2+153B_{22}^2n_2^2-378B_{12}B_{13}n_2n_3\\
&+306B_{22}B_{23}n_2n_3-189B_{13}^2n_3^2+153B_{23}^2n_3^2+B_{11}(-378B_{12}n_1n_2\\
&-378B_{13}n_1n_3)+B_{21}(306B_{22}n_1n_2+306B_{23}n_1n_3)))rs+e(24B_{11}n_1\\
&+24B_{12}n_2+24B_{13}n_3)rs^2)\cos 3M+(a^2e^2(48B_{11}n_1+48B_{12}n_2+48B_{13}n_3)\\
&+a\,e(48B_{11}^2n_1^2-48B_{21}^2n_1^2+48B_{12}^2n_2^2-48B_{22}^2n_2^2+96B_{12}B_{13}n_2n_3\\
&-96B_{22}B_{23}n_2n_3+48B_{13}^2n_3^2-48B_{23}^2n_3^2+B_{11}(96B_{12}n_1n_2+96B_{13}n_1n_3)\\
&+B_{21}(-96B_{22}n_1n_2-96B_{23}n_1n_3))rs+e^2(32B_{11}n_1+32B_{12}n_2\\
&+32B_{13}n_3)rs^2)\cos 4M+a\,e^2(75B_{11}^2n_1^2-75B_{21}^2n_1^2+75B_{12}^2n_2^2-75B_{22}^2n_2^2\\
&+150B_{12}B_{13}n_2n_3-150B_{22}B_{23}n_2n_3+75B_{13}^2n_3^2-75B_{23}^2n_3^2+B_{11}(150B_{12}n_1n_2\\
&+150B_{13}n_1n_3)+B_{21}(-150B_{22}n_1n_2-150B_{23}n_1n_3))rs\cos 5M\\
&+(a^2e(-144B_{21}n_1-144B_{22}n_2-144B_{23}n_3)+a(B_{11}(-144B_{21}n_1^2\\
&-144B_{22}n_1n_2-144B_{23}n_1n_3)+B_{12}(-144B_{21}n_1n_2-144B_{22}n_2^2\\
&-144B_{23}n_2n_3)+B_{13}(-144B_{21}n_1n_3-144B_{22}n_2n_3-144B_{23}n_3^2)\\
&+e^2(B_{11}(276B_{21}n_1^2+276B_{22}n_1n_2+276B_{23}n_1n_3)+B_{12}(276B_{21}n_1n_2\\
&+276B_{22}n_2^2+276B_{23}n_2n_3)+B_{13}(276B_{21}n_1n_3+276B_{22}n_2n_3+276B_{23}n_3^2)))rs\\
&+e(-40B_{21}n_1-40B_{22}n_2-40B_{23}n_3)rs^2)\sin M+(a^2(48B_{21}n_1+48B_{22}n_2\\
&+48B_{23}n_3+e^2(-120B_{21}n_1-120B_{22}n_2-120B_{23}n_3))+a\,e(B_{11}(-240B_{21}n_1^2\\
&-240B_{22}n_1n_2-240B_{23}n_1n_3)+B_{12}(-240B_{21}n_1n_2-240B_{22}n_2^2-240B_{23}n_2n_3)\\
&+B_{13}(-240B_{21}n_1n_3-240B_{22}n_2n_3-240B_{23}n_3^2))rs+(16B_{21}n_1+16B_{22}n_2\\
&+16B_{23}n_3+e^2(-40B_{21}n_1-40B_{22}n_2-40B_{23}n_3))rs^2)\sin 2M+(a^2e(48B_{21}n_1\\
&+48B_{22}n_2+48B_{23}n_3)+a(B_{11}(48B_{21}n_1^2+48B_{22}n_1n_2+48B_{23}n_1n_3)\\
&+B_{12}(48B_{21}n_1n_2+48B_{22}n_2^2+48B_{23}n_2n_3)+B_{13}(48B_{21}n_1n_3+48B_{22}n_2n_3\\
&+48B_{23}n_3^2)+e^2(B_{11}(-342B_{21}n_1^2-342B_{22}n_1n_2-342B_{23}n_1n_3)\\
&+B_{12}(-342B_{21}n_1n_2-342B_{22}n_2^2-342B_{23}n_2n_3)+B_{13}(-342B_{21}n_1n_3\\
&-342B_{22}n_2n_3-342B_{23}n_3^2)))rs+e(24B_{21}n_1+24B_{22}n_2+24B_{23}n_3)rs^2)\sin 3M\\
&+(a^2e^2(48B_{21}n_1+48B_{22}n_2+48B_{23}n_3)+a\,e(B_{11}(96B_{21}n_1^2+96B_{22}n_1n_2\\
&+96B_{23}n_1n_3)+B_{12}(96B_{21}n_1n_2+96B_{22}n_2^2+96B_{23}n_2n_3)+B_{13}(96B_{21}n_1n_3\\
&+96B_{22}n_2n_3+96B_{23}n_3^2))rs+e^2(32B_{21}n_1+32B_{22}n_2+32B_{23}n_3)rs^2)\sin 4M\\
&+a\,e^2(B_{11}(150B_{21}n_1^2+150B_{22}n_1n_2+150B_{23}n_1n_3)+B_{12}(150B_{21}n_1n_2\\
&+150B_{22}n_2^2+150B_{23}n_2n_3)+B_{13}(150B_{21}n_1n_3+150B_{22}n_2n_3\\
&+150B_{23}n_3^2))rs\sin 5M)
\end{aligned}
$$

$$\begin{aligned}\frac{\mathrm{d}i_1}{\mathrm{d}t} = &-\frac{1}{8rs}(B_{31}n_1 + B_{32}n_2 + B_{33}n_3)(-12cw\ e\ rs + a(12B_{11}cw\ n_1 + 12B_{12}cw\ n_2\\
&+ 12B_{13}cw\ n_3 - 12B_{21}n_1sw - 12B_{22}n_2sw - 12B_{23}n_3sw + e^2(48B_{11}cw\ n_1\\
&+ 48B_{12}cw\ n_2 + 48B_{13}cw\ n_3 + 12B_{21}n_1sw + 12B_{22}n_2sw + 12B_{23}n_3sw))\\
&+ ((8cw - 3cw\ e^2)rs + a\ e(-60B_{11}cw\ n_1 - 60B_{12}cw\ n_2 - 60B_{13}cw\ n_3\\
&- 12B_{21}n_1sw - 12B_{22}n_2sw - 12B_{23}n_3sw))\cos M + (4cw\ e\ rs\\
&+ a(12B_{11}cw\ n_1 + 12B_{12}cw\ n_2 + 12B_{13}cw\ n_3 + 12B_{21}n_1sw + 12B_{22}n_2sw\\
&+ 12B_{23}n_3sw + e^2(-36B_{11}cwn_1 - 36B_{12}cw\ n_2 - 36B_{13}cw\ n_3 - 24B_{21}n_1sw\\
&- 24B_{22}n_2sw - 24B_{23}n_3sw)))\cos 2M + (3cw\ e^2rs + a\ e(12B_{11}cw\ n_1\\
&+ 12B_{12}cw\ n_2 + 12B_{13}cw\ n_3 + 12B_{21}n_1sw + 12B_{22}n_2sw\\
&+ 12B_{23}n_3sw))\cos 3M + a\ e^2(12B_{11}cw\ n_1 + 12B_{12}cw\ n_2 + 12B_{13}cw\ n_3\\
&+ 12B_{21}n_1sw + 12B_{22}n_2sw + 12B_{23}n_3sw)\cos 4M + (rs(-8sw + 5e^2sw)\\
&+ ae(-36B_{21}cw\ n_1 - 36B_{22}cw\ n_2 - 36B_{23}cw\ n_3 + 36B_{11}n_1sw\\
&+ 36B_{12}n_2sw + 36B_{13}n_3sw))\sin M + (-4e\ rs\ sw + a(12B_{21}cw\ n_1\\
&+ 12B_{22}cw\ n_2 + 12B_{23}cw\ n_3 - 12B_{11}n_1sw - 12B_{12}n_2sw - 12B_{13}n_3sw\\
&+ e^2(-30B_{21}cw\ n_1 - 30B_{22}cw\ n_2 - 30B_{23}cw\ n_3 + 30B_{11}n_1sw + 30B_{12}n_2sw\\
&+ 30B_{13}n_3sw)))\sin 2M + (-3e^2rs\ sw + a\ e(12B_{21}cw\ n_1 + 12B_{22}cw\ n_2\\
&+ 12B_{23}cw\ n_3 - 12B_{11}n_1sw - 12B_{12}n_2sw - 12B_{13}n_3sw))\sin 3M\\
&+ a\ e^2(12B_{21}cw\ n_1 + 12B_{22}cw\ n_2 + 12B_{23}cw\ n_3 - 12B_{11}n_1sw\\
&- 12B_{12}n_2sw - 12B_{13}n_3sw)\sin 4M)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} = &\frac{1}{8rs}(B_{31}n_1 + B_{32}n_2 + B_{33}n_3)(12e\,rs\,sw + a(-12B_{21}cw\,n_1 - 12B_{22}cw\,n_2\\
&- 12B_{23}cw\,n_3 - 12B_{11}n_1sw - 12B_{12}n_2sw - 12B_{13}n_3sw + e^2(12B_{21}cw\,n_1\\
&+ 12B_{22}cw\,n_2 + 12B_{23}cw\,n_3 - 48B_{11}n_1sw - 48B_{12}n_2sw - 48B_{13}n_3sw))\\
&+ (rs(-8sw + 3e^2sw) + a\,e(-12B_{21}cw\,n_1 - 12B_{22}cw\,n_2 - 12B_{23}cw\,n_3\\
&+ 60B_{11}n_1sw + 60B_{12}n_2sw + 60B_{13}n_3sw))\cos M + (-4e\,rs\,sw\\
&+ a(12B_{21}cw\,n_1 + 12B_{22}cw\,n_2 + 12B_{23}cw\,n_3 - 12B_{11}n_1sw - 12B_{12}n_2sw\\
&- 12B_{13}n_3sw + e^2(-24B_{21}cw\,n_1 - 24B_{22}cw\,n_2 - 24B_{23}cw\,n_3\\
&+ 36B_{11}n_1sw + 36B_{12}n_2sw + 36B_{13}n_3sw)))\cos 2M + (-3e^2rs\,sw\\
&+ a\,e(12B_{21}cw\,n_1 + 12B_{22}cw\,n_2 + 12B_{23}cw\,n_3 - 12B_{11}n_1sw\\
&- 12B_{12}n_2sw - 12B_{13}n_3sw))\cos 3M + a\,e^2(12B_{21}cw\,n_1 + 12B_{22}cw\,n_2\\
&+ 12B_{23}\,cw\,n_3 - 12B_{11}n_1sw - 12B_{12}n_2sw - 12B_{13}n_3sw)\cos 4M\\
&+ ((-8cw + 5cw\,e^2)rs + a\,e(36B_{11}cw\,n_1 + 36B_{12}cw\,n_2\\
&+ 36B_{13}cw\,n_3 + 36B_{21}n_1sw + 36B_{22}n_2sw + 36B_{23}n_3sw))\sin M\\
&+ (-4cw\,e\,rs + a(-12B_{11}cw\,n_1 - 12B_{12}cw\,n_2 - 12B_{13}cw\,n_3\end{aligned}$$

$$
\begin{aligned}
&-12B_{21}n_1sw-12B_{22}n_2sw-12B_{23}n_3sw+e^2(30B_{11}cw\,n_1+30B_{12}cw\,n_2\\
&+30B_{13}cw\,n_3+30B_{21}n_1sw+30B_{22}n_2sw+30B_{23}n_3sw)))\sin 2M\\
&+(-3cw\,e^2rs+a\,e(-12B_{11}cw\,n_1-12B_{12}cw\,n_2-12B_{13}cw\,n_3\\
&-12B_{21}n_1sw-12B_{22}n_2sw-12B_{23}\,n_3sw))\sin 3M+a\,e^2(-12B_{11}cw\,n_1\\
&-12B_{12}cw\,n_2-12B_{13}cw\,n_3-12B_{21}n_1sw-12B_{22}n_2sw-12B_{23}n_3sw)\sin 4M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}M_1}{\mathrm{d}t}=&-\frac{1}{8rs^2}(a^2e(60B_{11}n_1+60B_{12}n_2+60B_{13}n_3)+a(-8+12B_{11}^2n_1^2+12B_{21}^2n_1^2\\
&+12B_{12}^2n_2^2+12B_{22}^2n_2^2+24B_{12}B_{13}n_2n_3+24B_{22}B_{23}n_2n_3+12B_{13}^2n_3^2\\
&+12B_{23}^2n_3^2+B_{11}(24B_{12}n_1n_2+24B_{13}n_1n_3)+B_{21}(24B_{22}n_1n_2+24B_{23}n_1n_3)\\
&+e^2(-20+60B_{11}^2n_1^2+60B_{12}^2n_2^2+120B_{12}B_{13}n_2n_3+60B_{13}^2n_3^2\\
&+B_{11}(120B_{12}n_1n_2+120B_{13}n_1n_3)))rs+e(-12B_{11}n_1-12B_{12}n_2\\
&-12B_{13}n_3)rs^2+(a^2(-24B_{11}n_1-24B_{12}n_2-24B_{13}n_3+e^2(-105B_{11}n_1\\
&-105B_{12}n_2-105B_{13}n_3))+a\,e(16-60B_{11}^2n_1^2+12B_{21}^2n_1^2-60B_{12}^2n_2^2\\
&+12B_{22}^2n_2^2-120B_{12}B_{13}n_2n_3+24B_{22}B_{23}n_2n_3-60B_{13}^2n_3^2+12B_{23}^2n_3^2\\
&+B_{11}(-120B_{12}n_1n_2-120B_{13}n_1n_3)+B_{21}(24B_{22}n_1n_2+24B_{23}n_1n_3))rs\\
&+(8B_{11}n_1+8B_{12}n_2+8B_{13}n_3+e^2(5B_{11}n_1+5B_{12}n_2+5B_{13}n_3))rs^2)\cos M\\
&+(a^2e(12B_{11}n_1+12B_{12}n_2+12B_{13}n_3)+a(12B_{11}^2n_1^2-12B_{21}^2n_1^2+12B_{12}^2n_2^2\\
&-12B_{22}^2n_2^2+24B_{12}B_{13}n_2n_3-24B_{22}B_{23}n_2n_3+12B_{13}^2n_3^2-12B_{23}^2n_3^2\\
&+B_{11}(24B_{12}n_1n_2+24B_{13}n_1n_3)+B_{21}(-24B_{22}n_1n_2-24B_{23}n_1n_3)\\
&+e^2(4-24B_{11}^2n_1^2+12B_{21}^2n_1^2-24B_{12}^2n_2^2+12B_{22}^2n_2^2-48B_{12}B_{13}n_2n_3\\
&+24B_{22}B_{23}n_2n_3-24B_{13}^2n_3^2+12B_{23}^2n_3^2+B_{11}(-48B_{12}n_1n_2-48B_{13}n_1n_3)\\
&+B_{21}(24B_{22}n_1n_2+24B_{23}n_1n_3)))rs+e(4B_{11}n_1+4B_{12}n_2+4B_{13}n_3)rs^2)\\
&\cos 2M+(a^2e^2(9B_{11}n_1+9B_{12}n_2+9B_{13}n_3)+a\,e(12B_{11}^2n_1^2-12B_{21}^2n_1^2\\
&+12B_{12}^2n_2^2-12B_{22}^2n_2^2+24B_{12}B_{13}n_2n_3-24B_{22}B_{23}n_2n_3+12B_{13}^2n_3^2\\
&-12B_{23}^2n_3^2+B_{11}(24B_{12}n_1n_2+24B_{13}n_1n_3)+B_{21}(-24B_{22}n_1n_2\\
&-24B_{23}n_1n_3))rs+e^2(3B_{11}n_1+3B_{12}n_2+3B_{13}n_3)rs^2)\cos 3M\\
&+a\,e^2(12B_{11}^2n_1^2-12B_{21}^2n_1^2+12B_{12}^2n_2^2-12B_{22}^2n_2^2+24B_{12}B_{13}n_2n_3\\
&-24B_{22}B_{23}n_2n_3+12B_{13}^2n_3^2-12B_{23}^2n_3^2+B_{11}(24B_{12}n_1n_2+24B_{13}n_1n_3)\\
&+B_{21}(-24B_{22}n_1n_2-24B_{23}n_1n_3))rs\,\cos 4M+(a^2(-24B_{21}n_1-24B_{22}n_2\\
&-24B_{23}n_3+e^2(-39B_{21}n_1-39B_{22}n_2-39B_{23}n_3))+a\,e(B_{11}(-72B_{21}n_1^2\\
&-72B_{22}n_1n_2-72B_{23}n_1n_3)+B_{12}(-72B_{21}n_1n_2-72B_{22}n_2^2-72B_{23}n_2n_3)\\
&+B_{13}(-72B_{21}n_1n_3-72B_{22}n_2n_3-72B_{23}n_3^2))rs+(8B_{21}n_1+8B_{22}n_2\\
&+8B_{23}n_3+e^2(3B_{21}n_1+3B_{22}n_2+3B_{23}n_3))rs^2)\sin M+(a^2e(12B_{21}n_1\\
&+12B_{22}n_2+12B_{23}n_3)+a(B_{11}(24B_{21}n_1^2+24B_{22}n_1n_2+24B_{23}n_1n_3)\\
&+B_{12}(24B_{21}n_1n_2+24B_{22}n_2^2+24B_{23}n_2n_3)+B_{13}(24B_{21}n_1n_3+24B_{22}n_2n_3\\
&+24B_{23}n_3^2)+e^2(B_{11}(-36B_{21}n_1^2-36B_{22}n_1n_2-36B_{23}n_1n_3)
\end{aligned}
$$

$$+B_{12}(-36B_{21}n_1n_2-36B_{22}n_2^2-36B_{23}n_2n_3)+B_{13}(-36B_{21}n_1n_3$$
$$-36B_{22}n_2n_3-36B_{23}n_3^2)))rs+e(4B_{21}n_1+4B_{22}n_2+4B_{23}n_3)rs^2)\sin 2M$$
$$+(a^2e^2(9B_{21}n_1+9B_{22}n_2+9B_{23}n_3)+a\,e(B_{11}(24B_{21}n_1^2+24B_{22}n_1n_2$$
$$+24B_{23}n_1n_3)+B_{12}(24B_{21}n_1n_2+24B_{22}n_2^2+24B_{23}n_2n_3)+B_{13}(24B_{21}n_1n_3$$
$$+24B_{22}n_2n_3+24B_{23}n_3^2))rs+B_{13}(24B_{21}n_1n_3+24B_{22}n_2n_3$$
$$+24B_{23}n_3^2))rs\,\sin 4M)$$

The final solutions can be obtained by substituting above formulas into (7.16). The long terms and long periodic terms are given as follows ($o=\Omega, w=\omega, si=\sin i$, $ci=\cos i$):

$$D_1=\frac{3}{4rs^2}a^2e((-1+ci)n_2\cos(o-w)+2n_3si\,\cos(w)+(1+ci)n_2\cos(o+w)$$
$$+(1-ci)n_1\sin(o-w)+(-1-ci)n_1\sin(o+w))$$

(7.43)

$$D_2=\frac{3}{8rs^2}((1-ci)n_2\cos(o-w)-2n_3si\cos(w)+(-1-ci)n_2\cos(o+w)$$
$$+(-1+ci)n_1\sin(o-w)+(1+ci)n_1\sin(o+w))(a^2(-2+2e^2)$$
$$+2rs^2+a(-5+5ci)e\,n_1rs\cos(o-w)+a(-5-5ci)e\,n_1rs\cos(o+w)$$
$$+a(-5+5ci)e\,n_2rs\sin(o-w)-10a\,e\,n_3\,rs\,si\sin(w)$$
$$+a(-5-5ci)e\,n_2rs\sin(o+w))$$

$$D_3=\frac{3}{16rs^2}(a\,e(8rs+(-6-6ci^2)n_1^2rs+(-6-6ci^2)n_2^2rs-12n_3^2rs\,si^2)$$
$$-24a\,ci\,e\,n_2n_3rs\,si\cos(o)+a\,e((-6+6ci^2)n_1^2rs+(6-6ci^2)n_2^2rs)\cos(2o)$$
$$+a\,e\,n_2n_3rs(-20si+20ci\,si)\cos(o-2w)+a\,e((-5+10ci-5ci^2)n_1^2rs$$
$$+(5-10ci+5ci^2)n_2^2rs)\cos(2o-2w)+(a^2((-4+4ci)n_1$$
$$+(-16+16ci)e^2n_1)+(4-4ci)n_1rs^2)\cos(o-w)+a\,e((-10+10ci^2)n_1^2rs$$
$$+(-10+10ci^2)n_2^2rs+20n_3^2rs\,si^2)\cos 2w+(a^2((-4-4ci)n_1$$
$$+(-16-16ci)e^2n_1)+(4+4ci)n_1rs^2)\cos(o+w)+a\,en_2n_3rs(20si+20ci\,si)$$
$$\cos(o+2w)+a\,e((-5-10ci-5ci^2)n_1^2rs+(5+10ci+5ci^2)n_2^2rs)$$
$$\cos(2o+2w)+24a\,ci\,e\,n_1n_3rs\,si\sin(o)+a(-12+12ci^2)e\,n_1n_2rs\sin(2o)$$
$$+a\,e\,n_1n_3rs(20si-20ci\,si)\sin(o-2w)+a(-10+20ci-10ci^2)e\,n_1n_2rs$$
$$\sin(2o-2w)+(a^2((-4+4ci)n_2+(-16+16ci)e^2n_2)+(4-4ci)n_2rs^2)$$
$$\sin(o-w)+(8n_3rs^2si+a^2(-8n_3si-32e^2n_3si))\sin(w)+(a^2((-4-4ci)n_2$$
$$+(-16-16ci)e^2n_2)+(4+4ci)n_2rs^2)\sin(o+w)$$
$$+a\,e\,n_1n_3rs(-20si-20ci\,si)\sin(o+2w)+a(-10-20ci-10ci^2)$$
$$e\,n_1n_2rs\sin(2o+2w))$$

$$D_4 = \frac{3}{8rs}(a(-4n_1 - 6e^2n_1)\cos(o) + a(-5+5ci)e^2n_1\cos(o-2w) + 4e\ rs\ \cos(w)$$
$$+ a(-5-5ci)e^2n_1\cos(o+2w) + a(-4n_2 - 6e^2n_2)\sin(o)$$
$$+ a(-5+5ci)e^2n_2\sin(o-2w) - 10a\ e^2n_3si\ \sin 2w$$
$$+ a(-5-5ci)e^2n_2\sin(o+2w))(ci\ n_3 - n_2si\ \cos(o) + n_1si\ \sin(o))$$

$$D_5 = \frac{3}{8rs}(ci\,n_3 - n_2si\cos(o) + n_1si\sin(o))(a(-4n_3si - 6e^2n_3si)$$
$$+ a(-4ci\,n_2 - 6ci\,e^2n_2)\cos(o) + a(-5+5ci)e^2n_2\cos(o-2w)$$
$$+ 10a\,e^2n_3si\cos(2w) + a(5+5ci)e^2n_2\cos(o+2w) + a(4ci\,n_1 + 6ci\,e^2n_1)$$
$$\sin(o) + a(5-5ci)e^2n_1\sin(o-2w) + 4e\,rs\sin(w)$$
$$+ a(-5-5ci)e^2n_1\sin(o+2w))$$

$$D_6 = -\frac{1}{16rs^2}(a(-16rs + (12+12ci^2)n_1^2rs + (12+12ci^2)n_2^2rs + 24n_3^2rs\,si^2$$
$$+ e^2(-40rs + (30+30ci^2)n_1^2rs + (30+30ci^2)n_2^2rs + 60n_3^2rs\,si^2))$$
$$+ a(48ci\,n_2n_3rs\,si + 120ci\,e^2n_2n_3\,rs\,si)\cos(o) + a((12-12ci^2)n_1^2rs$$
$$+ (-12+12ci^2)n_2^2rs + e^2((30-30ci^2)n_1^2rs + (-30+30ci^2)n_2^2rs))\cos(2o)$$
$$+ a\,e^2n_2n_3\,rs(60si - 60ci\,si)\cos(o-2w) + a\,e^2((15-30ci+15ci^2)n_1^2rs$$
$$+ (-15+30ci^2-15ci^2)n_2^2rs)\cos(2o-2w) + (a^2(60-60ci)e\,n_1$$
$$+ (-12+12ci)e\,n_1rs)\cos(o-w) + a\,e^2((30-30ci^2)n_1^2rs$$
$$+ (30-30ci^2)n_2^2rs - 60n_3^2rs\,si^2)\cos(2w) + (a^2(60+60ci)e\,n_1$$
$$+ (-12-12ci)e\,n_1rs)\cos(o+w) + a\,e^2n_2n_3rs(-60si - 60ci\,si)\cos(o+2w)$$
$$+ a\,e^2((15+30ci+15ci^2)n_1^2rs + (-15-30ci-15ci^2)n_2^2rs)\cos(2o+2w)$$
$$+ a(-48ci\,n_1n_3rs\,si - 120ci\,e^2\,n_1n_3rs\,si)\sin(o) + a((24-24ci^2)n_1n_2rs$$
$$+ (60-60ci^2)e^2n_1n_2rs)\sin(2o) + a\,e^2n_1n_3rs(-60si + 60ci\,si)\sin(o-2w)$$
$$+ a(30-60ci+30ci^2)e^2n_1n_2rs\sin(2o-2w) + (a^2(60-60ci)e\,n_2$$
$$+ (-12+12ci)e\,n_2rs)\sin(o-w) + (120a^2e\,n_3si - 24e\,n_3rs\,si)\sin(w)$$
$$+ (a^2(60+60ci)e\,n_2 + (-12-12ci)e\,n_2rs)\sin(o+w)$$
$$+ a\,e^2n_1n_3rs(60si + 60ci\,si)\sin(o+2w)$$
$$+ a(30+60ci+30ci^2)e^2n_1n_2rs\sin(2o-2w))$$

The long terms are

$$G_3 = -3ae\left(\frac{-8+9n_1^2+9n_2^2+6n_3^2}{16rs}\right) - \frac{3ae(3n_1^2+3n_2^2-6n_3^2)\cos 2i}{16rs}$$
$$= \frac{-3ae}{16rs}(1-3n_3^2)(1+3\cos 2i) \tag{7.44}$$

$$G_5 = a\left(\frac{3e^2(3n_1^2 + 3n_2^2 - 6n_3^2)}{16rs} - \frac{3(2n_1^2 + 2n_2^2 - 4n_3^2)}{16rs}\right)\sin 2i$$
$$= \frac{3a}{16rs}(1 - 3n_3^2)(3e^2 - 2)\sin 2i$$

$$G_6 = a\left(\frac{e^2(40 - 45n_1^2 - 45n_2^2 - 30n_3^2)}{16rs} + \frac{16 - 18n_1^2 - 18n_2^2 - 12n_3^2)}{16rs}\right)$$
$$+ a\left(\frac{-6n_1^2 - 6n_2^2 + 12n_3^2}{16rs} + \frac{e^2(-15n_1^2 - 15n_2^2 + 30n_3^2)}{16rs}\right)\cos 2i$$
$$= \frac{a}{16rs}(1 - 3n_3^2)(-5e^2 - 2)(1 + 3\cos 2i)$$

The above solutions are derived by using basic Gaussian equations of motion (7.17). Equation 7.43 shows that all Keplerian elements are disturbed by solar radiation long-periodically. The argument of perigee ω, the right ascension of ascending node Ω and the mean anomaly M are disturbed partly secularly. Substituting (7.44) into (7.16) and using the fact that $n_1^2 + n_2^2 + n_3^2 = 1$, yields:

$$\frac{d\omega}{dt} = h_3 G_3 - h_4(\cos i/\sin i)G_5$$
$$= h_3\frac{-3ae}{16rs}(1 - 3n_3^2)(1 + 3\cos 2i) - h_4\frac{\cos i}{\sin i}\frac{3a}{16rs}(1 - 3n_3^2)(3e^2 + 2)\sin 2i$$
$$= \frac{-3a}{16rs}(1 - 3n_3^2)(eh_3(1 + 3\cos 2i) + h_4(3e^2 + 2)2\cos^2 i)$$
$$\frac{d\Omega}{dt} = (h_4/\sin i)G_5 = h_4\frac{3a}{16rs}(1 - 3n_3^2)(3e^2 + 2)2\cos i$$
$$\frac{dM}{dt} = n - 2h_6 e G_6 - h_6 G_3 = n + h_6\frac{ae}{16rs}(1 - 3n_3^2)(1 + 3\cos 2i)(5e^2 + 3)$$

Above secular effects valid for the cases of non-circular and non-equatorial orbits. The secular solutions, i.e. the integrals of the equations given above, are obviously the right-hand side multiplied by time t.

The third component of solar unit vector n_3 is very small; therefore, the factor $(1 - 3n_3^2)$ is positive. Taking the coefficients h_j (see (7.16), all are positive) and the factor ξ into account, one has very interesting conclusions in the case of non-singularity: the solar radiation pressure will cause perigee precession or regression, which is depended on the inclination factor $(1 + 3\cos 2i)$. The mean motion delaying or advancing is also decided by the inclination factor $(1 + 3\cos 2i)$. The ascending node Ω will experience a linear disturbance and the direction is decided by the sign of $\cos i$.

When we compare the linear terms in (7.44) with (7.40), it is notable that a more precise solution seems necessary; the solutions (7.44) include additional linear effects that do not appear in (7.40). This indicates that the first two terms in (7.23) have meanwhile to be taken into account.

Results truncated to orders of $(e, a^2/r_s^3)$ and $(e^2, a^2/r_s^3)$ are also computed, respectively. The results confirm that the differences occurred only in the selected higher orders.

7.1.8 Summary

The singularity-free analytical solutions of satellite orbit disturbed by the solar radiation pressure are obtained. The related algorithms are also developed. Solutions are approximated to the second order through expansions and higher order solutions are similarly derived and studied for comparison. Conclusions are obtained theoretically: the solar radiation pressure force will lead to a linear disturbance in relation to the time on the argument of perigee ω and the mean anomaly M. The semi-major axis parameter a and the eccentricity of the orbital ellipse e as well as inclination angle i are disturbed long- and short-periodically. The solar radiation parameter ξ can be determined if necessary, and in such a case, the determination of disturbing effects turned out to be a linear fitting process and is much easier than using traditional numerical method to solve both the variation and observation equations. singularity-free solution could be extreme important for the on-board autonomous orbit determination of the ca. 100 circular and equatorial GNSS satellites in the future.

7.2 Discrete Solution of Satellite Orbit Disturbed by Solar Radiation Pressure

7.2.1 Three Approximations

The solar radiation force vector (7.1) is pointed from the sun to the satellite. If the shadow factor is known exactly, and the luminosity of the sun and the surface reflectivity of the satellite are considered constants, then the length of the solar force vector can be considered a constant, because (see (4.73))

$$\frac{r_{sun}^2}{(r_{sun}+r)^2} \le \frac{r_{\text{sun}}^2}{|\vec{r}-\vec{r}_{\text{sun}}|^2} \le \frac{r_{sun}^2}{(r_{sun}-r)^2}. \tag{7.45}$$

For GPS and GEO satellites there are

$$\frac{r_{sun}^2}{(r_{sun} \pm r)^2} = \left(\frac{r_{sun}}{r_{sun} \pm r}\right)^2 \approx \left(1 \mp \frac{r}{r_{sun}} \pm \ldots\right)^2 \approx 1 \mp \frac{2r}{r_{sun}} \approx 1 \mp 3.5 \times 10^{-4}$$

$$\text{and } \frac{r_{sun}^2}{(r_{sun} \pm r)^2} = \left(\frac{r_{sun}}{r_{sun} \pm r}\right)^2 \approx \left(1 \mp \frac{r}{r_{sun}} \pm \ldots\right)^2 \approx 1 \mp \frac{2r}{r_{sun}} \approx 1 \mp 5.6 \times 10^{-4}, \tag{7.46}$$

respectively. That is, the solar radiation force vector can be considered approximately as a vector, with constant length and changing direction. The approximation has a precision of better than third order and is precise enough for our purposes. For convenience, this approximation is called the first approximation later on.

The unit solar vector of the satellite $\vec{n}_{sun}$ can be approximated by

$$\vec{n}_{se} = \frac{\vec{r}_{se}}{|\vec{r}_{se}|}, \tag{7.47}$$

where index *se* denotes that the vector is pointing from the sun to the centre of the Earth. For GPS and GEO satellites the maximal angles between the above two unit vectors are 1.5×10^{-5} and 2.5×10^{-5} (rad), respectively. Therefore, such an approximation (called the second approximation) is allowed and is precise enough.

The third approximation is made for a suitable time duration of $\Delta t = t'_k - t'_{k-1}$ by

$$\vec{n}_{se}(t) \approx \vec{n}_{se}(t_k), \quad t_k = (t'_k + t'_{k-1})/2, \quad t \in [t'_{k-1}, t'_k]. \tag{7.48}$$

The discrete vector in this equation may be called an average vector of the time duration Δt. For $\Delta t = 5$ min, the third approximation has a precision of 3×10^{-5} (rad).

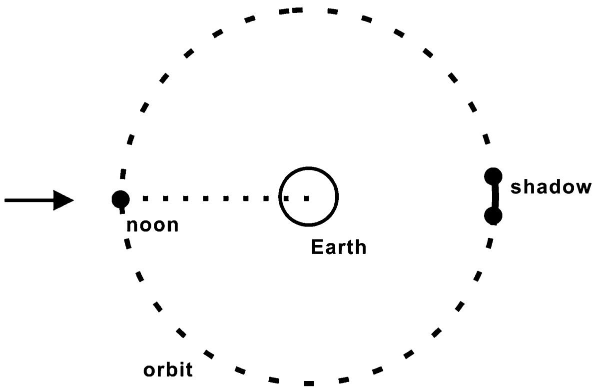
Note that the order of the solar radiation disturbance on a GPS satellite is about 50 m. For GPS satellite, all three approximations will lead to a precision of millimetre level. For the other satellite the precision of the approximations should be individually estimated.

7.2.2 Discretization and Solution

Denote the satellite period as T, and shadow access and exit points as t_a and t_e, respectively. The local noon is selected as the starting point of counting (see Fig. 7.4). A so-called sign function can be defined as

$$\delta(t) = \begin{cases} 1 & 0 \le t < T/2 \\ -1 & T/2 \le t \le T \end{cases}. \tag{7.49}$$

Fig. 7.4 Solar radiation pressure



The sign function shows that the solar radiation accelerates the satellite during the first half period and decelerates it during the second half period with respect to the nominal motion of the satellite. Then the duration of one period of 0 ~ T can be equally divided by Δt, i.e. by $t'_0, t'_1, \ldots, t'_k, \ldots, T$. The acceleration of the solar radiation of (7.1) is then discretized as

$$\vec{a}_{\text{solar}}(t) = \gamma P_s C_r \frac{S}{m} \vec{n}_{\text{se}}(t_k) \tag{7.50}$$

The disturbed velocity caused by the solar radiation is then

$$\vec{v}_{\text{solar}}(t) = \sum_{i=1}^{k} \gamma P_s C_r \frac{S}{m} \vec{n}_{\text{se}}(t_i) \delta(t_i) \Delta t \tag{7.51}$$

It is obvious that, the disturbed velocity of the satellite is not zero during the passing of the shadow. The disturbed position caused by the solar radiation is then

$$\vec{\rho}_{\text{solar}}(t) = \sum_{j=1}^{k} \vec{v}_{solar}(t_j) \Delta t \tag{7.52}$$

Equation 7.52 is the solution of the solar radiation disturbance on the orbit of the satellite.

7.2.3 *Properties of the Solution*

The integration (or summation) of the acceleration of solar radiation within a period T is nearly zero. However, the position disturbed by solar radiation during a period T is not zero. In other words, the disturbance of the solar radiation displays non-conservative behaviour. The disturbance may not be a periodic function of the orbit. The parameters of the force model, if they are not well known, can be determined using the expressions of the solution.

7.3 Analytical Solution of Satellite Orbit Disturbed by Atmospheric Drag

The text of this section is just slightly modified from Xu et al. (2010b). The basic Gaussian equations defined in Sect. 7.1 are used here instead of the so-called simplified Gaussian ones described in Xu et al. (2010b) due to the Xus' singularity-free theory (cf. Sect. 11.2.3 or Xu and Xu 2012). More precise solutions are given.

The analytical solution of a satellite orbit disturbed by the atmospheric drag is derived in this section. The disturbing force vector is first transformed and rotated to the orbital frame so that it can be used in the basic Gaussian equations of satellite motion. Then the force vector is expanded to trigonometric functions of the Keplerian angular elements and the disturbances are separated into three parts: short periodic terms with trigonometric functions of M, long periodic terms with trigonometric functions of ω and secular terms (non-periodic functions of (a, e, i)) with a program using mathematic symbolic operation software. The integrations are then carried out with respect to M, ω, and t, respectively, to obtain the analytical solutions of satellite orbits disturbed by atmospheric drag. Some interesting conclusions are obtained theoretically. The atmospheric disturbing force is not a function of Ω. The semi-major axis a of the orbital ellipse will be reduced in a constant and strong manner by the air disturbance; the shape of the ellipse (eccentricity e) will change towards a more circular orbit in a linear and weak manner. The right ascension of ascending node Ω and the mean anomaly M are influenced by the disturbance only short-periodically.

7.3.1 Introduction

The Earth satellite orbits disturbed by atmospheric drag are more complicated than orbits perturbed by multi-body gravitation disturbances because of the non-conservative property of the force (Boccaletti and Pucacco 2001; Brouwer and Clemence 1961; Chobotov 1991). The order of atmospheric drag acting on a satellite depends on the altitude of the satellite. The orbital behaviour of a low Earth orbit (LEO) satellite could be strongly influenced by atmospheric drag (Battin 1999; King-Hele 1964; Van Kamp 1967) and therefore, affects the quality of the satellite remote sensing (Vallado 2007; Herrick 1972). Traditionally, the orbit disturbance problem of atmospheric drag is solved by numerical integration (Montenbruck and Gill 2000; Xu 2007/2003; Seeber 2003; Meeus 1992). An analytical solution gives the theoretical integrals and shows the physical effects with very clear spectral properties (Licandro et al. 2008; Cui 1990; Schneider and Cui 2005; Kudryavtsev 2007; Lynden-Bell 2009; Pal 2009) and may give a direct insight into the physical disturbing phenomenon (Touma et al. 2009; Desmars et al. 2009; Torge 1991). A discretized solution is given in Xu (2008); however, an analytical solution is still missing. A method to solve the equations of satellite motion disturbed by solar and lunar gravitation has been developed (Xu et al. 2010a). In this section, a similar but independent method is developed and a solution of atmospheric disturbance is derived.

In order to be able to use Gaussian equations of motion to solve the atmospheric drag disturbance problem, the force vector is first transformed and represented in an orbital frame in the next section. The basic Gaussian equations are given and mathematical formulas are used to expand the right-hand sides of the equations into linear trigonometric functions of the Keplerian angular elements in Sect. 7.3.3. The solutions are derived with a program using mathematical symbolic operation software and are presented in Sect. 7.3.4. A brief summary is given at the end.

7.3.2 Disturbing Force of Atmospheric Drag

Atmospheric drag, caused by air, is a disturbing force acting on the satellite's surface and can be represented as (see, e.g. (4.75) or Battin 1999; Kaula 1966/2001; Montenbruck and Gill 2000)

$$\vec{f}_{\text{drag}} = -m\frac{1}{2}\left(\frac{C_{\text{d}}S}{m}\right)\sigma\left|\dot{\vec{r}} - \dot{\vec{r}}_{\text{air}}\right|^2 \vec{n}_a, \quad \vec{n}_a = \frac{\dot{\vec{r}} - \dot{\vec{r}}_{\text{air}}}{\left|\dot{\vec{r}} - \dot{\vec{r}}_{\text{air}}\right|}, \tag{7.53}$$

where S is the cross section (or effective area) of the satellite, C_{d} is the drag factor, m is the mass of the satellite, $\dot{\vec{r}}$ and $\dot{\vec{r}}_{air}$ are the geocentric velocity vectors of the satellite and the atmosphere and σ is the density of the atmosphere. Usually, S has a value of 1/4 of the outer surface area of the satellite and C_{d} has laboratory values of 2.2 ± 0.2. The unit vector $\vec{n}_a$ is in the direction of the drag force. r and r_{air} are the lengths of the related vectors. Using ξ to represent the coefficient part of the drag force vector (7.53) yields

$$\vec{f}_{\text{drag}} = \xi|\dot{\vec{r}} - \dot{\vec{r}}_{\text{air}}|(\dot{\vec{r}} - \dot{\vec{r}}_{\text{air}}). \tag{7.54}$$

The disturbance of atmospheric drag on the satellite can be solved by using Gaussian equations of motion. The drag force vector has to be rotated from the ECSF (Earth-Centred Space-Fixed) frame to the orbital coordinate frame by cf. e.g. (4.16), or Battin 1999; Kaula 1966/2001; Montenbruck and Gill 2000)

$$\begin{pmatrix} f_r \\ f_\alpha \\ f_h \end{pmatrix} = R_3(f)R_3(\omega)R_1(i)R_3(\Omega)\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}, \tag{7.55}$$

where $(f_r, f_\alpha, f_h)^T$ is a force vector with three orthogonal components in an orbital plane coordinate system, the first two components are in the orbital plane, f_r is the

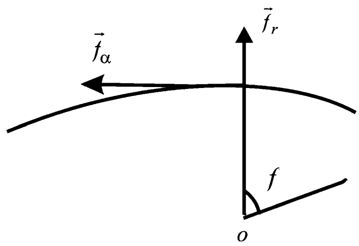


Fig. 7.5 Radial and its perpendicular components of atmospheric drag force in the orbital plane

radial force component, f_α is the force component perpendicular to f_r and pointed in the direction of satellite motion (see Fig. 7.5) and f_h completes a right-handed system. $\vec{f} = (f_x, f_y, f_z)^T$ is the same force vector in the ECSF system. R_k is the rotational matrix around the axis k. Keplerian elements (a, e, ω, i, Ω, M, f) are semi-major axis, eccentricity of the ellipse, argument of perigee, inclination angle, right ascension of ascending node, mean anomaly, true anomaly, respectively. Satellite position and velocity vectors in orbital frame are (see (3.41) and (3.42), or Battin 1999; Kaula 1966/2001; Montenbruck and Gill 2000)

$$\vec{q} = \begin{pmatrix} a(\cos E - e) \\ a\sqrt{1-e^2}\sin E \\ 0 \end{pmatrix} = \begin{pmatrix} r\cos f \\ r\sin f \\ 0 \end{pmatrix}, \tag{7.56}$$

$$\dot{\vec{q}} = \begin{pmatrix} -\sin E \\ \sqrt{1-e^2}\cos E \\ 0 \end{pmatrix} \frac{na}{1 - e\cos E} = \begin{pmatrix} -\sin f \\ e + \cos f \\ 0 \end{pmatrix} \frac{na}{\sqrt{1-e^2}} \tag{7.57}$$

Here E is the eccentric anomaly and n is the mean angular velocity. The vectors can be rotated from the orbital frame to the ECSF frame by (see (3.43), or Battin 1999; Kaula 1966/2001; Montenbruck and Gill 2000)

$$\begin{pmatrix} \vec{r} \\ \dot{\vec{r}} \end{pmatrix} = R_3(-\Omega)R_1(-i)R_3(-\omega)\begin{pmatrix} \vec{q} \\ \dot{\vec{q}} \end{pmatrix}. \tag{7.58}$$

The atmospheric velocity vector in the ECSF frame is given in (4.76):

$$\begin{aligned} \dot{\vec{r}}_{\text{air}} &= K\vec{\omega}_e \times \vec{r} = K\omega_e \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}, \\ &= K\omega_e \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = K\omega_e R_4 \vec{r} \end{aligned} \tag{7.59}$$

where $\vec{\omega}_e$ is the angular velocity vector of the Earth's rotation (index e is used to distinguish the Earth angular velocity from the Keplerian element ω) and $\omega_e = |\vec{\omega}_e|$, K is the atmospheric rotation factor. For the lower layer of the atmosphere, $K = 1$, i.e. the lower layer of the atmosphere is considered to be rotating with the Earth. For the higher layer, $K = 1.2$, because the higher ionosphere is accelerated by the Earth's magnetic field. For convenience of later use a matrix R_4 is defined in (7.59). Thus in the ECSF frame there is

$$\dot{\vec{r}} - \dot{\vec{r}}_{air} = R_3(-\Omega)R_1(-i)R_1(-\omega)\dot{\vec{q}} - K\omega_e R_4 R_3(-\Omega)R_1(-i)R_3(-\omega)\vec{q} \tag{7.60}$$

The vector can then be rotated to the orbital frame by (7.55). Denote the following matrix as R and its elements as R_{km}:

$$\begin{aligned} R &= R_3(\omega)R_1(i)R_3(\Omega)R_4R_3(-\Omega)R_1(-i)R_3(-\omega) \\ &= R_3(\omega)R_1(i)R_4R_1(-i)R_3(-\omega) \\ &= R_3(\omega)\begin{pmatrix} 0 & -\cos i & \sin i \\ \cos i & 0 & 0 \\ -\sin i & 0 & 0 \end{pmatrix} R_3(-\omega) \\ &= \begin{pmatrix} 0 & -\cos i & \sin i \cos\omega \\ \cos i & 0 & -\sin i \sin\omega \\ -\sin\cos\omega & \sin i \sin\omega & 0 \end{pmatrix}, \end{aligned} \tag{7.61}$$

and note that the length of a vector is an invariable under rotational transformations; one has the atmospheric drag force vector in orbital frame

$$\begin{aligned} \begin{pmatrix} f_r \\ f_\alpha \\ f_h \end{pmatrix} &= \xi\left|\dot{\vec{r}} - \dot{\vec{r}}_{air}\right| R_3(f)(\dot{\vec{q}} - K\omega_e R\vec{q}) \\ &= \xi\left|\dot{\vec{q}} - K\omega_e R\vec{q}\right| R_3(f)\left(\dot{\vec{q}} - K\omega_e R\vec{q}\right). \end{aligned} \tag{7.62}$$

The force vector (7.62) is represented completely in Keplerian elements and therefore can be used in Gaussian equations to solve the problem.

7.3.3 Basic Gaussian Equations and Expansions of Atmospheric Drag Force

The basic Gaussian equations of motion are given in Sect. 7.1.3.

The divisor $(1 + e\cos f)$ can be expanded by using (Bronstein and Semendjajew 1987; Wang et al. 1979)

$$(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + \ldots |x| < 1. \tag{7.63}$$

The vector part of (7.62) can be further simplified by noting the anti-symmetric property of R (see (7.61)):

$$\dot{\vec{q}} - K\omega_e R\vec{q} = \frac{1}{h_2}\begin{pmatrix} -\sin f \\ e + \cos f \\ 0 \end{pmatrix} - K\omega_e \begin{pmatrix} R_{12}\sin f \\ R_{12}\cos f \\ R_{13}\cos f - R_{23}\sin f \end{pmatrix} \frac{a(1-e^2)}{1+e\cos f}, \tag{7.64}$$

$$R_3(f)(\dot{\vec{q}} - K\omega_e R\vec{q}) = \frac{1}{h_2}\begin{pmatrix} e\sin f \\ 1 + e\cos f \\ 0 \end{pmatrix} - K\omega_e \begin{pmatrix} 0 \\ -R_{12} \\ -R_{13}\cos f - R_{23}\sin f \end{pmatrix} \frac{a(1-e^2)}{1+e\cos f}. \tag{7.65}$$

h_2 is defined in (7.17). According to the geometric relations of the vectors one has (see Fig. 7.6)

$$\begin{aligned} \dot{\vec{r}} &= \dot{\vec{r}}_{air} + (\dot{\vec{r}} - \dot{\vec{r}}_{air}) \\ \left|\dot{\vec{r}} - \dot{\vec{r}}_{air}\right|^2 &= \dot{r}^2 + \dot{r}_{air}^2 - 2\dot{r}\dot{r}_{air}\cos\alpha, \\ \dot{r}\dot{r}_{air}\cos\alpha &= \dot{\vec{r}}\cdot\dot{\vec{r}}_{air} \end{aligned} \tag{7.66}$$

where α is the angle between the geocentric velocity vectors of the satellite $\dot{\vec{r}}$ and the air $\dot{\vec{r}}_{air}$ and $\cos\alpha$ is the inner product of the two related unit vectors. Furthermore, there are (see (7.64) and (7.65))

$$\dot{r}^2 = \frac{1}{h_2^2}(1 + e^2 + 2e\cos f), \tag{7.67}$$

$$\dot{r} = \frac{1}{h_2}\left(1 + \frac{1}{2}(e^2 + 2e\cos f) - \frac{1}{2}e^2\cos^2 f\right), \tag{7.68}$$

$$\dot{r}_{air}^2 = K^2\omega_e^2\left(R_{12}^2 + (R_{13}\cos f + R_{23}\sin f)^2\right)\frac{a^2(1-e^2)^2}{(1+e\cos f)^2}, \tag{7.69}$$

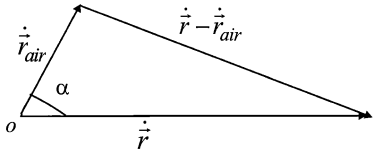


Fig. 7.6 Geometrical relationship between the geocentric velocity vectors of the satellite and the atmosphere

$$\begin{aligned}
\dot{r}\dot{r}_{air}\ \cos\alpha &= \left(R_3\ (f)\dot{\vec{q}}\right)\cdot\left(K\omega_e R_3(f)R\vec{q}\right)\\
&= \frac{1}{h_2}\begin{pmatrix} e\ \sin f\\ 1+e\cos f\\ 0\end{pmatrix}\cdot\begin{pmatrix} 0\\ -R_{12}\\ -R_{13}\cos f - R_{23}\sin f\end{pmatrix}K\omega_e\,\frac{a(1-e_2)}{1+e\cos f}.\\
&= -\frac{K\omega_e\,a(1-e^2)R_{12}}{h_2}
\end{aligned} \tag{7.70}$$

For convenience, the right-hand side of (7.70) is denoted as β. Equation 7.70 shows a very interesting property of the atmospheric drag disturbance, i.e. the projection of the satellite velocity vector onto the atmospheric velocity vector is a cosine function of the inclination angle i. In case of a polar orbit, $\beta = 0$, it means that the two vectors are perpendicular to each other. One can also note here that the disturbance of the atmospheric drag is not a function of the Keplerian element Ω. Using the expansion formula (Bronstein and Semendjajew 1987; Wang et al. 1979)

$$\sqrt{1\pm x} = 1 \pm \frac{1}{2}x - \frac{1.1}{2.4}x^2 \pm \frac{1.1.3}{2.4.6}x^3 - \frac{1.1.3.5}{2.4.6.8}x^4 \pm \ldots\ |x|<1 \tag{7.71}$$

one obtains

$$\begin{aligned}
\left|\dot{\vec{r}} - \dot{\vec{r}}_{air}\right| &= \dot{r}\sqrt{1+\left(\frac{\dot{r}_{air}^2 - 2\beta}{\dot{r}^2}\right)}\\
&= \dot{r}\left(1+\left(\frac{\dot{r}_{air}^2-2\beta}{2\dot{r}^2}\right)-\frac{1}{8}\left(\frac{\dot{r}_{air}^2-2\beta}{\dot{r}^2}\right)^2+\frac{1}{16}\left(\frac{\dot{r}_{air}^2-2\beta}{\dot{r}^2}\right)^3-\cdots\right)
\end{aligned} \tag{7.72}$$

All divisors above can be expanded using the formulas already given depending on the required precision of the approximation. Atmospheric disturbance only needs to be considered if the altitude of the satellite is lower than 1,000 km. For a LEO satellite such as CHAMP or GRACE (orbit height ca. 400 km with eccentricity of ca. 10^{-3}) the ratio of the velocities of the atmosphere and the satellite is smaller than 0.0785. Thus for second order solutions the approximations are precise enough if we neglect the e^2 and $(\dot{\vec{r}}/\dot{\vec{r}}_{air})^3$ terms. Therefore, under these circumstances, there are

$$\dot{r}^{-2} = h_2^2(1 - e^2 - 2e\cos f + 4e^2\cos^2 f), \tag{7.73}$$

$$\dot{r}^{-4} = h_2^4(1 - 2e^2 - 4e\cos f + 12e^2\cos^2 f), \tag{7.74}$$

$$\frac{1}{1 + e\cos f} = 1 - e\cos f + e^2\cos^2 f,$$

$$\frac{1}{(1 + e\cos f)^2} = 1 - 2e\cos f + 3e^2\cos^2 f, \tag{7.75}$$

$$\cos E = (e + \cos f)(1 - e\cos f + e^2\cos^2 f).$$

$$\left|\dot{\vec{r}} - \dot{\vec{r}}_{air}\right| \approx \dot{r}\left(1 + \frac{1}{2}\frac{(\dot{r}_{air}^2 - 2\beta)}{\dot{r}^2}\right) - \frac{1}{2}\left(\frac{\beta}{\dot{r}^2}\right).$$

7.3.4 Short Periodic Solutions

For convenience for the integration with respect to time t, the following functions are used to transform the trigonometric function of true anomaly f to mean anomaly M (see, e.g. (5.22), or Battin 1999; Kaula 1966/2001; Montenbruck and Gill 2000):

$$\sin f = \left(1 - \frac{7}{8}e^2\right)\sin M + e\left(1 - \frac{7}{6}e^2\right)\sin 2M + \frac{9}{8}e^2\sin 3M,$$

$$\cos f + e\left(1 - \frac{9}{8}e^2\right)\cos M + e\left(1 - \frac{4}{3}e^2\right)\cos 2M + \frac{9}{8}e^2\cos 3M. \tag{7.76}$$

Here terms smaller than e^2 are neglected. Substituting (7.73), (7.74), (7.75), and (7.65) into (7.62) allows the force vector to be represented completely by Keplerian elements. The basic Gaussian equations of satellite motion (7.17) can be transformed in terms of $\cos^m f \sin^k f$ and then $\{\cos mM, \cos kM, \sin mM, \sin kM\}$ using (7.76) and can be further reduced to a functional series of $\{\cos mM, \sin kM\}$, which can be integrated with respect to M to obtain the short periodic terms of solutions (the relation $M = nt$ is used). The terms that have nothing to do with M

can be reduced to trigonometric functions of ω and secular terms (non periodic terms). These transformations can be carried out with a program using mathematic symbolic operation software such as Mathematica or Maple. The form of the equations of motion is then

$$\frac{d\sigma_{1j}}{dt} = \xi\frac{dD_{1j}}{dt} + \xi\sum_{k=1,\ldots}(b_{jk}\cos kM + c_{jk}\sin kM), \tag{7.77}$$

where σ with index $1j$ denotes the jth Keplerian element in basic Gaussian equations (7.17); D, b, c are functions of (a, e, ω, i). All terms of b and c are short periodic perturbations and all D-terms include long periodic and linear perturbations. Or, to be more explicit (approximated to the order of e^2, the factor ξ is omitted, $kwe = K\omega_e$, $sw = \sin\omega$, $cw = \cos\omega$):

$$\begin{aligned}
\frac{da_1}{dt} = &-\frac{1}{32h_2}\Big(-\frac{32}{h_2}+\frac{40e^2}{h_2}+akwe(-64R_{12}+96e^2R_{12})\\
&+a^2h_2kwe^2(-32R_{12}^2-8R_{13}^2-8R_{23}^2+e^2(16R_{12}^2-7R_{13}^2-4R_{13}R_{23}+19R_{23}^2))\\
&+a^3h_2^2kwe^3(-8R_{12}R_{13}^2-8R_{12}R_{23}^2+e^2(24R_{12}^3-53R_{12}R_{13}^2+9R_{12}R_{23}^2))\\
&+(-\frac{96e}{h_2}-64ae\ kweR_{12}+a^2eh_2kwe^2(32R_{12}^2+28R_{13}^2-12R_{23}^2)\\
&+a^3eh_2^2kwe^3(52R_{12}R_{13}^2-4R_{12}R_{23}^2)\cos M+(-\frac{136e^2}{h_2}-64ae^2kweR_{12}\\
&+a^2h_2kwe^2(-8R_{13}^2+8R_{23}^2+e^2(16R_{12}^2+40R_{13}^2-36R_{23}^2))\\
&+a^3h_2^2kwe^3(-8R_{12}R_{13}^2+8R_{12}R_{23}^2+e^2(8R_{12}^3+8R_{12}R_{13}^2-12R_{12}R_{23}^2))\cos 2M\\
&+(a^2eh_2kwe^2(-12R_{13}^2+12R_{23}^2)+a^3eh_2^2kwe^3(-4R_{12}R_{13}^2+4R_{12}R_{23}^2))\cos 3M\\
&+(a^2e^2h_2kwe^2(-17R_{13}^2+4R_{13}R_{23}+17R_{23}^2)+a^3e^2h_2^2kwe^3(-3R_{12}R_{13}^2\\
&+3R_{12}R_{23}^2))\cos 4M+(40a^2eh_2kwe^2R_{13}R_{23}+56a^3eh_2^2kwe^3R_{12}R_{13}R_{23})\sin M\\
&+-\frac{16e^2}{h2}-16ae^2kweR_{12}+a^3h_2^2kwe^3(-16R_{12}R_{13}R_{23}+20e^2R_{12}R_{13}R_{23})\\
&+a^2h_2kwe^2(-16R_{13}R_{23}+e^2(-4R_{13}^2+76R_{13}R_{23}-4R_{23}^2))\sin 2M\\
&+(-24a^2eh_2kwe^2R_{13}R_{23}-8a^3e\ h_2^2kwe^3R_{12}R_{13}R_{23})\sin 3M\\
&+(-6a^3e^2h_2^2kwe^3R_{12}R_{13}R_{23}+a^2e^2h_2kwe^2(-2R_{13}^2-34R_{13}R_{23}+2R_{23}^2))\sin 4M\Big),
\end{aligned}$$

$$\begin{aligned}
\frac{de_1}{dt} = &\frac{1}{32h_2}\Big(\frac{32\,e}{h_2}-4a\,e\,kwe(20R_{12}+a\,h_2e(-28R_{12}^2-11R_{13}^2-h_2R_{23}^2))\\
&+a^3e\,h_2^2kwe^3R_{12}(-70R_{13}^2-18R_{23}^2)+(\frac{64}{h_2}-\frac{64\,e^2}{h_2}
\end{aligned}$$

$$+a\,kwe(128R_{12}-136e^2R_{12})+8a^2h_2kwe^2(8R_{12}^2+3R_{13}^2+R_{23}^2$$
$$+e^2(10R_{12}^2+7R_{13}^2-2R_{23}^2))+2a^3h_2^2kwe^3R_{12}(12R_{13}^2+4R_{23}^2$$
$$+e^2(-28R_{12}^2+119R_{13}^2-5R_{23}^2))\cos M+\frac{96\,e}{h_2}+80a\,e\,kwe\,R_{12}$$
$$+16a^2e\,h_2kwe^2(-R_{12}^2-2R_{13}^2+R_{23}^2)+8a^3e\,h_2^2kwe^3R_{12}(-8R_{13}^2+3R_{23}^2)$$
$$\cos 2M+8e^2(\frac{16}{h_2}+9a\,kwe\,R_{12}+h_2kwe^2(R_{13}^2-R_{23}^2$$
$$+e^2(-2R_{12}^2-5R_{13}^2+4R_{23}^2))+a^3h_2^2kwe^3R_{12}(8R_{13}^2-8R_{23}^2$$
$$+(-8R_{12}^2-19R_{13}^2+15R_{23}^2))\cos 3M+6a^2eh_2kwe^2(R_{13}^2-R_{23}^2)$$
$$(2+ah_2eR_{12})\cos 4M+a^2h_2kwe^2(e^2(R_{13}^2-R_{23}^2)(16+5ah_2eR_{12})\cos 5M$$
$$+2R_{13}R_{23}(2(2(2+3e^2)+ah_2eR_{12}(4+37e^2))\sin M$$
$$+(4\mathrm{e}(-6-11a\,h_2eR_{12})\sin 2M+((4(2-9e^2)+ah_2eR_{12}(8-17e^2))\sin 3M$$
$$+e(6(2+ah_2eR_{12})\sin 4M+e(16+5ah_2eR_{12})\sin 5M)))))$$

$$\frac{\mathrm{d}\omega_1}{\mathrm{d}t}=\frac{\sin(\frac{M}{2})\cos(\frac{M}{2})}{8h_2}\,(\frac{32}{h2}+\frac{16\,e^2}{h_2}+a(64kwe\,R_{12}+8e^2kwe\,R_{12})$$
$$+a^2(32h_2kwe^2R_{12}^2+8h_2kwe^2R_{13}^2+8h_2kwe^2R_{23}^2$$
$$+e^2(16h_2kwe^2R_{12}^2+20h_2kwe^2R_{13}^2-4h_2kwe^2R_{23}^2))+a^3(8h_2^2kwe^3R_{12}R_{13}^2$$
$$+8h_2^2kwe^3R_{12}R_{23}^2+e^2(-24h_2^2kwe^3R_{12}^2+51h_2^2kwe^3R_{12}R_{13}^2+h_2^2kwe^3R_{12}R_{23}^2))$$
$$+\frac{96e}{h_2}+80a\,e\,kwe\,R_{12}+a^2e(-16h_2kwe^2R_{12}^2-28h_2kwe^2R_{13}^2$$
$$+12h_2kwe^2R_{23}^2)+a^3e(-46h_2^2kwe^3R_{12}R_{13}^2+6h_2^2kwe^3R_{12}R_{23}^2)\cos M$$
$$+(\frac{128e^2}{h_2}+72a\,e^2kwe\,R_{12}+a^2(8h_2kwe^2R_{13}^2-8h_2kwe^2R_{23}^2$$
$$+e^2(-16h_2kwe^2R_{12}^2-28h_2kwe^2R_{13}^2+20h_2kwe^2R_{23}^2))+a^3(8h_2^2kwe^3R_{12}R_{13}^2$$
$$-8h_2^2kwe^3R_{12}R_{23}^2+e^2(-8h_2^2kwe^3R_{12}^3-8h_2^2kwe^3R_{12}R_{13}^2+4h_2^2kwe^3R_{12}R_{23}^2))$$
$$\times\cos 2M+(a^2e(12h_2kwe^2R_{13}^2-12h_2kwe^2R_{23}^2)+a^3e(6h_2^2kwe^3R_{12}R_{13}^2$$
$$-6h_2^2kwe^3R_{12}R_{23}^2))co3M+(a^2e^2(16h_2kwe^2R_{13}^2-16h_2kwe^2R_{23}^2)$$
$$+a^3e^2(5h_2^2kwe^3R_{12}R_{13}^2-5h_2^2kwe^3R_{12}R_{23}^2))\cos 4M$$
$$+(-40a^2e\,h_2kwe^2R_{13}R_{23}-52a^3e\,h_2^2kwe^3R_{12}R_{13}R_{23})\sin M$$
$$+(a^2(16h_2kwe^2R_{13}R_{23}-48e^2h_2kwe^2R_{13}R_{23})$$
$$+a^3(16h_2^2kwe^3R_{12}R_{13}R_{23}-12e^2h_2^2kwe^3R_{12}R_{13}R_{23}))\sin 2M$$
$$+(24a^2e\,h_2kwe^2R_{13}R_{23}+12a^3e\,h_2^2kwe^3R_{12}R_{13}R_{23})\sin 3M$$
$$+(32a^2e^2h_2kwe^2R_{13}R_{23}+10a^3e^2h_2^2kwe^3R_{12}R_{13}R_{23})\sin 4M)$$

$$
\begin{aligned}
\frac{\mathrm{d}i_1}{\mathrm{d}t} = & \frac{a}{128h_2} kwe(64cw\ R_{13} - 64R_{23}sw + e^2(72cw\ R_{13} + 104R_{23}sw) \\
& + a(64cw\ h_2kwe\ R_{12}R_{13} - 64h_2kwe\ R_{12}R_{23}sw + e^2(488cw\ h_2kwe\ R_{12}R_{13} \\
& + 8h_2kwe\ R_{12}R_{23}sw) + a^2(24cw\ h_2^2kwe^2R_{13}^3 + 24cw\ h_2^2kwe^2R_{13}R_{23}^2 \\
& - 24h_2^2kwe^2R_{13}^2R_{23}sw - 24h_2^2kwe^2R_{23}^2sw + e^2h_2^2(-56cw\ kwe^2R_{12}^2R_{13} \\
& + 426cw\ kwe^2R_{13}^3 + 198cw\ kwe^2R_{13}R_{23}^2 + 40kwe^2R_{12}^2R_{23}sw \\
& - 198kwe^2R_{13}^2R_{23}sw + 30kwe^2R_{23}^2sw)) + (e(-224cw\ R_{13} - 96R_{23}sw) \\
& + a\ e(-416cw\ h_2kwe\ R_{12}R_{13} - 32h_2kwe\ R_{12}R_{23}sw) \\
& + a^2h_2^2e(-264cw\ kwe^2R_{13}^2 - 120cw\ kwe^2R_{13}R_{23}^2 + 120kwe^2R_{13}^2R_{23}sw \\
& - 24kwe^2R_{23}^3sw))\cos M + (64cw\ R_{13} + 64R_{23}sw + e^2(-320cw\ R_{13} \\
& - 224R_{23}sw) + a(64cw\ h_2kwe\ R_{12}R_{13} + 64h_2kwe\ R_{12}R_{23}sw \\
& - 32e^2h_2kwe\ R_{12}R_{23}sw) + a^2(32cw\ {h_2}^2kwe^2R_{13}^3 + 32{h_2}^2kwe^2R_{23}^3sw \\
& + e^2h_2^2(-64cw\ kwe^2R_{12}^2R_{13} + 337cw\ kwe^2R_{13}^3 - 339cw\ kwe^2R_{13}R_{23}^2 \\
& - 32kwe^2R_{12}^2R_{23}sw + 339kwe^2R_{13}^2R_{23}sw - 65kwe^2R_{23}^3sw)))\cos 2M \\
& + (e(96cw\ R_{13} + 96R_{23}sw) + a\ e(32cw\ h_2kwe\ R_{12}R_{13} + 32h_2kwe\ R_{12}R_{23}sw) \\
& + a^2h_2^2e(-68cw\ kwe^2R_{13}^3 + 156cw\ kwe^2R_{13}R_{23}^2 - 156kwe^2R_{13}^2R_{23}sw \\
& + 36kwe^2R_{23}^3sw))\cos 3M + (e^2(120cw\ R_{13} + 120R_{23}sw) \\
& + a\ e^2(24cw\ h_2kwe\ R_{12}R_{13} + 24h_2kwe\ R_{12}R_{23}sw) + a^2h_2^2(8cw\ kwe^2R_{13}^3 \\
& - 24cw\ kwe^2R_{13}R_{23}^2 + 24kwe^2R_{13}^2R_{23}sw - 8kwe^2R_{23}^3sw \\
& + e^2h_2^2(-8cw\ kwe^2R_{12}^2R_{13} - 74cw\ kwe^2R_{13}^3 + 186cw\ kwe^2R_{13}R_{23}^2 \\
& - 8kwe^2R_{12}^2R_{23}sw - 186kwe^2R_{13}^2R_{23}sw + 50kwe^2R_{23}^3sw)))\cos 4M \\
& + a^2h_2^2e(12cw\ kwe^2R_{13}^3 - 36cw\ kwe^2R_{13}R_{23}^2 + 36h_2^2kwe^2R_{13}^2R_{23}sw \\
& - 12kwe^2R_{23}^3sw)\cos 5M + a^2e^2h_2^2(15cw\ kwe^2R_{13}^3 - 45cw\ kwe^2R_{13}R_{23}^2 \\
& + 45kwe^2R_{13}^2R_{23}sw - 15kwe^2R_{23}^3sw)\cos 6M + (e(-160cw\ R_{23} + 160R_{13}sw) \\
& + a\ e(-224cw\ h_2kwe\ R_{12}R_{23} + 224h_2kwe\ R_{12}R_{13}sw) \\
& + a^2h_2^2e(-216cw\ kwe^2R_{13}^2R_{23} - 72cw\ kwe^2R_{23}^3 + 72kwe^2R_{13}^3sw \\
& + 216kwe^2R_{13}R_{23}^2sw))\sin M + (64cwR_{23} - 64R_{13}sw \\
& + e^2(-272cw\ R_{23} + 272R_{13}sw) + a(64cw\ h_2kwe\ R_{12}R_{23} \\
& - 64h_2kwe\ R_{12}R_{13}sw + e^2(-16cw\ h_2kwe\ R_{12}R_{23} + 16h_2kwe\ R_{12}R_{13}sw)) \\
& + a^2h_2^2(48cw\ kwe^2R_{13}^2R_{23} + 16cw\ kwe^2R_{23}^3 - 16kwe^2R_{13}^3sw \\
& - 48kwe^2R_{13}R_{23}^2sw + e^2h_2^2(-48cw\ kwe^2R_{12}^2R_{23} + 609cw\ kwe^2R_{13}^2R_{23} \\
& - 67cw\ kwe^2R_{23}^3 + 48kwe^2R_{12}^2R_{13}sw - 203kwe^2R_{13}^3sw
\end{aligned}
$$

$$
\begin{aligned}
&+ 201kwe^2R_{13}R_{23}^2sw)))\sin 2M + (e(96cw\ R_{23} - 96R_{13}sw) \\
&+ a\ e(32cw\ h_2kwe\ R_{12}R_{23} - 32h_2kwe\ R_{12}R_{13}sw) \\
&+ a^2h_2^2e(-180cw\ kwe^2R_{13}^2R_{23} + 44cw\ kwe^2R_{23}^3 + 60kwe^2R_{13}^3sw \\
&- 132kwe^2R_{13}R_{23}^2sw))\sin 3M + (e^2(120cw\ R_{23} - 120R_{13}sw) \\
&+ a\ e^2(24cw\ h_2kwe\ R_{12}R_{23} - 24h_2kwe\ R_{12}R_{13}sw) + a^2h_2^2(24cw\ kwe^2R_{13}^2R_{23} \\
&- 8cw\ kwe^2R_{23}^3 - 8kwe^2R_{13}^3sw + 24kwe^2R_{13}R_{23}^2sw \\
&+ e^2h_2^2(-8cw\ kwe^2R_{12}^2R_{23} - 204cw\ kwe^2R_{13}^2R_{23} + 56cw\ kwe^2R_{23}^3 \\
&+ 8kwe^2R_{12}^2R_{13}sw + 68kwe^2R_{13}^3sw - 168kwe^2R_{13}R_{23}^2sw)))\sin 4M \\
&+ a^2h_2^2e(36cw\ kwe^2R_{13}^2R_{23} - 12cw\ kwe^2R_{23}^3 - 12kwe^2R_{13}^3sw \\
&+ 36kwe^2R_{13}R_{23}^2sw)\sin 5M + a^2e^2h_2^2(45cw\ kwe^2R_{13}^2R_{23} - 15cw\ kwe^2R_{23}^3 \\
&-15kwe^2R_{13}^3sw + 45kwe^2R_{13}R_{23}^2sw)\sin 6M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} = &-\frac{a}{128h_2}kwe(-64cw\ R_{23} - 64R_{13}sw + e^2(104cw\ R_{23} - 72R_{13}sw) \\
&+ a(-64cw\ h_2kwe\ R_{12}R_{23} - 64h_2kwe\ R_{12}R_{13}sw \\
&+ e^2(8cw\ h_2kwe\ R_{12}R_{23} - 488h_2kwe\ R_{12}R_{13}sw)) \\
&+ a^2h_2^2(-24cw\ kwe^2R_{13}^2R_{23} - 24cw\ kwe^2R_{23}^3 - 24kwe^2R_{13}^3sw \\
&- 24kwe^2R_{13}R_{23}^2sw + e^2(40cw\ kwe^2R_{12}^2R_{23} - 198cw\ kwe^2R_{13}^2R_{23} \\
&+ 30cw\ kwe^2R_{23}^3 + 56kwe^2R_{12}^2R_{13}sw - 426kwe^2R_{13}^3sw \\
&- 198kwe^2R_{13}R_{23}^2sw)) + (e(-96cw\ R_{23} + 224R_{13}sw) \\
&+ a\ e(-32cw\ h_2kwe\ R_{12}R_{23} + 416h_2kwe\ R_{12}R_{13}sw) \\
&+ a^2h_2^2e(120cw\ kwe^2R_{13}^2R_{23} - 24cw\ kwe^2R_{23}^3 + 264kwe^2R_{13}^3sw \\
&+ 120h_2^2kwe^2R_{13}R_{23}^2sw))\cos M + (64cw\ R_{23} - 64R_{13}sw \\
&+ e^2(-224cwR_{23} + 320R_{13}sw) + a(64cw\ h_2kwe\ R_{12}R_{23} \\
&- 32cw\ e^2h_2kwe\ R_{12}R_{23} - 64h_2kwe\ R_{12}R_{13}sw) \\
&+ a^2h_2^2(32cw\ kwe^2R_{23}^3 - 32kwe^2R_{13}^3sw + e^2(-32cw\ kwe^2R_{12}^2R_{23} \\
&+ 339cw\ kwe^2R_{13}^2R_{23} - 65cw\ kwe^2R_{23}^3 + 64kwe^2R_{12}^2R_{13}sw - 337kwe^2R_{13}^3sw \\
&+ 339kwe^2R_{13}R_{23}^2sw)))\cos 2M + (e(96cw\ R_{23} - 96R_{13}sw) \\
&+ a\ e(32cw\ h_2kwe\ R_{12}R_{23} - 32h_2kwe\ R_{12}R_{13}sw) \\
&+ a^2h_2^2e(-156cw\ kwe^2R_{13}^2R_{23} + 36cw\ kwe^2R_{23}^3 + 68kwe^2R_{13}^3sw \\
&- 156kwe^2R_{13}R_{23}^2sw))\cos 3M + (e^2(120cw\ R_{23} - 120R_{13}sw) \\
&+ a\ e^2(24cw\ h_2kwe\ R_{12}R_{23} - 24h_2kwe\ R_{12}R_{13}sw) \\
&+ a^2h_2^2(24cw\ kwe^2R_{13}^2R_{23} - 8cw\ kwe^2R_{23}^3 - 8kwe^2R_{13}^3sw
\end{aligned}
$$

$$
\begin{aligned}
&+24kwe^2R_{13}R_{23}^2sw+e^2(-8cw\,kwe^2R_{12}^2R_{23}-186cw\,kwe^2R_{13}^2R_{23}\\
&+50cw\,kwe^2R_{23}^3+8kwe^2R_{12}^2R_{13}sw+74kwe^2R_{13}^3sw\\
&-186kwe^2R_{13}R_{23}^2sw)))\cos 4M+a^2h_2^2e(36cw\,kwe^2R_{13}^2R_{23}-12cw\,kwe^2R_{23}^3\\
&-12kwe^2R_{13}^3sw+36kwe^2R_{13}R_{23}^2sw)\cos 5M+a^2e^2h_2^2(45cw\,kwe^2R_{13}^2R_{23}\\
&-15cw\,kwe^2R_{23}^3-15kwe^2R_{13}^3sw+45kwe^2R_{13}R_{23}^2sw)\cos 6M\\
&+(e(160cw\,R_{13}+160R_{23}swL+a\,e(224cw\,h_2kwe\,R_{12}R_{13}\\
&+224h_2kwe\,R_{12}R_{23}sw)+a^2h_2^2e(72cw\,kwe^2R_{13}^3+216cw\,kwe^2R_{13}R_{23}^2\\
&+216kwe^2R_{13}^2R_{23}sw+72kwe^2R_{23}^3sw))\sin M+(-64cw\,R_{13}-64R_{23}sw\\
&+e^2(272cw\,R_{13}+272R_{23}sw)+a(-64cw\,h_2kwe\,R_{12}R_{13}\\
&-64h_2kwe\,R_{12}R_{23}sw+e^2(16cw\,h_2kwe\,R_{12}R_{13}+16h_2kwe\,R_{12}R_{23}sw))\\
&+a^2h_2^2kwe^2(-16cw\,R_{13}^3-48cw\,R_{13}R_{23}^2-48R_{13}^2R_{23}sw-16R_{23}^3sw\\
&+e^2(48cw\,R_{12}^2R_{13}-203cw\,R_{13}^3+201cw\,R_{13}R_{23}^2+48R_{12}^2R_{23}sw\\
&-609R_{13}^2R_{23}sw+67R_{23}^3sw)))\sin 2M+(e(-96cw\,R_{13}-96R_{23}sw)\\
&+a\,e(-32cw\,h_2kwe\,R_{12}R_{13}-32h_2kwe\,R_{12}R_{23}sw)\\
&+a^2h_2^2kwe^2e(60cw\,R_{13}^3-132cw\,R_{13}R_{23}^2+180R_{13}^2R_{23}sw-44R_{23}^2sw))\sin 3M\\
&+(e^2(-120cw\,R_{13}-120R_{23}sw)+a\,e^2(-24cw\,h_2kwe\,R_{12}R_{13}\\
&-24h_2kwe\,R_{12}R_{23}sw)+a^2h_2^2kwe^2(-8cw\,R_{13}^3+24cw\,R_{13}R_{23}^2-24R_{13}^2R_{23}sw\\
&+8R_{23}^3sw+e^2(8cw\,R_{12}^2R_{13}+68cw\,R_{13}^3-168cw\,R_{13}R_{23}^2+8R_{12}^2R_{23}sw\\
&+204R_{13}^2R_{23}sw-56R_{23}^3sw)))\sin 4M+a^2h_2^2kwe^2e(-12cw\,R_{13}^3+36cw\,R_{13}R_{23}^2\\
&-36R_{13}^2R_{23}sw+12R_{23}^3sw)\sin 5M+a^2e^2h_2^2kwe^2(-15cw\,R_{13}^3\\
&+45cw\,R_{13}R_{23}^2-45R_{13}^2R_{23}sw+15R_{23}^3sw)\sin 6M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}M_1}{\mathrm{d}t}=&\frac{\sin(\frac{M}{2})\cos(\frac{M}{2})}{2h_2^2}e(4+4a\ h_2kwe\ R_{12}+a^2h_2^2kwe^2(R_{13}^2+R_{23}^2)\\
&+(8e+a^2h_2^2kwe^2e(-5R_{13}^2+R_{23}^2))\cos M+a^2h_2^2kwe^2(R_{13}^2-R_{23}^2)\cos 2M\\
&+a^2h_2^2kwe^2e(R_{13}^2-R_{23}^2)\cos 3M-6a^2e\ {h_2}^2kwe^2R_{13}R_{23}\sin M\\
&+2a^2{h_2}^2kwe^2R_{13}R_{23}\sin 2M+2a^2e\ {h_2}^2kwe^2R_{13}R_{23}\sin 3M)
\end{aligned}
\tag{7.78}
$$

The solutions (i.e. indefinite integrals) of the short periodic terms in (7.77) are then ($M = nt$ is used)

$$\Delta\sigma_{1j}=\xi\Delta D_j+\xi\sum_{k=1,\ldots}\frac{1}{kn}\left(b_{jk}\sin kM-c_{jk}\cos kM\right), \tag{7.79}$$

where $\xi\ \Delta D_j$ denotes the integrals of the long periodic and linear terms and will be discussed in the next section.

7.3.5 Long Periodic and Secular Solutions

The first terms on the right-hand side of (7.77) have the form

$$\xi \frac{\mathrm{d}D_j}{\mathrm{d}t} = \xi L_j + \xi \sum_k (g_{jk}\cos(k\omega) + q_{jk}\sin(k\omega)), \tag{7.80}$$

where g, q, L are functions of (a, e, i). All terms of g and q are long periodic terms of ω and L-terms are the secular effects. The indefinite integrals of (7.80) are then

$$\xi\Delta D_j = \xi L_j t + \xi \sum_k (g_{jk}\sin(k\omega) - q_{jk}\cos(k\omega))\frac{1}{kn_\omega}, \tag{7.81}$$

where n_ω is mean angular velocities of the Keplerian elements ω. Explicitly, (7.80) may be given as follows (factor ξ is omitted, $kwe = K\omega_e$, $w = \omega$, $si = \sin i$, $ci = \cos i$):

$$\begin{aligned}\frac{\mathrm{d}D_1}{\mathrm{d}t} = \frac{1}{32h_2}&(32 - 40e^2 + a(-64ci\ h_2kwe + 96ci\ e^2h_2kwe)\\ &+ a^2h_2^2kwe^2(32ci^2 + 8si^2 + e^2(-16ci^2 - 6si^2))\\ &+ a^3h_2^3kwe^3(-8ci\ si^2 + e^2(24ci^3 - 22ci\ si^2))\\ &+ h_2^2kwe^2(13a^2e^2si^2 - 31a^3ci\ e^2si^2)\cos(2w) - 2a^2e^2h_2^2kwe^2si^2\sin(2w))\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}D_2}{\mathrm{d}t} = \frac{e}{16h_2^2}&(16 + 40a\ ci\ h_2kwe + 22a^3ci\ h_2^3kwe^3si^2\\ &+ a^2h_2^2(-56ci^2kwe^2 - 12kwe^2si^2)\\ &+ (-10a^2h_2^2kwe^2si^2 + 13a^3ci\ h_2^3kwe^3si^2)\cos(2w))\end{aligned}$$

$$\frac{\mathrm{d}D_3}{\mathrm{d}t} = -\frac{1}{8}\sin(w)\cos(w)a^2e\,kwe^2si^2(-10 + 13a\ ci\ h_2kwe)$$

$$\begin{aligned}\frac{\mathrm{d}D_4}{\mathrm{d}t} = \frac{a}{64h_2}&kwe\ si(32 - 8e^2 + a\,h_2kwe(-32ci - 120ci\ e^2)\\ &+ a^2h_2^2kwe^2(12si^2 + e^2(-24ci^2 + 99si^2))\\ &+ (44e^2 - 124a\ ci\ e^2h_2kwe + a^2e^2h_2^2kwe^2(-4ci^2 + 114si^2))\cos(2w))\end{aligned}$$

$$\frac{\mathrm{d}D_5}{\mathrm{d}t} = \frac{\sin(w)\cos(w)}{32h_2}a\ e^2kwe\ si(44 - 124a\ ci\ h_2kwe + a^2h_2^2kwe^2(-4ci^2 + 57si^2))$$

$$\frac{\mathrm{d}D_6}{\mathrm{d}t} = \frac{3}{4}\sin(w)\cos(w)a^2e^2kwe^2si^2 \tag{7.82}$$

where

$$\begin{aligned} L_1 = & \frac{1}{32h_2}(32 - 40e^2 + a(-64 + 96e^2)ci\, h_2kwe \\ & + a^2h_2^2kwe^2(8 + 24ci^2 - e^2(10ci^2 + 6)) \\ & + a^3h_2^3kwe^3ci(-8si^2 + e^2(46ci^2 - 22))) \end{aligned}$$

$$L_2 = \frac{e}{16h_2^2}(16 + 40a\ ci\ h_2kwe + 22a^3ci\ h_2^3kwe^3si^2 + a^2h_2^2kwe^2(-56ci^2 - 12si^2))$$

$$\begin{aligned} L_4 = & \frac{a}{64h_2}kwe\ si(32 - 8e^2 + a\,h_2kwe\,ci(-32 - 120e^2) \\ & + a^2h_2^2kwe^2(12si^2 + e^2(-24ci^2 + 99si^2))) \end{aligned}$$

The final solutions of (7.16) can be directly formed from (7.79). Selecting the integration interval as the minimum common period of the long-periodic terms, integrals of long-periodic terms are zero. Taking the short period $k2\pi$ into account, there exists a minimum common interval over which the integrals of all periodic terms are zero. The most important terms are the secular effects which are represented in L and which are zero except for the above three terms.

The term L_1 is obviously a positive one. The term L_2 has a meaning of (squares) differences between the velocities of the satellite and atmosphere, and hence is also positive. Taking the negative coefficient ξ into account, interesting conclusions can be drawn. The atmospheric drag disturbance reduces both the semi-major axis and the eccentricity of the satellite orbit constantly! In other words, the atmospheric effects will linearly reduce the altitude (height) of the satellite and will constantly alter the satellite orbital ellipse towards a more circular orbit. The factor h_2 equals about $(1/a)h_1$ and L_2 equals about eL_1; therefore the semi-major axis a is strongly linearly disturbed and the eccentricity e is weakly linearly disturbed. The Keplerian (the inclination) i is also subjected to secular disturbance from the atmosphere and this is a unique phenomenon in the disturbing theory that inclination will be linearly disturbed by the atmosphere. The Keplerian elements (ω, Ω, M) are not perturbed linearly.

7.3.6 Summary

Analytical solutions of a satellite orbit disturbed by atmospheric drag are obtained. The solutions are approximated to the second order of e^2 through expansions. Using the same principles, higher order solutions can be similarly derived if desired. Conclusions are obtained theoretically: the atmospheric drag disturbance will lead to a linear change (a reduction) with time in the semi-major axis parameter a and the shape parameter e of the orbital ellipse. In other words, atmospheric drag will constantly reduce the altitude of the satellite and at the end will force the satellite down to the Earth; and the satellite elliptic orbit will be constantly changed towards a more circular orbit. The semi-major axis a is strongly disturbed linearly and the eccentricity e is weakly disturbed linearly. The atmospheric perturbation force is independent from the Keplerian element Ω; this indicates that the long periodic effect of the atmospheric disturbance is only the function of ω, The Keplerian element (the inclination) i is also subjected to secular disturbance from the atmosphere and this is a unique phenomenon in the disturbing theory that inclination will be linearly disturbed by the atmosphere. Some of the atmospheric parameters in ξ can be determined if desired in order to investigate the atmospheric behaviour acting on the satellite; in such a case, the determination of the disturbing effects of the air drag can be done with a linear fitting process and this is much easier than using the traditional numerical methods.

The intensive formulas given in this section are in electronic form available upon request (xu@gfz-potsdam.de, j.xu@arcor.de).

7.4 Discrete Solutions of Disturbance of Atmospheric Drag

Atmospheric drag, caused by the air, is the disturbance force acting on the satellite's surface. Air drag force can be represented as (see (4.75))

$$\vec{f}_{\text{drag}} = -m\frac{1}{2}\left(\frac{C_{\text{d}}S}{m}\right)\sigma\left|\dot{\vec{r}} - \dot{\vec{r}}_{\text{air}}\right|^2\vec{n}_a, \quad \vec{n}_a = \frac{\dot{\vec{r}} - \dot{\vec{r}}_{\text{air}}}{\left|\dot{\vec{r}} - \dot{\vec{r}}_{\text{air}}\right|}, \tag{7.83}$$

where the meanings of the symbols are the same as in (4.75) and unit vector $\vec{n}_a$ is the direction of the air drag force. For CHAMP satellite, with an orbit height of 400 km, the air drag force unit vector $\vec{n}_a$ changes its direction about 1.2×10^{-3} (rad s^{-1}). The amount of $\left|\dot{\vec{r}} - \dot{\vec{r}}_{\text{air}}\right|^2$ changes slower than the direction. In such a case the acceleration of the air drag can be discretized by

$$\vec{a}_{\text{drag}} = -\frac{1}{2}\left(\frac{C_{\text{d}}S}{m}\right)\sigma\left|\dot{\vec{r}}(t_k) - \dot{\vec{r}}_{\text{air}}(t_k)\right|^2\vec{n}_a(t_k)$$

The disturbed velocity caused by the atmospheric drag is then

$$\vec{v}_{\text{airdrag}}(t) = \sum_{i=1}^{k} -\frac{C_d S}{2m}\sigma\left|\dot{\vec{r}}(t_i) - \dot{\vec{r}}_{air}(t_i)\right|^2\vec{n}_{\text{a}}(t_i)\Delta t \tag{7.84}$$

The disturbed position caused by the solar radiation is then

$$\vec{\rho}_{\text{air}}(t) = \sum_{j=1}^{k} \vec{v}_{airdrag}(t_j)\Delta t \tag{7.85}$$

Equation 7.85 is the solution of the solar radiation disturbance on the orbit of the satellite.

For all satellites, with an orbit height greater than 1,000 km, the atmospheric drag is nearly zero and therefore this effect does not need to be taken into account.

7.5 Analytical Solutions of Satellite Orbit Disturbed by Lunar and Solar as Well as Planetary Gravitations

The text of this section is just slightly modified from Xu et al. (2010a). The basic Lagrangian equations defined in Sect. 6.3 (6.24) are used here instead of the so-called simplified Lagrangian ones stated in Xu et al. (2010a) due to the Xus' singularity-free theory (cf. Sect. 11.2.3 or Xu and Xu 2012). More precise solutions are given.

The analytical solution of a satellite orbit disturbed by lunar gravitational force is derived to the second order (5×10^{-6}) in this section. The force vector is first expanded to omit terms smaller than third order (10^{-9}). Then four terms of potential functions are derived from the expanded force vector and set into the basic Lagrangian equations of satellite motion to obtain the theoretical solutions. For the first term of potential functions the solutions are derived directly. For the second term, mathematical expansions and transformations are used to separate disturbances into three parts: short periodic terms with trigonometric functions of M, long periodic terms with trigonometric functions of (ω, Ω) and secular terms with non-periodic functions of (a, e, i). The integrations are then carried out with respect to M, (ω, Ω) and t, to obtain the analytical solutions of satellite orbits with a program using mathematic symbolic operation software. The third potential function differs from the second by a factor and the fourth is simpler than the second. Therefore the solutions are derived similarly using slightly modified programs, respectively. The results show that only two Keplerian elements (ω, M) are linearly perturbed by lunar gravitation; i.e. the lunar attracting force will cause a linear precession (delay) of the perigee (orientation of the ellipse) and a linear delay of the position (mean anomaly) on an Earth satellite. The Keplerian element a (semi-major axis of the ellipse) is not perturbed long-periodically as the others. The derived solutions are also valid for the solar and planetary gravitational disturbances. Because of the differences in distance between the moon, the sun, and the planets to the Earth or an Earth satellite, the solutions are of third and fourth order for the solar and planetary gravitational disturbances on an Earth satellite, respectively.

7.5.1 Introduction

The Earth satellite orbit disturbed by lunar, solar and planetary gravitational forces is a typical multi-body problem (Boccaletti and Pucacco 2001; Brouwer and Clemence 1961; Chobotov 1991; Eberle et al. 2008). A general multi-body problem in cases of more than three bodies does not have an applicable solution (Battin 1999; Van Kamp 1967; Diacu 1996). However, if the disturbance forces of the multi-body problem are two orders smaller (i.e. with a factor of 10^{-6}) than the Earth's attracting force, the effects of the multi-body disturbances can be considered as small additional motions added to the nominal Keplerian motion (Vallado 2007; Herrick 1972). This means that the multi-body disturbance problem can be dealt with by adding several three-body disturbance problems together. Therefore the solution of the Earth satellite orbit disturbed by the lunar or solar gravitation is a key which could lead to the solution of the multi-body problem and is meaningful to seek. Traditionally, the orbit disturbance problem of lunar and/or solar gravitation is solved by numerical integration (Montenbruck and Gill 2000; Xu 2007 2003; Seeber 2003; Meeus 1992). An analytical solution gives the theoretical integrals and shows the physical effects with very clear spectral properties (Licandro et al. 2008; Cui 1990; Schneider and Cui 2005; Kudryavtsev 2007; Lynden-Bell 2009; Pal 2009), and may give a direct insight into the physical disturbing phenomenon (Mysen 2009; Desmars et al. 2009; Torge 1991). Because the moon is much closer to the Earth satellite compared to the sun, the solutions of the lunar disturbances are more complicated than that of the solar case. Xu (2008) introduced a method to solve the equations of satellite motion disturbed by the solar gravitational force; however he did not give explicit solutions for the lunar disturbances. In this section, the complete method and solutions will be described and derived.

The potential functions of the expanded and simplified lunar (solar) gravitational force are derived in Sect. 7.5.2. The disturbed Lagrangian equations of satellite motion and the solutions of the first, second, third and fourth terms of potential functions are derived and given in the Sects. 7.5.3, 7.5.4, 7.5.5, and 7.5.6, respectively. A brief summary is given at the end.

7.5.2 Potential Functions of the Lunar and Solar Gravitational Forces

The disturbance forces of multiple point-masses acting on the satellite are (see, e.g. (4.50), or Battin 1999; Kaula 1966/2001; Montenbruck and Gill 2000)

$$\vec{f}_{mul} = -m \sum_j Gm(j) \left[\frac{\vec{r} - \vec{r}_{m(j)}}{\left| \vec{r} - \vec{r}_{m(j)} \right|^3} + \frac{\vec{r}_{m(j)}}{\vec{r}^{\,3}_{m(j)}} \right] \tag{7.86}$$

where G is the gravitational constant and $m(j)$ is the jth point-mass, $j = 1,2,\ldots$; $\vec{r}$ and $\vec{r}_{m(j)}$ are the geocentric vectors of the satellite and the point mass j and m is the

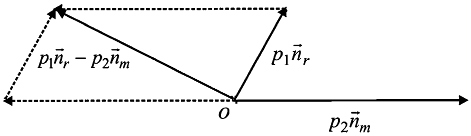


Fig. 7.7 Disturbance force vector and its two components (p_1 and p_2 are their lengths)

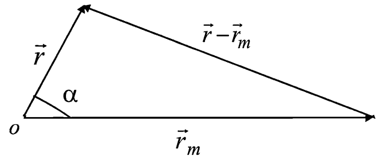


Fig. 7.8 Geometric relationship between the geocentric vectors of the satellite and the moon

mass of the satellite. The disturbance acceleration (indeed the force vector) of the moon is then

$$\begin{aligned}\vec{f}_m &= -m\mu_m\left(\frac{1}{|\vec{r}-\vec{r}_m|^2}\vec{n}_{ms} + \frac{1}{|\vec{r}_m|^2}\vec{n}_m\right)\\ &= -m\mu_m\left[\frac{r}{|\vec{r}-\vec{r}_m|^3}\vec{n}_r + \left(\frac{1}{r_m^2} - \frac{r_m}{|\vec{r}-\vec{r}_m|^3}\right)\vec{n}_m\right],\\ \vec{n}_{ms} &= \frac{\vec{r}-\vec{r}_m}{|\vec{r}-\vec{r}_m|}, \quad \vec{n}_m = \frac{\vec{r}_m}{|\vec{r}_m|}, \quad \vec{n}_r = \frac{\vec{r}}{r}\end{aligned} \tag{7.87}$$

where μ_m is the gravitational constant of the moon; $\vec{r}_m$ is the geocentric vector of the moon. r and r_m are the lengths of the related vectors. The unit vectors $\vec{n}_{ms}$, $\vec{n}_m$ and $\vec{n}_r$ represent the vector from the moon to the satellite, the geocentric vectors of the moon and satellite, respectively. The force vector of the moon acting on the satellite is the summation of two vectors in directions of $\vec{n}_r$ and $-\vec{n}_m$ (see (7.87) and Fig. 7.7). Sub-index m is used to denote the moon related constants, variables and vectors. Using sub-index s to denote the sun related constants, variables and vectors, derivations and formulas given here are generally valid for the solar case. According to the geometric relations of the vectors one has (see Fig. 7.8)

$$\begin{aligned}&\vec{r} = \vec{r}_m + (\vec{r}-\vec{r}_m)\\ &|\vec{r}-\vec{r}_m|^2 = r_m^2 + r^2 - 2rr_m\cos\alpha.\\ &\cos\alpha = \vec{n}_r \cdot \vec{n}_m\end{aligned} \tag{7.88}$$

In Fig. 7.8 α is the angle between the geocentric vectors of the satellite and the moon; and cosα is the inner product of the two unit vectors. For the sun, a similar discussion can be made. Using the well-known expansion formula (Bronstein and Semendjajew 1987; Wang et al. 1979)

$$\sqrt[q]{(1+x)^p} = 1 + \frac{p}{q}x + \frac{p(p-q)}{q\cdot 2q}x^2 + \frac{p(p-q)(p-2q)}{q\cdot 2q\cdot 3q}x^3 + \ldots$$
$$|x| \le 1, p > 0 \; or \; q > 0 \tag{7.89}$$

the lengths of the two vectors in (7.87) can be expanded as

$$\frac{r}{|\vec{r}-\vec{r}_m|^3} \approx \frac{r}{r_m^3}\left(1 + 3\frac{r}{r_m}\cos\alpha\right)$$
$$\frac{1}{r_m^2} - \frac{r_m}{|\vec{r}-\vec{r}_m|^3} \approx \frac{r}{r_m^3}\left(-3\cos\alpha - 1.5\frac{r}{r_m}(5\cos^2\alpha - 1)\right). \tag{7.90}$$

The geocentric distance rate (r/r_m) of the GPS satellite to the moon is about 6×10^{-2}. Therefore, for all satellites that are lower than the GPS satellite, the two expansions in (7.90) are precise enough (in the case of the second order solution), which yields for (7.89)

$$\vec{f}_m = -m\,\mu_m\left(\frac{r}{r_m^3}\left(1 + 3\frac{r}{r_m}\cos\alpha\right)\vec{n}_r - \frac{r}{r_m^3}\left(3\cos\alpha + \frac{3r}{2r_m}\left(5\cos^2\alpha - 1\right)\right)\vec{n}_m\right). \tag{7.91}$$

The potential function of the disturbing force (7.91) is

$$V_m = -m\,\mu_m\frac{r}{r_m^3} + m\mu_m\frac{3}{2r_m^3}r^2\cos^2\alpha - m\mu_m\frac{1}{2r_m^4}\left(-5r^3\cos^3\alpha + 3r^3\cos\alpha\right). \tag{7.92}$$

The geocentric distance ratios (r/r_s) of GPS and GEO satellites to the sun are about 1.77×10^{-4} and 2.8×10^{-4}, respectively. Therefore, for all satellites that are lower than the GEO satellite, the two expansions in (7.90) are precise enough to take only the first term into account (in the case of the second order solution). Then the force vector and potential function of the sun can also be represented by (7.91) and (7.92), respectively, however with the sub-index s instead of m. Furthermore, only the first two terms are needed in (7.92). The computation for the solar case shows that for a GPS satellite the force vector (7.87) and the first term of two components of (7.91) have differences of 3×10^{-10} (that is one order smaller than the values computed by using (7.87), see Fig. 7.9).

The correctness of the potential function can be verified directly by applying the gradient operation with respect to the satellite coordinates on (7.92) and comparing the results with the force vector (7.91).

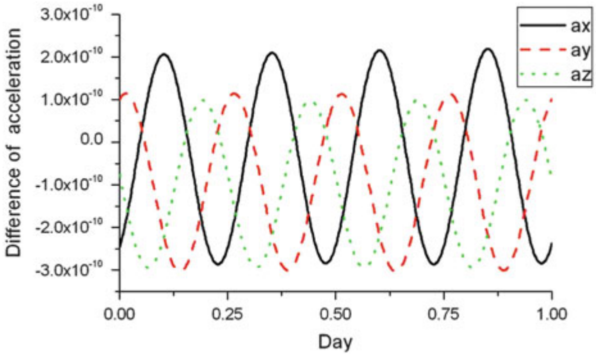


Fig. 7.9 Numerical difference of acceleration between (7.87) and the first term in two components of (7.91) in the solar case

7.5.3 Disturbed Equations of Motion and Solutions of the First Term

For convenience we rewrite the Lagrangian equations of satellite motion as follows (see, e.g. (4.11), or Battin 1999; Kaula 1966/2001; Montenbruck and Gill 2000):

$$
\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2}{na}\frac{\partial V}{\partial M} \\
\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{1-e^2}{na^2 e}\frac{\partial V}{\partial M} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial V}{\partial \omega} \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial V}{\partial e} - \frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial V}{\partial i} \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\left(\cos i\frac{\partial V}{\partial \omega} - \frac{\partial V}{\partial \Omega}\right) \\
\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial V}{\partial i} \\
\frac{\mathrm{d}M}{\mathrm{d}t} &= n - \frac{2}{na}\frac{\partial V}{\partial a} - \frac{1-e^2}{na^2 e}\frac{\partial V}{\partial a}.
\end{aligned}
\tag{7.93}
$$

The Lagrangian equations can be written as cf. (6.23))

$$
\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2}{na}\frac{\mathrm{d}a_1}{\mathrm{d}t} = h_1\frac{\mathrm{d}a_1}{\mathrm{d}t} \\
\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{1-e^2}{na^2e}\frac{\mathrm{d}a_1}{\mathrm{d}t} - \frac{\sqrt{1-e^2}}{na^2e}\frac{\mathrm{d}e_1}{\mathrm{d}t} = h_2\frac{\mathrm{d}a_1}{\mathrm{d}t} - h_3\frac{\mathrm{d}e_1}{\mathrm{d}t} \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{na^2e}\frac{\mathrm{d}\omega_1}{\mathrm{d}t} - \cos i\frac{\mathrm{d}\Omega}{\mathrm{d}t} = h_3\frac{\mathrm{d}\omega_1}{\mathrm{d}t} - h_5(\cos i/\sin i)\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} \\
\frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\left(\cos i\frac{de_1}{dt} - \frac{\mathrm{d}i_1}{\mathrm{d}t}\right) = h_4\frac{1}{\sin i}\left(\cos i\frac{de_1}{dt} - \frac{\mathrm{d}i_1}{\mathrm{d}t}\right) \\
\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} = h_5\frac{1}{\sin i}\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} \\
\frac{\mathrm{d}M}{\mathrm{d}t} &= n - \frac{2}{na}\frac{\mathrm{d}M_1}{\mathrm{d}t} - \frac{1-e^2}{na^2e}\frac{\mathrm{d}\omega_1}{\mathrm{d}t} = n - h_1\frac{\mathrm{d}M_1}{\mathrm{d}t} - h_2\frac{\mathrm{d}\omega_1}{\mathrm{d}t}
\end{aligned} \tag{7.94}
$$

where coefficients (h_1, h_2, h_3, h_4, h_5) are defined, and

$$
\begin{aligned}
&\frac{\mathrm{d}a_1}{\mathrm{d}t} = \frac{\partial V}{\partial M}, \quad \frac{\mathrm{d}e_1}{\mathrm{d}t} = \frac{\partial V}{\partial \omega}, \quad \frac{\mathrm{d}\omega_1}{\mathrm{d}t} = \frac{\partial V}{\partial e}, \\
&\frac{\mathrm{d}i_1}{\mathrm{d}t} = \frac{\partial V}{\partial \Omega}, \quad \frac{\mathrm{d}\Omega_1}{\mathrm{d}t} = \frac{\partial V}{\partial i}, \quad \frac{\mathrm{d}M_1}{\mathrm{d}t} = \frac{\partial V}{\partial a}.
\end{aligned} \tag{7.95}
$$

Keplerian elements (a, e, ω, i, Ω, M) are semi-major axis, eccentricity of the ellipse, argument of perigee, inclination angle, right ascension of ascending node, mean anomaly and mean angular velocity is denoted as n. V is the disturbance potential function (V_m in the lunar case and V_s in the solar case) and is given by (7.92).

Equation 7.95 are called basic Lagrangian equations of motion. By integrating (7.94) the mean value theorem for integration is used where the functions of variables (a, e, i) are considered constants. For solving the Lagrangian equations of motion (7.93) or (7.94), the simplest way is to solve the basic Lagrangian equations (7.95). Substituting solutions of (7.95) into (7.94) the solutions of (7.94) can be obtained. Therefore, without exception, the potential force disturbing orbit problem turns out to be the problem of solving the basic Lagrangian equations of motion (7.95). It is notable that the basic Lagrangian equations of motion are singularity-free.

Let us denote the first term of the potential function (7.92) by $V_{\mathrm{m}1}$ and note that $V_{\mathrm{m}1}$ is a function of the three Keplerian elements (a, e, M). The derivatives of the potential function with respect to the Keplerian elements are then (see (4.24))

$$
\frac{\partial V_{m1}}{\partial a} = \frac{\partial V_{m1}}{\partial r}\frac{\partial r}{\partial a} = \frac{1}{a}V_{m1}, \quad \frac{\partial V_{m1}}{\partial \Omega} = \frac{\partial V_{m1}}{\partial i} = \frac{\partial V_{ml}}{\partial \omega} = 0,
$$

$$
\frac{\partial V_{m1}}{\partial e} = \frac{\partial V_{m1}}{r}\frac{\partial r}{\partial e} = \frac{-a\cos f}{r}V_{m1}, \quad \frac{\partial V_{m1}}{\partial mM} = \frac{\partial V_{m1}}{r} = \frac{\partial r_{ml}}{\partial M} = \frac{ae\sin f}{r\sqrt{1-e^2}}V_{m1} \tag{7.96}
$$

where f is true anomaly. Substituting the derivatives (7.96) and V_{m1} into the equations of motion (7.93), and noting that (Bronstein and Semendjajew 1987; Wang et al. 1979)

$$(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + \ldots |x| < 1, \tag{7.97}$$

yields

$$\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{-2m\mu_m}{nr_m^3} \frac{e \sin f}{\sqrt{1-e^2}}, \\
\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{-m\mu_m\sqrt{1-e^2}}{na} \frac{\sin f}{r_m^3}, \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{-m\mu_m\sqrt{1-e^2}}{na} \frac{\sin f}{r_m^3}, \\
\frac{\mathrm{di}}{\mathrm{d}t} &= 0, \quad \frac{\mathrm{d}\Omega}{\mathrm{d}t} = 0, \\
\frac{\mathrm{d}M}{\mathrm{d}t} &= \frac{2}{na} \frac{m\mu_m}{r_m^3} \frac{\sqrt{1-e^2}}{1+e\cos f} - \frac{1-e^2}{nae} \frac{m\mu_m \cos f}{r_m^3} \\
&= \frac{m\mu_m\sqrt{1-e^2}}{naer_m^3}\left(\frac{1}{1+e\cos f} - \cos f\right). \\
&= \frac{m\mu_m\sqrt{1-e^2}}{naer_m^3}\left(2e - (1+2e^2)\cos f\right)
\end{aligned} \tag{7.98}$$

All terms smaller than order e^2 are neglected to keep the formula simpler. For convenience the mean angular velocity n in the last equation of (7.93) is omitted. Furthermore, there are

$$\frac{1}{r_m^3} = \frac{(1+e_m\cos f_m)^3}{a_m^3(1-e_m^2)^3} \approx \frac{1}{a_m^3}(1 + 3e_m^2 + 3e_m\cos f_m + 6e_m^2\cos^2 f_m + \ldots) \tag{7.99}$$

$$\frac{1}{r_m^4} = \frac{(1+e_m\cos f_m)^4}{a_m^4(1-e_m^2)^4} \approx \frac{1}{a_m^4}(1 + 4e_m^2 + 4e_m\cos f_m + 6e_m^2\cos^2 f_m + \ldots). \tag{7.100}$$

All lunar related Keplerian elements are denoted by the index m. Because of a small e_m (≈ 0.0549), using the daily average f_m to approximate the changing f_m will cause an angle error of maximally 0.228/2 rad/day which induces a maximal error of 3.98×10^{-3} for $6e_m^2\cos^2 f_m$. Therefore the geocentric lunar distance r_m can be considered as a daily constant, which makes the derivation process much easier.

For the solar case, e_s is small (≈ 0.0167) (it is the same as for the Earth). Using the daily average f_s to approximate the changing f_s will cause an angle error of maximally 0.017/2 rad/day, which corresponds to a maximal error of 4.26×10^{-4} for $3e_s \cos f_s$. Therefore the geocentric solar distance r_s can be considered as a daily constant.

For convenience for the integration with respect to time t the following functions are used to transform the trigonometric function of f to M (see, e.g. (5.22), or Battin 1999; Kaula 1966/2001; Montenbruck and Gill 2000):

$$\sin f = \left(1 - \frac{7}{8}e^2\right)\sin M + e\left(1 - \frac{7}{6}e^2\right)\sin 2M + \frac{9}{8}e^2 \sin 3M,$$

$$\cos f + e = \left(1 - \frac{9}{8}e^2\right)\cos M + e\left(1 - \frac{4}{3}e^2\right)\cos 2M + \frac{9}{8}e^2 \cos 3M \qquad (7.101)$$

This yields (terms smaller than order e^2 are neglected)

$$\delta S = \int \sin f dt = \frac{-1}{n}\left(\left(1 - \frac{7}{8}e^2\right)\cos M + \frac{e}{2}\left(1 - \frac{7}{6}e^2\right)\cos 2M + \frac{3}{8}e^2 \cos 3M\right),$$

$$\delta C = \int (\cos f + e)dt = \frac{1}{n}\left(\left(1 - \frac{9}{8}e^2\right)\sin M + \frac{e}{2}\left(1 - \frac{4}{3}e^2\right)\sin 2M + \frac{3}{8}e^2 \sin 3M\right) \qquad (7.102)$$

where the relation $M = nt$ is used (indeed n should include all linear effects of M). Then the solutions (i.e. indefinite integrals) of (7.98) are

$$\Delta a = \frac{-2m\mu_m}{nr_m^3}\frac{e}{\sqrt{1-e^2}}\partial S,$$

$$\Delta e = \frac{-m\mu_m\sqrt{1-e^2}}{nar_m^3}\partial S,$$

$$\Delta\omega = \frac{m\mu_m\sqrt{1-e^2}}{nar_m^3}(-et + \partial C),$$

$$\Delta i = const., \quad \Delta\Omega = const.,$$

$$\Delta M = \frac{m\mu_m(1-e^2)}{naer_m^3}\left(e(3+2e^2)t - (1+2e^2)\partial C\right). \qquad (7.103)$$

It is notable that the solutions of the first term multi-body perturbations on satellite orbits given above are solved by using Lagrangian equations of motion

(7.93) instead of using the basic ones (7.95). Equation 7.103 show that the orbital parameters ω and M are partly linearly perturbed by the moon. The orientation of the ellipse will rotate constantly in the direction of reducing ω; whereas the mean anomaly advances constantly. Selecting the integration interval as $2k\pi$ (k is any integer), integrals of (7.102) are zero (i.e. δC and δS are short periodic terms). Except for the two linear perturbations, all other perturbations are short periodic. The conclusions also hold in the case of solar gravitational disturbances.

7.5.4 Solutions of the Second Term

Denote the second term of the potential function (7.92) as $V_{\mathrm{m}2}$ and note that (see (7.58), or Battin 1999; Kaula 1966/2001; Montenbruck and Gill 2000)

$$\vec{n}_s = \begin{pmatrix} n_s1 \\ n_s2 \\ n_s3 \end{pmatrix}, \tag{7.104}$$

$$r\cos\alpha = \vec{n}_s \cdot \vec{r} = r(A\cos f + B\sin f). \tag{7.105}$$

$$\begin{aligned}
\vec{r} &= R_3(-\Omega)R_1(-i)R_3(-\omega)r\begin{pmatrix}\cos f\\ \sin f\\ 0\end{pmatrix} = rR\begin{pmatrix}\cos f\\ \sin f\\ 0\end{pmatrix}\\
&= r\begin{pmatrix}\cos\Omega\cos i\cos\omega-\sin\Omega\sin\omega & -\cos\Omega\cos i\sin\omega-\sin\Omega\sin\omega & \cos\Omega\sin i\\ \sin\Omega\sin i\cos\omega-\cos\Omega\sin\omega & -\sin\Omega\sin i\sin\omega-\cos\Omega\cos\omega & \sin\Omega\sin i\\ -\sin i\cos\omega & \sin i\sin\omega & \cos i\end{pmatrix}\\
&\quad\times\begin{pmatrix}\cos f\\ \sin f\\ 0\end{pmatrix}\\
&= r\begin{pmatrix}R_{11}\cos f+R_{12}\sin f\\ R_{21}\cos f+R_{22}\sin f\\ R_{31}\cos f+R_{32}\sin f\end{pmatrix}
\end{aligned} \tag{7.106}$$

The coefficients in (7.105) are

$$\begin{aligned}
A &= n_{m1}R_{11} + n_{m2}R_{21} + n_{m3}R_{31}\\
B &= n_{m1}R_{12} + n_{m2}R_{22} + n_{m3}R_{32}
\end{aligned} \tag{7.107}$$

where R with a single index k is the rotational matrix around the axis k, R with two indices are elements of the total rotational matrix R and they are trigonometric functions of (ω, i, Ω), lunar related elements are denoted with index m, n_{mk} is the kth component of the geocentric unit vector of the moon and A and B are functions of sinus and cosine of (ω, i, Ω) as well as components of the geocentric lunar unit vector. The derivatives of the second term potential functions with respect to the Keplerian elements are then

$$\frac{\partial V_{m2}}{\partial \sigma_j} = m\mu_m \frac{3}{r_m^3} r\cos\alpha \frac{\partial r\cos\alpha}{\partial \sigma_j} \tag{7.108}$$

where σ_j denotes the jth Keplerian element. Explicitly, the partial derivatives are (see (4.24))

$$\begin{aligned}
\frac{\partial V_{m2}}{\partial a} &= m\mu_m \frac{3}{r_m^3}\frac{r^2}{a}\cos^2\alpha,\\
\frac{\partial V_{m2}}{\partial a} &= m\mu_m \frac{3}{r_m^3} r\cos^2\alpha\left(\cos\alpha(-a\cos f) + r(-A\sin f + B\cos f)\frac{2+e\cos f}{1-e^2}\sin f\right)\\
\frac{\partial V_{m2}}{\partial \omega} &= m\mu_m \frac{3}{r_m^3} r^2\cos\alpha(A_\omega\cos f + B_\omega\sin f)\\
\frac{\partial V_{m2}}{\partial i} &= m\mu_m \frac{3}{r_m^3} r^2\cos\alpha(A_i\cos f + B_i\sin f)\\
\frac{\partial V_{m2}}{\partial \Omega} &= m\mu_m \frac{3}{r_m^3} r^2\cos\alpha(A_\Omega\cos f + B_\Omega\sin f),\ \text{and}\\
\frac{\partial V_{m2}}{\partial M} &= m\mu_m \frac{3}{r_m^3}\cos\alpha\left(\frac{ae}{\sqrt{1-e^2}} r\cos\alpha\sin f\right.\\
&\quad \left. + (-A\sin f + B\cos f)a^2\sqrt{1-e^2}\right).
\end{aligned} \tag{7.109}$$

In (7.109) A and B with indices (ω, i, Ω) are partial derivatives of A and B with respect to the related indices. Putting (7.109) into (7.95) and taking the following expansions (terms with order e^2 are neglected) into account (see (7.89)):

$$r = \frac{a(1-e^2)}{1+e\cos f} \approx a(1 - e\cos f - e^2\sin^2 f), \tag{7.110}$$

$$r^2 = \frac{a^2(1-e^2)^2}{(1+e\cos f)^2} \approx a^2(1 - 2e\cos f + 3e^2\cos^2 f - 2e^2), \tag{7.111}$$

the basic Lagrangian equations of satellite motion (7.95) can be transformed in terms of $\cos^m f \sin^k f$ and then in terms of products of $\{\cos mM, \cos kM, \sin mM, \sin kM\}$; in the end they can be reduced to a functional series of $\{\cos mM, \sin kM\}$ which can be integrated with respect to M to obtain the short periodic terms of solutions. The terms that have nothing to do with M are long periodic terms which can be reduced to trigonometric functions of ($k\omega + m\Omega$) and secular terms (non periodic terms) (k and m are integers). The transformation can be carried out by using mathematic symbolic operation software such as Mathematica or Maple and the equations of motion have the form

$$\frac{\mathrm{d}\sigma_{j1}}{\mathrm{d}t} = d_j + \sum_{k=1,\dots} (b_{jk}\cos kM + c_{jk}\sin kM). \tag{7.112}$$

Here d, b, c are functions of (a, e, ω, i, Ω). All terms of b and c are short periodic perturbations and all d terms include long periodic and linear perturbations. Or, to be more explicit (approximated to the order of e^2, the factor $3m\mu_m/r_m^3$ is omitted),

$$\begin{aligned}\frac{\mathrm{d}a_1}{\mathrm{dt}} = \frac{-1}{16}a^2(&-18AB\,e^2-8AB\,e\cos M+(-16AB+24AB\,e^2)\cos 2M\\&-24AB\,e\cos 3M-14AB\,e^2\cos 4M+(-4A^2-12B^2)e\sin M\\&+(8A^2-8B^2+(-6A^2-18B^2)e^2)\sin 2M+(12A^2-12B^2)e\sin 3M\\&+(7A^2-7B^2)e^2\sin 4M)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}e_1}{\mathrm{d}t} = -\frac{1}{16}a^2(&-8AA_\omega-8BB_\omega+(14AA_\omega-24BB_\omega)e_2\\&+(8AA_\omega-8BB_\omega)e\cos M+(-8AA_\omega-8BB_\omega+(40AA_\omega-32BB_\omega)e^2)\cos 2M\\&+(-8AA_\omega+8BB_\omega)e\cos 3M+(10AA_\omega+8BB_\omega)e^2\cos 4M\\&+(8A_\omega B+8AB_\omega)e\sin M+(-8A_\omega B-8AB_\omega+(18A_\omega B+18AB_\omega)e^2)\sin 2M\\&+(-8A_\omega B-8AB_\omega)e\sin 3M+(A_\omega B+AB_\omega)e^2\sin 4M)\end{aligned} \tag{7.113}$$

$$\begin{aligned}\frac{\mathrm{d}\omega_1}{\mathrm{d}t} = \frac{a^2}{16}(&-8A^2e+(-20A^2+4B^2+(22A^2-4B^2)e^2)\cos M\\&+(-32A^2+8B^2)e\cos 2M+(4A^2-4B^2+(-A^2+6B^2)e^2)\cos 3M\\&+(8A^2-8B^2)e\cos 4M+(11A^2-2B^2)e^2\cos 5M+(-24AB+38AB\,e^2)\sin M\\&-40AB\,e\sin 2M+(8AB-61AB\,e^2)\sin 3M+16AB\,e\sin 4M+13AB\,e^2\sin 5M)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}i_1}{\mathrm{d}t} = -\frac{a^2}{16}(&-8AA_\Omega-8BB_\Omega+(144AA_\Omega+8BB_\Omega)e^2+(8AA_\Omega-8BB_\Omega)e\cos M\\&+(-8AA_\Omega+8BB_\Omega+(40AA_\Omega-32BB_\Omega)e^2)\cos 2M+(-8AA_\Omega\\&+8BB_\Omega)e\cos 3M+(10AA_\Omega+8BB_\Omega)e^2\cos 4M+(8A_\Omega B+8AB_\Omega)e\sin M\\&+(-8A_\Omega B-8AB_\Omega+(18A_\Omega B+18AB_\Omega)e^2)\sin 2M\\&+(-8A_\Omega B-8AB_\Omega)e\sin 3M+(A_\Omega B+AB_\Omega)e^2\sin 4M)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} = -\frac{a^2}{16a^2}&(-8AA_i - 8BB_i + (144AA_i + 24BB_i)e^2 + (8AA_i - 8BB_i)e\cos M \\ &+ (-8AA_i + 8BBi + (40AA_i - 32BB_i)e^2)\cos 2M \\ &+ (-8AA_i + 8BB_i)e\cos 3M + (10AA_i + 8BB_i)e^2\cos 4M \\ &+ (8A_iB + 8AB_i)e\sin M + (-8A_iB - 8AB_i + (18A_iB + 18AB_i)e^2)\sin 2M \\ &+ (-8A_iB - 8AB_i)e\sin 3M + (A_iB + AB_i)e^2\sin 4M)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}M_1}{\mathrm{d}t} = -\frac{a}{8}&(-4A^2 - 4B^2 + (7A^2 + 12B^2)e^2 + (4A^2 - 4B^2)e\cos M \\ &+ (-4A^2 + 4B^2 + (20A^2 - 16B^2)e^2 + \cos 2M \\ &+ (-4A^2 + 4B^2)e\cos 3M + (5A^2 + 4B^2)\cos 4M + 8ABe\sin M \\ &+ (-8AB + 18ABe^2)\sin 2M - 8ABe\sin 3M + ABe^2\sin 4M)\end{aligned}$$

The short periodic terms in (7.112) can be easily integrated with respect to M by using the relation $M = nt$ (and considering secular and long-periodic terms as constants or by using the middle value theorem) and have the form of

$$\Delta\sigma_{j_1} = \Delta d_j + \sum_{k=1,\ldots}\frac{1}{kn}(b_{jk}\sin kM - c_{jk}\cos kM), \quad (7.114)$$

where Δd_j denote the integrals of the long periodic and linear terms.

It is obvious that the d terms in (7.112) are multiplications of {cosω, sinω, cosΩ, sinΩ}. They can be seen in (7.113) and can be reduced to a series of trigonometric functions of ($k\Omega + m\omega$) and secular terms. Therefore the d terms have the form of

$$d_j = L_j + \sum_{k,m}(g_{jkm}\cos(k\Omega + m\omega) + q_{jkm}\sin(k\Omega + m\omega)), \quad (7.115)$$

where g, q and L are functions of (a, e, i). All terms of g and q are long periodic terms; and the L terms are the secular effects. Or, to be more explicit (approximated to the order of e^2, the factor $3m\mu_m/r_m^3$ is omitted, $si = \sin i$, $ci = \cos i$, $o = \Omega$, $w = \omega$),

$$\begin{aligned}d_1 = -\frac{9}{32}a^2e^2&((n_1 + n_2(-1 + si))\cos(o - w) + (-n_1 + n_2(-1 - si))\cos(o + w) \\ &- ci\ n_1\sin(o - w) - 2n_3 si\ \sin(w) + ci\ n_1\sin(o + w))((-1 + ci)n_1\cos(o - w) \\ &- 2n_3 si\ \cos(w) + (1 + ci)n_1\cos(o + w) + n_2(-1 + si)\sin(o - w) \\ &+ n_2(1 + si)\sin(o + w))\end{aligned}$$

$$
\begin{aligned}
d_2 = & -\frac{1}{64}a^2((16n_1n_3si + e_2(-48n_1n_3si + n_2n_3(20si - 20si^2)))\cos(o - 2w) \\
& + (-8ci\, n_1^2 + n_1n_2(8 - 8si) + e^2(24ci\, n_1^2 + n_1n_2(-14 + 14si \\
& + ci(-10 + 10si))))\cos(2o - 2w) + (e^2n_1n_2(-20ci - 28si) + 16n_1n_2si)\cos 2w \\
& + (-16n_1n_3si + e^2(48n_1n_3si + n_2n_3(20si + 20si^2)))\cos(o + 2w) \\
& + (8ci\, n_1^2 + n_1n_2(-8 - 8si) + e^2(-24ci\, n_1^2 + n_1n_2(14 + ci(-10 - 10si) \\
& + 14si)))\cos(2o + 2w) + (-16n_1n_3si + e^2n_1n_3(28si + 20ci\, si))\sin(o - 2w) \\
& + ((-8 + 8ci)n_1^2 + n_1n_2(8 - 8si) + e^2((19 - 14ci - 5ci^2)n_1^2 \\
& + n_1n_2(-24 + 24si) + n_2^2(5 - 10si + 5si^2)))\sin(2o - 2w) \\
& + (-16n_1^2 - 16n_1n_2si + e^2((38 + 10ci^2)n_1^2 + 48n_1n_2si + 20n_3^2si^2 \\
& + n_2^2(-10 + 10si^2)))\sin 2w + (-16n_1n_3si + e^2n_1n_3(28si - 20ci\, si))\sin(o + 2w) \\
& + ((8 + 8ci)n_1^2 + n_1n_2(8 + 8si) + e^2((-19 - 14ci + 5ci^2)n_1^2 \\
& + n_1n_2(-24 - 24si) + n_2^2(-5 - 10si - 5si^2)))\sin(2o + 2w))
\end{aligned}
$$

$$
\begin{aligned}
d_3 = & -\frac{1}{8}a^2e((-1 + ci)n_1\cos(o - w) - 2n_3si\,\cos(w) + (1 + ci)n_1\cos(o + w) \\
& + n_2(-1 + si)\sin(o - w) + n_2(1 + si)\sin(o + w))^2
\end{aligned} \tag{7.116}
$$

$$
\begin{aligned}
d_4 = & -\frac{1}{64}a^2((16n_1n_3si + 32n_2n_3si^2 + e^2(-48n_1n_3si - 76n_2n_3si^2))\cos(o) \\
& + (-16ci\, n_1^2 + n_1n_2(16 - 32ci\, si) + e^2(48ci\, n_1^2 + n_1n_2(-28 + 76ci\, si)))\cos(2o) \\
& + (-8n_1n_3si + e^2(24n_1n_3si + n_2n_3(-10si + 10si^2)))\cos(o - 2w) \\
& + (8ci\, n_1^2 + e^2(-24ci\, n_1^2 + n_1n_2(14 + ci(10 - 10si) - 14si)) \\
& + n_1n_2(-8 + 8si))\cos(2o - 2w) + (-8n_1n_3si + e^2(24n_1n_3si \\
& + n_2n_3(10si + 10si^2)))\cos(o + 2w) + (8ci\, n_1^2 + n_1n_2(-8 - 8si) \\
& + e^2(-24ci\, n_1^2 + n_1n_2(14 + ci(-10 - 10si) + 14si)))\cos(2o + 2w) \\
& + (-32ci\, n_1n_3si + 76ci\, e^2n_1n_3si)\sin(o) + ((-16 + 16ci^2)n_1^2 - 16n_1n_2si \\
& + n_2^2(16 - 16si^2) + e^2((38 - 38ci^2)n_1^2 + 48n_1n_2si + n_2^2(-38 + 38si^2)))\sin(2o) \\
& + (8n_1n_3si + e^2n_1n_3(-14si - 10ci\, si))\sin(o - 2w) + ((8 - 8ci)n_1^2 \\
& + n_1n_2(-8 + 8si) + e^2((-19 + 14ci + 5ci^2)n_1^2 + n_1n_2(24 - 24si) \\
& + n_2^2(-5 + 10si - 5si^2)))\sin(2o - 2w) + (-8n_1n_3si + e^2n_1n_3(14si \\
& - 10ci\, si))\sin(o + 2w) + ((8 + 8ci)n_1^2 + n_1n_2(8 + 8si) + e^2((-19 - 14ci \\
& + 5ci^2)n_1^2 + n_1n_2(-24 - 24si) + n_2^2(-5 - 10si - 5si^2)))\sin(2o + 2w))
\end{aligned}
$$

$$
\begin{aligned}
d_5 = {} & \frac{a^2}{32}(-ci\, n_3 - n_1 si \cos(o) + ci\, n_2 \sin(o))(-16n_3 si + 38e^2 n_3 si \\
& + (16ci\, n_1 - 38ci\, e^2 n_1)\cos(o) + (-4n_1 + (7 + 5ci)e^2 n_1)\cos(o - 2w) \\
& - 10e^2 n_3 si \cos(2w) + (4n_1 + (-7 + 5ci)e^2 n_1)\cos(o + 2w) + (8n_1 + 16n_2 si \\
& + e^2(-24n_1 - 38n_2 si))\sin(o) + (-4n_1 + e^2(12n_1 + n_2(-5 + 5si)))\sin(o - 2w) \\
& + (-4n_1 + e^2(12n_1 + n_2(5 + 5si)))\sin(o + 2w))
\end{aligned}
$$

$$
\begin{aligned}
d_6 = {} & \frac{a}{64}((16 + 16ci_2)n_1^2 + 16n_1 n_2 si + 32n_3^2 si^2 + n_2^2(16 + 16si^2) \\
& + e^2((-38 - 38ci^2)n_1^2 - 48n_1 n_2 si - 76n_3^2 si^2 + n_2^2(-38 - 38si^2)) \\
& + (-64ci\, n_1 n_3 si + 152ci\, e^2 n_1 n_3 si)\cos(o) + ((-16 + 16ci^2)n_1^2 - 16n_1 n_2 si \\
& + n_2^2(16 - 16si^2) + e^2((38 - 38ci^2)n_1^2 + 48n_1 n_2 si + n_2^2(-38 + 38si^2)))\cos(2o) \\
& + (16n_1 n_3 si + e^2 n_1 n_3(-28si - 20ci\, si))\cos(o - 2w) + ((8 - 8ci)n_1^2 \\
& + n_1 n_2(-8 + 8si) + e^2((-19 + 14ci + 5ci^2)n_1^2 + n_1 n_2(24 - 24si) \\
& + n_2^2(-5 + 10si - 5si^2)))\cos(2o - 2w) + (-16n_1^2 - 16n_1 n_2 si \\
& + e^2((38 + 10ci^2)n_1^2 + 48n_1 n_2 si + 20n_2^2 si^2 + n_2^2(-10 + 10si^2)))\cos(2w) \\
& + (-16n_1 n_3 si + e^2 n_1 n_3(28si - 20ci\, si))\cos(o + 2w) + ((8 + 8ci)n_1^2 \\
& + n_1 n_2(8 + 8si) + e^2((-19 - 14ci + 5ci^2)n_1^2 + n_1 n_2(-24 - 24si) \\
& + n_2^2(-5 - 10si - 5si^2)))\cos(2o + 2w) + (-32n_1 n_3 si - 64n_2 n_3 si^2 \\
& + e^2(96n_1 n_3 si + 152n_2 n_3 si^2))\sin(o) + (16ci\, n_1^2 + n_1 n_2(-16 + 32ci\, si) \\
& + e^2(-48ci\, n_1^2 + n_1 n_2(28 - 76ci\, si)))\sin(2o) + (16n_1 n_3 si \\
& + e_2(-48n_1 n_3 si + n_2 n_3(20si - 20si^2)))\sin(o - 2w) + (-8ci\, n_1^2 + n_1 n_2(8 - 8si) \\
& + e^2(24ci\, n_1^2 + n_1 n_2(-14 + 14si + ci(-10 + 10si))))\sin(o - 2w) \\
& + (-16n_1 n_2 si + e^2 n_1 n_2(20ci + 28si))\sin(2w) + (16n_1 n_3 si + e^2(-48n_1 n_3 si \\
& + n_2 n_3(-20si - 20si^2)))\sin(o + 2w) + (-8ci\, n_1^2 + n_1 n_2(8 + 8si) \\
& + e^2(24ci\, n_1^2 + n_1 n_2(-14 - 14si + ci(10 + 10si))))\sin(2o + 2w))
\end{aligned}
$$

The most important terms are the secular effects included in (7.116) which are represented in L cf. (7.115)) and which are zero except for the following four terms (simplified using the fact that $n_1^2 + n_2^2 + n_3^2 = 1$):

$$L_1 = -n_1^2 - n_1 n_2 \sin i$$

$$
\begin{aligned}
L_3 &= a^2 e\left(-\frac{3n_1^2}{16} - \frac{3n_2^2}{16} - \frac{n_3^2}{8}\right) + a^2 e\left(-\frac{n_1^2}{16} + \frac{n_2^2}{16} + \frac{n_3^2}{8}\right)\cos 2i \\
&= \frac{a^2 e}{16}((-3 + n_3^2) + (-n_1^2 + n_2^2 + 2n_3^2)\cos 2i)
\end{aligned}
$$

$$L_5 = a^2\left(\frac{n_1n_2}{8} - \frac{3}{8}e^2n_1n_2\right)\cos i$$
$$+ a^2\left(-\frac{n_1^2}{8} + \frac{n_1^2}{8} + \frac{n_1^2}{4} + e^2\left(\frac{19n_1^2}{64} - \frac{19n_2^2}{64} - \frac{19n_3^2}{32}\right)\right)\sin 2i$$

$$L_6 = a\left(\frac{3n_1^2}{8} + \frac{3n_2^2}{8} + \frac{n_3^2}{4} + e^2\left(-\frac{57n_1^2}{64} - \frac{57n_2^2}{64} - \frac{19n_3^2}{32}\right)\right) + a\left(\frac{n_1^2}{8} - \frac{n_2^2}{8} - \frac{n_3^2}{4}\right.$$
$$\left.+e^2\left(-\frac{19n_1^2}{64} + \frac{19n_2^2}{64} + \frac{19n_3^2}{32}\right)\right)\cos 2i + a\left(\frac{n_1n_2}{4} - \frac{3}{4}e^2n_1n_2\right)\sin i$$
$$= a\left(\frac{3 - {n_3}^2}{8} + e^2\left(-\frac{57 + 19n_3^2}{64}\right)\right) + a\left(\frac{n_1^2}{8} - \frac{n_2^2}{8} - \frac{n_3^2}{4} + e^2\left(-\frac{19n_1^2}{64} + \frac{19n_2^2}{64}\right.\right.$$
$$\left.\left.+\frac{19n_3^2}{32}\right)\right)\cos 2i + a\left(\frac{n_1n_2}{4} - \frac{3}{4}e^2n_1n_2\right)\sin i \tag{7.117}$$

The integration of (7.117) can be made directly with respect to time t (the results of indefinite integrals are equal to L_jt). The results of the secular perturbations are combinations of (7.117) according to (7.94).

Coincidentally, similar to that of the solutions of the first term given in (7.103), both the orientation parameter ω and the mean anomaly are disturbed linearly. Comparing (7.103) with the combinations of (7.117) (according to (7.94)), it is notable that the linear terms in (7.103) are generally (by a factor of $1/(ae)$) smaller. This means that the linear effects are dominated by the combined (7.117) under the lunar (and/or solar) gravitational perturbation. Equation 7.117 indicates that the ascending node also subjected a linear disturbance. Most surprising is that the semi-major axis a of the orbital ellipse was also disturbed by multi-body linearly.

It is notable that the solutions of the second term multi-body perturbations on satellite orbits given above are solved by using basic Lagrangian equations of motion (7.95).

7.5.5 Solutions of the Third Term

Denote the third term of the potential function (7.92) as V_{m3}; then one has

$$\frac{\partial V_{m3}}{\partial \sigma_j} = m\mu_m \frac{15}{2r_m^4} r^2\cos^2\alpha \frac{\partial r\cos\alpha}{\partial \sigma_j}. \tag{7.118}$$

Comparing (7.118) with (7.108) one notes that using (7.118) and (7.108) will lead only to a factor difference in (7.109) and this factor is $5r\cos\alpha/(2r_m)$. Multiplying this factor by (7.109) and substituting it into (7.95), the basic Lagrangian equations of motion can be obtained. Because (7.114) are derived by using a program under symbolic mathematical operation software, to get the equations with respect to third term potential one just needs to modify the program slightly and the equations have the form of (7.112).

Or, to be explicit (approximated to the order of e^2, the factor $(15/2)m\mu_m/r_m^4$ is omitted),

$$\begin{aligned}\frac{\mathrm{d}a_1}{\mathrm{d}t} = &-\frac{1}{32}a^3((-16A^2B-16B^3)e+(-8A^2B-8B^3+(16A^2B-20B^3)e^2)\cos M\\&+(-24A^2B+8B^3+(82A^2B+8B^3)e^2)\cos 3M+(-48A^2B+16B^3)e\cos 4M\\&+(-18A^2B+12B^3)e^2\cos 5M+(8A^3+8AB^2+(2A^3-70AB^2)e^2)\sin M\\&+(8A^3-24AB^2+(-27A^3+47AB^2)e^2)\sin 3M+(16A^3-48AB^2)e\sin 4M\\&+(3A^3-27AB^2)e^2\sin 5M)\end{aligned} \tag{7.119}$$

$$\begin{aligned}\frac{\mathrm{d}e_1}{\mathrm{d}t} = &-\frac{1}{32}a^2((12A^2A_\omega+4A_\omega B^2+8ABB_\omega)+(-24A^2A_\omega-8A_\omega B^2-16ABB_\omega\\&+(112A^2A_\omega+28A_\omega B^2+56ABB_\omega)e^2)\cos M+(-16A_\omega B^2\\&-32ABB_\omega)e\cos 2M+(-8A^2A_\omega+8A_\omega B^2+16BB_\omega+(96A^2A_\omega-30A_\omega B^2\\&-60ABB_\omega)e^2)\cos 3M+(-12A^2A_\omega+12A_\omega B^2+24ABB_\omega)e\cos 4M\\&+(16A^2A_\omega+2A_\omega B^2+4ABB_\omega)e^2\cos 5M+(-16AA_\omega B-8A^2B_\omega\\&-24B^2B_\omega+(12AA_\omega B+6A^2B_\omega+98B^2B_\omega)e^2)\sin M+(16AA_\omega B+8A^2B_\omega\\&-24B^2B_\omega)e\sin 2M+(-16AA_\omega B-8A^2B_\omega+8B^2B_\omega+(90AA_\omega B\\&+45A^2B_\omega-51B^2B_\omega)e^2)\sin 3M+(-24AA_\omega B-12A^2B_\omega+12B^2B_\omega)e\sin 4M\\&+(14AA_\omega B+7A^2B_\omega+11B^2B_\omega)e^2\sin 5M)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}\omega_1}{\mathrm{d}t} = &\frac{a^3}{64}(-40A^3-40AB^2+(88A^3+84AB^2)e^2+(-72A^3-88AB^2)e\cos M\\&+(-32A^3+64AB^2+(204A^3-278AB^2)e^2)\cos 2M+(-76A^3\\&+148AB^2)e\cos 3M+(8A^3-24AB^2+(8A^3+236AB^2)e^2)\cos 4M\\&+(20A^3-60AB^2)e\cos 5M+(20A^3-42AB^2)e^2\cos 6M+(8A^2B-8B^3)e\sin M\\&+(-80A^2B+16B^3+(127A^2B-31B^3)e^2)\sin 2M+(-188A^2B+36B^3)e\sin 3M\\&+(24A^2B-8B^3+(-196A^2B+32B^3)e^2)\sin 4M+(60A^2B-20B^3)e\sin 5M\\&+(51A^2B-11B^3)e^2\sin 6M)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}i_1}{\mathrm{d}t} = \frac{1}{-32}a^3((12A^2A_\Omega + 4A_\Omega B^2 + 8ABB_\Omega)(-24A^2A_\Omega - 8A_\Omega B^2 - 16ABB_\Omega \\ + (112A^2A_\Omega + 28A_\Omega B^2 + 56ABB_\Omega)e^2)\cos M + (-16A_\Omega B^2 - 32ABB_\Omega e\cos 2M \\ + (-8A^2A_\Omega + 8A_\Omega B^2 + 16ABB_\Omega + (96A^2A_\Omega - 30A_\Omega B^2 - 60ABB_\Omega)e^2)\cos 3M \\ + (-12A^2A_\Omega + 12A_\Omega B^2 + 24ABB_\Omega)e\cos 4M + (16A^2A_\Omega + 2A_\Omega B^2 \\ + 4ABB_\Omega)e^2\cos 5M + (-16AA_\Omega B - 8A^2B_\Omega - 24B^2B_\Omega + (12AA_\Omega B + 6A^2B_\Omega \\ + 98B^2B_\Omega)e^2)\sin M + (16AA_\Omega B + 8A^2B_\Omega - 24B^2B_\Omega)e\sin 2M \\ + (-16AA_\Omega B - 8A^2B_\Omega + 8B^2B_\Omega + (90AA_\Omega B + 45A^2B_\Omega - 51B^2B_\Omega)e^2)\sin 3M \\ + (-24AA_\Omega B - 12A^2B_\Omega + 12B^2B_\Omega)e\sin 4M + (14AA_\Omega B + 7A^2B_\Omega \\ + 11B^2B_\Omega)e^2\sin 5M)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} = -\frac{1}{32}a^3((12A^2A_i + 4A_iB^2 + 8A\,B\,B_i)e + (-24A^2A_i - 8A_iB^2 - 16A\,B\,B_i \\ + (112A^2A_i + 28A_iB^2 + 56A\,B\,B_i)e^2)\cos M + (-16A_iB^2 - 32A\,B\,B_i)e\cos 2M \\ + (-8A^2A_i + 8A_iB^2 + 16A\,B\,B_i + (96A^2A_i - 30A_iB^2 - 60A\,B\,B_i)e^2)\cos 3M \\ + (-12A^2A_i + 12A_iB^2 + 24A\,B\,B_i)e\cos 4M + (16A^2A_i + 2A_iB^2 \\ + 4A\,B\,B_i)e^2\cos 5M + (-16A\,A_iB - 8A^2B_i - 24B^2B_i + (12A\,A_iB + 6A^2B_i \\ + 98B^2B_i)e^2)\sin M + (16A\,A_iB + 8A^2B_i - 24B^2B_i)e\sin 2M \\ + (-16A\,A_iB - 8A^2B_i + 8B^2B_i + (90A\,A_iB + 45A^2B_i - 51B^2B_i)e^2)\sin 3M \\ + (-24A\,A_iB - 12A^2B_i + 12B^2B_i)e\sin 4M + (14A\,A_iB + 7A^2B_i \\ + 11B^2B_i)e^2\sin 5M)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}M_1}{\mathrm{d}t} = -\frac{1}{16}a^2(A\cos M + B\sin M)(-8A^2 - 8B^2 + (16A^2 + 29B^2)e^2 \\ + (12A^2 - 12B^2)e\cos M + (-8A^2 + 8B^2 + (80A^2 - 40B^2)e^2)\cos 2M \\ + (-12A^2 + 12B^2)e\cos 3M + (16A^2 + 11B^2)e^2\cos 4M + 24A\,B\,e\sin M \\ + (-16A\,B + 66A\,B\,e^2)\sin 2M - 24A\,B\,e\sin 3M + 5A\,B\,e^2\sin 4M)\end{aligned}$$

The d-terms in (7.119) can be reduced to (approximated to the order of e^2, the factor $(15/2)m\mu_m/r_m^4$ is omitted, denote $i = \sqrt{-1}$)

$$
\begin{aligned}
d_1 = &-\frac{1}{16}a^3 e((((-1-i)+ci)n_1+n_2(i-i\,si))\cos(o-w)-2n_3 si\cos(w)\\
&+(((1+i)+ci)n_1+n_2(i+i\,si))\cos(o+w)+(i\,ci\,n_1+n_2(-1+si))\sin(o-w)\\
&+2i\,n_3\,si\sin(w)+(-i\,ci\,n_1+n_2(1+si))\sin(o+w))((((-1+i)+ci)n_1\\
&+n_2(-i+i\,si))\cos(o-w)-2n_3 si\cos(w)+(((1-i)+ci)n_1\\
&+n_2(-i-i\,si))\cos(o+w)+(-i\,ci\,n_1+n_2(-1+si))\sin(o-w)-2i\,n_3 si\sin(w)\\
&+(i\,ci\,n_1+n_2(1+si))\sin(o+w))((n_1+n_2(-1+si))\cos(o-w)+(-n_1\\
&+n_2(-1-si))\cos(o+w)-ci\,n_1\sin(o-w)-2n_3 si\sin(w)+ci\,n_1\sin(o+w))
\end{aligned}
\tag{7.120}
$$

$$
\begin{aligned}
d_2 = &\frac{a^3}{128e}(((-3ci-3ci^2)n_1^3+n_1^2n_2(6-18si+ci(6+6si))\\
&+n_1(-12n_3^2si^2+n_2^2(3+6si-9si^2)))\cos(o-3w)+(n_1^2n_3(-6si+12ci\ si)\\
&+n_1n_2n_3(-12si+12si^2))\cos(2o-3w)+((3ci-3ci^2)n_1^3\\
&+n_1^2n_2(-6+ci(6-6si)+6si)+n_1n_2^2(3-6si+3si^2))\cos(3o-3w)\\
&+((3ci+ci^2)n_1^3+n_2^3(-6+2si-2si^2+6si^3)+n_2n_3^2(-8si^2+24si^3)\\
&+n_1^2n_2(-6+6si+ci^2(-6+6si)+ci(-2+6si))\\
&+n_1(4n_3^2si^2+n_2^2(-1-2si+3si^2)))\cos(o-w)+(n_1^2n_3(6si-4ci\ si)\\
&+n_1n_2n_3(4si+12si^2+ci(8si-24si^2)))\cos(2o-w)+((-3ci+ci^2)n_1^3\\
&+n_1n_2^2(-1+2si-si^2)+n_2^3(-2+2si+2si^2-2si^3)\\
&+n_1^2n_2(6+ci(-2-6si)-2si+ci^2(-2+6si)))\cos(3o-w)\\
&+(-12n_1^2n_3si+n_1n_2n_3(16ci\ si-24si^2))\cos(w)\\
&+(12n_1^2n_3si-24n_1n_2n_3si^2)\cos(3w)+((3ci-ci^2)n_1^3\\
&+n_2n_3^2(-8si^2-24si^3)+n_2^3(-6-2si-2si^2-6si^3)\\
&+n_1^2n_2(-6+ci^2(-6-6si)-6si+ci(2+6si))\\
&+n_1(-4n_3^2si^2+n_2^2(1-2si-3si^2)))\cos(o+w)\\
&+(n_1^2n_3(6si+4ci\ si)+n_1n_2n_3(-4si+12si^2\\
&+ci(8si+24si^2)))\cos(2o+w)+((-3ci-ci^2)n_1^3+n_1^2n_2(6+ci^2(-2-6si)\\
&+ci(2-6si)+2si)+n_1n_2^2(1+2si+si^2)\\
&+n_2^3(-2-2si+2si^2+2si^3))\cos(3o+w)+((-3ci+3ci^2)n_1^3\\
&+n_1^2n_2(6+18si+ci(-6+6si))+n_1(12n_3^2si^2\\
&+n_2^2(-3+6si+9si^2)))\cos(o+3w)+(n_1^2n_3(-6si-12ci\ si)\\
&+n_1n_2n_3(12si+12si^2))\cos(2o+3w)+((3ci+3ci^2)n_1^3
\end{aligned}
$$

$$
\begin{aligned}
&+ n_1^2 n_2(-6 + ci(-6 - 6si) - 6si) + n_1 n_2^2(-3 - 6si - 3si^2))\cos(3o + 3w) \\
&+ ((-9 + 6ci + 3ci^2)n_1^3 + n_1^2 n_2(3 - 9si + ci(6 + 6si)) + n_1(12n_3^2 si^2 \\
&+ n_2^2(-3 - 6si + 9si^2)))\sin(o - 3w) + (n_1^2 n_3(12si - 12ci\ si) \\
&+ n_1 n_2 n_3(-12si + 12si^2))\sin(2o - 3w) + ((3 - 6ci + 3ci^2)n_1^3 \\
&+ n_1^2 n_2(-3 + ci(6 - 6si) + 3si) + n_1 n_2^2(-3 + 6si - 3si^2))\sin(3o - 3w) \\
&+ ((9 - 2ci + 3ci^2 - 6ci^3)n_1^3 + n_1^2 n_2(-1 + ci(-2 - 2si) + 9si) \\
&+ n_1(n_3^2(12si^2 - 24ci\ si^2) + n_2^2(5 + 2si + 9si^2 + ci(-6 - 4si \\
&- 6si^2))))\sin(o - w) + (n_1^2 n_3(-4si - 12ci\ si + 12ci^2 si) + n_1 n_2 n_3(4si \\
&- 4si^2) + n_2^2 n_3(4si + 8si^2 - 12si^3))\sin(2o - w) + ((-3 + 2ci + 3ci^2 \\
&- 2ci^3)n_1^3 + n_1^2 n_2(1 - 3si + ci(-2 + 2si)) + n_1 n_2^2(5 - 2si - 3si^2 \\
&+ ci(-2 - 4si + 6si^2)))\sin(3o - w) + (-8n_1 n_2 n_3 si^2 - 16n_3^3 si^3 + n_1^2 n_3(-8si \\
&- 24ci^2 si) + n_2^2 n_3(-8si - 24si^3))\sin(w) + (24n_1^2 n_3 si + 24n_1 n_2 n_3 si^2)\sin 3w \\
&+ ((9 + 2ci + 3ci^2 + 6ci^3)n_1^3 + n_1^2 n_2(1 + 9si + ci(-2 + 2si)) \\
&+ n_1(n_3^2(12si^2 + 24ci\ si^2) + n_2^2(5 - 2si + 9si^2 + ci(6 - 4si \\
&+ 6si^2))))\sin(o + w) + (n_1^2 n_3(4si - 12ci\ si - 12ci^2 si) \\
&+ n_1 n_2 n_3(4si + 4si^2) + n_2^2 n_3(-4si + 8si^2 + 12si^3))\sin(2o + w) \\
&+ ((-3 - 2ci + 3ci^2 + 2ci^3)n_1^3 + n_1^2 n_2(-1 + ci(-2 - 2si) - 3si) \\
&+ n_1 n_2^2(5 + 2si - 3si^2 + ci(2 - 4si - 6si^2)))\sin(3o + w) \\
&+ ((-9 - 6ci + 3ci^2)n_1^3 + n_1^2 n_2(-3 + ci(6 - 6si) - 9si) + n_1(12n_3^2 si^2 \\
&+ n_2^2(-3 + 6si + 9si^2)))\sin(o + 3w) + (n_1^2 n_3(-12si - 12ci\ si) \\
&+ n_1 n_2 n_3(-12si - 12si^2))\sin(2o + 3w) + ((3 + 6ci + 3ci^2)n_1^3 \\
&+ n_1 n_2^2(-3 - 6si - 3si^2) + n_1^2 n_2(3 + 3si + ci(6 + 6si)))\sin(3o + 3w))
\end{aligned}
$$

$$
\begin{aligned}
d_3 =& \frac{a^3}{256}((-1+ci)n_1\cos(o-w) - 2n_3 si\cos(w) + (1+ci)n_1\cos(o+w) \\
&+ n_2(-1+si)\sin(o-w) + n_2(1+si)\sin(o+w))((-40-40ci^2)n_1^2 - 40n_1 n_2 si \\
&- 80n_3^2 si^2 + n_2^2(-40-40si^2) + e^2((86+86ci^2)n_1^2 + 84n_1 n_2 si + 172n_3^2 si^2 \\
&+ n_2^2(86+86si^2)) + (160ci\, n_1 n_3 si - 344ci\, e^2 n_1 n_3 si)\cos(o) + ((40-40ci^2)n_1^2 \\
&+ 40n_1 n_2 si + n_2^2(-40+40si^2) + e^2((-86+86ci^2)n_1^2 - 84n_1 n_2 si \\
&+ n_2^2(86-86si^2)))\cos(2o) + (-40n_1 n_3 si + e^2 n_1 n_3(88si - 4ci\, si))\cos(o-2w) \\
&+ ((-20+20ci)n_1^2 + n_1 n_2(20-20si) + e^2((43-44ci+ci^2)n_1^2 \\
&+ n_1 n_2(-42+42si) + n_2^2(-1+2si-si^2)))\cos(2o-2w) + (40{n_1}^2 + 40n_1 n_2 si
\end{aligned}
$$

$$
\begin{aligned}
&+e^2((-86+2ci^2)n_1^2-84n_1n_2si+4n_3^2si^2+n_2^2(-2+2si^2)))\cos(2w)+(40n_1n_3si\\
&+e^2n_1n_3(-88si-4ci\,si))\cos(o+2w)+((-20-20ci)n_1^2+n_1n_2(-20-20si)\\
&+e^2((43+44ci+ci^2)n_1^2+n_1n_2(42+42si)+n_2^2(-1-2si-si^2)))\cos(2o+2w)\\
&+(80n_1n_3si+160n_2n_3si^2+e^2(-168n_1n_3si-344n_2n_3si^2))\sin(o)\\
&+(-40ci\,n_1^2+n_1n_2(40-80ci\,si)+e^2(84ci\,n_1^2+n_1n_2(-88+172ci\,si)))\sin(2o)\\
&+(-40n_1n_3si+e^2(84n_1n_3si+n_2n_3(4si-4si^2)))\sin(o-2w)+(20ci\,n_1^2\\
&+n_1n_2(-20+20si)+e^2(-42ci\,n_1^2+n_1n_2(44-44si+ci(-2\\
&+2si))))\sin(2o-2w)+(e^2n_1n_2(4ci-88si)+40n_1n_2si)\sin 2w\\
&+(-40n_1n_3si+e^2(84n_1n_3si+n_2n_3(-4si-4si^2)))\sin(o+2w)+(20ci\,n_1^2\\
&+n_1n_2(-20-20si)+e^2(-42ci\,n_1^2+n_1n_2(44+44si+ci(2+2si))))\sin(2o+2w)
\end{aligned}
$$

$$
\begin{aligned}
d_4=\frac{a^3e}{128}&(((ci+ci^2)n_1^3+n_1^2n_2(-2+ci(-2-2si)+6si)+n_1(4n_3^2si^2\\
&+n_2^2(-1-2si+3si^2)))\cos(o-3w)+(n_1^2n_3(4si-8ci\,si)\\
&+n_1n_2n_3(8si-8si^2))\cos(2o-3w)+((-3ci+3ci^2)n_1^3+n_1n_2^2(-3+6si-3si^2)\\
&+n_1^2n_2(6-6si+ci(-6+6si)))\cos(3o-3w)+((-3ci-ci^2)n_1^3+n_1^2n_2(6\\
&+ci(2-6si)+ci^2(6-6si)-6si)+n_2n_3^2(8si^2-24si^3)+n_2^3(6-2si+2si^2-6si^3)\\
&+n_1(-4n_3^2si^2+n_2^2(1+2si-3si^2)))\cos(o-w)+(n_1^2n_3(-12si+8ci\,si)\\
&+n_1n_2n_3(-8si-24si^2+ci(-16si+48si^2)))\cos(2o-w)+((9ci-3ci^2)n_1^3\\
&+n_1n_2^2(3-6si+3si^2)+n_2^3(6-6si-6si^2+6si^3)+n_1^2n_2(-18+ci^2(6-18si)\\
&+6si+ci(6+18si)))\cos(3o-w)+((3ci-ci^2)n_1^3+n_2n_3^2(-8si^2-24si^3)\\
&+n_2^3(-6-2si-2si^2-6si^3)+n_1^2n_2(-6+ci^2(-6-6si)-6si+ci(2+6si))\\
&+n_1(-4n_3^2si^2+n_2^2(1-2si-3si^2)))\cos(o+w)+(n_1^2n_3(12si+8ci\,si)\\
&+n_1n_2n_3(-8si+24si^2+ci(16si+48si^2)))\cos(2o+w)+((-9ci-3ci^2)n_1^3\\
&+n_1^2n_2(18+ci^2(-6-18si)+ci(6-18si)+6si)+n_1n_2^2(3+6si+3si^2)\\
&+n_2^3(-6-6si+6si^2+6si^3))\cos(3o+w)+((-ci+ci^2)n_1^3+n_1^2n_2(2+6si\\
&+ci(-2+2si))+n_1(4n_3^2si^2+n_2^2(-1+2si+3si^2)))\cos(o+3\ )\\
&+(n_1^2n_3(-4si-8ci\,si)+n_1n_2n_3(8si+8si^2))\cos(2o+3w)+((3ci+3ci^2)n_1^3\\
&+n_1^2n_2(-6+ci(-6-6si)-6si)+n_1n_2^2(-3-6si-3si^2))\cos(3o+3w)\\
&+((3-2ci-ci^2)n_1^3+n_1^2n_2(-1+ci(-2-2si)+3si)+n_1(-4n_3^2si^2\\
&+n_2^2(1+2si-3si^2)))\sin(o-3w)+(n_1^2n_3(-8si+8ci\,si)\\
&+n_1n_2n_3(8si-8si^2))\sin(2o-3w)+((-3+6ci-3ci^2)n_1^3+n_1n_2^2(3-6si+3si^2)
\end{aligned}
$$

$$
\begin{aligned}
&+n_1^2n_2(3-3si+ci(-6+6si)))\sin(3o-3w)+((-9+2ci-3ci^2\\
&+6ci^3)n_1^3+n_1^2n_2(1-9si+ci(2+2si))+n_1(n_3^2(-12si^2+24ci\,si^2)\\
&+n_2^2(-5-2si-9si^2+ci(6+4si+6si^2))))\sin(o-w)+(n_1^2n_3(8si\\
&+24ci\,si-24ci^2si)+n_1n_2n_3(-8si+8si^2)+n_2^2n_3(-8si-16si^2\\
&+24si^3))\sin(2o-w)+((9-6ci-9ci^2+6ci^3)n_1^3+n_1^2n_2(-3+ci(6-6si)\\
&+9si)+n_1n_2^2(-15+6si+9si^2+ci(6+12si-18si^2)))\sin(3o-w)\\
&+((9+2ci+3ci^2+6ci^3)n_1^3+n_1^2n_2(1+9si+ci(-2+2si))+n_1(n_3^2(12si^2\\
&+24ci\,si^2)+n_2^2(5-2si+9si^2+ci(6-4si+6si^2))))\sin(o+w)\\
&+(n_1^2n_3(8si-24ci\,si-24ci^2si)+n_1n_2n_3(8si+8si^2)+n_2^2n_3(-8si+16si^2\\
&+24si^3))\sin(2o+w)+((-9-6ci+9ci^2+6ci^3)n_1^3+n_1^2n_2(-3\\
&+ci(-6-6si)-9si)+n_1n_2^2(15+6si-9si^2+ci(6-12si\\
&-18si^2)))\sin(3o+w)+((-3-2ci+ci^2)n_1^3+n_1^2n_2(-1+ci(2-2si)-3si)\\
&+n_1(4n_3^2si^2+n_2^2(-1+2si+3si^2)))\sin(o+3w)+(n_1^2n_3(-8si-8ci\,si)\\
&+n_1n_2n_3(-8si-8si^2))\sin(2o+3w)+((3+6ci+3ci^2)n_1^3\\
&+n_1n_2^2(-3-6si-3si^2)+n_1^2n_2(3+3si+ci(6+6si)))\sin(3o+3w))
\end{aligned}
$$

$$
\begin{aligned}
d_5=&\frac{-1}{64}a^3e(-ci\,n_3-n_1si\cos(o)+ci\,n_2\sin(o))(4n_1n_3si\cos(o-3w)\\
&+((2-2ci)n_1^2+n_1n_2(-2+2si))\cos(2o-3w)+n_1n_3(12si-24ci\,si)\cos(o-w)\\
&+((-2-6ci+6ci^2)n_1^2+n_1n_2(2-2si)+n_2^2(2+4si-6si^2))\cos(2o-w)\\
&+((4+12ci^2)n_1^2+4n_1n_2si+24n_3^2si^2+n_2^2(4+12si^2))\cos(w)\\
&+(-4n_1^2-4n_1n_2si)\cos3w+n_1n_3(-12si-24ci\,si)\cos(o+w)\\
&+((-2+6ci+6ci^2)n_1^2+n_1n_2(-2-2si)+n_2^2(2-4si-6si^2))\cos(2o+w)\\
&-4n_1n_3si\cos(o+3w)+((2+2ci)n_1^2+n_1n_2(2+2si))\cos(2o+3w)\\
&+4n_1n_3si\sin(o-3w)+((1-2ci)n_1^2+n_1n_2(2-2si))\sin(2o-3w)\\
&+(-4n_1n_3si+n_2n_3(8si-24si^2))\sin(o-w)+((-3+2ci)n_1^2+n_1n_2(-2-6si\\
&+ci(-4+12si)))\sin(2o-w)+(-6n_1^2+n_1n_2(8ci-12si))\sin(w)\\
&+(2n_1^2-4n_1n_2si)\sin3w+(-4n_1n_3si+n_2n_3(-8si-24si^2))\sin(o+w)\\
&+((3+2ci)n_1^2+n_1n_2(-2+6si+ci(4+1\,si)))\sin(2o+w)+4n_1n_3si\sin(o+3w)\\
&+((-1-2ci)n_1^2+n_1n_2(2+2si))\sin(2o+3w))
\end{aligned}
$$

$$\begin{aligned}
d_6 = &-\frac{3}{64}a^2 e((((-1-i)+ci)n_1+n_2(i-i\,si))\cos(o-w)-2n_3 si\cos(w)\\
&+(((1+i)+ci)n_1+n_2(i+i\,si))\cos(o+w)+(i\,ci\,n_1+n_2(-1+si))\sin(o-w)\\
&+2i\,n_3\,si\sin(w)+(-i\,ci\,n_1+n_2(1+si))\sin(o+w))((((-1+i)+ci)n_1\\
&+n_2(-i+i\,si))\cos(o-w)-2n_3 si\cos(w)+(((1-i)+ci)n_1\\
&+n_2(-i-i\,si))\cos(o+w)+(-i\,ci\,n_1+n_2(-1+si))\sin(o-w)-2i\,n_3 si\sin(w)\\
&+(i\,ci\,n_1+n_2(1+si))\sin(o+w))((-1+ci)n_1\cos(o-w)-2n_3 si\cos(w)\\
&+(1+ci)n_1\cos(o+w)+n_2(-1+si)\sin(o-w)+n_2(1+si)\sin(o+w))
\end{aligned}$$

The above formulas shown that all terms in d are long-periodic terms and all secular terms are zero. This means that the disturbances of the third term of the potential function are nonlinear ones.

7.5.6 Solutions of the Fourth Term

Denote the fourth term of the potential function (7.92) as V_{m4}; then one has

$$\frac{\partial V_{m4}}{\partial \sigma_j} = -m\mu_m \frac{3}{2r_m^4}\left(3r^2\cos\alpha\frac{\partial r}{\partial \sigma}+r^3\frac{\partial\cos\alpha}{\partial\sigma}\right). \tag{7.121}$$

Explicitly, the partial derivatives are cf. (4.24))

$$\frac{\partial V_{m4}}{\partial a} = -m\mu_m \frac{9}{2r_m^4}\frac{r^3}{a}\cos\alpha,$$

$$\frac{\partial V_{m4}}{\partial e} = -m\mu_m \frac{3r^2}{2r_m^4}(3\cos\alpha(-a\cos f)+r(-A\sin f+B\cos f)\frac{2+e\cos f}{1-e^2}\sin f)$$

$$\frac{\partial V_{m4}}{\partial \omega} = -m\mu_m \frac{3}{2r_m^4}r^3(A_\omega\cos f+B_\omega\sin f),$$

$$\frac{\partial V_{m4}}{\partial i} = -m\mu_m \frac{3}{2r_m^4}r^3(A_i\cos f+B_i\sin f),$$

$$\begin{aligned}
\frac{\partial V_{m4}}{\partial \Omega} &= -m\mu_m \frac{3}{2r_m^4}r^3(A_\Omega\cos f+B_\Omega\sin f),\ \text{and}\\
\frac{\partial V_{m4}}{\partial M} &= -m\mu_m \frac{3r}{2r_m^4}(3r\cos\alpha\frac{ae}{\sqrt{1-e^2}}\sin f\\
&\quad+(-A\sin f+B\cos f)a^2\sqrt{1-e^2}).
\end{aligned} \tag{7.122}$$

Comparing (7.122) with (7.109) one notices that the equations at (7.122) are simpler than (7.109). Again, to get the equations with respect to the fourth term potential, one just needs to modify slightly the program used in Sect. 7.4 and the equations have the form of (7.112). Or, to be explicit (approximated to the order of e^2, the factor $(3/2)m\mu_m/r_m^4$ is omitted),

$$\frac{\mathrm{d}a_1}{\mathrm{d}t} = \frac{a^3}{8}(-8B\ e + (-8B + 6B\ e^2)\cos M + 8B\ e\ \cos 2M + 22B\ e^2\cos 3M + (8A - A\ e^2)\sin M - 8A\ e\ \sin 2M - 13A\ e^2\sin 3M) \tag{7.123}$$

$$\frac{\mathrm{d}e_1}{\mathrm{d}t} = \frac{a^3}{8}(12A_\omega e + (-8A_\omega + 16A_\omega e^2)\cos M + 4A_\omega e\ \cos 2M + 16A_\omega e^2\cos 3M + (-8B_\omega + 15B_\omega e^2)\sin M + 4B_\omega e\ \sin 2M + 7B_\omega e^2\sin 3M)$$

$$\frac{\mathrm{d}\omega_1}{\mathrm{d}t} = -\frac{a^3}{16}(-40A + 88A\ e^2 + 12A\ e\ \cos M + (-8A + 160A\ e^2)\cos 2M - 12A\ e\ \cos 3M + 40A\ e^2\cos 4M + 4B\ e\ \sin M + (-8B + 22B\ e^2)\sin 2M - 12B\ e\ \sin 3M - 5B\ e^2\sin 4M)$$

$$\frac{\mathrm{d}i_1}{\mathrm{d}t} = \frac{a^3}{8}(12A_\Omega e + (-8A_\Omega + 16A_\Omega e^2)\cos M + 4A_\Omega e\ \cos 2M + 16A_\Omega e^2\cos 3M + (-8B_\Omega + 15B_\Omega e^2)\sin M + 4B_\Omega e\ \sin 2M + 7B_\Omega e^2\sin 3M)$$

$$\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} = \frac{a^3}{8}(12A_i e + (-8A_i + 16A_i e^2)\cos M + 4A_i e\ \cos 2M + 16A_i e^2\cos 3M + (-8B_i + 15B_i e^2)\sin M + 4B_i e\ \sin 2M + 7B_i e^2\sin 3M)$$

$$\frac{\mathrm{d}M_1}{\mathrm{d}t} = \frac{a^2}{8}(12A\ e + (-8A + 16A\ e^2)\cos M + 4A\ e\ \cos 2M + 16A\ e^2\cos 3M + (-8B + 15B\ e^2)\sin M + 4B\ e\ \sin 2M + 7B\ e^2\sin 3M)$$

$$d_1 = \frac{a^3}{2}e((n_1 + n_2(-1 + si))\cos(o - w) + (-n_1 + n_2(-1 - si))\cos(0 + w) - ci\, n_1\sin(o - w) - 2n_3 si\,\sin(w) + ci\, n_1\sin(o + w))$$

$$d_2 = \frac{3a^3}{4}e(n_2(1 - si)\cos(o - w) + n_2(1 + si)\cos(o + w) + (-1 + ci)n_1\sin(o - w) + 2n_3 si\,\sin(w) + (-1 - ci)n_1\sin(o + w))$$

$$d_3 = -\frac{a^3}{4}(-5 + 11e^2)((-1 + ci)n_1\cos(o - w) - 2n_3 si\,\cos(w) + (-1 + ci)n_1\sin(o + w) + n_2(-1 + si)\sin(o - w) + n_2(1 + si)\sin(o + w))$$

$$d_4 = \frac{3a^3}{4} e(n_2(-1+si)\cos(o-w) + n_2(1+si)\cos(o+w) + (1-ci)n_1\sin(o-w) + (-1-ci)n_1\sin(o+w))$$

$$d_5 = -\frac{3a^3}{2} e\cos(w)(ci\, n_3 + n_1\, si\cos(o) - ci\ n_2\sin(o))$$

$$d_6 = \frac{9a^2}{4} e((-1+ci)n_1\cos(o-w) - 2n_3 si\cos(w) + (1+ci)n_1\cos(o+w) + n_2(-1+si)\sin(o-w) + n_2(1+si)\sin(o+w))$$

Reducing the d-terms in (7.123) shows that all terms in d are long-periodic terms and all secular terms are zero. This means that the disturbances of the fourth term of the potential function are nonlinear ones.

7.5.7 Summary

Adding the solutions derived in Sects. 7.5.3, 7.5.4, 7.5.5, and 7.5.6 together, the analytical solutions of the lunar gravitational disturbance of the second order are obtained. Conclusions are obtained theoretically: the lunar gravitational disturbance will lead to linear changing with time on the orientation parameter ω of the satellite orbital ellipse and the satellite position parameter M. The semi-major axis and ascending node of the satellite ellipse are also perturbed by the lunar gravitation linearly. The solutions of the lunar gravitational disturbances of the second order derived here are valid directly for the solar and planetary gravitational disturbances of the third and fourth order on an Earth satellite, respectively. The derived analytical solutions show the disturbing effects with very clear spectral properties and reflect exactly the disturbed effects with conservative behaviour. For nonlinear effects, because of the periodic properties, computations can be carried out efficiently within a suitable interval. The linear perturbation effect of the perigee remembers the effect of the well-known mercury perihelion precession.

7.6 Ephemeris of the Moon, the Sun and Planets

The ephemeris of the sun and the moon as well as planets are needed for the computation of shadow functions of the sun and moon (solar radiation pressure) and the disturbance forces of the sun, the moon and planets. The computation of the ephemeris of the sun and the moon can be simplified by considering the orbit of the sun (indeed it is the Earth!) and the moon as Keplerian motion. Consider the orbital right-handed coordinate system, the origin in the geocentre, the xy-plane as the orbital plane, the x-axis pointing to the perigee, and the z-axis pointing in the direction of $\vec{q} \times \dot{\vec{q}}$ where $\vec{q}$ and $\dot{\vec{q}}$ are the position and velocity vectors of the sun or the moon. The two vectors are (see (3.41), (3.42))

$$\vec{q} = \begin{pmatrix} a(\cos E - e) \\ a\sqrt{1-e^2}\sin E \\ 0 \end{pmatrix} = \begin{pmatrix} q\cos f \\ q\sin f \\ 0 \end{pmatrix}, \quad \dot{\vec{q}} = \begin{pmatrix} -\sin f \\ e + \cos f \\ 0 \end{pmatrix} \frac{na}{\sqrt{1-e^2}}, \tag{7.124}$$

where

$$q = \frac{a(1-e^2)}{1 + e\cos f} \tag{7.125}$$

The position and velocity vectors of the sun or the moon in the ECEI and ECSF coordinate systems are then (see Sect. 2.5 and (3.43))

$$\begin{aligned} \begin{pmatrix} \vec{p} \\ \dot{\vec{p}} \end{pmatrix} &= R_3(-\Omega)R_1(-i)R_3(-\omega)\begin{pmatrix} \vec{q} \\ \dot{\vec{q}} \end{pmatrix}, \\ \begin{pmatrix} \vec{r} \\ \dot{\vec{r}} \end{pmatrix} &= R_1(-\varepsilon)\begin{pmatrix} \vec{p} \\ \dot{\vec{p}} \end{pmatrix} \end{aligned} \tag{7.126}$$

where a and i are the semi-major axis of the orbit and the inclination angle of the orbital plane of the moon or the sun in the ecliptic coordinate system (ECEI). Ω is the ecliptic right ascension of the ascending node, e is the eccentricity of the ellipse, ω is the argument of perigee, f is the true anomaly of the moon or the sun and ε is the mean obliquity (the formula is given in Sect. 2.4). Because the sun moves along the ecliptic and the ascending node is defined as the equinox, parameters i and Ω are zero. True anomaly f, eccentric anomaly E and mean anomaly M are given by the Keplerian equation and by the following formulas:

$$\begin{aligned} E - e\sin E &= M \\ q\cos f &= a\cos E - ae \\ q\sin f &= b\sin E = a\sqrt{1-e^2}\sin E. \end{aligned} \tag{7.127}$$

For the moon, eccentricity $e_m = 0.05490$, inclination $i_m = 5.^\circ 145396$ and semi-major axis $a_m = 384401$ km. For the sun, eccentricity $e_s = 0.016709114$–$0.000042052\ T$–$0.000000126\ T^2$ and semi-major axis $a_s = 1.0000002$ AU. AU signifies the astronomical units (AU $= 1.49597870691 \times 10^8$ km). The fundamental arguments are given in the IERS Conventions (see McCarthy 1996) as follows:

$$\begin{aligned} l &= 134.^\circ 96340251 + 1717915923.''2178T + 31.''8792T^2 + 0.''051635T^3 \\ &\quad - 0.''00024470T^4 \end{aligned}$$

$$\begin{aligned} l' &= 357.^\circ 52910918 + 129596581.''0481T - 0.''5532T^2 + 0.''000136T^3 \\ &\quad - 0.''00001149T^4 \end{aligned}$$

$$\begin{aligned} F &= 93.^\circ 27209062 + 1739527262.''8478T - 12.''7512T^2 - 0.''001037T^3 \\ &\quad + 0.''00000417T^4 \end{aligned}$$

$$\begin{aligned} D &= 297.^\circ 85019547 + 1602961601.''2090T - 6.''3706T^2 + 0.''006593T^3 \\ &\quad - 0.''00003169T^4 \end{aligned}$$

$$\begin{aligned} \Omega &= 125.^\circ 04455501 - 6962890.''2665T + 7.''4722T^2 \\ &\quad + 0.''007702T^3 - 0.''00005939T^4 \end{aligned} \tag{7.128}$$

where l and l' are the mean anomalies of the moon and the sun, respectively. D is the mean elongation of the moon from the sun. Ω is the mean longitude of the ascending node of the moon. $F = L - \Omega$, L is the mean longitude of the moon (or L_{moon}), and T is the Julian centuries measured from epoch J2000.0. Formulas of (7.128) are the arguments used to compute the nutation. Mean angular velocities n of the sun and moon are the coefficients of the linear terms of l and l' (units: second/century), respectively.

For computation of the ephemeris of the sun, l' is set as M in (7.127) so that E and f of the sun can be computed. Using $D = L_{\text{moon}} - L_{\text{sun}} = F + \Omega - L_{\text{sun}}$, the mean longitude L_{sun} can be computed. ω can be computed by the relation $L_{\text{sun}} = \omega + f$.

For computation of the ephemeris of the moon, l is set as M in (7.127), so that E and f of the moon can be computed. ω can be computed by the spherical trigonometric formula

$$\tan(\omega + f) = \tan F / \cos i_{\text{m}} \tag{7.129}$$

where angles u $(= \omega + f)$ and F are in the same compartment.

Substituting the earlier-mentioned values of the moon and the sun into (7.124), (7.125), and (7.126) respectively, ephemeris of the moon and the sun are obtained in the ECSF coordinate system. For more precise computation of the ephemeris of the moon, several corrections have to be considered (see Meeus 1992; Montenbruck 1989). Equivalently, a correction $\mathrm{d}F$ can be added to F, and the change of $\mathrm{d}u$ in (7.129) can be considered $\mathrm{d}f$ and added to f, where $\mathrm{d}F$ has the form (units: seconds):

$$\begin{aligned} \mathrm{d}F &= 22640 \sin l + 769 \sin(2l) + 36 \sin(3l) - 125 \sin D + 2370 \sin(2D) - 668 \sin l' \\ &\quad - 412 \sin(2F) + 212 \sin(2D - 2l) + 4586 \sin(2D - l) + 192 \sin(2D + l) \\ &\quad + 165 \sin(2D - l') + 206 \sin(2D - l - l') - 110 \sin(l + l') + 148 \sin(l - l'). \end{aligned}$$

The orbits of the planets are given in the sun-centred ecliptic coordinate system by six Keplerian elements – the mean longitude (L) of the planet, the semi-major axis (a, units: AU) of the orbit of the planet, the eccentricity (e) of the orbit, the inclination (i) of the orbit to the ecliptic plane, the argument (ω) of the perihelion, and the longitude (Ω) of the ascending node. The orbital elements are expressed as a polynomial function of the instant of time T (Julian centuries) for Mercury, Venus, Mars, Jupiter, Saturn, Uranus and Neptune as follows (see Meeus, 1992, we use ω instead of π in this book. The argument of perihelion, ω can be obtained through the relation: $\omega = \pi - \Omega$):

$$\begin{pmatrix} L \\ a \\ e \\ i \\ \omega \\ \Omega \end{pmatrix}_{\text{Mercury}} = \begin{pmatrix} 252.250906 & 149474.0722491 & 0.00030397 & -0.00000002 \\ 0.38709831 & 0 & 0 & 0 \\ 0.20563175 & 0.000020406 & -0.0000000284 & -0.0000000002 \\ 7.0049860 & 0.0018215 & -0{,}00001809 & 0.000000053 \\ 29.1252260 & 0.3702885 & 0.00012002 & -0.000000155 \\ 48.3308930 & 1.1861890 & 0.00017587 & 0.000000211 \end{pmatrix} \begin{pmatrix} 1 \\ T \\ T^2 \\ T^3 \end{pmatrix}$$

$$\begin{pmatrix} L \\ a \\ e \\ i \\ \omega \\ \Omega \end{pmatrix}_{\text{Venus}} = \begin{pmatrix} 181.979801 & 58519.2130302 & 0.00031060 & 0.000000015 \\ 0.72332982 & 0 & 0 & 0 \\ 0.00677118 & -0.000047766 & 0.0000000975 & 0.00000000044 \\ 3.3946620 & 0.00100370 & -0{,}00000088 & -0.000000007 \\ 54.883787 & 0.50109980 & -0.00148002 & -0.000005235 \\ 76.6799200 & 0.90111900 & 0.00040665 & -0.00000008 \end{pmatrix} \begin{pmatrix} 1 \\ T \\ T^2 \\ T^3 \end{pmatrix}$$

$$\begin{pmatrix} L \\ a \\ e \\ i \\ \omega \\ \Omega \end{pmatrix}_{\text{Mars}} = \begin{pmatrix} 355.4332750 & 19141.6964746 & 0.00031097 & 0.000000015 \\ 1.523679342 & 0 & 0 & 0 \\ 0.09340062 & 0.000090483 & -0.0000000806 & -0.00000000035 \\ 1.8497260 & -0.0006010 & 0.00012760 & -0.000000006 \\ 286.502141 & 1.0689408 & 0.00011910 & -0.000002007 \\ 49.558093 & 0.7720923 & 0.00001605 & 0.000002325 \end{pmatrix} \begin{pmatrix} 1 \\ T \\ T^2 \\ T^3 \end{pmatrix}$$

$$\begin{pmatrix} L \\ a \\ e \\ i \\ \omega \\ \Omega \end{pmatrix}_{\text{Jupiter}} = \begin{pmatrix} 34.351484 & 3036.3027889 & 0.00022374 & 0.000000025 \\ 5.202603191 & 0.0000001913 & 0 & 0 \\ 0.04849485 & 0.000163244 & -0.0000004719 & -0.00000000197 \\ 1.303270 & -0.00549660 & 0.00000465 & -0.000000004 \\ 273.866868 & 0.5917118 & 0.00063010 & -0.000005138 \\ 100.464441 & 1.0209550 & 0.00040117 & 0.000000569 \end{pmatrix} \begin{pmatrix} 1 \\ T \\ T^2 \\ T^3 \end{pmatrix}$$

$$\begin{pmatrix} L \\ a \\ e \\ i \\ \omega \\ \Omega \end{pmatrix}_{\text{Saturn}} = \begin{pmatrix} 50.0774710 & 1223.5110141 & 0.00051952 & -0.000000003 \\ 9.554909596 & -0.0000021389 & 0 & 0 \\ 0.05550862 & -0.000346818 & -0.0000006456 & 0.00000000338 \\ 2.488878 & -0.0037363 & -0.00001516 & 0.000000089 \\ 339.391263 & 1.0866715 & 0.00095824 & 0.000007279 \\ 113.665524 & 0.8770979 & -0.00012067 & -0.00000238 \end{pmatrix} \begin{pmatrix} 1 \\ T \\ T^2 \\ T^3 \end{pmatrix}$$

Table 7.1 Gravitational constants of the sun, the moon and planets

	Gravitational constant (m^3s^{-2})
Sun	1.3271240000000E + 20
Moon	4.9027993000000E + 12
Earth	3.9860044180000E + 14
Mercury	2.2032070000000E + 13
Venus	3.2485850000000E + 14
Mars	4.2828300000000E + 13
Jupiter	1.2671270000000E + 17
Saturn	3.7940610000000E + 16
Uranus	5.8894334680000E + 15
Neptune	6.8364650040000E + 15

$$\begin{pmatrix} L \\ a \\ e \\ i \\ \omega \\ \Omega \end{pmatrix}_{\text{Uranus}} = \begin{pmatrix} 314.055005 & 429.8640561 & 0.00030434 & 0.000000026 \\ 19.218446062 & -0.0000000372 & 0.00000000098 & 0 \\ 0.04629590 & -0.000027337 & 0.0000000790 & 0.00000000025 \\ 0.773196 & 0.0007744 & 0.00003749 & -0.000000092 \\ 98.999212 & 0.9652526 & -0.00112532 & -0.000018083 \\ 74.005159 & 0.5211258 & 0.00133982 & 0.000018516 \end{pmatrix} \begin{pmatrix} 1 \\ T \\ T^2 \\ T^3 \end{pmatrix},$$

and

$$\begin{pmatrix} L \\ a \\ e \\ i \\ \omega \\ \Omega \end{pmatrix}_{Neptune} = \begin{pmatrix} 304.348655 & 219.8833092 & 0.00030926 & 0.000000018 \\ 30.110386869 & -0.0000001663 & 0.00000000069 & 0 \\ 0.00898809 & 0.000006408 & -0.0000000008 & -0.00000000005 \\ 1.769952 & -0.0093082 & -0.00000708 & 0.000000028 \\ 276.337634 & 0.3240620 & 0.00011912 & 0.000000633 \\ 131.784057 & 1.1022057 & 0.00026006 & -0.000000636 \end{pmatrix} \begin{pmatrix} 1 \\ T \\ T^2 \\ T^3 \end{pmatrix}$$

where, except for the semi-major axis a and eccentricity e, all other elements have units of degrees. $F = L - \Omega$, and f and E can be computed by using (7.129) and (7.127). Mean angular velocities n of the planets are the coefficients of the linear term of L (units: degree/century). The coordinate vector of the planet can then be computed using (7.124), (7.125), and (7.126). The results are in the sun-centred equatorial coordinate system. The results have to be transformed to the ECSF coordinate system by a translation:

$$\begin{pmatrix} \vec{r} \\ \dot{\vec{r}} \end{pmatrix}_{\text{ECSF}} = \begin{pmatrix} \vec{r} \\ \dot{\vec{r}} \end{pmatrix}_{\text{sun}} + \begin{pmatrix} \vec{r} \\ \dot{\vec{r}} \end{pmatrix}_{\text{SCES}} \tag{7.130}$$

where vectors with an index of sun and SCES are geocentric position and velocity vectors of the sun and the planet in the sun-centred equatorial system.

Gravitational constants of the sun, the moon and planets are given in Table 7.1.

Chapter 8
Numerical Orbit Determination

In this chapter the principle of numerical orbit determination will be outlined. An algebraic solution of the variation equation is derived. Numerical integration and interpolation algorithms as well as the related partial derivatives are given in detail.

8.1 Principle of GPS Precise Orbit Determination

Recalling the discussions in Sect. 4.1, the perturbed orbit of the satellite is the solution (or integration)

$$\vec{X}(t) = \vec{X}(t_0) + \int_{t_0}^{t} \vec{F} \mathrm{d}t, \tag{8.1}$$

which can be obtained by integrating the differential state equation under the initial condition

$$\begin{cases} \dot{\vec{X}}(t) = \vec{F} \\ \vec{X}(t_0) = \vec{X}_0 \end{cases}, \tag{8.2}$$

where $\vec{X}(t)$ is the instantaneous state vector of the satellite, $\vec{X}(t_0)$ is the initial state vector at time t_0 (denoted by $\vec{X}_0$), $\vec{F}$ is a function of the state vector $\vec{X}(t)$ and time t and

$$\vec{X} = \begin{pmatrix} \vec{r} \\ \dot{\vec{r}} \end{pmatrix} \quad \text{and} \quad \vec{F} = \begin{pmatrix} \dot{\vec{r}} \\ \vec{f}/m \end{pmatrix},$$

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where $\vec{f}$ is the summated force vector of all possible force vectors acting on the satellite, m is the mass of satellite and $\vec{r}$, $\dot{\vec{r}}$ are the position and velocity vectors of the satellite.

If the initial state vector and the force vectors are precisely known, then the precise orbits can be computed through the integration in (8.1). Expanding the integration time t into the future, the so-called forecasted orbits can be obtained. Therefore suitable numerical integration algorithms are required (see next section).

In practice, the precise initiate state vector and force models, which are related to the approximate initial state vector and force models, have to be determined. These can be realised through suitable parameterisation of the models in the GPS observation equations and then the parameters can be solved by adjustment or filtering.

We generally denote both the range and range rate together by ρ; their partial derivatives with respect to the orbit state vector (see Xu 2003, 2007) have the forms of

$$\frac{\partial \rho}{\partial \vec{r}}, \frac{\partial \rho}{\partial \dot{\vec{r}}}, \quad or \quad \frac{\partial \rho}{\partial \vec{X}}.$$

Therefore the orbit parameter related parts in the linearised GPS observation equation are

$$\frac{\partial \rho}{\partial (\vec{r}, \dot{\vec{r}})} \frac{\partial (\vec{r}, \dot{\vec{r}})}{\partial \vec{y}} \Delta \vec{y}^T, \quad or \quad \frac{\partial \rho}{\partial \vec{X}} \frac{\partial \vec{X}}{\partial \vec{y}} \Delta \vec{y}^T, \tag{8.3}$$

where

$$\vec{y} = \left(\vec{X}_0, \vec{Y}\right), \qquad \Delta \vec{y}^T = \left(\Delta \vec{X}_0, \Delta \vec{Y}\right)^T, \qquad \frac{\partial \vec{X}}{\partial \vec{y}} = \frac{\partial \vec{X}}{\partial (\vec{X}_0, \vec{Y})}.$$

$\vec{X}$, $\vec{Y}$ are the state vector of satellite and the parameter vector of the force models, and index 0 denotes the related initial vectors of time t_0. $\vec{y}$ is the total unknown vector of the orbit determination problem, the related correction vector is $\Delta \vec{y} = \vec{y} - \vec{y}_0$ and $\Delta \vec{X}_0$ is the correction vector of the initial state vector. The partial derivative of $\vec{X}$ with respect to $\vec{y}$ is called the transition matrix which has the dimension of $6 \times (6 + n)$, where n is the dimension of vector $\vec{Y}$. The partial derivatives of the equation of motion of the satellite (see (8.2)) with respect to the vector $\vec{y}$ are

$$\frac{\partial \dot{\vec{X}}(t)}{\partial \vec{y}} = \frac{\partial \vec{F}}{\partial \vec{y}} = \frac{\partial \vec{F}}{\partial \vec{X}} \frac{\partial \vec{X}}{\partial \vec{y}} + \left(\frac{\partial \vec{F}}{\partial \vec{y}}\right)^*, \tag{8.4}$$

where the superscript * denotes the partial derivatives of $\vec{F}$ with respect to the explicit parameter vector $\vec{y}$ in $\vec{F}$ and

$$D(t)=\left(\frac{\partial\vec{F}}{\partial\vec{X}}\right)=\begin{pmatrix}0_{3\times3} & E_{3\times3}\\ \frac{1}{m}\frac{\partial\vec{f}}{\partial\vec{r}} & \frac{1}{m}\frac{\partial\vec{f}}{\partial\dot{\vec{r}}}\end{pmatrix}=\begin{pmatrix}0_{3\times3} & E_{3\times3}\\ A(t) & B(t)\end{pmatrix},$$

$$C(t)=\left(\frac{\partial\vec{F}}{\partial\vec{y}}\right)^{*}=\begin{pmatrix}0_{3\times6} & 0_{3\times n}\\ 0_{3\times6} & \frac{1}{m}\frac{\partial\vec{f}}{\partial\vec{Y}}\end{pmatrix}=\begin{pmatrix}0_{3\times(6+n)}\\ G(t)\end{pmatrix}, \tag{8.5}$$

where E is an identity matrix; the partial derivatives will be discussed and derived in a later section in detail. Note that the force parameters are not functions of t. Therefore the order of the differentiations can be exchanged. Denoting the transition matrix by $\Phi(t, t_0)$, then (8.4) turns out to be

$$\frac{\mathrm{d}\Phi(t,t_0)}{\mathrm{d}t}=D(t)\Phi(t,t_0)+C(t). \tag{8.6}$$

Equation (8.6) is called a differential equation of the transition matrix or variation equation (see, e.g. Montenbruck and Gill 2000). Denoting

$$\Phi(t,t_0)=\begin{pmatrix}\Psi(t,t_0)\\ \dot{\Psi}(t,t_0)\end{pmatrix}, \tag{8.7}$$

an alternate expression of (8.6) can be obtained by substituting (8.7) and (8.5) into (8.6):

$$\frac{\mathrm{d}^2\Psi(t,t_0)}{\mathrm{d}t^2}=A(t)\Psi(t,t_0)+B(t)\frac{\mathrm{d}\Psi(t,t_0)}{\mathrm{d}t}+G(t). \tag{8.8}$$

The initial value matrix is (initial state vector does not depend on force parameters):

$$\Phi(t_0,t_0)=\begin{pmatrix}E_{6\times6} & 0_{6\times n}\end{pmatrix}. \tag{8.9}$$

That is, in the GPS observation equation, the transition matrix has to be obtained by solving the initial value problem of the variation equation (8.6) or (8.8). The problem is traditionally solved by integration.

8.1.1 Algebraic Solution of the Variation Equation

The variation equation can also be solved by numerical differentiation.

Equation (8.8) is a matrix differential equation system of size $3 \times (6 + n)$. Because $A(t)$ and $B(t)$ are 3×3 matrices, the differential equations are independent from column to column. That is, we need to discuss just the solution of the equation of a column. For column j, (8.8) and (8.9) are

$$\frac{\mathrm{d}^2\Psi_{ij}(t)}{\mathrm{d}t^2} = \sum_{k=1}^{3}\left(A_{ik}(t)\Psi_{kj}(t) + B_{ik}(t)\frac{\mathrm{d}\Psi_{kj}(t)}{\mathrm{d}t}\right) + G_{ij}(t), \quad i = 1,2,3,$$
$$\begin{pmatrix}\Psi_{ij}(t_0)\\ \dot{\Psi}_{ij}(t_0)\end{pmatrix} = \begin{pmatrix}\delta_{ij}\\ \delta_{(i+3)j}\end{pmatrix}, \quad i = 1,2,3, \quad \delta_{kj} = \begin{cases}1 & \text{if } k = j\\ 0 & \text{if } k \neq j\end{cases}, \tag{8.10}$$

where index ij denotes the related element of the matrix. For time interval $[t_0, t]$ and differentiation step $h = (t - t_0)/m$, one has $t_n = t_0 + nh$, $n = 1,\ldots,m$ and

$$\left.\frac{\mathrm{d}^2\Psi_{ij}(t)}{\mathrm{d}t^2}\right|_{t=t_n} = \frac{\Psi_{ij}(t_{n+1}) - 2\Psi_{ij}(t_n) + \Psi_{ij}(t_{n-1})}{h^2}, \quad i = 1,2,3,$$
$$\left.\frac{\mathrm{d}\Psi_{ij}(t)}{\mathrm{d}t}\right|_{t=t_n} = \frac{\Psi_{ij}(t_{n+1}) - \Psi_{ij}(t_{n-1})}{2h}, \quad \Psi_{ij}(t)\big|_{t=t_n} = \Psi_{ij}(t_n), \quad i = 1,2,3. \tag{8.11}$$

Then (8.10) turns out to be

$$\Psi_{ij}(t_0) = \Psi_{ij}(t_0), \quad \Psi_{ij}(t_1) = \Psi_{ij}(t_0) + h\dot{\Psi}_{ij}(t_0), \quad i = 1,2,3.$$
$$\frac{\Psi_{ij}(t_{n+1}) - 2\Psi_{ij}(t_n) + \Psi_{ij}(t_{n-1})}{h^2} =$$
$$\sum_{k=1}^{3}\left(A_{ik}(t_n)\Psi_{kj}(t_n) + B_{ik}(t_n)\frac{\Psi_{kj}(t_{n+1}) - \Psi_{kj}(t_{n-1})}{2h}\right) + G_{ij}(t_n), \quad i = 1,2,3, \tag{8.12}$$

where $n = 1, 2, \ldots, m-1$. For $i = 1, 2, 3$ and the sequential number n, there are three equations and three unknowns of time t_{n+1}, so that the initial value problem has a set of unique solutions sequentially. Equation (8.12) can be rewritten as

$$\left(\frac{E}{h^2} - \frac{B(t_n)}{2h}\right)\begin{pmatrix}\Psi_{1j}(t_{n+1})\\ \Psi_{2j}(t_{n+1})\\ \Psi_{3j}(t_{n+1})\end{pmatrix} = \begin{pmatrix}R_1\\ R_2\\ R_3\end{pmatrix}, \tag{8.13}$$

where

$$\begin{pmatrix}R_1\\ R_2\\ R_3\end{pmatrix} = \left(\frac{2E}{h^2} + A(t_n)\right)\begin{pmatrix}\Psi_{1j}(t_n)\\ \Psi_{2j}(t_n)\\ \Psi_{3j}(t_n)\end{pmatrix} - \left(\frac{E}{h^2} + \frac{B(t_n)}{2h}\right)\begin{pmatrix}\Psi_{1j}(t_{n-1})\\ \Psi_{2j}(t_{n-1})\\ \Psi_{3j}(t_{n-1})\end{pmatrix} + \begin{pmatrix}G_{1j}(t_n)\\ G_{2j}(t_n)\\ G_{3j}(t_n)\end{pmatrix}.$$

For $n = 1, \ldots, m-1$, this equation is solvable. Note that the three matrices

$$\left(\frac{E}{h^2} - \frac{B(t_n)}{2h}\right), \quad \left(\frac{2E}{h^2} + A(t_n)\right), \quad \left(\frac{E}{h^2} + \frac{B(t_n)}{2h}\right)$$

are independent from the column number j. The solutions of (8.13) are vectors

$$\begin{pmatrix} \Psi_{1j}(t_{n+1}) \\ \Psi_{2j}(t_{n+1}) \\ \Psi_{3j}(t_{n+1}) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \dot{\Psi}_{1j}(t_{n+1}) \\ \dot{\Psi}_{2j}(t_{n+1}) \\ \dot{\Psi}_{3j}(t_{n+1}) \end{pmatrix}, \quad n = 1, \ldots, m-1, \tag{8.14}$$

where the velocity vector can be computed using the definition of (8.11). Solving the equations of all column j, the solutions of the initial value problem of (8.8) and (8.9) can be obtained. Note that the required values are the values of t_n which can be computed by averaging the values of $t_{\mathrm{n}+1}$ and $t_{\mathrm{n}-1}$.

8.2 Numerical Integration and Interpolation Algorithms

The Runge–Kutta algorithm, Adams algorithm, Cowell algorithm and mixed algorithm as well as interpolation algorithms are discussed in this section (see, e.g. Brouwer and Clemence 1961; Bate et al. 1971; Herrick 1972; Xu 1994; Liu et al. 1996; Press et al. 1992).

8.2.1 Runge–Kutta Algorithms

The Runge–Kutta algorithm is a method that can be used to solve the initial value problem of

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t} &= F(t, X), \\ X(t_0) &= X_0 \end{aligned} \tag{8.15}$$

where X_0 is the initial value of variable X at time t_0, and F is the function of t and X. For step size h, the Runge–Kutta algorithm can be used to compute $X(t_0 + h)$. By repeating this process, a series of solutions can be obtained as $X(t_0 + h)$, $X(t_0 + 2\,h)$, …, $X(t_0 + nh)$, where n is an integer. Denoting $t_n = t_0 + nh$, $X(t_n + h)$ can be represented by the Taylor expansion at t_n by

$$X(t_n + h) = X(t_n) + h\frac{\mathrm{d}X}{\mathrm{d}t}\bigg|_{t=t_n} + \frac{h^2}{2}\frac{\mathrm{d}^2X}{\mathrm{d}t^2}\bigg|_{t=t_n} + \ldots + \frac{h^n}{n!}\frac{\mathrm{d}^nX}{\mathrm{d}t^n}\bigg|_{t=t_n} + \ldots, \tag{8.16}$$

where

$$\frac{\mathrm{d}X}{\mathrm{d}t} = F,$$

$$\frac{\mathrm{d}^2X}{\mathrm{d}t^2} = \frac{\mathrm{d}F(t,X)}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial X}\frac{\partial X}{\partial t} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial X}F,$$

$$\frac{\mathrm{d}^3X}{\mathrm{d}t^3} = \frac{\partial^2 F}{\partial t^2} + 2\frac{\partial^2 F}{\partial t \partial X}F + \frac{\partial^2 F}{\partial t \partial X} + \frac{\partial^2 F}{\partial X^2}F^2 + \left(\frac{\partial F}{\partial X}\right)^2 F \quad \text{and} \tag{8.17}$$

$$\frac{\mathrm{d}^4X}{\mathrm{d}t^4} = \frac{\partial^3 F}{\partial t^3} + \frac{\partial^3 F}{\partial t^2 \partial X}(3F+1) + \frac{\partial^3 F}{\partial t \partial X^2}(5F^2+2F) + 2\frac{\partial^2 F}{\partial t \partial X}\frac{\partial F}{\partial t} + 4\frac{\partial^3 F}{\partial X^3}F^3$$
$$+2\frac{\partial^2 F}{\partial X^2}\frac{\partial F}{\partial t}F + 4\frac{\partial F}{\partial X}\frac{\partial^2 F}{\partial t \partial X}F + 6\frac{\partial F}{\partial X}\frac{\partial^2 F}{\partial X^2}F^2 + \left(\frac{\partial F}{\partial X}\right)^2\frac{\partial F}{\partial t} + \left(\frac{\partial F}{\partial X}\right)^2\frac{\partial F}{\partial X}2F.$$

The principle of the Runge–Kutta algorithm is to use a set of combinations of the first order partial derivatives around the (t_n, $X(t_n)$) to replace the higher order derivatives in (8.16), that is,

$$X(t_{n+1}) = X(t_n) + \sum_{i=1}^{L} w_i K_i, \tag{8.18}$$

where

$$K_1 = hF(t_n, X(t_n))$$

$$K_i = hF\left(t_n + \alpha_i h, X(t_n) + \sum_{j=1}^{i-1}\beta_{ij}K_j\right),\ (i = 2, 3, \ldots), \tag{8.19}$$

where w_i, α_i and β_{ij} are constants to be determined, and L is an integer. The Taylor expansions of K_i (i = 2,3,…) at (t_n, $X(t_n)$) to the first order are

$$K_i = hF(t_n, X(t_n)) + h^2\alpha_i\frac{\partial F}{\partial t} + h\frac{\partial F}{\partial X}\sum_{j=1}^{i-1}\beta_{ij}K_j \quad \text{or} \tag{8.20}$$

$$K_2 = hF(t_n, X(t_n)) + h^2\left(\alpha_2\frac{\partial F}{\partial t} + \beta_{21}\frac{\partial F}{\partial X}F\right), \tag{8.21}$$

$$K_3 = hF + h^2\left(\alpha_3\frac{\partial F}{\partial t} + (\beta_{31}+\beta_{32})\frac{\partial F}{\partial X}F\right) + h^3\beta_{32}\frac{\partial F}{\partial X}\left(\alpha_2\frac{\partial F}{\partial t} + \beta_{21}\frac{\partial F}{\partial X}F\right),$$

$$
\begin{aligned}
K_4 &= hF + h^2\left(\alpha_4 \frac{\partial F}{\partial t} + (\beta_{41} + \beta_{42} + \beta_{43}) \frac{\partial F}{\partial X} F\right) \\
&+ h^3\left[(\beta_{42}\alpha_2 + \beta_{43}\alpha_3) \frac{\partial F}{\partial X} \frac{\partial F}{\partial t} + (\beta_{42}\beta_{21} + \beta_{43}(\beta_{31} + \beta_{32})) \frac{\partial F}{\partial X} \frac{\partial F}{\partial X} F\right] \\
&+ h^4 \beta_{43}\beta_{32} \frac{\partial F}{\partial X} \frac{\partial F}{\partial X}\left(\alpha_2 \frac{\partial F}{\partial t} + \beta_{21} \frac{\partial F}{\partial X} F\right),
\end{aligned}
$$

$$
\begin{aligned}
K_5 &= hF + h^2\left(\alpha_5 \frac{\partial F}{\partial t} + (\beta_{51} + \beta_{52} + \beta_{53} + \beta_{54}) \frac{\partial F}{\partial X} F\right) \\
&+ h^3 \frac{\partial F}{\partial X}\left(\begin{array}{l} \alpha_2 \frac{\partial F}{\partial t} + \beta_{21} \frac{\partial F}{\partial X} F + \beta_{53}\left(\alpha_3 \frac{\partial F}{\partial t} + (\beta_{31} + \beta_{32}) \frac{\partial F}{\partial X} F\right) \\ + \beta_{54}\left(\alpha_4 \frac{\partial F}{\partial t} + (\beta_{41} + \beta_{42} + \beta_{43}) \frac{\partial F}{\partial X} F\right) \end{array}\right) \\
&+ h^4 \frac{\partial F}{\partial X}\left(\begin{array}{l} (\beta_{53}\beta_{32}\alpha_2 + \beta_{54}(\beta_{42}\alpha_2 + \beta_{43}\alpha_3)) \frac{\partial F}{\partial X} \frac{\partial F}{\partial t} \\ + \left(\beta_{54}\left(\beta_{42}\beta_{21} + \beta_{43}(\beta_{31} + \beta_{32})\right) + \beta_{32}\beta_{21}\right) \frac{\partial F}{\partial X} \frac{\partial F}{\partial X} F \end{array}\right) \\
&+ h^5 \frac{\partial F}{\partial X} \beta_{54}\left(\beta_{43}\beta_{32} \frac{\partial F}{\partial X} \frac{\partial F}{\partial X}\left(\alpha_2 \frac{\partial F}{\partial t} + \beta_{21} \frac{\partial F}{\partial X} F\right)\right)
\end{aligned}
$$

...

where F and the related partial derivatives have values at $(t_n, X(t_n))$. Substituting these formulas into (8.18) and comparing the coefficients of h^n $(=1/n!)$ with (8.16), a group of equations of constants w_i, α_i and β_{ij} can be obtained by separating them through the partial derivative combinations. For example, for $L = 4$, one has

$$
\begin{aligned}
w_1 + w_2 + w_3 + w_4 &= 1, \\
w_2\alpha_2 + w_3\alpha_3 + w_4\alpha_4 &= \frac{1}{2}, \\
w_2\beta_{21} + w_3(\beta_{31} + \beta_{32}) + w_4(\beta_{41} + \beta_{42} + \beta_{43}) &= \frac{1}{2}, \\
w_3\alpha_2\beta_{32} + w_4(\alpha_2\beta_{42} + \alpha_3\beta_{43}) &= \frac{1}{6}, \\
w_3\beta_{21}\beta_{32} + w_4(\beta_{21}\beta_{42} + \beta_{31}\beta_{43} + \beta_{32}\beta_{43}) &= \frac{1}{6}, \\
w_4\alpha_2\beta_{43}\beta_{32} = \frac{1}{24} \quad &\text{and} \\
w_4\beta_{21}\beta_{43}\beta_{32} = \frac{1}{24}&.
\end{aligned}
\tag{8.22}
$$

There are 13 coefficients in the seven equations, so the solution set of (8.22) is not unique. Considering w as weight and α as the step factor, one may set, e.g. $w_1 = w_2 = w_3 = w_4 = 1/4$, $\alpha_2 = 1/3$, $\alpha_3 = 2/3$, $\alpha_4 = 1$ and substitute the same into (8.22) and have

$$\beta_{21} + \beta_{31} + \beta_{32} + \beta_{41} + \beta_{42} + \beta_{43} = 2,$$

$$\beta_{32} + \beta_{42} + 2\beta_{43} = 2,$$

$$\beta_{21}\beta_{32} + \beta_{21}\beta_{42} + \beta_{31}\beta_{43} + \beta_{32}\beta_{43} = \frac{2}{3},$$

$$\beta_{43}\beta_{32} = \frac{1}{2}$$

and

$$\beta_{21}\beta_{43}\beta_{32} = \frac{1}{6}.$$

Letting $\beta_{32} = 1$, one has $\beta_{42} = 0$, $\beta_{31} = -1/3$ and $\beta_{41} = 1/2$. Thus, a fourth-order Runge–Kutta formula is

$$X(t_{n+1}) = X(t_n) + \frac{1}{4}\sum_{i=1}^{4} K_i, \tag{8.23}$$

where

$$K_1 = hF(t_n, X(t_n)), \tag{8.24}$$

$$K_2 = hF\left(t_n + \frac{1}{3}h, X(t_n) + \frac{1}{3}K_1\right),$$

$$K_3 = hF\left(t_n + \frac{2}{3}h, X(t_n) - \frac{1}{3}K_1 + K_2\right)$$

and

$$K_4 = hF\left(t_n + h, X(t_n) + \frac{1}{2}K_1 + \frac{1}{2}K_3\right).$$

Similarly, a commonly used eighth order Runge–Kutta formula can be derived. It is quoted as follows (see Xu 1994; Liu et al. 1996):

$$X(t_{n+1})=X_n+\frac{1}{840}(41K_1+27K_4+272K_5+27K_6+216K_7+216K_9+41K_{10}), \quad (8.25)$$

where

$$K_1 = hF(t_n, X_n), \ X_n = X(t_n), \quad (8.26)$$

$$K_2 = hF\left(t_n + \frac{4}{27}h, X_n + \frac{4}{27}K_1\right),$$

$$K_3 = hF\left(t_n + \frac{2}{9}h, X_n + \frac{1}{18}K_1 + \frac{1}{6}K_2\right),$$

$$K_4 = hF\left(t_n + \frac{1}{3}h, X_n + \frac{1}{12}K_1 + \frac{1}{4}K_3\right),$$

$$K_5 = hF\left(t_n + \frac{1}{2}h, X_n + \frac{1}{8}K_1 + \frac{3}{8}K_4\right),$$

$$K_6 = hF\left(t_n + \frac{2}{3}h, X_n + \frac{1}{54}(13K_1 - 27K_3 + 42K_4 + 8K_5)\right),$$

$$K_7 = hF\left(t_n + \frac{1}{6}h, X_n + \frac{1}{4320}(389K_1 - 54K_3 + 966K_4 - 824K_5 + 243K_6)\right),$$

$$K_8 = hF\left(t_n + h, X_n + \frac{1}{20}(-231K_1 + 81K_3 - 1164K_4 + 656K_5 - 122K_6 + 800K_7)\right),$$

$$K_9 = hF\left(\begin{array}{r} t_n + \frac{5}{6}h, X_n + \frac{1}{288}(-127K_1 \ + 18K_3 - 678K_4 + 456K_5 \\ -9K_6 + 576K_7 + 4K_8) \end{array}\right)$$

and

$$K_{10} = hF\left(\begin{array}{r} t_n + h, X_n + \frac{1}{820}(1481K_1 \ - 81K_3 + 7104K_4 - 3376K_5 \\ +72K_6 - 5040K_7 - 60K_8 + 720K_9) \end{array}\right)$$

From the derivation process it is obvious that the Runge–Kutta algorithm is an approximation of the same order Taylor expansions. For every step of the solution, the function values of F have to be computed several times. The Runge–Kutta algorithm is also called the single step method and is commonly used for computing the start values for other multiple step methods.

Errors of the integration are dependent on the step size and the properties of function F. To ensure the required accuracy of the orbit integration, a step size adaptive control is also meaningful in computing efficiency (see Press et al. 1992). Because of the periodical motion of the orbit, the step control just needs to be made in a few special cycles of the motion. A step doubling method is suggested by Press et al. (1992). Integration is taken twice for each step, first with a full step, then independently with two half steps. By comparing the results, the step size can be adjusted to fit the accuracy requirement.

To apply the above formulas for solving the initial value problem of the equation of motion (8.2), (8.15) shall be rewritten as

$$\begin{array}{l}\frac{\mathrm{d}X_k}{\mathrm{d}t}=\dot{X}_k(t,X)\\ \frac{\mathrm{d}\dot{X}_k}{\mathrm{d}t}=f_k(t,X)/m\end{array},\qquad \begin{array}{l}X_k(t_0)=X_{k0}\\ \dot{X}_k(t_0)=\dot{X}_{k0}\end{array},\qquad k=1,2,3,$$

where $X=(X_1, X_2, X_3, \dot{X}_1, \dot{X}_2, \dot{X}_3)$. Using the Runge–Kutta algorithm to solve the above problem, an additional index k shall be added to all X and K in (8.25):

$$X_k(t_{n+1})=X_{kn}+\frac{1}{840}(41K_{k1}+27K_{k4}+272K_{k5}+27K_{k6}+216K_{k7}+216K_{k9}+41K_{k10}),$$

and the same index k shall be added to K on the left-hand side and F on the right-hand side of (8.26). For the last three equations, $F_k=f_k/m$, so $\dot{X}_k$ can be computed. For the first three equations, $F_k=\dot{X}_k$, so F_k can be computed through computing $\dot{X}_k$ at the needed coordinates t and X.

8.2.2 Adams Algorithms

For the initial value problem of

$$\begin{aligned}\frac{\mathrm{d}X}{\mathrm{d}t}&=F(t,X)\\ X(t_0)&=X_0,\end{aligned}\tag{8.27}$$

there exists

$$X(t_{n+1})=X(t_n)+\int_{t_n}^{t_{n+1}}F(t,X)\mathrm{d}t.\tag{8.28}$$

The Adams algorithm uses the Newtonian backward differential interpolation formula to represent the function F by

$$F(t,X) = F_n + \frac{t-t_n}{h}\nabla F_n + \frac{(t-t_n)(t-t_{n-1})}{2!h^2}\nabla^2 F_n + \ldots + \frac{(t-t_n)(t-t_{n-1})\cdots(t-t_{n-k+1})}{k!h^k}\nabla^k F_n, \tag{8.29}$$

where F_n is the value of F at the time t_n, h is the step size, $\nabla^k F$ is the kth order backward numerical difference of F and

$$\begin{aligned} \nabla F_n &= F_n - F_{n-1} \\ \nabla^2 F_n &= \nabla F_n - \nabla F_{n-1} = F_n - 2F_{n-1} + F_{n-2}, \\ &\ldots \\ \nabla^m F_n &= \sum_{j=0}^{m} (-1)^j C_m^j F_{n-j}, \qquad C_m^j = \frac{m!}{j!(m-j)!}, \end{aligned} \tag{8.30}$$

where C^j_{m} is the binomial coefficient. Letting $s = (t–t_n)/h$, then $\mathrm{d}t = h\mathrm{d}s$, $s = 0$ if $t = t_n$, $s = 1$ if $t = t_{n+1}$, so that (8.29) and (8.28) turn out to be

$$F(t,X) = \sum_{m=0}^{k} C_{s+m-1}^{m} \nabla^m F_n$$

and

$$X(t_{n+1}) = X(t_n) + \int_{t_n}^{t_{n+1}} \sum_{m=0}^{k} C_{s+m-1}^{m} \sum_{j=0}^{m} (-1)^j C_m^j F_{n-j} h\mathrm{d}s. \tag{8.31}$$

By denoting

$$\begin{aligned} \gamma_m &= \int_0^1 C_{s+m-1}^{m} \mathrm{d}s \\ \beta_j &= \sum_{m=j}^{k} (-1)^j C_m^j \gamma_m, \end{aligned} \tag{8.32}$$

one has

$$X(t_{n+1}) = X(t_n) + h\sum_{j=0}^{k} \beta_j F_{n-j}, \tag{8.33}$$

where the sequences of the two sequential summations have been changed. For the first equation of (8.32), there is (see Xu 1994)

$$\gamma_0 = 1, \qquad \gamma_m = 1 - \sum_{j=1}^{m} \frac{1}{j+1}\gamma_{m-j}, \qquad (m \geq 1). \tag{8.34}$$

Equation (8.33) is also called the Adams–Bashforth formula. It uses the function values of $\{F_{n-j}, j = 0,\ldots,k\}$ to compute the X_{n+1}. When the order of the algorithm is selected, the coefficients of β_j are constants. This makes the computation using (8.33) very simple. For every integration step, just one function value of F_n has to be computed. However, the Adams algorithm needs $\{F_{n-j}, j = 0,\ldots,k\}$ as initial values, but to compute those values, the states $\{X_{n-j}, j = 0,\ldots,k\}$ are needed. In other words, the Adams algorithm is not able to start the integration itself. The Runge–Kutta algorithm is usually used for computing the start values.

The Adams–Bashforth formula does not take the function value F_{n+1} into account. Using F_{n+1}, the Adams algorithm is expressed by the Adams–Moulton formula. Similar to the above discussions, function F can be represented by

$$\begin{aligned} F(t,X) = F_{n+1} &+ \frac{t-t_{n+1}}{h}\nabla F_{n+1} + \frac{(t-t_{n+1})(t-t_n)}{2!h^2}\nabla^2 F_{n+1} + \ldots \\ &+ \frac{(t-t_{n+1})(t-t_n)\cdots(t-t_{n-k+2})}{k!h^k}\nabla^k F_{n+1}, \end{aligned} \tag{8.35}$$

where

$$\nabla^m F_{n+1} = \sum_{j=0}^{m} (-1)^j C_m^j F_{n+1-j}. \tag{8.36}$$

If one lets $s = (t–t_{n+1})/h$, then $\mathrm{d}t = h\mathrm{d}s$, $s = -1$ if $t = t_n$ and $s = 0$ if $t = t_{n+1}$; formulas similar to (8.33) and (8.32) can be obtained:

$$X(t_{n+1}) = X(t_n) + h\sum_{j=0}^{k} \beta_j^* F_{n+1-j}, \tag{8.37}$$

$$\begin{aligned} \beta_j^* &= \sum_{m=j}^{k} (-1)^j C_m^j \gamma_m^* \\ \gamma_m^* &= \int_{-1}^{0} C_{s+m-1}^m \mathrm{d}s \end{aligned} \tag{8.38}$$

and (see Xu 1994)

$$\gamma_0^* = 1, \qquad \gamma_m^* = -\sum_{j=1}^{m} \frac{1}{j+1} \gamma_{m-j}^*, \qquad (m \geq 1). \tag{8.39}$$

Because of the use of F_{n+1} to approximate F, the Adams–Moulton formula may reach a higher accuracy than that of the Adams–Bashforth formula. However, before X_{n+1} has been computed, F_{n+1} might not have been computed exactly. So an iterative process is needed while using the Adams–Moulton formula. A simple way to use the Adams–Moulton formula is to use the Adams–Bashforth formula to compute X_{n+1} and F_{n+1}, and then to use the Adams–Moulton formula to compute the modified X_{n+1} using F_{n+1}. Experience shows that such a process will be accurate enough for many applications.

8.2.3 Cowell Algorithms

For the initial value problem of

$$\begin{aligned} \frac{\mathrm{d}^2 X}{\mathrm{d}t^2} &= F(t, X) \\ \dot{X}(t_0) &= \dot{X}_0 \\ X(t_0) &= X_0, \end{aligned} \tag{8.40}$$

there is

$$\dot{X}(t) = \dot{X}(t_n) + \int_{t_n}^{t} F(t, X)\mathrm{d}t. \tag{8.41}$$

Note that here X is the position coordinate of the satellite. In other words, the disturbing force F is not the function of the velocity of the satellite.

By integrating (8.41) in areas of [t_n, t_{n+1}] and [t_n, t_{n-1}] respectively, one has

$$X(t_{n+1}) - X(t_n) - \dot{X}(t_n)(t_{n+1} - t_n) = \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} F(t, X)\mathrm{d}t\mathrm{d}t \quad \text{and} \tag{8.42}$$

$$X(t_{n-1}) - X(t_n) - \dot{X}(t_n)(t_{n-1} - t_n) = \int_{t_n}^{t_{n-1}} \int_{t_n}^{t} F(t, X)\mathrm{d}t\mathrm{d}t, \tag{8.43}$$

where $(t_{n+1}–t_n) = h = (t_n–t_{n-1})$. Adding both equations together, one has

$$X(t_{n+1}) - 2X(t_n) + X(t_{n-1}) = \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} + \int_{t_n}^{t_{n-1}} \int_{t_n}^{t} F(t, X)\mathrm{d}t\mathrm{d}t. \tag{8.44}$$

Similar to the Adams–Bashforth formula, function F can be represented by

$$F(t,X) = F_n + \frac{t-t_n}{h}\nabla F_n + \frac{(t-t_n)(t-t_{n-1})}{2!h^2}\nabla^2 F_n + \ldots + \frac{(t-t_n)(t-t_{n-1})\cdots(t-t_{n-k+1})}{k!h^k}\nabla^k F_n. \tag{8.45}$$

Substituting (8.45) into (8.44), one has (similar to the derivation of Adams algorithms) (see Xu 1994)

$$X(t_{n+1}) = 2X(t_n) - X(t_{n-1}) + h^2\sum_{j=0}^{k}\beta_j F_{n-j}, \tag{8.46}$$

where

$$\beta_j = \sum_{m=j}^{k}(-1)^j C_m^j \sigma_m, \tag{8.47}$$

$$\sigma_0 = 1, \qquad \sigma_m = 1 - \sum_{j=1}^{m}\frac{2}{j+2}b_{j+1}\sigma_{m-j}, \quad (m \geq 1),$$

$$b_j = \sum_{i=1}^{j}\frac{1}{i}.$$

Equation (8.46) is called the Stormer formula. Similar to the discussions with the Adams algorithms, taking F_{n+1} into account, one has

$$F(t,X) = F_{n+1} + \frac{t-t_{n+1}}{h}\nabla F_{n+1} + \frac{(t-t_{n+1})(t-t_n)}{2!h^2}\nabla^2 F_{n+1} + \ldots + \frac{(t-t_{n+1})(t-t_n)\cdots(t-t_{n-k+2})}{k!h^k}\nabla^k F_{n+1}. \tag{8.48}$$

and (see Xu 1994)

$$X(t_{n+1}) = 2X(t_n) - X(t_{n-1}) + h^2\sum_{j=0}^{k}\beta_j^* F_{n+1-j}, \tag{8.49}$$

where

$$\beta_j^* = \sum_{m=j}^{k}(-1)^j C_m^j \sigma_m^*, \tag{8.50}$$

$$\sigma_0^* = 1, \qquad \sigma_m^* = -\sum_{j=1}^{m} \frac{2}{j+2} b_{j+1} \sigma_{m-j}^*, \qquad (m \geq 1), \qquad \text{and}$$

$$b_j = \sum_{i=1}^{j} \frac{1}{i}.$$

Equation (8.49) is called the Cowell formula. Because of the use of F_{n+1} to approximate F, the Cowell formula may reach a higher accuracy than that of the Stormer formula. However, before X_{n+1} has been computed, F_{n+1} may not be computed exactly. So an iterative process is needed while using the Cowell formula. A simple way is to use the Stormer formula first to compute X_{n+1} and F_{n+1}, and then the Cowell formula to compute the modified X_{n+1} using F_{n+1}. Experience shows that such a process will be accurate enough for many applications.

8.2.4 Mixed Algorithms and Discussions

Above we discussed three algorithms for solving the initial value problem of the orbit differential equation. The Runge–Kutta algorithm is a single step method. The formulas of different order Runge–Kutta algorithms do not have simple relationships, and even for a definite order the formulas are not unique. For every step of integration, several function values of F have to be computed. Interestingly, the Runge–Kutta algorithm is a self-starting method. Generally, it is used for providing the starting values for multiple-step algorithms.

Adams algorithms are multiple-step methods. The order of the formulas can be easily raised because of their sequential relationships. However, the Adams algorithms cannot start themselves. For every step of integration, only one function value has to be computed. The disturbing function is considered a function of time and the state of the satellite. So Adams methods can be used in orbit determination without having any problem with the disturbing function. For a higher accuracy requirement, mixed Adams–Bashforth and Adams–Moulton methods can be used in an iterative process.

Cowell algorithms are also multiple-step methods. The order can be changed easily. Cowell methods also need starting help from other methods. Analysis shows that Cowell algorithms have a higher accuracy than Adams algorithms when the same orders of formulas are used. However, Cowell formulas are only suitable for that kind of disturbing function F, which is the function of the time and the position of the satellite. It is well known that atmospheric drag is a disturbing force, which is a function of the velocity of the satellite. Therefore Cowell algorithms can be used only for integrating part of the disturbing forces. A mixed Cowell method still retains this property.

Obviously the forces of the equation of motion have to be separated into two parts: one includes the forces that are functions of the velocity of the satellite and the other all remaining forces. The first part can be integrated by Adams methods and the second by Cowell methods. The Runge–Kutta algorithm will be used for providing the needed starting values.

The selections of the order number and step size are dependent on the accuracy requirements and the orbit conditions. Usually the order and the step size are selectable input variables of the software, and can be properly selected after several test runs. Scheinert suggested using 8th-order Runge–Kutta algorithms, as well as 12th-order Adams and Cowell algorithms (see Scheinert 1996). Note that for order selection it is not the higher the order the higher the accuracy. For step size selection it is not the smaller the step size the better the results.

8.2.5 Interpolation Algorithms

Orbits are given through integration at the step points $t_0 + nh$ ($n = 0,1,\ldots$). For GPS satellites h is usually selected as 300 s. However, GPS observations are made, usually in IGS, every 15 s. For linearisation and formation of the GPS observation equations, the orbit data sometimes have to be interpolated to the needed epochs.

To obtain the ephemeris of any epoch, a Lagrange polynomial is used first to fit the given data and then to interpolate the data in the chosen epoch. The general Lagrange polynomial is (see Wang et al. 1979)

$$y(t) = \sum_{j=0}^{m} L_j(t) \cdot y(t_j), \tag{8.51}$$

where

$$L_j(t) = \prod_{k=0}^{m} \frac{(t - t_k)}{(t_j - t_k)}, \quad k \neq j, \tag{8.52}$$

where the symbol Π is a multiplying operator from $k = 0$ to $k = m$, m is the order of the polynomial, $y(t_j)$ are given data at the time t_j, $L_j(t)$ is called the base function of order m, and t is the time at which data will be interpolated. Generally speaking, t should be placed around the middle of the time duration (t_0, t_m) if possible. Therefore m is usually selected as an odd number. For IGS orbit interpolation, a standard m is selected as 7 or 9.

For the equal distance Lagrange interpolation there is

$$\begin{aligned} t_k &= t_0 + k\Delta t \\ t - t_k &= t - t_0 - k\Delta t \\ t_j - t_k &= (j - k)\Delta t, \end{aligned}$$

then

$$L_j(t) = \prod_{k=0}^{m} \frac{(t - t_0 - k\Delta t)}{(j - k)\Delta t}, \quad k \neq j, \tag{8.53}$$

where Δt is the data interval.

Usually the ephemeris of the sun and the moon are computed or forecast every half day (12 h). The ephemeris of the sun and the moon at a required epoch are interpolated from the data of the two adjacent epochs (t_1, t_2) by using a fifth-order polynomial:

$$f(t) = a + b(t - t_1) + c(t - t_1)^2 + d(t - t_1)^3 + e(t - t_1)^4 + f(t - t_1)^5.$$

For data at two epochs, e.g.

$$t_1 \; : \; x_1, y_1, z_1, \dot{x}_1, \dot{y}_1, \dot{z}_1, \ddot{x}_1, \ddot{y}_1, \ddot{z}_1$$

and

$$t_2 : \; x_2, y_2, z_2, \dot{x}_2, \dot{y}_2, \dot{z}_2, \ddot{x}_2, \ddot{y}_2, \ddot{z}_2,$$

where $\dot{x}$ and $\ddot{x}$ are the velocity and acceleration components related to x. Considering the formulas of $f(t)$, $\mathrm{d}f(t)/\mathrm{d}t$, $\mathrm{d}^2f(t)/\mathrm{d}t^2$ and letting $t = t_1$, one gets $a = x_1$, $b = \dot{x}_1$ and $c = \ddot{x}_1/2$. Letting $t = t_2$, coefficients of d, e, f can be derived theoretically, e.g. in the case of t_2–$t_1 = 0.5$:

$$d = 80(x_2 - x_1) - 16\dot{x}_2 - 24\dot{x}_1 + \ddot{x}_2 - 3\ddot{x}_1,$$

$$e = -240(x_2 - x_1) + 56\dot{x}_2 + 64\dot{x}_1 - 4\ddot{x}_2 + 6\ddot{x}_1,$$

$$f = 192(x_2 - x_1) - 48\dot{x}_2 - 48\dot{x}_1 + 4\ddot{x}_2 - 4\ddot{x}_1.$$

For components y and z the formulas are similar. Such an interpolating algorithm is accurate enough for using the given half-day ephemeris of the sun and moon to get the data at the required epoch. The computation of the ephemeris of the sun and moon are discussed in Sect. 7.8.

By deriving the Adams and Cowell algorithms, the Newtonian backward differentiation formula has been used to represent the disturbing function F. By simply considering F a function of t (t is any variable), one has

$$\begin{aligned} F(t) = F(t_n) + \frac{t - t_n}{h}\nabla F_n + \frac{(t - t_n)(t - t_{n-1})}{2!h^2}\nabla^2 F_n + \ldots \\ + \frac{(t - t_n)(t - t_{n-1})\cdots(t - t_{n-k+1})}{k!h^k}\nabla^k F_n. \end{aligned} \tag{8.54}$$

This is an interpolating formula of $F(t)$ using a set of function values of $\{\mathrm{F}_{n-j}, j = 0,\ldots,k\}$.

8.3 Orbit-Related Partial Derivatives

As mentioned in Sect. 8.1 the partial derivatives of

$$\frac{\partial \vec{f}}{\partial \vec{r}}, \quad \frac{\partial \vec{f}}{\partial \dot{\vec{r}}} \quad \text{and} \quad \frac{\partial \vec{f}}{\partial \vec{Y}} \tag{8.55}$$

will be derived in this section in detail, where the force vector is a summated vector of all disturbing forces in the ECSF coordinate system. If the force vector is given in the ECEF coordinate system, there is

$$\left(\frac{\partial \vec{f}}{\partial \vec{r}}, \frac{\partial \vec{f}}{\partial \dot{\vec{r}}}\right) = R_{\mathrm{P}}^{-1} R_{\mathrm{N}}^{-1} R_{\mathrm{S}}^{-1} R_{\mathrm{M}}^{-1} \left(\frac{\partial \vec{f}_{\mathrm{ECEF}}}{\partial \vec{r}}, \frac{\partial \vec{f}_{\mathrm{ECEF}}}{\partial \dot{\vec{r}}}\right). \tag{8.56}$$

Because

$$\vec{r} = R \cdot \vec{r}_{\mathrm{ECEF}}$$
$$\vec{f} = R \cdot \vec{f}_{\mathrm{ECEF}},$$

one may have the velocity transformation formula

$$\frac{d\vec{r}}{dt} = \frac{dR}{dt} \cdot \vec{r}_{ECEF} + R \cdot \frac{d\vec{r}_{ECEF}}{dt},$$

where

$$R = R_{\mathrm{P}}^{-1} R_{\mathrm{N}}^{-1} R_{\mathrm{S}}^{-1} R_{\mathrm{M}}^{-1}.$$

Therefore one has

$$\frac{\partial \vec{r}_{ECEF}}{\partial \vec{r}} = R^{-1},$$

$$\frac{\partial \dot{\vec{r}}_{ECEF}}{\partial \dot{\vec{r}}} = R^{-1},$$

and

$$\frac{\partial \vec{f}_{ECEF}}{\partial \vec{r}} = \frac{\partial \vec{f}_{ECEF}}{\partial \vec{r}_{ECEF}} \frac{\partial \vec{r}_{ECEF}}{\partial \vec{r}} = \frac{\partial \vec{f}_{ECEF}}{\partial \vec{r}_{ECEF}} R^{-1},$$

$$\frac{\partial \vec{f}_{ECEF}}{\partial \dot{\vec{r}}} = \frac{\partial \vec{f}_{ECEF}}{\partial \dot{\vec{r}}_{ECEF}} \frac{\partial \dot{\vec{r}}_{ECEF}}{\partial \dot{\vec{r}}} = \frac{\partial \vec{f}_{ECEF}}{\partial \dot{\vec{r}}_{ECEF}} R^{-1}.$$

8.3.1 Geopotential Disturbing Force

The geopotential disturbing force vector (see Sect. 4.2) has the form

$$\vec{f}_{\mathrm{ECEF}} = \begin{pmatrix} f_{x'} \\ f_{y'} \\ f_{z'} \end{pmatrix} = \begin{pmatrix} b_{11} \frac{\partial V}{\partial r} + b_{21} \frac{\partial V}{\partial \varphi} + b_{31} \frac{\partial V}{\partial \lambda} \\ b_{12} \frac{\partial V}{\partial r} + b_{22} \frac{\partial V}{\partial \varphi} + b_{32} \frac{\partial V}{\partial \lambda} \\ b_{13} \frac{\partial V}{\partial r} + b_{23} \frac{\partial V}{\partial \varphi} \end{pmatrix}, \tag{8.57}$$

where

$$\frac{\partial(r,\varphi,\lambda)}{\partial(x',y',z')} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} \cos\varphi\cos\lambda & \cos\varphi\sin\lambda & \sin\varphi \\ -\frac{1}{r}\sin\varphi\cos\lambda & -\frac{1}{r}\sin\varphi\sin\lambda & \frac{1}{r}\cos\varphi \\ -\frac{1}{r\cos\varphi}\sin\lambda & \frac{1}{r\cos\varphi}\cos\lambda & 0 \end{pmatrix},$$

and (x',y',z') are the three orthogonal Cartesian coordinates in the ECEF system. Thus

$$\frac{\partial \vec{f}_{\mathrm{ECEF}}}{\partial \vec{r}} = \begin{pmatrix} \frac{\partial f_{x'}}{\partial(x',y',z')} \\ \frac{\partial f_{y'}}{\partial(x',y',z')} \\ \frac{\partial f_{z'}}{\partial(x',y',z')} \end{pmatrix} = \left(\frac{\partial\left(f_{x'}, f_{y'}, f_{z'}\right)}{\partial(r,\varphi,\lambda)} \frac{\partial(r,\varphi,\lambda)}{\partial(x',y',z')} \right)^T. \tag{8.58}$$

Using index j (=1, 2, 3) to denote index (x', y', z'), one has

$$\frac{\partial f_j}{\partial(r,\varphi,\lambda)} = \begin{pmatrix} \frac{\partial b_{1j}}{\partial r}\frac{\partial V}{\partial r} + \frac{\partial b_{2j}}{\partial r}\frac{\partial V}{\partial \varphi} + \frac{\partial b_{3j}}{\partial r}\frac{\partial V}{\partial \lambda} + b_{1j}\frac{\partial^2 V}{\partial r^2} + b_{2j}\frac{\partial^2 V}{\partial r \partial \varphi} + b_{3j}\frac{\partial^2 V}{\partial r \partial \lambda} \\ \frac{\partial b_{1j}}{\partial \varphi}\frac{\partial V}{\partial r} + \frac{\partial b_{2j}}{\partial \varphi}\frac{\partial V}{\partial \varphi} + \frac{\partial b_{3j}}{\partial \varphi}\frac{\partial V}{\partial \lambda} + b_{1j}\frac{\partial^2 V}{\partial r \partial \varphi} + b_{2j}\frac{\partial^2 V}{\partial \varphi^2} + b_{3j}\frac{\partial^2 V}{\partial \varphi \partial \lambda} \\ \frac{\partial b_{1j}}{\partial \lambda}\frac{\partial V}{\partial r} + \frac{\partial b_{2j}}{\partial \lambda}\frac{\partial V}{\partial \varphi} + \frac{\partial b_{3j}}{\partial \lambda}\frac{\partial V}{\partial \lambda} + b_{1j}\frac{\partial^2 V}{\partial r \partial \lambda} + b_{2j}\frac{\partial^2 V}{\partial \varphi \partial \lambda} + b_{3j}\frac{\partial^2 V}{\partial \lambda^2} \end{pmatrix}^T, \tag{8.59}$$

where

$$\frac{\partial}{\partial r}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{r^2}\sin\varphi\cos\lambda & \frac{1}{r^2}\sin\varphi\sin\lambda & \frac{-1}{r^2}\cos\varphi \\ \frac{1}{r^2\cos\varphi}\sin\lambda & \frac{-1}{r^2\cos\varphi}\cos\lambda & 0 \end{pmatrix},$$

$$\frac{\partial}{\partial \varphi}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} -\sin\varphi\cos\lambda & -\sin\varphi\sin\lambda & \cos\varphi \\ -\frac{1}{r}\cos\varphi\cos\lambda & -\frac{1}{r}\cos\varphi\sin\lambda & -\frac{1}{r}\sin\varphi \\ -\frac{\sin\varphi}{r\cos^2\varphi}\sin\lambda & \frac{\sin\varphi}{r\cos^2\varphi}\cos\lambda & 0 \end{pmatrix}$$

and

$$\frac{\partial}{\partial \lambda}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} -\cos\varphi\sin\lambda & \cos\varphi\cos\lambda & 0 \\ \frac{1}{r}\sin\varphi\sin\lambda & -\frac{1}{r}\sin\varphi\cos\lambda & 0 \\ -\frac{1}{r\cos\varphi}\cos\lambda & -\frac{1}{r\cos\varphi}\sin\lambda & 0 \end{pmatrix} \tag{8.60}$$

and

$$\frac{\partial^2 V}{\partial r^2} = \frac{\mu}{r^3}\left[2 + \sum_{l=2}^{\infty}\sum_{m=0}^{l}(l+1)(l+2)\left(\frac{a}{r}\right)^l \bar{P}_{lm}(\sin\varphi)[\bar{C}_{lm}\cos m\lambda + \bar{S}_{lm}\sin m\lambda]\right],$$

$$\frac{\partial^2 V}{\partial r\partial\varphi} = -\frac{\mu}{r^2}\sum_{l=2}^{\infty}\sum_{m=0}^{l}(l+1)\left(\frac{a}{r}\right)^l \frac{\mathrm{d}\bar{P}_{lm}(\sin\varphi)}{\mathrm{d}\varphi}[\bar{C}_{lm}\cos m\lambda + \bar{S}_{lm}\sin m\lambda],$$

$$\frac{\partial^2 V}{\partial r\partial\lambda} = -\frac{\mu}{r^2}\left[\sum_{l=2}^{\infty}\sum_{m=0}^{l}(l+1)\left(\frac{a}{r}\right)^l \bar{P}_{lm}(\sin\varphi)m[-\bar{C}_{lm}\sin m\lambda + \bar{S}_{lm}\cos m\lambda]\right],$$

$$\frac{\partial^2 V}{\partial\varphi^2} = \frac{\mu}{r}\sum_{l=2}^{\infty}\sum_{m=0}^{l}\left(\frac{a}{r}\right)^l \frac{\mathrm{d}^2\bar{P}_{lm}(\sin\varphi)}{\mathrm{d}\varphi^2}[\bar{C}_{lm}\cos m\lambda + \bar{S}_{lm}\sin m\lambda],$$

$$\frac{\partial^2 V}{\partial\varphi\partial\lambda} = \frac{\mu}{r}\sum_{l=2}^{\infty}\sum_{m=0}^{l}\left(\frac{a}{r}\right)^l \frac{\mathrm{d}\bar{P}_{lm}(\sin\varphi)}{\mathrm{d}\varphi}m[-\bar{C}_{lm}\sin m\lambda + \bar{S}_{lm}\cos m\lambda]$$

and

$$\frac{\partial^2 V}{\partial\lambda^2} = -\frac{\mu}{r}\sum_{l=2}^{\infty}\sum_{m=0}^{l}m^2\left(\frac{a}{r}\right)^l \bar{P}_{lm}(\sin\varphi)[\bar{C}_{lm}\cos m\lambda + \bar{S}_{lm}\sin m\lambda], \tag{8.61}$$

where

$$\frac{\mathrm{d}\bar{P}_{lm}(\sin\varphi)}{\mathrm{d}\varphi} = \beta(m)\bar{P}_{l(m+1)}(\sin\varphi) - m\tan\varphi\bar{P}_{lm}(\sin\varphi),$$

$$\begin{aligned}\frac{\mathrm{d}^2\bar{P}_{lm}(\sin\varphi)}{\mathrm{d}\varphi^2} &= \beta(m)\frac{\mathrm{d}\bar{P}_{l(m+1)}(\sin\varphi)}{\mathrm{d}\varphi} - m\frac{1}{\cos^2\varphi}\bar{P}_{lm}(\sin\varphi) - m\tan\varphi\frac{\mathrm{d}\bar{P}_{lm}(\sin\varphi)}{\mathrm{d}\varphi} \\ &= \beta(m)\beta(m+1)\bar{P}_{l(m+2)}(\sin\varphi) - \beta(m)\tan\varphi(2m+1)\bar{P}_{l(m+1)}(\sin\varphi) \\ &\quad + \left(m^2\tan^2\varphi - m\frac{1}{\cos^2\varphi}\right)\bar{P}_{lm}(\sin\varphi),\end{aligned}$$

$$\beta(m) = \left[\frac{1}{2}(2-\delta_{0m})(l-m)(l+m+1)\right]^{1/2}$$

and

$$\beta(m+1) = \left[\frac{1}{2}(l-m-1)(l+m+2)\right]^{1/2}. \tag{8.62}$$

Other required functions are already given in Sect. 4.2. Because the force is not a function of velocity, it is obvious that

$$\frac{\partial\vec{f}_{\mathrm{ECEF}}}{\partial\dot{\vec{r}}} = [0]_{3\times 3}. \tag{8.63}$$

Only non-zero partial derivatives will be given in the text that follows.

Supposing the geopotential parameters $\bar{C}^N_{lm}, \bar{S}^N_{lm}$ are known (as initial values), $\bar{C}_{lm}$, $\bar{S}_{lm}$ are true values and $\Delta\bar{C}_{lm}$, $\Delta\bar{S}_{lm}$ are searched corrections (unknowns); then the geopotential force is

$$\begin{aligned}\vec{f}_{\mathrm{ECEF}}(\bar{C}_{lm},\bar{S}_{lm}) &= \vec{f}_{\mathrm{ECEF}}(\bar{C}^N_{lm},\bar{S}^N_{lm}) + \vec{f}_{\mathrm{ECEF}}(\bar{C}_{lm},\bar{S}_{lm}) - \vec{f}_{\mathrm{ECEF}}(\bar{C}^N_{lm},\bar{S}^N_{lm}) \\ &= \vec{f}_{\mathrm{ECEF}}(\bar{C}^N_{lm},\bar{S}^N_{lm}) + \vec{f}_{\mathrm{ECEF}}(\Delta\bar{C}_{lm},\Delta\bar{S}_{lm}),\end{aligned} \tag{8.64}$$

and

$$\frac{\partial\vec{f}_{\mathrm{ECEF}}}{\partial(\Delta\bar{C}_{lm},\Delta\bar{S}_{lm})} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33}\end{pmatrix}^T \frac{\partial}{\partial(\Delta\bar{C}_{lm},\Delta\bar{S}_{lm})}\begin{pmatrix}\frac{\partial V}{\partial r} \\ \frac{\partial V}{\partial\varphi} \\ \frac{\partial V}{\partial\lambda}\end{pmatrix},$$

$$\frac{\partial}{\partial(\Delta\bar{C}_{lm},\Delta\bar{S}_{lm})}\left(\frac{\partial V}{\partial r}\right)=-\frac{\mu}{r^2}(l+1)\left(\frac{a}{r}\right)^l\bar{P}_{lm}(\sin\varphi)(\cos m\lambda \quad \sin m\lambda),$$

$$\frac{\partial}{\partial(\Delta\bar{C}_{lm},\Delta\bar{S}_{lm})}\left(\frac{\partial V}{\partial \varphi}\right)=\frac{\mu}{r}\left(\frac{a}{r}\right)^l\frac{\mathrm{d}\bar{P}_{lm}(\sin\varphi)}{\mathrm{d}\varphi}(\cos m\lambda \quad \sin m\lambda)$$

and

$$\frac{\partial}{\partial(\Delta\bar{C}_{lm},\Delta\bar{S}_{lm})}\left(\frac{\partial V}{\partial \lambda}\right)=\frac{\mu}{r}m\left(\frac{a}{r}\right)^l\bar{P}_{lm}(\sin\varphi)(-\sin m\lambda \quad \cos m\lambda). \tag{8.65}$$

8.3.2 Perturbation Forces of the Sun and the Moon as well as Planets

The perturbation forces of the sun, the moon and the planets are given in Sect. 4.2 (see (4.50)) as

$$\vec{f}_m=-m\sum_j \mathrm{Gm}(j)\left[\frac{\vec{r}-\vec{r}_{m(j)}}{\left|\vec{r}-\vec{r}_{m(j)}\right|^3}+\frac{\vec{r}_{m(j)}}{r^3_{m(j)}}\right], \tag{8.66}$$

where Gm(j) are the gravitational constants of the sun and the moon as well as the planets, and the vectors with index $m(j)$ are the geocentric vectors of the sun, the moon and the planets. The partial derivatives of the perturbation force with respect to the satellite vector are then

$$\frac{\partial\vec{f}_m}{\partial\vec{r}}=-m\sum_j\frac{\mathrm{Gm}(j)}{\left|\vec{r}-\vec{r}_{m(j)}\right|^3}\left(E-\frac{3}{\left|\vec{r}-\vec{r}_{m(j)}\right|^2}\begin{pmatrix}x-x_{m(j)}\\ y-y_{m(j)}\\ z-z_{m(j)}\end{pmatrix}\begin{pmatrix}x-x_{m(j)}\\ y-y_{m(j)}\\ z-z_{m(j)}\end{pmatrix}^T\right), \tag{8.67}$$

where E is an identity matrix of size 3 × 3. The partial derivatives of the force vector with respect to the velocity vector of the satellite are zero. The disturbances of the sun, moon and planets are considered well-modelled; therefore no parameters will be adjusted. In other words, the partial derivatives of the force vector with respect to the model parameters do not exist.

8.3.3 Tidal Disturbing Forces

Similar to the geopotential attracting force, the tidal force (see Sect. 4.2.3) has the form

$$\vec{f}_{\mathrm{ECEF}} = \begin{pmatrix} f_{x'} \\ f_{y'} \\ f_{z'} \end{pmatrix} = \begin{pmatrix} b_{11}\frac{\partial V}{\partial r} + b_{21}\frac{\partial V}{\partial \varphi} + b_{31}\frac{\partial V}{\partial \lambda} \\ b_{12}\frac{\partial V}{\partial r} + b_{22}\frac{\partial V}{\partial \varphi} + b_{32}\frac{\partial V}{\partial \lambda} \\ b_{13}\frac{\partial V}{\partial r} + b_{23}\frac{\partial V}{\partial \varphi} \end{pmatrix}, \tag{8.68}$$

where $V = \delta V + \delta V_1 + \delta V_2$, which is a summation of the Earth tide potential and the two parts of ocean loading tide potentials. Equation (8.59) is still valid for this case. Other higher order partial derivatives can be derived as follows:

$$\frac{\partial^2 \delta V}{\partial r^2} = \sum_{j=1}^{2}\mu_j \sum_{n=2}^{N} k_n \frac{(n+1)(n+2)a_{\mathrm{e}}^{2n+1}}{r^{n+3} r_j^{n+1}} \begin{bmatrix} P_n(\sin\varphi)P_n(\sin\delta_j) \\ +2\sum_{k=1}^{n}\frac{(n-k)!}{(n+k)!}P_{nk}(\sin\varphi)P_{nk}(\sin\delta_j)\cos kh_j \end{bmatrix},$$

$$\frac{\partial^2 \delta V}{\partial r \partial \varphi} = \sum_{j=1}^{2}\mu_j \sum_{n=2}^{N} -k_n \frac{(n+1)a_{\mathrm{e}}^{2n+1}}{r^{n+2} r_j^{n+1}} \begin{bmatrix} \frac{\mathrm{d}P_n(\sin\varphi)}{\mathrm{d}\varphi}P_n(\sin\delta_j) \\ +2\sum_{k=1}^{n}\frac{(n-k)!}{(n+k)!}\frac{\mathrm{d}P_{nk}(\sin\varphi)}{\mathrm{d}\varphi}P_{nk}(\sin\delta_j)\cos kh_j \end{bmatrix},$$

$$\frac{\partial^2 \delta V}{\partial r \partial \lambda} = \sum_{j=1}^{2}\mu_j \sum_{n=2}^{N} -k_n \frac{(n+1)a_{\mathrm{e}}^{2n+1}}{r^{n+2} r_j^{n+1}} \left[2\sum_{k=1}^{n}\frac{(n-k)!}{(n+k)!}P_{nk}(\sin\varphi)P_{nk}(\sin\delta_j)k\sin kh_j \right],$$

$$\frac{\partial^2 \delta V}{\partial \varphi^2} = \sum_{j=1}^{2}\mu_j \sum_{n=2}^{N} k_n \frac{a_{\mathrm{e}}^{2n+1}}{r^{n+1} r_j^{n+1}} \begin{bmatrix} \frac{\mathrm{d}^2 P_n(\sin\varphi)}{\mathrm{d}\varphi^2}P_n(\sin\delta_j) \\ +2\sum_{k=1}^{n}\frac{(n-k)!}{(n+k)!}\frac{\mathrm{d}^2 P_{nk}(\sin\varphi)}{\mathrm{d}\varphi^2}P_{nk}(\sin\delta_j)\cos kh_j \end{bmatrix},$$

$$\frac{\partial^2 \delta V}{\partial \varphi \partial \lambda} = \sum_{j=1}^{2}\mu_j \sum_{n=2}^{N} k_n \frac{a_{\mathrm{e}}^{2n+1}}{r^{n+1} r_j^{n+1}} \left[2\sum_{k=1}^{n}\frac{(n-k)!}{(n+k)!}\frac{\mathrm{d}P_{nk}(\sin\varphi)}{\mathrm{d}\varphi}P_{nk}(\sin\delta_j)k\sin kh_j \right],$$

$$\frac{\partial^2 \delta V}{\partial \lambda^2} = \sum_{j=1}^{2}\mu_j \sum_{n=2}^{N} -k_n \frac{a_{\mathrm{e}}^{2n+1}}{r^{n+1} r_j^{n+1}} \left[2\sum_{k=1}^{n}\frac{(n-k)!}{(n+k)!}k^2 P_{nk}(\sin\varphi)P_{nk}(\sin\delta_j)\cos kh_j \right],$$

$$\frac{\partial^2 \delta V_1}{\partial r^2} = \oiint_O G\sigma H \sum_{n=0}^{\infty} (1+k_n') \frac{(n+1)(n+2)a_\mathrm{e}^n}{r^{n+3}} \times \left[\begin{array}{l} P_n(\sin\varphi) P_n(\sin\varphi_s) + (2-\delta_{0n}) \cdot \\ \sum_{k=0}^{n} \frac{(n-k)!}{(n+k)!} P_{nk}(\sin\varphi) P_{nk}(\sin\varphi_s) \cos k(\lambda_s - \lambda) \end{array} \right] \mathrm{d}s,$$

$$\frac{\partial^2 \delta V_1}{\partial r \partial\varphi} = \oiint_O G\sigma H \sum_{n=0}^{\infty} (1+k_n') \frac{-(n+1)a_\mathrm{e}^n}{r^{n+2}} \times \left[\begin{array}{l} \frac{\mathrm{d}P_n(\sin\varphi)}{\mathrm{d}\varphi} P_n(\sin\varphi_s) + (2-\delta_{0n}) \cdot \\ \sum_{k=0}^{n} \frac{(n-k)!}{(n+k)!} \frac{\mathrm{d}P_{nk}(\sin\varphi)}{\mathrm{d}\varphi} P_{nk}(\sin\varphi_s) \cos k(\lambda_s - \lambda) \end{array} \right] \mathrm{d}s,$$

$$\frac{\partial^2 \delta V_1}{\partial r \partial\lambda} = \oiint_O G\sigma H \sum_{n=0}^{\infty} (1+k_n') \frac{-(n+1)a_\mathrm{e}^n}{r^{n+2}} \times \left[\begin{array}{l} (2-\delta_{0n}) \cdot \\ \sum_{k=0}^{n} \frac{(n-k)!}{(n+k)!} P_{nk}(\sin\varphi) P_{nk}(\sin\varphi_s) k \sin k(\lambda_s - \lambda) \end{array} \right] \mathrm{d}s,$$

$$\frac{\partial^2 \delta V_1}{\partial\varphi^2} = \oiint_O G\sigma H \sum_{n=0}^{\infty} (1+k_n') \frac{a_\mathrm{e}^n}{r^{n+1}} \times \left[\begin{array}{l} \frac{\mathrm{d}^2 P_n(\sin\varphi)}{\mathrm{d}\varphi^2} P_n(\sin\varphi_s) + (2-\delta_{0n}) \cdot \\ \sum_{k=0}^{n} \frac{(n-k)!}{(n+k)!} \frac{\mathrm{d}^2 P_{nk}(\sin\varphi)}{\mathrm{d}\varphi^2} P_{nk}(\sin\varphi_s) \cos k(\lambda_s - \lambda) \end{array} \right] \mathrm{d}s,$$

$$\frac{\partial^2 \delta V_1}{\partial\varphi \partial\lambda} = \oiint_O G\sigma H \sum_{n=0}^{\infty} (1+k_n') \frac{a_\mathrm{e}^n}{r^{n+1}} \times \left[\begin{array}{l} (2-\delta_{0n}) \cdot \\ \sum_{k=0}^{n} \frac{(n-k)!}{(n+k)!} \frac{\mathrm{d}P_{nk}(\sin\varphi)}{\mathrm{d}\varphi} P_{nk}(\sin\varphi_s) k \sin k(\lambda_s - \lambda) \end{array} \right] \mathrm{d}s,$$

$$\frac{\partial^2 \delta V_1}{\partial \lambda^2} = \oiint_O G\sigma H \sum_{n=0}^{\infty} (1 + k'_n) \frac{a_e^n}{r^{n+1}} \times \left[\begin{array}{l} -(2-\delta_{0n})\cdot \\ \sum_{k=0}^{n} \frac{(n-k)!}{(n+k)!} P_{nk}(\sin\varphi) P_{nk}(\sin\varphi_s) k^2 \cos k(\lambda_s - \lambda) \end{array} \right] \mathrm{d}s,$$

$$\frac{\partial^2 \delta V_2}{\partial r^2} = \oiint_C G\sigma_e u_r \sum_{n=0}^{\infty} \frac{(n+1)(n+2)a_e^n}{r^{n+3}} \times \left[\begin{array}{l} P_n(\sin\varphi) P_n(\sin\varphi_s) + (2-\delta_{0n})\cdot \\ \sum_{k=0}^{n} \frac{(n-k)!}{(n+k)!} P_{nk}(\sin\varphi) P_{nk}(\sin\varphi_s) \cos k(\lambda_s - \lambda) \end{array} \right] \mathrm{d}s,$$

$$\frac{\partial^2 \delta V_2}{\partial r \partial \varphi} = \oiint_C G\sigma_e u_r \sum_{n=0}^{\infty} \frac{-(n+1)a_e^n}{r^{n+2}} \times \left[\begin{array}{l} \frac{\mathrm{d}P_n(\sin\varphi)}{\mathrm{d}\varphi} P_n(\sin\varphi_s) + (2-\delta_{0n})\cdot \\ \sum_{k=0}^{n} \frac{(n-k)!}{(n+k)!} \frac{\mathrm{d}P_{nk}(\sin\varphi)}{\mathrm{d}\varphi} P_{nk}(\sin\varphi_s) \cos k(\lambda_s - \lambda) \end{array} \right] \mathrm{d}s,$$

$$\frac{\partial^2 \delta V_2}{\partial r \partial \lambda} = \oiint_C G\sigma_e u_r \sum_{n=0}^{\infty} \frac{-(n+1)a_e^n}{r^{n+2}} \times \left[\begin{array}{l} (2-\delta_{0n})\cdot \\ \sum_{k=0}^{n} \frac{(n-k)!}{(n+k)!} P_{nk}(\sin\varphi) P_{nk}(\sin\varphi_s) k \sin k(\lambda_s - \lambda) \end{array} \right] \mathrm{d}s,$$

$$\frac{\partial^2 \delta V_2}{\partial \varphi^2} = \oiint_C G\sigma_e u_r \sum_{n=0}^{\infty} \frac{a_e^n}{r^{n+1}} \times \left[\begin{array}{l} \frac{\mathrm{d}^2 P_n(\sin\varphi)}{\mathrm{d}\varphi^2} P_n(\sin\varphi_s) + (2-\delta_{0n})\cdot \\ \sum_{k=0}^{n} \frac{(n-k)!}{(n+k)!} \frac{\mathrm{d}^2 P_{nk}(\sin\varphi)}{\mathrm{d}\varphi^2} P_{nk}(\sin\varphi_s) \cos k(\lambda_s - \lambda) \end{array} \right] \mathrm{d}s,$$

$$\frac{\partial^2 \delta V_2}{\partial \varphi \partial \lambda} = \oiint\limits_C G\sigma_e u_r \sum_{n=0}^{\infty} \frac{a_e^n}{r^{n+1}} \times \left[\begin{array}{l} (2-\delta_{0n}) \cdot \\ \sum_{k=0}^{n} \frac{(n-k)!}{(n+k)!} \frac{\mathrm{d}P_{nk}(\sin\varphi)}{\mathrm{d}\varphi} P_{nk}(\sin\varphi_s) k \sin k(\lambda_s - \lambda) \end{array} \right] \mathrm{d}s$$

and

$$\frac{\partial^2 \delta V_2}{\partial \lambda^2} = \oiint\limits_C G\sigma_e u_r \sum_{n=0}^{\infty} \frac{a_e^n}{r^{n+1}} \times \left[\begin{array}{l} -(2-\delta_{0n}) \cdot \\ \sum_{k=0}^{n} \frac{(n-k)!}{(n+k)!} P_{nk}(\sin\varphi) P_{nk}(\sin\varphi_s) k^2 \cos k(\lambda_s - \lambda) \end{array} \right] \mathrm{d}s, \tag{8.69}$$

where

$$\frac{\mathrm{d}P_n(\sin\varphi)}{\mathrm{d}\varphi} = \frac{n}{\cos\varphi} (P_{n-1}(\sin\varphi) - \sin\varphi P_n(\sin\varphi))$$

and

$$\frac{\mathrm{d}P_{nk}(\sin\varphi)}{\mathrm{d}\varphi} = P_{n(k+1)}(\sin\varphi) - k\tan\varphi P_{nk}(\sin\varphi). \tag{8.70}$$

8.3.4 Solar Radiation Pressure

If the solar radiation force acting on the satellite's surface (see Sect. 4.2.4) is

$$\vec{f}_{\text{solar}} = m\gamma P_s C_r r_{\text{sun}}^2 \frac{S}{m} \frac{\vec{r} - \vec{r}_{\text{sun}}}{|\vec{r} - \vec{r}_{\text{sun}}|^3}, \tag{8.71}$$

then the partial derivatives of the perturbation force with respect to the satellite vector are

$$\frac{\partial \vec{f}_{\text{solar}}}{\partial \vec{r}} = m\gamma P_s C_r r_{\text{sun}}^2 \frac{S}{m}$$
$$\times \frac{1}{|\vec{r} - \vec{r}_{\text{sun}}|^3} \left(E - \frac{3}{|\vec{r} - \vec{r}_{\text{sun}}|^2} \begin{pmatrix} x - x_{\text{sun}} \\ y - y_{\text{sun}} \\ z - z_{\text{sun}} \end{pmatrix} \begin{pmatrix} x - x_{\text{sun}} \\ y - y_{\text{sun}} \\ z - z_{\text{sun}} \end{pmatrix}^T \right), \tag{8.72}$$

where E is an identity matrix of size 3×3. The partial derivatives of the force vector with respect to the velocity vector of the satellite are zero. The disturbance of the solar radiation is considered not well-modelled; therefore, unknown parameters will also be adjusted. The total model (see Sect. 4.2.4) is

$$\vec{f}_{\text{solar-force}} = \vec{f}_{\text{solar}} + \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 \\ \cos u \\ \sin u \end{pmatrix}. \tag{8.73}$$

Thus,

$$\frac{\partial \vec{f}_{\text{solar-force}}}{\partial \vec{r}} = \frac{\partial \vec{f}_{\text{solar}}}{\partial \vec{r}} + \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 0 \\ -\sin u \\ \cos u \end{pmatrix} \frac{\partial u}{\partial \vec{r}}, \tag{8.74}$$

where

$$\frac{\partial u}{\partial \vec{r}} = \frac{\partial u}{\partial(\Omega, i, \omega, a, e, M)} \frac{\partial(\Omega, i, \omega, a, e, M)}{\partial(\vec{r}, \dot{\vec{r}})} \frac{\partial(\vec{r}, \dot{\vec{r}})}{\partial \vec{r}}. \tag{8.75}$$

On the right-hand side of above equation there are three matrices, the first is a 1×6 matrix (vector) and is given in Sect. 4.1.2 (see (4.24)), the second is given as its inverse in Sect. 5.4 (see (5.59) and (5.61)), and the third is a 6×3 matrix, or

$$\frac{\partial u}{\partial(\Omega, i, \omega, a, e, M)} = \left(0,\ 0,\ 1,\ 0,\ \frac{2 + e\cos f}{1 - e^2} \sin f,\ \left(\frac{a}{r}\right)^2 \sqrt{1 - e^2}\right),$$

$$\frac{\partial(\Omega, i, \omega, a, e, M)}{\partial(\vec{r}, \dot{\vec{r}})} = \left(\frac{\partial(\vec{r}, \dot{\vec{r}})}{\partial(\Omega, i, \omega, a, e, M)} \right)^{-1} = \begin{pmatrix} \frac{\partial R}{\partial(\Omega, i, \omega)} \vec{q} & R \frac{\partial \vec{q}}{\partial(a, e, M)} \\ \frac{\partial R}{\partial(\Omega, i, \omega)} \dot{\vec{q}} & R \frac{\partial \dot{\vec{q}}}{\partial(a, e, M)} \end{pmatrix}^{-1}$$

and

$$\frac{\partial(\vec{r}, \dot{\vec{r}})}{\partial \vec{r}} = \begin{pmatrix} E_{3\times3} \\ 0_{3\times3} \end{pmatrix}. \tag{8.76}$$

$$\frac{\partial u}{\partial \dot{\vec{r}}} = \frac{\partial u}{\partial(\Omega, i, \omega, a, e, M)} \frac{\partial(\Omega, i, \omega, a, e, M)}{\partial(\vec{r}, \dot{\vec{r}})} \frac{\partial(\vec{r}, \dot{\vec{r}})}{\partial \dot{\vec{r}}}$$

and

$$\frac{\partial(\vec{r}, \dot{\vec{r}})}{\partial \dot{\vec{r}}} = \begin{pmatrix} 0_{3\times3} \\ E_{3\times3} \end{pmatrix}. \tag{8.77}$$

The partial derivatives of the force vector with respect to the model parameters are (for $i = 1,2,3$)

$$\frac{\partial \vec{f}_{\text{solar-force}}}{\partial a_{ij}} = \begin{cases} 1 & \text{if } j = 1 \\ \cos u & \text{if } j = 2 \\ \sin u & \text{if } j = 3 \end{cases}. \tag{8.78}$$

If the model (4.74)

$$\alpha \vec{f}_{solar} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} \begin{pmatrix} 1 \\ t \end{pmatrix} \tag{8.79}$$

is used, then one has

$$\frac{\partial \vec{f}_{\text{solar-force}}}{\partial(a_i, b_i)} = (1,\ t), \quad i = 1, 2, 3. \tag{8.80}$$

8.3.5 Atmospheric Drag

Atmospheric drag force has the form (see Sect. 4.2.5)

$$\vec{f}_{\text{drag}} = -m\frac{1}{2}\left(\frac{C_d S}{m}\right)\sigma\left|\dot{\vec{r}} - \dot{\vec{r}}_{\text{air}}\right|\left(\dot{\vec{r}} - \dot{\vec{r}}_{\text{air}}\right), \tag{8.81}$$

and the air drag force model is

$$\vec{f}_{\text{air-drag}} = \vec{f}_{\text{drag}} + (1+q)\Delta \vec{f}_{drag}, \tag{8.82}$$

where (see (4.84) and (4.85))

$$\Delta\vec{f}_{drag} = [a + b\varphi(2\omega)\cos(2f) + c\varphi(3\omega)\cos(3f) + d\varphi(\omega)\cos f]\,\vec{p}, \tag{8.83}$$

$$\varphi(k\omega) = \begin{cases} \sin k\omega & \textit{if } \cos k\omega = 0 \\ \frac{1}{\cos k\omega} & \textit{if } \cos k\omega \neq 0 \end{cases}, \quad k = 1,\ 2,\ 3 \tag{8.84}$$

It is obvious that the partial derivatives of the air drag force with respect to the satellite position vector are zero and

$$\frac{\partial\vec{f}_{\text{drag}}}{\partial\dot{\vec{r}}} = -m\frac{1}{2}\left(\frac{C_{\text{d}}S}{m}\right)\sigma\left(\left|\dot{\vec{r}} - \dot{\vec{r}}_{\text{air}}\right|E + \frac{1}{\left|\dot{\vec{r}} - \dot{\vec{r}}_{\text{air}}\right|}\begin{pmatrix}\dot{x} - \dot{x}_{\text{air}} \\ \dot{y} - \dot{y}_{\text{air}} \\ \dot{z} - \dot{z}_{\text{air}}\end{pmatrix}\begin{pmatrix}\dot{x} - \dot{x}_{\text{air}} \\ \dot{y} - \dot{y}_{\text{air}} \\ \dot{z} - \dot{z}_{\text{air}}\end{pmatrix}^T\right), \tag{8.85}$$

$$\frac{\partial\Delta\vec{f}_{drag}}{\partial f} = [-2b\varphi(2\omega)\sin(2f) - c\varphi(3\omega)\sin(3f) - d\varphi(\omega)\sin f]\,\vec{p}, \tag{8.86}$$

$$\frac{\partial\Delta\vec{f}_{drag}}{\partial\omega} = \left[b\cos(2f)\frac{\partial\varphi(2\omega)}{\partial\omega} + c\cos(3f)\frac{\partial\varphi(3\omega)}{\partial\omega} + d\cos f\frac{\partial\varphi(\omega)}{\partial\omega}\right]\vec{p}, \tag{8.87}$$

$$\frac{\partial\varphi(k\omega)}{\partial\omega} = \begin{cases} k\cos k\omega & \textit{if } \cos k\omega = 0 \\ \frac{k\tan k\omega}{\cos k\omega} & \textit{if } \cos k\omega \neq 0 \end{cases}, \quad k = 1,\ 2,\ 3 \tag{8.88}$$

$$\frac{\partial\Delta\vec{f}_{drag}}{\partial(\vec{r},\,\dot{\vec{r}})} = \frac{\partial\Delta\vec{f}_{drag}}{\partial(\omega,f)}\,\frac{\partial(\omega,f)}{\partial(\Omega,i,\omega,a,e,M)}\,\frac{\partial(\Omega,i,\omega,a,e,M)}{\partial(\vec{r},\dot{\vec{r}})}\,\frac{\partial(\vec{r},\dot{\vec{r}})}{\partial(\vec{r},\,\dot{\vec{r}})}, \tag{8.89}$$

where

$$\frac{\partial\omega}{\partial(\Omega,i,\omega,a,e,M)} = (0,\ 0,\ 1,\ 0,\ 0,\ 0),$$

$$\frac{\partial f}{\partial(\Omega,i,\omega,a,e,M)} = \left(0,\ 0,\ 0,\ 0,\ \frac{2 + e\cos f}{1 - e^2}\sin f,\ \ \left(\frac{a}{r}\right)^2\sqrt{1 - e^2}\right)$$

Some of the formulas in this section have already been derived. The partial derivatives of the force vector with respect to the model parameters can be obtained from (8.82) and (8.83).

Chapter 9
Analytical Orbit Determination

Chapters 5, 6, and 7 covered the most important contents of analytical solutions of the disturbed equations of satellite motion. In this chapter, emphasis will be on the applications of the analytical orbit theory.

9.1 Principle of Analytical Orbit Determination

Orbit determination aims to determine the initial orbital elements (i.e. the initial state vector of the satellite) and the unknown model parameters. The technique of numerical orbit determination is developed in a situation that, on one hand, one needs the technique; however, on the other hand, one does not have analytical solutions of the disturbed equations of satellite motion. The key difference between the numerical and the analytical orbit determination is that the orbits are represented in the former algorithm by differential equations and in the latter algorithm by analytical formulas.

Recalling the discussions in Chaps. 5, 6, and 7, the perturbed orbit of the satellite is the solution (or integration)

$$\sigma_j(t) = \sigma_j(t_0) + (G_j(t) - G_j(t_0)) \quad \text{where } \; G_j(t) - G_j(t_0) = \int_{t_0}^{t} F_j \mathrm{d}t. \tag{9.1}$$

$G_j(t)$ are the indefinite integrations of the right functions of the equations of motion and are given explicitly by analytical formulas. Equation 9.1 have been obtained by integrating the disturbed equations of motion

$$\begin{cases} \dot{\sigma}_j(t) = F_j \\ \sigma_j(t_0) = \sigma_{j0} \end{cases}, \tag{9.2}$$

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where $\sigma_j(t)$ is the jth Keplerian element, $\sigma_j(t_0)$ is the related initial value at time t_0 and F_j is the related right function of the differential equation and is a function of disturbing forces.

If the initial Keplerian elements and the force functions are precisely known, then the precise orbits can be computed by using (9.1). Computing for time t in the future, the so-called forecasted orbits can be obtained. That is, for orbit determination using analytical solutions, the traditional numerical integration algorithms are not necessary any more (because the differential equations are theoretically integrated by deriving the solutions).

In practice, the precise initial Keplerian elements are not exactly known and the parameters of the force models have to be co-determined. These can be realised through suitable parameterisation of the models in the GPS observation equations and then solved by adjustment or filtering.

We generally denote both the range and range rate together by ρ; their partial derivatives with respect to the orbit state vector (see Xu 2003, 2007) have the form

$$\frac{\partial \rho}{\partial \vec{r}}, \frac{\partial \rho}{\partial \dot{\vec{r}}}. \tag{9.3}$$

Therefore the orbit parameter related parts in the linearised GPS observation equation are

$$\frac{\partial \rho}{\partial (\vec{r}, \dot{\vec{r}})} \frac{\partial (\vec{r}, \dot{\vec{r}})}{\partial (\sigma_j, j = 1, ..6)} \frac{\partial (\sigma_j, j = 1, \ldots 6)}{\partial \vec{y}} \Delta \vec{y}^T, \tag{9.4}$$

where

$$\vec{y} = \left(\vec{\sigma}_0, \vec{Y}\right), \qquad \Delta \vec{y}^T = \left(\Delta \vec{\sigma}_0, \Delta \vec{Y}\right)^T. \tag{9.5}$$

$\vec{\sigma}, \vec{Y}$ are the Keplerian element vector and the parameter vector of the force models, and index 0 denotes the related initial vectors of time t_0. $\vec{y}$ is the total unknown vector of the orbit determination problem, the related correction vector is $\Delta \vec{y} = \vec{y} - \vec{y}_0$ and $\Delta \vec{\sigma}_0$ is the correction vector of the initial Keplerian element vector. The partial derivatives of the satellite state vector with respect to the Keplerian element vector are known and can be found in Sect. 5.4. The partial derivative of the Keplerian element vector with respect to $\vec{y}$ is called transition matrix which has the dimension of $6 \times (6 + n)$, where n is the dimension of vector $\vec{Y}$. Because of the analytical solutions of the disturbed equations of motion, the partial derivatives of the Keplerian elements with respect to the vector $\vec{y}$ are almost given by the solutions explicitly. That is to say, by analytical orbit determination, the transition matrix is represented by analytical formulas instead of the so-called variation equations in the numerical algorithm. The variation equation has disappeared from the orbit determination process so the numerical integration algorithms traditionally used to solve the variation equation are no longer necessary.

Note that the orbit disturbances are mostly linear functions of the parameters of the force models. Therefore, the partial derivatives of Keplerian element vector with respect to parameter vector $\vec{y}$ of the force models are directly the coefficients of the related force parameters. No special derivations of the partial derivatives are needed.

Compared to numerical orbit determination (Chap. 8), in analytical orbit determination no variation equations need to be solved, no numerical integration algorithms are necessary and no special orbit-related partial derivatives have to be derived. These significant advantages should lead to more efficient algorithms and more accurate orbit determination.

9.2 Real Time Ability of Analytical Orbit Determination

9.2.1 Limitations of the Numerical Orbit Determination

Real time ability of the numerical orbit determination is limited first by the adjustment or filtering algorithms used.

If the classic least squares adjustment algorithm is used to solve the parameters of the orbit determination problem, it is not possible to obtain the solution in real time because of the size and dimension of the equations. The equations of IGS orbit determination are formed and solved daily. It takes from less than 1 h to several hours to compute the results depending, of course, on the computer used. The so-called rapid IGS orbits are partly computed using 23 h past data and 1 h updated data. In general, the classic least squares adjustment algorithm is not suitable for real time purpose.

Sequential least squares algorithms and Kalman filtering techniques are partly developed for real time applications. The sequential least squares algorithm is a special case of Kalman filtering; therefore the discussions will be focused on the filtering method. Kalman filtering solves the equations of every epoch or every epoch-block by taking into account the information from the past to obtain the results. In this way the problem can be solved epoch-wise or epoch-block-wise depending on the property of the problem. For equations of orbit determination the problem is insoluble (or singular) for a few epochs because of the dimension of the unknowns. The equations of orbit determination are generally solvable in 0.5 h (see Xu 2004) or longer. That is, the filtering technique and the property of the equations of orbit determination make the real time application of the numerical orbit determination very difficult.

Furthermore, in numerical orbit determination, the numerical integration algorithms have to be used to integrate the orbits and to solve the variation equations. The numerical integrator usually has a so-called integrator length. The selection of the integrator length depends on the accuracy requirement and the physical properties of functions that will be integrated and therefore is not a free choice. Usually in IGS orbit determination, the integrator length is selected as 5 min. This also restricts the real time application of the numerical orbit determination.

Because of the adjustment and filtering techniques and the use of the numerical integrator as well as the properties of the physical problem, numerical orbit determination is difficult in real time.

9.2.2 Real Time Ability of Analytical Orbit Determination

Using the analytical orbit theory the observation equation of the orbit determination problem can be formed easily epoch-wise. The equations are solvable for an epoch-block. Taking past information into account, the solvable equations of an epoch-block can be formed and solved in real time. Taking the information before the solved epoch-block into account, Kalman filtering techniques can be used to determine the orbit in real time. This is very significant for applications of satellite technology nowadays and should be further studied intensively.

9.3 Algorithms of Analytical Orbit Determination

9.3.1 Initial Time Selection

In numerical orbit determination, the initial time is a matter of free choice. For numerical integration it really does not matter from which time point one starts to integrate. However, in analytical orbit solution, nearly half of the formulas are functions of initial time point (another half of the formulas are indefinite integrations and functions of instantaneous time). In turn, the functions of the initial time point are in terms of sines and cosines. Of course, theoretically the initial time point of orbit determination can be freely selected. However, if the initial time point is selected at that point such that the sines or cosines of mean anomaly are zero, the intensity of the computations can be reduced by 25 %. That is, a suitable initial time selection is very important for analytical orbit determination.

9.3.2 Sines and Cosines Curve Tables

All long periodic and short periodic terms are functions of sines and cosines, computation of which could be read from tables computed before to avoid repeat computations.

9.3.3 Initial Time Dealing

For initial time t_0 and end time t, and for periodic function with period of P, the periodic property should be used. There exists an integer k so that

$t_0 < t_0 + kP < t < t_0 + (\mathrm{k} + 1)P$; then initial time could be considered as $t_0 + kP$. That is the full periodic time intervals can be omitted by the computation.

9.3.4 Summation of the Geopotential Disturbances

As shown in Chap. 6, the solutions of the second order geopotential disturbances are very long. For short periodic terms, the solutions are formed by a set of functions of

$$\{\sin kM,\ \cos kM,\ k = 1, \ldots, N\} \tag{9.6}$$

where M is the mean anomaly of the orbit; k is an integer index and has a truncation number N.

Similarly, for the long periodic terms of the second order geopotential disturbances, the solutions can be formed by the following sets of functions:

$$\{\sin k\omega,\ \cos k\omega,\ k = 1, \ldots, I\}$$

$$\{\sin k\Omega,\ \cos k\Omega,\ k = 1, \ldots, J\}$$

$$\{\sin(k\omega + m\Omega),\ \cos(k\omega + m\Omega),\ k, m = 1, \ldots, K\} \tag{9.7}$$

where k and m are integer indices; I, J and K are truncation numbers.

The general models of the solutions of the second order geopotential disturbances are then

$$\sum_{k=1}^{N}(A_k \cos kM + B_k \sin kM) + \sum_{k=1}^{I}(C_k \cos k\omega + D_k \sin k\omega)$$
$$+ \sum_{k=1}^{J}(E_k \cos k\Omega + F_k \sin k\Omega) + \sum_{k,m=1}^{K}(G_{km} \cos(k\omega + m\Omega) + H_{km} \sin(k\omega + m\Omega). \tag{9.8}$$

where coefficients (A_k, B_k, C_k, D_k, E_k, F_k, G_{km}, H_{km}) are functions of geopotential coefficients which can be considered well known. The coefficients can also be considered as unknown and should be codetermined by orbit determination. The truncation numbers of (I, J, K) could be different from N because of the long and short periodic properties and shall be suitably selected through practical experiments.

9.3.5 Summation of the Multi-Body Disturbances

The multi-body disturbances, i.e. the disturbances of the solar, lunar and planetary gravitations, are solved in daily. Therefore the integration should be done daily step-wise. The discussion will be made similar to the above geopotential disturbance.

9.3.6 Solar Radiation Disturbances

The solar radiation pressure disturbances are solved daily too. Therefore the integration should be done daily step-wise, taking into account shadowing time and sunshine time. Because of the uncertainties of the solar panel orientation and surface reflectivity, the effects of solar radiation should be parameterized and estimated.

9.3.7 Atmospheric Drag Disturbances

For MEO or GEO satellites the atmospheric drag does not need to be taken into account. For LEO satellites the atmospheric effects are extreme complicated and further study are necessary.

Chapter 10
Singularity-Free Theory and Discussions

The previous chapters of this book covered the most important contents of satellite orbit theory, including analytical solutions and applications. In particular the solutions of the geopotential and extraterrestrial disturbances of the second order were derived and the analytical applications of the theory were discussed. In this chapter emphasis will be on singularity-free theory and discussions. The contents of Sect. 10.1 are historical ones which are not perfect; however the first step which leads to the correctness. The contents of Sect. 10.2 are the study of Xu and Xu 2012 which are considered key breakthrough in the singularity problem in orbital mechanics. The contents of Sect. 10.3 are a comprehensive summary of the developed singularity-free theory.

10.1 Singularity-Free Orbit Theory of Xu 2008/2010

The singularity problem of the solutions of the geopotential disturbances is discussed first. Then the singularity-free disturbed equations of motion are given for three cases; i.e. for circular orbit, equatorial orbit, circular and equatorial orbit, respectively (cf. Xu 2008, 2010a). If the singularity-free disturbed equations of motion are used, then the derived orbit solutions are singularity-free.

10.1.1 Problem of Singularity of the Solutions

As already discussed in the properties of the solutions (Sect. 5.3), the derived solutions are singular in the cases of $e = 0$ and/or $\sin i = 0$. In other words, the solutions are not valid for the satellite with a circular or an equatorial orbit. An alternative method to overcome the problem of circular orbit has already been discussed in Sect. 4.1.1 and involves introducing new variables (see (4.12)). The new variables do not have clear geometric meanings and were used to replace the variables (ω, f), which could not be

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defined in a circular orbit. In the alternative equation of disturbance (4.15) the e factor in the dividend is then eliminated, i.e. the singularity of $e = 0$ disappears. Using another set of variables (a, $h = \sin i \cos \Omega$, $k = -\sin i \sin \Omega$, $\xi = e\cos(\omega + \Omega)$, $\eta = -e\sin(\omega + \Omega)$, $\lambda = M + \omega + \Omega$), both the singularities caused by $e = 0$ and $\sin i = 0$ may disappear. This means that the singularity is not a real problem of the orbits, but a consequence of poor parameterisation of the orbits. Another method to overcome the singularity problem is the canonical transformation. All these methods overcome the singularity problem on one hand and pay the price of losing the geometric meanings of the orbital variables on the other.

In the cases of $e = 0$ and/or $\sin i = 0$, the orbits become simpler in practice. However, the equations used to describe a simpler problem become more complicated. This is in conflict with basic scientific philosophy and common sense. A simpler problem should be able to be described in simpler terms.

Looking carefully into the solutions given in Chap. 5 it is obvious that the singular problem is not created by the partial derivations of the potential function with respect to the Keplerian variable. In other words, the singularity problem exists from the beginning in the Lagrangian perturbed equation system (4.11). This may be verified by derivations of (4.11) (see Kaula 1966/2001).

10.1.2 Disturbed Equations in the Case of Circular Orbit

In the case of a circular orbit, the eccentricity of the ellipse e is a constant of zero; the eccentric anomaly E, true anomaly f and mean anomaly M are identical. Note that in such a case the perigee of the orbit is arbitrary. Then the argument of the perigee ω and the true anomaly f (i.e. mean anomaly M) cannot be separated from each other. However, $\omega + f$, i.e. $\omega + M$, are defined and have the meaning of argument of the perigee plus true anomaly (or mean anomaly) counted from the ascending node of the orbit. For convenience we write ω and M further separated; in practice they should be added together. In this special case the orbit is simpler than the general one. The disturbed equations of motion – similar to (4.11) – can be similarly derived and have accordingly the following simpler forms:

$$\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2}{na}\frac{\partial R}{\partial M} \\
\frac{\mathrm{d}e}{\mathrm{d}t} &= 0 \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= -\frac{\cos i}{na^2 \sin i}\frac{\partial R}{\partial i} \\
\frac{\mathrm{d}i}{\mathrm{d}t} &= -\frac{1}{na^2 \sin i}\left(\cos i \frac{\partial R}{\partial \omega} - \frac{\partial R}{\partial \Omega}\right). \\
\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= \frac{1}{na^2 \sin i}\frac{\partial R}{\partial i} \\
\frac{\mathrm{d}M}{\mathrm{d}t} &= n - \frac{2}{na}\frac{\partial R}{\partial a}
\end{aligned} \tag{10.1}$$

10.1.3 Disturbed Equations in the Case of Equatorial Orbit

In the case of an equatorial orbit, sin$i$, the sine function of inclination angle i, is a constant of zero. Note that in such a case the ascending node is arbitrary. Then the right ascension of the ascending node Ω and the argument of the perigee ω cannot be separated from each other. However, $\Omega + \omega$ are defined and have the meaning of right ascension of the ascending node plus argument of the perigee counted from the vernal equinox. For convenience we write Ω and ω further separated; in practice they should be added together. In this special case the orbit is simpler than the general one. In particular, the transformed geopotential function with orbital variable is greatly simplified in such a case. The disturbed equations of motion – similar to (4.11) – can be similarly derived and have accordingly the following simpler forms:

$$
\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2}{na}\frac{\partial R}{\partial M} \\
\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial M} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \omega} \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e} \\
\frac{\mathrm{d}i}{\mathrm{d}t} &= 0 \\
\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= 0 \\
\frac{\mathrm{d}M}{\mathrm{d}t} &= n - \frac{2}{na}\frac{\partial R}{\partial a} - \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial e}.
\end{aligned}
\tag{10.2}
$$

10.1.4 Disturbed Equations in the Case of Circular and Equatorial Orbit

In the case of a circular and an equatorial orbit, the eccentricity of the ellipse e is a constant of zero; the eccentric anomaly E, true anomaly f and mean anomaly M are identical; sin$i$, the sine function of inclination angle i, is a constant of zero. Note that in such a case both the perigee and ascending node are arbitrary. Then the right ascension of the ascending node Ω and the argument of the perigee ω as well as the true anomaly f (i.e. mean anomaly M) cannot be separated from each other. However, $\Omega + \omega + f$, i.e. $\Omega + \omega + M$, are defined and have the meaning of right ascension of the ascending node plus argument of the perigee and plus the true anomaly (or mean anomaly) counted from the vernal equinox. For convenience we write Ω and ω as well as M further separated; in practice they should be added together. In this special case the orbit is the simplest one compared with the others. The disturbed equations of motion – similar to (4.11) – in this case can be similarly derived and have accordingly the following simpler forms:

$$\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2}{na}\frac{\partial R}{\partial M} \\
\frac{\mathrm{d}e}{\mathrm{d}t} &= 0 \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= 0 \\
\frac{\mathrm{d}i}{\mathrm{d}t} &= 0 \\
\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= 0 \\
\frac{\mathrm{d}M}{\mathrm{d}t} &= n - \frac{2}{na}\frac{\partial R}{\partial a}.
\end{aligned} \tag{10.3}$$

10.1.5 Singularity-Free Lagrangian Equations of Motion

Define two delta functions as

$$\delta_e = \begin{cases} 1 & \text{if } e \neq 0 \\ e^2 & \text{if } e = 0 \end{cases} \quad \text{and} \quad \delta_i = \begin{cases} 1 & \text{if } \sin i \neq 0 \\ \sin^2 i & \text{if } \sin i = 0 \end{cases}. \tag{10.4}$$

Then one has the singularity-free disturbed equations of motion

$$\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2}{na}\frac{\partial R}{\partial M} \\
\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial M}\delta_e - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \omega}\delta_e \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e}\delta_e - \frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i}\delta_i \\
\frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\left(\cos i\frac{\partial R}{\partial \omega} - \frac{\partial R}{\partial \Omega}\right)\delta_i \\
\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i}\delta_i \\
\frac{\mathrm{d}M}{\mathrm{d}t} &= n - \frac{2}{na}\frac{\partial R}{\partial a} - \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial e}\delta_e
\end{aligned} \tag{10.5}$$

Equation 10.5 are the singularity-free disturbed equations of motion. The solutions derived from these equations are singularity-free. For some reasons the solutions given in the original edition of this book are mostly derived from (4.11). To obtain the singularity-free solutions one has to add the two factors of the delta functions (10.4) into the given solutions and the interested readers may attempt these themselves.

10.1.6 Simplified Singularity-Free Lagrangian Equations

Similar to the simplified Gaussian disturbed equations of satellite motion (see (7.25)), the simplified singularity-free disturbed Lagrange equations of motion can be derived and written as

$$\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2}{na}\frac{\partial R}{\partial M} \\
\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{1-e^2}{2ae}\frac{\mathrm{d}a}{\mathrm{d}t}\delta_e - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \omega}\delta_e \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e}\delta_e - \cos i\frac{\mathrm{d}\Omega}{\mathrm{d}t} \\
\frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\left(\frac{na\cos i}{\sqrt{1-e^2}}\left(\frac{1-e^2}{2}\frac{da}{dt} - ae\frac{de}{dt}\right) - \frac{\partial R}{\partial \Omega}\right)\delta_i \\
\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i}\delta_i \\
\frac{\mathrm{d}M}{\mathrm{d}t} &= n - \frac{2}{na}\frac{\partial R}{\partial a} - \sqrt{1-e^2}\left(\frac{\mathrm{d}\omega}{\mathrm{d}t} + \cos i\frac{\mathrm{d}\Omega}{dt}\right)
\end{aligned} \tag{10.6}$$

It is obvious that such equations will lead to a simplified process of solving the problems.

10.1.7 Singularity-Free Gaussian Equations of Motion

Similarly, singularity-free Gaussian perturbed equations of motion are then cf. (4.26) and (10.4))

$$\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2}{n\sqrt{1-e^2}}[e\sin f\cdot f_r + (1+e\cos f)\cdot f_\alpha] \\
\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{na}[\sin f\cdot f_r + (\cos E+\cos f)\cdot f_\alpha] \\
\frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{(1-e\cos E)\cos u}{na\sqrt{1-e^2}}\cdot f_h \\
\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= \frac{(1-e\cos E)\sin u}{na\sqrt{1-e^2}\sin i}\delta_i\cdot f_h \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{nae}\left[-\cos f\cdot f_r + \frac{2+e\cos f}{1+e\cos f}\sin f\cdot f_\alpha\right]\delta_e - \cos i\frac{\mathrm{d}\Omega}{\mathrm{d}t} \\
\frac{\mathrm{d}M}{\mathrm{d}t} &= n - \frac{1-e^2}{nae}\left[-\left(\cos f - \frac{2e}{1+e\cos f}\right)\cdot f_r + \frac{2+e\cos f}{1+e\cos f}\sin f\cdot f_\alpha\right]\delta_e.
\end{aligned} \tag{10.7}$$

The solutions derived from (10.7) are singularity-free. For some reasons, the solutions given in Xu (2008) are partly derived from (4.26). To obtain the singularity-free solutions one has to add the two factors of the delta functions (10.4) into the given solutions and interested readers may easily attempt these themselves. Similar simplified equations can be derived if one wishes.

10.2 On the Singularity Problem in Orbital Mechanics

In this section an algorithm to deal with the singularity problem in orbital mechanics is proposed based on theory of Xu and Xu 2012 with minor modifications. Lagrangian and Gaussian equations of motion are transformed into singularity-free ones by multiplying the so-called singular factors by the individual equations and the intermediate solutions can be derived by indefinite integration. Two criteria are introduced to decide how to transform the intermediate solutions into the solutions of the original problems inversely. Two examples are given to show the applicability of the algorithm. They are orbit solutions disturbed by the solar oblateness and the solar radiation pressure. A similar method to solve the so-called critical inclination problem is also discussed. Comments on the traditional variable transformations used to solve the singularity problem are addressed.

10.2.1 Introduction

Lagrangian and Gaussian equations of satellite (and/or planet) motion describe the two-body problem disturbed by the potential function and the non-conservative force, respectively. The singularity problems exist in cases of circular, equatorial, and circular and equatorial orbits. The critical inclination problem exists in cases of long-periodic disturbances which are functions of ω. Using transformed Keplerian variables or canonical ones are well-known traditional methods to make the equations of motion singularity-free; however, the transformed variables do not have clear geometric properties anymore. Furthermore, philosophically speaking, after the transformed equations are solved, the solutions have to be transformed inversely back to Keplerian elements, either the singularity will still exist, or the inverse transformation does not exist. In other words, the variable transformation did not really solve the singularity problem. The singularity problem in the equations of satellite motion is a problem which has existed since the science of celestial mechanics was established and the satellite era began, and a satisfied and applicable algorithm to deal with the problem is still unavailable. Our intention is to propose a method to solve this problem from both theoretical and practical points of view.

Lagrangian and Gaussian equations of motion are given in the next section for convenience of discussion. The singularity-free equations are given together with discussions on how to deal with the singularity problem using two newly introduced criteria, and two analytical examples are given as validations of proposed algorithm in the third section, followed by a concluding summary.

10.2.2 Basic Lagrangian and Gaussian Equations of Motion

The Lagrangian equations of satellite motion are represented by (Battin 1999; Brouwer and Clemence 1961; Van Kamp 1967; Boccaletti and Pucacco 2001; Eberle et al. 2008)

$$
\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2}{na}\frac{\partial V}{\partial M} \\
\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{1-e^2}{na^2e}\frac{\partial V}{\partial M} - \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial V}{\partial \omega} \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial V}{\partial e} - \frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial V}{\partial i} \\
\frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\left(\cos i\frac{\partial v}{\partial w} - \frac{\partial v}{\partial \Omega}\right) \\
\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial v}{\partial i} \\
\frac{\mathrm{d}M}{\mathrm{d}t} &= n - \frac{2}{na}\frac{\partial v}{\partial a} - \frac{1-e^2}{na^2e}\frac{\partial v}{\partial e}
\end{aligned}
\tag{10.8}
$$

Here n is the mean angular velocity and will be omitted later on. The Keplerian elements $(a, e, \omega, i, \Omega, M, f)$ are the semi-major axis, the eccentricity of the ellipse, the argument of perigee, the inclination angle, the right ascension of ascending node, the mean anomaly, the true anomaly, respectively. V is the disturbing potential function.

The Lagrangian equations can be written as cf. (6.23) or (7.94))

$$\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2}{na}\frac{\mathrm{d}a_1}{\mathrm{d}t} = h_1\frac{\mathrm{d}a_1}{\mathrm{d}t} \\
\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{1-e^2}{na^2e}\frac{\mathrm{d}a_1}{\mathrm{d}t} - \frac{\sqrt{1-e^2}}{na^2e}\frac{\mathrm{d}e_1}{\mathrm{d}t} = h_2\frac{\mathrm{d}a_1}{\mathrm{d}t} - h_3\frac{\mathrm{d}e_1}{\mathrm{d}t} \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{na^2e}\frac{\mathrm{d}\omega_1}{\mathrm{d}t} - \cos i\frac{\mathrm{d}\Omega}{\mathrm{d}t} = h_3\frac{\mathrm{d}\omega_1}{\mathrm{d}t} - h_5(\cos i/\sin i)\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} \\
\frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\left(\cos i\frac{de_1}{dt} - \frac{\mathrm{d}i_1}{\mathrm{d}t}\right) = h_4\frac{1}{\sin i}\left(\cos i\frac{de_1}{dt} - \frac{\mathrm{d}i_1}{\mathrm{d}t}\right) \\
\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} = h_5\frac{1}{\sin i}\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} \\
\frac{\mathrm{d}M}{\mathrm{d}t} &= n - \frac{2}{na}\frac{\mathrm{d}M_1}{\mathrm{d}t} - \frac{1-e^2}{na^2e}\frac{\mathrm{d}\omega_1}{\mathrm{d}t} = n - h_1\frac{\mathrm{d}M_1}{\mathrm{d}t} - h_2\frac{\mathrm{d}\omega_1}{\mathrm{d}t}
\end{aligned} \tag{10.9}$$

where coefficients (h_1, h_2, h_3, h_4, h_5) are defined, and

$$\begin{aligned}
\frac{\mathrm{d}a_1}{\mathrm{d}t} &= \frac{\partial V}{\partial M}, \quad \frac{\mathrm{d}e_1}{\mathrm{d}t} = \frac{\partial V}{\partial \omega}, \quad \frac{\mathrm{d}\omega_1}{\mathrm{d}t} = \frac{\partial V}{\partial e}, \\
\frac{\mathrm{d}i_1}{\mathrm{d}t} &= \frac{\partial V}{\partial \Omega}, \quad \frac{\mathrm{d}\Omega_1}{\mathrm{d}t} = \frac{\partial V}{\partial i}, \quad \frac{\mathrm{d}M_1}{\mathrm{d}t} = \frac{\partial V}{\partial a}.
\end{aligned} \tag{10.10}$$

Equation 10.10 are called basic Lagrangian equations of motion. By integrating the equations (10.9) the mean value theorem for integration is used where the functions of variables (a, e, i) are considered constants. For solving the Lagrangian equations of motion (10.8) or (10.9), the simplest way is to solve the basic Lagrangian equations (10.10). Substituting solutions of (10.10) into (10.9) the solutions of (10.9) can be obtained. Therefore, without exception, the potential force disturbing orbit problem turns out to be the problem of solving the basic Lagrangian equations of motion (10.10). It is notable that the basic Lagrangian equations of motion are singularity-free.

The Gaussian perturbed equations of motion are (Kaula 1966/2001)

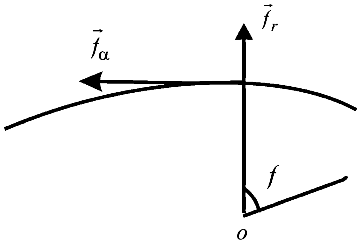

Fig. 10.1 Radial force vector and its perpendicular one in the orbital plane

$$
\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2}{n\sqrt{1-e^2}}[e\sin f \cdot f_r + (1+e\cos f)\cdot f_\alpha] \\
\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{na}[\sin f \cdot f_r + (\cos E + \cos f)\cdot f_\alpha] \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{nae}\left[-\cos f\cdot f_r + \frac{2+e\cos f}{1+e\cos f}\sin f\cdot f_\alpha\right] - \cos i\frac{\mathrm{d}\Omega}{\mathrm{d}t} \\
\frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{(1-e\cos E)\cos u}{na\sqrt{1-e^2}}\cdot f_h \\
\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= \frac{(1-e\cos E)\sin u}{na\sqrt{1-e^2}\sin i}\cdot f_h \\
\frac{\mathrm{d}M}{\mathrm{d}t} &= n - \frac{2}{na}\left(\frac{1-e^2}{1+e\cos f}\right)\cdot f_r - \frac{1-e^2}{nae}\left[-\cos f\cdot f_r + \frac{2+e\cos f}{1+e\cos f}\sin f\cdot f_\alpha\right]
\end{aligned}
\tag{10.11}
$$

and

$$
u = \omega + f, \quad \cos E = \frac{e+\cos f}{1+e\cos f}. \tag{10.12}
$$

Here $(f_r, f_\alpha, f_h)^T$ is a force vector in an orbital plane coordinate system, the first two components are in the orbital plane, f_r is the radial component, f_α is the component perpendicular to f_r and points in the direction of the satellite motion (see Fig. 10.1, where o is the focus of the orbital ellipse and f is the true anomaly counted from the perigee of the orbital ellipse) and f_h completes a right-handed system. E is the eccentric anomaly. The relation between eccentric anomaly E and true anomaly f is given in Fig. 10.2 (where S' is the vertical projection of the satellite S on the circle with a radius of a (semi-major axis of the ellipse), b is the semi-minor axis of the ellipse, O' is the centre of the circle and O is the focus of the ellipse).

The Gaussian equations can be rewritten as

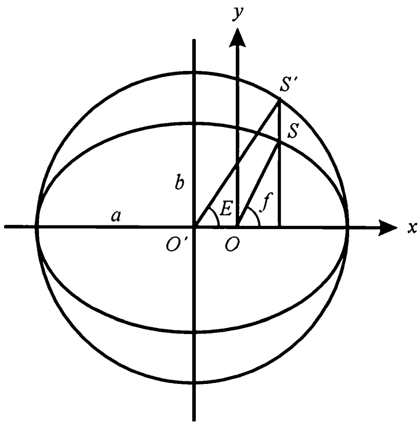

Fig. 10.2 The eccentric and true anomalies (f and E) of a satellite

$$
\begin{aligned}
\frac{da}{dt} &= \frac{2}{n\sqrt{1-e^2}}\frac{da_1}{dt} = h_1\frac{da_1}{dt}\\
\frac{de}{dt} &= \frac{\sqrt{1-e^2}}{na}\frac{de_1}{dt} = h_2\frac{de_1}{dt}\\
\frac{d\omega}{dt} &= \frac{\sqrt{1-e^2}}{nae}\frac{d\omega_1}{dt} - \cos i\frac{d\Omega}{dt} = h_3\frac{d\omega_1}{dt} - \cos i\frac{d\Omega}{dt} = h_3\frac{d\omega_1}{dt} - h_7\frac{d\Omega_1}{dt}\\
\frac{di}{dt} &= \frac{1}{na\sqrt{1-e^2}}\frac{di_1}{dt} = h_4\frac{di_1}{dt}\\
\frac{d\Omega}{dt} &= \frac{1}{na\sqrt{1-e^2}\sin i}\frac{d\Omega_1}{dt} = h_5\frac{d\Omega_1}{dt} = h_4\frac{1}{\sin i}\frac{d\Omega_1}{dt}\\
\frac{dM}{dt} &= n - 2\left(\frac{1-e^2}{na}\right)\frac{dM_1}{dt} - \frac{1-e^2}{nae}\frac{d\omega_1}{dt} = n - 2h_6 e\frac{dM_1}{dt} - h_6\frac{d\omega_1}{dt}
\end{aligned}
\tag{10.13}
$$

where coefficients of (h_1, h_2, h_3, h_4, h_5, h_6, $h_7 = h_5\cos i = h_4\cos i/\sin i$) are defined, and

$$
\begin{aligned}
\frac{da_1}{dt} &= e\sin f\cdot f_r + (1+e\cos f)\cdot f_\alpha\\
\frac{de_1}{dt} &= \sin f\cdot f_r + (\cos E+\cos f)\cdot f_\alpha\\
\frac{d\omega_1}{dt} &= -\cos f\cdot f_r + \frac{2+e\cos f}{1+e\cos f}\sin f\cdot f_\alpha\\
\frac{di_1}{dt} &= (1-e\cos E)\cos u\cdot f_h\\
\frac{d\Omega_1}{dt} &= (1-e\cos E)\sin u\cdot f_h\\
\frac{dM_1}{dt} &= \frac{1}{1+e\cos f}\cdot f_r
\end{aligned}
\tag{10.14}
$$

Equation 10.14 are called basic Gaussian equations of motion. By integrating the equations (10.13) the mean value theorem for integration is used where the functions of variables (a, e, i) are considered constants. For solving the Gaussian equations of motion (10.13), the simplest way is to solve the basic Gaussian equations (10.14). Substituting solutions of (10.14) into (10.13) the solutions of (10.13) can be obtained. Therefore, without exception, the non-conservative force disturbing orbit problem turns out to be the problem of solving the basic Gaussian equations of motion (10.14). It is notable that the basic Gaussian equations of motion are singularity-free.

In both the Lagrangian and Gaussian equations of motion (10.9) and (10.13) there exist singular problems. In the case of a small e ($e \ll 1$), the orbit is nearly circular, so that the perigee and the related Keplerian elements f and ω are not defined, and the problem is singular. To overcome this problem, traditionally, let $u = f + \omega$, and a parameter set of (a, i, Ω, $\xi = e\cos\omega$, $\eta = -e\sin\omega$, $\lambda = M + \omega$) can be used to describe the motion of the satellite. The related Lagrangian equations of motion can be derived and the singularity of e disappears. Using another set of variables (a, $h = \sin i \cos\Omega$, $k = -\sin i \sin\Omega$, $\xi = e\cos(\omega + \Omega)$, $\eta = -e\sin(\omega + \Omega)$, $\lambda = M + \omega + \Omega$), both the singularities caused by $e = 0$ and $\sin i = 0$ may disappear. Further a possible variable set is (a, $h = \tan(i/2)\sin\Omega$, $k = \tan(i/2)\cos\Omega$, $\xi = e\sin(\omega + \Omega)$, $\eta = e\cos(\omega + \Omega)$, $\lambda = M + \omega + \Omega$) (see, e.g. Chobotov 1991). Often also used are the canonical transformations using such as Hill variables (Cui 1990, 1997; Schneider and Cui 2005) and Delaunay elements (Kaula 2001; Wnuk 1990). These are traditional methods which may be used to solve the singularity problem; however, except for Keplerian variables, others do not have the clear geometric meanings.

In orbit determination practice for an equatorial satellite, to avoid the inclinational singularity problem, coordinate system will be rotated first so that the orbital plane will not coincide with the equatorial plane; after the orbits are determined the results will be rotated inversely.

10.2.3 Algorithm for Solving Singularity Problem

For convenience we call e and $\sin i$ singular factors. Any singular factor as divisor in the equations of motion could lead to a singularity; however it could also not. The partial derivatives of the potential function with respect to the Keplerian elements are not responsible for singularity (cf. Xu 2008, 2010a, Sect. 10.1). The singularity problem (if it exists) exists in the equations from the beginning.

Xu (2008, 2010a) suggested an alternative method to simplify the equations in case of singularity (cf. Sect. 10.1). However, the method is similar to that used for extending the Sundman's three-body solutions to the N-body's by omitting the singular equations (Sundman 1912; Wang 1991). This is, on one hand, reasonable because the Lagrangian equations are derived under the assumption that no singularity will occur; in cases of singularity, the related Lagrangian brackets are zero so that the related equation part disappears (Chobotov 1991; Xu 2008; Kaula 1966/2001). On the other hand, however, in the case of circular and equatorial orbits, only the terms of semi-major axis a and the mean anomaly M exist in the Lagrangian equations of motion (cf. (10.3)) so that the three dimensional perturbations are degraded to a two-dimensional (within the orbital plane) one and this is not acceptable and reasonable because not each equation with e and/or $\sin i$ as divisor may be singular (i.e. not each part of the equations with the singular factors is allowed to be omitted). This indicates that the singularity problem has to be further studied and discussed with care.

The analytical orbit solutions of Lagrangian equations (10.8) disturbed by the geopotentials have been derived (see, e.g. Chap. 6, or Wnuk 1990; Shapiro 1962; Kaula 1966/2001). By integration of the differential equations (10.8), the mean value theorem for integration is used and the coefficient functions of the short- and long-periodic terms (functions of M, ω and Ω) are considered as constants over the suitably selected time intervals (Wnuk 1990; Xu et al. 2010a, b; Xu et al. 2011). In other words, the functions of (a, e, i) are considered as constants by the indefinite integration. The time variations of the elements (a, e, i) are taken into account by integrating the equations step-wise.

Rewrite (10.8) or (10.9) as follows:

$$\begin{aligned}&\frac{da}{dt}=\delta_a,\quad \frac{de}{dt}=\frac{\delta_{e1}}{ae}+\frac{\delta_{e2}}{ae},\quad \frac{d\omega}{dt}=\frac{\delta_{\omega 1}}{ae}+\frac{\delta_{\omega 2}}{a\sin i},\\&\frac{di}{dt}=\frac{\delta_{i1}}{a\sin i}+\frac{\delta_{i2}}{a\sin i},\quad \frac{d\Omega}{dt}=\frac{\delta_{\Omega}}{a\sin i},\quad \frac{dM}{dt}=\delta_{M1}+\frac{\delta_{M2}}{ae}.\end{aligned}\tag{10.15}$$

Here δ with indices represent terms on the right-hand side of (10.8) or (10.9). The relationship between δ and the basic Lagrangian equations (10.10) can be seen by comparing (10.9) and (10.15). δs are in the same magnitude order (comparing (10.8) and (10.15)). Keplerian elements on the left-hand side of (10.8) will be represented by σ with indices later on. Multiplying e by the second and sixth equation of (10.15), $\sin i$ by the fourth and fifth, and $e\sin i$ by the third, it follows that

$$\begin{aligned}&\frac{da}{dt}=\delta_a,\quad e\frac{de}{dt}=\frac{\delta_{e1}}{a}+\frac{\delta_{e2}}{a},\quad e\sin i\frac{d\omega}{dt}=\sin i\frac{\delta_{\omega 1}}{a}+\frac{e\delta_{\omega 2}}{a},\\&\sin i\frac{di}{dt}=\frac{\delta_{i1}}{a}+\frac{\delta_{i2}}{a},\quad \sin i\frac{d\Omega}{dt}=\frac{\delta_{\Omega}}{a},\quad e\frac{dM}{dt}=e\delta_{M1}+\frac{\delta_{M2}}{a}.\end{aligned}\tag{10.16}$$

The above equations are singularity-free Lagrangian ones (n in the sixth equation is omitted). The solutions (i.e. the indefinite integrals of (10.16)) can be derived using methods described in Wnuk (1990); Xu et al. (2010a, b); Xu et al. (2011); Kaula (1966/2001) or formed using the solutions of the basic Lagrangian equations derived in this book, and have the forms

$$\Delta a = \Delta\delta_a, \quad e\Delta e = \frac{\Delta\delta_{e1}}{a} + \frac{\Delta\delta_{e2}}{a}, \quad e\sin i\Delta\omega = \frac{\sin i\Delta\delta_{\omega1}}{a} + \frac{e\Delta\delta_{\omega2}}{a},$$
$$\sin i\Delta i = \frac{\Delta\delta_{i1}}{a} + \frac{\Delta\delta_{i2}}{a}, \quad \sin i\Delta\Omega = \frac{\Delta\delta_{\Omega}}{a}, \quad e\Delta M = e\Delta\delta_{M1} + \frac{\Delta\delta_{M2}}{a}. \quad (10.17)$$

Here $\Delta\sigma$ and $\Delta\delta$ represent indefinite integrals of σ and δ, respectively. Deriving the integrals $\Delta\sigma$ from (10.8) or (10.15) turns out to be a task of deriving solutions $\Delta\sigma$ from (10.17). On the right-hand side, the integrals $\Delta\delta$ are the solutions of the basic Lagrangian equations (10.10) multiplying by factors which can be obtained by comparing (10.9), (10.10), and (10.16). The equations at (10.17) are called intermediate solutions. We see now why the basic Lagrangian equations are defined and solved systematically in this new edition of book. In the case of non-singularity, dividing e by the second and sixth equation of (10.17), sin$i$ by the fourth and fifth, and $e$sin$i$ by the third, it follows that

$$\Delta a = \Delta\delta_a, \quad \Delta e = \frac{\Delta\delta_{e1}}{ae} + \frac{\Delta\delta_{e2}}{ae}, \quad \Delta\omega = \frac{\Delta\delta_{\omega1}}{ae} + \frac{\Delta\delta_{\omega2}}{a\sin i},$$
$$\Delta i = \frac{\Delta\delta_{i1}}{a\sin i} + \frac{\Delta\delta_{i2}}{a\sin i}, \quad \Delta\Omega = \frac{\Delta\delta_{\Omega}}{a\sin i}, \quad \Delta M = \Delta\delta_{M1} + \frac{\Delta\delta_{M2}}{ae}. \quad (10.18)$$

The equations at (10.18) are the solutions of (10.15) in the case of non-singularity. The problem happens only in case of singularity, i.e. e and/or sin$i$ ($\approx i$) approaching zero. Dividing any extreme small number could unproportionally amplify the error in the result (lead to singularity) and this must be avoided.

To summarize, the singularity-free intermediate solutions of (10.17) can be obtained by the solutions of the basic Lagrangian equations. Then in the singularity-free case the final solutions (10.18) of the Lagrangian equations of motion can be formed. So the problem now is how to form the final solutions from the intermediate solutions in cases of singularities. Two criteria and three checking criteria are defined and introduced to distinguish the type of the singularity and decide how to form the related final solutions in different singular cases from (10.17) which are described in Xu and Xu (2012) in reasonable and rigorous detail.

Define radial and equatorial biases as

$$\varepsilon_{\mathrm{e}} = ae, \quad \varepsilon_{\mathrm{i}} = a\sin i. \quad (10.19)$$

Here the radial bias ae is the distance between the orbital geometric centre O and the geocentre O' (see Fig. 10.2). Expression of the satellite radius can be expanded to order e for the case of $f = 0$; it holds that (Bronstein and Semendjajew 1987; Wang et al. 1979)

$$r = \frac{a(1-e^2)}{1+e\cos f} \approx a(1-e\cos f) \overset{f=0}{\approx} a - ae. \quad (10.20)$$

Hence ae is also the maximal radius correction (error) which could be caused by e. The equatorial bias $a\sin i \approx ai$ is the maximal distance of the satellite vertical to the equator which could be caused by i (inclination i is the maximal latitude of the satellite). At the initial time of integration $t_0 = 0$, e and/or i are approaching zero, and therefore, at the integration time t, the correction of e is Δe and the correction of i is Δi. So (10.19) turns out to be

$$\varepsilon_{\mathrm{e}} = ae = a\Delta e, \quad \varepsilon_{\mathrm{i}} = a\sin i \approx a\Delta\sin i \approx a\Delta i. \tag{10.21}$$

An orbital bias can be defined as follows:

$$\begin{aligned} &\varepsilon_o = a(\Delta\omega + \Delta M), \quad \text{for circular orbit} \\ &\varepsilon_o = a(\Delta\Omega + \Delta\omega + \Delta M), \quad \text{for equatorial orbit} \\ &\varepsilon_o = a(\Delta\Omega + \Delta\omega + \Delta M), \quad \text{for circular and equatorial orbit.} \end{aligned} \tag{10.22}$$

ε_o has the meaning of the maximal bias in the orbital motion direction which could be caused by $\Delta\Omega$, $\Delta\omega$, ΔM. Therefore, the radial, equatorial and orbital biases are the maximal orbit errors in the three related components. The three biases represent the maximal error area of the orbit caused by the disturbance of (e,ω,i,Ω,M). In practice, for a required orbital precision (standard deviation or sigma) of 0.33 m, ε_{e}, ε_{i} and ε_{o} are ca. 1 m (three sigmas), respectively. This means that ε_{e}, ε_{i} and ε_{o} are predefined numbers which can be used as criteria.

In the case of (non-singularity)

$$ae \geq \varepsilon_e = 1, \quad a\sin i \geq \varepsilon_i = 1, \tag{10.23}$$

the solutions (10.17) are singularity-free and the equations of (10.18) are the final solutions.

In the case of (circular singularity)

$$ae < \varepsilon_e = 1, \quad a\sin i \geq \varepsilon_i = 1, \tag{10.24}$$

the first term on the right-hand side of the third equation of (10.17) is possibly responsible for the singularity. If there is a common factor e in the related $\Delta\delta$ then there is no singularity problem; otherwise the related term does not exist in the original Lagrangian equation and should be omitted in (10.17). Similarly, the second term on the right-hand side of the sixth equation of (10.17) is possibly responsible for the singularity. Again there are two possibilities. If there is a common factor e in the related $\Delta\delta$ then there is no singularity problem; otherwise the related term does not exist in the original Lagrangian equation and should be omitted in (10.17). Solutions (10.17) turn out to be

$$\Delta a = \Delta\delta_a, \quad \Delta e = \begin{cases} \dfrac{\Delta\delta_{e1}}{ae} + \dfrac{\Delta\delta_{e2}}{ae} & if\,(\Delta\delta_{e1}, e)\; and\; (\Delta\delta_{e2}, e) \\ \dfrac{1}{2} sqrt\left(\dfrac{\Delta\delta_{e1}}{a} + \dfrac{\Delta\delta_{e2}}{a}\right) & else \end{cases}$$
$$\Delta\omega = \begin{cases} \dfrac{\Delta\delta_{\omega 1}}{ae} + \dfrac{\Delta\delta_{\omega 2}}{a\sin i} & if\,(\Delta\delta_{\omega 1},\, e) \\ \dfrac{\Delta\delta_{\omega 2}}{a\sin i} & else \end{cases}, \quad \Delta i = \frac{\Delta\delta_{i1}}{a\sin i} + \frac{\Delta\delta_{i2}}{a\sin i}$$
$$\Delta\Omega = \frac{\Delta\delta_\Omega}{a\sin i}, \quad \Delta M = \begin{cases} \Delta\delta_{M1} + \dfrac{\Delta\delta_{M2}}{ae} & if\,(\Delta\delta_{M2},\, e) \\ \Delta\delta_{M1} & else \end{cases} \tag{10.25}$$

Here (F, e) means that in the function F there is a common factor e. In the case of (equatorial singularity)

$$ae \geq \varepsilon_e = 1, \quad a\sin i < \varepsilon_i = 1, \tag{10.26}$$

the second term on the right-hand side of the third equation of (10.17) is responsible for the singularity (therefore the related term does not exist in the original Lagrangian equation and should be omitted in (10.17)). Similarly, the term on the right-hand side of the fifth equation of (10.17) is responsible for the singularity (therefore should be omitted in (10.17)). Solutions (10.17) turn out to be

$$\Delta a = \Delta\delta_a, \quad \Delta e = \frac{\Delta\delta_{e1}}{ae} + \frac{\Delta\delta_{e2}}{ae}, \quad \Delta\omega = \begin{cases} \dfrac{\Delta\delta_{\omega 1}}{ae} + \dfrac{\Delta\delta_{\omega 2}}{a\sin i} & if\,(\Delta\delta_{\omega 2},\, \sin i) \\ \dfrac{\Delta\delta_{\omega 1}}{ae} & else \end{cases}$$
$$\Delta i = \begin{cases} \dfrac{\Delta\delta_{i1}}{a\sin i} + \dfrac{\Delta\delta_{i2}}{a\sin i} & if\,(\Delta\delta_{i1},\, \sin i)\; and\; (\Delta\delta_{i2},\, \sin i) \\ \cos^{-1}\left(\dfrac{-\Delta\delta_{i1}}{a} - \dfrac{\Delta\delta_{i2}}{a}\right) & else \end{cases}$$
$$\Delta\Omega = \begin{cases} \dfrac{\Delta\delta_\Omega}{a\sin i} & if\,(\Delta\delta_\Omega,\, \sin i) \\ 0 & else \end{cases}, \quad \Delta M = \Delta\delta_{M1} + \frac{\Delta\delta_{M2}}{ae} \tag{10.27}$$

Here $(F, \sin i)$ means in the function F there is a common factor $\sin i$. In the case of (circular and equatorial singularity)

$$ae < \varepsilon_e = 1, \quad a\sin i < \varepsilon_i = 1, \tag{10.28}$$

similarly, solutions (10.17) turn out to be

$$\Delta a = \Delta\delta_a, \quad \Delta e = \begin{cases} \dfrac{\Delta\delta_{e1}}{ae} + \dfrac{\Delta\delta_{e2}}{ae} & if(\Delta\delta_{e1}, e)\; and\; (\Delta\delta_{e2}, e) \\ \dfrac{1}{2} sqrt\left(\dfrac{\Delta\delta_{e1}}{a} + \dfrac{\Delta\delta_{e2}}{a}\right) & else \end{cases}$$

$$\Delta\omega = \begin{cases} \dfrac{\Delta\delta_{\omega 1}}{ae} + \dfrac{\Delta\delta_{\omega 2}}{a\sin i} & if(\Delta\delta_{\omega 1}, e)\; and\; (\Delta\delta_{\omega 2}, \sin i) \\ \dfrac{\Delta\delta_{\omega 1}}{ae} & if(\Delta\delta_{\omega 1}, e) \\ \dfrac{\Delta\delta_{\omega 2}}{a\sin i} & if(\Delta\delta_{\omega 2}, \sin i) \\ 0 & else \end{cases},$$

$$\Delta i = \begin{cases} \dfrac{\Delta\delta_{i1}}{a\sin i} + \dfrac{\Delta\delta_{i2}}{a\sin i} & if(\Delta\delta_{i1}, \sin i)\; and\; (\Delta\delta_{i2}, \sin i) \\ \cos^{-1}\left(\dfrac{-\Delta\delta_{i1}}{a} - \dfrac{\Delta\delta_{i2}}{a}\right) & else \end{cases}$$

$$\Delta\Omega = \begin{cases} \dfrac{\Delta\delta_{\Omega}}{a\sin i} & if(\Delta\delta_{\Omega}, \sin i) \\ 0 & else \end{cases}, \quad \Delta M = \begin{cases} \Delta\delta_{M1} + \dfrac{\Delta\delta_{M2}}{ae} & if(\Delta\delta_{M2}, e) \\ \Delta\delta_{M1} & else \end{cases}. \tag{10.29}$$

The criterion ε_o can be used to check whether the orbital bias computed is negligible.

The radial and equatorial biases are used as criteria to distinguish the kind of singularities and meanwhile decide how to transform the intermediate solutions (10.17) into the final solutions. In this way we developed an algorithm in which the Lagrangian equations (10.8) are first transformed into singularity-free Lagrangian ones (10.16) the solutions of which are the integrals (10.17), and depending on the cases of circular and/or equatorial orbits, the intermediate solutions (10.17) can be transformed to final solutions. The singularity problem is dealt with reasonably and rigorously.

Similar discussions can be made for the Gaussian equations of motion. Gaussian equations (10.15) can be similarly rewritten as

$$\begin{aligned} &\frac{da}{dt} = \delta_a, \quad \frac{de}{dt} = \delta_e, \quad \frac{d\omega}{dt} = \frac{\delta_{\omega 1}}{ae} + \frac{\delta_{\omega 2}}{a\sin i}, \\ &\frac{di}{dt} = \delta_i, \quad \frac{d\Omega}{dt} = \frac{\delta_{\Omega}}{a\sin i}, \quad \frac{dM}{dt} = \delta_{M1} + \frac{\delta_{M2}}{ae}. \end{aligned} \tag{10.30}$$

The related singularity-free Gaussian equations are then

$$\begin{aligned} &\frac{da}{dt} = \delta_a, \quad \frac{de}{dt} = \delta_e, \quad e\sin i\frac{d\omega}{dt} = \frac{\sin i\delta_{\omega 1}}{a} + \frac{e\delta_{\omega 2}}{a}, \\ &\frac{di}{dt} = \delta_i, \quad \sin i\frac{d\Omega}{dt} = \frac{\delta_{\Omega}}{a}, \quad e\frac{dM}{dt} = e\delta_{M1} + \frac{\delta_{M2}}{a}. \end{aligned} \tag{10.31}$$

The singularity-free intermediate solutions of (10.31) can be obtained by the solutions of the basic Gaussian equations as follows:

$$\Delta a = \Delta\delta_a, \quad \Delta e = \Delta\delta_e, \quad e\sin i\Delta\omega = \frac{\sin i\Delta\delta_{\omega 1}}{a} + \frac{e\Delta\delta_{\omega 2}}{a},$$
$$\Delta i = \Delta\delta_i, \quad \sin i\Delta\Omega = \frac{\Delta\delta_\Omega}{a}, \quad e\Delta M = e\Delta\delta_{M1} + \frac{\Delta\delta_{M2}}{a}. \tag{10.32}$$

In the case of non-singularity the final solutions can be derived from (10.32) as

$$\Delta a = \Delta\delta_a, \quad \Delta e = \Delta\delta_e, \quad \Delta\omega = \frac{\Delta\delta_{\omega 1}}{ae} + \frac{\Delta\delta_{\omega 2}}{a\sin i},$$
$$\Delta i = \Delta\delta_i, \quad \Delta\Omega = \frac{\Delta\delta_\Omega}{a\sin i}, \quad \Delta M = \Delta\delta_{M1} + \frac{\Delta\delta_{M2}}{ae}. \tag{10.33}$$

It should be noted that the singularity-free intermediate solutions of (10.32) can be obtained by the solutions of the basic Gaussian equations. Then in the singularity-free case the final solutions (10.33) of the Gaussian equations of motion can be formed. So the problem now is how to form the final solutions from the intermediate solutions in cases of singularities. Again two criteria and three checking criteria are defined and introduced to distinguish the type of singularity and decide how to form the related final solutions in different singular cases from (10.32) which are described in Xu and Xu (2012) in reasonable and rigorous detail.

In the case of (10.24) (circular singularity) the solutions are

$$\Delta a = \Delta\delta_a, \quad \Delta e = \Delta\delta_e, \quad \Delta\omega = \begin{cases} \dfrac{\Delta\delta_{\omega 1}}{ae} + \dfrac{\Delta\delta_{\omega 2}}{a\sin i} & if(\Delta\delta_{\omega 1}, e) \\ \dfrac{\Delta\delta_{\omega 2}}{a\sin i} & else \end{cases},$$
$$\Delta i = \Delta\delta_i, \quad \Delta\Omega = \frac{\Delta\delta_\Omega}{a\sin i}, \quad \Delta M = \begin{cases} \Delta\delta_{M1} + \dfrac{\Delta\delta_{M2}}{ae} & if(\Delta\delta_{M2}, e) \\ \Delta\delta_{M1} & else \end{cases}. \tag{10.34}$$

In the case of (10.26) (equatorial singularity) the solutions are

$$\Delta a = \Delta\delta_a, \quad \Delta e = \Delta\delta_e, \quad \Delta\omega = \begin{cases} \dfrac{\Delta\delta_{\omega 1}}{ae} + \dfrac{\Delta\delta_{\omega 2}}{a\sin i} & if(\Delta\delta_{\omega 2}, \sin i) \\ \dfrac{\Delta\delta_{\omega 1}}{ae} & else \end{cases},$$
$$\Delta i = \Delta\delta_i, \quad \Delta\Omega = \begin{cases} \dfrac{\Delta\delta_\Omega}{a\sin i} & if(\Delta\delta_\Omega, \sin i) \\ 0 & else \end{cases}, \quad \Delta M = \Delta\delta_{M1} + \frac{\Delta\delta_{M2}}{ae}. \tag{10.35}$$

In the case of (10.28) (circular and equatorial singularity) the solutions are

$$\Delta a = \Delta\delta_a, \quad \Delta e = \Delta\delta_e, \quad \Delta\omega = \begin{cases} \dfrac{\Delta\delta_{\omega 1}}{ae} + \dfrac{\Delta\delta_{\omega 2}}{a\sin i} & if\,(\Delta\delta_{\omega 1}, e)\, and\, (\Delta\delta_{\omega 2}, \sin i) \\ \dfrac{\Delta\delta_{\omega 1}}{ae} & if\,(\Delta\delta_{\omega 1}, e) \\ \dfrac{\Delta\delta_{\omega 2}}{a\sin i} & if\,(\Delta\delta_{\omega 2}, \sin i) \\ 0 & else \end{cases},$$

$$\Delta i = \Delta\delta_i, \quad \Delta\Omega = \begin{cases} \dfrac{\Delta\delta_\Omega}{a\sin i} & if\,(\Delta\delta_\Omega, \sin i) \\ 0 & else \end{cases}, \quad \Delta M = \begin{cases} \Delta\delta_{M1} + \dfrac{\Delta\delta_{M2}}{ae} & if\,(\Delta\delta_{M2}, e) \\ \Delta\delta_{M1} & else \end{cases}. \tag{10.36}$$

10.2.4 Application 1: Solutions of the Solar Oblateness Disturbance

Lagrangian equations (10.8) disturbed by the solar gravitational oblateness (Shapiro 1999; Hill et al. 1974; Gilvarry and Sturrock 1967; Sturrock and Gilvarry 1967; Wayte 2010; Boehme 1970; Fivian et al. 2008; Pitjeva 2005; Krivov et al. 1996; Kislik 1983) are solved for satellite (planet) orbit in Xu et al. (2011) (cf. Sect. 11.1) and the solutions are singular in the case of circular satellite orbit in the solar gravity field. The solar oblateness potential function is (Van Kamp 1967; Boccaletti and Pucacco 2001; Vallado 2007; Lynden-Bell 2009; Pal 2009)

$$V = \frac{-\mu_s a_s^2}{2r^3} J_2 (3\sin^2\varphi - 1). \tag{10.37}$$

Here μ_s is the solar gravitational constant, a_s is the mean ecliptic radius of the sun, r and φ are the heliocentric radius and latitude of the satellite and J_2 is an unnormalised coefficient. Substituting (10.37) into (10.15) the singularity-free Lagrangian equations (10.16) can be obtained. The solutions (10.17) (i.e. indefinite integrals of (10.16)) are explicitly given as follows (truncated to an order of e for terms of M and e^2 to the rest):

$$\Delta a = \frac{-3}{2n^2a^4}\mu_s a_s^2 J_2 \begin{bmatrix} \dfrac{-e}{2}\sin^2 i \sin 2\omega \sin M + \sin^2 i \sin 2\omega \sin 2M + \\ \dfrac{7e}{2}\sin^2 i \sin 2\omega \sin 3M - e\left(2 - \sin^2 i\left(\dfrac{5}{2} + \cos^2\omega\right)\right)\cos M \\ -\sin^2 i \cos 2\omega \cos 2M - \dfrac{7e}{2}\sin^2 i \cos 2\omega \cos 3M \end{bmatrix}, \tag{10.38}$$

$$e\Delta e = \frac{-\sqrt{1-e^2}}{8n^2a^5} e\mu_s a_s^2 J_2 \begin{bmatrix} 3\sin^2 i \sin 2\omega \sin M - 4(3 - \sin^2 i(5 + 2\sin^2\omega)) \cos M \\ +7\sin^2 i \sin 2\omega \sin 3M - 7\sin^2 i \cos 2\omega \cos 3M \end{bmatrix}, \tag{10.39}$$

$$e \sin i \Delta\omega = \frac{3\sqrt{1-e^2}}{8na^5} e \sin i \mu_s a_s^2 J_2 (3 + 5\cos 2i)t - \frac{\sqrt{1-e^2}}{n^2a^5} \sin i \mu_s a_s^2 J_2 \begin{bmatrix} \left(\frac{-3}{2} + \sin^2 i\left(\frac{3}{4}\cos^2\omega + \frac{15}{8}\right)\right) \sin M \\ + \frac{3e}{4}\left(-3 + \cos^2 i \cos 2\omega + \sin^2 i\left(5\cos^2\omega + \frac{1}{2}\right)\right) \sin 2M \\ -\frac{7}{8}\sin^2 i \cos 2\omega \sin 3M - \frac{51e}{16}\sin^2 i \cos 2\omega \sin 4M \\ +\frac{3}{8}\sin^2 i \sin 2\omega \cos M + \frac{3e}{4}\left(1 + \frac{3}{2}\sin^2 i\right) \sin 2\omega \cos 2M \\ -\frac{7}{8}\sin^2 i \sin 2\omega \cos 3M - \frac{51e}{16}\sin^2 i \sin 2\omega \cos 4M \end{bmatrix} \tag{10.40}$$

$$\sin i \Delta i = \frac{-1}{8n^2a^5\sqrt{1-e^2}} \sin i \sin 2i \mu_s a_s^2 J_2 \begin{bmatrix} -3e \sin 2\omega \sin M + 3\sin 2\omega \sin 2M \\ +7e \sin 2\omega \sin 3M + 3\cos 2\omega \cos M \\ -3\cos 2\omega \cos 2M - 7e \cos 2\omega \cos 3M \end{bmatrix} \tag{10.41}$$

$$\sin i \Delta\Omega = \frac{-3}{2na^5\sqrt{1-e^2}} \sin i \mu_s a_s^2 J_2 \left(1 + \frac{3}{2}e^2\right) t \cos i - \frac{\mu_s a_s^2 J_2 \sin i \cos i}{4n^2a^5\sqrt{1-e^2}} \begin{bmatrix} 3e(2\cos^2\omega + 5)\sin M - 3\cos 2\omega \sin 2M \\ -7e\cos 2\omega \sin 3M + 3e \sin 2\omega \cos M \\ -3\sin 2\omega \cos 2M - 7e \sin 2\omega \cos 3M \end{bmatrix}, \tag{10.42}$$

$$e\Delta M = \frac{3e\mu_s a_s^2 J_2}{8na^5}\left(\left(\left(1+\frac{21}{4}e^2\right)+\left(3+\frac{3}{4}e^2\right)\cos 2i\right)t - \int \frac{1}{2}e^2\sin^2 i\cos 2\omega dt\right)$$
$$-\frac{1}{n^2a^5}e\mu_s a_s^2 J_2$$
$$\begin{bmatrix} (\frac{3}{2e}-\frac{87e}{16}+\sin^2 i((\frac{-21}{8e}+\frac{351e}{32})\cos^2\omega+(\frac{-15}{8e}+\frac{171e}{32})\sin^2\omega))\sin M \\ +\frac{1}{2}(\frac{9}{2}+\sin^2 i(-15\cos^2\omega+\frac{3}{2}\sin^2\omega))\sin 2M \\ +(\frac{53e}{16}+\sin^2 i((\frac{7}{8e}-\frac{1079e}{64})\cos^2\omega+(\frac{-7}{8e}+\frac{443e}{64})\sin^2\omega))\sin 3M \\ +\frac{51}{16}\sin^2 i\cos 2\omega\sin 4M+\frac{507e}{64}\sin^2 i\cos 2\omega\sin 5M \\ -\sin^2 i(\frac{3}{8e}-\frac{87e}{32})\sin 2\omega\cos M-\frac{33}{8}\sin^2 i\sin 2\omega\cos 2M \\ -\sin^2 i(\frac{-7}{8e}+\frac{761e}{64})\sin 2\omega\cos 3M+\frac{51}{16}\sin^2 i\sin 2\omega\cos 4M \\ +\frac{507e}{64}\sin^2 i\sin 2\omega\cos 5M \end{bmatrix} \tag{10.43}$$

In (10.40), (10.41), and (10.42) both sides contain sin$i$; therefore there is no equatorial singularity problem. In (10.39) the factor e exists on both sides so there is no circular singularity problem. However, in short-periodic terms of (10.40) and (10.43), the factor e on the left-hand side could lead to a circular singularity.

According to (10.25) and noting that the circular singularity existed, the solutions of (10.40) and (10.43) should be

$$\Delta\omega = \frac{\Delta\delta_{\omega 2}}{a\sin i},$$
$$\Delta M = \Delta\delta_{M1}.$$

i.e.

$$\Delta\omega = -t\frac{\sqrt{1-e^2}}{na^5}\mu_s a_s^2 J_2\frac{3}{2}\cos^2 i - \frac{\sqrt{1-e^2}}{n^2a^5}\mu_s a_s^2 J_2$$
$$\begin{bmatrix} -\frac{3}{4}e\cos^2 i\sin 2\omega\cos M+\frac{3}{4}\cos^2 i\sin 2\omega\cos 2M \\ +\frac{7}{4}e\cos^2 i\sin 2\omega\cos 3M-\frac{3}{4}e\cos^2 i(7\cos^2\omega+5\sin^2\omega)\sin M \\ +\frac{3}{4}\cos^2 i\cos 2\omega\sin 2M+\frac{7}{4}e\cos^2 i\cos 2\omega\sin 3M \end{bmatrix}$$

$$\Delta M = -\frac{1}{na^5}\mu_s a_s^2 J_2\left(-(\frac{3}{4}+\frac{9}{8}e^2)(1+3\cos 2i)\right)t - \frac{1}{n^2a^5}\mu_s a_s^2 J_2$$
$$\left[\begin{array}{l} e(-9+\sin^2 i(\frac{63}{4}\cos^2\omega+\frac{45}{4}\sin^2\omega))\sin M \\ +\frac{9}{2}(\frac{-1}{2}\sin^2 i\cos 2\omega + e^2(\frac{-3}{2}+\sin^2 i(\frac{7}{2}\cos^2\omega+\sin^2\omega)))\sin 2M \\ -e\sin^2 i(\frac{31}{4}\cos 2\omega)\sin 3M - e^2\sin^2 i\frac{153}{16}\cos 2\omega\sin 4M \\ +\sin^2 i(\frac{9e}{4})\sin 2\omega\cos M - \frac{9}{4}(1-\frac{5}{2}e^2)\sin^2 i\sin 2\omega\cos 2M \\ -\sin^2 i(\frac{31e}{4})\sin 2\omega\cos 3M + \frac{153}{16}e^2\sin^2 i\sin 2\omega\cos 4M \end{array}\right]$$

The above two equations are solutions valid for the case of a circular orbit. It is notable that the long-periodic term in (10.43) is disappeared so that the critical inclination problem in the case of a circular orbit is also disappeared. The solutions (10.38, 10.39, 10.40, 10.41, 10.42, and 10.43) are in principle equivalent to the results given in Xu et al. 2011 (except a factor 2, cf. Sect. 11.1); however, the singularity problem is well dealt with here.

10.2.5 Application 2: Solutions of the Solar Radiation Pressure Disturbance

Gaussian equations (10.11) disturbed by solar radiation pressure are solved for satellite (planet) orbit in Xu (2008) (cf. also Sect. 11.2) and the solutions are singular in the case of a circular satellite orbit in the solar gravity field. Solar radiation force has the radial form (Kezerashvili and Vazquez-Poritz 2009; Herrick 1972; Hughes 1977; Kubo-oka and Sengoku 1999; Montenbruck and Gill 2000)

$$f_r = \xi\frac{1}{r^2}, \quad \xi = P_s\frac{\eta}{2\pi c}\frac{A}{m}. \tag{10.44}$$

Here r is the heliocentric radius of the satellite, ξ represents the coefficient part of the solar radiation pressure force, P_s is the solar luminosity, η is the surface reflectivity, A/m is the area to mass ratio of the satellite and c is the light velocity in vacuum. Substituting (10.44) into (10.30) the singularity-free Gaussian equations (10.31) can be obtained. The solutions (10.33) (i.e. transformed indefinite integrals of (10.31)) are obtained explicitly (truncated to an order of e^3) (cf. also Sect. 11.2):

$$\Delta a = \frac{-2e\xi}{n^2a^2\sqrt{1-e^2}}\begin{pmatrix}(1-\frac{5}{8}e^2)\cos M + (e-\frac{5}{6}e^3)\cos 2M \\ +\frac{9}{8}e^2\cos 3M + \frac{4}{3}e^3\cos 4M\end{pmatrix}, \tag{10.45}$$

$$e\Delta e = \frac{-e\xi\sqrt{1-e^2}}{n^2a^3}\begin{pmatrix}(1-\frac{5}{8}e^2)\cos M + (e-\frac{5}{6}e^3)\cos 2M \\ +\frac{9}{8}e^2\cos 3M + \frac{4}{3}e^3\cos 4M\end{pmatrix}, \tag{10.46}$$

$$e\sin i\Delta\omega = -\frac{\xi\sqrt{1-e^2}}{n^2a^3}\sin i\begin{pmatrix}(1-\frac{3}{8}e^2)\sin M + (e-\frac{2}{3}e^3)\sin 2M \\ +\frac{9}{8}e^2\sin 3M + \frac{4}{3}e^3\sin 4M\end{pmatrix}, \tag{10.47}$$

$$\Delta i = 0, \quad \Delta\Omega = 0,$$

$$e\Delta M = \frac{1-e^2}{n^2a^3}\xi\begin{pmatrix}n(-2e-2e^3)t+ \\ (1-\frac{19}{8}e^2)\sin M + (e-\frac{5}{3}e^3)\sin 2M \\ +\frac{9}{8}e^2\sin 3M + \frac{4}{3}e^3\sin 4M\end{pmatrix} \tag{10.48}$$

In (10.47), both sides, there exists sin$i$; therefore there is no equatorial singularity problem. In (10.46) the factor e exists in both sides; there is no circular singularity problem. However, in short-periodic terms of (10.47) and (10.48), the factor e on the left-hand sides could lead to a circular singularity. According to (10.34) and noting that the circular singularity existed, the solutions of (10.47) and (10.48) should be

$$\Delta\omega = \frac{\Delta\delta_{\omega 2}}{a\sin i},$$
$$\Delta M = \Delta\delta_{M1}.$$

i.e.

$$\Delta\omega = 0$$

$$\Delta M = \frac{1-e^2}{n^2a^3}\xi\begin{pmatrix}n(-2-2e^2)t+ \\ -2e\sin M - e^2\sin 2M\end{pmatrix}$$

The above two equations of solutions are valid for the case of a circular orbit.

10.2.6 Problems of the Critical Inclination

The secular term in (10.40) can be rewritten as

$$\Delta\omega = \frac{3\sqrt{1-e^2}}{4na^5}\mu_s a_s^2 J_2(5\cos^2 i - 1)t = n_\omega t. \tag{10.49}$$

The long-periodic term in (10.43) can be rewritten as

$$\Delta M = \frac{3\mu_s a_s^2 J_2}{-16na^5} e^2 \sin^2 i \int \cos 2\omega dt \tag{10.50}$$

The inclination factor ($5\cos^2 i - 1$) in the angular velocity of ω (denoted as n_ω) is called critical inclination because by integration of (10.50) the factor will be in the divisor which could approaching zero in the case of critical inclination. Within integration time t, a constant $\varepsilon_\omega = a\Delta\omega$ can be defined; for practical use, a value ε_ω could be estimated and used as a criterion, e.g. $\varepsilon_\omega = 1$ m; then if $a\Delta\omega \geq \varepsilon_\omega = 1$, the integral of (10.50) is

$$\Delta M = \frac{3\mu_s a_s^2 J_2}{-16na^5} e^2 \sin^2 i \frac{1}{2n_\omega} \sin 2\omega. \tag{10.51}$$

Equation 10.51 could not be "critical". If $a\Delta\omega < \varepsilon_\omega = 1$, i.e. n_ω is approaching zero (in other words, the ω is a constant), then the integral of (10.50) is

$$\Delta M = t\frac{3\mu_s a_s^2 J_2}{-16na^5} e^2 \sin^2 i \cos 2\omega. \tag{10.52}$$

In this way, the critical inclination problem disappears (well dealt with).

10.2.7 Problems of Traditional Singularity-Free Variables

The singularity problem in the equations of satellite motion is a problem which has existed since the start of the satellite era. Solving the equations directly, the solutions will be singular in the case of circular and/or equatorial orbits. Traditional transformations are really able to make the equations singularity-free and the solutions in new variables are also non-singular ones. However, for using the intermediate solutions they must be transformed inversely so that the solutions will be represented by the original variables. Hence the transformed solutions should still include singularity problem in the singular cases. The directly solved solutions and transformed-solved-inversely-transformed solutions must be equivalent because the original equations are the same and the solutions are represented in the same variables. So it seems that the singularity problem has been solved theoretically by traditional methods in orbital mechanics; however, in reality it is not. One of the advantages of using canonical transformations is that the singularity problem disappears and this advantage is not true anymore according to the above discussions.

10.2.8 Summary

An algorithm to deal with the singularity problem in the orbital mechanics is proposed. Lagrangian and Gaussian equations of motion are transformed into singularity-free ones by multiplying the so-called singular factors by the individual equations and the related solutions can be derived by indefinite integration. The so-called circular and equatorial criteria are introduced for making decisions on how the intermediate solutions will be transformed inversely to get the singularity-free final solutions. The algorithm is applied to derive orbit solutions disturbed by solar gravitational oblateness and solar radiation pressure using singularity-free Lagrangian and Gaussian equations, respectively. An algorithm to deal with the so-called critical inclination problem is also proposed. Comments on the traditional variable transformation for dealing with the singularity problem are also addressed.

10.3 Summary of Singularity-Free Theory

The above two sections described the development of the singularity-free theory. The breakthrough is obtained by Xu and Xu (2012) which give the first time an algorithm how to deal the singularity problem in orbital mechanics. Two minor modifications are made to the equations of (10.25), (10.27), and (10.29) (cf. Xu and Xu 2012). For easy understanding we give a short summary on the theory including the relationships between the singularity-free formulations given in Sect. 11.2 and the basic Lagrangian and Gaussian equations of motion, and the relationships between the singularity-free solutions and the related differential equations of motion in the cases of singularities.

10.3.1 Lagrange-Xu Equations of Satellite Motion

The basic Lagrangian equations of motion are defined by (10.10) which are singularity-free. The original Lagrangian equations of motion are given in (10.8) i.e. (10.9) which are valid in the case of non-singular orbits because the equations are derived under the assumptions. In practice the most communication and navigation satellite orbits are circular or equatorial ones and therefore are singular. Instead of trying to solve the singularity problem through parameter transformation as up to now commonly did, the studies of Xu and Xu (2012) try to derive the equations of satellite motion using the same method as Lagrange did; however, taking the singular cases into account. How the Lagrangian equations are derived can be found in Kaula (1966/2001), or Dvorak and Lhotka (2013) or other literature and can be stated as follows.

Lagrangian equations of motion are formed by all possible elements of

$$[\sigma_k, \sigma_j]\frac{\mathrm{d}\sigma_\mathrm{j}}{\mathrm{d}t} = \frac{\partial V}{\partial \sigma_k}, \quad j = 1, \ldots, 6, \quad k = 1, \ldots, 6$$

where σ with index represents Keplerian elements and V the potential function; [ ] is the so-called Lagrangian brackets. In the case of Lagrangian brackets [ ] = 0, the related term of partial derivative on the right-hand side vanishes. In other words, if the Lagrangian brackets [ ] = 0, the related partial derivative of the potential function with respect to the Keplerian element disappear. Then we have six Lagrangian equations which own maximal two terms on the right-hand side and they are valid for non-singular cases.

In the cases of singularities ($e = 0$ and/or $\sin i = 0$), additional [ ] = $e \bullet g$ and/or [ ] = $\sin i \bullet h$ could be zero, g and h are non-zero functions. However, the right-hand side partial derivative terms are not anymore allowed to be disappeared or omitted simply. Otherwise the Lagrangian equations will be degraded to a 2-D problem as we already discussed and pointed out (cf. Sect. 10.2).

The first, second and fourth Lagrangian-equations will never be singular. In the case of circular orbit, multiplying e to the second equation and noting that on the left-hand side there is $e\mathrm{d}e/\mathrm{d}t = \mathrm{d}e^2/2\mathrm{d}t$; that is the second equation will not be singular. In the case of equatorial singularity, multiplying $\sin i$ to the fourth Lagrangian equation and noting that on the left-hand side there is $\sin i \mathrm{d}i/\mathrm{d}t = -\mathrm{d}\cos i/\mathrm{d}t$; that is the fourth equation will never be singular either. These are parts of significant mathematical contributions of the singularity-free theory of Xu and Xu (2012).

The terms on the right-hand side of the 3rd, 5th and 6th Lagrangian-equations have to be checked for the singularity. If the term is singular, the term will disappear, otherwise not.

The derivation is a logical one. Mathematical dealing makes the 2nd and 4th Lagrangian-equations never singular. Lagrangian-equations are derived in the cases of singularity-free under the principal that "if [ ]=0 the related right-hand term will disappear". Analogously, in the cases of singularities, the 3rd, 5th and 6th Lagrangian-equations, the same principal are valid, i.e. "if term is singular, the term will disappear".

Comparing Lagrangian equations of motion (10.9)

$$
\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2}{na}\frac{\mathrm{d}a_1}{\mathrm{d}t} = h_1\frac{\mathrm{d}a_1}{\mathrm{d}t} \\
\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{1-e^2}{na^2e}\frac{\mathrm{d}a_1}{\mathrm{d}t} - \frac{\sqrt{1-e^2}}{na^2e}\frac{\mathrm{d}e_1}{\mathrm{d}t} = h_2\frac{\mathrm{d}a_1}{\mathrm{d}t} - h_3\frac{\mathrm{d}e_1}{\mathrm{d}t} \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{na^2e}\frac{\mathrm{d}\omega_1}{\mathrm{d}t} - \cos i\frac{\mathrm{d}\Omega}{\mathrm{d}t} = h_3\frac{\mathrm{d}\omega_1}{\mathrm{d}t} - h_5(\cos i/\sin i)\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} \\
\frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\left(\cos i\frac{\mathrm{d}e_1}{\mathrm{d}t} - \frac{\mathrm{d}i_1}{\mathrm{d}t}\right) = h_4\frac{1}{\sin i}\left(\cos i\frac{\mathrm{d}e_1}{\mathrm{d}t} - \frac{\mathrm{d}i_1}{\mathrm{d}t}\right) \\
\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} = h_5\frac{1}{\sin i}\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} \\
\frac{\mathrm{d}M}{\mathrm{d}t} &= n - \frac{2}{na}\frac{\mathrm{d}M_1}{\mathrm{d}t} - \frac{1-e^2}{na^2e}\frac{\mathrm{d}\omega_1}{\mathrm{d}t} = n - h_1\frac{\mathrm{d}M_1}{\mathrm{d}t} - h_2\frac{\mathrm{d}\omega_1}{\mathrm{d}t}
\end{aligned}
\tag{10.53}
$$

with (10.15)

$$
\begin{aligned}
&\frac{\mathrm{d}a}{\mathrm{d}t} = \delta_a, \quad \frac{\mathrm{d}e}{\mathrm{d}t} = \frac{\delta_{e1}}{ae} + \frac{\delta_{e2}}{ae}, \quad \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\delta_{\omega1}}{ae} + \frac{\delta_{\omega2}}{a\sin i}, \\
&\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{\delta_{i1}}{a\sin i} + \frac{\delta_{i2}}{a\sin i}, \quad \frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{\delta_\Omega}{a\sin i}, \quad \frac{\mathrm{d}M}{\mathrm{d}t} = \delta_{M1} + \frac{\delta_{M2}}{ae}.
\end{aligned}
\tag{10.54}
$$

yields

$$
\begin{aligned}
&\delta_a = h_1\frac{\mathrm{d}a_1}{\mathrm{d}t}, \quad \delta_{e1} = aeh_2\frac{\mathrm{d}a_1}{\mathrm{d}t}, \quad \delta_{e2} = -aeh_3\frac{\mathrm{d}e_1}{\mathrm{d}t} \\
&\delta_{\omega1} = aeh_3\frac{\mathrm{d}\omega_1}{\mathrm{d}t}, \quad \delta_{\omega2} = -ah_5\cos i\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} \\
&\delta_{i1} = ah_4\left(\cos i\frac{\mathrm{d}e_1}{\mathrm{d}t}\right), \quad \delta_{i2} = ah_4\left(-\frac{\mathrm{d}i_1}{\mathrm{d}t}\right). \\
&\delta_\Omega = ah_5\frac{\mathrm{d}\Omega_1}{\mathrm{d}t}, \quad \delta_{M1} = -h_1\frac{\mathrm{d}M_1}{\mathrm{d}t}, \quad \delta_{M2} = -aeh_2\frac{\mathrm{d}\omega_1}{\mathrm{d}t}
\end{aligned}
\tag{10.55}
$$

The terms on the right-hand side of (10.55) are factions of basic Lagrangian equations and the terms on the left-hand side are symbols used in equations and solutions of Xu and Xu (2012) (cf. Sect. 10.2). Taking (10.55) into account the differential equations (or called Lagrange-Xu equations of motion) of (10.25) have forms of (in the case of circular singularity)

$$\frac{\mathrm{d}a}{\mathrm{d}t} = h_1 \frac{\mathrm{d}a_1}{\mathrm{d}t}, \quad \frac{\mathrm{d}e}{\mathrm{d}t} = \begin{cases} h_2 \frac{\mathrm{d}a_1}{\mathrm{d}t} - h_3 \frac{\mathrm{d}e_1}{\mathrm{d}t} & \textit{if} \left(\frac{\mathrm{d}a_1}{\mathrm{d}t}, e \right) \textit{ and } \left(\frac{\mathrm{d}e_1}{\mathrm{d}t}, e \right) \\ \frac{1}{2} \mathrm{sqrt} \left(e h_2 \frac{\mathrm{d}a_1}{\mathrm{d}t} - e h_3 \frac{\mathrm{d}e_1}{\mathrm{d}t} \right) & \textit{else} \end{cases}$$
$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \begin{cases} h_3 \frac{\mathrm{d}\omega_1}{\mathrm{d}t} - h_5 \cos i \frac{\mathrm{d}\Omega_1}{\mathrm{d}t} \frac{1}{\sin i} & \textit{if} \left(\frac{\mathrm{d}\omega_1}{\mathrm{d}t}, e \right) \\ -h_5 \cos i \frac{\mathrm{d}\Omega_1}{\mathrm{d}t} \frac{1}{\sin i} & \textit{else} \end{cases}$$
$$\frac{\mathrm{d}i}{\mathrm{d}t} = h_4 \left(\cos i \frac{\mathrm{d}e_1}{\mathrm{d}t} \right) \frac{1}{\sin i} + h_4 \left(-\frac{\mathrm{d}i_1}{\mathrm{d}t} \right) \frac{1}{\sin i}$$
$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{1}{\sin i} h_5 \frac{\mathrm{d}\Omega_1}{\mathrm{d}t}, \quad \frac{\mathrm{d}M}{\mathrm{d}t} = \begin{cases} -h_1 \frac{\mathrm{d}M_1}{\mathrm{d}t} - h_2 \frac{\mathrm{d}\omega_1}{\mathrm{d}t} & \textit{if} \left(\frac{\mathrm{d}\omega_1}{\mathrm{d}t}, e \right) \\ -h_1 \frac{\mathrm{d}M_1}{\mathrm{d}t} & \textit{else} \end{cases}. \tag{10.56}$$

The above are the equations of motion valid in circular orbits, i.e. which are circular singularity-free, and could be called the first Lagrange-Xu equations of motion or circular ones.

The differential equations (or called Lagrange-Xu equations of motion) of (10.25) have forms of (in the case of equatorial singularity)

$$\frac{\mathrm{d}a}{\mathrm{d}t} = h_1 \frac{\mathrm{d}a_1}{\mathrm{d}t}, \quad \frac{\mathrm{d}e}{\mathrm{d}t} = h_2 \frac{\mathrm{d}a_1}{\mathrm{d}t} - h_3 \frac{\mathrm{d}e_1}{\mathrm{d}t}$$
$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \begin{cases} h_3 \frac{\mathrm{d}\omega_1}{\mathrm{d}t} - h_5 \cos i \frac{\mathrm{d}\Omega_1}{\mathrm{d}t} \frac{1}{\sin i} & \left(\frac{\mathrm{d}\Omega_1}{\mathrm{d}t}, \sin i \right) \\ h_3 \frac{\mathrm{d}\omega_1}{\mathrm{d}t} & \textit{else} \end{cases}$$
$$\frac{\mathrm{d}i}{\mathrm{d}t} = \begin{cases} h_4 \left(\cos i \frac{\mathrm{d}e_1}{\mathrm{d}t} \right) \frac{1}{\sin i} + h_4 \left(-\frac{\mathrm{d}i_1}{\mathrm{d}t} \right) \frac{1}{\sin i} & \textit{if} \left(\frac{\mathrm{d}e_1}{\mathrm{d}t}, \sin i \right) \textit{ and } \left(\frac{\mathrm{d}i_1}{\mathrm{d}t}, \sin i \right) \\ \cos^{-1} \left(-h_4 \left(\cos i \frac{\mathrm{d}e_1}{\mathrm{d}t} \right) - h_4 \left(-\frac{\mathrm{d}i_1}{\mathrm{d}t} \right) \right) & \textit{else} \end{cases}$$
$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \begin{cases} \frac{1}{\sin i} h_5 \frac{\mathrm{d}\Omega_1}{\mathrm{d}t} & \textit{if} \left(\frac{\mathrm{d}\Omega_1}{\mathrm{d}t}, \sin i \right) \\ 0 & \textit{else} \end{cases}, \quad \frac{\mathrm{d}M}{\mathrm{d}t} = -h_1 \frac{\mathrm{d}M_1}{\mathrm{d}t} - h_2 \frac{\mathrm{d}\omega_1}{\mathrm{d}t} \tag{10.57}$$

The above are the equations of motion valid in equatorial orbits, i.e. which are equatorial singularity-free, and could be called the second Lagrange-Xu equations of motion or equatorial ones.

The differential equations (or called Lagrange-Xu equations of motion) of (10.29) have forms of (in the case of circular and equatorial singularity)

$$\frac{\mathrm{d}a}{\mathrm{d}t} = h_1 \frac{\mathrm{d}a_1}{\mathrm{d}t}, \quad \frac{\mathrm{d}e}{\mathrm{d}t} = \begin{cases} h_2 \frac{\mathrm{d}a_1}{\mathrm{d}t} - h_3 \frac{\mathrm{d}e_1}{\mathrm{d}t} & if\left(\frac{\mathrm{d}a_1}{\mathrm{d}t}, e\right) \; and \; \left(\frac{\mathrm{d}e_1}{\mathrm{d}t}, e\right) \\ \frac{1}{2}\mathrm{sqrt}\left(eh_2 \frac{\mathrm{d}a_1}{\mathrm{d}t} - eh_3 \frac{\mathrm{d}e_1}{\mathrm{d}t}\right) & else \end{cases}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \begin{cases} h_3 \frac{\mathrm{d}\omega_1}{\mathrm{d}t} - h_5 \cos i \frac{\mathrm{d}\Omega_1}{\mathrm{d}t} \frac{1}{\sin i} & if\left(\frac{\mathrm{d}\omega_1}{\mathrm{d}t}, e\right) and \left(\frac{\mathrm{d}\Omega_1}{\mathrm{d}t}, \sin i\right) \\ h_3 \frac{\mathrm{d}\omega_1}{\mathrm{d}t} & if\left(\frac{\mathrm{d}\omega_1}{\mathrm{d}t}, e\right) \\ -h_5 \cos i \frac{\mathrm{d}\Omega_1}{\mathrm{d}t} \frac{1}{\sin i} & if\left(\frac{\mathrm{d}\Omega_1}{\mathrm{d}t}, \sin i\right) \\ 0 & else \end{cases},$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \begin{cases} h_4 \left(\cos i \frac{\mathrm{d}e_1}{\mathrm{d}t}\right) \frac{1}{\sin i} + h_4 \left(-\frac{\mathrm{d}i_1}{\mathrm{d}t}\right) \frac{1}{\sin i} & if\left(\frac{\mathrm{d}e_1}{\mathrm{d}t}, \sin i\right) \; and \; \left(\frac{\mathrm{d}i_1}{\mathrm{d}t}, \sin i\right) \\ \cos^{-1}\left(-h_4 \left(\cos i \frac{\mathrm{d}e_1}{\mathrm{d}t}\right) - h_4 \left(-\frac{\mathrm{d}i_1}{\mathrm{d}t}\right)\right) & else \end{cases}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \begin{cases} \frac{1}{\sin i} h_5 \frac{\mathrm{d}\Omega_1}{\mathrm{d}t} & if\left(\frac{\mathrm{d}\Omega_1}{\mathrm{d}t}, \sin i\right) \\ 0 & else \end{cases}, \quad \frac{\mathrm{d}M}{\mathrm{d}t} = \begin{cases} -h_1 \frac{\mathrm{d}M_1}{\mathrm{d}t} - h_2 \frac{\mathrm{d}\omega_1}{\mathrm{d}t} & if\left(\frac{\mathrm{d}\omega_1}{\mathrm{d}t}, e\right) \\ -h_1 \frac{\mathrm{d}M_1}{\mathrm{d}t} & else \end{cases}. \tag{10.58}$$

The above are the equations of motion valid in the circular and equatorial orbits, i.e. which are circular and equatorial singularity-free, and could be called the third Lagrange-Xu equations of motion or circular and equatorial ones. Lagrangian equations of motion are valid for non-singular orbits, whereas the three Lagrange-Xu equations are valid for the three cases of singular orbits. After the basic Lagrangian equations (10.10) are solved, the solutions of the above Lagrangian and three kinds of Lagrangian-Xu equations could be formed.

Equations 10.53, 10.56, 10.57 and 10.58 can be summarised as

$$\frac{\mathrm{d}a}{\mathrm{d}t} = h_1 \frac{\mathrm{d}a_1}{\mathrm{d}t}, \quad \frac{\mathrm{d}e}{\mathrm{d}t} = \begin{cases} h_2 \frac{\mathrm{d}a_1}{\mathrm{d}t} - h_3 \frac{\mathrm{d}e_1}{\mathrm{d}t} & if(\frac{\mathrm{d}a_1}{\mathrm{d}t}, e) \text{ and } (\frac{\mathrm{d}e_1}{\mathrm{d}t}, e) \text{ or } e \neq 0 \\ \frac{1}{2}\mathrm{sqrt}(eh_2 \frac{\mathrm{d}a_1}{\mathrm{d}t} - eh_3 \frac{\mathrm{d}e_1}{\mathrm{d}t}) & else \end{cases}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \begin{cases} h_3 \frac{\mathrm{d}\omega_1}{\mathrm{d}t} - h_5 \cos i \frac{\mathrm{d}\Omega_1}{\mathrm{d}t} \frac{1}{\sin i} & \begin{array}{l} if(\frac{\mathrm{d}\omega_1}{\mathrm{d}t}, e) \text{ and } (\frac{\mathrm{d}\Omega_1}{\mathrm{d}t}, \sin i) \text{ or } e \neq 0 \text{ and } i \neq 0 \\ \text{or } (\frac{\mathrm{d}\omega_1}{\mathrm{d}t}, e) \text{ and } i \neq 0 \text{ or } e \neq 0 \text{ and } \left(\frac{\mathrm{d}\Omega_1}{\mathrm{d}t}, \sin i\right) \end{array} \\ h_3 \frac{\mathrm{d}\omega_1}{\mathrm{d}t} & if(\frac{\mathrm{d}\omega_1}{\mathrm{d}t}, e) \text{ and } i = 0 \\ -h_5 \cos i \frac{\mathrm{d}\Omega_1}{\mathrm{d}t} \frac{1}{\sin i} & if(\frac{\mathrm{d}\Omega_1}{\mathrm{d}t}, \sin i) \text{ and } e = 0 \\ 0 & else \end{cases}$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \begin{cases} h_4 \left(\cos i \frac{\mathrm{d}e_1}{\mathrm{d}t} \right) \frac{1}{\sin i} + h_4 \left(-\frac{\mathrm{d}i_1}{\mathrm{d}t} \right) \frac{1}{\sin i} & if(\frac{\mathrm{d}e_1}{\mathrm{d}t}, \sin i) \text{ and } (\frac{\mathrm{d}i_1}{\mathrm{d}t}, \sin i) \text{ or } i \neq 0 \\ \cos^{-1}(-h_4 \left(\cos i \frac{\mathrm{d}e_1}{\mathrm{d}t} \right) - h_4 \left(-\frac{\mathrm{d}i_1}{\mathrm{d}t} \right)) & else \end{cases}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \begin{cases} \frac{1}{\sin i} h_5 \frac{\mathrm{d}\Omega_1}{\mathrm{d}t} & if(\frac{\mathrm{d}\Omega_1}{\mathrm{d}t}, \sin i) \text{ or } i \neq 0 \\ 0 & else \end{cases}, \quad \frac{\mathrm{d}M}{\mathrm{d}t} = \begin{cases} -h_1 \frac{\mathrm{d}M_1}{\mathrm{d}t} - h_2 \frac{\mathrm{d}\omega_1}{\mathrm{d}t} & if(\frac{\mathrm{d}\omega_1}{\mathrm{d}t}, e) \text{ or } e \neq 0 \\ -h_1 \frac{\mathrm{d}M_1}{\mathrm{d}t} & else \end{cases}.$$

Above are the general Lagrange–Xu equations of motion which are valid for any cases of orbits without singularity problem.

10.3.2 Gauss-Xu Equations of Satellite Motion

The basic Gaussian equations of motion are defined by (10.14) which are singularity-free. The original Gaussian equations of motion are given in (10.12) i.e. (10.13) which are valid in the case of non-singular orbits because the equations are derived from Lagrangian ones which are under the assumptions that no singularity will occur. In reality the most communication and navigation satellite orbits are circular or equatorial ones and therefore are singular. Instead of trying to solve the singularity problem through parameter transformation as up to now commonly did, the studies of Xu and Xu (2012) try to derive the Gaussian equations of satellite motion using the similar method; however, taking the singular cases into account. How the Gaussian equations are derived can be found in Kaula (2001), Xu (2008) and other literature.

Comparing Gaussian equations of motion (10.13)

$$
\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2}{n\sqrt{1-e^2}}\frac{\mathrm{d}a_1}{\mathrm{d}t} = h_1\frac{\mathrm{d}a_1}{\mathrm{d}t}\\
\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{na}\frac{\mathrm{d}e_1}{\mathrm{d}t} = h_2\frac{\mathrm{d}e_1}{\mathrm{d}t}\\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{nae}\frac{\mathrm{d}\omega_1}{\mathrm{d}t} - \cos i\frac{\mathrm{d}\Omega}{\mathrm{d}t} = h_3\frac{\mathrm{d}\omega_1}{\mathrm{d}t} - \cos i\frac{\mathrm{d}\Omega}{\mathrm{d}t} = h_3\frac{\mathrm{d}\omega_1}{\mathrm{d}t} - h_7\frac{\mathrm{d}\Omega_1}{\mathrm{d}t}\\
\frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{1}{na\sqrt{1-e^2}}\frac{\mathrm{d}i_1}{\mathrm{d}t} = h_4\frac{\mathrm{d}i_1}{\mathrm{d}t}\\
\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= \frac{1}{na\sqrt{1-e^2}\sin i}\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} = h_5\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} = h_4\frac{1}{\sin i}\frac{\mathrm{d}\Omega_1}{\mathrm{d}t}\\
\frac{\mathrm{d}M}{\mathrm{d}t} &= n - 2\left(\frac{1-e^2}{na}\right)\frac{\mathrm{d}M_1}{\mathrm{d}t} - \frac{1-e^2}{nae}\frac{\mathrm{d}\omega_1}{\mathrm{d}t} = n - 2h_6e\frac{\mathrm{d}M_1}{\mathrm{d}t} - h_6\frac{\mathrm{d}\omega_1}{\mathrm{d}t}
\end{aligned} \tag{10.59}
$$

with (10.30)

$$
\begin{aligned}
&\frac{\mathrm{d}a}{\mathrm{d}t} = \delta_a, \quad \frac{\mathrm{d}e}{\mathrm{d}t} = \delta_e, \quad \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\delta_{\omega 1}}{ae} + \frac{\delta_{\omega 2}}{a\sin i},\\
&\frac{\mathrm{d}i}{\mathrm{d}t} = \delta_i, \quad \frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{\delta_\Omega}{a\sin i}, \quad \frac{\mathrm{d}M}{\mathrm{d}t} = \delta_{M1} + \frac{\delta_{M2}}{ae}.
\end{aligned} \tag{10.60}
$$

yields

$$
\begin{aligned}
\delta_a &= h_1\frac{\mathrm{d}a_1}{\mathrm{d}t}, \quad \delta_e = h_2\frac{\mathrm{d}e_1}{\mathrm{d}t}\\
\delta_{\omega 1} &= aeh_3\frac{\mathrm{d}\omega_1}{\mathrm{d}t}, \quad \delta_{\omega 2} = -a\sin i h_7\frac{\mathrm{d}\Omega_1}{\mathrm{d}t}\\
\delta_i &= h_4\frac{\mathrm{d}i_1}{\mathrm{d}t}, \quad \delta_\Omega = ah_4\frac{\mathrm{d}\Omega_1}{\mathrm{d}t}\\
\delta_{M1} &= -2h_6e\frac{\mathrm{d}M_1}{\mathrm{d}t}, \quad \delta_{M2} = -aeh_6\frac{\mathrm{d}\omega_1}{\mathrm{d}t}
\end{aligned} \tag{10.61}
$$

The terms on the right-hand side of (10.61) are functions of basic Gaussian equations. Taking (10.59) into account the differential equations (or called Gauss-Xu equations of motion) of (10. 34) have forms of (in the case of circular orbits)

$$
\begin{aligned}
&\frac{\mathrm{d}a}{\mathrm{d}t} = h_1\frac{\mathrm{d}a_1}{\mathrm{d}t}, \quad \frac{\mathrm{d}e}{\mathrm{d}t} = h_2\frac{\mathrm{d}e_1}{\mathrm{d}t}\\
&\frac{\mathrm{d}\omega}{\mathrm{d}t} = \begin{cases} h_3\frac{\mathrm{d}\omega_1}{\mathrm{d}t} - h_7\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} & if\left(\frac{\mathrm{d}\omega_1}{\mathrm{d}t}, e\right)\\ -h_7\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} & else\end{cases}, \quad \frac{\mathrm{d}i}{\mathrm{d}t} = h_4\frac{\mathrm{d}i_1}{\mathrm{d}t}\\
&\frac{\mathrm{d}\Omega}{\mathrm{d}t} = h_4\frac{1}{\sin i}\frac{\mathrm{d}\Omega_1}{\mathrm{d}t}, \quad \frac{\mathrm{d}M}{\mathrm{d}t} = \begin{cases} -2h_6e\frac{\mathrm{d}M_1}{\mathrm{d}t} - h_6\frac{\mathrm{d}\omega_1}{\mathrm{d}t} & if\left(\frac{\mathrm{d}\omega_1}{\mathrm{d}t}, e\right)\\ -2h_6e\frac{\mathrm{d}M_1}{\mathrm{d}t} & else\end{cases}
\end{aligned} \tag{10.62}
$$

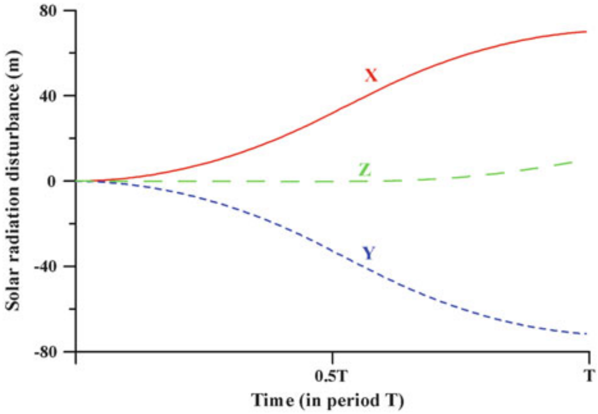

Fig. 10.3 Solar radiation disturbance on a GEO satellite

The above are the equations of motion valid in circular orbits, i.e. which are circular singularity-free, and could be called the first Gauss-Xu equations of motion or circular ones.

The differential equations (or called Gauss-Xu equations of motion) of (10.35) have forms of (in the case of equatorial singularity)

$$\begin{aligned}
&\frac{\mathrm{d}a}{\mathrm{d}t} = h_1 \frac{\mathrm{d}a_1}{\mathrm{d}t}, \quad \frac{\mathrm{d}e}{\mathrm{d}t} = h_2 \frac{\mathrm{d}e_1}{\mathrm{d}t} \\
&\frac{\mathrm{d}\omega}{\mathrm{d}t} = \begin{cases} h_3 \frac{\mathrm{d}\omega_1}{\mathrm{d}t} - h_7 \frac{\mathrm{d}\Omega_1}{\mathrm{d}t} & if\,(\frac{\mathrm{d}\Omega_1}{\mathrm{d}t}, \sin i) \\ h_3 \frac{\mathrm{d}\omega_1}{\mathrm{d}t} & else \end{cases}, \quad \frac{\mathrm{d}i}{\mathrm{d}t} = h_4 \frac{\mathrm{d}i_1}{\mathrm{d}t} \\
&\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \begin{cases} h_4 \frac{1}{\sin i} \frac{\mathrm{d}\Omega_1}{\mathrm{d}t} & if\,(\frac{\mathrm{d}\Omega_1}{\mathrm{d}t}, \sin i) \\ 0 & else \end{cases}, \quad \frac{\mathrm{d}M}{\mathrm{d}t} = -2h_6 e \frac{\mathrm{d}M_1}{\mathrm{d}t} - h_6 \frac{\mathrm{d}\omega_1}{\mathrm{d}t}
\end{aligned} \tag{10.63}$$

The above are the equations of motion valid in equatorial orbits, i.e. which are equatorial singularity-free, and could be called the second Gauss-Xu equations of motion or equatorial ones.

The differential equations (or called Gauss-Xu equations of motion) of (10.36) have forms of (in the case of circular and equatorial singularity)

$$\begin{aligned}
&\frac{\mathrm{d}a}{\mathrm{d}t} = h_1 \frac{\mathrm{d}a_1}{\mathrm{d}t}, \quad \frac{\mathrm{d}e}{\mathrm{d}t} = h_2 \frac{\mathrm{d}e_1}{\mathrm{d}t} \\
&\frac{\mathrm{d}\omega}{\mathrm{d}t} = \begin{cases} h_3 \frac{\mathrm{d}\omega_1}{\mathrm{d}t} - h_7 \frac{\mathrm{d}\Omega_1}{\mathrm{d}t} & if\,(\frac{\mathrm{d}\omega_1}{\mathrm{d}t}, e)\ and\ (\frac{\mathrm{d}\Omega_1}{\mathrm{d}t}, \sin i) \\ h_3 \frac{\mathrm{d}\omega_1}{\mathrm{d}t} & if\,(\frac{\mathrm{d}\omega_1}{\mathrm{d}t}, e) \\ -h_7 \frac{\mathrm{d}\Omega_1}{\mathrm{d}t} & if\,(\frac{\mathrm{d}\Omega_1}{\mathrm{d}t}, \sin i) \\ 0 & else \end{cases}, \quad \frac{\mathrm{d}i}{\mathrm{d}t} = h_4 \frac{\mathrm{d}i_1}{\mathrm{d}t} \\
&\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \begin{cases} h_4 \frac{1}{\sin i} \frac{\mathrm{d}\Omega_1}{\mathrm{d}t} & if\,(\frac{\mathrm{d}\Omega_1}{\mathrm{d}t}, \sin i) \\ 0 & else \end{cases}, \quad \frac{\mathrm{d}M}{\mathrm{d}t} = \begin{cases} -2h_6 e \frac{\mathrm{d}M_1}{\mathrm{d}t} - h_6 \frac{\mathrm{d}\omega_1}{\mathrm{d}t} & if\,(\frac{\mathrm{d}\omega_1}{\mathrm{d}t}, e) \\ -2h_6 e \frac{\mathrm{d}M_1}{\mathrm{d}t} & else \end{cases}
\end{aligned} \tag{10.64}$$

The above are the equations of motion valid in the circular and equatorial orbits, i.e. which are circular and equatorial singularity-free, and could be called the third Gauss-Xu equations of motion. Gaussian equations of motion are valid for non-singular orbits, whereas the three Gauss-Xu equations are valid for the three cases of singular orbits. After the basic Gaussian equations (10.14) are solved, the solutions of the above equations could be formed.

Equations 10.59, 10.62, 10.63 and 10.64 can be summarised as

$$\frac{da}{dt} = h_1 \frac{da_1}{dt}, \quad \frac{de}{dt} = h_2 \frac{de_1}{dt}$$

$$\frac{d\omega}{dt} = \begin{cases} h_3 \frac{d\omega_1}{dt} - h_7 \frac{d\Omega_1}{dt} & if(\frac{d\omega_1}{dt}, e) \text{ and } (\frac{d\Omega_1}{dt}, \sin i) \text{ or } e \neq 0 \text{ and } i \neq 0 \\ & \text{or } (\frac{d\omega_1}{dt}, e) \text{ and } i \neq 0 \text{ or } e \neq 0 \text{ and } (\frac{d\Omega_1}{dt}, \sin i) \\ h_3 \frac{d\omega_1}{dt} & if(\frac{d\omega_1}{dt}, e) \text{ and } i = 0 \\ -h_7 \frac{d\Omega_1}{dt} & if(\frac{d\Omega_1}{dt}, \sin i) \text{ and } e = 0 \\ 0 & else \end{cases}$$

$$\frac{di}{dt} = h_4 \frac{di_1}{dt}$$

$$\frac{d\Omega}{dt} = \begin{cases} h_4 \frac{1}{\sin i} \frac{d\Omega_1}{dt} & if(\frac{d\Omega_1}{dt}, \sin i) \text{ or } i \neq 0 \\ 0 & else \end{cases}$$

$$\frac{dM}{dt} = \begin{cases} -2h_6 e \frac{dM_1}{dt} - h_6 \frac{d\omega_1}{dt} & if(\frac{d\omega_1}{dt}, e) \text{ or } e \neq 0 \\ -2h_6 e \frac{dM_1}{dt} & else \end{cases}$$

Above are the general Gauss–Xu equations of motion which are valid for any cases of orbits without singularity problem.

10.3.3 Singularity-Free Solutions of all Disturbances of the 2nd Order

In Chaps. 6 and 7 the solutions of the basic Lagrangian and Gaussian equations of motion are solved for the disturbances of the geopotential, the solar radiation pressure, the atmospheric drag and the multi-body disturbances. According to the above discussed algorithms the related singularity-free solutions could be formed as the two examples shown in Sects. 10.2.4 and 10.2.5. The critical inclination problem could also be dealt with as shown in Sect. 10.2.6. Details are left for the readers. It could be interesting for readers to write the original Lagrangian equations of motion and the three Lagrange-Xu equations of motion together; the Gaussian equations of motion and the three Gauss-Xu equations of motion together as authors done before.

10.4 Equations of Motion in Non-Inertial Frame

It is well known that Newton's second law is valid in the inertial coordinate system. This is also the reason why the orbit problem is usually dealt with in the ECI frame. However, the geopotential force (or, say, geopotential function) is described in an Earth-Centre-Earth-Fixed non-inertial system. Without exception, one has to transform the geodetic coordinates of the potential function into orbital elements. However, for the equation of motion expressed in the inertial frame and for a simplified transformation between the CEI and ECEF,

$$\frac{\mathrm{d}^2 X_{ECI}}{\mathrm{d}t^2} = F_{ECI} \text{ and } X_{ECI} = R_3(\omega t) X_{ECEF}, \quad F_{ECI} = R_3(\omega t) F_{ECEF}, \tag{10.65}$$

the equation of motion expressed in ECEF frame can be derived:

$$\begin{cases} \frac{d^2x}{dt^2} + 2\omega \frac{dy}{dt} - \omega^2 x = f_x(x,y,z) \\ \frac{d^2y}{dt^2} - 2\omega \frac{dx}{dt} - \omega^2 y = f_y(x,y,z) \\ \frac{d^2z}{dt^2} = f_z(x,y,z) \end{cases} \tag{10.66}$$

where ω is the Earth's angular velocity. The homogenous solution of (10.65) is

$$\begin{cases} x = \frac{1}{2} e^{-i\omega t}[c_1(1+E+i\omega t - i\omega Et) - c_2(1-iE-\omega t-\omega Et) + c_4(t+Et) - c_5 i(t-Et)] \\ y = \frac{1}{2} e^{-i\omega t}[c_1(i-iE-\omega t-\omega Et) + c_2(1+E-i\omega t-i\omega Et) + c_4 i(t-Et) + c_5(t+Et)] \\ z = c_3 + c_6 t \end{cases}$$

where c_j is integral constants and $E = e^{2i\omega t}, i = \sqrt{-1}$.

As soon as any special solution of (10.62) is found, then the general solution of (10.66) is equal to the special solution plus the homogenous solution. It may be worthwhile to consider the problem of geopotential disturbance as an alternative.

10.5 Discussions

10.5.1 Confusion of Non Conservative Force with Conservative Effect

Solar radiation is a non-conservative disturbing force. It is said that such a non-conservative force has a conservative effect. This is confusion and is illustrated in Fig. 10.3 with an example of solar disturbance on a GEO satellite. One of the possible reasons for such confusion may come from the adjustment model of solar radiation used in the numerical orbit determination. The models (4.72) are periodic functions of the orbit. No matter what results are obtained from the adjustment, the results are periodic (or conservative). If the determined models are used to interpret the effects of solar radiation, the confusion is then the consequence. This shows that the parameterisation is very important and the parameterisation should be physically reasonable.

10.5.2 Long Term Effects in Extraterrestrial Disturbances

There exist long term effects in extraterrestrial disturbances (see (7.26)). Long term perturbations have to be taken into account in the transformation of integral variables. This shall be especially noticed in practical applications.

10.5.3 Further Studies

Further studies have to be carried out on the applications of the analytical theory (especially on analytical orbit determination), on the study of correlation of geopotential disturbances on orbits, on third order solutions disturbed by the Earth and ocean tides as well as relativity disturbance.

Chapter 11
Orbit in Solar Gravity Field

In this chapter, emphasis will be given to the analytical orbit theory in the solar gravity field. Section 11.1 is modified from the original paper authored by Xu et al. (2011) which first dealt with solar oblateness disturbance on planetary orbits with disturbing theory, taking the difference between the solar equator and ecliptic into account. It is worth mentioning that the work has been well valued and cited several times by a paper concerning the history of solar oblateness measurements and interpretation authored by Rozelot and Damiani (2011). Section 11.2 is a study which first dealt with the solar radiation disturbance on planetary orbits by using disturbing theory (cf. Xu and Xu 2013). By comparing with the results obtained by using the traditional mass correction method, interesting results (traditional mass correction method is imperfect) are obtained.

11.1 Solar Oblateness and Mercury's Perihelion Precession

The Keplerian laws of planet motion are solutions of two-body gravitational problems. Solar oblateness due to the rotation of the sun distorts the gravitational force acting on a planet and disturbs its Keplerian motion. An analytical solution of a planet orbit disturbed by solar gravitational oblateness is derived. Apart from short- and long-periodic disturbances, secular disturbances lead to perihelion precession and nodal regression as well as mean motion advancing. The magnitudes of the short-periodic perihelion precession could disturb the observation of the secular effect if the survey is made shorter than one Julian year. Transformation of the formulas from the solar equatorial plane to the ecliptic one is discussed. Numerical estimates of Mercury's, Venus' and Mars' secular perihelion precessions are in good agreement with published results, which confirm a correctness of our theory derived. Conversely, the solar oblateness could be determined through observation of perihelion precession of a planet. The solution is valid also for the satellite orbit in the solar gravity field.

G. Xu and J. Xu, *Orbits*, DOI 10.1007/978-3-642-32793-3_11,

11.1.1 Introduction

The measured perihelion precession of Mercury's orbit includes the effects of Earth's coordinates, general relativity and planetary gravitation attractions as well as solar gravitational oblateness (Shapiro 1999; Hill et al. 1974). The agreement between observation and theory was remarkably good around the 1910s (this fact has been accepted by the science community as confirmation of the correctness of the general relativity theory). However, the problem of how much of the perihelion advance could be contributed by solar oblateness remains and has been studied by numerous scientists during the past century (e.g. Shapiro 1999; Gilvarry and Sturrock 1967; Sturrock and Gilvarry 1967; Wayte 2010; Boehme 1970; Milani et al. 2001; Kuhn et al. 1998, 2009; Rozelot et al. 2001, 2004; Campbell and Moffat 1983; Campbell et al. 1983; Godier and Rozelot 1999, 2000; Pireaux and Rozelot 2003; Pireaux et al. 2006; Fivian et al. 2008, 2009; Dicke 1970; Dicke et al. 1987). The theoretical value has been estimated from the disturbed equation so far and restricted to the secular effects. Therefore, deriving analytical solutions of the equations of planet motion disturbed by the solar gravitational oblateness are also meaningful for investigating the non-secular effects.

Rotation of the sun introduces solar oblateness which distorts the gravitational force acting on a planet and disturbs the planetary Keplerian orbit. This is in principle similar to an Earth satellite orbit perturbed by the geopotential. Study shows that the solar oblateness is very small (0.00001, Fivian et al. 2008 and 0.00005, Sturrock and Gilvarry 1967). Compared to that of the Earth (1/298.245642) the sun is a very spherical mass-body. This indicates that it could be precise enough first to take only the J_2 term of heliopotential into account. Methods to solve the equations of satellite motion disturbed by solar and lunar gravitation, atmospheric drag, solar radiation pressure and geopotential have been developed and the solutions are given in Xu et al. (2010a, b) and Xu (2008). Hence, an analytical solution of a planet orbit disturbed by the zonal heliopotential terms of J_k, $k = 2, 3, 4, \ldots$, can be similarly derived.

The disturbing function of solar gravitational oblateness and the Lagrangian equations of planet motion are discussed in Sect. 11.1.2. The solutions for the oblateness disturbance are derived and coordinate transformation as well as numerical discussions are addressed in Sect. 11.1.3, followed by a concluding summary.

11.1.2 Disturbing Function of Solar Oblateness and Disturbed Equations of Motion

Heliopotential disturbing potential function, caused by solar oblateness, can be written as (see (4.35), or Vallado 2007; Lynden-Bell 2009; Pal 2009)

$$V = \frac{-\mu_s a_s^2}{2r^3} J_2 (3\sin^2\varphi - 1) = b\frac{1}{2r^3}(3\sin^2\varphi - 1)$$
$$b = -\mu_s a_s^2 J_2 \tag{11.1}$$

where solar variables are denoted with index s, μ_s is the solar gravitational constant, a_s is the mean equatorial radius of the sun and r and φ are the heliocentric radius and latitude of a planet. J_2 is an un-normalised coefficient which has different values from different authors due to the different measuring technologies and theoretical models used (e.g. Kuhn et al. 1998, 2009; Campbell et al. 1983; Rozelot et al. 2004, 2011). J_2 values are, e.g. $0 \sim 1.08 \times 10^{-5}$ (Kislik 1983), 1.46×10^{-7} (Fivian et al. 2008), 2×10^{-7} (Pireaux and Rozelot 2003; Pitjeva 2005), 2.3×10^{-7} (Shapiro 1999). A list of J_2 values determined by different authors can be found in Pireaux and Rozelot (2003) and Pitjeva (2005). A review of solar oblateness can be found in Rozelot et al. (2011).

Equation 11.1 can be represented in Keplerian variables using the following relations ((5.2) or Kaula 2001):

$$\sin\varphi = \sin i \sin u,\ u = \omega + f, \tag{11.2}$$

$$r = \frac{a(1-e^2)}{1 + e\cos f}. \tag{11.3}$$

The Keplerian elements (a, e, ω, i, Ω, M, f) are the semi-major axis, the eccentricity of the ellipse, the argument of perihelion, the inclination angle, the right ascension of ascending node, the mean anomaly and the true anomaly, respectively.

Then it follows that

$$\frac{\partial V}{\partial a} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial a} = \frac{-3}{a}\frac{b}{2r^3}(3\sin^2 i\sin^2 u - 1),$$

$$\frac{\partial V}{\partial \Omega} = 0,$$

$$\frac{\partial V}{\partial i} = \frac{b}{2r^3} 6\sin i\sin^2 u\cos i,$$

$$\frac{\partial V}{\partial \omega} = \frac{b}{2r^3} 6\sin u\cos u\sin^2 i,$$

$$\frac{\partial V}{\partial e} = \frac{-3b}{2r^4}(3\sin^2 i\sin^2 u - 1)\frac{\partial r}{\partial e} + \frac{6b}{2r^3}\sin u\cos u\sin^2 i\frac{\partial u}{\partial e}$$

and

$$\frac{\partial V}{\partial M} = \frac{-3b}{2r^4}\left(3\sin^2 i \sin^2 u - 1\right)\frac{\partial r}{\partial M} + \frac{6b}{2r^3}\sin u \cos u \sin^2 i \frac{\partial u}{\partial M} \tag{11.4}$$

Here partial derivatives are ((4.24), or Kaula 2001)

$$\begin{aligned}
\frac{\partial f}{\partial(e,M)} &= \left(\frac{2 + e\cos f}{1 - e^2}\sin f,\ \left(\frac{a}{r}\right)^2\sqrt{1-e^2}\right) \\
\frac{\partial r}{\partial(a,e,\omega,i,\Omega,M)} &= \left(\frac{r}{a}, -a\cos f, 0, 0, 0\ \frac{ae}{\sqrt{1-e^2}}\sin f\right)
\end{aligned} \tag{11.5}$$

Using mathematic expansion formulas (Wang et al. 1979; Bronstein and Semendjajew 1987) it yields (L = 2, 3, truncation to e^3)

$$\begin{aligned}
\frac{1}{r^L} &= \frac{(1 + e\cos f)^L}{a^L(1-e^2)^L} \\
&\approx \frac{1}{a^L}\left(1 + Le\cos f + \frac{L(L-1)}{2}e^2\cos^2 f + \frac{L(L-1)(L-2)}{2}e^3\cos^3 f\right)\left(1 + Le^2\right). \\
&\approx \frac{1}{a^L}\left((1 + Le\cos f)(1 + Le^2) + \frac{L(L-1)}{2}e^2\cos^2 f + \frac{L(L-1)(L-2)}{2}e^3\cos^3 f\right)
\end{aligned} \tag{11.6}$$

These formulas can be substituted into the following Lagrangian equations of planet motion ((4.11), or Battin 1999; Kaula 2001):

$$\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial V}{\partial M} \\
\frac{de}{dt} &= \frac{1-e^2}{na^2 e}\frac{\partial V}{\partial M} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial V}{\partial \omega}, \\
\frac{d\omega}{dt} &= \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial V}{\partial e} - \frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial V}{\partial i} \\
\frac{di}{dt} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\left(\cos i\frac{\partial V}{\partial \omega} - \frac{\partial V}{\partial \Omega}\right) \\
\frac{d\Omega}{dt} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial V}{\partial i}, \\
\frac{dM}{dt} &= n = \frac{2}{na}\frac{\partial V}{\partial a} - \frac{1-e^2}{na^2 e}\frac{\partial V}{\partial e}
\end{aligned} \tag{11.7}$$

Here n is the mean angular velocity. The first term "n" on the right-hand side of the last equation of (11.7) represents the Keplerian mean motion and will be omitted later on. The true anomaly f in the above formulas can be further transformed to the mean anomaly M using (see, e.g. (5.22), truncated to e^3)

$$\sin f = \left(1 - \frac{7}{8}e^2\right)\sin M + \left(e - \frac{7}{6}e^2\right)\sin 2M + \frac{9}{8}e^2\sin 3M + \frac{8}{6}e^3\sin 4M,$$
$$\cos f + e = \left(1 - \frac{9}{8}e^2\right)\cos M + \left(e - \frac{4}{3}e^2\right)\cos 2M + \frac{9}{8}e^2\cos 3M + \frac{4}{3}e^3\cos 4M. \tag{11.8}$$

Using software for mathematic symbolic operations, (11.7) can be transformed and reduced to a Fourier series with a form

$$\frac{\mathrm{d}\sigma_j}{\mathrm{d}t} = d_j + \sum_{k=1}^{6}\left(g_{jk}\cos kM + c_{jk}\sin kM\right). \tag{11.9}$$

Here σ_j is the jth Keplerian element, d, g, c are functions of (a,e,ω,i,Ω). All terms of M are short-periodic perturbations and d-terms are long-periodic and linear ones. We define individual factor h_j for the jth equation of (11.9) (i.e. (11.7)) as

$$h_1 = \frac{2b}{na^4} = 2h_6,\ \ h_2 = h_3 = \frac{\sqrt{1-e^2}}{na^5e}b,\quad h_4 = h_5 = \frac{b}{na^5\sqrt{1-e^2}}. \tag{11.10}$$

They are omitted later on and they are related by

$$h_3 \approx h_6/ae \approx h_5/e. \tag{11.11}$$

11.1.3 Solutions for Oblateness Disturbance

The short-periodic terms in (11.9) can easily be integrated with respect to M by using the relation $M = nt$ and the mean value theorem for integration (Wang et al. 1979; Bronstein and Semendjajew 1987):

$$\int_0^T p(y(t))q(M(t))dt = p(y(\xi))\int_0^T q(M(t))dt,\ 0 \le \xi \le T. \tag{11.12}$$

Here functions p (i.e. g and c in (11.9)) and q (i.e. $\sin kM$ and $\cos kM$ in (11.9)) can be integrated. The time interval [0, T] can be transformed into the mean motion angular intervals of $\{[2(j-1)\pi/k,\ 2j\pi/k], j = 1,\ldots J\}$ and $[2j\pi/k, nT]$ (for any k, the integer J can be obtained by relation $nT - 2J\pi/k < 2\pi/k$). The integrals of the trigonometric functions q (short-periodic terms) are zero, except over the rest (non-full cycle) intervals $[2J\pi/k, nT]$. Therefore, it holds that

$$\int_0^T p(y(t))q(M(t))dt = p(y(\xi)) \int_{2J\pi/k}^{nT} q(M)\frac{1}{n}dM. \tag{11.13}$$

Here y(t) are slow changing Keplerian elements (not include M). Selecting T so that p can be considered as constants over $[2J\pi/k, nT]$ (i.e. (11.13) is valid for $\xi = T$) then the integrals (11.13) are generally valid. For integration over time interval longer than T, the integration can be made step-wise and then accumulated.

The indefinite integrals (i.e. the solutions) of (11.9) disturbed by solar gravitational oblateness are then

$$\Delta\sigma_j = \Delta d_j + \sum_{k=1}^{6} \frac{1}{kn}(g_{jk}\sin kM - c_{jk}\sin kM), \tag{11.14}$$

where Δd_j denote symbolically the integrals of the secular and long-periodic terms. Or, explicitly (truncated to order of e for terms of M and e^2 to the rest, factor h_j omitted)

$$\begin{aligned}\triangle a = &\frac{-3e}{8n}\sin^2 i\sin 2\omega\sin M + \frac{3}{4n}\sin^2 i\sin 2\omega\sin 2M \\ &+ \frac{21e}{8n}\sin^2 i\sin 2\omega\sin 3M - \frac{3e}{2n}\left(1 - \sin^2 i\left(\frac{5}{4} + \frac{1}{2}\cos^2\omega\right)\right)\cos M \\ &- \frac{3}{4n}\sin^2 i\cos 2\omega\cos 2M - \frac{21e}{8n}\sin^2 i\cos 2\omega\cos 3M\end{aligned} \tag{11.15}$$

$$\begin{aligned}\triangle e = &\frac{3e}{8n}\sin^2 i\sin 2\omega\sin M - \frac{3e}{2n}\left(1 - \sin^2 i\left(\frac{5}{4} + \frac{1}{2}\sin^2\omega\right)\right)\cos M \\ &+ \frac{7e}{8n}\sin^2 i\sin 2\omega\sin 3M - \frac{7e}{8n}\sin^2 i\cos 2\omega\cos 3M\end{aligned} \tag{11.16}$$

$$\begin{aligned}\triangle\omega = &\left(\frac{-9e}{8} - \frac{15}{8}e\cos 2i\right)t + \frac{1}{n}\left(\frac{-3}{2} + \sin^2 i\left(\frac{3}{4}\cos^2\omega + \frac{15}{8}\right)\right)\sin M \\ &+ \frac{3e}{4n}\left(-3 + \cos^2 i\cos 2\omega + \sin^2 i\left(5\cos^2\omega + \frac{1}{2}\right)\right)\sin 2M \\ &- \frac{7}{8n}\sin^2 i\cos 2\omega\sin 3M - \frac{51e}{16n}\sin^2 i\cos 2\omega\sin 4M \\ &+ \frac{3}{8n}\sin^2 i\sin 2\omega\cos M + \frac{3e}{4n}\left(1 + \frac{3}{2}\sin^2 i\right)\sin 2\omega\cos 2M \\ &- \frac{7}{8n}\sin^2 i\sin 2\omega\cos 3M - \frac{51e}{16n}\sin^2 i\sin 2\omega\cos 4M\end{aligned} \tag{11.17}$$

$$\begin{aligned}\triangle i = &\frac{-3e}{8n}\sin 2i \sin 2\omega \sin M + \frac{3}{8n}\sin 2i \sin 2\omega \sin 2M \\ &+ \frac{7e}{8n}\sin 2i \sin 2\omega \sin 3M + \frac{3}{8n}\sin 2i \cos 2\omega \cos 2M \\ &- \frac{3}{8n}\sin 2i \cos 2\omega \cos 2M - \frac{7e}{8n}\sin 2i \cos 2\omega \cos 3M\end{aligned} \tag{11.18}$$

$$\begin{aligned}\triangle\Omega = &\left(\frac{3}{2}+\frac{9}{4}e^2\right)t\cos i + \frac{3e}{4n}(2\cos^2\omega + 5)\sin M \\ &- \frac{3}{4n}\cos i \cos 2\omega \sin 2M - \frac{7e}{4n}\cos i \cos 2\omega \sin 3M \\ &+ \frac{3e}{4n}\cos i \cos 2\omega \sin M - \frac{3}{4n}\cos i\ \sin 2\omega \cos 2M \\ &- \frac{7e}{8n}\cos i \sin 2\omega \cos 3M\end{aligned} \tag{11.19}$$

$$\begin{aligned}\triangle M = &\frac{-3}{8a}\left(\left(1+\frac{21}{4}e^2\right)+\left(3+\frac{3}{4}e^2\right)\cos 2i\right)t + \int\frac{3}{16}e^2\sin^2 i\cos 2\omega dt \\ &+\frac{1}{an}\left(\frac{3}{2e}-\frac{87e}{16}+\sin^2 i\left(\left(\frac{-21}{8e}+\frac{351e}{32}\right)\cos^2\omega+\left(\frac{-15}{8e}+\frac{171e}{32}\right)\sin^2\omega\right)\right)\sin 2M \\ &+\frac{1}{2an}\left(\frac{9}{2}+\sin^2 i\left(-15\cos^2\omega+\frac{3}{2}\sin^2\omega\right)\right)\sin 2M \\ &+\frac{1}{an}\left(\frac{53e}{16}+\sin^2 i\left(\left(\frac{7}{8e}-\frac{1079e}{64}\right)\cos^2\omega+\left(\frac{-7}{8e}+\frac{443e}{64}\right)\sin^2\omega\right)\right)\sin 3M \\ &+\frac{51}{16an}\sin^2 i\ \cos 2\omega\sin 4M+\frac{507e}{64an}\sin^2 i\ \cos 2\omega\sin 5M \\ &-\frac{1}{n}\sin^2 i\left(\frac{3}{8ae}-\frac{87e}{32a}\right)\sin 2\omega\cos M-\frac{33}{8na}\sin^2 i\ \sin 2\omega\cos 2M \\ &-\frac{1}{n}\sin^2 i\left(\frac{-7}{8ae}+\frac{761e}{64a}\right)\sin 2\omega\cos 3M+\frac{51}{16na}\sin^2\ i\ \sin 2\omega\cos 4M \\ &+\frac{507e}{16an}\sin^2 i\ \sin 2\omega\cos 5M\end{aligned} \tag{11.20}$$

All Keplerian elements are subjected to short-periodic disturbances. The mean anomaly is also disturbed long-periodically by (see (11.20), h_6 is taken into account)

$$\Delta M = h_6\int\frac{3}{16a}e^2\sin^2 i\ \cos 2\omega dt = \frac{3h_6 e^2}{32an_\omega}\sin^2 i\ \sin 2\omega\ . \tag{11.21}$$

The integration is done by using the relation $\omega = n_\omega t$ (n_ω is the secular motion of the perihelion). The magnitude of the secular disturbance (11.21) depends on the value n_ω and will be further discussed later on. The secular effects are (see (11.17), (11.19), and (11.20), h_j are taken into account)

$$\triangle\omega = h_3 \frac{3e}{8}(-3 - 5\cos 2i)t = n_\omega t, \tag{11.22}$$

$$\triangle\Omega = h_5 \frac{3}{2}\left(1 + \frac{3}{2}e^2\right) t \cos i = n_\Omega t, \tag{11.23}$$

$$\triangle M = h_6 \frac{-3}{8a}\left(\left(1 + \frac{21}{4}e^2\right) + \left(3 + \frac{3}{4}e^2\right)\cos 2i\right)t = n_M t\,. \tag{11.24}$$

Because of the small inclinations of the solar planets, cos$i$ and cos2i are both positive. Remember that the b in factor h_j is negative; then (11.22), (11.23), and (11.24) show that the solar oblateness (rotational ellipsoid form) will lead to a perihelion precession and an advancing of the mean motion of all planets. The right ascension of ascending node will experience a retrograde motion. Taking the values of the factors h_j in (11.10) and their relations (11.11) into account, the perihelion precession and the nodal regression as well as the mean motion advancing are of the same order of magnitude.

Comparing (11.15), (11.16), (11.17), (11.18), (11.19), and (11.20) with (11.14), the coefficients of g and c can easily be obtained; therefore the differential equations (11.9) are also explicitly given.

The above derived solution is also valid for satellite motion in the solar gravity field disturbed by solar oblateness.

11.1.3.1 Orbital Variables in Ecliptic and Equatorial Coordinate Systems

The solutions (11.15), (11.16), (11.17), (11.18), (11.19), and (11.20) are given in the solar equatorial coordinate system; whereas the planetary orbital elements and the results (e.g. observed perihelion precession from the Earth) are usually represented in the ecliptic coordinate system (denoted as $(a_1, e_1, \omega_1, i_1, \Omega_1, M_1)$). The relations of the Keplerian elements in the above-mentioned two coordinate systems can be obtained from the orbital sphere geometry (cf. Fig. 11.1, Wang et al. 1979; Bronstein and Semendjajew 1987)

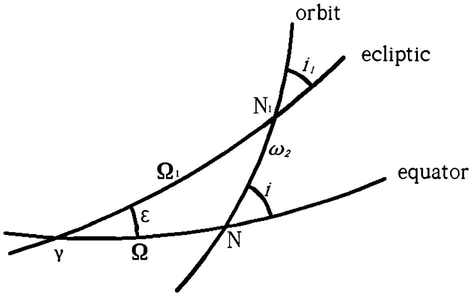

Fig. 11.1 Orbital sphere geometry of the ecliptic and the equator. Here γ is the equinox of date; N and N_1 are ascending nodes of the planet orbit on the equator and the ecliptic, respectively; ε is the inclination of the solar equator to the ecliptic

$$\begin{aligned}
&(a, e, \omega, M) = (a_1, e_1, \omega_1 + \omega_2, M_1),\\
&\tan\left(\frac{\Omega + \omega_2}{2}\right) = \left(\cos\left(\frac{i_1 - \varepsilon}{2}\right) / \cos\left(\frac{i_1 + \varepsilon}{2}\right)\right) \tan\left(\frac{\Omega_1}{2}\right),\\
&\tan\left(\frac{\Omega - \omega_2}{2}\right) = \left(\sin\left(\frac{i_1 - \varepsilon}{2}\right) / \sin\left(\frac{i_1 + \varepsilon}{2}\right)\right) \tan\left(\frac{\Omega_1}{2}\right),\\
&\tan\left(\frac{\pi - i}{2}\right) = \left(\cos\left(\frac{\Omega - \omega_2}{2}\right) / \cos\left(\frac{\Omega + \omega_2}{2}\right)\right) \mathrm{ctan}\left(\frac{i_I + \varepsilon}{2}\right).
\end{aligned} \tag{11.25}$$

Here ε is the inclination of the sun's equator to the ecliptic and can be found, e.g. in Bate et al. (1971). Therefore the orbital elements in the equatorial system can be obtained by using (11.25). Then the secular effects of (11.22), (11.23), and (11.24) can be computed. Taking only the secular effects into account and making a full derivative operation on (11.25), it yields

$$\begin{aligned}
&(\Delta\omega_1, \Delta M_1) = (\Delta\omega - \Delta\omega_2, \Delta M),\\
&\frac{\Delta\Omega + \Delta\omega_2}{2} \cos^{-2}\left(\frac{\Omega + \omega_2}{2}\right) = \frac{\Delta\Omega_1}{2} \left(\cos\left(\frac{i_1 - \varepsilon}{2}\right) / \cos\left(\frac{i_1 + \varepsilon}{2}\right)\right) \cos^{-2}\left(\frac{\Omega_1}{2}\right),\\
&\frac{\Delta\Omega + \Delta\omega_2}{2} \cos^{-2}\left(\frac{\Omega - \omega_2}{2}\right) = \frac{\Delta\Omega_1}{2} \left(\sin\left(\frac{i_1 - \varepsilon}{2}\right) / \sin\left(\frac{i_1 + \varepsilon}{2}\right)\right) \cos^{-2}\left(\frac{\Omega_1}{2}\right),
\end{aligned} \tag{11.26}$$

From the last two equations of (11.26), it follows that

$$\begin{aligned}
&\triangle\omega_2 = -\triangle\Omega \frac{\cos^{-2}\left(\frac{\Omega + \omega_2}{2}\right) - \beta \cos^{-2}\left(\frac{\Omega - \omega_2}{2}\right)}{\cos^{-2}\left(\frac{\Omega + \omega_2}{2}\right) + \beta \cos^{-2}\left(\frac{\Omega - \omega_2}{2}\right)}.\\
&\beta = \left(\mathrm{ctan}\left(\frac{i_1 - \varepsilon}{2}\right) / \mathrm{ctan}\left(\frac{i_1 + \varepsilon}{2}\right)\right)
\end{aligned} \tag{11.27}$$

Using $\triangle\omega_2$ of (11.27), $\triangle\Omega_1$ can be computed using one of the last two equations of (11.26). The traditional perihelion precession is related to the vernal equinox of date and has a form (Iorio 2005; Campbell et al. 1983)

$$\Delta\bar{\varpi}_1 = \Delta\omega_1 + \Delta\Omega_1 \cos i_1 \ . \tag{11.28}$$

Then we have all the formulas of the final secular effects of $(\Delta\bar{\varpi}_1, \Delta\Omega_1, \Delta M_1)$.

11.1.3.2 Secular Perihelion Precession of Mercury's Orbit

The perihelion precession of Mercury's orbit can be computed as follows. The masses of the sun and Mercury are 1.99×10^{30} and 3.30×10^{23} kg and the gravitational constant is 6.67×10^{-11} ($\mathrm{m^3kg^{-1}s^{-2}}$), so that $\mu_S \approx 1.33 \times 10^{20}$ and $\mu_{mercury} \approx 2.20 \times 10^{13}$ ($\mathrm{m^3s^{-2}}$). In J2000.0 Mercury's Keplerian elements are given in ecliptic coordinate (Xu 2008): $a = 0.38709831$ (AU), $e = 0.20563175$, $i_1 = 7^{\circ}.0049860$, $\omega_1 = 29^{\circ}.1252260$, $\Omega_1 = 48^{\circ}.3308930$, the mean longitude $L_1 = 252^{\circ}.250906$. 1 AU $= 1.49597870700 \times 10^{11}$ m, $a_s = 6.96 \times 10^8$ m, $t = 1$(Julian century (Jc)) $= 36{,}525 \times 86{,}400$ s. $\varepsilon = 7^{\circ}.25$ (Bate et al. 1971). Then it yields $\omega_2 = 24^{\circ}.7748$, $\Omega = 23^{\circ}.8888$, $i = 13^{\circ}.0005$. The mean angular velocity n of Mercury is represented by the Keplerian third law as

$$n = a^{-3/2}\mu^{1/2}. \tag{11.29}$$

Here μ is the masses of the sun and Mercury together multiplied by the gravitational constant. The secular effects of Mercury's orbit are then (units: arcsec/Jc)

$$\begin{aligned}\Delta\omega &= {\mu_s}^{1/2} a_s^2 J_2 \frac{3\sqrt{1-e^2}}{8a^{7/2}}(3+5\cos 2i)t = n_\omega t \\ &\approx 6.77452 \times 10^{-5} J_2 t = 2.13788 \times 10^5 J_2\end{aligned}$$

$$\begin{aligned}\Delta\Omega &= -{\mu_s}^{1/2} a_s^2 J_2 \frac{3}{2\sqrt{1-e^2}a^{7/2}}(1+\frac{3}{2}e^2)t\cos i = n_\Omega t \\ &\approx -3.91222 \times 10^{-5} J_2 t = -1.23460 \times 10^5 J_2\end{aligned} \tag{11.30}$$

$$\Delta M_1 = 1.17568 \times 10^5 J_2 \tag{11.31}$$

$$\Delta\omega_2 = -1.28652 \times 10^5 J_2$$

$$\Delta\omega_1 = \Delta\omega - \Delta\omega_2 = 3.42440 \times 10^5 J_2$$

$$\Delta\Omega_1 = -2.50820 \times 10^5 J_2 \tag{11.32}$$

Then it follows the perihelion precession of Mercury's orbit

$$\Delta\bar{\omega}_1 = 5.91388 \times 10^5 J_2 \tag{11.33}$$

A $J_2 = 1.47 \times 10^{-7}$ (Fivian et al. 2008) indicates a perihelion advance (related to the vernal equinox and the ecliptic plane) 0.0869 (as/Jc), and a $J_2 = 2 \times 10^{-7}$ (Pireaux and Rozelot 2003; Pitjeva 2005) a perihelion advance 0.1182 (as/Jc), and a $J_2 = 2.3 \times 10^{-7}$ (Shapiro 1999) an advance 0.1360 (as/Jc). The (11.30), (11.31), (11.32), and (11.33) are functions of J_2 which could also be useful for determining solar oblateness by surveying the perihelion precession of a planet's orbit.

11.1.3.3 Perihelion Precessions of Venus' and Mars' Orbits

For Venus, the gravitational constant and orbital Keplerian elements in J2000.0 are (Xu 2008) $\mu_{venus} \approx 3.248585 \times 10^{14}$ (m^3s^{-2}), $a = 0.72332982$ (AU), $e = 0.00677118$, $i_1 = 3^o.3946620$, $\omega_1 = 54^o.883787$, $\Omega_1 = 76^o.6799200$, $L_1 = 181^o.979801$. Then it yields $\omega_2 = 54^o.4478$, $\Omega = 22^o.4415$, $i = 8^o.68149$. The mean angular velocity n of Venus is represented by (11.29) where μ is the gravitational constants of the sun and Venus together. The secular effects of Venus' orbit are then (units: arcsec/Jc)

$$\Delta\omega = 2.5403 \times 10^4 \mathrm{J}_2 \ . \tag{11.34}$$

$$\Delta\Omega = -1.29255 \times 10^4 J_2,$$

$$\Delta M_1 = 1.26285 \times 10^4 J_2 \ . \tag{11.35}$$

$$\Delta\omega_2 = -4.37899 \times 10^4 J_2,$$

$$\Delta\omega_1 = \Delta\omega - \Delta\omega_2 = 6.91929 \times 10^4 J_2,$$

$$\Delta\Omega_1 = -5.66668 \times 10^4 J_2 \ . \tag{11.36}$$

It then follows the perihelion precession of Venus' orbit

$$\Delta\bar{\omega}_1 = 1.25760 \times 10^5 J_2 \ . \tag{11.37}$$

A $J_2 = 1.47 \times 10^{-7}$ (Fivian et al. 2008) indicates a perihelion advance (related to the vernal equinox and the ecliptic plane) 0.01848 (as/Jc), and a $J_2 = 2 \times 10^{-7}$ (Pireaux and Rozelot 2003; Pitjeva 2005) a perihelion advance 0.02515 (as/Jc), and a $J_2 = 2.3 \times 10^{-7}$ (Shapiro 1999) an advance 0.02892 (as/Jc).

For Mars the gravitational constant and orbital Keplerian elements in J2000.0 are (Xu 2008) $\mu_{mars} \approx 4.28283 \times 10^{13}$ (m^3s^{-2}), $a = 1.523679342$ (AU), $e = 0.09340062$, $i_1 = 1^o.8497260$, $\omega_1 = 286^o.502141$, $\Omega_1 = 49^o.558093$, $L_1 = 355^o.4332750$. It then yields $\omega_2 = 40^o.154$, $\Omega = 9^o.49336$, $i = 8^o.56572$. The mean angular velocity n of Mars is represented by (11.29) where μ is the gravitational constants of the sun and Mars together. The secular effects of Mars' orbit are then (units: arcsec/Jc)

$$\Delta\omega = 1.86585 \times 10^3 J_2 \,. \tag{11.38}$$

$$\Delta\Omega = -9.69704 \times 10^3 J_2,$$

$$\Delta M_1 = 9.48361 \times 10^2 J_2 \,. \tag{11.39}$$

$$\Delta\omega_2 = -4.8924 \times 10^3 J_2,$$

$$\Delta\omega_1 = \Delta\omega - \Delta\omega_2 = 6.7583 \times 10^3 J_2,$$

$$\Delta\Omega_1 = -5.8544 \times 10^3 J_2 \,. \tag{11.40}$$

It then follows the perihelion precession of Mars' orbit

$$\Delta\bar{\omega}_1 = 1.26096 \times 10^4 J_2 \,. \tag{11.41}$$

A $J_2 = 1.47 \times 10^{-7}$ (Fivian et al. 2008) indicates a perihelion advance (related to the vernal equinox and the ecliptic plane) 0.0019 (as/Jc), and a $J_2 = 2 \times 10^{-7}$ (Pireaux and Rozelot 2003; Pitjeva 2005) a perihelion advance 0.0025 (as/Jc), and a $J_2 = 2.3 \times 10^{-7}$ (Shapiro 1999) an advance 0.0029 (as/Jc).

11.1.3.4 Comparisons with Results of Iorio 2004

The solutions of (11.15), (11.16), (11.17), (11.18), (11.19), and (11.20) are newly derived including effects of short- and long-periodic terms. The secular terms of (11.22), (11.23), and (11.24) including (11.28) are nearly identical with the (11.12), (11.13), and (11.14) of Iorio (2004), except the factor functions of e. Our formulas are truncated to e^3 which indicate the formulas of Iorio have a precision of e. It seems that Iorio omitted the difference between the solar equatorial plane and the ecliptic one. The coefficient -123460 of J_2 in (11.30) has very good agreement with the value $-126{,}878.626$ given in Table 2 of Iorio. The coefficient 117568 of J_2 in (11.31) has good agreement with the value 123,703.132 of Iorio. These indicate that the values of Iorio are related to the solar equator ones. The coefficient 591388 of J_2 in (11.33) is different from the value 126404.437 of Iorio because of the difference of the equator and the ecliptic. The disagreements show that the difference between the solar equator and the ecliptic plane has to be dealt with precisely as outlined here (cf. also Xu et al. 2011). Comparisons between the results computed for Venus and Mars and given in Iorio showed a systematic consistency with the case of Mercury stated above.

11.1.3.5 Short-Periodic Perihelion Precession

For simplicity, the following discussions are made by considering the solar equator as the ecliptic. Because $\sin^2 i_1$ (≈ 0.015 and $e^2 \approx 0.042$) can be omitted in (11.17), the short periodic perihelion precession is dominated by

$$\begin{aligned}\Delta\omega &= \frac{3h_3}{n}\left(-\sin M - \frac{e}{2}((3-\cos^2 i\cos 2\omega)\sin 2M - \sin 2\omega\cos 2M)\right)\\ &= \frac{3a_s^2 J_2}{2a^2 e}\left(\sin M + \frac{e}{2}((3-\cos^2 i\cos 2\omega)\sin 2M - \sin 2\omega\cos 2M)\right)\\ &\approx 217.346 J_2(\sin M + 0.25515\sin 2M - 0.08743\cos 2M)\ .\end{aligned} \tag{11.42}$$

Comparing the magnitude of (11.42) with the mean velocity of the perihelion (11.22), one notes that the magnitude of the short periodic effect is much larger than the secular one (a factor of 6×10^6). Within one Julian year, the perihelion precession is 0.000508 arcsec. For $J_2 = 2.26804 \times 10^{-7}$ the amplitude of (11.42) is 0.0000492 arcsec and this is about 1/5 of the yearly advance. Therefore, the observations for the perihelion precession must be done yearly so as not to be disturbed by the short-periodic effects.

11.1.3.6 Secular Mean Motion Advancing

From (11.24) it holds that

$$\begin{aligned}n_M &= \frac{3na_s^2 J_2}{8a^2}\left((1+\frac{21}{4}e^2) + (3+\frac{3}{4}e^2)\cos 2i\right)\ .\\ &\approx 2.3037 \times 10^{-4} nJ_2\end{aligned} \tag{11.43}$$

Here n_M is a correction to the Keplerian third law (11.29). Due to the small J_2 the correction (11.43) is negligible.

11.1.3.7 Long-Periodic Mean Motion Disturbance

From (11.21) the unique long-periodic effect of solar oblateness, i.e. the mean motion disturbance has the form

$$\Delta M = \frac{-3na_s^2 J_2}{32a^2 n_\omega} e^2 \sin^2 i \sin 2\omega \ . \tag{11.44}$$

Taking (11.22) into account, it follows that

$$\begin{aligned}\Delta M &= \frac{-e^2 \sin^2 i}{4\sqrt{1-e^2}(3+5\cos 2i)} \sin 2\omega \\ &\approx -2.04629 \times 10^{-5} \sin 2\omega \ (\text{rad}) = -4.22078 \sin 2\omega \ (\text{as})\end{aligned} \tag{11.45}$$

A magnitude of -4.22078 is a notable mean motion effect.

11.1.4 Summary

The analytical solutions of a planet orbit disturbed by solar gravitational oblateness are derived including the short- and long-periodic terms. The secular terms are then applied for studying perihelion precessions of Mercury and Venus as well as Mars. It is notable that the amplitudes of the short periodic terms are much larger than the secular (linear) ones, which indicates that care has to be taken for determination of J_2 through planet orbit observation and the solutions derived in this chapter have to be used to fit the data. Comparison shows that the difference of the solar equator plane and the ecliptic one is not negligible.

11.2 On Orbital Disturbing Effects of the Solar Radiation

Solar radiation pressure acting on a planet's surface is traditionally considered as a sun's mass correction to the Keplerian motion. For a solar-orbit satellite, because the area to mass ratio varies due to the orientation uncertainty of the solar panel, the orbital disturbance should be considered using disturbing theory. An analytical solution of a solar-satellite orbit disturbed by the solar radiation pressure is derived. The mean anomaly M is subjected to a secular perturbation and there exist short periodic disturbances on all Keplerian elements. By comparing the results with that of the sun's mass correction, we found out that the result obtained by traditional sun's mass correction implicitly uses a focus, which is the common mass centre of the sun and the correcting mass and which rotates along with the satellite related to the solar mass centre, as a reference of the orbital ellipse. The focus is also not an inertial reference point.

11.2.1 Introduction

The effects of the solar radiation pressure on an Earth satellite orbit are discussed in numerous pieces of literature (e.g. Kudak et al. 2010; Levin 1968; Musen 1960; Parkinson et al. 1960; Vilhena De Moraes 1981; Saad et al. 2010; Touma et al. 2009; Vokrouhlicky and Milani 2000; Vokrouhlicky et al. 1993, 1994; Zizka and Vokrouhlicky 2011); short reviews on the research can be found, e.g. in Boulton (1983), Hughes (1977), Harwood and Swinerd (1995). The effects of the solar radiation pressure on a planet or a satellite in the solar gravitation field is traditionally dealt with so-called equivalent mass correction method (e.g. Kezerashvili and Vazquez-Poritz 2009; Hughes 1977). An analytical solution of the problem using disturbing theory is still missing and is desired for describing satellite motion in the solar gravity field because of the uncertainty of the area to mass ratio due to the solar panel miss-orientation. In this section, quite a similar (Xu et al. 2010a, b, 2011) but still independent way is developed to solve the dilemma of solar radiation disturbance on a planet (or solar satellite) motion. The solution is quite a simple one; however, a comparison of the result derived and the result of mass correction method shows confusion and imperfectness of the traditionally und commonly accepted result.

The solar radiation pressure model is discussed in Sect. 11.2.2. Gaussian equations of planet motion and the necessary mathematical formulas are given in Sect. 11.2.3. The solutions of the disturbed equations are derived and the comparisons are addressed in Sect. 11.2.4, followed by a concluding summary.

11.2.2 Disturbing Force of Solar Radiation Pressure

Solar radiation pressure, caused by the solar light, is a disturbing force (scaled by mass, or more exactly, acceleration) acting on a planet/satellite's surface which disturbs the orbit of the planet/satellite in the solar system and is represented as (Kezerashvili and Vazquez-Poritz 2009; Hughes 1977; Kubo-oka and Sengoku 1999; Montenbruck and Gill 2000; Seeber 2003)

$$\vec{f}_{\text{solar}} = P_s \frac{\eta}{2\pi c} \frac{A}{m} \frac{1}{r^2} \vec{n}, \quad \sigma = \frac{A}{m}, \quad \vec{n} = \frac{\vec{r}}{|\vec{r}|}, \tag{11.46}$$

where P_s is the solar luminosity ($= 3.842 \times 10^{26}$ W), η is the surface reflectivity ($0.5 \le \eta \le 1$, 0.5 corresponds to the total absorption of photons and 1 total reflection), σ is the area to mass ratio of the planet/satellite, $\vec{r}$ and r are the heliocentric vector and distance of the planet/satellite, $\vec{n}$ represents the heliocentric unit vector of the planet/satellite and c is the light velocity in vacuum. For any satellite design an equivalent σ can be computed, therefore model (11.46) is a general model.

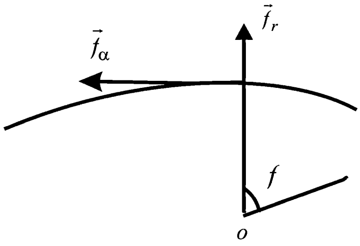

Fig. 11.2 Radial vector of the solar radiation pressure force (mass scaled) and its perpendicular components in the planet/satellite orbital plane

Using ξ (which could be slightly different in various pieces of literature because of different scales and/or units) to represent the coefficient part of the solar radiation pressure force vector (scaled by mass, or exactly, acceleration) (11.46), it yields

$$\vec{f}_{\text{solar}} = \xi\frac{1}{r^2}\vec{n} = \xi\frac{1}{r^3}\vec{r}, \quad \xi = P_s\frac{\eta}{2\pi c}\frac{A}{m}, \tag{11.47}$$

or

$$f_r = \xi\frac{1}{r^2} . \tag{11.48}$$

Here f_r is the radial acceleration component of vector $(f_r, f_\alpha, f_h)^T$ (see Fig. 11.2) and the other two components are zero (these lead to a significant simplification of the Gaussian equations of planet motion). Because of the same square-inverse ($1/r^2$) forms of the force (11.48) and solar central gravitational force, solar radiation disturbance is traditionally dealt with by a mass correction of the sun (see, e.g. Kezerashvili and Vazquez-Poritz 2009). However, for a satellite, ξ is an uncertain parameter caused by the uncertainty of satellite orientation (especially the solar panel) which could be precisely co-estimated by an orbit determination; therefore we will consider the problem using perturbation theory.

11.2.3 Gaussian Equations of Planet Motion

Denoting

$$h_1 = \frac{2e}{n\sqrt{1-e^2}}, \quad h_2 = \frac{\sqrt{1-e^2}}{na}, \quad h_3 = \frac{h_2}{e}, \quad h_6 = \frac{1-e^2}{nae}, \tag{11.49}$$

Gaussian equations of planet motion (see (4.26), or Montenbruck and Gill 2000; Battin 1999; Kaula 2001) turn out to be

$$\frac{\mathrm{d}a}{\mathrm{d}t} = h_1 \sin f f_r,\ \frac{\mathrm{d}e}{\mathrm{d}t} = h_2 \sin f f_r,\ \frac{\mathrm{d}\omega}{\mathrm{d}t} = -h_3 \cos f f_r,$$
$$\frac{\mathrm{d}i}{\mathrm{d}t} = 0,\ \frac{\mathrm{d}\Omega}{\mathrm{d}t} = 0,\ \frac{\mathrm{d}M}{\mathrm{d}t} = n - h_6\left(\frac{2e}{1+e\cos f} - \cos f\right)f_r. \tag{11.50}$$

Here Keplerian elements (a, e, ω, i, Ω, M, f) are the semi-major axis, the eccentricity of the ellipse, the argument of perihelion, the inclination angle, the right ascension of the ascending node, the mean anomaly and the true anomaly, respectively. The mean angular velocity n of a planet/satellite which represents the mean Keplerian motions is omitted in later derivations.

Using the mathematical expansion formula of (Bronstein and Semendjajew 1987; Wang et al. 1979)

$$(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + \ldots \quad |x| < 1, \tag{11.51}$$

yields

$$\frac{1}{1+e\ \cos f} = 1 - e\ \cos f + e^2\ \cos f - \ldots,$$
$$\frac{1}{r^2} = \frac{(1+e\ \cos f)^2}{a^2(1-e^2)^2} \approx (1+e\ \cos f)^2 a^{-2}(1+2e^2-\ldots). \tag{11.52}$$

Expansions (11.52) are valid only for circular and elliptical orbits. The following functions (see, e.g. (5.22), or Boccaletti and Pucacco 2001; Krivov et al. 1996; Pal 2009; Vallado 2007; Van Kamp 1967) are used to transform the trigonometric functions of the true anomaly f to the mean anomaly M:

$$\sin f = \left(1 - \frac{7}{8}e^2\right)\sin M + e\left(1 - \frac{7}{8}e^2\right)\sin 2M + \frac{9}{8}e^2 \sin 3M,$$
$$\cos f + e = \left(1 - \frac{9}{8}e^2\right)\cos M + e\left(1 - \frac{4}{3}e^2\right)\cos 2M + \frac{9}{8}e^2\cos 3M. \tag{11.53}$$

For keeping the results as simple as possible, the terms smaller than e^2 are neglected (truncated).

11.2.4 Solutions of the Disturbed Equations

Substituting (11.52) and (11.53) into (11.50), the equations of motion then have the forms of (truncated to the order of $h_j e^2$)

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{h_1\xi}{a^2}\left((1 - \frac{5}{8}e^2)\sin M + 2e\sin 2M + \frac{27}{8}e^2\sin 3M\right), \tag{11.54}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{h_2\xi}{a^2}\left((1-\frac{5}{8}e^2)\sin M + 2e\sin 2M + \frac{27}{8}e^2\sin 3M\right), \tag{11.55}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{h_3\xi}{a^2}\left((-1+\frac{3}{8}e^2)\cos M - 2e\cos 2M - \frac{27}{8}e^2\cos 3M\right), \tag{11.56}$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\Omega}{\mathrm{d}t} = 0, \tag{11.57}$$

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{h_6\xi}{a^2}\begin{pmatrix} (-2e-2e^3) + (1-\frac{19}{8}e^2)\cos M \\ +2e\cos 2M + \frac{27}{8}e^2\cos 3M \end{pmatrix}. \tag{11.58}$$

It is notable that all Keplerian elements are not disturbed by the solar radiation long-periodically (functions of ω and Ω). The inclinations angle i and the right ascension of ascending node Ω are not disturbed by solar radiation. The semi-major axis a, the eccentricity e and the argument of perigee ω are disturbed short-periodically (functions of M). Only the mean anomaly M is subjected to linear (secular) and short-periodic perturbations (annual, semi-annual …). In short-periodic terms of (11.56) and (11.58), the factor e in the coefficients h_3 and h_6 could lead to a circular singularity. According to singularity-free theory of Xu & Xu (2012, cf. Sect. 10.2) in such a case the equations (11.56) and (11.58) should be

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = 0,$$
$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{h_6\xi}{a^2}\left((-2e-2e^3) - 2e^2\cos M - 2e^3\cos 2M\right).$$

Using the relation $M = nt$ and the mean value theorem for integration (Bronstein and Semendjajew 1987; Wang et al. 1979), the solutions (i.e. indefinite integrals) of (11.54), (11.55), (11.56), (11.57), and (11.58) can be obtained by

$$\Delta a = \frac{-h_1\xi}{na^2}\left((1-\frac{5}{8}e^2)\cos M + e\cos 2M + \frac{9}{8}e^2\cos 3M\right), \tag{11.59}$$

$$\Delta e = \frac{-h_2\xi}{na^2}\left((1-\frac{5}{8}e^2)\cos M + e\cos 2M + \frac{9}{8}e^2\cos 3M\right), \tag{11.60}$$

$$\Delta\omega = \frac{h_3\xi}{na^2}\left((-1+\frac{3}{8}e^2)\sin M - e\sin 2M - \frac{9}{8}e^2\sin 3M\right), \tag{11.61}$$

$$\Delta i = 0, \quad \Delta\Omega = 0, \tag{11.62}$$

$$\Delta M = \frac{h_6 \xi}{na^2} \begin{pmatrix} n(-2e - 2e^3)t + (1 - \frac{19}{8}e^2)\sin M \\ +e \sin 2M + \frac{9}{8}e^2 \sin 3M \end{pmatrix}. \tag{11.63}$$

The magnitudes of the disturbances are depended on the coefficient h_j as defined in (11.49). The related singularity-free solutions of (11.56) and (11.58) should be

$$\begin{aligned} \triangle\omega &= 0, \\ \triangle M &= \frac{h_6 \xi}{na^2}\left(n(-2e - 2e^3)t - 2e^2 \sin M\right). \end{aligned}$$

Except for the linear change of the mean anomaly in (11.63) and the above singularity-free solution, all other disturbances are short periodic functions of M. Mean angular velocity n is changed (reduced) by solar radiation by an amount of (see (11.63) and (11.49), truncated to e^2)

$$\Delta n = \frac{-2\xi}{na^3} . \tag{11.64}$$

The solution derived above is both valid for planet/satellite motion in the solar gravity field disturbed by solar radiation. The factor 2 in (11.64) may be read directly from the last equation of (11.50). Equation 11.64 is an alternative description of the physical effect using disturbing theory.

Up to now, we derived the orbit solutions disturbed by the solar radiation for planet/satellite moving in the solar gravity filed.

11.2.5 A Modification of the Keplerian Third Law

Taking the Keplerian third law (see e.g. (3.32))

$$n = a^{-3/2}\mu^{1/2}, \tag{11.65}$$

into account, (11.64) and the ratio of (11.64) and (11.65) turn out to be

$$\Delta n = \frac{-2\xi}{\mu^{1/2}a^{3/2}}, \quad \frac{\Delta n}{n} = -\frac{2\xi}{\mu}, \tag{11.66}$$

so that the third Keplerian law of planet motion should have a term of solar radiation correction and should be

$$n_d = a^{-3/2}\mu^{1/2} - a^{-3/2}\mu^{-1/2}2\xi . \tag{11.67}$$

Here the index d denotes disturbed variable, ξ can generally be termed the solar radiation parameter and μ is the gravitational constant of the sun. The Keplerian third law states that the mean angular velocity of a planet/satellite motion in the solar gravity field is a function of the semi-major axis of the planet/satellite orbit. The modification derived here states that the mean angular velocity of a planet/satellite is additionally a function of the solar radiation parameter. Now (11.67) is termed the solar radiation corrected Keplerian third law. Solar radiation pressure leads to a slowing down of the mean motion of the planet/satellite. For a planet the modification is negligible; however, for a satellite located in the solar gravitational field the modification is significant. To be noted is that up to now the undisturbed Keplerian elements (without index d) are used to represent the results of disturbances.

11.2.6 Comparison with the Result of Traditional Mass Correction

Traditionally, solar radiation is equivalently taken into account by solar mass correction. The effective gravity constant is (see, e.g. Kezerashvili and Vazquez-Poritz 2009)

$$\mu_{eff} = \mu\left(1 - \frac{\xi}{\mu}\right) . \tag{11.68}$$

Then the Keplerian mean motion results (using (11.65)):

$$n_{dc} = a_{eff}^{-3/2}\mu^{1/2}\left(1 - \frac{\xi}{\mu}\right)^{1/2} \approx a_{eff}^{-3/2}\mu^{1/2}\left(1 - \frac{1}{2}\frac{\xi}{\mu}\right) . \tag{11.69}$$

The index c is used to distinguish the n_d derived from disturbing theory and mass correction method. The effective semi-major axis a_{eff} is related with the effective mass through the Keplerian third law (11.65). The relationship between a_{eff} and semi-major axis a will be derive through following discussion.

11.2.7 The Foci of the Orbital Ellipse

In a Sun-satellite/planet system, if the mass of the satellite/planet is negligible, the satellite/planet orbit is a Keplerian one and the mass centre of the Sun is located at the focus of the orbital ellipse (cf. Fig. 11.3: M, S are the Sun, satellite/planet. If S has no mass, then M and system mass centre N will coincide).

Fig. 11.3 Mass centre of a 2-body problem

Fig. 11.4 Mass centre of a 2-body problem with a negative mass S

In the case of the mass of the satellite/planet is not negligible, the satellite/planet orbit is further a Keplerian one; however, the focus is located at the mass centre of the system Sun-satellite/planet. That is, the focus of the ellipse is not any more the mass centre of the Sun, but the mass centre of the system (cf. Fig. 11.3: If S has a mass, the mass centre of M-S should be N located between M und S).

In disturbing theory (method we used), the mass of the satellite/planet is further considered negligible, i.e., the mass centre of the Sun is further located at the focus of the orbital ellipse; the solar radiation is considered a disturbing influence on the Keplerian motion.

In mass correction scenario, the mass of the satellite/planet is now considered a negative one (i.e. not negligible). The focus of the orbital ellipse of the satellite/planet is now located at the "mass centre" N of the Sun-(negative mass of the satellite/planet) system (cf. Fig. 11.4: If S has a negative mass, then the "mass centre" of M-S should be N which locates not between M and S. Note that N is not a true mass centre).

From above discussion, we could conclude that, disturbing method and mass correction method use two different foci as references. However, the both have a simple geometric relationship.

11.2.8 The Geometric Relationship Between the Two Foci

As shown in figure (cf. Figs. 11.5 and 11.6): M is the mass centre of the Sun. S is the satellite/planet. Because of the negative mass of S, the "mass centre" of the Sun-(negative mass satellite/planet) system is denoted by N.

Importances are:

- N is not the true mass centre of the sun + satellite/planet. Observing from M, the satellite S rotates around the M and the N does the same but with 180 ° latency. That is the N rotates with the motion of the satellite/planet around the true mass centre M. This can also explain why the periodic disturbing terms are disappeared in mass correction scenario; because a periodic moving focus is used as reference.

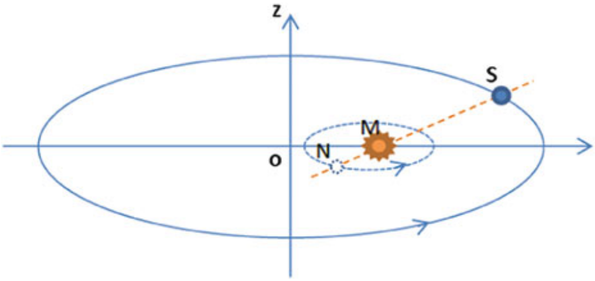

Fig. 11.5 Orbital ellipse with M as focus

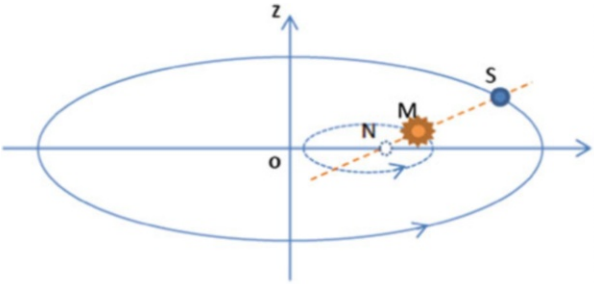

Fig. 11.6 Orbital ellipse with N as focus

- N is not suitable to be used as origin of an inertial frame; because N is not a true mass centre and any object at N will subject to gravitational acceleration.

  Geometric relations:
  To describe the orbit from M, one has (cf. Fig. 11.5: M is the focus):

$$r_{MS} = \frac{a(1-e^2)}{1+e\cos f} \approx a(1-e^2)(1-e\cos f). \tag{11.70}$$

  To describe the orbit from N, one has (cf. Fig. 11.6: N is the focus):

$$r_{NS} = \frac{a_{eff}(1-e^2)}{1+e\cos f} \approx a_{eff}(1-e^2)(1-e\cos f). \tag{11.71}$$

  Then one has (cf. Fig. 11.6 and formulas (11.70 and 11.71)):

$$\begin{aligned} r_{MS} &= r_{NS} - r_{NM} = r_{NS}(1 - r_{NM}/r_{NS}) \\ &= \left(1 - \frac{\xi}{\mu}\right)\frac{a_{eff}(1-e^2)}{1+e\cos f}, \end{aligned} \tag{11.72}$$

and (by comparing (11.70) and (11.72))

$$a = \left(1 - \frac{\xi}{\mu}\right)a_{eff} \quad \text{or} \quad a_{eff} = \left(1 + \frac{\xi}{\mu}\right)a. \tag{11.73}$$

In (11.72) the following mass centre relation is used:

$$r_{NM}/r_{NS} = \frac{\xi}{\mu}. \tag{11.74}$$

The effective parameters a_{eff} and μ_{eff} are related to (11.65). From (11.69) we have

$$\begin{aligned}
n_{dc} &= a_{eff}^{-3/2}\mu^{1/2}\left(1-\frac{\xi}{\mu}\right)^{1/2} \approx a_{eff}^{-3/2}\mu^{1/2}\left(1-\frac{1}{2}\frac{\xi}{\mu}\right) \\
&= \left(1+\frac{\xi}{\mu}\right)^{-3/2} a^{-3/2}\mu^{1/2}\left(1-\frac{1}{2}\frac{\xi}{\mu}\right) \\
&\approx a_{eff}^{-3/2}\mu^{1/2}\left(1-\frac{1}{2}\frac{\xi}{\mu}\right)\left(1-\frac{3}{2}\frac{\xi}{\mu}\right) \\
&= a^{-3/2}\mu^{1/2}\left(1-2\frac{\xi}{\mu}\right).
\end{aligned} \tag{11.75}$$

The right-hand side of (11.75) is identical with the right-hand side of (11.67). That is, the differences between the disturbing result (11.67) and the mass correction result (11.69) are caused through implicitly using the different foci as references to describe the ellipse.

11.2.9 Conclusions of Above Discussion and Derivation

Conclusions could be draw:

- In mass correction method, a moving focus which is not a true mass centre is used, and this leads to the different results of (11.67) and (11.69). The mean motion changing (11.67) derive from disturbing theory is a correct one.
- Accepting that the traditional mass correction method implicitly uses a rotating focus as a reference, then it is without wonder that the long term effect derived by disturbing theory is different from the traditional one, because the problem is seeing from different reference (focus).

One may summarise that, in disturbing theory, the focus of the ellipse remains the system's mass centre; whereas in the mass correction, the focus relocates a little bit and will rotate with the motion of the planet (satellite). A moving reference is used in the mass correction method without being noticed and a rotating focus around the true solar mass centre is not an inertial one. From these aspects, the result derived here using disturbing theory is preferred.

11.2.10 A Class of Non-Keplerian Orbits

To be noted is that, for a satellite in the solar gravity field, solar radiation correction to the Keplerian third law can be increased through increasing the area to mass ratio by satellite constructional design (Yonetoku et al. 2010). Denoting the mean angular velocity of a planet (Earth) and the semi-major axis of the planet (Earth) orbit as n_p and a_p, and setting the mean angular velocity of a satellite equal to n_p, it holds that

$$n = a^{-3/2}\mu^{1/2} - a^{-3/2}\mu^{-1/2}2\xi = n_p = a_p^{-3/2}\mu^{1/2}, \tag{11.76}$$

or

$$a^{-3/2}(\mu - 2\xi) = a_p^{-3/2}\mu \ . \tag{11.77}$$

For a given ξ, or area to mass ratio σ, a semi-major axis of the satellite can be found from (11.77) so that the satellite could have an orbit which has the same angular velocity as that of the planet (Earth). This is a kind of non-Keplerian orbit and many of these can be found in McInnes (1999). However, application of the disturbed theory is a kind of two-body approximation for multi-body problems.

A solar electric ion propulsion technique may also be used to produce a solar radiation-like force to assist the maintenance of the non-Keplerian orbit and to reduce the requirement of the area to mass ratio of the satellite as can be seen in McInnes (1999).

11.2.11 Summary

The analytical solutions of a planet/satellite orbit disturbed by solar radiation pressure are derived using disturbing theory. It is indicated that solar radiation pressure force (scaled by mass or acceleration) will lead to a linear disturbance in relation to the time on the mean anomaly M. Keplerian elements (i, Ω) are not disturbed by solar radiation, and (a, e, ω, M) are disturbed short periodically. There are no long periodic disturbances on all Keplerian elements. The linear term of disturbance of the solar radiation force (scaled by mass) on the mean motion of the planet leads to a modification of the Keplerian third law; mean angular velocity of a planet/satellite motion is a function of the semi-major axis a (Keplerian third law) minus a function (modification) of the solar radiation parameter (or area to mass ratio of the planet/satellite).

A comparison of results derived here with traditional ones is given, which shows that the traditional mass correction used implicitly a rotating focus related to the sun's mass centre as a reference to describe the solar radiation disturbed orbit and such confusion should be noted. From this aspect, using disturbing theory to deal with solar radiation disturbance is preferred.

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Appendix 1: IAU 1980 Theory of Nutation

Table A.1 Coefficients of the IAU 1980 theory of nutation (The units of A_i and B_i are 0.″0001, units of A_i' and B_i' are 0.″00001 cf. McCarthy 1996))

Coefficients of					Values of			
1	1′	F	D	Ω	A_i	A_i'	B_i	B_i'
0	0	0	0	1	-171,996	-1,742	92,025	89
0	0	2	-2	2	-13,187	16	5,736	-31
0	0	2	0	2	-2,274	-2	977	-5
0	0	0	0	2	2,062	2	-895	5
0	-1	0	0	0	-1,426	34	54	-1
1	0	0	0	0	712	1	-7	0
0	1	2	-2	2	-517	12	224	-6
0	0	2	0	1	-386	-4	200	0
1	0	2	0	2	-301	0	129	-1
0	-1	2	-2	2	217	-5	-95	3
-1	0	0	-2	0	158	0	-1	0
0	0	2	-2	1	129	1	-70	0
-1	0	2	0	2	123	0	-53	0
1	0	0	0	1	63	1	-33	0
0	0	0	2	0	63	0	-2	0
-1	0	2	2	2	-59	0	26	0
-1	0	0	0	1	-58	-1	32	0
1	0	2	0	1	-51	0	27	0
-2	0	0	2	0	-48	0	1	0
-2	0	2	0	1	46	0	-24	0
0	0	2	2	2	-38	0	16	0
2	0	2	0	2	-31	0	13	0
1	0	2	-2	2	29	0	-12	0
2	0	0	0	0	29	0	-1	0
0	0	2	0	0	26	0	-1	0
0	0	2	-2	0	-22	0	0	0
-1	0	2	0	1	21	0	-10	0
0	2	0	0	0	17	-1	0	0

(continued)

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Table A.1 (continued)

Coefficients of					Values of			
1	1′	F	D	Ω	A_i	A_i'	B_i	B_i'
-1	0	0	2	1	16	0	-8	0
0	2	2	-2	2	-16	1	7	0
0	1	0	0	1	-15	0	9	0
1	0	0	-2	1	-13	0	7	0
0	-1	0	0	1	-12	0	6	0
2	0	-2	0	0	11	0	0	0
-1	0	2	2	1	-10	0	5	0
1	0	2	2	2	-8	0	3	0
0	0	2	2	1	-7	0	3	0
0	-1	2	0	2	-7	0	3	0
0	1	2	0	2	7	0	-3	0
1	1	0	-2	0	-7	0	0	0
1	0	2	-2	1	6	0	-3	0
0	0	0	2	1	-6	0	3	0
2	0	2	-2	2	6	0	-3	0
1	0	0	2	0	6	0	0	0
-2	0	0	2	1	-6	0	3	0
2	0	2	0	1	-5	0	3	0
1	-1	0	0	0	5	0	0	0
0	0	0	-2	1	-5	0	3	0
0	-1	2	-2	1	-5	0	3	0
0	0	0	1	0	-4	0	0	0
1	0	-2	0	0	4	0	0	0
0	1	0	-2	0	-4	0	0	0
1	0	0	-1	0	-4	0	0	0
0	1	2	-2	1	4	0	-2	0
2	0	0	-2	1	4	0	-2	0
0	-1	2	2	2	-3	0	1	0
3	0	2	0	2	-3	0	1	0
-1	-1	2	2	2	-3	0	1	0
1	-1	2	0	2	-3	0	1	0
1	0	2	0	0	3	0	0	0
1	1	0	0	0	-3	0	0	0
1	-1	0	-1	0	-3	0	0	0
-2	0	2	0	2	-3	0	1	0
-1	0	2	4	2	-2	0	1	0
0	0	2	1	2	2	0	-1	0
3	0	0	0	0	2	0	0	0
1	0	0	0	2	-2	0	1	0
2	0	0	0	1	2	0	-1	0
-1	0	2	-2	1	-2	0	1	0
1	1	2	0	2	2	0	-1	0
-2	0	0	0	1	-2	0	1	0
0	-2	2	-2	1	-2	0	1	0
0	1	0	1	0	1	0	0	0

(continued)

Table A.1 (continued)

Coefficients of					Values of			
l	l′	F	D	Ω	A_i	A_i'	B_i	B_i'
0	0	2	4	2	−1	0	0	0
2	0	0	2	0	1	0	0	0
1	0	−2	2	0	−1	0	0	0
1	1	0	−2	1	−1	0	0	0
0	−1	2	0	1	−1	0	0	0
1	0	−2	−2	0	−1	0	0	0
0	1	0	2	0	−1	0	0	0
0	0	2	−1	2	−1	0	0	0
0	0	−2	0	1	−1	0	0	0
−1	−1	0	2	1	1	0	0	0
0	1	2	0	1	1	0	0	0
1	0	2	−2	0	−1	0	0	0
3	0	2	−2	2	1	0	0	0
0	0	4	−2	2	1	0	0	0
1	0	0	2	1	−1	0	0	0
2	0	2	2	2	−1	0	0	0
2	0	2	−2	1	1	0	−1	0
1	−1	0	−2	0	1	0	0	0
−1	0	4	0	2	1	0	0	0
−2	0	2	4	2	−1	0	1	0
1	0	2	2	1	−1	0	1	0
1	1	2	−2	2	1	0	−1	0
2	0	0	−4	0	−1	0	0	0
−2	0	2	2	2	1	0	−1	0
1	0	0	−4	0	−1	0	0	0
−1	0	0	0	2	1	0	−1	0
0	1	2	−2	0	−1	0	0	0
−1	0	0	1	1	1	0	0	0
0	1	0	0	2	1	0	0	0
0	1	−2	2	0	−1	0	0	0
0	0	−2	2	1	1	0	0	0
2	1	0	−2	0	1	0	0	0
2	0	−2	0	1	1	0	0	0

Appendix 2: Solutions of Geopotential $D_{43} \sim D_{88}$ Perturbations

Solutions of D_{43} Perturbation

$$
\begin{aligned}
\frac{\mathrm{d}a_1}{\mathrm{d}t} = \frac{105}{32} & si(co\,so^2(-36) + co^3(12) + ci^2(co^3(-12) + co\,so^2(36)) \\
& + e^2(co\,so^2(-360) + co^3(120) + ci(co^2so(-360cw\,sw) + so^3(120cw\,sw)) \\
& + ci^3(co^2so(-360cw\,sw) + so^3(120cw\,sw)) + ci^2(co^3(-240sw^2) \\
& + co\,so^2(720sw^2))) + e(co\,so^2(-264cw^4 - 504cw^2sw^2 - 240sw^4) \\
& + co^3(88cw^4 + 168cw^2sw^2 + 80sw^4) + ci^3(co^2so(-168cw^3sw - 168cw\,sw^3) \\
& + so^3(56cw^3sw + 56cw\,sw^3)) + ci(co^2so(-216cw^3sw - 216cw\,sw^3) \\
& + so^3(72cw^3sw + 72cw\,sw^3)) + ci^2(co^3(-24cw^4 - 168cw^2sw^2 - 144sw^4) \\
& + co\,so^2(72cw^4 + 504cw^2sw^2 + 432sw^4)))\cos M + (co^3(16cw^4 - 16sw^4) \\
& + co\,so^2(-48cw^4 + 48sw^4) + ci^3(so^3(-16cw^3sw - 16cw\,sw^3) \\
& + co^2so(48cw^3sw + 48cw\,sw^3)) + ci(co^2so(-144cw^3sw - 144cw\,sw^3) \\
& + so^3(48cw^3sw + 48cw\,sw^3)) \\
& + e^2(co\,so^2(-681cw^4 - 1242cw^2sw^2 - 585sw^4) \\
& + co^3(227cw^4 + 414cw^2sw^2 + 195sw^4) \\
& + ci^3(co^2so(-684cw^3sw - 660cw\,sw^3) + so^3(228cw^3sw + 220cw\,sw^3)) \\
& + ci(co^2so(-900cw^3sw - 828cw\,sw^3) + so^3(300cw^3sw + 276cw\,sw^3)) \\
& + ci^2(co^3(33cw^4 - 438cw^2sw^2 - 447sw^4) \\
& + co\,so^2(-99cw^4 + 1314cw^2sw^2 + 1341sw^4))))\cos 2M \\
& + e(co^3(96cw^4 + 72cw^2sw^2 - 120sw^4) \\
& + co\,so^2(-288cw^4 - 216cw^2sw^2 + 360sw^4) \\
& + ci^3(so^3(-216cw^3sw - 120cw\,sw^3) + co^2so(648cw^3sw + 360cw\,sw^3)) \\
& + ci(co^2so(-360cw^3sw - 1224cw\,sw^3) + so^3(120cw^3sw + 408cw\,sw^3))
\end{aligned}
$$

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$$
\begin{aligned}
&+ ci^2(co\ so^2(288\ cw^4 - 648cw^2sw^2 - 72sw^4) + co^3(-96cw^4 + 216cw^2sw^2\\
&+ 24sw^4)))\cos 3M + (co\ so^2(-12cw^4 + 72cw^2sw^2 - 12sw^4)\\
&+ co^3(4cw^4 - 24cw^2sw^2 + 4sw^4) + ci^3(so^3(16cw^3sw - 16cw\ sw^3)\\
&+ co^2so(-48cw^3sw + 48cw\ sw^3)) + ci(so^3(48cw^3sw - 48cw\ sw^3)\\
&+ co^2so(-144cw^3sw + 144cw\ sw^3)) + ci^2(co\ so^2(-36cw^4 + 216cw^2sw^2\\
&- 36sw^4) + co^3(12cw^4 - 72cw^2sw^2 + 12sw^4))\\
&+ e^2(co^3(320cw^4 + 624cw^2sw^2 - 528sw^4)\\
&+ co\ so^2(-960cw^4 - 1872cw^2sw^2 + 1584sw^4)\\
&+ci^3(so^3(-1200cw^3sw - 368cw\ sw^3) + co^2so(3600cw^3sw + 1104cw\ sw^3))\\
&+ ci(co^2so(1008cw^3sw - 6480cw\ sw^3) + so^3(-336cw^3sw + 2160cw\ sw^3))\\
&+ ci^2(co\ so^2(2016cw^4 - 5616cw^2sw^2 - 144sw^4)\\
&+ co^3(-672cw^4 + 1872cw^2sw^2 + 48sw^4))))\cos 4M\\
&+ e(co\ so^2(-120cw^4 + 720cw^2sw^2 - 120sw^4)\\
&+ co^3(40cw^4 - 240cw^2sw^2 + 40sw^4) + ci^3(so^3(160cw^3sw - 160cw\ sw^3)\\
&+ co^2so(-480cw^3sw + 480cw\ sw^3)) + ci(so^3(480cw^3sw - 480cw\ sw^3)\\
&+ co^2so(-1440cw^3sw + 1440cw\ sw^3))\\
&+ ci^2(co\ so^2(-360cw^4 + 2160cw^2sw^2 - 360sw^4)\\
&+ co^3(120cw^4 - 720cw^2sw^2 + 120sw^4)))\cos 5M\\
&+ e^2(co\ so^2(-639cw^4 + 3834cw^2sw^2 - 639sw^4)\\
&+co^3(213cw^4 - 1278cw^2sw^2 + 213sw^4) + ci^3(so^3(852cw^3sw - 852cw\ sw^3)\\
&+co^2so(-2556cw^3sw + 2556cw\ sw^3)) + ci(so^3(2556cw^3sw - 2556cw\ sw^3)\\
&+ co^2so(-7668cw^3sw + 7668cw\ sw^3))\\
&+ ci^2(co\ so^2(-1917cw^4 + 11502cw^2sw^2 - 1917sw^4)\\
&+ co^3(639cw^4 - 3834cw^2sw^2 + 639sw^4)))\cos 6M + e(co^3(-8cw\ sw)\\
&+ co\ so^2(24cw\ sw) + ci^2(co^3(-120cw\ sw) + co\ so^2(360cw\ sw))\\
&+ ci^3(co^2so(-264cw^4 - 360cw^2sw^2 - 96sw^4)\\
&+ so^3(88cw^4 + 120cw^2sw^2 + 32sw^4))\\
&+ ci(so^3(-24cw^4 - 120cw^2sw^2 - 96sw^4)\\
&+ co^2so(72cw^4 + 360cw^2sw^2 + 288sw^4)))\sin M + (co^3(-32cw\ sw)\\
&+ co\ so^2(96cw\ sw) + ci^3(co^2so(24cw^4 - 24sw^4) + so^3(-8cw^4 + 8sw^4))\\
&+ ci(so^3(24cw^4 - 24sw^4) + co^2so(-72cw^4 + 72sw^4))\\
&+ e^2(co^3(-36cw^3sw - 28cw\ sw^3) + co\ so^2(108cw^3sw + 84cw\ sw^3)
\end{aligned}
$$

$$
\begin{aligned}
&+ ci^2(co^3(-492cw^3sw - 468cw\,sw^3) + co\,so^2(1476cw^3sw + 1404cw\,sw^3))\\
&+ ci^3(co^2so(-1059cw^4 - 1422cw^2sw^2 - 387sw^4)\\
&+ so^3(353cw^4 + 474cw^2sw^2 + 129sw^4))\\
&+ ci(so^3(-93cw^4 - 498cw^2sw^2 - 381sw^4)\\
&+ co^2so(279cw^4 + 1494cw^2sw^2 + 1143sw^4))))\sin 2M\\
&+ e(co^3(-168cw^3sw - 264cw\,sw^3) + co\,so^2(504cw^3sw + 792cw\,sw^3)\\
&+ ci^2(co^3(264cw^3sw - 24cw\,sw^3) + co\,so^2(-792cw^3sw + 72cw\,sw^3))\\
&+ ci^3(co^2so(288cw^4 - 216cw^2sw^2 - 216sw^4)\\
&+ so^3(-96cw^4 + 72cw^2sw^2 + 72sw^4))\\
&+ ci(so^3(96cw^4 + 216cw^2sw^2 - 168sw^4)\\
&+ co^2so(-288cw^4 - 648cw^2sw^2 + 504sw^4)))\sin 3M\\
&+ (co\,so^2(48cw^3sw - 48cw\,sw^3) + co^3(-16cw^3sw + 16cw\,sw^3)\\
&+ ci^2(co\,so^2(144cw^3sw - 144cw\,sw^3) + co^3(-48cw^3sw + 48cw\,sw^3))\\
&+ ci^3(co^2so(-12cw^4 + 72cw^2sw^2 - 12sw^4) + so^3(4cw^4 - 24cw^2sw^2 + 4sw^4))\\
&+ ci(co^2so(-36cw^4 + 216cw^2sw^2 - 36sw^4)\\
&+ so^3(12cw^4 - 72cw^2sw^2 + 12sw^4)) + e^2(co^3(-432cw^3sw - 1264cw\,sw^3)\\
&+ co\,so^2(1296cw^3sw + 3792cw\,sw^3) + ci^2(co^3(1968cw^3sw - 528cw\,sw^3)\\
&+ co\,so^2(-5904cw^3sw + 1584cw\,sw^3))\\
&+ ci^3(co^2so(1488cw^4 - 1872cw^2sw^2 - 864sw^4)\\
&+ so^3(-496cw^4 + 624cw^2sw^2 + 288sw^4))\\
&+ ci(so^3(144cw^4 + 1872cw^2sw^2 - 768sw^4)\\
&+ co^2so(-432cw^4 - 5616cw^2sw^2 + 2304sw^4))))\sin 4M\\
&+ e(co\,so^2(480cw^3sw - 480cw\,sw^3) + co^3(-160cw^3sw + 160cw\,sw^3)\\
&+ ci^2(co\,so^2(1440cw^3sw - 1440cw\,sw^3) + co^3(-480cw^3sw + 480cw\,sw^3))\\
&+ ci^3(co^2so(-120cw^4 + 720cw^2sw^2 - 120sw^4)\\
&+ so^3(40cw^4 - 240cw^2sw^2 + 40sw^4))\\
&+ ci(co^2so(-360cw^4 + 2160cw^2sw^2 - 360sw^4)\\
&+ so^3(120cw^4 - 720cw^2sw^2 + 120sw^4)))\sin 5M)\\
&+ e^2(co\,so^2(2556cw^3sw - 2556cw\,sw^3) + co^3(-852\,cw^3sw + 852cw\,sw^3)\\
&+ ci^2(co\,so^2(7668cw^3sw - 7668cw\,sw^3) + co^3(-2556cw^3sw + 2556cw\,sw^3))\\
&+ ci^3(co^2so(-639cw^4 + 3834cw^2sw^2 - 639sw^4)\\
&+ so^3(213cw^4 - 1278cw^2sw^2 + 213sw^4))\\
&+ ci(co^2so(-1917cw^4 + 11502cw^2sw^2 - 1917sw^4)\\
&+ so^3(639cw^4 - 3834cw^2sw^2 + 639sw^4)))\sin 6M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}e_1}{\mathrm{d}t} = & -\frac{105}{8} si(ci^2(co^3(24cw^2 - 24cw^4 + (24 - 48cw^2)sw^2 - 24sw^4) \\
& + co\, so^2(-72cw^2 + 72cw^4 + (-72 + 144cw^2)sw^2 + 72sw^4)) \\
& + e^2(co\, so^2(9cw^4 - 9sw^4) + co^3(-3cw^4 + 3sw^4) \\
& + ci^3(co^2 so(-36cw^3 sw - 36cw\, sw^3) + so^3(12cw^3 sw + 12cw\, sw^3)) \\
& + ci(co^2 so((54cw - 54cw^3)sw - 54cw\, sw^3) \\
& + so^3((-18cw + 18cw^3)sw + 18cw\, sw^3)) \\
& + ci^2(co^3(102cw^2 - 93cw^4 + (138 - 240cw^2)sw^2 - 147sw^4) \\
& + co\, so^2(-306cw^2 + 279cw^4 + (-414 + 720cw^2)sw^2 + 441sw^4))) \\
& + e(co\, so^2(6cw^4 - 6sw^4) + co^3(-2cw^4 + 2sw^4) \\
& + ci^3(co^2 so(-24cw^3 sw - 24cw\, sw^3) + so^3(8cw^3 sw + 8cw\, sw^3)) \\
& + ci(co^2 so((36cw - 36cw^3)sw - 36cw\, sw^3) \\
& + so^3((-12cw + 12cw^3)sw + 12cw\, sw^3)) \\
& + ci^2(co^3(108cw^2 - 102cw^4 + (132 - 240cw^2)sw^2 - 138sw^4) \\
& + co\, so^2(-324cw^2 + 306cw^4 + (-396 + 720cw^2)sw^2 + 414sw^4))) \cos M \\
& + (co\, so^2(12cw^4 - 12sw^4) + co^3(-4cw^4 + 4sw^4) \\
& + ci^3(co^2 so(-48cw^3 sw - 48cw\, sw^3) + so^3(16cw^3 sw + 16cw\, sw^3)) \\
& + ci(co^2 so((72cw - 72cw^3)sw - 72cw\, sw^3) \\
& + so^3((-24cw + 24cw^3)sw + 24cw\, sw^3)) \\
& + ci^2(co^3(-24cw^2 + 36cw^4 + 24sw^2 - 36sw^4) \\
& + co\, so^2(72cw^2 - 108cw^4 - 72sw^2 + 108sw^4)) \\
& + e^2(co\, so^2(18cw^4 - 36cw^2 sw^2 - 6sw^4) + co^3(-6cw^4 + 12cw^2 sw^2 + 2sw^4) \\
& + ci^3(co^2 so(-24cw^3 sw - 72cw\, sw^3) + so^3(8cw^3 sw + 24cw\, sw^3)) \\
& + ci(co^2 so(72cw\, sw - 144cw\, sw^3) + so^3(-24cw\, sw + 48cw\, sw^3)) \\
& + ci^2(co^3(216cw^2 - 210cw^4 + (264 - 444cw^2)sw^2 - 282sw^4) \\
& + co\, so^2(-648\, cw^2 + 630cw^4 + (-792 + 1332cw^2)sw^2 + 846sw^4)))) \cos 2M \\
& + e(co\, so^2(36cw^4 + 108cw^2 sw^2 - 72sw^4) + co^3(-12cw^4 - 36cw^2 sw^2 \\
& + 24sw^4) + ci(co^2 so((324cw - 540cw^3)sw - 108cw\, sw^3) \\
& + so^3((-108cw + 180cw^3)sw + 36cw\, sw^3)) \\
& + ci^3(co^2 so(-288cw^3 sw - 144cw\, sw^3) + so^3(96cw^3 sw + 48cw\, sw^3)) \\
& + ci^2(co^3(-108cw^2 + 180cw^4 + (108 - 108cw^2)sw^2 - 144sw^4) \\
& + co\, so^2(324cw^2 - 540cw^4 + (-324 + 324cw^2)sw^2 + 432sw^4))) \cos 3M \\
& + (co^3(-4cw^4 + 24cw^2 sw^2 - 4sw^4) + co\, so^2(12cw^4 - 72cw^2 sw^2 + 12sw^4) \\
& + ci^3(co^2 so(48cw^3 sw - 48cw\, sw^3) + so^3(-16cw^3 sw + 16cw\, sw^3))
\end{aligned}
$$

$$
\begin{aligned}
&+ ci(co^2so(144cw^3sw - 144cw\ sw^3) + so^3(-48cw^3sw + 48cw\ sw^3)) \\
&+ ci^2(co^3(-12cw^4 + 72cw^2sw^2 - 12sw^4) + co\ so^2(36cw^4 - 216cw^2sw^2 \\
&+ 36sw^4)) + e^2(co\ so^2(27cw^4 + 792cw^2sw^2 - 291sw^4) + co^3(-9cw^4 \\
&- 264cw^2sw^2 + 97sw^4) + ci^3(co^2so(-1164cw^3sw - 108cw\ sw^3) \\
&+ so^3(388cw^3sw + 36cw\ sw^3)) + ci(so^3((-318cw + 846cw^3)sw \\
&- 210cw\ sw^3) + co^2so((954cw - 2538cw^3)sw + 630cw\ sw^3)) \\
&+ ci^2(co^3(-318cw^2 + 609cw^4 + (318 - 792cw^2)sw^2 - 345sw^4) \\
&+ co\ so^2(954cw^2 - 1827cw^4 + (-954 + 2376cw^2)sw^2 + 1035sw^4)))) \\
&\times \cos 4M + e(co^3(-26cw^4 + 156cw^2sw^2 - 26sw^4) + co\ so^2(78cw^4 - 468cw^2sw^2 \\
&+ 78sw^4) + ci^3(co^2so(312cw^3sw - 312cw\ sw^3) + so^3(-104cw^3sw \\
&+ 104cw\ sw^3)) + ci(co^2so(936cw^3sw - 936cw\ sw^3) + so^3(-312cw^3sw \\
&+ 312cw\ sw^3)) + ci^2(co^3(-78cw^4 + 468cw^2sw^2 - 78sw^4) \\
&+ co\ so^2(234cw^4 - 1404cw^2sw^2 + 234sw^4)))\cos 5M + e^2(co^3(-102cw^4 \\
&+ 612cw^2sw^2 - 102sw^4) + co\ so^2(306cw^4 - 1836cw^2sw^2 + 306sw^4) \\
&+ ci^3(co^2so(1224cw^3sw - 1224cw\ sw^3) + so^3(-408cw^3sw + 408cw\ sw^3)) \\
&+ ci(co^2so(3672cw^3sw - 3672cw\ sw^3) + so^3(-1224cw^3sw \\
&+ 1224cw\ sw^3)) + ci^2(co^3(-306cw^4 + 1836cw^2sw^2 - 306sw^4) \\
&+ co\ so^2(918cw^4 - 5508cw^2sw^2 + 918sw^4)))\cos 6M + e(co\ so^2(-12cw\ sw) \\
&+ co^3(4cw\ sw) + ci^2(co^3((24cw - 36cw^3)sw - 36cw\ sw^3) \\
&+ co\ so^2((-72cw + 108cw^3)sw + 108cw\ sw^3)) + ci^3(so^3(4cw^4 - 4sw^4) \\
&+ co^2so(-12cw^4 + 12sw^4)) + ci(so^3(-6cw^2 + 6cw^4 + 6sw^2 - 6sw^4) \\
&+ co^2so(18cw^2 - 18cw^4 - 18sw^2 + 18sw^4)))\sin M + (co\ so^2(-24cw\ sw) \\
&+ co^3(8cw\ sw) + ci^2(co^3((48cw - 72cw^3)sw - 72cw\ sw^3) \\
&+ co\ so^2((-144cw + 216cw^3)sw + 216cw\ sw^3)) + ci^3(so^3(8cw^4 - 8sw^4) \\
&+ co^2so(-24cw^4 + 24sw^4)) + ci(so^3(-12cw^2 + 12cw^4 + 12sw^2 - 12sw^4) \\
&+ co^2so(36cw^2 - 36cw^4 - 36sw^2 + 36sw^4)) + e^2(16co^3cw^3sw \\
&- 48co\ cw^3so^2sw + ci^2(co^3((48cw - 48cw^3)sw - 96cw\ sw^3) \\
&+ co\ so^2((-144cw + 144cw^3)sw + 288cw\ sw^3)) + ci^3(so^3(6cw^4 \\
&+ 12cw^2sw^2 - 10sw^4) + co^2so(-18cw^4 - 36cw^2sw^2 + 30sw^4)) \\
&+ ci(so^3(-12cw^2 + 6cw^4 + (12 + 36cw^2)sw^2 - 18sw^4) + co^2so(36cw^2 \\
&- 18cw^4 + (-36 - 108cw^2)sw^2 + 54sw^4))))\sin 2M + e(co\ so^2(-36cw^3sw \\
&- 180cw\ sw^3) + co^3(12cw^3sw + 60cw\ sw^3) + ci^2(co^3((216cw - 396cw^3)sw \\
&- 252cw\ sw^3) + co\ so^2((-648cw + 1188cw^3)sw + 756cw\ sw^3))
\end{aligned}
$$

$$+ ci^3(so^3(42cw^4 - 36cw^2sw^2 - 30sw^4) + co^2so(-126cw^4 + 108cw^2sw^2$$
$$+ 90sw^4)) + ci(so^3(-54cw^2 + 72cw^4 + (54 - 108cw^2)sw^2 - 36sw^4)$$
$$+ co^2so(162cw^2 - 216cw^4 + (-162 + 324cw^2)sw^2 + 108sw^4)))\sin 3M$$
$$+ (co^3(16cw^3sw - 16cw\ sw^3) + co\ so^2(-48cw^3sw + 48cw\ sw^3)$$
$$+ ci^2(co^3(48cw^3sw - 48cw\ sw^3) + co\ so^2(-144cw^3sw + 144cw\ sw^3))$$
$$+ ci^3(so^3(-4cw^4 + 24cw^2sw^2 - 4sw^4) + co^2so(12cw^4 - 72cw^2sw^2$$
$$+ 12sw^4)) + ci(so^3(-12cw^4 + 72cw^2sw^2 - 12sw^4) + co^2so(36cw^4$$
$$- 216cw^2sw^2 + 36sw^4)) + e^2(co\ so^2(210cw^3sw - 846cw\ sw^3)$$
$$+ co^3(-70cw^3sw + 282cw\ sw^3) + ci^2(co^3((636cw - 1482cw^3)sw$$
$$- 426cw\ sw^3) + co\ so^2((-1908cw + 4446cw^3)sw + 1278cw\ sw^3))$$
$$+ ci(so^3(-159cw^2 + 291cw^4 + (159 - 792cw^2)sw^2 - 27sw^4)$$
$$+ co^2so(477cw^2 - 873cw^4 + (-477 + 2376cw^2)sw^2 + 81sw^4))$$
$$+ ci^3(so^3(150cw^4 - 264cw^2sw^2 - 62sw^4) + co^2so(-450cw^4 + 792cw^2sw^2$$
$$+ 186sw^4))))\sin 4M + e(co^3(104cw^3sw - 104cw\ sw^3) + co\ so^2(-312cw^3sw$$
$$+ 312cw\ sw^3) + ci^2(co^3(312cw^3sw - 312cw\ sw^3) + \ so^2(-936cw^3sw$$
$$+ 936cw\ sw^3)) + ci^3(so^3(-26cw^4 + 156cw^2sw^2 - 26sw^4) + co^2so(78cw^4$$
$$- 468cw^2sw^2 + 78sw^4)) + ci(so^3(-78cw^4 + 468cw^2sw^2 - 78sw^4)$$
$$+ co^2so(234cw^4 - 1404cw^2sw^2 + 234sw^4)))\sin 5M + e^2(co^3(408cw^3sw$$
$$- 408cw\ sw^3) + co\ so^2(-1224cw^3sw + 1224cw\ sw^3) + ci^2(co^3(1224cw^3sw$$
$$- 1224cw\ sw^3) + co\ so^2(-3672cw^3sw + 3672cw\ sw^3)) + ci^3(so^3(-102cw^4$$
$$+ 612cw^2sw^2 - 102sw^4) + co^2so(306cw^4 - 1836cw^2sw^2 + 306sw^4))$$
$$+ ci(so^3(-306cw^4 + 1836cw^2sw^2 - 306sw^4) + co^2so(918cw^4$$
$$- 5508cw^2sw^2 + 918sw^4)))\sin 6M)$$

$$\frac{d\omega_1}{dt} = \frac{105}{128} si(e(co\ so^2(-288cw\ sw) + co^3(96cw\ sw) + ci^2(co^3(-960cw\ sw)$$
$$+ co\ so^2(2880cw\ sw)) + ci(co^2so(-1944cw^4 - 2880cw^2sw^2 - 936sw^4)$$
$$+ so^3(648cw^4 + 960cw^2sw^2 + 312sw^4)) + ci^3(so^3(-216cw^4 - 960cw^2sw^2$$
$$- 744sw^4) + co^2so(648cw^4 + 2880cw^2sw^2 + 2232sw^4)))$$
$$+ (co\ so^2(-96cw^3sw - 96cw\ sw^3) + co^3(32cw^3sw + 32cw\ sw^3)$$
$$+ ci^2(co^3(-480cw^3sw - 480cw\ sw^3) + co\ so^2(1440cw^3sw$$
$$+ 1440cw\ sw^3)) + ci(co^2so(-1008cw^4 - 1440cw^2sw^2 - 432sw^4)$$
$$+ so^3(336cw^4 + 480cw^2sw^2 + 144sw^4)) + ci^3(so^3(-112cw^4 - 480cw^2sw^2$$

$$
\begin{aligned}
&- 368sw^4) + co^2so(336cw^4 + 1440cw^2sw^2 + 1104sw^4)) \\
&+ e^2(co\ so^2(-1908cw^3sw - 1644cw\ sw^3) + co^3(636cw^3sw + 548cw\ sw^3) \\
&+ ci^2(co^3(-4908cw^3sw - 5172\ cw\ sw^3) + co\ so^2(14724\ cw^3sw \\
&+ 15516cw\ sw^3)) + ci(co^2so(-9639cw^4 - 15174\ cw^2sw^2 - 4743sw^4) \\
&+ so^3(3213cw^4 + 5058\ cw^2sw^2 + 1581sw^4)) + ci^3(so^3(-1033cw^4 \\
&- 4794cw^2sw^2 - 3849sw^4) + co^2so(3099cw^4 + 14382cw^2sw^2 \\
&+ 11547sw^4)))) \cos M + e(co\ so^2(-432cw^3sw - 336cw\ sw^3) \\
&+ co^3(144cw^3sw + 112cw\ sw^3) + ci^2(co^3(-1872cw^3sw - 1968cw\ sw^3) \\
&+ co\ so^2(5616cw^3sw + 5904cw\ sw^3)) + ci(co^2so(-3996cw^4 \\
&- 5976cw^2sw^2 - 1692sw^4) + so^3(1332cw^4 + 1992cw^2sw^2 + 564sw^4)) \\
&+ ci^3(so^3(-452cw^4 - 1896cw^2sw^2 - 1476sw^4) + co^2so(1356cw^4 \\
&+ 5688cw^2sw^2 + 4428sw^4))) \cos 2M + (co\ so^2(-1152cw^3sw - 576cw\ sw^3) \\
&+ co^3(384cw^3sw + 192cw\ sw^3) + ci^2(co^3(-192cw^3sw - 768cw\ sw^3) \\
&+ co\ so^2(576cw^3sw + 2304cw\ sw^3)) + ci(co^2so(504cw^4 - 1296cw^2sw^2 \\
&- 72sw^4) + so^3(-168cw^4 + 432cw^2sw^2 + 24sw^4)) + ci^3(so^3(168cw^4 \\
&+ 144cw^2sw^2 - 216sw^4) + co^2so(-504cw^4 - 432cw^2sw^2 + 648sw^4)) \\
&+ e^2(co^3(144cw^3sw - 216cw\ sw^3) + co\ so^2(-432cw^3sw + 648cw\ sw^3) \\
&+ ci^2(co^3(-3960cw^3sw - 5040cw\ sw^3) + co\ so^2(11880cw^3sw \\
&+ 15120cw\ sw^3)) + ci(co^2so(-10881cw^4 - 18090cw^2sw^2 - 3969sw^4) \\
&+ so^3(3627cw^4 + 6030cw^2sw^2 + 1323sw^4)) + ci^3(so^3(-1539cw^4 \\
&- 4950cw^2sw^2 - 3771sw^4) + co^2so(4617cw^4 + 14850cw^2sw^2 \\
&+ 11313sw^4)))) \cos 3M + e(co\ so^2(-5472cw^3sw - 4704cw\ sw^3) \\
&+ co^3(1824cw^3sw + 1568cw\ sw^3) + ci^2(co^3(-2496cw^3sw - 3264cw\ sw^3) \\
&+ co\ so^2(7488cw^3sw + 9792cw\ sw^3)) + ci(co^2so(1944cw^4 \\
&- 1728cw^2sw^2 - 1368sw^4) + so^3(-648cw^4 + 576cw^2sw^2 + 456sw^4)) \\
&+ ci^3(so^3(1112cw^4 + 192cw^2sw^2 - 1176sw^4) + co^2so(-3336cw^4 \\
&- 576cw^2sw^2 + 3528sw^4))) \cos 4M + (co^3(224cw^3sw - 224cw\ sw^3) \\
&+ co\ so^2(-672cw^3sw + 672cw\ sw^3) + ci^2(co^3(672cw^3sw - 672cw\ sw^3) \\
&+ co\ so^2(-2016cw^3sw + 2016cw\ sw^3)) + ci^3(so^3(-56cw^4 + 336cw^2sw^2 \\
&- 56sw^4) + co^2so(168cw^4 - 1008cw^2sw^2 + 168sw^4)) + ci(so^3(-168cw^4 \\
&+ 1008cw^2sw^2 - 168sw^4) + co^2so(504cw^4 - 3024cw^2sw^2 + 504sw^4)) \\
&+ e^2(co\ so^2(-13272cw^3sw - 23712cw\ sw^3) + co^3(4424cw^3sw \\
&+ 7904cw\ sw^3) + ci^2(co^3(-15840cw^3sw - 5400cw\ sw^3)
\end{aligned}
$$

$$+ co\ so^2(47520cw^3sw + 16200cw\ sw^3)) + ci(co^2so(1989cw^4$$
$$+ 23490cw^2sw^2 - 9819sw^4) + so^3(-663cw^4 - 7830cw^2sw^2 + 3273sw^4))$$
$$+ ci^3(so^3(4631cw^4 - 2610cw^2sw^2 - 3761sw^4) + co^2so(-13893cw^4$$
$$+ 7830cw^2sw^2 + 11283sw^4)))) \cos 5M + e(co^3(1776cw^3sw - 1776cw\ sw^3)$$
$$+ co\ so^2(-5328cw^3sw + 5328cw\ sw^3) + ci^2(co^3(5328cw^3sw$$
$$- 5328cw\ sw^3) + co\ so^2(-15984cw^3sw + 15984cw\ sw^3))$$
$$+ ci^3(so^3(-444cw^4 + 2664cw^2sw^2 - 444sw^4) + co^2so(1332cw^4$$
$$- 7992cw^2sw^2 + 1332sw^4)) + ci(so^3(-1332cw^4 + 7992cw^2sw^2 - 1332sw^4)$$
$$+ co^2so(3996cw^4 - 23976cw^2sw^2 + 3996\ sw^4))) \cos 6M$$
$$+ e^2(co^3(8236cw^3sw - 8236cw\ sw^3) + co\ so^2(-24708cw^3sw$$
$$+ 24708cw\ sw^3) + ci^2(co^3(24708cw^3sw - 24708cw\ sw^3)$$
$$+ co\ so^2(-74124cw^3sw + 74124cw\ sw^3)) + ci^3(so^3(-2059cw^4$$
$$+ 12354cw^2sw^2 - 2059sw^4) + co^2so(6177cw^4 - 37062cw^2sw^2$$
$$+ 6177sw^4)) + ci(so^3(-6177cw^4 + 37062cw^2sw^2 - 6177sw^4)$$
$$+ co^2so(18531cw^4 - 111186cw^2sw^2 + 18531sw^4))) \cos 7M$$
$$+ (co\ so^2(-336cw^4 - 576cw^2sw^2 - 240sw^4) + co^3(112cw^4 + 192cw^2sw^2$$
$$+ 80sw^4) + ci(so^3(-192cw^3sw - 192cw\ sw^3) + co^2so(576cw^3sw$$
$$+ 576cw\ sw^3)) + ci^3(so^3(-256cw\ sw) + co^2so(768cw\ sw))$$
$$+ ci^2(co\ so^2(1008cw^4 + 576cw^2sw^2 - 432sw^4) + co^3(-336cw^4$$
$$- 192cw^2sw^2 + 144sw^4)) + e^2(co\ so^2(-1125cw^4 - 1386cw^2sw^2 - 525sw^4)$$
$$+ co^3(375cw^4 + 462cw^2sw^2 + 175sw^4) + ci(so^3(-348cw^3sw$$
$$- 612cw\ sw^3) + co^2so(1044cw^3sw + 1836cw\ sw^3)) + ci^3(so^3(-836cw^3sw$$
$$- 924cw\ sw^3) + co^2so(2508cw^3sw + 2772cw\ sw^3)) + ci^2(co\ so^2(3033cw^4$$
$$+ 2178cw^2sw^2 - 1647sw^4) + co^3(-1011cw^4 - 726cw^2sw^2$$
$$+ 549sw^4))))\sin M + e(co\ so^2(-1284cw^4 - 2088cw^2sw^2 - 900sw^4)$$
$$+ co^3(428cw^4 + 696cw^2sw^2 + 300sw^4) + ci(so^3(-720cw^3sw - 816cw\ sw^3)$$
$$+ co^2so(2160cw^3sw + 2448cw\ sw^3)) + ci^3(so^3(-1008cw^3sw$$
$$- 1040cw\ sw^3) + co^2so(3024cw^3sw + 3120cw\ sw^3)) + ci^2(co\ so^2(3924cw^4$$
$$+ 2376cw^2sw^2 - 1836sw^4) + co^3(-1308cw^4 - 792cw^2sw^2$$
$$+ 612sw^4)))\sin 2M + (co^3(168cw^4 - 144cw^2sw^2 - 120sw^4)$$
$$+ co\ so^2(-504cw^4 + 432cw^2sw^2 + 360sw^4) + ci(so^3(480cw^3sw$$
$$- 96cw\ sw^3) + co^2so(-1440cw^3sw + 288cw\ sw^3))$$

$$
\begin{aligned}
&+ ci^3(so^3(-288cw^3sw - 480cw\ sw^3) + co^2so(864cw^3sw + 1440cw\ sw^3))\\
&+ ci^2(co\ so^2(504cw^4 + 1296cw^2sw^2 - 936sw^4) + co^3(-168cw^4\\
&- 432cw^2sw^2 + 312sw^4)) + e^2(co\ so^2(-2889cw^4 - 4806cw^2sw^2\\
&- 2997sw^4) + co^3(963cw^4 + 1602cw^2sw^2 + 999sw^4) + ci^3(so^3(-2052cw^3sw\\
&- 2412cw\ sw^3) + co^2so(6156cw^3sw + 7236cw\ sw^3)) + ci(so^3(-1764cw^3sw\\
&- 2844cw\ sw^3) + co^2so(5292cw^3sw + 8532cw\ sw^3)) + ci^2(co\ so^2(9153cw^4\\
&+ 8046cw^2sw^2 - 4347sw^4) + co^3(-3051cw^4 - 2682cw^2sw^2 + 1449sw^4))))\\
&\times \sin 3M + e(co^3(880cw^4 - 192cw^2sw^2 - 816sw^4) + co\ so^2(-2640cw^4\\
&+ 576cw^2sw^2 + 2448sw^4) + ci(co^2so(-4464cw^3sw - 2160cw\ sw^3)\\
&+ so^3(1488cw^3sw + 720cw\ sw^3)) + ci^3(so^3(-2160cw^3sw - 2416cw\ sw^3)\\
&+ co^2so(6480cw^3sw + 7248cw\ sw^3)) + ci^2(co\ so^2(4032cw^4 + 1728cw^2sw^2\\
&- 4608sw^4) + co^3(-1344cw^4 - 576cw^2sw^2 + 1536sw^4)))\sin 4M\\
&+ (co\ so^2(-168cw^4 + 1008cw^2sw^2 - 168sw^4) + co^3(56cw^4 - 336cw^2sw^2\\
&+ 56sw^4) + ci^3(so^3(224cw^3sw - 224cw\ sw^3) + co^2so(-672cw^3sw\\
&+ 672cw\ sw^3)) + ci(so^3(672cw^3sw - 672cw\ sw^3) + co^2so(-2016cw^3sw\\
&+ 2016cw\ sw^3)) + ci^2(co\ so^2(-504cw^4 + 3024cw^2sw^2 - 504sw^4)\\
&+ co^3(168cw^4 - 1008cw^2sw^2 + 168sw^4)) + e^2(co^3(2647cw^4 + 2610cw^2sw^2\\
&- 3517sw^4) + co\ so^2(-7941cw^4 - 7830cw^2sw^2 + 10551sw^4)\\
&+ ci(co^2so(3852cw^3sw - 27468cw\ sw^3) + so^3(-1284cw^3sw\\
&+ 9156cw\ sw^3)) + ci^3(so^3(-10132cw^3sw - 6652cw\ sw^3)\\
&+ co^2so(30396cw^3sw + 19956cw\ sw^3)) + ci^2(co\ so^2(19845cw^4\\
&- 23490cw^2sw^2 - 12015sw^4) + co^3(-6615cw^4 + 7830cw^2sw^2\\
&+ 4005sw^4))))\sin 5M + e(co\ so^2(-1332cw^4 + 7992cw^2sw^2 - 1332sw^4)\\
&+ co^3(444cw^4 - 2664cw^2sw^2 + 444sw^4) + ci^3(so^3(1776cw^3sw\\
&- 1776cw\ sw^3) + co^2so(-5328cw^3sw + 5328cw\ sw^3)) + ci(so^3(5328cw^3sw\\
&- 5328cw\ sw^3) + co^2so(-15984cw^3sw + 15984cw\ sw^3))\\
&+ ci^2(co\ so^2(-3996cw^4 + 23976cw^2sw^2 - 3996sw^4) + co^3(1332cw^4\\
&- 7992cw^2sw^2 + 1332sw^4)))\sin 6M + e^2(co\ so^2(-6177cw^4 + 37062cw^2sw^2\\
&- 6177sw^4) + co^3(2059cw^4 - 12354cw^2sw^2 + 2059sw^4) + ci^3(so^3(8236cw^3sw\\
&- 8236cw\ sw^3) + co^2so(-24708cw^3sw + 24708cw\ sw^3))\\
&+ ci(so^3(24708cw^3sw - 24708cw\ sw^3) + co^2so(-74124cw^3sw\\
&+ 74124cw\ sw^3)) + ci^2(co\ so^2(-18531cw^4 + 111186cw^2sw^2 - 18531sw^4)\\
&+ co^3(6177cw^4 - 37062\ cw^2sw^2 + 6177sw^4)))\sin 7M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}i_1}{\mathrm{d}t} = & \frac{315}{16} si(ci^3(co\ so^2(-18) + co^3(6)) + ci(co^3(-6) + co\ so^2(18)) \\
& + e^2(co^2so(-18cw\ sw) + so^3(6cw\ sw) + ci^2(so^3(-18cw\ sw) \\
& + co^2so(54cw\ sw)) + ci^3(co\ so^2(-72cw^4 - 180cw^2sw^2 - 108sw^4) \\
& + co^3(24cw^4 + 60cw^2sw^2 + 36sw^4)) + ci(co^3(-30cw^4 - 60cw^2sw^2 \\
& - 30sw^4) + co\ so^2(90cw^4 + 180cw^2sw^2 + 90sw^4))) + e(co^2so(-12cw^3sw \\
& - 12cw\ sw^3) + so^3(4cw^3sw + 4cw\ sw^3) + ci^2(so^3(-12cw^3sw - 12cw\ sw^3) \\
& + co^2so(36cw^3sw + 36cw\ sw^3)) + ci^3(co\ so^2(-78cw^4 - 180cw^2sw^2 \\
& - 102sw^4) + co^3(26cw^4 + 60cw^2sw^2 + 34sw^4)) + ci(co^3(-30cw^4 \\
& - 60cw^2sw^2 - 30sw^4) + co\ so^2(90cw^4 + 180cw^2sw^2 + 90sw^4)))\cos M \\
& + (co^2so(-24cw^3sw - 24cw\ sw^3) + so^3(8cw^3sw + 8cw\ sw^3) \\
& + ci^2(so^3(-24cw^3sw - 24cw\ sw^3) + co^2so(72cw^3sw + 72cw\ sw^3)) \\
& + ci^3(co\ so^2(24cw^4 - 24sw^4) + co^3(-8cw^4 + 8sw^4)) \\
& + e^2(co^2so(-36cw^3sw - 12cw\ sw^3) + so^3(12cw^3sw + 4cw\ sw^3) \\
& + ci^2(so^3(-12cw^3sw - 36cw\ sw^3) + co^2so(36cw^3sw + 108cw\ sw^3)) \\
& + ci^3(co\ so^2(-159cw^4 - 342cw^2sw^2 - 207sw^4) + co^3(53cw^4 + 114cw^2sw^2 \\
& + 69sw^4)) + ci(co^3(-57cw^4 - 138cw^2sw^2 - 57sw^4) + co\ so^2(171cw^4 \\
& + 414cw^2sw^2 + 171sw^4))))\cos 2M + e(co^2so(-72cw^3sw - 144cw\ sw^3) \\
& + so^3(24cw^3sw + 48cw\ sw^3) + ci^2(so^3(-144cw^3sw - 72cw\ sw^3) \\
& + co^2so(432cw^3sw + 216cw\ sw^3)) + ci(co^3(-9cw^4 + 54cw^2sw^2 - 9sw^4) \\
& + co\ so^2(27cw^4 - 162cw^2sw^2 + 27sw^4)) + ci^3(co\ so^2(117cw^4 \\
& - 54cw^2sw^2 - 99sw^4) + co^3(-39cw^4 + 18cw^2sw^2 + 33sw^4)))\cos 3M \\
& + (so^3(8cw^3sw - 8cw\ sw^3) + co^2so(-24cw^3sw + 24cw\ sw^3) \\
& + ci^2(so^3(24cw^3sw - 24cw\ sw^3) + co^2so(-72cw^3sw + 72cw\ sw^3)) \\
& + ci^3(co\ so^2(-6cw^4 + 36cw^2sw^2 - 6sw^4) + co^3(2cw^4 - 12cw^2sw^2 \\
& + 2sw^4)) + ci(co\ so^2(-18cw^4 + 108cw^2sw^2 - 18sw^4) + co^3(6cw^4 \\
& - 36cw^2sw^2 + 6sw^4)) + e^2(co^2so(-54cw^3sw - 582cw\ sw^3) + so^3(18cw^3sw \\
& + 194cw\ sw^3) + ci^2(so^3(-582cw^3sw - 54cw\ sw^3) + co^2so(1746cw^3sw \\
& + 162cw\ sw^3)) + ci^3(co\ so^2(384cw^4 - 396cw^2sw^2 - 252sw^4) \\
& + co^3(-128cw^4 + 132cw^2sw^2 + 84sw^4)) + ci(co^3(-66cw^4 + 396cw^2sw^2 \\
& - 66sw^4) + co\ so^2(198cw^4 - 1188cw^2sw^2 + 198sw^4))))\cos 4M \\
& + e(so^3(52cw^3sw - 52cw\ sw^3) + co^2so(-156cw^3sw + 156cw\ sw^3) \\
& + ci^2(so^3(156cw^3sw - 156cw\ sw^3) + co^2so(-468cw^3sw + 468cw\ sw^3))
\end{aligned}
$$

$$+ ci^3(co\ so^2(-39cw^4 + 234cw^2sw^2 - 39sw^4) + co^3(13cw^4 - 78cw^2sw^2$$
$$+ 13sw^4)) + ci(co\ so^2(-117cw^4 + 702cw^2sw^2 - 117sw^4) + co^3(39cw^4$$
$$- 234cw^2sw^2 + 39sw^4)))\cos 5M + e^2(so^3(204cw^3sw - 204cw\ sw^3)$$
$$+ co^2so(-612cw^3sw + 612cw\ sw^3) + ci^2(so^3(612cw^3sw - 612cw\ sw^3)$$
$$+ co^2so(-1836cw^3sw + 1836cw\ sw^3)) + ci^3(co\ so^2(-153cw^4 + 918cw^2sw^2$$
$$- 153sw^4) + co^3(51cw^4 - 306cw^2sw^2 + 51sw^4)) + ci(co\ so^2(-459cw^4$$
$$+ 2754cw^2sw^2 - 459sw^4) + co^3(153cw^4 - 918cw^2sw^2 + 153sw^4)))\cos 6M$$
$$+ e(so^3(2cw^4 - 2sw^4) + co^2so(-6cw^4 + 6sw^4) + ci^3(co\ so^2(-24cw^3sw$$
$$- 24cw\ sw^3) + co^3(8cw^3sw + 8cw\ sw^3)) + ci^2(co^2so(18cw^4 - 18sw^4)$$
$$+ so^3(-6cw^4 + 6sw^4)))\sin M + (so^3(4cw^4 - 4sw^4) + co^2so(-12cw^4$$
$$+ 12sw^4) + ci^3(co\ so^2(-48cw^3sw - 48cw\ sw^3) + co^3(16cw^3sw$$
$$+ 16cw\ sw^3)) + ci^2(co^2so(36cw^4 - 36sw^4) + so^3(-12cw^4 + 12sw^4))$$
$$+ e^2(so^3(5cw^4 - 6cw^2sw^2 - 3sw^4) + co^2so(-15cw^4 + 18cw^2sw^2 + 9sw^4)$$
$$+ ci(co\ so^2(36cw^3sw - 36cw\ sw^3) + co^3(-12cw^3sw + 12cw\ sw^3))$$
$$+ ci^3(co\ so^2(-36cw^3sw - 60cw\ sw^3) + co^3(12cw^3sw + 20cw\ sw^3))$$
$$+ ci^2(co^2so(27cw^4 + 54cw^2sw^2 - 45sw^4) + so^3(-9cw^4 - 18cw^2sw^2$$
$$+ 15sw^4))))\sin 2M + e(so^3(15cw^4 + 18cw^2sw^2 - 21sw^4) + co^2so(-45cw^4$$
$$- 54cw^2sw^2 + 63sw^4) + ci^3(co\ so^2(-252cw^3sw - 180cw\ sw^3)$$
$$+ co^3(84cw^3sw + 60cw\ sw^3)) + ci(co^3(36cw^3sw - 36cw\ sw^3)$$
$$+ co\ so^2(-108cw^3sw + 108cw\ sw^3)) + ci^2(co^2so(189cw^4 - 162cw^2sw^2$$
$$- 135sw^4) + so^3(-63cw^4 + 54cw^2sw^2 + 45sw^4)))\sin 3M + (co^2so(-6cw^4$$
$$+ 36cw^2sw^2 - 6sw^4) + so^3(2cw^4 - 12cw^2sw^2 + 2sw^4)$$
$$+ ci^3(co\ so^2(24cw^3sw - 24cw\ sw^3) + co^3(-8cw^3sw + 8cw\ sw^3))$$
$$+ ci(co\ so^2(72cw^3sw - 72cw\ sw^3) + co^3(-24cw^3sw + 24cw\ sw^3))$$
$$+ ci^2(co^2so(-18cw^4 + 108cw^2sw^2 - 18sw^4) + so^3(6cw^4 - 36cw^2sw^2$$
$$+ 6sw^4)) + e^2(so^3(31cw^4 + 132cw^2sw^2 - 75sw^4) + co^2so(-93cw^4$$
$$- 396cw^2sw^2 + 225sw^4) + ci^3(co\ so^2(-900cw^3sw - 372cw\ sw^3)$$
$$+ co^3(300cw^3sw + 124cw\ sw^3)) + ci(co^3(264cw^3sw - 264cw\ sw^3)$$
$$+ co\ so^2(-792cw^3sw + 792cw\ sw^3)) + ci^2(co^2so(675cw^4 - 1188cw^2sw^2$$
$$- 279sw^4) + so^3(-225cw^4 + 396cw^2sw^2 + 93sw^4))))\sin 4M$$
$$+ e(co^2so(-39cw^4 + 234cw^2sw^2 - 39sw^4) + so^3(13cw^4 - 78cw^2sw^2$$
$$+ 13sw^4) + ci^3(co\ so^2(156cw^3sw - 156cw\ sw^3) + co^3(-52cw^3sw$$
$$+ 52cw\ sw^3)) + ci(co\ so^2(468cw^3sw - 468cw\ sw^3) + co^3(-156cw^3sw$$

$$+ 156cw\ sw^3)) + ci^2(co^2so(-117cw^4 + 702cw^2sw^2 - 117sw^4) + so^3(39cw^4$$
$$- 234cw^2sw^2 + 39sw^4)))\sin 5M + e^2(co^2so(-153cw^4 + 918cw^2sw^2$$
$$- 153sw^4) + so^3(51cw^4 - 306cw^2sw^2 + 51sw^4) + ci^3(co\ so^2(612cw^3sw$$
$$- 612cw\ sw^3) + co^3(-204cw^3sw + 204cw\ sw^3)) + ci(co\ so^2(1836cw^3sw$$
$$- 1836cw\ sw^3) + co^3(-612cw^3sw + 612cw\ sw^3)) + ci^2(co^2so(-459cw^4$$
$$+ 2754cw^2sw^2 - 459sw^4) + so^3(153cw^4 - 918cw^2sw^2 + 153sw^4)))\sin 6M)$$

$$\frac{d\Omega_1}{dt} = \frac{105}{16}(ci^2(co^2so(-18) + so^3(6)) + ci^4(so^3(-6) + co^2so(18)) + si^2(so^3(-6)$$
$$+ co^2so(18) + ci^2(co^2so(-54) + so^3(18))) + e^2(ci(co\ so^2(-18cw\ sw)$$
$$+ co^3(6cw\ sw)) + ci^3(co^3(-18cw\ sw) + co\ so^2(54cw\ sw)) + ci^2(co^2so(-90)$$
$$+ so^3(30)) + ci^4(so^3(-24cw^4 - 60cw^2sw^2 - 36sw^4) + co^2so(72cw^4$$
$$+ 180cw^2sw^2 + 108sw^4)) + si^2(so^3(-30cw^4 - 60cw^2sw^2 - 30sw^4)$$
$$+ co^2so(90cw^4 + 180cw^2sw^2 + 90sw^4) + ci(co\ so^2(-108cw^3sw$$
$$- 108cw\ sw^3) + co^3(36cw^3sw + 36cw\ sw^3)) + ci^2(co^2so(-216cw^4$$
$$- 540cw^2sw^2 - 324sw^4) + so^3(72cw^4 + 180cw^2sw^2 + 108sw^4))))$$
$$+ e(ci(co\ so^2(-12cw^3sw - 12cw\ sw^3) + co^3(4cw^3sw + 4cw\ sw^3))$$
$$+ ci^3(co^3(-12cw^3sw - 12cw\ sw^3) + co\ so^2(36cw^3sw + 36cw\ sw^3))$$
$$+ ci^2(co^2so(-90cw^4 - 180cw^2sw^2 - 90sw^4) + so^3(30cw^4 + 60cw^2sw^2$$
$$+ 30sw^4)) + ci^4(so^3(-26cw^4 - 60cw^2sw^2 - 34sw^4) + co^2so(78cw^4$$
$$+ 180cw^2sw^2 + 102sw^4)) + si^2(so^3(-30cw^4 - 60cw^2sw^2 - 30sw^4)$$
$$+ co^2so(90cw^4 + 180cw^2sw^2 + 90sw^4) + ci(co\ so^2(-72cw^3sw$$
$$- 72cw\ sw^3) + co^3(24cw^3sw + 24cw\ sw^3)) + ci^2(co^2so(-234cw^4$$
$$- 540cw^2sw^2 - 306sw^4) + so^3(78cw^4 + 180cw^2sw^2 + 102sw^4))))\cos M$$
$$+ (ci(co\ so^2(-24cw^3sw - 24cw\ sw^3) + co^3(8cw^3sw + 8cw\ sw^3))$$
$$+ ci^3(co^3(-24cw^3sw - 24cw\ sw^3) + co\ so^2(72cw^3sw + 72cw\ sw^3))$$
$$+ ci^4(so^3(8cw^4 - 8sw^4) + co^2so(-24cw^4 + 24sw^4))$$
$$+ si^2(ci(co\ so^2(-144cw^3sw - 144cw\ sw^3) + co^3(48cw^3sw + 48cw\ sw^3))$$
$$+ ci^2(co^2so(72cw^4 - 72sw^4) + so^3(-24cw^4 + 24sw^4)))$$
$$+ e^2(ci(co\ so^2(-36cw^3sw - 12cw\ sw^3) + co^3(12cw^3sw + 4cw\ sw^3))$$
$$+ ci^3(co^3(-12cw^3sw - 36cw\ sw^3) + co\ so^2(36cw^3sw + 108cw\ sw^3))$$
$$+ ci^2(co^2so(-171cw^4 - 414cw^2sw^2 - 171sw^4) + so^3(57cw^4$$
$$+ 138cw^2sw^2 + 57sw^4)) + ci^4(so^3(-53cw^4 - 114cw^2sw^2 - 69sw^4)$$

$$
\begin{aligned}
&+ co^2so(159cw^4 + 342cw^2sw^2 + 207sw^4)) + si^2(so^3(-57cw^4 - 138cw^2sw^2 \\
&- 57sw^4) + co^2so(171cw^4 + 414cw^2sw^2 + 171sw^4) + ci(co\ so^2(-72cw^3sw \\
&- 216cw\ sw^3) + co^3(24cw^3sw + 72cw\ sw^3)) + ci^2(co^2so(-477cw^4 \\
&- 1026cw^2sw^2 - 621sw^4) + so^3(159cw^4 + 342cw^2sw^2 + 207sw^4)))))\cos 2M \\
&+ e(ci(co\ so^2(-72cw^3sw - 144cw\ sw^3) + co^3(24cw^3sw + 48cw\ sw^3)) \\
&+ ci^3(co^3(-144cw^3sw - 72cw\ sw^3) + co\ so^2(432cw^3sw + 216cw\ sw^3)) \\
&+ ci^2(co^2so(-27cw^4 + 162cw^2sw^2 - 27sw^4) + so^3(9cw^4 - 54cw^2sw^2 \\
&+ 9sw^4)) + ci^4(so^3(39cw^4 - 18cw^2sw^2 - 33sw^4) + co^2so(-117cw^4 \\
&+ 54cw^2sw^2 + 99sw^4)) + si^2(so^3(-9cw^4 + 54cw^2sw^2 - 9sw^4) \\
&+ co^2so(27cw^4 - 162cw^2sw^2 + 27sw^4) + ci(co\ so^2(-864cw^3sw \\
&- 432cw\ sw^3) + co^3(288cw^3sw + 144cw\ sw^3)) + ci^2(co^2so(351cw^4 \\
&- 162cw^2sw^2 - 297sw^4) + so^3(-117cw^4 + 54cw^2sw^2 + 99sw^4))))\cos 3M \\
&+ (ci(co^3(8cw^3sw - 8cw\ sw^3) + co\ so^2(-24cw^3sw + 24cw\ sw^3)) \\
&+ ci^3(co^3(24cw^3sw - 24cw\ sw^3) + co\ so^2(-72cw^3sw + 72cw\ sw^3)) \\
&+ ci^4(so^3(-2cw^4 + 12cw^2sw^2 - 2sw^4) + co^2so(6cw^4 - 36cw^2sw^2 + 6sw^4)) \\
&+ ci^2(so^3(-6cw^4 + 36cw^2sw^2 - 6sw^4) + co^2so(18cw^4 - 108cw^2sw^2 \\
&+ 18sw^4)) + si^2(co^2so(-18cw^4 + 108cw^2sw^2 - 18sw^4) + so^3(6cw^4 \\
&- 36cw^2sw^2 + 6sw^4) + ci(co\ so^2(144cw^3sw - 144cw\ sw^3) + co^3(-48cw^3sw \\
&+ 48cw\ sw^3)) + ci^2(co^2so(-18cw^4 + 108cw^2sw^2 - 18sw^4) + so^3(6cw^4 \\
&- 36cw^2sw^2 + 6sw^4))) + e^2(ci^3(co^3(-582cw^3sw - 54cw\ sw^3) \\
&+ co\ so^2(1746cw^3sw + 162cw\ sw^3)) + ci(co\ so^2(-54cw^3sw - 582cw\ sw^3) \\
&+ co^3(18cw^3sw + 194cw\ sw^3)) + ci^2(co^2so(-198cw^4 + 1188cw^2sw^2 \\
&- 198sw^4) + so^3(66cw^4 - 396cw^2sw^2 + 66sw^4)) + ci^4(so^3(128cw^4 \\
&- 132cw^2sw^2 - 84sw^4) + co^2so(-384cw^4 + 396cw^2sw^2 + 252sw^4)) \\
&+ si^2(so^3(-66cw^4 + 396cw^2sw^2 - 66sw^4) + co^2so(198cw^4 \\
&- 1188cw^2sw^2 + 198sw^4) + ci(co\ so^2(-3492cw^3sw - 324cw\ sw^3) \\
&+ co^3(1164cw^3sw + 108cw\ sw^3)) + ci^2(co^2so(1152cw^4 - 1188cw^2sw^2 \\
&- 756sw^4) + so^3(-384cw^4 + 396cw^2sw^2 + 252sw^4)))))\cos 4M \\
&+ e(ci(co^3(52cw^3sw - 52cw\ sw^3) + co\ so^2(-156cw^3sw + 156cw\ sw^3)) \\
&+ ci^3(co^3(156cw^3sw - 156cw\ sw^3) + co\ so^2(-468cw^3sw + 468cw\ sw^3)) \\
&+ ci^4(so^3(-13cw^4 + 78cw^2sw^2 - 13sw^4) + co^2so(39cw^4 - 234cw^2sw^2 \\
&+ 39sw^4)) + ci^2(so^3(-39cw^4 + 234cw^2sw^2 - 39sw^4) + co^2so(117cw^4 \\
&- 702cw^2sw^2 + 117sw^4)) + si^2(co^2so(-117cw^4 + 702cw^2sw^2 - 117sw^4)
\end{aligned}
$$

$$
\begin{aligned}
&+so^3(39cw^4-234cw^2sw^2+39sw^4)+ci(co\ so^2(936cw^3sw-936cw\ sw^3)\\
&+co^3(-312cw^3sw+312cw\ sw^3))+ci^2(co^2so(-117cw^4+702cw^2sw^2\\
&-117sw^4)+so^3(39cw^4-234cw^2sw^2+39sw^4))))\cos5M\\
&+e^2(ci(co^3(204cw^3sw-204cw\ sw^3)+co\ so^2(-612cw^3sw+612cw\ sw^3))\\
&+ci^3(co^3(612cw^3sw-612cw\ sw^3)+co\ so^2(-1836cw^3sw+1836cw\ sw^3))\\
&+ci^4(so^3(-51cw^4+306cw^2sw^2-51sw^4)+co^2so(153cw^4-918cw^2sw^2\\
&+153sw^4))+ci^2(so^3(-153cw^4+918cw^2sw^2-153sw^4)+co^2so(459cw^4\\
&-2754cw^2sw^2+459sw^4))+si^2(co^2so(-459cw^4+2754cw^2sw^2-459sw^4)\\
&+so^3(153cw^4-918cw^2sw^2+153sw^4)+ci(co\ so^2(3672cw^3sw\\
&-3672cw\ sw^3)+co^3(-1224cw^3sw+1224cw\ sw^3))+ci^2(co^2so(-459cw^4\\
&+2754cw^2sw^2-459sw^4)+so^3(153cw^4-918cw^2sw^2+153sw^4))))\cos6M\\
&+e(ci^4(so^3(-8cw\ sw)+co^2so(24cw\ sw))+ci^3(co\ so^2(18cw^4-18sw^4)\\
&+co^3(-6cw^4+6sw^4))+ci(co^3(2cw^4-2sw^4)+co\ so^2(-6cw^4+6sw^4))\\
&+si^2(ci^2(co^2so(-72cw\ sw)+so^3(24cw\ sw))+ci(co^3(12cw^4-12sw^4)\\
&+co\ so^2(-36cw^4+36sw^4))))\sin M+(ci^4(so^3(-16cw^3sw-16cw\ sw^3)\\
&+co^2so(48cw^3sw+48cw\ sw^3))+ci^3(co\ so^2(36cw^4-36sw^4)\\
&+co^3(-12cw^4+12sw^4))+ci(co^3(4cw^4-4sw^4)+co\ so^2(-12cw^4\\
&+12sw^4))+si^2(ci^2(co^2so(-144cw^3sw-144cw\ sw^3)+so^3(48cw^3sw\\
&+48cw\ sw^3))+\ ci(co^3(24cw^4-24sw^4)+co\ so^2(-72cw^4+72sw^4)))\\
&+e^2(ci^2(so^3(12cw^3sw-12cw\ sw^3)+co^2so(-36cw^3sw+36cw\ sw^3))\\
&+ci^4(so^3(-12cw^3sw-20cw\ sw^3)+co^2so(36cw^3sw+60cw\ sw^3))\\
&+ci(co^3(5cw^4-6cw^2sw^2-3sw^4)+co\ so^2(-15cw^4+18cw^2sw^2+9sw^4))\\
&+ci^3(co\ so^2(27cw^4+54cw^2sw^2-45sw^4)+co^3(-9cw^4-18cw^2sw^2\\
&+15sw^4))+si^2(co^2so(36cw^3sw-36cw\ sw^3)+so^3(-12cw^3sw+12cw\ sw^3)\\
&+ci^2(co^2so(-108cw^3sw-180cw\ sw^3)+so^3(36cw^3sw+60cw\ sw^3))\\
&+ci(co^3(18cw^4+36cw^2sw^2-30sw^4)+co\ so^2(-54cw^4-108cw^2sw^2\\
&+90sw^4)))))\sin2M+e(ci^2(co^2so(108cw^3sw-108cw\ sw^3)+so^3(-36cw^3sw\\
&+36cw\ sw^3))+ci^4(so^3(-84cw^3sw-60cw\ sw^3)+co^2so(252cw^3sw\\
&+180cw\ sw^3))+ci^3(co\ so^2(189cw^4-162cw^2sw^2-135sw^4)\\
&+co^3(-63cw^4+54cw^2sw^2+45sw^4))+ci(co^3(15cw^4+18cw^2sw^2-21sw^4)\\
&+co\ so^2(-45cw^4-54cw^2sw^2+63sw^4))+si^2(so^3(36cw^3sw-36cw\ sw^3)\\
&+co^2so(-108cw^3sw+108cw\ sw^3)+ci^2(co^2so(-756cw^3sw-540cw\ sw^3)
\end{aligned}
$$

$$
\begin{aligned}
&+ so^3(252cw^3sw + 180cw\ sw^3)) + ci(co^3(126cw^4 - 108cw^2sw^2 - 90sw^4)\\
&+ co\ so^2(-378cw^4 + 324cw^2sw^2 + 270sw^4))))\sin 3M + (ci^4(so^3(8cw^3sw\\
&- 8cw\ sw^3) + co^2so(-24cw^3sw + 24cw\ sw^3)) + ci^2(so^3(24cw^3sw\\
&- 24cw\ sw^3) + co^2so(-72cw^3sw + 72cw\ sw^3)) + ci(co\ so^2(-6cw^4\\
&+ 36cw^2sw^2 - 6sw^4) + co^3(2cw^4 - 12cw^2sw^2 + 2sw^4))\\
&+ ci^3(co\ so^2(-18cw^4 + 108cw^2sw^2 - 18sw^4) + co^3(6cw^4 - 36cw^2sw^2\\
&+ 6sw^4)) + si^2(co^2so(72cw^3sw - 72cw\ sw^3) + so^3(-24cw^3sw + 24cw\ sw^3)\\
&+ ci^2(co^2so(72cw^3sw - 72cw\ sw^3) + so^3(-24cw^3sw + 24cw\ sw^3))\\
&+ ci(co^3(-12cw^4 + 72cw^2sw^2 - 12sw^4) + co\ so^2(36cw^4 - 216cw^2sw^2\\
&+ 36sw^4))) + e^2(ci^2(co^2so(792cw^3sw - 792cw\ sw^3) + so^3(-264cw^3sw\\
&+ 264cw\ sw^3)) + ci^4(so^3(-300cw^3sw - 124cw\ sw^3) + co^2so(900cw^3sw\\
&+ 372cw\ sw^3)) + ci^3(co\ so^2(675cw^4 - 1188cw^2sw^2 - 279sw^4)\\
&+ co^3(-225cw^4 + 396cw^2sw^2 + 93sw^4)) + \ ci(co^3(31cw^4 + 132cw^2sw^2\\
&- 75sw^4) + co\ so^2(-93cw^4 - 396cw^2sw^2 + 225sw^4)) + si^2(so^3(264cw^3sw\\
&- 264cw\ sw^3) + co^2so(-792cw^3sw + 792cw\ sw^3) + ci^2(co^2so(-2700cw^3sw\\
&- 1116cw\ sw^3) + so^3(900cw^3sw + 372cw\ sw^3)) + ci(co^3(450cw^4\\
&- 792cw^2sw^2 - 186sw^4) + \ co\ so^2(-1350cw^4 + 2376cw^2sw^2\\
&+ 558sw^4)))))\sin 4M + e(ci^4(so^3(52cw^3sw - 52cw\ sw^3)\\
&+ co^2so(-156cw^3sw + 156cw\ sw^3)) + ci^3(co\ so^2(-117cw^4 + 702cw^2sw^2\\
&- 117sw^4) + co^3(39cw^4 - 234cw^2sw^2 + 39sw^4)) + si^2(co^2so(468cw^3sw\\
&- 468cw\ sw^3) + so^3(-156cw^3sw + 156cw\ sw^3) + ci^2(co^2so(468cw^3sw\\
&- 468cw\ sw^3) + so^3(-156cw^3sw + 156cw\ sw^3)) + ci(co^3(-78cw^4\\
&+ 468cw^2sw^2 - 78sw^4) + co\ so^2(234cw^4 - 1404cw^2sw^2 + 234sw^4))))\sin 5M\\
&+ e^2(ci^4(so^3(204cw^3sw - 204cw\ sw^3) + co^2so(-612cw^3sw + 612cw\ sw^3))\\
&+ ci^2(so^3(612cw^3sw - 612cw\ sw^3) + co^2so(-1836cw^3sw + 1836cw\ sw^3))\\
&+ ci(co\ so^2(-153cw^4 + 918cw^2sw^2 - 153sw^4) + co^3(51cw^4 - 306cw^2sw^2\\
&+ 51sw^4)) + ci^3(co\ so^2(-459cw^4 + 2754cw^2sw^2 - 459sw^4) + co^3(153cw^4\\
&- 918cw^2sw^2 + 153sw^4)) + si^2(co^2so(1836cw^3sw - 1836cw\ sw^3)\\
&+ so^3(-612cw^3sw + 612cw\ sw^3) + ci^2(co^2so(1836cw^3sw - 1836cw\ sw^3)\\
&+ so^3(-612cw^3sw + 612cw\ sw^3)) + ci(co^3(-306cw^4 + 1836cw^2sw^2\\
&- 306sw^4) + co\ so^2(918cw^4 - 5508cw^2sw^2 + 918sw^4))))\sin 6M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}M_1}{\mathrm{d}t} = & -\frac{525}{16a} si(ci(co^2 so(-18) + so^3(6)) + ci^3(so^3(-6) + co^2 so(18)) \\
& + e^2(co\ so^2(-18cw\ sw) + co^3(6cw\ sw) + ci^2(co^3(-18cw\ sw) \\
& + co\ so^2(54cw\ sw)) + ci(co^2 so(-90) + so^3(30)) + ci^3(so^3(-24cw^4 \\
& - 60cw^2 sw^2 - 36sw^4) + co^2 so(72cw^4 + 180cw^2 sw^2 + 108sw^4))) \\
& + e(co\ so^2(-12cw\ sw) + co^3(4cw\ sw) + ci^2(co^3(-12cw\ sw) \\
& + co\ so^2(36cw\ sw)) + ci(co^2 so(-90) + so^3(30)) + ci^3(so^3(-26cw^4 \\
& - 60cw^2 sw^2 - 34sw^4) + co^2 so(78cw^4 + 180cw^2 sw^2 + 102sw^4)))\cos M \\
& + (co\ so^2(-24cw\ sw) + co^3(8cw\ sw) + ci^2(co^3(-24cw\ sw) \\
& + co\ so^2(72cw\ sw)) + ci^3(so^3(8cw^4 - 8sw^4) + co^2 so(-24cw^4 + 24sw^4)) \\
& + e^2(co\ so^2(-36cw^3 sw - 12cw\ sw^3) + co^3(12cw^3 sw + 4cw\ sw^3) \\
& + ci^2(co^3(-12cw^3 sw - 36cw\ sw^3) + co\ so^2(36cw^3 sw + 108cw\ sw^3)) \\
& + ci(co^2 so(-171cw^4 - 414cw^2 sw^2 - 171sw^4) + so^3(57cw^4 + 138cw^2 sw^2 \\
& + 57sw^4)) + ci^3(so^3(-53cw^4 - 114cw^2 sw^2 - 69sw^4) + co^2 so(159cw^4 \\
& + 342cw^2 sw^2 + 207sw^4))))\cos 2M + e(co\ so^2(-72cw^3 sw - 144cw\ sw^3) \\
& + co^3(24cw^3 sw + 48cw\ sw^3) + ci^2(co^3(-144cw^3 sw - 72cw\ sw^3) \\
& + co\ so^2(432cw^3 sw + 216cw\ sw^3)) + ci(co^2 so(-27cw^4 + 162cw^2 sw^2 \\
& - 27sw^4) + so^3(9cw^4 - 54cw^2 sw^2 + 9sw^4)) + ci^3(so^3(39cw^4 - 18cw^2 sw^2 \\
& - 33sw^4) + co^2 so(-117cw^4 + 54cw^2 sw^2 + 99sw^4)))\cos 3M \\
& + (co^3(8cw^3 sw - 8cw\ sw^3) + co\ so^2(-24cw^3 sw + 24cw\ sw^3) \\
& + ci^2(co^3(24cw^3 sw - 24cw\ sw^3) + co\ so^2(-72cw^3 sw + 72cw\ sw^3)) \\
& + ci^3(so^3(-2cw^4 + 12cw^2 sw^2 - 2sw^4) + co^2 so(6cw^4 - 36cw^2 sw^2 \\
& + 6sw^4)) + ci(so^3(-6cw^4 + 36cw^2 sw^2 - 6sw^4) + co^2 so(18cw^4 \\
& - 108cw^2 sw^2 + 18sw^4)) + e^2(co\ so^2(-54cw^3 sw - 582cw\ sw^3) \\
& + co^3(18cw^3 sw + 194cw\ sw^3) + ci^2(co^3(-582cw^3 sw - 54cw\ sw^3) \\
& + co\ so^2(1746cw^3 sw + 162cw\ sw^3)) + ci(co^2 so(-198cw^4 + 1188cw^2 sw^2 \\
& - 198sw^4) + so^3(66cw^4 - 396cw^2 sw^2 + 66sw^4)) + ci^3(so^3(128cw^4 \\
& - 132cw^2 sw^2 - 84sw^4) + co^2 so(-384cw^4 + 396cw^2 sw^2 + 252sw^4)))) \\
& \times \cos 4M + e(co^3(52cw^3 sw - 52cw\ sw^3) + co\ so^2(-156cw^3 sw + 156cw\ sw^3) \\
& + ci^2(co^3(156cw^3 sw - 156cw\ sw^3) + co\ so^2(-468cw^3 sw + 468cw\ sw^3)) \\
& + ci^3(so^3(-13cw^4 + 78cw^2 sw^2 - 13sw^4) + co^2 so(39cw^4 - 234cw^2 sw^2 \\
& + 39sw^4)) + ci(so^3(-39cw^4 + 234cw^2 sw^2 - 39sw^4) \\
& + co^2 so(117cw^4 - 702cw^2 sw^2 + 117sw^4)))\cos 5M
\end{aligned}
$$

$$
\begin{aligned}
&+ e^2(co^3(204cw^3sw - 204cw\ sw^3) + co\ so^2(-612cw^3sw + 612cw\ sw^3)\\
&+ ci^2(co^3(612cw^3sw - 612cw\ sw^3) + co\ so^2(-1836cw^3sw + 1836cw\ sw^3))\\
&+ ci^3(so^3(-51cw^4 + 306cw^2sw^2 - 51sw^4) + co^2so(153cw^4 - 918cw^2sw^2\\
&+ 153sw^4)) + ci(so^3(-153cw^4 + 918cw^2sw^2 - 153sw^4)\\
&+ co^2so(459cw^4 - 2754cw^2sw^2 + 459sw^4)))\cos 6M + e(co^3(2cw^4 - 2sw^4)\\
&+ co\ so^2(-6cw^4 + 6sw^4) + ci^3(so^3(-8cw\ sw) + co^2so(24cw\ sw))\\
&+ ci^2(co\ so^2(18cw^4 - 18sw^4) + co^3(-6cw^4 + 6sw^4)))\sin M + (co^3(4cw^4\\
&- 4sw^4) + co\ so^2(-12cw^4 + 12sw^4) + ci^3(so^3(-16cw\ sw) + co^2so(48cw\ sw))\\
&+ ci^2(co\ so^2(36cw^4 - 36sw^4) + co^3(-12cw^4 + 12sw^4)) + e^2(co^3(5cw^4\\
&- 6cw^2sw^2 - 3sw^4) + co\ so^2(-15cw^4 + 18cw^2sw^2 + 9sw^4)+\\
&+ ci(so^3(12cw^3sw - 12cw\ sw^3) + co^2so(-36cw^3sw + 36cw\ sw^3))\\
&+ ci^3(so^3(-12cw^3sw - 20cw\ sw^3) + co^2so(36cw^3sw + 60cw\ sw^3))\\
&+ ci^2(co\ so^2(27cw^4 + 54cw^2sw^2 - 45sw^4) + co^3(-9cw^4 - 18cw^2sw^2\\
&+ 15sw^4))))\sin 2M + e(co^3(15cw^4 + 18cw^2sw^2 - 21sw^4) + co\ so^2(-45cw^4\\
&- 54cw^2sw^2 + 63sw^4) + ci(co^2so(108cw^3sw - 108cw\ sw^3)\\
&+ so^3(-36cw^3sw + 36cw\ sw^3)) + ci^3(so^3(-84cw^3sw - 60cw\ sw^3)\\
&+ co^2so(252cw^3sw + 180cw\ sw^3)) + ci^2(co\ so^2(189cw^4 - 162cw^2sw^2\\
&- 135sw^4) + co^3(-63cw^4 + 54cw^2sw^2 + 45sw^4)))\sin 3M + (co\ so^2(-6cw^4\\
&+ 36cw^2sw^2 - 6sw^4) + co^3(2cw^4 - 12cw^2sw^2 + 2sw^4) + ci^3(so^3(8cw^3sw\\
&- 8cw\ sw^3) + co^2so(-24cw^3sw + 24cw\ sw^3)) + ci(so^3(24cw^3sw\\
&- 24cw\ sw^3) + co^2so(-72cw^3sw + 72cw\ sw^3)) + ci^2(co\ so^2(-18cw^4\\
&+ 108cw^2sw^2 - 18sw^4) + co^3(6cw^4 - 36cw^2sw^2 + 6sw^4)) + e^2(co^3(31cw^4\\
&+ 132cw^2sw^2 - 75sw^4) + co\ so^2(-93cw^4 - 396cw^2sw^2 + 225sw^4)\\
&+ ci(co^2so(792cw^3sw - 792cw\ sw^3) + so^3(-264cw^3sw + 264cw\ sw^3))\\
&+ ci^3(so^3(-300cw^3sw - 124cw\ sw^3) + co^2so(900cw^3sw + 372cw\ sw^3))\\
&+ ci^2(co\ so^2(675cw^4 - 1188cw^2sw^2 - 279sw^4) + co^3(-225cw^4\\
&+ 396cw^2sw^2 + 93sw^4))))\sin 4M + e(co\ so^2(-39cw^4 + 234cw^2sw^2\\
&- 39sw^4) + co^3(13cw^4 - 78cw^2sw^2 + 13sw^4) + ci^3(so^3(52cw^3sw\\
&- 52cw\ sw^3) + co^2so(-156cw^3sw + 156cw\ sw^3)) + ci(so^3(156cw^3sw\\
&- 156cw\ sw^3) + co^2so(-468cw^3sw + 468cw\ sw^3)) + ci^2(co\ so^2(-117cw^4\\
&+ 702cw^2sw^2 - 117sw^4) + co^3(39cw^4 - 234cw^2sw^2 + 39sw^4)))\sin 5M\\
&+ e^2(co\ so^2(-153cw^4 + 918cw^2sw^2 - 153sw^4) + co^3(51cw^4 - 306cw^2sw^2\\
&+ 51sw^4) + ci^3(so^3(204cw^3sw - 204cw\ sw^3) + co^2so(-612cw^3sw
\end{aligned}
$$

$$+612cw\,sw^3)) + ci(so^3(612cw^3sw - 612cw\,sw^3) + co^2so(-1836cw^3sw + 1836cw\,sw^3)) + ci^2(co\,so^2(-459cw^4 + 2754cw^2sw^2 - 459sw^4) + co^3(153cw^4 - 918cw^2sw^2 + 153sw^4)))\sin 6M)$$

The long periodic terms included in above formulas are given as follows:

$$d_2 = -\frac{315}{16}e^2 si((-1+3ci^2-2ci^3)\cos(3o-2w)+(-1+3ci^2+2ci^3)\cos(3o+2w))$$

$$d_3 = -\frac{315}{32}e\,si((40ci-40ci^3)\sin(3o)+(2+7ci-20ci^2+11ci^3)\sin(3o-2w) + (-2+7ci+20ci^2+11ci^3)\sin(3o+2w))$$

$$d_4 = -\frac{945}{32}si((4ci-4ci^3+(20ci-20ci^3)e^2)\cos(3o)+(1-3ci^2+2ci^3)e^2\cos(3o-2w) + (-1+3ci^2+2ci^3)e^2\cos(3o+2w))$$

$$d_5 = -\frac{315}{32}((4ci^2-4ci^4+(20ci^2-20ci^4)e^2 + (-4+12ci^2+(-20+60ci^2)e^2)si^2)\sin(3o)+((ci-3ci^3+2ci^4)e^2 + (6ci-6ci^2)e^2si^2)\sin(3o-2w)+((-ci+3ci^3+2ci^4)e^2 + (-6ci-6ci^2)e^2si^2)\sin(3o+2w))$$

$$d_6 = \frac{1575}{32a}si((4ci-4ci^3+(20ci-20ci^3)e^2)\sin(3o) + (1-3ci^2+2ci^3)e^2\sin(3o-2w)+(-1+3ci^2+2ci^3)e^2\sin(3o+2w))$$

Solutions of D_{44} Perturbation

$$\frac{da_1}{dt} = -\frac{525}{4}\sin(M/2)\cos(M/2)e(co^2so^2(-18)+co^4(3)+so^4(3) + ci^4(co^2so^2(-18)+co^4(3)+so^4(3))+ci^2(co^4(-6)+so^4(-6) + co^2so^2(36))+e(co^2so^2(-192cw^4-288cw^2sw^2-96sw^4)+co^4(32cw^4 + 48cw^2sw^2+16sw^4)+so^4(32cw^4+48cw^2sw^2+16sw^4) + ci^3(co\,so^3(-128cw^3sw-128cw\,sw^3)+co^3so(128cw^3sw+128cw\,sw^3))$$

$$
\begin{aligned}
&+ci(co^3so(-128cw^3sw-128cw\,sw^3)+co\,so^3(128cw^3sw+128cw\,sw^3))\\
&+ci^4(co^2so^2(-96cw^4-288cw^2sw^2-192sw^4)+co^4(16cw^4+48cw^2sw^2\\
&+32sw^4)+so^4(16cw^4+48cw^2sw^2+32sw^4))+ci^2(co^4(-48cw^4\\
&-96cw^2sw^2-48sw^4)+so^4(-48cw^4-96cw^2sw^2-48sw^4)\\
&+co^2so^2(288cw^4+576cw^2sw^2+288sw^4)))\cos M+(co^4(4cw^4-4sw^4)\\
&+so^4(4cw^4-4sw^4)+co^2so^2(-24cw^4+24sw^4)+ci^3(co\,so^3(-64\,cw^3sw\\
&-64\,cw\,sw^3)+co^3so(64\,cw^3sw+64\,cw\,sw^3))+ci(co^3so(-64\,cw^3sw\\
&-64\,cw\,sw^3)+co\,so^3(64\,cw^3sw+64\,cw\,sw^3))+ci^4(co^2so^2(24cw^4-24sw^4)\\
&+co^4(-4cw^4+4sw^4)+so^4(-4cw^4+4sw^4)))\cos 2M+e(co^4(24cw^4\\
&-24sw^4)+so^4(24cw^4-24sw^4)+co^2so^2(-144cw^4+144sw^4)\\
&+ci^3(co\,so^3(-384cw^3sw-384cw\,sw^3)+co^3so(384\,cw^3sw+384\,cw\,sw^3))\\
&+ci(co^3so(-384cw^3sw-384cw\,sw^3)+co\,so^3(384cw^3sw+384cw\,sw^3))\\
&+ci^4(co^2so^2(144cw^4-144sw^4)+co^4(-24cw^4+24sw^4)+so^4(-24cw^4\\
&+24sw^4)))\cos 3M+(co^2so^2(-6\,cw^4+36\,cw^2sw^2-6\,sw^4)+co^4(cw^4\\
&-6\,cw^2sw^2+sw^4)+so^4(cw^4-6\,cw^2sw^2+sw^4)+ci(co\,so^3(64cw^3sw\\
&-64cw\,sw^3)+co^3so(-64cw^3sw+64cw\,sw^3))+ci^3(co\,so^3(64cw^3sw\\
&-64cw\,sw^3)+co^3so(-64cw^3sw+64cw\,sw^3))+ci^4(co^2so^2(-6cw^4\\
&+36cw^2sw^2-6sw^4)+co^4(cw^4-6cw^2sw^2+sw^4)+so^4(cw^4-6cw^2sw^2\\
&+sw^4))+ci^2(co^2so^2(-36cw^4+216cw^2sw^2-36sw^4)+co^4(6cw^4\\
&-36cw^2sw^2+6sw^4)+so^4(6cw^4-36cw^2sw^2+6sw^4)))\cos 4M\\
&+e(co^2so^2(-48cw^4+288cw^2sw^2-48sw^4)+co^4(8cw^4-48cw^2sw^2+8sw^4)\\
&+so^4(8cw^4-48cw^2sw^2+8sw^4)+ci(co\,so^3(512cw^3sw-512cw\,sw^3)\\
&+co^3so(-512cw^3sw+512cw\,sw^3))+ci^3(co\,so^3(512cw^3sw-512cw\,sw^3)\\
&+co^3so(-512cw^3sw+512cw\,sw^3))+ci^4(co^2so^2(-48cw^4+288cw^2sw^2\\
&-48sw^4)+co^4(8cw^4-48cw^2sw^2+8sw^4)+so^4(8cw^4-48cw^2sw^2+8sw^4))\\
&+ci^2(co^2so^2(-288cw^4+1728cw^2sw^2-288sw^4)+co^4(48cw^4-288cw^2sw^2\\
&+48sw^4)+so^4(48cw^4-288cw^2sw^2+48sw^4)))\cos 5M+e(co^4(-16cw\,sw)\\
&+so^4(-16cw\,sw)+co^2so^2(96cw\,sw)+ci^4(co^2so^2(-96cw\,sw)\\
&+co^4(16cw\,sw)+so^4(16cw\,sw))+ci(co\,so^3(64cw^4-64sw^4)\\
&+co^3so(-64cw^4+64sw^4))+ci^3(co^3so(64cw^4-64sw^4)+co\,so^3(-64cw^4\\
&+64sw^4)))\sin M+(co^4(-8cw^3sw-8cw\,sw^3)+so^4(-8cw^3sw-8cw\,sw^3)\\
&+co^2so^2(48cw^3sw+48cw\,sw^3)+ci^4(co^2so^2(-48cw^3sw-48cw\,sw^3))
\end{aligned}
$$

$$+co^4(8cw^3sw+8cw\,sw^3)+so^4(8\,cw^3sw+8\,cw\,sw^3))+ci(co\,so^3(32cw^4$$
$$-32sw^4)+co^3so(-32cw^4+32sw^4))+ci^3(co^3so(32\,cw^4-32sw^4)$$
$$+co\,so^3(-32cw^4+32sw^4)))\sin 2M+e(co^4(-48cw^3sw-48cw\,sw^3)$$
$$+so^4(-48cw^3sw-48cw\,sw^3)+co^2so^2(288cw^3sw+288cw\,sw^3)$$
$$+ci^4(co^2so^2(-288cw^3sw-288cw\,sw^3)+co^4(48cw^3sw+48cw\,sw^3)$$
$$+so^4(48cw^3sw+48cw\,sw^3))+ci(co\,so^3(192cw^4-192sw^4)$$
$$+co^3so(-192cw^4+192sw^4))+ci^3(co^3so(192cw^4-192sw^4)$$
$$+co\,so^3(-192cw^4+192sw^4)))\sin 3M+(co^2so^2(24cw^3sw-24cw\,sw^3)$$
$$+co^4(-4cw^3sw+4cw\,sw^3)+so^4(-4cw^3sw+4cw\,sw^3)$$
$$+ci^4(co^2so^2(24cw^3sw-24cw\,sw^3)+co^4(-4cw^3sw+4cw\,sw^3)$$
$$+so^4(-4cw^3sw+4cw\,sw^3))+ci^2(co^2so^2(144cw^3sw-144cw\,sw^3)$$
$$+co^4(-24cw^3sw+24cw\,sw^3)+so^4(-24cw^3sw+24cw\,sw^3))$$
$$+ci(co^3so(-16cw^4+96cw^2sw^2-16sw^4)+co\,so^3(16cw^4$$
$$-96cw^2sw^2+16sw^4))+ci^3(co^3so(-16cw^4+96cw^2sw^2-16sw^4)$$
$$+co\,so^3(16cw^4-96cw^2sw^2+16sw^4)))\sin 4M+e(co^2so^2(192cw^3sw$$
$$-192cw\,sw^3)+co^4(-32cw^3sw+32cw\,sw^3)+so^4(-32cw^3sw+32cw\,sw^3)$$
$$+ci^4(co^2so^2(192cw^3sw-192cw\,sw^3)+co^4(-32cw^3sw+32cw\,sw^3)$$
$$+so^4(-32cw^3sw+32cw\,sw^3))+ci^2(co^2so^2(1152cw^3sw-1152cw\,sw^3)$$
$$+co^4(-192cw^3sw+192cw\,sw^3)+so^4(-192cw^3sw+192cw\,sw^3))$$
$$+ci(co^3so(-128cw^4+768cw^2sw^2-128sw^4)+co\,so^3(128cw^4$$
$$-768cw^2sw^2+128sw^4))+ci^3(co^3so(-128cw^4+768cw^2sw^2-128sw^4)$$
$$+co\,so^3(128cw^4-768cw^2sw^2+128sw^4)))\sin 5M)$$

$$\frac{\mathrm{d}e_1}{\mathrm{d}t}=\frac{105}{4}(ci(co^3so(32cw^2-32cw^4+(32-64cw^2)sw^2-32sw^4)$$
$$+co\ so^3(-32cw^2+32cw^4+(-32+64\ cw^2)sw^2+32sw^4))$$
$$+ci^3(co\ so^3(96cw^2-96cw^4+(96-192cw^2)sw^2-96sw^4)$$
$$+co^3so(-96cw^2+96cw^4+(-96+192cw^2)sw^2+96sw^4))$$
$$+e^2(co^2so^2(-36cw\ sw)+co^4(6cw\ sw)+so^4(6cw\ sw)+ci^4(co^4(-6cw\ sw)$$
$$+so^4(-6cw\ sw)+co^2so^2(36cw\ sw))+ci^2(co^4((36cw-36cw^3)sw$$
$$-36cw\ sw^3)+so^4((36cw-36cw^3)sw-36cw\ sw^3)+co^2so^2((-216cw$$
$$+216cw^3)sw+216cw\ sw^3))+ci(co^3so(184cw^2-160cw^4$$
$$+(136-320cw^2)sw^2-160sw^4)+co\ so^3(-184cw^2+160cw^4$$

$$+ (-136 + 320cw^2)sw^2 + 160sw^4)) + ci^3(co\ so^3(408cw^2 - 384cw^4$$
$$+ (552 - 960cw^2)sw^2 - 576sw^4) + co^3so(-408cw^2 + 384cw^4 + (-552$$
$$+ 960cw^2)sw^2 + 576sw^4))) + e(co^2so^2(-24cw\ sw) + co^4(4cw\ sw)$$
$$+ so^4(4cw\ sw) + ci^4(co^4(-4cw\ sw) + so^4(-4cw\ sw) + co^2so^2(24cw\ sw))$$
$$+ ci^2(co^4((24cw - 24cw^3)sw - 24cw\ sw^3) + so^4((24cw - 24cw^3)sw$$
$$- 24cw\ sw^3) + co^2so^2((-144cw + 144cw^3)sw + 144cw\ sw^3))$$
$$+ ci(co^3so(176cw^2 - 160cw^4 + (144 - 320cw^2)sw^2 - 160sw^4)$$
$$+ co\ so^3(-176cw^2 + 160cw^4 + (-144 + 320cw^2)sw^2 + 160sw^4))$$
$$+ ci^3(co\ so^3(432cw^2 - 416cw^4 + (528 - 960cw^2)sw^2 - 544sw^4)$$
$$+ co^3so(-432cw^2 + 416cw^4 + (-528 + 960cw^2)sw^2 + 544sw^4)))\cos M$$
$$+ (co^2so^2(-48cw\ sw) + co^4(8cw\ sw) + so^4(8cw\ sw) + ci(co^3so(32cw^2$$
$$- 32sw^2) + co\ so^3(-32cw^2 + 32sw^2)) + ci^4(co^4(-8cw\ sw)$$
$$+ so^4(-8cw\ sw) + co^2so^2(48cw\ sw)) + ci^2(co^4((48cw - 48cw^3)sw$$
$$- 48cw\ sw^3) + so^4((48cw - 48cw^3)sw - 48cw\ sw^3) + co^2so^2((-288cw$$
$$+ 288cw^3)sw + 288cw\ sw^3)) + ci^3(co\ so^3(-96cw^2 + 128cw^4 + 96sw^2$$
$$- 128sw^4) + co^3so(96cw^2 - 128cw^4 - 96sw^2 + 128sw^4))$$
$$+ e^2(co^2so^2(-72cw^3sw - 24cw\ sw^3) + co^4(12cw^3sw + 4cw\ sw^3)$$
$$+ so^4(12cw^3sw + 4cw\ sw^3) + ci^4(co^4(-4cw^3sw - 12cw\ sw^3)$$
$$+ so^4(-4cw^3sw - 12cw\ sw^3) + co^2so^2(24cw^3sw + 72cw\ sw^3))$$
$$+ ci^2(co^4((48cw - 24cw^3)sw - 72cw\ sw^3) + so^4((48cw - 24cw^3)sw$$
$$- 72cw\ sw^3) + co^2so^2((-288cw + 144cw^3)sw + 432cw\ sw^3))$$
$$+ ci(co^3so(352cw^2 - 304cw^4 + (288 - 736cw^2)sw^2 - 304sw^4)$$
$$+ co\ so^3(-352cw^2 + 304cw^4 + (-288 + 736cw^2)sw^2 + 304sw^4))$$
$$+ ci^3(co\ so^3(864cw^2 - 848cw^4 + (1056 - 1824cw^2)sw^2 - 1104sw^4)$$
$$+ co^3so(-864cw^2 + 848cw^4 + (-1056 + 1824cw^2)sw^2 + 1104sw^4))))$$
$$\times \cos 2M + e(co^2so^2(-144cw^3sw - 288cw\ sw^3) + co^4(24cw^3sw + 48cw\ sw^3)$$
$$+ so^4(24cw^3sw + 48cw\ sw^3) + ci^4(co^4(-48cw^3sw - 24cw\ sw^3)$$
$$+ so^4(-48cw^3sw - 24cw\ sw^3) + co^2so^2(288cw^3sw + 144cw\ sw^3))$$
$$+ ci^2(co^4((216cw - 288cw^3)sw - 144cw\ sw^3) + so^4((216cw$$
$$- 288cw^3)sw - 144cw\ sw^3) + co^2so^2((-1296cw + 1728cw^3)sw$$
$$+ 864cw\ sw^3)) + ci(co^3so(144cw^2 - 48cw^4 + (-144 + 288cw^2)sw^2$$
$$- 48sw^4) + co\ so^3(-144cw^2 + 48cw^4 + (144 - 288cw^2)sw^2 + 48sw^4))$$
$$+ ci^3(co\ so^3(-432cw^2 + 624cw^4 + (432 - 288cw^2)sw^2 - 528sw^4)$$

$$+ co^3so(432cw^2 - 624cw^4 + (-432 + 288cw^2)sw^2 + 528sw^4)))\cos 3M$$
$$+ (co^4(8cw^3sw - 8cw\ sw^3) + so^4(8cw^3sw - 8cw\ sw^3) + co^2so^2(-48cw^3sw$$
$$+ 48cw\ sw^3) + ci^4(co^4(8cw^3sw - 8cw\ sw^3) + so^4(8cw^3sw - 8cw\ sw^3)$$
$$+ co^2so^2(-48cw^3sw + 48cw\ sw^3)) + ci^2(co^4(48cw^3sw - 48cw\ sw^3)$$
$$+ so^4(48cw^3sw - 48cw\ sw^3) + co^2so^2(-288cw^3sw + 288cw\ sw^3))$$
$$+ ci(co\ so^3(-32cw^4 + 192cw^2sw^2 - 32sw^4) + co^3so(32cw^4 - 192cw^2sw^2$$
$$+ 32sw^4)) + ci^3(co\ so^3(-32cw^4 + 192cw^2sw^2 - 32sw^4) + co^3so(32cw^4$$
$$- 192cw^2sw^2 + 32sw^4)) + e^2(co^2so^2(-108cw^3sw - 1164cw\ sw^3)$$
$$+ co^4(18cw^3sw + 194cw\ sw^3) + so^4(18cw^3sw + 194cw\ sw^3)$$
$$+ ci^4(co^4(-194cw^3sw - 18cw\ sw^3) + so^4(-194cw^3sw - 18cw\ sw^3)$$
$$+ co^2so^2(1164cw^3sw + 108cw\ sw^3)) + ci^2(co^4((636cw - 1164cw^3)sw$$
$$- 108cw\ sw^3) + so^4((636cw - 1164cw^3)sw - 108cw\ sw^3)$$
$$+ co^2so^2((-3816cw + 6984cw^3)sw + 648cw\ sw^3)) + ci(co^3so(424cw^2$$
$$- 352cw^4 + (-424 + 2112cw^2)sw^2 - 352sw^4) + co\ so^3(-424cw^2$$
$$+ 352cw^4 + (424 - 2112cw^2)sw^2 + 352sw^4)) + ci^3(co\ so^3(-1272cw^2$$
$$+ 2048cw^4 + (1272 - 2112cw^2)sw^2 - 1344sw^4) + co^3so(1272cw^2$$
$$- 2048cw^4 + (-1272 + 2112cw^2)sw^2 + 1344sw^4))))\cos 4M$$
$$+ e(co^4(52cw^3sw - 52cw\ sw^3) + so^4(52cw^3sw - 52cw\ sw^3)$$
$$+ co^2so^2(-312cw^3sw + 312cw\ sw^3) + ci^4(co^4(52cw^3sw - 52cw\ sw^3)$$
$$+ so^4(52cw^3sw - 52cw\ sw^3) + co^2so^2(-312cw^3sw + 312cw\ sw^3))$$
$$+ ci^2(co^4(312cw^3sw - 312cw\ sw^3) + so^4(312cw^3sw - 312cw\ sw^3)$$
$$+ co^2so^2(-1872cw^3sw + 1872cw\ sw^3)) + ci(co\ so^3(-208cw^4$$
$$+ 1248cw^2sw^2 - 208sw^4) + co^3so(208cw^4 - 1248cw^2sw^2 + 208sw^4))$$
$$+ ci^3(co\ so^3(-208cw^4 + 1248cw^2sw^2 - 208sw^4) + co^3so(208cw^4$$
$$- 1248cw^2sw^2 + 208sw^4)))\cos 5M + e^2(co^4(204\ cw^3sw - 204cw\ sw^3)$$
$$+ so^4(204cw^3sw - 204cw\ sw^3) + co^2so^2(-1224cw^3sw + 1224cw\ sw^3)$$
$$+ ci^4(co^4(204cw^3sw - 204cw\ sw^3) + so^4(204cw^3sw - 204cw\ sw^3)$$
$$+ co^2so^2(-1224cw^3sw + 1224cw\ sw^3)) + ci^2(co^4(1224cw^3sw$$
$$- 1224cw\ sw^3) + so^4(1224cw^3sw - 1224cw\ sw^3) + co^2so^2(-7344cw^3sw$$
$$+ 7344cw\ sw^3)) + ci(co\ so^3(-816cw^4 + 4896cw^2sw^2 - 816sw^4)$$
$$+ co^3so(816cw^4 - 4896cw^2sw^2 + 816sw^4)) + ci^3(co\ so^3(-816cw^4$$
$$+ 4896cw^2sw^2 - 816sw^4) + co^3so(816cw^4 - 4896cw^2sw^2$$
$$+ 816sw^4)))\cos 6M + e(ci(-32co^3cw\ so\ sw + 32co\ cw\ so^3sw)$$

$$
\begin{aligned}
&+ co^4(2cw^4 - 2sw^4) + so^4(2cw^4 - 2sw^4) + co^2so^2(-12cw^4 + 12sw^4) \\
&+ ci^3(co\ so^3((96cw - 128cw^3)sw - 128cw\ sw^3) + co^3so((-96cw \\
&+ 128cw^3)sw + 128cw\ sw^3)) + ci^4(co^2so^2(12cw^4 - 12sw^4) + co^4(-2cw^4 \\
&+ 2sw^4) + so^4(-2cw^4 + 2sw^4)) + ci^2(co^2so^2(-72cw^2 + 72cw^4 + 72sw^2 \\
&- 72sw^4) + co^4(12cw^2 - 12cw^4 - 12sw^2 + 12sw^4) + so^4(12cw^2 - 12cw^4 \\
&- 12sw^2 + 12sw^4)))\sin M + (ci(-64co^3cw\ so\ sw + 64co\ cw\ so^3sw) \\
&+ co^4(4cw^4 - 4sw^4) + so^4(4cw^4 - 4sw^4) + co^2so^2(-24cw^4 + 24sw^4) \\
&+ ci^3(co\ so^3((192cw - 256cw^3)sw - 256cw\ sw^3) + co^3so((-192cw \\
&+ 256cw^3)sw + 256cw\ sw^3)) + ci^4(co^2so^2(24cw^4 - 24sw^4) + co^4(-4cw^4 \\
&+ 4sw^4) + so^4(-4cw^4 + 4sw^4)) + ci^2(co^2so^2(-144cw^2 + 144cw^4 \\
&+ 144sw^2 - 144sw^4) + co^4(24cw^2 - 24cw^4 - 24sw^2 + 24sw^4) \\
&+ so^4(24cw^2 - 24cw^4 - 24sw^2 + 24sw^4)) + e^2(co^4(5cw^4 - 6cw^2sw^2 \\
&- 3sw^4) + so^4(5cw^4 - 6cw^2sw^2 - 3sw^4) + co^2so^2(-30cw^4 + 36cw^2sw^2 \\
&+ 18sw^4) + ci(co\ so^3((64cw + 64cw^3)sw - 64cw\ sw^3) + co^3so((-64cw \\
&- 64cw^3)sw + 64cw\ sw^3)) + ci^3(co\ so^3((192cw - 192cw^3)sw \\
&- 320cw\ sw^3) + co^3so((-192cw + 192cw^3)sw + 320cw\ sw^3)) \\
&+ ci^4(co^2so^2(18cw^4 + 36cw^2sw^2 - 30sw^4) + co^4(-3cw^4 - 6cw^2sw^2 \\
&+ 5sw^4) + so^4(-3cw^4 - 6cw^2sw^2 + 5sw^4)) + ci^2(co^2so^2(-144cw^2 \\
&+ 108cw^4 + (144 + 216cw^2)sw^2 - 180sw^4) + co^4(24cw^2 - 18cw^4 \\
&+ (-24 - 36cw^2)sw^2 + 30sw^4) + so^4(24cw^2 - 18cw^4 \\
&+ (-24 - 36cw^2)sw^2 + 30sw^4))))\sin 2M + e(co^4(15cw^4 + 18cw^2sw^2 \\
&- 21sw^4) + so^4(15cw^4 + 18cw^2sw^2 - 21sw^4) + co^2so^2(-90cw^4 \\
&- 108cw^2sw^2 + 126sw^4) + ci(co^3so((-288cw + 192cw^3)sw - 192cw\ sw^3) \\
&+ co\ so^3((288cw - 192cw^3)sw + 192cw\ sw^3)) + ci^3(co\ so^3((864cw \\
&- 1344cw^3)sw - 960cw\ sw^3) + co^3so((-864cw + 1344cw^3)sw \\
&+ 960cw\ sw^3)) + ci^4(co^2so^2(126cw^4 - 108cw^2sw^2 - 90sw^4) \\
&+ co^4(-21cw^4 + 18cw^2sw^2 + 15sw^4) + so^4(-21cw^4 + 18cw^2sw^2 \\
&+ 15sw^4)) + ci^2(co^2so^2(-648cw^2 + 756cw^4 + (648 - 648cw^2)sw^2 \\
&- 540sw^4) + co^4(108cw^2 - 126cw^4 + (-108 + 108cw^2)sw^2 + 90sw^4) \\
&+ so^4(108cw^2 - 126cw^4 + (-108 + 108cw^2)sw^2 + 90sw^4)))\sin 3M \\
&+ (co^2so^2(-12cw^4 + 72cw^2sw^2 - 12sw^4) + co^4(2cw^4 - 12cw^2sw^2 \\
&+ 2sw^4) + so^4(2cw^4 - 12cw^2sw^2 + 2sw^4) + ci(co\ so^3(128cw^3sw \\
&- 128cw\ sw^3) + co^3so(-128cw^3sw + 128cw\ sw^3)) + ci^3(co\ so^3(128cw^3sw
\end{aligned}
$$

$$
\begin{aligned}
&- 128cw\,sw^3) + co^3so(-128cw^3sw + 128cw\,sw^3)) + ci^4(co^2so^2(-12cw^4 \\
&+ 72cw^2sw^2 - 12sw^4) + co^4(2cw^4 - 12cw^2sw^2 + 2sw^4) + so^4(2cw^4 \\
&- 12cw^2sw^2 + 2sw^4)) + ci^2(co^2so^2(-72cw^4 + 432cw^2sw^2 - 72sw^4) \\
&+ co^4(12cw^4 - 72cw^2sw^2 + 12sw^4) + so^4(12cw^4 - 72cw^2sw^2 + 12sw^4)) \\
&+ e^2(co^4(31cw^4 + 132cw^2sw^2 - 75sw^4) + so^4(31cw^4 + 132cw^2sw^2 \\
&- 75sw^4) + co^2so^2(-186cw^4 - 792cw^2sw^2 + 450sw^4) \\
&+ ci(co^3so((-848cw + 1408cw^3)sw - 1408cw\,sw^3) + co\,so^3((848cw \\
&- 1408cw^3)sw + 1408cw\,sw^3)) + ci^3(co\,so^3((2544cw - 4800cw^3)sw \\
&- 1984cw\,sw^3) + co^3so((-2544cw + 4800cw^3)sw + 1984cw\,sw^3)) \\
&+ ci^4(co^2so^2(450cw^4 - 792cw^2sw^2 - 186sw^4) + co^4(-75cw^4 \\
&+ 132cw^2sw^2 + 31sw^4) + so^4(-75cw^4 + 132cw^2sw^2 + 31sw^4)) \\
&+ ci^2(co^2so^2(-1908cw^2 + 2700cw^4 + (1908 - 4752cw^2)sw^2 - 1116sw^4) \\
&+ co^4(318cw^2 - 450cw^4 + (-318 + 792cw^2)sw^2 + 186sw^4) + so^4(318cw^2 \\
&- 450cw^4 + (-318 + 792cw^2)sw^2 + 186sw^4))))\sin 4M + e(co^2so^2(-78cw^4 \\
&+ 468cw^2sw^2 - 78sw^4) + co^4(13cw^4 - 78cw^2sw^2 + 13sw^4) + so^4(13cw^4 \\
&- 78cw^2sw^2 + 13sw^4) + ci(co\,so^3(832cw^3sw - 832cw\,sw^3) \\
&+ co^3so(-832cw^3sw + 832cw\,sw^3)) + ci^3(co\,so^3(832cw^3sw \\
&- 832cw\,sw^3) + co^3so(-832cw^3sw + 832cw\,sw^3)) + ci^4(co^2so^2(-78cw^4 \\
&+ 468cw^2sw^2 - 78sw^4) + co^4(13cw^4 - 78cw^2sw^2 + 13sw^4) + so^4(13cw^4 \\
&- 78cw^2sw^2 + 13sw^4)) + ci^2(co^2so^2(-468cw^4 + 2808cw^2sw^2 - 468sw^4) \\
&+ co^4(78cw^4 - 468cw^2sw^2 + 78sw^4) + so^4(78cw^4 - 468cw^2sw^2 \\
&+ 78sw^4)))\sin 5M + e^2(co^2so^2(-306cw^4 + 1836cw^2sw^2 - 306sw^4) \\
&+ co^4(51cw^4 - 306cw^2sw^2 + 51sw^4) + so^4(51cw^4 - 306cw^2sw^2 + 51sw^4) \\
&+ ci(co\,so^3(3264cw^3sw - 3264cw\,sw^3) + co^3so(-3264cw^3sw \\
&+ 3264cw\,sw^3)) + ci^3(co\,so^3(3264cw^3sw - 3264cw\,sw^3) \\
&+ co^3so(-3264cw^3sw + 3264cw\,sw^3)) + ci^4(co^2so^2(-306cw^4 \\
&+ 1836cw^2sw^2 - 306sw^4) + co^4(51cw^4 - 306cw^2sw^2 + 51sw^4) \\
&+ so^4(51cw^4 - 306cw^2sw^2 + 51sw^4)) + ci^2(co^2so^2(-1836cw^4 \\
&+ 11\,016cw^2sw^2 - 1836sw^4) + co^4(306cw^4 - 1836cw^2sw^2 + 306sw^4) \\
&+ so^4(306cw^4 - 1836cw^2sw^2 + 306sw^4)))\sin 6M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\omega_1}{\mathrm{d}t} = & \frac{525}{128}(e(co^2so^2(-960cw^4 - 1152cw^2sw^2 - 192sw^4) + co^4(160cw^4 \\
& + 192cw^2sw^2 + 32sw^4) + so^4(160cw^4 192cw^2sw^2 + 32sw^4) \\
& + ci^3(co\,so^3(-1024cw\,sw) + co^3so(1024cw\,sw)) + ci(co^3so(-1024cw\,sw) \\
& + co\,so^3(1024cw\,sw)) + ci^4(co^2so^2(-192cw^4 - 1152cw^2sw^2 - 960sw^4) \\
& + co^4(32cw^4 + 192cw^2sw^2 + 160sw^4) + so^4(32cw^4 + 192cw^2sw^2 \\
& + 160sw^4)) + ci^2(co^4(-192cw^4 - 384cw^2sw^2 - 192sw^4) + so^4(-192cw^4 \\
& - 384cw^2sw^2 - 192sw^4) + co^2so^2(1152cw^4 + 2304cw^2sw^2 + 1152sw^4))) \\
& + (co^2so^2(-480cw^4 - 576cw^2sw^2 - 96sw^4) + co^4(80cw^4 + 96cw^2sw^2 \\
& + 16sw^4) + so^4(80cw^4 + 96cw^2sw^2 + 16sw^4) + ci^3(co\,so^3(-512cw\,sw) \\
& + co^3so(512cw\,sw)) + ci(co^3so(-512cw\,sw) + co\,so^3(512cw\,sw)) \\
& + ci^4(co^2so^2(-96cw^4 - 576cw^2sw^2 - 480sw^4) + co^4(16cw^4 + 96cw^2sw^2 \\
& + 80sw^4) + so^4(16cw^4 + 96cw^2sw^2 + 80sw^4)) + ci^2(co^4(-96cw^4 \\
& - 192cw^2sw^2 - 96sw^4) + so^4(-96cw^4 - 192cw^2sw^2 - 96sw^4) \\
& + co^2so^2(576cw^4 + 1152cw^2sw^2 + 576sw^4)) + e^2(co^2so^2(-4950cw^4 \\
& - 5724cw^2sw^2 - 918sw^4) + co^4(825cw^4 + 954cw^2sw^2 + 153sw^4) \\
& + so^4(825cw^4 + 954cw^2sw^2 + 153sw^4) + ci(co^3so(-5568cw^3sw \\
& - 5184cw\,sw^3) + co\,so^3(5568cw^3sw + 5184cw\,sw^3)) \\
& + ci^3(co\,so^3(-5184cw^3sw - 5568cw\,sw^3) + co^3so(5184cw^3sw \\
& + 5568cw\,sw^3)) + ci^4(co^2so^2(-918cw^4 - 5724cw^2sw^2 - 4950sw^4) \\
& + co^4(153cw^4 + 954cw^2sw^2 + 825sw^4) + so^4(153cw^4 + 954cw^2sw^2 \\
& + 825sw^4)) + ci^2(co^4(-954cw^4 - 2052cw^2sw^2 - 954sw^4) + so^4(-954cw^4 \\
& - 2052cw^2sw^2 - 954sw^4) + co^2so^2(5724cw^4 + 12\,312cw^2sw^2 \\
& + 5724sw^4))))\cos M + e(co^2so^2(-1920cw^4 - 2304cw^2sw^2 - 384sw^4) \\
& + co^4(320cw^4 + 384cw^2sw^2 + 64sw^4) + so^4(320cw^4 + 384cw^2sw^2 \\
& + 64sw^4) + ci^3(co\,so^3(-2048cw\,sw) + co^3so(2048cw\,sw)) \\
& + ci(co^3so(-2048cw\,sw) + co\,so^3(2048cw\,sw)) + ci^4(co^2so^2(-384cw^4 \\
& - 2304cw^2sw^2 - 1920sw^4) + co^4(64cw^4 + 384cw^2sw^2 + 320sw^4) \\
& + so^4(64cw^4 + 384cw^2sw^2 + 320sw^4)) + + ci^2(co^4(-384cw^4 - 768cw^2sw^2 \\
& - 384sw^4) + so^4(-384cw^4 - 768cw^2sw^2 - 384sw^4) + co^2so^2(2304cw^4 \\
& + 4608cw^2sw^2 + 2304sw^4)))\cos 2M + (ci(-1024co^3cw^3so\,sw \\
& + 1024co\,cw^3so^3sw) + ci^3(1024co^3cw\,so\,sw^3 - 1024co\,cw\,so^3sw^3) \\
& + co^4(40cw^4 - 48cw^2sw^2 - 24sw^4) + so^4(40cw^4 - 48cw^2sw^2 - 24sw^4)
\end{aligned}
$$

$$
\begin{aligned}
&+co^2so^2(-240cw^4+288cw^2sw^2+144sw^4)+ci^4(co^2so^2(144cw^4\\
&+288cw^2sw^2-240sw^4)+co^4(-24cw^4-48cw^2sw^2+40sw^4)\\
&+so^4(-24cw^4-48cw^2sw^2+40sw^4))\,+\,ci^2(co^2so^2(-288cw^4\\
&+1728cw^2sw^2-288sw^4)+co^4(48cw^4-288cw^2sw^2+48sw^4)\\
&+so^4(48cw^4-288cw^2sw^2+48sw^4))+e^2(co^2so^2(-4914cw^4\\
&-6372cw^2sw^2-1314sw^4)+co^4(819cw^4+1062cw^2sw^2+219sw^4)\\
&+so^4(819cw^4+1062cw^2sw^2+219sw^4)+ci^3(co\ so^3(-4992cw^3sw\\
&-4608cw\ sw^3)+co^3so(4992cw^3sw+4608cw\ sw^3))\\
&+ci(co^3so(-4608cw^3sw-4992cw\ sw^3)+co\ so^3(4608cw^3sw\\
&+4992cw\ sw^3))+ci^4(co^2so^2(-1314cw^4-6372cw^2sw^2-4914sw^4)\\
&+co^4(219cw^4+1062cw^2sw^2+819sw^4)+so^4(219cw^4+1062cw^2sw^2\\
&+819sw^4))+ci^2(co^4(-1062cw^4-1980cw^2sw^2-1062sw^4)\\
&+so^4(-1062cw^4-1980cw^2sw^2-1062sw^4)+co^2so^2(6372cw^4\\
&+11\,880cw^2sw^2+6372sw^4))))\cos 3M+e(co^4(224cw^4-192cw^2sw^2\\
&-160sw^4)+so^4(224cw^4-192cw^2sw^2-160sw^4)+co^2so^2(-1344cw^4\\
&+1152cw^2sw^2+960sw^4)+ci(co^3so(-5120cw^3sw-1024cw\ sw^3)\\
&+co\ so^3(5120cw^3sw+1024cw\ sw^3))+ci^3(co\ so^3(-1024cw^3sw\\
&-5120cw\ sw^3)+co^3so(1024cw^3sw+5120cw\ sw^3))\\
&+ci^2(co^2so^2(-1152cw^4+6912cw^2sw^2-1152sw^4)+co^4(192cw^4\\
&-1152cw^2sw^2+192sw^4)+so^4(192cw^4-1152cw^2sw^2+192sw^4))\\
&+ci^4(co^2so^2(960cw^4+1152cw^2sw^2-1344sw^4)+co^4(-160cw^4\\
&-192cw^2sw^2+224sw^4)+so^4(-160cw^4-192cw^2sw^2+224sw^4)))\cos 4M\\
&+(co^2so^2(-48cw^4+288cw^2sw^2-48sw^4)+co^4(8cw^4-48cw^2sw^2\\
&+8sw^4)+so^4(8cw^4-48cw^2sw^2+8sw^4)+ci(co\ so^3(512cw^3sw\\
&-512cw\ sw^3)+co^3so(-512cw^3sw+512cw\ sw^3))\\
&+ci^3(co\ so^3(512cw^3sw-512cw\ sw^3)+co^3so(-512cw^3sw+512cw\ sw^3))\\
&+ci^4(co^2so^2(-48cw^4+288cw^2sw^2-48sw^4)+co^4(8cw^4-48cw^2sw^2\\
&+8sw^4)+so^4(8cw^4-48cw^2sw^2+8sw^4))+ci^2(co^2so^2(-288cw^4\\
&+1728cw^2sw^2-288sw^4)+co^4(48cw^4-288cw^2sw^2+48sw^4)\\
&+so^4(48cw^4-288cw^2sw^2+48sw^4))+e^2(co^4(745cw^4-222cw^2sw^2\\
&-671sw^4)+so^4(745cw^4-222cw^2sw^2-671sw^4)+co^2so^2(-4470cw^4\\
&+1332cw^2sw^2+4026sw^4)+ci(co^3so(-13\,696cw^3sw-8960cw\ sw^3)
\end{aligned}
$$

$$
\begin{aligned}
&+ co\ so^3(13\,696cw^3sw + 8960cw\ sw^3)) + ci^3(co\ so^3(-8960cw^3sw \\
&- 13\,696cw\ sw^3) + co^3so(8960cw^3sw + 13\,696cw\ sw^3)) \\
&+ ci^2(co^2so^2(-1332cw^4 + 7992cw^2sw^2 - 1332sw^4) + co^4(222cw^4 \\
&- 1332cw^2sw^2 + 222sw^4) + so^4(222cw^4 - 1332cw^2sw^2 + 222sw^4)) \\
&+ ci^4(co^2so^2(4026cw^4 + 1332cw^2sw^2 - 4470sw^4) + co^4(-671cw^4 \\
&- 222cw^2sw^2 + 745sw^4) + so^4(-671cw^4 - 222cw^2sw^2 + 745sw^4)))) \\
&\times \cos 5M + e(co^2so^2(-384cw^4 + 2304cw^2sw^2 - 384sw^4) + co^4(64cw^4 \\
&- 384cw^2sw^2 + 64sw^4) + so^4(64cw^4 - 384cw^2sw^2 + 64sw^4) \\
&+ ci(co\ so^3(4096cw^3sw - 4096cw\ sw^3) + co^3so(-4096cw^3sw \\
&+ 4096cw\ sw^3)) + ci^3(co\ so^3(4096cw^3sw - 4096cw\ sw^3) \\
&+ co^3so(-4096cw^3sw + 4096cw\ sw^3)) + ci^4(co^2so^2(-384cw^4 \\
&+ 2304cw^2sw^2 - 384sw^4) + co^4(64cw^4 - 384cw^2sw^2 + 64sw^4) \\
&+ so^4(64cw^4 - 384cw^2sw^2 + 64sw^4)) + ci^2(co^2so^2(-2304cw^4 \\
&+ 13\,824cw^2sw^2 - 2304sw^4) + co^4(384cw^4 - 2304cw^2sw^2 + 384sw^4) \\
&+ so^4(384cw^4 - 2304cw^2sw^2 + 384sw^4)))\cos 6M + e^2(co^2so^2(-1794cw^4 \\
&+ 10764cw^2sw^2 - 1794sw^4) + co^4(299cw^4 - 1794cw^2sw^2 + 299sw^4) \\
&+ so^4(299cw^4 - 1794cw^2sw^2 + 299sw^4) + ci(co\ so^3(19136cw^3sw \\
&- 19136cw\ sw^3) + co^3so(-19136cw^3sw + 19136cw\ sw^3)) \\
&+ ci^3(co\ so^3(19136\ cw^3sw - 19136cw\ sw^3) + co^3so(-19136\ cw^3sw \\
&+ 19136cw\ sw^3)) + ci^4(co^2so^2(-1794cw^4 + 10764cw^2sw^2 - 1794sw^4) \\
&+ co^4(299cw^4 - 1794cw^2sw^2 + 299sw^4) + so^4(299cw^4 - 1794cw^2sw^2 \\
&+ 299sw^4)) + ci^2(co^2so^2(-10764cw^4 + 64584cw^2sw^2 - 10764sw^4) \\
&+ co^4(1794cw^4 - 10764cw^2sw^2 + 1794sw^4) + so^4(1794cw^4 \\
&- 10764cw^2sw^2 + 1794sw^4)))\cos 7M + (co^4(-64cw\ sw) + so^4(-64cw\ sw) \\
&+ co^2so^2(384cw\ sw) + ci^4(co^2so^2(-384cw\ sw) + co^4(64cw\ sw) \\
&+ so^4(64cw\ sw)) + ci(co\ so^3(256cw^4 - 256sw^4) + co^3so(-256cw^4 \\
&+ 256sw^4)) + ci^3(co^3so(256cw^4 - 256sw^4) + co\ so^3(-256cw^4 + 256sw^4)) \\
&+ e^2(co^4(-220cw^3sw - 196cw\ sw^3) + so^4(-220cw^3sw - 196cw\ sw^3) \\
&+ co^2so^2(1320cw^3sw + 1176cw\ sw^3) + ci^2(co^2so^2(432cw^3sw \\
&- 432cw\ sw^3) + co^4(-72cw^3sw + 72cw\ sw^3) + so^4(-72cw^3sw \\
&+ 72cw\ sw^3)) + ci^4(co^2so^2(-1176cw^3sw - 1320cw\ sw^3) + co^4(196cw^3sw \\
&+ 220cw\ sw^3) + so^4(196cw^3sw + 220cw\ sw^3)) + ci(co\ so^3(880cw^4
\end{aligned}
$$

$$- 288cw^2sw^2 - 784sw^4) + co^3so(-880cw^4 + 288cw^2sw^2 + 784sw^4))$$
$$+ ci^3(co^3so(784cw^4 + 288cw^2sw^2 - 880sw^4) + co\ so^3(-784cw^4$$
$$- 288cw^2sw^2 + 880sw^4))))\sin M + e(co^4(-256cw\ sw) + so^4(-256cw\ sw)$$
$$+ co^2so^2(1536cw\ sw) + ci^4(co^2so^2(-1536cw\ sw) + co^4(256cw\ sw)$$
$$+ so^4(256cw\ sw)) + ci(co\ so^3(1024cw^4 - 1024sw^4) + co^3so(-1024cw^4$$
$$+ 1024sw^4)) + ci^3(co^3so(1024cw^4 - 1024sw^4) + co\ so^3(-1024cw^4$$
$$+ 1024sw^4)))\sin 2M + (co^4(-96cw^3sw - 32cw\ sw^3) + so^4(-96cw^3sw$$
$$- 32cw\ sw^3) + co^2so^2(576cw^3sw + 192cw\ sw^3) + ci^4(co^2so^2(-192cw^3sw$$
$$- 576cw\ sw^3) + co^4(32cw^3sw + 96cw\ sw^3) + so^4(32cw^3sw + 96cw\ sw^3))$$
$$+ ci^2(co^2so^2 \times (1152cw^3sw - 1152cw\ sw^3) + co^4(-192cw^3sw$$
$$+ 192cw\ sw^3) + so^4(-192cw^3sw + 192cw\ sw^3)) + ci(co\ so^3(384cw^4$$
$$- 768cw^2sw^2 - 128sw^4) + co^3so(-384cw^4 + 768cw^2sw^2 + 128sw^4))$$
$$+ ci^3(co^3so(128cw^4 + 768cw^2sw^2 - 384sw^4) + co\ so^3(-128cw^4$$
$$- 768cw^2sw^2 + 384sw^4)) + e^2(co^4(-588cw^3sw - 612cw\ sw^3)$$
$$+ so^4(-588cw^3sw - 612cw\ sw^3) + co^2so^2(3528cw^3sw + 3672cw\ sw^3)$$
$$+ ci^2(co^4(72cw^3sw - 72cw\ sw^3) + so^4(72cw^3sw - 72cw\ sw^3)$$
$$+ co^2so^2(-432cw^3sw + 432cw\ sw^3)) + ci^4(co^2so^2(-3672cw^3sw$$
$$- 3528cw\ sw^3) + co^4(612cw^3sw + 588cw\ sw^3) + so^4(612cw^3sw$$
$$+ 588cw\ sw^3)) + ci^3(co^3so(2448cw^4 - 288cw^2sw^2 - 2352sw^4)$$
$$+ co\ so^3(-2448cw^4 + 288cw^2sw^2 + 2352sw^4)) + ci(co\ so^3(2352cw^4$$
$$+ 288cw^2sw^2 - 2448sw^4) + co^3so(-2352cw^4 - 288cw^2sw^2$$
$$+ 2448sw^4))))\sin 3M + e(co^4(-512cw^3sw - 256cw\ sw^3)$$
$$+ so^4(-512cw^3sw - 256cw\ sw^3) + co^2so^2(3072cw^3sw + 1536cw\ sw^3)$$
$$+ ci^4(co^2so^2(-1536cw^3sw - 3072cw\ sw^3) + co^4(256cw^3sw + 512cw\ sw^3)$$
$$+ so^4(256cw^3sw + 512cw\ sw^3)) + ci^2(co^2so^2(4608cw^3sw - 4608cw\ sw^3)$$
$$+ co^4(-768cw^3sw + 768cw\ sw^3) + so^4(-768cw^3sw + 768cw\ sw^3))$$
$$+ ci(co\ so^3(2048cw^4 - 3072cw^2sw^2 - 1024sw^4) + co^3so(-2048cw^4$$
$$+ 3072cw^2sw^2 + 1024sw^4)) + ci^3(co^3so(1024cw^4 + 3072cw^2sw^2$$
$$- 2048sw^4) + co\ so^3(-1024cw^4 - 3072cw^2sw^2 + 2048sw^4)))\sin 4M$$
$$+ (co^2so^2(192cw^3sw - 192cw\ sw^3) + co^4(-32cw^3sw + 32cw\ sw^3)$$
$$+ so^4(-32cw^3sw + 32cw\ sw^3) + ci^4(co^2so^2(192cw^3sw - 192cw\ sw^3)$$
$$+ co^4(-32cw^3sw + 32cw\ sw^3) + so^4(-32cw^3sw + 32cw\ sw^3))$$
$$+ ci^2(co^2so^2(1152cw^3sw - 1152cw\ sw^3) + co^4(-192cw^3sw$$

$$
\begin{aligned}
&+192cw\,sw^3)+so^4(-192cw^3sw+192cw\,sw^3))\\
&+ci(co^3so(-128cw^4+768cw^2sw^2-128sw^4)+co\,so^3(128cw^4\\
&-768cw^2sw^2+128sw^4))+ci^3(co^3so(-128cw^4+768cw^2sw^2-128sw^4)\\
&+co\,so^3(128cw^4-768cw^2sw^2+128sw^4))+e^2(co^4(-1564cw^3sw\\
&-1268cw\,sw^3)+so^4(-1564cw^3sw-1268cw\,sw^3)\\
&+co^2so^2(9384cw^3sw7608cw\,sw^3)+ci^2(co^2so^2(5328cw^3sw\\
&-5328cw\,sw^3)+co^4(-888cw^3sw+888cw\,sw^3)+so^4(-888\,cw^3sw\\
&+888cw\,sw^3))+ci^4(co^2so^2(-7608cw^3sw-9384cw\,sw^3)\\
&+co^4(1268cw^3sw+1564cw\,sw^3)+so^4(1268cw^3sw+1564cw\,sw^3))\\
&+ci(co\,so^3(6256cw^4-3552cw^2sw^2-5072sw^4)+co^3so(-6256cw^4\\
&+3552cw^2sw^2+5072sw^4))+ci^3(co^3so(5072cw^4+3552cw^2sw^2\\
&-6256sw^4)+co\,so^3(-5072cw^4-3552cw^2sw^2+6256sw^4))))\sin 5M\\
&+e(co^2so^2(1536cw^3sw-1536cw\,sw^3)+co^4(-256cw^3sw+256cw\,sw^3)\\
&+so^4(-256cw^3sw+256cw\,sw^3)+ci^4(co^2so^2(1536cw^3sw-1536cw\,sw^3)\\
&+co^4(-256cw^3sw+256cw\,sw^3)+so^4(-256cw^3sw+256cw\,sw^3))\\
&+ci^2(co^2so^2(9216cw^3sw-9216cw\,sw^3)+co^4(-1536cw^3sw\\
&+1536cw\,sw^3)+so^4(-1536cw^3sw+1536cw\,sw^3))+ci(co^3so(-1024cw^4\\
&+6144cw^2sw^2-1024sw^4)+co\,so^3(1024cw^4-6144cw^2sw^2+1024sw^4))\\
&+ci^3(co^3so(-1024cw^4+6144cw^2sw^2-1024sw^4)+co\,so^3(1024cw^4\\
&-6144cw^2sw^2+1024sw^4)))\sin 6M+e^2(co^2so^2(7176cw^3sw\\
&-7176cw\,sw^3)+co^4(-1196cw^3sw+1196cw\,sw^3)+so^4(-1196cw^3sw\\
&+1196cw\,sw^3)+ci^4(co^2so^2(7176cw^3sw-7176cw\,sw^3)\\
&+co^4(-1196cw^3sw+1196cw\,sw^3)+so^4(-1196cw^3sw+1196cw\,sw^3))\\
&+ci^2(co^2so^2(43056cw^3sw-43056cw\,sw^3)+co^4(-7176cw^3sw\\
&+7176cw\,sw^3)+so^4(-7176cw^3sw+7176cw\,sw^3))+ci(co^3so(-4784cw^4\\
&+28704cw^2sw^2-4784sw^4)+co\,so^3(4784cw^4-28704cw^2sw^2\\
&+4784sw^4))+ci^3(co^3so(-4784cw^4+28\,704cw^2sw^2-4784sw^4)\\
&+co\,so^3(4784cw^4-28704cw^2sw^2+4784sw^4)))\sin 7M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}i_1}{\mathrm{d}t} = & -105(co\ so^3(-6) + co^3so(6) + ci^4(co\ so^3(-6) + co^3so(6)) \\
& + ci^2(co^3so(-12) + co\ so^3(12)) + e^2(co\ so^3(-36cw^4 - 60cw^2sw^2 \\
& - 24sw^4) + co^3so(36cw^4 + 60cw^2sw^2 + 24sw^4) + ci(co^2so^2(-36cw\ sw) \\
& + co^4(6cw\ sw) + so^4(6cw\ sw)) + ci^3(co^4(-6cw\ sw) + so^4(-6cw\ sw) + co^2so^2 \\
& \times (36cw\ sw)) + ci^4(co\ so^3(-24cw^4 - 60cw^2sw^2 - 36sw^4) + co^3so(24cw^4 \\
& + 60cw^2sw^2 + 36sw^4)) + ci^2(co^3so(-60cw^4 - 120cw^2sw^2 - 60sw^4) \\
& + co\ so^3(60cw^4 + 120cw^2sw^2 + 60sw^4))) + e(co\ so^3(-34cw^4 - 60cw^2sw^2 \\
& - 26sw^4) + co^3so(34cw^4 + 60cw^2sw^2 + 26sw^4) + ci(co^2so^2(-24cw\ sw) \\
& + co^4(4cw\ sw) + so^4(4cw\ sw)) + ci^3(co^4(-4cw\ sw) + so^4(-4cw\ sw) \\
& + co^2so^2(24cw\ sw)) + ci^4(co\ so^3(-26cw^4 - 60cw^2sw^2 - 34sw^4) \\
& + co^3so(26cw^4 + 60cw^2sw^2 + 34sw^4)) + ci^2(co^3so(-60) + co\ so^3(60)))\cos M \\
& + (co^3so(8cw^4 - 8sw^4) + co\ so^3(-8cw^4 + 8sw^4) + ci(co^2so^2(-48cw\ sw) \\
& + co^4(8cw\ sw) + so^4(8cw\ sw)) + ci^3(co^4(-8cw\ sw) + so^4(-8cw\ sw) \\
& + co^2so^2(48cw\ sw)) + ci^4(co\ so^3(8cw^4 - 8sw^4) + co^3so(-8cw^4 + 8sw^4)) \\
& + e^2(co\ so^3(-69cw^4 - 114cw^2sw^2 - 53sw^4) + co^3so(69cw^4 + 114cw^2sw^2 \\
& + 53sw^4) + ci(co^2so^2(-72cw^3sw - 24cw\ sw^3) + co^4(12cw^3sw + 4cw\ sw^3) \\
& + so^4(12cw^3sw + 4cw\ sw^3)) + ci^3(co^4(-4cw^3sw - 12cw\ sw^3) \\
& + so^4(-4cw^3sw - 12cw\ sw^3) + co^2so^2(24cw^3sw + 72cw\ sw^3)) \\
& + ci^4(co\ so^3(-53cw^4 - 114cw^2sw^2 - 69sw^4) + co^3so(53cw^4 + 114cw^2sw^2 \\
& + 69sw^4)) + ci^2(co^3so(-114cw^4 - 276cw^2sw^2 - 114sw^4) + co\ so^3(114cw^4 \\
& + 276cw^2sw^2 + 114sw^4))))\cos 2M + e(co^3so(33cw^4 + 18cw^2sw^2 - 39sw^4) \\
& + co\ so^3(-33cw^4 - 18cw^2sw^2 + 39sw^4) + ci(co^2so^2(-144cw^3sw \\
& - 288cw\ sw^3) + co^4(24cw^3sw + 48cw\ sw^3) + so^4(24cw^3sw + 48cw\ sw^3)) \\
& + ci^3(co^4(-48cw^3sw - 24cw\ sw^3) + so^4(-48cw^3sw - 24cw\ sw^3) \\
& + co^2so^2(288cw^3sw + 144cw\ sw^3)) + ci^2(co^3so(-18cw^4 + 108cw^2sw^2 \\
& - 18sw^4) + co\ so^3(18cw^4 - 108cw^2sw^2 + 18sw^4)) + ci^4(co\ so^3(39cw^4 \\
& - 18cw^2sw^2 - 33sw^4) + co^3so(-39cw^4 + 18cw^2sw^2 + 33sw^4)))\cos 3M \\
& + (co\ so^3(-2cw^4 + 12cw^2sw^2 - 2sw^4) + co^3so(2cw^4 - 12cw^2sw^2 + 2sw^4) \\
& + ci(co^4(8cw^3sw - 8cw\ sw^3) + so^4(8cw^3sw - 8cw\ sw^3) \\
& + co^2so^2(-48cw^3sw + 48cw\ sw^3)) + ci^3(co^4(8cw^3sw - 8cw\ sw^3) \\
& + so^4(8cw^3sw - 8cw\ sw^3) + co^2so^2(-48cw^3sw + 48cw\ sw^3))
\end{aligned}
$$

$$
\begin{aligned}
&+ ci^4(co\ so^3(-2cw^4 + 12cw^2sw^2 - 2sw^4) + co^3so(2cw^4 - 12cw^2sw^2 \\
&+ 2sw^4)) + ci^2(co\ so^3(-12cw^4 + 72cw^2sw^2 - 12sw^4) + co^3so(12cw^4 \\
&- 72cw^2sw^2 + 12sw^4)) + e^2(co^3so(84cw^4 + 132cw^2sw^2 - 128sw^4) \\
&+ co\ so^3(-84cw^4 - 132cw^2sw^2 + 128sw^4) + ci^3(co^4(-194cw^3sw \\
&- 18cw\ sw^3) + so^4(-194cw^3sw - 18cw\ sw^3) + co^2so^2(1164cw^3sw \\
&+ 108cw\ sw^3)) + ci(co^2so^2(-108cw^3sw - 1164cw\ sw^3) + co^4(18cw^3sw \\
&+ 194cw\ sw^3) + so^4(18cw^3sw + 194cw\ sw^3)) + ci^4(co\ so^3 \times (128cw^4 \\
&- 132cw^2sw^2 - 84sw^4) + co^3so(-128cw^4 + 132cw^2sw^2 + 84sw^4)) \\
&+ ci^2(co^3so(-132cw^4 + 792cw^2sw^2 - 132sw^4) + co\ so^3(132cw^4 \\
&- 792cw^2sw^2 + 132sw^4))))\cos 4M + e(co\ so^3(-13cw^4 + 78cw^2sw^2 \\
&- 13sw^4) + co^3so(13cw^4 - 78cw^2sw^2 + 13sw^4) + ci(co^4(52cw^3sw \\
&- 52cw\ sw^3) + so^4(52cw^3sw - 52cw\ sw^3) + co^2so^2(-312cw^3sw \\
&+ 312cw\ sw^3)) + ci^3(co^4(52cw^3sw - 52cw\ sw^3) + so^4(52cw^3sw \\
&- 52cw\ sw^3) + co^2so^2(-312cw^3sw + 312cw\ sw^3)) + ci^4(co\ so^3(-13cw^4 \\
&+ 78cw^2sw^2 - 13sw^4) + co^3so(13cw^4 - 78cw^2sw^2 + 13sw^4)) \\
&+ ci^2(co\ so^3(-78cw^4 + 468cw^2sw^2 - 78sw^4) + co^3so(78cw^4 \\
&- 468cw^2sw^2 + 78sw^4)))\cos 5M + e^2(co\ so^3(-51cw^4 + 306cw^2sw^2 \\
&- 51sw^4) + co^3so(51cw^4 - 306cw^2sw^2 + 51sw^4) + ci(co^4(204cw^3sw \\
&- 204cw\ sw^3) + so^4(204cw^3sw - 204cw\ sw^3) + co^2so^2(-1224cw^3sw \\
&+ 1224cw\ sw^3)) + ci^3(co^4(204cw^3sw - 204cw\ sw^3) + so^4(204cw^3sw \\
&- 204cw\ sw^3) + co^2so^2(-1224cw^3sw + 1224cw\ sw^3)) \\
&+ ci^4(co\ so^3(-51cw^4 + 306cw^2sw^2 - 51sw^4) + co^3so(51cw^4 - 306cw^2sw^2 \\
&+ 51sw^4)) + ci^2(co\ so^3(-306cw^4 + 1836cw^2sw^2 - 306\ sw^4) \\
&+ co^3so(306\ cw^4 - 1836cw^2sw^2 + 306sw^4)))\cos 6M + e(co^3so(-8cw\ sw) \\
&+ co\ so^3(8cw\ sw) + ci^4(co\ so^3(-8cw\ sw) + co^3so(8cw\ sw)) \\
&+ ci^3(co^2so^2(12cw^4 - 12sw^4) + co^4(-2cw^4 + 2sw^4) + so^4(-2cw^4 + 2sw^4)) \\
&+ ci(co^4(2cw^4 - 2sw^4) + so^4(2cw^4 - 2sw^4) + co^2so^2(-12cw^4 + 12sw^4))) \\
&\times \sin M + (co^3so(-16cw\ sw) + co\ so^3(16cw\ sw) + ci^4(co\ so^3(-16cw\ sw) \\
&+ co^3so(16cw\ sw)) + ci^3(co^2so^2(24cw^4 - 24sw^4) + co^4(-4cw^4 + 4sw^4) \\
&+ so^4(-4cw^4 + 4sw^4)) + ci(co^4(4cw^4 - 4sw^4) + so^4(4cw^4 - 4sw^4) \\
&+ co^2so^2(-24cw^4 + 24sw^4)) + e^2(co^3so(-20cw^3sw - 12cw\ sw^3) \\
&+ co\ so^3(20cw^3sw + 12cw\ sw^3) + ci^4(co\ so^3(-12cw^3sw - 20cw\ sw^3) \\
&+ co^3so(12cw^3sw + 20cw\ sw^3)) + ci^2(co\ so^3(24cw^3sw - 24cw\ sw^3)
\end{aligned}
$$

$$
\begin{aligned}
&+ co^3so(-24cw^3sw + 24cw\ sw^3)) + ci^3(co^2so^2(18cw^4 + 36cw^2sw^2\\
&- 30sw^4) + co^4(-3cw^4 - 6cw^2sw^2 + 5sw^4) + so^4(-3cw^4 - 6cw^2sw^2\\
&+ 5sw^4)) + ci(co^4(5cw^4 - 6cw^2sw^2 - 3sw^4) + so^4(5cw^4 - 6cw^2sw^2\\
&- 3sw^4) + co^2so^2(-3cw^4 + 36cw^2sw^2 + 18sw^4))))\sin 2M\\
&+ e(co^3so(-60cw^3sw - 84cw\ sw^3) + co\ so^3(60cw^3sw + 84cw\ sw^3)\\
&+ ci^4(co\ so^3(-84cw^3sw - 60cw\ sw^3) + co^3so(84cw^3sw + 60cw\ sw^3))\\
&+ ci^2(co^3so(72cw^3sw - 72cw\ sw^3) + co\ so^3(-72cw^3sw + 72cw\ sw^3))\\
&+ ci^3(co^2so^2(126cw^4 - 108cw^2sw^2 - 90sw^4) + co^4(-21cw^4 + 18cw^2sw^2\\
&+ 15sw^4) + so^4(-21cw^4 + 18cw^2sw^2 + 15sw^4)) + ci(co^4(15cw^4 + 18cw^2sw^2\\
&- 21sw^4) + so^4(15cw^4 + 18cw^2sw^2 - 21sw^4) + co^2so^2(-90cw^4\\
&- 108cw^2sw^2 + 126sw^4)))\sin 3M + (co\ so^3(8cw^3sw - 8cw\ sw^3)\\
&+ co^3so(-8cw^3sw + 8cw\ sw^3) + ci^4(co\ so^3(8cw^3sw - 8cw\ sw^3)\\
&+ co^3so(-8cw^3sw + 8cw\ sw^3)) + ci^2(co\ so^3(48cw^3sw - 48cw\ sw^3)\\
&+ co^3so(-48cw^3sw + 48cw\ sw^3)) + ci(co^2so^2(-12cw^4 + 72cw^2sw^2\\
&- 12sw^4) + co^4(2cw^4 - 12cw^2sw^2 + 2sw^4) + so^4(2cw^4 - 12cw^2sw^2\\
&+ 2sw^4)) + ci^3(co^2so^2(-12cw^4 + 72cw^2sw^2 - 12sw^4) + co^4(2cw^4\\
&- 12cw^2sw^2 + 2sw^4) + so^4(2cw^4 - 12cw^2sw^2 + 2sw^4))\\
&+ e^2(co^3so(-124cw^3sw - 300cw\ sw^3) + co\ so^3(124cw^3sw + 300cw\ sw^3)\\
&+ ci^4(co\ so^3(-300cw^3sw - 124cw\ sw^3) + co^3so(300cw^3sw + 124cw\ sw^3))\\
&+ ci^2(co^3so(528cw^3sw - 528cw\ sw^3) + co\ so^3(-528cw^3sw + 528cw\ sw^3))\\
&+ ci^3(co^2so^2(450cw^4 - 792cw^2sw^2 - 186sw^4) + co^4(-75cw^4 + 132cw^2sw^2\\
&+ 31sw^4) + so^4(-75cw^4 + 132cw^2sw^2 + 31sw^4)) + ci(co^4(31cw^4\\
&+ 132cw^2sw^2 - 75sw^4) + so^4(31cw^4 + 132cw^2sw^2 - 75sw^4)\\
&+ co^2so^2(-186cw^4 - 792cw^2sw^2 + 450sw^4))))\sin 4M\\
&+ e(co\ so^3(52cw^3sw - 52cw\ sw^3) + co^3so(-52cw^3sw + 52cw\ sw^3)\\
&+ ci^4(co\ so^3(52cw^3sw - 52cw\ sw^3) + co^3so(-52cw^3sw + 52cw\ sw^3))\\
&+ ci^2(co\ so^3(312cw^3sw - 312cw\ sw^3) + co^3so(-312cw^3sw + 312cw\ sw^3))\\
&+ ci(co^2so^2(-78cw^4 + 468cw^2sw^2 - 78sw^4) + co^4(13cw^4 - 78cw^2sw^2\\
&+ 13sw^4) + so^4(13cw^4 - 78cw^2sw^2 + 13sw^4)) + ci^3(co^2so^2(-78cw^4\\
&+ 468cw^2sw^2 - 78sw^4) + co^4(13cw^4 - 78cw^2sw^2 + 13sw^4) + so^4(13cw^4\\
&- 78cw^2sw^2 + 13sw^4)))\sin 5M + e^2(co\ so^3(204cw^3sw - 204cw\ sw^3)\\
&+ co^3so(-204cw^3sw + 204cw\ sw^3) + ci^4(co\ so^3(204cw^3sw - 204cw\ sw^3)\\
&+ co^3so(-204cw^3sw + 204cw\ sw^3)) + ci^2(co\ so^3(1224cw^3sw
\end{aligned}
$$

$$
\begin{aligned}
&- 1224cw\, sw^3) + co^3so(-1224cw^3sw + 1224cw\, sw^3))\\
&+ ci(co^2so^2(-306cw^4 + 1836cw^2sw^2 - 306sw^4) + co^4(51cw^4 - 306cw^2sw^2\\
&+ 51sw^4) + so^4(51cw^4 - 306cw^2sw^2 + 51sw^4)) + ci^3(co^2so^2(-306cw^4\\
&+ 1836cw^2sw^2 - 306sw^4) + co^4(51cw^4 - 306cw^2sw^2 + 51sw^4)\\
&+ so^4(51cw^4 - 306cw^2sw^2 + 51sw^4)))\sin 6M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\Omega_1}{dt} = &-\frac{105}{4}si(ci^3(co^2so^2(-36) + co^4(6) + so^4(6)) + ci(co^4(-6) + so^4(-6)\\
&+ co^2so^2(36)) + e^2(co^3so(-24cw\, sw) + co\, so^3(24cw\, sw)\\
&+ ci^2(co\, so^3(-72cw\, sw) + co^3so(72cw\, sw)) + ci^3(co^2so^2(-144cw^4\\
&- 360cw^2sw^2 - 216sw^4) + co^4(24cw^4 + 60cw^2sw^2 + 36sw^4) + so^4(24cw^4\\
&+ 60cw^2sw^2 + 36sw^4)) + ci(co^4(-30) + so^4(-30) + co^2so^2(180)))\\
&+ e(co^3so(-16cw\, sw) + co\, so^3(16cw\, sw) + ci^2(co\, so^3(-48cw\, sw)\\
&+ co^3so(48cw\, sw)) + ci^3(co^2so^2(-156cw^4 - 360cw^2sw^2 - 204sw^4)\\
&+ co^4(26cw^4 + 60cw^2sw^2 + 34sw^4) + so^4(26cw^4 + 60cw^2sw^2 + 34sw^4))\\
&+ ci(co^4(-30) + so^4(-30) + co^2so^2(180)))\cos M + (co^3so(-32cw\, sw)\\
&+ co\, so^3(32cw\, sw) + ci^2(co\, so^3(-96cw\, sw) + co^3so(96cw\, sw))\\
&+ ci^3(co^2so^2(48cw^4 - 48sw^4) + co^4(-8cw^4 + 8sw^4) + so^4(-8cw^4\\
&+ 8sw^4)) + e^2(co^3so(-48cw^3sw - 16cw\, sw^3) + co\, so^3(48cw^3sw\\
&+ 16cw\, sw^3) + ci^2(co\, so^3(-48cw^3sw - 144cw\, sw^3) + co^3so(48cw^3sw\\
&+ 144cw\, sw^3)) + ci^3(co^2so^2(-318cw^4 - 684cw^2sw^2 - 414sw^4)\\
&+ co^4(53cw^4 + 114cw^2sw^2 + 69sw^4) + so^4(53cw^4 + 114cw^2sw^2 + 69sw^4))\\
&+ ci(co^4(-57cw^4 - 138cw^2sw^2 - 57sw^4) + so^4(-57cw^4 - 138cw^2sw^2\\
&- 57sw^4) + co^2so^2(342cw^4 + 828cw^2sw^2 + 342sw^4))))\cos 2M\\
&+ e(co^3so(-96cw^3sw - 192cw\, sw^3) + co\, so^3(96cw^3sw + 192cw\, sw^3)\\
&+ ci^2(co\, so^3(-576cw^3sw - 288cw\, sw^3) + co^3so(576cw^3sw + 288cw\, sw^3))\\
&+ ci^3(co^2so^2(234cw^4 - 108cw^2sw^2 - 198sw^4) + co^4(-39cw^4 + 18cw^2sw^2\\
&+ 33sw^4) + so^4(-39cw^4 + 18cw^2sw^2 + 33sw^4)) + ci(co^4(-9cw^4\\
&+ 54cw^2sw^2 - 9sw^4) + so^4(-9cw^4 + 54cw^2sw^2 - 9sw^4) + co^2so^2(54cw^4\\
&- 324cw^2sw^2 + 54sw^4)))\cos 3M + (co\, so^3(32cw^3sw - 32cw\, sw^3)\\
&+ co^3so(-32cw^3sw + 32cw\, sw^3) + ci^2(co\, so^3(96cw^3sw - 96cw\, sw^3)\\
&+ co^3so(-96cw^3sw + 96cw\, sw^3)) + ci^3(co^2so^2(-12cw^4 + 72cw^2sw^2
\end{aligned}
$$

$$- 12sw^4) + co^4(2cw^4 - 12cw^2sw^2 + 2sw^4) + so^4(2cw^4 - 12cw^2sw^2$$
$$+ 2sw^4)) + ci(co^2so^2(-36cw^4 + 216cw^2sw^2 - 36sw^4) + co^4(6cw^4$$
$$- 36cw^2sw^2 + 6sw^4) + so^4(6cw^4 - 36cw^2sw^2 + 6sw^4))$$
$$+ e^2(co^3so(-72cw^3sw - 776cw\,sw^3) + co\,so^3(72cw^3sw + 776cw\,sw^3)$$
$$+ ci^2(co\,so^3(-2328cw^3sw - 216cw\,sw^3) + co^3so(2328cw^3sw$$
$$+ 216cw\,sw^3)) + ci^3(co^2so^2(768cw^4 - 792cw^2sw^2 - 504sw^4)$$
$$+ co^4(-128cw^4 + 132cw^2sw^2 + 84sw^4) + so^4(-128cw^4 + 132cw^2sw^2$$
$$+ 84sw^4)) + ci(co^4(-66cw^4 + 396cw^2sw^2 - 66sw^4) + so^4(-66cw^4$$
$$+ 396cw^2sw^2 - 66sw^4) + co^2so^2(396cw^4 - 2376cw^2sw^2 + 396sw^4))))$$
$$\times \cos 4M + e(co\,so^3(208cw^3sw - 208cw\,sw^3) + co^3so(-208cw^3sw$$
$$+ 208cw\,sw^3) + ci^2(co\,so^3(624cw^3sw - 624cw\,sw^3) + co^3so(-624cw^3sw$$
$$+ 624cw\,sw^3)) + ci^3(co^2so^2(-78cw^4 + 468cw^2sw^2 - 78sw^4)$$
$$+ co^4(13cw^4 - 78cw^2sw^2 + 13sw^4) + so^4(13cw^4 - 78cw^2sw^2 + 13sw^4))$$
$$+ ci(co^2so^2(-234cw^4 + 1404cw^2sw^2 - 234sw^4) + co^4(39cw^4$$
$$- 234cw^2sw^2 + 39sw^4) + so^4(39cw^4 - 234cw^2sw^2 + 39sw^4)))\cos 5M$$
$$+ e^2(co\,so^3(816cw^3sw - 816cw\,sw^3) + co^3so(-816cw^3sw + 816cw\,sw^3)$$
$$+ ci^2(co\,so^3(2448cw^3sw - 2448cw\,sw^3) + co^3so(-2448cw^3sw$$
$$+ 2448cw\,sw^3)) + ci^3(co^2so^2(-306cw^4 + 1836cw^2sw^2 - 306sw^4)$$
$$+ co^4(51cw^4 - 306cw^2sw^2 + 51sw^4) + so^4(51cw^4 - 306cw^2sw^2 + 51sw^4))$$
$$+ ci(co^2so^2(-918cw^4 + 5508cw^2sw^2 - 918sw^4) + co^4(153cw^4$$
$$- 918cw^2sw^2 + 153sw^4) + so^4(153cw^4 - 918cw^2sw^2 + 153sw^4)))\cos 6M$$
$$+ e(co\,so^3(8cw^4 - 8sw^4) + co^3so(-8cw^4 + 8sw^4) + ci^3(co^2so^2(-48cw\,sw)$$
$$+ co^4(8cw\,sw) + so^4(8cw\,sw)) + ci^2(co^3so(24cw^4 - 24sw^4)$$
$$+ co\,so^3(-24cw^4 + 24sw^4)))\sin M + (co\,so^3(16cw^4 - 16sw^4)$$
$$+ co^3so(-16cw^4 + 16sw^4) + ci^3(co^2so^2(-96cw\,sw) + co^4(16cw\,sw)$$
$$+ so^4(16cw\,sw)) + ci^2(co^3so(48cw^4 - 48sw^4) + co\,so^3(-48cw^4$$
$$+ 48sw^4)) + e^2(co\,so^3(20cw^4 - 24cw^2sw^2 - 12sw^4) + co^3so(-20cw^4$$
$$+ 24cw^2sw^2 + 12sw^4) + ci(co^2so^2(72cw^3sw - 72cw\,sw^3) + co^4(-12cw^3sw$$
$$+ 12cw\,sw^3) + so^4(-12cw^3sw + 12cw\,sw^3)) + ci^3(co^2so^2(-72cw^3sw$$
$$- 120cw\,sw^3) + co^4(12cw^3sw + 20cw\,sw^3) + so^4(12cw^3sw + 20cw\,sw^3))$$
$$+ ci^2(co^3so(36cw^4 + 72cw^2sw^2 - 60sw^4) + co\,so^3(-36cw^4 - 72cw^2sw^2$$
$$+ 60sw^4))))\sin 2M + e(co\,so^3(60cw^4 + 72cw^2sw^2 - 84sw^4)$$

$$
\begin{aligned}
&+ co^3 so(-60cw^4 - 72cw^2sw^2 + 84sw^4) + ci^3(co^2so^2(-504cw^3sw \\
&- 360cw\ sw^3) + co^4(84cw^3sw + 60cw\ sw^3) + so^4(84cw^3sw + 60cw\ sw^3)) \\
&+ ci(co^4(36cw^3sw - 36cw\ sw^3) + so^4(36cw^3sw - 36cw\ sw^3) \\
&+ co^2so^2(-216cw^3sw + 216cw\ sw^3)) + ci^2(co^3so(252cw^4 - 216cw^2sw^2 \\
&- 180sw^4) + co\ so^3(-252cw^4 + 216cw^2sw^2 + 180sw^4)))\sin 3M \\
&+ (co^3so(-8cw^4 + 48cw^2sw^2 - 8sw^4) + co\ so^3(8cw^4 - 48cw^2sw^2 \\
&+ 8sw^4) + ci^3(co^2so^2(48cw^3sw - 48cw\ sw^3) + co^4(-8cw^3sw \\
&+ 8cw\ sw^3) + so^4(-8cw^3sw + 8cw\ sw^3)) + ci(co^2so^2(144cw^3sw \\
&- 144cw\ sw^3) + co^4(-24cw^3sw + 24cw\ sw^3) + so^4(-24cw^3sw \\
&+ 24cw\ sw^3)) + ci^2(co^3so(-24cw^4 + 144cw^2sw^2 - 24sw^4) \\
&+ co\ so^3(24cw^4 - 144cw^2sw^2 + 24sw^4)) + e^2(co\ so^3(124cw^4 \\
&+ 528cw^2sw^2 - 300sw^4) + co^3so(-124cw^4 - 528cw^2sw^2 + 300sw^4) \\
&+ ci^3(co^2so^2(-1800cw^3sw - 744cw\ sw^3) + co^4(300cw^3sw + 124cw\ sw^3) \\
&+ so^4(300cw^3sw + 124cw\ sw^3)) + ci(co^4(264cw^3sw - 264cw\ sw^3) \\
&+ so^4(264cw^3sw - 264cw\ sw^3) + co^2so^2(-1584cw^3sw + 1584cw\ sw^3)) \\
&+ ci^2(co^3so(900cw^4 - 1584cw^2sw^2 - 372sw^4) + co\ so^3(-900cw^4 \\
&+ 1584cw^2sw^2 + 372sw^4))))\sin(4M) + e(co^3so(-52cw^4 + 312cw^2sw^2 \\
&- 52sw^4) + co\ so^3(52cw^4 - 312cw^2sw^2 + 52sw^4) + ci^3(co^2so^2(312cw^3sw \\
&- 312cw\ sw^3) + co^4(-52cw^3sw + 52cw\ sw^3) + so^4(-52cw^3sw \\
&+ 52cw\ sw^3)) + ci(co^2so^2(936cw^3sw - 936cw\ sw^3) + co^4(-156cw^3sw \\
&+ 156cw\ sw^3) + so^4(-156cw^3sw + 156cw\ sw^3)) + ci^2(co^3so(-156cw^4 \\
&+ 936cw^2sw^2 - 156sw^4) + co\ so^3(156cw^4 - 936cw^2sw^2 + 156sw^4)))\sin 5M \\
&+ e^2(co^3so(-204cw^4 + 1224cw^2sw^2 - 204sw^4) + co\ so^3(204cw^4 \\
&- 1224cw^2sw^2 + 204sw^4) + ci^3(co^2so^2(1224cw^3sw - 1224cw\ sw^3) \\
&+ co^4(-204cw^3sw + 204cw\ sw^3) + so^4(-204cw^3sw + 204cw\ sw^3)) \\
&+ ci(co^2so^2(3672cw^3sw - 3672cw\ sw^3) + co^4(-612cw^3sw + 612cw\ sw^3) \\
&+ so^4(-612cw^3sw + 612cw\ sw^3)) + ci^2(co^3so(-612cw^4 + 3672cw^2sw^2 \\
&- 612sw^4) + co\ so^3(612cw^4 - 3672cw^2sw^2 + 612sw^4)))\sin 6M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}M_1}{\mathrm{d}t} = &-\frac{525}{16a}(co^2so^2(-36) + co^4(6) + so^4(6) + ci^4(co^2so^2(-36) + co^4(6) \\
&+ so^4(6)) + ci^2(co^4(-12) + so^4(-12) + co^2so^2(72)) + e^2(co^2so^2(-216cw^4 \\
&- 360cw^2sw^2 - 144sw^4) + co^4(36cw^4 + 60cw^2sw^2 + 24sw^4) + so^4(36cw^4
\end{aligned}
$$

$$
\begin{aligned}
&+60cw^2sw^2+24sw^4)+ci^3(co\,so^3(-96cw\,sw)+co^3so(96cw\,sw))\\
&+ci(co^3so(-96cw\,sw)+co\,so^3(96cw\,sw))+ci^4(co^2so^2(-144cw^4\\
&-360cw^2sw^2-216sw^4)+co^4(24cw^4+60cw^2sw^2+36sw^4)+so^4(24cw^4\\
&+60cw^2sw^2+36sw^4))+ci^2(co^4(-60)+so^4(-60)+co^2so^2(360)))\\
&+e(co^2so^2(-204cw^4-360cw^2sw^2-156sw^4)+co^4(34cw^4+60cw^2sw^2\\
&+26sw^4)+so^4(34cw^4+60cw^2sw^2+26sw^4)+ci^3(co\,so^3(-64cw\,sw)\\
&+co^3so(64cw\,sw))+ci(co^3so(-64cw\,sw)+co\,so^3(64cw\,sw))\\
&+ci^4(co^2so^2(-156cw^4-360cw^2sw^2-204sw^4)+co^4(26cw^4+60cw^2sw^2\\
&+34sw^4)+so^4(26cw^4+60cw^2sw^2+34sw^4))+ci^2(co^4(-60)\\
&+so^4(-60)+co^2so^2(360)))\cos M+(co^4(8cw^4-8sw^4)+so^4(8cw^4\\
&-8sw^4)+co^2so^2(-48cw^4+48sw^4)+ci^3(co\,so^3(-128cw\,sw)\\
&+co^3so(128cw\,sw))+ci(co^3so(-128cw\,sw)+co\,so^3(128cw\,sw))\\
&+ci^4(co^2so^2(48cw^4-48sw^4)+co^4(-8cw^4+8sw^4)+so^4(-8cw^4\\
&+8sw^4))+e^2(co^2so^2(-414cw^4-684cw^2sw^2-318sw^4)+co^4(69cw^4\\
&+114cw^2sw^2+53sw^4)+so^4(69cw^4+114cw^2sw^2+53sw^4)\\
&+ci(co^3so(-192cw^3sw-64cw\,sw^3)+co\,so^3(192cw^3sw+64cw\,sw^3))\\
&+ci^3(co\,so^3(-64cw^3sw-192cw\,sw^3)+co^3so(64cw^3sw+192cw\,sw^3))\\
&+ci^4(co^2so^2(-318cw^4-684cw^2sw^2-414sw^4)+co^4(53cw^4\\
&+114cw^2sw^2+69sw^4)+so^4(53cw^4+114cw^2sw^2+69sw^4))\\
&+ci^2(co^4(-114cw^4-276cw^2sw^2-114sw^4)+so^4(-114cw^4\\
&-276cw^2sw^2-114sw^4)+co^2so^2(684cw^4+1656cw^2sw^2+684sw^4))))\\
&\times\cos 2M+e(co^4(33cw^4+18cw^2sw^2-39sw^4)+so^4(33cw^4+18cw^2sw^2\\
&-39sw^4)+co^2so^2(-198cw^4-108cw^2sw^2+234sw^4)\\
&+ci^3(co\,so^3(-768cw^3sw-384cw\,sw^3)+co^3so(768cw^3sw+384cw\,sw^3))\\
&+ci(co^3so(-384cw^3sw-768cw\,sw^3)+co\,so^3(384cw^3sw+768cw\,sw^3))\\
&+ci^4(co^2so^2(234cw^4-108cw^2sw^2-198sw^4)+co^4(-39cw^4+18cw^2sw^2\\
&+33sw^4)+so^4(-39cw^4+18cw^2sw^2+33sw^4))+ci^2(co^4(-18cw^4\\
&+108cw^2sw^2-18sw^4)+so^4(-18cw^4+108cw^2sw^2-18sw^4)\\
&+co^2so^2(108cw^4-648cw^2sw^2+108sw^4)))\cos 3M+(co^2so^2(-12cw^4\\
&+72cw^2sw^2-12sw^4)+co^4(2cw^4-12cw^2sw^2+2sw^4)+so^4(2cw^4\\
&-12cw^2sw^2+2sw^4)+ci(co\,so^3(128cw^3sw-128cw\,sw^3)\\
&+co^3so(-128cw^3sw+128cw\,sw^3))+ci^3(co\,so^3(128cw^3sw-128cw\,sw^3)
\end{aligned}
$$

$$+co^3so(-128cw^3sw+128cw\,sw^3))+ci^4(co^2so^2(-12cw^4+72cw^2sw^2$$
$$-12sw^4)+co^4(2cw^4-12cw^2sw^2+2sw^4)+so^4(2cw^4-12cw^2sw^2$$
$$+2sw^4))+ci^2(co^2so^2(-72cw^4+432cw^2sw^2-72sw^4)+co^4(12cw^4$$
$$-72cw^2sw^2+12sw^4)+so^4(12\,cw^4-72cw^2sw^2+12sw^4))$$
$$+e^2(co^4(84cw^4+132cw^2sw^2-128sw^4)+so^4(84cw^4+132cw^2sw^2$$
$$-128sw^4)+co^2so^2(-504cw^4-792cw^2sw^2+768sw^4)$$
$$+ci^3(co\,so^3(-3104cw^3sw-288cw\,sw^3)+co^3so(3104cw^3sw$$
$$+288cw\,sw^3))+ci(co^3so(-288cw^3sw-3104cw\,sw^3)+co\,so^3(288cw^3sw$$
$$+3104cw\,sw^3))+ci^4(co^2so^2(768cw^4-792cw^2sw^2-504sw^4)$$
$$+co^4(-128cw^4+132cw^2sw^2+84sw^4)+so^4(-128cw^4+132cw^2sw^2$$
$$+84sw^4))+ci^2(co^4(-132cw^4+792cw^2sw^2-132sw^4)+so^4(-132cw^4$$
$$+792cw^2sw^2-132sw^4)+co^2so^2(792cw^4-4752cw^2sw^2+792sw^4))))$$
$$\times\cos 4M+e(co^2so^2(-78cw^4+468cw^2sw^2-78sw^4)+co^4(13cw^4$$
$$-78cw^2sw^2+13sw^4)+so^4(13cw^4-78cw^2sw^2+13sw^4)$$
$$+ci(co\,so^3(832cw^3sw-832cw\,sw^3)+co^3so(-832cw^3sw+832cw\,sw^3))$$
$$+ci^3(co\,so^3(832cw^3sw-832cw\,sw^3)+co^3so(-832cw^3sw+832cw\,sw^3))$$
$$+ci^4(co^2so^2(-78cw^4+468cw^2sw^2-78sw^4)+co^4(13cw^4-78cw^2sw^2$$
$$+13sw^4)+so^4(13cw^4-78cw^2sw^2+13sw^4))+ci^2(co^2so^2(-468cw^4$$
$$+2808cw^2sw^2-468sw^4)+co^4(78cw^4-468cw^2sw^2+78sw^4)$$
$$+so^4(78cw^4-468cw^2sw^2+78sw^4)))\cos 5M+e^2(co^2so^2(-306cw^4$$
$$+1836cw^2sw^2-306sw^4)+co^4(51cw^4-306cw^2sw^2+51sw^4)$$
$$+so^4(51cw^4-306cw^2sw^2+51sw^4)+ci(co\,so^3(3264cw^3sw$$
$$-3264cw\,sw^3)+co^3so(-3264cw^3sw+3264cw\,sw^3))$$
$$+ci^3(co\,so^3(3264cw^3sw-3264cw\,sw^3)+co^3so(-3264cw^3sw$$
$$+3264cw\,sw^3))+ci^4(co^2so^2(-306cw^4+1836cw^2sw^2-306sw^4)$$
$$+co^4(51cw^4-306cw^2sw^2+51sw^4)+so^4(51cw^4-306cw^2sw^2+51sw^4))$$
$$+ci^2(co^2so^2(-1836cw^4+11016cw^2sw^2-1836sw^4)+co^4(306cw^4$$
$$-1836cw^2sw^2+306sw^4)+so^4(306cw^4-1836cw^2sw^2+306sw^4)))\cos 6M$$
$$+e(co^4(-8cw\,sw)+so^4(-8cw\,sw)+co^2so^2(48cw\,sw)$$
$$+ci^4(co^2so^2(-48cw\,sw)+co^4(8cw\,sw)+so^4(8cw\,sw))$$
$$+ci(co\,so^3(32cw^4-32sw^4)+co^3so(-32cw^4+32sw^4))$$
$$+ci^3(co^3so(32cw^4-32sw^4)+co\,so^3(-32cw^4+32sw^4)))\sin M$$
$$+(co^4(-16cw\,sw)+so^4(-16cw\,sw)+co^2so^2(96cw\,sw)$$

$$
\begin{aligned}
&+ci^4(co^2so^2(-96cw\,sw)+co^4(16cw\,sw)+so^4(16cw\,sw))\\
&+ci(co\,so^3(64cw^4-64sw^4)+co^3so(-64cw^4+64sw^4))\\
&+ci^3(co^3so(64cw^4-64sw^4)+co\,so^3(-64cw^4+64sw^4))\\
&+e^2(co^4(-20cw^3sw-12cw\,sw^3)+so^4(-20cw^3sw-12cw\,sw^3)\\
&+co^2so^2(120cw^3sw+72cw\,sw^3)+ci^4(co^2so^2(-72cw^3sw-120cw\,sw^3)\\
&+co^4(12cw^3sw+20cw\,sw^3)+so^4(12cw^3sw+20cw\,sw^3))\\
&+ci^2(co^2so^2(144cw^3sw-144cw\,sw^3)+co^4(-24cw^3sw+24cw\,sw^3)\\
&+so^4(-24cw^3sw+24cw\,sw^3))+ci(co\,so^3(80cw^4-96cw^2sw^2-48sw^4)\\
&+co^3so(-80cw^4+96cw^2sw^2+48sw^4))+ci^3(co^3so(48cw^4+96cw^2sw^2\\
&-80sw^4)+co\,so^3(-48cw^4-96cw^2sw^2+80sw^4))))\sin 2M\\
&+e(co^4(-60cw^3sw-84cw\,sw^3)+so^4(-60cw^3sw-84cw\,sw^3)\\
&+co^2so^2(360cw^3sw+504cw\,sw^3)+ci^4(co^2so^2(-504cw^3sw\\
&-360cw\,sw^3)+co^4(84cw^3sw+60cw\,sw^3)+so^4(84cw^3sw+60cw\,sw^3))\\
&+ci^2(co^4(72cw^3sw-72cw\,sw^3)+so^4(72cw^3sw-72cw\,sw^3)\\
&+co^2so^2(-432cw^3sw+432cw\,sw^3))+ci^3(co^3so(336cw^4-288cw^2sw^2\\
&-240sw^4)+co\,so^3(-336cw^4+288cw^2sw^2+240sw^4))\\
&+ci(co\,so^3(240cw^4+288cw^2sw^2-336sw^4)+co^3so(-240cw^4\\
&-288cw^2sw^2+336sw^4)))\sin 3M+(co^2so^2(48cw^3sw-48cw\,sw^3)\\
&+co^4(-8cw^3sw+8cw\,sw^3)+so^4(-8cw^3sw+8cw\,sw^3)\\
&+ci^4(co^2so^2(48cw^3sw-48cw\,sw^3)+co^4(-8cw^3sw+8cw\,sw^3)\\
&+so^4(-8cw^3sw+8cw\,sw^3))+ci^2(co^2so^2(288cw^3sw-288cw\,sw^3)\\
&+co^4(-48cw^3sw+48cw\,sw^3)+so^4(-48cw^3sw+48cw\,sw^3))\\
&+ci(co^3so(-32cw^4+192cw^2sw^2-32sw^4)+co\,so^3(32cw^4-192cw^2sw^2\\
&+32sw^4))+ci^3(co^3so(-32cw^4+192cw^2sw^2-32sw^4)+co\,so^3(32cw^4\\
&-192cw^2sw^2+32sw^4))+e^2(co^4(-124cw^3sw-300cw\,sw^3)\\
&+so^4(-124cw^3sw-300cw\,sw^3)+co^2so^2(744cw^3sw+1800cw\,sw^3)\\
&+ci^4(co^2so^2(-1800cw^3sw-744cw\,sw^3)+co^4(300cw^3sw+124cw\,sw^3)\\
&+so^4(300cw^3sw+124cw\,sw^3))+ci^2(co^4(528cw^3sw-528cw\,sw^3)\\
&+so^4(528cw^3sw-528cw\,sw^3)+co^2so^2(-3168cw^3sw+3168cw\,sw^3))\\
&+ci^3(co^3so(1200cw^4-2112cw^2sw^2-496sw^4)+co\,so^3(-1200cw^4\\
&+2112cw^2sw^2+496sw^4))+ci(co\,so^3(496cw^4+2112cw^2sw^2-1200sw^4)\\
&+co^3so(-496cw^4-2112cw^2sw^2+1200sw^4))))\sin 4M\\
&+e(co^2so^2(312cw^3sw-312cw\,sw^3)+co^4(-52cw^3sw+52cw\,sw^3)
\end{aligned}
$$

$$
\begin{aligned}
&+so^4(-52cw^3sw+52cw\,sw^3)+ci^4(co^2so^2(312cw^3sw-312cw\,sw^3)\\
&+co^4(-52cw^3sw+52cw\,sw^3)+so^4(-52cw^3sw+52cw\,sw^3))\\
&+ci^2(co^2so^2(1872cw^3sw-1872cw\,sw^3)+co^4(-312cw^3sw+312cw\,sw^3)\\
&+so^4(-312cw^3sw+312cw\,sw^3))+ci(co^3so(-208cw^4+1248cw^2sw^2\\
&-208sw^4)+co\,so^3(208cw^4-1248cw^2sw^2+208sw^4))\\
&+ci^3(co^3so(-208cw^4+1248cw^2sw^2-208sw^4)+co\,so^3(208cw^4\\
&-1248cw^2sw^2+208sw^4)))\sin 5M+e^2(co^2so^2(1224cw^3sw-1224cw\,sw^3)\\
&+co^4(-204cw^3sw+204cw\,sw^3)+so^4(-204cw^3sw+204cw\,sw^3)\\
&+ci^4(co^2so^2(1224cw^3sw-1224cw\,sw^3)+co^4(-204cw^3sw+204cw\,sw^3)\\
&+so^4(-204cw^3sw+204cw\,sw^3))+ci^2(co^2so^2(7344cw^3sw\\
&-7344cw\,sw^3)+co^4(-1224cw^3sw+1224cw\,sw^3)+so^4(-1224cw^3sw\\
&+1224cw\,sw^3))+ci(co^3so(-816cw^4+4896cw^2sw^2-816sw^4)\\
&+co\,so^3(816cw^4-4896cw^2sw^2+816sw^4))+ci^3(co^3so(-816cw^4\\
&+4896cw^2sw^2-816sw^4)+co\,so^3(816cw^4-4896cw^2sw^2\\
&+816sw^4)))\sin 6M)
\end{aligned}
$$

The long periodic terms included in above formulas are given as follows:

$$
\begin{aligned}
d_2 =& \frac{315}{8}e^2(-1+ci)(1+ci)((-1+2ci-ci^2)\sin(4o-2w)+(1+2ci+ci^2)\\
&\sin(4o+2w))
\end{aligned}
$$

$$
\begin{aligned}
d_3 =& -\frac{525}{4}e(-1+ci)(1+ci)((3-3ci^2)\cos(4o)+(1-2ci+ci^2)\cos(4o-2w)\\
&+(1+2ci+ci^2)\cos(4o+2w))
\end{aligned}
$$

$$
\begin{aligned}
d_4 =& \frac{315}{4}(-1+ci)(1+ci)((2-2ci^2+(10-10ci^2)e^2)\sin(4o)+(1-2ci+ci^2)\\
&e^2\sin(4o-2w)+(1+2ci+ci^2)e^2\sin(4o+2w))
\end{aligned}
$$

$$
\begin{aligned}
d_5 =& \frac{315}{8}si((4ci-4ci^3+(20ci-20ci^3)e^2)\cos(4o)+(1-3ci^2+2ci^3)\\
&e^2\cos(4o-2w)+(-1+3ci^2+2ci^3)e^2\cos(4o+2w))
\end{aligned}
$$

$$
\begin{aligned}
d_6 =& \frac{1575}{16a}(-1+ci)(1+ci)((2-2ci^2+(10-10ci^2)e^2)\cos(4o)+(1-2ci+ci^2)\\
&e^2\cos(4o-2w)+(1+2ci+ci^2)e^2\cos(4o+2w))
\end{aligned}
$$

Solutions of D_{50} Perturbation

$$\begin{aligned}\frac{\mathrm{d}a_1}{\mathrm{d}t}=&\frac{3}{1024}si((640cw+si^2(-2240cw)+si^4(1680cw)+e^2(1840cw\\&+si^2(-6160cw^3-7280cw\,sw^2)+si^4(4515cw^5+10290cw^3sw^2\\&+5775cw\,sw^4)))\cos M+e(5120cw+si^2(-17920cw)\\&+si^4(13440cw))\cos 2M+(si^2(2240cw^3-6720cw\,sw^2)\\&+si^4(-2520cw^5+5040cw^3sw^2+7560cw\,sw^4)+e^2(20880cw\\&+si^2(-76440cw^3-63000cw\,sw^2)+si^4(59157cw^5+96390cw^3sw^2\\&+46305cw\,sw^4)))\cos 3M+e(si^2(17920cw^3-53760cw\,sw^2)\\&+si^4(-21504cw^5+53760cw^3sw^2+53760cw\,sw^4))\cos 4M+(si^4(840cw^5\\&-8400cw^3sw^2+4200cw\,sw^4)+e^2(si^2(82600cw^3-247800cw\,sw^2)\\&+si^4(-107625cw^5+332850cw^3sw^2+205275cw\,sw^4)))\cos 5M\\&+e\,si^4(8064cw^5-80640cw^3sw^2+40320cw\,sw^4)\cos 6M+e^2si^4(43953cw^5\\&-439530cw^3sw^2+219765cw\,sw^4)\cos 7M+(-640sw+si^2(2240sw)\\&+si^4(-1680sw)+e^2(-6480sw+si^2(21840cw^2sw+22960sw^3)\\&+si^4(-16065cw^4sw-33390cw^2sw^3-17325sw^5)))\sin M+e(-5120sw\\&+si^2(17920sw)+si^4(-13440sw))\sin 2M+(si^2(-6720cw^2sw+2240sw^3)\\&+si^4(7560cw^4sw+5040cw^2sw^3-2520sw^5)+e^2(-20880sw\\&+si^2(83160cw^2sw+69720sw^3)+si^4(-68985cw^4sw-111510cw^2sw^3\\&-51597sw^5)))\sin 3M+e(si^2(-53760cw^2sw+17920sw^3)\\&+si^4(67200cw^4sw+26880cw^2sw^3-18816sw^5))\sin 4M\\&+(si^4(-4200cw^4sw+8400cw^2sw^3-840sw^5)+e^2(si^2(-247800cw^2sw\\&+82600sw^3)+si^4(352275cw^4sw+38850cw^2sw^3-78225sw^5)))\sin 5M\\&+e\,si^4(-40320cw^4sw+80640cw^2sw^3-8064sw^5)\sin 6M\\&+e^2si^4(-219765cw^4sw+439530cw^2sw^3-43953sw^5)\sin 7M)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}e_1}{\mathrm{d}t}=&\frac{15}{1024}si(e(256cw+si^2(-896cw)+si^4(672cw))+(128cw+si^2(-448cw)\\&+si^4(336cw)+e^2(1296cw+si^2(-4368cw^3-5040cw\,sw^2)+si^4(3213cw^5\\&+7182cw^3sw^2+3969cw\,sw^4)))\cos M+e(512cw+si^2(-1792cw)\\&+si^4(1344cw))\cos 2M+(si^2(448cw^3-1344cw\,sw^2)+si^4(-504cw^5\\&+1008cw^3sw^2+1512cw\,sw^4)+e^2(1392cw+si^2(-5544cw^3-2856cw\,sw^2)\\&+si^4(4599cw^5+3906cw^3sw^2+2331cw\,sw^4)))\cos 3M+e(si^2(2688cw^3\\&-8064cw\,sw^2)+si^4(-3360cw^5+9408cw^3sw^2+7392cw\,sw^4))\cos 4M\end{aligned}$$

$$+ (si^4(168cw^5 - 1680cw^3sw^2 + 840cw\,sw^4) + e^2(si^2(9912cw^3 - 29736cw\,sw^2) + si^4(-14091cw^5 + 51702cw^3sw^2 + 18753cw\,sw^4))) \times \cos 5M + e\,si^4(1344cw^5 - 13440cw^3sw^2 + 6720cw\,sw^4)\cos 6M + e^2si^4(6279cw^5 - 62790cw^3sw^2 + 31395cw\,sw^4)\cos 7M + (-128sw + si^2(448sw) + si^4(-336cw^4sw - 672cw^2sw^3 - 36sw^5) + e^2(-368sw + si^2(784cw^2sw + 1456sw^3) + si^4(-399cw^4sw - 1554cw^2sw^3 - 1155sw^5)))\sin M + e(-512sw + si^2(1792sw) + si^4(-1344sw))\sin 2M + (si^2(-1344cw^2sw + 448sw^3) + si^4(1512cw^4sw + 1008cw^2sw^3 - 504sw^5) + e^2(-1392sw + si^2(6888cw^2sw + 4200sw^3) + si^4(-6867cw^4sw - 6930cw^2sw^3 - 3087sw^5)))\sin 3M + e(si^2(-8064cw^2sw + 2688sw^3) + si^4(10752cw^4sw + 2688cw^2sw^3 - 2688sw^5))\sin 4M + (si^4(-840cw^4sw + 1680cw^2sw^3 - 168sw^5) + e^2(si^2(-29736cw^2sw + 9912sw^3) + si^4(48153cw^4sw - 7098cw^2sw^3 - 8211sw^5)))\sin 5M + e\,si^4(-6720cw^4sw + 13440cw^2sw^3 - 1344sw^5)\sin 6M + e^2si^4(-31395cw^4sw + 62790cw^2sw^3 - 6279sw^5)\sin 7M)$$

$$\frac{d\omega_1}{dt} = \frac{3}{152}si(640sw + si^2(-2240sw) + si^4(1680sw) + e^2(10080sw + si^2(-33600cw^2sw - 35840sw^3) + si^4(24570cw^4sw + 51660cw^2sw^3 + 27090sw^5)) + e(6480sw + si^2(-21840cw^2sw - 22960sw^3) + si^4(16065cw^4sw + 33390cw^2sw^3 + 17325sw^5))\cos M + (1280sw + si^2(-4480sw) + si^4(3360sw) + e^2(19840sw + si^2(-64400cw^2sw - 71120sw^3) + si^4(46200cw^4sw + 100800cw^2sw^3 + 53928sw^5)))\cos 2M + e(6960sw + si^2(-27720cw^2sw - 23240sw^3) + si^4(22995cw^4sw + 37170cw^2sw^3 + 17199sw^5))\cos 3M + (si^2(6720cw^2sw - 2240sw^3) + si^4(-8400cw^4sw - 3360cw^2sw^3 + 2352sw^5) + e^2(23840sw + si^2(-118720cw^2sw - 71680sw^3) + si^4(116760cw^4sw + 122640cw^2sw^3 + 52248sw^5)))\cos 4M + e(si^2(49560cw^2sw - 16520sw^3) + si^4(-70455cw^4sw - 7770cw^2sw^3 + 15645sw^5))\cos 5M + (si^4(3360cw^4sw - 6720cw^2sw^3 + 672sw^5) + e^2(si^2(216720cw^2sw - 72240sw^3) + si^4(-355320cw^4sw + 60480cw^2sw^3 + 58968sw^5))) \times \cos 6M + e\,si^4(31395cw^4sw - 62790cw^2sw^3 + 6279sw^5)\cos 7M$$

$$
\begin{aligned}
&+e^2 si^4(167790cw^4 sw - 335580cw^2 sw^3 + 33558sw^5)\cos 8M + e(1840cw\\
&+si^2(-6160cw^3 - 7280cw\, sw^2) + si^4(4515cw^5 + 10290cw^3 sw^2\\
&+5775cw\, sw^4))\sin M + (1280cw + si^2(-4480cw) + si^4(3360cw)\\
&+e^2(8000cw + si^2(-26320cw^3 - 33040cw\, sw^2) + si^4(19068cw^5\\
&+46200cw^3 sw^2 + 26460cw\, sw^4)))\sin 2M + e(6960cw + si^2(-25480cw^3\\
&-21000cw\, sw^2) + si^4(19719cw^5 + 32130cw^3 sw^2 + 15435cw\, sw^4))\sin 3M\\
&+(si^2(2240cw^3 - 6720cw\, sw^2) + si^4(-2688cw^5 + 6720cw^3 sw^2\\
&+6720cw\, sw^4) + e^2(23840cw + si^2(-95200cw^3 - 48160cw\, sw^2)\\
&+si^4(78708cw^5 + 69720cw^3 sw^2 + 37380cw\, sw^4)))\sin 4M\\
&+e(si^2(16520cw^3 - 49560cw\, sw^2) + si^4(-21525cw^5 + 66570cw^3 sw^2\\
&+41055cw\, sw^4))\sin 5M + (si^4(672cw^5 - 6720cw^3 sw^2 + 3360cw\, sw^4)\\
&+e^2(si^2(72240cw^3 - 216720cw\, sw^2) + si^4(-103572cw^5 + 385560cw^3 sw^2\\
&+132300cw\, sw^4)))\sin 6M + e\, si^4(6279cw^5 - 62790cw^3 sw^2\\
&+31395cw\, sw^4)\sin 7M + e^2 si^4(33558cw^5 - 335580cw^3 sw^2\\
&+167790cw\, sw^4)\sin 8M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\Omega_1}{dt} = &\frac{15}{1024} ci(e(256sw + si^2(-2688sw) + si^4(3360cw^4 sw + 6720cw^2 sw^3\\
&+3360sw^5)) + (128sw + si^2(-1344sw) + si(1680sw) + e^2(1296sw\\
&+si^2(-13104cw^2 sw - 13776sw^3) + si^4(16065cw^4 sw + 33390cw^2 sw^3\\
&+17325sw^5)))\cos M + e(512sw + si^2(-5376sw - 5376sw) + si^4(6720sw\\
&+13440cw^2 sw^3 + 6720sw^5))\cos 2M + (si^2(1344cw^2 sw - 448sw^3)\\
&+si^4(-2520cw^4 sw - 1680cw^2 sw^3 + 840sw^5) + e^2(1392sw\\
&+si^2(-16632cw^2 sw - 13944sw^3) + si^4(22995cw^4 sw + 37170cw^2 sw^3\\
&+17199sw^5)))\cos 3M + e(si^2(8064cw^2 sw - 2688\, sw^3) + si^4(-16800cw^4 sw\\
&-6720cw^2 sw + 4704sw^5))\cos 4M + (si^4(840cw^4 sw - 1680cw^2 sw^3\\
&+168sw^5) + e^2(si^2(29736cw^2 sw - 9912sw^3) + si^4(-70455cw^4 sw\\
&-7770cw^2 sw^3 + 15645sw^5)))\cos 5M + e\, si^4(6720cw^4 sw - 13440cw^2 sw^3\\
&+1344sw^5)\cos 6M + e^2 si^4(31395cw^4 sw - 62790cw\, sw + 6279sw^5)\cos 7M\\
&+(128cw + si^2(-1344cw) + si^4(1680cw^5 + 3360cw^3 sw^2 + 1680cw\, sw^4)\\
&+e^2(368cw + si^2(-3696cw^3 - 4368cw\, sw^2) + si^4(4515cw^5 + 10290cw^3 sw^2\\
&+5775cw\, sw^4)))\sin M + e(512cw + si^2(-5376cw) + si^4(6720cw^5
\end{aligned}
$$

$$
\begin{aligned}
&+13440cw^3sw^2+6720cw\,sw^4))\sin2M+(si^2(448cw^3-1344cw\,sw^2)\\
&+si^4(-840cw^5+1680cw^3sw^2+2520cw\,sw^4)+e^2(1392cw\\
&+si^2(-15288cw^3-12600cw\,sw^2)+si^4(19719cw^5+32130cw^3sw^2\\
&+15435cw\,sw^4)))\sin3M+e(si^2(2688cw^3-8064cw\,sw^2)+si^4(-5376cw^5\\
&+13440cw^3sw^2+13440cw\,sw^4))\sin4M+(si^4(168cw^5-1680cw^3sw^2\\
&+840cw\,sw^4)+e^2(si^2(9912cw^3-29736cw\,sw^2)+si^4(-21525cw^5\\
&+66570cw^3sw^2+41055cw\,sw^4)))\sin5M+e\,si^4(1344cw^5-13440cw^3sw^2\\
&+6720cw\,sw^4)\sin6M+e^2si^4(6279cw^5-62790cw^3sw^2\\
&+31395cw\,sw^4)\sin7M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dM_1}{dt}=&-\frac{3}{512a}si(e(3840sw+si^2(-13440sw)+si^4(10080sw))+(1920sw\\
&+si^2(-6720sw)+si^4(5040sw)+e^2(19440sw+si^2(-65520cw^2sw\\
&-68880sw^3)+si^4(48195cw^4sw+100170cw^2sw^3+51975sw^5)))\cos M\\
&+e(7680sw+si^2(-26880sw)+si^4(20160sw))\cos2M+(si^2(6720cw^2sw\\
&-2240sw^3)+si^4(-7560cw^4sw-5040cw^2sw^3+2520sw^5)+e^2(20880sw\\
&+si^2(-83160cw^2sw-69720sw^3)+si^4(68985cw^4sw+111510cw^2sw^3\\
&+51597sw^5)))\cos3M+e(si^2(40320cw^2sw-13440sw^3)\\
&+si^4(-50400cw^4sw-20160cw^2sw^3+14112sw^5))\cos4M\\
&+(si^4(2520cw^4sw-5040cw^2sw^3+504sw^5)+e^2(si^2(148680cw^2sw\\
&-49560sw^3)+si^4(-211365cw^4sw-23310cw^2sw^3+46935sw^5)))\cos5M\\
&+e\,si^4(20160cw^4sw-4020cw^2sw^3+4032sw^5)\cos6M+e^2si^4(94185cw^4sw\\
&-188370cw^2sw^3+18837sw^5)\cos7M+(1920cw+si^2(-6720cw)\\
&+si^4(5040cw)+e^2(5520cw+si^2(-18480cw^3-21840cw\,sw^2)\\
&+si^4(13545cw^5+30870cw^3sw^2+17325cw\,sw^4)))\sin M+e(7680cw\\
&+si^2(-26880cw)+si^4(20160cw))\sin2M+(si^2(2240cw^3-6720cw\,sw^2)\\
&+si^4(-2520cw^5+5040cw^3sw^2+7560cw\,sw^4)+e^2(20880cw\\
&+si^2(-76440cw^3-63000cw\,sw^2)+si^4(59157cw^5+96390cw^3sw^2\\
&+46305cw\,sw^4)))\sin3M+e(si^2(13440cw^3-40320cw\,sw^2)\\
&+si^4(-16128cw^5+40320cw^3sw^2+40320cw\,sw^4))\sin4M\\
&+(si^4(504cw^5-5040cw^3sw^2+2520cw\,sw^4)+e^2(si^2(49560cw^3\\
&-148680cw\,sw^2)+si^4(-64575cw^5+199710cw^3sw^2\\
&+123165cw\,sw^4)))\sin5M+e\,si^4(4032cw^5-40320cw^3sw^2\\
&+20160cw\,sw^4)\sin6M+e^2si^4(18837cw^5-188370cw^3sw^2\\
&+94185cw\,sw^4)\sin7M)
\end{aligned}
$$

The long periodic terms and long terms included in above formulas are given as follows:

$$d_2 = \frac{120}{301}\cos(w)e\ si(8 - 28si^2 + 21si^4)$$

$$d_3 = \frac{-15}{256}\sin(w)si(-64 - 1008e^2 + (224 + 3472e^2)si^2 + (-168 - 2583e^2)si^4 + (-112e^2si^2 + 126e^2si^4)\cos 2w)$$

$$d_5 = \frac{15}{32}\sin(w)ci\ e(8 - 84si^2 + 105si^4)$$

$$d_6 = -\frac{45e\ si}{16a}\sin(w)(8 - 28si^2 + 21si^4)$$

Solutions of $D_{51} \sim D_{55}$ Perturbations

Solutions of D_{51} Perturbation

$$d_2 = -\frac{15}{64}e((-8 + 8ci + (28 - 84ci)si^2 + (-21 + 105ci)si^4)\sin(o - w) + (8 + 8ci + (-28 - 84ci)si^2 + (21 + 105ci)si^4)\sin(o + w))$$

$$d_3 = -\frac{15}{512}(((-392 + 392ci)e^2si^2 + (609 - 567ci)e^2si^4)\cos(o - w) + (-96 + 96ci + (-1248 + 1248ci)e^2 + (560 - 784ci + (6048 - 11424ci)e^2)si^2 + (-588 + 924ci + (-5796 + 13860ci)e^2)si^4)\cos(o - w) + (-96 - 96ci + (-1248 - 1248ci)e^2 + (560 + 784ci + (6048 + 11424ci)e^2)si^2 + (-588 - 924ci + (-5796 - 13860ci)e^2)si^4)\cos(o + w) + ((-392 - 392ci)e^2si^2 + (609 + 567ci)e^2si^4)\cos(o + 3w))$$

$$d_4 = -\frac{15}{64}e((8 - 8ci + (-28 + 84ci)si^2 + (21 - 105ci)si^4)\sin(o - w) + (8 + 8ci + (-28 - 84ci)si^2 + (21 + 105ci)si^4)\sin(o + w))$$

$$d_5 = -\frac{15}{64}e\ si((-8 + 56ci - 168ci^2 + (84 - 84ci + 420ci^2)si^2 - 105si^4)\cos(o - w) + (8 + 56ci + 168ci^2 + (-84 - 84ci - 420ci^2)si^2 + 105si^4)\cos(o + w))$$

$$d_6 = -\frac{45e}{32a}(8 - 8ci + (-28 + 84ci)si^2 + (21 - 105ci)si^4)\cos(o - w) + (8 + 8\,ci + (-28 - 84ci)si^2 + (21 + 105ci)si^4)\cos(o + w))$$

Solutions of D_{52} Perturbation

$$d_2 = -\frac{105}{16}e\ si((2 + 4ci - 6ci^2 + (-3 - 6ci + 15ci^2)si^2)\cos(2o - w) + (2 - 4ci - 6ci^2 + (-3 + 6ci + 15ci^2)si^2)\cos(2o + w))$$

$$d_3 = \frac{105}{128}si(((44 - 88ci + 44ci^2)e^2 + (-81 + 198ci - 117ci^2)e^2si^2)\sin(2o - 3w) + (64ci - 4ci^2 + (132 + 744ci - 876ci^2)e^2 + (-144ci + 144ci^2 + (-198 - 1476ci + 2070ci^2)e^2)si^2)\sin(2o - w) + (64ci + 64ci^2 + (-132 + 744ci + 876\,ci^2)e^2 + (-144ci - 144ci^2 + (198 - 1476ci - 2070ci^2)e^2)si^2) \times \sin(2o + w) + ((-44 - 88ci - 44ci^2)e^2 + (81 + 198ci + 117ci^2)e^2si^2) \times \sin(2o + 3w))$$

$$d_4 = -\frac{105}{8}e\ si((-2 - 4ci + 6ci^2 + (3 + 6ci - 15ci^2)si^2)\cos(2o - w) + (2 - 4ci - 6ci^2 + (-3 + 6ci + 15ci^2)si^2)\cos(2o + w))$$

$$d_5 = -\frac{105}{16}e((-2ci - 4ci^2 + 6ci^3 + (4 - 3ci + 18ci^2 - 45ci^3)si^2 + (-6 + 30ci)si^4) \times \sin(2o - w) + (2ci - 4ci^2 - 6ci^3 + (4 + 3ci + 18ci^2 + 45ci^3)si^2 + (-6 - 30ci)si^4)\sin(2o + w))$$

$$d_6 = \frac{315}{8a}e\ si((-2 - 4ci + 6ci^2 + (3 + 6ci - 15ci^2)si^2)\sin(2o - w) + (2 - 4ci - 6ci^2 + (-3 + 6ci + 15ci^2)si^2)\sin(2o + w))$$

Solutions of D_{53} Perturbation

$$d_2 = \frac{315}{16}e((-2 + 2ci + 2ci^2 - 2ci^3 + (3 - 9ci - 9ci^2 + 15ci^3)si^2)\sin(3o - w) + (2 + 2ci - 2ci^2 - 2ci^3 + (-3 - 9ci + 9ci^2 + 15ci^3)si^2)\sin(3o + w))$$

$$d_3 = \frac{315}{128}(((-20+60ci-60ci^2+20ci^3)e^2+(27-207ci+333ci^2-153ci^3)e^2si^2) \times \cos(3o-3w)+(-24+24ci+24ci^2-24ci^3+(-312+312ci+312ci^2 -312ci^3)e^2+(60-36ci-180ci^2+156ci^3+(648-864ci-1944ci^2 +2160ci^3)e^2)si^2)\cos(3o-w)+(-24-24ci+24ci^2+24ci^3+(-312 -312ci+312ci^2+312ci^3)e^2+(60+36ci-180ci^2-156ci^3+(648 +864ci-1944ci^2-2160ci^3)e^2)si^2)\cos(3o+w)+((-20-60ci-60ci^2 -20ci^3)e^2+(27+207ci+333ci^2+153ci^3)e^2si^2)\cos(3o+3w))$$

$$d_4 = \frac{945}{16}e((2-2ci-2ci^2+2ci^3+(-3+9ci+9ci^2-15ci^3)si^2)\sin(3o-w) +(2+2ci-2ci^2-2ci^3+(-3-9ci+9ci^2+15ci^3)si^2)\sin(3o+w))$$

$$d_5 = -\frac{105}{16}e((-2ci-4ci^2+6ci^3+(4-3ci+18ci^2-45ci^3)si^2+(-6 +30ci)si^4)\sin(2o-w)+(2ci-4ci^2-6ci^3+(4+3ci+18ci^2+4ci^3)si^2 +(-6-30ci)si^4)\sin(2o+w))$$

$$d_6 = \frac{945e}{8a}((2-2ci-2ci^2+2ci^3+(-3+9ci+9ci^2-15ci^3)si^2)\cos(3o-w) +(2+2ci-2ci^2-2ci^3+(-3-9ci+9ci^2+15ci^3)si^2)\cos(3o+w))$$

Solutions of D_{54} Perturbation

$$d_1 = -\frac{2835}{4}(-1+2\mathrm{ci}-2\mathrm{ci}^3+\mathrm{ci}^4)\mathrm{e\ si\ }\cos(4\mathrm{o}-\mathrm{w})-\frac{2835}{4}(-1-2\mathrm{ci}+2\mathrm{ci}^3 +\mathrm{ci}^4)\mathrm{e\ si\ }\cos(4\mathrm{o}+\mathrm{w})$$

$$d_2 = \frac{945}{8}e\ si(-1+ci)(1+ci)((-1-4ci+5ci^2)\cos(4o-w)+(-1+4ci +5ci^2)\cos(4o+w))$$

$$d_3 = -\frac{945}{64}si(e^2(25-12ci-114ci^2+164ci^3-63ci^4)\sin(4o-3w)+(-8 +64ci-48ci^2-64ci^3+56ci^4+(6+744ci-756ci^2-744ci^3 +750ci^4)e^2)\sin(4o-w)+(8+64ci+48ci^2-64ci^3-56ci^4+(-6+744ci +756ci^2-744ci^3-750ci^4)e^2)\sin(4o+w)+(-25-12ci+114ci^2+164ci^3 +63ci^4)e^2\sin(4o+3w))$$

$$d_4 = \frac{945}{2} e\ si(-1+ci)(1+ci)((1+4ci-5ci^2)\cos(4o-w)+(-1+4ci + 5ci^2)\cos(4o+w))$$

$$d_5 = \frac{945}{8} e((-ci-4ci^2+6ci^3+4ci^4-5ci^5+(4-12ci-12ci^2+20ci^3)si^2) \times \sin(4o-w)+(ci-4ci^2-6ci^3+4ci^4+5ci^5+(4+12ci-12ci^2 - 20ci^3)si^2)\sin(4o+w))$$

$$d_6 = -\frac{2835e\ si}{4a}(-1+ci)(1+ci)((1+4ci-5ci^2)\sin(4o-w)+(-1+4ci + 5ci^2)\sin(4o+w))$$

Solutions of D_{55} Perturbation

$$d_1 = \frac{14175}{16}(-1+ci+2ci^2-2ci^3-ci^4+ci^5)e\ \sin(5o-w)+\frac{14175}{16}(1+ci - 2ci^2 - 2ci^3 + ci^4 + ci^5)e\sin(5o+w)$$

$$d_2 = -\frac{4725}{8} e(-1+ci)^2(1+ci)^2((-1+ci)\sin(5o-w)+(1+ci)\sin(5o+w))$$

$$d_3 = \left(-\frac{14175}{64}(-1+ci)(1+ci)((5-15ci+15ci^2-5ci^3)e^2\cos(5o-3w) + (4-4ci-4ci^2+4ci^3+(52-52ci-52ci^2+52ci^3)e^2)\cos(5o-w) + (4+4ci-4ci^2-4ci^3+(52+52ci-52ci^2-52ci^3)e^2)\cos(5o+w) + (5+15ci+15ci^2+5ci^3)e^2\cos(5o+3w)\right)$$

$$d_4 = -\frac{23625}{8} e(-1+ci)^2(1+ci)^2((1-ci\ \sin(5o-w)+(1+ci)\sin(5o+w))$$

$$d_5 = -\frac{4725}{8} e\ si(-1+ci)(1+ci)((1+4ci-5ci^2)\cos(5o-w)+(-1+4ci + 5ci^2)\cos(5o+w))$$

$$d_6 = -\frac{14\ 175e}{4a}(-1+ci)^2(1+ci)^2((1-ci)\cos(5o-w)+(1+ci)\cos(5o+w))$$

Solutions of D_{60} Perturbations

$$
\begin{aligned}
\frac{\mathrm{d}a_1}{\mathrm{d}t} = & \frac{7}{1024}(e(1440cw\ si^2sw + si^4(-4320sw + si^6(2970cw\ sw))\cos M \\
& + (1920cw\ si^2sw + si^4(-5760cw\ sw) + si^6(3960cw\ sw))\cos 2M \\
& + e(15840cw\ si^2sw + si^4(-49680cw^3sw^3 - 45360cw\ sw^3) \\
& + si^6(35046cw^5sw + 65340cw^3sw^3 + 30294cw\ sw^5))\cos 3M \\
& + (si^4(5760cw^3sw - 5760cw\ sw^3) + si^6(-6336cw^5sw \\
& + 6336cw\ sw^5))\cos 4M + e(si^4(54000cw^3sw - 54000cw\ sw^3) \\
& + si^6(-64350cw^5sw + 16500cw^3sw^3 + 54450cw\ sw^5))\cos 5M \\
& + si^6(2376cw^5sw - 7920cw^3sw^3 + 2376cw\ sw^5)\cos 6M \\
& + e\ si^6(26334cw^5sw - 87780cw^3sw^3 + 26334cw\ sw^5)\cos 7M \\
& + e(320 + si^2(2640cw^2 - 4080sw^2) + si^4(5400cw^4 + 15120\ cw^2sw^2 \\
& + 9720sw^4) + si^6(-3135cw^6 - 12375cw^4sw^2 - 15345cw^2sw^4 \\
& - 6105sw^6))\sin M + (960cw^2 - 960sw^2) + si^4(-2880cw^4 + 2880sw^4) \\
& + si^6(1980cw^6 + 1980cw^4sw^2 - 1980cw^2sw^4 - 1980sw^6))\sin 2M \\
& + e(si^2(7920cw^2 - 7920sw^2) + si^4(-24300cw^4 + 3240cw^2sw^2 \\
& + 23220sw^4) + si^6(16929cw^6 + 13365cw^4sw^2 - 19305cw^2sw^4 \\
& - 15741sw^6))\sin 3M + (si^4(1440cw^4 - 8640cw^2sw^2 + 1440sw^4) \\
& + si^6(-1584cw^6 + 7920cw^4sw^2 + 7920cw^2sw^4 - 1584sw^6))\sin 4M \\
& + e(si^4(13500cw^4 - 81000cw^2sw^2 + 13500sw^4) + si^6(-15675cw^6 \\
& + 86625cw^4sw^2 + 61875cw^2sw^4 - 14025sw^6))\sin 5M + si^6(396cw^6 \\
& - 5940cw^4sw^2 + 5940cw^2sw^4 - 396sw^6)\sin 6M + e\ si^6(4389cw^6 \\
& - 65835cw^4sw^2 + 65835cw^2sw^4 - 4389sw^6)\sin 7M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}e_1}{\mathrm{d}t} = & \frac{21}{512}si^2(e(480cw\ sw + si^2(-1440cw\ sw + si^4(990cw\ sw))\cos M \\
& + (320cw\ sw + si^2(-960cw\ sw) + si^4(660cw\ sw))\cos 2M + e(1760cw\ sw \\
& + si^2(-5760cw^3sw^3 - 4800cw\ sw^3) + si^4(4158cw^5sw + 7260cw^3sw^3 \\
& + 3102cw\ sw^5))\ \cos 3M + (si^2(960cw^3sw - 960cw\ sw^3) + si^4(-1056cw^5sw \\
& + 1056cw\ sw^5))\cos 4M + e(si^2(7200cw^3sw - 7200cw\ sw^3) \\
& + si^4(-8910cw^5sw + 3300cw^3sw^3 + 6930cw\ sw^5))\cos 5M \\
& + si^4(396cw^5sw - 1320cw^3sw^3 + 396cw\ sw^5)\cos 6M + e\ si^4(3762cw^5sw \\
& - 12540cw^3sw^3 + 3762cw\ sw^5)\cos 7M + e(240cw^2 - 240sw^2 + si^2(-720cw^4
\end{aligned}
$$

$$+720sw^4)+si^4(495cw^6+495cw^4sw^2-495cw^2sw^4-495sw^6))\sin M$$
$$+(160cw^2-160sw^2+si^2(-480cw^4+480sw^4)+si^4(330cw^6+330cw^4sw^2$$
$$-330cw^2sw^4-330sw^6))\sin 2M+e(880cw^2-880sw^2+si^2(-2760cw^4$$
$$+720cw^2sw^2+2520sw^4)+si^4(1947cw^6+1155cw^4sw^2-2475cw^2sw^4$$
$$-1683sw^6))\sin 3M+(si^2 240cw^4-1440cw^2sw^2+240sw^4)+si^4(-264cw^6$$
$$+1320cw^4sw^2+1320cw^2sw^4-264sw^6))\sin 4M+e(si^2(1800cw^4$$
$$-10800cw^2sw^2+1800sw^4)+si^4(-2145cw^6+12375cw^4sw^2$$
$$+7425cw^2sw^4-1815sw^6))\sin 5M+si^4(66cw^6-990cw^4sw^2+990cw^2sw^4$$
$$-66sw^6)\sin 6M+e\,si^4(627cw^6-9405cw^4sw^2+9405cw^2sw^4$$
$$-627sw^6)\sin 7M)$$

$$\frac{\mathrm{d}\omega_1}{\mathrm{d}t}=-\frac{7}{1024}(e(960+si^2(-7680cw^2-12480sw^2)+si^4(15480cw^4$$
$$+45360cw^2sw^2+29880sw^4)+si^6(-8910cw^6-36630cw^4sw^2$$
$$-46530\,cw^2sw^4-18810\,sw^6))+(320+si^2(-2640cw^2-4080sw^2)$$
$$+si^4(5400cw^4+15120cw^2sw^2+9720sw^4)+si^6(-3135cw^6-12375cw^4sw^2$$
$$-15345cw^2sw^4-6105sw^6))\cos M+e(1600+si^2(-10560cw^2-23040sw^2)$$
$$+si^4(19260cw^4+74520cw^2sw^2+56700sw^4)+si^6(-10428cw^6$$
$$-55440cw^4sw^2-81180cw^2sw^4-36168sw^6))\cos 2M+(si^2(2640cw^2$$
$$-2640sw^2)+si^4(-8100cw^4+1080cw^2sw^2+7740sw^4)+si^6(5643\,cw^6$$
$$+4455\,cw^4sw^2-6435\,cw^2sw^4-5247sw^6))\cos 3M+e(si^2(18240cw^2$$
$$-18240sw^2)+si^4(-58680cw^4+23760cw^2sw^2+50760sw^4)$$
$$+si^6(42240cw^6+11880cw^4sw^2-55440cw^2sw^4-33528sw^6))\cos 4M$$
$$+(si^4(2700cw^4-16200cw^2sw^2+2700sw^4)+si^6(-3135cw^6$$
$$+17325cw^4sw^2+12375cw^2sw^4-2805sw^6))\cos 5M+e(si^4(23940cw^4$$
$$-143640cw^2sw^2+23940sw^4)+si^6(-29700cw^6+182160cw^4sw^2$$
$$+81180cw^2sw^4-22968sw^6))\cos 6M+si^6(627cw^6-9405cw^4sw^2$$
$$+9405cw^2sw^4-627sw^6)\cos 7M+e\,si^6(6798cw^6-101970cw^4sw^2$$
$$+101970cw^2sw^4-6798sw^6)\cos 8M+(-1440cw\,si^2sw+si^4(4320cw\,sw)$$
$$+si^6(-2970cw\,sw))\sin M+e(-12480cw\,si^2sw+si^4(36720cw^3sw$$
$$+38160cw\,sw^3)+si^6(-24948cw^5sw-51480cw^3sw^3$$
$$-26532cw\,sw^5))\sin 2M+(-5280cw\,si^2sw+si^4(16560cw^3sw$$
$$+15120cw\,sw^3)+si^6(-11682cw^5sw-21780cw^3sw^3$$

$$- 10098cw\,sw^5))\sin 3M + e(-36480cw\,si^2 sw + si^4(125280cw^3 sw + 93600cw\,sw^3) + si^6(-94248cw^5 sw - 145200cw^3 sw^3 - 59400cw\,sw^5))\sin 4M + (si^4(-10800cw^3 sw + 10800cw\,sw^3) + si^6(12870cw^5 sw - 3300cw^3 sw^3 - 10890\,cw\,sw^5))\sin 5M + e(si^4(-95760cw^3 sw + 95760cw\,sw^3) + si^6(125532cw^5 sw - 67320cw^3 sw^3 - 85140cw\,sw^5))\sin 6M + si^6(-3762cw^5 sw + 12540\,cw^3 sw^3 - 3762cw\,sw^5)\sin 7M + e\,si^6(-40788cw^5 sw + 135960cw^3 sw^3 - 40788cw\,sw^5)\sin 8M)$$

$$\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} = \frac{21}{256} ci\ si(sw\ \cos M + cw\ \sin M)(e(240sw + si^2(-1560sw) + si^4(1518sw)) + (160\ sw + si^2(-720sw) + si^4(660sw))\cos M + e(880\ sw + si^2(-3840cw^2 sw - 3360sw^3) + si^4(3201cw^4 sw + 6270cw^2 sw^3 + 3069sw^5))\cos 2M + (si^2(720cw^2 sw - 240sw^3) + si^4(-990cw^4 sw - 660cw^2 sw^3 + 330sw^5))\cos 3M + e(si^2(5400cw^2 sw - 1800sw^3) + si^4(-7854cw^4 sw - 3036cw^2 sw^3 + 2178sw^5))\cos 4M + si^4(330cw^4 sw - 660cw^2 sw^3 + 66sw^5)\cos 5M + e\ si^4(3135cw^4 sw - 6270cw^2 sw^3 + 627sw^5)\cos 6M + (160cw + si^2(-720cw) + si^4(660cw))\sin M + e(880cw + si^2(-3600\ cw^3 - 3120\ cw\ sw^2) + si^4(3135cw^5 + 6138cw^3 sw^2 + 3003cw\ sw^4))\sin 2M + (si^2(240cw^3 - 720cw\ sw^2) + si^4(-330cw^5 + 660cw^3 sw^2 + 990cw\ sw^4))\sin 3M + e(si^2(1800cw^3 - 5400cw\ sw^2) + si^4(-2508cw^5 + 6336cw^3 sw^2 + 6204cw\ sw^4))\sin 4M + si^4(66cw^5 - 660cw^3 sw^2 + 330cw\ sw^4)\sin 5M + e\ si^4(627cw^5 - 6270cw^3 sw^2 + 3135cw\ sw^4)\sin 6M)$$

$$\frac{\mathrm{d}M_1}{\mathrm{d}t} = \frac{7}{1024a}(320 + si^2(-3360) + si^4(7560) + si^6(-4620cw^6 - 13860cw^4 sw^2 - 13860cw^2 sw^4 - 4620sw^6) + e(2240 + si^2(-18480cw^2 - 28560sw^2) + si^4(37800cw^4 + 105840cw^2 sw^2 + 68040sw^4) + si^6(-21945cw^6 - 86625cw^4 sw^2 - 107415cw^2 sw^4 - 42735sw^6))\cos M + (si^2(3360cw^2 - 3360sw^2) + si^4(-10080cw^4 + 10080\ sw^4) + si^6(6930cw^6 + 6930cw^4 sw^2 - 6930\ cw^2 sw^4 - 6930sw^6))\cos 2M + e(si^2(18480cw^2 - 18480sw^2) + si^4(-56700cw^4 + 7560cw^2 sw^2 + 54180sw^4) + si^6(39501cw^6 + 31185cw^4 sw^2 - 45045cw^2 sw^4 - 36729sw^6))\ \cos 3M + (si^4(2520cw^4$$

$$- 15120cw^2sw^2 + 2520sw^4) + si^6(-2772cw^6 + 13860cw^4sw^2 + 13\,860cw^2sw^4 - 2772sw^6))\cos 4M + e(si^4(18900\,cw^4 - 113400\,cw^2sw^2 + 18900sw^4) + si^6(-21945\,cw^6 + 121275\,cw^4sw^2 + 86625cw^2sw^4 - 19635sw^6))\cos 5M + si^6(462cw^6 - 6930cw^4sw^2 + 6930cw^2sw^4 - 462sw^6)\cos 6M + e\,si^6(4389cw^6 - 65835\,cw^4sw^2 + 65835cw^2sw^4 - 4389sw^6)\cos 7M + e(-10080cw\,si^2sw + si^4(30240cw\,sw) + si^6(-20790cw\,sw))\sin M + (-6720cw\,si^2sw + si^4(20160cw\,sw) + si^6(-13860cw\,sw))\sin 2M + e(-36960cw\,si^2sw + si^4(115920cw^3sw + 105840cw\,sw^3) + si^6(-81774cw^5sw - 152460cw^3sw^3 - 70686cw\,sw^5))\sin 3M + (si^4(-10080cw^3sw + 10080cw\,sw^3) + si^6(11088cw^5sw - 11088cw\,sw^5))\sin 4M + e(si^4(-75600cw^3sw + 75600cw\,sw^3) + si^6(90090\,cw^5sw - 23100cw^3sw^3 - 76230cw\,sw^5))\,\sin 5M + si^6(-2772cw^5sw + 9240cw^3sw^3 - 2772cw\,sw^5)\sin 6M + e\,si^6(-26334cw^5sw + 87780cw^3sw^3 - 26334cw\,sw^5)\sin 7M)$$

$$d_3 = -\frac{105}{512}e(32 - 336si^2 + 756si^4 - 462si^6 + (80si^2 - 240si^4 + 165si^6)\cos(2w))$$

$$d_5 = \frac{105}{128}ci\ si(8 - 36si^2 + 33si^4)$$

$$d_6 = -\frac{35}{256a}(-16 + 168si^2 - 378si^4 + 231si^6)$$

Solutions of $D_{61} \sim D_{66}$ Perturbations

Solutions of D_{61} Perturbation

$$d_1 = \frac{105}{128}\cos(o)si(8 - 36si^2 + 33si^4)$$

$$d_3 = \frac{105}{2048}e\ si((-2688ci + 12096ci\ si^2 - 11088ci\ si^4)\sin(o) + (-496 + 496ci + (2016 - 2448ci)si^2 + (-1749 + 2343ci)si^4)\sin(o - 2w) + (496 + 496ci + (-2016 - 2448ci)si^2 + (1749 + 2343ci)si^4)\sin(o + 2w))$$

$$d_4 = -\frac{105}{128}\cos(o)ci\ si(8 - 36si^2 + 33si^4)$$

$$d_5 = -\frac{105}{128}\sin(o)(8ci^2 + (-8 - 108ci^2)si^2 + (36 + 165ci^2)si^4 - 33si^6)$$

$$d_6 = \frac{735}{128a}\sin(o)ci\ si(8 - 36si^2 + 33si^4)$$

Solutions of D_{62} Perturbation

$$d_1 = -\frac{315}{8}\sin(o)\cos(o)ci\ si(-6 + 11si^2)$$

$$\begin{aligned} d_3 = &\frac{315}{1024}e((448 - 448ci^2 + (-2016 + 6048ci^2)si^2 + (1848 - 9240ci^2)si^4)\cos(2o) \\ &+ (112 - 224\,ci + 112ci^2 + (-528 + 2016ci - 1488ci^2)si^2 + (495 - 2794ci \\ &+ 2255ci^2)si^4)\cos(2o - 2w) + (112 + 224ci + 112ci^2 + (-528 - 2016ci \\ &- 1488ci^2)si^2 + (495 + 2794ci + 2255ci^2)si^4)\cos(2o + 2w)) \end{aligned}$$

$$d_3 = \frac{105}{32}\sin(o)\cos(o)(-8 + 8ci^2 + (36 - 108ci^2)si^2 + (-33 + 165ci^2)si^4)$$

$$d_5 = \frac{105}{32}\sin(o + \frac{\pi}{4})\sin(o - \frac{\pi}{4})ci\ si(28 - 108ci^2 + (42 + 330ci^2)si^2 - 165si^4)$$

$$d_6 = -\frac{735}{64a}\sin(o + \frac{\pi}{4})\sin(o - \frac{\pi}{4})(-8 + 8ci^2 + (36 - 108ci^2)si^2 + (-33 + 165ci^2)si^4)$$

Solutions of D_{63} Perturbation

$$d_1 = -\frac{945}{32}\cos(o)(-1 + 2\cos(2o))si(6 - 6ci^2 + (-11 + 33ci^2)si^2)$$

$$\begin{aligned} d_3 = &-\frac{945}{512}e\ si((-2016ci + 2016ci^3 + (3696ci - 6160ci^3)si^2)\sin(3o) \\ &+ (-160 - 264ci + 1008ci^2 - 584ci^3 + (341 + 627ci - 2673ci^2 \\ &+ 1705ci^3)si^2)\sin(3o - 2w) + (160 - 264ci - 1008ci^2 - 584ci^3 \\ &+ (-341 + 627ci + 2673ci^2 + 1705ci^3)si^2)\sin(3o + 2w)) \end{aligned}$$

$$d_4 = \frac{945}{32}\cos(o)(-1+2\cos(2o))ci\ si(18-18ci^2+(-33+55ci^2)si^2)$$

$$d_5 = \frac{945}{32}\sin(o)(1+2\cos(2o))(6ci^2-6ci^4+(-6-15ci^2+55ci^4)si^2+(11-55ci^2)si^4)$$

$$d_6 = -\frac{2205}{32a}\sin(o)(1+2\cos(2o))ci\ si(18-18ci^2+(-33+55ci^2)si^2)$$

Solutions of D_{64} Perturbation

$$d_1 = -20790\sin(\mathrm{o})\cos(\mathrm{o})\sin(\mathrm{o}+\frac{\pi}{4})\sin(\mathrm{o}-\frac{\pi}{4})(-1+\mathrm{ci})\mathrm{ci}(1+\mathrm{ci})\mathrm{si}^2$$

$$\begin{aligned} d_3 = -\frac{945}{64}e((&252-504ci^2+252ci^4+(-462+2772ci^2-2310ci^4)si^2)\cos(4o) \\ &+(84-168ci+168ci^3-84ci^4+(-209+572ci+198ci^2-1276ci^3 \\ &+715ci^4)si^2)\cos(4o-2w)+(84+168ci-168ci^3-84ci^4+(-209-572ci \\ &+198ci^2+1276ci^3+715ci^4)si^2)\cos(4o+2w)) \end{aligned}$$

$$\begin{aligned} d_4 = 945\sin(o)\cos(o)\sin(o+\frac{\pi}{4})\sin(o-\frac{\pi}{4})(-1+ci)(1+ci)(6-6ci^2+(-11 \\ +55ci^2)si^2) \end{aligned}$$

$$d_5 = -\frac{945}{8}\sin(2o+\frac{\pi}{4})\sin(2o-\frac{\pi}{4})ci\ si(-1-54ci^2+55ci^4+(66-110ci^2)si^2)$$

$$d_6 = \frac{6615}{16a}\sin(2o+\frac{\pi}{4})\sin(2o-\frac{\pi}{4})(-1+ci)(1+ci)(6-6ci^2+(-11+55ci^2)si^2)$$

Solutions of D_{65} Perturbation

$$d_1 = \frac{51975}{16}\cos(o)(1-2\cos(2o)+2\cos(4o))si(-1+ci)^2(1+ci)^2$$

$$\begin{aligned} d_3 = \frac{51975}{256}(-1+ci)(1+ci)((&336ci-336ci^3)\sin(5o)+(9+97ci-221ci^2 \\ &+115ci^3)\sin(5o-2\,w)+(-9+97ci+221ci^2+115ci^3)\sin(5o+2w))\ e\ si \end{aligned}$$

$$d_4 = -\frac{259\,875}{16}\cos(o)(1 - 2\cos(2o) + 2\cos(4o))(-1 + ci)^2 ci(1 + ci)^2 si$$

$$d_5 = -\frac{51\,975}{16}\sin(o)(1 + 2\cos(2\,o) + 2\cos(4o))(-1 + ci)(1 + ci)(-ci^2 + ci^4 + (1 - 5ci^2)si^2)$$

$$d_6 = \frac{363\,825}{16a}\sin(o)(1 + 2\cos(2o) + 2\cos(4o))(-1 + ci)^2 ci(1 + ci)^2 si$$

Solutions of D_{66} Perturbation

$$d_3 = \frac{1091475}{128}e(-1 + ci)^2(1 + ci)^2 ((8 - 8ci^2)\cos(6o) + (3 - 6ci + 3ci^2)\cos(6o - 2w) + (3 + 6ci + 3ci^2) \times \cos(6o + 2w))$$

$$d_4 = \frac{155925}{4}\sin(o)\cos(o)(-1 + 2\cos(2o)(1 + 2\cos(2o))(-1 + ci)^3(1 + ci)^3$$

$$d_5 = \frac{155925}{4}\sin(o + \frac{\pi}{4})\ \sin(o - \frac{\pi}{4})(-1 + 2\sin(2o))(1 + 2\sin(2o))(-1 + ci)^2 \times ci(1 + ci)^2 si$$

$$d_6 = \frac{363825}{8a}\sin(\mathrm{o} + \frac{\pi}{4})\sin(\mathrm{o} - \frac{\pi}{4})(-1 + 2\sin(2\mathrm{o}))(1 + 2\ \sin(2\mathrm{o}))(-1 + \mathrm{ci})^3(1 + \mathrm{ci})^3$$

Solutions of D_{70} Perturbation

$$\frac{\mathrm{d}a_1}{\mathrm{d}t} = -\frac{1}{1024}si((2240cw + si^2(-15120cw) + si^4(27720cw) + si^6(-15015cw^7 - 45045cw^5sw^2 - 45045cw^3sw^4 - 15015cw\,sw^6))\cos M + e(22400cw + si^2(-141120cw^3 - 181440cw\,sw^2) + si^4(249480cw^5 + 609840cw^3sw^2 + 360360cw\,sw^4) + si^6(-132132cw^7 - 468468cw^5sw^2 - 540540cw^3sw^4 - 204204cw\,sw^6))\cos 2M + (si^2(15120cw^3 - 45360cw\,sw^2) + si^4(-41580cw^5 + 83160cw^3sw^2 + 124740cw\,sw^4) + si^6(27027cw^7$$

$$
\begin{aligned}
&-27027cw^5sw^2-135135cw^3sw^4-81081cw\,sw^6))\cos 3M\\
&+e(si^2(141120cw^3-423360cw\,sw^2)+si^4(-399168cw^5+887040cw^3sw^2\\
&+1108800cw\,sw^4)+si^6(264264cw^7-360360cw^5sw^2-1321320cw^3sw^4\\
&-696696cw\,sw^6))\cos 4M+(si^4(13860cw^5-138600cw^3sw^2\\
&+69300cw\,sw^4)+si^6(-15015cw^7+135135cw^5sw^2+75075cw^3sw^4\\
&-75075cw\,sw^6))\cos 5M+e(si^4(149688cw^5-1\,496880cw^3sw^2\\
&+748440cw\,sw^4)+si^6(-169884cw^7+1621620cw^5sw^2+540540cw^3sw^4\\
&-756756cw\,sw^6))\cos 6M+si^6(3003cw^7-63063cw^5sw^2+105105cw^3sw^4\\
&-21021cw\,sw^6)\cos 7M+e\,si^6(37752cw^7-792792cw^5sw^2\\
&+1321320cw^3sw^4-264264cw\,sw^6)\cos 8M+(-2240sw+si^2(15120sw)\\
&+si^4(-27720sw)+si^6(15015cw^6sw+45045cw^4sw^3+45045cw^2sw^5\\
&+15015sw^7))\sin M+e(-22400sw+si^2(120960cw^2sw+161280sw^3)\\
&+si^4(-194040cw^4sw-498960cw^2sw^3-304920sw^5)+si^6(96096cw^6sw\\
&+360360cw^4sw^3+432432cw^2sw^5+168168sw^7))\sin 2M\\
&+(si^2(-45360cw^2sw+15120sw^3)+si^4(124740cw^4sw+83160cw^2sw^3\\
&-41580sw^5)+si^6(-81081cw^6sw-135135cw^4sw^3-27027cw^2sw^5\\
&+27027sw^7))\sin 3M+e(si^2(-423360cw^2sw+141120sw^3)\\
&+si^4(1219680cw^4sw+665280cw^2sw^3-376992sw^5)+si^6(-816816cw^6sw\\
&-1201200cw^4sw^3-144144cw^2sw^5+240240sw^7))\sin 4M\\
&+(si^4(-69300cw^4sw+138600cw^2sw^3-13860sw^5)+si^6(75075cw^6sw\\
&-75075cw^4sw^3-135135cw^2sw^5+15015sw^7))\sin 5M\\
&+e(si^4(-748440cw^4sw+1496880cw^2sw^3-149688sw^5)\\
&+si^6(864864cw^6sw-1081080cw^4sw^3-1297296cw^2sw^5\\
&+154440sw^7))\sin 6M+si^6(-21021cw^6sw+105105cw^4sw^3\\
&-63063cw^2sw^5+3003sw^7)\sin 7M+e\,si^6(-264264cw^6sw\\
&+1321320cw^4sw^3-792792cw^2sw^5+37752sw^7)\sin 8M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}e_1}{\mathrm{d}t}=&-\frac{7}{1024}si(e(960cw+si^2(-6480cw)+si^4(11880cw)+si^6(-6435cw^7\\
&-19305\,cw^5sw^2-19305cw^3sw^4-6435cw\,sw^6))+(320cw\\
&+si^2(-2160cw)+si^4(3960cw)+si^6(-2145cw^7-6435cw^5sw^2\\
&-6435cw^3sw^4-2145cw\,sw^6))\cos M+e(1600cw+si^2(-8640cw^3\\
&-17280cw\,sw^2)+si^4(13860cw^5+51480cw^3sw^2+37620cw\,sw^4)
\end{aligned}
$$

$$+ si^6(-6864cw^7 - 36036cw^5sw^2 - 51480cw^3sw^4 - 22308cw\,sw^6))\cos 2M$$
$$+ (si^2(2160cw^3 - 6480cw\,sw^2) + si^4(-5940cw^5 + 11880cw^3sw^2$$
$$+ 17820cw\,sw^4) + si^6(3861cw^7 - 3861cw^5sw^2 - 19305\,cw^3sw^4$$
$$- 11\,583cw\,sw^6))\cos 3M + e(si^2(15120cw^3 - 45360cw\,sw^2)$$
$$+ si^4(-43560cw^5 + 102960cw^3sw^2 + 114840cw\,sw^4) + si^6(29172cw^7$$
$$- 46332cw^5sw^2 - 145860cw^3sw^4 - 70356cw\,sw^6))\,\cos 4M$$
$$+ (si^4(1980cw^5 - 19800\,cw^3sw^2 + 9900cw\,sw^4) + si^6(-2145cw^7$$
$$+ 19305cw^5sw^2 + 10725cw^3sw^4 - 10725cw\,sw^6))\cos 5M$$
$$+ e(si^4(17820cw^5 - 178200cw^3sw^2 + 89100cw\,sw^4) + si^6(-20592cw^7$$
$$+ 200772cw^5sw^2 + 51480cw^3sw^4 - 87516cw\,sw^6))\cos 6M$$
$$+ si^6(429cw^7 - 9009cw^5sw^2 + 15015cw^3sw^4 - 3003cw\,sw^6)\cos 7M$$
$$+ e\,si^6(4719cw^7 - 99099cw^5sw^2 + 165165cw^3sw^4$$
$$- 33033cw\,sw^6)\cos(8M) + (-320sw + si^2(2160csw) + si^4(-3960sw)$$
$$+ si^6(2145cw^6sw + 6435cw^4sw^3 + 6435cw^2sw^5 + 2145sw^7))\sin M$$
$$+ e(-1600sw + si^2(4320cw^2sw + 12960sw^3) + si^4(-1980cw^4sw$$
$$- 27720cw^2sw^3 - 25740sw^5) + si^6(-858cw^6sw + 12870cw^4sw^3$$
$$+ 28314cw^2sw^5 + 14586sw^7))\sin 2M + (si^2(-6480cw^2sw + 2160sw^3)$$
$$+ si^4(17820cw^4sw + 11880cw^2sw^3 - 5940sw^5) + si^6(-11583cw^6sw$$
$$- 19305cw^4sw^3 - 3861cw^2sw^5 + 3861sw^7))\sin 3M + e(si^2(-45360cw^2sw$$
$$+ 15120sw^3) + si^4(134640cw^4sw + 63360cw^2sw^3 - 39600sw^5)$$
$$+ si^6(-91806cw^6sw - 124410cw^4sw^3 - 7722cw^2sw^5 + 24882sw^7\sin 4M$$
$$+ (si^4(-9900cw^4sw + 19800cw^2sw^3 - 1980sw^5) + si^6(10725cw^6sw$$
$$- 10725cw^4sw^3 - 19305cw^2sw^5 + 2145sw^7))\sin 5M + e(si^4(-89100cw^4sw$$
$$+ 178200cw^2sw^3 - 17820sw^5) + si^6(105534cw^6sw - 141570cw^4sw^3$$
$$- 146718cw^2sw^5 + 18018sw^7))\sin 6M + si^6(-3003cw^6sw + 15015cw^4sw^3$$
$$- 9009cw^2sw^5 + 429sw^7)\sin 7M + e\,si^6(-33033cw^6sw + 165165cw^4sw^3$$
$$- 99099cw^2sw^5 + 4719sw^7)\sin 8M)$$

$$\frac{d\omega_1}{dt} = \frac{1}{4096}si(-26880sw + si^2(181440sw) + si^4(-332640sw)$$
$$+ si^6(180180cw^6sw + 540540cw^4sw^3 + 540540cw^2sw^5 + 180180sw^7)$$
$$+ e(-356160sw + si^2(2207520cw^2sw + 2469600sw^3)$$
$$+ si^4(-3866940cw^4sw - 8454600cw^2sw^3 - 4587660sw^5)$$

$$
\begin{aligned}
&+si^6(2036034cw^6sw+6576570cw^4sw^3+7045038cw^2sw^5\\
&+2504502sw^7))\cos M+(-44800sw+si^2(241920cw^2sw+322560sw^3)\\
&+si^4(-388080cw^4sw-997920cw^2sw^3-609840sw^5)+si^6(192192cw^6sw\\
&+720720cw^4sw^3+864864cw^2sw^5+336336sw^7))\cos 2M+e(-288960sw\\
&+si^2(1345680cw^2sw+2152080sw^3)+si^4(-1954260cw^4sw\\
&-5959800cw^2sw^3-4138596sw^5)+si^6(900900cw^6sw+3963960cw^4sw^3\\
&+5369364cw^2sw^5+2306304sw^7))\cos 3M+(si^2(-423360cw^2sw\\
&+141120sw^3)+si^4(1219680cw^4sw+665280cw^2sw^3-376992\,sw^5)\\
&+si^6(-816816cw^6sw-1201200cw^4sw^3-144144cw^2sw^5\\
&+240240\,sw^7))\cos 4M+e(si^2(-3553200cw^2sw+1184400sw^3)\\
&+si^4(10990980cw^4sw+4074840cw^2sw^3-3013164sw^5)\\
&+si^6(-7747740cw^6sw-8888880cw^4sw^3+36036cw^2sw^5\\
&+1863576sw^7))\cos 5M+(si^4(-498960cw^4sw+997920cw^2sw^3\\
&-99792sw^5)+si^6(576576cw^6sw-720720cw^4sw^3-864864cw^2sw^5\\
&+102960sw^7))\cos 6M+e(si^4(-5169780cw^4sw+10339560cw^2sw^3\\
&-1033956sw^5)+si^6(6441435cw^6sw-9804795cw^4sw^3-7558551cw^2sw^5\\
&+999999sw^7))\cos 7M+si^6(-132132cw^6sw+660660cw^4sw^3\\
&-396396cw^2sw^5+18876sw^7)\cos 8M+e\,si^6(-1630629cw^6sw\\
&+8153145cw^4sw^3-4891887cw^2sw^5+232947sw^7)\cos 9M+e(-109760cw\\
&+si^2(675360cw^3+937440cw\,sw^2)+si^4(-1178100cw^5-3076920cw^3sw^2\\
&-1898820cw\,sw^4)+si^6(618618cw^7+2324322cw^5sw^2+2792790cw^3sw^4\\
&+1087086cw\,sw^6))\sin M+(-44800cw+si^2(282240cw^3+362880cw\,sw^2)\\
&+si^4(-498960cw^5-1219680cw^3sw^2-720720cw\,sw^4)+si^6(264264cw^7\\
&+936936cw^5sw^2+1081080cw^3sw^4+408408cw\,sw^6))\sin 2M\\
&+e(-288960cw+si^2(1748880cw^3+2555280cw\,sw^2)+si^4(-3029796cw^5\\
&-8177400cw^3sw^2-5280660cw\,sw^4)+si^6(1585584cw^7+6090084cw^5sw^2\\
&+7567560cw^3sw^4+3063060cw\,sw^6))\sin 3M+(si^2(-141120cw^3\\
&+423360cw\,sw^2)+si^4(399168cw^5-887040cw^3sw^2-1108800cw\,sw^4)\\
&+si^6(-264264cw^7+360360cw^5sw^2+1321320cw^3sw^4+696696cw\,sw^6))\\
&\times \sin 4M+e(si^2(-1\,184400cw^3+3553200cw\,sw^2)+si^4(3501036cw^5\\
&-8953560cw^3sw^2-8551620cw\,sw^4)+si^6(-2392104cw^7\\
&+4720716cw^5sw^2+11531520cw^3sw^4+5105100cw\,sw^6))\sin 5M
\end{aligned}
$$

$$
\begin{aligned}
&+(si^4(-99792cw^5+997920cw^3sw^2-498960cw\,sw^4)+si^6(113256cw^7\\
&-1081080cw^5sw^2-360360cw^3sw^4+504504cw\,sw^6))\sin 6M\\
&+e(si^4(-1033956cw^5+10339560cw^3sw^2-5169780cw\,sw^4)\\
&+si^6(1240239cw^7-12603591cw^5sw^2-1396395cw^3sw^4\\
&+4759755cw\,sw^6))\sin 7M+si^6(-18876cw^7+396396cw^5sw^2\\
&-660660cw^3sw^4+132132cw\,sw^6)\sin 8M+e\,si^6(-232947cw^7\\
&+4891887cw^5sw^2-8153145cw^3sw^4+1630629cw\,sw^6)\sin 9M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\Omega_1}{dt}=&\frac{7}{1024}ci(e(-960sw+si^2(19440sw)+si^4(-59400sw)+si^6(45045cw^6sw\\
&+135135cw^4sw^3+135135cw^2sw^5+45045sw^7))+(-320sw\\
&+si^2(6480sw)+si^4(-19800sw)+si^6(15015cw^6sw+45045cw^4sw^3\\
&+45045cw^2sw^5+15015sw^7))\cos M+e(-1600sw+si^2(25920cw^2sw\\
&+34560sw^3)+si^4(-69300cw^4sw-178200cw^2sw^3-108900sw^5)\\
&+si^6(48048cw^6sw+180180cw^4sw^3+216216cw^2sw^5+84084sw^7))\cos 2M\\
&+(si^2(-6480cw^2sw+2160sw^3)+si^4(29700cw^4sw+19800cw^2sw^3\\
&-9900sw^5)+si^6(-27027cw^6sw-45045cw^4sw^3-9009cw^2sw^5\\
&+9009sw^7))\cos 3M+e(si^2(-45360cw^2sw+15120sw^3)+si^4(217800cw^4sw\\
&+118800cw^2sw^3-67320sw^5)+si^6(-20420cw^6sw-300300cw^4sw^3\\
&-36036cw^2sw^5+60060sw^7))\cos 4M+(si^4(-9900\,cw^4sw+19800cw^2sw^3\\
&-1980sw^5)+si^6(15015cw^6sw-15015cw^4sw^3-27027cw^2sw^5\\
&+3003sw^7))\cos 5M+e(si^4(-89100cw^4sw+178200cw^2sw^3-17820sw^5)\\
&+si^6(144144cw^6sw-180180cw^4sw^3-216216cw^2sw^5+25740\,sw^7))\cos 6M\\
&+si^6(-3003cw^6sw+15015cw^4sw^3-9009cw^2sw^5+429sw^7)\cos 7M\\
&+e\,si^6(-33033\,cw^6sw+165165cw^4sw^3-99099\,cw^2sw^5+4719\,sw^7)\cos 8M\\
&+(-320cw+si^2(6480cw)+si^4(-19800cw)+si^6(15015cw^7\\
&+45045cw^5sw^2+45045cw^3sw^4+15015cw\,sw^6))\sin M+e(-1600cw\\
&+si^2(30240cw^3+38880cw\,sw^2)+si^4(-89100cw^5-217800cw^3sw^2\\
&-128700cw\,sw^4)+si^6(66066cw^7+234234cw^5sw^2+270270cw^3sw^4\\
&+102102cw\,sw^6))\sin 2M+(si^2(-2160cw^3+6480cw\,sw^2)+si^4(9900cw^5\\
&-19800cw^3sw^2-29700cw\,sw^4)+si^6(-9009cw^7+9009cw^5sw^2\\
&+45045cw^3sw^4+27027cw\,sw^6))\sin 3M+e(si^2(-15120cw^3
\end{aligned}
$$

$$
\begin{aligned}
&+45360cw\,sw^2)+si^4(71280cw^5-158400cw^3sw^2-198000cw\,sw^4)\\
&+si^6(-66066cw^7+90090cw^5sw^2+330330cw^3sw^4+174174cw\,sw^6))\\
&\times\sin(4M)+(si^4(-1980cw^5+19800cw^3sw^2-9900cw\,sw^4)+si^6(3003cw^7\\
&-27027cw^5sw^2-15015cw^3sw^4+15015cw\,sw^6))\sin(5M)\\
&+e(si^4(-17820cw^5+178200cw^3sw^2-89100cw\,sw^4)\\
&+si^6(28314cw^7-270270cw^5sw^2-90090cw^3sw^4+126126cw\,sw^6))\\
&\times\sin(6M)+si^6(-429cw^7+9009cw^5sw^2-15015cw^3sw^4+3003cw\,sw^6)\\
&\times\sin(7M)+e\,si^6(-4719cw^7+99099cw^5sw^2-165165cw^3sw^4\\
&+33033cw\,sw^6)\sin(8M))
\end{aligned}
$$

$$
\begin{aligned}
\frac{dM_1}{dt}=&-\frac{1}{128a}si(e(-6720sw+si^2(45360sw)+si^4(-83160sw)+si^6(45045cw^6sw\\
&+135135cw^4sw^3+135135cw^2sw^5+45045sw^7))+(-2240sw\\
&+si^2(15120sw)+si^4(-27720sw)+si^6(15015cw^6sw+45045cw^4sw^3\\
&+45045\,cw^2sw^5+15015sw^7))\cos M+e(-11200sw+si^2(60480cw^2sw\\
&+80640sw^3)+si^4(-97020cw^4sw-249480cw^2sw^3-152460sw^5)\\
&+si^6(48048cw^6sw+180180cw^4sw^3+216216cw^2sw^5+84084\,sw^7))\cos 2M\\
&+(si^2(-15120cw^2sw+5040sw^3)+si^4(41580cw^4sw+27720cw^2sw^3\\
&-13860sw^5)+si^6(-27027cw^6sw-45045cw^4sw^3-9009cw^2sw^5\\
&+9009sw^7))\cos 3M+e(si^2(-105840cw^2sw+35280sw^3)\\
&+si^4(304920cw^4sw+166320cw^2sw^3-94248sw^5)+si^6(-204204cw^6sw\\
&-300300cw^4sw^3-36036cw^2sw^5+60060sw^7))\cos 4M\\
&+(si^4(-13860cw^4sw+27720cw^2sw^3-2772sw^5)+si^6(15015cw^6sw\\
&-15015cw^4sw^3-27027cw^2sw^5+3003sw^7))\cos 5M\\
&+e(si^4(-124740cw^4sw+249480cw^2sw^3-24948sw^5)+si^6(144144cw^6sw\\
&-180180cw^4sw^3-216216cw^2sw^5+25740sw^7))\cos 6M+si^6(-3003cw^6sw\\
&+15015cw^4sw^3-9009cw^2sw^5+429sw^7)\cos 7M+e\,si^6(-33033cw^6sw\\
&+165165cw^4sw^3-99099cw^2sw^5+4719sw^7)\cos 8M+(-2240cw\\
&+si^2(15120cw)+si^4(-27720cw)+si^6(15015cw^7+45045cw^5sw^2\\
&+45045cw^3sw^4+15015cw\,sw^6))\sin M+e(-11200cw+si^2(70560cw^3\\
&+90720cw\,sw^2)+si^4(-124740cw^5-304920cw^3sw^2-180180cw\,sw^4)\\
&+si^6(66066cw^7+234234cw^5sw^2+270270cw^3sw^4+102102cw\,sw^6))\\
&\times\sin 2M+(si^2(-5040cw^3+15120cw\,sw^2)+si^4(13860cw^5-27720cw^3sw^2
\end{aligned}
$$

$$-41580cw\,sw^4)+si^6(-9009cw^7+9009cw^5sw^2+45045cw^3sw^4$$
$$+27027cw\,sw^6))\sin 3M+e(si^2(-35280cw^3+105840cw\,sw^2)$$
$$+si^4(99792cw^5-221760cw^3sw^2-277200cw\,sw^4)+si^6(-66066cw^7$$
$$+90090cw^5sw^2+330330cw^3sw^4+174174cw\,sw^6))\sin 4M$$
$$+(si^4(-2772cw^5+27720cw^3sw^2-13860cw\,sw^4)+si^6(3003cw^7$$
$$-27027cw^5sw^2-15015cw^3sw^4+15015cw\,sw^6))\sin 5M$$
$$+e(si^4(-24948cw^5+249480cw^3sw^2-124740cw\,sw^4)+si^6(28314cw^7$$
$$-270270cw^5sw^2-90090cw^3sw^4+126126cw\,sw^6))\sin 6M$$
$$+si^6(-429cw^7+9009cw^5sw^2-15015cw^3sw^4+3003cw\,sw^6)\sin 7M$$
$$+e\,si^6(-4719cw^7+99099cw^5sw^2-165165cw^3sw^4+33033cw\,sw^6)\sin 8M)$$

$$d_2=\frac{105}{1024}\cos(w)e\;si(-64+432si^2-792si^4+429si^6)$$

$$d_3=\frac{105}{1024}\sin(w)si(-64+432si^2-792si^4+429si^6)$$

$$d_5=\frac{105}{1024}\sin(w)ci\;e(-64+1296si^2-3960si^4+3003si^6)$$

$$d_6=-\frac{105}{128a}\sin(w)e\;si(-64+432si^2-792si^4+429si^6)$$

Solutions of $D_{71} \sim D_{77}$ Perturbations

Solutions of D_{71} Perturbation

$$d_1=\frac{35}{512}e$$
$$((64-64ci+(-1296+432ci)si^2+(3960-792ci)si^4+(-3003$$
$$+429ci)si^6)\sin(o-w)+(-64-64ci+(1296+432ci)si^2+(-3960$$
$$-792ci)si^4+(3003+429ci)si^6)\sin(o+w))$$

$$d_2=-\frac{105}{2048}e$$
$$((64-64ci+(-432+1296ci)si^2+(792-3960ci)si^4+(-429$$
$$+3003ci)si^6)\sin(o-w)+(-64-64ci+(432+1296ci)si^2+(-792$$
$$-3960ci)si^4+(429+3003ci)si^6)\sin(o+w))$$

$$d_3 = \frac{35}{1024} ((-128 + 128ci + (1296 - 2160ci)si^2 + (-3168 + 6336ci)si^4 + (2145 - 4719ci)si^6)\cos(o - w) + (-128 - 128ci + (1296 + 2160ci)si^2 + (-3168 - 6336ci)si^4 + (2145 + 4719ci)si^6)\cos(o + w))$$

$$d_4 = -\frac{105}{2048} e ((-64 + 64ci + (432 - 1296ci)si^2 + (-792 + 3960ci)si^4 + (429 - 3003ci)si^6)\sin(o - w) + (-64 - 64ci + (432 + 1296ci)si^2 + (-792 - 3960ci)si^4 + (429 + 3003ci)si^6)\sin(o + w))$$

$$d_5 = -\frac{105}{2048} e\ si ((64 - 864ci + 2592ci^2 + (-1296 + 3168ci - 15840ci^2)si^2 + (3960 - 2574ci + 18018ci^2)si^4 - 3003si^6)\cos(o - w) + (-64 - 864ci - 2592ci^2 + (1296 + 3168ci + 15840ci^2)si^2 + (-3960 - 2574ci - 18018ci^2)si^4 + 3003si^6)\cos(o + w))$$

$$d_6 = -\frac{105}{256a} e ((-64 + 64ci + (432 - 1296ci)si^2 + (-792 + 3960ci)si^4 + (429 - 3003ci)si^6)\cos(o - w) + (-64 - 64ci + (432 + 1296ci)si^2 + (-792 - 3960ci)si^4 + (429 + 3003\ ci)si^6)\cos(o + w))$$

Solutions of D_{72} Perturbation

$$d_1 = \frac{315}{128} e\ si ((48 - 96ci + 48ci^2 + (-176 + 704ci - 176ci^2)si^2 + (143 - 858ci + 143ci^2)si^4)\cos(2o - w) + (48 + 96ci + 48ci^2 + (-176 - 704ci - 176ci^2)si^2 + (143 + 858ci + 143ci^2)si^4)\cos(2o + w))$$

$$d_2 = -\frac{945}{1024} e\ si ((-48 - 96ci + 144ci^2 + (176 + 352ci - 880ci^2)si^2 + (-143 - 286ci + 1001ci^2)si^4)\cos(2o - w) + (-48 + 96ci + 144ci^2 + (176 - 352ci - 880ci^2)si^2 + (-143 + 286ci + 1001ci^2)si^4)\cos(2o + w))$$

$$d_3 = -\frac{315}{1024} si$$
$$((48 + 480ci - 528ci^2 + (-176 - 2464ci + 2992ci^2)si^2 + (143 + 2574ci - 3289ci^2)si^4)\ \sin(2o - w) + (-48 + 480ci + 528ci^2 + (176 - 2464ci - 2992ci^2)si^2 + (-143 + 2574ci + 3289ci^2)si^4)\sin(2o + w))$$

$$d_4 = -\frac{945}{512} e\ si$$
$$((48 + 96ci - 144ci^2 + (-176 - 352ci + 880ci^2)si^2(143 + 286ci - 1001ci^2)si^4)\cos(2o - w) + (-48 + 96ci + 144ci^2 + (176 - 352ci - 880ci^2)si^2 + (-143 + 286ci + 1001ci^2)si^4)\cos(2o + w))$$

$$d_5 = -\frac{945}{1024} e$$
$$((48ci + 96ci^2 - 144ci^3 + (-96 - 240ci - 1056ci^2 + 2640ci^3)si^2 + (352 - 1045ci + 1430ci^2 - 5005ci^3)si^4 + (-286 + 2002ci)si^6)\sin(2o - w) + (-48ci + 96ci^2 + 144ci^3 + (-96 + 240ci - 1056ci^2 - 2640ci^3)si^2 + (352 + 1045ci + 1430ci^2 + 5005ci^3)si^4 + (-286 - 2002ci)si^6)\sin(2o + w))$$

$$d_6 = \frac{945}{128a} e\ si$$
$$((48 + 96ci - 144ci^2 + (-176 - 352ci + 880ci^2)si^2 + (143 + 286ci - 1001ci^2)si^4)\sin(2o - w) + (-48 + 96ci + 144ci^2 + (176 - 352ci - 880ci^2)si^2 + (-143 + 286ci + 1001ci^2)si^4)\sin(2o + w))$$

Solutions of D_{73} Perturbation

$$d_1 = -\frac{945}{256} e$$
$$((48 - 48ci - 48ci^2 + 48ci^3 + (-528 - 528ci + 1584ci^2 - 528ci^3)si^2 + (715 + 1001ci - 3575ci^2 + 715ci^3)si^4)\sin(3o - w) + (-48 - 48ci + 48ci^2 + 48ci^3 + (528 - 528ci - 1584ci^2 - 528ci^3)si^2 + (-715 + 1001ci + 3575ci^2 + 715ci^3)si^4)\sin(3o + w))$$

$$d_2 = \frac{945}{1024} e$$
$$((144 - 144ci - 144ci^2 + 144ci^3 + (-528 + 1584ci + 1584ci^2 - 2640ci^3)si^2$$
$$+ (429 - 2145ci - 2145ci^2 + 5005ci^3)si^4)\sin(3o - w) + (-144 - 144ci$$
$$+ 144ci^2 + 144ci^3 + (528 + 1584ci - 1584ci^2 - 2640ci^3)si^2 + (-429$$
$$- 2145ci + 2145ci^2 + 5005ci^3)si^4)\sin(3o + w))$$

$$d_3 = -\frac{945}{128}((1 - ci)\cos(3o - w) + (1 + ci)\cos(3o + w))$$
$$(-24 + 24ci^2 + (132 - 396ci^2)si^2 + (-143 + 715ci^2)si^4)$$

$$d_4 = \frac{2835}{1024} e$$
$$((-144 + 144ci + 144ci^2 - 144ci^3 + (528 - 1584ci - 1584ci^2 + 2640ci^3)si^2$$
$$+ (-429 + 2145ci + 2145ci^2 - 5005ci^3)si^4)\sin(3o - w) + (-144 - 144ci$$
$$+ 144ci^2 + 144ci^3 + (528 + 1584ci - 1584ci^2 - 2640ci^3)si^2 + (-429$$
$$- 2145ci + 2145ci^2 + 5005ci^3)si^4)\sin(3o + w))$$

$$d_5 = \frac{945}{1024} e\, si$$
$$((144 - 768ci + 2736ci^2 + 3168ci^3 - 5280ci^4 + (-1584 - 1452ci - 660ci^2$$
$$- 8580ci^3 + 20020ci^4)si^2 + (2145 + 4290ci - 15015\, ci^2)si^4)\cos(3o - w)$$
$$+ (-144 - 768ci - 2736ci^2 + 3168ci^3 + 5280ci^4 + (1584 - 1452ci + 660ci^2$$
$$- 8580ci^3 - 20020ci^4)si^2 + (-2145 + 4290ci + 15015ci^2)si^4)\cos(3o + w))$$

$$d_6 = \frac{945}{128a} e$$
$$((-144 + 144ci + 144ci^2 - 144ci^3 + (528 - 1584ci - 1584ci^2 + 2640ci^3)si^2$$
$$+ (-429 + 2145ci + 2145ci^2 - 5005ci^3)si^4)\cos(3o - w) + (-144 - 144ci$$
$$+ 144ci^2 + 144ci^3 + (528 + 1584ci - 1584ci^2 - 2640ci^3)si^2 + (-429$$
$$- 2145ci + 2145ci^2 + 5005ci^3)si^4)\cos(3o + w))$$

Solutions of D_{74} Perturbation

$$d_1 = -\frac{3465}{16} e\ si$$
$$((24 - 48ci + 48ci^3 - 24ci^4 + (-39 + 156ci + 78ci^2 - 260ci^3$$
$$+ 65ci^4)si^2)\cos(4o - w) + (24 + 48ci - 48ci^3 - 24ci^4 + (-39 - 156ci$$
$$+ 78ci^2 + 260ci^3 + 65ci^4)si^2)\cos(4o + w))$$

$$d_2 = \frac{10395}{256} e\ si$$
$$((-24 - 96ci + 144ci^2 + 96ci^3 - 120ci^4 + (39 + 156ci - 390ci^2 - 260ci^3$$
$$+ 455ci^4)si^2)\cos(4o - w) + (-24 + 96ci + 144ci^2 - 96ci^3 - 120ci^4 + (39$$
$$- 156ci - 390ci^2 + 260ci^3 + 455ci^4)si^2)\cos(4o + w))$$

$$d_3 = \frac{3465}{256} si$$
$$((-24 + 480ci - 432ci^2 - 480ci^3 + 456ci^4 + (39 - 1092ci + 858ci^2$$
$$+ 1820ci^3 - 1625ci^4)si^2)\ \sin(4o - w) + (24 + 480ci + 432ci^2 - 480ci^3$$
$$- 456ci^4 + (-39 - 1092ci - 858ci^2 + 1820ci^3 + 1625ci^4)si^2)\sin(4o + w))$$

$$d_4 = \frac{10395}{64} e\ si$$
$$((24 + 96ci - 144ci^2 - 96ci^3 + 120ci^4 + (-39 - 156ci + 390ci^2 + 260ci^3$$
$$- 455ci^4)si^2)\cos(4o - w) + (-24 + 96ci + 144ci^2 - 96ci^3 - 120ci^4$$
$$+ (39 - 156ci - 390ci^2 + 260ci^3 + 455ci^4)si^2)\cos(4o + w))$$

$$d_5 = \frac{10\ 395}{256} e$$
$$((24ci + 96ci^2 - 144ci^3 - 96ci^4 + 120ci^5 + (-96 + 171ci - 180ci^2$$
$$+ 690ci^3 + 780ci^4 - 1365ci^5)si^2 + (156 - 780ci - 780ci^2$$
$$+ 1820ci^3)si^4)\sin(4o - w) + (-24ci + 96ci^2 + 144ci^3 - 96ci^4 - 120ci^5$$
$$+ (-96 - 171ci - 180\ ci^2 - 690ci^3 + 780ci^4 + 1365ci^5)si^2 + (156$$
$$+ 780ci - 780ci^2 - 1820ci^3)si^4)\sin(4o + w))$$

$$d_6 = -\frac{10\ 395}{32a} e\ si$$
$$((24 + 96ci - 144ci^2 - 96ci^3 + 120ci^4 + (-39 - 156ci + 390ci^2 + 260ci^3$$
$$- 455ci^4)si^2)\ \sin(4o - w) + (-24 + 96ci + 144ci^2 - 96ci^3 - 120ci^4$$
$$+ (39 - 156ci - 390ci^2 + 260ci^3 + 455ci^4)si^2)\sin(4o + w))$$

Solutions of D_{75} Perturbation

$$d_1 = \frac{51975}{64} e(-1+ci)(1+ci)$$
$$((-8+8ci+8ci^2-8ci^3+(39+91ci-195ci^2+65ci^3)si^2)\sin(5o-w)$$
$$+(8+8ci-8ci^2-8ci^3+(-39+91ci+195ci^2+65ci^3)si^2)\sin(5o+w))$$

$$d_2 = -\frac{155925}{256} e(-1+ci)(1+ci)$$
$$((-8+8ci+8ci^2-8ci^3+(13-39ci-65ci^2+91ci^3)si^2)\sin(5o-w)$$
$$+(8+8ci-8ci^2-8ci^3+(-13-39ci+65ci^2+91ci^3)si^2)\sin(5o+w))$$

$$d_3 = \frac{51975}{128}(-1+ci)(1+ci)$$
$$((16-16ci-16ci^2+16ci^3+(-39+13ci+195ci^2-169ci^3)si^2)\cos(5o-w)$$
$$+(16+16ci-16ci^2-16ci^3+(-39-13ci+195ci^2+169ci^3)si^2)\cos(5o+w))$$

$$d_4 = -\frac{779625}{256} e(-1+ci)(1+ci)$$
$$(8-8ci-8ci^2+8ci^3+(-13+39ci+65ci^2-91ci^3)si^2)\sin(5o-w)$$
$$+(8+8ci-8ci^2-8ci^3+(-13-39ci+65ci^2+91ci^3)si^2)\sin(5o+w))$$

$$d_5 = -\frac{155925}{256} e\ si((8+6ci+30ci^2+124ci^3-220ci^4-130ci^5+182ci^6$$
$$+(-39-156ci+390ci^2+260ci^3-455ci^4)si^2)\cos(5o-w)$$
$$+(-8+6ci-30ci^2+124ci^3+220ci^4-130ci^5-182ci^6$$
$$+(39-156ci-390ci^2+260ci^3+455ci^4)si^2)\cos(5o+w))$$

$$d_6 = -\frac{155\,925}{32a} e(-1+ci)(1+ci)$$
$$((8-8ci-8ci^2+8ci^3+(-13+39ci+65ci^2-91ci^3)si^2)\cos(5o-w)$$
$$+(8+8ci-8ci^2-8ci^3+(-13-39ci+65ci^2+91ci^3)si^2)\cos(5o+w))$$

Solutions of D_{76} Perturbation

$$d_1 = \frac{2027025}{16} e\ si(-1+ci)^2(1+ci)^2((1-2ci+ci^2)\cos(6o-w) + (1+2ci+ci^2)\cos(6o+w))$$

$$d_2 = -\frac{2027025}{128} e\ si(-1+ci)^2(1+ci)^2((-1-6ci+7ci^2)\cos(6o-w) + (-1+6ci+7\ ci^2)\cos(6o+w))$$

$$d_3 = -\frac{2027025}{128} si(-1+ci)^2(1+ci)^2((-1+10ci-9ci^2)\sin(6o-w) + (1+10ci+9ci^2)\sin(6o+w))$$

$$d_4 = -\frac{6081075}{64} e\ si(-1+ci)^2(1+ci)^2((1+6ci-7ci^2)\cos(6o-w) + (-1+6ci+7ci^2)\cos(6o+w))$$

$$d_5 = -\frac{2027025}{128}(-1+ci)(1+ci)e \\ ((-ci-6ci^2+8ci^3+6ci^4-7ci^5+(6-18ci-30ci^2+42ci^3)si^2)\sin(6o-w) \\ +(ci-6ci^2-8ci^3+6ci^4+7ci^5+(6+18ci-30ci^2-42ci^3)si^2)\sin(6o+w))$$

$$d_6 = \frac{2027025}{16a} e\ si(-1+ci)^2(1+ci)^2((1+6ci-7ci^2)\sin(6o-w) + (-1+6ci+7ci^2)\sin(6o+w))$$

Solutions of D_{77} Perturbation

$$d_1 = -\frac{4729725}{32} e(-1+ci)^3(1+ci)^3((-1+ci)\sin(7o-w)+(1+ci)\sin(7o+w))$$

$$d_2 = \frac{14189175}{128} e(-1+ci)^3(1+ci)^3((-1+ci)\sin(7o-w)+(1+ci)\sin(7o+w))$$

$$d_3 = -\frac{4729725}{32}(-1+ci)^3(1+ci)^3((1-ci)\cos(7o-w)+(1+ci)\cos(7o+w))$$

$$d_4 = \frac{99324225}{128} e(-1+ci)^3(1+ci)^3((1-ci)\sin(7o-w)+(1+ci)\sin(7o+w))$$

$$d_5 = \frac{14189175}{128} e\ si(-1+ci)^2(1+ci)^2((1+6ci-7ci^2)\cos(7o-w) + (-1+6ci+7ci^2)\cos(7o+w))$$

$$d_6 = \frac{14189175}{16a} e(-1+ci)^3(1+ci)^3((1-ci)\cos(7o-w)+(1+ci)\cos(7o+w))$$

Solutions of D_{80} Perturbation

$$
\begin{aligned}
\frac{\mathrm{d}a_1}{\mathrm{d}t} = & -\frac{9}{32768}(e(89600cw\,si^2sw + si^4(-492800cw\,sw) + si^6(800800cw\,sw) \\
& + si^8(-400400cw^7sw - 1201200cw^5sw^3 - 1201200cw^3sw^5 - 400400cw\,sw^7))\cos M \\
& + (71680cw\,si^2sw + si^4(-394240cw\,sw) + si^6(640640cw\,sw) \\
& + si^8(-320320cw^7sw - 960960cw^5sw^3 - 960960cw^3sw^5 - 320320cw\,sw^7))\cos 2M \\
& + e(698880cw\,si^2sw + si^4(-3696000cw^3sw - 3991680cw\,sw^3) \\
& + si^6(5861856cw^5sw + 12492480cw^3sw^3 + 6630624cw\,sw^5) \\
& + si^8(-2882880cw^7sw - 9129120cw^5sw^3 - 9609600cw^3sw^5 - 3363360cw\,sw^7))\cos 3M \\
& + (si^4(394240cw^3sw - 394240cw\,sw^3) + si^6(-1025024cw^5sw + 1025024cw\,sw^5) \\
& + si^8(640640cw^7sw + 640640cw^5sw^3 - 640640cw^3sw^5 - 640640cw\,sw^7))\cos 4M \\
& + e(si^4(4188800cw^3sw - 4188800cw\,sw^3) + si^6(-11371360cw^5sw + 1601600cw^3sw^3 \\
& + 10410400cw\,sw^5) + si^8(7321600cw^7sw + 5\,605600cw^5sw^3 - 8008000cw^3sw^5 \\
& - 6292000cw\,sw^7))\cos 5M + (si^6(384384cw^5sw - 1281280cw^3sw^3 + 384384cw\,sw^5) \\
& + si^8(-411840cw^7sw + 960960cw^5sw^3 + 960960cw^3sw^5 - 411840cw\,sw^7))\cos 6M \\
& + e(si^6(4708704\,cw^5sw - 15695680cw^3sw^3 + 4708704cw\,sw^5) + si^8(-5325320cw^7sw \\
& + 13733720cw^5sw^3 + 9809800cw^3sw^5 - 4764760\,cw\,sw^7))\cos 7M + si^8(91520cw^7sw \\
& - 640640cw^5sw^3 + 640640cw^3sw^5 - 91520cw\,sw^7)\cos 8M + e\,si^8(1287000cw^7sw \\
& - 9009000cw^5sw^3 + 9009000cw^3sw^5 - 1287000cw\,sw^7)\cos 9M \\
& + e(8960 + si^2(-116480cw^2 - 206080sw^2) + si^4(418880cw^4 + 1\,330560\,cw^2sw^2 \\
& + 911680\,sw^4) + si^6(-560560\,cw^6 - 2482480cw^4sw^2 - 3283280cw^2sw^4 \\
& - 1361360sw^6) + si^8(250250cw^8 + 1401400cw^6sw^2 + 2702700cw^4sw^4 \\
& + 2202200cw^2sw^6 + 650650sw^8))\sin M + (si^2(35840cw^2 - 35840sw^2) \\
& + si^4(-197120cw^4 + 197120sw^4) + si^6(320320cw^6 + 320320cw^4sw^2 \\
& - 320320cw^2sw^4 - 320320sw^6) + si^8(-160160cw^8 - 320320cw^6sw^2 \\
& + 320320cw^2sw^6 + 160160sw^8))\sin 2M + e(si^2(349440cw^2 - 349440sw^2) \\
& + si^4(-1884960cw^4 - 221760cw^2sw^2 + 1958880sw^4) \\
& + si^6(3027024cw^6 + 3603600cw^4sw^2 - 2642640cw^2sw^4 - 3219216sw^6) \\
& + si^8(-1501500cw^8 - 3363360cw^6sw^2 - 600600cw^4sw^4 \\
& + 2882880cw^2sw^6 + 1621620sw^8))\sin 3M + (si^4(98560cw^4 - 591360cw^2sw^2 \\
& + 98560sw^4) + si^6(-256256cw^6 + 1281280cw^4sw^2 + 1281280cw^2sw^4 - 25625\,sw^6)) \\
& + si^8(160160cw^8 - 640640cw^6sw^2 - 1601600cw^4sw^4 - 640640cw^2sw^6 \\
& + 160160sw^8))\sin 4M + e(si^4(1047200cw^4 - 6283200cw^2sw^2 + 1047200sw^4)
\end{aligned}
$$

$$+si^6(-2802800cw^6 + 14814800cw^4sw^2 + 12412400cw^2sw^4 - 2642640sw^6)$$
$$+ si^8(1787500cw^8 - 8008000cw^6sw^2 - 17017000cw^4sw^4 - 5605600cw^2sw^6$$
$$+ 1615900sw^8)) \sin 5M + (si^6(64064cw^6 - 960960cw^4sw^2 + 960960cw^2sw^4$$
$$- 64064sw^6) + si^8(-68640cw^8 + 960960cw^6sw^2 - 960960cw^2sw^6 + 68640sw^8))\sin 6M$$
$$+ e(si^6(784784cw^6 - 11771760cw^4sw^2 + 11771760cw^2sw^4 - 784784sw^6)$$
$$+ si^8(-875875cw^8 + 12752740cw^6sw^2 - 2452450cw^4sw^4 - 10790780cw^2sw^6$$
$$+ 805805sw^8)) \sin 7M + si^8(11440cw^8 - 320320cw^6sw^2 + 800800cw^4sw^4$$
$$- 320320cw^2sw^6 + 11440sw^8)\sin 8M + e\, si^8(160875cw^8 - 4504500cw^6sw^2$$
$$+ 11261250cw^4sw^4 - 4504500cw^2sw^6 + 160875sw^8)\sin 9M)$$

$$\frac{de_1}{dt} = -\frac{9}{4096} si^2(e(22400cw\ sw + si^2(-123200cw\ sw) + si^4(200200cw\ sw)$$
$$+ si^6(-100100cw^7sw - 300300cw^5sw^3 - 300300cw^3sw^5 - 100100cw\ sw^7)) \cos M$$
$$+ (8960\ cw\ sw + si^2(-49280cw\ sw) + si^4(80080cw\ sw)$$
$$+ si^6(-40040cw^7sw - 120120cw^5sw^3 - 120120cw^3sw^5 - 40040cw\ sw^7))\cos 2M$$
$$+ e(58240cw\ sw + si^2(-295680cw^3sw - 344960cw\ sw^3)$$
$$+ si^4(456456cw^5sw + 1041040cw^3sw^3 + 584584cw\ sw^5)$$
$$+ si^6(-220220cw^7sw - 740740cw^5sw^3 - 820820cw^3sw^5 - 300300cw\ sw^7))\cos 3M$$
$$+ (si^2(49280cw^3sw - 49280cw\ sw^3) + si^4(-128128cw^5sw + 128128cw\ sw^5)$$
$$+ si^6(80080cw^7sw + 80080cw^5sw^3 - 80080cw^3sw^5 - 80080cw\ sw^7)) \cos 4M$$
$$+ e(si^2(418880cw^3sw - 418880cw\ sw^3) + si^4(-1161160\ cw^5sw + 240240\ cw^3sw^3$$
$$+ 1017016cw\ sw^5)si^6(757900cw^7sw + 500500cw^5sw^3 - 860860cw^3sw^5$$
$$+ -603460cw\ sw^7))\cos 5M + (si^4(48048cw^5sw - 160160cw^3sw^3 + 48048cw\ sw^5)$$
$$+ si^6(-51480cw^7sw + 120120cw^5sw^3 + 120120cw^3sw^5 - 51480cw\ sw^7))\cos 6M$$
$$+ e(si^4(504504cw^5sw - 1681680cw^3sw^3 + 504504cw\ sw^5) + si^6(-580580cw^7sw$$
$$+ 1541540cw^5sw^3 + 980980cw^3sw^5 - 500500cw\ sw^7))\cos 7M$$
$$+ si^6(11440cw^7sw - 80080cw^5sw^3 + 80080cw^3sw^5 - 11440cw\ sw^7)\cos 8M$$
$$+ e\ si^6(143000cw^7sw - 1001000cw^5sw^3 + 1001000cw^3sw^5 - 143000cw\ sw^7)\cos 9M$$
$$+ e(11200cw^2 - 11200sw^2 + si^2(-61600cw^4 + 61600sw^4)$$
$$+ si^4(100100cw^6 + 100100cw^4sw^2 - 100100cw^2sw^4 - 100100sw^6)$$
$$+ si^6(-50050cw^8 - 100100cw^6sw^2 + 100100cw^2sw^6 + 50050sw^8))\sin M$$
$$+ (4480cw^2 - 4480sw^2 + si^2(-24640cw^4 + 24640sw^4)$$
$$+ si^4(40040cw^6 + 40040cw^4sw^2 - 40040cw^2sw^4 - 40040\ sw^6)$$
$$+ si^6(-20020cw^8 - 40040cw^6sw^2 + 40040cw^2sw^6 + 20020sw^8))\sin 2M$$
$$+ e(29120cw^2 - 29120sw^2 + si^2(-154000cw^4 - 36960cw^2sw^2 + 166320sw^4)$$

$$
\begin{aligned}
&+ si^4(244244cw^6 + 340340cw^4sw^2 - 180180cw^2sw^4 - 276276sw^6)\\
&+ si^6(-120120cw^8 - 300300cw^6sw^2 - 100100cw^4sw^4 + 220220cw^2sw^6\\
&+ 140140sw^8))\sin 3M + (si^2(12320cw^4 - 73920cw^2sw^2 + 12320sw^4)\\
&+ si^4(-32032\,cw^6 + 160160cw^4sw^2 + 160160cw^2sw^4 - 32032sw^6)\\
&+ si^6(20020cw^8 - 80080cw^6sw^2 - 200200\,cw^4sw^4 - 80080\,cw^2sw^6\\
&+ 20020\,sw^8))\sin(4M) + e(si^2(104720cw^4 - 628320cw^2sw^2 + 104720sw^4)\\
&+ si^4(-284284cw^6 + 1541540cw^4sw^2 + 1181180cw^2sw^4 - 260260sw^6)\\
&+ si^6(183040cw^8 - 860860cw^6sw^2 - 1701700cw^4sw^4 - 500500cw^2sw^6\\
&+ 157300\,sw^8))\,\sin 5M + (si^4(8008cw^6 - 120120cw^4sw^2\\
&+ 120120cw^2sw^4 - 8008sw^6) + si^6(-8580cw^8\\
&+ 120120cw^6sw^2 - 120120cw^2sw^6 + 8580sw^8))\sin 6M\\
&+ e(si^4(84084cw^6 - 1261260cw^4sw^2 + 1261260cw^2sw^4 - 84084sw^6)\\
&+ si^6(-95095cw^8 + 1401400cw^6sw^2 - 350350cw^4sw^4 - 1121120cw^2sw^6\\
&+ 85085sw^8))\sin 7M + si^6(1430cw^8 - 40040cw^6sw^2 + 100100cw^4sw^4\\
&- 40040cw^2sw^6 + 1430sw^8)\sin 8M + e\,si^6(17875cw^8 - 500500cw^6sw^2\\
&+ 1251250cw^4sw^4 - 500500cw^2sw^6 + 17875sw^8)\sin 9M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\omega_1}{dt} = &\frac{9}{32768}(e(35840 + si^2(-456960cw^2 - 833280sw^2)\\
&+ si^4(1626240cw^4 + 5322240cw^2sw^2 + 3696000sw^4) + si^6(-2162160cw^6\\
&- 9849840cw^4sw^2 - 13213200cw^2sw^4 - 5525520sw^6) + si^8(960960cw^8\\
&+ 5525520cw^6sw^2 + 10810800cw^4sw^4 + 8888880cw^2sw^6 + 2642640\,sw^8))\\
&+ (8960 + si^2(-116480cw^2 - 206080sw^2) + si^4(418880cw^4 + 1330560cw^2sw^2\\
&+ 911680\,sw^4) + si^6(-560560cw^6 - 2482480cw^4sw^2 - 3283280cw^2sw^4\\
&- 1361360sw^6) + si^8(250250cw^8 + 1401400\,cw^6sw^2 + 2702700cw^4sw^4\\
&+ 2202200cw^2sw^6 + 650650sw^8))\cos M + e(53760 + si^2(-465920cw^2\\
&- 1469440sw^2) + si^4(1281280cw^4 + 7687680cw^2sw^2 + 6800640sw^4)\\
&+ si^6(-1409408cw^6 - 12172160cw^4sw^2 - 21141120cw^2sw^4 - 10378368sw^6)\\
&+ si^8(540540cw^8 + 6006000cw^6sw^2 + 15415400cw^4sw^4 + 14974960cw^2sw^6\\
&+ 5025020sw^8))\,\cos 2M + (si^2(116480cw^2 - 116480sw^2) + si^4(-628320cw^4\\
&- 73920cw^2sw^2 + 652960sw^4) + si^6(1009008cw^6 + 1201200cw^4sw^2\\
&- 880880cw^2sw^4 - 1073072sw^6) + si^8(-500500cw^8 - 1121120cw^6sw^2\\
&- 200200cw^4sw^4 + 960960cw^2sw^6 + 540\,540\,sw^8))\cos 3M + e(si^2(922880cw^2\\
&- 922880sw^2) + si^4(-4977280cw^4 - 591360cw^2sw^2 + 5174400sw^4)\\
&+ si^6(8008000cw^6 + 9289280cw^4sw^2 - 6726720cw^2sw^4 - 8520512sw^6)
\end{aligned}
$$

$$+ si^8(-3981120cw^8 - 8648640cw^6sw^2 - 1601600cw^4sw^4 + 7367360cw^2sw^6$$
$$+ 4301440sw^8))\ \cos 4M + (si^4(209440cw^4 - 1256640cw^2sw^2 + 209440sw^4)$$
$$+ si^6(-560560cw^6 + 2962960cw^4sw^2 + 2482480cw^2sw^4 - 528528sw^6)$$
$$+ si^8(357500cw^8 - 1601600cw^6sw^2 - 3403400cw^4sw^4$$
$$- 1121120cw^2sw^6 + 323180sw^8))\cos 5M + e(si^4(2069760cw^4$$
$$- 12418560cw^2sw^2 + 2069760sw^4) + si^6(-5765760cw^6 + 32672640cw^4sw^2$$
$$+ 21141120cw^2sw^4 - 4996992sw^6) + si^8(3788070cw^8 - 19579560cw^6sw^2$$
$$- 32732700cw^4sw^4 - 8048040cw^2sw^6 + 2964390sw^8))\ \cos 6M$$
$$+ (si^6(112112cw^6 - 1681680cw^4sw^2 + 1681680cw^2sw^4 - 112112sw^6)$$
$$+ si^8(-125125cw^8 + 1\ 821820cw^6sw^2 - 350350cw^4sw^4 - 1541540cw^2sw^6$$
$$+ 115115sw^8))\cos 7M + e(si^6(1329328cw^6 - 19939920cw^4sw^2$$
$$+ 19939920cw^2sw^4 - 1329328sw^6) + si^8(-1555840cw^8$$
$$+ 23623600\ cw^6sw^2 - 9209200cw^4sw^4 - 16256240cw^2sw^6$$
$$+ 1292720sw^8))\cos 8M + si^8(17875\ cw^8 - 500500\ cw^6sw^2$$
$$+ 1251250cw^4sw^4 - 500500cw^2sw^6 + 17875sw^8)\cos 9M$$
$$+ e\ si^8(247390cw^8 - 6926920cw^6sw^2 + 17317300cw^4sw^4$$
$$- 6926920cw^2sw^6 + 247390sw^8)\ \cos(10M) + (-89600cw\ si^2sw$$
$$+ si^4(492800cw\ sw) + si^6(-800800cw\ sw) + si^8(400400cw^7sw$$
$$+ 1201200cw^5sw^3 + 1201200cw^3sw^5 + 400400cw\ sw^7))\sin M$$
$$+ e(-1003520cw\ si^2sw + si^4(5322240cw^3sw + 5716480cw\ sw^3)$$
$$+ si^6(-8456448\ cw^5sw - 17937920cw^3sw^3 - 9481472cw\ sw^5)$$
$$+ si^8(4164160cw^7sw + 13133120cw^5sw^3 + 13773760cw^3sw^5$$
$$+ 4804800cw\ sw^7))\sin 2M + (-232960cw\ si^2sw + si^4(1232000cw^3sw$$
$$+ 1330560cw\ sw^3) + si^6(-1953952cw^5sw - 4164160cw^3sw^3 - 2210208cw\ sw^5)$$
$$+ si^8(960960cw^7sw + 3043040cw^5sw^3 + 3203200cw^3sw^5$$
$$+ 1121120cw\ sw^7))\sin 3M + e(-1845760cw\ si^2sw + si^4(9757440cw^3sw$$
$$+ 10545920cw\ sw^3) + (si^6\ (-15567552cw^5sw - 32672640cw^3sw^3$$
$$- 17617600cw\ sw^5) + si^8(7710560cw^7sw + 23863840\ cw^5sw^3$$
$$+ 25145120cw^3sw^5 + 8991840cw\ sw^7))\sin 4M + (si^4(-837760cw^3sw$$
$$+ 837760cw\ sw^3) + si^6(2274272cw^5sw - 320320cw^3sw^3 - 2082080cw\ sw^5)$$
$$+ si^8(-1464320cw^7sw - 1121120cw^5sw^3 + 1601600cw^3sw^5$$
$$+ 1258400cw\ sw^7))\ \sin 5M + e(si^4(-8279040cw^3sw + 8279040cw\ sw^3)$$
$$+ si^6(23831808cw^5sw - 7687680cw^3sw^3 - 19219200cw\ sw^5)$$
$$+ si^8(-16027440cw^7sw - 6966960cw^5sw^3 + 18498480cw^3sw^5$$

$$+ 11085360cw\, sw^7))\sin 6M + (si^6(-672672cw^5 sw + 2242240cw^3 sw^3$$
$$- 672672cw\, sw^5) + si^8(760760cw^7 sw - 1961960cw^5 sw^3$$
$$- 1401400cw^3 sw^5 + 680680cw\, sw^7))\sin 7M + e(si^6(-7975968\, cw^5 sw$$
$$+ 26586560cw^3 sw^3 - 7975968cw\, sw^5) + si^8(9598160cw^7 sw - 27307280cw^5 sw^3$$
$$- 12572560cw^3 sw^5 + 7493200cw\, sw^7))\sin 8M + si^8(-143000cw^7 sw$$
$$+ 1001000cw^5 sw^3 - 1001000cw^3 sw^5 + 143000cw\, sw^7)\sin 9M$$
$$+ e\, si^8(-1979120cw^7 sw + 13853840cw^5 sw^3 - 13853840cw^3 sw^5$$
$$+ 1979120cw\, sw^7)\sin 10M)$$

$$\frac{d\Omega_1}{dt} = -\frac{9}{2048} ci\, si(sw\cos M + cw\sin M)(e(11200sw + si^2(-117040sw)$$
$$+ si^4(264264sw) + si^6(-168025cw^6 sw - 504075cw^4 sw^3 - 504075cw^2 sw^5$$
$$- 168025sw^7)) + (4480sw + si^2(-36960sw) + si^4(80080sw)$$
$$+ si^6(-50050cw^6 sw - 150150cw^4 sw^3 - 150150cw^2 sw^5 - 50050sw^7))\cos M$$
$$+ e(29120sw + si^2(-197120cw^2 sw - 221760sw^3) + si^4(348348cw^4 sw$$
$$+ 840840cw^2 sw^3 + 492492\, sw^5) + si^6(-185900cw^6 sw - 686400cw^4 sw^3$$
$$- 815100cw^2 sw^5 - 314600sw^7))\cos 2M + (si^2(36960cw^2 sw - 12320sw^3)$$
$$+ si^4(-120120cw^4 sw - 80080cw^2 sw^3 + 40040sw^5) + si^6(90090cw^6 sw$$
$$+ 150150cw^4 sw^3 + 30030cw^2 sw^5 - 30030sw^7))\cos 3M$$
$$+ e(si^2(314160cw^2 sw - 104720sw^3) + si^4(-1033032cw^4 sw - 528528cw^2 sw^3$$
$$+ 312312sw^5) + si^6(775060cw^6 sw + 1\,129700cw^4 sw^3$$
$$+ 128700cw^2 sw^5 - 225940sw^7))\cos 4M + (si^4(40040cw^4 sw - 80080cw^2 sw^3$$
$$+ 8008sw^5) + si^6(-50050cw^6 sw + 50050cw^4 sw^3 + 90090cw^2 sw^5$$
$$- 10010sw^7))\cos 5M + e(si^4(420420cw^4 sw - 840840cw^2 sw^3 + 84084sw^5)$$
$$+ si^6(-546260cw^6 sw + 686400cw^4 sw^3 + 815100cw^2 sw^5$$
$$- 97240sw^7))\cos 6M + si^6(10010cw^6 sw - 50050cw^4 sw^3 + 30030cw^2 sw^5$$
$$- 1430\, sw^7)\cos 7M + e\, si^6(125125cw^6 sw - 625625cw^4 sw^3$$
$$+ 375375cw^2 sw^5 - 17875sw^7)\cos 8M + (4480cw + si^2(-36960cw)$$
$$+ si^4(80080cw) + si^6(-50050cw^7 - 150150cw^5 sw^2 - 150150cw^3 sw^4$$
$$- 50050cw\, sw^6))\sin M + e(29120cw + si^2(-209440cw^3 - 234080cw\, sw^2)$$
$$+ si^4(420420cw^5 + 984984cw^3 sw^2 + 564564cw\, sw^4) + si^6(-250250cw^7$$
$$- 879450cw^5 sw^2 - 1008150cw^3 sw^4 - 378950cw\, sw^6))\sin 2M$$
$$+ (si^2(12320cw^3 - 36960cw\, sw^2) + si^4(-40040\, cw^5 + 80080cw^3 sw^2$$

$$
\begin{aligned}
&+120120cw\,sw^4)+si^6(30030cw^7-30030cw^5sw^2-150150\,cw^3sw^4\\
&-90090cw\,sw^6))\sin 3M+e(si^2(104720cw^3-314160cw\,sw^2)\\
&+si^4(-336336cw^5+768768cw^3sw^2+912912cw\,sw^4)+si^6(250250cw^7\\
&-347490cw^5sw^2-1251250cw^3sw^4-653510cw\,sw^6))\sin 4M\\
&+(si^4(8008\,cw^5-80080cw^3sw^2+40040cw\,sw^4)+si^6(-10010cw^7\\
&+90090cw^5sw^2+50050cw^3sw^4-50050cw\,sw^6))\sin 5M+e(si^4(84084cw^5\\
&-840840cw^3sw^2+420420cw\,sw^4)+si^6(-107250cw^7\\
&+1025310cw^5sw^2+336050cw^3sw^4-476190cw\,sw^6))\sin 6M\\
&+si^6(1430cw^7-30030cw^5sw^2+50050cw^3sw^4-10010cw\,sw^6)\sin 7M\\
&+e\,si^6(17875cw^7-375375cw^5sw^2+625625cw^3sw^4\\
&-125125cw\,sw^6)\sin 8M)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dM_1}{dt}=&\frac{-9}{32\,768a}(8960+si^2(-161280)+si^4(665280)+si^6(-960960cw^6\\
&-2882880cw^4sw^2-2882880cw^2sw^4-960960sw^6)+si^8(450450cw^8\\
&+1801800\,cw^6sw^2+2702700cw^4sw^4+1801800cw^2sw^6+450450sw^8)\\
&+e(80640+si^2(-1048320cw^2-185472sw^2)+si^4(3769920cw^4\\
&+11975040cw^2sw^2+8205120sw^4)+si^6(-5045040cw^6\\
&-22342320cw^4sw^2-29549520cw^2sw^4-12252240sw^6)\\
&+si^8(2252250cw^8+12612600cw^6sw^2+24324300cw^4sw^4\\
&+19819800cw^2sw^6+5855850\;sw^8))\cos M+(si^2(161280cw^2\\
&-161280sw^2)+si^4(-887040cw^4+887040sw^4)+si^6(1441440cw^6\\
&+1441440cw^4sw^2-1441440cw^2sw^4-1441440sw^6)+si^8(-720720cw^8\\
&-1441440cw^6sw^2+1441440cw^2sw^6+720720sw^8))\cos 2M\\
&+e(si^2(1048320cw^2-1048320sw^2)+si^4(-5654880cw^4-665280cw^2sw^2\\
&+5876640sw^4)+si^6(9081072cw^6+10810800cw^4sw^2-7927920cw^2sw^4\\
&-9657648sw^6)+si^8(-4504500cw^8-10090080cw^6sw^2-1801800cw^4sw^4\\
&+8648640cw^2sw^6+4864860sw^8))\cos 3M+(si^4(221760cw^4\\
&-133560cw^2sw^2+221760sw^4)+si^6(-576576cw^6+2882880cw^4sw^2\\
&+2882880cw^2sw^4-576576sw^6)+si^8(360360\,cw^8-1441440cw^6sw^2\\
&-3603600cw^4sw^4-1441440cw^2sw^6+360360sw^8))\cos 4M\\
&+e(si^4(1884960cw^4-11309760cw^2sw^2+1884960sw^4)\\
&+si^6(-5045040cw^6+26666640cw^4sw^2+22342320cw^2sw^4
\end{aligned}
$$

$$
\begin{aligned}
&+ -4756752sw^6)si^8(3217500cw^8 - 14414400cw^6sw^2 - 30630600cw^4sw^4\\
&- 10090080cw^2sw^6 + 2908620sw^8))\cos 5M + (si^6(96096cw^6\\
&- 1441440cw^4sw^2 + 1441440cw^2sw^4 - 96096sw^6) + si^8(-102960cw^8\\
&+ 1441440cw^6sw^2 - 1441440cw^2sw^6 + 102960sw^8))\cos 6M\\
&+ e(si^6(1009008cw^6 - 15135120cw^4sw^2 + 15135120cw^2sw^4\\
&- 1009008sw^6) + si^8(-1126125cw^8 + 16396380\,cw^6sw^2\\
&- 3153150cw^4sw^4 - 13873860cw^2sw^6 + 1036035sw^8))\cos 7M\\
&+ si^8(12870cw^8 - 360360cw^6sw^2 + 900900cw^4sw^4 - 360360cw^2sw^6\\
&+ 12870sw^8)\cos 8M + e\,si^8(160875\,cw^8 - 4504500\,cw^6sw^2\\
&+ 11261250cw^4sw^4 - 4504500cw^2sw^6 + 160875\,sw^8)\cos 9M\\
&+ e(-806400cw\,si^2sw + si^4(4435200cw\,sw) + si^6(-7207200cw\,sw)\\
&+ si^8(3603600cw^7sw + 10810800cw^5sw^3 + 10810800cw^3sw^5\\
&+ 3603600\,cw\,sw^7))\sin M + (-322560cw\,si^2sw + si^4(1774080cw^3sw\\
&+ 1774\,080\,cw\,sw^3) + si^6(-2882880\,cw^5sw - 5765760cw^3sw^3\\
&- 2882880cw\,sw^5) + si^8(1441440cw^7sw + 4324320\,cw^5sw^3\\
&+ 4324320cw^3sw^5 + 1441440cw\,sw^7))\sin 2M + e(-2096640cw\,si^2sw\\
&+ si^4(11088000cw^3sw + 11975040cw\,sw^3) + si^6(-17585568cw^5sw\\
&- 37477440cw^3sw^3 - 19891872\,cw\,sw^5) + si^8(8648640cw^7sw\\
&+ 27387360cw^5sw^3 + 28828800cw^3sw^5 + 10090080cw\,sw^7))\sin 3M\\
&+ (si^4(-887040cw^3sw + 887040cw\,sw^3) + si^6(2306304cw^5sw\\
&- 2306304cw\,sw^5) + si^8(-1441440cw^7sw - 1\,441440\,cw^5sw^3\\
&+ 1441440cw^3sw^5 + 1441440cw\,sw^7))\sin 4M + e(si^4(-7539840cw^3sw\\
&+ 7539840cw\,sw^3) + si^6(20468448cw^5sw - 2882880cw^3sw^3\\
&- 18738720cw\,sw^5) + si^8(-13178880cw^7sw - 10090080cw^5sw^3\\
&+ 14414400\,cw^3sw^5 + 11325600\,cw\,sw^7))\sin 5M + (si^6(-576576cw^5sw\\
&+ 1921920cw^3sw^3 - 576576cw\,sw^5) + si^8(617760cw^7sw\\
&- 1441440\,cw^5sw^3 - 1441440cw^3sw^5 + 617760cw\,sw^7))\sin 6M\\
&+ e(si^6(-6054048cw^5sw + 20180160cw^3sw^3 - 6054048cw\,sw^5)\\
&+ si^8(6846840cw^7sw - 17657640cw^5sw^3 - 12612600cw^3sw^5\\
&+ 6126120cw\,sw^7))\sin 7M + si^8(-102960cw^7sw + 720720cw^5sw^3\\
&- 720720cw^3sw^5 + 102960\,cw\,sw^7)\sin 8M + e\,si^8(-1287000cw^7sw\\
&+ 9009000cw^5sw^3 - 9009000cw^3sw^5 + 1287000cw\,sw^7)\sin 9M)
\end{aligned}
$$

$$d_3 = -\frac{315}{4096} e(-128 + 2304si^2 - 9504si^4 + 13728si^6 - 6435si^8 + (-672si^2 + 3696si^4 - 6006si^6 + 3003si^8)\cos(2w))$$

$$d_5 = \frac{315}{2048} ci\ si(-64 + 528si^2 - 1144si^4 + 715si^6)$$

$$d_6 = -\frac{315}{16384a}(128 - 2304si^2 + 9504si^4 - 13728si^6 + 6435si^8)$$

Solutions of $D_{81} \sim D_{88}$ Perturbations

Solutions of D_{81} Perturbation

$$d_1 = \frac{315}{2048}(\cos(o)si(-64 + 528si^2 - 1144si^4 + 715si^6))$$

$$d_3 = \frac{945}{8192}(e\ si((3072ci - 25344ci\ si^2 + 54912ci\ si^4 - 34320ci\ si^6)\sin(o) + (608 - 608ci + (-4224 + 5808ci)si^2 + (8294 - 13442ci)si^4 + (-4862 + 8723ci)si^6)\sin(o - 2w) + (-608 - 608ci + (4224 + 5808\ ci)si^2 + (-8294 - 13442ci)si^4 + (4862 + 8723ci)si^6)\sin(o + 2w))))$$

$$d_4 = -\frac{315}{2048}(\cos(o)ci\ si(-64 + 528si^2 - 1144si^4 + 715si^6))$$

$$d_5 = \frac{-315}{2048}(\sin(o)(-64ci^2 + (64 + 1584ci^2)si^2 + (-528 - 5720ci^2)si^4 + (1144 + 5005ci^2)si^6 - 715si^8))$$

$$d_6 = \frac{2835}{2048a}(\sin(o)ci\ si(-64 + 528si^2 - 1144si^4 + 715si^6))$$

Solutions of D_{82} Perturbation

$$d_1 = -\frac{3465}{256}(\sin(o)\cos(o)ci\ si^2(48 - 208si^2 + 195si^4))$$

$$d_3 = \frac{945}{4096}(e((-1536 + 1536ci^2 + (12672 - 38016ci^2)si^2 + (-27456 + 137280ci^2)si^4 + (17160 - 120120ci^2)si^6)\cos(2o) + (-384 + 768ci - 384ci^2 + (2640 - 12672ci + 10032ci^2)si^2 + (-5148 + 40040ci - 37180ci^2)si^4 + (3003 - 32604ci + 33033ci^2)si^6)\cos(2o - 2w) + (-384 - 768ci - 384ci^2 + (2640 + 12672ci + 10032ci^2)si^2 + (-5148 - 40040ci - 37180ci^2)si^4 + (3003 + 32604ci + 33033ci^2)si^6)\cos(2o + 2w)))$$

$$d_4 = \frac{315}{512}(\sin(o)\cos(o)(64 - 64ci^2 + (-528 + 1584ci^2)si^2 + (1144 - 5720ci^2)si^4 + (-715 + 5005ci^2)si^6))$$

$$d_5 = \frac{315}{512}(\sin(o + \frac{\pi}{4})\sin(o - \frac{\pi}{4})ci\ si (-464 + 1584ci^2 + (704 - 11440ci^2)si^2 + (3575 + 15015ci^2)si^4 - 5005si^6))$$

$$d_6 = -\frac{2835}{1024a}(\sin(o + \frac{\pi}{4})\sin(o - \frac{\pi}{4}) (64 - 64ci^2 + (-528 + 1584ci^2)si^2 + (1144 - 5720ci^2)si^4 + (-715 + 5005ci^2)si^6))$$

Solutions of D_{83} Perturbation

$$d_1 = -\frac{10395}{1024}(\cos(o)(-1 + 2\cos(2o))si(-48 + 48ci^2 + (208 - 624ci^2)si^2 + (-195 + 975ci^2)si^4))$$

$$d_3 = -\frac{31185}{4096}(e\ si((2304ci - 2304ci^3 + (-9984ci + 16640ci^3)si^2 + (9360ci - 21840ci^3)si^4)\sin(3o) + (224 + 240ci - 1152ci^2 + 688ci^3 + (-988 - 1092ci + 7020ci^2 - 4940ci^3)si^2 + (936 + 1053ci - 8502ci^2 + 6461ci^3)si^4)\sin(3o - 2w) + (-224 + 240ci + 1152ci^2 + 688ci^3 + (988 - 1092ci - 7020ci^2 - 4940ci^3)si^2 + (-936 + 1053ci + 8502ci^2 + 6461ci^3)si^4)\sin(3o + 2w)))$$

$$d_4 = \frac{10395}{1024}(\cos(o)(-1 + 2\cos(2o))ci\ si(-144 + 144ci^2 + (624 - 1040ci^2)si^2 + (-585 + 1365ci^2)si^4))$$

$$d_5 = \frac{10395}{1024}(\sin(o)(1 + 2\cos(2o))(-48ci^2 + 48ci^4 + (48 + 480ci^2 - 1040ci^4)si^2 + (-208 + 65ci^2 + 2275ci^4)si^4 + (195 - 1365ci^2)si^6))$$

$$d_6 = -\frac{31185}{1024a}(\sin(o)(1 + 2\cos(2o))ci\ si(-144 + 144ci^2 + (624 - 1040ci^2)si^2 + (-585 + 1365ci^2)si^4))$$

Solutions of D_{84} Perturbation

$$d_1 = -\frac{135135}{8}(\sin(o)\cos(o)\sin(o + \frac{\pi}{4})\sin(o - \frac{\pi}{4})ci\ si^2(8 - 8ci^2 + (-15 + 25ci^2)si^2))$$

$$d_3 = -\frac{31185}{1024}(e((-576 + 1152ci^2 - 576ci^4 + (2496 - 14976ci^2 + 12480ci^4)si^2 + (-2340 + 23400ci^2 - 27300ci^4)si^4)\cos(4o) + (-192 + 384\ ci - 384ci^3 + 192ci^4 + (962 - 3224ci - 468ci^2 + 6760ci^3 - 4030ci^4)si^2 + (-975 + 4680ci + 1170ci^2 - 13520ci^3 + 8645\ ci^4)si^4)\cos(4o - 2w) + (-192 - 384ci + 384ci^3 + 192ci^4 + (962 + 3224ci - 468ci^2 - 6760ci^3 - 4030ci^4)si^2 + (-975 - 4680ci + 1170ci^2 + 13520ci^3 + 8645ci^4)si^4)\cos(4o + 2w)))$$

$$d_4 = \frac{10395}{32}(\sin(o)\cos(o)\sin(o+\frac{\pi}{4})\sin(o-\frac{\pi}{4})$$
$$(48 - 96ci^2 + 48ci^4 + (-208 + 1248ci^2 - 1040ci^4)si^2$$
$$+ (195 - 1950ci^2 + 2275ci^4)si^4))$$

$$d_5 = -\frac{10395}{128}(\sin(2o+\frac{\pi}{4})\sin(2o-\frac{\pi}{4})ci\ si$$
$$(-56 + 576ci^2 - 520ci^4 + (-429 - 910ci^2 + 2275ci^4)si^2$$
$$+ (975 - 2275ci^2)si^4))$$

$$d_6 = \frac{93555}{512a}(\sin(2o+\frac{\pi}{4})\sin(2o-\frac{\pi}{4})$$
$$(48 - 96ci^2 + 48ci^4 + (-208 + 1248ci^2 - 1040ci^4)si^2$$
$$+ (195 - 1950ci^2 + 2275ci^4)si^4))$$

Solutions of D_{85} Perturbation

$$d_1 = \frac{675675}{256}(\cos(o)(1 - 2\cos(2o) + 2\cos(4o))si(-1 + ci)(1 + ci)(8 - 8ci^2$$
$$+ (-15 + 75ci^2)si^2))$$

$$d_3 = \frac{2027025}{1024}(e\ si((384ci - 768ci^3 + 384ci^5 + (-720ci + 2400ci^3$$
$$- 1680ci^5)si^2)\sin(5o) + (18 + 98ci - 268ci^2 + 36ci^3 + 250ci^4 - 134ci^5$$
$$+ (-42 - 225ci + 720ci^2 - 30ci^3 - 990ci^4 + 567ci^5)si^2)\sin(5o - 2w)$$
$$+ (-18 + 98ci + 268ci^2 + 36ci^3 - 250ci^4 - 134ci^5 + (42 - 225ci - 720ci^2$$
$$- 30ci^3 + 990ci^4 + 567ci^5)si^2)\sin(5o + 2w)))$$

$$d_4 = -\frac{3378375}{256}(\cos(o)(1 - 2\cos(2o) + 2\cos(4o))$$
$$(-1 + ci)ci(1 + ci)si(8 - 8ci^2 + (-15 + 35ci^2)si^2)))$$

$$d_5 = -\frac{675675}{256}(\sin(o)(1 + 2\cos(2o) + 2\cos(4o))(-8ci^2 + 16ci^4 - 8ci^6$$
$$+ (8 - 3ci^2 - 110ci^4 + 105ci^6)si^2 + (-15 + 150ci^2 - 175ci^4)si^4))$$

$$d_6 = \frac{6081075}{256a}(\sin(o)(1 + 2\cos(2o) + 2\cos(4o))(-1 + ci)ci$$
$$(1 + ci)si(8 - 8ci^2 + (-15 + 35ci^2)si^2))$$

Solutions of D_{86} Perturbation

$$d_1 = \frac{-30405375}{32}(\sin(o)\cos(o)(-1+2\cos(2o))(1+2\cos(2o)) \times (-1+ci)^2 ci(1+ci)^2 si^2)$$

$$d_3 = \frac{6081075}{512}(e(-1+ci)(1+ci) (64-128ci^2+64ci^4+(-120+960ci^2-840ci^4)si^2)\cos(6o) + (24-48ci+48ci^3-24ci^4+(-59+156ci+166ci^2 - 564ci^3+301ci^4)si^2)\cos(6o-2w)+(24+48ci-48ci^3-24ci^4 + (-59-156ci+166ci^2+564ci^3+301ci^4)si^2)\cos(6o+2w)))$$

$$d_4 = \frac{2027025}{64}(\sin(o)\cos(o)(-1+2\cos(2o))(1+2\cos(2o))(-1+ci)^2(1+ci)^2 \times (8-8ci^2+(-15+105ci^2)si^2))$$

$$d_5 = \frac{-2027025}{64}(\sin(o+\frac{\pi}{4})\sin(o-\frac{\pi}{4})(-1+2\sin(2o))(1+2\sin(2o))(-1+ci) \times ci(1+ci)si(-3-32ci^2+35ci^4+(45-105ci^2)si^2))$$

$$d_6 = \frac{6081075}{128a}(\sin(o+\frac{\pi}{4})\sin(o-\frac{\pi}{4})(-1+2\sin(2o))(1+2\sin(2o)) \times (-1+ci)^2(1+ci)^2(8-8ci^2+(-15+105ci^2)si^2))$$

Solutions of D_{87} Perturbation

$$d_1 = \frac{-70945875}{128}(\cos(o)(-1+2\cos(2o)-2\cos(4o)+2\cos(6o)) \times si(-1+ci)^3(1+ci)^3)$$

$$d_3 = \frac{-42567525}{512}((-1+ci)^2(1+ci)^2((240ci-240ci^3)\sin(7o)+(4+83ci - 178ci^2+91ci^3)\sin(7o-2w)+(-4+83ci+178ci^2+91ci^3) \times \sin(7o+2w))e\ si))$$

$$d_4 = \frac{496621125}{128}(\cos(o)(-1 + 2\cos(2o) - 2\cos(4o) + 2\cos(6o))(-1 + ci)^3 ci(1 + ci)^3 si)$$

$$d_5 = \frac{70945875}{128}(\sin(o)(1 + 2\cos(2o) + 2\cos(4o) + 2\cos(6o))(-1 + ci)^2(1 + ci)^2 \times (-ci^2 + ci^4 + (1 - 7ci^2)si^2))$$

$$d_6 = \frac{-638512875}{128a}(\sin(o)(1 + 2\cos(2o) + 2\cos(4o) + 2\cos(6o))(-1 + ci)^3 \times ci(1 + ci)^3 si)$$

Solutions of D_{88} Perturbation

$$d_3 = \frac{-127702575}{32}\Big(e(-1 + ci)^3(1 + ci)^3\big((5 - 5ci^2)\cos(8o) + (2 - 4ci + 2ci^2)\cos(8o - 2w) + (2 + 4\ ci + 2\ ci^2)\cos(8o + 2w)))$$

$$d_4 = -141891750(\sin(o)\cos(o)\sin(o + \frac{\pi}{4})\sin(o - \frac{\pi}{4})\sin(2o + \frac{\pi}{4})\sin(2o - \frac{\pi}{4}) \times (-1 + ci)^4(1 + ci)^4\Big)$$

$$d_5 = \frac{70945875}{8}(\sin(4o + \frac{\pi}{4})\sin(4o - \frac{\pi}{4})(-1 + ci)^3 ci(1 + ci)^3 si\Big)$$

$$d_6 = \frac{638512875}{64a}(\sin(4o + \frac{\pi}{4})\sin(4o - \frac{\pi}{4})(-1 + ci)^4(1 + ci)^4\Big)$$

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