

STUDIES IN *FUZZINESS*  
AND *SOFT COMPUTING*

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John N. Mordeson  
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Mark J. Wierman

# Applying Fuzzy Mathematics to Formal Models in Comparative Politics

 Springer

Terry D. Clark, Jennifer M. Larson, John N. Mordeson, Joshua D. Potter,  
Mark J. Wierman

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Dedicated to Father Michael J. Proterra, S.J., Dean of the  
Creighton College of Arts and Sciences, 1988-2000

# First Foreword

“Applying Fuzzy Mathematics to Formal Models in Comparative Politics,” or FMCP for short, is a pioneering work which is focused on fuzzy mathematics and on novel application areas of fuzzy mathematics in the realm of soft sciences—sciences such as economics, political science, sociology, psychology, philosophy and linguistics. For me, FMCP has a special significance. A bit of history is needed to explain why and to throw some light on the genesis of fuzzy set theory.

Modern science rests on the foundation of classical, crisp-set-based mathematics. The brilliant successes of modern science are visible to all. We sent men to the moon. We can build telescopes which can explore the far reaches of the universe. We can build planes that can fly at supersonic speeds. We can clone animals and we can build computers that can execute billions of instructions per second. But when we take a closer look at successes, what we see is that successes have a much lower visibility in the realm of soft sciences—sciences such as economics, political science, philosophy, psychology and, more generally, sciences in which human judgment, perceptions and emotions play an important role. Is there an explanation for this disparity?

Science deals not with reality but with models of reality. In the realm of hard sciences, classical, crisp-set-based mathematics serves as a modeling language which is adequate in the sense that it can be employed to construct models which are both realistic and precise. This is not the case in the realm of soft sciences—sciences in which imprecision, uncertainty, incompleteness of information and partiality of truth lie at the center rather than on the periphery. It is the inadequacy of crisp-set-based mathematics as a modeling language that underlies the paucity of solid scientific theories in soft sciences.

I am not a mathematician by training but I have always been an admirer of mathematics and mathematicians, and a strong believer in the power of classical mathematics to solve almost all problems in almost all fields. My faith began to waver in the late fifties and early sixties of the last century when I began to realize that there is a wide gap between the precision of mathematics and the imprecision of the real world. A simple idea which occurred to me was that the

gap could be narrowed or even closed by formalizing the concept of a class with unsharp boundaries. This is what motivated my 1965 paper on fuzzy sets—sets which are graduated in the sense that membership in a fuzzy set is a matter of degree.

My expectation was that the theory of fuzzy sets would be welcomed by the soft sciences community. I was mistaken. Fuzzy is a word which is usually used in a pejorative sense. Partly for this reason, the concept of a fuzzy set was not taken seriously. During much of its early history, fuzzy set theory was an object of criticism and derision.

Contrary to my expectation, fuzzy set theory was welcomed in hard sciences and engineering. Today, the INSPEC database lists over 53,000 papers with “fuzzy” in title, and the math.sci.net database lists over 15,000. What is striking by comparison is the paucity of papers with “fuzzy” in title in the literature of soft sciences. What this reflects is the lack of recognition within the soft sciences community that concepts and techniques drawn from fuzzy mathematics can be of great value in solution of problems which are beyond the reach of crisp-set-based mathematics. Bringing about an awareness of the existence of fuzzy mathematics and its importance to soft sciences, may well be a major impact of the work of Professor John Mordeson and his collaborators.

Fuzzy mathematics is fuzzy-set-based. Since a crisp set is a special case of a fuzzy set, fuzzy mathematics is substantially more general than crisp-set-based mathematics. But given the conservatism of the mathematical establishment, achievement of recognition of the importance of fuzzy mathematics as a branch of mathematics has been and is likely to continue to be a slow process. The same applies even more to applications of fuzzy mathematics in soft sciences.

The high relevance of fuzzy mathematics to soft sciences is directly related to its high power as a modeling language, which in turn is closely related to its high power of precisiation of meaning. A clarification is in order.

In soft sciences, information is generally described in a natural language. To construct a model of meaning which lends itself to computation, it is necessary to precisiate the meaning of propositions which describe the model. More specifically, let  $p$  be a proposition which is an object of precisiation and let  $p^*$  be the result of meaning precisiation (m-precisiation).  $p$  and  $p^*$  are referred to as precisiend and m-precisiand, respectively. To be useful, m-precisiand should be mathematically well-defined, in which case it is referred to as mm-precisiand, and be a good model of m-precisiend,  $p$ . A measure of the goodness of model is referred to as cointension. It is in this sense that fuzzy mathematics, viewed as a modeling language, has a high power of cointensive mm-precisiation. This is a key feature of fuzzy mathematics and is the main reason why fuzzy mathematics is of high relevance to soft sciences.

Professor John Mordeson deserves major credit both for contributing so much and in so many important ways to the advancement of fuzzy mathematics, and his pioneering work on applications of fuzzy mathematics to soft sciences. He and his collaborators, Terry D. Clark, Jennifer M. Larson, Joshua D. Potter and Mark J. Wierman, have produced an important work which breaks new

ground in application of fuzzy mathematics to soft sciences and especially to comparative politics. FMCP opens the door to many further applications. In this perspective, the importance of FMCP is hard to exaggerate. Professor Mordeson, his co-authors and the publisher, Springer, deserve loud applause.

Lotfi A. Zadeh  
Berkeley, California  
September 19, 2007

## Second Foreword

“I can no other answer make but thanks,  
And thanks, and ever thanks.”

*Twelfth Night*

I am delighted to be asked to offer these few words as a foreword. This is a book that is well-written and replete with vision.

The philosopher, Blaise Pascal, says somewhere that in speaking of their works, authors will usually say: “My book.” They would do better, he thinks, to say: “Our book.” This is a very useful observation. This book, in particular, is a broadly collaborative one, though perhaps not evident at first. The primary researchers and writers are learned scholars, graduate students, and undergraduate students. Such collaboration is both exciting and productive. This book should become a model contribution to the desire for joint research efforts by senior scholars and students.

The task set for the authors was how to deal with uncertainty in political science beyond what probability theory can provide. Fuzzy mathematics permits them to analyze politics when both the preference orderings of individuals as well as the distance between them is vague. The limits of crisp set assumptions open up an opportunity to set theory.

This study is a welcome addition to a field that arguably is still in an embryonic stage. The most important literature in the field is brought to bear on the explorations; no doubt this study will take its place among such literature. Further, the authors condense the current literature with rigor and clarity, and bring a certain freshness to this material. This is an exacting book as it charts new, somewhat untested paths.

This book will serve as a guide to a new generation of scholars and students of political science. An excellent tool, I am confident that this work will enjoy broad distribution.

Michael Proterra, S.J.  
Parochial Vicar  
St Raphael Church and Schools  
Raleigh, North Carolina

# Preface

For much of the Cold War, a-theoretic approaches largely dominated comparative politics, the sub-field of political science devoted to the analysis and comparison of domestic political processes in non-US systems. While a number of scholars called upon comparativists to give greater attention to theories and models (see for example, [Hoffmann and Fleron, 1971](#); [Motyl, 1992](#)), the preponderance of research eschewed their use as well as the cross-regional comparisons they might have made possible. This was particularly true of scholarship on communist Europe where the hegemony of communist parties sustained by the military power of the Soviet Union imposed an orthodoxy on the region that largely justified an idiosyncratic analytical approach. Furthermore, the secrecy of communist regimes resulted in a paucity of data that reinforced the preference for interpretive analyses of contextual, historical, and cultural arguments.

Since the end of the Cold War, the argument for the uniqueness of regions and individual countries has been undermined as the number of countries that have transitioned or are transitioning into functioning democracies has grown. The emergence of a community of scholars studying democratic institutions (e.g., political parties, legislatures, committee systems, and regime type) across former regional divides has resulted in a renewed interest among comparativists in theoretically-informed research. This, in turn, has contributed to a growing literature focusing on the development and testing of formal models. (See [Clark 2002](#), for an explication of this trend.) While some continue to insist that only theories developed within the context peculiar to a given region are likely to produce valid results (see [Bunce, 1994, 1995a, 1999, 1995b](#); [Wiarda, 2002](#)), numerous analyses informed by general social science theory have appeared.

Not surprisingly, comparativists have developed and made use of formal models informed by the logic of crisp set theory, since they are virtually the only ones used in the social sciences. Crisp set theory assumes that an absolute value for a political phenomenon exists in nature. Any uncertainty regarding the phenomenon is related to the degree to which the observer is sure that (s)he has correctly determined that value. Probability theory is frequently employed to deal with these kinds of uncertainty in political science.

Crisp set assumptions are clearly justifiable when considering such phenomena as the percentage of the vote that a candidate for political office receives in an election, the number of seats that a given political party occupies in a legislature, the number of ministerial portfolios allocated to a party, or the percentage of trade between two states expressed as a proportion of the gross domestic product. However, there is a very large number of phenomena that involve an altogether different kind of uncertainty, uncertainty related to ambiguity and vagueness concerning the degree to which an observation demonstrates the qualities of a concept. Among concepts about which political observers might express such uncertainty are democracy, stability, the strength of committee systems, and the level of ethnic violence. Humans express the degree to which an observation elicits the qualities related to these concepts in vague and ambiguous terms.

Crisp set theory is not well suited to dealing with such uncertainty. Fuzzy set theory is. In contrast to crisp set theory, which views observations in either-or terms, fuzzy set theory works on the principle of the “missing middle” and assumes many observables are only partially included in a set. Hence, if crisp set theory views countries as either democratic or non-democratic, fuzzy set theory permits us to consider the degree to which they might be democratic or not democratic.

Given the rather strict assumption that crisp set theory makes about the preciseness with which the set membership of political phenomena can be determined, it is our contention that it may not serve well as a basis for constructing formal models regarding a very large number of applications and questions in political science. That is particularly the case regarding the *preferences* of individuals and collective institutions, which lie at the base of most formal models in comparative politics. The purpose of our book is to demonstrate how fuzzy set theory might be usefully applied to these models.

Our book project has its genesis in early efforts by Terry D. Clark, professor of political science, and Raivydas Simenas, then an undergraduate student at Creighton University (and winner of several European math competitions), to formally model the inherent instability of presidentialism, a recurring theme in the comparative literature (see for example, [Linz, 1990, 1994](#); [Mainwaring and Shugart, 1997](#); [Shugart and Carey, 1992](#)). Clark and Simenas recast the problem as one pitting self-interested (rational) actors in a contest for control over policy. In their reformulation, political actors are willing to adhere to the democratic rules only as long as they calculate there is a potential for a higher payoff (policy access afforded by political office) by doing so than by defecting. This can be captured in the simple cost-benefit calculation in Eq. [0.1](#), in which the *total* outcome of a political contest to each player  $i$ ,  $T_i$ , is the difference between the gain from successfully *defecting*,  $D_i$  (defecting from democratic rules, as in a coup attempt), and the sum of the gain from continuing to *cooperate*,  $C_i$  (playing by the democratic rules), and the loss from a *failed defection* effort,  $L_i$

$$T_i = (C_i + L_i) - D_i. \quad (0.1)$$

If the cost-benefit calculation  $T_i$  is greater than or equal to zero, then the given player will choose to continue to cooperate. Otherwise, the best strategy

is to seek to overturn the democratic rules in a bid to gain complete control of the policy process.

In other words, presidentialism's essential problem is that it biases the contest in favor of a single institution, the presidency. This induces a winner-take-all, zero-sum game from which those actors not likely to win the office of the presidency have an incentive to defect. Hence, to the degree that all actors are certain of their prospects for victory in presidential elections, presidentialism is unstable insofar as the losers are incentivized to seek policy control by other than democratic means (i.e., a resort to a palace coup or a street uprising).

The problem was that by using crisp set assumptions, Clark and Simenas were unable to deal with ambiguity in varying assessments among political actors concerning estimates of successful defection ( $D_i$ ), gains from cooperation ( $C_i$ ), and losses from a failed defection effort ( $L_i$ ). The problem carried over in subsequent efforts by Clark and Simenas and later Jennifer M. Larson (then a double major in political science and mathematics who subsequently went on to pursue the Ph.D. in political science at Harvard University) to predict the size of a governing legislative coalition. The puzzle they sought to resolve was under what conditions a surplus majority coalition forms<sup>1</sup>. Crisp set assumptions essentially place an upper bound on the number of parties in a coalition: the number of parties can not exceed the number of salient dimensions defining the policy space. However, there are numerous empirical examples of surplus coalitions comprising three or more parties in two-dimensional issue space.

These efforts lay bare the limitations of imposing a crisp set logic on formal models in comparative politics and prepared the ground for the fortuitous visit of John N. Mordeson with Clark in late spring 2005. Mordeson, a professor of mathematics who has devoted much of his career to the development and application of fuzzy set theory, quickly grasped the problem and suggested several strategies for resolving the impasses faced by Clark and Larson. Within a few short months after the three began work on these solutions, Mark J. Wierman, associate professor of computer science, joined the team, providing a much needed dimension in neural nets, genetic algorithms, and other soft computing and modeling approaches. A short time later, Joshua D. Potter, a talented political science major who had made yeoman contributions in the development of some early models, joined the team as its chief research assistant. The result is the book in front of you.

Each of us wishes to express our gratitude to those persons who play a very special role in our lives. John Mordeson wrote with his granddaughter Elizabeth in his mind and heart. Mark Wierman would like to acknowledge Mary Dobransky for the influence on his life, and George Klir for the influence on his work. Terry Clark is grateful to his wife, Marnie, for her love and faithfulness. Jennifer Larson thanks her parents for their unending support in everything she has ever

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<sup>1</sup> A surplus majority coalition is one uniting more parties than required to assure a majority of deputies in parliament. If any one of the parties to a coalition can be removed and the remaining number of deputies continues to constitute a majority, the coalition is a surplus majority coalition.

decided to do and hopes this “light read” on fuzzy mathematics is the book they have always wanted to display on their coffee table. Joshua Potter thanks David, Sheri, and Sarah Potter for their encouragement, advice, and company. He also extends his thanks to the staff and patrons of the Java Jay and Caffeine Dreams for their flexible hours and superb cappuccinos, respectively.

Collectively we are indebted to a very large number of persons who helped make this book possible. We wish to express our gratitude to the staff of Springer-Verlag, especially Janusz Kacprzyk, for their help and encouragement in writing and assembling this volume. Our special thanks go to Paul Wang, Duke University, for his strong support for research extending applications of fuzzy mathematics; Claudio Cioffi-Revilla, George Mason University, and Charles Taber, Stony Brook University, who provided much needed feedback and encouragement in the formulation of the project; and Nicholas Miller, University of Maryland, Baltimore County, who permitted us both to probe him on the voluminous work done on crisp formal modeling and to test our own ideas on him. At Creighton University, we are indebted to Dr. Robert Kennedy, Dean of the College of Arts and Sciences, and to Dr. and Mrs. George Haddix for their support of our work; and we gratefully acknowledge the support provided by a Creighton University Academic Affairs Research Initiative Grant.

We would be remiss were we not to acknowledge the involvement of a number of very talented Creighton University undergraduates in the research leading to this book. Creighton is a Jesuit Catholic university located in the Midwest with a strong liberal arts college. The college’s tradition of excellence in teaching has in recent years combined with a commitment to undergraduate research to attract a substantial core of gifted students. We have been fortunate to tap into this intellectual resource. Given the degree and quality of the participation of many of them in this book project, we have taken the unusual step of noting their names at the beginning of the chapters to which they contributed. We are grateful for their intellectual energy, dedication, and friendship.

Finally, we dedicate this book to Fr. Michael J. Proterra of the Society of Jesus, Dean of the College of Arts and Sciences at Creighton University, from 1988 to 2000. A visionary with a deep commitment to academic excellence, Dean Proterra possessed a particular interest in cross-disciplinary, collaborative work. During his tenure as Dean, he significantly expanded the number of tenure line faculty in the college, moved resources to innovative ideas and programs, and was instrumental in the founding of Creighton University’s Center for Research in Fuzzy Mathematics, which today is known as the Center of Mathematics for Uncertainty. Thanks to these efforts and many others, Creighton University today is a place where scholars engage one another across disciplinary boundaries. We are grateful to him for bringing us to Creighton and for keeping us at Creighton.

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# 1 Applying Fuzzy Set Theory to Comparative Politics

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In contrast to the discipline of economics - which adopted formal, deductive approaches grounded in rational choice assumptions more than a century ago - political science remained largely normative and inductive until the 1950s. Not surprisingly, many of the first works analyzing politics using mathematical models were written by economists (for example, [Downs, 1957](#); [Olson, 1965](#); [Buchanan and Tullock, 1962](#)). Since then the discipline has made significant progress in the application of mathematics to a wide number of questions as growing numbers of political scientists have embraced formal approaches. This has been particularly so in the sub-fields of American government, public policy, and international relations where rational choice models have been particularly popular. In contrast, the application of formal modeling to comparative politics has not had as much traction. Owing to an area studies tradition that emphasizes description and interpretive approaches over theory building and hypothesis testing, scholars analyzing the political systems of nation-states have largely eschewed cross-regional comparisons, let alone formal, deductive models.

The end of the Cold War and the upsurge in the number of democracies across the globe, however, have helped to fuel an increasing interest among comparativists in formal models. Indeed, this appears to be as propitious a moment to focus on improving and expanding the use of formal models in comparative politics as any in recent memory. The current international system begs for answers to questions concerning the design and implementation of democratic constitutions. While many argue that institutional design is trumped by political culture (i.e., democratic institutions can not function when publics and elites are not habituated to the norms that under-gird their functioning), it would be fool-hardy to write off institutional approaches rooted in formal mathematical analysis given the contributions they have made in the areas in which culturalist approaches fail us. Moreover, the behavioralist argument that culture must be changed before democracy can be stabilized is crippled by the extraordinarily high policy costs in time and money that are associated with failure.

The goal of our book is to demonstrate fuzzy set theory's capacity to enhance formal models in comparative politics. While the volume represents a collaborative effort involving scholars from across the disciplines of computer science, mathematics, and political science, the lead author is a political scientist and the approach and style reflect the norms of that discipline. Indeed, we are particularly keen on making the book accessible to formal modelers and comparativists in political science in the hopes of convincing the former of the elegance and intuitiveness to be had with fuzzy set theory and the latter of the utility and accessibility of fuzzy formal models. Toward that end, we endeavor as much as possible to approach modeling in the same manner and with the same assumptions that political scientists do. Nevertheless, we are mindful that we can not merely overlay a fuzzy approach on the existing crisp set approaches. Fuzzy set theory has a logic of its own, one that results in greater accessibility, elegance, and coherence. These are the very characteristics that commend a fuzzy approach to comparativists and formal modelers in political science.

While our primary audience is the political science community, we expect that this volume will be of interest as well to mathematicians and computer scientists who are interested in working on fuzzy set theory applications to problems as political scientists see and frame them. There is already a growing literature in journals devoted to fuzzy mathematics addressing issues related to formal modeling in political science. (See Nurmi & Kacprzyk 2007, for a review of this literature.) Among the topics addressed have been individual preferences (Nurmi, 1981b), collective preference relations (Bezdek et al., 1978, 1979; Kacprzyk et al., 1992; Orlovsky, 1978), and stability conditions under satisficing in two-dimensional spatial models (Nurmi, 1981a). We build upon this literature in presenting our approach to fuzzy modeling.

We are well aware of the criticism leveled at formal modeling in the social sciences, particularly that in the rational choice tradition. Models have all too often required the importation of assumptions that bear very little relationship to the real world. One of the most troubling assumptions of most models is that the preferences of political actors can be represented by Euclidean distance, with preferences declining monotonically from a single, ideal point. Political actors almost never approximate this ideal. Their preferences are a good deal vaguer. While alternative metrics and spaces have been proposed, their use has been limited, and we believe that fuzzy set theory gives us the means for dealing with the inherent ambiguity in human preferences.

## 1.1 Comparative Politics

If our goal is to enhance formal modeling efforts in comparative politics, we need to begin with an understanding of the boundaries of the discipline. In other words, what defines *comparative politics* as a field of inquiry? The question is not an easy one to answer. One of several sub-fields in political science, graduate students pursuing the Ph.D. are frequently told that comparative politics is not defined by the substance of research, in the manner of international relations

or American government, but rather by a method, the comparative method. However, anyone who has any degree of familiarity with comparative politics knows that there is no *one* methodological paradigm dominating the field. In fact, the very label *comparative* is misleading given that a substantial amount of the research being done comprises purely descriptive or traditional, interpretive case studies. Such research is the stock and trade of the area studies programs that focus on a single country or region. Rooted in traditional historical and philosophical epistemologies, interpretive case studies comprise one of the oldest methodological traditions in political science. Despite the widely proclaimed charge that the area studies programs failed to predict the demise of the Soviet Union, the approach continues to dominate much of the discipline, particularly in the post-9/11 era, in which Middle Eastern and Asian specialists are in high demand throughout academia.

Defenders of the area studies approach within comparative politics claim that a case study is implicitly comparative if the analysis is based on hypotheses derived from theory (Ragin, 1987). In this view, only the a-theoretical case study is excluded. The argument does little to detract from the reality that the a-theoretical case study remains very much in prominence. Indeed, many studies represent little more than ad hoc hypothesis testing; and in some area studies programs, leading scholars assert that theories must be developed based on the peculiarities and specifics of the country or region. They further contend that inductive theory building can not begin until sufficient descriptive data are first made available (for example Buncé, 1994, 1995).

In contrast to supporters of the a-theoretical single country study, those who argue that the testing of hypotheses derived from theory must lie at the core of all political science research (King et al., 1994) are at a minimum implicitly *comparative*. While proponents of such research claim that hypotheses must be rooted in theory, the theory is often quite under-developed and often as not merely implied. Hence, a good number of multivariate analyses in reality constitute ad hoc hypothesis testing. Among the proponents of both ad hoc hypothesis testing and theoretically informed research there is no agreed upon method. Instead there are two major competitors: multivariate statistical analyses and deductive modeling. Judging on the basis of the sheer number of published articles, the former is clearly the dominant of the two.

Multivariate statistical analyses emerged in the immediate aftermath of World War II when political science was transformed by the *behavioral revolution*, the central notion of which was that the social world, like the physical world, could be studied scientifically. Under the sway of the behavioral revolution, many leading scholars refocused their research agenda on the development of theory that would permit the search for causality through the testing of hypotheses. Multivariate statistical analyses in comparative politics frequently focus on causality within a single nation-state. While theoretically informed cross-national research is not rare, such studies seldom seek to explain differences in bivariate relations across nation-states as urged by Przeworski and Teune (1970).

Formal deductive modeling based on rational choice assumptions - a set of assumptions borrowed from economics - is the most recent approach in comparative politics. Scholars of American government and international relations seized upon formal deductive models in the 1970s. In contrast, formal approaches did not begin to make a noticeable appearance in the comparative literature until the early 1990s. This largely owed to the dominance of the area studies approach, an approach that eschewed theory in favor of substance and deductive logic in favor of inductive analysis.

If then the sub-field is not distinguished by a single method, what is *comparative politics*? Contrary to what students are told in graduate school, like all sub-fields of political science it is indeed distinguished by the substance of the research agenda, not a method. If the label itself does not give us any idea of what this substance is, comparisons with the other sub-fields does. American government, as it suggests, is devoted to analyses of U.S. politics. International relations is concerned with interactions across international borders. That would appear to leave analyses within the borders of all nation-states, with the exception of the United States, to the realm of what we call comparative politics. In fact, a cursory examination of the sub-field's major journals - *Comparative Politics*, *Comparative Political Studies*, and *World Politics* - would lead an impartial observer to that very conclusion. In our view, it is a fair judgment of the actual boundaries, vice self-proclaimed, of *comparative politics*.

Hence, *comparative politics* comprises a-theoretical and theoretically informed non-U.S. single country interpretive and multivariate regression analyses, cross-national multivariate analyses, formal deductive modeling of cross-national political phenomena, and tests of propositions derived from those models. Studies that include the United States are also included in the category of comparative politics if it is not the only country included in the analysis. If included, the analysis must be cross-national and explicitly comparative.

## 1.2 Fuzzy Mathematics and Political Science

Our book explores the potential of *fuzzy mathematics* in formal deductive modeling in comparative politics. In doing so, we do not mean to imply that such approaches are preferable to others. Rather, as the most rigorously theoretical of the approaches employed in comparative politics, we believe that formal modeling has the most to gain from the application of fuzzy set theory. Furthermore, while our purpose is not to contribute explicitly to the development of theoretically informed descriptive and interpretive studies or multivariate statistical analysis, scholars in these traditions may nonetheless find much of what follows in the pages of this book useful.

Fuzzy set theory was developed by [Zadeh \(1965\)](#) to address uncertainty. Most political scientists resort to probability theory when dealing with uncertainty. This is appropriate when the events associated with the political phenomenon being studied are random. In such cases, probability theory permits us to deal with the likelihood of something occurring or not occurring. For example,

suppose that we are concerned with the uncertainty associated with the toss of a coin. The probability distribution for a coin toss is  $p = \{\frac{1}{2}, \frac{1}{2}\}$  for  $X = \{H, T\}$ , where  $H$  is heads and  $T$  is tails. The probability statement contends that long term results of repeatedly tossing a coin produces a ratio of heads to total coin tosses that is about  $\frac{1}{2}$  and of tails to total coin tosses that is also around  $\frac{1}{2}$ .

While probability theory is useful for studying the uncertainty that is associated with random events, it is not very useful in considering uncertainty associated with events that occur partially. The coin toss always results in either a heads or a tails. Many events related to political phenomena, however, are not so easily categorized. In fact, it is often the case that an observation may partially share the characteristics of a particular phenomenon. For instance, it is not always the case that a given political system is or is not a democracy (that a coin toss will turn up either heads or tails). It is more often the case that it is a democracy to some degree. The same is true for a great many other political concepts. The vagueness or ambiguity inherent in such observations, is an altogether different type of uncertainty than that related to the likelihood of an exact event occurring. Fuzzy set theory permits us to deal precisely with this kind of uncertainty.

*Fuzzy mathematics* has not been widely used in political science. That is not to say that it has not been used at all. In fact, there is a small, but substantial literature employing fuzzy mathematics in international relations, a cognate field of inquiry to comparative politics. [Cioffi-Revilla \(1981\)](#) was the first to argue that fuzzy math could have important applications in international relations. Following a careful exposition of the fundamentals of fuzzy set theory, he demonstrated how it might be applied to issues of data measurement and bivariate correlations. In particular, he argued that fuzzy logic captured the inherent uncertainty in international relations better than the standard interpretation from statistical analyses, an approach frequently marked by measurement problems, data validity issues, and random error.

In response to Cioffi-Revilla, [Taber \(1992\)](#) developed a computer simulation of U.S. foreign policy decision-making in Asia using fuzzy logic. The simulation was based on an algorithm developed from the expert literature on U.S. foreign policy in Asia. The algorithm essentially comprised a series of “if-then” statements, each of which was assigned a certainty factor based on the degree of vagueness or ambiguity of the given statement. The ambiguity itself was inherent in the language used by the experts when writing about U.S. policy. When tested in simulations, the algorithm was able to predict U.S. responses with a substantial degree of accuracy.

Among others applying fuzzy logic to questions of decision-making in international relations are Mathias Koenig-Archibugi, Badredine Arfi, and Gregory S. Sanjian. [Koenig-Archibugi \(2004\)](#) used fuzzy logic to explain differences among European Union member states over the transfer of sovereignty to the EU in matters related to foreign policy and security. In contrast, to Koenig-Archibugi’s rather more traditional focus on determining the existence and strength of hypothesized relationships between variables, Arfi and Sanjian are concerned with

the development of general models from which hypotheses can be derived and tested. Arfi's (2005) highly generalized model of foreign policy decision-making, which she illustrates with a notional case, makes use of a *linguistic* fuzzy set. Sanjian, who has employed fuzzy mathematics more than any other scholar to date in political science, focused his efforts on explaining decision-making related to arms transfers. In the first in a series of articles on the subject, Sanjian (1988) developed a model explaining U.S. arms sales decisions during the Cold War. He subsequently extended the model to address arms sales by western powers (1991), U.S. and Soviet arms transfers to South Asia (1998), and the effect of arms sales on regional rivalries (1999, 2001). He also employed fuzzy logic to the 1998 NATO debate over the decision to modernize short range nuclear forces (1992).

In comparison to international relations, scholars have only relatively recently begun to argue for the utility of fuzzy mathematics in addressing issues in comparative politics. Ragin (2000) reformulated his earlier work (1987) in which he argued that Boolean logic could be applied to the test for multiple paths of causality (and data measurement) across a modest number of nation-states. In the reformulation, Ragin abandons the crisp set logic required by Boolean logic (yes-no membership in a set) in favor of the vague, or fuzzy set membership. He then demonstrates how fuzzy logic can be used to argue for necessary or sufficient conditions as well as multiple paths of causality between a set of conditions and an outcome. In so doing, he attempts to get beyond the single path of causality approach implicit in multivariate statistical analysis as well as the lack of rigor and replicability that attends comparative studies with too few cases to employ statistical analysis meaningfully. He further, emphasizes sufficient and necessary causes rather than a mere correlation between variables.

Thus far very few comparativists have taken Ragin up on the challenge to employ fuzzy mathematics in comparative politics. Among the exceptions Pennings (2003) asks how we can explain variation in the degree to which chief executives are held accountable by legislatures in forty-five parliamentary democracies. Identifying four potential explanatory conditions, he measures a state's degree of membership in each of the four for all forty-five states. He then looks for combinations of necessary and sufficient causes in these data following the procedure outlined by Ragin (2000).

Given that the single case interpretive approach and multivariate statistical analysis (both cross-national and within system) are the most widely practiced methods in comparative politics at the present time, it is not surprising that the application of fuzzy set theory to comparative politics has come in the form of a critique of the two. In fact Ragin's (2000) book is a purposeful effort to correct the deficiencies of both by melding them together in a single approach that employs fuzzy logic to identify sets (singular and plural) of necessary and sufficient conditions leading to a given political outcome. Even so, we are not aware of the application of fuzzy mathematics to other methods within any sub-field in political science. That is particularly the case for formal deductive models based on rational choice assumptions, an approach that arguably has come to

dominate the “commanding heights” of studies of American government judging by the preference of granting agencies and the discipline’s premier journals for such research.

Thus, with the exception of the contributions of [Taber \(1992\)](#) and [Seitz \(1994\)](#), fuzzy mathematics has not been systematically applied to formal modeling in any sub-field of political science. Most fuzzy mathematics applications have focused on testing hypothesized correlations between a set of causal factors and an outcome variable. Such tests rely on the assignment of membership grades in a set of observables based on the degree to which each demonstrates the characteristics of the set (e.g., democratic countries are “scored” based on the degree to which they are democratic). We contend that fuzzy math’s most important contribution to comparative politics is not likely to be in hypothesis testing but rather in its application to formal modeling.

A standard assumption of formal models in comparative politics is that political actors are rational; that is, they are utility maximizers. Given a choice between several policy options, they choose the one that comes closest to their ideal. By convention, these same models adopt a crisp set approach in tandem with the Euclidean distance assumption. As a consequence, actors are able to unambiguously identify a single, ideal policy position and determine their preferences over a set of alternatives based on how far each is removed from the ideal policy. In a fuzzy approach, actors’ preferences are vague and determined on the basis of the *degree* to which they derive utility from a given alternative. Consequently, they not only express indifference over their ideal point, but over large sets of alternatives at distances from it.

Formal models in comparative politics treat actors’ preferences as observables from which they attempt to predict political outcomes. In contrast to the crisp approach, the fuzzy approach requires that consideration be given to methods for aggregating individual preferences in spatial models (models in which political actors’ preferences involve more than one issue dimension) in order to permit comparisons.

There are a number of reasons why scholars might be better advised to analyze preferences in fuzzy rather than crisp terms. First, individual and collective players may possess little information about the available options and consequently are faced with a substantial degree of uncertainty. Second political actors may be faced with so many options near their ideal point that they are indifferent over a very large number of them. Third, they may be unable to discern any real differences among options lying somewhere in a “middle ground.” Fourth, issues may be complex and provoke deliberations about trade-offs on multiple dimensions. Fifth, bargaining may result in spin-off proposals that differ only marginally from one another. Finally, a political actor with a clearly discernible preference order over a finite set of policy proposals may be faced with a new proposal that does not fit easily into his existing preference ordering.

In the pages of this book we lay out the argument and demonstrate the utility of a fuzzy approach to formal modeling in comparative politics. We do so by critiquing and extending *The New Institutionalism*, a research paradigm that

lends itself to the formal modeling approach within the sub-field of comparative politics. Our thesis is that fuzzy set theory provides a much needed correction for the crisp set approach, which assumes that utility maximizing political actors possess clear and unambiguous ideal policy preferences. These assumptions permit little ambiguity or vagueness regarding the preferences of political actors. Were crisp set models able to make empirically valid predictions, a critique of the assumptions laying behind them would be pointless. However, crisp set models suffer from a number of serious problems, one of the most important of which is an inability to predict a stable political outcome (cycling ) without resort to highly restrictive assumptions. Moreover, even when models are able to make predictions, they are often not empirically valid.

### 1.3 The New Institutionalism

Formal modeling approaches within political science are dominated by rational choice assumptions. In their most terse form, these assumptions argue that individual political actors are self-interested, utility maximizers. Given a set of alternatives, each actor will arrange the alternatives in order of their respective ability to achieve the given actor's goals. Such a rank ordering of alternatives is typically referred to as a preference ordering. Preference relations over a rational actor's preferences orderings have the twin qualities of completeness and transitivity. Completeness simply means that any two alternatives in the preference order can be compared; that given any two alternatives, a political actor can state whether one is strictly preferred to the other or the actor is indifferent in a choice between the two. In formal terms, given a set of alternatives

$$X = \{x_1, x_2, \dots, x_n\} ,$$

over which a political actor must choose,

$$\text{for all } x_i, x_j \in X, x_i \neq x_j ,$$

either

$$\begin{aligned} x_i &> x_j, \\ x_i &< x_j, \text{ or} \\ x_i &= x_j . \end{aligned}$$

where  $>$  denotes strict preference and  $=$  denotes indifference. Transitivity means that if a rational actor views one option as strictly preferred to (or at least as good as) another, which in turn is strictly preferred to (or at least as good as) a third option, then the first option is strictly preferred to (or at least as good as) the third option. That is, if

$$x_i \geq x_j \geq x_k ,$$

then  $x_i \geq x_k$ .

**Table 1.1.** A Notional Preference Ordering for Three Individuals Faced with a Choice between Options  $A$ ,  $B$ , and  $C$ 

Person		
1	2	3
$A$	$B$	$C$
$B$	$C$	$A$
$C$	$A$	$B$
$A > B > C$	$B > C > A$	$C > A > B$

The problem in political life is that decisions are taken collectively. That is, unlike in economics where each individual makes separate, independent decisions on purchases, in politics the “purchases” comprise a collective good that can not be divided among individuals. Clean air, national security, minimum wage laws, health care policies, and the like are available to all and come in only one size and shape, the size and shape “purchased” by the collective via a decision reached through a process that we call politics. This can result in a paradox. While individual preference orders are transitive, those of collectivities (political systems uniting more than two individuals) are not.

Say, for example, that three individuals,  $N = \{1, 2, 3\}$ , faced with a choice between a set of options  $X = \{A, B, C\}$ , possess the respective preference orderings over  $X$  as shown in Table (1.1). All three individuals are rational. As a consequence, their preference orderings are both complete and transitive. Therefore, person one prefers  $A$  to  $B$ ,  $B$  to  $C$ , and  $A$  to  $C$  (which is the case for person 1). The collective decision ideally should be that choice that defeats both of the others. That is, it should reflect the majority will. However, in this case, given the notional preference orderings, no such choice exists. The reader should see for herself that  $A$  is preferred by the group to  $B$ , and  $B$  is preferred to  $C$ . But  $C$  is preferred to  $A$ !

This problem is known as the Condorcet Paradox, named after its discoverer in the eighteenth century. We say in such situations that there is no Condorcet winner. The problem arises when there is no one option that is majority preferred to all other options in a series of binary contests. Both Black (1958) and Arrow (1951) grappled with the issue in the middle of the twentieth century. In the absence of a Condorcet winner, it would seem that there is no such thing as a best social choice since there is no *one* choice preferred by the collective to all others. Say, for example, that our notional group (at Table 1.1) chooses between  $A$  and  $B$  (in which case, if each individual votes *sincerely* in accordance with her expressed preference ordering,  $A$  will win) followed by a choice between the winner of that contest, option  $A$ , and  $C$ . The result would be option  $C$ . However, at this point, person two should protest that since  $B$  defeats  $C$ , she would like a “recount.” Since  $C$  is the lowest priority for person one, she will support this move, and the new “winner” will be option  $B$ . At this point, persons one and

three should collude to put forward option *A*, which they both prefer to option *B*; and so, it will go on without any conceivable end.

What we have just demonstrated is what is known as a *majority cycling problem*. Riker (1980) argued that instability should be ubiquitous in the political world since the chance of cycling occurring increases greatly as the number of options increases (and somewhat as the members of the polity increase). However, that turns out not to be the case (Tullock, 1981). While cycling problems have been noted in a variety of legislative settings (Lagerspetz, 1993a,b), to include the Russian Supreme Soviet in 1993 (Andrews, 2002), they appear to be rare. That is because in the world of real politics, social choices emerge as the result of rules, which aggregate individual preferences to arrive at a collective choice. In essence, rules induce a stable outcome. However, even in a democracy there is no guarantee that the outcome will be a Condorcet winner (if it exists).

Take for instance our previous example of the social choice between options *A*, *B*, and *C*. If the rules stipulate that a vote will first be taken between *A* and *B* followed by a vote between the winner of that contest and *C*, then, if all individuals vote sincerely (casting a ballot for their preferred option at each stage of the process), *C* will emerge as the “democratic” collective choice. If on the other hand, the rules require that *B* first be considered against *C* followed by a contest between the winner and *A*, then *A* will be the “democratic” winner. Not to put too fine a point on it, an initial contest between *A* and *B* followed by one between the winner and *C* will result in *B* as the “democratic” choice. In short, rules matter; and what distinguishes democracy is not that the popular will determines policy but that the rules operating through institutions do so (versus individuals or oligarchic groups) (Riker, 1982).

Arrow (1951) established that no set of rules exists for aggregating collective preferences in order to arrive at collective decisions that do not violate at least one of four minimal conditions for majoritarian democracy (a system in which the majority will prevail in matters of collective choice). In effect, rules violate either the principle of unrestricted domain (all options are admissible), the Pareto principle (if everyone in the collectivity prefers option *A* to *B*, then option *A* must be preferred to option *B* by the collectivity), the independence condition (the introduction of irrelevant options does not change the collective preference between any two options), or the non-dictatorship principle (no one individual dictates the group’s choice). They may well violate more than one of these principles.

The importance of rules in inducing stable political outcomes has led scholars to study the impact of institutional design on political outcomes. The research agenda defining the boundaries of this effort, which is based in rational choice assumptions and the importance of institutions in aggregating the preferences of individuals to achieve a collective choice, is *The New Institutionalism*. Among the very first institutions considered was electoral law (Downs, 1957; Riker, 1980, 1982). The general argument is that single mandate, plurality (SMD-P) systems of the type used to elect members to the U.S. House of Representatives tends to a smaller number of political parties than do other systems for aggregating election outcomes, particularly proportional representation (PR).

## 1.4 Single Dimensional Models

Downs (1957) seminal work on the effect of electoral law on party systems makes use of a simple single dimension model. He begins by arguing that both voters and political parties are rational. Their ideal policy preference can be identified on a left-right political spectrum and located along this spectrum (dimension) relative to one another. Downs then assumes that the preference of each voter and party for any given policy can be related in Euclidean space with some distance metric. That is the further the proposal from the given actor's ideal point (in either direction), no matter how infinitesimally small the change, the less preferred the proposal is. One way to express this formally is: each player  $i$  assesses the utility,  $u_i$ , of a proposal  $x$  in reference to its ideal point  $x^*$ ,

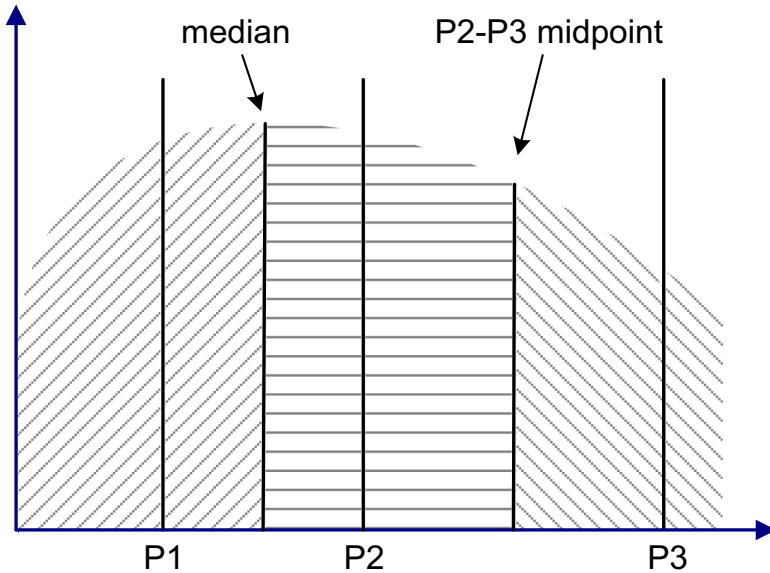
$$u_i(x) = -(x^* - x)^2.$$

The closer  $u_i$  is to zero, the closer the proposal is to the ideal point of player  $i$  and the more player  $i$  prefers the proposal.

Assuming that political parties wish to win elections (they are office-seeking), their best strategy in a winner-take-all election, in which only one candidate can win, is to place themselves along the single left-right policy spectrum as close to the median position of the voters as possible. Hence, in such an election, there is room for only two parties, both of which will attempt to establish their policy position close to the electorate's median position, one to the left and one to the right. Any party setting up further to the right or left will not only assure it loses the election but that the party closest to it does so as well, thereby delivering victory to a party further removed from it.

Figure (1.1) illustrates the argument. The x-axis represents a single issue along which the relative position of voters and parties can be arrayed. The y-axis represents the percent of the voting population (voters) whose ideal policy preference on the issue is at the corresponding location on the x-axis. All preferences are Euclidean. Two political parties,  $P_1$  and  $P_2$ , have located themselves closely to the median. Party  $P_1$  receives all the votes to its left as well as the votes half the distance between it and  $P_2$ .  $P_2$  receives the votes located half the distance between it and  $P_1$ , but it does not receive all of the votes to its right. Owing to the presence of  $P_3$ ,  $P_2$  receives only those votes located half the distance on its right between it and  $P_3$ . As a consequence,  $P_1$  wins the election.

The special property of the median position in one-dimensional space is summarized by Black's (1958) *median voter theorem*. The median position on any single dimension will be preferred by a majority to all other policy proposals. Each political actor will prefer a policy outcome that is close to her own ideal, and she will rank order all other preferences based on their (Euclidean) distance from this ideal. Consequently, a position at either of the polar extremes can not win against any position near the median. That is because every actor will vote for the proposal immediately adjacent to that of the position at the extreme, since it is closer to her own ideal point. Following this logic, the reader can work through the options to deduce that a central position held by the median voter



**Fig. 1.1.** A Notional Three-Party System in One-Dimension Issue Space

is the only choice that can defeat all other alternatives. In essence, the median position is a Condorcet winner, a stable equilibrium in single dimensional policy space.

## 1.5 Spatial (Multi-dimensional) Models

Another early area of research was coalition formation in parliamentary systems. Similarly to Downs, Riker's (1982) seminal work assumes that parties are office seeking. However, if in presidential systems they are interested in winning elections to achieve majorities in the legislature or gain control of the presidency, in parliamentarism they are primarily interested in taking positions in government. In order to maximize the payoffs from their participation in a government, parties enter into coalitions with the smallest number of deputies necessary to ensure a majority. Axelrod (1970), De Swaan (1973), and Dodd (1976) countered that policy is more important to political parties than political office. (Parties elicit policy-seeking behavior.) Hence, minimum-winning coalitions are not always possible as the potential partners may not be able to coalesce owing to substantial policy distance. In such circumstances, minority and surplus majority coalitions are more likely to form. Dodd (1976) further argued that surplus majority coalitions are more likely to result when parties are uncertain of their relative strength with voters.

The problem facing early scholars was that formal models can not predict a stable outcome for the government formation process unless negotiations over a

government between policy-seeking parties are simplified to a single dimension. [Plotf \(1967\)](#) demonstrated that a stable outcome in two-dimensions under Euclidean preference assumptions only existed if ideal points are arrayed in a radially symmetric pattern. In essence there is no stable outcome in multi-dimensional issue space with as few as two-dimensions. [McKelvey \(1976\)](#) later demonstrated that in the absence of radial symmetry, any point in two-dimensional space would be majority preferred by at least one other point. The implications for the government formation process were clear. Since bargaining was occurring over policy in more than one dimension, the outcome of the process could result in any policy position.

Empirically, the cabinet formation process is far more stable than the models suggested. Scholars have looked to institutions for the answer. Among the more rigorously formal approaches offered is Kenneth Shepsle and Michael Laver's [\(1996\)](#) *portfolio allocation model*, which borrows from Shepsle's much heralded work on the impact of the committee system on legislative voting outcomes in the U.S. Congress ([Shepsle and Weingast, 1984, 1987](#); [Shepsle, 1979](#)). The model begins by assuming that parties are policy-seeking. However, their internal structure permits them to put forward only a single position on each policy dimension. These policy positions collectively constitute the party's ideal preference in multi-dimensional space. Hence, bargaining over a government is reduced to the intersection of these ideal points, which constitutes a finite set of plausible cabinets. The reduction of the number of coalition cabinets from an infinite set to a finite set induces a stable outcome biased in favor of centrist parties that sit at the intersection of the dimension-by-dimension median.

The problem and Laver and Shepsle's solution are illustrated at [Figure 1.2](#). Ideal policy preferences of five notional parties are plotted at  $A$ ,  $B$ ,  $C$ ,  $D$ , and  $E$  in two-dimensional space. The relative size of each party is shown in parentheses. There are one hundred deputies in the parliament, so a coalition of parties with at least fifty-one members is required in order to form the government. Suppose a government with a policy position at "1" is agreed upon between parties  $A$  and  $C$ . This would require a policy compromise on the first dimension between the two and a surrender of Party  $A$  to the policy position of Party  $C$  on the second dimension. Circles are drawn with their centers at the ideal points of the three largest parties with a radius from the center to the policy position at "1" showing the set of points at the same Euclidean distance from the ideal points. The grey shaded petals are the resulting winsets. Any policy position in either petal is majority preferred by some coalition to that at "1." So, the policy position at "1" is unstable. In fact, as McKelvey argues and as the reader can confirm, any subsequent policy choice is equally unstable; and every policy position throughout the entire two-dimensional space is possible.

[Laver and Shepsle \(1996\)](#) argue that only those governments that represent the policy positions offered on each dimension by the parties are possible. In essence, a coalition agreement requires abdication on the part of one of the parties on at least one dimension. Only a one-party government would not experience abdication, in which case its ideal point on all dimensions would be the

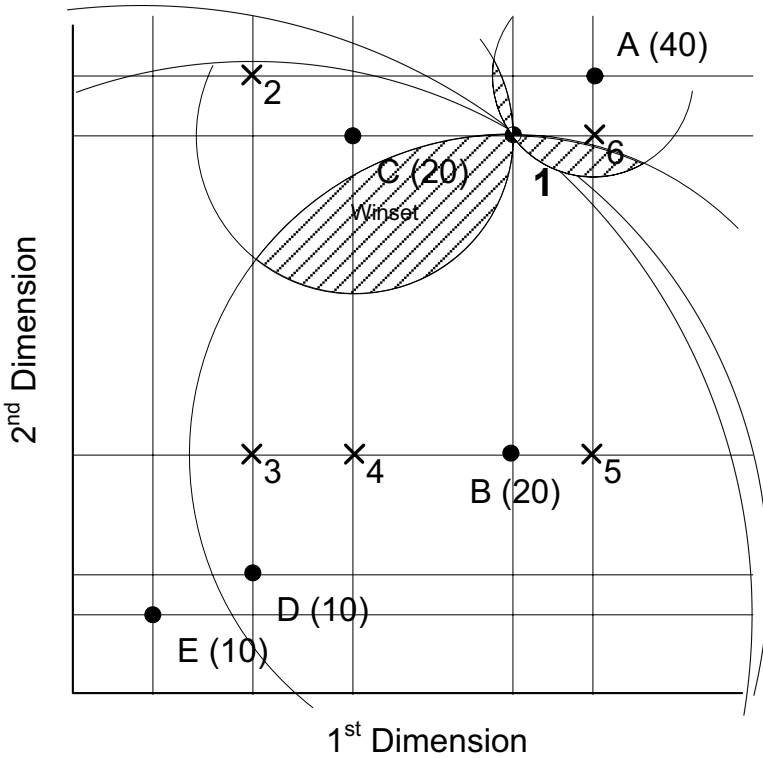


Fig. 1.2. A Stable Outcome to the Cabinet Formation Process

government’s policy. The set of possible governments is therefore reduced from an infinite set to a discrete set of those at the lattice points of the lines drawn through each party’s ideal point parallel to both axes. These include twenty-five governments, to include the five single-party minority governments at the ideal points of the parties and twenty coalitions, among them X<sub>2</sub>, X<sub>3</sub>, X<sub>4</sub>, X<sub>5</sub>, X<sub>6</sub>, and “1.” X<sub>2</sub> is a coalition uniting party D, with jurisdiction over the first dimension, and party A, with jurisdiction over the second dimension.

Let us suppose again that a coalition at “1” is once again chosen. This time, however, given the assumptions of the *portfolio allocation model* the parties to the coalition must be B and C, which together constitute a minority government with forty votes in the one hundred seat legislature. However, unlike before the government is a stable one because there is no lattice point (plausible government) in the grey petals. The limitation on the number of possible governments reduces the likelihood of a majority cycling problem (Arrow, 1951; Black, 1958; Buchanan and Tullock, 1962) and induces a set of equilibrium cabinets. No cabinet among the set of plausible cabinets is able to muster a majority against government “1.”

## 1.6 Democratic Consolidation

The end of the Cold War brought a renewed interest in theory with cross-national applications within *comparative politics*. (See for example, the research of Remington and Smith on parliamentary parties and factions [1998, 2001]; Ordeshook and Shvetsova on the impact of electoral law and social heterogeneity on the number of parties [1994], and the design of federal institutions for democratic stability [2004]; and Cox and Schoppa [2002], Ferrara and Herron [2005], and Herron and Nishikawa [2001] concerning the effect of mixed-tier electoral systems on party systems.) To be sure, the emphasis on theory with cross-national application was not warmly greeted by all (see for example, Bunce, [1994, 1995]; Roedel, [1999]; Terry, [1993]; Wiarda, [2002]), and led to concerns that theories developed by rational choice formal modelers in American government would fail to account properly for regional-specific contexts. Nonetheless, those area studies programs that had been traditionally more attuned to deductive theory, particularly Latin American studies, took the lead in transitioning to a more theoretically-informed, cross-national approach. They contended that theories that are generalizable to countries throughout the international system should be the aim uniting the disparate area studies programs into a unified and truly comparative field of inquiry within *comparative politics* (see for example, Schmitter and Karl, [1994; Karl and Schmitter, 1995; Munck, 1997]).

The shift within *comparative politics* to cross-national research came at the hands of two developments. First the prevailing sentiment among political scientists was that area studies programs had failed to predict the collapse of communism as a result of their insularity and idiosyncratic approach. Second, post-communist Europe was added to a growing list of democratizing states that spanned several regions. Well in advance of the transitions unleashed in Eastern and East Central Europe by the fall of communism, scholars had noted a trend away from authoritarianism toward democracy beginning in Southern Europe and extending to Latin America and later parts of Asia. The “third wave” of democratization (Huntington, [1991]), as it was referred to, became the source of a substantial degree of empirical research and theoretical development within *comparative politics*.

Scholars identified two key challenges in the democratization process: the *transition to democracy* (or more appropriately the break-down of an authoritarian regime) and *democratic consolidation*. While Przeworski’s ([1991]) groundbreaking work succeeded in formalizing the theoretical postulates concerning the *transition to democracy* emerging from the research conducted under the auspices of the Woodrow Wilson Center (see O’Donnell and Schmitter, [1986]), efforts to do the same for *democratic consolidation* were impeded by the widely accepted definition of the phenomenon as an end-state in which elites and the public are habituated to democratic norms and procedures (Linz, [1990]). In essence, democratic procedures and norms have become so deeply internalized in social, institutional, and psychological life that no salient political groups seriously consider pursuing their goals outside of these procedures. Given this definition of democratic consolidation, early theoretical and empirical efforts focused on the

attitudes and behaviors of elites and the public, (Diamond et al., 1997; Diamond, 1994) an approach that, while it gave impetus to a large number of statistical analyses of public survey data, proved largely impermeable to theorizing.

Linz's (1994) subsequent contention that presidentialism creates less stable conditions for the *democratic consolidation* research agenda to include a consideration of institutions. As Linz (1994) and others (Mainwaring and Shugart, 1997; Stepan and Skach, 1993) see it, the heart of the problem is presidentialism's tendency to induce a zero-sum, winner-take-all contest for a single political office, the presidency. Possessing a fixed term and a mandate separate from that of the legislature, the president is in essence constitutionally empowered to engage in a tug-of-war with the assembly over control of policy.

Linz's contention about the merits of parliamentarism versus presidentialism led a number of scholars to consider institutional designs that might lead to further instability or mitigate against it (Colton, 1995; Fish, 2000; Mainwaring, 1993; Samuels, 2000, 2003; Shugart, 1998). Most have concluded that power sharing arrangements that reduce the power of presidents are critical for stabilizing presidential systems. Both Linz's contention and the effect of power sharing arrangements fall squarely within *The New Institutionalism*. Furthermore, as we shall demonstrate next, the logic of Przeworski's (1991) formal treatment of the *transition to democracy* can be extended to model the core of the argument against presidentialism.

## 1.7 Modeling the “Problem of Presidentialism”

We assume that candidates for political office are lured by more than prospects of electoral victory. While they seek political office, they do so because of the access it affords them to the policy process. By implication, we assume that political actors will prefer those political positions that are salient in the policy process to those that are not. It follows that every player would prefer a system in which he/she is assured her/his most favorable outcome. Short of a dictatorship or an idealistic scenario in which all parties desire precisely the same outcomes, this is impossible. Hence, by using elections to determine who has access to policy decisions, democracy gives rise to an inherent instability. Some set of actors is invariably denied access to the policy process, as a consequence of which they have an incentive to defect from democracy altogether.

The problem can be dampened, however, by the degree to which parties to an election are unsure of the outcome (Przeworski, 1991). The greater the degree of uncertainty, the more likely it is that all potential candidates will judge their prospects of victory as acceptable. In essence, political actors are willing to adhere to the democratic rules only as long as they calculate there is a potential for a higher payoff (policy access afforded by political office) by doing so than by defecting. This can be captured in the simple cost-benefit calculation in equation (1.1), in which the *total* outcome of a political contest to each player  $i$ ,  $T_i$ , is the difference between the gain from successfully *defecting*,  $D_i$  (defecting from

democratic rules, as in a coup attempt), and the sum of the gain from continuing to *cooperate*,  $C_i$  (playing by the democratic rules), and the loss from a *failed defection* effort,  $L_i$ ,

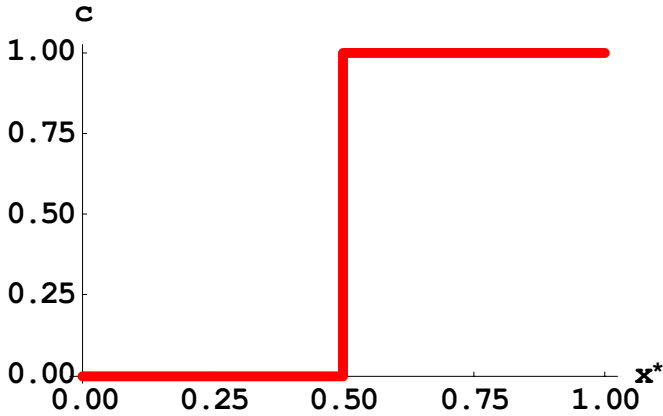
$$T_i = (C_i + L_i) - D_i. \quad (1.1)$$

If and only if the cost-benefit calculation  $T_i$  is greater than or equal to zero will the given player choose to continue to cooperate. Otherwise, the best strategy is to seek to overturn the democratic rules in a bid to gain complete control of the policy process.

Linz (1990) postulates that the fundamental problem of systems with strong presidents is that they substantially reduce the number of *relevant* players in the policy process. Presidents are invested with virtual control over policy. The payoff to any political party is thus dependent on its capturing or failing to capture the presidency in an election. If we assume that fifty percent of the votes cast in an election is required to accomplish this (the threshold for victory), then the party receiving the majority vote wins policy control. Theoretically, in a contest determined on the basis of plurality rules, this percentage could be less than fifty percent; however, the logic of the competition for power will push the threshold to fifty percent. Whatever the threshold is (we will call it  $z$ ), the payoff in presidential systems is a function of the proportion of the vote. Hence, for any player  $i$  the payoff  $C_i$  is 0 when the percentage of the vote,  $x$ , ranges from 0 to  $z$  (where  $z = 0.5$  in a contest determined on the basis of majority rules). It follows that the payoff  $C_i$  is 1 for all  $x$  greater than  $z$ . Since the game is also zero-sum (each side’s gains coming at the direct expense of the other, and the sum of the payoffs for both sides always equal to one), the other player in the winner-take-all game necessarily receives a payoff of 0 when the first player receives a vote total greater than  $z$ .

This highly unstable situation is demonstrated in the graphic representation of the relationship between the payoffs from cooperation for both players in Figure (L3). (Majority rules with a  $z$  equal to fifty percent are assumed.) One player will always be greatly enticed to disengage from the democratic process by very low payoffs from defection. Fortunately, this attains under assumptions of perfect information, which violates our assumption of uncertainty in electoral contests. (The idea that any side knows with a one hundred percent certainty that it will achieve a given electoral outcome is highly improbable in any political system, particularly a transitioning one.)

To incorporate the assumption of electoral uncertainty (imperfect information) into the model requires us to consider payoffs as calculations dependent upon the belief each side possesses regarding the odds it has of achieving a given vote outcome. Hence, the payoff from electoral success  $C_i$  can be expressed for each contestant as the product of the probability of success,  $p_i$  (winning the presidency), times the *benefit from success*,  $B_i$ . In a zero-sum, winner-take-all contest, the benefit is one (expressed as a proportion of the total possible benefit). Hence,



**Fig. 1.3.** A Graphic Display of the Relationship between the Payoff ( $c$ ) and the Proportion of the Vote ( $x^*$ ) for any Contestant in a Strong Presidential System, Assuming Perfect Information

$$C_i = p_i B_i = p_i(1). \tag{1.2}$$

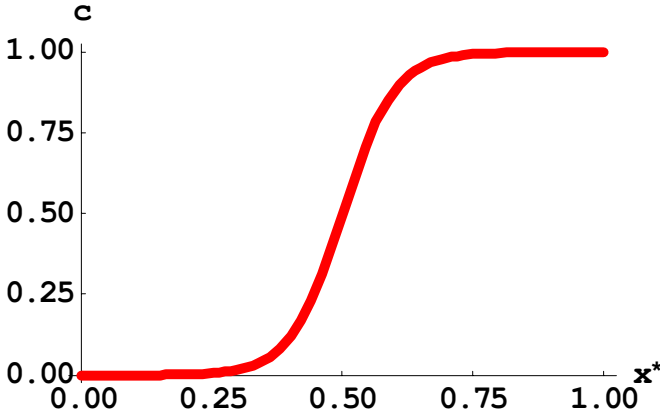
If we assume that the probability of electoral success,  $p_i$ , is normally distributed about some point  $k$  (the vote outcome most expected by a political player), then the probability of achieving a given vote outcome ( $x$ ),  $p_{ix}$ , is

$$p_{ix} = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma}(x-k)^2}. \tag{1.3}$$

In the winner-take-all, zero-sum game the payoff from cooperation on the left side of  $z$  is zero. By substitution,

$$\begin{aligned} C_i &= 0 \times \int_0^z \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma}(x-k)^2} dx \\ &\quad + 1 \times \int_z^1 \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma}(x-k)^2} dx \\ &= \int_z^1 \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma}(x-k)^2} dx. \end{aligned} \tag{1.4}$$

$C_i$  is a function of two variables: a party's best estimate of the electoral outcome (the percent of the vote it will receive),  $k$ , and the relative certainty of its estimate,  $\sigma$ . A low  $\sigma$  indicates high certainty; a high  $\sigma$  indicates low certainty. Hence, equation (1.4) accounts for imperfect information.



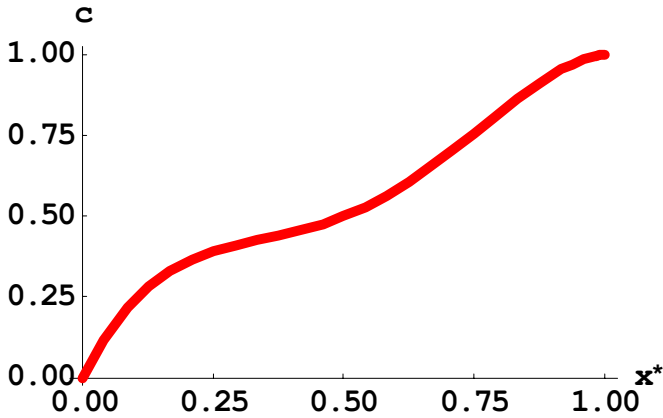
**Fig. 1.4.** A Graphic Display of the Relationship between the Payoff ( $c$ ) and the Proportion of the Vote ( $x^*$ ) for any Contestant in a Strong Presidential System, Assuming Imperfect Information

Figure (1.4) graphically displays the relationship between the payoff  $C_i$  versus the vote outcome ( $x$ ) under conditions of imperfect information for presidential democracy. The graph assumes a vote threshold  $z = 0.5$  required for electoral victory and a political contestant whose probability of vote outcomes is distributed about  $k$ . In contrast to the perfect information game depicted at Figure (1.3), there is now a range distributed about the threshold,  $z$ , within which a political contestant estimates less than a one hundred percent gain from victory, but more than zero. The range increases as the degree of uncertainty,  $\sigma$ , increases.

This is the most stable area for democracy because both sides are estimating a gain from playing by the democratic rules,  $C_i$ . Since the payoff from cooperation is a function of the degree of uncertainty,  $\sigma$ , and the estimated vote outcome,  $k$ , in equation (1.4), it follows that as uncertainty increases, the range about the threshold,  $z$ , within which both sides estimate a gain from cooperation increases. Hence, stability in a presidential democracy relies on the degree of uncertainty of electoral outcomes.

Further consideration of Figure (1.4) reveals that there are two ways to move away from dependence on electoral uncertainty to induce stability. Both involve institutions. The first solution is to *reduce* the area under the curve to the *right* of  $k$ . This is induced when elections to several institutions are decided on the basis of differing electoral laws (perhaps a single constituency, plurality rule for one institution, a majority rule for a second, and proportional representation for yet a third), therefore requiring a super-majority (pushing  $z$  to the right) in order to occupy all of the political offices that are salient in the policy process.

The second solution is to *increase* the area under the curve to the *left* of  $k$ , the expected electoral outcome for any contestant. The effect of increasing the area under the curve to the *left* of  $k$  is graphed at Figure (1.5). Contestants obtain a



**Fig. 1.5.** A Graphic Display of the Relationship between the Payoff ( $c$ ) and the Proportion of the Vote ( $x^*$ ) for any Contestant in a non-Majoritarian Parliamentary System

payoff for vote totals less than or equal to  $z$  ( $z = 0.5$  in Figure 1.5). The general form of the curve from 0 to  $z$  is modeled by Eq. (1.5).

$$C_{i,left} = x^n, \quad (1.5)$$

where  $x$  is a given vote outcome less than  $z$ , the threshold for total victory,  $n$  is some function related to the number of effective political parties in the political system, and  $C_{i,left}$  is the resultant payoff. As a consequence, the left side of equation (1.5) is no longer zero. The practical implication is that the range within which contestants receive higher payoffs from cooperation is increased. As a consequence, players are more likely to prefer to play by the democratic rules than attempt to defect.

In order for the area under the curve to the left of  $k$  to increase, political contests must no longer be winner-take-all, and access to the policy process may no longer be achieved by winning a decisive victory in an election to a single institution. Such an outcome is only possible when policy access is no longer concentrated in a single political office, but distributed across several institutions. This requires some form of power sharing arrangement that is not existent in strong presidential systems, which might include systems in which presidents allocate cabinet positions to a number of political parties in order to put together legislative support for their programs. Multiparty parliamentary systems by their very design make it possible for parties achieving less than a majority vote to gain a slice of the policy pie. Such systems require coalitions to form, the parties to which divide ministerial portfolios among themselves. In essence, such systems permit payoffs to a greater number of parties. Calculations concerning the payoff from continuing to play by democratic rules,  $C_i$ , are no longer based on the probability of winning or losing a majority of seats in the legislature. Instead, calculations reflect the probability of gaining entry into the

governing coalition, which is possible even when a small number of seats are gained in an electoral contest.

## 1.8 Veto Player Theory

Formalization of Linz’s argument has the additional benefit of helping to reveal that majoritarian parliamentarism induces the same problem as the worst case scenario for presidentialism that we have modeled to this point. Figure (L3) defines the situation in both of them. Since elections in a majoritarian party system return a single party majority that captures control of the government, the outcome is a winner-take-all, zero-sum contest between two parties to gain a legislative majority. Therefore, in both presidentialism and majoritarian parliamentarism stability is dependent on electoral uncertainty, and the solution in both is the introduction of institutions that essentially establish power sharing arrangements.

Taking a somewhat different approach, Tsebelis (1995, 1999, 2002) arrives at a similar conclusion, that the differences in constitutional design that are most consequential for the stability of democratic governance are not primarily connected with regime type. Tsebelis’ *veto player theory* focuses on policy stability and argues formally that the identification of the institutions whose consent is required for a political decision to be enacted (veto players), the policy distance between them, and the location of the status quo in the policy space establishes the likelihood that the existing law (status quo) can or can not be changed. The more veto players and the greater the policy distance between them, the greater the policy stability in the system. Cox and McCubbins (2001) point out the implications for regime stability; institutional conflict is the likely outcome under conditions of policy stability when veto points are in the hands of political forces with differing policy agenda.

## 1.9 Formal Models in Comparative Politics: A “Fuzzy” Critique

In our view, the introduction of formal approaches in *comparative politics* is a welcome occurrence that has served to strengthen the field. However, while some may read our enthusiasm to mean that we are opposed to traditional interpretive case studies or multivariate statistical analyses, that is not our position. We believe that a healthy competition of approaches is the best way to assure progress in both our substantive and theoretical knowledge. Having said that, it seems to us that the formal deductive approach grounded in rational choice assumptions is itself in need of critique - a critique that will serve to make it stronger. We offer such a critique in this book.

The formal models introduced by *The New Institutionalism* into *comparative politics* explicitly adopt rational choice assumptions that political actors have preference orderings over a set of alternatives. However, by adopting a crisp set

(Cantor set theory) approach, these models further make a number of implicit assumptions. The most important of these is that an exact point in nature exists for any political phenomenon under consideration, but it may not be knowable. The previous model of Linz's (1990) contention that parliamentarism is preferable to presidentialism to assure the stability of democratic regimes illustrates a classic case of just such a situation. The voting outcome both in terms of the percentage of the electorate that will cast a ballot for each party together with the winner(s) is a fixed point in nature, but unknown prior to the election. Faced with this kind of uncertainty, political scientists properly employ probability theory as our model did.

In contrast, both Laver and Shepsle's (1996) *portfolio allocation model* and Tsebelis' (2002) *veto player theory* deal with an altogether different kind of uncertainty. In these models, the crisp set approach forces the analyst to plot the exact policy preference of a political actor, whether an individual or a set of individuals, such as a political party. This assumes that the actors themselves possess a single, ideal point in the policy space. In effect, the probability that an actor may be indifferent over a broad set of alternatives, while admitted to by the completeness principle of rational choice theory is negated by the crisp set approach. There is minimal room for ambiguity or vagueness expressed as indifference over a set of ideal alternatives.

The inability of rational choice models to deal with ambiguity is further exacerbated by the convention of adopting Euclidean distance to determine which policy choices are more preferable. This essentially requires that preferences over alternatives monotonically decrease at the same rate in both directions from the ideal point. That is, any change away from an individuals' policy choice is less preferred, no matter how small the change. Indifference can only occur between two options in opposite directions from the political actor's ideal point. That is given an ideal budget, a political actor can be indifferent between a proposal that is a million dollars more and one that is a million dollars less. Further, the adoption of Euclidean distance results in the argument that both a defense budget that is one dollar more or one that is one dollar less than the exact amount proposed by the president are both less preferred by the president's ideal budget. In fact, they are equally less preferred.

We propose an application of fuzzy set theory to formal models in comparative politics that helps to remove these strong restrictions. Our incorporation of fuzzy set theory impacts the completeness of alternatives by permitting a greater degree of indifference, particularly near the ideal point. Fuzzy set logic permits us to model ambiguous individual preference orders in which political actors are indifferent over a wide set of policy options and proposals. This ambiguity or vagueness is reflected in the language that a policy maker uses when confronted with choices between options. The policy maker does not see them as crisp sets – “yes” or “no” options. Instead one option might be “highly likely” to achieve the goal while the other is considered “somewhat less likely” in comparison, or simply “likely.” In essence, fuzzy set theory permits us to replace a single, ideal point with a range of ideal points over which a political actor is indifferent.

As we will discuss in subsequent chapters, fuzzy set theory also permits us to jettison the Euclidean distance assumption, which has become the convention in formal models, in favor of the notion of the degree of membership in the set of acceptable policies. Just as fuzzy set theory permits us to reconsider whether a policy change moving the outcome closer to an ideal point is any more preferable than the status quo, it permits us to consider that policy changes away from an ideal set may be no less preferred to the status quo. Once a policy is outside of that set, all policies further removed are equally onerous to the political actor.

In the pages that follow, we develop a formal approach that addresses the political outcomes resulting from various institutional designs aggregating sets of fuzzy preference relations. We apply the model to some of the oldest conundrums in political science, among them majority cycling, policy stability, and government formation. For those readers who are unfamiliar with the literature on these subjects or the literature on fuzzy set theory, we have included an annotated bibliography.

The plan of the book is as follows. In the next chapter we lay the groundwork for the critique and reformulation of formal models in *comparative politics* by considering the basic concepts of fuzzy set theory. While we suspect that the chapter’s contents will be rather straightforward for the mathematician, we urge our political science readers to take the time to digest the materials. Chapter three extends fuzzy set theory to fuzzy geometry. This will prove to be the most difficult chapter for many readers. Therefore, we discuss the geometric concepts in more accessible terms in each of the ensuing chapters.

In chapter four we turn our attention to the development of a fuzzy one-dimensional model. We begin by considering the implications of the rationality assumption that political actors are utility maximizers. Given a set of alternatives, they will rank order the alternatives in that set on the basis of the degree to which each achieves the actor’s goals. Following the lead of Nurmi (1981b), we treat the policy preferences of political actors as fuzzy observables.

One of the more important concepts we develop in chapter four is the notion of a discrete fuzzy number, which permits the treatment of a significant degree of ambiguity and indifference throughout the policy space. In chapter five we extend this concept to spatial modeling in two-dimensions. This requires the aggregation of preferences across dimensions. We demonstrate our approach with a reformulation of Shepsle and Laver’s (1996) *portfolio allocation model*.

Chapter six gives further consideration to the use of fuzzy numbers to represent the preferences of political actors. We discuss whether in addition to providing information about utility payoffs, fuzzy numbers might not also be used to analyze the amount of information available to an actor regarding a policy or the intensity of an actor’s preference concerning a given policy. The chapter concludes with a discussion of various approaches to aggregating preferences across two or more dimensions.

Chapter seven returns to the recurring problem of cycling in formal modeling. We begin by discussing fuzzy alternatives that have been advanced in order to induce a set of equilibrium outcomes. After considering how well our own fuzzy

utility maximizing approach deals with the problem, we conclude with some general observations and implications for testing hypotheses derived from our models.

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## 2 Fuzzy Set Theory

In this chapter we present the reader with the fundamental concepts of fuzzy set theory. The basic primer on fuzzy set theory remains Zadeh's [1965] seminal work. A number of scholars have since discussed several aspects of fuzzy set theory pertinent to the social sciences. Perhaps the best overview is provided by [Smithson and Verkuilen] (2006). Among the more thoroughly discussed topics are the construction of fuzzy numbers ([Smithson and Verkuilen, 2006; Verkuilen, 2005; Bilgic and Turksen, 1995]) and fuzzy set operations ([Smithson and Verkuilen, 2006]).

This chapter begins with a discussion of the differences between traditional, crisp sets and fuzzy sets. Set theory provides a systematic way to consider collections of distinct objects. Most fields of mathematics can be understood in terms of sets of abstract objects. To harness the utility of sets as building blocks, we must have a way to precisely specify the elements that are members of a set. The notion of membership in a crisp set is simple: an object either is or is not a member of a set. Fuzzy sets allow the possibility of partial membership. An object may partially be a member of the fuzzy set, another object may be more a member of the set than the first object but still not fully a member. Fuzzy set membership can be partial and relative to other objects, so to fully specify a fuzzy set, it is necessary to not only list the objects that are at least partial members, but also to indicate the extent to which each object is a member of a set. Section 2.2 discusses membership functions and related notation.

Once we know which objects are at least partially elements of a set and how strongly each object is an element of a set, it is useful to have tools that allow us to further characterized fuzzy sets. Section 2.3 introduces a means of grouping and characterizing subsets of sets. By considering only elements that belong to a fuzzy set with a given minimum intensity, alpha- cuts offer a way to limit attention to a subset of a fuzzy set and also offer a complete characterization of a fuzzy set that can be easily compared to other fuzzy sets. The end of this section presents terms that will be necessary for the remainder of the book.

We are most interested in a certain class of fuzzy sets called fuzzy numbers. Fuzzy numbers are sets that satisfy some minimal properties that will allow

arithmetic operations and manipulation conducive to the geometry presented in later chapters. There are many types of fuzzy numbers; we present six types (and one non-type) in Section 2.4, with special attention on trapezoidal numbers which will be most important in the next chapter.

The material in this chapter (and the rest of the book) assumes that a fuzzy set exists. However, the difficulty that all social scientists face is that data sets of any form must be created. Section 2.5 describes a variety of methods for constructing fuzzy sets and for turning the fuzzy sets into the more useful fuzzy numbers. The first few methods are straightforward and parallel methods used in constructing any data set (fuzzy or not). The last few methods give the reader an introduction to more sophisticated methods available for most accurately representing data with fuzzy numbers. Our hope with these latter methods is to pique the more computer-savvy reader's interest and to provide the citations for such a reader to replicate the methods.

The final two sections introduce some basic and commonly used operations on fuzzy sets. We conclude with distance metrics that bring us to the cusp of fuzzy geometry. With these fundamentals of fuzzy set theory, we will be ready to delve into fuzzy geometry, the tools of spatial formal models in the social sciences. The approach that we present for formal modeling in the rest of this book makes extensive use of fuzzy geometry, in particular discrete fuzzy numbers, to map the preferences of individual and collective political actors. We demonstrate that our fuzzy approach improves the ability of formal models in comparative politics to make stable predictions.

## 2.1 Fuzzy Sets

Sets are the basis of all mathematics. A set is a collection of objects, the candidates for which comprise the universal set, most often labeled  $X$  or  $U$ . Typical universal sets are the real numbers or the natural numbers. The important aspect in defining a set is that the definition must enable one to determine which objects are in the set and which objects are not in the set. Sets are specified using one of three methods: explicit listing of elements, specification of the necessary properties, and the use of characteristic functions. For finite sets one can use a complete listing such as  $A = \{2, 4, 6\}$ . For countably infinite sets the ellipsis "... " denotes "and so on" as in the set of all natural numbers  $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ . Specifying a set by listing a property that all its contents must have can be used for any type of set. For example we can have  $A = \{x \mid x \text{ defines an even natural number less than } 7\}$  which is the same  $A$  as in the previous item. Another example is the infinite set  $P = \{x \mid x \text{ is a prime number}\}$ .

For the purposes of this book, the characteristic function method is the most important way of determining a set's contents. The characteristic function  $\chi_A$  of a set  $A$  is a mapping of objects into membership indicators.  $\chi_A$  maps object  $x$  to zero if  $x$  is not in the set  $A$  and one if  $x$  is in the set. For example, if  $A$  is the subset  $\{2, 4, 6\}$  of  $\mathbb{N}$ ,

$$\chi_A(x) = \begin{cases} 1 & \text{if } x = 2, 4, 6, \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic functions of classical (Cantor) set theory map elements of some universal set  $X$  into the binary set  $\mathbb{B} = \{0, 1\}$ . Traditional set theory says that every candidate  $x$  must be either in the set or not in the set. Classical set theory is useful for the characterization of objects for which complete precision is possible. Numbers, for example, can be precise. 2 is either even or not even. This dichotomy is typical of classical western thinking, whether it is *yes* and *no*, *true* and *false*, or *one* and *zero*. However, such exactness is not always possible. While most people would place chocolate in the set of tasty food,  $F$ , where would we place caviar? Caviar is salty and fishy and not to everyone's taste, yet those who do like it consider it a tasty food. The set  $F$  suffers because tasty food is not an easily defined notion, as opposed to the notion of a prime number which can be defined precisely. Another example is the exact dividing line between *rich* and *poor*. Where does the middle class fit into such a categorization?

Such ambiguity frequently pervades human thinking, and it is reflected in human language. For example, the response from most people to the question, "How old is Juan?", is not likely to be a numeric answer. Only his friends and relatives usually know his exact calendar age. An acquaintance will answer that he is a "teenager" or "young" or "adolescent." Furthermore, the age of the respondent will greatly influence the answer given. A person of seventy might say Juan is a "boy" whereas a contemporary might say, "He is the same age I am," a response that provides no direct numerical information.

Lotfi Zadeh (1965) created fuzzy set theory in order to mathematically represent and process such ambiguity. While the term "fuzzy" might carry the connotation to some that it is associated with fuzzy thinking, one might as well criticize probability theory for leading to random actions. If probability theory is a precise mathematical device for processing data whose source is a random event, fuzzy set theory is a precise mathematical tool for processing data that is derived from vague sources. While probability deals with uncertainty associated with randomness, fuzzy set logic is concerned with uncertainty associated with imprecision rooted in vagueness and ambiguity that has nothing to do with randomness. Rather it has to do with ambiguity concerning the exact characteristic, be it the quality, location, preference, etc., of a thing.

## 2.2 Membership Functions

The fundamental idea of fuzzy set theory is that real world phenomena cannot be divided cleanly into *black* and *white* divisions. In fuzzy set theory we extend the image set of the characteristic function from the binary set  $\mathbb{B} = \{0, 1\}$  which contains only two alternatives, to the unit interval  $\mathbb{U} = [0, 1]$  which has an infinite number of alternatives. To reflect this definition, we also relabel the characteristic function as the membership function, denoted by the symbol  $\mu$ , instead of  $\chi$ . This introduces a rich approach for measuring the world in shades

of grey. For instance, using fuzzy set theory we can characterize democracy as more than “democracy” or “not democracy.” We can choose gradations such as “very high,” “high” “uncertain,” “low,” or “very low” and assign membership values accordingly. These are not evaluations of the *probability* that a country is a democracy, rather they are measures of the degree to which it is a democracy.

A fuzzy set is essentially a function whose domain is some universal set  $X$ . Its range is the unit interval  $\mathbb{U}=[0, 1]$ . The notation,

$$\mu_A : X \rightarrow [0, 1],$$

which specifies the membership function, defines the fuzzy set. The image of  $\mu_A$  will be a set of membership values bounded by 0 and 1 indicating the degree of membership in  $A$  of each object  $x \in X$ . Hence, a fuzzy set consists of a set of objects and their membership values. (We could think of a crisp set in the same way, but the membership values for all elements of the set would be 1.) Consider the following example.

*Example 2.1.* Let  $X = \{a, b, c\}$  and set  $\mu_A(a) = 1.0$ ,  $\mu_A(b) = 0.7$ , and  $\mu_A(c) = 0.4$ . The example defines fuzzy set  $A$  with three objects  $\{a, b, c\}$ , and their respective membership values, 1, .7, and .4. Here  $\mu_A$  tells us that  $a$  is fully a member in the set  $A$ ,  $b$  is .7 a member, and  $c$  is .4 a member.

The tag of the fuzzy set,  $A$ , is difficult to read as a subscript, and the membership function of a fuzzy set with a subscript, such as  $A_i$  with membership function  $\mu_{A_i}$ , compounds the subscript problem. Therefore, it is often desirable to use the same tag for a set and its characteristic function. We will use the tag  $A$  to represent both the fuzzy set and its membership function. Thus, if  $A$  is a fuzzy set then we will also use  $A$  as the label of a function from a universe of discourse  $X$  into the unit interval  $\mathbb{U} = [0, 1]$

$$A : X \rightarrow [0, 1]. \quad (2.1)$$

The membership function of the fuzzy set in the previous example can be rewritten as  $A(a) = 1.0$ ,  $A(b) = 0.7$ , and  $A(c) = 0.4$ .

The notation used in books about fuzzy set theory is not yet standardized. A fuzzy set was initially indicated by the tilde above the character, such as  $\tilde{A}$ . Much of the early literature includes phrases such as “the fuzzy set  $\tilde{E}$ ”. We will use the notation  $A$  in this book to denote a fuzzy set and  $\mu_A$  to denote a fuzzy set membership function only if there is a chance of confusion. Synonyms for membership function are membership grade and characteristic function. We will also make use of Zadeh’s original fraction notation, which takes up less space. This notation is a list of fractions that use + as a separator. The + does not represent addition, it simply separates terms. Each fraction has a denominator which is the element of the fuzzy set under discussion and a numerator that specifies the membership value. Thus the fuzzy set  $A$  specified by  $A(a) = 1.0$ ,  $A(b) = 0.7$ , and  $A(c) = 0.4$  is given as

$$A = \frac{1.0}{a} + \frac{0.7}{b} + \frac{0.4}{c}. \quad (2.2)$$

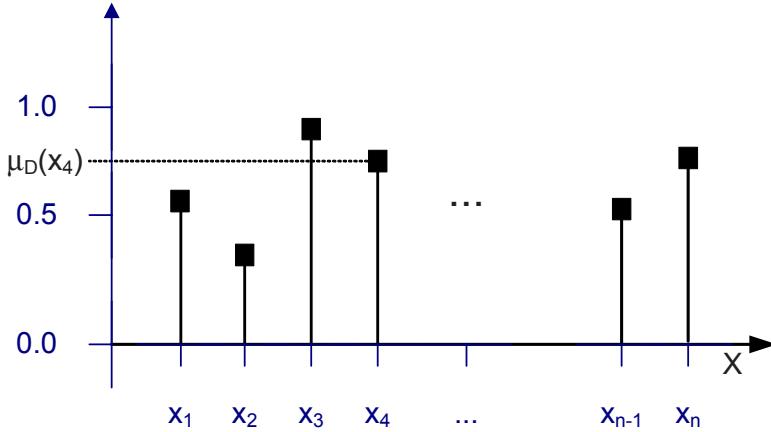


Fig. 2.1. A Fuzzy Set  $D$  on a Discrete Universe  $X = \{x_1, x_2, \dots, x_n\}$

The element below the bar has the membership grade equal to the value above the bar and the plus sign only separates members of the list. The statement  $A(b) = 0.7$  is interpreted as saying that the membership grade of  $b$  in the fuzzy set is seven-tenths.

*Example 2.2.* Let  $X = \{x_1, x_2, x_3, x_4, \dots, x_n\}$  and set  $\mu_D(x_1) = 0.5$ ,  $\mu_D(x_2) = 0.4$ ,  $\mu_D(x_3) = 0.8$ ,  $\mu_D(x_4) = 0.7, \dots, \mu_D(x_{n-1}) = 0.5$  and  $\mu_D(x_n) = 0.7$ . Then the discrete fuzzy set  $D$  is depicted at Figure 2.1.

*Example 2.3. Example 2.4.* Let  $X = [1, 10]$  and  $\mu_C(x_i) = [0, 1]$ . Then one example of a continuous fuzzy set  $C$  is depicted at Figure 2.2. Note that  $\mu_C : [1, 10] \rightarrow [0, 1]$ , but the mapping is not fully specified here. Figure 2.2 is one of an infinite number of fuzzy sets that use a membership value map like  $\mu_C$ .

The only difference between a traditional set and a fuzzy set is the image of their membership functions. A traditional set has its membership grades taking values in the set  $\{0, 1\}$  while a fuzzy set has its membership grades in the unit interval  $[0, 1]$ . A fuzzy set can include elements that are not fully included in the set, but that are not fully excluded. A standard set will be called a crisp set whenever it is necessary to distinguish it from a fuzzy set. The universal set  $X$  is always understood to be a crisp set.

A few more definitions are necessary to our discussion. Crisp set theory defines the class of all sets defined on a universe as the power set of the universe  $X$ ,  $\mathcal{P}(X)$ . Fuzzy set theory makes use of an analogous concept. The set of all fuzzy sets defined on the universal set  $X$  is called the fuzzy power set.

**Definition 2.5.** Suppose that  $X$  is a crisp universal set. Let the class of all fuzzy sets defined on  $X$  be denoted by  $\mathcal{F}(X)$ .  $\mathcal{F}(X)$  is called the fuzzy power set of  $X$ .

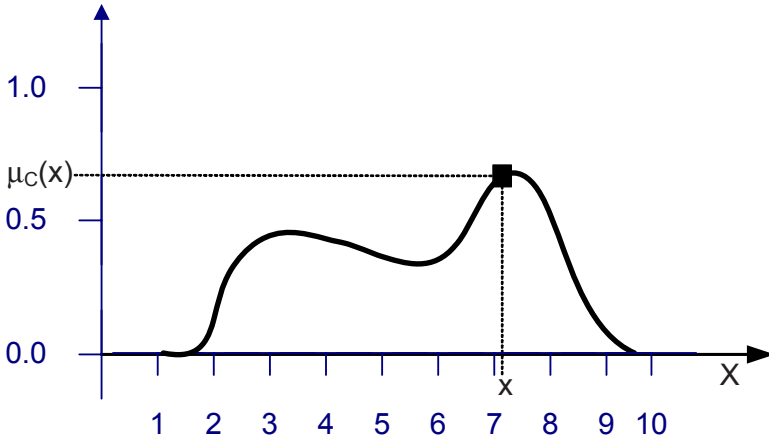


Fig. 2.2. A Fuzzy Set  $C$  on a Continuous Domain  $X = [1, 10]$

The scalar cardinality of a fuzzy set  $A$  is a count of the number of elements in  $A$ . Of course, some elements are not completely in fuzzy set  $A$  so the scalar cardinality of  $A$  is not necessarily an integer (as is the case for a discrete crisp set.)

**Definition 2.6.** *The scalar cardinality of  $A$  is the sum of the degree of membership of every element in  $A$  and is denoted  $|A|$ ,*

$$|A| = \sum_{x \in X} A(x). \tag{2.3}$$

*Example 2.7.* The scalar cardinality of  $A$ , whose membership function is indicated by Eq. (2.2) is

$$|A| = \sum_{x \in X} A(x) \tag{2.4}$$

$$= \sum_{x \in \{a,b,c\}} A(x) \tag{2.5}$$

$$= A(a) + A(b) + A(c) \tag{2.6}$$

$$= 1.0 + 0.7 + 0.4 \tag{2.7}$$

$$= 2.1 . \tag{2.8}$$

### 2.3 Alpha-Cut or $\alpha$ -Cut

Sometimes it is useful to discuss a subset associated with a fuzzy set. One particularly useful class of subsets comprises the elements of a fuzzy set with membership values larger than a given cutoff  $\alpha$ .

**Definition 2.8.** For every  $\alpha \in [0, 1]$ , a given fuzzy set  $A$  yields a crisp set

$$A^\alpha = \{x \in X \mid A(x) \geq \alpha\}$$

which is called an  $\alpha$ -cut of  $A$ .

Since  $\alpha_1 < \alpha_2$  implies  $A^{\alpha_1} \supseteq A^{\alpha_2}$  the set of all distinct  $\alpha$ -cuts of any fuzzy set forms a nested sequence of crisp sets.

**Definition 2.9.** We define the image set of  $A$ ,  $Im(A)$  as the image of the membership function  $\mu_A$ . It consists of all values  $\alpha$  in the unit interval such that  $A(x) = \alpha$  for some  $x \in X$ .

$$Im(A) = \{A(x) \mid x \in X\} . \quad (2.9)$$

The image set can simply be thought of as the set of membership values of the objects in  $X$ .

*Example 2.10.* The image set of  $A$  is  $Im(A) = \{0.4, 0.7, 1.0\}$ . Consider once more

$$A = \frac{1.0}{a} + \frac{0.7}{b} + \frac{0.4}{c} .$$

The  $\alpha$ -cut at  $\alpha = 0.5$  is

$$A^{0.5} = \{a, b\}$$

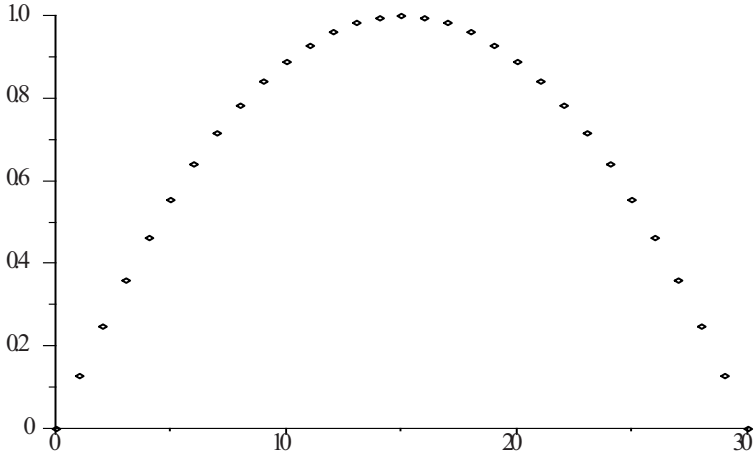
since  $\mu_A(a) = 1.0 \geq 0.5$  and  $\mu_A(b) = 0.7 \geq 0.5$ , but  $\mu_A(c) = 0.4 \not\geq 0.5$ . The image set of  $A$ ,  $Im(A) = \{1.0, 0.7, 0.4\}$ , gives all the membership values that generate distinct  $\alpha$ -cuts.

In fact, an arbitrary fuzzy set  $A$ , it is uniquely represented by the associated sequence of its distinct  $\alpha$ -cuts via the formula

$$A(x) = \sup_{\alpha \in [0,1]} \alpha \cdot \chi_{A^\alpha}(x), \quad (2.10)$$

where  $\chi_{A^\alpha}$  denotes the characteristic function of the crisp set  $A^\alpha$  and sup designates the supremum (or maximum, when the image set  $Im(A)$  is finite) which is the largest value the expression attains as  $\alpha$  ranges over the unit interval.

Equation (2.10), usually referred to as a *decomposition theorem* of fuzzy sets (Zadeh, 1971), establishes an important connection between fuzzy sets and crisp sets. This connection provides us with a criterion for generalizing properties of crisp sets into their fuzzy counterparts: when a fuzzy set satisfies a property that is claimed to be a generalization of a property established for crisp sets, this property usually should be preserved (in the crisp sense) in all  $\alpha$ -cuts of the fuzzy set. For example, all  $\alpha$ -cuts of a convex fuzzy set should be convex crisp sets; the cardinality of a fuzzy set should yield for each  $\alpha$  the cardinality of its  $\alpha$ -cut; each  $\alpha$ -cut of a properly defined fuzzy equivalence relation (or fuzzy compatibility relation, fuzzy ordering relation, etc.) should be an equivalence relation (or compatibility relation, ordering relation, etc., respectively) in the classical sense. Every property of a fuzzy set that is preserved in all its  $\alpha$ -cuts is called a *cutworthy property*.



**Fig. 2.3.** The Membership Grade of the  $x$  Values  $0, 1, 2, \dots, 30$  in the Fuzzy Set  $B$

*Example 2.11.* Let  $X = \{0, 1, 2, \dots, 29, 30\}$ . Define the fuzzy set  $B$  by  $\forall x \in X$ ,

$$B(x) = \frac{30x - x^2}{225}. \tag{2.11}$$

Then  $B(0) = 0$ ,  $B(5) = 0.56$ ,  $B(15) = 1$ ,  $B(20) = 0.89$ , etc. This fuzzy set is depicted at Figure 2.3.

The  $\alpha$ -cut of  $B$  at  $\alpha = 0.7$  is a crisp set containing all those elements  $x \in X$  whose membership grade is greater than or equal to 0.7, that is  $B^\alpha = \{7, 8, 9, \dots, 22, 23\}$ .

Two crisp sets are equal when the elements of both are identical,  $\{a, b, c\} = \{a, b, c\} \neq \{a, b, c, d\}$ . Likewise, two fuzzy sets  $A$  and  $B$  are equal if the membership function assigns the same membership function for all objects,  $A(x) = B(x) \forall x \in X$ . Another property of equivalence is stated in the following theorem.

**Theorem 2.12.** *Two fuzzy sets are equal if and only if all their corresponding  $\alpha$  cuts are equal.*

*Proof.* We are trying to prove that :

$$\forall \alpha : A^\alpha = B^\alpha \Leftrightarrow A = B, \tag{2.12}$$

where  $\Leftrightarrow$  indicates equivalence. We use arrows to indicate implication, where the two-directional arrow indicates that statement on the left implies that on the right and that on the right implies that on the left. The symbol  $\leftrightarrow$  indicates that the statement on the left implies that on the right, but not the converse.

First let us show that :  $A = B \Rightarrow A^\alpha = B^\alpha$ . Again,  $A = B$  means that their membership functions are identical for every  $x \in X$ . The  $\alpha$ -cuts can be written  $A^\alpha = \{x \mid A(x) \geq \alpha\}$  and  $B^\alpha = \{x \mid B(x) \geq \alpha\}$ . Since  $A(x) = B(x)$  for all  $x$ ,  $A^\alpha = B^\alpha$  for all  $\alpha$ .

To show the reverse implication, that :  $\forall \alpha A^\alpha = B^\alpha \Rightarrow A = B$ , we will derive a contraction from supposing that  $\forall \alpha A^\alpha = B^\alpha$  but  $A \neq B$ . But  $A \neq B$  if and only if there exists some  $y \in X$  such that  $A(y) \neq B(y)$ . Without loss of generality assume that  $A(y) < B(y)$  and let  $\gamma = B(y)$ . It must be that  $y \notin A^\gamma$  but  $y \in B^\gamma$ . Then the  $\alpha$ -cuts of  $A$  and  $B$  are not identical, a contradiction.

This shows that the alpha-cuts of a fuzzy set completely characterize that fuzzy set. We need a few more definitions before we can proceed to fuzzy numbers.

**Definition 2.13.** *The strong alpha cut,  $A^\alpha_\gt$ , is defined to be all those elements that have membership strictly greater than alpha,*

$$A^\alpha_\gt = \{x \mid A(x) > \alpha\}. \tag{2.13}$$

**Definition 2.14.** *The support of a fuzzy set is the strong alpha-cut at zero:*

$$S(A) = A^0_\gt \tag{2.14}$$

$$= \{x \mid A(x) > 0\} . \tag{2.15}$$

Hence it is that subset of the domain which has positive membership grade in a fuzzy set.

**Definition 2.15.** *The core, peak or mode of a fuzzy set is the alpha-cut at one:*

$$C(A) = A^1 \tag{2.16}$$

$$= \{x \mid A(x) = 1\} . \tag{2.17}$$

**Definition 2.16.** *The height of a fuzzy set  $A$ ,  $h(A)$ , is the maximum or supreme membership grade achieved by  $A(x)$  over the domain of  $X$  :*

$$h(A) = \sup_{x \in X} (x) . \tag{2.18}$$

*If the height of a fuzzy set is equal to 1 then we call that fuzzy set normal.*

**Definition 2.17.** *The plinth of a fuzzy set  $A$ ,  $p(A)$ , is the minimum, or infimum, membership grade achieved by  $A(x)$  over the domain of  $X$  :*

$$p(A) = \inf_{x \in X} (x) . \tag{2.19}$$

## 2.4 Fuzzy Numbers

The *true* building block of applied fuzzy set theory is the fuzzy number. A fuzzy number,  $T$ , is a convex normal fuzzy set defined upon the reals. Before we

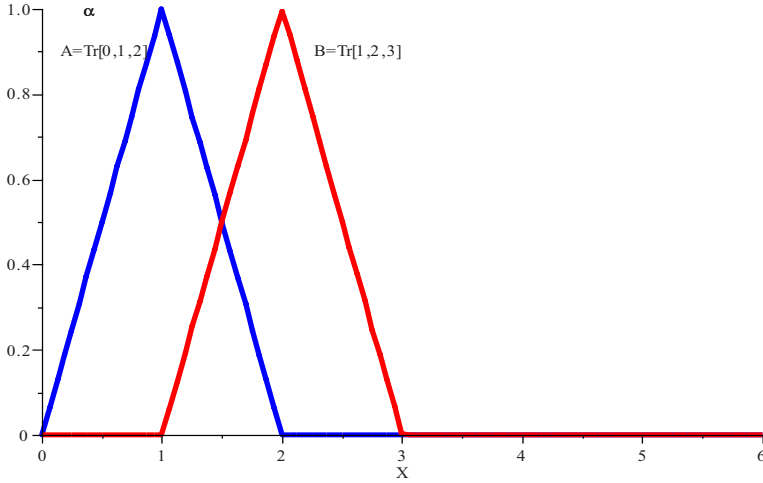


Fig. 2.4. The Fuzzy Numbers  $A$ , “about One” and  $B$ , “about Two”

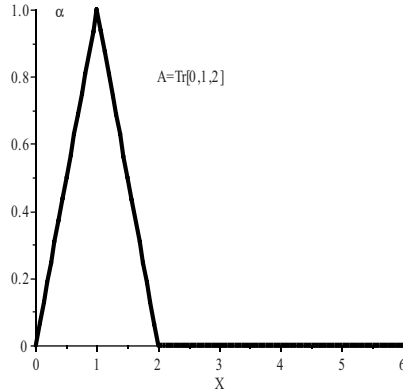
present a formal definition, let us start with a simple example. Eq. (2.20) and (2.21) define two fuzzy numbers  $A$  and  $B$ . The fuzzy number  $A$  represents the vague concept “about one” and  $B$  represents the vague concept “about two.” A fuzzy set is determined by its membership function so that the equation for fuzzy set  $A$  is

$$A(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1 \\ 2 - x & \text{for } 1 < x \leq 2 \end{cases} \tag{2.20}$$

and the equation for fuzzy set  $B$  is

$$B(x) = \begin{cases} x - 1 & \text{for } 1 \leq x \leq 2 \\ 3 - x & \text{for } 2 < x \leq 3. \end{cases} \tag{2.21}$$

If we examine a graph of  $A$  and  $B$ , at Figure 2.4, we see that the fuzzy numbers are represented by triangularly shaped functions made up of two line segments. Note also, that we suppressed mentioning the additional restriction that  $A(x) = 0$  if  $x$  is outside the closed interval  $[0, 2]$  and that  $B(x) = 0$  if  $x$  is outside the closed interval  $[1, 3]$ . From this moment on, we shall assume that every fuzzy number has zero as its membership grade for every domain value of  $x$  in the universe  $X$  not explicitly listed as having a nonzero membership grade. The fuzzy number  $A$  expresses the notion that “about one” includes any domain value greater than zero and less than two, but also indicates that we are more confident that 1.1 is “about one” (.9 membership in the set “about one”) than we are that .2 is “about one” (.2 membership in the set “about one”) and that 2.1 is not “about one” at all (0 set inclusion level). This is somewhat related to notions like accuracy and precision but on a sliding scale. For instance a ruler that is



**Fig. 2.5.** The Fuzzy Triangular Number  $A = T_r[0, 1, 2]$

marked in centimeters must be read to the nearest centimeter. A measurement of 34 centimeters says that the exact value is in the interval  $[33.5, 34.5]$ , however all values in this range may be regarded equally as 34 centimeters.

Formally, we have the following definition.

**Definition 2.18 (fuzzy number).** *A fuzzy number is a fuzzy set with domain  $\mathbb{R}$ , the real numbers, that is normal, has bounded support, and whose  $\alpha$ -cuts are closed intervals for positive  $\alpha$ .*

As defined previously, a fuzzy set  $T$  is termed normal if there exists an  $x \in X$  such that  $T(x) = 1$ . The support, or strong  $\alpha$ -cut at zero must be bounded, that is  $T^0 = [a, b]$  where neither  $a$  nor  $b$  is infinite. Finally, every  $\alpha$ -cut must be a closed interval (an interval that includes its end points) for  $\alpha > 0$ . Because every  $\alpha$ -cut of a fuzzy number is a closed interval of the real line, a useful property is that the  $\alpha$ -cuts of a fuzzy number are convex. A fuzzy set is convex if every  $\alpha$ -cut  $T^\alpha$ , for  $\alpha \in (0, 1]$ , is a convex subset of the domain, i.e., a continuous interval. An equivalent definition of convexity requires for any  $\lambda \in I$ ,  $I = [0, 1]$ , that  $\min[T(x), T(y)] \leq T(\lambda x + (1 - \lambda)y)$ . Hence every fuzzy number is a convex fuzzy set.

### 2.4.1 Triangular Fuzzy Numbers

A triangular fuzzy number,  $T_r$ , is named for its shape. Its membership function is given by two line segments, one rising from the point  $\langle a, 0 \rangle$  to  $\langle m, 1 \rangle$  and the second falling from  $\langle m, 1 \rangle$  to  $\langle b, 0 \rangle$ . Its domain is the closed interval  $[a, b]$ . An example is illustrated at Figure 2.5. A triangular fuzzy number can be specified by the ordered triple  $\langle a, m, b \rangle$  with  $a \leq m \leq b$  and its membership function is:

$$T_r[a, m, b](x) = \begin{cases} \frac{x-a}{m-a} & \text{for } a \leq x \leq m \\ \frac{x-b}{m-b} & \text{for } m < x \leq b. \end{cases}$$

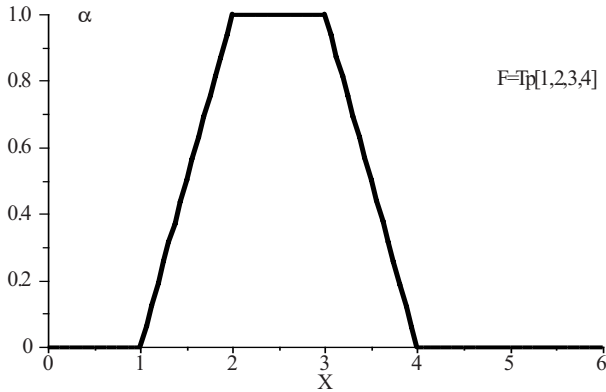


Fig. 2.6. The Fuzzy Trapezoidal Number  $F = Tp[1, 2, 3, 4]$

When an ordered pair  $\langle x, y \rangle$  is used as an argument to a function of two variables, it is almost always written as  $f(x, y)$  and not as  $f(\langle x, y \rangle)$ . The extra brackets only make the expression more difficult to interpret. Similarly, parameters are usually enclosed in square brackets,  $[p]$ , and  $Tr[\langle a, m, b \rangle]$  is abbreviated to  $Tr[a, m, b]$ . However the parameters of the triangular fuzzy number, and the other fuzzy numbers to come, are in order, and the square brackets do not represent a closed interval.

### 2.4.2 Trapezoidal Fuzzy Numbers

A trapezoidal fuzzy number,  $Tp$ , can be specified by an ordered quadruple  $\langle a, l, r, b \rangle$  with  $a \leq l \leq r \leq b$  and a membership function consisting of three line segments. The first segment rises from  $\langle a, 0 \rangle$  to  $\langle l, 1 \rangle$ , the second segment is a horizontal line that has a constant value of one and that stretches from  $\langle l, 1 \rangle$  to  $\langle r, 1 \rangle$ , and the third segment drops from  $\langle r, 1 \rangle$  to  $\langle b, 0 \rangle$ . An example of a trapezoidal number is illustrated at Figure 2.6.

The membership function for a trapezoidal fuzzy number is:

$$Tp[a, l, r, b](x) = \begin{cases} \frac{x-a}{l-a} & \text{for } a \leq x \leq l \\ 1 & \text{for } l < x < r \\ \frac{x-b}{r-b} & \text{for } r \leq x \leq b. \end{cases}$$

Not all fuzzy numbers are composed of line segments. Other standard fuzzy numbers are the differentiable piecewise quadratic, Gaussian bell shaped numbers, and the  $L - R$  fuzzy numbers.

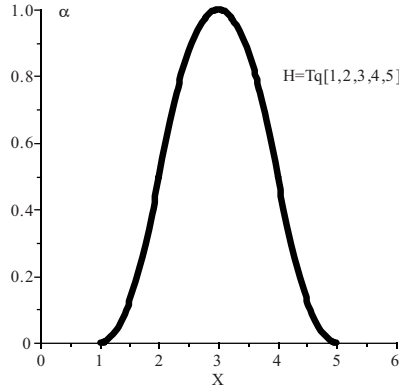


Fig. 2.7. The Piecewise Quadratic Fuzzy Number  $H = Tq [1, 2, 3, 4, 5]$

### 2.4.3 Differentiable Piecewise Quadratic Fuzzy Numbers

Differentiable piecewise quadratic numbers  $Tq$  (see Figure 2.7) consist of four quadratic pieces and are parameterized by five values  $a, l, m, r,$  and  $b$ . The values  $a$  and  $b$  are the left and right hand limits of the support,  $m$  is the mean value (core) where  $Tq$  assumes the value one, and  $l$  and  $r$  are the left and right hand crossover points (points of inflection). If  $Tq \langle a, l, m, r, b \rangle$  is a *continuous piecewise quadratic number* then the membership function of  $T_q$  is given by:

$$Tq[a, l, m, r, b](x) = \begin{cases} \frac{1}{2} \left( \frac{a-x}{a-l} \right)^2 & \text{for } a \leq x \leq l \\ 1 - \frac{1}{2} \left( \frac{m-x}{m-l} \right)^2 & \text{for } l < x \leq m \\ 1 - \frac{1}{2} \left( \frac{m-x}{m-r} \right)^2 & \text{for } m < x \leq r \\ \frac{1}{2} \left( \frac{b-x}{b-r} \right)^2 & \text{for } r < x \leq b. \end{cases}$$

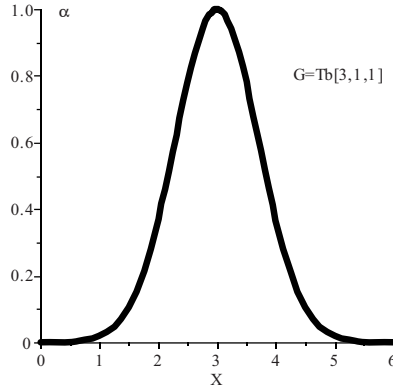
### 2.4.4 Gaussian Fuzzy Numbers

Gaussian numbers are parameterized by the mean  $m$ , the spread  $s$ , and the scale parameter  $\gamma$  that adjusts the height. The membership function of a *bell shaped* fuzzy number (see Figure 2.8) is:

$$Tb[m, s, \gamma](x) = \gamma e^{-(x-m)^2/s^2}.$$

### 2.4.5 L – R Fuzzy Numbers

An  $L - R$  fuzzy number (Dubois and Prade, 1980) is a unimodal fuzzy number on the reals that can be described in terms of two shape function, the left hand shape function,  $L$ , and the right hand shape function,  $R$ . A unimodal fuzzy set



**Fig. 2.8.** The Fuzzy Bell Number  $G = Tb[3, 1, 1]$

has its maximum value at a unique value  $m$  of the domain  $X$ . This insures that  $T(x) = 1$  if and only if  $x = m$ . (Triangular fuzzy numbers are modal; trapezoidal fuzzy numbers are not.) A shape function  $L$  (or  $R$ ) is a function from  $[0, \infty)$  to  $[0, 1]$  that is non-increasing, and such that  $L(0) = 1$  and  $L(1) = 0$ .

With these conventions an  $L - R$  fuzzy number can be described in terms of an ordered triple

$$Tlr = \langle \langle m, u, v \rangle, L, R \rangle,$$

where  $L$  and  $R$  are the shape functions,  $m$  is the mode of the fuzzy number and  $u$  and  $v$ , both greater than zero, are the left and right-hand spread of the function. The formal definition of the  $L - R$  fuzzy number is

$$Tlr(x) = \begin{cases} R\left(\frac{x-m}{v}\right) & \text{for } x \geq m \\ L\left(\frac{m-x}{u}\right) & \text{for } x < m. \end{cases}$$

Almost all unimodal fuzzy numbers can be expressed as  $L - R$  fuzzy numbers with an appropriate choice for the left and right shape functions,  $L$  and  $R$ , as well as for the parameters; the mode  $m$ , left spread  $u$ , and right spread  $v$ .

### 2.4.6 Impulse Fuzzy Numbers

The impulse fuzzy number  $Ti[m]$  (see Figure 2.9), also called a fuzzy point number, has membership function

$$Ti[m](x) = \begin{cases} 1 & \text{for } x = m \\ 0 & \text{otherwise} \end{cases} \quad (2.22)$$

An impulse number can be considered a degenerate case of a triangular (and other types) of fuzzy numbers since  $Ti[m] = Tr \langle m, m, m \rangle$ . This is also called a fuzzy singleton in the literature of fuzzy set theory.

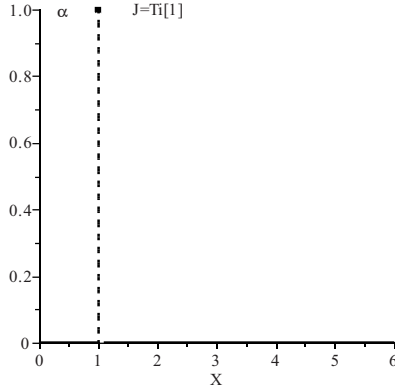


Fig. 2.9. The Fuzzy Singleton  $Ti[1]$

### 2.4.7 S-Shaped Fuzzy Numbers

Finally, we address S-shaped fuzzy numbers. The bounded support requirement for a fuzzy number means that what goes up must come down. While this is an important requirement for fuzzy numbers, many monotonic functions that asymptotically approach one are treated as if they are fuzzy numbers, though, technically, they are not. An asymptotic function gets very close to a limiting value (which is usually zero or one in this book) but never quite reaches it.

Monotonically increasing and monotonically decreasing functions as illustrated in Figure 2.10 are often called  $s$ -shaped fuzzy numbers, which is an abuse of language that occurs all too often. An increasing  $s$ -shaped fuzzy set  $S$  may consist of two quadratic pieces, line segments, exponential curves, or any monotonic function that achieves or is asymptotic to 1 at one end of its domain set, and achieves or is asymptotic to 0 at the other end of its domain set. Assume in all the following definitions of this section that  $a \leq l \leq m \leq r \leq b$  are all real numbers.

Some examples of increasing  $s$ -shaped fuzzy sets are the quadratic  $s$ -shaped fuzzy sets  $Sqi$ :

$$Sqi[a, l, m](x) = \begin{cases} \frac{1}{2} \left( \frac{a-x}{a-l} \right)^2 & \text{for } a \leq x \leq l \\ 1 - \frac{1}{2} \left( \frac{m-x}{m-l} \right)^2 & \text{for } l < x \leq m \end{cases}$$

and the linear  $s$ -shaped sets

$Sli$ :

$$Sli[a, l, r, b](x) = \begin{cases} 0 & \text{for } a \leq x \leq l \\ \frac{x-l}{r-l} & \text{for } l < x \leq r \\ 1 & \text{for } r < x \leq b. \end{cases}$$

Some examples of decreasing  $s$ -shaped fuzzy numbers are the quadratic  $s$ -shaped fuzzy sets  $Sqd$ :

$$Sq d [m, r, b] (x) = \begin{cases} 1 - \frac{1}{2} \left( \frac{m-x}{m-r} \right)^2 & \text{for } m \leq x \leq r \\ \frac{1}{2} \left( \frac{b-x}{b-r} \right)^2 & \text{for } r < x \leq b \end{cases}$$

and the linear  $s$ -shaped sets  $Sld$ :

$$Sld [a, l, r, b] (x) = \begin{cases} 1 & \text{for } a \leq x \leq l \\ \frac{x-r}{l-r} & \text{for } l < x \leq r \\ 0 & \text{for } r < x \leq b. \end{cases}$$

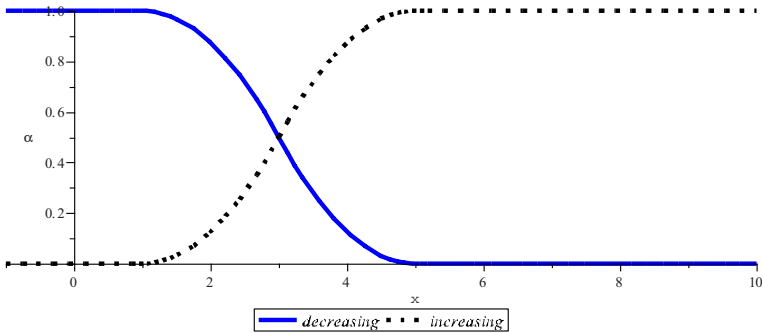
Other extremely popular increasing  $s$ -shaped fuzzy sets are those generated by the sigmoid functions,

$$S[\beta] (x) = (1 + e^{-\beta x})^{-1}.$$

with  $\beta \geq 0$ . This curve is illustrated in Figure 2.11, for various values of  $\beta > 0$ . If  $\beta$  is less than zero then the  $s$ -shaped fuzzy set becomes decreasing.

## 2.5 Constructing Fuzzy Sets

While fuzzy sets are not designed to be a precise representation, a poor representation will nonetheless create a system that is useless for any constructive purpose. How then do we construct good fuzzy sets from data? The construction and design of fuzzy set membership is known as fuzzification. We first consider the vertical and horizontal view of fuzzy data. We then turn our attention to methods for fuzzifying data.



**Fig. 2.10.** An Increasing and a Decreasing  $s$ -Shaped Fuzzy Set

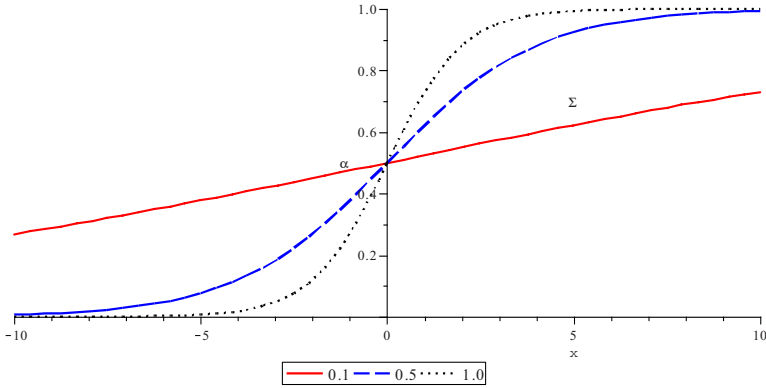


Fig. 2.11. Sigmoid Fuzzy Sets

### 2.5.1 General Views of Fuzzy Data

The vertical view of a fuzzy set is the membership function view. For each  $x \in X$  the membership function  $\mu_A(x)$  gives the degree of compatibility of the element  $x$  with the category or linguistic term  $A$  presently under discussion. It has always been the assumption that the greater the membership value,  $\mu_A(x)$ , the greater the compatibility of  $x$  with the concept represented by  $A$ . Therefore to construct  $\mu_A(x)$  vertically from information about  $x$ , it is necessary that the data allows one to order the compatibility of  $x$  with respect to the concept  $A$ .

Since the membership function of a fuzzy set always takes values in  $[0, 1]$  the standard vertical transformation of any function  $f : X \rightarrow \mathbb{R}$  (or data set) into a fuzzy set  $F$  with membership  $\mu_F$  is

$$\mu_F(x) = \frac{f(x) - f_{\min}}{f_{\max} - f_{\min}},$$

where  $f_{\min}$  and  $f_{\max}$  are the minimum and maximum values obtained by the function (or data) over the domain set  $X$ . If the function maps to the nonnegative reals,  $f : X \rightarrow \mathbb{R}^+$ , or the data are all positive, then it is sometimes assumed that  $f_{\min} = 0$  and the formula

$$\mu_F(x) = \frac{f(x)}{f_{\max}}$$

is used. Numeric data already assembled as measures of compatibility with a concept (such as survey responses to the question, "How favorably do you view candidate  $X$ ?") may be fuzzified in this straightforward manner.

The horizontal view of a fuzzy set is the  $\alpha$ -cut view. This is an interval associated with each  $\alpha \in [0, 1]$ . If we think of  $\alpha$  as a confidence factor or surety level, then we can interpret the  $\alpha$ -cut as the  $x$  values that we are at least  $\alpha$  sure are compatible with the concept or label  $A$ . The decomposition theorem

**2.10** ensures that the fuzzy set membership function can be constructed from the parametric family of its  $\alpha$ -cuts. Therefore, if one can determine the  $\alpha$ -cuts then one can determine the fuzzy set. When  $\alpha$  is 1 we need to determine the smallest and largest values of  $x$  that are completely compatible with the concept or category  $A$ . When  $\alpha$  is 0.5 we need to determine the smallest and largest values of  $x$  that are half-way compatible with or 50% surely possess the concept or category  $A$ .

If we determine enough  $\alpha$ -cuts,  $A^\alpha$ , we can produce a model of the membership function  $\mu_A(x)$  using the formula

$$\mu_A(x) = \sup_{\alpha \in [0,1]} \{\alpha \mid x \in A^\alpha\}.$$

### 2.5.2 Granularity

Defining fuzzy subsets over the domain of a variable is referred to as granulation, granularity, or variable granulation (in contrast to the division of a domain set into crisp subsets which is called quantization). Granulation results in the grouping of objects into imprecise clusters of fuzzy granules. The objects forming a granule are drawn together by similarity. In essence, granulation can be seen as a form of fuzzy data compression.

Often granulation is obtained manually through expert interviews. If expert knowledge on a domain is not available, an automatic granulation approach can be used. One such approach is the Likert-scale, which is commonly used in the social sciences. The scale divides the domain into an odd number of granules: strongly agree, agree, not sure, disagree, and strongly disagree. These granules can be used as surrogates for the preferences that players have for a policy alternative in a formal model.

### 2.5.3 Fuzzification Methods

Fuzzification involves defining a fuzzy subset over the domain of a variable. Assigning set level inclusion values for a variable is often straightforward at the upper and lower limits. For instance, if we were to take separate graphs of ages considered “young” (Figure 2.12) and those considered old (Figure 2.13), we can see that there is little ambiguity at either end of the spectrum. However, if we graph them together (Figure 2.14), there are a lot of ages that are not included in these two categories to any great extent. Indeed, no sharp boundaries exist between age groups. Fuzzification permit us to group events or observations into imprecise clusters, each of which draws together similar elements.

There are three different categories of fuzzification methods: manual, automatic, and adaptive. Manual methods are best suited to evidence obtained from human responses, automatic methods are primarily used for processing data sets to determine appropriate fuzzy set representation, and adaptive methods search for an optimal system design.

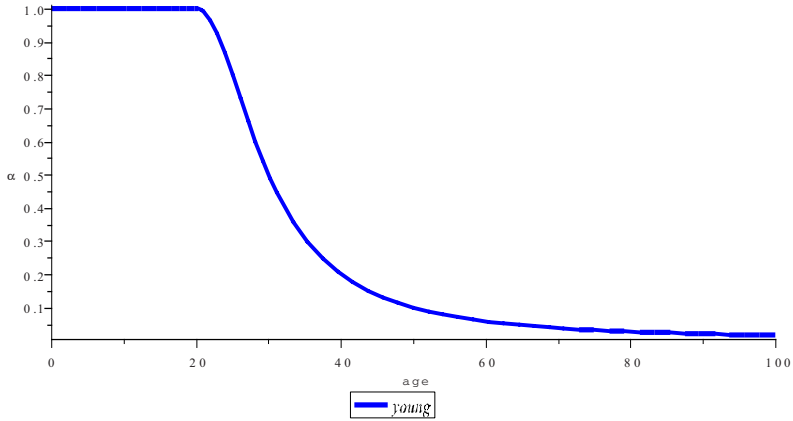


Fig. 2.12. The Linguistic Term “Young”

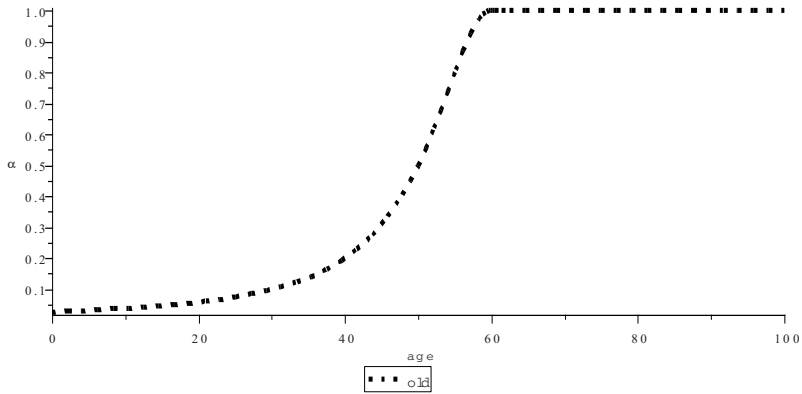


Fig. 2.13. The Linguistic Term “Old”

## Manual Methods

Various statistical techniques are often used in the determination of fuzzy set membership functions. In [Watanabe \(1993\)](#) the author asserts that these statistical techniques fall into two broad categories: the use of frequencies and direct estimation. The first methodology, the frequency method, obtains the membership function by measuring the percentage of people in a test group (typically experts in the particular domain under consideration) who answer affirmatively to a question about whether an object belongs to a particular set. The second methodology, direct estimation, derives its values from a sliding scale, it elicits a responses from experts that grade the compatibility of the object and the set. Experiments conducted by Watanabe came to the conclusion that direct estimation methods are superior to the frequency method. In [Turksen \(1991\)](#) the author examined four different approaches (that include Watanabe's) to the

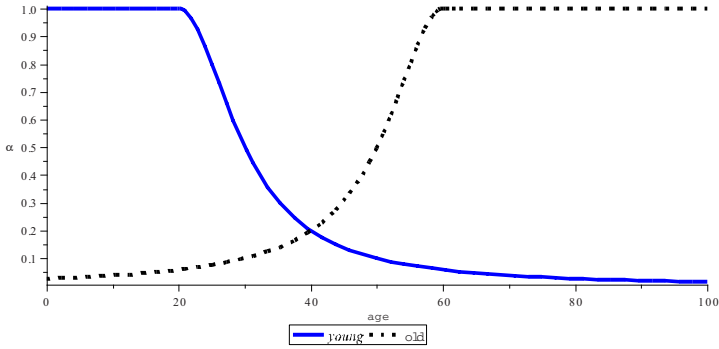


Fig. 2.14. The Linguistic Terms “Young” and “Old”

acquisition of membership functions: *direct rating*, *polling*, *set valued statistics* and *reverse rating*. His notation includes:

1. The set of elements  $\theta \in \Theta$ , the domain,
2.  $V$ , a linguistic variable which is a label for  $\theta \in \Theta$ ,
3.  $A$ , a linguistic term of a linguistic variable,
4. A measurable numerical assignment interval  $X \in [-\infty, \infty]$ ;
5.  $\mu_A(\theta)$ , the membership value representing the degree of membership of  $\theta$  to the set of elements determined by linguistic term  $A$ .

For example when considering the set of men we have a ‘man’,  $\theta$ , in the set ‘men’,  $\Theta$ , whose height,  $V$ , describes how ‘tall’,  $A$ , he is. For example the height might be in the range  $[0, 4]$  meters,  $X$ , and how tall Michael Jordan might be is the value of the membership function  $\mu_{\text{tall}}(MJ)$ . *Direct rating* presents randomly selected  $\theta \in \Theta$ , with values  $V(\theta) \in X$  to subjects who answer the question “How  $A$  is  $\theta(V(\theta))$ ?” . In other words the question put to the expert is “How tall is Michael Jordan?” (note that ‘tall’ is different from ‘height’) and they respond by using a simple indicator on a sliding scale. Then, using the expert’s opinion of the range of heights, a simple calculation reveals  $\mu_{\text{tall}}(MJ)$ . This experiment is repeated for other men and the same candidate, Michael Jordan, repeatedly to reduce error. *Polling* asks a different question “Do you agree that  $\theta$  is  $A$ ?” expecting a yes or no response. The ratio of yes responses to total responses is used to arrive at a proportion that is then used to help generate the membership function. So, in our example, many respondents would be asked “Do you agree that Michael Jordan is tall?”. *Set valued statistics* rely on the idea of combining ordinary sets using a frequentist approach based on observation. This method asks questions such as “Is the range 6 feet to 7 feet correctly termed tall?” Frequencies gained from asking this question repeatedly are then amalgamated to produce a fuzzy set membership function. *Reverse rating* takes a different approach by asking an expert to answer the following question “Identify  $\theta(V(\theta))$  that has the  $y$ -th degree of membership in fuzzy set  $A$ .” So we might ask an expert “Identify a man whose height indicates that he

possesses the degree 0.5 of membership in the fuzzy set tall. Again, providing the expert has an understanding of the upper and lower limits this allows for a ready representation of  $\mu_{\text{tall}}$ . Other authors (e.g. [Kempton \(1984\)](#)) employ various knowledge acquisition interviewing techniques for acquiring membership functions.

All the ‘manual’ approaches suffer from the same deficiency; they rely on the very subjective interpretation of words, the foibles of human experts, and generally, all the knowledge acquisition problems that are well documented with knowledge based systems.

## One Expert

If there is one expert available who can manage a system and the goal is to automate his skill before he retires then direct questioning is the best available method. However the elicitation of knowledge from an expert is a very difficult and specialized skill. An expert may be cooperative or antagonistic, and he may not consciously have numerical values associated with the rules he uses to control the process. He may not even have encapsulated his knowledge as rules, since he may not have trained a replacement. Furthermore, some rules may be used so seldom that only the dire circumstances of their necessity can bring them forth from long term memory.

This is very similar to the problems posed by constructing inference engine expert systems or knowledge based systems (KBS). In fact this problem is similar to the common computer science dilemma of requirements specification. The elicitation of knowledge from the problem domain and its experts, who speak their own language, is a difficulty common to all software engineering. As such, the literature on these subjects does give some guidelines.

First of all patience is required. The expert may be trying to be helpful, even if his answers are not taking the form that you like. It may be necessary to try many different approaches, and many different questions to infer the shape of a fuzzy set. Practice has shown that triangular and trapezoidal fuzzy sets work very well in many applications so that the elicitation of critical values with questions like:

What rate of change in velocity causes you to start paying close attention?

Ans: *a*.

What rate of change in velocity causes you to start action  $B_{ji}$ ?

Ans: *b*.

What rate of change in velocity causes you to start an action other than  $B_j$ ?

Ans: *c*.

What rate of change in velocity generally indicates the process is out of control?

Ans: *d*.

It is important *not* to put the questions in direct order, and possibly *not* to ask them all at the same time. Cognitive psychology has many things to say about the best methodologies of questioning people. However, if the answers to these questions form an order  $a \leq b \leq c \leq d$  then  $A_i = Tp[a, b, c, d]$  is a good fuzzy number to represent the critical velocity. Note that experts give us both knowledge of the fuzzy set memberships as well as (possibly) knowledge of the inference rules for the construction of an Approximate Reasoning system.

In other cases it may be difficult for the expert to give precise numerical values to such questions, however range values are almost as good, and sometimes all that is really needed is to find the core of the fuzzy numbers even if this is just a single value. A group of statements such as:

“When the temperature is about  $110^\circ$  then lower the boiler feed by about two liters of *foo*.”

“When the temperature is about  $125^\circ$  then lower the boiler feed by about three liters of *foo*.”

“When the temperature is about  $140^\circ$  then lower the boiler feed by about six liters of *foo*.”

gives the critical central values  $a_1 = 110$ ,  $a_2 = 125$ , and  $a_3 = 140$  for triangular fuzzy numbers  $A_i = Tr[a_i - s_i, a_i, a_i + s_i]$  and  $b_1 = 2$ ,  $b_2 = 3$ , and  $b_3 = 6$  for triangular fuzzy numbers  $B_j = Tr[b_j - s_j, b_j, b_j + s_j]$ . The spread values of  $s_i$  and  $s_j$ ,  $i, j \in \mathbb{N}_3 = \{1, 2, 3\}$ , can be determined graphically using the granularity heuristic that the fuzzy sets  $A_i$  should cover the critical range of the input space and have a 10–20% overlap with the neighbors on each side. Thus if the critical range of temperatures is  $100^\circ$ – $150$  then the three antecedent fuzzy sets

$$A_1 = Tp[100, 110, 120] \quad (2.23)$$

$$A_2 = Tr[115, 125, 135] \quad (2.24)$$

and

$$A_3 = Tp[130, 150, 150] \quad (2.25)$$

partition the input interval appropriately. The fuzzy sets  $B_1$ ,  $B_2$ , and  $B_3$ , can be similarly constructed once the interval of its domain set is determined.

## Multiple Experts

The problem of determining fuzzy set membership grades when many experts are available has both positive and negative aspects. On the plus side, a weighted average of values obtained from the group of experts certainly contains a greater amount of information than that derived from a single person. However, the determination of appropriate weights and the melding of potentially conflicting evidence presents difficulties.

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<sup>1</sup> As is the convention in computer science, we use the word “foo” in this chapter to mean “anything.”

Suppose that there are  $n$  experts available. Assume that the experts evidence comes as *polling*. We are asking the question “Does  $\theta$  belong to  $A$ ?” or, equivalently, “Does  $x$  belong to  $A$ ?” where  $\theta$  has value  $x$ . That is, Michael Jordan has height 6' 7" so one can ask, “Does Michael Jordan belong to tall?” or, equivalently, “Does 6' 7" belong to *tall*?” Then each expert  $i$  ( $i \in \mathbb{N}_n = \{1, 2, \dots, n\}$ ), gives a value  $a_i(x)$  for each  $x \in X$  that is either 0 if the expert does not believe the statement is true, or 1 if the expert does believe the statement is true. We can then use the simple approximation

$$\mu_A(x) = \frac{\sum_{i=1}^n a_i(x)}{n} \tag{2.26}$$

for the set membership grade. If on the other hand the values  $\tilde{a}_i(x)$  are derived from *direct polling* then they may not be values in the unit interval. However the linear transformation

$$a_i(x) = \frac{\tilde{a}_i(x) - a_{\min}}{a_{\max} - a_{\min}}, \tag{2.27}$$

where  $a_{\max}$  and  $a_{\min}$  represent the maximum and minimum values that the scale allows, will transform the data so that Eq. (2.26) is applicable.

In the absence of additional information we must assume that all experts are equally qualified. Of course there may be reason to trust more experienced experts more than their less seasoned colleagues. The uniform weight of  $c_i = \frac{1}{n}$  represents equal confidence in all experts, and  $c_i$  can be changed to reflect the most appropriate distribution of weights. We can then express the Eq. (2.26) in the form

$$\mu_A(x) = \sum_{i=1}^n c_i \cdot a_i(x). \tag{2.28}$$

However, if years of experience or breadth of knowledge is incorporated in a weighing system other than uniform then the values of  $c_i$ ,  $i \in \mathbb{N}_n$ , should be nonnegative and add to one. Here again the values of  $a_i(x)$  may come from polling or from direct estimation.

### Indirect Manual Methods

Both the method for one expert and the method for multiple experts can be extrapolated to the case where the data is specified as intervals rather than points. Turksen calls this *set valued statistics*. As mentioned previously, an expert will probably refuse to give information he believes is misleading. This may cause him to make statements such as;

“When the temperature is between 120° and 130° then lower the boiler feed rate by about three liters of *foo*.”

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<sup>2</sup> We remind the reader that in accordance with convention, we use the word “foo” in this chapter to mean “anything.”

We can gather data about the intervals as easily as we gathered data about exact values. For example, the question “Is the range 6 feet to 7 feet correctly termed tall?” can be asked of each expert and the number of agreements for each interval calculated. Assume that  $m$  intervals are used in the questions or presented by the experts. Relative frequencies can then be calculated so that each interval gets a weight  $l_i$  equal to the number of agreements that interval  $L_i$  is correctly termed  $A$  divided by the number of agreements to all intervals. Finally the fuzzy set membership grade is calculated as

$$\mu_A(x) = \frac{\sum_{x \in L_i} l_i}{\sum_{i=1}^m l_i}. \tag{2.29}$$

This methodology has a strong connection to random set theory (Goodman, 1982).

Another indirect method is based on the ability to compare objects. If there are  $n$  different values that we are going to use to determine the fuzzy set membership function, then it may be easier to get answers to questions of the type, “How much more  $A$  is  $x_i$  than  $x_j$ ?” For example, “How much more *Taller* is *Reggie Miller* than *Cheryl Miller*?” If we can determine a preference<sup>3</sup> matrix  $\mathbf{P} = [p_{ij}]_{nn}$  that is consistent (consistency means  $p_{ij} = p_{ik} \cdot p_{kj}$ ) then we may be able to determine the individual membership grades  $m_i = \mu_A(x_i)$  by assuming that  $p_{ij} = \frac{m_i}{m_j}$ . Under this assumption

$$\sum_{j=1}^n p_{ij} m_j = \sum_{j=1}^n m_i = n m_i \tag{2.30}$$

or in matrix form

$$\mathbf{Pm} = n\mathbf{m}, \tag{2.31}$$

where the vector  $\mathbf{m} = [m_i]_n^T$  are the membership grades to be determined. Equation (2.31) can be written

$$(\mathbf{P} - n\mathbf{I})\mathbf{m} = 0, \tag{2.32}$$

where  $\mathbf{I}$  is the identity matrix. This equation has a solution if and only if  $\mathbf{m}$  is an eigenvector of the matrix  $\mathbf{P} - n\mathbf{I}$  and  $n$  is an eigenvalue. A final assumption is necessary to complete the derivation. If  $\mathbf{m}$  is a solution to Eq. (2.32) than so is any scalar multiple. Hence a unique final solution comes only if we assume that  $\max m_i = 1$  or some smaller positive value. Another method is to assume that  $\sum m_i = 1$ . In this case

$$\sum_{i=1}^n p_{ij} = \sum_{i=1}^n \frac{m_i}{m_j} = \frac{1}{m_j} \sum_{i=1}^n m_i = \frac{1}{m_j} \tag{2.33}$$

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<sup>3</sup> This use of the term “preference” is to be distinguished from that which we will use later in the book when we discuss the choices of political players over a set of alternatives.

where

$$m_j = \frac{1}{\sum_{i=1}^n p_{ij}}. \quad (2.34)$$

If the data is slightly inconsistent then it may still be possible to find a solution to the linear equation

$$\mathbf{Pm} = \lambda \mathbf{m} \quad (2.35)$$

for some eigenvalue  $\lambda$  that is close to  $n$ . Then  $\frac{\lambda-n}{n}$  can be used as a measure of accuracy of the estimate. If this value is large the estimate is poor and the data are just too inconsistent.

#### 2.5.4 Automatic Methods

Automatic methods are data driven. They are used primarily for processing data sets to determine appropriate fuzzy set representation. Among the more common of such procedures are curve fitting and histograms.

#### Curve Fitting

If there is a data set that can be used as a basis for the construct of the fuzzy set membership functions then a host of methods are available. The first is the common mathematical methods of curve fitting. For example, besides triangular and trapezoidal curves, Gaussian bell shaped curves with generic formula

$$Tb[m, s, \gamma](x) = \gamma e^{-(x-m)^2/s^2} \quad (2.36)$$

$m$ ,  $s$ , and  $\gamma$  all positive, are common in mathematics. If we assume that there are  $n$  data points  $\langle a_i, b_i \rangle$  then the value

$$E = \sum_{i=1}^n \left( b_i - \gamma e^{-(a_i-m)^2/s^2} \right)^2 \quad (2.37)$$

is the total squared error between the curve  $Tb$  and the given data points. This can be minimized easily and rapidly on a computer using any standard numerical technique. The solution is a least squared error estimate of the fuzzy bell number. The result will not always be a fuzzy set for arbitrary  $\alpha$ ,  $\beta$ , and  $\gamma$ . It is then necessary to either truncate or scale the result so that it never exceeds the threshold of 1.0.

Entirely similar methods can be applied with triangular fuzzy numbers  $Tr[a, m, b]$  and trapezoidal fuzzy numbers  $Tp[a, l, r, b]$  to determine the values of the parameters  $a$ ,  $b$ ,  $\dots$ , that minimize the least squared error.

Almost any curve fitting methodology in the mathematical canon can be adapted to finding fuzzy set membership values. These include regression and Lagrangian interpolation.

## Histograms

Sometimes it makes more sense to fit the curve to the histogram of the data set. This is the case when we do not have input-output pairs, rather we have a large sample of the input values that are not correlated with a controlled output. This is the case when no successful control, human or machine, exists.

If we have  $n$  elements  $\langle a_i, b_i \rangle$  in a data set then the first step is to find  $a_{\min}$  and  $a_{\max}$ , the minimum and maximum value obtained by the first element in the ordered pair. The next step is to divide the interval  $a_{\min}$  to  $a_{\max}$  into  $m$  segments, where  $m$  is usually much smaller than  $n$ . For each interval

$$[a_{j-1}, a_j], \quad (2.38)$$

where  $a_0 = a_{\min}$  and for  $j \in \mathbb{N}_m$

$$a_j = a_0 + j \cdot \frac{a_{\max} - a_{\min}}{m} \quad (2.39)$$

we calculate the number,  $h_j$ , of points  $a_i$  that fall into the interval  $[a_{j-1}, a_j]$ . Let  $h_{\max}$  be the maximum value obtained by data set  $\{h_j\}$  and let

$$\hat{h}_j = \frac{h_j}{h_{\max}}. \quad (2.40)$$

Also calculate  $\hat{a}_j$ , the midpoint of each interval with

$$\hat{a}_j = \frac{a_{j-1} + a_j}{2}. \quad (2.41)$$

Finally we can now use the set of points  $\langle \hat{a}_j, \hat{h}_j \rangle$  for curve fitting as described in the previous section.

## Histograms for Discrete Data

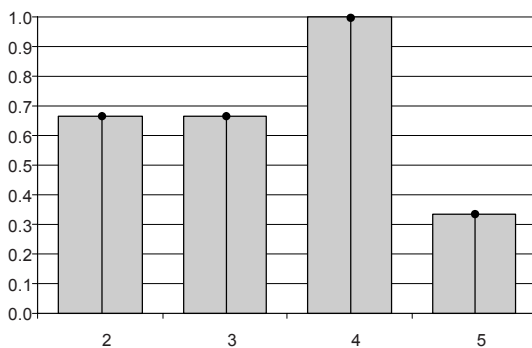
If the data can only take on limited values, the histogram method can be simplified. Suppose that we are using observational data. For example the data might be the number of cups of coffee that employees drank on a certain day at the office. For discrete data, that can only take on a fixed set of values, the standard method of fuzzification is to use frequency data.

Suppose that the data can only take values in  $X$  with  $|X| = n$ . For the coffee experiment, the employees only drank 2 to 5 cups, and the number of cups is always an integer. We can then process the the data to calculate frequencies  $f_i$  for  $i \in \mathbb{N}_n$ . The frequency  $f_i$  is just the count of the number of times that  $x_i$  occurs in the data set  $d$ .

*Example 2.19.* Let  $X = \{2, 3, 4, 5\}$ . Let us convert the information in  $data = [5, 3, 4, 2, 4, 3, 4, 2]$  into frequencies. The value 5 occurs once, the value 4 occurs three times, and the values 2 and 3 occur twice each. A graph of the frequency data is called a histogram. If we then divide the frequencies by the largest of the frequencies, 3, we produce a fuzzy set  $D$ . These results are summarized in Table (2.1). The graphical result is illustrated in Figure 2.15.

**Table 2.1.** Frequency Table

$x_i$	$f_i$	$D(x_i)$
2	2	0.667
3	2	0.667
4	3	1.0
5	1	0.333

**Fig. 2.15.** Using a Histogram to Generate a Fuzzy Set

### 2.5.5 Adaptive Methods

Adaptive methods search for an optimal system design as a means for calculating the fuzzy set membership functions from data. Among the most common of these methods is neural networks. A neural network is a (computer) simulation of the architecture used in the human brain. The brain of all animals in fact is made of a very large collection of processing units called neurons. The neurons are connected together in a large network that takes in information from our senses and processes it for recognition and subsequent action.

There are many advantages to neural networks. First, it makes no assumption about the appropriate shape to fit to an input-output data set  $\langle a_i, b_i \rangle$ . Secondly, it functions well empirically regardless of the nature and source of the data set. Third, it can be employed on-line, or off-line, to continuously refine the membership functions in the face of new behavior of the system that one is trying to control. This adaptive behavior is particularly desirable in the construction of robust systems. Neural networks have the additional advantage of robustness and adaptability. A neuro-fuzzy controller can be built that changes the fuzzy set membership functions over time as conditions change. This is what allows a fuzzy controller for a dam to deal with drought and flood as well as the typical situation.

Among other means for constructing fuzzy data are Genetic algorithms, deformable prototypes, gradient search, and inductive reasoning. Genetic algorithms are biologically inspired techniques to evolve better fuzzy sets. To

accomplish this evolution the computer needs a goal. The goal in fuzzy set theory is better fuzzy logic controllers.

An early approach by Bremermann is based on the idea of a deformable prototype. Devised initially for pattern recognition, it appears to be a potentially useful method for automatic determination of membership functions. It is based on the concept of taking an object which needs identifying and deforming to match a prototype. The amount of matching and distortion is measured by a distortion function and a matching function to combine to give a cybernetic functional. In other words there is a combination of the matching of an object to a prototype and the distortion required to deform the prototype. Using Bremermann's notation the matching function would be

$$\langle M(\Phi_i(p_1, p_2, \dots, p_n)), \phi \rangle, \tag{2.42}$$

where  $\phi$  is the object,  $\Phi_i$  is the  $i$ th prototype, and  $p_1, p_2, \dots, p_n$  are the parameters that control the distortion of the prototype. Denote

$$D(\Phi_i(p_1, p_2, \dots, p_n)) \tag{2.43}$$

to be the distortion function. Then the cybernetic functional for the  $i$ th prototype can be defined by

$$\langle \langle f_i, \phi \rangle \rangle = \min_{p_1, p_2, \dots, p_n} \langle F_i(p_1, p_2, \dots, p_n), \phi \rangle, \tag{2.44}$$

where

$$\langle F_i(p_1, p_2, \dots, p_n), \phi \rangle = \langle M(\Phi_i(p_1, p_2, \dots, p_n)), \phi \rangle \tag{2.45}$$

$$+ cD(\Phi_i(p_1, p_2, \dots, p_n)) \tag{2.46}$$

and  $c$  is a constant. Bremermann then goes on to explain that this cybernetic functional can be used to generate a membership function

$$\chi(\phi) = 1 - \frac{\langle f, \phi \rangle}{\max}, \tag{2.47}$$

where  $\max$  is the least upper bound for  $f$ . This initial suggestion seems to have been hampered by the speed of computers and techniques available at the time to carry out the optimization of the parameters. However ? reports on successful implementations for pattern recognition and in particular for fuzzy sets in ECG interpretation. This does raise the question of whether, for instance, genetic algorithms may be useful in conjunction with this technique. This work still raises questions about arriving at a suitable prototype and choosing the various functions.

Based on the original work of Procyck and Mamdani (1979), Burkhardt and Bonissone use gradient search methods to tune fuzzy knowledge bases. They fine tune the rules and also the membership function for a fuzzy logic controller. They assume first, as with most fuzzy logic controllers, that the membership

functions are triangular and then they use gradient search to determine the optimal scaling factor for the base of the triangles. This optimization is carried out on an application of fuzzy logic control to the cart-pole system where the goal is to maintain the time the pole is vertical on the cart while reducing overshoot (the pole falls off) and steady state error (the pole wobbles a lot). They present a variety of results and draw the conclusion that this approach out-performed a simple controller.

The approach adopted by [Kim and Russell \(1993\)](#) is to use inductive reasoning to generate the membership functions and the rules. They assume they have no information other than a set of data. The approach is to partition a set of data into classes based on minimizing the entropy. The entropy where only one outcome is true is the expected value of the information contained in the data set and is given by

$$S = k \sum_{i=1}^N [p_i \ln p_i + (1 - p_i) \ln(1 - p_i)], \quad (2.48)$$

where  $k$  is an arbitrary constant, the probability of the  $i$ -th sample to be true is  $p_i$ , and  $N$  is the number of samples. In their work they give a two class example where by iteratively partitioning the space they calculate an estimate for the entropy which leaves them with points in the region that are then used to determine triangular membership functions. This approach suffers from the fact that there is no way of knowing whether the membership functions are realistic and that the sets obtained are triangular.

## 2.6 Fuzzy Set Operations

The mathematics of fuzzy set theory is often more difficult than that of the traditional set theory because the continuous interval  $\mathbb{U} = [0, 1]$  is inherently more complex than the binary set  $\mathbb{B} = \{0, 1\}$ . Furthermore, if we work from the premise that both crisp sets and fuzzy sets can be defined using membership functions, crisp sets are, in a sense, a special case of fuzzy sets. The crisp set membership values zero and one are contained in the unit interval and crisp sets can be thought of as fuzzy sets with a restricted image set.

In this section we define what it means for a fuzzy set to be contained in another fuzzy set. We also provide operators for fuzzy sets that correspond to “and”, “or”, and “not” in human logic. We define a fuzzy intersection to represent “and”, a fuzzy union to represent “or”, and a fuzzy complement to represent “not”. All the definitions for complements, unions, and intersections are given in terms of membership functions. We use the same notation for fuzzy intersections, unions, and complements as those for crisp intersections, unions, and complements. The definition of a fuzzy intersection is crafted so that if it has  $\{0, 1\}$  as its image set then the fuzzy intersection behaves like the intersection of crisp sets. This will also be true for the fuzzy union, fuzzy complement, and the fuzzy subset relation. Hence, we need not add the term “fuzzy” to these operations. The fuzzy intersection is the crisp intersection whenever the sets involved correspond to crisp sets.

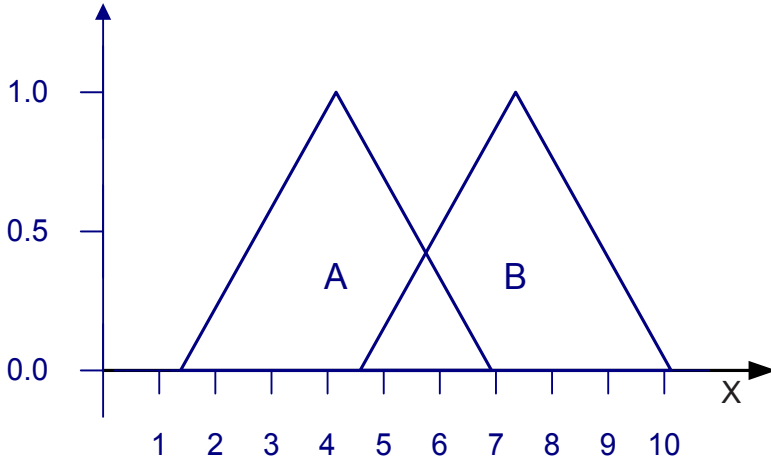


Fig. 2.16. Fuzzy Sets  $A$  and  $B$

**2.6.1 Subsets**

A fuzzy set  $A$  is a subset of a fuzzy set  $B$  if  $A(x)$  is less than or equal to  $B(x)$  for all  $x$  in  $X$ . Thus  $A \subseteq B$  if and only if

$$\forall x A(x) \leq B(x). \tag{2.49}$$

For  $A$  to be equal to  $B$  the membership values of  $A(x)$  and  $B(x)$  must be equal for all  $x$ . Therefore  $A = B$  if and only if

$$\forall x A(x) = B(x). \tag{2.50}$$

It is simple to show that  $A \subseteq B$  and  $A \supseteq B$  imply that  $A = B$ . The class of all fuzzy subsets of the universe  $X$  is the fuzzy power set  $\mathcal{F}(X)$ . There are an infinite number of fuzzy subsets of any non-empty universe.

**2.6.2 Intersection**

Suppose we have two continuous fuzzy numbers,  $A$  and  $B$ , as depicted at Figure 2.16.

The intersection of two fuzzy sets is the fuzzy set  $C$  defined by  $C(x) = (A \cap B)(x)$ , where

$$(A \cap B)(x) = \min\{A(x), B(x)\} \tag{2.51}$$

for all  $x$  in  $X$ . The intersection of fuzzy sets  $A$  and  $B$  is the hatched area in Figure 2.17.

The minimum operator is represented by the symbol “ $\wedge$ ” so that  $\min\{a, b\}$  can be written  $a \wedge b$  and the intersection membership function is often written

$$(A \cap B)(x) = A(x) \wedge B(x). \tag{2.52}$$

The intersection of two fuzzy sets contains all the elements of minimum membership grade in  $A$  and  $B$ .

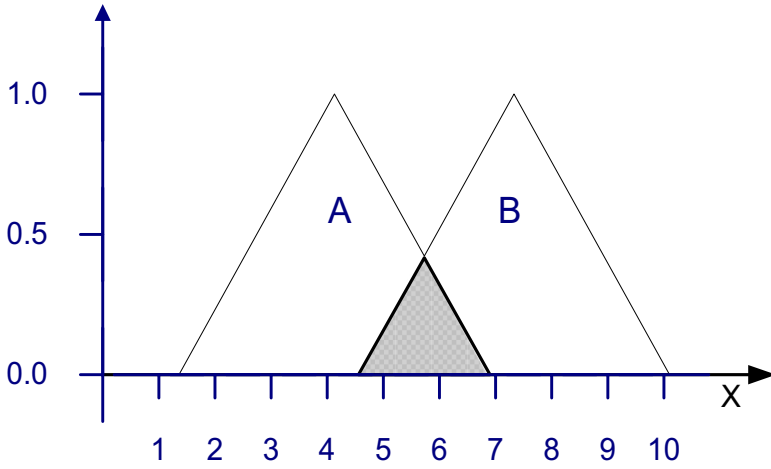


Fig. 2.17.  $A \cap B$ , the Intersection of Fuzzy Sets  $A$  and  $B$

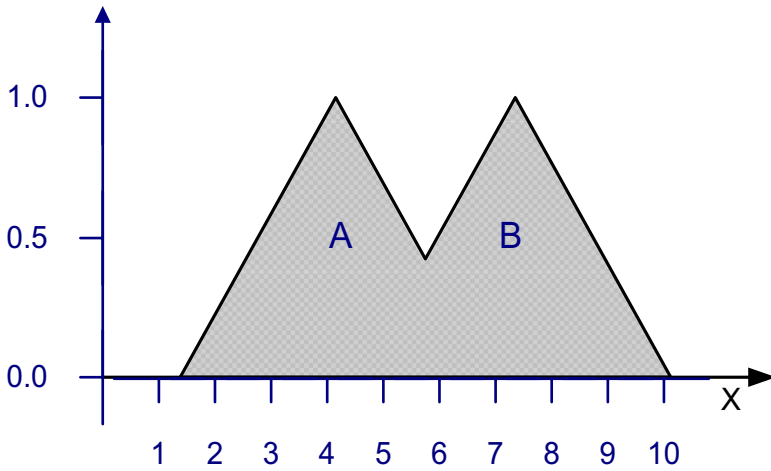


Fig. 2.18.  $A \cup B$ , the Union of Fuzzy Sets  $A$  and  $B$

### 2.6.3 Union

The union of two fuzzy sets is the fuzzy set  $C$  defined by  $C(x) = (A \cup B)(x)$ , where

$$(A \cup B)(x) = \max\{A(x), B(x)\} \quad (2.53)$$

for all  $x$  in  $X$ . The union of fuzzy sets  $A$  and  $B$  is the hatched area in Figure 2.18.

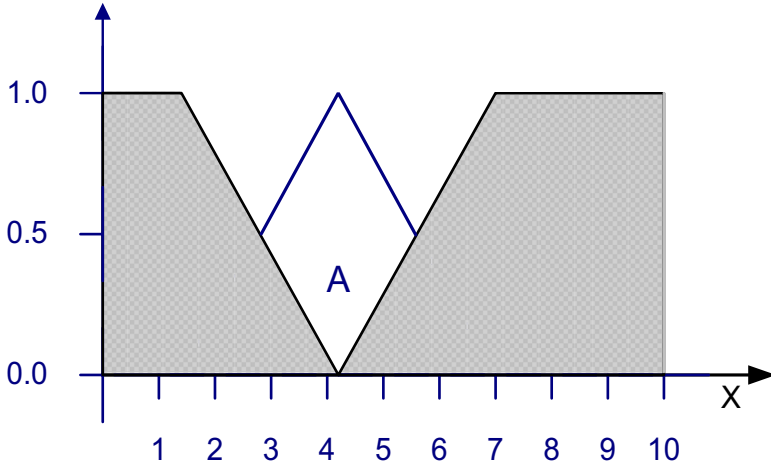


Fig. 2.19.  $A^c$ , the Complement of  $A$

The maximum operator is represented by the symbol “ $\vee$ ” so that  $\max\{a, b\}$  can be written  $a \vee b$  and the membership function for union can be written

$$(A \cup B)(x) = A(x) \vee B(x). \tag{2.54}$$

Thus the union of two fuzzy sets contains all the elements of the maximum degree consistent with  $A$  or  $B$ .

### 2.6.4 Complement

The complement represents the notion of “not” in human language. The complement of a set  $A$  is another set  $A^c$  that contains each element  $x$  in the universe  $X$  to the opposite degree that the original set contained  $x$ . The complement of fuzzy set  $A$  (in Figure 2.16) is depicted in Figure 2.19.

The standard complement operator is

$$A^c(x) = 1 - A(x). \tag{2.55}$$

This formula represents one of the most important differences between fuzzy set theory and standard set theory. In set theory it is always true that a set and its complement have nothing in common. In fuzzy set theory a set and its complement can be identical. Consider the fuzzy set  $A$  of Eq. (2.2). Then its complement is

$$A^c = \frac{0.0}{a} + \frac{0.3}{b} + \frac{0.6}{c} \tag{2.56}$$

and the membership grade of the intersection of  $A$  and  $A^c$  is

$$A \cap A^c = \frac{0.0}{a} + \frac{0.3}{b} + \frac{0.4}{c}. \tag{2.57}$$

This is a better picture of reality since, for instance, we could go to a movie and like and dislike it at the same time. People often hold contradictory feelings, opinions and evaluations about the same exact thing. We often prevaricate in our evaluations, for instance of an unusual dish ordered at a restaurant, jumping from one side of a decision to the other with monotonous regularity (or with each new bite).

### 2.7 Metrics for Fuzzy Numbers

When we are given two numbers, like  $a = 12$  and  $b = 7$ , it is quite easy to say how far apart they are, technically, the distance between  $a$  and  $b$ . Unless some unordinary measuring procedure or unordinary space is specified, they are 5 units apart and the formula for the distance is the absolute value of the difference:

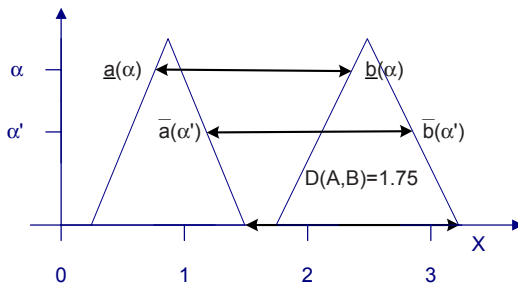
$$dist(a, b) = |a - b| .$$

For fuzzy numbers a more complex definition is needed. In fact, more than one type of distance has been proposed. The ones that are applied the most often are the Hausdorff,  $C_\infty$ , Hamming, and Discrete Hamming distances. In all of the following definitions  $A$  and  $B$  are fuzzy numbers, the  $\alpha$ -cuts of  $A$  and  $B$  are  $A^\alpha(x) = [\underline{a}(\alpha), \bar{a}(\alpha)]$  and  $B^\alpha = [\underline{b}(\alpha), \bar{b}(\alpha)]$  and the membership functions are  $A(x)$  and  $B(x)$ . The Hausdorff distance  $Dist_D(A, B)$  (see Figure 2.20) is the maximal distance between  $\alpha$  level sets of  $A$  and  $B$

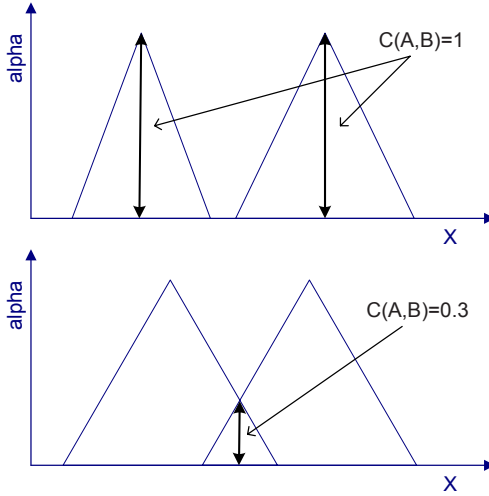
$$Dist_D(A, B) = \sup_{\alpha \in [0,1]} \max\{|\underline{a}(\alpha) - \underline{b}(\alpha)|, |\bar{a}(\alpha) - \bar{b}(\alpha)|\}.$$

The  $C_\infty$  distance  $Dist_{C_\infty}(A, B)$  (see Figure 2.21) is the maximal distance between the membership grades of  $A$  and  $B$

$$Dist_{C_\infty}(A, B) = \|A - B\|_\infty = \sup_{x \in X} |A(x) - B(x)| .$$



**Fig. 2.20.** Distance between Fuzzy Numbers  $A$  and  $B$  Using the Hausdorff Distance



**Fig. 2.21.** Distance between Fuzzy Numbers  $A$  and  $B$  Using distance  $C_\infty$

The Hamming distance  $H(A, B)$  is the integral of the distance between the membership grades of  $A$  and  $B$

$$Dist_H(A, B) = \int_X |A(x) - B(x)| dx.$$

The Discrete Hamming distance  $H(A, B)$  is the sum of the distance between the membership grades of  $A$  and  $B$  for each of the  $n$  elements of the discrete universe  $X = \{x_1, x_2, \dots, x_n\}$

$$Dist_H(A, B) = \sum_{i=1}^n |A(x_i) - B(x_i)| .$$

## 2.8 Fuzzy Geometry

In this chapter we have considered the basics of fuzzy set theory. These concepts will be sufficient for the single dimensional model we will develop in chapter four. However, a substantial number of models in comparative politics are multiple dimensional (spatial) models. We will need a two-dimensional fuzzy geometry for these models. We discuss fuzzy geometry in the next chapter.

The reader will find it to be the most rigorously formal chapter of the book. Those readers less interested in a rigorous treatment of fuzzy geometry may skip it without their understanding of later chapters suffering for it. We provide a substantial discussion in layman’s terms of the fuzzy geometry that we use in chapter five, and we are relatively certain that this will prove sufficient for most readers in understanding the spatial models that we develop in that chapter.

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### 3 Fuzzy Geometry

In this chapter we discuss a version of fuzzy plane geometry developed by Buckley and Eslami (1997a,b). The approach presented here is one in which the area, heights, width, diameter, and perimeter of fuzzy subsets are fuzzy numbers. The chapter lays the basic groundwork for the models that we develop in the ensuing chapters, particularly chapter five. Those readers not interested in the formalism behind the geometry of these models will find the discussions in the chapters sufficient.

We begin by reformulating the definition of a fuzzy number in a manner better suited for the geometry that follows. Understanding that fuzzy points in two dimensional space can be visualized as surfaces in three dimensions is the key to understanding fuzzy geometry. It is this property that will help model uncertainty in ways that a single crisp point could not.

Spatial models are useful because relationships can be visualized. Hence, we need a concept of distance between points and a concept of regions bounded by points. For the former, we build on the preliminary discussion of fuzzy distance at the end of Chapter 2. For the latter, we move toward defining fuzzy shapes with the definition of a fuzzy line. Just as a crisp line can be understood as a collection of points, a fuzzy line can be thought of as a collection of fuzzy points. As a result, a fuzzy line not only has length, but can be thick as well. We also define a measure of parallelness that indicates the extent to which two fuzzy lines can be said to be parallel.

From fuzzy lines, we move to fuzzy circles and their properties, and then to line segments. With these tools we can finally define generic fuzzy polygons. This chapter concludes with some geometry and trigonometry of fuzzy polygons and a note on the distinction between crisp and fuzzy shapes.

#### 3.1 Points and Lines

Let  $N$  be a fuzzy subset of  $\mathbb{R}$ . Then  $N$  is called a (real) fuzzy number if the following conditions hold:

- i.  $N$  is upper semi-continuous,
- ii. there exist  $c, d \in \mathbb{R}$  with  $c \leq d$  such that  $\forall x \notin [c, d], N(x) = 0$ ,

- iii. there exist  $a, b \in \mathbb{R}$  such that  $c \leq a \leq b \leq d$  and  $N$  is increasing on  $[c, a]$ ,  $N$  is decreasing on  $[b, d]$ , and  $N(x) = 1$  for all  $x \in [a, b]$ .

It follows that  $\forall \alpha \in [0, 1]$ , if  $N$  is a fuzzy number, then  $N^\alpha$  is a bounded closed interval.

**Theorem 3.1.** Klir and Yuan (1995). *Let  $N$  be a fuzzy subset of  $\mathbb{R}$ . Then  $N$  is a fuzzy number if and only if there exists  $c, a, b, d \in \mathbb{R}$  such that  $c \leq a \leq b \leq d$  such that  $N(x) = 1 \forall x \in [a, b]$  and there exists a function  $l : (-\infty, a) \rightarrow [0, 1]$  that is monotonically increasing, continuous from the right,  $l(x) = 0$  for all  $x \in (-\infty, c)$  and a function  $r : (b, \infty) \rightarrow [0, 1]$  that is monotonically decreasing, continuous from the left,  $r(x) = 0$  for all  $x \in (d, \infty)$ , and  $N(x) = l(x) \forall x \in (-\infty, a)$ ,  $N(x) = r(x) \forall x \in (b, \infty)$ .*

Suppose that  $N$  is a fuzzy subset of  $\mathbb{R}$  satisfying condition (ii) with  $a = b$  such that  $N(a) = 1$  and the graph of  $N$  is a straight line segment from  $c$  to  $a$  and a straight line segment from  $a$  to  $d$ . Then  $N$  is a fuzzy number and is called a triangular fuzzy number. We will use the notation  $(c/a/d)$  to specify a triangular fuzzy number.

A natural way to define a fuzzy point in the plane would be as an ordered pair of real fuzzy numbers. However this definition does not give good results for fuzzy lines. Also pictures of fuzzy points under this definition cannot be constructed. Hence the following definition of a fuzzy point is used. Let  $\langle a, b \rangle \in \mathbb{R}^2$ , the Euclidean plane, and let  $P$  be a fuzzy subset of  $\mathbb{R}^2$ . Then  $P$  is called a fuzzy point at  $\langle a, b \rangle$  if the following conditions hold:

- i.  $P$  is upper semi-continuous;
- ii.  $\forall \langle x, y \rangle \in \mathbb{R}^2$ ,  $P(x, y) = 1$  if and only if  $\langle x, y \rangle = \langle a, b \rangle$ ;
- iii.  $\forall \alpha \in [0, 1]$ ,  $P^\alpha$  is a compact, convex subset of  $\mathbb{R}^2$ .

If  $P$  is a fuzzy point at  $\langle a, b \rangle$ , we sometimes write  $P_{\langle a, b \rangle}$  for  $P$ .

This definition of fuzzy point is easily modified to a more general situation by replacing condition ii. with the condition that  $\langle a, b \rangle$  belong to  $P^1$ .

Let  $\langle a, b \rangle \in \mathbb{R}^2$  and let  $P$  be a fuzzy point at  $\langle a, b \rangle$ . Then we can visualize  $P$  as a surface in  $\mathbb{R}^3$  through the graph of the equation  $z = P(x, y)$ ,  $\langle x, y \rangle \in \mathbb{R}^2$ . (When a point  $\langle x, y \rangle$  is used as an argument, as in  $P(x, y)$ , we will follow convention and use parentheses.)

*Example 3.2.* Let  $X$  and  $Y$  be real fuzzy numbers, where  $X(x) = 1$  if and only if  $x = a$  and  $Y(y) = 1$  if and only if  $y = b$ . Then the fuzzy subset  $P$  of  $\mathbb{R}^2$  defined by

$$P(x, y) = X(x) \wedge Y(y) \quad \forall \langle x, y \rangle \in \mathbb{R}^2 \tag{3.1}$$

is a fuzzy point at  $\langle a, b \rangle$ .

In the following, we let  $d$  denote the usual Euclidean distance metric on  $\mathbb{R}^2$ . We now define the fuzzy distance between two fuzzy points. Let  $P_1$  and  $P_2$  be two fuzzy points.  $\forall \alpha \in [0, 1]$ , let

$$\Omega(\alpha) = \{d(u, v) \mid u \in (P_1)^\alpha \text{ and } v \in (P_2)^\alpha\}. \tag{3.2}$$

Define the fuzzy subset  $D(P_1, P_2)$  of  $\mathbb{R}$  by

$$D(P_1, P_2)(r) = \vee\{\alpha \mid r \in \Omega(\alpha)\} \tag{3.3}$$

for all  $r \in \mathbb{R}$ . Let  $\alpha \in [0, 1]$ . We note that  $\Omega(\alpha)$  is defined in terms of a pair of fuzzy points, say  $P_1, P_2$ . It follows that

$$\Omega(\alpha) = \{r \in \mathbb{R} \mid \exists u \in (P_1)^\alpha, \exists v \in (P_2)^\alpha \text{ such that } r = d(u, v)\}. \tag{3.4}$$

**Theorem 3.3.** *Let  $P_1$  and  $P_2$  be two fuzzy points. Then  $\forall \alpha \in [0, 1]$ ,*

$$D(P_1, P_2)^\alpha = \Omega(\alpha). \tag{3.5}$$

*Furthermore,  $D(P_1, P_2)$  is a fuzzy number.*

Let  $P$  and  $Q$  be fuzzy points at  $\langle a, b \rangle$  and  $\langle c, d \rangle$ , respectively. Suppose that  $D(P, Q)(r) = 0$  for all  $r > 0$ . Then

$$0 = \vee\{\alpha \mid \exists u \in (P)^\alpha, \exists v \in (PQ)^\alpha \text{ such that } r = d(u, v)\}. \tag{3.6}$$

Thus there do not exist  $u \in (P)^\alpha$  and  $v \in (PQ)^\alpha$  such that  $r = d(u, v)$  for any  $r > 0$  and for any  $\alpha \in (0, 1]$ . Hence  $\langle a, b \rangle = \langle c, d \rangle$  else for  $r = d(\langle a, b \rangle, \langle c, d \rangle)$ ,  $D(P, Q)(r) = 1$  and  $r > 0$ . Now suppose that  $D(P, Q)(r) = 1$  for all  $r > 0$ , where  $\langle a, b \rangle = \langle c, d \rangle$ . Suppose that  $Q$  is not crisp. Then  $\exists v \in (Q)^{\alpha_0}, v \neq \langle a, b \rangle$ , for some  $\alpha_0$  such that  $0 < \alpha_0 < 1$ . Now  $d(\langle a, b \rangle, v) > 0$ . Hence  $D(P, Q)(d(\langle a, b \rangle, v)) = 1$  by assumption and also

$$D(P, Q)(d(\langle a, b \rangle, \langle a, b \rangle)) = 1. \tag{3.7}$$

However this is impossible since  $D(P, Q)$  is a fuzzy number and thus attains the value 1 uniquely.

Now suppose that  $P$  and  $Q$  are fuzzy points at  $\langle a, b \rangle$  and  $\langle c, d \rangle$ , respectively. Let  $r > 0$ . Then  $P^\alpha = \{\langle a, b \rangle\}$  and  $Q^\alpha = \{\langle c, d \rangle\} \forall \alpha \in (0, 1]$ . Hence

$$D(P, Q)(r) = \vee\{\alpha \mid \exists u \in \{\langle a, b \rangle\}, \exists v \in \{\langle c, d \rangle\} \text{ such that } r = d(u, v)\}. \tag{3.8}$$

Thus

$$D(P, Q)(r) = \begin{cases} 1 & \text{if } r = d(\langle a, b \rangle, \langle c, d \rangle), \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

Hence  $D$  reduces to  $d$  if  $P$  and  $Q$  are crisp.

There are several possible ways to define a fuzzy line. First, we might define a fuzzy line to be the set of all pairs of fuzzy numbers  $(X, Y)$  which are solutions to  $AX + BY = C$  for given fuzzy numbers  $A, B, C$ . However this equation often has no solution for  $X$  and  $Y$  using standard fuzzy arithmetic.

Another possible method is to define a fuzzy line to be the set of all pairs of fuzzy numbers  $(X, Y)$  which are solutions to  $Y = MX + B$  for given fuzzy numbers  $M, B$ . With this method, one cannot construct pictures of this type of fuzzy line.

A third possible method is to use the following approach. Let  $A, B, C$  be fuzzy numbers. If  $A(1) = \{a\}$  and  $B(1) = \{b\}$ , we assume that  $a$  and  $b$  are not both zero. Let

$$\Omega_{11}(\alpha) = \{\langle x, y \rangle \mid ax + by = c, a \in A^\alpha, b \in B^\alpha, c \in C^\alpha\} \quad (3.10)$$

$\forall \alpha \in [0, 1]$ . Then we let  $L_{11}$  denote the fuzzy subset of  $\mathbb{R}^2$  defined by  $\forall \langle x, y \rangle \in \mathbb{R}^2$ ,

$$L_{11}(x, y) = \vee \{\alpha \mid \langle x, y \rangle \in \Omega_{11}(\alpha)\}. \quad (3.11)$$

If  $A(1) = 0$  and  $B(1) = 0$ , then  $\Omega_{11}(1)$  can be empty since we then have the equation  $0x + 0y = c, c \in C(1)$ , which will have no solution when  $c \neq 0$ .

Another possible method to define a fuzzy line is with the equation  $y = mx + b$ . Let  $M$  and  $B$  be fuzzy numbers. Let

$$\Omega_{12}(\alpha) = \{\langle x, y \rangle \mid y = mx + b, m \in M^\alpha, b \in B^\alpha\} \forall \alpha \in [0, 1]. \quad (3.12)$$

Then we let  $L_{12}$  denote the fuzzy subset of  $\mathbb{R}^2$  defined by  $\forall \langle x, y \rangle \in \mathbb{R}^2$ ,

$$L_{12}(x, y) = \vee \{\alpha \in [0, 1] \mid \langle x, y \rangle \in \Omega_{12}(\alpha)\}. \quad (3.13)$$

Still another possible method is by using a point-slope form. Let  $K$  be a fuzzy point in  $\mathbb{R}^2$  and let  $M$  be a fuzzy number. Let

$$\Omega_2(\alpha) = \{(x, y) \mid y - v = m(x - u), \langle u, v \rangle \in K^\alpha, m \in M^\alpha\} \forall \alpha \in [0, 1]. \quad (3.14)$$

Then we let  $L_2$  denote the fuzzy subset of  $\mathbb{R}^2$  defined by  $\forall \langle x, y \rangle \in \mathbb{R}^2$ ,

$$L_2(x, y) = \vee \{\alpha \in [0, 1] \mid \langle x, y \rangle \in \Omega_2(\alpha)\}. \quad (3.15)$$

Finally, another possibility is the two-point method. Let  $P_1$  and  $P_2$  be two fuzzy points in the plane. Let

$$\begin{aligned} \Omega_3(\alpha) = \{ \langle x, y \rangle \mid (y - v_1)/(x - u_1) &= (v_2 - v_1)/(u_2 - u_1), \\ \langle u_1, v_1 \rangle \in (P_1)^\alpha, \langle u_2, v_2 \rangle \in (P_2)^\alpha \} &\forall \alpha \in [0, 1]. \end{aligned}$$

Then we let  $L_3$  denote the fuzzy subset of  $\mathbb{R}^2$  defined by  $\forall \langle x, y \rangle \in \mathbb{R}^2$ ,

$$L_3(x, y) = \vee \{\alpha \in [0, 1] \mid \langle x, y \rangle \in \Omega_3(\alpha)\}. \quad (3.16)$$

We consider  $L_{11}, L_{12}, L_2$ , and  $L_3$  to be four different types of fuzzy lines.

**Theorem 3.4.**

$$(L_{11})^\alpha = \Omega_{11}(\alpha), \quad (3.17)$$

$$(L_{12})^\alpha = \Omega_{12}(\alpha), \quad (3.18)$$

$$(L_2)^\alpha = \Omega_2(\alpha), \quad (3.19)$$

$$(L_3)^\alpha = \Omega_3(\alpha) \quad (3.20)$$

for all  $\alpha \in [0, 1]$ .

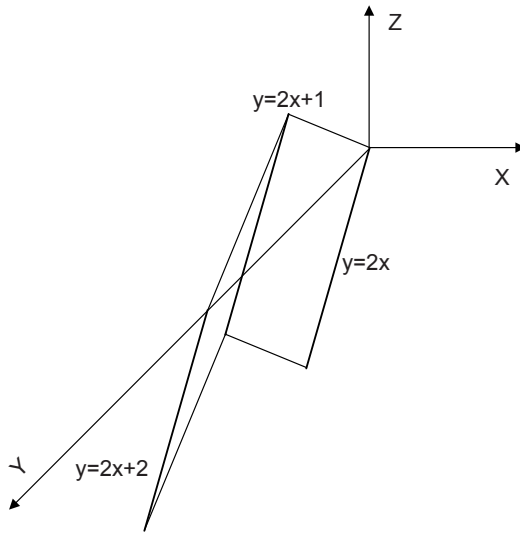


Fig. 3.1. Graph of  $L_{12}$

We now give some examples of fuzzy lines. The examples are from Buckley and Eslami (1997a).

*Example 3.5.* Let  $A = (-1, 0, 1)$ ,  $B = (-1, 1, 2)$ , and  $C = (0, 1, 2)$  be triangular fuzzy numbers. Then the support of  $L_{11}$ ,  $cl(\bigcup_{0 < \alpha \leq 1} L_{11}^\alpha)$ , is all of  $\mathbb{R}^2$ . Also the 1-cut  $(L_{11})^1$  is the crisp line  $y = 1$ .

*Example 3.6.* Let  $L_{12}$  be defined by  $y = 2x + B$ , where  $B = (0, 1, 2)$  is a fuzzy triangular number. Here  $M$  is the crisp number 2. Thus

$$\Omega_{12}(1) = \{\langle x, y \rangle \mid y = 2x + b, b \in B^1 = \{1\}\}. \quad (3.21)$$

Now  $B^\alpha = [\alpha, 2 - \alpha]$ . Hence

$$\Omega_{12}(\alpha) = \{\langle x, y \rangle \mid y = 2x + b, b \in [\alpha, 2 - \alpha]\}. \quad (3.22)$$

Thus  $L_{12}(x, y) = \vee\{\alpha \in [0, 1] \mid y = 2x + b, b \in [\alpha, 2 - \alpha]\}$ . Hence it follows that the graph of  $z = L_{12}(x, y)$  is generated by  $B$  on the  $y$ -axis, base on the interval  $[0, 2]$ , and “running” the triangle along the crisp line  $y = 2x + 1$ . This is illustrated in Figure 3.1.

*Example 3.7.* Let  $M$  denote the crisp real number 1 and let  $K$  be a fuzzy point at  $\langle 1, 1 \rangle$ . Then  $(L_2)^\alpha$  is the set of all lines with slope 1 and through a point in  $K^\alpha$ .  $(L_2)^1$  is the crisp line  $y = x$ .  $L_2$  is “thin” when  $cl(\bigcup_{0 < \alpha \leq 1} K^\alpha)$  is “small”.

*Example 3.8.* Let  $P_1(0, 0)$  and  $P_2(1, 1)$  be two fuzzy points whose graph is a right circular cone. The base of  $P_1(0, 0)$  is  $B_1 = \{\langle x, y \rangle \mid x^2 + y^2 \leq (1/3)^2\}$  and vertex

$\langle 0, 0 \rangle$ . The base of  $P_2(1, 1)$  is  $B_2 = \{ \langle x, y \rangle \mid (x - 1)^2 + (y - 1)^2 \leq (1/3)^2 \}$  and vertex  $\langle 1, 1 \rangle$ . Then the support of  $L_3$  is all lines through a point in  $B_1$  and a point in  $B_2$ .  $(L_3)^1$  is the line  $y = x$ .  $L_3$  is thin between  $B_1$  and  $B_2$ , but gets wider and wider as we move along  $y = x$  for  $x > 1$  or for  $x < 0$ .

A fuzzy line  $L$  is said to contain a fuzzy point  $P$  if and only if  $P \subseteq L$ .

Clearly  $L_2$  contains  $K$ . If  $P_{\langle c, d \rangle}$  is a fuzzy point at  $\langle c, d \rangle$  and  $L_2$  contains  $P_{\langle c, d \rangle}$ , then  $\langle c, d \rangle \in \Omega_2(1)$ . Let the 1-cut  $M^1 = [m_1, m_2]$  be an interval. We have that  $\Omega_2(1)$  is all lines through  $\langle a, b \rangle$  with slope  $m$ ,  $m_1 \leq m \leq m_2$ . If  $M$  is a triangular fuzzy number,  $M^1 = \{m\}$ , then  $\Omega_2(1)$  is the crisp line  $y - b = m(x - a)$ .

Let  $P_1$  and  $P_2$  be fuzzy points at  $\langle a_1, b_1 \rangle$  and  $\langle a_2, b_2 \rangle$ , respectively, which define  $L_3$ . Then  $L_3$  contains both  $P_1$  and  $P_2$ . Also  $(L_3)^1 = \Omega_3(1)$  will always be the crisp line through  $\langle a_1, b_1 \rangle$  and  $\langle a_2, b_2 \rangle$ . If  $L_3$  contains some other point  $Q$ , then  $Q^1$  must be on the line which is  $\Omega_3(1)$ .

It is shown in Buckley and Eslami (1997a) that under certain conditions,  $L_{11}$  is an  $L_{12}$ ,  $L_3$  is an  $L_2$ , and  $L_2$  is an  $L_{12}$ . Next we summarize these results. We first consider  $L_{11}$  and  $L_{12}$ .

Assume that zero does not belong to  $B^0$ . Define

$$\Omega_m(\alpha) = \{ -a/b \mid a \in A^\alpha, b \in B^\alpha \}, \quad 0 \leq \alpha \leq 1, \tag{3.23}$$

and define  $M$  by  $\forall x \in \mathbb{R}^2$ ,

$$M(x) = \vee \{ \alpha \mid x \in \Omega_m(\alpha) \}. \tag{3.24}$$

Next set

$$\Omega_b(\alpha) = \{ c/d \mid b \in B^\alpha, c \in C^\alpha \}, \quad 0 \leq \alpha \leq 1, \tag{3.25}$$

and define  $B_0$  by  $\forall x \in \mathbb{R}^2$ ,

$$B_0(x) = \vee \{ \alpha \mid x \in \Omega_b(\alpha) \}. \tag{3.26}$$

In the above definitions,  $A, B, C$  are the fuzzy numbers in the definition of  $L_{11}$ . It follows that  $M$  and  $B_0$  are also fuzzy numbers and that  $M^\alpha = \Omega_m(\alpha)$  and  $(B_0)^\alpha = \Omega_b(\alpha)$  for all  $\alpha$ . Thus let  $M$  and  $M_0$  be the fuzzy numbers in the definition of  $L_{12}$ .

**Theorem 3.9.**  $L_{11} = L_{12}$ .

We now consider  $L_3$  and  $L_2$ . Let  $P_1 = P(a_1, b_1)$ ,  $P_2 = P(a_2, b_2)$  be two fuzzy points which define  $L_3$ . Define  $Proj_x(Proj_y)$  to be the projection of a subset of the plane onto the x-axis (y-axis). Assume that  $Proj_x(P_1)^0 \cap Proj_x(P_2)^0 \neq \emptyset$ . Hence if  $\langle u_1, v_1 \rangle \in (P_1)^0$  and  $\langle u_2, v_2 \rangle \in (P_2)^0$ , then  $u_1 - u_2$  will never be zero. Define

$$\Omega_m(\alpha) = \{ m \mid m = \frac{v_2 - v_1}{u_2 - u_1}, \langle u_1, v_1 \rangle \in (P_1)^\alpha, \langle u_2, v_2 \rangle \in (P_2)^\alpha \} \tag{3.27}$$

for  $0 \leq \alpha \leq 1$ , and set  $M(x) = \vee \{ \alpha \mid x \in \Omega_m(\alpha) \}$ . It follows that  $M$  is a fuzzy number and  $M^\alpha = \Omega_m(\alpha)$  for all  $\alpha$ . Now let  $M$  and  $P_1$  be fuzzy subsets in the definition of  $L_2$ .

**Theorem 3.10.**  $L_3 = L_2$ .

We consider  $L_{12}$  and  $L_2$  next. We first show that given an  $L_{12}$ , we can define  $L_2$  so that  $L_2 = L_{12}$ . Let  $M, B$  be fuzzy numbers that define  $L_{12}$ . We use the same  $M$  for  $L_{12}$ . Recall that  $(L_{12})^\alpha = \Omega_{12}(\alpha)$ . We need to specify a fuzzy point  $K$  to completely define  $L_2$ . Let  $M^\alpha = [m_1, m_2]$  and  $B^\alpha = [b_1, b_2]$ . Assume that  $B(x) = 1$  if and only if  $x = b^*$  where  $b_1 < b^* < b_2$ . That is,  $B$  is normalized at only one point. Define  $K$  to be a fuzzy point at  $(0, b^*)$  so that its  $\alpha$ -cut lies in region  $R^\alpha$ . Then  $\Omega_2(\alpha) = \Omega_{12}(\alpha)$  and  $L_2$  will be the same as  $L_{12}$ .

Conversely, let  $M$  and  $K$  be the fuzzy subsets in the definition of  $L_2$ . Then  $K$  must be a fuzzy point on the  $y$ -axis. Let  $K$  be a fuzzy point at  $\langle 0, b^* \rangle$  so that its  $\alpha$ -cuts fit inside region  $R^\alpha$ .  $\text{Proj}_y \tilde{K}^\alpha$  will be  $[b_1, b_2]$ . Define the fuzzy number  $B$  so that  $B^\alpha = \text{Proj}_y \tilde{K}^\alpha$  for all  $\alpha$ . The same  $M$  is used for  $L_{12}$ . Then  $\Omega_{12}(\alpha) = \Omega_2(\alpha)$  for all  $\alpha$  so that  $L_{12} = L_2$ .

**Theorem 3.11.**  $L_{12} = L_2$ .

We consider some general properties, concerning fuzzy lines  $L$ . Let  $L$  be either  $L_{11}, L_{12}, L_2$ , or  $L_3$ . Then the following properties hold:

1.  $\alpha$ -cuts of  $L$  are closed, connected and arcwise connected, but not necessarily convex;
2.  $L$  is upper semi-continuous since  $\alpha$ -cuts are closed; and
3.  $L$  is normalized, or there is always at least one crisp line in  $L^1$ .

**Definition 3.12.** Let  $L_a, L_b$  be two fuzzy lines. A measure of parallelness ( $\rho$ ) of  $L_a$  and  $L_b$  is defined to be  $1 - \lambda$  where

$$\lambda = \vee \{L_a(x, y) \wedge L_b(x, y) \mid \langle x, y \rangle \in \mathbb{R}^2\}. \tag{3.28}$$

According to this Definition,  $\lambda$  is just the height of the intersection of  $L_a$  and  $L_b$ . Hence, if  $L_a \cap L_b$  is the empty set (completely parallel), then  $\lambda = 0$  and  $\rho = 1$ . Let  $l_a$  and  $l_b$  be a crisp lines in  $(L_a)^1$  and  $(L_b)^1$ , respectively. If  $l_a$  and  $l_b$  intersect, then  $\lambda = 1$  and  $\rho = 0$ . Thus  $\rho$  has some properties we would expect for a measure of parallelness.

Suppose  $L_a$  and  $L_b$  are both crisp lines. Then  $\rho = 1$  if and only if  $L_a$  and  $L_b$  are parallel.

Now let  $L_a$  be a crisp line and  $L_b$  be a fuzzy line. For example, let  $L_b = L_{12}$  or  $L_2$ . If  $M^1 = [m_1, m_2]$ ,  $m_1 < m_2$ , then  $L_a$  intersects a crisp line in  $(L_b)^1$  and  $\rho = 0$ . Hence  $\rho = 0$  for a crisp line and a fuzzy line.

**Definition 3.13.** Let  $L_a$  and  $L_b$  be two fuzzy lines. Assume  $\rho < 1$ . The fuzzy region  $R$  of intersection of  $L_a$  and  $L_b$  is  $R = L_a \cap L_b$ . Then  $R(x, y) = L_a(x, y) \wedge L_b(x, y)$ .

Clearly, if  $\rho = 1$ , then the fuzzy region of intersection  $R$  is the empty set.

### 3.2 Circles and Polygons

This section continues the development of fuzzy plane geometry of the previous section. The material is from [Buckley and Eslami \(1997b\)](#). We investigate fuzzy circles and fuzzy polygons. We show that the fuzzy area of a fuzzy circle, or a fuzzy polygon, is a fuzzy number. We also show that the fuzzy perimeter of a fuzzy circle, or a fuzzy polygon, is a fuzzy number.

We define a fuzzy circle and show how to obtain  $\alpha$ -cuts of a fuzzy circle. We also consider some examples of fuzzy circles. We define the fuzzy area of a fuzzy circle and show that it is a fuzzy number. We then define the fuzzy circumference of a fuzzy circle and prove that it is also a fuzzy number. We look at some examples of the fuzzy area and the circumference of fuzzy circles.

The equation  $x^2 + ax + y^2 + by = c$  defines a circle when  $4c > a^2 + b^2$ . Our first method is to fuzzify this procedure. Let  $A, B, C$  be fuzzy numbers. A fuzzy circle  $\mathcal{C}$  is all pairs of fuzzy numbers  $(X, Y)$  which are solutions to

$$(X)^2 + AX + (Y)^2 + BY = C,$$

where  $4C > (A)^2 + (B)^2$ .

However, the above equation usually has no solution (using standard fuzzy arithmetic) for  $X$  and  $Y$ . Therefore, we do not use this method in defining a fuzzy circle.

Another possible method in specifying a circle is to use the standard equation for a circle:  $(x - a)^2 + (y - b)^2 = c^2$ . This leads us to a second method of defining a fuzzy circle. Let  $A, B, C$  be fuzzy numbers. A fuzzy circle  $\mathcal{C}$  is all pairs of fuzzy numbers  $(X, Y)$  which are solutions to

$$(X - A)^2 + (Y - B)^2 = (C)^2.$$

Unfortunately, this equation also has few, if any, solutions for  $X$  and  $Y$ .

Let  $A, B, C$  be fuzzy numbers. Let

$$\Omega(\alpha) = \{ \langle x, y \rangle \mid (x - a)^2 + (y - b)^2 = c^2, a \in A^\alpha, b \in B^\alpha, c \in C^\alpha \}, \quad (3.29)$$

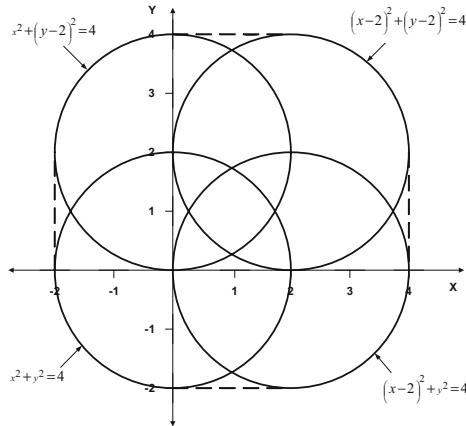
for  $0 \leq \alpha \leq 1$ . A fuzzy circle is defined as follows:  $\forall \langle x, y \rangle \in \mathbb{R}^2$ ,

$$\mathcal{C}(x, y) = \vee \{ \alpha \mid \langle x, y \rangle \in \Omega(\alpha) \}. \quad (3.30)$$

We will adopt this method to define a fuzzy circle. This type of fuzzy circle will be seen to have desirable properties including its fuzzy area and circumference being fuzzy numbers (or real numbers as a special case of fuzzy numbers). The next result gives the  $\alpha$ -cuts of fuzzy circles.

**Theorem 3.14.**  $\mathcal{C}^\alpha = \Omega(\alpha)$ ,  $0 \leq \alpha \leq 1$ .

*Example 3.15.* The fuzzy circle defined here is called “thick” or “fat.” Let  $A = (0/1/2) = B = C$  be triangular fuzzy numbers. The support of  $\mathcal{C}$ ,  $\mathcal{C}^0 = \Omega(0)$  is



**Fig. 3.2.** Graph of the Support of  $\mathcal{C}$

the rectangle  $[-2, 4] \times [-2, 4]$  with rounded edges. See Figure 3.2. In fact, all  $\alpha$ -cuts of  $\mathcal{C}$ ,  $0 \leq \alpha \leq 1$ , are rounded corner rectangles with  $\mathcal{C}^1$  the crisp circle  $(x-1)^2 + (y-1)^2 = 1$ . The graph of the membership function of  $\mathcal{C}$  is a four-sided pyramid, with rounded edges and vertex at  $\langle 1, 1 \rangle$ .

*Example 3.16.* By a “regular” fuzzy circle, we mean that  $\alpha$ -cuts,  $0 \leq \alpha < 1$ , will be disks. Let  $A = B = 1$  (real number one) and  $C = (1/2/3)$ . All  $\alpha$ -cuts,  $0 \leq \alpha < 1$ , of  $\mathcal{C}$  are disks with  $\mathcal{C}^0 = \{ \langle x, y \rangle \mid 1 \leq (x-1)^2 + (y-1)^2 \leq 9 \}$ .  $\mathcal{C}^1$  is the crisp circle  $(x-1)^2 + (y-1)^2 = 1$ .

We next consider the area of a fuzzy circle.

**Definition 3.17.** Let  $A, B, C$  be fuzzy numbers in the definition of the fuzzy circle  $\mathcal{C}$ . Set

$$\Omega_a(\alpha) = \{ \theta \mid \theta \text{ is the area of } (x-a)^2 + (y-b)^2 = c^2, a \in A^\alpha, b \in B^\alpha, c \in C^\alpha \}, 0 \leq \alpha \leq 1.$$

The area  $\Theta$  of  $\mathcal{C}$  is defined as follows:

$$\Theta(\theta) = \vee \{ \alpha \mid \theta \in \Omega_a(\alpha) \}. \tag{3.31}$$

There are some degenerate cases which must be considered when  $\Theta$  is a (crisp) real number (a special case of a fuzzy number). If  $A, B, C$  are all real numbers, then  $\mathcal{C}$  is a crisp circle and  $\Theta$  is a real number. Thus assume that at least one of  $A, B, C$  is a fuzzy number which is not a real number. The only case where  $\Theta$  can degenerate into a crisp number is when  $C$  is real. Let  $C = c, c > 0$ , a real number. Then  $\Theta = \pi c^2$ , a real number. Thus, in the next theorem, it is assumed that  $C$  is not a real number.

**Theorem 3.18.**  $\Theta^\alpha = \Omega_a(\alpha)$  for all  $\alpha$  and  $\Theta$  is a fuzzy number.

**Definition 3.19.** Let  $A, B, C$  be fuzzy numbers in the definition of  $\mathcal{C}$ . Define

$$\Omega_c(\alpha) = \{\Delta \mid \Delta \text{ is the circumference of } (x - a)^2 + (y - b)^2 = c^2, \\ a \in A^\alpha, b \in B^\alpha, c \in C^\alpha, 0 \leq \alpha \leq 1\}.$$

Define the circumference  $\delta$  as  $\delta(\Delta) = \vee\{\alpha \mid \Delta \in \Omega_c(\alpha)\}$ .

In the following theorem, assume that  $C$  is not a real number. If  $C$  is real, then  $\delta$  is a real number.

**Theorem 3.20.**  $\delta^\alpha = \Omega_c(\alpha)$  for all  $\alpha$  and  $\delta$  is a fuzzy number.

*Example 3.21.* Consider the “regular” fuzzy circle of Example 3.15. Then  $A = B = 1$  and  $C = (1/2/3)$  is a triangular fuzzy number. The  $\alpha$ -cuts of  $C$  are  $[\alpha + 1, 3 - \alpha]$ ,  $0 \leq \alpha \leq 1$ . Thus

$$\Theta^\alpha = [\pi(\alpha + 1)^2, \pi(3 - \alpha)^2], \tag{3.32}$$

$$\delta^\alpha = [2\pi(\alpha + 1), 2\pi(3 - \alpha)], \tag{3.33}$$

where  $0 \leq \alpha \leq 1$ . Hence, the fuzzy area  $\Theta$  is a triangular-shaped fuzzy number with support  $[\pi, 9\pi]$  and vertex at  $(4\pi, 0)$ . The fuzzy circumference  $\delta$  is a triangular fuzzy number  $(2\pi/4\pi/6\pi)$ .

*Example 3.22.* Consider Example 3.14, where  $A = B = C = (0/1/2)$ . The level sets of  $C$  are  $[\alpha, 2 - \alpha]$ ,  $0 \leq \alpha \leq 1$ . It follows that  $\Theta$  is a triangular-shaped fuzzy number with  $\alpha$ -cuts  $[\pi\alpha^2, \pi(2 - \alpha)^2]$ ,  $0 \leq \alpha \leq 1$ . Also,  $\delta$  is a triangular fuzzy number  $(0/\pi/4\pi)$ .

We next discuss what we mean by a regular  $n$ -sided polygon, define (regular)  $n$ -sided polygons, and compute  $\alpha$ -cuts of fuzzy polygons. We define the fuzzy area and perimeter of a fuzzy polygon and show that they are fuzzy numbers. We also look at two special cases as fuzzy rectangles and triangles.

A polygon is a rectilinear figure with  $n$  sides,  $n \geq 3$ . We allow  $n = 3$  and  $n = 4$  so that triangles and rectangles can be considered polygons. A regular polygon will have a convex interior.

**Definition 3.23.** Distinct points  $v_1, \dots, v_n \in \mathbb{R}^2$  are said to be convex independent if and only if any  $v_i$  does not belong to the convex hull (convex closure) of the rest of the  $v_j$ ,  $1 \leq j \leq n$ ,  $j \neq i$ .

Let  $v_1, \dots, v_n$  be convex independent. Assume they are numbered counterclockwise, i.e., as we travel from  $v_1$  to  $v_2, \dots, v_{n-1}$  to  $v_n$  we continually travel in a counterclockwise direction. We connect adjacent  $v_i$  with line segments in the following manner. We draw a line segment  $l_{12}$  from  $v_1$  to  $v_2, \dots, l_{n1}$  from  $v_n$  to  $v_1$ . The  $v_i$ , together with the line segments, define a regular  $n$ -sided polygon. We call such a polygon an  $n$ -gon. The interior of an  $n$ -gon is convex. When  $n = 3$  we have a triangle.

Let  $v_1, \dots, v_n$  be  $n$  distinct points in the plane that define an  $n$ -gon (a regular  $n$ -sided polygon). Next let  $P_i$  be a fuzzy point at  $v_i$ ,  $1 \leq i \leq n$ . We now need the definition of a fuzzy line segment.

**Definition 3.24.** Let  $P$  and  $Q$  be distinct fuzzy points. Let  $\Omega_l(\alpha)$  denote the set of all line segments from a point in  $P^\alpha$  to a point  $Q^\alpha$ . The fuzzy line segment  $L_{PQ}$ , from  $P$  to  $Q$  is defined as follows:

$$L_{PQ}(x, y) = \vee\{\alpha \mid \langle x, y \rangle \in l, l \in \Omega_l(\alpha)\}. \tag{3.34}$$

It can be shown that  $(L_{pq})^\alpha = \Omega_l(\alpha)$  for all  $\alpha$ .

*Example 3.25.* Let  $P_1$  be a fuzzy point at  $(1, 0)$  with base  $(x - 1)^2 + y^2 = 1/4$  and let  $P_2$  be a point at  $(3, 0)$  with base  $(x - 3)^2 + y^2 = 1/4$ . See Figure 3.3. Then  $\forall r \in \mathbb{R}, D(P_1, P_2)(r) = \vee\{\alpha \mid r \in \Omega(\alpha)\}$ , where  $\Omega(\alpha) = \{d(u, v) \mid u \in P_1^\alpha, v \in P_2^\alpha\}$ . The equation of the right circular cone that defines  $P_1$  is given by  $(x - 1)^2 + y^2 = (\frac{1}{2} - \frac{z}{2})^2$  and the equation of the right circular cone that defines  $P_2$  is given by  $(x - 3)^2 + y^2 = (\frac{1}{2} - \frac{z}{2})^2$ . It follows that

$$P_1(u_1, u_2) = \begin{cases} \alpha \text{ if } (u_1 - 1)^2 + u_2^2 = (\frac{1}{2} - \frac{\alpha}{2})^2, \alpha \in [0, 1], \\ 0 \text{ otherwise,} \end{cases}$$

and

$$P_2(v_1, v_2) = \begin{cases} \alpha \text{ if } (v_1 - 3)^2 + v_2^2 = (\frac{1}{2} - \frac{\alpha}{2})^2, \alpha \in [0, 1], \\ 0 \text{ otherwise.} \end{cases}$$

Let  $\alpha \in [0, 1]$ . Then

$$P_1^\alpha = \{\langle u_1, u_2 \rangle \in \mathbb{R}^2 \mid (u_1 - 1)^2 + u_2^2 \leq (\frac{1}{2} - \frac{\alpha}{2})^2\} \tag{3.35}$$

and

$$P_2^\alpha = \{\langle v_1, v_2 \rangle \in \mathbb{R}^2 \mid (v_1 - 3)^2 + v_2^2 \leq (\frac{1}{2} - \frac{\alpha}{2})^2\}. \tag{3.36}$$

Now

$$\Omega(\alpha) = \{d(\langle u_1, u_2 \rangle, \langle v_1, v_2 \rangle) \mid \langle u_1, u_2 \rangle \in P_1^\alpha \text{ and } \langle v_1, v_2 \rangle \in P_2^\alpha\} \tag{3.37}$$

and  $\langle 1 - (\frac{1}{2} - \frac{\alpha}{2}), 0 \rangle \in P_1^\alpha$  and  $\langle 3 + (\frac{1}{2} - \frac{\alpha}{2}), 0 \rangle \in P_2^\alpha$  are such that

$$d((1 - (\frac{1}{2} - \frac{\alpha}{2}), 0), (3 + (\frac{1}{2} - \frac{\alpha}{2}), 0)) = \vee\{d(u, v) \mid u \in P_1^\alpha \text{ and } v \in P_2^\alpha\}. \tag{3.38}$$

Clearly,

$$d((1 - (\frac{1}{2} - \frac{\alpha}{2}), 0), (3 + (\frac{1}{2} - \frac{\alpha}{2}), 0)) = 3 - \alpha. \tag{3.39}$$

Thus for  $r \in [2, 3], D(P_1, P_2)(r) = 3 - r$  and  $D(P_1, P_2)(r) = 0$  otherwise. Hence  $D(P_1, P_2)(2.5) = .5$ . See Figure 3.3.

Now consider the fuzzy line segment  $L$  from  $P_1$  to  $P_2$ . Then  $L^1 = \Omega_l(1) = \{l\}$ , where  $l$  is the line segment from  $\langle 1, 0 \rangle$  to  $\langle 3, 0 \rangle$ . We also have that  $L(2, 0) = 1$  and  $L(2, .5) = 0$ . We next determine  $L(2, .25)$ . The line through  $(2, .25)$  and

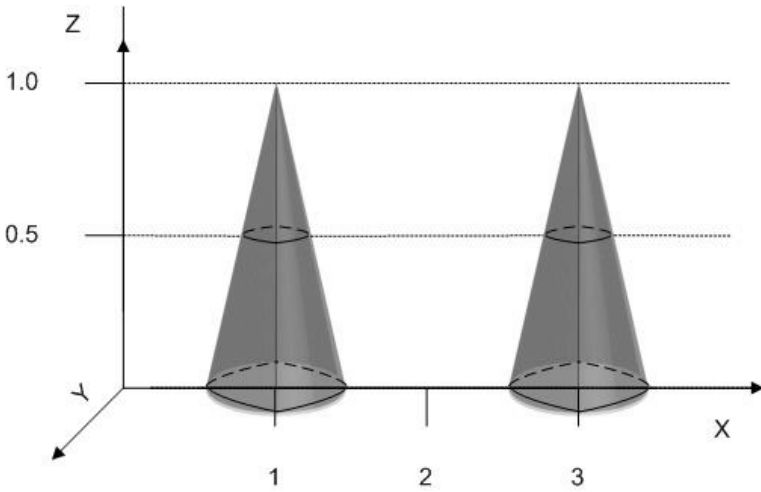


Fig. 3.3. Fuzzy Points  $P_1$  and  $P_2$

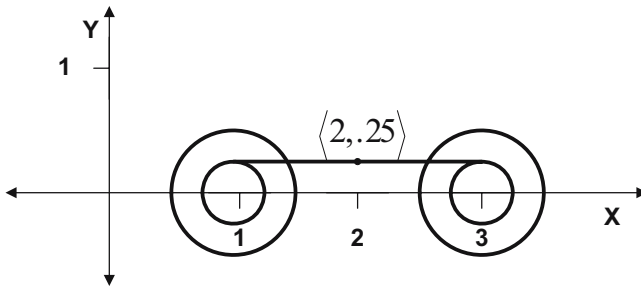


Fig. 3.4. Determination of  $L(2, .25)$

parallel to the  $x$ -axis intersects  $P_1^\alpha$  and  $P_2^\alpha$  at  $(1, .25)$  and  $(3, .25)$ , respectively when  $\alpha = .5$ . This follows since

$$\frac{1}{2} = \vee\{\alpha \in [0, 1] \mid (1 - 1)^2 + \left(\frac{1}{4}\right)^{\frac{1}{2}} \leq \left(\frac{1}{2} - \frac{\alpha}{2}\right)^2 \tag{3.40}$$

$$= \vee\{\alpha \in [0, 1] \mid (3 - 3)^2 + \left(\frac{1}{4}\right)^{\frac{1}{2}} \leq \left(\frac{1}{2} - \frac{\alpha}{2}\right)^2. \tag{3.41}$$

Thus  $L(2, .25) = .5$ . This is illustrated in Figure 3.4.

**Definition 3.26.** Let  $L_1, \dots, L_n$  be fuzzy line segments from  $P_1$  to  $P_2, \dots, P_n$  to  $P_1$ , respectively. Then a (regular)  $n$ -sided fuzzy polygon  $\mathcal{B}$  is defined to be  $\mathcal{B} = \cup_{i=1}^n L_i$ .

It follows that for all  $\langle x, y \rangle \in \mathbb{R}^2$ ,

$$\mathcal{B}(x, y) = \vee\{L_i(x, y) \mid i = 1, \dots, n\}. \tag{3.42}$$

If the two fuzzy points  $P_1$  and  $P_2$  are such that  $(P_2)^0$  is a subset of  $(P_1)^0$ , then we obtain a degenerate fuzzy polygon. In a degenerate fuzzy polygon the support,  $\mathcal{B}^0$ , does not show all the (fuzzy)  $n$  vertices.

*Example 3.27.* Let  $n = 3$  and the fuzzy points be all right circular cones.  $P_1$  has base

$$B_1 = \{ \langle x, y \rangle \mid x^2 + y^2 \leq (0.1)^2 \} \tag{3.43}$$

and vertex at  $\langle 0, 0 \rangle$ ,  $P_2$  has base

$$B_2 = \{ \langle x, y \rangle \mid (x - 1)^2 + y^2 \leq (0.1)^2 \} \tag{3.44}$$

with vertex at  $\langle 1, 0 \rangle$ , and  $P_3$  has base

$$B_3 = \{ \langle x, y \rangle \mid (x - 1)^2 + (y - 0.5)^2 \leq 1 \} \tag{3.45}$$

having vertex at  $\langle 1, 0.5 \rangle$ . Clearly,  $(P_2)^0 \subseteq (P_3)^0$  and  $(P_1)^0, (P_3)^0$  are disjoint. Thus,  $(L_2)^0 \subseteq (P_3)^0$  and  $\mathcal{B}$  is degenerate. Also,  $L_1(0) \subseteq L_3(0)$ . Hence the support of  $\mathcal{B}$  is  $(L_3)^0$ .

**Definition 3.28.** We call a regular  $n$ -sided fuzzy polygon a fuzzy  $n$ -gon. We say a fuzzy  $n$ -gon is non-degenerate if  $(P_i)^0$  is not a subset of  $(P_j)^0, j \neq i$ , for all  $i = 1, \dots, n$ . We say the fuzzy  $n$ -gon is strongly non-degenerate if the  $(P_i)^0, i = 1, \dots, n$ , are pairwise disjoint.

*Example 3.29.* Let  $n = 4$  and let  $P_1$  be a fuzzy point at  $\langle 0, 0 \rangle$ ,  $P_2$  a fuzzy point at  $\langle 1, 0 \rangle$ ,  $P_3$  a fuzzy point at  $\langle 1, 1 \rangle$ , and  $P_4$  a fuzzy point at  $\langle 0, 1 \rangle$ . Let  $P_1$  be a right circular cone with base  $B_1 = \{ \langle x, y \rangle \mid x^2 + y^2 \leq (0.1)^2 \}$  and vertex at  $\langle 0, 0 \rangle$ . The rest of the  $P_i, i > 1$ , will just be rigid translations of  $P_1$  to their position in the plane. Then  $\mathcal{B}$  is strongly non-degenerate; in fact, we would call  $\mathcal{B}$  a fuzzy rectangle.

**Theorem 3.30.**  $\mathcal{B}^\alpha = \cup_{i=1}^n (L_i)^\alpha$  for all  $\alpha$ .

We now consider the fuzzy area of a fuzzy  $n$ -gon.

**Definition 3.31.** Let  $\mathcal{B}$  be a strongly non-degenerate fuzzy  $n$ -gon defined by fuzzy points  $P_i, 1 \leq i \leq n$ . Define  $\Omega_a(\alpha)$  to be the set of all areas of  $n$ -gons such that  $v_i \in (P_i)^\alpha, 1 \leq i \leq n, 0 \leq \alpha \leq 1$ . The area  $\Theta$  of  $\mathcal{B}$  is defined to be

$$\tilde{\Theta}(\theta) = \vee \{ \alpha \mid \theta \in \Omega_a(\alpha) \}. \tag{3.46}$$

In a similar way, we can define the fuzzy perimeter of a fuzzy  $n$ -gon.

**Definition 3.32.** Let  $\mathcal{B}$  be a strongly non-degenerate fuzzy  $n$ -gon defined by fuzzy points  $P_i, 1 \leq i \leq n$ . Set  $\Omega_p(\alpha)$  equal to the set of all perimeters of  $n$ -gons defined by  $v_i$  in  $(P_i)^\alpha, 1 \leq i \leq n, 0 \leq \alpha \leq 1$ . Then the fuzzy perimeter  $\delta$  is

$$\delta(\Delta) = \vee \{ \alpha \mid \Delta \in \Omega_p(\alpha) \}. \tag{3.47}$$

**Theorem 3.33.**  $\Theta^t = \Omega_\alpha(\alpha)$  and  $\delta^\alpha = \Omega_p(\alpha)$  for all  $\alpha$ .

If we allow  $\mathcal{B}$  to be degenerate, we still obtain  $\Theta^\alpha = \Omega_\alpha(\alpha)$ ,  $\delta^\alpha = \Omega_p(\alpha)$ , for all  $\alpha$ , but  $\Theta$  and  $\delta$  may not be fuzzy numbers.

*Example 3.34.* Consider the degenerate fuzzy polygon in Example 3.29. Since some  $\alpha$ -cuts of  $P_2$  are inside the corresponding  $\alpha$ -cuts of  $P_3$ ,  $\Omega_a(\alpha)$  is the interval  $(0, r(\alpha)]$ ,  $0 \leq \alpha \leq \alpha^*$  for some  $0 < \alpha^* < 1$ . However  $\Theta^\alpha = \Omega_a(\alpha)$  so that  $\Theta^\alpha$  will not always be a closed interval. Hence,  $\Theta$  is not upper semi-continuous. Similar results hold for the fuzzy perimeter. Here neither  $\Theta$  nor  $\delta$  are fuzzy numbers.

If  $n = 3$  is a fuzzy  $n$ -gon, it is a fuzzy triangle. If it is strongly non-degenerate, its fuzzy area and perimeter are fuzzy numbers. Now, since we have a fuzzy triangle we may investigate the beginnings of fuzzy trigonometry. For the remainder of the section, we assume that the fuzzy triangle is strongly non-degenerate.

**Definition 3.35.** Let  $P_1, P_2, P_3$  be the fuzzy points that define the fuzzy triangle  $\mathcal{T}$ . Let  $L_{12}, L_{23}, L_{31}$  be the fuzzy line segments connecting  $P_1$  to  $P_2$ ,  $P_2$  to  $P_3$ , and  $P_3$  to  $P_1$ , respectively. Let  $\Omega_\pi(\alpha)$  denote the set of all angles (in radians) between  $l_{12}$  and  $l_{31}$  such that  $l_{12}$  is a line segment from a point in  $(P_1)^\alpha$  to a point in  $(P_2)^\alpha$ ,  $l_{31}$  is a line segment from  $(P_3)^\alpha$  to  $(P_1)^\alpha$ , and  $l_{12}, l_{31}$  initiate from the same point in  $(P_1)^\alpha$ ,  $0 \leq \alpha \leq 1$ . We define  $\Pi$ , the fuzzy angle between  $\tilde{L}_{12}$  and  $\tilde{L}_{31}$ , as follows:

$$\Pi(\pi) = \vee\{\alpha \mid \pi \in \Omega_\pi(\alpha)\}. \tag{3.48}$$

**Theorem 3.36.**  $\Pi^\alpha = \Omega_\pi(\alpha)$  for all  $\alpha$  and  $\Pi$  is a fuzzy number.

In the definition of  $\Pi$ , assume that  $P_i$  is a fuzzy point at  $v_i$  in  $\mathbb{R}^2$ ,  $i = 1, 2, 3$ .  $\Pi$  depends not only on the  $v_i$ , but also on the “size” of the fuzzy points. That is, if we substitute fuzzy point  $P'_2$  at  $v_2$  for  $P_2$  and  $(P_2)^\alpha \subseteq (P'_2)^\alpha$  for all  $\alpha$  and  $\Pi'$  is the resulting fuzzy angle, then we can get  $\Pi \subseteq \Pi'$ . Thus we write  $\Pi = \Pi(P_1, P_2, P_3)$  to show the dependence of  $\Pi$  on the fuzzy points.

Consider elementary fuzzy right triangle trigonometry. Consider fuzzy points  $P_1, P_2, P_3$  at  $v_1 = \langle 0, 0 \rangle$ ,  $v_2 = \langle a, 0 \rangle$ ,  $v_3 = \langle a, b \rangle$ ,  $a > 0$ ,  $b > 0$ , respectively.

The  $P_i$ ,  $1 \leq i \leq 3$ , form a right triangle.  $\Pi$  is the fuzzy angle between  $L_{12}$  and  $L_{31}$ . Define

$$\tan(v) = \frac{D(P_2, P_3)}{D(P_1, P_2)}, \tag{3.49}$$

where  $D(P, Q)$  is the fuzzy distance from fuzzy point  $P$  to  $Q$ . Now  $\tan(\Pi)$  is well-defined and is a fuzzy number, since (a) zero does not belong to the support of  $D(P_1, P_2)$  and (b)  $D(P_2, P_3)$  and  $D(P_1, P_2)$  are both fuzzy numbers. We can define  $\sin(\Pi)$  and  $\cos(\Pi)$  in a similar manner and both will be fuzzy numbers.

However, many properties of triangles and crisp trigonometry identities such as  $\cos^2(\theta) + \sin^2(\theta) = 1$  do not generalize to fuzzy triangles. The following example shows that the Pythagorean theorem may not hold for fuzzy right triangles.

*Example 3.37.* Let  $P_1, P_2, P_3$  be fuzzy points at  $v_1 = \langle 0, 0 \rangle$ ,  $v_2 = \langle 1, 0 \rangle$ ,  $v_3 = \langle 1, 1 \rangle$ , respectively. Assume each  $P_i$  is a right circular cone with base a circle of radius 0.1 centered at  $v_i$ , with vertex at  $v_i$ ,  $i = 1, 2, 3$ . Let  $A = D(P_1, P_2)$ ,  $B = D(P_2, P_3)$ ,  $C = D(P_2, P_3)$ . Now  $L_{12}$  and  $L_{23}$  are the two sides of the fuzzy right angle  $\mathcal{T}$  and  $L_{31}$  is the hypotenuse. The lengths of  $L_{12}, L_{23}, L_{31}$  are  $A, B, C$ , respectively. But  $(A)^2 + (B)^2 \neq (C)^2$  because  $A^0 = B^0 = [0.8, 1.2]$ ,  $C^0 = [\sqrt{2} - 0.2, \sqrt{2} + 0.2]$  so that the support of  $(A)^2 + (B)^2$  is  $[1.28, 2.88]$ , but the support of  $(C)^2$  is  $[1.474, 2.606]$ .

We now consider fuzzy rectangles. Let  $v_1 = \langle x, y \rangle$ ,  $v_2 = \langle x + r, y \rangle$ ,  $v_3 = \langle x + r, y + s \rangle$ , and  $v_4 = \langle x, y + s \rangle$ ,  $r > 0$ ,  $s > 0$  be four points in the plane. Let  $P_i$  be a fuzzy point at  $v_i$ ,  $i = 1, 2, 3, 4$ . The  $P_i$  define a fuzzy 4-gon. We assume it is strongly non-degenerate. We call this fuzzy 4-gon a fuzzy rectangle. We have seen above that we can define the area and perimeter of a fuzzy rectangle and they will be fuzzy numbers.

However, the fuzzy area of the fuzzy rectangle may not be the product of the side lengths and the fuzzy perimeter may not equal the sum of the side lengths. Let  $A = D(P_1, P_2)$ ,  $B = D(P_2, P_3)$ ,  $C = D(P_3, P_4)$  and  $D = D(P_4, P_1)$ . Since the  $\alpha$ -cuts of the fuzzy points may have different sizes, we may get  $A \neq C$  and  $B \neq D$ . Thus assume the fuzzy points are all the “same”, just centered at different points in the plane. The following example shows that still the area may not be the product of the side lengths and the perimeter may not be equal to the sum of the side lengths.

*Example 3.38.* Let  $P_1, P_2, P_3, P_4$  be fuzzy points at  $v_1 = \langle 0, 0 \rangle$ ,  $v_2 = \langle 0, 1 \rangle$ ,  $v_3 = \langle 1, 1 \rangle$ ,  $v_4 = \langle 1, 0 \rangle$ , respectively. Then we have a fuzzy square. Assume each  $P_i$  is a right circular cone with base a circle of radius 0.1 centered at  $v_i$ , and vertex at  $v_i$ ,  $1 \leq i \leq 4$ .

Let  $A = D(P_1, P_2)$ ,  $B = D(P_2, P_3)$ ,  $C = D(P_3, P_4)$ ,  $D = D(P_4, P_1)$ . It follows that  $A^0 = B^0 = C^0 = D^0 = [0.8, 1.2]$ . Thus,  $(AB)^0 = [0.64, 1.44]$  and  $(A+B+C+D)^0 = [3.2, 4.8]$ . Let  $\Omega_a(\alpha)$  denote the set of all areas of a rectangle with vertices in  $P_i(\alpha)$ ,  $1 \leq i \leq 4$ ,  $0 \leq \alpha \leq 1$ . Then the fuzzy area  $\Theta$  is such that  $\Theta(\theta) = \vee \{ \alpha \in [0, 1] \mid \theta \in \Omega_a(\alpha) \}$ .

Define the fuzzy perimeter  $\delta$  similarly. It follows that the left end point of  $\Theta^0$  is greater than 0.64 and the left end point of  $\delta^0$  is more than 3.2. Hence,  $\Theta \neq AB$ ,  $\delta \neq A + B + C + D$ .

### 3.3 Looking Ahead

In the previous chapter, we discussed the fundamentals of fuzzy set theory. In this chapter we extended that discussion to cover fuzzy geometry. We now have the concepts that we need in order to present a fuzzy approach to formal modeling in comparative politics.

We begin the next chapter by reviewing some of the assumptions behind crisp formal models that we discussed in the opening chapter of this book. After

demonstrating the crisp model in one-dimensional space, we then build a fuzzy one-dimensional model, relying on a fuzzy number to represent the preferences of political actors. The fuzzy model retains the basic rationality principle, but the logic of fuzzy set theory permits us to jettison the conventional Euclidean preference assumption in crisp models in comparative politics. In chapter five we extend the argument to spatial models ( $n$ -dimensional models, where  $n > 1$ ).

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## 4 Fuzzy One-Dimensional Models

With Christopher W. Cook, Alan J. Hickle, and Eric M. Ruchensky

The aim of formal modeling under the research agenda of The New Institutionalism is to predict political outcomes on the basis of the preferences of political actors and the design of institutions that aggregate those preferences into a single policy choice. Spatial models plot preferences and policies in  $n$ -dimensional issue space and use rules of geometry over that space to make predictions. In this chapter, we introduce the single dimensional model.

To contrast crisp with fuzzy methods, as well as continuous fuzzy with discrete fuzzy methods, we rely on an example of presidential veto offered by [Kiewiet and McCubbins \(1988\)](#). The first section introduces the example as Kiewiet and McCubbins do, in its original crisp form. By assuming that all actors have a single ideal point, prefer points closer to the ideal point to points farther away in either direction, and have strict preferences over any two distinct policies however similar, the model yields predictions about when a president would accept a proposal and when she would exercise a veto. We conclude the crisp section by explaining limitations implicit in such assumptions, limitations that a fuzzy reformulation can bypass.

The remainder of the chapter reconsiders the veto example first using continuous fuzzy representations of preferences, then using discrete fuzzy representations. Both approaches jettison the single point and the Euclidean distance assumptions, and the discrete approach also relaxes the perfectly specified strict preference assumption. From here, we increase the dimensionality of the issue space and consider multi-dimensional models in the next chapter.

In order for formal models to make stable predictions, a non-empty maximal set must exist. A maximal set comprises those alternatives that are majority preferred to all others and over which the political actor herself is indifferent. (No one alternative in the maximal set is majority preferred to another.) While rules other than majority can be accommodated by the models, the generic model assumes majority rule. Formally,  $M(R, X) = \{x \in X \mid \forall y \in X, x R y\}$  ([Austen-Smith and Banks, 1999](#), p. 3), where  $R$  is a binary relationship,  $X$  is the set containing all alternatives over which political actors are making binary

choices, and  $P$  is a binary relationship that denotes strict preference (i.e.,  $x$  is majority preferred to  $y$ ). If there is only one alternative in the maximal set, then it is the Condorcet winner. (See Chapter 1). In the absence of a maximal set, there is no equilibrium, majority cycling results, and formal models in comparative politics cannot predict outcomes.

One-dimensional models are frequently employed in formal modeling in political science. Such models map actors' policy preferences along a single issue. What makes them particularly appealing is that the conditions under which a maximal set exists are less restrictive than in spatial models ( $n$ -dimensional models, where  $n \geq 2$ ). We begin this chapter by considering the conditions for a non-empty maximal set in one-dimensional space. Next, we illustrate a conventional one-dimensional crisp set approach, [Kiewiet and McCubbins' \(1988\)](#) representation of the U.S. presidential veto. We then adopt fuzzy set theory to reformulate the model. Our reformulation, which relies on a fuzzy number to represent the preference of political actors, permits us to consider the effect of discarding two assumptions common to crisp one-dimension models: 1) the assumption of a single, ideal point and 2) mapping the preferences of political actors from their ideal points based on Euclidean distance (the Euclidean preference assumption). The combination of an exact ideal point and the Euclidean preference assumption imposes a set of very strict restrictions that may not often, if at all, attain in reality and that may distort propositions derived from the models. In the ensuing chapter, we look at the likelihood of the existence of a non-empty maximal set in two-dimensional, spatial models. We do not tamper with the rationality principle, the assumption that political actors are utility maximizers.

## 4.1 Crisp One-Dimensional Models

As we discussed in the opening chapter of this book, rational choice models assume that individuals are self-interested, *utility maximizers*. Given a set of alternatives, they rank order the alternatives on the basis of which best serve their interests (achieve their goals). The resulting preference order induced over the set of alternatives by the selection criterion has the properties of *completeness* and *transitivity*. Completeness means that given any two alternatives, the political actor can state if she prefers one to the other or is indifferent between them. Transitivity requires that given three options,  $X = \{x, y, z\}$ , if person one strictly prefers  $x$  to  $y$  and  $y$  to  $z$  ( $x P_1 y P_1 z$ ), then she must strictly prefer  $x$  to  $z$  ( $x P_1 z$ ), where strict preference for person  $i$  is denoted by  $P_i$ .

While the preference orders of rational individuals demonstrate both properties, the resulting social preferences of collectives of rational individuals may not. The preference rankings of collectives most often will be complete. Formally,  $\forall x, y \in X, x R y, y R x$ , or  $x I y$ , where  $R$  means "at least as good as." However, they are often not transitive. Transitivity requires that  $\forall x, y, z \in X, x R y$  and  $y R z \implies x R z$ , where  $\implies$  denotes implication.

The lack of transitivity introduces the possibility of cycling, which results in an empty maximal set and renders formal models incapable of predicting an

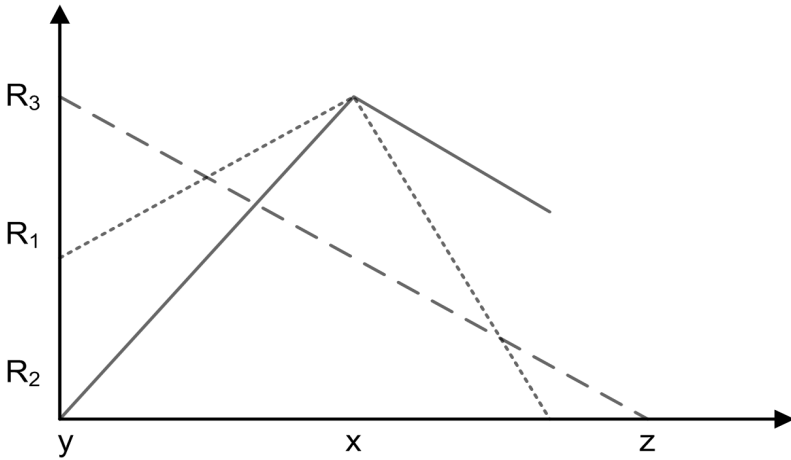


Fig. 4.1. A Single Peaked Preference Profile

equilibrium outcome. As it turns out, however, a weaker condition than transitivity, acyclicity, together with completeness is sufficient to assure the existence of a maximal set. Formally, acyclicity requires  $\forall u, v, w, \dots, x, y, z \in X, u P v \ \& \ v P w \dots \& \ y P z \implies u R z$  (Austen-Smith and Banks, 1999, p. 4). The reader will note that acyclicity permits the possibility of indifference between a great many alternatives, e.g.,  $v I x$  or  $w I z$ .

One of the most commonly studied conditions for assuring completeness and acyclicity in one-dimensional models (and hence the existence of a non-empty maximal set) is single-peakedness. Single-peakedness requires that the preference orders of all political actors are able to be arranged together along a one-dimensional graph so that each individual preference has an ideal point from which the player’s preferences decrease monotonically in both directions. It is the preference profile  $\rho$  of the group, not individual preference orders considered separately from each other, that must be single-peaked. Let  $X = \{x_{t-n}, \dots, x_t, \dots, x_{t+n}\}$ . Then, player  $i$ ’s individual preference order is single-peaked if there is an  $x_t$  such that  $x_t P_i x_{t+1} P_i \dots P_i x_{t+n}$  and  $x_t P_i x_{t-1} P_i \dots P_i x_{t-n}$ . A preference profile  $\rho$  is single-peaked on  $X$  if the preference profile for every player  $i \in N$ , where  $N$  is the set of political actors, is single-peaked (Austen-Smith and Banks, 1999, , pp. 94–95).

Let  $X = \{x, y, z\}$  with a strict preference order  $Q = y Q x Q z$  (alternatives are arrayed from left to right in this order) and  $N = \{1, 2, 3\}$  with preference profile  $\rho = \{R_1, R_2, R_3\}$ , where  $R_1 = x P_1 y P_1 z$ ,  $R_2 = x P_2 z P_2 y$ , and  $R_3 = y P_3 x P_3 z$  over  $X$ . Figure 4.1 illustrates that  $\rho$  is single-peaked with respect to  $X$  since the preference orders for political actors one ( $R_1$ ) and two ( $R_2$ ) strictly decrease from  $x$  and that for player 3 ( $R_3$ ) monotonically decreases from  $y$  when the three are plotted on  $Q$ .

Figure 4.2 illustrates a non-single-peaked profile. Note that in Figure 4.2 one of the preference orders of an individual player ( $R_2$ ) does not have a single ideal

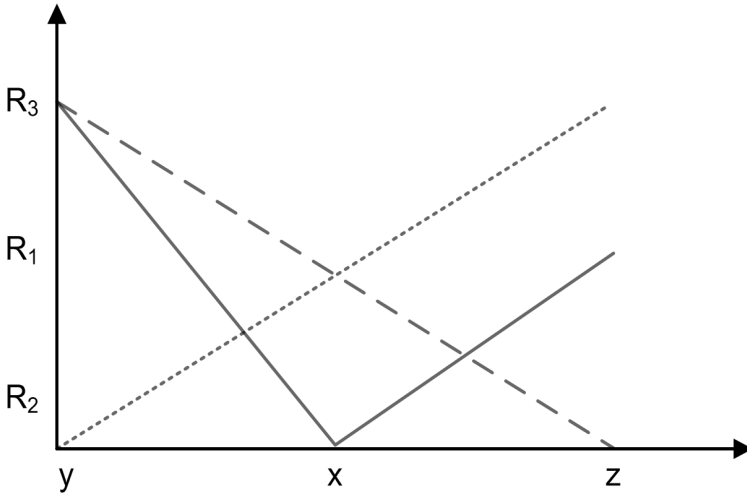


Fig. 4.2. A Non-Single Peaked Preference Profile

point from which his preferences decrease monotonically. This preference profile is subject to cycling.

When preference profiles are single-peaked, then it is possible to predict a stable outcome. Black’s (1958) median voter theorem states that if a preference profile is one-dimensional and single-peaked then the equilibrium is at the median point. Hence, the three political players in Figure 4.1 would choose alternative  $x$ ; and since alternative  $x$  is the only alternative in the maximal set, it is a Condorcet winner.

We discussed Downs’ (1957) use of Black’s (1958) median voter theorem in chapter one. The reader will recall that Downs’ model demonstrates that parties should position themselves as close as possible to the median elector in order to assure themselves of electoral victory.

There is an important limitation to Black’s median voter theorem. An equilibrium may not exist if a significant degree of indifference is permitted. In Figure 4.3 political actor three (preference order  $R_3$ ) is indifferent over a subset of alternatives on the same side of the ideal point. This can be problematic for one-dimensional crisp models. While no cycling will occur under standard majority rule, rules such as plurality can result in cycling. (Austen-Smith and Banks, 1999, pp. 94–99). The possibility of indifference at the ideal point, illustrated by the preference curve for political actor two (preference order  $R_2$ ) is not as problematic.

The conventional adoption of the Euclidean distance assumption permits one-dimensional models to avoid the likelihood of cycling resulting from indifference. The assumption contends that each player  $i$  assesses the utility,  $u_i$ , of a proposal  $x$  in reference to its ideal point  $x^*$ . The closer  $u_i$  is to zero, the closer the proposal is to the ideal point of player  $i$  and the more player  $i$  prefers the proposal. Essentially, the preferences of political actors decrease strictly and at identical

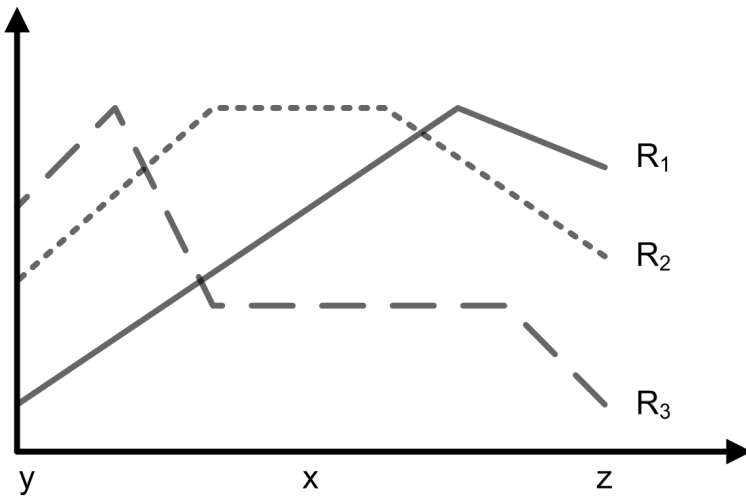


Fig. 4.3. Indifference in a Preference Profile

rates in both directions from the ideal points for all political players. No matter how small the change in the location of an alternative in relation to the ideal point, it must result in either greater or lesser preference on the part of the given political actor. If a player's ideal defense budget is \$200 billion, one dollar more or less from that ideal is less preferred. Furthermore, whatever the unit change in the alternative, it has no greater or lesser effect on the preference of any one political actor.

The Euclidean distance assumption affects the completeness of individual preference orders by placing restraints on the type of indifference that individuals may express over a set of alternatives. Most preferences involve some degree of indifference, which the Euclidean assumption does not in any way harm. For instance, actor one, whose preference curve ( $R_1$ ) is mapped in Figure 4.3, is indifferent over all alternatives equidistant from her ideal point in opposite directions. As she moves in both directions away from her ideal point, there are an infinite number of subsets uniting pairs of points over which she is indifferent. What the Euclidean distance assumption negates is the possibility of flat areas in the preference curves, such as those in  $R_2$  and  $R_3$  in Figure 4.3. (Political actors are indifferent over all alternatives in these areas.) While this restriction on indifference can sometimes be weakened, doing so may result in an empty maximal set.

With the restrictions placed on models by single-peakedness and the Euclidean distance assumption, political scientists map the ideal positions of all relevant political actors in one-dimensional space. However, in most instances what is being mapped are the ideal points of institutions, not individuals. This is all the more the case as The New Institutionalism argues that institutional design is a key, if not the key, factor in mitigating against cycling. Nonetheless, as just discussed, given the assumptions of modeling in one-dimensional space, institutions

often do not perform this role, rather the assumptions behind one-dimensional modeling do.

Institutions that are headed by a single player, such as the presidency, are easy to map. One need only consider the ideal policy of the position's occupant. At first glance, collective institutions, such as legislatures, committees, or courts would appear to be a great deal more difficult. However, if we assume that these institutions are located in one-dimensional issue space and that decisions will be made by majority rule by individuals with single-peaked preferences, then the collective position will be that of the median individual in the institution.

Institutional ideal points are mapped relative to one another as points along a line, or policy continuum, representing the full range of feasible outcomes in one-dimensional policy space. We define this policy continuum  $X \in \mathbf{R}$  where any point  $x$  can be chosen such that  $x$  represents a policy outcome over which an institutional actor  $i$  can hold preferences. This continuum can formally be defined as  $X = [0, \infty)$  or the interval bounded on the left at  $x = 0$  and unbounded on the right as  $x$  approaches infinity. The unbounded characteristic of the interval's right side remains purely theoretical; in reality, a stop point will be imposed based on the policy preferences of the right-most actor.

Let us suppose that an institution  $i$  has a preference order over several outcomes,  $X = \{x_1, x_2, x_3\}$ ,  $\rho_i = x_1 P_i x_2 P_i x_3$ , indicating that the (median individual in the) institution strictly prefers the first outcome,  $x_1$ , to the second,  $x_2$ , and the second outcome to the third,  $x_3$ . Let us further suppose that some alternative  $SQ$  is the current status quo policy. The set of alternatives most preferred to  $SQ$  by actor  $i$  can be defined as the maximal set in relation to  $SQ$ , or formally,

$$M_i(SQ) = \{x_j \in X | \forall x \in X, x_j P_i x\}. \quad (4.1)$$

$M_i(SQ)$  is the set of alternative policy proposals an institutional actor would accept in a vote against the current policy. In the one-dimensional case, an actor prefers any alternative located closer to her ideal point than the status quo. The status quo position constitutes a yardstick of sorts against which the political actors weigh any proposed legislation.

In a crisp one-dimensional model under the Euclidean preference assumption, any institution  $i$  derives more utility from any  $x_j$  located in  $M_i(SQ)$  to  $SQ$  or any point outside of  $M_i(SQ)$ . Additionally, any  $x_k$  in  $M_i(SQ)$  closer to  $i$ 's ideal point than  $x_j$  is more preferred. Figure 4.4 illustrates the point. Institutional actors whose ideal policy positions are located at  $x_1$ ,  $x_2$ , and  $x_3$  are plotted along a one-dimensional policy continuum. The status quo is located at point  $SQ$ . The  $y$ -axis depicts the expected utility in some utile measure over the interval  $[0, 1]$  with 1 representing maximal utility at the ideal point. The Euclidean preference ordering of the institution with its most preferred policy position at  $x_2$  is shown by the dashed curve with its peak (maximum utility at the ideal point) at  $x_2$ . Any point in the interval  $(a, c)$  is absolutely preferred by the institution at  $x_2$  to  $SQ$ . If offered any policy within this segment of the continuum,  $x_2$  will choose it over the status quo. Further, the closer any proposal is to  $x_2$ , the more preferred it is by the institution whose ideal policy is located at  $x_2$ .

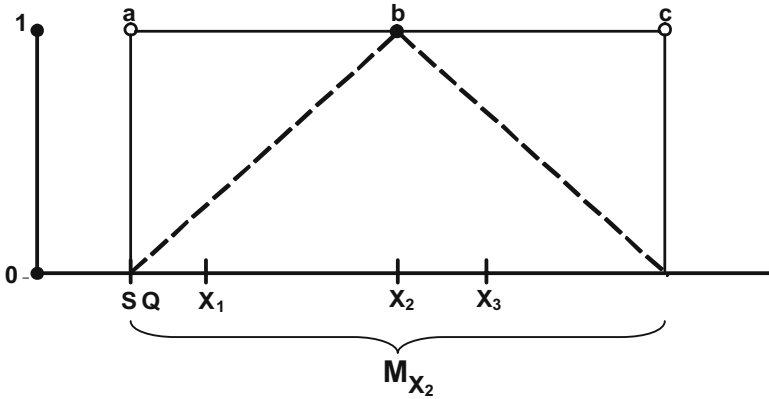


Fig. 4.4. Maximal Set of an Institutional Actor at  $x_2$ .

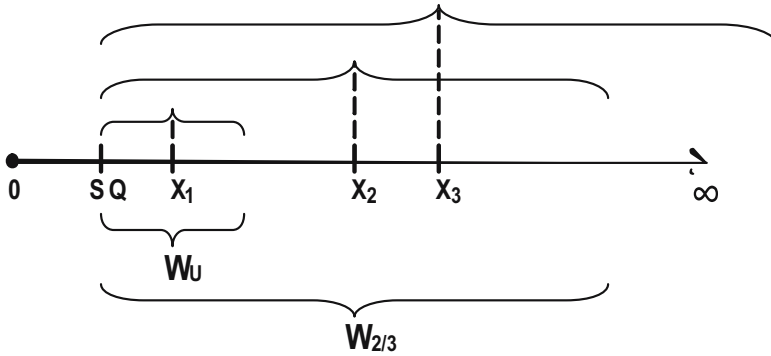


Fig. 4.5. Winsets of Political Actors with Ideal Points at  $x_i$

In most non-authoritarian systems, the final policy choice does not reside with a single institution, but rather is the consequence of choices between several institutions. Figure 4.5 depicts three institutions with ideal points at  $x_1$ ,  $x_2$ , and  $x_3$  along a one-dimensional policy continuum. The status quo is located at point  $SQ$ , and the maximal sets of the respective actors are depicted with brackets above the continuum. The maximal sets are  $2d$  in length, where  $d$  is the distance between an actor's ideal point and the status quo. Under the Euclidean distance assumption, each institutional actor is indifferent between any two points located at equal distances to either side of its ideal point; and each institution prefers to move the policy anywhere closer to its position within the outer limits of the bracketed space defining its maximal set.

Figure 4.5 further develops the concept of winsets, depicted by brackets below the continuum. The winset of a particular policy, in this case the status quo, is the set of alternatives that can defeat the policy. The existence, size, and location of a winset is partially a function of the rules under which the decision for or against an alternative is made by the institutional players

involved in the process. Formally, the winset of the status quo under majority rule is:

$$W_m(SQ) = \{(M_{x_1}(SQ) \cap M_{x_2}(SQ)) \cup (M_{x_2}(SQ) \cap M_{x_3}(SQ)) \cup (M_{x_1}(SQ) \cap M_{x_3}(SQ))\}. \quad (4.2)$$

Alternatively, the winset of the status quo under unanimity rules is:

$$W_u(SQ) = \{M_{x_1}(SQ) \cap M_{x_2}(SQ) \cap M_{x_3}(SQ)\}. \quad (4.3)$$

When the outcome is determined by unanimity, the policy position of the actor closest to the status quo is favored. Since every institution effectively exercises a veto over any change in the policy that is not more to its liking, the only change possible in Figure 4.5 is one that locates a new policy within the maximal set of the player at  $x_1$ . In contrast, under majority rule the outcome will be the alternative closest to  $x_2$ , the median, which defeats all other alternatives in a head-to-head vote.

In the section that follows, we will illustrate the utility of the concepts developed to this point with a one-dimensional model developed by Kiewiet and McCubbins (1988) that explains the conditions under which the United States President is more likely to veto a legislative appropriations bill. We will then fuzzify Kiewiet and McCubbins' model.

## 4.2 Modeling the Presidential Veto Game

Kiewiet and McCubbins (1988) model the veto game between the U.S. president and Congress. The latter is treated as a single chamber (one player with one ideal point) to simplify the analysis. The positions of the president ( $P$ ), the median Congressional legislator ( $x_2$ ), the one-thirds (left-edge) veto threshold legislator ( $x_1$ ), and the two-thirds (right-edge) veto threshold legislator ( $x_3$ ) are mapped relative to one another. The status quo, in this case, is represented by a continuing resolution, or the current year's allocation of funds at  $C$ . The funding level, or bill, that Congress proposes is labeled  $b$ . The game progresses as follows. Congress reaches a funding allocation level at the position of the median legislator,  $x_2$ , that is presented to the president. Depending on the proximity of his ideal point  $P$  relative to the status quo and that of the Congress, the president chooses to either sign the bill or veto it.

In the event of a veto, Congress must decide to either sustain or override the veto. A veto override requires a two-thirds majority. Therefore, the congressional decision depends upon the policy preferences of the legislators located at the one-third and two-thirds median. If the median Congressional proposal is more attractive than  $C$  to either of these two legislators, then the veto is overridden. If, however, both of these actors lay in closer proximity to  $C$  than  $b$ ,

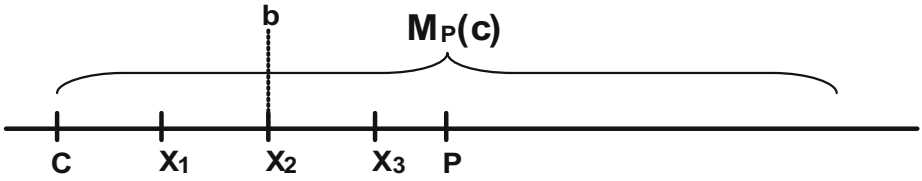


Fig. 4.6. The President at  $P$  with a Legislative Proposal at  $x_2$

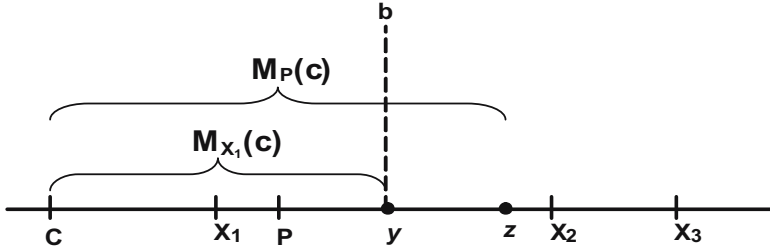


Fig. 4.7. The President at  $P$  with a Legislative Proposal at  $y$

then they will sustain the veto and the game ends in favor of the continuing resolution,  $C$ .

Figure 4.6 illustrates the model’s analysis of the presidential veto under one set of assumptions about the relative policy preferences of the key actors. In this case, the president is located further from the reversion point than the median legislator is. Congress proposes  $b$  at  $x_2$ , which is closer to  $P$  than  $C$  and well within the president’s maximal set. Accordingly, he will not veto and the game ends with his signing the new legislation.

Consider, however, the arrangement depicted at Figure 4.7. Here, the president is located closer to the reversion point than the median legislator,  $x_2$ , and the most-preferred set of the one-third legislator at  $x_1$  is largely consonant with that of the president, making his position much stronger. The most advantageous proposal Congress can make is at  $z$ , where the president is at least indifferent between this point and the continuing resolution. If the president vetoes, Congress will not be able to muster the necessary two-thirds to override the presidential veto, and the continuing resolution will prevail, because the one-third legislator (and every legislator to his left) prefers  $C$  to  $z$ . The same situation results for every policy on the interval from  $z$  to  $y$ . However, a congressional proposal at  $y$  renders the one-third median legislator indifferent between  $b$  and  $C$ . At this juncture, Congress can override a presidential veto, and thus the president must accept a Congressional bill at  $y$ , the extent of his ability to influence the outcome having ceased at this point. In effect,  $y$  constitutes the absolute minimum that the president and the one-third median legislator, as utility maximizers, are willing to accept. Congress could make winning proposals to the left of  $y$ , but it would not do so as they would be further removed from the median legislator’s ideal point.

### 4.3 The Case for a New Approach

The crisp approach in one-dimensional policy space can be critiqued on a number of grounds. To begin with, by placing an actor's ideal point at an exact location along a policy continuum, crisp logic imposes an extraordinarily restrictive assumption of the precision with which political phenomena can be measured that often does not obtain in the social world. While the number of parties in a legislature is subject to fairly precise measurement, whether that number represents a large party system or a small one is not. That is all the more the case concerning the policy preferences of individuals and collective institutions. Preferences – similarly to the evaluation of the size of the party system or whether or country is democratic – are marked by ambiguity and imprecision that defy exactness. The failure to recognize this ambiguity results in projecting a bipolarity onto the universe that is not tractable in many cases. The answer is either yes or no, a thing either is or isn't, an individual is either pro-life or pro-choice, a political orientation is either liberal or conservative.

The inability to deal with gradation, the middle ground, or shades of gray is compounded in many instances by the Euclidean distance assumption. Its adoption means that formal models essentially posit that political actors are satisfied with absolute gains of (quite literally) next to nothing. Actors are able to rank-order their preferences in such a way as to discern between infinitesimal differences. An actor knows he prefers \$5.02 to \$5.01 to \$5.00. While this is conceivable when considering the division of a few dollars, such fine tuned calibrations in deliberations over a multi-billion dollar budget are implausible. Further, actors evaluate proposals well removed from their ideal position in the same manner; any finite distance is important, and any infinitesimal gain is preferred. As a consequence, given the endless arrangements allowed by the mutability of preferences, it is the status quo policy that is central to the model and not the actors themselves.

Fuzzy set theory deals with these problems by moving away from the strictures of finite points and measurements and substituting ranges, intervals, areas, and other broad measurements in their place. In the standard crisp set approach to formal modeling, an actor's ideal policy comprises a single point. For example, a preferred budget is a set with one element, say  $\{\$1,500,482,299.04\}$ . In contrast, fuzzy set theory invites our attention to the likelihood that an actor exercising preferences over a multi-billion dollar continuum may find not just a single dollar amount, but rather a range of funding levels to be (at a minimum) of some utility and (maximally) of equal utility. These ideas often transfer well to the social sciences when the study of human thoughts and interactions are being analyzed. Such phenomena are vague and ambiguous by nature, and they lend themselves to a fuzzy approach. That is not to say that fuzzy set theory is less rigorous than the crisp approach, but rather that it permits the precise study of things that are inherently imprecise.

## 4.4 Toward a Fuzzy One-Dimensional Model

Fuzzy set theory makes no assumption about the existence of an exact point. Rather, it is concerned with the degree to which any given position approximates an ideal point. It does so by assigning membership values over the interval  $[0, 1]$  to elements in a set indicating their degree of membership in the set based on a given characteristic. Formally, a fuzzy subset  $A$  of a set  $X$  is a function of  $X$  into the closed interval  $[0, 1]$ . For  $\alpha \in [0, 1]$ , an alpha-cut ( $\alpha$ -cut) of  $A$ , written  $A^\alpha$ , is the set  $\{x \in X \mid A(x) \geq \alpha\}$ . That is, the  $\alpha$ -cut selects those elements of the domain whose membership grade is at least  $\alpha$ . The strong  $\alpha$ -cut of  $A$  comprises those elements that have a membership grade strictly greater than  $\alpha$ ,  $A^\alpha_\gt = \{x \mid A(x) > \alpha\}$ . The  $\alpha$ -cut at one ( $\alpha = 1$ ) is the core and the strong  $\alpha$ -cut at zero ( $\alpha = 0$ ) is the support of a fuzzy set  $A$ . If  $A(x) = \alpha \in [0, 1]$  we say that  $x$  has alpha-level ( $\alpha$ -level) membership in  $A$ .

Suppose that we wish to assign membership grades to countries based on the degree to which they are democratic. An alpha cut at  $\alpha = 1$  would include all elements that are perfectly included in the set (countries that are unequivocally democratic). Descending lower into the fuzzy number, an alpha cut at  $\alpha = 0.75$  would also include all of those elements “more in the set than out” (countries that are more democratic than not). The alpha-cut at 0.50 would also include those elements that are neither more in nor out of the set (countries that are no more democratic than not democratic), the alpha-cut at 0.25 would also indicate those sets of elements that are “more out of the set than in” (countries that are less democratic than non-democratic), and an alpha cut at  $\alpha = 0.00$  would also include all elements outside of the set (all non-democracies).

Formal models in comparative politics map the preferences of political actors. Hence, we must assign membership values to those preferences. There are two approaches to doing so, the most common of which is to assign fuzzy values to collective preference relations (Bezdek et al., 1978, 1979; Kacprzyk et al., 1992; Orlovsky, 1978). Essentially, membership values indicate the degree to which a set of individuals prefers one alternative over another. It is not clear how these collective preferences would be determined apart from individual preferences. Furthermore, formal models in political science are premised on the notion of methodological individualism: “the individual is the basic unit of analysis” (Austen-Smith and Banks, 1999, pp. 94–95). (Shepsle and Bonchek, 1997, p. 19). Therefore, we follow the lead of Nurmi (1981b) and use fuzzy individual preferences from which we “aggregate collective preferences” (Nurmi and Kacprzyk, 2007). Membership grades can be continuous or discrete. In this book, we will most often use a set of discrete grades categorized in the following way. We map the preferences of political actors by assigning membership grades in the set of excellent (most preferred) policies. “Excellent policies” would be given a membership grade of 1.00 ( $\alpha = 1.00$ ), “very good” policies would have a membership grade of 0.75 ( $\alpha = 0.75$ ), “good policies” would have a membership grade of 0.50 ( $\alpha = 0.50$ ), “poor policies” would have a membership grade of 0.25 ( $\alpha = 0.25$ ), and unacceptable policies would have a membership grade of 0 ( $\alpha = 0.00$ ). It is possible that an actor may well consider a wide range of policy options to be

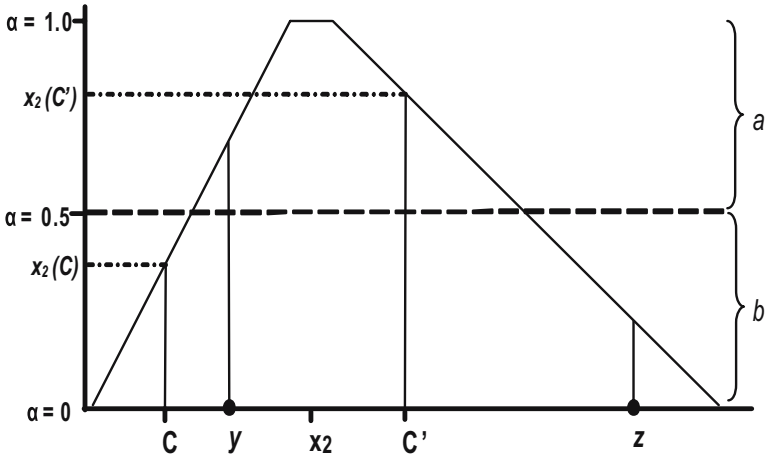


Fig. 4.8. A Fuzzy Number,  $x_2$ , and  $\alpha$ -levels

included in each of the categories. As a consequence, instead of mapping an actor’s ideal positions as a single, exact point in the policy space, fuzzy set theory permits us to consider an interval over which an actor is indifferent to alternatives. We do so by using the concept of a fuzzy number.

In crisp logic, a number either is or is not equal to 6. In contrast, in fuzzy set theory, 5 has a relatively high degree of membership in the set of numbers that are 6. While 5 is not equal to 6, it is pretty close. On the other hand, 10 is not very close, and 100 is not 6. Hence, the membership grade for about 6 for each number in the set  $\{5, 10, 100\}$  is  $\{0.75, 0.25, 0.00\}$ . The membership grades are the  $\alpha$ -levels for the number 6. If we were to assign membership grades for 6 along the entire continuum of real numbers, we would be able to map the fuzzy number 6. In synonymous fashion, we can use a fuzzy number to represent membership grades in such concepts as democracy, power, autonomy, or preferences over a set of alternatives. We simply treat the concept as a fuzzy number, assign membership grades for each value, and plot the number.

Figure 4.8 illustrates a fuzzy number,  $x_2$ . The shape of the fuzzy number may take many forms. A crisp number would be represented by a single vertical line at the ideal point with a height ( $\alpha$ -level) of 1. No other values to the left or right of the ideal point would be included in the set. In contrast to a crisp number,  $x_2$  in Figure 4.8 is not located at a single point, rather it spans almost the full range along the  $x$ -axis shown. The  $y$ -axis represents the  $\alpha$ -level for a given value of  $x$ , that is the degree to which a value along the  $x$ -axis is included in the set belonging to  $x_2$ . The core of  $x_2$  is the area of maximal set inclusion ( $\alpha = 1$ ). The core of a fuzzy number need not comprise a single point. The entire fuzzy number includes all elements at  $\alpha > 0$  (some degree of inclusion in the set). This interval is referred to as the support. The sloping lines from the core of the fuzzy number  $x_2$  that extend out to the support form the bounds of values included in each  $\alpha$ -cut.

We can use fuzzy numbers to represent the degree of membership in the set of alternatives that comprise an “ideal” outcome. In such a case, an actor is completely indifferent across all alternatives within the core (alternatives with an  $\alpha$ -level of 1 being the set of excellent policies, and alternatives with an  $\alpha$ -level at 0 being all policies that are not excellent to any degree). This conforms to political reality, where most actors are willing to compromise in small amounts, move a few dollars away from their ideal allocation, or acknowledge that at a certain minute level, subtle variation does not change the outcome substantially. The satisfaction that the actor has with positions outside of the core corresponds to lower levels of set inclusion ( $\alpha$ -levels). The lower the  $\alpha$ -level, the less ideal the position is. An  $\alpha$ -level of 0.75 would indicate that a policy held a 0.75 degree of membership in the set of excellent policies. Consider the proposal at  $C$ . The level of set inclusion is  $x_2(C) \approx 0.40$ . Now suppose that a counter proposal is made at  $C'$ . The new proposal is more acceptable at an alpha level  $x_2(C') \approx 0.70$ , indicating a higher utility to the political player.

While the size of alpha-cuts generally increases at lower levels, the rate of increase need not be uniform. Indeed, as Figure 4.8 illustrates, the rate of increase is not symmetrical in both directions from the core. In effect, a fuzzy approach necessarily jettisons the Euclidean distance assumption. As a consequence, proximity is no longer the primary determinant in the model’s predictions, but rather the slopes of the fuzzy numbers and the relative membership levels they represent. The analysis becomes much more than simply locating a proposal in relation to a given actor’s ideal point. The model returns the actor herself to center stage by focusing on the “shape” of the fuzzy number defining her preferences, to include the size of the support and core and their relative positions to one another.

Let us consider a political actor whose policy preferences are mapped by Figure 4.8. The core is located in a region centered at  $x_2$ . The dashed line perpendicular to the  $y$ -axis at  $\alpha = 0.5$  divides the fuzzy number into two zones:  $a = \alpha > 0.5$  and  $b = \alpha < 0.5$ . In the standard crisp approach assuming Euclidean distance, the slope of the lines on either side of  $x_2$  would be symmetric, and points equidistant in both directions from the core would be equally preferred. That is not the case for the fuzzy number at Figure 4.8, nor is it the case for most fuzzy numbers. The alternative at position  $z$  is quite a bit further removed from the core than is  $y$ . In the crisp approach this would be enough to conclude that  $z$  is preferred to  $y$ . While in fact  $z$  is also preferred to  $y$ , this conclusion can only be reached after determining that  $z$  lies at an  $\alpha$ -level lower than that for  $y$ . Now consider  $y$  and  $C'$ . While  $y$  and  $C'$  are equidistant from  $x_2$ ,  $C'$  is at a higher  $\alpha$ -level and therefore preferred to  $y$ . In the crisp model, the political actor whose preferences are mapped at Figure 4.8 would be indifferent in a choice between  $y$  and  $C'$ .

The shape of the fuzzy number is an indication that a political actor is more sensitive to change in one direction than another. In Figure 4.8, the political actor is less willing to make concessions over the interval  $[C, x_2]$  than she is over  $[x_2, z]$ .

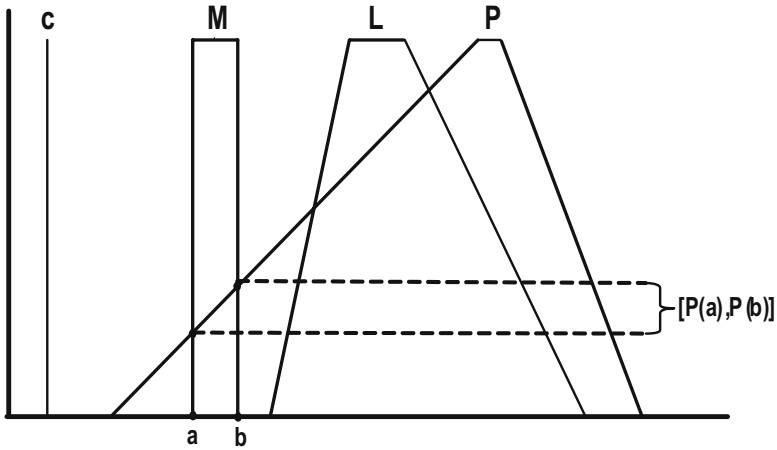


Fig. 4.9. The President Accepts a Proposal at  $M$

At  $\alpha = 0.5$ , the actor’s acceptable set of positions extends further to the right of the core (in the direction of  $z$ ) than it does to the left (in direction of  $y$ ).

There is yet another reason that the shape of the fuzzy number is important. In the standard crisp approach, preferences extend out to near infinity in both directions (a consequence of the Euclidean distance assumption). A fuzzy approach recognizes that there is an infinite number of alternatives outside of the support ( $\alpha = 0$ ). This has the appealing quality of permitting us to model regions of alternatives that once reached are simply unacceptable to a player. Being closer to the player’s ideal point does not in-and-of-itself make the alternative more attractive. Hence, if a person finds a defense budget at greater than 3% of GDP not in anyway appealing, she will not see any difference for a proposal of 4% or 4.3%. Both are equally repulsive.

In contrast to its crisp counterpart, a fuzzy approach reveals a deeper level of information about a political actor’s preferences. Fuzzy set theory returns the focus of the model back to the political actors. It reveals better and more detailed information about these actors’ preferences and in so doing potentially offers us a method more capable of furnishing tractable explanations of the political world.

### 4.5 Fuzzifying Kiewiet and McCubbin’s Presidential Veto Model

We now turn to the task of demonstrating a fuzzy set theory approach in one-dimensional modeling. We do so by fuzzifying Kiewiet and McCubbins’ (1988) presidential veto model presented earlier in this chapter. As before, we plot the policy preferences of three institutional actors along a one-dimensional policy spectrum ( $x$ -axis): the President ( $P$ ), the median legislator (here  $M$ ), and the relevant two-thirds median legislator, represented by  $L$ . This time, however,

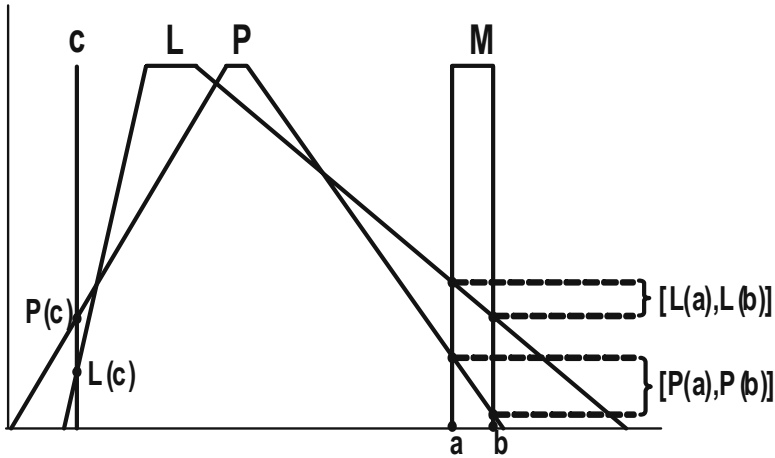


Fig. 4.10. A Presidential Veto is Over-ridden by *L*

the ideal points, represented as points in crisp analysis, are replaced by fuzzy numbers. The slopes of *M* are not included, as the median legislator has proposal power and will therefore put forth a policy somewhere in the set of his best options. Therefore, *M* is represented simply by its core, the interval  $[a, b]$  where  $a$  is the left edge at  $\alpha = 1$  and  $b$  is the right edge at  $\alpha = 1$ . The  $y$ -axis represents the respective alpha-levels; and we retain  $c$  as the reversion point.

The analysis hinges on a comparison of where  $c$  appears in an actor’s fuzzy set relative to where the range of legislative proposals appears in the set. *M* can make a proposal anywhere in the interval  $[a, b]$  as all points are of equal value to him. For each actor,  $c$  can appear in the set above  $[a, b]$ , below it, or somewhere within the interval itself. It is important to note that changing the slopes (“shape”) of *P* and *L* yields different points of intersection along  $c$  and  $[a, b]$  and, thus, different levels of membership in the respective sets.

The president first evaluates the relationship between  $c$  and  $[a, b]$  in his set and makes a decision. If he accepts, the proposal becomes law. If he vetoes, *L* is determinant in sustaining or overriding the veto. As we discussed previously, a mere consideration of relative placement along the policy space is no longer sufficient to determine the outcome of the game. Players located in close proximity to a policy may find it less preferable than a point located quite distant to them.

The arrangement at Figure 4.9 results in the president accepting the proposal of the median legislator at *M*. When *M* is located between  $c$  and *P*, a monotonically decreasing slope of a fuzzy number makes it impossible for *P* to find  $c$  at a higher alpha level than any proposal *M* might introduce. There are, however, some nuances here. *M* now has an interval in which he can make winning proposals. In rare cases where  $c$  falls between  $a$  and  $b$  for the president, the median legislator’s choice of proposal becomes crucial. This phenomenon becomes more likely as *M*’s (core)  $\alpha = 1$  becomes larger, thus increasing the probability that  $c$  will fall somewhere in this interval. As depicted in Figure 4.9, however,

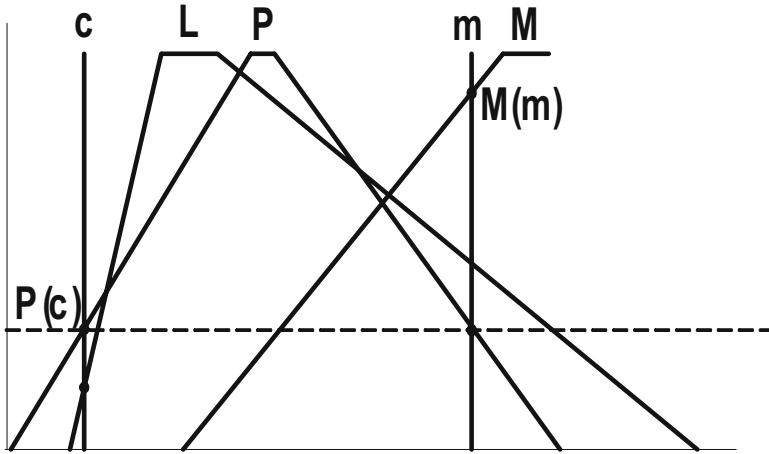


Fig. 4.11. A Winning Proposal from  $M$

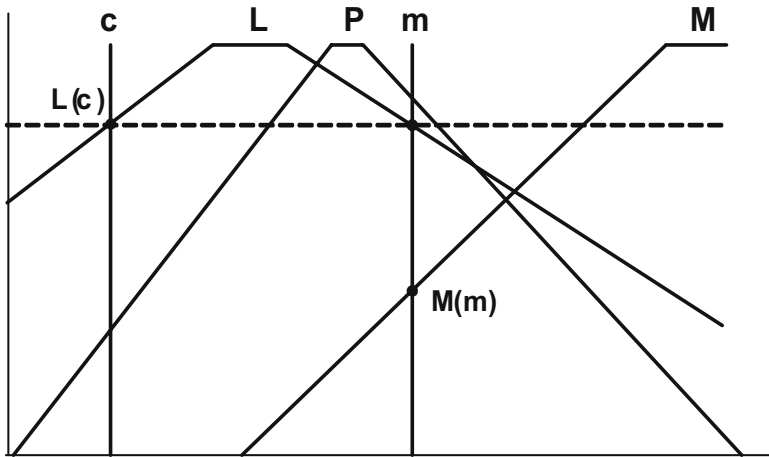


Fig. 4.12. The Effect of Changing the Shape of  $L$ 's Fuzzy Number

the median legislator ( $M$ ) has free reign to locate the policy anywhere in his  $\alpha = 1$ .

Figure 4.10 depicts a veto over-ride that differs substantially from that of the crisp model. Here,  $P(c) > [P(a), P(b)]$ , the president derives greater benefit from  $c$ , which is at a higher alpha-level than anything  $M$  can propose in the interval  $[a, b]$ . He accordingly vetoes any legislative proposal in  $[a, b]$ . However, the one-third legislator ( $L$ ), while being closer to  $c$  nonetheless prefers any proposal from  $M$  in the interval  $[a, b]$  to the reversion point. Thus Congress can muster an override, and  $M$ 's proposal results.

Where would  $M$  have to propose to avoid a presidential veto? The answer to that question is  $M$ 's proposal must be at least as good as or better than  $P(c)$ .

Figure 4.11 depicts a point  $m$  (or the left of  $m$ ) where a proposal made by  $M$  will be accepted by the president. However, the slope of  $M$  demonstrates that it loses utility with  $M(m)$ .

Should  $L$  prefer  $c$  at a much higher alpha-level than before, as illustrated at Figure 4.12, it will be more willing to sustain a presidential veto of a proposal from  $M$  in the interval  $[a, b]$ . As a consequence, the minimum winning point for  $M$ ,  $m$ , now shifts dramatically to the left as the president will veto (and Congress will not be able to over-ride) any proposal located to the right of  $m$ . In order to make a proposal that  $L$  will find at least as good as  $c$ ,  $M$  must give up more of its utility as the point  $m$  appears at a much lower alpha-level. As long as the alpha-level remains higher than that for  $c$ , however,  $m$  remains a viable option.

In the fuzzy model, spatial arrangement and distance are now of secondary importance to membership grades of any two points. As it turns out, fuzzy set theory permits us to compare membership, thereby reducing the entire fuzzy model to six data points ( $P(a)$ ,  $P(b)$ ,  $P(c)$ ,  $L(a)$ ,  $L(b)$ , and  $L(c)$ ), that can be compared on the basis of the five inequalities in Table 4.1. These data points along with the table of inequalities predict the outcome of the presidential veto given the relationship between the fuzzy numbers of the president ( $P$ ), the median legislator ( $M$ ), and the relevant one-third or two-thirds legislator ( $L$ ). We discuss them in the order they are described in Table 4.1.

Scenario I. It is possible that neither  $C$  nor  $[a, b]$  appear at  $\alpha = 0$  for  $P$ . In this case,  $P$  exercises no preferences over either, as both lack membership grades in the set. The model can make no prediction as to the president's decision and thus no result obtains.

Scenario II. Whenever  $C$  is less preferred than the legislative proposal, the president accepts and the new policy is located somewhere in  $[a, b]$ .

Scenario III. Whenever  $C$  appears in a higher alpha-cut for  $P$  than any value in  $[a, b]$  the president vetoes and the analysis then is determined on the basis of the one-third legislator's response. In such a case, four outcomes emerge from any of the five scenarios displayed at Table 4.1. An action is said to be "ambiguous" when  $C$  appears in the interval  $[a, b]$  for either actor. As it is unknown where the proposal will actually be located in this interval, the response of the legislator and the prediction of the model are also unknown.

Scenario IV. Scenario IV considers situations in which  $C$  is located between points  $a$  and  $b$  within the fuzzy set of  $P$ . Similar to previous scenarios, it is unknown where the exact proposal will emerge, so the outcome is unknown.

Scenario V. Scenario V portrays the same basic arrangement and identical outcomes, but reorders  $a$  and  $b$  to allow for a spatial arrangement that would put  $P$  and  $L$  to the right of  $M$ . Representing the potential  $M$  proposal as a range of options might force upon  $P$  too abstract a choice between  $c$  and  $M$ . In actuality, he will never have to weigh  $c$  (a point) against an  $M$  proposal (an interval). Indeed, he is not required to cope with  $M$ 's area of consideration so much as it is an actual proposal point originating somewhere within  $[a, b]$ . Once the median legislator chooses an actual point,  $P$  can evaluate accordingly and rarely will he find the two proposals to be exactly equivalent.

Table 4.1. A Model with Unknown Legislative Proposal M

	P Preferences	L Preferences	P Action	L Action	Result
<b>Group I</b>					
	If $P(c) = (P(a), P(b)) = 0$		then	No prediction	No result
<b>Group II</b>					
	If $P(c) < (P(a), P(b))$		then	Accepts	In core(M)
<b>Group III</b>					
	If $P(c) > (P(a), P(b))$	and $L(c) > (L(a), L(b))$	then	Vetoes	Sustains
	If $P(c) > (P(a), P(b))$	and $L(c) < (L(a), L(b))$	then	Vetoes	Overrides
	If $P(c) > (P(a), P(b))$	and $L(a) < L(c) < L(b)$	then	Vetoes	Ambiguous
	If $P(c) > (P(a), P(b))$	and $L(b) < L(c) < L(a)$	then	Vetoes	Ambiguous
	If $P(c) > (P(a), P(b))$	and $L(c) = (L(a), L(b)) = 0$	then	Vetoes	No prediction
<b>Group IV</b>					
	If $P(a) < P(c) < P(b)$	and $L(c) > (L(a), L(b))$	then	Ambiguous	Unknown
	If $P(a) < P(c) < P(b)$	and $L(c) < (L(a), L(b))$	then	Ambiguous	Unknown
	If $P(a) < P(c) < P(b)$	and $L(a) < L(c) < L(b)$	then	Ambiguous	Ambiguous
	If $P(a) < P(c) < P(b)$	and $L(b) < L(c) < L(a)$	then	Ambiguous	Ambiguous
	If $P(a) < P(c) < P(b)$	and $L(c) = (L(a), L(b)) = 0$	then	Ambiguous	No prediction
<b>Group V</b>					
	If $P(b) < P(c) < P(a)$	and $L(c) > (L(a), L(b))$	then	Ambiguous	Unknown
	If $P(b) < P(c) < P(a)$	and $L(c) < (L(a), L(b))$	then	Ambiguous	Unknown
	If $P(b) < P(c) < P(a)$	and $L(a) < L(c) < L(b)$	then	Ambiguous	Ambiguous
	If $P(b) < P(c) < P(a)$	and $L(b) < L(c) < L(a)$	then	Ambiguous	Ambiguous
	If $P(b) < P(c) < P(a)$	and $L(c) = (L(a), L(b)) = 0$	then	Ambiguous	No prediction

Table 4.2. A Model with Known Legislative Proposal M

	P Preferences	L Preferences		P Action	L Action	Result
<b>Group I</b>						
	If $P(c) = P(m)$		then	No Prediction		No Result
<b>Group II</b>						
	If $P(c) > P(m)$	and $L(c) > L(m)$	then	Vetoes	Sustains	C
	If $P(c) > P(m)$	and $L(c) < L(m)$	then	Vetoes	Overrides	M
	If $P(c) > P(m)$	and $L(c) = L(m)$	then	Vetoes	No Prediction	No Result
<b>Group III</b>						
	If $P(c) < P(m)$	and	then	Accepts		M

Table 4.2 constructs the model once we know the actual legislative proposal  $M$ . The table complements Table 4.1. In some cases Table 4.1 is sufficient to predict the outcome of the presidential veto game. When it is not, and ambiguity results, once  $M$  is known, Table 4.2 permits a prediction. Hence, between them, Tables 4.1 and 4.2 present a complete predictive model.

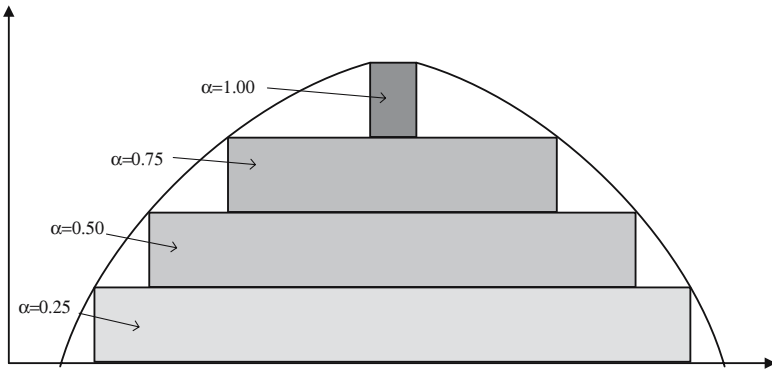
## 4.6 Discrete Fuzzy Numbers

We began the previous section by criticizing the standard crisp set approach to formal modeling in comparative politics for the exactitude it imposes on political actors' ideal points as well as the fine-tuned distinctions that it requires between options that are extraordinarily close to one another. The careful reader will have noted, however, that our fuzzy set approach dealt with only the former, leaving the latter relatively intact. While moving away from the rigid symmetry of the Euclidean distance assumption (which requires that actors are indifferent over alternatives equidistant from the ideal in both directions), the asymmetric smooth slopes of the fuzzy number still assume that political actors are capable of making fine tuned distinctions between a .750 and a .751 level of set inclusion. We now develop the notion of a discrete fuzzy number that permits us to jettison the requirement that political actors make fine tuned distinctions between alternatives at any level of set inclusion, not just at  $\alpha = 1$ .

In this section we are using *discrete*, *continuous*, and *granularity* in reference to the image of a fuzzy set. Here, "discrete fuzzy set" means a fuzzy set whose image is a discrete set. A "continuous fuzzy set" means a fuzzy set whose image is a continuum.

Figure 4.13 illustrates a fuzzy point in one dimensional space. Were political actors able to precisely determine preferences among options, the smooth curve within which several levels of rectangular boxes are fit would define the actor's preferences, with the  $x$ -axis representing a policy position and the  $y$ -axis representing the degree to which the actor prefers the given policy. The rectangular boxes stacked in stair step fashion represent a discrete fuzzy number, which is able to incorporate indifference resulting from ambiguity over preferences. Each box represents an  $\alpha$ -cut. The rectangular box at the top of the stair step fuzzy number in Figure 4.13 is the core. Those in between the core and the  $x$ -axis are  $\alpha$ -cuts at 0.75, 0.50, and 0.25 (descending from the core to the support). All domain points within the core are equally preferred to one another. The same is not true for the other  $\alpha$ -cuts; only positions encompassed by the rectangular box for a given  $\alpha$ -cut that are not included in those at a higher  $\alpha$ -cut are equally preferred at that level. In fuzzy geometry, the core is a fuzzy point. Alternatively, we might call this a fuzzy point at  $\alpha = 1$ . Fuzzy points are also located at the  $\alpha$ -cuts of 0.75, 0.50, 0.25, and 0. Each of these fuzzy points no longer defines a single, exact position, but rather an interval.

There is no fundamental difference between a discrete and a continuous fuzzy number. A continuous fuzzy number is simply a discrete fuzzy number with an infinitely high level of granularity. Based on the degree of granularity that we



**Fig. 4.13.** A Discrete Fuzzy Number

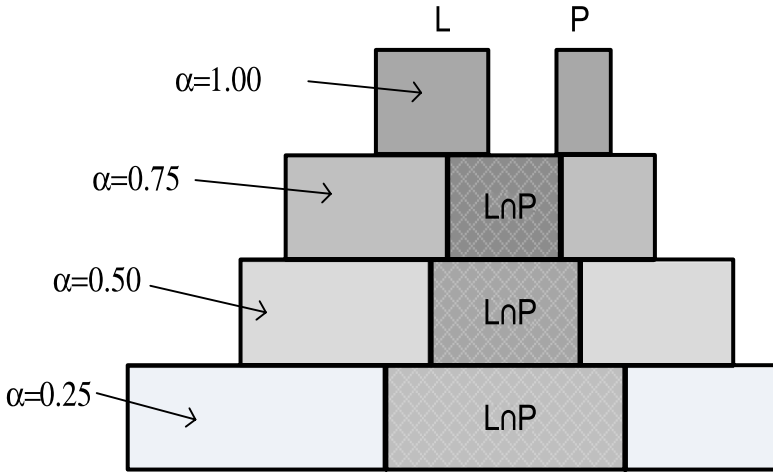
desire, we can construct a discrete fuzzy number,  $A$ , from a continuous fuzzy number,  $B$ . Increasing the granularity of a fuzzy number indicates that the actor has a more nuanced or informed set of preferences over an area.

The following algorithm would create a discrete fuzzy number with five preference levels. The resulting discrete fuzzy number will fit within the boundaries of the corresponding continuous fuzzy number. We thereby avoid committing the error of stipulating more information over the  $x$ -axis than that which we actually possess.

$$\mu_A = \begin{cases} 0.00 & \text{if } \mu_B < 0.25 \\ 0.25 & \text{if } 0.25 \leq \mu_B < 0.50 \\ 0.50 & \text{if } 0.50 \leq \mu_B < 0.75 \\ 0.75 & \text{if } 0.75 \leq \mu_B < 1.00 \\ 1.00 & \text{if } \mu_B = 1.00. \end{cases} \quad (4.4)$$

Similarly to continuous fuzzy numbers, we can use the actual shape of a fuzzy point at different  $\alpha$ -levels to predict that contests between political actors will be resolved at the intersection of two fuzzy numbers at the highest level of set inclusion ( $\alpha$ -level). Figure 4.14 demonstrates an area of overlap between the fuzzy set representation of the policy preference of two political actors,  $L$  and  $P$ . Their preferences do not intersect at  $\alpha = 1$ . However, they do intersect at  $\alpha = .75$ ,  $\alpha = .50$ , and  $\alpha = .25$ . Since the highest level at which the two agree is  $\alpha = .75$ , they will agree on a policy in the range outlined by the intersection at that level. This is the area in which they both get a policy at the highest level of utility (expressed as the level of set inclusion in the set of “excellent” policies).

The table of inequalities (Tables 4.1 and 4.2) for the analysis of the one-dimensional game still holds when using discrete fuzzy numbers. Because the analysis of the one dimensional game is based on six  $\alpha$ -levels, we simply find the  $\alpha$ -levels at points  $P(a)$ ,  $P(b)$ ,  $P(c)$ ,  $L(a)$ ,  $L(b)$ , and  $L(c)$  and compare values in the tables. It does not matter what the shape or arrangement of the fuzzy



**Fig. 4.14.** The Intersection of Two Discrete Fuzzy Numbers

numbers are, it simply matters what the  $\alpha$ -levels are at the six points. With the discrete preferences, there could be an instance where the actor has the same preference level over her three points but it does not matter in the final analysis. The table will give the predicted result without respect for the type of fuzzy number as long as the six points can be identified.

A major benefit of discrete fuzzy numbers is the degree of control they afford in representing the granularity of actors' preferences. Take, for instance, the average U.S. citizen who is randomly selected to answer questions concerning his political position on a set of specific policy issues. He likely possesses little more than a vague notion of exactly where he stands on each issue. Consequently, his responses comprise general impressions and vague ideas. In such cases, a Likert-scale, making use of five discrete response categories is best suited to mapping the respondent's preferences since he himself is unable to furnish us with much information beyond "I agree strongly" or "I neither agree nor disagree" as he is unable to discern between more nuanced positions. Clearly a continuous fuzzy number would be a gross overstatement of his ability to identify his preferences. We cannot fault him for his ignorance, but we also cannot model his preferences with a finer grain than that which he himself possesses. Contrast this with the professional politician who has dedicated decades to studying public issues. If put to the task of assigning values to any number of policy proposals in reference to her own ideal, a five-point Likert-scale fails to offer her the level of fine tuned discernment she undoubtedly possesses. For this scenario, the researcher would be wise to present her with some scoring mechanism that accounts for higher granularity of preference. The comparison of these two individuals reveals the power of the discrete approach. Allowing the research design to closely model the granularity characteristic of the fuzzy preference relation moves formal models away from unrealistic assumptions.

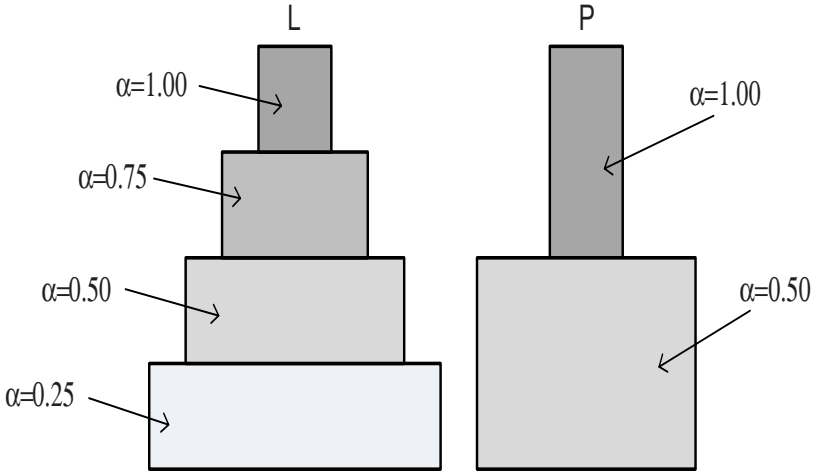


Fig. 4.15. Two Players with Different Granularity

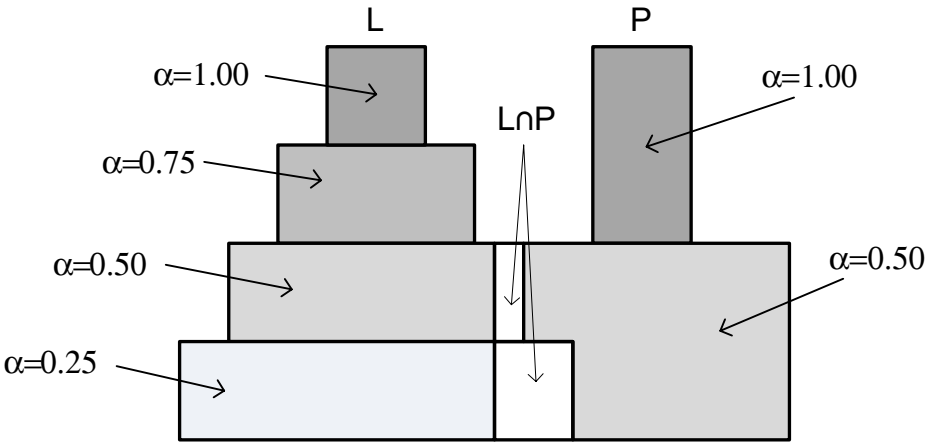


Fig. 4.16. Consensus between Legislators  $L$  and  $P$

More importantly, differences in granularity can have important implications for the predictions made by models. Consider two actors,  $L$  with alpha-levels at .25, .50, .75, and 1 and  $P$  with alpha-levels at .50 and 1 as depicted in Figure 4.15. The first actor's preferences are more granular, or finely tuned, as he is more sensitive to policy shifts over the same continuum. The latter's are less granular.

Suppose that  $L$  and  $P$  are two legislators and that Figure 4.15 represents their policy preferences on defense. Furthermore, suppose that legislator  $L$  represents a district with a major military installation and that legislator  $P$  is an agricultural specialist from a rural state with very little experience on defense issues. Figure

4.15 demonstrates that  $L$ 's preferences jump two levels (i.e. garnering him greater payoff) over a given interval for legislator  $P$ . This can work to the advantage of the more acutely aware actor, legislator  $L$ , as seen in Figure 4.16. Notice that the two legislators agree at  $\alpha = .50$  for  $P$  and either  $\alpha = .25$  or  $\alpha = .50$  for  $L$ . In this situation, legislator  $L$  can obtain the concurrence of legislator  $P$  to move the policy closer to himself and obtain the higher relative payoff at  $\alpha = .50$  since the move does not in any way reduce  $P$ 's payoff. Furthermore, if  $P$ 's discrete fuzzy number were moved just slightly further to the left in Figure 4.16, then  $L$  would be able to enact a policy receiving a membership grade of .75. In essence, legislator  $L$  can tweak the policy to his liking without altering the payoff to the less discerning legislator.

## 4.7 A One-Dimensional Model with Discrete Fuzzy Numbers

We illustrate the use of discrete fuzzy numbers in Kiewit and McCubbins' presidential model and compare the results achieved with those from the continuous fuzzy number model. Figure 4.17 reproduces the situation (in Figure 4.9) in which the president,  $P$ , accepts the legislative proposal (the policy preference of the median legislator at  $M$ ). The core of  $M$ 's fuzzy number is as shown. The preferences of the president ( $P$ ) and one-third majority ( $L$ ) are now represented by discrete fuzzy numbers with alpha-levels at  $\alpha = \{1, .75, .50, .25, 0\}$ .  $L$  is depicted by the small dots, and  $P$  is outlined by the dark dashes. The intersections at the players' highest alpha-levels are in the dark shaded areas. In the continuous fuzzy number model, the preferences of the one-third legislator,  $L$ , and  $P$  intersect at a unique point. In the discrete fuzzy number model, they intersect across a continuum of policies at  $\alpha = .75$ , illustrating that modeling political interaction with discrete fuzzy numbers (especially those with coarse granularity) results in more ambiguity among and between actors. Nonetheless, the result does not change:  $P$  accepts the legislative proposal at  $M$  (a proposal opposed by the one-third legislator,  $L$  [at  $\alpha = 0$ ], whose opposition has been rendered irrelevant since the president has not exercised the veto).

It is not always the case that the results of an analysis using continuous (fine granularity) or discrete (coarse granularity) numbers will be the same. In fact, depending on the difference in the degree of granularity between the two, the results can change. Take for instance, the situation illustrated in Figure 4.10, in which a model using continuous fuzzy numbers predicts that a presidential veto will be over-ridden by a legislative majority. The slope of the continuous number does not tell us the shape of the discrete fuzzy numbers (in particular, the cutoffs for each alpha-cut). Suppose that they are those shown in Figure 4.18.  $P$  does not intersect with the median legislator,  $M$ , but does intersect with the status quo,  $C$ . Furthermore, the one-third legislator ( $L$ ) intersects with  $M$ , but not  $C$ . In this case, the model predicts that a presidential veto will be over-ridden. Hence, the prediction is the same as that for the continuous fuzzy number.

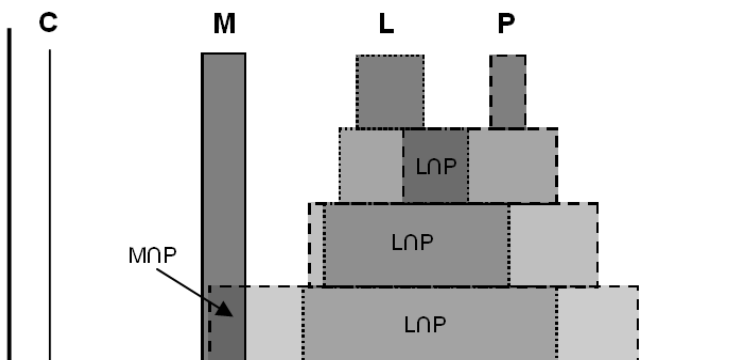


Fig. 4.17. *P* Accepts a Proposal at *M*, Discrete Fuzzy Numbers

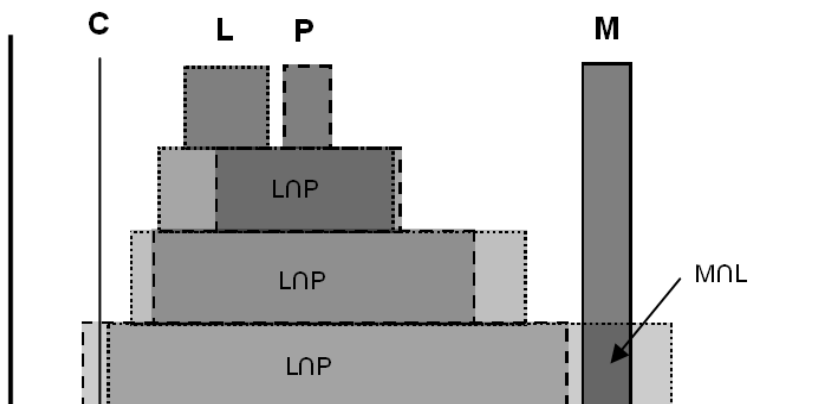
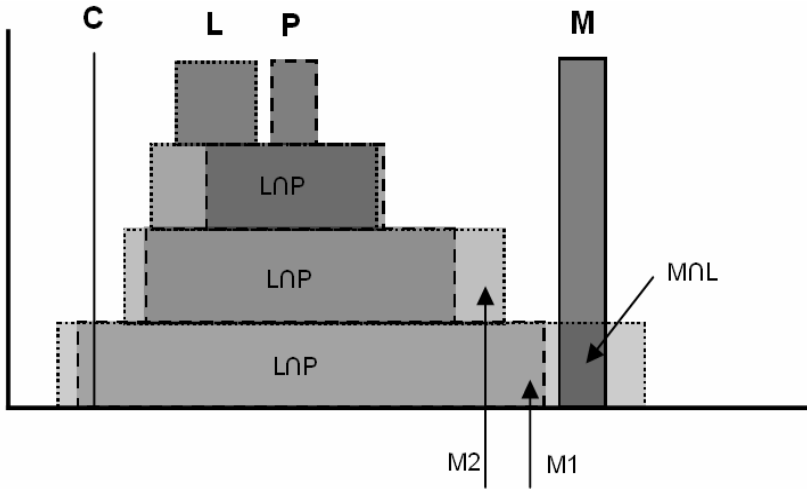


Fig. 4.18. *L* Over-rides a Presidential Veto, Discrete Fuzzy Numbers

Suppose, however, that *L*'s discrete fuzzy number does intersect *C* at the  $\alpha = .25$  level, as depicted in Figure 4.19. This is entirely plausible given the shape of the continuous fuzzy number in Figure 4.10. (Whether it does so for a discrete fuzzy number depends entirely on the choice of granularity.) In this case, a presidential veto would be sustained as the one-third median legislator (*L*) would be indifferent between the status quo and a legislative proposal (*M*). Depending on the shape of the median legislator's discrete fuzzy number (representing his preferences), *M* might choose to make a proposal outside of its core in the lowest level of  $L \cap P$  (for instance, at  $M_1$ ). *P* would sign the bill into law.

What is equally interesting is that the one-third legislator might be able to work with the legislative majority to achieve a proposal at  $M_2$  that, depending on the shape of *M*'s discrete fuzzy number, might be of equal preference to  $M_1$ . Such a proposal would be no more preferred by *P* to  $M_1$ , but it would move *L* from  $\alpha = .25$  to  $\alpha = .50$ . In the continuous fuzzy number analysis, both *L* and *P*'s slopes increase monotonically from right to left over this range. Accordingly,



**Fig. 4.19.** A Presidential Veto Is Sustained, Discrete Fuzzy Numbers

a gain for one is a gain for both. In the discrete case, however, it is possible for *L* to gain with no gain accruing to *P*. Furthermore, *M*'s collusion with *L* would have rendered the president's veto useless.

### 4.8 Comparing Results: Fuzzy Advantages

In this chapter we applied the basic principles of fuzzy set theory and fuzzy geometry to one-dimensional models. We began by developing the idea that a fuzzy number could be used to represent the full range of preferences for a political actor. After demonstrating the utility of a continuous fuzzy number, we presented a discrete fuzzy number, which permits the modeling of actors who are less discerning and vaguer in their preferences. Hence, they are indifferent over a large segment of the issue space.

The fuzzy set approach that we developed in this chapter places the focus of analysis more squarely on individual actors. Moreover, it releases formal models from the precision and stringency that a crisp approach demands of political actors, something that hardly every attains in politics. This permits us to analyze and understand otherwise incomprehensible outcomes, demonstrating that an approach to formal modeling based on fuzzy set theory is a promising avenue for social science inquiry. Nevertheless, while the predictions generated by the crisp and fuzzy approaches may differ under a number of circumstances, both the crisp and the fuzzy approach in most cases lead to stable predictions. In essence, our argument for jettisoning the assumptions implicit in the crisp approach as well as Euclidean preference is premised on the argument that the resulting model *may* produce more empirically valid predictions.

In the next chapter, we demonstrate that the argument for the fuzzy approach is stronger for *n*-dimensional, spatial models. While the crisp approach hardly

ever achieves a stable prediction in spatial models, fuzzy models often do. Hence, the justification for incorporating fuzzy set theory into these models is to be had at a more fundamental level.

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## 5 Fuzzy Spatial Models

With Peter C. Casey, Christopher W. Cook, Meghan C. Freeman, Alan J. Hickle, Danae D. Mercer, Eric M. Ruchensky, and Raivydas Simenas

Most political issues are more than one-dimensional in scope. For example, budget bills contain funding across a number of issue areas, and political parties engaged in cabinet formation must concern themselves with several issue dimensions in determining a government program capable of uniting a legislative majority. Hence, spatial models must incorporate  $n > 1$  dimensions to be useful. Unfortunately, increasing the space to include multiple dimensions opens the possibility of cycling. The conditions under which cycling can occur have been the object of study for decades, and the conclusion is that the possibility of cycling is pervasive. Plotki (1967), for instance, found that a maximal set<sup>1</sup> exists in two-dimensional space only when ideal points are arrayed symmetrically to one another (the radial symmetry condition); and McKelvey (1976) found that in the absence of a maximal set, cycling is possible over the entire two-dimensional space. In order to reduce the likelihood of cycling, scholars have adopted increasingly restrictive assumptions in their models. While this has permitted the models to predict outcomes, increasingly restrictive assumptions remove the models further from reality, and empirical tests have often falsified the predictions. As a consequence, formal models have come under increasing criticism for the gap between their predictions and their empirical implications. The empirical implications in theoretical models (EITM) movement is one reflection of these criticisms (see Achen et al., 2002; De Marchi, 2005).

In what follows we begin by discussing the extent of the cycling problem in two-dimensional spatial modeling. Next we consider an example of how scholars resolve the cycling problem in crisp analysis. We then demonstrate fuzzy set theory's ability to reduce the cycling problem and strengthen the ability of models to predict more empirically sound outcomes. The demonstration relies partly on the fuzzification of one of the more interesting spatial models in The

<sup>1</sup> A maximal set in two-dimensional formal models is most often referred to as a core. A core in this sense is not the same as the core of a fuzzy number. To avoid confusion, we will use the term maximal set when referring to a core in formal modeling.

New Institutionalism's repertoire, Laver and Shepsle's (1996) *portfolio allocation model*. Our general argument is that fuzzy theory offers a more elegant approach to formal modeling. As was the case with one-dimensional models, a fuzzy approach jettisons two common assumptions of spatial models: 1) the assumption of a single, ideal point and 2) the assumption that preferences can be thought of as a function of Euclidean distance from that ideal point. These assumptions impose a set of restrictions that may not often, if at all, attain in reality and that may distort propositions derived from the models. More importantly, however, in the case of spatial models, the fuzzy approach results in more stable predictions.

## 5.1 The Cycling Problem in Crisp Two-Dimensional Spatial Models

Spatial models begin with the fundamental rationality assumptions that individuals are self-interested, utility maximizers and that their preference orders over a set of alternatives are complete and transitive. They further adopt the two assumptions common to one-dimensional models: the exact ideal point and Euclidean preference. As we discussed in the previous chapter, the first assumption is a consequence of the crisp set approach. Crisp set logic dictates that ideal points have an exact location, about which there is no ambivalence on the part of the actor. In effect, political actors possess a single ideal point; they are not indifferent over a set of ideal points. Rational choice modelers typically extend the assumption of exactness by adopting the perfect information assumption that a political actor not only knows the location of her own exact ideal point but she knows those of the other political actors as well. Indifference over alternatives is further reduced by the conventional adoption of the Euclidean distance assumption, as a consequence of which only those points lying on a circle equidistant from the ideal point (at the center of the circle) are equally preferred. Points further from the circle, no matter how infinitesimally small the distance, are less preferred, and those within the circle are more preferred.

Taken together, these assumptions have the effect of reducing indifference to a very small subset of the set of alternatives. There is nothing about the rationality assumption that requires this. In fact, rationality permits indifference over large sets of alternatives. However, in practice the conventional crisp set approach and the conventional Euclidean distance assumption virtually negate the possibility.

Crisp set formal models that assume that players are capable of making the fine-grained distinctions between alternatives demanded by the Euclidean preference assumption are plagued by the conclusion that individually transitive preference orders of rational political actors frequently result in intransitivity of collective choice, cycling. There is no set of alternatives that are immune to defeat under majority rule; no alternative is majority preferred to every other

alternative. Hence, the maximal set is empty.<sup>2</sup> Worse, as McKelvey (1976) demonstrated, if all players vote sincerely, a top cycle set exists that encompasses the entirety of the two-dimensional space. Formally (Austen-Smith and Banks, 1999, p. 169), a top cycle set is defined as:

$$T(\rho) = \{x \in X \mid \forall y \in X \setminus \{x\}, \exists \{a_0, a_1, \dots, a_r\} \subseteq X, \quad (5.1)$$

such that  $a_0 = x, a_r = y, r < \infty$   
and,  $\forall t \leq r - 1, a_t P a_{t+1}\}$ ,

where  $\rho$  is a set of preference orders, or a *preference profile*,<sup>3</sup> held by individuals making choices over a set of alternatives. To say a top cycle set encompasses all of two-dimensional space is to say that any point is majority preferred by some other point, given a large enough number of immediate pairwise competitions. Therefore, cycling is not restricted to a region, but occurs throughout the entire dimensional space.

Scholars have tried to ascertain the conditions under which cycling is not likely to occur. Plott (1967) found that the conditions are extraordinarily restrictive. They require that ideal points be arrayed in two-dimensional space in radial symmetry. Such a pattern is displayed at Figure 5.1. The ideal policy preferences of five political players ( $A, B, C, D,$  and  $E$ ) are plotted in two-dimensional space. Limiting median lines are drawn with dashes.<sup>4</sup> These limiting median lines all intersect at one unique point,  $C$ . Plott (1967) found they do so because the remaining points are arrayed about  $C$  in symmetric pairs (a line through  $C$  can be drawn that passes through a pair of points).

McKelvey (1976) demonstrated that only such an arrangement will yield a stable result. In the situation depicted at Figure 5.1, the stable outcome will be position  $C$ , which is defined as the maximal set. Since crisp logic combined with Euclidean preferences limits players to a single, exact ideal point, the maximal set is singleton; it has only one element. In essence, it is a Condorcet winner in two-dimensional space.

No maximal set exists in two-dimensional space in the absence of radial symmetry. If any one ideal point at Figure 5.1 is moved even slightly, an outcome can no longer be predicted. In Figure 5.2 the ideal point of political player  $C$  has

<sup>2</sup> As we discussed in chapter four, a maximal set comprises those alternatives that are majority preferred to all others and over which the political actor herself is indifferent. (No one alternative in the maximal set is majority preferred to another.) Formally,  $M(R, X) = \{x_i \in X \mid \forall x_j \in X, x_i R x_j\}$ , where  $R$  is a binary relationship and  $X$  is the set containing all alternatives over which political actors make binary choices. Players are indifferent to all alternatives in the maximal set; and all alternatives,  $x_i$ , in the maximal set are at least as good as all alternatives,  $x_j$ , outside of the maximal set. If there is only one alternative in the maximal set, then it is the Condorcet winner.

<sup>3</sup> An individual has a preference order. A collective has a preference profile. A set of preference orders characterizing a set of individuals is a preference profile.

<sup>4</sup> Limiting median lines mark the limits of all median lines that divide the ideal points such that there are majorities on both sides.

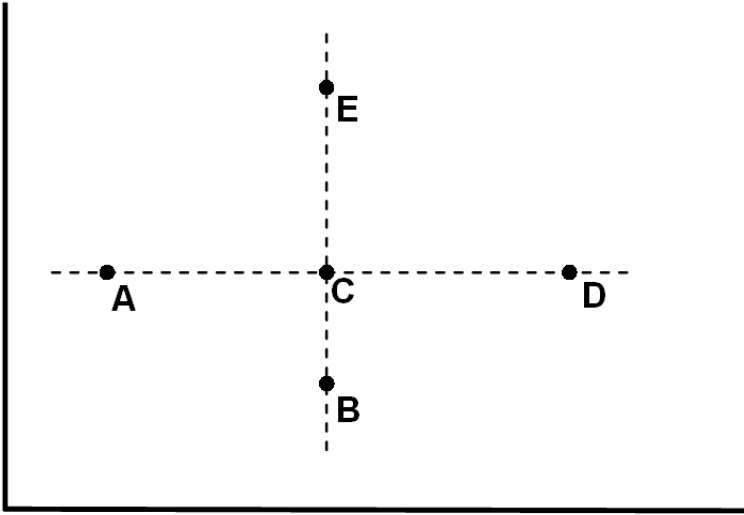


Fig. 5.1. Radial Symmetry with a Maximal Set at  $C$

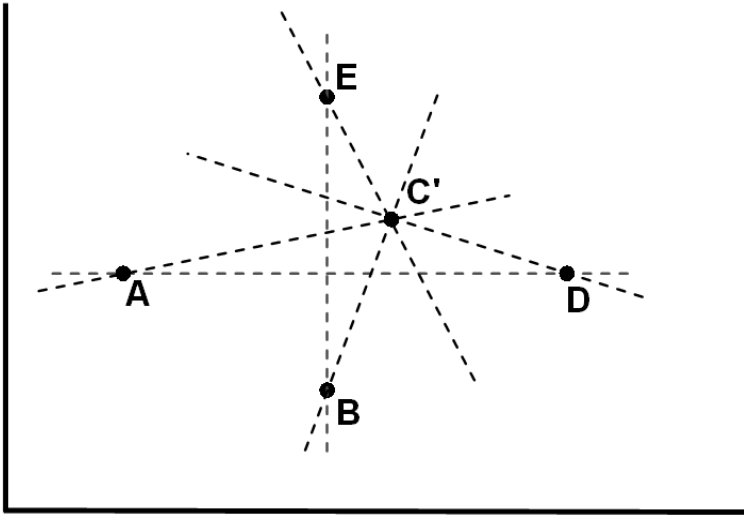


Fig. 5.2. An Empty Maximal Set

shifted to  $C'$ , as a consequence of which the ideal points are no longer arrayed in radial symmetry and the limiting lines no longer pass through a single unique point. In such situations, there is no majority preferred outcome.

The effect of an empty maximal set is demonstrated at Figures 5.3 and 5.4. The ideal policy preferences of three political actors are mapped in two-dimensional space at positions  $A$ ,  $B$ , and  $C$ . By the majority rule principle, an

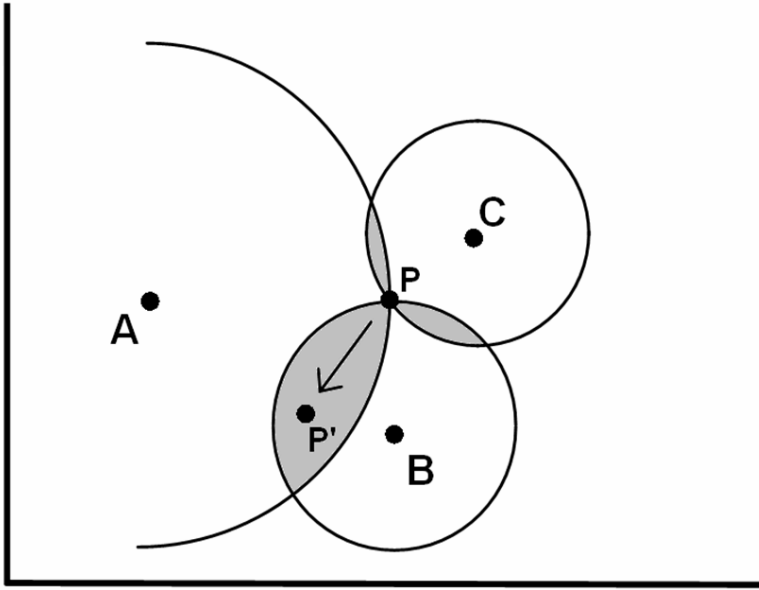


Fig. 5.3. Winset of  $P$

outcome requires the agreement of any two actors. Furthermore, the rationality principle requires that players seek the highest utility payoff possible. Suppose that political actors with ideal policy positions at  $A$ ,  $B$ , and  $C$  are considering alternatives to position  $P$  in the space between them. Euclidean distance circles with their centers at each of the ideal points and their radii from the ideal point to  $P$ , result in the three shaded petals that bound the set of positions majority preferred to  $P$  by  $A$  and  $B$ ,  $A$  and  $C$ , and  $B$  and  $C$ . These petals define the winset of  $P$ ,  $W(P)$ . Any point within  $W(P)$  in Figure 5.3, to include  $P'$ , is predicted by the model.

Figure 5.4 demonstrates that  $P'$  also has a winset within which  $P''$  resides. The reader can check for herself that  $P'''$  subsequently lies within the winset of  $P''$ . In fact, as McKelvey (1976) demonstrates, in the absence of a maximal set, any point in the two-dimensional space at Figures 5.3 and 5.4 can be reached. In effect, the entire dimensional space is in the top cycle set, a consequence of an empty maximal set.

Adding either actors or dimensions to policy space results in a greater likelihood of non-radial symmetric arrangements. Hence, non-radial symmetry is the norm, radial symmetry is rare, and cycling is predicted by crisp spatial models. However, the problem is nowhere near as ubiquitous as the models suggest. In fact, it has been difficult to find empirical evidence of cycling (Tullock, 1981).

Rational choice formal modelers have taken two approaches to address the gap between theory and empirical evidence. The first is to study the effect of different assumptions about the voting behavior of political actors. The most promising of these is encompassed in a research agenda that is focused on the uncovered set.

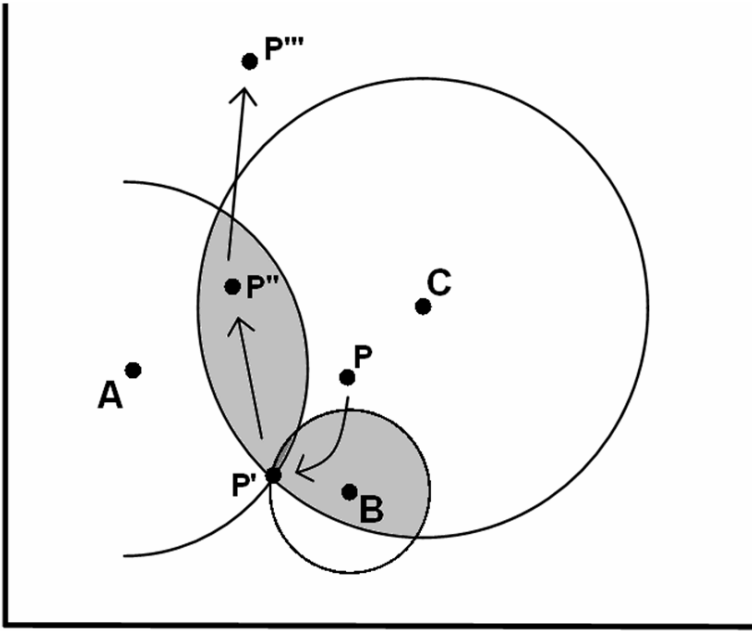


Fig. 5.4. Cycling in the Absence of a Maximal Set

McKelvey’s (1976) finding that cycling occurs across the entire two-dimensional space depends on the assumption that political players vote sincerely at each round of the political process. They are essentially myopic and do not consider the current choice of options in the context of previous or future choices. They vote for their best possible outcome merely based on their preference orders at each round. However, if we assume that political actors are more sophisticated, then they might vote strategically at each round. That is, they would cast their votes not based strictly on their preferences over the choices at a given round, but rather with the intent of maximizing the final outcome.

Take for example, the structure of the cycling problem that we discussed in the opening chapter of this book. That structure, presented in Table 1.1, is reproduced in Table 5.1 below. The social preference, or preference profile, of the group is:  $A$  is strictly preferred to  $B$ ,  $B$  is strictly preferred to  $C$ , and  $C$  is strictly preferred to  $A$ . If it is known ahead of time that the voting order will pit alternative  $A$  against  $B$  in the first round and the winner of that contest against alternative  $C$  in the final round, then if all players vote sincerely,  $C$  will be the social choice. Political actor 3 is clearly satisfied with the result and has no incentive to vote other than sincerely. However, both actors 1 and 2 can do better if actor 1 votes strategically in the first round, casting his vote for alternative  $B$ , his second choice, rather than alternative  $A$ , his first choice. The result will be that  $B$  will win the first round and defeat  $C$  in a sincere vote in the final round. Actor 1 will receive his second best payoff, instead of his worst payoff, and actor 2 her best payoff, instead of her second best payoff.

**Table 5.1.** A Notional Preference Ordering for Three Individuals Faced with a Choice between Options *A*, *B*, and *C*

Person		
1	2	3
<i>A</i>	<i>B</i>	<i>C</i>
<i>B</i>	<i>C</i>	<i>A</i>
<i>C</i>	<i>A</i>	<i>B</i>
$A > B > C$	$B > C > A$	$C > A > B$

If we assume that political players vote strategically, that their preference orders are strict and monotonically decreasing, and that decisions will be made by majority rule, then the set of stable majority preferred options can be reduced to an identifiable region of two-dimensional space. The shape and size of the region is defined by the *uncovered set*, the set of alternatives that are not *covered* by another point. A point *x* is said to cover *y*,  $xPy$ , if:

$$xPy \text{ and for all } z \text{ in } X, yPz \text{ implies } xPz. \tag{5.2}$$

In words, *x* covers *y* if *x* beats *y* and *x* beats any point that *y* beats (Miller, 1980). Thus, the winset of *x* (all the points which beat *x* in one step),  $W(x)$ , is a subset of the winset of *y*,  $W(y)$ . Formally, *x* covers *y* if and only if  $x \in W(y)$  &  $W(x) \subseteq W(y)$ .

The *uncovered set* of *x*,  $UC(x)$ , is the set of all points not covered by *x*. Unfortunately, there are several definitions of the uncovered set. Miller (1980, 2007) defines  $UC(x)$  as comprising all points *y* in policy space *X* for which  $yPx$  or such that there is some *z* such that  $yPzPx$ . Thus *y* either beats *x* or *y* beats some *z* that then beats *x*. Shepsle and Weingast (1984), define  $UC(x)$  as the set of all points *y* in policy *X* where  $yPx$ , or  $yPz$  whenever  $xPz$ . That is, *y* beats *x* or *y* beats any point *x* beats. As Penn (2006) has shown, if political actors are indifferent over any subset of alternatives, each definition will lead to different results.

What is fundamental to all definitions of the uncovered set, however, is that  $UC(x)$  contains all the points that can beat *x* in one or two steps. This is the “two-step principle” which gives the uncovered set its essential importance. Those readers wishing a further discussion of the concept are referred to McKelvey (1986), Miller (1980), Miller (2007), and Shepsle and Weingast (1984).

The concept most useful in helping to limit cycling to a subset of policy alternatives is the uncovered set of the entire policy space *X*,  $UC(X)$ .  $UC(X)$  is the set of all uncovered points in the policy space, *X*: those alternatives in two-dimensional space that are *not* covered by some other point (see Miller, 1980). In other words,  $UC(X)$  is the set of all points that beat every other point in either one or two steps (Miller, 1980, 2007). Hence,  $UC(X) = \bigcap UC(x_i) \forall x \in X$ . Formally,

$$x \in UC(X) \Leftrightarrow \forall y \in X \setminus \{x\} (xPy \text{ or } \exists z (xPz \text{ and } zPy)). \tag{5.3}$$

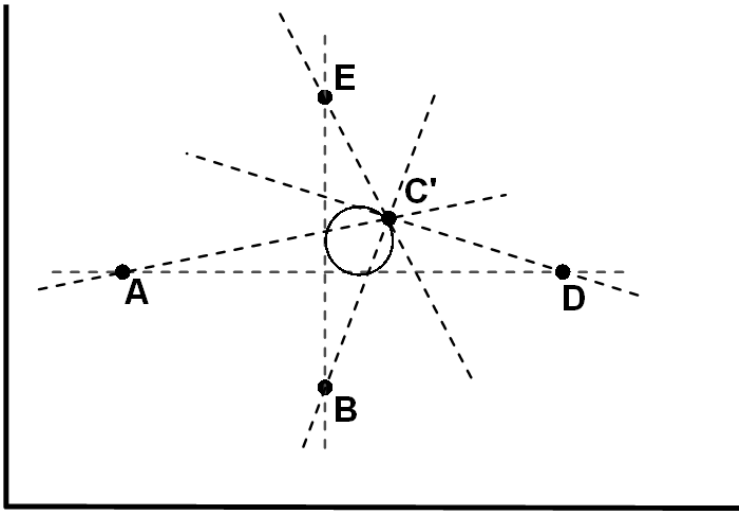


Fig. 5.5. A Yolk

The uncovered set,  $UC(X)$ , is a subset of the top cycle set that contains only Pareto efficient alternatives in the policy space (Austen-Smith and Banks, 2005, p. 135). It comprises the set of points that might be the final choice in a sophisticated voting game, the actual choice depending on the path that voting takes. While the uncovered set provides a solution to cycling by adopting a highly defensible assumption of sophisticated (strategic) voting on the part of political actors, its utility is considerably undermined by the difficulty of calculating the uncovered set. McKelvey (1986) found that the area within which  $UC(X)$  will be constrained is a function of the size of the yolk, the smallest circle intersecting all limiting median lines. The yolk for the array of ideal positions of political actors in Figure 5.2 is drawn in Figure 5.5. For a yolk of radius  $r$ , the uncovered set is contained within a circle whose center is located at the center of the yolk with a radius four times that of the yolk ( $4r$ ). The size of the yolk is a function of the number of political actors and their positions relative to one another. In general, as the number of political players increases, the size of the yolk decreases. As the size of the yolk decreases, the uncovered set decreases. If there is a maximal set, then the maximal set is singleton, the yolk is a single, exact point and the uncovered set and the maximal set are the same point. (For intuition for this claim, recall that radial symmetry is a necessary condition for the existence of a maximal set, and as Figure 5.1 makes clear, both the maximal set and the yolk in such case are singleton.)

Miller (2005) has demonstrated that the  $4r$  distance is overly generous and that typically  $x$  covers all points  $y$   $3.5r$  further from the yolk than it is, and  $x$  covers most points  $z$  that are  $2.5r$  further from the yolk. In any event, these are only *limitations* on the size of the uncovered set. The actual size, shape, and location of the uncovered set are difficult to determine. While software is

beginning to emerge on the market with the capacity for more precisely locating the uncovered set - *CyberSenate* is the most promising of these – the results are not at all intuitive and often quite strange. Furthermore, calculations depend on the Euclidean preference assumption and are greatly complicated unless issue dimensions are assumed to be equal in salience and independent of one another (separable).

A second, more elegant approach to dealing with cycling in order to bridge the gap between theory and empirical results is to consider the effect of varying institutional designs, an approach firmly located within The New Institutionalism. Kenneth A. Shepsle is among the pioneers of this approach. One form of institutional rules that induces stable outcomes divides jurisdiction over issues, so the dimension-by-dimension median (the intersection of the medians on each dimension) becomes the maximal set. Shepsle's ground-breaking work on the U.S. Congress argues that work on legislative bills is divided among committees and that the germaneness rule constrains committees to deal only with that part of proposed legislation over which they exercise jurisdiction (Shepsle, 1979). As a consequence, committees treat issue dimensions independently of one another; no bargaining or trade-offs occur relative to other dimensions. In effect, this induces issue *separability*. Furthermore, since the set of alternatives from which a committee may choose is restricted to those available on the single dimension within its jurisdiction, by Black's median voter theorem, the choice is that alternative at the median. The final bill is an assemblage of the work done in separate committees, the whole house voting on a bill at the intersection of the one-dimensional outcomes on each of the dimensions (the dimension-by-dimension median). (See Shepsle and Weingast 1995 for a succinct review of the literature on this approach.)

Shepsle subsequently applied the same approach to a model of cabinet formation in parliamentary systems in a work co-authored with Michael Laver. The portfolio allocation model (Laver and Shepsle, 1996) induces a stable cabinet at the dimension-by-dimension median by assuming that the act of allocating a ministry to a party is an act of abdication on the policy dimension by all other parties to the coalition. The model further assumes issue separability: parties do not consider trade-offs when allocating ministries.

Figure 5.6 demonstrates Laver and Shepsle's (1996) model. Suppose that three political parties, *A*, *B*, and *C*, each with an equal number of deputies, occupy policy positions in two-dimensional space as depicted. To avoid cycling, the government that is predicted by the model must be majority preferred to any other plausible government. Since parties abdicate control over policy on the dimensions over which the respective ministries allotted to other parties in the government exercise jurisdiction, a finite set of governments exists. Government formation is restricted to the intersection of the policy positions of the respective parties. Vertical lattice lines indicate the position of each party on the *x*-dimension, and horizontal lattice lines indicate their positions on the *y*-dimension. Nine alternative governments lie at the intersection of the lattice lines. In addition to abdication, Laver and Shepsle (1996) also assume that issue

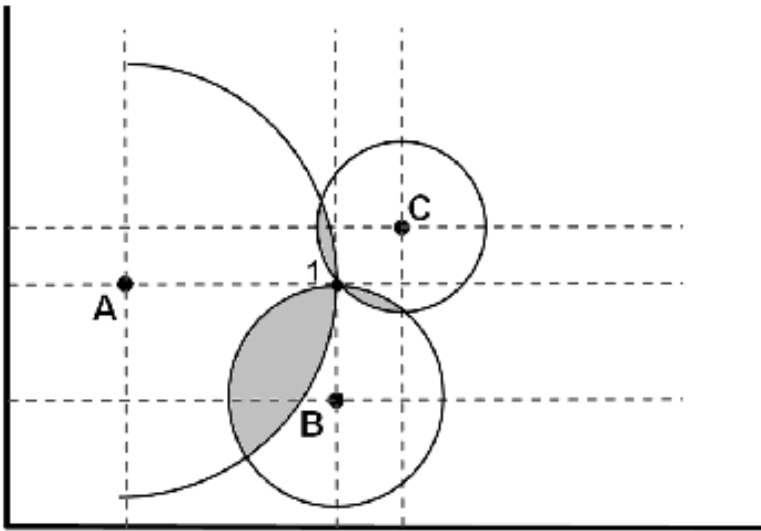


Fig. 5.6. A Crisp Outcome Predicted at Government 1

dimensions are independent of one another, or separable. These two assumptions permit the portfolio allocation model to predict a unique, stable outcome at the dimension-by-dimension median.

Since player *B* occupies the median position on the *x*-dimension, and player *A* does so on the *y*-dimension, Laver and Shepsle's (1996) model predicts a government at position 1 in Figure 5.6. This is a government in which party *B* takes the ministerial portfolios associated with the *x*-dimension and party *A* takes those on the *y*-dimension. Euclidean distance circles drawn through the government at 1 with their centers at the ideal points of each of the three parties demonstrate that none of the remaining eight governments in the finite set of plausible cabinets lies in any of the winsets of a government at that location in the policy space.

The outcome at position 1 is stable. No player has an incentive to change her position since doing so will not make her better off on either dimension. Notice, however, that the possible governments at the lattice points assume separability; that is preferences on the *x*-dimension are independent of preferences on the *y*-dimension. Separability is key to the model's results, and relaxing this assumption invalidates some of the theoretical backbone Laver and Shepsle offer.

As was the case with one-dimensional models, crisp *n*-dimensional models generally map the ideal point of an institution as a single, exact point (as it does the ideal point of an individual). Such exactness is difficult to justify given the cycling problem that is likely to attend collective choice in institutions such as legislatures and courts. While most models simply ignore the problem, as is the case with one-dimensional models, Tsebelis (2002) employs the logic of the uncovered set to argue that in large institutional settings, the outcome will be reduced to that approximating an exact, single point in the two-dimensional space.

Both the uncovered set and the analysis of institutional design have helped scholars to produce spatial models that restrict or mitigate against cycling. However, the former approach lacks elegance and the latter requires a substantial number of restrictive assumptions. In what follows, we demonstrate the ability of fuzzy set theory to constrain cycling in formal models more elegantly and under less restrictive assumptions. Furthermore, fuzzy models can predict stable outcomes under an assumption of either sincere or sophisticated voting.

## 5.2 A Fuzzy Set Theory Approach to Two-Dimensional Models

We use Laver and Shepsle's (1996) portfolio allocation model to demonstrate the utility of the fuzzy approach, in particular its capacity for dealing with cycling without having to import the same restrictive assumptions necessary in the crisp approach. Our first pass at "fuzzifying" the model will be little more than an overlay of fuzzy concepts onto the crisp analysis. Subsequent revisions weaken the model's assumptions. As we lift these assumptions, the fuzzy set approach becomes progressively better able than its crisp set counterpart to predict stable outcomes. The final model that we illustrate is essentially a generic model, one that is most plagued by cycling in the crisp approach.

We remind the reader that fuzzy set theory, developed by Lotfi Zadeh (1965), assigns membership values over the interval  $[0, 1]$  to elements in a set indicating their degree of membership in the set. Formally, a fuzzy subset  $A$  of a set  $X$  is a function mapping elements of  $X$  into the closed interval  $[0, 1]$ . For  $\alpha \in [0, 1]$ , the  $\alpha$ -cut of a fuzzy set  $F$  is the set of all elements that have a membership grade of at least  $\alpha$ ,  $F^\alpha = \{x \mid F(x) \geq \alpha\}$ . The strong  $\alpha$ -cut of  $F$  comprises those elements that have a membership grade strictly greater than  $\alpha$ ,  $F^\alpha_> = \{x \mid F(x) > \alpha\}$ . The  $\alpha$ -cut at one ( $\alpha = 1$ ) is the core<sup>5</sup> and the strong  $\alpha$ -cut at zero ( $\alpha = 0$ ) is the support of a fuzzy set  $F$ . If  $F(x) = \alpha \in [0, 1]$  we say that  $x$  has alpha-level ( $\alpha$ -level) membership in  $F$ .

We generalize the single-dimensional fuzzy model developed in the last chapter to a two-dimensional spatial model. In so doing, we retain the primary advantage of fuzzy one-dimensional models: release from the twin assumptions of perfect information and Euclidean preference. The first assumption is related to a crisp set approach and assumes the existence of exact, ideal points known to all political actors. The second limits uncertainty to the set of points lying on a circle equidistant from the ideal point. Both assumptions are questionable. Under these assumptions, a political actor whose ideal budget appropriation is \$1,000 (expressed in millions) would be strictly less supportive of counter-proposals at \$9,090 and \$1,010, but indifferent in a choice between the two of them. It is highly implausible that humans know their preferences with such exactitude and that they can make such fine-grained distinctions. It is instead far more likely

<sup>5</sup> We remind the reader that a core of a fuzzy number is not the same as the core, or maximal set, of a multi-dimensional formal model.

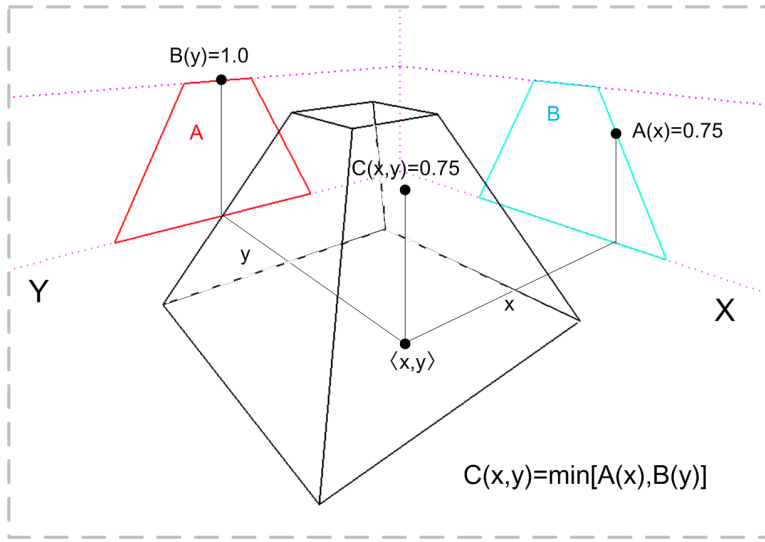


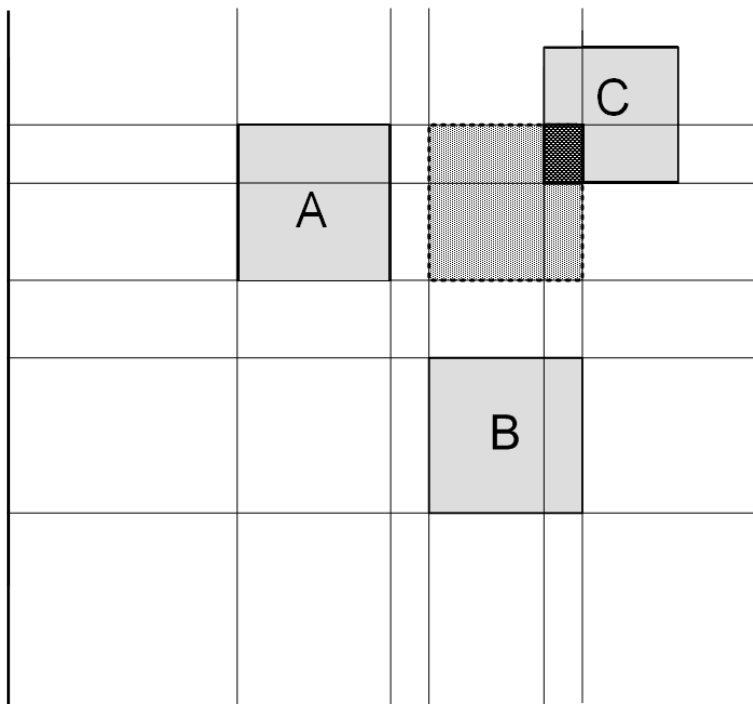
Fig. 5.7. A Fuzzy Number in Separable Dimensional Space

that their preferences over relatively large sets of options are characterized by ambiguity: a political actor is likely to be indifferent to a final budget lying anywhere along a substantial range of possibilities.

### 5.2.1 Separability and Finite Sets of Alternatives

We begin by redoing the analysis illustrated at Figure 5.6 making use of fuzzy numbers, which permit us to assume that parties are ambiguous about their ideal positions in the two-dimensional policy space. Laver and Shepsle's (1996) portfolio allocation model restricts outcomes to a finite set of governments by assuming 1) abdication and 2) issue separability. We can model the separability assumption in fuzzy set theory as follows. A natural way to define a fuzzy point in the plane would be as an ordered pair of real fuzzy numbers  $\langle A, B \rangle$ . This approach does not yield good results for fuzzy lines and is difficult to use in the construction of pictures. However, we can use  $A$  and  $B$  to construct a fuzzy point  $C$  whose membership function for a point  $\langle x, y \rangle$  is simply the minimum of  $A(x)$  and  $B(y)$ . That is, we define  $P : \mathbb{R}^2 \rightarrow [0, 1]$ , by  $C(x, y) = \min[A(x), B(y)] \forall \langle x, y \rangle \in \mathbb{R}^2$ . In fuzzy set theory, this is sometimes written  $C = A \times B$ . The resulting function looks like a truncated pyramid illustrated in Figure 5.7. This construction is reversible, and thus  $C = A \times B$  is separable. If we project  $C$  into the first dimension using the supremum operator, we recover  $A$  and  $B$ ;  $A(x) = \sup_{y \in Y} C(x, y)$  and  $B(y) = \sup_{x \in X} C(x, y)$ .

We can see that this type of fuzzy point models separability from the fact that the level sets of the fuzzy point (and their interiors) are rectangles, the sides of which are parallel to the  $x$ - and  $y$ -axes. Therefore, the sides of the rectangles



**Fig. 5.8.** A Fuzzy Maximal Set Solution

are defined by equations of the form  $x = a$  and  $y = b$ , which clearly exhibit the independence of  $x$  and  $y$ .

A fuzzy line is constructed from two fuzzy points  $P$  and  $Q$  in the following manner. Let  $\Omega_{PQ}(\alpha)$  denote the set of all line segments from a point in  $P^\alpha$  to a point in  $Q^\alpha$  for a fixed  $\alpha \in [0, 1]$ . The fuzzy line segment  $L_{PQ}$  from  $P$  to  $Q$  is defined as follows:  $\forall (x, y) \in \mathbb{R}^2, L_{PQ}(x, y) = \sup\{\alpha \mid (x, y) \in l \ \& \ l \in \Omega_{PQ}(\alpha)\}$ .

It is known that for all  $\alpha \in [0, 1]$ ,  $L_{PQ}^\alpha = \Omega_{PQ}(\alpha)$ .

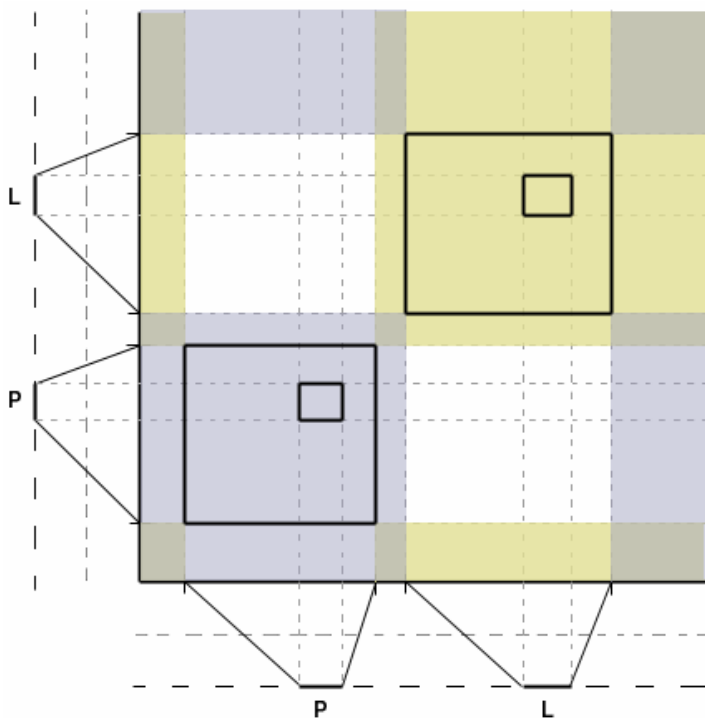
The intersection of the fuzzy lines results in rectangles, as shown in Figure 5.8. Essentially, we have oriented the fuzzy number in Figure 5.7 in a such a manner that we are looking down on the  $x, y$  plane from which the pyramid is rising. The fuzzy lattice lines now intersect in a unique area that includes more than one government. In fact, the number of governments that the fuzzy model predicts is considerably expanded from that of the crisp model. Among the governments included in the region of the intersection is a minority government in which  $C$  takes all portfolios.  $C$  can also choose to form a government with  $B$  in which it leaves the first dimension portfolios to the latter, or it might also form a government with  $A$  in which it leaves the second dimension portfolios to its coalition partner. Alternatively,  $C$  can enter into a grand coalition with  $A$  and  $B$ , in which they divide the portfolios on each of the dimensions. Finally, it is possible that  $A$  and  $B$  could agree to a government without  $C$ .

The crisp model (laden with assumptions) predicts a single, unique outcome in the maximal set while the fuzzy model operating under the same set of assumptions predicts a maximal set of five alternative governments. While party  $C$  is included in four of the five governments, its absence in one (uniting  $B$  on the first dimension and  $A$  on the second) means it can not exercise a veto in determining the outcome. Hence, in this case, we can not offer a more precise prediction than a maximal set of five (from a possible ten, when the grand coalition is included). Every one of the five alternatives is majority preferred to any alternative outside the set, and none in the set is majority preferred to any other in the set.

The skeptical reader will likely argue that we have made the case for the greater utility of the crisp model. After all, how useful can a model be that predicts that five of ten alternatives are equally likely? One could just as soon make predictions on the basis of a coin toss. While we confess that the argument is not without merit, we note that among the outcomes predicted by the fuzzy model is a grand coalition uniting all three parties. The crisp portfolio allocation model can not predict a three-party coalition in two-dimensional space. In fact, the number of parties that the crisp model can predict will form a cabinet can not exceed the number of issue dimensions. This excludes not only a grand coalition, but any surplus majority coalition as well.

Furthermore, we note that the fuzzy model will not predict a maximal set with more than one outcome in every case. Whether or not it does so depends on more than simply the relative size of a coalition (in terms of numbers of deputies) and location in the policy space, which are the only two concerns in the crisp model. The prediction is also sensitive to the ambiguity of the political players, the relative size of the policy space over which they are indifferent. As this ambiguity increases relative to the location of the political actors, fuzzy points are likely to overlap on dimensions and the size of the predicted set will increase.

Finally, we note that the crisp model operating under the restraint of the Euclidean distance assumption, which our fuzzy approach has jettisoned, forces the modeler to argue that any position, no matter how far away, is acceptable for an actor as long as it is the closest outcome possible. This seems highly implausible. Our fuzzy models permit us to consider the effect of areas outside of which no bargain can be made. These are areas at  $\alpha = 0$ , indicating that the position is unacceptable to a political player. Suppose that two players,  $L$  and  $P$  have rectangular preferences as shown in Figure 5.9, constructed from their fuzzy preferences on each of two dimensions (represented by the fuzzy numbers at the bottom of the  $x$ -axis and to the left of the  $y$ -axis). In the absence of a dictator that can insist on getting her ideal position on both dimensions, the crisp approach would indicate two plausible compromise outcomes, one in which  $L$  gets the first dimension and  $P$  takes the second dimension and the other in which the two players divide the two dimensions in the opposite manner. Euclidean distance does not permit us to consider the effect of relative distance on preferences in any nuanced fashion. In the fuzzy case, however, neither compromise position is plausible since the opponent's position on a given dimension is outside the limits of acceptability for each player. Thus, our fuzzy model would predict gridlock



**Fig. 5.9.** Institutional Instability

likely leading to institutional instability (conflict), rather than policy instability (cycling), which the crisp model can not address.

### 5.2.2 Separability and Bargaining over Outcomes

A stronger argument for a fuzzy approach attains once we begin to release models from some of the more restrictive restraints under which crisp approaches typically operate. As we do so, we leave Laver and Shepsle's (1996) model and begin to develop a more generic model of majority preference relations. We begin by releasing the model from the abdication assumption, permitting actors to bargain over the entire policy space. Abandoning the abdication assumption in crisp models results in the set of collective outcomes no longer being restricted to a finite set; and cycling results over the entire two-dimensional space. To counter this, crisp models typically make resort to the cumbersome uncovered set. The fuzzy set approach demonstrates its far greater elegance and utility at this point.

Suppose that the shape of one-dimensional fuzzy points is that argued at the end of the previous chapter. That is, a fuzzy point can be represented by a discrete fuzzy number. For the reader's convenience, we reproduce the general shape at Figure 5.10. Then, a fuzzy point in two-dimensional, separable issue space no longer resembles the smooth-sided pyramid depicted in Figure 5.7, but

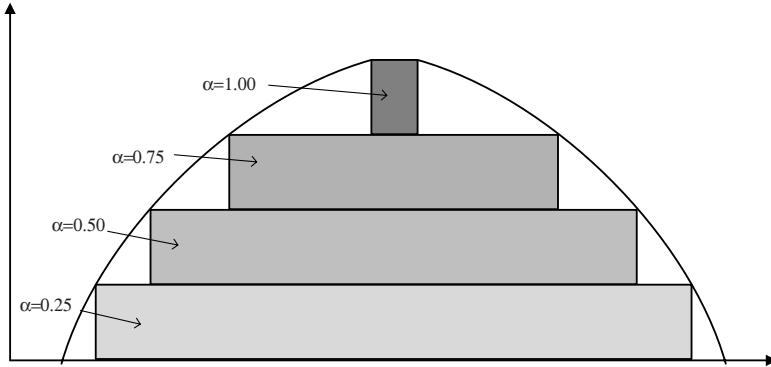


Fig. 5.10. A Discrete Fuzzy Number

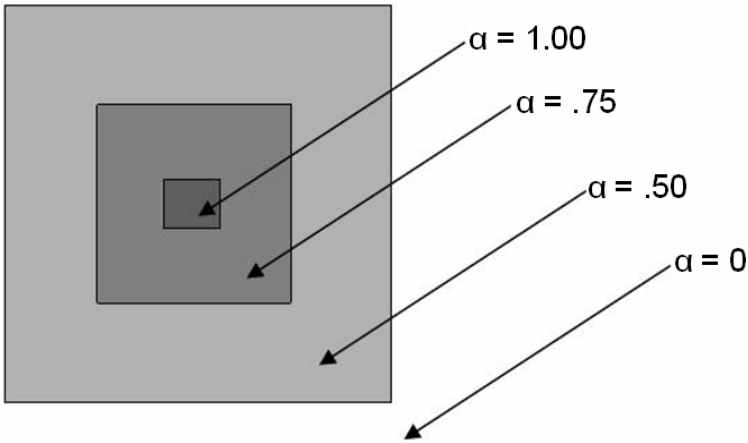


Fig. 5.11. A Fuzzy Number and Various  $\alpha$ -cuts in the  $x, y$  Plane

rather a stair-stepped pyramid. If we look down on these pyramids onto the  $x$ - and  $y$ -axes, we see a series of “stairs” or levels at various  $\alpha$ -levels as shown in Figure 5.11.

Suppose that the preferences of three political actors are those depicted by the fuzzy numbers in Figure 5.12. This is a classic case of cycling in crisp models. In our fuzzy model, the actors have alpha-levels at  $\alpha = 1$  and  $\alpha = 0.75$ . Alternatives outside of these two set inclusion levels are unacceptable to them. Since the supports ( $\alpha > 0$ ) of the players’ respective fuzzy numbers do not overlap (intersect), they can not come to an agreement by majority rule on any position that is agreeable to any two of them. In this case, we would expect indecisiveness or institutional gridlock that could result in political conflict.

Consider the arrangement of the preferences of three political actors as depicted by the discrete fuzzy numbers with alpha-levels at  $\alpha = 1$  and  $0.75$  at Figure 5.13. In the crisp case, in the absence of restrictive assumptions such

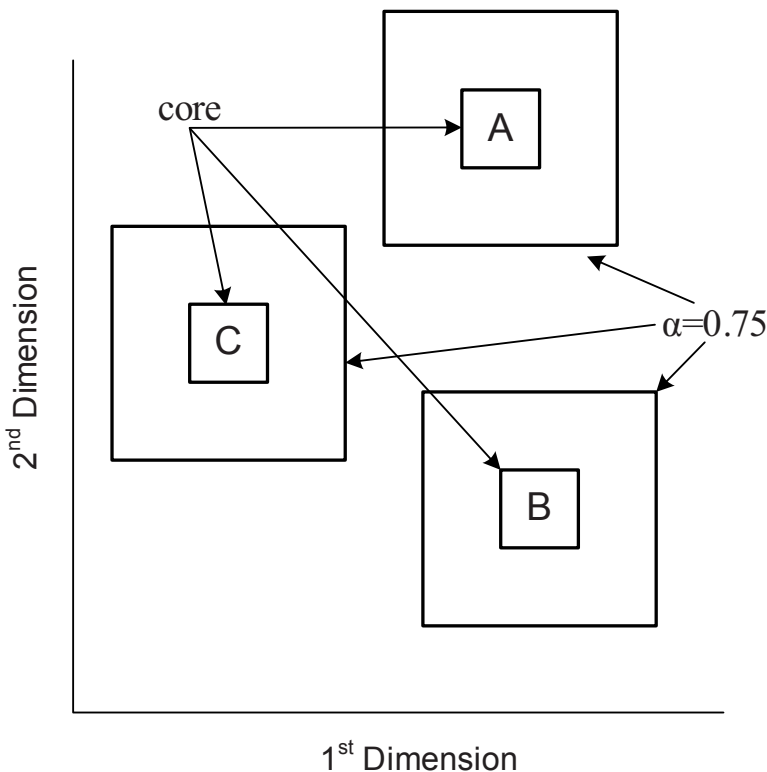


Fig. 5.12. Gridlock

as abdication, the set of available alternatives is infinite. In the fuzzy case, the set of alternatives is limited to the number of areas at which the three players' preferences intersect at different alpha-levels. Each of these intersections is a fuzzy point that represents a distinct alternative. Since their alpha-levels are identical within this region, players are perfectly indifferent over all policies, and cycling does not occur within a fuzzy point.

The set of alternatives available to three players is represented by the set of three-tuple payoffs  $(A(x), B(X), C(x))$ , each payoff being an alpha-level<sup>6</sup>. Points at which the three players'  $\alpha$ -cuts do not intersect, such as  $(1, 1, 1)$  and  $(.75, .75, 0)$ , are not included in the set of alternatives in Figure 5.13. Included in the set are

$$A = \{(0, 0, 0), (1, 0, 0), (.75, 0, 0), (.75, 0, .75), (0, 1, 0), (0, .75, 0), (0, 0, 1), (0, 0, .75)\}. \tag{5.4}$$

The point at which a majority intersects at the highest level of set inclusion is highlighted in gray in Figure 5.13. The point  $(.75, 0, .75)$  unites players A and

<sup>6</sup> Since we are using alpha-levels as a measure of utility, the payoff for a given outcome for each player is the respective alpha-level for that outcome.

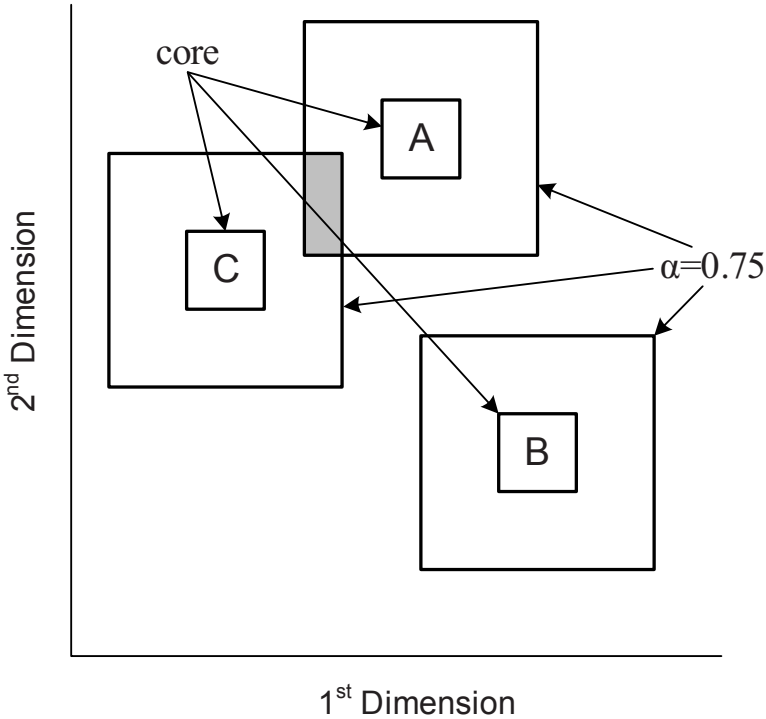


Fig. 5.13. A Large Maximal Set

$C$  at  $\alpha = .75$ . While this alternative is in the maximal set, it is not singleton. There are four other alternatives that can not be defeated by a majority. A contest between any two points in the maximal set results in a tie because two players are indifferent. Hence, the maximal set

$$MS = \{ ( 1, 0, 0), (.75, 0, 0), (.75, 0, .75), ( 0, 0, 1), (0, 0, .75) \}. \tag{5.5}$$

It is tempting to argue that player  $A$  would insist on  $(1, 0, 0)$  instead of  $(.75, 0, 0)$  and that neither  $A$  nor  $C$  would settle for anything less than at least a payoff at  $\alpha = 0.75$ , in which case the only plausible outcome is  $(.75, 0, .75)$ . However, to do so violates the assumption that political actors are *utility maximizers*, an assumption that lies at the heart of all rational choice formal models in comparative politics. If political actors are utility maximizers, we can not exclude outcomes at  $(1, 0, 0)$  or  $(0, 0, 1)$  merely on the basis of the argument that  $A$  will be forced to compromise with  $C$  and reach an agreement with a three-tuple payoff of  $(0.75, 0, 0.75)$ . If  $A$  can get an outcome with a payoff of  $(1, 0, 0)$ , she will do so. As a consequence, we must calculate the maximal set on the basis of what is preferred by a majority of utility maximizing actors. That of course leaves us with a prediction that five of eight alternatives are possible.

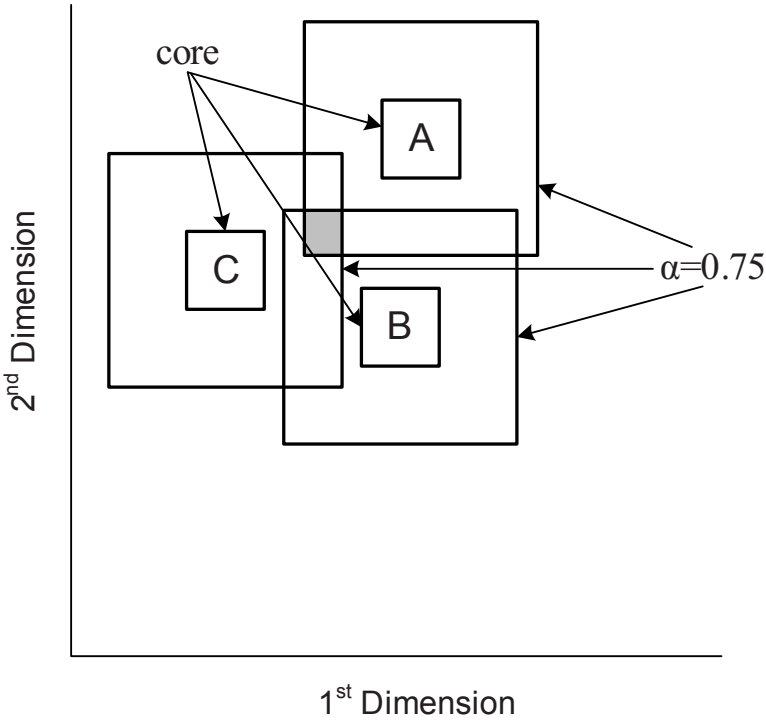


Fig. 5.14. A Small Maximal Set

Nonetheless, the fuzzy model restricts the outcomes to a well-defined maximal set, which is superior to the crisp case where the uncovered set is both difficult to calculate and hard to envision.

Consider further the arrangement in Figure 5.14. The players' alpha-cuts overlap at  $(0.75, 0.75, 0.75)$ , highlighted in gray. The set of alternatives

$$\begin{aligned}
 A = \{ & (0, 0, 0), (1, 0, 0), (0.75, 0, 0), (0.75, 0.75, 0), \\
 & (0.75, 0, 0.75), (0.75, 0.75, 0.75), (0, 1, 0), (0, 0.75, 0), \\
 & (0, 0.75, 0.75), (0, 0, 1), (0, 0, 0.75)\}.
 \end{aligned} \tag{5.6}$$

In this scenario the maximal set (the set of undominated alternatives) is limited to a smaller, well-defined subset

$$MS = \{(0.75, 0.75, 0.75), (0.75, 0.75, 0), (0.75, 0, 0.75), (0, 0.75, 0.75)\}. \tag{5.7}$$

It is easy to see how these fuzzy concepts can be applied to the cabinet formation process. Suppose that the same three parties in Figure 5.8 must form a government, however, abdication is not assumed, and they may bargain over government policy anywhere in the issue space. To determine the likely area of agreement, we need to map the fuzzy numbers of the respective parties in two-dimensional space at the various  $\alpha$ -levels. Suppose that players have preferences

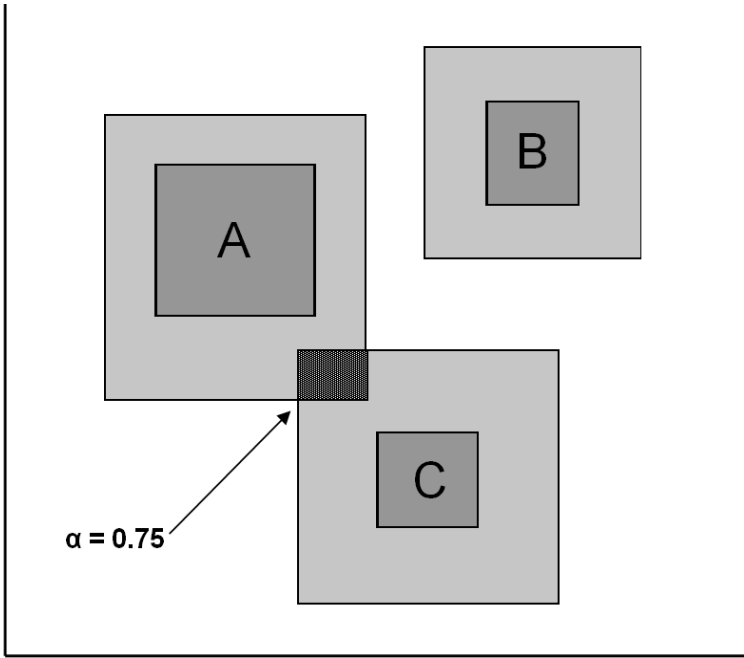
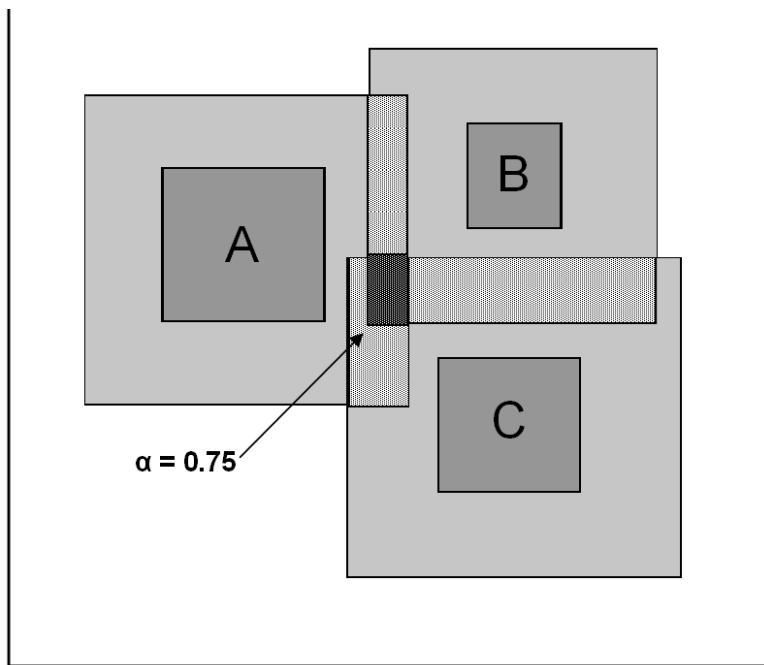


Fig. 5.15. A Two-Party Coalition

at only three levels of set inclusion,  $a = 1$ ,  $a = 0.75$ , and  $a = 0$  as shown in Figure 5.15. (Everything outside of the  $a = 0.75$  rectangle is at the  $a = 0$  level.) In this case, parties A and B and parties B and C can not form a coalition, since there is no area in the policy space on which they are in agreement. The prediction is reduced to a maximal set with three plausible governments: a two party coalition uniting parties A and C with a policy program at the three-tuple payoff of  $(0.75, 0, 0.75)$ <sup>7</sup> or any of two minority governments with a policy program at the three-tuple payoff of  $(1, 0, 0)$  or  $(0.75, 0, 0)$  at A, or  $(0, 0, 1)$  or  $(0, 0, 0.75)$  at C. None of these outcomes is strictly preferred by a majority to any other in the set, and all three are (strictly) majority preferred to any other outcome outside of the set.

If we change the shapes of the discrete fuzzy numbers representing the policy preferences of the three parties to those depicted in Figure 5.16, then the maximal set contains four possible governments: a three party super majority coalition with a three-tuple payoff of  $(0.75, 0.75, 0.75)$ , a government uniting parties A and B with a three-tuple payoff of  $(0.75, 0.75, 0)$ , one uniting parties A and C with a payoff of  $(0.75, 0, 0.75)$ , and a government uniting B and C  $(0, 0.75, 0.75)$ . Minority governments at A, B, or C are not predicted since three of the four governments in the maximal set are majority preferred to each.

<sup>7</sup> The three-tuple payoff indicates the alpha-cut level for a given outcome for players A, B, and C respectively.



**Fig. 5.16.** A Three-Party Super Majority Coalition

While the maximal set now contains four plausible coalitions, we remind the reader that a crisp model would only be able to estimate the outcome to an uncovered set, the size and shape of which is neither intuitive nor easy to calculate. This is an important point that we dare not gloss over. Moreover, the governments in the fuzzy maximal set are not regions within which an infinite number of points might be reached. Rather, each of the four plausible governments predicted by the fuzzy model pursues a discrete policy program. Furthermore, political players are not able to distinguish between policies represented by a fuzzy point. Hence, no internal cycling occurs within each point. Finally, the fuzzy points do not constitute a top cycle set. Each is a single, unique government, none of which is majority preferred to the other. Once a decision is made to adopt any one of them, there will be no cycling.

### 5.2.3 Non-separability

The elegance and utility of the fuzzy approach are even more apparent once we drop the separability assumption. As we discussed earlier in this chapter, the problem of cycling is unmanageable for crisp set approaches under assumptions of non-separability (interdependence of the issue dimensions). Moreover, non-separability renders the determination of the location of the uncovered set

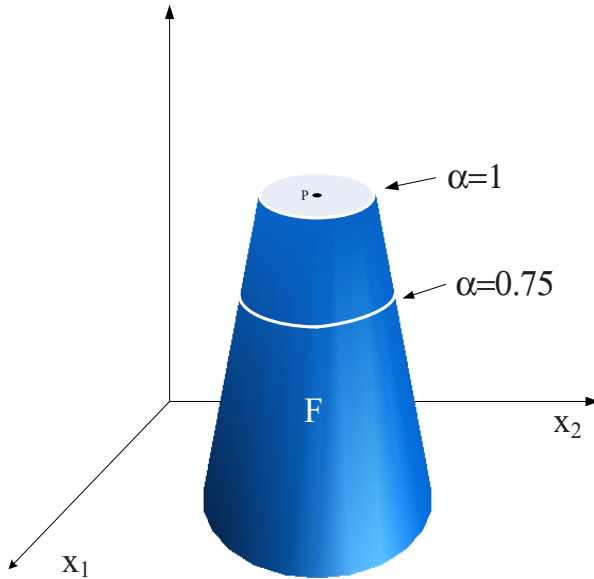
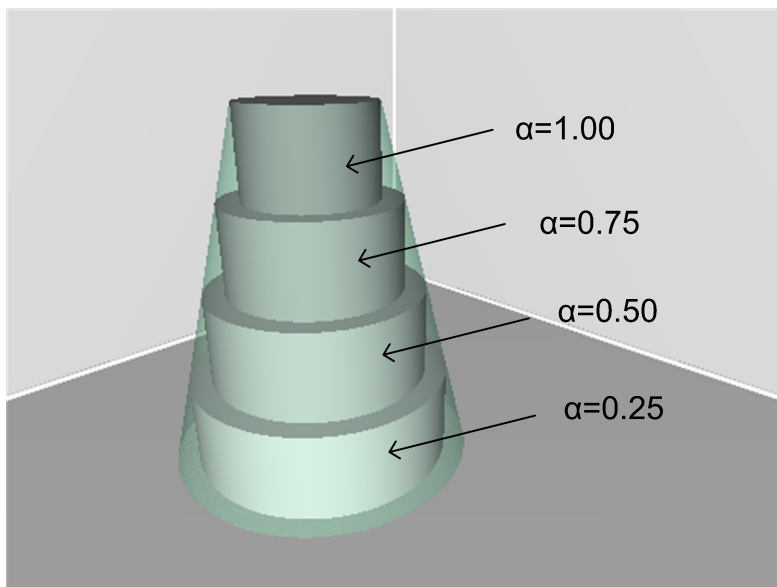


Fig. 5.17. A Non-separable Fuzzy Number

virtually impossible. In contrast, a fuzzy set approach increases the probability of predicting a maximal set solution. In fact, the logic is identical to that just demonstrated, but the shape of the fuzzy number can vary from that of a rectangle (a shape imposed on the fuzzy number by the logic of the separability assumption).

Without loss of generality, we will use a frustum to represent the shape of a two-dimensional fuzzy point. (In fact, it will take the shape that best defines the preferences of the relevant political party at various levels of satisfaction [fuzzy set inclusion or  $\alpha$ -level] with a policy region in two-dimensional space.) In a two-dimensional distribution of points in fuzzy geometry, a point  $f$  is represented as a three-dimensional frustum,  $F$ . (See Figure 5.17). The top of the frustum is a circle within which objects possess fuzzy membership in the set of “ideal” policy positions ( $\alpha = 1$ ). A political actor is indifferent over all policy positions within this circle, which in fuzzy geometry constitutes a fuzzy point. Planes intersecting the frustum at levels less than full membership in the set of ideal points (below  $\alpha = 1$ ) represent less than the ideal position. The slice of the frustum formed by the plane at  $\alpha = .75$  is drawn in Figure 5.17.

Like the smooth curve in the one-dimensional model, the slope of the frustum in Figure 5.17 indicates that there is an infinite range of levels of policy acceptability between these two levels of set inclusion (delineating the set of policies that are unacceptable,  $\alpha = 0$ , and those that are ideal,  $\alpha = 1$ ). For most purposes, this is simply too fine grained to be plausible. Figure 5.18 illustrates a two-dimensional version of the discrete fuzzy number that we developed



**Fig. 5.18.** A Two-Dimensional Non-separable Stair-step Fuzzy Number

in the previous chapter. This is a more appropriate representation of political actors, who are more likely to make coarse distinctions between policy positions, rather than fine-grained ones. The slope of the frustum is not a monotonically decreasing curve, but rather assumes the shape of a wedding cake as in the one-dimensional approach discussed in the previous chapter.

Figure 5.19 maps the fuzzy points at  $\alpha = 1, .75,$  and  $.50$  for parties  $A, B$  and  $C$  in two-dimensional policy space. (The inner-most ellipse is the  $\alpha = 1$  level for each of the parties.) The shapes of the fuzzy numbers are what one would see at these respective  $\alpha$ -levels if she were looking directly down on the  $x$ - and  $y$ -axes. (The reader will note that the preferences are not concentric and that we are not using a symmetrical frustum to depict the players' preferences.) Suppose that each party has an equal number of members and that they must form a coalition government. This is essentially a classic case of cycling in crisp models. Indeed, we have stripped away all of the additional assumptions made by the portfolio allocation model (Laver and Shepsle, 1996) and are left with a generic rational choice model of collective decision making among utility maximizing players under majority rule.

We have numbered each fuzzy point (alternative) and identified its corresponding three-tuple payoff  $(A(x), B(X), C(x))$  in parentheses. (For ease of presentation, we have not labeled points at which two or more players intersect at  $\alpha = 0$ . These points are defeated by a number of other points, and they can not defeat any other point by majority rule. Hence, they clearly are not

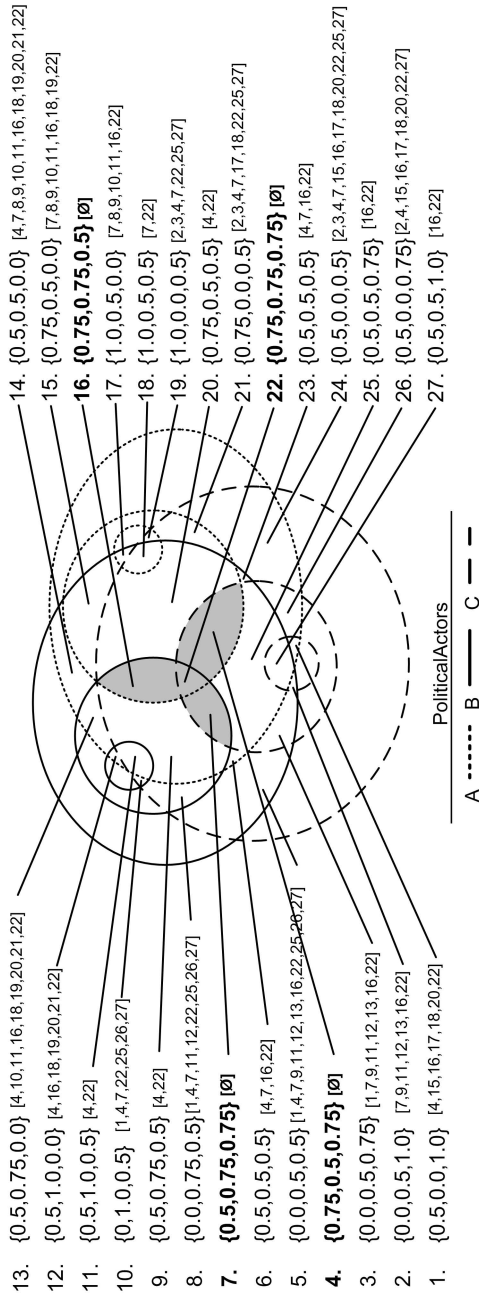


Fig. 5.19. Maximal Set Assuming Issue Non-Separability

among the elements in any maximal set.<sup>8</sup>) Furthermore, we have identified in brackets the set of options that are majority preferred [winset] to each numbered alternative. The set of undominated alternatives, whose winsets are empty, are in the maximal set and highlighted in gray in the figure. There are four governments in the maximal set circumscribed by this area: one uniting parties *A* and *B* with a policy position at a three-tuple payoff of  $(.75, .75, .50)$ , one uniting parties *A* and *C* with a policy position at a three-tuple payoff of one uniting parties  $(.75, .50, .75)$ , one uniting parties *B* and *C* with a policy position at a three-tuple payoff of and  $(.50, .75, .75)$ , and a grand coalition uniting all three parties with a policy position at a three-tuple payoff of  $(.75, .75, .75)$ .

### 5.3 Implications

The utility of the fuzzy set approach lies clearly with the increased probability of generating a maximal set. Even if the maximal set consists of several alternatives, this is superior to what a crisp approach can do. In the absence of Plott's (1967) radial symmetry, a crisp approach can not predict a maximal set under assumptions of sincere voting. To resolve this, resort typically is made to restrictive assumptions. Alternatively, crisp models limit predictions to the extraordinarily complicated uncovered set under assumptions of Euclidean preferences and sophisticated voting. A fuzzy approach increases the probability that formal models can make predictions without these assumptions. Furthermore, it permits scholars to consider institutional stability and not just policy stability.

Just as importantly, a fuzzy set approach accomplishes these tasks while also permitting consideration of indifference over large subsets of the policy space. The indifference that fuzzy geometry permits us to capture is not related to the uncertainty that actors have over their own ideal position or those of others. A number of approaches have already been developed that permit researchers to deal with incomplete information. Rather, it deals with political actors' ambiguity about their ideal policy positions. Actors may be certain of their policy positions and at the same time indifferent over a large set of options. Such ambiguity or vagueness, as opposed to uncertainty, seems to us to be far more likely in politics than a strict assumption that reduces the set of ideal policy positions to a single alternative.

The fuzzy approach that we have developed in this chapter also gives us more information about political outcomes than the typical crisp approach. Take the government formation process that we have used as an illustration. Not only do we know what parties are in the coalition, but we know the coalition's negotiated policy position on the salient issues. Crisp models that permit negotiation over policy are not able to predict stable outcomes. Moreover, the alpha-levels provide information that potentially may be useful in simplifying a model. We have already noted that a player will not vote for any proposal outside of his support (at  $\alpha = 0$ ). But what about at levels below  $\alpha = .50$ ? Set inclusion at these levels is low. Might this not mean that a player potentially is more opposed to

<sup>8</sup> The points include  $(0, 0, 0)$ ,  $(0, 0, .5)$ ,  $(0, 0, .75)$ ,  $(0, 0, 1)$ ,  $(0, .5, 0)$ ,  $(0, .75, 0)$ ,  $(0, 1, 0)$ ,  $(.5, 0, 0)$ ,  $(.75, 0, 0)$ , and  $(1, 0, 0)$ .

an alternative than she is in favor of it? We consider these and other issues in the ensuing chapter.

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## 6 Estimating Fuzzy Policy Preferences

With Peter C. Casey, Alan J. Hickle, Christopher W. Cook, Meghan C. Freeman, Danae D. Mercer, and Eric M. Ruchensky

The spatial models presented in the last two chapters used overlapping  $\alpha$ -cuts to determine compromises between individuals. In this chapter, we take a closer look at the assumptions implicit in such a method. In particular, we consider ways in which individuals rank alternatives in multiple dimensions.

The chapter begins by returning to the interpretation of fuzzy membership scores. While in earlier chapters the scores on the interval  $[0, 1]$  represented the degree of inclusion in the set ‘excellent policies,’ here we propose that membership scores also convey information about the intensity of preferences. Not only might a score of 1 assigned to a policy mean that a person prefers that policy to one assigned a score of .5; the score of 1 might also mean that the person cares more about the policy assigned a 1. Systematically accounting for intensity may be fruitful for formal models, but we leave the issue open and instead use an understanding of intensity to guide our choice of aggregation operators, discussed in the next section.

Spatial models are useful because they provide a visual representation of preferences. To derive predictions from a spatial model, we need to know how actors make compromises. If all actors could have policies exactly at their ideal points, spatial models would be simple. Plotting the ideal points of actors would suffice to show the resulting policies. Politics in the real world, however, do not allow all actors to have their way and so require trade-offs and compromise. One way to model compromise is to use straightforward Euclidean distance, so that two actors would be giving up an equal amount if a chosen policy was equidistant from their ideal points. We have spoken at length about how simple Euclidean distance may be an overly restrictive assumption. In this chapter we present a collection of aggregation operators that each offer a different interpretation of trade-offs an actor is willing to make between dimensions and so each result in a slightly different method of modeling compromise. These operators are considered in more detail in the appendix following the chapter, along with other fuzzy operators.

## 6.1 Information Conveyed in Membership Grades

We begin by considering the implications of using a fuzzy number to represent preferred policy outcomes. Given a certain policy and that policy's membership grade within the set, how much does the actor know about that policy? Suppose there exists a number continuum from 0 to 1,000. In this case, we could classify 0 "absolutely small" or "smallest" and 1,000 "absolutely large" or "largest." Let us pick 11 numbers within this range at equal intervals:

$$\{0\ 100\ 200\ 300\ 400\ 500\ 600\ 700\ 800\ 900\ 1000\}. \quad (6.1)$$

At the most basic level, we can construct a fuzzy set "numbers between and including 0 and 1,000" and, as the universe of numbers we have specified all perfectly meet this criterion, we obtain membership grades that fail to differentiate one number from another:

$$\{1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\}. \quad (6.2)$$

At the other extreme, we can construct a fuzzy set "the number 1000" and obtain nearly the exact reversal, where only one element in the set meets the criterion and all others fail:

$$\{0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\}. \quad (6.3)$$

Neither of these tells us much about the nature of the numbers. The number 1,000 is the only perfect match in a field of entries with otherwise no inclusion. Hence, we can assume that the number 1,000 exhibits a characteristic in its entirety that the other entries in the set lack entirely. We have a piece of complete information; we know that the number 1,000 is exactly the number 1,000. That we know the numbers 100, 200, etc. are perfectly not 1,000 is not as informative since they could be any one of several other things. This, of course, prompts the question, just what are these other numbers? With the given membership grades, we cannot answer that question to any capacity.

Let us now construct a set "large numbers" that reveals a bit more about the numbers in the set (recall, that our continuum is bounded by 0 and 1,000). In that case, 1,000 would be perfectly large while the number 900 might be very large and so on. Keeping the math simple, we obtain the following matrix of membership grades:

$$\{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}. \quad (6.4)$$

In this case, we are prepared to share a bit more information about a few of the numbers. Any entry greater than .50 stipulates that the number is more in the set than out and, thus, exhibits the given characteristic more than it lacks it. On a fundamental level, higher membership grades mean that a number either exhibits the characteristic perfectly or approximates it to varying degrees. The lower scores, however, indicate numbers for which we have comparatively less

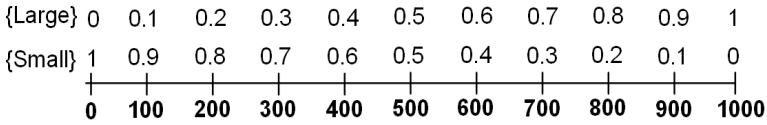


Fig. 6.1. Membership Grades for a Large and Small Set

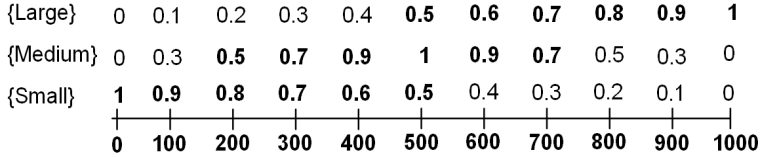


Fig. 6.2. Membership Grades for a Large, Medium, and Small Set

information. With a 1.0, we can definitely say that “Yes! It is this thing!” while for 0.2 we must be much less certain: “We know it is mostly not this thing. It could be any number of other things.” We could, however, construct a fuzzy set “small numbers” to counteract this.

In Figure 6.1, the membership grades for both our “large” and “small” sets are given. Whereas only considering one set (large) gave us 6 entries with grades equal to or larger than 0.5, considering both these sets simultaneously gives us numbers equal to or greater than 0.5 for all of the numbers; we just have to choose the set that reveals more information about whatever element with which we are particularly concerned. If we add a third fuzzy set, “medium sized numbers,” we obtain Figure 6.2.

All numbers equal to or greater than 0.5 are in bold type to illustrate that we have more information on numbers closer to the center than on the periphery. Admittedly, this information becomes redundant; once a perfect membership score appears in a column, we know all we could possibly know about an entry. For that reason, 0, 500, and 1,000 are not particularly interesting. What the addition of more fuzzy sets provides for us, however, is an increasingly detailed list of criteria for those numbers that do not perfectly match any one set. Consider the number 300, for instance with 0.7 in “Medium” and 0.7 in “Small.” It does not fit either set perfectly, but given the information, we can reasonably assert that it might fit extremely well into another set we might add entitled “numbers of medium-small-sized value.” Consider the number 600. It is relatively easy to deduce that this number would belong much better in a “slightly-larger-than medium-sized” fuzzy set than it would in a “slightly-smaller-than-medium-sized set.”

To express this more succinctly, consider a universe of elements  $U$  (homes, fruits, democracies, and so forth) and assume that we have the capacity to identify all the elements  $n$  in the universal set. The level of information we possess about the characteristics of every element in  $U$  is directly related to the number of non-trivial

fuzzy sets  $N$  we construct to describe them.<sup>1</sup> For instance, if the universe of citrus fruits  $U$  is comprised of 15 elements  $n$ , then three fuzzy sets  $N$  (describing characteristics such as size, pH level, and color) will provide more information than a single fuzzy set describing color would. That is, we can evaluate an orange's membership grades in fuzzy sets large-highly-acidic-yellow, small-highly-acidic-orange, and small-moderately-acidic-green and derive a reasonably well informed understanding about the extent to which an orange embodies any of these characteristics in actuality, outside of the set theoretical framework. If we were working with a smaller  $N$ , say just one set, our information would be less complete.

Of course, this idea squares rather well with the inherent differences between continuous and discrete fuzzy numbers discussed in chapter four. In fact, the information conveyed in membership scores is an issue of granularity. It is difficult and time consuming to construct classificatory systems of such fine granularity for large  $n$ . For the possessor of preferences over a range of options, it is similarly daunting to assemble a relation that closely approximates smooth continuity. But there are also marked differences between the way one might express information using fuzzy sets and how a political actor might feel about a range of policies scoring at particular levels in his fuzzy preference relation. We now ask, what do membership grades reveal about the elements in a fuzzy set when we recast a fuzzy set as a fuzzy preference relation? What can we say about the utility the actor derives from a policy and how intensely might she feel about the policy?

## 6.2 Fuzzy Preference Relations: Membership as Utility and Intensity

The standard crisp approach equates preference relations with varying degrees of utility: the higher the preference, the greater the utility that a political actor derives from an alternative. Our use of fuzzy geometry has made extensive use of a fuzzy set theory analog of this conceptualization of utility. We have used alpha-levels to indicate the degree of membership of policy alternatives in the set of "excellent" policy choices. Set membership at  $\alpha = 1$  indicates an excellent policy, set membership at  $\alpha = 0.75$  indicates a "very good policy" (one that is more excellent than not), set membership at  $\alpha = 0.50$  indicates a so-so policy (one that is neither excellent or not excellent), set membership at  $\alpha = 0.25$  indicates a poor policy (one that is less excellent than excellent), and set membership at  $\alpha = 0$  indicates a policy that is not excellent. Since a political actor derives maximum utility from an excellent policy, these same set inclusion levels (alpha-levels) are surrogates for utility. The higher the alpha-level (membership grade), the greater the utility.

<sup>1</sup> Of course, adding fuzzy sets for which the membership values are all zero results in no additional information. A more precise characterization is that the level of information we possess is weakly increasing with the number of fuzzy sets; that is, the level is non-decreasing.

Set membership level also gives us an ordinal measure of intensity of preference. Crisp set approaches that rank order discrete choices typically do not deal with intensity. For instance, given a set of alternatives,  $X = \{a, b, c\}$ , player  $i$  might have a preference order  $a P_i b P_i c$ . Hence, given a dichotomous choice between  $a$  and  $b$  she prefers  $a$ . We have no idea how much more intensely she prefers  $a$  to  $b$ , or  $b$  to  $c$ . While crisp set geometric models overcome this limitation by adopting the Euclidean distance assumption, the result is that intensity is measured in overly precise terms, and then only by assumption. Furthermore, as we have demonstrated in previous chapters, this comes at a cost: the ability to deal with indifference is virtually negated. Membership grades in fuzzy set theory permit treatment of ordinal differences between policy alternatives in an actor's preference order. Hence, scholars may consider relative intensity of preferences that are non-Euclidean. Instead of merely stating that  $a P_i b P_i c$ , we can state binary preferences as  $p(a, b) = 0.75$ ,  $p(b, c) = 0.50$ ,  $p(a, c) = 0.25$ . In this case  $a$  is preferred to  $b$  at the relatively intense .75 level,  $b$  is preferred to  $c$  at the moderately intense .50 level, and  $a$  is preferred to  $c$  at the low intensity .25 level. As our granularity improves, we can achieve even greater levels of precision in determining intensity of preferences. For instance, we might conceivably be able to determine that  $p(a, b) = 0.90$ ,  $p(b, c) = 0.55$ ,  $p(a, c) = 0.15$ .

The middling value 0.50, which is most often characteristic of an element "just as much in the set as not," further improves our understanding of intensity of preferences. In terms of the information conveyed by membership grades, this is the point at which we have the poorest information. At high membership grades, we know that an element embodies most of the characteristics of the set and at low grades we know that an element lacks most of those characteristics; but at 0.50, we are unclear. A policy scoring 0.50 for an actor would find that individual evaluating the policy as being both equally in and equally out of his set of preferred policies. Accordingly, we would not expect him to muster much emotion or intense preference over that policy and would most likely cede the ground.

If we take 0.50 as a qualitative benchmark, policies at higher  $a$ -cuts can be identified as those that an actor would like to see enacted at higher levels of intensity. Similarly, scores below the 0.50 level can be identified as grades indicating that an actor would rather not see a given policy enacted with lower grades revealing higher levels of distaste for a policy. Moving away from the conceptualization of utility we presented above, membership grades can be recast primarily as indicators of intensity of preference where degrees of intensity, for or against an option, increase as one moves higher or lower in a fuzzy set away from the 0.50 point of qualitative ambivalence.

Hence, we can use fuzzy numbers to measure utility or intensity of preference. In the first case, if an actor is presented with two options, one which he scores at the 0.50 level of set inclusion in the set of excellent policies and the other at the .10 level, we might argue that he prefers the first to the latter based on its higher level of utility. Therefore, we expect him to cast his vote for the first alternative. On the other hand, if we focus on the intensity with which he rank orders the two options, we might argue that he will feel rather strongly that the .10 policy

should not be enacted while the 0.50 policy fails to foster in him much emotion at all. In this scenario, the actor is likely to focus his political efforts on ensuring that the .10 policy will never become law, perhaps ignoring the 0.50 altogether. Hence, we expect him to be far more likely to cast his ballot when the second option comes up for a vote than when the first alternative's turn comes. Formal models in comparative politics have thus far focused on utility calculations, but calculations of the intensity with which political actors hold their preferences opens up an intriguing set of considerations.

### 6.3 Aggregating Fuzzy Preferences in Spatial Analysis

An important challenge for fuzzy spatial models is how best to determine an individual actor's preference order over a set of (discrete or continuous) alternatives. In  $n$ -dimensional space ( $n > 1$ ), alternatives comprise two or more issue dimensions. The Euclidean distance assumption permits crisp modelers to compare and rank order any point  $\langle x_i, y_i \rangle$  with any other point  $\langle x_j, y_j \rangle$  based on the absolute distance of each from an ideal point. In order to compare and rank order points in two-dimensional fuzzy space, we must create an aggregate score across the two dimensions. The modeler can then compare aggregate scores to determine an individual actor's preference order.

A consideration of intensity might help to establish the characteristics of the scores that we would want from an aggregation operator. We begin by assuming that both dimensions (represented by the  $x$ - and  $y$ -axes) are weighted equally. We further assume that,

1. actors are ambivalent (due to uncertainty) about a point at  $(0.50, 0.50)$ ,
2. 0.25 and 0.75 both indicate moderate levels of preference intensity, and
3. 1 and 0 both indicate extreme levels of preference intensity.

Our discussion of intensity argues that an actor will be relatively ambivalent concerning policies that are qualitatively "just as much in the set as out." Therefore, ordered pairs of two-dimensional membership grades where at least one of the dimensions is at  $\alpha = 0.50$  should result in the aggregation operator defining a score very near the other number to compensate for the indifference on the 0.50 dimension. For instance, given  $a = 0.90$  and  $b = 0.50$ , the resulting value should be quite close to 0.90, as the actor holding this preference feels very intensely along one dimension and not at all along the other. By the same token, when an actor is faced with a status quo point at  $\alpha = 0.25$  and a proposal at  $\alpha = 0.50$ , he will choose the latter option even if its status in the set is fairly ambiguous. After all, ambiguity (or indifference) is better than definitive repugnance.

In making comparisons across several pairs of membership grades, we can construct a set of heuristic rules from which we can deduce a theoretically sound aggregation technique. There are fifteen possible combinations of membership grades. (Since the dimensions are weighted identically, the  $x$  and  $y$  values can be switched without changing the aggregate value.)

$$\begin{aligned}
 &\langle 1.0, 1.0 \rangle, \quad \langle 1.0, 0.75 \rangle, \langle 0.75, 0.75 \rangle, \langle 1.0, 0.5 \rangle, \\
 &\langle 0.75, 0.5 \rangle, \quad \langle 1.0, 0.25 \rangle, \langle 0.5, 0.5 \rangle, \quad \langle 0.25, 0.75 \rangle, \\
 &\langle 1.0, 0.0 \rangle, \quad \langle 0.5, 0.25 \rangle, \langle 0.75, 0.0 \rangle, \quad \langle 0.25, 0.0 \rangle, \\
 &\langle 0.25, 0.25 \rangle, \langle 0.5, 0.0 \rangle, \quad \langle 0.0, 0.0 \rangle.
 \end{aligned} \tag{6.5}$$

Based on our assumptions about intensity, we can rank order these ordered pairs as follows:

1.  $\langle 1, 1 \rangle$  best with an aggregate value very close or equal to 1
2.  $\langle 1, 0.75 \rangle$  higher than .75, as we have both moderate and extreme present
3.  $\langle 0.75, 0.75 \rangle$  right at 0.75, as we have the same value
4.  $\langle 1, 0.50 \rangle$  very close to 1 as we are dealing with a high value and ambivalence
5.  $\langle 0.75, 0.50 \rangle$  around 0.75
6.  $\langle 1, 0.25 \rangle$  closer to 1 than to 0.25 and greater than .05
7.  $\langle 0.50, 0.50 \rangle$  the same value obtains resulting in ambivalence
8.  $\langle 0.25, 0.75 \rangle$  the two scores cancel each out, resulting in ambivalence
9.  $\langle 1, 0 \rangle$  extremes on opposite ends, resulting in ambivalence
10.  $\langle 0.75, 0 \rangle$  closer to 0 than to .75 and less than .5
11.  $\langle 0.50, 0.25 \rangle$  around 0.25
12.  $\langle 0.25, 0.25 \rangle$  right at 0.25
13.  $\langle 0.25, 0 \rangle$  less than .25, with both moderate and extreme present
14.  $\langle 0.50, 0 \rangle$  very close to 0 as we are dealing with an extreme and ambivalence
15.  $\langle 0, 0 \rangle$  worst case with a value of 0

To use the numerical listing as proxies for the ordered pairs, the formal preference relation can be constructed as:

$$1 > 2 > 3 > 4 > 5 > 6 > 7 = 8 = 9 > 10 > 11 > 12 > 13 > 14 > 15. \tag{6.6}$$

## 6.4 Aggregation Operators

Just what aggregation operator might we select that would approximate this ordering of preferences? An entire class of aggregation operators, known as *t*-norms, permits us to consider how individual actors might aggregate their set membership scores on two or more dimensions in order to permit comparisons of alternatives. We discuss *t*-norms in the appendix to this chapter. While a number of scholars have addressed the use of aggregation operators on fuzzy data (see for example, Smithson and Verkuilen, 2006), none have done so for purposes of determining cross-dimensional individual preferences.

The simplest methods for aggregating a single score across dimensions are the fuzzy intersection and union operators, both of which were discussed at length in chapter two. Given a policy point with two dimensions of consideration, the intersection operator chooses the smallest membership grade across the two dimensions. In essence, the resulting aggregate score for a policy along both

**Table 6.1.** Aggregated Preference Scores for the Intersection Operator

	<b>0.000</b>	<b>0.250</b>	<b>0.500</b>	<b>0.750</b>	<b>1.000</b>
<b>0.000</b>	0.000	0.000	0.000	0.000	0.000
<b>0.250</b>	0.000	0.250	0.250	0.250	0.250
<b>0.500</b>	0.000	0.250	0.500	0.500	0.500
<b>0.750</b>	0.000	0.250	0.500	0.750	0.750
<b>1.000</b>	0.000	0.250	0.500	0.750	1.000

**Table 6.2.** Aggregated Preference Scores for the Union Operator

	<b>0.000</b>	<b>0.250</b>	<b>0.500</b>	<b>0.750</b>	<b>1.000</b>
<b>0.000</b>	0.000	0.250	0.500	0.750	1.000
<b>0.250</b>	0.250	0.250	0.500	0.750	1.000
<b>0.500</b>	0.500	0.500	0.500	0.750	1.000
<b>0.750</b>	0.750	0.750	0.750	0.750	1.000
<b>1.000</b>	1.000	1.000	1.000	1.000	1.000

dimensions is the score at which the individual is minimally satisfied. Implicit in the intersection operator is that the individual ignores additional gains she might obtain on the dimension where she has the higher utility. Hence, if we choose to adopt the minimum aggregation principle (intersection operator), we are assuming that political players are very conservative. They seek to assure themselves some minimal threshold on all dimensions.

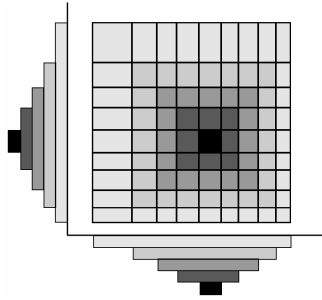
The union operator takes the highest membership grade between the two dimensions. In this way, a union operator settles upon the value that makes the individual maximally satisfied on one of the dimensions. Adoption of the union operator implies that political players are unconcerned with losses on the dimension with the lower utility. Such players are extremists concerned with optimizing their gains on one dimension of the policy issue and not willing to engage in bartering for higher gains on the other.

Given discrete membership grades in the set

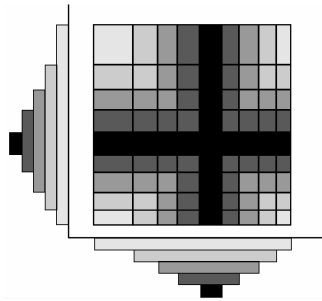
$$M = \{0.00, 0.25, 0.50, 0.75, 1.00\} \tag{6.7}$$

for a set of alternatives in two-dimensional space, Tables 6.1 and 6.2 exhaust the set of possible  $\langle x, y \rangle$  coordinates. The columns list the membership grade on the  $x$ -dimension of the alternative and the rows list the membership grade on the  $y$ -dimension. Aggregated membership grades for the intersection (minimum) operator are given in the cells of Table 6.1 lying at the intersection of the membership grade on the  $x$ -dimension and the membership grade on the  $y$ -dimension. Those for the union (maximum) operator are in Table 6.2.

The resulting geometric representations of assuming that players aggregate a single score across two dimensions using the intersection (minimum) or union (maximum) operator are quite different as demonstrated in Figures 6.3 and 6.4. The series of boxes below the  $x$ -axis and to the left of the  $y$ -axis give the shape



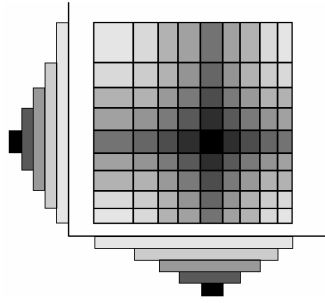
**Fig. 6.3.** An Individual Player’s Preferences in Two-Dimensional Space Using the Intersection Operator



**Fig. 6.4.** An Individual Player’s Preferences in Two-Dimensional Space Using the Union Operator

of the discrete fuzzy number at alpha-cuts of 1, 0.75, 0.50, 0.25, and 0 on each dimension of a given policy. The resulting alpha-level in two-dimensional space is then determined based on the given aggregator. The area of space at  $\alpha = 1$  is the dark area in the figure. The area of space at  $\alpha = 0.75$  is slightly lighter, and so on to  $\alpha = 0$ . Figure 6.3 demonstrates that using the intersection operator results in the representation of an individual player’s preferences on a two-dimensional issue as a series of symmetrical, concentric rectangles, with  $\alpha = 1$  located in the middle and each alpha-level at a lower level wrapped around it. In contrast, Figure 6.4 demonstrates that the union operator results in a series of symmetrical, concentric crosses.

The reader will note that this representation of an individual fuzzy number in two-dimensional space is the one that we adopted in the previous chapter. The cross-shape induced by the union operator complicates predictions. Apart from an argument for utility, however, the union operator appears to be modeling extremist players who will not bargain or compromise on any dimension of a policy issue and are most concerned about the dimension on which they have the highest satisfaction. While this might be useful for modeling some types of political situations, we believe the majority involve a more moderate type of



**Fig. 6.5.** An Individual Player's Preferences in Two-Dimensional Space Using the Average Operator

politics, particularly in democratic political systems. Furthermore, we find many scenarios implausible. For example, in a choice between two alternatives, one of which has a membership grade in two-dimensional  $\langle x, y \rangle$  space of  $\langle 1, 0 \rangle$  and the other  $\langle 0.75, 0.75 \rangle$ , the union operator argues the individual will select the first option. In essence, the possibility of bargaining across dimensions is negated.

Having said that, the principle of selecting based totally on the minimum membership grade is equally unsatisfying for much the same reason. While the rectangular shape of individual preferences at different alpha-level levels is quite useful for geometric modeling, the assumptions concerning political behavior are overly conservative and in many cases equally implausible. For example, using the intersection operator a player is judged indifferent in a choice between  $\langle 0.1, 0.5 \rangle$  and  $\langle 0.5, 0.5 \rangle$ . Furthermore, as with the union operator, cross-dimensional trade-offs are not permitted.

At first glance, the average operator offers an attractive solution, if for no other reason than it takes into account the possibility of trade-offs from one dimension to the other to achieve higher gains on a dimension with a lower utility payoff. (The aggregate scores for the average operator are at Table 6.3.) However, as Figure 6.5 demonstrates, the resulting representation of individual preferences in two-dimensional space takes an irregular shape past  $\alpha = 0.75$ . More importantly, trade-offs across dimensions are balanced. Players value both the dimension on which they have the highest utility and that with the lower utility equally. This accords with the argument that dimensions are equally valued and therefore, equally weighted. It seems to us that a stronger argument can be made that a political actor is more sensitive to losses on a dimension with lower utility. While this is the general principle behind the conservatism of the intersection (minimum) operator, the intersection (maximum) operator permits no trade-offs.

We contend that a formula weighting the dimension with the lower utility more heavily than the maximum utility dimension is a better option. While assuming risk aversion on the part of political actors, it permits trade-offs across dimensions, particularly those that increase utility on the dimension with a lower utility. The general form of the operator is:

$$a(\min\langle x, y \rangle) + b(\max\langle x, y \rangle), \quad (6.8)$$

**Table 6.3.** Aggregated Preference Scores for the Average Operator

	<b>0.000</b>	<b>0.250</b>	<b>0.500</b>	<b>0.750</b>	<b>1.000</b>
<b>0.000</b>	0.000	0.125	0.250	0.375	0.500
<b>0.250</b>	0.125	0.250	0.375	0.500	0.625
<b>0.500</b>	0.250	0.375	0.500	0.625	0.750
<b>0.750</b>	0.375	0.500	0.625	0.750	0.875
<b>1.000</b>	0.500	0.625	0.750	0.875	1.000

where

$$a + b = 1. \tag{6.9}$$

Equation [6.8](#), also known as ordered weighted averaging (OWA) is in fact a general description of the intersection and union operators. In the case of the intersection operator,  $a = 1$  and  $b = 0$ , whereas for the union operator,  $a = 0$  and  $b = 1$ . In the case of the average operator (arithmetic mean<sup>2</sup>),  $a = b = 0.50$ . An operator in which  $a = .75$  or more and  $b = .25$  or less seems the best choice. Trade-offs are permitted that favor the lower utility dimension and the resulting representation of individual preferences in two-dimensional space is virtually identical to that of the intersection (minimum) operator. For these reasons, we employed such an operator in the geometric models we developed in the previous chapter.

For certain there are other aggregation operators among which scholars may choose. However, we believe that the utility and elegance of the OWA, equation [6.8](#), with  $a \geq .75$ , commends itself. It represents preferences in a form that is both intuitive and easy to analyze while permitting us to model political players who are willing to make trades across the dimensions of an issue but at the same time are particularly concerned about losses on the dimension on which they have the least utility. Nonetheless, the importance of the choice of an aggregation operator cannot be overstated. Depending on how we aggregate preferences at the individual level, a person’s preference relation might actually change. For example, suppose that an individual is evaluating options  $a, b$ , and  $c$  and that his preferences are defined as  $a = \langle 0.40, 0.60 \rangle$ ,  $b = \langle 0.10, 0.90 \rangle$ , and  $c = \langle 0.50, 0.50 \rangle$ . The intersection operator returns a preference ordering of  $c P a P b$ , the union operator results in  $b P a P c$ , and the arithmetic mean operator results in  $a P b P c$ . Furthermore, the prediction of the collective choice will in turn be effected by the choice of the operator that we choose for aggregating an individual’s preferences across dimensions. Therefore, the choice of an aggregation operator is more than a mere matter of taste, and scholars need to give their choice serious consideration.

Toward that end, we discuss a few additional operators that might be considered, depending on the assumptions made about the calculations of political players in determining the relative merits of one alternative over another in

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<sup>2</sup> Smithson and Verkuilen (2006, pp. 14-15) briefly discuss the arithmetic mean, but they do not explicitly apply it to preference aggregation.

**Table 6.4.** Aggregated Preference Scores for the Quadratic Mean Operator

	<b>0.000</b>	<b>0.250</b>	<b>0.500</b>	<b>0.750</b>	<b>1.000</b>
<b>0.000</b>	0.000	0.177	0.354	0.530	0.707
<b>0.250</b>	0.177	0.250	0.395	0.559	0.729
<b>0.500</b>	0.354	0.395	0.500	0.637	0.791
<b>0.750</b>	0.530	0.559	0.637	0.750	0.884
<b>1.000</b>	0.707	0.729	0.791	0.884	1.000

two-dimensional space. (We also discuss the class of t-norms that may prove useful for aggregation purposes in the appendix to this chapter.)

One operator that yields a range of plausible aggregated values is the quadratic mean, a variation on the arithmetic mean:

$$\sqrt{\frac{a^2 + b^2}{2}}. \tag{6.10}$$

The aggregate scores for the quadratic mean are in Table 6.4. Generally, the quadratic mean yields values more biased toward the higher of the two constituent membership grades, lending itself more to a theory of utility maximization rather than minimal satisfaction.

Two other operators are the geometric mean<sup>3</sup>

$$\sqrt{a * b} \tag{6.11}$$

and the harmonic mean:

$$\frac{2}{\frac{1}{a} + \frac{1}{b}} \text{ for } a, b \neq 0, 0 \text{ when } a = 0 \text{ or } b = 0. \tag{6.12}$$

The aggregate scores for the geometric mean are in Table 6.5 and those for the harmonic mean are in Table 6.6. Both operators are less compelling to our way of thinking as they return to one of the major problems of the intersection operator: they wash out information and subtle nuances between options at the fringes of the fuzzy preference relations.

In spite of the fact that different aggregation operators return different results, there are some clearly discernible patterns. In every case, mirrored ordered pairs such as  $\langle 75, 0.75 \rangle$  or  $\langle 1, 1 \rangle$  yield an aggregate value equal to their constituent values. (That is  $\langle 0.75, 0.75 \rangle$  always returns an aggregate value of 0.75, and  $\langle 1, 1 \rangle$  always returns a value of 1.) Furthermore, while there are marked differences in the range of values (for instance, the range for the quadratic mean is 0.707 while that for the average operator is 0.750), the largest variations between aggregate values for membership grades in two-dimensional space occur with ordered pairs where one or the other value is either very high or very low. However, when

<sup>3</sup> Smithson and Verkuilen (2006, p. 14) briefly discuss the geometric mean, but they do not explicitly apply it to preference aggregation.

**Table 6.5.** Aggregated Preference Scores for the Geometric Mean Operator

	<b>0.000</b>	<b>0.250</b>	<b>0.500</b>	<b>0.750</b>	<b>1.000</b>
<b>0.000</b>	0.000	0.000	0.000	0.000	0.000
<b>0.250</b>	0.000	0.250	0.354	0.433	0.500
<b>0.500</b>	0.000	0.354	0.500	0.612	0.707
<b>0.750</b>	0.000	0.433	0.612	0.750	0.866
<b>1.000</b>	0.000	0.500	0.707	0.866	1.000

**Table 6.6.** Aggregated Preference Scores for the Harmonic Mean Operator

	<b>0.000</b>	<b>0.250</b>	<b>0.500</b>	<b>0.750</b>	<b>1.000</b>
<b>0.000</b>	0.000	0.000	0.000	0.000	0.000
<b>0.250</b>	0.000	0.250	0.333	0.375	0.400
<b>0.500</b>	0.000	0.333	0.500	0.600	0.667
<b>0.750</b>	0.000	0.375	0.600	0.750	0.857
<b>1.000</b>	0.000	0.400	0.667	0.857	1.000

dealing with middle range values or values similar to each other, there is much less deviation between the sets of scores the two operators produce.

Take, for example two points:  $a$  at  $\langle 0.75, 0.20 \rangle$  and  $b$  at  $\langle 0.25, 0.10 \rangle$ . Using different sets of operators on these two points, we derive the following aggregated values:

Operator	Points	
	$\mathbf{a} = \langle 0.75, 0.20 \rangle$	$\mathbf{b} = \langle 0.25, 0.10 \rangle$
Intersection	0.20	0.10
Union	0.75	0.25
Average	0.475	0.175

Notice the variability in aggregate scores on  $a$ , where the range between the two values in the ordered pair is much larger than those for point  $b$ . When choosing a method for modeling fuzzy preference aggregations then, political scientists must consider what sort of derived values best fit their models. Do we want greater or less variability among scores? Do we accept bias toward the low end or the higher end of the spectrum?

## 6.5 Summary

This chapter has explored what membership grades reveal about actors' preferences, intensity, and knowledge regarding a range of policies. In particular we considered the insights into the nature of political preferences that a fuzzy set's middle value offers. We discussed its role as an indicator of information deficit, low preference of intensity, and possible inaction on the part of a political actor.

A fuzzy set conceptualization of preferences, then, offers several advantages over the traditional crisp approach. Instead of simple rank orderings of policies, where all we know is which policy is better than another, a listing of fuzzy set membership grades provides us with an ordinal ranking of policies and an indication of the amount of information that a political player might possess about the policy and the intensity with which she might prefer the policy. Couple these characteristics with fuzzy set theory's ability to model preference granularity into membership grades as well and we are left with an approach to the modeling of political preferences that is potentially more informative than that advanced by traditional crisp set theory.

That being said, the construction of individual fuzzy preference relations requires a great deal of consideration and a careful examination and justification of assumptions. In this chapter, we explored one construction technique that remains central to spatial analysis: aggregating preference values across dimensions. Given that social scientists will be working with a broad range of political situations, we thought it best to present a conceptual "tool kit" of fuzzy set operators rather than forcefully advocate any one approach. One of the benefits of using the fuzzy analytical approach in the construction of models using preferences is the flexibility it affords the researcher and this is a point we want to stress. In general, however, we believe that the choice of operator should at least loosely adhere to the table of heuristics we outlined at the onset of the discussion.

In the final chapter, we consider the implications of the granularity and informational issues we have raised throughout this book on the classic majority cycling problem. We demonstrate the implications of different sets of assumptions on resulting collective outcomes and, in doing so, arrive at some interesting observations about the nature of cycling in fuzzy models.

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## A Appendix: Fuzzy Operators

### A.1 Fuzzy Operators

Operators take one, two, or more fuzzy sets and produce another fuzzy set. An operator that takes only one argument is called a unary operator. The standard fuzzy complement is a unary operator. Those that take two fuzzy sets and produce a third are called binary operators. Chapter two discussed several binary operators, to include fuzzy intersection and union. If the fuzzy sets are fuzzy numbers then addition is another binary operator that takes two fuzzy numbers and produces their sum.

A single fuzzy set can be operated on by the application of a *fuzzy complement*. Two fuzzy sets can be combined by aggregation operations, which include *fuzzy intersection*, *union*, and *averaging operations*. Multiple fuzzy sets can be combined with *aggregation* operators. Operations of each of these types are not unique. The whole scope of operations of each type can be conveniently captured by a class of functions distinguished from one another by distinct values of a parameter taken from a specific range of values. The choice of a particular operation is determined by the purpose for which it is used.

By far, the most important and common fuzzy complement, intersection and union operations are those defined by the formulas

$$A^c(x) = 1 - A(x), \quad (6.13a)$$

$$(A \cap B)(x) = \min[A(x), B(x)], \quad (6.13b)$$

$$(A \cup B)(x) = \max[A(x), B(x)]. \quad (6.13c)$$

Axiomatic characterization of these operations, which are usually referred to as *standard fuzzy operations*, was investigated by [Bellman and Gierz \(1973\)](#). Any property generalized from classical set theory into the domain of fuzzy set theory that is preserved in all  $\alpha$ -cuts for  $\alpha \in (0, 1]$  is called *cutworthy*. An

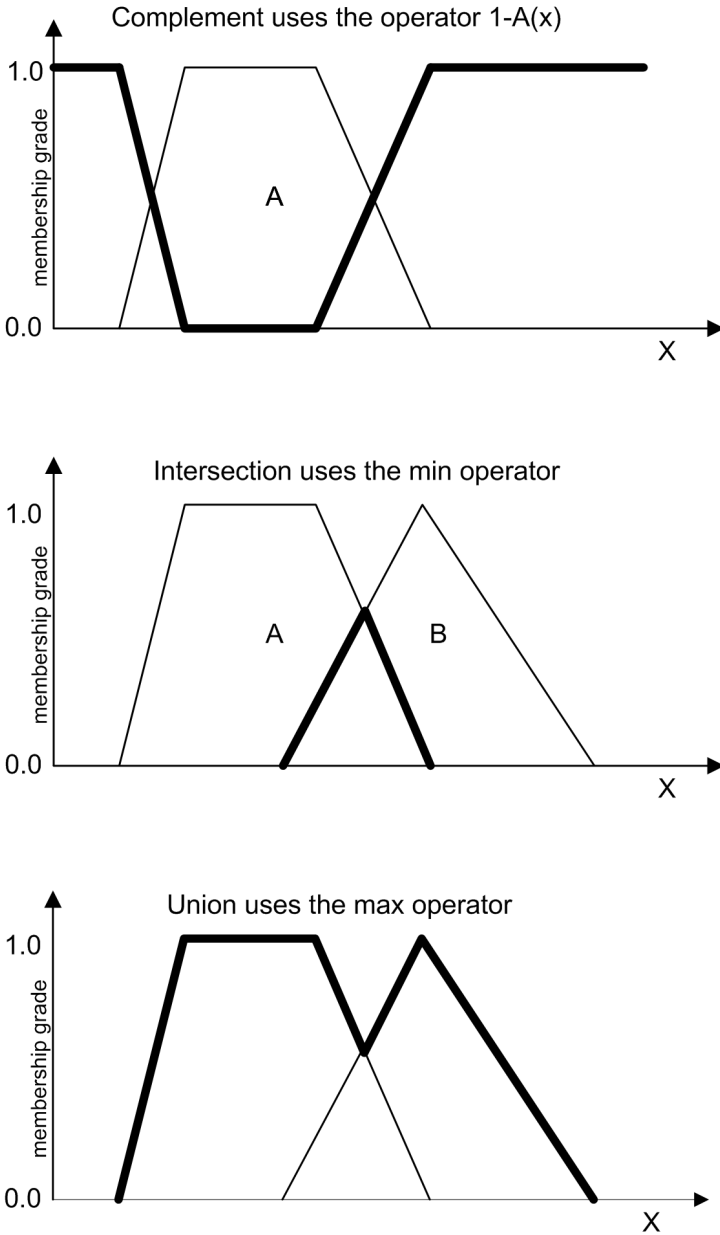
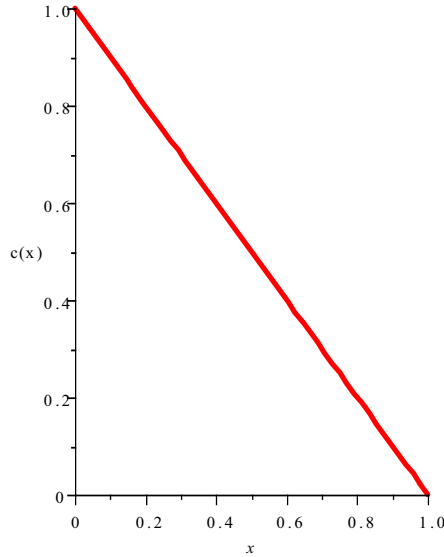


Fig. 6.6. Illustrating the Standard Fuzzy Operators

arbitrary binary operator  $\star$  is *idempotent* if  $a \star a = a$ . The minimum operation is the only fuzzy intersection that is idempotent and cutworthy; similarly, the maximum operation is the only union that is idempotent and cutworthy. No fuzzy complement is cutworthy.



**Fig. 6.7.** Standard Complement Function

### A.1.1 Fuzzy Complement

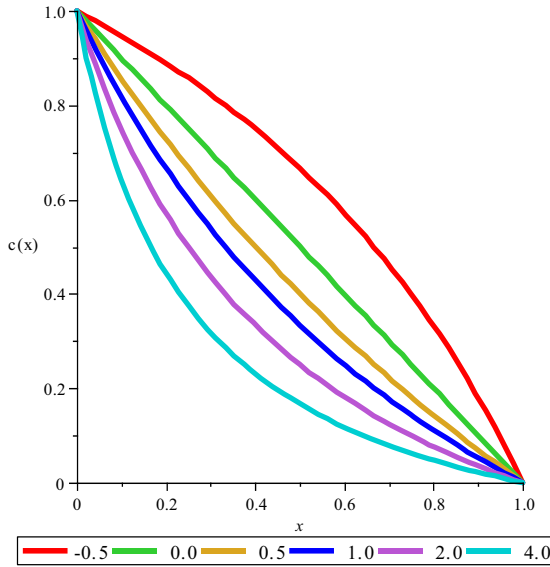
An arbitrary complement operator,  $c : [0, 1] \rightarrow [0, 1]$ , must satisfy the following three axioms:

- (c1) Membership dependency — The membership grade of  $x$  in the complement of  $A$  depends only on the membership grade of  $x$  in  $A$ . When a fuzzy set has all its membership values in the set  $\{0, 1\}$  then the fuzzy set is essentially a crisp set. In this case, the complement should be the same as the complement of a crisp set, so that the complement of zero is one and vice versa.
- (c2) Boundary condition —  $c(0) = 1$  and  $c(1) = 0$ , that is  $c$  behaves as the ordinary complement for crisp sets. The membership degree of an element  $x$  in the complement of a fuzzy set  $A$  should be a local phenomena, that is, it should depend only on the degree of membership of  $x$  in the original fuzzy set  $A$ .
- (c3) Monotonicity — For all  $a, b \in [0, 1]$ , if  $a < b$ , then  $c(a) \geq c(b)$ , that is  $c$  is monotonic non-increasing. The more  $x$  is in  $A$  the less  $x$  is in the complement of  $A$ .

Two additional axioms, which are usually considered desirable, constrain the large family of functions that would be permitted by the above three axioms; they are:

- (c4) Continuity —  $c$  is continuous.
- (c5) Involution —  $c$  is involutive, that is  $c(c(a)) = a$ .

The standard fuzzy complement function is:  $c(a) = 1 - a$  and it is graphed in Figure (6.7).



**Fig. 6.8.** Sugeno Complement Functions

Some of the functions that conform to these five axioms besides the standard fuzzy complement are in the Sugeno class of fuzzy complements defined for all  $a \in [0, 1]$  by

$$c[\lambda](a) = \frac{1 - a}{1 + \lambda a}, \tag{6.14}$$

with  $\lambda \in (-1, \infty)$ . The curves generated by the Sugeno complement for various values of  $\lambda$  are illustrated in Figure (6.8). Notice that the standard fuzzy complement is a Sugeno complement when  $\lambda = 0$ .

The Yager class of fuzzy complements are defined for all  $a \in [0, 1]$  by

$$c[w](a) = (1 - a^w)^{1/w} \tag{6.15}$$

with  $w \in (0, \infty)$ . The curves generated by the Yager complement for various values of  $w$  are illustrated in Figure (6.9). Notice again that the standard fuzzy complement is a Yager complement when  $w = 1$ .

Observe that the standard fuzzy complement,  $c(a) = 1 - a$ , is obtained as the Sugeno complement at zero,  $c[\lambda = 0]$  and as the Yager complement at one,  $c[w = 1]$ .

An example of fuzzy complements that conform to (c1)–(c3) but not to (c4) and (c5) are the threshold fuzzy complements

$$c[t](a) = \begin{cases} 1 & \text{when } a \in [0, t] \\ 0 & \text{when } a \in (t, 1] \end{cases} \tag{6.16}$$

with  $t \in [0, 1]$ .

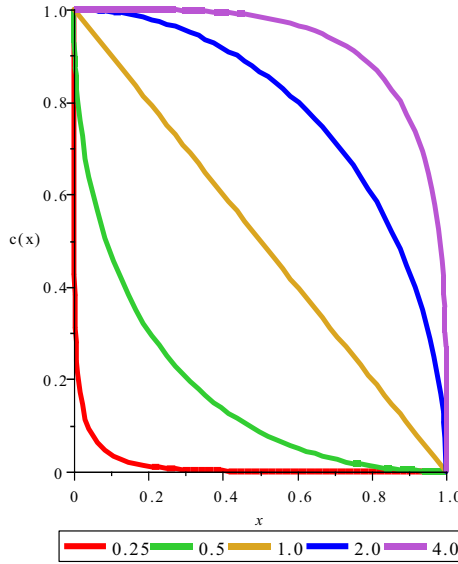


Fig. 6.9. Yager Complement Functions

Subsequently, we shall write  $A^c$  for an arbitrary complement of the fuzzy set  $A$ ; its membership function is  $A^c(x) = c(A(x))$ .

An equilibrium,  $e_c$ , of a fuzzy complement  $c$ , if it exists, is a number in  $[0, 1]$  for which  $c(e_c) = e_c$ . Every fuzzy complement has at most one fuzzy equilibrium and if a fuzzy complement is continuous (i.e., if it satisfies axioms (c1)–(c4)), the existence of its equilibrium is guaranteed (Klir and Yuan, 1996). For example, the equilibria of fuzzy complements in the Yager class (6.15) are

$$e_c[w] = 0.5^{\frac{1}{w}} \tag{6.17}$$

for each  $w \in (0, \infty)$ .

### A.1.2 Fuzzy Set Intersections

Complementation can be based on a function  $c$  that manipulates membership values. Using the same line of reasoning, intersection of fuzzy sets  $A$  and  $B$  can be based upon a function  $t$  that takes two arguments. The function  $t$  takes the membership grade of  $x$  in the fuzzy set  $A$  and the membership grade of  $x$  in the fuzzy set  $B$  and returns the membership grade of  $x$  in the fuzzy set  $A \cap B$ .

Fuzzy set intersection works the same as crisp intersection whenever the fuzzy sets have membership grades restricted to the set  $\{0, 1\}$ . Since  $A \cap B = B \cap A$  the function should be commutative. Since  $(A \cap B) \cap C = A \cap (B \cap C)$  the function should be associative. Lastly the more  $x$  is in fuzzy set  $A$  and  $B$  the more it should be in  $A \cap B$ .

The intersection of two fuzzy sets is a function that maps pairs of numbers in the unit interval into the unit interval  $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$ . It is now

well established that *triangular norms* or *t-norms* do possess all properties that are intuitively associated with fuzzy intersections. These functions are, for all  $a, b, c, d \in [0, 1]$ , characterized by the following axioms:

- (i1) Boundary condition —  $t(1, a) = a$ .
- (i2) Monotonicity —  $t(a, b) \leq t(c, d)$  whenever  $a \leq c$  and  $b \leq d$ .
- (i3) Commutativity —  $t(a, b) = t(b, a)$ .
- (i4) Associativity —  $t(a, t(b, c)) = t(t(a, b), c)$ .

Functions that obey these four rules have been extensively studied in the literature of probability before the creation of fuzzy set theory. The intersection functions get the name *t-norm* from this literature (Menger, 1942).

The largest *t-norm* is the minimum function and the smallest is the drastic product

$$t_{\min}(a, b) = \begin{cases} a & \text{when } b = 1 \\ b & \text{when } a = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{6.18}$$

If  $t$  is any *t-norm* then for any  $a, b \in [0, 1]$ ,

$$t_{\min}(a, b) \leq t(a, b) \leq \min(a, b). \tag{6.19}$$

One of the most commonly applied alternative *t-norms* is the algebraic product:

$$t_p(a, b) = a \cdot b, \tag{6.20}$$

which is calculated by simply multiplying  $a$  and  $b$ . Since multiplication is commutative and associative, axioms **i3** and **i4** are satisfied by  $t_p$ . Multiplication of nonnegative numbers is monotonic so axiom **i2** is satisfied (since all membership values are non-negative). Finally  $t_p(1, a) = 1 \cdot a = a$  so that axiom **i1** is satisfied. Thus  $t_p$  is a *t-norm*.

Another common alternative to the standard intersection operator  $\min$  is the bounded difference:

$$t_b(a, b) = \max(0, a + b - 1). \tag{6.21}$$

Since  $t_b(1, a) = \max(0, 1 + a - 1) = a$  the bounded difference satisfies axiom **i1**. The function  $\max(0, a + b - 1)$  is obviously commutative (since  $a + b - 1 = b + a - 1$ ) and since  $a + b - 1$  and  $\max$  are both monotone, their composition is monotone. Thus the bounded difference satisfies axioms **i3** and **i2**.

It can be proven that  $t_b(a, b, c) = \max(0, a + b + c - 2)$ . Therefore, the bounded difference is associative, and satisfies axiom **i4**.

It can also be shown that the basic *t-norms* have the following order:

$$t_{\min}(a, b) \leq \max(0, a + b - 1) \leq ab \leq \min(a, b) \tag{6.22}$$

The most common *t-norms* are given in Table (6.7).

It is sometimes convenient to think of a *t-norm* as binary operator, and use an in-line notation when applying it to a specific problem. Thus instead of writing  $t(a, b)$  we would write  $a \wedge_t b$ , or  $a \mathop{\wedge}_t b$ , or just  $a \wedge b$ , with the understanding that  $\wedge$  is some intersection operator modeled by a *t-norm*  $t$ .

**Table 6.7.** Common t-norms

Name	t-norm
standard intersection	$t(a, b) = \min(a, b)$
algebraic product	$t_p(a, b) = ab$
bounded difference	$t_b(a, b) = \max(0, a + b - 1)$
drastic product	$t_{\min}(a, b) = \begin{cases} a & \text{when } b = 1 \\ b & \text{when } a = 1 \\ 0 & \text{otherwise} \end{cases}$

### A.1.3 Fuzzy Set Unions

Similar ideas from the discussion of fuzzy set intersections lead to the following axiomatic skeleton for a function  $s$  to model the union operator. The union of two fuzzy sets must be a function that maps pairs of numbers in the unit interval into the unit interval,  $s: [0, 1] \times [0, 1] \rightarrow [0, 1]$ . Functions known as *triangular conorms* or t-conorms, possess all the properties that are intuitively associated with fuzzy unions. They are characterized for all  $a, b, c, d \in [0, 1]$  by the following axioms:

- (u1) Boundary condition —  $s(0, a) = a$ .
- (u2) Monotonicity —  $s(a, b) \leq s(c, d)$  whenever  $a \leq c$  and  $b \leq d$ .
- (u3) Commutativity —  $s(a, b) = s(b, a)$ .
- (u4) Associativity —  $s(a, s(b, c)) = s(s(a, b), c)$ .

The smallest t-conorm is the maximum function and the largest is the drastic sum sometimes called the drastic union:

$$s_{\max}(a, b) = \begin{cases} a & \text{when } b = 0 \\ b & \text{when } a = 0 \\ 1 & \text{otherwise.} \end{cases} \tag{6.23}$$

If  $s$  is any t-conorm then for any  $a, b \in [0, 1]$

$$\max(a, b) \leq s(a, b) \leq s_{\max}(a, b). \tag{6.24}$$

One of the most commonly applied alternative t-conorms is the algebraic sum:

$$s_p(a, b) = a + b - ab, \tag{6.25}$$

which is also called the probabilistic sum.

Another common alternative to the standard union operator  $\max$  is the bounded sum:

$$s_b(a, b) = \min(1, a + b). \tag{6.26}$$

It can also be shown that the basic t-conorms have the following order:

$$s_{\max}(a, b) \geq \min(1, a + b) \geq a + b - ab \geq \max(a, b). \tag{6.27}$$

**Table 6.8.** Common t-conorms

Name	t-conorm
standard union	$s(a, b) = \min(a, b)$
algebraic sum	$s_p(a, b) = a + b - ab$
bounded sum	$s_b(a, b) = \min(1, a + b)$
drastic sum	$s_{\max}(a, b) = \begin{cases} a & \text{when } b = 0 \\ b & \text{when } a = 0 \\ 1 & \text{otherwise} \end{cases}$

The most common t-conorms are given in Table (6.8).

It is sometimes convenient to think of a t-conorm (sometimes called an s-norm) as a binary operator, and use an in-line notation when applying it to a specific problem. Thus instead of writing  $s(a, b)$  we would write  $a \vee_s b$ , or  $a \mathbf{s} b$ , or just  $a \vee b$ , with the understanding that  $\vee$  is some union operator modeled by a t-conorm  $s$ .

### A.2 Generating Functions

Most t-norms, t-conorms and complements can be generated from some underlying function, aptly named a generating function. We start with the simplest:

**Theorem 6.1 (First Characterization Theorem of Fuzzy Complements)**  
*Let  $c$  be a function from  $[0, 1]$  to  $[0, 1]$ . Then  $c$  is an involutive fuzzy complement if and only if there exists a continuous function  $g$  from  $[0, 1]$  on  $\mathbb{R}$  such that  $g(0) = 0$ ,  $g$  is strictly increasing, and*

$$c(a) = g^{-1}(g(1) - g(a)), \tag{6.28}$$

where  $g^{-1}$  is the inverse of  $g$  for all  $a$  in  $[0, 1]$ .

Functions  $g$  are usually called *increasing generators*. Each function that qualifies as an increasing generator determines an involutive fuzzy complement by the equation above.

For a Standard Fuzzy Complement the increasing generator is

$$g(a) = a. \tag{6.29}$$

For the Sugeno class of complements the increasing generator is

$$g_\lambda(a) = \frac{1}{\lambda} \ln(1 + \lambda a) \tag{6.30}$$

with  $\lambda > -1$ .

For the Yager class of complements, the increasing generator is

$$g_w(a) = a^w \tag{6.31}$$

with  $w > 0$ .

We can combine these to give a double parameterized increasing generator:

$$g_{\lambda,w}(a) = \frac{1}{\lambda} \ln(1 + \lambda a^w) \tag{6.32}$$

with  $\lambda > -1$  and  $w > 0$ . This yields

$$c_{\lambda,w}(a) = \left( \frac{1 - a^w}{1 + \lambda a^w} \right)^{1/w} \tag{6.33}$$

which contains both the Sugeno class and the Yager class as special subclasses.

As one more example, the increasing generator

$$g_{\gamma}(a) = \frac{a}{\gamma + (1 - \gamma)a} \tag{6.34}$$

with  $\gamma > 0$  produces the class of involutive fuzzy complements

$$c_{\gamma}(a) = \frac{\gamma^2(1 - a)}{a + \gamma^2(1 - a)} \tag{6.35}$$

with  $\gamma > 0$ .

Involutive fuzzy complements can also be produced by *decreasing generators*.

**Theorem 6.2 (Second Characterization Theorem of Fuzzy Complements)**

Let  $c$  be a function from  $[0, 1]$  to  $[0, 1]$ . Then  $c$  is an involutive fuzzy complement iff there exists a continuous function  $f$  from  $[0, 1]$  to  $\mathbb{R}$  such that  $f(1) = 0$ ,  $f$  is strictly decreasing, and

$$c(a) = f^{-1}(f(0) - f(a)) \tag{6.36}$$

where  $f^{-1}$  is the inverse of  $f$  for all  $a$  in  $[0, 1]$ .

For a Standard Fuzzy Complement the decreasing generator is

$$f(a) = -ka + k \tag{6.37}$$

for all  $k > 0$ .

For the Yager class of complements the generating function is

$$f_w(a) = 1 - a^w \tag{6.38}$$

with  $w > 0$ .

The *pseudo-inverse* of a decreasing generator  $f$ , denoted  $f^{(-1)}$ , is a function from  $\mathbb{R}$  to  $[0, 1]$  given by

$$f^{(-1)}(a) = \begin{cases} 1 & \text{for } a \in (-\infty, 0) \\ f^{-1}(a) & \text{for } a \in [0, f(0)] \\ 0 & \text{for } a \in (f(0), \infty) \end{cases} \tag{6.39}$$

Consider two examples of generating functions:

$$f_1(a) = 1 - a^p \text{ with } a \in [0, 1] \text{ and } p > 0 \tag{6.40}$$

$$f_2(a) = -\ln a \text{ with } a \in [0, 1] \text{ and } f_2(0) = \infty. \tag{6.41}$$

These functions have pseudo-inverses

$$f_1^{(-1)}(a) = \begin{cases} 1 & \text{for } a \in (-\infty, 0) \\ (1 - a)^{1/p} & \text{for } a \in [0, 1] \\ 0 & \text{for } a \in (1, \infty) \end{cases} \tag{6.42}$$

and

$$f_2^{(-1)}(a) = \begin{cases} 1 & \text{for } a \in (-\infty, 0) \\ e^{-a} & \text{for } a \in [0, \infty) \end{cases}. \tag{6.43}$$

A decreasing generator and its pseudo-inverse satisfy  $f^{(-1)}(f(a)) = a$  for any  $a$  in  $[0, 1]$  and

$$f\left(f^{(-1)}(a)\right) = \begin{cases} 0 & \text{for } a \in (-\infty, 0) \\ a & \text{for } a \in [0, f(0)] \\ f(0) & \text{for } a \in (f(0), \infty) \end{cases}. \tag{6.44}$$

The pseudo-inverse of an increasing generator  $g$ , denoted  $g^{(-1)}$ , is a function from  $\mathbb{R}$  to  $[0, 1]$  defined by

$$g^{(-1)}(a) = \begin{cases} 0 & \text{for } a \in (-\infty, 0) \\ g^{-1}(a) & \text{for } a \in [0, g(1)] \\ 1 & \text{for } a \in (g(1), \infty) \end{cases}. \tag{6.45}$$

Some examples are

$$g_1(a) = a^p \text{ with } a \in [0, 1] \text{ and } p > 0 \tag{6.46}$$

$$g_2(a) = -\ln(1 - a) \text{ with } a \in [0, 1] \text{ and } g_2(1) = \infty. \tag{6.47}$$

These functions have pseudo-inverses

$$g_1^{(-1)}(a) = \begin{cases} 0 & \text{for } a \in (-\infty, 0) \\ a^{1/p} & \text{for } a \in [0, 1] \\ 1 & \text{for } a \in (1, \infty) \end{cases} \tag{6.48}$$

and

$$g_2^{(-1)}(a) = \begin{cases} 1 & \text{for } a \in (-\infty, 0) \\ 1 - e^{-a} & \text{for } a \in [0, \infty) \end{cases} \dots \tag{6.49}$$

A decreasing generator and its pseudo-inverse satisfy  $g^{(-1)}(g(a)) = a$  for any  $a$  in  $[0, 1]$  and

$$g\left(g^{(-1)}(a)\right) = \begin{cases} 0 & \text{for } a \in (-\infty, 0) \\ a & \text{for } a \in [0, g(1)] \\ g(1) & \text{for } a \in (g(1), \infty) \end{cases} \quad (6.50)$$

**Lemma 6.3.** *Let  $f$  be a decreasing generator. Then a function  $g$  defined by  $g(a) = f(0) - f(a)$  for any  $a$  in  $[0, 1]$  is an increasing generator with  $g(1) = f(0)$ , and its pseudo-inverse  $g^{(-1)}$  is given by*

$$g^{(-1)}(a) = f^{(-1)}(f(0) - a) \quad (6.51)$$

for any  $a$  in  $\mathbb{R}$ .

**Lemma 6.4.** *Let  $g$  be an increasing generator. Then a function  $f$  defined by  $f(a) = g(1) - g(a)$  for any  $a$  in  $[0, 1]$  is a decreasing generator with  $f(0) = g(1)$ , and its pseudo-inverse  $f^{(-1)}$  is given by*

$$f^{(-1)}(a) = g^{(-1)}(g(1) - a) \quad (6.52)$$

for any  $a$  in  $\mathbb{R}$ .

**Theorem 6.5 (Characterization Theorem of  $t$ -norms).** *Let  $t$  be a binary operation on the unit interval. Then,  $t$  is an Archimedean  $t$ -norm iff there exists a decreasing generator  $f$  such that*

$$t(a, b) = f^{(-1)}(f(a) + f(b)) \quad (6.53)$$

for all  $a, b$  in  $[0, 1]$ .

**Theorem 6.6 (Characterization Theorem of  $t$ -conorms).** *Let  $s$  be a binary operation on the unit interval. Then,  $s$  is an Archimedean  $t$ -conorm iff there exists an increasing generator  $g$  such that*

$$t(a, b) = g^{(-1)}(g(a) + g(b)) \quad (6.54)$$

for all  $a, b$  in  $[0, 1]$ .

Below are various examples.

*Example 6.7.* (Schweizer and Sklar, 1963) The class of decreasing generators parameterized by  $p$  can be written

$$f_p(a) = 1 - a^p \quad (6.55)$$

where  $p \neq 0$ , generate the pseudo-inverse

$$f_p^{(-1)}(z) = \begin{cases} 1 & \text{for } z \in (-\infty, 0) \\ (1 - z)^{1/p} & \text{for } z \in [0, 1] \\ 0 & \text{for } z \in (1, \infty) \end{cases} .$$

This pseudo-inverse in turn generates the following t-norm.

$$\begin{aligned}
 t_p(a, b) &= f_p^{(-1)}(f_p(a) + f_p(b)) \\
 &= f_p^{(-1)}(2 - a^p - b^p) \\
 &= \begin{cases} (a^p + b^p - 1)^{1/p}, & 2 - a^p - b^p \text{ is in } [0, 1] \\ 0 & \text{otherwise} \end{cases} \\
 &= \max \left[ 0, (a^p + b^p - 1)^{1/p} \right].
 \end{aligned}
 \tag{6.56}$$

*Example 6.8.* (Yager, 1980) The class of decreasing generators parameterized by  $w$  can be written

$$f_w(a) = (1 - a)^w, \tag{6.57}$$

where  $w > 0$ , generate the pseudo-inverse

$$f_w^{(-1)}(z) = \begin{cases} 1 & \text{for } z \in (-\infty, 0) \\ 1 - z^{1/w} & \text{for } z \in [0, 1] \\ 0 & \text{for } z \in (1, \infty). \end{cases}
 \tag{6.58}$$

This pseudo-inverse in turn generates the following t-norm.

$$\begin{aligned}
 t_w(a, b) &= f_w^{(-1)}(f_w(a) + f_w(b)) \\
 &= f_w^{(-1)}((1 - a)^w + (1 - b)^w) \\
 &= \begin{cases} 1 - ((1 - a)^w + (1 - b)^w)^{1/w}, & (1 - a)^w + (1 - b)^w \text{ is in } [0, 1] \\ 0 & \text{otherwise} \end{cases} \\
 &= 1 - \min(1, [(1 - a)^w + (1 - b)^w]^{1/w}).
 \end{aligned}
 \tag{6.59}$$

*Example 6.9.* (Frank, 1979) The class of decreasing generators parameterized by  $s$  can be written

$$f_s(a) = -\ln \frac{s^a - 1}{s - 1}$$

with  $s > 0$  and  $s \neq 1$ , generates the pseudo-inverse

$$f^{(-1)}(z) = \log(s)(1 + (s - 1)e^{-z}). \tag{6.60}$$

This pseudo-inverse in turn generates the following t-norm

$$\begin{aligned}
 t_s(a, b) &= f_s^{(-1)}(f_s(a) + f_s(b)) \\
 &= f_s^{(-1)} \left( -\ln \frac{(s^a - 1)(s^b - 1)}{(s - 1)^2} \right) \\
 &= \log_s \left[ 1 + (s - 1) \frac{(s^a - 1)(s^b - 1)}{(s - 1)^2} \right] \\
 &= \log_s \left[ 1 + \frac{(s^a - 1)(s^b - 1)}{s - 1} \right].
 \end{aligned}
 \tag{6.61}$$

**Table 6.9.** Some Classes of  $\mathbf{t}$ -norms

FORMULA	PARAMETER	ORIGINATOR YEAR
$\mathbf{t}_{\min}$		
$[\max(0, a^p + b^p - 1)]^{\frac{1}{p}}$	$p \neq 0$	Schweizer&Sklar 1983
$\frac{ab}{\gamma + (1 - \gamma)(a + b - ab)}$	$\gamma \in (0, \infty)$	Hamacher 1978
$\log_s \left[ 1 + \frac{(s^a - 1)(s^b - 1)}{s - 1} \right]$	$s \in (0, \infty),$ $s \neq 1$	Frank 1979
$1 - \min \left[ 1, ((1 - a)^w + (1 - a)^w)^{\frac{1}{w}} \right]$	$w \in (0, \infty)$	Yager 1980a
$\frac{ab}{\max(a, b, \alpha)}$	$\alpha \in [0, 1]$	Dubois&Prade 1980b
$\left[ 1 + \left[ \left( \frac{1}{a} - 1 \right)^\lambda + \left( \frac{1}{b} - 1 \right)^\lambda \right]^{\frac{1}{\lambda}} \right]^{-1}$	$\lambda \in (0, \infty)$	Dombi 1982

The Yager class of  $\mathbf{t}$ -norms

$$\mathbf{t}_w(a, b) = 1 - \min \left[ 1, ((1 - a)^w + (1 - b)^w)^{1/w} \right], \text{ with } w > 0, \quad (6.62)$$

covers the full range of  $\mathbf{t}$ -norms expressed in the following Theorem.

**Theorem 6.10.** *Let  $\mathbf{t}_w$  denote the class of Yager  $\mathbf{t}$ -norms defined above. Then*

$$\mathbf{t}_{\min}(a, b) \leq \mathbf{t}_w(a, b) \leq \min(a, b). \quad (6.63)$$

For detailed proofs of all these results see [Klir and Yuan \(1996\)](#).

A host of  $\mathbf{t}$ -norms and  $\mathbf{t}$ -conorms have been proposed to deal with specific problems. A selection of some well-known parametric classes of  $\mathbf{t}$ -norms are given in Table [\(6.9\)](#), and the same for  $\mathbf{t}$ -conorms are given in Table [\(6.10\)](#). Various experimental procedures are available to select the appropriate  $\mathbf{t}$ -norm for a particular application [Klir and Yuan \(1996\)](#).

### A.3 Combinations of Operations

In classical set theory, the operations of intersection and union are dual with respect to the complement in that they satisfy the De Morgan laws.

**Table 6.10.** Some Classes of t-conorms

FORMULA	PARAMETER	ORIGINATOR YEAR
$s_{\max}$		
$1 - [\max(0, (1 - a)^p + (1 - b)^p - 1)]^{\frac{1}{p}}$	$p \neq 0$	Schweizer&Sklar 1983
$\frac{a + b - (\gamma - 2)ab}{1 + (\gamma - 1)ab}$	$\gamma \in (0, \infty)$	Hamacher 1978
$1 - \log_s \left[ 1 + \frac{(s^{1-a} - 1)(s^{1-b} - 1)}{s - 1} \right]$	$s \in (0, \infty),$ $s \neq 1$	Frank 1979
$\min \left[ 1, (a^w + b^w)^{\frac{1}{w}} \right]$	$w \in (0, \infty)$	Yager 1980a
$1 - \frac{(1 - a)(1 - b)}{\max(a, b, 1 - \alpha)}$	$\alpha \in [0, 1]$	Dubois&Prade 1980b
$\left[ 1 + \left[ \left( \frac{1}{a} - 1 \right)^{-\lambda} + \left( \frac{1}{b} - 1 \right)^{-\lambda} \right]^{-\frac{1}{\lambda}} \right]^{-1}$	$\lambda \in (0, \infty)$	Dombi 1982

1. The complement of the intersection of  $A$  and  $B$  equals the union of the complement of  $A$  and the complement of  $B$ .
2. The complement of the union of  $A$  and  $B$  equals the intersection of the complement of  $A$  and the complement of  $B$ .

$$\begin{aligned} (A \cap B)^c &= A^c \cup B^c \\ (A \cup B)^c &= A^c \cap B^c \end{aligned}$$

Obviously, only certain combinations of t-norms, t-conorms, and fuzzy complements satisfy the duality. We say that a t-norm  $t$  and a t-conorm  $s$  are “dual with respect to a fuzzy complement  $c$ ” if and only if

$$c(t(a, b)) = s(c(a), c(b)) \tag{6.64}$$

and

$$c(s(a, b)) = t(c(a), c(b)). \tag{6.65}$$

These equations define the De Morgan laws for fuzzy sets. Let the triple  $(t, s, c)$  denote that  $t$  and  $s$  are dual with respect to  $c$ , and let any such triple be called a “dual triple”.

The following t-norms and t-conorms are dual with respect to the Standard Fuzzy Complement  $c$  (i.e., dual triples):

**Table 6.11.** Triples of Operators

t-norms	t-conorms	complement
$t(a, b) = \min(a, b)$	$s(a, b) = \max(a, b)$	$c(a) = 1 - a$
$t(a, b) = ab$	$s(a, b) = a + b - ab$	$c(a) = 1 - a$
$t(a, b) = \max(0, a + b - 1)$	$s(a, b) = \min(1, a + b)$	$c(a) = 1 - a$
$t(a, b) = t_{\min}(a, b)$	$s(a, b) = s_{\max}(a, b)$	$c(a) = 1 - a$

**Theorem 6.11.** *The triples  $(\min, \max, c)$  and  $(t_{\min}, s_{\max}, c)$  are dual with respect to any fuzzy complement  $c$ .*

**Theorem 6.12.** *Given a t-norm  $t$  and an involutive fuzzy complement  $c$ , the binary operation  $s$  on  $[0, 1]$  defined by*

$$s(a, b) = c(t(c(a), c(b))) \tag{6.66}$$

for all  $a, b$  in  $[0, 1]$  is a t-conorm such that  $(t, s, c)$  is a dual triple.

**Theorem 6.13.** *Given a t-conorm  $s$  and an involutive fuzzy complement  $c$ , the binary operation  $t$  on  $[0, 1]$  defined by*

$$t(a, b) = c(s(c(a), c(b))) \tag{6.67}$$

for all  $a, b$  in  $[0, 1]$  is a t-norm such that  $(t, s, c)$  is a dual triple.

**Theorem 6.14.** *Given an involutive fuzzy complement  $c$  and an increasing generator  $g$  of  $c$ , the t-norm and t-conorm generated by  $g$  are dual with respect to  $c$ .*

**Theorem 6.15.** *Let  $(t, s, c)$  be a dual triple generated by Theorem 6.14. Then, the fuzzy operations  $t, s, c$  satisfy the law of excluded middle*

$$s(a, c(a)) = 1 \tag{6.68}$$

and the law of contradiction

$$t(a, c(a)) = 0. \tag{6.69}$$

**Theorem 6.16.** *Let  $(i, u, c)$  be a dual triple that satisfies the law of excluded middle and the law of contradiction. Then  $(i, u, c)$  does not satisfy the distributive laws. This means that  $t(a, s(b, d))$  is not equal to  $s(t(a, b), t(a, d))$  for all  $a, b, d$  in  $[0, 1]$ .*

For detailed proofs of all these results see [Klir and Yuan \(1996\)](#).

If we would like a dual triple, which makes the algebraic manipulation of fuzzy sets a little simpler, then we can create such a dual triple from one generating function  $g$ . In application, the complement is the simplest function to analyze. Suppose we can generate some data that gives us a feel for how people feel subjectively about membership and non-membership of some object in a fuzzy set. If we can fit a Sugeno or Yager complement to this data, we can retrieve its generating function  $g$  and then use it to generate corresponding (dual triple) union,  $s$ , and intersection,  $t$ , operators.

**Table 6.12.** Averaging Operators

Name	Operator	Generator
arithmetic mean	$\frac{a+b}{2}$	$x$
generalized $p$ -mean	$\sqrt[p]{\frac{a^p+b^p}{2}}$	$x^p$
harmonic mean	$\frac{2ab}{a+b}$	$\frac{1}{x}$
geometric mean	$\sqrt{xy}$	$\log x$
dual of geometric mean	$1 - \sqrt{(1-x)(1-y)}$	$\log(1-x)$

### A.4 Averaging Operator

The averaging operators are a third class of binary operators used to average the arguments  $a$  and  $b$ . Since all intersection operators produce results that are equal to or less than the minimum of  $a$  and  $b$  and all union operators produce results that are greater than or equal to the maximum of  $a$  and  $b$ , there is a large range of values that are excluded by these two classes of operators. Into this gap we now introduce averaging operators  $h(a, b)$ . These operators do not correspond exactly to any logical connective, the way intersection operators model **and** and union operators model **or**. These averaging operators take two arguments and produce a result that is greater than or equal to the  $\min(a, b)$  and less than or equal to  $\max(a, b)$ . An averaging operator is a function  $h : [0, 1] \times [0, 1] \rightarrow [0, 1]$  such that following axioms hold.

- (h1) Idempotency —  $h(a, a) = a$ .
- (h2) Monotonicity —  $h(a, b) \leq h(c, d)$  whenever  $a \leq c$  and  $b \leq d$ .
- (h3) Commutativity —  $h(a, b) = h(b, a)$ .
- (h4) Continuity —  $h$  is a continuous function.

The following properties can be listed as a condition, but they are consequences of the previous axioms.

- (g5) Extremes —  $h(0, 0) = 0$  and  $h(1, 1) = 1$ .
- (g6) Boundary conditons —  $\min(a, b) \leq h(a, b) \leq \max(a, b)$ .

Averaging operators allow for an interaction between the values of two fuzzy sets. It allows the resultant averaged value to be greater than the worst case but less than the best case. In fact the average value is often right in the middle, which should come as no surprise. However there are other averaging operators beside the geometric mean, such as the harmonic mean. Assume  $f$  is any continuous strictly monotone function. All continuous strictly monotone functions have inverses so we know that  $f^{-1}$  exists. Then

$$h(a, b) = f^{-1} \left[ \frac{f(a) + f(b)}{2} \right]$$

is called a quasi-arithmetic mean. Such a function  $h$  is always an averaging operator (Aczél, 1966). Let  $\alpha \in [0, 1]$ . Then

$$h(a, b) = f^{-1} [\alpha f(a) + (1 - \alpha)f(b)]$$

is a more general form of quasi-arithmetic operators (Aczél, 1966). Some of the more important averaging operators are given in Table (6.12).

### A.4.1 Aggregation Operations

The idea of an averaging operator can be extended to aggregation operators over  $m$  elements. Since averaging operations are in general not associative, they must be defined as functions of  $m$  arguments ( $m \geq 2$ ). That is, an averaging operation  $h$  is a function of the form

$$h : [0, 1]^m \rightarrow [0, 1]. \tag{6.70}$$

Averaging operations are characterized by the following set of axioms:

(h1) Idempotency — for all  $a \in [0, 1]$ ,

$$h(a, a, a, \dots, a) = a. \tag{6.71}$$

(h2) Monotonicity — for any pair of  $m$ -tuples of real numbers in  $[0, 1]$ ,  $\langle a_1, a_2, a_3, \dots, a_m \rangle$  and  $\langle b_1, b_2, b_3, \dots, b_m \rangle$ , if  $a_k \leq b_k$  for all  $k \in \mathbb{N}_m$  then

$$h(a_1, a_2, a_3, \dots, a_m) \leq h(b_1, b_2, b_3, \dots, b_m). \tag{6.72}$$

It is significant that any function  $h$  that satisfies these axioms produces numbers that, for any  $m$ -tuple  $\langle a_1, a_2, a_3, \dots, a_m \rangle \in [0, 1]^m$ , lie in the closed interval defined by the inequalities

$$\min(a_1, a_2, a_3, \dots, a_m) \leq h(a_1, a_2, a_3, \dots, a_m) \leq \max(a_1, a_2, a_3, \dots, a_m). \tag{6.73}$$

The min and max operations qualify, due to their idempotency, not only as fuzzy counterparts of classical set intersection and union, respectively, but also as extreme averaging operations.

An example of a class of symmetric averaging operations are *generalized means*, which are defined for all  $m$ -tuples  $\langle a_1, a_2, a_3, \dots, a_m \rangle$  in  $[0, 1]^m$  by the formula

$$h_p(a_1, a_2, a_3, \dots, a_m) = \frac{1}{m}(a_1^p + a_2^p + a_3^p + \dots + a_m^p)^{\frac{1}{p}}, \tag{6.74}$$

where  $p$  is a parameter whose range is the set of all real numbers excluding 0. For  $p = 0$ ,  $h_p$  is not defined; however for  $p \rightarrow 0$ ,  $h_p$  converges to the well known geometric mean. That is, we take

$$h_0(a_1, a_2, a_3, \dots, a_m) = (a_1 a_2 a_3 \dots a_m)^{\frac{1}{m}}. \tag{6.75}$$

For  $p \rightarrow -\infty$  and  $p \rightarrow \infty$ ,  $h_p$  converges to the min and max operations, respectively.

Assume again that  $f$  is any continuous strictly monotone function, then

$$h(a_1, a_2, \dots, a_m) = f^{-1} \left[ \frac{1}{m} \sum_{i=1}^m f(a_i) \right]$$

is still called a quasi-arithmetic mean. Let  $\langle w_1, w_2, \dots, w_m \rangle$  be weights with  $w_i \in [0, 1]$ . Then

$$h(a_1, a_2, \dots, a_m) = f^{-1} \left[ \sum_{i=1}^m w_i f(a_i) \right]$$

is a more general form of quasi-arithmetic operator (Aczél, 1966).

#### A.4.2 OWA Operators

Yager (1988) introduced ordered weighted averaging (OWA) operators. They are by nature averaging operators that treat a fuzzy set in its possibility theory interpretation. Let  $\mathbf{a} = \langle a_1, a_2, a_3, \dots, a_m \rangle$  be an  $m$ -dimensional vector of values and let  $\mathbf{w} = \langle w_1, w_2, w_3, \dots, w_m \rangle$  be an  $m$ -dimensional vector of weights, with both  $a_i \in [0, 1]$  and  $w_i \in [0, 1]$ , for  $1 \leq i \leq m$ . Define the vector  $\mathbf{b} = \langle b_1, b_2, b_3, \dots, b_m \rangle$  to be the vector  $\mathbf{a}$  sorted in decreasing order of magnitude, so that  $b_i \geq b_{i+1}$ . Then the OWA average of  $\mathbf{a}$  is

$$OWA_{\mathbf{w}}(\mathbf{a}) = \sum_{i=1}^m w_i b_i.$$

At first it would seem that OWA operators are very artificial. However, let us examine three special cases of OWA operators

Let  $w^* = \langle 1, 0, 0, \dots, 0 \rangle$  then

$$OWA_{w^*}(\mathbf{a}) = b_1 = \max[a_1, a_2, a_3, \dots, a_m].$$

Let  $w_* = \langle 0, 0, 0, \dots, 1 \rangle$  then

$$OWA_{w_*}(\mathbf{a}) = b_m = \min[a_1, a_2, a_3, \dots, a_m].$$

Let  $w_* = \langle \frac{1}{m}, \frac{1}{m}, \frac{1}{m}, \dots, \frac{1}{m} \rangle$  then

$$OWA_{w_*}(\mathbf{a}) = \frac{1}{m} \sum_{i=1}^m b_i = \frac{1}{m} \sum_{i=1}^m a_i.$$

Thus OWA operators allow us to perform a delicate mix of values emphasizing either large values in  $\mathbf{a}$  by making  $w_i$  big for low values of  $i$  and tiny for higher values of  $i$  or vice versa.

#### A.5 Notes

Most of the important early work on t-norms comes from early, and quite complex, works on probabilistic metric spaces. The term is originally due to Menger (1942). Most of the developmental work was done in Schweizer and Sklar (1961) and Schweizer and Sklar (1963). The book Schweizer and Sklar (1983) provides a complete development of the subject. Characterization of the union and intersection operators for fuzzy set theory was presented in Bellman and Giertz (1973). An excellent overview of aggregation operators is in Dubois and Prade (1985).

## 7 Cycling in Fuzzy Spatial Models

With Peter C. Casey, Alan J. Hickle, Danae D. Mercer, and Raivydas Simenas

As we discussed in the opening chapter of this volume, cycling is a persistent and troubling problem for formal models in comparative politics. While the problem is not as acute in single-dimensional problems as it is in multi-dimensional models, chapter five demonstrated that a fuzzy set theory approach can reduce the problem substantially in spatial models. In this concluding chapter we give consideration to the degree to which fuzzy spatial models are plagued by cycling. We argue that while the use of fuzzy math reduces the extent of the problem in a number of circumstance, it does not do away entirely with cycling. We conclude by identifying future avenues for research on this important topic.

### 7.1 Fuzzy Cycling

There is a small, but substantial, literature in journals devoted to fuzzy mathematics that addresses the problems of cycling (see [Nurmi and Kacprzyk 2007](#) for a review of this literature). The general conclusion is that cycling is as persistent a problem in fuzzy approaches as in the crisp case ([Barrett et al., 1992](#)). However, a number of scholars have considered the possible effect on cycling exercised by fuzzy set theory's capacity to arrange individual preferences ordinally in terms of the degree to which one is preferred to another (see [George et al., 2007](#); [Kacprzyk et al., 1992](#); [Nurmi, 1981b,a](#)). Take, for instance, the crisp preference profile of three political actors,  $N = \{1, 2, 3\}$ , over three alternatives,  $X = \{a, b, c\}$ , depicted in [Table 7.1](#). This preference profile results in the classic three-player cycling problem in crisp models. Each player's preference order is acyclic, but the social preference is cyclical under majority rule.

As we discussed in the previous chapter, the standard crisp approach gives us little information about the actors' preferences. We do not know how much each actor prefers each alternative; we only know in a dichotomous choice between any two which one will be chosen. Fuzzy set theory's ability to consider the degree

**Table 7.1.** A Cyclical Crisp Preference Profile

Political Actors		
	1	2 3
1st choice	a	c b
2nd choice	b	a c
3rd choice	c	b a
Collective Outcome:		
<i>a P b, b P c, and c P a</i>		

**Table 7.2.** Cycling in a Fuzzy Preference Profile

Political Actors			
	1	2	3
1st choice	a (1.00)	c (1.00)	b (1.00)
2nd choice	b (0.50)	a (0.50)	c (0.50)
3rd choice	c (0.00)	b (0.00)	a (0.00)
Collective Outcome: <i>a P b, b P c, and c P a</i>			

to which each actor prefers a given alternative breaks away from dichotomous binary relations and might offer a solution.

Suppose that we replace the crisp data with fuzzy data by replacing dichotomous rankings with ordinal rankings based on  $\alpha$ -levels (membership grades in each individual’s fuzzy preference relation). The  $\alpha$ -levels are essentially surrogates for the ordinal rank ordering. The players’ individual crisp preferences orders in Table 7.1 might conceivably be those in Table 7.2 once the degree to which they prefer each alternative is incorporated (as displayed in the parentheses next to each alternative). The first choice in each actor’s preference ordering is assigned a value of  $\alpha = 1$ , the second choice is assigned a value of  $\alpha = 0.50$ , and the third choice is assigned a value of  $\alpha = 0$ .

The result is exactly the same as before, cycling. No one option is preferred over any other since the maximal set contains all elements in the alternative set. Unsurprisingly, the result is the same as in the crisp case since we have utilized membership grades in roughly the same manner. Essentially both the crisp approach and our initial fuzzy set approach determine majority choice based on ordinal rankings in a manner analogous to a Borda count. It may be that cycling occurs less frequently in fuzzy set theory as the granularity and the degree of precision in determining alpha-levels increase.

To check for this possibility, suppose that the membership grades over a set of options are those shown in Table 7.3.

The top cycle set still contains all three alternatives. In fact, only in the case in which one actor is indifferent between two alternatives does a singleton maximal set attain (a Condorcet winner). That this is so is quite easy to demonstrate. Suppose that we have three players with the preference profile shown in Table 7.1. Let us assign the numerical values  $a_1$  to represent the alpha-level of the first

**Table 7.3.** Continued Cycling in a Fuzzy Preference Profile

Political Actors			
	1	2	3
1st choice	a (1.00)	c (1.00)	b (1.00)
2nd choice	b (0.50)	a (0.75)	c (0.50)
3rd choice	c (0.00)	b (0.25)	a (0.50)

Collective Outcome:  $a P b, b P c$ , and  $c P a$

choice of player one,  $b_1$  to that of his second choice, and  $c_1$  to that of his third choice. In similar fashion, player two’s first choice is assigned an alpha-level of  $c_2$ , her second choice  $a_2$ , and her third choice  $b_2$ ; and player three’s first choice is assigned an alpha-level of  $b_3$ , her second choice  $a_3$ , and her third choice  $c_3$ . Then for any assignment of  $\alpha$ -levels reflecting the preference profile in Table 7.1 such that

$$\begin{aligned}
 a_1 &> b_1 > c_1 & (7.1) \\
 b_2 &> c_2 > a_2 \\
 &\text{and} \\
 c_3 &> a_3 > b_3
 \end{aligned}$$

cycling exists. If the potential outcome is  $A$ , then the utility is  $(a_1, a_2, a_3)$ , but  $c_2 > a_2$  and  $c_3 > a_3$  so that coalition  $\{2, 3\}$  prefers  $A$  to  $C$ . If the potential outcome is  $B$ , then the utility is  $(b_1, b_2, b_3)$ , but  $a_1 > b_1$  and  $a_3 > b_3$  so that coalition  $\{1, 3\}$  prefers  $A$  to  $B$ . If the potential outcome is  $C$ , then the utility is  $(c_1, c_2, c_3)$ , but  $b_1 > c_1$  and  $b_2 > c_2$  so that coalition  $\{1, 2\}$  prefers  $B$  to  $C$ .

## 7.2 Modeling Consensus

Kacprzyk et al. (1992) and Nurmi (1981a) have argued that fuzzy set theory permits scholars to re-frame the cycling problem within the context of satisficing, versus utility maximizing. While they offer a number of ways to predict outcomes, we demonstrate the general approach as modeling consensus. Satisficing biases outcomes in favor of the least, least preferred outcome, which approximates the manner in which consensus rules operates.

Unlike the crisp case where we consider which option is majority preferred to all others, we calculate the maximum in the intersection of the membership grades for each of the three options; that is the alternative with the maximum of the minimum membership grades assigned by all political actors. Formally,

$$M(S) = \max \bigcap_{i=1}^n s_i = \max \min_{i=1}^n (s_i) , \tag{7.2}$$

in which the maximal set (most preferred alternative(s)) contains the alternative with the largest (maximum)  $\alpha$ -level value for each alternative ( $s_i$ ) of the smallest (minimum)  $\alpha$ -level value across the political actors.

Suppose that three political actors possess the preferences in Table 7.4 over alternatives in  $X = \{a, b, c\}$  with fuzzy membership levels in parentheses. The notation “ $1a \cap 2a$ ” means “the intersection of the membership grades regarding Policy  $a$  in the fuzzy preference relations of actors 1 and 2.” Majority rule implies that actors first seek coalitions in which they are mutually minimally satisfied. They then choose the one coalition in which they are most satisfied.

In the case presented in Table 7.4, two coalitions emerge at two separate policy points. Both involve actor 1. The first unites actor 1 and 3 in support of policy  $a$  at an  $\alpha$ -level of 0.75. The second unites actor 1 and 2 in support of policy  $b$  at an  $\alpha$ -level of 0.75. No other coalitions are minimally satisfactory to prospective members, so these two are the only two likely. Under majority rule with three actors, a coalition of two is decisive. Since actor 1 is in both decisive coalitions, she forms a collegium of one and will cast the deciding vote. Since policy  $a$  is her most preferred choice, she will most assuredly choose a coalition with actor 3 in support of alternative  $a$  at a payoff of  $\alpha = 1$ .

Table 7.5 presents a situation in which a collegium of one once again emerges, but which exhibits much different characteristics than the one depicted in Table 7.4. Here, actor 2 is present in two decisive coalitions that are equally desirable by the collective participants privy to each coalition, but, if given a choice, she must settle upon a policy that is surprisingly not her first choice. While policy  $b$  grants her the most utility, there is no potential partner that will offer her such an option. Given this situation, she will choose  $c$  in coalition with actor 3 with a payoff at the  $\alpha = 0.75$  level.

While the fuzzy satisficing approach predicts a maximal set, the number of elements in the maximal set is dependent on the number of alternatives for which any political actors assign the lowest  $\alpha$ -level among all alternatives. This is an artifact of the manner in which fuzzy intersection operates on the data. Hence, cycling can be overcome using the intersection operator to determine the social choice only in cases where the minimal value for all alternatives is not the same.

While the satisficing rule may give us a potential solution to cycling, we are not sure that it will do so when individual preferences are aggregated across dimensions. In chapter five we considered different aggregation operators capable of generating scores that are surrogates for the utility ( $\alpha$ -level) that an individual player assigns to each alternative  $\langle x_i, y_i \rangle$  in two-dimensional space. These utility scores, which are assigned to fuzzy points that encompass regions of the policy

**Table 7.4.** Actor 1 as a Collegium with an Optimal Choice

	Political Actors		
	<b>1</b>	<b>2</b>	<b>3</b>
1st choice	a (1.00)	b (1.00)	c (0.75)
2nd choice	b (0.75)	c (0.50)	a (0.75)
3rd choice	c (0.50)	a (0.25)	b (0.00)
	$1a \cap 2a = 0.25$	$1a \cap 3a = 0.75$	$2a \cap 3a = 0.25$
	$1b \cap 2b = 0.75$	$1b \cap 3b = 0.0$	$2b \cap 3b = 0.0$
	$1c \cap 2c = 0.5$	$1c \cap 3c = 0.5$	$2c \cap 3c = 0.5$

**Table 7.5.** Actor 2 as a Collegium with a Non-Optimal Choice

Political Actors			
	<b>1</b>	<b>2</b>	<b>3</b>
1st choice	A (0.80)	B (1.00)	C (0.60)
2nd choice	B (0.50)	C (0.75)	A (0.50)
3rd choice	C (0.30)	A (0.60)	B (0.40)
	$\mathbf{1A} \cap \mathbf{2A} = \mathbf{0.6}$	$\mathbf{1A} \cap \mathbf{3A} = \mathbf{0.5}$	$\mathbf{2A} \cap \mathbf{3A} = \mathbf{0.5}$
	$\mathbf{1B} \cap \mathbf{2B} = \mathbf{0.5}$	$\mathbf{1B} \cap \mathbf{3B} = \mathbf{0.4}$	$\mathbf{2B} \cap \mathbf{3B} = \mathbf{0.4}$
	$\mathbf{1C} \cap \mathbf{2C} = \mathbf{0.3}$	$\mathbf{1C} \cap \mathbf{3C} = \mathbf{0.3}$	$\mathbf{2C} \cap \mathbf{3C} = \mathbf{0.6}$

**Table 7.6.** General Preference Profile for Three Political Actors

Political Actors			
	<i>1</i>	<i>2</i>	<i>3</i>
1st choice	$\langle 1Ax, 1Ay \rangle$	$\langle 2Bx, 2By \rangle$	$\langle 3Cx, 3Cy \rangle$
2nd choice	$\langle 1Bx, 1By \rangle$	$\langle 2Cx, 2Cy \rangle$	$\langle 3Ax, 3Ay \rangle$
3rd choice	$\langle 1Cx, 1Cy \rangle$	$\langle 2Ax, 2Ay \rangle$	$\langle 3Bx, 3By \rangle$

**Table 7.7.** An Example of a Randomly Generated Preference Profile

Political Actors			
	<b>1</b>	<b>2</b>	<b>3</b>
1st choice	A $\langle 0.55, 0.90 \rangle$	B $\langle 0.45, 0.70 \rangle$	C $\langle 1.00, 0.95 \rangle$
2nd choice	B $\langle 0.35, 0.60 \rangle$	C $\langle 0.40, 0.50 \rangle$	A $\langle 0.20, 0.85 \rangle$
3rd choice	C $\langle 0.30, 0.15 \rangle$	A $\langle 0.25, 0.40 \rangle$	B $\langle 0.15, 0.30 \rangle$

space over which the actor is indifferent, can be used to determine each individual player’s fuzzy preference order over a set of alternatives. The crucial question is whether the resulting fuzzy preference profiles (the preference orders of all political actors) are cyclical under satisficing.

We ran four simulations thirty times each under different scenarios to see if we could find a case of cycling. For each scenario, we randomly generated thirty sets of membership grades along two dimensions for three actor who were assumed to have the preference profile in Table 7.6, which in the crisp case would result in cycling.

For example, a randomly generated set of two-dimensional  $\langle x, y \rangle$  preference orders for each player might be that in Table 7.7. We then checked to see if there was any cycling.

The first scenario used an aggregation operator in which the maximum dimension was weighted 0.2, the minimum was weighted 0.80, and the discrete fuzzy number had 21 alpha-levels. The scenario essentially assumes risk averse actors who are acutely aware of the effect on their preferences of relatively small variations in the positioning of alternatives and who harbor little ambiguity or indifference over the policy space. In the second scenario, the granularity remained the same, but the OWA weights were changed to 0.70 for the

maximum and 0.30 for the minimum. Hence, we are modeling actors who are far more aggressive in their political behavior but no less ambiguous or indifferent about their preferences. Each of the sixty simulations across both scenarios resulted in a prediction of a single, stable winning coalition.

We then redid the analysis reducing the granularity to five alpha-levels. The third scenario used OWA weights of 0.20 for the maximum and 0.80 for the minimum. It models political actors who are risk averse and relatively ambiguous about their preferences. The fourth scenario used OWA weights of 0.70 for the maximum and 0.30 for the minimum. It models risk-takers who are ambiguous about their preferences. In several cases, the models for the third and fourth scenarios predicted two plausible majority coalitions. However, in all instances the same actor was in both coalitions, thereby comprising a collegium. This political actor would have chosen the higher utility coalition in each case. Hence, the models do not predict cycling.

A consistent principle emerges. It is not the mathematical underpinnings of aggregation that result in small maximal sets or single point solutions, but rather the degree of ambiguity and indifference over preferences. The ability to predict a single outcome in the maximal set is dependent on the degree of granularity rather than the aggregation operator by which individual multi-dimensional preference orders are calculated. Nonetheless, even when more than one outcome is plausible, the emergence of a collegium assures a stable, predictable outcome.

In addition to its ability to generate stable outcomes, there is a strong empirical case to be made for adopting satisficing. While theoretically in the state of nature, collectives of humans might prefer to choose options that are majority preferred to all others (Condorcet winners), in those institution-less settings that we can observe (e.g., room mates in a dormitory, faculty in a committee, friends on a outing), consensus is the actual rule in operation. Collectives choose those options against which there is the least opposition. Hence, the outcomes can be counter-intuitive. A single actor with extremely low scores for an option can veto an otherwise widely-acceptable policy. Furthermore, an alternative enjoying moderate levels of support can defeat one with very high levels of support among a super-majority of political actors.

Despite the argument that can be made for satisficing, we are quite certain that most rational choice scholars will not be persuaded to adopt it. The most important reason is it represents an explicit rejection of the utility maximization assumption that lies at the base of the rationality principle. Utility maximization requires that political actors are not satisfied with an outcome that is merely good enough. They want the best outcome. In that vein, satisficing represents a fundamental departure from the approach to determining social preference than the one that rational choice theorists imply with the maximal set. The maximal set comprises all alternatives that are majority preferred in binary relations across the full set of alternatives. When the maximal set is singleton, it is the Condorcet winner (see Chapter 1). Such a social preference, which may be known only to God in the state of nature, bears some similarity to Rousseau's normative general will. For that reason, it has been the subject of

much consideration by scholars, who understandably are quite concerned with a concept of democracy that should find a relationship between social preference (the existence of a Condorcet winner) and social choice (outcomes induced by institutional rules and design in democratic systems).

### 7.3 Cycling in Spatial Models

While we might imply from the discussion to this point that fuzzy set theory offers no relief from cycling in spatial models under the rationality assumption (utility maximizing behavior), chapter five suggests otherwise. Furthermore, Nurmi's (1981b) application of fuzzy set theory to formal, geometric models does so as well. Since models constitute the primary focus of this book, it is important that we address the question head on. We extend Nurmi's argument to reach some important tentative conclusions.

Nurmi (1981b) uses the core of a fuzzy number to represent all alternatives that are satisfactory to a political actor. Those outside the core are not satisfactory. This is essentially a dichotomous crisp approach with a large indifference set at the ideal point (the core). Take for example, Figure 7.1. Since none of the players' preferences intersect at a level satisfactory to any of them, there is no maximal set and the model predicts gridlock.

On the other hand, Figure 7.2 depicts a situation in which  $A$  and  $C$  can reach an agreement, the maximal set being the fuzzy point at the intersection of their respective cores. The approach is no different from what that which

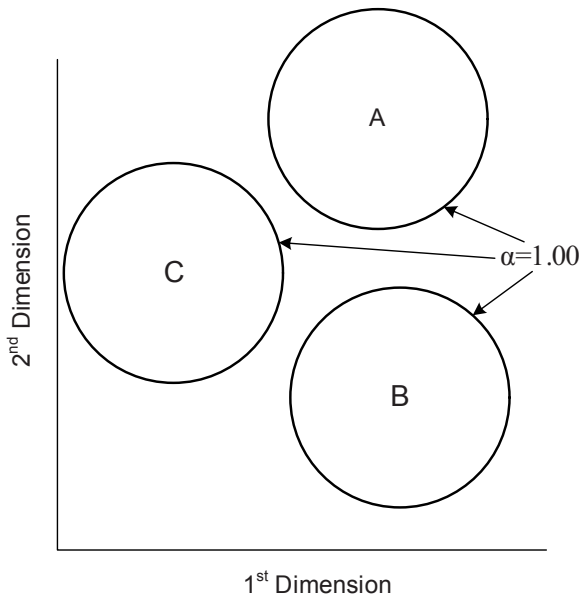
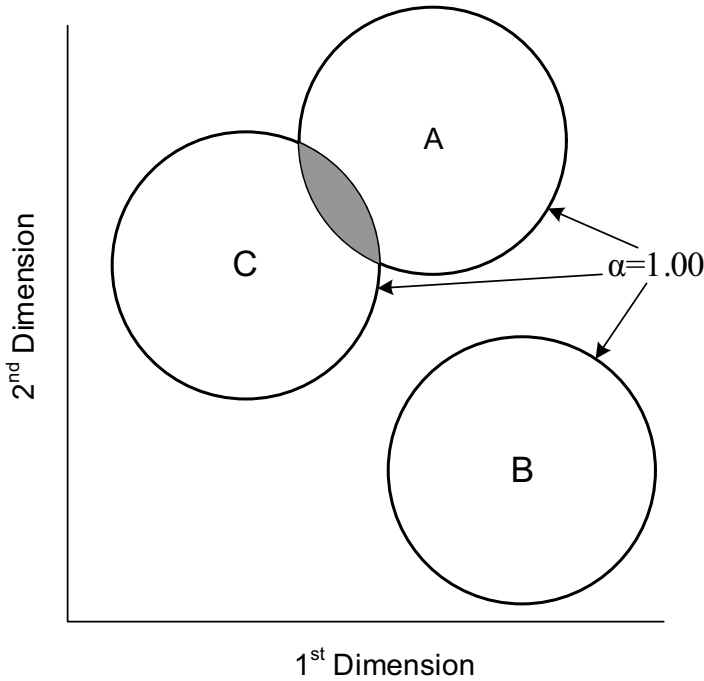


Fig. 7.1. An Empty Maximal Set



**Fig. 7.2.** A Maximal Set

we demonstrated in chapter five; it has merely been simplified by reducing the number of alpha-levels.

As we demonstrated in chapter five, we can uncover a maximal set under more complicated scenarios, which demonstrates the power of a fuzzy set approach to formal modeling. The crisp approach would predict cycling under an assumption of sincere voting. Alternatively it would have reduced the outcome to a difficult to determine set of points within the uncovered set under an assumption of sophisticated or strategic voting. Whether players vote sincerely or strategically a fuzzy model potentially predicts a centrally located and relatively easy to calculate maximal set. Furthermore, it appears to do so under a broad number of scenarios.

Take for instance, the preferences of three players,  $N = \{A, B, C\}$ , represented by the fuzzy numbers in Figure 7.3, Figure 7.4, and Figure 7.5. We have numbered and labeled the fuzzy points in each figure with their respective three-tuple payoff  $\{A(x), B(X), C(x)\}$  in brackets in order to make it easier to determine the maximal set. (Once again, for ease of presentation, we have not labeled points at which two or more players intersect at  $\alpha = 0$ . These points are defeated by a number of other points, and they can not defeat any other point by majority rule. Hence, they clearly are not among the elements in any maximal set.) We have noted the winset (the set of alternatives that are majority preferred) for each alternative in brackets. Those alternatives that are undominated (their winsets are empty) comprise the maximal set. They are highlighted in gray in each figure.

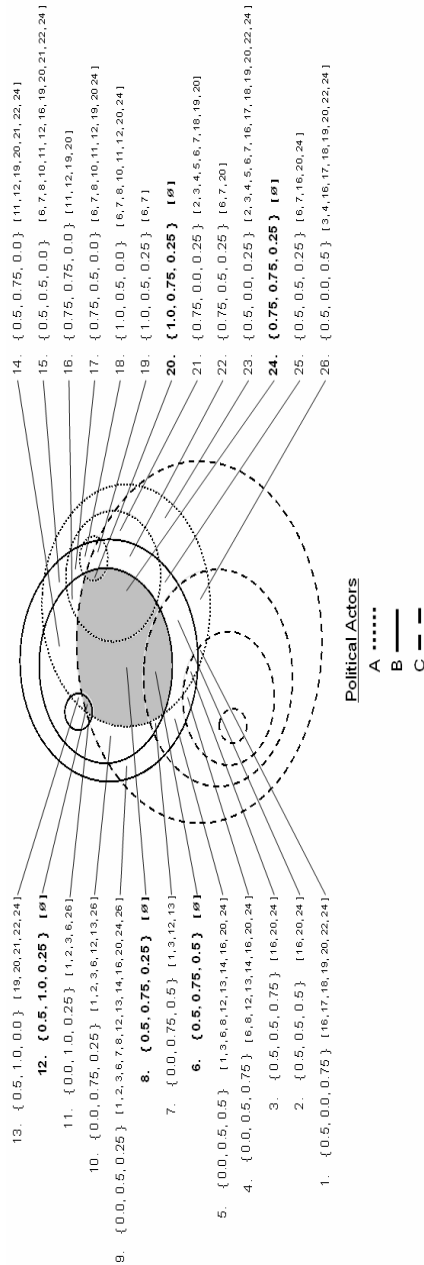
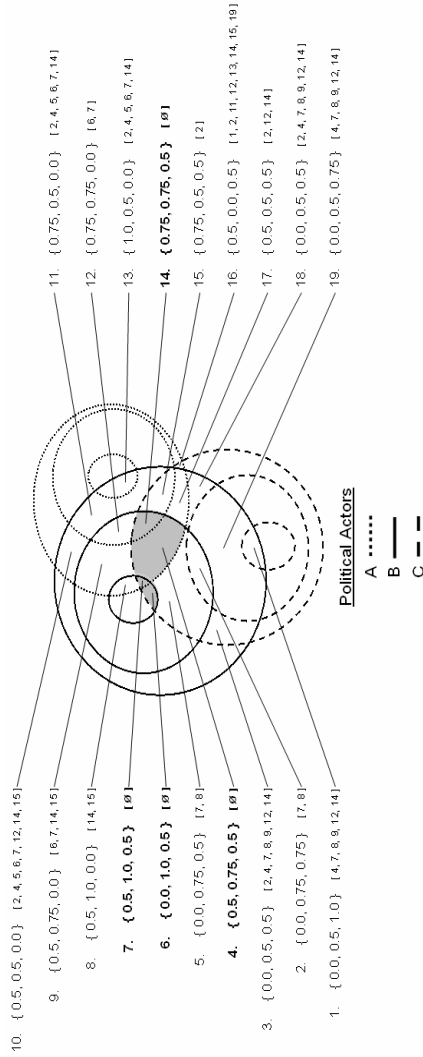


Fig. 7.3. A Relatively Large Maximal Set

All three models result in cycling in the crisp approach. Each produces a maximal set in the fuzzy approach. Furthermore, the maximal set has an intuitive appeal. Its location tends to be central, falling within an area circumscribed by



**Fig. 7.4.** A Moderate-Sized Maximal Set

the cores of the fuzzy numbers representing the preferences of the players, and its size decreases as the highest alpha-levels at the intersection between two or more players increases.

Do fuzzy set theory models always predict a well-defined maximal set? Unfortunately, the answer is no. Figure 7.6 maps the fuzzy points at  $\alpha = 1$  and  $.50$  for player *A*,  $\alpha = 1, .75$ , and  $.50$  for player *B*, and  $\alpha = 1$  and  $0.75$  for player *C* in two-dimensional policy space. We assume as before that each player has an equal weighted vote and that a decision requires a majority, and we have labeled the three-tuple payoffs for each fuzzy point in the set of alternatives. In this

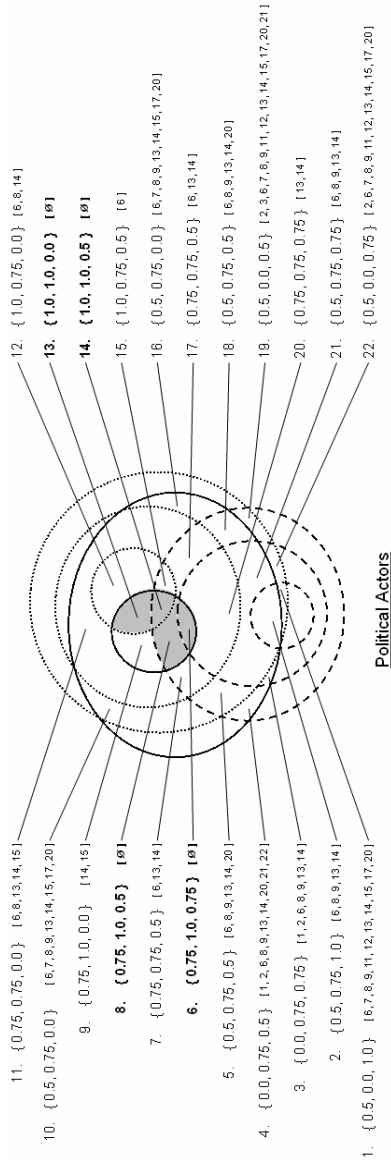
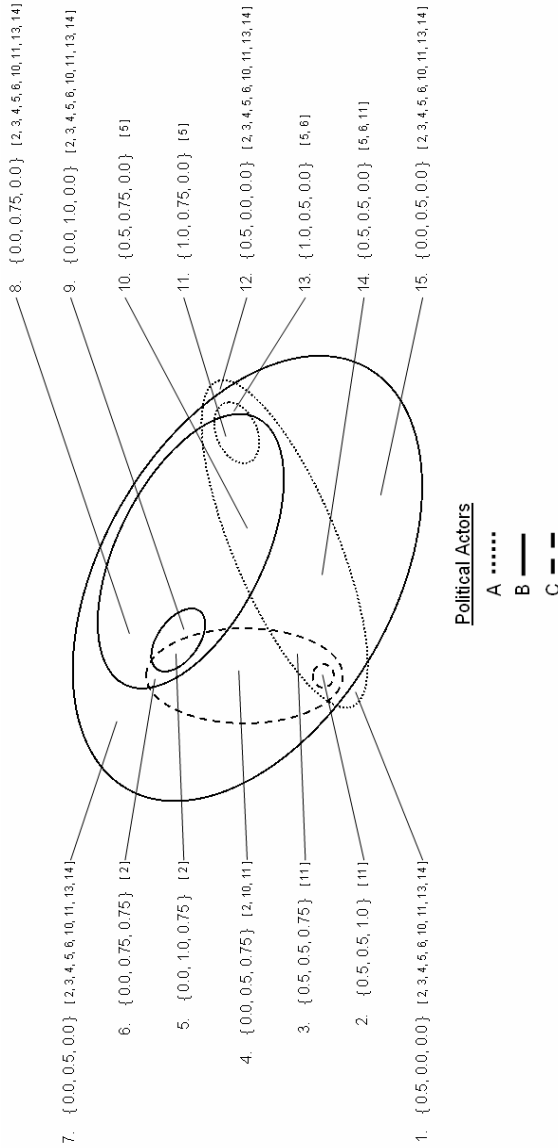


Fig. 7.5. A Small Maximal Set

case, there is no maximal set. All points are majority preferred by at least one other point. However, alternative 2 is majority preferred to alternative 5, which is majority preferred to alternative 11, which is majority preferred to alternative 2. Hence, there is a top cycle set, comprising the ideal fuzzy points of the three players (the cores of their fuzzy numbers). No alternative outside of this set is majority preferred to any alternative within it, and all alternatives outside



**Fig. 7.6.** A Top Cycle Set

of it are dominated by a point within it (to include alternatives 3, 6, and 10). Although we can not return a maximal set in this scenario, the top cycle set is relatively small and confined to the ideal points.

This juncture brings us to the frontier of our investigation. We have not exhausted all sets of conditions to be able to say definitively when a fuzzy model

will produce a maximal set or a top cycle. Nonetheless, it is interesting to note that the preferences in Figure 7.6 map to those in Table 7.3. While this results in a well-defined top cycle set when mapped in two-dimensional space, preferences such as those in Table 7.2 do not. The crisp approach allows cycling in virtually all of two-dimensional space.

We are also not yet certain how well or poorly behaved the cycling problem might be. While the complexity of the scenarios we have outlined gives us some confidence, absent formal proof, we can not argue definitively. Nevertheless, we dare not under-emphasize the point that a crisp approach would predict cycling in both of these scenarios (as well as the simpler scenarios we considered at the end of chapter five). The fuzzy approach does not.

## 7.4 Conclusions

We have endeavored in this book to extend fuzzy set theory to formal modeling in comparative politics. In particular, we have sought to contribute to The New Institutionalism, a research agenda that is at the forefront of rational choice modeling in comparative politics. We are aware that we have provided no more than an introduction to the possibilities that fuzzy set theory offers, and in that vein we are quite certain that our book will not be the final word on such applications. Indeed, it is our ardent hope that it will not be so.

Our primary concern has been to introduce a concept of applying fuzzy set theory to formal models in comparative politics in as accessible a manner as possible for comparativists and political scientists. For that reason as much as possible, we have attempted to hone as closely to the vocabulary and assumptions of formal modelers in political science as possible. The approach has been motivated by a desire to see fuzzy set theory more broadly applied in that discipline. Indeed, given the often ambiguous and vague nature of political phenomena, many of which are subjective, human constructs, fuzzy set theory seems ideally suited to the study of politics. That is most certainly the case for human preferences, which lie at the core of rational choice models.

Our central argument is that a fuzzy set approach to formal modeling in comparative politics offers a more elegant, less cumbersome, and fruitful alternative to crisp approaches, particularly when the political phenomena we are modeling are vague or ambiguous. That is almost always the case for preferences. A major challenge, however, will be to determine the set level inclusion (alpha-levels) of the players over the alternatives. Nonetheless, we believe that the consequent reduction in the likelihood of cycling and freedom from unrealistic assumptions will justify the effort. We have demonstrated that we can avoid cycling under conditions that are far less strict than Plott's (1967) radial symmetry. It is the case that  $\alpha = 0$  establishes an outer boundary on possible alternatives, and that together with the fact that the intersections of players alpha-levels define unique fuzzy points, reduces the number of alternatives to a finite set. In the absence of formal proof of a set of theorems, we can only speculate that this may induce the greater likelihood of a maximal set.

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# List of Symbols

$\subseteq$	Subset, page 58
$A$	The labels of fuzzy sets, page 32
$A^\alpha$	Alpha-cut of A, page 35
$A_\alpha^>$	Strong alpha-cut of A, page 37
$D(P_1, P_2)$	Distance between fuzzy points, page 69
$Dist(A, B)$	Distance between fuzzy numbers, page 62
$Dist_H(A, B)$	Hamming distance, page 63
$Dist_{C_\infty}(A, B)$	Maximum vertical distance, page 63
$Dist_D(A, B)$	Hausdorff distance, page 63
$I$	Indifference relation, page 86
$Im(A)$	Image of A, page 36
$L$	A fuzzy line, page 70
$M(R, X)$	Maximal set, page 86
$M_i(SQ)$	Maximal set for i relative to the status quo, page 90
$OWA_w$	Ordered Weighted Averaging function, page 177
$P$	A fuzzy point, page 68
$P$	Strict preference relation, page 86
$R$	Preference relation, page 86
$S$	S-shaped fuzzy number, page 44
$S\langle\beta\rangle$	Sigmoid fuzzy set, page 45
$Sld$	linear decreasing s-shaped fuzzy set, page 45
$Sli$	linear increasing s-shaped fuzzy set, page 44
$Sqd$	quadratic decreasing s-shaped fuzzy set, page 44
$Sqi$	quadratic increasing s-shaped fuzzy set, page 44
$T(\rho)$	Top cycle, page 117

$Tb\langle m, s, \gamma \rangle$	Gaussian bell fuzzy number $m$ , page 42
$Ti\langle m \rangle$	Impulse fuzzy number $m$ , page 43
$Tr$	Left, right fuzzy number, page 43
$Tp\langle a, l, r, b \rangle$	Triapezoidal fuzzy number (interval), page 41
$Tq\langle a, l, m, r, b \rangle$	Piecewise quadratic fuzzy number $m$ , page 41
$Tr\langle a, m, b \rangle$	Triangular fuzzy number $m$ , page 40
$UC(X)$	Uncovered set, page 122
$W_m(SQ)$	Majority win set relative to the status quo, page 92
$W_m(SQ)$	Unanimity win set relative to the status quo, page 92
$\alpha$ -level	A fuzzy set membership grade, page 95
$\frac{1.0}{a} + \frac{0.7}{b}$	Fraction notation for fuzzy set membership function, page 33
$\mathbb{B}$	The Boole set, page 59
$\mathbb{U}$	The unit interval, page 59
$\mathcal{C}$	A fuzzy circle, page 75
$\mathcal{F}(X)$	Fuzzy power set, page 34
$s(a, b)$	t-conorm union function, page 165
$t(a, b)$	t-norm intersection function, page 164
$\mu_A(x)$	The membership function of a fuzzy set, page 32
$\vee$	Fuzzy union, ususally max, page 60
$\wedge$	Fuzzy intersection, usually min, page 60
$c(a)$	complement function, page 162
$f^{(-1)}$	Pseudo-inverse function of $f$ , page 168
$f^{-1}$	Inverse function of $f$ , page 168
$h(a, b)$	Averaging function, page 175
$h(a_1, a_2, \dots, a_m)$	Averaging function, page 175

## Greek Alphabet

<i>Letters</i>	<i>Names</i>	<i>Letters</i>	<i>Names</i>	<i>Letters</i>	<i>Names</i>
$A$	$\alpha$ Alpha	$I$	$\iota$ Iota	$P$	$\rho$ Rho
$B$	$\beta$ Beta	$K$	$\kappa$ Kappa	$\Sigma$	$\sigma$ Sigma
$\Gamma$	$\gamma$ Gamma	$\Lambda$	$\lambda$ Lambda	$T$	$\tau$ Tau
$\Delta$	$\delta$ Delta	$M$	$\mu$ Mu	$\Upsilon$	$\upsilon$ Upsilon
$E$	$\varepsilon$ Epsilon	$N$	$\nu$ Nu	$\Phi$	$\phi$ Phi
$Z$	$\zeta$ Zeta	$\Xi$	$\xi$ Xi	$X$	$\chi$ Chi
$M$	$\eta$ Eta	$O$	$o$ Omicron	$\Psi$	$\psi$ Psi
$\Theta$	$\theta$ Theta	$\Pi$	$\pi$ Pi	$\Omega$	$\omega$ Omega

## Fuzzy Sets and Fuzzy Logic

English translation	Logical connective	Set operation	Fuzzy operator	Standard operator
not A	$\neg A$	$A^c$	complement	$1 - a$
A and B	$A \wedge B$	$A \cap B$	t-norm	$\min(a, b)$
A or B	$A \vee B$	$A \cup B$	t-conorm	$\max(a, b)$
If A then B	$A \rightarrow B$	$A \subseteq B$	implication	none

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Retrieved June 12, 2007. This is a National Science Foundation report written by experts on formal modeling that calls for greater efforts to demonstrate the empirical relevance of formal models in political science.
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Austen-Smith, D., Banks, J.S.: *Positive Political Theory II: Strategy and Structure*. University of Michigan Press, Ann Arbor (2005)

This second volume expands upon the basic arguments presented in the first by situating them firmly in institutional and strategic frameworks. Depending heavily on game theoretical analysis and the authors extensive use of the set theoretical approach, Austen-Smith and Banks consider many of the real world processes that generate political outcomes.

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Buckley, J.J., Eslami, E.: Fuzzy plane geometry I: Points and lines. *Fuzzy Sets and Systems* 86, 179–187 (1997a)

This paper initiated the study of fuzzy geometry by introducing the notions of fuzzy points and fuzzy lines in plane geometry. The paper began the comprehensive study of fuzzy geometry in a unified manner. Previous to this paper, the notions of area, height, width, diameter, and perimeter of a fuzzy subset of the plane were real numbers. In this paper, the fuzzy distance between two points is a fuzzy real number.

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Bunce, V.: Comparing east and south. *Journal of Democracy* 6(3), 87–100 (1994)

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Bunce, V.: Paper curtains and paper tigers. *Slavic Review* 54(4), 979–987 (1995a)

Bunce challenges the applicability of theories of democratic transitions developed in Latin America to democratic transitions in post-communist Europe.

Bunce, V.: Should transitologists be grounded? *Slavic Review* 54(1), 111–127 (1995b)

Bunce rejects the primacy of cross-regional comparative theories and argues instead that theories must account for regional peculiarities.

Bunce, V.: The political economy of postsocialism. *Slavic Review* 58(4), 756–793 (1999)

The author analyzes the transformation of political economy in postcommunist Europe and argues that they were much similar to each other than those occurring earlier in Southern Europe or Latin America.

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Burkhardt and Bonissone use gradient search methods to tune fuzzy knowledge bases. They use the standard control problem of balancing an inverted pole. The fuzzy controller starts with rules based on triangular fuzzy sets. They then apply gradient search to determine the optimal scaling factor for the base of the triangles.

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This was the first article to advocate the use of fuzzy set theory in the study of international relations. It is an excellent explication of the basic underpinnings of fuzzy set membership functions and their potential utility in classifying political data. The article closes with some interesting suggestions for the course of future research both in international relations and other areas of political science.

Clark, T.D.: *Beyond post-communist studies: Political science and the new democracies*, M.E. Sharpe, Armonk, New York (2002)

Clark argues that the literature on democratic consolidation lacks theoretical coherence. He presents a rational choice model of democratic consolidation and argues for its utility in post-communist Europe.

Colton, T.J.: Superpresidentialism and Russia's backward state. *Post-Soviet Affairs* 11(2), 114–148 (1995)

Colton gives consideration to the effects of Russia's extraordinarily powerful president on other political institutions and the quality of democracy.

Cox, G.W., McCubbins, M.D.: The institutional determinants of economic policy outcomes. In: Haggard, S., McCubbins, M.D. (eds.) *Presidents, Parliaments, and Policy*, pp. 21–63. Cambridge University Press, Cambridge (2001)

The volume contains a collection of essays on economic reform in presidential systems in Asia, Latin America, and Central Europe. The thesis is that presidentialism does not always lead to gridlock and institutional instability.

Cox, K.E., Schoppa, L.J.: Interaction Effects in Mixed-Member Electoral Systems: Theory and Evidence From Germany, Japan, and Italy. *Comparative Political Studies* 35(9), 1027–1053 (2002)

The authors argue that simultaneous balloting in mixed-member electoral systems does not lead to results that agree with Duverger's Theory that proportional representation tends to a larger number of parties, while single mandate systems lead to two party systems. They demonstrate empirically that the number of parties emerging in the single mandate tier is considerably larger. Hence, mixed electoral systems produce unintended effects.

De Marchi, S.: *Computational and mathematical modeling in the social sciences*. Cambridge University Press, Cambridge (2005)

The book discusses game theory, statistics, and computational methods and proposes an alternative framework for using mathematics in political science.

De Swaan, A.: *Coalition Theories and Cabinet Formations*. Jossey-Bass, Inc., San Francisco (1973)

DeSwaan engages in a comprehensive review and test of formal theories on coalition formation using case studies of nine European parliamentary democracies.

Diamond, L., Plattner, M.F., Yun-han, C., Hung-mao, T.: *Consolidating the Third Wave Democracies: Themes and Perspectives*. Johns Hopkins University Press, Baltimore (1997)

This book is a collection of essays on democratic consolidation.

Diamond, L.J.: Rethinking civil society: Toward democratic consolidation. *Journal of Democracy* 5(3), 4–17 (1994)

Diamond questions the dominant approach to democratic consolidation, which emphasizes the choices made by strategically situated elites engaged in bargaining. He offers that civil society is an overlooked, but important factor in successful democratization. The article makes a contribution to the literature that seeks to bring society back into the study of democracy.

Dodd, L.C.: *Coalitions in Parliamentary Governments*. Princeton University Press, Princeton (1976)

Dodd demonstrates the ability of coalition theory to predict outcomes of the government formation process.

Dombi, J.: A general class of fuzzy operators, the De Morgan class of fuzzy operators and fuzziness measures induced by fuzzy operators. *Fuzzy Sets and Systems* 8(2), 149–163 (1982)

Introduces one of the important classes of  $t$ -norms and  $t$ -conorms.

Downs, A.: *An Economic Theory of Democracy*. Harper & Row, New York (1957)

Downs' classic is one of the earliest attempts to extend formal rational choice modeling to political science. He develops a model that demonstrates the effect of voter uncertainty on party systems.

Dubois, D., Prade, H.: *Fuzzy Sets and Systems: Theory and Applications*. Academic Press, New York (1980a)

The review in SIAM by J. A. Goguen is the best summary of this work. "As the authors clearly state, this is a "research compendium" based upon about 550 publications in English, French, and German available prior to 1978, rather than a textbook as such. Moreover, it develops topics "unequally according to (the author's] state of knowledge and fields of interest", in particular omitting all results requiring advanced mathematical tools such as category theory. Subject to these limitations, this book is the best available survey of the theoretical literature on fuzzy sets, and often successfully integrates works of diverse origin, notation, and approach."

Dubois, D., Prade, H.: New results about properties and semantics of fuzzy set-theoretic operations. In: Wang, P.P., Chang, S.K. (eds.) *Fuzzy Sets: Theory and Applications to Policy Analysis and Information Systems*, pp. 59–75. Plenum Press, New York (1980b)

Introduces one of the important classes of  $t$ -norms and  $t$ -conorms.

Dubois, D., Prade, H.: On distances between fuzzy points and their use for plausible reasoning. In: *Proceedings of the International Conference on Systems Man and Cybernetics*, pp. 300–303 (1983)

The authors present a method for measuring distance between fuzzy points.

Dubois, D., Prade, H.: An overview of aggregation operators. *Information Sciences* 36(1,2), 85–121 (1985)

An excellent overview and comparison of aggregation operators introduced up until 1985.

Eslami, E., Nemat, F., Buckley, J.: Fuzzy space geometry I: Points, lines, planes. *Journal of Fuzzy Mathematics* 9, 659–675 (2001a)

This paper extends the definition of a fuzzy point in (Buckley and Eslami, 1997a) to three dimensional space. A direction vector is used to define fuzzy lines in space. The definition of a fuzzy line is given using two methods, a parametric form and a symmetric form. This paper also defines and examines fuzzy planes.

Eslami, E., Nemat, F., Buckley, J.: Fuzzy space geometry II: Subpoints and sublimes. *Journal of Fuzzy Mathematics* 9, 693–700 (2001b)

This paper shows in three dimensional space that there may not exist a fuzzy line passing through two fuzzy points and that two intersecting fuzzy lines do not necessarily determine a fuzzy line. In order to examine these deficiencies further, the authors introduce the notions of fuzzy subpoints and fuzzy sublimes.

Ferrara, F., Herron, E.S.: Going it alone? strategic entry under mixed electoral rules. *American Journal of Political Science* 49(1), 16–31 (2005)

The article considers the psychological effect of mixed electoral systems on pre-electoral alliance behavior of political parties.

Filippov, M., Ordeshook, P.C., Shvetsova, O.: *Designing Federalism: A Theory of Self-Sustainable Federal Institutions*. Cambridge University Press, Cambridge (2004)

The authors argue against the prevailing view in the literature on federalism that the success of federal state is contingent upon a careful delineation of policy jurisdiction between center and periphery. They argue instead that the content of regional charters and the extent to which political offices are elected instead of appointed are decisive.

Fish, S.M.: The executive deception: Superpresidentialism and the degradation of russian politics. In: Sperling, V. (ed.) *Building the Russian State: Institutional Crisis and the Quest for Democratic Governance*, Boulder, CO, pp. 177–192. Westview Press (2000)

Fish questions the ability of Russias state institutions to function effectively.

Frank, M.J.: On the simultaneous associativity of  $f(x, y)$  and  $x+y-f(x, y)$ . *Aequationes Mathematicae* 19(2,3), 194–226 (1979)

Introduces one of the important classes of  $t$ -norms and  $t$ -conorms.

George, M., Kuriakose, S., Thomaskutty, P.: A fuzzy preference based choice function. *The Journal of Fuzzy Mathematics* 15(1), 239–246 (2007)

The paper offers a fuzzy set preference relationship as a means for overcoming cycling in social choice theory.

Goodman, I.R.: Fuzzy sets as equivalence classes of random sets. In: Yager, R. (ed.) *Fuzzy Sets and Possibility Theory*, pp. 327–342. Pergamon Press, Oxford (1982)

The author offers a rigorous consideration of fuzzy sets as equivalence classes of random sets.

Hamacher, H.: Über logische verknüpfungen unscharfer aussagen und deren zugehörige bewertungs-funktionen. In: *Progress in Cybernetics and Systems Research*, pp. 276–287 (1978)

Introduces one of the important classes of  $t$ -norms and  $t$ -conorms.

Herron, E.S., Nishikawa, M.: Contamination effects and the number of parties in mixed-superpositional electoral systems. *Electoral Studies* 20(1), 63–86 (2001)

The article argues that mixed electoral systems are more than the sum of their parts. The two tiers contaminate each other, as a result of which the psychological effects of each tier imposes itself on electoral outcomes in the other.

Hoffmann, E.P., Flernon, F.J. (eds.): *The Conduct of Soviet Foreign Policy*, Chicago, Atherton (1971)

The volume provides a broad overview of the sources and methods of Soviet foreign policy behavior.

Huntington, S.P.: *The Third Wave: Democratization in the Late Twentieth Century*. University of Oklahoma Press, Norman, OK (1991)

Huntington discusses the issues involved in successful democratization.

- Kacprzyk, J., Fedrizzi, M., Nurmi, H.: Group decision making and consensus under fuzzy preferences and fuzzy majority. *Fuzzy Sets and Systems* 49, 21–31 (1992)  
The paper offers a formal proof of the existence of a maximal set when fuzzy preferences are aggregated under consensus decision-making rules.
- Karl, T.L., Schmitter, P.C.: From an iron curtain to a paper curtain: Grounding transitologists or students of postcommunism? *Slavic Review* 54(4), 965–978 (1995)  
This response to Bunce's "Should Transitologists Be Grounded?" argues for a cross-regional comparative approach to theory building and testing.
- Kempton, W.: Interview methods for eliciting fuzzy categories. *Fuzzy Sets and Systems* 14(1), 43–64 (1984)
- Kiewiet, D.R., McCubbins, M.D.: Presidential influence on congressional appropriations decisions. *American Journal of Political Science* 32(3), 713–736 (1988)  
The authors develop a one-dimensional model of the influence of the U.S. presidential veto over Congressional appropriations bills. The model predicts that the president can reduce Congressional spending, but he can not increase it.
- Kim, C.J., Russell, D.: Automatic Generation of Membership Function and Fuzzy Rule Using Inductive Reasoning. In: *IFIS 1993*, pp. 93–96 (1993)  
This paper discusses the automatic generation of membership function and fuzzy rule. The generation of them are accomplished by utilizing the essential characteristic of the inductive reasoning which derives a general consensus from the particular. The induction is performed by the entropy minimization principle which clusters most optimally the parameters corresponding to the output classes. The rule derivation also provide the average probability of each step of rule, which is no other than the rule weight. The generation scheme is illustrated for practical use.
- King, G., Keohane, R.O., Verba, S.: *Designing Social Inquiry: Scientific Inference in Qualitative Research*. Princeton University Press, Princeton (1994)  
This text argues that whether research in political science is qualitative or quantitative in design, the same canon for theorizing, hypothesizing, operationalizing data, choosing cases, and inferring from data apply.
- Klir, G.J., Yuan, B.: *Fuzzy Sets and Fuzzy Logic; Theory and Applications*. Prentice Hall, Upper Saddle River, N.Y. (1995)  
The most comprehensive textbook in the field of fuzzy set theory. Written at the graduate school level.
- Klir, G.J., Yuan, B.: *Fuzzy Sets, Fuzzy Logic, and Fuzzy Systems: Selected Papers by Lotfi A. Zadeh*, 1st edn. World Scientific, Singapore (1996)
- Koenig-Archibugi, M.: Explaining Government Preferences for Institutional Change in EU Foreign and Security Policy. *International Organization* 58(01), 137–174 (2004)  
The paper explains the variation in support for a single European Union (EU) foreign and security policy using a fuzzy set logic approach.
- Lagerspetz, E.: Social choice in the real world. *Scandinavian Political Studies* 16(1), 1–23 (1993a)  
Lagerspetz considers the problem of cycling in the empirical world.
- Lagerspetz, E.: Social choice in the real world ii: Cyclical preferences and strategic voting in the finnish presidential elections. *Scandinavian Political Studies* 20(1), 53–67 (1993b)  
The paper catalogs an empirical incident of cycling in politics.

- Laver, M., Shepsle, K.A.: *Making and Breaking Governments: Cabinets and Legislatures in Parliamentary Governments*. Cambridge University Press, Cambridge (1996)  
Laver and Shepsle present a formal model predicting the outcome of the government formation process in parliamentary systems. The model assumes Euclidean preference, sincere voting, and abdication on the part of political parties.
- Linz, J.J.: *Transitions to democracy*. *Washington Quarterly* 13, 156 (1990)  
Linz makes his well-known observation that parliamentarism is more likely to assure the success of democratization than is presidentialism in newly emerging democracies.
- Linz, J.J.: *Presidential or parliamentary democracy: Does it make a difference?* In: Linz, J.J., Valenzuela, A. (eds.) *The Failure of Presidential Democracy*, pp. 3–87. Johns Hopkins University Press, Baltimore (1994)  
The volume offers essays on the inherent problems of presidentialism for democracy.
- Mainwaring, S.: *Presidentialism, Multipartyism, and Democracy: The Difficult Combination*. *Comparative Political Studies* 26(2), 198 (1993)  
The authors contend that it is not presidentialism that is prejudicial to democracy, but rather presidentialism in combination with multipartyism.
- Mainwaring, S., Shugart, M.S.: *Juan linz, presidentialism, and democracy: A critical appraisal*. *Comparative Politics* 29(4), 449–471 (1997)  
The authors challenge Linz's thesis that parliamentarism is better able to assure the success of democratization than presidentialism. They argue that under certain conditions presidentialism has clear advantages. Furthermore, the case for parliamentarism rests on its having been adopted in countries with significant economic and political advantages.
- McKelvey, R.D.: *Intransitivities in multidimensional voting models*. *Journal of Economic Theory* 12, 472–482 (1976)  
McKelvey's seminal article demonstrates that there is a path to any outcome in two-dimensional policy space in the absence of Plott's condition under sincere voting. In effect, cycling is endemic to these models.
- McKelvey, R.D.: *Governing, dominance and institution-free properties of social choice*. *American Journal of Political Science* 30, 283–314 (1986)
- Menger, K.: *Statistical metrics*. *Proc. Nat. Acad. Sci.* 28, 535–537 (1942)  
Introduces the concepts of  $t$ -norms and  $t$ -conorms which are so important in fuzzy set theory today.
- Miller, N.R.: *A new solution set for tournaments and majority voting*. *American Journal of Political Science* 24, 68–96 (1980)  
The article argues that the set of undominated proposals in multidimensional policy space may be quite large. It proposes the uncovered set as an alternative solution.
- Miller, N.R.: *In search of the uncovered set*. *Political Analysis* 15, 21–45 (2007)  
The article offers an exploration of properties of the uncovered set with the assistance of CyberSenate.
- Motyl, A.J.: *Thinking Theoretically about Soviet Nationalities: History and Comparison in the Study of the USSR*. Columbia University Press, New York (1992)  
Motyl argues that Soviet studies has suffered from a lack of attention to theory. He contends that substantive understanding in the field will only increase if competing theories are developed and tested.

- Munck, G.: Bringing postcommunist societies into democratization studies. *Slavic Review* 56(3), 542–550 (1997)  
 Munck argues in favor of applying cross-regional comparative theories to the study of democratization in post-communist Europe.
- Nurmi, H.: Approaches to collective decision making with fuzzy preference relations. *Fuzzy Sets and Systems* 6, 249–259 (1981a)  
 Nurmi explores the predictions made under fuzzy preference relations. He demonstrates that satisficing results in a maximal set under certain conditions.
- Nurmi, H.: A fuzzy solution to a majority voting game. *Fuzzy Sets and Systems* 5, 187–198 (1981b)  
 Experiments have reached different conclusions on a number of dimensions from those predicted by social choice theory. Nurmi offers a fuzzy set theoretical explanation of these anomalies.
- Nurmi, H., Kacprzyk, J.: Fuzzy sets in political science: An overview. *New Mathematics and Natural Computation* 3(3), 1–19 (2007)  
 This is a concise review of the literature on the application of fuzzy set theory in political science.
- O'Donnell, G., Schmitter, P.C.: *Transitions from Authoritarian Rule: Tentative Conclusions from Uncertain Democracies*. The Johns Hopkins University Press, Baltimore (1986)  
 The volume presents a summary of the findings of case studies of the process of democratization in Southern Europe and Latin America conducted under the auspices of the Woodrow Wilson Center.
- Olson, M.: *The Logic of Collective Action: Public Goods and the Theory of Groups*. Harvard University Press, Cambridge (1965)  
 Olson's seminal book offers a rational choice analysis of the dynamic of group and organizational behavior. It argues that large organizations incentivize free-riding on the part of their members.
- Ordeshook, P.C., Shvetsova, O.V.: Ethnic heterogeneity, district magnitude, and the number of parties. *American Journal of Political Science* 38(1), 100–123 (1994)  
 The paper tests hypotheses related to institutional design and social heterogeneity that are hypothesized to account for party system fragmentation.
- Orlovsky, S.: Decision-making with a fuzzy preference relation. *Fuzzy Sets and Systems* 1, 155–167 (1978)  
 The paper defines fuzzy strict preference relations as a non-fuzzy set of alternatives that introduce the fuzzy set of undominated alternatives. It demonstrates that an undominated solution exists under certain conditions.
- Penn, E.M.: Alternate definitions of the uncovered set and their implications. *Social Choice and Welfare* 27(1), 83–87 (2006)  
 Penn analyzes the different formal definitions of the uncovered set. She finds that they are functionally equivalent as long as individuals hold strict preferences.
- Pennings, P.: Beyond dichotomous explanations: Explaining constitutional control of the executive with fuzzy-sets. *European Journal of Political Research* 42(4), 541–568 (2003)  
 The article tests competing hypotheses explaining variation in constitutional controls of the executive in forty-five parliamentary democracies using Ragins fuzzy set qualitative comparative analysis (FsQCA) method.

Plott, C.R.: A notion of equilibrium and its possibility under majority rule. *American Economic Review* 57, 787–806 (1967)

This is Plott's seminal argument in which he demonstrates that a core in two-dimensional policy space only exists under conditions of radial symmetry.

Procyk, T.S., Mamdani, E.H.: A Linguistic Self-Organizing Process Controller. *Automatica* 15, 15–30 (1979)

Przeworski, A.: *Democracy and the Market: Political and Economic Reforms in Eastern Europe and Latin America*. Cambridge University Press, Cambridge (1991)

Przeworski explores the effects of political and economic transformation in post-communist Europe. He contributes a formal model of the transition to democracy process unearthed by the inductive case studies of democratization in Southern Europe and Latin America conducted under the auspices of the Woodrow Wilson Center. He also argues that liberal economic reform can undermine democratic political reform.

Przeworski, A., Teune, H.: *The Logic of Comparative Social Inquiry*. John Wiley and Sons, Inc., Malabar, FL (1970)

Przeworski and Teune discuss the comparative method. They argue that comparative studies must be theoretically driven, and that they must seek to explain variation in bivariate relations cross-nationally.

Ragin, C.C.: *The Comparative Method: Moving Beyond Qualitative and Quantitative Strategies*. University of California Press, Berkeley (1987)

Ragin offers a Boolean algebra based method for isolating necessary and sufficient causes in a small number of cases. The method also finds multiple paths of causality.

Ragin, C.C.: *Fuzzy-Set Social Science*. University of Chicago Press, Chicago (2000)

Ragin's book develops a fuzzy set theoretical approach to qualitative research. The method focuses on determining combinations of necessary and sufficient conditions for a political outcome given a relatively small number of cases.

Remington, T.F., Smith, S.S.: Theories of legislative institutions and the organization of the Russian Duma. *American Journal of Political Science* 42, 546–572 (1998)

Remington, T.F., Smith, S.S.: *The Politics of Institutional Choice: The Formation of the Russian State Duma*. Princeton University Press, Princeton (2001)

Remington and Smith offer an authoritative study of several legislative systems in the Russian State Duma, to include the links between electoral system design, the party system, and legislative organization. Much of the questions asked are based on rational choice theories on legislatures.

Riker, W.H.: *The Theory of Political Coalitions*. Yale University Press, New Haven (1962)

Riker offers a rational choice theory of the government formation process in parliamentary systems. Starting with the premise that political parties are motivated by the desire to achieve office, he argues that coalitions should be the smallest possible assuring a majority vote in the legislature. Such coalitions are highly stable as they offer office-seeking parties the highest possible payoff.

Riker, W.H.: Implications from the disequilibrium of majority rule for the study of institutions. *American Political Science Review* 74, 432–446 (1980)

The article argues for an emphasis on institutional analysis, rather than the political values of groups, as the primary means for discovering generalizations in political science. The article further argues that efforts must be undertaken to both examples of and solutions to the problem of cycling that is predicted by formal models of politics.

- Riker, W.H.: *Liberalism Against Populism*. Freeman, San Francisco (1982)  
 Riker's seminal work argues that the rules by which individual preferences are aggregated, rather than popular will, are determinant in collective choices.
- Robertson, J.D.: Economic polarization and cabinet formation in western europe. *Legislative Studies Quarterly* 11, 533–549 (1986)  
 Robertson considers the effect of political economy on cabinet government.
- Roeder, P.G.: The revolution of 1989: Postcommunism and the social sciences. *Slavic Review* 58(4), 743–755 (1999)  
 The article urges that social scientists devote efforts to build theories capable of explaining the economic, political, and social transformations occurring in post-communist Europe.
- Rosenfeld, A.: The diameter of a fuzzy set. *Fuzzy Sets and Systems* 13, 241–246 (1984)  
 This paper introduces definitions of intrinsic and extrinsic diameter for fuzzy subsets. These definitions reduce to the ordinary definitions when the subsets are crisp. The height and width for a fuzzy subset are defined and it is shown how these notions relate to the area. For convex fuzzy subsets it is shown that the intrinsic diameter cannot exceed the extrinsic diameter and can, in fact, be smaller. It is shown for piecewise constant convex fuzzy subsets that the intrinsic diameter cannot exceed half the fuzzy perimeter. These notions have meaning in pattern recognition.
- Rosenfeld, A.: Fuzzy rectangles. *Pattern Recognition Letters* 11, 677–679 (1990)  
 The notion of a convex polygon is defined in terms of the infimum of fuzzy half planes. In a separable fuzzy subset of the plane, the notions of fuzzy connectedness, convexity, and orthoconvexity are equivalent. A fuzzy subset with these properties is called a fuzzy rectangle.
- Rosenfeld, A.: Fuzzy plane geometry: Triangles. In: *Proceedings of the Third IEEE International Conference in Fuzzy Systems, Orlando, vol. II, pp. 891–893 (June 1994)*  
 A fuzzy triangle  $T$  with a finite-valued membership function is defined as a nest of parallel-sided triangle  $T_i$  with progressively higher membership values. The side lengths and altitudes of  $T$  are also defined as weighted sums obtained from projections. The perimeter of  $T$  is the sum of the side lengths which are related to the vertex angles of the Law of Sines. However there is no simple relationship between the area of  $T$  and the products of the side lengths and altitudes.
- Rosenfeld, A., Haber, S.: The perimeter of a fuzzy set. *Pattern Recognition* 18, 125–130 (1985)  
 The area and perimeter in terms of integrals of a fuzzy subset and its gradient are provided.
- Samuels, D.J.: The gubernatorial coattails effect: Federalism and congressional elections in brazil. *The Journal of Politics* 62(1), 240–253 (2000)  
 The article argues that federalism can shape the number of parties placing candidates in elections. The empirical analysis reveals that gubernatorial races rather than presidential races have a greater effect on the number of parties contesting in Brazil.
- Samuels, D.J.: Fiscal straitjacket: The politics of macroeconomic reform in brazil. *Journal of Latin American Studies* 35(3), 545–569 (2003)  
 This article analyzes the causes of Brazils pre-1995 macroeconomic difficulties and the consequences of President Cardoso's national plan for dealing with them.

- Sanjian, G.S.: Fuzzy Set Theory and US Arms Transfers: Modeling the Decision-Making Process. *American Journal of Political Science* 32(4), 1018–1046 (1988)  
Sanjian develops a fuzzy set model of arms export decision making and tests it predictions against U.S. arms transfers decisions.
- Sanjian, G.S.: Great power arms transfers: Modeling the decision-making processes of hegemonic, industrial, and restrictive exporters. *International Studies Quarterly* 35(2), 173–193 (1991)  
The paper develops and tests a fuzzy multi-criteria model of arms transfer decision-making. It tests the model against the arms transfer strategies of the United States, France, and West Germany from 1950 to 1976.
- Sanjian, G.S.: A fuzzy set model of nato decision-making: The case of short range nuclear forces in europe. *Journal of Peace Research* 29(3), 271–285 (1992)  
The article presents a fuzzy set model to a collective decision making situation that predicts the policy outlook of each of the major players in NATO.
- Sanjian, G.S.: Cold War Imperatives and Quarrelsome Clients: Modeling US and USSR Arms Transfers to India and Pakistan. *The Journal of Conflict Resolution* 42(1), 97–127 (1998)  
Sanjian tests a fuzzy systems model of the stratetic consequences of great power arms transfers using U.S.-Pakistani and U.S.S.R.-India arms transfer deals as case studies.
- Sanjian, G.S.: Promoting Stability or Instability? Arms Transfers and Regional Rivalries, 1950–1991. *International Studies Quarterly* 43(4), 641–670 (1999)  
The paper advances a fuzzy model of the effect of arms transfers on regional stability. It tests the model with case studies of arms transfers in South and Southwest Asia.
- Sanjian, G.S.: Arms and Arguments: Modeling the Effects of Weapons Transfers on Subsystem Relationships. *Political Research Quarterly* 54(2), 285–309 (2001)  
The article tests the effects of arms transfers on India-Pakistani and Iran-Iraq relations during the Cold War.
- Schmitter, P.C., Karl, T.L.: The conceptual travels of transitologists and consolidologists: How far to the east should they attempt to go? *Slavic Review* 53(1), 173–185 (1994)  
Schmitter and Karl argue in favor of using cross-national theories to the study of democratization in post-communist Europe.
- Schweizer, B., Sklar, A.: Associate functions and statistical triangle inequalities. *Publicationes Mathematicae, Debrecen* 8, 169–186 (1961)  
Early and important results concerning  $t$ -norms and  $t$ -conorms.
- Schweizer, B., Sklar, A.: Associative functions and abstract semigroups. *Publicationes Mathematicae, Debrecen* 10, 69–81 (1963)  
More, early and important results concerning  $t$ -norms and  $t$ -conorms.
- Schweizer, B., Sklar, A.: *Probability Metric Spaces*. North-Holland, New York (1983)  
Complete coverage of  $t$ -norms and  $t$ -conorms in their original context of probabilistic metric spaces.
- Seitz, S.T.: Apollo's oracle: Strategizing for peace. *Synthese* 100(3), 461–495 (1994)  
The article develops a fuzzy inference engine for computationally simulating systemic polarity and balance of power and tests these systems against propensity for conflict or peace.

- Shepsle, K.A.: Institutional arrangements and equilibrium in multidimensional voting models. *American Journal of Political Science* 23, 27–59 (1979)  
 Shepsle's seminal article offers an insitutional explanation for the lack of cycling in the U.S. Congress. His thesis is that the committee system induces a stable outcome at the median-by-median point.
- Shepsle, K.A., Bonchek, M.S.: *Analyzing Politics: Rationality, Behavior, and Institutions*. W.W. Norton & Co., NTL (1997)  
 A catch-all textbook to the study of The New Institutionalism, the book presents in straightforward terms many of the tenants of social choice theory, spatial models, game theoretical bargaining situations, and basic institutional analysis.
- Shepsle, K.A., Weingast, B.R.: Uncovered sets and sophisticated voting outcomes with implications for agenda institutions. *American Journal of Political Science* 28, 49–74 (1984)  
 The paper argues that institutions place constraints on outcomes under majority rule and strategic voting.
- Shepsle, K.A., Weingast, B.R.: The institutional foundations of committee power. *American Political Science Review* 81, 85–104 (1987)  
 The article advances a formal model of committee power.
- Shepsle, K.A., Weingast, B.R.: *Positive Theories of Congressional Institutions*. University of Michigan Press, Ann Arbor (1995)
- Shugart, M.S.: The inverse relationship between party strength and executive strength: A theory of politician's constitutional choices. *British Journal of Political Science* 28, 1–29 (1998)  
 Shugart argues that strong executives are associated with weak party systems and weak executives are associated with strong party systems.
- Shugart, M.S., Carey, J.M.: *Presidents and Assemblies: Constitutional Design and Electoral Dynamic*. Cambridge University Press, Cambridge (1992)  
 The book offers a schema for classifying types of presidential systems. It argues that not all types of presidentialism are prejudicial to democracy.
- Smithson, M., Verkuilen, J.: *Fuzzy Set Theory: Applications in the Social Sciences*. Sage Publications, Thousand Oaks, CA (2006)  
 The text discusses measurement issues and correlation methods in fuzzy set logic as applied to the social sciences.
- Stepan, A., Skach, C.: Constitutional frameworks and democratic consolidation. *World Politics* 46, 1–22 (1993)  
 The authors offer an empirical analysis of the effect of several institutional designs on democratic consolidation. The general argument is that parliamentary systems are more highly associated with democratic stability than are presidential systems.
- Taber, C.S.: POLI: An Expert System Model of US Foreign Policy Belief Systems. *The American Political Science Review* 86(4), 888–904 (1992)  
 Taber presents a fuzzy expert model for analyzing foreign policy.
- Terry, S.M.: Thinking about post-communist transitions: How different are they? *Slavic Review* 52(2), 333–337 (1993)  
 The article contributes to the debate on whether scholars can meaningfully apply theories developed in other regions to the transformations in post-communist Europe.

Tsebelis, G.: Decision making in political systems: Veto players in presidentialism, parliamentarism, multicameralism, and multipartyism. *British Journal of Political Science* 25, 289–326 (1995)

The article lays out a model of democratic political systems in which policy change is a function of the number of institutional actors whose consent is required to make policy change (veto players), the congruence of their policy positions, and the policy cohesion of the veto players. Tsebelis argues that his model permits comparison across democratic political systems and accounts for government stability as well as policy stability.

Tsebelis, G.: Veto players and law production in parliamentary democracies: An empirical analysis. *American Political Science Review* 93(3), 591–608 (1999)

The article finds empirical support for hypotheses derived from veto player theory that an increase in the number of parties in government and the policy distance between them will decrease the number of significant laws that the government and its parliamentary majority will be able to pass. The number of significant laws is also found to decrease as government durability increases.

Tsebelis, G.: *Veto Players: How Political Institutions Work*. Russell Sage Foundation, New York (2002)

Tsebelis argues that the primary differences among democratic systems do not owe to regime type (presidentialism versus parliamentarism) but rather the number of constitutional players whose consent is necessary in order to effect a policy change (veto players). He lays out a theory in which the number of veto players, the location of the status quo, and the distance between the veto players in two-dimensional policy space establishes whether a policy can be changed and to what degree.

Tullock, G.: Why so much stability? *Public Choice* 37(2), 189–202 (1981)

This is the seminal article in which Tullock asks why there is such a wide gulf between rational choice formal models that predict cycling and the empirical world in which examples of cycling are hard to find.

Turksen, I.B.: Measurement of membership functions and their acquisition. *Fuzzy Sets and Systems* 40(1), 5–38 (1991)

Three basic views of the representation of previous termmembership functions are reviewed, together with fundamental previous termmeasurement of linguistic terms of linguistic variables. The conclusion is that such previous termmeasurements are either on an ordinal or an interval scale based on whether the appropriate axioms are validated by the empirical data, with an allowance for stochastic variation. The conjoint previous termmeasurement is introduced for the case of multidimensional linguistic variables whose linguistic terms are compositions of two or finitely many more component linguistic terms of distinct (independent) linguistic variables. It is shown that any composition of distinct (independent) component linguistic terms of component linguistic variables by any t-norm or s-norm or any finite convex linear combination preserves the monotonic weak order property of the components in the composite. Once the scale properties of the previous termmeasurement values of particular terms are validated, composition procedures may be applied to the experimental data to obtain compound previous termmembership functions of fuzzy sets induced by meaningful representations of compositions of linguistic terms of linguistic variables. Finally, four methods of previous termmembership acquisition and construction are reviewed from the perspective of fundamental previous termmeasurement.

- Verkuilen, J.: Assigning membership in a fuzzy set analysis. *Sociological Methods & Research* 33(4), 462–496 (2005)  
 The article discusses the problem of assigning set membership values and approaches for dealing with the problem.
- Warwick, P.V.: Getting the assumptions right: A reply to Laver and Shepsle. *British Journal of Political Science* 29(2), 402–412 (1999)  
 Warwick challenges Laver and Shepsles' abdication assumption in coalition formation in parliamentary systems. He finds empirical evidence that parties do not abdicate as part of formal agreements on government formation.
- Watanabe, N.: Statistical Methods for Estimating Membership Functions. *Japanese Journal of Fuzzy Theory and Systems* 5(4), 589–601 (1993)
- Wiarda, H.J.: Southern Europe, Eastern Europe, and comparative politics: Transitology and the need for new theory. *East European Politics and Societies* 15(3), 485–501 (2002)  
 Wiarda argues that theories of democratization developed in Southern Europe and Latin America are deeply awed. Therefore, apart from the issue of whether the transformations in post-communist Europe are substantially different from those in Southern Europe and Latin America that they require an altogether different set of theories, the existent theories are incapable of providing a guide to research.
- Yager, R.R.: Ordered weighted averaging aggregation operators in multi-criteria decision making. *IEEE Trans. on Systems, Man and Cybernetics* 18, 183–190 (1988)  
 Introduces one of the important classes of  $t$ -norms and  $t$ -conorms.
- Yager, R.R.: On a general class of fuzzy connectives. *Fuzzy Sets and Systems* 4(3), 235–242 (1980)  
 Introduces ordered weighted averaging (OWA). OWAs can perform as an average, max, min, median, or as a mixture of any and all of these operators. A paper that grows in importance.
- Yuan, X., Shen, Z.: Notes on fuzzy plane geometry I and II. *Fuzzy Sets and Systems* 121, 545–547 (2001)  
 Membership functions are given for the concepts of fuzzy distance, fuzzy lines, fuzzy circles, fuzzy area, and fuzzy circumference of a fuzzy circle found in  $\square$  and  $\square$ .
- Zadeh, L.A.: Fuzzy sets. *Information and Control* 8(3), 338–353 (1965)  
 Seldom is a paper so groundbreaking as lucid and as masterful as “Fuzzy Sets”. In this seminal paper fuzzy set theory appears in its complete form, somewhat like Athena from the head of Zeus. The paper is approachable by anyone with the mathematical maturity of a high school senior. Required reading for anyone in the field.
- Zadeh, L.A.: Similarity relations and fuzzy orderings. *Information Sciences* 3(2), 177–200 (1971)  
 Fuzzy relations are introduced in the seminal Zadeh, (1965). This article is devoted to fuzzy relations on a fixed set  $X$ , and in particular to generalizations of equivalence and linear ordering relations. Zadeh defines a similarity relation  $R$  to be a reflexive, transitive, and symmetric fuzzy relation. A fuzzy partial ordering is reflexive, transitive, and antisymmetric. Zadeh also introduces fuzzy linear order and fuzzy preorder.

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