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Handbook of Continued Fractions for Special Functions

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PREFACE

The idea to write a *Handbook of Continued fractions for Special functions* originated more than 15 years ago, but the project only got started end of 2001 when a pair of Belgian and a pair of Norwegian authors agreed to join forces with the initiator W.B. Jones. The book splits naturally into three parts: *Part I* discussing the concept, correspondence and convergence of continued fractions as well as the relation to Padé approximants and orthogonal polynomials, *Part II* on the numerical computation of the continued fraction elements and approximants, the truncation and round-off error bounds and finally *Part III* on the families of special functions for which we present continued fraction representations.

Special functions are pervasive in all fields of science and industry. The most well-known application areas are in physics, engineering, chemistry, computer science and statistics. Because of their importance, several books and websites (see for instance `functions.wolfram.com`) and a large collection of papers have been devoted to these functions. Of the standard work on the subject, the *Handbook of mathematical functions with formulas, graphs and mathematical tables* edited by Milton Abramowitz and Irene Stegun, the American National Institute of Standards and Technology claims to have sold over 700 000 copies (over 150 000 directly and more than fourfold that number through commercial publishers)! But so far no project has been devoted to the systematic study of continued fraction representations for these functions. This handbook is the result of such an endeavour. We emphasise that only 10% of the continued fractions contained in this book, can also be found in the Abramowitz and Stegun project or at the Wolfram website!

The fact that the Belgian and Norwegian authors could collaborate in pairs at their respective home institutes in Antwerp (Belgium) and Trondheim (Norway) offered clear advantages. Nevertheless, most progress with the manuscript was booked during the so-called handbook workshops which were organised at regular intervals, three to four times a year, by the first four authors A. Cuyt, V. B. Petersen, B. Verdonk and H. Waadeland. They got together a staggering 16 times, at different host institutes, for a total of 28 weeks to compose, streamline and discuss the contents of the different chapters.

The Belgian and Norwegian pair were also welcomed for two or more weeks at the MFO (Oberwolfach, Germany), CWI (Amsterdam, The Netherlands), University of La Laguna (Tenerife, Spain), the University of Stellenbosch (South-Africa) and last, but certainly not least, the University of

Antwerp and the Norwegian University of Science and Technology. Without the inspiring environment and marvellous library facilities offered by our supportive colleagues G.-M. Greuel, N. Temme, P. Gonzalez-Vera and J.A.C. Weideman a lot of the work contained in this book would not have been possible. In addition, three meetings were held at hotels, in 2002 in Montelupo Fiorentino (Italy) and in 2003 and 2005 in Røros (Norway). At the occasion of the first two of these meetings W.B. Jones joined his European colleagues. In addition to his input and encouragement, his former student Cathy Bonan-Hamada contributed to the handbook as a principal author of *Chapter* 5 and to a lesser extent in a few chapters on special functions.

Several collaborators at the University of Antwerp have also been extremely helpful. The authors have greatly benefitted from the input of S. Becuwe with respect to several TEX-issues, the spell checking, the proof reading and especially, the generation of the tables and numerical verification of all formulas in the book. For the latter, use was made of a Maple library for continued fractions developed by F. Backeljauw [BC07]. Thanks are due to T. Docx for the help with the graphics, for which software was made available by J. Tupper [BCJ+05]. My daughter A. Van Soom was an invaluable help with the entering and management of almost 4600 BibTEX entries, from which only a selection is printed in the reference list.

Financial support was received from the FWO-Flanders (Fonds voor Wetenschappelijk Onderzoek, Belgium) and its Scientific Research Network *Advanced numerical methods for mathematical modelling*, the Department of Mathematics of the Norwegian University of Science and Technology (Trondheim), the Sør Trondelag University College (Trondheim), the Royal Norwegian Society of Science and Letters, and the National Science Foundation (USA).

Thanks are also due to our patient publisher: after many promises the team finally met its own requirements and turned in the manuscript. We apologise to our dear readers: any mistakes found in the book are ours and we take joint responsibility for them.

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February 2007
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NOTATION

AS	continued fraction also available in [AS64]
	relative truncation error is tabulated
	error is reliably graphed
$\circ$	composition
$\equiv$	equivalent continued fractions
$\not\equiv$	not identically equal to
$\approx$	asymptotic expansion
$\lfloor\cdot\rfloor$	floor function
$\|\|\cdot\|\|$	norm
$\langle\cdot,\cdot\rangle$	inner product
$\{\cdot_n\}$	sequence
$\|\cdot\|_s$	signed modulus
A_n	n^{th} numerator
a_m	m^{th} partial numerator
$(a)_k$	Pochhammer symbol
$(a;q)_k$	generalised Pochhammer symbol
$\operatorname{Arg} z$	argument, $-\pi < \operatorname{Arg} z \leq \pi$
$\arg z$	$\operatorname{Arg} z \pm 2k\pi, k \in \mathbb{N}_0$
(a,b)	open interval $a < x < b$
$[a,b]$	closed interval $a \leq x \leq b$
$B(a,b)$	beta function
$B_q(a,b)$	q-beta function
$B_x(a,b)$	incomplete beta function
B_n	n^{th} denominator
b_m	m^{th} partial denominator
$\mathbb{C}$	set of complex numbers
$\widehat{\mathbb{C}}$	$\mathbb{C} \cup \{\infty\}$
$C(z)$	Fresnel cosine integral
$\operatorname{Ci}(z)$	cosine integral
$C_n^{(\alpha)}(x)$	Gegenbauer (or ultraspherical) polynomial
$\hat{C}_n^{(\alpha)}(x)$	monic Gegenbauer polynomial
cdf	cumulative distribution function
CMP, CSMP, CHMP	classical moment problems
$\Gamma(z)$	gamma function
$\Gamma(a,z)$	complementary incomplete gamma function
$\Gamma_q(z)$	q-gamma function
$\gamma(a,z)$	(lower) incomplete gamma function

$D_\nu(z)$	parabolic cylinder function
∂	degree
$\mathrm{Ei}(z)$	exponential integral
$\mathrm{Ein}(z)$	exponential integral
$E_n(z)$	exponential integral ($n \in \mathbb{N}_0$)
$E_\nu(z)$	exponential integral ($\nu \in \mathbb{C}$)
$E[X]$	expectation value of X
$\mathrm{erf}(z)$	error function
$\mathrm{erfc}(z)$	complementary error function
$\mathbb{F}, \mathbb{F}(\beta, t, L, U)$	set of floating-point numbers
${}_pF_q(\dots, a_p; \dots, b_q; z)$	hypergeometric series
${}_2F_1(a, b; c; z)$	Gauss hypergeometric series
${}_1F_1(a; b; z)$	confluent hypergeometric function
${}_2F_0(a, b; z)$	confluent hypergeometric series
${}_0F_1(; b; z)$	confluent hypergeometric limit function
$\mathtt{F}_n(z; w_n)$	computed approximation of $f_n(z; w_n)$
$f_n, f_n(z)$	n^{th} approximant
$f_n(w_n), f_n(z; w_n)$	n^{th} modified approximant
$f_n^{(M)}$	n^{th} approximant of M^{th} tail
$f^{(n)}, g^{(n)}, \dots$	n^{th} tail
FLS	formal Laurent series
FPS, FTS	formal power series, formal Taylor series
$\Phi(t)$	distribution function
$\phi(t)$	weight function
${}_r\phi_s(\dots, a_r; \dots, b_s; q; z)$	basic hypergeometric series
${}_2\phi_1(q^\alpha, q^\beta; q^\gamma; q; z)$	Heine series
$\varphi_\ell[z_0, \dots, z_\ell]$	inverse difference
$g_\nu^{(1)}(z), g_\nu^{(2)}(z)$	modified spherical Bessel function 3^{rd} kind
$H_n(x)$	Hermite polynomial
$\hat{H}_n(x)$	monic Hermite polynomial
$H_k^{(m)}(c)$	Hankel determinant for the (bi)sequence c
$H_\nu^{(1)}(z), H_\nu^{(2)}(z)$	Hankel function, Bessel function 3^{rd} kind
$h_\nu^{(1)}(z), h_\nu^{(2)}(z)$	spherical Bessel function 3^{rd} kind
$I_k(x)$	repeated integral of the probability integral
$I_x(a, b)$	regularised (incomplete) beta function
$I_\nu(z)$	modified Bessel function 1^{st} kind
$I^k\,\mathrm{erfc}(z)$	repeated integral of $\mathrm{erfc}(z)$ for $k \geq -1$
$i_\nu(z)$	modified spherical Bessel function 1^{st} kind
i	imaginary number $\sqrt{-1}$
$\Im z$	imaginary part of z

$J(z)$	Binet function
$J_\nu(z)$	Bessel function 1^{st} kind
$j_\nu(z)$	spherical Bessel function 1^{st} kind
$K_\nu(z)$	modified Bessel function 2^{nd} kind
$\mathrm{K}\,(a_m/b_m)$	continued fraction
$k_\nu(z)$	modified spherical Bessel function 2^{nd} kind
$\mathrm{Ln}(z)$	principal branch of natural logarithm
$L_n^{(\alpha)}(x)$	generalised Laguerre polynomial
$\hat{L}_n^{(\alpha)}(x)$	monic generalised Laguerre polynomial
$\mathrm{li}(x)$	logarithmic integral
$\lambda(L)$	order of FPS $L(z)$
$\Lambda_0(f) = f_{(0)}(z)$	Laurent expansion in deleted neighbourhood of 0
$\Lambda_\infty(f) = f_{(\infty)}(z)$	Laurent expansion in deleted neighbourhood of ∞
$M(a, b, z)$	Kummer function 1^{st} kind
$M_{\kappa,\mu}(z)$	Whittaker function
μ_k	k^{th} moment
μ_k'	k^{th} central moment
$\mathbb{N}$	$\{1, 2, 3, \dots\}$
$\mathbb{N}_0$	$\{0, 1, 2, 3, \dots\}$
$N(\mu, \sigma^2)$	normal distribution
$[n]_q$	q-analogue of n
$[n]_q!$	q-factorial
$P_n(x)$	Legendre polynomial
$\hat{P}_n(x)$	monic Legendre polynomial
$P_n^{(\alpha,\beta)}(x)$	Jacobi polynomial
$\hat{P}_n^{(\alpha,\beta)}(x)$	monic Jacobi polynomial
pdf	probability density function
$\mathcal{P}_n(L)$	partial sum of degree n of FTS $L(z)$
$\psi_k(z)$	polygamma functions ($k \geq 0$)
$\mathbb{R}$	set of real numbers
$\mathbb{R}[x]$	ring of polynomials with coefficients in $\mathbb{R}$
$R(x)$	Mills ratio
$\Re z$	real part of z
$r_{m,n}(z)$	Padé approximant
$r_{k,\ell}^{(2)}(z)$	two-point Padé approximant
$\rho_\ell[z_0, \dots, z_\ell]$	reciprocal difference
$S(z)$	Fresnel sine integral
$\mathrm{Si}(z)$	sine integral
$S_n(w_n), S_n(z; w_n)$	modified approximant
$s_n(w_n), s_n(z; w_n)$	linear fractional transformation

SSMP, SHMP	strong moment problems
σ	standard deviation
σ^2	variance
$T_n(x)$	Chebyshev polynomial 1^{st} kind
$\hat{T}_n(x)$	monic Chebyshev polynomial 1^{st} kind
TMP	trigonometric moment problem
$U_n(x)$	Chebyshev polynomial 2^{nd} kind
$\hat{U}_n(x)$	monic Chebyshev polynomial 2^{nd} kind
$U(a, b, z)$	Kummer function 2^{nd} kind
`ulp`	unit in the last place
$\overline{V}$	set closure
V_n	value set
$W_{\kappa,\mu}(z)$	Whittaker function
$w_n(z)$	n^{th} modification for $\mathrm{K}_{m=1}^{\infty}(a_m/1)$
$\tilde{w}_n(z)$	n^{th} modification for $\mathrm{K}_{m=1}^{\infty}(a_m/b_m)$
$w_n^{(1)}(z)$	improved n^{th} modification for $\mathrm{K}_{m=1}^{\infty}(a_m/1)$
$\tilde{w}_n^{(1)}(z)$	improved n^{th} modification for $\mathrm{K}_{m=1}^{\infty}(a_m/b_m)$
$Y_\nu(z)$	Bessel function 2^{nd} kind
$y_\nu(z)$	spherical Bessel function 2^{nd} kind
$\overline{z}$	complex conjugate of z
$\mathbb{Z}$	$\{\dots, -2, -1, 0, 1, 2, \dots\}$
$\mathbb{Z}^-$	$\{-1, -2, -3, \dots\}$
$\mathbb{Z}_0^-$	$\{0, -1, -2, -3, \dots\}$
$\zeta(z)$	Riemann zeta function

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General considerations

The purpose of this chapter is to explain the general organisation of the book, despite the fact that we hope the handbook is accessible to an unprepared reader. For the customary mathematical notations used throughout the book we refer to the list of notations following the preface.

To scientists novice in the subject of continued fractions we recommend the following order of reading in *Part I* and *Part II*:

- first the *Chapters* 1 through 3 on the fundamental theory of continued fractions,
- then *Chapter* 6, with excursions to *Chapter* 4, on algorithms to construct continued fraction representations,
- and finally the *Chapters* 7 and 8, with *Chapter* 5 as background material, for truncation and round-off error bounds.

0.1 Part one

Part I comprises the necessary theoretic background about continued fractions, when used as a tool to approximate functions. Its concepts and theorems are heavily used later on in the handbook. We deal with three term recurrence relations, linear fractional transformations, equivalence transformations, limit periodicity, continued fraction tails and minimal solutions. The connection between continued fractions and series is worked out in detail, especially the correspondence with formal power series at 0 and ∞.

The continued fraction representations of functions are grouped into several families, the main ones being the S-fractions, C-fractions, P-fractions, J-fractions, T-fractions, M-fractions and Thiele interpolating continued fractions. Most classical convergence results are given, formulated in terms of element and value regions. The connection between C- and P-fractions and Padé approximants on the one hand, and between M-fractions and two-point Padé approximants on the other hand is discussed. To conclude, several moment problems, their link with Stieltjes integral transform representations and the concept of orthogonality are presented.

0.2 Part two

In *Part II* the reader is offered algorithms to construct different continued fraction representations of functions, known either by one or more formal series representations or by a set of function values. The qd-algorithm constructs C-fractions, the $\alpha\beta$- and FG-algorithms respectively deliver J- and T-fraction representations, and inverse or reciprocal differences serve to construct Thiele interpolating fractions. Also Thiele continued fraction expansions can be obtained as a limiting form.
When evaluating a continued fraction representation, only a finite part of the fraction can be taken into account. Several algorithms exist to compute continued fraction approximants. Each of them can make use of an estimate of the continued fraction tail to improve the convergence. A priori and a posteriori truncation error bounds are developed and accurate round-off error bounds are given.

0.3 Part three

The families of special functions discussed in the separate chapters in *Part III* are the bulk of the handbook and its main goal. We present series and continued fraction representations for several mathematical constants, the elementary functions, functions related to the gamma function, the error function, the exponential integrals, the Bessel functions and also several probability functions. All can be formulated in terms of either hypergeometric or confluent hypergeometric functions. We conclude with a brief discussion of the q-hypergeometric function and its continued fraction representations.
Each chapter in *Part III* is more or less structured in the same way, depending on the availability of the material. We now discuss the general organisation of such a chapter and the conventions adopted in the presentation of the formulas.
All tables and graphs in *Part III* are labelled and preceded by an extensive caption. Detailed information on their use and interpretation is given in the *Sections* 9.2 and 9.3, respectively.

Definitions and elementary properties. The nomenclature of the special functions is not unique. In the first section of each chapter the reader is presented with the different names attached to a single function. The variable z is consistently used to denote a complex argument and x for a real argument.

In a function definition the sign := is used to indicate that the left hand side denotes the function expression at the right hand side, on the domain given in the equation:

$$J(z) := \mathrm{Ln}(\Gamma(z)) - \left(z - \frac{1}{2}\right)\mathrm{Ln}(z) + z - \ln(\sqrt{2\pi}).$$

Here the principal branch of a multivalued complex function is indicated with a capital letter, as in Ln, while the real-valued and multivalued function are indicated with lower case letters, as in ln. The function definition is complemented with symmetry properties, such as mirror, reflection or translation formulas:

$$\mathrm{Ln}(\bar{z}) = \overline{\mathrm{Ln}(z)}.$$

Recurrence relations. Continued fractions are closely related to three-term recurrence relations, also called second order linear difference equations. Hence these are almost omnipresent, as in:

$$\begin{aligned} A_{-1} &:= 1, \qquad A_0 := 0, \\ A_n &:= a_n A_{n-1} + b_n A_{n-2}, \qquad n = 1, 2, 3, \ldots \end{aligned}$$

or

$$\begin{aligned} {}_2F_1(a, b; c+1; z) = &-\frac{c(c-1-(2c-a-b-1)z)}{(c-a)(c-b)z}\,{}_2F_1(a, b; c; z) \\ &- \frac{c(c-1)(z-1)}{(c-a)(c-b)z}\,{}_2F_1(a, b; c-1; z). \end{aligned}$$

The recurrence relations immediately connected to continued fraction theory are listed. Other recurrences may be found in the literature, but may not serve our purpose.

Series expansion. Representations as infinite series are given with the associated domain of convergence. Often these series are power series as in (2.2.2) or (2.2.6). The series in the right hand side and the function in the left hand side coincide, denoted by the equality sign =, on the domain given in the right hand side:

$$\tan(z) = \sum_{k=1}^{\infty} \frac{4^k(4^k-1)|B_{2k}|}{(2k)!} z^{2k-1}, \qquad |z| < \pi/2.$$

Asymptotic series expansion. Asymptotic expansions of the form (2.2.4) or (2.2.8) are given, if available, with the set of arguments where they are valid. Now the equation sign is replaced by the sign $\approx$:

$$J(z) \approx z^{-1} \sum_{k=0}^{\infty} \frac{B_{2k+2}}{(2k+1)(2k+2)} z^{-2k}, \qquad z \to \infty, \quad |\arg z| < \frac{\pi}{2}.$$

Stieltjes transform. For functions that can be represented as Stieltjes integral transforms, or equivalently as convergent S-fractions, positive T-fractions or real J-fractions, specific sharp truncation error estimates exist and the relative round-off error exhibits a slow growth rate when evaluating the continued fraction representation of the function by means of the backward algorithm.
Hence, if possible, the function under consideration or a closely related function is written as a Stieltjes integral transform:

$$\frac{\Gamma(a,z)}{z^a e^{-z}} = \frac{1}{\Gamma(1-a)} \int_0^{\infty} \frac{e^{-t} t^{-a}}{z+t}\, dt, \qquad |\arg z| < \pi, \quad -\infty < a < 1.$$

The conditions on the right hand side of the integral representation, here $|\arg z| < \pi$, $-\infty < a < 1$, are inherited from the function definition.

S-fraction, regular C-fraction and Padé approximants. S-fraction representations are usually found from the solution of the classical Stieltjes moment problem:

$$e^z E_n(z) = \frac{1/z}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{a_m/z}{1} \right), \qquad a_{2k} = n + k - 1, \quad a_{2k+1} = k,$$
$$|\arg z| < \pi, \quad n \in \mathbb{N}.$$

The equality sign = between the left and right hand sides here has to be interpreted in the following way. The convergence of the continued fraction in the right hand side is uniform on compact subsets of the given convergence domain, here $|\arg z| < \pi$, excluding the poles of the function in the left hand side. When the convergence domain of the continued fraction in the right hand side is larger than the domain of the function in the left hand side, it may be regarded as an analytic continuation of that function.
C-fractions can be obtained for instance, by dropping some conditions that ensure the positivity of the coefficients a_m:

$$e^z E_\nu(z) = \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m(\nu) z^{-1}}{1} \right), \qquad |\arg z| < \pi, \quad \nu \in \mathbb{C},$$
$$a_1(\nu) = 1, \quad a_{2j}(\nu) = j + \nu - 1, \quad a_{2j+1}(\nu) = j, \quad j \in \mathbb{N}.$$

A C-fraction is intimately connected with Padé approximants, since its successive approximants equal Padé approximants on a staircase in the Padé table. When available, explicit formulas for the Padé approximants in part or all of the table are given. With the operator $\mathcal{P}_k$ defined as in (15.4.1),

$$r_{m+1,n}(z) = \frac{z^{-1}\mathcal{P}_{m+n}\left({}_2F_0(\nu,1;-z^{-1})\ {}_2F_0(-\nu-m,-n;z^{-1})\right)}{{}_2F_0(-\nu-m,-n;z^{-1})}, \qquad m+1 \geq n.$$

T-fraction, M-fraction and two-point Padé approximants. M-fractions correspond simultaneously to series expansions at 0 and at ∞. For instance, the fraction in the right hand side of

$$\frac{{}_1F_1(a+1;b+1;z)}{{}_1F_1(a;b;z)} = \frac{b}{b-z} + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{(a+m)z}{b+m-z}\right), \qquad z \in \mathbb{C}, \quad a \in \mathbb{C}, \quad b \in \mathbb{C}\setminus\mathbb{Z}_0^-$$

corresponds at 0 to the series representation of the function in the left hand side and corresponds at $z=\infty$ to the series representation of

$$-\frac{b}{z}\frac{{}_2F_0(a+1,a-b+1;-1/z)}{{}_2F_0(a,a-b+1;-1/z)}.$$

The two-point Padé approximants $r^{(2)}_{n+k,n-k}(z)$ corresponding to the same series at $z=0$ and at $z=\infty$, are given by

$$r^{(2)}_{n+k,n-k}(z) = \frac{P_{n-1,k}(\infty,a+1,b,z)}{P_{n,k}(\infty,a,b,z)}, \qquad 0 \leq k \leq n,$$

where

$$\begin{aligned} P_{n,k}(\infty,b,c,z) &:= \lim_{a\to\infty} P_{n,k}(a,b,c,z/a), \qquad 0 \leq k \leq n, \\ &= \mathcal{P}_n({}_1F_1(b;c;z)\ {}_1F_1(-b-n;1-c-k-n;-z)), \end{aligned}$$

for $P_{n,k}(a,b,c,z)$ given by (15.4.9) and the operator $\mathcal{P}_n$ defined in (15.4.1).

Real J-fraction and other continued fractions. Contractions of some continued fractions may result in J-fraction representations. Or minimal solutions of some recurrence relation may lead to yet another continued fraction representation. If closed formulas exist for the partial numerators and denominators of these fractions, these are listed after the usual families of S-, C- and T- or M-fractions. In general, we do not list different equivalent forms of a continued fraction.

Significant digits. Traditionally, the goal in designing mathematical approximations for use in function evaluations or implementations is to minimise the computation time. Our emphasis is on accuracy instead of speed. Therefore our numerical and graphical illustrations essentially focus on the presentation of the number of significant digits achieved by the series and continued fraction approximants. All output is reliable and correctly rounded.

By the presentation of tables and graphs for different approximants, also the speed of convergence in different regions of the complex plane is illustrated. More information on the tables and graphs in this handbook can be found in *Chapter* 9.

Reliability. All series and continued fraction representations in the handbook were verified numerically. So when encountering a slightly different formula from the one given in the original reference, it was corrected because the original work most probably contained a typo.

Further reading

- Similar formula books for different families of functions are [AS64; Ext78; SO87; GR00].
- Books discussing some of the special functions treated in this work are [Luk75; Luk69; AAR99].

Part I

BASIC THEORY

1

Basics

We develop some basic tools to handle continued fractions with complex numbers as elements. These include recurrence relations, equivalence transformations, the Euler connection with series, and a study of the tail behaviour of continued fractions which is quite different from that of series. Starting *Section* 1.10 we also deal with continued fractions in which the elements depend on a complex variable z. The representation of functions is further developed from *Chapter* 2 on.

1.1 Symbols and notation

The expression

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} \tag{1.1.1a}$$

is called a continued fraction, where a_m and b_m are complex numbers and $a_m \neq 0$ for all m. More recently, for convenience, other symbols are used to denote the same continued fraction. These include the following:

$$b_0 + \frac{a_1 |}{| b_1} + \frac{a_2 |}{| b_2} + \frac{a_3 |}{| b_3} + \ldots, \tag{1.1.1b}$$

$$b_0 + \frac{a_1}{b_1} \genfrac{}{}{0pt}{}{}{+} \frac{a_2}{b_2} \genfrac{}{}{0pt}{}{}{+} \frac{a_3}{b_3} \genfrac{}{}{0pt}{}{}{+} \ldots \tag{1.1.1c}$$

and

$$b_0 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m}{b_m} \right), \tag{1.1.1d}$$

or for short

$$b_0 + \mathrm{K} \left(\frac{a_m}{b_m} \right). \tag{1.1.1e}$$

The symbol K in (1.1.1d) and (1.1.1e) for (infinite) fraction, from the German word Kettenbruch, is the analogue of Σ for (infinite) sum.
Correspondingly the n^{th} approximant f_n of the continued fraction is expressed by

$$f_n = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots + \cfrac{a_n}{b_n}}}}, \tag{1.1.2a}$$

$$f_n = b_0 + \frac{a_1 |}{| b_1} + \frac{a_2 |}{| b_2} + \frac{a_3 |}{| b_3} + \cdots + \frac{a_n |}{| b_n}, \tag{1.1.2b}$$

$$f_n = b_0 + \frac{a_1}{b_1 +}\,\frac{a_2}{b_2 +}\,\frac{a_3}{b_3 +} \cdots \frac{a_n}{+ b_n} \tag{1.1.2c}$$

and

$$f_n = b_0 + \operatorname*{K}_{m=1}^{n} \left(\frac{a_m}{b_m} \right) . \tag{1.1.2d}$$

Only the symbols (1.1.1c), (1.1.1d), (1.1.1e) and (1.1.2c), (1.1.2d) are used in the present book.
The continued fraction (1.1.1) is more than just the sequence of approximants $\{f_n\}$ or the limit of this sequence, if it exists. In fact, the continued fraction is the mapping of the ordered pair of sequences $(\{a_m\}, \{b_m\})$ onto the sequence $\{f_n\}$. This concept is made more precise in the definition of continued fraction in the following section.

1.2 Definitions

The complex plane is denoted by $\mathbb{C}$ and the extended complex plane by

$$\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} .$$

The symbols $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets

$$\mathbb{N} := \{1, 2, 3, \ldots\}, \qquad \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}.$$

Continued fraction. An ordered pair of sequences $(\{a_m\}_{m\in\mathbb{N}}, \{b_m\}_{m\in\mathbb{N}_0})$ of complex numbers, with $a_m \neq 0$ for $m \geq 1$, gives rise to sequences $\{s_n(w)\}_{n\in\mathbb{N}_0}$ and $\{S_n(w)\}_{n\in\mathbb{N}_0}$ of *linear fractional transformations*

$$s_0(w) := b_0 + w, \quad s_n(w) := \frac{a_n}{b_n + w}, \quad n = 1, 2, 3, \ldots, \tag{1.2.1a}$$

$$S_0(w) := s_0(w), \quad S_n(w) := S_{n-1}(s_n(w)), \quad n = 1, 2, 3, \dots \tag{1.2.1b}$$

and to a sequence $\{f_n\}$, given by

$$f_n = S_n(0) \in \widehat{\mathbb{C}}, \qquad n = 0, 1, 2, \dots . \tag{1.2.2}$$

The ordered pair [Hen77, p. 474]

$$((\{a_m\}, \{b_m\}), \{f_n\}) \tag{1.2.3}$$

is the *continued fraction* denoted by the five symbols in (1.1.1). The numbers a_m and b_m are called m^{th} *partial numerator* and *partial denominator*, respectively, of the continued fraction. The value f_n is called the n^{th} *approximant* and is denoted by the four symbols (1.1.2). Some authors use the term *convergent* where we use approximant. A common name for partial numerator and denominator is *element*.
The linear fractional transformation $S_n(w)$ can be expressed as

$$S_n(w) = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots + \cfrac{a_n}{b_n + w}}}}, \tag{1.2.4a}$$

or more conveniently as

$$S_n(w) = b_0 + \frac{a_1}{b_1} \,+\, \frac{a_2}{b_2} \,+\, \frac{a_3}{b_3} \,+\, \cdots \,+\, \frac{a_{n-1}}{b_{n-1}} \,+\, \frac{a_n}{b_n + w} \, . \tag{1.2.4b}$$

Equivalently,

$$S_n(w) = s_0 \circ s_1 \circ s_2 \circ \cdots \circ s_n(w), \tag{1.2.5}$$

where $\circ$ denotes *composition* such as in

$$s_0 \circ s_1(w) := s_0(s_1(w)) \, .$$

In particular,

$$s^n(w) := \underbrace{s \circ \cdots \circ s}_{n \text{ times}}(w) \, .$$

For a given sequence $\{w_n\}_{n\in\mathbb{N}_0}$, the number

$$S_n(w_n) \in \widehat{\mathbb{C}} \tag{1.2.6}$$

is called an n^{th} *modified approximant.*

Convergence. A continued fraction $b_0 + \mathrm{K}\,(a_m/b_m)$ is said to *converge* if and only if the sequence of approximants $\{f_n\} = \{S_n(0)\}$ converges to a limit $f \in \widehat{\mathbb{C}}$. In this case f is called the *value* of the continued fraction. Note that convergence to ∞ is accepted. If the continued fraction is convergent to f, then the symbols (1.1.1) are used to represent both the ordered pair (1.2.3) and the value f. That is, we may write

$$f = \lim_{n\to\infty} S_n(0) = b_0 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m}{b_m}\right) . \tag{1.2.7}$$

Sometimes (1.2.7) is called *classical convergence.*

General convergence. A continued fraction *converges generally* [Jac86; LW92, p. 43] to an extended complex number f if and only if there exist two sequences $\{v_n\}$ and $\{w_n\}$ in $\widehat{\mathbb{C}}$ such that

$$\liminf_{n\to\infty} d(v_n, w_n) > 0$$

and

$$\lim_{n\to\infty} S_n(v_n) = \lim_{n\to\infty} S_n(w_n) = f.$$

Here $d(z, w)$ denotes the *chordal metric* defined by

$$d(z, w) := \frac{|z - w|}{\sqrt{1 + |z|^2}\sqrt{1 + |w|^2}}, \qquad z, w \in \mathbb{C}$$

and

$$d(\infty, w) := \frac{1}{\sqrt{1 + |w|^2}}, \qquad w \in \mathbb{C}\,.$$

The value f is unique. Convergence to f implies general convergence to f since

$$S_n(\infty) = S_{n-1}(0)$$

but general convergence does not imply convergence.

Example 1.2.1: The continued fraction

$$\frac{2}{1} + \frac{1}{1} + \frac{-1}{1} + \frac{2}{1} + \frac{1}{1} + \frac{-1}{1} + \ldots$$

diverges. By using the recurrence relations (1.3.1), we find for $n \geq 1$ that

$$\begin{aligned} A_{3n-2} &= 2^n, & A_{3n-1} &= 2^n, & A_{3n} &= 0, \\ B_{3n-2} &= 2^{n+1} - 3, & B_{3n-1} &= 2^{n+1} - 2, & B_{3n} &= 1. \end{aligned}$$

For the modified approximants $S_n(w_n)$ we find from (1.3.2) that

$$S_{3n-2}(w_{3n-2}) = \frac{2^n + w_{3n-2} \cdot 0}{(2^{n+1} - 3) + w_{3n-2} \cdot 1},$$

which converges to 1/2 if the sequence $\{w_{3n-2}\}$ is bounded. Similarly, we find that the sequence

$$S_{3n-1}(w_{3n-1}) = \frac{2^n + w_{3n-1} \cdot 2^n}{(2^{n+1} - 2) + w_{3n-1}(2^{n+1} - 3)}$$

converges to 1/2 if the sequence $\{w_{3n-1}\}$ is bounded away from -1 and the sequence

$$S_{3n}(w_{3n}) = \frac{0 + w_{3n} \cdot 2^n}{1 + w_{3n}(2^{n+1} - 2)}$$

converges to 1/2 if the sequence $\{w_{3n}\}$ is bounded away from 0. Hence we have that the continued fraction converges generally.

1.3 Recurrence relations

The n^{th} *numerator* A_n and the n^{th} *denominator* B_n of a continued fraction $b_0 + \mathrm{K}\,(a_m/b_m)$ are defined by the *recurrence relations* (second order linear difference equations)

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} := b_n \begin{bmatrix} A_{n-1} \\ B_{n-1} \end{bmatrix} + a_n \begin{bmatrix} A_{n-2} \\ B_{n-2} \end{bmatrix}, \qquad n = 1, 2, 3, \ldots \tag{1.3.1a}$$

with initial conditions

$$A_{-1} := 1, \quad B_{-1} := 0, \quad A_0 := b_0, \quad B_0 := 1\,. \tag{1.3.1b}$$

The modified approximant $S_n(w_n)$ in (1.2.6) can then be written as

$$S_n(w_n) = \frac{A_n + A_{n-1}w_n}{B_n + B_{n-1}w_n}, \qquad n = 0, 1, 2, \ldots \tag{1.3.2}$$

and hence for the n^{th} approximant f_n we have

$$f_n = S_n(0) = \frac{A_n}{B_n}, \qquad f_{n-1} = S_n(\infty) = \frac{A_{n-1}}{B_{n-1}}\,. \tag{1.3.3}$$

Determinant formula. The n^{th} numerator and denominator satisfy the *determinant formula*

$$\begin{aligned} \begin{vmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{vmatrix} &= A_n B_{n-1} - A_{n-1} B_n \\ &= (-1)^{n-1} \prod_{m=1}^{n} a_m, \qquad n = 1, 2, 3, \ldots . \end{aligned} \tag{1.3.4}$$

Matrix connection with continued fractions. Let $\mathrm{K}\,(a_m/b_m)$ be a given continued fraction with n^{th} numerator A_n and n^{th} denominator B_n. Let

$$s_m(w) := \frac{a_m}{b_m + w}, \qquad x_m := \begin{pmatrix} 0 & a_m \\ 1 & b_m \end{pmatrix}, \qquad m = 1, 2, 3, \ldots .$$

Then the linear fractional transformation $S_n(w)$ given by (1.2.5) and (1.3.2) leads to

$$X_n := x_1 x_2 x_3 \cdots x_n = \begin{pmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{pmatrix}, \qquad n = 1, 2, 3, \ldots .$$

Therefore multiplication of 2×2 matrices can be used to construct the sequences $\{A_n\}, \{B_n\}$ and $\{f_n\}$, where f_n is given by (1.2.2) and (1.3.3). More generally, if

$$t_m(w) := \frac{a_m + c_m w}{b_m + d_m w}, \qquad y_m := \begin{pmatrix} c_m & a_m \\ d_m & b_m \end{pmatrix}, \qquad m = 1, 2, 3, \ldots$$

then

$$T_n(w) := t_1 \circ t_2 \circ t_3 \circ \cdots \circ t_n(w) = \frac{A_n + C_n w}{B_n + D_n w}, \qquad n = 1, 2, 3, \ldots$$

and

$$Y_n := y_1 y_2 y_3 \cdots y_n = \begin{pmatrix} C_n & A_n \\ D_n & B_n \end{pmatrix}, \qquad n = 1, 2, 3, \ldots .$$

1.4 Equivalence transformations

Two continued fractions $b_0 + \mathrm{K}\,(a_m/b_m)$ and $d_0 + \mathrm{K}\,(c_m/d_m)$ are said to be *equivalent* if and only if they have the same sequence of approximants. This is written

$$b_0 + \mathop{\mathrm{K}}_{m=1}^{\infty} (a_m/b_m) \equiv d_0 + \mathop{\mathrm{K}}_{m=1}^{\infty} (c_m/d_m). \tag{1.4.1}$$

The equivalence (1.4.1) holds if and only if there exists a sequence of complex numbers $\{r_m\}$, with $r_0 = 1$ and $r_m \neq 0$ for $m \geq 1$, such that

$$d_0 = b_0, \quad c_m = r_m r_{m-1} a_m, \quad d_m = r_m b_m, \quad m = 1, 2, 3, \ldots. \tag{1.4.2}$$

Equations (1.4.2) define an *equivalence transformation*. Since $a_m \neq 0$ for $m \geq 1$, one can always choose

$$r_m = \prod_{k=1}^{m} a_k^{(-1)^{m+1-k}} = \left(\frac{\prod\limits_{k=1}^{\lfloor m/2 \rfloor} a_{2k}}{\prod\limits_{k=1}^{\lfloor (m+1)/2 \rfloor} a_{2k-1}} \right)^{(-1)^{m-1}}, \qquad m = 1, 2, 3, \ldots,$$

which yields the equivalence transformation

$$\begin{aligned} b_0 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m}{b_m} \right) &\equiv b_0 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{1}{d_m} \right) \\ &= b_0 + \frac{1}{b_1/a_1} {+} \frac{1}{b_2 a_1/a_2} {+} \frac{1}{b_3 a_2/(a_1 a_3)} {+} \ldots, \end{aligned}$$

where in general

$$\begin{aligned} d_1 &= \frac{b_1}{a_1}, \\ d_{2m} &= b_{2m} \frac{a_1 a_3 \cdots a_{2m-1}}{a_2 a_4 \cdots a_{2m}}, \qquad m = 1, 2, 3, \ldots, \\ d_{2m+1} &= b_{2m+1} \frac{a_2 a_4 \cdots a_{2m}}{a_1 a_3 \cdots a_{2m+1}}, \qquad m = 1, 2, 3, \ldots. \end{aligned}$$

Hence, in studying continued fractions there is no loss of generality in the restriction to continued fractions $\mathrm{K}\,(1/d_m)$. On the other hand, if

$$b_m \neq 0, \qquad m = 1, 2, 3, \ldots,$$

then one can obtain an equivalence transformation of the form

$$b_0 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m}{b_m}\right) \equiv b_0 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{c_m}{1}\right)$$
$$= b_0 + \frac{a_1/b_1}{1} \underset{+}{} \frac{a_2/(b_1 b_2)}{1} \underset{+}{} \frac{a_3/(b_2 b_3)}{1} \underset{+}{} \ldots,$$

where in general

$$r_m = \frac{1}{b_m}, \qquad m = 1, 2, 3, \ldots,$$

$$c_1 = \frac{a_1}{b_1}, \qquad c_m = \frac{a_m}{b_{m-1} b_m}, \qquad m = 2, 3, 4, \ldots.$$

Hence, in studying continued fractions there is only little loss of generality in the restriction to continued fractions $\mathrm{K}\,(c_m/1)$.

1.5 Contractions and extensions

In this section we let A_n, B_n and f_n denote the n^{th} numerator, denominator and approximant, respectively of a continued fraction $b_0 + \mathrm{K}\,(a_m/b_m)$ and we let C_n, D_n and g_n denote the n^{th} numerator, denominator and approximant, respectively, of a continued fraction $d_0 + \mathrm{K}\,(c_m/d_m)$. Then $d_0 + \mathrm{K}\,(c_m/d_m)$ is called a *contraction* of $b_0 + \mathrm{K}\,(a_m/b_m)$ if and only if there exists a sequence $\{n_k\}$ such that

$$g_k = f_{n_k}, \qquad k = 0, 1, 2, \ldots. \tag{1.5.1}$$

The continued fraction $b_0 + \mathrm{K}\,(a_m/b_m)$ is then called an *extension* of $d_0 + \mathrm{K}\,(c_m/d_m)$.

Canonical contraction. If in addition to (1.5.1) there exists a sequence $\{n_k\}$ such that

$$C_k = A_{n_k}, \qquad D_k = B_{n_k}, \qquad k = 0, 1, 2, \ldots, \tag{1.5.2}$$

then $d_0 + \mathrm{K}\,(c_m/d_m)$ is called a *canonical contraction* of $b_0 + \mathrm{K}\,(a_m/b_m)$.

Even contraction. A continued fraction $d_0 + \mathrm{K}\,(c_m/d_m)$ is called an *even contraction* or *even part* of $b_0 + \mathrm{K}\,(a_m/b_m)$ if and only if

$$g_n = f_{2n}, \qquad n = 0, 1, 2, \ldots$$

and it is called the *even canonical contraction* of $b_0 + \mathrm{K}\,(a_m/b_m)$ if and only if

$$C_n = A_{2n}, \qquad D_n = B_{2n}, \qquad n = 0, 1, 2, \ldots.$$

An even canonical contraction of $b_0 + \mathrm{K}\,(a_m/b_m)$ exists if and only if

$$b_{2k} \neq 0, \qquad k = 1, 2, 3, \ldots .$$

When it exists, the even canonical contraction of $b_0 + \mathrm{K}\,(a_m/b_m)$ is given by

$$d_0 + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{c_m}{d_m}\right) = b_0 + \frac{a_1 b_2}{a_2 + b_1 b_2} - \frac{a_2 a_3 b_4/b_2}{a_4 + b_3 b_4 + a_3 b_4/b_2} - \frac{a_4 a_5 b_6/b_4}{a_6 + b_5 b_6 + a_5 b_6/b_4} - \ldots \tag{1.5.3a}$$

where

$$\begin{aligned} &d_0 = b_0, \quad c_1 = a_1 b_2, \quad d_1 = a_2 + b_1 b_2, \\ &c_m = -\frac{a_{2m-2} a_{2m-1} b_{2m}}{b_{2m-2}}, \qquad m = 2, 3, 4, \ldots, \\ &d_m = a_{2m} + b_{2m-1} b_{2m} + \frac{a_{2m-1} b_{2m}}{b_{2m-2}}, \qquad m = 2, 3, 4, \ldots . \end{aligned} \tag{1.5.3b}$$

Odd contraction. A continued fraction $d_0 + \mathrm{K}\,(c_m/d_m)$ is called an *odd contraction* or *odd part* of $b_0 + \mathrm{K}\,(a_m/b_m)$ if and only if

$$g_n = f_{2n+1}, \qquad n = 0, 1, 2, \ldots$$

and it is called an *odd canonical contraction* if and only if

$$\begin{aligned} &C_0 = \frac{A_1}{B_1}, \qquad D_0 = 1, \\ &C_n = A_{2n+1}, \qquad D_n = B_{2n+1}, \qquad n = 1, 2, 3, \ldots . \end{aligned}$$

An odd canonical contraction of $b_0 + \mathrm{K}\,(a_m/b_m)$ exists if and only if

$$b_{2k+1} \neq 0, \qquad k = 0, 1, 2, \ldots .$$

If it exists, an odd canonical contraction of $b_0 + \mathrm{K}\,(a_m/b_m)$ is given by

$$d_0 + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{c_m}{d_m}\right) = \frac{a_1 + b_0 b_1}{b_1} - \frac{a_1 a_2 b_3/b_1}{b_1(a_3 + b_2 b_3) + a_2 b_3} - \frac{a_3 a_4 b_1 b_5/b_3}{a_5 + b_4 b_5 + a_4 b_5/b_3} - \frac{a_5 a_6 b_7/b_5}{a_7 + b_6 b_7 + a_6 b_7/b_5} - \ldots \tag{1.5.4a}$$

where

$$\begin{aligned}
c_1 &= -\frac{a_1 a_2 b_3}{b_1}, \qquad c_2 = -\frac{a_3 a_4 b_1 b_5}{b_3},\\
d_0 &= \frac{a_1 + b_0 b_1}{b_1}, \qquad d_1 = b_1(a_3 + b_2 b_3) + a_2 b_3,\\
c_m &= -\frac{a_{2m-1} a_{2m} b_{2m+1}}{b_{2m-1}}, \qquad m = 3, 4, 5, \ldots,\\
d_m &= a_{2m+1} + b_{2m} b_{2m+1} + a_{2m} b_{2m+1}/b_{2m-1}, \qquad m = 2, 3, 4, \ldots.
\end{aligned} \tag{1.5.4b}$$

1.6 Continued fractions with prescribed approximants

A sequence $\{f_n\}$ in $\widehat{\mathbb{C}}$ can be the sequence of approximants of a continued fraction if and only if

$$f_0 \neq \infty, \qquad f_n \neq f_{n-1}, \qquad n = 1, 2, 3, \ldots. \tag{1.6.1}$$

A sequence $\{f_n\}$ in $\widehat{\mathbb{C}}$ can be the sequence of approximants of a continued fraction of the form $b_0 + \mathrm{K}\,(a_m/1)$ if and only if

$$f_0 \neq \infty, \qquad f_n \neq f_{n-1}, \qquad f_{n+1} \neq f_{n-1}, \qquad n = 1, 2, 3, \ldots. \tag{1.6.2}$$

Let $\{A_n\}$ and $\{B_n\}$ be given sequences in $\mathbb{C}$. Then there exists a continued fraction $b_0 + \mathrm{K}\,(a_m/b_m)$ with n^{th} numerator A_n and n^{th} denominator B_n, for $n \geq 0$, if and only if

$$B_0 = 1, \qquad A_n B_{n-1} - A_{n-1} B_n \neq 0, \qquad n = 1, 2, 3, \ldots. \tag{1.6.3}$$

If (1.6.3) holds then the elements a_m and b_m of $b_0 + \mathrm{K}\,(a_m/b_m)$ are given by

$$b_0 = A_0, \qquad a_1 = A_1 - A_0 B_1, \qquad b_1 = B_1, \tag{1.6.4a}$$

$$a_m = \frac{A_{m-1} B_m - A_m B_{m-1}}{A_{m-1} B_{m-2} - A_{m-2} B_{m-1}}, \qquad m = 2, 3, 4, \ldots, \tag{1.6.4b}$$

$$b_m = \frac{A_m B_{m-2} - A_{m-2} B_m}{A_{m-1} B_{m-2} - A_{m-2} B_{m-1}}, \qquad m = 2, 3, 4, \ldots. \tag{1.6.4c}$$

1.7 Connection between continued fractions and series

The Euler connection. Let $\{c_k\}$ be a sequence in $\mathbb{C}\backslash\{0\}$ and

$$f_n = \sum_{k=0}^{n} c_k, \qquad n = 0, 1, 2, \ldots. \tag{1.7.1}$$

Since $f_n \neq f_{n-1}$ for $n \geq 1$, it follows from (1.6.1) that there exists a continued fraction $b_0 + \mathrm{K}\,(a_m/b_m)$ with n^{th} approximant f_n for all n. Applying (1.6.4), we find that this continued fraction is given by

$$c_0 + \frac{c_1}{1} \, + \, \frac{-c_2/c_1}{1 + c_2/c_1} \, + \cdots + \, \frac{-c_m/c_{m-1}}{1 + c_m/c_{m-1}} \, + \ldots \,. \tag{1.7.2}$$

EXAMPLE 1.7.1: For $c_k = (-1)^k/(k+1)$, $k \geq 0$, we have that the n^{th} partial sum of

$$\ln(2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \tag{1.7.3}$$

and the n^{th} approximant of

$$\ln(2) = 1 + \frac{-\frac{1}{2}}{1} \, + \, \frac{\frac{2}{3}}{1 - \frac{2}{3}} \, + \, \frac{\frac{3}{4}}{1 - \frac{3}{4}} \, + \cdots + \, \frac{\frac{m}{m+1}}{1 - \frac{m}{m+1}} \, + \ldots \tag{1.7.4}$$

are equal.

Conversely, suppose that $b_0 + \mathrm{K}\,(a_m/b_m)$ is a given continued fraction with finite approximants. Let the sequence $\{c_k\}$ be defined by

$$c_0 := b_0, \qquad c_k := \frac{(-1)^{k-1} \prod_{j=1}^{k} a_j}{B_k B_{k-1}}, \qquad k = 1, 2, 3, \ldots, \tag{1.7.5}$$

where B_k denotes the k^{th} denominator of the continued fraction. Then the n^{th} approximant f_n of $b_0 + \mathrm{K}\,(a_m/b_m)$ satisfies (1.7.1) [Eul48]. The connection between continued fractions and series described above is only of limited interest, since in this situation both have exactly the same approximants and hence the same convergence or divergence behaviour.

The method of Viskovatov. Let $\{c_{0k}\}$ and $\{c_{1k}\}$ be sequences in $\mathbb{C}$ and consider the quotient

$$\frac{c_{10}+c_{11}+c_{12}+\dots}{c_{00}+c_{01}+c_{02}+\dots}\,. \tag{1.7.6}$$

This can be rewritten as

$$\begin{aligned}\frac{c_{10}+c_{11}+c_{12}+\dots}{c_{00}+c_{01}+c_{02}+\dots} &= \cfrac{1}{\frac{c_{00}}{c_{10}}+\frac{c_{00}+c_{01}+c_{02}+\dots}{c_{10}+c_{11}+c_{12}+\dots}-\frac{c_{00}}{c_{10}}}\\ &= \cfrac{c_{10}}{c_{00}+\cfrac{(c_{10}c_{01}-c_{00}c_{11})+(c_{10}c_{02}-c_{00}c_{12})+\dots}{c_{10}+c_{11}+c_{12}+\dots}}\\ &= \cfrac{c_{10}}{c_{00}+\cfrac{c_{20}+c_{21}+c_{22}+\dots}{c_{10}+c_{11}+c_{12}+\dots}}\end{aligned}$$

where $c_{2i}=c_{10}c_{0,i+1}-c_{00}c_{1,i+1}$ for $i\geq 0$. If we repeat this process and let

$$c_{kj}=c_{k-1,0}c_{k-2,j+1}-c_{k-2,0}c_{k-1,j+1}, \qquad k\geq 2, \quad j\geq 0, \tag{1.7.7}$$

we obtain the continued fraction [Vis06]

$$\mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{c_{m0}}{c_{m-1,0}}\right). \tag{1.7.8}$$

If in (1.7.6) we consider the special case $c_{00}=1$, $c_{0k}=0$ for $k\geq 1$, then the method of Viskovatov is a means of connecting continued fractions and series. The difference with the Euler connection is that, in general, the n^{th} approximant of (1.7.8) is not equal to

$$f_n=\sum_{k=0}^{n}c_{1k}\,.$$

As is indicated in more detail in *Chapter* 6, the method of Viskovatov often permits the convergence of the continued fraction (1.7.8) to be more favourable than that of the corresponding series.
In case $c_{00}=1$ and $c_{0k}=0$ for $k\geq 1$, it may also be more convenient to start the Viskovatov algorithm in a slightly different way:

$$c_{10}+c_{11}+c_{12}+\dots = c_{10}+\cfrac{c_{11}}{1+\cfrac{-c_{12}-c_{13}-\dots}{c_{11}+c_{12}+c_{13}+\dots}}$$

Applying the Viskovatov algorithm (1.7.7) to the sequences $\{\tilde{c}_{0k}\}$ and $\{\tilde{c}_{1k}\}$ given by

$$\begin{aligned} &\tilde{c}_{00} := 1, \qquad \tilde{c}_{0k} := 0, \qquad k > 0, \\ &\tilde{c}_{1k} := c_{1,k+1}, \qquad k \geq 0, \end{aligned} \tag{1.7.9}$$

leads to the continued fraction

$$c_{10} + \operatorname*{K}_{m=1}^{\infty} \left(\frac{\tilde{c}_{m,0}}{\tilde{c}_{m-1,0}} \right) . \tag{1.7.10}$$

EXAMPLE 1.7.2: Consider again the series (1.7.3). If we start the method of Viskovatov with $c_{1k} = (-1)^k/(k+1)$ for $k \geq 0$, $c_{00} = 1$, $c_{0k} = 0$ for $k \geq 1$, we obtain the continued fraction

$$\frac{1}{1} + \frac{1/2}{1} + \frac{1/12}{1/2} + \frac{1/72}{1/12} + \dots . \tag{1.7.11}$$

Observe that the first few approximants of (1.7.11) indicate faster convergence to $\ln(2)$ than the approximants of (1.7.4). For the given series, the alternative form (1.7.10) looks like

$$1 + \frac{-1/2}{1} + \frac{-1/3}{-1/2} + \frac{1/72}{-1/3} + \dots .$$

1.8 Periodic and limit periodic continued fractions

Periodic continued fractions. The very simplest periodic continued fraction is the 1-periodic continued fraction, where the period starts at the beginning. It has the form

$$\frac{a}{b} + \frac{a}{b} + \frac{a}{b} + \dots + \frac{a}{b} + \dots, \qquad a \neq 0.$$

A related continued fraction is the 1-periodic continued fraction where the period starts later:

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_m}{b_m} + \frac{a}{b} + \frac{a}{b} + \frac{a}{b} + \dots .$$

More generally we may have periods of any length. If the continued fraction has a period of length k which starts at the beginning, it has the form

$$\frac{a_1}{b_1} + \cdots + \frac{a_k}{b_k} + \frac{a_1}{b_1} + \cdots + \frac{a_k}{b_k} + \frac{a_1}{b_1} + \cdots + \frac{a_k}{b_k} + \ldots,$$

and is called a k-periodic continued fraction. Also here we may have a later start of the period.

More formally, a continued fraction $\mathrm{K}\,(a_m/b_m)$ is called *periodic* with *period* k or *k-periodic* if the sequences of elements $\{a_m\}$ and $\{b_m\}$ are k-periodic after the first N elements. That is

$$a_{N+pk+q} = a_{N+q} =: a_q^*, \qquad b_{N+pk+q} = b_{N+q} =: b_q^* \tag{1.8.1}$$

where N is a fixed non-negative integer, k is a fixed positive integer, $p \geq 1$ and $q = 1, 2, \ldots, k$.

Usually N and k are taken to be the minimal numbers for which (1.8.1) holds. The linear fractional transformation $S_n(w)$ of a k-periodic continued fraction $\mathrm{K}\,(a_m/b_m)$ of the form (1.8.1) is given by

$$S_{N+pk+q}(w) = S_N \circ T_k^p \circ T_q(w), \qquad p = 1, 2, 3, \ldots, \quad q = 1, 2, \ldots, k, \tag{1.8.2a}$$

where

$$T_q(w) := \frac{a_1^*}{b_1^*} + \frac{a_2^*}{b_2^*} + \cdots + \frac{a_{q-1}^*}{b_{q-1}^*} + \frac{a_q^*}{b_q^* + w}, \qquad q = 1, 2, \ldots, k. \tag{1.8.2b}$$

Example 1.8.1: A special case of a 1-periodic continued fraction, with $N = 0$, $a_m = b_m = 1$ is given by

$$\mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{1}{1}\right) = \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \ldots. \tag{1.8.3}$$

If f_n denotes the n^{th} approximant of (1.8.3), then we obtain the inverse of the golden ratio

$$\lim_{n\to\infty} f_n = \tilde{\phi} := \frac{\sqrt{5}-1}{2} = \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \ldots.$$

Limit periodic continued fractions. A continued fraction $\mathrm{K}\,(a_m/b_m)$ is called *limit periodic* with *period* k if the sequences of elements $\{a_m\}$ and $\{b_m\}$ are limit k-periodic. That is, the limits

$$\lim_{p\to\infty} a_{pk+q} = a_q^*, \qquad \lim_{p\to\infty} b_{pk+q} = b_q^* \tag{1.8.4}$$

exist in $\widehat{\mathbb{C}}$. Here again k is a fixed positive integer and $1 \le q \le k$.

EXAMPLE 1.8.2: A special case of a limit periodic continued fraction with period $k = 1$ is given by

$$\ln(2) = \frac{1}{1} \underset{+}{} \frac{1/2}{1} \underset{+}{} \frac{1/6}{1} \underset{+}{} \frac{2/6}{1} \underset{+}{} \frac{2/10}{1} \underset{+}{} \frac{3/10}{1} \underset{+}{} \cdots \underset{+}{} \frac{a_m}{1} \underset{+}{} \cdots, \tag{1.8.5a}$$

where

$$a_1 = 1, \quad a_{2m} = \frac{m}{2(2m-1)}, \quad a_{2m+1} = \frac{m}{2(2m+1)}, \quad m = 1, 2, 3, \ldots, \tag{1.8.5b}$$

and

$$\lim_{m\to\infty} a_m = \frac{1}{4}\,.$$

Observe that the continued fraction (1.8.5) is equivalent to the continued fraction (1.7.11) constructed by the method of Viskovatov in *Example* 1.7.2.

1.9 Tails of continued fractions

The M^{th} *tail* of a continued fraction $\mathrm{K}\,(a_m/b_m)$ is the continued fraction

$$\underset{m=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{a_{M+m}}{b_{M+m}}\right) = \underset{m=M+1}{\overset{\infty}{\mathrm{K}}} \left(\frac{a_m}{b_m}\right) = \frac{a_{M+1}}{b_{M+1}} \underset{+}{} \frac{a_{M+2}}{b_{M+2}} \underset{+}{} \frac{a_{M+3}}{b_{M+3}} \underset{+}{} \cdots, \qquad M = 0, 1, 2, \ldots. \tag{1.9.1}$$

The n^{th} numerator, n^{th} denominator and n^{th} approximant of the M^{th} tail are denoted by $A_n^{(M)}, B_n^{(M)}$ and $f_n^{(M)}$. If (1.9.1) converges, its value is denoted by $f^{(M)}$. The same linear recurrence relations hold, only with $b_0 = 0$ in the initial conditions, an additional superscript $^{(M)}$ on all numerators and denominators, and a_m, b_m replaced by a_{M+m}, b_{M+m}. The determinant formula holds, with the same obvious adjustments.

Tails of convergent continued fractions. Let $\mathrm{K}\,(a_m/b_m)$ be a continued fraction converging to a value $f \in \widehat{\mathbb{C}}$ so that

$$f = \lim_{n\to\infty} f_n = \lim_{n\to\infty} S_n(0) = \operatorname*{K}_{m=1}^{\infty} \left(\frac{a_m}{b_m}\right). \tag{1.9.2}$$

Then, for $M \geq 0$, the M^{th} tail of $\mathrm{K}\,(a_m/b_m)$ converges to a value $f^{(M)} \in \widehat{\mathbb{C}}$ where

$$f^{(M)} = \frac{a_{M+1}}{b_{M+1} + f^{(M+1)}} = \operatorname*{K}_{m=M+1}^{\infty} \left(\frac{a_m}{b_m}\right), \qquad M = 0, 1, 2, \ldots \tag{1.9.3}$$

and hence

$$f = f^{(0)} = S_M(f^{(M)}), \qquad M = 0, 1, 2, \ldots. \tag{1.9.4}$$

By determining an approximation $\hat{f}^{(M)}$ of the M^{th} tail $f^{(M)}$, it is sometimes possible to have

$$\lim_{M\to\infty} \frac{|f - S_M(\hat{f}^{(M)})|}{|f - S_M(0)|} = 0,$$

which means that the sequence $\{S_M(\hat{f}^{(M)})\}$ of modified approximants converges to f faster than $\{S_M(0)\}$. Hence appropriate choices for $\hat{f}^{(M)}$ can accelerate the convergence to f.
Note that the sequence $\{f^{(M)}\}$ of tails of a convergent continued fraction may not converge at all, and if it converges, the limit is 0 only in very special cases [Syl89]. This is in sharp contrast with convergent series where

$$\lim_{n\to\infty} \sum_{k=n+1}^{\infty} c_k = 0,$$

and convergent infinite products where

$$\lim_{n\to\infty} \prod_{k=n+1}^{\infty} p_k = 1.$$

Example 1.9.1: Consider the convergent 2-periodic continued fraction

$$\sqrt{2} - 1 = \operatorname*{K}_{m=1}^{\infty} \left(\frac{(3 + (-1)^m)/2}{1}\right) = \frac{1}{1} + \frac{2}{1} + \frac{1}{1} + \frac{2}{1} + \ldots.$$

It can easily be seen that

$$f^{(2M)} = \sqrt{2} - 1, \qquad f^{(2M+1)} = \sqrt{2}, \qquad M = 0, 1, 2, \ldots$$

and hence the sequence $\{f^{(M)}\}$ does not converge.

EXAMPLE 1.9.2: We study the continued fraction

$$\operatorname*{K}_{m=1}^{\infty} \left(\frac{m(m+2)}{1} \right) = \frac{1 \cdot 3}{1} \, {}_{+} \, \frac{2 \cdot 4}{1} \, {}_{+} \, \frac{3 \cdot 5}{1} \, {}_{+} \, \ldots \,. \tag{1.9.5}$$

Since all the elements in (1.9.5) are positive, we have

$$f_2 < f_4 < f_6 < \cdots < f_5 < f_3 < f_1.$$

Moreover $f_{2n+1} - f_{2n} \to 0$ as $n \to \infty$. Hence the continued fraction converges. One can prove that the value of (1.9.5) is $f = f^{(0)} = 1$. From $f^{(0)} = 3/(1 + f^{(1)})$ we find $f^{(1)} = 2$. By induction it follows that the M^{th} tail equals $M + 1$. Hence the sequence of tails $\{f^{(M)}\}$ converges to ∞.

The tails of 1-periodic continued fractions are the simplest ones. For the convergent continued fraction

$$\frac{a}{1} \, {}_{+} \, \frac{a}{1} \, {}_{+} \, \frac{a}{1} \, {}_{+} \, \ldots \,, \qquad a > 0, \tag{1.9.6}$$

the tail $f^{(1)}$ is given by

$$f^{(0)} = \frac{a}{1 + f^{(1)}} \,.$$

Since $f^{(0)} = f^{(1)}$, the tail $f^{(1)}$ and all further tails satisfy

$$f^{(n)} = \frac{a}{1 + f^{(n)}}, \qquad n = 0, 1, 2, \ldots.$$

Since $a > 0$ in (1.9.6), it follows that

$$f^{(n)} = \frac{\sqrt{1 + 4a} - 1}{2}, \qquad n = 0, 1, 2, \ldots.$$

Tail sequence. A sequence $\{t_n\}$ in $\widehat{\mathbb{C}}$ is called a *tail sequence* of a continued fraction $\mathrm{K}\,(a_m/b_m)$ if and only if, for a starting value $t_0 \in \widehat{\mathbb{C}}$,

$$t_{n-1} = s_n(t_n) = \frac{a_n}{b_n + t_n}, \qquad n = 1, 2, 3, \ldots. \tag{1.9.7}$$

In other words,

$$t_0 = s_1 \circ s_2 \circ \cdots \circ s_n(t_n) = S_n(t_n) \tag{1.9.8}$$

and hence

$$\begin{aligned} t_n &= S_n^{-1}(t_0) = s_n^{-1} \circ s_{n-1}^{-1} \circ \cdots \circ s_1^{-1}(t_0) \\ &= -\left(b_n + \frac{a_n}{b_{n-1}} + \cdots + \frac{a_2}{b_1} + \frac{a_1}{(-t_0)}\right), \qquad n = 1, 2, 3, \ldots . \end{aligned} \tag{1.9.9}$$

It follows from (1.9.3) that the sequence of tails $\{f^{(M)}\}$ is a particular tail sequence of $\mathrm{K}\,(a_m/b_m)$ and in view of (1.9.4) we call $\{f^{(M)}\}$ the *right tail sequence.*
Another tail sequence of particular importance is given by $\{-h_n\}$, where

$$\begin{aligned} h_n &:= -S_n^{-1}(\infty) = \frac{B_n}{B_{n-1}} \\ &= b_n + \frac{a_n}{b_{n-1}} + \frac{a_{n-1}}{b_{n-2}} + \cdots + \frac{a_2}{b_1}, \qquad n = 1, 2, 3, \ldots \end{aligned} \tag{1.9.10}$$

and B_n denotes the n^{th} denominator of $\mathrm{K}\,(a_m/b_m)$. The sequence $\{-h_n\}$, which has starting value ∞, is called the *critical tail sequence* of $\mathrm{K}\,(a_m/b_m)$, because of the following theorem.

THEOREM 1.9.1: [LW92, P. 67]

The continued fraction $b_0 + \mathrm{K}\,(a_m/b_m)$ *converges generally to* f *if and only if* $\lim S_n(u_n) = f$ *for every sequence* $\{u_n\}$ *in* $\widehat{\mathbb{C}}$ *satisfying*

$$\liminf_{n\to\infty} d(u_n, -h_n) > 0$$

when $f \neq \infty$ *and*

$$\liminf_{n\to\infty} d(u_n, -A_n/A_{n-1}) > 0$$

when $f = \infty$.

1.10 Continued fractions over normed fields

The definition of continued fraction in *Section* 1.2 is extended to include continued fractions

$$b_0 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m}{b_m}\right)$$

in which the elements a_m and b_m belong to a normed field, for instance when they are certain types of complex valued functions of a complex variable z. Of primary interest is the special case in which a_m and b_m are polynomials in z. Let us recall the notion of a normed field.

Normed field. Let $\mathbb{F}$ denote a field and let us adjoin to $\mathbb{F}$ an additional element called infinity and denote it by ∞. The extended field $\widehat{\mathbb{F}}$ is given by

$$\widehat{\mathbb{F}} := \mathbb{F} \cup \{\infty\}\,.$$

We denote by 0 the neutral element for addition in $\mathbb{F}$. Operations $+$ and $\cdot$ on $\widehat{\mathbb{F}}$ involving ∞ are defined as follows. For all $a, b \in \mathbb{F}$ with $a \neq 0$,

$$a \cdot \infty := \infty, \quad \frac{a}{\infty} := 0, \quad \frac{a}{0} := \infty, \quad b + \infty := \infty.$$

The field $\mathbb{F}$ is called a *normed field* if, for each $x \in \mathbb{F}$, there is defined a unique real number designated by $\|x\|$ with the following properties. For $x, y \in \mathbb{F}$,

$$\|x\| \geq 0, \tag{1.10.1a}$$

$$\|x\| = 0 \Leftrightarrow x = 0, \tag{1.10.1b}$$

$$\|xy\| \leq \|x\| \cdot \|y\|, \tag{1.10.1c}$$

$$\|x + y\| \leq \|x\| + \|y\|. \tag{1.10.1d}$$

The number $\|x\|$ is called the *norm of* x. If $z \in \mathbb{C}$, then $\|z\| := |z|$, the absolute value or modulus of z.

Convergence in $\widehat{\mathbb{F}}$. A sequence $\{x_n\}$ in $\widehat{\mathbb{F}}$ is said to converge to $x \in \mathbb{F}$ if, for n sufficiently large, $x_n \in \mathbb{F}$ and

$$\lim_{n\to\infty} \|x - x_n\| = 0.$$

A sequence $\{x_n\}$ in $\mathbb{F}$ is said to converge to $\infty \in \widehat{\mathbb{F}}$ if, for all n sufficiently large, $1/x_n \in \mathbb{F}$ and

$$\lim_{n\to\infty} \|1/x_n\| = 0.$$

If a sequence $\{x_n\}$ in $\widehat{\mathbb{F}}$ converges to $x \in \widehat{\mathbb{F}}$, this is designated by writing

$$\lim_{n\to\infty} x_n = x.$$

The following rules for limits hold. If $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ are convergent sequences in $\mathbb{F}$ to elements in $\mathbb{F}$ and if $\lim_{n\to\infty} u_n \neq 0$, then

$$\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n,$$

$$\lim_{n\to\infty} (x_n y_n) = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right),$$

$$\lim_{n\to\infty} \left(\frac{1}{u_n}\right) = \frac{1}{\lim\limits_{n\to\infty} u_n}.$$

A continued fraction over a normed field $\mathbb{F}$,

$$b_0 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m}{b_m} \right), \qquad a_m \in \mathbb{F} \setminus \{0\}, \quad b_m \in \mathbb{F}$$

is defined in a manner completely analogous to the definition of continued fraction over $\mathbb{C}$ given in *Section* 1.2. Analogues of properties given in *Section* 1.1 to *Section* 1.9 for continued fractions over $\mathbb{C}$ also hold for continued fractions over a normed field $\mathbb{F}$.

1.11 Generalisations of continued fractions

Generalisations arise when the partial numerators and denominators of the continued fraction are:

- vectors in $\mathbb{C}^n$ [AK87; BGM96; dBJ87; LF96; LVBB94; Par87; Rob02; Smi02],
- square matrices with complex elements [BB83; BB80; BVB90; Chu01; Fie84; Gu03; LB96; SVI99],
- operators in a Hilbert space [BF79; Cuy84; Fai72; Hay74; Sch96],
- multivariate expressions and/or continued fractions themselves [Cha86; Cuy83; Cuy88; CV88a; CV88b; GS81; HS84; KS87; Kuc78; Kuc87; MO78; O'D74; Sem78; Sie80].

These *multidimensional* and *multivariate* generalisations are not straightforward because non-commutativity and division may cause problems. The generalisation where the partial numerators and denominators of the continued fraction are themselves continued fractions gives rise to so-called *branched continued fractions*.

When replacing the second-order linear difference equations (1.3.1) by n^{th}-order linear difference equations, the recurrence yields *generalised continued fractions*. The approximants in this case are n-dimensional vectors. In the same way as there is a close relation between the theory of continued fractions and that of Padé approximation (see *Chapter* 4), generalised continued fractions are connected to Padé-Hermite approximation.

Further reading

- Basic references on the topic of continued fractions are, among others, [Per29; Wal48; Per54; Khi56; Per57; Old63; Kho63; Hen77; JT80; LW92; BGM96].
- Several volumes in the series Lecture Notes in Mathematics are devoted to the proceedings of conferences and workshops on continued fractions and related topics.

2

Continued fraction representation of functions

To represent functions of a complex variable z in continued fraction form, we need to have continued fractions with elements $a_m(z)$ and $b_m(z)$ that depend on z. In *Section* 2.3 the most important families of continued fractions are described. Most of them are so-called corresponding continued fractions, either to series developments at one point or at two points. These correspondence properties are further detailed in *Section* 2.4 and *Section* 4.3 for C-fractions, in *Section* 2.5 and *Section* 4.4 for P-fractions, in *Section* 2.6 and *Section* 4.6 for T-fractions and in *Section* 2.6 for J-fractions.

2.1 Symbols and notation

Let the functions $f(z)$ and $g(z)$ be defined for $z \in D$ where D is a subset of the complex plane and let $u \in \overline{D}$. We write for z tending to the limit point u,

$$f(z) \sim g(z) \Leftrightarrow \lim_{z \to u} f(z)/g(z) = 1. \tag{2.1.1a}$$

The symbols $o()$ and $O()$ are used to denote

$$f(z) = o(g(z)) \Leftrightarrow \lim_{z \to u} f(z)/g(z) = 0, \tag{2.1.1b}$$

$$f(z) = O(g(z)) \Leftrightarrow \exists K \in \mathbb{R}^+ : |f(z)/g(z)| \leq K, \; z \to u. \tag{2.1.1c}$$

The symbol $O()$ can also apply to the whole set D instead of to $z \to u$.

EXAMPLE 2.1.1: For $f(z) = \tanh(z), D = \mathbb{C}, u = 0$ and $g(z) = z$ we find

$$\tanh(z) \sim z, \qquad z \to 0 .$$

For $f(z) = \exp(-z), D = \mathbb{R}, u = +\infty$ and $g(z) = 1$ we can write

$$\exp(-z) = o(1), \qquad z \to +\infty \;.$$

For $f(z) = \sin(z), D = \mathbb{R}, u = 0$ and $g(z) = z$ we have

$$\sin(z) = O(z), \qquad z \to 0 \;.$$

2.2 Correspondence

An important application of continued fractions is the representation of holomorphic functions of a complex variable z by continued fractions

$$b_0(z) + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m(z)}{b_m(z)} \right), \tag{2.2.1}$$

where the elements $a_m(z)$ and $b_m(z)$ are polynomials in z or $1/z$. To indicate the dependence on z, we denote the n^{th} approximant of (2.2.1) by $f_n(z)$, the n^{th} numerator and denominator by $A_n(z)$ and $B_n(z)$ and the n^{th} modified approximant by $S_n(z; w)$.

Formal power series at $z = 0$. A series $L(z)$ is called a *formal power series* (FPS) *at* $z = 0$ if and only if $L(z)$ has the form

$$L(z) = \sum_{k=m}^{\infty} c_k z^k, \qquad c_k \in \mathbb{C}, \quad m \in \mathbb{Z}, \tag{2.2.2}$$

where $c_m \neq 0$ or all $c_k = 0$. The neutral element for the addition of FPS is denoted by the symbol 0 as usual. The set $\mathbb{L}_0$ of all FPS at $z = 0$ is a field over $\mathbb{C}$ with the usual operations of addition and multiplication. A series (2.2.2) is called a *formal Taylor series* (FTS) *at* $z = 0$ if $m \geq 0$ and it is called a *formal Laurent series* (FLS) *at* $z = 0$ if $m < 0$.
For all $L(z) \in \mathbb{L}_0$ we define $\lambda(L)$ by

$$\lambda(L) := \begin{cases} m, & L(z) = \displaystyle\sum_{k=m}^{\infty} c_k z^k, \qquad c_m \neq 0, \\ \infty, & L(z) = 0 \;. \end{cases} \tag{2.2.3}$$

A norm $\| \cdot \|$ defined on $\mathbb{L}_0$ is given by

$$\|L\| := 2^{-\lambda(L)}, \qquad L(z) \in \mathbb{L}_0$$

where $2^{-\infty} = 0$. In fact it is readily shown that $\|\cdot\|$ verifies the properties (1.10.1), with equality in (1.10.1c). Hence $\mathbb{L}_0$ is a normed field and, by *Section* 1.10, a continued fraction of the form (2.2.1) is defined if its elements $a_m(z)$ and $b_m(z)$ are in $\mathbb{L}_0$ and $a_m(z) \neq 0$. The approximants are all in $\mathbb{L}_0$ or equal ∞.

A FTS is an *asymptotic expansion* of a function $f(z)$ *at* $z = 0$, with respect to a region D in $\mathbb{C}$ with $0 \in \overline{D}$, if

$$\forall n \in \mathbb{N}_0, \exists \rho_n > 0, \eta_n > 0 : \left| f(z) - \sum_{k=m}^{n} c_k z^k \right| \leq \eta_n |z|^{n+1}, \qquad |z| < \rho_n, \quad z \in D, \quad m \geq 0,$$

or equivalently

$$\forall n \in \mathbb{N}_0, \exists \rho_n > 0 : f(z) - \sum_{k=m}^{n} c_k z^k = O\left(z^{n+1}\right), \qquad |z| < \rho_n, \quad z \in D, \quad m \geq 0.$$

This is denoted by

$$f(z) \approx \sum_{k=m}^{\infty} c_k z^k, \qquad z \to 0. \tag{2.2.4}$$

Correspondence to a FPS at $z = 0$. Although our interest is mainly in the case where $L(z)$ is the convergent or asymptotic expansion of a function $f(z)$, we begin by assuming $L(z)$ is an arbitrary non-zero FPS at $z = 0$, as in (2.2.2). Let $R(z)$ be a function meromorphic at $z = 0$. Let the mapping

$$\Lambda_0 : R(z) \to \Lambda_0(R)$$

associate with $R(z)$ its Laurent expansion in a deleted neighbourhood of the origin. A sequence $\{R_n(z)\}$ of functions meromorphic at the origin is said to *correspond to a FPS* $L(z)$ *at* $z = 0$ if and only if

$$\nu_n := \lambda(L - \Lambda_0(R_n)) \to \infty. \tag{2.2.5}$$

By the definition of λ in (2.2.3), the series L and $\Lambda_0(R_n)$ agree term-by-term up to and including the term involving $z^{\nu_n - 1}$. We can write this as

$$L(z) - \Lambda_0(R_n(z)) = O\left(z^{\nu_n}\right).$$

The integer ν_n is called the *order of correspondence* of $R_n(z)$ to $L(z)$.
A continued fraction of the form (2.2.1) is said to correspond to $L(z)$ at $z = 0$ if and only if the sequence of its approximants $\{f_n(z)\}$ corresponds to $L(z)$ at $z = 0$.

Correspondence to a function at $z = 0$. A sequence $\{R_n(z)\}$ or a continued fraction (2.2.1), is said to correspond at $z = 0$ to a function $f(z)$ meromorphic at the origin if and only if it corresponds to the FPS $\Lambda_0(f(z))$ at $z = 0$.

Formal power series at $z = \infty$. A series $L(z)$ is called a *formal power series at $z = \infty$* if and only if $L(z)$ has the form

$$L(z) = \sum_{k=m}^{\infty} c_{-k} z^{-k}, \qquad c_{-k} \in \mathbb{C}, \quad m \in \mathbb{Z}, \tag{2.2.6}$$

where $c_{-m} \neq 0$ or all $c_{-k} = 0$. The set $\mathbb{L}_\infty$ of all FPS at $z = \infty$ is a field over $\mathbb{C}$ with the usual operations of addition and multiplication. A series (2.2.6) is called a *formal Taylor series at $z = \infty$* if $m \geq 0$ and it is called a *formal Laurent series at $z = \infty$* if $m < 0$.
For all $L(z) \in \mathbb{L}_\infty$ we define

$$\lambda(L) := \begin{cases} m, & L(z) = \sum_{k=m}^{\infty} c_{-k} z^{-k}, \qquad c_{-m} \neq 0, \\ \infty, & L(z) = 0\ . \end{cases} \tag{2.2.7}$$

Note that when $c_{-m} \neq 0$, then $\lambda(L)$ is the degree in $1/z$ of the first non-zero term of $L(z)$. A norm $\|\cdot\|$ defined on $\mathbb{L}_\infty$ is given by

$$\|L\| := 2^{-\lambda(L)}, \qquad L(z) \in \mathbb{L}_\infty$$

with $2^{-\infty} = 0$. It is easy to show that $\|\cdot\|$ verifies the same properties as the norm on $\mathbb{L}_0$. Therefore $\mathbb{L}_\infty$ is a normed field and a continued fraction (2.2.1) is defined if its elements $a_m(z)$ and $b_m(z)$ are in $\mathbb{L}_\infty$ and $a_m(z) \neq 0$. The approximants are all in $\mathbb{L}_\infty$ or equal ∞.
A FTS is an *asymptotic expansion* of a function $f(z)$ *at $z = \infty$*, with respect to a region D in $\mathbb{C}$, if

$$\forall n \in \mathbb{N}_0, \exists \rho_n > 0, \eta_n > 0 : \left| f(z) - \sum_{k=m}^{n} c_{-k} z^{-k} \right| \leq \eta_n |z|^{-n-1},$$
$$|z| > \rho_n, \quad z \in D, \quad m \geq 0,$$

or equivalently

$$\forall n \in \mathbb{N}_0, \exists \rho_n > 0 : f(z) - \sum_{k=m}^{n} c_{-k} z^{-k} = O\left(z^{-n-1}\right),$$
$$|z| > \rho_n, \quad z \in D, \quad m \geq 0.$$

We denote this by

$$f(z) \approx \sum_{k=m}^{\infty} c_{-k} z^{-k}, \qquad z \to \infty. \tag{2.2.8}$$

Correspondence to a FPS at $z = \infty$. For a function $R(z)$ meromorphic at $z = \infty$, we denote its Laurent expansion in a deleted neighbourhood of $z = \infty$ by $\Lambda_\infty(R)$. A sequence of functions $\{R_n(z)\}$ meromorphic at $z = \infty$ is said to correspond to a FPS $L(z)$ at $z = \infty$ if the sequence $\{R_n(1/w)\}$ corresponds at $w = 0$ to the FPS $L(1/w)$. A continued fraction corresponds at $z = \infty$ to a FPS $L(z)$ at $z = \infty$ if and only if the sequence of approximants corresponds to $L(z)$ at $z = \infty$.
In a similar manner correspondence at $z = a$ where $a \in \mathbb{C}$, can be defined by considering $z = w + a$ which gives rise to a FPS in $z - a$.

Correspondence to a function at $z = \infty$. A sequence $\{R_n(z)\}$ or a continued fraction is said to correspond at $z = \infty$ to a function $f(z)$ meromorphic at infinity, if and only if it corresponds to $\Lambda_\infty(f(z))$ at $z = \infty$.

Simultaneous correspondence at 0 and ∞. Consider the FPS $\Lambda_0(f(z))$ and $\Lambda_\infty(f(z))$ at $z = 0$ and $z = \infty$ of a function $f(z)$ meromorphic at the origin and at infinity. A sequence $\{R_n(z)\}$ of functions meromorphic at the origin and at infinity is said to *correspond simultaneously* to $\Lambda_0(f(z))$ and $\Lambda_\infty(f(z))$ if and only if both

$$\lambda\left(\Lambda_0(f - R_n)\right) \to \infty,$$
$$\lambda\left(\Lambda_\infty(f - R_n)\right) \to \infty.$$

A continued fraction is said to correspond simultaneously to $\Lambda_0(f(z))$ and $\Lambda_\infty(f(z))$ if its sequence of approximants corresponds simultaneously to $\Lambda_0(f(z))$ and $\Lambda_\infty(f(z))$.

Criteria for correspondence. Theorems stated in this chapter help to answer the following questions.

- For a given continued fraction, does there exist a FPS $L(z)$ to which the continued fraction corresponds?
- For a given FPS $L(z)$, can we find a corresponding continued fraction of the form (2.2.1)?

The theorems and examples apply to correspondence at $z = 0$. Related results hold for correspondence at $z = \infty$. *Theorem* 2.2.1 asserts the existence of a FPS $L(z)$ corresponding at $z = 0$ to a given sequence $\{f_n(z)\}$ of functions meromorphic at $z = 0$. In the special case where $\{f_n(z)\}$ is the sequence of approximants of a continued fraction, we find a FPS to which it corresponds.

The construction of a corresponding continued fraction for a given FPS $L(z)$ is treated in the *Sections* 2.4 through 2.7.

THEOREM 2.2.1: [JT80, pp. 151–152]

Let $\{f_n(z)\}$ be a sequence of functions meromorphic at $z = 0$. Then:

(A) There exists a FPS $L(z)$ at $z = 0$ such that $\{f_n(z)\}$ corresponds to $L(z)$ at $z = 0$ if and only if for $k_n := \lambda\left(\Lambda_0(f_{n+1} - f_n)\right)$

$$\lim_{n\to\infty} k_n = \infty. \tag{2.2.9}$$

(B) If (2.2.9) holds then the FPS $L(z)$ to which $\{f_n(z)\}$ corresponds is uniquely determined and the order of correspondence ν_n of $f_n(z)$ as defined in (2.2.5) satisfies

$$k_n \le \nu_n, \qquad n = 1, 2, 3, \ldots. \tag{2.2.10}$$

(C) Moreover, if the sequence $\{k_n\}$ tends monotonically to ∞, then

$$\nu_n = k_n, \qquad n = 1, 2, 3, \ldots. \tag{2.2.11}$$

The inequality (2.2.10) follows immediately from

$$k_n = \lambda(\Lambda_0(f_{n+1} - f_n)) \le \lambda(\Lambda_0(L - f_n)) = \nu_n, \qquad n = 1, 2, 3, \ldots.$$

Let $f_n(z)$, $A_n(z)$, $B_n(z)$ denote the n^{th} approximant, numerator and denominator, respectively, of a continued fraction (2.2.1) where the elements $a_m(z)$ and $b_m(z)$ are FPS at $z = 0$. Then by the determinant formulas (1.3.4)

$$\begin{aligned} f_{n+1}(z) - f_n(z) &= \frac{A_{n+1}(z)B_n(z) - A_n(z)B_{n+1}(z)}{B_n(z)B_{n+1}(z)} \\ &= \frac{(-1)^n \prod_{m=1}^{n+1} a_m(z)}{B_n(z)B_{n+1}(z)}, \qquad n = 0, 1, 2, \ldots. \end{aligned} \tag{2.2.12}$$

In many cases enough information is known about the functions $a_m(z)$, $B_n(z)$ and $B_{n+1}(z)$ in (2.2.12) so that a sequence $\{k_n\}$ can be determined such that (2.2.9) follows from (2.2.12).

EXAMPLE 2.2.1: Consider the 1-periodic continued fraction

$$1 + \frac{z}{1} \underset{+}{} \frac{z}{1} \underset{+}{} \frac{z}{1} \underset{+}{} \dots,$$

so that $a_m(z) = z$ and $b_m(z) = 1$. From the recurrence formulas (1.3.1) one can verify that the n^{th} numerator $A_n(z)$ and the n^{th} denominator $B_n(z)$ are polynomials in z of the forms

$$\begin{aligned} A_{2n-1}(z) = B_{2n}(z) = z^n + \cdots + 1, \qquad n = 1, 2, 3, \dots, \\ A_{2n}(z) = B_{2n+1}(z) = (n+1)z^n + \cdots + 1, \qquad n = 1, 2, 3, \dots. \end{aligned} \tag{2.2.13}$$

It follows readily from (2.2.12) and (2.2.13) that

$$k_n = \lambda\left(\Lambda_0(f_{n+1} - f_n)\right) = n + 1, \qquad n = 1, 2, 3, \dots,$$

and hence there exists a unique FPS $L(z) \in \mathbb{L}_0$ such that

$$\nu_n = \lambda\left(L - \Lambda_0(f_n)\right) = n + 1, \qquad n = 1, 2, 3, \dots.$$

2.3 Families of continued fractions

C-fractions. A continued fraction of the form

$$b_0 + \underset{m=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{a_m z^{\alpha_m}}{1}\right), \qquad a_m \in \mathbb{C} \setminus \{0\}, \quad \alpha_m \in \mathbb{N} \tag{2.3.1}$$

is called a *C-fraction* [LS39]. The name C-fraction comes from the property of these fractions to correspond to FPS at $z = 0$. If $\alpha_m = 1$ for $m \geq 1$, then (2.3.1) is called a *regular C-fraction*. More information on regular C-factions can be found in *Section* 2.4 and in *Chapter* 4.

S-fractions. A continued fraction of the form

$$F(z) = \underset{m=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{a_m z}{1}\right), \qquad a_m > 0 \tag{2.3.2}$$

is called a Stieltjes fraction or *S-fraction* and any continued fraction that is equivalent to (2.3.2) is also called an S-fraction [Sti95]. For example

$$E(z) = \underset{m=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{z}{b_m}\right), \qquad b_m > 0$$

is an S-fraction since with $a_1 = 1/b_1$ and $a_m = 1/(b_{m-1}b_m)$ for $m > 1$, one finds $F(z) \equiv E(z)$.
A continued fraction $C(z)$ is called a *modified S-fraction* if there exist transformations $C(z) \to B(C(z))$ and $z \to a(z)$ such that the resulting continued fraction $B(C(a(z)))$ is an S-fraction. In the sequel the term modified is given the same meaning when applied to other families of continued fractions.
Examples of modified S-fractions are

$$G(z) = \frac{a_1}{z} \, \underset{+}{} \, \frac{a_2}{1} \, \underset{+}{} \, \frac{a_3}{z} \, \underset{+}{} \, \frac{a_4}{1} \, \underset{+}{} \, \ldots, \qquad a_m > 0, \tag{2.3.3a}$$

$$H(z) = \frac{a_1}{1} \, \underset{+}{} \, \frac{a_2}{z} \, \underset{+}{} \, \frac{a_3}{1} \, \underset{+}{} \, \frac{a_4}{z} \, \underset{+}{} \, \ldots, \qquad a_m > 0, \tag{2.3.3b}$$

$$D(z) = \frac{a_1}{z} \, \underset{+}{} \, \frac{a_2}{z} \, \underset{+}{} \, \frac{a_3}{z} \, \underset{+}{} \, \frac{a_4}{z} \, \underset{+}{} \, \ldots, \qquad a_m > 0. \tag{2.3.3c}$$

In case of convergence we find the relationships

$$F(z) \equiv G(1/z) \equiv zH(1/z) \equiv \sqrt{z}D(1/\sqrt{z}), \qquad |\arg z| < \pi. \tag{2.3.4}$$

Associated continued fractions. A continued fraction of the form

$$\frac{\alpha_1 z}{1 + \beta_1 z} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-\alpha_m z^2}{1 + \beta_m z} \right), \qquad \alpha_m \in \mathbb{C}\backslash\{0\}, \quad \beta_m \in \mathbb{C} \tag{2.3.5}$$

is called an *associated continued fraction*. The even part of a regular C-fraction is an associated continued fraction, but the converse does not always hold.

P-fractions. Continued fractions of the form

$$b_0(z) + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{1}{b_m(z)} \right), \tag{2.3.6}$$

where each $b_m(z)$ is a polynomial in $1/z$,

$$b_m(z) = \sum_{k=-N_m}^{0} c_k^{(m)} z^k, \qquad c_{-N_m}^{(m)} \neq 0, \quad N_0 \geq 0, \quad N_m \geq 1, \quad m \in \mathbb{N}, \tag{2.3.7}$$

are called *P-fractions* [Mag62a; Mag62b]. The name P-fraction stands for principal part continued fraction expansion, as we explain in *Section* 2.5.

J-fractions. Continued fractions of the form

$$\frac{\alpha_1}{\beta_1 + z} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-\alpha_m}{\beta_m + z} \right), \qquad \alpha_m \in \mathbb{C}\backslash\{0\}, \quad \beta_m \in \mathbb{C}, \tag{2.3.8}$$

are called *J-fractions* and were introduced by Jacobi. The even contraction of a modified regular C-fraction

$$\frac{a_1}{z} + \frac{a_2}{1} + \frac{a_3}{z} + \frac{a_4}{1} + \dots \tag{2.3.9}$$

is a J-fraction, but the converse does not always hold. The continued fraction (2.3.8) is called a *real J-fraction* if $\alpha_m > 0$ and β_m is real. These conditions on the coefficients α_m and β_m are satisfied if the modified regular C-fraction (2.3.9) is a modified S-fraction (2.3.3a). J-fractions play an important role in moment theory for which we refer to *Chapter* 5.

T-fractions. Continued fractions of the form

$$\mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{F_m z}{1 + G_m z} \right), \qquad F_m \in \mathbb{C}\backslash\{0\}, \quad G_m \in \mathbb{C}, \tag{2.3.10}$$

are called Thron fractions or *general T-fractions* [Thr48; Per57]. If all $G_m \neq 0$ the general T-fraction corresponds simultaneously to FPS at 0 and ∞ as is explained in *Section* 2.6 and *Chapter* 4. Equivalent forms of general T-fractions are

$$\mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{c_m z}{e_m + d_m z} \right), \qquad c_m, e_m \in \mathbb{C}\backslash\{0\}, \quad d_m \in \mathbb{C}, \tag{2.3.11}$$

and

$$\frac{\lambda_1}{\frac{1}{\beta_0 z} + \beta_1} + \frac{\lambda_2}{\frac{z}{\beta_1} + \beta_2} + \frac{\lambda_3}{\frac{1}{\beta_2 z} + \beta_3} + \frac{\lambda_4}{\frac{z}{\beta_3} + \beta_4} + \dots, \qquad \lambda_m, \beta_m \in \mathbb{C}\backslash\{0\}. \tag{2.3.12}$$

If all $F_m = 1$ in (2.3.10) then it is called a *T-fraction*, without further specification.

Following are several important subfamilies of general T-fractions. When all F_m and G_m are strictly positive then the general T-fraction is called a *positive T-fraction*. When all F_m and G_m are real and nonzero and in addition

$$F_{2m-1}F_{2m} > 0, \qquad F_{2m-1}/G_{2m-1} > 0, \tag{2.3.13}$$

the general T-fraction is called an *APT-fraction*, which stands for alternating positive term fraction [JNT83b]. The conditions (2.3.13) formulated in terms of the coefficients λ_m and β_m in (2.3.12) are

$$\frac{\lambda_m \beta_{m-1}}{\beta_m} > 0, \qquad m \in \mathbb{N}.$$

M-fractions. When $F_1 z$ in (2.3.10) is replaced by F_1, we obtain the continued fraction

$$\frac{F_1}{1 + G_1 z} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{F_m z}{1 + G_m z} \right), \qquad F_m \in \mathbb{C} \backslash \{0\}, \quad G_m \in \mathbb{C}, \tag{2.3.14}$$

which is called an *M-fraction* after Murphy and Mc Cabe [Mur71; MC75; MCM76]. Observe that the special case

$$\frac{1}{1} + \operatorname*{K}_{m=1}^{\infty} \left(\frac{F_m z}{1 - F_m z} \right)$$

of the M-fraction (2.3.14) is the Euler continued fraction (1.7.2) for the sequence of approximants

$$f_n = 1 + \sum_{k=0}^{n} \left(\prod_{j=1}^{k} (-F_j) \right) z^k.$$

PC-fractions. Perron-Carathéodory or *PC-fractions* are of the form

$$\beta_0 + \frac{\alpha_1}{\beta_1} + \frac{1}{\beta_2 z} + \frac{\alpha_3 z}{\beta_3} + \frac{1}{\beta_4 z} + \frac{\alpha_5 z}{\beta_5} + \frac{1}{\beta_6 z} + \dots,$$
$$\alpha_{2m+1}, \beta_m \in \mathbb{C}, \quad \alpha_{2m+1} = 1 - \beta_{2m}\beta_{2m+1} \neq 0.$$

The special case of the form

$$\delta_0 - \frac{2\delta_0}{1} + \frac{1}{\overline{\delta}_1 z} + \frac{(1 - |\delta_1|^2) z}{\delta_1} + \frac{1}{\overline{\delta}_2 z} + \frac{(1 - |\delta_2|^2) z}{\delta_2} + \dots,$$
$$\delta_0 > 0, \quad \delta_m \in \mathbb{C}, \quad |\delta_m| < 1 \tag{2.3.15}$$

is called a positive Perron–Carathéodory continued fraction or a *PPC-fraction*. It naturally arises in the solution of the trigonometric moment problem [JNT86a; JNT89] which is discussed in *Section* 5.1.

Thiele continued fractions. A *Thiele interpolating continued fraction* [Thi09] is of the form

$$b_0 + \operatorname*{K}_{m=1}^{\infty} \left(\frac{z - z_{m-1}}{b_m} \right), \qquad b_m \in \mathbb{C}, \quad z_m \in \mathbb{C}. \tag{2.3.16}$$

Rather than being a corresponding continued fraction, it interpolates function data. The computation of the b_m from the interpolation conditions is given in *Chapter* 6.

2.4 Correspondence of C-fractions

There is a one-to-one correspondence between the set of all C-fractions, including terminating C-fractions, and the set of FTS $L(z)$ at $z = 0$.

THEOREM 2.4.1: [LS39; JT80, pp. 156–157; LW92, p. 253]

(A) *Every C-fraction (2.3.1) corresponds to a unique FTS $L(z)$ at $z = 0$ and the order of correspondence of the n^{th} approximant $f_n(z)$ is*

$$\nu_n = \sum_{k=1}^{n+1} \alpha_k. \tag{2.4.1}$$

(B) *Let $L(z)$ be a given FTS at $z = 0$ with $L(0) = c_0$. Then either there exists a C-fraction (2.3.1) corresponding to $L(z)$ at $z = 0$, or for some $n \in \mathbb{N}$ there exists a terminating C-fraction*

$$f_n(z) = c_0 + \mathop{\mathrm{K}}_{m=1}^{n} \left(\frac{a_m z^{\alpha_m}}{1} \right), \tag{2.4.2}$$

such that

$$L(z) = \Lambda_0(f_n(z)). \tag{2.4.3}$$

(C) *If $f(z)$ is a rational function holomorphic at $z = 0$ and if $L(z) = \Lambda_0(f(z))$ is the Taylor series expansion of $f(z)$ about $z = 0$, then there exists a terminating C-fraction $f_n(z)$ of the form (2.4.2) such that (2.4.3) holds.*

It follows from *Theorem* 2.4.1 that the S-fraction (2.3.2) corresponds to the unique series

$$\Lambda_0(F(z)) = \sum_{k=0}^{\infty} c_k z^{k+1} \tag{2.4.4a}$$

with order of correspondence $\nu_n = n + 1$. Then the correspondence and order of correspondence of the modified S-fractions $G(z)$, $H(z)$ and $D(z)$ introduced in (2.3.3) are respectively given by

$$\Lambda_\infty(G(z)) = \sum_{k=0}^{\infty} c_k z^{-k-1}, \qquad \nu_n = n + 1, \tag{2.4.4b}$$

$$\Lambda_\infty(H(z)) = \sum_{k=0}^{\infty} c_k z^{-k}, \qquad \nu_n = n, \tag{2.4.4c}$$

$$\Lambda_\infty(D(z)) = \sum_{k=0}^{\infty} c_k z^{-2k-1}, \qquad \nu_n = 2n + 1. \tag{2.4.4d}$$

2.5 Correspondence of P-fractions

Let $L_0(z)$ be a FPS as in (2.2.2) which we write in the form

$$L_0(z) = \sum_{k=-N_0}^{\infty} c_k^{(0)} z^k = b_0(z) + \frac{1}{L_1(z)},$$

where

$$b_0(z) = \sum_{k=-N_0}^{0} c_k^{(0)} z^k$$

and

$$L_1(z) = \Lambda_0 \left(\frac{1}{\sum_{k=1}^{\infty} c_k^{(0)} z^k} \right).$$

We note that $b_0(z)$ is the principal part of $L_0(z)$ plus the constant term $c_0^{(0)}$ and that $L_1(z)$ is a new FPS. Now write the FPS $L_1(z)$ in the form

$$L_1(z) = b_1(z) + \frac{1}{L_2(z)}$$

where

$$b_1(z) = \sum_{k=-N_1}^{0} c_k^{(1)} z^k, \qquad c_{-N_1}^{(1)} \neq 0, \qquad N_1 \geq 1,$$

is the principal part of $L_1(z)$ and $L_2(z)$ is again a FPS. By continuing in this manner one obtains the P-fraction representation of $L_0(z)$.

THEOREM 2.5.1: [Mag74; JT80, pp. 159–160]

(A) Every P-fraction (2.3.6) corresponds at $z = 0$ to a unique FPS $L_0(z)$. The order of correspondence of the n^{th} approximant $f_n(z)$ is

$$\nu_n = 2\sum_{k=1}^{n} N_k + N_{n+1}, \qquad n = 0, 1, 2, \ldots. \tag{2.5.1}$$

(B) Conversely, let $L_0(z)$ be a given FPS. Then either there exists a P-fraction (2.3.6) corresponding to $L_0(z)$ at $z = 0$, or else there exists a terminating P-fraction

$$f_n(z) = b_0(z) + \frac{1}{b_1(z)} \, {}_{+} \, \frac{1}{b_2(z)} \, {}_{+} \cdots {}_{+} \, \frac{1}{b_n(z)}, \qquad n = 0, 1, 2, \ldots, \tag{2.5.2}$$

such that

$$L_0(z) = \Lambda_0(f_n(z)) \tag{2.5.3}$$

where each $b_m(z)$ is a polynomial in $1/z$.

From *Theorem* 2.5.1 it follows that there is a one-to-one correspondence between the set of FPS at $z = 0$ and the set of all P-fractions, including terminating P-fractions.

2.6 Correspondence of J-fractions and T-fractions

THEOREM 2.6.1: [JT80, pp. 249–250; LW92, p. 346]

Every J-fraction of the form (2.3.8) corresponds to a FPS at $z = \infty$. The order of correspondence of the n^{th} approximant $f_n(z)$ is

$$\nu_n = 2n + 1.$$

The existence of a J-fraction corresponding to an arbitrary FPS at $z = \infty$ is not guaranteed. Necessary conditions on the coefficients of the FPS are given in *Theorem* 6.5.1 for associated continued fractions.

THEOREM 2.6.2: [JT80, pp. 259–261]

Every T-fraction (2.3.10) corresponds to a unique FTS $L_0(z)$ at $z = 0$ and the order of correspondence of the n^{th} approximant $f_n(z)$ is

$$\nu_n = n + 1.$$

If, in addition, all $G_n \neq 0$ in (2.3.10), the T-fraction (2.3.10) also corresponds to a unique FTS $L_\infty(z)$ at $z = \infty$. The order of correspondence of the n^{th} approximant $f_n(z)$ is

$$\nu_n = n.$$

The existence of a T-fraction corresponding to an arbitrary pair of FPS at $z = 0$ and at $z = \infty$ is only guaranteed under certain conditions on the coefficients of the pair of FPS. These conditions are made explicit in *Chapter* 6 for M-fractions.

EXAMPLE 2.6.1: For the T-fraction

$$\mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{z}{1 - z} \right) \tag{2.6.1}$$

it can be proved by induction that the n^{th} approximant of (2.6.1) is given by

$$f_n(z) = \frac{z(1-(-z)^n)}{1-(-z)^{n+1}}.$$

Since

$$\Lambda_0(f_n(z)) = z + O(z^{n+1}),$$
$$\Lambda_\infty(f_n(z)) = -1 + O(z^{-n}),$$

the continued fraction (2.6.1) corresponds simultaneously to $L_0(z) = z$ and $L_\infty(z) = -1$.

2.7 Correspondence and three-term recurrences

Three-term recurrence relations are used in *Section* 1.3 to define the n^{th} numerator A_n and n^{th} denominator B_n of a continued fraction. In this section such recurrence relations play a basic role in continued fraction correspondence to FPS and continued fraction representations of functions.

THEOREM 2.7.1: [JT80, pp. 160–161]

Let $\{a_m(z)\}$, $\{b_m(z)\}$ and $\{P_m(z)\}$ be sequences in $\mathbb{L}_0$ such that

$$a_m(z) \not\equiv 0, \quad m \geq 1, \qquad P_m(z) \not\equiv 0, \quad m \geq 0 \tag{2.7.1a}$$

and

$$P_m(z) = b_m(z)P_{m+1}(z) + a_{m+1}(z)P_{m+2}(z), \qquad m = 0, 1, 2, \ldots. \tag{2.7.1b}$$

Let

$$L_m(z) := \Lambda_0\left(\frac{P_m}{P_{m+1}}(z)\right), \qquad m = 0, 1, 2, \ldots.$$

Then the continued fraction (2.2.1) corresponds at $z = 0$ to the FPS $L_0(z)$ provided

$$\lambda(a_m(z)) \geq 1, \quad \lambda(b_{m-1}(z)) \leq 0, \quad \lambda(L_m(z)) \leq 0, \quad m = 1, 2, 3, \ldots$$

or

$$\lambda(a_m(z)) \geq 0, \quad \lambda(b_{m-1}(z)) \leq -1, \quad \lambda(L_m(z)) \leq 0, \quad m = 1, 2, 3, \ldots.$$

The order of correspondence ν_n of the n^{th} approximant of (2.2.1) is given by

$$\nu_0 = \lambda(a_1(z)) - \lambda(L_1(z)),$$
$$\nu_n = \sum_{k=1}^{n+1} \lambda(a_k(z)) - 2\sum_{k=1}^{n} \lambda(b_k(z)) - \lambda(L_{n+1}(z)), \qquad n = 1, 2, 3, \ldots. \tag{2.7.2}$$

EXAMPLE 2.7.1: Let

$$b_0(z) = 1, \quad a_m(z) = z, \quad b_m(z) = 1, \quad m = 1, 2, 3, \ldots.$$

Then one solution of the three-term recurrence relation (2.7.1b) is

$$\{P_m(z)\} = \left\{ \left(\frac{\sqrt{1+4z} - 1}{2z} \right)^m \right\}$$

with

$$L_m(z) = \Lambda_0 \left(\frac{P_m}{P_{m+1}}(z) \right) = \frac{2z}{\sqrt{1+4z} - 1} = 1 + z + \ldots.$$

Since

$$\lambda(a_m(z)) = \lambda(z) = 1, \qquad m = 1, 2, 3, \ldots,$$
$$\lambda(b_m(z)) = \lambda(1) = 0, \qquad m = 1, 2, 3, \ldots,$$
$$\lambda(L_m(z)) = \lambda(1 + z + \ldots) = 0, \qquad m = 1, 2, 3, \ldots,$$

the conditions of *Theorem* 2.7.1 are satisfied and the continued fraction

$$b_0(z) + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m(z)}{b_m(z)} \right) = 1 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{z}{1} \right)$$

corresponds at $z = 0$ to the FPS $L_0(z) = 1 + z + \ldots$.

3

Convergence criteria

Since a continued fraction is a non-terminating expression, it is important to know if it converges and at which rate. For a continued fraction with elements $a_m(z)$ and $b_m(z)$ which are functions of z, it is also important to know where in the complex plane it converges. The issue of the speed of convergence and the development of sharp truncation error bounds are dealt with in *Chapter* 7. The latter is in a way the dual of the convergence problem: while we want to obtain convergence in as wide a region as possible and for as many fractions as possible, truncation error bounds are only useful if they are as specific and sharp as we can get them.

3.1 Some classical theorems

We refer to *Section* 1.2 for the definitions of convergence and general convergence. We recall that convergence to ∞ is accepted.

THEOREM 3.1.1: WORPITZKY [Wor65; JT80, p. 94; LW92, p. 35]

Let $|a_m| \leq 1/4$ *for all* $m \in \mathbb{N}$. *Then the continued fraction*

$$\mathrm{K}\left(\frac{a_m}{1}\right) = \frac{a_1}{1} + \frac{a_2}{1} + \cdots + \frac{a_m}{1} + \ldots$$

converges, all approximants f_n *are in the disk* $|w| < 1/2$, *and the value* f *is in the disk* $|w| \leq 1/2$.

THEOREM 3.1.2: ŚLESZYŃSKI-PRINGSHEIM [Sle88; Pri99; JT80, p. 92; LW92, p. 30]

The continued fraction

$$\mathrm{K}\left(\frac{a_m}{b_m}\right) = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_m}{b_m} + \ldots$$

converges if

$$|b_m| \geq |a_m| + 1, \qquad m \geq 1.$$

Under the same condition the property $|f_n| < 1$ *holds for all approximants* f_n*, and* $|f| \leq 1$ *for the value of the continued fraction.*

THEOREM 3.1.3: VAN VLECK [VV01; JT80, pp. 88–89; LW92, pp. 32–33]
Let $0 < \epsilon < \pi/2$ *and let* b_m *satisfy*

$$-\frac{\pi}{2} + \epsilon < \arg b_m < \frac{\pi}{2} - \epsilon, \qquad m \geq 1.$$

Then all approximants f_n *of the continued fraction*

$$\mathrm{K}\left(\frac{1}{b_m}\right) = \frac{1}{b_1} \genfrac{}{}{0pt}{}{}{+} \frac{1}{b_2} \genfrac{}{}{0pt}{}{}{+} \cdots \genfrac{}{}{0pt}{}{}{+} \frac{1}{b_m} \genfrac{}{}{0pt}{}{}{+} \ldots$$

are finite and in the angular domain

$$-\frac{\pi}{2} + \epsilon < \arg f_n < \frac{\pi}{2} - \epsilon\ .$$

The sequences $\{f_{2n}\}$ *and* $\{f_{2n+1}\}$ *converge to finite values. The continued fraction* $\mathrm{K}\,(1/b_m)$ *converges if and only if, in addition,*

$$\sum_{m=1}^{\infty} |b_m| = \infty.$$

When convergent, the value f *is finite and satisfies* $|\arg f| < \pi/2$.

The *Stern-Stolz series* of a continued fraction $\mathrm{K}\,(a_m/b_m)$, given by

$$\sum_{m=1}^{\infty} \left| b_m \prod_{k=1}^{m} a_k^{(-1)^{m-k+1}} \right| \tag{3.1.1}$$

plays an important role in establishing convergence. The divergence of the Stern-Stolz series (3.1.1) is a necessary condition for the convergence of $\mathrm{K}\,(a_m/b_m)$. The series (3.1.1) is invariant under equivalence transformations. The following theorem shows that for continued fractions $\mathrm{K}\,(a_m/1)$, the divergence of the Stern-Stolz series may be replaced by a simpler condition.

THEOREM 3.1.4:

Let $\{a_m\}$ be a sequence of complex numbers. If

$$|a_m| < M, \qquad m \geq 1, \tag{3.1.2}$$

or

$$\sum_{m=1}^{\infty} (m|a_m|)^{-1} = \infty \tag{3.1.3}$$

then the Stern-Stolz series with all $b_m = 1$ diverges.

A convergence result for S-fractions. Many of the important functions used in applications are S-fractions, which are defined in *Section* 2.3.

THEOREM 3.1.5:

An S-fraction $\mathrm{K}\,(a_m z/1)$ corresponding at $z = 0$ to $L(z) = \sum_{k=1}^{\infty} c_k z^k$ is convergent in $\{z \in \mathbb{C} : |\arg z| < \pi\}$ if one of the following conditions holds:
(A)

$$a_m \leq M, \qquad m = 1, 2, 3, \ldots, \tag{3.1.4}$$

(B)

$$\sum_{m=1}^{\infty} \frac{1}{\sqrt{a_m}} = \infty, \tag{3.1.5}$$

(C) Carleman criterion

$$\sum_{k=1}^{\infty} \frac{1}{|c_k|^{\frac{1}{2k}}} = \infty. \tag{3.1.6}$$

If the S-fraction $\mathrm{K}\,(a_m z/1)$ is convergent, then it converges to a finite value.

Proofs of the above results can be found respectively in [JT80, p. 136; Per57, p. 77; Wal48, p. 330]. The convergence of S-fractions to a finite value follows from *Theorem* 3.4.2.

3.2 Convergence sets and value sets

A *convergence set* is a subset Ω of $\mathbb{C} \times \mathbb{C}$ such that the continued fraction

$$\mathrm{K}\left(\frac{a_m}{b_m}\right) = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_m}{b_m} + \ldots$$

converges whenever all $(a_m, b_m) \in \Omega$. Examples are

$$\Omega = \{(a, b) \in \mathbb{C} \times \mathbb{C} : |b| \geq |a| + 1\} \tag{3.2.1}$$

in Ślesyński-Pringsheim's theorem and

$$\Omega = \{a \in \mathbb{C} : |a| \leq 1/4\} \times \{1\} \tag{3.2.2}$$

in Worpitzky's theorem.

A set $\Omega \in \mathbb{C} \times \mathbb{C}$ is called a *conditional convergence set* if $\mathrm{K}\,(a_m/b_m)$ with $(a_m, b_m) \in \Omega$ converges if and only if the Stern-Stolz series (3.1.1) diverges. The set Ω is called a *uniform convergence set* if the convergence is uniform with respect to the family of continued fractions defined by $(a_m, b_m) \in \Omega$. If, in the definitions, convergence is replaced by general convergence, we get *general convergence sets* and *uniform general convergence sets* [Jac86]. Sometimes we need a sequence $\{\Omega_m\}$ of convergence sets rather than merely a single convergence set.

The sequence $\{V_n\}$ is a sequence of *value sets* for $\mathrm{K}\,(a_m/b_m)$ [JT80, p. 64; LW92, p. 110] if all sets V_n are non-empty subsets of the extended complex plane and

$$s_n(V_n) = \frac{a_n}{b_n + V_n} \subseteq V_{n-1}, \qquad n = 1, 2, 3, \ldots. \tag{3.2.3}$$

If (3.2.3) holds for $(a_n, b_n) \in E_n$, the sequence $\{E_n\}$ is called a sequence of *element sets* corresponding to the sequence $\{V_n\}$ of value sets. The sequence $\{E_n\}$ may or may not be a sequence of convergence sets. An important special case is when all V_n are equal to V for all n, in which case we say that V is a value set. Similarly E is an element set, possibly a convergence set, if all $E_n = E$. The following are some important properties of value sets for $\mathrm{K}\,(a_m/b_m)$:

- for $m \geq 0, k \geq 1$ and all $w_{m+k} \in V_{m+k}$ we have

$$S_k^{(m)}(w_{m+k}) = \frac{a_{m+1}}{b_{m+1}} + \frac{a_{m+2}}{b_{m+2}} + \cdots + \frac{a_{m+k}}{b_{m+k} + w_{m+k}} \in V_m;$$

- for any n all approximants of the n^{th} tail are located in V_n;
- in case of convergence the value of the n^{th} tail is in the closure $\overline{V}_n$;
- in particular the value of the continued fraction itself is in $\overline{V}_0$.

3.3 Parabola and oval theorems

THEOREM 3.3.1: PARABOLA THEOREM [Thr43; Thr63; LW92, pp. 130–135]

Let $\alpha \in (-\pi/2, \pi/2)$ be fixed and let P_α be the parabolic region given by

$$P_\alpha := \{a \in \mathbb{C} : |a| - \Re(ae^{-2\alpha \mathrm{i}}) \leq {}^1\!/\!_2 \cos^2(\alpha)\}. \qquad (3.3.1)$$

Let $\{a_m\}$ be such that all a_m are in P_α. Then the even and odd parts of the continued fraction

$$\mathop{\mathrm{K}}\left(\frac{a_m}{1}\right) = \frac{a_1}{1} + \frac{a_2}{1} + \cdots + \frac{a_m}{1} + \ldots$$

converge to finite values. The continued fraction itself converges if and only if the Stern-Stolz series (3.1.1) with all $b_m = 1$ diverges. This holds in particular if either (3.1.2) or (3.1.3) hold. The approximants $f_n = S_n(0)$ of the continued fraction are in the half plane

$$V_\alpha := \{w \in \mathbb{C} : \Re(we^{-\mathrm{i}\alpha}) > -{}^1\!/\!_2 \cos(\alpha)\}. \qquad (3.3.2)$$

In case of convergence, the value f is in the closure of the half plane.

Alternative versions of the parabola theorem are given in [JT80, pp. 105–106; LW92, pp. 130–131].
Observe that the theorem also covers the case $a_n \in P_\alpha, a_n \to \infty$.
In the simplest case $\alpha = 0$, the parabolic region P_0 as well as the half-plane V_0 are symmetric with respect to the real axis. If, in addition, all a_m are real, the element set is the ray $[-1/4, \infty)$ of the real axis, and the value set is the ray $[-1/2, \infty)$. In the parabola theorem we have in case of convergence that for any sequence $\{w_n\} \subset \overline{V}_\alpha$ the sequence $\{S_n(w_n)\}$ converges to the value f of the continued fraction. An upper bound for the truncation error $|f - S_n(w_n)|$ is given in *Theorem* 7.1.1.

THEOREM 3.3.2: UNIFORM PARABOLA THEOREM [Thr58; JT80, p. 99]

Let P_α be given by (3.3.1) and assume that the continued fraction

$$\mathop{\mathrm{K}}\left(\frac{a_m}{1}\right) = \frac{a_1}{1} + \frac{a_2}{1} + \cdots + \frac{a_m}{1} + \ldots, \qquad a_m \in P_\alpha, \quad m \geq 1, \qquad (3.3.3)$$

converges to $f \neq \infty$. Then with $w_n \in V_\alpha$ given by (3.3.2), the modified approximants $S_n(w_n)$ of the continued fraction are all located in $\overline{V}_\alpha$. Moreover,

$$\lim_{n\to\infty} S_n(w_n) = f, \qquad w_n \in V_\alpha,$$

and

$$|f - S_n(w_n)| \le \lambda_n \to 0,$$

where λ_n *is independent of the choice of* $w_n \in V_\alpha$.

This independence indicates uniformity with respect to w_n. Also, any bounded set $|a_m| \le M$ of the parabolic region P_α is a uniform convergence set.

THEOREM 3.3.3: OVAL THEOREM [JT86; LW92, p. 141]

Let $C \in \mathbb{C}$ *with* $\Re C > -1/2$ *and* $r \in \mathbb{R}$ *such that* $0 < r < |1 + C|$. *Then the set*

$$E := \left\{a \in \mathbb{C} : |a(1 + \overline{C}) - C(|1 + C|^2 - r^2)| + r|a| \le r(|1 + C|^2 - r^2)\right\} \tag{3.3.4}$$

is a convergence set for the continued fractions $\mathrm{K}\,(a_m/1)$, *and*

$$V := \{w \in \mathbb{C} : |w - C| < r\} \tag{3.3.5}$$

is a value set for E. *Moreover, with*

$$M := \max_{w \in \overline{V}} \left| \frac{w}{1 + w} \right|$$

we have

$$|f - S_n(w)| \le 2r \frac{|C| + r}{|1 + C| - r} M^{n-1}, \qquad w \in \overline{V} \tag{3.3.6}$$

for every continued fraction $\mathrm{K}\,(a_m/1)$ *with* $a_m \in E$, *where* f *is the value of* $\mathrm{K}\,(a_m/1)$.

Some important remarks can be made about the oval theorem.

- The boundary of E is a *Cartesian oval*. For $C = 0$ the set E is the disk $|a| \le r(1-r)$ and the set V the disk $|w| < r$. With $r = 1/2$, *Theorem* 3.3.3 reduces to Worpitzky's theorem.
- For real C we have symmetry with respect to the real axis for E as well as for V. If in addition we have a continued fraction with real elements, we can replace the oval E and the disk V by intervals on the real axis.
- With $0 < p < q$ and

$$\begin{aligned} X &:= \frac{p}{1} + \frac{q}{1} + \frac{p}{1} + \frac{q}{1} + \dots \\ &= \frac{1}{2}\left(\sqrt{(1 + p + q)^2 - 4pq} - 1 - q + p\right), \\ Y &:= \frac{q}{1} + \frac{p}{1} + \frac{q}{1} + \frac{p}{1} + \dots \\ &= \frac{1}{2}\left(\sqrt{(1 + p + q)^2 - 4pq} - 1 - p + q\right) \end{aligned} \tag{3.3.7}$$

it follows from the oval theorem that the interval $[p,q]$ is the convergence set E corresponding to the value set $[X,Y]$. Here we have $X = r - C$, $Y = r + C$ and for all $w \in [X,Y]$,

$$|f - S_n(w)| \le (Y - X)\frac{Y}{1+X}\left(\frac{Y}{1+Y}\right)^{n-1}. \tag{3.3.8}$$

- If in addition to (3.3.7), we have $Y < 2X$, then for all f and any $u \in [X,Y]$ we find
$$|f - S_n(0)| > |f - S_n(u)|.$$

Theorem 3.3.4: Oval sequence theorem [LW92, p. 145]

Let $C_n \in \mathbb{C}$ and $r_n \in \mathbb{R}$, with

$$0 < r_n < |1 + C_n|, \qquad n = 0, 1, 2, \ldots \tag{3.3.9a}$$
$$|C_{n-1}| r_n \le |1 + C_n| r_{n-1}, \qquad n = 1, 2, 3, \ldots. \tag{3.3.9b}$$

Then

$$V_n := \{w \in \mathbb{C} : |w - C_n| < r_n\}, \qquad n = 0, 1, 2, \ldots \tag{3.3.10}$$

defines a sequence of value sets for the sequence of element sets defined by

$$\begin{aligned} E_n := \big\{a \in \mathbb{C} : &|a(1 + \overline{C}_n) - C_{n-1}(|1 + C_n|^2 - r_n^2)| + r_n|a| \\ &\le r_{n-1}(|1 + C_n|^2 - r_n^2)\big\}, \qquad n = 1, 2, 3, \ldots. \end{aligned} \tag{3.3.11}$$

For all continued fractions $\mathrm{K}\,(a_m/1)$ *with* $a_m \in E_m$ *and for all* $w_k \in \overline{V}_k$*,*

$$\begin{aligned} |S_{n+j}(w_{n+j}) - S_n(w_n)| &\le 2r_n \frac{|C_0| + r_0}{|1 + C_n| - r_n} \prod_{k=1}^{n-1} M_k, \\ &n = 1, 2, 3, \ldots, \quad j = 1, 2, 3, \ldots, \end{aligned} \tag{3.3.12}$$

where

$$M_k := \max_{w \in \overline{V}_k} \left|\frac{w}{1+w}\right|.$$

We remark that condition (3.3.9b) is equivalent with $E_n \neq \emptyset$.
The oval sequence theorem is very useful, in particular for limit periodic continued fractions, where we have shrinking E_n and V_n. We see this theorem in use in *Chapter* 7.

3.4 Correspondence and uniform convergence

We now deal with continued fractions of the form

$$b_0(z) + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m(z)}{b_m(z)} \right), \tag{3.4.1}$$

where a_m and b_m are polynomials with complex coefficients and z is a complex variable.

A set $D \subseteq \mathbb{C}$ is called a *domain* if and only if D is open and connected.

A sequence $\{f_n(z)\}$ of functions meromorphic in a domain D is said to *converge uniformly* on a compact subset K of D if and only if:

- there exists $N_K \in \mathbb{N}$ such that $f_n(z)$ is holomorphic in some domain containing K for all $n \geq N_K$, and
- given $\epsilon > 0$, there exists $N_\epsilon > N_K$ such that

$$\sup_{z \in K} |f_{n+m}(z) - f_n(z)| < \epsilon, \qquad n \geq N_\epsilon, \qquad m = 0, 1, 2, \ldots.$$

A continued fraction with n^{th} approximant $f_n(z)$ is said to converge uniformly on a compact subset K of a domain D if and only if $\{f_n(z)\}$ satisfies the conditions above.

A sequence $\{f_n(z)\}$ of functions meromorphic in a domain D is said to be *uniformly bounded* on a compact subset K of D if and only if there exist N_K and B_K such that

$$\sup_{z \in K} |f_n(z)| \leq B_K, \qquad n \geq N_K.$$

Theorem 3.4.1: A correspondence/convergence theorem [JT80, p. 181]

Assume that the continued fraction (3.4.1) corresponds at 0 *to a FTS* $L(z)$*. Let D be a domain containing the origin* $z = 0$*. Then the continued fraction (3.4.1) converges uniformly on any compact subset of D to a holomorphic function* $f(z)$ *if and only if the sequence of approximants of (3.4.1) is uniformly bounded on every compact subset of D. The series* $L(z)$ *is the FTS at* $z = 0$ *of* $f(z)$*.*

If D is properly larger than the disk of convergence for $L(z)$, then the continued fraction provides an analytic continuation of $f(z)$ to D. An analogous result holds when the role of $z = 0$ is replaced by $z = \infty$.

Correspondence alone does not imply convergence. But, as seen in *Theorem* 3.4.1, a certain boundedness property in addition to correspondence leads to convergence.

THEOREM 3.4.2: [Sti95; LW92, p. 138]

Let $\mathrm{K}\,(a_m z/1)$ *be an S-fraction and let* $D = \{z : |\arg z| < \pi\}$ *be the complex plane cut along the negative real axis. Then the following statements hold.*

(A) The even and odd parts of the S-fraction converge locally uniformly in D *to holomorphic functions.*
(B) The continued fraction itself converges to a holomorphic function in D *if and only if either (3.1.4), (3.1.5) or the Stern-Stolz series with all* $b_m = 1$ *diverges.*
(C) The continued fraction diverges for all $z \in D$ *if the Stern-Stolz series (3.1.1) converges.*

3.5 Periodic and limit periodic continued fractions

We refer to *Section* 1.8 for definitions and notation.

Convergence of 1-periodic continued fractions. The continued fraction

$$\frac{z}{1} + \frac{z}{1} + \frac{z}{1} + \cdots + \frac{z}{1} + \ldots \tag{3.5.1}$$

converges for all complex z except for those on the ray $(-\infty, -1/4)$ of the negative real axis. The value is

$$f = \frac{\sqrt{1+4z} - 1}{2}, \tag{3.5.2}$$

where the branch of the root is the one with positive real part. The value f is the root of

$$f = \frac{z}{1+f}$$

of smallest modulus. A generalisation of this result for k-periodic continued fractions is given in *Theorem* 3.5.1. We know from *Section* 1.4 that by an equivalence transformation we can restrict ourselves to $\mathrm{K}\,(a/1)$ rather than $\mathrm{K}\,(a/b)$, without any loss of generality.
In case the period starts later, as in

$$\frac{a_1}{1} + \frac{a_2}{1} + \cdots + \frac{a_N}{1} + \frac{z}{1} + \frac{z}{1} + \cdots + \frac{z}{1} + \ldots,$$

the continued fraction converges under the same condition. The value is

$$F = S_N(f) = \frac{A_N + A_{N-1} f}{B_N + B_{N-1} f}, \tag{3.5.3}$$

where f is as in (3.5.2).

Linear fractional transformations. We need to recall some known facts about linear fractional transformations

$$T(w) := \frac{Dw+E}{Fw+G}, \qquad DG - FE \neq 0. \tag{3.5.4}$$

The fixpoints of $T(w)$ are the solutions w_1 and w_2 of the quadratic equation $w = T(w)$. If $F = 0$ the equation is not properly quadratic, and one of the fixpoints is ∞. Assume first that we have two distinct, finite fixpoints w_1 and w_2. If

$$\left|\frac{Fw_1+G}{Fw_2+G}\right| > 1 \tag{3.5.5}$$

we have for any $w \neq w_2$ that

$$\lim_{n\to\infty} T^n(w) = w_1. \tag{3.5.6}$$

In this case w_1 is called the *attractive fixpoint*, and w_2 is called the *repulsive fixpoint*. If the absolute value of the ratio in (3.5.5) is 1, the limit in (3.5.6) does not exist.
In case $w_1 = w_2$ in (3.5.4) the limit exists and is equal to w_1.

Convergence of *k*-periodic continued fractions.

Theorem 3.5.1: [JT80, p. 53]

Consider a k-periodic continued fraction with $b_0 = 0$ and with the period starting at a_1/b_1,

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_k}{b_k} + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_k}{b_k} + \ldots,$$

and let $S_k(w)$ equal

$$S_k(w) = \frac{A_k + A_{k-1}w}{B_k + B_{k-1}w} = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_k}{b_k + w}$$

with fixpoints w_1 and w_2. Then the following statements hold.
(A) If S_k is the identity transformation, the continued fraction diverges.
(B) If $w_1 = w_2$, then the transformation S_k is called parabolic. The continued fraction converges to w_1.
(C) If $w_1 \neq w_2$ and

$$\left|\frac{B_k + B_{k-1}w_1}{B_k + B_{k-1}w_2}\right| = 1, \tag{3.5.7}$$

then the transformation is called elliptic and the continued fraction diverges.

(D) *If* $w_1 \neq w_2$ *and*

$$\left|\frac{B_k + B_{k-1}w_1}{B_k + B_{k-1}w_2}\right| > 1, \tag{3.5.8}$$

then the transformation is called loxodromic. The continued fraction converges to the attractive fixpoint w_1.

Convergence of limit periodic continued fractions.

THEOREM 3.5.2: [Per57, p. 93; LW92, p. 151]

Let $\mathrm{K}\,(a_m/1)$ *be a continued fraction where*

$$\lim_{m\to\infty} a_m = a \neq \infty.$$

Then the following holds.

(A) *If*

$$\left|\arg(a + {}^1\!/\!_4)\right| < \pi, \qquad a \neq -\frac{1}{4},$$

then the continued fraction converges, possibly to ∞.

(B) *The sequence* $f^{(N)}$ *of tails converges to*

$$\lim_{N\to\infty} f^{(N)} = \lim_{N\to\infty}\left(\mathop{\mathrm{K}}_{m=N+1}^{\infty}\left(\frac{a_m}{1}\right)\right) = \mathrm{K}\left(\frac{a}{1}\right) = \frac{\sqrt{1+4a}-1}{2},$$

which is the attractive fixpoint of the transformation $w \to a/(1+w)$, *or equivalently, the root of* $w(1+w) - a = 0$ *of smallest modulus.*

The case $a_n \to \infty$ is covered by *Theorem* 3.3.1 under the additional condition that $a_n \in P_\alpha$ given by (3.3.1) from a certain n on. A simple consequence of *Theorem* 3.5.2 is the following result for limit periodic regular C-fractions.

COROLLARY 3.5.1: [Per57, p. 95]

Consider

$$\mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{a_m z}{1}\right) \tag{3.5.9}$$

with

$$\lim_{m\to\infty} a_m = a \neq \infty$$

and let R_a be defined by

$$R_a := \{z \in \mathbb{C} : |\arg(az + 1/4)| < \pi\}. \tag{3.5.10}$$

Then the continued fraction (3.5.9) converges to a function $f(z)$, meromorphic in R_a. The convergence is uniform on any compact subset of R_a without poles of $f(z)$. The function is holomorphic at $z = 0$.

The region R_a is the complement of the ray from $-1/(4a)$ to ∞, which is part of the ray from 0 to ∞ through $-1/(4a)$.
In case $a = 0$ the continued fraction converges to a function $f(z)$ which is meromorphic. To any $r > 0$ there is an n_r, such that the continued fraction

$$\frac{a_{n_r+1}z}{1} + \frac{a_{n_r+2}z}{1} + \frac{a_{n_r+3}z}{1} + \dots$$

converges to a holomorphic function on $|z| < r$, uniformly on any compact subset of that disk [JT80, p. 131].

3.6 Convergence and minimal solutions

In this section a connection is established between the set of solutions of a system of three-term recurrence relations and convergence properties of continued fractions.
Let $\{a_n\}$ and $\{b_n\}$ be sequences in a normed field $\mathbb{F}$ with $a_n \neq 0$ for $n \geq 1$. Here 0 denotes the zero element of $\mathbb{F}$. Of interest to us are the cases where $\mathbb{F}$ is either $\mathbb{C}$ of $\mathbb{L}_c$, the set of formal power series at $z = c$. The set of solutions $\{y_n\}$ in $\mathbb{F}$ of the system of three-term recurrence relations

$$y_n = b_n y_{n-1} + a_n y_{n-2}, \qquad n = 1, 2, 3, \dots \tag{3.6.1}$$

is a linear vector space V of dimension 2 over the field $\mathbb{F}$. From (1.3.1) we know that the sequence of numerators $\{A_n\}$ and denominators $\{B_n\}$ of the continued fraction $\mathrm{K}\,(a_m/b_m)$ are solutions of the three-term recurrence relations (3.6.1). They form a basis for the linear space V [LW92, p. 192].
We say that $\{u_n\}$ is a *minimal solution* of (3.6.1) if $\{u_n\} \neq \{0\}$ and there exists a solution $\{v_n\}$ of (3.6.1) such that

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 0. \tag{3.6.2}$$

The solution $\{v_n\}$ is called a *dominant solution* of (3.6.1). In general a system (3.6.1) may not have a minimal solution. If $\{u_n\}$ is a minimal solution and $\{v_n\}$ is a dominant solution of (3.6.1), then the sequences $\{u_n\}$ and $\{v_n\}$ form a basis for V and all solutions $c\{u_n\}$, $c \in \mathbb{F} \setminus \{0\}$, are also minimal solutions.

THEOREM 3.6.1: PINCHERLE GENERALISED [JT80, p. 164; LW92, p. 202]

Let $\{a_n\}$ *and* $\{b_n\}$ *be sequences in a normed field* $\mathbb{F}$ *with* $a_n \neq 0$ *for* $n \geq 1$. *Then:*

(A) The system of three-term recurrence relations (3.6.1) has a minimal solution if and only if the continued fraction $\mathrm{K}\,(a_m/b_m)$ *over the field* $\mathbb{F}$ *converges to a value in* $\widehat{\mathbb{F}} = \mathbb{F} \cup \{\infty\}$.

(B) Suppose that there exists a minimal solution $\{u_n\}$ *of (3.6.1) in* $\mathbb{F}$. *Then*

$$-\frac{u_{n-1}}{u_{n-2}} = \frac{a_n}{b_n} \genfrac{}{}{0pt}{}{}{+} \frac{a_{n+1}}{b_{n+1}} \genfrac{}{}{0pt}{}{}{+} \genfrac{}{}{0pt}{}{}{\ldots}, \qquad n = 1, 2, 3, \ldots. \tag{3.6.3}$$

By (3.6.3) we mean the following.

- If $u_{n-2} = 0$, then $u_{n-1} \neq 0$ and the continued fraction in (3.6.3) converges to ∞.
- If $u_{n-2} \neq 0$, then the continued fraction in (3.6.3) converges to the finite limit $-u_{n-1}/u_{n-2} \in \mathbb{F}$.

EXAMPLE 3.6.1: For the continued fraction

$$\mathrm{K}\left(\frac{a_m(z)}{b_m(z)}\right) = \frac{z+1}{z-1} \genfrac{}{}{0pt}{}{}{+} \frac{z+2}{z} \genfrac{}{}{0pt}{}{}{+} \frac{z+3}{z+1} \genfrac{}{}{0pt}{}{}{+} \genfrac{}{}{0pt}{}{}{\ldots}, \qquad z \in \mathbb{C} \setminus \{-1\}, \tag{3.6.4}$$

where

$$a_m(z) = z + m, \qquad b_m(z) = z + m - 2, \qquad m = 1, 2, 3, \ldots,$$

the three-term recurrence relation (3.6.1) has the solution

$$u_n = (-1)^n (z + n + 2), \qquad n = -1, 0, 1, \ldots.$$

This solution can be proved to be a minimal solution. Hence, by Pincherle's theorem, the continued fraction (3.6.4) converges to $-u_0/u_{-1}$. In other words,

$$\frac{z+2}{z+1} = \frac{z+1}{z-1} \genfrac{}{}{0pt}{}{}{+} \frac{z+2}{z} \genfrac{}{}{0pt}{}{}{+} \frac{z+3}{z+1} \genfrac{}{}{0pt}{}{}{+} \genfrac{}{}{0pt}{}{}{\ldots}, \qquad z \in \mathbb{C} \setminus \{-1\}\,.$$

Theorem 3.6.2: Auric generalised [JT80, p. 173; LW92, p. 207]

Let $\{a_n\}$ *and* $\{b_n\}$ *be sequences in a normed field* $\mathbb{F}$ *with* $a_n \neq 0$ *for all* n*. Let* A_n *and* B_n *denote the* n^{th} *numerator and denominator, respectively, of the continued fraction* $\mathrm{K}\,(a_m/b_m)$ *over* $\mathbb{F}$*. Let the sequence* $\{y_n\}$ *in* $\mathbb{F}$ *be a solution of the system of three-term recurrence relations (3.6.1). If* $y_n \neq 0$ *for all* $n \geq -1$*, the continued fraction* $\mathrm{K}\,(a_m/b_m)$ *converges to the finite limit*

$$-\frac{y_0}{y_{-1}} = \lim_{n\to\infty} \frac{A_n}{B_n}$$

if and only if

$$\lim_{n\to\infty} \|R_n\| = \infty, \qquad R_n = \sum_{k=1}^{n} \frac{(-1)^k \prod_{m=1}^{k} a_m}{y_k y_{k-1}} \in \mathbb{F}\,.$$

4

Padé approximants

Padé approximants, either at one finite point or at ∞ and a finite point, are closely related to continued fractions, since Padé approximants are rational functions satisfying some order of correspondence and can be obtained as continued fraction approximants. For this matter we refer to the *Sections* 4.3, 4.4 and 4.6.

The convergence of a sequence of Padé approximants is detailed in a number of additional theorems, which are most useful for functions meromorphic in a substantial region of the complex plane.

Padé approximation theory is also connected to the theory of orthogonal polynomials which is further developed in *Chapter* 5. This connection is explained in *Section* 4.8.

4.1 Definition and notation

Let the FTS

$$f(z) = \sum_{j=0}^{\infty} c_j z^j, \qquad c_j \in \mathbb{C}, \quad c_0 \neq 0 \tag{4.1.1}$$

be given. For simplicity, the symbol $f(z)$ denotes both the FTS $\Lambda_0(f)$ and its limit function f when it exists, unless otherwise indicated. The *Padé approximant* of order (m,n) for $f(z)$ is the irreducible form $r_{m,n}(z) = p_{m,n}(z)/q_{m,n}(z)$ with $q_{m,n}(0) = 1$ of the rational function $p(z)/q(z)$ satisfying

$$\begin{aligned} p(z) &= \sum_{j=0}^{m} a_j z^j, \qquad a_j \in \mathbb{C}, \\ q(z) &= \sum_{j=0}^{n} b_j z^j, \qquad b_j \in \mathbb{C}, \\ \lambda(fq - p) &\geq m + n + 1. \end{aligned} \tag{4.1.2}$$

Because different solutions p/q and $\tilde{p}/\tilde{q}$ of (4.1.2) are equivalent in the sense that $p\tilde{q} = \tilde{p}q$, the Padé approximant $r_{m,n}(z)$ is unique. Usually $r_{m,n}(z)$ is normalised such that $q_{m,n}(0) = 1$. This is always possible because by (4.1.2),

$$\lambda(p) \geq \lambda(q).$$

Let ∂p denote the exact degree of the polynomial $p(z)$. For $p_{m,n}(z)$ and $q_{m,n}(z)$ the exact order of correspondence is given by

$$\lambda(fq_{m,n} - p_{m,n}) = \partial p_{m,n} + \partial q_{m,n} + t + 1, \qquad t \geq 0\ . \tag{4.1.3}$$

Possibly $\partial p_{m,n}+\partial q_{m,n}+t+1 < m+n+1$ and then $p_{m,n}(z)$ and $q_{m,n}(z)$ not necessarily satisfy (4.1.2), although $p(z)$ and $q(z)$ do. Nevertheless for $p_{m,n}$ and $q_{m,n}$ there exists an integer s with $0 \leq s \leq \min(m-\partial p_{m,n}, n-\partial q_{m,n})$ such that $p(z) = z^s p_{m,n}(z)$ and $q(z) = z^s q_{m,n}(z)$ satisfy (4.1.2). On the other hand, while $p(z)$ and $q(z)$ may start with higher order terms in z, we deduce from (4.1.3) that the exact order of correspondence of $r_{m,n}(z)$ is

$$\lambda\left(f - \frac{p_{m,n}}{q_{m,n}}\right) = \partial p_{m,n} + \partial q_{m,n} + t + 1, \qquad t \geq 0.$$

When the FTS of $f(z)$ is given at a finite point u different from the origin, then in all of the above z is replaced by $z - u$.
We introduce the notation

$$T_{m,n+1} := \begin{pmatrix} c_m & \cdots & c_{m-n} \\ \vdots & \ddots & \vdots \\ c_{m+n} & \cdots & c_m \end{pmatrix}, \qquad n = 0, 1, \ldots, \quad m = 0, 1, \ldots \tag{4.1.4}$$

for the $(n+1) \times (n+1)$ *Toeplitz matrix*, which is fully determined by its first row $(c_m, \ldots, c_{m-n})$ and its first column $(c_m, \ldots, c_{m+n})^T$. Here $c_k = 0$ for $k < 0$. The $(n+1) \times (n+1)$ *Toeplitz determinant*, is denoted by

$$T_{n+1}^{(m)} := \det T_{m,n+1}. \tag{4.1.5}$$

4.2 Fundamental properties

The Padé approximants $r_{m,n}(z)$ for $f(z)$ are arranged in the *Padé table* as follows:

$$\begin{array}{lllll} r_{0,0} & r_{0,1} & r_{0,2} & r_{0,3} & \cdots \\ r_{1,0} & r_{1,1} & r_{1,2} & \cdots & \\ r_{2,0} & r_{2,1} & \ddots & & \\ r_{3,0} & \vdots & & & \\ \vdots & & & & \end{array}$$

The first column $\{r_{m,0}\}$ of this table contains the successive partial sums of the series (4.1.1).

Reciprocal covariance. Let $r_{m,n}(z) = p_{m,n}(z)/q_{m,n}(z)$ be the Padé approximant of order (m,n) for $f(z)$. If $c_0 \neq 0$, then

$$\frac{q_{m,n}(z)/c_0}{p_{m,n}(z)/c_0}$$

is the Padé approximant of order (n,m) for $1/f(z)$. Hence, when $c_0 \neq 0$, the first row $\{r_{0,m}\}$ of the Padé table consists of the reciprocals of the partial sums of $1/f(z)$.

Block structure of the Padé table. A remarkable property of the Padé table is that identical entries in the table always appear in a single coherent square block. An entry does never reappear in the table outside its block.

THEOREM 4.2.1: [Pad92]

With t defined by (4.1.3), the Padé approximants $r_{k,\ell}(z)$ for $f(z)$ with $\partial p_{m,n} \leq k \leq \partial p_{m,n} + t$ and $\partial q_{m,n} \leq \ell \leq \partial q_{m,n} + t$ satisfy

$$r_{k,\ell}(z) = r_{m,n}(z)$$

and hence appear in square blocks of size

$$t + 1 = \lambda(f q_{m,n} - p_{m,n}) - \partial p_{m,n} - \partial q_{m,n} \ .$$

Also $m \leq \partial p_{m,n} + t$ and $n \leq \partial q_{m,n} + t$ and hence $r_{m,n}(z)$ itself belongs to the above block.

EXAMPLE 4.2.1: Let

$$f(z) = 1 + \sin(z) = 1 + \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \ .$$

For $m = 2k$ and $n = 0$ we find that

$$\partial p_{2k,0} = 2k - 1,$$
$$\lambda(f q_{2k,0} - p_{2k,0}) = 2k + 1 \ .$$

Hence for all k:

$$r_{2k-1,0}(z) = r_{2k,0}(z) = r_{2k-1,1}(z) = r_{2k,1}(z), \qquad k \geq 1 \ .$$

Normality. A Padé approximant $r_{m,n}(z)$ is called *normal* if it occurs only once in the Padé table. In other words, its block in the Padé table is of size $t + 1 = 1$.

THEOREM 4.2.2: [Pad92]

Let $r_{m,n}(z)$ be the Padé approximant of order (m, n) for a FTS $f(z)$ given by (4.1.1). The following statements, where $T_n^{(m)}$ is defined by (4.1.5), are equivalent:
(A) $r_{m,n}(z)$ is normal;
(B) $\partial p_{m,n} = m, \partial q_{m,n} = n$ and $\lambda(fq_{m,n} - p_{m,n}) = m + n + 1$;
(C) $T_n^{(m)}\, T_n^{(m+1)}\, T_{n+1}^{(m)}\, T_{n+1}^{(m+1)} \neq 0$.

EXAMPLE 4.2.2: Let

$$f(z) = \exp(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!} \;.$$

In [Per57, p. 432] explicit formulas for the Padé numerator and denominator for the exponential function are given:

$$p_{m,n}(z) = \sum_{j=0}^{m} \frac{m(m-1)\cdots(m-j+1)}{(m+n)(m+n-1)\cdots(m+n-j+1)} \frac{z^j}{j!}, \tag{4.2.1}$$

$$q_{m,n}(z) = \sum_{j=0}^{n} (-1)^j \frac{n(n-1)\cdots(n-j+1)}{(m+n)(m+n-1)\cdots(m+n-j+1)} \frac{z^j}{j!} \;. \tag{4.2.2}$$

Here the products appearing in the coefficients equal 1 when empty. All Padé approximants $r_{m,n}(z)$ for the exponential function are normal as can be seen from (B) in *Theorem* 4.2.2. From (4.2.1) and (4.2.2) it is immediately clear that $\partial p_{m,n} = m$ and $\partial q_{m,n} = n$. The remaining condition $\lambda(fq_{m,n} - p_{m,n}) = m + n + 1$ is easy to verify.

Let $f(z)$ be defined by

$$f(z) = \int_0^\infty \frac{d\Phi(t)}{1+zt}, \qquad |\arg z| < \pi, \tag{4.2.3}$$

where $\Phi(t)$ is a classical moment distribution function on $(0, \infty)$ as defined in *Section* 5.1. Here it suffices to note that

$$\Lambda_0(f) = \sum_{j=0}^{\infty} (-1)^j \left(\int_0^\infty t^j \, d\Phi(t) \right) z^j \;.$$

For functions of the form (4.2.3), which are called Stieltjes functions, the following remarkable theorem holds.

Theorem 4.2.3: [Gra74]

Let $f(z)$ be a Stieltjes function, as in (4.2.3). Then for all $m, n \geq 0$, the Padé approximant $r_{m,n}(z)$ for $f(z)$ is normal.

Normality of the Padé table highly simplifies the computation of C-fraction representations, as can be seen in *Chapter* 6. Because of the following result, normality can be achieved by approximating $f(z)$ from its power series at some shifted origin u near zero.

Theorem 4.2.4: Shifting origin Kronecker theorem [Lub88]

Let $f(z)$ be analytic in an open connected set D containing the origin and let $f(z)$ not be a rational function. Then there exists an at most countable set S such that if $u \in D \setminus S$:

$$T_n^{(m)}(u) \neq 0, \qquad m, n = 0, 1, 2, \ldots$$

where $T_n^{(m)}(u)$ is defined as $T_n^{(m)}$ but at the shifted origin u.

Recurrence relations. A well-known recurrence relation for Padé approximants is the 5-term star identity (4.2.4e) [Fro81]. We list it here together with some three-term relations. The fact that the numerators and denominators of neighbouring approximants in the Padé table obey the same recurrence relations, is the key to the connection with the theory of continued fractions.

Theorem 4.2.5: [BGM96, pp. 81–89]

Let $r_{m,n} = p_{m,n}/q_{m,n}$ be the Padé approximant of order (m, n) for a FTS $f(z)$ given by (4.1.1). If all of the involved Padé approximants are normal, then:
(A)

$$\begin{aligned}(-1)^{n-1}\, T_{n-1}^{(m+1)}\, p_{m,n} &= T_n^{(m)} p_{m+1,n-1} - z\, T_n^{(m+1)}\, p_{m,n-1},\\ (-1)^{n-1}\, T_{n-1}^{(m+1)}\, q_{m,n} &= T_n^{(m)} q_{m+1,n-1} - z\, T_n^{(m+1)} q_{m,n-1}\,.\end{aligned} \tag{4.2.4a}$$

(B)

$$\begin{aligned}(-1)^{n-1}\, T_{n-1}^{(m)}\, p_{m,n} &= T_n^{(m+1)} p_{m,n-1} - z\, T_n^{(m+1)} p_{m-1,n-1},\\ (-1)^{n-1}\, T_{n-1}^{(m)}\, q_{m,n} &= T_n^{(m+1)} q_{m,n-1} - z\, T_n^{(m+1)} q_{m-1,n-1}\,.\end{aligned} \tag{4.2.4b}$$

(C)

$$T_n^{(m)}\,p_{m,n} = T_n^{(m+1)} p_{m-1,n} - (-1)^n\, T_{n+1}^{(m)} p_{m,n-1},$$
$$T_n^{(m)}\,q_{m,n} = T_n^{(m+1)} q_{m-1,n} - (-1)^n\, T_{n+1}^{(m)} q_{m,n-1}\ . \tag{4.2.4c}$$

(D)

$$T_n^{(m-1)} p_{m,n} = T_n^{(m)} p_{m-1,n} - (-1)^n\, z\, T_{n+1}^{(m)} p_{m-1,n-1},$$
$$T_n^{(m-1)} q_{m,n} = T_n^{(m)} q_{m-1,n} - (-1)^n\, z\, T_{n+1}^{(m)} q_{m-1,n-1}\ . \tag{4.2.4d}$$

(E)

$$(r_{m+1,n} - r_{m,n})^{-1} + (r_{m-1,n} - r_{m,n})^{-1} =$$
$$(r_{m,n+1} - r_{m,n})^{-1} + (r_{m,n-1} - r_{m,n})^{-1}\ . \tag{4.2.4e}$$

Identities involving normal Padé approximants on downward or upward sloping diagonals in the Padé table are obtained by applying the three-term identities repeatedly. An easy way to remember (4.2.4e) is to associate each of the Padé approximants with a geographical direction:

$$r_{m-1,n}(z) = N$$

$$r_{m,n-1}(z) = W \qquad r_{m,n}(z) = C \qquad r_{m,n+1}(z) = E$$

$$r_{m+1,n}(z) = S$$

Then the 5-term *star identity* (4.2.4e) becomes

$$(N - C)^{-1} + (S - C)^{-1} = (E - C)^{-1} + (W - C)^{-1}\ .$$

4.3 Connection with regular C-fractions

We consider the *descending staircase*

$$T_k := \{r_{k,0}(z), r_{k+1,0}(z), r_{k+1,1}(z), r_{k+2,1}(z), \dots\}, \qquad k \geq 0 \tag{4.3.1}$$

of approximants in the Padé table. The following result generalises *Theorem* 2.4.1 which holds for $k = 0$.

THEOREM 4.3.1: [CW87, p. 77]

Let $f(z)$ be a given FTS as in (4.1.1). If every three consecutive elements in the sequence T_k, given by (4.3.1), of Padé approximants for $f(z)$ are distinct, then there exists a continued fraction of the form

$$r_{k,0}(z) + z^k \operatorname*{K}_{m=1}^{\infty} \left(\frac{a_m^{(k)} z}{1} \right), \qquad a_m^{(k)} \neq 0, \qquad k \geq 0 \tag{4.3.2}$$

such that the n^{th} approximant of (4.3.2) equals the $(n+1)^{\text{th}}$ element of T_k.

Continued fractions of the form (4.3.2) relate to the Padé approximants on or below the main diagonal in the Padé table. For the right upper half of the table one can use the reciprocal covariance property given in *Section* 4.2. Because the elements of T_k satisfy (4.1.2), a particular result is obtained for T_0.

COROLLARY 4.3.1: [CW87, p. 78]

Let $f(z)$ be a given FTS as in (4.1.1). If every three consecutive elements in the sequence T_0, given by (4.3.1), of Padé approximants for $f(z)$ are distinct, then there exists a regular C-fraction

$$c_0 + \operatorname*{K}_{m=1}^{\infty} \left(\frac{a_m^{(0)} z}{1} \right), \qquad a_m^{(0)} \neq 0 \tag{4.3.3}$$

corresponding to $f(z)$.

The algorithm for the computation of the coefficients $a_m^{(k)}$, both for $k > 0$ and $k = 0$ is given in *Chapter* 6.

4.4 Connection with P-fractions

Let for $s \in \mathbb{Z}$,

$$b_0^{(s)}(z) + \operatorname*{K}_{m=1}^{\infty} \left(\frac{1}{b_m^{(s)}(z)} \right) \tag{4.4.1}$$

be the P-fraction representation of $z^s f(z)$ with $f(z)$ given by (4.1.1) where $b_m^{(s)}(z)$ and $N_m^{(s)}$ for $m \geq 0$ are as in (2.3.7).

EXAMPLE 4.4.1: For $f(z) = \exp(z)$ and $s = 0$ the P-fraction representation is given by

$$\begin{aligned} b_0^{(0)}(z) &= 1, \\ b_1^{(0)}(z) &= \frac{1}{z} - \frac{1}{2}, \\ b_{2k}^{(0)}(z) &= \frac{4(4k-1)}{z}, \qquad k \geq 1, \\ b_{2k+1}^{(0)}(z) &= \frac{4k+1}{z}, \qquad k \geq 1. \end{aligned}$$

Let $A_n^{(s)}(z)$ and $B_n^{(s)}(z)$ denote the n^{th} numerator and denominator of (4.4.1). After multiplication by a suitable power of z, these are polynomials in z. For $A_n^{(s)}/B_n^{(s)}$ a more general result than the one formulated in *Theorem* 2.5.1, which covers the case $s = 0$, can be formulated.

THEOREM 4.4.1: [Mag62b; Mag62a]

Let $A_n^{(s)}(z)/B_n^{(s)}(z)$ be the n^{th} approximant of the P-fraction (4.4.1). Then:
(A) for $s > 0$,

$$\begin{aligned} r_{0,s}(z) &= \frac{A_1^{(s)}(z)}{z^{N_1^{(s)}} B_1^{(s)}(z)}, \\ r_{N_2^{(s)}+\cdots+N_n^{(s)},\, s+N_2^{(s)}+\cdots+N_n^{(s)}}(z) &= \frac{z^{N_2^{(s)}+\cdots+N_n^{(s)}} A_n^{(s)}(z)}{z^{N_1^{(s)}+\cdots+N_n^{(s)}} B_n^{(s)}(z)}, \end{aligned} \tag{4.4.2}$$

(B) for $s \leq 0$,

$$\begin{aligned} r_{-s,0}(z) &= \frac{z^{N_0^{(s)}} A_0^{(s)}(z)}{B_0^{(s)}(z)}, \\ r_{-s+N_1^{(s)}+\cdots+N_n^{(s)},\, N_1^{(s)}+\cdots+N_n^{(s)}}(z) &= \frac{z^{N_0^{(s)}+\cdots+N_n^{(s)}} A_n^{(s)}(z)}{z^{N_1^{(s)}+\cdots+N_n^{(s)}} B_n^{(s)}(z)}. \end{aligned} \tag{4.4.3}$$

Note that in (2.5.1) $N_{n+1}^{(0)}$ stands for the size $t+1$ of the block in the Padé table that contains $A_n^{(0)}/B_n^{(0)}$, while $\nu_n - N_{n+1}^{(0)} = 2\sum_{k=1}^{n} N_k^{(0)}$ is the sum of the numerator and denominator degrees of the Padé approximant. It is now easily shown that every distinct entry in the Padé table of $f(z)$ is one of the entries constructed in *Theorem* 4.4.1. So, while the entries in a normal Padé table are closely connected to regular C-fractions, the different entries in a non-normal Padé table are retrieved by the P-fractions (4.4.1).

4.5 Extension of the Padé table

Approximants of continued fractions of the form (4.3.2) which correspond to a single FPS, given at a point, are Padé approximants. Continued fractions which correspond simultaneously to FPS at two points, give rise to two-point Padé approximants, which we formally define here. We restrict ourselves to the points 0 and ∞ because the application of a bilinear transformation to the variable z leads to analogous results for expansions about two finite points.

Let the FPS $L_0(z)$ at 0 equal

$$L_0(z) = \Lambda_0(f(z)) = \sum_{j=0}^{\infty} c_j z^j, \qquad c_j \in \mathbb{C}, \qquad c_0 \neq 0 \tag{4.5.1}$$

while the FPS $L_\infty(z)$ at ∞ is given by

$$L_\infty(z) = \Lambda_\infty(f(z)) = -\sum_{j=1}^{\infty} c_{-j} z^{-j}, \qquad c_{-j} \in \mathbb{C}, \qquad c_{-1} \neq 0. \tag{4.5.2}$$

Clearly, to have any agreement of rational functions with (4.5.2), the numerator degree must be one less than the denominator degree. The *two-point Padé approximant* $r_{k,\ell}^{(2)}(z)$ is the unique irreducible form of the rational function $p(z)/q(z)$ satisfying

$$\begin{aligned} p(z) &= \sum_{j=0}^{m-1} a_j z^j, \\ q(z) &= \sum_{j=0}^{m} b_j z^j, \\ \lambda(L_0 q - p) &\geq k, \\ \lambda(L_\infty q - p) &\geq \ell + 1 - m, \\ k + \ell &= 2m. \end{aligned} \tag{4.5.3}$$

Hence the $2m+1$ coefficients of p/q are such that when $L_0 q - p$ is expanded as a FPS in z, there is agreement with the terms $c_j z^j$ up to and including $j = k-1$, and when $L_\infty q - p$ is expanded as a FPS in $1/z$ there is agreement with the terms $c_{-j} z^{-j}$ up to and including $j = \ell - m$. If $b_m \neq 0$, then $r_{k,\ell}^{(2)}(z)$ has order of correspondence k to $L_0(z)$ and order of correspondence $\ell + 1$ to $L_\infty(z)$.

4.6 Connection with M-fractions and the M-table

Let $L_0(z)$ and $L_\infty(z)$ be given by (4.5.1) and (4.5.2). Then coefficients $F_i^{(0)}$ and $G_i^{(0)}$ can be computed [MCM76] such that successive approximants of the M-fraction

$$M_0(z) = \frac{F_1^{(0)}}{1 + G_1^{(0)} z} + \mathop{\mathrm{K}}_{j=2}^{\infty} \left(\frac{F_j^{(0)} z}{1 + G_j^{(0)} z} \right)$$

fit equal numbers of terms of L_0 and L_∞. More generally, consider the expressions

$$M_s(z) = \sum_{j=0}^{s-1} c_j z^j + \frac{F_1^{(s)} z^s}{1 + G_1^{(s)} z} + \mathop{\mathrm{K}}_{j=2}^{\infty} \left(\frac{F_j^{(s)} z}{1 + G_j^{(s)} z} \right), \qquad s \geq 0, \tag{4.6.1}$$

$$M_{-s}(z) = -\sum_{j=1}^{s} c_{-j} z^{-j} + \frac{F_1^{(-s)} z^{-s}}{1 + d_1^{(-s)} z} + \mathop{\mathrm{K}}_{j=2}^{\infty} \left(\frac{F_j^{(-s)} z}{1 + G_j^{(-s)} z} \right), \qquad s > 0 \tag{4.6.2}$$

involving M-fractions. Denote by $M_{s,n}(z)$ the n^{th} approximants of the continued fractions (4.6.1) and (4.6.2) respectively, where $M_{0,0}(z) = 0$. The entries $M_{s,n}(z)$ are arranged in a table as

$$\begin{array}{ccccc}
\vdots & \vdots & \ldots & & \\
M_{-2,0}(z) & M_{-2,1}(z) & \ldots & & \\
M_{-1,0}(z) & M_{-1,1}(z) & M_{-1,2}(z) & \ldots & \\
M_{0,0}(z) & M_{0,1}(z) & M_{0,2}(z) & M_{0,3}(z) & \ldots \\
M_{1,0}(z) & M_{1,1}(z) & M_{1,2}(z) & \ldots & \\
M_{2,0}(z) & M_{2,1}(z) & M_{2,2}(z) & \ldots & \\
\vdots & \vdots & \ddots & &
\end{array}$$

THEOREM 4.6.1: [MC75]

Let $L_0(z)$ and $L_\infty(z)$ be given by (4.5.1) and (4.5.2) respectively. Under the conditions of Theorem 6.6.1, there exist M-fractions (4.6.1) and (4.6.2), such that the approximants $M_{s,n}(z)$ of these continued fractions satisfy the following properties. For $s \in \mathbb{Z}$ and $n > |s|$, $M_{s,n}(z)$ equals the two-point Padé approximant $r^{(2)}_{n+s,n-s}(z)$ of degree $n-1$ in the numerator and n in *the denominator. For $s \geq 0$ and $n \leq s$, the rational function $M_{s,n}(z)$ equals the Padé approximant $r_{s-1,n}(z)$ of (4.5.1).*

The table of rational functions $M_{s,n}(z)$ satisfying the properties stated in *Theorem* 4.6.1 is referred to as the M-table.

EXAMPLE 4.6.1: Consider

$$L_0(z) = 1 - \frac{z}{2!} + \frac{z^2}{3!} - \frac{z^3}{4!} + \dots,$$
$$L_\infty(z) = \frac{1}{z}.$$

For $s = 1$, the entries $M_{1,0}(z)$ and $M_{1,1}(z)$ equal the Padé approximants $r_{0,0}(z)$ and $r_{0,1}(z)$ of $L_0(z)$. For $s = 2$, we need the Padé approximants $r_{1,0}, r_{1,1}$ and $r_{1,2}$ of (4.5.1) to fill the positions $M_{2,0}, M_{2,1}$ and $M_{2,2}$. Further, the entries $M_{0,1}, M_{1,2}$ and $M_{2,3}$ are obtained as the first, second and third approximant of the respective M-fractions

$$M_0(z) = \frac{1}{1 + z} + \dots,$$
$$M_1(z) = 1 - \frac{z/2}{1 + z/2} - \frac{z/6}{1 + z/3} + \dots,$$
$$M_2(z) = 1 - \frac{z}{2} + \frac{z^2/6}{1 + z/3} - \frac{z/12}{1 + z/4} - \frac{z/10}{1 + z/5} + \dots,$$

of which the computation is fully detailed in *Chapter* 6. The approximants $M_{0,1}, M_{1,2}$ and $M_{2,3}$ respectively equal the two-point Padé approximants $r^{(2)}_{1,1}$, $r^{(2)}_{3,1}$ and $r^{(2)}_{5,1}$ for $L_0(z)$ and $L_\infty(z)$. So far the M-table, starting with the row $M_{0,n}$, looks like:

$$\begin{array}{cccccc}
 & \vdots & \vdots & \vdots & \iddots & \\
s=0 & 0 & \dfrac{1}{1+z} & \cdots & & \\
s=1 & 1 & \dfrac{2}{2+z} & \dfrac{6+z}{6+4z+z^2} & \cdots & \\
s=2 & 1-\dfrac{z}{2} & \dfrac{6-z}{6+2z} & \dfrac{12}{12+6z+z^2} & \dfrac{60+6z+z^2}{60+36z+9z^2+z^3} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

4.7 Convergence of Padé approximants

If the limit of a sequence of Padé approximants is to be at all useful, this limit has to be meromorphic in some substantial region of the complex plane. We consider two different cases:

- the convergence of sequences $\{r_{m,N}\}$ with N fixed,
- the convergence of *paradiagonal sequences* $\{r_{m+j,m}\}$ with $j \in \mathbb{Z}$ fixed and *ray sequences* $\{r_{m,n}\}$ with $m/n = K$ and $0 < K < \infty$.

Especially the paradiagonal sequences are closely connected with continued fractions, because of *Theorem* 4.3.1.

THEOREM 4.7.1: DE MONTESSUS DE BALLORE [dM05]

Let $f(z)$ be meromorphic in $B(0,r)$ with distinct poles z_j of total multiplicity N arranged in order of increasing modulus:

$$0 < |z_1| \leq \cdots \leq |z_k| < r \ .$$

Then the sequence $\{r_{m,N}\}$ converges uniformly to f on every closed and bounded subset of $B(0,r) \setminus \{z_1, \ldots, z_k\}$.

EXAMPLE 4.7.1: Let

$$f(z) = \left(\frac{1}{1-z^3} + \frac{1}{(2e^{\mathrm{i}\pi/4} - z)^2} + \frac{1}{2\mathrm{i} - z} \right) \exp(z),$$

which is a meromorphic function with 3 simple poles of modulus 1 and 3 poles of modulus 2, one simple and one double. From *Theorem* 4.7.1 we find that

$$\lim_{m\to\infty} r_{m,3}(z) = f(z), \qquad |z| < 2, z^3 \neq 1,$$
$$\lim_{m\to\infty} r_{m,6}(z) = f(z), \qquad z \notin \left\{1, e^{\mathrm{i}\frac{2\pi}{3}}, e^{\mathrm{i}\frac{4\pi}{3}}, 2e^{\mathrm{i}\frac{\pi}{4}}, 2\mathrm{i}\right\}.$$

We say that a sequence of Padé approximants $\{r_{m_k,n_k}\}$ *converges to $f(z)$ in measure* on $B(0,r)$ if, for given small positive ϵ and δ, there exists a K such that for $k \geq K$ and for all z in $B(0,r) \setminus D_k$, where D_k is a set of points of measure less than δ,

$$|f(z) - r_{m_k,n_k}(z)| < \epsilon, \qquad z \in B(0,r) \setminus D_k, \qquad k \geq K \ .$$

In the next theorem we denote the multiplicity of the pole z_j by μ_j.

THEOREM 4.7.2: ZINN-JUSTIN [ZJ71]

Let $f(z)$ be meromorphic in $B(0,r)$ with distinct poles z_j of total multiplicity N arranged in order of increasing modulus:

$$0 < |z_1| \le \cdots \le |z_k| < r \, .$$

For fixed $M > N$ the sequence $\{r_{m,M}\}$ converges to $f(z)$ in measure on $B(0,r)$.

One may wonder whether sequences $\{r_{m,N}\}$ with N fixed, of Padé approximants for a function $f(z)$ analytic in the disk $B(0,r)$ but not in any larger disk, converge in $B(0,r)$. The following counterexample illustrates that convergence in this case cannot be secured. *Theorem* 4.7.1 guarantees the convergence of the sequence $\{r_{m,0}\}$ since $N = 0$ in this case. By *Theorem* 4.7.2 only convergence in measure of $\{r_{m,N}\}$ is guaranteed for $N > 0$.

EXAMPLE 4.7.2: Let $f(z)$ be given by

$$f(z) = \frac{1+\sqrt[3]{2}z}{1-z^3},$$
$$\Lambda_0(f) = 1 + \sqrt[3]{2}z + z^3 + \sqrt[3]{2}z^4 + \ldots,$$

which is analytic in $|z| < 1$. Despite the analyticity of $f(z)$ in $B(0,1)$, every approximant $r_{m,2}(z)$ has a pole in $B(0,1)$ [BGS84]:

$$\begin{array}{llll} m = 3k: & q_{m,2}(z_{1,2}) = 0, & z_{1,2} = \left(\frac{1}{2} \pm \mathrm{i}\frac{\sqrt{3}}{2}\right)/\sqrt[3]{2}, & |z_{1,2}| < 1, \\ m = 3k+1: & q_{m,2}(z_1) = 0, & z_1 = -1/\sqrt[3]{2}, & |z_1| < 1, \\ m = 3k+2: & q_{m,2}(z_1) = 0, & z_1 = \dfrac{-\sqrt[3]{4} + \sqrt{6}\sqrt[6]{2}}{2}, & |z_1| < 1. \end{array}$$

Because Padé approximants are constructed from a FTS given at a point, the following question arises. Does there always exist a small neighbourhood of that point in which a sequence of Padé approximants converges? The answer is no, not even when $f(z)$ defined by the FTS (4.1.1) is an entire function.

EXAMPLE 4.7.3: Let $\{z_n\}$ be a sequence of complex points and let the FTS $f(z)$ be given by (4.1.1) with the c_j defined in triples by [Per29, p. 467]

$$|z_j| \leq 1 \Rightarrow \begin{cases} c_{3j} = z_j/(3j+2)! \\ c_{3j+1} = 1/(3j+2)! \\ c_{3j+2} = 1/(3j+2)! \end{cases}, \quad |z_j| > 1 \Rightarrow \begin{cases} c_{3j} = 1/(3j+2)! \\ c_{3j+1} = 1/(3j+2)! \\ c_{3j+2} = z_j^{-1}/(3j+2)! \end{cases}.$$

The FTS $f(z)$ defined in this way represents an entire function. And either $r_{3n,1}(z)$ or $r_{3n+1,1}(z)$ has a pole at $z = z_n$. Since the $\{z_n\}$ can be chosen dense in $\mathbb{C}$, the $\{r_{n,1}\}$ cannot converge in any open set in $\mathbb{C}$, however small.

For paradiagonals and rays in the Padé table the following results hold.

THEOREM 4.7.3: [Bak75, pp. 213–217]

The sequence $\{r_{m+j,m}\}$ with $j \geq -1$ of Padé approximants to a Stieltjes series analytic in $|z| < r$, converges uniformly to $f(z)$ on Δ_δ where Δ_δ is a bounded region of the complex plane which is at least at a distance δ from the cut $-\infty < z \leq 0$ along the negative real axis.

THEOREM 4.7.4: NUTTALL-POMMERENKE [Nut70; Pom73]

Let $f(z)$ be analytic at the origin and in the entire complex plane, except for at most a countable number of poles and isolated essential singularities. Then the sequence $\{r_{m,n}\}$ with $m/n = K$ and $0 < K < \infty$, converges in measure to $f(z)$ on any closed and bounded subset of the complex plane.

4.8 Formal orthogonality property

If we associate with (4.1.1) a linear functional c, defined on the space $\mathbb{C}[t]$ of polynomials in the variable t with complex coefficients, by

$$c(t^j) = c_j, \qquad j = 0, 1, \ldots,$$

then $f(z)$ can formally be viewed as

$$f(z) = \sum_{j=0}^{\infty} c(t^j) z^j = c\left(\frac{1}{1 - tz}\right).$$

For instance, when

$$c_j = \int_{-\infty}^{+\infty} t^j \, d\Phi(t)$$

then, at least formally, $f(z)$ is the integral transform

$$f(z) = \int_{-\infty}^{+\infty} \frac{d\Phi(t)}{1 - zt}.$$

This choice for c coincides with the inner product

$$c(g(t)) = \langle 1, g(t) \rangle_\Phi = \int_{-\infty}^{+\infty} g(t) \, d\Phi(t)$$

introduced in *Section* 5.4. When the polynomial $\tilde{q}(z)$ of degree m satisfies the orthogonality conditions

$$c\left(t^j \tilde{q}(t)\right) = 0, \qquad j = 0, \ldots, m-1 \tag{4.8.1}$$

and the polynomial $\tilde{p}(z)$ of degree $m-1$ is defined by

$$\tilde{p}(z) = c\left(\frac{\tilde{q}(t) - \tilde{q}(z)}{t - z}\right), \tag{4.8.2}$$

then for

$$\begin{aligned} p(z) &= z^{m-1}\tilde{p}(1/z), \\ q(z) &= z^m \tilde{q}(1/z), \end{aligned} \tag{4.8.3}$$

the conditions

$$\lambda(fq - p) \geq 2m$$

hold. In other words, the irreducible form of p/q computed from $\tilde{p}$ and $\tilde{q}$ satisfying (4.8.1) and (4.8.2), is the Padé approximant of order $(m-1, m)$ for $f(z)$ [Bre80, pp. 32–39]. The polynomial $\tilde{p}(z)$ in (4.8.2) is called the associated polynomial in *Section* 5.5, while the polynomial $\tilde{q}(z)$ is the orthogonal polynomial of degree m for the distribution function Φ.

EXAMPLE 4.8.1: For $c_j = \int_{-1}^{1} u^j \, du$, the orthogonality conditions (4.8.1) amount to

$$\int_{-1}^{1} \tilde{q}(t) t^j \, dt = 0, \qquad j < m, \tag{4.8.4}$$

where the left hand side of (4.8.4) equals the inner product of $\tilde{q}(t)$ and t^j as defined in (5.4.1). The polynomial $\tilde{q}(z)$ of degree m satisfying (4.8.4) is the Legendre polynomial of degree m for which further properties are given in (5.5.19a) and (5.5.19b). Thus the Legendre polynomials are the Padé denominators of $r_{m-1,m}(z)$ for the function

$$\sum_{j=0}^{\infty} \left(\int_{-1}^{1} t^j \, dt \right) z^j = \int_{-1}^{1} \frac{dt}{1-zt} = \frac{1}{z} \operatorname{Ln} \left(\frac{1+z}{1-z} \right) .$$

The construction of the Padé approximant of order $(m+k, m)$ with $k \geq -1$, follows the same lines. Note that the Padé approximants $r_{m+k,m}(z)$ are also the even-numbered entries on the descending staircase T_k given in (4.3.1). The formal power series (4.1.1) can be rewritten as

$$f(z) = \sum_{j=0}^{k} c_j z^j + z^{k+1} \tilde{f}(z)$$

with

$$\tilde{f}(z) = \sum_{j=0}^{\infty} c_{k+1+j} z^j .$$

If we define the functional $c^{(k+1)}$ by

$$c^{(k+1)}(t^j) = c_{k+1+j}$$

and the polynomials $\tilde{q}(z)$ and $\tilde{p}(z)$ by

$$\begin{aligned} & c^{(k+1)} \left(t^j \tilde{q}(t) \right) = 0, \qquad j = 0, \ldots, m-1, \\ & \tilde{p}(z) = c^{(k+1)} \left(\frac{\tilde{q}(t) - \tilde{q}(z)}{t-z} \right), \end{aligned} \tag{4.8.5}$$

then for $p(z)$ and $q(z)$ given by (4.8.3) the conditions

$$\lambda(\tilde{f} q - p) \geq 2m$$

are satisfied. The Padé approximant $r_{m+k,m}(z)$ is the irreducible form of

$$\sum_{j=0}^{k} c_j z^j + z^{k+1} \frac{p(z)}{q(z)} .$$

For instance, when

$$c^{(k+1)}(t^j) = \int_{-\infty}^{+\infty} t^{k+j+1}\, d\Phi(t)$$

the orthogonality conditions (4.8.5) result from multiplying the weight function $d\Phi(t)$ in (4.8.1) by t^{k+1}. Put another way, the denominators of the sequence $\{r_{m-1,m}\}_{m\in\mathbb{N}}$ of Padé approximants are orthogonal to each other with respect to the weight function $d\Phi(t)$ which produces the sequence $\{c_j\}_{j\in\mathbb{N}}$, and the denominators of the sequence $\{r_{m+k,m}\}_{m\in\mathbb{N}}$ for $k \geq -1$ are also orthogonal but now relative to the weight function $t^{k+1}\, d\Phi(t)$.

Further reading

- Simple proofs of several of the above properties are given in [CW87]. Additional information on Padé approximants can be found in the encyclopedic volume [BGM96].
- Generalisations of the notion of Padé approximant to matrix-valued and multivariate functions are extensively described in [XB90; Cuy99].

5

Moment theory and orthogonal functions

5.1 Moment theory

A function Φ is called a *distribution function* on an interval (a, b) where $-\infty \leq a < b \leq \infty$ if Φ is bounded and non-decreasing with infinitely many points of increase on (a, b). If Φ is a distribution function on (a, b), we say the k^{th} *moment* for Φ exists if the Riemann–Stieltjes integral

$$\int_a^b t^k \, d\Phi(t) \tag{5.1.1}$$

converges. In that case (5.1.1) is called the k^{th} moment for Φ.

EXAMPLE 5.1.1: It is straight forward to verify that the distribution function $\Phi(t) = t$ on the interval $(0, 1)$ generates the sequence of moments $\{1/(k+1)\}_{k=0}^{\infty}$.

A *moment problem* is to determine when a sequence of numbers is the sequence of moments for some distribution function. More specifically, a moment problem for a sequence $\{\mu_k\}_{k=0}^{\infty}$ or a *bisequence* $\{\mu_k\}_{k=-\infty}^{\infty}$ of real numbers is to find conditions on $\{\mu_k\}_{k=0}^{\infty}$ or $\{\mu_k\}_{k=-\infty}^{\infty}$ to ensure the existence of a distribution function Φ on (a, b) such that the k^{th} moment (5.1.1) for Φ exists for all $k \in \mathbb{N}_0$ or $k \in \mathbb{Z}$, respectively, and equals the k^{th} term in the sequence or bisequence. That is, for all $k \in \mathbb{N}_0$ or $k \in \mathbb{Z}$, as appropriate,

$$\mu_k = \int_a^b t^k \, d\Phi(t). \tag{5.1.2}$$

When $a = 0$ and $b = \infty$, the moment problem for the sequence $\{\mu_k\}_{k=0}^{\infty}$ of real numbers is called the *classical Stieltjes moment problem* (CSMP), and the moment problem for the bisequence $\{\mu_k\}_{k=-\infty}^{\infty}$ of real numbers is called the *strong Stieltjes moment problem* (SSMP).
When $a = -\infty$ and $b = \infty$, the moment problem for the sequence $\{\mu_k\}_{k=0}^{\infty}$ of real numbers is called the *classical Hamburger moment problem* (CHMP), and the moment problem for the bisequence $\{\mu_k\}_{k=-\infty}^{\infty}$ of real numbers is called the *strong Hamburger moment problem* (SHMP).
When $a = 0$ and $b = 1$, the moment problem for the sequence $\{\mu_k\}_{k=0}^{\infty}$ is called the *Hausdorff moment problem* (HDMP). When a and b are finite but different from 0 and 1 respectively, the moment problem is related to the Hausdorff moment problem.
A distribution function Φ satisfying (5.1.2) for all $k \in \mathbb{N}_0$ is called a solution to the classical moment problem for the sequence $\{\mu_k\}_{k=0}^{\infty}$ on (a, b). In that case Φ is called a classical moment distribution function on (a, b). If Φ is a classical moment distribution function on (a, b) and if Φ is absolutely continuous then we call

$$\phi(x) := \Phi'(x) \geq 0, \qquad x \in (a, b)$$

a *weight function* on (a, b).
A distribution function Φ satisfying (5.1.2) for all $k \in \mathbb{Z}$ is called a solution to the strong moment problem for the bisequence $\{\mu_k\}_{k=-\infty}^{\infty}$ on (a, b). In that case Φ is called a *strong moment distribution function* on (a, b).
The term strong is used to describe moment problems for bisequences since the requirements for a solution to a moment problem for a bisequence are stronger than the requirements for a solution to a moment problem for the associated sequence.
A distribution function Φ that solves the moment problem for $\{\mu_k\}_{k=0}^{\infty}$ on (a, b) is also a solution of the CHMP for the sequence $\{\mu_k\}_{k=0}^{\infty}$. It suffices to set $d\Phi(t) = 0$ for $-\infty < t < a$ and $b < t < \infty$. In addition, if a is nonnegative and b is positive, the distribution function Φ is also a solution of the CSMP for the sequence $\{\mu_k\}_{k=0}^{\infty}$.
One can also look at moment problems on the unit circle. The *trigonometric moment problem* (TMP) for a sequence $\{\mu_k\}_{k=0}^{\infty}$ of complex numbers is to find conditions on the sequence $\{\mu_k\}_{k=0}^{\infty}$ to ensure the existence of a distribution function function Φ on $(-\pi, \pi)$ such that

$$\mu_k = \int_{-\pi}^{\pi} e^{-\mathrm{i}k\theta}\, d\Phi(\theta), \qquad k \in \mathbb{N}_0. \tag{5.1.3}$$

A distribution function Φ satisfying (5.1.3) is called a solution to the TMP for $\{\mu_k\}_{k=0}^{\infty}$.

Two solutions of a moment problem are considered equivalent if their difference is a constant at the set of all points where the difference is continuous. A solvable moment problem is called *determinate* if all solutions to the moment problem are equivalent. If a moment problem is determinate, we call its solution unique. It is called *indeterminate* if there exist non-equivalent solutions. Note that if Φ_1 and Φ_2 are two distinct solutions of a moment problem, then

$$\Phi(t) := \alpha\Phi_1(t) + (1-\alpha)\Phi_2(t), \qquad 0 < \alpha < 1,$$

is also a solution and hence there exist infinitely many solutions.

Existence and uniqueness results for moment problems. Existence and uniqueness results for solutions to several moment problems can be expressed in terms of conditions on continued fractions. We present such results for the CSMP, HDMP, CHMP, SSMP and TMP. However, there is no known simple family of continued fractions that can be used to determine existence and uniqueness results for the SHMP.

THEOREM 5.1.1: EXISTENCE/UNIQUENESS OF SOLUTIONS OF CSMP [Sti95; LW92, p. 357]

The CSMP for a sequence $\{\mu_k\}_{k=0}^{\infty}$ of real numbers has a solution if and only if there exists a modified S-fraction of the form

$$\frac{a_1}{z} \, \underset{+}{} \, \frac{a_2}{1} \, \underset{+}{} \, \frac{a_3}{z} \, \underset{+}{} \, \frac{a_4}{1} \, \underset{+}{} \, \underset{\ldots}{}, \qquad a_m > 0, \quad m \in \mathbb{N}, \tag{5.1.4}$$

which corresponds to the FTS

$$L(z) = z^{-1}\sum_{k=0}^{\infty}(-1)^k \mu_k z^{-k} \tag{5.1.5}$$

at $z = \infty$ with order of correspondence $n+1$. The solution is unique if and only if (5.1.4) converges to a function $G(z)$ holomorphic in the cut plane $|\arg z| < \pi$.

The condition on the modified S-fraction in *Theorem* 5.1.1 can also be expressed in terms of the S-fraction $F(z)$ and modified S-fractions $H(z)$ and $D(z)$

$$F(z) = \frac{a_1 z}{1} \, \underset{+}{} \, \frac{a_2 z}{1} \, \underset{+}{} \, \frac{a_3 z}{1} \, \underset{+}{} \, \frac{a_4 z}{1} \, \underset{+}{} \, \underset{\ldots}{}, \qquad a_m > 0, \tag{5.1.6a}$$

$$H(z) = \frac{a_1}{1} \, \underset{+}{} \, \frac{a_2}{z} \, \underset{+}{} \, \frac{a_3}{1} \, \underset{+}{} \, \frac{a_4}{z} \, \underset{+}{} \, \underset{\ldots}{}, \qquad a_m > 0, \tag{5.1.6b}$$

$$D(z) = \frac{a_1}{z} \, \underset{+}{} \, \frac{a_2}{z} \, \underset{+}{} \, \frac{a_3}{z} \, \underset{+}{} \, \frac{a_4}{z} \, \underset{+}{} \, \underset{\ldots}{}, \qquad a_m > 0, \tag{5.1.6c}$$

introduced in *Section* 2.3, corresponding respectively to the FTS

$$z\sum_{k=0}^{\infty}(-1)^k\mu_k z^k, \tag{5.1.7a}$$

$$\sum_{k=0}^{\infty}(-1)^k\mu_k z^{-k}, \tag{5.1.7b}$$

$$z^{-1}\sum_{k=0}^{\infty}(-1)^k\mu_k z^{-2k}, \tag{5.1.7c}$$

with order of correspondence $n+1$ at $z=0$, n at $z=\infty$ and $2n+1$ at $z=\infty$, respectively.

Theorem 5.1.2: Existence/Uniqueness of solutions of HDMP [Wal48, p. 263]

The HDMP for a sequence $\{\mu_k\}_{k=0}^{\infty}$ of real numbers has a solution if and only if there exists a modified S-fraction of the form

$$\frac{\mu_0}{z}+\frac{(1-g_0)g_1}{1}+\frac{(1-g_1)g_2}{z}+\frac{(1-g_2)g_3}{1}+\dots, \qquad \mu_0>0,\quad 0\le g_m\le 1,$$

which corresponds to the FTS

$$L(z)=z^{-1}\sum_{k=0}^{\infty}(-1)^k\mu_k z^{-k}$$

at $z=\infty$ with order of correspondence $n+1$. The solution of a solvable HDMP is unique.

Theorem 5.1.3: Existence/Uniqueness of solutions of CHMP [Ham21]

The CHMP for a sequence $\{\mu_k\}_{k=0}^{\infty}$ of real numbers has a solution if and only if there exists a real J-fraction of the form

$$\frac{\alpha_1}{\beta_1+z}+\operatorname*{K}_{m=2}^{\infty}\left(\frac{-\alpha_m}{\beta_m+z}\right), \qquad \alpha_m>0,\quad \beta_m\in\mathbb{R},\quad m\in\mathbb{N}, \tag{5.1.8}$$

which corresponds to the FTS

$$L(z)=z^{-1}\sum_{k=0}^{\infty}(-1)^k\mu_k z^{-k}$$

at $z = \infty$ with order of correspondence $2n + 1$. The solution is unique if and only if the coefficients of the real J-fraction satisfy

$$\sum_{m=1}^{\infty} \frac{P_m^2(0)}{\alpha_1 \alpha_2 \cdots \alpha_{m+1}} = \infty \qquad \text{or} \qquad \sum_{m=1}^{\infty} \frac{Q_m^2(0)}{\alpha_1 \alpha_2 \cdots \alpha_{m+1}} = \infty \tag{5.1.9}$$

where $P_m(z)$ and $Q_m(z)$ denote the m^{th} numerator and the m^{th} denominator, respectively, of (5.1.8).

Note that convergence of the real J-fraction (5.1.8) in the cut complex plane $|\arg z| < \pi$ does not imply uniqueness of a solution of the CHMP. Also note [Per57, p. 234] that there are cases where the CSMP is determinate, but the CHMP for the same sequence is indeterminate.

Theorem 5.1.4: Existence/Uniqueness of solutions of SSMP [JTW80]

The SSMP for a bisequence $\{\mu_k\}_{k=-\infty}^{\infty}$ of real numbers has a solution if and only if there exists a positive T-fraction of the form

$$\operatorname*{K}_{m=1}^{\infty} \left(\frac{z}{e_m + d_m z} \right), \qquad e_m > 0, \quad d_m > 0, \quad m \in \mathbb{N}, \tag{5.1.10}$$

which corresponds to the pair of FTS

$$L_0(z) = -\sum_{k=1}^{\infty} (-1)^k \mu_{-k} z^k, \qquad L_\infty(z) = \sum_{k=0}^{\infty} (-1)^k \mu_k z^{-k}$$

at $z = 0$ and $z = \infty$ with orders of correspondence $n+1$ and n, respectively. The solution is unique if and only if (5.1.10) converges to a function $G(z)$ holomorphic in the cut plane $|\arg z| < \pi$, in which case the convergence is locally uniform.

A positive T-fraction of the form (5.1.10) converges if and only if the coefficients e_m and d_m satisfy

$$\sum_{m=1}^{\infty} (e_m + d_m) = \infty.$$

THEOREM 5.1.5: EXISTENCE/UNIQUENESS OF SOLUTIONS OF THE TMP [JNT89]

The TMP for a sequence $\{\mu_k\}_{k=0}^{\infty}$ *of complex numbers has a solution if and only if there exists a PPC-fraction of the form*

$$\delta_0 - \frac{2\delta_0}{1} + \frac{1}{\overline{\delta}_1 z} + \frac{(1-|\delta_1|^2)z}{\delta_1} + \frac{1}{\overline{\delta}_2 z} + \frac{(1-|\delta_2|^2)z}{\delta_2} + \dots,$$
$$\delta_0 > 0, \quad \delta_m \in \mathbb{C}, \quad |\delta_m| < 1, \quad m \in \mathbb{N}, \qquad (5.1.11)$$

which corresponds to the pair of FTS

$$L_0(z) = \mu_0 + 2\sum_{k=1}^{\infty} \mu_k z^k, \qquad L_\infty(z) = -\mu_0 - 2\sum_{k=1}^{\infty} \overline{\mu_k} z^{-k}$$

at $z = 0$ *and* $z = \infty$, *both with order of correspondence* $n+1$. *The solution of a solvable TMP is unique.*

The continued fraction occurring in each of the above theorems is said to *correspond* to the distribution function Φ determined by the sequence $\{\mu_k\}_{k=0}^{\infty}$ or $\{\mu_k\}_{k=-\infty}^{\infty}$.
Necessary and sufficient conditions for existence of solutions to the classical, strong, and trigonometric moment problems can be given in terms of Hankel determinants $H_k^{(m)}(\mu)$ associated with the sequence $\{\mu_k\}_{k=0}^{\infty}$ or bisequence $\{\mu_k\}_{k=-\infty}^{\infty}$, where

$$H_0^{(m)}(\mu) := 1, \qquad H_k^{(m)}(\mu) := \begin{vmatrix} \mu_m & \mu_{m+1} & \cdots & \mu_{m+k-1} \\ \mu_{m+1} & \mu_{m+2} & \cdots & \mu_{m+k} \\ \vdots & \vdots & & \vdots \\ \mu_{m+k-1} & \mu_{m+k} & \cdots & \mu_{m+2k-2} \end{vmatrix},$$
$$m \in \mathbb{Z}, \quad k \in \mathbb{N}. \quad (5.1.12)$$

If the Hankel determinant $H_k^{(m)}(\mu)$ is associated with the sequence $\{\mu_k\}_{k=0}^{\infty}$ and $m \in \mathbb{Z}^-$, then it is assumed that $\mu_i = 0$ for $i < 0$.

THEOREM 5.1.6: HANKEL DETERMINANT CONDITIONS [Sti95; Ham21; JTW80; JTN84; JNT86b; JNT83b]

(A) *The CSMP for a sequence* $\{\mu_k\}_{k=0}^{\infty}$ *of real numbers has a solution if and only if the Hankel determinants associated with* $\{\mu_k\}_{k=0}^{\infty}$ *satisfy*

$$H_n^{(0)}(\mu) > 0, \qquad H_n^{(1)}(\mu) > 0, \qquad n \in \mathbb{N}.$$

(B) The CHMP for a sequence $\{\mu_k\}_{k=0}^{\infty}$ *of real numbers has a solution if and only if the Hankel determinants associated with* $\{\mu_k\}_{k=0}^{\infty}$ *satisfy*

$$H_n^{(0)}(\mu) > 0, \qquad n \in \mathbb{N}.$$

(C) The SSMP for a bisequence $\{\mu_k\}_{k=-\infty}^{\infty}$ *of real numbers has a solution if and only if the Hankel determinants associated with* $\{\mu_k\}_{k=-\infty}^{\infty}$ *satisfy*

$$\begin{aligned} H_{2n}^{(-2n)}(\mu) > 0, &\qquad H_{2n+1}^{(-2n)}(\mu) > 0, \qquad n \in \mathbb{N}_0, \\ H_{2n}^{(-2n+1)}(\mu) > 0, &\qquad H_{2n+1}^{(-2n-1)}(\mu) > 0, \qquad n \in \mathbb{N}_0. \end{aligned}$$

(D) The SHMP for a bisequence $\{\mu_k\}_{k=-\infty}^{\infty}$ *of real numbers has a solution if and only if the Hankel determinants associated with* $\{\mu_k\}_{k=-\infty}^{\infty}$ *satisfy*

$$H_{2n}^{(-2n)}(\mu) > 0, \qquad H_{2n+1}^{(-2n)}(\mu) > 0, \qquad n \in \mathbb{N}_0.$$

(E) The TMP for a sequence $\{\mu_k\}_{k=0}^{\infty}$ *of complex numbers has a solution if and only if the Hankel determinants associated with* $\{\mu_k\}_{k=0}^{\infty}$ *satisfy*

$$(-1)^{n(n+1)/2} H_{n+1}^{(-n)}(\mu) > 0, \qquad n \in \mathbb{N},$$

where we define

$$\mu_{-k} := \overline{\mu_k}, \qquad k \in \mathbb{N}.$$

Observe that the Hankel determinant conditions (A) through (E) ensure the existence of continued fractions corresponding to FTS in the respective theorems 5.1.1 through 5.1.5. For (A) this is elaborated upon in *Section* 6.3 where conditions for the existence of a corresponding S-fraction are given for a FTS with a constant term:

$$\sum_{k=0}^{\infty} c_k z^k = c_0 + z \sum_{k=0}^{\infty} (-1)^k \mu_k z^k, \qquad c_k := (-1)^{k-1} \mu_{k-1}, \quad k \in \mathbb{N} \tag{5.1.13a}$$

or

$$\sum_{k=0}^{\infty} (-1)^k \gamma_k z^k = \gamma_0 + z \sum_{k=0}^{\infty} (-1)^{k-1} \mu_k z^k \qquad \gamma_k := \mu_{k-1}, \quad k \in \mathbb{N}. \tag{5.1.13b}$$

When the sequences $\{c_k\}_{k=1}^{\infty}$ and $\{\gamma_k\}_{k=1}^{\infty}$ are related to the sequence $\{\mu_k\}_{k=0}^{\infty}$ as in (5.1.13), then

$$\begin{aligned} H_k^{(2m+1)}(c) &= H_k^{(2m)}(\mu), \\ H_k^{(2m+2)}(c) &= (-1)^k H_k^{(2m+1)}(\mu), \qquad m \in \mathbb{N}_0, \quad k \in \mathbb{N}_0. \\ H_k^{(m+1)}(\gamma) &= H_k^{(m)}(\mu), \end{aligned} \tag{5.1.14}$$

The conditions in part (C) of *Theorem* 5.1.6 ensure the existence of a positive T-fraction corresponding to two power series at $z = 0$ and at $z = \infty$. In *Section* 6.7 these conditions are also given, there in terms of Hankel determinants for the sequence $\{\mu_k\}_{k=-\infty}^{\infty} = \{(-1)^k c_k\}_{k=-\infty}^{\infty}$. We have

$$\begin{aligned} H_k^{(2m)}(c) &= H_k^{(2m)}(\mu), \\ H_k^{(2m+1)}(c) &= (-1)^k H_k^{(2m+1)}(\mu), \end{aligned} \qquad m \in \mathbb{Z}, \quad k \geq 0. \tag{5.1.15}$$

Uniqueness results for classical and strong moment problems can be given in terms of conditions on the associated sequences of moments.

Theorem 5.1.7: Carleman criteria for moment problems [Car23; Wal48, p. 330; Car26; Wal48, p. 330; Ald87]

(A) If $\{\mu_k\}_{k=0}^{\infty}$ is a sequence of real numbers for which the CSMP has a solution, then this moment problem is determinate if

$$\sum_{k=1}^{\infty} \left(\frac{1}{\mu_k}\right)^{1/(2k)} = \infty. \tag{5.1.16a}$$

(B) If $\{\mu_k\}_{k=0}^{\infty}$ is a sequence of real numbers for which the CHMP has a solution, then this moment problem is determinate if

$$\sum_{k=1}^{\infty} \left(\frac{1}{\mu_{2k}}\right)^{1/(2k)} = \infty. \tag{5.1.16b}$$

(C) If $\{\mu_k\}_{k=-\infty}^{\infty}$ is a bisequence of real numbers for which the SSMP has a solution, then this moment problem is determinate if

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left(\frac{1}{\mu_k}\right)^{1/(2|k|)} = \infty. \tag{5.1.16c}$$

(D) If $\{\mu_k\}_{k=-\infty}^{\infty}$ is a bisequence of real numbers for which the SHMP has a solution, then this moment problem is determinate if

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left(\frac{1}{\mu_{2k}}\right)^{1/(2|k|)} = \infty. \tag{5.1.16d}$$

EXAMPLE 5.1.2: Let $\{\mu_k\}_{k=0}^{\infty}$ be defined by

$$\mu_k := k!, \qquad k \in \mathbb{N}_0.$$

Since

$$\int_0^{\infty} t^k e^{-t}\, dt = k!, \qquad k \in \mathbb{N}_0,$$

the CSMP for $\{k!\}_{k=0}^{\infty}$ has a solution,

$$\Phi(t) = -e^{-t}, \qquad 0 \le t < \infty.$$

By Carleman's criterion (5.1.16a), the CSMP is determinate since

$$\sum_{k=0}^{\infty} \left(\frac{1}{k!}\right)^{1/(2k)} = \infty.$$

THEOREM 5.1.8: [Wal48, p. 267]

The HDMP has a unique solution if and only if the sequence $\{\mu_k\}_{k=0}^{\infty}$ is a totally monotone sequence, meaning that

$$\mu_k \ge 0, \qquad \triangle\mu_k = \mu_k - \mu_{k+1} \ge 0, \qquad k = 0, 1, 2, \ldots$$
$$\triangle^n\mu_k = \triangle^{n-1}\mu_k - \triangle^{n-1}\mu_{k+1} \ge 0, \qquad \triangle^0\mu_k = \mu_k, \qquad k = 0, 1, 2, \ldots.$$

5.2 Stieltjes transforms

For certain moment problems, the continued fraction related to the moment problem can be represented by a Stieltjes integral transform.

THEOREM 5.2.1:

Let Φ be a classical moment distribution function on $(0, \infty)$ for the sequence $\{\mu_k\}_{k=0}^{\infty}$. Then:

(A) The Stieltjes integral transform

$$\int_0^\infty \frac{d\Phi(t)}{z+t} \tag{5.2.1}$$

is a holomorphic function in the cut plane $|\arg z| < \pi$.

(B) If the modified S-fraction (5.1.4) corresponding at $z = \infty$ to the FTS (5.1.5) is convergent, then its limit is represented by the Stieltjes integral transform

$$\frac{a_1}{z} \,\underset{+}{}\, \frac{a_2}{1} \,\underset{+}{}\, \frac{a_3}{z} \,\underset{+}{}\, \frac{a_4}{1} \,\underset{+}{}\, \ldots = \int_0^\infty \frac{d\Phi(t)}{z+t}, \qquad |\arg z| < \pi. \tag{5.2.2}$$

A truncation error bound for the n^{th} partial sum of (5.1.5) is given by

$$\left| \int_0^\infty \frac{d\Phi(t)}{z+t} - \sum_{k=0}^{n-1} (-1)^k \mu_k z^{-k-1} \right| \le \begin{cases} \mu_n |z|^{-n-1}, & |\arg z| \le \dfrac{\pi}{2}, \\[2mm] \dfrac{\mu_n |z|^{-n-1}}{|\sin(\arg z)|}, & \dfrac{\pi}{2} < |\arg z| < \pi, \end{cases} \qquad n \ge 1. \tag{5.2.3}$$

References for the results in *Theorem* 5.2.1 are [Cop62, pp. 110–115] for (A) and [Hen77, p. 617] for (B). A truncation error bound for the n^{th} approximant of (5.1.4) is given in *Theorem* 7.5.3.

If the modified S-fraction (5.1.4) is convergent, then it can be expressed as the Stieltjes integral transform (5.2.2). In that case we also have for the S-fractions (5.1.6) that

$$F(z) = \int_0^\infty \frac{z\, d\Phi(t)}{1+zt}, \qquad |\arg z| < \pi, \tag{5.2.4a}$$

$$H(z) = \int_0^\infty \frac{z\, d\Phi(t)}{z+t}, \qquad |\arg z| < \pi, \tag{5.2.4b}$$

$$D(z) = \int_0^\infty \frac{z\, d\Phi(t)}{z^2+t}, \qquad |\arg z| < \frac{\pi}{2} \tag{5.2.4c}$$

and

$$\left| \int_0^\infty \frac{z\, d\Phi(t)}{1+zt} - z \sum_{k=0}^{n-1} (-1)^k \mu_k z^k \right| \le \begin{cases} \mu_n |z|^{n+1}, & |\arg z| \le \dfrac{\pi}{2}, \\ \dfrac{\mu_n |z|^{n+1}}{|\sin(\arg z)|}, & \dfrac{\pi}{2} < |\arg z| < \pi, \end{cases} \quad n \ge 1, \tag{5.2.5a}$$

$$\left| \int_0^\infty \frac{z\, d\Phi(t)}{z+t} - \sum_{k=0}^{n-1} (-1)^k \mu_k z^{-k} \right| \le \begin{cases} \mu_n |z|^{-n}, & |\arg z| \le \dfrac{\pi}{2}, \\ \dfrac{\mu_n |z|^{-n}}{|\sin(\arg z)|}, & \dfrac{\pi}{2} < |\arg z| < \pi, \end{cases} \quad n \ge 1, \tag{5.2.5b}$$

$$\left| \int_0^\infty \frac{z\, d\Phi(t)}{z^2+t} - z^{-1} \sum_{k=0}^{n-1} (-1)^k \mu_k z^{-2k} \right| \le \mu_n |z|^{-2n-1}, \qquad |\arg z| < \frac{\pi}{2}, \quad n \ge 1. \tag{5.2.5c}$$

Example 5.2.1: Since

$$\int_0^\infty t^k e^{-t}\, dt = k!, \qquad k \in \mathbb{N}_0,$$

the CSMP for the sequence $\{k!\}_{k=0}^\infty$ has a solution Φ that satisfies $d\Phi(t) = e^{-t}\, dt$. The modified S-fraction corresponding to the FTS

$$L(z) = z^{-1} \sum_{k=0}^\infty (-1)^k k!\, z^{-k}$$

at $z = \infty$ with order of correspondence $n+1$ is given by

$$\frac{1}{z} \underset{+}{} \frac{1}{1} \underset{+}{} \frac{1}{z} \underset{+}{} \frac{2}{1} \underset{+}{} \frac{2}{z} \underset{+}{} \frac{3}{1} \underset{+}{} \frac{3}{z} \underset{+}{} \frac{4}{1} \underset{+}{} \frac{4}{z} \underset{+}{} \ldots, \qquad |\arg z| < \pi.$$

The coefficients of the continued fraction satisfy (3.1.5) and so the continued fraction converges by *Theorem* 3.1.5. It follows from *Theorem* 5.2.1 that its limit is represented by the Stieltjes integral transform

$$\frac{1}{z} \underset{+}{} \frac{1}{1} \underset{+}{} \frac{1}{z} \underset{+}{} \frac{2}{1} \underset{+}{} \frac{2}{z} \underset{+}{} \frac{3}{1} \underset{+}{} \frac{3}{z} \underset{+}{} \ldots = \int_0^\infty \frac{e^{-t}}{z+t}\, dt, \qquad |\arg z| < \pi.$$

According to *Theorem* 5.1.1 the solution to the CSMP for the sequence $\{k!\}_{k=0}^{\infty}$ is unique.

We include here a result valid for a particular family of distribution functions. For the purpose of defining this family, we consider real valued, even functions $Q(x)$ with $x \in \mathbb{R}$ and derivatives for $x > 0$ up to and including order 3. Moreover the following is required:

- there exist numbers $M > 0$ and $\epsilon > 0$ such that

$$|xQ'(x)| \leq M, \qquad 0 < x < \epsilon;$$

- there exist numbers $X > 0$ and $B > 0$ such that, for all $x > X$

$$Q'(x) > 0, \qquad \left|\frac{x^2 Q'''(x)}{Q'(x)}\right| \leq B;$$

- the limit

$$\lim_{x\to\infty} \frac{xQ''(x)}{Q'(x)}$$

  exists.

For $\alpha > 0, \delta > 0$ and $c > 0$ the class $\mathbb{Q}(\alpha, \delta, c)$ is defined as all $Q(x)$ for which

$$Q'(x) = cx^{\alpha-1} + O(x^{\alpha-\delta-1}), \qquad x \to \infty.$$

For $\alpha \geq 1$ the O-term may be replaced by $o(1)$.

Theorem 5.2.2: [JVA98; JS99]

Let $f(z)$ be defined by a Stieltjes transform

$$f(z) = \int_0^\infty \frac{z\phi(t)}{1+zt}\, dt, \qquad |\arg z| < \pi,$$

where $\phi(t)$ is a positive weight function on $(0, \infty)$ such that, for some $\alpha > 0$, $\delta > 0$ and $c > 0$, the function

$$Q(x) := -\operatorname{Ln}(|x|\phi(x^2)), \qquad x \in \mathbb{R}\backslash\{0\}$$

belongs to the class $\mathbb{Q}(\alpha, \delta, c)$. Then:
(A) The moments

$$\mu_k = \int_0^\infty t^k \phi(t)\, dt, \qquad k = 0, 1, 2, \ldots$$

exist.

(B) *The coefficients* a_m *of the S-fraction* $\mathrm{K}\,(a_m z/1)$ *corresponding to the sequence* $\{\mu_k\}_{k=0}^{\infty}$ *satisfy*

$$a_m \sim d m^{2/\alpha}, \qquad m \to \infty, \quad d = \frac{1}{4}\left(\frac{\alpha\sqrt{\pi}\,\Gamma(\frac{\alpha}{2})}{c\,\Gamma(\frac{\alpha+1}{2})}\right)^{2/\alpha}.$$

(C) *If* $\alpha \geq 1$*, then* $\mathrm{K}\,(a_m z/1)$ *is convergent and*

$$f(z) = \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{a_m z}{1}\right), \qquad |\arg z| < \pi.$$

Analogous statements can be made for modified S-fractions introduced in (2.3.3).

THEOREM 5.2.3:

Let Φ *be a classical moment distribution function for the sequence* $\{\mu_k\}_{k=0}^{\infty}$ *on* (a, b) *where* $-\infty \leq a < b \leq +\infty$*. Then:*

(A) *The Stieltjes transform*

$$\int_a^b \frac{d\Phi(t)}{z+t}$$

represents holomorphic functions $F^+(z)$ *in* $\{z \in \mathbb{C} : \Im z > 0\}$ *and* $F^-(z)$ *in* $\{z \in \mathbb{C} : \Im z < 0\}$*.*

(B) *If (5.1.8) is the real J-fraction corresponding to*

$$L(z) = z^{-1} \sum_{k=0}^{\infty} (-1)^k \mu_k z^{-k}$$

at $z = \infty$ *and if (5.1.9) holds, then the real J-fraction converges to the holomorphic function* $F^+(z)$ *for* $\Im z > 0$ *and to* $F^-(z)$ *for* $\Im z < 0$*.*

(C) *If (5.1.8) is the real J-fraction corresponding to* $L(z)$ *at* $z = \infty$ *and if* (a, b) *is a finite interval, then the real J-fraction converges to a function holomorphic in the region* $\mathbb{C} \setminus [-b, -a]$ *and*

$$\frac{\alpha_1}{\beta_1 + z} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{-\alpha_m}{\beta_m + z}\right) = \int_a^b \frac{d\Phi(t)}{z+t}, \qquad z \in \mathbb{C} \setminus [-b, -a].$$

References are [Wal48, p. 247] for (A), [Wal48, p. 114] for (B) and [Mar95] for (C). A truncation error bound for the n^{th} approximant $f_n(z)$ of the real J-fraction is given in *Theorem* 7.5.4.

Theorem 5.2.4: [JTW80]

Let Φ *be a strong moment distribution function for* $\{\mu_k\}_{k=-\infty}^{\infty}$ *on* (a, b) *where* $0 \leq a < b \leq +\infty$ *and let (5.1.10) be the positive T-fraction corresponding to the pair of FTS*

$$L_0(z) = -\sum_{k=1}^{\infty}(-1)^k \mu_{-k} z^k, \qquad L_\infty(z) = \sum_{k=0}^{\infty}(-1)^k \mu_k z^{-k}$$

at $z = 0$ *and* $z = \infty$*. If the continued fraction (5.1.10) converges, then its limit is represented by*

$$\mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{z}{e_m + d_m z}\right) = \int_a^b \frac{z\, d\Phi(t)}{z+t}, \qquad z \in \mathbb{C} \setminus [-b, -a],$$

the convergence being locally uniform on $\mathbb{C} \setminus [-b, -a]$.

A truncation error bound for the n^{th} approximant f_n of the positive T-fraction is given in *Theorem* 7.5.5. Some examples to illustrate *Theorem* 5.2.4 are given in [JNT83a].

5.3 Construction of solutions

One technique that may be utilised to construct a solution to a solvable moment problem uses approximants of the corresponding continued fraction. We do not discuss this technique in general but outline it here only for the SSMP. Suppose the SSMP for a bisequence $\{\mu_k\}_{k=-\infty}^{\infty}$ has a solution. Let $A_n(z)$ and $B_n(z)$ denote the n^{th} numerator and denominator of the corresponding positive T-fraction (5.1.10). Then $A_n(z)$ and $B_n(z)$ are polynomials in z of degree n and the zeros $r_m^{(n)}$ of $B_n(z)$ are all distinct and negative and can be arranged in order such that

$$0 < -r_1^{(n)} < -r_2^{(n)} < \cdots < -r_n^{(n)}.$$

The n^{th} approximant $A_n(z)/B_n(z)$ has the partial fraction decomposition

$$\frac{A_n(z)}{B_n(z)} = \sum_{m=1}^{n} \frac{z p_m^{(n)}}{z - r_m^{(n)}}, \qquad n \in \mathbb{N},$$

where $p_m^{(n)} > 0$ for $m \geq 1$ and $\sum_{m=1}^{n} p_m^{(n)} = 1/d_1$ where d_1 is a coefficient of the positive T-fraction (5.1.10). Define

$$\Phi_n(t) := \begin{cases} 0, & 0 \leq t \leq -r_1^{(n)}, \\ \displaystyle\sum_{m=1}^{k} p_m^{(n)}, & -r_k^{(n)} < t \leq -r_{k+1}^{(n)}, \quad 1 \leq k \leq n, \\ \dfrac{1}{d_1}, & -r_n^{(n)} < t < \infty. \end{cases}$$

Then

$$\frac{A_n(z)}{B_n(z)} = \int_0^\infty \frac{z\,d\Phi_n(t)}{z+t}, \qquad n \in \mathbb{N}. \tag{5.3.1}$$

If the positive T-fraction (5.1.10) converges, then (5.3.1) converges to

$$G(z) = \int_0^\infty \frac{z\,d\Phi(t)}{z+t} \tag{5.3.2}$$

where $\Phi(t)$ is the unique solution of the SSMP. The solution $\Phi(t)$ can be determined from (5.3.2) by using the Stieltjes inversion formula [Chi78, p. 90]

$$\Phi(t) - \Phi(s) = -\frac{1}{\pi} \lim_{y \to 0^+} \int_s^t \Im(G(x + \mathrm{i}y))\,dx.$$

If the positive T-fraction does not converge, then the SSMP has infinitely many solutions. Using the fact that the even and odd parts of the positive T-fraction converge, the above procedure can be applied to $A_{2n}(z)/B_{2n}(z)$ and to $A_{2n+1}(z)/B_{2n+1}(z)$ separately to find two, and hence infinitely many, solutions.

5.4 Orthogonal polynomials

The origins of the field of orthogonal polynomials can be found in the theory of continued fractions [Tch58; Sti95]. It is the purpose of this and the next section to indicate some of the connections between orthogonal polynomials and continued fractions.

Let Φ be a classical moment distribution function on an interval (a, b) where $-\infty \leq a < b \leq +\infty$. Then an *inner product* $\langle \cdot, \cdot \rangle_\Phi$ over the space $\mathbb{R}[x]$ of real polynomials is defined by

$$\langle f, g \rangle_\Phi := \int_a^b f(x)g(x)\,d\Phi(x), \qquad f, g \in \mathbb{R}[x]. \tag{5.4.1}$$

The norm of $R \in \mathbb{R}[x]$ is given by

$$\|R\|_\Phi := (\langle R, R \rangle_\Phi)^{1/2}.$$

A sequence of real polynomials $\{R_n(x)\}_{n=0}^\infty$ is called an *orthogonal polynomial* sequence for Φ if, for $m, n \in \mathbb{N}_0$,

$$\begin{aligned} \partial R_n &= n, \\ \langle R_m, R_n \rangle_\Phi &= 0, \qquad m \neq n, \\ \langle R_n, R_n \rangle_\Phi &= \|R_n\|_\Phi^2 > 0. \end{aligned}$$

Using the notation

$$R_n(x) = k_{n,n}x^n + k_{n,n-1}x^{n-1} + \cdots + k_{n,0}, \qquad k_{n,n} \neq 0, \quad n \in \mathbb{N}_0,$$

an orthogonal polynomial sequence $\{R_n(x)\}_{n=0}^{\infty}$ satisfies a recurrence relation of the form

$$R_{-1}(x) = 0, \qquad R_0(x) = k_{0,0} > 0, \tag{5.4.2a}$$

$$R_{n+1}(x) = (b_n + c_n x)R_n(x) - a_n R_{n-1}(x), \qquad n \in \mathbb{N}_0, \tag{5.4.2b}$$

where the connection between the coefficients a_n, b_n, c_n and $k_{n,i}$ is given by

$$c_0 = \frac{k_{1,1}}{k_{0,0}}, \quad b_0 = \frac{k_{1,0}}{k_{0,0}}, \quad a_0 = 0,$$

$$c_n = \frac{k_{n+1,n+1}}{k_{n,n}}, \quad b_n = c_n\left(\frac{k_{n+1,n}}{k_{n+1,n+1}} - \frac{k_{n,n-1}}{k_{n,n}}\right), \quad n \in \mathbb{N},$$

$$a_n = \frac{k_{n+1,n+1}k_{n-1,n-1}\|R_n\|_\Phi^2}{k_{n,n}^2\|R_{n-1}\|_\Phi^2}, \quad n \in \mathbb{N}.$$

5.5 Monic orthogonal polynomials on $\mathbb{R}$ and J-fractions

An orthogonal polynomial sequence $\{Q_n(x)\}_{n=0}^{\infty}$ is called *monic* if each polynomial $Q_n(x)$ in the sequence has leading coefficient 1. Given an orthogonal polynomial sequence $\{R_n(x)\}_{n=0}^{\infty}$, a related monic orthogonal polynomial sequence $\{Q_n(x)\}_{n=0}^{\infty}$ can be constructed by setting

$$Q_n(x) = \left(\frac{1}{k_{n,n}}\right)R_n(x), \qquad n \in \mathbb{N}_0.$$

From the following two theorems it is seen that every monic orthogonal polynomial sequence on the real line is the sequence of denominators of a real J-fraction (5.1.8). Conversely, the sequence of denominators of any real J-fraction is a monic orthogonal polynomial sequence for some distribution function Φ on $(-\infty, \infty)$.

THEOREM 5.5.1: [Chi78, pp. 85–86]

Let $\{Q_n(x)\}_{n=0}^{\infty}$ be a monic orthogonal polynomial sequence with respect to a classical moment distribution function Φ on (a, b). Then $\{Q_n(x)\}_{n=0}^{\infty}$ is the sequence of denominators of the real J-fraction

$$\frac{\alpha_1}{\beta_1 + x} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{-\alpha_m}{\beta_m + x}\right) \tag{5.5.1}$$

with coefficients

$$\alpha_1 = 1, \quad \beta_1 = k_{1,0},$$

$$\alpha_n = \frac{\|Q_{n-1}\|_\Phi^2}{\|Q_{n-2}\|_\Phi^2} > 0, \quad \beta_n = k_{n,n-1} - k_{n-1,n-2} \in \mathbb{R}, \quad n \geq 2,$$

where $k_{m,m-1}$ *is the coefficient of* x^{m-1} *for* $Q_m(x)$.

Theorem 5.5.2: [JT80, pp. 252–253]

Let $Q_n(x)$ *denote the* n^{th} *denominator of the real J-fraction (5.5.1) and let* Φ *denote a corresponding classical moment distribution function. Then* $\{Q_n(x)\}_{n=0}^{\infty}$ *is the monic orthogonal polynomial sequence for* Φ.

Since the orthogonal polynomials $Q_n(x)$ in *Theorem* 5.5.2 are the denominators of the real J-fraction (5.5.1) they can be constructed using the basic recurrence relations (1.3.1)

$$\begin{gathered} Q_{-1}(x) = 0, \qquad Q_0(x) = 1, \\ Q_n(x) = (\beta_n + x)Q_{n-1}(x) - \alpha_n Q_{n-2}(x), \qquad n \in \mathbb{N}, \end{gathered} \tag{5.5.2}$$

where the α_n and β_n are the coefficients of the real J-fraction (5.5.1). We remark that *Theorem* 5.5.2 follows from a more general theorem called *Favard's theorem.*

Theorem 5.5.3: Favard [Fav35]

Let $\{\alpha_m\}_{m=1}^{\infty}$ *and* $\{\beta_m\}_{m=1}^{\infty}$ *be any sequences that satisfy*

$$\alpha_m > 0, \qquad \beta_m \in \mathbb{R}, \qquad m \in \mathbb{N}, \tag{5.5.3}$$

and let $\{Q_n(x)\}_{n=0}^{\infty}$ *be defined by the three-term recurrence relations (5.5.2). Then there exists a classical moment distribution function* Φ *on* (a, b) *such that* $\{Q_n(x)\}_{n=0}^{\infty}$ *is the monic orthogonal polynomial sequence for* Φ.

The polynomials $Q_n(x)$ in *Theorem* 5.5.2 can also be represented by the determinant formulas

$$Q_0(x) = 1,$$

$$Q_n(x) = \frac{1}{H_n^{(0)}(\mu)} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}, \qquad n \in \mathbb{N},$$

where the μ_k are the moments for Φ given in (5.1.2) and the $H_n^{(0)}(\mu)$ are Hankel determinants (5.1.12) associated with the sequence $\{\mu_k\}_{k=0}^\infty$.
The sequence of numerators of a real J-fraction can also be used to define a monic orthogonal polynomial sequence. Let $P_n(x)$ denote the n^{th} numerator of the real J-fraction (5.5.1) corresponding to a moment distribution function Φ. Let $\{\tilde{P}_n(x)\}_{n=0}^\infty$ be defined by

$$\tilde{P}_n(x) := \alpha_1^{-1} P_{n+1}(x), \qquad n \geq -1.$$

Then the sequence $\{\tilde{P}_n(x)\}_{n=0}^\infty$ satisfies

$$\tilde{P}_{-1}(x) = 0, \qquad \tilde{P}_0(x) = 1,$$

$$\tilde{P}_n(x) = (\beta_{n+1} + x)\tilde{P}_{n-1}(x) - \alpha_{n+1}\tilde{P}_{n-2}(x), \qquad n \geq 1,$$

and the coefficients β_{n+1} and α_{n+1} of the J-fraction satisfy (5.5.3). Hence by *Theorem* 5.5.3 there exists a moment distribution function Ψ such that $\{\tilde{P}_n(x)\}_{n=0}^\infty$ is the monic orthogonal polynomial sequence for Ψ.
If Φ is the classical moment distribution function corresponding to a real J-fraction, the polynomial numerators $P_n(x)$ of the real J-fraction can be expressed in terms of the polynomial denominators $Q_n(x)$ by

$$P_n(x) = \int_a^b \frac{Q_n(x) - Q_n(t)}{x - t}\, d\Phi(t), \qquad n \geq 0.$$

The numerator polynomials are often called the *associated polynomials*.
Next we deal with certain properties of so-called classical orthogonal polynomial sequences, named after Hermite, Laguerre and Jacobi, and their connection to a special family of continued fractions, the J-fractions. Special cases of Jacobi polynomials are Legendre, Chebyshev and Gegenbauer polynomials.
Let $\{R_n(x)\}_{n=0}^\infty$ be an orthogonal polynomial sequence for a classical moment distribution function Φ on the interval (a, b). If Φ is absolutely continuous, then $\phi(x) = \Phi'(x)$ is a weight function and we say that $\{R_n(x)\}_{n=0}^\infty$ is an orthogonal polynomial sequence for the weight function $\phi(x)$ on (a, b).

Hermite polynomials. The sequence $\{H_n(x)\}_{n=0}^\infty$ of Hermite polynomials is an orthogonal polynomial sequence for the weight function $\phi(x) = e^{-x^2}$ on the interval $(-\infty, \infty)$.
The Hermite polynomials satisfy the three-term recurrence relations

$$H_{-1}(x) = 0, \qquad H_0(x) = 1, \tag{5.5.4a}$$

$$H_n(x) = 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x), \qquad n \in \mathbb{N}. \tag{5.5.4b}$$

An explicit formula is

$$H_n(x) = n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^m \frac{(2x)^{n-2m}}{m!\,(n-2m)!}, \qquad n \in \mathbb{N}_0. \tag{5.5.5}$$

The monic Hermite polynomials $\hat{H}_n(x)$ given by

$$\hat{H}_n(x) = 2^{-n} H_n(x), \qquad n \in \mathbb{N}_0, \tag{5.5.6}$$

satisfy the recurrence relations

$$\hat{H}_{-1}(x) = 0, \qquad \hat{H}_0(x) = 1, \tag{5.5.7a}$$

$$\hat{H}_n(x) = x\hat{H}_{n-1}(x) - \frac{n-1}{2}\hat{H}_{n-2}(x), \qquad n \in \mathbb{N}. \tag{5.5.7b}$$

The monic orthogonal Hermite polynomial sequence $\{\hat{H}_n(x)\}_{n=1}^{\infty}$ forms the sequence of denominators of the real J-fraction

$$\frac{1}{x} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-(m-1)/2}{x} \right). \tag{5.5.8}$$

Laguerre polynomials. For fixed $\alpha > -1$, the sequence $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ of generalised Laguerre polynomials is an orthogonal polynomial sequence for the weight function $\phi(x) = e^{-x}x^{\alpha}$ on the interval $[0, \infty)$.
The generalised Laguerre polynomials satisfy the three-term recurrence relations

$$L_{-1}^{(\alpha)}(x) = 0, \qquad L_0^{(\alpha)}(x) = 1, \tag{5.5.9a}$$

$$L_n^{(\alpha)}(x) = \frac{(2n+\alpha-1-x)}{n} L_{n-1}^{(\alpha)}(x) - \frac{(n+\alpha-1)}{n} L_{n-2}^{(\alpha)}(x), \qquad n \in \mathbb{N}. \tag{5.5.9b}$$

An explicit formula is

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n} (-1)^m \binom{n+\alpha}{n-m} \frac{x^m}{m!}, \qquad n \in \mathbb{N}_0. \tag{5.5.10}$$

The monic generalised Laguerre polynomials $\hat{L}_n^{(\alpha)}(x)$ given by the formula

$$\hat{L}_n^{(\alpha)}(x) = (-1)^n n!\, L_n^{(\alpha)}(x), \qquad n \in \mathbb{N}_0, \tag{5.5.11}$$

satisfy the recurrence relations

$$\hat{L}_{-1}^{(\alpha)}(x) = 0, \qquad \hat{L}_{0}^{(\alpha)}(x) = 1, \tag{5.5.12a}$$

$$\hat{L}_{n}^{(\alpha)}(x) = (1 - 2n - \alpha + x)\hat{L}_{n-1}^{(\alpha)}(x) - (n-1)(n+\alpha-1)\hat{L}_{n-2}^{(\alpha)}(x), \quad n \in \mathbb{N}. \tag{5.5.12b}$$

The monic orthogonal Laguerre polynomial sequence $\{\hat{L}_n^{(\alpha)}(x)\}_{n=1}^{\infty}$ forms the sequence of denominators of the real J-fraction

$$\frac{1}{-1-\alpha+x} + \operatorname*{K}_{m=2}^{\infty}\left(\frac{-(m-1)(m-1+\alpha)}{1-2m-\alpha+x}\right). \tag{5.5.13}$$

Jacobi polynomials. For fixed $\alpha > -1$ and $\beta > -1$, the sequence $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ of Jacobi polynomials is an orthogonal polynomial sequence for the weight function $\phi(x) = (1-x)^{\alpha}(1+x)^{\beta}$ on the interval $[-1, 1]$. The Jacobi polynomials satisfy the three-term recurrence relations

$$P_{-1}^{(\alpha,\beta)}(x) = 0, \qquad P_{0}^{(\alpha,\beta)}(x) = 1, \tag{5.5.14a}$$

$$P_{n}^{(\alpha,\beta)}(x) = (b_n + c_n x)P_{n-1}^{(\alpha,\beta)}(x) - a_n P_{n-2}^{(\alpha,\beta)}(x), \qquad n \geq 1, \tag{5.5.14b}$$

where

$$b_n = \frac{(\alpha^2-\beta^2)(2n+\alpha+\beta-1)}{2n(n+\alpha+\beta)(2n+\alpha+\beta-2)}, \qquad n \geq 1, \tag{5.5.14c}$$

$$c_n = \frac{(2n+\alpha+\beta)(2n+\alpha+\beta-1)}{2n(n+\alpha+\beta)}, \qquad n \geq 1, \tag{5.5.14d}$$

$$a_1 = 1, \qquad a_n = \frac{(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)}{n(n+\alpha+\beta)(2n+\alpha+\beta-2)}, \qquad n \geq 2. \tag{5.5.14e}$$

An explicit formula is

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n}\sum_{m=0}^{n}\binom{n+\alpha}{m}\binom{n+\beta}{n-m}(x-1)^{n-m}(x+1)^m, \qquad n \in \mathbb{N}_0. \tag{5.5.15}$$

The monic Jacobi polynomials $\hat{P}_n^{(\alpha,\beta)}(x)$ given by

$$\hat{P}_n^{(\alpha,\beta)}(x) = \frac{2^n}{\binom{2n+\alpha+\beta}{n}}P_n^{(\alpha,\beta)}(x), \qquad n \in \mathbb{N}_0, \tag{5.5.16}$$

satisfy the recurrence relations

$$\hat{P}_{-1}^{(\alpha,\beta)}(x) = 0, \qquad \hat{P}_{0}^{(\alpha,\beta)}(x) = 1, \tag{5.5.17a}$$

$$\hat{P}_{n}^{(\alpha,\beta)}(x) = (\beta_n + x)\hat{P}_{n-1}^{(\alpha,\beta)}(x) - \alpha_n \hat{P}_{n-2}^{(\alpha,\beta)}(x), \qquad n \in \mathbb{N}, \tag{5.5.17b}$$

where

$$\beta_n = \frac{\alpha^2 - \beta^2}{(2n+\alpha+\beta-2)(2n+\alpha+\beta)}, \qquad n \in \mathbb{N}, \tag{5.5.17c}$$

$$\alpha_1 = 1, \qquad \alpha_2 = \frac{4(\alpha+1)(\beta+1)}{(\alpha+\beta+2)^2(\alpha+\beta+3)}, \tag{5.5.17d}$$

$$\alpha_n = \frac{4(n-1)(n+\alpha-1)(n+\beta-1)(n+\alpha+\beta-1)}{(2n+\alpha+\beta-2)^2(2n+\alpha+\beta-1)(2n+\alpha+\beta-3)}, \qquad n \geq 3, \tag{5.5.17e}$$

except that when $\alpha = -\beta$ we have $\beta_1 = (\alpha - \beta)/(\alpha + \beta + 2)$.
The monic Jacobi polynomial sequence $\{\hat{P}_n^{(\alpha,\beta)}(x)\}_{n=1}^{\infty}$ forms the sequence of denominators of the real J-fraction

$$\frac{1}{\beta_1 + x} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{-\alpha_m}{\beta_m + x} \right), \tag{5.5.18}$$

where the α_n and β_n are given by (5.5.17).

Legendre polynomials. In the case $\alpha = \beta = 0$, the Jacobi polynomials $P_n^{(0,0)}(x)$ are called the Legendre polynomials and are denoted $P_n(x)$. The monic Legendre polynomials $\hat{P}_n(x)$ are orthogonal on the interval $[-1, 1]$ with respect to the weight function $\phi(x) = 1$. They satisfy the three term recurrence relations

$$\hat{P}_{-1}(x) = 0, \qquad \hat{P}_0(x) = 1, \tag{5.5.19a}$$

$$\hat{P}_n(x) = x\hat{P}_{n-1}(x) - \frac{(n-1)^2}{(2n-1)(2n-3)}\hat{P}_{n-2}(x), \qquad n \in \mathbb{N}, \tag{5.5.19b}$$

and $\{\hat{P}_n(x)\}_{n=1}^{\infty}$ forms the sequence of denominators of the real J-fraction

$$\frac{1}{x} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{-\frac{(m-1)^2}{(2m-1)(2m-3)}}{x} \right). \tag{5.5.20}$$

Chebyshev polynomials of the first kind. In the case $\alpha = \beta = -1/2$, the Jacobi polynomials $P_n^{(-1/2,-1/2)}(x)$ are called the Chebyshev polynomials of the first kind and are denoted $T_n(x)$. The monic Chebyshev polynomials of the first kind $\hat{T}_n(x)$ are orthogonal on the interval $[-1, 1]$ with respect to the weight function $\phi(x) = (1 - x^2)^{-1/2}$. They satisfy the three term recurrence relations

$$\hat{T}_{-1}(x) = 0, \qquad \hat{T}_0(x) = 1, \tag{5.5.21a}$$

$$\hat{T}_n(x) = x\hat{T}_{n-1}(x) - \alpha_n \hat{T}_{n-2}(x), \qquad n \in \mathbb{N}, \tag{5.5.21b}$$

where

$$\alpha_1 = 1, \quad \alpha_2 = \frac{1}{2}, \quad \alpha_n = \frac{1}{4}, \quad n \geq 3, \tag{5.5.21c}$$

and form the sequence of denominators of the real J-fraction

$$\frac{1}{x+} \, \frac{-1/2}{x} + \operatorname*{K}_{m=1}^{\infty} \left(\frac{-1/4}{x} \right). \tag{5.5.22}$$

Chebyshev polynomials of the second kind. In the case $\alpha = \beta = 1/2$, the Jacobi polynomials $P_n^{(1/2,1/2)}(x)$ are called the Chebyshev polynomials of the second kind and are denoted $U_n(x)$. The monic Chebyshev polynomials of the second kind $\hat{U}_n(x)$ are orthogonal on the interval $[-1, 1]$ with respect to the weight function $\phi(x) = (1 - x^2)^{1/2}$. They satisfy the three term recurrence relations

$$\hat{U}_{-1}(x) = 0, \qquad \hat{U}_0(x) = 1, \tag{5.5.23a}$$

$$\hat{U}_n(x) = x\hat{U}_{n-1}(x) - \alpha_n \hat{U}_{n-2}(x), \qquad n \in \mathbb{N}, \tag{5.5.23b}$$

where

$$\alpha_1 = 1, \quad \alpha_n = \frac{1}{4}, \quad n \geq 2, \tag{5.5.23c}$$

and form the sequence of denominators of the real J-fraction

$$\frac{1}{x+} \operatorname*{K}_{m=1}^{\infty} \left(\frac{-1/4}{x} \right). \tag{5.5.24}$$

Ultraspherical or Gegenbauer polynomials. In the case $\beta = \alpha$, the Jacobi polynomials $P_n^{(\alpha,\alpha)}(x)$ are called the ultraspherical polynomials or Gegenbauer polynomials and are denoted $C_n^{(\alpha)}(x)$. The monic Gegenbauer polynomials $\hat{C}_n^{(\alpha)}(x)$ are orthogonal on the interval $[-1, 1]$ with respect to the weight function $\phi(x) = (1-x^2)^\alpha$. They satisfy the three term recurrence relations

$$\hat{C}_{-1}^{(\alpha)}(x) = 0, \qquad \hat{C}_0^{(\alpha)}(x) = 1, \tag{5.5.25a}$$

$$\hat{C}_n^{(\alpha)}(x) = x\hat{C}_{n-1}^{(\alpha)}(x) - \frac{(n-1)(n+2\alpha-2)}{4(\alpha+n-1)(\alpha+n-2)}\hat{C}_{n-2}^{(\alpha)}(x), \qquad n \in \mathbb{N}, \tag{5.5.25b}$$

and form the sequence of denominators of the real J-fraction

$$\frac{1}{x} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{-\frac{(m-1)(m+2\alpha-2)}{4(\alpha+m-1)(\alpha+m-2)}}{x}\right). \tag{5.5.26}$$

Gaussian quadrature. The Gaussian quadrature formula described in the next theorem provides an efficient method for the numerical approximation of integrals. Choosing the n zeros of the n^{th} denominator of a real J-fraction as the nodes in the quadrature formula results in a greater degree of exactness than for other choices of the nodes.

THEOREM 5.5.4: GAUSSIAN QUADRATURE

Let Φ be a classical moment distribution function on (a,b) and let $P_n(z)$ and $Q_n(z)$ denote the n^{th} numerator and denominator, respectively, of the real J-fraction (5.1.8) corresponding to Φ. Then:

(A) The n zeros $x_k^{(n)}$, $1 \le k \le n$, of $Q_n(z)$ are real, simple and contained in the interval (a,b).

(B) The error term $E_n(f)$ in the quadrature formula

$$\int_a^b f(x)\,d\Phi(x) = \sum_{k=1}^{n} \lambda_k^{(n)} f(x_k^{(n)}) + E_n(f), \tag{5.5.27a}$$

satisfies

$$E_n(f) = 0, \qquad f \in \mathbb{R}[x], \qquad \partial f \le 2n-1, \tag{5.5.27b}$$

where the constants $\lambda_k^{(n)}$ are called the Christoffel numbers and are given by

$$\lambda_k^{(n)} := \frac{P_n(x_k^{(n)})}{Q_n'(x_k^{(n)})} > 0, \qquad 1 \le k \le n, \tag{5.5.27c}$$

and

$$\sum_{k=1}^{n} \lambda_k^{(n)} = \mu_0 = \int_a^b d\Phi(x). \tag{5.5.27d}$$

The number $2n-1$ in (5.5.27b), called the degree of exactness, is the best result possible with n nodes. Numerically stable algorithms for computing the zeros $x_k^{(n)}$, $1 \le k \le n$, are given in [SD72; Gau81; GW69].

5.6 Szegő polynomials and PPC-fractions

Szegő polynomials arise as the denominators of PPC-fractions (5.1.11) and are closely related to the trigonometric moment problem discussed in *Section* 5.1. Let Φ be a distribution function on $(-\pi, \pi)$. Then an inner product $\langle \cdot, \cdot \rangle_\Phi$ over the space $\mathbb{C}[z]$ of complex polynomials is given by

$$\langle f, g \rangle_\Phi := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{\mathrm{i}\theta}) \overline{g(e^{\mathrm{i}\theta})}\, d\Phi(\theta), \qquad f, g \in \mathbb{C}[z]. \tag{5.6.1}$$

Theorem 5.6.1: [JNT89]

Let $Q_n(z)$ denote the n^{th} denominator of the PPC-fraction (5.1.11) and let Φ denote the corresponding distribution function on $(-\pi, \pi)$. Then $\{Q_{2n+1}(z)\}_{n=0}^{\infty}$ is a monic polynomial sequence orthogonal with respect to the inner product (5.6.1).

The denominator polynomials $Q_{2n+1}(z)$ and $Q_{2n}(z)$ of the PPC-fraction in *Theorem* 5.6.1 are called, respectively, the n^{th} *Szegő polynomial* and n^{th} *reciprocal polynomial* for Φ. We use the notation

$$\rho_n(z) := Q_{2n+1}(z), \qquad \rho_n^*(z) := Q_{2n}(z), \qquad n \in \mathbb{N}_0. \tag{5.6.2}$$

The Szegő and reciprocal polynomials (5.6.2) satisfy the relations

$$\rho_n^*(z) = z^n \overline{\rho_n(1/\bar{z})}, \qquad n \in \mathbb{N}_0,$$

and the recurrence relations

$$\begin{aligned}
&\rho_0(z) = 1, \qquad \rho_0^*(z) = 1, \\
&\rho_n(z) = z\rho_{n-1}(z) + \delta_n \rho_{n-1}^*(z), \qquad n \in \mathbb{N}, \\
&\rho_n^*(z) = \bar{\delta}_n z \rho_{n-1}(z) + \rho_{n-1}^*(z), \qquad n \in \mathbb{N},
\end{aligned}$$

where the δ_n, which are called the *reflection coefficients*, are the coefficients of the PPC-fraction in (5.1.11).

We recall from *Chapter* 4 the notation $T_k^{(m)}$ for the *Toeplitz determinant* associated with a sequence $\{\mu_k\}_{k=0}^{\infty}$:

$$T_k^{(m)} = \begin{vmatrix} \mu_m & \mu_{m+1} & \cdots & \mu_{m+k-1} \\ \mu_{m-1} & \mu_m & \cdots & \mu_{m+k-2} \\ \vdots & \vdots & & \vdots \\ \mu_{m-k+1} & \mu_{m-k+2} & \cdots & \mu_m \end{vmatrix}, \qquad k \in \mathbb{N}, \quad m \in \mathbb{Z},$$

where

$$\mu_{-k} = \overline{\mu_k}, \qquad k \in \mathbb{N}.$$

The Szegő polynomials and the reciprocal polynomials (5.6.2) can also be expressed by the determinant formulas

$$\rho_n(z) = \frac{1}{T_n^{(0)}} \begin{vmatrix} \mu_0 & \mu_{-1} & \cdots & \mu_{-n} \\ \mu_1 & \mu_0 & \cdots & \mu_{-n+1} \\ \vdots & \vdots & & \vdots \\ \mu_{n-1} & \mu_{n-2} & \cdots & \mu_{-1} \\ 1 & z & \cdots & z^n \end{vmatrix}, \qquad n \in \mathbb{N}_0,$$

$$\rho_n^*(z) = \frac{1}{T_n^{(0)}} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_{-1} & \mu_0 & \cdots & \mu_{n-1} \\ \vdots & \vdots & & \vdots \\ \mu_{-n+1} & \mu_{-n+2} & \cdots & \mu_1 \\ z^n & z^{n-1} & \cdots & 1 \end{vmatrix}, \qquad n \in \mathbb{N}_0,$$

where for $k = 0, 1, 2, \ldots$ the μ_k are the moments for the distribution function Φ on $(-\pi, \pi)$.

For all $n \in \mathbb{N}$, the Szegő polynomials and the reciprocal polynomials (5.6.2) satisfy the orthogonality conditions

$$\langle \rho_n, z^m \rangle_\Phi = \begin{cases} 0, & 0 \le m \le n-1, \\ T_{n+1}^{(0)}/T_n^{(0)}, & m = n, \end{cases}$$

$$\langle \rho_n^*, z^m \rangle_\Phi = \begin{cases} T_{n+1}^{(0)}/T_n^{(0)}, & m = 0, \\ 0, & 1 \le m \le n. \end{cases}$$

Using the relation

$$B_n(z) = \rho_n(z) + \tau \rho_n^*(z), \qquad |\tau| = 1$$

we obtain the *para-orthogonal polynomials*. Since the n zeros of $\rho_n(z)$ lie in the open disk $|z| < 1$, the zeros of $B_n(z)$ lie on the unit circle. This property can be used to obtain a quadrature formula on the unit circle [JNT89].

5.7 Orthogonal Laurent polynomials and APT-fractions

Let Λ denote the space of real Laurent polynomials (L-polynomials) given by

$$\Lambda := \left\{ \sum_{j=p}^{q} c_j x^j : c_j \in \mathbb{R},\, p, q \in \mathbb{Z},\, p \leq q \right\},$$

and let Λ_{2n} and Λ_{2n+1} denote the subsets of Λ given by

$$\Lambda_{2n} := \left\{ \sum_{j=-n}^{n} c_j x^j : c_n \neq 0 \right\}, \qquad n \in \mathbb{N}_0,$$

$$\Lambda_{2n+1} := \left\{ \sum_{j=-n-1}^{n} c_j x^j : c_{-n-1} \neq 0 \right\}, \qquad n \in \mathbb{N}_0.$$

For an L-polynomial $\sum_{j=-n}^{n} c_j x^j$ in Λ_{2n} we call c_n the leading coefficient and c_{-n} the trailing coefficient. For an L-polynomial $\sum_{j=-n-1}^{n} c_j x^j$ in Λ_{2n+1} we call c_{-n-1} the leading coefficient and c_n the trailing coefficient. An L-polynomial is called monic if its leading coefficient is one. If the trailing coefficient of an L-polynomial is nonzero the L-polynomial is called regular. Otherwise it is called *singular*.

An L-polynomial is said to have L-degree n if the L-polynomial is in Λ_n. We denote the L-degree of an L-polynomial R by $L\partial(R)$.

Two methods are widely used to define an inner product on Λ. One is by means of a positive definite strong linear functional and the other by means of a strong moment distribution function. The latter method is used here. Let Φ be a strong moment distribution function on (a, b), $-\infty \leq a < b \leq \infty$. An inner product $\langle \cdot, \cdot \rangle_\Phi$ over the space Λ of L-polynomials is defined by

$$\langle f, g \rangle_\Phi := \int_a^b f(t) g(t)\, d\Phi(t), \qquad f, g \in \Lambda. \tag{5.7.1}$$

The norm of $R \in \Lambda$ is given by

$$\|R\|_\Phi := (\langle R, R \rangle_\Phi)^{1/2}.$$

A sequence of real L-polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ is called an *orthogonal L-polynomial* sequence for a strong distribution function Φ on (a, b) if, for $m, n \in \mathbb{N}_0$,

$$L\partial(Q_n) = n,$$
$$\langle Q_n, Q_m\rangle_\Phi = 0, \qquad m \neq n,$$
$$\langle Q_n, Q_n\rangle_\Phi = \|Q_n\|_\Phi^2 > 0.$$

An orthogonal L-polynomial sequence $\{Q_n(x)\}_{n=0}^{\infty}$ for a strong moment distribution function Φ on (a,b) is said to be monic if $Q_n(x)$ is monic for each $n \in \mathbb{N}_0$. It can be shown that if Φ is a strong moment distribution function on (a,b) with moments $\{\mu_k\}_{k=-\infty}^{\infty}$ given by (5.1.2), then there exists a monic orthogonal L-polynomial sequence $\{Q_n(x)\}_{n=0}^{\infty}$ for Φ. Formulas for the L-polynomials in terms of the moments μ_k, and associated Hankel determinants can be found in [JN99].

The possible occurrence of singular L-polynomials in an orthogonal L-polynomial sequence renders the theories of orthogonal polynomials and orthogonal L-polynomials significantly different. For instance while every orthogonal polynomial sequence satisfies a system of three-term recurrence relations of the form (5.4.2), there exist sequences of orthogonal L-polynomials that only satisfy four or five-term recurrence relations and other orthogonal L-polynomial sequences that satisfy three-term recurrence relations. There are, however, similarities between the theories of orthogonal polynomial sequences and regular orthogonal L-polynomial sequences. In particular, a monic orthogonal L-polynomial sequence $\{Q_n(x)\}_{n=0}^{\infty}$ satisfies a system of three-term recurrence relations if and only if $\{Q_n(x)\}_{n=0}^{\infty}$ is regular. Two more similarities can be seen by comparing *Theorem* 5.5.1 and *Theorem* 5.5.2 to the next two theorems.

THEOREM 5.7.1:

Let $\{Q_n(x)\}_{n=0}^{\infty}$ be a regular monic orthogonal L-polynomial sequence for a strong moment distribution function Φ. Then, for $n \in \mathbb{N}$, $Q_n(x)$ is the n^{th} denominator of the modified APT-fraction

$$\frac{\lambda_1}{\frac{1}{\beta_0 x} + \beta_1} + \frac{\lambda_2}{\frac{x}{\beta_1} + \beta_2} + \frac{\lambda_3}{\frac{1}{\beta_2 x} + \beta_3} + \frac{\lambda_4}{\frac{x}{\beta_3} + \beta_4} + \dots, \tag{5.7.2a}$$

$$\frac{\lambda_n \beta_{n-1}}{\beta_n} > 0, \qquad n \in \mathbb{N}, \tag{5.7.2b}$$

with coefficients given by

$$\begin{aligned}
\beta_0 &:= 1, \qquad \lambda_1 := -\mu_{-1} = -H_1^{(-1)}(\mu), \\
\beta_{2m} &:= \frac{H_{2m}^{(-2m+1)}(\mu)}{H_{2m}^{(-2m)}(\mu)} \neq 0, \qquad m \geq 1, \\
\beta_{2m+1} &:= -\frac{H_{2m+1}^{(-2m-1)}(\mu)}{H_{2m+1}^{(-2m)}(\mu)} \neq 0, \qquad m \geq 0, \\
\lambda_{2m+1} &:= \frac{-H_{2m+1}^{(-2m-1)}(\mu) H_{2m-1}^{(-2m+2)}(\mu)}{H_{2m-1}^{(-2m)}(\mu) H_{2m}^{(-2m+1)}(\mu)} \neq 0, \qquad m \geq 1, \\
\lambda_{2m+2} &:= \frac{-H_{2m+2}^{(-2m-1)}(\mu) H_{2m}^{(-2m)}(\mu)}{H_{2m+1}^{(-2m)}(\mu) H_{2m+1}^{(-2m-1)}}(\mu) \neq 0, \qquad m \geq 0.
\end{aligned} \tag{5.7.2c}$$

Recall that the $H_k^{(n)}(\mu)$ *are the Hankel determinants (5.1.12) associated with the bisequence of moments* $\{\mu_k\}_{k=-\infty}^{\infty}$ *given by (5.1.2) for the strong moment distribution function* Φ*.*

THEOREM 5.7.2:

Let $Q_n(x)$ *denote the* n^{th} *denominator of a modified APT-fraction (5.7.2). Then there exists a strong moment distribution function* Φ *such that the sequence* $\{Q_n(x)\}_{n=0}^{\infty}$ *of denominators of (5.7.2) is the monic orthogonal L-polynomial sequence for* Φ*.*

L-polynomial analogues of the classical orthogonal polynomials and Gaussian quadrature can be found in [dAD98; HJT98; Hen90; JT81; Njå89].

Further reading

- Basic references on classical moment problems include [Akh65; ST43; Per57; Wal48; BGM96; Chi78].
- Basic references on strong moment problems include [JN99; Njå96].
- Basic references on the trigonometric moment problem include [Akh65; Fra71; Ger61; GS58; JNT89].
- Basic references on orthogonal polynomials include [Sze67; Sze68; Fra71; Chi78; VA87; NT89].
- Basic references on orthogonal Laurent polynomials include [HvR86; JN99].

Part II

NUMERICS

6

Continued fraction construction

Algorithms are developed to construct different continued fraction representations of functions, known either by one or more formal series representations or by a set of function values. The qd-algorithm constructs C-fractions, the $\alpha\beta$- and FG-algorithms respectively deliver J- and M-fraction representations, and inverse or reciprocal differences serve to construct Thiele interpolating fractions. Also Thiele continued fraction expansions can be obtained as a limiting form.

6.1 Regular C-fractions

Consider the FTS

$$L_0(z) = \sum_{k=0}^{\infty} c_k z^k, \qquad c_k \in \mathbb{C} \tag{6.1.1}$$

and the regular C-fraction

$$f(z) = c_0 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m z}{1} \right), \qquad c_0 \in \mathbb{C}, \quad a_m \in \mathbb{C}\backslash\{0\}, \quad m \geq 1. \tag{6.1.2}$$

In the Hankel determinants $H_k^{(m)}(c)$ defined by (5.1.12) we put $c_k = 0$ for $k < 0$.

THEOREM 6.1.1: [JT80, p. 223]

For the FTS $L_0(z)$ given by (6.1.1) there exists a regular C-fraction (6.1.2) corresponding to $L_0(z)$ at $z = 0$ if and only if

$$H_k^{(1)}(c) \neq 0, \qquad H_k^{(2)}(c) \neq 0, \qquad k \geq 1. \tag{6.1.3}$$

As indicated in *Theorem* 6.1.2, the coefficients a_m can be obtained from the series coefficients c_k using Rutishauser's *qd-algorithm* which we give here in two basic forms: a standard form which is unstable and a progressive form which is more stable. A variant of the Viskovatov algorithm in *Section* 1.7, called the normalised Viskovatov algorithm, can also be used.

The qd-algorithm. The qd-table consists of values $q_\ell^{(k)}$ and $e_\ell^{(k)}$ where the superscript indicates a downward sloping diagonal and the subscript a column:

$$\begin{array}{ccccccccc}
 & q_1^{(0)} & & q_2^{(-1)} & & q_3^{(-2)} & \cdots & & \\
e_0^{(1)} & & e_1^{(0)} & & e_2^{(-1)} & & e_3^{(-2)} & \cdots & \\
 & q_1^{(1)} & & q_2^{(0)} & & q_3^{(-1)} & & \ddots & \\
e_0^{(2)} & & e_1^{(1)} & & e_2^{(0)} & & e_3^{(-1)} & & \\
 & q_1^{(2)} & & q_2^{(1)} & & q_3^{(0)} & & \ddots & \\
e_0^{(3)} & & e_1^{(2)} & & e_2^{(1)} & & e_3^{(0)} & & \\
 & q_1^{(3)} & & q_2^{(2)} & & \ddots & & \ddots & \\
e_0^{(4)} & \vdots & e_1^{(3)} & \vdots & e_2^{(2)} & & & & \\
\vdots & & \vdots & & \vdots & \ddots & & &
\end{array} \tag{6.1.4}$$

In its standard form, the qd-algorithm or quotient-difference algorithm [Hen74, p. 609] associates with the FTS $L_0(z)$ given by (6.1.1), the values

$$e_0^{(k+1)} = 0, \qquad k \geq 0, \tag{6.1.5a}$$

$$q_1^{(k)} = \frac{c_{k+1}}{c_k}, \qquad k \geq 0, \tag{6.1.5b}$$

$$e_\ell^{(k)} = q_\ell^{(k+1)} - q_\ell^{(k)} + e_{\ell-1}^{(k+1)}, \qquad \ell \geq 1, \quad k \geq 1, \tag{6.1.5c}$$

$$q_{\ell+1}^{(k)} = \frac{e_\ell^{(k+1)}}{e_\ell^{(k)}} q_\ell^{(k+1)}, \qquad \ell \geq 1, \quad k \geq 1 \tag{6.1.5d}$$

which are computed from left to right and fill up the lower left half of table (6.1.4), meaning under the principal diagonal with superscript $^{(1)}$. The starting values (6.1.5b) and (6.1.5c) fill the first two columns. Equations (6.1.5c) and (6.1.5d) are called the *rhombus rules* for the qd-algorithm because each connects four elements, either by addition or by multiplication, which form a rhombus in the qd-table:

$$\begin{array}{ccccc}
 & q_\ell^{(k)} & & & \\
 & & + & & \\
e_{\ell-1}^{(k+1)} & & & e_\ell^{(k)} & \\
 & + & & & \times \\
 & q_\ell^{(k+1)} & & & q_{\ell+1}^{(k)} \\
 & & \times & & \\
 & & & e_\ell^{(k+1)} &
\end{array} \tag{6.1.6}$$

The rhombus rules are used to compute the elements $e_\ell^{(k)}$ and $q_{\ell+1}^{(k)}$ in the furthermost right corner of each rhombus.

EXAMPLE 6.1.1: In some of the following examples we consider the exponential function,

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}, \qquad z \in \mathbb{C}. \tag{6.1.7}$$

Initialising $e_0^{(k+1)} = 0$ and $q_1^{(k)} = 1/(k+1)$ for $k \geq 0$ and using (6.1.5) delivers the following elements in the qd-table on and below the diagonal with superscript $^{(1)}$:

$$\begin{array}{cccccc}
0 & & & & & \\
 & \frac{1}{2} & & & & \\
0 & & -\frac{1}{6} & & & \\
 & \frac{1}{3} & & \frac{1}{6} & & \\
0 & & -\frac{1}{12} & & -\frac{1}{10} & \\
 & \frac{1}{4} & & \frac{3}{20} & & \ddots \\
0 & & -\frac{1}{20} & & \vdots & \\
 & \frac{1}{5} & & \vdots & & \\
0 & & \vdots & & & \\
 & \vdots & & & &
\end{array}$$

Each additional value $q_1^{(k)}$ allows to compute one more upward sloping diagonal in the qd-table.

In its progressive form, which is numerically more stable, the qd-algorithm fills up the upper right half of the table instead of the lower left half and computes all values in the table from top to bottom. This form needs the coefficients in the FTS of $1/L_0(z)$. With

$$\Lambda_0(1/L_0) = \sum_{k=0}^{\infty} d_k z^k, \tag{6.1.8a}$$

the progressive form of the qd-algorithm [Hen74, p. 614] associates with the FTS $L_0(z)$ given by (6.1.1), the values

$$q_1^{(0)} = -\frac{d_1}{d_0}, \qquad q_{k+1}^{(-k)} = 0, \qquad k \geq 1, \tag{6.1.8b}$$

$$e_0^{(-1)} = 0, \qquad e_1^{(0)} = \frac{d_2}{d_1}, \qquad e_{k+1}^{(-k)} = \frac{d_{k+2}}{d_{k+1}}, \qquad k \geq 1, \tag{6.1.8c}$$

$$e_\ell^{(k+1)} = \frac{q_{\ell+1}^{(k)}}{q_\ell^{(k+1)}} e_\ell^{(k)}, \qquad \ell \geq 1, \quad k \geq 1, \tag{6.1.8d}$$

$$q_\ell^{(k+1)} = q_\ell^{(k)} + e_\ell^{(k)} - e_{\ell-1}^{(k+1)}, \qquad \ell \geq 1, \quad k \geq 1. \tag{6.1.8e}$$

The starting values (6.1.8b) and (6.1.8c) fill the first two rows.

EXAMPLE 6.1.2: We continue *Example* 6.1.1. Initialising the qd-algorithm with $q_{k+1}^{(-k)} = 0$ and $e_{k+1}^{(-k)} = d_{k+2}/d_{k+1}$ where $L_0(z) = \Lambda_0(\exp(z))$, delivers the following part of the qd-table:

$$\begin{array}{cccccccccccc}
 & 1 & & 0 & & 0 & & 0 & & 0 & & \cdots \\
0 & & -\frac{1}{2} & & -\frac{1}{3} & & -\frac{1}{4} & & -\frac{1}{5} & & \cdots & \\
 & \frac{1}{2} & & \frac{1}{6} & & \frac{1}{12} & & \frac{1}{20} & & \cdots & & \\
 & & -\frac{1}{6} & & -\frac{1}{6} & & -\frac{3}{20} & & \cdots & & & \\
 & & & \frac{1}{6} & & \frac{1}{10} & & \cdots & & & & \\
 & & & & -\frac{1}{10} & & \cdots & & & & & \\
 & & & & & \ddots & & & & & &
\end{array}$$

The bottom downward sloping diagonal now contains the elements with superscript $^{(1)}$.

THEOREM 6.1.2: [JT80, p. 229]

Let $L_0(z)$ be a given FTS (6.1.1). If (6.1.3) holds and if $e_\ell^{(k)}, q_\ell^{(k)}$ satisfy (6.1.5) or (6.1.8), then the coefficients of the regular C-fraction (6.1.2) corresponding to $L_0(z)$, are given by

$$a_1 = c_1, \quad a_{2\ell} = -q_\ell^{(1)}, \quad a_{2\ell+1} = -e_\ell^{(1)}, \quad \ell \geq 1. \tag{6.1.9}$$

The following result explains why condition (6.1.3) is required in *Theorem* 6.1.2: it guarantees that the qd-algorithm doesn't break down.

THEOREM 6.1.3: [Hen74, p. 610]

Let (6.1.1) be given. If there exists a positive integer n such that $H_k^{(m)}(c) \neq 0$ for $k = 0, 1, \ldots, n$ and $m \geq 0$, then the values $q_\ell^{(m)}$ and $e_\ell^{(m)}$ exist for $\ell = 0, 1, \ldots, n$ and $m \geq 0$ and they are given by

$$q_\ell^{(m)} = \frac{H_{\ell-1}^{(m)}(c) H_\ell^{(m+1)}(c)}{H_\ell^{(m)}(c) H_{\ell-1}^{(m+1)}(c)}, \qquad e_\ell^{(m)} = \frac{H_{\ell+1}^{(m)}(c) H_{\ell-1}^{(m+1)}(c)}{H_\ell^{(m)}(c) H_\ell^{(m+1)}(c)}, \qquad \ell \geq 1. \tag{6.1.10}$$

Taking $m = 1$, we find that the values $q_\ell^{(1)}$ and $e_\ell^{(1)}$ in (6.1.10) exist under the condition (6.1.3) and we obtain a determinant representation for the coefficients in (6.1.2).

EXAMPLE 6.1.3: For

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}, \qquad z \in \mathbb{C}$$

we have $c_k = 1/k!$ and find that, for $k \geq 0$ and $\ell \geq 1$,

$$q_\ell^{(k)} = \frac{k + \ell - 1}{(k + 2\ell - 2)(k + 2\ell - 1)}, \qquad e_\ell^{(k)} = \frac{-\ell}{(k + 2\ell - 1)(k + 2\ell)} .$$

The regular C-fraction representation of $\exp(z)$ is

$$\exp(z) = 1 + \frac{z}{1} + \frac{-q_1^{(1)} z}{1} + \frac{-e_1^{(1)} z}{1} + \frac{-q_2^{(1)} z}{1} + \ldots, \qquad z \in \mathbb{C}.$$

EXAMPLE 6.1.4: Applying the qd-algorithm to

$$\sqrt{z}\,\mathrm{Arctan}\left(\sqrt{z}\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} z^k, \qquad |z| \le 1, \quad z \ne -1,$$

yields a corresponding regular C-fraction

$$\begin{aligned} &\mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m z}{1}\right), \qquad \sqrt{z} \notin (-\infty,-1) \cup (1,+\infty), \\ &a_1 = 1, \quad a_m = \frac{(m-1)^2}{(2m-3)(2m-1)}, \quad m \ge 2. \end{aligned} \tag{6.1.11}$$

Since all $a_m > 0$, (6.1.11) is actually an S-fraction.

Explicit formulas for $q_\ell^{(k)}$ and $e_\ell^{(k)}$ with more general expressions for c_k can be found in [BGM96, pp. 150–152].

The normalised Viskovatov algorithm. Let us adapt the method of Viskovatov given in (1.7.7) and (1.7.9) as follows. From the FTS (6.1.1), one defines

$$\tilde{c}_{0j} = \begin{cases} 1, & j = 0, \\ 0, & j > 0, \end{cases} \tag{6.1.12a}$$

$$\tilde{c}_{1j} = c_{j+1} z^{j+1}, \qquad j \ge 0, \tag{6.1.12b}$$

and computes

$$\tilde{c}_{m,j} = \frac{\tilde{c}_{m-2,j+1}}{\tilde{c}_{m-2,0}} - \frac{\tilde{c}_{m-1,j+1}}{\tilde{c}_{m-1,0}}, \qquad m \ge 2, \quad j \ge 0. \tag{6.1.12c}$$

The new values $\tilde{c}_{m,j}$ differ from the partial numerators in (1.7.10) only by an equivalence transformation chosen such that the partial denominators equal 1.

THEOREM 6.1.4: [BGM96, pp. 133–134]

Let $L_0(z)$ be a given FTS (6.1.1). If (6.1.3) holds and if the coefficients $\tilde{c}_{m,j}$ satisfy (6.1.12) with $\tilde{c}_{m0} \ne 0$ for $m \ge 1$, then the partial numerators of the regular C-fraction (6.1.2) corresponding to $L_0(z)$, are given by

$$a_m z = \tilde{c}_{m0}, \qquad m \ge 1\ .$$

6.2 C-fractions

If in *Theorem* 6.1.4 some of the $\tilde{c}_{m0}$ equal zero, Viskovatov's method generates [BGM96, pp. 134–135] a C-fraction of the form

$$b_0 + \frac{a_1 z^{\alpha_1}}{1} + \frac{a_2 z^{\alpha_2}}{1} + \frac{a_3 z^{\alpha_3}}{1} + \dots \tag{6.2.1}$$

Let

$$L_0(z) = c_0 + \sum_{j=1}^{\infty} c_{k_j} z^{k_j}, \qquad k_{j+1} \geq k_j$$

and put

$$\tilde{c}_{0j} = \begin{cases} 1, & j = 0, \\ 0, & j > 0, \end{cases}$$
$$\tilde{c}_{10} = c_{k_1},$$
$$\tilde{c}_{1j} = c_{k_{j+1}} z^{k_{j+1}-k_1}, \qquad j > 0.$$

The normalised Viskovatov algorithm (6.1.12c) now leads to a C-fraction with

$$b_0 = c_0, \qquad a_1 z^{\alpha_1} = \tilde{c}_{10} z^{k_1}, \qquad a_m z^{\alpha_m} = \tilde{c}_{m0}, \qquad m \geq 2.$$

Example 6.2.1: Consider $f(z) = 1 + \sin(z)$. From

$$L_0(z) = 1 + z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 - \dots, \qquad z \in \mathbb{C} \tag{6.2.2}$$

we initialise

$$\tilde{c}_{10} = 1, \quad \tilde{c}_{11} = -z^2/6, \quad \tilde{c}_{12} = z^4/120, \quad \dots$$

and from (6.1.12c),

$$1 + \sin(z) = 1 + \frac{z}{1} + \frac{z^2/6}{1} + \frac{-7z^2/60}{1} + \dots .$$

6.3 S-fractions

Let us consider the S-fraction

$$f(z) = \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m z}{1}\right), \qquad a_m > 0, \quad m \geq 1. \tag{6.3.1}$$

Since the S-fraction is a special case of a regular C-fraction, all theorems of *Section* 6.1 apply to S-fractions, but more can be said. It is easier to use the coefficients γ_k in the expansion

$$L_0(z) = \sum_{k=0}^{\infty} (-1)^k c_k (-z)^k = \sum_{k=0}^{\infty} \gamma_k (-z)^k, \tag{6.3.2}$$

rather than the standard c_k because the following determinant conditions take on a simpler form.

THEOREM 6.3.1: [BGM96, pp. 197–199]

Let $L_0(z)$ be a FTS of the form (6.1.1) with $c_0 = 0$. Then there exists an S-fraction (6.3.1) corresponding to $L_0(z)$ if and only if the Hankel determinants associated with $\{\gamma_k\}$ satisfy

$$H_k^{(1)}(\gamma) > 0, \qquad H_k^{(2)}(\gamma) > 0, \qquad k \geq 1. \tag{6.3.3}$$

The determinants $H_k^{(m)}(c)$ and $H_k^{(m)}(\gamma)$ are related as in (5.1.14) and the conditions on the Hankel determinants in *Theorem* 6.3.1 therefore coincide with the conditions in part (A) of *Theorem* 5.1.6.

6.4 P-fractions

Condition (6.1.3) expresses that all elements in the sequence T_0, defined in (4.3.1), of Padé approximants to $L_0(z)$ are distinct. If (6.1.3) does not hold, then the Padé table contains square blocks of equal elements that are traversed by the staircase T_0. In that case the qd-algorithm breaks down and the representation of $L_0(z)$ by (6.1.2) does not hold anymore.

EXAMPLE 6.4.1: We reconsider (6.2.2) for $f(z) = 1 + \sin(z)$. The qd-algorithm cannot be initialised because (6.1.5b) breaks down. For the purpose of the subsequent examples we give part of the Padé table for

$f(z)$:

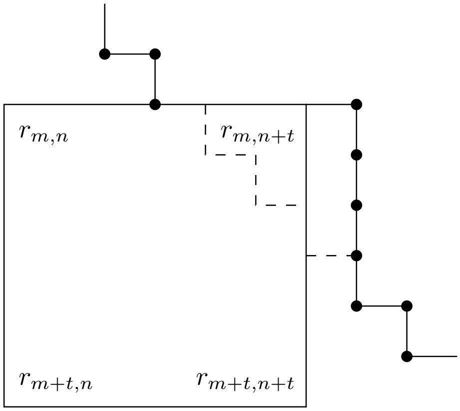
$$
\begin{array}{ccccc}
1 & \dfrac{1}{1-z} & \dfrac{1}{1-z+z^2} & \cdots & \\
1+z & 1+z & \dfrac{1+\frac{5}{6}z}{1-\frac{1}{6}z+\frac{1}{6}z^2} & \cdots & \\
1+z & 1+z & \dfrac{1+z+\frac{1}{6}z^2}{1+\frac{1}{6}z^2} & \cdots & \\
1+z-\frac{1}{6}z^3 & 1+z-\frac{1}{6}z^3 & \dfrac{1+z+\frac{1}{20}z^2-\frac{7}{60}z^3}{1+\frac{1}{20}z^2} & \cdots & \\
1+z-\frac{1}{6}z^3 & 1+z-\frac{1}{6}z^3 & \cdots & & \\
1+z-\frac{1}{6}z^3+\frac{1}{120}z^5 & \vdots & & & \\
\vdots & & & &
\end{array}
\qquad (6.4.1)
$$

From *Theorem* 4.4.1 we know that the sequence of P-fraction approximants picks up one Padé approximant per block in the Padé table. But P-fractions cannot directly be constructed using the qd-algorithm. We first construct a related continued fraction.

The qd-algorithm revisited. It is possible to define staircases, that jump over square blocks in the Padé table and of which the elements can be obtained as successive approximants of a continued fraction [CW79]. When the staircase T_0 traverses a block of size $t+1$ with corner elements $r_{m,n}$, $r_{m,n+t}$, $r_{m+t,n}$ and $r_{m+t,n+t}$, and all other elements in the Padé table are distinct, then we consider the adapted staircase

$$
\begin{aligned}
T_0^* = \{ & r_{0,0}, r_{1,0}, \ldots, r_{n+t-k+1,n+t-k}, r_{n+t-k+1,n+t+1}, \\
& r_{n+t-k+2,n+t+1}, \ldots, r_{n+t+2,n+t+1}, r_{n+t+2,n+t+2}, \ldots \}, \qquad 1 \le k \le t
\end{aligned}
$$

FIGURE 6.4.1: Adapted staircase T_0^* with $k = 3$.



and a continued fraction of the form

$$c_0 + \frac{c_1 z}{1} + \frac{-q_1^{(1)} z}{1} + \frac{-e_1^{(1)} z}{1} + \cdots + \frac{-q_{m-1}^{(1)} z}{1} + \frac{-e_{m-1}^{(1)} z}{1} + \frac{-v_{k,1}^{(t+1)} z^{k+1}}{1 - v_{k,k+1}^{(t+1)} z - \cdots - v_{k,2}^{(t+1)} z^k} + \frac{-v_{k,k+2}^{(t+1)} z}{1} + \mathop{\mathrm{K}}_{i=1}^{k} \left(\frac{-v_{k,k+i+2}^{(t+1)} z}{1 + v_{k,k+i+2}^{(t+1)} z} \right) + \frac{-q_{n+t+2}^{(1)} z}{1} + \frac{-e_{n+t+2}^{(1)} z}{1} + \frac{-q_{n+t+3}^{(1)} z}{1} + \frac{-e_{n+t+3}^{(1)} z}{1} + \ldots \quad (6.4.2)$$

This staircase hits the block in the Padé table from above at the entry $r_{m,n+t-k}$ with $m = n + t - k + 1$, then skips the other block entries and goes down column $n+t+1$ to recapture the old staircase T_0 at $r_{m+k+1,n+t}$. It is this vertical movement down column $n + t + 1$ that introduces the v-values in (6.4.2). Similar continued fractions can be constructed of which the convergents are the elements on a special staircase hitting the block of equal entries from the left, and passing below the block to move horizontally along row $m + t + 1$ while introducing h-values [CW79].

EXAMPLE 6.4.2: With $f(z) = 1 + \sin(z)$ the staircase T_0 hits a block in the Padé table (6.4.1) for $n = 0, t = 1$ and $k = 1$. The adapted staircase T_0^* proceeds via

$$T_0^* = \{r_{0,0}, r_{1,0}, r_{1,2}, r_{2,2}, r_{3,2}, r_{3,3}, \ldots\} .$$

Since the qd-algorithm breaks down when (6.1.3) does not hold, we need new rules to compute $q_{n+t}^{(m-n-t)}$, $e_{n+t}^{(m-n-t)}$ and $q_{n+t+2}^{(m-n-t+i-1)}$ for $1 \leq i \leq t+1$. These are:

$$q_{n+t+1}^{(m-n-t)} \prod_{j=1}^{t+1} e_{n+j-1}^{(m-n-j+1)} = e_n^{(m-n+t+1)} \prod_{j=1}^{t+1} q_n^{(m-n+j)}, \tag{6.4.3a}$$

$$q_{n+t+1}^{(m-n-t)} + e_{n+t+1}^{(m-n-t)} = e_n^{(m-n+t+1)} + q_{n+1}^{(m-n+t+1)}, \tag{6.4.3b}$$

and for $k = 1, 2, \ldots, t$,

$$q_{n+t+1}^{(m-n-t)} \prod_{j=1}^{k} e_{n+t-j+1}^{(m-n-t+j-1)} + e_{n+t+1}^{(m-n+t+2)} \prod_{j=1}^{k} q_{n+t+2}^{(m-n-t+j-1)} =$$

$$e_n^{(m-n+t+1)} \prod_{j=1}^{k} q_n^{(m-n+t-j+2)} + q_{n+1}^{(m-n+t+1)} \prod_{j=1}^{k} e_{n+j}^{(m-n+t-j+2)}, \tag{6.4.3c}$$

$$e_{n+t+1}^{(m-n+t+2)} \prod_{j=1}^{t+1} q_{n+t+2}^{(m-n-t+j-1)} = q_{n+1}^{(m-n+t+1)} \prod_{j=1}^{t+1} e_{n+j}^{(m-n+t-j+2)}. \tag{6.4.3d}$$

We now identify the new values $v_{k,i}^{(t+1)}$ in (6.4.2).

THEOREM 6.4.1: [CW79]

Let the Padé table for $L_0(z)$ contain a block of size $t+1$ with corners $r_{m,n}$, $r_{m,n+t}$, $r_{m+t,n}$ and $r_{m+t,n+t}$. Then

$$v_{1,1}^{(t+1)} = e_{n+t}^{(m-n-t)} q_{n+t+1}^{(m-n-t)}, \tag{6.4.4a}$$

$$v_{1,2}^{(t+1)} = q_{n+t+1}^{(m-n-t)}, \tag{6.4.4b}$$

$$v_{1,3}^{(t+1)} = e_{n+t+1}^{(m-n-t)}, \tag{6.4.4c}$$

$$v_{1,4}^{(t+1)} = q_{n+t+2}^{(m-n-t)}, \tag{6.4.4d}$$

and for $k > 1$ and $i = 2, 3, \ldots, 2k+1$:

$$v_{k,1}^{(t+1)} = e_{n+t-k+1}^{(m-n-t+k-1)} v_{k-1,1}^{(t+1)}, \tag{6.4.4e}$$

$$v_{k,i}^{(t+1)} = v_{k-1,i-1}^{(t+1)}, \tag{6.4.4f}$$

$$v_{k,2k+2}^{(t+1)} = q_{n+t+2}^{(m-n-t+k-1)}. \tag{6.4.4g}$$

EXAMPLE 6.4.3: We recall from *Example* 6.4.2 that for $f(z) = 1 + \sin(z)$ we have $n = 0, t = 1, k = 1$ and hence $m = 1$. Its continued fraction representation of the form (6.4.2) is given by

$$1 + \sin(z) = 1 + \frac{z}{1} + \frac{z^2/6}{1 - z/6} + \frac{z/6}{1} + \frac{7z/10}{1 - 7z/10} + \frac{-q_3^{(1)} z}{1} + \dots$$

and the first few approximants are

$$\begin{aligned} f_0 &= 1, \\ f_1 &= 1 + z, \\ f_2 &= \frac{1 + \frac{5}{6}z}{1 - \frac{1}{6}z + \frac{1}{6}z^2}, \\ f_3 &= \frac{1 + z + \frac{1}{6}z^2}{1 + \frac{1}{6}z^2}, \\ f_4 &= \frac{1 + z + \frac{1}{20}z^2 - \frac{7}{60}z^3}{1 + \frac{1}{20}z^2} . \end{aligned} \tag{6.4.5}$$

Obtaining the P-fraction. By means of the formulas (1.5.1) and (1.6.4), a suitable contraction of the continued fraction that picks up one element per block in the Padé table along the staircase-like path T_0^*, delivers the P-fraction representation of $L_0(z)$.

EXAMPLE 6.4.4: The P-fraction representation of $f(z) = 1 + \sin(z)$ is given by

$$f(z) = 1 + \frac{1}{1/z} + \frac{1}{6/z} + \frac{1}{-10/7z} + \dots .$$

Its first four approximants equal $r_{0,0}, r_{1,0}, r_{2,2}$ and $r_{3,2}$ of (6.4.1), respectively. They also equal the approximants f_0, f_1, f_3 and f_4 of (6.4.2) given in (6.4.5).

Generalised Viskovatov algorithm. P-fractions are equivalent to fractions of the form

$$\pi_0(z) + \operatorname*{K}_{m=1}^{\infty} \left(\frac{z^{\alpha_m}}{\pi_m(z)} \right) \tag{6.4.6}$$

where for $m \geq 0$ the $\pi_m(z)$ are polynomials in z of degree β_m and where $\alpha_m \geq 0$. A continued fraction of the form (6.4.6), corresponding to (6.1.1), can be constructed using a generalised form of Viskovatov's algorithm (1.7.9) [BGM96, p. 135; Mag62b]. Define $L_1(z) = 1$, choose $\beta_0 \geq 0$ and denote the partial sum of degree n of a FTS $L(z)$ by $\mathcal{P}_n(L(z))$. We recall that the order of a FTS $L(z)$, which is the degree of its first non-zero term, is denoted by $\lambda(L)$ and defined in (2.2.7). Start with

$$\begin{aligned} \pi_0(z) &= \mathcal{P}_{\beta_0}\left(L_0/L_1\right), \\ \alpha_1 &= \lambda\left(L_0 - \pi_0 L_1\right), \\ \beta_1 &= \alpha_1 - \beta_0, \end{aligned} \tag{6.4.7a}$$

and compute for $m \geq 1$,

$$\begin{aligned} L_{m+1}(z) &= z^{-\alpha_m}\left(L_{m-1} - \pi_{m-1}L_m\right)(z), \\ \pi_m(z) &= \mathcal{P}_{\beta_m}\left(L_m/L_{m+1}\right), \\ \alpha_{m+1} &= \lambda\left(L_m - \pi_m L_{m+1}\right), \\ \beta_{m+1} &= \alpha_{m+1} - \beta_m \ . \end{aligned} \tag{6.4.7b}$$

The n^{th} approximant of (6.4.6) is the Padé approximant of degree $\beta_0 + \sum_{m=1}^{n} \beta_m$ in the numerator and degree $\sum_{m=1}^{n} \beta_m$ in the denominator.

EXAMPLE 6.4.5: We reconsider $f(z) = 1 + \sin(z)$ for which

$$L_0(z) = 1 + z - \frac{z^3}{6} + \frac{z^5}{120} - \dots .$$

From (6.4.7) we find for $\beta_0 = 1$ and $L_1(z) = 1$,

$$\pi_0(z) = 1 + z, \quad \alpha_1 = 3, \quad \beta_1 = 2$$

$$\pi_1(z) = -6 - \frac{3}{10}z^2, \quad \alpha_2 = 2, \quad \beta_2 = 0$$

resulting in the corresponding continued fraction

$$f(z) = 1 + z + \frac{z^3}{-6 - \frac{3}{10}z^2 + \dots} .$$

Its first two approximants are the Padé approximants $r_{1,0}(z)$ and $r_{3,2}(z)$ of (6.4.1).

6.5 J-fractions

J-fractions can be obtained through their relationship with associated continued fractions: if in the associated continued fraction (6.5.1) we let $z = 1/w$, omit the initial term c_0 and make an equivalence transformation, we obtain the J-fraction (2.3.8). A necessary condition for the existence of an associated continued fraction is weaker than for regular C-fractions.

THEOREM 6.5.1: [JT80, p. 244]

If for a given FTS $L_0(z)$ there exists an associated continued fraction

$$c_0 + \frac{\alpha_1 z}{1+\beta_1 z} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-\alpha_m z^2}{1+\beta_m z} \right), \qquad \alpha_m \in \mathbb{C}\backslash\{0\}, \quad \beta_m \in \mathbb{C}, \quad (6.5.1)$$

which corresponds to $L_0(z)$, then

$$H_k^{(1)}(c) \neq 0, \qquad k \geq 1. \qquad (6.5.2)$$

The coefficients α_m and β_m in the associated continued fraction (6.5.1), can be computed as follows. Set [JT80, p. 248]

$$\gamma_{-1} = 1, \quad \delta_{-1} = 0, \quad b_{0,0} = 1 \qquad (6.5.3a)$$

and compute for $m \geq 0$ the values

$$\gamma_m = \sum_{j=0}^{m} b_{m,j} c_{2m+1-j}, \qquad (6.5.3b)$$

$$\delta_m = \frac{1}{\gamma_m} \left(\sum_{j=0}^{m} b_{m,j} c_{2m+2-j} \right), \qquad (6.5.3c)$$

$$\alpha_{m+1} = \frac{\gamma_m}{\gamma_{m-1}}, \qquad (6.5.3d)$$

$$\beta_{m+1} = \delta_{m-1} - \delta_m, \qquad (6.5.3e)$$

$$b_{m-1,-1} = 0, \quad b_{m,m+1} = 0, \quad b_{m+1,0} = 1, \qquad (6.5.3f)$$

$$b_{m+1,j} = b_{m,j} + \beta_{m+1} b_{m,j-1} - \alpha_{m+1} b_{m-1,j-2}, \qquad j = 1, 2, \ldots, m+1. \qquad (6.5.3g)$$

Algorithm (6.5.3) is more general than the qd-algorithm and only requires condition (6.5.2). But in practice it turns out that the values α_m and β_m are ill-conditioned functions of the sequence of coefficients c_k.

EXAMPLE 6.5.1: The function

$$f(z) = 1 + \sqrt{z}\,\mathrm{Arctan}\left(\sqrt{z}\right)$$

has a FTS given by

$$L_0(z) = 1 + z - \frac{z^2}{3} + \frac{z^3}{5} - \frac{z^4}{7} + \frac{z^5}{9} - \dots, \qquad |z| \le 1, \quad z \ne -1.$$

Algorithm (6.5.3) delivers the following coefficients:

m	α_m	β_m	γ_m	δ_m	$b_{m,0}$	$b_{m,1}$	$b_{m,2}$
-1			1	0			
0			1	$-\frac{1}{3}$	1		
1	1	$\frac{1}{3}$	$\frac{4}{45}$	$-\frac{6}{7}$	1	$\frac{1}{3}$	
2	$\frac{4}{45}$	$\frac{11}{21}$	$\frac{64}{11025}$	$-\frac{15}{11}$	1	$\frac{6}{7}$	$\frac{3}{35}$
3	$\frac{16}{245}$	$\frac{39}{77}$	$\dots$		$\vdots$		
$\vdots$	$\vdots$	$\vdots$					

The J-fraction representation of $f(z)$ is

$$f(z) = 1 + \frac{1}{1 + z/3} + \frac{-4z^2/45}{1 + 11z/21} + \dots .$$

For a determinant representation of α_m and β_m we introduce the values

$$h_0^{(1)} = 0, \quad h_1^{(1)} = c_1, \quad h_k^{(1)} = \begin{vmatrix} c_1 & c_2 & \dots & c_{k-1} & c_{k+1} \\ \vdots & \vdots & & \vdots & \vdots \\ c_k & c_{k+1} & \dots & c_{2k-2} & c_{2k} \end{vmatrix}, \quad k \ge 2. \tag{6.5.4}$$

THEOREM 6.5.2: [JT80, p. 245]

Let $L_0(z)$ be given by (6.1.1). If (6.5.2) holds then the coefficients α_m and β_m of the associated continued fraction (6.5.1) are given by

$$\alpha_m = \frac{H_m^{(1)}(c) H_{m-2}^{(1)}(c)}{\left(H_{m-1}^{(1)}(c)\right)^2}, \qquad \beta_m = \frac{h_{m-1}^{(1)}}{H_{m-1}^{(1)}}(c) - \frac{h_m^{(1)}}{H_m^{(1)}}(c), \qquad m \ge 1. \tag{6.5.5}$$

6.6 M-fractions

We now study the case where the continued fraction approximant corresponds to two given power series, one at $z = 0$ and one at $z = \infty$. An appealing situation is that the order of correspondence of the n^{th} approximant equals n at $z = 0$ and $n + 1$ at $z = \infty$. We assume that we are given a pair of FPS

$$L_0(z) = \sum_{k=0}^{\infty} c_k z^k, \tag{6.6.1a}$$

$$L_\infty(z) = -\sum_{k=1}^{\infty} c_{-k} z^{-k}. \tag{6.6.1b}$$

The Hankel determinants $H_k^{(m)}(c)$ introduced in (5.1.12) are now associated with the bisequence $\{c_k\}_{k=-\infty}^{\infty}$.

Theorem 6.6.1: [BGM96, pp. 359–360]

Let (6.6.1) be given. An M-fraction representation (2.3.14) corresponding to $L_0(z)$ and $L_\infty(z)$ with $F_m \neq 0$ and $G_m \neq 0$ can be constructed if

$$H_m^{(-m+1)}(c) \neq 0, \qquad H_m^{(-m)}(c) \neq 0, \qquad m \geq 1. \tag{6.6.2}$$

The FG-algorithm. Under the conditions of *Theorem* 6.6.1, the following qd-type algorithm developed in [MCM76] computes the elements of the corresponding M-fraction [BGM96, p. 359]. The FG-table consists of entries arranged as

$$\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \\
F_1^{(-1)} & G_1^{(-1)} & F_2^{(-1)} & G_2^{(-1)} & \dots \\
F_1^{(0)} & G_1^{(0)} & F_2^{(0)} & G_2^{(0)} & \dots \\
F_1^{(1)} & G_1^{(1)} & F_2^{(1)} & G_2^{(1)} & \dots \\
\vdots & \vdots & \vdots & \vdots &
\end{array} \tag{6.6.3}$$

As can be seen from *Theorem* 6.6.2, the principal row in the table is the one with superscript ${}^{(0)}$. The first three columns are determined by the initialisation

$$F_1^{(s)} = c_s, \quad G_1^{(s)} = -c_s/c_{s-1}, \quad F_2^{(s)} = G_1^{(s+1)} - G_1^{(s)}, \quad s = 0, \pm 1, \pm 2, \dots \tag{6.6.4a}$$

and the remaining columns by the recursions

$$G_{j+1}^{(s+1)} = F_{j+1}^{(s+1)} G_j^{(s)} / F_{j+1}^{(s)}, \qquad j \geq 1, \tag{6.6.4b}$$

$$s = 0, \pm 1, \pm 2, \ldots .$$

$$F_{j+1}^{(s)} = F_j^{(s+1)} + G_j^{(s+1)} - G_j^{(s)}, \qquad j \geq 2, \tag{6.6.4c}$$

THEOREM 6.6.2: [BGM96, pp. 359–360]

Let the FTS $L_0(z)$ and $L_\infty(z)$ be given by (6.6.1). If (6.6.2) holds and if $F_m^{(s)}$ and $G_m^{(s)}$ satisfy (6.6.4), then the coefficients in the M-fraction representation (2.3.14) corresponding to $L_0(z)$ with order of correspondence n and to $L_\infty(z)$ with order of correspondence $n+1$, are given by

$$F_m = F_m^{(0)}, \qquad G_m = G_m^{(0)}, \qquad m \geq 1.$$

In addition, the entries $F_m^{(s)}$ and $G_m^{(s)}$ with $s \neq 0$ are the coefficients in the M-fractions (4.6.1) and (4.6.2), of which the correspondence properties to $L_0(z)$ and $L_\infty(z)$ are detailed in *Theorem* 4.6.1.
The determinant representation for $F_m^{(s)}$ and $G_m^{(s)}$ explains why we need condition (6.6.2) for *Theorem* 6.6.2.

THEOREM 6.6.3: [BGM96, p. 360]

If (6.6.2) holds, then

$$F_m^{(s)} = \frac{-H_m^{(s-m+1)}(c) H_{m-2}^{(s-m+2)}(c)}{H_{m-1}^{(s-m+2)}(c) H_{m-1}^{(s-m+1)}(c)}, \tag{6.6.5a}$$

$$G_m^{(s)} = \frac{-H_m^{(s-m+1)}(c) H_{m-1}^{(s-m+1)}(c)}{H_{m-1}^{(s-m+2)}(c) H_m^{(s-m)}(c)}, \tag{6.6.5b}$$

for all $m \geq 1$ and $s \in \mathbb{Z}$, where $H_{-1}^{(s)} = 1$.

EXAMPLE 6.6.1: Let ${}_1F_1(a;b;z)$ be defined as in (16.1.2). An M-fraction expansion for ${}_1F_1(a;b+1;z)/{}_1F_1(a;b;z)$ is given in [Dij77]:

$$\frac{{}_1F_1(a;b+1;z)}{{}_1F_1(a;b;z)} = \frac{b}{b+z} - \frac{z(b+1-a)}{b+1+z} - \cdots - \frac{z(b+n-a)}{b+n+z} - \ldots,$$

$$a, b \geq 0, \quad z \geq 0 \, .$$

It corresponds to the FTS about $z = 0$ and to the asymptotic expansion about $z = +\infty$ of the left-hand side, and converges to the left-hand side on the positive real axis. The special case with $b = a$ is

$$\frac{{}_1F_1(a; a+1; z)}{\exp(z)} = \frac{a}{a - z} - \frac{z}{a+1+z} - \cdots - \frac{nz}{a+n+z} - \ldots, \qquad a \geq 0, \quad z \geq 0\,.$$

EXAMPLE 6.6.2: The following illustrates that condition (6.6.2) is sufficient but not necessary. For

$$L_0(z) = 1, \qquad |z| < 1,$$
$$L_\infty(z) = -1/z, \qquad |z| > 1$$

we obtain the M-fraction

$$\frac{1}{1-z} + \frac{z}{1-z} + \frac{z}{1-z} + \ldots\,.$$

All the poles and zeroes of its approximants lie on the unit circle.

6.7 Positive T-fractions

When considering correspondence to two power series, the roles of 0 and ∞ are sometimes interchanged. Instead of (6.6.1), we then consider

$$\tilde{L}_0(z) = -\sum_{k=1}^{\infty} c_{-k} z^k, \tag{6.7.1a}$$

$$\tilde{L}_\infty(z) = \sum_{k=0}^{\infty} c_k z^{-k}. \tag{6.7.1b}$$

THEOREM 6.7.1: [JTW80]

Let $\tilde{L}_0(z)$ and $\tilde{L}_\infty(z)$ be given by (6.7.1). There exists a positive T-fraction of the form (2.3.10) with

$$F_m > 0, \qquad G_m > 0, \qquad m \geq 1,$$

corresponding to $\tilde{L}_0(z)$ at $z = 0$ and to $\tilde{L}_\infty(z)$ at $z = \infty$ if and only if the Hankel determinants satisfy

$$\begin{aligned} H_m^{(-m+1)}(c) &> 0, \qquad m \geq 1, \\ H_{2m}^{(-2m)}(c) &> 0, \qquad m \geq 1, \\ H_{2m-1}^{(-2m+1)}(c) &< 0, \qquad m \geq 1. \end{aligned} \tag{6.7.2}$$

The Hankel determinants $H_k^{(m)}(c)$ are related to the Hankel determinants for the sequence $\{\mu_k\}_{k=-\infty}^{\infty} = \{(-1)^k c_k\}_{k=-\infty}^{\infty}$ by (5.1.15). The conditions on the Hankel determinants in *Theorem* 6.7.1 therefore coincide with the conditions in part (C) of *Theorem* 5.1.6.

6.8 Thiele fractions

Let $f(z)$ be known at the distinct points $\{z_0, z_1, z_2, \dots\}$. Inverse differences for $f(z)$ are given by

$$\varphi_0[z_k] := f(z_k), \qquad k \geq 0, \tag{6.8.1a}$$

$$\varphi_1[z_k, z_\ell] := \frac{z_\ell - z_k}{\varphi_0[z_\ell] - \varphi_0[z_k]}, \qquad \ell > k \geq 0, \tag{6.8.1b}$$

$$\varphi_\ell[z_0, \dots, z_\ell] := \frac{z_\ell - z_{\ell-1}}{\varphi_{\ell-1}[z_0, \dots, z_{\ell-2}, z_\ell] - \varphi_{\ell-1}[z_0, \dots, z_{\ell-1}]}, \qquad \ell \geq 1\ . \tag{6.8.1c}$$

The continued fraction

$$t(z) = \varphi_0[z_0] + \operatorname*{K}_{m=1}^{\infty} \left(\frac{z - z_{m-1}}{\varphi_m[z_0, \dots, z_m]} \right) \tag{6.8.2}$$

is a Thiele interpolating continued fraction for $f(z)$ [Thi06; BGM96, pp. 343–344], satisfying

$$t(z_k) = f(z_k), \qquad k = 0, 1, \dots.$$

Instead of inverse differences one can also compute reciprocal differences for $f(z)$:

$$\rho_0[z_k] := f(z_k), \qquad k \geq 0, \tag{6.8.3a}$$

$$\rho_1[z_k, z_\ell] := \frac{z_\ell - z_k}{f(z_\ell) - f(z_k)}, \qquad k \geq 0, \quad \ell \geq 0, \quad k \neq \ell, \tag{6.8.3b}$$

$$\rho_\ell[z_0, \dots, z_\ell] := \rho_{\ell-2}[z_0, \dots, z_{\ell-2}] + \frac{z_\ell - z_{\ell-1}}{\rho_{\ell-1}[z_0, \dots, z_{\ell-2}, z_\ell] - \rho_{\ell-1}[z_0, \dots, z_{\ell-1}]}, \qquad \ell \geq 2. \tag{6.8.3c}$$

The reciprocal differences are related to the inverse differences by

$$\begin{aligned} \varphi_0[z_k] &= \rho_0[z_k], \qquad k \geq 0, \\ \varphi_1[z_k, z_\ell] &= \rho_1[z_k, z_\ell], \qquad k \geq 0, \quad \ell \geq 0, \quad k \neq \ell, \\ \varphi_\ell[z_0, \dots, z_\ell] &= \rho_\ell[z_0, \dots, z_\ell] - \rho_{\ell-2}[z_0, \dots, z_{\ell-2}], \qquad \ell \geq 2. \end{aligned}$$

THEOREM 6.8.1: [MT51, p. 111]

A determinant formula for the reciprocal differences is given for $\ell \geq 1$, *by*

$$\rho_{2\ell-1}[z_0, \dots, z_{2\ell-1}] = \tag{6.8.4a}$$

$$\frac{\begin{vmatrix} 1 & f(z_0) & \dots & z_0^{\ell-2} & z_0^{\ell-2} f(z_0) & z_0^{\ell-1} & z_0^{\ell} \\ \vdots & \vdots & & & & \vdots & \vdots \\ 1 & f(z_{2\ell-1}) & \dots & z_{2\ell-1}^{\ell-2} & z_{2\ell-1}^{\ell-2} f(z_{2\ell-1}) & z_{2\ell-1}^{\ell-1} & z_{2\ell-1}^{\ell} \end{vmatrix}}{\begin{vmatrix} 1 & f(z_0) & \dots & z_0^{\ell-2} & z_0^{\ell-2} f(z_0) & z_0^{\ell-1} & z_0^{\ell-1} f(z_0) \\ \vdots & \vdots & & & & \vdots & \vdots \\ 1 & f(z_{2\ell-1}) & \dots & z_{2\ell-1}^{\ell-2} & z_{2\ell-1}^{\ell-2} f(z_{2\ell-1}) & z_{2\ell-1}^{\ell-1} & z_{2\ell-1}^{\ell-1} f(z_{2\ell-1}) \end{vmatrix}},$$

$$\rho_{2\ell}[z_0, \dots, z_{2\ell}] = \tag{6.8.4b}$$

$$\frac{\begin{vmatrix} 1 & f(z_0) & \dots & z_0^{\ell-1} & z_0^{\ell-1} f(z_0) & z_0^{\ell} f(z_0) \\ \vdots & \vdots & & & & \vdots \\ 1 & f(z_{2\ell}) & \dots & z_{2\ell}^{\ell-1} & z_{2\ell}^{\ell-1} f(z_{2\ell}) & z_{2\ell}^{\ell} f(z_{2\ell}) \end{vmatrix}}{\begin{vmatrix} 1 & f(z_0) & \dots & z_0^{\ell-1} & z_0^{\ell-1} f(z_0) & z_0^{\ell} \\ \vdots & \vdots & & & & \vdots \\ 1 & f(z_{2\ell}) & \dots & z_{2\ell}^{\ell-1} & z_{2\ell}^{\ell-1} f(z_{2\ell}) & z_{2\ell}^{\ell} \end{vmatrix}}.$$

We can see from *Theorem* 6.8.1 that the reciprocal differences offer the advantage that they do not depend on the numbering of their arguments $z_0, \dots, z_\ell$.

A continued fraction expansion for $f(z)$ at $z = u$ is obtained as the limiting value of (6.8.2) where all $z_k \to u$ [MT51, pp. 120–121]:

$$\lim_{\substack{z_k \to u \\ k \geq 0}} t(z) = \varphi_0(u) + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{z-u}{\varphi_m(u)} \right). \tag{6.8.5}$$

Here

$$\varphi_m(u) := \lim_{\substack{z_i \to u \\ i=0,\dots,m}} \varphi_m[z_0, \dots, z_m], \qquad m \geq 0.$$

The recursive scheme for the values $\varphi_\ell(u)$ is given by [MT51, pp. 117–119]

$$\varphi_0(u) = f(u) =: \rho_0(u) \tag{6.8.6a}$$

$$\varphi_1(u) = \left(\frac{df}{dz}\right)^{-1}_{\big|_{z=u}} =: \rho_1(u) \tag{6.8.6b}$$

$$\varphi_\ell(u) = \ell\left(\frac{d\rho_{\ell-1}(z)}{dz}\right)^{-1}_{\big|_{z=u}}, \qquad \ell \geq 2, \tag{6.8.6c}$$

where

$$\rho_\ell(z) := \lim_{\substack{z_i \to z \\ i=0,\ldots,\ell}} \rho_\ell[z_0, \ldots, z_\ell],$$

and hence

$$\rho_\ell(z) = \varphi_\ell(z) + \rho_{\ell-2}(z), \qquad \ell \geq 2. \tag{6.8.6d}$$

An alternative to this scheme for the construction of a *Thiele continued fraction expansion* is based on Viskovatov's algorithm. From the FTS of $f(z)$ at u,

$$f(z) = c_0^{(0)} + c_1^{(0)}(z-u) + c_2^{(0)}(z-u)^2 + \ldots$$

the coefficients $\varphi_\ell(u)$ in (6.8.6) can numerically be computed as follows

$$\varphi_0(u) = c_0^{(0)}, \tag{6.8.7a}$$

$$\varphi_1(u) = 1/c_1^{(0)}, \tag{6.8.7b}$$

$$c_k^{(1)} = -\varphi_1(u)c_{k+1}^{(0)}, \qquad k \geq 1, \tag{6.8.7c}$$

$$\varphi_\ell(u) = c_1^{(\ell-2)}/c_1^{(\ell-1)}, \qquad \ell \geq 2, \tag{6.8.7d}$$

$$c_k^{(\ell)} = c_{k+1}^{(\ell-2)} - \varphi_\ell(u)c_{k+1}^{(\ell-1)}, \qquad k \geq 1, \quad \ell \geq 2. \tag{6.8.7e}$$

For instance, $\varphi_2(u)$ and $\varphi_3(u)$ are given by

$$\varphi_2(u) = \frac{-\left(c_1^{(0)}\right)^2}{c_2^{(0)}},$$

$$\varphi_3(u) = \frac{-\left(c_2^{(0)}\right)^2/c_1^{(0)}}{\left(c_2^{(0)}\right)^2 - c_1^{(0)}c_3^{(0)}}.$$

Example 6.8.1: Take

$$f(z) = \frac{\mathrm{Ln}(1+z)}{z} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} z^k \,.$$

Applying (6.8.7) to the FTS of $f(z)$ at $z = 0$ delivers the Thiele continued fraction expansion of $f(z)$ at $z = 0$,

$$\begin{aligned} \frac{\mathrm{Ln}(1+z)}{z} &= 1 + \frac{z}{-2} + \frac{z}{-3/4} + \frac{z}{-16} + \frac{z}{-5/36} + \dots \\ &= 1 + \frac{-z/2}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\frac{a_m z}{m(m+1)}}{1} \right), \\ & \qquad a_{2k} = (k+1)^2, \quad a_{2k+1} = k^2, \quad k = 1, 2, \dots . \end{aligned} \tag{6.8.8}$$

7

Truncation error bounds

When investigating the convergence of continued fractions for functions $f(z)$, as in *Chapter* 3, the goal is to find largest possible convergence sets. In the present chapter the approach is different. Starting from a given continued fraction, the aim is to obtain smallest possible truncation error bounds for $|f(z) - S_n(z; w_n)|$. The truncation error bounds are often derived for a convergence set which contains the elements of the given continued fraction. The smaller this convergence set, the sharper the truncation error bounds.

There are two types of *truncation error* bounds. The *a priori* bound depends upon the elements of the continued fraction, whereas the *a posteriori* bound depends upon $f_n - f_{n-1}$, where f_n is the n^{th} approximant. In this chapter the main emphasis is on a priori bounds, even though a posteriori bounds can be sharper because they exploit the information contained in computed approximants.

7.1 Parabola theorems

We recall the parabola theorem from *Section* 3.3 now emphasising the truncation error bound. The parabola theorem deals with continued fractions $\mathrm{K}(a_m/1)$ where all elements a_m are located in a parabolic region

$$P_\alpha = \left\{a \in \mathbb{C} : |a| - \Re(ae^{-2\alpha i}) \leq {}^1\!/_2 \cos^2(\alpha)\right\}, \qquad |\alpha| < \pi/2. \tag{7.1.1}$$

THEOREM 7.1.1: PARABOLA THEOREM [LW92, p. 131]

The even and odd parts of $\mathrm{K}(a_m/1)$ *with all* $a_m \in P_\alpha$ *converge to finite values. The half plane* V_α *given by*

$$V_\alpha = \left\{w \in \mathbb{C} : \Re(we^{-\alpha i}) > -{}^1\!/_2 \cos(\alpha)\right\}$$

is a value set for P_α*. If, in addition, the continued fraction converges to a value* f *then*

$$|f - S_n(w_n)| \leq \frac{2|a_1|/\cos(\alpha)}{\prod_{k=2}^{n}\left(1 + \frac{\cos^2(\alpha)}{4(k-1)|a_k|}\right)}, \qquad w_n \in \overline{V}_\alpha\,. \tag{7.1.2}$$

The bound (7.1.2) holds for all w_n in the half plane, in particular for $w_n = 0$ which is always in V_α.

EXAMPLE 7.1.1: We apply the parabola theorem to

$$z\cot(z) - 1 = \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{-z^2/(4m^2-1)}{1}\right) = \frac{-z^2/3}{1} + \frac{-z^2/15}{1} + \frac{-z^2/35}{1} + \dots \qquad (7.1.3)$$

with $z = 1/2$. For any $\alpha \in (-\pi/2, \pi/2)$, all the elements of (7.1.3) are in P_α. For $\alpha = 0$ all elements of (7.1.3) are on the axis of the parabola, more specifically in the interval $(-1/12, 0)$ of the negative real axis, and the error bound (7.1.2) is minimal. For the fifth approximant of (7.1.3), the truncation error bound (7.1.2) gives for $z = 1/2$

$$|(z\cot(z) - 1) - S_5(z;w)| \le 0.994 \times 10^{-6}, \qquad \Re w > -1/2 \qquad (7.1.4)$$

while

$$(z\cot(z) - 1) - S_5(z;0) = -0.185\ldots \times 10^{-12}\ .$$

The error bound is a rather rough estimate.

EXAMPLE 7.1.2: Let $z = 1.17(1 - \mathrm{i})$ in (7.1.3). Then we cannot choose $\alpha = 0$ in (7.1.1). We observe that if $a_1 = -z^2/3 = 0.9126\mathrm{i} \in P_\alpha$, then also $a_m \in P_\alpha$ for $m \ge 1$. We therefore determine α such that $a_1 \in P_\alpha$ and such that $\cos(\alpha)$ is large in order to minimise (7.1.2). The value $\alpha = \pi/12$ satisfies these conditions. For the fifth approximant of (7.1.3), the truncation error bound (7.1.2) for $z = 1.17(1 - \mathrm{i})$ gives

$$|(z\cot(z) - 1) - S_5(z;0)| < 0.385 \times 10^{-1},$$

while

$$|(z\cot(z) - 1) - S_5(z;0)| = 0.277\ldots \times 10^{-6}\ .$$

The error bound again largely overestimates the true error.

There exists a generalisation of *Theorem* 7.1.1 where a single parabola is replaced by a sequence of parabolas determined by parameters $g_n \in (0,1)$ for $n \ge 1$ [LW92, pp. 136–137]. In (7.1.1) $1/2\ \cos^2(\alpha)$ is then replaced by $2g_{n-1}(1 - g_n)\cos^2(\alpha)$. This influences the width of the parabolas. If $g_n \to 0$ the parabolas degenerate to a ray from the origin at the angle 2α.

7.2 The oval sequence theorem

We recall from *Section* 3.3 that $\{V_n\}$ with

$$V_n = \{w \in \mathbb{C} : |w - C_n| < r_n\}, \qquad n = 0, 1, 2, \dots \tag{7.2.1}$$

is a sequence of value sets for the sequence $\{E_n\}$ of element sets given by

$$E_n = \big\{a \in \mathbb{C} : |a(1+\overline{C}_n) - C_{n-1}(|1+C_n|^2 - r_n^2)| + r_n|a| \le r_{n-1}(|1+C_n|^2 - r_n^2)\big\}, \qquad n = 1, 2, 3, \dots \tag{7.2.2}$$

if

$$0 < r_n < |1+C_n|, \qquad n = 0, 1, 2, \dots \tag{7.2.3a}$$

$$|C_{n-1}|r_n \le |1+C_n|r_{n-1}, \qquad n = 1, 2, 3, \dots. \tag{7.2.3b}$$

The sets V_n in (7.2.1) are disks, the sets E_n in (7.2.2) are Cartesian ovals.

THEOREM 7.2.1: OVAL SEQUENCE THEOREM [LW92, pp. 145–146]

Let $K(a_m/1)$ converge to the finite value f. If $a_n \in E_n$ for $n \in \mathbb{N}$ and $w_n \in \overline{V}_n$ for $n \in \mathbb{N}_0$, then

$$|f - S_n(w_n)| \le 2r_n \frac{|C_0| + r_0}{|1+C_n| - r_n} \prod_{k=1}^{n-1} M_k, \qquad n \ge 1, \tag{7.2.4}$$

where

$$M_k = \max_{w \in \overline{V}_k} \left| \frac{w}{1+w} \right| .$$

Note that, while the true truncation error $f - S_n(w_n)$ varies with w_n, the truncation error upper bound in *Theorem* 7.2.1 holds for all $w_n \in V_n$. The proof of the oval sequence theorem can be adapted to deliver a relative truncation error:

$$\left| \frac{f - S_n(w_n)}{f} \right| \le \frac{2r_n}{|1+C_n| - r_n} \prod_{k=1}^{n-1} M_k, \qquad n \ge 1. \tag{7.2.5}$$

The bound in *Theorem* 7.2.1 is an upper bound for $|f - S_n(0)|$ if $0 \in V_n$ from a certain n on. If not, nothing can be inferred about $|f - S_n(0)|$ from (7.2.4). Enlarging V_n such that $0 \in V_n$ yields less sharp truncation error bounds if the tails $f^{(n)} \in \overline{V}_n$ are not close to zero. However, a truncation error bound for $S_n(0)$ can be obtained from a truncation error bound for $S_n(w)$ and vice versa. Based on (1.3.2), (1.3.3) and the determinant formula (1.3.4), we have

$$S_n(w) - S_n(0) = \frac{(-1)^n w \prod_{k=1}^n a_k}{(B_n + wB_{n-1})B_n} .$$

where B_n is the n^{th} denominator of $K(a_m/1)$.
We now give an explicit formula for M_k in (7.2.4), obtained directly from basic properties of linear fractional transformations.

LEMMA 7.2.1:

If $0 < r$ *and* $C \in \mathbb{C}$ *with* $r < |1 + C|$, *then*

$$\max_{w\in\overline{V}} \left| \frac{w}{1+w} \right| = \frac{\left|C + |C|^2 - r^2\right| + r}{|1+C|^2 - r^2}, \tag{7.2.6}$$

where $V = \{w : |w - C| < r\}$.

For a given continued fraction, the sharpness of the truncation error bound (7.2.4) in the oval sequence theorem depends on the choice of C_n and r_n. The smaller the element set E_n containing the partial numerator a_n, the sharper the truncation error bound becomes. The difficulty in applying *Theorem* 7.2.1 is to find good values of C_n and r_n. We now discuss this issue in more detail.
When all a_n are in $(-1/4, +\infty)$, the oval and the disk are reduced to intervals on the real axis, and we can in many cases obtain the best oval. This important special case is discussed in *Section* 7.3.
In the other cases, there is no general rule for how to proceed in order to find best possible C_n and r_n. The following can be said.

- We know from *Section* 3.2 that, in case of convergence, the n^{th} tail of the continued fraction is in $\overline{V}_n$, and that all approximants of the n^{th} tail are located in V_n. So it is natural to choose the centre C_n of V_n to be an approximant or an approximation of the n^{th} tail. Let $\{f^{(n)}\}$ be the sequence of tails for the continued fraction $\mathrm{K}(a_m/1)$. Then $f^{(n-1)} = a_n/(1 + f^{(n)})$ for all n, or equivalently, if we exclude the case $f^{(n)} = -1$, it holds that $f^{(n-1)}(1 + f^{(n)}) - a_n = 0$. This suggests to choose $\{C_n\}_{n\in\mathbb{N}}$ such that $C_{n-1}(1 + C_n) - a_n$ is small. In the special case that we have a convergent limit periodic continued fraction $\mathrm{K}(a_m/1)$ with $\lim_{m\to\infty} a_m = a$, we know from *Theorem* 3.5.2 that the sequence $\{f^{(n)}\}$ converges to $c = (\sqrt{1+4a} - 1)/2$. Thus we can choose $C_n = c$.

EXAMPLE 7.2.1: Let f be defined by the limit periodic S-fraction

$$f(z) = \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m z}{1}\right), \quad a_m = 1 + \delta_m, \quad \delta_m \geq 0, \quad \lim_{m\to\infty} \delta_m = 0\,. \quad (7.2.7)$$

For $z = -4+2\mathrm{i}$, the sequence of tails of (7.2.7) converges to $c = (\sqrt{1+4z}-1)/2 = 2\mathrm{i}$. When choosing all $C_n = 2\mathrm{i}$, the sets V_n shrink in the limit to the point $2\mathrm{i}$ because of the limit periodicity and the factor M_k in (7.2.4) tends to $|2\mathrm{i}/(1+2\mathrm{i})| = 2/\sqrt{5} \simeq 0.894$. The rate at which a_m tends to $\lim_{m\to\infty} a_m$ is not crucial. The determining factors in the truncation error upper bound (7.2.4) are $C_n = c$ and r_n, as can be seen from (7.2.6).

- Remains to determine, for chosen C_n, the values r_n such that $a_n \in E_n$ given by (7.2.2) and such that the conditions (7.2.3) are satisfied. The following lemma helps to find a suitable sequence $\{r_n\}$.

LEMMA 7.2.2: [Lor03]

Let $K(a_m/1)$ be given and let $\{C_n\}$ be a sequence of complex numbers such that $|1+C_n| - |C_{n-1}| > 0$ for all $n \in \mathbb{N}$. If

$$r_0 = r_1, \qquad r_n = \sup_{m\geq n} \frac{2\left|C_{m-1} - \dfrac{a_m}{1+C_m}\right|}{1 - \left|\dfrac{C_{m-1}}{1+C_m}\right|}, \qquad n \in \mathbb{N},$$

satisfies

$$r_n \leq \frac{|1+C_n| - |C_{n-1}|}{2}, \qquad n \in \mathbb{N}, \qquad (7.2.8)$$

then $a_n \in E_n$ for all $n \in \mathbb{N}$ with E_n defined by (7.2.2), and (7.2.3) holds. Automatically $\{V_n\}$ with V_n given by (7.2.1) is a sequence of value sets for $\{E_n\}$.

EXAMPLE 7.2.2: We reconsider the continued fraction in *Example* 7.2.1 in order to illustrate *Lemma* 7.2.2. With $z = -4 + 2\mathrm{i}$ and $C_n = c = 2\mathrm{i}$ the values r_n are given by

$$r_0 = r_1, \qquad r_n = \frac{4\sqrt{5}}{\sqrt{5}-2}\delta_n, \qquad n = 1, 2, 3, \ldots.$$

If

$$\delta_n \le (\sqrt{5}-2)^2 \frac{\sqrt{5}}{40}, \qquad n = 1, 2, 3, \dots \tag{7.2.9}$$

we find

$$r_n \le \frac{|1+c| - |c|}{2} = \frac{\sqrt{5}-2}{2}, \qquad n = 1, 2, 3, \dots$$

and (7.2.8) is satisfied. With $C_n = c = 2\mathrm{i}$ and $r_n = r = 1/2(\sqrt{5}-2)$, we obtain from *Lemma* 7.2.1,

$$M_n = \frac{\left|c + |c|^2 - r^2\right| + r}{|1+c|^2 - r^2} = 0.918\dots, \qquad n \ge 1.$$

Finally, we obtain from the oval sequence theorem for $z = -4 + 2\mathrm{i}$,

$$|f(z) - S_{11}(z; w_{11})| < 2r \frac{|c| + r}{|1+c| - r} M_1^{10} < 0.1005, \qquad |w_{11} - 2\mathrm{i}| \le 1/2(\sqrt{5}-2)\ . \tag{7.2.10}$$

This bound holds for all continued fractions (7.2.7) for which (7.2.9) holds. In case all $\delta_m = 0$ and $f(z) = 2\mathrm{i}$, the true truncation error, for a few choices of w_{11}, equals

$$\left| f(z) - S_{11}\left(z; 2\mathrm{i} + 1/2(\sqrt{5}-2)\right)\right| \simeq 0.0346,$$
$$\left| f(z) - S_{11}\left(z; 2\mathrm{i} + 1/2(\sqrt{5}-2)\mathrm{i}\right)\right| \simeq 0.0335\ .$$

Note that (7.2.10) does not yield an upper bound for the approximant $S_{11}(z; 0)$.

The following result is a corollary of *Theorem* 7.2.1 for the choice $C_n = c$, where c is the limit value of the tails of $K(a_m/1)$.

COROLLARY 7.2.1: [LW92, pp. 151–154]

Let $K(a_m/1)$ be a limit periodic continued fraction converging to the finite value f with $\lim_{m\to\infty} a_m = a$ and $c = 1/2\left(\sqrt{1+4a} - 1\right)$. Let

$$\Delta = |1 + c| - |c|,$$
$$d_n = \sup\{|a_m - a| : m \ge n\}, \qquad n \ge 1.$$

If $d_2 < \Delta^2/4$, *and*

$$r_0 = \frac{2d_1 + |c|(\Delta - \sqrt{\Delta^2 - 4d_2})}{|1+c| + |c| + \sqrt{\Delta^2 - 4d_2}},$$
$$r_n = (\Delta - \sqrt{\Delta^2 - 4d_{n+1}})/2, \qquad n \geq 1,$$
$$C_n = c, \qquad n \geq 0,$$

then for E_n *and* V_n *given by (7.2.2) and (7.2.1) we have*

$$|f - S_n(w_n)| \leq 2r_n \frac{|c| + r_0}{|1+c| + |c| + \sqrt{\Delta^2 - 4d_{n+1}}} \prod_{k=1}^{n-1} M_k,$$
$$|w_n - c| \leq r_n, \quad n \geq 1,$$

where

$$M_k = \max\left\{\left|\frac{w}{1+w}\right| : |w - c| \leq r_k\right\}$$
$$\leq \frac{|1+c| + |c| - \sqrt{\Delta^2 - 4d_{k+1}}}{|1+c| + |c| + \sqrt{\Delta^2 - 4d_{k+1}}}, \qquad k \geq 1\ .$$

The choice $C_n = 0$ for $a = 0$ leads to the following corollary of the oval sequence theorem.

Corollary 7.2.2:

Let $0 < r_0 < 1$ *and let* $\{r_n\}_{n=0}^{\infty}$ *be a non-increasing sequence of positive numbers, then the disks* $V_n = \{w : |w| < r_n\}$ *form a sequence of value sets for the element sets* $E_n = \{a : |a| \leq r_{n-1}(1 - r_n)\}$*. Any continued fraction* $K(a_m/1)$ *with* $|a_m| \leq r_{m-1}(1 - r_m)$ *converges to a finite value* f *and*

$$|f - S_n(w_n)| \leq 2r_n \frac{r_0}{1 - r_n} \prod_{k=1}^{n-1} \frac{r_k}{1 - r_k}, \qquad |w_n| < r_n. \tag{7.2.11}$$

The *Corollary* 7.2.2 is particularly useful in case $\lim_{m\to\infty} a_m = a$ with $a = 0$ or $|a|$ sufficiently small.

Example 7.2.3: We reconsider the continued fraction (7.1.3) for $z = 1/2$. The partial numerators satisfy

$$|a_m| = \frac{1}{4(4m^2 - 1)} \qquad m \geq 1\ .$$

Since $\lim_{m\to\infty} a_m = 0$, we choose all $C_n = 0$. By an application of *Lemma* 7.2.2 we find

$$r_0 = r_1 = 2|a_1| = \frac{1}{6}, \qquad r_n = 2|a_n| = \frac{1}{2(4n^2-1)}, \qquad n \geq 1\ .$$

It follows from (7.2.11) that for $z = 1/2$

$$|(z\cot(z) - 1) - S_5(z; w_5)| \leq 0.136 \times 10^{-8}, \qquad |w_5| \leq r_5\ .$$

This bound is significantly sharper than the bound (7.1.4) obtained from the parabola theorem.

7.3 The interval sequence theorem

The oval sequence theorem can be simplified when formulated for continued fractions with real elements larger than $-1/4$.

THEOREM 7.3.1: INTERVAL SEQUENCE THEOREM [CVW06]

Let the real numbers L_n *and* R_n *satisfy*

$$-1/2 \leq L_n \leq R_n < \infty, \qquad n \in \mathbb{N}_0,$$

and let

$$\begin{aligned} b_n &:= (1 + \operatorname{sgn}(L_n)\max(|L_n|, |R_n|))L_{n-1}, \\ c_n &:= (1 + \operatorname{sgn}(L_n)\min(|L_n|, |R_n|))R_{n-1}, \end{aligned} \qquad n \in \mathbb{N},$$

be such that

$$b_n \leq c_n, \qquad 0 \leq b_n c_n, \qquad n \in \mathbb{N}.$$

Then

$$V_n := [L_n, R_n], \qquad n \in \mathbb{N}_0,$$

defines a sequence of value sets for the sequence of element sets

$$E_n := [b_n, c_n] = \begin{cases} [(1+R_n)L_{n-1}, (1+L_n)R_{n-1}], & b_n \geq 0, \\ [(1+L_n)L_{n-1}, (1+R_n)R_{n-1}], & b_n \leq 0, \end{cases} \qquad n \in \mathbb{N}.$$

If the continued fraction $\mathrm{K}\,(a_m/1)$ *with* $a_m \in E_m$ *converges to* f, *we have*

$$|f - S_n(w_n)| \leq (R_n - L_n)\frac{R_0}{1+L_n}\prod_{k=1}^{n-1} M_k, \qquad w_n \in V_n, \qquad (7.3.1)$$

where

$$M_k = \max\left\{\left|\frac{L_k}{1+L_k}\right|, \left|\frac{R_k}{1+R_k}\right|\right\}, \qquad k \in \mathbb{N}. \tag{7.3.2}$$

In the same way a bound on the relative truncation error can be proved:

$$\left|\frac{f - S_n(w_n)}{f}\right| \le \frac{R_n - L_n}{1+L_n} \prod_{k=1}^{n-1} M_k, \qquad w_n \in V_n. \tag{7.3.3}$$

In *Theorem* 7.3.1 the element sets E_n are determined from the value sets $V_n = [L_n, R_n]$. It is also possible, starting from given sets $E_n = [b_n, c_n]$, to determine the bounds L_n and R_n of the value sets V_n [CVW06]. For simplicity we assume that the sign of b_n is identical for all n. In the more general case, the principle remains the same and the formulas for L_n and R_n only become notationally more complicated. In case all $b_n \ge 0$,

$$L_n = \frac{b_{n+1}}{1} + \frac{c_{n+2}}{1} + \frac{b_{n+3}}{1} + \frac{c_{n+4}}{1} + \ldots,$$
$$R_n = \frac{c_{n+1}}{1} + \frac{b_{n+2}}{1} + \frac{c_{n+3}}{1} + \frac{b_{n+4}}{1} + \ldots,$$

and when all $b_n \le 0$,

$$L_n = \frac{b_{n+1}}{1} + \frac{b_{n+2}}{1} + \frac{b_{n+3}}{1} + \frac{b_{n+4}}{1} + \ldots,$$
$$R_n = \frac{c_{n+1}}{1} + \frac{c_{n+2}}{1} + \frac{c_{n+3}}{1} + \frac{c_{n+4}}{1} + \ldots.$$

Observe that L_n and R_n are tails of

$$\hat{D} = \frac{b_1}{1} + \frac{c_2}{1} + \frac{b_3}{1} + \frac{c_4}{1} + \ldots,$$
$$\hat{U} = \frac{c_1}{1} + \frac{b_2}{1} + \frac{c_3}{1} + \frac{b_4}{1} + \ldots,$$
$$\check{D} = \frac{b_1}{1} + \frac{b_2}{1} + \frac{b_3}{1} + \frac{b_4}{1} + \ldots,$$
$$\check{U} = \frac{c_1}{1} + \frac{c_2}{1} + \frac{c_3}{1} + \frac{c_4}{1} + \ldots.$$

More precisely, for $b_n \ge 0$ we have

$$L_{2k} = D^{(2k)}, \qquad L_{2k+1} = U^{(2k+1)}, \qquad k \ge 0,$$
$$R_{2k} = U^{(2k)}, \qquad R_{2k+1} = D^{(2k+1)}, \qquad k \ge 0,$$

and for $b_n \leq 0$ we have

$$L_n = \check{D}^{(n)}, \qquad R_n = \check{U}^{(n)}, \qquad n \geq 0.$$

In the special case that $E_n = E = [p, q]$ with $p \geq 0$, the bounds L_n and R_n reduce to the closed form expressions X and Y given by (3.3.7). In general, the bounds L_n and R_n in (7.3.1) are not computable because they are infinite expressions. We therefore have to compute suitable approximants of L_n and R_n which themselves bound L_n and R_n from below or above [CVW06].

7.4 Specific a priori bounds for S-fractions

Truncation error bounds as stated in the parabola and oval theorems apply to continued fractions of the form $\mathrm{K}(a_m(z)/1)$ and hence certainly to regular C-fractions $\mathrm{K}(a_m z/1)$ and to S-fractions $\mathrm{K}(a_m z/1)$ with $a_m > 0$. Specific truncation error bounds for S-fractions only hold for the classical approximants $f_n = S_n(0)$.

Truncation error bounds for continued fractions which are contractions of S-fractions, in particular real J-fractions, can be obtained from results given for S-fractions. The same holds for modified S-fractions. Necessary and sufficient conditions for the convergence of S-fractions are given in *Theorem* 3.1.5.

THEOREM 7.4.1: THRON/GRAGG-WARNER BOUND [GW83; Thr81]

Let $\mathrm{K}(a_m z/1)$ *be an S-fraction converging to* $f(z)$ *and let* $z = \rho e^{2\alpha \mathrm{i}}$ *with* $|\alpha| < \pi/2$. *Then*

$$|f(z) - f_n(z)| \leq 2 \frac{a_1 \rho}{\cos(\alpha)} \prod_{k=2}^{n} \frac{\sqrt{1 + 4a_k \rho / \cos^2(\alpha)} - 1}{\sqrt{1 + 4a_k \rho / \cos^2(\alpha)} + 1}, \qquad n \geq 2\ . \quad (7.4.1)$$

COROLLARY 7.4.1: [BHJ05]

Let $\mathrm{K}\left(a_m z/1\right)$ *be an S-fraction converging to* $f(z)$ *and let the coefficients* a_m *satisfy*

$$a_m \sim bm, \qquad m \to \infty,$$

or

$$a_m \sim bm^2, \qquad m \to \infty$$

for some constant $b > 0$. *Then there exist constants* $A > 0$, $B > 0$ *and* $C > 1$ *such that for* $n \geq 1$,

$$|f(z) - f_n(z)| \leq \begin{cases} \dfrac{A}{C^{\sqrt{n}}}, & a_m \sim bm, \quad m \to \infty, \quad |\arg z| < \pi, \\ \dfrac{A}{n^B}, & a_m \sim bm^2, \quad m \to \infty, \quad |\arg z| < \pi. \end{cases} \quad (7.4.2)$$

If the S-fraction is limit periodic and if the S-fraction coefficients satisfy certain monotonicity properties, the product in (7.4.1) can be replaced by a power. This is useful to determine in advance which approximant satisfies

$$|f(z) - f_n(z)| \le \epsilon,$$

but in doing so the truncation error bound becomes less sharp.

COROLLARY 7.4.2:

Let $\mathrm{K}\,(a_m z/1)$ *be a convergent S-fraction where* $\lim_{m\to\infty} a_m = a < \infty$. *Then for any* $p \in \mathbb{N}$ *and* $n \in \mathbb{N}, n \ge p$ *the following holds.*
(A) If $\{a_m\}$ *is an increasing sequence, then*

$$|f(z) - f_n(z)| \le \frac{2a_1\rho}{\cos(\alpha)} \prod_{k=2}^{p} \frac{\sqrt{1+\frac{4a_k\rho}{\cos^2(\alpha)}}-1}{\sqrt{1+\frac{4a_k\rho}{\cos^2(\alpha)}}+1} \left(\frac{\sqrt{1+4a\frac{\rho}{\cos^2(\alpha)}}-1}{\sqrt{1+4a\frac{\rho}{\cos^2(\alpha)}}+1}\right)^{n-p}$$

(B) If $\{a_m\}$ *is a decreasing sequence, then*

$$|f(z) - f_n(z)| \le \frac{2a_1\rho}{\cos(\alpha)} \prod_{k=2}^{p} \frac{\sqrt{1+\frac{4a_k\rho}{\cos^2(\alpha)}}-1}{\sqrt{1+\frac{4a_k\rho}{\cos^2(\alpha)}}+1} \left(\frac{\sqrt{1+4a_{p+1}\frac{\rho}{\cos^2(\alpha)}}-1}{\sqrt{1+4a_{p+1}\frac{\rho}{\cos^2(\alpha)}}+1}\right)^{n-p}$$

EXAMPLE 7.4.1: We consider the S-fraction of *Example* 7.2.1 and let $\delta_m = 0$ for all m. For $z = -4 + 2\mathrm{i}$ we have $f(z) = 2\mathrm{i}, \rho = \sqrt{20}$ and $1/\cos^2(\alpha) = 10 + 4\sqrt{5}$. We find from *Theorem* 7.4.1 for $n = 11$,

$$|f(z) - S_{11}(z;0)| = |f(z) - f_{11}(z)| < 38.93 \times (0.897)^{10} \le 13.143$$

while $|f(z) - f_{11}(z)| \simeq 1.316$. Compared to *Example* 7.2.2, the modified approximants $S_{11}(z; w_{11})$ do a better job, but in both cases the error bounds are of the correct magnitude.

7.5 A posteriori truncation error bounds

An *a posteriori truncation error bound*

$$|f - f_n| \le K_n |f_n - f_{n-1}| \tag{7.5.1}$$

can be determined only after the approximants f_n and f_{n-1} are computed. There is a difference in use between a priori and a posteriori error bounds. With a priori bounds we can determine in advance the index n for which f_n achieves a desired accuracy. We then only have to compute f_n for the particular index n. An a posteriori bound is a stopping criterion. One computes $f_1, f_2, f_3, \dots$ until the right hand side of (7.5.1) is sufficiently small. In some cases the a posteriori bound is more accurate, and stops the process at a lower value than the one determined by the a priori bound. A simple a posteriori error bound can be given for continued fractions with positive elements.

THEOREM 7.5.1: [LW92, p. 97]

Let $K(a_m/1)$ be a convergent continued fraction and let $a_m > 0$ for all m. Then the sequence $\{f_{2k+1}\}$ of odd order approximants is decreasing, and the sequence $\{f_{2k}\}$ of even order approximants is increasing. Every odd order approximant is larger than any even order approximant and hence

$$\left| f - \frac{f_{n-1} + f_n}{2} \right| \le \frac{|f_n - f_{n-1}|}{2}, \qquad n \ge 2 \ .$$

EXAMPLE 7.5.1: Consider a continued fraction coming from an evaluation of the complementary incomplete gamma function,

$$\begin{aligned} f &= 1/\left(e\Gamma(0,1)\right) - 1 \\ &= \frac{1}{1} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \frac{2}{1} \, {}_{+} \, \frac{2}{1} \, {}_{+} \, \frac{3}{1} \, {}_{+} \, \frac{3}{1} \, {}_{+} \, \frac{4}{1} \, {}_{+} \, \frac{4}{1} \, {}_{+} \, \dots \\ &= 0.676875028 \dots . \end{aligned}$$

This continued fraction satisfies the conditions of *Theorem* 7.5.1. We find

$$f_{10} = 0.67396 \ldots < f < f_{11} = 0.67846 \ldots$$

which leads to the approximation $(f_{10} + f_{11})/2 \simeq 0.676$ and the truncation error bound $(f_{11} - f_{10})/2 \simeq 0.00225$.

The next result deals with continued fractions with elements in the Worpitzky disk.

THEOREM 7.5.2: [JT88]

If the elements of $\mathrm{K}(a_m/1)$ *satisfy* $|a_m| \leq 1/4 - \epsilon$, *where* $0 < \epsilon < 1/4$, *then the continued fraction converges to a finite limit* f *and*

$$|f - f_n| \leq \frac{1 - 2\sqrt{\epsilon}}{4\sqrt{\epsilon}} |f_n - f_{n-1}|, \qquad n \geq 2 \ .$$

For S-fractions the following particular result can be given.

THEOREM 7.5.3: HENRICI-PFLUGER BOUND [HP66]

Let $K(a_m z/1)$ *be an S-fraction converging to a finite value* $f(z)$. *Then*

$$|f(z) - f_n(z)| \leq \begin{cases} |f_n(z) - f_{n-1}(z)|\,, & |\arg z| \leq \pi/2, \\ \left| \dfrac{f_n(z) - f_{n-1}(z)}{\sin(\arg z)} \right|, & \pi/2 < |\arg z| \leq \pi. \end{cases} \tag{7.5.2}$$

EXAMPLE 7.5.2: Consider

$$\mathrm{Ln}(1+z) = \frac{z}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{a_m z}{1} \right), \qquad |\,\mathrm{Arg}(1+z)| < \pi$$

where

$$a_{2k} = \frac{k}{2(2k-1)}, \quad a_{2k+1} = \frac{k}{2(2k+1)}, \quad k \geq 0.$$

We find for $n = 5$ and $z = (1+\mathrm{i})/2$

$$\begin{aligned} |f(z) - f_5(z)| &= 5.27\ldots \times 10^{-5} \\ &\leq |f_5(z) - f_4(z)| < 3.454 \times 10^{-4} \end{aligned}$$

and for $z = (-1+\mathrm{i})/2$

$$\begin{aligned} |f(z) - f_5(z)| &= 6.87\ldots \times 10^{-4} \\ &\leq \sqrt{2} |f_5(z) - f_4(z)| < 3.545 \times 10^{-3}. \end{aligned}$$

For real J-fractions and positive T-fractions converging to functions represented by Stieltjes transforms, specific a posteriori bounds can be given.

Theorem 7.5.4: [CJT93]

Let the real J-fraction

$$\frac{\alpha_1}{\beta_1 + z} + \overset{\infty}{\underset{m=2}{\mathrm{K}}} \left(\frac{-\alpha_m}{\beta_m + z} \right), \qquad \alpha_m > 0, \quad \beta_m \in \mathbb{R}, \quad m \in \mathbb{N}$$

converge to

$$\int_a^b \frac{d\Phi(t)}{z+t}, \qquad -\infty \leq a < b \leq +\infty$$

which represents holomorphic functions $F^+(z)$ *in* $\{z \in \mathbb{C} : \Im z > 0\}$ *and* $F^-(z)$ *in* $\{z \in \mathbb{C} : \Im z < 0\}$*. Then*

$$\left| \int_a^b \frac{d\Phi(t)}{z+t} - f_n(z) \right| \leq \frac{\alpha_1 |h_n(z)|}{|\Im(h_n(z))|} |f_n(z) - f_{n-1}(z)|, \qquad n \geq 2,$$

where $f_n(z)$ *is the* n^{th} *approximant of the real J-fraction,* $B_n(z)$ *its* n^{th} *denominator and* $h_n(z) = B_n(z)/B_{n-1}(z)$*.*

Theorem 7.5.5: [Jon77; Gra80]

Let the positive T-fraction

$$\overset{\infty}{\underset{m=1}{\mathrm{K}}} \left(\frac{z}{e_m + d_m z} \right)$$

converge to

$$\int_a^b \frac{z}{z+t} \, d\Phi(t)$$

in $z \in \mathbb{C} \setminus [-b, -a]$*. Then*

$$\left| \int_a^b \frac{z}{z+t} \, d\Phi(t) - f_n(z) \right| \leq \begin{cases} |f_n(z) - f_{n-1}(z)|, & |\arg z| \leq \dfrac{\pi}{2}, \\ \left| \dfrac{f_n(z) - f_{n-1}(z)}{\sin(\arg z)} \right|, & \dfrac{\pi}{2} < |\arg z| < \pi, \end{cases} \qquad n \geq 2,$$

where $f_n(z)$ *denotes the* n^{th} *approximant of the positive T-fraction.*

7.6 Tails and truncation error bounds

From the truncation error bound for a tail of a continued fraction, a bound for the fraction itself can be inferred. We assume that all fractions in question are converging to finite values, and that $n \geq k+1$:

$$f^{(k)} = \frac{a_{k+1}}{1} + \frac{a_{k+2}}{1} + \cdots + \frac{a_n}{1} + \ldots, \qquad f^{(0)} = f$$
$$S^{(k)}_{n-k}(w) = \frac{a_{k+1}}{1} + \frac{a_{k+2}}{1} + \cdots + \frac{a_n}{1+w}, \qquad S^{(0)}_n(w) = S_n(w)\ .$$

Then, from (1.3.2), (1.3.3) and the determinant formula (1.3.4) we obtain

$$f - S_n(w) = \frac{(-1)^k \prod_{j=1}^k a_j}{(B_k + B_{k-1} f^{(k)})(B_k + B_{k-1} S^{(k)}_{n-k}(w))} (f^{(k)} - S^{(k)}_{n-k}(w)), \tag{7.6.1}$$

where B_k is the k^{th} denominator of f. Formula (7.6.1) can also be expressed in terms of the critical tail sequence $\{-h_n\} = \{-B_n/B_{n-1}\}$ introduced in (1.9.10):

$$f - S_n(w) = \frac{(-1)^k \prod_{j=1}^k a_j}{B^2_{k-1}(h_k + f^{(k)})(h_k + S^{(k)}_{n-k}(w))} (f^{(k)} - S^{(k)}_{n-k}(w))\ . \tag{7.6.2}$$

In (7.6.1) and (7.6.2) the parameters $a_1, \ldots, a_k, B_k, B_{k-1}, h_k$ and $S^{(k)}_{n-k}(w)$ are known or can be computed. From the oval sequence theorem we obtain an upper bound for $|f^{(k)} - S^{(k)}_{n-k}(w)|$ and we know that $f^{(k)}$ and $S^{(k)}_{n-k}(w)$ are in the value set V_k. Therefore, using the notation of *Theorem* 7.2.1, expression (7.6.2) becomes

$$|f - S_n(w)| \leq \frac{\prod_{j=1}^k |a_j|}{|B^2_{k-1}|(\min\limits_{u \in \overline{V}_k} |h_k + u|)^2} \, \frac{2r_n(|C_k| + r_k)}{|1 + C_n| - r_n} \prod_{j=k+1}^{n-1} M_j,$$
$$n \geq k+1 \geq 2\ .$$

7.7 Choice of modification

To minimise the truncation error, we want the index n of the approximant to be large. To minimise the rounding error and for efficiency, we want the computation to be stable and n to be small. As we have already indicated in the *Chapters* 1 and 3, with a good modification w the truncation error for $S_n(w)$ is smaller than the truncation error for $S_n(0)$. Alternatively, both $S_n(0)$ and $S_m(w)$ with $m < n$ yield the same truncation error. When

w is chosen instead of 0, computing $S_n(w)$ involves one more addition than $S_n(0)$ and requires the same number of operations as $S_{n+1}(0)$ if $b_{n+1} = 1$ when using the backward recurrence algorithm described in *Section* 8.2. In this section we discuss several choices for the modification w.

General case. To quantify the improvement obtained by different modifications, we recall from *Section* 1.3 that

$$f - S_n(w) = \frac{(-1)^n \prod_{k=1}^n a_k}{(B_n + wB_{n-1})(B_n + f^{(n)}B_{n-1})}(f^{(n)} - w) \,. \qquad (7.7.1)$$

Hence

$$\frac{f - S_n(w)}{f - S_n(u)} = \frac{B_n + uB_{n-1}}{B_n + wB_{n-1}} \, \frac{f^{(n)} - w}{f^{(n)} - u} \,. \qquad (7.7.2)$$

Writing this in terms of the critical tail sequence $\{-h_n\} = \{-B_n/B_{n-1}\}$ defined by (1.9.10), we find

$$\frac{f - S_n(w)}{f - S_n(u)} = \frac{h_n + u}{h_n + w} \, \frac{f^{(n)} - w}{f^{(n)} - u} \,. \qquad (7.7.3)$$

Useful upperbounds for the left hand side of (7.7.3) are given in (7.7.7), (7.7.9) and (7.7.11).

For continued fractions of the form $K(c_m/d_m)$, a suitable modification can be obtained by setting

$$\tilde{w}_n = d_n w_n, \qquad n \geq 1, \qquad (7.7.4)$$

where w_n is a modification for the continued fraction $K(a_m/1)$ which is equivalent to $K(c_m/d_m)$, as described in *Section* 1.4. This follows from the fact that the tails of $K(c_m/d_m)$ and $K(a_m/1)$ satisfy

$$\mathop{\mathrm{K}}_{m=n+1}^{\infty} \left(\frac{c_m}{d_m}\right) = d_n \mathop{\mathrm{K}}_{m=n+1}^{\infty} \left(\frac{c_m/(d_m d_{m-1})}{1}\right) = d_n \mathop{\mathrm{K}}_{m=n+1}^{\infty} \left(\frac{a_m}{1}\right).$$

For $\tilde{w}_n$ given by (7.7.4), the n^{th} modified approximant of $K(c_m/d_m)$ with modification $\tilde{w}_n$ equals the n^{th} modified approximant of $K(a_m/1)$ with modification w_n.

Limit periodic case. Since many of the special functions have limit periodic continued fraction expansions, we discuss modifications only for limit periodic continued fractions $K(a_m/1)$. We distinguish between three cases.

- Since $\lim_{m\to\infty} a_m = a$, we can replace the n^{th} tail by the value of the periodic continued fraction $K(a/1)$,

$$w_n = w := \frac{a}{1} + \frac{a}{1} + \frac{a}{1} + \dots = \frac{\sqrt{1+4a}-1}{2}, \qquad a \notin (-\infty, -1/4). \tag{7.7.5}$$

Here we choose $\sqrt{1+4a}$ with $\Re\left(\sqrt{1+4a}\right) > 0$. When $a \notin (-\infty, -1/4] \cup \{0, \infty\}$ and

$$|a_n - a| \le \min\left(\frac{|1/4 + a| + 1/4 - |a|}{2}, |a|\right), \tag{7.7.6}$$

it is proved in [TW80a] that

$$\left|\frac{f - S_n(w)}{f - S_n(0)}\right| \le \max_{m\ge n} |a_m - a| \; \frac{2|a| + \left|1 + 2a + \sqrt{1+4a}\right|}{|a|\left(|1/4 + a| + 1/4 - |a|\right)}. \tag{7.7.7}$$

For a limit periodic continued fraction, condition (7.7.6) is always satisfied from a certain n on. Replacing $S_n(0)$ by $S_n((\sqrt{1+4a}-1)/2)$ then accelerates the convergence.
For $a = 0$ we get $w = 0$ in (7.7.5). The case $a = -1/4$ is more complicated and discussed in [TW80a]. For $a = \infty$, the choice (7.7.5) does not make sense since $S_n(\infty) = S_{n-1}(0)$.
- As an alternative to (7.7.5) we can choose

$$w_n := \frac{a_{n+1}}{1} + \frac{a_{n+1}}{1} + \frac{a_{n+1}}{1} + \dots = \frac{\sqrt{1+4a_{n+1}}-1}{2}, \qquad a_{n+1} \notin (-\infty, -1/4). \tag{7.7.8}$$

Again we take $\sqrt{1+4a_{n+1}}$ with $\Re\left(\sqrt{1+4a_{n+1}}\right) > 0$. This modification, called the *square root modification*, improves the convergence as follows [JJW87]. Under the conditions that the partial numerators a_n are eventually contained in some parabolic region

$$P_\alpha := \left\{z \in \mathbb{C} : |z| - \Re\left(ze^{-2\alpha i}\right) \le 1/2 \cos^2(\alpha)\right\}, \qquad |\alpha| < \pi/2$$

and that the sequence $\{a_{n+1} - a_n\}_{n\in\mathbb{N}}$ is bounded and has its limit points contained in some disk

$$\left\{z \in \mathbb{C} : \left|z - 2\rho^2 e^{2\alpha i}\right| \le 2R\right\}, \qquad 0 < R < \rho\cos(\alpha),$$

then

$$\left|\frac{f - S_n\left(1/2(\sqrt{1+4a_{n+1}} - 1)\right)}{f - S_n(0)}\right| \leq \frac{4\rho}{\sqrt{4a_{n+1}+1} - 2\rho - 1}\,. \tag{7.7.9}$$

This choice for w_n may be of use when $a_n \to \infty$.

- Finally, we can improve the modification w given in (7.7.5) as follows. Let $\mathrm{K}\,(a_m/1)$ be limit periodic and let $\lim_{m\to\infty} a_m = a \in \mathbb{C}$. If

$$s = \lim_{n\to\infty} \frac{a_{n+1} - a}{a_n - a}$$

exists, then the modification

$$w_n^{(1)} = w + \frac{a_{n+1} - a}{1 + (s+1)w} \tag{7.7.10}$$

accelerates the convergence under the following conditions [JW88]. Let for

$$\begin{aligned}
d &:= |1+w| - |sw|,\\
e &:= |1+w| - |w|,\\
\epsilon_n &:= \max_{m\geq n}\left|\frac{a_{m+1}-a}{a_m - a} - s\right|,\\
\delta_n &:= |a|\epsilon_n + 2|w|(\epsilon_n + |s|)\frac{a_{n+1}-a}{e + \sqrt{e^2 - 4(a_{n+1}-a)}},\\
\gamma_n &:= |w|\epsilon_n + 2\frac{a_{n+1}-a}{e + \sqrt{e^2 - 4(a_{n+1}-a)}}
\end{aligned}$$

the inequalities

$$\begin{aligned}
\alpha_{n+2} &:= \max_{m\geq n+2} |a_m - a| \leq e^2/4,\\
\beta_n &:= \min\left(\sqrt{d^2 - 8\delta_{n+1}}, d - 2\gamma_{n+1} - 2\sqrt{\gamma_{n+1}^2 + 4\gamma_{n+1}|ws|}\right) \geq 0
\end{aligned}$$

be satisfied, which in case of limit periodicity always hold from a certain n on. Then

$$\left|\frac{f - S_n(w_n^{(1)})}{f - S_n(w)}\right| \leq \left|\frac{h_n + w}{h_n + w_n^{(1)}}\right| \frac{d - \beta_n + \dfrac{2\alpha_{n+2}}{e + \sqrt{e^2 - 4\alpha_{n+2}}}}{4(1 + (1+s)w)}. \tag{7.7.11}$$

The results (7.7.7), (7.7.9) and (7.7.11) emphasise the importance of truncation error bounds for modified approximants rather than for classical approximants.

EXAMPLE 7.7.1: We reconsider the continued fraction of *Example* 7.5.1, which is limit periodic with $a_n \to \infty$ and converges to $f = 0.676875028\ldots$. Formula (7.7.8) recommends the modification

$$w_{2n-2} = w_{2n-1} = \frac{\sqrt{4n+1}-1}{2}, \qquad n \geq 1$$

for the evaluation of the continued fraction. Indeed the modified approximant clearly yields better results. For $n = 6$ we get $S_{10}(w_{10}) = S_{10}(2) \simeq 0.67670$ while $S_{10}(0) \simeq 0.67396$, and for $n = 20$ we find $S_{38}(w_{38}) = S_{38}(4) \simeq 0.67687501$ while $S_{38}(0) \simeq 0.67687417$.

EXAMPLE 7.7.2: Consider

$$f(x) = \frac{\Gamma(x+1)}{\Gamma\left(x+{}^1\!/\!_2\right)} = \operatorname*{K}_{m=1}^{\infty}\left(\frac{a_m}{1}\right), \qquad x+1 \in \mathbb{R} \setminus \mathbb{Z}_0^-$$

with

$$\begin{aligned} a_{2k+1} &= \frac{-k(k-x)}{2(x+2k-1)(x+2k)}, \\ a_{2k+2} &= \frac{-(x+k)(2x+k)}{2(x+2k)(x+2k+1)}, \end{aligned} \qquad k = 0,1,2,\ldots.$$

Use of the modifications w and $w_n^{(1)}$ is illustrated in the *Tables* 15.3.4 and 15.3.6. Here $a = -{}^1\!/\!_8$. For $x = 1$ condition (7.7.6) holds from $n = 7$ on because

$$|a_{2k+1} - a| = |a_{2k+2} - a| = \frac{3}{8(2k+1)}, \qquad k = 0,1,2,\ldots.$$

For $x = 100$ though, condition (7.7.6) is only satisfied from $n = 771$ on. The lack of improvement by plugging in the modification w in that region for x is also noticeable from the tables.

More general types of modification exist. Assume that we want to compute the value of a continued fraction $\mathrm{K}(a_m/1)$, which is near a well-known continued fraction $\mathrm{K}(b_m/1)$ in the sense that $a_m - b_m \to 0$ when $m \to \infty$. If we know the value g and the values of all the tails $g^{(n)}$ of $\mathrm{K}(b_m/1)$, then we can use the tails of the latter as modifications in the computation of the former [Jac87].

8

Continued fraction evaluation

After selecting an appropriate continued fraction approximant, the effect of finite precision machine arithmetic comes into play when programming the evaluation of this approximant. Let $f_n(w_n)$ or $S_n(z; w_n)$ denote the n^{th} modified approximant of the continued fraction representation of the function $f(z)$ and let $\mathtt{F}_n(z; w_n)$ denote the value obtained for $f_n(w_n)$ from the evaluation where all operations are replaced by their respective machine operations. Algorithms for the computation of $f_n(w_n)$ can be selected from *Section* 8.2. A detailed round-off error analysis is presented in *Section* 8.4.

8.1 The effect of finite precision arithmetic

Standard IEEE arithmetic. With respect to the underlying machine arithmetic, we assume that it is fully compliant with the IEEE 754-854 standard [Flo87] for floating-point arithmetic, by which we mean the following.

- Let us denote by β the base, by t the precision and by $[L, U]$ the exponent range of the IEEE floating-point arithmetic in use. The set $\mathbb{F}(\beta, t, L, U)$ of finite precision floating-point numbers, often denoted by $\mathbb{F}$, is then given by

$$\begin{aligned}\mathbb{F}(\beta, t, L, U) := &\{\pm d_0 \,.\, d_1 \ldots d_{t-1} \times \beta^e : d_0 \neq 0, 0 \leq d_i \leq \beta - 1, L \leq e \leq U\} \\ &\cup \{\pm 0 \,.\, d_1 \ldots d_{t-1} \times \beta^L : 0 \leq d_i \leq \beta - 1\} \\ &\cup \{+0, -0, +\infty, -\infty, \mathtt{NaN}\}.\end{aligned}$$

  Here `NaN` denotes the pattern that is returned for an undefined or irrepresentable result.
- Each of the four (nearest, upward, downward, toward zero) possible rounding functions

$$\bigcirc : \mathbb{R} \to \mathbb{F} : x \to \bigcirc(x)$$

satisfies

$$x \in \mathbb{F} \Rightarrow \bigcirc(x) = x,$$
$$x \leq y \Rightarrow \bigcirc(x) \leq \bigcirc(y) \ .$$

The rounding is either to the nearest floating-point neighbour, in which case

$$\frac{|x - \bigcirc(x)|}{|x|} \leq \frac{1}{2}\beta^{-t+1}, \qquad x \neq 0, \tag{8.1.1}$$

or is consistently upward, downward or toward zero, with

$$\frac{|x - \bigcirc(x)|}{|x|} \leq \beta^{-t+1}, \qquad x \neq 0. \tag{8.1.2}$$

The quantity β^{-t+1} is also called 1 `ulp` or *unit in the last place.*

- For maximal accuracy, each of the binary operations $* \in \{+, -, \times, \div,$ mod$\}$ is implemented such that

$$x \circledast y = \bigcirc(x * y), \qquad x, y \in \mathbb{F}, \tag{8.1.3}$$

where $\circledast$ denotes the machine version of the mathematical operation $*$. Again either (8.1.1) or (8.1.2) apply, depending on the rounding function, now with x replaced by $x * y$.

- Let d2b (decimal-to-base) and b2d (base-to-decimal) denote the conversions between decimal and base β representations. Each of the unary operations $* \in \{\sqrt{\ }, \mathrm{mod}, \mathrm{b2d}\}$ is supported such that

$$\circledast(x) = \bigcirc\left(*x\right), \qquad x \in \mathbb{F}, \tag{8.1.4}$$

while (8.1.4) is somewhat relaxed for the operation d2b in the sense that it does not have to hold for the entire range of real decimal numbers. Either (8.1.1) or (8.1.2) apply to $*x$, depending on the rounding function.

Error build-up. When evaluating the n^{th} approximant, plugging in the continued fraction's tail estimate w_n and replacing all mathematical operations by machine operations, several errors come into play. The relative truncation error ϵ_T and *round-off error* ϵ_R are defined by

$$\epsilon_T := \frac{|f(z) - S_n(z; w_n)|}{|f(z)|}, \tag{8.1.5a}$$

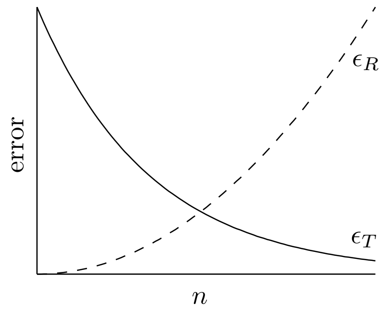
$$\epsilon_R := \frac{|S_n(z; w_n) - \mathtt{F}_n(z; w_n)|}{|f(z)|} \ . \tag{8.1.5b}$$

Both ϵ_T and ϵ_R depend on several parameters, among which f, z, n, w_n and the parameters of the floating-point arithmetic in use. For the compound error we write

$$\epsilon_C(f, z, n, w_n, \mathbb{F}) := \frac{|f(z) - \mathtt{F}_n(z; w_n)|}{|f(z)|} \leq \epsilon_T + \epsilon_R \,. \tag{8.1.5c}$$

When n increases in (8.1.5a), the truncation error ϵ_T decreases, but the number of operations in the computation of $f_n(w_n) = S_n(z; w_n)$ and hence also the accumulated round-off error ϵ_R increase. A typical situation is illustrated in *Figure* 8.1.1. For simplicity we assume that the argument z is an exact floating-point number.

FIGURE 8.1.1: Typical evolution of truncation and round-off error.



When targeting a composite error threshold $\epsilon \geq \epsilon_C$, the following approach is used. It is based on the fact that the round-off error ϵ_R does not only depend on n but also very much on the finite floating-point precision t. An increase in t implies a decrease in ϵ_R over all n. In practice, in order to guarantee a maximal relative error of ϵ:

1. we determine n such that $\epsilon_T \leq \epsilon/2$;
2. we specify a precision t for the computation of $f_n(w)$ to guarantee $\epsilon_R \leq \epsilon/2$.

Significant digits. When $\hat{f} = \pm d_0 \,.\, d_1 \ldots d_{t-1} \times \beta^e$ is a computed floating-point approximation for a nonzero value f, then the k^{th} digit d_{k-1} of $\hat{f}$ is called a *significant digit* for f if

$$\frac{|\hat{f} - f|}{|f|} \leq \frac{\beta}{2}\beta^{-k} = \frac{1}{2}\beta^{-k+1}.$$

Usually $\hat{f}$ is subject to some truncation and round-off error and f is not known. Then one can only get some information on the number of significant digits in $\hat{f}$ from the knowledge of truncation and round-off error upperbounds. Of course, since these bounds are mostly pessimistic, in this way one underestimates the correct number of significant digits.

8.2 Evaluation of approximants

In many applications of continued fractions $\mathrm{K}\,(a_m/b_m)$ the elements a_m and b_m are given and one must evaluate the n^{th} approximant

$$f_n(0) = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}, \qquad a_m \in \mathbb{C}\backslash\{0\}, \quad b_m \in \mathbb{C}.$$

Based on the fact that the n^{th} approximant f_n equals the first unknown x_1 of the linear system [Mik76]

$$\begin{pmatrix} b_1 & -1 & 0 & \dots & & 0 \\ a_2 & b_2 & -1 & 0 & \dots & 0 \\ 0 & a_3 & b_3 & -1 & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ & & & a_{n-1} & b_{n-1} & -1 \\ 0 & \dots & & 0 & a_n & b_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{8.2.1}$$

several algorithms can be devised for the computation of f_n. Four algorithms for computing f_n are described here, of which the forward and backward recurrence are the most popular.

Euler-Minding summation. Reducing the tridiagonal matrix in (8.2.1) to an upper triangular form by Gaussian elimination, is equivalent to using the series representation (1.7.5) of f_n [Bla64]:

$$\begin{aligned} h_1 &= b_1, \\ h_k &= b_k + \frac{a_k}{h_{k-1}}, \qquad k \geq 2, \\ f_1 &= a_1/b_1, \\ f_n &= \sum_{k=1}^{n} (-1)^{k-1} \frac{a_1 \cdots a_k}{h_1^2 \cdots h_{k-1}^2 h_k}, \qquad n > 1. \end{aligned}$$

Here $\{h_n\}_{n\in\mathbb{N}}$ coincides with the critical tail sequence (1.9.10). If $\omega(f_n)$ denotes the number of basic operations (addition, multiplication and division) required to compute f_n, and $\omega(f_1, \ldots, f_{n-1} \to f_n)$ denotes the number of operations required to obtain f_n from $f_1, \ldots, f_{n-1}$, then $\omega(f_1) = 1, \omega(f_2) = 6$ and

$$\omega(f_{i-2}, f_{i-1} \to f_i) = 5, \qquad i \geq 3.$$

Hence

$$\omega(f_1, \ldots, f_n) = 5n - 3, \qquad n \geq 2. \tag{8.2.2}$$

Backward recurrence. Reducing the tridiagonal matrix in (8.2.1) to a lower triangular form by Gaussian elimination, leads to a very efficient algorithm to compute a single approximant:

$$\begin{aligned} F_{n+1}^{(n)} &= 0 \\ F_k^{(n)} &= \frac{a_k}{b_k + F_{k+1}^{(n)}}, \qquad k = n, n-1, \ldots, 1 \\ f_n &= F_1^{(n)}. \end{aligned} \tag{8.2.3}$$

The arithmetic complexity of the backward algorithm is

$$\omega(f_1) = 1, \qquad \omega(f_n) = 2n - 1, \qquad n \geq 2$$

and

$$\omega(f_1, \ldots, f_n) = \sum_{k=1}^{n} \omega(f_k) = n^2, \qquad n \geq 1.$$

When computing a modified approximant $f_n(w_n)$ instead of a classical approximant $f_n(0)$, then $F_{n+1}^{(n)} = w_n$.

Forward recurrence. Let A_n and B_n denote the n^{th} numerator and n^{th} denominator, respectively, of $\mathrm{K}\,(a_m/b_m)$. Then by the recurrence relations (1.3.1) one computes $A_1, B_1, A_2, B_2, \ldots, A_n, B_n$ and

$$f_n = \frac{A_n}{B_n}. \tag{8.2.4}$$

It is readily seen that $\omega(f_1) = 1$, $\omega(f_2) = 4$, and

$$\omega(f_n) = 6n - 8, \qquad n \geq 3.$$

This assumes that we set $A_1 = a_1$, $B_1 = b_1$, $A_2 = b_2 A_1$ and $B_2 = a_2 + b_2 B_1$. The number of operations required to compute $f_1, f_2, \ldots, f_n$ equals the number of operations required to get A_n and B_n and the additional n divisions A_n/B_n:

$$\omega(f_1, \ldots, f_n) = (6n - 9) + n = 7n - 9, \qquad n \geq 3.$$

From the approximants A_{n-1}/B_{n-1} and A_n/B_n the modified approximant $f_n(w_n)$ can be obtained by using (1.3.2). Thus in terms of the number of arithmetic operations required, the backward recurrence is more efficient than the forward recurrence if one computes a single approximant f_n.

Product form. Introducing the related linear system

$$\begin{pmatrix} b_2 & -1 & 0 & \dots & & 0 \\ a_3 & b_3 & -1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & & \\ \vdots & \ddots & & & & \\ & & a_{n-1} & b_{n-1} & -1 \\ 0 & \dots & 0 & a_n & b_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} a_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad (8.2.5)$$

allows to write $f_n = x_n/y_{n-1}$. Reducing the coefficient matrices in both (8.2.1) and (8.2.5) to upper triangular form, leads to [CVDC85]:

$$\begin{aligned} g_1 &= a_1, \qquad g_2 = b_2 \\ g_k &= b_k + a_k/g_{k-1}, \qquad k > 2 \\ h_1 &= b_1, \\ h_k &= b_k + a_k/h_{k-1}, \qquad k > 1 \\ f_n &= \prod_{k=1}^{n} \frac{g_k}{h_k} \end{aligned}$$

where one implicitly makes use of the critical tail sequence $\{h_n\}_{n\in\mathbb{N}}$ (1.9.10). Assuming that one stores a_1/b_1 during the computation of f_2, we obtain $\omega(f_1) = 1, \omega(f_2) = 4$ and $\omega(f_n) = 6n - 8$ for $n \geq 3$, just as for the forward algorithm. Moreover,

$$\omega(f_{n-1} \to f_n) = 6, \qquad n \geq 3$$

and hence

$$\omega(f_1, \dots, f_n) = 6n - 7.$$

8.3 The forward recurrence and minimal solutions

As explained in *Section* 3.6, the forward recurrence algorithm

$$y_n = b_n y_{n-1} + a_n y_{n-2}, \qquad n = 1, 2, 3, \dots \qquad (8.3.1)$$

where we start with initial values y_{-1} and y_0 and compute $y_1, y_2, y_3, \dots$ using (8.3.1), is numerically *stable* for the computation of dominant solutions $\{v_n\}$ of (8.3.1), but numerically unstable for minimal solutions $\{u_n\}$, meaning that small errors in the initial terms may lead to unbounded errors in later terms. We denote by $\hat{y}_1, \hat{y}_2, \ldots,$ the numbers obtained when replacing y_{-1} and y_0 by approximations $\hat{y}_{-1}$ and $\hat{y}_0$ and executing (8.3.1) in finite precision. Since, in general, $\{\hat{y}_n\}$ is not proportional to $\{y_n\}$, it is a dominant solution and so

$$\lim_{n\to\infty} \frac{y_n}{\hat{y}_n} = 0$$

and hence

$$\lim_{n\to\infty} \left| \frac{\hat{y}_n - y_n}{y_n} \right| = \infty \, .$$

This implies that the forward recurrence is unstable for the computation of minimal solutions.

A stable algorithm for computing minimal solutions of the system of three-term recurrence relations (8.3.1) is based on continued fractions. If $\{u_n\}$ is a minimal solution with $u_{-1} \neq 0$, we have from *Theorem* 3.6.1 that

$$u_n = u_{-1} \prod_{j=0}^{n} \frac{u_j}{u_{j-1}} = u_{-1} \prod_{j=0}^{n} (-f^{(j)}), \qquad n = 0, 1, 2, \ldots$$

with

$$f = -\frac{u_0}{u_{-1}} = \frac{a_1}{b_1} \, \underset{+}{} \, \frac{a_2}{b_2} \, \underset{+}{} \, \ldots$$

and its tails $f^{(j)}$ defined in (1.9.3). To approximate u_n we start with one initial approximation $\hat{u}_{-1}$ of u_{-1} and approximate successive ratios u_j/u_{j-1} by

$$-f_k^{(j)} = -\frac{a_{j+1}}{b_{j+1}} \, \underset{+}{} \, \frac{a_{j+2}}{b_{j+2}} \, \underset{+}{} \, \cdots \, \underset{+}{} \, \frac{a_{j+k}}{b_{j+k}}, \qquad j = 0, 1, \ldots, \quad k = 1, 2, \ldots.$$

Then, for any $k \in \mathbb{N}$, an approximation $\hat{u}_n$ of u_n is given by

$$\hat{u}_n = \hat{u}_{-1} \prod_{j=0}^{n} (-f_k^{(j)}), \qquad n = 0, 1, 2, \ldots.$$

The quality of the approximation $\hat{u}_n$ depends on how well $f_k^{(n)}$ approximates $-u_n/u_{n-1}$ and on the stability and precision of the computation of $f_k^{(n)}$.

In [PFTV92, p. 181] is explained how an initial approximation $\hat{u}_{-1}$ can be obtained from a so-called normalisation, such as a formula for the sum of the u_n's. The technique is often referred to as *Miller's algorithm*. Assume that besides the three-term recurrence and the continued fraction (3.6.3) for $-u_{n-1}/u_{n-2}$ we also have at our disposal some information of the form

$$\omega_{-1}u_{-1} + \sum_{j=0}^{\infty} \omega_j u_j = \omega. \tag{8.3.2}$$

If we put

$$\omega^{(n)} = \frac{1}{u_n} \sum_{j=n+1}^{\infty} \omega_j u_j,$$

then

$$\begin{aligned} f^{(n-1)} &= \frac{a_n}{b_n + f^{(n)}}, \\ \omega^{(n-1)} &= -f^{(n)} \left(\omega_n + \omega^{(n)} \right) . \end{aligned} \tag{8.3.3}$$

Starting with an approximation $\hat{\omega}^{(n)} = 0$ and applying the backward scheme (8.3.3) ultimately delivers $f^{(0)}$ and $\hat{\omega}^{(-1)}$ and from there

$$\hat{u}_{-1} = \frac{\omega}{\hat{\omega}^{(-1)} + \omega_{-1}}.$$

A computed version of the minimal solution $\{u_n\}$ is then again obtained from

$$\hat{u}_n = -f_{k_n}^{(n)} \hat{u}_{n-1}, \qquad n = 0, 1, 2, \ldots, \quad k_n \geq 0.$$

8.4 Round-off error in the backward recurrence

Round-off error occurs in the computation of an approximant

$$f_n = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}, \qquad a_m, b_m \in \mathbb{C} \setminus \{0\}$$

in the context of finite precision arithmetic. The relative round-off error in the computed approximation $\hat{f}_n = \mathtt{F}_n(z; w_n)$ of f_n is denoted by

$$\epsilon_n := \frac{|f_n - \hat{f}_n|}{|f_n|}. \tag{8.4.1}$$

Note that $\epsilon_R = \epsilon_n |f_n/f|$.

In this section we give some rigorous error bounds for ϵ_n. Fortunately, for some of the most important families of continued fractions, such as S-fractions, positive T-fractions and real J-fractions, the relative round-off error ϵ_n has a slow rate of growth when the backward recurrence (8.2.3) is used to compute $\hat{f}_n$ [JT74].

Relative round-off error bounds. Serious problems arise if one attempts to compute a minimal solution of the recurrence relation (1.3.1). If we compute such a solution $\{A_n\}$, using only approximate starting values A_1 and A_2, due to rounding for example, we obtain a solution $\{\hat{A}_n\}$ that is, in general, linearly independent of $\{A_n\}$. Hence $\lim_{n\to\infty} A_n/\hat{A}_n = 0$ and

$$\lim_{n\to\infty} \left| \frac{\hat{A}_n - A_n}{A_n} \right| = \infty$$

meaning that the relative error of the computed $\hat{A}_n$, the intended approximation to A_n, becomes arbitrarily large. Therefore the forward recurrence is not guaranteed to be a stable procedure [Gau67].

When implementing $f_n(w)$, we need to take into account that each basic operation $\star \in \{+, -, \times, \div\}$ is being replaced by a machine operation $\circledast \in \{\oplus, \ominus, \otimes, \oslash\}$ and hence subject to a relative error as indicated in (8.1.1) or (8.1.2). Also each partial numerator a_m and denominator b_m needs to be converted to machine numbers $\hat{a}_m$ and $\hat{b}_m$, thereby entailing relative rounding errors $\epsilon_m^{(a)}$ and $\epsilon_m^{(b)}$ given by

$$\begin{aligned} \hat{a}_m &= a_m(1 + \epsilon_m^{(a)}), \\ \hat{b}_m &= b_m(1 + \epsilon_m^{(b)}). \end{aligned}$$

Here $|\epsilon_m^{(a)}|$ and $|\epsilon_m^{(b)}|$ are bounded by $1/2$ `ulp` in round-to-nearest and only if a_m and b_m are not compound expressions. Otherwise they may be somewhat larger. Without loss of generality, we assume that w is a machine number estimating the tail t_n. When executing the backward recurrence, each computed $\hat{F}_k^{(n)}$ differs from the true $F_k^{(n)}$ by a relative rounding error $\epsilon_k^{(n)}$, and this for $k = n, \ldots, 1$:

$$\begin{aligned} \hat{F}_{n+1}^{(n)} &= w, \qquad \epsilon_{n+1}^{(n)} = 0, \\ \hat{F}_k^{(n)} &= \hat{a}_k \oslash \left(\hat{b}_k \oplus \hat{F}_{k+1}^{(n)} \right) \\ &= \frac{\hat{a}_k}{\hat{b}_k + \hat{F}_{k+1}^{(n)}} (1 + \delta_k) \\ &= F_k^{(n)}(1 + \epsilon_k^{(n)}), \qquad k = n, \ldots, 1, \\ \hat{F}_1^{(n)} &= F_1^{(n)}(1 + \epsilon_1^{(n)}). \end{aligned}$$

Here δ_k is the relative rounding error introduced in step k of the algorithm. The main question is: how large is $|\epsilon_1^{(n)}|$? This question is answered in *Theorem* 8.4.1, the latter being a slight generalisation of a result proved in [JT74] where $F_{n+1}^{(n)} = 0$. Let us introduce the notation

$$\gamma_k^{(n)} = F_{k+1}^{(n)}/(b_k + F_{k+1}^{(n)}), \qquad k = 1, \ldots, n. \tag{8.4.2}$$

THEOREM 8.4.1: [JT74]
Let $F_{n+1}^{(n)} = w$ be a machine number and let for $k = 1, \ldots, n$

$$\begin{aligned} |\epsilon_k^{(a)}| &\le \epsilon^{(a)} \ \mathtt{ulp}, \\ |\epsilon_k^{(b)}| &\le \epsilon^{(b)} \ \mathtt{ulp}, \\ |\delta_k| &\le \delta \ \mathtt{ulp}, \\ |\gamma_k^{(n)}| &\le \gamma_n, \qquad G(n) = \sum_{j=0}^{n-1} \gamma_n^j. \end{aligned}$$

Let the base β and precision t of the IEEE arithmetic in use satisfy

$$\left(1 + 2\epsilon^{(b)}(1+\gamma_n) + (1 + 2\epsilon^{(a)} + 2\epsilon^{(b)}(1+\gamma_n) + 2\delta)(G(n)-1)\right)^2 < \beta^{t-1}.$$

Then

$$|\epsilon_1^{(n)}| \le \frac{1 + 2\epsilon^{(a)} + 2\epsilon^{(b)}(1+\gamma_n) + 2\delta}{2} G(n) \ \mathtt{ulp}\,. \tag{8.4.3}$$

It is clear that applications of *Theorem* 8.4.1 require realistic estimates of the quantities $\gamma_n^{(k)}$ defined by (8.4.2). In the remainder of this section we describe how to obtain upper bounds of $|\gamma_k^{(n)}|$.

Methods for estimating $\boldsymbol{\gamma_k^{(n)}}$. It is important to note that the quantity $\gamma_n^{(k)}$ is invariant under equivalence transformations of continued fractions. The significance of this is that we do not need to search for an optimal form of a continued fraction from a point of view of minimising the $\gamma_n^{(k)}$. The following is a slight improvement of a result found in [JT74] in the sense that we do not assume $0 \in V_n$.

THEOREM 8.4.2: [JT74]

Let the subsets $V_1, \ldots, V_n$ *of the extended complex plane satisfy*

$$f^{(n)} \in V_n,$$
$$\frac{a_k}{b_k + V_k} \subseteq V_{k-1}, \qquad k = 2, 3, \ldots, n$$

and let

$$A^{(n)} = \max_{2 \le m \le n} |a_m|,$$
$$B^{(n)} = \min_{1 \le m \le n} d(-b_m, V_m),$$
$$M^{(n)} = \sup \{|w| : w \in V_m/(b_m + V_m)\}.$$

Then
(A) $|\gamma_k^{(n)}| \le A^{(n)}/(B^{(n)})^2$, *and*
(B) $|\gamma_k^{(n)}| \le M^{(n)}$.

Part III

SPECIAL FUNCTIONS

9

On tables and graphs

In the *Chapters* 10 to 19, we study several families of special functions and their various series and continued fraction representations. Only a small number of these representations is also found in [AS64]. The latter are marked with the symbol AS in the margin.
The collected formulas are further illustrated numerically and graphically. We now explain how to interpret and use the tables and graphs. In the sequel we consistently use z for a complex argument and x for a real argument.

9.1 Introduction

While we mention the domain of convergence with every continued fraction in the next chapters, the precise convergence behaviour is not described. Since, in practice, it is the initial convergence behaviour that matters and not the asymptotic one, we illustrate the convergence rate empirically.
This is done,

- either numerically, in tables, where we evaluate different continued fraction representations for a large range of arguments,
- or graphically, by presenting level curves of significant digits, or graphing the evolution of the approximants' accuracy.

The former is detailed in *Section* 9.2 and the latter in *Section* 9.3. All tables and graphs are labelled and preceded by an extensive caption.

9.2 Comparative tables

In the next chapters all formulas which are evaluated in one of the tables, are marked with the symbol ▦ in the right margin. For formulas that are not marked in that way no numerical illustration of their behaviour is given. All tables are composed in the same way. The two leftmost columns contain the function argument and the function value. The function value is the correctly rounded mathematical value, verified in a variety of programming environments. In case the function value $f(z)$ is a complex value, only its *signed modulus*

$$|f(z)|_s = \operatorname{sgn}(\Re f(z))\ |f(z)| \tag{9.2.1}$$

is returned. The sign of $\Re(f(z))$ indicates whether the complex value $f(z)$ lies in the right or the left half-plane. The other columns contain the relative truncation error

$$\left|\frac{f(z) - f_n(z)}{f(z)}\right| \tag{9.2.2}$$

incurred when using a certain partial sum or continued fraction approximant $f_n(z)$ instead of the function $f(z)$ under investigation. The continued fraction approximant $f_n(z)$ can be either a classical approximant $f_n(z;0)$ or a modified approximant $f_n(z;w_n)$.
The evaluation of the special function for the selected arguments is exactly rounded to 7 decimal digits and the truncation errors are upward rounded to 2 decimal digits. Since the modulus of the truncation error (9.2.2) is always positive, the sign is omitted here.
The approximant number n doesn't appear in the table but is mentioned in the caption. By tabulating the truncation error for different n, also the speed of convergence is illustrated.
The function arguments are selected in the intersection of the domains associated with each of the formulas evaluated in the table (with a slight exception for some series representations). The resulting set of arguments is traversed in the following way, if applicable: from the positive real axis over the first quadrant to the positive imaginary axis, then through the second quadrant of the complex plane to the negative real axis and so on. As a rule templates of all possible function arguments are tabulated, for increasing modulus, except when function evaluations for different arguments are related by symmetry relations. The numerical illustration of the elementary functions forms an exception: since these are thoroughly illustrated graphically, evaluations in the tables are restricted to real arguments only.
When evaluating the approximants of a limit periodic continued fraction $\mathrm{K}_{m=1}^{\infty}(a_m/1)$ of which the partial numerators do not tend to zero, use of one or more modifications may be appropriate. In that case the evaluations without modification and with use of the different modifications are tabulated. We clearly indicate in the caption of the table which column in the table illustrates which modification.
When the upward rounded relative truncation error satisfies

$$C = \triangle\left(\left|\frac{f(z) - f_n(z)}{f(z)}\right|\right) \le 5 \times 10^{-s}, \qquad s \in \mathbb{N}, \tag{9.2.3}$$

then the approximation $f_n(z)$ guarantees s significant decimal digits compared to the exact value $f(z)$. When $C \simeq 10^k$ with $k \geq 0$, care must be taken in interpreting the quality of the approximation $f_n(z)$. For $k > 1$ we find $|f_n(z)| \simeq 10^k |f(z)|$, while for $k = 0$ we can very well have $|f_n(z)| \ll |f(z)|$. In both cases $f_n(z)$ can be way off, even missing to predict the magnitude of $f(z)$. In general

$$|f(z) - f_n(z)| \leq C|f(z)| \Longrightarrow |f_n(z)| \in |f(z)| \, [1 - C, 1 + C].$$

All printed values in the tables are verified and therefore reliable. Where IEEE 754 arithmetic was insufficient because of overflow or underflow, multiprecision interval arithmetic or high precision computer algebra implementations were used.

Example 9.2.1: Consider

$$_2F_1\left(1/2, 1; 3/2; z\right) = \frac{1}{2\sqrt{z}} \operatorname{Ln}\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

which has the regular C-fraction representation (15.3.7) given by

$$z \, _2F_1\left(1/2, 1; 3/2; z\right) = \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{c_m z}{1}\right), \qquad z \in \mathbb{C} \setminus [1, +\infty),$$

$$c_1 = 1, \quad c_m = \frac{-(m-1)^2}{4(m-1)^2 - 1}, \quad m \geq 2.$$

The function also has the M-fraction representation (15.3.12) given by

$$\frac{1/2}{1/2 + z/2} - \frac{z}{3/2 + 3z/2} - \frac{4z}{5/2 + 5z/2} - \dots, \qquad |z| < 1$$

and the so-called Nörlund fraction representation (15.3.17) given by

$$\frac{1}{1-z} + \frac{z(1-z)}{3/2 - 5/2 z} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{m(m - 1/2)z(1-z)}{(m + 1/2) - (2m + 1/2)z}\right), \qquad \Re z < 1/2.$$

The intersection of the domain of $f(z) = {_2F_1}(1/2, 1; 3/2; z)$ with the convergence domains of the three continued fractions is the set

$$(\{z : |z| < 1\} \cap \{z : \Re z < 1/2\}) \setminus \{z : \operatorname{Arg} z = \pi\}.$$

So we can choose arguments:

- on the positive real axis in the interval $[0, 1/2)$,
- in all four quadrants as long as we remain inside the unit circle and have the real part less than $1/2$,
- and on the imaginary axis in the interval $(-\mathrm{i}, \mathrm{i})$.

TABLE 9.2.1: Because of the symmetry property $f(x+\mathrm{i}x) = f(x-\mathrm{i}x)$, we can restrict ourselves to the upper half-plane, which we traverse in counterclockwise direction as explained. We evaluate the 20[th] classical approximant $f_{20}(z;0)$ of each fraction and compare it to the function evaluation at the argument. Remember that for complex arguments only the signed modulus $|f(z)|_s$, as defined in (9.2.1), is displayed instead of the complex function value $f(z)$.

x	${}_2F_1(1/2, 1; 3/2; x)$	(15.1.4)	(15.3.7)	(15.3.12)	(15.3.17)
0.1	1.035488e+00	2.5e−23	4.0e−32	1.5e−20	1.5e−20
0.2	1.076022e+00	5.6e−17	1.3e−25	1.5e−14	1.7e−13
0.3	1.123054e+00	3.0e−13	1.4e−21	4.8e−11	7.6e−09
0.4	1.178736e+00	1.4e−10	1.8e−18	1.4e−08	5.0e−05

x	$\lvert{}_2F_1(1/2, 1; 3/2; x+\mathrm{i}x)\rvert_s$	(15.1.4)	(15.3.7)	(15.3.12)	(15.3.17)
0.1	1.033684e+00	3.6e−20	3.9e−29	1.5e−17	1.4e−17
0.2	1.066938e+00	8.0e−14	1.1e−22	1.6e−11	9.3e−11
0.3	1.097258e+00	4.2e−10	8.4e−19	5.0e−08	1.5e−06
0.4	1.121184e+00	1.8e−07	5.5e−16	1.6e−05	1.4e−03

x	$\lvert{}_2F_1(1/2, 1; 3/2; \mathrm{i}x)\rvert_s$	(15.1.4)	(15.3.7)	(15.3.12)	(15.3.17)
0.1	9.985628e−01	2.3e−23	1.4e−32	1.6e−20	1.8e−21
0.3	9.875589e−01	2.4e−13	3.6e−23	5.5e−11	2.9e−12
0.5	9.678199e−01	1.0e−08	5.8e−19	1.5e−06	2.1e−08
0.7	9.425900e−01	1.1e−05	2.4e−16	1.3e−03	3.1e−06
0.9	9.147830e−01	2.1e−03	1.6e−14	1.9e−01	6.8e−05

x	$\lvert{}_2F_1(1/2, 1; 3/2; x-\mathrm{i}x)\rvert_s$	(15.1.4)	(15.3.7)	(15.3.12)	(15.3.17)
−0.1	9.673650e−01	3.2e−20	5.4e−30	1.6e−17	2.8e−19
−0.3	9.077224e−01	2.9e−10	2.8e−21	6.1e−08	2.4e−11
−0.5	8.563213e−01	1.2e−05	1.3e−17	1.8e−03	2.3e−08
−0.7	8.123036e−01	1.3e−02	2.1e−15	3.9e+00	9.9e−07

We see that, for the real argument $x = 0.3$, the 20[th] approximant $f_{20}(x;0)$ of the C-fraction (15.3.7) ensures 21 significant decimal digits because

$$\left|\frac{{}_2F_1(1/2, 1; 3/2; x) - f_{20}(x;0)}{{}_2F_1(1/2, 1; 3/2; x)}\right| \le 1.4 \times 10^{-21}, \qquad x = 0.3\,.$$

Clearly, here the C-fraction delivers the better approximant. The evaluation of (15.3.7) can further be improved with the use of the modifications (15.3.5), $w(z) = 1/2\left(\sqrt{1-z}-1\right)$, and (15.3.6),

$$w_n^{(1)}(z) = w(z) + \frac{c_{n+1}z + z/4}{1+2w(z)} .$$

TABLE 9.2.2: The approximant $f_{20}(x)$ of (15.3.7) is first evaluated without modification and subsequently with the mentioned modifications. Note that the first truncation error column equals the first truncation error column of *Table* 9.2.1 for real arguments x, since both concern the classical approximant $f_{20}(x;0)$ of the C-fraction.

x	${}_2F_1(1/2, 1; 3/2; x)$	(15.3.7)	(15.3.7)	(15.3.7)
0.1	1.035488e+00	4.0e−32	2.6e−35	6.5e−38
0.2	1.076022e+00	1.3e−25	8.8e−29	4.8e−31
0.3	1.123054e+00	1.4e−21	1.1e−24	9.6e−27
0.4	1.178736e+00	1.8e−18	1.4e−21	1.9e−23

With use of the modifications, the truncation error in $x = 0.3$ decreases to

$$\left|\frac{{}_2F_1(1/2, 1; 3/2; x) - f_{20}(x; w(x))}{{}_2F_1(1/2, 1; 3/2; x)}\right| \le 1.1 \times 10^{-24}, \quad x = 0.3,$$

$$\left|\frac{{}_2F_1(1/2, 1; 3/2; x) - f_{20}(x; w_{20}^{(1)}(x))}{{}_2F_1(1/2, 1; 3/2; x)}\right| \le 9.6 \times 10^{-27}, \quad x = 0.3,$$

respectively ensuring 21, 24 and 26 significant decimal digits.

EXAMPLE 9.2.2: The ratio of parabolic cylinder functions $U(1, x)/U(0, x)$ has the C-fraction representation (16.5.7),

$$\frac{U(1,x)}{U(0,x)} = \frac{1}{x} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{m-\frac{1}{2}}{x}\right), \qquad x > 0.$$

Since the partial numerators tend to infinity, use of the modification

$$\tilde{w}_n(x) = \frac{-x + \sqrt{4(n+1/2) + x^2}}{2}$$

is recommended when evaluating the approximants.

TABLE 9.2.3: We tabulate the relative error of the 5^{th} approximants $f_5(x)$ which are first evaluated without modification and then with the modification $w_5(x)$.

x	$U(a,x)/U(a-1,x)$	(16.5.7)	(16.5.7)
0.25	8.329323e−01	1.4e+00	2.0e−02
0.75	6.485192e−01	1.8e−01	4.4e−03
1.25	5.211635e−01	4.0e−02	1.0e−03
5.25	1.813514e−01	7.3e−06	1.2e−07
20.25	4.920381e−02	2.6e−11	5.9e−14
50.25	1.988869e−02	3.1e−15	1.2e−18
100.25	9.973574e−03	3.2e−18	3.1e−22

The tables in the handbook are not discussed, only presented, because the conclusions are obvious most of the times. The speed of convergence can be observed. The variation in the magnitude of the truncation error throughout the complex plane is clear. Specific observations, such as extremely slow convergence, are confirmed in the existing literature.

Other counter-intuitive behaviour can be understood by taking a closer look at the formulas involved. For instance, while T-fractions correspond to series developments both for small and large z (around 0 and ∞), this behaviour is not confirmed (at first sight) in the evaluation of (17.1.48). Take a look at the approximation of $J_\nu(x)$ in the *Tables* 17.1.1 and 17.1.2. For real-valued arguments the Bessel function is real-valued. But here the T-fraction (exceptionally) introduces an imaginary part in the approximation of $J_{\nu+1}(x)/J_\nu(x)$. Of course, this disturbs the quality of the approximation on the real axis. Because of (17.2.2) a similar observation can be made for the Bessel function $I_\nu(\mathrm{i}x)$ evaluated on the imaginary axis. In other parts of the complex plane the behaviour is as expected.

9.3 Reliable graphs

For graphical illustrations of the specific behaviour of special functions in subsets of their domain, we refer to [AS64] and [SO87]. Also several websites are devoted to the visualisation of special functions, both for real and complex variables. Our interest is in the approximation power of series representations and continued fraction representations and hence in the visualisation of the truncation error incurred when using these approximations. We therefore show *level curves* of s such that

$$\left|\frac{f(z) - f_n(z)}{f(z)}\right| \leq 5 \times 10^{-s}, \qquad z \in \mathbb{C}, \quad s \in \mathbb{N} \tag{9.3.1}$$

or

$$\left|\frac{f(x) - f_n(x)}{f(x)}\right| = 5 \times 10^{-s}, \qquad x \in \mathbb{R}, \quad s \in \mathbb{R}_0^+ \tag{9.3.2}$$

where $f_n(z)$ is the n^{th} approximant of a series or continued fraction representation for the function $f(z)$. Continued fraction representations depicted in graphs are marked throughout the chapters with  in the right margin.

In all our plots the grid lines are drawn one unit apart, both in the horizontal and the vertical direction (aspect ratio 1 for each unit square). If the x- or y-axis belong to the plot, they are shown as a solid line. For instance, in the area $[1, 19] \times [0, 9]$ only the x-axis is shown, while in the area $[-2, 2] \times [-2, 2]$ both axes appear. Since the axes do not always appear in the picture, we have preferred not to label them. The caption provides sufficient information on the plotted area.

So all our graphs show implicit relations $R_{s,t}(x, y) = 0$ involving at most two real unknowns x and y and some real parameters s and/or t. Given that this kind of graphing problem has been discussed for centuries, it is unsurprising that there is an abundance of (partial) solutions to it. It is, however, surprising that until recently [Tup91] there was no computer method capable of reliably solving this problem.

The algorithm implemented in `GrafEq` (pronounced "graphic") correctly graphs mathematical formulas involving the basic operations, inequalities and known elementary functions [Tup04]. When applied to a difficult formula that is beyond its capabilities, the algorithm clearly marks the pixels that it cannot decide to belong to the graph, as uncertain. At no point does the algorithm use any approximations that may cause it to produce an incorrect graph. We summarise the internal workings of `GrafEq` below.

In order for `GrafEq` to be useful in the context of the continued fraction handbook, two extensions were developed [BCJ$^+$05]:

- since expressions in a complex variable z may get quite complicated when manually converted to a relation in $x = \Re z$ and $y = \Im z$, it is necessary to add the direct handling of complex variables to `GrafEq`'s interface;
- since `GrafEq` only has implementations of the elementary functions and none of the special functions, we need to be able to dynamically extend the list of functions known by `GrafEq` by providing their implementation.

Internal working of GrafEq. Any formula $R(x, y)$, when evaluated with specific real numbers x and y, is always either false (`F`) or true (`T`). Given a mathematical formula $R(x, y)$ and a rectangular region $[L, R] \times [B, T]$ of the Cartesian plane $\mathbb{R}^2$, `GrafEq` produces an illustration that consists of a $W \times H$ rectangular array of pixels. Each pixel herein represents a closed rectangular region of the plane. Since no algorithm can produce correct black and white graphs, black meaning that there is at least one solution of $R(x, y)$ within the pixel and white meaning that there are no solutions within the pixel, we allow to color some pixels "uncertain", meaning that there may or may not be solutions of $R(x, y)$ within the pixel.
Even if the bounds L, R, B and T of the graphing area are given as machine numbers, the bounds of individual pixels may not be representable exactly. Therefore `GrafEq` uses inner and outer bounds of the rectangular region corresponding to the pixel. The inner bounds are used to show the guaranteed existence of solutions and the outer bounds to show the absence of solutions. Further `GrafEq` makes use of interval arithmetic with boolean values to represent and process the result of formula evaluations. Three boolean intervals are possible, $\langle \texttt{F, F}\rangle, \langle \texttt{F, T}\rangle, \langle \texttt{T, T}\rangle$ with $\texttt{F} < \texttt{T}$. The boolean intervals provide:

- domain tracking, by keeping track of whether or not a quantity such as $\sqrt{x}$ with $x < 0$, is well-defined,
- continuity tracking, by providing information on whether a quantity is continuous or not within the given bounds,
- branch tracking, by tracing to which branch each piece belongs when breaking a discontinuous evaluation apart into pieces.

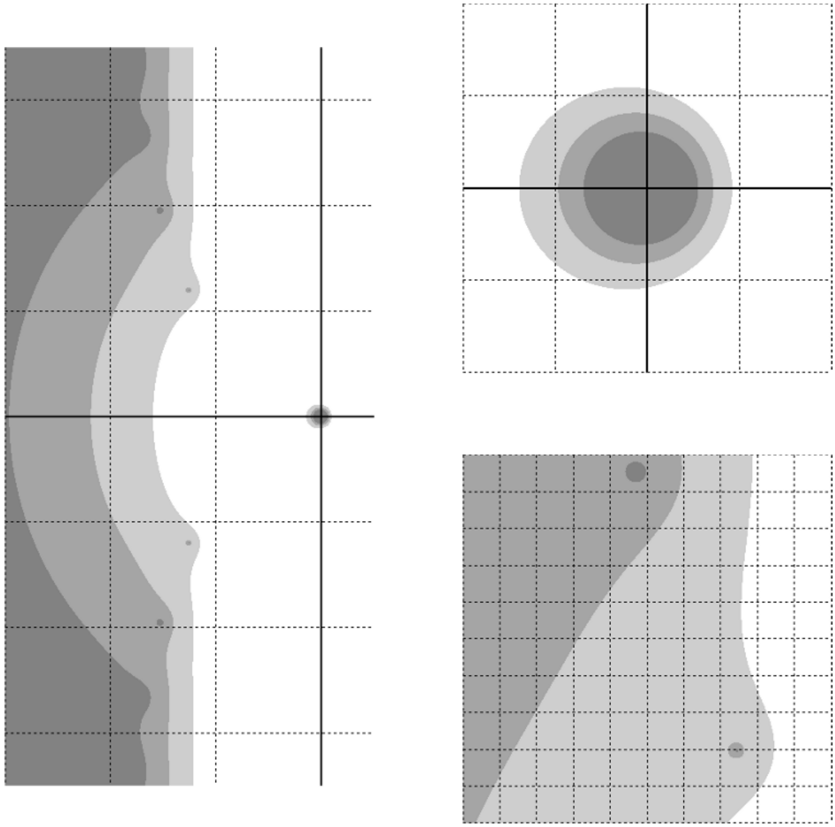
Plotting special functions. Additional function implementations must also return interval evaluations and support the domain and continuity boolean intervals required by `GrafEq`'s internal engine. To guarantee the reliability of the function evaluations, the results in [CVW06] on the implementation of special functions are used.

EXAMPLE 9.3.1: We show a simple illustration of (9.3.1) for a T-fraction approximant of the exponential function, namely

$$\begin{aligned} f(z) &= \exp(z) - 1, \\ f_n(z) &= \frac{z}{1-z+} \mathop{\mathrm{K}}_{m=2}^{n} \left(\frac{(m-1)z}{m-z} \right), \qquad n = 8, \\ \frac{|f(z) - f_n(z)|}{|f(z)|} &\le 5 \times 10^{-s}, \qquad s = 6, 7, 8. \end{aligned}$$

T-fractions have the property that, besides being useful for relatively small z, they approximate well for small $1/z$ at the same time.

FIGURE 9.3.1: We consider the region $-30 \leq \Re z \leq 5, |\Im z| \leq 35$ (exceptionally, in this figure the grid-lines are 10 units apart) and zoom in on the regions $|\Re z| \leq 2, |\Im z| \leq 2$ and $-20 \leq \Re z \leq -10, 10 \leq \Im z \leq 20$ (in these figures the grid-lines are 1 unit apart, as usual). The regions corresponding to $s = 6, 7, 8$ are respectively coloured light-grey, medium-grey and dark-grey, respectively. Axes and grid-lines are coloured black.



Note the very small isolated regions in the left half-plane of the larger drawing, which are impossible to locate without a reliable graphing method. The resolution of the small insets is 192×192 pixels while that of the larger figure is 192×384 pixels. With the same number of 73728 plot-points for the latter, the computer algebra system Maple (version 10) is unable to produce the correct graph for $s = 8$.

EXAMPLE 9.3.2: The following is an illustration of (9.3.2) taken from *Chapter* 14. In contrast to the above example where $s \in \mathbb{N}$, here s can take on all positive real values. Let the exponential integrals $E_n(z)$ be defined in $\Re z > 0$ by

$$E_n(z) := \int_1^\infty \frac{e^{-zt}}{t^n}\, dt, \qquad n \in \mathbb{N}.$$

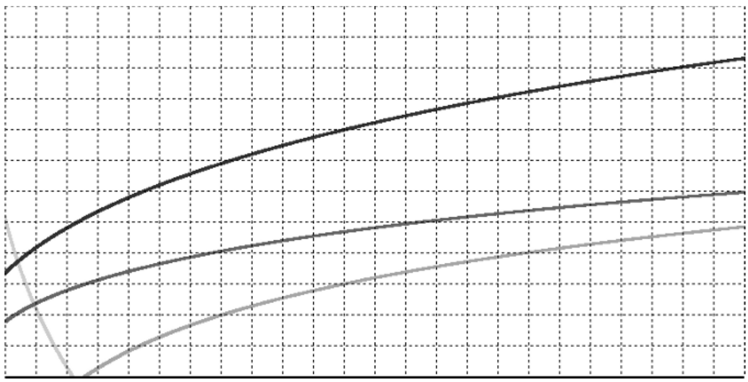
Analytic continuation of $E_n(z)$ to the cut plane $|\arg z| < \pi$ extends the definition and yields a single-valued function. Except on the negative real axis, they can be represented by the series representation (14.1.11), the asymptotic series expansion (14.1.13), the S-fraction (14.1.16),

$$\begin{aligned} e^z E_n(z) &= \frac{1}{z} \, {}_{+} \, \frac{n}{1} \, {}_{+} \, \frac{1}{z} \, {}_{+} \, \frac{n+1}{1} \, {}_{+} \, \frac{2}{z} \, {}_{+} \, \frac{n+2}{1} \, {}_{+} \, \dots, \qquad n \in \mathbb{N}, \\ &= \frac{1/z}{1} \, {}_{+} \, \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{a_m/z}{1} \right), \qquad a_{2k} = n + k - 1, \quad a_{2k+1} = k, \\ &\qquad |\arg z| < \pi \end{aligned}$$

and the real J-fraction representation (14.1.23),

$$e^z E_n(z) = \frac{1}{n+z} \, {}_{+} \, \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{(2-m)(n+m-2)}{n+2m-2+z} \right), \qquad |\arg z| < \pi.$$

FIGURE 9.3.2: On the vertical axis we display the value s in (9.3.2) in the range $0 \le s \le 12$, where $f_n(z)$ respectively equals the 5^{th} partial sum of the series development (14.1.11) (lightest), the 5^{th} partial sum of the asymptotic series (14.1.13) (second lightest), the 5^{th} approximant of the S-fraction (second darkest) and the 5^{th} approximant of the real J-fraction (darkest) of $f(z) = E_3(z)$, all for real $z = x$ with $1 \le x \le 25$.

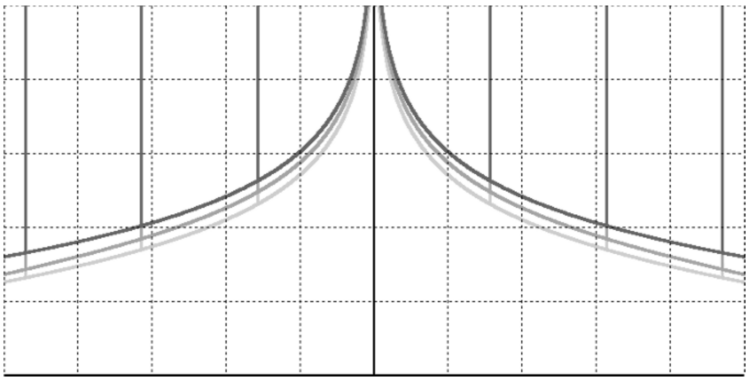


It is easy to see how the continued fraction representations outperform the series developments. From *Table* 14.1.1 one also sees that in the interval $[1, 25]$ the 5^{th} modified approximant of the S-fraction guarantees on average one more significant decimal digit than the classical 5^{th} approximant.

Besides the graphs shown in *Example* 9.3.1, for which we needed reliable graphing software, it is also interesting to take a look at the next figure, in which we show the number of significant digits delivered by successive approximants $f_n(x)$ of the Thiele interpolating continued fraction (11.3.9)

$$\tan(z) = \frac{z}{1} + \frac{-4\pi^{-2}z^2}{1} + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{m^4 - 4\pi^{-2}m^2z^2}{2m+1}\right), \quad z \in \mathbb{C} \setminus \{\pi/2 + k\pi : k \in \mathbb{Z}\}$$

for real arguments x, where from light to dark $n = 5, 6$ and 7. At the interpolation points $\pm m\pi/2$, $m \in \mathbb{N}_0$, the accuracy and hence the number of significant digits is infinite, but in the neighbourhood of the interpolation points the peaks in the graph are so steep that the traditional device independent graphing tools miss each of them. Fortunately `GrafEq` does not!



10

Mathematical constants

The calculation of mathematical constants has been a topic of investigation for mathematicians throughout the centuries. Several methods are developed, such as integral representations, series, products and continued fractions. In this chapter we show how continued fractions can be relevant in connection with some of the important mathematical constants.

10.1 Regular continued fractions

A continued fraction of the form

$$b_0 + \mathop{\mathrm{K}}\left(\frac{1}{b_m}\right) = b_0 + \frac{1}{b_1} \, {}_{+} \, \frac{1}{b_2} \, {}_{+} \, \frac{1}{b_3} \, {}_{+} \, \frac{1}{b_4} \, {}_{+} \, \ldots, \qquad b_m \in \mathbb{N} \qquad (10.1.1)$$

is called a *regular continued fraction* [Per54, p. 23]. When using a large number of elements, it can be denoted by

$$[b_0, b_1, b_2, b_3, b_4, \ldots] := b_0 + \mathop{\mathrm{K}}\left(\frac{1}{b_m}\right), \qquad b_m \in \mathbb{N}\,. \qquad (10.1.2)$$

For any positive irrational number x, there exists a unique regular continued fraction converging to x, which is called the regular continued fraction expansion of x. Since a continued fraction with positive elements oscillates as formulated in *Theorem* 7.5.1, it provides bounds for its value as well as truncation error estimates.

Let x be a positive irrational number represented by its regular continued fraction f. Then each approximant $f_n = A_n/B_n$ is a *best rational approximant* to x in the sense that [Per54, p. 44; MK85]

$$\forall p, q \in \mathbb{N}, q \leq B_n, pB_n - qA_n \neq 0 : \left| x - \frac{p}{q} \right| > \left| x - \frac{A_n}{B_n} \right| \,.$$

10.2 Archimedes' constant, symbol π

The number π is one of the most important mathematical constants. It is defined as the area enclosed by the unit circle,

$$\pi := 4 \int_0^1 \sqrt{1-x^2}\, dx = 3.141592653589793238\ldots.$$

By calculating the areas of regular inscribed and circumscribed polygons with 96 sides, Archimedes established the inequalities [Sha93, p. 140]

$$\frac{10}{71} < \pi - 3 < \frac{10}{70}\,.$$

In the sixteenth century the mathematician Ludolph van Ceulen devoted much of his life to the calculation of π, and he was able to determine 35 correct digits. The number π is sometimes called *Ludolph's constant* [Huy95, pp. 60–61], but a more frequently used name is *Archimedes' constant*. The number π was proved to be irrational by Lambert (1767) [Lam68] and transcendental by Lindemann (1882) [Lin82, pp. 213–225].

Leibniz' formula. A famous expression for π is the simple series representation [EP98, p. 656]

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots, \qquad (10.2.1)$$

called *Leibniz' formula*.

Machin's formula. A simple formula based upon the fact that the expression $4\arctan(1/5)$ is close to $\pi/4$, is a formula commonly known as *Machin's formula* [Jon06, p. 263; EP98, p. 656]:

$$\frac{\pi}{4} = 4\arctan(1/5) - \arctan(1/239)\,. \qquad (10.2.2)$$

Using the partial sum of degree 7 of the FTS (11.3.3) for $\arctan(z)$ we get only six correct digits,

$$\pi = 3.14159177\ldots.$$

Machin was the first to calculate the 100 initial digits of π. Currently π is computed to more than a trillion hexadecimal and decimal digits. The number π appears in many connections, among others in Buffon's *needle experiment* in probability theory [Sch74, pp. 183–186].

Wallis' formula. A well-known product representation of π is *Wallis' formula* [Wal56; AS64, p. 258]

$$\pi = 2\prod_{k=1}^{\infty} \frac{4k^2}{(2k-1)(2k+1)}\,. \tag{10.2.3}$$

Here the partial products increase very slowly towards π. With k running up to 15 in the partial product we obtain the rather poor result

$$\pi = 3.0913\ldots.$$

Regular continued fraction. The regular continued fraction expansion f for π is given by [Per54, pp. 35–36]

$$\begin{aligned}\pi &= 3 + \frac{1}{7} \, \genfrac{}{}{0pt}{}{}{+} \, \frac{1}{15} \, \genfrac{}{}{0pt}{}{}{+} \, \frac{1}{1} \, \genfrac{}{}{0pt}{}{}{+} \, \frac{1}{292} \, \genfrac{}{}{0pt}{}{}{+} \, \frac{1}{1} \, \genfrac{}{}{0pt}{}{}{+} \, \frac{1}{1} \, \genfrac{}{}{0pt}{}{}{+} \, \frac{1}{1} \, \genfrac{}{}{0pt}{}{}{+} \, \frac{1}{2} \, \genfrac{}{}{0pt}{}{}{+} \, \genfrac{}{}{0pt}{}{}{\ldots}\\ &= [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, \ldots]\,.\end{aligned} \tag{10.2.4}$$

The value π is enclosed by

$$3.141592653 < f_4 = \frac{103993}{33102} < \pi < f_5 = \frac{104348}{33215} < 3.141592654\,.$$

The fact that $b_4 = 292$ is large implies that f_4 is a very good approximant, with

$$|f_4 - f_5| \leq 9.1 \times 10^{-10}\,.$$

The same accuracy using Leibniz' formula is only achieved after more than 2700 terms. Often, a disadvantage of regular continued fraction expansions is the lack of a known pattern.

Special cases of S-fractions. The expression

$$\pi = \frac{4}{1} \, \genfrac{}{}{0pt}{}{}{+} \, \frac{1^2}{3} \, \genfrac{}{}{0pt}{}{}{+} \, \frac{2^2}{5} \, \genfrac{}{}{0pt}{}{}{+} \, \frac{3^2}{7} \, \genfrac{}{}{0pt}{}{}{+} \, \genfrac{}{}{0pt}{}{}{\ldots} \tag{10.2.5}$$

follows immediately from the modified S-fraction expansion

$$\begin{gathered}\sqrt{z}\arctan(\sqrt{z}) = \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{a_m z}{b_m}\right), \qquad |\arg z| < \pi,\\ a_1 = b_1 = 1, \quad a_m = (m-1)^2, b_m = 2m-1, \quad m \geq 2\end{gathered}$$

for $z = 1$ [JT80, p. 202; LW92, p. 561]. Here the value π is enclosed by

$$3.1415925404 < f_{10} = \frac{4317632}{1374345} < \pi < f_9 = \frac{3763456}{1197945} < 3.1415933119$$

with

$$|f_9 - f_{10}| \leq 7.8 \times 10^{-7} .$$

If we consider the S-fraction (12.1.12) for a special ratio of two gamma values and let $z = 4k + 1$, then we obtain

$$\left(\frac{(2k)!}{k!^2}\right)^2 \frac{\pi}{2^{4k}} = \frac{4}{4k+1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{(2m-1)^2}{8k+2}\right), \qquad k \in \mathbb{N}_0.$$

The special cases $k = 0$ [Lan99] and $k = 1$ yield

$$\pi = \frac{4}{1} + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{(2m-1)^2}{2}\right),$$

$$\pi = \frac{16}{5} + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{(2m-1)^2}{10}\right).$$

If we let $z = 3$ in (12.1.12), we obtain the slowly converging and not regular continued fraction

$$3 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{(2m-1)^2}{6}\right) = 3 + \frac{1^2}{6} \, \genfrac{}{}{0pt}{}{}{+} \, \frac{3^2}{6} \, \genfrac{}{}{0pt}{}{}{+} \, \frac{5^2}{6} \, \genfrac{}{}{0pt}{}{}{+} \, \frac{7^2}{6} \, \genfrac{}{}{0pt}{}{}{+} \, \frac{9^2}{6} \, \genfrac{}{}{0pt}{}{}{+} \, \genfrac{}{}{0pt}{}{}{\ldots} \; . \qquad (10.2.6)$$

For the approximants f_9 and f_{10} we obtain

$$3.1414067 < f_{10} = \frac{45706007}{14549535} < \pi < f_9 = \frac{91424611}{29099070} < 3.14183962,$$

with

$$|f_9 - f_{10}| \leq 4.4 \times 10^{-4}.$$

10.3 Euler's number, base of the natural logarithm

The base of the natural logarithm, named e for Euler, sometimes known as *Euler's number*, is given by

$$e := \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \qquad (10.3.1a)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \qquad (10.3.1b)$$

$$= 2.7182818284590 \ldots . \qquad (10.3.1c)$$

Using (10.3.1a) with $n = 100$ we obtain only two significant digits:

$$\left(1 + \frac{1}{100}\right)^{100} = 2.7048138294\ldots.$$

With $n = 10$ in (10.3.1b) we obtain seven significant digits:

$$\sum_{n=0}^{10} \frac{1}{n!} = \frac{9864101}{362880} = 2.718281801\ldots.$$

The number e is irrational (Euler, 1737) [Eul48; HW79] and transcendental (Hermite, 1873) [Her73]. It is related to the trigonometric functions through

$$e^{\mathrm{i}\theta} = \cos(\theta) + \mathrm{i}\sin(\theta)\ . \tag{10.3.2}$$

Euler's simple relationship is

$$e^{\mathrm{i}\pi} = -1\ . \tag{10.3.3}$$

An interesting example of the appearance of e occurs in probability theory. If real numbers are selected at random from the interval (0,1) until the sum exceeds 1, the expected number of selections is e [Fin03, p. 13].

An unusual limit representation. A limit formula for e is given by [BK98, pp. 25–29]

$$e = \lim_{n\to\infty}\left[\frac{(n+1)^{n+1}}{n^n} - \frac{n^n}{(n-1)^{n-1}}\right]. \tag{10.3.4}$$

With $n = 100$ we get five significant digits:

$$\frac{101^{101}}{100^{100}} - \frac{100^{100}}{99^{99}} = 2.718293155\ldots.$$

Regular continued fractions. The regular continued fraction representation f of e is [LW92, p. 562]

$$\begin{aligned} e &= 2 + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{1}{b_m}\right), \qquad b_{3j-2} = 1, b_{3j-1} = \frac{1}{2j}, b_{3j} = 1 \\ &= [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots]\ . \end{aligned} \tag{10.3.5}$$

The value e is enclosed by

$$2.718281828445 < f_{14} = \frac{517656}{190435} < e < f_{13} = \frac{49171}{18089} < 2.718281828736$$

with

$$|f_{13} - f_{14}| \leq 3.0 \times 10^{-10} \,.$$

This continued fraction does not converge fast since it has no large terms. Another regular continued fraction involving e is [Old63, pp. 135–136]

$$\frac{e-1}{e+1} = \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{1}{4m-2} \right) = [0, 2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, \ldots] \,. \tag{10.3.6}$$

The approximants f_5 and f_6 produce the inequalities

$$\frac{342762}{741721} = f_6 < \frac{e-1}{e+1} < f_5 = \frac{15541}{33630},$$

and hence

$$2.71828182845 < e < 2.71828182874.$$

10.4 Integer powers and roots of π and e

It turns out that powers and roots of e enjoy regular continued fraction representations with very nice patterns. For instance,

$$\begin{aligned} \sqrt{e} &= 1 + \frac{1}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{1}{b_m} \right), \qquad b_{3j-1} = 1, b_{3j} = 1, b_{3j+1} = 4j + 1 \\ &= [1, 1, 1, 1, 5, 1, 1, 9, 1, 1, 13, 1, 1, 17, \ldots], \end{aligned} \tag{10.4.1}$$

or more generally [Per54, p. 124]

$$\begin{aligned} e^{\frac{1}{k}} &= 1 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{1}{b_m} \right), \qquad b_{3j-2} = (2j-1)k - 1, b_{3j-1} = 1, b_{3j} = 1 \\ &= [1, k-1, 1, 1, 3k-1, 1, 1, 5k-1, 1, 1, \ldots], \qquad k > 1 \,. \end{aligned} \tag{10.4.2}$$

Faster converging continued fractions for integer roots of e are [Hur96]

$$\begin{aligned} e^{\frac{1}{k}} &= \frac{k+1}{k} + \frac{1}{k} \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{1}{b_m} \right), \qquad k > 1, \\ &\qquad b_{3j-2} = 2k - 1, b_{3j-1} = 2j, b_{3j} = 1, \end{aligned} \tag{10.4.3}$$

$$\begin{aligned} &= \frac{k}{k-1} + \frac{1}{2k} + \mathop{\mathrm{K}}_{m=3}^{\infty} \left(\frac{1}{b_m} \right), \qquad k > 1, \\ &\qquad b_{3j} = 1, b_{3j+1} = 2j, b_{3j+2} = 2k - 1. \end{aligned} \tag{10.4.4}$$

Furthermore we have [Per54, p. 125]

$$e^2 = 7 + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{1}{b_m}\right),$$
$$b_{5j-4} = 3j-1, b_{5j-3} = b_{5j-2} = 1, b_{5j-1} = 3j, b_{5j} = 12j+6. \quad (10.4.5)$$

The regular continued fraction for $\sqrt{\pi}$ is

$$\sqrt{\pi} = [1,1,3,2,1,1,6,1,28,13,1,1,2,18,\ldots],$$

which unfortunately has no special pattern.
The constant $\pi^2/12$ however, which equals $\zeta(2)/2$ where $\zeta(z)$ is the Riemann zeta function (10.11.1), can be obtained from [Ber89, p. 150]

$$\frac{\pi^2}{12} = \frac{1}{1} + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{m^4}{2m+1}\right). \quad (10.4.6)$$

An alternating continued fraction for $\pi^2/12$ is given in *Example* 15.6.1.
A continued fraction with a very nice pattern [Per57, p. 157] is

$$\frac{e^{\frac{2k}{\ell}}-1}{e^{\frac{2k}{\ell}}+1} = \frac{k}{\ell} + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{k^2}{(2m+1)\ell}\right), \qquad k,\ell \in \mathbb{Z}\setminus\{0\}. \quad (10.4.7)$$

With $\ell = 2k$ an equivalence transformation returns (10.3.6).

TABLE 10.4.1: In tabulating approximants for powers and roots of π and e we choose $k=3$ in (10.4.3) and $k=3, \ell=2$ in (10.4.7). The relative errors for the 5th, 10th, 15th, 20th and 25th approximants are given.

	exact	f_5	f_{10}	f_{15}	f_{20}	f_{25}
(10.4.1)	1.648721e+00	7.8e−04	3.0e−09	4.9e−14	4.0e−19	2.5e−26
(10.4.3)	1.395612e+00	1.2e−06	2.0e−12	8.3e−18	9.2e−25	1.5e−31
(10.4.5)	7.389056e+00	2.2e−07	3.0e−14	4.8e−22	1.6e−30	1.5e−39
(10.4.6)	8.224670e−01	2.0e−02	5.5e−03	2.5e−03	1.4e−03	9.3e−04
(10.4.7)	9.051483e−01	2.2e−06	1.2e−16	5.2e−29	8.3e−43	1.1e−57

10.5 The natural logarithm, ln(2)

The logarithm having base e is defined by

$$\ln(x) := \int_1^x \frac{1}{t}\,dt, \qquad x > 0. \quad (10.5.1)$$

It is called the natural logarithm. Substituting $x = 2$ in the integral representation gives

$$\ln(2) = 0.6931471805599\ldots.$$

The number $\ln(2)$ is transcendental (Weierstrass, 1885) [HW79, p. 162].

Taylor series. A FTS for $\ln(x)$ is the alternating series [Mer68]

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^k x^{k-1}}{k}, \qquad -1 < x \le 1, \tag{10.5.2}$$

called the *Mercator series*. For $x = 1$ the sum converges very slowly.

Continued fraction. A continued fraction representation discovered by Euler is [Eul48]

$$\begin{aligned} \ln(2) &= \frac{1}{1} + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{m^2}{1}\right) \\ &= \frac{1}{1} \, {}_{+} \, \frac{1^2}{1} \, {}_{+} \, \frac{2^2}{1} \, {}_{+} \, \frac{3^2}{1} \, {}_{+} \, \frac{4^2}{1} \, {}_{+} \, \frac{5^2}{1} \, {}_{+} \, \ldots \end{aligned} \tag{10.5.3}$$

which also converges slowly. Another continued fraction f for $\ln(2)$ arises from the representation of $\ln(1+x)$ with $x = 1$ [LW92, pp. 17–18]:

$$\mathrm{K}\left(\frac{a_m}{1}\right) = \frac{1}{1} \, {}_{+} \, \frac{1/2}{1} \, {}_{+} \, \frac{1/6}{1} \, {}_{+} \, \frac{2/6}{1} \, {}_{+} \, \frac{2/10}{1} \, {}_{+} \, \frac{3/10}{1} \, {}_{+} \, \ldots \tag{10.5.4}$$

with

$$a_1 = 1, a_{2k} = \frac{k}{2(2k-1)}, a_{2k+1} = \frac{k}{2(2k+1)}. \tag{10.5.5}$$

Approximant f_7 equals

$$\frac{1073}{1548} = 0.69315245478\ldots.$$

Since $\lim_{m\to\infty} a_m = 1/4$, the tail of (10.5.4) converges to

$$\frac{1/4}{1} \, {}_{+} \, \frac{1/4}{1} \, {}_{+} \, \frac{1/4}{1} \, {}_{+} \, \frac{1/4}{1} \, {}_{+} \, \ldots = \frac{\sqrt{2}-1}{2}. \tag{10.5.6}$$

The 6$^{\text{th}}$ modified approximant $S_6(w)$ with w given by (10.5.6) equals

$$\begin{aligned} S_6\left(\frac{\sqrt{2}-1}{2}\right) &= \frac{1}{1} \, {}_{+} \, \frac{1/2}{1} \, {}_{+} \, \frac{1/6}{1} \, {}_{+} \, \frac{2/6}{1} \, {}_{+} \, \frac{2/10}{1} \, {}_{+} \, \frac{3/10}{1 + \frac{\sqrt{2}-1}{2}} \\ &= 0.69315156969\ldots. \end{aligned} \tag{10.5.7}$$

10.6 Pythagoras' constant, the square root of two

An important irrational number is the diagonal $\sqrt{2}$ of a unit square, sometimes called *Pythagoras' constant*. The numerical value of this algebraic number is

$$\sqrt{2} = 1.414213562373\ldots.$$

Regular continued fraction and Pell numbers. Pythagoras' constant is connected to the *Pell numbers*, which are defined by the three-term recurrence relation [McD96, pp. 105–107]

$$P_n = 2P_{n-1} + P_{n-2}, \qquad P_0 = 0, \quad P_1 = 1, \quad n \geq 2. \tag{10.6.1}$$

The relation entails the limit

$$\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \sqrt{2} + 1. \tag{10.6.2}$$

The regular continued fraction f associated with the three term recurrence relation (10.6.1) is [LW92, p. 10; Wei03, p. 971]

$$\sqrt{2} + 1 = 2 + \mathrm{K}\left(\frac{1}{2}\right) = 2 + \frac{1}{2} \,{}_{+}\, \frac{1}{2} \,{}_{+}\, \frac{1}{2} \,{}_{+}\, \frac{1}{2} \,{}_{+}\, \ldots\,. \tag{10.6.3}$$

The approximants f_9 and f_{10} yield

$$2.414213551 < \frac{13860}{5741} = f_{10} < \sqrt{2} + 1 < f_9 = \frac{5741}{2378} < 2.414213625$$

where

$$|f_{10} - f_{11}| \leq 7.4 \times 10^{-8}\,.$$

Several non-regular continued fractions represent the same constant.

10.7 The cube root of two

The numerical value of the algebraic number $\sqrt[3]{2}$ is

$$\sqrt[3]{2} = 1.25992104989\ldots.$$

Regular continued fraction. The cube root of two is represented by the regular continued fraction [LT72, pp. 112–134]

$$[1, 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 4, 12, 2, \ldots] \tag{10.7.1}$$

which converges rather slowly.

Branched continued fraction. The number $2+\sqrt[3]{2}$ can be represented by a branched continued fraction. One way of describing it is as follows. Let

$$C = 3 + \frac{1}{3} {}_{+} \frac{C}{3} {}_{+} \frac{C}{3} {}_{+} \frac{C}{3} {}_{+} \ldots$$

where C is again the same continued fraction, such that we may write

$$2+\sqrt[3]{2} = C = 3 + \frac{1}{3} {}_{+} \frac{3 + \frac{1}{3} {}_{+} \frac{C}{3} {}_{+} \frac{C}{3} {}_{+} \ldots}{3} {}_{+} \ldots .$$

The approximants are recursively defined by

$$\begin{aligned}
C_0 &= 3, \\
C_1 &= 3 + \frac{1}{3}, \\
C_n &= 3 + \frac{1}{3} {}_{+} \frac{C_{n-2}}{3} {}_{+} \cdots {}_{+} \frac{C_0}{3} = 3 + \frac{1}{3 + C_{n-2}(C_{n-1} - 3)} .
\end{aligned}$$

We have in particular

$$\begin{aligned}
C_4 &= \frac{577}{177} = 3.2598870\ldots, \\
C_5 &= \frac{740}{227} = 3.25991189\ldots, \\
C_6 &= \frac{8541}{2620} = 3.25992366\ldots.
\end{aligned}$$

The sequence $\{C_n\}_{n\in\mathbb{N}}$ is known to converge and the limit is given by

$$C = 3 + \frac{1}{3} {}_{+} \frac{C}{3} {}_{+} \frac{C}{3} {}_{+} \frac{C}{3} {}_{+} \ldots = 3 + \frac{1}{3+U}$$

where

$$U = \frac{-3}{2} + \sqrt{\frac{9}{4} + C.}$$

This leads to an algebraic equation of degree 4 for C. Since the constant term is missing, the degree can be reduced to 3. Substituting $T = C - 2$ then leads to the equation $T^3 = 2$.

10.8 Euler's constant, symbol γ

Euler's constant was first introduced by Euler in 1734 as the limit [Eul48]

$$\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right) = 0.57721566490153\ldots. \tag{10.8.1}$$

It is also known as the *Euler-Mascheroni constant*. It is closely related to the gamma function $\Gamma(z)$ discussed in *Chapter* 12, through the *Weierstrass product formula* [Hen77, p. 25]

$$\frac{1}{\Gamma(z)} = z\, e^{\gamma z}\prod_{n=1}^{\infty}\Big((1+z/n)e^{-z/n}\Big), \qquad z \in \mathbb{C}\,. \tag{10.8.2}$$

From this formula, which is also referred to as *Euler's infinite product*, we obtain the relation

$$\gamma = -\frac{d\Gamma(x)}{dx}\Big|_{x=1}\,. \tag{10.8.3}$$

It is not known whether γ is irrational or transcendental.

Regular continued fraction. The regular continued fraction f for γ converges slowly [Knu62]:

$$\gamma = [0,1,1,2,1,2,1,4,3,13,5,1,1,8,1,2,4,1,1,40,\ldots]\,.$$

Euler's constant plays an important role in analysis through its relation to the gamma function, the Bessel functions and number theory.

10.9 Golden ratio, symbol ϕ

The *golden ratio* is also known as the golden mean, the golden section or the divine proportion because of its relation to geometric figures, natural phenomena and art. The geometric property can be described as follows. The golden ratio is the division of a given unit of length into two parts such that the ratio of the shorter to the longer equals the ratio of the longer part to the whole. Calling the longer part ϕ and accordingly the shorter part $1-\phi$, we get

$$\frac{1}{\phi} = \frac{\phi}{1-\phi}$$

and

$$\phi^2 + \phi - 1 = 0$$

where the positive root is called the golden ratio [Hen77, p. 25],

$$\phi = \frac{\sqrt{5}+1}{2} = 1.61803398874989\ldots. \tag{10.9.1}$$

The golden ratio and the Fibonacci sequence. In the way Pythagoras' constant is closely related to the Pell numbers, the golden ratio is connected to the *Fibonacci sequence* [Fin03, pp. 5–6]

$$F_n = F_{n-1} + F_{n-2}, \qquad F_0 = 0, \quad F_1 = 1, \quad n \geq 2 \tag{10.9.2}$$

with

$$\phi = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} \,. \tag{10.9.3}$$

Regular continued fraction. The continued fraction f for ϕ,

$$\phi = 1 + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \ldots \tag{10.9.4}$$

is already introduced in *Example* 1.8.1. The approximants f_{14} and f_{15} yield the bounds

$$1.6180327 < \frac{987}{610} = f_{14} < \phi < f_{15} = \frac{1597}{987} < 1.6180345$$

where

$$|f_{14} - f_{15}| \leq 1.7 \times 10^{-6} \,.$$

The convergence is rather slow due to the small values of the partial denominators.

10.10 The rabbit constant, symbol ρ

The original problem investigated by Fibonacci in 1202 was about how fast a rabbit breeds under ideal and hypothetical circumstances where the rabbits reproduce and never die. The reproduction is given by the recurrence relation (10.9.2) which generates the Fibonacci sequence. The Fibonacci sequence is also called the *golden sequence* because of its relation to the golden ratio. Let the substitution

$$0 \to 1$$

correspond to young rabbits growing old and

$$1 \to 10$$

to old rabbits producing young rabbits. Starting with 0 and iterating, we get the sequence

$$1, 10, 101, 10110, 10110101, 1011010110110, \ldots .$$

When input as binary numbers, this sequence equals

$$1, 2, 5, 22, 181, \ldots$$

with the n^{th} term given by the recurrence relation

$$R_n = 2^{F_{n-1}} R_{n-1} + R_{n-2}, \qquad R_0 = 0, \quad R_1 = 1 \tag{10.10.1}$$

where F_n is the n^{th} Fibonacci number. Considering $\lim_{n\to\infty} R_n$ as the binary fraction of a number, defines the *rabbit constant* ρ:

$$\begin{aligned} \rho &= 0.1011010110110\ldots \\ &= 0.709803442861291\ldots \\ &= \sum_{k=1}^{\infty} 2^{-a_k} \end{aligned} \tag{10.10.2}$$

with [Gar89; Sch91]

$$a_k = \lfloor k\phi \rfloor = \left\lfloor k \frac{\sqrt{5}+1}{2} \right\rfloor = \lfloor k \times (1.61803398874989\ldots) \rfloor . \tag{10.10.3}$$

The rabbit constant is a transcendental number (Böhmer, 1926) [Böh26]. From $a_k \geq k$ follows that

$$\sum_{k=n+1}^{\infty} 2^{-a_k} \leq \sum_{k=n+1}^{\infty} 2^{-k} = \frac{1}{2^n} \tag{10.10.4}$$

which gives a rough error estimate. The partial sum using 12 terms of (10.10.2) equals

$$\rho = 0.709802627563\ldots$$

with an error of less than

$$\frac{1}{2^{12}} < 2.5 \times 10^{-4} .$$

Regular continued fraction. Another interesting property is that the rabbit constant can be expressed by the regular continued fraction f given by [Sch91; Gar89]

$$\operatorname*{K}_{m=1}^{\infty} \left(\frac{1}{2^{F_m}} \right) = \frac{1}{2^{F_0}} + \frac{1}{2^{F_1}} + \frac{1}{2^{F_2}} + \frac{1}{2^{F_3}} + \frac{1}{2^{F_4}} + \frac{1}{2^{F_5}} + \frac{1}{2^{F_6}} + \ldots \tag{10.10.5}$$

where F_n is the n^{th} Fibonacci number. Approximant f_{11} equals

$$f_{11} = 0.70980344286129\ldots .$$

Approximant f_{12} produces the same initial digits. A closer investigation shows a difference of only

$$3.3 \times 10^{-114}.$$

10.11 Apéry's constant, $\zeta(3)$

The value of the Riemann zeta function

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z} \tag{10.11.1}$$

at $z = 3$,

$$\zeta(3) = 1.202056903159594285399738\ldots \tag{10.11.2}$$

is called *Apéry's constant.* The number is irrational (Apéry, 1979) [Apé79; vdP79] but it is not known whether it is transcendental.

Series representation. A rapidly converging series for $\zeta(3)$ is [AZ97]

$$\zeta(3) = \sum_{n=0}^{\infty} (-1)^n \frac{(n!)^{10}(205n^2 + 250n + 77)}{64((2n+1)!)^5} \,. \tag{10.11.3}$$

The sum of the first five terms is 1.20205690315959428... which has as good as 16 significant digits.

Continued fraction representations. The number $\zeta(3)$ can be expressed by the regular continued fraction f [AZ97] given by

$$\begin{aligned}\zeta(3) &= 1 + \frac{1}{4} {+} \frac{1}{1} {+} \frac{1}{18} {+} \frac{1}{1} {+} \frac{1}{1} {+} \frac{1}{1} {+} \frac{1}{4} {+} \ldots \\ &= [1, 4, 1, 18, 1, 1, 1, 4, 1, 9, 9, 2, 1, 1, 1, 2, 7, 1, 1, 7, 11, \ldots] \,.\end{aligned} \tag{10.11.4}$$

To obtain the same number of correct digits as with (10.11.3), using the continued fraction (10.11.4), we need approximant f_{17}.

Another continued fraction representation is the remarkable [vdP79]

$$\zeta(3) = \frac{6}{5} + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{-m^6}{34m^3 + 51m^2 + 27m + 5} \right) . \tag{10.11.5}$$

In addition, Apéry's constant has the following property. Given three integers chosen at random, the probability that no common factor divides them all is $1/\zeta(3)$ [Wei03, p. 94].

10.12 Catalan's constant, symbol C

Catalan's constant is defined by [Ber89, p. 153]

$$C := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.915965594\ldots \tag{10.12.1}$$

and equals the value $\beta(2)$ of Dirichlet's beta function

$$\beta(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^x} \; . \tag{10.12.2}$$

Sometimes Dirichlet's beta function is called Catalan's beta function. It is an open question if Catalan's constant is irrational.

Continued fractions. The continued fraction [Ber89, p. 153]

$$2C = 1 + \frac{1}{1/2} + \frac{1^2}{1/2} + \frac{1\cdot 2}{1/2} + \frac{2^2}{1/2} + \frac{2\cdot 3}{1/2} + \frac{3^2}{1/2} + \frac{3\cdot 4}{1/2} + \ldots \tag{10.12.3}$$

converges very slowly. The regular continued fraction f has no special pattern and it starts with

$$[0, 1, 10, 1, 8, 1, 88, 4, 1, 1, 7, 22, 1, 2, 3, \ldots] \; . \tag{10.12.4}$$

The value C of the regular continued fraction is enclosed by

$$0.91596559399 < f_8 = \frac{48559}{53014} < C < f_7 = \frac{38869}{42435} < 0.91596559444$$

with

$$|f_7 - f_8| \le 4.5 \times 10^{-10} \; .$$

A rapidly converging continued fraction for Catalan's constant is given by [Zud03]

$$\begin{aligned} C &= \frac{13/2}{q(0)} + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{(2m-1)^4 (2m)^4 p(m-1) p(m+1)}{q(m)} \right), \\ p(m) &= 20m^2 - 8m + 1, \\ q(m) &= 3520m^6 + 5632m^5 + 2064m^4 - 384m^3 - 156m^2 + 16m + 7 \; . \end{aligned} \tag{10.12.5}$$

Using (10.12.5) we can enclose C by

$$0.91596559417721898 < f_8 < C < f_7 < 0.91596559417722331$$

with

$$|f_7 - f_8| \le 4.4 \times 10^{-15} \; .$$

In statistical mechanics C arises as part of the exact solution of the so-called *dimer problem* [Bec64, p. 105].

10.13 Gompertz' constant, symbol G

The integral

$$G := \int_0^\infty \frac{e^{-u}}{1+u}\, du = eE_1(1) = -e\,\mathrm{Ei}(-1) = 0.596347362\ldots$$

where $E_1(x)$ and $\mathrm{Ei}(x)$ are the exponential integrals discussed in *Chapter* 14, is called *Gompertz' constant* [LL83, p. 29].

Continued fraction. Stieltjes showed that the continued fraction expansion f of Gompertz constant is given by [Wei03, p. 1213]

$$G = \frac{1}{2} + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{-m^2}{2(m+1)}\right) = \frac{1}{2+}\,\frac{-1^2}{4+}\,\frac{-2^2}{6+}\,\frac{-3^2}{8+\ldots}. \tag{10.13.1}$$

The approximants f_{20} and f_{21} equal

$$f_{20} = \frac{6058867628609513926 0}{10159967541436156691 3}$$

and

$$f_{21} = \frac{1328430141356219649982 0}{2217611858134646955714 1}$$

with

$$0.59634714 < f_{20} < G < f_{21} < 0.59634723$$

and

$$|f_{22} - f_{21}| \leq 0.8 \times 10^{-7}.$$

10.14 Khinchin's constant, symbol K

Let x be a positive number and

$$x = [a_0, a_1, a_2, \ldots]$$

be its regular continued fraction representation. We assume that x is irrational, otherwise the continued fraction terminates. Khinchin (1934) [Khi34] considered the following problem. For almost every x the limit of the geometric mean

$$K_n(x) = \lim_{n\to\infty} (a_1 a_2 \cdots a_n)^{1/n} \tag{10.14.1}$$

exists and is surprisingly enough independent of x [JT80, pp. 4–5]. The value of *Khinchin's constant* is

$$K = 2.685452001\ldots.$$

It is not known if K is irrational.
There are several representations for K. We just mention explicitly [SW59, p. 93; Khi97, pp. 86–94]

$$K = \prod_{n=1}^{\infty} \left[1 + \frac{1}{n(n+2)}\right]^{\ln(n)/\ln(2)} . \qquad (10.14.2)$$

Other representations involve, for instance, the Riemann zeta function $\zeta(k)$ [BBC97],

$$K = \exp\left(\frac{1}{\ln(2)} \sum_{k=1}^{\infty} \frac{H'_{2k-1}\left(\zeta(2k) - 1\right)}{k}\right), \qquad (10.14.3)$$

and the k^{th} *alternating harmonic number* H'_k defined by

$$H'_k = \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} .$$

The start of the regular continued fraction of K is

$$[2, 1, 2, 5, 1, 1, 2, 1, 1, \ldots] .$$

11

Elementary functions

The elementary functions are grouped into a number of smaller families. For every function we list several continued fraction representations, each with their domain of convergence in the complex plane. Most continued fraction representations are limit periodic. The speed of convergence of each listed continued fraction formula is illustrated with some typical evaluations.

In the sequel we consistently use z for a complex argument and x for a real argument.

11.1 The exponential function

The *exponential function* is an entire function without zeroes. For $z \in \mathbb{C}$,

$$\exp(z) = \exp(|z|) \left(\cos(\arg z) + \mathrm{i} \sin(\arg z)\right) .$$

Formal series expansion.

$$\exp(z) = \sum_{k=0}^{\infty} \frac{1}{k!} z^k, \qquad z \in \mathbb{C}. \tag{11.1.1}$$

Continued fraction representations. A regular C-fraction [Wal48, p. 348] is given in (11.1.3) and a general T-fraction for $\exp(z) - 1$ [Kho63, p. 113] in (11.1.4). The related function $-2 + z + 2z/(\exp(z) - 1)$ [Kho63, p. 114], when viewed as a function of z^2, gives rise to the S-fraction (11.1.2).

All converge throughout the entire complex plane:

$$\exp(z) = 1 + \frac{2z}{2 - z +} \frac{z^2/6}{1} + \mathop{\mathrm{K}}_{m=3}^{\infty} \left(\frac{a_m z^2}{1} \right), \qquad z \in \mathbb{C}, \tag{11.1.2}$$

$$a_m = \frac{1}{4(2m-3)(2m-1)}$$

$$= 1 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m z}{1} \right), \qquad z \in \mathbb{C}, \tag{11.1.3}$$

$$a_1 = 1, \quad a_{2k} = \frac{-1}{2(2k-1)}, \quad a_{2k+1} = \frac{1}{2(2k+1)}$$

$$= 1 + \frac{z}{1 - z +} \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{(m-1)z}{m - z} \right), \qquad z \in \mathbb{C}. \tag{11.1.4}$$

TABLE 11.1.1: Relative error of 5th partial sum and 5th approximants.

x	$\exp(x)$	(11.1.1)	(11.1.2)	(11.1.3)	(11.1.4)
−30	9.357623e−14	1.8e+18	1.5e+12	6.1e+13	4.4e+07
−10	4.539993e−05	1.2e+07	8.3e+01	1.5e+04	1.4e+01
−5	6.737947e−03	1.8e+03	8.6e−03	1.3e+01	6.2e−01
1	2.718282e+00	5.9e−04	1.0e−10	1.2e−04	3.3e−03
2	7.389056e+00	1.7e−02	2.2e−07	7.5e−03	1.0e+00
5	1.484132e+02	3.8e−01	8.6e−03	7.3e−01	1.0e+00
15	3.269017e+06	1.0e+00	1.0e+00	1.0e+00	1.0e+00

TABLE 11.1.2: Relative error of 20th partial sum and 20th approximants.

x	$\exp(x)$	(11.1.1)	(11.1.2)	(11.1.3)	(11.1.4)
−30	9.357623e−14	9.1e+23	1.4e+02	9.0e+09	2.7e+01
−10	4.539993e−05	3.0e+05	4.0e−20	3.4e−04	1.6e−03
−5	6.737947e−03	1.1e−03	1.1e−32	6.8e−11	8.1e−08
1	2.718282e+00	7.5e−21	2.2e−61	1.1e−25	5.1e−20
2	7.389056e+00	6.1e−15	4.9e−49	2.3e−19	2.8e−13
5	1.484132e+02	8.1e−08	1.1e−32	6.8e−11	1.1e−03
15	3.269017e+06	8.3e−02	1.4e−12	8.7e−01	1.0e+00

FIGURE 11.1.1: Complex region where $f_8(z;0)$ of (11.1.2) guarantees k significant digits for $\exp(z)$ (from light to dark $k = 6, 7, 8, 9$).

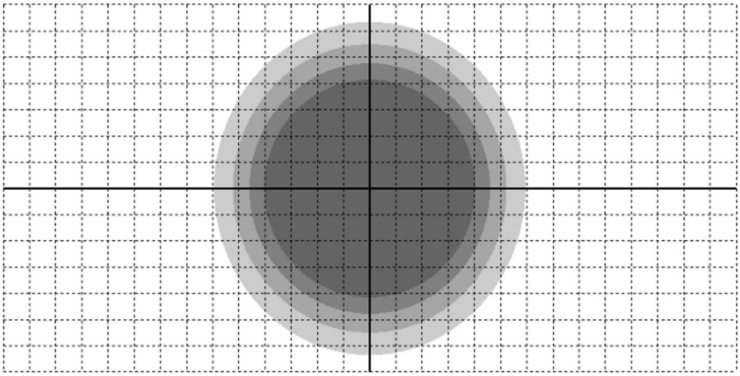
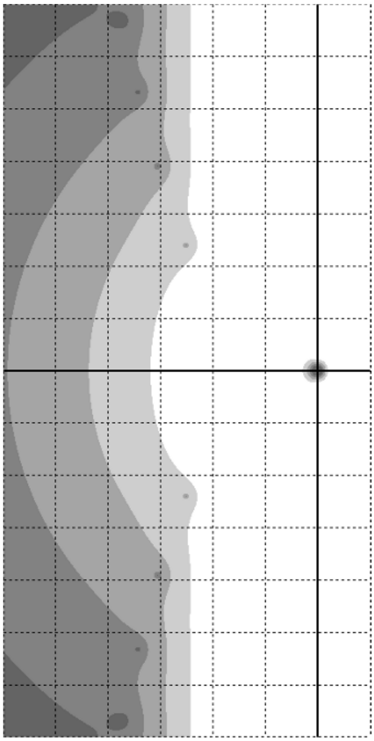


FIGURE 11.1.2: Complex region where $f_8(z;0) - 1$ of (11.1.4) guarantees k significant digits for $\exp(z) - 1$ (from light to dark $k = 6, 7, 8, 9$). Exceptionally the grid lines are 5 units apart on the real and imaginary axis.



11.2 The natural logarithm

The *logarithmic function* is a many-valued function. Its principal value or branch is given by

$$\operatorname{Ln}(z) = \ln(|z|) + \mathrm{i}\operatorname{Arg} z, \qquad \operatorname{Arg} z \in (-\pi, \pi], \quad z \neq 0.$$

Formal series expansion.

$$\operatorname{Ln}(1+z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} z^k, \qquad |z| < 1. \tag{11.2.1}$$

Continued fraction representations. An S-fraction for $\operatorname{Ln}(1+z)$ is given in (11.2.2) [Wal48, p. 342]. Formula (11.2.3) [Kho63, p. 111] is the even contraction of the continued fraction that can be constructed from the series (11.2.1) through the Euler connection (1.7.2). An S-fraction for the related function $-1+2z/\operatorname{Ln}\left((1+z)/(1-z)\right)$ in the variable $(\mathrm{i}z)^2$ [Wal48, p. 343] can be found in (11.2.4):

$$\operatorname{Ln}(1+z) = \frac{z}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{a_m z}{1}\right), \qquad |\operatorname{Arg}(1+z)| < \pi, \tag{11.2.2}$$

$$a_{2k} = \frac{k}{2(2k-1)}, \quad a_{2k+1} = \frac{k}{2(2k+1)}$$

$$= \frac{2z}{2+z} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-(m-1)^2 z^2}{(2m-1)(2+z)}\right), \tag{11.2.3}$$

$$|\operatorname{Arg}\left(1 - z^2/(2+z)^2\right)| < \pi$$

$$\operatorname{Ln}\left(\frac{1+z}{1-z}\right) = \frac{2z}{1} + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m z^2}{1}\right), \qquad |\operatorname{Arg}(1-z^2)| < \pi, \tag{11.2.4}$$

$$a_m = \frac{-m^2}{(2m-1)(2m+1)}.$$

Since in (11.2.2), $\lim_{m\to\infty} a_m z = z/4$ and

$$\lim_{m\to\infty} \frac{a_{m+1} - \frac{1}{4}}{a_m - \frac{1}{4}} = -1,$$

the modifications (7.7.5) and (7.7.10), given here by

$$w(z) = \frac{-1 + \sqrt{1+z}}{2}$$

and

$$\begin{cases} w_{2k}^{(1)}(z) = w(z) + \dfrac{kz}{2(2k+1)} - \dfrac{z}{4}, \\ w_{2k+1}^{(1)}(z) = w(z) + \dfrac{(k+1)z}{2(2k+1)} - \dfrac{z}{4}, \end{cases}$$

can be used. In the same way, the respective modifications

$$\tilde{w}_n(z) = (2n-1)\frac{-(2+z)+2\sqrt{1+z}}{2},$$
$$\tilde{w}_n^{(1)}(z) = \tilde{w}_n(z) - \frac{z^2}{8(2n+1)\sqrt{1+z}}$$

can be used for the evaluation of (11.2.3). The results are displayed in the *Tables* 11.2.2 and 11.2.4, where they can also be compared to the unmodified approximants copied from the *Tables* 11.2.1 and 11.2.3. A numerical illustration of (11.2.4) is given in the *Tables* 15.3.1 and 15.3.2 in *Chapter* 15.

A Thiele continued fraction expansion for $\mathrm{Ln}(1+z)$ is given in (6.8.8) and compared with in the *Tables* 11.2.1 and 11.2.3. Its evaluation can be combined with the modification

$$w(z) = \frac{-1+\sqrt{z+1}}{2}. \tag{11.2.5}$$

TABLE 11.2.1: Relative error of 5th partial sum and 5th approximants.

x	$\mathrm{Ln}(1+x)$	(11.2.1)	(11.2.2)	(11.2.3)	(6.8.8)
−0.9	−2.302585e+00	1.7e−01	4.8e−02	1.9e−03	2.9e−02
−0.4	−5.108256e−01	7.1e−04	5.4e−05	1.6e−09	9.5e−06
0.1	9.531018e−02	1.4e−07	1.3e−08	8.8e−17	4.9e−10
0.5	4.054651e−01	1.9e−03	1.9e−05	1.7e−10	3.1e−06
1.1	7.419373e−01	1.9e−01	3.8e−04	6.4e−08	1.3e−04
5	1.791759e+00	1.2e+03	2.6e−02	2.4e−04	3.0e−02
10	2.397895e+00	6.2e+04	9.9e−02	2.6e−03	2.0e−01
100	4.615121e+00	3.6e+10	1.8e+00	1.2e−01	2.6e+01

TABLE 11.2.2: Relative error of 5$^{\text{th}}$ (modified) approximants.

x	$\text{Ln}(1+x)$	(11.2.2)	(11.2.2)	(11.2.2)
-0.9	$-2.302585e{+}00$	$4.8e{-}02$	$1.2e{-}02$	$1.1e{-}03$
-0.4	$-5.108256e{-}01$	$5.4e{-}05$	$9.4e{-}06$	$1.7e{-}07$
0.1	$9.531018e{-}02$	$1.3e{-}08$	$2.2e{-}09$	$7.4e{-}12$
0.5	$4.054651e{-}01$	$1.9e{-}05$	$3.0e{-}06$	$4.4e{-}08$
1.1	$7.419373e{-}01$	$3.8e{-}04$	$6.1e{-}05$	$1.6e{-}06$
5	$1.791759e{+}00$	$2.6e{-}02$	$4.3e{-}03$	$2.7e{-}04$
10	$2.397895e{+}00$	$9.9e{-}02$	$1.6e{-}02$	$1.4e{-}03$
100	$4.615121e{+}00$	$1.8e{+}00$	$2.4e{-}01$	$3.4e{-}02$

x	$\text{Ln}(1+x)$	(11.2.3)	(11.2.3)	(11.2.3)
-0.9	$-2.302585e{+}00$	$1.9e{-}03$	$3.2e{-}05$	$3.4e{-}06$
-0.4	$-5.108256e{-}01$	$1.6e{-}09$	$1.7e{-}11$	$8.2e{-}14$
0.1	$9.531018e{-}02$	$8.8e{-}17$	$8.8e{-}19$	$1.5e{-}22$
0.5	$4.054651e{-}01$	$1.7e{-}10$	$1.7e{-}12$	$5.2e{-}15$
1.1	$7.419373e{-}01$	$6.4e{-}08$	$6.7e{-}10$	$7.0e{-}12$
5	$1.791759e{+}00$	$2.4e{-}04$	$3.3e{-}06$	$2.1e{-}07$
10	$2.397895e{+}00$	$2.6e{-}03$	$4.5e{-}05$	$5.3e{-}06$
100	$4.615121e{+}00$	$1.2e{-}01$	$7.2e{-}03$	$4.1e{-}03$

TABLE 11.2.3: Relative error of 20$^{\text{th}}$ partial sum and 20$^{\text{th}}$ approximants.

x	$\text{Ln}(1+x)$	(11.2.1)	(11.2.2)	(11.2.3)	(6.8.8)
-0.9	$-2.302585e{+}00$	$1.5e{-}02$	$2.8e{-}06$	$5.8e{-}12$	$1.5e{-}06$
-0.4	$-5.108256e{-}01$	$2.5e{-}10$	$1.8e{-}18$	$2.2e{-}36$	$2.4e{-}19$
0.1	$9.531018e{-}02$	$4.4e{-}23$	$5.3e{-}33$	$1.9e{-}65$	$1.3e{-}34$
0.5	$4.054651e{-}01$	$1.8e{-}08$	$1.9e{-}20$	$2.3e{-}40$	$2.0e{-}21$
1.1	$7.419373e{-}01$	$2.4e{-}01$	$2.8e{-}15$	$5.3e{-}30$	$5.5e{-}16$
5	$1.791759e{+}00$	$1.0e{+}13$	$4.2e{-}08$	$1.3e{-}15$	$2.0e{-}08$
10	$2.397895e{+}00$	$1.8e{+}19$	$5.3e{-}06$	$2.1e{-}11$	$3.3e{-}06$
100	$4.615121e{+}00$	$1.0e{+}40$	$1.8e{-}02$	$3.6e{-}04$	$2.5e{-}02$

TABLE 11.2.4: Relative error of 20^{th} (modified) approximants.

x	$\text{Ln}(1+x)$	(11.2.2)	(11.2.2)	(11.2.2)
-0.9	$-2.302585e{+}00$	$2.8e{-}06$	$1.9e{-}07$	$2.9e{-}10$
-0.4	$-5.108256e{-}01$	$1.8e{-}18$	$9.3e{-}20$	$4.3e{-}22$
0.1	$9.531018e{-}02$	$5.3e{-}33$	$2.7e{-}34$	$3.2e{-}37$
0.5	$4.054651e{-}01$	$1.9e{-}20$	$9.5e{-}22$	$5.5e{-}24$
1.1	$7.419373e{-}01$	$2.8e{-}15$	$1.5e{-}16$	$1.8e{-}18$
5	$1.791759e{+}00$	$4.2e{-}08$	$2.6e{-}09$	$1.1e{-}10$
10	$2.397895e{+}00$	$5.3e{-}06$	$3.7e{-}07$	$2.8e{-}08$
100	$4.615121e{+}00$	$1.8e{-}02$	$2.9e{-}03$	$9.2e{-}04$

x	$\text{Ln}(1+x)$	(11.2.3)	(11.2.3)	(11.2.3)
-0.9	$-2.302585e{+}00$	$5.8e{-}12$	$6.6e{-}15$	$2.2e{-}16$
-0.4	$-5.108256e{-}01$	$2.2e{-}36$	$1.4e{-}39$	$2.2e{-}42$
0.1	$9.531018e{-}02$	$1.9e{-}65$	$1.2e{-}68$	$6.1e{-}73$
0.5	$4.054651e{-}01$	$2.3e{-}40$	$1.5e{-}43$	$1.4e{-}46$
1.1	$7.419373e{-}01$	$5.3e{-}30$	$3.5e{-}33$	$1.1e{-}35$
5	$1.791759e{+}00$	$1.3e{-}15$	$1.1e{-}18$	$2.2e{-}20$
10	$2.397895e{+}00$	$2.1e{-}11$	$2.5e{-}14$	$9.3e{-}16$
100	$4.615121e{+}00$	$3.6e{-}04$	$1.7e{-}06$	$3.1e{-}07$

FIGURE 11.2.1: Complex region where $f_8(z;0)$ of (11.2.2) guarantees k significant digits for $\text{Ln}(1+z)$ (from light to dark $k = 6, 7, 8, 9$).

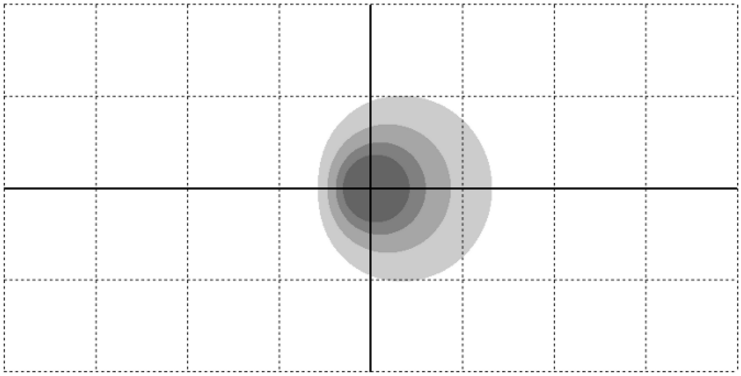
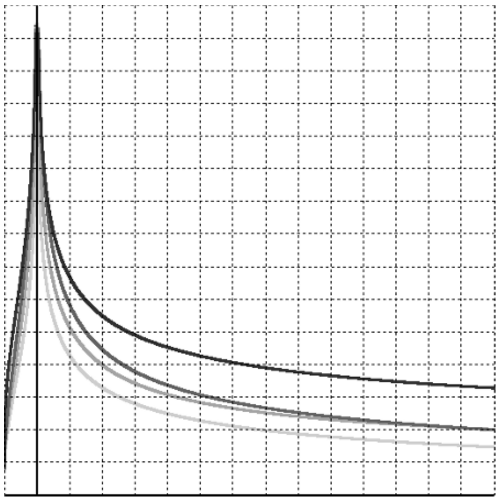


FIGURE 11.2.2: Number of significant digits guaranteed by the n^{th} classical approximant of (11.2.2) (from light to dark $n = 5, 6, 7$) and the 5^{th} modified approximant evaluated with $w_5^{(1)}(z)$ (darkest).



11.3 Trigonometric functions

Of the six *trigonometric functions* we discuss only three, namely the sine, cosine and tangent functions:

$$
\begin{aligned}
\sin(z) &= \frac{\exp(\mathrm{i}z) - \exp(-\mathrm{i}z)}{2\mathrm{i}}, \\
\sin(x + \mathrm{i}y) &= \sin(x)\cosh(y) + \mathrm{i}\cos(x)\sinh(y), \\
\cos(z) &= \frac{\exp(\mathrm{i}z) + \exp(-\mathrm{i}z)}{2}, \\
\cos(x + \mathrm{i}y) &= \cos(x)\cosh(y) - \mathrm{i}\sin(x)\sinh(y), \\
\tan(z) &= \frac{\sin(z)}{\cos(z)}, \qquad \cos(z) \neq 0, \\
\tan(x + \mathrm{i}y) &= \frac{\sin(2x) + \mathrm{i}\sinh(2y)}{\cos(2x) + \cosh(2y)} .
\end{aligned}
$$

The other three functions are the reciprocals of these:

$$
\begin{aligned}
\sec(z) &= 1/\cos(z), \qquad \cos(z) \neq 0, \\
\csc(z) &= 1/\sin(z), \qquad \sin(z) \neq 0, \\
\cot(z) &= 1/\tan(z), \qquad \sin(z) \neq 0.
\end{aligned}
$$

Formal series expansion.

$$\sin(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)!} z^{2k-1}, \qquad z \in \mathbb{C}, \tag{11.3.1}$$

$$\cos(z) = \sum_{k=0}^{\infty} \frac{(-1)^{k}}{(2k)!} z^{2k}, \qquad z \in \mathbb{C}, \tag{11.3.2}$$

$$\tan(z) = \sum_{k=1}^{\infty} \frac{4^k(4^k-1)|B_{2k}|}{(2k)!} z^{2k-1}, \qquad |z| < \pi/2. \tag{11.3.3}$$

where B_{2k} is the $(2k)^{\text{th}}$ *Bernoulli number* given by the recursion

$$\begin{aligned} B_0 &= 1, \qquad B_1 = -1/2, \\ B_{2m} &= \frac{1}{2} - \frac{1}{2m+1} \sum_{k=0}^{m-1} \binom{2m+1}{2k} B_{2k}, \qquad m \geq 1 \\ &= \frac{1}{2} - \frac{1}{2m+1} - \frac{1}{2m+1} \sum_{k=0}^{m-1} \binom{2m+1}{k} B_{2k}, \\ B_{2m+1} &= 0, \qquad m \geq 1. \end{aligned} \tag{11.3.4}$$

The Bernoulli numbers B_{2m} also satisfy

$$\frac{2(2m)!}{(2\pi)^{2m}} < (-1)^{m+1} B_{2m} < \frac{2(2m)!}{(2\pi)^{2m}} \left(\frac{1}{1-2^{1-2m}} \right), \qquad m \geq 1 \tag{11.3.5}$$

and hence

$$|B_{2m}| \sim \frac{2(2m)!}{(2\pi)^{2m}}, \qquad m \to \infty. \tag{11.3.6}$$

Continued fraction representations. Using the Euler connection (1.7.2) general T-fractions can be obtained for $\sin(z)$ and $\cos(z)$. They do however not offer any advantage over the series representations (11.3.1) and (11.3.2). For $\tan(z)$ several Thiele interpolating continued fractions exist [Per54, p. 35; ABJL92, p. 50], as well as the S-fraction (11.3.7) in $-z^2$ for $-1 + z/\tan(z)$ [Wal48, p. 349]. All converge everywhere $\tan(z)$ is defined:

$$\tan(z) = \frac{z}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-a_m z^2}{1} \right), \qquad a_m = \frac{1}{(2m-3)(2m-1)},$$
$$z \in \mathbb{C} \setminus \{\pi/2 + k\pi : k \in \mathbb{Z}\} \qquad (11.3.7)$$

$$\tan(\pi z/4) = \frac{z}{1} + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{(2m-1)^2 - z^2}{2} \right),$$
$$z \in \mathbb{C} \setminus \{\pi/2 + k\pi : k \in \mathbb{Z}\} \qquad (11.3.8)$$

$$\tan(z) = \frac{z}{1} + \frac{-4\pi^{-2}z^2}{1} + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{m^4 - 4\pi^{-2}m^2z^2}{2m+1} \right),$$
$$z \in \mathbb{C} \setminus \{\pi/2 + k\pi : k \in \mathbb{Z}\}. \qquad (11.3.9)$$

Note that the satisfaction of more interpolation conditions by higher approximants of (11.3.9) does not guarantee any additional significant digits in the immediate neighbourhood of the interpolation points $\pm m\pi/2$ (see *Figure* 11.3.2) when compared to previous approximants. At the interpolation points the number of significant digits is infinite.

TABLE 11.3.1: Relative error of 5^{th} partial sum and 5^{th} approximants.

x	$\tan(x)$	(11.3.3)	(11.3.7)	(11.3.9)
−1.5	−1.410142e+01	5.6e−01	1.0e−04	2.2e−02
−0.75	−9.315965e−01	1.2e−04	8.2e−09	5.5e−03
−0.25	−2.553419e−01	2.2e−10	1.0e−13	6.0e−04
0.1	1.003347e−01	3.6e−15	1.0e−17	9.7e−05
0.3	3.093362e−01	1.9e−09	6.3e−13	8.7e−04
0.6	6.841368e−01	8.0e−06	7.7e−10	3.5e−03
1	1.557408e+00	3.9e−03	2.0e−07	9.7e−03

TABLE 11.3.2: Relative error of 20^{th} partial sum and 20^{th} approximants.

x	$\tan(x)$	(11.3.3)	(11.3.7)	(11.3.9)
−1.5	$-1.410142e{+}01$	$1.4e{-}01$	$5.3e{-}41$	$1.2e{-}03$
−0.75	$-9.315965e{-}01$	$2.8e{-}14$	$3.6e{-}54$	$3.0e{-}04$
−0.25	$-2.553419e{-}01$	$2.4e{-}34$	$2.1e{-}73$	$3.3e{-}05$
0.1	$1.003347e{-}01$	$4.7e{-}51$	$2.4e{-}89$	$5.3e{-}06$
0.3	$3.093362e{-}01$	$5.2e{-}31$	$3.1e{-}70$	$4.8e{-}05$
0.6	$6.841368e{-}01$	$2.3e{-}18$	$4.1e{-}58$	$1.9e{-}04$
1	$1.557408e{+}00$	$5.1e{-}09$	$5.1e{-}49$	$5.3e{-}04$

FIGURE 11.3.1: Complex region where $f_8(z;0)$ of (11.3.7) guarantees k significant digits for $\tan(z)$ (from light to dark $k = 6, 7, 8, 9$).

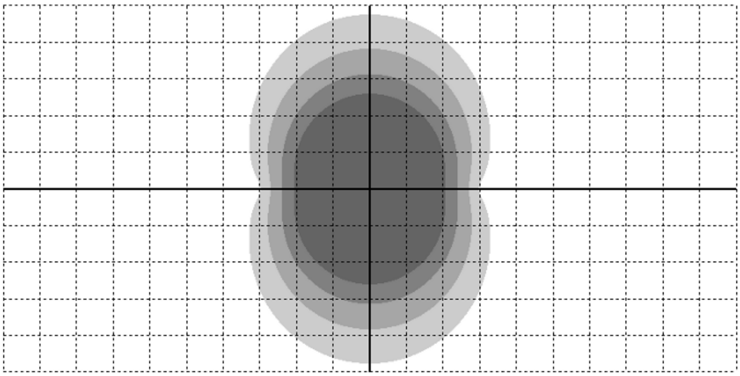
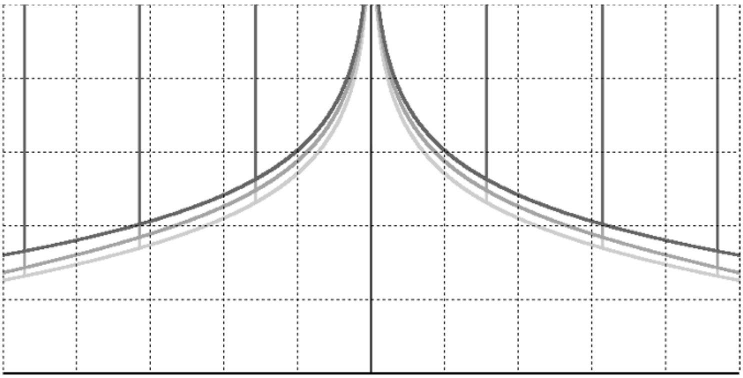


FIGURE 11.3.2: Number of significant digits guaranteed by the n^{th} approximant of (11.3.9) (from light to dark $n = 5, 6, 7$).



11.4 Inverse trigonometric functions

A variety of ways exists to extend the inverse trigonometric functions to multi-valued functions of a complex argument. They can be expressed in terms of the natural logarithm function as follows:

$$\begin{aligned}
\operatorname{Arcsin}(z) &= \frac{1}{\mathrm{i}} \operatorname{Ln}\left(\mathrm{i}z + \sqrt{1-z^2}\right),\\
\operatorname{Arccos}(z) &= \frac{1}{\mathrm{i}} \operatorname{Ln}\left(z + \sqrt{z^2-1}\right),\\
\operatorname{Arctan}(z) &= \frac{1}{2\mathrm{i}} \operatorname{Ln}\left(\frac{1+\mathrm{i}z}{1-\mathrm{i}z}\right),\\
\operatorname{Arccot}(z) &= \frac{1}{2\mathrm{i}} \operatorname{Ln}\left(\frac{z+\mathrm{i}}{z-\mathrm{i}}\right).
\end{aligned}$$

They are closely related to the principal branch of the inverse hyperbolic functions by

$$\begin{aligned}
\operatorname{Asinh}(\mathrm{i}z) &= \mathrm{i}\operatorname{Arcsin}(z),\\
\operatorname{Acosh}(z) &= \mathrm{i}\operatorname{Arccos}(z),\\
\operatorname{Atanh}(\mathrm{i}z) &= \mathrm{i}\operatorname{Arctan}(z),\\
\operatorname{Acoth}(\mathrm{i}z) &= -\mathrm{i}\operatorname{Arccot}(z).
\end{aligned}$$

In addition, a lot of relationships exist among the six inverse trigonometric functions. We therefore only present the FTS for $\operatorname{Arcsin}(z)$ and $\operatorname{Arctan}(z)$.

Formal series expansion.

$$\operatorname{Arcsin}(z) = z + \sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k)!!\,(2k+1)} z^{2k+1}, \qquad |z|<1, \qquad (11.4.1)$$

$$\operatorname{Arccos}(z) = \frac{\pi}{2} - \operatorname{Arcsin}(z), \qquad |z|<1, \qquad (11.4.2)$$

$$\operatorname{Arctan}(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} z^{2k-1}, \qquad |z|<1, \qquad (11.4.3)$$

$$\begin{aligned}
\operatorname{Arccot}(z) &= \operatorname{Arctan}(1/z),\\
\operatorname{Arcsec}(z) &= \operatorname{Arccos}(1/z),\\
\operatorname{Arccsc}(z) &= \operatorname{Arcsin}(1/z).
\end{aligned}$$

where the *double factorial* is defined by

$$0!! = 1,$$
$$(2k)!! = \prod_{j=1}^{k}(2j),$$
$$(2k+1)!! = \prod_{j=0}^{k}(2j+1).$$

Continued fraction representations. Several S-fraction like representations can be given [Kho63, pp. 118–121; Wal48, pp. 343–345]. Note that the even contractions of (11.4.4) and (11.4.5) are equal:

$$\operatorname{Arcsin}(z) = \frac{z/\sqrt{1-z^2}}{1} + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{m^2 z^2/(1-z^2)}{2m+1}\right), \quad |\operatorname{Arg}(1-z^2)| < \pi \qquad (11.4.4)$$

$$= \frac{z\sqrt{1-z^2}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{a_m z^2}{1}\right), \qquad |\operatorname{Arg}(1-z^2)| < \pi,$$
$$a_{2k} = \frac{-2k(2k-1)}{(4k-1)(4k-3)}, \quad a_{2k+1} = \frac{-2k(2k-1)}{(4k+1)(4k-1)}. \qquad (11.4.5)$$

TABLE 11.4.1: Relative error of 5th partial sum and 5th approximants.

x	Arcsin(x)	(11.4.1)	(11.4.4)	(11.4.5)
-0.9	$-1.119770e{+}00$	$1.2e{-}02$	$1.6e{-}02$	$1.7e{-}02$
-0.5	$-5.235988e{-}01$	$5.1e{-}06$	$2.9e{-}06$	$3.5e{-}06$
-0.2	$-2.013579e{-}01$	$7.3e{-}11$	$1.7e{-}10$	$2.0e{-}10$
0.1	$1.001674e{-}01$	$1.7e{-}14$	$1.5e{-}13$	$1.8e{-}13$
0.3	$3.046927e{-}01$	$9.8e{-}09$	$1.1e{-}08$	$1.3e{-}08$
0.6	$6.435011e{-}01$	$5.0e{-}05$	$2.7e{-}05$	$3.1e{-}05$
0.8	$9.272952e{-}01$	$2.2e{-}03$	$1.6e{-}03$	$1.8e{-}03$

TABLE 11.4.2: Relative error of 20^{th} partial sum and 20^{th} approximants.

x	Arcsin(x)	(11.4.1)	(11.4.4)	(11.4.5)
−0.9	−1.119770e+00	1.2e−04	1.3e−08	1.3e−08
−0.5	−5.235988e−01	8.1e−16	2.1e−23	2.1e−23
−0.2	−2.013579e−01	1.3e−32	2.3e−40	2.3e−40
0.1	1.001674e−01	2.9e−45	1.6e−52	1.6e−52
0.3	3.046927e−01	3.3e−25	4.4e−33	4.4e−33
0.6	6.435011e−01	1.9e−12	1.3e−19	1.3e−19
0.8	9.272952e−01	5.2e−07	1.5e−12	1.5e−12

Similarly, the even contractions of (11.4.6) and (11.4.7) are equal, and those of the continued fractions (11.4.8) and (11.4.9) too:

$$\operatorname{Arccos}(z) = \frac{\sqrt{1-z^2}/z}{1} + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{m^2(1-z^2)/z^2}{2m+1} \right), \qquad \Re z > 0, \quad (1-z^2)/z^2 \notin (-\infty, -1] \tag{11.4.6}$$

$$= \frac{z\sqrt{1-z^2}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-a_m(1-z^2)}{1} \right), \qquad \Re z > 0,$$
$$a_{2k} = \frac{2k(2k-1)}{(4k-1)(4k-3)}, \quad a_{2k+1} = \frac{2k(2k-1)}{(4k+1)(4k-1)}. \tag{11.4.7}$$

TABLE 11.4.3: Relative error of 5^{th} partial sum and 5^{th} approximants.

x	Arccos(x)	(11.4.2)	(11.4.6)	(11.4.7)
0.1	1.470629e+00	1.2e−15	1.1e+00	5.4e−01
0.3	1.266104e+00	2.4e−09	8.4e−02	8.3e−02
0.5	1.047198e+00	2.5e−06	7.0e−03	7.6e−03
0.7	7.953988e−01	3.5e−04	2.8e−04	3.1e−04
0.9	4.510268e−01	3.1e−02	6.2e−07	7.3e−07

TABLE 11.4.4: Relative error of 20th partial sum and 20th approximants.

x	Arccos(x)	(11.4.2)	(11.4.6)	(11.4.7)
0.1	1.470629e+00	2.0e−46	3.4e−02	3.4e−02
0.3	1.266104e+00	8.1e−26	7.6e−06	7.6e−06
0.5	1.047198e+00	4.0e−16	4.9e−10	4.9e−10
0.7	7.953988e−01	1.4e−09	1.4e−15	1.4e−15
0.9	4.510268e−01	2.9e−04	4.2e−26	4.2e−26

$$\operatorname{Arctan}(z) = \frac{z}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{(m-1)^2 z^2}{2m-1}\right), \qquad iz \notin (-\infty,-1) \cup (1,+\infty) \tag{11.4.8}$$

$$= \frac{z/(1+z^2)}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{-a_m z^2/(1+z^2)}{1}\right),$$
$$a_{2k} = \frac{2k(2k-1)}{(4k-3)(4k-1)}, \quad a_{2k+1} = \frac{2k(2k-1)}{(4k-1)(4k+1)}, \qquad iz \notin (-\infty,-1) \cup (1,+\infty). \tag{11.4.9}$$

Both (11.4.8) and (11.4.9) are limit periodic and hence the modification (7.7.5), given respectively by

$$\tilde{w}_n(z) = \frac{2n-1}{2}\left(-1+\sqrt{1+z^2}\right)$$

and

$$w(z) = \frac{1}{2}\left(-1+\sqrt{1-\frac{z^2}{1+z^2}}\right)$$

can be used. For (11.4.8) and (11.4.9) we respectively have

$$\lim_{m\to\infty} \frac{\frac{m^2}{4m^2-1} - \frac{1}{4}}{\frac{(m-1)^2}{(2m-1)(2m-3)} - \frac{1}{4}} = 1$$

and

$$\lim_{m\to\infty} \frac{-a_{m+1} + \frac{1}{4}}{-a_m + \frac{1}{4}} = -1,$$

and then the improved modification (7.7.10), given here by

$$\tilde{w}_n^{(1)}(z) = \tilde{w}_n(z) + \frac{z^2}{4(2n+1)\sqrt{1+z^2}}$$

and

$$w_{2k-1}^{(1)}(z) = w(z) + \frac{-2k+\frac{3}{4}}{16k^2-16k+3}\,\frac{z^2}{1+z^2},$$
$$w_{2k}^{(1)}(z) = w(z) + \frac{2k-\frac{1}{4}}{16k^2-1}\,\frac{z^2}{1+z^2}$$

respectively, is useful. In the *Tables* 11.4.6 and 11.4.8 the continued fraction representations (11.4.8) and (11.4.9) are first evaluated with $w = 0$ and subsequently with the modifications $w(z)$ and $w_n^{(1)}(z)$. In the *Tables* 11.4.5 and 11.4.7 all approximants are evaluated without any use of modification.

TABLE 11.4.5: Relative error of 5th partial sum and 5th approximants.

x	Arctan(x)	(11.4.3)	(11.4.8)	(11.4.9)
−0.9	−7.328151e−01	1.6e−02	1.1e−04	1.3e−04
−0.5	−4.636476e−01	1.7e−05	8.3e−07	9.7e−07
−0.2	−1.973956e−01	3.1e−10	1.4e−10	1.6e−10
0.1	9.966865e−02	7.7e−14	1.4e−13	1.7e−13
0.3	2.914568e−01	3.9e−08	7.0e−09	8.4e−09
0.7	6.107260e−01	8.6e−04	1.5e−05	1.8e−05

TABLE 11.4.6: Relative error of 5th (modified) approximants.

x	Arctan(x)	(11.4.8)	(11.4.8)	(11.4.8)
−0.9	−7.328151e−01	1.1e−04	8.7e−07	3.4e−08
−0.5	−4.636476e−01	8.3e−07	7.5e−09	1.2e−10
−0.2	−1.973956e−01	1.4e−10	1.3e−12	3.9e−15
0.1	9.966865e−02	1.4e−13	1.4e−15	1.1e−18
0.3	2.914568e−01	7.0e−09	6.8e−11	4.3e−13
0.7	6.107260e−01	1.5e−05	1.3e−07	3.5e−09

x	Arctan(x)	(11.4.9)	(11.4.9)	(11.4.9)
−0.9	−7.328151e−01	1.3e−04	2.3e−05	2.5e−07
−0.5	−4.636476e−01	9.7e−07	1.7e−07	7.5e−10
−0.2	−1.973956e−01	1.6e−10	2.9e−11	2.3e−14
0.1	9.966865e−02	1.7e−13	3.0e−14	6.0e−18
0.3	2.914568e−01	8.4e−09	1.5e−09	2.5e−12
0.7	6.107260e−01	1.8e−05	3.2e−06	2.4e−08

TABLE 11.4.7: Relative error of 20^{th} partial sum and 20^{th} approximants.

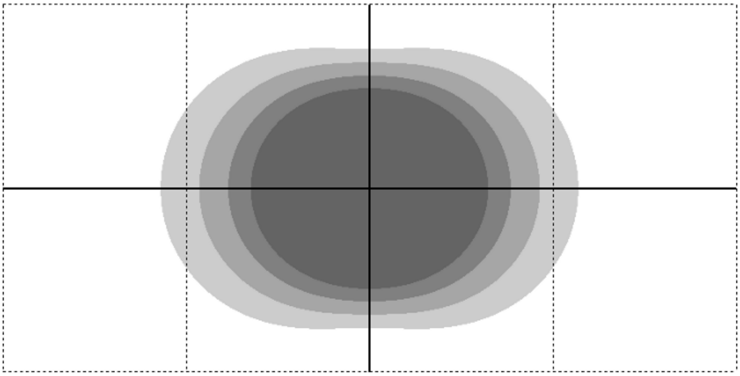
x	Arctan(x)	(11.4.3)	(11.4.8)	(11.4.9)
−0.9	−7.328151e−01	1.9e−04	3.7e−17	3.7e−17
−0.5	−4.636476e−01	4.6e−15	1.3e−25	1.3e−25
−0.2	−1.973956e−01	1.0e−31	1.0e−40	1.0e−40
0.1	9.966865e−02	2.3e−44	1.3e−52	1.3e−52
0.3	2.914568e−01	2.4e−24	7.2e−34	7.2e−34
0.7	6.107260e−01	5.7e−09	1.4e−20	1.4e−20

TABLE 11.4.8: Relative error of 20^{th} (modified) approximants.

x	Arctan(x)	(11.4.8)	(11.4.8)	(11.4.8)
−0.9	−7.328151e−01	3.7e−17	1.8e−20	2.1e−22
−0.5	−4.636476e−01	1.3e−25	7.4e−29	3.6e−31
−0.2	−1.973956e−01	1.0e−40	6.4e−44	5.8e−47
0.1	9.966865e−02	1.3e−52	8.0e−56	1.8e−59
0.3	2.914568e−01	7.2e−34	4.3e−37	8.5e−40
0.7	6.107260e−01	1.4e−20	7.4e−24	6.2e−26

x	Arctan(x)	(11.4.9)	(11.4.9)	(11.4.9)
−0.9	−7.328151e−01	3.7e−17	2.0e−18	1.8e−20
−0.5	−4.636476e−01	1.3e−25	6.9e−27	2.6e−29
−0.2	−1.973956e−01	1.0e−40	5.5e−42	3.8e−45
0.1	9.966865e−02	1.3e−52	6.6e−54	1.2e−57
0.3	2.914568e−01	7.2e−34	3.8e−35	5.7e−38
0.7	6.107260e−01	1.4e−20	7.4e−22	4.9e−24

FIGURE 11.4.1: Complex region where $f_8(z;0)$ of (11.4.8) guarantees k significant digits for $\operatorname{Arctan}(z)$ (from light to dark $k = 6, 7, 8, 9$).



11.5 Hyperbolic functions

The *hyperbolic* sine, cosine, tangent and cotangent *functions* are defined in terms of the exponential function:

$$\begin{aligned}
\sinh(z) &= \frac{\exp(z) - \exp(-z)}{2}, \\
\sinh(x + \mathrm{i}y) &= \sinh(x)\cos(y) + \mathrm{i}\cosh(x)\sin(y), \\
\cosh(z) &= \frac{\exp(z) + \exp(-z)}{2}, \\
\cosh(x + \mathrm{i}y) &= \cosh(x)\cos(y) + \mathrm{i}\sinh(x)\sin(y), \\
\tanh(z) &= \frac{\exp(2z) - 1}{\exp(2z) + 1}, \\
\tanh(x + \mathrm{i}y) &= \frac{\sinh(2x) + \mathrm{i}\sin(2y)}{\cosh(2x) + \cos(2y)}, \\
\coth(z) &= \frac{\exp(2z) + 1}{\exp(2z) - 1}, \\
\coth(x + \mathrm{i}y) &= \frac{\sinh(2x) - \mathrm{i}\sin(2y)}{\cosh(2x) - \cos(2y)}.
\end{aligned}$$

Formal series expansion.

$$\sinh(z) = \sum_{k=1}^{\infty} \frac{1}{(2k-1)!} z^{2k-1}, \qquad z \in \mathbb{C}, \tag{11.5.1}$$

$$\cosh(z) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} z^{2k}, \qquad z \in \mathbb{C}, \tag{11.5.2}$$

$$\tanh(z) = \sum_{k=1}^{\infty} \frac{4^k(4^k-1)B_{2k}}{(2k)!} z^{2k-1}, \qquad |z| < \pi/2, \tag{11.5.3}$$

$$\coth(z) = \sum_{k=0}^{\infty} \frac{4^k B_{2k}}{(2k)!} z^{2k-1}, \qquad |z| < \pi. \tag{11.5.4}$$

Continued fraction representations. From the Euler connection (1.7.2) we can obtain general T-fractions for $\sinh(z)$ and $\cosh(z)$, which have the same convergence behaviour as (11.5.1) and (11.5.2) though. For $-1 + z/\tanh(z)$ an S-fraction in z^2 [Kho63, p. 123] is given in (11.5.5) and for $z\coth(z) - 1$ a Thiele interpolating continued fraction [ABJL92, p. 50] in (11.5.6):

$$\tanh(z) = \frac{z}{1} + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m z^2}{1}\right), \qquad a_m = \frac{1}{(2m-1)(2m+1)}, \tag{11.5.5}$$
$$z \in \mathbb{C} \setminus \{\mathrm{i}(\pi/2 + k\pi) : k \in \mathbb{Z}\},$$

$$\coth(z) = \frac{1}{z} + \frac{4\pi^{-2}z}{1} + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{m^2(m^2+4\pi^{-2}z^2)}{(2m+1)}\right), \qquad z \in \mathbb{C}. \tag{11.5.6}$$

In (11.5.6) $a_m(z)/(b_{m-1}b_m) \to \infty$. In that case the modification (7.7.8) combined with (7.7.4) is recommended:

$$\tilde{w}_n(z) = \frac{2n-1}{2}\left(-1 + \sqrt{4\,\frac{n^2\,(n^2+4\pi^{-2}z^2)}{4n^2-1} + 1}\right).$$

Since

$$\tilde{w}_n(z) \simeq \hat{w}_n(z) = \frac{2n-1}{2}\left(-1 + \sqrt{n^2 + 4\pi^{-2}z^2 + 1}\right) \tag{11.5.7}$$

we also show the results when using (11.5.7). In the *Tables* 11.5.3 and 11.5.4 the continued fraction approximants are evaluated with $w = 0, w = \tilde{w}_n(z)$ and $w = \hat{w}_n(z)$ respectively.

TABLE 11.5.1: Relative error of 5th partial sum and 5th approximants.

x	$\tanh(x)$	(11.5.3)	(11.5.5)
-20	$-1.000000e{+}00$	$1.8e{+}12$	$5.8e{-}01$
-12	$-1.000000e{+}00$	$6.5e{+}09$	$1.9e{-}01$
-5	$-9.999092e{-}01$	$3.9e{+}05$	$7.5e{-}03$
-1.6	$-9.216686e{-}01$	$8.6e{-}01$	$3.7e{-}06$
-0.4	$-3.799490e{-}01$	$6.0e{-}08$	$9.8e{-}12$
-0.2	$-1.973753e{-}01$	$1.5e{-}11$	$1.0e{-}14$
0.1	$9.966799e{-}02$	$3.6e{-}15$	$1.0e{-}17$
0.3	$2.913126e{-}01$	$1.9e{-}09$	$5.7e{-}13$
0.8	$6.640368e{-}01$	$2.4e{-}04$	$7.8e{-}09$
3	$9.950548e{-}01$	$1.2e{+}03$	$4.0e{-}04$
10	$1.000000e{+}00$	$8.6e{+}08$	$1.1e{-}01$
15	$1.000000e{+}00$	$7.6e{+}10$	$3.2e{-}01$

TABLE 11.5.2: Relative error of 20th partial sum and 20th approximants.

x	$\tanh(x)$	(11.5.3)	(11.5.5)
-20	$-1.000000e{+}00$	$2.5e{+}45$	$6.3e{-}09$
-12	$-1.000000e{+}00$	$2.0e{+}36$	$1.9e{-}13$
-5	$-9.999092e{-}01$	$4.8e{+}20$	$3.6e{-}24$
-1.6	$-9.216686e{-}01$	$1.5e{+}00$	$9.7e{-}42$
-0.4	$-3.799490e{-}01$	$9.0e{-}26$	$2.6e{-}65$
-0.2	$-1.973753e{-}01$	$2.1e{-}38$	$2.6e{-}77$
0.1	$9.966799e{-}02$	$4.7e{-}51$	$2.4e{-}89$
0.3	$2.913126e{-}01$	$5.1e{-}31$	$2.7e{-}70$
0.8	$6.640368e{-}01$	$3.8e{-}13$	$2.2e{-}53$
3	$9.950548e{-}01$	$3.3e{+}11$	$1.1e{-}31$
10	$1.000000e{+}00$	$1.1e{+}33$	$2.1e{-}15$
15	$1.000000e{+}00$	$1.9e{+}40$	$2.6e{-}11$

FIGURE 11.5.1: Complex region where $f_8(z;0)$ of (11.5.5) guarantees k significant digits for $\tanh(z)$ (from light to dark $k = 6, 7, 8, 9$).

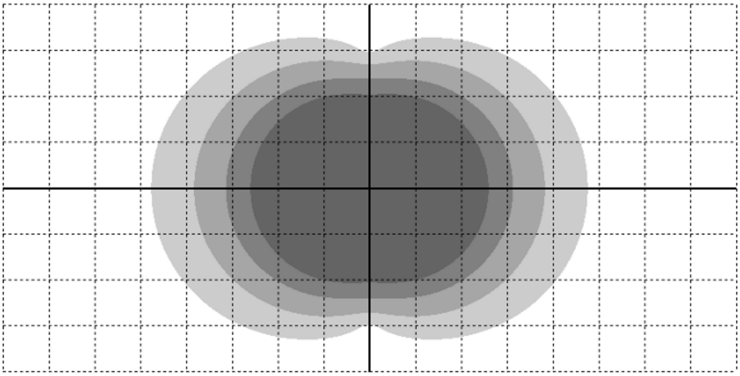


TABLE 11.5.3: Relative error of 5^{th} partial sum and 5^{th} approximants.

x	$\coth(x)$	(11.5.4)	(11.5.6)	(11.5.6)	(11.5.6)
-20	$-1.000000e{+}00$	$1.1e{+}07$	$1.5e{+}00$	$5.7e{-}03$	$1.9e{-}03$
-1.6	$-1.084989e{+}00$	$2.8e{-}04$	$1.7e{-}02$	$8.8e{-}04$	$5.9e{-}04$
0.1	$1.003331e{+}01$	$2.2e{-}18$	$6.5e{-}05$	$3.6e{-}06$	$2.4e{-}06$
0.4	$2.631932e{+}00$	$3.4e{-}11$	$1.0e{-}03$	$5.8e{-}05$	$3.9e{-}05$
3.2	$1.003329e{+}00$	$3.8e{-}01$	$6.5e{-}02$	$3.0e{-}03$	$2.0e{-}03$
10	$1.000000e{+}00$	$1.9e{+}04$	$5.2e{-}01$	$8.5e{-}03$	$5.2e{-}03$

TABLE 11.5.4: Relative error of 20^{th} partial sum and 20^{th} approximants.

x	$\coth(x)$	(11.5.4)	(11.5.6)	(11.5.6)	(11.5.6)
-20	$-1.000000e{+}00$	$1.4e{+}31$	$1.5e{-}01$	$1.5e{-}03$	$1.4e{-}03$
-1.6	$-1.084989e{+}00$	$4.5e{-}13$	$1.2e{-}03$	$1.6e{-}05$	$1.5e{-}05$
0.1	$1.003331e{+}01$	$2.6e{-}63$	$4.8e{-}06$	$6.3e{-}08$	$5.9e{-}08$
0.4	$2.631932e{+}00$	$4.8e{-}38$	$7.7e{-}05$	$1.0e{-}06$	$9.4e{-}07$
3.2	$1.003329e{+}00$	$6.6e{-}01$	$4.9e{-}03$	$6.3e{-}05$	$5.9e{-}05$
10	$1.000000e{+}00$	$2.4e{+}19$	$4.5e{-}02$	$5.4e{-}04$	$5.1e{-}04$

11.6 Inverse hyperbolic functions

The prefix “a” actually means “area” and its pertinence can be appreciated by reference to the relation between the *inverse hyperbolic functions* and the logarithm function, which leads to integral representations for these functions:

$$\operatorname{Asinh}(z) = \operatorname{Ln}\left(z + \sqrt{z^2+1}\right),$$
$$\operatorname{Acosh}(z) = \operatorname{Ln}\left(z + \sqrt{z^2-1}\right),$$
$$\operatorname{Atanh}(z) = \frac{1}{2}\left(\operatorname{Ln}(1+z) - \operatorname{Ln}(1-z)\right),$$
$$\operatorname{Acoth}(z) = \frac{1}{2}\left(\operatorname{Ln}(z+1) - \operatorname{Ln}(z-1)\right).$$

Formal series expansion.

$$\operatorname{Asinh}(z) = z + \sum_{k=1}^{\infty} \frac{(-1)^k (2k-1)!!}{(2k)!!\,(2k+1)} z^{2k+1}, \qquad |z| < 1, \quad (11.6.1)$$

$$\operatorname{Acosh}(1/z) = \operatorname{Ln}(2/z) + \sum_{k=1}^{\infty} \frac{-(2k-1)!!}{(2k)!!\,(2k)} z^{2k}, \qquad |z| < 1, \quad (11.6.2)$$

$$\operatorname{Atanh}(z) = \sum_{k=1}^{\infty} \frac{1}{2k-1} z^{2k-1}, \qquad |z| < 1, \qquad (11.6.3)$$
$$\operatorname{Acoth}(z) = \operatorname{Atanh}(1/z).$$

Continued fraction representations. Several S-fraction like representations can be given [Kho63, pp. 117–122]. Pairwise the even contractions of the fractions (11.6.4) and (11.6.5) for Asinh, (11.6.6) and (11.6.7) for Acosh, and (11.6.8) and (11.6.9) for Atanh are equal:

$$\operatorname{Asinh}(z) = \frac{z\sqrt{1+z^2}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{a_m z^2}{1}\right), \qquad iz \notin (-\infty,-1)\cup(1,+\infty),$$
$$a_{2k} = \frac{2k(2k-1)}{(4k-3)(4k-1)}, \quad a_{2k+1} = \frac{2k(2k-1)}{(4k-1)(4k+1)} \qquad (11.6.4)$$

AS

$$= \frac{z/\sqrt{1+z^2}}{1} + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{-a_m z^2/(1+z^2)}{1}\right),$$
$$a_m = \frac{m^2}{(2m-1)(2m+1)}, \quad iz \notin (-\infty,-1)\cup(1,+\infty). \qquad (11.6.5)$$

TABLE 11.6.1: Relative error of 5th partial sum and 5th approximants.

x	Asinh(x)	(11.6.1)	(11.6.4)	(11.6.5)
-4	$-2.094713e{+}00$	$4.1e{+}04$	$1.9e{-}01$	$8.0e{-}02$
-2	$-1.443635e{+}00$	$2.4e{+}01$	$1.5e{-}02$	$9.8e{-}03$
-0.8	$-7.326683e{-}01$	$8.6e{-}04$	$5.1e{-}05$	$4.0e{-}05$
-0.2	$-1.986901e{-}01$	$6.9e{-}11$	$1.6e{-}10$	$1.4e{-}10$
0.1	$9.983408e{-}02$	$1.7e{-}14$	$1.7e{-}13$	$1.4e{-}13$
0.5	$4.812118e{-}01$	$3.7e{-}06$	$9.7e{-}07$	$7.9e{-}07$
1	$8.813736e{-}01$	$1.1e{-}02$	$2.7e{-}04$	$2.1e{-}04$
3	$1.818446e{+}00$	$1.9e{+}03$	$7.7e{-}02$	$4.0e{-}02$

TABLE 11.6.2: Relative error of 20th partial sum and 20th approximants.

x	Asinh(x)	(11.6.1)	(11.6.4)	(11.6.5)
-4	$-2.094713e{+}00$	$6.6e{+}21$	$5.6e{-}05$	$5.6e{-}05$
-2	$-1.443635e{+}00$	$3.7e{+}09$	$5.7e{-}09$	$5.7e{-}09$
-0.8	$-7.326683e{-}01$	$1.7e{-}07$	$9.4e{-}19$	$9.4e{-}19$
-0.2	$-1.986901e{-}01$	$1.2e{-}32$	$1.0e{-}40$	$1.0e{-}40$
0.1	$9.983408e{-}02$	$2.8e{-}45$	$1.3e{-}52$	$1.3e{-}52$
0.5	$4.812118e{-}01$	$5.5e{-}16$	$1.3e{-}25$	$1.3e{-}25$
1	$8.813736e{-}01$	$1.7e{-}03$	$7.1e{-}16$	$7.1e{-}16$
3	$1.818446e{+}00$	$5.5e{+}16$	$2.5e{-}06$	$2.5e{-}06$

$$\operatorname{Acosh}(z) = \frac{z\sqrt{z^2-1}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{a_m(z^2-1)}{1}\right), \qquad \Re z > 0, \tag{11.6.6}$$

$$a_{2k} = \frac{2k(2k-1)}{(4k-3)(4k-1)}, \quad a_{2k+1} = \frac{2k(2k-1)}{(4k-1)(4k+1)}$$

$$= \frac{\sqrt{z^2-1}/z}{1} + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{-a_m(z^2-1)/z^2}{1}\right), \tag{11.6.7}$$

$$a_m = \frac{m^2}{(2m-1)(2m+1)}, \quad |\operatorname{Arg}(1/z^2)| < \pi.$$

TABLE 11.6.3: Relative error of 5$^{\text{th}}$ partial sum and 5$^{\text{th}}$ approximants.

x	Acosh(x)	(11.6.2)	(11.6.6)	(11.6.7)
1.1	4.435683e−01	4.4e−02	4.4e−07	3.6e−07
2	1.316958e+00	4.4e−06	7.6e−03	5.1e−03
4	2.063437e+00	5.7e−10	1.8e−01	7.5e−02
10	2.993223e+00	6.3e−15	1.7e+00	2.6e−01

TABLE 11.6.4: Relative error of 20$^{\text{th}}$ partial sum and 20$^{\text{th}}$ approximants.

x	Acosh(x)	(11.6.2)	(11.6.6)	(11.6.7)
1.1	4.435683e−01	5.4e−04	5.5e−27	5.5e−27
2	1.316958e+00	6.6e−16	3.9e−10	3.9e−10
4	2.063437e+00	7.8e−29	4.1e−05	4.1e−05
10	2.993223e+00	9.8e−46	1.5e−02	1.5e−02

$$\operatorname{Atanh}(z) = \frac{z/(1-z^2)}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{a_m z^2/(1-z^2)}{1}\right), \qquad |\operatorname{Arg}(1-z^2)| < \pi,$$

$$a_{2k} = \frac{2k(2k-1)}{(4k-3)(4k-1)}, \quad a_{2k+1} = \frac{2k(2k-1)}{(4k-1)(4k+1)} \tag{11.6.8}$$

$$= \frac{z}{1} + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{-m^2z^2/(4m^2-1)}{1}\right), \qquad |\operatorname{Arg}(1-z^2)| < \pi. \tag{11.6.9}$$

TABLE 11.6.5: Relative error of 5$^{\text{th}}$ partial sum and 5$^{\text{th}}$ approximants.

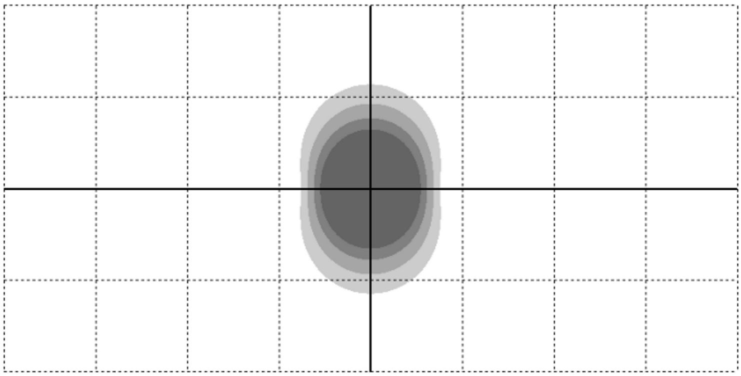
x	Atanh(x)	(11.6.3)	(11.6.8)	(11.6.9)
−0.9	−1.472219e+00	4.8e−02	1.8e−02	1.1e−02
−0.5	−5.493061e−01	2.2e−05	3.4e−06	2.8e−06
0.1	1.003353e−01	7.7e−14	1.8e−13	1.5e−13
0.3	3.095196e−01	4.3e−08	1.3e−08	1.1e−08
0.7	8.673005e−01	1.5e−03	2.3e−04	1.8e−04

TABLE 11.6.6: Relative error of 20th partial sum and 20th approximants.

x	Atanh(x)	(11.6.3)	(11.6.8)	(11.6.9)
−0.9	−1.472219e+00	7.7e−04	1.0e−08	1.0e−08
−0.5	−5.493061e−01	6.3e−15	2.0e−23	2.0e−23
0.1	1.003353e−01	2.3e−44	1.6e−52	1.6e−52
0.3	3.095196e−01	2.7e−24	4.3e−33	4.3e−33
0.7	8.673005e−01	1.1e−08	4.0e−16	4.0e−16

The continued fraction representations (11.6.4) through (11.6.9), when viewed with positive a_m, all have $a_m \to 1/4$, but different limits for their partial numerators. As illustrated before, the modifications (7.7.5) and (7.7.10) can be used.

FIGURE 11.6.1: Complex region where $f_8(z;0)$ of (11.6.8) guarantees k significant digits for Atanh(z) (from light to dark $k = 6, 7, 8, 9$).



11.7 The power function

The *power function* or general binomial function $(1+z)^\alpha$ equals the hypergeometric function ${}_2F_1(-\alpha, 1; 1; -z)$ which is further discussed in *Chapter* 15.

Continued fraction representations. Regular C-fraction representations for $(1+z)^\alpha$ are [Per29, p. 348]:

$$(1+z)^\alpha = 1 + \frac{\alpha z}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{a_m z}{1}\right), \qquad |\operatorname{Arg}(z+1)| < \pi, \tag{11.7.1}$$
$$a_{2k} = \frac{(k-\alpha)}{2(2k-1)}, \quad a_{2k+1} = \frac{(k+\alpha)}{2(2k+1)}$$

$$= \frac{1}{1} + \frac{-\alpha z}{1} + \mathop{\mathrm{K}}_{m=3}^{\infty}\left(\frac{a_m z}{1}\right), \qquad |\operatorname{Arg}(z+1)| < \pi, \tag{11.7.2}$$
$$a_{2k} = \frac{k-1-\alpha}{2(2k-1)}, \quad a_{2k+1} = \frac{k+\alpha}{2(2k-1)}$$

$$= \frac{1}{1} + \frac{-\alpha z/(1+z)}{1} + \mathop{\mathrm{K}}_{m=3}^{\infty}\left(\frac{a_m}{1}\right), \qquad |\operatorname{Arg}(z+1)| < \pi,$$
$$a_{2k} = \frac{(-\alpha-k+1)z}{2(2k-1)(1+z)}, \quad a_{2k+1} = \frac{(\alpha-k)z}{2(2k-1)(1+z)}. \tag{11.7.3}$$

For the continued fractions (11.7.2) and (11.7.3) we find from (7.7.5) that the respective modifications

$$w(z) = \frac{1}{2}\left(-1+\sqrt{1+z}\right)$$

and

$$w(z) = \frac{1}{2}\left(-1+\frac{1}{\sqrt{1+z}}\right)$$

may be useful. Their use is illustrated in the *Tables* 11.7.1 and 11.7.2, where the tabulated continued fraction approximants are first evaluated with $w = 0$ and subsequently with $w = w(z)$.

TABLE 11.7.1: Relative error of 5th partial sum and 5th approximants for $\alpha = 2.5$ and $\alpha = 9.5$.

x	$(x+1)^\alpha$	(11.7.1)	(11.7.2)	(11.7.2)	(11.7.3)	(11.7.3)
0.001	1.002502e+00	2.0e−21	4.1e−17	2.5e−16	4.1e−17	1.8e−17
0.1	1.269059e+00	1.5e−09	3.2e−07	1.9e−06	3.2e−07	1.4e−07
0.5	2.755676e+00	8.0e−06	4.7e−04	2.9e−03	4.7e−04	2.1e−04
1.1	6.390697e+00	2.7e−04	1.0e−02	6.6e−02	1.0e−02	4.5e−03
5	8.818163e+01	4.6e−02	6.1e−01	9.4e−01	6.1e−01	3.6e−01

x	$(x+1)^\alpha$	(11.7.1)	(11.7.2)	(11.7.2)	(11.7.3)	(11.7.3)
0.001	1.009540e+00	6.6e−17	1.0e−13	1.4e−13	1.0e−13	7.9e−14
0.1	2.473036e+00	4.4e−05	8.4e−04	1.1e−03	8.4e−04	6.5e−04
0.5	4.708331e+01	2.2e−01	7.0e−01	7.7e−01	7.0e−01	6.3e−01
1.1	1.151021e+03	9.6e−01	9.9e−01	1.0e+00	9.9e−01	9.9e−01
5	2.468521e+07	1.0e+00	1.0e+00	1.0e+00	1.0e+00	1.0e+00

TABLE 11.7.2: Relative error of 20th partial sum and 20th approximants for $\alpha = 2.5$ and $\alpha = 9.5$.

x	$(x+1)^\alpha$	(11.7.1)	(11.7.2)	(11.7.2)	(11.7.3)	(11.7.3)
0.001	1.002502e+00	8.0e−76	4.5e−72	1.1e−72	2.7e−72	7.2e−73
0.1	1.269059e+00	2.9e−34	1.8e−32	4.2e−33	1.0e−32	2.8e−33
0.5	2.755676e+00	4.5e−21	6.6e−20	1.6e−20	3.8e−20	1.1e−20
1.1	6.390697e+00	1.3e−15	1.1e−14	2.6e−15	6.1e−15	1.7e−15
5	8.818163e+01	5.6e−08	2.9e−07	7.2e−08	1.3e−07	4.7e−08

x	$(x+1)^\alpha$	(11.7.1)	(11.7.2)	(11.7.2)	(11.7.3)	(11.7.3)
0.001	1.009540e+00	1.6e−71	1.3e−66	6.9e−67	3.4e−68	6.2e−67
0.1	2.473036e+00	5.9e−30	5.5e−27	2.8e−27	1.4e−28	2.5e−27
0.5	4.708331e+01	1.1e−16	2.8e−14	1.4e−14	6.7e−16	1.2e−14
1.1	1.151021e+03	4.4e−11	8.0e−09	3.9e−09	1.7e−10	3.5e−09
5	2.468521e+07	2.7e−02	8.4e−01	6.9e−01	6.8e−02	6.5e−01

A regular C-fraction representation in $1/z^2$ for a function related to the power $((z+1)/(z-1))^\alpha$ is given by [Per29, p. 350]:

$$\left(\frac{z+1}{z-1}\right)^\alpha = 1 + \frac{2\alpha/z}{1-\alpha/z} + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{a_m/z^2}{1}\right), \qquad z \in \mathbb{C}\setminus[-1,1],$$

$$a_m = \frac{\alpha^2 - m^2}{(2m-1)(2m+1)}. \quad (11.7.4)$$

TABLE 11.7.3: Relative error of 5th approximants for $\alpha = 2.5$ and $\alpha = 9.5$.

x	$\left(\frac{x+1}{x-1}\right)^\alpha$	(11.7.4)
1.1	2.020916e+03	1.7e−01
5	2.755676e+00	7.1e−11
15	1.396304e+00	3.5e−16
50	1.105186e+00	6.2e−22
90	1.057130e+00	9.6e−25

x	$\left(\frac{x+1}{x-1}\right)^\alpha$	(11.7.4)
1.1	3.639848e+12	1.0e+00
5	4.708331e+01	1.9e−04
15	3.555687e+00	7.1e−10
50	1.462359e+00	1.2e−15
90	1.235060e+00	1.9e−18

TABLE 11.7.4: Relative error of 20th approximants for $\alpha = 2.5$ and $\alpha = 9.5$.

x	$\left(\frac{x+1}{x-1}\right)^\alpha$	(11.7.4)
1.1	2.020916e+03	5.1e−08
5	2.755676e+00	4.1e−41
15	1.396304e+00	7.7e−61
50	1.105186e+00	2.7e−82
90	1.057130e+00	9.2e−93

x	$\left(\frac{x+1}{x-1}\right)^\alpha$	(11.7.4)
1.1	3.639848e+12	6.9e−04
5	4.708331e+01	3.2e−39
15	3.555687e+00	5.6e−59
50	1.462359e+00	1.9e−80
90	1.235060e+00	6.6e−91

12

Gamma function and related functions

The *gamma function* $\Gamma(z)$ is the most important special function of classical analysis after the so-called elementary functions. It is an extension of the factorial $n!$ to real and complex arguments. It is related to the factorial by $\Gamma(n) = (n-1)!$. The present chapter contains continued fraction representations of functions related to the gamma function, its logarithmic derivatives $\psi_k(z)$, also called the polygamma functions, and the incomplete gamma functions $\gamma(a, z)$ and $\Gamma(a, z)$.
We often derive approximations for the function $\gamma(a, z)z^{-a}/\Gamma(a)$, which is a single-valued analytic function of a and z possessing no finite singularities.

12.1 Gamma function

Definitions and elementary properties. The gamma function $\Gamma(z)$ is defined by the *Euler integral*

$$\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}\,dt, \qquad \Re z > 0. \tag{12.1.1}$$

It is continued analytically by *Euler's formula*

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\,n^z}{z(z+1)\cdots(z+n)}, \qquad z \in \mathbb{C}\backslash\mathbb{Z}_0^- \tag{12.1.2}$$

and by *Euler's infinite product* (10.8.2), repeated here for convenience,

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(\left(1+\frac{z}{n}\right)e^{-z/n}\right), \qquad z \in \mathbb{C} \tag{12.1.3}$$

where γ is the Euler constant defined in (10.8.1). The function $\Gamma(z)$ is meromorphic in $\mathbb{C}$ with poles at $z = -n$ for $n \in \mathbb{N}_0$, all of which are simple. The residue of $\Gamma(z)$ at $z = -n$ is given by

$$\operatorname{Res}(\Gamma(z); z = -n) = \frac{(-1)^n}{n!}, \qquad n = 0, 1, 2, \ldots. \tag{12.1.4}$$

From the recurrence formula

$$\Gamma(z+1) = z\Gamma(z), \qquad z \in \mathbb{C}\backslash\mathbb{Z}_0^-, \tag{12.1.5}$$

and $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$ and $0! = 1$, we obtain

$$\Gamma(n+1) = n!, \qquad n = 0, 1, 2, \ldots, \tag{12.1.6a}$$

$$\Gamma(n+z) = (z)_n \Gamma(z), \qquad n = 0, 1, 2, \ldots, \tag{12.1.6b}$$

$$\Gamma(n+1/2) = \sqrt{\pi}\,(1/2)_n, \qquad n = 0, 1, 2, \ldots, \tag{12.1.6c}$$

where $(a)_n$ is the *Pochhammer symbol* or *shifted factorial* defined by

$$\begin{aligned} &(a)_0 := 1, \\ &(a)_n := a(a+1)\cdots(a+n-1), \qquad n = 1, 2, 3, \ldots. \end{aligned} \tag{12.1.7}$$

The gamma function $\Gamma(z)$ is related to the Riemann zeta function $\zeta(z)$ of (10.11.1) by [Hav03, p. 60]

$$\Gamma(z)\zeta(z) = \int_0^\infty \frac{u^{z-1}}{e^u - 1}\,du, \qquad \Re z > 1. \tag{12.1.8}$$

The gamma function also satisfies the mirror property

$$\Gamma(\bar{z}) = \overline{\Gamma(z)}. \tag{12.1.9}$$

Series expansion. The coefficients in the series expansion of $\Gamma(z+1)$ are obtained from

$$\Gamma(z+1) = \sum_{k=0}^{\infty} c_k z^k, \qquad |z| < 1, \tag{12.1.10}$$

$$c_0 = 1, \quad c_k = -\frac{1}{k}\sum_{j=1}^{k}(-1)^{j+1} b_j c_{k-j},$$

$$b_1 = \gamma, \quad b_k = \zeta(k) = \sum_{j=1}^{\infty} j^{-k}, \quad k > 1.$$

Series expansions for $1/\Gamma(z), 1/\Gamma(z+1)$ and $\mathrm{Ln}(\Gamma(z+1))$ can also be given [Luk75, pp. 1–7].

Asymptotic series expansion. From the definition of the gamma function we obtain

$$\Gamma(z) \approx e^{-z} z^{z-\frac{1}{2}} \sqrt{2\pi} \sum_{k=0}^{\infty} d_k z^{-k}, \qquad z \to \infty, \quad |\arg z| < \pi \tag{12.1.11}$$

where the values for d_k have been given in [Wre68; Spi71].

S-fraction. A special ratio of two Γ-values has the modified S-fraction representation [Bau72; BR95, p. 47]

$$\left(\frac{\Gamma\left(\frac{z+1}{4}\right)}{\Gamma\left(\frac{z+3}{4}\right)} \right)^2 = \frac{4}{z} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{(2m-1)^2}{2z} \right), \qquad z \in \mathbb{C}. \tag{12.1.12}$$

C-fraction. Making use of (12.6.1), (12.6.17) and (12.6.23) the gamma function $\Gamma(z)$ can be written as the sum of two regular C-fractions. The separate C-fractions do not have the same speed of convergence and some care needs to be taken when using this relationship to approximate $\Gamma(z)$ [Luk75, p. 100].

Other rational approximations. The following expansion for $\Gamma(z+1)$ is due to [Lan64]:

$$\Gamma(z+1) = \sqrt{2\pi}\,(z + \sigma + 1/2)^{z+\frac{1}{2}} \exp(-z - \sigma - 1/2) \times$$
$$\sum_{k=0}^{\infty} d_k \frac{z(z-1)\cdots(z-k+1)}{(z+1)(z+2)\cdots(z+k)}, \qquad \Re(z + \sigma + 1/2) > 0,$$
$$d_0 = \frac{\exp(\sigma)}{\sqrt{2\pi}} \sqrt{\frac{e}{\sigma + 1/2}},$$
$$d_k = 2\frac{(-1)^k \exp(\sigma)}{\sqrt{2\pi}} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k)_j \left(\frac{e}{j + \sigma + 1/2} \right)^{j+\frac{1}{2}},$$
$$k = 1, 2, \ldots .$$

The infinite series portion behaves like a partial fraction decomposition.

12.2 Binet function

Definition and elementary properties. The *Binet function* $J(z)$ is closely related to $\Gamma(z)$ and defined by [Hen77, p. 39]

$$J(z) := \ln(\Gamma(z)) - \left(z - \frac{1}{2}\right)\ln(z) + z - \ln(\sqrt{2\pi}), \tag{12.2.1}$$

or, equivalently, by

$$\Gamma(z) = \sqrt{2\pi}\, z^{z-\frac{1}{2}} e^{-z} e^{J(z)}. \tag{12.2.2}$$

The function $\ln(\Gamma(z))$ in definition (12.2.1) is called the log-gamma function. Throughout this chapter principal branches are taken for multiple valued functions. From (12.2.2) *Stirling's approximation* of $n!$ for large n is obtained,

$$n! \sim \sqrt{2\pi}(n+1)^{n+\frac{1}{2}} e^{-(n+1)}, \qquad n \to \infty, \tag{12.2.3}$$

by setting $z = n+1$ and replacing $J(n+1)$ by 0 since

$$J(x) \le \frac{1}{12x}, \qquad 0 < x < \infty.$$

More generally, (12.2.2) yields the approximation

$$\Gamma(z) \sim \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z}, \qquad z \to \infty, \quad |\arg z| \le \theta < \pi \tag{12.2.4}$$

since for every θ with $0 \le \theta < \pi$, there exists a constant $\kappa(\theta)$ such that [Hen77, p. 39]

$$|J(z)| \le \frac{\kappa(\theta)}{|z|}, \qquad |\arg z| \le \theta < \pi. \tag{12.2.5}$$

Asymptotic series expansion. For the Binet function $J(z)$ we have [AS64, p. 257]

$$J(z) \approx z^{-1} \sum_{k=0}^{\infty} \frac{B_{2k+2}}{(2k+1)(2k+2)} z^{-2k}, \qquad z \to \infty, \quad |\arg z| < \pi \tag{12.2.6}$$

where B_{2n} denotes the $2n^{\text{th}}$ Bernoulli number, defined in (11.3.4).

Stieltjes transform. It can be shown [Hen77, p. 624] that

$$\frac{J(\sqrt{z})}{\sqrt{z}} = \int_0^\infty \frac{\phi(t)}{z+t}\, dt, \qquad |\arg z| < \pi \tag{12.2.7a}$$

where

$$\phi(t) := \frac{1}{2\pi}\int_0^t \frac{1}{\sqrt{t}}\,\mathrm{Ln}\left(\frac{1}{1-e^{-2\pi\sqrt{t}}}\right)dt, \qquad 0 < t < \infty. \tag{12.2.7b}$$

The k^{th} moment μ_k with respect to the weight function $\phi(t)$ is given by

$$\mu_k = \int_0^\infty t^k \phi(t)\,dt = \frac{(-1)^k B_{2k+2}}{(2k+1)(2k+2)}, \qquad k = 0, 1, 2, \ldots. \tag{12.2.8}$$

S-fraction. Since the classical Stieltjes moment problem has a solution $\phi(t)$ for μ_k given by (12.2.8), it follows from *Theorem* 5.1.1 that there exists a modified S-fraction of the form

$$\frac{a_1}{z} \, \frac{}{+} \, \frac{a_2}{1} \, \frac{}{+} \, \frac{a_3}{z} \, \frac{}{+} \, \frac{a_4}{1} \, \frac{}{+} \ldots, \qquad a_m > 0, \quad m \in \mathbb{N}, \tag{12.2.9}$$

corresponding at $z = \infty$ to the FPS

$$L(z) = z^{-1}\sum_{k=0}^{\infty}(-1)^k \mu_k z^{-k} = z^{-1}\sum_{k=0}^{\infty}\frac{B_{2k+2}}{(2k+1)(2k+2)}z^{-k}.$$

It follows from the asymptotic behaviour (11.3.6) of the Bernoulli numbers B_{2n} that the moments μ_k given by (12.2.8) satisfy Carleman's criterion (5.1.16a). Hence the solution $\phi(t)$ of the classical Stieltjes moment problem for the sequence $\{\mu_k\}$ is unique. Thus by *Theorem* 5.2.1 the modified S-fraction (12.2.9) converges to (12.2.7a):

$$\frac{J(\sqrt{z})}{\sqrt{z}} = \frac{a_1}{z} \, \frac{}{+} \, \frac{a_2}{1} \, \frac{}{+} \, \frac{a_3}{z} \, \frac{}{+} \, \frac{a_4}{1} \, \frac{}{+} \ldots, \qquad |\arg z| < \pi. \tag{12.2.10}$$

A transformation of the form (2.3.4) yields the S-fraction representation [Sti95]

$$J(z) = \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{a_m}{z}\right) = \int_0^\infty \frac{z\phi(t)}{z^2+t}\,dt, \qquad |\arg z| < \pi, \tag{12.2.11}$$

and also the S-fraction representation

$$\sqrt{z}J\left(\frac{1}{\sqrt{z}}\right) = \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{a_m z}{1}\right) = \int_0^\infty \frac{z\phi(t)}{1+zt}\,dt, \qquad |\arg z| < \pi, \tag{12.2.12}$$

where $\phi(t)$ is given by (12.2.7b). There is no known closed-form expression for the coefficients a_m in (12.2.10) through (12.2.12). The first few are given by

$$a_1 = \frac{1}{12}, \quad a_2 = \frac{1}{30}, \quad a_3 = \frac{53}{210}, \quad a_4 = \frac{195}{371}, \quad \ldots.$$

By applying *Theorem* 5.2.2 to (12.2.7b) with $\alpha = \delta = 1$, $c = 2\pi$ and $d = 1/16$, it is shown in [ČV82; JVA98] that the coefficients a_m of the continued fractions (12.2.10) through (12.2.12) satisfy

$$a_m \sim \frac{m^2}{16}, \qquad m \to \infty. \tag{12.2.13}$$

TABLE 12.2.1: Illustration of (12.2.13).

m	a_m	$a_m/(m^2/16)$
10	5.002768e+00	8.004429e−01
20	2.247047e+01	8.988189e−01
30	5.244129e+01	9.322896e−01
40	9.491384e+01	9.491384e−01
50	1.498876e+02	9.592803e−01
100	6.122662e+02	9.796259e−01
150	1.387154e+03	9.864208e−01
200	2.474548e+03	9.898191e−01
250	3.874445e+03	9.918579e−01
300	5.586845e+03	9.932168e−01
400	9.949151e+03	9.949151e−01
500	1.556146e+04	9.959336e−01
600	2.242378e+04	9.966125e−01
700	3.053610e+04	9.970972e−01
800	3.989842e+04	9.974606e−01
900	5.051075e+04	9.977432e−01
1000	6.237308e+04	9.979693e−01
1500	1.404348e+05	9.986471e−01
2000	2.497465e+05	9.989858e−01
2500	3.903082e+05	9.991889e−01

TABLE 12.2.2: Comparison of the 5^{th} partial sum of the asymptotic series (12.2.6), in $|\arg z| < \pi$, with the 5^{th} approximant of the S-fraction (12.2.11). The partial numerators are computed using the qd-algorithm from *Chapter* 6. Making use of (12.1.9) it suffices to explore only the first and second quadrant.

x	$J(x)$	(12.2.6)	(12.2.11)
1	8.106147e−02	1.7e−02	2.4e−04
5	1.664469e−02	2.7e−10	5.3e−10
50	1.666644e−03	3.1e−22	6.9e−20
100	8.333306e−04	7.7e−26	6.8e−23
500	1.666666e−04	3.2e−34	6.9e−30

x	$\|J(x+\mathrm{i}x)\|_s$	(12.2.6)	(12.2.11)
1	5.881245e−02	5.4e−04	6.4e−05
5	1.178507e−02	4.9e−12	2.2e−11
50	1.178511e−03	4.9e−24	2.2e−21
100	5.892557e−04	1.2e−27	2.1e−24
500	1.178511e−04	4.9e−36	2.2e−31

x	$\|J(\mathrm{i}x)\|_s$	(12.2.6)	(12.2.11)
1	8.704350e−02	3.9e−02	1.4e−02
5	1.668915e−02	3.9e−10	9.7e−10
50	1.666689e−03	3.2e−22	7.0e−20
100	8.333361e−04	7.7e−26	6.8e−23
500	1.666667e−04	3.2e−34	7.0e−30

x	$\|J(x-\mathrm{i}x)\|_s$	(12.2.6)	(12.2.11)
1	5.881245e−02	5.4e−04	6.4e−05
5	1.178507e−02	4.9e−12	2.2e−11
50	1.178511e−03	4.9e−24	2.2e−21
100	5.892557e−04	1.2e−27	2.1e−24
500	1.178511e−04	4.9e−36	2.2e−31

TABLE 12.2.3: Comparison of the 20th partial sum of the asymptotic series (12.2.6), in $|\arg z| < \pi$, with the 20th approximant of the S-fraction (12.2.11). The partial numerators are computed using the qd-algorithm from *Chapter* 6. Making use of (12.1.9) it suffices to explore only the first and second quadrant.

x	$J(x)$	(12.2.6)	(12.2.11)
1	$8.106147e{-}02$	$5.9e{+}15$	$1.4e{-}06$
5	$1.664469e{-}02$	$4.0e{-}13$	$4.8e{-}20$
50	$1.666644e{-}03$	$1.1e{-}54$	$2.3e{-}56$
100	$8.333306e{-}04$	$2.5e{-}67$	$2.3e{-}68$
500	$1.666666e{-}04$	$1.1e{-}96$	$2.6e{-}96$

x	$\lvert J(x+\mathrm{i}x)\rvert_s$	(12.2.6)	(12.2.11)
1	$5.881245e{-}02$	$5.6e{+}09$	$3.2e{-}07$
5	$1.178507e{-}02$	$3.9e{-}19$	$3.0e{-}23$
50	$1.178511e{-}03$	$5.4e{-}61$	$2.5e{-}62$
100	$5.892557e{-}04$	$1.2e{-}73$	$2.3e{-}74$
500	$1.178511e{-}04$	$5.4e{-}103$	$2.5e{-}102$

x	$\lvert J(\mathrm{i}x)\rvert_s$	(12.2.6)	(12.2.11)
1	$8.704350e{-}02$	$5.8e{+}15$	$4.6e{-}02$
5	$1.668915e{-}02$	$1.9e{-}12$	$2.0e{-}12$
50	$1.666689e{-}03$	$1.1e{-}54$	$3.0e{-}56$
100	$8.333361e{-}04$	$2.6e{-}67$	$2.5e{-}68$
500	$1.666667e{-}04$	$1.1e{-}96$	$2.6e{-}96$

x	$\lvert J(x-\mathrm{i}x)\rvert_s$	(12.2.6)	(12.2.11)
1	$5.881245e{-}02$	$5.6e{+}09$	$3.2e{-}07$
5	$1.178507e{-}02$	$3.9e{-}19$	$3.0e{-}23$
50	$1.178511e{-}03$	$5.4e{-}61$	$2.5e{-}62$
100	$5.892557e{-}04$	$1.2e{-}73$	$2.3e{-}74$
500	$1.178511e{-}04$	$5.4e{-}103$	$2.5e{-}102$

12.3 Polygamma functions

Definition and representations. The $(k+1)^{\text{th}}$ derivative of the log-gamma function is called the *polygamma function* $\psi_k(z)$:

$$\psi_k(z) := \frac{d^{k+1}}{dz^{k+1}} \ln(\Gamma(z)), \qquad k \in \mathbb{N}_0. \tag{12.3.1}$$

The polygamma functions have the representation

$$\begin{aligned} \psi_0(z) &= -\gamma + \sum_{m=0}^{\infty} \left(\frac{1}{m+1} - \frac{1}{z+m} \right), \qquad z \notin \mathbb{Z}_0^-, \\ \psi_k(z) &= (-1)^{k+1} k! \sum_{m=0}^{\infty} \frac{1}{(z+m)^{k+1}}, \qquad z \notin \mathbb{Z}_0^-, \quad k \in \mathbb{N} \end{aligned} \tag{12.3.2}$$

where γ is the Euler constant (10.8.1). The function $\psi_0(z)$ is referred to as the digamma or psi function and often denoted $\Psi(z)$ instead of $\psi_0(z)$. It follows from (12.3.2) that

$$\psi_k(n+1) = (-1)^k k! \left[-\zeta(k+1) + 1 + \frac{1}{2^{k+1}} + \cdots + \frac{1}{n^{k+1}} \right], \qquad k \geq 1 \tag{12.3.3}$$

where $\zeta(z)$ denotes the Riemann zeta function (10.11.1).
The polygamma functions satisfy

$$\psi_k(\overline{z}) = \overline{\psi_k(z)}, \tag{12.3.4}$$

the recurrence relation

$$\psi_k(z+1) = \psi_k(z) + \frac{(-1)^k k!}{z^{k+1}}, \qquad k = 1, 2, 3, \ldots \tag{12.3.5}$$

and the reflection formula

$$\psi_k(1-z) + (-1)^{k+1} \psi_k(z) = (-1)^k \pi \frac{d^k}{dz^k} \cot(\pi z), \qquad k = 1, 2, 3, \ldots. \tag{12.3.6}$$

Asymptotic series expansion. We have [AS64, p. 260]

$$\psi_0(z) \approx \ln(z) - \frac{1}{2z} - z^{-2} \sum_{m=0}^{\infty} \frac{B_{2m+2}}{2m+2} z^{-2m}, \qquad z \to \infty, \quad |\arg z| < \pi, \tag{12.3.7}$$

$$\begin{aligned} \psi_k(z) \approx (-1)^{k-1} \Bigg(\frac{(k-1)!}{z^k} + \frac{k!}{2z^{k+1}} + \\ z^{-2} \sum_{m=0}^{\infty} \frac{B_{2m+2}(2m+k+1)!}{(2m+2)!} z^{-(2m+k)} \Bigg), \\ z \to \infty, \quad |\arg z| < \pi, \quad k \geq 1, \end{aligned} \tag{12.3.8}$$

where B_{2n} are the Bernoulli numbers. Some special values of $\psi_k(z)$ are

$$\psi_k(1) = (-1)^{k+1}k!\,\zeta(k+1), \qquad k \geq 1, \tag{12.3.9}$$

$$\psi_k\,(1/2) = (-1)^{k+1}k!\,(2^{k+1}-1)\zeta(k+1), \qquad k \geq 1. \tag{12.3.10}$$

Stieltjes transform. A Stieltjes transform for a function related to the polygamma function $\psi_k(z)$ can be given [SB71]. Let

$$\psi_0(z) = \ln(z) - \frac{1}{2z} - g_0(z), \tag{12.3.11}$$

$$\psi_k(z) = (-1)^{k-1}\left(\frac{(k-1)!}{z^k} + \frac{k!}{2z^{k+1}} + \left(\frac{2\pi}{z}\right)^k g_k(z)\right), \qquad k \geq 1. \tag{12.3.12}$$

From (12.3.7) and (12.3.8) we have

$$g_0(z) \approx z^{-2}\sum_{m=0}^{\infty}\frac{B_{2m+2}}{2m+2}z^{-2m}, \qquad z \to \infty, \quad |\arg z| < \pi, \tag{12.3.13}$$

$$g_k(z) \approx \left(\frac{1}{2\pi}\right)^k z^{-2}\sum_{m=0}^{\infty}\frac{B_{2m+2}(2m+k+1)!}{(2m+2)!}z^{-2m}, \qquad z \to \infty, \quad |\arg z| < \pi, \quad k \geq 1. \tag{12.3.14}$$

The function $g_k(z)$ has the Stieltjes transform representation

$$g_k(z) = \int_0^\infty \frac{\phi_k(t)}{z^2+t}\,dt, \qquad |\arg z| < \frac{\pi}{2}, \quad k \geq 0 \tag{12.3.15}$$

where the weight function is given by

$$\phi_k(t) := \frac{t^{\frac{k}{2}}\phi_k(u)}{(u-1)^{k+1}}, \qquad u = e^{2\pi\sqrt{t}}, \quad 0 < t < \infty. \tag{12.3.16}$$

Here the functions $\phi_k(u)$ satisfy

$$\phi_0(u) = 1, \tag{12.3.17a}$$

$$\phi_k(u) = u(1-u)\frac{d}{du}\phi_{k-1}(u) + ku\phi_{k-1}(u), \qquad k \geq 1. \tag{12.3.17b}$$

For every $k \geq 1$ the function $\phi_k(u)$ is a monic polynomial in u of degree k, with $\phi_k(0) = 0$. Its coefficients $p_\ell^{(k)}$ are positive and symmetric in the sense that

$$p_1^{(k)} = p_k^{(k)} = 1, \qquad p_{k-\ell}^{(k)} = p_{\ell+1}^{(k)}, \quad \ell = 0, 1, \ldots, \lfloor (k-1)/2 \rfloor. \tag{12.3.18}$$

The first few polynomials are

$$\phi_1(u) = u, \quad \phi_2(u) = u + u^2, \quad \phi_3(u) = u + 4u^2 + u^3. \tag{12.3.19}$$

The moments $\mu_m^{(k)}$ for $\phi_k(t)$ are given by

$$\mu_m^{(k)} = \int_0^\infty t^m \phi_k(t)\, dt = \frac{(-1)^m (2m+k+1)!\, B_{2m+2}}{(2\pi)^k (2m+2)!}, \qquad m \geq 0. \tag{12.3.20}$$

S-fractions. Since the classical Stieltjes moment problem for $\mu_m^{(k)}$ given by (12.3.20) has a solution $\phi_k(t)$ for each $k \geq 0$, it follows from *Theorem* 5.1.1 that there exists a modified S-fraction of the form

$$\frac{a_1^{(k)}}{z^2} + \frac{a_2^{(k)}}{1} + \frac{a_3^{(k)}}{z^2} + \frac{a_4^{(k)}}{1} + \dots, \qquad a_m^{(k)} > 0, \tag{12.3.21}$$

corresponding to the series

$$L_k(z) = z^{-2} \sum_{m=0}^{\infty} (-1)^m \mu_m^{(k)} z^{-2m}, \qquad k \geq 0. \tag{12.3.22}$$

The moments (12.3.20) satisfy Carleman's criterion (5.1.16a) and thus the solution of the classical Stieltjes moment problem for the sequence $\{\mu_m^{(k)}\}$ with k fixed, is unique. Hence from *Theorem* 5.2.1, the modified S-fraction (12.3.21) is convergent and

$$g_k(z) = \frac{a_1^{(k)}}{z^2} + \frac{a_2^{(k)}}{1} + \frac{a_3^{(k)}}{z^2} + \frac{a_4^{(k)}}{1} + \dots, \qquad |\arg z| < \frac{\pi}{2}, \quad k \geq 0. \tag{12.3.23}$$

In general, there is no known closed-form expression for the coefficients $a_m^{(k)}$ of the modified S-fractions (12.3.23). By applying *Theorem* 5.2.2 with $\alpha = \delta = 1$, $c = 2\pi$ and $d = 1/16$, we find the following asymptotic behaviour of the coefficients of (12.3.23):

$$a_m^{(k)} \sim \frac{m^2}{16}, \qquad m \to \infty, \quad k \geq 0. \tag{12.3.24}$$

There are two special cases of the polygamma functions, the trigamma function $\psi_1(z)$ and the tetragamma function $\psi_2(z)$, for which there exist explicit formulas for the partial numerators $a_m^{(k)}$. Since $a_m^{(k)} \to \infty$, the modification (7.7.8) given by

$$\tilde{w}_n(z) = b_n \frac{-1 + \sqrt{4a_{n+1}^{(k)} z^{-2} + 1}}{2}, \qquad b_{2k-1} = z^2, \quad b_{2k} = 1, \quad k \geq 1 \tag{12.3.25}$$

can be useful when evaluating (12.3.23). For $\psi_1(z)$ and $\psi_2(z)$ this is illustrated in the *Tables* 12.4.1, 12.4.2, 12.5.1 and 12.5.2.
It is proved in [Ber98, pp. 50–51] that

$$\psi_0(z+1) - \psi_0(1+z/3) + \frac{1}{z} - \ln(3) \\ = \frac{a_1}{b_1 z^2} \underset{+}{} \frac{a_2}{b_2} \underset{+}{} \frac{a_3}{b_3 z^2} \underset{+}{} \frac{a_4}{b_4} \underset{+}{} \frac{a_5}{b_5 z^2} \underset{+}{} \ldots, \qquad \Re z > 0, \quad (12.3.26)$$

where the coefficients a_m and b_m equal

$$a_1 = \frac{2}{3}, \quad a_{2k} = a_{2k+1} = k^3 - k, \quad k \geq 1, \\ b_{2k-1} = k, \quad b_{2k} = 6, \quad k \geq 1.$$

12.4 Trigamma function

S-fraction. From (12.3.12) we find [Rog07; Lan94, pp. 241–243]

$$\psi_1(z) = \frac{1}{z} + \frac{1}{2z^2} + \frac{2\pi}{z} g_1(z). \qquad (12.4.1a)$$

The coefficients in the modified S-fraction representation (12.3.23) for the function $g_1(z)$ can be given explicitly. We have

$$g_1(z) = \frac{a_1^{(1)}}{z^2} \underset{+}{} \frac{a_2^{(1)}}{1} \underset{+}{} \frac{a_3^{(1)}}{z^2} \underset{+}{} \frac{a_4^{(1)}}{1} \underset{+}{} \ldots, \qquad |\arg z| < \frac{\pi}{2} \qquad (12.4.1b)$$

where

$$a_1^{(1)} = \frac{1}{12\pi}, \quad a_m^{(1)} = \frac{m^2(m^2-1)}{4(4m^2-1)}, \quad m \geq 2. \qquad (12.4.1c)$$

In the *Tables* 12.4.1 and 12.4.2 the S-fraction representation (12.4.1) is first evaluated without modification and afterwards with the modification (12.3.25).

C-fraction. The C-fraction representation in $1/z$ for $\psi_1(z)$ is given by [Lan94, pp. 241–245] ,

$$\psi_1(z) = \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{c_m z^{-1}}{1} \right), \qquad \Re z > \frac{1}{2} \qquad (12.4.2a)$$

with

$$c_1 = 1, \quad c_{2j} = \frac{-j^2}{2(2j-1)}, \quad c_{2j+1} = \frac{j^2}{2(2j+1)}, \quad j \geq 1. \qquad (12.4.2b)$$

J-fraction. A J-fraction representation in $1/z$ for $\psi_1(z)$ can be obtained [Sti90, p. 387; Wal48, p. 373; Lan94, pp. 240–242] by an even contraction of (12.4.2)

$$\psi_1(z) = \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{\alpha_m}{-\frac{1}{2} + z} \right), \qquad \Re z > \frac{1}{2}, \tag{12.4.3a}$$

where

$$\alpha_1 = 1, \quad \alpha_m = \frac{(m-1)^4}{4(2m-3)(2m-1)}, \quad m \geq 2. \tag{12.4.3b}$$

We note that (12.4.3) is also a modified S-fraction in $w = z - 1/2$.

TABLE 12.4.1: In combination with property (12.3.4), the following sequence of tables describes the relative error of the 5^{th} approximants and the 5^{th} partial sum throughout the region $\Re z > 1/2$. The S-fraction representation (12.4.1) is first evaluated without modification and afterwards with the modification (12.3.25).

x	$\psi_1(x)$	(12.4.1)	(12.4.1)	(12.4.2)	(12.4.3)	(12.3.8)
0.6	3.636210e+00	5.4e−03	2.6e−04	1.6e−01	8.8e−01	5.0e+00
1.1	1.433299e+00	1.8e−04	8.6e−06	1.3e−02	9.3e−03	1.3e−02
5	2.213230e−01	1.3e−10	2.8e−12	7.4e−06	4.3e−09	8.0e−10
10	1.051663e−01	4.6e−14	3.3e−16	2.2e−07	3.2e−12	2.3e−13
20	5.127082e−02	1.3e−17	2.5e−20	6.5e−09	2.5e−15	6.0e−17
50	2.020133e−02	2.2e−22	7.1e−26	6.5e−11	2.3e−19	1.0e−21
95	1.058191e−02	1.0e−25	9.0e−30	2.6e−12	3.6e−22	4.7e−25

x	$\lvert\psi_1(x+\mathrm{i}x)\rvert_s$	(12.4.1)	(12.4.1)	(12.4.2)	(12.4.3)	(12.3.8)
0.6	1.867764e+00	3.0e−03	1.5e−04	6.2e−02	2.0e−01	2.3e−01
1.1	8.145666e−01	4.4e−05	2.5e−06	3.1e−03	1.1e−03	5.1e−04
5	1.486857e−01	3.3e−12	5.1e−14	1.2e−06	1.1e−10	1.5e−11
10	7.250152e−02	8.3e−16	3.4e−18	3.7e−08	8.3e−14	3.9e−15
20	3.580010e−02	2.1e−19	2.1e−22	1.1e−09	7.1e−17	9.5e−19
50	1.421302e−02	3.5e−24	5.6e−28	1.1e−11	6.9e−21	1.6e−23
95	7.462843e−03	1.6e−27	7.1e−32	4.6e−13	1.1e−23	7.3e−27

x	$\lvert\psi_1(1+\mathrm{i}x)\rvert_s$	(12.4.1)	(12.4.1)	(12.4.2)	(12.4.3)	(12.3.8)
0.6	1.255548e+00	2.1e−04	1.1e−05	1.2e−02	9.0e−03	8.2e−03
1.1	8.525365e−01	7.4e−05	4.3e−06	3.9e−03	2.0e−03	8.6e−04
5	1.996599e−01	2.7e−10	1.5e−11	6.3e−06	2.8e−09	9.7e−10
10	9.995813e−02	5.7e−14	5.3e−16	2.0e−07	2.2e−12	2.5e−13
20	4.999479e−02	1.3e−17	2.8e−20	6.2e−09	2.0e−15	6.2e−17
50	1.999967e−02	2.2e−22	7.3e−26	6.4e−11	2.1e−19	1.0e−21
95	1.052627e−02	1.0e−25	9.1e−30	2.6e−12	3.4e−22	4.7e−25

TABLE 12.4.2: In combination with property (12.3.4), the following sequence of tables describes the relative error of the 20^{th} approximants and the 20^{th} partial sum throughout the region $\Re z > 1/2$. The S-fraction representation (12.4.1) is first evaluated without modification and afterwards with the modification (12.3.25).

x	$\psi_1(x)$	(12.4.1)	(12.4.1)	(12.4.2)	(12.4.3)	(12.3.8)
0.6	3.636210e+00	3.1e−04	3.9e−06	4.5e−01	3.6e−01	6.6e+24
1.1	1.433299e+00	9.4e−07	1.2e−08	2.0e−03	4.1e−04	2.6e+14
5	2.213230e−01	6.0e−20	8.2e−22	2.3e−13	2.7e−18	1.2e−12
10	1.051663e−01	7.1e−30	7.5e−32	3.1e−19	2.3e−28	5.5e−25
20	5.127082e−02	1.8e−41	9.0e−44	2.6e−25	7.8e−40	1.7e−37
50	2.020133e−02	8.1e−58	8.2e−61	2.3e−33	1.1e−55	3.6e−54
95	1.058191e−02	1.8e−69	5.3e−73	5.7e−39	7.0e−67	7.2e−66

x	$\lvert\psi_1(x+\mathrm{i}x)\rvert_s$	(12.4.1)	(12.4.1)	(12.4.2)	(12.4.3)	(12.3.8)
0.6	1.867764e+00	1.7e−04	2.1e−06	1.9e−01	1.2e−01	8.7e+18
1.1	8.145666e−01	2.1e−07	2.8e−09	2.3e−04	4.5e−05	3.2e+08
5	1.486857e−01	3.5e−23	6.0e−25	4.9e−16	8.3e−22	1.2e−18
10	7.250152e−02	7.9e−35	8.8e−37	3.0e−22	1.7e−33	3.8e−31
20	3.580010e−02	2.3e−47	7.7e−50	2.2e−28	1.1e−45	9.0e−44
50	1.421302e−02	4.6e−64	2.5e−67	2.1e−36	9.7e−62	1.8e−60
95	7.462843e−03	9.0e−76	1.3e−79	5.3e−42	6.3e−73	3.5e−72

x	$\lvert\psi_1(1+\mathrm{i}x)\rvert_s$	(12.4.1)	(12.4.1)	(12.4.2)	(12.4.3)	(12.3.8)
0.6	1.255548e+00	1.7e−06	2.3e−08	2.5e−03	6.5e−04	2.8e+13
1.1	8.525365e−01	5.8e−07	7.6e−09	5.2e−04	1.3e−04	2.0e+09
5	1.996599e−01	1.7e−14	3.2e−16	1.9e−12	3.2e−13	1.8e−12
10	9.995813e−02	3.4e−26	4.6e−27	3.1e−19	1.1e−25	1.3e−24
20	4.999479e−02	1.4e−40	1.5e−42	2.0e−25	1.8e−39	2.1e−37
50	1.999967e−02	1.1e−57	1.3e−60	2.0e−33	9.6e−56	3.7e−54
95	1.052627e−02	2.0e−69	6.0e−73	5.2e−39	6.2e−67	7.3e−66

12.5 Tetragamma function

S-fraction. From (12.3.12) we have [Lan94, pp. 245–249]

$$\psi_2(z) = -\frac{1}{z^2} - \frac{1}{z^3} - \left(\frac{2\pi}{z}\right)^2 g_2(z). \qquad (12.5.1a)$$

The coefficients in the modified S-fraction representation (12.3.23) for the function $g_2(z)$ can be given explicitly. We have

$$g_2(z) = \frac{a_1^{(2)}}{z^2} \,{+}\, \frac{a_2^{(2)}}{1} \,{+}\, \frac{a_3^{(2)}}{z^2} \,{+}\, \frac{a_4^{(2)}}{1} \,{+}\, \ldots, \qquad |\arg z| < \frac{\pi}{2} \qquad (12.5.1b)$$

where

$$a_1^{(2)} = \frac{1}{8\pi^2}, \quad a_{2j}^{(2)} = \frac{j^2(j+1)}{2(2j+1)}, \quad a_{2j+1}^{(2)} = \frac{j(j+1)^2}{2(2j+1)}, \quad j \geq 1. \quad (12.5.1c)$$

An S-fraction representation for $-\psi_2(z)$ in $1/z(z-1)$ [Lan94, pp. 245–249] is given by

$$-\psi_2(z) = \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m/z(z-1)}{m}\right), \qquad \Re z > 1/2, \quad z \notin (1/2, 1], \quad (12.5.2a)$$

where

$$a_1 = 1, \quad a_{2j} = a_{2j+1} = j^4, \quad j \geq 1. \qquad (12.5.2b)$$

In the *Tables* 12.5.1 and 12.5.2, the S-fraction representation (12.5.1) is first evaluated without modification and afterwards with the modification (12.3.25).

C-fraction. From the asymptotic series expansion (12.3.8) we obtain the regular C-fraction representation in $1/z$ for $\psi_2(z)$ [Lan94, pp. 245–249],

$$\psi_2(z) = -\frac{1}{z} + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{c_m z^{-1}}{1} \right), \qquad \Re z > 1, \tag{12.5.3a}$$

where

$$\begin{aligned} c_1 = 1, \quad c_{4j-2} = c_{4j-1} - 1 &= \frac{j^2 - 2j + 2}{2j - 1}, \\ c_{4j} = -c_{4j+1} &= \frac{j^3}{2(j^2+1)}, \quad j \geq 1. \end{aligned} \tag{12.5.3b}$$

TABLE 12.5.1: In combination with property (12.3.4), the following sequence of tables describes the relative error of the 5th approximants and the 5th partial sum throughout the region $\Re z > 1$. The S-fraction representation (12.5.1) is first evaluated without modification and afterwards with the modification (12.3.25).

x	$\psi_2(x)$	(12.5.1)	(12.5.1)	(12.5.2)	(12.5.3)	(12.3.8)
1.1	$-1.861457e{+}00$	$7.8e{-}04$	$3.4e{-}05$	$2.8e{-}01$	$1.2e{-}01$	$1.0e{-}01$
2	$-4.041138e{-}01$	$1.0e{-}05$	$4.2e{-}07$	$5.1e{-}04$	$1.0e{-}02$	$2.3e{-}04$
5	$-4.878973e{-}02$	$1.3e{-}09$	$2.7e{-}11$	$4.9e{-}08$	$1.9e{-}04$	$9.2e{-}09$
10	$-1.104983e{-}02$	$4.9e{-}13$	$3.6e{-}15$	$3.7e{-}11$	$1.0e{-}05$	$2.8e{-}12$
20	$-2.628122e{-}03$	$1.4e{-}16$	$2.9e{-}19$	$3.0e{-}14$	$5.7e{-}07$	$7.5e{-}16$
50	$-4.080800e{-}04$	$2.5e{-}21$	$8.4e{-}25$	$2.7e{-}18$	$1.4e{-}08$	$1.3e{-}20$
95	$-1.119758e{-}04$	$1.1e{-}24$	$1.1e{-}28$	$4.2e{-}21$	$1.0e{-}09$	$6.0e{-}24$

x	$\lvert\psi_2(x+\mathrm{i}x)\rvert_s$	(12.5.1)	(12.5.1)	(12.5.2)	(12.5.3)	(12.3.8)
1.1	$6.678989e{-}01$	$2.7e{-}04$	$1.4e{-}05$	$9.1e{-}03$	$3.4e{-}02$	$4.7e{-}03$
2	$1.613576e{-}01$	$1.2e{-}06$	$5.9e{-}08$	$2.3e{-}05$	$2.2e{-}03$	$7.9e{-}06$
5	$2.211105e{-}02$	$3.5e{-}11$	$5.8e{-}13$	$1.3e{-}09$	$4.1e{-}05$	$1.9e{-}10$
10	$5.256579e{-}03$	$9.2e{-}15$	$3.9e{-}17$	$9.6e{-}13$	$2.3e{-}06$	$4.9e{-}14$
20	$1.281651e{-}03$	$2.3e{-}18$	$2.5e{-}21$	$8.3e{-}16$	$1.4e{-}07$	$1.2e{-}17$
50	$2.020101e{-}04$	$3.9e{-}23$	$6.7e{-}27$	$8.1e{-}20$	$3.4e{-}09$	$2.1e{-}22$
95	$5.569402e{-}05$	$1.8e{-}26$	$8.4e{-}31$	$1.3e{-}22$	$2.6e{-}10$	$9.5e{-}26$

x	$\|\psi_2(1.5+\mathrm{i}x)\|_s$	(12.5.1)	(12.5.1)	(12.5.2)	(12.5.3)	(12.3.8)
1.1	$-4.482870e{-}01$	$4.1e{-}05$	$2.0e{-}06$	$1.4e{-}03$	$1.5e{-}02$	$6.4e{-}04$
2	$2.055887e{-}01$	$6.3e{-}06$	$3.8e{-}07$	$1.0e{-}04$	$3.8e{-}03$	$3.7e{-}05$
5	$3.880917e{-}02$	$2.2e{-}09$	$1.1e{-}10$	$2.8e{-}08$	$1.3e{-}04$	$9.0e{-}09$
10	$9.925145e{-}03$	$6.1e{-}13$	$5.9e{-}15$	$2.5e{-}11$	$8.3e{-}06$	$3.0e{-}12$
20	$2.495315e{-}03$	$1.5e{-}16$	$3.3e{-}19$	$2.4e{-}14$	$5.2e{-}07$	$7.8e{-}16$
50	$3.998800e{-}04$	$2.5e{-}21$	$8.7e{-}25$	$2.5e{-}18$	$1.3e{-}08$	$1.3e{-}20$
95	$1.107941e{-}04$	$1.1e{-}24$	$1.1e{-}28$	$4.0e{-}21$	$1.0e{-}09$	$6.1e{-}24$

TABLE 12.5.2: In combination with property (12.3.4), the following sequence of tables describes the relative error of the 20^{th} approximants and the 20^{th} partial sum throughout the region $\Re z > 1$. The S-fraction representation (12.5.1) is first evaluated without modification and afterwards with the modification (12.3.25).

x	$\psi_2(x)$	(12.5.1)	(12.5.1)	(12.5.2)	(12.5.3)	(12.3.8)
1.1	$-1.861457e{+}00$	$5.2e{-}06$	$6.6e{-}08$	$1.3e{-}03$	$1.1e{-}03$	$7.5e{+}15$
2	$-4.041138e{-}01$	$1.3e{-}09$	$1.7e{-}11$	$1.5e{-}07$	$2.1e{-}06$	$4.1e{+}05$
5	$-4.878973e{-}02$	$1.2e{-}18$	$1.6e{-}20$	$4.8e{-}17$	$2.8e{-}12$	$4.5e{-}11$
10	$-1.104983e{-}02$	$1.9e{-}28$	$2.0e{-}30$	$5.8e{-}27$	$1.2e{-}17$	$2.2e{-}23$
20	$-2.628122e{-}03$	$5.9e{-}40$	$2.9e{-}42$	$2.3e{-}38$	$2.6e{-}23$	$7.0e{-}36$
50	$-4.080800e{-}04$	$2.8e{-}56$	$2.9e{-}59$	$3.4e{-}54$	$7.0e{-}31$	$1.5e{-}52$
95	$-1.119758e{-}04$	$6.2e{-}68$	$1.9e{-}71$	$2.2e{-}65$	$3.5e{-}36$	$3.1e{-}64$

x	$\|\psi_2(x+\mathrm{i}x)\|_s$	(12.5.1)	(12.5.1)	(12.5.2)	(12.5.3)	(12.3.8)
1.1	$6.678989e{-}01$	$1.6e{-}06$	$2.1e{-}08$	$3.0e{-}04$	$2.7e{-}04$	$1.0e{+}10$
2	$1.613576e{-}01$	$8.7e{-}11$	$1.2e{-}12$	$6.1e{-}09$	$1.3e{-}07$	$5.2e{-}01$
5	$2.211105e{-}02$	$9.5e{-}22$	$1.6e{-}23$	$2.1e{-}20$	$1.3e{-}14$	$4.8e{-}17$
10	$5.256579e{-}03$	$2.6e{-}33$	$2.9e{-}35$	$5.3e{-}32$	$2.1e{-}20$	$1.6e{-}29$
20	$1.281651e{-}03$	$8.0e{-}46$	$2.7e{-}48$	$3.4e{-}44$	$3.7e{-}26$	$3.8e{-}42$
50	$2.020101e{-}04$	$1.6e{-}62$	$8.8e{-}66$	$3.1e{-}60$	$9.5e{-}34$	$7.5e{-}59$
95	$5.569402e{-}05$	$3.1e{-}74$	$4.8e{-}78$	$2.0e{-}71$	$4.7e{-}39$	$1.5e{-}70$

x	$\lvert\psi_2(1.5+\mathrm{i}x)\rvert_s$	(12.5.1)	(12.5.1)	(12.5.2)	(12.5.3)	(12.3.8)
1.1	$-4.482870e{-}01$	$4.0e{-}08$	$5.3e{-}10$	$5.4e{-}06$	$1.8e{-}05$	$8.2e{+}06$
2	$2.055887e{-}01$	$4.6e{-}09$	$6.4e{-}11$	$3.8e{-}07$	$1.8e{-}06$	$7.7e{+}01$
5	$3.880917e{-}02$	$3.1e{-}14$	$5.7e{-}16$	$6.0e{-}13$	$1.8e{-}11$	$2.3e{-}11$
10	$9.925145e{-}03$	$7.1e{-}25$	$6.0e{-}26$	$2.7e{-}24$	$1.8e{-}17$	$4.1e{-}23$
20	$2.495315e{-}03$	$5.0e{-}39$	$5.3e{-}41$	$5.9e{-}38$	$2.6e{-}23$	$8.3e{-}36$
50	$3.998800e{-}04$	$4.0e{-}56$	$4.6e{-}59$	$3.1e{-}54$	$6.7e{-}31$	$1.6e{-}52$
95	$1.107941e{-}04$	$6.9e{-}68$	$2.1e{-}71$	$2.0e{-}65$	$3.3e{-}36$	$3.1e{-}64$

12.6 Incomplete gamma functions

Definitions and elementary properties. The gamma function $\Gamma(z)$ can be generalised to the *incomplete gamma function* $\gamma(a,z)$ and the *complementary incomplete gamma function* $\Gamma(a,z)$. The generalisations satisfy the relation

$$\Gamma(a,z)+\gamma(a,z)=\Gamma(a), \qquad \Re a>0, \quad |\arg z|<\pi. \tag{12.6.1}$$

The incomplete gamma function $\gamma(a,z)$ is defined by

$$\gamma(a,z):=\int_0^z e^{-t}t^{a-1}\,dt, \qquad \Re a>0, \quad z\in\mathbb{C}, \tag{12.6.2}$$

where the path of integration is the line segment $t=z\tau$, $0<\tau<1$. Therefore

$$\gamma(a,z)=z^a\int_0^1 e^{-zt}t^{a-1}\,dt, \qquad \Re a>0, \quad z\in\mathbb{C}. \tag{12.6.3}$$

The complementary incomplete gamma function $\Gamma(a,z)$ is defined by

$$\Gamma(a,z):=\int_z^\infty e^{-t}t^{a-1}\,dt, \qquad a\in\mathbb{C}, \quad |\arg z|<\pi, \tag{12.6.4}$$

where the path of integration is $t=z+\tau$, $0\le\tau<\infty$. Hence

$$\Gamma(a,z)=e^{-z}\int_0^\infty \frac{e^{-t}}{(z+t)^{1-a}}\,dt, \qquad a\in\mathbb{C}, \quad |\arg z|<\pi. \tag{12.6.5}$$

Both incomplete gamma functions satisfy the same symmetry relations

$$\begin{aligned}\gamma(\overline{a},\overline{z})&=\overline{\gamma(a,z)},\\ \Gamma(\overline{a},\overline{z})&=\overline{\Gamma(a,z)}.\end{aligned} \tag{12.6.6}$$

Series expansions. The incomplete gamma function is closely related to the confluent hypergeometric function or Kummer's confluent hypergeometric function of the first kind ${}_1F_1(a;b;z)$, also denoted $M(a,b,z)$ and introduced in (16.1.2). We have

$$\begin{aligned}\gamma(a,z) &= z^a \sum_{k=0}^{\infty} \frac{(-z)^k}{(a+k)k!}, \qquad \Re a > 0, \quad z \in \mathbb{C} \\ &= \frac{z^a}{a}\ {}_1F_1(a;a+1;-z). \qquad (12.6.7)\end{aligned}$$

An alternative series representation for $\gamma(a,z)$ is given by

$$\begin{aligned}\gamma(a,z) &= \frac{z^a e^{-z}}{a} \sum_{k=0}^{\infty} \frac{z^k}{(1+a)_k}, \qquad \Re a > 0, \quad z \in \mathbb{C} \\ &= \frac{z^a e^{-z}}{a}\ {}_1F_1(1;1+a;z). \qquad (12.6.8)\end{aligned}$$

The complementary incomplete gamma function is closely related to Kummer's confluent hypergeometric function of the second kind $U(a,b,z)$ introduced in (16.1.4):

$$\Gamma(a,z) = e^{-z} U(1-a, 1-a, z), \qquad a \in \mathbb{C}, \quad |\arg z| < \pi. \quad (12.6.9)$$

Asymptotic series expansions. The asymptotic expansion for the complementary incomplete gamma function is given by [AS64, p. 263]

$$\begin{aligned}\frac{\Gamma(a,z)}{z^a e^{-z}} &\approx z^{-1} \sum_{k=0}^{\infty} (-1)^k (1-a)_k z^{-k}, \qquad a \in \mathbb{C}, \quad z \to \infty, \quad |\arg z| < \pi, \\ &= z^{-1}\ {}_2F_0(1, 1-a; -z^{-1}), \qquad (12.6.10)\end{aligned}$$

where $(a)_k$ is the Pochhammer symbol and ${}_2F_0(a,b;z)$ is the confluent hypergeometric series introduced in (16.1.12). Formula (12.6.10) also follows from the asymptotic expansion (16.1.11) for the function $U(a,b,z)$.

Stieltjes transform. For the complementary incomplete gamma function we obtain a Stieltjes transform representation [LW92, p. 576] by applying the identity

$$\frac{z^\alpha}{\Gamma(\beta)} \int_0^\infty \frac{e^{-t} t^{\beta-1}}{(z+t)^\alpha}\, dt = \frac{z^\beta}{\Gamma(\alpha)} \int_0^\infty \frac{e^{-t} t^{\alpha-1}}{(z+t)^\beta}\, dt, \qquad \alpha > 0, \quad \beta > 0, \qquad (12.6.11)$$

to (12.6.5). This gives

$$\frac{\Gamma(a,z)}{z^a e^{-z}} = \frac{1}{\Gamma(1-a)} \int_0^\infty \frac{e^{-t} t^{-a}}{z+t}\, dt, \qquad |\arg z| < \pi, \quad -\infty < a < 1, \tag{12.6.12}$$

with the weight function defined by

$$\phi_a(t) := \frac{e^{-t} t^{-a}}{\Gamma(1-a)}, \qquad 0 < t < \infty, \quad -\infty < a < 1. \tag{12.6.13}$$

The moments $\mu_k(a)$ for $\phi_a(t)$ are given by

$$\mu_k(a) = \int_0^\infty t^k \phi_a(t)\, dt = (1-a)_k, \qquad k \geq 0. \tag{12.6.14}$$

S-fraction. Since the classical Stieltjes moment problem has a solution $\phi_a(t)$ for $\mu_k(a)$ given by (12.6.14), it follows from *Theorem* 5.1.1 that there exists a modified S-fraction of the form

$$\frac{a_1(a)}{z} \underset{+}{} \frac{a_2(a)}{1} \underset{+}{} \frac{a_3(a)}{z} \underset{+}{} \frac{a_4(a)}{1} \underset{+}{} \ldots, \qquad -\infty < a < 1 \tag{12.6.15a}$$

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corresponding to the asymptotic series (12.6.10). The coefficients are given by [Wal48, p. 356]

$$a_1(a) = 1, \quad a_{2j}(a) = j - a, \quad a_{2j+1}(a) = j, \quad j \geq 1. \tag{12.6.15b}$$

Since the coefficients $a_m(a)$ satisfy

$$a_m(a) \sim \frac{m}{2}, \qquad m \to \infty \tag{12.6.16}$$

it follows from *Theorem* 3.1.5 that the S-fraction (12.6.15) is convergent.

C-fractions. When dropping the condition $-\infty < a < 1$, the continued fraction in (12.6.15) becomes a corresponding modified C-fraction. After an equivalence transformation we find

$$\frac{\Gamma(a,z)}{z^a e^{-z}} = \operatorname*{K}_{m=1}^{\infty} \left(\frac{a_m(a)/z}{1} \right), \qquad a \in \mathbb{C}, \quad |\arg z| < \pi \tag{12.6.17}$$

with $a_m(a)$ given by (12.6.15b). Again $a_m(a) \to \infty$ in which case use of the modification (7.7.8) can be recommended. For (12.6.17) $w_n(z)$ equals

$$w_{2j-1}(z) = \frac{-1 + \sqrt{4(j-a)/z + 1}}{2}, \quad w_{2j}(z) = \frac{-1 + \sqrt{4j/z + 1}}{2}, \quad j \geq 1. \tag{12.6.18}$$

In the *Tables* 12.6.3 and 12.6.4 the C-fraction (12.6.17) is evaluated with $w_n(z) = 0$ and $w_n(z)$ given by (12.6.18) respectively.
A more general result follows from the relations in [JT85]:

$$\frac{\Gamma(a,z)}{z^a e^{-z}} = -\sum_{k=0}^{\ell-1} \frac{z^k}{(a)_{k+1}} + \frac{z^\ell}{(a)_\ell} \frac{\Gamma(a+\ell,z)}{z^{a+\ell}e^{-z}},$$
$$|\arg z| < \pi, \quad a \in \mathbb{C}, \quad (a)_\ell \neq 0, \quad \ell \in \mathbb{N}_0 \quad (12.6.19)$$

and

$$\frac{\Gamma(a,z)}{z^a e^{-z}} = -\sum_{k=1}^{\ell} \frac{(1-a)_{k-1}}{(-z)^k} + \frac{(1-a)_\ell}{(-z)^\ell} \frac{\Gamma(a-\ell,z)}{z^{a-\ell}e^{-z}},$$
$$|\arg z| < \pi, \quad a \in \mathbb{C}, \quad \ell \in \mathbb{N}_0. \quad (12.6.20)$$

Applying (12.6.17) to (12.6.19) and (12.6.20) we find

$$\frac{\Gamma(a,z)}{z^a e^{-z}} = -\sum_{k=0}^{\ell-1} \frac{z^k}{(a)_{k+1}} + \frac{z^\ell}{(a)_\ell} \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m(a+\ell)/z}{1} \right),$$
$$|\arg z| < \pi, \quad (a)_\ell \neq 0, \quad \ell \in \mathbb{N}_0, \quad (12.6.21)$$

$$\frac{\Gamma(a,z)}{z^a e^{-z}} = -\sum_{k=1}^{\ell} \frac{(1-a)_{k-1}}{(-z)^k} + \frac{(1-a)_\ell}{(-z)^\ell} \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m(a-\ell)/z}{1} \right),$$
$$|\arg z| < \pi, \quad \ell \in \mathbb{N}_0, \quad (12.6.22)$$

where $a_m(a+\ell)$ and $a_m(a-\ell)$ are given by (12.6.15b). If $w = 1/z$ then the successive approximants of (12.6.22) are the Padé approximants of the function $\Gamma(a, 1/w)e^{1/w}w^{a-1}$ on the staircase $T_{\ell-1}$ defined in (4.3.1).
From the series representation (12.6.8) for $\gamma(a,z)$ and (16.1.14), we obtain the corresponding regular C-fraction [Wal48, p. 347]

$$\frac{\gamma(a,z)}{z^a e^{-z}} = \frac{1}{z} \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{c_m(a)z}{1} \right), \qquad z \in \mathbb{C}, \quad \Re a > 0, \qquad (12.6.23a)$$

where the coefficients are given by

$$c_1(a) = \frac{1}{a}, \qquad c_{2j}(a) = \frac{-(a+j-1)}{(a+2j-2)(a+2j-1)}, \quad j \geq 1,$$
$$c_{2j+1}(a) = \frac{j}{(a+2j-1)(a+2j)}, \quad j \geq 1. \qquad (12.6.23b)$$

Since

$$|c_m(a)| \sim \frac{1}{2m}, \qquad m \to \infty,$$

it follows from *Corollary* 3.5.1 that the C-fraction converges to a function holomorphic at $z = 0$ and meromorphic in $\mathbb{C}$, for each $a \in \mathbb{C}\backslash\mathbb{Z}_0^-$. It follows from *Theorem* 3.4.1 that this function equals $\gamma(a,z)z^{-a}e^z$ if $\Re a > 0$ and $z \in \mathbb{C}$. Therefore the right hand side of (12.6.23a) provides the analytic continuation of $\gamma(a,z)z^{-a}e^z$ from $\Re a > 0$ to $a \in \mathbb{C}\backslash\mathbb{Z}_0^-$.
Another regular C-fraction expansion for the complementary incomplete gamma function $\Gamma(a,z)$ is

$$\frac{\Gamma(a,z)}{z^a e^{-z}} = \frac{\Gamma(a)}{z^a e^{-z}} - \frac{1}{z}\mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{c_m(a)z}{1}\right), \qquad |\arg z| < \pi, \quad a \in \mathbb{C}\backslash\mathbb{Z}_0^-, \tag{12.6.24}$$

where the coefficients $c_m(a)$ are defined by (12.6.23b). In the same way as above, the function $\Gamma(a,z)z^{-a}e^z$ is continued analytically to $z \in \mathbb{C}\backslash\{0\}$ for $a \in \mathbb{C}\backslash\mathbb{Z}_0^-$ by the right hand side of (12.6.24). Applying (12.6.24) to (12.6.19) and (12.6.20) yields

$$\frac{\Gamma(a,z)}{z^a e^{-z}} = \frac{\Gamma(a+\ell)}{(a)_\ell z^a e^{-z}} - \sum_{k=0}^{\ell-1}\frac{z^k}{(a)_{k+1}} - \frac{z^{\ell-1}}{(a)_\ell}\mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{c_m(a+\ell)z}{1}\right),$$
$$|\arg z| < \pi, \quad a \in \mathbb{C}\backslash\mathbb{Z}_0^-, \quad \ell \in \mathbb{N}_0, \tag{12.6.25}$$

$$\frac{\Gamma(a,z)}{z^a e^{-z}} = \frac{(1-a)_\ell\Gamma(a-\ell)}{(-1)^\ell z^a e^{-z}} - \sum_{k=1}^{\ell}\frac{(1-a)_{k-1}}{(-z)^k} - \frac{(1-a)_\ell}{(-1)^\ell z^{\ell+1}}\mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{c_m(a-\ell)z}{1}\right),$$
$$|\arg z| < \pi, \quad a \in \mathbb{C}\backslash\mathbb{Z}_0^-, \quad \ell \in \mathbb{N}_0, \tag{12.6.26}$$

where the coefficients $c_m(a+\ell)$ and $c_m(a-\ell)$ are given by (12.6.23b). The right hand side of both (12.6.25) and (12.6.26) is an analytic continuation of $\Gamma(a,z)z^{-a}e^z$ from $|\arg z| < \pi$ to $z \in \mathbb{C} \setminus \{0\}$ for $a \in \mathbb{C}\backslash\mathbb{Z}_0^-$.
The successive approximants of (12.6.25) are the Padé approximants of the function

$$\frac{\Gamma(a,z)}{z^a e^{-z}} - \frac{\Gamma(a)}{z^a e^{-z}}$$

on the staircase $T_{\ell-1}$ defined in (4.3.1).
The analytic continuations (12.6.23) for $\gamma(a,z)$ from $\Re a > 0$ to $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and (12.6.24) for $\Gamma(a,z)$ from $|\arg z| < \pi$ to $z \in \mathbb{C} \setminus \{0\}$ allow us to generalise (12.6.1) to

$$\Gamma(a,z) + \gamma(a,z) = \Gamma(a), \qquad z \in \mathbb{C} \setminus \{0\}, \quad a \in \mathbb{C} \setminus \mathbb{Z}_0^-. \tag{12.6.27}$$

Padé approximants. As a special case of (16.2.5), explicit formulas can be given for Padé approximants of $\Gamma(a,z)z^{1-a}e^z$ at $z=\infty$:

$$r_{m,n}(z) = \frac{\mathcal{P}_{m+n}({}_2F_0(1-a,1;-z^{-1})\ {}_2F_0(a-m-1,-n,z^{-1}))}{{}_2F_0(a-m-1,-n,z^{-1})}, \qquad m \ge n-1, \quad (12.6.28)$$

where the operator $\mathcal{P}_k$ is defined in (15.4.1). In [Luk75, pp. 82–83] more explicit formulas for the numerator of $r_{m,n}$ are given in case $m=n$ or $m=n-1$, in terms of hypergeometric series ${}_2F_2$.

Similarly, as a special case of (16.1.15), explicit formulas can be given for Padé approximants of $a\gamma(a,z)z^{-a}e^z$ at $z=0$:

$$r_{m,n}(z) = \frac{\mathcal{P}_{m+n}({}_1F_1(1;a+1;z)\ {}_1F_1(-n;-a-m-n;-z))}{{}_1F_1(-n;-a-m-n;-z)}, \qquad m \ge n-1. \quad (12.6.29)$$

In [Luk75, pp. 79–80] more explicit formulas for the numerator of $r_{m,n}$ are given in case $m=n$ or $m=n-1$, in terms of hypergeometric series ${}_3F_1$.

M-fractions. From the series representation (12.6.8) and (16.1.17) we obtain the M-fraction representation of the form (2.3.14) [JT85],

$$\frac{\gamma(a,z)}{z^a e^{-z}} = \frac{1}{a-z} + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{mz}{a+m-z}\right), \qquad z \in \mathbb{C}, \quad a \in \mathbb{C}\setminus\mathbb{Z}_0^-. \quad (12.6.30)$$

The continued fraction corresponds at $z=0$ to the series representation

$$\frac{1}{a}\ {}_1F_1(1;a+1;z)$$

and at $z=\infty$ to

$$-\frac{1}{z}\ {}_2F_0(1,1-a;-1/z).$$

An M-fraction representation for $\Gamma(a,z)$ follows from (12.6.1) and (12.6.30),

$$\frac{\Gamma(a,z)}{z^a e^{-z}} = \frac{\Gamma(a)}{z^a e^{-z}} - \frac{1}{a-z} + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{mz}{a+m-z}\right), \qquad |\arg z| < \pi, \quad \Re a > 0. \quad (12.6.31)$$

Applying (12.6.31) to (12.6.19) and (12.6.20) yields [JT85]

$$\frac{\Gamma(a,z)}{z^a e^{-z}} = \frac{\Gamma(a+\ell)}{(a)_\ell z^a e^{-z}} - \sum_{k=0}^{\ell-1} \frac{z^k}{(a)_{k+1}} - \frac{z^\ell}{(a)_\ell} \left(\frac{1}{a+\ell-z} + \operatorname*{K}_{m=1}^{\infty} \left(\frac{mz}{a+\ell+m-z} \right) \right),$$
$$|\arg z| < \pi, \quad \Re a > 0, \quad \ell \in \mathbb{N}_0, \tag{12.6.32}$$

$$\frac{\Gamma(a,z)}{z^a e^{-z}} = \frac{(1-a)_\ell \Gamma(a-\ell)}{(-1)^\ell z^a e^{-z}} - \sum_{k=1}^{\ell} \frac{(1-a)_{k-1}}{(-z)^k} - \frac{(1-a)_\ell}{(-1)^\ell z^\ell} \left(\frac{1}{a-\ell-z} + \operatorname*{K}_{m=1}^{\infty} \left(\frac{mz}{a-\ell+m-z} \right) \right),$$
$$|\arg z| < \pi, \quad \Re a > 0, \quad a \notin \{1,2,\ldots,\ell\}, \quad \ell \in \mathbb{N}_0. \tag{12.6.33}$$

The right hand sides of (12.6.31), (12.6.32) and (12.6.33) are also analytic continuations of $\Gamma(a,z)z^{-a}e^z$ from $|\arg z| < \pi$ to $z \in \mathbb{C} \setminus \{0\}$.

J-fractions. The even part of the modified C-fraction (12.6.17) is the J-fraction [JT88, p. 195]

$$\frac{\Gamma(a,z)}{z^a e^{-z}} = \frac{1}{1+z-a} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{(1-m)(m-1-a)}{(2m-1)+z-a} \right),$$
$$a \in \mathbb{C}, \quad |\arg z| < \pi. \tag{12.6.34}$$

The odd part of the modified C-fraction (12.6.17) is also a J-fraction and is given by

$$\frac{\Gamma(a,z)}{z^a e^{-z}} = \frac{1}{z} \left(1 + \frac{a-1}{2+z-a} + \operatorname*{K}_{m=1}^{\infty} \left(\frac{(1-m)(m-a)}{2m+z-a} \right) \right),$$
$$a \in \mathbb{C}, \quad |\arg z| < \pi. \tag{12.6.35}$$

TABLE 12.6.1: In combination with property (12.6.6) the following sequence of tables describes the relative error of the 5th partial sums and 5th approximants in case $a = 3/2$.

x	$\gamma(a, x)$	(12.6.7)	(12.6.8)	(12.6.23)	(12.6.30)
0.01	6.626809e−04	2.8e−16	9.4e−17	5.8e−15	2.7e−14
0.1	1.986097e−02	2.9e−10	9.2e−11	6.0e−10	3.0e−09
0.5	1.761359e−01	5.5e−06	1.3e−06	2.2e−06	1.6e−05
1	3.789447e−01	4.3e−04	6.9e−05	8.3e−05	9.5e−04
2	6.545104e−01	4.1e−02	3.1e−03	3.9e−03	1.2e−01
3	7.873149e−01	6.4e−01	2.3e−02	4.2e−02	1.9e+00
5	8.697731e−01	2.2e+01	1.8e−01	5.7e−01	1.0e+00
15	8.862257e−01	4.5e+04	9.9e−01	1.0e+00	1.0e+00

x	$\lvert\gamma(a, x + \mathrm{i}x)\rvert_s$	(12.6.7)	(12.6.8)	(12.6.23)	(12.6.30)
0.01	1.114488e−03	2.2e−15	7.6e−16	3.3e−14	1.5e−13
0.1	3.339048e−02	2.3e−09	7.4e−10	3.4e−09	1.7e−08
0.5	2.936018e−01	4.4e−05	1.0e−05	1.2e−05	8.9e−05
1	6.145774e−01	3.6e−03	5.7e−04	4.7e−04	5.5e−03
2	9.533397e−01	3.7e−01	2.7e−02	2.1e−02	1.6e+00
3	9.867346e−01	6.6e+00	2.3e−01	2.0e−01	9.1e−01
5	8.875841e−01	2.7e+02	1.8e+00	8.5e−01	1.0e+00
15	8.862276e−01	5.0e+05	1.1e+00	1.0e+00	1.0e+00

x	$\lvert\gamma(a, \mathrm{i}x)\rvert_s$	(12.6.7)	(12.6.8)	(12.6.23)	(12.6.30)
0.01	−6.666644e−04	2.8e−16	9.5e−17	5.7e−15	2.6e−14
0.1	−2.107462e−02	2.8e−10	9.5e−11	5.7e−10	2.6e−09
0.5	−2.336879e−01	4.4e−06	1.5e−06	1.8e−06	8.2e−06
1	−6.440760e−01	2.9e−04	9.8e−05	5.7e−05	2.7e−04
2	1.638901e+00	2.0e−02	6.8e−03	1.8e−03	9.1e−03
3	2.503061e+00	2.7e−01	9.1e−02	1.3e−02	9.6e−02
5	2.737259e+00	1.0e+01	3.5e+00	1.9e−01	3.3e−01
15	4.073550e+00	1.3e+04	4.8e+03	2.0e+00	2.2e−01

x	$\|\gamma(a, x - \mathrm{i}x)\|_s$	(12.6.7)	(12.6.8)	(12.6.23)	(12.6.30)
−0.01	−1.127943e−03	2.2e−15	7.6e−16	3.2e−14	1.5e−13
−0.1	−3.764782e−02	2.1e−09	7.8e−10	3.1e−09	1.3e−08
−0.5	−5.352469e−01	2.7e−05	1.4e−05	8.5e−06	2.4e−05
−1	−2.048481e+00	1.4e−03	1.0e−03	2.3e−04	4.0e−04
−2	−1.080189e+01	5.3e−02	8.3e−02	5.0e−03	3.1e−03
−3	3.869274e+01	3.5e−01	1.2e+00	2.6e−02	5.2e−03
−5	3.766683e+02	1.9e+00	3.4e+01	1.5e−01	3.1e−03
−15	−1.480821e+07	1.0e+00	3.5e+04	1.7e+00	2.0e−05

TABLE 12.6.2: In combination with property (12.6.6) the following sequence of tables describes the relative error of the 20th partial sums and 20th approximants in case $a = 3/2$.

x	$\gamma(a, x)$	(12.6.7)	(12.6.8)	(12.6.23)	(12.6.30)
0.01	6.626809e−04	1.3e−63	2.5e−64	2.2e−66	1.1e−61
0.1	1.986097e−02	1.4e−42	2.4e−43	2.3e−46	1.3e−41
0.5	1.761359e−01	8.1e−28	9.8e−29	2.7e−32	2.2e−27
1	3.789447e−01	2.2e−21	1.7e−22	3.8e−26	4.7e−21
2	6.545104e−01	7.3e−15	2.2e−16	6.6e−20	2.0e−14
3	7.873149e−01	5.3e−11	6.5e−13	3.5e−16	2.6e−10
5	8.697731e−01	4.4e−06	8.6e−09	2.1e−11	8.7e−05
15	8.862257e−01	1.7e+05	4.2e−02	7.3e−01	1.0e+00

x	$\|\gamma(a, x + \mathrm{i}x)\|_s$	(12.6.7)	(12.6.8)	(12.6.23)	(12.6.30)
0.01	1.114488e−03	1.9e−60	3.6e−61	2.2e−63	1.1e−58
0.1	3.339048e−02	2.0e−39	3.5e−40	2.4e−43	1.3e−38
0.5	2.936018e−01	1.2e−24	1.4e−25	2.8e−29	2.3e−24
1	6.145774e−01	3.3e−18	2.5e−19	4.0e−23	5.0e−18
2	9.533397e−01	1.2e−11	3.7e−13	7.4e−17	2.4e−11
3	9.867346e−01	1.0e−07	1.3e−09	4.3e−13	3.5e−07
5	8.875841e−01	1.0e−02	2.0e−05	2.7e−08	1.6e−01
15	8.862276e−01	3.9e+08	5.9e+01	1.0e+00	1.0e+00

x	$\lvert\gamma(a, \mathrm{i}x)\rvert_s$	(12.6.7)	(12.6.8)	(12.6.23)	(12.6.30)
0.01	$-6.666644e{-04}$	$1.3e{-63}$	$2.5e{-64}$	$2.2e{-66}$	$1.1e{-61}$
0.1	$-2.107462e{-02}$	$1.3e{-42}$	$2.5e{-43}$	$2.2e{-46}$	$1.1e{-41}$
0.5	$-2.336879e{-01}$	$6.3e{-28}$	$1.2e{-28}$	$2.1e{-32}$	$1.0e{-27}$
1	$-6.440760e{-01}$	$1.3e{-21}$	$2.6e{-22}$	$2.2e{-26}$	$1.1e{-21}$
2	$1.638901e{+00}$	$3.1e{-15}$	$6.0e{-16}$	$2.5e{-20}$	$1.3e{-15}$
3	$2.503061e{+00}$	$1.9e{-11}$	$3.6e{-12}$	$9.4e{-17}$	$5.1e{-12}$
5	$2.737259e{+00}$	$1.7e{-06}$	$3.2e{-07}$	$4.2e{-12}$	$2.7e{-07}$
15	$4.073550e{+00}$	$5.2e{+04}$	$1.0e{+04}$	$3.9e{-03}$	$2.2e{-01}$

x	$\lvert\gamma(a, x - \mathrm{i}x)\rvert_s$	(12.6.7)	(12.6.8)	(12.6.23)	(12.6.30)
-0.01	$-1.127943e{-03}$	$1.9e{-60}$	$3.6e{-61}$	$2.2e{-63}$	$1.1e{-58}$
-0.1	$-3.764782e{-02}$	$1.8e{-39}$	$3.7e{-40}$	$2.1e{-43}$	$9.6e{-39}$
-0.5	$-5.352469e{-01}$	$6.8e{-25}$	$2.0e{-25}$	$1.6e{-29}$	$5.0e{-25}$
-1	$-2.048481e{+00}$	$1.1e{-18}$	$5.1e{-19}$	$1.2e{-23}$	$2.5e{-19}$
-2	$-1.080189e{+01}$	$1.3e{-12}$	$1.5e{-12}$	$7.2e{-18}$	$5.5e{-14}$
-3	$3.869274e{+01}$	$3.4e{-09}$	$1.0e{-08}$	$1.3e{-14}$	$3.8e{-11}$
-5	$3.766683e{+02}$	$3.7e{-05}$	$6.9e{-04}$	$8.0e{-11}$	$3.7e{-08}$
-15	$-1.480821e{+07}$	$6.0e{+01}$	$1.5e{+07}$	$8.5e{-05}$	$5.7e{-08}$

TABLE 12.6.3: In combination with property (12.6.6), the following tables describe the relative error of the 5th approximants and 5th partial sum in $\lvert \arg z\rvert < \pi$ for $a = 1/2$. The C-fraction (12.6.17) is first evaluated without modification and then with $w_n(z)$ given by (12.6.18).

x	$\Gamma(a, x)$	(12.6.17)	(12.6.17)	(12.6.34)	(12.6.10)
0.01	$1.573119e{+00}$	$2.4e{+00}$	$9.7e{-03}$	$6.4e{-01}$	$1.9e{+12}$
0.1	$1.160462e{+00}$	$4.2e{-01}$	$1.4e{-02}$	$1.6e{-01}$	$7.1e{+06}$
0.5	$5.624182e{-01}$	$5.6e{-02}$	$9.1e{-03}$	$9.4e{-03}$	$1.3e{+03}$
2	$8.064712e{-02}$	$2.8e{-03}$	$1.6e{-03}$	$8.4e{-05}$	$7.7e{-01}$
10	$1.372627e{-05}$	$1.1e{-05}$	$3.7e{-05}$	$7.6e{-09}$	$1.1e{-04}$
50	$2.701168e{-23}$	$9.0e{-09}$	$1.7e{-07}$	$1.3e{-14}$	$9.3e{-09}$
100	$3.701748e{-45}$	$3.2e{-10}$	$1.3e{-08}$	$2.1e{-17}$	$1.5e{-10}$
500	$3.183031e{-219}$	$1.2e{-13}$	$2.3e{-11}$	$3.3e{-24}$	$1.0e{-14}$

x	$\lvert\Gamma(a, x+\mathrm{i}x)\rvert_s$	(12.6.17)	(12.6.17)	(12.6.34)	(12.6.10)
0.01	1.555753e+00	2.0e+00	1.2e−02	6.2e−01	2.8e+11
0.1	1.121398e+00	3.3e−01	1.8e−02	1.3e−01	1.1e+06
0.5	5.242904e−01	3.8e−02	8.3e−03	5.4e−03	2.1e+02
2	−7.200584e−02	1.3e−03	1.1e−03	2.8e−05	1.2e−01
10	−1.177770e−05	3.2e−06	1.6e−05	9.2e−10	1.5e−05
50	2.282250e−23	1.8e−09	5.0e−08	6.6e−16	1.2e−09
100	3.120389e−45	6.2e−11	3.4e−09	8.4e−19	2.0e−11
500	−2.677931e−219	2.1e−14	5.9e−12	1.1e−25	1.3e−15

x	$\lvert\Gamma(a, \mathrm{i}x)\rvert_s$	(12.6.17)	(12.6.17)	(12.6.34)	(12.6.10)
0.01	1.636643e+00	2.6e+00	2.5e−02	7.8e−01	1.8e+12
0.1	1.380103e+00	5.8e−01	2.9e−02	3.5e−01	6.8e+06
0.5	1.029521e+00	1.2e−01	2.0e−02	3.6e−02	1.3e+03
2	−6.541871e−01	8.5e−03	4.4e−03	6.5e−04	9.0e−01
10	−3.143992e−01	2.6e−05	8.6e−05	6.2e−08	1.3e−04
50	1.413861e−01	1.2e−08	2.3e−07	3.2e−14	1.0e−08
100	9.999376e−02	3.7e−10	1.5e−08	3.4e−17	1.6e−10
500	−4.472125e−02	1.2e−13	2.4e−11	3.6e−24	1.0e−14

x	$\lvert\Gamma(a, x-\mathrm{i}x)\rvert_s$	(12.6.17)	(12.6.17)	(12.6.34)	(12.6.10)
−0.01	1.694757e+00	2.3e+00	4.1e−02	9.4e−01	2.6e+11
−0.1	1.613954e+00	6.6e−01	5.8e−02	8.0e−01	9.7e+05
−0.5	1.792530e+00	2.6e−01	4.4e−02	1.0e−01	2.0e+02
−2	−4.684326e+00	1.6e−02	9.2e−03	2.9e−03	1.9e−01
−10	6.002169e+03	1.2e−05	7.3e−05	5.5e−08	2.5e−05
−50	6.196574e+20	2.5e−09	7.2e−08	2.0e−15	1.4e−09
−100	2.266085e+42	7.1e−11	4.1e−09	1.5e−18	2.1e−11
−500	5.280993e+215	2.2e−14	6.1e−12	1.2e−25	1.3e−15

TABLE 12.6.4: In combination with property (12.6.6), the following tables describe the relative error of the 20th approximants and 20th partial sum in $|\arg z| < \pi$ for $a = 1/2$. The C-fraction (12.6.17) is first evaluated without modification and then with $w_n(z)$ given by (12.6.18).

x	$\Gamma(a,x)$	(12.6.17)	(12.6.17)	(12.6.34)	(12.6.10)
0.01	1.573119e+00	4.9e−01	1.2e−03	3.2e−01	1.9e+58
0.1	1.160462e+00	5.2e−02	1.2e−03	1.0e−02	7.5e+37
0.5	5.624182e−01	7.2e−04	4.8e−05	1.8e−05	4.8e+23
2	8.064712e−02	5.2e−07	9.5e−08	3.5e−10	3.1e+11
10	1.372627e−05	1.4e−13	1.0e−13	1.6e−20	2.1e−03
50	2.701168e−23	7.0e−24	2.9e−23	3.9e−38	9.3e−18
100	3.701748e−45	3.4e−29	3.0e−28	7.2e−48	5.2e−24
500	3.183031e−219	1.6e−42	7.8e−41	1.7e−73	1.3e−38

x	$\|\Gamma(a,x+\mathrm{i}x)\|_s$	(12.6.17)	(12.6.17)	(12.6.34)	(12.6.10)
0.01	1.555753e+00	4.6e−01	2.6e−03	2.9e−01	1.6e+55
0.1	1.121398e+00	3.7e−02	1.0e−03	6.1e−03	6.4e+34
0.5	5.242904e−01	3.2e−04	2.7e−05	5.7e−06	4.2e+20
2	−7.200584e−02	1.0e−07	2.4e−08	3.3e−11	2.9e+08
10	−1.177770e−05	4.7e−15	4.8e−15	9.6e−23	1.9e−06
50	2.282250e−23	3.1e−26	1.8e−25	3.6e−42	7.4e−21
100	3.120389e−45	8.0e−32	1.0e−30	1.3e−52	3.9e−27
500	−2.677931e−219	1.9e−45	1.3e−43	3.4e−79	8.9e−42

x	$\|\Gamma(a,\mathrm{i}x)\|_s$	(12.6.17)	(12.6.17)	(12.6.34)	(12.6.10)
0.01	1.636643e+00	6.8e−01	5.1e−03	5.4e−01	1.9e+58
0.1	1.380103e+00	1.6e−01	3.5e−03	4.6e−02	7.0e+37
0.5	1.029521e+00	5.8e−03	3.7e−04	4.3e−04	4.4e+23
2	−6.541871e−01	1.6e−05	2.7e−06	8.9e−08	3.1e+11
10	−3.143992e−01	1.3e−11	7.9e−12	8.4e−17	2.7e−03
50	1.413861e−01	1.1e−22	4.7e−22	8.1e−35	1.2e−17
100	9.999376e−02	1.9e−28	1.8e−27	1.7e−45	6.1e−24
500	−4.472125e−02	2.4e−42	1.2e−40	7.6e−73	1.3e−38

x	$\lvert\Gamma(a, x - \mathrm{i}x)\rvert_s$	(12.6.17)	(12.6.17)	(12.6.34)	(12.6.10)
−0.01	1.694757e+00	9.6e−01	1.1e−02	1.0e+00	1.5e+55
−0.1	1.613954e+00	3.6e−01	1.1e−02	1.5e−01	5.5e+34
−0.5	1.792530e+00	3.3e−02	2.5e−03	6.2e−03	3.5e+20
−2	−4.684326e+00	2.5e−04	4.4e−05	8.3e−06	2.9e+08
−10	6.002169e+03	8.6e−11	5.7e−11	2.1e−14	3.9e−06
−50	6.196574e+20	1.7e−24	1.3e−23	3.7e−36	1.1e−20
−100	2.266085e+42	6.4e−31	9.8e−30	3.5e−49	4.8e−27
−500	5.280993e+215	2.9e−45	2.1e−43	1.8e−78	9.2e−42

Convergence speed. The difference in speed of convergence between the continued fractions (12.6.15) and (12.6.23) is explained by the following truncation error upper bounds.

Let $f_n(a, z; 0)$ denote the n^{th} approximant of the regular C-fraction (12.6.23) for $\gamma(a, z)z^{-a}e^z$. Then by [JT88, p. 195] there exist constants $A(z) > 0$ and $B(z) > 0$ dependent on z, such that

$$\left| \frac{\gamma(a, z)}{z^a e^{-z}} - f_n(a, z; 0) \right| \leq A(z) \left(\frac{B(z)}{n} \right)^{n+\frac{3}{2}}, \qquad n \geq 2,$$
$$z \in \mathbb{C}, \quad a \in \mathbb{C} \backslash \mathbb{Z}_0^-. \quad (12.6.36)$$

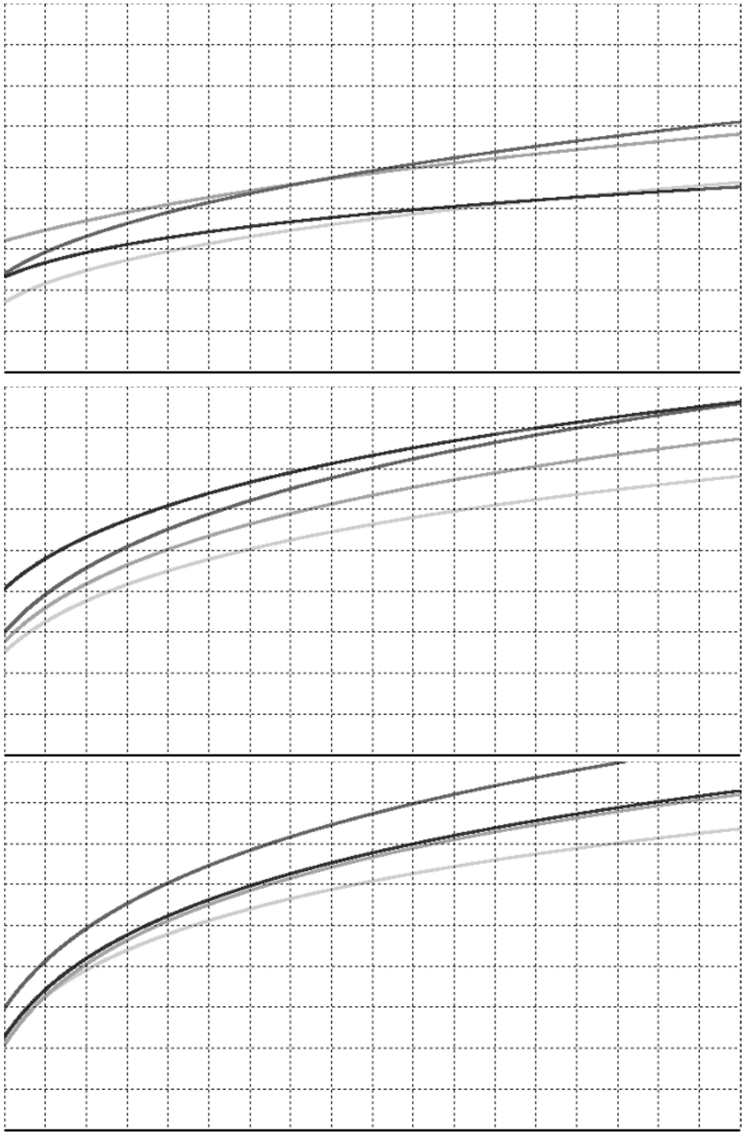
This implies fast convergence of (12.6.23), as illustrated in the *Tables* 12.6.1 and 12.6.2.

Let $f_n(a, z; 0)$ denote the n^{th} approximant of the S-fraction (12.6.15) for $\Gamma(a, z)z^{-a}e^z$. Since the coefficients satisfy (12.6.16), it follows from *Corollary* 7.4.1 that there exist constants $A(z) > 0$ and $C(z) > 1$ dependent on z such that

$$\left| \frac{\Gamma(a, z)}{z^a e^{-z}} - f_n(a, z; 0) \right| \leq A(z)/C(z)^{\sqrt{n}}, \qquad n \geq 1,$$
$$|\arg z| < \pi, \quad -\infty < a < 1. \quad (12.6.37)$$

This bound explains the slow convergence of (12.6.15) shown in the *Tables* 12.6.3 and 12.6.4.

FIGURE 12.6.1: Significant digits guaranteed by the n^{th} classical approximant of (12.6.17) (from light to dark $n = 5, 6, 7$) and the 5^{th} modified approximant with $w_5(z)$ given by (12.6.18) (darkest), for $a = -7.9, 0.5$ and 2.5 respectively. On the horizontal axis we have real $z, 1 \le z \le 19$ and on the vertical axis we find the number of significant digits (from 0 to 9).



13

Error function and related integrals

In this chapter we deal with some special functions defined by integrals which cannot be evaluated in closed form in terms of elementary functions. These include the error function, the complementary and complex error function, repeated integrals of the error function, Dawson's integral and the Fresnel integrals. All are entire functions defined in the whole complex plane.

The error functions and Dawson's integral are special cases of the incomplete and complementary incomplete gamma functions $\gamma(1/2, z)$ and $\Gamma(1/2, z)$.

13.1 Error function and Dawson's integral

Definitions and elementary properties. The *error function* $\operatorname{erf}(z)$ is an entire function and is defined by

$$\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt, \qquad z \in \mathbb{C} \tag{13.1.1}$$

and the related *Dawson's integral* by

$$e^{-z^2} \int_0^z e^{t^2}\, dt := \frac{\mathrm{i}\sqrt{\pi}}{2} e^{-z^2} \operatorname{erf}(-\mathrm{i}z), \qquad z \in \mathbb{C}. \tag{13.1.2}$$

The path of integration in both (13.1.1) and (13.1.2) is a straight line segment from 0 to z. We have the symmetry relations

$$\operatorname{erf}(-z) = -\operatorname{erf}(z), \tag{13.1.3a}$$

$$\operatorname{erf}(\bar{z}) = \overline{\operatorname{erf}(z)}, \tag{13.1.3b}$$

and the property

$$\operatorname{erf}(z) \to 1, \qquad z \to \infty, \quad |\arg z| < \frac{\pi}{4}. \tag{13.1.4}$$

The error function and Dawson's integral are special instances of the incomplete gamma function $\gamma(a,z)$ introduced in *Section* 12.6:

$$\operatorname{erf}(z) = \frac{1}{\sqrt{\pi}}\gamma\left(1/2, z^2\right), \tag{13.1.5}$$

$$e^{-z^2}\int_0^z e^{t^2}\,dt = \frac{\mathrm{i}}{2}e^{-z^2}\gamma\left(1/2, -z^2\right). \tag{13.1.6}$$

The relationship with the incomplete gamma function is crucial in deriving all the representations for the error function and for Dawson's integral given in this section.

Series expansions. From (13.1.5) and the series expansion for $\gamma(a,z)$ given in *Section* 12.6 we find two alternative series representations for the error function:

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+1}}{(2k+1)k!} = \frac{2z}{\sqrt{\pi}}\,{}_1F_1\left(\frac{1}{2};\frac{3}{2};-z^2\right), \qquad z \in \mathbb{C} \tag{13.1.7}$$

and

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}e^{-z^2}\sum_{k=0}^{\infty}\frac{z^{2k+1}}{(\frac{3}{2})_k} = \frac{2z}{\sqrt{\pi}}e^{-z^2}\,{}_1F_1\left(1;\frac{3}{2};z^2\right), \qquad z \in \mathbb{C} \tag{13.1.8}$$

where ${}_1F_1(a;b;z)$ is the confluent hypergeometric series (16.1.2). Similarly, from (13.1.6) we find for Dawson's integral

$$\begin{aligned} e^{-z^2}\int_0^z e^{t^2}\,dt &= e^{-z^2}\sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)k!} \\ &= ze^{-z^2}\,{}_1F_1\left(\frac{1}{2};\frac{3}{2};z^2\right), \qquad z \in \mathbb{C}, \end{aligned} \tag{13.1.9}$$

or alternatively

$$\begin{aligned} e^{-z^2}\int_0^z e^{t^2}\,dt &= \sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+1}}{(\frac{3}{2})_k} \\ &= z\,{}_1F_1\left(1;\frac{3}{2};-z^2\right), \qquad z \in \mathbb{C}. \end{aligned} \tag{13.1.10}$$

C-fractions. We obtain the regular C-fraction representation in z^2 [Wal48, p. 348] for the error function and Dawson's integral as a special case of (12.6.23):

$$\sqrt{\pi} z e^{z^2} \operatorname{erf}(z) = \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{c_m z^2}{1} \right), \qquad z \in \mathbb{C}, \tag{13.1.11a}$$

$$-2z e^{-z^2} \int_0^z e^{t^2}\, dt = \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{-c_m z^2}{1} \right), \qquad z \in \mathbb{C}, \tag{13.1.11b}$$

where the coefficients c_m are given by

$$c_1 = 2, \quad c_{2k} = \frac{-2(2k-1)}{(4k-3)(4k-1)}, \quad c_{2k+1} = \frac{4k}{(4k-1)(4k+1)}, \quad k \geq 1. \tag{13.1.11c}$$

Padé approximants. The series expansions (13.1.8) for the error function and (13.1.10) for Dawson's integral allows us to write down explicit formulas for the Padé approximants of these respective functions as a special case of (16.1.15). The Padé approximants $r_{m,n}(z^2)$ for $(\sqrt{\pi}/2) z^{-1} e^{z^2} \operatorname{erf}(z)$ are given by

$$r_{m,n}(z^2) = \frac{\mathcal{P}_{m+n}\left({}_1F_1(1; 3/2; z^2)\ {}_1F_1(-n; -m-n-1/2; -z^2) \right)}{{}_1F_1(-n; -m-n-1/2; -z^2)}, \qquad m \geq n-1. \tag{13.1.12}$$

When replacing z^2 by $-z^2$ in (13.1.12), we obtain the Padé approximants $r_{m,n}(-z^2)$ for $z^{-1} e^{-z^2} \int_0^z e^{t^2}\, dt$.

T-fractions. We obtain the general T-fraction representation in z^2 [JT80, p. 282; Dij77] for the error function and Dawson's integral as a special case of (12.6.30):

$$\sqrt{\pi} z e^{z^2} \operatorname{erf}(z) = \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{F_m z^2}{1 + G_m z^2} \right), \qquad z \in \mathbb{C}, \tag{13.1.13a}$$

$$-2z e^{-z^2} \int_0^z e^{t^2}\, dt = \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{-F_m z^2}{1 - G_m z^2} \right), \qquad z \in \mathbb{C}, \tag{13.1.13b}$$

where

$$\begin{aligned} F_1 &= 2, \quad F_m = \frac{4(m-1)}{(2m-3)(2m-1)}, \quad m \geq 2, \\ G_m &= \frac{-2}{2m-1}, \quad m \geq 1. \end{aligned} \tag{13.1.13c}$$

TABLE 13.1.1: The symmetry properties (13.1.3) reduce an investigation of approximations for the error function in the complex plane to the first quadrant. The following tables give the relative error of the 5th partial sums and 5th approximants.

x	$\mathrm{erf}(x)$	(13.1.7)	(13.1.8)	(13.1.11)	(13.1.13)
0.05	5.637198e−02	2.6e−20	1.2e−19	3.8e−17	1.2e−15
0.1	1.124629e−01	1.1e−16	4.7e−16	3.9e−14	1.3e−12
0.25	2.763264e−01	6.5e−12	2.7e−11	3.8e−10	1.3e−08
0.5	5.204999e−01	2.7e−08	1.0e−07	4.1e−07	1.7e−05
0.75	7.111556e−01	3.8e−06	1.1e−05	2.5e−05	1.5e−03
1	8.427008e−01	1.3e−04	2.7e−04	5.0e−04	4.9e−02
1.5	9.661051e−01	1.9e−02	1.6e−02	4.0e−02	1.1e+00
2	9.953223e−01	6.5e−01	1.6e−01	5.8e−01	1.0e+00
2.5	9.995930e−01	9.9e+00	5.1e−01	9.7e−01	1.0e+00
5	1.000000e+00	3.3e+04	1.0e+00	1.0e+00	1.0e+00
10	1.000000e+00	8.1e+07	1.0e+00	1.0e+00	1.0e+00
50	1.000000e+00	4.2e+15	1.0e+00	1.0e+00	1.0e+00

x	$\lvert\mathrm{erf}(x+\mathrm{i}x)\rvert_s$	(13.1.7)	(13.1.8)	(13.1.11)	(13.1.13)
0.05	7.978837e−02	1.7e−18	7.4e−18	1.2e−15	3.9e−14
0.1	1.595741e−01	6.8e−15	3.0e−14	1.3e−12	4.0e−11
0.25	3.986653e−01	4.1e−10	1.8e−09	1.2e−08	3.8e−07
0.5	7.890543e−01	1.7e−06	7.5e−06	1.2e−05	4.0e−04
0.75	1.130852e+00	2.3e−04	1.0e−03	7.0e−04	2.4e−02
1	1.329860e+00	8.0e−03	3.5e−02	1.2e−02	3.7e−01
1.5	9.115563e−01	2.1e+00	9.1e+00	6.6e−01	1.1e+00
2	1.158326e+00	5.5e+01	2.4e+02	2.2e+00	8.6e−01
2.5	8.820421e−01	9.6e+02	4.0e+03	1.3e+00	1.1e+00
5	9.311940e−01	2.0e+06	8.2e+06	3.9e+00	1.1e+00
10	9.617121e−01	4.0e+09	1.6e+10	9.1e+00	1.0e+00
50	9.935650e−01	1.9e+17	7.7e+17	4.3e+01	1.0e+00

x	$\lvert \mathrm{erf}(\mathrm{i}x)\rvert_s$	(13.1.7)	(13.1.8)	(13.1.11)	(13.1.13)
0.05	5.646601e−02	2.6e−20	1.2e−19	3.8e−17	1.2e−15
0.1	1.132152e−01	1.1e−16	4.8e−16	3.9e−14	1.2e−12
0.25	2.880836e−01	6.3e−12	2.9e−11	3.7e−10	1.1e−08
0.5	6.149521e−01	2.5e−08	1.3e−07	3.6e−07	8.6e−06
0.75	1.035757e+00	3.0e−06	2.0e−05	1.9e−05	3.2e−04
1	1.650426e+00	8.3e−05	7.8e−04	3.1e−04	3.0e−03
1.5	4.584733e+00	7.0e−03	1.6e−01	1.3e−02	2.4e−02
2	1.856480e+01	9.8e−02	8.3e+00	1.4e−01	2.7e−02
2.5	1.303958e+02	4.1e−01	1.7e+02	6.7e−01	1.0e−02
5	8.298274e+09	1.0e+00	1.2e+06	1.5e+01	1.2e−05
10	1.524307e+42	1.0e+00	5.8e+09	9.2e+01	1.2e−08
50	6.148182e+1083	1.0e+00	1.5e+18	2.7e+03	1.2e−15

TABLE 13.1.2: The symmetry properties (13.1.3) reduce an investigation of approximations for the error function in the complex plane to the first quadrant. The following tables give the relative error of the 20^{th} partial sums and 20^{th} approximants. Note that, despite the convergence of (13.1.7) throughout the entire complex plain, the value of n from where on the relative truncation error of the n^{th} partial sum of (13.1.7) decreases, depends on x.

x	$\mathrm{erf}(x)$	(13.1.7)	(13.1.8)	(13.1.11)	(13.1.13)
0.05	5.637198e−02	1.0e−76	8.4e−76	5.5e−77	5.8e−71
0.1	1.124629e−01	4.6e−64	3.7e−63	6.1e−65	6.5e−59
0.25	2.763264e−01	2.4e−47	1.8e−46	5.1e−49	5.7e−43
0.5	5.204999e−01	1.1e−34	7.2e−34	6.0e−37	8.1e−31
0.75	7.111556e−01	3.0e−27	1.5e−26	7.4e−30	1.3e−23
1	8.427008e−01	5.8e−22	1.9e−21	8.4e−25	2.3e−18
1.5	9.661051e−01	1.8e−14	1.9e−14	1.3e−17	1.2e−10
2	9.953223e−01	3.9e−09	8.2e−10	2.0e−12	8.7e−05
2.5	9.995930e−01	5.2e−05	1.4e−06	2.7e−08	8.9e−01
5	1.000000e+00	2.8e+08	7.8e−01	1.0e+00	1.0e+00
10	1.000000e+00	9.3e+20	1.0e+00	1.0e+00	1.0e+00
50	1.000000e+00	5.1e+49	1.0e+00	1.0e+00	1.0e+00

x	$\lvert \operatorname{erf}(x+\mathrm{i}x)\rvert_s$	(13.1.7)	(13.1.8)	(13.1.11)	(13.1.13)
0.05	7.978837e−02	2.2e−70	1.8e−69	5.8e−71	6.1e−65
0.1	1.595741e−01	9.5e−58	7.8e−57	6.4e−59	6.7e−53
0.25	3.986653e−01	4.9e−41	4.0e−40	5.3e−43	5.5e−37
0.5	7.890543e−01	2.2e−28	1.8e−27	5.8e−31	6.2e−25
0.75	1.130852e+00	5.7e−21	4.7e−20	6.7e−24	7.1e−18
1	1.329860e+00	1.1e−15	9.3e−15	7.3e−19	8.0e−13
1.5	9.115563e−01	6.1e−08	5.0e−07	1.4e−11	1.9e−05
2	1.158326e+00	1.1e−02	8.9e−02	8.6e−07	1.2e+00
2.5	8.820421e−01	2.0e+02	1.6e+03	3.1e−03	1.1e+00
5	9.311940e−01	7.6e+14	6.1e+15	1.2e+00	1.1e+00
10	9.617121e−01	1.7e+27	1.4e+28	8.9e−01	1.0e+00
50	9.935650e−01	7.7e+55	6.1e+56	9.7e−01	1.0e+00

x	$\lvert \operatorname{erf}(\mathrm{i}x)\rvert_s$	(13.1.7)	(13.1.8)	(13.1.11)	(13.1.13)
0.05	5.646601e−02	1.0e−76	8.5e−76	5.5e−77	5.8e−71
0.1	1.132152e−01	4.5e−64	3.7e−63	6.1e−65	6.3e−59
0.25	2.880836e−01	2.3e−47	2.0e−46	4.9e−49	4.9e−43
0.5	6.149521e−01	9.6e−35	9.9e−34	5.0e−37	4.1e−31
0.75	1.035757e+00	2.2e−27	2.9e−26	4.9e−30	3.0e−24
1	1.650426e+00	3.3e−22	6.6e−21	4.1e−25	1.6e−19
1.5	4.584733e+00	4.6e−15	2.9e−13	2.5e−18	2.7e−13
2	1.856480e+01	2.9e−10	9.2e−08	9.0e−14	1.5e−09
2.5	1.303958e+02	7.0e−07	1.7e−03	1.5e−10	2.1e−07
5	8.298274e+09	7.5e−01	1.9e+10	1.7e−03	3.1e−10
10	1.524307e+42	1.0e+00	1.3e+23	7.0e−01	2.4e−22
50	6.148182e+1083	1.0e+00	3.6e+52	5.2e+00	2.7e−50

TABLE 13.1.3: Dawson's integral satisfies the same symmetry properties (13.1.3) and therefore an investigation of approximations in the complex plane can be reduced to the first quadrant. The following tables give the relative error of the 5^{th} partial sums and 5^{th} approximants.

x	Dawson(x)	(13.1.9)	(13.1.10)	(13.1.11)	(13.1.13)
0.25	2.398392e−01	6.3e−12	2.9e−11	3.7e−10	1.1e−08
0.5	4.244364e−01	2.5e−08	1.3e−07	3.6e−07	8.6e−06
1	5.380795e−01	8.3e−05	7.8e−04	3.1e−04	3.0e−03
1.5	4.282491e−01	7.0e−03	1.6e−01	1.3e−02	2.4e−02
2	3.013404e−01	9.8e−02	8.3e+00	1.4e−01	2.7e−02
2.5	2.230837e−01	4.1e−01	1.7e+02	6.7e−01	1.0e−02
5	1.021341e−01	1.0e+00	1.2e+06	1.5e+01	1.2e−05
10	5.025385e−02	1.0e+00	5.8e+09	9.2e+01	1.2e−08

x	$\lvert$ Dawson$(x + \mathrm{i}x)\rvert_s$	(13.1.9)	(13.1.10)	(13.1.11)	(13.1.13)
0.25	3.533079e−01	4.1e−10	1.8e−09	1.2e−08	3.8e−07
0.5	6.992812e−01	1.7e−06	7.5e−06	1.2e−05	4.0e−04
1	1.178557e+00	8.0e−03	3.5e−02	1.2e−02	3.7e−01
1.5	−8.078457e−01	2.1e+00	9.1e+00	6.6e−01	1.1e+00
2	1.026540e+00	5.5e+01	2.4e+02	2.2e+00	8.6e−01
2.5	7.816894e−01	9.6e+02	4.0e+03	1.3e+00	1.1e+00
5	−8.252492e−01	2.0e+06	8.2e+06	3.9e+00	1.1e+00
10	−8.522952e−01	4.0e+09	1.6e+10	9.1e+00	1.0e+00

x	$\lvert$ Dawson$(\mathrm{i}x)\rvert_s$	(13.1.9)	(13.1.10)	(13.1.11)	(13.1.13)
0.25	2.606818e−01	6.5e−12	2.7e−11	3.8e−10	1.3e−08
0.5	5.922965e−01	2.7e−08	1.0e−07	4.1e−07	1.7e−05
1	2.030078e+00	1.3e−04	2.7e−04	5.0e−04	4.9e−02
1.5	8.123289e+00	1.9e−02	1.6e−02	4.0e−02	1.1e+00
2	4.816001e+01	6.5e−01	1.6e−01	5.8e−01	1.0e+00
2.5	4.588901e+02	9.9e+00	5.1e−01	9.7e−01	1.0e+00
5	6.381268e+10	3.3e+04	1.0e+00	1.0e+00	1.0e+00
10	2.382282e+43	8.1e+07	1.0e+00	1.0e+00	1.0e+00

Table 13.1.4: Dawson's integral satisfies the same symmetry properties (13.1.3) and therefore an investigation of approximations in the complex plane can be reduced to the first quadrant. The following tables give the relative error of the 20th partial sums and 20th approximants.

x	Dawson(x)	(13.1.9)	(13.1.10)	(13.1.11)	(13.1.13)
0.25	2.398392e−01	2.3e−47	2.0e−46	4.9e−49	4.9e−43
0.5	4.244364e−01	9.6e−35	9.9e−34	5.0e−37	4.1e−31
1	5.380795e−01	3.3e−22	6.6e−21	4.1e−25	1.6e−19
1.5	4.282491e−01	4.6e−15	2.9e−13	2.5e−18	2.7e−13
2	3.013404e−01	2.9e−10	9.2e−08	9.0e−14	1.5e−09
2.5	2.230837e−01	7.0e−07	1.7e−03	1.5e−10	2.1e−07
5	1.021341e−01	7.5e−01	1.9e+10	1.7e−03	3.1e−10
10	5.025385e−02	1.0e+00	1.3e+23	7.0e−01	2.4e−22

x	$\lvert$ Dawson$(x+\mathrm{i}x)\rvert_s$	(13.1.9)	(13.1.10)	(13.1.11)	(13.1.13)
0.25	3.533079e−01	4.9e−41	4.0e−40	5.3e−43	5.5e−37
0.5	6.992812e−01	2.2e−28	1.8e−27	5.8e−31	6.2e−25
1	1.178557e+00	1.1e−15	9.3e−15	7.3e−19	8.0e−13
1.5	−8.078457e−01	6.1e−08	5.0e−07	1.4e−11	1.9e−05
2	1.026540e+00	1.1e−02	8.9e−02	8.6e−07	1.2e+00
2.5	7.816894e−01	2.0e+02	1.6e+03	3.1e−03	1.1e+00
5	−8.252492e−01	7.6e+14	6.1e+15	1.2e+00	1.1e+00
10	−8.522952e−01	1.7e+27	1.4e+28	8.9e−01	1.0e+00

x	$\lvert$ Dawson$(\mathrm{i}x)\rvert_s$	(13.1.9)	(13.1.10)	(13.1.11)	(13.1.13)
0.25	2.606818e−01	2.4e−47	1.8e−46	5.1e−49	5.7e−43
0.5	5.922965e−01	1.1e−34	7.2e−34	6.0e−37	8.1e−31
1	2.030078e+00	5.8e−22	1.9e−21	8.4e−25	2.3e−18
1.5	8.123289e+00	1.8e−14	1.9e−14	1.3e−17	1.2e−10
2	4.816001e+01	3.9e−09	8.2e−10	2.0e−12	8.7e−05
2.5	4.588901e+02	5.2e−05	1.4e−06	2.7e−08	8.9e−01
5	6.381268e+10	2.8e+08	7.8e−01	1.0e+00	1.0e+00
10	2.382282e+43	9.3e+20	1.0e+00	1.0e+00	1.0e+00

13.2 Complementary and complex error function

Definitions and elementary properties. The *complementary error function* $\operatorname{erfc}(z)$ is defined by

$$\operatorname{erfc}(z) := \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt, \qquad z \in \mathbb{C}, \tag{13.2.1}$$

where the path of integration is subject to the restriction $\arg t \to \alpha$ with $|\alpha| < \pi/4$ as $t \to \infty$ along the path. The value $\alpha = \pi/4$ is allowed if there exists a constant $M > 0$ such that $\Re(t^2) > -M$ on the path of integration. These conditions are satisfied if the path of integration is the horizontal line $t = z + \tau$, $0 \le \tau < \infty$. The error function and the complementary error function are related by

$$\operatorname{erfc}(z) = 1 - \operatorname{erf}(z). \tag{13.2.2}$$

The *complex error function* $w(z)$ defined by

$$w(z) := e^{-z^2} \left(1 + \frac{2i}{\sqrt{\pi}} \int_0^z e^{t^2}\, dt\right), \qquad z \in \mathbb{C}, \tag{13.2.3}$$

is related to the complementary error function by

$$w(z) = e^{-z^2} \operatorname{erfc}(-iz). \tag{13.2.4}$$

We have the symmetry properties

$$\operatorname{erfc}(\bar{z}) = \overline{\operatorname{erfc}(z)} \tag{13.2.5}$$

and

$$w(-z) = 2e^{-z^2} - w(z), \tag{13.2.6a}$$

$$w(\bar{z}) = \overline{w(-z)}. \tag{13.2.6b}$$

The complementary and the complex error function are special cases of the complementary incomplete gamma function introduced in *Section* 12.6:

$$\operatorname{erfc}(z) = \frac{1}{\sqrt{\pi}} \Gamma\left(1/2, z^2\right), \tag{13.2.7}$$

$$w(z) = \frac{e^{-z^2}}{\sqrt{\pi}} \Gamma\left(1/2, -z^2\right). \tag{13.2.8}$$

The relationship with $\Gamma(a, z)$ is essential in deriving the representations for the complementary error function and for the function $w(z)$ given in this section.

Series expansion. From (13.2.4) and the series expansion (13.1.8) for the error function, we find

$$\operatorname{erfc}(z) = e^{-z^2} \sum_{k=0}^{\infty} \frac{(-z)^k}{\Gamma(\frac{k}{2}+1)}, \qquad z \in \mathbb{C}, \tag{13.2.9}$$

$$w(z) = \sum_{k=0}^{\infty} \frac{(\mathrm{i}z)^k}{\Gamma(\frac{k}{2}+1)}, \qquad z \in \mathbb{C}. \tag{13.2.10}$$

Asymptotic series expansion. From (13.2.7) and the asymptotic expansion (12.6.10) for $\Gamma(a, z)$ given in *Section* 12.6 we find [Hen77, p. 393]

$$\begin{aligned} \sqrt{\pi} z e^{z^2} \operatorname{erfc}(z) &\approx 1 + \sum_{k=1}^{\infty} (-1)^k \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{(2z^2)^k} \\ &= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} (-1)^k \Gamma\left(k + \frac{1}{2}\right) z^{-2k} \\ &= {}_2F_0\left(1, {}^1\!/_2; -z^{-2}\right), \qquad z \to \infty, \quad |\arg z| < 3\pi/4, \end{aligned} \tag{13.2.11}$$

$$-\mathrm{i}\sqrt{\pi} z w(z) \approx {}_2F_0\left(1, {}^1\!/_2; z^{-2}\right), \qquad z \to \infty, \quad |\arg(-\mathrm{i}z)| < 3\pi/4 \tag{13.2.12}$$

where ${}_2F_0(a, b; z)$ is the confluent hypergeometric series defined in (16.1.12).

Stieltjes transforms. Based on (13.2.7) and the Stieltjes transform (12.6.12) for $\Gamma(a, z)$ we find

$$\operatorname{erfc}(z) = \frac{z e^{-z^2}}{\pi} \int_0^{\infty} \frac{e^{-t}}{\sqrt{t}(z^2 + t)}\, dt, \qquad \Re z > 0, \tag{13.2.13}$$

$$w(z) = \frac{-\mathrm{i}z}{\pi} \int_0^{\infty} \frac{e^{-t}}{\sqrt{t}(-z^2 + t)}\, dt, \qquad \Im z > 0. \tag{13.2.14}$$

The weight function

$$\phi(t) = \frac{e^{-t}}{\sqrt{t}}, \qquad 0 < t < \infty, \tag{13.2.15}$$

is a solution to the Stieltjes moment problem for the sequence of moments μ_k given by

$$\mu_k = \int_0^\infty t^k \frac{e^{-t}}{\sqrt{t}}\, dt$$
$$= \Gamma\left(k + \frac{1}{2}\right), \qquad k = 0, 1, 2, \ldots. \tag{13.2.16}$$

The integral representation (13.2.14) can also be written as

$$w(z) = \frac{\mathrm{i}}{\pi} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{z+t}\, dt, \qquad \Im z > 0. \tag{13.2.17}$$

The weight function

$$\phi(t) = e^{-t^2}, \qquad -\infty < t < \infty, \tag{13.2.18}$$

is a solution to the Hamburger moment problem for the sequence of moments μ_k given by

$$\mu_k = \int_{-\infty}^{\infty} t^k e^{-t^2}\, dt$$

which are known explicitly by

$$\mu_{2j} = \Gamma\left(j + \frac{1}{2}\right), \quad \mu_{2j+1} = 0, \quad j \geq 0. \tag{13.2.19}$$

Using Stirling's approximation (12.2.4) for $\Gamma(z)$ we find

$$\sqrt[2j]{\mu_{2j}} \sim \sqrt{\frac{j}{e}}, \qquad j \to \infty.$$

This implies that Carleman's criterion (5.1.16b) is satisfied for the sequence $\{\mu_k\}_{k=0}^\infty$ and hence the classical Hamburger moment problem for $\{\mu_k\}_{k=0}^\infty$ is determinate. Its unique solution is given by $\phi(t)$.

S-fractions. Based on (13.2.7) and the modified S-fraction representation for the complementary incomplete gamma function (12.6.15), we find [Wal48, p. 356]

$$\operatorname{erfc}(z) = \frac{z}{\sqrt{\pi}} e^{-z^2} \left(\frac{a_1}{z^2} + \frac{a_2}{1} + \frac{a_3}{z^2} + \frac{a_4}{1} + \ldots \right), \qquad \Re z > 0, \tag{13.2.20a}$$

$$w(z) = -\frac{\mathrm{i}z}{\sqrt{\pi}} \left(\frac{a_1}{-z^2} + \frac{a_2}{1} + \frac{a_3}{-z^2} + \frac{a_4}{1} + \ldots \right), \qquad \Im z > 0, \tag{13.2.20b}$$

where the coefficients are given by

$$a_1 = 1, \quad a_m = \frac{m-1}{2}, \quad m \geq 2. \tag{13.2.20c}$$

With $a_m \to \infty$, the modification

$$\tilde{w}_n(z) = b_n \frac{-1+\sqrt{1+2nz^{-2}}}{2} \tag{13.2.21}$$

can be used, where for $\operatorname{erfc}(z)$ we have $b_{2k-1} = z^2$ and $b_{2k} = 1$ with $k \geq 1$ and for $w(z)$ we have $b_{2k-1} = -z^2$ and $b_{2k} = 1$ with $k \geq 1$.

EXAMPLE 13.2.1: For $z = 1/\sqrt{2}$, we find from (13.2.20) after an equivalence transformation

$$\operatorname{erfc}(1/\sqrt{2}) = \sqrt{\frac{2}{e\pi}} \left(\frac{1}{1} + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{m}{1} \right) \right).$$

Padé approximants. The asymptotic expansion (13.2.11) for the complementary error function allows us to obtain explicit formulas for the Padé approximants of $\operatorname{erfc}(z)$ as a special case of (16.2.5). The Padé approximants $r_{m,n}(z^2)$ at $z = \infty$ for $\sqrt{\pi} z e^{z^2} \operatorname{erfc}(z)$ are given by

$$r_{m,n}(z^2) = \frac{\mathcal{P}_{m+n}({}_2F_0(1, 1/2; -z^{-2})\, {}_2F_0(-m - 1/2, -n; z^{-2}))}{{}_2F_0(-m - 1/2, -n; z^{-2})}, \quad m \geq n-1. \tag{13.2.22}$$

When replacing z^2 by $-z^2$ in (13.2.22) we obtain the Padé approximants $r_{m,n}(-z^2)$ for the function $\mathrm{i}\sqrt{\pi} z w(z)$.

J-fractions. A real J-fraction representation for the complementary error function can be obtained by an even contraction of (13.2.20a):

$$\operatorname{erfc}(z) = \frac{e^{-z^2}}{\sqrt{\pi}} \left(\frac{2z}{1+2z^2} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-(2m-3)(2m-2)}{4m-3+2z^2} \right) \right), \qquad \Re z > 0. \tag{13.2.23a}$$

For the complex error function it is obtained by an even contraction of (13.2.20b):

$$w(z) = \frac{1}{\sqrt{\pi}} \frac{\mathrm{i}z}{z^2 - \frac{1}{2}} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-(m-\frac{3}{2})(m-1)}{2m - \frac{3}{2} - z^2} \right), \qquad \Im z > 0. \tag{13.2.23b}$$

TABLE 13.2.1: Combined with property (13.2.5), the following tables give the relative error of the 5th approximants and 5th partial sum in the right half-plane. The continued fraction approximants of (13.2.20) are first evaluated without modification and next with the modification (13.2.21).

x	$\operatorname{erfc}(x)$	(13.2.9)	(13.2.20)	(13.2.20)	(13.2.23)	(13.2.11)
0.01	9.887166e−01	1.7e−13	2.9e+01	4.6e−02	9.7e−01	1.7e+23
0.05	9.436280e−01	2.7e−09	5.4e+00	3.8e−02	8.2e−01	3.6e+15
0.25	7.236736e−01	4.7e−05	6.5e−01	1.4e−02	2.5e−01	8.9e+07
0.5	4.795001e−01	3.4e−03	1.5e−01	4.1e−03	4.2e−02	5.3e+04
0.75	2.888444e−01	4.2e−02	4.6e−02	1.3e−03	6.9e−03	6.9e+02
1	1.572992e−01	2.6e−01	1.5e−02	4.3e−04	1.2e−03	3.2e+01
1.5	3.389485e−02	3.3e+00	2.0e−03	5.6e−05	4.9e−05	4.1e−01
2	4.677735e−03	2.0e+01	3.4e−04	8.9e−06	2.6e−06	1.8e−02
2.5	4.069520e−04	8.2e+01	7.0e−05	1.7e−06	1.8e−07	1.5e−03
5	1.537460e−12	6.3e+03	2.2e−07	2.9e−09	5.9e−12	5.4e−07
10	2.088488e−45	4.6e+05	3.2e−10	1.4e−12	2.1e−17	1.5e−10
50	2.070921e−1088	8.1e+09	3.8e−17	7.6e−21	3.6e−31	6.6e−19

x	$\lvert \operatorname{erfc}(x+\mathrm{i}x)\rvert_s$	(13.2.9)	(13.2.20)	(13.2.20)	(13.2.23)	(13.2.11)
0.01	9.887798e−01	1.3e−12	2.1e+01	4.6e−02	9.7e−01	3.7e+21
0.05	9.451669e−01	2.1e−08	3.8e+00	3.8e−02	8.4e−01	8.0e+13
0.25	7.564018e−01	3.8e−04	4.9e−01	1.3e−02	2.9e−01	2.0e+06
0.5	5.808450e−01	2.8e−02	1.2e−01	3.6e−03	3.6e−02	1.3e+03
0.75	−4.568231e−01	3.6e−01	3.3e−02	9.9e−04	4.8e−03	1.8e+01
1	−3.690856e−01	2.2e+00	8.5e−03	2.8e−04	6.5e−04	9.0e−01
1.5	2.597263e−01	3.0e+01	6.7e−04	2.3e−05	1.3e−05	1.2e−02
2	−1.977325e−01	1.9e+02	6.9e−05	2.2e−06	3.3e−07	4.8e−04
2.5	1.589711e−01	7.6e+02	9.6e−06	2.7e−07	1.1e−08	3.7e−05
5	7.976858e−02	5.7e+04	1.2e−08	1.1e−10	3.2e−14	1.0e−08
10	3.989360e−02	3.9e+06	1.2e−11	2.9e−14	3.4e−20	2.5e−12
50	7.978845e−03	6.6e+10	1.2e−18	1.2e−22	3.6e−34	1.0e−20

x	$\lvert \operatorname{erfc}(1+\mathrm{i}x)\rvert_s$	(13.2.9)	(13.2.20)	(13.2.20)	(13.2.23)	(13.2.11)
0.01	1.573125e−01	2.6e−01	1.5e−02	4.3e−04	1.2e−03	3.2e+01
0.05	1.576311e−01	2.6e−01	1.5e−02	4.3e−04	1.2e−03	3.2e+01
0.25	1.658167e−01	3.1e−01	1.4e−02	4.2e−04	1.2e−03	2.3e+01
0.5	1.943285e−01	5.1e−01	1.3e−02	3.8e−04	1.1e−03	1.0e+01
0.75	−2.535415e−01	1.0e+00	1.1e−02	3.3e−04	8.6e−04	3.2e+00
1	−3.690856e−01	2.2e+00	8.5e−03	2.8e−04	6.5e−04	9.0e−01
1.5	−1.099560e+00	1.1e+01	4.2e−03	1.6e−04	2.9e−04	7.8e−02
2	5.277796e+00	4.3e+01	1.5e−03	7.1e−05	8.9e−05	9.0e−03
2.5	4.180092e+01	1.4e+02	4.2e−04	2.6e−05	1.9e−05	1.3e−03
5	2.985464e+09	8.3e+03	5.7e−07	1.9e−08	2.4e−10	6.7e−07
10	−5.578925e+41	5.3e+05	4.1e−10	2.4e−12	5.6e−17	1.6e−10
50	2.261337e+1083	8.3e+09	3.9e−17	7.7e−21	3.8e−31	6.7e−19

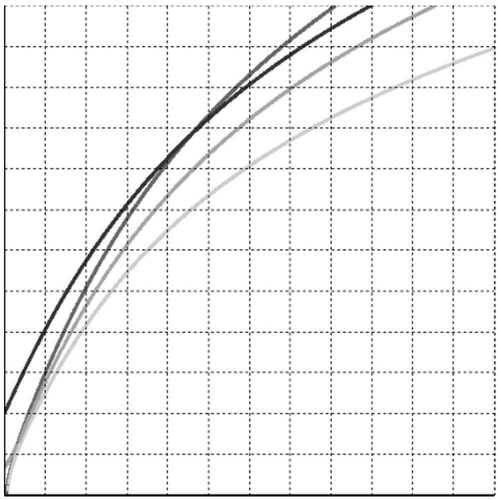
TABLE 13.2.2: Combined with property (13.2.5), the following tables give the relative error of the 20^{th} approximants and 20^{th} partial sum in the right half-plane. The continued fraction approximants of (13.2.20) are first evaluated without modification and next with the modification (13.2.21).

x	$\operatorname{erfc}(x)$	(13.2.9)	(13.2.20)	(13.2.20)	(13.2.23)	(13.2.11)
0.01	9.887166e−01	8.5e−50	9.5e−01	1.1e−02	9.2e−01	1.7e+99
0.05	9.436280e−01	4.2e−35	7.3e−01	7.1e−03	6.1e−01	4.0e+70
0.25	7.236736e−01	2.3e−20	1.1e−01	7.3e−04	3.0e−02	1.1e+42
0.5	4.795001e−01	5.7e−14	6.8e−03	4.6e−05	5.1e−04	6.1e+29
0.75	2.888444e−01	3.2e−10	4.6e−04	3.1e−06	9.3e−06	4.4e+22
1	1.572992e−01	1.5e−07	3.3e−05	2.3e−07	1.9e−07	3.8e+17
1.5	3.389485e−02	9.0e−04	2.2e−07	1.6e−09	9.8e−11	2.9e+10
2	4.677735e−03	4.3e−01	2.1e−09	1.6e−11	7.4e−14	2.5e+05
2.5	4.069520e−04	5.2e+01	2.8e−11	2.0e−13	7.9e−17	3.0e+01
5	1.537460e−12	1.4e+08	4.9e−19	3.0e−21	1.3e−29	1.5e−11
10	2.088488e−45	3.7e+14	3.4e−29	1.1e−31	7.2e−48	5.2e−24
50	2.070921e−1088	2.2e+29	2.3e−56	4.6e−60	6.5e−101	2.7e−53

x	$\lvert \operatorname{erfc}(x+\mathrm{i}x)\rvert_s$	(13.2.9)	(13.2.20)	(13.2.20)	(13.2.23)	(13.2.11)
0.01	9.887798$e-$01	1.2$e-$46	9.5$e-$01	1.1$e-$02	9.2$e-$01	1.2$e+$93
0.05	9.451669$e-$01	6.0$e-$32	7.7$e-$01	7.0$e-$03	6.7$e-$01	2.7$e+$64
0.25	7.564018$e-$01	3.4$e-$17	1.1$e-$01	7.0$e-$04	2.9$e-$02	7.4$e+$35
0.5	5.808450$e-$01	8.6$e-$11	5.8$e-$03	3.9$e-$05	4.3$e-$04	4.4$e+$23
0.75	$-$4.568231$e-$01	5.1$e-$07	3.1$e-$04	2.2$e-$06	6.2$e-$06	3.4$e+$16
1	$-$3.690856$e-$01	2.5$e-$04	1.6$e-$05	1.2$e-$07	8.9$e-$08	3.1$e+$11
1.5	2.597263$e-$01	1.5$e+$00	4.9$e-$08	3.9$e-$10	1.9$e-$11	2.6$e+$04
2	$-$1.977325$e-$01	7.5$e+$02	1.7$e-$10	1.4$e-$12	4.2$e-$15	2.5$e-$01
2.5	1.589711$e-$01	9.1$e+$04	7.3$e-$13	6.4$e-$15	1.1$e-$18	3.0$e-$05
5	7.976858$e-$02	2.5$e+$11	1.1$e-$22	8.1$e-$25	8.1$e-$35	1.2$e-$17
10	3.989360$e-$02	6.0$e+$17	2.1$e-$34	5.1$e-$37	4.5$e-$57	3.0$e-$30
50	7.978845$e-$03	3.3$e+$32	2.4$e-$62	2.4$e-$66	8.2$e-$113	1.3$e-$59

x	$\lvert \operatorname{erfc}(1+\mathrm{i}x)\rvert_s$	(13.2.9)	(13.2.20)	(13.2.20)	(13.2.23)	(13.2.11)
0.01	1.573125$e-$01	1.5$e-$07	3.3$e-$05	2.3$e-$07	1.9$e-$07	3.8$e+$17
0.05	1.576311$e-$01	1.6$e-$07	3.3$e-$05	2.3$e-$07	1.9$e-$07	3.6$e+$17
0.25	1.658167$e-$01	2.9$e-$07	3.2$e-$05	2.2$e-$07	1.8$e-$07	1.1$e+$17
0.5	1.943285$e-$01	1.6$e-$06	2.8$e-$05	2.0$e-$07	1.6$e-$07	4.1$e+$15
0.75	$-$2.535415$e-$01	1.8$e-$05	2.2$e-$05	1.6$e-$07	1.2$e-$07	4.5$e+$13
1	$-$3.690856$e-$01	2.5$e-$04	1.6$e-$05	1.2$e-$07	8.9$e-$08	3.1$e+$11
1.5	$-$1.099560$e+$00	4.6$e-$02	6.7$e-$06	5.1$e-$08	3.4$e-$08	1.8$e+$07
2	5.277796$e+$00	4.9$e+$00	1.9$e-$06	1.5$e-$08	8.7$e-$09	3.5$e+$03
2.5	4.180092$e+$01	2.8$e+$02	3.4$e-$07	3.0$e-$09	1.4$e-$09	2.3$e+$00
5	2.985464$e+$09	2.7$e+$08	1.8$e-$13	4.2$e-$15	2.0$e-$16	2.6$e-$11
10	$-$5.578925$e+$41	5.0$e+$14	1.8$e-$27	1.7$e-$29	1.3$e-$40	6.4$e-$24
50	2.261337$e+$1083	2.3$e+$29	2.8$e-$56	5.6$e-$60	1.2$e-$100	2.8$e-$53

FIGURE 13.2.1: Number of significant digits (between 0 and 12) guaranteed by the n^{th} classical approximant of (13.2.20) (from light to dark $n = 5, 6, 7$) and the 5^{th} modified approximant with $w_5(z)$ given by (13.2.21) (darkest). On the horizontal axis we have z real, $0 \le z \le 12$.



13.3 Repeated integrals

Definition and elementary properties. The *repeated integrals of the complementary error function* are defined recursively as follows

$$I^{-1}\,\mathrm{erfc}(z) := \frac{2}{\sqrt{\pi}} e^{-z^2}, \tag{13.3.1a}$$

$$I^{0}\,\mathrm{erfc}(z) := \mathrm{erfc}(z), \tag{13.3.1b}$$

$$I^{k}\,\mathrm{erfc}(z) := \int_z^\infty I^{k-1}\,\mathrm{erfc}(t)\,dt, \qquad k = 1, 2, \ldots. \tag{13.3.1c}$$

An explicit formula for $I^k\,\mathrm{erfc}(z)$ for $k \ge 1$ is

$$I^k\,\mathrm{erfc}(z) = \frac{2}{k!\,\sqrt{\pi}} \int_z^\infty (t-z)^k e^{-t^2}\,dt, \qquad z \in \mathbb{C}.$$

Asymptotic series expansion. We have [AS64, p. 300]

$$I^k\,\mathrm{erfc}(z) \approx \frac{2}{\sqrt{\pi}} \frac{e^{-z^2}}{(2z)^{k+1}} \sum_{m=0}^\infty \frac{(-1)^m (2m+k)!}{k!\,m!\,(2z)^{2m}},$$
$$z \to \infty, \quad |\arg z| < 3\pi/4. \tag{13.3.2}$$

S-fraction. The sequence

$$e^{z^2} I^k \operatorname{erfc}(z), \qquad k = -1, 0, 1, 2, \ldots, \tag{13.3.3}$$

is a minimal solution of the system of three-term recurrence relations [Gau67]

$$y_k = -\frac{z}{k} y_{k-1} + \frac{1}{2k} y_{k-2}, \qquad k = 1, 2, \ldots. \tag{13.3.4}$$

Hence by Pincherle's *Theorem* 3.6.1 we obtain the modified S-fraction

$$\frac{I^k \operatorname{erfc}(z)}{I^{k-1} \operatorname{erfc}(z)} = \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m^{(k)}}{z} \right), \qquad \Re z > 0, \quad k \geq 0, \tag{13.3.5a}$$

where the coefficients are given by

$$a_1^{(k)} = \frac{1}{2}, \qquad a_m^{(k)} = \frac{k + m - 1}{2}, \quad m \geq 2, \quad k \geq 0. \tag{13.3.5b}$$

13.4 Fresnel integrals

Definitions and elementary properties. The *Fresnel cosine* and *sine integral functions* are defined by

$$C(z) := \int_0^z \cos\left(\frac{\pi}{2} t^2\right) dt, \qquad z \in \mathbb{C}, \tag{13.4.1a}$$

$$S(z) := \int_0^z \sin\left(\frac{\pi}{2} t^2\right) dt, \qquad z \in \mathbb{C}, \tag{13.4.1b}$$

where the path of integration is the straight line segment from 0 to z.
We have the symmetry properties

$$C(-z) = -C(z), \quad S(-z) = -S(z), \tag{13.4.2a}$$

$$C(\mathrm{i}z) = \mathrm{i}C(z), \quad S(\mathrm{i}z) = -\mathrm{i}S(z), \tag{13.4.2b}$$

$$C(\bar{z}) = \overline{C(z)}, \quad S(\bar{z}) = \overline{S(z)}. \tag{13.4.2c}$$

Also

$$\lim_{x \to +\infty} C(x) = \lim_{x \to +\infty} S(x) = \frac{1}{2}. \tag{13.4.3}$$

The Fresnel integral functions $C(z)$ and $S(z)$ are related to the error function by

$$C(z) + \mathrm{i}S(z) = \frac{1 + \mathrm{i}}{2} \operatorname{erf}\left(\frac{\sqrt{\pi}}{2} (1 - \mathrm{i}) z \right), \tag{13.4.4}$$

$$C(z) - \mathrm{i}S(z) = \frac{1 - \mathrm{i}}{2} \operatorname{erf}\left(\frac{\sqrt{\pi}}{2} (1 + \mathrm{i}) z \right). \tag{13.4.5}$$

The relationship with the error function, in particular the connection between the series representations (13.1.8) and (13.4.8), is crucial in deriving all the representations for $C(z) + \mathrm{i}S(z)$ given in this section. Likewise representations for $C(z) - \mathrm{i}S(z)$ can be derived and hence the Fresnel integral functions can also be represented as sums of continued fractions [LS62].

Series expansions. We have

$$C(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (\pi/2)^{2k}}{(2k)!\,(4k+1)} z^{4k+1}, \qquad z \in \mathbb{C}, \tag{13.4.6a}$$

$$S(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (\pi/2)^{2k+1}}{(2k+1)!\,(4k+3)} z^{4k+3}, \qquad z \in \mathbb{C}, \tag{13.4.6b}$$

and

$$C(z) + \mathrm{i}S(z) = z\ {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; \mathrm{i}\frac{\pi}{2}z^2\right), \tag{13.4.7}$$

or alternatively

$$C(z) + \mathrm{i}S(z) = ze^{\mathrm{i}\pi z^2/2}\ {}_1F_1\left(1; \frac{3}{2}; -\mathrm{i}\frac{\pi}{2}z^2\right). \tag{13.4.8}$$

C-fractions. From (13.1.11) we obtain the modified regular C-fraction

$$C(z) + \mathrm{i}S(z) = \frac{e^{\mathrm{i}\pi z^2/2}}{z} \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{c_m z^2}{1}\right), \qquad z \in \mathbb{C}, \tag{13.4.9a}$$

where

$$\begin{aligned} c_1 = 1, \quad c_{2k} &= \frac{\mathrm{i}\pi(2k-1)}{(4k-3)(4k-1)}, \quad k \geq 1, \\ c_{2k+1} &= \frac{-\mathrm{i}\pi 2k}{(4k-1)(4k+1)}, \quad k \geq 1. \end{aligned} \tag{13.4.9b}$$

T-fractions. From (13.1.13) we find

$$C(z) + \mathrm{i}S(z) = \frac{e^{\mathrm{i}z^2\pi/2}}{z} \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{F_m z^2}{1 + G_m z^2}\right), \qquad z \in \mathbb{C}, \tag{13.4.10a}$$

where

$$\begin{aligned} F_1 = 1, \quad F_m &= \frac{-2\mathrm{i}\pi(m-1)}{(2m-3)(2m-1)}, \quad m \geq 2, \\ G_m &= \frac{\mathrm{i}\pi}{2m-1}, \quad m \geq 1. \end{aligned} \tag{13.4.10b}$$

TABLE 13.4.1: Because of the symmetry properties (13.4.2) it is sufficient to study the approximations in the first quadrant of the complex plane (excluding the imaginary axis). The following tables give the relative error of the 5th partial sum and 5th approximants.

x	$C(x)$	(13.4.6)	(13.4.9)	(13.4.10)
0.05	4.999992e−02	1.1e−39	1.3e−19	4.7e−17
0.1	9.999753e−02	1.9e−32	5.5e−16	1.9e−13
0.2	1.999211e−01	3.2e−25	2.3e−12	8.0e−10
0.5	4.923442e−01	1.1e−15	1.4e−07	4.7e−05
1	7.798934e−01	2.4e−08	6.5e−04	1.4e−01
2	4.882534e−01	1.1e+00	2.4e+00	1.0e+00
4	4.984260e−01	8.9e+06	2.6e+00	1.0e+00

x	$\|C(x+\mathrm{i}x)\|_s$	(13.4.6)	(13.4.9)	(13.4.10)
0.05	7.071111e−02	4.6e−36	7.1e−01	7.1e−01
0.1	1.414353e−01	7.7e−29	7.1e−01	7.1e−01
0.2	2.832897e−01	1.3e−21	7.1e−01	7.1e−01
0.5	7.519882e−01	4.3e−12	7.3e−01	7.3e−01
1	3.614438e+00	3.2e−05	9.3e−01	1.0e+00
2	1.687705e+04	5.4e−01	1.0e+00	1.0e+00
4	1.921763e+20	1.0e+00	1.0e+00	1.0e+00

TABLE 13.4.2: Because of the symmetry properties (13.4.2) it is sufficient to study the approximations in the first quadrant of the complex plane (excluding the imaginary axis). The following tables give the relative error of the 20th partial sum and 20th approximants.

x	$C(x)$	(13.4.6)	(13.4.9)	(13.4.10)
0.05	4.999992e−02	7.5e−155	4.6e−73	4.9e−67
0.1	9.999753e−02	1.4e−129	5.1e−61	5.3e−55
0.2	1.999211e−01	2.8e−104	5.6e−49	5.9e−43
0.5	4.923442e−01	7.6e−71	4.7e−33	4.6e−27
1	7.798934e−01	1.9e−45	6.3e−21	2.5e−17
2	4.882534e−01	1.1e−19	1.4e−08	2.4e−02
4	4.984260e−01	3.3e+06	1.6e+00	1.0e+00

x	$\lvert C(x+\mathrm{i}x)\rvert_s$	(13.4.6)	(13.4.9)	(13.4.10)
0.05	7.071111e−02	3.3e−142	7.1e−01	7.1e−01
0.1	1.414353e−01	6.4e−117	7.1e−01	7.1e−01
0.2	2.832897e−01	1.2e−91	7.1e−01	7.1e−01
0.5	7.519882e−01	3.1e−58	7.3e−01	7.3e−01
1	3.614438e+00	2.5e−33	9.1e−01	9.1e−01
2	1.687705e+04	2.2e−11	1.0e+00	1.0e+00
4	1.921763e+20	8.8e−01	1.0e+00	1.0e+00

Table 13.4.3: Because of the symmetry properties (13.4.2) it is sufficient to study the approximations in the first quadrant of the complex plane (excluding the imaginary axis). The following tables give the relative error of the 5^{th} partial sum and 5^{th} approximants.

x	$S(x)$	(13.4.6)	(13.4.9)	(13.4.10)
0.05	6.544977e−05	2.4e−40	2.8e−13	8.9e−12
0.1	5.235895e−04	4.0e−33	7.1e−11	2.3e−09
0.2	4.187609e−03	6.8e−26	1.8e−08	5.8e−07
0.5	6.473243e−02	2.4e−16	2.8e−05	8.3e−04
1	4.382591e−01	4.8e−09	7.4e−03	1.8e−02
2	3.434157e−01	7.0e−01	7.6e−01	1.5e+00
4	4.205158e−01	2.1e+07	1.6e+00	1.2e+00

x	$\lvert S(x+\mathrm{i}x)\rvert_s$	(13.4.6)	(13.4.9)	(13.4.10)
0.05	−1.851209e−04	9.8e−37	2.7e+02	2.7e+02
0.1	−1.481065e−03	1.6e−29	6.8e+01	6.8e+01
0.2	−1.186106e−02	2.8e−22	1.7e+01	1.7e+01
0.5	−1.934385e−01	9.4e−13	2.8e+00	2.9e+00
1	−2.915950e+00	8.7e−06	1.1e+00	1.0e+00
2	−1.687634e+04	4.3e−01	1.0e+00	1.0e+00
4	−1.921763e+20	1.0e+00	1.0e+00	1.0e+00

TABLE 13.4.4: Because of the symmetry properties (13.4.2) it is sufficient to study the approximations in the first quadrant of the complex plane (excluding the imaginary axis). The following tables give the relative error of the 20$^{\text{th}}$ partial sum and 20$^{\text{th}}$ approximants.

x	$S(x)$	(13.4.6)	(13.4.9)	(13.4.10)
0.05	$6.544977e{-05}$	$5.1e{-156}$	$3.4e{-74}$	$1.5e{-66}$
0.1	$5.235895e{-04}$	$9.9e{-131}$	$3.7e{-62}$	$1.6e{-54}$
0.2	$4.187609e{-03}$	$1.9e{-105}$	$4.1e{-50}$	$1.8e{-42}$
0.5	$6.473243e{-02}$	$5.1e{-72}$	$3.4e{-34}$	$1.4e{-26}$
1	$4.382591e{-01}$	$1.2e{-46}$	$4.3e{-22}$	$1.2e{-14}$
2	$3.434157e{-01}$	$2.3e{-20}$	$3.1e{-09}$	$1.9e{-06}$
4	$4.205158e{-01}$	$2.2e{+06}$	$2.2e{+00}$	$1.2e{+00}$

x	$\lvert S(x+\mathrm{i}x)\rvert_s$	(13.4.6)	(13.4.9)	(13.4.10)
0.05	$-1.851209e{-04}$	$2.2e{-143}$	$2.7e{+02}$	$2.7e{+02}$
0.1	$-1.481065e{-03}$	$4.3e{-118}$	$6.8e{+01}$	$6.8e{+01}$
0.2	$-1.186106e{-02}$	$8.4e{-93}$	$1.7e{+01}$	$1.7e{+01}$
0.5	$-1.934385e{-01}$	$2.1e{-59}$	$2.8e{+00}$	$2.8e{+00}$
1	$-2.915950e{+00}$	$2.2e{-34}$	$1.1e{+00}$	$1.1e{+00}$
2	$-1.687634e{+04}$	$6.4e{-12}$	$1.0e{+00}$	$1.0e{+00}$
4	$-1.921763e{+20}$	$8.5e{-01}$	$1.0e{+00}$	$1.0e{+00}$

14

Exponential integrals and related functions

The exponential integrals $E_n(z)$ and $\mathrm{Ei}(z)$, and the logarithmic, sine and cosine integral form another family of special hypergeometric functions. They are closely related to the complementary incomplete gamma functions $\Gamma(1-n, z)$ and $\Gamma(0, z)$ and hence to the confluent hypergeometric functions. The analytic continuation $E_\nu(z)$ for complex ν is an entire function of ν for fixed z.

14.1 Exponential integrals

Definitions and representations. The *exponential integrals* $E_n(z)$ are defined by

$$E_n(z) := \int_1^\infty \frac{e^{-zt}}{t^n}\,dt, \qquad \Re z > 0, \quad n \in \mathbb{N}. \tag{14.1.1}$$

Analytic continuation of $E_n(z)$ to the cut plane $|\arg z| < \pi$ extends the definition and yields a single-valued function. The functions $E_n(z)$ are related to the complementary incomplete gamma function $\Gamma(a, z)$ introduced in (12.6.4) by

$$E_n(z) = z^{n-1}\Gamma(1-n, z), \qquad |\arg z| < \pi, \quad n \in \mathbb{N}, \tag{14.1.2}$$

$$= z^{n-1}\int_z^\infty \frac{e^{-t}}{t^n}\,dt. \tag{14.1.3}$$

The relationship with the complementary incomplete gamma function is crucial in deriving all representations for the exponential integrals and related functions. Since $\Gamma(a, z)$ is defined for all $a \in \mathbb{C}$, the exponential integrals $E_n(z)$ can be continued analytically by

$$E_\nu(z) = z^{\nu-1}\Gamma(1-\nu, z), \qquad |\arg z| < \pi, \quad \nu \in \mathbb{C}. \tag{14.1.4}$$

Combining (12.6.27) and (14.1.4) gives

$$E_\nu(z) = z^{\nu-1}\Gamma(1-\nu) - z^{\nu-1}\gamma(1-\nu, z), \qquad z \in \mathbb{C}\backslash\{0\}, \quad \nu \in \mathbb{C}\backslash\mathbb{N}. \tag{14.1.5}$$

Here $\gamma(1-\nu, z)$ denotes the analytic continuation of the incomplete gamma function to $\nu \in \mathbb{C}\backslash\mathbb{N}$ for $z \in \mathbb{C}\backslash\{0\}$. For (14.1.1) and the analytic continuation (14.1.4) we have

$$E_n(\bar{z}) = \overline{E_n(z)}, \qquad n \in \mathbb{N}, \tag{14.1.6}$$

$$E_{\bar{\nu}}(\bar{z}) = \overline{E_\nu(z)}, \qquad \nu \in \mathbb{C}. \tag{14.1.7}$$

Recurrence relations. The functions $E_n(z)$ satisfy the recurrence relation

$$E_{n+1}(z) = \frac{e^{-z}}{n} - \frac{z}{n}E_n(z), \qquad n \in \mathbb{N}, \tag{14.1.8}$$

and the relation

$$E_n(z) = e^{-z}\sum_{k=0}^{r-1}\frac{(-1)^k (n)_k}{z^{k+1}} + \frac{(-1)^r (n)_r}{z^r}E_{n+r}(z), \qquad n \in \mathbb{N}, \quad r \in \mathbb{N}, \tag{14.1.9}$$

where $(n)_r$ denotes the Pochhammer symbol introduced in (12.1.7).

Series expansions. The exponential integrals have the series expansion

$$E_1(z) = -\gamma - \mathrm{Ln}(z) - \sum_{k=1}^{\infty}\frac{(-1)^k z^k}{(k!)k}, \qquad |\arg z| < \pi, \tag{14.1.10}$$

$$E_n(z) = \frac{(-1)^{n-1}z^{n-1}}{(n-1)!}\left(-\gamma - \mathrm{Ln}(z) + \sum_{k=1}^{n-1}\frac{1}{k}\right) - \sum_{\substack{k=0\\k\neq n-1}}^{\infty}\frac{(-1)^k z^k}{k!\,(k-n+1)}, \qquad |\arg z| < \pi, \quad n \in \mathbb{N}, \tag{14.1.11}$$

where γ is Euler's constant (10.8.1). From (14.1.5) and the series representation (12.6.7) for $\gamma(a, z)$ we find

$$E_\nu(z) = \Gamma(1-\nu)z^{\nu-1} - \sum_{k=0}^{\infty}\frac{(-1)^k z^k}{(1-\nu+k)k!}, \qquad z \in \mathbb{C}\setminus\{0\}, \quad \nu \in \mathbb{C}\setminus\mathbb{N}. \tag{14.1.12}$$

Asymptotic series expansion. From the relation (14.1.4) and the asymptotic expansion (12.6.10) we find an asymptotic representation for

$E_\nu(z)$:

$$E_\nu(z) \approx e^{-z} z^{-1} \sum_{k=0}^{\infty} (-1)^k (\nu)_k z^{-k}, \qquad z \to \infty, \quad |\arg z| < \pi, \quad \nu \in \mathbb{C},$$
$$= z^{-1} e^{-z} \, {}_2F_0(1, \nu; -z^{-1}) \tag{14.1.13}$$

where ${}_2F_0(a, b; z)$ is the confluent hypergeometric series (16.1.12).

Stieltjes transform. From (12.6.12) we obtain the Stieltjes transform [Hen77, p. 622]

$$e^z E_n(z) = \int_0^\infty \frac{\phi_n(t)}{z+t} \, dt, \qquad |\arg z| < \pi, \quad n \in \mathbb{N}, \tag{14.1.14a}$$

where the weight function $\phi_n(t)$ is given by

$$\phi_n(t) = \frac{e^{-t} t^{n-1}}{(n-1)!}, \qquad 0 < t < \infty, \quad n \in \mathbb{N}. \tag{14.1.14b}$$

The weight function (14.1.14b) is the solution to the Stieltjes moment problem for the sequence of moments $\mu_k(n)$ given by

$$\mu_k(n) = \frac{1}{(n-1)!} \int_0^\infty e^{-t} t^{k+n-1} \, dt = \frac{(k+n-1)!}{(n-1)!} = (n)_k,$$
$$k \in \mathbb{N}_0, \quad n \in \mathbb{N}. \tag{14.1.15}$$

The formulas (14.1.14) and (14.1.15) also hold when $n \in \mathbb{R}^+$.

S-fractions. Based on the S-fraction representation for the incomplete gamma function (12.6.15), we get the modified S-fraction representation [Sti95, p. 721]

$$e^z E_n(z) = \frac{1}{z} \genfrac{}{}{0pt}{}{}{+} \frac{n}{1} \genfrac{}{}{0pt}{}{}{+} \frac{1}{z} \genfrac{}{}{0pt}{}{}{+} \frac{n+1}{1} \genfrac{}{}{0pt}{}{}{+} \frac{2}{z} \genfrac{}{}{0pt}{}{}{+} \frac{n+2}{1} \genfrac{}{}{0pt}{}{}{+} \dots, \qquad |\arg z| < \pi, \quad n \in \mathbb{N},$$
$$= \frac{1/z}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{a_m/z}{1} \right), \qquad |\arg z| < \pi, \quad n \in \mathbb{N},$$
$$a_{2j} = j + n - 1, \quad a_{2j+1} = j, \quad j \geq 1. \tag{14.1.16}$$

Since $\lim_{m\to\infty} a_m = +\infty$, the modification

$$w_{2k}(z) = \frac{-1 + \sqrt{4kz^{-1} + 1}}{2}, \qquad w_{2k+1}(z) = \frac{-1 + \sqrt{4(n+k)z^{-1} + 1}}{2} \tag{14.1.17}$$

can be useful in the evaluation of the approximants of (14.1.16). In the *Tables* 14.1.3 and 14.1.4 the approximants of (14.1.16) are first evaluated without modification and afterwards with the use of (14.1.17).
Combining (14.1.9) and (14.1.16) yields

$$e^z E_n(z) = z^{-1} \sum_{k=0}^{r-1} (-1)^k (n)_k z^{-k} + (-1)^r (n)_r z^{-r} \left(\frac{1}{z} \genfrac{}{}{0pt}{}{}{+} \frac{n+r}{1} \genfrac{}{}{0pt}{}{}{+} \frac{1}{z} \genfrac{}{}{0pt}{}{}{+} \frac{n+r+1}{1} \genfrac{}{}{0pt}{}{}{+} \frac{2}{z} \genfrac{}{}{0pt}{}{}{+} \dots \right), \quad |\arg z| < \pi, \quad n \in \mathbb{N}, \quad r \in \mathbb{N}. \tag{14.1.18}$$

The continued fraction representations (14.1.16) and (14.1.18) also hold when $n \in \mathbb{R}^+$.

C-fractions. From (14.1.4) and the C-fraction representation (12.6.17) for $\Gamma(a, z)$ we obtain [Gau13; Wal48, p. 348]

$$E_\nu(z) = e^{-z} \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m(\nu) z^{-1}}{1} \right), \qquad |\arg z| < \pi, \quad \nu \in \mathbb{C}, \tag{14.1.19a}$$

where the coefficients are given by

$$a_1(\nu) = 1, \quad a_{2j}(\nu) = j + \nu - 1, \quad a_{2j+1}(\nu) = j, \quad j \in \mathbb{N}. \tag{14.1.19b}$$

The same modification (14.1.17) applies with n replaced by ν.
Regular C-fractions can also be obtained for $E_\nu(z)$ by substituting the C-fraction (12.6.23) for the incomplete gamma function into (14.1.5). An additional equivalence transformation leads to

$$E_\nu(z) = z^{\nu-1}\Gamma(1-\nu) - e^{-z} z^{-1} \mathop{\mathrm{K}}_{k=1}^{\infty} \left(\frac{a_k(\nu) z}{k - \nu} \right), \qquad z \in \mathbb{C}\backslash\{0\}, \quad \nu \in \mathbb{C}\backslash\mathbb{N}, \tag{14.1.20a}$$

where

$$a_1(\nu) = 1, \quad a_{2j}(\nu) = \nu - j, \quad a_{2j+1}(\nu) = j, \quad j \in \mathbb{N}. \tag{14.1.20b}$$

In a similar way as for the complementary incomplete gamma function, other regular C-fraction representations can be obtained by setting $a = 1 - \nu$ in (12.6.21), (12.6.22), (12.6.25) and (12.6.26).

Padé approximants. By replacing a by $1 - \nu$ in (12.6.28) we obtain explicit formulas for the Padé approximants of $e^z E_\nu(z)$ at $z = \infty$:

$$r_{m+1,n}(z) = \frac{z^{-1} \mathcal{P}_{m+n}\left({}_2F_0(\nu, 1; -z^{-1}) \; {}_2F_0(-\nu - m, -n; z^{-1}) \right)}{{}_2F_0(-\nu - m, -n; z^{-1})}, \quad m + 1 \geq n. \tag{14.1.21}$$

The operator $\mathcal{P}_k$ is defined in (15.4.1). We recall that the $(2n)^{\text{th}}$ approximant of the continued fraction in (14.1.19) equals $r_{n,n}(z)$ and its $(2n+1)^{\text{th}}$ approximant equals $r_{n+1,n}(z)$.

M-fractions. Using (14.1.5), we obtain M-fractions for $E_\nu(z)$ in the same way as C-fractions are obtained. From (12.6.30) we find

$$E_\nu(z) = z^{\nu-1}\Gamma(1-\nu) - e^{-z}\left(\frac{1}{1-\nu-z} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{(m-1)z}{m-\nu-z}\right)\right), \qquad z \in \mathbb{C}\backslash\{0\}, \quad \nu \in \mathbb{C}\setminus\mathbb{N}. \quad (14.1.22)$$

As for the regular C-fractions, other M-fraction representations can be obtained by setting $a = 1-\nu$ in (12.6.32) and (12.6.33).

J-fractions. A J-fraction representation can be obtained by taking the even part of the C-fraction (14.1.19), or equivalently, by using the relation (14.1.2) and the J-fraction representation for the incomplete gamma function (12.6.34). It is given by [Gau73]

$$e^z E_\nu(z) = \frac{1}{\nu+z} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{(1-m)(\nu+m-2)}{\nu+2m-2+z}\right), \qquad |\arg z| < \pi, \quad \nu \in \mathbb{C}. \quad (14.1.23)$$

Another J-fraction representation can be obtained by taking the odd part of the C-fraction (14.1.19). The resulting J-fraction is a special case of (12.6.35) and is given by [Gau73]

$$e^z E_\nu(z) = \frac{1}{z}\left(1 - \frac{\nu}{\nu+1+z} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{(1-m)(\nu+m-1)}{\nu+2m-1+z}\right)\right), \qquad |\arg z| < \pi, \quad \nu \in \mathbb{C}. \quad (14.1.24)$$

For $n \in \mathbb{R}^+$ the fractions (14.1.23) and (14.1.24) are real J-fractions.

TABLE 14.1.1: Together with the symmetry property (14.1.6) the following tables let us investigate the relative error of the 5^{th} partial sum and the 5^{th} approximants throughout the cut complex plane for $n = 3$. The fraction (14.1.16) is first evaluated without modification and afterwards with.

x	$E_n(x)$	(14.1.11)	(14.1.16)	(14.1.16)	(14.1.23)	(14.1.13)
0.1	4.162915e−01	9.8e−14	1.8e+00	2.5e−01	2.9e−02	5.4e+09
0.5	2.216044e−01	7.0e−08	2.5e−01	4.3e−02	7.4e−03	4.1e+05
1	1.096920e−01	3.4e−05	8.2e−02	1.4e−02	2.1e−03	7.4e+03
2.5	1.629537e−02	3.1e−01	1.2e−02	1.8e−03	1.5e−04	3.9e+01
5	8.778009e−04	1.2e+03	1.9e−03	2.3e−04	7.8e−06	7.4e−01
15	1.714014e−08	2.5e+11	4.2e−05	3.1e−06	1.1e−08	1.3e−03
50	3.642909e−24	7.2e+30	2.4e−07	7.7e−09	7.5e−13	1.2e−06

x	$\|E_n(x+\mathrm{i}x)\|_s$	(14.1.11)	(14.1.16)	(14.1.16)	(14.1.23)	(14.1.13)
0.1	4.136467e−01	1.6e−12	1.3e+00	2.0e−01	2.8e−02	6.8e+08
0.5	2.131214e−01	1.2e−06	1.7e−01	3.1e−02	6.0e−03	5.3e+04
1	1.020957e−01	5.9e−04	5.3e−02	9.2e−03	1.4e−03	9.9e+02
2.5	−1.423028e−02	5.6e+00	6.7e−03	9.6e−04	6.9e−05	5.4e+00
5	7.242069e−04	2.3e+04	8.4e−04	9.5e−05	2.3e−06	1.0e−01
15	−1.307015e−08	4.4e+12	1.3e−05	7.9e−07	1.4e−09	1.8e−04
50	2.647205e−24	1.2e+32	5.3e−08	1.3e−09	4.4e−14	1.5e−07

x	$\|E_n(\mathrm{i}x)\|_s$	(14.1.11)	(14.1.16)	(14.1.16)	(14.1.23)	(14.1.13)
0.1	4.925981e−01	8.4e−14	2.0e+00	3.2e−01	4.2e−02	5.1e+09
0.5	4.396266e−01	3.7e−08	3.8e−01	8.1e−02	2.3e−02	3.7e+05
1	3.789634e−01	1.1e−05	1.6e−01	3.4e−02	1.1e−02	6.6e+03
2.5	−2.607690e−01	2.4e−02	3.6e−02	6.5e−03	1.4e−03	3.7e+01
5	1.667005e−01	8.8e+00	6.3e−03	9.5e−04	9.9e−05	7.8e−01
15	−6.468238e−02	9.2e+04	1.1e−04	1.0e−05	1.1e−07	1.5e−03
50	1.994070e−02	1.5e+09	3.7e−07	1.4e−08	2.5e−12	1.3e−06

x	$\|E_n(x-\mathrm{i}x)\|_s$	(14.1.11)	(14.1.16)	(14.1.16)	(14.1.23)	(14.1.13)
−0.1	5.909108e−01	1.1e−12	1.6e+00	3.4e−01	6.6e−02	6.0e+08
−0.5	9.012207e−01	3.0e−07	4.5e−01	1.3e−01	9.1e−02	4.0e+04
−1	−1.328200e+00	5.5e−05	2.8e−01	7.5e−02	6.5e−02	7.4e+02
−2.5	−3.834542e+00	3.3e−02	9.1e−02	1.8e−02	1.2e−02	5.2e+00
−5	2.543372e+01	1.5e+00	1.2e−02	2.2e−03	7.0e−04	1.4e−01
−15	−1.697030e+05	1.6e+00	5.8e−05	7.1e−06	1.1e−07	2.6e−04
−50	−7.555020e+19	1.0e+00	8.7e−08	2.8e−09	2.1e−13	1.7e−07

TABLE 14.1.2: Together with the symmetry property (14.1.6) the following tables let us investigate the relative error of the 20^{th} partial sum and the 20^{th} approximants throughout the cut complex plane for $n = 3$. The fraction (14.1.16) is first evaluated without modification and afterwards with.

x	$E_n(x)$	(14.1.11)	(14.1.16)	(14.1.16)	(14.1.23)	(14.1.13)
0.1	4.162915e−01	4.4e−47	5.9e−03	7.0e−03	7.7e−04	1.2e+42
0.5	2.216044e−01	9.7e−31	5.2e−04	2.3e−04	1.3e−05	3.2e+27
1	1.096920e−01	1.6e−23	6.1e−05	1.8e−05	3.7e−07	1.8e+21
2.5	1.629537e−02	1.5e−13	6.9e−07	1.2e−07	2.5e−10	1.1e+13
5	8.778009e−04	2.1e−05	4.5e−09	5.8e−10	7.0e−14	7.4e+06
15	1.714014e−08	7.5e+10	4.8e−14	3.6e−15	3.8e−22	1.2e−03
50	3.642909e−24	2.0e+38	1.2e−21	4.6e−23	2.0e−35	1.9e−14

x	$\|E_n(x+\mathrm{i}x)\|_s$	(14.1.11)	(14.1.16)	(14.1.16)	(14.1.23)	(14.1.13)
0.1	4.136467e−01	1.3e−43	5.4e−03	5.2e−03	6.6e−04	8.5e+38
0.5	2.131214e−01	2.9e−27	3.6e−04	1.2e−04	6.3e−06	2.3e+24
1	1.020957e−01	5.0e−20	3.1e−05	7.3e−06	1.2e−07	1.3e+18
2.5	−1.423028e−02	4.8e−10	1.9e−07	2.8e−08	3.1e−11	8.8e+09
5	7.242069e−04	7.2e−02	6.2e−10	6.7e−11	3.1e−15	6.1e+03
15	−1.307015e−08	2.7e+14	1.5e−15	9.9e−17	1.6e−24	1.0e−06
50	2.647205e−24	6.5e+41	7.1e−24	2.2e−25	2.8e−39	1.5e−17

x	$\lvert E_n(\mathrm{i}x)\rvert_s$	(14.1.11)	(14.1.16)	(14.1.16)	(14.1.23)	(14.1.13)
0.1	4.925981e−01	3.7e−47	1.2e−02	1.4e−02	2.5e−03	1.1e+42
0.5	4.396266e−01	5.0e−31	3.6e−03	1.4e−03	2.5e−04	2.7e+27
1	3.789634e−01	4.9e−24	8.7e−04	2.3e−04	2.3e−05	1.5e+21
2.5	−2.607690e−01	1.0e−14	2.9e−05	4.8e−06	9.9e−08	9.4e+12
5	1.667005e−01	1.3e−07	4.0e−07	4.9e−08	1.3e−10	6.9e+06
15	−6.468238e−02	2.8e+04	6.1e−12	5.0e−13	4.3e−18	1.4e−03
50	1.994070e−02	4.9e+16	3.2e−20	1.5e−21	7.2e−32	2.4e−14

x	$\lvert E_n(x-\mathrm{i}x)\rvert_s$	(14.1.11)	(14.1.16)	(14.1.16)	(14.1.23)	(14.1.13)
−0.1	5.909108e−01	9.1e−44	2.6e−02	2.5e−02	1.0e−02	7.3e+38
−0.5	9.012207e−01	7.2e−28	3.0e−02	8.2e−03	5.4e−03	1.5e+24
−1	−1.328200e+00	4.2e−21	1.3e−02	2.5e−03	1.2e−03	8.3e+17
−2.5	−3.834542e+00	2.2e−12	9.0e−04	1.2e−04	2.2e−05	6.1e+09
−5	2.543372e+01	3.0e−06	1.6e−05	1.7e−06	7.8e−08	5.9e+03
−15	−1.697030e+05	5.2e+01	4.2e−11	3.7e−12	1.6e−15	1.9e−06
−50	−7.555020e+19	9.7e−01	9.9e−22	5.4e−23	1.1e−32	2.3e−17

FIGURE 14.1.1: Number of significant digits (between 0 and 12) guaranteed by the 5^{th} partial sum of (14.1.11) (lightest), the 5^{th} partial sum of (14.1.13) (second lightest), the 5^{th} approximant of (14.1.16) (second darkest) and the 5^{th} approximant of (14.1.23) (darkest) of $E_3(x)$, in the region $1 \le x \le 25$.

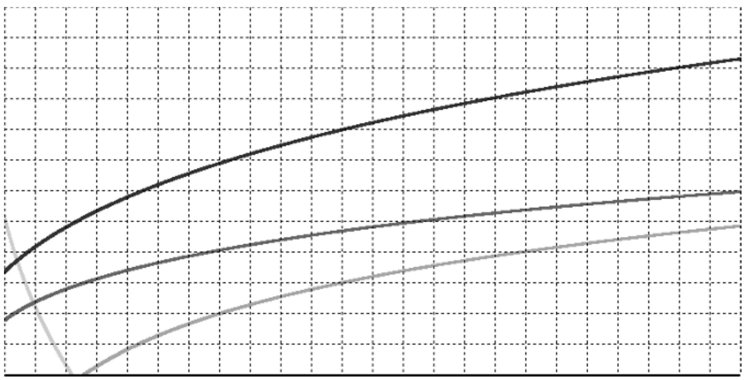


TABLE 14.1.3: Together with the symmetry property (14.1.7) the following tables let us investigate the relative error of the 5^{th} partial sum and the 5^{th} approximants for $E_\nu(z)$ with $z = 9$ and $\nu \in \mathbb{C} \setminus \mathbb{N}$.

x	$\|E_{x+\mathrm{i}x}(9)\|_s$	(14.1.12)	(14.1.19)	(14.1.19)	(14.1.20)	(14.1.22)
0.5	1.303847e−05	4.0e+06	2.6e−05	1.0e−06	2.9e+04	2.9e+04
1.5	1.180128e−05	5.1e+06	1.6e−04	1.4e−05	3.8e+04	3.8e+04
5.5	8.206623e−06	6.8e+06	1.4e−03	2.9e−04	1.2e+02	1.2e+02
20	3.577491e−06	3.6e+06	2.9e−03	7.5e−04	6.8e−06	5.2e−09
40	1.974316e−06	3.0e+06	2.4e−03	5.5e−04	5.8e−08	1.1e−11
60	1.360663e−06	2.8e+06	1.9e−03	3.9e−04	3.5e−09	2.3e−13

x	$\|E_{\mathrm{i}x}(9)\|_s$	(14.1.12)	(14.1.19)	(14.1.19)	(14.1.20)	(14.1.22)
0.5	1.369754e−05	3.5e+06	1.2e−05	8.4e−07	6.7e+03	6.7e+03
1.5	1.358139e−05	3.4e+06	7.1e−05	8.2e−06	2.4e+03	2.4e+03
5.5	1.214787e−05	3.0e+06	1.5e−03	4.1e−04	9.3e+00	9.5e+00
20	5.815821e−06	2.5e+06	7.2e−03	2.5e−03	1.0e−04	9.1e−07
40	3.040867e−06	2.5e+06	5.0e−03	1.5e−03	7.4e−07	7.2e−10
60	2.043685e−06	2.5e+06	3.5e−03	9.2e−04	4.3e−08	1.2e−11

x	$\|E_{x-\mathrm{i}x}(9)\|_s$	(14.1.12)	(14.1.19)	(14.1.19)	(14.1.20)	(14.1.22)
−0.5	1.442067e−05	3.1e+06	1.0e−05	1.1e−06	2.0e+03	2.0e+03
−1.5	1.592640e−05	2.4e+06	9.0e−05	2.2e−05	2.1e+02	2.1e+02
−5.5	2.135553e−05	1.1e+06	1.7e−02	1.3e−02	2.2e−01	6.4e−01
−20	5.279119e−06	1.8e+06	7.9e−01	7.8e−01	9.9e−06	9.1e−08
−40	−1.735177e+01	2.9e−01	1.0e+00	1.0e+00	9.7e−15	5.9e−18
−60	−2.246944e+12	1.5e−12	1.0e+00	1.0e+00	2.7e−27	4.0e−31

x	$E_x(9)$	(14.1.12)	(14.1.19)	(14.1.19)	(14.1.20)	(14.1.22)
−0.5	1.443728e−05	3.1e+06	4.7e−06	4.8e−07	2.3e+03	2.3e+03
−1.5	1.611841e−05	2.4e+06	6.6e−06	1.5e−06	3.3e+02	3.4e+02
−5.5	2.844313e−05	9.1e+05	7.2e−03	1.7e−02	1.9e+00	7.3e+00
−20	2.222497e−02	5.2e+02	1.0e+00	1.0e+00	5.1e−08	3.0e−09
−40	6.133413e+08	1.1e−08	1.0e+00	1.0e+00	5.2e−21	1.5e−23
−60	5.144957e+23	9.0e−24	1.0e+00	1.0e+00	2.2e−37	1.3e−40

TABLE 14.1.4: Together with the symmetry property (14.1.7) the following tables let us investigate the relative error of the 20^{th} partial sum and the 20^{th} approximants for $E_\nu(z)$ with $z = 9$ and $\nu \in \mathbb{C} \setminus \mathbb{N}$.

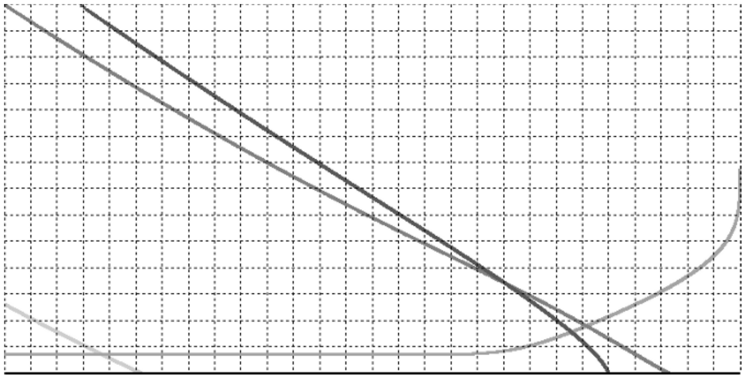
x	$\lvert E_{x+\mathrm{i}x}(9)\rvert_s$	(14.1.12)	(14.1.19)	(14.1.19)	(14.1.20)	(14.1.22)
0.5	1.303847e−05	5.5e+03	7.4e−13	1.3e−14	2.3e+00	2.9e+04
1.5	1.180128e−05	6.3e+03	7.9e−12	5.2e−13	2.3e+01	3.8e+04
5.5	8.206623e−06	1.1e+04	1.8e−10	5.2e−11	5.4e+01	1.2e+02
20	3.577491e−06	2.1e+04	4.0e−11	5.6e−11	4.2e−16	7.8e−17
40	1.974316e−06	1.7e+04	6.4e−13	2.1e−12	1.6e−26	7.1e−33
60	1.360663e−06	1.6e+04	2.9e−14	1.5e−13	3.4e−32	4.3e−40

x	$\lvert E_{\mathrm{i}x}(9)\rvert_s$	(14.1.12)	(14.1.19)	(14.1.19)	(14.1.20)	(14.1.22)
0.5	1.369754e−05	5.1e+03	2.6e−13	4.9e−15	2.0e−01	6.7e+03
1.5	1.358139e−05	5.1e+03	2.0e−12	9.7e−14	6.8e−02	2.4e+03
5.5	1.214787e−05	5.6e+03	3.2e−10	5.8e−11	1.7e−04	9.6e+00
20	5.815821e−06	8.8e+03	2.2e−08	2.0e−08	2.7e−14	2.6e−13
40	3.040867e−06	1.1e+04	2.0e−10	4.9e−10	4.2e−23	1.6e−26
60	2.043685e−06	1.2e+04	4.8e−12	1.8e−11	2.1e−28	5.2e−34

x	$\lvert E_{x-\mathrm{i}x}(9)\rvert_s$	(14.1.12)	(14.1.19)	(14.1.19)	(14.1.20)	(14.1.22)
−0.5	1.442067e−05	4.7e+03	1.5e−13	4.4e−15	2.2e−02	2.0e+03
−1.5	1.592640e−05	4.1e+03	9.2e−13	6.7e−14	3.1e−04	1.9e+02
−5.5	2.135553e−05	2.6e+03	2.4e−08	5.3e−09	5.9e−10	3.5e−06
−20	5.279119e−06	6.2e+03	7.7e−01	7.7e−01	6.1e−21	1.4e−22
−40	−1.735177e+01	1.2e−03	1.0e+00	1.0e+00	1.2e−35	1.0e−40
−60	−2.246944e+12	6.6e−15	1.0e+00	1.0e+00	7.5e−52	7.2e−59

x	$E_x(9)$	(14.1.12)	(14.1.19)	(14.1.19)	(14.1.20)	(14.1.22)
−0.5	1.443728e−05	4.7e+03	6.7e−14	1.7e−15	2.5e−02	2.3e+03
−1.5	1.611841e−05	4.0e+03	3.3e−14	1.8e−15	5.3e−04	3.2e+02
−5.5	2.844313e−05	2.0e+03	4.8e−13	7.6e−14	4.7e−09	1.2e−04
−20	2.222497e−02	1.6e+00	1.0e+00	1.0e+00	8.7e−22	4.7e−22
−40	6.133413e+08	4.0e−11	1.0e+00	1.0e+00	1.4e−39	5.4e−43
−60	5.144957e+23	3.6e−26	1.0e+00	1.0e+00	3.0e−59	1.4e−64

FIGURE 14.1.2: Number of significant digits (between 0 and 14) guaranteed by (from light to dark) the 5^{th} partial sum of (14.1.12), the 5^{th} approximant of (14.1.19), the 5^{th} approximant of (14.1.20) and the 5^{th} approximant of (14.1.22) of $E_\nu(9)$, in the region $-30 \le \nu \le -2$.



14.2 Related functions

Definitions and elementary properties. The function $\mathrm{Ei}(x)$ defined by

$$\mathrm{Ei}(x) := -⨍_{-x}^{\infty} \frac{e^{-t}}{t}\, dt, \qquad x > 0, \tag{14.2.1}$$

where $⨍$ denotes the principal value of the integral, is often also called an exponential integral. It is extended to the negative real axis by

$$\mathrm{Ei}(-x) := -E_1(x), \qquad x > 0. \tag{14.2.2}$$

We recall from *Section* 10.13 that Gompertz' constant equals $-e\,\mathrm{Ei}(-1)$. The *logarithmic integral* $\mathrm{li}(x)$ is defined by the principal value of the integral

$$\mathrm{li}(x) := ⨍_{0}^{x} \frac{dt}{\ln(t)}, \qquad x > 1, \tag{14.2.3}$$

and is related to the exponential integrals $\mathrm{Ei}(x)$ by

$$\mathrm{li}(x) = \mathrm{Ei}(\ln(x)), \qquad x > 1. \tag{14.2.4}$$

The entire function $\mathrm{Ein}(z)$ defined by

$$\mathrm{Ein}(z) := \int_0^z \frac{1 - e^{-t}}{t}\, dt, \qquad z \in \mathbb{C}, \tag{14.2.5}$$

is related to the exponential integral $E_1(z)$ by

$$\mathrm{Ein}(z) = E_1(z) + \gamma + \mathrm{Ln}(z), \qquad |\arg z| < \pi \tag{14.2.6}$$

where γ is Euler's constant (10.8.1).

The entire functions *sine integral* $\mathrm{Si}(z)$ and *cosine integral* $\mathrm{Ci}(z)$, are defined by

$$\mathrm{Si}(z) := \int_0^z \frac{\sin(t)}{t}\,dt, \qquad z \in \mathbb{C}, \tag{14.2.7}$$

$$\mathrm{Ci}(z) := \gamma + \mathrm{Ln}(z) + \int_0^z \frac{\cos(t) - 1}{t}\,dt, \qquad |\arg z| < \pi, \tag{14.2.8}$$

where again γ is Euler's constant (10.8.1). The sine integral (14.2.7) is related to the exponential integrals $E_1(z)$ and $\mathrm{Ei}(z)$ by

$$\mathrm{Si}(z) = \frac{1}{2\mathrm{i}}\left(E_1(\mathrm{i}z) - E_1(-\mathrm{i}z)\right) + \frac{\pi}{2}, \qquad |\arg z| < \frac{\pi}{2}, \tag{14.2.9}$$

$$\mathrm{Si}(\mathrm{i}x) = \frac{\mathrm{i}}{2}\left(\mathrm{Ei}(x) + E_1(x)\right), \qquad x > 0 \tag{14.2.10}$$

and the cosine integral (14.2.8) satisfies the relations

$$\mathrm{Ci}(z) = -\frac{1}{2}\left(E_1(\mathrm{i}z) + E_1(-\mathrm{i}z)\right), \qquad |\arg z| < \frac{\pi}{2}, \tag{14.2.11}$$

$$\mathrm{Ci}(\mathrm{i}x) = \frac{1}{2}\left(\mathrm{Ei}(x) - E_1(x)\right) + \frac{\mathrm{i}\pi}{2}, \qquad x > 0. \tag{14.2.12}$$

From (14.2.9) and (14.2.11) it is easily seen that

$$E_1(\mathrm{i}z) = -\,\mathrm{Ci}(z) + \mathrm{i}\left(\mathrm{Si}(z) - \frac{\pi}{2}\right), \qquad |\arg z| < \pi/2. \tag{14.2.13}$$

Series expansions. From the relations above, we obtain the series representations

$$\mathrm{Ei}(x) = \gamma + \ln(x) + \sum_{k=1}^{\infty} \frac{x^k}{(k!)k}, \qquad x > 0, \tag{14.2.14}$$

$$\mathrm{li}(x) = \gamma + \ln(\ln(x)) + \sum_{k=1}^{\infty} \frac{(\ln(x))^k}{(k!)k}, \qquad x > 1, \tag{14.2.15}$$

$$\mathrm{Ein}(z) = -\sum_{k=1}^{\infty} \frac{(-1)^k z^k}{(k!)k}, \qquad z \in \mathbb{C}, \tag{14.2.16}$$

$$\mathrm{Si}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)(2k+1)!}, \qquad z \in \mathbb{C}, \tag{14.2.17}$$

$$\mathrm{Ci}(z) = \gamma + \mathrm{Ln}(z) + \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)(2k)!}, \qquad |\arg z| < \pi, \tag{14.2.18}$$

where γ is Euler's constant (10.8.1).

Asymptotic expansion. From (14.2.2) and the asymptotic expansion (14.1.13) we find

$$\mathrm{Ei}(x) \approx e^x x^{-1} \sum_{k=0}^{\infty} k!\, x^{-k}, \qquad x \to \infty. \tag{14.2.19}$$

S-fractions. From (14.1.10) and (14.2.14) we find

$$\mathrm{Ei}(x) = 2\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)(2k+1)!} - E_1(x), \qquad x > 0. \tag{14.2.20}$$

Combined with the modified S-fraction representation (14.1.16) for $E_1(z)$, this gives

$$\mathrm{Ei}(x) = 2\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)(2k+1)!} - e^{-x}\left(\frac{x^{-1}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{\lfloor\frac{m}{2}\rfloor x^{-1}}{1}\right)\right), \qquad x > 0. \tag{14.2.21}$$

Since the infinite series in (14.2.21) converges for $z \in \mathbb{C}$ and the S-fraction converges to $E_1(z)$ for $|\arg z| < \pi$, the function $\mathrm{Ei}(x)$ can be continued analytically from $\mathbb{R}^+$ to the cut plane $|\arg z| < \pi$ by the right hand side of (14.2.21).

By replacing x by $\ln(x)$ in (14.2.21), we find

$$\mathrm{li}(x) = 2\sum_{k=0}^{\infty} \frac{(\ln(x))^{2k+1}}{(2k+1)(2k+1)!} - \frac{1}{x}\left(\frac{(\ln(x))^{-1}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{\lfloor\frac{m}{2}\rfloor (\ln(x))^{-1}}{1}\right)\right), \qquad x > 1. \tag{14.2.22}$$

As for $\mathrm{Ei}(x)$, the function $\mathrm{li}(x)$ can be continued analytically into that part of $\mathbb{C}$ for which $|\arg(\mathrm{Ln}(z))| < \pi$ by the right hand side of (14.2.22).

From (14.2.6) and (14.1.16) we have

$$\mathrm{Ein}(z) = \gamma + \mathrm{Ln}(z) + e^{-z}\left(\frac{z^{-1}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{\lfloor\frac{m}{2}\rfloor z^{-1}}{1}\right)\right), \qquad |\arg z| < \pi. \tag{14.2.23}$$

With $\lim_{m\to\infty}\lfloor\frac{m}{2}\rfloor = +\infty$ in (14.2.23), the modification

$$w_{2k}(z) = \frac{-1+\sqrt{4kz^{-1}+1}}{2}, \qquad w_{2k+1}(z) = \frac{-1+\sqrt{4(k+1)z^{-1}+1}}{2}$$

can be useful when evaluating the approximants of (14.2.23).
Additional S-fraction expansions of $\mathrm{Ei}(z)$, $\mathrm{li}(z)$ and $\mathrm{Ein}(z)$ can be obtained from (14.1.18) with $n = 1$.

C-fraction. From (14.2.2) and the S-fraction representation (14.1.16) for the exponential integral $E_1(z)$, we obtain a C-fraction representation for the continuation of the function $\mathrm{Ei}(x)$ on the negative real axis:

$$\mathrm{Ei}(-x) = \frac{x^{-1}e^{-x}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\lfloor \frac{m}{2} \rfloor x^{-1}}{1} \right), \qquad x > 0. \tag{14.2.24}$$

TABLE 14.2.1: Together with the symmetry property (14.1.6) the following tables let us investigate the relative error of the 5^{th} partial sum and the 5^{th} approximants of $E_1(x)$, which is at the heart of all functions in *Section* 14.2, throughout the cut complex plane. The fraction (14.1.16) is first evaluated without modification and afterwards with.

x	$E_1(x)$	(14.1.10)	(14.1.16)	(14.1.16)	(14.1.23)	(14.1.13)
0.1	1.822924e+00	1.3e−10	8.8e−01	8.0e−02	1.5e−01	5.8e+07
0.5	5.597736e−01	6.1e−06	1.1e−01	7.2e−03	1.3e−02	7.6e+03
1	2.193839e−01	9.4e−04	3.2e−02	1.5e−03	2.1e−03	1.7e+02
2.5	2.491492e−02	1.7e+00	3.5e−03	1.0e−04	6.6e−05	1.1e+00
5	1.148296e−03	1.9e+03	4.1e−04	6.8e−06	1.9e−06	2.4e−02
15	1.918628e−08	4.6e+10	6.1e−06	3.2e−08	1.2e−09	4.6e−05
50	3.783264e−24	1.2e+29	2.8e−08	2.3e−11	5.0e−14	4.1e−08
90	9.005474e−42	1.0e+48	1.7e−09	5.1e−13	2.2e−16	1.3e−09

x	$\lvert E_1(x + \mathrm{i}x)\rvert_s$	(14.1.10)	(14.1.16)	(14.1.16)	(14.1.23)	(14.1.13)
0.1	1.631875e+00	1.1e−09	6.6e−01	6.0e−02	1.3e−01	8.2e+06
0.5	4.731368e−01	5.8e−05	7.6e−02	4.5e−03	8.4e−03	1.1e+03
1	1.793248e−01	9.1e−03	1.9e−02	8.2e−04	1.1e−03	2.5e+01
2.5	−1.941653e−02	1.7e+01	1.7e−03	4.0e−05	2.2e−05	1.7e−01
5	8.659610e−04	1.9e+04	1.5e−04	2.1e−06	4.1e−07	3.5e−03
15	−1.395107e−08	4.2e+11	1.6e−06	5.8e−09	1.2e−10	6.5e−06
50	2.700545e−24	1.0e+30	5.7e−09	2.8e−12	2.5e−15	5.4e−09
90	−6.402167e−42	8.4e+48	3.3e−10	5.4e−14	9.3e−18	1.6e−10

x	$\lvert E_1(\mathrm{i}x)\rvert_s$	(14.1.10)	(14.1.16)	(14.1.16)	(14.1.23)	(14.1.13)
0.1	2.269126e+00	1.0e−10	1.1e+00	1.2e−01	2.8e−01	5.3e+07
0.5	1.092255e+00	3.3e−06	2.2e−01	1.7e−02	5.1e−02	7.0e+03
1	−7.100057e−01	3.2e−04	8.1e−02	4.8e−03	1.2e−02	1.6e+02
2.5	−3.533717e−01	1.5e−01	1.1e−02	4.2e−04	5.8e−04	1.2e+00
5	1.911718e−01	1.6e+01	1.2e−03	3.2e−05	2.0e−05	2.9e−02
15	−6.624423e−02	1.8e+04	1.3e−05	1.1e−07	8.5e−09	5.7e−05
50	1.998806e−02	2.6e+07	3.8e−08	4.3e−11	1.3e−13	4.6e−08
90	−1.110906e−02	8.8e+08	2.0e−09	7.4e−13	4.0e−16	1.3e−09

x	$\lvert E_1(x-\mathrm{i}x)\rvert_s$	(14.1.10)	(14.1.16)	(14.1.16)	(14.1.23)	(14.1.13)
−0.1	2.589013e+00	7.2e−10	1.1e+00	1.5e−01	5.9e−01	6.5e+06
−0.5	−1.860020e+00	1.7e−05	3.9e−01	3.3e−02	1.6e−01	9.4e+02
−1	−1.918894e+00	1.1e−03	1.8e−01	1.1e−02	5.1e−02	2.5e+01
−2.5	−3.832519e+00	1.6e−01	2.1e−02	1.0e−03	3.0e−03	2.4e−01
−5	2.285582e+01	2.1e+00	1.5e−03	6.0e−05	7.0e−05	6.1e−03
−15	1.592592e+05	1.0e+00	4.7e−06	5.1e−08	3.7e−09	9.4e−06
−50	−7.405902e+19	1.0e+00	8.0e−09	5.9e−12	8.3e−15	6.1e−09
−90	−9.641796e+36	1.0e+00	3.9e−10	8.2e−14	1.8e−17	1.8e−10

TABLE 14.2.2: Together with the symmetry property (14.1.6) the following tables let us investigate the relative error of the 20^{th} partial sum and the 20^{th} approximants of $E_1(x)$ which is at the heart of all functions in *Section* 14.2, throughout the cut complex plane. The fraction (14.1.16) is first evaluated without modification and afterwards with.

x	$E_1(x)$	(14.1.10)	(14.1.16)	(14.1.16)	(14.1.23)	(14.1.13)
0.1	1.822924e+00	5.1e−43	5.0e−02	7.3e−03	1.0e−02	1.2e+39
0.5	5.597736e−01	7.8e−28	1.1e−03	7.5e−05	2.9e−05	5.4e+24
1	2.193839e−01	4.1e−21	6.1e−05	3.2e−06	3.7e−07	3.9e+18
2.5	2.491492e−02	7.7e−12	2.5e−07	9.4e−09	7.8e−11	3.1e+10
5	1.148296e−03	3.2e−04	7.9e−10	2.3e−11	9.3e−15	2.4e+04
15	1.918628e−08	1.5e+11	3.0e−15	5.9e−17	1.4e−23	4.5e−06
50	3.783264e−24	3.6e+37	3.4e−23	3.9e−25	2.4e−37	7.6e−17
90	9.005474e−42	2.2e+60	1.2e−27	9.2e−30	1.7e−45	3.8e−22

x	$\lvert E_1(x+\mathrm{i}x)\rvert_s$	(14.1.10)	(14.1.16)	(14.1.16)	(14.1.23)	(14.1.13)
0.1	1.631875e+00	8.2e−40	3.8e−02	4.6e−03	6.6e−03	9.3e+35
0.5	4.731368e−01	1.3e−24	5.3e−04	3.1e−05	9.9e−06	4.4e+21
1	1.793248e−01	7.2e−18	2.1e−05	9.7e−07	7.6e−08	3.3e+15
2.5	−1.941653e−02	1.4e−08	4.6e−08	1.5e−09	6.3e−12	2.8e+07
5	8.659610e−04	6.0e−01	7.2e−11	1.9e−12	2.6e−16	2.2e+01
15	−1.395107e−08	2.7e+14	6.6e−17	1.2e−18	3.7e−26	3.9e−09
50	2.700545e−24	6.0e+40	1.6e−25	1.5e−27	2.4e−41	6.0e−20
90	−6.402167e−42	3.6e+63	3.2e−30	2.0e−32	4.1e−50	2.9e−25

x	$\lvert E_1(\mathrm{i}x)\rvert_s$	(14.1.10)	(14.1.16)	(14.1.16)	(14.1.23)	(14.1.13)
0.1	2.269126e+00	4.1e−43	1.4e−01	1.9e−02	4.6e−02	1.1e+39
0.5	1.092255e+00	4.1e−28	8.6e−03	5.6e−04	6.6e−04	4.7e+24
1	−7.100057e−01	1.3e−21	9.3e−04	4.6e−05	2.4e−05	3.4e+18
2.5	−3.533717e−01	6.0e−13	1.0e−05	3.6e−07	3.1e−08	3.0e+10
5	1.911718e−01	2.3e−06	6.1e−08	1.8e−09	1.6e−11	2.6e+04
15	−6.624423e−02	5.9e+04	2.6e−13	5.9e−15	1.2e−19	5.9e−06
50	1.998806e−02	9.1e+15	6.1e−22	9.2e−24	5.6e−34	9.8e−17
90	−1.110906e−02	2.2e+21	8.2e−27	8.2e−29	6.4e−43	4.5e−22

x	$\lvert E_1(x-\mathrm{i}x)\rvert_s$	(14.1.10)	(14.1.16)	(14.1.16)	(14.1.23)	(14.1.13)
−0.1	2.589013e+00	5.2e−40	4.2e−01	3.9e−02	1.8e−01	7.2e+35
−0.5	−1.860020e+00	3.5e−25	5.3e−02	2.8e−03	1.0e−02	3.2e+21
−1	−1.918894e+00	7.3e−19	9.6e−03	3.8e−04	9.0e−04	2.5e+15
−2.5	−3.832519e+00	8.9e−11	2.0e−04	5.8e−06	4.5e−06	2.7e+07
−5	2.285582e+01	3.5e−05	1.3e−06	3.2e−08	5.2e−09	2.9e+01
−15	1.592592e+05	6.2e+01	7.0e−13	1.9e−14	1.7e−17	8.4e−09
−50	−7.405902e+19	1.0e+00	1.1e−23	2.1e−25	3.3e−35	9.0e−20
−90	−9.641796e+36	1.0e+00	3.5e−29	3.4e−31	3.0e−46	3.6e−25

15

Hypergeometric functions

A hypergeometric series or function is a series for which the ratio of successive terms in the series is a rational function of the index of the term. The function ${}_2F_1(a, b; c; x)$ is the first hypergeometric function to be studied and, in general, arises the most frequently in physical problems. It is generally known as Gauss's hypergeometric function.
Many of the special functions in mathematics, physics and engineering are hypergeometric functions, or can be expressed in terms of them. We show how the contiguous relations for the ${}_2F_1$ functions lead to several continued fraction representations.

15.1 Definition and basic properties

The *hypergeometric series* is defined by

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}, \tag{15.1.1} $$

$$ a_i \in \mathbb{C}, \quad b_j \in \mathbb{C} \setminus \mathbb{Z}_0^-, \quad 1 \le i \le p, \quad 1 \le j \le q $$

where

$$ (a)_0 = 1, \quad (a)_k = a(a+1)(a+2)\cdots(a+k-1), \quad a \in \mathbb{C}, \quad k \in \mathbb{N} $$

is the Pochhammer symbol or shifted factorial defined in (12.1.7). The subscripts p and q indicate the number of parameters in numerator and denominator of the coefficients. Assuming that also all $a_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$, the following holds for the convergence of (15.1.1):

- $p < q + 1$: the series converges absolutely for $z \in \mathbb{C}$,
- $p = q + 1$: the series converges absolutely for $|z| < 1$ and diverges for $|z| > 1$, and for $|z| = 1$ it converges absolutely for $\Re(\sum_{k=1}^{q} b_k - \sum_{k=1}^{p} a_k) > 0$,
- $p > q + 1$: the series converges only for $z = 0$.

The hypergeometric series (15.1.1) is a solution of the differential equation [AAR99, p. 188]

$$z\frac{d}{dz}\prod_{j=1}^{q}\left(z\frac{dy}{dz}+(b_j-1)y\right)-z\prod_{j=1}^{p}\left(z\frac{dy}{dz}+a_jy\right)=0. \tag{15.1.2}$$

For $p=2$ and $q=1$, the differential equation (15.1.2) is the second-order differential equation

$$z(1-z)\frac{d^2y}{dz^2}+(c-(a+b+1)z)\frac{dy}{dz}-aby=0 \tag{15.1.3}$$

which is called the *hypergeometric differential equation.* Equation (15.1.3) has three regular singular points at $z=0$, $z=1$ and $z=\infty$. The solution of (15.1.3) with initial conditions $y(0)=1$ and $(dy/dz)(0)=ab/c$ is called the *Gauss hypergeometric series* ${}_2F_1(a,b;c;z)$ and is given by [Gau12]

$${}_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!},\qquad a,b\in\mathbb{C},\quad c\in\mathbb{C}\setminus\mathbb{Z}_0^-\,. \tag{15.1.4}$$

The series ${}_2F_1(a,b;c;z)$ converges for $|z|<1$ and diverges for $|z|>1$. In case of convergence we use the term *Gauss hypergeometric function.* The Gauss hypergeometric function is an analytical function of a,b,c and z. For fixed b,c and z it is an entire function of a. For fixed a,c and z it is an entire function of b. If $a\in\mathbb{Z}_0^-$ or $b\in\mathbb{Z}_0^-$, (15.1.4) reduces to a polynomial in z. In particular,

$${}_2F_1(0,b;c;z)={}_2F_1(a,0;c;z)=1. \tag{15.1.5}$$

If the parameters a,b,c satisfy

$$c\in\mathbb{C}\setminus\mathbb{Z},\quad c-a-b\in\mathbb{C}\setminus\mathbb{Z},\quad a-b\in\mathbb{C}\setminus\mathbb{Z},$$

there are three sets of two linearly independent solutions of (15.1.3) corresponding to $z=0$, $z=1$ and $z=\infty$. They are given by [AS64, p. 563]

$${}_2F_1(a,b;c;z), \tag{15.1.6}$$

$$z^{1-c}\,{}_2F_1(b+1-c,a+1-c;2-c;z), \tag{15.1.7}$$

$${}_2F_1(a,b;a+b+1-c;1-z), \tag{15.1.8}$$

$$(1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c+1-a-b;1-z), \tag{15.1.9}$$

$$z^{-a}\,{}_2F_1(a,a+1-c;a+1-b;z^{-1}), \tag{15.1.10}$$

$$z^{-b}\,{}_2F_1(b+1-c,b;b+1-a;z^{-1}). \tag{15.1.11}$$

Many elementary and special functions can be expressed in terms of hypergeometric functions [AS64, p. 556].

EXAMPLE 15.1.1:

$$
\begin{aligned}
&{}_2F_1(1,1;2;z) = -z^{-1}\,\mathrm{Ln}(1-z),\\
&{}_2F_1(1/2,1;3/2;z^2) = \frac{1}{2z}\,\mathrm{Ln}\left(\frac{1+z}{1-z}\right),\\
&{}_2F_1(1/2,1;3/2;-z^2) = z^{-1}\,\mathrm{Arctan}(z),\\
&{}_2F_1(1/2,1/2;3/2;-z^2) = \sqrt{1+z^2}\;{}_2F_1(1,1;3/2;-z^2) = z^{-1}\,\mathrm{Ln}\left(z+\sqrt{1+z^2}\right),\\
&z\;{}_2F_1(1/n,1;1+1/n;-z^n) = \int_0^z \frac{dt}{1+t^n}, \qquad z^n \in \mathbb{C}\setminus(-\infty,-1],\\
&\frac{{}_2F_1(1/2,-1/2;1/2;z^2)}{{}_2F_1(1/2,1/2;3/2;z^2)} = \frac{z\sqrt{1-z^2}}{\mathrm{Arcsin}(z)},\\
&\frac{x^a}{a}\;{}_2F_1(a,1-b;a+1;x) = B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt,\\
&\hspace{20em} a,b\in\mathbb{R}^+, \quad 0\le x\le 1
\end{aligned}
$$

where $B_x(a,b)$ is the incomplete beta function introduced in (18.5.3).

The derivative of the hypergeometric series ${}_2F_1(a,b;c;z)$ is given by

$$\frac{d}{dz}\,{}_2F_1(a,b;c;z) = \frac{ab}{c}\,{}_2F_1(a+1,b+1;c+1;z).$$

Contiguous and recurrence relations. Two hypergeometric series ${}_pF_q$ of the same variable and whose corresponding parameters differ by integers, are called *contiguous* and are linearly related. Three-term recurrence relations are examples of contiguous relations.
For a fixed triplet (s_1,s_2,s_3) with $s_i\in\{-1,0,1\}$ and not all s_i zero, the Gauss hypergeometric functions

$$y_n = {}_2F_1(a+s_1n, b+s_2n; c+s_3n; z), \qquad n\in\mathbb{N}, \tag{15.1.12}$$

satisfy a three-term recurrence relation of the form

$$A_n y_{n+1} = B_n y_n + C_n y_{n-1}. \tag{15.1.13}$$

There exist 26 triplets of this kind and hence 26 three-term recurrence relations of the form (15.1.13). These can be reduced to four basic recurrence relations [GST06a] by using the symmetry relation

$$ {}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z) \tag{15.1.14a} $$

and the transformation formulas [AS64, p. 563]

$$ {}_2F_1(a, b; c; z) = (1-z)^{-a}\, {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right), \tag{15.1.14b} $$

$$ {}_2F_1(a, b; c; z) = (1-z)^{-b}\, {}_2F_1\left(c-a, b; c; \frac{z}{z-1}\right), \tag{15.1.14c} $$

$$ {}_2F_1(a, b; c; z) = (1-z)^{c-a-b}\, {}_2F_1\left(c-a, c-b; c; z\right). \tag{15.1.14d} $$

Formulas (15.1.14b) and (15.1.14c) are referred to as *Pfaff's transformation* and (15.1.14d) is referred to as *Euler's transformation*.
All continued fraction representations for the Gauss hypergeometric function given in this chapter are derived from either the contiguous relation [AAR99, p. 97]

$$ {}_2F_1(a, b; c; z) = {}_2F_1(a, b+1; c+1; z) - \frac{a(c-b)}{c(c+1)}\, {}_2F_1(a+1, b+1; c+2; z) \tag{15.1.15} $$

or the basic form [GST06a]

$$ \begin{aligned} {}_2F_1(a, b; c+1; z) = &-\frac{c(c-1-(2c-a-b-1)z)}{(c-a)(c-b)z}\, {}_2F_1(a, b; c; z) \\ &- \frac{c(c-1)(z-1)}{(c-a)(c-b)z}\, {}_2F_1(a, b; c-1; z) \end{aligned} \tag{15.1.16a} $$

associated with the triplet $(0, 0, 1)$, where we have taken $n = 0$. By means of (15.1.14) the following recurrence relations can be obtained from (15.1.16a) [AAR99, p. 94]:

$$ \begin{aligned} {}_2F_1(a, b+1; c+1; z) = &\frac{c(c-1+(b-a)z)}{b(c-a)z}\, {}_2F_1(a, b; c; z) \\ &- \frac{c(c-1)}{b(c-a)z}\, {}_2F_1(a, b-1; c-1; z), \end{aligned} \tag{15.1.16b} $$

$$ \begin{aligned} {}_2F_1(a+1, b+1; c+1; z) = &\frac{c(-c+1+(a+b-1)z)}{abz(1-z)}\, {}_2F_1(a, b; c; z) \\ &+ \frac{c(c-1)}{abz(1-z)}\, {}_2F_1(a-1, b-1; c-1; z) \end{aligned} \tag{15.1.16c} $$

and another relation in which the roles of a and b in (15.1.16b) are switched. All recurrence relations in (15.1.16) can be associated with triplets of the form $(s_1, s_2, 1)$ where $s_1, s_2 \neq -1$.

15.2 Stieltjes transform

The Gauss hypergeometric series (15.1.4) has the following integral representation due to Euler [AAR99, p. 65]:

$$
{}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{-a+c-1}(1-tz)^{-b}\, dt,
$$
$$
z \in \mathbb{C} \setminus [1, \infty), \quad b, c-a \in \mathbb{C} \setminus \mathbb{Z}_0^-, \quad \Re c > \Re a > 0. \quad (15.2.1)
$$

Observe that ${}_2F_1(a, b; c; z)$ is symmetric in a and b, but the right-hand side of (15.2.1) is not. For the remainder of the section we assume $a, b, c \in \mathbb{R}^+$ and $c > a$. If $b = 1$ and z is replaced by $-z$, we find

$$
z\, {}_2F_1(a, 1; c; -z) = \int_0^1 \frac{z\phi(t)}{1+tz}\, dt, \qquad z \in \mathbb{C} \setminus (-\infty, -1], \qquad (15.2.2)
$$

where

$$
\phi(t) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} t^{a-1}(1-t)^{-a+c-1}. \qquad (15.2.3)
$$

From the conditions on the parameters a and c, it follows that $\phi(t)$ is well-defined and positive, and (15.2.2) is the Stieltjes integral transform (5.2.4a). From the expansion

$$
z\, {}_2F_1(a, 1; c; -z) = \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k}{(c)_k} z^{k+1}, \qquad (15.2.4)
$$

we obtain the moments

$$
\mu_0 = 1, \quad \mu_k = \int_0^1 t^k \phi(t)\, dt = \frac{(a)_k}{(c)_k}, \quad k = 1, 2, 3, \ldots. \qquad (15.2.5)
$$

15.3 Continued fraction representations

In this section continued fraction representations are given for ratios of hypergeometric functions of the form ${}_2F_1(a, b; c; z)/{}_2F_1(a, b+1; c+1; z)$. Continued fraction representations for other ratios of hypergeometric functions can be obtained from these by applying the transformation formulas (15.1.14). As an example, we give the Nörlund fraction for the ratio

${}_2F_1(a,b;c;z)/{}_2F_1(a+1,b+1;c+1;z)$, which is related to the T-fraction for ${}_2F_1(a,b;c;z)/{}_2F_1(a,b+1;c+1;z)$ by (15.1.14c).

S-fraction. Since the classical Stieltjes moment problem has a solution $\phi(t)$ for μ_k given by (15.2.5), it follows from *Theorem* 5.1.1 that there exists an S-fraction of the form

$$\operatorname*{K}_{m=1}^{\infty}\left(\frac{d_m z}{1}\right), \qquad d_m > 0, \tag{15.3.1}$$

corresponding to the asymptotic series (15.2.4).
The moments μ_k satisfy Carleman's criterion (5.1.16a) and thus the solution to the Stieltjes moment problem for the sequence μ_k is unique. Hence from *Theorem* 5.2.1, the S-fraction (15.3.1) is convergent, and

$$z\,{}_2F_1(a,1;c;-z) = \int_0^\infty \frac{z\phi(t)}{1+zt}\,dt = \operatorname*{K}_{m=1}^{\infty}\left(\frac{d_m z}{1}\right),$$
$$z \in \mathbb{C}\setminus(-\infty,-1], \quad c > a > 0. \tag{15.3.2}$$

Observe that we here have a larger domain of convergence than in *Theorem* 5.2.1. Explicit formulas for the coefficients d_m can be obtained by considering (15.3.2) for $-z\,{}_2F_1(a,1;c+1;z)$ and comparing it to (15.3.4).

C-fractions. From (15.1.15) we obtain the regular C-fraction [AAR99, pp. 97–98; JT80, pp. 199–201]

$$\frac{{}_2F_1(a,b;c;z)}{{}_2F_1(a,b+1;c+1;z)} = 1 + \operatorname*{K}_{m=1}^{\infty}\left(\frac{a_m z}{1}\right), \quad z \in \mathbb{C}\setminus[1,+\infty),$$
$$a,b \in \mathbb{C}, \quad c \in \mathbb{C}\setminus\mathbb{Z}_0^-, \tag{15.3.3a}$$

where the coefficients a_m are given by

$$a_{2k+1} = \frac{-(a+k)(c-b+k)}{(c+2k)(c+2k+1)}, \qquad k \in \mathbb{N}_0, \tag{15.3.3b}$$
$$a_{2k} = \frac{-(b+k)(c-a+k)}{(c+2k-1)(c+2k)}, \qquad k \in \mathbb{N}. \tag{15.3.3c}$$

The continued fraction (15.3.3) is called the *Gauss continued fraction.* For $a,b,c \in \mathbb{R}$ in (15.3.3), the continued fraction is an S-fraction in $(-z)$ from a certain m on. From (15.1.5) and (15.3.3) we obtain the C-fraction representation

$$z\,{}_2F_1(a,1;c+1;z) = \operatorname*{K}_{m=1}^{\infty}\left(\frac{c_m z}{1}\right), \quad z \in \mathbb{C}\setminus[1,+\infty),$$
$$a \in \mathbb{C}, \quad c \in \mathbb{C}\setminus\mathbb{Z}_0^-, \tag{15.3.4a}$$

where the coefficients c_m are given by

$$c_1 = 1, \qquad c_{2k+2} = \frac{-(a+k)(c+k)}{(c+2k)(c+2k+1)}, \qquad k \in \mathbb{N}_0, \quad (15.3.4b)$$

$$c_{2k+1} = \frac{-k(c-a+k)}{(c+2k-1)(c+2k)}, \qquad k \in \mathbb{N}. \quad (15.3.4c)$$

The continued fractions (15.3.3) and (15.3.4) are limit periodic with

$$\lim_{m\to\infty} a_m = -\frac{1}{4} = \lim_{m\to\infty} c_m.$$

We recall from (7.7.7) that the modification

$$w(z) = \frac{\sqrt{1-z}-1}{2} \qquad (15.3.5)$$

may be useful. From (7.7.11) we also find that $w(z)$ can be improved if

$$r = \lim_{n\to\infty} \frac{a_{n+1}+1/4}{a_n+1/4}$$

exists. Then use of the modification

$$w_n^{(1)}(z) = w(z) + \frac{a_{n+1}z + z/4}{1+(r+1)w(z)} \qquad (15.3.6)$$

is recommended. For (15.3.3) we find $r = -1$ if $a - b \neq 1/2$ and $r = 1$ if $a - b = 1/2$. The modification $w(z)$ can also be used for (15.3.4). The modification $w_1^{(n)}(z)$ applies with a_{n+1} replaced by c_{n+1} and $r = -1$ if $a \neq 1/2$ and $r = 1$ if $a = 1/2$.

EXAMPLE 15.3.1: Consider

$${}_2F_1\left(1/2, 1; 3/2; z\right) = \frac{1}{2\sqrt{z}} \operatorname{Ln}\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

which by (15.3.4) has the regular C-fraction representation

$$z\,{}_2F_1\left(1/2, 1; 3/2; z\right) = \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{c_m z}{1}\right), \qquad z \in \mathbb{C} \setminus [1, +\infty) \qquad (15.3.7a)$$

with

$$c_1 = 1, \quad c_m = \frac{-(m-1)^2}{4(m-1)^2 - 1}, \quad m \geq 2. \tag{15.3.7b}$$

Use of the modifications $w(z) = {}^1\!/\!_2 \left(\sqrt{1-z} - 1\right)$ and, with $r = 1$,

$$w_n^{(1)}(z) = w(z) + \frac{c_{n+1}z + z/4}{1 + 2w(z)}$$

is illustrated in the *Tables* 15.3.1 and 15.3.2. The approximants of (15.3.4) are first evaluated without modification and subsequently with the modifications given by (15.3.5) and (15.3.6).

T-fractions. For the ratio of hypergeometric series in (15.3.3) a T-fraction representation, already found by Euler, can be obtained from the recurrence relation (15.1.16b) [AAR99, p. 98]. The correspondence and convergence of this general T-fraction are given in [CJM88]:

$$\frac{{}_2F_1(a,b;c;z)}{{}_2F_1(a,b+1;c+1;z)} = \frac{c + (b-a+1)z}{c} + \frac{1}{c} \operatorname*{K}_{m=1}^{\infty} \left(\frac{c_m z}{e_m + d_m z} \right),$$
$$|z| < 1, \quad a, b \in \mathbb{C}, \quad c \in \mathbb{C} \setminus \mathbb{Z}_0^-, \tag{15.3.8a}$$

$$\frac{(b-a+1)z}{c} \frac{{}_2F_1(b-c+1, b; b-a+1; 1/z)}{{}_2F_1(b-c+1, b+1; b-a+2; 1/z)} =$$
$$\frac{c + (b-a+1)z}{c} + \frac{1}{c} \operatorname*{K}_{m=1}^{\infty} \left(\frac{c_m z}{e_m + d_m z} \right),$$
$$|z| > 1, \quad b - a \neq -2, -3, \ldots, \quad c \neq 0, \tag{15.3.8b}$$

where

$$c_m = -(c-a+m)(b+m), \quad e_m = c+m, \quad d_m = b-a+m+1, \quad m \geq 1. \tag{15.3.8c}$$

For $b = 0$ in (15.3.8) we find the M-fraction representation

$${}_2F_1(a, 1; c+1; z) = \frac{c}{c + (1-a)z} + \operatorname*{K}_{m=1}^{\infty} \left(\frac{c_m z}{e_m + d_m z} \right),$$
$$|z| < 1, \quad a \in \mathbb{C}, \quad c \in \mathbb{C} \setminus \mathbb{Z}_0^-, \tag{15.3.9a}$$

$$\frac{cz^{-1}}{(1-a)} \, {}_2F_1(1-c, 1; 2-a; 1/z) = \frac{c}{c + (1-a)z} + \operatorname*{K}_{m=1}^{\infty} \left(\frac{c_m z}{e_m + d_m z} \right),$$
$$|z| > 1, \quad a \neq 2, 3, 4, \ldots, \quad c \in \mathbb{C}, \tag{15.3.9b}$$

where

$$c_m = -m(c-a+m), \quad e_m = c+m, \quad d_m = m+1-a, \quad m \geq 1. \quad (15.3.9c)$$

The infinite fraction parts in (15.3.8) and (15.3.9) are limit periodic. A suitable modification is found by combining (7.7.4) and (7.7.5), and is given by

$$\tilde{w}_n(z) = (e_n + d_n z)w(z) \qquad (15.3.10a)$$

with

$$w(z) = \mathrm{K}\left(\frac{-z/(1+z)^2}{1}\right) = \begin{cases} \dfrac{-z}{1+z}, & |z| < 1, \\ \dfrac{-1}{1+z}, & |z| > 1. \end{cases} \qquad (15.3.10b)$$

For $|z| = 1$ the continued fraction in (15.3.10b) diverges, except for $z = 1$, where it converges to $-1/2$. The modification (15.3.10) for the continued fractions in (15.3.8) and (15.3.9) can be improved by combining (7.7.4) and (7.7.10) into

$$\tilde{w}_n^{(1)}(z) = (e_n + d_n z)\left(w(z) + \frac{\frac{c_{n+1}z}{(e_{n+1}+d_{n+1}z)(e_n+d_n z)} + \frac{z}{(1+z)^2}}{1 + (r+1)w(z)}\right), \quad r = 1. \qquad (15.3.11)$$

EXAMPLE 15.3.2: With $a = c = 1/2$ in (15.3.9) we find the M-fraction representation

$$\frac{1/2}{1/2 + z/2} - \frac{z}{3/2 + 3z/2} - \frac{4z}{5/2 + 5z/2} - \ldots, \qquad |z| < 1, \qquad (15.3.12)$$

which corresponds at $z = 0$ to

$$ {}_2F_1(1/2, 1; 3/2; z) = \frac{1}{2\sqrt{z}} \mathrm{Ln}\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)$$

and at $z = \infty$ to

$$z^{-1}\ {}_2F_1(1/2, 1; 3/2; z^{-1}).$$

In the *Tables* 15.3.1 and 15.3.2 the approximants of (15.3.12) are first evaluated without modification and then with the modifications (15.3.10) and (15.3.11), respectively.

Another T-fraction for the ratio ${}_2F_1(a,b;c;z)/{}_2F_1(a,b+1;c+1;z)$ which is given in [Fra56; CJM88] reduces to an Euler fraction (1.7.2) for ${}_2F_1(a,1;c+1;z)$ when $b=0$. In [Fra56] T-fraction representations are given for several other ratios of hypergeometric functions. All these T-fractions can be derived from (15.3.8) and the transformation formulas (15.1.14).

Nörlund fractions. By applying (15.1.14c) to the T-fraction (15.3.8), we obtain the fraction (15.3.13) for ${}_2F_1(a,b;c;z)/{}_2F_1(a+1,b+1;c+1;z)$. This continued fraction can also be obtained from the recurrence relation (15.1.16c) [Nör24; LW92, pp. 304–306]:

$$\frac{{}_2F_1(a,b;c;z)}{{}_2F_1(a+1,b+1;c+1;z)} = \frac{c-(a+b+1)z}{c} + \frac{1}{c}\operatorname*{K}_{m=1}^{\infty}\left(\frac{c_m(z-z^2)}{e_m+d_m z}\right),$$
$$\Re z < 1/2, \quad a,b\in\mathbb{C}, \quad c\in\mathbb{C}\setminus\mathbb{Z}_0^-, \qquad (15.3.13a)$$

where

$$c_m=(a+m)(b+m), \quad e_m=c+m, \quad d_m=-(a+b+2m+1), \quad m\geq 1. \qquad (15.3.13b)$$

The continued fraction (15.3.13) is called the *Nörlund fraction*. It corresponds at $z=0$ to the left-hand side of (15.3.13) with order of correspondence n.

From (15.1.5) and (15.3.13), we obtain the continued fraction representation

$${}_2F_1(a+1,1;c+1;z) = \frac{c}{c-(a+1)z} + \operatorname*{K}_{m=1}^{\infty}\left(\frac{c_m(z-z^2)}{e_m+d_m z}\right) \qquad (15.3.14a)$$

where

$$c_m=(a+m)m, \quad e_m=c+m, \quad d_m=-(a+2m+1), \quad m\geq 1. \qquad (15.3.14b)$$

The infinite fraction parts in (15.3.13) and (15.3.14) are limit periodic. Hence, by combining (7.7.4) and (7.7.5), we find for (15.3.13) and (15.3.14) the modification

$$\tilde{w}_n(z) = (e_n+d_n z)w(z) \qquad (15.3.15a)$$

with

$$w(z) = \mathop{\mathrm{K}}\left(\frac{(z-z^2)/(1-2z)^2}{1}\right) = \begin{cases} \dfrac{z}{1-2z}, & \Re z < 1/2, \\ \dfrac{z-1}{1-2z}, & \Re z > 1/2. \end{cases} \tag{15.3.15b}$$

The above modification can be improved by combining (7.7.4) and (7.7.10) into

$$\tilde{w}_n^{(1)}(z) = (e_n + d_n z)\left(w(z) + \frac{\frac{c_{n+1}(z-z^2)}{(e_{n+1}+d_{n+1}z)(e_n+d_n z)} - \frac{z-z^2}{(1-2z)^2}}{1+(r+1)w(z)}\right), \qquad r = 1. \tag{15.3.16}$$

Applying the transformation formula (15.1.14d), a Nörlund-like fraction can be obtained for the ratio ${}_2F_1(a,b;c;z)/{}_2F_1(a,b;c+1;z)$. Another way to derive this continued fraction is based on the fact that the sequence $\{{}_2F_1(a,b;c+n;z)\}_n$ is a minimal solution of (15.1.16a) for $\Re z < 1/2$ [GST06a] and on Pincherle's *Theorem* 3.6.1.

EXAMPLE 15.3.3: For $a = -1/2$ and $c = 1/2$ we find from (15.3.14) and (15.1.14d) that

$$\begin{aligned} {}_2F_1(1/2, 1; 3/2; z) &= \frac{1}{1-z}\frac{{}_2F_1(1, 1/2; 3/2; z)}{{}_2F_1(1, 1/2; 1/2; z)} \\ &= \frac{1}{2\sqrt{z}}\,\mathrm{Ln}\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right) \\ &= \frac{1}{1-z} + \frac{z(1-z)}{3/2 - 5/2 z} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{m(m-1/2)z(1-z)}{(m+1/2)-(2m+1/2)z}\right), \\ &\qquad \Re z < 1/2. \end{aligned} \tag{15.3.17}$$

In the *Tables* 15.3.1 and 15.3.2 the fractions (15.3.7), (15.3.12) and (15.3.17), which are special cases of

- the C-fraction (15.3.4),
- the M-fraction (15.3.9) and
- the Nörlund fraction (15.3.14),

respectively, are evaluated without and with the suggested modifications.

TABLE 15.3.1: Relative error of the 5^{th} partial sum and 5^{th} (modified) approximants. More details can be found in the *Examples* 15.3.1, 15.3.2 and 15.3.3.

x	${}_2F_1(1/2, 1; 3/2; x)$	(15.3.7)	(15.3.12)	(15.3.17)
0.1	1.035488e+00	1.9e−08	1.4e−05	6.1e−06
0.2	1.076022e+00	7.9e−07	4.4e−04	3.4e−04
0.3	1.123054e+00	8.0e−06	3.1e−03	4.9e−03
0.4	1.178736e+00	4.7e−05	1.2e−02	4.3e−02

x	${}_2F_1(1/2, 1; 3/2; x)$	(15.3.7)	(15.3.7)	(15.3.7)
0.1	1.035488e+00	1.9e−08	2.0e−10	1.6e−12
0.2	1.076022e+00	7.9e−07	8.7e−09	1.5e−10
0.3	1.123054e+00	8.0e−06	9.4e−08	2.7e−09
0.4	1.178736e+00	4.7e−05	5.9e−07	2.5e−08

x	${}_2F_1(1/2, 1; 3/2; x)$	(15.3.12)	(15.3.12)	(15.3.12)
0.1	1.035488e+00	1.4e−05	1.7e−07	5.7e−09
0.2	1.076022e+00	4.4e−04	6.3e−06	4.7e−07
0.3	1.123054e+00	3.1e−03	5.7e−05	7.0e−06
0.4	1.178736e+00	1.2e−02	2.9e−04	5.5e−05

x	${}_2F_1(1/2, 1; 3/2; x)$	(15.3.17)	(15.3.17)	(15.3.17)
0.1	1.035488e+00	6.1e−06	6.4e−07	1.6e−08
0.2	1.076022e+00	3.4e−04	3.9e−05	2.1e−06
0.3	1.123054e+00	4.9e−03	6.6e−04	6.1e−05
0.4	1.178736e+00	4.3e−02	8.3e−03	1.4e−03

TABLE 15.3.2: Relative error of the 20th partial sum and 20th (modified) approximants. More details can be found in the *Examples* 15.3.1, 15.3.2 and 15.3.3.

x	${}_2F_1(1/2, 1; 3/2; x)$	(15.3.7)	(15.3.12)	(15.3.17)
0.1	1.035488e+00	4.0e−32	1.5e−20	1.5e−20
0.2	1.076022e+00	1.3e−25	1.5e−14	1.7e−13
0.3	1.123054e+00	1.4e−21	4.8e−11	7.6e−09
0.4	1.178736e+00	1.8e−18	1.4e−08	5.0e−05

x	${}_2F_1(1/2, 1; 3/2; x)$	(15.3.7)	(15.3.7)	(15.3.7)
0.1	1.035488e+00	4.0e−32	2.6e−35	6.5e−38
0.2	1.076022e+00	1.3e−25	8.8e−29	4.8e−31
0.3	1.123054e+00	1.4e−21	1.1e−24	9.6e−27
0.4	1.178736e+00	1.8e−18	1.4e−21	1.9e−23

x	${}_2F_1(1/2, 1; 3/2; x)$	(15.3.12)	(15.3.12)	(15.3.12)
0.1	1.035488e+00	1.5e−20	1.1e−23	1.2e−25
0.2	1.076022e+00	1.5e−14	1.4e−17	3.3e−19
0.3	1.123054e+00	4.8e−11	5.9e−14	2.3e−15
0.4	1.178736e+00	1.4e−08	2.3e−11	1.4e−12

x	${}_2F_1(1/2, 1; 3/2; x)$	(15.3.17)	(15.3.17)	(15.3.17)
0.1	1.035488e+00	1.5e−20	4.0e−22	2.9e−24
0.2	1.076022e+00	1.7e−13	4.5e−15	7.1e−17
0.3	1.123054e+00	7.6e−09	2.4e−10	6.4e−12
0.4	1.178736e+00	5.0e−05	2.3e−06	1.1e−07

EXAMPLE 15.3.4: One of the special values of the Gauss hypergeometric function is

$$ {}_2F_1(2a, 1; a+1; 1/2) = \sqrt{\pi}\frac{\Gamma(a+1)}{\Gamma(a+1/2)}, \qquad a+1 \in \mathbb{C} \setminus \mathbb{Z}_0^-. $$

It is illustrated in the *Tables* 15.3.3, 15.3.4, 15.3.5 and 15.3.6.

TABLE 15.3.3: Relative error of the 5th partial sum and 5th approximants for *Example* 15.3.4.

x	${}_2F_1(2x, 1; x+1; 1/2)$	(15.1.4)	(15.3.4)	(15.3.9)
−400.25	3.544908e+01	8.3e−01	7.1e−01	3.3e+00
−100.25	1.772457e+01	6.5e−01	3.7e−01	1.9e+00
−50.25	1.253322e+01	4.8e−01	2.2e−02	1.5e+00
−10.25	5.605866e+00	9.8e−01	5.9e+00	5.1e−03
−3.25	3.075227e+00	3.9e+00	8.4e−01	2.9e+00
−1.25	1.797210e+00	2.8e−03	2.9e−02	1.0e+00
0.25	1.311029e+00	2.8e−03	2.6e−04	7.8e−02
2.25	2.809347e+00	4.6e−02	2.9e−04	2.6e−05
5.25	4.158909e+00	1.2e−01	1.3e−02	2.3e−05
20.25	8.025424e+00	3.7e−01	1.5e−01	4.0e−02
40.25	1.127993e+01	5.2e−01	2.9e−01	1.4e−01
200.25	2.509761e+01	7.7e−01	6.2e−01	8.9e−01
500.25	3.965309e+01	8.5e−01	7.5e−01	1.8e+00

x	$\lvert {}_2F_1(2(\mathrm{i}x), 1; \mathrm{i}x+1; 1/2)\rvert_s$	(15.1.4)	(15.3.4)	(15.3.9)
−400.25	3.546015e+01	8.9e−01	8.2e−01	1.8e+00
−100.25	1.774668e+01	7.9e−01	6.7e−01	6.7e−01
−50.25	1.256444e+01	7.1e−01	5.5e−01	3.8e−01
−10.25	5.674621e+00	4.0e−01	1.8e−01	7.9e−02
−3.25	3.195337e+00	1.0e−01	2.8e−02	3.4e−02
−1.25	1.982433e+00	1.9e−02	4.3e−03	4.8e−02
0.25	1.094362e+00	2.4e−03	4.4e−04	2.9e−01
2.25	2.658683e+00	5.5e−02	1.3e−02	3.4e−02
5.25	4.061202e+00	2.1e−01	6.6e−02	4.3e−02
20.25	7.976042e+00	5.6e−01	3.5e−01	1.6e−01
40.25	1.124496e+01	6.8e−01	5.1e−01	3.1e−01
200.25	2.508194e+01	8.4e−01	7.5e−01	1.1e+00
500.25	3.964318e+01	9.0e−01	8.4e−01	2.1e+00

TABLE 15.3.4: Relative error of the 5^{th} (modified) approximants for *Example* 15.3.4. For (15.3.4) the modifications (15.3.5) and (15.3.6) are used, in that order. For (15.3.9) we show the modifications (15.3.10) and (15.3.11).

x	${}_2F_1(2x, 1; x+1; 1/2)$	(15.3.4)	(15.3.4)	(15.3.4)
−400.25	3.544908e+01	7.1e−01	7.0e−01	2.1e+00
−100.25	1.772457e+01	3.7e−01	3.4e−01	1.7e+00
−50.25	1.253322e+01	2.2e−02	7.6e−02	1.4e+00
−10.25	5.605866e+00	5.9e+00	5.0e+00	2.8e−01
−3.25	3.075227e+00	8.4e−01	8.8e−01	1.2e+00
−1.25	1.797210e+00	2.9e−02	2.6e−01	3.3e−02
0.25	1.311029e+00	2.6e−04	2.9e−06	7.3e−07
2.25	2.809347e+00	2.9e−04	1.8e−04	1.8e−05
5.25	4.158909e+00	1.3e−02	1.1e−02	2.4e−03
20.25	8.025424e+00	1.5e−01	1.4e−01	7.9e−02
40.25	1.127993e+01	2.9e−01	2.7e−01	2.5e−01
200.25	2.509761e+01	6.2e−01	6.1e−01	2.6e+00
500.25	3.965309e+01	7.5e−01	7.4e−01	1.4e+01

x	$\lvert {}_2F_1(2(\mathrm{i}x), 1; \mathrm{i}x+1; 1/2)\rvert_s$	(15.3.4)	(15.3.4)	(15.3.4)
−400.25	3.546015e+01	8.2e−01	8.1e−01	1.9e+00
−100.25	1.774668e+01	6.7e−01	6.6e−01	8.7e−01
−50.25	1.256444e+01	5.5e−01	5.4e−01	4.8e−01
−10.25	5.674621e+00	1.8e−01	1.7e−01	7.0e−02
−3.25	3.195337e+00	2.8e−02	2.6e−02	5.8e−03
−1.25	1.982433e+00	4.3e−03	3.5e−03	3.6e−04
0.25	1.094362e+00	4.4e−04	1.6e−04	6.5e−06
2.25	2.658683e+00	1.3e−02	1.2e−02	2.1e−03
5.25	4.061202e+00	6.6e−02	6.1e−02	1.9e−02
20.25	7.976042e+00	3.5e−01	3.3e−01	1.8e−01
40.25	1.124496e+01	5.1e−01	4.9e−01	3.9e−01
200.25	2.508194e+01	7.5e−01	7.5e−01	1.4e+00
500.25	3.964318e+01	8.4e−01	8.3e−01	2.0e+00

x	${}_2F_1(2x, 1; x+1; 1/2)$	(15.3.9)	(15.3.9)	(15.3.9)
−400.25	3.544908e+01	3.3e+00	3.8e+00	6.0e−01
−100.25	1.772457e+01	1.9e+00	2.2e+00	5.7e−02
−50.25	1.253322e+01	1.5e+00	1.7e+00	7.8e−01
−10.25	5.605866e+00	5.1e−03	5.5e−01	2.3e+00
−3.25	3.075227e+00	2.9e+00	5.6e+01	5.1e−01
−1.25	1.797210e+00	1.0e+00	1.5e+00	4.1e+00
0.25	1.311029e+00	7.8e−02	7.6e−03	1.4e−03
2.25	2.809347e+00	2.6e−05	4.4e−05	1.7e−05
5.25	4.158909e+00	2.3e−05	9.2e−04	4.9e−04
20.25	8.025424e+00	4.0e−02	8.6e−02	5.7e−02
40.25	1.127993e+01	1.4e−01	2.5e−01	1.6e−01
200.25	2.509761e+01	8.9e−01	1.2e+00	5.1e−01
500.25	3.965309e+01	1.8e+00	2.4e+00	6.7e−01

x	$\lvert {}_2F_1(2(\mathrm{i}x), 1; \mathrm{i}x+1; 1/2)\rvert_s$	(15.3.9)	(15.3.9)	(15.3.9)
−400.25	3.546015e+01	1.8e+00	2.3e+00	7.6e−01
−100.25	1.774668e+01	6.7e−01	9.0e−01	5.7e−01
−50.25	1.256444e+01	3.8e−01	5.3e−01	4.3e−01
−10.25	5.674621e+00	7.9e−02	1.2e−01	9.5e−02
−3.25	3.195337e+00	3.4e−02	4.4e−02	2.3e−02
−1.25	1.982433e+00	4.8e−02	3.5e−02	1.3e−02
0.25	1.094362e+00	2.9e−01	8.7e−02	1.0e−02
2.25	2.658683e+00	3.4e−02	3.7e−02	1.7e−02
5.25	4.061202e+00	4.3e−02	6.3e−02	3.9e−02
20.25	7.976042e+00	1.6e−01	2.3e−01	2.2e−01
40.25	1.124496e+01	3.1e−01	4.4e−01	3.8e−01
200.25	2.508194e+01	1.1e+00	1.5e+00	6.8e−01
500.25	3.964318e+01	2.1e+00	2.6e+00	7.9e−01

TABLE 15.3.5: Relative error of the 20$^{\text{th}}$ partial sum and 20$^{\text{th}}$ approximants for *Example* 15.3.4.

x	${}_2F_1(2x, 1; x+1; 1/2)$	(15.1.4)	(15.3.4)	(15.3.9)
−400.25	3.544908e+01	3.4e−01	2.1e+00	9.7e−01
−100.25	1.772457e+01	1.0e+00	6.3e+00	1.1e+00
−50.25	1.253322e+01	5.9e+00	8.1e−01	2.6e+00
−10.25	5.605866e+00	1.7e−01	1.3e+00	1.1e+00
−3.25	3.075227e+00	2.0e−09	1.4e−11	1.5e+00
−1.25	1.797210e+00	3.3e−09	2.2e−14	1.0e−01
0.25	1.311029e+00	3.6e−08	8.8e−16	4.0e−06
2.25	2.809347e+00	4.2e−06	5.2e−17	1.0e−12
5.25	4.158909e+00	7.1e−05	1.0e−17	1.9e−17
20.25	8.025424e+00	6.4e−03	1.5e−09	4.6e−19
40.25	1.127993e+01	3.6e−02	9.2e−07	2.5e−10
200.25	2.509761e+01	3.1e−01	3.0e−03	1.9e−04
500.25	3.965309e+01	5.1e−01	3.3e−02	6.2e−03

x	$\lvert {}_2F_1(2(\mathrm{i}x), 1; \mathrm{i}x+1; 1/2) \rvert_s$	(15.1.4)	(15.3.4)	(15.3.9)
−400.25	3.546015e+01	6.6e−01	8.2e−02	2.3e−02
−100.25	1.774668e+01	4.1e−01	3.1e−03	3.5e−04
−50.25	1.256444e+01	2.3e−01	2.3e−04	1.8e−05
−10.25	5.674621e+00	1.6e−03	1.2e−08	7.2e−08
−3.25	3.195337e+00	4.8e−06	3.1e−12	2.3e−07
−1.25	1.982433e+00	2.5e−07	3.6e−14	2.1e−06
0.25	1.094362e+00	2.3e−08	1.7e−15	3.2e−05
2.25	2.658683e+00	1.2e−06	4.0e−13	5.4e−07
5.25	4.061202e+00	4.5e−05	7.8e−11	9.3e−08
20.25	7.976042e+00	3.3e−02	1.6e−06	3.4e−07
40.25	1.124496e+01	1.7e−01	8.0e−05	6.3e−06
200.25	2.508194e+01	5.5e−01	2.1e−02	3.8e−03
500.25	3.964318e+01	6.9e−01	1.2e−01	3.6e−02

TABLE 15.3.6: Relative error of the 20$^{\text{th}}$ (modified) approximants for *Example* 15.3.4, for (15.3.4) with the modifications (15.3.5) and (15.3.6) in that order, and for (15.3.9) with the modifications (15.3.10) and (15.3.11).

x	${}_2F_1(2x, 1; x+1; 1/2)$	(15.3.4)	(15.3.4)	(15.3.4)
−400.25	3.544908e+01	2.1e+00	1.1e+00	1.7e+00
−100.25	1.772457e+01	6.3e+00	5.2e+00	7.6e+00
−50.25	1.253322e+01	8.1e−01	6.1e−01	3.9e−01
−10.25	5.605866e+00	1.3e+00	1.3e+00	8.2e−01
−3.25	3.075227e+00	1.4e−11	7.5e−12	5.6e−13
−1.25	1.797210e+00	2.2e−14	6.0e−15	1.2e−16
0.25	1.311029e+00	8.8e−16	6.0e−19	1.7e−19
2.25	2.809347e+00	5.2e−17	2.8e−17	1.2e−18
5.25	4.158909e+00	1.0e−17	2.3e−17	1.8e−18
20.25	8.025424e+00	1.5e−09	7.3e−09	1.2e−09
40.25	1.127993e+01	9.2e−07	3.6e−06	6.9e−07
200.25	2.509761e+01	3.0e−03	3.3e−02	3.1e−03
500.25	3.965309e+01	3.3e−02	4.0e−01	4.2e−02

x	$\lvert {}_2F_1(2(\mathrm{i}x), 1; \mathrm{i}x+1; 1/2)\rvert_s$	(15.3.4)	(15.3.4)	(15.3.4)
−400.25	3.546015e+01	8.2e−02	2.0e−01	9.3e−02
−100.25	1.774668e+01	3.1e−03	7.7e−03	2.7e−03
−50.25	1.256444e+01	2.3e−04	5.0e−04	1.6e−04
−10.25	5.674621e+00	1.2e−08	1.8e−08	3.0e−09
−3.25	3.195337e+00	3.1e−12	2.0e−12	1.2e−13
−1.25	1.982433e+00	3.6e−14	9.3e−15	2.3e−16
0.25	1.094362e+00	1.7e−15	1.2e−16	1.2e−18
2.25	2.658683e+00	4.0e−13	1.8e−13	7.8e−15
5.25	4.061202e+00	7.8e−11	7.5e−11	7.1e−12
20.25	7.976042e+00	1.6e−06	3.0e−06	7.3e−07
40.25	1.124496e+01	8.0e−05	1.7e−04	5.3e−05
200.25	2.508194e+01	2.1e−02	5.4e−02	2.0e−02
500.25	3.964318e+01	1.2e−01	2.8e−01	1.4e−01

x	${}_2F_1(2x, 1; x+1; 1/2)$	(15.3.9)	(15.3.9)	(15.3.9)
−400.25	3.544908e+01	9.7e−01	1.1e+00	2.2e+01
−100.25	1.772457e+01	1.1e+00	1.4e+00	3.0e+00
−50.25	1.253322e+01	2.6e+00	5.8e+00	5.4e−01
−10.25	5.605866e+00	1.1e+00	1.7e+00	7.3e+00
−3.25	3.075227e+00	1.5e+00	2.1e+00	5.5e+00
−1.25	1.797210e+00	1.0e−01	3.5e−02	5.3e−03
0.25	1.311029e+00	4.0e−06	7.3e−08	3.2e−09
2.25	2.809347e+00	1.0e−12	3.6e−13	6.1e−14
5.25	4.158909e+00	1.9e−17	1.9e−17	5.3e−18
20.25	8.025424e+00	4.6e−19	8.1e−17	4.1e−17
40.25	1.127993e+01	2.5e−10	9.5e−10	5.8e−10
200.25	2.509761e+01	1.9e−04	3.1e−04	2.5e−04
500.25	3.965309e+01	6.2e−03	8.2e−03	7.2e−03

x	$\lvert {}_2F_1(2(\mathrm{i}x), 1; \mathrm{i}x+1; 1/2)\rvert_s$	(15.3.9)	(15.3.9)	(15.3.9)
−400.25	3.546015e+01	2.3e−02	2.8e−02	2.5e−02
−100.25	1.774668e+01	3.5e−04	5.1e−04	4.1e−04
−50.25	1.256444e+01	1.8e−05	2.8e−05	2.1e−05
−10.25	5.674621e+00	7.2e−08	8.8e−08	4.3e−08
−3.25	3.195337e+00	2.3e−07	1.3e−07	3.4e−08
−1.25	1.982433e+00	2.1e−06	5.3e−07	6.3e−08
0.25	1.094362e+00	3.2e−05	2.7e−06	8.1e−08
2.25	2.658683e+00	5.4e−07	2.2e−07	4.4e−08
5.25	4.061202e+00	9.3e−08	7.4e−08	2.6e−08
20.25	7.976042e+00	3.4e−07	5.3e−07	3.2e−07
40.25	1.124496e+01	6.3e−06	1.0e−05	7.2e−06
200.25	2.508194e+01	3.8e−03	5.1e−03	4.4e−03
500.25	3.964318e+01	3.6e−02	4.4e−02	3.9e−02

15.4 Padé approximants

Basic polynomials. Explicit formulas for Padé approximants of ratios of hypergeometric functions ${}_2F_1(a, b; c; z)$ are developed in [WB93]. We define the operator $\mathcal{P}_k$ by

$$\mathcal{P}_k\left(\sum_{j=0}^{\infty} c_j z^j\right) := \sum_{j=0}^{k} c_j z^j. \tag{15.4.1}$$

For any pair $m, n \in \mathbb{N}_0$, we associate with the hypergeometric series ${}_2F_1(a, b; c; z)$ the polynomial

$$\begin{aligned} &V_{m,n}(a, b; c; z) := \\ &\quad \mathcal{P}_{m+n+1}\big({}_2F_1(a, b; c; z)\, {}_2F_1(-a-m, -b-n; -c-m-n; z)\big). \end{aligned} \tag{15.4.2}$$

The degree of $V_{m,n}$ is less than or equal to $\max(m, n)$. Certain identities of the polynomials $V_{m,n}$ are useful in the computation of Padé approximants [WB93]:

$$\begin{aligned} V_{m,n}(a, b; c; z) &= V_{m,n}(-a-m, -b-n; -c-m-n; z) \\ &= V_{n,m}(b, a; c; z), && (15.4.3) \\ V_{m,n}(b, a; c; z) &= V_{m,n}(c-b, c-a; c; z), && (15.4.4) \\ V_{m,n}(c, b; c; z) &= {}_2F_1(-n, b-c-m; -c-m-n; z), && (15.4.5) \\ V_{m,n}(a, 0; c; z) &= {}_2F_1(-n, -a-m; -c-m-n; z). && (15.4.6) \end{aligned}$$

Explicit formulas.

THEOREM 15.4.1: [WB93]

For $s \in \{-1, 0, 1\}$ and $m, n \in \mathbb{N}_0$ let

$$F_s(z) := \frac{{}_2F_1(a+s, b+1; c+s+1; z)}{{}_2F_1(a, b; c; z)} = \sum_{i=0}^{\infty} d_i z^i.$$

Furthermore let $m \geq n-1$ and, in case $b(c-a) \neq 0$, also $m \leq n-s$. Then the Padé approximants $r_{m,n}(z)$ for $F_s(z)$ are given by the irreducible form of:

$$r_{m,0}(z) = \mathcal{P}_m(F_s(z)), \tag{15.4.7}$$

$$r_{m,n}(z) = \frac{V_{m,n-1}(a+s, b+1; c+s+1; z)}{V_{m+s,n}(a, b; c; z)}, \qquad n \geq 1. \tag{15.4.8}$$

When $b = 0$ or $a = c$ the above theorem gives an explicit formula for the Padé approximants $r_{m,n}(z)$ for $m \geq n-1$. This case is also given in [Wal48, p. 341].

Because of the connection between Padé tables and C- and P-fractions, we obtain from *Theorem* 15.4.1 explicit expressions in terms of the polynomials $V_{m,n}(z)$ for the approximants of the C-fraction representation of $F_s(z)$. In case $s = 0$, the regular C-fraction for the reciprocal of $F_s(z)$ is given by (15.3.3). The $2n^{\text{th}}$ and $(2n+1)^{\text{th}}$ approximants of (15.3.3) are given by $1/r_{n,n}(z)$ and $1/r_{n,n+1}(z)$ respectively, after appropriate normalisation of the Padé approximant.

EXAMPLE 15.4.1: With $a = c = 1$, $b = 0, s = 0$ and z replaced by $-z$ we have

$$F_0(-z) = {}_2F_1(1,1;2;-z) = \frac{\text{Ln}(1+z)}{z}.$$

From *Theorem* 15.4.1 we find

$$r_{m,n}(-z) = \frac{V_{m,n-1}(1,1;2;-z)}{{}_2F_1(-n,-1-m;-1-m-n;-z)}, \qquad m \geq n-1 \geq 0,$$

and the following excerpt of the Padé table for $\text{Ln}(1+z)/z$.

$m\backslash n$	1	2	3
1	$\frac{1+\frac{z}{6}}{1+\frac{2z}{3}}$	$\frac{1+\frac{z}{2}}{1+z+\frac{z^2}{6}}$	
2	$\frac{1+\frac{z}{4}-\frac{z^2}{24}}{1+\frac{3z}{4}}$	$\frac{1+\frac{7z}{10}+\frac{z^2}{30}}{1+\frac{6z}{5}+\frac{3z^2}{10}}$	$\frac{1+z+\frac{11z^2}{60}}{1+\frac{3z}{2}+\frac{3z^2}{5}+\frac{z^3}{20}}$
3	$\frac{1+\frac{3z}{10}-\frac{z^2}{15}+\frac{z^3}{60}}{1+\frac{4z}{5}}$	$\frac{1+\frac{5z}{6}+\frac{z^2}{15}-\frac{z^3}{180}}{1+\frac{4z}{3}+\frac{2z^2}{5}}$	$\frac{1+\frac{17z}{14}+\frac{z^2}{3}+\frac{z^3}{140}}{1+\frac{12z}{7}+\frac{6z^2}{7}+\frac{4z^3}{35}}$

EXAMPLE 15.4.2: With $a = 2, b = 0, c = 1, s = 0$ we get the function

$$F_0(z) = \frac{{}_2F_1(2,1;2;z)}{{}_2F_1(2,0;1;z)} = \frac{1}{1-z}.$$

Except for the leftmost column in the Padé table, where the entries are the Taylor polynomials $1, 1+z, 1+z+z^2, \ldots$, the rest of the table is an infinite block where all entries are $1/(1-z)$. In this case the numerator and denominator polynomials in *Theorem* 15.4.1 have common factors.

Normality. The following theorems give results on the normality of the Padé table for the hypergeometric function ${}_2F_1(a,b;c;z)$ for different values of the parameters a,b and c. Some of these normality results follow naturally from the fact that ${}_2F_1(a,1;c;-z)$ has a Stieltjes fraction representation when $c > a > 0$.

THEOREM 15.4.2: [Wal48, pp. 389–390]

The Padé table for the hypergeometric function ${}_2F_1(a,1;c;z)$ *with* $c > a > 0$ *is normal.*

THEOREM 15.4.3: [dB77]

For $m \geq n$ *the Padé approximants* $r_{m,n}(z)$ *for the hypergeometric function* ${}_2F_1(a,1;c;z)$ *with* $a,c,c-a \notin \mathbb{Z}_0^-$ *are normal.*

Two-point Padé approximants. We associate with the hypergeometric series ${}_2F_1(a,b;c;z)$ the polynomial

$$P_{n,k}(a,b,c,z) := \mathcal{P}_n\big({}_2F_1(a,b;c;z){}_2F_1(1-a-k,-b-n;1-c-k-n;z)\big),\\ 0 \leq k \leq n, \quad (15.4.9)$$

where the operator $\mathcal{P}_n$ is defined in (15.4.1). It is shown in [WB95] that $P_{n,k}(a,b,c,z)$ is a polynomial in z of exact degree n when $0 \leq k \leq n$.

THEOREM 15.4.4: [WB95]

Let

$$L_0(z) = \Lambda_0\left(\frac{{}_2F_1(a,b+1;c+1;z)}{{}_2F_1(a,b;c;z)}\right),$$
$$L_\infty(z) = \Lambda_\infty\left(\frac{cz^{-1}\ {}_2F_1(b-c+1,b+1;b-a+2;z^{-1})}{(b-a+1)\ {}_2F_1(b-c+1,b;b-a+1;z^{-1})}\right).$$

Then the two-point Padé approximant $r^{(2)}_{n+k,n-k}(z)$ *defined by (4.5.3) which corresponds to* $L_0(z)$ *and* $L_\infty(z)$*, is given by*

$$r^{(2)}_{n+k,n-k}(z) = \frac{P_{n-1,k}(a,b+1,c,z)}{P_{n,k}(a,b,c,z)}, \qquad 0 \leq k \leq n.$$

When $b = 0$ in *Theorem* 15.4.4, the series $L_0(z)$ and $L_\infty(z)$ are given by

$$L_0(z) = {}_2F_1(a,1;c+1;z),$$
$$L_\infty(z) = \frac{cz^{-1}}{1-a}\ {}_2F_1(1-c,1;2-a;z^{-1}).$$

Because of the connection between two-point Padé approximants and M-fractions indicated in *Theorem* 4.6.1, we obtain from *Theorem* 15.4.4 explicit expressions for the approximants of the M-fraction corresponding to $L_0(z)$ and $L_\infty(z)$ in terms of the polynomials $P_{n,k}(a,b,c,z)$. For $L_0(z)$ and $L_\infty(z)$ given in *Theorem* 15.4.4, this M-fraction is the reciprocal of the continued fraction given in (15.3.8). The n^{th} approximant of this M-fraction is given by $r^{(2)}_{n,n}(z)$.

EXAMPLE 15.4.3: Let $a = c = 1/2$, $b = 0$ in *Theorem* 15.4.4. Then $L_0(z)$ and $L_\infty(z)$ are the same series at $z = 0$ and $z = \infty$ as in *Example* 15.3.2:

$$L_0(z) = {}_2F_1(1/2, 1; 3/2; z),$$
$$L_\infty(z) = z^{-1}{}_2F_1(1/2, 1; 3/2; z^{-1}).$$

For $k = 0$ and $n = 2$, the two-point Padé approximant $r^{(2)}_{2,2}(z)$ corresponding to $L_0(z)$ and $L_\infty(z)$ is given by

$$r^{(2)}_{2,2}(z) = \frac{P_{1,0}(1/2, 1, 1/2, z)}{P_{2,0}(1/2, 0, 1/2, z)} = \frac{1+z}{1+2/3z+z^2},$$

with order of correspondence

$$L_0(z) - \frac{P_{1,0}(1/2, 1, 1/2, z)}{P_{2,0}(1/2, 0, 1/2, z)} = O(z^2)$$

at $z = 0$, and

$$L_\infty(z) - \frac{P_{1,0}(1/2, 1, 1/2, z)}{P_{2,0}(1/2, 0, 1/2, z)} = O(z^{-3})$$

at $z = \infty$. Observe that $r^{(2)}_{2,2}(z)$ is the second approximant of the M-fraction (15.3.12) in *Example* 15.3.2.

15.5 Monotonicity properties

Limit periodicity in combination with monotonicity properties of S-fraction coefficients can be used to simplify truncation error bounds as seen in *Corollary* 7.4.2. Many ratios of hypergeometric functions with real parameters have such properties. We consider the continued fraction obtained by replacing z by $-z$ in the C-fraction (15.3.3)

$$\frac{{}_2F_1(a,b;c;-z)}{{}_2F_1(a,b+1;c+1;-z)} = 1 + \operatorname*{K}_{m=1}^{\infty}\left(\frac{c_m z}{1}\right), \tag{15.5.1}$$

where the coefficients $c_m = -a_m$ are given in (15.3.3).

THEOREM 15.5.1: [Waa05]

If $0 \le a < c+1$, $0 \le b < c$, then the continued fraction (15.5.1) is an S-fraction. Let

$$\alpha = a - b - 1/2,$$
$$\beta = (2b - c + 1)(2b - c),$$
$$\gamma = (2b - c + 1)(2b - c)(2a - c)(2a - c - 1).$$

(A1) If $\alpha = 0$ and $\beta > 0$, then the sequence $\{c_m\}$ is monotonely increasing.
(A2) If $\alpha = 0$ and $\beta < 0$, then the sequence $\{c_m\}$ is monotonely decreasing.
(B1) If $\alpha < 0$ and $\gamma \le 0$, then the sequences $\{c_{2k}\}$ and $\{c_{2k+1}\}$ are monotonely decreasing and increasing respectively.
(B2) If $\alpha > 0$ and $\gamma \le 0$, then the sequences $\{c_{2k}\}$ and $\{c_{2k+1}\}$ are monotonely increasing and decreasing respectively.
(C1) If $\alpha < 0$ and $\gamma > 0$, then there exists $M \in \mathbb{N}$ such that the sequences $\{c_{2M+2k}\}$ and $\{c_{2M+2k-1}\}$ are monotonely decreasing and increasing respectively.
(C2) If $\alpha > 0$ and $\gamma > 0$, then there exists $M \in \mathbb{N}$ such that the sequences $\{c_{2M+2k}\}$ and $\{c_{2M+2k-1}\}$ are monotonely increasing and decreasing respectively.

EXAMPLE 15.5.1: For $a = 1, b = 2$ and $c = 3$ case (C1) of *Theorem* 15.5.1 applies with $M = 1$. This is easily verified from

$$c_{2k+1} = \frac{(1+k)^2}{(3+2k)(4+2k)}, \quad k \in \mathbb{N}_0, \quad c_{2k} = \frac{(2+k)^2}{(2+2k)(3+2k)}, \quad k \in \mathbb{N}.$$

For $a = 1, b = 1/3$ and $c = 4$ we have case (C2) with

$$c_{2k+1} = \frac{(1+k)(11/3+k)}{(4+2k)(5+2k)}, \quad k \in \mathbb{N}_0, \quad c_{2k} = \frac{(1/3+k)(3+k)}{(3+2k)(4+2k)}, \quad k \in \mathbb{N}.$$

The sequence $\{c_{2k+1}\}$ is increasing from $k = 0$ to $k = 18$ and decreasing from $k = 18$ on. The sequence $\{c_{2k}\}$ is increasing from $k = 1$ on. Hence $M = 18$ in case (C2).

15.6 Hypergeometric series ${}_pF_q$

The hypergeometric series ${}_pF_q$ is defined in (15.1.1). For the special case $a_p = a_{q+1} = a$ and $b_q = 1$ we get the series representation

$$ {}_{q+1}F_q(a, a, \ldots, a; 1, 1, \ldots, 1; z) = 1 + \sum_{n=1}^{\infty} \left(\frac{(a)_n}{n!} \right)^{q+1} z^n, \qquad q \in \mathbb{N}. \tag{15.6.1} $$

Euler's integral (15.2.1) for the function ${}_2F_1(a, b; c; z)$ can be generalised to [AAR99, p. 67]

$$ {}_{p+1}F_{q+1}(a_1, \ldots, a_{p+1}; b_1, \ldots, b_{q+1}; z) = \frac{\Gamma(b_{q+1})}{\Gamma(a_{p+1})\Gamma(b_{q+1} - a_{p+1})} \times \int_0^1 t^{a_{p+1}-1}(1-t)^{b_{q+1}-a_{p+1}-1} \; {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; zt)\, dt, \quad \Re b_{q+1} > \Re a_{p+1}. \tag{15.6.2} $$

The special case $p = 1, q = 0$ and $a_2 = 1$ leads to the Stieltjes transform (15.2.2), as discussed in *Section* 15.2. The derivative of the hypergeometric series ${}_pF_q$ is given by

$$ \frac{d}{dz}\left({}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)\right) = \frac{\prod_{k=1}^{p} a_k}{\prod_{k=1}^{q} b_k} \; {}_pF_q((a_1+1), \ldots, (a_p+1); (b_1+1), \ldots, (b_q+1); z). \tag{15.6.3} $$

Continued fractions for the hypergeometric series ${}_3F_2$. For hypergeometric series ${}_pF_q$ with $p \le q + 2$ [AAR99] $(q+2)$-term recurrence relations exist. Under certain conditions, these relations become three-term recurrence relations. Such three-term recurrence relations can be used to obtain continued fraction representations for ratios of contiguous hypergeometric series, as described in *Section* 3.6. In particular, for the series ${}_3F_2(a, b, c; d, e; z)$ there exist four-term recurrence relations which, for the special case $z = 1$, reduce to three-term recurrence relations for contiguous series ${}_3F_2(a, b, c; d, e; 1)$ [DS00]. Observe that the series ${}_3F_2(a, b, c; d, e; 1)$ converges for

$$ \Re(d + e - a - b - c) > 0. $$

From these recurrence relations the following continued fraction representations can be obtained (the original paper contains several typographical errors which are hereby removed):

$$\frac{{}_3F_2(a,b,c;d,e;1)}{{}_3F_2(a+1,b,c;d,e;1)} = 1 - \frac{bc/d}{e-a-1\;+}\;\frac{(a+1)(d-b)(d-c)/d(d+1)}{1\;-}\;\frac{(d-a)(b+1)(c+1)/(d+1)(d+2)}{e-a-1\;+}\;\frac{(a+2)(d-b+1)(d-c+1)/(d+2)(d+3)}{1\;-}\;\frac{(d-a+1)(b+2)(c+2)/(d+3)(d+4)}{e-a-1\;+\;\ldots}, \quad (15.6.4)$$

$$\frac{{}_3F_2(a,b,c;d,e;1)}{{}_3F_2(a,b,c;d+1,e;1)} = 1 + \frac{abc/d(d+1)}{d+e-a-b-c\;-}\;\frac{(1+d-a)(1+d-b)(1+d-c)/(d+1)(d+2)}{1\;+}\;\frac{(a+1)(b+1)(c+1)/(d+2)(d+3)}{d+e-a-b-c\;-}\;\frac{(2+d-a)(2+d-b)(2+d-c)/(d+3)(d+4)}{1\;+}\;\frac{(a+2)(b+2)(c+2)/(d+4)(d+5)}{d+e-a-b-c\;-\;\ldots}, \quad (15.6.5)$$

$$\frac{{}_3F_2(a,b,c;d,e;1)}{{}_3F_2(a+1,b,c;d+1,e;1)} = 1 - \frac{(d-a)bc/d(d+1)}{(e-a-1)\;+}\;\frac{(a+1)(d-b+1)(d-c+1)/(d+1)(d+2)}{1\;-}\;\frac{(1+d-a)(1+b)(1+c)/(d+2)(d+3)}{(e-a-1)\;+\;\ldots}\;\frac{(a+2)(d-b+2)(d-c+2)/(d+3)(d+4)}{1\;-\;\ldots}, \quad (15.6.6)$$

$$\frac{{}_3F_2(a,b,c;d,e;1)}{{}_3F_2(a,b+1,c+1;d+1,e+1;1)} = \frac{e-a}{e}+ \cfrac{a(d-b)(d-c)/ed(d+1)}{1 - \cfrac{(d-a+1)(b+1)(c+1)/(e+1)(d+1)(d+2)}{(e-a)/(e+1) + \cfrac{(a+1)(d-b+1)(d-c+1)/(e+1)(d+2)(d+3)}{1 - \cfrac{(d-a+2)(b+2)(c+2)/(e+2)(d+3)(d+4)}{(e-a)/(e+2) + \dots}}}}. \quad (15.6.7)$$

EXAMPLE 15.6.1: Let $a = 0$, $b = c = d = 1$ and $e = 2$ in the continued fraction representation (15.6.6). Then we get

$$\begin{aligned}{}_3F_2(1,1,1;2,2;1) &= \sum_{m=1}^{\infty} \frac{1}{m^2} \\ &= \frac{\pi^2}{6} = \zeta(2) = \frac{1}{1-}\,\frac{1/2}{1+}\,\frac{1/6}{1-}\,\frac{2/3}{1+}\,\frac{2/5}{1-}\dots .\end{aligned}$$

16

Confluent hypergeometric functions

The confluent hypergeometric function ${}_1F_1(a;b;z)$ can be obtained as the result of a limit process applied to the hypergeometric function ${}_2F_1(a,b;c;z)$ introduced in *Chapter* 15. It is closely related to the Kummer functions, the Whittaker functions and the parabolic cylinder functions discussed here, and to the incomplete gamma functions discussed in *Chapter* 12.
Likewise the confluent hypergeometric limit function ${}_0F_1(;b;z)$ is obtained by applying a limit process to ${}_1F_1(a;b;z)$, and the formal confluent hypergeometric series ${}_2F_0(a,b;z)$ is the result of another limit process applied to ${}_2F_1(a,b;c;z)$.

16.1 Kummer functions

Definitions and elementary properties. The second-order differential equation [AS64, p. 504]

$$z\frac{d^2w}{dz^2} + (b-z)\frac{dw}{dz} - aw = 0, \qquad a \in \mathbb{C}, \quad b \in \mathbb{C} \setminus \mathbb{Z}_0^- \qquad (16.1.1)$$

is called the *confluent hypergeometric differential equation* or *Kummer's differential equation*. It can be obtained from the differential equation (15.1.2) by letting $p = 1$ and $q = 1$, or from the hypergeometric differential equation (15.1.3) by replacing z with z/a and taking the limit $a \to \infty$, or by replacing z with z/b and taking the limit $b \to \infty$.
Equation (16.1.1) has a regular singularity at the origin and an irregular singularity at infinity. Among the solutions are the *Kummer functions* $M(a,b,z)$ and $U(a,b,z)$.
The solution $M(a,b,z)$ with initial conditions $w(0) = 0$ and $(dw/dz)(0) = a/b$ is called the *confluent hypergeometric function of the first kind* or *Kummer's confluent hypergeometric function of the first kind*. It has a hypergeometric series representation with one parameter in the numerator, here

denoted a, and one parameter in the denominator, here denoted b. The series is given by

$$M(a,b,z) := {}_1F_1(a;b;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}, \qquad z \in \mathbb{C}, \quad a \in \mathbb{C}, \quad b \in \mathbb{C} \setminus \mathbb{Z}_0^-. \tag{16.1.2}$$

It converges locally uniformly in $\mathbb{C}$ to an entire function [AS64, p. 504].
The confluent hypergeometric function of the first kind (16.1.2) can be obtained from the Gauss hypergeometric series

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \qquad a,b \in \mathbb{C}, \quad c \in \mathbb{C} \setminus \mathbb{Z}_0^-$$

introduced in (15.1.4). The limit process used to obtain the confluent hypergeometric differential equation (16.1.1) gives [SO87, p. 461]

$$\lim_{a\to\infty} {}_2F_1\left(a,b;c;\frac{z}{a}\right) = M(b,c,z) = {}_1F_1(b;c;z), \tag{16.1.3a}$$

$$\lim_{b\to\infty} {}_2F_1\left(a,b;c;\frac{z}{b}\right) = M(a,c,z) = {}_1F_1(a;c;z). \tag{16.1.3b}$$

Observe that we use the notation ${}_1F_1(a;b;z)$.
If we apply the same limit process to the two linearly independent solutions (15.1.6) and (15.1.7) of (15.1.3), we find the two linearly independent solutions ${}_1F_1(a;b;z)$ and $z^{1-b}\,{}_1F_1(a-b+1;2-b;z)$ of (16.1.1). The function $M(a,b,z)$ is the first of these. The function $U(a,b,z)$ is a linear combination of these two solutions and is given by [SO87, p. 471]

$$U(a,b,z) := \frac{\Gamma(1-b)}{\Gamma(a-b+1)}\,{}_1F_1(a;b;z) + z^{1-b}\frac{\Gamma(b-1)}{\Gamma(a)}\,{}_1F_1(a-b+1;2-b;z), \qquad z \in \mathbb{C}, \quad a \in \mathbb{C}, \quad b \in \mathbb{C} \setminus \mathbb{Z}. \tag{16.1.4}$$

The function $U(a,b,z)$ is called the *confluent hypergeometric function of the second kind* or *Kummer's confluent hypergeometric function of the second kind*. Other frequently used names are the *Tricomi function* and the *Gordon function*.
We have

$$ {}_1F_1(0;b;z) = U(0,b,z) = 1. \tag{16.1.5}$$

The identities

$$ {}_1F_1(a;b;z) = e^z\,{}_1F_1(b-a;b;-z), \tag{16.1.6a}$$

$$U(a,b,z) = z^{1-b}U(a-b+1,2-b,z) \tag{16.1.6b}$$

are known as *Kummer transformations* [AS64, p. 505].
There are several connections between the confluent hypergeometric functions and the elementary functions as well as the error function, the logarithmic integral and functions related to the gamma function.

EXAMPLE 16.1.1:

$$ {}_1F_1(1; a+1; z) = a z^{-a} e^z \gamma(a, z), \tag{16.1.7a} $$

$$ {}_1F_1(a; a+1; -z) = a z^{-a} \gamma(a, z), \tag{16.1.7b} $$

$$ {}_1F_1(a; a; z) = e^z, \tag{16.1.7c} $$

$$ U(a, a+1, z) = z^{-a}, \tag{16.1.7d} $$

$$ {}_1F_1(1; 2; 2z) = \frac{e^z}{z} \sinh(z), \tag{16.1.7e} $$

$$ U(a, a, z) = e^z \Gamma(1-a, z), \tag{16.1.7f} $$

$$ U(1, 1, z) = -e^z \operatorname{li}(e^{-z}), \tag{16.1.7g} $$

$$ {}_1F_1\left(1/2; 3/2; -z^2\right) = \frac{\sqrt{\pi}}{2z} \operatorname{erf}(z). \tag{16.1.7h} $$

The derivative of the functions ${}_1F_1(a; b; z)$ and $U(a, b, z)$ is given by [AS64, p. 507]

$$ \frac{d}{dz}\, {}_1F_1(a; b; z) = \frac{a}{b}\, {}_1F_1(a+1; b+1; z), \tag{16.1.8a} $$

$$ \frac{d}{dz}\, U(a, b, z) = -a\, U(a+1, b+1, z). \tag{16.1.8b} $$

Recurrence relations. Recurrence relations for the function ${}_1F_1(a, b; z)$ are given by [AS64, pp. 506–507]:

$$ {}_1F_1(a+1; b; z) = \tfrac{(2a-b+z)}{a}\, {}_1F_1(a; b; z) + \tfrac{(b-a)}{a}\, {}_1F_1(a-1; b; z), \tag{16.1.9a} $$

$$ {}_1F_1(a; b+1; z) = \tfrac{b(b-1+z)}{(b-a)z}\, {}_1F_1(a; b; z) - \tfrac{b(b-1)}{(b-a)z}\, {}_1F_1(a; b-1; z), \tag{16.1.9b} $$

$$ {}_1F_1(a+1; b+1; z) = \tfrac{b(1-b+z)}{az}\, {}_1F_1(a; b; z) + \tfrac{b(b-1)}{az}\, {}_1F_1(a-1; b-1; z). \tag{16.1.9c} $$

The function $U(a, b, z)$ satisfies the recurrence relations [AS64, pp. 506–507]

$$U(a+1, b, z) = \frac{(2a-b+z)}{a(a-b+1)} U(a, b, z) - \frac{1}{a(a-b+1)} U(a-1, b, z), \tag{16.1.10a}$$

$$U(a, b+1, z) = \frac{(b-1+z)}{z} U(a, b, z) + \frac{(a-b+1)}{z} U(a, b-1, z), \tag{16.1.10b}$$

$$U(a+1, b+1, z) = \frac{(b-1+z)}{az} U(a, b, z) - \frac{1}{az} U(a-1, b-1, z). \tag{16.1.10c}$$

Asymptotic series expansion. An asymptotic series expansion of the confluent hypergeometric function of the second kind is given by [SO87, p. 474; AS64, p. 508]

$$U(a, b, z) \approx z^{-a} \, {}_2F_0\big(a, a-b+1; -z^{-1}\big), \qquad z \to \infty, \quad |\arg z| < \frac{3\pi}{2}, \tag{16.1.11}$$

where ${}_2F_0(a, b; z)$ is the divergent hypergeometric series

$${}_2F_0(a, b; z) = \sum_{k=0}^{\infty} (a)_k (b)_k \frac{z^k}{k!}, \qquad a, b \in \mathbb{C}. \tag{16.1.12}$$

For Kummer functions of the first kind, the continued fraction representations given in this section are obtained by applying the limit process (16.1.3) to the continued fraction representations for ratios of Gauss hypergeometric series given in *Chapter* 15. The continued fractions are given for the ratio ${}_1F_1(a; b; z)/{}_1F_1(a+1; b+1; z)$, which is closely related to the logarithmic derivative of ${}_1F_1(a; b; z)$ because of (16.1.8). As a special case, continued fractions for ${}_1F_1(1; b+1; z)$ can be derived. Continued fraction representations for other ratios of Kummer functions of the first kind can be obtained by applying the Kummer transformations (16.1.6) to the continued fractions in this section.

C-fractions. From applying the limit process (16.1.3a) to the continued fraction representation (15.3.3) for the ratio ${}_2F_1(a, b; c; z/a)/{}_2F_1(a, b+1; c+1; z/a)$, we get the regular C-fraction expansion [JT80, p. 206]

$$\frac{{}_1F_1(a; b; z)}{{}_1F_1(a+1; b+1; z)} = 1 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m z}{1} \right), \qquad z \in \mathbb{C},$$

$$a \in \mathbb{C}, \quad b \in \mathbb{C} \setminus \mathbb{Z}_0^-, \tag{16.1.13a}$$

where the coefficients a_m are given by

$$a_{2k+1} = \frac{-(b-a+k)}{(b+2k)(b+2k+1)}, \qquad k \geq 0,$$
$$a_{2k} = \frac{a+k}{(b+2k-1)(b+2k)}, \qquad k \geq 1. \tag{16.1.13b}$$

Since $\lim_{m\to\infty} a_m = 0$, use of a modification when evaluating (16.1.13) is not interesting.
From (16.1.5) and (16.1.13) we obtain the C-fraction representation

$$z\,{}_1F_1(1;b+1;z) = \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{c_m z}{1}\right), \qquad z \in \mathbb{C}, \quad b \in \mathbb{C} \setminus \mathbb{Z}_0^-, \tag{16.1.14a}$$

where

$$c_1 = 1, \qquad c_{2k} = \frac{-(b+k-1)}{(b+2k-2)(b+2k-1)}, \qquad k \geq 1,$$
$$c_{2k+1} = \frac{k}{(b+2k-1)(b+2k)}, \qquad k \geq 1. \tag{16.1.14b}$$

It is illustrated in the *Tables* 16.1.1 and 16.1.2, for $z = 1$ and varying b. For $\Re b > 0$, the continued fraction (16.1.14) is already given in (12.6.23a) and represents the function $bz^{1-b}e^z\gamma(b,z)$.

EXAMPLE 16.1.2: Let $a = 0$ and $b = 1$ in (16.1.13). Then we obtain the C-fraction representation given in (11.1.3),

$${}_1F_1(1;2;z) = \frac{e^z - 1}{z} = 2\sinh\left(\frac{z}{2}\right) = \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{c_m z}{1}\right), \quad z \in \mathbb{C},$$

with

$$c_1 = 1, \quad c_{2k+2} = \frac{-1}{2(2k+1)}, \quad k \geq 0, \quad c_{2k+1} = \frac{1}{2(2k+1)}, \quad k \geq 1.$$

Padé approximants. We get Padé approximants $r_{m,n}(z)$ for the ratio ${}_1F_1(b+1;c+s+1;z)/{}_1F_1(b;c;z)$ of confluent hypergeometric functions by using the limit process (16.1.3) and *Theorem* 15.4.1 [WB93]. Let

$V_{m,n}(a,b;c;z)$ be given by (15.4.2). Then for $m \geq n-1$ and, if $b \neq 0$ also for $m \leq n-s$, we find

$$r_{m,n}(z) = \frac{V_{m,n-1}(\infty, b+1; c+s+1; z)}{V_{m+s,n}(\infty, b; c; z)}, \qquad s \in \{-1, 0, 1\}, \tag{16.1.15a}$$

where

$$\begin{aligned} V_{m,n}(\infty, b; c; z) &:= \lim_{a\to\infty} V_{m,n}\left(a, b; c; \frac{z}{a}\right) \\ &= \mathcal{P}_{m+n+1}\left({}_1F_1(b;c;z)\, {}_1F_1(-b-n; -c-m-n; -z)\right). \end{aligned} \tag{16.1.15b}$$

For the confluent hypergeometric function ${}_1F_1(1;b;z)$ the following normality result can be stated.

THEOREM 16.1.1: [dB77]

The Padé approximants $r_{m,n}(z)$ for the confluent hypergeometric function ${}_1F_1(1;b;z)$ with $m \geq n$ and $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$ are normal.

T-fractions. Applying the limit process (16.1.3a) to the general T-fraction (15.3.8), we obtain the T-fraction [JT80, pp. 278–281]

$$\frac{{}_1F_1(a;b;z)}{{}_1F_1(a+1;b+1;z)} = \frac{b-z}{b} + \frac{1}{b} \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{(a+m)z}{b+m-z} \right), \qquad z \in \mathbb{C},$$
$$a \in \mathbb{C}, \quad b \in \mathbb{C} \setminus \mathbb{Z}_0^-. \tag{16.1.16a}$$

The continued fraction in (16.1.16a) corresponds at $z = \infty$ to

$$-\frac{z}{b} \frac{{}_2F_0\left(a, a-b+1; -z^{-1}\right)}{{}_2F_0(a+1, a-b+1; -z^{-1})}, \tag{16.1.16b}$$

where the series ${}_2F_0(a,b;z)$ is given by (16.1.12). Applying the limit process (16.1.3) to the Nörlund fraction (15.3.13) also leads to the continued fraction representation (16.1.16).
From (16.1.5) and (16.1.16) we obtain the M-fraction representation

$${}_1F_1(1; b+1; z) = \frac{b}{b-z} + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{mz}{b+m-z} \right), \qquad z \in \mathbb{C}, \quad b \in \mathbb{C} \setminus \mathbb{Z}_0^-. \tag{16.1.17}$$

It is illustrated for $z = 1$ and varying b in the *Tables* 16.1.1 and 16.1.2. For $\Re b > 0$ the continued fraction (16.1.17) is already given in (12.6.30) and represents the function $bz^{-b}e^z\gamma(b,z)$.

For neither (16.1.16) nor (16.1.17), use of a modification is recommended, since the partial numerators in the equivalent representations $\mathrm{K}_{m=1}^{\infty}\left(a_m/1\right)$ tend to zero.

EXAMPLE 16.1.3: For $a = 0$ and $b = 1/2$ in (16.1.16) we obtain the M-fraction

$$\frac{1/2}{1/2 - z^2} + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{mz^2}{m + 1/2 - z^2}\right), \qquad z \in \mathbb{C}. \tag{16.1.18}$$

The continued fraction (16.1.18) corresponds at $z = 0$ to the convergent series (13.1.8),

$${}_1F_1\left(1; 3/2; z^2\right) = \frac{1}{2z}\sqrt{\pi}e^{z^2}\,\mathrm{erf}(z),$$

and at $z = \infty$ to

$$-\frac{1}{2}z^{-2}\;{}_2F_0(1, 1/2; -z^{-2}).$$

The continued fraction (16.1.18) multiplied by $2z^2$ is equivalent to the T-fraction expansion (13.1.13a) given in *Chapter* 13.

Two-point Padé approximants. Let $P_{n,k}(a, b, c, z)$ be given by (15.4.9) and define

$$\begin{aligned} P_{n,k}(\infty, b, c, z) &:= \lim_{a\to\infty} P_{n,k}(a, b, c, z/a), \qquad 0 \le k \le n \\ &= \mathcal{P}_n({}_1F_1(b; c; z)\;{}_1F_1(-b - n; 1 - c - k - n; -z)), \end{aligned}$$

where the operator $\mathcal{P}_n$ is defined in (15.4.1). The two-point Padé approximant $r^{(2)}_{n+k,n-k}(z)$ corresponding at $z = 0$ and at $z = \infty$ respectively to

$$\begin{aligned} L_0(z) &= \Lambda_0\left(\frac{{}_1F_1(a + 1; b + 1; z)}{{}_1F_1(a; b; z)}\right), \\ L_\infty(z) &= \Lambda_\infty\left(-\frac{b}{z}\frac{{}_2F_0(a + 1, a - b + 1; -z^{-1})}{{}_2F_0(a, a - b + 1; -z^{-1})}\right) \end{aligned}$$

is given by [WB95]

$$r^{(2)}_{n+k,n-k}(z) = \frac{P_{n-1,k}(\infty, a + 1, b, z)}{P_{n,k}(\infty, a, b, z)}, \qquad 0 \le k \le n.$$

J-fractions. Let $u_k = (a)_k U(a+k,b,z)$ where $(a)_k$ is the Pochhammer symbol defined in (12.1.7). Then it follows from the recurrence relation (16.1.10a) that [Tem83]

$$u_{n+1} = \frac{2a-b+2n+z}{a-b+n+1} u_n - \frac{a+n-1}{a-b+n+1} u_{n-1}, \qquad n \geq 1. \quad (16.1.19)$$

The sequence $\{u_n\}_{n\in\mathbb{N}}$ is a minimal solution of this three-term recurrence relation [Tem83]. From applying Pincherle's *Theorem* 3.6.1 we find a J-fraction for the ratio

$$\frac{U(a,b,z)}{U(a+1,b,z)} = 2a-b+2+z - \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{(a+m)(b-a-m-1)}{b-2a-2m-2-z} \right),$$
$$z \in \mathbb{C}, \quad a \in \mathbb{C}, \quad b \in \mathbb{C} \setminus \mathbb{Z}. \quad (16.1.20)$$

Combined with the relation [AS64, p. 507]

$$U(a+1,b,z) = \frac{1}{1+a-b} U(a,b,z) + \frac{z}{a(1+a-b)} U'(a,b,z) \quad (16.1.21)$$

a J-fraction for the logarithmic derivative of $U(a,b,z)$ is obtained:

$$\frac{dU(a,b,z)/dz}{U(a,b,z)} = -\frac{a}{z} + \frac{a(1+a-b)/z}{2a-b+2+z} \, - \, \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{(a+m)(b-a-m-1)}{b-2a-2m-2-z} \right),$$
$$z \in \mathbb{C}, \quad a \in \mathbb{C}, \quad b \in \mathbb{C} \setminus \mathbb{Z}. \quad (16.1.22)$$

TABLE 16.1.1: Relative error of the 5^{th} partial sum of (16.1.2) and the 5^{th} approximants of (16.1.14) and (16.1.17), for $a = 0$ and $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

x	${}_1F_1(1; x; 1)$	(16.1.2)	(16.1.14)	(16.1.17)
−100.25	9.901244e−01	1.1e−12	2.2e−14	1.2e−18
−50.25	9.804954e−01	8.4e−11	3.1e−12	1.3e−15
−10.25	9.118584e−01	2.8e−06	2.5e−06	2.0e−06
−5.25	9.097331e−01	7.5e−02	7.2e−02	7.2e−02
−1.25	1.128766e+01	2.6e−02	4.4e−01	9.4e−01
−0.75	−1.258138e+01	1.3e−02	1.5e−01	1.1e+00
0.15	1.726993e+01	2.8e−03	1.3e−02	9.3e−01
0.95	2.832945e+00	6.5e−04	1.6e−03	7.7e−01
3.25	1.396433e+00	3.0e−05	2.7e−05	1.2e−04
7.25	1.156651e+00	1.2e−06	4.5e−07	1.2e−07
20.25	1.051816e+00	7.3e−09	8.4e−10	7.8e−12
70.25	1.014437e+00	6.7e−12	2.0e−13	3.8e−17
200.25	1.005019e+00	1.4e−14	1.5e−16	1.1e−21

x	$\lvert{}_1F_1(1; x + ix; 1)\rvert_s$	(16.1.2)	(16.1.14)	(16.1.17)
−100.25	9.950247e−01	1.3e−13	1.8e−15	3.7e−20
−50.25	9.900978e−01	9.0e−12	2.4e−13	3.8e−17
−10.25	9.522613e−01	2.2e−07	2.6e−08	3.8e−10
−5.25	9.084042e−01	2.2e−05	5.1e−06	5.0e−07
−1.25	7.380543e−01	2.0e−02	2.7e−01	6.1e−01
−0.75	−1.617730e+00	1.8e−02	1.9e−01	1.3e+00
0.15	1.205642e+01	2.8e−03	1.3e−02	9.5e−01
0.95	1.910744e+00	5.5e−04	1.3e−03	9.1e−02
3.25	1.172973e+00	1.3e−05	8.3e−06	1.0e−05
7.25	1.071925e+00	3.1e−07	7.8e−08	6.4e−09
20.25	1.025020e+00	1.3e−09	9.6e−11	2.9e−13
70.25	1.007143e+00	9.3e−13	1.9e−14	1.2e−18
200.25	1.002500e+00	1.9e−15	1.3e−17	3.6e−23

x	$\lvert {}_1F_1(1; \mathrm{i}x; 1)\rvert_s$	(16.1.2)	(16.1.14)	(16.1.17)
−100.25	9.999502e−01	9.8e−13	2.0e−14	1.2e−18
−50.25	9.998019e−01	6.2e−11	2.5e−12	1.2e−15
−10.25	9.951842e−01	7.1e−07	1.6e−07	1.1e−08
−5.25	9.810739e−01	2.5e−05	1.5e−05	1.1e−05
−1.25	1.153210e+00	3.1e−03	1.2e−02	4.7e−01
−0.75	−2.525707e+00	3.7e−03	1.7e−02	1.2e+00
0.15	−1.780178e+01	3.7e−03	1.9e−02	1.0e+00
0.95	1.729564e+00	3.6e−03	1.6e−02	9.9e−01
3.25	9.492770e−01	2.1e−04	2.8e−04	1.3e−03
7.25	9.902626e−01	4.7e−06	1.7e−06	3.8e−07
20.25	9.987770e−01	1.4e−08	1.4e−09	1.1e−11
70.25	9.998987e−01	8.3e−12	2.4e−13	4.1e−17
200.25	9.999875e−01	1.6e−14	1.5e−16	1.2e−21

x	$\lvert {}_1F_1(1; x - \mathrm{i}x; 1)\rvert_s$	(16.1.2)	(16.1.14)	(16.1.17)
−100.25	9.950247e−01	1.3e−13	1.8e−15	3.7e−20
−50.25	9.900978e−01	9.0e−12	2.4e−13	3.8e−17
−10.25	9.522613e−01	2.2e−07	2.6e−08	3.8e−10
−5.25	9.084042e−01	2.2e−05	5.1e−06	5.0e−07
−1.25	7.380543e−01	2.0e−02	2.7e−01	6.1e−01
−0.75	−1.617730e+00	1.8e−02	1.9e−01	1.3e+00
0.15	1.205642e+01	2.8e−03	1.3e−02	9.5e−01
0.95	1.910744e+00	5.5e−04	1.3e−03	9.1e−02
3.25	1.172973e+00	1.3e−05	8.3e−06	1.0e−05
7.25	1.071925e+00	3.1e−07	7.8e−08	6.4e−09
20.25	1.025020e+00	1.3e−09	9.6e−11	2.9e−13
70.25	1.007143e+00	9.3e−13	1.9e−14	1.2e−18
200.25	1.002500e+00	1.9e−15	1.3e−17	3.6e−23

TABLE 16.1.2: Relative error of the 20th partial sum of (16.1.2) and the 20th approximants of (16.1.14) and (16.1.17), for $a = 0$ and $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

x	${}_1F_1(1; x; 1)$	(16.1.2)	(16.1.14)	(16.1.17)
−100.25	9.901244e−01	9.0e−42	8.6e−53	6.3e−61
−50.25	9.804954e−01	2.5e−34	3.4e−42	2.5e−47
−10.25	9.118584e−01	4.1e−13	2.3e−08	2.0e−06
−5.25	9.097331e−01	4.3e−14	1.4e−14	3.2e−05
−1.25	1.128766e+01	6.3e−18	2.4e−20	2.7e−11
−0.75	−1.258138e+01	1.6e−18	3.9e−21	2.5e−12
0.15	1.726993e+01	9.9e−20	1.2e−22	2.9e−14
0.95	2.832945e+00	8.7e−21	5.7e−24	6.5e−16
3.25	1.396433e+00	3.3e−23	4.7e−27	1.2e−22
7.25	1.156651e+00	4.1e−26	6.6e−31	6.7e−29
20.25	1.051816e+00	1.2e−31	1.6e−38	3.4e−40
70.25	1.014437e+00	1.1e−40	2.6e−51	4.6e−58
200.25	1.005019e+00	1.7e−49	7.4e−64	4.4e−75

x	$\lvert {}_1F_1(1; x + \mathrm{i}x; 1) \rvert_s$	(16.1.2)	(16.1.14)	(16.1.17)
−100.25	9.950247e−01	1.9e−45	4.7e−58	1.0e−67
−50.25	9.900978e−01	1.0e−38	2.1e−48	5.0e−55
−10.25	9.522613e−01	3.7e−23	1.8e−25	4.1e−24
−5.25	9.084042e−01	8.5e−20	2.6e−21	8.6e−17
−1.25	7.380543e−01	4.2e−18	1.5e−20	6.9e−13
−0.75	−1.617730e+00	2.0e−18	4.9e−21	5.0e−13
0.15	1.205642e+01	9.9e−20	1.2e−22	2.0e−14
0.95	1.910744e+00	7.0e−21	4.5e−24	1.5e−17
3.25	1.172973e+00	9.9e−24	1.1e−27	4.9e−24
7.25	1.071925e+00	4.0e−27	3.4e−32	3.7e−31
20.25	1.025020e+00	1.8e−33	5.1e−41	5.8e−44
70.25	1.007143e+00	2.6e−43	5.1e−55	3.3e−63
200.25	1.002500e+00	1.9e−52	4.6e−68	9.0e−81

x	$\lvert {}_1F_1(1; \mathrm{i}x; 1)\rvert_s$	(16.1.2)	(16.1.14)	(16.1.17)
-100.25	$9.999502e{-}01$	$8.3e{-}43$	$2.8e{-}54$	$1.9e{-}62$
-50.25	$9.998019e{-}01$	$1.1e{-}36$	$1.7e{-}45$	$1.2e{-}50$
-10.25	$9.951842e{-}01$	$4.0e{-}25$	$4.0e{-}29$	$2.1e{-}27$
-5.25	$9.810739e{-}01$	$2.4e{-}22$	$1.4e{-}25$	$1.5e{-}21$
-1.25	$1.153210e{+}00$	$1.2e{-}19$	$1.5e{-}22$	$1.6e{-}15$
-0.75	$-2.525707e{+}00$	$1.6e{-}19$	$2.0e{-}22$	$7.0e{-}15$
0.15	$-1.780178e{+}01$	$1.6e{-}19$	$2.1e{-}22$	$6.4e{-}14$
0.95	$1.729564e{+}00$	$1.5e{-}19$	$1.9e{-}22$	$4.0e{-}15$
3.25	$9.492770e{-}01$	$4.9e{-}21$	$4.7e{-}24$	$9.8e{-}19$
7.25	$9.902626e{-}01$	$1.6e{-}23$	$4.9e{-}27$	$4.8e{-}24$
20.25	$9.987770e{-}01$	$2.4e{-}29$	$5.4e{-}35$	$4.1e{-}36$
70.25	$9.998987e{-}01$	$1.3e{-}39$	$1.0e{-}49$	$2.4e{-}56$
200.25	$9.999875e{-}01$	$4.5e{-}49$	$3.1e{-}63$	$2.0e{-}74$

x	$\lvert {}_1F_1(1; x - \mathrm{i}x; 1)\rvert_s$	(16.1.2)	(16.1.14)	(16.1.17)
-100.25	$9.950247e{-}01$	$1.9e{-}45$	$4.7e{-}58$	$1.0e{-}67$
-50.25	$9.900978e{-}01$	$1.0e{-}38$	$2.1e{-}48$	$5.0e{-}55$
-10.25	$9.522613e{-}01$	$3.7e{-}23$	$1.8e{-}25$	$4.1e{-}24$
-5.25	$9.084042e{-}01$	$8.5e{-}20$	$2.6e{-}21$	$8.6e{-}17$
-1.25	$7.380543e{-}01$	$4.2e{-}18$	$1.5e{-}20$	$6.9e{-}13$
-0.75	$-1.617730e{+}00$	$2.0e{-}18$	$4.9e{-}21$	$5.0e{-}13$
0.15	$1.205642e{+}01$	$9.9e{-}20$	$1.2e{-}22$	$2.0e{-}14$
0.95	$1.910744e{+}00$	$7.0e{-}21$	$4.5e{-}24$	$1.5e{-}17$
3.25	$1.172973e{+}00$	$9.9e{-}24$	$1.1e{-}27$	$4.9e{-}24$
7.25	$1.071925e{+}00$	$4.0e{-}27$	$3.4e{-}32$	$3.7e{-}31$
20.25	$1.025020e{+}00$	$1.8e{-}33$	$5.1e{-}41$	$5.8e{-}44$
70.25	$1.007143e{+}00$	$2.6e{-}43$	$5.1e{-}55$	$3.3e{-}63$
200.25	$1.002500e{+}00$	$1.9e{-}52$	$4.6e{-}68$	$9.0e{-}81$

16.2 Confluent hypergeometric series ${}_2F_0$

The confluent hypergeometric series ${}_2F_0(a, b; z)$ can be obtained from the hypergeometric series ${}_2F_1(a, b; c; z)$ by taking the limit termwise [LW92,

p. 316],

$$\lim_{c\to\infty} {}_2F_1(a,b;c;cz) = {}_2F_0(a,b;z). \tag{16.2.1}$$

It is easy to verify that

$$ {}_2F_0(0,b;z) = {}_2F_0(a,0;z) = 1. \tag{16.2.2}$$

Recurrence relations. The hypergeometric series ${}_2F_0(a,b;z)$ satisfies the recurrence relations

$$ {}_2F_0(a+1,b;z) = \tfrac{1+(a-b)z}{az}\, {}_2F_0(a,b;z) - \tfrac{1}{az}\, {}_2F_0(a-1,b;z), \tag{16.2.3a}$$

$$ {}_2F_0(a+1,b+1;z) = \tfrac{1-(a+b-1)z}{abz^2}\, {}_2F_0(a,b;z) - \tfrac{1}{abz^2}\, {}_2F_0(a-1,b-1;z). \tag{16.2.3b}$$

Applying the limit process (16.2.1), when possible, to the continued fraction representations for ratios of Gauss hypergeometric series given in *Chapter* 15, leads to continued fraction representations for ratios of hypergeometric series ${}_2F_0(a,b;z)$.

C-fraction. Applying the limit process (16.2.1) to the C-fraction representation (15.3.3) for the ratio ${}_2F_1(a,b;c;cz)/{}_2F_1(a,b+1;c+1;cz)$, we get the regular C-fraction [LW92, p. 316]

$$1 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m z}{1}\right), \qquad z \in \mathbb{C} \setminus (0,+\infty), \quad a,b \in \mathbb{C} \tag{16.2.4a}$$

where

$$a_1 = -a, \quad a_{2k} = -(b+k), \quad a_{2k+1} = -(a+k), \quad k \geq 1. \tag{16.2.4b}$$

The C-fraction (16.2.4) corresponds at $z=0$ to the ratio

$$\frac{{}_2F_0(a,b;z)}{{}_2F_0(a,b+1;z)}$$

and converges to a meromorphic function in $\mathbb{C} \setminus (0,+\infty)$.
As a special case, the C-fraction corresponding to $z^{-1}\, {}_2F_0(1,1-a;-z^{-1})$ at $z=\infty$ is given in (12.6.17). It converges to $z^{-a}e^{z}\Gamma(a,z)$.

Padé approximants. Applying the limit process (16.2.1), we find in a similar way as for the function ${}_1F_1(a;b;z)$ that for $m \geq n-1$ and, if $b \neq 0$ also for $m \leq n-s$, the Padé approximants $r_{m,n}(z)$ for the ratio

${}_2F_0(a+s,b+1;z)/{}_2F_0(a,b;z)$ of confluent hypergeometric series are given by

$$r_{m,n}(z) = \frac{V_{m,n-1}(a+s,b+1;\infty;z)}{V_{m+s,n}(a,b;\infty;z)}, \qquad s \in \{-1,0,1\}, \tag{16.2.5a}$$

where

$$\begin{aligned} V_{m,n}(a,b;\infty;z) &:= \lim_{c\to\infty} V_{m,n}\,(a,b;c;cz) \\ &= \mathcal{P}_{m+n+1}\left({}_2F_0(a;b;z)\;{}_2F_0(-a-m;-b-n;-z)\right). \end{aligned} \tag{16.2.5b}$$

The following theorems give results on the normality of the Padé table for the hypergeometric series ${}_2F_0(a,1;z)$.

THEOREM 16.2.1: [dB77]

The Padé table for the confluent hypergeometric series ${}_2F_0(a,1;z)$ with $a > 0$ is normal.

THEOREM 16.2.2: [dB77]

The Padé approximants $r_{m,n}(z)$ for the confluent hypergeometric series ${}_2F_0(a,1;z)$ with $m \geq n$ and $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$ are normal.

Another continued fraction. Applying the limit process (16.2.1) to the Nörlund fraction (15.3.13) gives the continued fraction

$$1-(a+b+1)z + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{-(a+m)(b+m)z^2}{1-(a+b+2m+1)z}\right), \qquad |\arg(-z)| < \pi/2, \quad a,b \in \mathbb{C}, \tag{16.2.6a}$$

corresponding at $z = \infty$ to

$$\frac{{}_2F_0(a,b;z)}{{}_2F_0(a+1,b+1;z)} \tag{16.2.6b}$$

and converging to a meromorphic function for $|\arg(-z)| < \pi/2$. This continued fraction can also be obtained from the J-fraction (16.1.20) for a ratio of Kummer functions of the second kind by using the relation (16.1.11).

16.3 Confluent hypergeometric limit function

The solution to the differential equation [Wei03, p. 512]

$$z\frac{d^2w}{dz^2} + b\frac{dw}{dz} - w = 0$$

with initial conditions $w(0) = 1$ and $(dw/dz)(0) = 1/b$ is called the *confluent hypergeometric limit function*. Its Taylor series expansion at $z = 0$ is given by

$${}_0F_1(;b;z) = \sum_{m=0}^{\infty} \frac{z^m}{(b)_m m!}, \qquad z \in \mathbb{C}, \quad b \in \mathbb{C} \setminus \mathbb{Z}_0^-. \tag{16.3.1}$$

In a similar way as for the confluent hypergeometric functions, the function ${}_0F_1(;b;z)$ can be obtained from the limit process

$$\lim_{a\to\infty} {}_1F_1\left(a;b;\frac{z}{a}\right) = {}_0F_1(;b;z), \qquad b \in \mathbb{C} \setminus \mathbb{Z}_0^-. \tag{16.3.2}$$

A relation between ${}_1F_1(a;b;z)$ and ${}_0F_1(;a;z)$ is given by the Kummer transformation [AAR99]

$${}_1F_1(a;2a;4z) = e^{2z}\,{}_0F_1(;a+{}^1\!/_2;z^2). \tag{16.3.3}$$

Recurrence relation. The confluent hypergeometric limit function satisfies the recurrence relation

$${}_0F_1(;b+1;z) = \frac{b(b-1)}{z}\,{}_0F_1(;b;z) - \frac{b(b-1)}{z}\,{}_0F_1(;b-1;z).$$

C-fraction. Applying the limit process (16.3.2) to all continued fraction representations for ratios of Kummer functions of the first kind given in *Section* 16.1 leads to the C-fraction representation [Wal48, p. 347; JT88, pp. 209–210]:

$$\frac{{}_0F_1(;b;z)}{{}_0F_1(;b+1;z)} = 1 + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{\frac{1}{(b-1+m)(b+m)}z}{1}\right), \qquad z \in \mathbb{C}, \quad b \in \mathbb{C} \setminus \mathbb{Z}_0^-. \tag{16.3.4}$$

Padé approximants. We also find Padé approximants $r_{m,n}(z)$ for the ratio ${}_0F_1(;c+s+1;z)/{}_0F_1(;c;z)$ by using (16.1.15) and the limit process (16.3.2):

$$r_{m,n}(z) = \frac{V_{m,n-1}(\infty,\infty;c+s+1;z)}{V_{m+s,n}(\infty,\infty;c;z)},$$
$$n-1 \le m \le n-s, \quad s \in \{0,1\}, \tag{16.3.5a}$$

with

$$V_{m,n}(\infty,\infty;c;z) := \lim_{b\to\infty} V_{m,n}(\infty,b;c;\frac{z}{b}) \\ = \mathcal{P}_{m+n+1}\left({}_0F_1(;c;z)\,{}_0F_1(;-c-m-n;z)\right). \tag{16.3.5b}$$

T-fraction. By applying the Kummer transformation (16.1.6a) to the T-fraction (16.1.16), we find a T-fraction for ${}_1F_1(a;b;z)/{}_1F_1(a;b+1;z)$. Using (16.3.3), this leads to

$$\frac{{}_0F_1(;b;z)}{{}_0F_1(;b+1;z)} = 1 + \frac{\sqrt{z}}{b} + \frac{1}{2b}\mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{c_m\sqrt{z}}{e_m + d_m\sqrt{z}}\right), \qquad z \in \mathbb{C}, \tag{16.3.6a}$$

with

$$c_m = -2(2b+2m-1), \quad e_m = 2b+m, \quad d_m = 4, \quad m \ge 1. \tag{16.3.6b}$$

16.4 Whittaker functions

Definitions and elementary properties. *Whittaker's differential equation* [WW80, p. 337]

$$\frac{d^2W}{dz^2} + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{\frac{1}{4}-\mu^2}{z^2}\right)W = 0 \tag{16.4.1}$$

can be obtained from Kummer's differential equation (16.1.1) by the substitution $W(z) = e^{-\frac{z}{2}}z^{\mu+\frac{1}{2}}w(z)$, $\kappa = b/2 - a$ and $\mu = (b-1)/2$. Standard solutions are [AS64, p. 505]

$$M_{\kappa,\mu}(z) = e^{-\frac{z}{2}}z^{\mu+\frac{1}{2}}M(\mu-\kappa+1/2;1+2\mu,z), \\ -\pi < \arg z \le \pi, \quad \kappa \in \mathbb{C}, \quad 2\mu \in \mathbb{C}\setminus\mathbb{Z}^- \tag{16.4.2a}$$

and

$$W_{\kappa,\mu}(z) = e^{-\frac{z}{2}}z^{\mu+\frac{1}{2}}U(\mu-\kappa+1/2,1+2\mu,z), \\ -\pi < \arg z \le \pi, \quad \kappa \in \mathbb{C}, \quad 2\mu \in \mathbb{C}\setminus\mathbb{Z}^- \tag{16.4.2b}$$

where $M(a,b,z)$ and $U(a,b,z)$ are the Kummer functions (16.1.2) and (16.1.4) respectively. Conversely we have

$$M(a,b,z) = e^{\frac{z}{2}}z^{-\frac{b}{2}}M_{\frac{b-2a}{2},\frac{b-1}{2}}(z), \tag{16.4.3a}$$

$$U(a,b,z) = e^{\frac{z}{2}}z^{-\frac{b}{2}}W_{\frac{b-2a}{2},\frac{b-1}{2}}(z). \tag{16.4.3b}$$

As a special case of (16.4.3) we find

$$zM(1, b+1, z) = e^{\frac{z}{2}} z^{\frac{1-b}{2}} M_{\frac{b-1}{2}, \frac{b}{2}}(z) = bz^{-b+1} e^z \gamma(b, z), \qquad (16.4.4)$$

where $\gamma(a, z)$ is the incomplete gamma function (12.6.2).
The following relations also hold:

$$W_{\kappa,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2} - \mu - \kappa)} M_{\kappa,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2} + \mu - \kappa)} M_{\kappa,-\mu}(z),$$
$$|\arg z| < \frac{3\pi}{2}, \quad 2\mu \in \mathbb{C} \setminus \mathbb{Z}, \qquad (16.4.5)$$

$$W_{-\kappa,\mu}(-z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2} - \mu - \kappa)} M_{\kappa,\mu}(-z) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2} + \mu + \kappa)} M_{-\kappa,-\mu}(-z),$$
$$|\arg(-z)| < \frac{3\pi}{2}, \quad 2\mu \in \mathbb{C} \setminus \mathbb{Z}. \quad (16.4.6)$$

Recurrence relations. The functions $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$ satisfy the recurrence relations [AS64, p. 507]

$$M_{\kappa+1,\mu}(z) = \frac{2(2\kappa - z)}{(1 + 2\mu + 2\kappa)} M_{\kappa,\mu}(z) + \frac{(1 + 2\mu - 2\kappa)}{(1 + 2\mu + 2\kappa)} M_{\kappa-1,\mu}(z),$$
$$W_{\kappa+1,\mu}(z) = (z - 2\kappa) W_{\kappa,\mu}(z) + (\mu - \kappa + 1/2)(\mu + \kappa - 1/2) W_{\kappa-1,\mu}(z) .$$

Asymptotic series expansion. The asymptotic series expansion for $W_{\kappa,\mu}(z)$ is given by [WW80, p. 343]

$$W_{\kappa,\mu}(z) \approx e^{-z/2} z^{\kappa} \sum_{j=0}^{\infty} (-\kappa - \mu + 1/2)_j (-\kappa + \mu + 1/2)_j \frac{(-z)^{-j}}{j!}$$
$$= e^{-z/2} z^{\kappa} {}_2F_0(-\kappa - \mu + 1/2, -\kappa + \mu + 1/2; -1/z),$$
$$z \to \infty, \quad |\arg z| < \frac{3\pi}{2}. \quad (16.4.7)$$

Stieltjes transform. A function closely related to the Whittaker function (16.4.2b) and defined by

$$\Psi_{\alpha,\beta}(z) := z^{(\alpha+\beta)/2-1} e^{z/2} W_{-(\alpha+\beta)/2,(\beta-\alpha)/2}(z), \qquad (16.4.8)$$

can be expressed as the Stieltjes transform (5.2.1) [GH67]:

$$\Psi_{\alpha,\beta}(z) = \int_0^{\infty} \frac{\phi_{\alpha,\beta}(t)}{z + t} \, dt, \qquad |\arg z| < \pi \qquad (16.4.9a)$$

where $\phi_{\alpha,\beta}(t)$ is the weight function

$$\phi_{\alpha,\beta}(t) = \frac{t^{\alpha+\beta}e^{-t}}{\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})}\Psi_{-\alpha,-\beta}(t), \qquad 0 < t < \infty. \tag{16.4.9b}$$

The parameters α and β satisfy either

$$-\frac{1}{2} < \alpha, \qquad -\frac{1}{2} < \beta \leq \frac{1}{2}, \tag{16.4.10a}$$

or

$$-\frac{1}{2} < \beta, \qquad -\frac{1}{2} < \alpha \leq \frac{1}{2}. \tag{16.4.10b}$$

The moments $\mu_k(\alpha,\beta)$ with respect to the weight function $\phi_{\alpha,\beta}(t)$ are given by

$$\mu_k(\alpha,\beta) = \int_0^\infty t^k \phi_{\alpha,\beta}(t)\,dt = \frac{(-1)^k(\alpha+\frac{1}{2})_k(\beta+\frac{1}{2})_k}{k!}, \qquad k \geq 0. \tag{16.4.11}$$

S-fraction. Since the classical Stieltjes moment problem has a solution $\phi_{\alpha,\beta}(t)$ for $\mu_k(\alpha,\beta)$ given by (16.4.11), it follows from *Theorem* 5.1.1 that there exists an S-fraction of the form

$$\frac{a_1}{z} \, \frac{}{+} \, \frac{a_2}{1} \, \frac{}{+} \, \frac{a_3}{z} \, \frac{}{+} \, \frac{a_4}{1} \, \frac{}{+} \, \ldots, \qquad a_m > 0$$

corresponding to the asymptotic series

$$\Psi_{\alpha,\beta}(z) \approx \sum_{k=0}^\infty (-1)^k \mu_k(\alpha,\beta) z^{-k-1} = \sum_{k=0}^\infty (\alpha + 1/2)_k\,(\beta + 1/2)_k\,\frac{z^{-k-1}}{k!}. \tag{16.4.12}$$

The moments $\mu_k(\alpha,\beta)$ satisfy Carleman's criterion (5.1.16a) and thus the solution $\phi_{\alpha,\beta}(t)$ to the Stieltjes moment problem for the sequence $\mu_k(\alpha,\beta)$ is unique. Hence from *Theorem* 5.2.1, provided (16.4.10) is satisfied, the S-fraction is convergent and

$$\Psi_{\alpha,\beta}(z) = \int_0^\infty \frac{\phi_{\alpha,\beta}(t)\,dt}{z+t} = \frac{a_1}{z} \, \frac{}{+} \, \frac{a_2}{1} \, \frac{}{+} \, \frac{a_3}{z} \, \frac{}{+} \, \frac{a_4}{1} \, \frac{}{+} \, \ldots, \qquad |\arg z| < \pi. \tag{16.4.13}$$

There is no known closed expression for the coefficients a_m of the S-fraction (16.4.13), but the coefficients a_m satisfy the asymptotic behaviour [JS99]

$$a_m \sim \frac{m}{2}, \qquad m \to \infty. \tag{16.4.14}$$

Other continued fraction representations. Because of the close relations between Kummer functions and Whittaker functions given in (16.4.2) and (16.4.3), we can obtain continued fraction representations for special ratios of Whittaker functions. From (16.4.3a) we get

$$\frac{{}_1F_1(a;b;z)}{{}_1F_1(a+1;b+1;z)} = \sqrt{z}\frac{M_{\frac{b-2a}{2},\frac{b-1}{2}}(z)}{M_{\frac{b-2a-1}{2},\frac{b}{2}}(z)}, \tag{16.4.15a}$$

$$\frac{{}_1F_1(a;b;z)}{{}_1F_1(a;b+1;z)} = \sqrt{z}\frac{M_{\frac{b-2a}{2},\frac{b-1}{2}}(z)}{M_{\frac{b-2a+1}{2},\frac{b}{2}}(z)}. \tag{16.4.15b}$$

The first ratio has the C-fraction representation (16.1.13) and the T-fraction representation (16.1.16). The C-fraction and T-fraction representation for the second ratio can be obtained from (16.1.13) and (16.1.16) by applying the Kummer transformation (16.1.6). Continued fraction representations for the special case $M_{\frac{b-1}{2},\frac{b}{2}}(z)$ are given in *Section* 12.6 and obtained from (16.4.4).

EXAMPLE 16.4.1: For $b = 1$ in (16.4.4) and (12.6.23a), we obtain the C-fraction representation

$$M_{0,\frac{1}{2}}(z) = e^{z/2} - e^{-z/2} = ze^{-z/2} \operatorname*{K}_{m=1}^{\infty} \left(\frac{c_m z}{1}\right)$$

where the coefficients are given in *Example* 16.1.2.

16.5 Parabolic cylinder functions

Definitions and elementary properties. The parabolic cylinder functions arise in the solution of several practical problems expressed in cylindrical coordinates. There are a number of slightly different definitions in use by various authors.

One way of defining the parabolic cylinder functions is as solutions to the *Weber differential equation* [WW80, p. 347; GR00, p. 1021]

$$\frac{d^2D_\nu(z)}{dz^2} + \left(\nu + \frac{1}{2} - \frac{z^2}{4}\right)D_\nu(z) = 0, \qquad \nu \in \mathbb{C}.$$

Two independent solutions are $D_\nu(z)$ and $D_{-\nu-1}(\mathrm{i}z)$ where

$$D_\nu(z) = 2^{\nu/2+1/4} z^{-1/2} W_{\frac{\nu}{2}+\frac{1}{4},-\frac{1}{4}}\left(\frac{z^2}{2}\right) \tag{16.5.1a}$$

$$= 2^{\nu/2} e^{-z^2/4} (-\mathrm{i}z)^{1/4} (\mathrm{i}z)^{1/4} z^{1/2} U\left(-\frac{\nu}{2}, \frac{1}{2}, \frac{z^2}{2}\right). \tag{16.5.1b}$$

Here $W_{\kappa,\mu}(z)$ is the Whittaker function (16.4.2) and $U(a,b,z)$ is the Kummer function of the second kind (16.1.4) which are related by (16.4.2b). In the right half plane (16.5.1a) is equivalent to

$$D_\nu(z) = 2^{\nu/2} e^{-z^2/4} U\left(-\frac{\nu}{2}, \frac{1}{2}, \frac{z^2}{2}\right), \qquad \Re z > 0. \tag{16.5.2}$$

The solution $D_{-\nu-1}(\mathrm{i}z)$ is valid in the range $-3\pi/4 \le \arg z \le \pi/4$.
The solution $D_\nu(z)$ can be written in terms of the confluent hypergeometric function ${}_1F_1(a;b;z)$ as [GR00, p. 1018]

$$D_\nu(z) = 2^{\frac{\nu}{2}} e^{-\frac{z^2}{4}} \left(\frac{\sqrt{\pi}}{\Gamma(\frac{1-\nu}{2})}\, {}_1F_1(-\tfrac{\nu}{2}; \tfrac{1}{2}; \tfrac{z^2}{2}) - \frac{\sqrt{2\pi} z}{\Gamma(-\frac{\nu}{2})}\, {}_1F_1(\tfrac{1-\nu}{2}; \tfrac{3}{2}; \tfrac{z^2}{2}) \right),$$
$$|\arg z| \le \frac{3}{4}\pi. \tag{16.5.3}$$

The function $D_\nu(z)$ is related to two functions, denoted $U(a,z)$ and $V(a,z)$ and defined by

$$U(a,z) := D_{-a-\frac{1}{2}}(z), \tag{16.5.4}$$

$$V(a,z) := \frac{1}{\pi}\Gamma\left(a+\tfrac{1}{2}\right)\Big(\sin(\pi a) D_{-a-\frac{1}{2}}(z) + D_{-a-\frac{1}{2}}(-z)\Big). \tag{16.5.5}$$

The following examples illustrate the connection with the Kummer functions (16.1.2) and (16.1.4) as well as the Whittaker functions (16.4.2).

Example 16.5.1: For $b = 1/2$ and $b = 3/2$ the Kummer functions (16.1.2) and (16.1.4) reduce to

$$U\left(\frac{2a+1}{4}, \frac{1}{2}, \frac{z^2}{2}\right) = 2^{a/2+1/4} e^{z^2/4} U(a,z),$$

$$U\left(\frac{2a+3}{4}, \frac{3}{2}, \frac{z^2}{2}\right) = 2^{a/2+3/4} e^{z^2/4} \frac{1}{z} U(a,z),$$

$$M\left(\frac{2a+1}{4}; \frac{1}{2}, \frac{z^2}{2}\right) = 2^{a/2-5/4} \Gamma\left(\frac{2a+3}{4}\right) e^{z^2/4} (U(a,z) + U(a,-z)),$$

$$M\left(\frac{2a+3}{4}; \frac{3}{2}, \frac{z^2}{2}\right) = 2^{a/2-7/4} \Gamma\left(\frac{2a+1}{4}\right) \frac{e^{z^2/4}}{z} (U(a,-z) - U(a,z)).$$

EXAMPLE 16.5.2: For $\mu = 1/4$ and $\mu = -1/4$ the Whittaker functions (16.4.2) can be written as

$$W_{-\frac{a}{2},-\frac{1}{4}}\left(\frac{z^2}{2}\right) = 2^{a/2}\sqrt{z}U(a,z),$$
$$W_{-\frac{a}{2},\frac{1}{4}}\left(\frac{z^2}{2}\right) = 2^{a/2}\sqrt{z}U(a,z),$$
$$M_{-\frac{a}{2},-\frac{1}{4}}\left(\frac{z^2}{2}\right) = 2^{a/2-1}\Gamma\left(\frac{2a+3}{4}\right)\sqrt{\frac{z}{\pi}}(U(a,z)+U(a,-z)),$$
$$M_{-\frac{a}{2},\frac{1}{4}}\left(\frac{z^2}{2}\right) = 2^{a/2-2}\Gamma\left(\frac{2a+1}{4}\right)\sqrt{\frac{z}{\pi}}(U(a,-z)+U(a,z)).$$

A relation between the parabolic cylinder functions and the repeated integral for the complementary error function (13.3.1) is given by [AS64, p. 301]

$$I^n\mathrm{erfc}(z) = e^{-z^2/2}\left(2^n-1\right)^{-1/2} D_{-n-1/2}(z\sqrt{2}).$$

For special values of ν the parabolic cylinder functions are related to the Hermite polynomials $H_\nu(x)$ defined in (5.5.5), the error function $\mathrm{erf}(z)$ defined in (13.1.1) and the modified Bessel functions of the second kind $K_\nu(z)$, defined in (17.2.7):

$$D_\nu(x) = 2^{-\nu/2}e^{-x^2/4}H_\nu\left(\frac{x}{\sqrt{2}}\right), \qquad \nu \in \mathbb{C}\setminus\mathbb{Z}_0^-,$$
$$D_{-1}(x) = e^{z^2/4}\left(\frac{\pi}{2}\right)^{1/2}\mathrm{erf}\left(\frac{z}{\sqrt{2}}\right),$$
$$D_{-1/2}(z) = \left(\frac{z}{2\pi}\right)^{1/2}K_{1/4}\left(\frac{z^2}{4}\right).$$

Recurrence relations.

The parabolic cylinder functions $D_\nu(z)$ satisfy the recurrence relations

$$D_{\nu+1}(z) = zD_\nu(z) - \nu D_{\nu-1}(z),$$
$$\frac{dD_\nu}{dz} = -\frac{z}{2}D_\nu(z) + \nu D_{\nu-1}(z).$$

The functions $U(a,x)$ and $V(a,x)$ defined by (16.5.4) and (16.5.5), respectively, satisfy the recurrence relations [GST06b]

$$U(a-1,x) = xU(a,x) + (a+\frac{1}{2})U(a+1,x), \tag{16.5.6a}$$
$$V(a+1,x) = xV(a,x) + (a-\frac{1}{2})V(a-1,x). \tag{16.5.6b}$$

C-fraction. It can be shown [GST06b] that $\{U(a+n,x)\}_{n\in\mathbb{N}}$ is a minimal solution of (16.5.6a) for $x>0$. Hence, applying Pincherle's *Theorem* 3.6.1 to the recurrence relation (16.5.6a), we obtain [SG98]

$$\frac{U(a,x)}{U(a-1,x)} = \frac{1}{x} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{a+m-3/2}{x}\right),$$
$$x>0, \quad a \neq -k+1/2, \quad k\in\mathbb{N}. \tag{16.5.7}$$

Since the partial numerators tend to infinity, use of the modification

$$\tilde{w}_n(x) = \frac{-x+\sqrt{4(a+n-1/2)+x^2}}{2} \tag{16.5.8}$$

can be worthwhile when evaluating the approximants of (16.5.7).

Continued fractions representing special ratios of the parabolic cylinder functions can also be obtained from the continued fraction representations for the Kummer functions and the Whittaker functions because of the close connections given in (16.5.1b), (16.5.1a) and (16.5.2).

EXAMPLE 16.5.3: The ratio of parabolic cylinder functions

$$\frac{D_{-3/2}(x)}{D_{-1/2}(x)} = \frac{U(1,x)}{U(0,x)}$$

can be expressed in terms of modified Bessel functions,

$$\frac{D_{-3/2}(x)}{D_{-1/2}(x)} = \sqrt{2}x\left(1-\frac{K_{-3/4}(x^2/4)}{K_{1/4}(x^2/4)}\right) \tag{16.5.9}$$

and because of (16.5.1a) and (16.5.2) it also equals a specific ratio of Kummer functions of the second kind and Whittaker functions,

$$\frac{D_{-3/2}(x)}{D_{-1/2}(x)} = \frac{U(3/4,1/2,x^2/2)}{U(1/4,1/2,x^2/2)} = \frac{1}{\sqrt{2}}\frac{W_{-1/2,1/4}(x^2/2)}{W_{0,1/4}(x^2/2)}\,.$$

TABLE 16.5.1: Relative error of the 5^{th} approximants of (16.5.7) for $a = 1$, more precisely (16.5.9). The approximants are first evaluated with $w_5 = 0$ and then with w_5 given by (16.5.8).

x	$U(a,x)/U(a-1,x)$	(16.5.7)	(16.5.7)
0.25	8.329323e−01	1.4e+00	2.0e−02
0.75	6.485192e−01	1.8e−01	4.4e−03
1.25	5.211635e−01	4.0e−02	1.0e−03
5.25	1.813514e−01	7.3e−06	1.2e−07
20.25	4.920381e−02	2.6e−11	5.9e−14
50.25	1.988869e−02	3.1e−15	1.2e−18
100.25	9.973574e−03	3.2e−18	3.1e−22

TABLE 16.5.2: Relative error of the 20^{th} approximants of (16.5.7) for $a = 1$, more precisely (16.5.9). The approximants are first evaluated with $w_{20} = 0$ and then with w_{20} given by (16.5.8).

x	$U(a,x)/U(a-1,x)$	(16.5.7)	(16.5.7)
0.25	8.329323e−01	2.7e−01	1.9e−03
0.75	6.485192e−01	7.2e−03	4.7e−05
1.25	5.211635e−01	2.0e−04	1.3e−06
5.25	1.813514e−01	8.3e−15	5.6e−17
20.25	4.920381e−02	2.5e−34	4.8e−37
50.25	1.988869e−02	9.5e−50	3.6e−53
100.25	9.973574e−03	1.1e−61	1.1e−65

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Bessel functions

Solutions of boundary value problems are often expressed as linear combinations of the Bessel functions $J_n(z)$ for $n \geq 0$. Evaluating the solution of such a differential equation therefore requires the computation of the functions $J_n(z)$.
The Bessel functions $J_\nu(z), Y_\nu(z), I_\nu(z)$ and $K_\nu(z)$, which are particular forms of the confluent hypergeometric function, are analytic functions of z for $|\arg z| < \pi$, and for fixed nonzero z they are entire functions of ν. For $\nu \in \mathbb{Z}$, the functions $J_\nu(z)$ and $I_\nu(z)$ are entire functions of z.
The Bessel functions $J_\nu(z)$ and $Y_\nu(z)$ and their derivatives can be computed either by means of the recurrence relations combined with the use of continued fractions and Wronskian relations, or by making use of the continued fraction representations for the logarithmic derivatives of $J_\nu(z)$ and $J_\nu(z) + \mathrm{i}Y_\nu(z)$ with a Wronskian relation connecting $J_\nu(z), Y_\nu(z)$ and their derivatives.

17.1 Bessel functions

Definitions and elementary properties. The second order differential equation

$$z^2 \frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - \nu^2)w = 0, \qquad \nu \in \mathbb{C} \tag{17.1.1}$$

is called Bessel's differential equation. Among the solutions are the *Bessel functions of the first kind* $J_\nu(z)$, and the *Bessel functions of the second kind* $Y_\nu(z)$. Here ν denotes the order. The Bessel functions $J_\nu(z)$ and $Y_\nu(z)$ are defined by

$$J_\nu(z) := \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{2k}, \qquad |\arg z| < \pi, \quad \nu \in \mathbb{C}, \tag{17.1.2a}$$

$$Y_\nu(z) := \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}, \qquad |\arg z| < \pi, \quad \nu \in \mathbb{C} \setminus \mathbb{Z}, \tag{17.1.2b}$$

where $\Gamma(z)$ is the gamma function introduced in (12.1.1). For integer order $n \in \mathbb{Z}$ the right hand side of (17.1.2b) is replaced by its limit

$$Y_n(z) = \lim_{\nu \to n} \left(J_\nu(z) \cot(\nu\pi) - J_{-\nu}(z) \csc(\nu\pi) \right), \qquad n \in \mathbb{Z}. \tag{17.1.3}$$

The functions $J_\nu(z)$ and $Y_\nu(z)$ are linearly independent for all $\nu \in \mathbb{C}$, and the functions $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent for $\nu \in \mathbb{C} \setminus \mathbb{Z}$.
When discussing Bessel functions, we use the notation ν for complex order and n for integer order. The function $J_\nu(z)$ is sometimes called *cylinder function*, and the function $Y_\nu(z)$ is also called the *Weber function* or the *Neumann function*. In the special case $n \in \mathbb{N}_0$, the Bessel functions of the first kind $J_n(z)$ are also known as *Bessel coefficients*.
The Bessel functions $J_\nu(z)$ and $Y_\nu(z)$ satisfy the symmetry property

$$J_\nu(\bar{z}) = \overline{J_\nu(z)}, \quad Y_\nu(\bar{z}) = \overline{Y_\nu(z)}, \qquad \nu \in \mathbb{R}. \tag{17.1.4}$$

The Bessel functions $J_n(z)$ and $Y_n(z)$ satisfy the reflection formulas

$$\begin{aligned} J_{-n}(z) &= (-1)^n J_n(z), \qquad n \in \mathbb{Z}, \\ Y_{-n}(z) &= (-1)^n Y_n(z), \qquad n \in \mathbb{Z}. \end{aligned} \tag{17.1.5}$$

In the particular case $\nu \in \mathbb{R}$, the function $J_\nu(z)$ has infinitely many real zeros, all of which are simple with the possible exception of $z = 0$. If in addition $\nu \geq 0$, the positive zeros $j_{\nu,k}$ of $J_\nu(z)$ interlace with the positive zeros $j_{\nu+1,k}$ of $J_{\nu+1}(z)$ so that

$$0 < j_{\nu,1} < j_{\nu+1,1} < j_{\nu,2} < j_{\nu+1,2} < j_{\nu,3} < j_{\nu+1,3} < \dots,$$

and the negative zeros of $J_\nu(z)$ are given by $-j_{\nu,k}$. In this special case an infinite product representation of $J_\nu(z)$ is given by

$$J_\nu(z) = \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{j_{\nu,k}^2}\right), \qquad |\arg z| < \pi, \quad \nu \geq 0. \tag{17.1.6}$$

The relation

$$e^{\mathrm{i}z\cos(\theta)} = \sum_{n=-\infty}^{\infty} \mathrm{i}^n e^{\mathrm{i}n\theta} J_n(z) = J_0(z) + 2\sum_{n=0}^{\infty} \mathrm{i}^n \cos(n\theta) J_n(z), \qquad n \in \mathbb{Z} \tag{17.1.7}$$

is called the *Jacobi-Anger identity* [AAR99, p. 211].

The relation

$$\frac{2}{\pi z} = J_\nu(z)\frac{d}{dz}Y_\nu(z) - Y_\nu(z)\frac{d}{dz}J_\nu(z) = J_{\nu+1}(z)Y_\nu(z) - J_\nu(z)Y_{\nu+1}(z) \tag{17.1.8}$$

is a *Wronskian* relation.

Hankel functions. Another pair of solutions of the differential equation (17.1.1) is given by

$$H_\nu^{(1)}(z) = J_\nu(z) + \mathrm{i}Y_\nu(z), \qquad \nu \in \mathbb{C}, \tag{17.1.9a}$$

$$H_\nu^{(2)}(z) = J_\nu(z) - \mathrm{i}Y_\nu(z), \qquad \nu \in \mathbb{C}. \tag{17.1.9b}$$

The functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are called *Bessel functions of the third kind* or *Hankel functions.*

The Hankel functions satisfy the symmetry properties

$$H_{-\nu}^{(1)}(z) = e^{\mathrm{i}\nu\pi}H_\nu^{(1)}(z), \qquad \nu \in \mathbb{C}, \tag{17.1.10a}$$

$$H_{-\nu}^{(2)}(z) = e^{-\mathrm{i}\nu\pi}H_\nu^{(2)}(z), \qquad \nu \in \mathbb{C}, \tag{17.1.10b}$$

$$H_\nu^{(1)}(\bar{z}) = \overline{H_\nu^{(2)}(z)}, \qquad \nu \in \mathbb{R}, \tag{17.1.10c}$$

$$H_\nu^{(2)}(\bar{z}) = \overline{H_\nu^{(1)}(z)}, \qquad \nu \in \mathbb{R}. \tag{17.1.10d}$$

Spherical Bessel functions. The second order differential equation

$$z^2\frac{d^2w}{dz^2} + 2z\frac{dw}{dz} + (z^2 - n(n+1))w = 0, \qquad n \in \mathbb{Z} \tag{17.1.11}$$

is called the spherical Bessel differential equation. Among the solutions are the *spherical Bessel functions of the first kind* $j_n(z)$, the *spherical Bessel functions of the second kind* $y_n(z)$, and the *spherical Bessel functions of the third kind* $h_n^{(1)}(z)$ and $h_n^{(2)}(z)$. The spherical Bessel functions are closely related to the Bessel functions $J_n(z)$ and $Y_n(z)$ and the Hankel functions $H_n^{(1)}(z)$ and $H_n^{(2)}(z)$ by

$$j_n(z) := \sqrt{\frac{\pi}{2z}}J_{n+\frac{1}{2}}(z), \qquad n \in \mathbb{Z}, \tag{17.1.12a}$$

$$y_n(z) := \sqrt{\frac{\pi}{2z}}Y_{n+\frac{1}{2}}(z), \qquad n \in \mathbb{Z}, \tag{17.1.12b}$$

$$h_n^{(1)}(z) := j_n(z) + \mathrm{i}y_n(z) = \sqrt{\frac{\pi}{2z}}H_{n+\frac{1}{2}}^{(1)}(z), \qquad n \in \mathbb{Z}, \tag{17.1.12c}$$

$$h_n^{(2)}(z) := j_n(z) - \mathrm{i}y_n(z) = \sqrt{\frac{\pi}{2z}}H_{n+\frac{1}{2}}^{(2)}(z), \qquad n \in \mathbb{Z}. \tag{17.1.12d}$$

When $n = 0$ in (17.1.12) we find

$$J_{\frac{1}{2}}(z) = Y_{-\frac{1}{2}}(z) = \sqrt{\frac{2z}{\pi}} j_0(z) = \sqrt{\frac{2}{\pi z}} \sin(z), \tag{17.1.13a}$$

$$Y_{\frac{1}{2}}(z) = -J_{-\frac{1}{2}}(z) = \sqrt{\frac{2z}{\pi}} y_0(z) = -\sqrt{\frac{2}{\pi z}} \cos(z). \tag{17.1.13b}$$

The Wronskian determinant relation is

$$y_n(z) j_{n+1}(z) - y_{n+1}(z) j_n(z) = \frac{1}{z^2}. \tag{17.1.14}$$

Recurrence relations. Let $G_\nu(z)$ denote one of the functions $J_\nu(z)$, $Y_\nu(z)$, $H_\nu^{(1)}(z)$ or $H_\nu^{(2)}(z)$. Then $G_\nu(z)$ satisfies the recurrence relations [AS64, p. 361]

$$\frac{2\nu}{z} G_\nu(z) = G_{\nu-1}(z) + G_{\nu+1}(z), \tag{17.1.15}$$

$$2\frac{d}{dz} G_\nu(z) = G_{\nu-1}(z) - G_{\nu+1}(z). \tag{17.1.16}$$

Let $g_n(z)$ denote one of the functions $j_n(z)$, $y_n(z)$, $h_n^{(1)}(z)$ or $h_n^{(2)}(z)$. Then because of (17.1.12) the spherical function $g_n(z)$ satisfies the recurrence relations [AS64, p. 439]

$$\frac{2n+1}{z} g_n(z) = g_{n-1}(z) + g_{n+1}(z), \qquad n \in \mathbb{Z}, \tag{17.1.17}$$

$$(2n+1)\frac{d}{dz} g_n(z) = n g_{n-1}(z) - (n+1) g_{n+1}(z), \qquad n \in \mathbb{Z}. \tag{17.1.18}$$

Combining (17.1.16) with (17.1.15) and (17.1.17) with (17.1.18) gives

$$\frac{d}{dz} G_\nu(z) = \frac{\nu}{z} G_\nu(z) - G_{\nu+1}(z), \qquad \nu \in \mathbb{C}, \tag{17.1.19}$$

$$\frac{d}{dz} g_n(z) = \frac{n}{z} g_n(z) - g_{n+1}(z), \qquad n \in \mathbb{Z}. \tag{17.1.20}$$

Series expansions. The function $J_\nu(z)$ is defined by the series representation (17.1.2a) which can be rewritten as [AS64, p. 362]

$$J_\nu(z) = \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu {}_0F_1\left(;\nu+1; -z^2/4\right), \qquad |\arg z| < \pi, \quad \nu \in \mathbb{C} \tag{17.1.21}$$

where ${}_0F_1(;b;z)$ is the confluent hypergeometric limit function (16.3.1). Using Kummer's transformation (16.3.3) we obtain the representation

$$J_\nu(z) = \frac{e^{-\mathrm{i}z}}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(\nu+1/2)_k (2\mathrm{i})^k}{(2\nu+1)_k k!} z^k, \qquad |\arg z| < \pi, \quad \nu \in \mathbb{C}$$

$$= \frac{e^{-\mathrm{i}z}}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu {}_1F_1\left(\nu+1/2; 2\nu+1; 2\mathrm{i}z\right), \qquad (17.1.22)$$

where ${}_1F_1(a;b;z)$ is the confluent hypergeometric series (16.1.2).
The Hankel functions can be expressed in terms of the Kummer function of the second kind $U(a,b,z)$ introduced in (16.1.4) [AS64, p. 510],

$$H_\nu^{(1)}(z) = \frac{2}{\sqrt{\pi}} e^{-\mathrm{i}(\pi(\nu+\frac{1}{2})-z)} (2z)^\nu U\left(\nu+1/2, 2\nu+1, -2\mathrm{i}z\right), \qquad |\arg z| < \pi, \quad 2\nu \in \mathbb{C}\setminus\mathbb{Z}, \qquad (17.1.23)$$

$$H_\nu^{(2)}(z) = \frac{2}{\sqrt{\pi}} e^{\mathrm{i}(\pi(\nu+\frac{1}{2})-z)} (2z)^\nu U\left(\nu+1/2, 2\nu+1, 2\mathrm{i}z\right), \qquad |\arg z| < \pi, \quad 2\nu \in \mathbb{C}\setminus\mathbb{Z}. \qquad (17.1.24)$$

Combining (17.1.12a) with each of the series representations (17.1.21) and (17.1.22) for $J_\nu(z)$, we find

$$j_n(z) = \frac{\sqrt{\pi}}{(2n+1)\Gamma(n+1/2)}\left(\frac{z}{2}\right)^n {}_0F_1\left(;n+3/2; -z^2/4\right), \qquad |\arg z| < \pi, \qquad n \in \mathbb{Z} \qquad (17.1.25)$$

and

$$j_n(z) = \frac{\sqrt{\pi}e^{-\mathrm{i}z}}{(2n+1)\Gamma(n+1/2)}\left(\frac{z}{2}\right)^n {}_1F_1(n+1; 2n+2; 2\mathrm{i}z), \qquad |\arg z| < \pi, \qquad n \in \mathbb{Z}. \qquad (17.1.26)$$

Asymptotic series expansions. *Hankel's symbol* (ν, k) is frequently used in representing the coefficients in the asymptotic expansions of Bessel functions:

$$(\nu,k) = (-1)^k \frac{(\nu+1/2)_k(-\nu+1/2)_k}{k!} = \frac{\Gamma\left(\nu+k+1/2\right)}{k!\,\Gamma\left(\nu-k+1/2\right)}, \qquad k = 0,1,2,\ldots. \qquad (17.1.27)$$

We have $(\nu, 0) = 1$ and the recursion

$$(\nu, k+1) = \frac{\nu^2 - (k + 1/2)^2}{k+1}(\nu, k), \qquad k = 0, 1, 2, \ldots.$$

Let $P(\nu, z)$ and $Q(\nu, z)$ have the following asymptotic expansions

$$P(\nu, z) \approx \sum_{k=0}^{\infty}(-1)^k \frac{(\nu, 2k)}{(2z)^{2k}}, \qquad z \to \infty,$$

$$Q(\nu, z) \approx \sum_{k=0}^{\infty}(-1)^k \frac{(\nu, 2k+1)}{(2z)^{2k+1}}, \qquad z \to \infty.$$

Then, for the Bessel functions of the first, second and third kind we have the asymptotic expansions [Tem96, p. 239]

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \left(P(\nu, z) \cos\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right) - Q(\nu, z) \sin\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right)\right), \quad z \to \infty, \quad |\arg z| < \pi, \tag{17.1.28}$$

$$Y_\nu(z) = \sqrt{\frac{2}{\pi z}} \left(P(\nu, z) \sin\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right) + Q(\nu, z) \cos\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right)\right), \quad z \to \infty, \quad |\arg z| < \pi, \tag{17.1.29}$$

$$H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi z}} \left(P(\nu, z) + \mathrm{i}Q(\nu, z)\right) e^{\mathrm{i}\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right)}, \quad z \to \infty, \quad -\pi < \arg z < 2\pi, \tag{17.1.30}$$

$$H_\nu^{(2)}(z) = \sqrt{\frac{2}{\pi z}} \left(P(\nu, z) - \mathrm{i}Q(\nu, z)\right) e^{-\mathrm{i}\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right)}, \quad z \to \infty, \quad -2\pi < \arg z < \pi. \tag{17.1.31}$$

An alternative asymptotic representation [Tem96, p. 239] for the Hankel functions is

$$H_\nu^{(1)}(z) \approx \sqrt{\frac{2}{\pi z}} e^{z\mathrm{i} - \nu\pi\mathrm{i}/2 - \pi\mathrm{i}/4} \sum_{k=0}^{\infty} \frac{(\nu, k)}{(-2\mathrm{i}z)^k}, \quad z \to \infty, \quad -\pi < \arg z < 2\pi,$$

$$H_\nu^{(2)}(z) \approx \sqrt{\frac{2}{\pi z}} e^{-z\mathrm{i} + \nu\pi\mathrm{i}/2 + \pi\mathrm{i}/4} \sum_{k=0}^{\infty} \frac{(\nu, k)}{(2\mathrm{i}z)^k}, \quad z \to \infty, \quad -2\pi < \arg z < \pi.$$

Stieltjes transform. A Stieltjes transform representation for a particular ratio of Bessel functions,

$$g(\nu,z) := -1 + \frac{1}{4(\nu+1/2)\sqrt{-z}} \frac{J_{\nu-\frac{1}{2}}\left(\frac{1}{2\sqrt{-z}}\right)}{J_{\nu+\frac{1}{2}}\left(\frac{1}{2\sqrt{-z}}\right)}, \qquad \nu + 1/2 > 0, \quad (17.1.32)$$

can be given in terms of the zeros $q_{\nu,k}$ of the Bessel function $J_{\nu+\frac{1}{2}}\left(\frac{1}{2\sqrt{-z}}\right)$. For ν real the zeros $q_{\nu,k}$ are all simple and negative, and can be arranged so that

$$q_{\nu,1} < q_{\nu,2} < q_{\nu,3} < \cdots < 0, \qquad \nu \in \mathbb{R}. \quad (17.1.33)$$

Then [JTW94]

$$g(\nu,z) = \int_0^\infty \frac{d\Phi(\nu,t)}{z+t} = -\frac{2}{2\nu+1} \sum_{k=1}^\infty \frac{q_{\nu,k}}{z - q_{\nu,k}}, \qquad z \neq q_{\nu,k}, \quad \nu + 1/2 > 0. \quad (17.1.34)$$

Here the distribution function $\Phi(\nu,t)$ is the step function

$$\Phi(\nu,t) = \begin{cases} -\dfrac{2}{2\nu+1} \displaystyle\sum_{k=n+1}^\infty q_{\nu,k}, & -q_{\nu,n+1} < t < -q_{\nu,n}, \\ -\dfrac{2}{2\nu+1} \displaystyle\sum_{k=1}^\infty q_{\nu,k}, & -q_{\nu,1} < t < \infty. \end{cases} \quad (17.1.35)$$

The m^{th} moment $\mu_m(\nu)$ with respect to the distribution function $\Phi(\nu,t)$ is given by

$$\mu_m(\nu) = \int_0^\infty t^m \, d\Phi(\nu,t) = \frac{2}{2\nu+1} \sum_{k=1}^\infty (-q_{\nu,k})^{m+1}. \quad (17.1.36)$$

Further information about the discrete distribution $\Phi(\nu,t)$ can be found in [Sch39; Dic54].

For the Hankel functions also a Stieltjes transform representation can be given. This representation follows from the relation (17.2.7) with the modified Bessel functions of the second kind, and the Stieltjes transform representation (17.2.28) for the function $K_\nu(z)$.

S-fractions. Since the classical Stieltjes moment problem for the moments $\mu_m(\nu)$ given by (17.1.36) has the solution $\Phi(\nu,t)$, it follows from the results in *Chapter* 5 that there exists a modified S-fraction corresponding to the asymptotic series representation

$$g(\nu, z) \approx z^{-1} \sum_{m=0}^{\infty} (-1)^m \mu_m(\nu) z^{-m}, \qquad z \to \infty, \quad \nu + 1/2 > 0,$$

where $g(\nu, z)$ is defined in (17.1.32). The explicit coefficients of this modified S-fraction and of the S-fraction representation for a more general ratio of Bessel functions can be obtained from the series expansion (17.1.21) and the continued fraction (16.3.4) for the confluent hypergeometric limit function [Wal48, p. 349; AS64, p. 363; JT80, pp. 183–184]. Because of (17.1.19) we find at the same time an S-fraction representation for the logarithmic derivative of $J_\nu(z)$:

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = \frac{\nu}{z} - \frac{J'_\nu(z)}{J_\nu(z)} \tag{17.1.37}$$

$$= \frac{z/(2\nu+2)}{1} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{a_m(\nu)(\mathrm{i}z)^2}{1} \right), \qquad z \in \mathbb{C}, \quad \nu \geq 0, \tag{17.1.38a}$$

$$a_m(\nu) = \frac{1}{4(\nu+m-1)(\nu+m)}, \qquad m \geq 2. \tag{17.1.38b}$$

The coefficients $a_m(\nu)$ satisfy the asymptotic behaviour

$$a_m(\nu) \sim \frac{1}{4m^2}, \qquad m \to \infty.$$

For the spherical Bessel functions $j_n(z)$ we have

$$\frac{j_{n+1}(z)}{j_n(z)} = \frac{z/(2n+3)}{1} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{a_m(n+1/2)(\mathrm{i}z)^2}{1} \right), \qquad z \in \mathbb{C}, \quad n \in \mathbb{N}_0, \tag{17.1.39}$$

where the coefficients $a_m(n + 1/2)$ are given by (17.1.38b). The continued fractions (17.1.37), (17.1.38) and (17.1.39) are S-fractions in $-z^2$ from $m = 2$ on.

The S-fraction representation (17.1.38) can also be obtained in the following alternative way. Since the sequence of functions $J_{\nu+n}(z)$ is a minimal solution of the three term recurrence relation (17.1.15) [JT80, pp. 167–168], we know from *Theorem* 3.6.1 that

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = -\operatorname*{K}_{m=1}^{\infty} \left(\frac{-1}{2(\nu+m)/z} \right), \qquad z \in \mathbb{C}, \quad \nu \geq 0. \tag{17.1.40}$$

The same holds for the ratio of spherical Bessel functions $j_{n+1}(z)/j_n(z)$.

An S-fraction representation for the Bessel coefficients $J_0(x)$ and $Y_0(x)$ exists and follows from (17.2.9) and (17.2.30) in the next section,

$$J_0(x) = -\frac{2}{\pi}\Im\left(\frac{a_1}{2x\mathrm{i}} \underset{+}{} \frac{a_2}{1} \underset{+}{} \frac{a_3}{2x\mathrm{i}} \underset{+}{} \frac{a_4}{1} \underset{+}{} \dots\right), \qquad x > 0, \tag{17.1.41a}$$

$$Y_0(x) = -\frac{2}{\pi}\Re\left(\frac{a_1}{2x\mathrm{i}} \underset{+}{} \frac{a_2}{1} \underset{+}{} \frac{a_3}{2x\mathrm{i}} \underset{+}{} \frac{a_4}{1} \underset{+}{} \dots\right), \qquad x > 0, \tag{17.1.41b}$$

but without closed formula for the coefficients a_m.

S-fraction representations for the Hankel functions follow from the relation (17.2.7) and the S-fraction (17.2.30) for the modified Bessel function $K_\nu(z)$:

$$H_\nu^{(1)}(z) = 2e^{-\mathrm{i}\nu\pi/2+\mathrm{i}z}\sqrt{\frac{2\mathrm{i}z}{\pi}}\left(\frac{a_1}{-2\mathrm{i}z} \underset{+}{} \frac{a_2}{1} \underset{+}{} \frac{a_3}{-2\mathrm{i}z} \underset{+}{} \frac{a_4}{1} \underset{+}{} \dots\right), \quad -1 < \nu < 1, \tag{17.1.42a}$$

$$H_\nu^{(2)}(z) = 2\mathrm{i}ze^{\mathrm{i}\nu\pi/2-\mathrm{i}z}\sqrt{\frac{2\mathrm{i}z}{\pi}}\left(\frac{a_1}{2\mathrm{i}z} \underset{+}{} \frac{a_2}{1} \underset{+}{} \frac{a_3}{2\mathrm{i}z} \underset{+}{} \frac{a_4}{1} \underset{+}{} \dots\right), \quad -1 < \nu < 1. \tag{17.1.42b}$$

No known closed formula exists for the coefficients a_m in the S-fraction (17.1.42). Their asymptotic behaviour is given by [JS99]

$$a_m \sim \frac{m}{2}, \qquad m \to \infty.$$

C-fractions. If we relax the condition on ν in (17.1.38) and allow $\nu \in \mathbb{C} \setminus \mathbb{Z}^-$, the S-fraction becomes a C-fraction [JT80, pp. 183–184].

According to (17.2.7) the Hankel functions are closely related to the modified Bessel functions $K_\nu(z)$ of the second kind introduced in *Section* 17.2. Hence a C-fraction representation can be obtained from (17.2.35). Because of (17.1.19) this gives at the same time a C-fraction representation of the logarithmic derivative of the Hankel functions:

$$\frac{H^{(1)}_{\nu+1}(z)}{H^{(1)}_{\nu}(z)} = \frac{\nu}{z} - \frac{dH^{(1)}_{\nu}(z)/dz}{H^{(1)}_{\nu}(z)} \tag{17.1.43}$$

$$= \frac{-1}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{c_m(\nu)/(-2\mathrm{i}z)}{1}\right), \qquad |\arg(-\mathrm{i}z)| < \pi, \quad \nu \in \mathbb{C}, \tag{17.1.44}$$

$$\frac{H^{(2)}_{\nu+1}(z)}{H^{(2)}_{\nu}(z)} = \frac{\nu}{z} - \frac{dH^{(2)}_{\nu}(z)/dz}{H^{(2)}_{\nu}(z)} \tag{17.1.45}$$

$$= \frac{1}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{c_m(\nu)/(2\mathrm{i}z)}{1}\right), \qquad |\arg(\mathrm{i}z)| < \pi, \quad \nu \in \mathbb{C}, \tag{17.1.46}$$

$$c_{2k}(\nu) = 2k - 3 - 2\nu, \quad c_{2k+1}(\nu) = 2k + 1 + 2\nu, \quad k \geq 1.$$

In *Section* 7.7 the respective modifications

$$w_n^{\mp}(z) = \frac{-1 + \sqrt{1 + 4c_{n+1}(\nu)/(\mp \mathrm{i}z)}}{2}$$

are suggested for the evaluation of (17.1.44) and (17.1.46). Here the minus sign goes with the former and the plus sign with the latter continued fraction.

T-fractions. From the series representation (17.1.21) and the T-fraction representation (16.3.6) for a ratio of confluent hypergeometric limit functions, we obtain a T-fraction representation for a ratio of Bessel functions. Because of (17.1.19) it is at the same time a T-fraction representation of the logarithmic derivative of $J_\nu(z)$:

$$\frac{J_{\nu+1}(z)}{J_{\nu}(z)} = \frac{\nu}{z} - \frac{J'_{\nu}(z)}{J_{\nu}(z)} \tag{17.1.47}$$

$$= \frac{z}{2\nu + 2 - \mathrm{i}z} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{c_m z}{e_m + d_m z}\right), \qquad z \in \mathbb{C}, \quad \nu \in \mathbb{C} \setminus \mathbb{Z}^-, \tag{17.1.48a}$$

$$c_m = (2\nu + 2m - 1)\mathrm{i}, \quad e_m = 2\nu + m + 1, \quad d_m = -2\mathrm{i}, \quad m \geq 2. \tag{17.1.48b}$$

The T-fraction representation for a ratio of spherical Bessel functions is given by

$$\frac{j_{n+1}(z)}{j_n(z)} = \frac{z}{2n+3-\mathrm{i}z} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{\tilde{c}_m z}{\tilde{e}_m + \tilde{d}_m z}\right), \qquad z \in \mathbb{C}, \quad n \in \mathbb{N}_0, \tag{17.1.49a}$$

where the coefficients are given by

$$\tilde{c}_m = 2(n+m)\mathrm{i}, \quad \tilde{e}_m = 2n+m+2, \quad \tilde{d}_m = -2\mathrm{i}, \quad m \geq 2. \tag{17.1.49b}$$

J-fractions. From the J-fraction representation (17.2.41) for a ratio of modified Bessel functions we obtain [Hit68, p. 109] J-fraction representations for ratios of both Hankel functions. Because of (17.1.19) this gives at the same time a J-fraction representation of the logarithmic derivative of the Hankel functions:

$$\frac{H^{(1)}_{\nu+1}(z)}{H^{(1)}_{\nu}(z)} = \frac{\nu}{z} - \frac{dH^{(1)}_{\nu}(z)/dz}{H^{(1)}_{\nu}(z)} \tag{17.1.50}$$

$$= \frac{2\nu+1-2\mathrm{i}z}{2z} - \frac{1}{z}\mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{\nu^2-(2m-1)^2/4}{2(\mathrm{i}z-m)}\right),$$

$$|\arg(-\mathrm{i}z)| < \pi, \quad \nu \in \mathbb{C}, \tag{17.1.51}$$

$$\frac{H^{(2)}_{\nu+1}(z)}{H^{(2)}_{\nu}(z)} = \frac{\nu}{z} - \frac{dH^{(2)}_{\nu}(z)/dz}{H^{(2)}_{\nu}(z)} \tag{17.1.52}$$

$$= \frac{2\nu+1+2\mathrm{i}z}{2z} + \frac{1}{z}\mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{\nu^2-(2m-1)^2/4}{2(\mathrm{i}z+m)}\right),$$

$$|\arg(\mathrm{i}z)| < \pi, \quad \nu \in \mathbb{C}. \tag{17.1.53}$$

In *Section* 7.7 the respective modifications

$$\tilde{w}_n^{\mp}(z) = -(\mathrm{i}z \mp n)$$

are suggested for the evaluation of the continued fractions in (17.1.51) and (17.1.53). The minus sign in $\tilde{w}_n^{\mp}(z)$ goes with the former and the plus sign with the latter.

TABLE 17.1.1: Relative error of the 5^{th} partial sum of the series (17.1.2) and the asymptotic series (17.1.28) for $J_{85/2}(z)$, and relative error of the product $J_{1/2}(z)\prod_{k=0}^{41}\left(J_{k+3/2}(z)/J_{k+1/2}(z)\right)$ where each ratio $J_{k+3/2}/J_{k+1/2}$ is evaluated by means of the 5^{th} approximant of the continued fractions (17.1.38) and (17.1.48). The factor $J_{1/2}(z) = \sqrt{2/(z\pi)}\sin(z)$ is supplied with 30 decimal digits accuracy so that the truncation errors in the approximation of the factors $J_{k+3/2}/J_{k+1/2}$ dominate. All results are compared with the recurrence (17.1.15) which is unstable for $\nu/x > 1$. Here the second starting value $J_{-1/2}(z)$ is supplied exactly. Making use of the symmetry properties (17.1.4) and (17.1.5), we can restrict our investigation to the first quadrant, including the positive real and imaginary axes.

x	$J_\nu(x)$	(17.1.2)	(17.1.15)	(17.1.38)	(17.1.48)	(17.1.28)
0.01	1.750149e−150	3.6e−41	7.9e+264	2.3e−29	1.6e−13	2.9e+197
0.5	2.809806e−78	8.8e−21	8.6e+121	2.4e−12	5.2e−05	4.6e+105
1	1.740120e−65	3.6e−17	7.8e+96	2.8e−09	1.9e−03	1.6e+89
2	1.063809e−52	1.5e−13	3.5e+71	5.6e−06	1.3e−01	6.5e+72
5	7.705496e−36	1.0e−08	6.9e+37	1.6e−01	1.2e+00	1.3e+51
15	4.538163e−16	1.5e−02	4.1e−02	3.3e−01	1.0e+00	2.0e+26
50	−5.735702e−02	1.4e+12	2.3e−30	1.0e+00	1.8e+00	1.4e+06
100	−5.912167e−02	1.1e+28	7.4e−31	1.0e+00	3.6e−01	3.0e+01
500	2.680276e−02	1.4e+65	1.2e−31	2.3e+25	1.3e+00	1.4e−06
1000	−2.407106e−02	9.6e+80	9.7e−31	2.7e+38	8.2e−01	4.0e−10

x	$\lvert J_\nu(x+\mathrm{i}x)\rvert_s$	(17.1.2)	(17.1.15)	(17.1.38)	(17.1.48)	(17.1.28)
0.01	−4.364782e−144	2.3e−39	3.1e+252	7.5e−28	8.9e−13	2.2e+189
0.5	−7.017587e−72	5.6e−19	6.2e+109	7.3e−11	1.5e−04	4.0e+97
1	−4.364789e−59	2.3e−15	1.6e+85	7.3e−08	2.5e−03	2.8e+81
2	−2.714862e−46	9.4e−12	3.8e+60	5.4e−05	1.3e−02	4.3e+65
5	−2.221215e−29	5.6e−07	1.3e+29	3.3e−02	4.7e−03	2.7e+45
15	4.531313e−09	2.7e−01	1.1e−03	9.2e−01	9.0e−03	8.4e+23
50	−2.285875e+16	4.0e+02	1.6e−22	3.7e+01	5.1e−03	1.7e+08
100	9.539441e+39	1.0e+00	7.8e−27	2.6e+06	5.4e−04	4.8e+02
500	8.533405e+214	1.0e+00	6.6e−30	4.6e+32	4.2e−07	6.6e−08
1000	−1.330564e+432	1.0e+00	5.0e−30	1.3e+45	1.5e−08	1.1e−11

x	$\lvert J_\nu(\mathrm{i}x)\rvert_s$	(17.1.2)	(17.1.15)	(17.1.38)	(17.1.48)	(17.1.28)
0.01	$-1.750151e{-150}$	$3.6e{-41}$	$1.0e{+265}$	$2.4e{-29}$	$1.6e{-13}$	$2.9e{+197}$
0.5	$-2.817892e{-78}$	$8.8e{-21}$	$1.1e{+122}$	$2.2e{-12}$	$2.6e{-05}$	$5.9e{+105}$
1	$-1.760236e{-65}$	$3.6e{-17}$	$1.3e{+98}$	$2.0e{-09}$	$4.1e{-04}$	$4.4e{+89}$
2	$-1.113862e{-52}$	$1.4e{-13}$	$7.3e{+72}$	$1.3e{-06}$	$2.9e{-03}$	$5.8e{+73}$
5	$-1.027067e{-35}$	$7.7e{-09}$	$9.5e{+41}$	$1.5e{-03}$	$6.1e{-03}$	$3.2e{+53}$
15	$-6.034981e{-15}$	$1.5e{-03}$	$4.7e{+08}$	$3.2e{-01}$	$6.7e{-03}$	$3.4e{+31}$
50	$-9.052865e{+12}$	$9.8e{-01}$	$3.3e{-16}$	$1.0e{+02}$	$4.5e{-03}$	$2.5e{+13}$
100	$-1.401375e{+38}$	$1.0e{+00}$	$9.5e{-23}$	$6.7e{+05}$	$1.0e{-03}$	$2.0e{+06}$
500	$-4.111670e{+214}$	$1.0e{+00}$	$9.1e{-29}$	$1.2e{+27}$	$1.9e{-06}$	$9.8e{-06}$
1000	$-1.007133e{+432}$	$1.0e{+00}$	$4.5e{-30}$	$1.2e{+39}$	$7.4e{-08}$	$1.0e{-09}$

TABLE 17.1.2: Relative error of the 20^{th} partial sum of the series (17.1.2) and the asymptotic series (17.1.28) for $J_{85/2}(z)$, and relative error of the product $J_{1/2}(z)\prod_{k=0}^{41}\left(J_{k+3/2}(z)/J_{k+1/2}(z)\right)$ where each ratio is again evaluated by means of the 20^{th} approximant of the continued fractions (17.1.38) and (17.1.48). The factor $J_{1/2}(z)=\sqrt{2/(z\pi)}\sin(z)$ is supplied with 135 decimal digits accuracy so that the truncation errors in the approximation of the factors $J_{k+3/2}/J_{k+1/2}$ dominate. All results are compared with the recurrence (17.1.15) which is unstable for $\nu/x>1$. Here the second starting value $J_{-1/2}(z)$ is supplied exactly. Making use of the symmetry properties (17.1.4) and (17.1.5), we can restrict our investigation to the first quadrant, including the positive real and imaginary axes.

x	$J_\nu(x)$	(17.1.2)	(17.1.15)	(17.1.38)	(17.1.48)	(17.1.28)
0.01	$1.750149e{-150}$	$2.6e{-153}$	$5.8e{+159}$	$4.1e{-132}$	$2.2e{-57}$	$2.4e{+295}$
0.5	$2.809806e{-78}$	$5.9e{-82}$	$2.3e{+17}$	$4.0e{-64}$	$2.3e{-23}$	$4.5e{+152}$
1	$1.740120e{-65}$	$2.6e{-69}$	$5.7e{-08}$	$5.3e{-52}$	$2.9e{-17}$	$2.1e{+127}$
2	$1.063809e{-52}$	$1.2e{-56}$	$3.4e{-34}$	$1.4e{-39}$	$7.5e{-11}$	$5.9e{+101}$
5	$7.705496e{-36}$	$6.8e{-40}$	$2.2e{-67}$	$1.8e{-22}$	$1.6e{-01}$	$1.0e{+68}$
15	$4.538163e{-16}$	$2.3e{-19}$	$1.9e{-106}$	$2.7e{-04}$	$4.6e{+00}$	$6.4e{+27}$
50	$-5.735702e{-02}$	$2.0e{+11}$	$2.6e{-135}$	$2.2e{+00}$	$1.6e{+00}$	$1.2e{-09}$
100	$-5.912167e{-02}$	$2.7e{+36}$	$2.5e{-135}$	$9.0e{-01}$	$3.5e{-01}$	$3.7e{-22}$
500	$2.680276e{-02}$	$4.1e{+94}$	$9.0e{-136}$	$8.3e{+16}$	$1.3e{+00}$	$1.5e{-51}$
1000	$-2.407106e{-02}$	$3.2e{+119}$	$1.6e{-135}$	$1.0e{+00}$	$8.2e{-01}$	$4.6e{-64}$

x	$\lvert J_\nu(x+\mathrm{i}x)\rvert_s$	(17.1.2)	(17.1.15)	(17.1.38)	(17.1.48)	(17.1.28)
0.01	−4.364782e−144	5.4e−147	2.2e+147	4.3e−126	2.2e−54	5.6e+282
0.5	−7.017587e−72	1.2e−75	9.9e+04	3.9e−58	9.1e−21	1.1e+140
1	−4.364789e−59	5.4e−63	2.3e−20	4.2e−46	3.9e−15	5.7e+114
2	−2.714862e−46	2.4e−50	8.1e−45	3.3e−34	5.0e−10	3.8e+89
5	−2.221215e−29	1.2e−33	5.2e−76	9.4e−20	7.0e−06	1.1e+57
15	4.531313e−09	1.2e−13	1.9e−108	2.0e−07	1.0e−12	6.4e+20
50	−2.285875e+16	2.9e+06	2.9e−127	4.1e−02	1.0e−12	1.2e−11
100	9.539441e+39	6.4e+07	2.6e−131	1.1e+00	1.6e−16	2.4e−26
500	8.533405e+214	1.0e+00	1.8e−134	1.0e+00	5.6e−29	2.8e−57
1000	−1.330564e+432	1.0e+00	5.5e−135	1.0e+00	8.2e−35	4.0e−70

x	$\lvert J_\nu(\mathrm{i}x)\rvert_s$	(17.1.2)	(17.1.15)	(17.1.38)	(17.1.48)	(17.1.28)
0.01	−1.750151e−150	2.6e−153	8.6e+159	4.1e−132	2.2e−57	2.4e+295
0.5	−2.817892e−78	5.9e−82	1.7e+17	3.5e−64	8.4e−24	4.9e+152
1	−1.760236e−65	2.6e−69	2.4e−07	3.2e−52	3.1e−18	2.9e+127
2	−1.113862e−52	1.1e−56	3.0e−32	1.9e−40	2.9e−13	2.2e+102
5	−1.027067e−35	5.1e−40	4.1e−64	6.4e−26	4.9e−09	6.1e+69
15	−6.034981e−15	1.9e−20	6.8e−97	3.3e−12	9.1e−11	1.2e+33
50	−9.052865e+12	3.2e−03	1.2e−120	4.1e−04	2.1e−13	7.6e−02
100	−1.401375e+38	1.0e+00	1.3e−128	5.9e−02	1.3e−15	4.1e−18
500	−4.111670e+214	1.0e+00	1.0e−133	1.0e+00	2.4e−26	1.4e−50
1000	−1.007133e+432	1.0e+00	1.6e−134	1.0e+00	5.4e−32	1.3e−63

17.2 Modified Bessel functions

Definitions and elementary properties. The second order differential equation

$$z^2 \frac{d^2 w}{dz^2} + z\frac{dw}{dz} - (z^2+\nu^2)w = 0, \qquad \nu \in \mathbb{C} \tag{17.2.1}$$

is called the modified Bessel differential equation. The solutions $I_\nu(z)$ and $K_\nu(z)$ are called the modified Bessel functions. The *modified Bessel function of the first kind* $I_\nu(z)$, is defined in terms of the Bessel function

$J_\nu(z)$,

$$I_\nu(z) := e^{-i\nu\pi/2} J_\nu(iz), \qquad -\pi < \arg z \leq \frac{\pi}{2}, \quad \nu \in \mathbb{C}, \tag{17.2.2a}$$

$$I_\nu(z) := e^{i3\nu\pi/2} J_\nu(iz), \qquad \frac{\pi}{2} < \arg z \leq \pi, \quad \nu \in \mathbb{C}, \tag{17.2.2b}$$

and the *modified Bessel function of the second kind* $K_\nu(z)$, is defined by

$$K_\nu(z) := \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}, \qquad \nu \in \mathbb{C} \setminus \mathbb{Z}. \tag{17.2.3}$$

For integer order $n \in \mathbb{Z}$ the right hand side of (17.2.3) is replaced by its limit

$$K_n(z) = \lim_{\nu \to n} \frac{\pi}{2} \csc(\nu\pi) \left(I_{-\nu}(z) - I_\nu(z)\right), \qquad n \in \mathbb{Z}. \tag{17.2.4}$$

The functions $I_\nu(z)$ and $K_\nu(z)$ are linearly independent for all $\nu \in \mathbb{C}$, and the functions $I_\nu(z)$ and $I_{-\nu}(z)$ are linearly independent for $\nu \in \mathbb{C} \setminus \mathbb{Z}$. The function $I_\nu(z)$ is sometimes called the *hyperbolic Bessel function*, and the function $K_\nu(z)$ is also called the *Basset function* or the *Macdonald function*. The modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ satisfy the symmetry property

$$I_\nu(\bar{z}) = \overline{I_\nu(z)}, \quad K_\nu(\bar{z}) = \overline{K_\nu(z)}, \qquad \nu \in \mathbb{R} \tag{17.2.5}$$

and the relations [SO87, p. 489; AS64, p. 375]

$$\begin{aligned} I_\nu(-x) &= (-1)^\nu I_\nu(x), \qquad x > 0, \quad \nu \in \mathbb{C}, \\ I_{-n}(z) &= I_n(z), \qquad n \in \mathbb{Z}, \\ K_{-\nu}(z) &= K_\nu(z), \qquad \nu \in \mathbb{C}. \end{aligned} \tag{17.2.6}$$

The modified Bessel function of the second kind $K_\nu(z)$, can be expressed in terms of the Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ by the relations

$$K_\nu(z) = \frac{\pi}{2} i e^{i\nu\pi/2} H_\nu^{(1)}(iz), \qquad -\pi < \arg z \leq \frac{\pi}{2}, \tag{17.2.7a}$$

$$K_\nu(z) = -\frac{\pi}{2} i e^{-i\nu\pi/2} H_\nu^{(2)}(-iz), \qquad \frac{\pi}{2} < \arg z \leq \pi. \tag{17.2.7b}$$

From (17.1.9) and (17.2.7), we get

$$J_\nu(z) = -\frac{1}{\pi i} \left(e^{i\nu\pi/2} K_\nu(iz) - e^{-i\nu\pi/2} K_\nu(-iz) \right), \qquad |\arg z| < \frac{\pi}{2}, \tag{17.2.8a}$$

$$Y_\nu(z) = -\frac{1}{\pi} \left(e^{i\nu\pi/2} K_\nu(iz) + e^{-i\nu\pi/2} K_\nu(-iz) \right), \qquad |\arg z| < \frac{\pi}{2}, \tag{17.2.8b}$$

and hence we obtain

$$J_\nu(x) = -\frac{2}{\pi}\Im\left(e^{i\nu\pi/2}K_\nu(ix)\right), \qquad x > 0, \tag{17.2.9a}$$

$$Y_\nu(x) = -\frac{2}{\pi}\Re\left(e^{-i\nu\pi/2}K_\nu(ix)\right), \qquad x > 0. \tag{17.2.9b}$$

The Wronskian relation is

$$\begin{aligned} -1/z &= I_\nu(z)\frac{d}{dz}K_\nu(z) - K_\nu(z)\frac{d}{dz}I_\nu(z) \\ &= -I_\nu(z)K_{\nu+1}(z) - I_{\nu+1}(z)K_\nu(z) \end{aligned} \tag{17.2.10}$$

Because of the close connection between the Bessel functions $J_\nu(z)$ and the modified Bessel functions $I_\nu(z)$ given in (17.2.2), the results established for the functions $J_\nu(z)$ can be used to derive continued fraction representations for $I_\nu(z)$.

Modified spherical Bessel functions. The second order differential equation

$$z^2\frac{d^2w}{dz^2} + 2z\frac{dw}{dz} - (z^2 + n(n+1))w = 0, \qquad n \in \mathbb{Z} \tag{17.2.11}$$

is called the modified spherical Bessel differential equation. Among the solutions are the *modified spherical Bessel functions of the first kind* $i_n(z)$, the *modified spherical Bessel functions of the second kind* $k_n(z)$, and the *modified spherical Bessel functions of the third kind* $g_\nu^{(1)}(z)$ and $g_\nu^{(2)}(z)$. The spherical Bessel functions are closely related to the modified Bessel functions $I_n(z)$ and $K_n(z)$ and the Hankel functions $H_n^{(1)}(z)$ and $H_n^{(2)}(z)$ and are given by

$$i_n(z) := \sqrt{\frac{\pi}{2z}}I_{n+\frac{1}{2}}(z), \qquad n \in \mathbb{Z}, \tag{17.2.12a}$$

$$k_n(z) := \sqrt{\frac{\pi}{2z}}K_{n+\frac{1}{2}}(z), \qquad n \in \mathbb{Z}, \tag{17.2.12b}$$

$$g_n^{(1)}(z) := i_n(z) + ik_n(z) = \sqrt{\frac{\pi}{2z}}H^{(1)}_{n+\frac{1}{2}}(z), \qquad n \in \mathbb{Z}, \tag{17.2.12c}$$

$$g_n^{(2)}(z) := i_n(z) - ik_n(z) = \sqrt{\frac{\pi}{2z}}H^{(2)}_{n+\frac{1}{2}}(z), \qquad n \in \mathbb{Z}. \tag{17.2.12d}$$

The Wronskian relation is

$$i_n(z)k_{n+1}(z) + i_{n+1}(z)k_n(z) = \frac{\pi}{2z^2}. \qquad (17.2.13)$$

Recurrence relations. Let $L_\nu(z)$ denote one of the functions $I_\nu(z)$ or $e^{i\nu\pi}K_\nu(z)$. The function $L_\nu(z)$ satisfies the recurrence relations [AS64, p. 376]

$$\frac{2\nu}{z}L_\nu(z) = L_{\nu-1}(z) - L_{\nu+1}(z), \qquad (17.2.14)$$

$$2\frac{d}{dz}L_\nu(z) = L_{\nu-1}(z) + L_{\nu+1}(z). \qquad (17.2.15)$$

Combined with (17.2.12) these lead to recurrence relations for the modified spherical Bessel functions [AS64, p. 444]. If $\ell_n(z)$ denotes one of the functions $i_n(z)$ or $(-1)^{n+1}k_n(z)$, then

$$\frac{2n+1}{z}\ell_n(z) = \ell_{n-1}(z) - \ell_{n+1}(z), \qquad n \in \mathbb{Z}, \qquad (17.2.16)$$

$$\frac{2n+1}{z}\frac{d}{dz}\ell_n(z) = n\ell_{n-1}(z) + (n+1)\ell_{n+1}(z), \qquad n \in \mathbb{Z}. \qquad (17.2.17)$$

Combining (17.2.14) with (17.2.15) and (17.2.16) with (17.2.17) gives

$$\frac{d}{dz}L_\nu(z) = \frac{\nu}{z}L_\nu(z) + L_{\nu+1}(z), \qquad \nu \in \mathbb{C}, \qquad (17.2.18)$$

$$\frac{1}{z}\frac{d}{dz}\ell_n(z) = \frac{n}{z}\ell_n(z) + \frac{2n+1}{2(n+1)}\ell_{n+1}(z), \qquad n \in \mathbb{Z}. \qquad (17.2.19)$$

Series expansions. From (17.2.2) and (17.1.2a) we find the series representation

$$\begin{aligned} I_\nu(z) &= \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{2k}, \qquad |\arg z| < \pi, \quad \nu \in \mathbb{C} \\ &= \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu {}_0F_1\left(;\nu+1; z^2/4\right), \end{aligned} \qquad (17.2.20)$$

where ${}_0F_1(;b;z)$ is the confluent hypergeometric limit function (16.3.1). By using Kummer's transformation (16.3.3) we get

$$\begin{aligned} I_\nu(z) &= \frac{e^{-z}}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(\nu+1/2)_k 2^k}{(2\nu+1)_k k!} z^k, \qquad |\arg z| < \pi, \quad \nu \in \mathbb{C} \\ &= \frac{e^{-z}}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu {}_1F_1\left(\nu+1/2; 2\nu+1; 2z\right) \end{aligned} \qquad (17.2.21)$$

where ${}_1F_1(a;b;z)$ denotes the confluent hypergeometric series (16.1.2). Combining the relation (17.2.12a) with each of the series representations (17.2.20) and (17.2.21) for $I_\nu(z)$, we find

$$i_n(z) = \frac{\sqrt{\pi}}{(2n+1)\Gamma(n+1/2)}\left(\frac{z}{2}\right)^n {}_0F_1\left(;n+3/2;z^2/4\right), \quad |\arg z|<\pi, \quad n\in\mathbb{Z}, \tag{17.2.22}$$

and

$$i_n(z) = \frac{\sqrt{\pi}e^{-iz}}{(2n+1)\Gamma(n+1/2)}\left(\frac{z}{2}\right)^n {}_1F_1(n+1;2n+2;2z), \quad |\arg z|<\pi, \quad n\in\mathbb{Z}. \tag{17.2.23}$$

Asymptotic series expansions. The Hankel symbol (ν,k) is defined in (17.1.27). For the function $I_\nu(z)$ we combine (17.2.2) with the asymptotic expansion (17.1.28) for $J_\nu(z)$ [Tem96, p. 240]:

$$I_\nu(z) \approx \frac{e^z}{\sqrt{2\pi z}}\sum_{k=0}^{\infty}(-1)^k\frac{(\nu,k)}{(2z)^k} + \frac{e^{-z+i(2\nu+1)\pi/2}}{\sqrt{2\pi z}}\sum_{k=0}^{\infty}\frac{(\nu,k)}{(2z)^k}, \tag{17.2.24}$$

$$z\to\infty, \quad -\frac{\pi}{2}<\arg z<\frac{3\pi}{2}, \quad \nu\in\mathbb{C},$$

$$I_\nu(z) \approx \frac{e^z}{\sqrt{2\pi z}}\sum_{k=0}^{\infty}(-1)^k\frac{(\nu,k)}{(2z)^k} + \frac{e^{-z-i(2\nu+1)\pi/2}}{\sqrt{2\pi z}}\sum_{k=0}^{\infty}\frac{(\nu,k)}{(2z)^k}, \tag{17.2.25}$$

$$z\to\infty, \quad -\frac{3\pi}{2}<\arg z<\frac{\pi}{2}, \quad \nu\in\mathbb{C}.$$

The function $K_\nu(z)$ is related to the Whittaker function $W_{0,\nu}(z)$ given in (16.4.2b), by [AS64, p. 377]

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}W_{0,\nu}(2z). \tag{17.2.26}$$

From the asymptotic expansion (16.4.7) we find

$$\begin{aligned} K_\nu(z) &\approx \sqrt{\frac{\pi}{2z}}e^{-z}\sum_{k=0}^{\infty}\frac{(\nu+1/2)_k(-\nu+1/2)_k}{k!}(-2z)^{-k} \\ &= \sqrt{\frac{\pi}{2z}}e^{-z}\sum_{k=0}^{\infty}\frac{(\nu,k)}{(-2z)^k}, \qquad z\to\infty, \quad |\arg z|<\frac{3\pi}{2} \qquad (17.2.27) \\ &= \sqrt{\frac{\pi}{2z}}e^{-z}\ {}_2F_0(\nu+1/2,-\nu+1/2;-1/(2z)). \end{aligned}$$

Stieltjes transform. Relation (17.2.26) between $K_\nu(z)$ and $W_{0,\nu}(2z)$ leads to

$$K_\nu(z) = e^{-z}\sqrt{2\pi z}\,\Psi_{\nu,-\nu}(2z), \quad -1 < \nu < 1,$$

where $\Psi_{\alpha,\beta}$ is introduced in (16.4.8). Hence by (16.4.13) we get the Stieltjes transform representation

$$K_\nu(z) = e^{-z}\sqrt{2\pi z}\int_0^\infty \frac{\phi(\nu,t)}{2z+t}\,dt, \qquad |\arg z| < \pi \qquad (17.2.28a)$$

where the weight function $\phi(\nu, t)$ is given by

$$\phi(\nu,t) = \frac{1}{\pi}\sin(\pi(\nu + 1/2))e^{-t}\Psi_{-\nu,\nu}(t), \qquad 0 < t < \infty. \qquad (17.2.28b)$$

The moments $\mu_k(\nu)$ for $\phi(\nu, t)$ are

$$\mu_k(\nu) = \int_0^\infty t^k\phi(\nu,t)\,dt = \frac{(-1)^k(\nu+1/2)_k(-\nu+1/2)_k}{k!}, \qquad k \geq 0. \qquad (17.2.29)$$

S-fractions. In a similar way as in *Section* 16.4, a modified S-fraction representation can be obtained from the Stieltjes integral (17.2.28),

$$K_\nu(z) = e^{-z}\sqrt{2\pi z}\left(\frac{a_1}{2z} \underset{+}{} \frac{a_2}{1} \underset{+}{} \frac{a_3}{2z} \underset{+}{} \frac{a_4}{1} \underset{+}{} \dots\right), \qquad -1 < \nu < 1. \qquad (17.2.30)$$

No known closed formula exists for the coefficients a_m in the S-fraction representation (17.2.30). The coefficients satisfy the asymptotic behaviour [JS99]

$$a_m \sim \frac{m}{2}, \qquad m \to \infty.$$

The S-fraction representation for a ratio of modified Bessel functions $I_\nu(z)$ can be obtained from the S-fraction (17.1.38) and the relation (17.2.2) [GS78]. Because of (17.2.18) at the same time an S-fraction representation for the logarithmic derivative of $I_\nu(z)$ is obtained:

$$\frac{I_{\nu+1}(z)}{I_\nu(z)} = -\frac{\nu}{z} + \frac{I'_\nu(z)}{I_\nu(z)} \tag{17.2.31}$$

$$= \frac{z/(2\nu+2)}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{a_m(\nu)z^2}{1} \right), \qquad z \in \mathbb{C}, \quad \nu \geq 0, \tag{17.2.32a}$$

$$a_m(\nu) = \frac{1}{4(\nu+m-1)(\nu+m)}, \quad m \geq 2. \tag{17.2.32b}$$

The coefficients $a_m(\nu)$ behave asymptotically as

$$a_m(\nu) \sim \frac{1}{4m^2}, \quad m \to \infty.$$

For the modified spherical Bessel functions we have

$$\frac{i_{n+1}(z)}{i_n(z)} = \frac{z/(2n+3)}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{a_m(n+1/2)z^2}{1} \right), \qquad z \in \mathbb{C}, \quad n \in \mathbb{N}_0, \tag{17.2.33}$$

where the coefficients $a_m(n + 1/2)$ are given by (17.2.32b). The continued fractions (17.2.32) and (17.2.33) are often used in equivalent forms similar to (17.1.40).

EXAMPLE 17.2.1: For $\nu = 0$ and $z = 2$ in (17.2.32) we obtain the simple continued fraction [Rob95]

$$\frac{I_1(2)}{I_0(2)} = \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{1}{m} \right).$$

C-fractions. If we relax the condition on ν in (17.2.32) to allow $\nu \in \mathbb{C}\backslash\mathbb{Z}^-$, the S-fraction becomes a C-fraction.

From (17.2.27), and formula (16.2.4) we obtain C-fraction representations for the ratio $K_{\nu+1}(z)/K_\nu(z)$ [Hit68, p. 108] as well as the logarithmic derivative of $K_\nu(z)$:

$$\frac{K_{\nu+1}(z)}{K_\nu(z)} = \frac{\nu}{z} - \frac{K'_\nu(z)}{K_\nu(z)} \tag{17.2.34}$$

$$= \frac{1}{1} + \frac{-(1+2\nu)/(2z)}{1} + \mathop{\mathrm{K}}_{m=3}^{\infty}\left(\frac{c_m(\nu)/(2z)}{1}\right),$$

$$|\arg z| < \pi, \quad \nu \in \mathbb{C} \tag{17.2.35a}$$

$$c_{2k+2}(\nu) = \frac{2k-1}{2} - \nu, \quad c_{2k+1}(\nu) = \frac{2k+1}{2} + \nu, \quad k \geq 1. \tag{17.2.35b}$$

In *Section* 7.7 the modification

$$w_n(z) = \frac{-1 + \sqrt{1 + 4c_{n+1}(\nu)/(2z)}}{2} \tag{17.2.36}$$

is suggested for the evaluation of (17.2.34) and (17.2.35b). For real z and $\nu \geq 0$, n is best taken even or sufficiently large. Use of this modification is illustrated in the *Tables* 17.2.1 and 17.2.2.

T-fractions. The T-fraction representation for a ratio of modified Bessel functions $I_\nu(z)$ follows from the relation (17.2.2) and the T-fraction representation (17.1.48) [GS78; Gau77]. Because of (17.2.18) we obtain at the same time a T-fraction representation for the logarithmic derivative:

$$\frac{I_{\nu+1}(z)}{I_\nu(z)} = -\frac{\nu}{z} + \frac{I'_\nu(z)}{I_\nu(z)} \tag{17.2.37}$$

$$= \frac{z}{2\nu + 2 + z} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{c_m z}{e_m + d_m z}\right), \qquad z \in \mathbb{C}, \quad \nu \in \mathbb{C} \setminus \mathbb{Z}^-, \tag{17.2.38a}$$

$$c_m = -(2\nu + 2m - 1), \quad e_m = 2\nu + m + 1, \quad d_m = 2, \quad m \geq 2. \tag{17.2.38b}$$

The T-fraction representation for a ratio of modified spherical Bessel functions of the first kind is given by

$$\frac{i_{n+1}(z)}{i_n(z)} = \frac{z}{2n + 3 + z} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{\tilde{c}_m z}{\tilde{e}_m + \tilde{d}_m z}\right), \qquad z \in \mathbb{C}, \quad n \in \mathbb{N}_0, \tag{17.2.39a}$$

with

$$\tilde{c}_m = -2(n+m), \quad \tilde{e}_m = 2n + m + 2, \quad \tilde{d}_m = 2, \quad m \geq 2. \tag{17.2.39b}$$

J-fractions. From the relation [PFTV92, p. 246]

$$\frac{K_{\nu+1}(z)}{K_\nu(z)} = \frac{2\nu+1+2z}{2z} + \frac{1}{z}(\nu^2 - {}^1\!/\!_4)\frac{U(\nu+{}^3\!/\!_2, 2\nu+1, 2z)}{U(\nu+{}^1\!/\!_2, 2\nu+1, 2z)}$$

and the continued fraction (16.1.20) for a ratio of Kummer functions of the second kind, we obtain a J-fraction representation. Because of (17.2.40) this is at the same time a J-fraction representation for the logarithmic derivative of $K_\nu(z)$ [Hit68, p. 109]:

$$\frac{K_{\nu+1}(z)}{K_\nu(z)} = \frac{\nu}{z} - \frac{K_\nu'(z)}{K_\nu(z)} \tag{17.2.40}$$

$$= \frac{2\nu+1+2z}{2z} + \frac{1}{z}\mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{\nu^2-(2m-1)^2/4}{2(z+m)}\right), \quad |\arg z| < \pi, \quad \nu \in \mathbb{C}. \tag{17.2.41}$$

In (7.7.4) the modification

$$\tilde{w}_n(z) = -(z+n) \tag{17.2.42}$$

is suggested. It is illustrated in the *Tables* 17.2.1 and 17.2.2.

FIGURE 17.2.1: Region in the (x,ν)-plane where approximant $f_8(x;0)$ of (17.2.32) with real x guarantees k significant digits for $I_{\nu+1}(x)/I_\nu(x)$ (from light to dark $k = 8, 9, 10, 11$). We investigate $0 \le x \le 20$ and $0 \le \nu \le 10$.

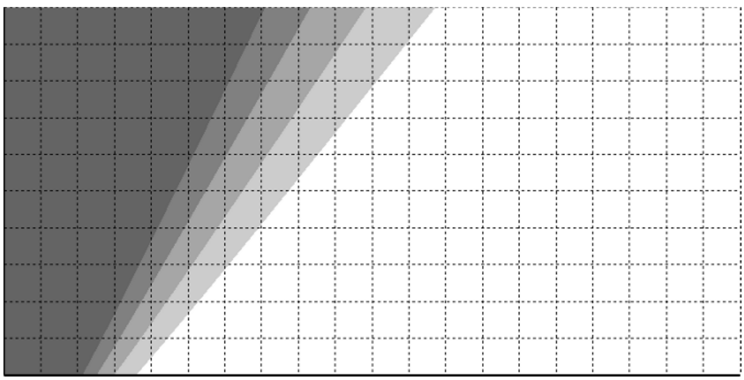


TABLE 17.2.1: Relative error of the 5^{th} approximants of the continued fraction representations (17.2.34) and (17.2.40) for the logarithmic derivative $d/dz(\operatorname{Ln} K_{10}(z))$, first evaluated without modification and second with use of the respective modifications $w_5(z)$ given by (17.2.36) and (17.2.42). Because of (17.2.5) it is sufficient to explore the upper half plane.

x	$(\operatorname{Ln} K_\nu(x))'$	(17.2.34)	(17.2.34)	(17.2.40)	(17.2.40)
0.01	−1.000001e+03	2.0e+00	2.0e+00	4.3e−02	1.7e−01
2	−5.109615e+00	1.4e+00	2.0e+00	4.2e−03	2.3e−02
5	−2.257443e+00	4.0e−01	1.2e+00	1.8e−04	1.6e−03
15	−1.225079e+00	8.0e−03	2.3e−02	9.1e−08	2.5e−06
50	−1.029382e+00	2.5e−05	7.1e−06	4.0e−13	7.4e−11
500	−1.001199e+00	2.5e−10	5.5e−12	8.3e−25	1.2e−20
1000	−1.000550e+00	7.6e−12	8.5e−14	2.1e−28	1.2e−23

x	$\lvert(\operatorname{Ln} K_\nu(x+\mathrm{i}x))'\rvert_s$	(17.2.34)	(17.2.34)	(17.2.40)	(17.2.40)
0.01	−7.071068e+02	2.0e+00	2.0e+00	4.3e−02	1.7e−01
2	−3.543348e+00	1.8e+00	1.7e+00	3.8e−03	2.1e−02
5	−1.518459e+00	3.1e−01	5.1e−01	9.5e−05	9.7e−04
15	−1.032498e+00	1.9e−03	1.4e−03	7.1e−09	2.8e−07
50	−1.005255e+00	4.5e−06	7.8e−07	9.7e−15	3.2e−12
500	−1.000500e+00	4.3e−11	6.8e−13	1.3e−26	3.9e−22
1000	−1.000250e+00	1.3e−12	1.1e−14	3.3e−30	3.9e−25

x	$\lvert(\operatorname{Ln} K_\nu(\mathrm{i}x))'\rvert_s$	(17.2.34)	(17.2.34)	(17.2.40)	(17.2.40)
0.01	−9.999994e+02	2.0e+00	2.0e+00	4.3e−02	1.8e−01
2	−4.887294e+00	2.3e+00	2.0e+00	3.9e−02	1.5e−01
5	−1.691040e+00	2.6e+00	2.6e+00	2.6e−02	1.2e−01
15	−7.534162e−01	1.9e−02	1.4e−02	3.9e−06	5.7e−05
50	−9.799154e−01	2.6e−05	5.6e−06	9.6e−13	1.4e−10
500	−9.998010e−01	2.4e−10	5.4e−12	8.9e−25	1.3e−20
1000	−9.999502e−01	7.6e−12	8.4e−14	2.2e−28	1.2e−23

x	$\lvert(\operatorname{Ln} K_\nu(x - \mathrm{i}x))'\rvert_s$	(17.2.34)	(17.2.34)	(17.2.40)	(17.2.40)
−0.01	7.071068e+02	2.0e+00	2.0e+00	4.4e−02	1.8e−01
−2	3.543348e+00	2.2e+00	2.1e+00	4.0e−01	8.7e−01
−5	8.645909e−01	2.5e+00	2.5e+00	2.7e+00	2.6e+00
−15	−9.911570e−01	1.6e−03	5.4e−04	9.6e−08	1.5e−06
−50	−9.949619e−01	4.2e−06	5.7e−07	2.0e−14	5.1e−12
−500	−9.995000e−01	4.3e−11	6.6e−13	1.4e−26	4.1e−22
−1000	−9.997500e−01	1.3e−12	1.0e−14	3.5e−30	4.0e−25

TABLE 17.2.2: Relative error of the 20th approximants of the continued fraction representations (17.2.34) and (17.2.40) for the logarithmic derivative $d/dz(\operatorname{Ln} K_{10}(z))$, first evaluated without modification and second with use of the respective modifications $w_{20}(z)$ given by (17.2.36) and (17.2.42). Because of (17.2.5) it is sufficient to explore the upper half plane.

x	$(\operatorname{Ln} K_\nu(x))'$	(17.2.34)	(17.2.34)	(17.2.40)	(17.2.40)
0.01	−1.000001e+03	2.0e+00	1.3e+00	6.3e−15	2.0e−14
2	−5.109615e+00	9.9e−05	2.2e−05	3.1e−18	1.4e−17
5	−2.257443e+00	1.6e−07	4.0e−08	2.4e−22	1.5e−21
15	−1.225079e+00	3.3e−13	7.1e−14	7.0e−32	1.0e−30
50	−1.029382e+00	5.0e−22	6.4e−23	4.9e−48	3.0e−46
500	−1.001199e+00	2.1e−41	4.2e−43	4.7e−87	1.6e−83
1000	−1.000550e+00	2.2e−47	2.2e−49	1.6e−99	2.1e−95

x	$\lvert(\operatorname{Ln} K_\nu(x + \mathrm{i}x))'\rvert_s$	(17.2.34)	(17.2.34)	(17.2.40)	(17.2.40)
0.01	−7.071068e+02	2.0e+00	1.2e+00	6.3e−15	2.0e−14
2	−3.543348e+00	4.1e−05	1.0e−05	1.8e−18	8.3e−18
5	−1.518459e+00	3.1e−08	8.2e−09	2.7e−23	1.9e−22
15	−1.032498e+00	4.5e−15	9.8e−16	8.1e−35	1.4e−33
50	−1.005255e+00	1.1e−24	1.2e−25	7.0e−53	6.6e−51
500	−1.000500e+00	2.3e−44	3.2e−46	3.4e−93	2.2e−89
1000	−1.000250e+00	2.2e−50	1.6e−52	9.7e−106	2.5e−101

x	$\lvert(\mathrm{Ln}\, K_\nu(\mathrm{i}x))'\rvert_s$	(17.2.34)	(17.2.34)	(17.2.40)	(17.2.40)
0.01	$-9.999994e{+}02$	$2.0e{+}00$	$2.9e{+}00$	$6.6e{-}15$	$2.1e{-}14$
2	$-4.887294e{+}00$	$2.7e{-}03$	$7.1e{-}04$	$2.9e{-}15$	$1.0e{-}14$
5	$-1.691040e{+}00$	$1.7e{-}04$	$5.4e{-}05$	$9.6e{-}17$	$4.1e{-}16$
15	$-7.534162e{-}01$	$1.0e{-}10$	$3.4e{-}11$	$3.3e{-}26$	$2.7e{-}25$
50	$-9.799154e{-}01$	$2.6e{-}21$	$4.7e{-}22$	$3.1e{-}45$	$1.2e{-}43$
500	$-9.998010e{-}01$	$2.5e{-}41$	$5.2e{-}43$	$1.1e{-}86$	$3.4e{-}83$
1000	$-9.999502e{-}01$	$2.4e{-}47$	$2.5e{-}49$	$2.5e{-}99$	$3.2e{-}95$

x	$\lvert(\mathrm{Ln}\, K_\nu(x-\mathrm{i}x))'\rvert_s$	(17.2.34)	(17.2.34)	(17.2.40)	(17.2.40)
-0.01	$7.071068e{+}02$	$2.0e{+}00$	$4.5e{+}00$	$6.9e{-}15$	$2.2e{-}14$
-2	$3.543348e{+}00$	$6.3e{-}02$	$2.3e{-}02$	$1.5e{-}11$	$3.8e{-}11$
-5	$8.645909e{-}01$	$1.4e{+}01$	$3.1e{+}00$	$8.6e{-}09$	$2.2e{-}08$
-15	$-9.911570e{-}01$	$1.1e{-}12$	$6.0e{-}13$	$1.4e{-}25$	$6.7e{-}25$
-50	$-9.949619e{-}01$	$5.4e{-}24$	$1.1e{-}24$	$3.7e{-}49$	$1.6e{-}47$
-500	$-9.995000e{-}01$	$2.6e{-}44$	$4.0e{-}46$	$8.3e{-}93$	$4.9e{-}89$
-1000	$-9.997500e{-}01$	$2.4e{-}50$	$1.8e{-}52$	$1.5e{-}105$	$3.7e{-}101$

TABLE 17.2.3: Relative error of the 5^{th} partial sum of the series representation (17.2.22) for $i_{20}(z)$, and the computation of $i_{20}(z)$ using the 5^{th} approximants of the continued fraction representations (17.2.33) and (17.2.39) for the factors $i_{n+1}(z)/i_n(z), n = 0, \ldots, 19$. The factor $i_0(z) = \sinh(z)/z$ is provided with 30 decimal digits accuracy so that the approximation error in the factors i_{n+1}/i_n dominates. Making use of the symmetry properties (17.2.5) and (17.2.6), we can reduce our investigation to the first quadrant, including the positive real and imaginary axis.

x	$i_n(x)$	(17.2.22)	(17.1.17)	(17.2.33)	(17.2.39)
0.01	$7.625988e{-}66$	$1.8e{-}39$	$4.9e{+}98$	$2.2e{-}29$	$1.6e{-}13$
0.5	$7.293871e{-}32$	$4.4e{-}19$	$1.5e{+}31$	$2.2e{-}12$	$2.6e{-}05$
1	$7.715148e{-}26$	$1.8e{-}15$	$2.8e{+}19$	$2.0e{-}09$	$4.1e{-}04$
2	$8.376728e{-}20$	$7.1e{-}12$	$8.2e{+}06$	$1.3e{-}06$	$2.9e{-}03$
10	$2.371544e{-}05$	$6.6e{-}04$	$2.0e{-}15$	$6.3e{-}02$	$5.6e{-}03$
50	$7.904304e{+}17$	$1.0e{+}00$	$2.5e{-}27$	$9.5e{+}01$	$6.6e{-}04$
500	$9.218923e{+}213$	$1.0e{+}00$	$1.1e{-}30$	$2.8e{+}17$	$4.5e{-}08$

x	$\lvert i_n(x+\mathrm{i}x)\rvert_s$	(17.2.22)	(17.1.17)	(17.2.33)	(17.2.39)
0.01	$-7.809003e{-63}$	$1.2e{-37}$	$2.5e{+92}$	$7.5e{-28}$	$8.9e{-13}$
0.5	$-7.447251e{-29}$	$2.8e{-17}$	$1.3e{+25}$	$7.3e{-11}$	$1.5e{-04}$
1	$-7.809096e{-23}$	$1.2e{-13}$	$8.2e{+12}$	$7.3e{-08}$	$2.5e{-03}$
2	$-8.189907e{-17}$	$4.7e{-10}$	$3.4e{+01}$	$5.4e{-05}$	$1.3e{-02}$
10	$8.782339e{-03}$	$1.0e{-01}$	$5.4e{-21}$	$3.7e{-01}$	$7.4e{-03}$
50	$4.422559e{+18}$	$1.6e{+00}$	$8.9e{-29}$	$1.0e{+02}$	$3.3e{-04}$
500	$-8.044847e{+213}$	$1.0e{+00}$	$6.9e{-31}$	$2.0e{+20}$	$8.8e{-09}$

x	$\lvert i_n(\mathrm{i}x)\rvert_s$	(17.2.22)	(17.1.17)	(17.2.33)	(17.2.39)
0.01	$7.625970e{-66}$	$1.8e{-39}$	$1.7e{+99}$	$1.9e{-29}$	$1.6e{-13}$
0.5	$7.251588e{-32}$	$4.4e{-19}$	$1.6e{+31}$	$2.4e{-12}$	$5.2e{-05}$
1	$7.537796e{-26}$	$1.8e{-15}$	$6.3e{+18}$	$2.8e{-09}$	$1.9e{-03}$
2	$7.632641e{-20}$	$7.7e{-12}$	$3.7e{+06}$	$5.6e{-06}$	$1.3e{-01}$
10	$2.308372e{-06}$	$5.3e{-03}$	$6.1e{-22}$	$1.6e{-01}$	$1.0e{+00}$
50	$-1.578503e{-02}$	$4.3e{+15}$	$1.1e{-30}$	$1.0e{+00}$	$1.2e{+00}$
500	$-1.575766e{-03}$	$5.2e{+46}$	$4.6e{-30}$	$8.5e{+16}$	$5.2e{-01}$

TABLE 17.2.4: Relative error of the 20^{th} partial sum of the series representation (17.2.22) for $i_3(z)$, and the computation of $i_3(z)$ using the 20^{th} approximants of the continued fraction representations (17.2.33) and (17.2.39) for the factors $i_{n+1}(z)/i_n(z), n = 0, \dots, 19$. The factor $i_0(z) = \sinh(z)/z$ is provided with 135 decimal digits accuracy so that the approximation error in the factors i_{n+1}/i_n dominates. Making use of the symmetry properties (17.2.5) and (17.2.6), we can reduce our investigation to the first quadrant, including the positive real and imaginary axis.

x	$i_n(x)$	(17.2.22)	(17.1.17)	(17.2.33)	(17.2.39)
0.01	$7.625988e{-66}$	$2.3e{-148}$	$2.2e{-07}$	$4.1e{-132}$	$2.2e{-57}$
0.5	$7.293871e{-32}$	$5.2e{-77}$	$1.4e{-74}$	$3.5e{-64}$	$8.4e{-24}$
1	$7.715148e{-26}$	$2.3e{-64}$	$1.7e{-86}$	$3.2e{-52}$	$3.1e{-18}$
2	$8.376728e{-20}$	$9.6e{-52}$	$6.6e{-98}$	$1.9e{-40}$	$2.9e{-13}$
10	$2.371544e{-05}$	$7.5e{-23}$	$4.7e{-121}$	$1.3e{-16}$	$2.5e{-09}$
50	$7.904304e{+17}$	$1.2e{-01}$	$1.7e{-132}$	$4.1e{-04}$	$1.4e{-14}$
500	$9.218923e{+213}$	$1.0e{+00}$	$2.7e{-135}$	$1.0e{+00}$	$5.7e{-31}$

x	$\lvert i_n(x+\mathrm{i}x)\rvert_s$	(17.2.22)	(17.1.17)	(17.2.33)	(17.2.39)
0.01	$-7.809003e{-63}$	$4.8e{-142}$	$2.1e{-13}$	$4.3e{-126}$	$2.2e{-54}$
0.5	$-7.447251e{-29}$	$1.1e{-70}$	$5.8e{-81}$	$3.9e{-58}$	$9.1e{-21}$
1	$-7.809096e{-23}$	$4.8e{-58}$	$2.3e{-92}$	$4.2e{-46}$	$3.9e{-15}$
2	$-8.189907e{-17}$	$2.1e{-45}$	$9.1e{-104}$	$3.3e{-34}$	$5.0e{-10}$
10	$8.782339e{-03}$	$4.2e{-16}$	$1.2e{-125}$	$4.6e{-11}$	$2.1e{-09}$
50	$4.422559e{+18}$	$1.1e{+07}$	$2.0e{-133}$	$4.1e{-02}$	$1.9e{-15}$
500	$-8.044847e{+213}$	$1.0e{+00}$	$3.9e{-135}$	$1.0e{+00}$	$8.5e{-34}$

x	$\lvert i_n(\mathrm{i}x)\rvert_s$	(17.2.22)	(17.1.17)	(17.2.33)	(17.2.39)
0.01	$7.625970e{-66}$	$2.3e{-148}$	$1.1e{-06}$	$4.1e{-132}$	$2.2e{-57}$
0.5	$7.251588e{-32}$	$5.2e{-77}$	$5.8e{-75}$	$4.0e{-64}$	$2.3e{-23}$
1	$7.537796e{-26}$	$2.3e{-64}$	$7.2e{-87}$	$5.3e{-52}$	$2.9e{-17}$
2	$7.632641e{-20}$	$1.1e{-51}$	$3.4e{-99}$	$1.4e{-39}$	$7.5e{-11}$
10	$2.308372e{-06}$	$7.3e{-22}$	$3.1e{-127}$	$2.8e{-10}$	$1.2e{+00}$
50	$-1.578503e{-02}$	$1.4e{+18}$	$8.0e{-136}$	$1.2e{+00}$	$1.2e{+00}$
500	$-1.575766e{-03}$	$3.3e{+79}$	$2.0e{-135}$	$1.4e{+01}$	$5.2e{-01}$

18

Probability functions

Several probability distribution functions can be expressed in terms of special functions. Therefore, continued fractions representations can be obtained from the continued fractions given in the previous chapters. We discuss the normal and log-normal distribution, the gamma, exponential and chi-square distribution, and the beta, F- and Student's t-distribution.

18.1 Definitions and elementary properties

Probability distribution. A function $F(x)$ which satisfies [AS64, p. 927]

$$F(x_1) \leq F(x_2), \quad x_1 \leq x_2, \qquad x \in \mathbb{R}, \tag{18.1.1a}$$

$$F(x) = \lim_{\epsilon \to 0^+} F(x+\epsilon), \qquad x \in \mathbb{R}, \tag{18.1.1b}$$

$$F(-\infty) = 0, \qquad F(\infty) = 1, \qquad x \in \mathbb{R}, \tag{18.1.1c}$$

is called a *cumulative distribution function*, abbreviated cdf. It is also called a *probability distribution function* or *probability distribution* for short. If X denotes a random variable, and $P(X \leq x)$ represents the probability of the event $X \leq x$, then the cdf $F(x)$ gives the probability that the variable takes a value less than or equal to x:

$$F(x) = P(X \leq x). \tag{18.1.2}$$

There are two principal types of probability distributions: the *continuous probability distribution* and the *discrete probability distribution*. Also probability distributions that are neither continuous nor discrete exist.

Continuous probability distributions. If $F(x)$ is absolutely continuous so that the derivative of the cdf $F'(x) = f(x)$ exists a.e., then [AS64, p. 927]

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt = \int_{-\infty}^{x} dF(t). \tag{18.1.3}$$

The function f is called the *probability density function*, abbreviated pdf. Other used names are *probability function* or *frequency function*. The probability density function is constrained by the normalisation condition

$$\int_{-\infty}^{\infty} f(x)\,dx = 1. \tag{18.1.4}$$

Discrete probability distributions. For discrete probability distributions the random variable X lies in a denumerable set $\{\dots, x_{-1}, x_0, x_1, \dots\}$ with point probabilities [AS64, p. 927]

$$0 \le p_n = P(X = x_n), \qquad \sum_{n=-\infty}^{\infty} p_n = 1. \tag{18.1.5}$$

The cdf has the form

$$F(x) = P(X \le x) = \sum_{x_n \le x} p_n = \int_{-\infty}^{x} dF(t) \tag{18.1.6}$$

where the sum is over all p_n for which $x_n \le x$.

Basic terminology. We introduce some basic terminology for continuous probability distributions [AS64, p. 928]. By using Stieltjes integrals we obtain the same expression for discrete probability distributions. The k^{th} *moment about the origin* or the k^{th} *raw moment* of a probability distribution $F(x)$ is defined by

$$\mu_k := \int_{-\infty}^{\infty} x^k\,dF(x). \tag{18.1.7}$$

The first raw moment μ_1 is called the *mean* or *expectation value* of X, denoted $E[X]$ or μ, and given by

$$\mu := E[X] := \mu_1 = \int_{-\infty}^{\infty} x\,dF(x). \tag{18.1.8}$$

The expectation value $E[g(X)]$ of a function $g(X)$ is given by

$$E[g(X)] := \int_{-\infty}^{\infty} g(x)\,dF(x). \tag{18.1.9}$$

The moments are usually taken about the mean. These are called k^{th} *central moments*, denoted μ'_k, and defined by

$$\mu'_k := \int_{-\infty}^{\infty} (x-\mu)^k\,dF(x), \qquad \mu'_1 = 0. \tag{18.1.10}$$

The second moment about the mean is called the *variance*

$$\sigma^2 := \int_{-\infty}^{\infty} (x-\mu)^2 \, dF(x), \tag{18.1.11}$$

and

$$\sigma = \sqrt{\mu_2'}$$

is called the *standard deviation*.

18.2 Normal and log-normal distributions

Definitions and elementary properties. A probability distribution with mean μ and variance σ^2 is called a *normal distribution* $N(\mu, \sigma^2)$, if it has probability density function [AS64, p. 931; Wei03, p. 2036]

$$f(x;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \qquad \mu, x \in \mathbb{R}, \quad \sigma > 0. \tag{18.2.1}$$

The graph of $f(x;\mu,\sigma^2)$ is symmetrical, bell-shaped and centred at the mean μ. The cdf for the normal distribution is

$$\begin{aligned} F(x;\mu,\sigma^2) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(t-\mu)^2/2\sigma^2} \, dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-\mu)/\sigma} e^{-t^2/2} \, dt = F\left(\frac{x-\mu}{\sigma}; 0, 1\right). \end{aligned} \tag{18.2.2}$$

An alternative way of representing the cdf (18.2.2) is given by

$$F(x;\mu,\sigma^2) = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right), \tag{18.2.3}$$

where $\operatorname{erf}(x)$ denotes the error function (13.1.1).
The simplest case of the normal distribution (18.2.2) with $\mu = 0$ and $\sigma^2 = 1$ is known as *standard normal distribution* or *Gaussian distribution*, [Wei03, p. 2037]

$$F(x;0,1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, dt = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right), \qquad x \in \mathbb{R}. \tag{18.2.4}$$

The *standard normal probability density function* or *Gaussian probability density function* is given by

$$f(x;0,1) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad x \in \mathbb{R}. \tag{18.2.5}$$

A function closely related to $F(x;0,1)$, denoted $Q(x;0,1)$, is defined by

$$Q(x;0,1) := P(X > x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt, \qquad x \in \mathbb{R}. \tag{18.2.6}$$

Then by (18.1.3) and (18.1.4)

$$F(x;0,1) + Q(x;0,1) = 1, \qquad x \in \mathbb{R}. \tag{18.2.7}$$

From (18.2.4) and (18.2.7) we obtain

$$Q(x;0,1) = \frac{1}{2}\left(1 - \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right) = \frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right), \tag{18.2.8}$$

where $\operatorname{erfc}(x)$ is the complementary error function (13.2.1).

Log-normal distribution. A probability distribution closely related to the normal distribution $N(\mu, \sigma^2)$ is defined in the following way. Let X be a random variable such that $\ln(X)$ has a normal distribution with mean μ and variance σ^2. Then the probability distribution of X is called a *log-normal distribution*, denoted $\log\text{-}N(\mu, \sigma^2)$. If X is a random variable with a normal distribution, then $Y = e^X$ has a log-normal distribution. The log-normal distribution has pdf [WMMY07, p. 201]

$$f(\ln(x);\mu,\sigma^2) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-(\ln(x)-\mu)^2/2\sigma^2}, \qquad \mu \in \mathbb{R}, \quad x, \sigma > 0. \tag{18.2.9}$$

The cdf of the log-normal distribution is

$$F(\ln(x);\mu,\sigma^2) = P(X \le x) = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{\ln(x) - \mu}{\sigma\sqrt{2}}\right)\right). \tag{18.2.10}$$

A special case of the log-normal distribution is obtained by taking $\mu = 0$ and $\sigma = 1$ in (18.2.9). From (18.2.4) and (18.2.8) we obtain

$$F(\ln(x);0,1) = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{\ln(x)}{\sqrt{2}}\right)\right), \qquad x > 0, \tag{18.2.11}$$

$$Q(\ln(x);0,1) = \frac{1}{2}\operatorname{erfc}\left(\frac{\ln(x)}{\sqrt{2}}\right), \qquad x > 0. \tag{18.2.12}$$

The probability distribution $\log\text{-}N(0,1)$ is also called *Gibrat distribution* [Wei03, p. 1194].

All the results given for the normal distribution also apply to the log-normal distribution by replacing x by $\ln(x)$ in all formulas.

Series expansion. From the series representation (13.1.7) for the error function and the relations (18.2.4) and (18.2.8) we obtain respectively

$$F(x;0,1) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^k(2k+1)k!}, \qquad x \in \mathbb{R}, \quad (18.2.13a)$$

$$Q(x;0,1) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^k(2k+1)k!}, \qquad x \in \mathbb{R}. \quad (18.2.13b)$$

Asymptotic series expansion. From the asymptotic series expansion (13.2.11) of the complementary error function and the relation (18.2.8) we obtain

$$\begin{aligned} Q(x;0,1) &\approx \frac{e^{-x^2/2}}{x\sqrt{2\pi}} \left(1 + \sum_{k=1}^{\infty} (-1)^k \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{x^{2k}} \right) \\ &= \frac{e^{-x^2/2}}{x\sqrt{2\pi}} \, {}_2F_0\left(1, \frac{1}{2}; \frac{-2}{x^2}\right), \qquad x \to \infty. \end{aligned} \quad (18.2.14)$$

Stieltjes transform. It follows from (18.2.8) and (13.2.13) that $Q(x;0,1)$ can be expressed by the Stieltjes transform

$$Q(x;0,1) = \frac{xe^{-x^2/2}}{2\sqrt{2\pi}} \int_0^{\infty} \frac{e^{-t}}{\sqrt{t}(t+x^2/2)} \, dt, \qquad x > 0. \quad (18.2.15)$$

S-fraction. From the relation (18.2.8) and the S-fraction (13.2.20) for the complementary error function, we obtain a modified S-fraction for the function $Q(x;0,1)$, given by

$$Q(x;0,1) = \frac{xe^{-x^2/2}}{\sqrt{2\pi}} \left(\frac{a_1}{x^2} \, {}_{+} \, \frac{a_2}{1} \, {}_{+} \, \frac{a_3}{x^2} \, {}_{+} \, \frac{a_4}{1} \, {}_{+} \, \ldots \right), \qquad x > 0, \quad (18.2.16a)$$

where the coefficients a_m satisfy

$$a_1 = 1, \qquad a_m = m - 1, \quad m \geq 2. \quad (18.2.16b)$$

An equivalence transformation leads to

$$Q(x;0,1) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \left(\frac{1}{x} \, {}_{+} \, \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{m-1}{x} \right) \right). \quad (18.2.17)$$

C-fraction. A C-fraction representation for $Q(x;0,1)$ follows from (18.2.8) and (13.1.11) [AS64, pp. 931–932],

$$Q(x;0,1) = \frac{1}{2} - \frac{e^{-x^2/2}}{x\sqrt{2\pi}} \left(\frac{x^2}{1} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{c_m x^2}{1} \right) \right), \qquad x \in \mathbb{R} \quad (18.2.18a)$$

where the coefficients c_m are

$$c_m = \frac{(-1)^{m-1}(m-1)}{(2m-3)(2m-1)}, \qquad m \geq 2. \quad (18.2.18b)$$

J-fraction. A J-fraction representation for $Q(x;0,1)$ can be obtained from the relation (18.2.8) and the J-fraction (13.2.23a)

$$Q(x;0,1) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \left(\frac{x}{1+x^2} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{-(2m-3)(2m-2)}{4m-3+x^2} \right) \right), \qquad x > 0. \quad (18.2.19)$$

Continued fractions for Mills ratio. We introduce the function $R(x)$ defined by [Les95]

$$R(x) := \frac{1-F(x;0,1)}{F'(x;0,1)} = \sqrt{2\pi} e^{x^2/2} Q(x;0,1) = e^{x^2/2} \int_x^{\infty} e^{-t^2/2}\, dt, \quad (18.2.20)$$

which is called *Mills ratio*. An S-fraction representation of $R(x)$, introduced by Laplace, follows immediately from (18.2.17) [Lap05; Les95]

$$R(x) = \frac{1}{x} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{m-1}{x} \right), \qquad x > 0. \quad (18.2.21)$$

A C-fraction expansion for a function related to $R(x)$ [She54; Les95] follows directly from (18.2.18),

$$\sqrt{\frac{\pi}{2}} e^{x^2/2} - R(x) = \frac{x}{1} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{a_m x^2}{b_m} \right), \qquad x > 0, \quad (18.2.22a)$$

$$a_m = (-1)^{m-1}(m-1), \qquad b_m = 2m-1. \quad (18.2.22b)$$

18.3 Repeated integrals

Definition and elementary properties. The *repeated integrals of the probability integral* are defined recursively by [AS64, p. 934]

$$I_{-1}(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad (18.3.1a)$$

$$I_0(x) := Q(x;0,1) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\,dt, \qquad (18.3.1b)$$

$$I_k(x) := \int_x^\infty I_{k-1}(t)\,dt, \qquad k \geq 1. \qquad (18.3.1c)$$

It follows from (18.2.8) and (13.3.1c) that

$$I_k(x) = 2^{\frac{k}{2}-1} I^k \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right), \qquad k \geq 0, \qquad (18.3.2)$$

where $I^k \operatorname{erfc}(x)$ denotes the repeated integral of the error function.

Asymptotic series expansion. From the asymptotic series expansion (13.3.2) and (18.3.2) we obtain

$$I_k(x) \approx \frac{e^{-x^2/2}}{\sqrt{2\pi} x^{k+1}} \sum_{m=0}^\infty \frac{(-1)^m (2m+k)!}{k!\, m!\, 2^m x^{2m}}, \qquad x \to \infty. \qquad (18.3.3)$$

S-fraction. A modified S-fraction follows from (13.3.5),

$$\frac{I_k(x)}{I_{k-1}(x)} = \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m^{(k)}}{x}\right), \qquad k \geq 1, \quad x > 0, \qquad (18.3.4a)$$

where

$$a_1^{(k)} = 1, \qquad a_m^{(k)} = k + m - 1, \quad m \geq 2. \qquad (18.3.4b)$$

18.4 Gamma and chi-square distribution

Definitions and elementary properties. A probability distribution with pdf [Wei03, p. 1135]

$$f(x;\alpha,\theta)=\frac{1}{\theta^{\alpha}\Gamma(\alpha)}e^{-x/\theta}x^{\alpha-1},\qquad \alpha,\theta>0,\quad x\geq 0,\tag{18.4.1}$$

with parameters α and θ, is called a *gamma distribution*.
For $\theta=1$ we obtain

$$f(x;\alpha,1)=\frac{1}{\Gamma(\alpha)}e^{-x}x^{\alpha-1},\qquad \alpha>0,\quad x\geq 0,\tag{18.4.2}$$

which is called the *standard gamma distribution*. The cdf for the gamma distribution (18.4.1) is

$$P(x;\alpha,\theta)=\frac{1}{\theta^{\alpha}\Gamma(\alpha)}\int_0^x e^{-t/\theta}t^{\alpha-1}\,dt,\qquad \alpha,\theta>0,\quad x\geq 0.\tag{18.4.3}$$

For the function $Q(x;\alpha,\theta)$, we get

$$Q(x;\alpha,\theta)=1-P(x;\alpha,\theta)=\frac{1}{\theta^{\alpha}\Gamma(\alpha)}\int_x^\infty e^{-t/\theta}t^{\alpha-1}\,dt,\qquad \alpha,\theta>0,\quad x\geq 0.\tag{18.4.4}$$

Both the functions $P(x;\alpha,\theta)$ and $Q(x;\alpha,\theta)$ are related to the incomplete gamma function (12.6.2) and the complementary incomplete gamma function (12.6.4) by

$$P(x;\alpha,\theta)=\frac{\gamma\left(\alpha,x/\theta\right)}{\Gamma\left(\alpha\right)},\tag{18.4.5a}$$

$$Q(x;\alpha,\theta)=\frac{\Gamma\left(\alpha,x/\theta\right)}{\Gamma\left(\alpha\right)}.\tag{18.4.5b}$$

The functions (18.4.5) are called the *regularised gamma functions* [Wei03, pp. 2526–2527].
For integer values of the parameter α, the gamma distribution is also known as the *Erlang distribution* [Wei03, p. 1135]. For $\alpha=1$ the pdf (18.4.1) reduces to

$$f(x;1,\theta)=\frac{1}{\theta}e^{-x/\theta},\qquad \theta>0,\quad x\geq 0,\tag{18.4.6}$$

which is the probability density function for the *exponential distribution* [AS64, p. 930]. From the relations (18.4.6) and (18.4.5a) we obtain

$$P(x;1,\theta) = \frac{1}{\theta}\int_0^x e^{-t/\theta}\,dt = \gamma\Big(1,\frac{x}{\theta}\Big) = 1 - e^{-x/\theta}, \qquad \theta > 0, \quad x \geq 0. \tag{18.4.7}$$

Chi-square distribution. Let $X_1, X_2, \ldots, X_\nu$ be random variables that are varied independently and normally distributed with mean $\mu = 0$ and variance $\sigma = 1$. Then the probability distribution of the random variable [AS64, p. 940]

$$\chi^2 = \sum_{j=1}^{\nu} X_j^2, \tag{18.4.8}$$

is called a *chi-square distribution* with ν degrees of freedom. The pdf is given by

$$f(x,\nu) = \frac{1}{2\Gamma\left(\frac{\nu}{2}\right)} e^{-x/2}\left(\frac{x}{2}\right)^{\nu/2-1}, \qquad \nu \in \mathbb{N}, \quad x \geq 0, \tag{18.4.9}$$

and the cdf is

$$P(x^2,\nu) = P(\chi^2 \leq x^2) = \frac{1}{2\Gamma\left(\frac{\nu}{2}\right)} \int_0^{x^2} e^{-t/2}\left(\frac{t}{2}\right)^{\nu/2-1} dt, \qquad \nu \in \mathbb{N}, \quad x \geq 0. \tag{18.4.10}$$

The related function $Q(x^2,\nu)$ is

$$Q(x^2,\nu) = 1 - P(x^2,\nu) = \frac{1}{2\Gamma\left(\frac{\nu}{2}\right)} \int_{x^2}^{\infty} e^{-t/2}\left(\frac{t}{2}\right)^{\nu/2-1} dt, \qquad \nu \in \mathbb{N}, \quad x \geq 0. \tag{18.4.11}$$

Hence the chi-square distribution is a special case of the gamma distribution (18.4.1) with $\alpha = \nu/2$, $\theta = 2$ and x replaced by x^2. Therefore all the results given for the gamma distribution apply to the chi-square distribution with these substitutions and

$$P(x^2;\nu) = \frac{\gamma\left(\nu/2, x/2\right)}{\Gamma\left(\nu/2\right)}, \tag{18.4.12a}$$

$$Q(x^2;\nu) = \frac{\Gamma\left(\nu/2, x/2\right)}{\Gamma\left(\nu/2\right)}. \tag{18.4.12b}$$

From the integrals (18.4.11) we observe that for $\nu = 1$ we have the connection

$$P(x^2, 1) = 2F(x; 0, 1) - 1, \tag{18.4.13}$$

$$Q(x^2, 1) = 2Q(x; 0, 1), \tag{18.4.14}$$

where $F(x; 0, 1)$ is the standard normal distribution (18.2.4) and $Q(x; 0, 1)$ is the function (18.2.6).

Series expansions. From equation (18.4.5a) and (12.6.7) we obtain the series expansion

$$\begin{aligned} P(x; \alpha, \theta) &= \frac{(x/\theta)^\alpha}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{(-x/\theta)^k}{(\alpha + k)k!} \\ &= \frac{(x/\theta)^\alpha}{\Gamma(\alpha + 1)} \, {}_1F_1(\alpha; \alpha + 1; -x/\theta) \qquad \alpha, \theta > 0, \quad x \geq 0, \end{aligned} \tag{18.4.15}$$

where ${}_1F_1(a; b; z)$ is the confluent hypergeometric function (16.1.2). An alternative series representation follows from (18.4.5b) and (12.6.8):

$$\begin{aligned} P(x; \alpha, \theta) &= \frac{(x/\theta)^\alpha e^{-x/\theta}}{\Gamma(\alpha + 1)} \sum_{k=0}^{\infty} \frac{(x/\theta)^k}{(\alpha + 1)_k} \\ &= \frac{(x/\theta)^\alpha e^{-x/\theta}}{\Gamma(\alpha + 1)} \, {}_1F_1(1; 1 + \alpha; x/\theta), \qquad \alpha, \theta > 0, \quad x \geq 0. \end{aligned} \tag{18.4.16}$$

Asymptotic series expansion. An asymptotic series expansion for the function $Q(x; \alpha, \theta)$ follows from (18.4.5b) and (12.6.10)

$$\begin{aligned} \frac{Q(x; \alpha, \theta)\Gamma(\alpha)}{(x/\theta)^\alpha e^{-x/\theta}} &\approx \left(\frac{x}{\theta}\right)^{-1} \sum_{k=0}^{\infty} (-1)^k (1 - \alpha)_k \left(\frac{x}{\theta}\right)^{-k} \\ &= \left(\frac{x}{\theta}\right)^{-1} {}_2F_0(1, 1 - \alpha; -\theta/x), \qquad \alpha, \theta > 0, \quad x \to \infty, \end{aligned} \tag{18.4.17}$$

where ${}_2F_0(a; b; z)$ is the confluent hypergeometric series (16.1.12).

Stieltjes transform. From (18.4.5b) and the Stieltjes transform (12.6.5) we obtain

$$\frac{Q(x; \alpha, \theta)\Gamma(\alpha)}{(x/\theta)^\alpha e^{-x/\theta}} = \frac{1}{\Gamma(1 - \alpha)} \int_0^\infty \frac{e^{-t} t^{-\alpha}}{t + x/\theta} \, dt, \qquad \alpha, \theta > 0, \quad x \geq 0. \tag{18.4.18}$$

S-fraction. From (18.4.18) and the S-fraction (12.6.15) we get a modified S-fraction of the form

$$\frac{Q(x;\alpha,\theta)\Gamma(\alpha)}{(x/\theta)^{\alpha}e^{-x/\theta}} = \frac{a_1}{x/\theta} + \frac{a_2}{1} + \frac{a_3}{x/\theta} + \frac{a_4}{1} + \dots, \qquad \alpha\in(0,1), \quad \theta>0, \quad x>0, \tag{18.4.19a}$$

corresponding to the asymptotic series (18.4.17). The coefficients are

$$a_1 = 1, \qquad a_{2m} = m-\alpha, \quad a_{2m+1} = m, \quad m \geq 1. \tag{18.4.19b}$$

C-fractions. From (18.4.5a) and (12.6.23a), we obtain the regular C-fraction

$$\frac{P(x;\alpha,\theta)\Gamma(\alpha)}{(x/\theta)^{\alpha-1}e^{-x/\theta}} = \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{c_m\,(x/\theta)}{1}\right), \qquad \alpha,\theta>0, \quad x\geq 0, \tag{18.4.20a}$$

where the coefficients are

$$\begin{aligned} c_1 = \frac{1}{\alpha}, \qquad c_{2j} &= \frac{-(\alpha+j-1)}{(\alpha+2j-2)(\alpha+2j-1)}, \quad j\geq 1,\\ c_{2j+1} &= \frac{j}{(\alpha+2j-1)(\alpha+2j)}, \quad j\geq 1. \end{aligned} \tag{18.4.20b}$$

From (18.4.5b) and (12.6.24) we obtain the C-fraction

$$\frac{Q(x;\alpha,\theta)\Gamma(\alpha)}{(x/\theta)^{\alpha-1}e^{-x/\theta}} = \frac{\Gamma(\alpha)}{(x/\theta)^{\alpha-1}e^{-x/\theta}} - \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{c_m\,(x/\theta)}{1}\right), \qquad \alpha,\theta>0, \quad x\geq 0, \tag{18.4.21}$$

where the coefficients c_m are given in (18.4.20b).

Padé approximants. From (18.4.5b) and (12.6.28) we obtain explicit formulas for the Padé approximants of

$$Q(x;\alpha,\theta)\Gamma(\alpha)\left(\frac{x}{\theta}\right)^{1-\alpha}e^{x/\theta}.$$

The Padé approximants $r_{m,n}(x/\theta)$ at $x=\infty$ are obtained by setting $a=\alpha$ and $z=x/\theta$ in (12.6.28):

$$r_{m,n}\left(\frac{x}{\theta}\right) = \frac{\mathcal{P}_{m+n}\left({}_2F_0\left(1-\alpha,1;-\theta/x\right)\ {}_2F_0\left(\alpha-m-1,-n;\theta/x\right)\right)}{{}_2F_0\left(\alpha-m-1,-n;\theta/x\right)}, \qquad m\geq n-1. \tag{18.4.22}$$

From the relation (18.4.5a) and (12.6.29) we obtain explicit formulas for the Padé approximants of

$$P(x;\alpha,\theta)\Gamma(\alpha+1)\left(\frac{x}{\theta}\right)^{-\alpha} e^{x/\theta}.$$

The Padé approximants $r_{m,n}(x/\theta)$ at $x=0$ are obtained by setting $a=\alpha$ and $z=x/\theta$ in (12.6.29):

$$r_{m,n}\left(\frac{x}{\theta}\right) = \frac{\mathcal{P}_{m+n}\left({}_1F_1\left(1;1+\alpha;x/\theta\right)\ {}_1F_1\left(-n;-\alpha-m-n;-x/\theta\right)\right)}{{}_1F_1\left(-n;-\alpha-m-n;-x/\theta\right)}, \quad m \geq n-1. \tag{18.4.23}$$

M-fractions. From (18.4.5a) and (12.6.30) we obtain the M-fraction representation

$$\frac{P(x;\alpha,\theta)\Gamma(\alpha)}{(x/\theta)^{\alpha}e^{-x/\theta}} = \frac{1}{\alpha - x/\theta} + \operatorname*{K}_{m=2}^{\infty}\left(\frac{(m-1)x/\theta}{\alpha+m-1-x/\theta}\right), \quad \alpha,\theta>0, \quad x \geq 0. \tag{18.4.24}$$

The continued fraction (18.4.24) corresponds at $x=0$ to the series representation

$$\frac{1}{\alpha}\ {}_1F_1\left(1;\alpha+1;\frac{x}{\theta}\right)$$

and at $x=\infty$ to

$$-\left(\frac{x}{\theta}\right)^{-1}\ {}_2F_0\left(1,1-\alpha;-\frac{\theta}{x}\right).$$

From (18.4.5b) and (18.4.24) we obtain the M-fraction

$$\frac{Q(x;\alpha,\theta)\Gamma(\alpha)}{(x/\theta)^{\alpha}e^{-x/\theta}} = \frac{\Gamma(\alpha)}{(x/\theta)^{\alpha}e^{-x/\theta}} - \frac{1}{\alpha - x/\theta} + \operatorname*{K}_{m=2}^{\infty}\left(\frac{(m-1)x/\theta}{\alpha+m-1-x/\theta}\right), \quad \alpha,\theta>0, \quad x \geq 0. \tag{18.4.25}$$

18.5 Beta, F- and Student's *t*-distributions

Definitions and elementary properties. The *beta function* $B(a,b)$ is defined by [AS64, p. 258]

$$B(a,b) := \int_0^1 t^{a-1}(1-t)^{b-1}\,dt, \qquad \Re a>0, \quad \Re b>0, \tag{18.5.1}$$

where t^{a-1} and $(1-t)^{b-1}$ have their principal values. The beta function is a special case of the hypergeometric series ${}_2F_1(a,b;c;z)$ introduced in (15.1.4),

$$B(a,b) = \frac{1}{a}\,{}_2F_1(a,1-b;a+1;1) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \tag{18.5.2}$$

where $\Gamma(z)$ is the gamma function (12.1.1). The *incomplete beta function* is defined by [Wei03, p. 1473]

$$B_x(a,b) := \int_0^x t^{a-1}(1-t)^{b-1}\,dt, \qquad 0 \le x \le 1. \tag{18.5.3}$$

The function $I_x(a,b)$ defined by [Wei03, p. 2526]

$$\begin{aligned} I_x(a,b) &:= \frac{B_x(a,b)}{B(a,b)} \\ &= \frac{1}{B(a,b)}\int_0^x t^{a-1}(1-t)^{b-1}\,dt, \end{aligned} \tag{18.5.4}$$

is called the *regularised beta function* or the *regularised incomplete beta function*. Observe that

$$B(a,b) = B(b,a), \tag{18.5.5}$$

$$I_x(a,b) = 1 - I_{1-x}(b,a). \tag{18.5.6}$$

The *beta distribution* has pdf [Wei03, p. 206]

$$f(x) = \frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}, \qquad a,b>0, \quad 0 \le x < 1, \tag{18.5.7}$$

and the cdf is

$$F(x) = I_x(a,b), \qquad a,b>0, \quad 0 \le x < 1. \tag{18.5.8}$$

F-distribution. Let χ_1^2 and χ_2^2 be independent chi-square distributions with ν_1 and ν_2 degrees of freedom respectively. The probability distribution of the ratio [AS64, p. 946]

$$\frac{\chi_1^2/\nu_1}{\chi_2^2/\nu_2} \tag{18.5.9}$$

is called an *F-distribution* with ν_1 and ν_2 degrees of freedom. The pdf for the F-distribution is given by

$$f(F;\nu_1,\nu_2) = \frac{\nu_1^{\nu_1/2}\nu_2^{\nu_2/2}F^{\nu_1/2-1}}{(\nu_2+\nu_1 F)^{(\nu_1+\nu_2)/2}B\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)}, \qquad \nu_1,\nu_2\in\mathbb{N}, \quad F\geq 0, \tag{18.5.10}$$

where $B(a,b)$ is the beta function (18.5.1). The cdf is

$$P(F;\nu_1,\nu_2) = P\left(\frac{X_1^2/\nu_1}{X_2^2/\nu_2}\leq F\right) = \frac{\nu_1^{\nu_1/2}\nu_2^{\nu_2/2}}{B\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)}\int_0^F \frac{t^{\nu_1/2-1}}{(\nu_2+\nu_1 t)^{(\nu_1+\nu_2)/2}}\,dt, \quad \nu_1,\nu_2\in\mathbb{N}, \quad F\geq 0. \tag{18.5.11}$$

The F-distribution is related to the beta distribution by

$$P(F;\nu_1,\nu_2) = I_x\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right), \qquad \nu_1,\nu_2\in\mathbb{N}, \tag{18.5.12a}$$

where

$$x = \frac{\nu_1 F}{\nu_2+\nu_1 F} \tag{18.5.12b}$$

and $I_x(a,b)$ is the regularised beta function (18.5.4). Hence the results for the beta distribution can be carried over to the F-distribution, taking into account (18.5.12).

Student's t-distribution. Let X be a random variable with a normal distribution having mean $\mu = 0$ and variance $\sigma = 1$. Let χ^2 be a random variable with an independent chi-square distribution with ν degrees of freedom. Then the probability distribution of the ratio [AS64, p. 948]

$$\frac{X}{\sqrt{\chi^2/\nu}}$$

is called *Student's t-distribution* or *t-distribution* with ν degrees of freedom. The pdf is defined by

$$f_\nu(t) = \frac{1}{\sqrt{\nu}B(\frac{1}{2},\frac{\nu}{2})}\left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2}, \qquad \nu\in\mathbb{N}, \quad t\in\mathbb{R}, \tag{18.5.13}$$

where $B(a,b)$ is the beta function introduced in (18.5.1). The cdf for Student's t-distribution is

$$\begin{aligned} F_\nu(t) &= P\left(\frac{X}{\sqrt{\chi^2/\nu}}\leq t\right) \\ &= \frac{1}{\sqrt{\nu}B(\frac{1}{2},\frac{\nu}{2})}\int_{-\infty}^t \left(1+\frac{x^2}{\nu}\right)^{-(\nu+1)/2} dx, \qquad \nu\in\mathbb{N}, \quad t\in\mathbb{R}. \end{aligned} \tag{18.5.14}$$

The Student t-distribution is related to the beta distribution by

$$F_\nu(t) = \begin{cases} 1 - \frac{1}{2} I_x\left(\frac{\nu}{2}, \frac{1}{2}\right), & t > 0, \quad \nu \in \mathbb{N} \\ \frac{1}{2} I_x\left(\frac{\nu}{2}, \frac{1}{2}\right), & t \leq 0, \quad \nu \in \mathbb{N} \end{cases} \tag{18.5.15a}$$

where $I_x(a, b)$ is the cdf of the beta distribution and

$$x = \frac{\nu}{\nu + t^2}, \qquad t \in \mathbb{R}. \tag{18.5.15b}$$

The t-distribution is related to the F-distribution as follows: if X has a Student's t-distribution with ν degrees of freedom, then $Y = X^2$ is distributed as F with 1 and ν degrees of freedom. For large ν the Student's t-distribution approaches a normal distribution.

Series expansions. The probability distribution (18.5.8) has the series representation [AS64, p. 945]

$$I_x(a, b) = \frac{x^a}{aB(a, b)}\, {}_2F_1(a, 1 - b; a + 1; x), \qquad a, b > 0, \quad 0 \leq x < 1, \tag{18.5.16a}$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric series (15.1.4). Alternatively, by using the relation (15.1.14d), we get

$$I_x(a, b) = \frac{x^a (1 - x)^b}{aB(a, b)}\, {}_2F_1(1, a + b; a + 1; x), \qquad a, b > 0, \quad 0 \leq x < 1. \tag{18.5.16b}$$

C-fraction. From the series representation (18.5.16b) and the C-fraction (15.3.4), we obtain a regular C-fraction for $I_x(a, b)$ [AS64, p. 944; JT80, p. 132],

$$I_x(a, b) = \frac{x^{a-1} (1 - x)^b}{aB(a, b)} \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m x}{1}\right), \qquad a, b > 0, \quad 0 \leq x < 1, \tag{18.5.17a}$$

where

$$\begin{aligned} a_1 &= 1, \qquad a_{2m+2} = -\frac{(a + m)(a + b + m)}{(a + 2m)(a + 2m + 1)}, \quad m \geq 0, \\ a_{2m+1} &= \frac{m(b - m)}{(a + 2m - 1)(a + 2m)}, \quad m \geq 1. \end{aligned} \tag{18.5.17b}$$

Since (18.5.17) is limit periodic, use of the modification (7.7.5), given here by $w(x) = (-1 + \sqrt{1-x})/2$, is recommended when the order of the approximant is larger than $2b + 1$.
In [AS64, p. 944] a C-fraction representation for $I_x(a, b)$ in the variable $x/(1 - x)$ is given. It can be obtained from (18.5.17) by applying the transformation formula (15.1.14b).
An associated continued fraction representation for $I_x(a, b)$ is given in [TW80b].

M-fraction. From the series representation (18.5.16b) and the M-fraction representation (15.3.9), we obtain

$$\frac{B(a,b)I_x(a,b)}{x^a(1-x)^b} = \frac{1}{a+(1-a-b)x} + \operatorname*{K}_{m=2}^{\infty}\left(\frac{c_m x}{e_m + d_m x}\right), \quad a, b > 0, \quad 0 \le x < 1, \tag{18.5.18a}$$

where

$$c_m = (m-1)(b-m+1), \quad e_m = a+m-1, \quad d_m = -(a+b-m), \quad m \ge 2. \tag{18.5.18b}$$

Because of the limit periodicity of (18.5.18), use of the modification

$$\tilde{w}_n(x) = -\frac{(e_n + d_n x)x}{1+x}, \qquad 0 \le x < 1,$$

as explained in (7.7.4) and (7.7.5), may be worthwhile when evaluating the continued fraction. Its usefulness depends on the values a and b compared to the order of the approximant.

Other continued fraction representations. From the series representation (18.5.16b) and the special case (15.3.14) of the Nörlund fraction, we obtain the continued fraction representation

$$\frac{B(a,b)I_x(a,b)}{x^a(1-x)^b} = \frac{1}{a-(a+b)x} + \operatorname*{K}_{m=2}^{\infty}\left(\frac{c_m(x-x^2)}{e_m + d_m x}\right), \quad a, b > 0, \quad 0 \le x < 1/2, \tag{18.5.19a}$$

where

$$c_m = (a+b+m-2)(m-1), \quad e_m = a+m-1, \quad d_m = -(a+b+2m-2), \quad m \ge 2. \tag{18.5.19b}$$

For (18.5.19) the modification

$$\tilde{w}_n(x) = (e_n + d_n x)\frac{x}{1-2x}, \qquad 0 \le x < 1/2,$$

obtained from a combination of (7.7.4) and (7.7.5), can be used.
A more complicated continued fraction representation is given by [DDM92]

$$\frac{B(a,b)I_x(a,b)}{x^a(1-x)^b} = \operatorname*{K}_{m=1}^{\infty}\left(\frac{\alpha_m(x)}{\beta_m(x)}\right), \qquad a,b>0, \quad 0 \le x < 1, \quad (18.5.20a)$$

where the coefficients $\alpha_m(x)$ and $\beta_m(x)$ are

$$\begin{aligned}
\alpha_1(x) &= 1,\\
\alpha_{m+1}(x) &= \frac{(a+m-1)(a+b+m-1)(b-m)m}{(a+2m-1)^2}x^2, \qquad m \ge 1,\\
\beta_{m+1}(x) &= a+2m+\left(\frac{m(b-m)}{a+2m-1} - \frac{(a+m)(a+b+m)}{a+2m+1}\right)x,\\
&\qquad\qquad m \ge 0.
\end{aligned} \quad (18.5.20b)$$

Because of (18.5.6) it is sufficient to explore the series and continued fraction representations of $I_x(a,b)$ for $0 \le x \le 1/2$. In view of the fact that (18.5.20) is most useful when $x \le a/(a+b)$, the role of a and b and x and $1-x$ may need to be interchanged when evaluating representations of $I_x(a,b)$.

TABLE 18.5.1: Relative error of the 5th partial sum of (18.5.16b) and the 5th approximants of (18.5.17), (18.5.18), (18.5.19) and (18.5.20) for $a = 20$ and $b = 25$. The approximants of (18.5.17), (18.5.18) and (18.5.19) are all evaluated without modification.

x	$I_x(a,b)$	(18.5.16)	(18.5.17)	(18.5.18)	(18.5.19)	(18.5.20)
0.01	$1.399753e{-28}$	$6.9e{-11}$	$3.3e{-12}$	$2.2e{-15}$	$9.8e{-14}$	$1.1e{-24}$
0.1	$1.606253e{-09}$	$6.7e{-05}$	$6.8e{-07}$	$1.2e{-09}$	$4.3e{-08}$	$5.1e{-14}$
0.2	$1.206101e{-04}$	$4.1e{-03}$	$5.8e{-05}$	$3.8e{-07}$	$1.0e{-05}$	$4.0e{-10}$
0.3	$2.175738e{-02}$	$4.3e{-02}$	$1.6e{-03}$	$4.5e{-05}$	$1.1e{-03}$	$2.7e{-07}$
0.4	$2.773366e{-01}$	$2.1e{-01}$	$3.7e{-02}$	$5.9e{-03}$	$1.5e{-01}$	$9.9e{-05}$

TABLE 18.5.2: Relative error of the 20^{th} partial sum of (18.5.16b) and the 20^{th} approximants of (18.5.17), (18.5.18), (18.5.19) and (18.5.20) for $a = 20$ and $b = 25$. The approximants of (18.5.17), (18.5.18) and (18.5.19) are all evaluated without modification.

x	$I_x(a,b)$	(18.5.16)	(18.5.17)	(18.5.18)	(18.5.19)	(18.5.20)
0.01	1.399753e−28	2.2e−37	3.8e−49	1.7e−58	3.2e−46	2.0e−102
0.1	1.606253e−09	2.1e−16	4.9e−28	1.1e−36	2.3e−24	1.1e−60
0.2	1.206101e−04	4.0e−10	1.3e−20	1.9e−28	6.0e−16	1.6e−46
0.3	2.175738e−02	1.7e−06	1.8e−15	2.0e−22	1.3e−09	4.6e−37
0.4	2.773366e−01	5.2e−04	4.3e−11	4.4e−17	8.9e−04	2.5e−29

FIGURE 18.5.1: Region in the (a,b)-plane where the 6^{th} approximant of (18.5.20) for the incomplete beta function $B_x(a,b)$ attains k significant digits when $x = 0.25$ (from light to dark: $k = 10, 11, 12$ and 13). For (a,b) we explore the region $[5,25] \times [5,25]$.

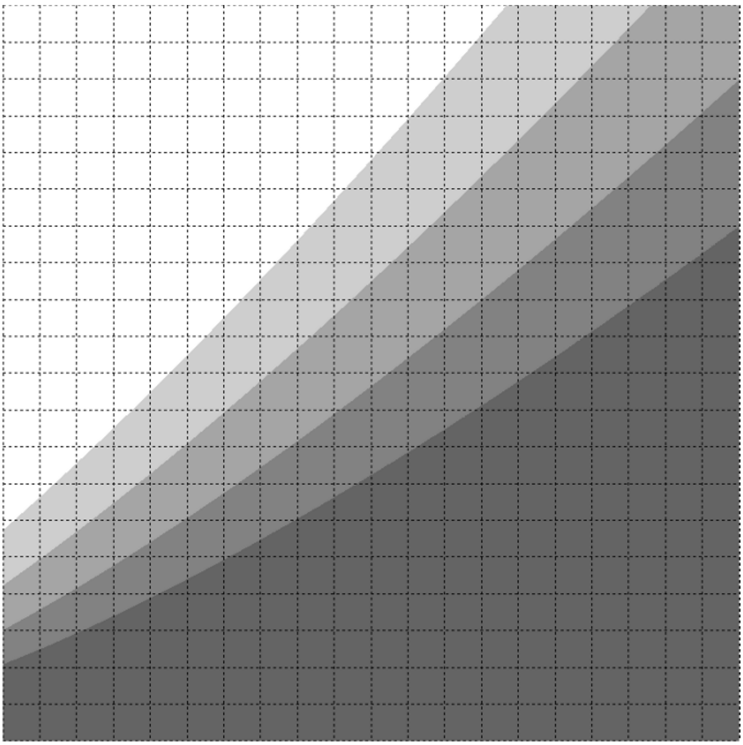
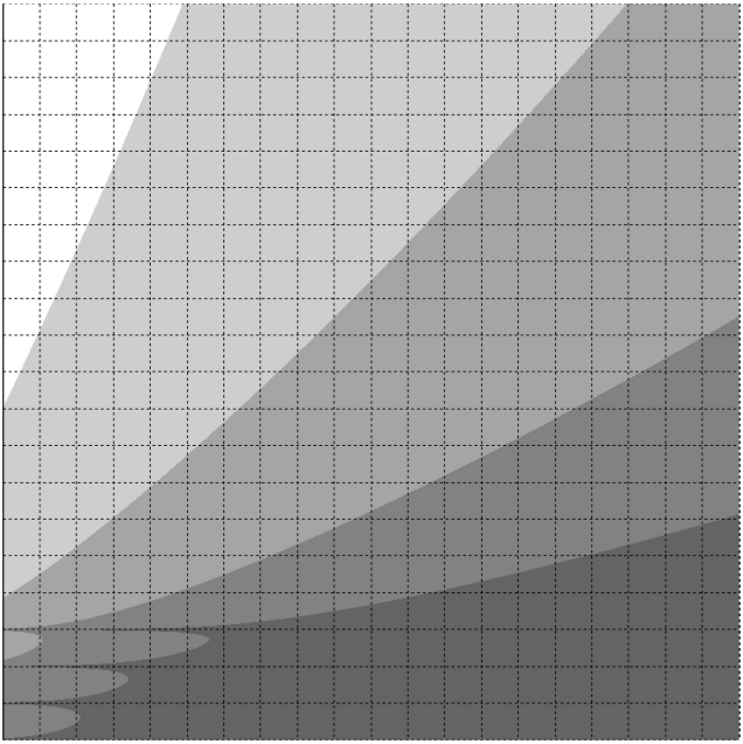


FIGURE 18.5.2: Region in the (a, b)-plane where the 6^{th} approximant of (18.5.20) for the incomplete beta function $B_x(a, b)$ attains 10 significant digits for $x = k/8$ (from light to dark: $k = 1, 2, 3$ and 4). For (a, b) we explore the region $[0, 20] \times [3, 23]$.



19

Basic hypergeometric functions

A *q-analogue* or *q-bracket* is a mathematical expression that generalises a known expression with a given parameter q. The q-analogue reduces to the original expression by taking the limit $q \to 1$ for q inside the unit circle. The q-analogue is also called the *q-extension* or the *q-generalisation.* The earliest q-analogue studied in detail is the q-hypergeometric series which was developed by Heine in 1846.

19.1 Definition and basic properties

The *q-hypergeometric series* or *basic hypergeometric series* is defined by [GR04, p. 4]

$$
\begin{aligned}
&{}_r\phi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q; z) := \\
&\qquad \sum_{m=0}^{\infty} \frac{(a_1;q)_m \cdots (a_r;q)_m}{(b_1;q)_m \cdots (b_s;q)_m} \frac{z^m}{(q;q)_m} \left((-1)^m q^{\binom{m}{2}}\right)^{1+s-r}, \qquad (19.1.1) \\
&\qquad a_k, b_k \in \mathbb{C}, \quad b_k \neq q^{-n}, \quad k = 1, \ldots, s, \quad n \in \mathbb{N}_0, \quad 0 < |q| < 1,
\end{aligned}
$$

where $(a;q)_k$ is the *generalised Pochhammer symbol* or *q-shifted factorial* defined by

$$
\begin{aligned}
&(a;q)_0 := 1, \\
&(a;q)_k := (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{k-1}), \qquad k \in \mathbb{N}.
\end{aligned} \qquad (19.1.2)
$$

The product (19.1.2) is also defined for $k = \infty$ [AAR99, p. 488]:

$$
(a;q)_\infty := \prod_{k=1}^{\infty} (1 - aq^{k-1}). \qquad (19.1.3)
$$

Since

$$
(a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty}, \qquad |q| < 1, \quad k \in \mathbb{N}_0,
$$

the definition of $(a,q)_k$ can be extended to

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, \qquad |q|<1, \quad \alpha\in\mathbb{C},$$

where the principal value of q^α is taken when $q\neq 0$.
The left hand side of (19.1.1) represents the q-hypergeometric function ${}_r\phi_s$ where the series converges. Assuming $0<|q|<1$, the following holds for the convergence of (19.1.1) [GR04, p. 5]:

- $r<s+1$: the series converges absolutely for $z\in\mathbb{C}$,
- $r=s+1$: the series converges for $|z|<1$,
- $r>s+1$: the series converges only for $z=0$, unless it terminates.

In case of convergence, the q-hypergeometric function ${}_r\phi_s$ represents an analytic function of z in the convergence region. Furthermore, the q-hypergeometric function ${}_r\phi_s$ terminates if one of the numerator parameters a_k equals q^{-m} with $m\in\mathbb{N}_0$. In particular,

$${}_r\phi_s(\ldots,a_{k-1},1,a_{k+1},\ldots;b_1,\ldots,b_s;q;z)=1. \tag{19.1.4}$$

A special case of the q-hypergeometric series is:

$${}_2\phi_1(a,b;c;q;z) := \sum_{m=0}^{\infty}\frac{(a;q)_m(b;q)_m}{(c;q)_m}\frac{z^m}{(q;q)_m},$$
$$|z|<1, \quad a,b,c\in\mathbb{C}, \quad c\neq q^{-n}, \quad n\in\mathbb{N}_0, \quad 0<|q|<1. \tag{19.1.5}$$

If $a=q^\alpha, b=q^\beta$ and $c=q^\gamma$ in (19.1.5) then we obtain the *Heine series* [And86, p. 10] defined by

$$\sum_{m=0}^{\infty}\frac{(q^\alpha;q)_m(q^\beta;q)_m}{(q^\gamma;q)_m}\frac{z^m}{(q;q)_m}, \qquad \gamma\notin\mathbb{Z}_0^-, \quad 0<|q|<1. \tag{19.1.6}$$

Using the definition of the generalised Pochhammer symbol (19.1.2), the Heine series (19.1.6) can be rewritten as

$$1+\frac{(1-q^\alpha)(1-q^\beta)}{(1-q^\gamma)(1-q)}z+\frac{(1-q^\alpha)(1-q^{\alpha+1})(1-q^\beta)(1-q^{\beta+1})}{(1-q^\gamma)(1-q^{\gamma+1})(1-q)(1-q^2)}z^2+\ldots,$$
$$\gamma\notin\mathbb{Z}_0^-, \quad 0<|q|<1. \tag{19.1.7}$$

Since

$$\lim_{q\to 1}\frac{1-q^s}{1-q^t}=\frac{s}{t} \tag{19.1.8}$$

the Heine series reduces to the hypergeometric series

$$ {}_2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{z^n}{n!} \tag{19.1.9} $$

when $q \to 1$. The q-analogue of the natural number n, denoted $[n]_q$, is defined by

$$ [n]_q = \frac{(1-q^n)}{(1-q)} = 1 + q + q^2 + \cdots + q^{n-1}, \qquad n \in \mathbb{N}, \quad 0 < |q| < 1. \tag{19.1.10} $$

The *q-factorial* is defined by

$$ \begin{aligned} [0]_q! &:= 1, \\ [n]_q! &:= \frac{(1-q)\cdots(1-q^n)}{(1-q)^n} \\ &= \frac{(q;q)_n}{(1-q)^n} = \frac{(q;q)_\infty}{(1-q)^n (q^{n+1};q)_\infty}, \qquad n \in \mathbb{N}, \quad 0 < |q| < 1. \end{aligned} \tag{19.1.11} $$

From (19.1.8) we get

$$ \lim_{q \to 1} [n]_q! = n!\,. \tag{19.1.12} $$

The *q-gamma function* generalises the q-factorial and is defined by

$$ \begin{aligned} \Gamma_q(1) &:= 1, \\ \Gamma_q(z) &:= \frac{(q;q)_\infty}{(q^z;q)_\infty} (1-q)^{1-z}. \end{aligned} \tag{19.1.13} $$

Here we take the principal values of q^z and $(1-q)^{1-z}$ [GR04, p. 21]. Then $\Gamma_q(z)$ is a meromorphic function with poles at $z = -n \pm 2\pi i k / \ln(q)$ where $k, n \in \mathbb{N}$ [AAR99, p. 493]. For $\Gamma_q(z)$ the recurrence

$$ \Gamma_q(z+1) = \frac{1-q^z}{1-q} \Gamma_q(z) \tag{19.1.14} $$

holds [AAR99, p. 494], which reduces to the recurrence relation (12.1.5) for $q \to 1$.

Recurrence relations. Continued fraction representations for ratios of the hypergeometric series ${}_2F_1(a, b; c; z)$ are obtained by using the contiguous relation (15.1.15) and the recurrence relations (15.1.16b) and (15.1.16c). Similarly, we obtain continued fractions for ratios of the q-hypergeometric series ${}_2\phi_1(a,b;c;q;z)$ from the q-analogue of these relations. The contiguous relation (15.1.15) for the ${}_2\phi_1(a,b;c;z)$ series is [LW92, p. 320]

$$ {}_2\phi_1(a,b;c;q;z) = {}_2\phi_1(a,bq;cq;q;z) + \frac{(1-a)(c-b)}{(1-c)(1-cq)} z\, {}_2\phi_1(aq,bq;cq^2;q;z). \tag{19.1.15} $$

The q-analogue of the recurrence relations (15.1.16) is given by [VS01; AR93; LW92, p. 321]

$$ \begin{aligned} {}_2\phi_1(a,b;cq^2;q;q^2z) = \\ -\frac{(1-cq)(1-(a+b)z+c(qz+z-1))}{(a-cq)(b-cq)z}\, {}_2\phi_1(a,b;cq;q;qz) \\ + \frac{(1-c)(1-cq)(1-z)}{(a-cq)(b-cq)z}\, {}_2\phi_1(a,b;c;q;z), \end{aligned} \tag{19.1.16a} $$

$$ \begin{aligned} {}_2\phi_1(a,bq^2;cq^2;q;z) = \frac{((1-c)q+(a-bq)z)(1-cq)}{(1-bq)(a-cq)z}\, {}_2\phi_1(a,bq;cq;q;z) \\ - \frac{(1-c)q(1-cq)}{(1-bq)(a-cq)z}\, {}_2\phi_1(a,b;c;q;z), \end{aligned} \tag{19.1.16b} $$

$$ \begin{aligned} {}_2\phi_1(aq^2,bq^2;cq^2;q;z) = \\ \frac{(1-c-(a+b-ab-abq)z)(cq-1)}{(1-aq)(1-bq)(c-abqz)z}\, {}_2\phi_1(aq,bq;cq;q;z) \\ + \frac{(1-c)(1-cq)}{(1-aq)(1-bq)(c-abqz)z}\, {}_2\phi_1(a,b;c;q;z). \end{aligned} \tag{19.1.16c} $$

Integral representations. The *q-integral* is defined by

$$ \int_0^1 f(x)\, d_q(x) = (1-q) \sum_{m=0}^{\infty} f(q^m) q^m. $$

The q-analogue of Euler's integral representation (15.2.1) for ${}_2F_1(a,b;c;z)$ is given by [And86, p. 11]

$$ {}_2\phi_1(q^\alpha,q^\beta;q^\gamma;q;z) = \frac{\Gamma_q(\gamma)}{\Gamma_q(\beta)\Gamma_q(\gamma-\beta)} \int_0^1 \frac{t^{\beta-1}(qt;q)_{\gamma-\beta-1}}{(zt;q)_\alpha}\, d_q(t), $$
$$ |z|<1, \quad \Re\beta>0, \quad \gamma-\beta \in \mathbb{C}\setminus\mathbb{Z}_0^-, \quad 0<|q|<1. $$

A *q-beta function* which is a q-analogue of the beta function (18.5.1) is defined by

$$ B_q(a,b) := \int_0^1 t^{b-1}(qt;q)_{a-1}\, d_q(t) = \frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(a+b)}, \qquad \Re a>0, \quad b\in\mathbb{C}\backslash\mathbb{Z}_0^-. \tag{19.1.17} $$

19.2 Continued fraction representations

C-fraction. The contiguous relation (19.1.15) gives rise to the Heine continued fraction [Ber91, p. 21; AR93]

$$\frac{{}_2\phi_1(a,b;c;q;z)}{{}_2\phi_1(a,bq;cq;q;z)} = 1 + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{a_m z}{1}\right),$$
$$z \in \mathbb{C}, \quad a,b,c \in \mathbb{C}, \quad c \neq q^{-n}, \quad n \in \mathbb{N}_0, \quad 0 < |q| < 1, \quad (19.2.1a)$$

with

$$a_{2k+1} = \frac{(1-aq^k)(cq^k-b)q^k}{(1-cq^{2k})(1-cq^{2k+1})}, \qquad k \in \mathbb{N}_0,$$
$$a_{2k} = \frac{(1-bq^k)(cq^k-a)q^{k-1}}{(1-cq^{2k-1})(1-cq^{2k})}, \qquad k \in \mathbb{N}. \quad (19.2.1b)$$

The C-fraction (19.2.1) is the q-analogue of the Gauss continued fraction (15.3.3). Putting $b = 1$ in (19.2.1) and using (19.1.4), we find

$$z\ {}_2\phi_1(a,q;cq;q;z) = \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{c_m z}{1}\right),$$
$$z \in \mathbb{C}, \quad a,c \in \mathbb{C}, \quad c \neq q^{-n}, \quad n \in \mathbb{N}_0, \quad 0 < |q| < 1, \quad (19.2.2a)$$

with

$$c_1 = 1, \qquad c_{2k+2} = \frac{(1-aq^k)(cq^k-1)q^k}{(1-cq^{2k})(1-cq^{2k+1})}, \qquad k \in \mathbb{N}_0,$$
$$c_{2k+1} = \frac{(1-q^k)(cq^k-a)q^{k-1}}{(1-cq^{2k-1})(1-cq^{2k})}, \qquad k \in \mathbb{N}. \quad (19.2.2b)$$

Example 19.2.1: Setting $a = c = q$ in (19.2.2) leads to

$${}_2\phi_1(q,q;q^2;q;-z) = 1 + \sum_{m=1}^{\infty}\frac{(1-q)}{1-q^{m+1}}(-z)^m = z^{-1}\mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{d_m z}{1}\right), \quad (19.2.3a)$$

with

$$d_1 = 1, \qquad d_{2k+2} = \frac{(1-q^{k+1})q^k}{(1+q^{k+1})(1-q^{2k+1})}, \quad k \in \mathbb{N}_0,$$
$$d_{2k+1} = \frac{(1-q^k)q^k}{(1+q^k)(1-q^{2k+1})}, \quad k \in \mathbb{N}. \quad (19.2.3b)$$

For $q \to 1$ the series tends to the Taylor series expansion of $\mathrm{Ln}(1+z)/z$ at the origin and the continued fraction tends to the S-fraction representation (11.2.2) of $\mathrm{Ln}(1+z)/z$.

Padé approximants. The Padé approximant $r_{m,n}(z)$ for ${}_2\phi_1(a,q;cq;q;z)$ is given by the irreducible form of $p(z)/q(z)$ where [Aga94]

$$\begin{aligned} p(z) &= \frac{(az/c)^m (q^{-n};q)_m (q^{-m}/a;q)_m}{(q;q)_m (q^{-m-n}/c;q)_m} \sum_{k=0}^{m} \frac{(q^{-m};q)_k (a;q)_k (cq^{n+1};q)_k q^k}{(cq;q)_k (aq;q)_k (q^{n-m+1};q)_k} \times \\ &\qquad {}_3\phi_2(q^{-m+k}, cq^{n+k+1}, q; q^{n-m+k+1}, aq^{k+1}; q/z), \\ q(z) &= {}_2\phi_1(q^{-n}, q^{-m}/a; q^{-m-n}/c; az/c). \end{aligned} \tag{19.2.4}$$

Observe that when $m = n$, the above formulas give us explicit expressions for the $(2n+1)^{\mathrm{th}}$ approximants of the C-fraction (19.2.2). Similarly, when $m = n-1$, we obtain explicit expressions for the $(2n)^{\mathrm{th}}$ approximant of (19.2.2).

T-fractions. The q-analogue of the T-fraction (15.3.8) is obtained from the recurrence relation (19.1.16b), and given in [AR93]:

$$\frac{{}_2\phi_1(a,b;c;q;z)}{{}_2\phi_1(a,bq;cq;q;z)} = \frac{q(1-c)+(a-bq)z}{q(1-c)} + \frac{1}{q(1-c)} \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{c_m z}{e_m + d_m z} \right),$$
$$|z| < |q/a|, \quad a,b,c \in \mathbb{C}, \quad c \neq q^{-n}, \quad n \in \mathbb{N}_0, \quad 0 < |q| < 1, \tag{19.2.5a}$$

$$\begin{aligned} &\frac{a-bq}{q(1-c)} z \frac{{}_2\phi_1(bq/c, b; bq/a; q; cq/(abz))}{{}_2\phi_1(bq/c, bq; bq^2/a; q; cq/(abz))} = \\ &\qquad \frac{q(1-c)+(a-bq)z}{q(1-c)} + \frac{1}{q(1-c)} \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{c_m z}{e_m + d_m z} \right), \\ &\qquad |z| > |q/a|, \quad a,b,c \in \mathbb{C}, \quad 0 < |q| < 1, \end{aligned} \tag{19.2.5b}$$

where

$$\begin{aligned} c_m &= q(1-bq^m)(cq^m - a), \\ e_m &= q(1-cq^m), \qquad d_m = a - bq^{m+1}, \end{aligned} \qquad m \geq 1. \tag{19.2.5c}$$

The T-fraction (19.2.5) is limit periodic. A suitable modification for this fraction is found by combining (7.7.4) and (7.7.5), and is given by

$$\tilde{w}_n(z) = (e_n + d_n z) w(z) \tag{19.2.6a}$$

with

$$w(z) = \mathrm{K}\left(\frac{-aqz/(q+az)^2}{1}\right) = \begin{cases} \dfrac{-q}{q+az}, & |z| > \dfrac{|q|}{|a|}, \\ \dfrac{-az}{q+az}, & |z| < \dfrac{|q|}{|a|}. \end{cases} \tag{19.2.6b}$$

q-analogue of the Nörlund fraction. The q-analogue of the Nörlund fraction (15.3.13) [Fra60; IL89] can be derived from the recurrence relation (19.1.16c) and is given by

$$\frac{{}_2\phi_1(a,b;c;q;z)}{{}_2\phi_1(aq,bq;cq;q;z)} = \frac{1-c-(a+b-ab-abq)z}{1-c} + \frac{1}{1-c}\mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{c_m(z)}{e_m+d_m z}\right),$$
$$z \in \mathbb{C}, \quad a,b,c \in \mathbb{C}, \quad c \neq q^{-n}, \quad n \in \mathbb{N}_0, \quad 0 < |q| < 1, \tag{19.2.7a}$$

with

$$\begin{aligned} c_m(z) &= (1-aq^m)(1-bq^m)(cz-abq^m z^2)q^{m-1}, \\ e_m &= 1-cq^m, \qquad\qquad m \geq 1. \\ d_m &= -(a+b-abq^m-abq^{m+1})q^m, \end{aligned} \tag{19.2.7b}$$

Another continued fraction. Let the function $h(a,b;c;q;z)$ be defined by [VS01]

$$h(a,b;c;q;z) = (c;q)_\infty (z;q)_\infty \, {}_2\phi_1(aq,b;c;q;z). \tag{19.2.8}$$

The function (19.2.8) satisfies the relations

$$\begin{aligned} h(a,b;c;q;z) &= h(aq,b;c;q;z) + az(b-1)h(aq,bq;cq;q;z), \\ h(aq,b;c;q;z) &= (1-z)h(a,b;c;q;qz) + z(1-b)h(aq,bq;cq;q;z), \\ h(aq,b;c;q;qz) &= \frac{(aqz-c)}{(abqz-c)}h(a,b;c;q;qz) + \frac{aqz(b-1)}{(abqz-c)}h(aq,bq;cq;q;z). \end{aligned} \tag{19.2.9}$$

The relations (19.2.9) lead to the continued fraction

$$\frac{{}_2\phi_1(a,b;c;q;z)}{{}_2\phi_1(aq,b;c;q;z)} = 1 + \frac{a}{-1} + \frac{\alpha_0(z)}{\beta_0(z)} + \frac{\gamma_0(z)}{1} + \frac{a}{-1} + \frac{\alpha_1(z)}{\beta_1(z)} + \frac{\gamma_1(z)}{1} + \dots$$
$$|z| < 1, \quad a,b,c \in \mathbb{C}, \quad c \neq q^{-n}, \quad n \in \mathbb{N}_0, \quad 0 < |q| < 1, \tag{19.2.10a}$$

where

$$\begin{aligned} \alpha_m(z) &= aq(1-q^m z), \\ \beta_m(z) &= c - aq^{m+1}z, \qquad m \geq 0. \\ \gamma_m(z) &= abq^{m+1}z - c, \end{aligned} \tag{19.2.10b}$$

Continued fractions in qz. The function ${}_2\phi_1(a,b;c;q;z)$ also satisfies the recurrence relation

$${}_2\phi_1(a,b;c;q;q^2z) = \frac{-q-c+(a+b)qz}{abqz-c}\,{}_2\phi_1(a,b;c;q;qz) + \frac{(1-z)q}{abqz-c}\,{}_2\phi_1(a,b;c;q;z),$$

which leads to a continued fraction for ${}_2\phi_1(a,b;c;q;qz)/{}_2\phi_1(a,b;c;q;z)$ given by [IL89]

$$\frac{{}_2\phi_1(a,b;c;q;qz)}{{}_2\phi_1(a,b;c;q;z)} = \frac{1-z}{1+cq^{-1}-(a+b)z} + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{a_m(z)}{b_m(z)}\right),$$
$$z \in \mathbb{C}, \quad a,b,c \in \mathbb{C}, \quad |c/q| \neq 1, \quad 0 < |q| < 1. \tag{19.2.11a}$$

The coefficients are

$$\begin{aligned} a_m(z) &= -(cq^{-1} - abq^{m-1}z)(1-q^m z), \\ b_m(z) &= 1 + cq^{-1} - (a+b)q^m z, \end{aligned} \qquad m \geq 1. \tag{19.2.11b}$$

The continued fraction in (19.2.11) converges to the left-hand side if (a,b,c) belongs to a neighbourhood of $(0,0,0)$, $|z|<1$ and z is not a pole of the right-hand side [IL89]. A continued fraction for the reciprocal of the ratio in (19.2.11) is given by [Den84]

$$\frac{{}_2\phi_1(a,b;c;q;z)}{{}_2\phi_1(a,b;c;q;qz)} = 1 + \frac{\alpha_0 z}{1-z} + \frac{\gamma_0(z)}{1} + \frac{\alpha_1 z}{1-z} + \frac{\gamma_1(z)}{1} + \dots,$$
$$z \neq 1, \quad a,b,c \in \mathbb{C}, \quad c \neq q^{-n}, \quad n \in \mathbb{N}_0, \quad 0 < |q| < 1 \tag{19.2.12a}$$

where

$$\begin{aligned} \alpha_m &= (1-aq^m)(1-bq^m), \\ \gamma_m(z) &= abq^{2m+1}z - cq^m, \end{aligned} \qquad m \geq 0. \tag{19.2.12b}$$

A continued fraction for the ratio ${}_2\phi_1(aq, b; c; q; z)/{}_2\phi_1(a, b; c; q; z)$ is given by [VDSR87]

$$\frac{{}_2\phi_1(aq, b; c; q; z)}{{}_2\phi_1(a, b; c; q; qz)} = 1 + \frac{\alpha_0 z}{(1-a)(1-z)} \,\frac{}{+}\, \frac{\gamma_0}{1} \,\frac{}{+}\, \frac{\alpha_1 z}{(1-a)(1-z)} \,\frac{}{+}\, \frac{\gamma_1}{1} \,\frac{}{+}\, \dots$$
$$\left|\frac{z}{(1-a)(1-z)}\right| < \frac{1}{4}, \quad \left|\frac{a}{(1-a)(1-z)}\right| < \frac{1}{4},$$
$$a, b, c \in \mathbb{C}, \quad c \neq q^{-n}, \quad n \in \mathbb{N}_0, \quad 0 < |q| < 1 \qquad (19.2.13a)$$

with

$$\begin{aligned} \alpha_m &= (1 - bq^m), \\ \gamma_m &= a - cq^m, \end{aligned} \qquad m \geq 0. \qquad (19.2.13b)$$

19.3 Higher order basic hypergeometric functions

The q-hypergeometric series ${}_r\phi_s$ is defined in (19.1.1). There are results on continued fraction representations for higher order basic hypergeometric functions, ratios of such functions or other expressions involving such functions [Mas95].

We restrict ourselves here to a continued fraction representation for two ratios of ${}_3\phi_2(a, b, c; e, f; q; z)$ given in [VDSR87]. For the first ratio we have

$$\frac{{}_3\phi_2(a, b, c; e, f; q; ef/(abc))}{{}_3\phi_2(a, b, c; eq, f; q; efq/(abc))} = 1 + \frac{\alpha_0}{\beta_0} \,\frac{}{+}\, \frac{\gamma_0}{1} \,\frac{}{+}\, \frac{\alpha_1}{\beta_1} \,\frac{}{+}\, \frac{\gamma_1}{1} \,\frac{}{+}\, \dots \qquad (19.3.1a)$$

where the coefficients α_m, β_m and γ_m, are given by

$$\begin{aligned} \alpha_m &= \frac{(1 - aq^m)(1 - bq^m)(1 - cq^m)ef/(abc)}{(1 - eq^{2m})(1 - eq^{2m+1})(1 - fq^m)}, \\ \beta_m &= \frac{1 - ef/(abc)}{1 - fq^m}, \qquad\qquad m \geq 0. \\ \gamma_m &= \frac{-(1 - eq^{m+1}/a)(1 - eq^{m+1}/b)(1 - eq^{m+1}/c)fq^m}{(1 - eq^{2m+1})(1 - eq^{2m+2})(1 - fq^m)}, \end{aligned} \qquad (19.3.1b)$$

Eliminating c by setting $c = ef/(abz)$ in (19.3.1) and taking the limit $e \to 0$, we obtain the continued fraction (19.2.12) as a special case. For the second ratio we have

$$\frac{{}_3\phi_2(aq, b, c; e, f; q; ef/(abcq))}{{}_3\phi_2(a, b, c; e, f; q; ef/(abc))} = 1 + \frac{\alpha_0}{\beta} \,\frac{}{+}\, \frac{\gamma_0}{1} \,\frac{}{+}\, \frac{\alpha_1}{\beta} \,\frac{}{+}\, \frac{\gamma_1}{1} \,\frac{}{+}\, \dots \qquad (19.3.2a)$$

where the coefficients α_m, β and γ_m, are given by

$$\begin{aligned}
\alpha_m &= (1-bq^m)(1-cq^m)\frac{ef}{abcq}, \\
\beta &= (1-a)\left(1-\frac{ef}{abcq}\right), \qquad\qquad m \geq 0. \\
\gamma_m &= a\left(1-\frac{eq^m}{a}\right)\left(1-\frac{fq^m}{a}\right),
\end{aligned} \tag{19.3.2b}$$

Eliminating c by setting $c = ef/(abqz)$ in (19.3.2) and taking the limit $e \to 0$, we obtain the continued fraction (19.2.13).

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