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The H-Function

Theory and Applications

Springer

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Hans J. Haubold is the chief scientist at the outer space division of the United Nations, situated at Vienna, Austria. His research contribution is mainly in the area of theoretical physics. He has published over 300 papers, over 30 of them are jointly with A.M. Mathai on stellar and solar models, energy generation, neutrino problem, gravitational instability problem, etc. He has authored a number of papers jointly with A.M. Mathai and R.K. Saxena on applications of fractional calculus to reaction–diffusion problems. In the beginning of 2008 he has published the book *Special Functions for Applied Scientists*, jointly with A.M. Mathai (Springer, New York). His research collaboration with A.M. Mathai goes back to 1984.

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Preface

The H-function or popularly known in the literature as Fox's H-function has recently found applications in a large variety of problems connected with reaction, diffusion, reaction–diffusion, engineering and communication, fractional differential and integral equations, many areas of theoretical physics, statistical distribution theory, etc. One of the standard books and most cited book on the topic is the 1978 book of Mathai and Saxena. Since then, the subject has grown a lot, mainly in the fields of applications. Due to popular demand, the authors were requested to upgrade and bring out a revised edition of the 1978 book. It was decided to bring out a new book, mostly dealing with recent applications in statistical distributions, pathway models, nonextensive statistical mechanics, astrophysics problems, fractional calculus, etc. and to make use of the expertise of Hans J. Haubold in astrophysics area also.

It was decided to confine the discussion to H-function of one scalar variable only. Matrix variable cases and many variable cases are not discussed in detail, but an insight into these areas is given. When going from one variable to many variables, there is nothing called a unique bivariate or multivariate analogue of a given function. Whatever be the criteria used, there may be many different functions qualified to be bivariate or multivariate analogues of a given univariate function. Some of the bivariate and multivariate H-functions, currently in the literature, are also questioned by many authors. Hence, it was decided to concentrate on one variable case and to put some multivariable situations in an appendix; only the definitions and immediate properties are given here.

Chapter 1 gives the definitions, various contours, existence conditions, and some particular cases. Chapter 2 deals with various types of transforms such as Laplace, Fourier, Hankel, etc. on H-functions, their properties, and some relationships among them. Chapter 3 goes into fractional calculus and their connections to H-functions. All the popular fractional differential and fractional integral operators are examined in this chapter.

Chapter 4 is on the applications of H-function in various areas of statistical distribution theory, various structures of random variables, generalized distributions, Mathai's pathway models, a versatile integral which is connected to different fields, etc. Chapter 5 gives a glimpse into functions of matrix argument, mainly real-valued scalar functions of matrix argument when the matrices are real or Hermitian positive definite. H-function of matrix argument is defined only in the form of a class of functions satisfying a certain integral equation and hence a detailed discussion is not attempted here.

Chapter 6 examines applications of H-function into various problems in physics. The problems examined are the following: solar and stellar models, gravitational instability problem, energy generation, solar neutrino problem, generalized entropies, Tsallis statistics, superstatistics, Mathai's pathway analysis, input–output models, kinetic equations, reaction, diffusion, and reaction–diffusion problems where H-functions prop up in the analytic solutions to these problems.

The book is intended as a reference source for teachers and researchers, and it can also be used as a textbook in a one-semester graduate (post-graduate) course on H-function. In this context, a more or less exhaustive and up-to-date bibliography on H-function is included in the book.

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Contents

Chapter 1
On the H-Function With Applications

1.1 A Brief Historical Background

Mellin–Barnes integrals are discovered by Salvatore Pincherle, an Italian mathematician in the year 1888. These integrals are based on the duality principle between linear differential equations and linear difference equations with rational coefficients. The theory of these integrals has been developed by Mellin (1910) and has been used in the development of the theory of hypergeometric functions by Barnes (1908). Important contributions of Salvatore Pincherle are recently given in a paper by Mainardi and Pagnini (2003). In the year 1946, these integrals were used by Meijer to introduce the G-function into mathematical analysis. From 1956 to 1970 lot of work has been done on this function, which can be seen from the bibliography of the book by Mathai and Saxena (1973a).

In the year 1961, in an attempt to discover a most generalized symmetrical Fourier kernel, Charles Fox (1961) defined a new function involving Mellin–Barnes integrals, which is a generalization of the G-function of Meijer. This function is called Fox's H-function or the H-function. The importance of this function is realized by the scientists, engineers and statisticians due to its vast potential of its applications in diversified fields of science and engineering. This function includes, among others, the functions considered by Boersma (1962), Mittag-Leffler (1903), generalized Bessel function due to Wright (1934), the generalization of the hypergeometric functions studied by Fox (1928), and Wright (1935, 1940), Krätzel function (Krätzel 1979), generalized Mittag-Leffler function due to Dzherbashyan (1960), generalized Mittag-Leffler function due to Prabhakar (1971) and multi-index Mittag-Leffler function due to Kiryakova (2000), etc. Except the functions of Boersma (1962), the aforesaid functions cannot be obtained as special cases of the G-function of Meijer (1946), hence a study of the H-function will cover wider range than the G-function and gives general, deeper, and useful results directly applicable in various problems of physical, biological, engineering and earth sciences, such as fluid flow, rheology, diffusion in porous media, kinematics in viscoelastic media, relaxation and diffusion processes in complex systems, propagation of seismic waves, anomalous diffusion and turbulence, etc. see, Caputo (1969), Glöckle

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and Nonnenmacher (1993), Mainardi et al. (2001), Saichev and Zaslavsky (1997), Hilfer (2000), Metzler and Klafter (2000), Podlubny (1999), Schneider (1986) and Schneider and Wyss (1989) and others.

1.2 The H-Function

Notation 1.1.

$$H(x) = H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\text{: H-function.} \tag{1.1}$$

Definition 1.1. The H-function is defined by means of a Mellin–Barnes type integral in the following manner (Mathai and Saxena 1978)

$$\begin{aligned} H(x) &= H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right] \\ &= \frac{1}{2\pi i}\int_L \Theta(s) z^{-s} \mathrm{d}s, \end{aligned} \tag{1.2}$$

where $i = (-1)^{\frac{1}{2}}, z \neq 0$, and $z^{-s} = \exp[-s\{\ln|z| + i \arg z\}]$, where $\ln|z|$ represents the natural logarithm of $|z|$ and $\arg z$ is not necessarily the principal value. Here

$$\Theta(s) = \frac{\{\prod_{j=1}^{m}\Gamma(b_j + B_j s)\}\{\prod_{j=1}^{n}\Gamma(1 - a_j - A_j s)\}}{\{\prod_{j=m+1}^{q}\Gamma(1 - b_j - B_j s)\}\{\prod_{j=n+1}^{p}\Gamma(a_j + A_j s)\}}. \tag{1.3}$$

An empty product is always interpreted as unity; $m, n, p, q \in N_0$ with $0 \leq n \leq p$, $1 \leq m \leq q, A_i, B_j \in R_+, a_i, b_j \in R$, or C, $i = 1, \ldots, p; j = 1, \ldots, q$. L is a suitable contour separating the poles

$$\zeta_{j\nu} = -\left(\frac{b_j + \nu}{B_j}\right), \; j = 1, \ldots, m; \quad \nu = 0, 1, 2, \ldots \tag{1.4}$$

of the gamma functions $\Gamma(b_j + sB_j)$ from the poles

$$\omega_{\lambda k} = \left(\frac{1 - a_\lambda + k}{A_\lambda}\right), \; \lambda = 1, \ldots, n; \quad k = 0, 1, 2, \ldots \tag{1.5}$$

of the gamma functions $\Gamma(1 - a_\lambda - sA_\lambda)$, that is

$$A_\lambda(b_j + \nu) \neq B_j(a_\lambda - k - 1), \; j = 1, \cdots, m; \lambda = 1, \ldots, n; \; \nu, k = 0, 1, 2, \ldots \tag{1.6}$$

The contour L exists on account of (1.6). These assumptions will be retained throughout. The contour L is either $L_{-\infty}, L_{+\infty}$ or $L_{i\gamma\infty}$. The following are the definitions of these contours.

(i) $L = L_{-\infty}$ is a loop beginning and ending at $-\infty$ and encircling all the poles of $\Gamma(b_j + B_j s), j = 1, \ldots, m$ once in the positive direction but none of the poles of $\Gamma(1 - a_\lambda - A_\lambda s), \lambda = 1, \ldots, n$. The integral converges for all z if $\mu > 0$ and $z \neq 0$; or $\mu = 0$ and $0 < |z| < \beta$. The integral also converges if

$$\mu = 0, |z| = \beta \quad \text{and} \quad \Re(\delta) < -1, \tag{1.7}$$

where

$$\beta = \left\{\prod_{j=1}^{p} (A_j)^{-A_j}\right\} \left\{\prod_{j=1}^{q} (B_j)^{B_j}\right\}, \tag{1.8}$$

$$\mu = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j, \text{ and} \tag{1.9}$$

$$\delta = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{p-q}{2}. \tag{1.10}$$

(ii) $L = L_{+\infty}$ is a loop beginning and ending at $+\infty$ and encircling all the poles of $\Gamma(1 - a_\lambda - A_\lambda s), \lambda = 1, \ldots, n$ once in the negative direction but none of the poles of $\Gamma(b_j + B_j s), j = 1, \ldots, m$. The integral converges for all z if

$$\mu < 0 \text{ and } z \neq 0 \text{ or } \mu = 0 \text{ and } |z| > \beta. \tag{1.11}$$

The integral also converges if the conditions given in (1.7) are satisfied.

(iii) $L = L_{i\gamma\infty}$ is a contour starting at the point $\gamma - i\infty$ and going to $\gamma + i\infty$ where $\gamma \in R = (-\infty, +\infty)$ such that all the poles of $\Gamma(b_j + B_j s), j = 1, \ldots, m$ are separated from those of $\Gamma(1 - a_\lambda - A_\lambda s), \lambda = 1, \ldots, n$. The integral converges if

$$\alpha > 0, |\arg z| < \frac{1}{2}\pi\alpha, \ a \neq 0. \tag{1.12}$$

The integral also converges if $\alpha = 0$, $\gamma\mu + \Re(\delta) < -1$, $\arg z = 0$ and $z \neq 0$ where

$$\alpha = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j. \tag{1.13}$$

A detailed and comprehensive account of the H-function is available from the monographs Mathai and Saxena (1978), Prudnikov et al. (1990), Kilbas and Saigo (2004).

Existence conditions for the *H*-function. In many applied problems associated with fractional differential equations and fractional integral equations, the solutions of certain problems are obtained in terms of the H-function. The H-function naturally occurs as solutions of such equations. In order to find the existence conditions of the solution of the problem, we therefore need the existence conditions for the H-function. The existence conditions for the H-function are enumerated below. It is presumed that the condition (1.6) is satisfied throughout this book unless otherwise stated.

Theorem 1.1. *The H-function is an analytic function of z and exists in the following cases:*

$$\textit{Case 1}: q \geq 1, \mu > 0, \ \textit{H-function exists for all } z \neq 0, \tag{1.14}$$

$$\textit{Case 2}: q \geq 1, \mu = 0, \ \textit{H-function exists for } 0 < |z| < \beta, \tag{1.15}$$

$$\textit{Case 3}: q \geq 1, \mu = 0, \Re(\delta) < -1, \ \textit{H-function exists for } |z| = \beta, \tag{1.16}$$

$$\textit{Case 4}: p \geq 1, \mu < 0, \ \textit{H-function exists for all } z, z \neq 0, \tag{1.17}$$

$$\textit{Case 5}: p \geq 1, \mu = 0, \ \textit{H-function exists for } |z| > \beta, \tag{1.18}$$

$$\textit{Case 6}: p \geq 1, \mu = 0 \textit{ and } \Re(\delta) < -1, \ \textit{H-function exists for } |z| = \beta, \tag{1.19}$$

$$\textit{Case 7}: \alpha > 0, |\arg z| < \frac{1}{2}\pi\alpha, \ \textit{H-function exists for all } z \neq 0, \tag{1.20}$$

$$\textit{Case 8}: \alpha = 0, \gamma\mu + \Re(\delta) < -1, \ \textit{H-function exists for } \arg z = 0 \textit{ and } z \neq 0. \tag{1.21}$$

In what follows

$$c^* = m + n - \frac{1}{2}p - \frac{1}{2}q. \tag{1.22}$$

Proof 1.1. The proof of the existence conditions can be obtained by finding the convergence of the integral (1.2), which depends on the asymptotic estimate of $\Theta(s)$ at infinity. Such a result can be found by using the following asymptotic relation for the gamma function $\Gamma(z), z = x + iy, \ x, y \in R$ at infinity on lines parallel to the coordinate axes given by Kilbas and Saigo (1999, p. 193):

$$|x + iy| \sim \sqrt{2\pi}|x|^{x-\frac{1}{2}} \exp[-x - x(1 - \mathrm{sign}(x))y/2], \quad |x| \to \infty, \tag{1.23}$$

and

$$|x + iy| \sim \sqrt{2\pi}|y|^{x-\frac{1}{2}} \mathrm{e}^{-x-\pi|y|/2}, \quad |y| \to \infty. \tag{1.24}$$

The proof of the above results (1.23) and (1.24) can be developed by making use of the Stirling formula (Erdélyi et al. 1953, p. 47, 1.18(2))

$$\Gamma(z) \sim \sqrt{2\pi}\, z^{z-\frac{1}{2}} \mathrm{e}^{-z} \left[1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - 0(z^{-4})\right], \quad |\arg z| < \pi. \tag{1.25}$$

For details of the proof, see Kilbas and Saigo (1999).

In order to prove Theorem 1.1, we first establish the following two lemmas. These lemmas will then be applied in finding the asymptotic relations along the lines τ_1, τ_2 and τ_γ, defined by

$$\tau_1 = \{t + i\varphi_1 : t \in R\}, \tau_2 = \{t + \varphi_2 : t \in R\}, \tau_\gamma = \{\gamma + it : t \in R\}, \quad (1.26)$$

where $\varphi_1, \varphi_2, \gamma \in R$. □

Lemma 1.1. *For $\sigma, t \in R$, there holds the asymptotic estimate*

$$\Theta(t + i\sigma)| \sim A \left(\frac{\mathrm{e}}{t}\right)^{\mu t} \beta^{-t} t^{\Re(\delta)}, t \to \infty, \quad (1.27)$$

where

$$A = (2\pi)^{c^*} \mathrm{e}^{q-m-n} \frac{\left\{\prod_{j=1}^{q} [B_j^{\Re(b_j)-\frac{1}{2}} \mathrm{e}^{-\Re(b_j)}]\right\} \left\{\prod_{j=1}^{n} \mathrm{e}^{\pi[\sigma A_j + \mathrm{Im}(a_j)]}\right\}}{\left\{\prod_{j=1}^{p} [A_j^{\Re(a_j)-\frac{1}{2}} \mathrm{e}^{-\Re(a_j)}]\right\} \left\{\prod_{j=1}^{n} \mathrm{e}^{\pi[\sigma B_j + \mathrm{Im}(b_j)]}\right\}}, \quad (1.28)$$

and

$$|\Theta(t + i\sigma)| \sim B \left(\frac{\mathrm{e}}{|t|}\right)^{\mu |t|} \beta^{-|t|} |t|^{\Re(\delta)}, \; t \to -\infty, \quad (1.29)$$

where

$$B = (2\pi)^{c^*} \mathrm{e}^{q-m-n} \frac{\left\{\prod_{j=1}^{q} [B_j^{\Re(b_j)-\frac{1}{2}} \mathrm{e}^{-\Re(b_j)}]\right\} \left\{\prod_{j=n+1}^{p} \mathrm{e}^{\pi[\sigma A_j + \mathrm{Im}(a_j)]}\right\}}{\left\{\prod_{j=1}^{p} [A_j^{\Re(a_j)-\frac{1}{2}} \mathrm{e}^{-\Re(a_j)}]\right\} \left\{\prod_{j=1}^{m} \mathrm{e}^{\pi[\sigma B_j + \mathrm{Im}(b_j)]}\right\}}, \quad (1.30)$$

and β, μ and δ are defined in (1.8), (1.9), and (1.10) respectively.

Lemma 1.2. *For $\sigma, t \in R$ there holds the asymptotic relation*

$$|\Theta(\sigma + it) \sim C |t|^{\mu\sigma + \Re(\delta)} \exp[-\pi\{|t|\alpha + \mathrm{Im}(\nu) sign(t)\}/2], |t| \to \infty, \quad (1.31)$$

uniformly on σ on any bounded interval in R, where

$$C = (2\pi)^{c^*} \exp\{-c^* - \mu\sigma - \Re(\delta)\} \beta^{\sigma} \left\{\prod_{j=1}^{p} A_j^{\frac{1}{2}-a_j}\right\} \left\{\prod_{j=1}^{q} B_j^{b_j - \frac{1}{2}}\right\}, \quad (1.32)$$

where μ, δ, c^* are defined in (1.9), (1.10) and (1.22) respectively, and

$$\nu = \sum_{j=1}^{n} a_j - \sum_{j=n+1}^{p} a_j + \sum_{j=1}^{m} b_j - \sum_{j=m+1}^{q} b_j.$$

The Lemma 1.1 and Lemma 1.2 follow from (1.3), (1.19) and (1.20). By virtue of the above Lemmas 1.1 and 1.2. it is not difficult to derive the following asymptotic relations at infinity of the integrand of (1.2):

$$|\Theta(z)z^{-s}| \sim B_i \mathrm{e}^{\phi_j \arg z} \left(\frac{\mathrm{e}}{|t|}\right)^{\mu|t|} \left(\frac{|z|}{\beta}\right)^{|t|} |t|^{\Re(\delta)},\ s = t + i\phi_j \in \tau_j, j = 1, 2, \tag{1.33}$$

as $t \to -\infty$;

$$|\Theta(z)z^{-s}| \sim A_j \mathrm{e}^{\phi_j \arg z} \left(\frac{\mathrm{e}}{|t|}\right)^{-\mu|t|} \left(\frac{\beta}{|z|}\right)^{t} |t|^{\Re(\delta)},\ s = t + i\phi_j \in \tau_j, j = 1, 2, \tag{1.34}$$

as $t \to +\infty$;

$$|\Theta(z)z^{-s}| \sim C_1 \exp[-\gamma \log |z| + \pi \operatorname{Im}(\nu)\, sign(t)/2]|t|^{\gamma\mu+\Re(\delta)}, \tag{1.35}$$

$$\times \exp[-\pi|t|\frac{\alpha}{2} + t \arg z],\ s = \gamma + it \in \tau_\gamma, \tag{1.36}$$

as $|t| \to \infty$. Here A_1 and A_2, B_1 and B_2 are defined in (1.24) and (1.28) with σ replaced by ϕ_1 and ϕ_2 respectively, and C_1 by (1.28) with σ replaced by γ.

The conditions for the existence of the H-function then follow as a consequence of these asymptotic relations.

Remark 1.1. Existence conditions for the H-function are given by Braaksma (1964, p. 240), Mathai (1993c) and Kilbas and Saigo (2004). The conditions described here are based on the results given by Kilbas and Saigo (1998, p. 44), also see Kilbas and Saigo (2004); which provide slight improvement over the conditions given in the theorem initially given by Prudnikov, Brychkov, and Marichev (1990, Sect. 8.3.1, p. 627).

Note 1.1. Due to the presence of the factor z^{-s} in the integrand of (1.2), the H-function is, in general, multivalued but one-valued on the Riemann surface of $\ln z$ (Braaksma 1964).

Note 1.2. The convergence of a general Mellin–Barnes integral is already given in the book by Erdélyi et al. (1953, pp. 49–50). Asymptotic estimates for the function $\Theta(\sigma + it)$ and its derivative $\Theta'(\sigma + it)$ as $|t| \to \infty$ are given by Kilbas et al. (1993).

Remark 1.2. An extension of the definition of the H-function has been given by Skibinski (1970), Inayat-Hussain (1987), and Südland et al. (1998). Definition of some of these extensions will be presented in the Appendix.

1.3 Illustrative Examples

The simplest examples of the H-function involve the exponential function, Mittag-Leffler functions (Erdélyi et al. (1955, Sect. 18.1); Mittag-Leffler (1903)), and generalized Mittag-Leffler function (Prabhakar 1971), which are directly applicable in fractional reaction, fractional relaxation and fractional reaction–diffusion problems of science and engineering. These functions will be introduced with the help of the following examples:

Example 1.1. Evaluate

$$f(z) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(s) z^{-s} \mathrm{d}s, \ (|\arg z| < \frac{1}{2}\pi; z \neq 0), \tag{1.37}$$

where the path of integration is a straight line $\Re(s) = \gamma, \gamma > 0$, lying on the right of the poles of $\Gamma(s)$ given by $s = -\nu,\ \nu = 0, 1, 2, \ldots$ and express it in terms of the H-function.

Solution 1.1. Evaluating the integral as the sum of residues we have

$$\begin{aligned} f(z) &= \sum_{\nu=o}^{\infty} \lim_{s\to-\nu} (s+\nu)\Gamma(s) z^{-s} \\ &= \sum_{\nu=0}^{\infty} \lim_{s\to-\nu} \frac{(s+\nu)(s+\nu-1)\ldots s}{(s+\nu-1)\cdots s} \Gamma(s) z^{-s} \\ &= \sum_{\nu=0}^{\infty} \lim_{s\to-\nu} \frac{\Gamma(s+\nu+1)}{(s+\nu-1)\ldots s} z^{-s} = \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{\nu!} z^{\nu} = \mathrm{e}^{-z}. \end{aligned} \tag{1.38}$$

On comparing the equation (1.37) with the definition of the H-function (1.2), we obtain the relation

$$\mathrm{e}^{-z} = H_{0,1}^{1,0}\left[z\big|_{(0,1)} \right]. \tag{1.39}$$

Note 1.3. Equation (1.37) gives the Mellin–Barnes integral for the exponential function e^{-z}. This integral is called Cahen–Mellin integral and is very useful in evaluating integrals involving product of two exponential functions or one exponential function and one special function in a compact form. This integral is also useful in the study of statistical distributions.

Example 1.2. Prove that

$$(1-z)^{-a} = \frac{1}{2\pi i\ \Gamma(a)} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(-s)\Gamma(s+a)(-z)^{s} \mathrm{d}s,\ |\arg(-z)| < \pi, \tag{1.40}$$

where $0 < \Re(\gamma) < \Re(a)$ and the contour is a straight line $\Re(s) = \gamma$, separating the poles of $\Gamma(-s)$ at the points $-s = -\nu,\ \nu = 0, 1, \ldots$ from those of $\Gamma(s+a)$ at the points $s = -a - \nu,\ \nu = 0, 1, \ldots$.

Solution 1.2. As in the preceding example, evaluating the integral as the sum of residues we have

$$\frac{1}{2\pi i\ \Gamma(a)}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(-s)\Gamma(s+a)(-z)^s\mathrm{d}s$$
$$=\frac{1}{\Gamma(a)}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu\Gamma(a+\nu)(-z)^\nu}{\nu!}=\sum_{\nu=0}^{\infty}\frac{(a)_\nu}{\nu!}z^\nu$$
$$={}_1F_0(a;\ ;z)=(1-z)^{-a},|z|<1, \tag{1.41}$$

where $(a)_k,\ a\in C, k\in N_0$, is the Pochhammer symbol or shifted factorial, defined by

$$(a)_0=1,\ (a)_k=a(a+1)\dots(a+k-1), a\neq 0$$
$$=\frac{\Gamma(a+k)}{\Gamma(a)}, \tag{1.42}$$

when $\Gamma(a)$ is defined.

The result (1.42) can be expressed in terms of the H-function as

$$(1-z)^{-a}=\frac{1}{\Gamma(a)}H_{1,1}^{1,1}\left[-z\Big|_{(0,1)}^{(1-a,1)}\right]. \tag{1.43}$$

Notation 1.2. $E_\alpha(z)$: Mittag-Leffler function (Mittag-Leffler 1903).

Definition 1.2.

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \alpha\in C, \Re(\alpha)>0, z\in C. \tag{1.44}$$

Notation 1.3. $E_{\alpha,\beta}(z)$: Generalized Mittag-Leffler function (Erdélyi et al. (1955), Sect. 18.1, Wiman (1905)).

Definition 1.3.

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \alpha,\beta\in C, \Re(\alpha)>0, \Re(\beta)>0, z\in C. \tag{1.45}$$

Note 1.4. Both the functions defined by (1.44) and (1.45) are entire functions of order.

$$\rho=\frac{1}{\alpha}\quad\text{and}\quad\text{type}\ \sigma=1.$$

Notation 1.4. $E^{\gamma}_{\alpha,\beta}(z)$: Generalized Mittag-Leffler function.

Definition 1.4.

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\alpha k + \beta) k!}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, z \in C. \tag{1.46}$$

This function is also an entire function with $\rho = \frac{1}{\Re(\alpha)}$, see Prabhakar (1971).

Example 1.3. Evaluate the Mellin–Barnes integral

$$f(z) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1-\alpha s)} (-z)^{-s} \mathrm{d}s, |\arg z| < \pi, \tag{1.47}$$

where $\alpha \in R^+$ and show that $f(z)$ is the Mittag-Leffler function $E_\alpha(z)$ defined by the series (1.44).

Solution 1.3. We have

$$\begin{aligned} f(z) &= \sum_{\nu=0}^{\infty} \lim_{s \to -\nu} \frac{(s+\nu)\Gamma(s)\Gamma(1-s)}{\Gamma(1-\alpha s)} (-z)^{-s} = \sum_{\nu=0}^{\infty} \frac{z^\nu}{\Gamma(\alpha\nu + 1)} \\ &= E_\alpha(z) = H^{1,1}_{1,2}\left[-z\Big|^{(0,1)}_{(0,1),(0,\alpha)}\right], \end{aligned} \tag{1.48}$$

on comparing the results (1.2) and (1.48).

Example 1.4. Establish the Mellin–Barnes integral

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-\alpha s)} (-z)^{-s} \mathrm{d}s, \ |\arg z| < \pi, \tag{1.49}$$

where $\alpha \in R^+, \beta \in C, \Re(\beta) > 0$ and $E_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function defined by the series (1.45).

Solution 1.4. Evaluating the contour integral as a sum of residues, we find that

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma - \infty}^{\gamma + i\infty} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-\alpha s)} (-z)^{-s} \mathrm{d}s &= \sum_{\nu=0}^{\infty} \lim_{s \to -\nu} \frac{(s+\nu)\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-\alpha s)} (-z)^{-s} \\ &= \sum_{\nu=0}^{\infty} \frac{z^\nu}{\Gamma(\alpha\nu+\beta)} = E_{\alpha,\beta}(z) \\ &= H^{1,1}_{1,2}\left[-z\Big|^{(0,1)}_{(0,1),(1-\beta,\alpha)}\right], \end{aligned} \tag{1.50}$$

where we have used the definition of the generalized Mittag-Leffler function (1.45) and the definition of the H-function (1.2).

In a similar manner, we can prove the next example.

Example 1.5. Prove that the generalized Mittag-Leffler function $E^{\gamma}_{\alpha,\beta}(z)$ defined by (1.46) is represented as a Mellin–Barnes integral in the form

$$E^{\gamma}_{\alpha,\beta}(z) = \frac{1}{2\pi i\ \Gamma(\gamma)} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\Gamma(\gamma-s)}{\Gamma(\beta-\alpha s)}(-z)^{-s}\mathrm{d}s,\ |\arg z| < \pi, \tag{1.51}$$

where $\alpha \in R^{+}, \beta, \gamma \in C, \Re(\beta) > 0, \gamma \neq 0, -1, -2, \ldots$

Solution 1.5. Proceed as in Solution 1.4 to establish the result.

Note 1.5. Applications of the generalized Mittag-Leffler function $E^{\gamma}_{\alpha,\beta}(z)$ in finite-size scaling in anisotropic systems can be found in the papers by Tonchev (2005, 2007) and Chamati and Tonchev (2006). This function is studied by Prabhakar (1971), Kilbas et al. (2002, 2004) and Saxena and Saigo (2005).

Example 1.6. Evaluate the following reaction rate integral of physics in terms of the H-function.

$$I(a,b,c;\rho) = \int_0^{\infty} t^{a-1}\exp(-bt - ct^{-\rho})\mathrm{d}t, \tag{1.52}$$

where $a, b, c > 0$.

Solution 1.6. Expressing the right hand side of the above expression with the help of the convolution property of the Mellin transform and then taking the inverse Mellin transform one has

$$\begin{aligned}\int_0^{\infty} t^{a-1}\exp(-bt - ct^{-\rho})\mathrm{d}t &= \frac{1}{\rho b^a}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(\alpha+s)\Gamma\left(\frac{s}{\rho}\right)(bc^{\frac{1}{\rho}})^{-s}\mathrm{d}s \\ &= \frac{1}{\rho b^a}H^{2,0}_{0,2}\left[bc^{\frac{1}{\rho}}\big|_{(0,1),(0,\frac{1}{\rho})}\right].\end{aligned} \tag{1.53}$$

Remark 1.3. The integral of this example defines the Krätzel function (Krätzel 1979). For a detailed account of this function, the reader may consult the book by Kilbas and Saigo (2004). Further, this integral is useful in the study of nuclear reaction rates in astrophysics, see Anderson et al. (1994), Haubold and Mathai (1986), Mathai and Haubold (1988) and Saxena et al. (2004), etc.

Following a similar procedure, it is not difficult to prove the next example.

Example 1.7. Prove that the Mellin–Barnes integral

$$J_{\nu}(z) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(s)}{\Gamma(1+\nu-s)}\left(\frac{1}{2}z\right)^{\nu-2s}\mathrm{d}s, \gamma > 0, \tag{1.54}$$

defines the Bessel function of the first kind, $J_\nu(z)$, defined by

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1+\nu+k)k!} \left(\frac{z}{2}\right)^{\nu+2k}. \tag{1.55}$$

1.4 Some Identities of the H-Function

This section deals with certain basic properties of the H-function. Many authors have investigated various properties of this function, and the researches carried out by Braaksma (1964), Gupta (1965), Gupta and Jain (1966, 1968, 1969), Bajpai (1969a), Lawrynowicz (1969), Anandani (1969a, 1969b), Kilbas and Saigo (2004), Chaurasia (1976b) and Skibinski (1970) will be discussed here.

The results of this section follow as a consequence of the definition of the H-function (1.2) by the application of certain properties of gamma functions, hence their proofs are omitted.

Property 1.1. *The H-function is symmetric in the pairs $(a_1, A_1), \ldots, (a_n, A_n)$, likewise $(a_{n+1}, A_{n+1}), \ldots, (a_p, A_p)$; in $(b_1, B_1), \ldots, (b_m, B_m)$ and in $(b_{m+1}, B_{m+1}), \ldots, (b_q, B_q)$.*

Property 1.2. *If one of the (a_j, A_j), $j = 1, \ldots, n$ is equal to one of the (b_j, B_j), $j = m+1, \ldots, q$ or one of the (b_j, B_j), $j = 1, \ldots, m$ is equal to one of the (a_j, A_j), $j = n+1, \ldots, p$ then the H-function reduces to one of the lower order p and q, and n (or m) decrease by unity.*

Thus we have the following reduction formulae:

$$H_{p,q}^{m,n}\left[z\middle|_{(b_1,B_1),\ldots,(b_{q-1},B_{q-1}),(a_1,A_1)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right] = H_{p-1,q-1}^{m,n-1}\left[z\middle|_{(b_1,B_1),\ldots,(b_{q-1},B_{q-1})}^{(a_2,A_2),\ldots,(a_p,A_p)}\right], \tag{1.56}$$

provided $n \geq 1$ and $q > m$; and

$$H_{p,q}^{m,n}\left[z\middle|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_{p-1},A_{p-1}),(b_1,B_1)}\right] = H_{p-1,q-1}^{m-1,n}\left[z\middle|_{(b_2,B_2),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_{p-1},A_{p-1})}\right], \tag{1.57}$$

provided $m \geq 1$ and $p > n$.

Property 1.3. *There holds the formula:*

$$H_{p,q}^{m,n}\left[z\middle|_{(b_q,B_q)}^{(a_p,A_p)}\right] = H_{q,p}^{n,m}\left[\frac{1}{z}\middle|_{(1-a_p,A_p)}^{(1-b_q,B_q)}\right]. \tag{1.58}$$

This is an important property of the H-function because it enables us to transform a H-function with $\mu = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0$ and arg$z$ *to one with $\mu < 0$ and* arg$\frac{1}{z}$ *and vice versa. It also helps in deducing the asymptotic expansion for the H-function for the case $\mu < 0$ from the given result for this function for $\mu > 0$ and vice versa.*

Property 1.4. *The following result holds:*

$$H_{p,q}^{m,n}\left[z\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = k\ H_{p,q}^{m,n}\left[z^k\Big|_{(b_q,kB_q)}^{(a_p,kA_p)}\right], \tag{1.59}$$

where $k > 0$.

Property 1.5. *There holds the formula*

$$z^{\sigma} H_{p,q}^{m,n}\left[z\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = H_{p,q}^{m,n}\left[z\Big|_{(b_q+\sigma B_q,B_q)}^{(a_p+\sigma A_p,A_p)}\right], \tag{1.60}$$

where $\sigma \in C$.

Property 1.6. *The following relation holds:*

$$H_{p+1,q+1}^{m,n+1}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q),(r,\gamma)}^{(0,\gamma),(a_1,A_1),\ldots,(a_p,A_p)}\right] = (-1)^r H_{p+1,q+1}^{m+1,n}\left[z\Big|_{(r,\gamma),(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p),(0,\gamma)}\right], \tag{1.61}$$

where $p \le q, \gamma > 0$.

Property 1.7. *The following relation holds:*

$$H_{p+1,q+1}^{m+1,n}\left[z\Big|_{(1,\gamma),(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p),(1-r,\gamma)}\right]=(-1)^r H_{p+1,q+1}^{m,n+1}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q),(1,\gamma)}^{(1-r,\gamma),(a_1,A_1),\ldots,(a_p,A_p)}\right], \tag{1.62}$$

where $p \le q, \gamma > 0$.

Note 1.6. In the above results (1.58) to (1.62), the branches of the H-function are suitably chosen.

Property 1.8. *The multiplication formula for the H-function is given by:*

$$H_{p,q}^{m,n}\left[z\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = (2\pi)^{(1-t)c^*} t^{\delta+1} H_{tp,tq}^{tm,tn}\left[(zt^{-\mu})^t\Big|_{(\Delta(t,b_q),B_q)}^{(\Delta(t,a_p),A_p)}\right], \tag{1.63}$$

where t is a positive integer, μ, δ and c^* are defined in (1.9), (1.10), and (1.22) respectively, and $(\Delta(t,\delta_r),\gamma_r)$ represents the sequence of parameters

$$\left(\frac{\delta_r}{t},\gamma_r\right), \left(\frac{\delta_r+1}{t},\gamma_r\right), \ldots, \left(\frac{\delta_r+t-1}{t},\gamma_r\right). \tag{1.64}$$

For similar results see Gupta and Jain (1969). The following properties of the H-function follow from the definition itself.

Property 1.9. *For $a, b, c \in C$, there holds the formulae:*

$$H_{p,q}^{m,n}\left[z\Big|_{(b_q,B_q)}^{(a,0),(a_2,A_2),\ldots,(a_p,A_p)}\right] = \Gamma(1-a)H_{p-1,q}^{m,n-1}\left[z\Big|_{(b_q,B_q)}^{(a_2,A_2),\ldots,(a_p,A_p)}\right], \quad (1.65)$$

where $\Re(a) < 1$ and $n \geq 1$;

$$H_{p,q}^{m,n}\left[z\Big|_{(b,0),(b_2,B_2),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right] = \Gamma(b)H_{p,q-1}^{m-1,n}\left[z\Big|_{(b_2,B_2),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right], \quad (1.66)$$

where $\Re(b) > 0$ and $m \geq 1$;

$$H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_{p-1},A_{p-1}),(a,0)}\right] = \frac{1}{\Gamma(a)}H_{p-1,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_{p-1},A_{p-1})}\right], \quad (1.67)$$

where $\Re(a) > 0$ and $p > n$.

$$H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_{q-1},B_{q-1}),(b,0)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right] = \frac{1}{\Gamma(1-b)}H_{p,q-1}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_{q-1},B_{q-1})}^{(a_1,A_1),\ldots,(a_p,A_p)}\right], \quad (1.68)$$

where $\Re(b) < 1$ and $q > m$.

1.4.1 Derivatives of the H-Function

The following formulas immediately follow from the definition of the H-function and are useful in the study of fractional integrals and derivatives of the H-function.

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \left\{z^{\rho-1}H_{p,q}^{m,n}\left[az^{\sigma}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right\} = z^{\rho-n-1}H_{p+1,q+1}^{m,n+1}\left[az\Big|_{(b_q,B_q),(1-\rho+n,\sigma)}^{(1-\rho,\sigma),(a_p,A_p)}\right]$$
$$= (-1)^n z^{\rho-n-1}H_{p+1,q+1}^{m+1,n}\left[az^{\sigma}\Big|_{(1-\rho+n,\sigma),(b_q,B_q)}^{(a_p,A_p),(1-\rho,\sigma)}\right], \quad (1.69)$$

where $a, \sigma \in C, \sigma > 0$.

Lawrynowich (1969) has given the following four formulae for the successive derivatives of the H-function:

$$\frac{\mathrm{d}^r}{\mathrm{d}z^r}\left\{z^{-(\gamma\frac{b_1}{B_1})}H_{p,q}^{m,n}\left[z^{\gamma}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$= \left(-\frac{\gamma}{B_1}\right)^r z^{-(r+\gamma\frac{b_1}{B_1})}H_{p,q}^{m,n}\left[z^{\gamma}\Big|_{(r+b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right], \quad (1.70)$$

where $m \geq 1, \gamma = B_1$ for $r > 1$;

$$\frac{\mathrm{d}^r}{\mathrm{d}z^r}\left\{z^{-(\gamma\frac{b_q}{B_q})}H_{p,q}^{m,n}\left[z^{\gamma}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$= \left(\frac{\gamma}{B_q}\right)^r z^{-(r+\gamma\frac{b_q}{B_q})}H_{p,q}^{m,n}\left[z^{\gamma}\Big|_{(b_1,B_1),\ldots,(b_{q-1},B_{q-1}),(r+b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right], \quad (1.71)$$

where $m < q, \gamma = B_q$ for $r > 1$;

$$\frac{\mathrm{d}^r}{\mathrm{d}z^r}\left\{z^{-(\gamma\frac{(1-a_1)}{A_1})}H_{p,q}^{m,n}\left[z^{-\gamma}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$= \left(-\frac{\gamma}{A_1}\right)^r z^{-(r+\gamma\frac{(1-a_1)}{A_1})}H_{p,q}^{m,n}\left[z^{-\gamma}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1-r,A_1),\ldots,(a_p,A_p)}\right], \quad (1.72)$$

where $n \geq 1, \gamma = A_1$ for $r > 1$;

$$\frac{\mathrm{d}^r}{\mathrm{d}z^r}\left\{z^{-(\gamma\frac{(1-a_p)}{A_p})}H_{p,q}^{m,n}\left[z^{-\gamma}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$= \left(\frac{\gamma}{A_p}\right)^r z^{-(r+\gamma\frac{(1-a_p)}{A_p})}H_{p,q}^{m,n}\left[z^{-\gamma}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_{p-1},A_{p-1}),(a_p-r,A_p)}\right], \quad (1.73)$$

where $p > n, \gamma = A_p$ for $r > 1$.

The results (1.70) to (1.73) for $r = 1$ are immediate consequences of the differential formulae given by Anandani (1969a).

Remark 1.4. The results of Lawrynowicz cited above are in a compact form and are convenient for practical application.

Next we give three-term differentiation formulae for the H-function.

$$z\frac{\mathrm{d}}{\mathrm{d}z}\left\{H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$= \frac{\eta(a_1-1)}{A_1}\left\{H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$+ \frac{\eta}{A_1}H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1-1,A_1),(a_2,A_2),\ldots,(a_p,A_p)}\right], \quad (1.74)$$

where $n \geq 1$;

$$z\frac{\mathrm{d}}{\mathrm{d}z}\left\{H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$= \frac{\eta(a_p-1)}{A_p}\left\{H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$- \frac{\eta}{A_p}H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_{p-1},A_{p-1}),(a_p-1,A_p)}\right], \quad (1.75)$$

where $n \leq p-1$;

$$z\frac{\mathrm{d}}{\mathrm{d}z}\left\{H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$= \frac{\eta b_1}{B_1}\left\{H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$- \frac{\eta}{B_1}H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(1+b_1,B_1),(b_2,B_2),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right], \tag{1.76}$$

where $m \geq 1$;

$$z\frac{\mathrm{d}}{\mathrm{d}z}\left\{H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$= \frac{\eta b_q}{B_q}\left\{H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$+ \frac{\eta}{B_q}H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(b_1,B_1),\ldots,(b_{q-1},B_{q-1}),(b_q+1,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right], \tag{1.77}$$

where $m \leq q-1$.

The above results can be proved with the help of the following formulae:

$$-A_1 s\Gamma(1-a_1-A_1 s) = (a_1-1)\Gamma(1-a_1-A_1 s) + \Gamma(2-a_1-A_1 s), \tag{1.78}$$
$$-\frac{A_p s}{\Gamma(a_p+A_p s)} = \frac{a_p-1}{\Gamma(a_p+A_p s)} - \frac{1}{\Gamma(a_p-1+A_p s)}, \tag{1.79}$$
$$-B_1 s\Gamma(b_1+B_1 s) = b_1\Gamma(b_1+B_1 s) - \Gamma(1+b_1+B_1 s), \tag{1.80}$$

and

$$-\frac{B_q s}{\Gamma(1-b_q-B_q s)} = \frac{b_q}{\Gamma(1-b_q-B_q s)} + \frac{1}{\Gamma(-b_q-B_q s)}, \tag{1.81}$$

which readily follow from the property of the gamma function

$$\Gamma(z+1) = z\Gamma(z). \tag{1.82}$$

Nair (1972, 1973) has given four formulae for the derivative of the H-function. His results are the extensions of the formulae proved earlier by Gupta and Jain (1968). One of the formulae proved by Nair (1972) is the following:

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}-c_1\right)\cdots\left(x\frac{\mathrm{d}}{\mathrm{d}x}-c_r\right)\left\{x^s H_{p,q}^{m,n}\left[zx^h\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$= x^s H_{p+r,q+r}^{m,n+r}\left[zx^h\Big|_{(b_1,B_1),\ldots,(b_q,B_q),(c_1-s+1,h),\ldots,(c_r-s+1,h)}^{(c_1-s,h),\ldots,(c_r-s,h),(a_1,A_1),\ldots,(a_p,A_p)}\right], \tag{1.83}$$

where $h > 0$.

When $c_1 = c_2 = \cdots = c_r = 0$, (1.83) reduces to a result due to Gupta and Jain (1968, p. 191). Oliver and Kalla (1971) have derived four differentiation formulae for the H-function which extend the results of Anandani (1970c), which itself are the generalization of the results due to Goyal and Goyal (1967a). One of the results proved by Oliver and Kalla is the following:

$$\frac{\mathrm{d}^r}{\mathrm{d}x^r}\left\{H_{p,q}^{m,n}\left[(cx+d)^h\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$= \frac{c^r}{(cx+d)^r}H_{p+1,q+1}^{m,n+1}\left[(cx+d)^h\Big|_{(b_1,B_1),\ldots,(b_q,B_q),(r,h)}^{(0,h),(a_1,A_1),\ldots,(a_p,A_p)}\right], \tag{1.84}$$

where c and d are complex numbers and h is real and positive.

Note 1.7. We note that partial derivatives of the H-function with respect to the parameters are investigated by Buschman (1974b).

1.5 Recurrence Relations for the H-Function

Gupta (1965) has obtained four recurrence formulae for the H-function by the method of integral transforms due to Meijer (1940, 1941). One of his results is given below.

$$(a_1-a_2)H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right] = H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),(a_2-1,A_1),(a_3,A_3),\ldots,(a_p,A_p)}\right]$$
$$- H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1-1,A_1),(a_2,A_2),\ldots,(a_p,A_p)}\right], \tag{1.85}$$

where $n \geq 2$.

Anandani (1989) has given six recurrence relations for the H-function which follow as a consequence of the definition of the H-function (1.2). Two such results are enumerated below:

$$(b_1A_1 - a_1B_1 + B_1)H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]$$
$$= B_1H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1-1,A_1),(a_2,A_2),\ldots,(a_p,A_p)}\right]$$
$$+ A_1H_{p,q}^{m,n}\left[z\Big|_{(1+b_1,B_1),(b_2,B_2),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right], \tag{1.86}$$

where $m, n \geq 1$;

$$(b_q A_q - a_q B_q + B_q) H_{p,q}^{m,n} \left[z \Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)} \right] = B_q H_{p,q}^{m,n} \left[z \Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_q-1,A_q),(a_2,A_2),\ldots,(a_p,A_p)} \right] - A_q H_{p,q}^{m,n} \left[z \Big|_{(b_1,B_1),\ldots,(b_{q-1},B_{q-1}),(b_q+1,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)} \right], \tag{1.87}$$

where $n \geq 1, 1 \leq m \leq q - 1$.

For further results on recurrence relations of the H-function, see the work of Bora and Kalla (1971a), Jain (1967), Srivastava and Gupta (1970, 1971), Raina (1976), and Raina and Koul (1977). A set of contiguous relations for the H-function are given by Buschman (1974b).

1.6 Expansion Formulae for the H-Function

Expansion formulae for the H-function are given by Lawrynowich (1969), Raina (1979), and Kilbas and Saigo (2004). The four expansion formulae for the G-function due to Meijer (1941a) have been extended to H-functions by Lawrynowicz (1969) by using a method analogous to the one adopted by Meijer (1941a) for the G-function. The results are the following:

(i) Let m, n, p, and q be nonnegative integers such that $1 \leq m \leq q,\ 0 \leq n \leq p$. Further, let $A_j,\ j = 1, \ldots, p$ and $B_j,\ j = 1, \ldots, q$ be positive numbers and $a_j,\ j = 1, \ldots, p$ and $b_j,\ j = 1, \ldots, q$ be complex numbers satisfying the condition (1.6) and $\mu > 0$, where μ is defined in (1.9). Then if ω and η are complex numbers such that $\omega \neq 0$ and $\eta \neq 0$, then the following results hold:

$$H_{p,q}^{m,n} \left[\eta\omega \Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)} \right] = \eta^{\frac{b_1}{B_1}} \sum_{r=0}^{\infty} \frac{(1 - \eta^{\frac{1}{B_1}})^r}{r!} H_{p,q}^{m,n} \left[\omega \Big|_{(r+b_1,B_1),(b_2,B_2),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)} \right], \tag{1.88}$$

where η is arbitrary for $m = 1$, and for $m > 1$, $|\eta^{\frac{1}{B_1}} - 1| < 1$, $\arg(\eta\omega) = B_1 \arg(\eta^{\frac{1}{B_1}}) + \arg\omega$ and $|\arg(\eta^{\frac{1}{B_1}})| < \frac{\pi}{2}$;

$$H_{p,q}^{m,n} \left[\eta\omega \Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)} \right] = \eta^{\left(\frac{b_q}{B_q}\right)} \sum_{r=0}^{\infty} \frac{(\eta^{\frac{1}{B_q}} - 1)^r}{r!} H_{p,q}^{m,n} \left[\omega \Big|_{(b_1,B_1),\ldots,(b_{q-1},B_{q-1}),(r+b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)} \right], \tag{1.89}$$

where $q > m, |\eta^{\frac{1}{B_q}} - 1| < 1$ $\arg(\eta\omega) = B_q \arg(\eta^{\frac{1}{B_q}}) + \arg\omega$, and $|\arg(\eta^{\frac{1}{B_q}}| < \frac{\pi}{2}$;

$$H^{m,n}_{p,q}\left[z\Big|^{(a_1,A_1),\ldots,(a_p,A_p)}_{(b_1,B_1),\ldots,(b_q,B_q)}\right] = \eta^{\frac{(a_1-1)}{A_1}} \sum_{r=0}^{\infty} \frac{(1-\eta^{-\frac{1}{A_1}})^r}{r!} H^{m,n}_{p,q}\left[\omega\Big|^{(a_1-r,A_1),(a_2,A_2),\ldots,(a_p,A_p)}_{(b_1,B_1),\ldots(b_q,B_q)}\right], \tag{1.90}$$

where $n > 0, \Re(\eta^{\frac{1}{A_1}}) > \frac{1}{2}$, $\arg(\eta\omega) = A_1 \arg(\eta^{\frac{1}{A_1}}) + \arg\omega$ and $|\arg(\eta^{\frac{1}{A_1}})| < \frac{\pi}{2}$;

$$H^{m,n}_{p,q}\left[\eta\omega\Big|^{(a_1,A_1),\ldots,(a_p,A_p)}_{(b_1,B_1),\ldots,(b_q,B_q)}\right] = \eta^{\frac{(a_p-1)}{A_p}} \sum_{r=0}^{\infty} \frac{(\eta^{-\frac{1}{A_p}}-1)^r}{r!} H^{m,n}_{p,q}\left[\omega\Big|^{(a_1,A_1),\ldots,(a_{p-1},A_{p-1}),(a_p-r,A_p)}_{(b_1,B_1),\ldots,(b_q,B_q)}\right], \tag{1.91}$$

where $p > n, \Re(\eta^{\frac{1}{A_p}}) > \frac{1}{2}$, $\arg(\eta\omega) = A_p \arg(\eta^{\frac{1}{A_p}}) + \arg\omega$ and $|\arg(\eta^{\frac{1}{A_p}})| < \frac{\pi}{2}$. By virtue of the following transformation formula for the Gauss hypergeometric function (Erdélyi et al. 1953, 2.10(1))

$$\begin{aligned} {}_2F_1(a,b;c;z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a,b;a+b-c+1;1-z) \\ &\quad + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} {}_2F_1(c-a,c-b;c-a-b;1-z), \end{aligned} \tag{1.92}$$

for $|\arg(1-z)| < \pi$ we find that

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n H^{m+1,n}_{p+1,q+1}\left[z\Big|^{(a_p,A_p),(c+n,\gamma)}_{(b+n,\gamma),(b_q,B_q)}\right] \\ &= \frac{\Gamma(c-a-b)}{\Gamma(c-b)} \sum_{n=0}^{\infty} \frac{(a)_n}{(a+b-c+1)_n} \frac{(1-z)^n}{n!} H^{m+1,n}_{p+1,q+1}\left[z\Big|^{(a_p,A_p),(c-a,\gamma)}_{(c+n,\gamma),(b_q,B_q)}\right] \\ &\quad + \frac{\Gamma(a+b-c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{(c-b)_n}{(c-a-b+1)_n} \frac{(1-z)^{c-a-b+n}}{n!} \\ &\quad \times H^{m+1,n}_{p+1,q+1}\left[z\Big|^{(a_p,A_p),(b,\gamma)}_{(c+n,\gamma),(b_q,B_q)}\right], \end{aligned} \tag{1.93}$$

where $a, b, c \in C, \gamma > 0, |\arg(1-z)| < \pi, \Re(c-a-b) > 0$ if $z = 1$.

1.7 Asymptotic Expansions

The behavior of the H-function for small and large values of the argument has been discussed by Braaksma (1964) in detail. Explicit power and power-logarithmic series expansions for the H-fucntion are given by Kilbas and Saigo (1999, 2004). In this section we present some of their results which are useful in applied problems. Asymptotic expansions of the H-function are discussed by Dixon and Ferrar (1936). Convergence of the Mellin–Barnes integrals are recently discussed by Paris and Kaminski (2001, p. 63).

Theorem 1.2. *Let α and μ be as given in (1.13) and (1.9) and let the condition (1.6) be satisfied. Then there holds the following results:*

(i) If $\mu \geq 0$ or $\mu < 0, \alpha > 0, |\arg z| < \frac{1}{2}\pi\alpha$ then the H-function has either the asymptotic expansion at zero given by

$$H_{p,q}^{m,n}(z) = O(z^c),\ |z| \to 0,\ \textit{or} \tag{1.94}$$

$$H_{p,q}^{m,n}(z) = O(z^c|\ln(z)|^{N-1}),\ |z| \to 0. \tag{1.95}$$

Here,

$$c = \min_{1\leq j\leq m}\left[\frac{\Re(b_j)}{B_j}\right], \tag{1.96}$$

and N is the order of the poles $\zeta_{j\nu}$ in (1.4) to which some other poles of $\Gamma(b_j + B_j s),\ j = 1,\ldots,m$ coincide. Also for $\mu < 0, \alpha = 0$

$$H_{p,q}^{m,n}(z) = O(z^\sigma),\ |z| \to 0, |\arg(z) \leq \epsilon^*, \tag{1.97}$$

$$\sigma = \min_{1\leq j\leq m}\left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta) + \frac{1}{2}}{\mu}\right], \tag{1.98}$$

and ϵ^* is a constant such that

$$0 < \epsilon^* < \frac{\pi}{2}\min_{1\leq j\leq m; m+1\leq k\leq q}(A_j, B_k). \tag{1.99}$$

(ii) If $\mu \leq 0$ or $\mu > 0, \alpha > 0$ then the H-function has either the asymptotic expansion at infinity given by

$$H_{p,q}^{m,n}(z) = O(z^d),\ |z| \to \infty,\ \textit{or} \tag{1.100}$$

$$H_{p,q}^{m,n}(z) = O(z^d|\ln(z)|^{M-1}),\ |z| \to \infty, \tag{1.101}$$

$$d = \min_{1\leq j\leq n}\left[\frac{\Re(a_j) - 1}{A_j}\right], \tag{1.102}$$

and M is the order of the poles $\omega_{\lambda k}$ in (1.5) to which some of the poles of $\Gamma(1-a_j-A_js),\ j=1,\dots,n$ coincide. Also for $\mu>0,\alpha=0$

$$H_{p,q}^{m,n}(z)=O(z^{\rho}),\ |z|\to\infty,|\arg(z)|\le\epsilon, \tag{1.103}$$

$$\rho=\max_{1\le j\le n}\left[\frac{\Re(a_j)-1}{A_j},\frac{\Re(\delta)+\frac{1}{2}}{\mu}\right], \tag{1.104}$$

and ϵ is a constant such that

$$0<\epsilon<\frac{\pi}{2}\min_{n+1\le j\le p;1\le k\le m}(A_j,B_k). \tag{1.105}$$

For $n=0$ the H-function, for real argument x, vanishes exponentially for large x in certain cases. The case $m=0$ is also discussed. Let

$$\tau=\sum_{j=1}^{m}B_j-\sum_{j=n+1}^{p}A_j. \tag{1.106}$$

Theorem 1.3.

(i) Let $n=0,\alpha,\beta,\mu,\delta$ and τ be given by (1.13), (1.8), (1.9), (1.10), and (1.106) respectively. Further, let $\mu>0,\alpha\ge 0,\epsilon$ be a constant such that $0<\epsilon<\frac{\pi\mu}{2}$, and the condition (1.6) and $A_j(1-a_i+k)\neq A_i(1-a_j+\lambda),i\neq j,j=1,\dots,n;k,\lambda\in N_0$ are satisfied then for real x there holds the following assertion: We have

$$H_{p,q}^{m,0}(x)=O\left(x^{[\Re(\delta)+\frac{1}{2}]/\mu}\right)\exp\left[\cos\left(\frac{\tau\pi}{\mu}\right)\mu\beta^{-\frac{1}{\mu}}x^{\frac{1}{\mu}}\right],\ x\to\infty. \tag{1.107}$$

In particular,

$$H_{p,q}^{q,0}(x)=O\left(x^{[\Re(\delta)+\frac{1}{2}]/\mu}\right)\exp\left[-\mu\beta^{-\frac{1}{\mu}}x^{\frac{1}{\mu}}\right],\ x\to\infty. \tag{1.108}$$

(ii) Let $m=0,\alpha,\beta,\mu,$ and δ be given by (1.13),(1.8),(1.9) and (1.10) respectively. Further, let $\mu<0,\alpha\ge 0;\epsilon^*$ be a constant such that $0<\epsilon^*<\frac{\pi|\mu|}{2}$, and the condition (1.6) and $B_j(b_i+k)\neq B_i(b_j+\lambda),i\neq j;i,j=1,\dots,m;k,\lambda\in N_0$ are satisfied. Then for real x there holds the following assertion: We have

$$H_{p,q}^{0,n}(x)=O\left(x^{-[\Re(\delta)+\frac{1}{2}]/|\mu|}\right)\exp\left[\cos\left(\frac{\zeta\pi}{|\mu|}\right)|\mu|\beta^{\frac{1}{|\mu|}}x^{-\frac{1}{|\mu|}}\right],x\to 0+, \tag{1.109}$$

$$\zeta=\sum_{j=1}^{n}A_j-\sum_{j=m+1}^{q}B_j.$$

In particular,

$$H_{p,q}^{0,p}(x) = O\left(x^{-[\Re(\delta)+\frac{1}{2}]/|\mu|}\right)\exp\left[-|\mu|\beta^{\frac{1}{|\mu|}}x^{-\frac{1}{|\mu|}}\right],\ x \to 0+. \quad (1.110)$$

Remark 1.5. Power logarithmic expansions in particular cases of the H-function $H_{0,p}^{p,0}$ and $H_{p,p}^{p,0}$ are investigated by Mathai (1973).

1.8 Some Special Cases of the H-Function

Notation 1.5.

$$G(z) = G_{p,q}^{m,n}(z) = G_{p,q}^{m,n}\left(z\Big|_{b_q}^{a_p}\right) = G_{p,q}^{m,n}\left(z\Big|_{b_1,\ldots,b_q}^{a_1,\ldots,a_p}\right):\ \text{Meijer's } G\text{-function}$$

or the G-function. (1.111)

Definition 1.5.

$$G(z) = G_{p,q}^{m,n}(z) = G_{p,q}^{m,n}\left(z\Big|_{b_q}^{a_p}\right) = G_{p,q}^{m,n}\left(z\Big|_{b_1,\ldots,b_q}^{a_1,\ldots,a_p}\right)$$

$$= \frac{1}{2\pi i}\int_L \frac{\left\{\prod_{j=1}^m \Gamma(b_j+s)\right\}\left\{\prod_{j=1}^n \Gamma(1-a_j-s)\right\}}{\left\{\prod_{j=m+1}^q \Gamma(1-b_j-s)\right\}\left\{\prod_{j=n+1}^p \Gamma(a_j+s)\right\}} z^{-s}\mathrm{d}s, \quad (1.112)$$

where $0 \le m \le q, 0 \le n \le q; a_j, j = 1,\ldots,p$ and $b_j, j = 1,\ldots,q$ are complex numbers and are such that

$$a_j - b_h \ne 0,1,\ldots; j = 1,\ldots,n; h = 1,\ldots,m. \quad (1.113)$$

The parameters are such that the points

$$s = -(b_j + v), j = 1,\ldots,m; v \in N_0, \quad (1.114)$$

and

$$s = -(a_j - v - 1), j = 1,\ldots,n; v \in N_0, \quad (1.115)$$

are separated. Here L is the same contour taken for the H-function defined by (1.2).

A detailed and comprehensive account of the theory and applications of the G-function is available from the monographs written by Erdélyi et al. (1953, Sects. 5.3–5.6), Luke (1969), Mathai and Saxena (1973), Mathai (1993c), Prudnikov et al. (1990, Sects. 8.2 and 8.4). The G-function itself is a generalization of a number of known special functions occurring in applied mathematics and mathematical physics. Special cases of the G-function can be found in Erdélyi et al. (1953, Sect. 5.6), Luke (1969, Sects. 6.4, 6.5), Mathai and Saxena (1973a, Chap. II), and Mathai (1993c).

Notation 1.6. ${}_pF_q(z) = {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)$: Generalized hypergeometric series.

Definition 1.6.

$$ {}_pF_q(z) = {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) = \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!}, \tag{1.116} $$

where $(a)_k$ is the Pochhammer symbol defined in (1.42); $a_i, b_j \in C, i = 1,\ldots,p$; $j = 1,\ldots,q; b_j \neq -\nu, \nu \in N_0$.

Notation 1.7. $E(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z)$: MacRobert's E-function (Erdélyi et al. 1953, p. 203).

Definition 1.7.

$$ \begin{aligned} E(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z) &= G^{p,1}_{q+1,p}\left[z\Big|^{1,\beta_1,\ldots,\beta_q}_{\alpha_1,\ldots,\alpha_p}\right] \\ &= \frac{1}{2\pi i}\int_L \frac{\Gamma(-s)\prod_{j=1}^{p}\Gamma(\alpha_j+s)}{\prod_{j=1}^{q}\Gamma(\beta_j+s)} z^{-s}\mathrm{d}s. \end{aligned} \tag{1.117} $$

Notation 1.8. $J_\nu^\mu(z)$: Bessel–Maitland function or Maitland–Bessel function (Marichev, 1982, Eq. (8.3)).

Definition 1.8.

$$ J_\nu^\mu(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma(\nu+n\mu+1)\,n!}. \tag{1.118} $$

Notation 1.9. $J_{\nu,\lambda}^\mu(z)$: Generalized Bessel–Maitland function (Marichev 1983, (8.2)).

Definition 1.9.

$$ J_{\nu,\lambda}^\mu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\nu+n\mu+\lambda+1)\Gamma(n+\lambda+1)}\left(\frac{z}{2}\right)^{\nu+2\lambda+2n}. \tag{1.119} $$

Notation 1.10. $Z_\rho^\nu(z)$: Krätzel function (Krätzel 1979).

Definition 1.10.

$$ Z_\rho^\nu(z) = \int_0^\infty t^{\nu-1}\exp\left[-t^\rho - \frac{z}{t}\right]\mathrm{d}t,\ \nu \in C, \rho > 0, \Re(z) > 0. \tag{1.120} $$

Notation 1.11. $K_\nu(z)$: Modified Bessel function of the third kind or Macdonald function, see also Sect. 1.8.1.

Definition 1.11.

$$K_\nu(z) = \frac{1}{2}\int_0^\infty \exp[-\frac{z}{2}(t+\frac{1}{t})]t^{-\nu-1}\mathrm{d}t, \Re\nu < \frac{1}{2}, \tag{1.121}$$

$$= \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}-\nu\right)}\left(\frac{2}{z}\right)^\nu \int_1^\infty \mathrm{e}^{-zt}(t^2-1)^{-\nu-\frac{1}{2}}\mathrm{d}t,\ \Re(z) > 0, \tag{1.122}$$

see Sect. 1.8.1 for more details.

Notation 1.12.

$${}_p\Psi_q(z) = {}_p\Psi_q\left[z\Big|^{(a_p,A_p)}_{(b_q,B_q)}\right]$$: Wright generalized hypergeometric function (Wright (1935)).

Definition 1.12.

$${}_p\Psi_q\left[z\Big|^{(a_p,A_p)}_{(b_q,B_q)}\right] = \sum_{n=0}^\infty \frac{\prod_{j=1}^p \Gamma(a_j+nA_j)}{\prod_{j=1}^q \Gamma(b_j+nB_j)}\frac{z^n}{n!}, \tag{1.123}$$

where $a_i, b_j \in C$ and $A_i, B_j \in R = (-\infty,\infty)$; $A_i, B_j \neq 0, i = 1,\ldots,p, j = 1,\ldots,q$; $\sum_{j=1}^q B_j - \sum_{j=1}^p A_j > -1$.

Notation 1.13. $\phi(a,b;z), {}_0\Psi_1(z)$: Wright function

Definition 1.13.

$$\phi(a,b;z) = {}_0\Psi_1\left[z\big|_{(b,a)}\right] = \sum_{n=0}^\infty \frac{1}{\Gamma(an+b)}\frac{z^n}{n!},\ b,z \in C; a \in R, a \neq 0. \tag{1.124}$$

The H-function in the generalized form contains a vast number of analytic functions as special cases. These analytic functions appear in various problems arising in theoretical and applied branches of mathematics, statistics, and engineering sciences. We present here a few interesting special cases of the H-function, which may be useful for workers on integral transforms, fractional calculus, special functions, applied statistics, physical and engineering sciences, astrophysics, etc.

$$H^{1,0}_{0,1}\left[z\big|_{(b,B)}\right] = B^{-1}z^{\frac{b}{B}}\exp\left(-z^{\frac{1}{B}}\right), \tag{1.125}$$

$$H^{1,1}_{1,1}\left[z\Big|^{(1-\nu,1)}_{(0,1)}\right] = \Gamma(\nu)(1+z)^{-\nu} = \Gamma(\nu){}_1F_0(\nu;\ ;-z), |z| < 1 \tag{1.126}$$

$$H^{1,0}_{0,2}\left[\frac{z^2}{4}\Big|_{\left(\frac{a+\nu}{2},1\right),\left(\frac{a-\nu}{2},1\right)}\right] = \left(\frac{z}{2}\right)^a J_\nu(z), \tag{1.127}$$

where $J_\nu(z)$ is the ordinary Bessel function of the first kind, see also Sect. 1.8.1.

$$H_{0,2}^{2,0}\left[\frac{z^2}{4}\Big|_{\left(\frac{a+\nu}{2},1\right),\left(\frac{a-\nu}{2},1\right)}\right] = 2\left(\frac{z}{2}\right)^a K_\nu(z), \tag{1.128}$$

where $K_\nu(z)$ is the modified Bessel function of the third kind or Macdonald function, see also Sect. 1.8.1.

$$H_{1,3}^{2,0}\left[\frac{z^2}{4}\Big|_{\left(\frac{a-\nu}{2},1\right),\left(\frac{a+\nu}{2},1\right),\left(\frac{a-\nu-1}{2},1\right)}^{\left(\frac{a-\nu-1}{2},1\right)}\right] = \left(\frac{z}{2}\right)^a Y_\nu(z), \tag{1.129}$$

where $Y_\nu(z)$ is the modified Bessel function of the second kind or the Neumann function, see also Sect. 1.8.1.

$$H_{1,2}^{1,1}\left[z\Big|_{(0,1),(1-c,1)}^{(1-a,1)}\right] = \frac{\Gamma(a)}{\Gamma(c)}\Phi(a;c;-z) = \frac{\Gamma(a)}{\Gamma(c)}{}_1F_1(a;c;-z), \tag{1.130}$$

which are called the Kummer's confluent hypergeometric functions.

$$H_{2,2}^{1,2}\left[z\Big|_{(0,1),(1-c,1)}^{(1-a,1),(1-b,1)}\right] = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}F(a,b;c;-z), \tag{1.131}$$

$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}{}_2F_1(b,a;c;-z), \tag{1.132}$$

which are called the Gauss' hypergeometric functions. The relation connecting H-function and MacRobert's E-function is given by

$$H_{q+1,p}^{p,1}\left[z\Big|_{(\alpha_1,1),\ldots,(\alpha_p,1)}^{(1,1),(\beta_1,1),\ldots,(\beta_q,1)}\right] = E(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z). \tag{1.133}$$

The relation connecting Whittaker function and the H-function is given by

$$H_{1,2}^{2,0}\left[\frac{z^2}{4}\Big|_{(\rho+m+\frac{1}{2}),(\rho-m+\frac{1}{2},1)}^{(\rho-k+1,1)}\right] = z^\rho e^{-\frac{z}{2}} W_{k,m}(z), \tag{1.134}$$

see also Sect. 1.8.1. We now give the special cases of the H-function which cannot be obtained from the G-function:

$$H_{1,2}^{1,1}\left[-z\Big|_{(0,1),(0,\alpha)}^{(0,1)}\right] = E_\alpha(z), \tag{1.135}$$

where $E_\alpha(z)$ is the Mittag-Leffler function (Mittag-Leffler 1903).

$$H_{1,2}^{1,1}\left[-z\Big|_{(0,1),(1-\beta,\alpha)}^{(0,1)}\right] = E_{\alpha,\beta}(z), \tag{1.136}$$

where $E_{\alpha,\beta}(z)$ is also the Mittag-Leffler function (Mittag-Leffler 1903).

$$\frac{1}{\Gamma(\gamma)} H_{1,2}^{1,1}\left[-z\Big|_{(0,1),(1-\beta,\alpha)}^{(1-\gamma,1)}\right] = E_{\alpha,\beta}^{\gamma}(z), \Re(\gamma) > 0, \tag{1.137}$$

where $E_{\alpha,\beta}^{\gamma}(z)$ is the generalized Mittag-Leffler function.

$$H_{0,2}^{1,0}\left[z\Big|_{(0,1),(-\nu,\mu)}\right] = J_{\nu}^{\mu}(z), \tag{1.138}$$

where $J_{\nu}^{\mu}(z)$ is the Bessel–Maitland function or Maitland Bessel function (see Marichev 1983, (8.3)).

$$H_{1,3}^{1,1}\left[\frac{z^2}{4}\Big|_{(\lambda+\frac{\nu}{2},1),(\frac{\nu}{2},1),(\mu(\lambda+\frac{\nu}{2})-\lambda-\nu,\mu)}^{(\lambda+\frac{\nu}{2},1)}\right] = J_{\nu,\lambda}^{\mu}(z), \tag{1.139}$$

where $J_{\nu,\lambda}^{\mu}(z)$ is the generalized Bessel–Maitland function (Marichev 1983, p. 128, (8.2)),

$$H_{p,q+1}^{1,p}\left[-z\Big|_{(0,1),(1-b_1,B_1),\ldots,(1-b_q,B_q)}^{(1-a_1,A_1),\ldots,(1-a_p,A_p)}\right] = {}_p\Psi_q\left[z\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]$$
$$= \frac{1}{2\pi i}\int_L \frac{\Gamma(s)\prod_{j=1}^{p}\Gamma(a_j - A_j s)}{\prod_{j=1}^{q}\Gamma(b_j - B_j s)}(-z)^{-s}\mathrm{d}s, \tag{1.140}$$

where ${}_p\Psi_q(z)$ is the Wright generalized hypergeometric function (Wright 1935).

$$H_{0,2}^{2,0}\left[z\Big|_{(0,1),(\frac{\nu}{\rho},\frac{1}{\rho})}\right] = \rho Z_{\rho}^{\nu}(z), z \in C, \rho > 0, \nu \in C, \tag{1.141}$$

where $Z_{\rho}^{\nu}(z)$ is the Krätzel function (Krätzel, 1979). The following special cases of the *H*-function occur in the study of certain statistical distributions.

$$H_{2,2}^{2,0}\left[z\Big|_{(\alpha_1-1,1),(\alpha_2-1,1)}^{(\alpha_1+\beta_1-1,1),(\alpha_2+\beta_2-1,1)}\right] = \frac{z^{\alpha_2-1}(1-z)^{\beta_1+\beta_2-1}}{\Gamma(\beta_1+\beta_2)}$$
$$\times {}_2F_1(\alpha_2+\beta_2-\alpha_1, \beta_1; \beta_1+\beta_2; 1-z), |z| < 1, \tag{1.142}$$

$$H_{1,1}^{1,0}\left[z\Big|_{(\alpha,1)}^{(\alpha+\frac{1}{2},1)}\right] = \pi^{-\frac{1}{2}} z^{\alpha}(1-z)^{-\frac{1}{2}}, |z| < 1, \tag{1.143}$$

$$H_{2,2}^{2,0}\left[z\Big|_{(\alpha,1),(\alpha,1)}^{(\alpha+\frac{1}{3},1),(\alpha+\frac{2}{3},1)}\right] = z^{\alpha}{}_2F_1\left(\frac{2}{3},\frac{1}{3};1;1-z\right), \quad |1-z| < 1. \tag{1.144}$$

1.8.1 Some Commonly Used Special Cases of the H-Function

(i) Psi function

$$\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}(\ln \Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)}, \tag{1.145}$$

$$= \int_0^\infty [t^{-1}\mathrm{e}^{-t} - (1-\mathrm{e}^{-t})^{-1}\mathrm{e}^{-tz}]\mathrm{d}t, \Re(z) > 0, \tag{1.146}$$

$$= -\gamma + (z-1)\sum_{k=0}^{\infty}[(k+1)(z+k)]^{-1}, \gamma \approx 0.5772156649\ldots, \tag{1.147}$$

(ii) Zeta function (Riemann zeta function)

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-\rho}, \Re(\rho) > 1, \tag{1.148}$$

$$\zeta(\rho, a) = \sum_{n=0}^{\infty}(n+a)^{-\rho}, \Re(\rho) > 1, a \neq 0, -1, -2, \ldots, \tag{1.149}$$

(iii) Whittaker functions

$$M_{\mu,\nu}(z) = z^{\nu+\frac{1}{2}}\mathrm{e}^{-z/2}{}_1F_1\left(\frac{1}{2} - \mu + \nu; 2\nu + 1; z\right) \tag{1.150}$$

$$= z^{\nu\frac{1}{2}}\mathrm{e}^{z/2}{}_1F_1\left(\frac{1}{2} + \mu + \nu; 2\nu + 1; -z\right) \tag{1.151}$$

$$= \frac{\Gamma(1+2\nu)}{\Gamma\left(\frac{1}{2}+\nu+\mu\right)\Gamma\left(\frac{1}{2}+\nu-\mu\right)}\mathrm{e}^{-z/2}z^{\nu+\frac{1}{2}}\int_0^1 \mathrm{e}^{-zt}t^{\nu-\mu-\frac{1}{2}}$$

$$\times (1-t)^{\nu+\mu-\frac{1}{2}}\mathrm{d}t, \Re\left(\frac{1}{2}+\nu \pm \mu\right) > 0, |\arg z| < \pi \tag{1.152}$$

$$= \frac{\Gamma(1+2\nu)}{\Gamma\left(\frac{1}{2}+\nu-\mu\right)}\mathrm{e}^{-z/2}z^{\nu+\frac{1}{2}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}$$

$$\times \frac{\Gamma(s)\Gamma\left(\frac{1}{2}+\nu-\mu-s\right)}{\Gamma(1+2\nu-s)}(-z)^{-s}\mathrm{d}s, |\arg z| < \pi/2, 2\nu \neq -1, -2, \ldots \tag{1.153}$$

$$W_{\mu,\nu}(z) = \frac{\Gamma(-2\nu)}{\Gamma\left(\frac{1}{2}-\mu-\nu\right)}M_{\mu,\nu}(z) + \frac{\Gamma(2\nu)}{\Gamma\left(\frac{1}{2}-\mu+\nu\right)}M_{\mu,-\nu}(z), \qquad (1.154)$$

$$\frac{1}{2}-\mu \neq \nu \neq 0,-1,-2,\ldots,2\nu \neq 0,\pm 1,\ldots$$

$$= W_{\mu,-\nu}(z) \qquad (1.155)$$

$$= \frac{z^{\mu}\mathrm{e}^{-z/2}}{\Gamma\left(\frac{1}{2}+\nu-\mu\right)}\int_0^{\infty} \mathrm{e}^{-t}t^{\nu-\mu-\frac{1}{2}}(1+\frac{t}{z})^{\nu+\mu-\frac{1}{2}}\,\mathrm{d}t, \qquad (1.156)$$

$$\Re\left(\frac{1}{2}+\nu-\mu\right) > 0, |\arg z| < \pi,$$

$$= \frac{z^{\mu}\mathrm{e}^{-z/2}}{\Gamma\left(\frac{1}{2}+\nu-\mu\right)\Gamma\left(\frac{1}{2}-\nu-\mu\right)}\frac{1}{2\pi i}\int_{c-\imath\infty}^{c+\imath\infty}\Gamma(-s)$$

$$\times\,\Gamma\left(\frac{1}{2}+\nu-\mu+s\right)\Gamma\left(\frac{1}{2}-\nu-\mu+s\right)z^{-s}\mathrm{d}s \qquad (1.157)$$

$$|\arg z| < \frac{3\pi}{2}, -\frac{1}{2}+\mu\pm\nu \neq 0,1,2,\ldots$$

(iv) Parabolic cylinder function

$$D_\nu(z) = 2^{\frac{\nu}{2}+\frac{1}{4}}z^{-\frac{1}{2}}W_{\frac{\nu}{2}+\frac{1}{4},\frac{1}{4}}\left(\frac{z^2}{2}\right) \qquad (1.158)$$

$$= (-1)^n \mathrm{e}^{z^2/4}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(\mathrm{e}^{-\frac{z^2}{2}}\right) \qquad (1.159)$$

$$= 2^{\frac{\nu}{2}+\frac{1}{4}}\mathrm{e}^{z/2}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(-\frac{1}{4}+s\right)\Gamma\left(\frac{1}{4}+s\right)}{\Gamma(s-\frac{\nu}{2}+\frac{1}{4})}\left(\frac{z^2}{2}\right)^{-s}\mathrm{d}s, \quad (1.160)$$

$$|\arg z| < \frac{\pi}{4}.$$

(v) Bessel and associated functions

$$J_\nu(z) = \sum_{r=0}^{\infty}\frac{(-1)^r(z/2)^{\nu+2r}}{r!\Gamma(\nu+r+1)} = \frac{(z/2)^\nu}{\Gamma(\nu+1)}{}_0F_1\left(\ ;1+\nu;-\frac{z^2}{4}\right) \qquad (1.161)$$

$$= \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{\nu+s}{2}\right)}{\Gamma\left(1+\frac{\nu-s}{2}\right)}\left(\frac{z}{2}\right)^{-s}\mathrm{d}s, -\Re(\nu) < 1, |\arg z| < \pi$$

$$(1.162)$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(-s)}{\Gamma(\nu+s+1)}\left(\frac{z}{2}\right)^{\nu+2s}\mathrm{d}s, z > 0, \Re(\nu) > -1. \qquad (1.163)$$

$$I_\nu(z) = \sum_{r=0}^{\infty}\frac{(z/2)^{\nu+2r}}{r!\Gamma(\nu+r+1)} = \frac{(z/2)^\nu}{\Gamma(\nu+1)}{}_0F_1\left(\ ;1+\nu;\frac{z^2}{4}\right) \qquad (1.164)$$

$$= \mathrm{e}^{-i\nu\pi/2}J_\nu(z\mathrm{e}^{i\pi/2}), -\pi < \arg z \leq \pi/2. \qquad (1.165)$$

$$I_m\left(\frac{z}{2}\right) = \frac{2^{-2m}z^{-\frac{1}{2}}}{\Gamma(m+1)}M_{0,m}(z). \tag{1.166}$$

$$Y_\nu(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(s-\frac{\nu}{2}\right)\Gamma\left(s+\frac{\nu}{2}\right)}{\Gamma\left(s-\frac{\nu+1}{2}\right)\Gamma\left(\frac{3+\nu}{2}-s\right)}\left(\frac{z^2}{4}\right)^{-s} \mathrm{d}s \tag{1.167}$$

$$-3 < \Re(\nu) < -1,$$

$$K_\nu(z) = \left(\frac{2z}{\pi}\right)^{-\frac{1}{2}} W_{0,\nu}(2z) \tag{1.168}$$

$$= \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma\left(s+\frac{\nu}{2}\right)\Gamma\left(s-\frac{\nu}{2}\right)\left(\frac{z^2}{4}\right)^{-s}\mathrm{d}s, |\arg z| < \frac{\pi}{2}. \tag{1.169}$$

(vi) Struve's function

$$H_\nu(z) = \sum_{r=0}^{\infty}\frac{(-1)^r(z/2)^{\nu+2r+1}}{\Gamma\left(r+\frac{3}{2}\right)\Gamma\left(\nu+r+\frac{3}{2}\right)} \tag{1.170}$$

$$= \frac{(z/2)^{\nu+1}}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\nu+\frac{3}{2}\right)}{}_1F_2\left(1;\frac{3}{2},\nu+\frac{3}{2};-\frac{z^2}{4}\right). \tag{1.171}$$

(vii) Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}{}_2F_1\left(-n,n+\lambda;\alpha+1;\frac{1-x}{2}\right) \tag{1.172}$$

$$= \frac{(-1)^n(\beta+1)_n}{n!}{}_2F_1\left(-n,n+\lambda;\beta+1;\frac{1+x}{2}\right) \tag{1.173}$$

$$= \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\{(1-x)^{\alpha+n}(1+x)^{\beta+n}\} \tag{1.174}$$

$$= 2^{-n}\sum_{k=0}^{n}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}(x-1)^{n-k}(x+1)^k, \lambda = \alpha+\beta+1. \tag{1.175}$$

(viii) Shifted Jacobi polynomial

$$R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x-1). \tag{1.176}$$

(ix) Legendre polynomials

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n = P_n^{(0,0)}(x). \tag{1.177}$$

(x) Gegenbauer polynomial

$$C_n^{(\alpha+\frac{1}{2})} = \frac{(2\alpha+1)_n}{(\alpha+1)_n} P_n^{(\alpha,\alpha)}(x). \tag{1.178}$$

(xi) Chebyshev polynomials

$$T_n(x) = \frac{n!}{(1/2)_n} P_n^{(-\frac{1}{2},-\frac{1}{2})}(x) \tag{1.179}$$

$$= \cos(n \cos^{-1} x). \tag{1.180}$$

$$T_n^*(x) = T_n(2x-1). \tag{1.181}$$

$$U_n(x) = \frac{(n+1)!}{(3/2)_n} P_n^{(\frac{1}{2},\frac{1}{2})}(x). \tag{1.182}$$

$$U_n^*(x) = U_n(2x-1). \tag{1.183}$$

(xii) Laguerre polynomials

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!} \frac{d^n}{dx^n}(e^{-x} x^{n+\alpha}) \tag{1.184}$$

$$= \frac{(\alpha+1)_n}{n!} {}_1F_1(-n; \alpha+1; x) \tag{1.185}$$

$$= \lim_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1 - \frac{2x}{\beta}\right). \tag{1.186}$$

$$L_n^{(0)}(x) = L_n(x). \tag{1.187}$$

(xiii) Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}). \tag{1.188}$$

$$H_{e_n}(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2}). \tag{1.189}$$

1.9 Generalized Wright Functions

In this section, generalized Wright function is studied. Its existence conditions are presented. In the preceding section the representations of the generalized Wright function in terms of the Mellin–Barnes integral and the H-function were given. Conditions for such representations are proved by Kilbas et al. (2002), also see Kilbas et al. (2006).

1.9.1 Existence Conditions

Existence conditions for the generalized Wright function are given by Braaksma (1964, p. 326), also see Kilbas et al. (2002). In this section we will prove the existence conditions for the generalized Wright function. The main result is given in the form of the following:

Theorem 1.4. *Let $p, q \in N_0$. Further, let $a_i, b_j \in C$ and $A_i, B_j \in R_+, i = 1, \ldots, p; j = 1, \ldots, q$*

(i) If $\mu > -1$ then the series in (1.190) is absolutely convergent for all $z \in C$.
(ii) If $\mu = -1$ then the series in (1.190) is absolutely convergent for all values of $|z| < \beta$ and for $|z| = \beta, \Re(\delta) > \frac{1}{2}$ where μ and δ are defined in (1.9) and (1.10) respectively.

Proof 1.2. Equation (1.190) is a power series

$$ {}_p\Psi_q \left[z \Big|_{(b_q, B_q)}^{(a_p, A_p)} \right] = \sum_{n=0}^{\infty} c_n z^n, \tag{1.190} $$

$$ c_n = \frac{\prod_{i=1}^{p} \Gamma(a_i + A_i n)}{\prod_{j=1}^{q} \Gamma(b_j + B_j n)\, n!}, n \in N_0. \tag{1.191} $$

In order to investigate the asymptotic behavior of c_n when $n \to \infty$ we use the Stirling formula for the gamma function (1.25) to obtain the following relations:

$$ \Gamma(a_i + nA_i) \sim P_i \left(\frac{n}{\mathrm{e}}\right)^{nA_i} A_i^{nA_i} n^{a_i - \frac{1}{2}},\ P_i = (2\pi)^{\frac{1}{2}} A_i^{a_i - \frac{1}{2}} \mathrm{e}^{-a_i}, \tag{1.192} $$

as $n \to \infty$ for $i = 1, \ldots, p$;

$$ \Gamma(b_j + B_j n) \sim Q_j \left(\frac{n}{\mathrm{e}}\right)^{nB_j} B_j^{nB_j} n^{b_j - \frac{1}{2}}, Q_j = (2\pi)^{\frac{1}{2}} B_j^{b_j - \frac{1}{2}} \mathrm{e}^{-b_j}, \tag{1.193} $$

as $n \to \infty$ for $j = 1, \ldots, q$; and

$$ n! \sim (2\pi)^{\frac{1}{2}} \left(\frac{n}{\mathrm{e}}\right)^{n} n^{\frac{1}{2}} \mathrm{e}, n \to \infty. \tag{1.194} $$

Using the results (1.192), (1.193), and (1.194) into (1.191) it yields the estimate for c_n in the form

$$ c_n \sim R \left(\frac{n}{\mathrm{e}}\right)^{-n(\mu+1)} \left\{ \left[\prod_{j=1}^{p} A_j^{A_j} \right] \left[\prod_{j=1}^{q} B_j^{-B_j} \right] \right\}^{n} n^{-[\delta + \frac{1}{2}]},\ n \to \infty, \tag{1.195} $$

where μ and δ are defined in (1.17) and (1.18) respectively and

$$R = (2\pi)^{\frac{(p-q-1)}{2}} \frac{\prod_{j=1}^{p}(A_j^{a_j-\frac{1}{2}} e^{-a_j})}{e\prod_{j=1}^{q}(B_j^{b_j-\frac{1}{2}} e^{-b_j})}. \tag{1.196}$$

The theorem now follows from the known convergence principles of the power series in (1.190). □

Corollary 1.1. *Let $p, q \in N_0$. Let $a_i, b_j \in C, A_i, B_j \in R_+, i = 1, \ldots, p; j = 1, \ldots, q$ be such that the condition $\mu > -1$ is satisfied. Then the generalized Wright function ${}_p\Psi_q(z)$ is an entire function of z, where μ is defined in (1.9).*

Corollary 1.2. *Let a be real and $b \in C$ in the Wright function $\phi(a; b; z)$ of (1.124).*

(i) If $a > -1$ then the series in (1.124) is absolutely convergent for all $z \in C$.
(ii) If $a = -1$ then the series in (1.124) is absolutely convergent for all $|z| < 1$ and for $|z| = 1, \Re(\beta) > 1$ where μ is defined in (1.9).

Corollary 1.3. *If $a > -1$ and $b \in C$ then the Wright function $\phi(a, b; z)$ defined by (1.124) is an entire function of z.*

Corollary 1.4. *If $\mu > -1$ and $\nu \in C$ then the Bessel–Maitland function $J_\nu^\mu(z)$ defined by (1.118) is an entire function of z.*

1.9.2 Representation of Generalized Wright Function

Notation 1.14.

$$ {}_2R_1(a, b; c, \omega; \mu; z): \text{ Dotsenko function (Dotsenko 1991, 1993)} \tag{1.197}$$

Definition 1.14.

$$ {}_2R_1(a, b; c, \omega; \mu; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k\frac{\omega}{\mu})}{\Gamma(c+k\frac{\omega}{\mu})} \frac{z^k}{k!} \tag{1.198}$$

$$ = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} {}_2\Psi_1\left[z\Big|^{(a,1),(b,\frac{\omega}{\mu})}_{(c,\frac{\omega}{\mu})}\right]. \tag{1.199}$$

The existence of the generalized Wright function ${}_p\Psi_q(z)$ defined by means of the Mellin–Barnes integral (1.140) is given by the following results which yield different conditions for the representation (1.140) with the contours $L = L_{-\infty}, L = L_{+\infty}$ and $L = L_{i\gamma\infty}$. By following a procedure similar to that adopted in proving the existence conditions of the H-function in Theorem 1.1, the following theorems can be established on the contours L_{∞}, $L_{-\infty}$ and $L_{i\gamma\infty}$ (defined in Sect. 1.1). For a detailed proof of these theorems, one can refer to Kilbas, Saigo, and Trujillo (2002) and also to a recent article by Kilbas et al. (2006).

Theorem 1.5. *Let $p, q \in N_0$. Let $a_i, b_j \in C$ and $A_i, B_j \in R_+, i = 1, \dots, p; j = 1, \dots, q$ and be such that the conditions $\frac{a_i+k}{A_i} \neq -\nu; k, \nu \in N_0, i = 1, \dots, p$ and $(a_i + k)A_j \neq (a_j + m)A_i, i \neq j, j = 1, \dots, p; k, m \in N_0$ be satisfied. Let either of the following conditions hold:*

$$\mu > -1,\ z \neq 0, \tag{1.200}$$

$$\mu = -1,\ 0 < |z| < \beta, \tag{1.201}$$

$$\mu = -1, |z| = \beta, \Re(\delta) > \frac{1}{2}. \tag{1.202}$$

Then there exists the generalized Wright function ${}_p\Psi_q(z)$ defined by means of the Mellin–Barnes integral (1.140), where the path of integration $L = L_{-\infty}$ separates all poles given in $s = -\nu, \nu \in N_0$ to the left and all poles given by $s = \frac{a_i+k}{A_i}, i = 1, \dots, n; k \in N_0$ to the right.

Theorem 1.6. *Let $p, q \in N_0, a_i, b_j \in C$ and $A_i, B_j \in R_+, i = 1, \dots, p; j = 1, \dots, q$ and be such that the conditions on the parameters in Theorem 1.5 are satisfied. Let either of the following conditions hold:*

$$\mu < -1,\ z \neq 0, \tag{1.203}$$

$$\mu = -1, |z| > \beta, \tag{1.204}$$

$$\mu = -1, |z| = \beta, \Re(\delta) > \frac{1}{2}. \tag{1.205}$$

Then there exists the generalized Wright function ${}_p\Psi_q(z)$ defined by means of Mellin–Barnes integral (1.140), where the path of integration $L = L_{+\infty}$ separates all poles as stated in Theorem 1.5.

Theorem 1.7. *Let $p, q \in N_0, a_i, b_j \in C$ and $A_i, B_j \in R_+, i = 1, \dots, p; j = 1, \dots, q$ and be such that the conditions on the parameters as stated in Theorem 1.5 be satisfied. Let either of the following conditions hold:*

$$\mu < 1,\ |\arg(-z)| < \frac{(1-\mu)\pi}{2}, z \neq 0, \tag{1.206}$$

$$\mu = 1, (1+\mu)\gamma + \frac{1}{2} < \Re(\delta), \arg(-z) = 0, z \neq 0. \tag{1.207}$$

Then there exists the generalized Wright function ${}_p\Psi_q(z)$ defined by means of Mellin–Barnes integral (1.140), where the path of integration $L = L_{i\gamma\infty}$ separates all poles as stated as in the Theorem 1.5.

If we combine the Theorems 1.5–1.7 then we arrive at the following theorem given by Kilbas et al. (2006, p. 125), which gives the conditions under which the generalized Wright function can be represented as an H-function by (1.140).

Theorem 1.8. *Let $p, q \in N_0, a_i, b_j \in C$ and $A_i, B_j \in R_+, i = 1, \ldots, p; j = 1, \ldots, q$ and be such that the conditions in Theorem 1.5 be satisfied, and let $\gamma \in R$. Let L be the contour which separates all poles as given in Theorem 1.5. Further, let either of the following conditions hold:*

$$(i) \quad L = L_{-\infty} \text{ and either (1.200), (1.201) or (1.202) holds} \tag{1.208}$$

$$(ii) \quad L = L_{+\infty} \text{ and either (1.203), (1.204) or (1.205) holds} \tag{1.209}$$

$$(iii) \quad L = L_{\gamma\infty} \text{ and either (1.206), or (1.207) holds} \tag{1.210}$$

Then the generalized Wright function ${}_p\Psi_q(z)$ defined by (1.123) is represented as an H-function by (1.140).

The utility and importance of the generalized Wright function is realized in recent years due to its occurrence in certain problems of applied character. This function is in the proximity of the H-function so its utility is further increased. Nearly all the Mittag-Leffler functions and their generalizations can be expressed in terms of this function; in this connection one can refer to the paper by Kilbas et al. (2002). Various properties of the Wright function are studied by many authors in a series of papers, some of which are enumerated below.

Wright (1933) showed the application of the results obtained for the function $\phi(a, b; z)$ defined by (1.124) to the asymptotic theory of partitions. Dotsenko (1991) developed fractional relations for the Wright function. Asymptotic relations and distribution of the zeros of this function $\phi(a, b; z)$ are investigated by Luchko (2000, 2001). Application of this function in operational calculus is given by Mikusinski (1959) and in integral transform of Hankel type by Gajic and Stankovic (1976) and Stankovic (1970). Mainardi (1994) derived the solution of fractional diffusion-wave equation in terms of the Wright function. In this connection, the interested reader can also refer to the book by Podlubny (1999, Sect. 4.12) and to the survey paper Mainardi (1997). Scale-variant solutions of some partial differential equations of fractional order are given in terms of the special cases of the generalized Wright function by Buckwar and Buckwar and Luchko (1998), Luchko and Gorenflo (1998) and Gorenflo et al. (2000). Analytic properties of the Wright function with applications are obtained by Gorenflo et al. (1999). Existence conditions and representations of the generalized Wright function in terms of Mellin–Barnes integrals and the H-function are obtained by Kilbas et al. (2002). Wright function representations of the Krätzel function are investigated recently by Kilbas et al. (2006). Generalized Wright function has been used in the study of generalized gamma functions by Srivastava et al. (2003). Generalized Wright function as a kernel of an integral transform is recently studied by Saxena et al. (2006). Analytical continuation formulae and asymptotic formulae for the generalized Wright function are investigated by Kilbas et al. (2006).

Exercises

1.1. Prove that if $\Re(\delta) > 0$, then

$$\text{(i)} \quad f(x;\delta,\alpha,\gamma,1) = 2\left(\frac{\gamma x}{\alpha}\right)^{\frac{\delta}{2}} K_\delta[2(\alpha\gamma x)^{\frac{1}{2}}],$$

$$\text{(ii)} \quad f(x;\delta,\alpha,\gamma,-1) = \Gamma(\delta)(\alpha+\gamma x)^{-\delta}, \Re(\alpha+\gamma x) > 0, |\frac{\gamma x}{\alpha}| < 1,$$

$$\text{(iii)} \quad f(x;\delta,\alpha,\gamma,-\frac{1}{2}) = 2^{1-\delta}\Gamma(2\delta)\alpha^{-\delta}\exp\left(\frac{\alpha^{-1}\gamma^2 x}{8}\right) D_{-2\delta}[(2\alpha)^{-\frac{1}{2}}\gamma x],$$

$$\text{(iv)} \quad f(x;\delta,\alpha,\gamma,-2) = \Gamma(\delta)(2\gamma x)^{-\frac{\delta}{2}}\exp[-\frac{\alpha^2}{\delta\gamma x}]D_{-\delta}[\alpha(2\gamma x)^{-\frac{1}{2}}],$$

where $f(x;\delta,\alpha,\gamma,\phi) = \alpha^{-\delta} H_{0,2}^{2,0}[\alpha^\phi \gamma x|(\delta,\phi),(0,1)]$ (Buschman 1974a).

1.2. Prove that

$$B_1 z^{-b_1} H_{p,q}^{1,n}\left[z^{B_1}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]$$
$$= \sum_{v=0}^{\infty}\frac{(-z)^v}{v!}\frac{\prod_{j=1}^{n}\Gamma\left[1-a_j+A_j\left(\frac{b_1+v}{B_1}\right)\right]}{\left\{\prod_{j=2}^{q}\Gamma\left[1-b_j+B_j\left(\frac{b_1+v}{B_1}\right)\right]\right\}\left\{\prod_{j=n+1}^{p}\Gamma\left[a_j-A_j\left(\frac{b_1+v}{B_1}\right)\right]\right\}}.$$

(Braaksma 1964, p. 279)

1.3. Prove that

$$\text{(i)} \quad z^r\frac{\mathrm{d}^r}{\mathrm{d}z^r}\left\{H_{p,q}^{m,n}\left[x^\delta\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right\} = H_{p+1,q+1}^{m,n+1}\left[z^\delta\Big|_{(b_q,B_q),(r,\delta)}^{(0,\delta),(a_p,A_p)}\right],$$

$$\text{(ii)} \quad z^r\frac{\mathrm{d}^r}{\mathrm{d}z^r}\left\{H_{p,q}^{m,n}\left[z^{-\delta}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right\} = (-1)^r H_{p+1,q+1}^{m,n+1}\left[z^{-\delta}\Big|_{(b_q,B_q),(1,\delta)}^{(1-r,\delta),(a_p,A_p)}\right],$$

giving the conditions of validity of the result. Hint: use the formulae

$$z^r\frac{\mathrm{d}^r}{\mathrm{d}z^r}(z^{s\delta}) = \frac{\Gamma(1+s\delta)}{\Gamma(1+s\delta-r)}z^{s\delta},$$

and

$$z^r\frac{\mathrm{d}^r}{\mathrm{d}z^r}(z^{-s\delta}) = \frac{(-1)^r\Gamma(r+s\delta)}{\Gamma(s\delta)}z^{-s\delta}.$$

Show that

$$\frac{d^r}{dz^r}\left\{z^\lambda H^{m,n}_{p,q}\left[\beta z^\delta \Big|^{(a_p,A_p)}_{(b_q,B_q)}\right]\right\} = z^{\lambda-r} H^{m,n+1}_{p+1,q+1}\left[\beta z^\delta \Big|^{(-\lambda,\delta),(a_p,A_p)}_{(b_q,B_q),(r-\lambda,\delta)}\right].$$

(Anandani 1970)

1.4. Establish the following identities:

(i) $$H^{m,n+1}_{p+1,q+1}\left[z\Big|^{(\alpha,\delta),(a_p,A_p)}_{(b_q,B_q),(\alpha+r,\delta)}\right] = (-1)^r H^{m+1,n}_{p+1,q+1}\left[z\Big|^{(a_p,A_p),(\alpha,\delta)}_{(\alpha+r,\delta),(b_q,B_q)}\right].$$

(Anandani 1970, p. 191)

(ii) $$H^{4,0}_{2,4}\left[z\Big|^{(\frac{1}{2}+a,1),(\frac{1}{2}-a,1)}_{(0,1),(\frac{1}{2},1),(b,1),(-b,1)}\right] = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} W_{a,b}(2z^{\frac{1}{2}})W_{-a,b}(2z^{\frac{1}{2}}),$$

where $W_{a,b}(z)$ and $W_{-a,b}(z)$ are Whittaker functions.

(iii) $$H^{m,n+2}_{p+2,q+2}\left[z\Big|^{(-\sigma,h),(\alpha-\sigma,h),(a_p,A_p)}_{(b_q,B_q),(\alpha-\sigma-\nu,h),(-1-\beta-\sigma-\nu,h)}\right]$$
$$= (-1)^\nu H^{m+1,n+1}_{p+2,q+2}\left[z\Big|^{(-\sigma,h),(a_p,A_p),(\alpha-\sigma,h)}_{(\alpha-\sigma-\nu,h),(b_q,B_q),(-1-\beta-\sigma-\nu,h)}\right],$$

(iv) $$H^{m+1,n}_{p+1,q+1}\left[x\Big|^{(a_p,A_p),(\alpha-\beta-1,h)}_{(\alpha-\beta,h),(b_q,B_q)}\right]$$
$$= H^{m+1,n}_{p+1,q+1}\left[x\Big|^{(a_p,A_p),(\alpha+1,h)}_{(\alpha+2,h),(b_q,B_q)}\right] - (\beta+2)H^{m,n}_{p,q}\left[x\Big|^{(a_p,A_p)}_{(b_q,B_q)}\right]$$

(Anandani 1969).

1.5. Prove that

$$\left(\frac{d}{dx}x - c_1\right)\cdots\left(\frac{d}{dx}x - c_r\right)\left\{x^\delta H^{m,n}_{p,q}\left[zx^h\Big|^{(a_p,A_p)}_{(b_q,B_q)}\right]\right\}$$
$$= x^\delta H^{m,n+r}_{p+r,q+r}\left[zx^h\Big|^{(c_r-\delta-1,h),\ldots,(c_1-\delta-1,h),(a_p,A_p)}_{(b_q,B_q),(c_r-\delta,h),\ldots,(c_1-\delta,h)}\right],$$

where $h > 0$ and the symbol $\frac{d}{dx}x$ indicates that the function of x in front of it is first multiplied by x and then the product is differentiated with respect to x. Hence deduce the following result:

$$\left(\frac{d}{dx}x - c\right)\left(\frac{d}{dx}x - c + e\right)\cdots\left(\frac{d}{dx}x - c + (r-1)e\right)$$
$$\times\left\{x^{\delta e+c-1} H^{m,n}_{p,q}\left[zx^{he}\Big|^{(a_p,A_p)}_{(b_q,B_q)}\right]\right\}$$
$$= e^r x^{\delta e+c-1} H^{m,n+1}_{p+1,q+1}\left[zx^{he}\Big|^{(1-r-\delta,h),(a_p,A_p)}_{(b_q,B_q),(1-\delta,h)}\right],$$

provided $e \neq 0, h > 0$.

(Nair 1972)

1.6. Establish the following differentiation formulae:

$$\text{(i)}\quad \frac{\mathrm{d}^r}{\mathrm{d}x^r} H_{p,q}^{m,n}\left[(cx+d)^h\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = \frac{(-c)^r}{(cx+d)^r} H_{p+1,q+1}^{m+1,n}\left[(cx+d)^h\Big|_{(r,h),(b_q,B_q)}^{(a_p,A_p),(0,h)}\right],$$

$$\text{(ii)}\quad \frac{\mathrm{d}^r}{\mathrm{d}x^r} H_{p,q}^{m,n}\left[\frac{1}{(cx+d)^h}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = \frac{c^r}{(cx+d)^r} H_{p+1,q+1}^{m,n+1}\left[\frac{1}{(cx+d)^h}\Big|_{(1,h),(b_q,B_q)}^{(a_p,A_p),(1-r,h)}\right],$$

$$\text{(iii)}\quad \frac{\mathrm{d}^r}{\mathrm{d}x^r} H_{p,q}^{m,n}\left[\frac{1}{(cx+d)^h}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = \frac{(-c)^r}{(cx+d)^r} H_{p+1,q+1}^{m,n+1}\left[\frac{1}{(cx+d)^h}\Big|_{(b_q,B_q),(1,h)}^{(1-r,h),(a_p,A_p)}\right],$$

where c and d are complex numbers, r is a positive integer and $h > 0$. (Oliver and Kalla 1971).

1.7. Prove the following results:

$$\text{(i)}\quad H_{p,q}^{m,n}\left[z\lambda^\sigma\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,\sigma),\ldots,(a_{p-1},A_{p-1}),(a_p,\mu\sigma)}\right] = \lambda^{a_1-1}\sum_{r=0}^{\infty}\frac{1}{r!}\left(1-\frac{1}{\lambda}\right)^r \times H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1-r,\sigma),(a_2,A_2),\ldots,(a_{p-1},A_{p-1}),(a_p,\mu\sigma)}\right],$$

where $1 \le n \le p-1$, $\mu > 0, \sigma > 0$ and λ and z are complex numbers.

$$\text{(ii)}\quad (a_p-\mu a_1)H_{p,q}^{m,n}\left[x\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(1+a_1,\sigma),(a_2,A_2),\ldots,(a_{p-1},A_{p-1}),(a_p,\mu\sigma)}\right] = H_{p,q}^{m,n}\left[x\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(1+a_1,\sigma),(a_2,A_2),\ldots,(a_{p-1},A_{p-1}),(a_p,\mu\sigma)}\right] + \mu H_{p,q}^{m,n}\left[x\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,\sigma),(a_2,A_2),\ldots,(a_{p-1},A_{p-1}),(a_p+1,\mu\sigma)}\right],$$

where $1 \le n \le p-1$ and $\mu > 0$.

$$\text{(iii)}\quad H_{p+1,q+1}^{m+1,n}\left[x\Big|_{(a_1+\nu+1,\sigma),(b_1,B_1),\ldots,(b_q,B_q)}^{(1+a_1,\sigma),(a_2,A_2),\ldots,(a_{p-1},A_{p-1}),(a_p,\mu\sigma),(a_1+\nu,\sigma)}\right] = \nu H_{p,q}^{m,n}\left[x\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(1+a_1,\sigma),(a_2,A_2),\ldots,(a_{p-1},A_{p-1}),(a_p,\mu\sigma)}\right] - H_{p,q}^{m,n}\left[x\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,\sigma),(a_2,A_2),\ldots,(a_{p-1},A_{p-1}),(a_p,\mu\sigma)}\right],$$

where $1 \le n \le p-1$ and $\mu > 0$.

(iv) $$\left[x^{1-\frac{1}{\mu}}\frac{\mathrm{d}}{\mathrm{d}x}x^{\frac{(a_p-1)}{\mu}}\right]^r H_{p,q}^{m,n}\left[zx^{-\sigma}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,\sigma),(a_2,A_2),\ldots,(a_{p-1},A_{p-1}),(a_p,\mu\sigma)}\right]$$
$$= \left(\frac{1}{\mu}\right)^r x^{\frac{(a_p-r-1)}{\mu}}$$
$$\times H_{p,q}^{m,n}\left[zx^{-\sigma}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,\sigma),(a_2,A_2),\ldots,(a_{p-1},A_{p-1}),(a_p-r,\mu\sigma)}\right],$$
where $1 \le n \le p-1$ and $\mu > 0$.

(Srivastava and Gupta 1970)

Hint: The above results can be proved by representing the H-functions on the right by their Mellin–Barnes representations, taking the common factors out and then combining the terms.

1.8. Let
$$d(b_1, a_p - k) = \det\begin{bmatrix} b_1 & a_p - k \\ B_1 & A_p \end{bmatrix},$$
in which the first row of the determinant is written by our notation. The second row of the determinant is always to be completed with the appropriate A's and B's corresponding to the a's and b's of the first row. Further, we employ the notation $H[b_1 + 1]$ to denote the contiguous function in which b_1 is replaced by $b_1 + 1$, but with all other parameters left unchanged. Similar meanings hold for all other contiguous H-functions occurring in this problem. In the following results H will denote the H-function. Prove the following relations of contiguity for the H-function.

$$A_pH[b_1 + 1] - B_1[a_p - 1] = d(b_1, a_p - 1)H. \tag{1.211}$$
$$A_pH[a_1 - 1] + A_1H[a_p - 1] = -d(a_1 - 1, a_p - 1)H. \tag{1.212}$$

$$B_qH[a_1 - 1] - A_1H[b_q + 1] = -d(a_1 - 1, b_q)H. \tag{1.213}$$
$$B_qH[b_1 + 1] + B_1H[b_q + 1] = d(b_1, b_q)H. \tag{1.214}$$
$$A_1H[b_1 + 1] + B_1H[a_1 - 1] = d(b_1, a_1 - 1)H. \tag{1.215}$$
$$B_qH[a_p - 1] + A_pH[b_q + 1] = d(a_p - 1, b_q)H. \tag{1.216}$$

$$B_qH[b_1 + 1] - B_1H[b_2 + 1] = d(b_1, b_2)H. \tag{1.217}$$
$$A_2H[a_1 - 1] - A_1H[a_2 - 1] = -d(a_1 - 1, a_2 - 1)H. \tag{1.218}$$

$$A_{p-1}H[a_p-1]-A_pH[a_{p-1}-1]=d(a_p-1,a_{p-1}-1)H. \tag{1.219}$$

$$B_{q-1}H[b_q+1]-B_qH[b_{q-1}+1]=-d(b_q,b_{q-1})H. \tag{1.220}$$

$$d(a_p-1,b_q)H[a_1-1]-d(b_q-a_1-1)H[a_p-1] \\ =-d(a_1-1,a_p-1)H[b_q+1]. \tag{1.221}$$

$$d(a_p-1,b_q)H[b_1+1]+d(b_q,b_1)H[a_p-1] \\ =d(b_1,a_p-1)H[b_q+1]. \tag{1.222}$$

$$d(a_1-1,b_q)H[b_1+1]-d(b_q,b_1)H[a_1-1] \\ =d(b_1,a_1-1)H[b_q+1]. \tag{1.223}$$

$$d(a_1-1,b_q)H[b_1+1]-d(b_q,b_1)H[a_1-1] \\ =d(b_1,a_1-1)H[b_q+1]. \tag{1.224}$$

$$d(a_1-1,a_p-1)H[b_1+1]-d(a_p-1,b_1)H[a_1-1] \tag{1.225}$$

$$=-d(b_1,a_1-1)H[a_p-1]. \tag{1.226}$$

$$d(b_2,b_3)H[b_1+1]+d(b_3,b_1)H[b_2+1) \\ =-d(b_1,b_2)H[b_3+1]. \tag{1.227}$$

$$d(a_2-1,a_3-1)H[a_1-1]+d(a_3-1,a_2-1)H[a_2-1] \\ =-d(a_1-1,a_2-1)H[a_3-1]. \tag{1.228}$$

$$d(a_{p-1}-1,a_{p-2}-1)H[a_p-1]+d(a_{p-2}-1,a_p-1)H[a_{p-1}-1] \\ =-d(a_p-1,a_{p-1}-1)H[a_{p-2}-1]. \tag{1.229}$$

$$d(b_{q-1},b_{q-2})H[b_q+1]+d(b_q-2,b_q)H[b_{q-1}+1] \\ =-d(b_q,b_{q-1})H[b_{q-2}+1]. \tag{1.230}$$

$$d(a_p-1,b_1)H[a_{p-1}-1]+d(b_1,a_{p-1}-1)H[a_p-1] \\ =-d(a_{p-1}-1,a_p-1)H[b_1+1]. \tag{1.231}$$

$$d(b_q,a_1-1)H[b_{q-1}+1]+d(a_1-1,b_{q-1})H[b_q+1] \\ =-d(b_{q-1},b_q)H[a_1-1]. \tag{1.232}$$

$$d(a_2-1,a_p-1)H[a_1-1]+d(a_p-1,a_1-1)H[a_2-1] \\ = d(a_1-1,a_2-1)H[a_p-1]. \tag{1.233}$$

$$d(b_2,b_q)H[b_1+1]+d(b_q,b_1)H[b_2+1] \\ = d(b_1,b_2)H[b_q+1]. \tag{1.234}$$

$$d(a_{p-1}-1,a_1-1)H[a_p-1]+d(a_1-1,a_p-1)H[a_{p-1}-1] \\ = d(a_{p-1},a_{p-1}-1)H[a_1-1]. \tag{1.235}$$

$$d(b_{q-1},b_1)H[b_q+1]+d(b_1,b_q)H[b_{q-1}+1] \\ = d(b_q,b_{q-1})H[b_1+1]. \tag{1.236}$$

$$d(a_2-1,b_q)H[a_1-1]+d(b_q,a_1-1)H[a_2-1] \\ = -d(a_1-1,a_2-1)H[b_q+1]. \tag{1.237}$$

$$d(b_2,a_{p-1})H[b_1+1]+d(a_{p-1},b_1)H[b_2+1] \\ = -d(b_1,b_2)H[a_p-1]. \tag{1.238}$$

$$d(a_2-1,b_1)H[a_1-1]+d(b_1,a_1-1)H[a_2-1] \\ = d(a_1-1,a_2-1)H[b_1+1]. \tag{1.239}$$

$$d(b_2,a_1-1)H[b_1+1]+d(a_1-1,b_1)H[b_2+1] \\ = d(b_1,b_2)H[a_1-1]. \tag{1.240}$$

$$d(a_{p-1}-1,b_q)H[a_p-1]+d(b_q,a_p-1)H[a_{p-1}-1] \\ = d(a_p-1,a_{p-1}-1)H[b_q+1]. \tag{1.241}$$

$$d(b_{q-1},a_p-1)H[b_q+1]+d(a_p-1,b_q)H[b_{q-1}+1] \\ = d(b_q,b_{q-1})H[a_p-1]. \tag{1.242}$$

(Buschman 1972)

Hint: First establish the basic relations (1.211) and (1.212) given above and then derive all the others from two of them and using the transformation formula of $H(x)$ going to $H(\frac{1}{x})$.

1.9. Establish the following results associated with the Mellin transforms of the partial derivatives of the H-function with respect to their parameters.

$$\text{(i)} \quad M\left\{\frac{\partial}{\partial b_1}H_{p,q}^{m,n}(x)\right\} = \chi(-s)\psi(b_1+B_1 s),\ m>0$$

$$\text{(ii)} \quad M\left\{\frac{\partial}{\partial a_1}H_{p,q}^{m,n}(x)\right\} = -\chi(-s)\psi(1-a_1-A_1 s), n>0$$

$$\text{(iii)}\quad M\left\{\frac{\partial}{\partial a_p}H_{p,q}^{m,n}(x)\right\} = -\chi(-s)\psi(a_p + A_p s), n < p$$

$$\text{(iv)}\quad M\left\{\frac{\partial}{\partial b_q}H_{p,q}^{m,n}(x)\right\} = \chi(-s)\psi(1 - b_q - B_q s), m < q$$

$$\text{(v)}\quad M\left\{\frac{\partial}{\partial B_1}H_{p,q}^{m,n}(x)\right\} = s\chi(-s)\psi(b_1 + B_1 s), m > 0$$

$$\text{(vi)}\quad M\left\{\frac{\partial}{\partial A_1}H_{p,q}^{m,n}(x)\right\} = -s\chi(-s)\psi(1 - a_1 - A_1 s), n > 0$$

$$\text{(vii)}\quad M\left\{\frac{\partial}{\partial A_p}H_{p,q}^{m,n}(x)\right\} = -s\chi(-s)\psi(a_p + A_p s), n < p$$

$$\text{(viii)}\quad M\left\{\frac{\partial}{\partial B_q}H_{p,q}^{m,n}(x)\right\} = s\chi(-s)\psi(1 - b_q - B_q s), m < q$$

where M denotes the Mellin transform, ψ is the psi-function and $\chi(s)$ is given as $\Theta(s)$ in (1.3). (Buschman 1974a, p. 151).

1.10. Prove that

$$H_{1,2}^{1,1}\left[z\Big|_{(a,A),(0,1)}^{(a,A)}\right] = A^{-1}\sum_{k=0}^{\infty}\frac{(-1)^k z^{\frac{(k+a)}{A}}}{\Gamma\left(1 + \frac{(k+a)}{A}\right)},$$

where $A > 0$.

1.11. Prove that

$$H_{2,1}^{1,1}\left[z\Big|_{(1-a,A)}^{(1-a,A),(1,1)}\right] = A^{-1}\sum_{k=0}^{\infty}\frac{(-1)^k z^{-\frac{(a-k-1)}{A}}}{\Gamma\left(1 + \frac{(a-k-1)}{A}\right)},$$

where $A > 0$.

1.12. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}z}H_{1,2}^{1,1}\left[z\Big|_{(a,A),(0,1)}^{(a,A)}\right] = H_{1,2}^{1,1}\left[z\Big|_{(a-A,A),(0,1)}^{(a-A,A)}\right],\ A > 0.$$

1.13. Prove that

$$\frac{1}{2\pi i}\int_L \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta - \alpha s)}(-z)^{-s}\mathrm{d}s = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta + \alpha k)}.$$

1.14. Prove that

$$Z_\rho^\nu(z) = -\frac{1}{\rho} H_{1,1}^{1,1}\left[z\Big|_{(0,1)}^{(1-\frac{\nu}{\rho},-\frac{1}{\rho})}\right], z \in C, z \neq 0, \rho < 0, \Re(\nu) < 0.$$

1.15. Evaluate

$$f(z) = \frac{1}{2\pi i}\int_C \Gamma(s-a) z^{-s} \mathrm{d}s,$$

where C is a loop which embraces all the poles of $\Gamma(s-a)$ at the points $s = a - \nu, \nu \in N_0$.

1.16. Prove that the Mellin–Barnes integral (Paris and Kaminski 2001, p. 113)

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(-s)}{s+a} z^{a+s} \mathrm{d}s,$$

defines the incomplete gamma function $\gamma(a,z)$ defined by $\gamma(a,z) = \int_0^z t^{a-1}\mathrm{e}^{-t}\mathrm{d}t$, where $|\arg(z)| < \frac{\pi}{2}$ and the contour C separates the poles at $s = -\nu, \nu \in N_0$ from the pole $s = -a$ (a is not a positive integer).

1.17. Prove that the Wright function (or Dotsenko function) ${}_2R_1(a,b;c;\omega,\mu;z)$ can be expressed by the Mellin–Barnes integral (Kilbas et al. 2006, p. 123) in the form

$${}_2R_1(a,b;c;\omega,\mu;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i}\int_L \frac{\Gamma(s)\Gamma(a-s)\Gamma(b-\frac{\omega s}{\mu})}{\Gamma(c-\frac{\omega s}{\mu})}(-z)^{-s}\mathrm{d}s,$$

where the contour of integration $L = L_{-\infty}$ separates all poles of $\Gamma(s)$ to the left and all the poles of $\Gamma(1-s)$ and $\Gamma(b-\frac{\omega s}{\mu})$ to the right.

1.18. Prove that (Braaksma 1964, p. 289)

$$G_{p,q}^{q,0}\left[x\Big|_{(b_q)}^{(a_p)}\right] = \frac{(2\pi)^{\frac{1}{2}(a-1)}}{\sqrt{(a)}} x^{\frac{(1-b)}{a}} \mathrm{e}^{-ax^{\frac{1}{a}}}[1 + O(x^{-\frac{1}{a}})], \text{ as } x \to \infty,$$

$$a = q - p, b = \sum_{i=1}^{p} a_i - \sum_{j=1}^{q} b_j + \frac{1}{2}(q - p + 1).$$

1.19. Prove that the function $\lambda_\gamma^{(n)}(z)$ defined by the integral

$$\lambda_\gamma^{(n)}(z) = \frac{(2\pi)^{\frac{(n-1)}{2}}\sqrt{n}}{\Gamma(\gamma+1-\frac{1}{n})}\left(\frac{z}{n}\right)^{n\gamma}\int_1^\infty (t^n - 1)^{\gamma-\frac{1}{n}}\mathrm{e}^{-zt}\mathrm{d}t,$$

for $n \in N, \Re(\gamma) > \frac{1}{n} - 1, \Re(z) > 0$ can be expressed in terms of the H-function as

$$H_{1,2}^{2,0}\left[z\Big|_{(n\gamma,1),(0,\frac{1}{n})}^{(\gamma+1-\frac{1}{n},\frac{1}{n})}\right] = (2\pi)^{\frac{(1-n)}{2}} n^{n\gamma+\frac{1}{2}} \lambda_\gamma^{(n)}(z).$$

1.20. Prove that the function $\lambda_{\gamma,\sigma}^{(\beta)}(z)$ defined by the integral

$$\lambda_{\gamma,\sigma}^{(\beta)}(z) = \frac{\beta}{\Gamma(\gamma+1-\frac{1}{\beta})} \int_1^\infty (t^\beta - 1)^{\gamma-\frac{1}{\beta}} \mathrm{e}^{-zt} \mathrm{d}t,$$

for $\beta > 0, \Re(\gamma) > \frac{1}{\beta} - 1, \Re(z) > 0, \sigma \in C$, can be expressed in terms of the H-function as

$$H_{1,2}^{2,0}\left[z\Big|_{(n\gamma,1),(-\gamma-\frac{\sigma}{\beta},\frac{1}{\beta})}^{(1-\frac{(\sigma+1)}{\beta},\frac{1}{\beta})}\right] = \lambda_{\gamma,\sigma}^{(\beta)}(z).$$

1.21. Prove the following results:

$$\lambda_\gamma^{(2)}(z) = 2\left(\frac{z}{2}\right)^\gamma K_{-\gamma}(z),$$

and

$$\lambda_{\gamma,0}^{2}(z) = \frac{2}{\sqrt{\pi}}\left(\frac{2}{z}\right)^\gamma K_{-\gamma}(z),$$

where $K_{-\gamma}(z)$ is the modified Bessel function of the third kind.

Notation 1.15. Multi-index Mittag-Leffler functions: $E_{\left(\frac{1}{\rho_i}\right),(\mu_i)}(z)$.

Definition 1.15. Let $m = 1$ be an integer, $\rho_1, \ldots, \rho_m > 0$ and $\mu_1, \ldots, \mu_m$ be arbitrary real numbers. By means of "multi-indices" $(\rho_i), (\mu_i)$, the so-called multi-index (m-tuple, multiple) Mittag-Leffler functions are introduced (Kiryakova 2000, p. 244) as

$$E_{\left(\frac{1}{\rho_i}\right),(\mu_i)}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots\Gamma(\mu_m+\frac{k}{\rho_m})}. \tag{1.243}$$

1.22. Prove that the multi-index Mittag-Leffler functions in Definition 1.15 can be expressed as follows:

$$\begin{aligned} E_{\left(\frac{1}{\rho_i}\right),(\mu_i)}(z) &= {}_1\Psi_m\left[z\Big|_{(\mu_1,\frac{1}{\rho_1}),\ldots,(\mu_m,\frac{1}{\rho_m})}^{(1,1)}\right] \\ &= H_{1,m+1}^{1,1}\left[-z\Big|_{(0,1),(1-\mu_1,\frac{1}{\rho_1}),\ldots,(1-\mu_m,\frac{1}{\rho_m})}^{(0,1)}\right]. \end{aligned}$$

1.23. For the multi-index function $E_{(\frac{1}{\rho_1}),(\mu_i)}(z)$ prove the following result (Saxena et al. 2003, p. 369): For $\rho_i > 0, \mu_i > 0, i = 1, \ldots, m, r \in N$ there holds the formula

$$z^r E_{(\frac{1}{\rho_i}),(\mu_i+\frac{r}{\rho_i})}(z) = E_{(\frac{1}{\rho_i}),(\mu_i)}(z) - \sum_{h=0}^{r-1} \frac{z^h}{\prod_{j=1}^m \Gamma(\mu_j + \frac{h}{\rho_j})}.$$

1.24. Prove the following asymptotic estimates for the Mittag-Leffler function $E_\alpha(z)$. For $0 < \alpha < 2$ show that

$$E_\alpha(z) \sim \begin{cases} \frac{1}{\alpha}\exp(z^{\frac{1}{\alpha}}) - \sum_{k=1}^\infty \frac{z^{-k}}{\Gamma(1-\alpha k)}, |\arg(z)| < \frac{3}{2}\pi\alpha \\ -\sum_{k=1}^\infty \frac{z^{-k}}{\Gamma(1-\alpha k)}, |\arg(-z)| < \frac{1}{2}\pi(2-\alpha) \end{cases},$$

as $|z| \to \infty$. Further, show that for $\alpha > 2$ the following asymptotic estimate holds:

$$E_\alpha(z) \sim \frac{1}{\alpha} \sum_{r=-N}^{N} \exp\{z^{\frac{1}{2}} e^{\frac{2\pi i r}{\alpha}}\} - \sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\alpha k)}, -\pi < \arg(z) \le \pi,$$

as $|z| \to \infty$, where $N = \left[\frac{1}{2}\alpha - \frac{1}{2}\right]$, (Paris and Kaminski 2001, p. 189).

Chapter 2
H-Function in Science and Engineering

2.1 Integrals Involving H-Functions

This chapter deals with integrals involving H-functions. We propose to present the results for Mellin, Laplace, Hankel, Bessel, and Euler transforms of the H-functions. Further, on account of the importance and considerable popularity achieved by fractional calculus, that is, the calculus of fractional integrals and fractional derivatives of arbitrary real or complex orders, during the last four decades due to its applications in various fields of science and engineering, such as fluid flow rheology, diffusive transport akin to diffusion, electric networks and probability, the discussion of H-function is more relevant. In this connection, one can refer to the work of Phillips (1989, 1990), Bagley (1990), Bagley and Torvik (1986) and Somorjai and Bishop (1970) and the book by Podlubny (1999). In the present book, fractional integration and fractional differentiation of the H-functions will be discussed. A long list of papers on integrals of the H-functions is available from the bibliography of the books by Mathai and Saxena (1978), Srivastava et al. (1982), Prudnikov et al. (1990) and Kilbas and Saigo (2004).

2.2 Integral Transforms of the H-Function

2.2.1 Mellin Transform

In order to present the results of this section, a few notations and definitions are given first

Notation 2.1. $M\{f(t) \ : \ s\}, f^*(s)$, Mellin transform of f with respect to a parameter s.

Notation 2.2. $M^{-1}\{f^*(s); x\}$: Inverse Mellin transform

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Definition 2.1. The Mellin transform of a function $f(t)$, denoted by $f^*(s)$, is defined by

$$f^*(s) = M[f(t); s] = \int_0^\infty t^{s-1} f(t)\mathrm{d}t, \quad t > 0, \tag{2.1}$$

provided that the integral converges. The inverse Mellin transform is given by the contour integral

$$f(x) = M^{-1}\{f^*(s); x\} = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} f^*(s)x^{-s}\mathrm{d}s. \tag{2.2}$$

If $f^*(s)$ is analytic in the relevant strip then $f(x)$ is uniquely determined by $f^*(s)$ by using the formula (2.2).

2.2.2 Illustrative Examples

Example 2.1. Find the Mellin transform of Gauss hypergeometric function ${}_2F_1$.

Solution 2.1. By definition (2.1), we have to evaluate the integral

$$I = \int_0^\infty t^{s-1}{}_2F_1(a, b : c : -t)\mathrm{d}t,$$

where $a, b, c \in C$, $\min\{\Re(a), \Re(b)\} > \Re(s) > 0$.

If we use Euler integral representation of the hypergeometric function then the given integral becomes

$$\begin{aligned} I &= \int_0^\infty t^{s-1} \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1}(1-u)^{c-b-1}(1+tu)^{-a}\mathrm{d}u\mathrm{d}t \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1}(1-u)^{c-b-1} \int_0^\infty \frac{t^{s-1}}{(1+tu)^a}\mathrm{d}t\mathrm{d}u \\ &= \frac{\Gamma(s)\Gamma(a-s)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-s-1}(1-u)^{c-b-1}\mathrm{d}u = \frac{\Gamma(s)\Gamma(a-s)\Gamma(c)\Gamma(b-s)}{\Gamma(a)\Gamma(b)\Gamma(c-s)}, \end{aligned} \tag{2.3}$$

for $\Re(s) > 0, \Re(a-s) > 0, \Re(b-s) > 0, \Re(c-s) > 0$. The interchange of the order of integration in the above steps is justified under the conditions given along with the integral. This completes the solution of Example 2.1.

Example 2.2. Find the inverse Mellin transform of the right side in (2.3).

Solution 2.2. By virtue of the results (2.2) and (2.3), we find that

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\Gamma(a-s)\Gamma(b-s)}{\Gamma(c-s)} (-z)^{-s} ds, \quad (2.4) $$

where $a,b,c \in C$, $\min\{\Re(a),\Re(b)\} > \gamma > 0$ and $c \neq 0,-1,-2,\cdots.;|arg(-z)| < \pi$. The path of integration separates the poles at $s = a+m,\ s = b+m$ from the poles at $s = -m, m \in N_0$.

Example 2.3. Prove that the Mellin transform of the generalized Mittag-Leffler function $E^{\gamma}_{\alpha,\beta}(z)$, defined by (1.47), is given by

$$ M\left\{E^{\gamma}_{\alpha,\beta}(-z);s\right\} = \frac{\Gamma(s)\Gamma(\gamma-s)}{\Gamma(\gamma)\Gamma(\beta-\alpha s)}, \quad (2.5) $$

where $\Re(s) > 0, \Re(\gamma-s) > 0; \alpha \in R^+, \beta,\gamma \in C, \beta \neq 0,-1,-2,\ldots$ and when $\Gamma(\gamma)$ is defined.

The result (2.5) follows from Example 1.5, and (2.2).

Note 2.1. From (2.5), we see that the Mellin transforms of the Mittag Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$, defined by (1.44) and (1.45) respectively, are given by

$$ M\left\{E_\alpha(-z);s\right\} = \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1-as)}, 0 < \Re(s) < 1, \quad (2.6) $$

and

$$ M\{E_{\alpha,\beta}(-z);s\} = \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-as)}, 0 < \Re(s) < 1. \quad (2.7) $$

In what follows, the H-functions considered satisfy the condition Eq. (1.6) and α,β,μ and δ have the values given in (1.13), (1.8), (1.9), and (1.10), respectively.

2.2.3 Mellin Transform of the H-Function

In view of the Mellin inversion formula (see, Titchmarsh (1986), Sect. 1.5) the Mellin transform of the H-function follows from the Definition 1.1). We have

$$ \int_0^\infty x^{s-1} H^{m,n}_{p,q}\left[ax\middle|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right] dx $$
$$ = a^{-s} \frac{\left[\prod_{j=1}^{m}\Gamma(b_j+B_js)\right]\left[\prod_{j=1}^{n}\Gamma(1-a_j-A_js)\right]}{\left[\prod_{j=m+1}^{q}\Gamma(1-b_j-B_js)\right]\left[\prod_{j=n+1}^{p}\Gamma(a_j+A_js)\right]}, \quad (2.8) $$

where $a, s \in C; -\min_{1\le j\le m} \Re\left(\frac{b_j}{B_j}\right) < \Re(s) < \max_{1\le i\le n}\left[\frac{1-\Re(a_j)}{A_j}\right]$, $|\arg\, a| < \frac{1}{2}\pi\alpha$, $\alpha > 0$. Further, $\mu\Re(s) + \Re(\delta) < -1$, when $\alpha = 0, \arg a = 0$ and $a \neq 0$.

2.2.4 Mellin Transform of the G-Function

If we set $A_j = B_j = 1$, for all i and j and use the identity (1.112), we obtain the Mellin transform of the G-function.

$$\int_0^\infty x^{s-1} G_{p,q}^{m,n}\left[ax\middle|\begin{matrix}(a_p)\\(b_q)\end{matrix}\right] \mathrm{d}x = a^{-s}\frac{\left[\prod_{j=1}^m \Gamma(b_j+s)\right]\left[\prod_{j=1}^n \Gamma(1-a_j-s)\right]}{\left[\prod_{j=m+1}^q \Gamma(1-b_j-s)\right]\left[\prod_{j=n+1}^p \Gamma(a_j+s)\right]}, \tag{2.9}$$

where $a, s \in C, -\min_{1\le j\le m} \Re(b_j) < \Re(s) < 1 - \max_{1\le i\le n} \Re(a_i)$, $|\arg\, a| < \frac{1}{2}\pi c^*, c^* > 0$ and c^* is defined in (1.22).

2.2.5 Mellin Transform of the Wright Function

$$\int_0^\infty x^{s-1}\ {}_p\Psi_q\left[\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\middle| -ax\right] \mathrm{d}x = a^{-s}\frac{\Gamma(s)\left[\prod_{j=1}^p \Gamma(a_j - A_j s)\right]}{\left[\prod_{j=1}^q \Gamma(b_j - B_j s)\right]}, \tag{2.10}$$

where $s \in C, \Re(s) > 0, \Re(a_j - A_j s) > 0, j = 1, \cdots, p$, $|\arg\, a| < \frac{1}{2}\pi b$, $b = 1 + \sum_{i=1}^p A_i - \sum_{j-1}^q B_j$; $\mu > -1$ and μ is defined in (1.9).

2.2.6 Laplace Transform

Notation 2.3. $F(s) = L\{f(t); s\} = (Lf)(s)$: Laplace transform of $f(t)$ with parameter s

Notation 2.4. $L^{-1}\{F(s); t\}$: Inverse Laplace transform

Definition 2.2. The Laplace transform of a function $f(t)$, denoted by $F(s)$, is defined by the integral equation

$$F(s) = L\{f(t); s\} = (L\ f)(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\mathrm{d}t, \tag{2.11}$$

where $\Re(s) > 0$, which may be symbolically written as

$$F(s) = L\{f(t); s\} \text{ or } f(t) = L^{-1}\{F(s); t\},$$

provided that the function $f(t)$ is continuous for $t \geq 0$, it being tacitly assumed that the integral in (2.11) exists.

Definition 2.3. The inverse Laplace transform is given by the contour integral

$$f(t) = L^{-1}\{F(s); t\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} F(s)ds. \tag{2.12}$$

2.2.7 *Illustrative Examples*

Example 2.4. Find the Laplace transform of the Mittag-Leffler function $x^{\beta-1}E_{\alpha,\beta}(ax^{\alpha})$.

Solution 2.3. We have

$$\begin{aligned} L\{t^{\beta-1}E_{\alpha,\beta}(ax^{\alpha}); s\} &= \int_0^{\infty} e^{-sx}x^{\beta-1}E_{\alpha,\beta}(ax^{\alpha})dx \\ &= \int_0^{\infty} x^{\beta-1}e^{-sx}\sum_{k=0}^{\infty}\frac{a^k x^{\alpha k}}{\Gamma(ak+\beta)}dx \\ &= \sum_{k=0}^{\infty}\frac{a^k}{\Gamma(ak+\beta)}\int_0^{\infty} e^{-sx}x^{ak+\beta-1}dx \\ &= \frac{s^{\alpha-\beta}}{s^{\alpha}-a}, \Re(\alpha) > 0, \Re(\beta) > 0, |as^{-\alpha}| < 1. \end{aligned} \tag{2.13}$$

Note 2.2. We note from the above result that

$$L\{E_{\alpha}(ax^{\alpha}; s)\} = \frac{s^{\alpha-1}}{s^{\alpha}-a}, \tag{2.14}$$

where $a, s, \alpha \in C, \Re(\alpha) > 0, \Re(s) > 0$, and $|as^{-\alpha}| < 1$.

Example 2.5. Find the inverse Laplace transform of $s^{-\beta}(1-as^{-\alpha})^{-\gamma}$.

Solution 2.4. We have

$$L^{-1}\{s^{-\beta}(1-as^{-\alpha})^{-\gamma}; x\} = L^{-1}\left\{\sum_{k=0}^{\infty}\frac{(\gamma)_k a^k s^{-ak-\beta}}{k!}; x\right\}.$$

Applying the formula

$$L^{-1}\{s^{-\rho};x\}=\frac{x^{\rho-1}}{\Gamma(\rho)},\rho,s\in C,\Re(s)>0,\Re(\rho)>0, \tag{2.15}$$

the above line reduces to

$$x^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k(ax^{\alpha})^k}{\Gamma(\alpha k+\beta)k!}=x^{\beta-1}E^{\gamma}_{\alpha,\beta}(ax^{\alpha}), \tag{2.16}$$

where $\alpha,a,\beta,\gamma\epsilon C,\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0,|as^{-\alpha}|<1$ is the generalized Mittag-Leffler function defined in (1.46).

Remark 2.1. When $\gamma=1$, Example 2.5 gives the interesting transform pair

$$L^{-1}\{s^{-\beta}(1-as^{-\alpha})^{-1};x\}=x^{\beta-1}E_{\alpha,\beta}(at^{\alpha}), \tag{2.17}$$

where $\alpha,\beta,a\epsilon C,\Re(\alpha)>0,\Re(\beta)>0$, and $|as^{-\alpha}|<1$. For $\beta=1$, (2.17) reduces to

$$L^{-1}\{s^{-1}(1-as^{-\alpha})^{-1};x\}=E_{\alpha}(ax^{\alpha}), \tag{2.18}$$

where $a,\ \alpha\epsilon C,\Re(\alpha)>0,|as^{-\alpha}|<1$.

2.2.8 Laplace Transform of the H-Function

Let either $\alpha>0$, $|\arg a|<\frac{1}{2}\pi\alpha$ or $\alpha=0$ and $\Re(\delta)<-1$. Further assume that $\alpha>0;\ \rho,\alpha,s\ \epsilon\ C,\ \sigma>0$, satisfy the condition

$$\Re(\rho)+\sigma\min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}\right]>0 \text{ for } \alpha>0 \text{ or } \alpha=0,\mu\ge 0; \text{ and}$$

$$\Re(\rho)+\sigma\min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}+\frac{\Re(\delta)+\frac{1}{2}}{\mu}\right]>0,$$

for $\alpha=0$ and $\mu<0$. Then for $\Re(s)>0$, there holds the formula

$$L\left\{x^{\rho-1}H^{m,n}_{p,q}\left[ax^{\sigma}\left|\begin{matrix}(a_p,A_p)\\ \\ (b_q;b_q)\end{matrix}\right.\right];s\right\}=s^{-\rho}H^{m,n+1}_{p+1,q}\left[as^{-\sigma}\left|\begin{matrix}(1-\rho,\sigma),(a_p,A_p)\\ \\ (b_q,B_q)\end{matrix}\right.\right], \tag{2.19}$$

for $\Re(s)>0,s\in C$. (2.19) can be established by virtue of the definition of the H-function (1.2) and the well-known gamma function formula.

2.2.9 Inverse Laplace Transform of the H-Function

Due to the importance and utility of inverse Laplace transforms of special functions in physical problems, we present the inverse Laplace transform of the H-function in this section.

By virtue of the cancelation law for the H-function (1.56), the result (2.19) can be written in the form

$$L\left\{x^{\rho-1}H_{p,q+1}^{m,n}\left[ax^{\sigma}\Big|_{(b_q;b_q),(1-\rho,\sigma)}^{(a_p,A_p)}\right];s\right\} = s^{-\rho}H_{p,q}^{m,n}\left[as^{-\sigma}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right], \tag{2.20}$$

If we use the property of the H-function from Mathai and Saxena (1978, p. 4, Eq. (1.38)) then the desired result follows:

$$L^{-1}\left\{s^{-\rho}H_{p,q}^{m,n}\left[as^{\sigma}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right];t\right\} = t^{\rho-1}H_{p+1,q}^{m,n}\left[at^{-\sigma}\Big|_{(b_q,B_q)}^{(a_p,A_p),(\rho,\sigma)}\right], \tag{2.21}$$

where $\rho, a, s \in C, \Re(s) > 0, \sigma > 0, \Re(\rho)+\sigma \max_{1\le i\le n}\left[\frac{1}{A_i} - \frac{\Re(a_i)}{A_i}\right] > 0, |\arg a| < \frac{1}{2}\pi\theta, \theta = \alpha - \sigma$. Two interesting special cases of (2.21), which are applicable in fractional diffusion problems, are given below. If we use the identity

$$H_{0,1}^{1,0}\left[x\Big|_{(\alpha,1)}\right] = x^{\alpha}\mathrm{e}^{-x}, \tag{2.22}$$

we obtain

$$L^{-1}\{s^{-\rho}\exp(-as^{\sigma});t\} = t^{\rho-1}H_{1,1}^{1,0}\left[at^{-\sigma}\Big|_{(0,1)}^{(\rho,\sigma)}\right], \tag{2.23}$$

where $\Re(s) > 0, \Re(a) > 0, \sigma > 0$. Further, if we employ the identity

$$H_{0,2}^{2,0}\left[x\Big|_{(\frac{\nu}{2},1),(-\frac{\nu}{2},1)}\right] = 2K_{\nu}(2x^{\frac{1}{2}}), \tag{2.24}$$

we obtain

$$2L^{-1}\{s^{-\rho}K_{\nu}(as^{\sigma});x\} = x^{\rho-1}H_{1,2}^{2,0}\left[\frac{a^2x^{-2\sigma}}{4}\Big|_{(\frac{\nu}{2},1),(-\frac{\nu}{2},1)}^{(\rho,2\sigma)}\right], \tag{2.25}$$

where $\Re(s) > 0, \Re(a^2) > 0, \sigma > 0$, and $K_{\nu}(x)$ is the modified Bessel function of the third kind or Macdonald function.

Remark 2.2. It will not be out of place to mention here that one-sided Lévy stable density can be obtained from the above result by virtue of the identity (Mathai and Saxena 1973a)

$$K_{\pm\frac{1}{2}}(x) = \left[\frac{\pi}{2x}\right]^{\frac{1}{2}} \mathrm{e}^{-x}, \tag{2.26}$$

and can be conveniently expressed in terms of the Laplace transform

$$\int_0^\infty \mathrm{e}^{-sx}\Phi_\rho(x)\mathrm{d}x = \exp(-s^\rho), \rho, s \in C,\ \Re(s) > 0, \Re(\rho) > 0, \tag{2.27}$$

where

$$\Phi_\rho(x) = \frac{1}{\rho} H_{1,1}^{1,0}\left[\frac{1}{x}\Big|_{\left(\frac{1}{\rho},\frac{1}{\rho}\right)}^{(1,1)}\right],\ \rho > 0. \tag{2.28}$$

This result is obtained earlier by Schneider and Wyss (1989) by following a different procedure. Asymptotic expansion of $\Phi_\alpha(x)$ is given by Schneider (1986).

2.2.10 Laplace Transform of the G-Function

In what follows, the G-functions involved satisfy the existence conditions. When $A_i = B_j = 1$ for all i and j, the H-function reduces to a G-function and consequently we arrive at the following result:

$$L\left\{x^{\rho-1} G_{p,q}^{m,n}\left[ax^\sigma\Big|_{b_q}^{a_p}\right]; s\right\} = s^{-\rho} H_{p+1,q}^{m,n+1}\left[as^{-\sigma}\Big|_{(b_q,1)}^{(1-\rho,\sigma),(a_p,1)}\right], \tag{2.29}$$

where $\rho, s \in C, \Re(s) > 0, \sigma > 0, \Re(\rho) + \sigma \min_{1\le j\le m} \Re(b_j) > 0, |\arg \alpha| < \frac{1}{2}c^*, c^* > 0, c^*$ is defined in (1.21).

If we set $\sigma = \frac{k}{\lambda}, k, \lambda \in N$ in (2.29), we arrive at a result given by Saxena (1960, p. 402):

$$\begin{aligned} &L\left\{x^{\rho-1} G_{p,q}^{m,n}\left[ax^{\frac{k}{\lambda}}\Big|_{b_q}^{a_p}\right]; s\right\} \\ &= s^{-\rho}(2\pi)^{(1-\lambda)c^*+\frac{1}{2}(1-k)}\lambda^{\delta+1}k^{\rho-\frac{1}{2}} \\ &\times H_{\lambda p+k,\lambda q}^{\lambda m,\lambda n+k}\left[\frac{k^k a^\lambda s^{-k}}{\lambda^{(q-p)\lambda}}\Big|_{\Delta(\lambda;b_1),\ldots,\Delta(\lambda;b_q)}^{\Delta(k;1-\rho),\Delta(\lambda,a_1),\ldots,\Delta(\lambda;a_p)}\right], \end{aligned} \tag{2.30}$$

where $\rho, s \in C, \Re(s) > 0, \Re(\rho) + \left(\frac{k}{\lambda}\right) \min_{1\le j\le m} \Re(b_j) > 0, |\arg a| < \frac{1}{2}\pi c^*, c^* > 0, c^*$ is defined in equation (1.21) and the existence conditions of the G-function are satisfied. Here, $\Delta(k; b)$ represents the sequence

$$\frac{b}{k}, \frac{b+1}{k}, \ldots, \frac{b+k-1}{k}, k \in N.$$

Several special cases of the general result (2.30) can be obtained by using the tables of the special cases of the G-function (Mathai and Saxena 1973a; Mathai 1993c) but for brevity one interesting case is presented here, associated with the Whittaker function, given by Saxena (1960, p. 404, Eq. (15))

$$L\left\{x^{\rho-1}\exp\left(-\frac{1}{2}ax^{-\frac{k}{\lambda}}\right)W_{\tau,\nu}(ax^{-\frac{k}{\lambda}});s\right\} = s^{-\rho}(2\pi)^{\frac{1}{2}(2-k-\lambda)}\lambda^{\tau+\frac{1}{2}}k^{\rho-\frac{1}{2}} \times G^{2\lambda+k,0}_{\lambda,2\lambda+k}\left[\frac{a^{\lambda}s^{k}}{\lambda^{\lambda}k^{k}}\Bigg|^{\Delta(\lambda;1-\tau)}_{\Delta(2\lambda;1\pm 2\nu),\Delta(k;\rho)}\right], \tag{2.31}$$

where $\rho, s \in C, \Re(a) > 0, \Re(s) > 0$. One interesting particular case of (2.31) can be obtained by using the identity

$$W_{0,\pm\frac{1}{2}}(x) = \exp\left(-\frac{1}{2}x\right).$$

That is,

$$L\left\{x^{\rho-1}\exp(-ax^{-\frac{k}{\lambda}});s\right\} = s^{-\rho}(2\pi)^{\frac{1}{2}(2-k-\lambda)}k^{\rho-\frac{1}{2}}\lambda^{\frac{1}{2}} \times G^{\lambda+k,0}_{0,\lambda+k}\left[\frac{a^{\lambda}s^{k}}{\lambda^{\lambda}k^{k}}\Bigg|_{\Delta(\lambda;0),\Delta(k;\rho)}\right], \tag{2.32}$$

where $\rho, s \in C, \Re(a) > 0, \Re(s) > 0$.

Remark 2.3. The result (2.32) is very useful in problems of physics. Regarding its application in nuclear and neutrino astrophysics, one can refer to the monograph of Mathai and Haubold (1988). An alternative derivation of this result based on statistical techniques is given by Mathai (1971).

2.2.11 K-Transform

Notation 2.5. $R_{\nu}\{f(x);p\}$: K-Transform

Definition 2.4. The transform defined by the following integral equation

$$R_{\nu}\{f(x);p\} = g(p;\nu) = \int_0^{\infty}(px)^{\frac{1}{2}}K_{\nu}(px)f(x)\mathrm{d}x, \tag{2.33}$$

is called the K-transform with p as a complex parameter.

This transform was defined by Meijer (1940) who obtained its inversion formula and representation theorems. Its inversion formula is given by

$$G(p) = \frac{1}{\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} (px)^{\frac{1}{2}} I_v(xp) g(p) \mathrm{d}p, \tag{2.34}$$

where $I_v(x)$ is Bessel function of the first kind defined by

$$I_v(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{v+2k}}{k!\Gamma(v+k+1)}. \tag{2.35}$$

2.2.12 K-Transform of the H-Function

Let us assume that either $\alpha > 0, |\arg b| < \frac{1}{2}\pi\alpha$ or $\alpha = 0$ and $\Re(\delta) < -1$. Further assume that $\alpha > 0; \rho, v, a, b \in C, \sigma > 0$ satisfy the condition

$$\Re(\rho) + |\Re(v)| + \sigma \min_{1 \le j \le m} \left[\frac{\Re(b_j)}{B_j} \right] > 0,$$

for $\alpha > 0, |\arg b| < \frac{1}{2}\pi\alpha$ or $\alpha = 0, \mu \ge 0$; and

$$\Re(\rho) + |\Re(v)| + \sigma \min_{1 \le j \le m} \left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta) + \frac{1}{2}}{\mu} \right] > 0,$$

for $\alpha = 0$ and $\mu < 0$. Then for $\Re(a) > 0, \sigma > 0$ there holds the formula

$$\int_0^\infty x^{\rho-1} K_v(ax) H_{p,q}^{m,n} \left[bx^\sigma \Big|_{(b_q, B_q)}^{(a_p, A_p)} \right] \mathrm{d}x$$
$$= 2^{\rho-2} a^{-\rho} H_{p+2,q}^{m,n+2} \left[b \left(\frac{2}{a} \right)^\sigma \Big|_{(b_q, B_q)}^{(1-\frac{1}{2}(\rho-v), \frac{\sigma}{2}), (1-\frac{1}{2}(\rho+v), \frac{\sigma}{2}), (a_p, A_p)} \right]. \tag{2.36}$$

The result (2.36) can be computed directly from the definition of the H-function (1.2) if we use the formula (Mathai and Saxena 1973, p. 78)

$$\int_0^\infty x^{\rho-1} K_v(ax) \mathrm{d}x = 2^{\rho-2} a^{-\rho} \Gamma\left(\frac{\rho \pm v}{2} \right), \tag{2.37}$$

where $\Re(s) > |\Re(v)|, \Re(a) > 0$. If we apply the definition (1.2) of the H-function to the given integral then we have

$$\int_0^\infty x^{\rho-1} K_\nu(ax) H_{p,q}^{m,n}\left[bx^\sigma \Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] \mathrm{d}x$$
$$= \frac{1}{2\pi i}\int_{L_{i\gamma\infty}} \Theta(s) b^{-s} \int_0^\infty x^{\rho-s\sigma-1} K_\nu(ax)\mathrm{d}x\mathrm{d}s$$
$$= 2^{\rho-2} a^{-\rho} \frac{1}{2\pi i}\int_{L_{i\gamma\infty}} \Gamma\left(\frac{\rho \pm \nu - \sigma s}{2}\right) \Theta(s) b^{-s} \left(\frac{2}{a}\right)^{-\sigma s} \mathrm{d}s,$$

and the result (2.36) readily follows from the definition of the H-function (1.2).

Remark 2.4. When $\nu = \pm\frac{1}{2}$ in (2.36) then by virtue of the identity (2.26) one can obtain the Laplace transform of the H-function with argument $bx^\sigma, \sigma > 0$.

2.2.13 Varma Transform

Notation 2.6. $V(f, k, m, s)$: Varma transform

Definition 2.5. Varma transform is defined by the integral equation

$$V(f, k, m; s) = \int_0^\infty (sx)^{m-\frac{1}{2}} \exp\left(-\frac{1}{2}sx\right) W_{k,m}(sx) f(x)\mathrm{d}x, \quad \Re(s) > 0, \tag{2.38}$$

where $W_{k,m}$ represents a Whittaker function, defined by

$$W_{k,m}(z) = \sum_{m,-m} \frac{\Gamma(-2m)}{\Gamma\left(\frac{1}{2}-k-m\right)} M_{k,m}(z), \tag{2.39}$$

where the summation symbol indicates that the expression following it, a similar expression with m replaced by $-m$ is to be added. For the definition of $M_{k,m}(z)$ see, Sect. 1.8.1. This transform is introduced by Varma (1951), who gave some inversion formulae for this transform.

2.2.14 Varma Transform of the H-Function

Let $\alpha > 0, |\arg b| < \frac{1}{2}\pi\alpha$ or $\alpha = 0$ and $\Re(\delta) < -1$; further, $\nu, a, k, b\rho \in C, \sigma > 0$,

$$\Re(\rho) + |\Re(\nu)| + \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}\right] > -\frac{1}{2},$$

for $\alpha > 0, |\arg b| < \frac{1}{2}\pi\alpha$ or $\alpha = 0, \mu \ge 0$ and

$$\Re(\rho) + |\Re(\nu)| + \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta)+\frac{1}{3}}{\mu}\right] > -\frac{1}{2},$$

for $\alpha = 0$ and $\mu < 0$ then for $\Re(a) > 0$, the following result holds:

$$\int_0^\infty x^{\rho-1} \exp\left(-\frac{1}{2}ax\right) W_{k,\nu}(ax) H_{p,q}^{m,n}\left[bx^\sigma \middle|_{(b_q,B_q)}^{(a_p,A_p)}\right] \mathrm{d}x$$

$$= a^{-\rho} H_{p+2,q+1}^{m,n+2}\left[\frac{b}{a^\sigma} \middle|_{(b_q,B_q),(k-\rho,\sigma)}^{(\frac{1}{2}-\nu-\rho,\sigma),(\frac{1}{2}+\nu-\rho,\sigma),(a_p,A_p)}\right], \quad (2.40)$$

which can be computed directly from the definition of the H-function (1.2) and from the following formula (Mathai and Saxena 1973, p. 79):

$$\int_0^\infty x^{\rho-1} \exp\left(-\frac{1}{2}ax\right) W_{k,\nu}(ax)\mathrm{d}x = a^{-\rho}\frac{\Gamma(\rho+\nu+\frac{1}{2})\Gamma(\rho-\nu+\frac{1}{2})}{\Gamma(1-k+\rho)}, \quad (2.41)$$

where $\Re(a) > 0, \Re(\rho \pm \nu) > -\frac{1}{2}$.

Remark 2.5. It is interesting to observe that for $k = \nu + \frac{1}{2}$ the Varma transform defined by (2.38) reduces to the Laplace transform (2.11) by virtue of the identity

$$W_{\nu+\frac{1}{2},\pm\nu}(x) = x^{\nu+\frac{1}{2}} \exp\left(-\frac{1}{2}x\right). \quad (2.42)$$

Consequently the Laplace transform of the H-function (2.19) can be derived from the result (2.40) by taking $k = \nu+\frac{1}{2}$. Certain properties of the Varma transform involving Meijer's G-functions and Whittaker functions are investigated by Saxena in a series of papers in Saxena (1960, 1962, 1964).

2.2.15 Hankel Transform

Notation 2.7. $H_\nu\{f(x);\rho\}$: Hankel transform of order ν of $f(x)$.

Definition 2.6. The Hankel transform of a function $f(x)$, denoted by $g(p;\nu)$ or in short by simply $g(p)$ is defined as

$$g(p;\nu) = \int_0^\infty (px)^{\frac{1}{2}} J_\nu(px) f(x)\mathrm{d}x,\ p > 0. \quad (2.43)$$

The inverse Hankel transform is given by

$$f(x) = \int_0^\infty (xp)^{\frac{1}{2}} J_\nu(xp) g(p)\mathrm{d}p,\quad \Re(\nu) > -1. \quad (2.44)$$

Remark 2.6. This transform is self-reciprocal. It is used in solving problems of applied mathematics and physical sciences.

2.2.16 Hankel Transform of the H-Function

Suppose that $\alpha > 0$ or $\alpha = \mu = 0$ and $\Re(\delta) < -1$. Then if $\rho, v, b \in C, \sigma > 0$ satisfy the conditions

$$\Re(\rho) + \Re(v) + \sigma \min_{1 \le j \le m} \left[\frac{\Re(b_j)}{B_j} \right] > -1,$$

and

$$\Re(\rho) + \sigma \max_{1 \le j \le n} \left[\frac{1}{A_j} - \frac{\Re(a_j)}{A_j} \right] < \frac{3}{2},$$

for $\Re(v) > -\frac{1}{2}$. Then for $a, b > 0$ there holds the formula

$$\begin{aligned} &\int_0^\infty x^{\rho-1} J_v(ax) H_{p,q}^{m,n} \left[bx^\sigma \middle|^{(a_p,A_p)}_{(b_q,B_q)} \right] \mathrm{d}x \\ &= \frac{2^{\rho-1}}{a^\rho} H_{p+2,q}^{m,n+1} \left[b \left(\frac{2}{a} \right)^\sigma \middle|^{(1-\frac{(\rho+v)}{2}, \frac{\sigma}{2}),(a_p,A_p),(1-\frac{(\rho-v)}{2}, \frac{\sigma}{2})}_{(b_q,B_q)} \right]. \end{aligned} \tag{2.45}$$

The result (2.45) can be established with the help of the definition of the H-function (1.2) and the formula

$$\int_0^\infty x^{\lambda-1} J_v(ax) \mathrm{d}x = 2^{\lambda-1} a^{-\lambda} \frac{\Gamma\left(\frac{\lambda+v}{2}\right)}{\Gamma\left(1 + \frac{v-\lambda}{2}\right)}, \tag{2.46}$$

where $a > 0, -\Re(v) < \Re(\lambda) < \frac{3}{2}$. If we use the definition of the H-function (1.2) and the result (2.46), then

$$\begin{aligned} &\int_0^\infty x^{\rho-1} J_v(ax) H_{p,q}^{m,n} \left[bx^\sigma \middle|^{(a_p,A_p)}_{(b_q,B_q)} \right] \mathrm{d}x \\ &= \frac{1}{2\pi i} \int_{L_{i\gamma\infty}} \Theta(s) b^{-s} \int_0^\infty x^{\rho-\sigma s-1} J_v(ax) \mathrm{d}x \mathrm{d}s \\ &= 2^{\rho-1} a^{-\rho} \frac{1}{2\pi i} \int_{L_{i\gamma\infty}} \Theta(s) \frac{\Gamma\left(\frac{\rho+v-\sigma s}{2}\right)}{\Gamma\left(1 + \frac{v-\rho+\sigma s}{2}\right)} b^{-s} \left(\frac{2}{a}\right)^{-\sigma s} \mathrm{d}s. \end{aligned}$$

Interpreting the above result with the help of (1.2), the result (2.45) readily follows. When $v = \pm\frac{1}{2}$ then by using the identities

$$J_{\frac{1}{2}}(x) = \sqrt{\left(\frac{2}{\pi x}\right)} \sin(x), \tag{2.47}$$

and

$$J_{-\frac{1}{2}}(x) = \sqrt{\left(\frac{2}{\pi x}\right)} \cos(x), \tag{2.48}$$

we arrive at the following results which provide the sine and cosine transforms of the H-function

$$\int_0^\infty x^{\rho-1} \sin(ax) H_{p,q}^{m,n}\left[bx^\sigma \middle|_{(b_q,B_q)}^{(a_p,A_p)}\right] \mathrm{d}x$$
$$= \frac{2^{\rho-1}\sqrt{\pi}}{a^\rho} H_{p+2,q}^{m,n+1}\left[b\left(\frac{2}{a}\right)^\sigma \middle|_{(b_q,B_q)}^{\left(\frac{(1-\rho)}{2},\frac{\sigma}{2}\right),(a_p,A_p),\left(\frac{(2-\rho)}{2},\frac{\sigma}{2}\right)}\right], \tag{2.49}$$

where $a, \alpha, \sigma > 0, \rho, b \in C; |\arg b| < \frac{1}{2}\pi\alpha;$

$$\Re(\rho) + \sigma \min_{1\le j\le m} \Re\left(\frac{b_j}{B_j}\right) > -1; \Re(\rho) + \sigma \max_{1\le j\le n} [\frac{(a_j-1)}{A_j}] < 1.$$

$$\int_0^\infty x^{\rho-1} \cos(ax) H_{p,q}^{m,n}\left[bx^\sigma \middle|_{(b_q,B_q)}^{(a_p,A_p)}\right] \mathrm{d}x = \frac{2^{\rho-1}\sqrt{\pi}}{a^\rho}$$
$$\times H_{p+2,q}^{m,n+1}\left[b\left(\frac{2}{a}\right)^\sigma \middle|_{(b_q,B_q)}^{\left(\frac{(2-\rho)}{2},\frac{\sigma}{2}\right),(a_p,A_p),\left(\frac{1-\rho}{2},\frac{\sigma}{2}\right)}\right], \tag{2.50}$$

where $a, \alpha, \sigma > 0, \rho, b \in C; |\arg b| < \frac{1}{2}\pi\alpha;$

$$\Re(\rho) + \sigma \min_{1\le j\le m} \Re\left(\frac{b_j}{B_j}\right) > 0; \Re(\rho) + \sigma \max_{1\le j\le n} \left[\frac{(a_j-1)}{A_j}\right] < 1.$$

2.2.17 Euler Transform of the H-Function

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1} H_{p,q}^{m,n}\left[bx^k(t-x)^\tau \middle|_{(b_q,B_q)}^{(a_p,A_p)}\right] \mathrm{d}x$$
$$= t^{\rho+\sigma-1} H_{p+2,q+1}^{m,n+2}\left[bt^{k+\tau} \middle|_{(b_q,B_q),(1-\rho-\sigma,k+\tau)}^{(1-\rho,k),(1-\sigma,\tau),(a_p,A_p)}\right], \tag{2.51}$$

Let $\alpha > 0$ and $|\arg b| < \frac{1}{2}\pi\alpha$ or $\alpha > 0, \Re(\delta) < -1$. The result (2.51) holds provided the parameters $\rho, \sigma, b \in C, k$ and $\tau > 0$ satisfy

$$\Re(\rho) + k \min_{1\le j\le m}\left[\Re\left(\frac{b_j}{B_j}\right)\right] > 0,\ \Re(\rho) + \tau \min_{1\le j\le m}\left[\Re\left(\frac{b_j}{B_j}\right)\right] > 0,$$

for $\alpha > 0, |\arg b| < \frac{1}{2}\pi\alpha$ or $\alpha = 0, \mu \ge 0$, and

$$\Re(\rho) + k \min_{1\le j\le m} \Re\left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta)+\frac{1}{2}}{\mu}\right] > 0, \Re(\rho) + \tau \min_{1\le j\le m} \Re\left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta)+\frac{1}{2}}{\mu}\right] > 0,$$

for $\alpha = 0$ and $\mu < 0$. The result (2.51) can be proved in the same way if we use the well-known beta function formula

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = B(\alpha,\beta). \tag{2.52}$$

where $\min\{\Re(\alpha), \Re(\beta)\} > 0$. The result (2.51) has been given by Goyal (1969). As $\tau \to 0$ in (2.51), it yields the Euler transform of the H-function:

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1} H_{p,q}^{m,n}\left[bx^k \Big|_{(b_q,B_q)}^{(a_p.A_p)}\right]\mathrm{d}x = t^{\rho+\sigma-1}\Gamma(\sigma) H_{p+1,q+1}^{m,n+1}\left[bt^k \Big|_{(b_q,B_q),(1-\rho-\sigma,k)}^{(1-\rho,k),(a_p,A_p)}\right], \tag{2.53}$$

which holds under the same condition as given with the result (2.51) with $\tau = 0$. By an obvious change of variable in (2.53) we arrive at its companion the integral

$$\int_t^\infty x^{\rho-1}(x-t)^{\sigma-1} H_{p,q}^{m,n}\left[bx^k \Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}x = t^{\rho+\sigma-1}\Gamma(\sigma) H_{p+1,q+1}^{m+1,n}\left[bt^k \Big|_{(1-\rho-\sigma,k),(b_q,B_q)}^{(a_p,A_p),(1-\rho,k)}\right]; \tag{2.54}$$

which holds for $\rho, b, \sigma \in C, \Re(\sigma) > 0, \rho \in C$ and $k > 0$; either $\alpha > 0, |\arg b| < \frac{1}{2}\pi\alpha$ or $\alpha = 0, \Re(\delta) < -1$ and the following conditions are satisfied:

$$k \max_{1\le j\le n}\left[\frac{\Re(a_j)-1}{A_j}\right] + \Re(\rho) + \Re(\sigma) < 1,$$

for $\alpha > 0, |\arg b| < \frac{1}{2}\pi\alpha$ or $\alpha = 0, \mu \le 0$; and

$$k \max_{1\le i\le n}\left[\frac{\Re(a_i)-1}{A_i}, \frac{\Re(\delta)+\frac{1}{2}}{\mu}\right] + \Re(\rho) + \Re(\sigma) < 1,$$

for $\alpha = 0$ and $\mu < 0$.

2.3 Mellin Transform of the Product of Two H-Functions

$$\int_0^\infty x^{s-1} H_{p,q}^{m,n}\left[zx^\sigma \middle| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix}\right] H_{p_1,q_1}^{m_1,n_1}\left[\eta x \middle| \begin{matrix} (d_{p_1}, D_{p_1}) \\ (e_{q_1}, E_{q_1}) \end{matrix}\right] \mathrm{d}x$$

$$= \eta^{-s} H_{p+q_1,q+p_1}^{m+n_1,n+m_1}\left[z\eta^{-\sigma} \middle| \begin{matrix} (1-e_{q_1}-sE_{q_1}, \sigma E_{q_1}), (a_p, A_p) \\ (b_m, B_m), (1-d_{p_1}-sD_{p_1}, \sigma D_{p_1}), (b_{m+1}, B_{m+1}), \ldots, (b_q, B_q) \end{matrix}\right],$$

where $\eta, s, z, \in C, \sigma > 0, \alpha > 0, \mu > 0, |\arg z| < \frac{1}{2}\pi\alpha, |\arg\eta| < \frac{1}{2}\pi k, k > 0$;

$$k = \sum_{i=1}^{n_1} D_i - \sum_{i=n+1}^{p_1} D_i - \sum_{i=1}^{m_1} E_i - \sum_{i=m+1}^{q_1} E_i > 0,$$

$$-\sigma \min_{1\le h\le m}\left[\frac{\Re(b_h)}{B_h}\right] - \min_{1\le j\le m_1}\left[\frac{\Re(e_j)}{E_j}\right] < \Re(s)$$

$$< \sigma \max_{1\le j\le n}\left[\frac{1-\Re(a_j)}{A_j}\right] + \max_{1\le j\le n_1}\left[\frac{1-\Re(d_j)}{D_j}\right].$$

Remark 2.7. For the applications of this result in the theory of statistical distributions, see the work of Mathai and Saxena (1969). It can be established with the help of the definition of the H-function and the result (2.8).

2.3.1 Eulerian Integrals for the H-Function

In this section, certain Eulerian integrals for the H-functions will be evaluated in terms of the H-function of two variables. In order to present the results, we need the definition of the H-function of two complex variables introduced earlier by Mittal and Gupta (1972). The analysis developed here is based on the work of Saxena and Nishimoto (1994), Saigo and Saxena (1998). To unify and extend the existing results on Riemann-Liouville fractional integrals available in the literature, certain new Eulerian integrals associated with the H-function are investigated by Saxena and Nishimoto (1994). The importance of the derived results lies in the fact that a table of Riemann-Liouville fractional integrals can be prepared by using the tables of the special cases of the H-function given in the monograph by Mathai and Saxena (1978, pp. 145–151). Further special cases of these integrals can be used in studying statistical density functions.

Notation 2.8.

$$H\begin{bmatrix} x \\ y \end{bmatrix} = H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix} x \\ y \end{matrix}\middle|\begin{matrix} (a_i;\alpha_i,A_i)_{1,p_1}:(c_i,\gamma_i)_{1,p_2};(e_i,E_i)_{1,p_3} \\ (b_j;\beta_j,B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2};(f_j,F_j)_{1,q_3} \end{matrix}\right]: \quad (2.55)$$

The H-function of two variables.

Definition 2.7. (Srivastava et al. 1982, pp. 82–83; also see Srivastava and Panda 1976)

$$H\begin{bmatrix} x \\ y \end{bmatrix} = H^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{matrix} x \\ y \end{matrix}\middle|\begin{matrix} (a_i;\alpha_i,A_i)_{1,p_1}:(c_i,\gamma_i)_{1,p_2};(e_i,E_i)_{1,p_3} \\ (b_j;\beta_j,B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2};(f_j,F_j)_{1,q_3} \end{matrix}\right] \quad (2.56)$$

$$= -\frac{1}{4\pi^2}\int_{L_1}\int_{L_2} \phi(s,t)\phi_1(s)\phi_2(t)x^s y^t \,\mathrm{d}s\mathrm{d}t, \quad (2.57)$$

where x and y are not equal to zero. For convenience the parameters $(a_i;\alpha_i,A_i)_1,p_1$ and $(c_i,\gamma_i)_1,p_2$ will abbreviate the sequence of the parameters $(a_1;\alpha_1,A_1),\ldots,$ $(a_{p_1};\alpha_{p_1},A_{p_1})$ and $(c_1,\gamma_1),\ldots,(c_{p_2},\gamma_{p_2})$ respectively, and similar meanings hold for the other parameters $(b_j;\beta_j,B_j)_1,q_1$ and $(d_j,\delta_j)_1,q_2$, etc. Here

$$\phi(s,t) = \frac{\prod_{i=1}^{n_1}\Gamma(1-a_i+\alpha_i s+A_i t)}{[\prod_{i=n_1+1}^{p_1}\Gamma(a_i-\alpha_i s-A_i t)][\prod_{j=1}^{q_1}\Gamma(1-b_j+\beta_j s+B_j t)]} \quad (2.58)$$

$$\phi_1(s) = \frac{[\prod_{j=1}^{m_2}\Gamma(d_j-\delta_j s)][\prod_{i=1}^{n_2}\Gamma(1-c_i+\gamma_i s)]}{[\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\delta_j s)][\prod_{i=n_2+1}^{p_2}\Gamma(c_i-\gamma_i s)]}, \quad (2.59)$$

$$\phi_2(t) = \frac{[\prod_{j=1}^{m_3}\Gamma(f_j-F_j t)][\prod_{i=1}^{n_3}\Gamma(1-e_i+E_i t)]}{[\prod_{j=m_3+1}^{q_3}\Gamma(1-f_j+F_j t)][\prod_{i=n_3+1}^{p_3}\Gamma(e_i-E_i t)]}. \quad (2.60)$$

It is assumed that all the poles of the integrand are simple. An empty product is interpreted as unity. Further, we suppose that all the parameters a_i, b_j, c_i, d_j, e_i and f_j be complex numbers and associated coefficients $\alpha_i, A_i, \beta_j, B_j, \gamma_i, \delta_j, E_i$ and F_j be real and positive for the standardization purposes, such that

$$\rho_1 = \sum_{i=1}^{p_1} \alpha_i + \sum_{i=1}^{p_2} \gamma_i - \sum_{j=1}^{q_1} \beta_j - \sum_{j=1}^{q_2} \delta_j \leq 0, \tag{2.61}$$

$$\rho_2 = \sum_{i=1}^{p_1} A_i + \sum_{i=1}^{p_2} E_i - \sum_{j=1}^{q_1} B_j - \sum_{j=1}^{q_2} F_j \leq 0, \tag{2.62}$$

$$\begin{aligned}\Omega_1 = &- \sum_{i=n_1+1}^{p_1} \alpha_i - \sum_{j=1}^{q_1} \beta_j + \sum_{j=1}^{m_2} \delta_j \\ &- \sum_{j=m_2+1}^{p_2} \delta_j + \sum_{i=1}^{n_2} \gamma_i - \sum_{i=n_2+1}^{p_2} \gamma_i > 0,\end{aligned} \tag{2.63}$$

$$\begin{aligned}\Omega_2 = &- \sum_{i=n_1+1}^{p_1} A_i - \sum_{j=1}^{q_1} B_j + \sum_{j=1}^{m_3} F_j \\ &- \sum_{j=m_3+1}^{p_3} F_j + \sum_{i=1}^{n_3} E_i - \sum_{i=n_3+1}^{p_3} E_i > 0.\end{aligned} \tag{2.64}$$

It can be seen that the contour integral (2.56) converges absolutely under the conditions (2.61)–(2.64) and defines an analytic function of two complex variables x and y inside the sectors given by

$$|\arg x| < \frac{1}{2}\pi\Omega_1 \ \text{ and } \ |\arg y| < \frac{1}{2}\pi\Omega_2, \tag{2.65}$$

the points $x = 0$ and $y = 0$ being tacitly excluded.

The conditions given here from (2.61) to (2.65) are the sufficient conditions for the convergence of the Mellin–Barnes double integral (2.57), for details the reader is referred to the book by Srivastava et al. (1982).

Remark 2.8. In a series of papers Buschman (1978) has given a detailed analysis of the sufficient conditions for the convergence of H-function of two variables of a general character. Simple criteria are provided for the determination of the convergence of certain double Mellin–Barnes integrals in terms of their parameters by Hai et al. (1992). A systematic and comprehensive account of the double Mellin–Barnes type integrals or rather H-function of two variables can be found in the book by Hai and Yakubovich (1992).

2.3.2 Fractional Integration of a H-Function

Theorem 2.1. *If* $\min\{\Re(\alpha), \Re(\beta)\} > 0, \alpha, \beta, \gamma \in C, \mu \geq 0, b \neq a, \left|\frac{(a-b)c}{ac+d}\right| < 1,$ $|\arg(d+cb)/(d+ca)| < \pi; \phi, \eta > 0, |\arg k| < \frac{1}{2}\pi\phi$, *then there holds the formula*

$$\int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}(cx+d)^{\gamma} H_{p,q}^{m,n}\left[k(cx+d)^{-\eta}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}x$$
$$= (b-a)^{\alpha+\beta-1}(ac+d)^{\gamma}\Gamma(\beta)$$
$$\times H_{1,0:p,q+1;1,2}^{0,1:m,n;1,1}\left[\begin{matrix}\frac{k}{(ac+d)^{\eta}}\\ \frac{c(b-a)}{(ac+d)}\end{matrix}\left|\begin{matrix}(1+\gamma:\eta,1):-\\ -\\ (a_1,A_1),\ldots,(a_p,A_p);(1-\alpha,1)\\ (b_1,B_1),\ldots,(b_a,B_q),(1+\gamma,\eta);(0,1),(1-\alpha-\beta,1)\end{matrix}\right.\right], \tag{2.66}$$

$$\phi = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j > 0.$$

Proof 2.1. To establish (2.66) we express the H-function in terms of the contour integral (1.2), interchange the order of integration, which is permissible due to absolute convergence of the integrals involved in the process, and evaluate the x-integral by means of the integral (Prudnikov et al. 1986, p. 301):

$$\int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}(cx+d)^{\gamma}\mathrm{d}x$$
$$= (ac+d)^{\gamma}(b-a)^{\alpha+\beta-1}B(\alpha,\beta)\,{}_2F_1\left(\alpha,-\gamma;\alpha+\beta;\frac{c(a-b)}{(ac+d)}\right), \tag{2.67}$$

where $\Re(\alpha)>0, \Re(\beta)>0, |c(a-b)/(ac+d)|<1, |\arg(((d+bc)/(ac+d))|<1,$ the integral transforms into the form

$$(b-a)^{\alpha+\beta-1}(ac+d)^{\gamma}\frac{1}{2\pi i}\int_L \Theta(s)(ac+d)^{s\eta}\,{}_2F_1\left(\alpha,-\gamma-s\eta;\alpha+\beta;\frac{c(a-b)}{ac+d}\right)\mathrm{d}s.$$

If we now employ the formula

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma_i\infty}\frac{\Gamma(-s)\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)}(-z)^s\mathrm{d}s, \tag{2.68}$$

where $|\arg(-z)| < \pi$ and the path of integration separates the poles at $s = 0,1,2,\ldots$ from the poles at $s = -a-n, s = -b-n, n = 0,1,\ldots,$ the result readily follows from (2.57). □

On applying the identity (1.58), we obtain the following theorem:

Theorem 2.2. *If* $\min\{\Re(\alpha),\Re(\beta)\} > 0,\ \alpha,\beta,\gamma \in C, \mu \geq 0, b \neq a,\ |[(a-b)c]/(ac+d)| < 1,\ |\arg(d+cb)/(d+ca)| < \pi,\ \phi,\eta > 0,\ |\arg k| < \frac{1}{2}\pi\phi,$ *then there holds the formula*

$$\int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}(cx+d)^{\gamma} H_{p,q}^{m,n}\left[k(cx+d)^{\eta}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] \mathrm{d}x$$
$$= (b-a)^{\alpha+\beta-1}(ac+d)^{\gamma}\Gamma(\beta)$$
$$\times H_{1,0:q,p+1;1,2}^{0,1:n,m;1,1}\left[\begin{matrix} k^{-1}(ac+d)^{-\eta} \\ \frac{c(b-a)}{ac+d}\end{matrix}\middle| \begin{matrix} (1+\gamma:\eta,1):- \\ - \\ (1-b_1,B_1),\ldots,(1-b_q,B_q);(1-\alpha,1) \\ (1-a_1,A_1),\ldots,(1-a_p,A_p),(1+\gamma,\eta);(0,1),(1-\alpha-\beta,1)\end{matrix}\right]. \tag{2.69}$$

When $A_i = B_j = 1$ for all i and j, then we obtain the following corollaries from the above theorems, involving Meijer G-function.

Corollary 2.1. *If* $\min\{\Re(\alpha),\Re(\beta)\} > 0, \alpha,\beta,\gamma \in C, \mu \geq 0, b \neq a,$ $|[(a-b)c]/(ac+d)| < 1,$ $|\arg(d+cb)/(d+ca)| < \pi;$ $c^* > 0, |\arg k| < \frac{1}{2}\pi c^*$, *then there holds the formula*

$$\int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}(cx+d)^{\gamma} G_{p,q}^{m,n}\left[k(cx+d)^{-\eta}\Big|_{b_q}^{a_p}\right] \mathrm{d}x$$
$$= (b-a)^{\alpha+\beta-1}(ac+d)^{\gamma}\Gamma(\beta)$$
$$\times H_{1,0:p,q+1;1,2}^{0,1:m,n;1,1}\left[\begin{matrix} \frac{k}{(ac+d)^{\eta}} \\ \frac{c(b-a)}{ac+d}\end{matrix}\middle| \begin{matrix} (1+\gamma:\eta,1):- \\ - \\ (a_1,1),\ldots,(a_p,1);(1-\alpha,1) \\ (b_1,1),\ldots,(b_q,1),(1+\gamma,\eta);(0,1),(1-\alpha-\beta,1)\end{matrix}\right]. \tag{2.70}$$

Corollary 2.2. *If* $\min\{\Re(\alpha),\Re(\beta)\} > 0, \alpha,\beta,\gamma \in C, \mu \geq 0, b \neq a,$ $|[(a-b)c]/(ac+d)| < 1,$ $|\arg(d+cb)/(d+ca)| < \pi;$ $c^* > 0, |\arg k| < \frac{1}{2}\pi c^*$, *then there holds the formula*

$$\int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}(cx+d)^{\gamma} G_{p,q}^{m,n}\left[k(cx+d)^{\eta}\Big|_{(b_q)}^{(a_p)}\right] \mathrm{d}x$$
$$= (b-a)^{\alpha+\beta-1}(ac+d)^{\gamma}\Gamma(\beta)$$
$$\times H_{1,0:q,p+1;1,2}^{0,1:n,m;1,1}\left[\begin{matrix} \frac{k^{-1}}{(ac+d)^{\eta}} \\ \frac{c(b-a)}{ac+d}\end{matrix}\middle| \begin{matrix} (1+\gamma:\eta,1):- \\ - \\ (1-b_1,1),\ldots,(1-b_q,1):(1-\alpha,1) \\ (1-a_1,1),\ldots,(1-a_p,1),(1+\gamma,\eta);(0,1),(1-\alpha-\beta,1)\end{matrix}\right], \tag{2.71}$$

where c^* *is defined in (1.22).*

On the other hand, if we use the identity (Mathai and Saxena 1978, p. 4) then we arrive at the following corollaries associated with Wright generalized hypergeometric functions.

Corollary 2.3. *If* $\min\{\Re(\alpha),\Re(\beta)\} > 0, \alpha,\beta,\gamma \in C, \mu \geq 0, b \neq a, |[c(a-b)]/(ac+d)| < 1,$ $|\arg(d+cb)/(d+ca)| < \pi;$ $\Omega = \mu+1 > 0, \eta > 0$ *and* $|\arg k| < \frac{1}{2}\pi\Omega$ *then there holds the formula*

$$\int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}(cx+d)^{\gamma}\,{}_p\Psi_q\left[-k(cx+d)^{-\eta}\Big|^{(a_1,A_1),\ldots,(a_p,A_p)}_{(b_1,B_1),\ldots,(b_q,B_q)}\right]\mathrm{d}x$$
$$= (b-a)^{\alpha+\beta-1}(ac+d)^{\gamma}\Gamma(\beta)$$
$$\times H^{0,1:1,p;1,1}_{1,0:p,q+1;1,2}\left[\begin{matrix}\frac{k}{(ac+d)^{\eta}}\\ \frac{c(b-a)}{ac+d}\end{matrix}\left|\begin{matrix}(1+\gamma:\eta,1):-\\ -\\ (1-a_1,A_1),\ldots,(1-a_p,A_p):(1-\alpha,1)\\ (0,1),(1-b_1,B_1),\ldots,(1-b_q,B_q),(1+\gamma,\eta);(0,1),(1-\alpha-\beta,1)\end{matrix}\right.\right], \tag{2.72}$$

where μ is defined in (1.9).

Corollary 2.4. *If* $\min\{\Re(\alpha),\Re(\beta)\}>0,\alpha,\beta,\gamma\in C,\mu\geq 0,k\neq 0,b\neq a,$ $|[c(a-b)]/(ac+d)|<1$, $|\arg(d+cb)/(d+ca)|<\pi$*;* $\Omega=\mu+1>0,\eta>0$ *and* $|\arg k|<\frac{1}{2}\pi\Omega$ *then there holds the formula*

$$\int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}(cx+d)^{\gamma}\,{}_p\Psi_q\left[-k(cx+d)^{\eta}\Big|^{(a_1,A_1),\ldots,(a_p,A_p)}_{(b_1,B_1),\ldots,(b_q,B_q)}\right]\mathrm{d}x$$
$$= (b-a)^{\alpha+\beta-1}(ac+d)^{\gamma}\Gamma(\beta)$$
$$\times H^{0,1:p,1;1,1}_{1,0:q+1,p+1;1,2}\left[\begin{matrix}\frac{1}{k(ac+d)^{\eta}}\\ \frac{c(b-a)}{ac+d}\end{matrix}\left|\begin{matrix}(1+\gamma:\eta,1):-\\ -\\ (1,1),(b_1,B_1),\ldots,(b_q,B_q):(1-\alpha,1)\\ (a_1,A_1),\ldots,(a_p,A_p),(1+\gamma,\eta);(0,1),(1-\alpha-\beta,1)\end{matrix}\right.\right], \tag{2.73}$$

where μ is defined in (1.9).

When $d=0$, (2.66), (2.69) give rise to the following theorems:

Theorem 2.3. *If* $\min\{\Re(\alpha),\Re(\beta)\}>0,\alpha,\beta,\gamma\in C,\mu\geq 0,b\neq a,a\neq 0,$ $|1-\frac{b}{a}|<1$, $|\arg(b/a)|<\pi$, $\phi,\eta>0,|\arg k|<\frac{1}{2}\pi\phi$*, then there holds the formula*

$$\int_a^b x^{\gamma}(x-a)^{\alpha-1}(b-x)^{\beta-1}H^{m,n}_{p,q}\left[kx^{-\eta}\Big|^{(a_p,A_p)}_{(b_q,B_q)}\right]\mathrm{d}x$$
$$= a^{\gamma}(b-a)^{\alpha+\beta-1}\Gamma(\beta)$$
$$\times H^{0,1:m,n;1,1}_{1,0:p,q+1;1,2}\left[\begin{matrix}\frac{k}{a^{\eta}}\\ \frac{b}{a}-1\end{matrix}\left|\begin{matrix}(1+\gamma:\eta,1):-\\ -\\ (a_p,A_p);(1-\alpha,1)\\ (b_q,B_q),(1+\gamma,\eta);(0,1),(1-\alpha-\beta,1)\end{matrix}\right.\right]. \tag{2.74}$$

Theorem 2.4. *If* $\min\{\Re(\alpha),\Re(\beta)\}>0,\alpha,\beta,\gamma\in C,\mu\geq 0,b\neq a,a\neq 0,k\neq 0,$ $|1-\frac{b}{a}|<1,|\arg(b/a)|<\pi$*;* $\phi,\eta>0$, $|\arg k|<\frac{1}{2}\pi\phi$*, then there holds the formula*

$$\int_a^b x^\gamma (x-a)^{\alpha-1}(b-x)^{\beta-1} H_{p,q}^{m,n}\left[kx^\eta \Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] \mathrm{d}x$$
$$= a^\gamma (b-a)^{\alpha+\beta-1}\Gamma(\beta)$$
$$\times H_{1,0:q,p+1;1,2}^{0,1:n,m;1,1}\left[\begin{matrix}\frac{1}{ka^\eta}\\ \frac{b}{a}-1\end{matrix}\left|\begin{matrix}(1+\gamma:\eta,1):-\\ -\\ (1-b_1,B_1),\ldots,(1-b_q,B_q);(1-\alpha,1)\\ (1-a_1,A_1),\ldots,(1-a_p,A_p),(1+\gamma,\eta);(0,1),(1-\alpha-\beta,1)\end{matrix}\right.\right]. \tag{2.75}$$

Alternative form of Theorem 2.1. *Let*

$$f(z) = (z-a)^{\beta-1}(cz+d)^\gamma H_{p,q}^{m,n}[k(cz+d)^{-\nu}],$$

then there holds the formula

$${}_aD_z^{-\alpha}[f(z)] = (z-a)^{\alpha+\beta-1}(ac+d)^\gamma$$
$$\times H_{1,0:p,q+1;1,2}^{0,1:m,n;1,1}\left[\begin{matrix}\frac{k}{(ac+d)^\nu}\\ \frac{c(z-a)}{ac+d}\end{matrix}\left|\begin{matrix}(1+\gamma:\nu,1):-\\ -\\ (a_1,A_1),\ldots,(a_p,A_p);(1-\beta,1)\\ (b_1,B_1),\ldots,(b_q,B_q),(1+\gamma,\nu);(0,1),(1-\alpha-\beta,1)\end{matrix}\right.\right], \tag{2.76}$$

under the conditions stated along with (2.66) with b replaced by z and η replaced by $+\nu$, where ${}_0D_z^{-\alpha}$ is the fractional integral operator, see Chap. 3 for a discussion of fractional integrals and fractional derivatives.

Alternative form of Theorem 2.2. *Let*

$$f(z) = (z-a)^{\beta-1}(cz+d)^\gamma H_{p,q}^{m,n}[k(cz+d)^{\nu}],$$

then there holds the formula

$${}_aD_z^{-\alpha}[f(z)] = (z-a)^{\alpha+\beta-1}(ac+d)^\gamma$$
$$\times H_{1,0:q,p+1;1,2}^{0,1:n,m;1,1}\left[\begin{matrix}\frac{1}{k(ac+d)^\nu}\\ \frac{c(z-a)}{ac+d}\end{matrix}\left|\begin{matrix}(1+\gamma:\nu,1):-\\ -\\ (1-b_1,B_1),\ldots,(1-b_q,B_q);(1-\beta,1)\\ (1-a_1,A_1),\ldots,(1-a_p.A_p),(1+\gamma,\nu);(0,1),(1-\alpha-\beta,1)\end{matrix}\right.\right], \tag{2.77}$$

under the conditions stated along with (2.69) with b replaced by z and η replaced by $+\nu$.

It may be mentioned here that for generalization of the results of this section, one can refer to the papers by Saxena and Saigo (1998), Saigo and Saxena (1999, 1999a, 2001) and Srivastava and Hussain (1995).

Remark 2.9. On the integration of H-functions with respect to their parameters, see the works of Nair (1973), Nair and Nambudiripad (1973), Anandani (1970b), Taxak (1971). Golas (1968) and Pendse (1970). Integration of products of generalized Legendre functions and H-functions with respect to a parameter is discussed by Anandani (1970b, 1971d).

2.4 H-Function and Exponential Functions

The following integrals are evaluated by Bajpai (1970) with the help of the integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta)^{\alpha-1}\exp(i\beta\theta)\mathrm{d}\theta = \frac{\pi\Gamma(\alpha)}{2^{\alpha-1}\Gamma\left(\frac{\alpha+\beta+1}{2}\right)\Gamma\left(\frac{\alpha-\beta+1}{2}\right)}, \tag{2.78}$$

where $\Re(\alpha) > 0$.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta)^{k+\lambda-2}\exp[i(k-\lambda)\theta] H_{p,q}^{m,n}\left[z(\mathrm{e}^{i\theta}\cos\theta)^{-h}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}\theta$$
$$= \frac{\pi}{2^{k+\lambda-2}\Gamma(\lambda)} H_{p+1,q+1}^{m+1,n}\left[2^h z\Big|_{(k+\lambda-1,h),(b_q,B_q)}^{(a_p,A_p),(k,h)}\right], \tag{2.79}$$

where $\Re(k+\lambda-h\frac{a_i}{A_i}) > 1-\frac{h}{A_i}, i = 1,\ldots,n, h > 0, k,\lambda \in C, \mu \geq 0, \alpha > 0$, $|\arg z| < \frac{1}{2}\pi\alpha$.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta)^{k+\lambda-2}\exp[i(k-\lambda)\theta] H_{p,q}^{m,n}\left[z\mathrm{e}^{ih\theta}(\sec\theta)^{h}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}\theta$$
$$= \frac{\pi}{2^{k+\lambda-2}\Gamma(k)} H_{p+1,q+1}^{m+1,n}\left[2^h z\Big|_{(k+\lambda-1,h),(b_q,B_q)}^{(a_p,A_p),(\lambda,h)}\right], \tag{2.80}$$

where $\Re(k+\lambda-h\frac{a_i}{A_i}) > 1-\frac{h}{A_i}, i = 1,\ldots,n, h > 0, k,\lambda \in C, \mu \geq 0, \alpha > 0$, $|\arg z| < \frac{1}{2}\pi\alpha$.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta)^{k+\lambda-2}\exp[i(k-\lambda)\theta] H_{p,q}^{m,n}\left[z(\sec\theta)^{2h}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}\theta$$
$$= \frac{\pi}{2^{k+\lambda-2}} H_{p+2,q+1}^{m+1,n}\left[2^{2h} z\Big|_{(k+\lambda-1,2h),(b_q,B_q)}^{(a_p,A_p),(k,h),(\lambda,h)}\right], \tag{2.81}$$

where $\Re(k+\lambda-h\frac{a_i}{A_i}) > 1-\frac{h}{A_i}, i = 1,\ldots,n, h > 0, k,\lambda \in C, \mu \geq 0, \alpha > 0$, $|\arg z| < \frac{1}{2}\pi\alpha$. By means of the following integral (Nielson 1906, p. 158)

$$\int_0^{\pi} (\sin t)^{\alpha} e^{-\beta t} dt = \frac{\pi e^{-\frac{\pi\beta}{2}} \Gamma(\alpha+1)}{2^{\alpha}\Gamma\left(1+\frac{\alpha+i\beta}{2}\right)\Gamma\left(1+\frac{\alpha-i\beta}{2}\right)}, \tag{2.82}$$

where $\Re(\alpha) > -1$, Saxena (1971a) has established the following results:

(i) Let $\alpha > 0, |\arg z| < \frac{1}{2}\pi\alpha$ or $\alpha = 0, \Re(\delta) < -1$ then there holds the formula

$$\int_0^{\pi} (\sin\theta)^{\gamma-1} e^{-\eta\theta} H_{p,q}^{m,n}\left[z(\sin\theta)^{2h}\middle|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right] d\theta$$
$$= \frac{\pi}{2^{\gamma-1}} \exp\left(-\frac{\pi\eta}{2}\right) H_{p+1,q+2}^{m,n+1}\left[\frac{z}{4^h}\middle|\begin{matrix}(1-\gamma,2h),(a_p,A_p)\\(b_q,B_q),\left(\frac{1-\gamma+i\eta}{2},h\right),\left(\frac{1-\gamma-i\eta}{2},h\right)\end{matrix}\right], \tag{2.83}$$

where $\gamma, \eta \in C, h > 0$ are such that $\Re(\gamma) + 2h \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}\right] > 0$ for $\alpha > 0$, $|\arg z| < \frac{1}{2}\pi\alpha$ or $\alpha = 0, \mu \ge 0$, and $\Re(\gamma) + 2h \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta)+\frac{1}{2}}{\mu}\right]$ > 0 for $\alpha = 0$ and $\mu < 0$.

(ii)

$$\int_0^{\pi} (\sin\theta)^{\gamma-1} e^{-\eta\theta} H_{p,q}^{m,n}\left[ze^{i2h\theta}\middle|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right] d\theta$$
$$= \frac{\pi\Gamma(\gamma)}{2^{\gamma-1}} \exp\left(-\frac{\pi\eta}{2}]\right) H_{p+1,q+1}^{m,n}\left[\frac{ze^{i\pi h}}{4^h}\middle|\begin{matrix}(a_p,A_p),\left(\frac{1+\gamma-i\eta}{2},h\right)\\(b_q,B_q),\left(\frac{1-\gamma-i\eta}{2},h\right)\end{matrix}\right], \tag{2.84}$$

where $\gamma, \eta \in C, h > 0, |\arg z| < \frac{1}{2}\pi\alpha, \alpha > 0, \Re(\gamma) > 0$.

(iii) Let $\alpha > 0, \Re(\gamma) > 0$ or $\alpha = 0, \Re(\delta) < -1$ then

$$\int_0^{\pi} (\sin\theta)^{\gamma-1} e^{-\eta\theta} H_{p,q}^{m,n}\left[z(\sin\theta)^{2\lambda}e^{i2h\theta}\middle|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right] d\theta$$
$$= \frac{\pi}{2^{\gamma-1}} \exp\left(-\frac{\pi\eta}{2}\right) H_{p+1,q+2}^{m,n+1}\left[\frac{ze^{i\pi h}}{4^{\lambda}}\middle|\begin{matrix}(1-\gamma,2\lambda),(a_p,A_p)\\(b_q,B_q),\left(\frac{1-\gamma-i\eta}{2},\lambda+h\right),\left(\frac{1-\gamma+i\eta}{2},\lambda-h\right)\end{matrix}\right], \tag{2.85}$$

holds for $h > 0, \lambda > h, \gamma, \eta \in C$, such that $\Re(\gamma) + 2\lambda \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}\right] > 0$ for $\alpha > 0$ or $\alpha = 0$ and $\mu \ge 0$; and $\Re(\gamma) + 2\lambda \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta)+\frac{1}{2}}{\mu}\right] > 0$ for $\alpha = 0$ and $\mu < 0$. When $h = 1$ and $A_i = B_j = 1$ for all i and j, then the H-function reduces to a G-function and from the results (2.83) and (2.85) we find that

$$\int_0^{\pi} (\sin\theta)^{\gamma-1} e^{-\eta\theta} G_{p,q}^{m,n}\left[z\sin^2\theta \Big|_{(b_q)}^{(a_p)} \right] d\theta$$
$$= \sqrt{\pi}\exp\left(-\frac{\pi\eta}{2}\right) G_{p+2,q+2}^{m,n+2}\left[z\Big|_{b_q,\frac{1-\gamma+i\eta}{2},\frac{1-\gamma-i\eta}{2}}^{\frac{1-\gamma}{2},\frac{2-\lambda}{2},a_p} \right], \tag{2.86}$$

where $\Re(\gamma) + 2\min_{1\le j\le m}\Re(b_j) > 0, \gamma,\eta \in C, c^* > 0, |\arg z| < \frac{1}{2}\pi c^*$, where c^* is defined in (1.22) and

$$\int_0^{\pi} (\sin\theta)^{\gamma-1} e^{-\eta\theta} G_{p,q}^{m,n}\left[ze^{2i\theta} \Big|_{b_q}^{a_p} \right] d\theta$$
$$= \frac{\pi}{2^{\gamma-1}}\Gamma(\gamma)\exp\left(-\frac{\pi\eta}{2}\right) G_{p+1,q+1}^{m,n}\left[ze^{i\pi}\Big|_{(b_q),\frac{1-\gamma-i\eta}{2}}^{a_p,\frac{1+\gamma-i\eta}{2}} \right], \tag{2.87}$$

where $\Re(\gamma) > 0, |\arg z| < \frac{1}{2}\pi c^*, c^* > 0$.

2.5 Legendre Function and the H-Function

Let $\rho, z \in C, \alpha > 0, |\arg z| < \frac{1}{2}\pi\alpha$ or $\alpha = 0, \Re(\mu) < -1$. Further, let $\rho \in C, k > 0$ satisfy the conditions

$$\Re(\rho) + k\min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}\right] > \frac{1}{2}|\Re(\mu)|$$

for $\alpha > 0, |\arg z| < \frac{1}{2}\pi\alpha$ or $\alpha = 0, \mu \ge 0$ and

$$\Re(\rho) + k\min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta)+\frac{1}{2}}{\mu}\right] > \frac{1}{2}|\Re(\mu)|,$$

for $\alpha = 0, \mu < 0$ then there holds the formula (Singh and Varma 1972)

$$\int_{-1}^{1} (1-x^2)^{\rho-1} P_\nu^\lambda(x) H_{p,q}^{m,n}\left[z(1-x^2)^k \Big|_{(b_q,B_q)}^{(a_p,A_p)} \right] dx$$
$$= \frac{2^\lambda \pi}{\Gamma\left(\frac{2+\nu-\lambda}{2}\right)\Gamma\left(\frac{1-\nu-\lambda}{2}\right)}$$
$$\times H_{p+2,q+2}^{m,n+2}\left[z\Big|_{(b_q,B_q),(1-\rho+\frac{\nu}{2},k),(-\rho-\frac{\nu}{2},k)}^{(1-\rho\pm\frac{\lambda}{2},k),(a_p,A_p)} \right]. \tag{2.88}$$

For a definition of $P^{\lambda}_{nu}(x)$ see Sect. 1.8.1. On making use of finite difference operator E (Milne-Thomson 1933, p. 33 with $\omega = 1$), which has the following properties:

$$E_a f(a) = f(a+1) \tag{2.89}$$

$$E_a^n f(a) = E_a[E_a^{n-1} f(a)]. \tag{2.90}$$

Singh and Varma (1972) have further shown that

$$\begin{aligned}
&\int_{-1}^{1} (1-x^2)^{\rho-1} P_\nu^\lambda(x) {}_U F_V(\alpha_1,\ldots,\alpha_U;\beta_1,\ldots,\beta_V;c(1-x^2)^d) \\
&\quad\times H_{p,q}^{m,n}\left[z(1-x^2)^k \Big|_{(b_q.B_q)}^{(a_p,A_p)}\right] \mathrm{d}x \\
&= \frac{2^\lambda \pi}{\Gamma\left(\frac{2+\nu-\lambda}{2}\right)\Gamma\left(\frac{1-\nu-\lambda}{2}\right)} \sum_{r=0}^{\infty} \frac{(\alpha_1)_r\cdots(\alpha_U)_r}{(\beta_1)_r\cdots(\beta_V)_r}\frac{c^r}{r!} \\
&\quad\times H_{p+2,q+2}^{m,n+2}\left[z\Big|_{(b_q,B_q),(1-\rho-rd+\frac{\nu}{2},k),(-\rho-rd-\frac{\nu}{2},k)}^{(1-\rho-rd\pm\frac{\lambda}{2},k),(a_p,A_p)}\right],
\end{aligned} \tag{2.91}$$

which holds under the conditions given with the result along with the conditions that k and d are positive integers, $U < V$ or $U = V + 1$ and $|c| < 1$ and none of $\beta_j, j = 1,\ldots,V$ is a negative integer or zero. In case $\lambda = 0$ and $\nu = \lambda$, where λ is a positive integer, then the result (2.91) reduces to

$$\begin{aligned}
&\int_{-1}^{1} (1-x^2)^{\rho-1} P_\lambda(x) {}_U F_V(\alpha_1,\ldots,\alpha_U;\beta_1,\ldots,\beta_V;c(1-x^2)^d) \\
&\quad\times H_{p,q}^{m,n}\left[z(1-x^2)^k \Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] \mathrm{d}x \\
&= \frac{\pi}{\Gamma\left(\frac{2+\lambda}{2}\right)\Gamma\left(\frac{1-\lambda}{2}\right)} \sum_{r=0}^{\infty} \frac{(\alpha_1)_r\cdots(\alpha_U)_r}{(\beta_1)_r\cdots(\beta_V)_r}\frac{c^r}{r!} \\
&\quad\times H_{p+2,q+2}^{m,n+2}\left[z\Big|_{(b_q,B_q),(1-\rho-rd+\frac{\lambda}{2},k),(-\rho-rd-\frac{\lambda}{2},k)}^{(1-\rho-rd,k),(1-\rho-rd,k),(a_p,A_p)}\right],
\end{aligned} \tag{2.92}$$

where $P_\lambda(x)$ is the Legendre polynomial and the conditions of the validity are the same as stated in (2.91) with $\lambda = 0$ and ν replaced by λ.

2.6 Generalized Laguerre Polynomials

From the integral (Mathai and Saxena 1973, p. 76) it can be easily shown that

$$\int_0^\infty x^\gamma \mathrm{e}^{-x} L_k^{(\sigma)}(x) H_{p,q}^{m,n}\left[zx^\eta \Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}x$$
$$= \frac{(2\pi)^{\frac{(1-\eta)}{2}}\eta^{\gamma+k+\frac{1}{2}}}{k!}$$
$$\times H_{p+2\eta,q+\eta}^{m+\eta,n+\eta}\left[z\eta^\eta \Big|_{(\Delta(\eta;\sigma-\gamma+k),1),(b_q,B_q)}^{\Delta(\eta;-\gamma),1),(\Delta(\eta;\sigma-\gamma);1),(a_p,A_p)}\right], \tag{2.93}$$

where η is a positive integer, either $\alpha > 0, |\arg z| < \frac{1}{2}\pi\alpha$ or $\alpha = 0, \Re(\delta) < -1$. Further, the parameters $z, \gamma, \sigma \in C$ are such that $\Re(\gamma) + \eta \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}\right] > -1$ for $\alpha > 0, |\arg z| < \frac{1}{2}\pi\alpha$ or $\alpha=0, \mu \ge 0$; and $\Re(\gamma)+\eta \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta)+\frac{1}{2}}{\mu}\right] > -1$ for $\alpha = 0$ and $\mu < 0$. For a definition of Laguerre polynomials see Sect. 1.8.1.

Remark 2.10. Solutions of certain integral equations involving general H-function were developed by Galué et al. (1993). It is interesting to observe that the results given earlier by Kalla and Kiryakova (1990) for the Erdélyi–Kober and Weyl operators follow easily from the results of this section.

Exercises

2.1. Prove that

$$\int_0^t x^{\rho-1}(t-x)^{c-1} {}_2F_1(a,b;c;1-\frac{x}{t}) H_{p,q}^{m,n}\left[dx^\sigma\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}x$$
$$= t^{\rho+c-1}\Gamma(c) H_{p+2,q+2}^{m,n+2}\left[dt^\sigma\Big|_{(b_q,B_q),(1+a-c-\rho,\sigma),(1+b-c-\rho,\sigma)}^{(1-\rho,\sigma),(1+a+b-c-\rho,\sigma),(a_p,A_p)}\right], \tag{2.94}$$

where either $\alpha > 0, |\arg d| < \frac{1}{2}\pi\alpha$ or $\alpha = 0, \Re(\delta) < -1$ and the parameters $\rho, a, b, c, d \in C, \Re(c) > 0, \sigma > 0$ be such that $\Re(\rho + c - a - b) + \min_{1\le k\le m}\left[\frac{\Re(b_j)}{B_j}\right] > 0$ for $\alpha > 0, |\arg z| < \frac{1}{2}\pi\alpha$ or $\alpha = 0, \mu \ge 0$ and $\Re(\rho + c - a - b) + \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta)+\frac{1}{2}}{\mu}\right] > 0$ for $\alpha = 0$ and $\mu < 0$.

2.2. Establish the following integrals:

(i)

$$\prod_{r=1}^{t}\int_0^1 x_r^{\alpha_r-1}(1-x_r)^{-\frac{1}{2}}T_{n_r}(2x_r-1)H_{p,q}^{m,n}\left[\frac{z}{(x_1x_2\cdots x_t)^h}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}x_r$$
$$=\pi^{\frac{t}{2}}H_{p+2t,q+2t}^{m+2t,n}\left[z\left|\begin{matrix}(a_p,A_p),(\alpha_1-n_1+\frac{1}{2},h),(\alpha_1+n_1+\frac{1}{2},h)\\ \cdots(\alpha_t-n_t+\frac{1}{2},h),(\alpha_t+n_t+\frac{1}{2},h)\\ (b_q,B_q),(\alpha_1,h),(\alpha_1+\frac{1}{2},h),\ldots\\ (\alpha_t,h),(\alpha_t+\frac{1}{2},h)\end{matrix}\right.\right],$$

where $z,\alpha_r\in C$, either $\alpha>0,|\arg z|<\frac{1}{2}\pi\alpha$ or $\alpha>0,\Re(\delta)<-1$. [For a definition of $T_n(x)$ see Sect. 1.8.1]. Further, the parameter h is such that $\Re(\alpha_r)>h\max_{1\le j\le n}\left[\frac{\Re(a_j)-1}{A_j}\right]>0, r=1,\ldots,t$ for $\alpha>0$ or $\alpha=0,\mu\le 0$ and $\Re(\alpha_r)+\min_{1\le j\le m}\left[\frac{\Re(a_j)-1}{A_j},\frac{\Re(\delta)+\frac{1}{2}}{\mu}\right]>0, r=1,\ldots,t$ for $\alpha=0$ and $\mu>0$.

(ii)

$$\prod_{r=1}^{t}\int_0^1 x_r^{\alpha_r-1}(1-x_r)^{-\frac{1}{2}}T_{n_r}(2x_r-1)H_{p,q}^{m,n}\left[z(x_1x_2\cdots x_t)^h\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}x_r$$
$$=\pi^{\frac{t}{2}}H_{p+2t,q+2t}^{m,n+2t}\left[z\left|\begin{matrix}(1-\alpha_1,h),\left(\frac{1}{2}-\alpha_1,h\right),\\ \ldots,(1-\alpha_t,h),\left(\frac{1}{2}-\alpha_t,h\right),(a_p,A_p)\\ (b_q,B_q),\left(\frac{1}{2}-\alpha_1-n_1,h\right),\left(\frac{1}{2}-\alpha_1+n_1,h\right),\ldots\\ \left(\frac{1}{2}-\alpha_t-n_t,h\right),\left(\frac{1}{2}-\alpha_t+n_t,h\right)\end{matrix}\right.\right],$$

where $z,\alpha_r\in C$, either $\alpha>0,|\arg z|<\frac{1}{2}\pi\alpha$ or $\alpha=0,\Re(\delta)<-1$. Further, the parameter $h>0$ is such that $\Re(\alpha_r)+h\min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}\right]>-1$, $r=1,\ldots,t$ for $\alpha>0$ or $\alpha=0,\mu\ge 0$, and $\Re(\rho+c-a-b)+\min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j},\frac{\Re(\delta)+\frac{1}{2}}{\mu}\right]>0, r=1,\ldots,t$ for $\alpha=0$ and $\mu<0$. Hint: use the integral (Prudnikov et al. 1990, p. 681, Eq. (8.4.31.1))

2.3. Let $\alpha,\beta,\gamma\in C$, either $\alpha>0,|\arg y|<\frac{1}{2}\pi\alpha$ or $\alpha=0,\Re(\delta)<-1$. Further, let $\eta\ge 0, b\ne a, \left|\frac{(b-a)c}{ac+d}\right|<1, \left|\frac{y(b-a)^{\lambda+\eta}}{(ac+d)^{\nu}}\right|<1, |\arg(d+cb)/(d+ca)|<\pi$ be such that $\Re(\alpha)+\lambda\min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}\right]>0, \Re(\alpha)+\eta\min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}\right]>0$ for $\alpha>0$ or $\alpha=0,\mu\ge 0$ and $\Re(\alpha)+\lambda\min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j},\frac{\Re(\delta)+\frac{1}{2}}{\mu}\right]>0, \Re(\alpha)+\eta\min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j},\frac{\Re(\delta)+\frac{1}{2}}{\mu}\right]>0$ for $\alpha=0,\mu<0$ then there holds the formula

$$\int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}(cx+d)^{\gamma} H_{p,q}^{m,n}\left[y(x-a)^{\lambda}(b-x)^{\eta}(cx+d)^{-\nu}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}x$$
$$= (b-a)^{\alpha+\beta-1}(ac+d)^{\gamma}$$
$$\times H_{2,1:p+1,q+1;0,1}^{0,2:m,n+1;1,0}\left[\begin{array}{c}\frac{y(b-a)^{\lambda+\eta}}{(ac+d)^{\nu}}\\ \frac{c(b-a)}{ac+d}\end{array}\Bigg| \begin{array}{c}(1-\alpha;\lambda,1),(1+\gamma;\nu,1):-\\ -\\ (a_1,A_1),\ldots,(a_p,A_p);(1-\beta,\eta)\\ (1-\alpha-\beta;\lambda+\eta,1);(b_1,B_1),\ldots,(b_q,B_q),(1+\gamma,\nu);(0,1)\end{array}\right].$$

Hence or otherwise show that

$$\int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}x^{\gamma} H_{p,q}^{m,n}\left[yx^{-\nu}(x-a)^{\lambda}(b-x)^{\eta}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}x$$
$$= (b-a)^{\alpha+\beta-1}a^{\gamma}$$
$$\times H_{2,1:p+1,q+1;0,1}^{0,2:m,n+1;1,0}\left[\begin{array}{c}\frac{y(b-a)^{\lambda+\eta}}{a^{\nu}}\\ \frac{c(b-a)}{a}\end{array}\Bigg| \begin{array}{c}(1-\alpha;\lambda,1),(1+\gamma;\nu,1):-\\ -\\ (a_1,A_1),\ldots,(a_p,A_p);(1-\beta,\eta);-\\ (1-\alpha-\beta;\lambda+\eta,1):(b_1,B_1),\ldots,(b_q,B_q),(1+\gamma,\nu);(0,1)\end{array}\right],$$

and give its conditions of validity (Saxena and Saigo 1998).

2.4. *Notation 2.9.* F_3: Appell function of the third kind

Definition 2.8. The Appell function of the third kind is defined in the form

$$F_3(a,a',b,b';c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}} \frac{x^m y^n}{m!n!}$$
$$= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} {}_2F_1(a',b';c';y)\frac{x^m}{m!},$$

where $\max\{|x|,|y|\} < 1$. Prove the following result:

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(yt+z)^{\eta}\mathrm{d}t$$
$$= (b-a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\eta}B(\alpha,\beta)$$
$$\times F_3\left(a,\beta,-\gamma,-\eta;\alpha+\beta;-\frac{(b-a)u}{au+v},\frac{(b-a)y}{by+z}\right),$$

where for convergence

$$\max\left\{\left|\frac{(b-a)u}{au+v}\right|,\left|\frac{(b-a)y}{by+z}\right|\right\} < 1; b \neq a, \min\{\Re(\alpha),\Re(\beta)\} > 0.$$

2.5. Show that

$$\int_{-1}^{1}(1+t)^{\rho-1}(1-t)^{\lambda-1}P_{\nu}^{(\alpha,\beta)}\left[1-\frac{\sigma y}{2}(1-t)\right]H_{p,q}^{m,n}\left[z(1-t)^{h}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}t$$
$$=\frac{2^{\lambda+\rho-1}(\alpha+1)_{\nu}\Gamma(\rho)}{\nu!}\sum_{r=0}^{\nu}\frac{(-\nu)_r}{r!}\frac{(1+\alpha+\beta_1)_r}{(1+\alpha)_r}\left(\frac{\sigma y}{2}\right)^{r}$$
$$\times H_{p+1,q+1}^{m,n+1}\left[2^{h}z\Big|_{(b_q,B_q),(1-\lambda-\rho-\nu,h)}^{(1-\lambda-r,h),(a_p,A_p)}\right],$$

where $z,\lambda,\rho\in C, h>0, \mu\geq 0, \alpha>0, |\arg z|<\frac{1}{2}\pi\alpha$; $\Re(\rho)>0, \Re(\lambda)+h\min_{1\leq j\leq m}\left[\frac{\Re(b_j)}{B_j}\right]>0$.

2.6. Prove that

$$\int_{0}^{\infty}t^{-\rho}J_{\nu}(t)J_{w}(t)H_{p,q}^{m,n}\left[at^{2h}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}t$$
$$=2^{-\rho}H_{p+4,q+1}^{m+1,n+1}\left[zt^{2h}\Big|_{(\rho,2h),(b_q,B_q)}^{\left(\frac{1+\rho-w-\nu}{2},h\right),(a_p,A_p),\left(\frac{1+\rho+\nu\pm w}{2},h\right),\left(\frac{\rho+w-\nu+1}{2},h\right)}\right],$$

where $J_{\nu}(\cdot)$ is the ordinary Bessel function, $h>0, \rho,\nu,\omega\in C, \mu\geq 0, \alpha>0$, $|\arg a|<\frac{1}{2}\pi\alpha$, $\Re(w+\nu-\rho+2h)\min_{1\leq j\leq m}\left[\frac{\Re(b_j)}{B_j}\right]>-1$ and $\Re(\rho)>2h\max_{1\leq i\leq n}\left[\frac{\Re(a_i)-1}{A_i}\right]$.

Chapter 3
Fractional Calculus

3.1 Introduction

The subject of fractional calculus deals with the investigations of integrals and derivatives of any arbitrary real or complex order, which unify and extend the notions of integer-order derivative and n-fold integral. It has gained importance and popularity during the last four decades or so, mainly due to its vast potential of demonstrated applications in various seemingly diversified fields of science and engineering, such as fluid flow, rheology, diffusion, relaxation, oscillation, anomalous diffusion, reaction-diffusion, turbulence, diffusive transport akin to diffusion, electric networks, polymer physics, chemical physics, electrochemistry of corrosion, relaxation processes in complex systems, propagation of seismic waves, dynamical processes in self-similar and porous structures and others. In this connection, one can refer to Caputo (1967), Glöckle and Nonnenmacher (1991), Mainardi (1995, 1996), Mainardi and Tomirotti (1997), Metzler et al. (1994), and monographs by Podlubny (1999), Dzherbashyan (1966), Oldham and Spanier (1974), Miller and Ross (1993), Hilfer (2000), Kilbas et al. (2006) and references therein.

The importance of this subject further lies in the fact that during the last three decades, three international conferences dedicated exclusively to fractional calculus and its applications were held in the University of New Haven in 1974, University of Strathclyde, Glasgow, Scotland in 1984, and the third in Nihon University in Tokyo, Japan in 1989 in which various workers presented their investigations dealing with the theory and applications of fractional calculus (see, for details, Ross (1975), McBride and Roach (1985), and Nishimoto (1991)). The works of Srivastava and Owa (1989), Kalia (1993), Rusev et al. (1995, 1997), Gaishun et al. (1996) also deal especially with the subject of fractional calculus.

A comprehensive account of fractional calculus and its applications can be found in the monographs written by Kiryakova (1994), McBride (1985), Oldham and Spanier (1974), Miller and Ross (1993), and Ross (1975). In particular, the five volumes work published recently by Nishimoto (1984, 1987, 1989, 1991, 1996) contains an interesting account of the theory and applications of fractional calculus in a number of areas of mathematical analysis, such as ordinary and partial differential equations, summation of series, special functions, etc.

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This chapter deals with the definition and basic properties of various operators of fractional integration and fractional differentiation of arbitrary order. Among the various operators studied, it involves the Riemann–Liouville fractional integration operators, Riemann–Liouville fractional differentiation operators, Weyl operators, Kober operators, Saigo operators, etc. Besides the basic properties of these operators, their behavior under Laplace, Fourier, and Mellin transforms are also presented. Application of Riemann–Liouville fractional calculus operators in the solution of kinetic equations, fractional reaction, fractional diffusion and fractional reaction–diffusion equations, etc. are demonstrated. The results are mostly derived in a closed form in terms of the H-functions and Mittag-Leffler functions suitable for numerical computation.

3.2 A Brief Historical Background

In order to give a meaning to the notation $\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$ for the nth order derivative, when n is any number: fractional, irrational or complex, fractional calculus came into existence. In fact G.A. l'Hopital wrote to G. W. Leibnitz to know the meaning of $\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$, when $n = \frac{1}{2}$. Leibnitz replied in a letter of 30 September 1695 to l'Hopital that " $\mathrm{d}^{\frac{1}{2}}x$ will be equal to $x\sqrt{\mathrm{d}x : x}$, an apparent paradox from which one day useful consequences will be drawn". The name "fractional calculus" is probably due to l'Hopital's question "what if n is $\frac{1}{2}$?" In another letter of Leibniz to J. Wallis dated 28 May 1697, Leibniz discusses Wallis' infinite product for π, mentions differential calculus and uses the notation $\mathrm{d}^{\frac{1}{2}}y$ to denote a derivative of order $\frac{1}{2}$.

Lacroix (1819) observed that

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}x^n = \frac{n!}{(n-m)!}x^{n-m},\; n \in N = 1, 2, 3, \ldots;\;\; m \in N_0 = N \cup \{0\};\;\; n \geq m. \tag{3.1}$$

Since $n! = \Gamma(n+1)$ and $(n-m)! = \Gamma(n-m+1)$, the above equation was written by Lacroix (1819) in terms of the gamma function in the form

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}x^n = \frac{\Gamma(n+1)}{\Gamma(n-m+1)}x^{n-m}, \tag{3.2}$$

and then set $m = \frac{1}{2}$ and $n = 1$ to obtain

$$\frac{\mathrm{d}^{\frac{1}{2}}x}{\mathrm{d}x^{\frac{1}{2}}} = \frac{2x^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}.$$

During the eighteenth century, several mathematicians have contributed to the development of fractional calculus, which includes Fourier (1822), Abel (1823–1826), Liouville (1822–1837), and Riemann (1847).

Gruñwald (1867) defined the differintegration in terms of the following infinite series:

$$\frac{d^q f}{[d(x-a)]^q} = \lim_{N\to\infty} \left\{ \frac{[(x-a)/N]^{-q}}{\Gamma(-q)} \sum_{k=0}^{N-1} \frac{\Gamma(k-q)}{\Gamma(k+1)} f\left(x - k\left[\frac{x-a}{N}\right]\right)\right\}, \tag{3.3}$$

where q is arbitrary. The above definition was further generalized by Post (1930) to the form

$$\frac{d^n f}{dx^n} = \lim_{\delta x\to 0} \left\{ (\delta x)^{-n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x-k\delta x)\right\}, \; n \in N_0, \tag{3.4}$$

where,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The theory of fractional calculus by complex integral transformations approach has been developed by many mathematicians including Augustin–Louis Cauchy (1789–1857) and Edward Goursat (1858–1936). Further, Sonin in 1869 wrote a paper entitled "On differentiation with arbitrary index" from which the present definition of the Riemann–Liouville operator appears to follow. Letnikov (1872) in his four papers presented an explanation of the main concept of theory of differentiation of an arbitrary index which provides extension of Sonin's work. A detailed account of the origin of the Riemann–Liouville definition and its applications can be found in the monograph of Miller and Ross (1993). The works of Davis (1927, 1936), Love (1936–1996), Erdélyi (1939–1965), Kober (1940), Riesz (1949), Gelfand and Shilov (1959–1964), and Caputo (1969) may also be mentioned in this connection.

A chronological bibliography of fractional calculus given by Ross is available from the monograph of Oldham and Spanier (1974, pp. 1–15). Ross (1975) has also given a brief history and exposition of the fundamental theory of fractional calculus.

3.3 Fractional Integrals

Notation 3.1. ${}_aI_x^n, {}_aD_x^{-n}; n \in N_0$: Fractional integral of integer order n.

Definition 3.1.

$$({}_aI_x^n f)(x) = {}_aD_x^{-n} f(x) = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t)dt, x > a, \tag{3.5}$$

where $n \in N_0$.

We begin our study by introducing a fractional integral of order n in the form (Cauchy formula):

$$({}_aD_x^{-n}f)(x) = \frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1}f(t)\mathrm{d}t. \tag{3.6}$$

It will be shown that the above integral can be expressed in terms of n-fold integral, that is,

$$({}_aD_x^{-n}f)(x) = \int_a^x \mathrm{d}x_1 \int_a^{x_1} \mathrm{d}x_2 \int_a^{x_2} \mathrm{d}x_3 \cdots \int_a^{x_{n-1}} f(t)\mathrm{d}t, \tag{3.7}$$

Proof 3.1. When $n = 2$, then using the well-known Dirichlet formula, namely

$$\int_a^b \mathrm{d}x \int_a^x f(x,y)\mathrm{d}y = \int_a^b \mathrm{d}y \int_y^b f(x,y)\mathrm{d}x \tag{3.8}$$

Equation (3.7) becomes

$$\begin{aligned}\int_a^x \mathrm{d}x_1 \int_a^{x_1} f(t)\mathrm{d}t &= \int_a^x \mathrm{d}t\, f(t) \int_t^x \mathrm{d}x_1 \\ &= \int_a^x (x-t)f(t)\mathrm{d}t.\end{aligned} \tag{3.9}$$

This shows that the twofold integral can be reduced to a simple integral with the help of Dirichlet formula. For $n = 3$, the integral in (3.7) gives

$$\begin{aligned}({}_aD_x^{-3}f)(x) &= \int_a^x \mathrm{d}x_1 \int_a^{x_1} \mathrm{d}x_2 \int_a^{x_2} f(t)\mathrm{d}t \\ &= \int_a^x \mathrm{d}x_1 \left[\int_a^{x_1} \mathrm{d}x_2 \int_a^{x_2} f(t)\mathrm{d}t\right].\end{aligned} \tag{3.10}$$

Using the result (3.9) the integrals within big bracket simplify to yield

$$({}_aD_x^{-3}f)(x) = \int_a^x \mathrm{d}x_1 \left[\int_a^{x_1} (x_1 - t)f(t)\mathrm{d}t\right]. \tag{3.11}$$

If we use (3.8), then the above line reduces to

$$({}_aD_x^{-3}f)(x) = \int_a^x \mathrm{d}t\, f(t)\left[\int_t^x (x_1 - t)\mathrm{d}x_1\right] = \int_a^x \frac{(x-t)^2}{2!}f(t)\mathrm{d}t, \tag{3.12}$$

□

Continuing this process, we finally obtain

$$({}_aD_x^{-n} f)(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\mathrm{d}t = \frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1} f(t)\mathrm{d}t. \tag{3.13}$$

It is evident that the last integral in (3.13) is meaningful for any number n provided its real part is greater than zero.

3.3.1 Riemann–Liouville Fractional Integrals

Notation 3.2. ${}_aI_x^{\alpha}$, ${}_aD_x^{-\alpha}$; I_{a+}^{α} : Riemann–Liouville left-sided fractional integral of order α.

Notation 3.3. ${}_xI_b^{\alpha}, {}_xD_b^{-\alpha}; I_{b-}^{\alpha}$: Riemann–Liouville right-sided fractional integral of order α.

Notation 3.4. $L(a,b)$: Space of Lebesgue measurable real or complex valued functions.

Definition 3.2. $L(a,b)$ consists of Lebesgue measurable real or complex valued function $f(x)$ on $[a,b]$:

$$L(a,b) = \left\{ f : ||f||_1 = \int_a^b |f(t)|\mathrm{d}t < +\infty \right\}. \tag{3.14}$$

Definition 3.3. Let $f(x) \in L(a,b), \alpha \in C, \Re(\alpha) > 0$, then

$${}_aI_x^{\alpha} f(x) = {}_aD_x^{-\alpha} f(x) = I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, x > a, \tag{3.15}$$

is called the Riemann–Liouville left-sided fractional integral of order α.

Definition 3.4. Let $f(x) \in L(a,b), \alpha \in C, \Re(\alpha) > 0$, then

$${}_xI_b^{\alpha} f(x) = {}_xD_b^{-\alpha} f(x) = I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t, x < b, \tag{3.16}$$

is called the Riemann–Liouville right-sided fractional integral of order α.

3.3.2 Basic Properties of Fractional Integrals

Proposition 3.1. *Fractional integrals obey the following property:*

$$\begin{aligned}({}_aI_x^{\alpha}\,{}_aI_x^{\beta}\varphi)(x) &= ({}_aI_x^{\alpha+\beta}\varphi)(x) = ({}_aI_x^{\beta}\,{}_aI_x^{\alpha}\varphi)(x).\\ ({}_xI_b^{\alpha}\,{}_xI_b^{\beta}\varphi)(x) &= ({}_xI_b^{\alpha+\beta}\varphi)(x) = ({}_xI_b^{\beta}\,{}_xI_b^{\alpha}\varphi)(x).\end{aligned} \tag{3.17}$$

Proof 3.2. By virtue of the definition (3.14) and the Dirichlet formula (3.8), it follows that

$$\begin{aligned}({}_aI_x^{\alpha}\ {}_aI_x^{\beta}\varphi)(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\mathrm{d}t}{(x-t)^{1-\alpha}}\frac{1}{\Gamma(\beta)}\int_a^t \frac{\varphi(u)\mathrm{d}u}{(t-u)^{1-\beta}} \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x \mathrm{d}u\varphi(u)\int_u^x \frac{\mathrm{d}t}{(x-t)^{1-\alpha}(t-u)^{1-\beta}}, \end{aligned} \tag{3.18}$$

If we use the substitution $y = \frac{t-u}{x-u}$, the value of the second integral is

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)(x-u)^{1-\alpha-\beta}}\int_0^1 y^{\beta-1}(1-y)^{\alpha-1}\mathrm{d}y = \frac{(x-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)},$$

which when substituted in (3.18) yields the first part of (3.17). The second part can be similarly established. In particular,

$${}_aI_x^{n+\alpha}f(x) = ({}_aI_x^{n}\ {}_aI_x^{\alpha}f)(x), \quad n \in N, \Re(\alpha) > 0, \tag{3.19}$$

which shows that the n-fold differentiation

$$\left(\frac{\mathrm{d}^n}{\mathrm{d}x^n}I_x^{n+\alpha}f\right)(x) = {}_aI_x^{\alpha}f(x), n \in N, \Re(\alpha) > 0, \tag{3.20}$$

for all x. When $\alpha = 0$, we obtain

$$({}_aI_x^{0}f)(x) = f(x); \quad ({}_aI_x^{-n}f)(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}f(x) = f^{(n)}(x). \tag{3.21}$$

□

Note 3.1. The property given in (3.17) is called the semigroup property of fractional integration.

Proposition 3.2. *The following result holds:*

$$\int_a^b f(x)({}_aI_x^{\alpha}g)\mathrm{d}x = \int_a^b g(x)({}_xI_b^{\alpha}f)\mathrm{d}x. \tag{3.22}$$

The result (3.22) can be established by interchanging the order of integration in the integral on the left of (3.22) and then by using the Dirichlet formula (3.8).

Remark 3.1. Stanislavsky (2004) derived a specific interpretation of fractional calculus. It was shown that there exists a relation between stable probability distribution and the fractional integral. The relation investigated shows that the parameter of the stable distribution coincides with the exponent of the fractional integral.

3.3.3 Illustrative Examples

Example 3.1. If $f(x) = (x-a)^{\beta-1}$, then find the value of ${}_aI_x^\alpha f(x)$.

Solution 3.1. We have

$$({}_aI_x^\alpha f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}(t-a)^{\beta-1}\mathrm{d}t.$$

If we substitute $t = a + y(x-a)$ in the above integral, it reduces to

$$\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1},$$

where $\Re(\beta) > 0$. Thus,

$$({}_aI_x^\alpha f)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}, \tag{3.23}$$

provided $\alpha, \beta \in C$, $\min\{\Re(\alpha), \Re(\beta)\} > 0$.

Example 3.2. It can be similarly shown that

$$({}_xI_b^\alpha g)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(b-x)^{\alpha+\beta-1}, \quad x < b, \tag{3.24}$$

where $g(x) = (b-x)^{\beta-1}, \alpha, \beta \in C; \min\{\Re(\alpha), \Re(\beta)\} > 0$.

Note 3.2. It may be noted that (3.23) and (3.24) give the Riemann–Liouville integrals of the power functions $f(x) = (x-a)^{\beta-1}$ and $g(x) = (b-x)^{\beta-1}$, $\min\{\Re(\alpha), (\Re(\beta)\} > 0$.

Exercises 3.2

3.2.1. Prove that

$$({}_aI_x^\alpha[x \pm c]^{\gamma-1})(x) = \frac{(a \pm c)^{\gamma-1}}{\Gamma(\alpha+1)}(x-a)^\alpha {}_2F_1\left(1, 1-\gamma; \alpha+1; \frac{a-x}{a \pm c}\right),$$

where $\Re(\alpha) > 0, \alpha, \gamma \in C, \left|\frac{a-x}{a \pm c}\right| < 1$.

3.2.2. Prove that

$$\left({}_aI_x^\alpha(x-a)^{\beta-1}(b-x)^{\gamma-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{(x-a)^{\alpha+\beta-1}}{(b-a)^{1-\gamma}}{}_2F_1(\beta, 1-\gamma; \alpha+\beta; \frac{a-x}{b-a}),$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, \alpha, \beta, \gamma \in C, \left|\frac{a-x}{b-a}\right| < 1$.

3.2.3. Prove that

$$\left({}_aI_x^\alpha[\mathrm{e}^{\lambda x}]\right)(x) = \mathrm{e}^{\lambda a}(x-a)^\alpha E_{1,\alpha+1}(\lambda x - \lambda a).$$

where $x > a, \alpha, \lambda \in C, \Re(\alpha) > 0$ and $E_{1,\alpha+1}(.)$ is the Mittag-Leffler function.

3.2.4. Prove that

$$\left({}_aI_x^\alpha[\mathrm{e}^{\lambda x}(x-a)^{\beta-1}]\right)(x) = \mathrm{e}^{\lambda a}\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}{}_1F_1(\beta;\alpha+\beta;\lambda x-\lambda a),$$

where $\alpha, \beta \in C, \min\{\Re(\alpha), \Re(\beta)\} > 0.$

3.2.5. Prove that

$$\left({}_aI_x^\alpha([x-a]^{\beta-1}\ln(x-a)\right)(x)$$
$$= \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}[\ln(x-a) + \psi(\beta) - \psi(\alpha+\beta)],$$

where $\alpha, \beta \in C, \min\{\Re(\alpha), \Re(\beta)\} > 0$; and $\psi(.)$ is the logarithmic derivative of the gamma function.

3.2.6. Prove that

$$\left({}_aI_x^\alpha[(x-a)^{\frac{\nu}{2}}J_\nu[\lambda\sqrt{x-a}]\right)(x) = \left(\frac{2}{\lambda}\right)^\alpha (x-a)^{(\alpha+\nu)/2}J_{\alpha+\nu}(\lambda\sqrt{x-a}),$$

where $\alpha, \nu \in C, \Re(\alpha) > 0, \Re(\nu) > -1.$

3.2.7. Prove that

$$\left({}_aI_x^\nu((x-a)^{\beta-1}E_{\mu,\beta}[(x-a)]^\mu\right)(x) = (x-a)^{\nu+\beta-1}E_{\mu,\nu+\beta}[(x-a)^\mu],$$

where $\beta, \mu, \nu \in C, \Re(\nu) > 0.$

3.2.8. Prove that

$$\left({}_0I_x^\nu[x^{\mu-1}\sin ax]\right)(x)$$
$$= \frac{x^{\mu+\nu-1}}{2i}\frac{\Gamma(\mu)}{\Gamma(\mu+\nu)}\left[{}_1F_1(\mu;\mu+\nu;iax) - {}_1F_1(\mu;\mu+\nu;-iax)\right],$$

where $\beta, \nu \in C, a > 0, \min\{\Re(\nu), \Re(\mu)\} > 0.$

3.2.9. Prove that

$$(I_{b-}^{n} g)(x) = \int_x^b \mathrm{d}t_1 \int_{t_1}^b \mathrm{d}t_2 \cdots \int_{t_{n-1}}^b g(t_n)\mathrm{d}t_n = \frac{1}{(n-1)!}\int_x^b (t-x)^{n-1} g(t)\mathrm{d}t,\ n \in \mathrm{N}.$$

3.2.10. Prove that Riemann–Liouville fractional integrals ${}_aI_x^\alpha$ and ${}_xI_b^\alpha$ with $\Re(\alpha) > 0$ are bounded in $L_1[a,b]$. That is

$$||{}_aI_x^\alpha h||_1 \le \frac{(b-a)^{\Re(\alpha)}}{|\Gamma(\alpha)|\Re(\alpha)}||h||_1, ||{}_xI_b^\alpha h||_1 \le \frac{(b-a)^{\Re(\alpha)}}{|\Gamma(\alpha)|\Re(\alpha)}||h||_1, \tag{3.25}$$

where $\alpha \in C, \Re(\alpha) > 0$.

3.2.11. Prove that the Riemann–Liouville fractional integral I_{0+}^α of the H-function exists and the following result holds:

$$\left(I_{0+}^\alpha t^{\rho-1} H_{p,q}^{m,n}\left[t^\sigma \Big|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right]\right)(x) = x^{\rho+\alpha-1} H_{p+1,q+1}^{m,n+1}\left[x^\sigma \Big|\begin{matrix}(1-\rho,\sigma),(a_p,A_p)\\(b_q,B_q),(1-\rho-\alpha,\sigma)\end{matrix}\right],$$

provided $\alpha \in C, \Re(\alpha) > 0, a_i, b_j \in C, A_i, B_j > 0, i = 1,\ldots,p; j = 1,\ldots,q$, $\rho \in C, \sigma > 0$. Further let

$$\alpha^* = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j > 0, \tag{3.26}$$

or

$$\alpha^* = 0, \gamma\mu + \Re(\delta) < -1; \sigma \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}\right] + \Re(\rho) > 0, \tag{3.27}$$

and

$$\gamma\sigma < \Re(\rho), \quad \text{where the contour of integration is } L = L_{i\gamma\infty}. \tag{3.28}$$

3.4 Riemann–Liouville Fractional Derivatives

Notation 3.5. $\{\alpha\}$ means the fractional part of number $\alpha, 0 \le \{\alpha\} < 1$.

Notation 3.6. $[\alpha]$ means the integral part of number α.

Note 3.3. We note that

$$\alpha = \{\alpha\} + [\alpha].$$

Notation 3.7. ${}_aD_x^\alpha\ \varphi;\ D_{a+}^\alpha\ \varphi$: Riemann–Liouville left-sided fractional derivative of the function $\varphi(x)$ of order α.

Notation 3.8. ${}_bD_x^\alpha\ \varphi,\ I_{b-}^\alpha\ \varphi$: Riemann–Liouville right-sided fractional derivative of the function $\varphi(x)$ of order α.

Definition 3.5. The left-sided Riemann–Liouville fractional derivative of order $\alpha \in C, \Re(\alpha) \geq 0$ of the function $\varphi(x)$ is defined by

$$({}_aD_x^\alpha\, \varphi)(x) = (D_{a+}^\alpha\, \varphi)(x)$$
$$= \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_a^x \frac{\varphi(t)\mathrm{d}t}{(x-t)^{\alpha-n+1}}, \quad n = [\Re(\alpha)] + 1;\ x > a, \quad (3.29)$$

where $[\Re(\alpha)]$ means the integral part of $\Re(\alpha)$.

Definition 3.6. The right-sided Riemann–Liouville fractional derivative of order $\alpha \in C,\ \Re(\alpha) \geq 0$ of the function $\varphi(x)$is defined by

$$({}_xD_b^\alpha\, \varphi)(x) = (D_{b-}^\alpha\, \varphi)(x)$$
$$= \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_x^b \frac{\varphi(t)\mathrm{d}t}{(t-x)^{\alpha-n+1}}, \quad n = [\Re(\alpha)] + 1;\ x < b. \quad (3.30)$$

In short, one can express (3.29) in the form

$$({}_aD_x^\alpha \varphi)(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} ({}_aI_x^{n-\alpha} \varphi)(x), \quad (3.31)$$

and (3.30) as

$$({}_xD_b^\alpha\, \varphi)(x) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} ({}_xI_b^{n-\alpha}\, \varphi)(x). \quad (3.32)$$

For $\alpha \in R^+$, the equations (3.29) and (3.30) take the forms

$$({}_aD_x^\alpha\, \varphi)(x) = (D_{a+}^\alpha\, \varphi)(x)$$
$$= \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_a^x \frac{\varphi(t)\mathrm{d}t}{(x-t)^{\alpha-n+1}}, \quad n = [\alpha] + 1;\ x < b \quad (3.33)$$

and

$${}_xD_b^\alpha\, \varphi)(x) = (D_{b-}^\alpha\, \varphi)(x)$$
$$= \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_x^b \frac{\varphi(t)\mathrm{d}t}{(t-x)^{\alpha-n+1}}, \quad n = [\alpha] + 1;\ x < b. \quad (3.34)$$

We shall also employ the notations

$$({}_aD_x^\alpha)\, \varphi = ({}_aI_x^{-\alpha})\, \varphi = ({}_aI_x^\alpha)^{-1} \varphi;\ \alpha \geq 0.$$

Similarly, we have

$$({}_xD_b^\alpha)\varphi = ({}_xI_b^{-\alpha})\varphi = ({}_xI_b^\alpha)^{-1} \varphi;\ \alpha \geq 0.$$

Remark 3.2. Geometric and physical interpretations of fractional integration and fractional differentiation were given by Podlubny (2002), also see Nigmatullin (1992).

Notation 3.9. $\Omega = [a, b], -\infty < a < b < \infty, \Omega$ may be a finite interval, a half line or a whole line.

Notation 3.10. $AC(\Omega)$, the space of absolutely continuous functions.

Notation 3.11. $AC^n(\Omega)$. If $n \in N$, the space of complex-valued functions $h(x)$ which have continuous derivatives up to order $n-1$ on $[a, b]$ with $h^{(n-1)}(x) \in AC[a, b]$ is denoted by $AC^n[a, b]$. That is

$$AC^n[a, b] = \left\{h : [a, b] \to C \text{ and } \left(D^{n-1}h\right)(x) \in AC[a, b]\right\}, \ D = \frac{\mathrm{d}}{\mathrm{d}x}, \tag{3.35}$$

where C is the set of complex numbers. It is evident that $AC^1[a, b] = AC[a, b]$.

We now present some properties of the operators defined by (3.29) and (3.30) (see Samko et al. (1993)).

Proposition 3.3. *Let $AC[a, b]$ be the space of absolutely continuous functions h on $[a, b]$. It is known [see Kolmogorov and Fomin 1984, p. 338] that $AC[a, b]$ coincides with the space of primitives of Lebesgue summable functions:*

$$h(x) \in AC[a, b] \Leftrightarrow h(x) = c + \int_a^x \varphi(t)\mathrm{d}t, \ \varphi(t) \in L(a, b). \tag{3.36}$$

Hence absolutely continuous function $h(x)$ has a summable derivative $h'(x) = \varphi(x)$ almost everywhere on $[a, b]$. Thus (3.36) gives

$$\varphi(t) = h'(t) \text{ and } c = h(a). \tag{3.37}$$

The following lemma can be established with the help of (3.36), which provides the characterization of the space $AC^n[a, b]$.

Lemma 3.1. *The space $AC^n[a, b]$ consists of those and only those functions $h(x)$, which are represented in the form*

$$h(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}\varphi(t)\mathrm{d}t + \sum_{r=0}^{n-1} c_r(x-a)^r, \tag{3.38}$$

where $\varphi(x) \in L(a, b)$ and $c_r, r = 0, 1, \ldots, n-1$ are arbitrary constants. It follows from (3.38) that

$$\varphi(x) = h^{(n)}(x) \ \text{ and } c_r = \frac{h^{(r)}(a)}{r!}, \ r = 0, 1, \ldots, n-1. \tag{3.39}$$

The next theorem characterizes the conditions for the existence of the fractional derivatives in the space $AC^n[a,b]$, defined by (3.35)

Theorem 3.1. *If* $\alpha \in C, \Re(\alpha) \geq 0; n = |\Re(\alpha)| + 1,$ *and* $h(x) \in AC^n[a.b]$*, then the fractional differentiation operators* ${}_aD_x^\alpha h$ *and* ${}_xD_b^\alpha h$ *exist almost everywhere on* $[a, b]$ *and may be represented in the forms*

$$({}_aD_x^\alpha h)(x) = \sum_{r=o}^{n-1} \frac{h^{(r)}(a)}{\Gamma(1+r-\alpha)}(x-a)^{r-\alpha} + \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{h^{(n)}(t)\mathrm{d}t}{(x-t)^{\alpha-n+1}}, \tag{3.40}$$

and

$$({}_xD_b^\alpha h)(x) = \sum_{r=o}^{n-1} \frac{(-1)^r h^{(r)}(b)}{\Gamma(1+r-\alpha)}(b-x)^{r-\alpha} + \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{h^{(n)}(t)\mathrm{d}t}{(t-x)^{\alpha-n+1}}, \tag{3.41}$$

respectively.

To prove the first part of the theorem, we observe that since $h(x) \in AC^n$, consequently the representation (3.38) holds. Using this in the definition of the fractional derivative ${}_aD_x^\alpha h$ (3.29) and taking (3.39) into account, the result (3.40) follows. The second part can be proved similarly by using the Definition 3.6 and the representation for the function $f(x) \in AC^n[a,b]$ of the form (3.38):

$$f(x) = \frac{(-1)^n}{(n-1)!} \int_x^b (t-x)^{n-1}\theta(t)\mathrm{d}t + \sum_{r=0}^{n-1} (-1)^r \mathrm{e}_r (b-x)^r, \tag{3.42}$$

where

$$\theta(t) = f^{(n)}(t) \text{ and } \mathrm{e}_r = \frac{f^{(r)}(b)}{r!}. \tag{3.43}$$

Corollary 3.1. *If* $\alpha \in C, 0 \leq \Re(\alpha) < 1, \alpha \neq 0,$ *and* $h(x) \in AC[a,b]$*, then there holds the relations*

$$({}_aD_x^\alpha h)(x) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{h(a)}{(x-a)^\alpha} + \int_a^x \frac{h'(t)\mathrm{d}t}{(x-t)^\alpha} \right], \tag{3.44}$$

and

$$({}_xD_b^\alpha h)(x) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{h(b)}{(b-x)^\alpha} - \int_x^b \frac{h'(t)\mathrm{d}t}{(t-x)^\alpha} \right]. \tag{3.45}$$

Lemma 3.2. *If* $\alpha \in C, \Re(\alpha) > 0$ *and* $h(x) \in L_p(a,b), 1 \leq p < \infty$, *then the following formulae*

$$({}_aD_x^\alpha \ {}_aI_x^\alpha h)(x) = h(x) \text{ and } ({}_xD_b^\alpha \ {}_xI_b^\alpha h)(x) = h(x), \ \Re(\alpha) > 0 \tag{3.46}$$

hold almost everywhere on $[a,b]$.

Remark 3.3. The above assertion shows that the fractional differentiation is an operation inverse to fractional integration from the left.

Lemma 3.3. *If* $\alpha, \beta \in C, \Re(\alpha) > \Re(\beta) > 0$, *then for* $h(x) \in L_p(a,b), \ 1 \leq p < \infty$, *the composition relations*

$$({}_aD_x^\beta \ {}_aI_x^\alpha h)(x) = {}_aI_x^{\alpha-\beta} h(x) \quad \text{and} \quad ({}_xD_b^\beta \ {}_xI_b^\alpha h)(x) = {}_xI_b^{\alpha-\beta} h(x), \tag{3.47}$$

hold almost everywhere on $[a,b]$.

The first part in (3.47) readily follows from the results (3.40), Theorem 3.2 and Lemma 3.2. The second part can be proved similarly.

Notation 3.12. ${}_aI_x^\alpha(L_p)$: Space of functions.

Notation 3.13. ${}_xI_b^\alpha(L_p)$: Space of functions,

Definition 3.7. The space of functions ${}_aI_x^\alpha(L_p)$ is defined by

$${}_aI_x^\alpha(L_p) = \{h : h = {}_aI_x^\alpha \varphi; \varphi \in L_p(a,b)\}, \tag{3.48}$$

for $\alpha \in C, \Re(\alpha) > 0$ and$1 \leq p < \infty$.

Definition 3.8. The space of functions ${}_xI_b^\alpha(L_p)$ is defined by

$${}_xI_b^\alpha(L_p) = \{h : h = {}_xI_b^\alpha \varphi; \varphi \in L_p(a,b)\}, \tag{3.49}$$

for $\alpha \in C, \Re(\alpha) > 0$ and $1 \leq p \leq \infty$.

Theorem 3.2. *Let* $\alpha \in C, \Re(\alpha) > 0, n = |\Re(\alpha)| + 1$ *and let* $h_{n-\alpha}(x) = ({}_aI_x^{n-\alpha} h)(x)$ *be the fractional integral of order* $n - \alpha$, *defined by (3.15). Then the following results hold:*

(i) If $h(x) \in {}_aI_x^\alpha(L_p), \ 1 \leq p < \infty$, *then*

$$\left({}_aI_x^\alpha {}_aD_x^\alpha h\right)(x) = h(x). \tag{3.50}$$

(ii) If $h(x) \in L_1[a.b]$ *and* $h_{n-\alpha}(x) \in AC^n[a.b]$, *then the formula*

$$\left({}_aI_x^\alpha {}_aD_x^\alpha h\right)(x) = h(x) - \sum_{j=1}^{n} \frac{h_{n-\alpha}^{(n-j)}(a)}{\Gamma(n-j+1)}(x-a)^{n-j}, \tag{3.51}$$

holds almost everywhere on $[a,b]$.

3.4.1 Illustrative Examples

Example 3.3. Prove that

$$({}_0D_x^{\alpha}[t^{\gamma}])(x) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}x^{\gamma-\alpha}, \alpha \geq 0, \gamma \in C, \Re(\gamma) > -1, x > 0, \quad (3.52)$$

Solution 3.2. We have

$$\begin{aligned}({}_0D_x^{\alpha}[t^{\gamma}])(x) &= \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_0^x t^{\gamma}(x-t)^{n-\alpha-1}\mathrm{d}t \\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+n+1-\alpha)}(\gamma-\alpha+1)_n x^{\gamma-\alpha} \\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}x^{\gamma-\alpha}, \end{aligned} \quad (3.53)$$

for $\gamma \in C, \Re(\gamma) > -1$.

Note 3.4. It is interesting to observe that for $\gamma = 0$, (3.53) yields

$$({}_0D_x^{\alpha}1)(x) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}; \alpha \neq 1, 2, \ldots, \quad (3.54)$$

which is a surprising result and indicates that the fractional derivative of a constant is, in general, not equal to zero. Thus it is not difficult to show that

$$({}_aD_x^{\alpha}1)(x) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} \text{ and } ({}_xD_b^{\alpha}1)(x) = \frac{(b-x)^{-\alpha}}{\Gamma(1-\alpha)}; 0 < \Re(\alpha) < 1. \quad (3.55)$$

Example 3.4. Prove that

$$({}_0I_x^{\nu}[\ln t])(x) = \frac{x^{\nu}}{\Gamma(\nu+1)}[\ln x - \gamma - \psi(\nu+1)],$$

where γ is the Euler's constant and $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$.

Solution 3.3. We have

$$({}_0I_x^{\nu}[\ln t])(x) = \frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}\ln t \, \mathrm{d}t.$$

If we use the substitution $t = xu$, then

$$\begin{aligned}({}_0I_x^{\nu}[\ln t])(x) &= \frac{1}{\Gamma(\nu)}\int_0^1 x^{\nu}(1-u)^{\nu-1}(\ln x + \ln u)\mathrm{d}u \\ &= \frac{x^{\nu}\ln x}{\Gamma(\nu+1)} + \frac{x^{\nu}}{\Gamma(\nu)}\int_0^1 (1-u)^{\nu-1}\ln u\mathrm{d}u.\end{aligned}$$

We know that

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\ln t\,\mathrm{d}t = B(\alpha,\beta)[\psi(\alpha)-\psi(\alpha+\beta)], \tag{3.56}$$

where $\alpha,\beta \in C, \Re(\alpha) > 0, \Re(\beta) > 0$. Applying the formula (3.56) for $\alpha = 1$ and noting that $\psi(1) = -\gamma$, we see that

$$({}_0I_x^{\alpha}[\ln t])(x) = \frac{x^{\nu}}{\Gamma(\nu+1)}[\ln x - \gamma - \psi(\nu+1)].$$

Similarly, we can prove the result in the next example.

Example 3.5. Prove that

$$({}_0D_x^{\alpha}[\ln t])(x) = \frac{x^{-\nu}}{\Gamma(1-\nu)}[\ln x - \gamma - \psi(-\nu+1)].$$

Example 3.6.

$$({}_0D_x^{\alpha}[\mathrm{e}^{at}])(x) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}{}_1F_1(1;1-\alpha;ax).$$

Solution 3.4. We have

$$\begin{aligned}({}_0D_x^{\alpha}[\exp(at)])(x) &= \sum_{r=0}^{\infty}\frac{a^r}{r!}{}_0D_x^{\alpha}(x^r)\\ &= \sum_{r=0}^{\infty}\frac{a^r}{r!}\frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)}x^{r-\alpha}\\ &= \frac{x^{-\alpha}}{\Gamma(1-\alpha)}{}_1F_1\ (1;1-\alpha;ax).\end{aligned}$$

Remark 3.4. One can unify the definitions of Riemann-Liouville fractional integral defined by (3.15) and Riemann–Liouville fractional derivative defined by (3.29) of arbitrary order $\alpha, \alpha \in C, \Re(\alpha) \neq 0, n \in N$, in the form

$$({}_aD_x^{\alpha}f)(x) = \begin{cases}\frac{1}{\Gamma(-\alpha)}\int_a^x (x-t)^{-\alpha-1}f(t)\mathrm{d}t, \ \Re(\alpha) < 0\\ \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n ({}_aI_x^{n-\alpha}f)(x), \ \Re(\alpha) > 0; n-1 \leq \Re(\alpha) < n,\end{cases} \tag{3.57}$$

which is called differintegral of f of order α. This process is also called fractional integro–differentiation. (Butzer and Westphal 2000)

Exercises 3.3

3.3.1. Prove that

$$\left({}_0D_x^{\alpha}[x^p \exp(ax)]\right)(x) = \frac{\Gamma(p+1)x^{p-\alpha}}{\Gamma(p-\alpha+1)}\,{}_1F_1(p+1; p-\alpha+1; ax),$$

where $\alpha, p \in C, \Re(p) > -1$.

3.3.2. Prove that

$$J_\nu(z) = \pi^{-1/2}2^{1-\nu}z^{-\nu}\,{}_0D_z^{-\nu+(1/2)}(\sin z).$$

3.3.3. Prove that

$$\psi(x) = -\gamma + \ln z - \Gamma(x)z^{1-x}\,{}_0D_z^{1-x}(\ln z),$$

where $\psi(x)$ is the logarithmic derivative of the gamma function and γ is the Euler's constant.

3.3.4. Prove that

$$\gamma(a,z) = \Gamma(a)\mathrm{e}^{-z}\,{}_0D_z^{-a}(\exp z),$$

where $\gamma(a,z)$ is the incomplete gamma function.

3.3.5. Prove that

$$\left({}_0D_x^{\nu}[x^{\mu/2}J_\mu(x^{\frac{1}{2}})]\right)(x) = 2^{-\nu}x^{\frac{1}{2}(\mu-\nu)}J_{\mu-\nu}(x^{\frac{1}{2}}).$$

where $\mu \in C, \Re(\mu) > -1$.

3.3.6. Prove that

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)}z^{1-c}\,{}_0D_z^{b-c}[z^{b-1}(1-z)^{-a}].$$

3.3.7. Establish the result

$$({}_0D_x^{\nu}[x^{\lambda}\,{}_2F_1(a,b;c;x)])(x) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\nu+1)}x^{\lambda-\nu}\,{}_3F_2(\lambda+1,a,b;c;\lambda-\nu+1;x),$$

where $\lambda, \nu, a, b, c \in C, \Re(\lambda) > -1, \Re(\lambda-\nu) > -1$ and $c \neq 0, -1, -2, \ldots;$ and $|x| < 1$.

3.3.8. Prove that

$$({}_aD_x^{\alpha}\;{}_aI_x^{\alpha}h)(x) = h(x).$$

3.3.9. Prove that

$$({}_aD_x^{\beta}\ {}_aI_x^{\alpha}h)(x) = {}_aI_x^{\alpha-\beta}h(x),$$

where $\alpha, \beta \in C$, $\min\{\Re(\alpha), \Re(\beta)\} > 0$, $h(x) \in L(a,b)$.

3.5 The Weyl Integral

Notation 3.14. ${}_xW_{\infty}^{\alpha}, {}_xI_{\infty}^{\alpha}, I_{-}^{\alpha}$ Weyl integral of order α.

Definition 3.9. The Weyl integral of $f(x)$ of order α, denoted by ${}_xW_{\infty}^{\alpha}$, is defined by

$$\begin{aligned}({}_xW_{\infty}^{\alpha}f)(x) &= ({}_xI_{\infty}^{\alpha}f)(x) = (I_{-}^{\alpha}f)(x)\\ &= \frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}f(t)\mathrm{d}t,\ -\infty < x < \infty \end{aligned} \tag{3.58}$$

where $\alpha \in C, \Re(\alpha) > 0$.

Notation 3.15. ${}_xD_{\infty}^{\alpha},\ {}_xD_{-}^{\alpha}$: Weyl fractional derivative.

Definition 3.10. The Weyl fractional derivative of $f(x)$ of order α, denoted by ${}_xD_{\infty}^{\alpha}$, is defined by

$$\begin{aligned}({}_xD_{\infty}^{\alpha}f)(x) &= (D_{-}^{\alpha}f)(x) = (-1)^m\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m\left({}_xW_{\infty}^{m-\alpha}f(x)\right)\\ &= (-1)^m\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m\frac{1}{\Gamma(m-\alpha)}\int_x^{\infty}\frac{f(t)\mathrm{d}t}{(t-x)^{1+\alpha-m}}, -\infty < x < \infty\end{aligned} \tag{3.59}$$

where $m-1 \le \alpha < m;\ \ m \in N, \alpha \in C$.

3.5.1 Basic Properties of Weyl Integrals

Proposition 3.4. *The following result holds.*

$$\int_0^{\infty}\varphi(x)\left({}_0I_x^{\alpha}\psi(x)\right)\mathrm{d}x = \int_0^{\infty}\psi(x)\left({}_xW_{\infty}^{\alpha}\varphi(x)\right)\mathrm{d}x. \tag{3.60}$$

Equation (3.60) is called the formula for fractional integration by parts. It is also called the Parseval equality. Equation (3.60) can be established by interchanging the order of integration.

Proposition 3.5. *Weyl fractional integrals obey the semigroup property. That is*

$$\left({}_xW_\infty^\alpha\ {}_xW_\infty^\beta f\right)(x) = ({}_xW_\infty^{\alpha+\beta} f)(x) = \left({}_xW_\infty^\beta\ {}_xW_\infty^\alpha f\right)(x). \tag{3.61}$$

Proof 3.3. We have

$$\left({}_xW_\infty^\alpha\ {}_xW_\infty^\beta f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty \mathrm{d}t\,(t-x)^{\alpha-1} \times \frac{1}{\Gamma(\beta)} \int_t^\infty (u-t)^{\beta-1} f(u)\mathrm{d}u.$$

Using the modified form of Dirichlet formula (3.8), namely

$$\int_x^a \mathrm{d}t\,(t-x)^{\alpha-1} \int_t^a (u-t)^{\beta-1} f(u)\mathrm{d}u = B(\alpha,\beta) \int_t^a (u-t)^{\alpha+\beta-1}\mathrm{d}u \tag{3.62}$$

and letting $a \to \infty$, (3.62) yields the desired result

$$\left({}_xW_\infty^\alpha\ {}_xW_\infty^\beta f\right)(x) = ({}_xW_\infty^{\alpha+\beta} f)(x). \tag{3.63}$$

The second part of Eq. (3.61) can be similarly proved. □

3.5.2 Illustrative Examples

Example 3.7. Prove that

$$\left({}_xW_\infty^\alpha[\mathrm{e}^{-\lambda x}]\right)(x) = \frac{\mathrm{e}^{-\lambda x}}{\lambda^\alpha}$$

where $\Re(\alpha) > 0$.

Solution 3.5. We have

$$({}_xW_\infty^\alpha[\mathrm{e}^{-\lambda x}])(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1}\mathrm{e}^{-\lambda t}\mathrm{d}t, \lambda > 0,$$

$$= \frac{\mathrm{e}^{-\lambda x}}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-1}\mathrm{e}^{-\lambda u}\mathrm{d}u = \frac{\mathrm{e}^{-\lambda x}}{\lambda^\alpha}.$$

Example 3.8. Find the value of $({}_xD_\infty^\alpha[\mathrm{e}^{-\lambda x}])(x), \lambda > 0$.

Solution 3.6. We have

$$\begin{aligned}({}_xD^{\alpha}_{\infty}[\mathrm{e}^{-\lambda x}])(x) &= (-1)^m\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m {}_xW^{m-\alpha}_{\infty}\mathrm{e}^{-\lambda x}\\ &= (-1)^m\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \lambda^{-(m-\alpha)}\mathrm{e}^{-\lambda x} = \lambda^{\alpha}\mathrm{e}^{-\lambda x}.\end{aligned}$$

Exercises 3.4

3.4.1. Prove that

$$({}_xW^{\nu}_{\infty}[x^{-\lambda}\exp(a/x)])(x) = \frac{\Gamma(\lambda-\nu)}{\Gamma(\lambda)}x^{\nu-\lambda}\Phi(\lambda-\nu,\lambda;a/x),$$

where $\lambda,\nu\in C, 0<\Re(\nu)<\Re(\lambda)$.

3.4.2. Prove that

$$({}_xW^{\nu}_{\infty}[x^{\nu-1}\exp(-ax)])(x) = \pi^{-\frac{1}{2}}(x/a)^{\nu-\frac{1}{2}}\exp(-ax/2)K_{\nu-\frac{1}{2}}(ax/2),$$

where $\Re(ax)>0, \nu\in C, \Re(\nu)>0$.

3.4.3. Prove that

$$({}_xW^{\alpha}_{\infty}[x^{-\alpha-\gamma}E^{\delta}_{\beta,\gamma}(ax^{-\beta})])(x) = x^{-\gamma}E^{\delta}_{\beta,\alpha+\gamma}(ax^{-\beta}),$$

where $\min\{\Re(\alpha),\Re(\beta),\Re(\gamma)\}>0;\ \ \alpha,\beta,\gamma\in C, a\in R$.

3.4.4. Prove that the Riemann–Liouville fractional integral I^{α}_{-} of the H-function exists and the following relation holds:

$$\begin{aligned}&\left(I^{\alpha}_{-}t^{\rho-1}H^{m,n}_{p,q}\left[t^{\sigma}\middle|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right]\right)(x) = x^{\rho+\alpha-1}\\ &\qquad\times H^{m+1,n}_{p+1,q+1}\left[x^{\sigma}\middle|\begin{matrix}(a_p,A_p),(1-\rho,\sigma)\\(1-\rho-\alpha,\sigma),(b_q,B_q)\end{matrix}\right],\end{aligned}$$

provided $\alpha\in C,\Re(\alpha)>0$ and further the constants $a_i,b_j\in C, A_i,B_j>0$ $i=1,\ldots,p;\ j=1,\ldots,q,\rho\in C,\sigma>0$ satisfy

$$\sigma\max_{1\le j\le n}\left[\frac{\Re(a_j)-1}{A_j}\right]+\Re(\rho)+\Re(\alpha)<1,$$

and $1+\gamma\sigma>\Re(\rho)+\Re(\alpha)$; the contour of integration being $L=L_{i\gamma\infty}$.

3.6 Laplace Transform

In this section, we derive the Laplace transforms of fractional integrals and fractional derivatives which are applicable in certain problems associated with fractional reaction, fractional diffusion fractional reaction–diffusion, etc.

3.6.1 Laplace Transform of Fractional Integrals

We have

$$({}_0I_x^\nu f)(x) = I_{0+}^\nu f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)\mathrm{d}t, \tag{3.64}$$

where $\nu \in C, \Re(\nu) > 0$.

Application of the convolution theorem of the Laplace transform to (3.64) gives

$$\begin{aligned} L\{{}_0I_x^\nu f; s\} &= L\left\{\frac{t^{\nu-1}}{\Gamma(\nu)}; s\right\} L\{f(t); s\} \\ &= s^{-\nu} F(s), \end{aligned} \tag{3.65}$$

where $s, \nu \in C, \Re(s) > 0, \Re(\nu) > 0$.

3.6.2 Laplace Transform of Fractional Derivatives

Let $n \in N$, then by the theory of the Laplace transform, we know that

$$L\left\{\frac{\mathrm{d}^n}{\mathrm{d}x^n} f; s\right\} = s^n F(s) - \sum_{r=0}^{n-1} s^{n-r-1} f^{(r)}(0_+) \tag{3.66}$$

$$= s^n F(s) - \sum_{r=0}^{n-1} s^r f^{(n-r-1)}(0_+), \tag{3.67}$$

where $s \in C, \Re(s) > 0$ and $F(s)$ is the Laplace transform of $f(t)$.

By virtue of the definition of the Riemann–Liouville fractional derivative, we find that

$$L\left[{}_0D_x^{\alpha} f; s\right] = L\left\{\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,{}_0I_x^{n-\alpha} f; s\right\}$$

$$= s^n L\left[{}_0I_x^{n-\alpha} f; s\right] - \sum_{r=0}^{n-1} s^r \frac{\mathrm{d}^{n-r-1}}{\mathrm{d}x^{n-r-1}}\,{}_0I_x^{n-\alpha} f(0_+)$$

$$= s^{\alpha} F(s) - \sum_{r=0}^{n-1} s^r \frac{\mathrm{d}^{\alpha-r-1}}{\mathrm{d}x^{\alpha-r-1}} f(0_+) \tag{3.68}$$

$$= s^{\alpha} F(s) - \sum_{r=1}^{n} s^{r-1} \frac{\mathrm{d}^{\alpha-r}}{\mathrm{d}x^{\alpha-r}} f(0_+), \tag{3.69}$$

$$= s^{\alpha} F(s) - \sum_{r=1}^{n} s^{r-1} D^{\alpha-r} f(0_+), \left(D = \frac{\mathrm{d}}{\mathrm{d}x}\right), \quad n-1 < \alpha \le n, \tag{3.70}$$

where $\Re(s) > 0$.

3.6.3 Laplace Transform of Caputo Derivative

Notation 3.16. ${}_a^C D_x^{\alpha} f$: Caputo fractional derivative of $f(t)$.

Definition 3.11. The Caputo fractional derivative of a casual function $f(t)$ (that is $f(t) = 0$ for $t < 0$) with $\alpha > 0$ was defined by Caputo (1969) in connection with certain boundary value problems arising in the theory of viscoelasticity and the hereditary solid mechanics in the form

$$({}_a^C D_x^{\alpha} f)(x) = {}_aI_x^{n-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x) = {}_aD_x^{-(n-\alpha)} f^{(n)}(x) \tag{3.71}$$

$$= \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\mathrm{d}t, n-1 < \alpha < n \tag{3.72}$$

$$= \frac{\mathrm{d}^n f}{\mathrm{d}x^n}, \text{ if } \alpha = n, n \in N. \tag{3.73}$$

From the Eqs. (3.65), (3.67) and (3.71), it follows that

$$L\left\{{}_0^C D_x^{\alpha} f; s\right\} = s^{-(n-\alpha)} L\{f^{(n)}(t)\}. \tag{3.74}$$

On using (3.66) and (3.73), we see that

$$L\left\{{}_0^C D_x^{\alpha} f; s\right\} = s^{-(n-\alpha)}\left[s^n F(s) - \sum_{r=0}^{n-1} s^{n-r-1} f^{(r)}(0_+)\right]$$
$$= s^{\alpha} F(s) - \sum_{r=0}^{n-1} s^{\alpha-r-1} f^{(r)}(0_+), n-1 < \alpha \leq n, \tag{3.75}$$

where $\alpha, s \in C, \Re(s) > 0, \Re(\alpha) > 0$.

Note 3.5. From (3.71), it can be seen that

$${}_0^C D_x^{\alpha} A = 0, \tag{3.76}$$

where A is a constant, and whereas the Riemann–Liouville derivative

$${}_0 D_x^{\alpha} A = \frac{A t^{-\alpha}}{\Gamma(1-\alpha)}, \quad \alpha \neq 1, 2, \ldots, \tag{3.77}$$

which is a surprising result.

Remark 3.5. In a recent paper, Freed and Diethelm (2007) have extended the Fung's elastic law to one that is appropriate for the viscoelastic representation of soft biological tissues, and whose kinetics are of fractional order.

3.7 Mellin Transforms

Notation 3.17. $M_p(0, \infty)$, : a subspace of $L_p(0, \infty)$.

Definition of the subspace $M_p(0, \infty)$: $M_p(0, \infty)$ denotes the class of all functions $f(x)$ of $L_p(0, \infty)$, with $p > 2$, which are inverse Mellin transforms of functions of $L_q(-\infty, \infty)$; $q = \frac{p}{p-1}$.

Theorem 3.3. *The following result holds true.*

$$M\left({}_0 I_x^{\alpha} f\right)(s) = \frac{\Gamma(1-\alpha-s)}{\Gamma(1-s)} f^*(s+\alpha), \tag{3.78}$$

where $s, \alpha \in C, \Re(\alpha) > 0$ *and* $\Re(\alpha + s) < 1$.

Proof 3.4. We have

$$M\left({}_0 I_x^{\alpha} f\right)(s) = \int_0^{\infty} z^{s-1} \frac{1}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} f(t) \mathrm{d}t \mathrm{d}z \tag{3.79}$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^{\infty} f(t) \mathrm{d}t \int_t^{\infty} z^{s-1} (z-t)^{\alpha-1} \mathrm{d}z.$$

Setting $z = t/u$, the z-integral becomes

$$t^{\alpha+s-1}\int_0^1 u^{-\alpha-s}(1-u)^{\alpha-1}\mathrm{d}u = t^{\alpha+s-1}B(\alpha, 1-\alpha-s), \tag{3.80}$$

where $\Re(\alpha) > 0, \Re(\alpha+s) < 1$. The result (3.78) now follows from (3.80). □

Similarly, we can establish the following result:

Theorem 3.4. *The following result holds true:*

$$M\left({}_xI_\infty^\alpha f\right)(s) = \frac{\Gamma(s)}{\Gamma(s+\alpha)}M\left(t^\alpha f(t); s\right) \tag{3.81}$$

$$= \frac{\Gamma(s)}{\Gamma(s+\alpha)}f^*(s+\alpha), \tag{3.82}$$

where $s, \alpha \in C, \Re(\alpha) > 0, \Re(s) > 0.$

3.7.1 Mellin Transform of the nth Derivative

Theorem 3.5. *If* $n \in N$, *then*

$$M\left\{f^{(n)}(t); s\right\} = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)}M\left\{f(t); s-n\right\}, \tag{3.83}$$

where $s \in C, \Re(s-n) > 0$

Equation (3.83) can be proved by integrating by parts and using the definition of the Mellin transform.

3.7.2 Illustrative Examples

Example 3.9. Find the Mellin transform of the Riemann–Liouville fractional derivative ${}_0D_t^\alpha$.

Solution 3.7. We have

$${}_0D_t^\alpha f = ({}_0D_t^n\, {}_0D_t^{\alpha-n})f = ({}_0D_t^n\, {}_0I_t^{n-\alpha})f. \tag{3.84}$$

Therefore,

$$M\left({}_0D_t^{\alpha} f\right)(s) = \frac{(-1)^n \Gamma(s)}{\Gamma(s-n)} M\left({}_0I_t^{n-\alpha} f\right)(s-n),\ n-1 \le \Re(\alpha) < n \tag{3.85}$$

$$= \frac{(-1)^n \Gamma(s)\Gamma(1-(s-\alpha))}{\Gamma(s-n)\Gamma(1-s+n)} M\left\{f(t); s-\alpha\right\}, \tag{3.86}$$

where $\alpha, s \in C, \Re(s) > 0,\ \Re(s) < 1 + \Re(\alpha)$.

Example 3.10. In a similar manner, we can prove

$$M\left({}_0D_t^{\alpha} f\right)(s) = \frac{(-1)^n \Gamma(s)\sin[\pi(s-n)]}{\Gamma(s-\alpha)\sin[\pi(s-\alpha)]} M\left\{f(t); s-\alpha\right\}, \tag{3.87}$$

where $\alpha, s \in C, \Re(s) > 0, \Re(\alpha - s) > -1$.

Exercises 3.6

3.6.1. Find the Mellin transform of the Caputo derivative.

3.8 Kober Operators

Kober operators are the generalization of Riemann-Liouville and Weyl operators. These operators have been used by many authors in deriving the solution of single, dual, and triple integral equations possessing special functions of mathematical physics, as their kernels. These operators $(I_{(\alpha,\eta)} f)(x)$, are also called Erdélyi–Kober operators.

3.8.1 Erdélyi–Kober Operators

These operators are applicable in deriving the solution of certain integral equations involving special functions of mathematical physics which possess a Mellin–Barnes type integral representation. In this connection, refer to the works of Fox (1961, 1963, 1965, 1971), Saxena (1966, 1967, 1967a), Narain (1965, 1967), Nasim (1983), Habibullah (1977), and others. For further details see the survey paper entitled "Operators of fractional integration and their applications" by Srivastva and Saxena (2001).

Notation 3.18. $I[f(x)],\ I[\alpha,\eta;\ f(x)], E_{0,x}^{\alpha,\eta} f, I_x^{\eta,\alpha} f, (I_{\eta,\alpha}^{+} f)(x)$: Erdélyi–Kober fractional integral of the first kind.

Notation 3.19. R[f(x)], $R[\alpha,\zeta; f(x)]$, $K_{x,\infty}^{\alpha,\zeta} f$, $K_x^{\zeta,\alpha} f$, $(K_{\zeta,\alpha}^{-} f)(x)$, $(K(\alpha,\zeta f))(x)$: Erdélyi–Kober fractional integral of the second kind.

Definition 3.12.

$$I[f(x)] = I[\alpha,\eta; f(x)] = E_{0,x}^{\alpha,\eta} f = I_x^{\eta,\alpha} f = (I_{\eta,\alpha}^{+} f)(x) = (I(\alpha,\eta)f)(x)$$
$$= \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x t^{\eta}(x-t)^{\alpha-1} f(t)\mathrm{d}t, \alpha,\eta \in C; \Re(\alpha) > 0, \tag{3.88}$$

Definition 3.13.

$$R[f(x)] = R[\alpha,\zeta; f(x)] = K_{x,\infty}^{\alpha,\zeta} f = K_x^{\zeta,\alpha} f = (K_{\zeta,\alpha}^{-} f)(x) = (K(\alpha,\zeta)f)(x)$$
$$= \frac{x^{\zeta}}{\Gamma(\alpha)} \int_x^{\infty} t^{-\zeta-\alpha}(t-x)^{\alpha-1} f(t)\mathrm{d}t, \alpha,\zeta \in C; \Re(\alpha) > 0. \tag{3.89}$$

Equations (3.88) and (3.89) exist under the following set of conditions :

$$f \in L_p(0,\infty), \Re(\alpha) > 0, \Re(\eta) > -\frac{1}{q}, \Re(\zeta) > -\frac{1}{p}, \frac{1}{p} + \frac{1}{q} = 1, p \geq 1.$$

When $\eta = 0$, (3.88) reduces to Riemann-Liouville operator. That is

$$I_x^{0,\alpha} f = x^{-\alpha} {}_0I_x^{\alpha} f. \tag{3.90}$$

For $\zeta = 0$, (3.89) yields the Weyl operator of the function $t^{-\alpha} f(t)$. That is

$$K_x^{0,\alpha} f = {}_xW_{\infty}^{\alpha} t^{-\alpha} f(t). \tag{3.91}$$

Theorem 3.6. *(Kober 1940) If $\alpha,\eta,s \in C, \Re(\alpha) > 0, \Re(\eta - s) > -1, f \in L_p(0,\infty)$, $1 \leq p \leq 2$ (or $f \in M_p(0,\infty)$, a subspace of $L_p(0,\infty)$ and $p > 2$), $\Re(\eta) > -\frac{1}{q}; \frac{1}{p} + \frac{1}{q} = 1$, then there holds the formula*

$$M\{I(\alpha,\eta)f\}(s) = \frac{\Gamma(1+\eta-s)}{\Gamma(\alpha+\eta+1-s)} M\{f(x);s\}. \tag{3.92}$$

The proof of (3.92) can be developed on similar lines to that of Theorem 3.3. In a similar manner, we can establish

Theorem 3.7. *(Kober 1940) If $\alpha,s,\zeta \in C, \Re(\alpha) > 0, \Re(\zeta + s) > 0, f \in L_p(0,\infty), 1 \leq p \leq 2$ (or $f \in M_p(0,\infty)$, a subspace of $L_p(0,\infty)$ and $p > 2$), $\Re(\zeta) > -\frac{1}{p}; \frac{1}{p} + \frac{1}{q} = 1$, then there holds the formula*

$$M\{R(\alpha,\zeta)f\}(s) = \frac{\Gamma(\zeta+s)}{\Gamma(\alpha+\zeta+s)} M\{f(x);s\}. \tag{3.93}$$

Semigroup property of the Erdélyi–Kober operators has been given in the form of the following theorem, which can be proved in the same way:

Theorem 3.8. *If* $\alpha, \eta \in C, \Re(\alpha) > 0, \Re(\eta) > \max\{-\frac{1}{p}, -\frac{1}{q}\}$; $f \in L_p(0,\infty), g \in L_q(0,\infty), 1 \le p \le 2$ *(or* $f \in M_p(0,\infty)$, *a subspace of* $L_p(0,\infty)$ *and* $p > 2$*),* $\frac{1}{p} + \frac{1}{q} = 1$, *then there holds the formula*

$$\int_0^\infty g(x)\,(I(\alpha,\eta; f))\,(x)\mathrm{d}x = \int_0^\infty f(x)\,(R(\alpha,\eta; g))\,(x)\mathrm{d}x. \tag{3.94}$$

Remark 3.6. Operators more general than the operators defined by (3.88) and (3.89) are defined by Galué et al. (2000) in the form

$$(I_{0+}^{\alpha,0,\eta} f)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_a^x t^\eta (x-t)^{\alpha-1} f(t)\mathrm{d}t, \alpha, \eta \in C; \Re(\alpha) > 0. \tag{3.95}$$

Exercises 3.7

3.7.1. For the Erdélyi–Kober operators defined by

$$I_{\eta,\alpha}^+ f(x) = \frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int_0^x (x^2 - t^2)^{\alpha-1} t^{2\eta+1} f(t)\mathrm{d}t,$$

where $\Re(\alpha) > 0$, establish the following results (Sneddon 1975):

(i) $I_{\eta,\alpha}^+ x^{2\beta} f(x) = x^{2\beta} I_{\eta+\beta,\alpha}^+ f(x)$.
(ii) $I_{\eta,\alpha}^+ I_{\eta+\alpha,\beta}^+ = I_{\eta,\alpha+\beta}^+ = I_{\eta+\alpha.\beta}^+ I_{\eta,\alpha}^+$.
(iii) $(I_{\eta,\alpha}^+)^{-1} = I_{\eta+\alpha,-\alpha}^+$

Remark 3.7. The results of Exercise 3.7.1 also hold for the operator, defined by

$$K_{\eta,\alpha}^- f(x) = \frac{2x^{2\eta}}{\Gamma(\alpha)} \int_x^\infty (t^2 - x^2)^{\alpha-1} t^{-2\alpha-2\eta+1} f(t)dt,$$

where $\Re(\alpha) > 0$.

3.7.2. Prove that the Erdélyi–Kober fractional integral $I_{\eta,\alpha}^+$ of the H-function exists and the following result holds:

$$\left(I_{\eta,\alpha}^+ t^{\rho-1} H_{p,q}^{m,n}\left[t^\sigma \middle| \begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right]\right)(x)$$
$$= x^{\rho-1} H_{p+1,q+1}^{m,n+1}\left[x^\sigma \middle| \begin{matrix}(1-\rho-\eta,\sigma), (a_p, A_p)\\(b_q, B_q), (1-\rho-\alpha-\eta,\sigma)\end{matrix}\right],$$

provided $\alpha, \eta \in C, \Re(\alpha) > 0$, and further the constants $a_i, b_j \in C, A_i, B_j > 0$, $i = 1, \dots, p; j = 1, \dots, q, \rho \in C, \sigma > 0$ satisfy

$$\sigma \min_{1 \le j \le m} \left[\frac{\Re(b_j)}{B_j} \right] + \Re(\rho) + \min[0, \Re(\eta)] > 0.$$

and $\gamma\sigma < -\Re(\rho) - \min[0, \Re(\eta)]$. (Here the contour of integration is $L = L_{i\gamma\infty}$.

3.7.3. Prove that the Erdélyi–Kober fractional integral $K^{-}_{\eta,\alpha}$ of the H-function exists and the following relation holds:

$$\left(K^{-}_{\eta,\alpha} t^{\rho-1} H^{m,n}_{p,q} \left[t^{\sigma} \middle| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right] \right)(x)$$
$$= x^{\rho-1} H^{m+1,n}_{p+1,q+1} \left[x^{\sigma} \middle| \begin{matrix} (a_p, A_p), (1-\rho+\alpha+\eta, \sigma) \\ (1-\rho+\eta, \sigma), (b_q, B_q) \end{matrix} \right],$$

provided $\alpha, \eta \in C, \Re(\alpha) > 0$, and further the constants $a_i, b_j \in C, A_i, B_j > 0$, $i = 1, \dots, p; j = 1, \dots, q, \rho \in C, \sigma > 0$ satisfy

$$\sigma \max_{1 \le j \le n} \left[\frac{\Re(a_j) - 1}{A_j} \right] + \Re(\rho) < 1 + \Re(\eta)$$

and $1 - \gamma\sigma > \Re(\rho) - \Re(\eta)$.

3.9 Generalized Kober Operators

Notation 3.20. $I[\alpha, \beta, \gamma : m, k, \eta, a : f(x)], I[f(x)]$

Notation 3.21. $J[\alpha, \beta, \gamma : m, k, \delta, a : f(x)], K[f(x)]$

Notation 3.22. $R[f(x)], R \left[\begin{matrix} \alpha, \beta, \gamma : \\ \sigma, \rho, a : \end{matrix} f(x) \right]$

Notation 3.23. $K[f(x)], K \left[\begin{matrix} \alpha, \beta, \gamma : \\ \zeta, \rho, a : \end{matrix} f(x) \right]$

Let $\alpha, \beta, \gamma, k, \eta, \zeta, \sigma, \rho \in C, x \in R_+$.

Definition 3.14.

$$I[f(x)] = I[\alpha, \beta, \gamma : m, k, \eta, a : f(x)];$$
$$= \frac{k x^{-\eta-1}}{\Gamma(1-\alpha)} \int_0^x {}_2F_1\left(\alpha, \beta+m; \gamma; \frac{a t^k}{x^k}\right) t^{\eta} f(t) \mathrm{d}t, \tag{3.96}$$

where ${}_2F_1(\cdot)$ is the Gauss hypergeometric function.

Definition 3.15.

$$K[f(x)] = K[\alpha, \beta, \gamma : m, k, \zeta, a : f(x)]$$
$$= \frac{k x^{\zeta}}{\Gamma(1-\alpha)} \int_x^{\infty} {}_2F_1\left(\alpha, \beta + m; \gamma; \frac{a x^k}{t^k}\right) t^{-\zeta-1} f(t) \mathrm{d}t. \quad (3.97)$$

Operators defined by (3.96) and (3.97) exist under the following conditions:

(i) $p \geq 1, q < \infty, \frac{1}{p} + \frac{1}{q} = 1, |\arg(1-a)| < \pi, k > 0.$
(ii) $\Re(1-\alpha) > -m, \Re(\eta) > -1/q, \Re(\zeta) > -1/p, \Re(\gamma - \alpha - \beta - m) > -1, m \in N_0; \gamma \neq 0, -1, -2, \ldots$
(iii) $f \in L_p(0, \infty).$

The equations (3.96) and (3.97) are introduced by Kalla and Saxena (1969).

Remark 3.8. It is interesting to note that for $\gamma = \beta, a = k = 1$, the equations (3.96) and (3.97) reduce to the generalized Kober operators introduced and studied by Saxena (1967b).

Definition 3.16.

$$R[f(x)] = R\left[\begin{matrix}\alpha, \beta, \gamma : \\ \sigma, \rho, a :\end{matrix} f(x)\right]$$
$$= \frac{x^{-\sigma-\rho}}{\Gamma(\rho)} \int_0^x t^{\sigma}(x-t)^{\rho-1} {}_2F_1\left[\alpha, \beta; \gamma; a\left(1 - \frac{t}{x}\right)\right] f(t) \mathrm{d}t. \quad (3.98)$$

Definition 3.17.

$$K[f(x)] = K\left[\begin{matrix}\alpha, \beta, \gamma : \\ \zeta, \rho, a :\end{matrix} f(x)\right]$$
$$= \frac{x^{\zeta}}{\Gamma(\rho)} \int_x^{\infty} t^{-\zeta-\rho}(t-x)^{\rho-1} {}_2F_1\left[\alpha, \beta; \gamma; a\left(1 - \frac{x}{t}\right)\right] f(t) \mathrm{d}t. \quad (3.99)$$

The conditions of the validity of the operators (3.98) and (3.99) are given below:

(i) $p \geq 1, q < \infty, \frac{1}{p} + \frac{1}{q} = 1, |\arg(1-a)| < \pi$
(ii) $\Re(\sigma) > -1/q, \Re(\zeta) > -1/p, \Re(\gamma - \alpha - \beta) > 0, \Re(\rho) > 0; \gamma \neq 0, -1, -2, \ldots$
(iii) $f \in L_p(0, \infty).$

Remark 3.9. The operators defined by (3.98) and (3.99) are given by Saxena and Kumbhat (1973). For multidimensional generalized Kober operators associated with Gauss hypergeometric function, which provides an elegant multivariate analogue of the operators (3.98) and (3.99), see Saxena et al. (1990).

When a is replaced by a/α and $\alpha \to \infty$, the operators defined by (3.98) and (3.99) reduce to the following operators associated with confluent hypergeometric functions:

Definition 3.18.

$$R[f(x)] = R\left[\begin{matrix}\beta,\gamma: \\ \sigma,\rho,a:\end{matrix} f(x)\right] = \lim_{\alpha\to\infty} R\left[\begin{matrix}\alpha,\beta,\gamma: \\ \sigma,\rho,a/\alpha:\end{matrix} f(x)\right]$$

$$= \frac{x^{-\sigma-\rho}}{\Gamma(\rho)} \int_0^x t^{\sigma}(x-t)^{\rho-1}\Phi\left[\beta;\gamma;a\left(1-\frac{t}{x}\right)\right] f(t)\mathrm{d}t. \tag{3.100}$$

Definition 3.19.

$$K[f(x)] = K\left[\begin{matrix}\beta,\gamma: \\ \zeta,\rho,a:\end{matrix} f(x)\right] = \lim_{\alpha\to\infty} K\left[\begin{matrix}\alpha,\beta,\gamma: \\ \zeta,\rho,a/\alpha:\end{matrix} f(x)\right]$$

$$= \frac{x^{\zeta}}{\Gamma(\rho)} \int_x^{\infty} t^{-\zeta-\rho}(t-x)^{\rho-1}\Phi\left[\beta;\gamma;a\left(1-\frac{x}{t}\right)\right] f(t)\mathrm{d}t, \tag{3.101}$$

where $\Re(\rho) > 0, \Re(\zeta) > 0$; and $\Phi(\beta,\gamma;z)$ is the confluent hypergeometric function (Erdélyi et al. 1953, p. 248).

Many interesting and useful properties of the operators defined by (3.98) and (3.99) are investigated by Saxena and Kumbhat (1975), which deal with relations of these operators with well-known integral transforms, such as Laplace, Mellin, and Hankel transforms. Equation (3.98) was first considered by Love (1967).

Remark 3.10. In the special case, $\sigma = 0$, when α is replaced by $\alpha+\beta, \gamma$ by α and β by $-\eta$, then (3.98) reduces to the operator (3.102) considered by Saigo (1978). Similarly, (3.99) reduces to another operator (3.104) introduced by Saigo (1978).

3.10 Saigo Operators

An interesting extension of both the Riemann–Liouville and Erdélyi–Kober fractional integration operators was introduced by Saigo (1978) in terms of Gauss's hypergeometric function. In a series of papers, Saigo (1978, 1979, 1980,1981), Saigo et al. (1992, 1992a), Saigo and Raina (1991), Srivastava and Saigo (1987), Saigo and Saxena (1998), and others obtained several interesting properties of these operators and then applied in many problems. In this section, we present definitions and certain important properties of Saigo operators. Following Saigo (1978), we define the following generalized fractional calculus operators associated with Gauss hypergeometric function in the kernel.

Notation 3.24. $I_{0+}^{\alpha,\beta,\gamma}$: Left-sided generalized fractional integral operator.

Notation 3.25. $I_{-}^{\alpha,\beta,\gamma}$: Right-sided generalized fractional integral operator.

Notation 3.26. $D_{0+}^{\alpha,\beta,\gamma}$: Left-sided generalized fractional derivative operator.

Notation 3.27. $D_{-}^{\alpha,\beta,\gamma}$: Right-sided generalized fractional derivative operator.

Let $\alpha, \beta, \eta \in C$, and let $x \in \Re_+$ the generalized fractional integral and generalized fractional derivative of a function $f(x)$ on $\Re_+$ are defined in the following forms:

Definition 3.20.

$$(I_{0+}^{\alpha,\beta,\eta} f)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)\mathrm{d}t, \Re(\alpha) > 0 \tag{3.102}$$

$$= \frac{\mathrm{d}^n}{\mathrm{d}x^n}(I_{0+}^{\alpha+n,\beta-n,\eta-n} f)(x), \Re(\alpha) \le 0; n = [\Re(-\alpha)] + 1. \tag{3.103}$$

Definition 3.21.

$$(I_{-}^{\alpha,\beta,\eta} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}\right) f(t)\mathrm{d}t, \Re(\alpha) > 0 \tag{3.104}$$

$$= (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n}(I_{-}^{\alpha+n,\beta-n,\eta} f)(x), \Re(\alpha) \le 0; n = [\Re(-\alpha)] + 1. \tag{3.105}$$

Definition 3.22.

$$(D_{0+}^{\alpha,\beta,\gamma} f)(x) = (I_{0+}^{-\alpha,-\beta,\alpha+\eta} f)(x)$$
$$= \left(\frac{\mathrm{d}}{\mathrm{dx}}\right)^n (I_{0+}^{-\alpha+n,-\beta-n,\alpha+\eta-n} f)(x), \Re(\alpha) > 0; n = [\Re(\alpha)] + 1. \tag{3.106}$$

Definition 3.23.

$$(D_{-}^{\alpha,\beta,\gamma} f)(x) = (I_{-}^{-\alpha,-\beta,\alpha+\eta} f)(x)$$
$$= \left(-\frac{\mathrm{d}}{\mathrm{dx}}\right)^n (I_{-}^{-\alpha+n,-\beta-n,\alpha+\eta} f)(x), \Re(\alpha) > 0; n = [\Re(\alpha)] + 1. \tag{3.107}$$

For $\beta = -\alpha$, the operators defined by (3.102), (3.104), (3.106) and (3.107) reduce to the classical Riemann–Liouville fractional calculus operators for $\Re(\alpha) > 0$, namely the Riemann–Liouville operator I_{0+}^{α}, defined in Sect. 3.2 by the equation (3.15), the Weyl operator I_{-}^{α}, defined in Sect. 3.4 by the equation (3.58) and the fractional derivative operators D_{0+}^{α} and D_{-}^{α}, defined below by the Eqs. (3.109) and (3.111) respectively.

Notation 3.28. D^{α}_{0+} : Riemann–Liouville left-sided fractional derivative of order α.

Notation 3.29. D^{α}_{-} : Riemann–Liouville right-sided fractional derivative of order α.

Definition 3.24.

$$(D^{\alpha,-\alpha,\eta}_{0+}f)(x) = \left(D^{\alpha}_{0+}f\right)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\Re(\alpha)]+1}\left(I^{1-\alpha+[\Re(\alpha)]}_{0+}f\right)(x), x > 0 \tag{3.108}$$

$$= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\Re(\alpha)]+1}\frac{1}{\Gamma(1-\alpha+[\Re(\alpha)])}\int_0^x \frac{f(t)}{(x-t)^{\alpha-[\Re(\alpha)]}}\mathrm{d}t, x > 0. \tag{3.109}$$

Definition 3.25.

$$(D^{\alpha,-\alpha,\eta}_{-}f)(x) = \left(D^{\alpha}_{-}f\right)(x) = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\Re(\alpha)]+1}\left(I^{1-\alpha+[\Re(\alpha)]}_{-}f\right)(x), x > 0 \tag{3.110}$$

$$= \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\Re(\alpha)]+1}\frac{1}{\Gamma(1-\alpha+[\Re(\alpha)])}\int_x^\infty \frac{f(t)}{(t-x)^{\alpha-[\Re(\alpha)]}}\mathrm{d}t, x>0, \tag{3.111}$$

where the symbol $[\varsigma]$ means the integral part of a real positive number ς that is the largest integer not exceeding ς. In particular for real $\alpha > 0$, D^{α}_{0+} and D^{α}_{-} take the interesting forms

$$\begin{aligned}(D^{\alpha,-\alpha,\eta}_{0+}f)(x) &= \left(D^{\alpha}_{0+}f\right)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1}\left(I^{1-\{\alpha\}}_{0+}f\right)(x),\\ &= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1}\frac{1}{\Gamma(1-\{\alpha\})}\int_0^x \frac{f(t)}{(x-t)^{\{\alpha\}}}\mathrm{d}t, \quad x > 0,\end{aligned} \tag{3.112}$$

and

$$\begin{aligned}(D^{\alpha,-\alpha,\eta}_{-}f)(x) &= (D^{\alpha}_{-}f)(x) = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1}\left(I^{1-\{\alpha\}}_{-}f\right)(x)\\ &= \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1}\frac{1}{\Gamma(1-\{\alpha\})}\int_x^\infty \frac{f(t)}{(t-x)^{\{\alpha\}}}\mathrm{d}t, \quad x > 0,\end{aligned} \tag{3.113}$$

where $\{\varsigma\}$ denotes the fractional part of ς, that is $\{\varsigma\} = \varsigma - [\varsigma]$.

If we set $\beta = 0$, then the operators defined by (3.102) and (3.104) yield the Erdélyi–Kober operators, defined by (3.88) and (3.89) respectively.

3.10.1 Relations Among the Operators

We note that the relation connecting the operators (3.102) and (3.104) is given by

$$\left(I_{-}^{\alpha,\beta,\eta} f\left[\frac{1}{t}\right]\right)(x) = x^{-\beta-1}\left(I_{0+}^{\alpha,\beta,\eta}\left[t^{\beta-1} f(t)\right]\right)\left(\frac{1}{x}\right). \tag{3.114}$$

To prove the result (3.114), we observe that if we start from its left hand side then by a simple change of variable, we obtain the desired result.

When $\beta = -\alpha$, in (3.114), it gives the relation between the operators (3.111) and (3.58) given by Kilbas (2005):

$$\left(I_{-}^{\alpha,-\alpha,\eta} f\left[\frac{1}{t}\right]\right)(x) = \left(I_{-}^{\alpha} f\left[\frac{1}{t}\right]\right)(x) = \left(W_{x,\infty}^{\alpha} f\left[\frac{1}{t}\right]\right)(x)$$
$$= x^{\alpha-1}\left(I_{0+}^{\alpha,-\alpha,\eta}\left[t^{-\alpha-1} f(t)\right]\right)\left(\frac{1}{x}\right) = x^{\alpha-1}\left(I_{0+}^{\alpha}\left[t^{-\alpha-1} f(t)\right]\right)\left(\frac{1}{x}\right). \tag{3.115}$$

On the other hand, for $\beta = 0$, we obtain the relation between the operators (3.88) and (3.89) as

$$\left(I_{-}^{\alpha,0,\eta} f\left[\frac{1}{t}\right]\right)(x) = \left(K_{\eta,\alpha}^{-} f\left[\frac{1}{t}\right]\right)(x) = x^{-1}\left(I_{0+}^{\alpha,0,\eta}\left[t^{-1} f(t)\right]\right)\left(\frac{1}{x}\right)$$
$$= x^{-1}\left(I_{\eta,\alpha}^{+}\left[t^{-1} f(t)\right]\right)\left(\frac{1}{x}\right). \tag{3.116}$$

Note 3.6. We observe that the operators (3.106) and (3.107) are inverse to the operators (3.102) and (3.104):

$$D_{0+}^{\alpha,\beta,\eta} = (I_{0+}^{\alpha,\beta,\eta})^{-1} \text{ and } D_{-}^{\alpha,\beta,\eta} = (I_{-}^{\alpha,\beta,\eta})^{-1}. \tag{3.117}$$

3.10.2 Power Function Formulae

By making use of the following integral

$$\int_0^t x^{\rho-1}(t-x)^{c-1}{}_2F_1\left(a,b;c;1-\frac{x}{t}\right)\mathrm{d}x = \frac{\Gamma(c)\Gamma(\rho)\Gamma(\rho+c-a-b)}{\Gamma(\rho+c-a)\Gamma(\rho+c-b)}t^{\rho+c-1}, \tag{3.118}$$

where $\rho, a, b, c \in C, \Re(\rho) > 0, \Re(c) > 0, \Re(\rho + c - a - b) > 0$ and

$$\int_t^{\infty} x^{\rho-1}(x-t)^{c-1}{}_2F_1(a, b; c; 1-\frac{t}{x})\mathrm{d}x = \frac{\Gamma(c)\Gamma(1-\rho-c)\Gamma(1-\rho-a-b)}{\Gamma(1-\rho-a)\Gamma(1-\rho-b)}t^{\rho+c-1}, \tag{3.119}$$

where $\rho, a, b, c \in C; \Re(c) > 0, \Re(\rho + c) < 1, \Re(\rho + a + b) < 1$, we obtain the following power function formulae for the operators $\left(I_{0+}^{\alpha,\beta,\eta}\right)$ and $\left(I_{-}^{\alpha,\beta,\eta}\right)$:

$$\left(I_{0+}^{\alpha,\beta,\eta}t^{\lambda}\right)(x) = \frac{\Gamma(1+\lambda)\Gamma(1+\lambda+\eta-\beta)}{\Gamma(1+\lambda-\beta)\Gamma(1+\lambda+\alpha+\eta)}x^{\lambda-\beta}, \tag{3.120}$$

where α, β, η and $\lambda \in C, \Re(\alpha) > 0$ and $\Re(\lambda) > \max\{0, \Re(\beta - \eta)\} - 1$;

$$\left(I_{-}^{\alpha,\beta,\eta}t^{\lambda}\right)(x) = \frac{\Gamma(\beta-\lambda)\Gamma(\eta-\lambda)}{\Gamma(-\lambda)\Gamma(\alpha+\beta+\eta-\lambda)}x^{\lambda-\beta}, \tag{3.121}$$

where α, β, η, and $\lambda \in C$, $\Re(\alpha) > 0$, $\Re(\lambda) < \min\{\Re(\beta), \Re(\eta)\}$, or if $\Re(\alpha) \leq 0$; $0 < \Re(\alpha) + n \leq 1$ and $\Re(\lambda) < \min\{\Re(\beta) - n, \Re(\eta)\}$, where n is a positive integer.

For $\beta = -\alpha$, (3.120) and (3.121) give rise to the formulae

$$\left(I_{0+}^{\alpha}t^{\lambda}\right)(x) = \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda+\alpha)}x^{\lambda+\alpha}, \tag{3.122}$$

where $\alpha, \lambda \in C, \Re(\alpha) > 0, \Re(\lambda) > -1$; and

$$\left(I_{-}^{\alpha}t^{-\lambda}\right)(x) = \frac{\Gamma(\lambda-\alpha)}{\Gamma(\lambda)}x^{\alpha-\lambda}, \tag{3.123}$$

where $\alpha, \lambda \in C, \Re(\lambda) > \Re(\alpha) > 0$.

Similarly, for $\beta = 0$, we obtain

$$\left(I_{\eta,\alpha}^{+}t^{\lambda}\right)(x) = \frac{\Gamma(1+\lambda+\eta)}{\Gamma(1+\alpha+\lambda+\eta)}x^{\lambda}, \tag{3.124}$$

where $\alpha, \lambda, \eta \in C, \Re(\lambda + \eta) > -1$ and

$$\left(K_{\eta,\alpha}^{-}t^{\lambda}\right)(x) = \frac{\Gamma(\eta-\lambda)}{\Gamma(\alpha+\eta-\lambda)}x^{\lambda}, \tag{3.125}$$

where $\alpha, \lambda, \eta \in C, \Re(\alpha) > 0, \Re(\eta) > \Re(\lambda)$.

The discussion in the next two sections is based on the work of Saigo et al. (1992).

3.10.3 Mellin Transform of Saigo Operators

Theorem 3.9. *If* $\alpha, \beta, \eta \in C, \Re(\alpha) > 0$, *and* $\Re(s) < 1 + \min[0, \Re(\eta - \beta)]$, *then the following formula holds for* $f(x) \in L_p(0,\infty)$ *with* $1 \leq p \leq 2$ *or* $f(x) \in M_p(0,\infty)$ *with* $p > 2$*:*

$$M\left\{x^{\beta} I_{0+}^{\alpha,\beta,\eta} f; s\right\} = \frac{\Gamma(1-s)\Gamma(\eta-\beta+1-s)}{\Gamma(1-s-\beta)\,\Gamma(\alpha+\eta+1-s)} M\{f(x); s\}, \tag{3.126}$$

where $M_p(0,\infty)$*is defined in Sect. 3.6.*

Theorem 3.10. *If* $\alpha, \beta, \eta \in C, \Re(\alpha) > 0$, *and* $\Re(s) > -\min[\Re(\beta), \Re(\eta)]$, *then the following formula holds for* $f(x) \in L_p(0,\infty)$ *with* $1 \leq p \leq 2$ *or* $f(x) \in M_p(0,\infty)$ *with* $p > 2$*:*

$$M\left\{x^{\beta} I_{-}^{\alpha,\beta,\eta} f; s\right\} = \frac{\Gamma(\beta+s)\Gamma(\eta+s)}{\Gamma(s)\,\Gamma(\alpha+\beta+\eta+s)} M\{f(x); s\}. \tag{3.127}$$

3.10.4 Representation of Saigo Operators

A representation of Erdélyi–Kober operators (3.88) and (3.89) in terms of the Laplace transform operator L and its inverse L^{-1} was given by Fox (1971, 1972). Certain relations connecting L and L^{-1} operators, and fractional integration operators of Saxena (1967) were derived by Kumbhat and Saxena (1975) generalizing the results of Fox (1971, 1972). In this section we present certain representations of the Saigo operators by L and L^{-1}.

Theorem 3.11. *Let* $\alpha, \beta, \eta \in C, \Re(\alpha) > 0, \Re(\eta - \beta) > 0$ *and* $\Re(\eta) < 0$. *If a function* $f(x)$ *satisfies the following conditions:*

(i) $f(x) \in L(0,\infty)$
(ii) $y^{-\frac{1}{2}} f(y) \in L(0,\infty)$, *where* $f(y)$ *is of bounded variation near the point* $y = x$
(iii) $M\{f(x); s\} = F(s) \in L\left(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty\right)$
(iv) $y^{\beta-\frac{1}{2}} I_{0+}^{\alpha,\beta,\eta} f \in L(0,\infty)$ *and* $y^{\beta} I_{0+}^{\alpha,\beta,\eta} f$ *is of bounded variation near the point* $y = x$, *then there holds the relation*

$$I_{0+}^{\alpha,\beta,\eta} f = x^{-\alpha-\beta-\eta} L^{-1}\left[t^{-\alpha-\eta} L\left\{x^{\beta} L^{-1}\left[t^{\eta} L\left\{x^{\eta-\beta} f(x)\right\}\right]\right\}\right]. \tag{3.128}$$

Remark 3.11. For $\beta = 0$, (3.128) reduces to a result given by Fox (1972, p. 198).

Theorem 3.12. *Let* $\alpha, \beta, \eta \in C, \Re(\alpha) > 0, \Re(\beta) > 0$ *and* $\Re(\eta) > 0$. *If a function* $f(x)$ *satisfies the following conditions:*

(i) $f(x) \in L(0, \infty)$
(ii) $y^{-\frac{1}{2}} f(y) \in L(0, \infty)$*, where* $f(y)$ *is of bounded variation near the point* $y = x$
(iii) $M\{f(x);\} = F(s) \in L\left(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty\right)$
(iv) $y^{\beta-\frac{1}{2}} I_{-}^{\alpha,\beta,\eta} f \in L(0, \infty)$ *and* $y^{\beta} I_{-}^{\alpha,\beta,\eta} f$ *is of bounded variation near the point* $y = x$*, then there holds the relation*

$$I_{-}^{\alpha,\beta,\eta} f = x^{-\alpha-2\beta-\eta+1} L^{-1}\left[t^{-\alpha-\eta} L\left\{x^{\beta} L^{-1}\left[t^{\eta} L\left\{x^{\eta-1} f\left(\frac{1}{x}\right)\right\}\right]\right\}\right]_{x=\frac{1}{X}}. \tag{3.129}$$

Remark 3.12. For $\beta = 0$, (3.129) reduces to a result given by Fox (1972, p. 199).

Exercises 3.9

3.9.1. Let $\alpha^* > 0$ or $\alpha^* = 0$ and $\gamma\mu + \Re(\delta) < -1$. Further, let $\alpha, \beta, \eta \in C$, $\Re(\alpha) > 0, \Re(\beta) \neq \Re(\eta); \rho \in C$ and $\kappa > 0$ satisfy the conditions

$$\Re(\rho) + \kappa \min_{1 \le j \le m}\left[\frac{\Re(b_j)}{B_j}\right] + \min[0, \Re(\eta - \beta)] > 0.$$

for $\alpha^* > 0$ or $\alpha^* = 0, \mu \geq 0$, and

$$\Re(\rho) + \kappa \min_{1 \le j \le m}\left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta) + \frac{1}{2}}{\mu}\right] + \min[0, \Re(\eta - \beta)] > 0,$$

for $\alpha^* = 0$ and $\mu < 0$. Then show that the generalized fractional integration $I_{0+}^{\alpha,\beta,\eta}$ of the H-function exists and there holds the formula

$$\left(I_{0+}^{\alpha,\beta,\eta} t^{\rho-1} H_{p,q}^{m,n}\left[t^{\kappa}\middle|\begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right]\right)(x)$$
$$= x^{\rho-\beta-1} H_{p+2,q+2}^{m,n+2}\left[x^{\kappa}\middle|\begin{matrix}(1-\rho,\kappa), (1+\beta-\eta-\rho,\kappa), (a_p, A_p)\\(b_q, B_q), (1+\beta-\rho,\kappa), (1-\rho-\alpha-\eta,\kappa)\end{matrix}\right], \tag{3.130}$$

where μ, δ, α^* and γ are defined by (1.17), (1.18), (3.26), and (3.28) respectively.

3.9.2. Let either $\alpha^* > 0$ or $\alpha^* = 0$ and $\gamma\mu + \Re(\delta) < -1$. Further let $\alpha, \beta, \eta \in C, \Re(\alpha) > 0, \Re(\beta) \neq \Re(\eta); \rho \in C$ and $\kappa > 0$ satisfy the conditions

$$\Re(\rho) + \kappa \max_{1 \le i \le n}\left[\frac{\Re(a_i) - 1}{A_i}\right] < 1 + \min[\Re(\beta), \Re(\eta)],$$

for $\alpha^* > 0$ or $\alpha^* = 0, \mu \leq 0$, and

$$\Re(\rho) + \kappa \max_{1 \leq i \leq n} \left[\frac{\Re(a_i) - 1}{A_i}, \frac{\Re(\delta) + \frac{1}{2}}{\mu} \right] < 1 + \min[\Re(\beta), \Re(\eta)],$$

for $\alpha^* = 0$ and $\mu > 0$. Then show that the generalized fractional integration $I_{-}^{\alpha,\beta,\eta}$ of the H-function exists and there holds the formula

$$\left(I_{-}^{\alpha,\beta,\eta} t^{\rho-1} H_{p,q}^{m,n} \left[t^{\kappa} \middle| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right] \right)(x)$$
$$= x^{\rho-\beta-1} H_{p+2,q+2}^{m+2,n} \left[x^{\kappa} \middle| \begin{matrix} (a_p, A_p), (1-\rho, \kappa), (1+\alpha+\beta+\eta-\rho, \kappa) \\ (1-\rho+\beta, \kappa), (1-\rho+\eta, \kappa), (b_q, B_q) \end{matrix} \right], \tag{3.131}$$

where α^* is defined in (3.26).

Note 3.7. In Exercise 3.9.1, left-sided generalized fractional integral $I_{0+}^{\alpha,\beta,\eta}$ of the H-function is considered, whereas Exercise 3.9.2 gives the right-sided generalized fractional integral $I_{-}^{\alpha,\beta,\eta}$ of the H-function.

3.9.3. Let $\alpha, \beta, \eta \in C, \Re(\alpha) > 0, \Re(\alpha+\beta+\eta) \neq 0, \rho \in C, \kappa > 0$. Let $\alpha^* > 0$ or $\alpha^* = 0$, and $\gamma\mu + \Re(\delta) < -1$ satisfy the conditions

$$\Re(\rho) + \kappa \min_{1 \leq j \leq m} \left[\frac{\Re(b_j)}{B_j} \right] + \min[0, \Re(\alpha+\beta+\eta)] > 0.$$

for $\alpha^* > 0$, or $\alpha^* = 0, \mu \geq 0$, and

$$\Re(\rho) + \kappa \min_{1 \leq j \leq m} \left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta) + \frac{1}{2}}{\mu} \right] + \min[0, \Re(\alpha+\beta+\eta)] > 0,$$

for $\alpha^* > 0$ and $\mu < 0$. Then show that the generalized fractional differentiation $D_{0+}^{\alpha,\beta,\eta}$ of the H-function exists and there holds the formula

$$\left(D_{0+}^{\alpha,\beta,\eta} t^{\rho-1} H_{p,q}^{m,n} \left[t^{\kappa} \middle| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right] \right)(x)$$
$$= x^{\rho+\beta-1} H_{p+2,q+2}^{m,n+2} \left[x^{\kappa} \middle| \begin{matrix} (1-\rho, \kappa), (1-\rho-\eta-\alpha-\beta, \kappa), (a_p, A_p) \\ (b_q, B_q), (1-\rho-\beta, \kappa), (1-\rho-\eta, \kappa) \end{matrix} \right], \tag{3.132}$$

where α^* is defined in (3.26). Hence or otherwise show that the Riemann-Liouville fractional derivative D_{0+}^{α} of the H-function exists and the following result holds:

$$\left(D_{0+}^{\alpha}t^{\rho-1}H_{p,q}^{m,n}\left[t^{\sigma}\left|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right.\right]\right)(x)$$

$$= x^{\rho-\alpha-1}H_{p+1,q+1}^{m,n+1}\left[x^{\sigma}\left|\begin{matrix}(1-\rho,\sigma),(a_p,A_p)\\(b_q,B_q),(1-\rho+\alpha,\sigma)\end{matrix}\right.\right], \tag{3.133}$$

provided $\alpha \in C, \Re(\alpha) > 0$, and further $\alpha.\beta, \eta \in C, \Re(\alpha) > 0, \Re(\alpha+\beta+\eta) \neq 0$, $\rho \in C, \sigma > 0$; either $\alpha^* > 0$ or $\alpha^* = 0$, and $\gamma\mu + \Re(\delta) < -1$ satisfy the conditions

$$\Re(\rho) + \sigma \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}\right] > 0$$

for $\alpha^* > 0$ and $\alpha^* = 0, \mu \geq 0$, and

$$\Re(\rho) + \sigma \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta)+\frac{1}{2}}{\mu}\right] > 0.$$

for $\alpha^* = 0, \mu < 0, \alpha^*$ is defined in (3.26). (Kilbas and Saigo 1998)

3.9.4. Let either $\alpha^* > 0$ or $\alpha^* = 0$ and $\gamma\mu + \Re(\delta) < -1$. Further let $\alpha, \beta, \eta \in C$, $\Re(\alpha) > 0, \rho \in C$; $\Re(\alpha+\beta+\eta) + [\Re(\alpha)] + 1 \neq 0$, and $\kappa > 0$ satisfy the conditions

$$\Re(\rho) + \max[\Re(\beta), [\Re(\alpha)] + 1, -\Re(\alpha+\eta)] + \kappa \max_{1\le i\le n}\left[\frac{\Re(a_i)-1}{A_i}\right] < 1,$$

for $\alpha^* > 0$ or $\alpha^* = 0, \mu \leq 0$, and

$$\Re(\rho) + \max[\Re(\beta), [\Re(\alpha)] + 1, -\Re(\alpha+\eta)] + \kappa \max_{1\le i\le n}\left[\frac{\Re(a_i)-1}{A_i}, \frac{\Re(\delta)+\frac{1}{2}}{\mu}\right] < 1,$$

for $\alpha^* = 0$ and $\mu > 0$. Then show that the generalized fractional differentiation $D_{-}^{\alpha,\beta,\eta}$ of the H-function exists and there holds the formula

$$\left(D_{-}^{\alpha,\beta,\eta}t^{\rho-1}H_{p,q}^{m,n}\left[t^{\kappa}\left|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right.\right]\right)(x) = (-1)^{[\Re(\alpha)]+1}x^{\rho+\beta-1}H_{p+2,q+2}^{m+2,n}$$

$$\times\left[x^{\kappa}\left|\begin{matrix}(a_p,A_p),(1-\rho,\kappa),(1-\rho-\beta+\eta,\kappa)\\(1-\rho-\beta,\kappa),(1-\rho+\alpha+\eta,\kappa),(b_q,B_q)\end{matrix}\right.\right]. \tag{3.134}$$

Hence or otherwise show that the Riemann–Liouville fractional derivative D_{-}^{α} of the H-function exists and the following relation holds:

$$\left(D_{-}^{\alpha} t^{\rho-1} H_{p,q}^{m,n}\left[t^{\sigma}\,\middle|\, \begin{matrix}(a_p, A_p)\\ (b_q, B_q)\end{matrix}\right]\right)(x)$$
$$= (-1)^{[\Re(\alpha)]+1} x^{\rho-\alpha-1} H_{p+1,q+1}^{m+1,n}\left[x^{\sigma}\,\middle|\, \begin{matrix}(a_p, A_p), (1-\rho, \sigma)\\ (1-\rho+\alpha, \sigma), (b_q, B_q)\end{matrix}\right], \quad (3.135)$$

provided $\alpha, \beta, \eta \in C, \Re(\alpha) > 0, \Re(\alpha+\beta+\eta) \neq 0, \rho \in C, \sigma > 0$; further $\alpha^* > 0$ or $\alpha^* = 0$, and $\gamma\mu + \Re(\delta) < -1$ satisfy the conditions

$$\Re(\rho) + \sigma \max_{1\le j\le n}\left[\frac{\Re(a_j)-1}{A_j}\right] - \{\Re(\alpha)\} > 0,$$

for $\alpha^* > 0$ or $\alpha^* = 0, \mu \le 0$; while

$$\Re(\rho) + \sigma \max_{1\le j\le n}\left[\frac{\Re(a_j)-1}{A_j}, \frac{\Re(\delta)+\frac{1}{2}}{\mu}\right] - \{\Re(\alpha)\} > 0.$$

for $\alpha^* = 0$ and $\mu > 0$ where $\{\Re(\alpha)\}$ is the fractional part of $\Re(\alpha)$.

(Kilbas and Saigo 1999)

Note 3.8. In Exercise 3.9.3, we consider the left-sided generalized fractional derivative $D_{0+}^{\alpha,\beta,\eta}$ of the H-function, whereas Exercise 3.9.4 provides the right-sided generalized fractional derivative $D_{-}^{\alpha,\beta,\eta}$ of the H-function.

Note 3.9. It is observed that the result of Exercise 3.9.1 also holds for the generalized fractional integro–differentiation $I_{0+}^{\alpha,\beta,\eta}$ of the H-function defined by (3.103). Similarly the result of Exercise 3.9.2 also gives the generalized fractional integro–differentiation $I_{-}^{\alpha,\beta,\eta}$ of the H-function defined by (3.105).

Remark 3.13. Certain properties of the Riemann–Liouville fractional calculus operators associated with generalized Mittag-Leffler function are obtained by Saxena and Saigo (2005). Saigo-Maeda operators of fractional calculus associated with Appell function F_3 (Saigo-Maeda 1998), which are the generalizations of Saigo fractional Calculus operators, are studied by Saxena and Saigo (2001), which provide the extensions of the theorems given in this section (Exercises 3.9.1–3.9.4). For further results on Saigo-Maeda fractional calculus operators, refer to the papers by Saxena et al. (2002) and Kiryakova (2006).

3.9.5. With the help of the following chain rules for the Saigo operators

$$(I_{0+}^{\alpha,\beta,\eta} I_{0+}^{\gamma,\delta,\alpha+\eta} f)(x) = (I_{0+}^{\alpha+\gamma,\beta+\delta,\eta} f)(x),$$

and

$$(I_{-}^{\alpha,\beta,\eta} I_{-}^{\gamma,\delta,\alpha+\eta} f)(x) = (I_{-}^{\alpha+\gamma,\beta+\delta,\eta} f)(x),$$

derive the inverses

$$\left(I_{0_+}^{\alpha,\beta\eta}\right)^{-1} = I_{0_+}^{-\alpha,-\beta,\alpha+\eta}, \tag{3.136}$$

and

$$\left(I_{-}^{\alpha,\beta\eta}\right)^{-1} = I_{-}^{-\alpha,-\beta,\alpha+\eta}. \tag{3.137}$$

3.9.6. Establish the following property of Saigo operators called "integration by parts"

$$\int_0^\infty f(x)\left(I_{0_+}^{\alpha,\beta,\eta} g\right)(x)\mathrm{d}x = \int_0^\infty g(x)\left(I_{-}^{\alpha,\beta,\eta} f\right)(x)\mathrm{d}x. \tag{3.138}$$

3.9.7. Show that

$$\left(I_{0_+}^{\alpha,\beta,\eta} x^{\sigma-1}(a+bx)^c\right)(x) = a^c \frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\alpha+\eta)} \times {}_3F_2\left(1, \eta-\beta+1, -c; 1-\beta, \alpha+\eta+1; -\frac{bx}{a}\right).$$

Also give the conditions of validity of this result.

3.11 Multiple Erdélyi–Kober Operators

Fractional integration operators associated with the H-functions are studied by Saxena et al. (1974), Kalla (1969), Kalla and Kiryakova (1990), Srivastava and Buschman (1973). A detailed and comprehensive account of fractional integration operators and their applications studied by various authors during the last four decades can be found in the paper of Srivastava and Saxena (2001). The discussion in this section is based on the work of Galué et al. (1993).

Notation 3.30. $I_{(\beta_k),(\lambda_k),m}^{(\gamma_k),(\delta_k)}$: Multiple Erdélyi–Kober operator of Riemann–Liouville type.

Notation 3.31. C_α: Space of continuous functions.

Notation 3.32. $K_{(\varepsilon_k),(\xi_k),n}^{(\tau_k),(\alpha_k)} f(x)$: Multiple Erdélyi–Kober operator of Weyl type.

Notation 3.33. $C_{\alpha^*}^*$: Space of continuous functions.

Definition 3.26. Space of functions C_α is defined as

$$C_\alpha = \{f(x) = x^p f^{\sim}(x) : p > \alpha, f^{\sim}(x) \in C[0,\infty)\}$$
$$\text{with } \alpha =^{\max}_{1\le k\le m} [-\beta(\gamma_k+1)] \tag{3.139}$$

Definition 3.27. Space of functions $C^*_{\alpha^*}$ is defined as

$$C^*_{\alpha^*} = \left\{ f(x) = x^q g(x); q < \alpha^*, g \in C(0,\infty); |g| \le A_g \right\} \text{ with } \alpha^* = \min_{1\le k\le m} (\beta \tau_k) \tag{3.140}$$

Definition 3.28. A multiple Erdélyi–Kober operator of Riemann–Liouville type is defined in the form

$$I[f(x)] = I^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m} f(x) = \begin{cases} \int_0^1 H^{m,0}_{m,m}\left[u \middle| \begin{matrix} (\gamma_k+\delta_k+1-\frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \\ (\gamma_k+1-\frac{1}{\lambda_k}, \frac{1}{\lambda_k})_1^m \end{matrix} \right] f(xu)\mathrm{d}u \\ \text{if } \sum_1^m \delta_k > 0. \\ f(x), \quad \text{if } \delta_k = 0 \text{ and } \lambda_k = \beta_k, k = 1, \ldots, m, \end{cases} \tag{3.141}$$

where $m \in Z^+, \beta_k > 0, \delta_k \ge 0$, and $\gamma_k, k = 1, \ldots, m$ are real numbers. Furthermore

$$\sum_{k=1}^{m} \frac{1}{\lambda_k} \ge \sum_{k=1}^{m} \frac{1}{\beta_k},$$

and $f(x) \in C_\alpha$, where C_α is defined by (3.139), and

$$\alpha \ge \max_{1\le k\le m} [-\lambda_k(\gamma_k + 1)].$$

The definition (3.141) can be rewritten in the familiar form :

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m} f(x) = \frac{1}{x}\int_0^x H^{m,0}_{m,m}\left[\frac{t}{x} \middle| \begin{matrix} \left(\gamma_k + \delta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_1^m \\ \left(\gamma_k + 1 - \frac{1}{\lambda_k}, \frac{1}{\lambda_k}\right)_1^m \end{matrix} \right] f(t)\mathrm{d}t. \tag{3.142}$$

Remark 3.14. It is interesting to note that for $\lambda_k = \beta_k, k = 1, 2, \ldots, m$, we obtain the operator defined by Kalla and Kiryakova (1990). If, however, we set $m = 1$ and $\beta_k = \lambda_k, k = 1, \ldots, m$, we obtain a slight variant form of Erdélyi–Kober operator defined in (3.88). The following properties of this operator holds.

3.11.1 A Mellin Transform

$$M\left\{ I^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m} f(x); s \right\} = \prod_{k=1}^{m} \frac{\Gamma(\gamma_k + 1 - \frac{s}{\lambda_k})}{\Gamma(\gamma_k + \delta_k + 1 - \frac{s}{\lambda_k})} M\{f(x); s\}, \tag{3.143}$$

where

$$\sum_{k=1}^{m}\frac{1}{\lambda_k}-\sum_{k=1}^{m}\frac{1}{\beta_k}>0 \text{ and } \Re(s)<\min_{1\le k\le m}[\lambda_k(1+\gamma_k)].$$

3.11.2 Properties of the Operators

Some basic properties of the operator defined by (3.141) are given below:

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m}x^{\rho}=\prod_{k=1}^{m}\frac{\Gamma(\gamma_k+1+\frac{\rho}{\lambda_k})}{\Gamma(\gamma_k+\delta_k+1+\frac{\rho}{\lambda_k})}x^{\rho}, \tag{3.144}$$

where $\Re(\rho)+\max_{1\le k\le m}[\lambda_k(1+\gamma_k)]>0$.

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m}I^{(\tau_k),(\alpha_k)}_{(\varepsilon_k),(\xi_k),n}f(x)=I^{((\gamma_k)_1^m,(\tau_k)_1^n),((\delta_k)_1^m,(\alpha_k)_1^n)}_{((\beta_k)_1^m,(\varepsilon_k)_1^n),((\lambda_k)_1^m,(\xi_k)_1^n),m+n}f(x), \tag{3.145}$$

where

$$\sum_{k=1}^{n}\frac{1}{\xi_k}-\sum_{k=1}^{n}\frac{1}{\varepsilon_k}>0,\ A=\sum_{k=1}^{m}\frac{1}{\lambda_k}-\sum_{k=1}^{m}\frac{1}{\beta_k}>0,$$

$\max_{1\le k\le m}[1-\lambda_k(\gamma_k+1)]<0<\max_{1\le k\le n}[\xi_k(\tau_k+1)-1]$, and

$$\left|\arg\left(\frac{1}{x}\right)\right|<(\pi A/2).$$

The inverse of the operator $I^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m}$ is given by

$$\left(I^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m}\right)^{-1}f(x)=I^{(\gamma_k+\delta_k),(-\delta_k)}_{(\lambda_k),(\beta_k),m}f(x). \tag{3.146}$$

The results (3.143) and (3.146) are useful in deriving the solutions of a certain class of integral equations.

Definition 3.29. Another multiple Erdélyi–Kober fractional integral operator of Weyl type is defined by

$$K[f(x)]=K^{(\tau_k),(\alpha_k)}_{(\varepsilon_k),(\xi_k)n}f(x)=\int_x^{\infty}H^{n,0}_{n,n}\left[\frac{1}{u}\left|\begin{matrix}(\tau_k+\alpha_k+\frac{1}{\varepsilon_k},\frac{1}{\varepsilon_k})_1^n\\(\tau_k+\frac{1}{\xi_k},\frac{1}{\xi_k})_1^n\end{matrix}\right.\right]f(xu)\mathrm{d}u, \tag{3.147}$$

if $\sum_1^n\alpha_k>0$, and $f(x)$, if $\alpha_k=0$ and $\varepsilon_k=\xi_k, k=1,\ldots,n$, where

$$\sum_{k=1}^{n}\frac{1}{\xi_k}-\sum_{k=1}^{n}\frac{1}{\varepsilon_k}>0,$$

$n \in N, \varepsilon_k > 0, \xi_k > 0, \alpha_k \geq 0$ and $\tau_k, k = 1, \ldots, n$ are real numbers, $f(x) \in C^*_{\alpha^*}$, where $C^*_{\alpha^*}$ is defined by (3.140) and

$$\alpha^* \leq \min_{1 \leq k \leq n} (\xi_k \tau_k).$$

The definition (3.147) can easily be put in the familiar form :

$$K^{(\tau_k),(\alpha_k)}_{(\varepsilon_k),(\xi_k),n} f(x) = \frac{1}{x} \int_x^\infty H^{n,0}_{n,n} \left[\frac{x}{t} \middle| \begin{matrix} (\tau_k + \alpha_k + \frac{1}{\varepsilon_k}, \frac{1}{\varepsilon_k})_1^n \\ (\tau_k + \frac{1}{\xi_k}, \frac{1}{\xi_k})_1^n \end{matrix} \right] f(t) \mathrm{d}t, \qquad (3.148)$$

provided that

$$\sum_{k=1}^{n} \alpha_k > 0.$$

Remark 3.15. It is interesting to note that for $\varepsilon_k = \xi_k, k = 1, 2, \ldots, n$, we obtain the operator defined by Kalla and Kiryakova (1990). If, however, we set $n = 1$ and $\varepsilon_k = \xi_k, k = 1, \ldots, n$, we obtain a slight variation of the Erdélyi–Kober operator of Weyl type defined in (3.89). The following properties of this operator holds.

3.11.3 Mellin Transform of a Generalized Operator

It can be easily seen with the help of the Mellin transform of the H-function given by the equation (2.8) that

$$M\left\{K^{(\tau_k),(\alpha_k)}_{(\varepsilon_k),(\xi_k),n} f(x); s\right\} = \prod_{k=1}^{n} \frac{\Gamma(\tau_k + \frac{s}{\xi_k})}{\Gamma(\tau_k + \alpha_k + \frac{s}{\varepsilon_k})} M\{f(x); s\}, \qquad (3.149)$$

where

$$\sum_{k=1}^{n} \frac{1}{\xi_k} - \sum_{k=1}^{n} \frac{1}{\varepsilon_k} > 0 \text{ and } \max_{1 \leq k \leq n} (-\xi_k \tau_k) < \Re(s).$$

The power function formula for the operator $K^{(\tau_k),(\alpha_k)}_{(\varepsilon_k),(\xi_k),n}$ is given by

$$K^{(\tau_k),(\alpha_k)}_{(\varepsilon_k),(\xi_k),n} x^\rho = \prod_{k=1}^{m} \frac{\Gamma(\tau_k - \frac{\rho}{\xi_k})}{\Gamma(\tau_k + \alpha_k - \frac{\rho}{\varepsilon_k})} x^\rho, \qquad (3.150)$$

where $\Re(\rho) < \min_{1 \leq k \leq m} [\tau_k \xi_k]$. Further

$$K^{(\tau_k),(\alpha_k)}_{(\varepsilon_k),(\xi_k),n} K^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m} f(x) = K^{((\tau_k)_1^n,(\gamma_k)_1^m),((\alpha_k)_1^n,(\delta_k)_1^m)}_{((\varepsilon_k)_1^n,(\beta_k)_1^m),((\xi_k)_1^n,(\lambda_k)_1^m),m+n} f(x), \qquad (3.151)$$

where

$$B = \sum_{k=1}^{n} \frac{1}{\xi_k} - \sum_{k=1}^{n} \frac{1}{\varepsilon_k} > 0, \sum_{k=1}^{m} \frac{1}{\lambda_k} - \sum_{k=1}^{m} \frac{1}{\beta_k} > 0,$$
$$\max_{1 \le k \le m} (-\lambda_k \gamma_k - 1) < 0 < \min_{1 \le k \le n} (\xi_k \tau_k + 1),$$

and

$$|\arg x| < \frac{1}{2}\pi B.$$

Finally, the inverse of the operator $K_{(\varepsilon_k),(\xi_k),n}^{(\tau_k),(\alpha_k)}$ is given by

$$\left(K_{(\varepsilon_k),(\xi_k),n}^{(\tau_k),(\alpha_k)}\right)^{-1} f(x) = K_{(\xi_k),(\varepsilon_k),n}^{(\tau_k+\alpha_k),(-\alpha_k)} f(x). \tag{3.152}$$

Remark 3.16. Solutions of certain dual integral equations involving general H-functions have been developed by Galué et al. (1993) by the application of the operators (3.141) and (3.147). It is interesting to observe that the results given earlier by Kalla and Kiryakova (1990) for the multiple Erdélyi–Kober and Weyl operators follow easily from the results of this section.

Remark 3.17. Representations of fractional integration operators of multiple Riemann–Liouville and Weyl type defined by (3.141) and (3.147), in terms of the Laplace and inverse Laplace transforms, are recently obtained by Saxena et al. (2006). Integral formulae for the H-function generalized fractional integration operators discussed in this section are derived by Saxena et al. (2004a, 2007). Integral formulas for the generalized Erdélyi–Kober operator of Weyl type, defined by the equation (3.147), are recently evaluated by Saxena et al. (2005).

Chapter 4
Applications in Statistics

4.1 Introduction

Special functions are used in almost all areas of statistics. Statistical densities are basically elementary special functions or product of such functions. Hence, the theory of special functions is directly applicable to statistical distribution theory. While studying generalized densities, structural properties of densities, Bayesian inference, distributions of test statistics, characterization of densities and related studies of probability theory, stochastic processes and time series problems, and special functions and generalized special functions in the categories of Meijer's G-functions and H-functions come in naturally.

When looking at multivariate and matrix-variate distributions, the theory of special functions of matrix argument is directly applicable. Functions of matrix argument in the categories of matrix variable gamma, type-1 beta and type-2 beta, are the most commonly used special functions in current statistical literature.

In this chapter, a brief introduction to the applications of H-functions in statistical distribution theory will be given. Problems which fit directly into the definition of an H-function are dealt with in this chapter. With the knowledge of the basic materials discussed in this chapter, the reader will be able to tackle more complicated situations of applications of special functions in statistics. Only the real variable case is discussed in this chapter.

4.2 General Structures

General structures in statistical literature where H-function will be applicable are many. The simplest of the structures are products and ratios of statistically independently distributed positive real scalar random variables. A real scalar random variable x is said to have a generalized gamma density when the density is of the form

$$f(x) = \begin{cases} \frac{\beta a^{\frac{\alpha}{\beta}}}{\Gamma\left(\frac{\alpha}{\beta}\right)} x^{\alpha-1} \mathrm{e}^{-a x^{\beta}}, & x > 0, a > 0, \alpha > 0, \beta > 0 \\ 0, \text{elsewhere.} \end{cases} \tag{4.1}$$

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Note 4.1. Usually, in statistical problems, the parameters are real; hence, we will assume that the parameters $a, \alpha,$ *and* β are real.

Let

$$u = x_1 x_2 \cdots x_k, \tag{4.2}$$

where x_j has the density in (4.1) with the parameters $a_j > 0, \alpha_j > 0, \beta_j > 0, j = 1, 2, \cdots, k$ and let $x_1, \cdots, x_k$ be statistically independently distributed. Note that for $\beta_j = 1$ in (4.1), one has the standard gamma density. Hence, if y_1 has the density in (4.1) with $\beta_j = 1$, then a density of the structure in (4.1) can be created by considering $x_j = y_j^{\beta_j}, j = 1, \cdots, k$. Hence,

$$u^* = y_1^{\beta_1} \cdots y_k^{\beta_k}, \tag{4.3}$$

and u in (4.2) can be studied by using the same procedures. If one is interested in deriving the exact density of (4.2), then one of the methods, and possibly the easiest way, is to compute the Mellin transform of the density of u. If the unknown density of u is denoted by $g(u)$, one can evaluate the Mellin transform of $g(u)$, without knowing $g(u)$, by making use of the independence properties of $x_1, \cdots, x_k$. In the standard terminology in statistical literature, let E denote the mathematical expectation, then $E(x^h)$, when x has the density in (4.1), is given by

$$E(x^h) = \frac{\Gamma\left(\frac{\alpha+h}{\beta}\right)}{\Gamma\left(\frac{\alpha}{\beta}\right) a^{\frac{h}{\beta}}}, \quad \text{for } \Re(\alpha + h) > 0, \tag{4.4}$$

where $\Re(\cdot)$ denotes the real part of $(\cdot)$. Thus, when α and h are real, this expected value or the hth moment of x can exist for some negative value of h also such that $\alpha + h > 0$. Due to statistical independence,

$$\begin{aligned} E(u^h) &= [E(x_1^h)][E(x_2^h)] \cdots [E(x_k^h)] \\ &= \prod_{j=1}^{k} \frac{\Gamma\left(\frac{\alpha_j+h}{\beta_j}\right)}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right) a_j^{\frac{h}{\beta_j}}}, \quad \Re(\alpha_j + h) > 0, \quad j = 1, \ldots, k. \end{aligned} \tag{4.5}$$

But, with h replaced by $s - 1$, one has the Mellin transform of $g(u)$. That is,

$$\begin{aligned} E(u^{s-1}) &= \int_0^\infty u^{s-1} g(u) \mathrm{d}u \\ &= \prod_{j=1}^{k} \frac{\Gamma\left(\frac{\alpha_{j-1}}{\beta_j} + \frac{s}{\beta_j}\right)}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right)} \frac{a_j^{\frac{1}{\beta_j}}}{a_j^{\frac{s}{\beta_j}}}, \quad \Re(\alpha_j + s - 1) > 0, \quad j = 1, \ldots, k. \end{aligned} \tag{4.6}$$

Then, the unknown density $g(u)$ of u is available from the inverse Mellin transform. That is,

$$\begin{aligned} g(u) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[E(u^{s-1})\right]u^{-s}\mathrm{d}s,\ i=\sqrt{-1},\ c>-\alpha_j+1,\ j=1,\cdots,k \\ &= \left\{\prod_{j=1}^{k}\frac{a_j^{\frac{1}{\beta_j}}}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right)}\right\}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left\{\prod_{j=1}^{k}\Gamma\left(\frac{\alpha_j-1}{\beta_j}+\frac{s}{\beta_j}\right)\right\}\left[\left(\prod_{j=1}^{k}a_j^{\frac{1}{\beta_j}}\right)u\right]^{-s}\mathrm{d}s \\ &= \left\{\prod_{j=1}^{k}\frac{a_j^{\frac{1}{\beta_j}}}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right)}\right\}H_{0,k}^{k,0}\left[a_1^{\frac{1}{\beta_1}}\cdots a_k^{\frac{1}{\beta_k}}u\Big|_{\left(\frac{\alpha_j-1}{\beta_j},\frac{1}{\beta_j}\right),j=1,\ldots,k}\right],\ \ 0<u<\infty, \end{aligned} \tag{4.7}$$

and 0 elsewhere, is the density of u.

Note that for $\beta_j = 1,\ j = 1,\cdots,k$, the H-function in (4.7) reduces to a Meijer's G-function $G_{0,k}^{k,0}(\cdot)$. Further, for special values of k, one can evaluate (4.7) in terms of elementary special functions.

Note 4.2. Since statistical densities, in general, can be written in terms of elementary special functions and the H-function is a very generalized special function, one can represent almost all densities, in current use, in terms of H-functions.

Note 4.3. Special cases of the gamma density in (4.1) include the following:

(a) Weibull density ($\beta = \alpha$);
(b) chisquare density ($\beta = 1, a = \frac{1}{2}, \alpha = \frac{m}{2}, m = 1, 2, \cdots$);
(c) standard gamma density ($\beta = 1$);
(d) exponential density ($\beta = 1, \alpha = 1$);
(e) folded Gaussian ($\beta = 2, \alpha = 1$);
(f) chi density ($\beta = 2, \alpha = 1, n = 1, 2, \cdots$);
(g) Helley's density ($\beta = 1, \alpha = 1, a = \frac{mg}{KT}$);
(h) Helmert's density ($\beta = 2, a = \frac{n}{2\sigma^2}, \alpha = n - 1 > 0$);
(i) Maxwell–Boltzmann density ($\beta = 2, \alpha = 3$);
(j) Rayleigh density ($\beta = 2, \alpha = 2$).

Note 4.4. When x in (4.1) is replaced by $|x|, -\infty < x < \infty$, we obtain more generalized densities. The most important special cases will then be the Gaussian ($\beta = 2, \alpha = 1$) and the Laplace density ($\beta = 1, \alpha = 1$).

4.2.1 Product of Type-1 Beta Random Variables

A real scalar random variable is said to have a real type-1 beta distribution, if the density is of the following form:

$$f_1(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}, 0 < x < 1, \alpha > 0, \beta > 0 \\ 0, \text{elsewhere}, \end{cases} \tag{4.8}$$

where the parameters α and β are assumed to be real. The following discussion holds even when α and β are complex quantities. In that case, the condition becomes $\Re(\alpha) > 0$ and $\Re(\beta) > 0$ where $\Re(\cdot)$ means the real part of $(\cdot)$. The hth moment of x, when x has the density in (4.8), is given by

$$E(x^h) = \frac{\Gamma(\alpha + h)}{\Gamma(\alpha)} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha + \beta + h)}, \Re(\alpha + h) > 0. \tag{4.9}$$

When α and h are real, the moments can exist for some negative values of h also such that $\alpha + h > 0$. The Mellin transform of $f_1(x)$ is obtained from (4.9), by replacing h by $s - 1$ for some complex s.

Consider a set of real scalar random variables $x_1, \cdots, x_k$, mutually independently distributed, where x_j has the density in (4.8) with the parameters (α_j, β_j), $j = 1, \cdots, k$ and consider the product

$$u_1 = x_1 x_2 \cdots x_k. \tag{4.10}$$

Then, the Mellin transform of the density $g_1(u)$ of u_1 is obtained from the property of statistical independence and is given by,

$$\begin{aligned}\int_0^\infty u^{s-1} g_1(u) \mathrm{d}u &= E(u_1^{s-1}) = [E(x_1^{s-1})] \cdots [E(x_k^{s-1})] \\ &= \prod_{j=1}^{k} \frac{\Gamma(\alpha_j + s - 1)}{\Gamma(\alpha_j)} \frac{\Gamma(\alpha_j + \beta_j)}{\Gamma(\alpha_j + \beta_j + s - 1)} \\ &= \left[\prod_{j=1}^{k} \frac{\Gamma(\alpha_j + \beta_j)}{\Gamma(\alpha_j)}\right] \left[\prod_{j=1}^{k} \frac{\Gamma(\alpha_j + s - 1)}{\Gamma(\alpha_j + \beta_j + s - 1)}\right]. \end{aligned} \tag{4.11}$$

Then, the unknown density $g_1(u)$ is available by taking the inverse Mellin transform of (4.11). This can be written in terms of a Meijer's G-function of the type $G_{k,k}^{k,0}(\cdot)$. We can consider more general structures in the same category. For example, consider the structure

$$u_2 = x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_k^{\gamma_k}, \gamma_j > 0, j = 1, \ldots, k \tag{4.12}$$

where $x_1, \ldots, x_k$ are mutually independently distributed as in (4.10). Then, observing that

$$E(u_2^{s-1}) = E(x_1^{\gamma_1(s-1)}) E(x_2^{\gamma_2(s-1)}) E(x_k^{\gamma_k(s-1)}) \tag{4.13}$$

$$= \left\{\prod_{j=1}^{k} \frac{\Gamma(\alpha_j + \beta_j)}{\Gamma(\alpha_j)}\right\} \left\{\prod_{j=1}^{k} \frac{\Gamma(\alpha_j - \gamma_j + \gamma_j s)}{\Gamma(\alpha_j + \beta_j - \gamma_j + \gamma_j s)}\right\}, \tag{4.14}$$

$$\Re(\alpha_j - \gamma_j + \gamma_j s) > 0, j = 1, \cdots, k,$$

the density $g_2(u_2)$ of u_2 is available by taking the inverse Mellin transform, that is,

$$\begin{aligned} g_2(u_2) &= \left\{\prod_{j=1}^{k} \frac{\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j)}\right\} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \prod_{j=1}^{k} \frac{\Gamma(\alpha_j-\gamma_j+\gamma_j s)}{\Gamma(\alpha_j+\beta_j-\gamma_j+\gamma_j s)} u_2^{-s}\mathrm{d}s \\ &= \left\{\prod_{j=1}^{k} \frac{\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j)}\right\} H_{k,k}^{k,0}\left[u_2\Big|_{(\alpha_j-\gamma_j,\gamma_j),j=1,\cdots,k}^{(\alpha_j+\beta_j-\gamma_j,\gamma_j),j=1,\cdots,k}\right], 0<u_2<1. \end{aligned} \tag{4.15}$$

Observe that when $\gamma_j = 1, j = 1, \cdots, k$, the H-function reduces to the G-function. The case in (4.15) is slightly different from x_j having a generalized type-1 beta density and then considering the product $x_1 \cdots x_k$. Suppose x_j has a generalized type-1 beta density given by

$$f_2(x) = \begin{cases} \frac{\gamma a^{\frac{\alpha}{\gamma}}}{B\left(\frac{\alpha}{\gamma},\beta\right)} x^{\alpha-1}(1-ax^{\gamma})^{\beta-1}, 0<x<a^{-\frac{1}{\gamma}}, \alpha>0, \beta>0, \gamma>0, a>0, \\ \quad 1-ax^{\gamma}>0, \\ 0, \text{elsewhere}, \end{cases} \tag{4.16}$$

where $B(\cdot,\cdot)$ is a beta function

$$B\left(\frac{\alpha}{\gamma},\beta\right) = \frac{\Gamma\left(\frac{\alpha}{\gamma}\right)\Gamma(\beta)}{\Gamma\left(\frac{\alpha}{\gamma}+\beta\right)}, \alpha>0, \beta>0, \gamma>0.$$

If x follows the density in (4.16), then the $(s-1)$th moment of x is given by,

$$E(x^{s-1}) = \int_0^{a^{-\frac{1}{\gamma}}} x^{s-1} f_2(x)\mathrm{d}x = \frac{\Gamma\left(\frac{\alpha+s-1}{\gamma}\right)}{a^{\frac{s-1}{\gamma}}\Gamma\left(\frac{\alpha}{\gamma}\right)} \frac{\Gamma\left(\frac{\alpha}{\gamma}+\beta\right)}{\Gamma\left(\frac{\alpha+s-1}{\gamma}+\beta\right)}. \tag{4.17}$$

Let, x_j have the density in (4.16) with parameters $(a_j, \alpha_j, \beta_j, \gamma_j), j = 1, \cdots, k$ and let $x_1, \cdots, x_k$ be independently distributed. Then, if

$$u_3 = x_1 \cdots x_k, \tag{4.18}$$

then

$$E(u_3^{s-1}) = \prod_{j=1}^{k} \left\{ \frac{1}{a_j^{\frac{s-1}{\gamma_j}}} \frac{\Gamma\left(\frac{\alpha_j+s-1}{\gamma_j}\right)}{\Gamma\left(\frac{\alpha_j}{\gamma_j}\right)} \frac{\Gamma\left(\frac{\alpha_j}{\gamma_j}+\beta_j\right)}{\Gamma\left(\frac{\alpha_j+s-1}{\gamma_j}+\beta_j\right)} \right\}. \tag{4.19}$$

The density of u_3, denoted by $g_3(u_3)$, is available from the inverse Mellin transform in (4.19). That is,

$$g_3(u_3)=\left\{\prod_{j=1}^{k}\frac{a_j^{\frac{1}{\gamma_j}}\Gamma\left(\frac{\alpha_j}{\gamma_j}+\beta_j\right)}{\Gamma\left(\frac{\alpha_j}{\gamma_j}\right)}\right\}H_{k,k}^{k,0}\left[a_1^{\frac{1}{\gamma_1}}\cdots a_k^{\frac{1}{\gamma_k}}u_3\Big|_{\left(\frac{\alpha_j-1}{\gamma_j},\frac{1}{\gamma_j}\right),j=1,\ldots,k}^{\left(\frac{\alpha_j-1}{\gamma_j}+\beta_j,\frac{1}{\gamma_j}\right),j=1,\cdots,k}\right]$$

$$0<a_1^{\frac{1}{\gamma_1}}\cdots a_k^{\frac{1}{\gamma_k}}u_3<1. \tag{4.20}$$

Note that (4.20) is different from (4.15).

4.2.2 Real Scalar Type-2 Beta Structure

A real scalar random variable x is said to have a type-2 beta density, if x has the density

$$f_3(x)=\begin{cases}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1+x)^{-(\alpha+\beta)}, 0<x<\infty, \alpha>0, \beta>0\\ 0, \text{elsewhere.}\end{cases} \tag{4.21}$$

Then, the Mellin transform of $f_3(x)$ is given by,

$$\int_0^\infty x^{s-1}f_3(x)\mathrm{d}x=E(x^{s-1})=\frac{\Gamma(\alpha+s-1)}{\Gamma(\alpha)}\frac{\Gamma(\beta-s+1)}{\Gamma(\beta)}$$
$$\text{for } \Re(\alpha+s-1)>0, \Re(\beta-s+1)>0. \tag{4.22}$$

This is obtained from the normalizing constant in (4.21) by observing that $\alpha+\beta=(\alpha+s-1)+(\beta-s+1)$. As in the previous cases, consider

$$u_4=x_1^{\gamma_1}\cdots x_k^{\gamma_k}, \tag{4.23}$$

where $\gamma_1>0,\cdots,\gamma_k>0$, with x_j having the density in (4.21) with the parameters (α_j,β_j), $j=1,\cdots,k$ and $x_1,\cdots,x_k$ are independently distributed. Then, as in the previous situations, the Mellin transform of the density $g_4(u_4)$ of u_4 is given by

$$\int_0^\infty u_4^{s-1}g_4(u_4)\mathrm{d}u_4=E(u_4^{s-1})=\prod_{j=1}^{k}\frac{\Gamma(\alpha_j+\gamma_j s-\gamma_j)}{\Gamma(\alpha_j)}\frac{\Gamma(\beta_j-\gamma_j s+\gamma_j)}{\Gamma(\beta_j)}, \tag{4.24}$$

for $\Re(\alpha_j-\gamma_j+\gamma_j s)>0,\ \Re(\beta_j+\gamma_j-\gamma_j s)>0,\ j=1,\cdots,k.$

Then, by taking the inverse Mellin transform in (4.24), one has the density,

$$g_4(u_4) = \left\{\prod_{j=1}^{k} \frac{1}{\Gamma(\alpha_j)\Gamma(\beta_j)}\right\} H_{k,k}^{k,k}\left[u_4\Big|_{(\alpha_j-\gamma_j,\gamma_j),\ j=1,\cdots,k}^{(1-\beta_j-\gamma_j,\gamma_j),\ j=1,\cdots,k}\right],\ 0 < u_4 < \infty. \tag{4.25}$$

As illustrated before, the density of a product of generalized type-2 beta random variables will be different from (4.25). A generalized type-2 beta density has the form

$$f_4(x) = \begin{cases} \frac{\gamma\, a^{\frac{\alpha}{\gamma}}\Gamma\left(\frac{\alpha}{\gamma}+\beta\right)}{\Gamma(\frac{\alpha}{\gamma})\Gamma(\beta)} x^{\alpha-1}(1+ax^{\gamma})^{-(\alpha+\beta)}, 0 < x < \infty, \alpha > 0, \\ \quad \beta > 0, a > 0, \gamma > 0, \\ 0,\ \text{elsewhere}. \end{cases} \tag{4.26}$$

Thus, for all such special cases mentioned in Notes 4.2 and 4.3, the procedure discussed in this section is applicable. Observing that negative moments of the form $E(x^{-h}), h > 0$ are available from $E(x^h)$ with h replaced by $-h$ if $E(x^{-h})$ exists.

4.2.3 A More General Structure

We can consider more general structures. Let,

$$w = \frac{x_1 x_2 \cdots x_r}{x_{r+1}\cdots x_k}, \tag{4.27}$$

where $x_1, \cdots, x_k$ are mutually independently distributed real random variables having the density in (4.1) with x_j having parameters $a_j,\ \alpha_j,\ \beta_j,\ j = 1, \cdots, k$. Then,

$$E(w^h) = E(x_1^h)E(x_2^h)\cdots E(x_r^h)E(x_{r+1}^{-h})\cdots E(x_k^{-h}), \tag{4.28}$$

provided the right side in (4.28) exists. Then, from (4.4) we have,

$$E(w^{s-1}) = \left\{\prod_{j=1}^{r} \frac{\Gamma\left(\frac{\alpha_j+h}{\beta_j}\right)}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right) a_j^{\frac{h}{\beta_j}}}\right\} \left\{\prod_{j=r+1}^{k} \frac{\Gamma\left(\frac{\alpha_j-h}{\beta_j}\right)}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right) a_j^{-\frac{h}{\beta_j}}}\right\},\ h = s-1 \tag{4.29}$$

$$= \left\{\prod_{j=1}^{r} \frac{a_j^{\frac{1}{\beta_j}}}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right)}\right\} \left\{\prod_{j=r+1}^{k} \frac{1}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right) a_j^{\frac{1}{\beta_j}}}\right\}$$

$$\times \left\{\prod_{j=1}^{r} \Gamma\left(\frac{\alpha_j-1}{\beta_j}+\frac{s}{\beta_j}\right)\frac{1}{a_j^{\frac{s}{\beta_j}}}\right\} \left\{\prod_{j=r+1}^{k} \Gamma\left(\frac{\alpha_j+1}{\beta_j}-\frac{s}{\beta_j}\right) a_j^{\frac{s}{\beta_j}}\right\}, \tag{4.30}$$

for $\alpha_j + s - 1 > 0,\ j = 1,\cdots,r,\ \alpha_j - s + 1 > 0,\ j = r+1,\cdots,k$. Hence, the density of w, denoted by $g^*(w)$, is available from the inverse Mellin transform. That is,

$$g^*(w) = c^*\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left\{\prod_{j=1}^{r}\Gamma\left(\frac{\alpha_j-1}{\beta_j}+\frac{s}{\beta_j}\right)\right\}\left\{\prod_{j=r+1}^{k}\Gamma\left(\frac{\alpha_j+1}{\beta_j}-\frac{s}{\beta_j}\right)\right\}$$

$$\times\left[\frac{\prod_{j=1}^{r}a_j^{\frac{1}{\beta_j}}}{\prod_{j=r+1}^{k}a_j^{\frac{1}{\beta_j}}}w\right]^{-s}\mathrm{d}s,\ i=\sqrt{-1},\ \max_{j=1,\cdots,r}(1-\alpha_j)<c<\min_{j=r+1,\cdots,k}(\alpha_j+1)$$

$$= H_{k-r,r}^{r,k-r}\left[\delta u\Big|_{\left(\frac{\alpha_j-1}{\beta_j},\frac{1}{\beta_j}\right),\ j=1,\cdots,r}^{(1-\frac{\alpha_j+1}{\beta_j},\frac{1}{\beta_j}),\ j=r+1,\cdots,k}\right],\ 0<u<\infty \tag{4.31}$$

$$\text{where}\quad \delta = \frac{\prod_{j=1}^{r}a_j^{\frac{1}{\beta_j}}}{\prod_{j=r+1}^{k}a_j^{\frac{1}{\beta_j}}},\ c^* = \frac{\delta}{[\prod_{j=1}^{k}\Gamma\left(\frac{\alpha_j}{\beta_j}\right)]}. \tag{4.32}$$

Remark 4.1. In statistical applications, sometimes, the variables $x_1,\cdots,x_k$ are independently and identically distributed. In this case, the parameters in (4.28)–(4.32) will be such that $a_j = a,\ \beta_j = \beta,\ \alpha_j = \alpha$ for some $a,\ \beta,\ \alpha$ and for $j = 1,\cdots,k$.

Remark 4.2. In (4.27), we took x_j's belonging to the generalized gamma density in (4.1). But, we could have considered w consisting of x_j's belonging to (4.1), (4.8), (4.12), (4.21), (4.23), and (4.26) or mixed cases provided $E(w^{s-1})$ exists. Then, the density of such a general structure will be available by taking the inverse Mellin transform of $E(w^{s-1})$. The density of w can be written in terms of an H-function. More of such cases are contained in the pathway model to be discussed in the next section.

Exercises 4.1

4.1.1. If x is a real scalar variable having a generalized gamma density then evaluate the Laplace transform of the density of $\frac{1}{x}$ and show that this Laplace transform can be written as a H-function. [Hint: Evaluate the integral

$$c\int_0^\infty \mathrm{e}^{-\frac{p}{x}}x^{\gamma-1}\mathrm{e}^{-bx^\rho}\mathrm{d}x,$$

where c is the normalizing constant and p is the Laplace parameter.]

4.1.2. Show that

$$c\int_0^\infty x^{-\gamma-1}\mathrm{e}^{-px-bx^{-\rho}}\mathrm{d}x,$$

also leads to the same result as in Exercise 4.1.1.

4.1.3. Show that the Laplace transform of $\frac{1}{x}$ in a generalized type-2 beta density, that is

$$c\int_0^\infty \mathrm{e}^{-\frac{p}{x}}x^{\gamma-1}[1+a(\alpha-1)x^\delta]^{-\frac{1}{\alpha-1}}\mathrm{d}x,$$

for $a>0, \delta>0, \alpha>1, \gamma>-1, \frac{1}{\alpha-1}-\frac{\gamma+1}{\delta}>0$, is an H-function, where c is a normalizing constant in the density.

4.1.4. Evaluate the integral

$$c\int_0^\infty \mathrm{e}^{-px}x^{-\gamma-1}[1+a(\alpha-1)x^{-\delta}]^{-\frac{1}{\alpha-1}}\mathrm{d}x,$$

for $\alpha>1, a>0, \delta>0$ and write down the conditions for the existence of the integral. Interpret it as a Laplace transform.

4.1.5. Let x_1 and x_2 be independently distributed type-1 beta random variables with the parameters (α_1,β_1), *and* (α_2,β_2), respectively. Let $u=x_1^{\gamma_1}x_2^{\gamma_2}$. Give the conditions under which u is distributed as a power of a type-1 beta random variable.

4.3 A Pathway Model

A general density that was introduced by Mathai (2005) is a matrix-variate pathway density. The scalar version of the pathway density in the real case is the following:

$$f_x(x)=c\,|x|^\gamma[1-a(1-\alpha)|x|^\delta]^{\frac{\eta}{1-\alpha}},\ \delta>0,\ \eta>0,\ a>0,\ 1-a(1-\alpha)|x|^\delta>0, \tag{4.33}$$

and $f_x(x)=0$ elsewhere, where c is the normalizing constant. When $\alpha<1$ the range of x is

$$-\frac{1}{[a(1-\alpha)]^{\frac{1}{\delta}}}<x<\frac{1}{[a(1-\alpha)]^{\frac{1}{\delta}}}. \tag{4.34}$$

As α moves toward 1, the range becomes larger and larger, and eventually $-\infty<x<\infty$ when $\alpha\to 1$. Thus, for $\alpha<1$, (4.33) remains as a generalized type-1 beta family of densities. When $\alpha>1$, we can write $1-\alpha=-(\alpha-1),\ \alpha>1$, and then $1-a(1-\alpha)|x|^\delta=1+a(\alpha-1)|x|^\delta, -\infty<x<\infty$; then, the density in (4.33) becomes a generalized type-2 beta family of densities. When $\alpha\to 1$, either from the left or from the right,

$$\lim_{\alpha\to 1}[1-a(1-\alpha)|x|^\delta]^{\frac{\eta}{1-\alpha}}=\mathrm{e}^{-a\eta|x|^\delta}. \tag{4.35}$$

In this case, (4.33) becomes a generalized version of the density in (4.1). Thus, the model in (4.33) switches into three different families of densities, represented by three different functional forms, namely the generalized type-1 beta, type-2 beta, and gamma families. Then, α becomes a pathway parameter. As can be expected, c in (4.33) will be different for the three cases $\alpha < 1, \alpha > 1$, and $\alpha \to 1$, and the respective densities are the following:

$$f_1(x) = c_1\,|x|^{\gamma}[1-a(1-\alpha)|x|^{\delta}]^{\frac{\eta}{1-\alpha}},\ \alpha < 1,\ a > 0,\ \delta > 0,\ \eta > 0, \tag{4.36}$$
$$-\frac{1}{[a(1-\alpha)]^{\frac{1}{\delta}}} < x < \frac{1}{[a(1-\alpha)]^{\frac{1}{\delta}}},\ \text{and}\ f_1(x) = 0,\ \text{elsewhere},$$
$$f_2(x) = c_2\,|x|^{\gamma}[1+a(\alpha-1)|x|^{\delta}]^{-\frac{\eta}{\alpha-1}},\ a > 0,\ \delta > 0,\ \eta > 0,\ \alpha > 1,$$
$$-\infty < x < \infty, \tag{4.37}$$
$$f_3(x) = c_3\,|x|^{\gamma}\mathrm{e}^{-a\eta|x|^{\delta}},\ a > 0,\ \eta > 0,\ \delta > 0,\ -\infty < x < \infty, \tag{4.38}$$

where the conditions on γ will be available from the normalizing constants $c_1, c_2,$ *and* c_3, and these constants are evaluated with the help of type-1 beta integral, type-2 beta integral, and gamma integral, respectively, and they are the following:

$$c_1 = \frac{\delta\,[a(1-\alpha)]^{\frac{\gamma+1}{\delta}}\,\Gamma\left(\frac{\gamma+1}{\delta}+\frac{\eta}{1-\alpha}+1\right)}{2\Gamma\left(\frac{\gamma+1}{\delta}\right)\Gamma\left(\frac{\eta}{1-\alpha}+1\right)}, \alpha < 1, \gamma > -1, a > 0, \eta > 0, \delta > 0, \tag{4.39}$$

$$c_2 = \frac{\delta\,[a(\alpha-1)]^{\frac{\gamma+1}{\delta}}\,\Gamma\left(\frac{\eta}{\alpha-1}\right)}{2\,\Gamma\left(\frac{\gamma+1}{\delta}\right)\Gamma\left(\frac{\eta}{\alpha-1}-\frac{\gamma+1}{\delta}\right)}, \alpha > 1, \gamma > -1, \frac{\eta}{\alpha-1}-\frac{\gamma+1}{\delta} > 0,$$
$$\delta > 0, \eta > 0, a > 0, \tag{4.40}$$

$$c_3 = \frac{\delta\,(a\eta)^{\frac{\gamma+1}{\delta}}}{2\,\Gamma\left(\frac{\gamma+1}{\delta}\right)},\ \delta > 0, a > 0, \gamma > -1, \eta > 0. \tag{4.41}$$

4.3.1 Independent Variables Obeying a Pathway Model

Consider k-independent real scalar variables, distributed according to the pathway density in (4.33) with different parameters. Let, $u = x_1x_2\cdots x_k$. We can compute the density of u by following the procedure in Sect. 4.1. To this end, let us look at the $(s-1)$th moment of x in (4.33). This will have three different forms depending upon the cases $\alpha < 1$, $\alpha > 1$, and $\alpha \to 1$, and these are available from (4.39), (4.40), and (4.41), respectively. That is,

$$E(|x|^{s-1}) = \frac{1}{[a(1-\alpha)]^{\frac{s-1}{\delta}}} \frac{\Gamma\left(\frac{\gamma+s}{\delta}\right)}{\Gamma\left(\frac{\gamma+1}{\delta}\right)} \frac{\Gamma\left(\frac{\gamma+1}{\delta}+\frac{\eta}{1-\alpha}+1\right)}{\Gamma\left(\frac{\eta}{1-\alpha}+1+\frac{\gamma+s}{\delta}\right)},$$

$$\text{for}\, \alpha<1,\ a>0,\ \eta>0,\ \gamma+s>0,\ \delta>0,\ \gamma+1>0, \quad (4.42)$$

$$= \frac{1}{[a(\alpha-1)]^{\frac{s-1}{\delta}}} \frac{\Gamma\left(\frac{\gamma+s}{\delta}\right)}{\Gamma\left(\frac{\gamma+1}{\delta}\right)} \frac{\Gamma\left(\frac{\eta}{\alpha-1}-\frac{\gamma+1}{\delta}\right)}{\Gamma\left(\frac{\eta}{\alpha-1}-\frac{\gamma+s}{\delta}\right)}$$

$$\text{for}\ \ \alpha>1,\ a>0,\ \eta>0,\ \gamma+s>0,$$

$$\frac{\eta}{\alpha-1}-\frac{\gamma+s}{\delta}>0,\ \frac{\eta}{\alpha-1}-\frac{\gamma+1}{\delta}>0,\ \gamma+1>0, \quad (4.43)$$

$$= \frac{1}{(a\eta)^{\frac{s-1}{\delta}}} \frac{\Gamma\left(\frac{\gamma+s}{\delta}\right)}{\Gamma\left(\frac{\gamma+1}{\delta}\right)}\ \text{for}\, a>0,\ \eta>0, \gamma+s>0, \gamma+1>0. \quad (4.44)$$

The density of $|u| = |x_1 \cdots x_k| = |x_1| \cdots |x_k|$ is available by inverting

$$E|u|^{s-1} = E|x_1|^{s-1} E|x_2|^{s-1} \cdots E|x_k|^{s-1}.$$

Let the densities of $|u|$ for $\alpha<1,\ \alpha>1$ and $\alpha \to 1$ be denoted by $g_1(|u|), g_2(|u|)$, and $g_3(|u|)$, respectively. Then,

$$g_1(|u|) = \left\{\prod_{j=1}^{k} \frac{[a_j(1-\alpha)]^{\frac{1}{\delta_j}}}{\Gamma\left(\frac{\gamma_j+1}{\delta_j}\right)} \Gamma\left(\frac{\gamma_j+1}{\delta_j}+\frac{\eta_j}{1-\alpha}+1\right)\right\}$$

$$\times \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left\{\prod_{j=1}^{k} \Gamma\left(\frac{\gamma_j+s}{\delta_j}\right) \frac{1}{\Gamma\left(\frac{\gamma_j+s}{\delta_j}+\frac{\eta_j}{1-\alpha}+1\right)}\right\}$$

$$\times \frac{|u|^{-s}}{[\prod_{j=1}^{k} a_j(1-\alpha)]^{\frac{s}{\delta_j}}} \mathrm{ds}$$

$$= \left\{\prod_{j=1}^{k} \frac{[a_j(1-\alpha)]^{\frac{1}{\delta_j}}}{\Gamma\left(\frac{\gamma_j+1}{\delta_j}\right)} \Gamma\left(\frac{\gamma_j+1}{\delta_j}+\frac{\eta_j}{1-\alpha}+1\right)\right\}$$

$$\times H^{k,0}_{k,k}\left[\left[\prod_{j=1}^{k} a_j^{\frac{1}{\delta_j}}(1-\alpha)^{\frac{1}{\delta_j}}\right]|u|\Big|^{\left(\frac{\gamma_j}{\delta_j}+\frac{\eta_j}{1-\alpha}+1, \frac{1}{\delta_j}\right), j=1,\cdots,k}_{\left(\frac{\gamma_j}{\delta_j}, \frac{1}{\delta_j}\right), j=1,\cdots,k}\right] \quad (4.45)$$

$$\text{for}\ \ -\frac{1}{(\prod_{j=1}^{k} a_j^{\frac{1}{\delta_j}}(1-\alpha)^{\frac{1}{\delta_j}})} < u < \frac{1}{(\prod_{j=1}^{k} a_j^{\frac{1}{\delta_j}}(1-\alpha)^{\frac{1}{\delta_j}})},$$

$$\alpha<1, a_j>0, \delta_j>0, \gamma_j+1>0, \eta_j>0, j=1,\cdots,k,$$

and 0 elsewhere.

$$g_2(|u|) = \left\{\prod_{j=1}^{k} \frac{[a_j(\alpha-1)]^{\frac{1}{\delta_j}}}{\Gamma\left(\frac{\gamma_j+1}{\delta_j}\right)} \frac{1}{\Gamma\left(\frac{\eta_j}{\alpha-1}-\frac{\gamma_j+1}{\delta_j}\right)}\right\}$$
$$\times H_{k,k}^{k,k}\left[\left(\prod_{j=1}^{k} a_j^{\frac{1}{\delta_j}}(\alpha-1)^{\frac{1}{\delta_j}}\right)|u|\Big|_{\left(\frac{\gamma_j}{\delta_j},\frac{1}{\delta_j}\right),j=1,\cdots,k}^{(1-\frac{\eta_j}{\alpha-1}+\frac{\gamma_j}{\delta_j},\frac{1}{\delta_j}),j=1,\cdots,k}\right], \quad (4.46)$$
$$-\infty<u<\infty, \alpha>1, a_j>0, \delta_j>0, \gamma_j+1>0, \eta_j>0, j=1,\cdots,k.$$

$$g_3(|u|) = \left\{\prod_{j=1}^{k} \frac{(a_j\eta_j)^{\frac{1}{\delta_j}}}{\Gamma\left(\frac{\gamma_j+1}{\delta_j}\right)}\right\} H_{0,k}^{k,0}\left[(\prod_{j=1}^{k}(a_j\eta_j)^{\frac{1}{\delta_j}})|u|\Big|_{\left(\frac{\gamma_j}{\delta_j},\frac{1}{\delta_j}\right),j=1,\ldots,k}\right],$$
$$-\infty<u<\infty, a_j>0, \eta_j>0, \gamma_j+1>0, \delta_j>0, j=1,\cdots,k. \quad (4.47)$$

Remark 4.3. When $\delta_j = 1, j = 1,\cdots,k$ or when $\frac{1}{\delta_j} = m_j, m_j = 1,2,\cdots$, the H-functions in (4.45)–(4.47) become Meijer's G-functions. When $\frac{1}{\delta_j} = m_j, m_j = 1,2,\cdots$, one can expand $\Gamma(m_j s)$ and $\Gamma(m_j\gamma_j + \frac{\eta_j}{1-\alpha} + 1 + m_j s)$ in (4.45), $\Gamma(m_j s)$ and $\Gamma\left(\frac{\eta_j}{\alpha-1} - m_j(\gamma+s)\right)$ in (4.46), and $\Gamma(m_j s)$ in (4.47) by using the multiplication formula for gamma functions. Then, the coefficients of s in all gammas become ± 1, thereby the H-functions reduce to G-functions.

Exercises 4.2

4.2.1. Let α be the pathway parameter in a real scalar version of the pathway model. By using Maple/Mathematica, draw the graphs of the model for varying values of α and for fixed values of the other parameters.

4.2.2. Show that

$$f(x) = cx^{\gamma-1}[1+a_1(\alpha_1-1)x^{\delta_1}]^{-\frac{1}{\alpha_1-1}}[1+a_2(\alpha_2-1)x^{-\delta_2}]^{-\frac{1}{\alpha_2-1}},$$

where $x>0, \alpha_1>1, \alpha_2>1, a_1>0, a_2>0, \delta_1>0, \delta_2>0$ and $f(x)=0$ for $x\le 0$ can create a statistical density. Then, evaluate the normalizing constant c.

4.2.3. In Exercise 4.2.2, let $\alpha_1<1$ and $\alpha_2>1$. Then, can $f(x)$ still form a density? If so, evaluate the normalizing constant c.

4.2.4. In Exercise 4.2.2, show that

$$\lim_{\alpha_1\to 1} f(x), \lim_{\alpha_2\to 1} f(x), \lim_{\alpha_1\to 1,\alpha_2\to 1} f(x),$$

can create statistical densities. Evaluate the normalizing constants in each case.

4.2.5. Consider the normalizing constant c in Exercise 4.2.2. Show that c goes to the normalizing constants in each case in Exercise 4.2.4 under the respective conditions.

4.4 A Versatile Integral

This section deals with a general class of integrals, the particular cases of which are connected to a large number of problems in different disciplines. Reaction rate probability integrals in the theory of nuclear reaction rates, Krätzel integrals in applied analysis, inverse Gaussian distribution, generalized type-1, type-2, and gamma families of distributions in statistical distribution theory, Tsallis statistics and superstatistics in statistical mechanics, and the general pathway model are all shown to be connected to the integral under consideration. Representations of the integral in terms of generalized special functions such as Meijer's G-function and Fox's H-function are also given.

Consider the following integral:

$$f(z_2|z_1) = \int_0^\infty x^{\gamma-1}[1+z_1^\delta(\alpha-1)x^\delta]^{-\frac{1}{\alpha-1}}[1+z_2^\rho(\beta-1)x^{-\rho}]^{-\frac{1}{\beta-1}} \tag{4.48}$$

$$\text{for } \alpha>1, \beta>1, z_1\geq 0, z_2\geq 0, \delta>0, \rho>0, \Re(\gamma+1)>0,$$
$$\Re\left(\frac{1}{\alpha-1}-\frac{\gamma+1}{\delta}\right)>0, \Re\left(\frac{1}{\beta-1}-\frac{1}{\rho}\right)>0$$

$$= \int_0^\infty \frac{1}{x} f_1(x) f_2\left(\frac{z_2}{x}\right)\mathrm{d}x, \tag{4.49}$$

where $\Re(\cdot)$ denotes the real part of $(\cdot)$.

$$f_1(x) = x^\gamma[1+z_1^\delta(\alpha-1)x^\delta]^{-\frac{1}{\alpha-1}},\ f_2(x) = [1+(\beta-1)x^\rho]^{-\frac{1}{\beta-1}}, \tag{4.50}$$

with Mellin transforms

$$M_{f_1}(s) = [\delta z_1^{\gamma+s}(\alpha-1)^{\frac{\gamma+s}{\delta}}]^{-1}\frac{\Gamma\left(\frac{\gamma+s}{\delta}\right)\Gamma\left(\frac{1}{\alpha-1}-\frac{\gamma+s}{\delta}\right)}{\Gamma\left(\frac{1}{\alpha-1}\right)}, \tag{4.51}$$
$$\Re(\gamma+s)>0, \Re\left(\frac{1}{\alpha-1}-\frac{\gamma+s}{\delta}\right)>0,$$

and

$$M_{f_2}(s) = [\rho(\beta-1)^{\frac{s}{\rho}}]^{-1}\frac{\Gamma\left(\frac{s}{\rho}\right)\Gamma\left(\frac{1}{\beta-1}-\frac{s}{\rho}\right)}{\Gamma\left(\frac{1}{\beta-1}\right)} \tag{4.52}$$
$$\Re(s)>0, \Re\left(\frac{1}{\alpha-1}-\frac{s}{\rho}\right)>0.$$

Hence, the Mellin transform of $f(z_2|z_1)$, as a function of z_2, with parameter s is the following:

$$
\begin{aligned}
M_{f(z_2|z_1)}(s) &= M_{f_1}(s)M_{f_2}(s) \\
&= \frac{1}{\delta\, z_1^{\gamma+s}(\alpha-1)^{\frac{\gamma+s}{\delta}}} \frac{\Gamma\left(\frac{\gamma+s}{\delta}\right)\Gamma\left(\frac{1}{\alpha-1}-\frac{\gamma+s}{\delta}\right)}{\Gamma\left(\frac{1}{\alpha-1}\right)} \\
&\quad\times \frac{1}{\rho(\beta-1)^{\frac{s}{\rho}}} \frac{\Gamma\left(\frac{s}{\rho}\right)\Gamma\left(\frac{1}{\beta-1}-\frac{s}{\rho}\right)}{\Gamma\left(\frac{1}{\beta-1}\right)} \\
&\quad \text{for } \Re(\gamma+s)>0, \Re\left(\frac{1}{\alpha-1}-\frac{\gamma+s}{\delta}\right)>0, \Re(s)>0, \\
&\quad \Re\left(\frac{1}{\beta-1}-\frac{s}{\rho}\right)>0, z_1>0, z_2>0.
\end{aligned} \tag{4.53}
$$

Putting $y = \frac{1}{x}$ in (4.48), we have

$$
f(z_1|z_2) = \int_0^\infty \frac{y^{-\gamma}}{y}[1+z_1^\delta(\alpha-1)y^{-\delta}]^{-\frac{1}{\alpha-1}}[1+z_2^\rho(\beta-1)y^\rho]^{-\frac{1}{\beta-1}}\mathrm{d}y. \tag{4.54}
$$

Evaluating the Mellin transform of (4.54) with parameter s and treating it as a function of z_1, we have exactly the same expression in (4.53). Hence,

$$
M_{f(z_2|z_1)}(s) = M_{f(z_1|z_2)}(s) = \text{right side in (4.53)}. \tag{4.55}
$$

By taking the inverse Mellin transform of $M_{f(z_2|z_1)}(s)$, one can get the integral $f(z_2|z_1)$ as an H-function as follows:

Theorem 4.1.

$$
f(z_2|z_1) = c^{-1}H_{2,2}^{2,2}\left[z_1z_2(\alpha-1)^{\frac{1}{\delta}}(\beta-1)^{\frac{1}{\rho}}\Big|_{(\frac{\gamma}{\delta},\frac{1}{\delta}),(0,\frac{1}{\rho})}^{(1-\frac{1}{\alpha-1}+\frac{\gamma}{\delta},\frac{1}{\delta}),(1-\frac{1}{\beta-1},\frac{1}{\rho})}\right] \tag{4.56}
$$

where

$$
c = \delta\rho z_1^\gamma(\alpha-1)^{\frac{\gamma}{\delta}},
$$

and $H_{p,q}^{m,n}(\cdot)$ is a H-function.

The integral in (4.48) is connected to reaction rate probability integral in nuclear reaction rate theory in the nonresonant case, Tsallis statistics in nonextensive statistical mechanics, superstatistics in astrophysics, generalized type-2, type-1 beta, and gamma families of densities and the density of a product of two real positive random variables in statistical literature, Krätzel integrals in applied analysis, inverse Gaussian distribution in stochastic processes, and the like. Special cases include a wide range of functions appearing in different disciplines.

Observe that $f_1(x)$ and $f_2(x)$ in (4.50), multiplied by the appropriate normalizing constants, can produce statistical densities. Further, $f_1(x)$ and $f_2(x)$ are defined for $-\infty < \alpha < \infty, -\infty < \beta < \infty$. When $\alpha > 1$ and $z_1 > 0, \delta > 0$, $f_1(x)$ multiplied by the normalizing constant stays in the generalized type-2 beta family. When $\alpha < 1$, writing $\alpha - 1 = -(1-\alpha), \alpha < 1$, the function $f_1(x)$ switches into a generalized type-1 beta family and when $\alpha \to 1$,

$$\lim_{\alpha\to 1} f_1(x) = \mathrm{e}^{-z_1^\delta x^\delta}, \tag{4.57}$$

and hence $f_1(x)$ goes into a generalized gamma family. Similar is the behavior of $f_2(x)$ when β ranges from $-\infty$ to ∞. Thus, the parameters α and β create pathways to switch into different functional forms or different families of functions. Hence, we will call α and β pathway parameters in this case. Let us look into some interesting special cases. Take the special case $\beta \to 1$,

$$f_1(z_2|z_1) = \int_0^\infty x^{\gamma-1}[1 + z_1^\delta(\alpha-1)x^\delta]^{-\frac{1}{\alpha-1}}\mathrm{e}^{-z_2^\rho x^{-\rho}}\mathrm{d}x \tag{4.58}$$

$\alpha > 1, z_1 > 0, z_2 > 0, \delta > 0, \rho > 0$. Put $y = \frac{1}{x}$

$$f_1(z_1|z_2) = \int_0^\infty y^{-\gamma-1}[1 + z_1^\delta(\alpha-1)y^{-\delta}]^{-\frac{1}{\alpha-1}}\mathrm{e}^{-z_2^\rho y^{\rho}}\mathrm{d}y \tag{4.59}$$

$\alpha > 1, z_1 > 0, z_2 > 0, \delta > 0, \rho > 0$. Let $\alpha \to 1$ in (1)

$$f_2(z_2|z_1) = \int_0^\infty x^{\gamma-1}\mathrm{e}^{-z_1^\delta x^\delta}[1 + z_2^\rho(\beta-1)x^{-\rho}]^{-\frac{1}{\beta-1}}\mathrm{d}x \tag{4.60}$$

$\beta > 1, z_1 > 0, z_2 > 0, \delta > 0, \rho > 0$.

$$f_2(z_1|z_2) = \int_0^\infty x^{-\gamma-1}\mathrm{e}^{-z_1^\delta x^{-\delta}}[1 + z_2^\rho(\beta-1)x^{\rho}]^{-\frac{1}{\beta-1}}\mathrm{d}x \tag{4.61}$$

$\beta > 1, z_1 > 0, z_2 > 0, \delta > 0, \rho > 0$. Take $\alpha \to 1, \beta \to 1$ in (1)

$$f_3(z_2|z_1) = \int_0^\infty x^{\gamma-1}\mathrm{e}^{-z_1^\delta x^\delta - z_2^\rho x^{-\rho}}\mathrm{d}x \tag{4.62}$$

$z_1 > 0, z_2 > 0, \delta > 0, \rho > 0$.

$$f_3(z_1|z_2) = \int_0^\infty x^{-\gamma-1}\mathrm{e}^{-z_1^\delta x^{-\delta} - z_2^\rho x^{\rho}}\mathrm{d}x \tag{4.63}$$

$z_1 > 0, z_2 > 0, \delta > 0, \rho > 0$.

4.4.1 Case of $\alpha < 1$ or $\beta < 1$

When $\alpha < 1$, writing $\alpha - 1 = -(1-\alpha)$, we can define the function

$$g_1(x) = x^\gamma[1 + z_1^\delta(\alpha-1)x^\delta]^{-\frac{1}{\alpha-1}} = x^\gamma[1 - z_1^\delta(1-\alpha)x^\delta]^{\frac{1}{1-\alpha}}, \alpha < 1, \tag{4.64}$$

for $[1-z_1^{\delta}(1-\alpha)x^{\delta}]>0, \alpha<1 \Rightarrow x<\frac{1}{z_1(1-\alpha)^{\frac{1}{\delta}}}$ and $g_1(x)=0$ elsewhere. In this case, the Mellin transform of $g_1(x)$ is the following:

$$h_1(s)=\int_0^{\infty} x^{s-1} g_1(x)\mathrm{d}x=\int_0^{\frac{1}{z_1(1-\alpha)^{\frac{1}{\delta}}}} x^{\gamma+s-1}[1-z_1^{\delta}(1-\alpha)x^{\delta}]^{\frac{1}{1-\alpha}}\mathrm{d}x \tag{4.65}$$

$$=\frac{1}{\delta[z_1(1-\alpha)^{\frac{1}{\delta}}]^{\gamma+s}}\frac{\Gamma\left(\frac{\gamma+s}{\delta}\right)\Gamma\left(\frac{1}{1-\alpha}+1\right)}{\Gamma\left(\frac{1}{1-\alpha}+1+\frac{\gamma+s}{\delta}\right)}, \Re(\gamma+s)>0, \alpha<1, \delta>0. \tag{4.66}$$

Then, the Mellin transform of $f(z_2|z_1)$ for $\alpha<1, \beta>1$ is given by

$$M_{z_2|z_1}(s)=\frac{\Gamma\left(\frac{1}{1-\alpha}+1\right)}{\delta\rho z_2^s z_1^{\gamma+s}(\beta-1)^{\frac{s}{\rho}}(1-\alpha)^{\frac{\gamma+s}{\delta}}}\frac{\Gamma\left(\frac{\gamma+s}{\delta}\right)}{\Gamma\left(\frac{\gamma+s}{\delta}+\frac{1}{1-\alpha}+1\right)}\frac{\Gamma\left(\frac{s}{\rho}\right)\Gamma\left(\frac{1}{\beta-1}-\frac{s}{\rho}\right)}{\Gamma\left(\frac{1}{\beta-1}\right)}, \tag{4.67}$$

$$\Re(\gamma+s)>0, \Re(s)>0, \Re\left(\frac{1}{\beta-1}-\frac{s}{\rho}\right)>0.$$

Hence, the inverse Mellin transform for $\alpha<1, \beta>1$ is given in

Theorem 4.2. *For* $\alpha<1, \beta>1$

$$f(z_2|z_1)=\frac{\Gamma\left(\frac{1}{1-\alpha}+1\right)}{\delta\rho z_1^{\gamma}(1-\alpha)^{\frac{\gamma}{\delta}}\Gamma\left(\frac{1}{\beta-1}\right)} \times H_{2,2}^{2,1}\left[z_1z_2(1-\alpha)^{\frac{1}{\delta}}(\beta-1)^{\frac{1}{\rho}}\Big|_{\left(0,\frac{1}{\rho}\right),\left(\frac{\gamma}{\delta},\frac{1}{\delta}\right)}^{\left(1-\frac{1}{\beta-1},\frac{1}{\rho}\right),\left(1+\frac{1}{1-\alpha}+\frac{\gamma}{\delta},\frac{1}{\delta}\right)}\right], \tag{4.68}$$

$$\lim_{\beta\to 1} f(z_2|z_1)=\frac{\Gamma\left(\frac{1}{1-\alpha}+1\right)}{\rho\delta z_1^{\gamma}(1-\alpha)^{\frac{\gamma}{\delta}}}H_{1,2}^{2,0}\left[z_1z_2(1-\alpha)^{\frac{1}{\delta}}\Big|_{\left(0,\frac{1}{\delta}\right),\left(\frac{\gamma}{\delta},\frac{1}{\delta}\right)}^{\left(1+\frac{1}{1-\alpha}+\frac{\gamma}{\delta},\frac{1}{\delta}\right)}\right], \tag{4.69}$$

$$\lim_{\alpha\to 1} f(z_2|z_1)=\frac{1}{\rho\delta\Gamma\left(\frac{1}{\beta-1}\right)z_1^{\gamma}}H_{1,2}^{2,1}\left[z_1z_2(\beta-1)^{\frac{1}{\rho}}\Big|_{\left(0,\frac{1}{\rho}\right),\left(\frac{\gamma}{\delta},\frac{1}{\delta}\right)}^{\left(1-\frac{1}{\beta-1},\frac{1}{\rho}\right)}\right], \tag{4.70}$$

$$\lim_{\alpha\to 1,\beta\to 1} f(z_2|z_1)=\frac{1}{\rho\delta z_1^{\gamma}}H_{0,2}^{2,0}\left[z_1z_2\Big|_{\left(0,\frac{1}{\rho}\right),\left(\frac{\gamma}{\delta},\frac{1}{\delta}\right)}\right]. \tag{4.71}$$

In $f(z_2|z_1)$, if $\beta<1$, we may write $\beta-1=-(1-\beta)$, and if we assume $[1-z_2^{\rho}(1-\beta)x^{-\rho}]^{\frac{1}{1-\beta}}>0 \Rightarrow x>z_2(1-\beta)^{\frac{1}{\rho}}$, then the corresponding integrals can also be evaluated as H-functions. But, if $\alpha<1$ and $\beta<1$, then from the conditions

$$1-z_1^{\delta}(1-\alpha)x^{\delta} > 0 \Rightarrow x < \frac{1}{z_1(1-\alpha)^{\frac{1}{\delta}}} \text{ and } 1-z_2^{\rho}(1-\beta)x^{-\rho} > 0 \Rightarrow x > z_2(1-\beta)^{\frac{1}{\rho}}$$

the resulting integral may be zero. Hence, except this case of $\alpha < 1$ and $\beta < 1$, all other cases of $\alpha > 1, \beta > 1; \alpha < 1, \beta > 1; \alpha > 1, \beta < 1$ can be given meaningful interpretations as H-functions. Further, all these situations can be connected to practical problems. A few such practical situations will be considered next.

Remark 4.4. In the integrals in (4.48), (4.58)–(4.63), the exponents of x are taken as $(\delta, -\rho)$ or $(-\delta, \rho)$ with $\delta > 0, \rho > 0$. The cases where the exponents of x are (δ, ρ), $(-\delta, -\rho)$ with $\delta > 0, \rho > 0$ are not considered so far. But, these cases can be done by using the convolution property

$$g(z_1) = \int_0^{\infty} x f_1(z_1 x) f_2(x) \mathrm{d}x. \tag{4.72}$$

Remark 4.5. The convolution integrals in (4.49) and (4.72) can be interpreted easily in terms of independently distributed real scalar positive random variables when f_1 and f_2 are densities. Let x_1 and x_2 be statistically independently distributed real scalar positive random variables with densities $f_1(x_1)$ and $f_2(x_2)$ respectively. Let $u = x_1 x_2$ and $v = \frac{x_1}{x_2}$. Then, the densities of u and v are respectively given by

$$g_u(u) = \int_x \frac{1}{x} f_1(x) f_2\left(\frac{u}{x}\right) \mathrm{d}x \tag{4.73}$$

and

$$g_v(v) = \int_x x f_1(vx) f_2(x) \mathrm{d}x. \tag{4.74}$$

These are the two convolution formulae in (4.49) and (4.72), respectively. The densities $g_u(u)$ and $g_v(v)$ are available from the inverse Mellin transforms also. That is, whenever the Mellin transforms exist and invertible,

$$E(u^{s-1}) = E(x_1^{s-1})E(x_2^{s-1}) = h_1(s), \text{ say} \tag{4.75}$$

$$E(v^{s-1}) = E(x_1^{s-1})E(x_2^{1-s}) = h_2(s), \text{ say.} \tag{4.76}$$

Then

$$g_u(u) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} h_1(s) u^{-s} \mathrm{d}s, \tag{4.77}$$

and

$$g_v(v) = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} h_2(s) v^{-s} \mathrm{d}s. \tag{4.78}$$

4.4.2 Some Practical Situations

(a). Krätzel Integral

For $\delta = 1, z_2^{\rho} = z, z_1 = 1$ in $f_3(z_2|z_1)$ gives the Krätzel integral

$$f_3(z_2|z_1) = \int_0^{\infty} x^{\gamma-1} \mathrm{e}^{-x-zx^{-\rho}} \mathrm{d}x, \tag{4.79}$$

which was studied in detail by Krätzel (1979). Hence, f_3 can be considered as generalization of Krätzel integral. An additional property that can be seen from Krätzel integral as f_3 is that it can be written as a H-function of the type $H_{0,2}^{2,0}(\cdot)$. Hence all the properties of H-function can now be made use of to study this integral further.

(b). Inverse Gaussian Density in Statistics

Inverse Gaussian density is a popular density, which is used in many disciplines including stochastic processes. One form of the density is the following (Mathai 1993c, page 33):

$$f(x) = c\, x^{-\frac{3}{2}} \mathrm{e}^{-\frac{\lambda}{2}\left(\frac{x}{\mu^2}+\frac{1}{x}\right)}, \mu \neq 0, x > 0, \lambda > 0, \tag{4.80}$$

where $c = \pi^{-\frac{1}{2}} \mathrm{e}^{\frac{\lambda}{|\mu|}}$. Comparing this with our case $f_3(z_1|z_2)$, we see that the inverse Gaussian density is the integrand in $f_3(z_1|z_2)$ for $\gamma = \frac{1}{2}, \rho = 1, z_2 = \frac{\lambda}{2}\left(\frac{1}{\mu^2}\right)$, $\delta = 1$, $z_1 = \frac{\lambda}{2}$. Hence, f_3 can be used directly to evaluate the moments or Mellin transform in inverse Gaussian density.

(c). Reaction Rate Probability Integral in Astrophysics

In a series of papers Haubold and Mathai studied modifications of Maxwell–Boltzmann theory of reaction rates, a summary is given in Mathai and Haubold (1988). The basic reaction rate probability integral that appears there is the following:

$$I_1 = \int_0^{\infty} x^{\gamma-1} \mathrm{e}^{-ax-zx^{-\frac{1}{2}}} \mathrm{d}x. \tag{4.81}$$

This is the case in the nonresonant case of nuclear reactions. Compare integral I_1 with $f_3(z_2|z_1)$. The reaction rate probability integral I_1 is $f_3(z_2|z_1)$ for $\delta = 1, \rho = \frac{1}{2}, z_2^{\frac{1}{2}} = z$. The basic integral I_1 is generalized in many different forms for various situations of resonant and nonresonant cases of reactions, depletion of high energy tail, cut off of the high energy tail, and so on. Dozens of published papers are there in this area.

(d). Tsallis Statistics and Superstatistics

It is estimated that on Tsallis statistics in nonextensive statistical mechanics, over 1200 papers were published during the period 1990 to 2007. Tsallis statistics is of the following form:

$$f_x(x) = c_1[1 + (\alpha - 1)x]^{-\frac{1}{\alpha-1}}. \tag{4.82}$$

Compare $f_x(x)$ with the integrand in (1). For $z_2 = 0, \delta = 1,\ and\,\gamma = 1$, the integrand in (4.48) agrees with Tsallis statistics $f_x(x)$ given above. The three different forms of Tsallis statistics are available from $f_x(x)$ for $\alpha > 1, \alpha < 1,\ and\,\alpha \to 1$. The starting paper in nonextensive statistical mechanics may be seen from Tsallis (1988). But, the integrand in (4.48) with $z_2 = 0, z_1 = 1, \alpha > 1$ is the superstatistics of Beck and Cohen, see for example Beck and Cohen (2003), Beck (2006). In statistical language, this superstatistics is the unconditional density in a generalized gamma case when the scale parameter has a prior density belonging to the same class of generalized gamma density.

(e). Pathway Model

Mathai (2005) considered a rectangular matrix-variate function in the real case from where one can obtain almost all matrix-variate densities in current use in statistical and other disciplines. The corresponding version, when the elements are in the complex domain, is given in Mathai and Provost (2006). For the real scalar case, the function is of the following form:

$$f(x) = c^*|x|^\gamma[1 - a(1-\alpha)|x|^\delta]^{\frac{\eta}{1-\alpha}}, \tag{4.83}$$

for $-\infty < x < \infty, a > 0, \eta > 0, \delta > 0$, and c^* is the normalizing constant. Here, $f(x)$ for $\alpha < 1$ stays in the generalized type-1 beta family when $[1 - a(1-\alpha)|x|^\delta]^{\frac{\eta}{1-\alpha}} > 0$. When $\alpha > 1$, the function switches into a generalized type-2 beta family and when $\alpha \to 1$, it goes into a generalized gamma family of functions. Here α behaves as a pathway parameter, and hence the model is called a pathway model. Observe that the integrand in (4.48) is a product of two such pathway functions so that the corresponding integral is more versatile than a pathway model. Thus, for $z_2 = 0$ in (4.48), the integrand produces the pathway model of Mathai (2005).

Exercises 4.3

4.3.1. By normalizing the integrals in (4.58) to (4.63), create statistical densities corresponding to the integrands in the six equations.

4.3.2. Evaluate the hth moments for the six densities in Exercise 4.3.1.

4.3.3. Write down the hth moments in Exercise 4.3.2 for $h = 1, 2$ and compute the variances of the corresponding random variables.

4.3.4. Using Stirling's approximation on the gammas in (4.67), derive the corresponding Mellin–Barnes representations in (4.69)–(4.71).

4.3.5. Evaluate the series form in (4.71) for $\frac{1}{\rho} = 2, \frac{1}{\delta} = 3$.

Chapter 5
Functions of Matrix Argument

5.1 Introduction

Particular cases of a H-function with matrix argument are available for real as well as for complex matrices. For the general H-function only a class of functions is available analogous to the scalar variable H-function. Real-valued scalar functions of matrix argument is developed when the argument matrix is a real symmetric positive definite matrix or for hermitian positive definite matrices. We consider only real matrices here.

We will use the standard notations to denote matrices. The transpose of a matrix $X = (x_{ij})$ will be denoted by X' and trace of X by $\mathrm{tr}(X) =$ sum of the eigenvalues $=$ sum of the leading diagonal elements. Determinant of X will be denoted by $|X|$, a null matrix by a big O and an identity matrix by $I = I_n$. A diagonal matrix will be written as $\mathrm{diag}(\lambda_1, \ldots, \lambda_p)$ where $\lambda_1, \ldots, \lambda_p$ are the diagonal elements. $X > 0$ will mean the real symmetric matrix $X = X'$ is positive definite. Definiteness is defined only for symmetric matrices when real and hermitian matrices when complex, $X \geq 0$ (positive semidefinite), $X < 0$ (negative definite), $X \leq 0$ (negative semidefinite). A matrix which does not fall in the categories $X > 0, X \geq 0, X < 0, X \leq 0$ is called indefinite. $\int_X f(X)\mathrm{d}X$ means the integral over X. $\int_A^B f(X)\mathrm{d}X$ means the integral over $0 < A < X < B$, that is, $X = X' > 0, A = A' > 0, B = B' > 0, X - A > 0, B - X > 0$ and the integral is taken over all such X.

It is difficult to develop the theory of a real-valued scalar function of a general matrix X. Even for a square matrix A rational powers will create problems. For example even for an identity matrix, even a simple item such as a square root will create difficulties. If the existence of a matrix B such that $B^2 = A$ is taken as the square root of A then consider

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},\ A_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ A_4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We have then

$$A_1^2 = I_2,\ A_2^2 = I_2,\ A_3^2 = I_2,\ A_4^2 = I_2.$$

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Thus A_1, A_2, A_3, A_4 are all candidates for the square root of a nice matrix like an identity matrix. But if we confine our discussion to the class of positive definite matrices, when real, and hermitian positive definite matrices, when complex, then A_1 is the only candidate for the square root of I_2. In this class of positive definiteness, several items can be defined uniquely. Hence the theory is developed when the matrices are positive definite when real.

5.2 Exponential Function of Matrix Argument

Hypergeometric functions, in the scalar case, are special cases of a H-function. For example

$$ {}_0F_0(\,;\,;\pm x) = \mathrm{e}^{\pm x}, \tag{5.1}$$

when x is scalar. The corresponding function of matrix argument is

$$ {}_0F_0(\,;\,;\pm X) = \mathrm{e}^{\pm \mathrm{tr}(X)}, \tag{5.2}$$

where X is a $p \times p$ positive definite matrix. For any type of integral operations on (5.2) we need to define differential elements and wedge product of differentials.

Definition 5.1. Wedge product of differentials. Wedge product or skew symmetric product of differential elements $\mathrm{d}x$ and $\mathrm{d}y$ will be denoted by $\mathrm{d}x \wedge \mathrm{d}y$, where $\wedge$ = wedge, and will be defined by the relation

$$\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x. \tag{5.3}$$

That is, if the order is changed then it is to be multiplied by (-1). This will then imply that

$$\mathrm{d}x \wedge \mathrm{d}x = 0,\ \mathrm{d}x \wedge \mathrm{d}x \wedge \mathrm{d}x = 0,\ \mathrm{d}y \wedge \mathrm{d}y = 0,$$

and so on. An interesting consequence is there when products of differentials are taken. If X is a $p \times q$ matrix, $X = (x_{ij})$ then the wedge product of differentials is the following:

Notation 5.1.

$$\mathrm{d}X = \mathrm{d}x_{11} \wedge \mathrm{d}x_{12} \wedge \cdots \wedge \mathrm{d}x_{1q} \wedge \mathrm{d}x_{21} \wedge \cdots \wedge \mathrm{d}x_{pq}. \tag{5.4}$$

If $X = X'$ and $p \times p$ then there are only $\frac{p(p+1)}{2}$ free elements in X because $x_{ij} = x_{ji}$ for all i and j, and then

$$\mathrm{d}X = \mathrm{d}x_{11} \wedge \cdots \wedge \mathrm{d}x_{1p} \wedge \mathrm{d}x_{22} \wedge \cdots \wedge \mathrm{d}x_{2p} \wedge \cdots \wedge \mathrm{d}x_{pp}. \tag{5.5}$$

Thus

$$\int_X f(X)\mathrm{d}X = \int_{X=X'>0} f(X)\mathrm{d}X = \int_{X>0} f(X)\mathrm{d}X$$

will mean that the integral is taken over all $X > 0$.

$$\int_{O<X<I} f(X)\mathrm{d}X = \int_0^I f(X)\mathrm{d}X$$

will mean the integral over all $X = X' > 0$ such that $I - X > 0$. Now we are in a position to define an integral analogous to a gamma integral in the scalar case. Consider

$$\Gamma_p(\alpha) = \int_{X=X'>0} |X|^{\alpha-\frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(X)}\mathrm{d}X, \tag{5.6}$$

where X is $p \times p$ real symmetric and positive definite. For $p = 1$, (5.6) corresponds to the gamma integral. How can we evaluate (5.6)? This requires some matrix transformations and the associated Jacobians. For simplicity let us look at functions of two scalar variables x_1 and x_2. Let

$$y_1 = f_1(x_1, x_2) \text{ and } y_2 = f_2(x_1, x_2).$$

Then from basic calculus

$$\mathrm{d}y_1 = \frac{\partial f_1}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f_1}{\partial x_2}\mathrm{d}x_2 \text{ and } \mathrm{d}y_2 = \frac{\partial f_2}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f_2}{\partial x_2}\mathrm{d}x_2.$$

Now if we take the wedge product of the differentials we have

$$\begin{aligned}\mathrm{d}y_1 \wedge \mathrm{d}y_2 &= \left[\frac{\partial f_1}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f_1}{\partial x_2}\mathrm{d}x_2\right] \wedge \left[\frac{\partial f_2}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f_2}{\partial x_2}\mathrm{d}x_2\right]\\ &= \left[\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1}\right]\mathrm{d}x_1 \wedge \mathrm{d}x_2 + 0 + 0,\end{aligned}$$

by using the results $\mathrm{d}x_1 \wedge \mathrm{d}x_1 = 0$ and $\mathrm{d}x_2 \wedge \mathrm{d}x_1 = -\mathrm{d}x_1 \wedge \mathrm{d}x_2$. Then

$$\begin{aligned}\mathrm{d}y_1 \wedge \mathrm{d}y_2 &= \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{vmatrix} \mathrm{d}x_1 \wedge \mathrm{d}x_2\\ &\Rightarrow \mathrm{d}Y = J\,\mathrm{d}X,\end{aligned} \tag{5.7}$$

where $\mathrm{d}Y = \mathrm{d}y_1 \wedge \mathrm{d}y_2$, $\mathrm{d}X = \mathrm{d}x_1 \wedge \mathrm{d}x_2$ and J is the Jacobian or the determinant of the matrix of partial derivatives. In general, if we have a transformation of $x_1, \ldots, x_p$ going to $y_1, \ldots, y_p$ then

$$\mathrm{d}Y = \mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \cdots \wedge \mathrm{d}y_p = J\,\mathrm{d}X,\ \mathrm{d}X = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_p,$$

and

$$J = \left|\left(\frac{\partial y_i}{\partial x_j}\right)\right|, \tag{5.8}$$

where the (i, j)th element in the matrix is the partial derivative of y_i with respect to x_j.

Example 5.1. Evaluate the Jacobian in the linear transformation $Y = AX$ where X is $p \times 1$, Y is $p \times 1$, A is $p \times p$ nonsingular constant matrix and X is of distinct real scalar variables. Verify the result for

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}.$$

Solution 5.1. When $|A| \neq 0$ the transformation is unique or one-to-one. $Y = AX \Rightarrow X = A^{-1}Y$ where A^{-1} is the unique inverse of A. The transformation is of the form

$$\begin{aligned} y_1 &= a_{11}x_1 + \cdots + a_{1p}x_p \\ &\vdots \\ y_p &= a_{p1}x_1 + \cdots + a_{pp}x_p \end{aligned} \quad \Rightarrow \frac{\partial y_i}{\partial x_j} = a_{ij}$$

$$\Rightarrow \frac{\partial Y}{\partial X} = A \Rightarrow J = |A|.$$

That is,

$$\mathrm{d}Y = |A|\mathrm{d}X \text{ or } \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_p = |A|\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_p.$$

When

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}, \ |A| = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 1.$$

Hence, in this case,

$$\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_p = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_p.$$

This may be stated as a theorem.

Theorem 5.1.

$$Y = AX \Rightarrow \mathrm{d}Y = |A|\mathrm{d}X, \tag{5.9}$$

where X and Y are $p \times 1, |A| \neq 0, X$ is of distinct real scalar variables.

If X is a $p \times q$ matrix of distinct real scalar variables then the wedge product of the differentials in X, denoted by $\mathrm{d}X$, is given by

$$\mathrm{d}X = \mathrm{d}x_{11} \wedge \mathrm{d}x_{12} \wedge \cdots \wedge \mathrm{d}x_{pq}. \tag{5.10}$$

5.3 Jacobians of Matrix Transformations

We considered one linear transformation involving a vector of variables X going to a vector of variables Y through a nonsingular linear transformation $Y = AX, |A| \neq 0$ and we found the Jacobian to be $|A|$. Now we consider a few more elaborate linear transformations and some nonlinear transformations.

Let X be a $m \times n$ matrix of distinct real scalar variables and let A be a $m \times m$ nonsingular matrix of constants. Consider the transformation $Y = AX$. Let $X^{(1)}, \ldots, X^{(n)}$ be the columns of X. Then

$$\begin{aligned} Y &= AX = A(X^{(1)}, \ldots, X^{(n)}) = (AX^{(1)}, \ldots, AX^{(n)}) \\ &= (Y^{(1)}, \ldots, Y^{(n)}), \end{aligned}$$

where $Y^{(1)}, \ldots, Y^{(n)}$ are the columns of Y. Then we can look at the transformation as

$$\begin{bmatrix} Y^{(1)} \\ \vdots \\ Y^{(n)} \end{bmatrix} = \begin{bmatrix} AX^{(1)} \\ \vdots \\ AX^{(n)} \end{bmatrix}.$$

Then from Theorem 5.1,

$$\frac{\partial Y^{(i)}}{\partial X^{(i)}} = A,\ i = 1, \ldots, n,\ \frac{\partial Y^{(i)}}{\partial X^{(j)}} = O, i \neq j.$$

The matrix of partial derivatives is of the following form:

$$\begin{bmatrix} A & O & \cdots & O \\ O & A & \cdots & O \\ \vdots & \vdots & \ldots & \vdots \\ O & O & \cdots & A \end{bmatrix} \Rightarrow \begin{vmatrix} A & O & \cdots & O \\ \vdots & \vdots & \ldots & \vdots \\ O & O & \cdots & A \end{vmatrix} = |A|^n.$$

Hence we have the following theorem:

Theorem 5.2. *Let X be a $m \times n$ matrix of distinct real scalar variables or functionally independent real variables. Let A be a $m \times m$ nonsingular constant matrix. Then*

$$Y = AX \Rightarrow \mathrm{d}Y = |A|^n \mathrm{d}X. \tag{5.11}$$

In Theorem 5.2 we had a premultiplication of X by a constant nonsingular matrix A. Now let us consider a postmultiplication. Let B be a $n \times n$ nonsingular constant matrix. Then what will be the Jacobian in the transformation $Y = XB$? This can be computed exactly the same way by considering the rows of X. Let $X_{(1)}, \ldots, X_{(m)}$ be the m rows of X. Then

$$Y = XB = \begin{bmatrix} X_{(1)} \\ \vdots \\ X_{(m)} \end{bmatrix} B = \begin{bmatrix} X_{(1)}B \\ \vdots \\ X_{(m)}B \end{bmatrix} = \begin{bmatrix} Y_{(1)} \\ \vdots \\ Y_{(m)} \end{bmatrix},$$

where $Y_{(1)}, \ldots, Y_{(m)}$ are the m rows of Y. Now we can look at the long string

$$\begin{bmatrix} Y'_{(1)} \\ \vdots \\ Y'_{(m)} \end{bmatrix} = \begin{bmatrix} B'X'_{(1)} \\ \vdots \\ B'X'_{(m)} \end{bmatrix},$$

and apply Theorem 5.1. The matrix of partial derivatives will be

$$\begin{bmatrix} B' & O & \cdots & O \\ \vdots & \vdots & \ldots & \vdots \\ O & O & \cdots & B' \end{bmatrix} \Rightarrow \begin{vmatrix} B' & O & \cdots & O \\ \vdots & \vdots & \ldots & \vdots \\ O & O & \cdots & B' \end{vmatrix} = |B'|^m = |B|^m.$$

Hence we have the following result:

Theorem 5.3. *Let X be as in Theorem 5.2 and let B be a nonsingular $n \times n$ constant matrix. Then*

$$Y = XB \Rightarrow \mathrm{d}Y = |B|^m \mathrm{d}X. \tag{5.12}$$

Combining Theorems 5.2 and 5.3 we have the following result:

Theorem 5.4. *Let X, A, B be as in Theorems 5.2 and 5.3. Then*

$$Y = AXB \Rightarrow \mathrm{d}Y = |A|^n |B|^m \mathrm{d}X. \tag{5.13}$$

Example 5.2. Let X be a $m \times n$ matrix of functionally independent real variables. Let M, A, B be constant matrices where M is $m \times n$, A is $m \times m$, B is $n \times n$ with $|A| \neq 0, |B| \neq 0$ and further, let $A = A' > 0, B = B' > 0$ (positive definite matrices). Consider the function

$$f(X) = c\, \mathrm{e}^{-\mathrm{tr}[A(X-M)B(X-M)']}, \tag{5.14}$$

where f is a real-valued scalar function of X, c is a scalar constant and $\mathrm{tr}(\cdot)$ denotes the trace of $(\cdot)$. Evaluate $\int_X f(X)\mathrm{d}X$.

Solution 5.2. We wish to evaluate the total integral of $f(X)$ over all such $m \times n$ matrices X. From the theory of matrices we know that a positive definite matrix A (definiteness is defined only for symmetric matrices when real and hermitian matrices when complex) can be written as $A = A_1 A_1'$ with $|A_1| \neq 0$ where $A_1' =$ transpose of A_1. We also know that for any two matrices P and Q,

$$\mathrm{tr}(PQ) = \mathrm{tr}(QP),$$

whenever PQ and QP are defined, where PQ need not be equal to QP. By using these two results we can write

$$\begin{aligned}
\mathrm{tr}[A(X-M)B(X-M)'] &= \mathrm{tr}[A_1 A_1'(X-M)B_1 B_1'(X-M)'] \\
&= \mathrm{tr}[A_1'(X-M)B_1 B_1'(X-M)'A_1] \\
&= \mathrm{tr}(YY') = \sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij}^2,
\end{aligned}$$

where

$$Y = A_1'(X-M)B_1 \Rightarrow \mathrm{d}Y = |A_1|^n |B_1|^m \mathrm{d}(X-M) = |A|^{\frac{n}{2}}|B|^{\frac{m}{2}}\mathrm{d}X$$

by using Theorem 5.4. Note that

$$|A| = |A_1 A_1'| = |A_1||A_1'| = |A_1|^2 = |A_1'|^2,\ \mathrm{d}(X-M) = \mathrm{d}X,$$

since M is a constant matrix. Also from the theory of matrices we know that for any matrix G, $\mathrm{tr}(GG') =$ sum of squares of all elements in G. Hence

$$\begin{aligned}
\int_X f(X)\mathrm{d}X &= c\int_X \mathrm{e}^{-\mathrm{tr}[A(X-M)B(X-M)']} \\
&= c|A|^{-\frac{n}{2}}|B|^{-\frac{m}{2}}\int_Y \mathrm{e}^{-\mathrm{tr}(YY')}\mathrm{d}Y \\
&= c|A|^{-\frac{n}{2}}|B|^{-\frac{m}{2}}\prod_{i=1}^{m}\prod_{j=1}^{n}\int_{-\infty}^{\infty}\mathrm{e}^{-y_{ij}^2}\mathrm{d}y_{ij} \\
&= c|A|^{-\frac{n}{2}}|B|^{-\frac{m}{2}}\pi^{\frac{mn}{2}},
\end{aligned}$$

since

$$\int_{-\infty}^{\infty}\mathrm{e}^{-t^2}\mathrm{d}t = \sqrt{\pi}.$$

Hence if $f(X)$ is a density function then

$$c = \frac{|A|^{\frac{n}{2}}|B|^{\frac{m}{2}}}{\pi^{\frac{mn}{2}}}, \tag{5.15}$$

then the total integral is 1 and c in that case is called a normalizing constant and with this c, $f(X)$ becomes a density because by definition $f(X) > 0$ for all X, when $c > 0$ since it is an exponential function. The density in (5.14) with c in (5.15) is called a real matrix-variate Gaussian density.

In Theorem 5.4 our matrix X was rectangular. If $m = n$ then X is a square matrix with m^2 real scalar variables. If X is symmetric then there are only $\frac{m(m+1)}{2}$ distinct elements in X because $x_{ij} = x_{ji}$ for all i and j. What will happen to the Jacobian if we have a transformation of the type $Y = AXA', |A| \neq 0, X = X'$? This result will be stated here without proof.

Theorem 5.5. *Let $X = X'$ be $p \times p$ and of functionally independent real variables except for the condition $X = X'$. Let A be a $p \times p$ nonsingular constant matrix. Then*

$$Y = AXA' \Rightarrow \mathrm{d}Y = |A|^{p+1}\mathrm{d}X. \tag{5.16}$$

One way of proving this result is to represent the nonsingular matrix A as a product of basic elementary matrices and then look at the transformations successively. For example, let $A = E_1 E_2 \cdots E_k$ where $E_1, \ldots, E_k$ are some basic elementary matrices. Then

$$Y = AXA' = E_1 \cdots E_k X E_k' \cdots E_1'.$$

Now look at the transformations

$$Y_1 = E_k X E_k',\ Y_2 = E_{k-1} Y_1 E_{k-1}', \ldots, Y_k = E_1 Y_{k-1} E_1',$$

and evaluate the Jacobians successively.

$$\mathrm{d}Y_1 = J_1 \mathrm{d}X,\ \mathrm{d}Y_2 = J_2 \mathrm{d}Y_1 = J_2 J_1 \mathrm{d}X,$$

and so on. For more details on this and for other Jacobians see Mathai (1997).

5.4 Jacobians in Nonlinear Transformations

For a $p \times p$ positive definite matrix X of functionally independent real scalar variables consider the integral

$$\Gamma_p(\alpha) = \int_{X=X'>0} |X|^{\alpha - \frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(X)} \mathrm{d}X. \tag{5.17}$$

For $p = 1$, obviously (5.17) is the integral representation for the gamma function $\Gamma(\alpha)$. Hence $\Gamma_p(\alpha)$ in (5.17) is a matrix-variate version of $\Gamma(\alpha)$. The integral in (5.17) can be evaluated by using a triangular decomposition of X as $X = TT'$ where T is a lower triangular matrix. This transformation $X = TT'$ is not one-to-one. There can be many values for t_{ij}'s for given x_{ij}'s. But if we assume that the diagonal elements in T are positive, that is, $t_{jj} > 0, j = 1,\ldots,p$ then the transformation can be shown to be one-to-one. Take a case of $p = 3$, write X, T, TT' explicitly and verify this fact. Taking the x-variables in the order $x_{11}, x_{12}, \ldots, x_{1p}, x_{22}, \ldots, x_{2p}, \cdots, x_{pp}$ and the t-variables in the order $t_{11}, t_{21}, \ldots, t_{p1}, t_{22}, t_{32}, \ldots, t_{pp}$ we can easily see that the matrix of partial derivatives is of a triangular format with the diagonal elements t_{11} appearing p times, t_{22} appearing $p-1$ times and so on and t_{pp} appearing only once and a 2 appearing a total of p times in the diagonal. Hence we have the following result:

Theorem 5.6. *Let $X = X'$ be a positive definite matrix of functionally independent real scalar variables except for the symmetry condition. Let T be a lower triangular matrix with distinct real elements with the diagonal elements $t_{jj} > 0, j = 1,\ldots,p$. Then*

$$X = TT' \Rightarrow \mathrm{d}X = 2^p \left\{ \prod_{j=1}^{p} t_{jj}^{p+1-j} \right\} \mathrm{d}T.$$

Then by applying this triangular decomposition of X into t_{ij}'s, observing that

$$|X| = |TT'| = \left(\prod_{j=1}^{p} t_{jj}^2 \right),$$

$$\mathrm{tr}(X) = \mathrm{tr}(TT') = t_{11}^2 + \left(t_{21}^2 + t_{22}^2\right) + \cdots + \left(t_{p1}^2 + \cdots + t_{pp}^2\right),$$

and integrating out one has the following result:

$$\Gamma_p(\alpha) = \pi^{\frac{p(p-1)}{4}} \Gamma(\alpha)\Gamma\left(\alpha - \frac{1}{2}\right) \cdots \Gamma\left(\alpha - \frac{p-1}{2}\right), \Re(\alpha) > \frac{p-1}{2}. \quad (5.18)$$

Notation 5.2. $\Gamma_p(\alpha)$: **Real matrix-variate gamma function**.

Definition 5.2. Real matrix-variate gamma function is defined by (5.18).

The equation in (5.17) gives the integral representation for the real matrix-variate gamma function, where $\Re(\cdot)$ means the real part of $(\cdot)$.

In a similar fashion one can define a real matrix-variate beta function. To this end we can start with

$$\Gamma_p(\alpha)\Gamma_p(\beta) = [\int_{X>0} |X|^{\alpha - \frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(X)} \mathrm{d}X][\int_{Y>0} |Y|^{\beta - \frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(Y)} \mathrm{d}Y],$$

where X and Y are $p \times p$ positive definite matrices. Then

$$\Gamma_p(\alpha)\Gamma_p(\beta) = \int_{X>0} \int_{Y>0} |X|^{\alpha - \frac{p+1}{2}} |Y|^{\beta - \frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(X+Y)} \mathrm{d}X \wedge \mathrm{d}Y.$$

Make the transformation $U = X + Y$. Then

$$Y = U - X \Rightarrow |Y| = |U - X| = |U||I - U^{-\frac{1}{2}} X U^{-\frac{1}{2}}|.$$

Then put $Z = U^{-\frac{1}{2}} X U^{-\frac{1}{2}}$ for fixed U and integrate out X to obtain

$$\Gamma_p(\alpha)\Gamma_p(\beta) = \Gamma_p(\alpha + \beta) \int_Z |Z|^{\alpha - \frac{p+1}{2}} |I - Z|^{\beta - \frac{p+1}{2}} \mathrm{d}Z. \tag{5.19}$$

Notation 5.3. $B_p(\alpha, \beta)$: **Real matrix-variate beta function**.

Definition 5.3. $B_p(\alpha, \beta)$ is defined as

$$B_p(\alpha, \beta) = \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha + \beta)} = B_p(\beta, \alpha), \Re(\alpha) > \frac{p-1}{2}, \Re(\beta) > \frac{p-1}{2}. \tag{5.20}$$

One integral representation is given in (5.19). By changing $Z = I - W$ we can have one more representation. That is,

$$B_p(\alpha, \beta) = \int_{O<W<I} |W|^{\beta - \frac{p+1}{2}} |I - W|^{\alpha - \frac{p+1}{2}} \tag{5.21}$$

$$= \int_{O<Z<I} |Z|^{\alpha - \frac{p+1}{2}} |I - Z|^{\beta - \frac{p+1}{2}}. \tag{5.22}$$

These two representations are known as type-1 integral representations for a real matrix-variate beta.

Another nonlinear transformation that we need is about a nonsingular matrix going to its unique inverse. The result will be given here without proof.

Theorem 5.7. *Let X be a $p \times p$ nonsingular matrix of functionally independent real variables. Let $Y = X^{-1}$, the regular inverse of X. Then, ignoring the sign,*

$$\begin{aligned} Y = X^{-1} \Rightarrow \mathrm{d}Y &= |X|^{-2p} \mathrm{d}X \textit{ for a general } X \\ &= |X|^{-(p+1)} \mathrm{d}X \textit{ for } X = X' \\ &= |X|^{-(p-1)} \mathrm{d}X \textit{ for } X = -X'. \end{aligned} \tag{5.23}$$

This can be proved by making the following observations: For any θ,

$$XX^{-1} = I \Rightarrow \frac{\partial}{\partial \theta}(XX^{-1}) = \frac{\partial}{\partial \theta}(I) = O.$$

But

$$\frac{\partial}{\partial \theta}(XX^{-1}) = X[\frac{\partial}{\partial \theta} X^{-1}] + [\frac{\partial}{\partial \theta} X] X^{-1} = O.$$

Hence by taking differentials on both sides the matrices of differentials, denoted by $(\mathrm{d}X)$ and $(\mathrm{d}X^{-1})$ are connected by the relation

$$(\mathrm{d}X)X^{-1} + X(\mathrm{d}X^{-1}) = O \Rightarrow (\mathrm{d}X^{-1}) = -X^{-1}(\mathrm{d}X)X^{-1}. \quad (5.24)$$

Then by taking the wedge product of the differentials on both sides, keeping in mind that X^{-1} does not contain differentials and hence behaves like a constant when taking wedge product of differentials on both sides, we have the result in (5.23).

With the help of Theorem 5.7 one can have other representations for real matrix-variate beta function from the type-1 integral representations in (5.21) and (5.22). For the W or call it X in (5.21) consider the transformations

$$U = (I - X)^{-\frac{1}{2}} X (I - X)^{-\frac{1}{2}} \text{ and } V = U^{-1}.$$

Then the integral representations for $B_p(\alpha, \beta)$ reduce to the following:

$$\begin{aligned} B_p(\alpha, \beta) &= \int_{U=U'>0} |U|^{\alpha - \frac{p+1}{2}} |I + U|^{-(\alpha+\beta)} \mathrm{d}U \\ &= \int_{V=V'>0} |V|^{\beta - \frac{p+1}{2}} |I + V|^{-(\alpha+\beta)} \mathrm{d}V. \end{aligned} \quad (5.25)$$

The representations in (5.25) are called type-2 integral representations for a real matrix-variate beta function.

5.5 The Binomial Function

In the real scalar case, when we take the Laplace transform of a negative exponential function or a gamma function we obtain the binomial function. For example, for the scalar variable $x > 0$ and for the scalar parameter t

$$L_{f_1}(t) = \int_0^\infty \mathrm{e}^{-tx} f_1(x) \mathrm{d}x, \quad (5.26)$$

is the Laplace transform of $f_1(x)$ defined for $x > 0$. If we take the Laplace transform of the gamma type function

$$f_2(x) = \frac{x^{\alpha-1} \mathrm{e}^{-x}}{\Gamma(\alpha)}, x > 0,$$

we have

$$\begin{aligned} L_{f_2}(t) &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \mathrm{e}^{-tx} x^{\alpha-1} \mathrm{e}^{-x} \mathrm{d}x \\ &= (1 + t)^{-\alpha} \text{ for } 1 + t > 0. \end{aligned}$$

This is the binomial function or ${}_1F_0$ hypergeometric function. The Laplace transform in the matrix-variate case, analogous to the multivariate Laplace transform, is defined as

$$L_f(T^*) = \int_{X=X'>0} \frac{|X|^{\alpha-\frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(X)}}{\Gamma_p(\alpha)} \mathrm{e}^{-\mathrm{tr}(T^* X)} \mathrm{d}X, \tag{5.27}$$

where

$$T^* = (t_{ij}^*),\ t_{ij}^* = \frac{1}{2} t_{ij}, i \neq j,\ t_{jj}^* = t_{jj}, t_{ij} = t_{ji},$$

for all $i, j = 1, \ldots, p$. Then

$$L_f(T^*) = |I + T^*|^{-\alpha} = |T^*|^{-\alpha} |I + T^{*-1}|^{-\alpha}, \tag{5.28}$$

for $T^* = T^{*\prime} > 0$ and $I + T^* > 0$. Then the hypergeometric function ${}_1F_0$ with matrix argument U will be defined as

$${}_1F_0(\alpha;\ ; U) = |I - U|^{-\alpha} \text{ for } O < U < I. \tag{5.29}$$

Observe that $O < U < I$ implies that $U = U' > 0, I - U > 0$ which means that the eigenvalues of U are in the open interval $(0, 1)$. We can make one more observation on

$${}_0F_0(\ ;\ ; -X) = \mathrm{e}^{-\mathrm{tr}(X)},$$

and

$${}_1F_0(\alpha;\ ; -X) = |I + X|^{-\alpha},$$

that we obtained so far. Consider the integral of the following type:

$$\int_{X=X'>0} |X|^{\rho-\frac{p+1}{2}} f(X) \mathrm{d}X = \int_{X>0} |X|^{\rho-\frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(X)} \mathrm{d}X = \Gamma_p(\rho), \tag{5.30}$$

and

$$\int_{X>0} |X|^{\rho-\frac{p+1}{2}} |I + X|^{-\alpha} \mathrm{d}X = \frac{\Gamma_p(\rho)\Gamma_p(\alpha-\rho)}{\Gamma_p(\alpha)}, \tag{5.31}$$

for $\Re(\rho) > \frac{p-1}{2}$, $\Re(\alpha - \rho) > \frac{p-1}{2}$. The integral in (5.31) is evaluated by using the type-2 integral representation for a beta function in (5.25).

Notation 5.4. $M_f(\rho)$: M-transform of f.

Definition 5.4. The generalized matrix transform or M-transform of a real-valued scalar function of the real $p \times p$ matrix $X = X > 0$ is defined as

$$M_f(\rho) = \int_{X=X'>0} |X|^{\rho-\frac{p+1}{2}} f(X) \mathrm{d}X, \tag{5.32}$$

whenever $M_f(\rho)$ exists, where f is a symmetric function in the sense $f(AB) = f(BA)$ for all matrices A and B where AB and BA are defined.

Thus a class of functions f will have the M-transform $M_f(\rho)$ for the arbitrary parameter ρ. For example, when f is the ${}_0F_0$ or ${}_1F_0$ we have the M-transforms given in (5.30) and (5.31).

5.6 Hypergeometric Function and M-transforms

Notation 5.5. ${}_rF_s(a_1,\ldots,a_r;b_1,\ldots b_s;-X)$: **Hypergeometric function of matrix argument** $-X$.

Definition 5.5. A hypergeometric function of matrix argument $-X$ with r upper and s lower parameters is defined as the class of symmetric functions f having the following M-transform:

$$M_f(\rho) = \frac{\left\{\prod_{j=1}^{s}\Gamma_p(b_j)\right\}}{\left\{\prod_{j=1}^{r}\Gamma_p(a_j)\right\}}\Gamma_p(\rho)\frac{\left\{\prod_{j=1}^{r}\Gamma_p(a_j-\rho)\right\}}{\left\{\prod_{j=1}^{s}\Gamma_p(b_j-\rho)\right\}} \tag{5.33}$$

whenever the gammas on the right exist, where ρ is a parameter, and $a_1,\ldots,a_r$ and $b_1,\ldots,b_s$ are the upper and lower parameters of the hypergeometric function, which will be written as

$$f = {}_rF_s(a_1,\ldots,a_r;b_1,\ldots,b_s;-X).$$

In (5.33) it is assumed that f is a symmetric function in the sense $f(AB) = f(BA)$ for all A and B whenever AB and BA are defined. An implication of this condition is the following: Let Q be an orthonormal matrix such that $QQ' = I = Q'Q$ and $Q'XQ = \text{diag}(\lambda_1,\ldots,\lambda_p)$ where $\lambda_1,\ldots,\lambda_p$ are the eigenvalues of X, where it is assumed that the eigenvalues are distinct, then

$$\begin{aligned} f(X) &= f(XI) = f(XQQ') = f(Q'XQ) \\ &= f(D),\ D = \text{diag}(\lambda_1,\ldots,\lambda_p), \end{aligned} \tag{5.34}$$

or $f(X)$ becomes a function of the p eigenvalues only. Thus, under the condition of symmetry on $f(X)$, this function of the $\frac{p(p+1)}{2}$ real scalar variables in X becomes a function of p variables, namely the p eigenvalues of X, which by assumption are real, distinct and positive.

There are other definitions for a hypergeometric function of matrix argument. All definitions have the basic assumption that the function is symmetric in the above sense. One definition based on the Laplace and inverse Laplace pair gives

${}_rF_s(a_1,\ldots,a_r;b_1,\ldots,b_s\ :\ X)$ as that function satisfying the following pair of integral equations:

$$
\begin{aligned}
&{}_{r+1}F_s(a_1,\ldots,a_r,c;b_1,\ldots,b_s;-\Lambda^{-1})|\Lambda|^{-c}\\
&=\frac{1}{\Gamma_p(c)}\int_{U=U'>0}\mathrm{e}^{-\mathrm{tr}(\Lambda U)}{}_rF_s(a_1,\ldots,a_r;b_1,\ldots,b_s;-U)|U|^{c-\frac{p+1}{2}}\mathrm{d}U
\end{aligned}
\tag{5.35}
$$

$$
\begin{aligned}
&{}_rF_{s+1}(a_1,\ldots,a_r;b_1,\ldots,b_s,c;-\Lambda)|\Lambda|^{c-\frac{p+1}{2}}\\
&=\frac{\Gamma_p(c)}{(2\pi i)^{\frac{p(p+1)}{2}}}\int_{\Re(Z)=Z_0>0}\mathrm{e}^{\mathrm{tr}(\Lambda Z)}{}_rF_s(a_1,\ldots,a_r;b_1,\ldots,b_s;-Z^{-1})|Z|^{-c}\mathrm{d}Z.
\end{aligned}
\tag{5.36}
$$

Under certain conditions the function ${}_rF_s$ defined through (5.35) and (5.36) can be shown to be unique. From this definition also the explicit forms are available only for ${}_0F_0$ and ${}_1F_0$. Others will remain as the solutions of a pair of integral equations.

The third definition available is in terms of zonal polynomials, which are certain symmetric functions in the eigenvalues of $X = X' > 0$. For zonal polynomials and their properties see Mathai, Provost and Hayakawa (1995). Here ${}_rF_s$ will be defined as the following series:

$$
\begin{aligned}
&{}_rF_s(a_1,\ldots,a_r;b_1,\ldots,b_s;X)\\
&=\sum_{k=0}^{\infty}\sum_K\frac{(a_1)_K\cdots(a_r)_K}{(b_1)_K\cdots(b_s)_K}\frac{C_K(X)}{k!},
\end{aligned}
\tag{5.37}
$$

where $K=(k_1,\ldots,k_p), k=k_1+\cdots+k_p$

$$
(a)_K=\prod_{j=1}^{p}\left(a-\frac{j-1}{2}\right)_{k_j},
$$

and $C_K(X)$ is the zonal polynomial of order k. In this definition, ${}_rF_s$ is available explicitly for all r and s but zonal polynomials of higher orders are extremely difficult to evaluate and hence the practical utility of (5.37) is limited. The uniqueness of ${}_rF_s$, defined through (5.37), can be established by showing that (5.37) satisfies the pair of integral equations (5.35) and (5.36). For more details on (5.35), (5.36) and (5.37) and some applications see Mathai (1997).

Observe that H-functions and Meijer's G-functions, in the scalar cases, are defined in terms of their Mellin–Barnes representations. If we want a series representation then we have to take into account all the poles of the integrands in the Mellin–Barnes representations. Obviously the poles can be of all sorts of higher orders and then the series representations will be quite complicated involving, gamma, psi and generalized zeta functions as well as logarithmic terms. For a general expansion for the G-function see Mathai (1993c). The same procedure can be followed to obtain a series expansion for a H-function. This will be more complicated. Hence if we wish to extend the definition in (5.37) to a H-function of matrix argument it is extremely difficult because the series form need not correspond to the same in the scalar variable case. For the very special case of simple poles for the integrand in a Meijer's G-function one can obtain a series form in terms of hypergeometric series in the scalar case. If the series form is replaced by (5.37), the series form in zonal polynomials, still the procedure will not be correct because in $\Gamma_p(\alpha + s)$ itself the alternate gammas produce poles of higher orders, namely the poles of $\Gamma(\alpha+s), \Gamma(\alpha+s-1), \ldots$ are of higher orders and similar is the case for the poles of $\Gamma(\alpha+s-\frac{1}{2}), \Gamma(\alpha+s-\frac{3}{2}), \ldots$ Hence the procedure of making use of (5.37) is also not suitable for extending the definition to matrix variable case for a H-function. Therefore looking for a class of functions by using M-transforms may be the most convenient way of extending the definition to a matrix-variate H-function.

The above considerations lead to one important question. Is there a unique function which can be called the multivariate version of a given univariate function? The answer is obviously a big "no". There can be infinitely many multivariate functions, where the marginal functions yield your specified univariate functions. We can construct many examples.

Example 5.3. **Nonuniqueness of multivariate analogues**. Show that the following two bivariate functions

$$\text{(i)}\quad f_1(x,y,\rho) = \frac{1}{\pi\sqrt{1-\rho^2}} \mathrm{e}^{-\frac{(x^2-2\rho xy+y^2)}{1-\rho^2}},$$

for $1-\rho^2 > 0, -\infty < x < \infty, -\infty < y < \infty$ where ρ is a constant and

$$\text{(ii)}\quad f_2(x,y) = \alpha_1 f_1(x,y,\rho_1) + \cdots + \alpha_k f_k(x,y,\rho_k),$$

for $0 < \alpha_i < 1, 1-\rho_i^2 > 0, i = 1,\ldots,k, \alpha_1+\cdots+\alpha_k = 1$ yield the same marginal functions

$$f(x) = \frac{\mathrm{e}^{-x^2}}{\sqrt{\pi}}, \text{ and } g(y) = \frac{\mathrm{e}^{-y^2}}{\sqrt{\pi}},$$

for $-\infty < x < \infty, -\infty < y < \infty$.

Solution 5.3. Let us consider the marginal function of x from $f_1(x,y,\rho)$ by integrating out y. Consider the exponent, excluding -1.

$$\begin{aligned}\frac{1}{1-\rho^2}[x^2 - 2\rho xy + y^2] &= x^2 + \left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)^2\\ &= x^2 + u^2,\ u = \frac{y-\rho x}{\sqrt{1-\rho^2}}\\ \mathrm{d}y &= \sqrt{1-\rho^2}\mathrm{d}u \text{ for fixed } x.\end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} \frac{1}{\pi\sqrt{1-\rho^2}} \mathrm{e}^{-\frac{1}{1-\rho^2}(x^2-2\rho xy+y^2)}\mathrm{d}y$$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi\sqrt{1-\rho^2}} \mathrm{e}^{-[x^2+\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)^2]}\mathrm{d}y$$

$$= \frac{\mathrm{e}^{-x^2}}{\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-u^2}\mathrm{d}u = \frac{\mathrm{e}^{-x^2}}{\sqrt{\pi}}.$$

Similarly

$$\int_{-\infty}^{\infty} f_1(x,y,\rho)\mathrm{d}x = \frac{\mathrm{e}^{-y^2}}{\sqrt{\pi}}.$$

Thus for the given function

$$f(x) = \frac{\mathrm{e}^{-x^2}}{\sqrt{\pi}}, \quad -\infty < x < \infty,$$

one can take $f_1(x,y,\rho)$ for all ρ such that $1-\rho^2 > 0$ as a bivariate analogue.

Now, look at the process above. When we integrate out y from $f_2(x,y,\rho)$ we obtain

$$\alpha_1\frac{\mathrm{e}^{-x^2}}{\sqrt{\pi}} + \cdots + \alpha_k\frac{\mathrm{e}^{-x^2}}{\sqrt{\pi}} = \frac{\mathrm{e}^{-x^2}}{\sqrt{\pi}},$$

since $\alpha_1 + \cdots + \alpha_k = 1$. Thus all the classes of functions defined by f_2 can also be considered as bivariate extensions of the univariate function $f(x)$.

This example shows that for a given univariate function there is nothing called a unique bivariate or multivariate analogue. There will be several classes of functions which can all be legitimately called the multivariate analogues. Hence looking for a unique multivariate analogue for a given univariate H-function is a meaningless attempt. Looking for a nonempty class of matrix variable functions, where when the matrix is 1×1 or a scalar quantity the functions reduce to the one variable H-function, is the proper procedure. Keeping this in mind, the following classes of functions are defined as G and H-functions of matrix argument.

5.7 Meijer's G-Function of Matrix Argument

Let $f_1(X)$ be a symmetric function in the sense $f(AB) = f(BA)$ for all matrices A and B whenever AB and BA are defined. Let X be a $p \times p$ real positive definite matrix with distinct eigenvalues $\lambda_1 > \cdots > \lambda_p > 0$. Consider the following M-transform with the arbitrary parameter ρ.

Definition 5.6. Meijer's *G*-function of matrix argument in the real case. Let $f_1(X)$ be such that

$$\int_{X>0} |X|^{\rho-\frac{p+1}{2}} f_1(X)\mathrm{d}X = \phi(\rho) \tag{5.38}$$

where

$$\phi(\rho) = \frac{\left\{\prod_{j=1}^{m} \Gamma_p(b_j+\rho)\right\}\left\{\prod_{j=1}^{n} \Gamma_p\left(\frac{p+1}{2}-a_j-\rho\right)\right\}}{\left\{\prod_{j=m+1}^{s} \Gamma_p\left(\frac{p+1}{2}-b_j-\rho\right)\right\}\left\{\prod_{j=n+1}^{r} \Gamma_p(a_j+\rho)\right\}}. \tag{5.39}$$

Whenever the right side exists the class of functions defined by (5.38) and (5.39) will be called Meijer's G-function of matrix argument in the real case where $\Gamma_p(\cdot)$ is the real matrix-variate gamma function.

Note that when $p = 1$, $f_1(X)$ reduces to Meijer's G-function in the real scalar variable case. One can extend the same idea and define a H-function of matrix argument as follows:

Definition 5.7. *H*-function of matrix argument in the real case. Let $f_2(X)$ be a symmetric function in the sense $f_2(AB) = f_2(BA)$ for all matrices A and B whenever AB and BA are defined. Let X be a $p\times p$ real symmetric positive definite matrix with distinct eigenvalues $\lambda_1 > \cdots > \lambda_p > 0$. Let ρ be an arbitrary parameter. Consider the following integral equation:

$$\int_{X>0} |X|^{\rho-\frac{p+1}{2}} f_2(X)\mathrm{d}X = \psi(\rho), \tag{5.40}$$

$$\psi(\rho) = \frac{\left\{\prod_{j=1}^{m} \Gamma_p(b_j+\beta_j\rho)\right\}\left\{\prod_{j=1}^{n} \Gamma_p\left(\frac{p+1}{2}-a_j-\alpha_j\rho\right)\right\}}{\left\{\prod_{j=m+1}^{s} \Gamma_p\left(\frac{p+1}{2}-b_j-\beta_j\rho\right)\right\}\left\{\prod_{j=n+1}^{r} \Gamma_p(a_j+\alpha_j\rho)\right\}}, \tag{5.41}$$

with $\alpha_j, j = 1,\ldots,r$ and $\beta_j, j = 1,\ldots,s$ real and positive. Whenever the right side in (5.41) exists the class of functions $f_2(X)$ determined by (5.40) and (5.41) will be called the H-function of matrix argument in the real case.

For $p = 1$, (5.40) reduces to H-function in the real scalar case. For $\alpha_j = 1, j = 1,\ldots,r$ and $\beta_j = 1, j = 1,\ldots,s$ the class of functions $f_2(X)$ reduces to the class of functions $f_1(X)$ defined through (5.38) and (5.39) and the H-function reduces to a G-function.

5.7.1 Some Special Cases

When $m = 1, n = 0, r = 0, s = 1, b_1 = 0, \beta_1 = 1$, (5.40) reduces to the equation

$$\int_{X>0} |X|^{\rho-\frac{p+1}{2}} f_2(X)\mathrm{d}X = \Gamma_p(\rho). \tag{5.42}$$

One solution for (5.42) is obvious, namely,

$$f_2(X) = \mathrm{e}^{-\mathrm{tr}(X)}$$

because for $\Re(\rho) > \frac{p-1}{2}$

$$\int_{X>0} |X|^{\rho-\frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(X)} \mathrm{d}X = \Gamma_p(\rho).$$

Hence we may define ${}_0F_0(\,;\,;-X)$ by the integral equation in (5.42). On the other hand if $m = 1, n = 0, r = 0, s = 1, b_1 = \alpha, \beta_1 = 1$, then (5.41) reduces to $\Gamma_p(\alpha + \rho)$. Then the equation

$$\int_{X>0} |X|^{\rho-\frac{p+1}{2}} f_2(X)\mathrm{d}X = \Gamma_p(\alpha+\rho), \text{ for } \Re(\alpha+\rho) > \frac{p-1}{2}$$

gives one solution as

$$f_2(X) = |X|^{\alpha}{}_0F_0(\,;\,;-X).$$

Let $m = 1, b_1 = 0, \beta_1 = 1, n = r, s$ is replaced by $s + 1$ then (5.40) becomes

$$\int_{X>0} |X|^{\rho-\frac{p+1}{2}} f_2(X)\mathrm{d}X = \Gamma_p(\rho)\frac{\left\{\prod_{j=1}^{r} \Gamma_p\left(\frac{p+1}{2} - a_j - \alpha_j\rho\right)\right\}}{\left\{\prod_{j=1}^{s} \Gamma_p\left(\frac{p+1}{2} - b_j - \beta_j\rho\right)\right\}}. \tag{5.43}$$

For $p = 1$, (5.43) corresponds to Wright's function and hence we will call the class of functions $f_2(X)$ determined by (5.43) as the Wright's function of matrix argument in the real case.

When $\alpha_j = 1, j = 1, \ldots, r$ and $\beta_j = 1, j = 1, \ldots, s$ then comparing (5.43) with (5.33) we have

$$f_2(X) = \frac{\left\{\prod_{j=1}^{r} \Gamma_p\left(\frac{p+1}{2} - a_j\right)\right\}}{\left\{\prod_{j=1}^{s} \Gamma_p\left(\frac{p+1}{2} - b_j\right)\right\}} \times {}_rF_s\left(\frac{p+1}{2} - a_1, \ldots, \frac{p+1}{2} - a_p; \frac{p+1}{2} - b_1, \ldots, \frac{p+1}{2} - b_s; -X\right) \tag{5.44}$$

or the hypergeometric function of matrix argument in the real case. When $r = 1$, $s = 1$ in (5.43) we may call the corresponding $f_2(X)$ as the generalized Mittag-Leffler function in the real matrix-variate case. Classes of other elementary functions can be defined by taking special cases in (5.39)–(5.44). The theory of H-functions of matrix argument can be extended to complex cases also, that is, when the matrices are hermitian positive definite. Some preliminaries in this direction may be seen from Mathai (1997).

Exercises

5.1. Let $x_1,\ldots,x_p$ be real scalar variables. Let $y_1 = x_1+\cdots+x_p$, $y_2 = x_1x_2 + x_1x_3+\cdots+x_{p-1}x_p$ (sum of products taken two at a time), $\cdots$, $y_p = x_1x_2\cdots x_p$. For $x_j > 0$, $j = 1,\ldots,p$ show that

$$\mathrm{d}y_1\wedge\cdots\wedge\mathrm{d}y_p = \left\{\prod_{i=1}^{p-1}\prod_{j=i+1}^{p}|x_i-x_j|\right\}\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_p.$$

5.2. Consider the general polar coordinate transformation

$$\begin{aligned}
x_1 &= r\sin\theta_1,\\
x_j &= r\cos\theta_1\cos\theta_2\cdots\cos\theta_{j-1}\sin\theta_j,\ j=2,\ldots,p-1,\\
x_p &= r\cos\theta_1\cos\theta_2\cdots\cos\theta_{p-1},
\end{aligned}$$

for $r > 0, -\frac{\pi}{2} < \theta_j \le \frac{\pi}{2}, j = 1,\ldots,p-2, -\pi < \theta_{p-1} \le \pi$. Compute $\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_p$ in terms of $\mathrm{d}r\wedge\mathrm{d}\theta_1\wedge\cdots\wedge\mathrm{d}\theta_{p-1}$.

5.3. For $X = X' > 0, Y = Y' > 0$ and $p\times p$ show that

$$\lim_{a\to\infty}|I+\frac{XY}{a}|^{-a} = \mathrm{e}^{-\mathrm{tr}(XY)} = \lim_{a\to\infty}|I-\frac{XY}{a}|^{a}.$$

5.4. Let X and A be $p\times p$ lower triangular matrices of distinct elements. Let $A = (a_{ij})$ be a constant matrix such that $a_{jj} > 0$, $j = 1,\ldots,p$. Then show that

$$Y = XA \Rightarrow \mathrm{d}Y = \left\{\prod_{j=1}^{p}a_{jj}^{p+1-j}\right\}\mathrm{d}X.$$

5.5. For the same X and A in Exercise 5.4 evaluate the Jacobians in the transformations (i) $Y = AX$, (ii) $Y = aX$ where a is a scalar quantity.

5.6. Redo Exercises 5.4 and 5.5 if the matrices X and A are upper triangular.

5.7. For real $X = X' > 0$ and $p\times p$ evaluate the following integrals:

$$\text{(i)}\ f_1 = \int_X \mathrm{d}X;\ \ \text{(ii)}\ f_2 = \int_X |X|\mathrm{d}X;\ \ \text{(iii)}\ f_3 = \int_X |I-X|\mathrm{d}X$$

$$\text{(iv)}\ f_4 = \int_X |X|^{\alpha}\mathrm{d}X;\ \ \text{(v)}\ \int_X |I-X|^{\alpha}\mathrm{d}X$$

and evaluate these explicitly for (vi) $p = 2$; (vii) $p = 3$.

5.8. By showing that both sides have the same M-transforms establish the following results for the class of functions defined through (5.44) where all are $p \times p$ real symmetric positive definite matrices.

(i) $${}_1F_1(a;c;-X) = \frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(c-a)}|X|^{-(c-\frac{p+1}{2})} \times \int_{O<Y<X} \mathrm{e}^{-\mathrm{tr}(Y)}|Y|^{a-\frac{p+1}{2}}|X-Y|^{c-a-\frac{p+1}{2}}\mathrm{d}Y$$

(ii) $${}_2F_1(a,b;c;-X) = \frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(c-a)}\int_{O<X<I}|Y|^{a-\frac{p+1}{2}}|I-Y|^{c-a-\frac{p+1}{2}} \times |I+YX|^{-b}\mathrm{d}Y$$

(iii) $${}_2F_1(a,b;c;-X) = |I-X|^{-b}{}_2F_1(c-a,b;c;-X(I-X)^{-1}).$$

5.9. It is seen that for real $X = X' > 0$ and $p \times p$

$$\int_{X>0}|X|^{\rho-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(X)}\mathrm{d}X = \Gamma_p(\rho),$$

for arbitrary ρ such that $\Re(\rho) > \frac{p-1}{2}$. Suppose that

$$\int_{X>0}|X|^{\rho-\frac{p+1}{2}}f(X)\mathrm{d}X = \Gamma_p(\rho) \text{ for } \Re(\rho) > \frac{p-1}{2},$$

establish a set of sufficient conditions so that $f(X)$ is uniquely determined as $f(X) = \mathrm{e}^{-\mathrm{tr}(X)}$.

5.10. For real $X = X' > 0$ and $p \times p$ consider the equation

$$\int_{X>0}|X|^{\rho-\frac{p+1}{2}}f(X)\mathrm{d}X = \frac{\Gamma_p(\rho)\Gamma_p(\alpha-\rho)}{\Gamma_p(\alpha)},$$

for $\Re(\rho) > \frac{p-1}{2}, \Re(\alpha-\rho) > \frac{p-1}{2}$. One solution for $f(X)$ is seen to be

$$f(X) = |I+X|^{-\alpha}.$$

What are the sufficient conditions on f such that this is the only solution?

Chapter 6
Applications in Astrophysics Problems

6.1 Introduction

There are many areas in astrophysics where Meijer's G-function and H-function appear naturally. Some of these areas are analytic solar and stellar models, nuclear reaction rate theory and energy generation in stars, gravitational instability problems, nonextensive statistical mechanics, pathway analysis, input-output models and reaction-diffusion problems. Brief introductions to these areas will be given here so that the readers can develop the areas further and tackle more general and more complex situations.

6.2 Analytic Solar Model

The numerical approach to the study of solar structure is to go for the numerical solutions of the underlying system of differential equations. Even for a simple main sequence star in hydrostatic equilibrium at least four nonlinear differential equations are to be dealt with to obtain a good picture of the internal structure of the star. Our Sun is such a main-sequence star.

The simplest analytical procedure is to start with a simple mathematical model for the matter density distribution in the core of the Sun. Then, from there develop formulae for the mass, pressure, temperature, luminosity and other such critical parameters. Several such models were considered in a series of papers by Haubold and Mathai, some details may be seen from Mathai and Haubold (1988). A two-parameter model considered by them for the density $\rho(r)$, at an arbitrary distance of r from the center of the Sun is the following:

$$\rho(r) = \rho_c \left[1 - \left(\frac{r}{R_\odot}\right)^\delta\right]^\gamma, \delta > 0, \tag{6.1}$$

γ is a positive integer, where ρ_c is the central density, $R_\odot$ is the radius of the Sun. Let $y = \frac{r}{R_\odot}$. Then for the solar core, that is, $0 \le y \le 0.3$, it is seen that the

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$\delta = 1.28$ and $\gamma = 10$ give a good fit to the observational data. Then the model for $u = \frac{\rho(r)}{\rho_c}$ is given by

$$u = (1 - y^{\delta})^{\gamma}, \text{ with } \delta = 1.28, \text{ and } \gamma = 10. \tag{6.2}$$

This is shown to give good estimates for the solar mass $M(r)$, pressure $P(r)$, temperature $T(r)$, and luminosity $\epsilon(r)$. From standard formula we have

$$\frac{\mathrm{d}}{\mathrm{d}r} M(r) = 4\pi r^2 \rho(r) \tag{6.3}$$

where $M(r)$ is the mass at the distance r from the center.

$$\begin{aligned} M(r) &= 4\pi \int_0^r t^2 \rho(t) \mathrm{d}t \\ &= 4\pi\rho_c \int_0^r t^2 \left[1 - \left(\frac{t}{R_\odot}\right)^{\delta}\right]^{\gamma} \mathrm{d}t \qquad (6.4) \\ &= \frac{4\pi\rho_c}{3} R_\odot^3 \left(\frac{r}{R_\odot}\right)^3 {}_2F_1\left[-\gamma, \frac{3}{\delta}; \frac{3}{\delta} + 1; \left(\frac{r}{R_\odot}\right)^{\delta}\right], \qquad (6.5) \end{aligned}$$

where ${}_2F_1$ is a Gauss' hypergeometric function, which is a special case of a H-function. For $r = R_\odot$ in (6.4) we have the total mass of the Sun, which works out to be the following:

$$\begin{aligned} M(R_\odot) &= \frac{4\pi\rho_c R_\odot^3}{3} {}_2F_1\left(-\gamma, \frac{3}{\delta}; \frac{3}{\delta} + 1; 1\right) \qquad (6.6) \\ &= \frac{4\pi\rho_c}{3} \frac{\gamma!}{\left(\frac{3}{\delta} + 1\right)\left(\frac{3}{\delta} + 2\right) \cdots \left(\frac{3}{\delta} + \gamma\right)}, \qquad (6.7) \end{aligned}$$

by using he expansion formula

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}. \tag{6.8}$$

where $a > 0, c - a - b > 0$. Then

$$\frac{M(r)}{M(R_\odot)} = \frac{\left(\frac{3}{\delta} + 1\right) \cdots \left(\frac{3}{\delta} + \gamma\right)}{\gamma!} \left(\frac{r}{R_\odot}\right)^3 {}_2F_1\left(-\gamma, \frac{3}{\delta}; \frac{3}{\delta} + 1; \left(\frac{r}{R_\odot}\right)^{\delta}\right). \tag{6.9}$$

This is seen to be in good agreement with observational data for $\delta = 1.28$ and $\gamma = 10$. The internal pressure at arbitrary distance r from the center is available from standard formula

$$
\begin{aligned}
P(r) &= P_c - G\int_0^r \frac{M(t)\rho(t)}{t^2}\mathrm{d}t \\
&= P_c - \frac{4\pi G}{\delta^2}\rho_c^2 R_\odot^2 \sum_{m=0}^{\gamma} \frac{(-\gamma)_m \left(\frac{r}{R_\odot}\right)^{m\delta+2}}{\left(\frac{3}{\delta}+m\right)\left(\frac{2}{\delta}+m\right)} \\
&\quad \times {}_2F_1\left(-\gamma, \frac{2}{\delta}+m; \frac{2}{\delta}+m+1; \left(\frac{r}{R_\odot}\right)^{\delta}\right),
\end{aligned} \tag{6.10}
$$

where P_c is the pressure at the center and G is the gravitational constant. By using the fact that $P(R_\odot) = 0$ we can compute the pressure at the center P_c. Opening up the hypergeometric function we can write $P(r)$ in a closed form:

$$
\begin{aligned}
P(r) &= P_c - \frac{2}{3}\pi G\rho_c^2 r^2 \\
&\quad \times F_{1:2:0}^{1:3:1}\left[\begin{pmatrix}\left(\frac{r}{R_\odot}\right)^{\delta} \\ \left(\frac{r}{R_\odot}\right)^{\delta}\end{pmatrix}\middle| \begin{matrix}\frac{2}{3}; -\gamma, \frac{3}{\delta}, \frac{2}{\delta}; -\gamma \\ \frac{2}{\delta}+1; \frac{3}{\delta}+1, \frac{2}{\delta}+1;\end{matrix}\right],
\end{aligned} \tag{6.11}
$$

where $F_{1:2:0}^{1:3:1}(\cdot)$ is a Kampé de Fériet's function, see Srivastava and Karlsson (1985). The standard equation for temperature is the following:

$$
T(r) = \frac{\mu}{kN_A}P(r)\rho(r), \tag{6.12}
$$

where μ is the mean molecular weight, k is Boltzmann's constant and N_A is Avogadro's number. For the model in (6.1) it can be seen that

$$
T(r) = \frac{\mu}{kN_A}4\pi G\rho_c R_\odot^2 \frac{g(r)}{[1-(\frac{r}{R_\odot})^{\delta}]^{\gamma}}, \tag{6.13}
$$

where

$$
\begin{aligned}
g(r) &= \frac{1}{\delta^2}\sum_{m=0}^{\gamma}\frac{(-\gamma)_m}{m!}\frac{1}{\left(\frac{3}{\delta}+m\right)\left(\frac{2}{\delta}+m\right)} \\
&\quad \times \left[\frac{\gamma!}{\left(\frac{2}{\delta}+m+1\right)\cdots\left(\frac{2}{\delta}+m+\gamma\right)}\right. \\
&\quad \left. -\left(\frac{r}{R_\odot}\right)^{m\delta+2} {}_2F_1\left(-\gamma, \frac{2}{\delta}+m; \frac{2}{\delta}+m+1; \left(\frac{r}{R_\odot}\right)^{\delta}\right)\right].
\end{aligned} \tag{6.14}
$$

From the computations in Haubold and Mathai (1994) it is seen that $M(r)$, $P(r)$, $T(r)$ and luminosity $L(r)$ are in good agreement with observational data for the model in (6.1) with $\delta = 1.28$ and $\gamma = 10$. Further details may be seen from Haubold and Mathai (1994).

Exercises 6.1

6.1.1. From the model in (6.1) derive expressions for solar mass $M(r)$, pressure $P(r)$, temperature $T(r)$ and luminosity $L(r)$ at an arbitrary distance r from the center.

6.1.2. Let $u = \frac{\rho(r)}{\rho_c}$ where $\rho(r)$ is the matter density at a distance r from the center of the Sun and ρ_c is the density at the center. Let $y = \frac{r}{R_\odot}$ for $0 \leq y \leq 0.3$, where $R_\odot$ is the solar radius. The following is the data from Sear (1964)

$$y : 0.0864, 0.1153, 0.1441, 0.1873, 0.2161, 0.2450, 0.2882$$
$$u : 0.6519, 0.5253, 0.3856, 0.2810, 0.1994, 0.1424, 0.0962$$

By using the method of least squares fit a polynomial of degree 3 to this data and show that the polynomial model is

$$y = 1 - 0.940y + 6.67y^2 - 2.73y^3.$$

Compute u by using this model and compare with Sear's data.

6.1.3. Consider the following three models for u of Exercise 6.1.2

$$u = 1 - 4y + 2y^2 + 2y^3 - y^4,$$
$$u = (1 - \sqrt{y})(1 - y^3)^{64},$$
$$u = (1 - y^{\frac{3}{2}})^{16}.$$

Compute u under these models and compare with Sear's data.

6.1.4. Consider the following four models for u in Exercise 6.1.2.

$$u = (1 - \sqrt{y})(1 - y^3)^{64}(1 - y),$$
$$u = (1 - y^{1.48})^{14},$$
$$u = (1 - y^{1.48})^{13},$$
$$u = (1 - y^{1.28})^{10}.$$

Compute u under these models and compare with Sear's data.

6.1.5. Show that the last model in Exercise 6.1.4 is the best among all the eight models in Exercises 6.1.2, 6.1.3 and 6.1.4.

6.3 Thermonuclear Reaction Rates

In nuclear reaction rate theory one comes across the following four reaction probability integrals in nonresonant reactions, reactions with high energy tail cut off, in screened case and in the depleted case:

$$I_1 = \int_0^\infty y^\nu e^{-\left(y+zy^{-\frac{1}{2}}\right)} dy \tag{6.15}$$

$$I_2 = \int_0^d y^\nu e^{-\left(y+zy^{-\frac{1}{2}}\right)} dy \tag{6.16}$$

$$I_3 = \int_0^\infty y^\nu e^{-\left(y+\frac{z}{\sqrt{y+t}}\right)} dy \tag{6.17}$$

$$I_4 = \int_0^\infty y^\nu e^{-\left(y+by^\delta+zy^{-\frac{1}{2}}\right)} dy. \tag{6.18}$$

These are the reaction rate probability integrals dealt with in Anderson et al. (1994). A more general case of I_1 is the following:

$$I_5 = \int_0^\infty y^\nu e^{-(ay^\delta+by^{-\rho})} dy \tag{6.19}$$

for $a > 0, b > 0, \delta > 0, \rho > 0$, where for $a = 1, b = z, \delta = 1, \rho = \frac{1}{2}$ we have the integral I_1. Observe that (6.19) is the limiting form of the versatile integral discussed in Chap. 4. Writing

$$f_1(x) = x^{\nu+1} e^{-ax^\delta} \text{ and } f_2(x) = e^{-x^\rho},$$

the integral in (6.19) can be written as

$$I_5 = \int_{v=0}^\infty \frac{1}{v} f_1(v) f_2\left(\frac{u}{v}\right) dv, u = b^{\frac{1}{\rho}}. \tag{6.20}$$

Hence from Mellin convolution property, the Mellin transform of I_5 is the product of the Mellin transforms of $f_1(x)$ and $f_2(x)$ respectively. Denoting the Mellin transforms by $g_1(s)$ and $g_2(s)$, with s being the Mellin parameter, one has,

$$g_1(s) = \int_0^\infty x^{s-1} x^{\nu+1} e^{-ax^\delta} dx = \frac{1}{\delta} \frac{\Gamma\left(\frac{\nu+1+s}{\delta}\right)}{a^{\frac{\nu+1+s}{\delta}}}, \tag{6.21}$$

where $\Re(\nu + 1 + s) > 0$ and

$$g_2(s) = \int_0^\infty x^{s-1} e^{-x^\rho} dx = \frac{1}{\rho}\Gamma\left(\frac{s}{\rho}\right), \Re(s) > 0. \tag{6.22}$$

Then I_5 is available from the inverse Mellin transform of $g_1(s)g_2(s)$. That is,

$$I_5 = \frac{1}{2\pi i}\int_L \frac{1}{\delta\rho}\frac{\Gamma\left(\frac{\nu+1+s}{\delta}\right)}{a^{\frac{\nu+1+s}{\delta}}}\Gamma\left(\frac{s}{\rho}\right)u^{-s}\mathrm{d}s \tag{6.23}$$

$$= \frac{1}{\delta\rho a^{\frac{\nu+1}{\delta}}}\int_{c-i\infty}^{c+i\infty}\Gamma\left(\frac{s}{\rho}\right)\Gamma\left(\frac{\nu+1}{\delta}+\frac{s}{\delta}\right)\left(a^{\frac{1}{\delta}}u\right)^{-s}\mathrm{d}s$$

$$= \frac{1}{\delta\rho a^{\frac{\nu+1}{\delta}}}H_{0,2}^{2,0}\left[a^{\frac{1}{\delta}}b^{\frac{1}{\rho}}\Big|_{(0,\frac{1}{\rho}),\left(\frac{\nu+1}{\delta},\frac{1}{\delta}\right)}\right]. \tag{6.24}$$

Thus the special cases of the integral in (6.20) are the special cases of the H-function in (6.24).

Some interesting special cases are the situations where (i): $\frac{1}{\delta}=m, \frac{1}{\rho}=n, m,n=1,2,\ldots$; (ii): $\frac{\rho}{\delta}=\lambda, \lambda=1,2,\ldots$; (iii) $\frac{\delta}{\rho}=\mu, \mu=1,2,\ldots$. In all these cases one can reduce the H-function in (6.24) to Meijer's G-function with the help of the multiplication formula for gamma functions, some details and computable representations are available from Mathai and Haubold (1988).

Exercises 6.2

6.2.1. Show that the reaction rate probability integral

$$\int_0^\infty x^{\nu-1}\mathrm{e}^{-ax-zx^{-\rho}}\mathrm{d}x = \frac{a^{-\nu}}{\rho}H_{0,2}^{2,0}\left[az^{\frac{1}{\rho}}\big|_{(0,\frac{1}{\rho}),(\nu,1)}\right],$$

for $a>0, z>0, \rho>0$.

6.2.2. For $\rho=\frac{1}{2}, a=1$ in Exercise 6.2.1 show that

$$\int_0^\infty x^{\nu-1}\mathrm{e}^{-x-zx^{-\frac{1}{2}}}\mathrm{d}x = \pi^{-\frac{1}{2}}G_{0,3}^{3,0}\left[\frac{z^2}{4}\Big|_{0,\frac{1}{2},\nu}\right]$$

$$= \pi^{-\frac{1}{2}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma\left(\frac{1}{2}+s\right)\Gamma(\nu+s)\left(\frac{z^2}{4}\right)^{-s}\mathrm{d}s.$$

6.2.3. Write down the conditions for the poles of the integrand in the Mellin–Barnes integral in Exercise 6.2.2 to be simple. Evaluate the Mellin–Barnes integral in Exercise 6.2.2 in the case of simple poles.

6.2.4. Write down the integral in Exercise 6.2.2 in series form when ν is an integer thereby the poles of the integrand can be up to order 2.

6.2.5. Write down the integral in Exercise 6.2.2 in series form when ν is a half-integer thereby the poles of the integrand can be up to order 2.

6.4 Gravitational Instability Problem

Gravitational condensation is believed to be the reason for the formation of the basic building blocks of the universe, that is, the stars and galaxies and systems of them at various scales. The universe is a multi-component medium. The influence of the components' relative motions upon the gravitational instability was investigated by many authors. Gravitational instability in a multi-component medium in an expanding universe under Newtonian approximation was studied by Mathai et al. (1988). Exact solutions of the differential equations connected with the gravitational instability problems in a two-component, and then in a multi-component medium, were considered by Mathai et al. (1988) by converting the basic equations to the equations satisfied by a Meijer's G-function. After a few substitutions, see Mathai et al. (1988), the basic equations for a two-component medium can be written as follows:

$$\Delta^2\delta_1 + (2\eta - 1)\Delta\delta_1 + k_1^2 t^{\alpha_1}\delta_1 = \frac{2}{3}(\Omega_1\delta_1 + \Omega_2\delta_2), \tag{6.25}$$

$$\Delta^2\delta_2 + (2\eta - 1)\Delta\delta_2 + k_2^2 t^{\alpha_2}\delta_2 = \frac{2}{3}(\Omega_1\delta_1 + \Omega_2\delta_2), \tag{6.26}$$

where Δ is the operator $\Delta = t\frac{\mathrm{d}}{\mathrm{d}t}$, $\Omega = \Omega_1 + \Omega_2 = 1, \Omega_i, i = 1, 2$ are constants, and other parameters have physical interpretations. Solving for δ_2 from (6.25) and then substituting for $\delta_2, \Delta\delta_2, \Delta^2\delta_2$ in (6.26) one has the following fourth degree equation:

$$\begin{aligned}&\Delta^4\delta_1 + 2(2\eta - 1)\Delta^3\delta_1 + \left[k_1^2 t^{\alpha_1} + k_2^2 t^{\alpha_2} - \frac{2}{3} + (2\eta - 1)^2\right]\Delta^2\delta_1\\&+\left[(2\eta - 1)k_1^2 t^{\alpha_1} + (2\eta - 1)k_2^2 t^{\alpha_2} + 2k_1^2\alpha_1 t^{\alpha_1} - (2\eta - 1)\frac{2}{3}\right]\Delta\delta_1\\&+\left[k_1^2\alpha_1^2 t^{\alpha_1} + (2\eta - 1)k_1^2\alpha_1 t^{\alpha_1} - \frac{2}{3}\Omega_2 k_1^2 t^{\alpha_1}\right.\\&\left.-\frac{2}{3}\Omega_1 k_2^2 t^{\alpha_2} + k_1^2 k_2^2 t^{\alpha_1+\alpha_2}\right]\delta_1 = 0.\end{aligned} \tag{6.27}$$

Here (6.27) is the equation governing the growth and decay of gravitational condensation in the expanding two-fluid universe. An equation for δ_2, corresponding to (6.27) is available from symmetry. The following special cases of (6.27) have interesting solutions. We consider the following cases: (i) $k_1 = k_2 = 0$; (ii) $\alpha_1 = \alpha_2 = 0, k_1, k_2$ arbitrary; (iii) $\alpha_2 \neq 0, k_1 = 0$; (iv) $\alpha_1 \neq 0, k_2 = 0$; (v) $\alpha_1 = 0, \alpha_2 \neq 0$; (vi) $\alpha_1 \neq 0, \alpha_2 = 0$; (vii) $\alpha_2 = \alpha_1 = \alpha \neq 0, k_1 = k_2 = k \neq 0$.

In case (iii) by changing t to $x = \frac{k_2 t^{\alpha_2}}{\alpha_2^2}$ and $\tilde{\Delta} = x\frac{\mathrm{d}}{\mathrm{d}x}$, Eq. (6.27) reduces to

$$\begin{aligned}&\left\{(\tilde{\Delta} - b_1)(\tilde{\Delta} - b_2)(\tilde{\Delta} - b_3)(\tilde{\Delta} - b_4)\right\}\delta_1\\&+ x\left\{(\tilde{\Delta} - a_1)(\tilde{\Delta} - a_2)\right\}\delta_1 = 0,\end{aligned} \tag{6.28}$$

where

$$b_1 = 0, b_2 = -\frac{(2\eta-1)}{\alpha_2},\ b_3 = \frac{\left(\frac{1}{2}-\eta\right)}{\alpha_2} - \left[\frac{(\eta-\frac{1}{2})^2}{\alpha_2^2} + \frac{2}{3\alpha_2^2}\right]^{\frac{1}{2}}$$

$$b_4 = \frac{\left(\frac{1}{2}-\eta\right)}{\alpha_2} + \left[\frac{(\eta-\frac{1}{2})^2}{\alpha_2^2} + \frac{2}{3\alpha_2^2}\right]^{\frac{1}{2}}, a_1 = \frac{\left(\frac{1}{2}-\eta\right)}{\alpha_2} - \left[\frac{\left(\frac{1}{2}-\eta\right)^2}{\alpha_2^2} + \frac{1\Omega_1}{3\alpha_2^2}\right]^{\frac{1}{2}}$$

$$a_2 = \frac{\left(\frac{1}{2}-\eta\right)}{\alpha_2} + \left[\frac{\left(\frac{1}{2}-\eta\right)^2}{\alpha_2^2} + \frac{2\Omega_1}{3\alpha_2^2}\right]^{\frac{1}{2}}.$$

Observe that (6.28) is a special case of the differential equation satisfied by a Meijer's G-function, see for example Mathai (1993c), so that the theory of G-function can be applied to (6.28). In all the particular cases it is seen that equation (6.27) reduces to the form

$$\{(\Delta - b_1)(\Delta - b_2)(\Delta - b_3)(\Delta - b_4)\}\delta_1 + x\{(\Delta - a_1)(\Delta - a_2)\}\delta_1 = 0 \quad (6.29)$$

where $a_1, a_2, b_1, \ldots, b_4$ and x change from case to case. Comparing (6.29) with a G-function differential equation for $G_{p,q}^{m,n}\left(x\Big|_{b_1,\ldots,b_q}^{a_1,\ldots,a_p}\right)$ we have $q = 4, p = 2$, $(-1)^{p-m-n} = -1, a_1, a_2, b_1, \ldots, b_4$. From the standard solutions of the G-function equation, the solution near $x = 0$ is given by

$$\delta_1 = c_1G_1 + c_2G_2 + c_3G_3 + c_4G_4,$$

where c_1, c_2, c_3, c_4 are arbitrary constants and

$$G_j = \frac{\left[\prod_{k=1}^{2}\Gamma(-a_k + b_j)\right]}{\left[\prod_{k=1}^{4}\Gamma(1 - b_k + b_j)\right]}x^{b_j}$$
$$\times {}_2F_3(-a_1 + b_j, -a_2 + b_j; 1 - b_1 + b_j, \ldots, *, \ldots, 1 - b_4 + b_j; -x), \quad (6.30)$$

where the $*$ indicates that parameter of the type $1 - b_j + b_j$ and the corresponding gamma are absent, and it is assumed that $b_i - b_j \neq 0, \pm 1, \pm 2, \ldots$ for all $i \neq j = 1, \ldots, 4$ and ${}_2F_3$ is a hypergeometric function.

Here in (6.29) the G-function parameters are $q = 4, p = 2$ or $q > p$. Hence the 4 fundamental solutions and the general solution for $x \to \infty$ are the following:

$$\delta_1 = c_1G_1 + c_2G_2 + c_3G_3 + c_4G_4, \quad (6.31)$$

where c_1, c_2, c_3, c_4 are arbitrary constants and

$$G_1 = G^{4,1}_{2,4}\left[x\Big|^{1+a_1,1+a_2}_{b_1,\ldots,b_4}\right],$$

$$G_2 = G^{4,1}_{2,4}\left[x\Big|^{1+a_2,1+a_1}_{b_1,\ldots,b_4}\right],$$

$$G_3 = G^{4,0}_{2,4}\left[xe^{i\pi}\Big|^{1+a_1,1+a_2}_{b_1,\ldots,b_4}\right],$$

$$G_4 = G^{4,0}_{2,4}\left[xe^{-i\pi}\Big|^{1+a_1,1+a_2}_{b_1,\ldots,b_4}\right], \quad i = \sqrt{-1}.$$

Computable series forms as well as explicit solutions for various cases of 3-component medium are available from Mathai et al. (1988).

Exercises 6.3

6.3.1. Derive Eq. (6.27) from Eqs. (6.25) and (6.26).

6.3.2. Derive an equation for δ_2 from Eq. (6.27).

6.3.3. Under the special case $k_1 = 0, k_2 = 0$ show that (6.27) reduces to the form $\{(\Delta - a_1)(\Delta - a_2)(\Delta - a_3)(\Delta_4 - a_4)\}\delta_1 = 0$ so that the general solution is $\delta_1 = c_1 + c_2 t^{a_2} + c_3 t^{a_3} + c_4 t^{a_4}$ where

$$a_1 = 0, a_2 = -(2\eta - 1), a_3 = \left(\frac{1}{2} - \eta\right) - \left[\left(\frac{1}{2} - \eta\right)^2 + \frac{2}{3}\right]^{\frac{1}{2}},$$

$$a_4 = \left(\frac{1}{2} - \eta\right) + \left[\left(\frac{1}{2} - \eta\right)^2 + \frac{2}{3}\right]^{\frac{1}{2}}.$$

6.3.4. Show that under case (iv): $\alpha_1 \neq 0, k_2 = 0$ Eq. (6.27) reduces to

$$\{(\tilde{\Delta} - b_1')(\tilde{\Delta} - b_2')(\tilde{\Delta} - b_3')(\tilde{\Delta} - b_4')\}\delta_1 + x\{(\tilde{\Delta} - a_1')(\tilde{\Delta} - a_2')\}\delta_1 = 0,$$

where $\tilde{\Delta} = x\frac{\mathrm{d}}{\mathrm{d}x}, \tilde{x} = \frac{k_1^2 t^{\alpha_1}}{\alpha_1^2}$. Show that

$$a_1' = \left[-1 + \frac{\left(\frac{1}{2} - \eta\right)}{\alpha_1}\right] - \left[\frac{\left(\eta - \frac{1}{2}\right)^2}{\alpha_1^2} + \frac{2\Omega_2}{3\alpha_1^2}\right]^{\frac{1}{2}},$$

$$a_2' = \left[-1 + \frac{\left(\frac{1}{2} - \eta\right)}{\alpha_1}\right] + \left[\frac{\left(\eta - \frac{1}{2}\right)}{\alpha_1^2} + \frac{2\Omega_2}{3\alpha_1^2}\right]^{\frac{1}{2}},$$

and $b_i' = b_i$ of case (iii).

6.3.5. Show that under Case (v): $\alpha_1 = 0, \alpha_2 \neq 0$ Eq. (6.27) reduces to the same form as in Exercise 6.3.4 with

$$a_1 = \frac{\left(\frac{1}{2} - \eta\right)}{\alpha_2^2} - \left[\frac{\left(\frac{1}{2} - \eta\right)^2}{\alpha_2^2} + \frac{2\Omega_1}{3\alpha_2^2} - \frac{k_1^2}{\alpha_2^2}\right]^{\frac{1}{2}},$$

$$a_2 = \frac{\left(\frac{1}{2} - \eta\right)}{\alpha_2^2} + \left[\frac{\left(\frac{1}{2} - \eta\right)^2}{\alpha_2^2} + \frac{2\Omega_1}{3\alpha_2^2} - \frac{k_1^2}{\alpha_2^2}\right]^{\frac{1}{2}},$$

and the b_i's are the solutions of the equation

$$\alpha_2^4 b^4 + 2(2\eta - 1)\alpha_2^3 b^3 + \left[(2\eta - 1)^2 - \frac{2}{3} + k_1^2\right]\alpha_2^2 b^2 + \left[-\frac{2}{3}(2\eta - 1) + (2\eta - 1)k_1^2\right]\alpha_2 b - \frac{2}{3}\Omega_2 k_1^2 = 0.$$

6.5 Generalized Entropies in Astrophysics Problems

Entropy is a measure of uncertainty in a probability scheme or in a probability density. If $P = (p_1, \ldots, p_k), p_i \geq 0, i = 1, \ldots, k, p_1 + \cdots + p_k = 1$ be the probabilities in a set $A = \{A_1, \ldots, A_k\}$ of mutually exclusive and totally exhaustive events then a measure of uncertainty in this scheme (A, P), proposed by Shannon in 1948, was

$$S = -G \sum_{i=1}^{k} p_i \ln p_i, \qquad (6.32)$$

where G is a constant and ln is logarithm to the base e.

6.5.1 Generalizations of Shannon Entropy

Generalizations to Shannon's entropy were considered by many authors. A few of these are the following:

$$H - C = \frac{\sum_{i=1}^{k} p_i^{\alpha} - 1}{2^{1-\alpha} - 1}, \quad \alpha \geq 0,\ \alpha \neq 1 \text{ (Havrda-Chárvat)}, \tag{6.33}$$

$$R = \frac{\ln\left(\sum_{i=1}^{k} p_i^{\alpha}\right)}{1-\alpha}, \quad \alpha \geq 0,\ \alpha \neq 1 \text{ (Rényi)}, \tag{6.34}$$

$$T = \frac{\sum_{i=1}^{k} p_i^{q} - 1}{1-q}, \quad q \geq 0,\ q \neq 1, \text{ (Tsallis)}, \tag{6.35}$$

$$M = \frac{\sum_{i=1}^{k} p_i^{2-\alpha} - 1}{\alpha - 1}, \quad \alpha \leq 2,\ \alpha \neq 1 \text{ (Mathai)}. \tag{6.36}$$

All the α-generalized analogues, $H - C, R, T, M$ go to Shannon's entropy S when $\alpha \to 1$ and in this sense they are generalizations. Tsallis' entropy T is the basis for the current hot topic of nonextensive statistical mechanics and q-calculus. The corresponding measures in a probability density $f(x)$, $[f(x) \geq 0$ for all x, $\int_x f(x)\mathrm{d}x = 1]$ are the following:

$$S = -G \int_x f(x) \ln f(x)\mathrm{d}x, \tag{6.37}$$

$$H - C = \frac{\int_x [f(x)]^{\alpha}\mathrm{d}x - 1}{2^{1-\alpha} - 1}, \quad \alpha \geq 0,\ \alpha \neq 1, \tag{6.38}$$

$$R = \frac{\ln \int_x [f(x)]^{\alpha}\mathrm{d}x}{1-\alpha}, \quad \alpha \geq 0,\ \alpha \neq 1, \tag{6.39}$$

$$T = \frac{\int_x [f(x)]^{q}\mathrm{d}x - 1}{1-q}, \quad q \geq 0,\ q \neq 1, \tag{6.40}$$

$$M = \frac{\int_x [f(x)]^{2-\alpha}\mathrm{d}x - 1}{\alpha - 1}, \quad 0 \leq \alpha \leq 2,\ \alpha \neq 1. \tag{6.41}$$

Tsallis' q-exponential function is derived from T of (6.40) by optimizing T subject to the conditions $\int_x f(x)\mathrm{d}x = 1$ and that the first moment is pre-assigned, that is, $\int_x x f(x)\mathrm{d}x =$ given. If the optimization of T is done in the escort density

$$g(x) = \frac{[f(x)]^q}{\int_x [f(x)]^q \mathrm{d}x}, \tag{6.42}$$

then one obtains Tsallis density or known as Tsallis' statistics

$$f_1(x) = c_1[1 - (1-q)x]^{\frac{1}{1-q}}, \tag{6.43}$$

where c_1 is the normalizing constant such that $\int_x f(x)\mathrm{d}x = 1$. If Mathai's entropy (6.41) is optimized under the conditions of preassigning the δ-th moment and $(\gamma + \delta)$-th moment for some δ and γ, by using calculus of variation techniques, then one obtains a particular case of Mathai's pathway model in the scalar case

$$f_2(x) = c_2 x^{\gamma}[1 - a(1-\alpha)x^{\delta}]^{\frac{1}{1-\alpha}}, \delta > 0, a > 0 \tag{6.44}$$

where c_2 is the normalizing constant. Observe that c_2 will be different for the three cases $\alpha < 1, \alpha > 1, \alpha \to 1$. When $\alpha < 1$ then $f_2(x)$ for $1 - a(1-\alpha)x^{\delta} > 0$ remains in the generalized type-1 beta family of densities and when $\alpha > 1$, writing $1 - \alpha = -(\alpha - 1)$, $f_2(x)$ goes into the generalized type-2 beta family of densities. When $\alpha \to 1$, then $f_2(x)$ goes to $f_3(x)$ where

$$f_3(x) = c_3 x^{\gamma} \mathrm{e}^{-ax^{\delta}} \tag{6.45}$$

where c_3 is the normalizing constant. It may be mentioned here that M in (6.36) is also connected to the measure of directed divergence in discrete distributions. Observe that for $g_1(x) = f_1(x)/c_1$

$$\frac{\mathrm{d}}{\mathrm{d}x} g_1(x) = -[g_1(x)]^q \tag{6.46}$$

and hence $f_1(x)$, as a model, can describe situations of power function behavior, meaning that the rate of change of $g_1(x)$ is proportional to a power of $g_1(x)$.

When we study the properties (6.45), H-function comes in naturally as illustrated in Chap. 4, Sect. 4.3. These properties will not be repeated here. Thus, H-functions prop up when dealing with problems in nonextensive statistical mechanics, power laws, pathway analysis, generalized entropies and related areas.

Exercises 6.4

6.4.1. Consider the entropy measure in (6.41). By using calculus of variation techniques optimize M under the condition that the functional $f(x)$ is such that $f(x) \geq 0$ and $\int_x f(x)\mathrm{d}x = 1$ and show that the solution is a uniform density.

6.4.2. Optimize M in (6.41) for all densities $f(x)$ such that the first moment is a given or preassigned quantity. Show that the pathway model for $\gamma = 0$ and $\delta = 1$ is the resulting $f(x)$.

6.4.3. Redo Exercise 6.4.2 under the conditions $E(x^{\delta})$ and $E(x^{\delta+\gamma})$ are preassigned, where E denotes the expected value or δ-th moment and $(\delta+\gamma)$-th moments respectively. Show that the resulting density is the pathway model for the positive real scalar variable case.

6.4.4. Derive the density of $u = xy$ if x and y are independently distributed real scalar positive random variables where x is having the density in (6.44) with parameters as given there and y has the density in (6.45) with parameters $(\gamma_1, a_1, \delta_1)$.

6.4.5. Repeat Exercise 6.4.4 if x and y have the densities of the form in (6.44) with different parameters.

6.6 Input–Output Analysis

Input–output situations are many in nature. In a dam or storage capacity there is inflow and outflow and the difference or the residual part is the storage. In nuclear reactions, energy is produced and part of it is dissipated, destroyed or emitted out and the residual part is what is left out. In a human body a chemical called melatonin is produced every day. The production starts by evening, peaks by 1 am and the level of the chemical is back to normal by the morning. The body consumes or converts what is produced. There is a positive residual part during the night and the residual part is zero by the morning. In a growth–decay mechanism an item grows and part of it decays, and the residual part is the difference. In a stochastic process there is an input variable and after the process there is an output. In an industrial production process the total money value of raw materials plus operational cost is the input variable and the money value of the final product is the output variable.

A simple input–output model can be considered as a structure such as

$$u = x - y, \tag{6.47}$$

where x is the input variable and y is the output variable and u can be taken as the residual. Stochastic situations when x and y are independently distributed random variables, scalar variables or matrix variables, are considered by Mathai (1993c). Connections of a structure such as the one in (6.47) to distributions of bilinear forms and covariance structures are also established in Mathai (1993c). A model such as the one in (6.47) when both the input and output variables are gamma random variables can be used to model solar neutrino production or other such residual processes (Haubold and Mathai 1994).

In a reaction–diffusion process if $N(t)$ is the number density at time t and if the production rate is proportional to the original number, then

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = \lambda N(t),\ \lambda > 0, \tag{6.48}$$

where λ is the rate of production. If the consumption or destruction rate is also proportional to the original number then

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = -\mu N(t),\ \mu > 0, \tag{6.49}$$

where μ is the destruction rate. Then the residual part is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = -cN(t), \ c = \mu - \lambda. \tag{6.50}$$

If c is free of t then the solution is the exponential model

$$N(t) = N_0 \mathrm{e}^{-ct}, N_0 = N(t) \text{ at } t = t_0, \tag{6.51}$$

where t_0 is the starting time. Instead of the total derivative in (6.48)–(6.50) if we consider fractional derivative or fractional nature of reactions, that is, if we consider an equation of the form

$$N(t) - N_0 = -c^{\nu} {}_0D_t{}^{-\nu} N(t), \tag{6.52}$$

where ${}_0D_t{}^{-\nu}$ is the standard Riemann–Liouville fractional integral operator, then the solution for $N(t)$ is a Mittag-Leffler function

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k (ct)^{\nu k}}{\Gamma(\nu k + 1)} = N_0 E_{\nu}(-(ct)^{\nu}) \tag{6.53}$$

where $E_{\nu}(\cdot)$ is the Mittag-Leffler function, which is a special case of a H-function. That is,

$$N(t) = N_0 \frac{1}{2\pi i} \int_L \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1-\nu s)} [(ct)^{\nu}]^{-s} \mathrm{d}s = N_0 H_{1,2}^{1,1} \left[(ct)^{\nu} \Big|_{(0,1),(0,\nu)}^{(0,1)} \right], \tag{6.54}$$

where L is a suitable contour. In such input–output models one can notice that under fractional rate of input or output can produce particular cases of H-functions as illustrated in (6.52)–(6.54). More of such situations will be examined in detail in the coming sections.

Exercises 6.5

6.5.1. Work out the density of $u = x - y$ if x and y are independently distributed with exponential densities with different parameters.

6.5.2. Repeat Exercise 6.5.1 if x has a gamma density and y has an exponential density.

6.5.3. Repeat Exercise 6.5.1 if both x and y have gamma densities with different parameters, for the cases (1): $x - y > 0$ and (2): general, where $x - y$ can be negative also.

6.5.4. Let x_j have the density

$$f_j(x_j) = c_j x_j^{\gamma_j - 1} e^{-a_j x_j^{\delta_j}}, x_j > 0, a_j > 0, \delta_j > 0, j = 1, 2,$$

where $c_j, j = 1, 2$ are the normalizing constants. Let $u = \ln x_1 - \ln x_2$. When x_1 and x_2 are statistically independently distributed, evaluate the density of u by using Laplace transform of the density of u. Show that the density of u can be written as a H-function.

6.5.5. Work out the special cases in Exercise 6.5.4 when (1) x_j's are Weibull distributed with different parameters, (2) Weibull distributed with the same parameters, (3) gamma distributed (a) with different parameters, (b) with identical parameters, (4) exponentially distributed with (a) different parameters, (b) with identical parameters. Show that all the densities can be written as special cases of H-functions.

6.7 Application to Kinetic Equations

Fractional kinetic equations are studied to determine certain physical phenomena governing diffusion in porous media, reaction and relaxation processes in complex systems and anomalous diffusion, etc. In this connection, one can refer to the monographs by Hilfer (2000), Kilbas et al. (2006), Podlubny (1999), and the various works cited therein. Fractional kinetic equations are studied by Hille and Tamarkin (1930), Glöckle and Nonnenmacher (1991), Saichev and Zaslavsky (1997), Zaslavsky (1994) and Saxena et al. (2002, 2004, 2004b), among others, for their importance in the solution of certain applied problems. We now proceed to prove the following:

Theorem 6.1. *If $c > 0, \nu > 0$, then the solution of the integral equation*

$$N(t) - N_0 f(t) = -c^{\nu}{}_0D_t^{-\nu} N(t), \tag{6.55}$$

where $f(t)$ is any integrable function on the finite interval $[0.b]$, there holds the formula

$$N(t) = cN_0 \int_0^t H_{1.2}^{1,1}\left[c^{\nu}(t-\tau)^{\nu}\middle|_{(-\frac{1}{\nu},1),(0,\nu)}^{(-\frac{1}{\nu},1)}\right] f(\tau)\mathrm{d}\tau, \tag{6.56}$$

where $H_{1,2}^{1,1}(.)$ is the H-function defined by (1.2).

Proof 6.1. Applying the Laplace transform to (6.55) and using (3.65), it gives,

$$\tilde{N}(s) = L[N(t); s] = N_0 \frac{F(s)}{1 + (c/s)^{\nu}}. \tag{6.57}$$

Since (Mathai and Saxena 1978, p. 152)

$$\frac{s^\nu}{s^\nu + c^\nu} = H_{1,1}^{1,1}\left[(s/c)^\nu \,\middle|\, \begin{matrix}(1,1)\\(1,1)\end{matrix}\right], \tag{6.58}$$

then using (2.22), we obtain

$$L^{-1}\left[H_{1,1}^{1,1}\left[(s/c)^\nu \,\middle|\, \begin{matrix}(1,1)\\(1,1)\end{matrix}\right]\right] = t^{-1}H_{2,1}^{1,1}\left[(ct)^{-\nu} \,\middle|\, \begin{matrix}(1,1),(0,\nu)\\(1,1)\end{matrix}\right]. \tag{6.59}$$

If we use the property of the H-function (1.58), the above equation becomes

$$L^{-1}\left[H_{1,1}^{1,1}\left[(s/c)^\nu \,\middle|\, \begin{matrix}(1,1)\\(1,1)\end{matrix}\right]\right] = t^{-1}H_{1,2}^{1,1}\left[(ct)^{\nu} \,\middle|\, \begin{matrix}(0,1)\\(0,1),(1,\nu)\end{matrix}\right] \tag{6.60}$$

$$= cH_{1,2}^{1,1}\left[(ct)^{\nu} \,\middle|\, \begin{matrix}(-1/\nu,1)\\(-1/\nu,1),(0,\nu)\end{matrix}\right]. \tag{6.61}$$

□

The result (6.61) follows from (6.60), if we use the formula (1.60). Taking the inverse Laplace transform of (6.57) and applying the convolution theorem of the Laplace transform, we arrive at the desired result (6.56).

If we set $f(t) = t^{\mu-1}$, we obtain the result given by Saxena et al. (2002, p. 283, Eq. (15)). Theorem 6.1 was proved by Saxena et al. (2004).

Note 6.1. An alternative method for deriving the solution of fractional kinetic equations is recently given by Saxena and Kalla (2008).

6.8 Fickean Diffusion

We consider Fick's diffusion and establish the following:

Theorem 6.2. *The solution of the diffusion equation*

$$\frac{\partial}{\partial t}N(x,t) = C_1\frac{\partial^2}{\partial x^2}N(x,t), \tag{6.62}$$

with initial condition $N(x, t = 0) = \delta(x)$, where $\delta(x)$ is the Dirac delta function, is given by

$$N(x,t) = \frac{1}{\sqrt{(4\pi C_1 t)}}\exp\left(-\frac{x^2}{4C_1 t}\right). \tag{6.63}$$

Proof 6.2. Applying Laplace transform to (6.62) with respect to the variable t and applying the given condition, it gives

$$s\tilde{N}(x,s) - \delta(x) = C_1 \frac{\partial^2}{\partial x^2}\tilde{N}(x,s). \tag{6.64}$$

Applying Fourier transform to the above equation with respect to x, we obtain

$$s\tilde{N}^*(k,s) - 1 = C_1(-k^2)\tilde{N}^*(k,s). \tag{6.65}$$

Solving for $\tilde{N}^*(k,s)$, it gives

$$N^{\sim *}(k,s) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{k^2 C_1}{s}\right)^r s^{-1}. \tag{6.66}$$

On inverting (6.66), the desired result (6.63) is obtained, where we have used the inverse Fourier transform formula

$$F^{-1}\left\{e^{-ak^2};x\right\} = \frac{1}{\sqrt{4\pi a}}\exp\left(-\frac{x^2}{4a}\right). \tag{6.67}$$

□

Remark 6.1. Standard diffusion processes are described with the help of Fick's second law. The diffusion equation (6.62) can be derived by combining the continuity equation

$$\frac{\partial}{\partial t}N(x,t) = -S_x(x,t), \tag{6.68}$$

and the constitutive equation

$$S(x,t) = -C_1 N_x(x,t), \tag{6.69}$$

which is also called as Fick's first law. Here, $S(x,t)$ represents the flux, $N(x,t)$ the distribution function of the diffusing quantity, and C_1 a diffusion constant which is assumed to be a constant.

6.8.1 Application to Time-Fractional Diffusion

Theorem 6.3. *Consider the following time-fractional diffusion equation*

$$\frac{\partial^\alpha N(x,t)}{\partial t^\alpha} = D\frac{\partial^2 N(x,t)}{\partial x^2}, 0 < \alpha < 1, x \in R, R = (-\infty,\infty), \tag{6.70}$$

where D is the diffusion constant and $\in R\backslash\{0\}$

$$N(x,t=0)=\delta(x),\ \lim_{x\to\pm\infty} N(x,t)=0, \tag{6.71}$$

$\frac{\partial^\alpha}{\partial t^\alpha}$ *is the Caputo fractional derivative defined by (6.114) and* $\delta(x)$ *is the Dirac delta function. Then its fundamental solution is given by*

$$N(x,t)=\frac{1}{|x|}H_{1,1}^{1,0}\left[\frac{|x|^2}{Dt^\alpha}\Big|_{(1,2)}^{(1,\alpha)}\right]. \tag{6.72}$$

Remark 6.2. It can be seen that Brownian motion takes place at $\alpha=1$, which is irreversible. Wave propagation takes place at $\alpha=2$ which is reversible.

Proof 6.3. In order to find a closed form representation of the solution of the equation (6.70) in terms of the H-function, we use the method of joint Laplace–Fourier transform, defined by

$$\tilde{N}^*(k,s)=\int_0^\infty\int_{-\infty}^\infty \mathrm{e}^{-st+ikx}N(x,t)\mathrm{d}x\mathrm{d}t, \tag{6.73}$$

where, according to the convention followed, "$\sim$" will denote the Laplace transform and "$*$", the Fourier transform. Applying the Laplace transform with respect to time variable t, Fourier transform with respect to space variable x, using (3.75) and the given condition (6.71), we find that

$$s^\alpha\tilde{N}^*(k,s)-s^{\alpha-1}=-Dk^2N^{\sim *}(k,s).$$

Solving for $N^{\sim *}(k,s)$, it gives

$$\tilde{N}^*(k,s)=\frac{s^{\alpha-1}}{s^\alpha+Dk^2}.$$

Inverting the Laplace transform, it yields

$$N^*(k,t)=L^{-1}\left[\frac{s^{\alpha-1}}{s^\alpha+Dk^2}\right]=E_\alpha(-Dk^2t^\alpha), \tag{6.74}$$

where $E_\alpha(.)$, is the Mittag-Leffler function defined by (1.44). □

In order to invert the Fourier transform, we will make use of the integral

$$\int_0^\infty \cos(kt)E_{\alpha,\beta}(-at^2)\mathrm{d}t=\frac{\pi}{k}H_{1,1}^{1,0}\left[\frac{k^2}{a}\Big|_{(1,2)}^{(\beta,\alpha)}\right], \tag{6.75}$$

which follows from (2.51); where $\Re(\alpha) > 0, \Re(\beta) > 0, k > 0, a > 0$; and the formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-ikx} f(k)\mathrm{d}k = \frac{1}{\pi}\int_{0}^{\infty} f(k)\cos(kx)\mathrm{d}k, \tag{6.76}$$

then it yields the required solution.

Note 6.2. When $\alpha = 1$, (6.72) reduces to (6.63) as

$$\begin{aligned} N(x,t) &= \frac{1}{|x|}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(1-2s)}{\Gamma(1-s)}\left(\frac{|x|^2}{Dt}\right)^s \mathrm{d}s \\ &= \frac{1}{|x|}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma\left(\frac{1}{2}-s\right)\Gamma(1-s)2^{-2s}\pi^{-\frac{1}{2}}}{\Gamma(1-s)}\left(\frac{|x|^2}{Dt}\right)^s \mathrm{d}s \\ &= \frac{1}{(4\pi Dt)^{\frac{1}{2}}}\exp\left(-\frac{|x|^2}{4Dt}\right), \end{aligned} \tag{6.77}$$

which is a Gaussian density.

6.9 Application to Space-Fractional Diffusion

Notation 6.1. $\frac{\partial^\alpha}{\partial x^\alpha}N(x,t)$: Liouville fractional derivative of order α

Definition 6.1. The Liouville fractional derivative of order α is defined by

$$\frac{\partial^\alpha}{\partial x^\alpha}N(x,t)=\frac{1}{\Gamma(m-\alpha)}\left(\frac{\partial}{\partial x}\right)^m\int_{-\infty}^{x}\frac{N(t,y)}{(x-y)^{\alpha-m+1}}\mathrm{d}y, x\in R, \alpha>0, m=[\alpha]+1, \tag{6.78}$$

where $[\alpha]$ is the integral part of α.

Note 6.3. The operator defined by (6.78) is also denoted by

$${}_{-\infty}D_x^\alpha N(x,t).$$

Its Fourier transform is given by

$$F\{{}_{-\infty}D_x^\alpha f(x,t)\} = (ik)^\alpha \Psi(k,t), \alpha > 0, \tag{6.79}$$

where $\Psi(k,t)$ is the Fourier transform of $f(x,t)$ with respect to the variable x of $f(x,t)$. Following the convention initiated by Compte (1996), we suppress the imaginary unit in Fourier space by adopting the slightly modified form of the above result in our investigations

$$F\{{}_{-\infty}D_x^\alpha f(x,t)\} = -|k|^\alpha \Psi(k,t), \alpha > 0 \tag{6.80}$$

instead of (6.79).

In this section, we will investigate the solution of the equation (6.81). The result is given in the form of the following:

Theorem 6.4. *Consider the following space-fractional diffusion equation*

$$\frac{\partial N(x,t)}{\partial t} = D\frac{\partial^\alpha N(x,t)}{\partial x^\alpha}, 0 < \alpha < 1, x \in R, \tag{6.81}$$

where D is the diffusion constant and $\in R\backslash\{0\}$, $\frac{\partial^\alpha}{\partial x^\alpha}N(x,t)$ is the Liouville fractional derivative of order α; $N(x,t=0)=\delta(x)$, where $\delta(x)$ is the Dirac delta function and $\lim_{x\to\pm\infty} N(x,t)=0$. Then its fundamental solution is given by

$$N(x,t) = \frac{1}{\alpha|x|}H_{2,2}^{1,1}\left[\frac{|x|}{(Dt)^{1/\alpha}}\left|\begin{matrix}(1,1/\alpha),(1,\frac{1}{2})\\(1,1),(1,\frac{1}{2})\end{matrix}\right.\right]. \tag{6.82}$$

Proof 6.4. Applying the Laplace transform with respect to the time variable t, Fourier transform with respect to space variable x and using the given condition and the Eq. (6.80), it gives

$$s\tilde{N}^*(k,s) - 1 = -D|k|^\alpha \tilde{N}^*(k,s).$$

Solving for $\tilde{N}^*(k,s)$ and inverting the Laplace transform, it is seen that

$$\begin{aligned} N^*(k,t) &= L^{-1}\left[\sum_{r=0}^{\infty}(-1)^r s^{-r-1}(D|k|^\alpha)^r\right] = \sum_{r=0}^{\infty}\frac{(-1)^r t^r (D|k|^\alpha)^r}{\Gamma(r+1)} \\ &= \exp(-Dt|k|^\alpha) = H_{0,1}^{1,0}\left[Dt|k|^\alpha\left|\begin{matrix}-\\(0,1)\end{matrix}\right.\right]. \end{aligned} \tag{6.83}$$

If we invert the Fourier transform with $\beta=\gamma=1, \theta=0$, the result (6.82) follows. □

6.10 Application to Fractional Diffusion Equation

In this section we present an alternative shorter method for deriving the solution of a diffusion equation discussed earlier by Kochubei (1990).

Theorem 6.5. *Consider the Cauchy problem*

$${}_0D_t^\alpha N(x,t) = -c^\nu \Delta N(x,t), 0<\alpha<1; x\in\Re^n; 0<t\leq T, \tag{6.84}$$

with

$$N(x,t=0)=\delta(x), x\in\Re, \lim_{x\to\pm\infty} N(x,t)=0. \tag{6.85}$$

$_0D_t^\alpha$ *is the regularized Caputo (1969) partial fractional derivative with respect to t, defined by*

$$_0D_t^\alpha N(x,t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{\partial}{\partial t}\int_0^t \frac{N(x,s)\mathrm{d}s}{(t-x)^\alpha} - \frac{N(x,0)}{t^\alpha}\right],$$

and Δ *is the Laplacian. The fundamental solution of the above Cauchy problem is given by*

$$N(x,t) = |x|^{-n}\pi^{-\frac{n}{2}} H_{1,2}^{2,0}\left[\frac{|x|^2 t^{-\alpha}}{4c^\nu}\middle|_{(\frac{n}{2},1),(1,1)}^{(1,\alpha)}\right], \tag{6.86}$$

where $H_{1,2}^{2,0}(.)$ *is the* H*-function (1.2).*

Proof 6.5. Applying the Laplace transform with respect to t, Fourier transform with respect to x to (6.84) and using the result (3.75), it gives

$$s^\alpha \tilde{N}^*(k,s) - s^{\alpha-1} = -c^\nu |k|^2 \tilde{N}^*(k,s),$$

where the symbol "$\sim$" indicates the Laplace transform with respect to the time variable t and the symbol "$*$", the Fourier transform with respect to the space variable x. Solving for $\overset{\sim *}{N}(k,s)$, we have

$$\tilde{N}^*(k,s) = \frac{s^{\alpha-1}}{s^\alpha + c^\nu |k|^2}. \tag{6.87}$$

By virtue of the following Fourier transform formula

$$\left(F_x\left[|x|^{(2-n)/2}K_{(n-2)/2}(a|x|)\right]\right)(\tau) = \left(\frac{2\pi}{a}\right)^{n/2}\frac{a}{a^2+|\tau|^2},\ \tau\in\Re^n;\quad n\in \mathrm{N},\ a>0, \tag{6.88}$$

where the multidimensional Fourier transform with respect to $x \in \Re^n$ is defined by

$$(F_x N)(\tau,t) = \int_{\Re^n} N(x,t)\mathrm{e}^{ix\tau}\mathrm{d}x,\ \tau\in\Re^n,\ t>0, \tag{6.89}$$

and $K_\nu(.)$ is the modified Bessel function of the second kind, it yields

$$\tilde{N}(x,s) = c^{-\nu}s^{\alpha-1}(2\pi)^{-\frac{n}{2}}\left(\frac{|x|c^{\frac{\nu}{2}}}{s^{\frac{\alpha}{2}}}\right)^{1-\frac{n}{2}} K_{\frac{n-2}{2}}\left[\frac{|s^{\frac{\alpha}{2}}|x|}{c^{\frac{\nu}{2}}}\right]. \tag{6.90}$$

In order to invert the Laplace transform, we employ the following result given by the authors (Saxena et al. 2006)

$$L^{-1}\{s^{-\rho}K_\nu(zs^\sigma);t\} = \frac{1}{2}t^{\rho-1}H_{1,2}^{2,0}\left[\frac{z^2t^{-2\sigma}}{4}\middle|_{(\frac{\nu}{2},1),(-\frac{\nu}{2},1)}^{(\rho,2\sigma)}\right], \tag{6.91}$$

where $K_\nu(x)$ is the modified Bessel function of the second kind, $\Re(z^2) > 0$, $\Re(s) > 0$. Thus we obtain the solution in a closed form

$$N(x,t) = \frac{1}{2}(2\pi)^{-\frac{n}{2}} c^{-\frac{\nu}{2}-\frac{n\nu}{4}} |x|^{1-\frac{n}{2}} t^{-\frac{\alpha}{2}-\frac{\alpha n}{4}} H_{1,2}^{2,0}\left[\frac{t^{-\alpha}|x|^2}{4c^\nu}\middle|\begin{matrix}(1-\frac{\alpha}{2}-\frac{\alpha n}{4},\alpha)\\(\frac{n-2}{4},1),(\frac{2-n}{4},1)\end{matrix}\right]. \tag{6.92}$$

By virtue of the H function identity (1.60), the power of the expression $\left[\frac{\{t^{-\nu}|x|^2\}}{4c^\nu}\right]$ can be absorbed inside the H-function and consequently we obtain

$$N(x,t) = |\pi^{\frac{1}{2}} x|^{-n} H_{1,2}^{2,0}\left[\frac{t^{-\alpha}|x|^2}{4c^\nu}\middle|\begin{matrix}(1,\alpha)\\(\frac{n}{2},1),(1,1)\end{matrix}\right]. \tag{6.93}$$

□

Remark 6.3. If we employ the identity (1.58), the solution given by (6.93) can be expressed in the form

$$N(x,t) = \frac{1}{\alpha}|\pi^{\frac{1}{2}} x|^{-n} H_{1,2}^{2,0}\left[\frac{t^{-1}|x|^{2/\alpha}}{(4c^\nu)^{\frac{1}{\alpha}}}\middle|\begin{matrix}(1,1)\\(\frac{n}{2},\frac{1}{\alpha}),(1,\frac{1}{\alpha})\end{matrix}\right], \tag{6.94}$$

where $\alpha > 0$.

Note 6.4. We note that the above form of the solution is due to Schneider and Wyss (1989). There is one importance of our result (6.91) that it includes the Lévy stable density in terms of the H-function as shown in (6.102). Similarly, using the identity (1.59), we arrive at

$$N(x,t) = \frac{1}{2}|\pi^{\frac{1}{2}} x|^{-n} H_{1,2}^{2,0}\left[\frac{t^{-\frac{\alpha}{2}}|x|}{2c^{\frac{\nu}{2}}}\middle|\begin{matrix}(1,\frac{\alpha}{2})\\(\frac{n}{2},\frac{1}{2}),(1,\frac{1}{2})\end{matrix}\right], \tag{6.95}$$

where n is not an even integer. This form of the H-function is useful in determining its expansion in powers of x. Due to importance of the solution, we also discuss its series representation and behavior.

6.10.1 Series Representation of the Solution

Using the series expansion for the H-function given in the monograph (Mathai and Saxena, 1978), it follows that

$$H_{1,2}^{2,0}\left[x\middle|\begin{matrix}(1,1)\\(\frac{n}{2},\frac{1}{\alpha}),(1,\frac{1}{\alpha})\end{matrix}\right] = \frac{1}{2\pi i}\int_L \frac{\Gamma\left(\frac{n}{2}-\frac{s}{\alpha}\right)\Gamma\left(1-\frac{s}{\alpha}\right)}{\Gamma(1-s)} x^s \mathrm{d}s$$

$$= \alpha\left\{\sum_{\lambda=0}^{\infty}\frac{\Gamma\left(1-\frac{n}{2}-\lambda\right)(-1)^\lambda x^{\alpha(\frac{n}{2}+\lambda)}}{\Gamma\left(1-\frac{an}{2}-\alpha\lambda\right)(\lambda!)} + \sum_{\lambda=0}^{\infty}\frac{\Gamma\left(\frac{n}{2}-1-\lambda\right)(-1)^\lambda x^{\alpha(1+\lambda)}}{\Gamma(1-\alpha-\alpha\lambda)(\lambda!)}\right\}, \tag{6.96}$$

where n is not an even integer. Thus for $n = 1$, we find that

$$N(x,t) = \frac{1}{2t^{\frac{\alpha}{2}}} \sum_{\lambda=0}^{\infty} (-1)^{\lambda} \frac{A^{\frac{\lambda}{2}}}{\Gamma(1-\alpha(\lambda+1)/2)(\lambda!)}, \tag{6.97}$$

where $A = \frac{x^2}{t^{\alpha}}$ and the duplication formula for the gamma function is used. For $n = 2$, H-function of (6.95) is singular and in this case, the result is explicitly given by Saichev (Barkai 2001) in the form

$$N(x,t) \sim \frac{1}{\pi\Gamma(1-\alpha)t^{\alpha}} \ln\left[\frac{t^{\alpha/2}}{x}\right]. \tag{6.98}$$

For $n = 3$, the series expansion is given by

$$N(x,t) = \frac{1}{4\pi t^{\frac{3\alpha}{2}} A^{\frac{1}{2}}} \sum_{\lambda=0}^{\infty} \frac{(-1)^{\lambda} A^{\frac{\lambda}{2}}}{(\lambda)!\Gamma\left[1-\alpha\left(1+\frac{\lambda}{2}\right)\right]}. \tag{6.99}$$

From above it readily follows that for $n = 3$ and $\alpha \neq 1$,

$$N(x,t) \sim \frac{1}{x} \text{ as } x \to \infty. \tag{6.100}$$

It will not be out of place to mention that the one sided Lévy stable density $\varphi_{\rho}(t)$ can be obtained from Laplace inversion formula (6.91) by virtue of the identity

$$K_{\pm\frac{1}{2}}(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} \mathrm{e}^{-x}, \tag{6.101}$$

and can be conveniently expressed in terms of the Laplace transform as

$$\int_0^{\infty} \mathrm{e}^{-ut} \varphi_{\rho}(t) \mathrm{d}t = \mathrm{e}^{-u^{\rho}}, \Re(u) > 0, \Re(\rho) > 0. \tag{6.102}$$

The result is,

$$\varphi_{\rho}(t) = \frac{1}{\rho} H_{1,1}^{1,0}\left[\frac{1}{t}\left|\begin{matrix}(1,1)\\ \left(\frac{1}{\rho},\frac{1}{\rho}\right)\end{matrix}\right.\right], \rho > 0. \tag{6.103}$$

Note 6.5. This result is obtained earlier by Schneider and Wyss (1989) by following a different procedure. Asymptotic behavior of $\varphi_{\alpha}(t)$ is also given by Schneider (1986).

6.11 Application to Generalized Reaction-Diffusion Model

6.11.1 Motivation

It is a known fact that reaction–diffusion models play a very important role in pattern formation in biology, chemistry and physics, see Wilhelmsson and Lazzaro (2001) and Frank (2005). These systems indicate that diffusion can produce the spontaneous formation of spatio-temporal patterns. For details, one can refer to the work of Nicolis and Prigogine (1977) and Haken (2004). A general model for reaction–diffusion systems is investigated by Henry and Wearne (2000, 2002) and Henry et al. (2005).

The simplest reaction–diffusion models are of the form

$$\frac{\partial N}{\partial t} = D\frac{\partial^2 N}{\partial x^2} + F(N), N = N(x,t), \tag{6.104}$$

where D is the diffusion constant and $F(N)$ is a nonlinear function representing reaction kinetics. It is interesting to observe that for $F(N) = \gamma N(1-N)$, (6.104) reduces to Fisher–Kolmogorov equation and if, however, we set $F(N) = \gamma N(1-N^2)$, it gives rise to the real Ginsburg–Landau equation. Del-Castillo-Negrete et al. (2002) studied the front propagation and segregation in a system of reaction–diffusion equations with cross-diffusion. Recently Del-Castillo-Negrete et al. (2003) discussed the dynamics in reaction–diffusion systems with non-Gaussian diffusion caused by asymmetric Lévy flights and solved the following model:

$$\frac{\partial N}{\partial t} = {}_{\eta}D_x^{\alpha} N + F(N), \quad N = N(x,t), \quad F(0) = 0. \tag{6.105}$$

Remark 6.4. It is interesting to observe that the Eq. (6.104) also represents the classical reproduction-dispersal equation for the growth and dispersal of biological species (Fisher 1937; Kolomogorov et al. 1937).

In this section, we present a solution of a more general model of fractional reaction–diffusion system (6.105) in which $\frac{\partial N}{\partial t}$ has been replaced by the Riemann–Liouville fractional derivative ${}_0D_t^{\beta}, \beta > 0$. The results derived are of general nature than those investigated earlier by many authors notably by Jespersen et al. (1999) for anomalous diffusion and by Del-Castillo-Negrete et al. (2003) for the reaction–diffusion systems with Lévy flights and fractional diffusion equation by Kilbas et al. (2004). The solution has been developed in terms of the H-function in a compact and elegant form with the help of Laplace and Fourier transforms and their inverses. Most of the results obtained are in a form suitable for numerical computation. The results reported in this section are in continuation of our earlier investigations, Haubold (1998), Haubold and Mathai (2000) and Saxena et al. (2002, 2004, 2004a,b, 2006, 2006a).

6.11.2 Mathematical Prerequisites

In order to present the results of this section, the definitions of the well-known Laplace and Fourier transforms of a function $N(x,t)$ and their inverses are described below:

Notation 6.2. $L\{N(x,s)\}$: Laplace transform of a function $N(x,t)$ with respect to t.

Notation 6.3. $F\{N(x,t)\}$: The Fourier transform of a function $N(x,t)$ with respect to x.

Definition 6.2. The Laplace transform of a function $N(x,t)$ with respect to t is defined by

$$\tilde{N}(x,s) = L\{N(x,t)\} = \int_0^\infty \mathrm{e}^{-st} N(x,t)\mathrm{d}t, \quad t > 0, \quad x \in R, \tag{6.106}$$

where $\Re(s) > 0$, and its inverse transform with respect to s is given by

$$L^{-1}\{\tilde{N}(x,s)\} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \mathrm{e}^{st} \tilde{N}(x,s)\mathrm{d}s, \tag{6.107}$$

γ being a fixed real number.

Definition 6.3. The Fourier transform of a function $N(x,t)$ with respect to x is defined by

$$N^*(k,t) = F\{N(x,t)\} = \int_{-\infty}^{\infty} \mathrm{e}^{ikx} N(x,t)\mathrm{d}x \quad i = \sqrt{-1}. \tag{6.108}$$

The inverse Fourier transform with respect to k is given by the formula

$$N(x,t) = F^{-1}\{N^*(k,t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-ikx} N^*(k,t)\mathrm{d}k. \tag{6.109}$$

The space of functions for which the transforms defined by (6.106) and (6.108) exist is denoted by $LF = L(R_+) \times F(R)$.

Notation 6.4. ${}_0D_t^{-\nu} N(x,t)$: The Riemann–Liouville fractional integral of order ν.

Definition 6.4. The Riemann–Liouville fractional integral of order ν is defined by

$${}_0D_t^{-\nu} N(x,t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1} N(x,u)\mathrm{d}u, \tag{6.110}$$

where $\Re(\nu) > 0$.

Notation 6.5. ${}_0D_t^{\alpha} N(x,t)$: The Riemann–Liouville fractional derivative of order $\alpha > 0$.

Definition 6.5. Following Samko et al. (1993, p. 37) we define the fractional derivative of order $\alpha > 0$ in the form

$$ {}_0D_t^{\alpha} N(x,t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{N(x,u)}{(t-u)^{\alpha-n+1}} \mathrm{d}u, \quad t > 0, \quad n = [\alpha]+1, \quad (6.111) $$

where $[\alpha]$ means the integral part of the number α. From Erdélyi et al. (1954, Vol. II, p. 182) we have

$$ L\{{}_0D_t^{-\nu} N(x,t)\} = s^{-\nu} \tilde{N}(x,s), \quad (6.112) $$

where $\tilde{N}(x,s)$ is the Laplace transform with respect to t of $N(x,t)$, $\Re(s) > 0$ and $\Re(\nu) > 0$.

The Laplace transform of the fractional derivative, defined by (6.111) is given by Oldham and Spanier (1974, p. 134, Eq. (8.1.3)):

$$ L\{{}_0D_t^{\alpha} N(x,t)\} = s^{\alpha} \tilde{N}(x,s) - \sum_{r=1}^{n} s^{r-1} {}_0D_t^{\alpha-r} N(x,t)|_{t=0}, \quad n-1 < \alpha \leq n. \quad (6.113) $$

Notation 6.6. ${}^{C}{}_0D_t^{\alpha} f(x,t)$: Caputo fractional derivative of order $\alpha > 0$.

Definition 6.6. The following fractional derivative of order $\alpha > 0$ is introduced by Caputo (1969) in the form

$$ {}^{C}{}_0D_t^{\alpha} f(x,t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(x,\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha+1-m}} \mathrm{d}\tau, \quad m-1 < \alpha \leq m. $$

The above formula is useful in deriving the solution of differential and integral equations of fractional order governing certain physical problems of reaction and diffusion. The Laplace transform of the Caputo derivative is given by

$$ L\{{}^{C}{}_0D_t^{\alpha} f(x,t)\} = s^{\alpha} \tilde{f}(x,s) - \sum_{r=0}^{n-1} s^{\alpha-r-1} f^{(r)}(x,0_+), n-1 < \alpha \leq n, \quad (6.114) $$

where $\alpha, s \in C, \Re(s) > 0, \Re(\alpha) > 0$.

Note 6.6. If there is no confusion, then this derivative ${}^{C}{}_0D_t^{\alpha}$ for simplicity will be denoted by ${}_0D_t^{\alpha}$.

Remark 6.5. Recently, Bagley (2007) has given the equivalence of Riemann–Liouville and Caputo fractional order derivatives in connection with modeling of linear viscoelastic materials.

6.11.3 Fractional Reaction–Diffusion Equation

In this section, we will investigate the solution of the generalized reaction–diffusion equation (6.115). The result is given in the form of the following result:

Theorem 6.6. *Consider the generalized fractional reaction–diffusion model*

$$ {}_0D_t^{\beta} N(x,t) = \eta_{-\infty} D_x^{\alpha} N(x,t) + \phi(x,t), \tag{6.115}$$

where $\eta > 0, t > 0, x \in R, 1 < \beta \le 2,\ 0 \le \alpha \le 1$, *with the initial conditions*

$$ [{}_0D_r^{\beta-1} N(x,0)] = f(x),\ [{}_0D_t^{\beta-2} N(x,0)] = g(x), x \in R,\ \lim_{x\to\pm\infty} N(x,t) = 0, \tag{6.116}$$

where ${}_{-\infty}D_x^{\alpha} N(x,t)$ *is defined in (6.78);* $[{}_0D_t^{\beta-1} N(x,0)]$ *means the Riemann-Liouville fractional derivative of order* $\beta - 1$ *with respect to* t *evaluated at* $t = 0$*. Similarly* $[{}_0D_t^{\beta-2} N(x,0)]$ *means the Riemann–Liouville fractional derivative of order* $\beta - 2$ *with respect to* t *evaluated at* $t = 0$*.* η *is a diffusion constant and* $\phi(x,t)$ *is a nonlinear function belonging to the area of reaction kinetics. Then for the solution of (6.115), subject to the initial conditions (6.116), there holds the formula*

$$\begin{aligned} N(x,t) &= \frac{t^{\beta-1}}{2\pi}\int_{-\infty}^{\infty} f^*(k) E_{\beta,\beta}(-\eta)|k|^{\alpha} t^{\beta} \exp(-ikx)\mathrm{d}k \\ &+ \frac{t^{\beta-2}}{2\pi}\int_{-\infty}^{\infty} g^*(k) E_{\beta,\beta-1}(-\eta|k|^{\alpha} t^{\beta}) \exp(-ikx)\mathrm{d}x \\ &+ \frac{1}{2\pi}\int_0^t \zeta^{\beta-1}\int_{-\infty}^{\infty} \tilde{\phi}(k,t-\zeta) E_{\beta,\beta}(-\eta|k|^{\alpha}\zeta^{\beta}) \exp(-ikx)\mathrm{d}k\mathrm{d}\zeta, \end{aligned} \tag{6.117}$$

where $*$ *indicates the Fourier transform with respect to space variable* x*.*

Proof 6.6. If we apply the Laplace transform with respect to the time variable t and use the formula (6.113), the given equation (6.115) becomes

$$ s^{\beta}\tilde{N}(x,s) - f(x) - sg(x) = \eta_{-\infty} D_x^{\alpha}\tilde{N}(x,s) + \tilde{\phi}(x,s). \tag{6.118}$$

□

As is customary, it is convenient to employ the symbol $\tilde{N}(x,s)$ to indicate the Laplace transform of $N(x,t)$ with respect to the variable t.

Now we apply the Fourier transform with respect to space variable x to the above equation, use the initial conditions and the result (6.80), then the above equation transforms into the form

$$ \tilde{N}^*(k,s) = \frac{f^*(k)}{s^{\beta} + \eta|k|^{\alpha}} + \frac{sg^*(k)}{s^{\beta} + \eta|k|^{\alpha}} + \frac{\tilde{\phi}^*(k)}{s^{\beta} + \eta|k|^{\alpha}}. \tag{6.119}$$

On taking the inverse Laplace transform of (6.119) and using the result

$$L^{-1}\left\{\frac{s^{\beta-1}}{a+s^{\alpha}};t\right\} = t^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(-at^{\alpha}), \tag{6.120}$$

where $\Re(s) > 0, \Re(\alpha-\beta) > -1$, it is seen that

$$\begin{aligned} N^{*}(k,t) = {} & f^{*}(k)t^{\beta-1}E_{\beta,\beta}(-\eta|k|^{\alpha}t^{\beta}) + g^{*}(k)t^{\beta-2}E_{\beta,\beta-1}(-\eta|k|^{\alpha}t^{\beta}) \\ & + \int_0^t \tilde{\phi}(k,t-\zeta)\zeta^{\beta-1}E_{\beta,\beta}(-\eta|k|^{\alpha}\zeta^{\beta})\mathrm{d}\zeta. \end{aligned} \tag{6.121}$$

The required solution (6.121) now readily follows by taking the inverse Fourier transform of (6.117). Thus, we have

$$\begin{aligned} N(x,t) = {} & \frac{t^{\beta-1}}{2\pi}\int_{-\infty}^{\infty} f^{*}(k)E_{\beta,\beta}(-\eta|k|^{\alpha}t^{\beta})\exp(-ikx)\mathrm{d}k \\ & + \frac{t^{\beta-2}}{2\pi}\int_{-\infty}^{\infty} g^{*}(k)E_{\beta,\beta-1}(-\eta|k|^{\alpha}t^{\beta})\exp(-ikx)\mathrm{d}k \\ & + \frac{1}{2\pi}\int_0^t \zeta^{\beta-1}\int_{-\infty}^{\infty}\tilde{\phi}(k,t-\zeta)E_{\beta,\beta}(-\eta|k|^{\alpha}\zeta^{\beta})\exp(-ikx)\mathrm{d}k\mathrm{d}\zeta. \end{aligned} \tag{6.122}$$

This completes the proof of the Theorem 6.6.

Note 6.7. It may be noted here that by virtue of the identity (1.136), the solution (6.117) can be expressed in terms of the H-function as can be seen from the solutions given in the special cases of the theorem in the next section. Further, we observe that (6.117) is not an explicit solution, special cases are interesting, general solution is not.

6.11.4 Some Special Cases

When $g(x) = 0$, then applying the convolution theorem of the Fourier transform to the solution (6.117), the theorem yields the following result:

Corollary 6.1. *The solution of fractional reaction-diffusion equation*

$${}_0D_t^{\beta}N(x,t) = \eta\,{}_{-\infty}D_x^{\alpha}N(x,t) + \phi(x,t), \quad t > 0, \eta > 0, \tag{6.123}$$

subject to the conditions

$$[{}_0D_t^{\beta-1}N(x,t)]_{t=0} = f(x), \quad [{}_0D_t^{\beta-2}N(x,t)]_{t=0} = 0, \tag{6.124}$$

for $x \in R, \lim_{x\to\pm\infty} N(x,t) = 0,\ 1 < \beta \le 2,\ 0 \le \alpha \le 1$, *when* η *is a diffusion constant and* $\phi(x,t)$ *is a nonlinear function belonging to the area of reaction kinetics is given by*

$$N(x,t) = \int_{-\infty}^{\infty} G_1(x-\tau,t) f(\tau)\mathrm{d}\tau + \int_0^t (t-\zeta)^{\beta-1} \int_0^x G_2(x-\tau,t-\zeta)\phi(\tau,\zeta)\mathrm{d}\tau\mathrm{d}\zeta, \tag{6.125}$$

where,

$$\begin{aligned} G_1(x,t) &= \frac{t^{\beta-1}}{2\pi}\int_{-\infty}^{\infty} \exp(-ikx)E_{\beta,\beta}(-\eta|k|^{\alpha}t^{\beta})\mathrm{d}k \\ &= \frac{t^{\beta-1}}{\pi\alpha}\int_0^{\infty} \cos(kx) H_{1,2}^{1,1}\left[k\eta^{\frac{1}{\alpha}}t^{\frac{\beta}{\alpha}}\left|\begin{matrix}(0,\frac{1}{\alpha})\\(0,\frac{1}{\alpha}),(1-\beta,\frac{\beta}{\alpha})\end{matrix}\right.\right]\mathrm{d}k \\ &= \frac{t^{\beta-1}}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{\eta^{\frac{1}{\alpha}}t^{\frac{\beta}{\alpha}}}\left|\begin{matrix}(1,\frac{1}{\alpha}),(\beta,\frac{\beta}{\alpha}),(1,\frac{1}{2})\\(1,1),(1,\frac{1}{\alpha}),(1,\frac{1}{2})\end{matrix}\right.\right], \Re(\alpha) > 0, \end{aligned} \tag{6.126}$$

$$\begin{aligned} G_2(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ikx)E_{\beta,\beta}(-\eta|k|^{\alpha}t^{\beta})\mathrm{d}k \\ &= \frac{1}{\pi\alpha}\int_0^{\infty} \cos(kx) H_{1,2}^{1,1}\left[k\eta^{\frac{1}{\alpha}}t^{\frac{\beta}{\alpha}}\left|\begin{matrix}(0,\frac{1}{\alpha})\\(0,\frac{1}{\alpha}),(1-\beta,\frac{\beta}{\alpha})\end{matrix}\right.\right]\mathrm{d}k \\ &= \frac{1}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{\eta^{\frac{1}{\alpha}}t^{\frac{\beta}{\alpha}}}\left|\begin{matrix}(1,\frac{1}{\alpha}),(\beta,\frac{\beta}{\alpha}),(1,\frac{1}{2})\\(1,1),(1,\frac{1}{\alpha}),(1,\frac{1}{2})\end{matrix}\right.\right],\ \Re(\alpha) > 0. \end{aligned} \tag{6.127}$$

If we set $f(x) = \delta(x),\ \phi = 0$, where $\delta(x)$ is the Dirac-delta function, then we arrive at the following result:

Corollary 6.2. *Consider the following reaction–diffusion model*

$$\frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}}N(x,t) = \eta\,{}_{-\infty}D_x^{\alpha}N(x,t),\quad \eta > 0,\quad x \in R, \tag{6.128}$$

with the initial condition

$$[{}_0D_t^{\beta-1}N(x,t)]_{t=0} = \delta(x),\quad \lim_{x\to\pm\infty} N(x,t) = 0,\ 0 < \beta \le 1,$$

where η *is a diffusion constant and* $\delta(x)$ *is the Dirac-delta function. Then the fundamental solution of (6.128) under the given initial conditions is given by*

$$N(x,t) = \frac{t^{\beta-1}}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{(\eta t^{\beta})^{1/\alpha}}\left|\begin{matrix}(1,1/\alpha),(\beta,\beta/\alpha),(1,1/2)\\(1,1),(1,1/\alpha),(1,1/2)\end{matrix}\right.\right], \tag{6.129}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0.$

When $\beta = \frac{1}{2}$ the above corollary reduces to the following interesting result: Consider the following reaction–diffusion model

$$\frac{\mathrm{d}^{\frac{1}{2}}}{\mathrm{d}t^{\frac{1}{2}}}N(x,t) = \eta_{-\infty}D_x^{\alpha}N(x,t),\ \eta > 0, x \in R, \tag{6.130}$$

with the initial condition

$$[{}_0D_t^{-\frac{1}{2}}N(x,t)]_{t=0} = \delta(x),\quad \lim_{x\to\pm\infty} N(x,t) = 0,$$

where η is a diffusion constant and $\delta(x)$ is the Dirac-delta function. Then the fundamental solution of (6.130) under the given initial conditions is given by

$$N(x,t) = \frac{1}{\alpha|x|t^{1/2}}H_{3,3}^{2,1}\left[\frac{|x|}{(\eta t^{1/2})^{1/\alpha}}\middle|_{(1,1),(1,\frac{1}{\alpha}),(1,\frac{1}{2})}^{(1,\frac{1}{\alpha}),(\frac{1}{2},\frac{1}{2\alpha}),(1,\frac{1}{2})}\right], \tag{6.131}$$

where $\Re(\alpha) > 0$.

Remark 6.6. The solution of the Eq. (6.128), as given by Kilbas et al. (2004) is in terms of the inverse Laplace and inverse Fourier transforms of certain functions whereas the solution of the same equation is obtained here in an explicit closed form in terms of the H-function.

An interesting case occurs when $\beta \to 1$. Then in view of the cancelation law for the H-function (1.57), the equation (6.128) provides the following result given by Jespersen et al. (1999) and recently by Del-Castillo-Negrete et al. (2003) in an entirely different form.

For the solution of fractional reaction–diffusion equation

$$\frac{\mathrm{d}}{\mathrm{d}t}N(x,t) = \eta_{-\infty}D_x^{\alpha}N(x,t), \tag{6.132}$$

with initial condition

$$N(x,t=0) = \delta(x),\quad \lim_{x\to\pm\infty} N(x,t) = 0, \tag{6.133}$$

there holds the relation

$$N(x,t) = \frac{1}{\alpha|x|}H_{2,2}^{1,1}\left[\frac{|x|}{\eta^{\frac{1}{\alpha}}t^{\frac{1}{\alpha}}}\middle|_{(1,1),(1,\frac{1}{2})}^{(1,\frac{1}{\alpha}),(1,\frac{1}{2})}\right], \tag{6.134}$$

where $\Re(\alpha) > 0$. In passing, it may be noted that the equation (6.134) is a closed form representation of a Lévy stable law, see Metzler and Klafter (2000, 2004). It is interesting to note that as $\alpha \to 2$, the classical Gaussian solution is recovered as

$$N(x,t) = \frac{1}{2|x|} H_{2,2}^{1,1}\left[\frac{|x|}{(\eta t)^{\frac{1}{2}}}\middle|_{(1,1),(1,\frac{1}{2})}^{(1,\frac{1}{2})}\right]$$
$$= \frac{1}{2|x|} H_{1,1}^{1,0}\left[\frac{|x|}{(\eta t)^{\frac{1}{2}}}\middle|_{(1,1)}^{(1,\frac{1}{2})}\right]$$
$$= (4\pi\eta t)^{-\frac{1}{2}} \exp\left(-\frac{|x|^2}{4\eta t}\right). \tag{6.135}$$

It is useful to study the solution (6.131) due to its occurrence in certain fractional diffusion models. Now we will find the fractional order moments of (6.131) in the next section.

Remark 6.7. Applying Fourier transform with respect to x in (6.128), it is found that

$$\frac{d^\beta}{dt^\beta}\Psi(k,t) = -\eta|k|^\alpha\Psi(k,t),\ 0 < \beta \leq 1, \tag{6.136}$$

which is the generalized Fourier transformed diffusion equation, since for $\alpha = 2$ and for $\beta \to 1$, it reduces to Fourier transformed diffusion equation

$$\frac{d}{dt}\Psi(k,t) = -\eta|k|^2\Psi(k,t), \tag{6.137}$$

being a diffusion equation, for a fixed wave number k (Metzler and Klafter 2000, 2004). Here $\Psi(k,t)$ is the Fourier transform of $N(x,t)$ with respect to x.

Remark 6.8. It is interesting to observe that the method employed for deriving the solution of the Eqs. (6.115) and (6.116) in the space $LF = L(R_+) \times F(R)$ can also be applied in the space $LF' = L'(R_+) \times F'$, where $F' = F'(R)$ is the space of Fourier transforms of generalized functions. As an illustration, we can choose $F' = S'$ or $F' = D'$. The Fourier transforms in S' and D' are introduced by Gelfand and Shilov (1964). S' is the dual of the space S, which is the space of all infinitely differentiable functions which together with their derivatives approach zero more rapidly than any power of $1/|x|$ as $|x| \to \infty$. D' is the dual of the space D, which consists of all smooth functions with compact supports. In this connection, see the monographs by Gelfand and Shilov (1964) and Brychkov and Prudnikov (1989).

6.11.5 Fractional Order Moments

In this section, we will calculate the fractional order moments, defined by

$$< |x|^\delta > = \int_{-\infty}^{\infty} |x|^\delta N(x,t)dx. \tag{6.138}$$

Using the definition of the Mellin transform

$$M\{f(t);s)\} = \int_0^\infty t^{s-1} f(t)\mathrm{d}t, \tag{6.139}$$

we find from (6.138) that

$$< |x(t)|^\delta > = \int_{-\infty}^{\infty} |x|^\delta N(x,t)\mathrm{d}x. \tag{6.140}$$

$$< |x|^\delta(t) > = \frac{2t^{\beta-1}}{\alpha}\int_0^\infty x^{\delta-1} H_{3,3}^{2,1}\left[\frac{|x|}{\eta^{\frac{1}{\alpha}} t^{\frac{\beta}{\alpha}}}\left|\begin{matrix}(1,\frac{1}{\alpha}),(\beta,\frac{\beta}{\alpha}),(1,\frac{1}{2})\\(1,1),(1,\frac{1}{\alpha}),(1,\frac{1}{2})\end{matrix}\right.\right]\mathrm{d}x. \tag{6.141}$$

Applying the Mellin transform formula for the H-function (2.8) we see that

$$< |x|^\delta(t) >= \frac{2}{\alpha}\eta^{\frac{\delta}{\alpha}} t^{\beta\left(\frac{\delta}{\alpha}+1-\frac{1}{\beta}\right)} \frac{\Gamma\left(-\frac{\delta}{\alpha}\right)\Gamma(1+\delta)\Gamma\left(1+\frac{\delta}{\alpha}\right)}{\Gamma\left(-\frac{\delta}{2}\right)\Gamma\left(\beta+\frac{\beta\delta}{\alpha}\right)\Gamma\left(1+\frac{\delta}{2}\right)}, \tag{6.142}$$

whenever the gammas exist, $\Re(\delta) > -1$ and $\Re(\delta+\alpha) > 0$.

Two interesting special cases of (6.142) are worth mentioning.

(i) As $\delta \to 0$, then by using the result $\frac{1}{\Gamma(z)} \sim z$ for $z << 1$, we find that

$$\lim_{\delta\to 0} < |x|^\delta(t)| >= \beta t^{\beta-1}. \tag{6.143}$$

(ii) When $\alpha = 2, \delta = 2$, the linear time dependence

$$\lim_{\delta\to 2,\alpha\to 2} < |x(t)|^\delta >= \frac{2\eta t^{2\beta-1}}{\Gamma(2\beta)}, \tag{6.144}$$

of the mean squared displacement is recovered.

6.11.6 Some Further Applications

This section deals with the investigation of the solution of an unified fractional reaction–diffusion equation associated with the Caputo derivative as the time-derivative and Riesz–Feller fractional derivative as the space-derivative. The solution is derived by the application of the Laplace and Fourier transforms in a compact and closed form in terms of the H-function.

6.11.7 Background

The theory and applications of reaction–diffusion systems are contained in many books and articles. In recent works (Saxena et al. 2006a–c), the authors have demonstrated the depth of mathematics and related physical issues of reaction–diffusion equations such as nonlinear phenomena, stationary and spatio-temporal dissipative pattern formation, oscillation, waves, etc. (Frank 2005; Grafiychuk et al. 2006, 2007). In recent time, interest in fractional reaction–diffusion equations has increased because the equation exhibits self-organization phenomena and introduces a new parameter, the fractional index, into the equation. Additionally, the analysis of fractional reaction–diffusion equations is of great interest from the analytic and numerical point of view.

The object of this section is to derive the solution of an unified model of reaction–diffusion system, associated with the Caputo derivative and the Riesz–Feller derivative. This new model provides the extension of the models discussed earlier by Mainardi et al. (2001), Mainardi et al. (2005), and Saxena et al. (2006). The advantage of using Riesz–Feller derivative lies in the fact that the solution of the fractional reaction–diffusion equation containing this derivative includes the fundamental solution for space-time fractional diffusion, which itself is a generalization of neutral fractional diffusion, space-fractional diffusion, and time-fractional diffusion. These specialized type of diffusions can be interpreted as spatial probability density functions evolving in time and are expressible in terms of the H-functions in compact form.

Notation 6.7. ${}_xD_0^{\alpha}$: Riesz–Feller space-fractional derivative of order α.

Definition 6.7. Following Feller (1952, 1966) it is conventional to define the Riesz–Feller space-fractional derivative of order α and skewness θ in terms of its Fourier transform as

$$F\{{}_xD_\theta^{\alpha};k\} = -\psi_\alpha^{\theta}(k) f^*(k), \tag{6.145}$$

where,

$$\psi_\alpha^{\theta}(k) = |k|^{\alpha} \exp[i(sign\ k)\frac{\theta\pi}{2}],\ \ 0 < \alpha \le 2,\ \ |\theta| \le \min\{\alpha, 2-\alpha\}. \tag{6.146}$$

When $\theta = 0$, then (6.145) reduces to

$$F\{{}_xD_0^{\alpha} f(x);k\} = -|k|^{\alpha} f^*(k), \tag{6.147}$$

which is the Fourier transform of the Liouville fractional derivative, defined by

$${}_{-\infty}D_x^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{-\infty}^{t} \frac{f(u)}{(t-u)^{\alpha-n+1}} \mathrm{d}u. \tag{6.148}$$

This shows that Riesz–Feller space-fractional derivative may be regarded as a generalization of Liouville fractional derivative.

Note 6.8. Further, when $\theta = 0$, we have a symmetric operator with respect to x which can be interpreted as

$$ {}_xD_0^{\alpha} = -\left[-\frac{d^2}{dx^2}\right]^{\frac{\alpha}{2}}. \tag{6.149}$$

This can be formally deduced by writing $-(k)^{\alpha} = -(k^2)^{\frac{\alpha}{2}}$. For $0 < \alpha < 2$ and $|\theta| \leq \min\{\alpha, 2-\alpha\}$, the Riesz–Feller derivative can be shown to possess the following integral representation in x domain:

$$ {}_xD_\theta^{\alpha} f(x) = \frac{\Gamma(1+\alpha)}{\pi}\left\{ \sin[(\alpha+\theta)\frac{\pi}{2}] \int_0^\infty \frac{f(x+\zeta)-f(x)}{\zeta^{1+\alpha}} d\zeta \right.$$
$$\left. + \sin\left[(\alpha-\theta)\frac{\pi}{2}\right] \int_0^\infty \frac{f(x-\zeta)-f(x)}{\zeta^{1+\alpha}} d\zeta \right\}. \tag{6.150}$$

6.11.8 Unified Fractional Reaction–Diffusion Equation

In this section, we will investigate the solution of the reaction–diffusion equation (6.151) under the initial conditions (6.153). The result is given in the form of the following result:

Theorem 6.7. *Consider the following unified fractional reaction–diffusion model*

$$ {}_0D_t^{\beta} N(x,t) = \eta\, {}_xD_\theta^{\alpha} N(x,t) + \phi(x,t), \tag{6.151}$$

where $\eta, t > 0, x \in R; \alpha, \theta, \beta$ are real parameters with the constraints

$$ 0 < \alpha \leq 2, |\theta| \leq \min(\alpha, 2-\alpha),\ 0 < \beta \leq 2, \tag{6.152}$$

and the initial conditions

$$ N(x,0) = f(x), N_i(x,0) = g(x)\, for\, x \in R,\ \lim_{x\to\pm\infty} N(x,t) = 0, t > 0. \tag{6.153}$$

Here $N_i(x,0)$ means the first partial derivative of $N(x,t)$ with respect to t evaluated at $t = 0$, η is a diffusion constant and $\phi(x,t)$ is a nonlinear function belonging to the area of reaction–diffusion. Further, ${}_xD_\theta^{\alpha}$ is Riesz–Feller space-fractional derivative of order α and asymmetry θ. ${}_0D_t^{\beta}$ is the Caputo time-fractional derivative of order β. Then for the solution of (6.151), subject to the above constraints, there holds the formula

$$N(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f^*(k)E_{\beta,1}(-\eta t^\beta \Psi_\alpha^\theta(k))\exp(-ikx)\mathrm{d}k$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty} tg^*(k)E_{\beta,2}(-\eta k^\alpha t^\beta \Psi_\alpha^\theta(k))\exp(-ikx)\mathrm{d}k$$
$$+\frac{1}{2\pi}\int_0^t \zeta^{\beta-1}\int_{-\infty}^{\infty} \phi^*(k,t-\zeta)E_{\beta,\beta}(-\eta k^\alpha t^\beta \Psi_\alpha^\theta(k))\exp(-ikx)\mathrm{d}k\,\mathrm{d}\zeta. \tag{6.154}$$

Proof 6.7. If we apply the Laplace transform with respect to the time variable t, Fourier transform with respect to the space variable x, and use the initial conditions (6.153) and the formulae (6.114) and (6.147), then the given equation transforms into the form

$$s^\beta \tilde{N}^*(k,s) - s^{\beta-1}f^*(k) - s^{\beta-2}g^*(k) = -\eta\Psi_\alpha^\theta(k)\tilde{N}^*(k,s) + \tilde{\phi}^*(k,s), \tag{6.155}$$

where according to the conventions followed, the symbol $\tilde{N}(x,s)$ will stand for the Laplace transform with respect to time variable t and $*$ represents the Fourier transform with respect to space variable x. Solving for $\tilde{N}^*(k,s)$, it yields

$$\tilde{N}^*(k,s) = \frac{f^*(k)s^{\beta-1}}{s^\beta + \eta\Psi_\alpha^\theta(k)} + \frac{g^*(k)s^{\beta-2}}{s^\beta + \eta\Psi_\alpha^\theta(k)} + \frac{\tilde{\phi}^*(k)}{s^\beta + \eta\Psi_\alpha^\theta(k)}. \tag{6.156}$$

On taking the inverse Laplace transform of (6.156) and applying the formula (6.120), it is seen that

$$N^*(k,t) = f^*(k)E_{\beta,1}(-\eta t^\beta \Psi_\alpha^\theta(k)) + g^*(k)tE_{\beta,2}(-\eta t^\beta \Psi_\alpha^\theta(k))$$
$$+\int_0^t \phi^*(k,t-\zeta)\zeta^{\beta-1}E_{\beta,\beta}(-\eta\Psi_\alpha^\theta(k)\zeta^\beta)\mathrm{d}\zeta. \tag{6.157}$$

□

The required solution (6.154) is now obtained by taking the inverse Fourier transform of (6.157). This completes the proof of the Theorem 6.7.

6.11.9 Some Special Cases

When $g(x) = 0$ then by the application of the convolution theorem of the Fourier transform to the solution (6.154) of the Theorem 6.7, it readily yields the following result:

Corollary 6.3. *The solution of fractional reaction–diffusion equation*

$$\frac{\partial^\beta}{\partial t^\beta}N(x,t) - \eta\frac{\partial^\alpha}{\partial x^\alpha}N(x,t) = \phi(x,t), x \in R, t > 0, \eta > 0, \tag{6.158}$$

with initial conditions

$$N(x,0) = f(x), N(x,0) = 0 \, for \, x \in R, 1 < \beta \leq 2, \lim_{x \to \pm\infty} N(x,t) = 0, t > 0, \tag{6.159}$$

where η is a diffusion constant and $\phi(x,t)$ is a nonlinear function belonging to the area of reaction–diffusion, is given by

$$N(x,t) = \int_{-\infty}^{\infty} G_1(x-\tau,t) f(\tau) \mathrm{d}\tau + \int_0^t (t-\zeta)^{\beta-1} \int_0^x G_2(x-\tau,t-\zeta)\phi(\tau,\zeta) \mathrm{d}\tau \, \mathrm{d}\zeta, \tag{6.160}$$

where,

$$\rho = \frac{\alpha-\theta}{2\alpha}$$
$$G_1(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ikx) E_{\beta,1}\left(-\eta t^{\beta} \Psi_{\alpha}^{\theta}(k)\right) \mathrm{d}k = \frac{1}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{\eta^{\frac{1}{\alpha}} t^{\frac{\beta}{\alpha}}} \middle| \begin{matrix} (1,\frac{1}{\alpha}),(1,\frac{\beta}{\alpha}),(1,\rho) \\ (1,\frac{1}{\alpha}),(1,1),(1,\rho) \end{matrix}\right], \alpha > 0, \tag{6.161}$$

and

$$G_2(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ikx) E_{\beta,\beta}(-\eta t^{\beta} \Psi_{\alpha}^{\theta}(k)) \mathrm{d}k = \frac{1}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{\eta^{\frac{1}{\alpha}} t^{\frac{\beta}{\alpha}}} \middle| \begin{matrix} (1,\frac{1}{\alpha}),(\beta,\frac{\beta}{\alpha}),(1,\rho) \\ (1,\frac{1}{\alpha}),(1,1),(1,\rho) \end{matrix}\right], \alpha > 0. \tag{6.162}$$

In deriving the above results, we have used the inverse Fourier transform formula

$$F^{-1}[E_{\beta,\gamma}(-\eta t^{\beta} \Psi_{\theta}^{\alpha}(k)); x] = \frac{1}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{\eta^{\frac{1}{\alpha}} t^{\frac{\beta}{\alpha}}} \middle| \begin{matrix} (1,\frac{1}{\alpha}),(\gamma,\frac{\beta}{\alpha}),(1,\rho) \\ (1,\frac{1}{\alpha}),(1,1),(1,\rho) \end{matrix}\right] \tag{6.163}$$

where $\rho = \frac{\alpha-\theta}{2\alpha}, \Re(\beta) > 0, \Re(\gamma) > 0$, which can be established by following a procedure similar to that employed by Mainardi et al. (2001).

Next, if we set $f(x) = \delta(x), \phi = 0, g(x) = 0$, where $\delta(x)$ is the Dirac delta function, then we arrive at the following interesting result given by Mainardi et al. (2005).

Corollary 6.4. *Consider the following space-time fractional diffusion model*

$$\frac{\partial^{\beta}}{\partial t^{\beta}} N(x,t) = \eta \; {}_xD_{\theta}^{\alpha} N(x,t), \; \eta > 0, x \in R, 0 < \beta \leq 2, \tag{6.164}$$

with the initial conditions $N(x,t=0)=\delta(x)$, $N_t(x,0)=0$, $\lim_{x\to\pm\infty}N(x,t)=0$ *where* η *is a diffusion constant and* $\delta(x)$ *is the Dirac delta function. Then for the fundamental solution of (6.164) with initial conditions, there holds the formula*

$$N(x,t)=\frac{1}{\alpha|x|}H^{2,1}_{3,3}\left[\frac{|x|}{(\eta t^{\beta})^{\frac{1}{\alpha}}}\middle|^{(1,\frac{1}{\alpha}),(1,\frac{\beta}{\alpha}),(1,\rho)}_{(1,\frac{1}{\alpha}),(1,1),(1,\rho)}\right],\rho=\frac{\alpha-\theta}{2\alpha}. \quad (6.165)$$

Some interesting special cases of (6.164) are enumerated below.

(i) We note that for $\alpha=\beta$, Mainardi et al. (2005) have shown that the corresponding solution of (6.165), denoted by N^{θ}_{α}, which we call as the neutral fractional diffusion, can be expressed in terms of elementary function and can be defined for $x>0$ as **Neutral fractional diffusion**: $0<\alpha=\beta<2;\theta\leq\min\{\alpha,2-\alpha\}$,

$$N^{\theta}_{\alpha}(x)=\frac{1}{\pi}\frac{x^{\alpha-1}\sin[\left(\frac{\pi}{2}\right)(\alpha-\theta)]}{1+2x^{\alpha}\cos[\left(\frac{\pi}{2}\right)(\alpha-\theta)]+x^{2\alpha}}. \quad (6.166)$$

The neutral fractional diffusion is not studied at length in the literature.
Next we derive some stable densities in terms of the H-functions as special cases of the solution of the equation (6.164).

(ii) If we set $\beta=1,\ 0<\alpha<2;\theta\leq\min\{\alpha,2-\alpha\}$, then (6.164) reduces to space-fractional diffusion equation, which we denote by $L^{\theta}_{\alpha}(x)$, and we obtain the fundamental solution of the following **space-time fractional diffusion model**:

$$\frac{\partial}{\partial t}N(x,t)=\eta\,{}_xD^{\alpha}_{\theta}N(x,t),\ \eta>0,x\in R, \quad (6.167)$$

with the initial conditions $N(x,t=0)=\delta(x)$, $\lim_{x\to\pm\infty}N(x,t)=0$, where η is a diffusion constant and $\delta(x)$ is the Dirac delta function. Hence for the fundamental solution of (6.167) there holds the formula

$$L^{\theta}_{\alpha}(x)=\frac{1}{\alpha(\eta t)^{\frac{1}{\alpha}}}H^{1,1}_{2,2}\left[\frac{(\eta t)^{\frac{1}{\alpha}}}{|x|}\middle|^{(1,1),(\rho,\rho)}_{(\frac{1}{\alpha},\frac{1}{\alpha}),(\rho,\rho)}\right],0<\alpha<1,|\theta|\leq\alpha, \quad (6.168)$$

where $\rho=\frac{\alpha-\theta}{2\alpha}$. The density represented by the above expression is known as α-stable Lévy density. Another form of this density is given by

$$L^{\theta}_{\alpha}(x)=\frac{1}{\alpha(\eta t)^{\frac{1}{\alpha}}}H^{1,1}_{2,2}\left[\frac{|x|}{(\eta t)^{\frac{1}{\alpha}}}\middle|^{(1-\frac{1}{\alpha},\frac{1}{\alpha}),(1-\rho,\rho)}_{(0,1),(1-\rho,\rho)}\right], \quad (6.169)$$

where $1<\alpha<2,|\theta|\leq 2-\alpha$.

(iii) Next, if we take $\alpha=2,\ 0<\beta<2,\theta=0$ then we obtain the time-fractional diffusion, which is governed by the following time fractional diffusion model:

$$\frac{\partial^\beta}{\partial t^\beta}N(x,t) = \eta\frac{\partial^2}{\partial x^2}N(x,t),\ \eta > 0, x \in R, 0 < \beta \le 2, \tag{6.170}$$

with the initial conditions $N(x,t = 0) = \delta(x), N_t(x,0) = 0, \lim_{x\to\pm\infty} N(x,t) = 0$ where η is a diffusion constant and $\delta(x)$ is the Dirac delta function, whose fundamental solution is given by the equation

$$N(x,t) = \frac{1}{2|x|}H_{1,1}^{1,0}\left[\frac{|x|}{(\eta t^\beta)^{\frac{1}{2}}}\middle|^{(1,\frac{\beta}{2})}_{(1,1)}\right] \tag{6.171}$$

which is same as (6.72).

(iv) Further, if we set $\alpha = 2, \beta = 1$, and $\theta \to 0$ then for the fundamental solution of the standard diffusion equation

$$\frac{\partial}{\partial t}N(x,t) = \eta\frac{\partial^2}{\partial x^2}N(x,t), \tag{6.172}$$

with initial condition

$$N(x,t = 0) = \delta(x),\ \lim_{x\to\pm\infty} N(x,t) = 0, \tag{6.173}$$

there holds the formula

$$N(x,t) = \frac{1}{2|x|}H_{1,1}^{1,0}\left[\frac{|x|}{\eta^{\frac{1}{2}}t^{\frac{1}{2}}}\middle|^{(1,\frac{1}{2})}_{(1,1)}\right] = (4\pi\eta t)^{-\frac{1}{2}}\exp\left[-\frac{|x|^2}{4\eta t}\right], \tag{6.174}$$

which is the classical Gaussian density. For further details and importance of these special cases based on the Green function, one can refer to the papers by Mainardi et al. (2001, 2005).

Remark 6.9. Fractional order moments and the asymptotic expansion of the solution (6.165) are discussed by Mainardi et al. (2001).

Finally, for $\beta = \frac{1}{2}$ and $g(x) = 0$ in (6.151) we arrive at the following result:

Corollary 6.5. *Consider the following fractional reaction–diffusion model*

$$D^{\frac{1}{2}}N(x,t) = \eta_x D^\alpha_\theta N(x,t) + \phi(x,t), \tag{6.175}$$

where $\eta, t > 0, x \in R; \alpha, \theta$ are real parameters with the constraints $0 < \alpha \le 2$, $|\theta| \le \min(\alpha, 2-\alpha)$, and the initial conditions

$$N(x,0) = f(x), N_t(x,0) = 0\ for\ x \in R,\ \lim_{x\to\pm\infty} N(x,t) = 0. \tag{6.176}$$

Here η is a diffusion constant and $\phi(x,t)$ is a nonlinear function belonging to the area of reaction–diffusion. Further, ${}_xD^\alpha_\theta$ is the Riesz–Feller space fractional *derivative of order* α *and asymmetry* θ *and* $D_t^{\frac{1}{2}}$ *is the Caputo time-fractional derivative of order* $\frac{1}{2}$. *Then for the solution of (6.175), subject to the above constraints, there holds the formula*

$$N(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f^*(k)E_{\frac{1}{2}}(-\eta t^{\frac{1}{2}}\Psi_\alpha^\theta(k))\exp(-ikx)\mathrm{d}x + \frac{1}{2\pi}\int_0^t \zeta^{-\frac{1}{2}}\int_{-\infty}^{\infty}\phi^*(k,t-\zeta)E_{\frac{1}{2},\frac{1}{2}}(-\eta k^\alpha t^{\frac{1}{2}}\Psi_\alpha^\theta(k))\exp(-ikx)\mathrm{d}k\,\mathrm{d}\zeta. \tag{6.177}$$

If we set $\theta = 0$ in Theorem 6.7, then it reduces to the result recently obtained by Saxena et al. (2006) for the fractional reaction–diffusion equation.

Following a similar procedure, we can derive the solution of the fractional reaction–diffusion system (6.178) given below under the given initial conditions (6.179) associated with Riemann–Liouville fractional derivative and the Riesz–Feller fractional derivative. The result is given in the form of the following result:

Theorem 6.8. *Consider the unified fractional reaction–diffusion model associated with Riemann–Liouville fractional derivative* ${}_0D_t^\alpha$ *defined by (6.111) and the Riesz–Feller space fractional derivative* ${}_xD_\theta^\alpha$ *of order* α *and asymmetry* θ *defined by (6.145) in the form*

$${}_0D_t^\beta N(x,t) = \eta\,{}_xD_\theta^\alpha N(x,t) + \phi(x,t), \tag{6.178}$$

where $\eta, t > 0, x \in R, \alpha, \theta, \beta$ *are real parameters with the constraints* $0 < \alpha \le 2$, $|\theta| \le \min(\alpha, 2-\alpha)$, $1 < \beta \le 2$, *and the initial conditions*

$$[{}_0D_t^{\beta-1}N(x,0)] = f(x),\ \ [{}_0D_t^{\beta-2}N(x,0)] = g(x)\ \text{for}\ x \in R,$$
$$\lim_{|x|\to\infty} N(x,t) = 0, t > 0. \tag{6.179}$$

Here $[{}_0D_t^{\beta-1}N(x,0)]$ *means the Riemann–Liouville fractional partial derivative of* $N(x,t)$ *with respect to* t *of order* $\beta-1$ *evaluated at* $t=0$. *Similarly,* $[{}_0D_t^{\beta-2}N(x,0)]$ *is the Riemann–Liouville fractional partial derivative of* $N(x,t)$ *with respect to* t *of order* $\beta-2$ *evaluated at* $t = 0$, η *is a diffusion constant and* $\phi(x,t)$ *is a nonlinear function belonging to the area of reaction–diffusion. Then for the solution of (6.178), subject to the above constraints, there holds the formula*

$$N(x,t) = \frac{t^{\beta-1}}{2\pi}\int_{-\infty}^{\infty} f^*(k)E_{\beta,\beta}(-\eta t^\beta\Psi_\alpha^\theta(k))\exp(-ikx)\mathrm{d}k + \frac{t^{\beta-2}}{2\pi}\int_{-\infty}^{\infty} tg^*(k)E_{\beta,\beta-1}(-\eta t^\beta\Psi_\alpha^\theta(k))\exp(-ikx)\mathrm{d}k + \frac{1}{2\pi}\int_0^t \zeta^{\beta-1}\int_{-\infty}^{\infty}\phi^*(k,t-\zeta)E_{\beta,\beta}(-\eta\zeta^\beta\Psi_\alpha^\theta(k))\exp(-ikx)\mathrm{d}k\,\mathrm{d}\zeta. \tag{6.180}$$

6.11.10 More Special Cases

When $g(x) = 0$ then by the application of the convolution theorem of the Fourier transform to the solution (6.180) of the theorem, it readily yields the following result:

Corollary 6.6. *The solution of fractional reaction–diffusion equation*

$$ {}_0D_t^{\beta} N(x,t) - \eta_x D_\theta^\alpha N(x,t) = \phi(x,t), \; x \in R, t > 0, \eta > 0, \tag{6.181}$$

with initial conditions

$$ \begin{aligned} &[{}_0D_t^{\beta-1} N(x,t)] = f(x), \; [{}_0D_t^{\beta-2} N(x,0)] = 0 \, for \; x \in R, \\ &0 \le \alpha \le 1, 1 < \beta \le 2, \quad \lim_{x \to \pm\infty} N(x,t) = 0, \end{aligned} \tag{6.182}$$

where η is a diffusion constant and $\phi(x,t)$ is a nonlinear function belonging to the area of reaction–diffusion; $\eta, t > 0, x \in R; \alpha, \theta, \beta$ are real parameters with the constraints $0 < \alpha \le 2$, $|\theta| \le \min(\alpha, 2-\alpha)$, $1 < \beta \le 2$, *is given by*

$$ \begin{aligned} N(x,t) = &\int_{-\infty}^{\infty} G_1(x-\tau,t) f(\tau) \mathrm{d}\tau \\ &+ \int_0^t (t-\zeta)^{\beta-1} \int_0^x G_2(x-\tau, t-\zeta) \phi(\tau,\zeta) \mathrm{d}\tau \, \mathrm{d}\zeta, \end{aligned} \tag{6.183}$$

where $\rho = \frac{\alpha-\theta}{2\alpha}$*;*

$$ \begin{aligned} G_1(x,t) &= \frac{t^{\beta-1}}{2\pi} \int_{-\infty}^{\infty} \exp(-ikx) E_{\beta,\beta}(-\eta t^\beta \Psi_\alpha^\theta(k)) \mathrm{d}k \\ &= \frac{t^{\beta-1}}{\alpha|x|} H_{3,3}^{2,1} \left[\frac{|x|}{\eta^{\frac{1}{\alpha}} t^{\frac{\beta}{\alpha}}} \middle| \begin{matrix} (1,\frac{1}{\alpha}),(\beta,\frac{\beta}{\alpha}),(1,\rho) \\ (1,\frac{1}{\alpha}),(1,1),(1,\rho) \end{matrix} \right], \alpha > 0, \end{aligned} \tag{6.184}$$

and

$$ \begin{aligned} G_2(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-ikx) E_{\beta,\beta}(-\eta t^\beta \Psi_\alpha^\theta(k)) \mathrm{d}k \\ &= \frac{1}{\alpha|x|} H_{3,3}^{2,1} \left[\frac{|x|}{\eta^{\frac{1}{\alpha}} t^{\frac{\beta}{\alpha}}} \middle| \begin{matrix} (1,\frac{1}{\alpha}),(\beta,\frac{\beta}{\alpha}),(1,\rho) \\ (1,\frac{1}{\alpha}),(1,1),(1,\rho) \end{matrix} \right], \alpha > 0. \end{aligned} \tag{6.185}$$

In deriving the above results, we have used the inverse Fourier transform formula (6.163) given by Haubold et al. (2007).

Remark 6.10. It is interesting to observe that for $\theta = 0$, Theorem 6.8 reduces to (6.117) given by the authors Saxena et al. (2006b). On the other hand, if we set $f(x) = \delta(x)$, where $\delta(x)$ is the Dirac delta function, it yields the following result:

Corollary 6.7. *Consider the following reaction–diffusion model*

$$ {}_0D_t^{\beta} N(x,t) = \eta_x D_{\theta}^{\alpha} N(x,t), \tag{6.186}$$

with the initial conditions

$$ [{}_0D_t^{\beta-1} N(x,0) = \delta(x), 0 \leq \beta \leq 1, \quad \lim_{x \to \pm\infty} N(x,t) = 0, \tag{6.187}$$

where η is a diffusion constant; $\eta, t > 0, x \in R; \alpha, \theta, \beta$ are real parameters with the constraints $0 < \alpha \leq 2$, $|\theta| \leq \min(\alpha, 2-\alpha)$, and $\delta(x)$ is the Dirac delta function. Then for the fundamental solution of (6.186) with initial conditions in (6.187), there holds the formula

$$ N(x,t) = \frac{t^{\beta-1}}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{(\eta t^{\beta})^{\frac{1}{\alpha}}} \middle| \begin{matrix} (1,\frac{1}{\alpha}),(\beta,\frac{\beta}{\alpha}),(1,\rho) \\ (1,\frac{1}{\alpha}),(1,1),(1,\rho) \end{matrix} \right], \alpha > 0, \tag{6.188}$$

where $\rho = \frac{\alpha-\theta}{2\alpha}$.

Exercises 6.10

6.10.1. Consider the fractional reaction–diffusion equation connected with nonlinear waves

$$ \begin{aligned} &{}_0D_t^{\alpha} N(x,t) + \alpha_0 D_t^{\beta} N(x,t) \\ &\quad = \nu^2 {}_{-\infty}D_x^{\gamma} N(x,t) + \zeta^2 N(x,t) + \phi(x,t), \end{aligned} $$

for $x \in R, t > 0, 0 \leq \alpha \leq 1, 0 \leq \beta \leq 1$ with initial conditions

$$ N(x,0) = f(x), \quad \lim_{x \to \pm\infty} N(x,t) = 0, \quad x \in R $$

where the operator ${}_{-\infty}D_x^{\gamma}$ is defined in (6.78); ${}_0D_t^{\alpha}$ and ${}_0D_t^{\beta}$ are the Caputo fractional order derivatives, ν^2 is a diffusion constant, ζ is a constant which describes the nonlinearity in the system, and $\phi(x,t)$ is nonlinear function which belongs to the area of reaction–diffusion, then show that there holds the following formula for the solution of the above mentioned reaction–diffusion model.

$$N(x,t) = \sum_{r=0}^{\infty} \frac{(-a)^r}{2\pi} \int_{-\infty}^{\infty} t^{(\alpha-\beta)r} f^*(k) \exp(-ikx)$$
$$\times \left[E_{\alpha,(\alpha-\beta)r+1}(-bt^{\alpha}) + t^{\alpha-\beta} E_{\alpha,(\alpha-\beta)(r+1)+1}(-bt^{\alpha}) \right] \mathrm{d}k$$
$$+ \sum_{r=0}^{\infty} \frac{(-a)^r}{2\pi} \int_0^t \zeta^{\alpha+(\alpha-\beta)r-1}$$
$$\times \int_{-\infty}^{\infty} \phi(k, t-\zeta) \exp(-ikx) E_{\alpha,\alpha+(\alpha-\beta)r}(-b\zeta^{\alpha}) \mathrm{d}k \mathrm{d}\zeta,$$

where $\alpha > \beta$ and $E^{\delta}_{\beta,\gamma}(\cdot)$ is the generalized Mittag-Leffler function, defined by (1.39) and $b = \nu^2|k|^{\gamma} - \zeta^2$.

6.10.2. Consider the following fractional reaction–diffusion model

$$\frac{\partial^{\beta}}{\partial t^{\beta}} N(x,t) = \eta \; {}_{-\infty}D^{\alpha}_{x} N(x,t) + \phi(x,t); \eta, t > 0, x \in R, 0 < \beta \le 2,$$

with the initial conditions

$$N(x,0) = f(x), \;\; N_t(x,0) = g(x), \;\; x \in R, \;\; \lim_{x \to \pm\infty} N(x,t) = 0,$$

where the operator ${}_{-\infty}D^{\alpha}_{x}$ is defined in (6.78); $N_t(x,0)$ means the first derivative of $N(x,t)$ with respect to t evaluated at $t = 0$, η is a diffusion constant, $\phi(x,t)$ is a nonlinear function belonging to the area of reaction diffusion and $\frac{\partial^{\beta}}{\partial t^{\beta}}$ is the Caputo fractional derivative. Then show that for the solution of reaction–diffusion model, subject to the initial conditions, there holds the formula

$$N(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f^*(k) E_{\beta,1}(-\eta|k|^{\alpha} t^{\beta}) \exp(-ikx) \mathrm{d}k$$
$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} t g^*(k) E_{\beta,2}(-\eta|k|^{\alpha} t^{\beta}) \exp(-ikx) \mathrm{d}k$$
$$+ \frac{1}{2\pi} \int_0^t \zeta^{\beta-1} \int_{-\infty}^{\infty} \tilde{\phi}(k, t-\zeta) E_{\beta,\beta}(-\eta|k|^{\alpha} \zeta^{\beta}) \exp(-ikx) \mathrm{d}k \mathrm{d}\zeta.$$

Hence or otherwise derive the solution of the next exercise.

6.10.3. Consider the following reaction–diffusion model

$$\frac{\partial^{\beta}}{\partial t^{\beta}} N(x,t) = \eta \; {}_{-\infty}D^{\alpha}_{x} N(x,t), \alpha > 0, -\infty < x < \infty, 0 < \beta \le 1,$$

with the initial condition $N(x, t = 0) = \delta(x)$, $\lim_{x\to\pm\infty} N(x,t) = 0$, $\frac{\partial^\beta}{\partial t^\beta}$ is the Caputo fractional derivative, the operator ${}_{-\infty}D_x^\gamma$ is defined in (6.78), η is a diffusion constant and $\delta(x)$ is the Dirac delta function. Then show that for the solution of the above equation there holds the formula

$$N(x,t) = \frac{1}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{(\eta t^\beta)^{\frac{1}{\alpha}}}\left|\begin{matrix}(1,\frac{1}{\alpha}),(1,\frac{\beta}{\alpha}),(1\frac{1}{2})\\(1,1),(1,\frac{1}{\alpha}),(1,\frac{1}{2})\end{matrix}\right.\right].$$

6.10.4. Show that the solution of the following boundary value problem for the one-dimensional fractional diffusion equation associated with the Riemann–Liouville fractional derivative ${}_0D_t^\alpha$

$${}_0D_t^\alpha N(x,t) = \lambda^2 \frac{\partial^2}{\partial x^2} N(x,t), t > 0, -\infty < x < \infty,$$

with the initial conditions

$$\lim_{x\to\pm\infty} N(x,t) = 0, \ \ [{}_0D_t^{\alpha-1} N(x,t)]_{t=0} = \phi(x), 0 < \alpha < 1,$$

is given by

$$N(x,t) = \int_{-\infty}^{\infty} G(x-\zeta, t)\phi(\zeta)\mathrm{d}\zeta,$$

where

$$G(x,t) = \frac{t^{\alpha-1}}{|x|} H_{1,1}^{1,0}\left[\frac{|x|^2}{\lambda^2 t^\alpha}\left|\begin{matrix}(\alpha,\alpha)\\(1,2)\end{matrix}\right.\right].$$

(Nigmatullin 1986)

Remark 6.11. Nigmatullin (1986) derived the solution of the above fractional diffusion equation in terms of the following integral:

$$G(x,t) = \frac{1}{\pi}\int_0^\infty t^{\alpha-1} E_{\alpha,\alpha}(-\lambda^2 k^2 t^\alpha)\cos(kx)\mathrm{d}k,$$

whereas, the solution of this problem given here is in terms of the H-function in an explicit form.

6.10.5. Consider the fractional diffusion equation

$${}_0D_t^\nu N(x,t) - \frac{t^{-\nu}}{\Gamma(1-\nu)}\delta(x) = c^\nu \frac{\partial^2}{\partial x^2} N(x,t),$$

with the initial condition

$$D_t^{\nu-k} N(x,t)|_{t=0} = 0, k = 1, \ldots, n,$$

where $n = [\Re(\nu)] + 1$, c^ν is a diffusion constant and $\delta(x)$ is a Dirac delta function. Then show that for the solution of the diffusion equation, there exists the formula

$$N(x,t) = \frac{1}{(4\pi c^\nu t^\nu)^{\frac{1}{2}}} H_{1,2}^{2,0}\left[\frac{|x|^2}{4c^\nu t^\nu}\middle|\begin{matrix}(1-\frac{\nu}{2},\nu)\\(0,1),(\frac{1}{2},1)\end{matrix}\right].$$

(Metzler and Klafter 2000; Jorgenson and Lang 2001).

6.10.6. Consider the generalized free electron laser equation

$$_0D_\tau^\alpha f(\tau) = \lambda \int_0^\tau t^\sigma f(\tau - t)\phi(b, \sigma+1; i\nu t)\mathrm{d}t + k\tau^\gamma \phi(\beta, \gamma+1; i\nu t),\ 0 \le \tau \le 1 \tag{6.189}$$

with $\lambda, k \in C; \nu, b, \beta \in R, \alpha > 0, \gamma > -1, \sigma > -1$ with initial condition

$$_0D_\tau^{\alpha-\tau} f(\tau)|_{\tau=0} = b_r, r = 1, \ldots, N, \tag{6.190}$$

where $N = [\alpha] + 1$ is a positive integer, $N - 1 \le \alpha < N$ and b_r's are real numbers. Then show that there exists an unique solution of the Cauchy-type problem (6.189)–(6.190), given by

$$f(\tau) = f_0(\tau) + \int_0^\tau f(\zeta)\left[\sum_{m=1}^{\infty} P_1(m,\tau,\zeta)\right]\mathrm{d}\zeta + k\Gamma(\gamma+1)\sum_{m=0}^{\infty} P_2(m,\tau), \tag{6.191}$$

where,

$$f_0(\tau) = \sum_{j=1}^{N} \frac{b_j}{\Gamma(\alpha - j + 1)}\tau^{N-j}, \tag{6.192}$$

$$P_1(m,\tau,\zeta) = [\lambda\Gamma(\sigma+1)]^m (\tau-\zeta)^{m(\alpha+\sigma+1)-1}\phi^*[bm, m(\alpha+\sigma+1); i\nu(\tau-\zeta)], \tag{6.193}$$

$$\begin{aligned}P_2(m,\tau) = [\lambda\Gamma(\sigma+1)]^m \tau^{\alpha(m+1)+m(\sigma+1)+\gamma}\phi^*(bm+\beta, \alpha(m+1)\\ + m(\sigma+1)+\gamma+1; i\nu\tau),\end{aligned} \tag{6.194}$$

and

$$\phi^*(a,c;z) = \frac{1}{\Gamma(c)}\phi(a,c;z). \tag{6.195}$$

(Saxena and Kalla 2003).

6.10.7. Let $\alpha, \rho, \sigma, \gamma, \omega, \lambda \in C, \min\{\Re(\alpha), \Re(\rho), \Re(\sigma)\} > 0$. If $f(x) \in L(a,b)$, then show that the Cauchy-type problem

$$(D_{a+}^{\alpha} f)(x) = \lambda \int_a^x (x-t)^{\sigma-1} E_{\rho,\sigma}^{\gamma}[\omega(x-t)^{\rho}] f(t) \mathrm{d}t + h(x), a \leq x \leq b, \tag{6.196}$$

and

$$\lim_{x \to +a} (D_{a+}^{\alpha-r} f)(x) = b_r, r = 1, \dots, n = -[-\Re(\alpha)], \tag{6.197}$$

is solvable in the space $L(a, b)$ and its unique solution is given by

$$f(x) = \sum_{r=1}^{n} b_r f_r(x) + \int_a^x \Omega(x-t) h(t) \mathrm{d}t, \tag{6.198}$$

where

$$f_r(x) = (x-a)^{\alpha-r} \sum_{j=0}^{\infty} \lambda^j (x-a)^{\sigma+\alpha} \times E_{\rho+(\sigma+\alpha)j+\alpha-r+1}^{\gamma j}[\omega(x-a)^{\rho}], r = 1, \dots, n, \tag{6.199}$$

and

$$\Omega(u) = \sum_{j=0}^{\infty} \lambda^j u^{(\sigma+\alpha)j+\alpha-1} E_{\rho,(\sigma+\alpha)j+\alpha}^{\gamma j}[\omega u^{\rho}], \tag{6.200}$$

where $E_{\rho,\sigma}^{\gamma}(z)$ is the generalized Mittag-Leffler function defined in (1.46) (Kilbas et al. 2002).

Appendix

A.1 H-Function of Several Complex Variables

Notation A.1. $H(z_1, \ldots, z_n)$: Multivariable H-function or H-function of several complex variables.

Definition A.1. The multivariable H-function is defined in terms of multiple Mellin–Barnes type contour integral as

$$H[z_1, \ldots, z_r] = H^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\left|\begin{matrix} (a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)})_{1,p}:(c_j^{(1)},\gamma_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ \\ (b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)})_{1,q}:(d_j^{(1)},\delta_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r} \end{matrix}\right.\right]$$

$$= \frac{1}{(2\pi w)^r}\int_{L_1}\cdots\int_{L_r}\Psi(\zeta_1,\ldots,\zeta_r)\left\{\prod_{i=1}^{r}\phi_i(\zeta_i)z_i^{\zeta_i}\right\}\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_r, \quad \text{(A.1)}$$

where

$$\Psi(\zeta_1,\ldots,\zeta_r) = \frac{\prod_{j=1}^{n}\Gamma(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}\zeta_i)}{\left[\prod_{j=n+1}^{p}\Gamma(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}\zeta_i)\right]\left[\prod_{j=1}^{q}\Gamma(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}\zeta_i)\right]}, \quad \text{(A.2)}$$

$$\phi_i(\zeta_i) = \frac{\left[\prod_{\lambda=1}^{m_i}\Gamma(d_\lambda^{(i)}-\delta_\lambda^{(i)}\zeta_i)\right]\left[\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}\zeta_i)\right]}{\left[\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-\gamma_j^{(i)}\zeta_i)\right]\left[\prod_{\lambda=m_i+1}^{q_i}\Gamma(1-d_\lambda^{(i)}+\delta_\lambda^{(i)}\zeta_i)\right]}, \quad \text{(A.3)}$$

for $i = 1, \ldots, r$, and $L_i = L_{w\tau_i\infty}$, $w = (-1)^{\frac{1}{2}}$ represents the contours which start at the point $\tau_i - w\infty$ and goes to the point $\tau_i + w\infty$ with $\tau_i \in R = (-\infty, \infty)$, $i = 1, \ldots, r$ such that all the poles of $\Gamma(d_j^{(i)} - \delta_j^{(i)}\zeta_i)$, $j = 1, \ldots, m_i$; $i = 1, \ldots, r$ are separated from those of $\Gamma(1 - c_j^{(i)} - \gamma_j^{(i)}\zeta_i)$, $j = 1, \ldots, n_i$; $i = 1, \ldots, r$ and $\Gamma(1 - a_j + \sum_{i=1}^{r}\alpha_j^{(i)}\zeta_i)$, $j = 1, \ldots, n$. Here, the integers n, p, q, m_i, n_i, p_i and

q_i, satisfy the inequalities $0 \leq n \leq p; q \geq 0, 1 \leq m_i \leq q_i$ and $1 \leq n_i \leq p_i, i = 1, \ldots, r$. Further, we suppose that the parameters

$$\begin{aligned} &a_j, j = 1, \ldots, p; c_j^{(i)}, j = 1, \ldots, p_i; i = 1, \ldots, r, \\ &b_j, j = 1, \ldots, q; d_j^{(i)}, j = 1, \ldots, q_i; i = 1, \ldots, r, \end{aligned} \tag{A.4}$$

are complex numbers and the associated coefficients

$$\begin{aligned} &\alpha_j^{(i)}, j = 1, \ldots, p; i = 1, \ldots, r; \gamma_j^{(i)}, j = 1, \ldots, p_i, i = 1, \ldots, r, \\ &\beta_j^{(i)}, j = 1, \ldots, q; i = 1, \ldots, r; \delta_j^{(i)}, j = 1, \ldots, q_i; i = 1, \ldots, r, \end{aligned} \tag{A.5}$$

are positive real numbers, such that

$$\begin{aligned} \Lambda_i &= \sum_{j=1}^{p} \alpha_j^{(i)} + \sum_{j=1}^{p_i} \gamma_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} - \sum_{j=1}^{q_i} \delta_j^{(i)} \leq 0, \\ i &= 1, \ldots, r \end{aligned} \tag{A.6}$$

$$\begin{aligned} \Omega_i = &\sum_{j=1}^{n} \alpha_j^{(i)} - \sum_{j=n+1}^{p} \alpha_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=1}^{n_i} \gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} \gamma_j^{(i)} \\ &+ \sum_{j=1}^{m_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} \delta_j^{(i)} > 0, i = 1, \ldots, r. \end{aligned} \tag{A.7}$$

It is assumed that the poles of the integrand of (A.1) are simple. We know that the integral in (A.1) converges absolutely, under the conditions (A.7), (Srivastava et al. (1982), p. 251) with

$$|\arg(z_i)| < \frac{\pi}{2}\Omega_i, i = 1, \ldots, r, \tag{A.8}$$

and the points $z_i = 0, i = 1, \ldots, r$ and various exceptional paramater values being tacitly excluded. From Srivastava and Panda (1976b, p. 131) we have

$$H(z_1, \ldots, z_r) = O(|z_1|^{e_1}, \ldots, |z_r|^{e_r}), \max_{1 \leq j \leq r} [|z_j|] \to 0, \tag{A.9}$$

where

$$e_i = \min_{1 \leq j \leq m_i} \left[\frac{\Re(d_j^{(i)})}{\delta_j^{(i)}} \right], i = 1, \ldots, r. \tag{A.10}$$

For $n = 0$, there holds the following asymptotic expansion (Srivastava and Panda, 1976b, p. 131):

$$H[z_1, \ldots, z_r] = O(|z_1|^{g_1}, \ldots, |z_r|^{g_r}), \min_{1 \leq j \leq r} [|z_j|] \to \infty, \tag{A.11}$$

where

$$g_i = \max_{1 \le j \le n_i} \left[\frac{\Re(c_j^{(i)}) - 1}{\gamma_j^{(i)}} \right], i = 1, \ldots, r, \tag{A.12}$$

provided that each of the inequalities in (A.6), (A.7), and (A.8) hold true.

Remark A.1. When $n = 2$ the multivariable H-function defined by (A.1) reduces to the H-function of two variables studied by Mittal and Gupta (1972). H-function of two variables are also defined and studied by Munot and Kalla (1971) and Verma (1971), and others. A comprehensive and detailed account of the H-function of two variables is available from the monograph by Hai and Yakubovich (1992).

It is interesting to observe that for $n = p = q = 0$, the multivariable H-function breaks up into product of r H-functions and consequently there holds the following result (Saxena 1977):

$$H^{0,0:m_1,n_1;\cdots;m_r,n_r}_{0,0:p_1,q_1;\cdots;p_r,q_r} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \left| \begin{matrix} -:(c_j^{(1)},\gamma_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ -:(d_j^{(1)},\delta_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r} \end{matrix} \right. \right] = \prod_{i=1}^{r} H^{m_i,n_i}_{p_i,q_i} \left[z_i \left| \begin{matrix} (c_j^{(i)},\gamma_j^{(i)})_{1,p_i} \\ (d_j^{(i)},\delta_j^{(i)})_{1,q_i} \end{matrix} \right. \right]. \tag{A.13}$$

Remark A.2. The function defined by (A.1) was introduced and studied by Srivastava and Panda (1976a, p. 271).

When $\alpha_j^{(1)} = \cdots = \alpha_j^{(r)}, j = 1, \ldots, p; \beta_j^{(1)} = \cdots = \beta_j^{(r)}, j = 1, \ldots, q$ in (A.1) the multivariable H-function defined and studied by Saxena (1974, 1977) is obtained. In case all the Greek letters are assumed to be unity, the H-function of several complex variables (A.1) reduces to the G-function of several complex variables studied by Khadia and Goyal (1970, 1975).

Remark A.3. Fractional integrals involving multivariable H-functions are given in a series of papers by Saigo and Saxena (1999, 1999a, 2001). Srivastava and Hussain (1995) Saigo et al. (2005), and others.

A.2 Kampé de Fériet Function and Lauricella Functions

A.2.1 Kampé de Fériet Series in the Generalized Form

Definition A.2. Kampé de Fériet series in the generalized form is defined by

$$F^{p;q;r}_{k;m;n} \left[\begin{matrix} (a_p): & (b_q), & (c_r) & ; \\ (d_k) & :(e_m), & (g_n) & ; \end{matrix} x, y \right] = \sum_{\tau,\nu=0}^{\infty} \frac{\left[\prod_{j=1}^{p}(a_j)_{\tau+\nu}\right]\left[\prod_{j=1}^{q}(b_j)_{\tau}\right]\left[\prod_{j=1}^{r}(c_j)_{\nu}\right]}{\left[\prod_{j=1}^{k}(d_j)_{\tau+\nu}\right]\left[\prod_{j=1}^{m}(e_j)_{\tau}\right]\left[\prod_{j=1}^{n}(g_j)_{\nu}\right]} \frac{x^\tau}{\tau!} \frac{y^\nu}{\nu!}, \tag{A.14}$$

where, for convergence

$$(i)\ \ p+q<k+m+1; p+r<k+n+1, |x|<\infty, |y|<\infty, \qquad \text{(A.15)}$$

or

$$(ii)\ \ p+q=k+m+1; p+r=k+n+1,$$

and

$$\begin{cases} |x|^{1/(p-k)}+|y|^{1/(p-k)}<1, \text{ if } p>k, \\ \max\{|x|,|y|\}<1, \text{ if } p\leq k. \end{cases} \qquad \text{(A.16)}$$

The above series reduces to the original Kampé de Fériet series (Kampé de Fériet 1921), when $q=r$ and $m=n$, and is also called Kampé de Fériet series.

Remark A.4. A generalization of the series (A.14) is given by Srivastava and Daoust (1969), which is indeed the extension of Wright's generalized hypergeometric series ${}_p\Psi_q(z)$. This generalization is further extended by Srivastava and Daoust (1969a), which is described in the next subsection.

Three interesting special cases of the reducibility of (A.14) to generalized hypergeometric series ${}_pF_q(z)$, are given below. For further cases of reducibility of the series defined by (A.14) in terms of the generalized hypergeometric series, see the monograph by Srivastava and Karlsson (1985, pp. 28–32) and references of special cases given therein.

$$F^{p:0,0}_{q:0,0}\left[\begin{matrix} a_1,\cdots,a_p; & ; & ; \\ b_1,\cdots,b_q; & ; & ; \end{matrix} x,y\right] = {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x+y), \qquad \text{(A.17)}$$

$$F^{0:p,m}_{0:q,n}\left[\begin{matrix} ;a_1,\cdots,a_p; & c_1,\cdots,c_m \\ ;b_1,\cdots,b_q; & d_1,\cdots,d_n; \end{matrix} x,y\right] = {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x) \times {}_mF_n(c_1,\ldots,c_m;d_1,\ldots,d_n;y), \qquad \text{(A.18)}$$

$$F^{p:1,1}_{q:0,0}\left[\begin{matrix} a_1,\cdots,a_p; & c; & d; \\ b_1,\cdots,b_q; & ; & ; \end{matrix} x,x\right] = {}_{p+1}F_q(a_1,\ldots,a_p;c+d;b_1,\ldots,b_q;x). \qquad \text{(A.19)}$$

A.2.2 Generalized Lauricella Function

Notation A.2. $F^{A:B^{(1)};\cdots;B^{(n)}}_{C:D^{(1)};\cdots;D^{(n)}}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$: Generalized Lauricella function of n complex variables.

Definition A.3. The generalized Lauricella series (Srivastava and Daoust 1969a) is defined in the following manner:

$$F^{A:B^{(1)};\cdots;B^{(n)}}_{C:D^{(1)};\cdots;D^{(n)}}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= F^{A:B^{(1)};\cdots;B^{(n)}}_{C:D^{(1)};\cdots;D^{(n)}}\left[\begin{matrix} [(a):\theta^{(1)},\ldots,\theta^{(n)}]:[(b^{(1)}):\phi^{(1)})];\cdots;[(b^{(n)});\phi^{(n)}) \\ [(c):\psi^{(1)},\ldots,\psi^{(n)}]:[(d^{(1)}:(\delta^{(1)})];\cdots;[(d^{(n)});\delta^{(n)}) \end{matrix} x_1,\ldots,x_n\right]$$

$$= \sum_{m_1=0,\ldots,m_n=0}^{\infty} \chi(m_1,\ldots,m_n)\frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!}, \tag{A.20}$$

where, for convenience,

$$\chi(m_1,\ldots,m_n)$$
$$= \frac{[\prod_{j=1}^{A}(a_j)_{m_1\theta_1^{(1)}+\cdots+m_n\theta_j^{(n)}}][\prod_{j=1}^{B^{(1)}}(b_j^{(1)})_{m_1\phi_1^{(1)}}]\cdots[\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}]}{[\prod_{j=1}^{C}(c_j)_{m_1\psi_1^{(1)}+\cdots+m_n\psi_j^{(n)}}][\prod_{j=1}^{D^{(1)}}(d_j^{(1)})_{m_1\delta_j^{(1)}}]\cdots[\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}]}, \tag{A.21}$$

the coefficients

$$\begin{cases} \theta_j^{(k)}, j=1,\ldots,A; \phi_j^{(k)}, j=1,\ldots,B^{(k)}; \Psi_k^{(j)}, j=1,\ldots,C, \\ \delta_j^{(k)}, j=1,\ldots,D^{(k)}, k=1,\ldots,n, \end{cases} \tag{A.22}$$

are real and positive, and (a) abbreviates the array of A parameters $a_1,\ldots,a_A;(b^{(k)})$ abbreviates the array of $B^{(k)}$ parameters

$$b_j^{(k)}, j=1,\ldots,B^{(k)}, k=1,\ldots,n. \tag{A.23}$$

Similar interpretations hold for the remaining parameters. For precise conditions under which this multiple series (A.20) converges, see Srivastava and Daoust (1972, pp. 153–157), also see Exton (1976, Sect. 3.7) and Exton (1978, Sect. 1.4).

When each of the positive numbers given in (A.22) takes the value unity, the generalized Lauricella series (A.20) gives rise to a direct multivariable extension of Kampé de Fériet series (A.14). Thus the multivariable generalization of the Kampé de Fériet series defined by (A.14) is given by (see, Srivastava and Panda, 1975, p. 1127; Srivastava and Karlsson 1985, p. 38):

$$F^{p:p_1,\dots,p_n}_{k:q_1,\dots,q_n}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = F^{p:p_1,\dots,p_n}_{k:q_1,\dots,q_n}\left[\begin{matrix}(a_p):(b^{(1)}_{p_1});\cdots;(b^{(n)}_{p_n}): \\ (\alpha_k):(\beta^{(1)}_{q_1});\cdots;(\beta^{(n)}_{q_n}); \end{matrix} x_1,\dots,x_n\right], \tag{A.24}$$

$$= \sum_{m_1=0,\dots,m_n=0}^{\infty} \theta(m_1,\dots,m_n)\frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!}, \tag{A.25}$$

where

$$\theta(m_1,\dots,m_n) = \frac{[\prod_{j=1}^{p}(a_j)_{m_1+\cdots+m_n}]\left[\prod_{j=1}^{p_1}(b_j^{(1)})_{m_1}\right]\cdots\left[\prod_{j=1}^{p_n}(b_j^{(n)})_{m_n}\right]}{\left[\prod_{j=1}^{k}(\alpha_j)_{m_1+\cdots+m_n}\right]\left[\prod_{j=1}^{q_1}(\beta_j^{(1)})_{m_1}\right]\cdots\left[\prod_{j=1}^{q_n}(\beta_j^{(n)})_{m_n}\right]}, \tag{A.26}$$

and, for convergence of the series (A.25),

$$1 + k + q_r - p - p_r \geq 0, r = 1,\dots,n. \tag{A.27}$$

The equality holds when, in addition, either

$$p > k \text{ and } |x_1|^{1/(p-k)} + \cdots + |x_n|^{1/(p-k)} < 1, \tag{A.28}$$

or

$$p \leq k \text{ and } \max\{|x_1|,\dots,|x_n|\} < 1. \tag{A.29}$$

Remark A.5. Karlsson (1973) has considered a special case of (A.24) when

$$p_r = q, q_r = m_r, r = 1,\dots,n. \tag{A.30}$$

A relation connecting generalized Lauricella function and the multivariable H-function is given by Srivastava and Panda (1976a, p. 272)

$$H^{0,p:1,p_1;\cdots;1,p_r}_{p,q:p_1,q_1+1;\cdots;p_r,q_r+1}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle| \begin{matrix}(a_j;\alpha_j^{(1)},\dots,\alpha_j^{(r)})_{1,p}:(c_j^{(1)},\gamma_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ (b_j;\beta_j^{(1)},\dots,\beta_j^{(r)})_{1,q}:(0,1),(d_j^{(1)},\delta_j^{(1)})_{1,q_1};\cdots;(0,1),(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix}\right]$$

$$= \frac{[\prod_{j=1}^{p}\Gamma(1-a_j)][\prod_{j=1}^{p_1}\Gamma(1-c_j^{(1)})]\cdots[\prod_{j=1}^{p_r}\Gamma(1-c_j^{(r)})]}{[\prod_{j=1}^{q}\Gamma(1-b_j)][\prod_{j=1}^{q_1}\Gamma(1-d_j^{(1)})]\cdots[\prod_{j=1}^{q_r}\Gamma(1-d_j^{(r)})]}$$

$$\times F^{p:p_1;\cdots;p_r}_{q:q_1;\cdots;q_r}\left[\begin{matrix}[(1-a_j:\alpha_j^{(1)},\dots,\alpha_j^{(r)})]_{1,p}:[(1-c_j^{(1)},\gamma_j^{(1)})]_{1,p_1};\cdots \\ [(1-b_j:\beta_j^{(1)},\dots,\beta_j^{(r)})]_{1,q}:[(1-d_j^{(1)},\delta_j^{(1)})]_{1,q_1};\cdots\end{matrix}\right.$$

$$\left.\begin{matrix};[(1-c_j^{(r)};\gamma_j^{(r)})]_{1,p_r} \\ ;[(1-d_j^{(r)};\delta_j^{(r)})]_{1,q_r}\end{matrix} -z_1,\dots,-z_n\right]. \tag{A.31}$$

A.3 Appell Series

Notation A.3. $F_1(a,b,b';c;x,y)$: Appell function of the first kind.

Notation A.4. $F_2(a,b,b';c,c';x,y)$: Appell function of the second kind.

Notation A.5. $F_3(a,b,b';c;x,y)$: Appell function of the third kind.

Notation A.6. $F_4(a,b';c,c';x,y)$: Appell function of the fourth kind.

Following Appell (1880) we define the four Appell series as follows:

Definition A.4.

$$\begin{aligned} F_1(a,b,b';c;x,y) &= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}} \frac{x^m y^n}{m!n!} \\ &= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} {}_2F_1(a+m,b';c+m;y)\frac{x^m}{m!}, \end{aligned} \quad \text{(A.32)}$$

where $\max\{|x|,|y|\} < 1$.

Definition A.5.

$$\begin{aligned} F_2(a,b,b';c,c';x,y) &= \sum_{m=0,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n} \frac{x^m y^n}{m!n!} \\ &= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} {}_2F_1(a+m,b';c';y)\frac{x^m}{m!}, \end{aligned} \quad \text{(A.33)}$$

where $|x|+|y| < 1$.

Definition A.6.

$$\begin{aligned} F_3(a,b,b';c;x,y) &= \sum_{m=0,n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}} \frac{x^m y^n}{m!n!} \\ &= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} {}_2F_1(a',b';c+m;y)\frac{x^m}{m!}, \end{aligned} \quad \text{(A.34)}$$

where $\max\{|x|,|y|\} < 1$.

Definition A.7.

$$\begin{aligned} F_4(a,b';c,c';x,y) &= \sum_{m=0,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n} \frac{x^m y^n}{m!n!} \\ &= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} {}_2F_1(a+m,b+m;c';y)\frac{x^m}{m!}, \end{aligned} \quad \text{(A.35)}$$

where $\sqrt{|x|}+\sqrt{|y|} < 1$. Here the denominator parameters c and c' are neither zero nor a negative integer.

The above defined functions are discovered while considering the product of two Gauss series. In this analysis, we also come across the following interesting result:

$$ {}_2F_1(a,b;c;x+y) = \sum_{m=0,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_{m+n}} \frac{x^m y^n}{m!n!}. \tag{A.36}$$

A multiple integral representation for the generalized hypergeometric series is given by (Saigo and Saxena 1999)

$$\begin{aligned} &\frac{\prod_{j=1}^{P}\Gamma(A_j)}{\prod_{j=1}^{Q}\Gamma(B_j)}\,{}_PF_Q[(A_P);(B_Q);-(x_1+\cdots+x_n)] \\ &= \left(\frac{1}{2\pi i}\right)^n \int_{L_1}\cdots\int_{L_n} \frac{[\prod_{j=1}^{P}\Gamma(A_j+s_1+\cdots+s_n)]}{[\prod_{j=1}^{Q}\Gamma(B_j+s_1+\cdots+s_n)} \\ &\quad \times \Gamma(-s_1)\cdots\Gamma(-s_r)x_1^{s_1}\cdots s_n^{s_n}\mathrm{d}s_1\cdots\mathrm{d}s_n, \end{aligned} \tag{A.37}$$

where the contours are of Barnes type with indentations, if necessary, such that the poles of $\Gamma(A_j+s_1+\cdots+s_n), j = 1,\ldots,p$ are separated from those of $\Gamma(-s_j), j = 1,\ldots,n$.

A.3.1 Confluent Hypergeometric Function of Two Variables

Definition A.8.

$$\phi_1(a,b;c;x,y) = \sum_{m=0,n=0}^{\infty} \frac{(a)_{m+n}(b)_m}{(c)_{m+n}} \frac{x^m y^n}{m!n!}, |x| < 1, |y| < \infty. \tag{A.38}$$

Definition A.9.

$$\phi_2(b,b';c;x,y) = \sum_{m=0,n=0}^{\infty} \frac{(b)_m(b')_m}{(c)_{m+n}} \frac{x^m y^n}{m!n!}, |x| < \infty, |y| < \infty. \tag{A.39}$$

Definition A.10.

$$\phi_3(b;c;x,y) = \sum_{m=0,n=0}^{\infty} \frac{(b)_m}{(c)_{m+n}} x^m y^n, |x| < \infty, |y| < \infty. \tag{A.40}$$

Definition A.11.

$$\psi_1(a,b;c,c';x,y)=\sum_{m=0,n=0}^{\infty}\frac{(a)_{m+n}(b)_m}{(c)_m(c')_n}\frac{x^m y^n}{m!n!},|x|<1,|y|<\infty. \quad \text{(A.41)}$$

Definition A.12.

$$\psi_2(a;c,c';x,y)=\sum_{m=0,n=0}^{\infty}\frac{(a)_{m+n}}{(c)_m(c')_n}\frac{x^m y^n}{m!n!},|x|<\infty,|y|<\infty. \quad \text{(A.42)}$$

Definition A.13.

$$\Xi_1(a,a',b;c;x,y)=\sum_{m=0,n=0}^{\infty}\frac{(a)_m(a')_n(b)_m}{(c)_{m+n}}\frac{x^m y^n}{m!n!},|x|<1,|y|<\infty. \quad \text{(A.43)}$$

Definition A.14.

$$\Xi_2(a,b;c;x,y)=\sum_{m=0,n=0}^{\infty}\frac{(a)_m(b)_m}{(c)_{m+n}}\frac{x^m y^n}{m!n!},|x|<1,|y|<\infty. \quad \text{(A.44)}$$

A.4 Lauricella Functions of Several Variables

The four Appell series F_1, F_2, F_3, F_4 are generalized by Lauricella (1893) in terms of multiple hypergeometric series as given below.

Definition A.15.

$$F_A^{(n)}[a,b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n] = \sum_{m_1=0,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n}}\frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!}, \quad \text{(A.45)}$$

where $|x_1|+|x_2|+\cdots+|x_n|<1$.

Definition A.16.

$$F_B^{(n)}[a_1,\ldots,a_n;b_1,\ldots,b_n;c;x_1,\ldots,x_n] = \sum_{m_1=0,\ldots,m_n=0}^{\infty}\frac{(a_1)_{m_1}\cdots(a_n)_{m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}}\frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!}, \quad \text{(A.46)}$$

where $\max\{|x_1|,\ldots,|x_n|\}<1$.

Definition A.17.

$$F_C^{(n)}[a,b;c_1,\dots,c_n;x_1,\dots,x_n] = \sum_{m_1=0,\dots,m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b)_{m_1+\cdots+m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n}} \frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!}, \tag{A.47}$$

where $\sqrt{|x_1|}+\cdots+\sqrt{|x_n|}<1$.

Definition A.18.

$$F_D^{(n)}[a,b_1,\dots,b_n;c;x_1,\dots,x_n] = \sum_{m_1=0,\dots,m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!}, \tag{A.48}$$

where $\max\{|x_1|,\dots,|x_n|\}<1$.

For $n=2$ we have the following relations:

$$F_A^{(2)}=F_2,\, F_B^{(2)}=F_3,\, F_C^{(2)}=F_4,\, F_D^{(2)}=F_1, \tag{A.49}$$

where the Appell series are defined in the previous section. An interesting result is the following reduction formula (Lauricella, 1893)

$$F_D^{(n)}[a,b_1,\dots,b_n;c;x,\dots,x] = {}_2F_1(a,b_1+\cdots+b_n;c;x). \tag{A.50}$$

We also have (Lauricella 1893)

$$F_D^{(n)}[a,b_1,\dots,b_n;c;1,\dots,1] = \frac{\Gamma(c)\Gamma(c-a-b_1-\cdots-b_n)}{\Gamma(c-a)\Gamma(c-b_1-\cdots-b_n)}, \tag{A.51}$$

where $c\neq 0,-1,-2,\dots;\ \Re(c-a-b_1-\cdots-b_n)>0$.

Single integral representations for the function $F_D^{(n)}$ is given by

$$F_D^{(n)}[a,b_1,\dots,b_n;c;x_1,\dots,x_n] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 u^{a-1}(1-u)^{c-a-1}(1-ux_1)^{-b_1}\cdots(1-ux_n)^{-b_n}\,du, \tag{A.52}$$

where $\Re(a)>0, \Re(c-a)>0$.

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(f_1 t+g_1)^{\sigma_1}\cdots(f_k t+g_k)^{\sigma_k}\,\mathrm{d}t$$
$$= (b-a)^{\alpha+\beta-1}B(\alpha,\beta)(af_1+g_1)^{\sigma_1}\cdots(af_k+g_k)^{\sigma_k}$$
$$\times F_D^{(n)}\left[\alpha,-\sigma_1,\ldots,-\sigma_k;\alpha+\beta;-\frac{(b-a)f_1}{af_1+g_1},\ldots,-\frac{(b-a)f_k}{af_k+g_k}\right], \quad \text{(A.53)}$$

where $a,b\in\Re(a<b)$, $f_i,g_i,\sigma_i\in C, i=1,\ldots,k$, $\min\{\Re(\alpha),\Re(\beta)\}>0$ and

$$\max\left[\left|\frac{(b-a)f_1}{af_1+g_1}\right|,\ldots,\left|\frac{(b-a)f_k}{af_k+g_k}\right|\right]<1.$$

A.4.1 Confluent form of Lauricella Series

Definition A.19.

$$\Phi_2^{(n)}[b_1,\ldots,b_n;c;x_1,\ldots,x_n] = \sum_{m_1=0,\ldots,m_n=0}^{\infty}\frac{(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}}\frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!}. \quad \text{(A.54)}$$

Definition A.20.

$$\Psi_2^{(n)}[a;c_1,\ldots,c_n;x_1,\ldots,x_n] = \sum_{m_1=0,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\cdots+m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n}}\frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!}. \quad \text{(A.55)}$$

Definition A.21.

$$M_{k,\mu_1,\ldots,\mu_n}(x_1,\ldots,x_n) = x_1^{\mu_1+\frac{1}{2}}\cdots x_n^{\mu_n+\frac{1}{2}}\exp\left[-\frac{x_1+\cdots+x_n}{2}\right]$$
$$\times\Psi_2^{(n)}[\mu_1+\cdots+\mu_n-k+\frac{n}{2};2\mu_1+1,\ldots,2\mu_n+1;x_1,\ldots,x_n]. \quad \text{(A.56)}$$

For a detailed definition and other properties of these functions, see the original paper by Pierre Humbert [La fonction $W_{k,\mu_1,\ldots,\mu_n}(x_1,\ldots,x_n)$. Comptes Rendus.t. CLXXI, 1920, p. 328] and Appell and Kampé de Fériet (1926).

A.5 The Generalized H-Function (The $\bar{H}$-Function)

Notation A.7. $\bar{H}(z)$, $\bar{H}_{p,q}^{m,n}[x]$; H bar function

Definition A.22. In an attempt to derive certain Feynman integrals in two different ways which arise in perturbation calculations of the equilibrium properties of a magnetic model of phase transitions, Inayat-Hussain (1987b) investigated a generalization of the H-function as

$$\bar{H}(z) = \bar{H}^{m,n}_{p,q}(z) = \bar{H}^{m,n}_{p,q}\left[x\left|\begin{matrix}(\alpha_j,A_j,a_j)_{1,n},(\alpha_j,A_j)_{n+1,p}\\(\beta_j,B_j)_{1,m},(\beta_j,B_j,b_j)_{m+1,q}\end{matrix}\right.\right] \tag{A.57}$$

$$= \frac{1}{2\pi i}\int_L \chi(s)z^s \mathrm{d}s, \tag{A.58}$$

where

$$\chi(s) = \frac{\left[\prod_{j=1}^{m}\Gamma(\beta_j - B_j s)\right]\left[\prod_{j=1}^{n}\{\Gamma(1-\alpha_j + A_j)\}^{a_j}\right]}{\left[\prod_{j=M+1}^{q}\{\Gamma(1-\beta_j + B_j s)\}^{b_j}\right]\left[\prod_{j=n+1}^{p}\Gamma(\alpha_j - A_j)\right]}, \tag{A.59}$$

which contains fractional powers of some of the gamma functions. $L = L_{i\tau\infty}$ is a contour starting at the point $\tau - i\infty$, and going to the point $\tau + i\infty$ with $\gamma \in R = (-\infty, \infty)$. For a detailed definition, convergence and existence conditions, and for the computable representation of the $\bar{H}$-function, the reader is referred to the original papers of Buschman and Srivastava (1990) and Saxena (1998). It is interesting to note that for $a_j = b_j = 1$ for all j, the $\bar{H}$-function reduces to the familiar H-function defined by Fox (1961), see also Mathai and Saxena (1978) and Kilbas and Saigo (2004).

A.5.1 Special Cases of $\bar{H}$-Function

A few interesting special cases of the $\bar{H}$-function, which cannot be obtained from the H-function are given below.

$$g_1 = (-1)^p g(\gamma,\eta,\zeta,p;z) = \frac{K_{d-1}\Gamma(1+p)\Gamma(1+\frac{\zeta}{2})B\left(\frac{1}{2},\frac{1}{2}+\frac{\zeta}{2}\right)}{2^{2+p}\pi\Gamma(\gamma)\Gamma(\gamma-\frac{\zeta}{2})}$$
$$\times\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\mathrm{d}s(-z)^s\Gamma(-s)\Gamma(\gamma+s)\Gamma(\gamma-\frac{\zeta}{2}+s)}{(\eta+s)^{1+p}\Gamma(1+\frac{\zeta}{2}+s)} \tag{A.60}$$

$$= \frac{K_{d-1}\Gamma(1+p)\Gamma(\frac{1}{2}+\frac{\zeta}{2})}{2^{2+p}\pi\Gamma(\gamma)\Gamma(\gamma-\frac{\zeta}{2})}$$
$$\times\bar{H}^{1,3}_{3,3}\left[-z\left|\begin{matrix}(1-\gamma,1;1),(1-\gamma+\frac{\zeta}{2},1;1),(1-\eta,1;1+p)\\(0,1),\left(-\frac{\zeta}{2},1;1\right),(-\eta,1;1+p)\end{matrix}\right.\right], \tag{A.61}$$

where $K_d = [2^{1-d}\pi^{-\frac{d}{2}}/\Gamma\left(\frac{d}{2}\right)]$ (Inayat-Hussain 1987a, Eq. (5)). The above integral is connected with certain class of Feynman integrals.

$$\beta F(d,\epsilon) = -\frac{1}{4\pi^{\frac{d}{2}}(1+\epsilon)^2} \times \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\mathrm{d}s[-(1+\epsilon)^{-2}]^s\Gamma(-s)[\Gamma(1+s)]^2[\Gamma\left(\frac{3}{2}+s\right)]^d}{[\Gamma(2+s)]^{1+d}} \tag{A.62}$$

$$= -\frac{1}{4\pi^{\frac{d}{2}}(1+\epsilon)^2}\bar{H}^{1,2}_{2,2}\left[-(1+\epsilon)^{-2}\left|\begin{matrix}(0,1;2),(-\frac{1}{2},1;d)\\(0,1),(-1,1;1+d)\end{matrix}\right.\right] \tag{A.63}$$

$$= -\frac{1}{4\pi^{\frac{d}{2}}(1+\epsilon)^2}\bar{H}^{1,3}_{3,2}\left[-(1+\epsilon)^{-2}\left|\begin{matrix}(0,1;1),(0,1;1),(-\frac{1}{2},1;d)\\(0,1),(-1,1;1+d)\end{matrix}\right.\right]. \tag{A.64}$$

The above function is the exact partition function of the Gaussian model in statistical mechanics.

For further example of a function, which is not a special case of the H-function is the poly-logarithm of complex order ν, denoted by $L^\nu(z)$. Its relation with $\bar{H}$-function is given by Saxena (1998, eq. (1.12)) as

$$L^\nu(z) = \bar{H}^{1,2}_{2,2}\left[-z\left|\begin{matrix}(0,1,1),(1,1;\nu)\\(0,1),(0,1;\nu-1)\end{matrix}\right.\right]. \tag{A.65}$$

An account of $L^\nu(z)$ is available from the book by Marichev (1983).

The function due to Nagarsenker and Pillai (1973, 1974) also furnishes an example of a function, which is not a special case of Fox's H-function. Yet another function, which is not a special case of the H-function is the generalized Riemann-zeta function defined by

$$\phi(z,q,\eta) = \sum_{k=0}^{\infty}\frac{z^k}{(\eta+k)^q} = \bar{H}^{1,2}_{2,2}\left[-z\left|\begin{matrix}(0,1,1),(1-\eta,1,q)\\(0,1),(-\eta,1,q)\end{matrix}\right.\right]. \tag{A.66}$$

The above function is a generalization of the well-known generalized (Hurwitz's) zeta function $\zeta(q,\eta), q \neq 0,-1,-2,\ldots$ and the Riemann zeta function $\zeta(q)$, $\Re(q) > 1$. It has been shown by Buschman and Srivastava (1990, p. 4708) that the sufficient condition for absolute convergence of the contour integral (A.58) is given by

$$A = \sum_{j=1}^{m}|B_j| + \sum_{j=1}^{n}|a_jA_j| - \sum_{j=m+1}^{q}|b_jB_j| - \sum_{j=n+1}^{p}|A_j| > 0. \tag{A.67}$$

This condition provides exponential decay of the integrand in (A.58), and region of absolute convergence of the contour integral (A.58) is given by

$$|\arg z| < \frac{\pi A}{2}. \tag{A.68}$$

Remark A.6. In a series of papers, abelian theorems, complex inversion formulas and characterizations for the distributional $\bar{H}$-function transformation are established by Saxena and Gupta (1994, 1995, 1997). Functional relations for the $\bar{H}$-function are given by Saxena (1998). Unified fractional integration operators associated with the $\bar{H}$-function are defined and studied by Saxena and Soni (1997). Fractional integral formulas for this function are investigated by Gupta and Soni (2001). Fractional integral formulas associated with Saigo–Maeda operators of fractional integration are given by Saxena et al. (2002). Application of this function in bivariate probability distributions is demonstrated by Saxena et al. (2002).

A.6 Representation of an H-Function in Computable Form

Case I: When the poles of $\prod_{j=1}^{m} \Gamma(b_j - sB_j)$ are simple, that is, where $B_h(b_j + \lambda) \neq B_j(b_h + \nu)$ for $j \neq h, j, h = 1, \ldots, m; \lambda, \nu = 0, 1, 2, \ldots$; then we obtain the following expansion for the H-function.

$$\begin{aligned} H_{p,q}^{m,n}(z) &= \sum_{h=1}^{m} \sum_{\nu=0}^{\infty} \frac{[\prod_{j=1, j\neq h}^{m} \Gamma(b_j - B_j(b_h+\nu)/B_h)][\prod_{j=1}^{n} \Gamma(1-a_j - A_j(b_h+\nu)/B_h)]}{[\prod_{j=m+1}^{q} \Gamma(b_j - B_j(b_h+\nu)/B_h)][\prod_{j=n+1}^{p} \Gamma(a_j - A_j(b_h+\nu)/B_h)]} \\ &\quad \times \frac{(-1)^{\nu} z^{(b_h+\nu)/B_h}}{\nu! B_h}, \end{aligned} \tag{A.69}$$

which exists for all $z \neq 0$ if $\mu > 0$ and for $0 < |z| < \frac{1}{\beta}$ if $\mu = 0$, where β and μ are defined in (1.8) and (1.9) respectively.

Case II. When the poles of $\prod_{j=1}^{n} \Gamma(1-a_j+sA_j)$ are simple, that is, where $A_h(1-a_j+\nu) \neq A_j(1-a_h+\lambda)$ for $j \neq h, j, h = 1, \ldots, n; \lambda, \nu = 0, 1, 2, \ldots$ then we obtain the following expansion for the H-function.

$$\begin{aligned} H_{p,q}^{m,n}(z) &= \sum_{h=1}^{n} \sum_{\nu=0}^{\infty} \frac{[\prod_{j=1, j\neq h}^{n} \Gamma(1-a_j - A_j(1-a_h+\nu)/A_h)]}{[\prod_{j=m+1}^{q} \Gamma(1-b_j - B_j(1-a_h+\nu)/A_h)]} \\ &\quad \times \frac{[\prod_{j=1}^{m} \Gamma(b_j + B_j(1-a_h+\nu)/A_h)]}{[\prod_{j=n+1}^{p} \Gamma(a_j + A_j(1-a_h+\nu)/A_h)]} \frac{(-1)^{\nu} \left(\frac{1}{z}\right)^{\frac{1-a_h+\nu}{A_h}}}{\nu! A_h}, \end{aligned} \tag{A.70}$$

which exists for all $z \neq 0$ if $\mu < 0$ and for $|z| > \frac{1}{\beta}$ if $\mu = 0$, β and μ are defined in (1.8) and (1.9) respectively.

A.7 Further Generalizations of the H-Function

Notation A.8. **I-function**:

$$I_{p_i,q_i}^{m,n}[z], \quad I_{p_i,q_i}^{m,n}\left[z\left|\begin{matrix}(a_j,A_j)_{1,n},\ldots,(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m},\ldots,(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right].$$

Definition A.23. The I-function is defined, like the H-function in terms of a Mellin–Barnes type integral in the following form (Saxena 1982):

$$I_{p_i,q_i}^{m,n}\left[z\left|\begin{matrix}(a_j,A_j)_{1,n},\ldots,(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m},\ldots,(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]=\frac{1}{2\pi i w}\int_L \chi(s)z^{-s}\mathrm{d}s, \tag{A.71}$$

where

$$\chi(s)=\frac{\left[\prod_{j=1}^{m}\Gamma(b_j+B_js)\right]\left[\prod_{j=1}^{n}\Gamma(1-a_j-A_js)\right]}{\left[\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-B_{ji}s)\right]\left[\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)\right]\right]}, \tag{A.72}$$

where m,n,p_i,q_i are nonnegative integers satisfying $0\le n\le p_i, 1\le m\le q_i$, $i=1,\ldots,r$ with r being finite and $w=(-1)^{\frac{1}{2}}$. The existing conditions for the defining integral (A.71) are given below:

$$(i)\qquad \alpha_i>0, |\arg z|<\frac{1}{2}\alpha_i\pi, \tag{A.73}$$

$$(ii)\qquad \alpha_i\ge 0, |\arg z|\le\frac{1}{2}\alpha_i\pi \text{ and } \Re(\beta+1)<0, \tag{A.74}$$

where

$$\alpha_i=\sum_{j=1}^{n}A_j-\sum_{j=n+1}^{p_i}A_{ji}+\sum_{j=1}^{m}B_j-\sum_{j=m+1}^{q_i}B_{ji},\ \ i=1,\ldots,r, \tag{A.75}$$

and

$$\beta=\sum_{j=1}^{m}b_j+\sum_{j=m+1}^{q_i}-\sum_{j=1}^{n}a_j-\sum_{j=n+1}^{p_i}a_{ji}+\frac{1}{2}(p_i-q_i), i=1,\ldots,r. \tag{A.76}$$

Note A.1. For $r=1$ in (A.72), the definition of the H-function (1.2) is recovered.

Note A.2. We note that integral operators involving I-function are defined and studied by Saxena et al. (1993). A basic analogue of the I-function is given by Saxena et al. (1995). Saigo–Maeda operators of the product of I-function and a general class of polynomials are discussed by Saxena et al. (2002).

Remark A.7. $\bar{I}$-function is further generalized by Südland et al. (1998) in a different notation with a modified definition of slightly general nature and call it Aleph functions. Aleph functions occur naturally in certain problems of fractional driftless Fokker–Planck equations. For further details in this regard, one can refer to the original paper Südland et al. (2001).

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Glossary of Symbols

$H^{m,n}_{p,q}[z]$	H-function	Definition 1.1	2
$L_{+\infty}, L_{-\infty}, L_{i\gamma\infty}$	contours for H-function	Definition 1.1	2
$(a)_k$	Pochhammer symbol	Example 1.2	7
$E_\alpha(z)$	Mittag-Leffler function	Definition 1.2	8
$E_{\alpha,\beta}(z)$	Mittag-Leffler function	Definition 1.3	8
$E^{\gamma}_{\alpha,\beta}(z)$	Mittag-Leffler function	Definition 1.4	9
$G^{m,n}_{p,q}[z]$	Meijer's G-function	Section 1.8	21
${}_pF_q(z)$	hypergeometric function	Definition 1.6	22
$E(.,\ldots,.;.,\ldots,.:z)$	MacRobert's function	Definition 1.7	22
$J^{\mu}_{\nu}(z)$	Bessel-Maitland function	Definition 1.8	22
$Z^{\nu}_{\rho}(z)$	Krätzel function	Definition 1.10	22
${}_p\psi_q(z)$	Wright function	Definition 1.12	23
${}_2R_1$	Dotsenko function	Definition 1.14	31
$M\{f(t):s\}, f^*(s)$	Mellin transform	Section 2.2	45
$M^{-1}\{f^*(s);x\}$	inverse Mellin transform	Section 2.2	45
$L\{f(t):s\}, (Lf)(s)$	Laplace transform	Section 2.2.6	48
$L^{-1}[F(s);t]$	inverse Laplace transform	Section 2.2.6	48
$R_\nu\{f(x);p\}$	K-transform	Section 2.2.11	53
$V\{f,k,m;s\}$	Varma transform	Section 2.2.13	55
$H_\nu\{f(x):\rho\}$	Hankel transform	Section 2.2.15	56
${}_aI^{\alpha}_{x}, {}_aD^{-\alpha}_{x}, I^{\alpha}_{a+}$	fractional integrals	Section 3.3.1	79
${}_xI^{\alpha}_{b}, {}_xD^{-\alpha}_{b}, I^{\alpha}_{b-}$	fractional integrals	Section 3.3.1	79
$({}_xW^{\alpha}_{\infty}, {}_xI^{\alpha}_{\infty}, I^{\alpha}_{-}$	Weyl integrals	Section 3.5	91
${}_xD^{\alpha}_{\infty}$	Weyl derivative	Definition 3.10	91
${}^{c}{}_aD_x{}^{\alpha}$	Caputo derivative	Section 3.6.3	95

$I(\alpha,\eta;f)$	Erdélyi-Kober operator	Section 3.8.1	98
$E_{0,x}^{\eta,\alpha}(f)$	Erdélyi-Kober operator	Section 3.8.1	98
$I_{\eta,\alpha}^{+}f$	Erdélyi-Kober operator	Section 3.8.1	98
$K_{x,\infty}^{\alpha,\zeta}, K_{x}^{\zeta,\alpha}$	Erdélyi-Kober operator	Section 3.8.1	98
$K_{\zeta,\alpha}^{-}$	Erdélyi-Kober operator	Section 3.8.1	98
$K(\alpha,\zeta,f)$	Erdélyi-Kober operator	Section 3.8.1	98
$I(\alpha,\beta,\gamma;m,\mu,\alpha;f)$	generalized Kober operator	Section 3.9	101
$J(.,.,.;.,.;f)$	generalized Kober operator	Section 3.9	101
$K[f(x)], K\begin{bmatrix}\alpha,\beta,\gamma : & \\ \delta,\rho,a : & f\end{bmatrix}$	Kober operators	Section 3.9	101
$I_{0+}^{\alpha,\beta,\gamma}, I_{-}^{\alpha,\beta,\gamma}$	Saigo integral operators	Section 3.10	103
$D_{0+}^{\alpha,\beta,\gamma}, D_{-}^{\alpha,\beta,\gamma}$	Saigo differential operators	Section 3.10	103
$I_{.,.,m}^{.;.}$	multiple Erdélyi-Kober operator	Section 3.11	113
$E(\cdot)$	expected value	Section 4.2	119
$B(\alpha,\beta)$	beta function	Section 4.2	119
$\mathrm{tr}(\cdot)$	trace of the matrix $(\cdot)$	Section 5.1	139
$\int_{A}^{B} f(X)\mathrm{d}X$	integral over matrices	Section 5.1	139
$\mathrm{d}x \wedge \mathrm{d}y$	wedge product of differentials	Section 5.1	139
$\Gamma_p(\alpha)$	real matrix-variate gamma	Section 5.1	139
J	Jacobian	Section 5.1	139
$B_p(\alpha,\beta)$	real matrix-variate beta	Section 5.4	146
$(\mathrm{d}X)$	matrix of differentials	Section 5.4	146
$M(r), P(r), T(r)$	mass, pressure, temperature	Section 6.2	159
$< . >$	expected value	Section 6.11.5	189
$H_{.,.;\dots}^{.,.;\dots}$	H-function of many variables	Appendix A.1	205
$F_{:;;}^{:;;}$	Kampé de Fériet function	Appendix A.2	207
F_1, F_2, F_3, F_4	Appell functions	Appendix A.3	211
F_A, F_B, F_C, F_D	Lauricella functions	Appendix A.4	213
$\bar{H}$	H-bar function	Appendix A.5	215
$I_{p_i,q_i}^{m,n}$	I-function	Appendix A.7	219

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