Fundamental Equations of Statics and Mechanics of Materials

Cartesian Vector

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

Magnitude

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Directions

$$\mathbf{u}_A = \frac{\mathbf{A}}{A} = \frac{A_x}{A}\mathbf{i} + \frac{A_y}{A}\mathbf{j} + \frac{A_z}{A}\mathbf{k}$$

$$= \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Dot Product

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$$

$$= A_xB_x + A_yB_y + A_zB_z$$

Cross Product

$$\mathbf{C} = \mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Cartesian Position Vector

$$\mathbf{r} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

Cartesian Force Vector

$$\mathbf{F} = F\mathbf{u} = F\left(\frac{\mathbf{r}}{r}\right)$$

Moment of a Force

$$M_O = Fd$$

$$\mathbf{M}_O = \mathbf{r}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

Moment of a Force About a Specified Axis

$$M_a = \mathbf{u}\cdot\mathbf{r}\times\mathbf{F} = \begin{vmatrix} u_x & u_y & u_y \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

Simplification of a Force and Couple System

$$\mathbf{F}_R = \Sigma\mathbf{F}$$

$$(\mathbf{M}_R)_O = \Sigma\mathbf{M}_c + \Sigma\mathbf{M}_O$$

Equilibrium

Particle

$$\Sigma F_x = 0,\ \Sigma F_y = 0,\ \Sigma F_z = 0$$

Rigid Body—Two Dimensions

$$\Sigma F_x = 0,\ \Sigma F_y = 0,\ \Sigma M_O = 0$$

Rigid Body—Three Dimensions

$$\Sigma F_x = 0,\ \Sigma F_y = 0,\ \Sigma F_z = 0$$

$$\Sigma M_{x'} = 0,\ \Sigma M_{y'} = 0,\ \Sigma M_{z'} = 0$$

Friction

Static (maximum) $F_s = \mu_s N$

Kinetic $F_k = \mu_k N$

Center of Gravity

Particles or Discrete Parts

$$\bar{r} = \frac{\Sigma\tilde{r}\,W}{\Sigma W}$$

Body

$$\bar{r} = \frac{\int\tilde{r}\,dW}{\int dW}$$

Area Moment of Inertia

$$I = \int r^2 dA$$

Parallel-Axis Theorem

$$I = \bar{I} + Ad^2$$

Radius of Gyration

$$k = \sqrt{\frac{I}{A}}$$

Axial Load

Normal Stress

$$\sigma = \frac{P}{A}$$

Displacement

$$\delta = \int_0^L \frac{P(x)dx}{A(x)E},\ \delta = \Sigma\frac{PL}{AE},\ \delta_T = \alpha\,\Delta TL$$

Torsion

Shear Stress in Circular Shaft

$$\tau = \frac{T\rho}{J}$$

where

$$J = \frac{\pi}{2}c^4 \text{ solid cross section}$$

$$J = \frac{\pi}{2}(c_o^4 - c_i^4) \text{ tubular cross section}$$

Power

$$P = T\omega = 2\pi fT$$

Angle of Twist

$$\phi = \int_0^L \frac{T(x)dx}{J(x)G} \qquad \phi = \Sigma\frac{TL}{JG}$$

Average Shear Stress in a Thin-Walled Tube

$$\tau_{\text{avg}} = \frac{T}{2tA_m}$$

Shear Flow

$$q = \tau_{\text{avg}}t = \frac{T}{2A_m}$$

Bending

Normal Stress

$$\sigma = \frac{My}{I}$$

Unsymmetric Bending

$$\sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}, \qquad \tan\alpha = \frac{I_z}{I_y}\tan\theta$$

Shear

Average Direct Shear Stress

$$\tau_{\text{avg}} = \frac{V}{A}$$

Transverse Shear Stress

$$\tau = \frac{VQ}{It}$$

Shear Flow

$$q = \tau t = \frac{VQ}{I}$$

Stress in Thin-Walled Pressure Vessel

Cylinder

$$\sigma_1 = \frac{pr}{t}, \qquad \sigma_2 = \frac{pr}{2t}$$

Sphere

$$\sigma_1 = \sigma_2 = \frac{pr}{2t}$$

Stress Transformation Equations

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Principal Stress

$$\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2_{xy}}$$

Maximum In-Plane Shear Stress

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)/2}{\tau_{xy}}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2_{xy}}$$

$$\sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2}$$

Absolute Maximum Shear Stress

$$\tau_{\substack{\text{abs}\\\max}} = \frac{\sigma_{\max} - \sigma_{\min}}{2}$$

$$\sigma_{\text{avg}} = \frac{\sigma_{\max} + \sigma_{\min}}{2}$$

Material Property Relations

Poisson's Ratio

$$\nu = -\frac{\epsilon_{\text{lat}}}{\epsilon_{\text{long}}}$$

Generalized Hooke's Law

$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)]$$

$$\epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}, \quad \gamma_{yz} = \frac{1}{G}\tau_{yz}, \quad \gamma_{zx} = \frac{1}{G}\tau_{zx}$$

where

$$G = \frac{E}{2(1+\nu)}$$

Relations Between w, V, M

$$\frac{dV}{dx} = w(x), \qquad \frac{dM}{dx} = V$$

Elastic Curve

$$\frac{1}{\rho} = \frac{M}{EI}$$

$$EI\frac{d^4v}{dx^4} = w(x)$$

$$EI\frac{d^3v}{dx^3} = V(x)$$

$$EI\frac{d^2v}{dx^2} = M(x)$$

Buckling

Critical Axial Load

$$P_{\text{cr}} = \frac{\pi^2 EI}{(KL)^2}$$

Critical Stress

$$\sigma_{\text{cr}} = \frac{\pi^2 E}{(KL/r)^2}, \quad r = \sqrt{I/A}$$

Secant Formula

$$\sigma_{\max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right]$$

STATICS AND MECHANICS OF MATERIALS

FOURTH EDITION

R. C. HIBBELER

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To the Student

With the hope that this work will stimulate
an interest in Engineering Mechanics and
Mechanics of Materials and provide
an acceptable guide to its understanding.

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PREFACE

This book represents a combined abridged version of two of the author's books, namely *Engineering Mechanics: Statics, Thirteenth Edition* and *Mechanics of Materials, Ninth Edition*. It provides a clear and thorough presentation of both the theory and application of the important fundamental topics of these subjects, that are often used in many engineering disciplines. The development emphasizes the importance of satisfying equilibrium, compatibility of deformation, and material behavior requirements. The hallmark of the book, however, remains the same as the author's unabridged versions, and that is, strong emphasis is placed on drawing a free-body diagram, and the importance of selecting an appropriate coordinate system and an associated sign convention whenever the equations of mechanics are applied. Throughout the book, many analysis and design applications are presented, which involve mechanical elements and structural members often encountered in engineering practice.

New to This Edition

- **New Problems.** There are approximately 65% new problems in this edition. These problems relate to applications in many different fields of engineering. Also, an increase in algebraic type problems has been added, so that a generalized solution can be obtained.
- **Content Revisions.** Each section of the text was carefully reviewed and, in some areas, the material has been redeveloped to better explain the concepts.
- **New & Revised Example Problems.** Throughout the book examples have been altered or enhanced in an attempt to help clarify concepts for students. Where appropriate a new example has been added in order to emphasize important concepts that were needed.
- **Additional Fundamental Problems.** These problem sets serve as extended example problems since their solutions are given in the back of the book. Additional problems have been added, in some areas of the book where they are needed.
- **Expanded Solutions.** Some of the fundamental problems now have more detailed solutions, including some artwork, for better clarification. Also, some of the more difficult problems have additional hints along with its answer when given in the back of the book.
- **New Photos.** The relevance of knowing the subject matter is reflected by the real-world applications depicted in any new or updated photos placed throughout the book. These photos generally are used to explain how the relevant principles apply to real-world situations and how materials behave under load.

Hallmark Features

Besides the new features mentioned above, other outstanding features that define the contents of the text include the following.

Organization and Approach. Each chapter is organized into well-defined sections that contain an explanation of specific topics, illustrative example problems, and a set of homework problems. The topics within each section are placed into subgroups defined by boldface titles. The purpose of this is to present a structured method for introducing each new definition or concept and to make the book convenient for later reference and review.

Chapter Contents. Each chapter begins with a photo demonstrating a broad-range application of the material within the chapter. A bulleted list of the chapter contents is provided to give a general overview of the material that will be covered.

Emphasis on Free-Body Diagrams. Drawing a free-body diagram is particularly important when solving problems, and for this reason this step is strongly emphasized throughout the book. In particular, within the statics coverage some sections are devoted to show how to draw free-body diagrams. Specific homework problems have also been added to develop this practice.

Procedures for Analysis. A general procedure for analyzing any mechanics problem is presented at the end of the first chapter. Then this procedure is customized to relate to specific types of problems that are covered throughout the book. This unique feature provides the student with a logical and orderly method to follow when applying the theory. The example problems are solved using this outlined method in order to clarify its numerical application. Realize, however, that once the relevant principles have been mastered and enough confidence and judgment have been obtained, the student can then develop his or her own procedures for solving problems.

Important Points. This feature provides a review or summary of the most important concepts in a section and highlights the most significant points that should be realized when applying the theory to solve problems.

Conceptual Understanding. Through the use of photographs placed throughout the book, the theory is applied in a simplified way in order to illustrate some of its more important conceptual features and instill the physical meaning of many of the terms used in the equations. These simplified applications increase interest in the subject matter and better prepare the student to understand the examples and solve problems.

Fundamental Problems. These problem sets are located just after each group of example problems. They offer students simple applications of the concepts covered in each section and, therefore, provide them with the chance to develop their problem-solving skills before attempting to solve any of the standard problems that follow. The fundamental problems may be considered as extended examples, since the key equations and answers are all listed in the back of the book. Additionally, when assigned, these problems offer students an excellent means of preparing for exams, and they can be used at a later time as a review when studying for the Fundamentals of Engineering Exam.

Conceptual Problems. Throughout the text, usually at the end of each chapter, there is a set of problems that involve conceptual situations related to the application of the principles contained in the chapter. These analysis and design problems are intended to engage students in thinking through a real-life situation as depicted in a photo. They can be assigned after the students have developed some expertise in the subject matter and they work well either for individual or team projects.

Homework Problems. Apart from the Fundamental and Conceptual type problems mentioned previously, other types of problems contained in the book include the following:

- **General Analysis and Design Problems.** The majority of problems in the book depict realistic situations encountered in engineering practice. Some of these problems come from actual products used in industry. It is hoped that this realism will both stimulate the student's interest in engineering mechanics and provide a means for developing the skill to reduce any such problem from its physical description to a model or symbolic representation to which the principles of mechanics may be applied.

  Throughout the book, there is an approximate balance of problems using either SI of FPS units. Furthermore, in any set, an attempt has been made to arrange the problems in order of increasing difficulty, except for the end of chapter review problems, which are presented in random order. Problems that are simply indicated by a problem number have an answer given in the back of the book. However, an asterisk (*) before every fourth problem number indicates a problem without an answer.

- **Computer Problems.** An effort has been made to include some problems that may be solved using a numerical procedure executed on either a desktop computer or a programmable pocket calculator. The intent here is to broaden the student's capacity for using other forms of mathematical analysis without sacrificing the time needed to focus on the application of the principles of mechanics.

Accuracy. In addition to the author, the text and problem solutions have been thoroughly checked for accuracy by four other parties: Scott Hendricks, Virginia Polytechnic Institute and State University; Karim Nohra, University of South Florida; Kurt Norlin, Laurel Tech Integrated Publishing Services; and finally Kai Beng Yap, a practicing engineer.

Contents

The book is divided into two parts, and the material is covered in the traditional manner.

Statics. The subject of statics is presented in 6 chapters. The text begins in Chapter 1 with an introduction to mechanics and a discussion of units. The notion of a vector and the properties of a concurrent force system are introduced in Chapter 2. Chapter 3 contains a general discussion of concentrated force systems and the methods used to simplify them. The principles of rigid-body equilibrium are developed in Chapter 4 and then applied to specific problems involving the equilibrium of trusses, frames, and machines in Chapter 5. Finally, topics related to the center of gravity, centroid, and moment of inertia are treated in Chapter 6.

Mechanics of Materials. This portion of the text is covered in 10 chapters. Chapter 7 begins with a formal definition of both normal and shear stress, and a discussion of normal stress in axially loaded members and average shear stress caused by direct shear; finally, normal and shear strain are defined. In Chapter 8 a discussion of some of the important mechanical properties of materials is given. Separate treatments of axial load, torsion, bending, and transverse shear are presented in Chapters 9, 10, 11, and 12, respectively. Chapter 13 provides a partial review of the material covered in the previous chapters, in which the state of stress resulting from combined loadings is discussed. In Chapter 14 the concepts for transforming stress and strain are presented. Chapter 15 provides a means for a further summary and review of previous material by covering design of beams based on allowable stress. In Chapter 16 various methods for computing deflections of beams are presented, including the method for finding the reactions on these members if they are statically indeterminate. Lastly, Chapter 17 provides a discussion of column buckling.

Sections of the book that contain more advanced material are indicated by a star (*). Time permitting, some of these topics may be included in the course. Furthermore, this material provides a suitable reference for basic principles when it is covered in other courses, and it can be used as a basis for assigning special projects.

Alternative Method for Coverage of Mechanics of Materials. It is possible to cover many of the topics in the text in several different sequences. For example, some instructors prefer to cover stress and strain transformations *first,* before discussing specific

applications of axial load, torsion, bending, and shear. One possible method for doing this would be to first to cover stress and strain and its transformations, Chapter 7 and Chapter 14 then Chapters 8 through 13 can be covered with no loss in continuity.

Problems. Numerous problems in the book depict realistic situations encountered in engineering practice. It is hoped that this realism will both stimulate the student's interest in the subject and provide a means for developing the skill to reduce any such problem from its physical description to a model or symbolic representation to which the principles may be applied.

Throughout the text there is an approximate balance of problems using either SI of FPS units. Furthermore, in any set, an attempt has been made to arrange the problems in order of increasing difficulty. The answers to all but every fourth problem are listed in the back of the book. To alert the user to a problem without a reported answer, an asterisk (*) is placed before the problem number. Answers are reported to three significant figures, even though the data for material properties may be known with less accuracy. Although this might appear to be poor practice, it is done simply to be consistent and to allow the student a better chance to validate his or her solution. All the problems and their solutions have been independently checked four times for accuracy.

Acknowledgments

Over the years, this text has been shaped by the suggestions and comments of many of my colleagues in the teaching profession. Their encouragement and willingness to provide constructive criticism are very much appreciated and it is hoped that they will accept this anonymous recognition. A note of thanks is also given to the reviewers of both my *Statics* and *Mechanics of Materials* texts. Their comments have guided the improvement of this book as well.

During the production process I am thankful for the assistance of Rose Kernan, my production editor for many years, and to my wife, Conny, and daughter, Mary Ann, for their help in proofreading and typing, that was needed to prepare the manuscript for publication.

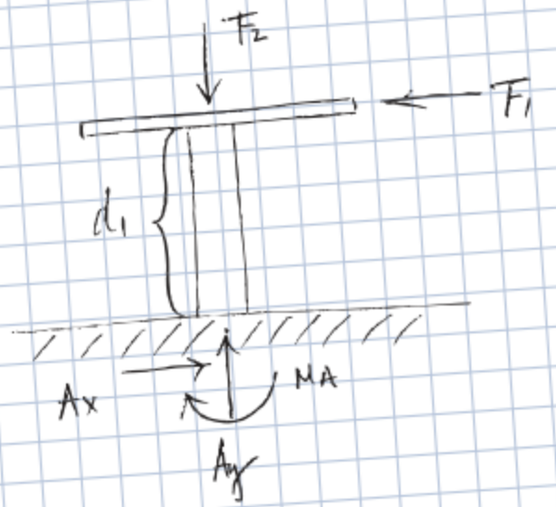
I would also like to thank all my students who have used the previous edition and have made comments to improve its contents; including those in the teaching profession who have taken the time to e-mail me their comments.

I would greatly appreciate hearing from you if at any time you have any comments or suggestions regarding the contents of this edition.

Russell Charles Hibbeler
hibbeler@bellsouth.net

your work...

PART A



Given: $F_1 = 55.0\,lb$, $F_2 = 170\,lb$, $M_A = 160\,lb \cdot ft$

$$\sum M_A = 0 \Rightarrow (F_1 + F_2) \times d_1 - M_A = 0$$

$$(F_1 + F_2) \times d_1 = M_A$$

$$d_1 = \frac{M_A}{F_1 + F_2}$$

$$= \frac{160\ lb \cdot ft}{55.0\ lb + 170\ lb}$$

$$\boxed{d_1 = 0.711\ ft}$$

your answer specific feedback

Try Again; 5 attempts remaining

Feedback Close

Both forces do not contribute to the moment about point *A*. The magnitude of the moment about *A* is equal to the force multiplied by the perpendicular distance between point *A* and the line of action of the force. What is the perpendicular distance between each force's line of action and point *A*?

Resources for Instructors

• **MasteringEngineering.** This online Tutorial Homework program allows you to integrate dynamic homework with automatic grading and adaptive tutoring. MasteringEngineering allows you to easily track the performance of your entire class on an assignment-by-assignment basis, or the detailed work of an individual student.

• **Instructor's Solutions Manual.** This supplement provides complete solutions supported by problem statements and problem figures. The thirteenth edition manual was revised to improve readability and was triple accuracy checked. The Instructor's Solutions Manual is available on Pearson Higher Education website: www.pearsonhighered.com.

• **Instructor's Resource.** Visual resources to accompany the text are located on the Pearson Higher Education website: www.pearsonhighered.com. If you are in need of a login and password for this site, please contact your local Pearson representative. Visual resources include all art from the text, available in PowerPoint slide and JPEG format.

• **Video Solutions.** Developed by Professor Edward Berger, University of Virginia, video solutions are located on the Companion Website for the text and offer step-by-step solution walkthroughs of representative homework problems from each section of the text. Make efficient use of class time and office hours by showing students the complete and concise problem-solving approaches that they can access any time and view at their own pace. The videos are designed to be a flexible resource to be used however each instructor and student prefers. A valuable tutorial resource, the videos are also helpful for student self-evaluation as students can pause the videos to check their understanding and work alongside the video. Access the videos at www.pearsonhighered.com/hibbeler/ and follow the links for the *Engineering Mechanics: Statics*, Thirteenth Edition text.

Resources for Students

• **MasteringEngineering.** Tutorial homework problems emulate the instructor's office-hour environment, guiding students through engineering concepts with self-paced individualized coaching. These in-depth tutorial homework problems are designed to coach students with feedback specific to their errors and optional hints that break problems down into simpler steps.

• **Statics Study Pack.** This supplement contains chapter-by-chapter study materials, a Free-Body Diagram Workbook and access to the Companion Website where additional tutorial resources are located.

• **Companion Website.** The Companion Website, located at www.pearsonhighered.com/hibbeler/, includes opportunities for practice and review including:

  • **Video Solutions**—Complete, step-by-step solution walkthroughs of representative homework problems from each section. Videos offer fully worked solutions that show every step of representative homework problems—this helps students make vital connections between concepts.

• **Statics Practice Problems Workbook.** This workbook contains additional worked problems. The problems are partially solved and are designed to help guide students through difficult topics.

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CONTENTS

1 General Principles 3

Chapter Objectives 3

1.1 Mechanics 3

1.2 Fundamental Concepts 4

1.3 Units of Measurement 7

1.4 The International System of Units 9

1.5 Numerical Calculations 10

1.6 General Procedure for Analysis 12

2 Force Vectors 17

Chapter Objectives 17

2.1 Scalars and Vectors 17

2.2 Vector Operations 18

2.3 Vector Addition of Forces 20

2.4 Addition of a System of Coplanar Forces 30

2.5 Cartesian Vectors 38

2.6 Addition of Cartesian Vectors 41

2.7 Position Vectors 50

2.8 Force Vector Directed Along a Line 53

2.9 Dot Product 60

3 Force System Resultants 75

Chapter Objectives 75

3.1 Moment of a Force—Scalar Formulation 75

3.2 Cross Product 79

3.3 Moment of a Force—Vector Formulation 82

3.4 Principle of Moments 86

3.5 Moment of a Force about a Specified Axis 96

3.6 Moment of a Couple 103

3.7 Simplification of a Force and Couple System 112

3.8 Further Simplification of a Force and Couple System 122

4 Equilibrium of a Rigid Body 139

Chapter Objectives 139

4.1 Conditions for Rigid-Body Equilibrium 139

4.2 Free-Body Diagrams 141

4.3 Equations of Equilibrium 151

4.4 Two- and Three-Force Members 157

4.5 Free-Body Diagrams 167

4.6 Equations of Equilibrium 172

4.7 Characteristics of Dry Friction 180

4.8 Problems Involving Dry Friction 184

4.9 Frictional Forces on Flat Belts 197

4.10 Frictional Forces on Screws 200

5 Structural Analysis 215

Chapter Objectives 215

5.1 Simple Trusses 215

5.2 The Method of Joints 218

5.3 Zero-Force Members 224

5.4 The Method of Sections 231

5.5 Frames and Machines 240

6 Center of Gravity, Centroid, and Moment of Inertia 261

Chapter Objectives 261

6.1 Center of Gravity, Center of Mass, and the Centroid of a Body 261

6.2 Composite Bodies 273

6.3 Resultant of a Distributed Loading 281

6.4 Moments of Inertia for Areas 290

6.5 Parallel-Axis Theorem for an Area 291

6.6 Moments of Inertia for Composite Areas 298

7 Stress and Strain 309

Chapter Objectives 309

7.1 Introduction 309

7.2 Internal Resultant Loadings 310

7.3 Stress 322

7.4 Average Normal Stress in an Axially Loaded Bar 324

7.5 Average Shear Stress 331

7.6 Allowable Stress 342

7.7 Design of Simple Connections 343

7.8 Deformation 355

7.9 Strain 356

8 Mechanical Properties of Materials 373

Chapter Objectives 373

8.1 The Tension and Compression Test 373

8.2 The Stress–Strain Diagram 375

8.3 Stress–Strain Behavior of Ductile and Brittle Materials 379

8.4 Hooke's Law 382

8.5 Strain Energy 384

8.6 Poisson's Ratio 392

8.7 The Shear Stress–Strain Diagram 394

9 Axial Load 405

Chapter Objectives 405

9.1 Saint-Venant's Principle 405

9.2 Elastic Deformation of an Axially Loaded Member 408

9.3 Principle of Superposition 421

9.4 Statically Indeterminate Axially Loaded Member 422

9.5 The Force Method of Analysis for Axially Loaded Members 428

9.6 Thermal Stress 434

9.7 Stress Concentrations 440

10 Torsion 451

Chapter Objectives 451

10.1 Torsional Deformation of a Circular Shaft 451

10.2 The Torsion Formula 454

10.3 Power Transmission 461

10.4 Angle of Twist 468

10.5 Statically Indeterminate Torque-Loaded Members 481

10.6 Solid Noncircular Shafts 488

10.7 Stress Concentration 492

11 Bending 501

Chapter Objectives 501

11.1 Shear and Moment Diagrams 501

11.2 Graphical Method for Constructing Shear and Moment Diagrams 508

11.3 Bending Deformation of a Straight Member 525

11.4 The Flexure Formula 529

11.5 Unsymmetric Bending 542

11.6 Stress Concentrations 550

12 Transverse Shear 559

Chapter Objectives 559

12.1 Shear in Straight Members 559

12.2 The Shear Formula 561

12.3 Shear Flow in Built-Up Members 578

13 Combined Loadings 591

Chapter Objectives 591

13.1 Thin-Walled Pressure Vessels 591

13.2 State of Stress Caused by Combined Loadings 598

14 Stress and Strain Transformation 619

Chapter Objectives 619

14.1 Plane-Stress Transformation 619

14.2 General Equations of Plane-Stress Transformation 624

14.3 Principal Stresses and Maximum In-Plane Shear Stress 627

14.4 Mohr's Circle—Plane Stress 639

14.5 Absolute Maximum Shear Stress 650

14.6 Plane Strain 657

14.7 General Equations of Plane-Strain Transformation 658

14.8 Mohr's Circle—Plane Strain 666

14.9 Strain Rosettes 674

14.10 Material-Property Relationships 676

15 Design of Beams and Shafts 693

Chapter Objectives 693

15.1 Basis for Beam Design 693

15.2 Prismatic Beam Design 696

15.3 Fully Stressed Beams 710

16 Deflection of Beams and Shafts 717

Chapter Objectives 717

16.1 The Elastic Curve 717

16.2 Slope and Displacement by Integration 721

16.3 Discontinuity Functions 735

16.4 Method of Superposition 745

16.5 Statically Indeterminate Beams and Shafts–Method of Superposition 752

17 Buckling of Columns 769

Chapter Objectives 769

17.1 Critical Load 769

17.2 Ideal Column with Pin Supports 772

17.3 Columns Having Various Types of Supports 778

17.4 The Secant Formula 788

17.5 Inelastic Buckling 794

Appendix

A. Mathematical Review and Expressions 804

B. Geometric Properties of An Area and Volume 808

C. Geometric Properties of Wide-Flange Sections 810

D. Slopes and Deflections of Beams 814

Fundamental Problems Partial Solutions and Answers 816

Answers to Selected Problems 845

Index 868

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STATICS AND MECHANICS OF MATERIALS

Chapter 1

Large cranes such as this one are required to lift extremely large loads. Their design is based on the basic principles of statics and dynamics, which form the subject matter of engineering mechanics.

General Principles

CHAPTER OBJECTIVES

- To provide an introduction to the basic quantities and idealizations of mechanics.
- To give a statement of Newton's Laws of Motion and Gravitation.
- To review the principles for applying the SI system of units.
- To examine the standard procedures for performing numerical calculations.
- To present a general guide for solving problems.

1.1 Mechanics

Mechanics can be defined as that branch of the physical sciences concerned with the state of rest or motion of bodies that are subjected to the action of forces. In this book we will study two very important branches of mechanics, namely, statics and mechanics of materials. These subjects form a suitable basis for the design and analysis of many types of structural, mechanical, or electrical devices encountered in engineering.

Statics deals with the equilibrium of bodies, that is, it is used to determine the forces acting either external to the body or within it necessary to keep the body either at rest or moving with a constant velocity. *Mechanics of materials* studies the relationships between the external loads and the intensity of internal forces acting within the body. This subject is also concerned with computing the deformations of the body, and it provides a study of the body's stability when the body is subjected to external forces. In this book we will first study the principles of statics since for the design and analysis of any structural or mechanical element, it is *first* necessary to determine the forces acting both on and within its various members. Once these internal forces are determined, the size of the members, their deflection, and their stability can then be determined using the fundamentals of mechanics of materials, which will be covered later.

1

Historical Development. The subject of statics developed very early in history because its principles can be formulated simply from measurements of geometry and force. For example, the writings of Archimedes (287–212 B.C.) deal with the principle of the lever. Studies of the pulley and inclined plane are also recorded in ancient writings—at times when the requirements for engineering were limited primarily to building construction.

The origin of mechanics of materials dates back to the beginning of the seventeenth century, when Galileo performed experiments to study the effects of loads on rods and beams made of various materials. However, at the beginning of the eighteenth century, experimental methods for testing materials were vastly improved, and at that time many experimental and theoretical studies in this subject were undertaken primarily in France, by such notables as Saint-Venant, Poisson, Lamé, and Navier.

Over the years, after many of the fundamental problems of mechanics of materials had been solved, it became necessary to use advanced mathematical and computer techniques to solve more complex problems. As a result, this subject expanded into other areas of mechanics, such as the *theory of elasticity* and the *theory of plasticity*. Research in these fields is ongoing, in order to meet the demands for solving more advanced problems in engineering.

1.2 Fundamental Concepts

Before we begin our study of engineering mechanics, it is important to understand the meaning of certain fundamental concepts and principles.

Basic Quantities. The following four quantities are used throughout mechanics.

Length. *Length* is used to locate the position of a point in space and thereby describe the size of a physical system. Once a standard unit of length is defined, one can then use it to define distances and geometric properties of a body as multiples of this unit.

Time. *Time* is conceived as a succession of events. Although the principles of statics are time independent, this quantity plays an important role in the study of dynamics.

Mass. *Mass* is a measure of a quantity of matter that is used to compare the action of one body with that of another. This property manifests itself as a gravitational attraction between two bodies and provides a measure of the resistance of matter to a change in velocity.

Force. In general, *force* is considered as a "push" or "pull" exerted by one body on another. This interaction can occur when there is direct contact between the bodies, such as a person pushing on a wall, or it can occur through a distance when the bodies are physically separated.

Examples of the latter type include gravitational, electrical, and magnetic forces. In any case, a force is completely characterized by its magnitude, direction, and point of application.

Idealizations. Models or idealizations are used in mechanics in order to simplify application of the theory. Here we will consider three important idealizations.

Particle. A *particle* has a mass, but a size that can be neglected. For example, the size of the earth is insignificant compared to the size of its orbit, and therefore the earth can be modeled as a particle when studying its orbital motion. When a body is idealized as a particle, the principles of mechanics reduce to a rather simplified form since the geometry of the body *will not be involved* in the analysis of the problem.

Three forces act on the ring. Since these forces all meet at a point, then for any force analysis, we can assume the ring to be represented as a particle.

Rigid Body. A *rigid body* can be considered as a combination of a large number of particles in which all the particles remain at a fixed distance from one another, both before and after applying a load. This model is important because the material properties of any body that is assumed to be rigid will not have to be considered when studying the effects of forces acting on the body. In most cases the actual deformations occurring in structures, machines, mechanisms, and the like are relatively small, and the rigid-body assumption is suitable for analysis.

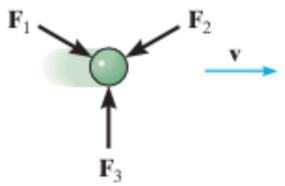
Concentrated Force. A *concentrated force* represents the effect of a loading which is assumed to act at a point on a body. We can represent a load by a concentrated force, provided the area over which the load is applied is very small compared to the overall size of the body. An example would be the contact force between a wheel and the ground.

Steel is a common engineering material that does not deform very much under load. Therefore, we can consider this railroad wheel to be a rigid body acted upon by the concentrated force of the rail.

1

Newton's Three Laws of Motion. The entire subject of rigid body mechanics is formulated on the basis of Newton's three laws of motion, the validity of which is based on experimental observation. These laws apply to the motion of a particle as measured from a *nonaccelerating* reference frame. They may be briefly stated as follows.

First Law. A particle originally at rest, or moving in a straight line with constant velocity, tends to remain in this state provided the particle is *not* subjected to an unbalanced force, Fig. 1–1*a*.



Equilibrium

(a)

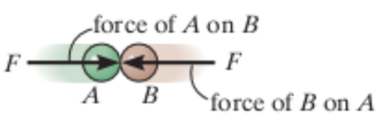
Second Law. A particle acted upon by an *unbalanced force* **F** experiences an acceleration **a** that has the same direction as the force and a magnitude that is directly proportional to the force, Fig. 1–1*b*.* If **F** is applied to a particle of mass m, this law may be expressed mathematically as

$$\mathbf{F} = m\mathbf{a} \tag{1–1}$$

Accelerated motion

(b)

Third Law. The mutual forces of action and reaction between two particles are equal, opposite, and collinear, Fig. 1–1*c*.



Action – reaction

(c)

Fig. 1–1

*Stated another way, the unbalanced force acting on the particle is proportional to the time rate of change of the particle's linear momentum.

Newton's Law of Gravitational Attraction. Shortly after formulating his three laws of motion, Newton postulated a law governing the gravitational attraction between any two particles. Stated mathematically,

$$F = G\frac{m_1 m_2}{r^2} \tag{1–2}$$

where

F = force of gravitation between the two particles

G = universal constant of gravitation; according to experimental evidence, $G = 66.73(10^{-12})\ \mathrm{m^3/(kg \cdot s^2)}$

m_1, m_2 = mass of each of the two particles

r = distance between the two particles

Weight. According to Eq. 1–2, any two particles or bodies have a mutual attractive (gravitational) force acting between them. In the case of a particle located at or near the surface of the earth, however, the only gravitational force having any sizable magnitude is that between the earth and the particle. Consequently, this force, termed the *weight*, will be the only gravitational force considered in our study of mechanics.

From Eq. 1–2, we can develop an approximate expression for finding the weight W of a particle having a mass $m_1 = m$. If we assume the earth to be a nonrotating sphere of constant density and having a mass $m_2 = M_e$, then if r is the distance between the earth's center and the particle, we have

$$W = G\frac{mM_e}{r^2}$$

Letting $g = GM_e/r^2$ yields

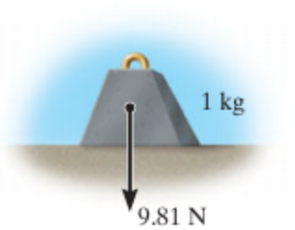
$$\boxed{W = mg} \tag{1–3}$$

By comparison with $\mathbf{F} = m\mathbf{a}$, we can see that g is the acceleration due to gravity. Since it depends on r, then the weight of a body is *not* an absolute quantity. Instead, its magnitude is determined from where the measurement was made. For most engineering calculations, however, g is determined at sea level and at a latitude of 45°, which is considered the "standard location."

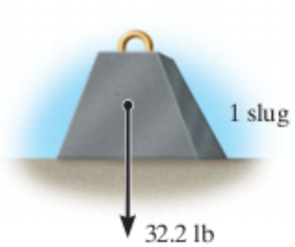
The astronaut's weight is diminished, since she is far removed from the gravitational field of the earth.

1.3 Units of Measurement

The four basic quantities—length, time, mass, and force—are not all independent from one another; in fact, they are *related* by Newton's second law of motion, $\mathbf{F} = m\mathbf{a}$. Because of this, the *units* used to measure these quantities cannot *all* be selected arbitrarily. The equality $\mathbf{F} = m\mathbf{a}$ is maintained only if three of the four units, called *base units*, are *defined* and the fourth unit is then *derived* from the equation.



(a)



(b)

Fig. 1–2

SI Units. The International System of units, abbreviated SI after the French "Système International d'Unités," is a modern version of the metric system which has received worldwide recognition. As shown in Table 1–1, the SI system defines length in meters (m), time in seconds (s), and mass in kilograms (kg). The unit of force, called a newton (N), is *derived* from $\mathbf{F} = m\mathbf{a}$. Thus, 1 newton is equal to a force required to give 1 kilogram of mass an acceleration of 1 m/s^2 ($N = kg \cdot m/s^2$).

If the weight of a body located at the "standard location" is to be determined in newtons, then Eq. 1–3 must be applied. Here measurements give $g = 9.806\,65\ m/s^2$; however, for calculations, the value $g = 9.81\ m/s^2$ will be used. Thus,

$$W = mg \quad (g = 9.81\ m/s^2) \tag{1–4}$$

Therefore, a body of mass 1 kg has a weight of 9.81 N, a 2-kg body weighs 19.62 N, and so on, Fig. 1–2*a*.

U.S. Customary. In the U.S. Customary system of units (FPS) length is measured in feet (ft), time in seconds (s), and force in pounds (lb), Table 1–1. The unit of mass, called a *slug*, is *derived* from $\mathbf{F} = m\mathbf{a}$. Hence, 1 slug is equal to the amount of matter accelerated at 1 ft/s^2 when acted upon by a force of 1 lb ($slug = lb \cdot s^2/ft$).

Therefore, if the measurements are made at the "standard location," where $g = 32.2\ ft/s^2$, then from Eq. 1–3,

$$m = \frac{W}{g} \quad (g = 32.2\ ft/s^2) \tag{1–5}$$

And so a body weighing 32.2 lb has a mass of 1 slug, a 64.4-lb body has a mass of 2 slugs, and so on, Fig. 1–2*b*.

TABLE 1–1 Systems of Units

Name	Length	Time	Mass	Force
International System of Units SI	meter	second	kilogram	newton*
	m	s	kg	$N \left(\frac{kg \cdot m}{s^2}\right)$
U.S. Customary FPS	foot	second	slug*	pound
	ft	s	$\left(\frac{lb \cdot s^2}{ft}\right)$	lb

*Derived unit.

Conversion of Units. Table 1–2 provides a set of direct conversion factors between FPS and SI units for the basic quantities. Also, in the FPS system, recall that 1 ft = 12 in. (inches), 5280 ft = 1 mi (mile), 1000 lb = 1 kip (kilo-pound), and 2000 lb = 1 ton.

TABLE 1–2 Conversion Factors

Quantity	Unit of Measurement (FPS)	Equals	Unit of Measurement (SI)
Force	lb		4.448 N
Mass	slug		14.59 kg
Length	ft		0.3048 m

1.4 The International System of Units

The SI system of units is used extensively in this book since it is intended to become the worldwide standard for measurement. Therefore, we will now present some of the rules for its use and some of its terminology relevant to engineering mechanics.

Prefixes. When a numerical quantity is either very large or very small, the units used to define its size may be modified by using a prefix. Some of the prefixes used in the SI system are shown in Table 1–3. Each represents a multiple or submultiple of a unit which, if applied successively, moves the decimal point of a numerical quantity to every third place.* For example, 4 000 000 N = 4 000 kN (kilo-newton) = 4 MN (mega-newton), or 0.005 m = 5 mm (milli-meter). Notice that the SI system does not include the multiple deca (10) or the submultiple centi (0.01), which form part of the metric system. Except for some volume and area measurements, the use of these prefixes is to be avoided in science and engineering.

TABLE 1–3 Prefixes

	Exponential Form	Prefix	SI Symbol
Multiple			
1 000 000 000	10^9	giga	G
1 000 000	10^6	mega	M
1 000	10^3	kilo	k
Submultiple			
0.001	10^{-3}	milli	m
0.000 001	10^{-6}	micro	μ
0.000 000 001	10^{-9}	nano	n

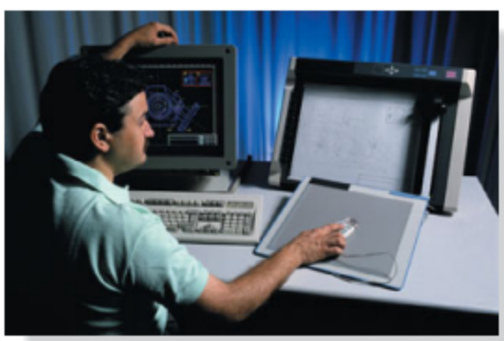
*The kilogram is the only base unit that is defined with a prefix.

Rules for Use. Here are a few of the important rules that describe the proper use of the various SI symbols:

- Quantities defined by several units which are multiples of one another are separated by a *dot* to avoid confusion with prefix notation, as indicated by $\mathrm{N} = \mathrm{kg \cdot m/s^2} = \mathrm{kg \cdot m \cdot s^{-2}}$. Also, $\mathrm{m \cdot s}$ (meter-second), whereas ms (milli-second).
- The exponential power on a unit having a prefix refers to both the unit *and* its prefix. For example, $\mu\mathrm{N}^2 = (\mu\mathrm{N})^2 = \mu\mathrm{N} \cdot \mu\mathrm{N}$. Likewise, mm^2 represents $(\mathrm{mm})^2 = \mathrm{mm \cdot mm}$.
- In general, avoid the use of a prefix in the denominator of composite units, with the exception of the base unit the kilogram. For example, do not write N/mm, but rather kN/m; also, m/mg should be written as Mm/kg.
- When performing calculations, represent the numbers in terms of their *base or derived units* by converting all prefixes to powers of 10. The final result should then be expressed using a *single prefix*. Also, after calculation, it is best to keep numerical values between 0.1 and 1000; otherwise, a suitable prefix should be chosen. For example,

$$\begin{aligned}(50\ \mathrm{kN})(60\ \mathrm{nm}) &= [50(10^3)\ \mathrm{N}][60(10^{-9})\ \mathrm{m}] \\ &= 3000(10^{-6})\ \mathrm{N \cdot m} = 3(10^{-3})\ \mathrm{N \cdot m} = 3\ \mathrm{mN \cdot m}\end{aligned}$$

1.5 Numerical Calculations



Computers are often used in engineering for advanced design and analysis.

Numerical work in engineering practice is most often performed by using handheld calculators and computers. It is important, however, that the answers to any problem be reported with justifiable accuracy using appropriate significant figures. In this section we will discuss these topics together with some other important aspects involved in all engineering calculations.

Dimensional Homogeneity. The terms of any equation used to describe a physical process must be *dimensionally homogeneous;* that is, each term must be expressed in the same units. Provided this is the case, all the terms of an equation can then be combined if numerical values are substituted for the variables. Consider, for example, the equation $s = vt + \frac{1}{2}at^2$, where, in SI units, s is the position in meters, m, t is time in seconds, s, v is velocity in m/s, and a is acceleration in $\mathrm{m/s^2}$. Regardless of how this equation is evaluated, it maintains its dimensional homogeneity. In the form stated, each of the three terms is expressed in meters [m, $(\mathrm{m/s})\,\mathrm{s}$, $(\mathrm{m/s^2})/\mathrm{s^2}$,] or solving for a, $a = 2s/t^2 - 2v/t$, the terms are each expressed in units of $\mathrm{m/s^2}$ [$\mathrm{m/s^2}$, $\mathrm{m/s^2}$, $(\mathrm{m/s})/\mathrm{s}$].

Keep in mind that problems in mechanics always involve the solution of dimensionally homogeneous equations, and so this fact can then be used as a partial check for algebraic manipulations of an equation.

Significant Figures. The number of significant figures contained in any number determines the accuracy of the number. For instance, the number 4981 contains four significant figures. However, if zeros occur at the end of a whole number, it may be unclear as to how many significant figures the number represents. For example, 23 400 might have three (234), four (2340), or five (23 400) significant figures. To avoid these ambiguities, we will use *engineering notation* to report a result. This requires that numbers be rounded off to the appropriate number of significant digits and then expressed in multiples of (10^3), such as (10^3), (10^6), or (10^{-9}). For instance, if 23 400 has five significant figures, it is written as $23.400(10^3)$, but if it has only three significant figures, it is written as $23.4(10^3)$.

If zeros occur at the beginning of a number that is less than one, then the zeros are not significant. For example, 0.008 21 has three significant figures. Using engineering notation, this number is expressed as $8.21(10^{-3})$. Likewise, 0.000 582 can be expressed as $0.582(10^{-3})$ or $582(10^{-6})$.

Rounding Off Numbers. Rounding off a number is necessary so that the accuracy of the result will be the same as that of the problem data. As a general rule, any numerical figure ending in five or greater is rounded up and a number less than five is rounded down. The rules for rounding off numbers are best illustrated by examples. Suppose the number 3.5587 is to be rounded off to *three* significant figures. Because the fourth digit (8) is *greater than* 5, the third number is rounded up to 3.56. Likewise 0.5896 becomes 0.590 and 9.3866 becomes 9.39. If we round off 1.341 to three significant figures, because the fourth digit (1) is *less than* 5, then we get 1.34. Likewise 0.3762 becomes 0.376 and 9.871 becomes 9.87. There is a special case for any number that ends in a 5. As a general rule, if the digit preceding the 5 is an *even number*, then this digit is *not* rounded up. If the digit preceding the 5 is an *odd number*, then it is rounded up. For example, 75.25 rounded off to three significant digits becomes 75.2, 0.1275 becomes 0.128, and 0.2555 becomes 0.256.

Calculations. When a sequence of calculations is performed, it is best to store the intermediate results in the calculator. In other words, do not round off calculations until expressing the final result. This procedure maintains precision throughout the series of steps to the final solution. In this text we will generally round off the answers to three significant figures since most of the data in engineering mechanics, such as geometry and loads, may be reliably measured to this accuracy.

1

When solving problems, do the work as neatly as possible. Being neat will stimulate clear and orderly thinking, and vice versa.

1.6 General Procedure for Analysis

Attending a lecture, reading this book, and studying the example problems helps, but **the most effective way of learning the principles of engineering mechanics is to *solve problems***. To be successful at this, it is important to always present the work in a *logical* and *orderly manner*, as suggested by the following sequence of steps:

- Read the problem carefully and try to correlate the actual physical situation with the theory studied.
- Tabulate the problem data and *draw to a large scale* any necessary diagrams.
- Apply the relevant principles, generally in mathematical form. When writing any equations, be sure they are dimensionally homogeneous.
- Solve the necessary equations, and report the answer with no more than three significant figures.
- Study the answer with technical judgment and common sense to determine whether or not it seems reasonable.

Important Points

- Statics is the study of bodies that are at rest or move with constant velocity.
- A particle has a mass but a size that can be neglected.
- A rigid body does not deform under load.
- Concentrated forces are assumed to act at a point on a body.
- Newton's three laws of motion should be memorized.
- Mass is measure of a quantity of matter that does not change from one location to another.
- Weight refers to the gravitational attraction of the earth on a body or quantity of mass. Its magnitude depends upon the elevation at which the mass is located.
- In the SI system the unit of force, the newton, is a derived unit. The meter, second, and kilogram are base units.
- Prefixes G, M, k, m, μ, and n are used to represent large and small numerical quantities. Their exponential size should be known, along with the rules for using the SI units.
- Perform numerical calculations with several significant figures, and then report the final answer to three significant figures.
- Algebraic manipulations of an equation can be checked in part by verifying that the equation remains dimensionally homogeneous.
- Know the rules for rounding off numbers.

EXAMPLE 1.1

Convert 2 km/h to m/s How many ft/s is this?

SOLUTION

Since 1 km = 1000 m and 1 h = 3600 s, the factors of conversion are arranged in the following order, so that a cancellation of the units can be applied:

$$2\ \text{km/h} = \frac{2\ \text{km}}{1\ \text{h}}\left(\frac{1000\ \text{m}}{1\ \text{km}}\right)\left(\frac{1\ \text{h}}{3600\ \text{s}}\right)$$

$$= \frac{2000\ \text{m}}{3600\ \text{s}} = 0.556\ \text{m/s} \qquad \textit{Ans.}$$

From Table 1–2, 1 ft = 0.3048 m. Thus,

$$0.556\ \text{m/s} = \left(\frac{0.556\ \text{m}}{\text{s}}\right)\left(\frac{1\ \text{ft}}{0.3048\ \text{m}}\right)$$

$$= 1.82\ \text{ft/s} \qquad \textit{Ans.}$$

NOTE: Remember to round off the final answer to three significant figures.

EXAMPLE 1.2

Convert the quantities 300 lb · s and 52 slug/ft^3 to appropriate SI units.

SOLUTION

Using Table 1–2, 1 lb = 4.448 N.

$$300\ \text{lb}\cdot\text{s} = 300\ \text{lb}\cdot\text{s}\left(\frac{4.448\ \text{N}}{1\ \text{lb}}\right)$$

$$= 1334.5\ \text{N}\cdot\text{s} = 1.33\ \text{kN}\cdot\text{s} \qquad \textit{Ans.}$$

Since 1 slug = 14.59 kg and 1 ft = 0.3048 m, then

$$52\ \text{slug/ft}^3 = \frac{52\ \text{slug}}{1\ \text{ft}^3}\left(\frac{14.59\ \text{kg}}{1\ \text{slug}}\right)\left(\frac{1\ \text{ft}}{0.3048\ \text{m}}\right)^3$$

$$= 26.8(10^3)\ \text{kg/m}^3$$

$$= 26.8\ \text{Mg/m}^3 \qquad \textit{Ans.}$$

EXAMPLE 1.3

Evaluate each of the following and express with SI units having an appropriate prefix: (a) (50 mN)(6 GN), (b) (400 mm)(0.6 MN)^2, (c) 45 MN^3/900 Gg.

SOLUTION

First convert each number to base units, perform the indicated operations, then choose an appropriate prefix.

Part (a)

$$\begin{aligned}(50\text{ mN})(6\text{ GN}) &= [50(10^{-3})\text{ N}][6(10^{9})\text{ N}] \\ &= 300(10^{6})\text{ N}^2 \\ &= 300(10^{6})\,\text{N}^{\cancel{2}}\left(\frac{1\text{ kN}}{10^{3}\,\cancel{\text{N}}}\right)\left(\frac{1\text{ kN}}{10^{3}\,\cancel{\text{N}}}\right) \\ &= 300\text{ kN}^2 \qquad \textit{Ans.}\end{aligned}$$

NOTE: Keep in mind the convention $\text{kN}^2 = (\text{kN})^2 = 10^6\,\text{N}^2$.

Part (b)

$$\begin{aligned}(400\text{ mm})(0.6\text{ MN})^2 &= [400(10^{-3})\text{ m}][0.6(10^{6})\text{ N}]^2 \\ &= [400(10^{-3})\text{ m}][0.36(10^{12})\text{ N}^2] \\ &= 144(10^{9})\text{ m}\cdot\text{N}^2 \\ &= 144\text{ Gm}\cdot\text{N}^2 \qquad \textit{Ans.}\end{aligned}$$

We can also write

$$\begin{aligned}144(10^{9})\text{ m}\cdot\text{N}^2 &= 144(10^{9})\text{ m}\cdot\cancel{\text{N}^2}\left(\frac{1\text{ MN}}{10^{6}\,\cancel{\text{N}}}\right)\left(\frac{1\text{ MN}}{10^{6}\,\cancel{\text{N}}}\right) \\ &= 0.144\text{ m}\cdot\text{MN}^2 \qquad \textit{Ans.}\end{aligned}$$

Part (c)

$$\begin{aligned}\frac{45\text{ MN}^3}{900\text{ Gg}} &= \frac{45(10^{6}\text{ N})^3}{900(10^{6})\text{ kg}} \\ &= 50(10^{9})\text{ N}^3/\text{kg} \\ &= 50(10^{9})\,\cancel{\text{N}^3}\left(\frac{1\text{ kN}}{10^{3}\,\cancel{\text{N}}}\right)^3\frac{1}{\text{kg}} \\ &= 50\text{ kN}^3/\text{kg} \qquad \textit{Ans.}\end{aligned}$$

PROBLEMS

1–1. Round off the following numbers to three significant figures: (a) 58 342 m, (b) 68.534 s, (c) 2553 N, and (d) 7555 kg.

1–2. Wood has a density of 4.70 slug/ft^3. What is its density expressed in SI units?

1–3. Represent each of the following combinations of units in the correct SI form: (a) kN/μs, (b) Mg/mN, and (c) MN/(kg · ms).

***1–4.** Represent each of the following combinations of units in the correct SI form using an appropriate prefix: (a) m/ms, (b) μkm, (c) ks/mg, and (d) km · μN.

1–5. Represent each of the following quantities in the correct SI form using an appropriate prefix: (a) 0.000 431 kg, (b) 35.3(10^3) N, (c) 0.005 32 km.

1–6. If a car is traveling at 55 mi/h, determine its speed in kilometers per hour and meters per second.

1–7. The *pascal* (Pa) is actually a very small unit of pressure. To show this, convert 1 Pa = 1 N/m^2 to lb/ft^2. Atmosphere pressure at sea level is 14.7 lb/in^2. How many pascals is this?

***1–8.** The specific weight (wt./vol) of brass is 520 lb/ft^3. Determine its density (mass/vol) in SI units. Use an appropriate prefix.

1–9. A rocket has a mass 250(10^3) slugs on earth. Specify (a) its mass in SI units, and (b) its weight in SI units. If the rocket is on the moon, where the acceleration due to gravity is $g_m = 5.30$ ft/s^2, determine to three significant figures (c) its weight in SI units, and (d) its mass in SI units.

1–10. Evaluate each of the following to three significant figures and express each answer in SI units using an appropriate prefix: (a) (0.631 Mm)/(8.60 kg)^2, (b) (35 mm)^2 (48 kg)^3.

1–11. Evaluate each of the following to three significant figures and express each answer in SI units using an appropriate prefix: (a) 354 mg(45 km)/(0.0356 kN), (b) (0.004 53 Mg) (201 ms), (c) 435 MN/23.2 mm.

***1–12.** Convert each of the following and express the answer using an appropriate prefix: (a) 175 lb/ft^3 to kN/m^3, (b) 6 ft/h to mm/s, and (c) 835 lb · ft to kN · m.

1–13. Convert each of the following to three significant figures. (a) 20 lb · ft to N · m, (b) 450 lb/ft^3 to kN/m^3, and (c) 15 ft/h to mm/s.

1–14. Evaluate each of the following and express with an appropriate prefix: (a) (430 kg)^2, (b) (0.002 mg)^2, and (c) (230 m)^3.

1–15. Determine the mass of an object that has a weight of (a) 20 mN, (b) 150 kN, (c) 60 MN. Express the answer to three significant figures.

***1–16.** What is the weight in newtons of an object that has a mass of: (a) 10 kg, (b) 0.5 g, (c) 4.50 Mg? Express the result to three significant figures. Use an appropriate prefix.

1–17. If an object has a mass of 40 slugs, determine its mass in kilograms.

1–18. Using the SI system of units, show that Eq. 1–2 is a dimensionally homogeneous equation which gives F in newtons. Determine to three significant figures the gravitational force acting between two spheres that are touching each other. The mass of each sphere is 200 kg and the radius is 300 mm.

1–19. Water has a density of 1.94 slug/ft^3. What is the density expressed in SI units? Express the answer to three significant figures.

***1–20.** Two particles have a mass of 8 kg and 12 kg, respectively. If they are 800 mm apart, determine the force of gravity acting between them. Compare this result with the weight of each particle.

1–21. If a man weighs 155 lb on earth, specify (a) his mass in slugs, (b) his mass in kilograms, and (c) his weight in newtons. If the man is on the moon, where the acceleration due to gravity is $g_m = 5.30$ ft/s^2, determine (d) his weight in pounds, and (e) his mass in kilograms.

Chapter 2

This electric transmission tower is stabilized by cables that exert forces on the tower at their points of connection. In this chapter we will show how to express these forces as Cartesian vectors, and then determined their resultant.

Force Vectors

CHAPTER OBJECTIVES

- To show how to add forces and resolve them into components using the Parallelogram Law.
- To express force and position in Cartesian vector form and explain how to determine the vector's magnitude and direction.
- To introduce the dot product in order to determine the angle between two vectors or the projection of one vector onto another.

2.1 Scalars and Vectors

All physical quantities in engineering mechanics are measured using either scalars or vectors.

Scalar. A *scalar* is any positive or negative physical quantity that can be completely specified by its *magnitude*. Examples of scalar quantities include length, mass, and time.

Vector. A *vector* is any physical quantity that requires both a *magnitude* and a *direction* for its complete description. Examples of vectors encountered in statics are force, position, and moment. A vector is shown graphically by an arrow. The length of the arrow represents the *magnitude* of the vector, and the angle θ between the vector and a fixed axis defines the *direction of its line of action*. The head or tip of the arrow indicates the *sense of direction* of the vector, Fig. 2–1.

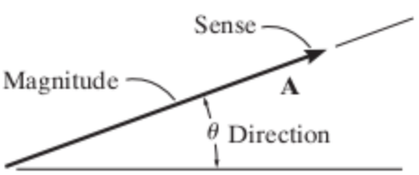
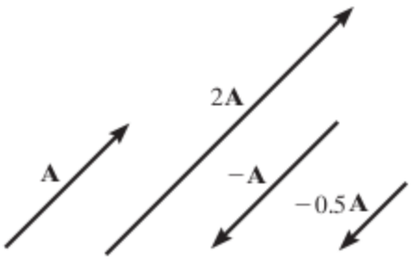


Fig. 2–1

In print, vector quantities are represented by boldfaced letters such as **A**, and the magnitude of a vector is italicized, A. For handwritten work, it is often convenient to denote a vector quantity by simply drawing an arrow above it, $\overrightarrow{A}$.

2.2 Vector Operations



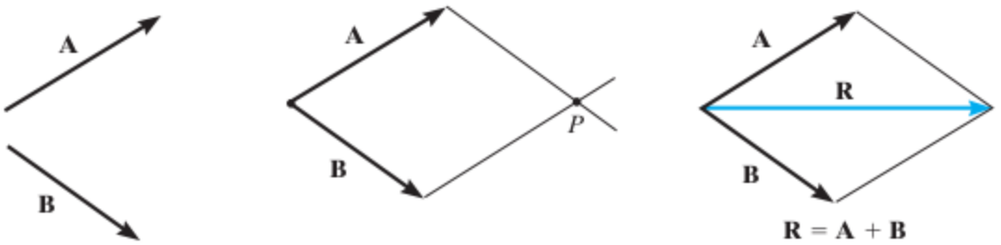
Scalar multiplication and division

Fig. 2–2

Multiplication and Division of a Vector by a Scalar. If a vector is multiplied by a positive scalar, its magnitude is increased by that amount. Multiplying by a negative scalar will also change the directional sense of the vector. Graphic examples of these operations are shown in Fig. 2–2.

Vector Addition. All vector quantities obey the *parallelogram law of addition*. To illustrate, the two "*component*" *vectors* **A** and **B** in Fig. 2–3*a* are added to form a "*resultant*" *vector* $\mathbf{R} = \mathbf{A} + \mathbf{B}$ using the following procedure:

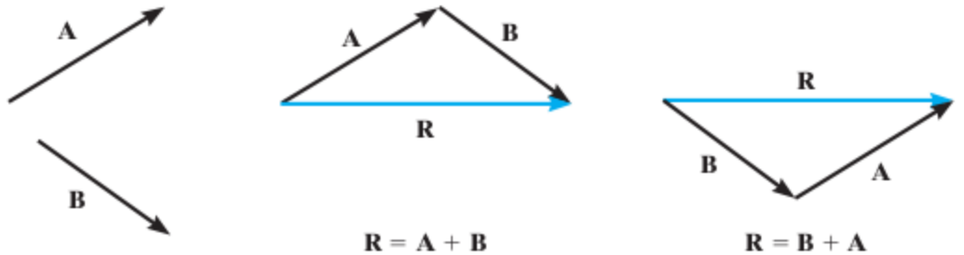
- First join the tails of the components at a point to make them concurrent, Fig. 2–3*b*.
- From the head of **B**, draw a line parallel to **A**. Draw another line from the head of **A** that is parallel to **B**. These two lines intersect at point *P* to form the adjacent sides of a parallelogram.
- The diagonal of this parallelogram that extends to *P* forms **R**, which then represents the resultant vector $\mathbf{R} = \mathbf{A} + \mathbf{B}$, Fig. 2–3*c*.



(a) (b) (c)

Fig. 2–3

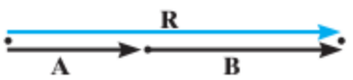
We can also add **B** to **A**, Fig. 2–4*a*, using the *triangle rule*, which is a special case of the parallelogram law, whereby vector **B** is added to vector **A** in a "head-to-tail" fashion, i.e., by connecting the head of **A** to the tail of **B**, Fig. 2–4*b*. The resultant **R** extends from the tail of **A** to the head of **B**. In a similar manner, **R** can also be obtained by adding **A** to **B**, Fig. 2–4*c*. By comparison, it is seen that vector addition is commutative; in other words, the vectors can be added in either order, i.e., $\mathbf{R} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.



(a) $\mathbf{R} = \mathbf{A} + \mathbf{B}$ Triangle rule (b) $\mathbf{R} = \mathbf{B} + \mathbf{A}$ Triangle rule (c)

Fig. 2–4

As a special case, if the two vectors **A** and **B** are *collinear*, i.e., both have the same line of action, the parallelogram law reduces to an *algebraic* or *scalar addition* $R = A + B$, as shown in Fig. 2–5.



$R = A + B$

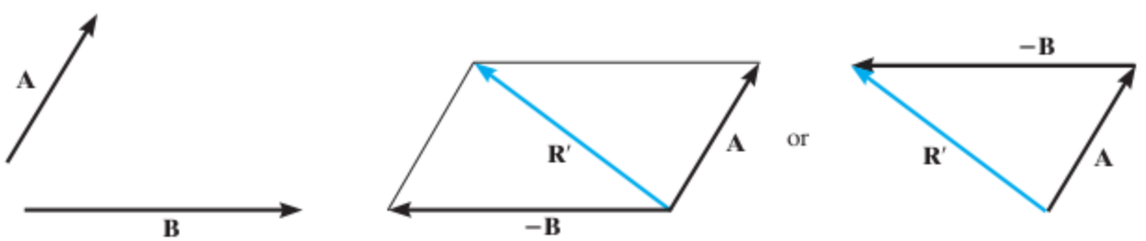
Addition of collinear vectors

Fig. 2–5

Vector Subtraction. The resultant of the *difference* between two vectors **A** and **B** of the same type may be expressed as

$$\mathbf{R}' = \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$

This vector sum is shown graphically in Fig. 2–6. Subtraction is therefore defined as a special case of addition, so the rules of vector addition also apply to vector subtraction.

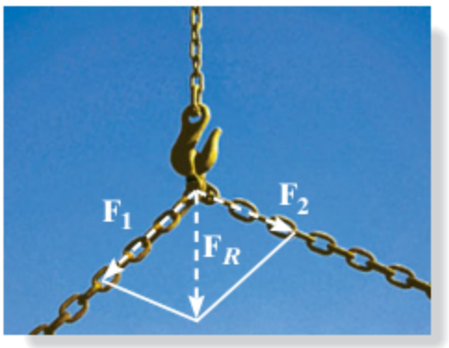


Parallelogram law Triangle construction

Vector subtraction

Fig. 2–6

2



The parallelogram law must be used to determine the resultant of the two forces acting on the hook.

2.3 Vector Addition of Forces

Experimental evidence has shown that a force is a vector quantity since it has a specified magnitude, direction, and sense and it adds according to the parallelogram law. Two common problems in statics involve either finding the resultant force, knowing its components, or resolving a known force into two components. We will now describe how each of these problems is solved using the parallelogram law.

Finding a Resultant Force. The two component forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting on the pin in Fig. 2–7*a* can be added together to form the resultant force $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$, as shown in Fig. 2–7*b*. From this construction, or using the triangle rule, Fig. 2–7*c*, we can apply the law of cosines or the law of sines to the triangle in order to obtain the magnitude of the resultant force and its direction.

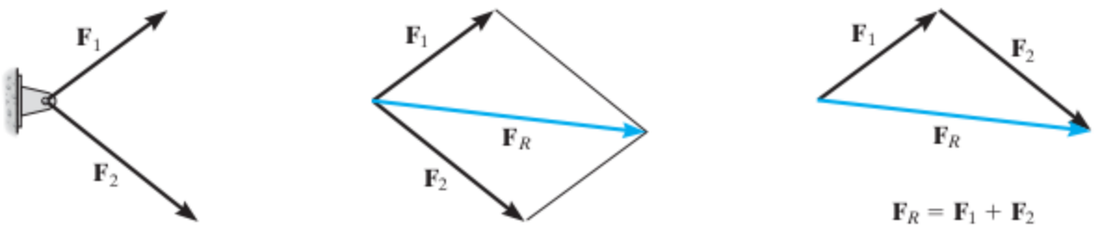


Fig. 2–7

Using the parallelogram law the supporting force **F** can be resolved into components acting along the *u* and *v* axes.

Finding the Components of a Force. Sometimes it is necessary to resolve a force into two *components* in order to study its pulling or pushing effect in two specific directions. For example, in Fig. 2–8*a*, **F** is to be resolved into two components along the two members, defined by the *u* and *v* axes. In order to determine the magnitude of each component, a parallelogram is constructed first, by drawing lines starting from the tip of **F**, one line parallel to *u*, and the other line parallel to *v*. These lines then intersect with the *v* and *u* axes, forming a parallelogram. The force components $\mathbf{F}_u$ and $\mathbf{F}_v$ are then established by simply joining the tail of **F** to the intersection points on the *u* and *v* axes, Fig. 2–8*b*. This parallelogram can then be reduced to a triangle, which represents the triangle rule, Fig. 2–8*c*. From this, the law of sines can then be applied to determine the unknown magnitudes of the components.

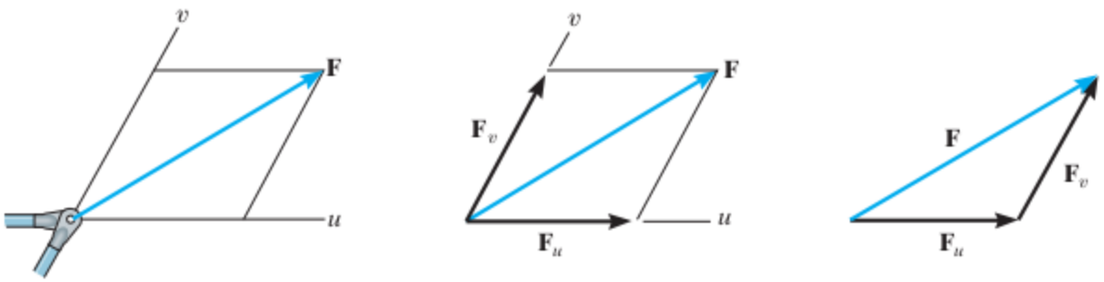


Fig. 2–8

Addition of Several Forces. If more than two forces are to be added, successive applications of the parallelogram law can be carried out in order to obtain the resultant force. For example, if three forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ act at a point O, Fig. 2–9, the resultant of any two of the forces is found, say, $\mathbf{F}_1 + \mathbf{F}_2$—and then this resultant is added to the third force, yielding the resultant of all three forces; i.e., $\mathbf{F}_R = (\mathbf{F}_1 + \mathbf{F}_2) + \mathbf{F}_3$. Using the parallelogram law to add more than two forces, as shown here, often requires extensive geometric and trigonometric calculation to determine the numerical values for the magnitude and direction of the resultant. Instead, problems of this type are easily solved by using the "rectangular-component method," which is explained in Sec. 2.4.

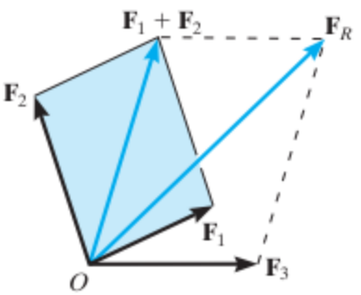
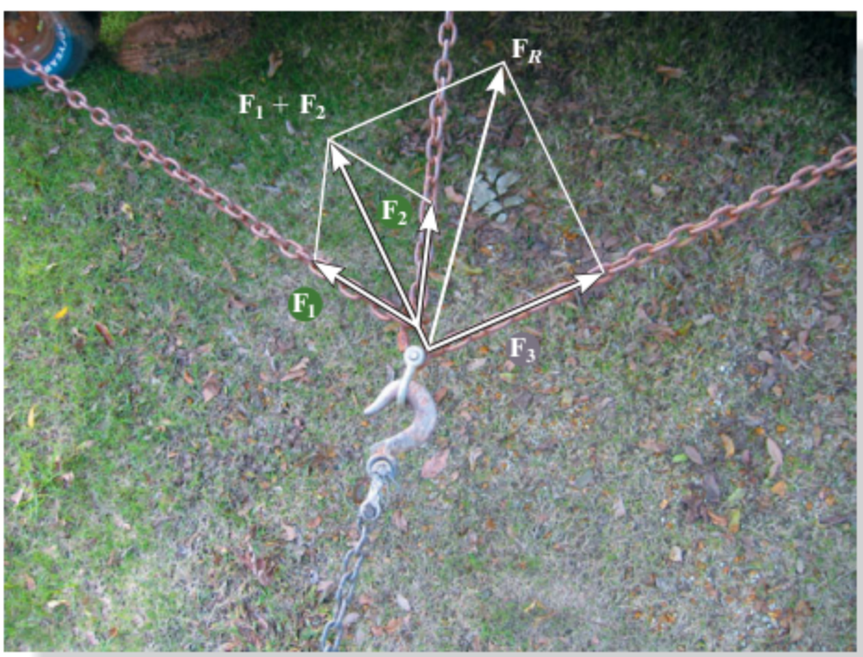
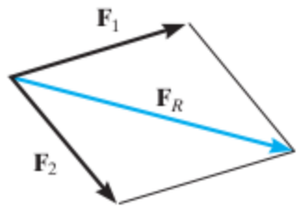


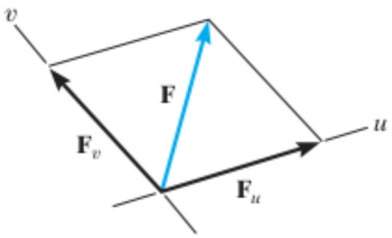
Fig. 2–9



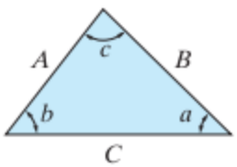
The resultant force $\mathbf{F}_R$ on the hook requires the addition of $\mathbf{F}_1 + \mathbf{F}_2$, then this resultant is added to $\mathbf{F}_3$.



(a)



(b)



Cosine law:
$C = \sqrt{A^2 + B^2 - 2AB\cos c}$
Sine law:
$\frac{A}{\sin a} = \frac{B}{\sin b} = \frac{C}{\sin c}$

(c)

Fig. 2–10

Procedure for Analysis

Problems that involve the addition of two forces can be solved as follows:

Parallelogram Law.

- Two "component" forces $\mathbf{F}_1$ and $\mathbf{F}_2$ in Fig. 2–10*a* add according to the parallelogram law, yielding a *resultant* force $\mathbf{F}_R$ that forms the diagonal of the parallelogram.
- If a force $\mathbf{F}$ is to be resolved into *components* along two axes u and v, Fig. 2–10*b*, then start at the head of force $\mathbf{F}$ and construct lines parallel to the axes, thereby forming the parallelogram. The sides of the parallelogram represent the components, $\mathbf{F}_u$ and $\mathbf{F}_v$.
- Label all the known and unknown force magnitudes and the angles on the sketch and identify the two unknowns as the magnitude and direction of $\mathbf{F}_R$, or the magnitudes of its components.

Trigonometry.

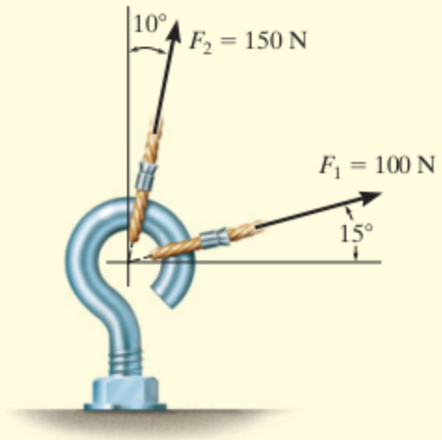
- Redraw a half portion of the parallelogram to illustrate the triangular head-to-tail addition of the components.
- From this triangle, the magnitude of the resultant force can be determined using the law of cosines, and its direction is determined from the law of sines. The magnitudes of two force components are determined from the law of sines. The formulas are given in Fig. 2–10*c*.

Important Points

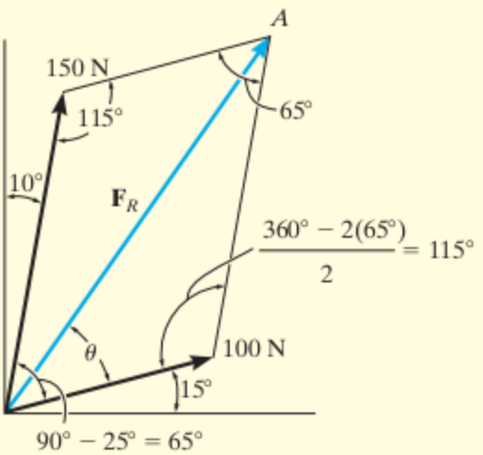
- A scalar is a positive or negative number.
- A vector is a quantity that has a magnitude, direction, and sense.
- Multiplication or division of a vector by a scalar will change the magnitude of the vector. The sense of the vector will change if the scalar is negative.
- As a special case, if the vectors are collinear, the resultant is formed by an algebraic or scalar addition.

EXAMPLE 2.1

The screw eye in Fig. 2–11*a* is subjected to two forces, $\mathbf{F}_1$ and $\mathbf{F}_2$. Determine the magnitude and direction of the resultant force.



(a)



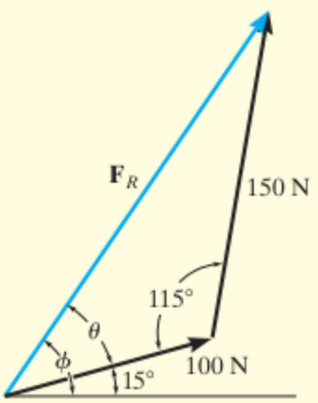
(b)

SOLUTION

Parallelogram Law. The parallelogram is formed by drawing a line from the head of $\mathbf{F}_1$ that is parallel to $\mathbf{F}_2$, and another line from the head of $\mathbf{F}_2$ that is parallel to $\mathbf{F}_1$. The resultant force $\mathbf{F}_R$ extends to where these lines intersect at point A, Fig. 2–11*b*. The two unknowns are the magnitude of $\mathbf{F}_R$ and the angle θ (theta).

Trigonometry. From the parallelogram, the vector triangle is constructed, Fig. 2–11*c*. Using the law of cosines

$$F_R = \sqrt{(100\text{ N})^2 + (150\text{ N})^2 - 2(100\text{ N})(150\text{ N})\cos 115°}$$
$$= \sqrt{10\,000 + 22\,500 - 30\,000(-0.4226)} = 212.6\text{ N}$$
$$= 213\text{ N} \qquad \textit{Ans.}$$



(c)

Fig. 2–11

Applying the law of sines to determine θ,

$$\frac{150\text{ N}}{\sin\theta} = \frac{212.6\text{ N}}{\sin 115°} \qquad \sin\theta = \frac{150\text{ N}}{212.6\text{ N}}(\sin 115°)$$
$$\theta = 39.8°$$

Thus, the direction ϕ (phi) of $\mathbf{F}_R$, measured from the horizontal, is

$$\phi = 39.8° + 15.0° = 54.8° \qquad \textit{Ans.}$$

NOTE: The results seem reasonable, since Fig. 2–11*b* shows $\mathbf{F}_R$ to have a magnitude larger than its components and a direction that is between them.

EXAMPLE 2.2

Resolve the horizontal 600-lb force in Fig. 2–12*a* into components acting along the *u* and *v* axes and determine the magnitudes of these components.

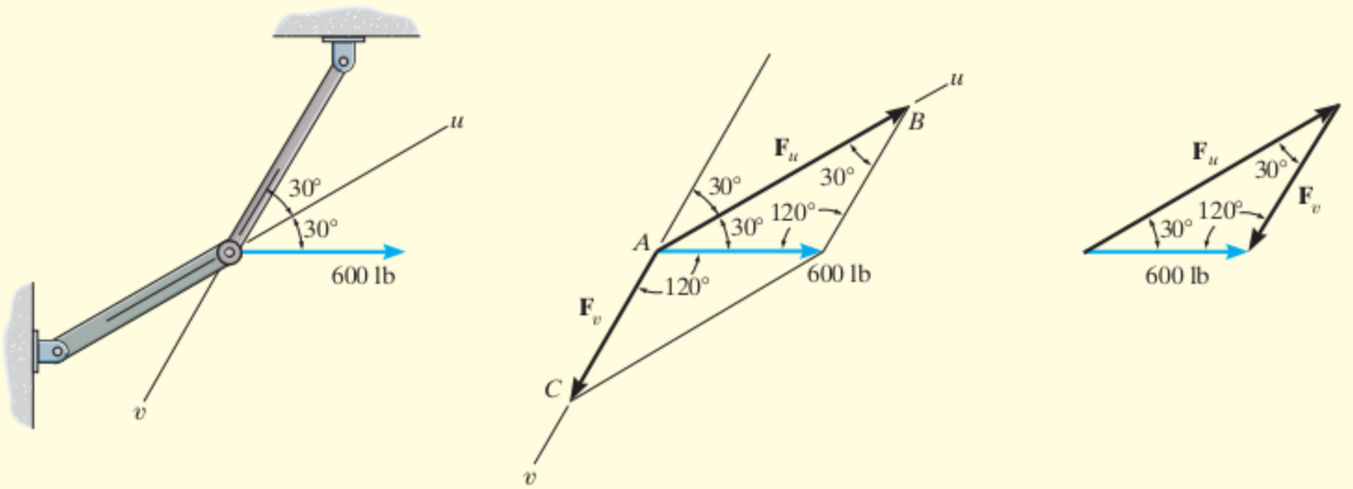


Fig. 2–12

SOLUTION

The parallelogram is constructed by extending a line from the *head* of the 600-lb force parallel to the *v* axis until it intersects the *u* axis at point *B*, Fig. 2–12*b*. The arrow from *A* to *B* represents $\mathbf{F}_u$. Similarly, the line extended from the head of the 600-lb force drawn parallel to the *u* axis intersects the *v* axis at point *C*, which gives $\mathbf{F}_v$.

The vector addition using the triangle rule is shown in Fig. 2–12*c*. The two unknowns are the magnitudes of $\mathbf{F}_u$ and $\mathbf{F}_v$. Applying the law of sines,

$$\frac{F_u}{\sin 120^\circ} = \frac{600\ \text{lb}}{\sin 30^\circ}$$

$$F_u = 1039\ \text{lb} \qquad \textit{Ans.}$$

$$\frac{F_v}{\sin 30^\circ} = \frac{600\ \text{lb}}{\sin 30^\circ}$$

$$F_v = 600\ \text{lb} \qquad \textit{Ans.}$$

NOTE: The result for F_u shows that sometimes a component can have a greater magnitude than the resultant.

EXAMPLE 2.3

Determine the magnitude of the component force **F** in Fig. 2–13*a* and the magnitude of the resultant force $\mathbf{F}_R$ if $\mathbf{F}_R$ is directed along the positive *y* axis.

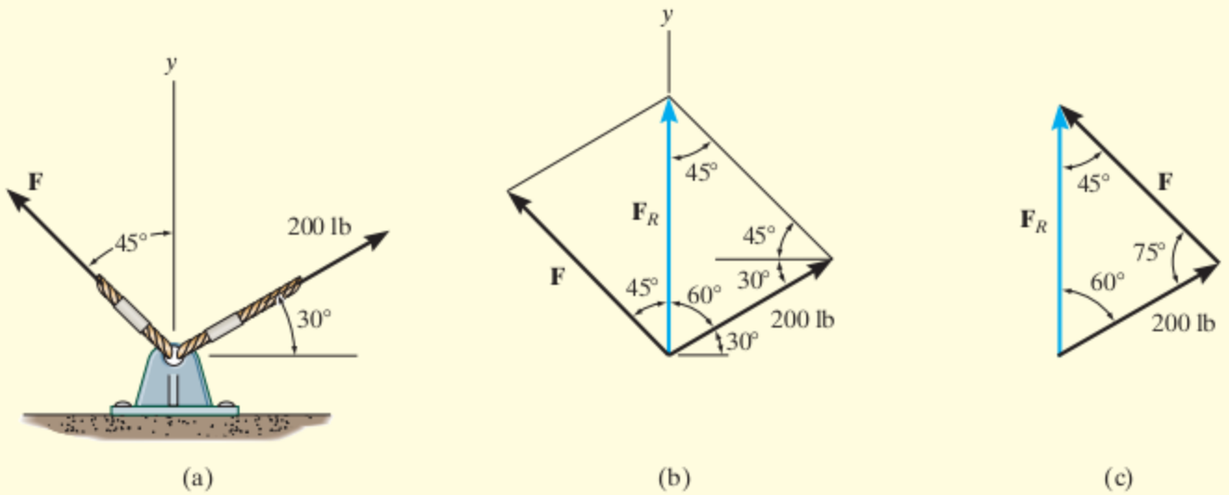


Fig. 2–13

SOLUTION

The parallelogram law of addition is shown in Fig. 2–13*b*, and the triangle rule is shown in Fig. 2–13*c*. The magnitudes of $\mathbf{F}_R$ and **F** are the two unknowns. They can be determined by applying the law of sines.

$$\frac{F}{\sin 60^\circ} = \frac{200\ \text{lb}}{\sin 45^\circ}$$

$$F = 245\ \text{lb} \qquad \textit{Ans.}$$

$$\frac{F_R}{\sin 75^\circ} = \frac{200\ \text{lb}}{\sin 45^\circ}$$

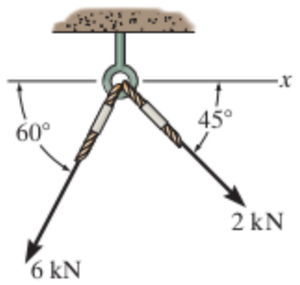
$$F_R = 273\ \text{lb} \qquad \textit{Ans.}$$

It is strongly suggested that you test yourself on the solutions to these examples by covering them over and then trying to draw the parallelogram law, and thinking about how the sine and cosine laws are used to determine the unknowns. Then before solving any of the problems, try and solve some of the Fundamental Problems given on the next page. The solutions and answers to these are given in the back of the book. Doing this throughout the book will help immensely in developing your problem-solving skills.

2

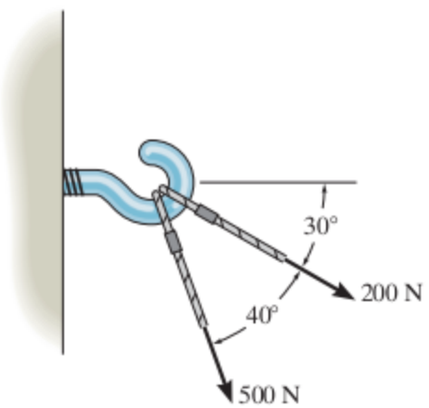
FUNDAMENTAL PROBLEMS*

F2–1. Determine the magnitude of the resultant force acting on the screw eye and its direction measured clockwise from the x axis.



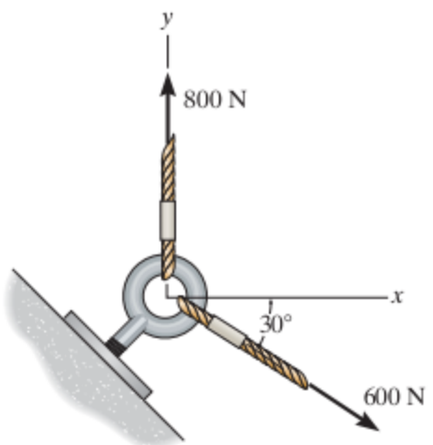
F2–1

F2–2. Two forces act on the hook. Determine the magnitude of the resultant force.



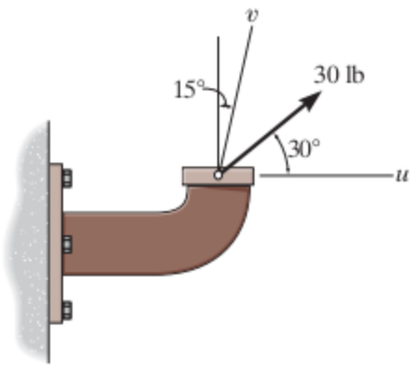
F2–2

F2–3. Determine the magnitude of the resultant force and its direction measured counterclockwise from the positive x axis.



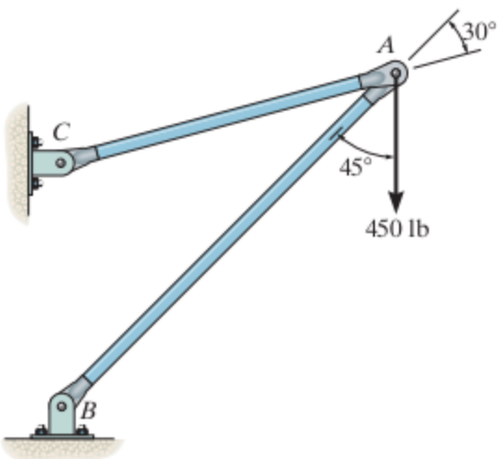
F2–3

F2–4. Resolve the 30-lb force into components along the u and v axes, and determine the magnitude of each of these components.



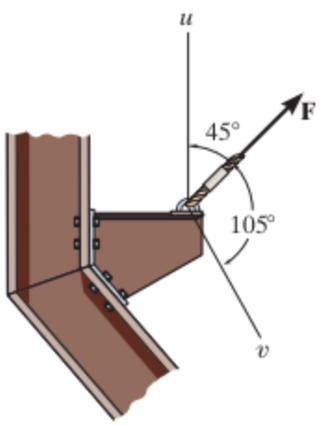
F2–4

F2–5. The force $F = 450$ lb acts on the frame. Resolve this force into components acting along members AB and AC, and determine the magnitude of each component.



F2–5

F2–6. If force **F** is to have a component along the u axis of $F_u = 6$ kN, determine the magnitude of **F** and the magnitude of its component $\mathbf{F}_v$ along the v axis.



F2–6

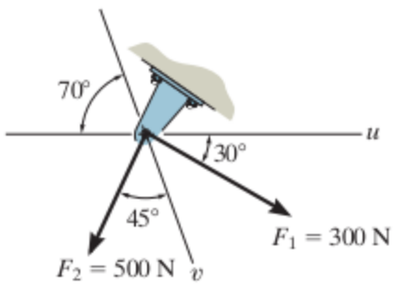
*Partial solutions and answers to all Fundamental Problems are given in the back of the book.

PROBLEMS

2–1. Determine the magnitude of the resultant force $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$ and its direction, measured clockwise from the positive u axis.

2–2. Resolve the force $\mathbf{F}_1$ into components acting along the u and v axes and determine the magnitudes of the components.

2–3. Resolve the force $\mathbf{F}_2$ into components acting along the u and v axes and determine the magnitudes of the components.

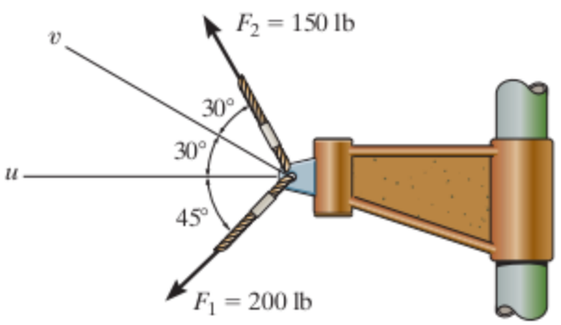


Probs. 2–1/2/3

*2–4. Determine the magnitude of the resultant force acting on the bracket and its direction measured counterclockwise from the positive u axis.

2–5. Resolve $\mathbf{F}_1$ into components along the u and v axes, and determine the magnitudes of these components.

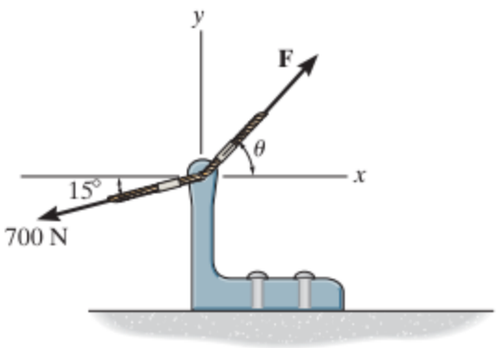
2–6. Resolve $\mathbf{F}_2$ into components along the u and v axes, and determine the magnitudes of these components.



Probs. 2–4/5/6

2–7. If $\theta = 60°$ and $F = 450$ N, determine the magnitude of the resultant force and its direction, measured counterclockwise from the positive x axis.

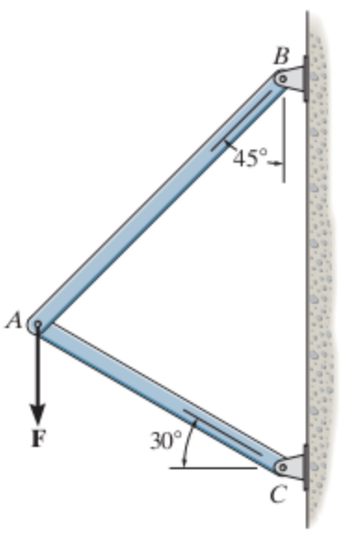
*2–8. If the magnitude of the resultant force is to be 500 N, directed along the positive y axis, determine the magnitude of force $\mathbf{F}$ and its direction θ.



Probs. 2–7/8

2–9. The vertical force $\mathbf{F}$ acts downward at A on the two-membered frame. Determine the magnitudes of the two components of $\mathbf{F}$ directed along the axes of AB and AC. Set $F = 500$ N.

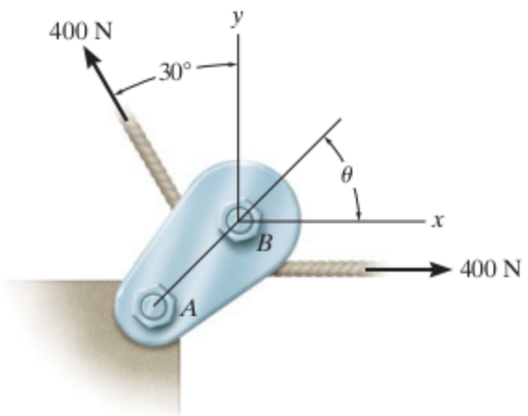
2–10. Solve Prob. 2–9 with $F = 350$ lb.



Probs. 2–9/10

2

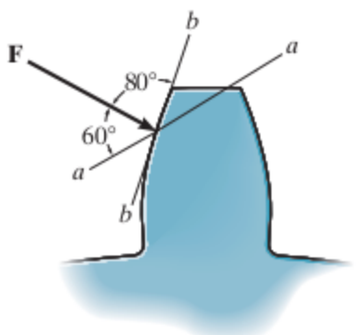
2–11. If the tension in the cable is 400 N, determine the magnitude and direction of the resultant force acting on the pulley. This angle is the same angle θ of line AB on the tailboard block.



Prob. 2–11

*2–12. The force acting on the gear tooth is $F = 20$ lb. Resolve this force into two components acting along the lines aa and bb.

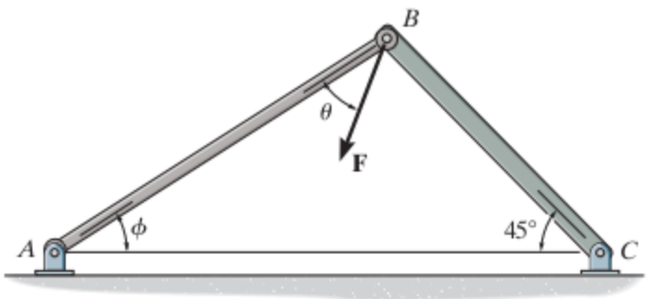
2–13. The component of force **F** acting along line aa is required to be 30 lb. Determine the magnitude of **F** and its component along line bb.



Probs. 2–12/13

2–14. Force **F** acts on the frame such that its component acting along member AB is 650 lb, directed from B towards A, and the component acting along member BC is 500 lb, directed from B towards C. Determine the magnitude of **F** and its direction θ. Set $\phi = 60°$.

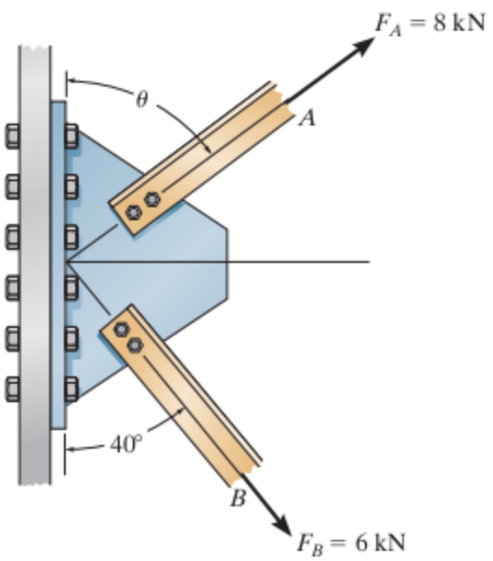
2–15. Force **F** acts on the frame such that its component acting along member AB is 650 lb, directed from B towards A. Determine the required angle ϕ ($0° \le \phi \le 90°$) and the component acting along member BC. Set $F = 850$ lb and $\theta = 30°$.



Probs. 2–14/15

*2–16. The plate is subjected to the two forces at A and B as shown. If $\theta = 60°$, determine the magnitude of the resultant of these two forces and its direction measured clockwise from the horizontal.

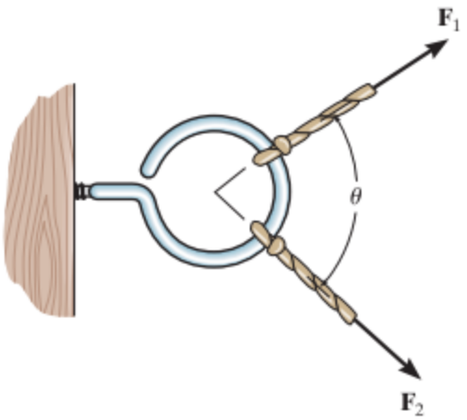
2–17. Determine the angle θ for connecting member A to the plate so that the resultant force of $\mathbf{F}_A$ and $\mathbf{F}_B$ is directed horizontally to the right. Also, what is the magnitude of the resultant force?



Probs. 2–16/17

2–18. Two forces act on the screw eye. If $F_1 = 400$ N and $F_2 = 600$ N, determine the angle θ ($0° \le \theta \le 180°$) between them, so that the resultant force has a magnitude of $F_R = 800$ N.

2–19. Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on the screw eye. If their lines of action are at an angle θ apart and the magnitude of each force is $F_1 = F_2 = F$, determine the magnitude of the resultant force $\mathbf{F}_R$ and the angle between $\mathbf{F}_R$ and $\mathbf{F}_1$.

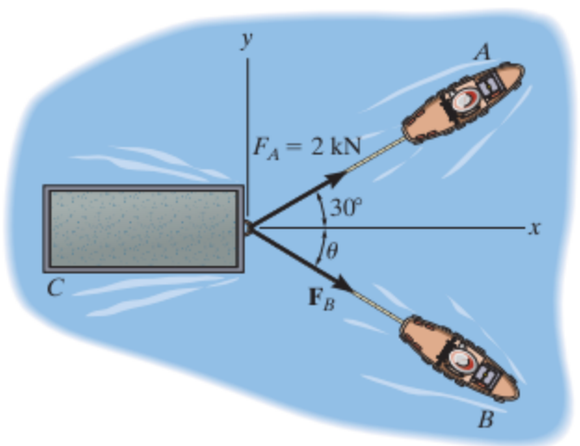


Probs. 2–18/19

*2–20. If the resultant force of the two tugboats is 3 kN, directed along the positive x axis, determine the required magnitude of force $\mathbf{F}_B$ and its direction θ.

2–21. If $F_B = 3$ kN and $\theta = 45°$, determine the magnitude of the resultant force of the two tugboats and its direction measured clockwise from the positive x axis.

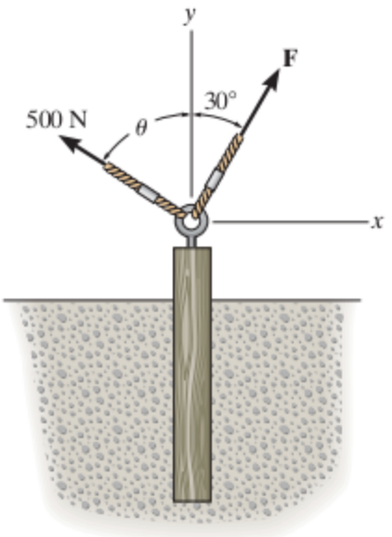
2–22. If the resultant force of the two tugboats is required to be directed towards the positive x axis, and $\mathbf{F}_B$ is to be a minimum, determine the magnitude of $\mathbf{F}_R$ and $\mathbf{F}_B$ and the angle θ.



Probs. 2–20/21/22

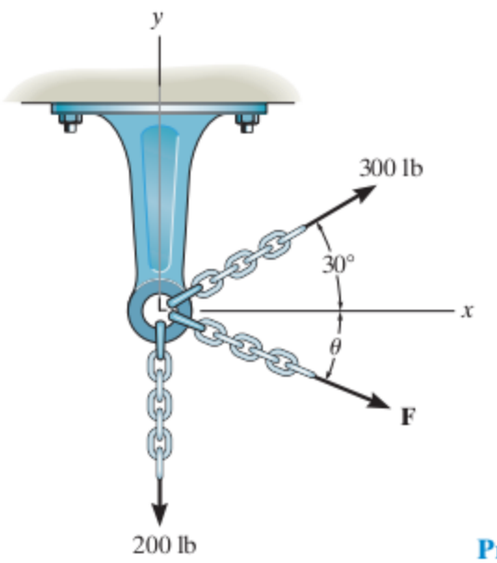
2–23. Two forces act on the screw eye. If $F = 600$ N, determine the magnitude of the resultant force and the angle θ if the resultant force is directed vertically upward.

*2–24. Two forces are applied at the end of a screw eye in order to remove the post. Determine the angle θ ($0° \le \theta \le 90°$) and the magnitude of force $\mathbf{F}$ so that the resultant force acting on the post is directed vertically upward and has a magnitude of 750 N.



Probs. 2–23/24

2–25. Three chains act on the bracket such that they create a resultant force having a magnitude of 500 lb. If two of the chains are subjected to known forces, as shown, determine the angle θ of the third chain measured clockwise from the positive x axis, so that the magnitude of force $\mathbf{F}$ in this chain is a *minimum*. All forces lie in the x–y plane. What is the magnitude of $\mathbf{F}$? *Hint*: First find the resultant of the two known forces. Force $\mathbf{F}$ acts in this direction.



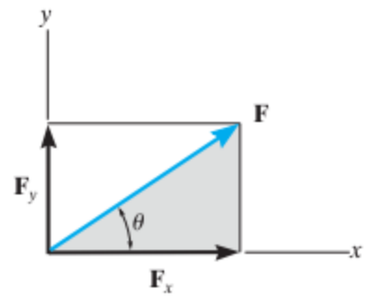
Prob. 2–25

2.4 Addition of a System of Coplanar Forces

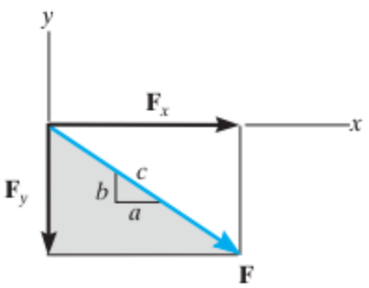
When a force is resolved into two components along the x and y axes, the components are then called *rectangular components*. For analytical work we can represent these components in one of two ways, using either scalar notation or Cartesian vector notation.

Scalar Notation. The rectangular components of force $\mathbf{F}$ shown in Fig. 2–14*a* are found using the parallelogram law, so that $\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y$. Because these components form a right triangle, they can be determined from

$$F_x = F\cos\theta \qquad \text{and} \qquad F_y = F\sin\theta$$



(a)



(b)

Fig. 2–14

Instead of using the angle θ, however, the direction of $\mathbf{F}$ can also be defined using a small "slope" triangle, as in the example shown in Fig. 2–14*b*. Since this triangle and the larger shaded triangle are similar, the proportional length of the sides gives

$$\frac{F_x}{F} = \frac{a}{c}$$

or

$$F_x = F\left(\frac{a}{c}\right)$$

and

$$\frac{F_y}{F} = \frac{b}{c}$$

or

$$F_y = -F\left(\frac{b}{c}\right)$$

Here the y component is a negative scalar since $\mathbf{F}_y$ is directed along the negative y axis.

It is important to keep in mind that this positive and negative scalar notation is to be used only for computational purposes, not for graphical representations in figures. Throughout the book, the *head of a vector arrow* in any figure indicates the sense of the vector *graphically*; algebraic signs are not used for this purpose. Thus, the vectors in Figs. 2–14*a* and 2–14*b* are designated by using boldface (vector) notation.* Whenever italic symbols are written near vector arrows in figures, they indicate the *magnitude* of the vector, which is *always* a *positive* quantity.

*Negative signs are used only in figures with boldface notation when showing equal but opposite pairs of vectors, as in Fig. 2–2.

Cartesian Vector Notation. It is also possible to represent the x and y components of a force in terms of Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$. They are called unit vectors because they have dimensionless magnitude of 1, and so they can be used to designate the *directions* of the x and y axes, respectively, Fig. 2–15.*

Since the *magnitude* of each component of $\mathbf{F}$ is *always a positive quantity*, which is represented by the (positive) scalars F_x and F_y, then we can express $\mathbf{F}$ as a *Cartesian vector*,

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j}$$

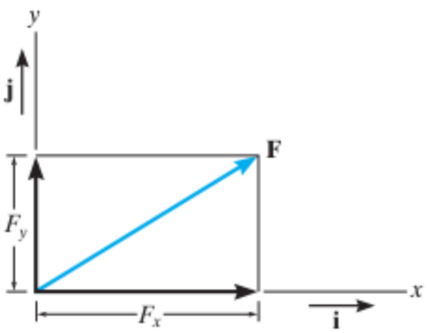
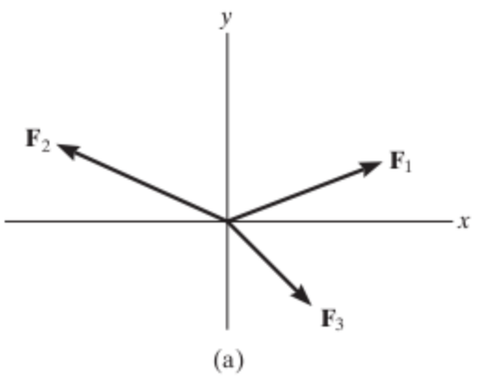


Fig. 2–15

Coplanar Force Resultants. We can use either of the two methods just described to determine the resultant of several *coplanar forces*. To do this, each force is first resolved into its x and y components, and then the respective components are added using *scalar algebra* since they are collinear. The resultant force is then formed by adding the resultant components using the parallelogram law. For example, consider the three concurrent forces in Fig. 2–16*a*, which have x and y components shown in Fig. 2–16*b*. Using *Cartesian vector notation*, each force is first represented as a Cartesian vector, i.e.,

$$\mathbf{F}_1 = F_{1x}\mathbf{i} + F_{1y}\mathbf{j}$$
$$\mathbf{F}_2 = -F_{2x}\mathbf{i} + F_{2y}\mathbf{j}$$
$$\mathbf{F}_3 = F_{3x}\mathbf{i} - F_{3y}\mathbf{j}$$



The vector resultant is therefore

$$\begin{aligned}\mathbf{F}_R &= \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\\ &= F_{1x}\mathbf{i} + F_{1y}\mathbf{j} - F_{2x}\mathbf{i} + F_{2y}\mathbf{j} + F_{3x}\mathbf{i} - F_{3y}\mathbf{j}\\ &= (F_{1x} - F_{2x} + F_{3x})\mathbf{i} + (F_{1y} + F_{2y} - F_{3y})\mathbf{j}\\ &= (F_R)_x\mathbf{i} + (F_R)_y\mathbf{j}\end{aligned}$$

If *scalar notation* is used, then we have

$$(\stackrel{+}{\rightarrow}) \qquad (F_R)_x = F_{1x} - F_{2x} + F_{3x}$$
$$(+\uparrow) \qquad (F_R)_y = F_{1y} + F_{2y} - F_{3y}$$

These are the *same* results as the $\mathbf{i}$ and $\mathbf{j}$ components of $\mathbf{F}_R$ determined above.

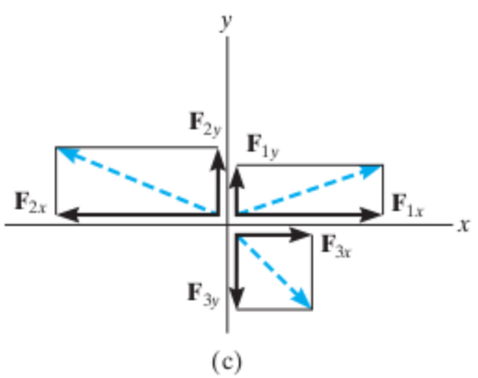
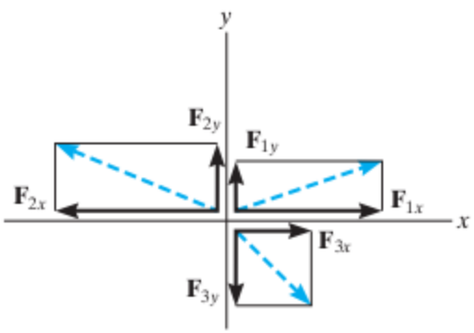


Fig. 2–16

*For handwritten work, unit vectors are usually indicated using a circumflex, e.g., $\hat{i}$ and $\hat{j}$. Also, realize that F_x and F_y in Fig. 2–15 represent the *magnitudes* of the components, which are *always positive scalars*. The directions are defined by $\mathbf{i}$ and $\mathbf{j}$. If instead we used scalar notation, then F_x and F_y could be positive or negative scalars, since they would account for *both* the magnitude and the direction of the components.

2



(c)

Fig. 2–16 (cont.)

We can represent the components of the resultant force of any number of coplanar forces symbolically by the algebraic sum of the x and y components of all the forces, i.e.,

$$\begin{aligned}(F_R)_x &= \Sigma F_x \\ (F_R)_y &= \Sigma F_y\end{aligned} \tag{2–1}$$

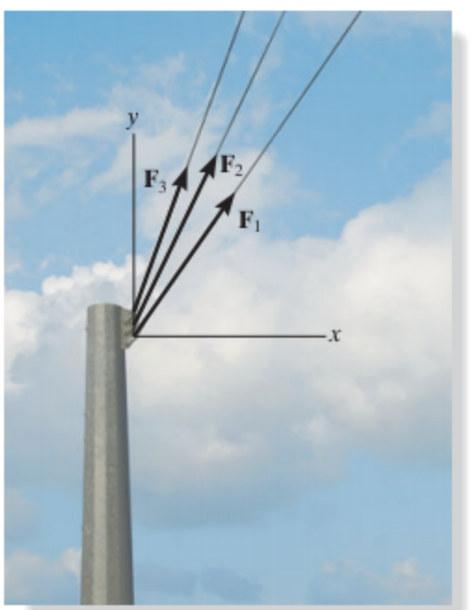
Once these components are determined, they may be sketched along the x and y axes with their proper sense of direction, and the resultant force can be determined from vector addition, as shown in Fig. 2–16*c*. From this sketch, the magnitude of $\mathbf{F}_R$ is then found from the Pythagorean theorem; that is,

$$F_R = \sqrt{(F_R)_x^2 + (F_R)_y^2}$$

Also, the angle θ, which specifies the direction of the resultant force, is determined from trigonometry:

$$\theta = \tan^{-1}\left|\frac{(F_R)_y}{(F_R)_x}\right|$$

The above concepts are illustrated numerically in the examples which follow.



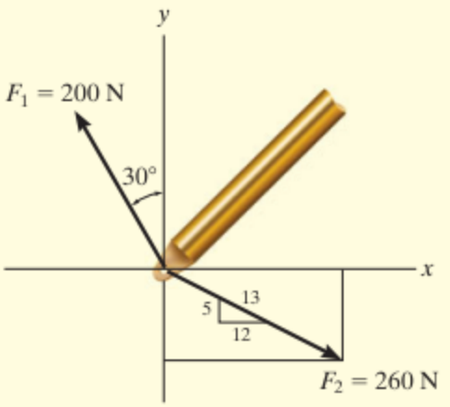
The resultant force of the three cable forces acting on the post can be determined by adding algebraically the separate x and y components of each cable force. This resultant $\mathbf{F}_R$ produces the *same pulling effect* on the post as all three cables.

Important Points

- The resultant of several coplanar forces can easily be determined if an x, y coordinate system is established and the forces are resolved along the axes.
- The direction of each force is specified by the angle its line of action makes with one of the axes, or by a slope triangle.
- The orientation of the x and y axes is arbitrary, and their positive direction can be specified by the Cartesian unit vectors **i** and **j**.
- The x and y components of the *resultant force* are simply the algebraic addition of the components of all the coplanar forces.
- The magnitude of the resultant force is determined from the Pythagorean theorem, and when the components are sketched on the x and y axes, the direction can be determined from trigonometry.

EXAMPLE 2.4

Determine the x and y components of $\mathbf{F}_1$ and $\mathbf{F}_2$ acting on the boom shown in Fig. 2–17*a*. Express each force as a Cartesian vector.



(a)

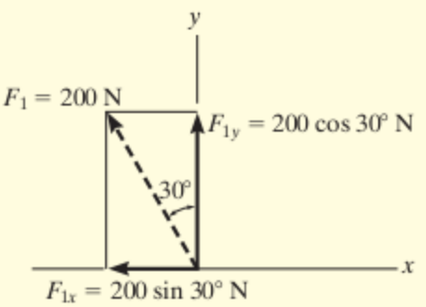
SOLUTION

Scalar Notation. By the parallelogram law, $\mathbf{F}_1$ is resolved into x and y components, Fig. 2–17*b*. Since $\mathbf{F}_{1x}$ acts in the $-x$ direction, and $\mathbf{F}_{1y}$ acts in the $+y$ direction, we have

$$F_{1x} = -200 \sin 30^\circ \text{ N} = -100 \text{ N} = 100 \text{ N} \leftarrow \qquad \textit{Ans.}$$

$$F_{1y} = 200 \cos 30^\circ \text{ N} = 173 \text{ N} = 173 \text{ N} \uparrow \qquad \textit{Ans.}$$

The force $\mathbf{F}_2$ is resolved into its x and y components as shown in Fig. 2–17*c*. Here the *slope* of the line of action for the force is indicated. From this "slope triangle" we could obtain the angle θ, e.g., $\theta = \tan^{-1}(\frac{5}{12})$, and then proceed to determine the magnitudes of the components in the same manner as for $\mathbf{F}_1$. The easier method, however, consists of using proportional parts of similar triangles, i.e.,



(b)

$$\frac{F_{2x}}{260 \text{ N}} = \frac{12}{13} \qquad F_{2x} = 260 \text{ N}\left(\frac{12}{13}\right) = 240 \text{ N}$$

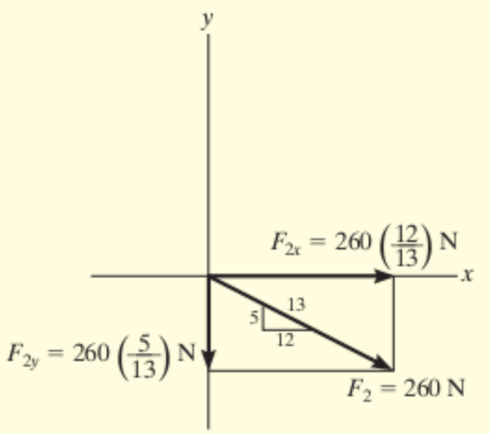
Similarly,

$$F_{2y} = 260 \text{ N}\left(\frac{5}{13}\right) = 100 \text{ N}$$

Notice how the magnitude of the *horizontal component*, $\mathbf{F}_{2x}$, was obtained by multiplying the force magnitude by the ratio of the *horizontal leg* of the slope triangle divided by the hypotenuse; whereas the magnitude of the *vertical component*, $\mathbf{F}_{2y}$, was obtained by multiplying the force magnitude by the ratio of the *vertical leg* divided by the hypotenuse. Hence, using scalar notation to represent these components, we have

$$F_{2x} = 240 \text{ N} = 240 \text{ N} \rightarrow \qquad \textit{Ans.}$$

$$F_{2y} = -100 \text{ N} = 100 \text{ N} \downarrow \qquad \textit{Ans.}$$



(c)

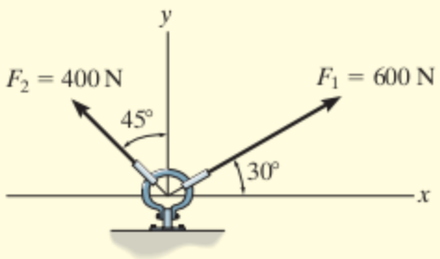
Fig. 2–17

Cartesian Vector Notation. Having determined the magnitudes and directions of the components of each force, we can express each force as a Cartesian vector.

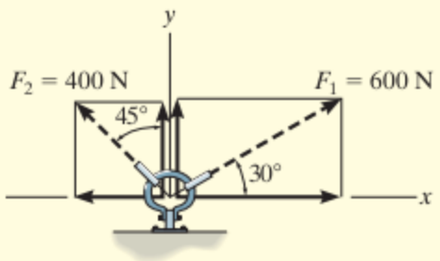
$$\mathbf{F}_1 = \{-100\mathbf{i} + 173\mathbf{j}\} \text{ N} \qquad \textit{Ans.}$$

$$\mathbf{F}_2 = \{240\mathbf{i} - 100\mathbf{j}\} \text{ N} \qquad \textit{Ans.}$$

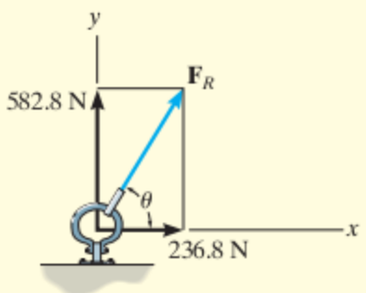
EXAMPLE 2.5



(a)



(b)



(c)

Fig. 2–18

The link in Fig. 2–18*a* is subjected to two forces $\mathbf{F}_1$ and $\mathbf{F}_2$. Determine the magnitude and direction of the resultant force.

SOLUTION I

Scalar Notation. First we resolve each force into its x and y components, Fig. 2–18*b*, then we sum these components algebraically.

$$\overset{+}{\rightarrow} (F_R)_x = \Sigma F_x; \qquad (F_R)_x = 600\cos 30^\circ \text{ N} - 400\sin 45^\circ \text{ N}$$

$$= 236.8 \text{ N} \rightarrow$$

$$+\uparrow (F_R)_y = \Sigma F_y; \qquad (F_R)_y = 600\sin 30^\circ \text{ N} + 400\cos 45^\circ \text{ N}$$

$$= 582.8 \text{ N} \uparrow$$

The resultant force, shown in Fig. 2–18*c*, has a *magnitude* of

$$F_R = \sqrt{(236.8 \text{ N})^2 + (582.8 \text{ N})^2}$$

$$= 629 \text{ N} \qquad \textit{Ans.}$$

From the vector addition,

$$\theta = \tan^{-1}\left(\frac{582.8 \text{ N}}{236.8 \text{ N}}\right) = 67.9^\circ \qquad \textit{Ans.}$$

SOLUTION II

Cartesian Vector Notation. From Fig. 2–18*b*, each force is first expressed as a Cartesian vector.

$$\mathbf{F}_1 = \{600\cos 30^\circ \mathbf{i} + 600\sin 30^\circ \mathbf{j}\} \text{ N}$$

$$\mathbf{F}_2 = \{-400\sin 45^\circ \mathbf{i} + 400\cos 45^\circ \mathbf{j}\} \text{ N}$$

Then,

$$\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2 = (600\cos 30^\circ \text{ N} - 400\sin 45^\circ \text{ N})\mathbf{i}$$

$$+ (600\sin 30^\circ \text{ N} + 400\cos 45^\circ \text{ N})\mathbf{j}$$

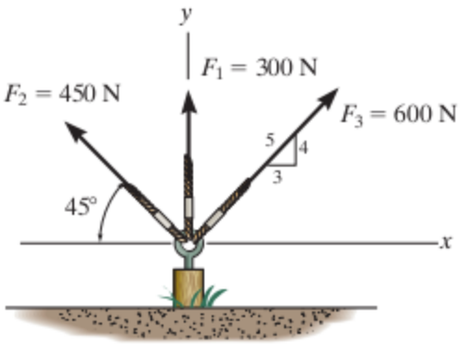
$$= \{236.8\mathbf{i} + 582.8\mathbf{j}\} \text{ N}$$

The magnitude and direction of $\mathbf{F}_R$ are determined in the same manner as before.

NOTE: Comparing the two methods of solution, notice that the use of scalar notation is more efficient since the components can be found *directly*, without first having to express each force as a Cartesian vector before adding the components. Later, however, we will show that Cartesian vector analysis is very beneficial for solving three-dimensional problems.

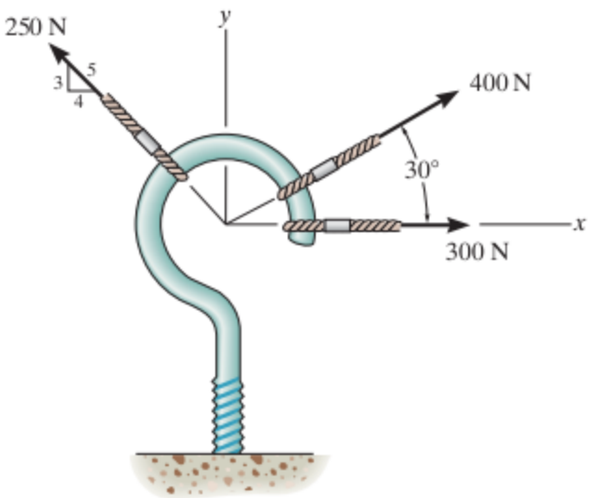
FUNDAMENTAL PROBLEMS

F2–7. Resolve each force acting on the post into its x and y components.



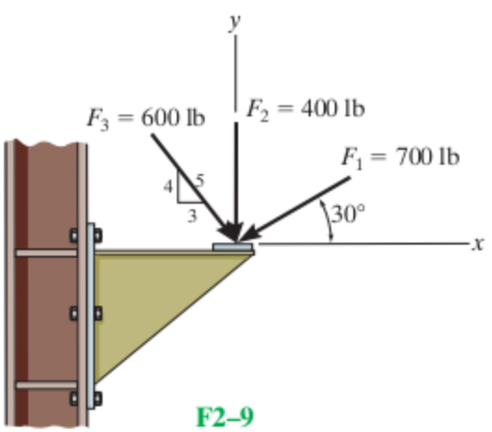
F2–7

F2–8. Determine the magnitude and direction of the resultant force.



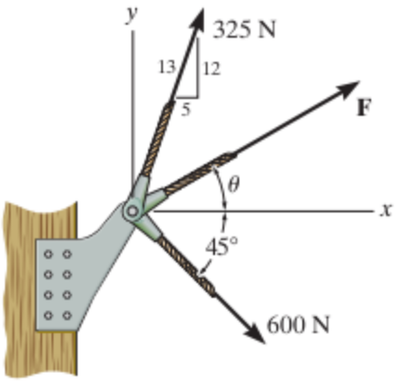
F2–8

F2–9. Determine the magnitude of the resultant force acting on the corbel and its direction θ measured counterclockwise from the x axis.



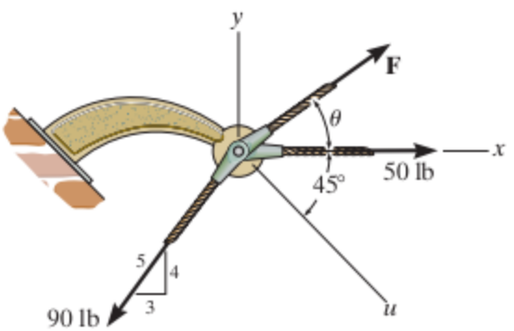
F2–9

F2–10. If the resultant force acting on the bracket is to be 750 N directed along the positive x axis, determine the magnitude of $\mathbf{F}$ and its direction θ.



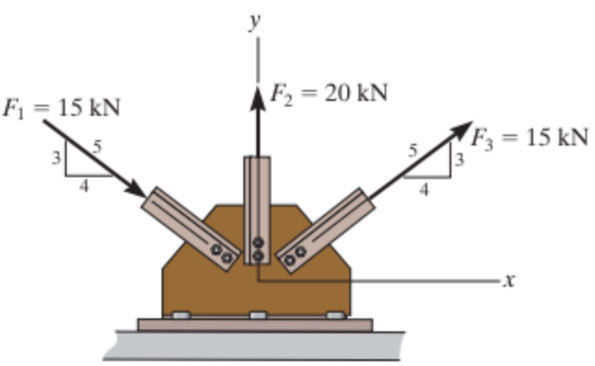
F2–10

F2–11. If the magnitude of the resultant force acting on the bracket is to be 80 lb directed along the u axis, determine the magnitude of $\mathbf{F}$ and its direction θ.



F2–11

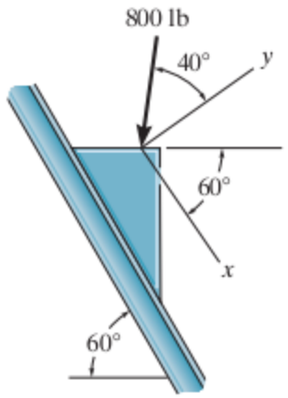
F2–12. Determine the magnitude of the resultant force and its direction θ measured counterclockwise from the positive x axis.



F2–12

PROBLEMS

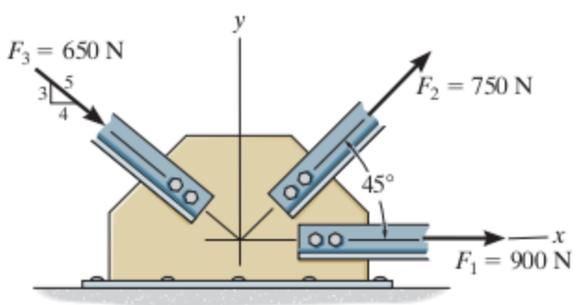
2–26. Determine the x and y components of the 800-lb force.



Prob. 2–26

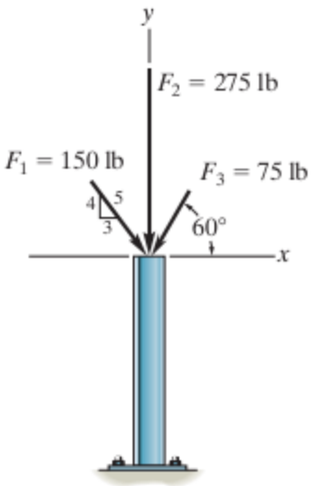
2–27. Resolve each force acting on the gusset plate into its x and y components, and express each force as a Cartesian vector.

***2–28.** Determine the magnitude of the resultant force acting on the plate and its direction, measured counterclockwise from the positive x axis.



Probs. 2–27/28

2–29. Express each of the three forces acting on the column in Cartesian vector form and compute the magnitude of the resultant force.

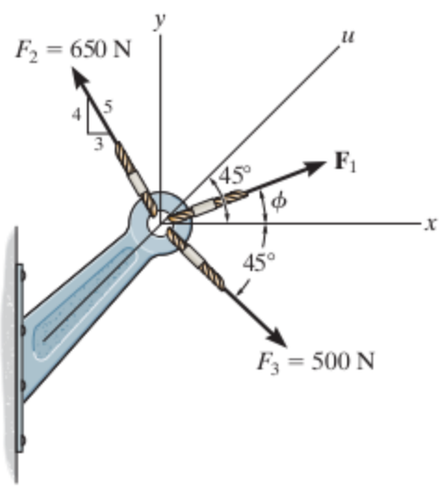


Prob. 2–29

2–30. The magnitude of the resultant force acting on the bracket is to be 400 N. Determine the magnitude of $\mathbf{F}_1$ if $\phi = 30°$.

2–31. If the resultant force acting on the bracket is to be directed along the positive u axis, and the magnitude of $\mathbf{F}_1$ is required to be *minimum*, determine the magnitudes of the resultant force and $\mathbf{F}_1$.

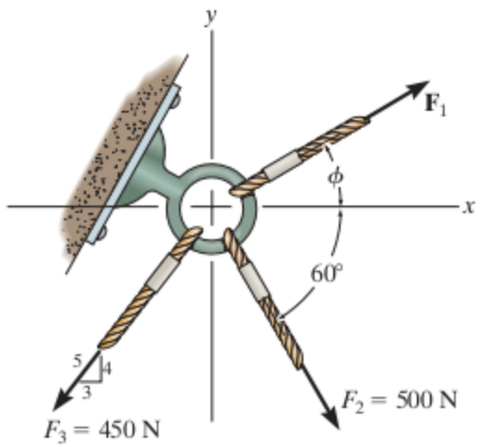
***2–32.** If the magnitude of the resultant force acting on the bracket is 600 N, directed along the positive u axis, determine the magnitude of $\mathbf{F}$ and its direction ϕ.



Probs. 2–30/31/32

2–33. If $F_1 = 600$ N and $\phi = 30°$, determine the magnitude of the resultant force acting on the eyebolt and its direction, measured clockwise from the positive x axis.

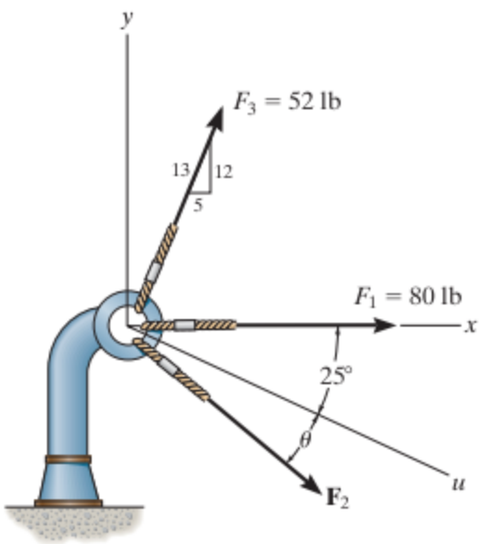
2–34. If the magnitude of the resultant force acting on the eyebolt is 600 N and its direction measured clockwise from the positive x axis is $\theta = 30°$, determine the magnitude of $\mathbf{F}_1$ and the angle ϕ.



Probs. 2–33/34

2–35. Three forces act on the bracket. Determine the magnitude and direction θ of $\mathbf{F}_2$ so that the resultant force is directed along the positive u axis and has a magnitude of 50 lb.

***2–36.** If $F_2 = 150$ lb and $\theta = 55°$, determine the magnitude and direction measured clockwise from the positive x axis, of the resultant force of the three forces acting on the bracket.

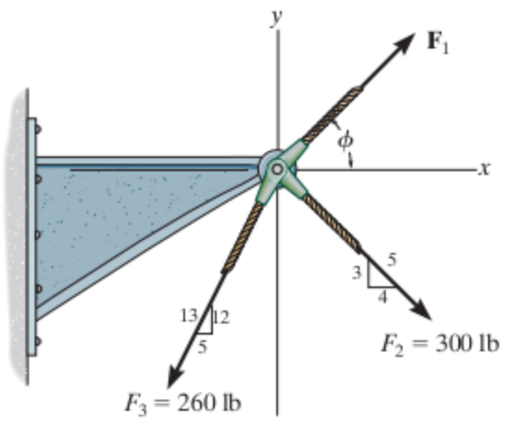


Probs. 2–35/36

2–37. If $\phi = 30°$ and $F_1 = 250$ lb, determine the magnitude of the resultant force acting on the bracket and its direction measured clockwise from the positive x axis.

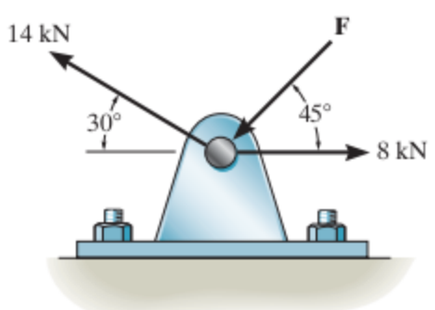
2–38. If the magnitude of the resultant force acting on the bracket is 400 lb directed along the positive x axis, determine the magnitude of $\mathbf{F}_1$ and its direction ϕ.

2–39. If the resultant force acting on the bracket is to be directed along the positive x axis and the magnitude of $\mathbf{F}_1$ is required to be a minimum, determine the magnitudes of the resultant force and $\mathbf{F}_1$.



Probs. 2–37/38/39

***2–40.** Determine the magnitude of force $\mathbf{F}$ so that the resultant force of the three forces is as small as possible. What is the magnitude of the resultant force?



Prob. 2–40

2.5 Cartesian Vectors

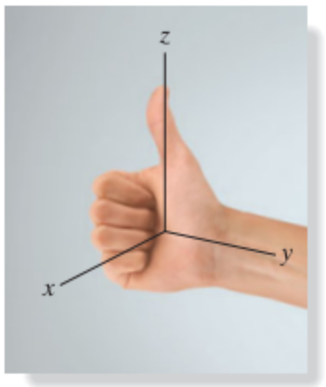


Fig. 2–19

The operations of vector algebra, when applied to solving problems in *three dimensions*, are greatly simplified if the vectors are first represented in Cartesian vector form. In this section we will present a general method for doing this; then in the next section we will use this method for finding the resultant force of a system of concurrent forces.

Right-Handed Coordinate System. We will use a right-handed coordinate system to develop the theory of vector algebra that follows. A rectangular coordinate system is said to be *right-handed* if the thumb of the right hand points in the direction of the positive z axis when the right-hand fingers are curled about this axis and directed from the positive x towards the positive y axis, Fig. 2–19.

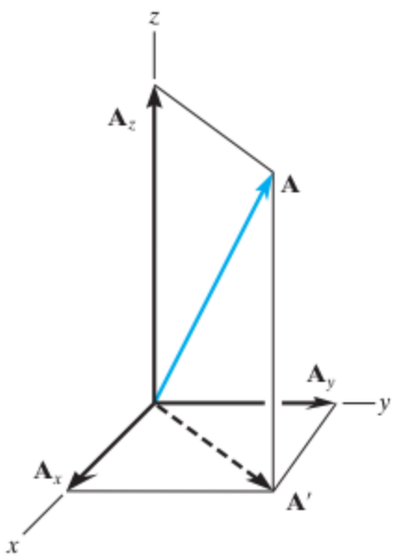


Fig. 2–20

Rectangular Components of a Vector. A vector $\mathbf{A}$ may have one, two, or three rectangular components along the x, y, z coordinate axes, depending on how the vector is oriented relative to the axes. In general, though, when $\mathbf{A}$ is directed within an octant of the x, y, z frame, Fig. 2–20, then by two successive applications of the parallelogram law, we may resolve the vector into components as $\mathbf{A} = \mathbf{A}' + \mathbf{A}_z$ and then $\mathbf{A}' = \mathbf{A}_x + \mathbf{A}_y$. Combining these equations, to eliminate $\mathbf{A}'$, $\mathbf{A}$ is represented by the vector sum of its *three* rectangular components,

$$\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y + \mathbf{A}_z \tag{2–2}$$

Cartesian Unit Vectors. In three dimensions, the set of Cartesian unit vectors, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is used to designate the directions of the x, y, z axes, respectively. As stated in Sec. 2.4, the *sense* (or arrowhead) of these vectors will be represented analytically by a plus or minus sign, depending on whether they are directed along the positive or negative x, y, or z axes. The positive Cartesian unit vectors are shown in Fig. 2–21.

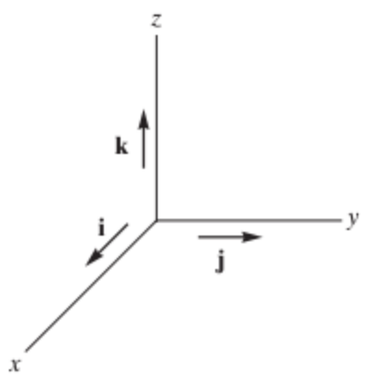


Fig. 2–21

Cartesian Vector Representation. Since the three components of **A** in Eq. 2–2 act in the positive **i**, **j**, and **k** directions, Fig. 2–22, we can write **A** in Cartesian vector form as

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \tag{2–3}$$

There is a distinct advantage to writing vectors in this manner. Separating the *magnitude* and *direction* of each *component vector* will simplify the operations of vector algebra, particularly in three dimensions.

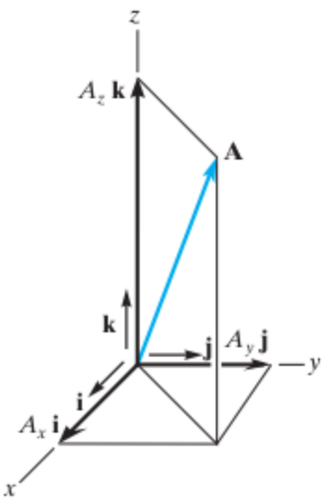


Fig. 2–22

Magnitude of a Cartesian Vector. It is always possible to obtain the magnitude of **A** provided it is expressed in Cartesian vector form. As shown in Fig. 2–23, from the blue right triangle, $A = \sqrt{A'^2 + A_z^2}$, and from the gray right triangle, $A' = \sqrt{A_x^2 + A_y^2}$. Combining these equations to eliminate A' yields

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{2–4}$$

Hence, the magnitude of **A** *is equal to the positive square root of the sum of the squares of its components.*

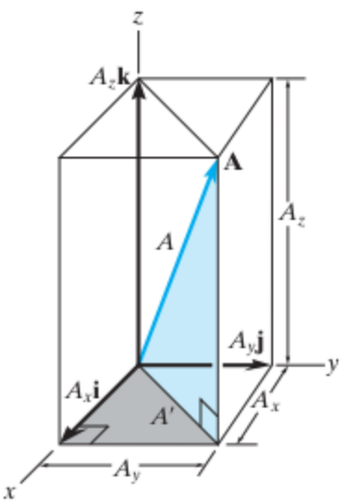


Fig. 2–23

Direction of a Cartesian Vector. We will define the *direction* of **A** by the *coordinate direction angles* α (alpha), β (beta), and γ (gamma), measured between the *tail* of **A** and the *positive* x, y, z axes provided they are located at the tail of **A**, Fig. 2–24. Note that regardless of where **A** is directed, each of these angles will be between 0° and 180°.

To determine α, β, and γ, consider the projection of **A** onto the x, y, z axes, Fig. 2–25. Referring to the blue colored right triangles shown in each figure, we have

$$\cos\alpha = \frac{A_x}{A} \quad \cos\beta = \frac{A_y}{A} \quad \cos\gamma = \frac{A_z}{A} \tag{2–5}$$

These numbers are known as the *direction cosines* of **A**. Once they have been obtained, the coordinate direction angles α, β, γ can then be determined from the inverse cosines.

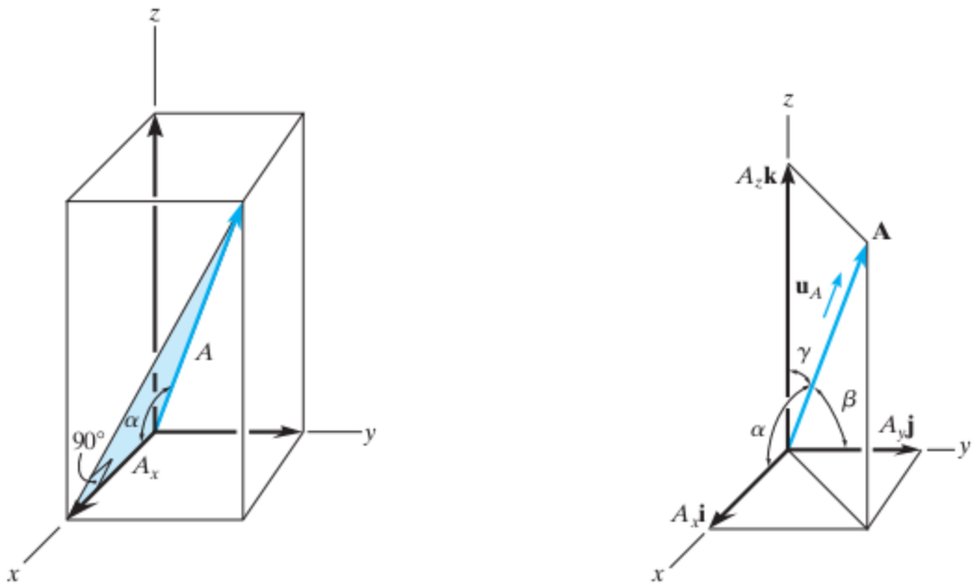


Fig. 2–24

An easy way of obtaining these direction cosines is to form a unit vector $\mathbf{u}_A$ in the direction of $\mathbf{A}$, Fig. 2–24. If $\mathbf{A}$ is expressed in Cartesian vector form, $\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$, then $\mathbf{u}_A$ will have a magnitude of one and be dimensionless provided $\mathbf{A}$ is divided by its magnitude, i.e.,

$$\mathbf{u}_A = \frac{\mathbf{A}}{A} = \frac{A_x}{A}\mathbf{i} + \frac{A_y}{A}\mathbf{j} + \frac{A_z}{A}\mathbf{k} \tag{2–6}$$

where $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$. By comparison with Eqs. 2–5, it is seen that *the* $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ *components of* $\mathbf{u}_A$ *represent the direction cosines of* $\mathbf{A}$, Fig. 2–25, i.e.,

$$\mathbf{u}_A = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k} \tag{2–7}$$

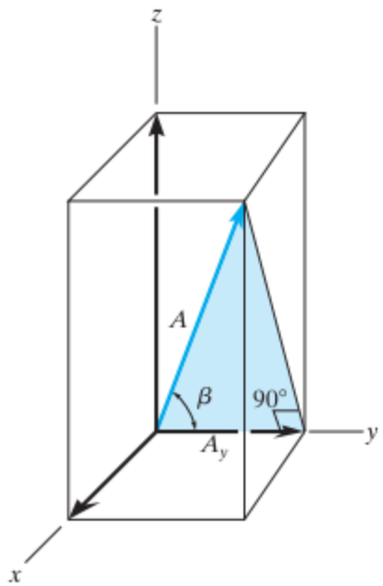
Since the magnitude of a vector is equal to the positive square root of the sum of the squares of the magnitudes of its components, and $\mathbf{u}_A$ has a magnitude of one, then from the above equation an important relation among the direction cosines can be formulated as

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{2–8}$$

Here we can see that if only *two* of the coordinate angles are known, the third angle can be found using this equation.

Finally, if the magnitude and coordinate direction angles of $\mathbf{A}$ are known, then $\mathbf{A}$ may be expressed in Cartesian vector form as

$$\begin{aligned}\mathbf{A} &= A\mathbf{u}_A \\ &= A\cos\alpha\,\mathbf{i} + A\cos\beta\,\mathbf{j} + A\cos\gamma\,\mathbf{k} \\ &= A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}\end{aligned} \tag{2–9}$$



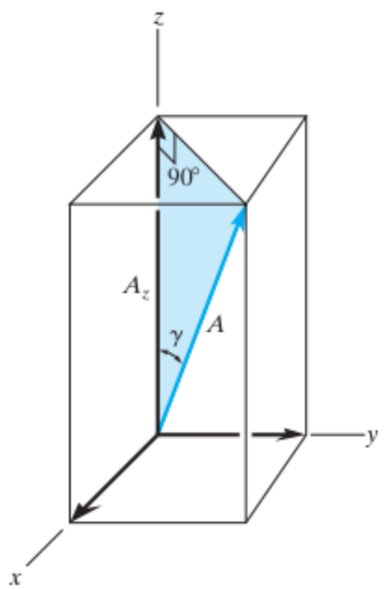


Fig. 2–25

Sometimes, the direction of **A** can be specified using two angles, θ and ϕ (phi), such as shown in Fig. 2–26. The components of **A** can then be determined by applying trigonometry first to the blue right triangle, which yields

$$A_z = A \cos \phi$$

and

$$A' = A \sin \phi$$

Now applying trigonometry to the grey shaded right triangle,

$$A_x = A' \cos \theta = A \sin \phi \cos \theta$$

$$A_y = A' \sin \theta = A \sin \phi \sin \theta$$

Therefore **A** written in Cartesian vector form becomes

$$\mathbf{A} = A \sin \phi \cos \theta \, \mathbf{i} + A \sin \phi \sin \theta \, \mathbf{j} + A \cos \phi \, \mathbf{k}$$

You should not memorize this equation, rather it is important to understand how the components were determined using trigonometry.

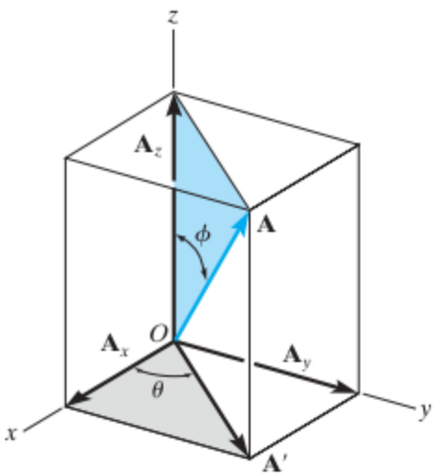


Fig. 2–26

2.6 Addition of Cartesian Vectors

The addition (or subtraction) of two or more vectors is greatly simplified if the vectors are expressed in terms of their Cartesian components. For example, if $\mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}$ and $\mathbf{B} = B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k}$, Fig. 2–27, then the resultant vector, **R**, has components which are the scalar sums of the **i**, **j**, **k** components of **A** and **B**, i.e.,

$$\mathbf{R} = \mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}$$

If this is generalized and applied to a system of several concurrent forces, then the force resultant is the vector sum of all the forces in the system and can be written as

$$\boxed{\mathbf{F}_R = \Sigma \mathbf{F} = \Sigma F_x \mathbf{i} + \Sigma F_y \mathbf{j} + \Sigma F_z \mathbf{k}} \qquad (2\text{–}10)$$

Here ΣF_x, ΣF_y, and ΣF_z represent the algebraic sums of the respective x, y, z or **i**, **j**, **k** components of each force in the system.

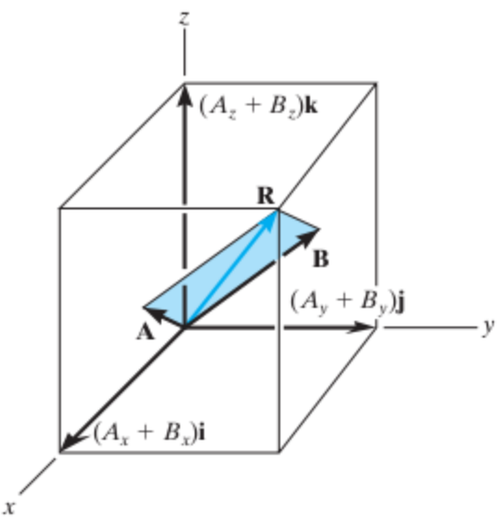


Fig. 2–27

The resultant force acting on the bow of the ship can be determined by first representing each rope force as a Cartesian vector and then summing the **i**, **j**, and **k** components.

Important Points

- Cartesian vector analysis is often used to solve problems in three dimensions.
- The positive directions of the x, y, z axes are defined by the Cartesian unit vectors **i**, **j**, **k**, respectively.
- The *magnitude* of a Cartesian vector is $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$.
- The *direction* of a Cartesian vector is specified using coordinate direction angles α, β, γ which the tail of the vector makes with the positive x, y, z axes, respectively. The components of the unit vector $\mathbf{u}_A = \mathbf{A}/A$ represent the direction cosines of α, β, γ. Only two of the angles α, β, γ have to be specified. The third angle is determined from the relationship $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.
- Sometimes the direction of a vector is defined using the two angles θ and ϕ as in Fig. 2–26. In this case the vector components are obtained by vector resolution using trigonometry.
- To find the *resultant* of a concurrent force system, express each force as a Cartesian vector and add the **i**, **j**, **k** components of all the forces in the system.

EXAMPLE 2.6

Express the force **F** shown in Fig. 2–28 as a Cartesian vector.

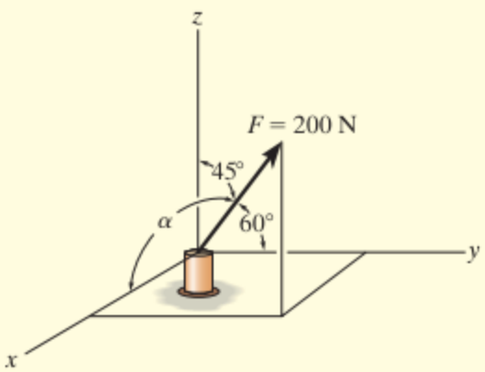


Fig. 2–28

SOLUTION

Since only two coordinate direction angles are specified, the third angle α must be determined from Eq. 2–8; i.e.,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$
$$\cos^2\alpha + \cos^2 60° + \cos^2 45° = 1$$
$$\cos\alpha = \sqrt{1 - (0.5)^2 - (0.707)^2} = \pm 0.5$$

Hence, two possibilities exist, namely,

$$\alpha = \cos^{-1}(0.5) = 60° \quad \text{or} \quad \alpha = \cos^{-1}(-0.5) = 120°$$

By inspection it is necessary that $\alpha = 60°$, since $\mathbf{F}_x$ must be in the $+x$ direction.

Using Eq. 2–9, with $F = 200$ N, we have

$$\begin{aligned}\mathbf{F} &= F\cos\alpha\mathbf{i} + F\cos\beta\mathbf{j} + F\cos\gamma\mathbf{k}\\ &= (200\cos 60°\ \text{N})\mathbf{i} + (200\cos 60°\ \text{N})\mathbf{j} + (200\cos 45°\ \text{N})\mathbf{k}\\ &= \{100.0\mathbf{i} + 100.0\mathbf{j} + 141.4\mathbf{k}\}\ \text{N}\end{aligned}$$ *Ans.*

Show that indeed the magnitude of $F = 200$ N.

EXAMPLE 2.7

Determine the magnitude and the coordinate direction angles of the resultant force acting on the ring in Fig. 2–29*a*.

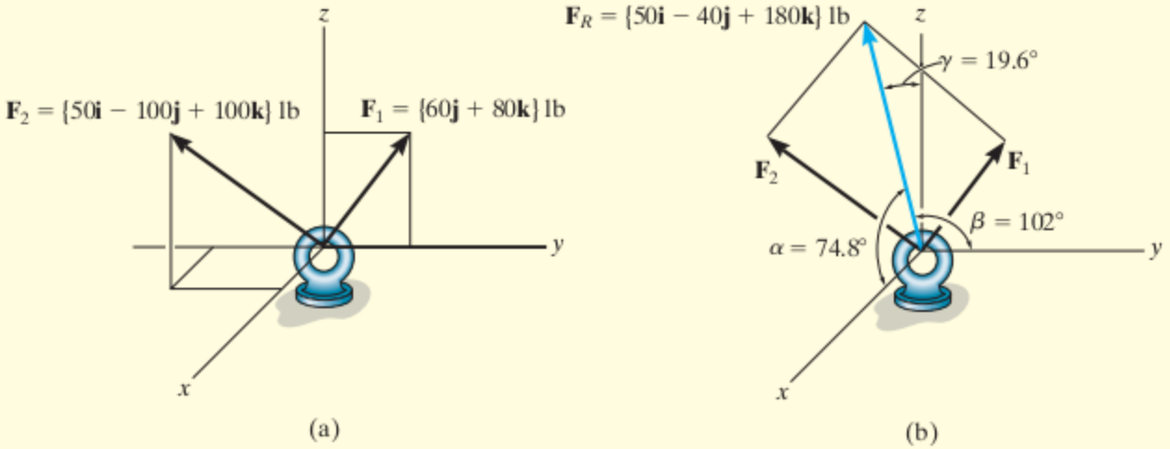


Fig. 2–29

SOLUTION

Since each force is represented in Cartesian vector form, the resultant force, shown in Fig. 2–29*b*, is

$$\mathbf{F}_R = \Sigma\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \{60\mathbf{j} + 80\mathbf{k}\}\text{ lb} + \{50\mathbf{i} - 100\mathbf{j} + 100\mathbf{k}\}\text{ lb}$$
$$= \{50\mathbf{i} - 40\mathbf{j} + 180\mathbf{k}\}\text{ lb}$$

The magnitude of $\mathbf{F}_R$ is

$$F_R = \sqrt{(50\text{ lb})^2 + (-40\text{ lb})^2 + (180\text{ lb})^2} = 191.0\text{ lb}$$
$$= 191\text{ lb} \qquad \textit{Ans.}$$

The coordinate direction angles α, β, γ are determined from the components of the unit vector acting in the direction of $\mathbf{F}_R$.

$$\mathbf{u}_{F_R} = \frac{\mathbf{F}_R}{F_R} = \frac{50}{191.0}\mathbf{i} - \frac{40}{191.0}\mathbf{j} + \frac{180}{191.0}\mathbf{k}$$
$$= 0.2617\mathbf{i} - 0.2094\mathbf{j} + 0.9422\mathbf{k}$$

so that

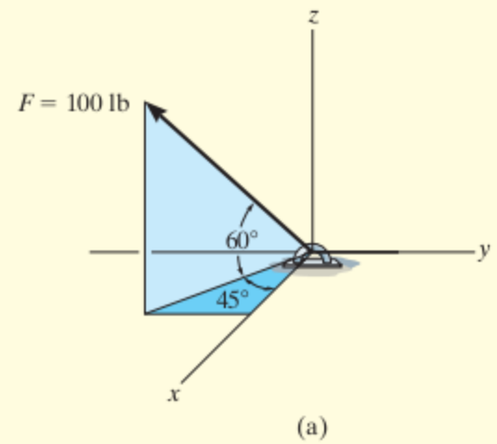
$$\cos\alpha = 0.2617 \qquad \alpha = 74.8° \qquad \textit{Ans.}$$
$$\cos\beta = -0.2094 \qquad \beta = 102° \qquad \textit{Ans.}$$
$$\cos\gamma = 0.9422 \qquad \gamma = 19.6° \qquad \textit{Ans.}$$

These angles are shown in Fig. 2–29*b*.

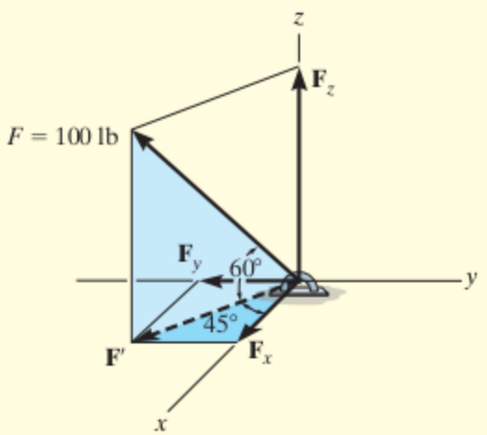
NOTE: In particular, notice that $\beta > 90°$ since the $\mathbf{j}$ component of $\mathbf{u}_{F_R}$ is negative. This seems reasonable considering how $\mathbf{F}_1$ and $\mathbf{F}_2$ add according to the parallelogram law.

EXAMPLE 2.8

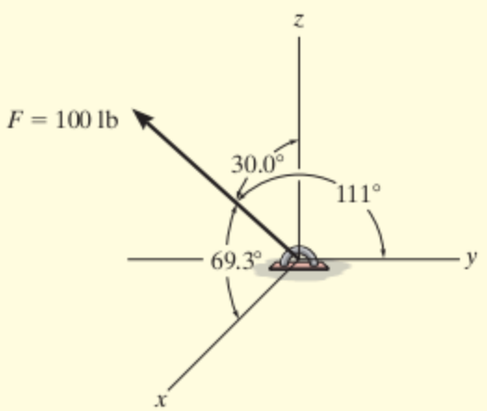
Express the force **F** shown in Fig. 2–30*a* as a Cartesian vector.



(a)



(b)



(c)

Fig. 2–30

SOLUTION

The angles of 60° and 45° defining the direction of **F** are *not* coordinate direction angles. Two successive applications of the parallelogram law are needed to resolve **F** into its x, y, z components. First $\mathbf{F} = \mathbf{F}' + \mathbf{F}_z$, then $\mathbf{F}' = \mathbf{F}_x + \mathbf{F}_y$, Fig. 2–30*b*. By trigonometry, the magnitudes of the components are

$$F_z = 100 \sin 60^\circ \text{ lb} = 86.6 \text{ lb}$$

$$F' = 100 \cos 60^\circ \text{ lb} = 50 \text{ lb}$$

$$F_x = F' \cos 45^\circ = 50 \cos 45^\circ \text{ lb} = 35.4 \text{ lb}$$

$$F_y = F' \sin 45^\circ = 50 \sin 45^\circ \text{ lb} = 35.4 \text{ lb}$$

Realizing that $\mathbf{F}_y$ has a direction defined by $-\mathbf{j}$, we have

$$\mathbf{F} = \{35.4\mathbf{i} - 35.4\mathbf{j} + 86.6\mathbf{k}\} \text{ lb} \qquad \textit{Ans.}$$

To show that the magnitude of this vector is indeed 100 lb, apply Eq. 2–4,

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

$$= \sqrt{(35.4)^2 + (35.4)^2 + (86.6)^2} = 100 \text{ lb}$$

If needed, the coordinate direction angles of **F** can be determined from the components of the unit vector acting in the direction of **F**. Hence,

$$\mathbf{u} = \frac{\mathbf{F}}{F} = \frac{\mathbf{F}_x}{\mathbf{F}}\mathbf{i} + \frac{F_y}{F}\mathbf{j} + \frac{F_z}{F}\mathbf{k}$$

$$= \frac{35.4}{100}\mathbf{i} - \frac{35.4}{100}\mathbf{j} + \frac{86.6}{100}\mathbf{k}$$

$$= 0.354\mathbf{i} - 0.354\mathbf{j} + 0.866\mathbf{k}$$

so that

$$\alpha = \cos^{-1}(0.354) = 69.3^\circ$$

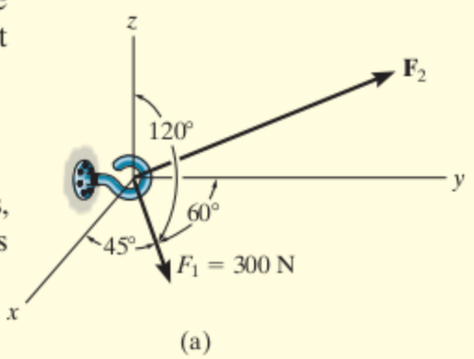
$$\beta = \cos^{-1}(-0.354) = 111^\circ$$

$$\gamma = \cos^{-1}(0.866) = 30.0^\circ$$

These results are shown in Fig. 2–30*c*.

EXAMPLE 2.9

Two forces act on the hook shown in Fig. 2–31*a*. Specify the magnitude of $\mathbf{F}_2$ and its coordinate direction angles of $\mathbf{F}_2$ that the resultant force $\mathbf{F}_R$ acts along the positive y axis and has a magnitude of 800 N.



(a)

SOLUTION

To solve this problem, the resultant force $\mathbf{F}_R$ and its two components, $\mathbf{F}_1$ and $\mathbf{F}_2$, will each be expressed in Cartesian vector form. Then, as shown in Fig. 2–31*a*, it is necessary that $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$.

Applying Eq. 2–9,

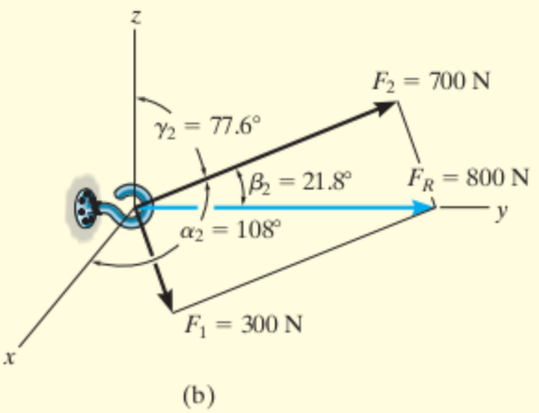
$$\begin{aligned}\mathbf{F}_1 &= F_1 \cos\alpha_1\mathbf{i} + F_1 \cos\beta_1\mathbf{j} + F_1 \cos\gamma_1\mathbf{k}\\ &= 300\cos 45°\mathbf{i} + 300\cos 60°\mathbf{j} + 300\cos 120°\mathbf{k}\\ &= \{212.1\mathbf{i} + 150\mathbf{j} - 150\mathbf{k}\}\,\text{N}\\ \mathbf{F}_2 &= F_{2x}\mathbf{i} + F_{2y}\mathbf{j} + F_{2z}\mathbf{k}\end{aligned}$$

Since $\mathbf{F}_R$ has a magnitude of 800 N and acts in the $+\mathbf{j}$ direction,

$$\mathbf{F}_R = (800\ \text{N})(+\mathbf{j}) = \{800\mathbf{j}\}\ \text{N}$$

We require

$$\begin{aligned}\mathbf{F}_R &= \mathbf{F}_1 + \mathbf{F}_2\\ 800\mathbf{j} &= 212.1\mathbf{i} + 150\mathbf{j} - 150\mathbf{k} + F_{2x}\mathbf{i} + F_{2y}\mathbf{j} + F_{2z}\mathbf{k}\\ 800\mathbf{j} &= (212.1 + F_{2x})\mathbf{i} + (150 + F_{2y})\mathbf{j} + (-150 + F_{2z})\mathbf{k}\end{aligned}$$



(b)

Fig. 2–31

To satisfy this equation the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components of $\mathbf{F}_R$ must be equal to the corresponding $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components of $(\mathbf{F}_1 + \mathbf{F}_2)$. Hence,

$$\begin{aligned}0 &= 212.1 + F_{2x} & F_{2x} &= -212.1\ \text{N}\\ 800 &= 150 + F_{2y} & F_{2y} &= 650\ \text{N}\\ 0 &= -150 + F_{2z} & F_{2z} &= 150\ \text{N}\end{aligned}$$

The magnitude of $\mathbf{F}_2$ is thus

$$\begin{aligned}F_2 &= \sqrt{(-212.1\ \text{N})^2 + (650\ \text{N})^2 + (150\ \text{N})^2}\\ &= 700\ \text{N}\end{aligned}$$ *Ans.*

We can use Eq. 2–9 to determine $\alpha_2, \beta_2, \gamma_2$.

$$\cos\alpha_2 = \frac{-212.1}{700}; \qquad \alpha_2 = 108°$$ *Ans.*

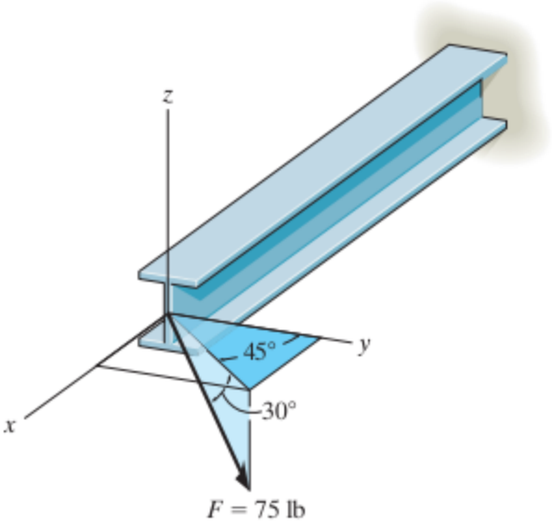
$$\cos\beta_2 = \frac{650}{700}; \qquad \beta_2 = 21.8°$$ *Ans.*

$$\cos\gamma_2 = \frac{150}{700}; \qquad \gamma_2 = 77.6°$$ *Ans.*

These results are shown in Fig. 2–31*b*.

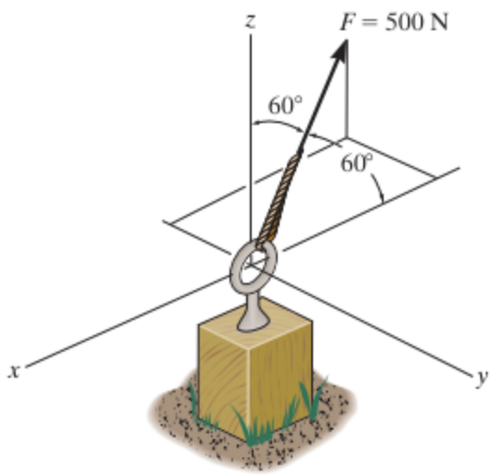
FUNDAMENTAL PROBLEMS

F2–13. Determine the coordinate direction angles of the force.



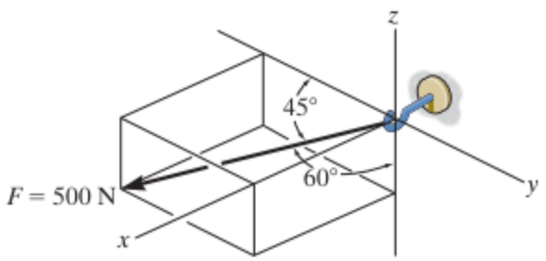
F2–13

F2–14. Express the force as a Cartesian vector.



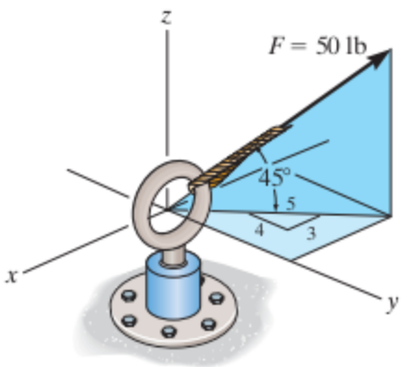
F2–14

F2–15. Express the force as a Cartesian vector.



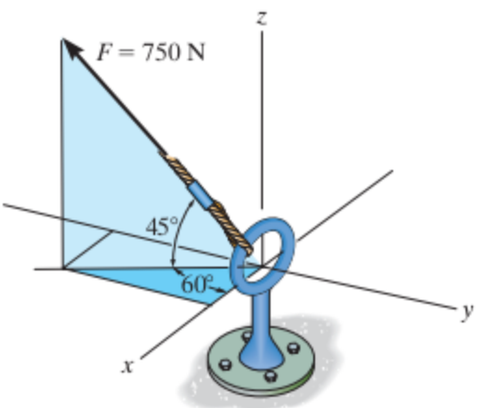
F2–15

F2–16. Express the force as a Cartesian vector.



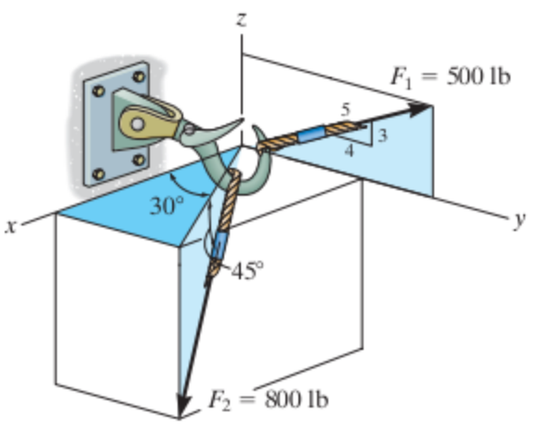
F2–16

F2–17. Express the force as a Cartesian vector.



F2–17

F2–18. Determine the resultant force acting on the hook.

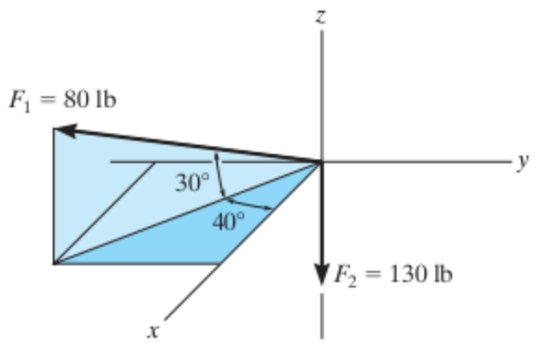


F2–18

PROBLEMS

2–41. Determine the magnitude and coordinate direction angles of the resultant force and sketch this vector on the coordinate system.

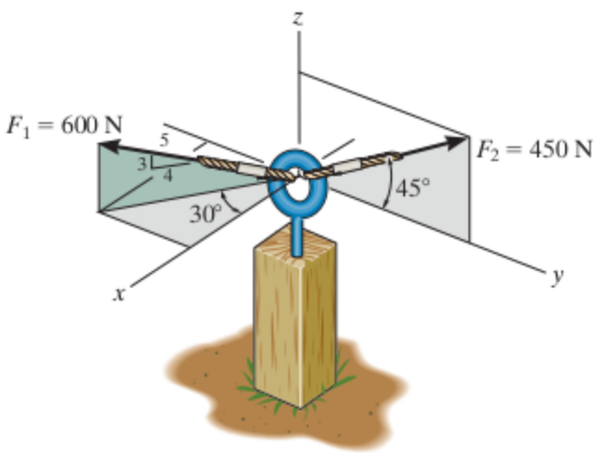
2–42. Specify the coordinate direction angles of $\mathbf{F}_1$ and $\mathbf{F}_2$ and express each force as a Cartesian vector.



Probs. 2–41/42

2–43. Determine the coordinate direction angles of force $\mathbf{F}_1$.

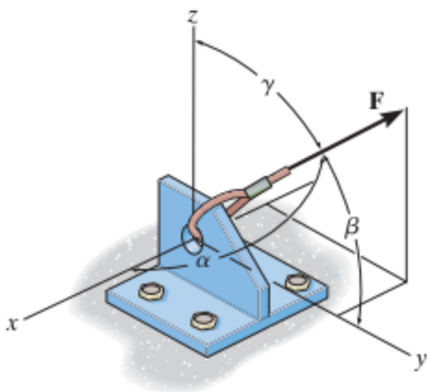
***2–44.** Determine the magnitude and coordinate direction angles of the resultant force acting on the eyebolt.



Probs. 2–43/44

2–45. The force $\mathbf{F}$ acts on the bracket within the octant shown. If $F = 400$ N, $\beta = 60°$, and $\gamma = 45°$, determine the x, y, z components of $\mathbf{F}$.

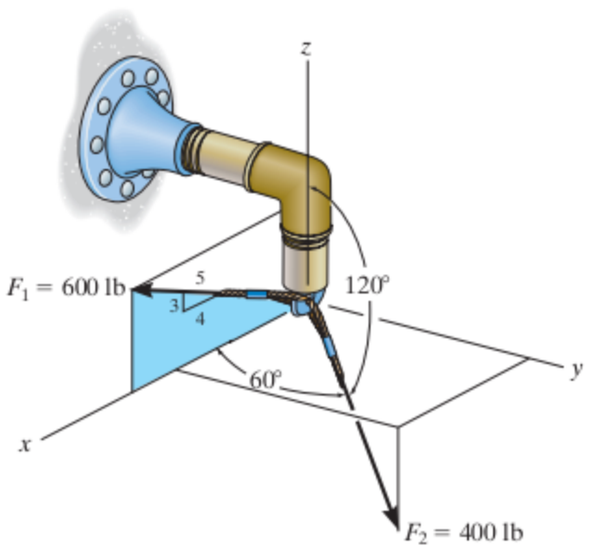
2–46. The force $\mathbf{F}$ acts on the bracket within the octant shown. If the magnitudes of the x and z components of $\mathbf{F}$ are $F_x = 300$ N and $F_z = 600$ N, respectively, and $\beta = 60°$, determine the magnitude of $\mathbf{F}$ and its y component. Also, find the coordinate direction angles α and γ.



Probs. 2–45/46

2–47. Express each force acting on the pipe assembly in Cartesian vector form.

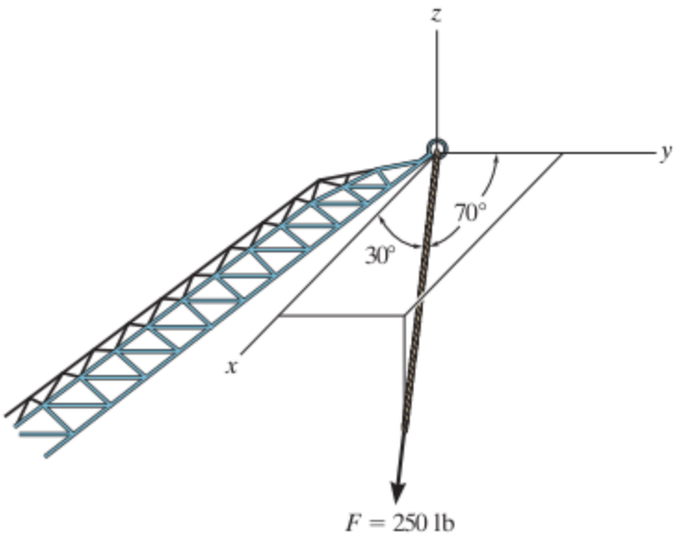
***2–48.** Determine the magnitude and the direction of the resultant force acting on the pipe assembly.



Probs. 2–47/48

2–49. Determine the magnitude and coordinate direction angles of $\mathbf{F}_1 = \{60\mathbf{i} - 50\mathbf{j} + 40\mathbf{k}\}$ N and $\mathbf{F}_2 = \{-40\mathbf{i} - 85\mathbf{j} + 30\mathbf{k}\}$ N. Sketch each force on an x, y, z reference frame.

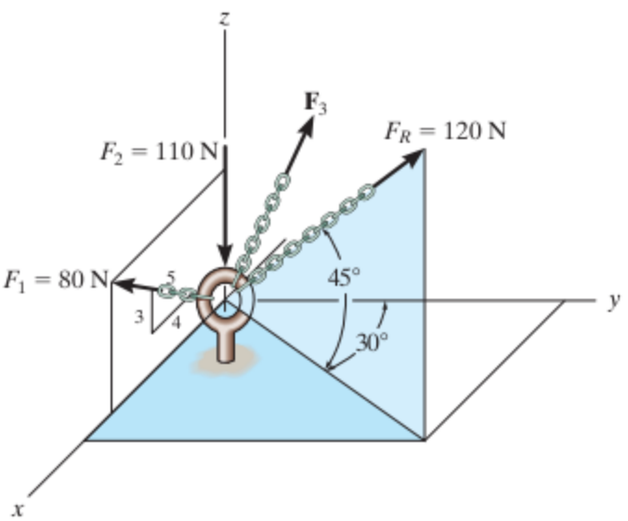
2–50. The cable at the end of the crane boom exerts a force of 250 lb on the boom as shown. Express $\mathbf{F}$ as a Cartesian vector.



Probs. 2–49/50

2–51. Three forces act on the ring. If the resultant force $\mathbf{F}_R$ has a magnitude and direction as shown, determine the magnitude and the coordinate direction angles of force $\mathbf{F}_3$.

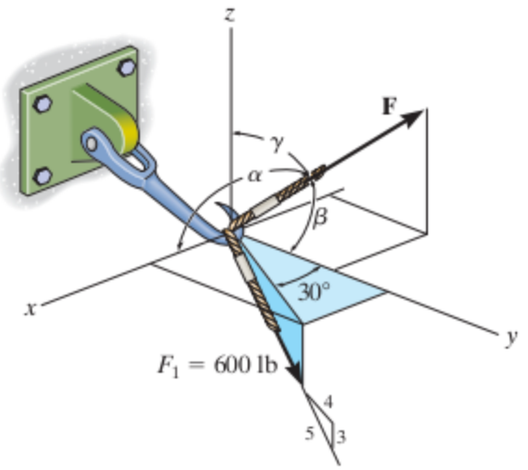
2–52. Determine the coordinate direction angles of $\mathbf{F}_1$ and $\mathbf{F}_R$.



Probs. 2–51/52

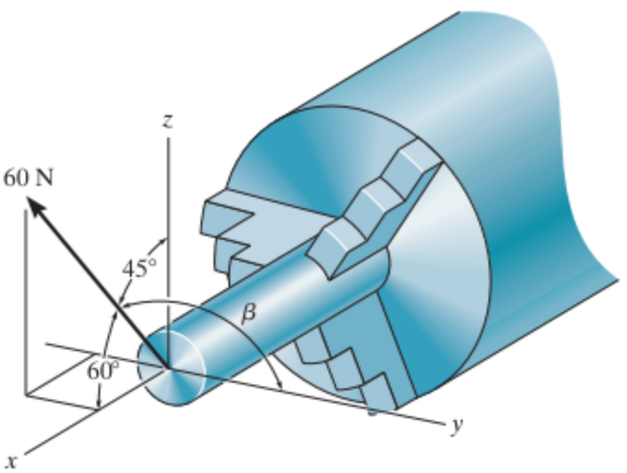
2–53. If $\alpha = 120°$, $\beta < 90°$, $\gamma = 60°$, and $F = 400$ lb, determine the magnitude and coordinate direction angles of the resultant force acting on the hook.

2–54. If the resultant force acting on the hook is $\mathbf{F}_R = \{-200\mathbf{i} + 800\mathbf{j} + 150\mathbf{k}\}$ lb, determine the magnitude and coordinate direction angles of $\mathbf{F}$.



Probs. 2–53/54

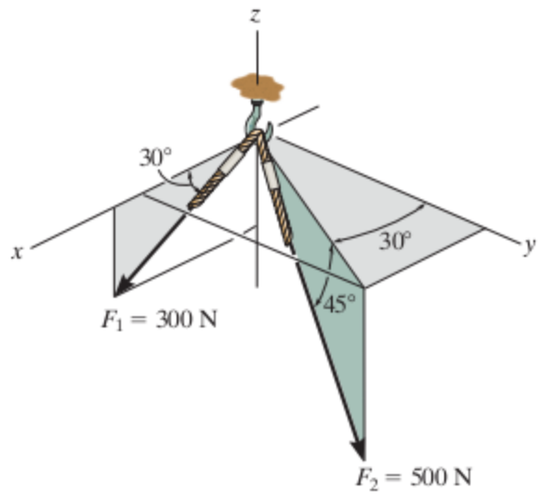
2–55. The stock mounted on the lathe is subjected to a force of 60 N. Determine the coordinate direction angle β and express the force as a Cartesian vector.



Prob. 2–55

*2–56. Express each force as a Cartesian vector.

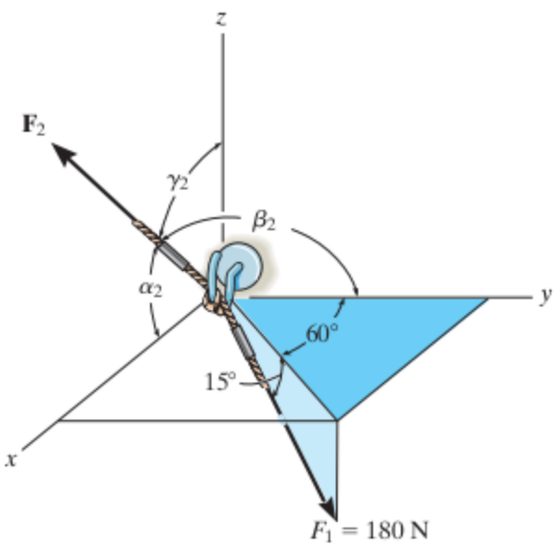
2–57. Determine the magnitude and coordinate direction angles of the resultant force acting on the hook.



Probs. 2–56/57

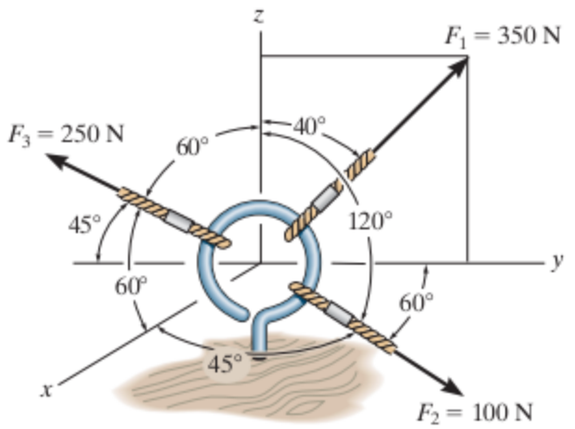
2–58. Determine the magnitude and coordinate direction angles of $\mathbf{F}_2$ so that the resultant of the two forces acts along the positive x axis and has a magnitude of 500 N.

2–59. Determine the magnitude and coordinate direction angles of $\mathbf{F}_2$ so that the resultant of the two forces is zero.



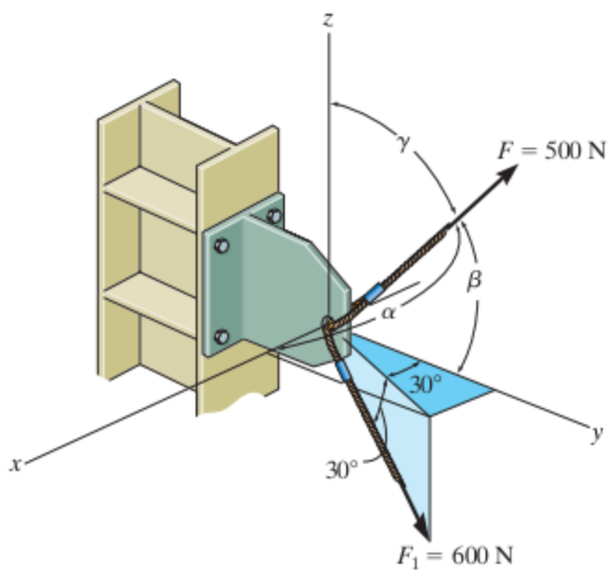
Probs. 2–58/59

*2–60. The cables attached to the screw eye are subjected to the three forces shown. Express each force in Cartesian vector form and determine the magnitude and coordinate direction angles of the resultant force.



Prob. 2–60

2–61. If the resultant force acting on the bracket is directed along the positive y axis, determine the magnitude of the resultant force and the coordinate direction angles of $\mathbf{F}$ so that $\beta < 90°$.



Prob. 2–61

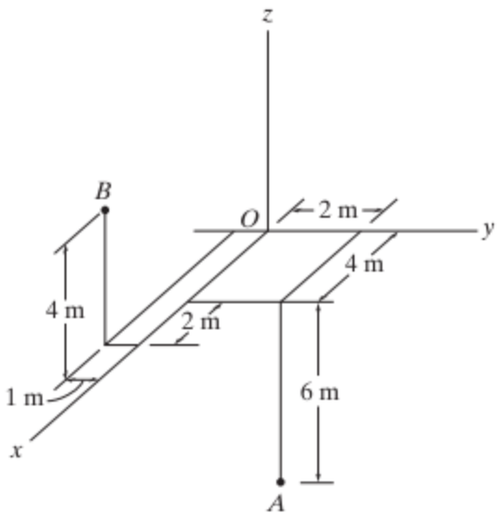


Fig. 2–32

2.7 Position Vectors

In this section we will introduce the concept of a position vector. It will be shown that this vector is of importance in formulating a Cartesian force vector directed between two points in space.

x, y, z Coordinates. Throughout the book we will use a *right-handed* coordinate system to reference the location of points in space. We will also use the convention followed in many technical books, which requires the positive z axis to be directed *upward* (the zenith direction) so that it measures the height of an object or the altitude of a point. The x, y axes then lie in the horizontal plane, Fig. 2–32. Points in space are located relative to the origin of coordinates, O, by successive measurements along the x, y, z axes. For example, the coordinates of point A are obtained by starting at O and measuring $x_A = +4$ m along the x axis, then $y_A = +2$ m along the y axis, and finally $z_A = -6$ m along the z axis. Thus, $A(4\text{ m}, 2\text{ m}, -6\text{ m})$. In a similar manner, measurements along the x, y, z axes from O to B yield the coordinates of B, i.e., $B(6\text{ m}, -1\text{ m}, 4\text{ m})$.

Position Vector. A *position vector* $\mathbf{r}$ is defined as a fixed vector which locates a point in space relative to another point. For example, if $\mathbf{r}$ extends from the origin of coordinates, O, to point $P(x, y, z)$, Fig. 2–33*a*, then $\mathbf{r}$ can be expressed in Cartesian vector form as

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Note how the head-to-tail vector addition of the three components yields vector $\mathbf{r}$, Fig. 2–33*b*. Starting at the origin O, one "travels" x in the $+\mathbf{i}$ direction, then y in the $+\mathbf{j}$ direction, and finally z in the $+\mathbf{k}$ direction to arrive at point $P(x, y, z)$.

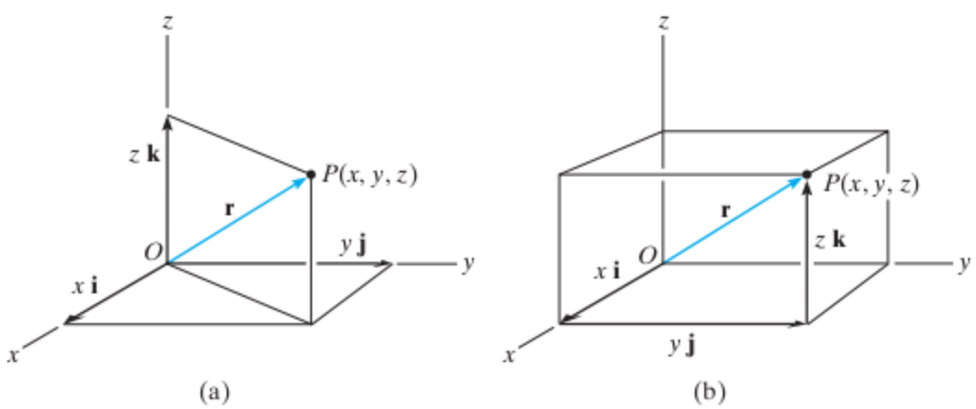


Fig. 2–33

In the more general case, the position vector may be directed from point A to point B in space, Fig. 2–34*a*. This vector is also designated by the symbol $\mathbf{r}$. As a matter of convention, we will *sometimes* refer to this vector with *two subscripts* to indicate from and to the point where it is directed. Thus, $\mathbf{r}$ can also be designated as $\mathbf{r}_{AB}$. Also, note that $\mathbf{r}_A$ and $\mathbf{r}_B$ in Fig. 2–34*a* are referenced with only one subscript since they extend from the origin of coordinates.

From Fig. 2–34*a*, by the head-to-tail vector addition, using the triangle rule, we require

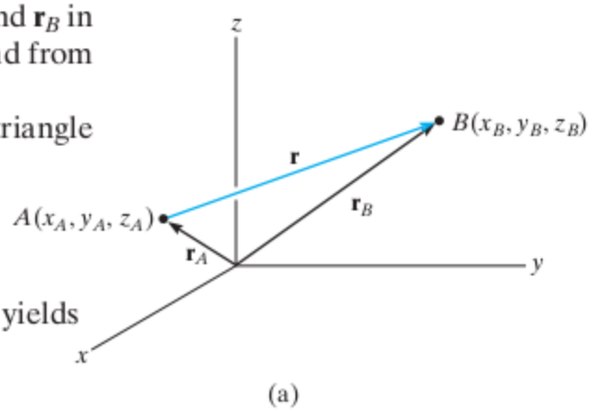
$$\mathbf{r}_A + \mathbf{r} = \mathbf{r}_B$$

Solving for $\mathbf{r}$ and expressing $\mathbf{r}_A$ and $\mathbf{r}_B$ in Cartesian vector form yields

$$\mathbf{r} = \mathbf{r}_B - \mathbf{r}_A = (x_B\mathbf{i} + y_B\mathbf{j} + z_B\mathbf{k}) - (x_A\mathbf{i} + y_A\mathbf{j} + z_A\mathbf{k})$$

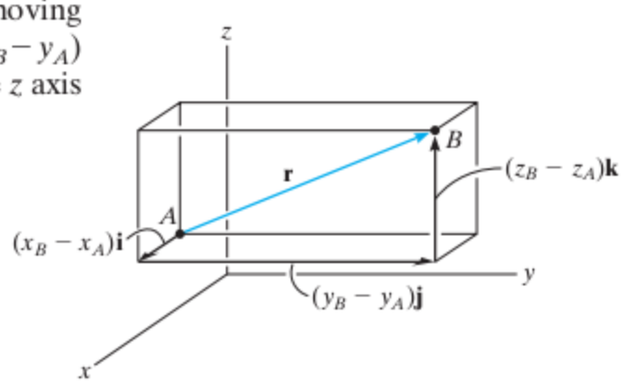
or

$$\mathbf{r} = (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k} \tag{2–11}$$



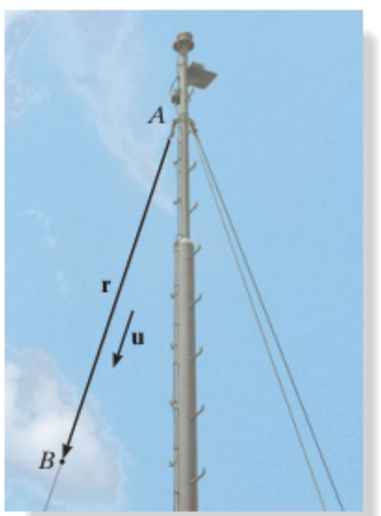
(a)

Thus, the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ *components of the position vector* $\mathbf{r}$ *may be formed by taking the coordinates of the tail of the vector* $A(x_A, y_A, z_A)$ *and subtracting them from the corresponding coordinates of the head* $B(x_B, y_B, z_B)$. We can also form these components *directly*, Fig. 2–34*b*, by starting at A and moving through a distance of $(x_B - x_A)$ along the positive x axis $(+\mathbf{i})$, then $(y_B - y_A)$ along the positive y axis $(+\mathbf{j})$, and finally $(z_B - z_A)$ along the positive z axis $(+\mathbf{k})$ to get to B.



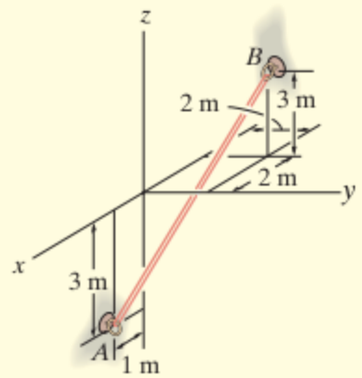
(b)

Fig. 2–34

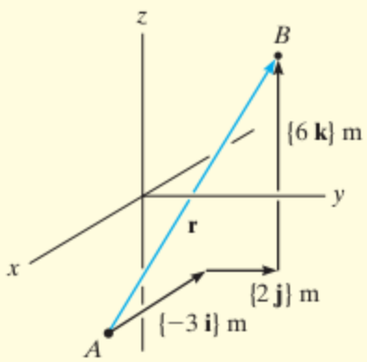


If an x, y, z coordinate system is established, then the coordinates of two points A and B on the cable can be determined. From this the position vector $\mathbf{r}$ acting along the cable can be formulated. Its magnitude represents the distance from A to B, and its unit vector, $\mathbf{u} = \mathbf{r}/r$, gives the direction defined by α, β, γ.

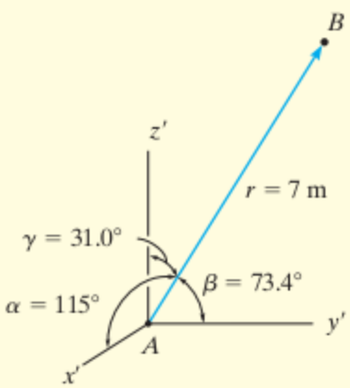
EXAMPLE 2.10



(a)



(b)



(c)

Fig. 2–35

An elastic rubber band is attached to points A and B as shown in Fig. 2–35*a*. Determine its length and its direction measured from A toward B.

SOLUTION

We first establish a position vector from A to B, Fig. 2–35*b*. In accordance with Eq. 2–11, the coordinates of the tail $A(1\text{ m}, 0, -3\text{ m})$ are subtracted from the coordinates of the head $B(-2\text{ m}, 2\text{ m}, 3\text{ m})$, which yields

$$\begin{aligned}\mathbf{r} &= [-2\text{ m} - 1\text{ m}]\mathbf{i} + [2\text{ m} - 0]\mathbf{j} + [3\text{ m} - (-3\text{ m})]\mathbf{k}\\ &= \{-3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}\}\text{ m}\end{aligned}$$

These components of $\mathbf{r}$ can also be determined *directly* by realizing that they represent the direction and distance one must travel along each axis in order to move from A to B, i.e., along the x axis $\{-3\mathbf{i}\}$ m, along the y axis $\{2\mathbf{j}\}$ m, and finally along the z axis $\{6\mathbf{k}\}$ m.

The length of the rubber band is therefore

$$r = \sqrt{(-3\text{ m})^2 + (2\text{ m})^2 + (6\text{ m})^2} = 7\text{ m} \qquad \textit{Ans.}$$

Formulating a unit vector in the direction of $\mathbf{r}$, we have

$$\mathbf{u} = \frac{\mathbf{r}}{r} = -\frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$$

The components of this unit vector give the coordinate direction angles

$$\alpha = \cos^{-1}\left(-\frac{3}{7}\right) = 115^\circ \qquad \textit{Ans.}$$

$$\beta = \cos^{-1}\left(\frac{2}{7}\right) = 73.4^\circ \qquad \textit{Ans.}$$

$$\gamma = \cos^{-1}\left(\frac{6}{7}\right) = 31.0^\circ \qquad \textit{Ans.}$$

NOTE: These angles are measured from the *positive axes* of a localized coordinate system placed at the tail of $\mathbf{r}$, as shown in Fig. 2–35*c*.

2.8 Force Vector Directed Along a Line

Quite often in three-dimensional statics problems, the direction of a force is specified by two points through which its line of action passes. Such a situation is shown in Fig. 2–36, where the force **F** is directed along the cord *AB*. We can formulate **F** as a Cartesian vector by realizing that it has the *same direction* and *sense* as the position vector **r** directed from point *A* to point *B* on the cord. This common direction is specified by the *unit vector* $\mathbf{u} = \mathbf{r}/r$. Hence,

$$\mathbf{F} = F\mathbf{u} = F\left(\frac{\mathbf{r}}{r}\right) = F\left(\frac{(x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}\right)$$

Although we have represented **F** symbolically in Fig. 2–36, note that it has *units of force*, unlike **r**, which has units of length.

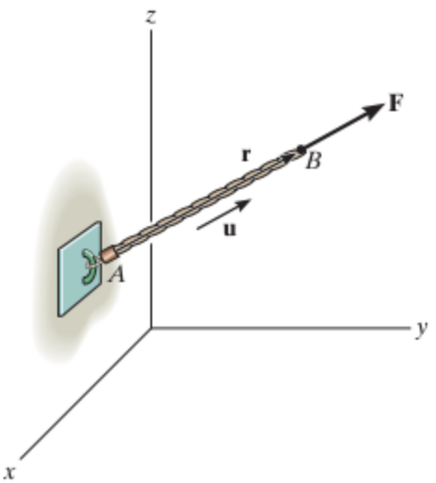


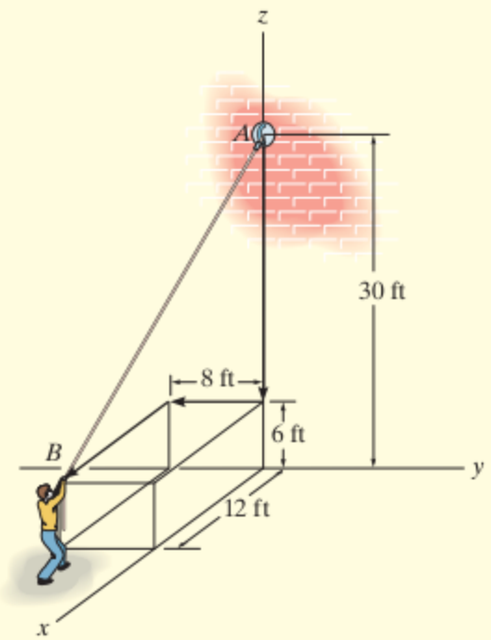
Fig. 2–36

The force **F** acting along the rope can be represented as a Cartesian vector by establishing *x*, *y*, *z* axes and first forming a position vector **r** along the length of the rope. Then the corresponding unit vector $\mathbf{u} = \mathbf{r}/r$ that defines the direction of both the rope and the force can be determined. Finally, the magnitude of the force is combined with its direction, $\mathbf{F} = F\mathbf{u}$.

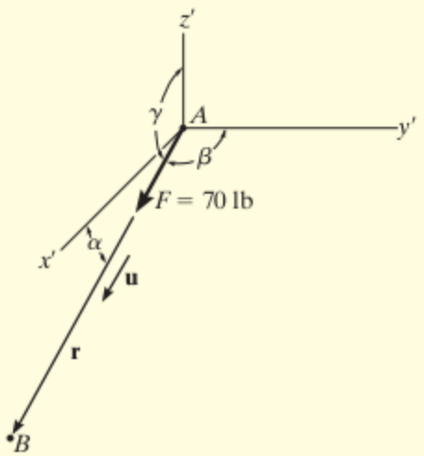
Important Points

- A position vector locates one point in space relative to another point.
- The easiest way to formulate the components of a position vector is to determine the distance and direction that must be traveled along the *x*, *y*, *z* directions—going from the tail to the head of the vector.
- A force **F** acting in the direction of a position vector **r** can be represented in Cartesian form if the unit vector **u** of the position vector is determined and it is multiplied by the magnitude of the force, i.e., $\mathbf{F} = F\mathbf{u} = F(\mathbf{r}/r)$.

EXAMPLE 2.11



(a)



(b)

Fig. 2–37

The man shown in Fig. 2–37*a* pulls on the cord with a force of 70 lb. Represent this force acting on the support *A* as a Cartesian vector and determine its direction.

SOLUTION

Force **F** is shown in Fig. 2–37*b*. The *direction* of this vector, **u**, is determined from the position vector **r**, which extends from *A* to *B*. Rather than using the coordinates of the end points of the cord, **r** can be determined *directly* by noting in Fig. 2–37*a* that one must travel from *A* $\{-24\mathbf{k}\}$ ft, then $\{-8\mathbf{j}\}$ ft, and finally $\{12\mathbf{i}\}$ ft to get to *B*. Thus,

$$\mathbf{r} = \{12\mathbf{i} - 8\mathbf{j} - 24\mathbf{k}\}\ \text{ft}$$

The magnitude of **r**, which represents the *length* of cord *AB*, is

$$r = \sqrt{(12\ \text{ft})^2 + (-8\ \text{ft})^2 + (-24\ \text{ft})^2} = 28\ \text{ft}$$

Forming the unit vector that defines the direction and sense of both **r** and **F**, we have

$$\mathbf{u} = \frac{\mathbf{r}}{r} = \frac{12}{28}\mathbf{i} - \frac{8}{28}\mathbf{j} - \frac{24}{28}\mathbf{k}$$

Since **F**, has a *magnitude* of 70 lb and a *direction* specified by **u**, then

$$\mathbf{F} = F\mathbf{u} = 70\ \text{lb}\left(\frac{12}{28}\mathbf{i} - \frac{8}{28}\mathbf{j} - \frac{24}{28}\mathbf{k}\right)$$

$$= \{30\mathbf{i} - 20\mathbf{j} - 60\mathbf{k}\}\ \text{lb} \qquad \textit{Ans.}$$

The coordinate direction angles are measured between **r** (or **F**) and the *positive axes* of a localized coordinate system with origin placed at *A*, Fig. 2–37*b*. From the components of the unit vector:

$$\alpha = \cos^{-1}\left(\frac{12}{28}\right) = 64.6^\circ \qquad \textit{Ans.}$$

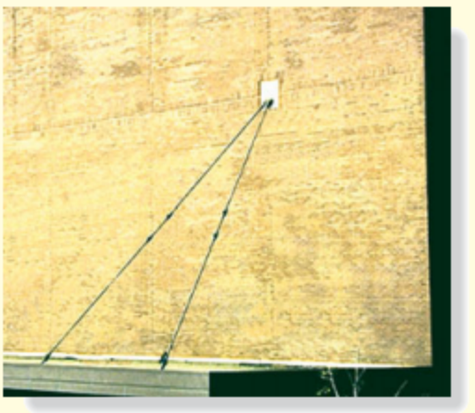
$$\beta = \cos^{-1}\left(\frac{-8}{28}\right) = 107^\circ \qquad \textit{Ans.}$$

$$\gamma = \cos^{-1}\left(\frac{-24}{28}\right) = 149^\circ \qquad \textit{Ans.}$$

NOTE: These results make sense when compared with the angles identified in Fig. 2–37*b*.

EXAMPLE 2.12

The roof is supported by cables as shown in the photo. If the cables exert forces $F_{AB} = 100$ N and $F_{AC} = 120$ N on the wall hook at A as shown in Fig. 2–38*a*, determine the resultant force acting at A. Express the result as a Cartesian vector.



SOLUTION

The resultant force $\mathbf{F}_R$ is shown graphically in Fig. 2–38*b*. We can express this force as a Cartesian vector by first formulating $\mathbf{F}_{AB}$ and $\mathbf{F}_{AC}$ as Cartesian vectors and then adding their components. The directions of $\mathbf{F}_{AB}$ and $\mathbf{F}_{AC}$ are specified by forming unit vectors $\mathbf{u}_{AB}$ and $\mathbf{u}_{AC}$ along the cables. These unit vectors are obtained from the associated position vectors $\mathbf{r}_{AB}$ and $\mathbf{r}_{AC}$. With reference to Fig. 2–38*a*, to go from A to B, we must travel $\{-4\mathbf{k}\}$ m and, then $\{-4\mathbf{i}\}$ m. Thus,

$$\mathbf{r}_{AB} = \{4\mathbf{i} - 4\mathbf{k}\}\ \text{m}$$

$$r_{AB} = \sqrt{(4\ \text{m})^2 + (-4\ \text{m})^2} = 5.66\ \text{m}$$

$$\mathbf{F}_{AB} = F_{AB}\left(\frac{\mathbf{r}_{AB}}{r_{AB}}\right) = (100\ \text{N})\left(\frac{4}{5.66}\mathbf{i} - \frac{4}{5.66}\mathbf{k}\right)$$

$$\mathbf{F}_{AB} = \{70.7\mathbf{i} - 70.7\mathbf{k}\}\ \text{N}$$

To go from A to C, we must travel $\{-4\mathbf{k}\}$ m, then $\{2\mathbf{j}\}$ m, and finally $\{4\mathbf{j}\}$. Thus,

$$\mathbf{r}_{AC} = \{4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}\}\ \text{m}$$

$$r_{AC} = \sqrt{(4\ \text{m})^2 + (2\ \text{m})^2 + (-4\ \text{m})^2} = 6\ \text{m}$$

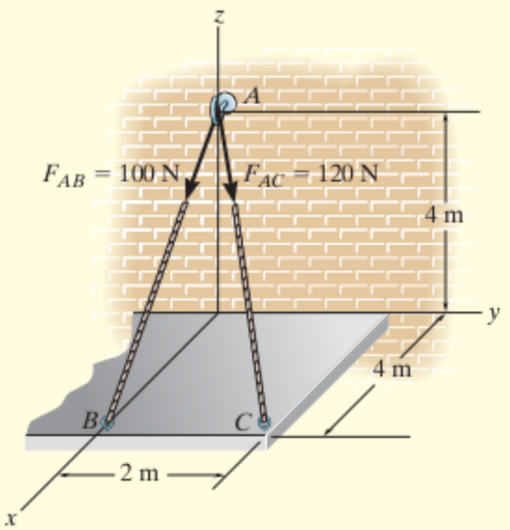
$$\mathbf{F}_{AC} = F_{AC}\left(\frac{\mathbf{r}_{AC}}{r_{AC}}\right) = (120\ \text{N})\left(\frac{4}{6}\mathbf{i} + \frac{2}{6}\mathbf{j} - \frac{4}{6}\mathbf{k}\right)$$

$$= \{80\mathbf{i} + 40\mathbf{j} - 80\mathbf{k}\}\ \text{N}$$

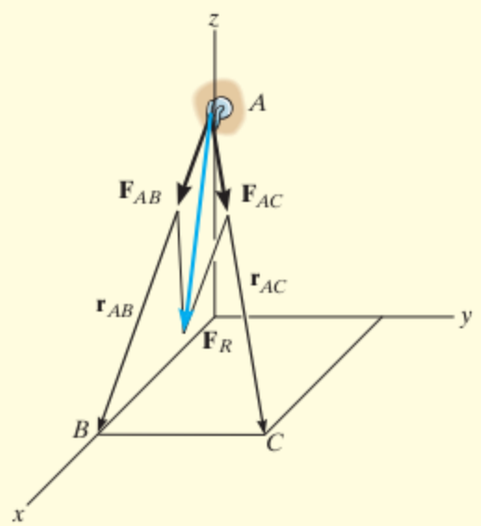
The resultant force is therefore

$$\mathbf{F}_R = \mathbf{F}_{AB} + \mathbf{F}_{AC} = \{70.7\mathbf{i} - 70.7\mathbf{k}\}\ \text{N} + \{80\mathbf{i} + 40\mathbf{j} - 80\mathbf{k}\}\ \text{N}$$

$$= \{151\mathbf{i} + 40\mathbf{j} - 151\mathbf{k}\}\ \text{N} \qquad \textit{Ans.}$$



(a)

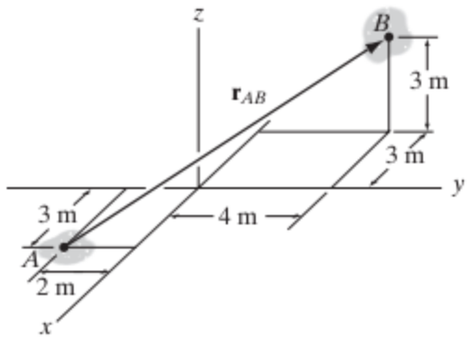


(b)

Fig. 2–38

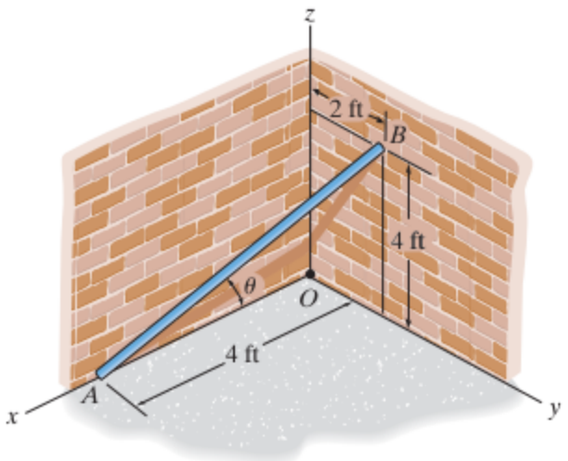
FUNDAMENTAL PROBLEMS

F2–19. Express the position vector $\mathbf{r}_{AB}$ in Cartesian vector form, then determine its magnitude and coordinate direction angles.



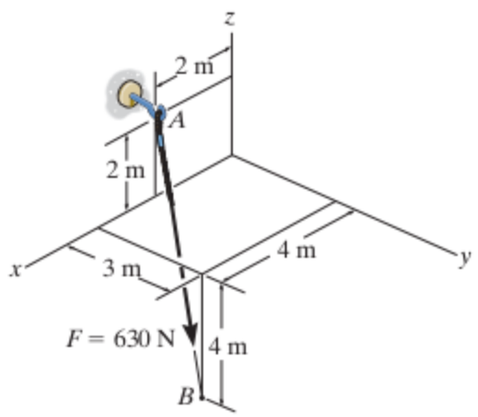
F2–19

F2–20. Determine the length of the rod and the position vector directed from A to B. What is the angle θ?



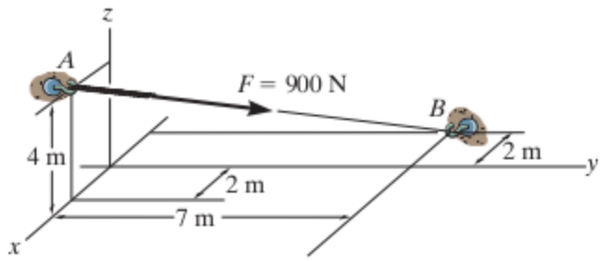
F2–20

F2–21. Express the force as a Cartesian vector.



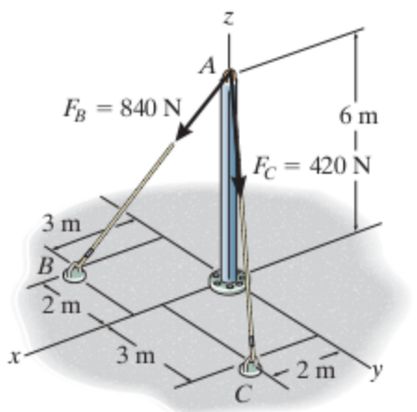
F2–21

F2–22. Express the force as a Cartesian vector.



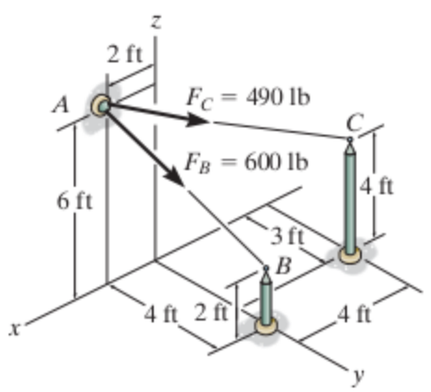
F2–22

F2–23. Determine the magnitude of the resultant force at A.



F2–23

F2–24. Determine the resultant force at A.

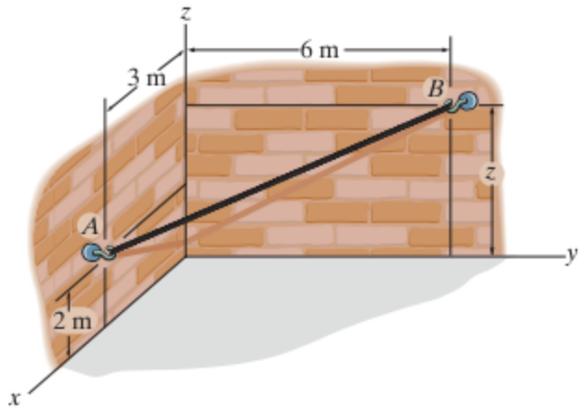


F2–24

PROBLEMS

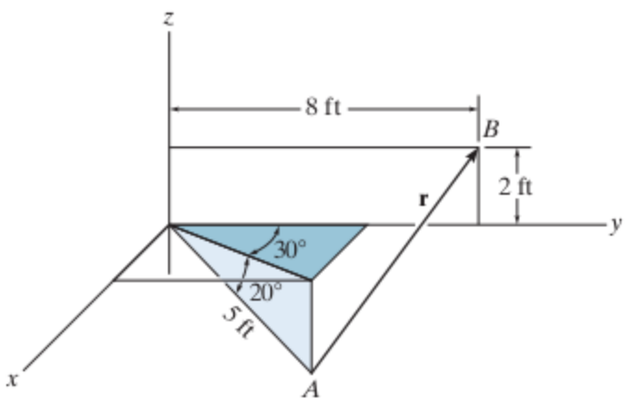
2–62. Determine the position vector **r** directed from point A to point B and the length of cord AB. Take $z = 4$ m.

2–63. If the cord AB is 7.5 m long, determine the coordinate position $+z$ of point B.



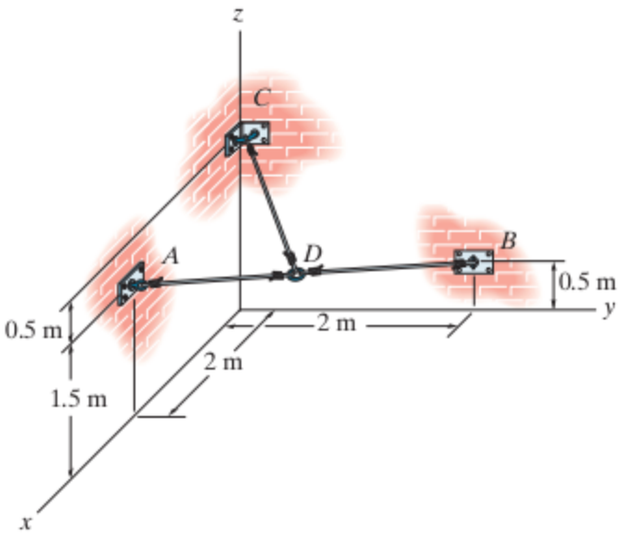
Probs. 2–62/63

*2–64. Express the position vector **r** in Cartesian vector form; then determine its magnitude and coordinate direction angles.



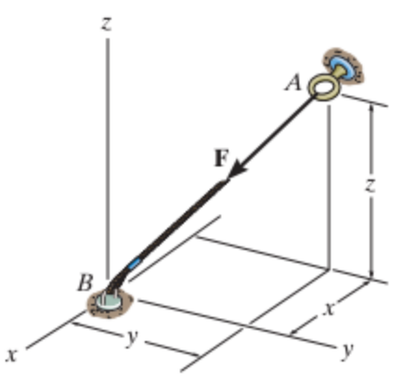
Prob. 2–64

2–65. Determine the lengths of wires AD, BD, and CD. The ring at D is midway between A and B.



Prob. 2–65

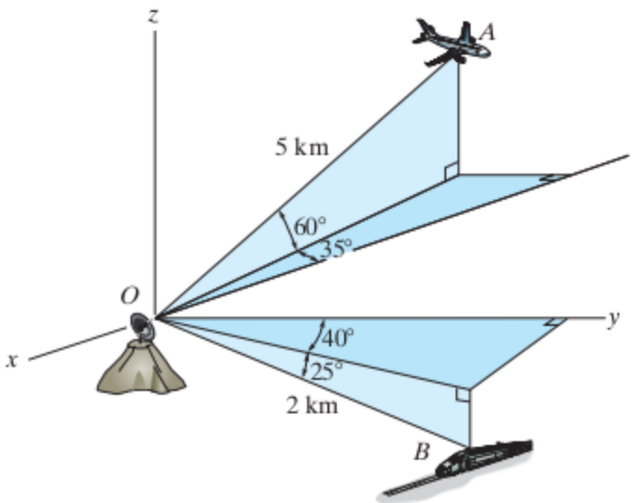
2–66. If $\mathbf{F} = \{350\mathbf{i} - 250\mathbf{j} - 450\mathbf{k}\}$ N and cable AB is 9 m long, determine the x, y, z coordinates of point A.



Prob. 2–66

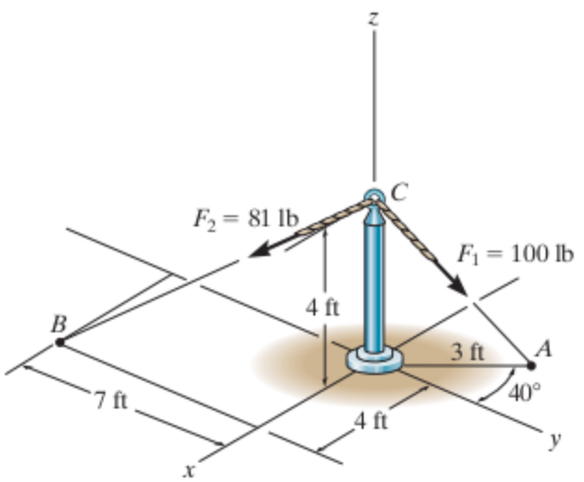
2

2–67. At a given instant, the position of a plane at A and a train at B are measured relative to a radar antenna at O. Determine the distance d between A and B at this instant. To solve the problem, formulate a position vector, directed from A to B, and then determine its magnitude.



Prob. 2–67

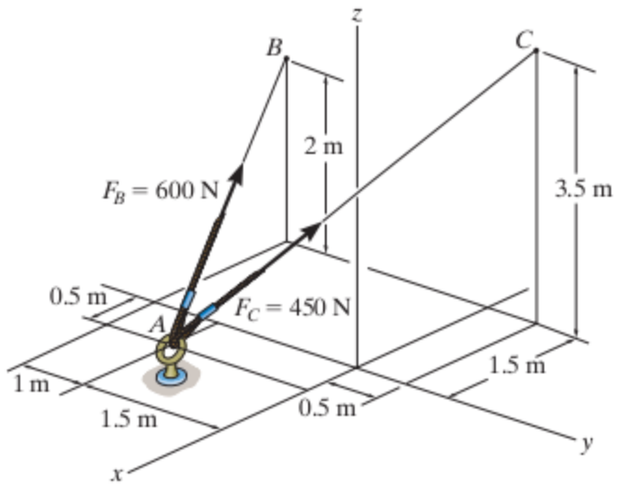
***2–68.** Determine the magnitude and coordinate direction angles of the resultant force.



Prob. 2–68

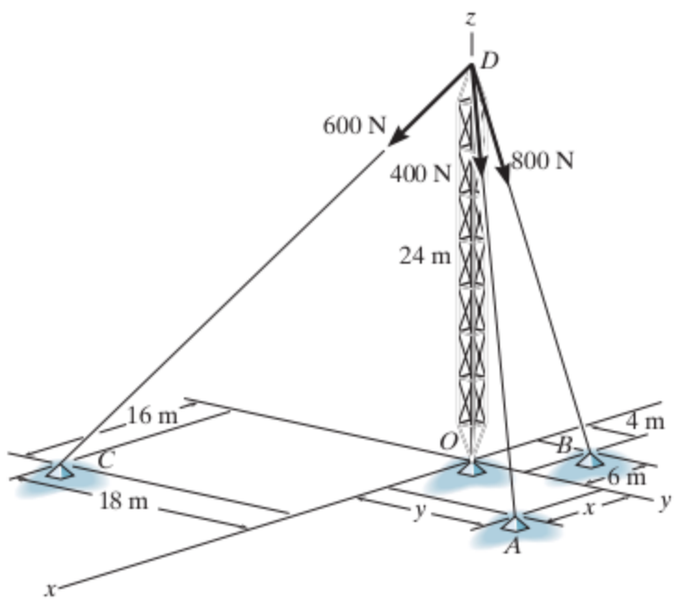
2–69. Express $\mathbf{F}_B$ and $\mathbf{F}_C$ in Cartesian vector form.

2–70. Determine the magnitude and coordinate direction angles of the resultant force acting at A.



Probs. 2–69/70

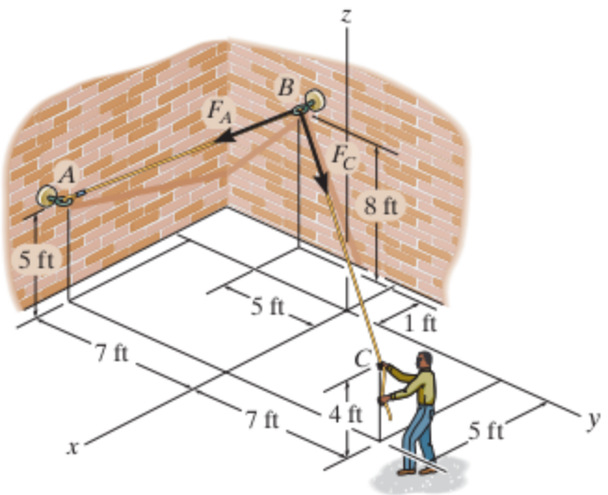
2–71. The tower is held in place by three cables. If the force of each cable acting on the tower is shown, determine the magnitude and coordinate direction angles α, β, γ of the resultant force. Take $x = 20$ m, $y = 15$ m.



Prob. 2–71

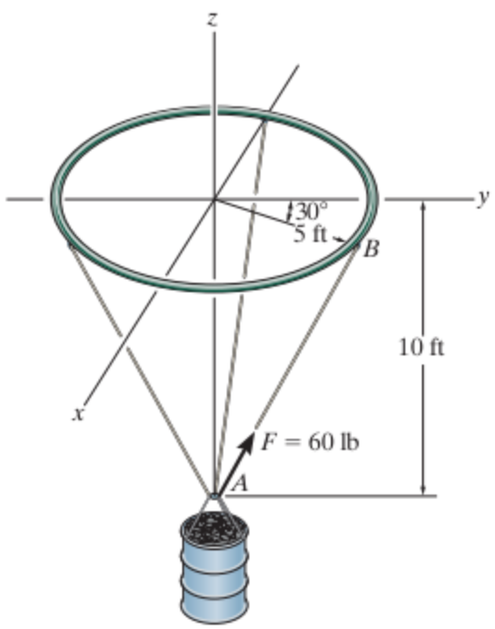
*2–72. The man pulls on the rope at C with a force of 70 lb which causes the forces $\mathbf{F}_A$ and $\mathbf{F}_C$ at B to have this same magnitude. Express each of these two forces as Cartesian vectors.

2–73. The man pulls on the rope at C with a force of 70 lb which causes the forces $\mathbf{F}_A$ and $\mathbf{F}_C$ at B to have this same magnitude. Determine the magnitude and coordinate direction angles of the resultant force acting at B.



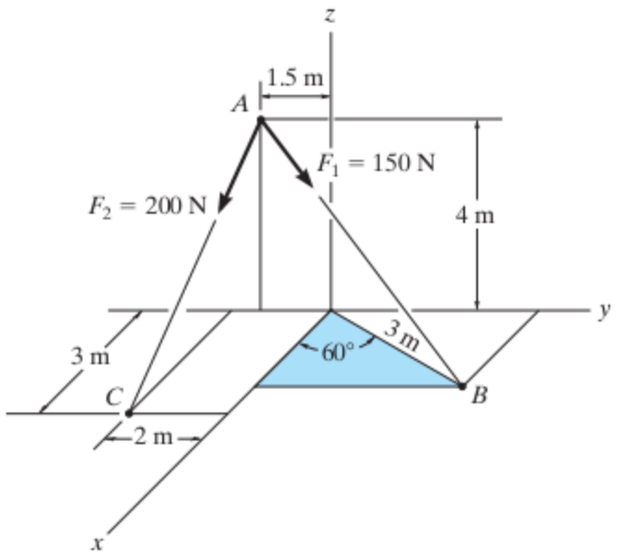
Probs. 2–72/73

2–74. The load at A creates a force of 60 lb in wire AB. Express this force as a Cartesian vector acting on A and directed toward B as shown.



Prob. 2–74

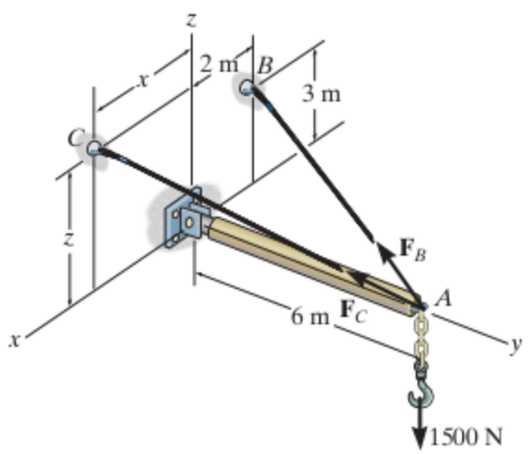
2–75. Determine the magnitude and coordinate direction angles of the resultant force acting at point A.



Prob. 2–75

*2–76. Two cables are used to secure the overhang boom in position and support the 1500-N load. If the resultant force is directed along the boom from point A towards O, determine the magnitudes of the resultant force and forces $\mathbf{F}_B$ and $\mathbf{F}_C$. Set $x = 3$ m and $z = 2$ m.

2–77. Two cables are used to secure the overhang boom in position and support the 1500-N load. If the resultant force is directed along the boom from point A towards O, determine the values of x and z for the coordinates of point C and the magnitude of the resultant force. Set $F_B = 1610$ N and $F_C = 2400$ N.



Probs. 2–76/77

2.9 Dot Product

Occasionally in statics one has to find the angle between two lines or the components of a force parallel and perpendicular to a line. In two dimensions, these problems can readily be solved by trigonometry since the geometry is easy to visualize. In three dimensions, however, this is often difficult, and consequently vector methods should be employed for the solution. The dot product, which defines a particular method for "multiplying" two vectors, will be used to solve the above-mentioned problems.

The *dot product* of vectors **A** and **B**, written **A** · **B**, and read "**A** dot **B**" is defined as the product of the magnitudes of **A** and **B** and the cosine of the angle θ between their tails, Fig. 2–39. Expressed in equation form,

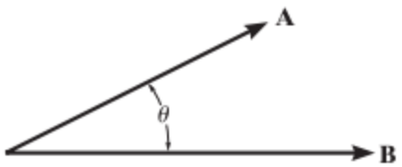


Fig. 2–39

$$\mathbf{A} \cdot \mathbf{B} = AB \cos\theta \tag{2–12}$$

where $0° \leq \theta \leq 180°$. The dot product is often referred to as the *scalar product* of vectors since the result is a *scalar* and not a vector.

Laws of Operation.

1. Commutative law: $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$
2. Multiplication by a scalar: $a(\mathbf{A} \cdot \mathbf{B}) = (a\mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (a\mathbf{B})$
3. Distributive law: $\mathbf{A} \cdot (\mathbf{B} + \mathbf{D}) = (\mathbf{A} \cdot \mathbf{B}) + (\mathbf{A} \cdot \mathbf{D})$

It is easy to prove the first and second laws by using Eq. 2–12. The proof of the distributive law is left as an exercise (see Prob. 2–78).

Cartesian Vector Formulation. Equation 2–12 must be used to find the dot product for any two Cartesian unit vectors. For example, $\mathbf{i} \cdot \mathbf{i} = (1)(1)\cos 0° = 1$ and $\mathbf{i} \cdot \mathbf{j} = (1)(1)\cos 90° = 0$. If we want to find the dot product of two general vectors **A** and **B** that are expressed in Cartesian vector form, then we have

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \cdot (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \\ &= A_xB_x\,(\mathbf{i} \cdot \mathbf{i}) + A_xB_y\,(\mathbf{i} \cdot \mathbf{j}) + A_xB_z\,(\mathbf{i} \cdot \mathbf{k}) \\ &\quad + A_yB_x\,(\mathbf{j} \cdot \mathbf{i}) + (A_yB_y\,(\mathbf{j} \cdot \mathbf{j}) + A_yB_z\,(\mathbf{j} \cdot \mathbf{k}) \\ &\quad + A_zB_x\,(\mathbf{k} \cdot \mathbf{i}) + A_zB_y\,(\mathbf{k} \cdot \mathbf{j}) + A_zB_z\,(\mathbf{k} \cdot \mathbf{k})\end{aligned}$$

Carrying out the dot-product operations, the final result becomes

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z \tag{2–13}$$

Thus, to determine the dot product of two Cartesian vectors, multiply their corresponding x, y, z components and sum these products algebraically. Note that the result will be either a positive or negative *scalar*.

Applications. The dot product has two important applications in mechanics.

- ***The angle formed between two vectors or intersecting lines.*** The angle θ between the tails of vectors **A** and **B** in Fig. 2–39 can be determined from Eq. 2–12 and written as

$$\theta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{AB}\right) \qquad 0^\circ \le \theta \le 180^\circ$$

  Here $\mathbf{A}\cdot\mathbf{B}$ is found from Eq. 2–13. In particular, notice that if $\mathbf{A}\cdot\mathbf{B} = 0, \theta = \cos^{-1} 0 = 90^\circ$ so that **A** will be *perpendicular* to **B**.

- ***The components of a vector parallel and perpendicular to a line.*** The component of vector **A** parallel to or collinear with the line *aa'* in Fig. 2–40 is defined by A_a where $A_a = A\cos\theta$. This component is sometimes referred to as the *projection* of **A** onto the line, since a *right angle* is formed in the construction. If the *direction* of the line is specified by the unit vector $\mathbf{u}_a$, and since $u_a = 1$, we can determine the magnitude of A_a directly from the dot product (Eq. 2–12); i.e.,

$$A_a = A\cos\theta = \mathbf{A}\cdot\mathbf{u}_a$$

  Hence, the scalar projection of **A** *along a line is determined from the dot product of* **A** *and the unit vector* $\mathbf{u}_a$ *which defines the direction of the line.* Notice that if this result is positive, then $\mathbf{A}_a$ has a directional sense which is the same as $\mathbf{u}_a$, whereas if A_a is a negative scalar, then $\mathbf{A}_a$ has the opposite sense of direction to $\mathbf{u}_a$

  The component $\mathbf{A}_a$ represented as a *vector* is therefore

$$\mathbf{A}_a = A_a\,\mathbf{u}_a$$

  The component of **A** that is perpendicular to line *aa* can also be obtained, Fig. 2–40. Since $\mathbf{A} = \mathbf{A}_a + \mathbf{A}_\perp$, then $\mathbf{A}_\perp = \mathbf{A} - \mathbf{A}_a$. There are two possible ways of obtaining $A_\perp$. One way would be to determine θ from the dot product, $\theta = \cos^{-1}(\mathbf{A}\cdot\mathbf{u}_A/A)$, then $A_\perp = A\sin\theta$. Alternatively, if A_a is known, then by Pythagorean's theorem we can also write $A_\perp = \sqrt{A^2 - A_a^2}$.

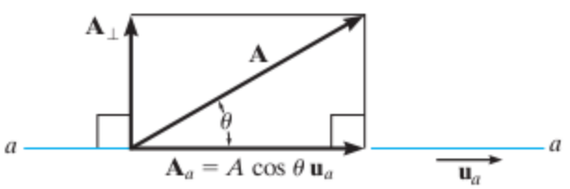
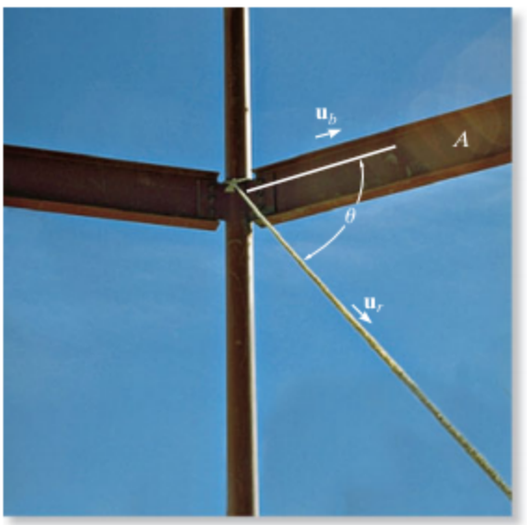
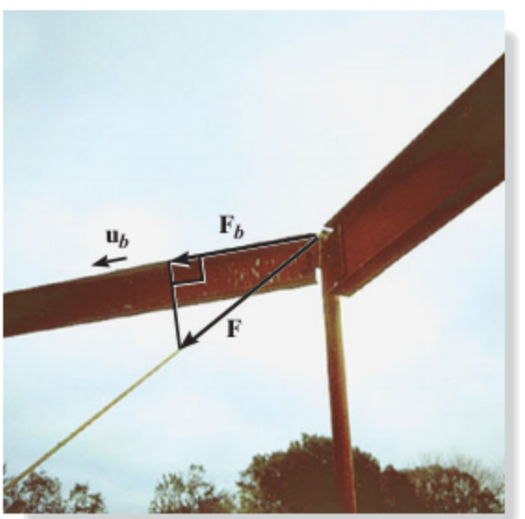


Fig. 2–40



The angle θ between the rope and the connecting beam can be determined by formulating unit vectors along the beam and rope and then using the dot product $\mathbf{u}_b\cdot\mathbf{u}_r = (1)(1)\cos\theta$.



The projection of the cable force **F** along the beam can be determined by first finding the unit vector $\mathbf{u}_b$ that defines this direction. Then apply the dot product, $F_b = \mathbf{F}\cdot\mathbf{u}_b$.

2

Important Points

- The dot product is used to determine the angle between two vectors or the projection of a vector in a specified direction.
- If vectors **A** and **B** are expressed in Cartesian vector form, the dot product is determined by multiplying the respective x, y, z scalar components and algebraically adding the results, i.e., $\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$.
- From the definition of the dot product, the angle formed between the tails of vectors **A** and **B** is $\theta = \cos^{-1}(\mathbf{A} \cdot \mathbf{B}/AB)$.
- The magnitude of the projection of vector **A** along a line aa whose direction is specified by $\mathbf{u}_a$ is determined from the dot product $A_a = \mathbf{A} \cdot \mathbf{u}_a$.

EXAMPLE 2.13

Determine the magnitudes of the projection of the force **F** in Fig. 2–41 onto the u and v axes.

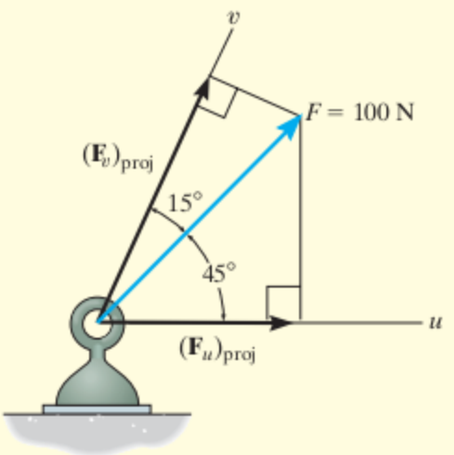


Fig. 2–41

SOLUTION

Projections of Force. The graphical representation of the *projections* is shown in Fig. 2–41. From this figure, the magnitudes of the projections of **F** onto the u and v axes can be obtained by trigonometry:

$$(F_u)_{\text{proj}} = (100\ \text{N})\cos 45° = 70.7\ \text{N} \quad \textit{Ans.}$$

$$(F_v)_{\text{proj}} = (100\ \text{N})\cos 15° = 96.6\ \text{N} \quad \textit{Ans.}$$

NOTE: These projections are not equal to the magnitudes of the components of force **F** along the u and v axes found from the parallelogram law. They will only be equal if the u and v axes are *perpendicular to one another.*

EXAMPLE 2.14

The frame shown in Fig. 2–42*a* is subjected to a horizontal force $\mathbf{F} = \{300\mathbf{j}\}$. Determine the magnitude of the components of this force parallel and perpendicular to member AB.

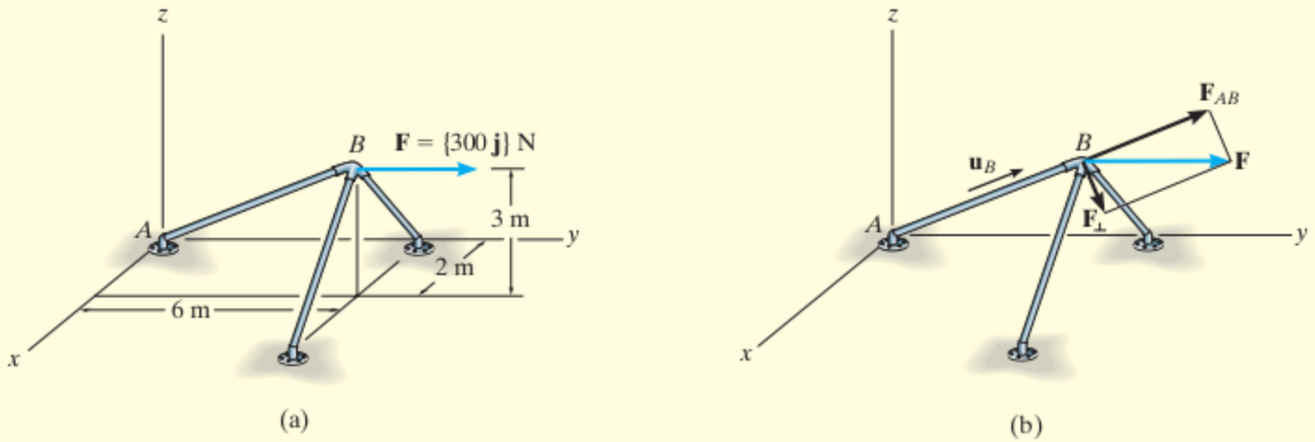


Fig. 2–42

SOLUTION

The magnitude of the component of $\mathbf{F}$ along AB is equal to the dot product of $\mathbf{F}$ and the unit vector $\mathbf{u}_B$, which defines the direction of AB, Fig. 2–42*b*. Since

$$\mathbf{u}_B = \frac{\mathbf{r}_B}{r_B} = \frac{2\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}}{\sqrt{(2)^2 + (6)^2 + (3)^2}} = 0.286\mathbf{i} + 0.857\mathbf{j} + 0.429\mathbf{k}$$

then

$$\begin{aligned} F_{AB} &= F\cos\theta = \mathbf{F}\cdot\mathbf{u}_B = (300\mathbf{j})\cdot(0.286\mathbf{i} + 0.857\mathbf{j} + 0.429\mathbf{k}) \\ &= (0)(0.286) + (300)(0.857) + (0)(0.429) \\ &= 257.1\ \text{N} \end{aligned}$$ *Ans.*

Since the result is a positive scalar, $\mathbf{F}_{AB}$ has the same sense of direction as $\mathbf{u}_B$, Fig. 2–42*b*.

Expressing $\mathbf{F}_{AB}$ in Cartesian vector form, we have

$$\begin{aligned} \mathbf{F}_{AB} &= F_{AB}\mathbf{u}_B = (257.1\ \text{N})(0.286\mathbf{i} + 0.857\mathbf{j} + 0.429\mathbf{k}) \\ &= \{73.5\mathbf{i} + 220\mathbf{j} + 110\mathbf{k}\}\ \text{N} \end{aligned}$$ *Ans.*

The perpendicular component, Fig. 2–42*b*, is therefore

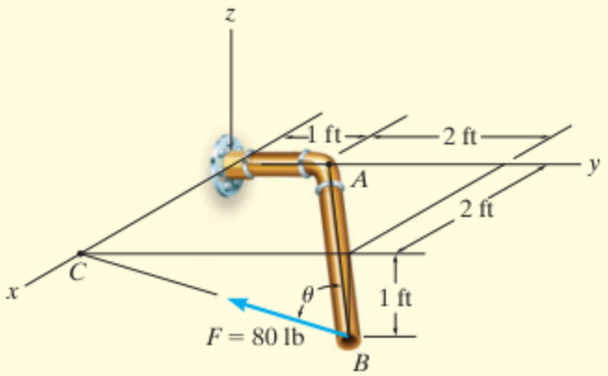
$$\begin{aligned} \mathbf{F}_\perp &= \mathbf{F} - \mathbf{F}_{AB} = 300\mathbf{j} - (73.5\mathbf{i} + 220\mathbf{j} + 110\mathbf{k}) \\ &= \{-73.5\mathbf{i} + 80\mathbf{j} - 110\mathbf{k}\}\ \text{N} \end{aligned}$$

Its magnitude can be determined either from this vector or by using the Pythagorean theorem, Fig. 2–42*b*:

$$\begin{aligned} F_\perp &= \sqrt{F^2 - F_{AB}^2} = \sqrt{(300\ \text{N})^2 - (257.1\ \text{N})^2} \\ &= 155\ \text{N} \end{aligned}$$ *Ans.*

EXAMPLE 2.15

The pipe in Fig. 2–43*a* is subjected to the force of $F = 80$ lb. Determine the angle θ between **F** and the pipe segment *BA* and the projection of **F** along this segment.



(a)

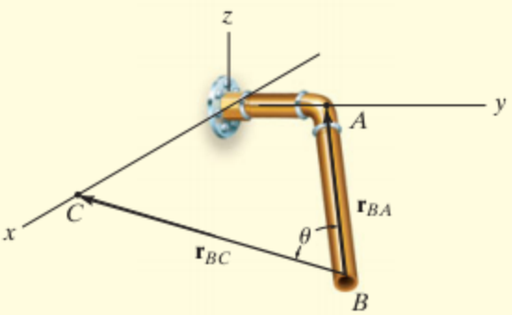
SOLUTION

Angle θ. First we will establish position vectors from *B* to *A* and *B* to *C*; Fig. 2–43*b*. Then we will determine the angle θ between the tails of these two vectors.

$$\mathbf{r}_{BA} = \{-2\mathbf{i} - 2\mathbf{j} + 1\mathbf{k}\}\ \text{ft},\quad r_{BA} = 3\ \text{ft}$$
$$\mathbf{r}_{BC} = \{-3\mathbf{j} + 1\mathbf{k}\}\ \text{ft},\quad r_{BC} = \sqrt{10}\ \text{ft}$$

Thus,

$$\cos\theta = \frac{\mathbf{r}_{BA}\cdot\mathbf{r}_{BC}}{\mathbf{r}_{BA}\mathbf{r}_{BC}} = \frac{(-2)(0) + (-2)(-3) + (1)(1)}{3\sqrt{10}} = 0.7379$$
$$\theta = 42.5° \qquad \textit{Ans.}$$



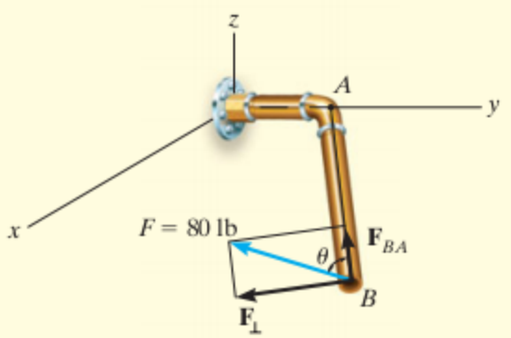
(b)

Components of F. The component of **F** along *BA* is shown in Fig. 2–43*b*. We must first formulate the unit vector along *BA* and force **F** as Cartesian vectors.

$$\mathbf{u}_{BA} = \frac{\mathbf{r}_{BA}}{r_{BA}} = \frac{(-2\mathbf{i} - 2\mathbf{j} + 1\mathbf{k})}{3} = -\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$$
$$\mathbf{F} = 80\ \text{lb}\left(\frac{\mathbf{r}_{BC}}{r_{BC}}\right) = 80\left(\frac{-3\mathbf{j} + 1\mathbf{k}}{\sqrt{10}}\right) = -75.89\mathbf{j} + 25.30\mathbf{k}$$

Thus,

$$F_{BA} = \mathbf{F}\cdot\mathbf{u}_{BA} = (-75.89\mathbf{j} + 25.30\mathbf{k})\cdot\left(-\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}\right)$$
$$= 0\left(-\frac{2}{3}\right) + (-75.89)\left(-\frac{2}{3}\right) + (25.30)\left(\frac{1}{3}\right)$$
$$= 59.0\ \text{lb} \qquad \textit{Ans.}$$



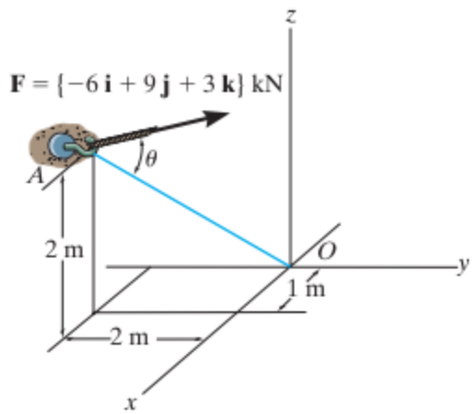
(c)

Fig. 2–43

NOTE: Since θ is known, then also, $F_{BA} = F\cos\theta = 80\ \text{lb}\cos 42.5° = 59.0\ \text{lb}$.

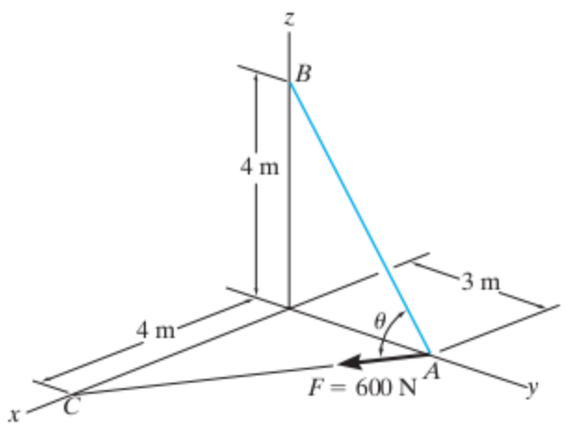
FUNDAMENTAL PROBLEMS

F2–25. Determine the angle θ between the force and the line AO.



F2–25

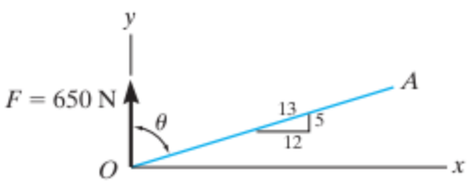
F2–26. Determine the angle θ between the force and the line AB.



F2–26

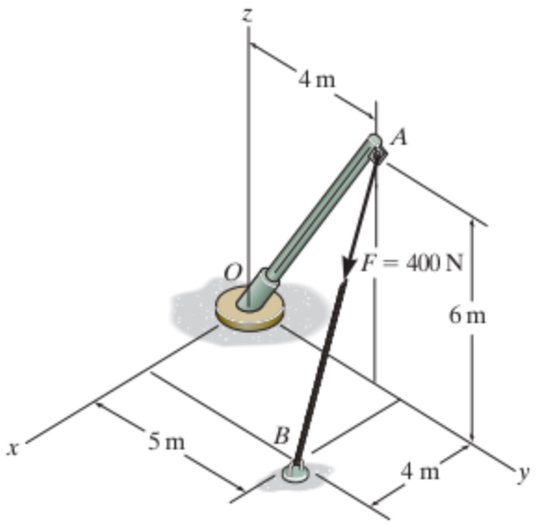
F2–27. Determine the angle θ between the force and the line OA.

F2–28. Determine the component of projection of the force along the line OA.



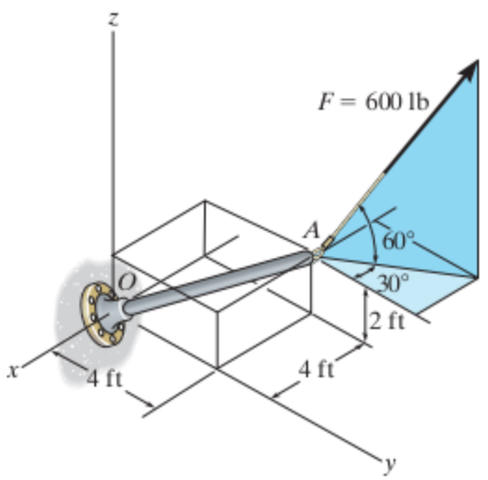
F2–27/28

F2–29. Find the magnitude of the projected component of the force along the pipe.



F2–29

F2–30. Determine the components of the force acting parallel and perpendicular to the axis of the pole.

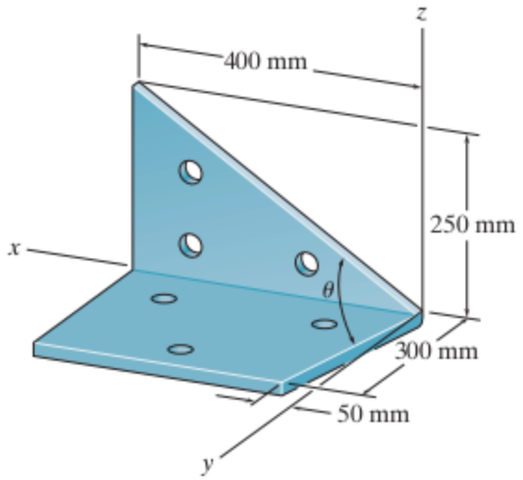


F2–30

PROBLEMS

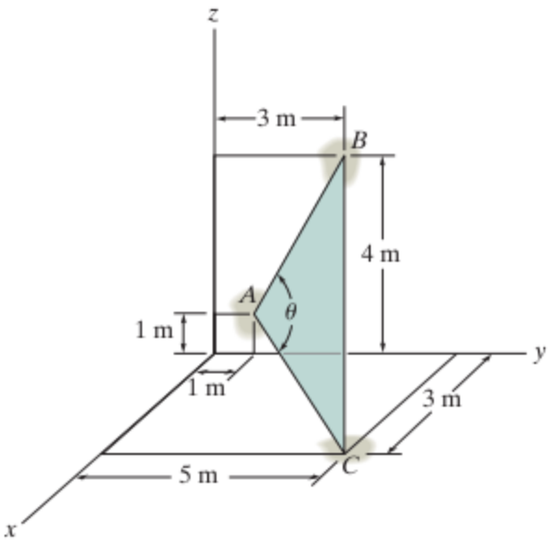
2–78. Given the three vectors **A**, **B**, and **D**, show that $\mathbf{A}\cdot(\mathbf{B}+\mathbf{D}) = (\mathbf{A}\cdot\mathbf{B}) + (\mathbf{A}\cdot\mathbf{D})$.

2–79. Determine the angle θ between the edges of the sheet-metal bracket.



Probs. 2–78/79

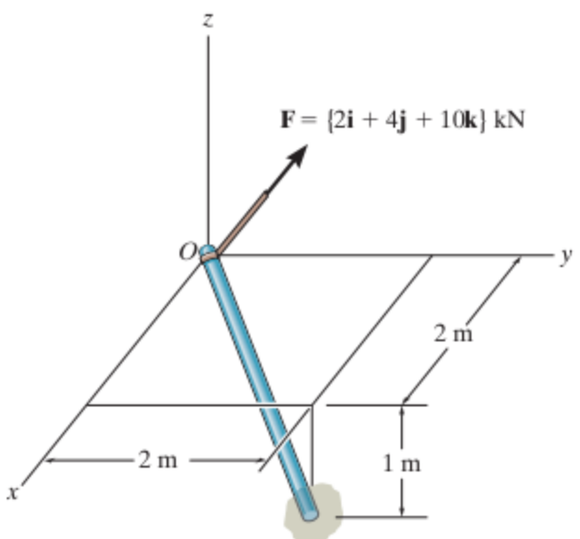
2–81. Determine the length of side BC of the triangular plate. Solve the problem by finding the magnitude of $\mathbf{r}_{BC}$; then check the result by first finding θ, r_{AB}, and r_{AC} and then using the cosine law.



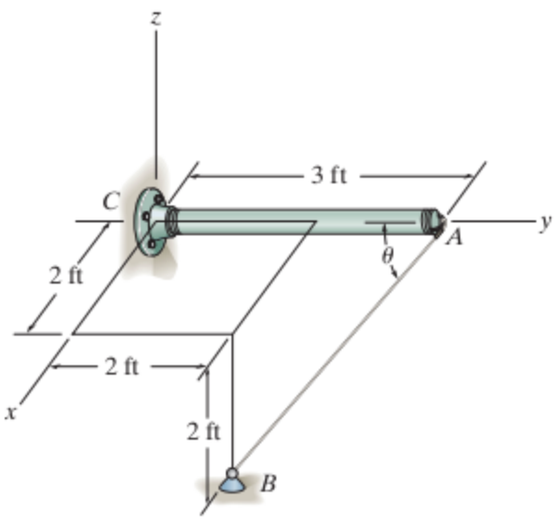
Prob. 2–81

2–82. Determine the angle θ between the y axis of the pole and the wire AB.

***2–80.** Determine the projection of the force **F** along the pole.



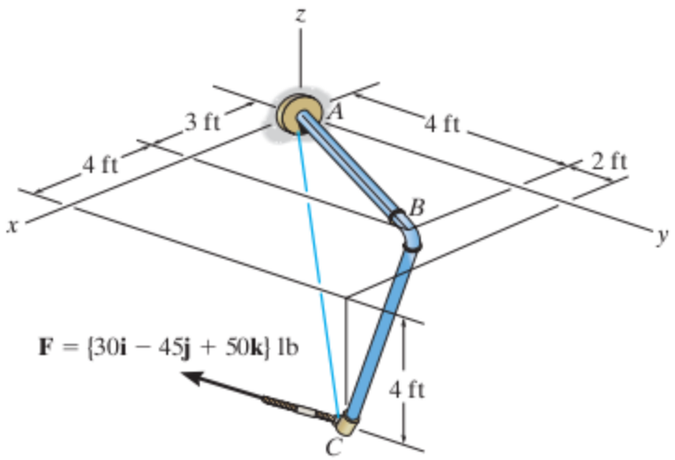
Prob. 2–80



Prob. 2–82

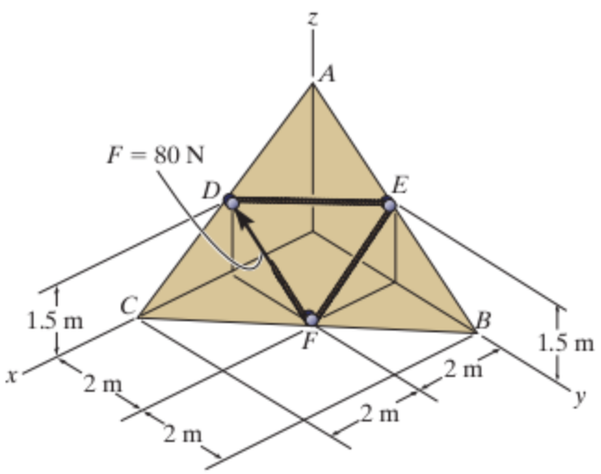
2–83. Determine the magnitudes of the components of **F** acting along and perpendicular to segment *BC* of the pipe assembly.

***2–84.** Determine the magnitude of the projected component of **F** along *AC*. Express this component as a Cartesian vector.



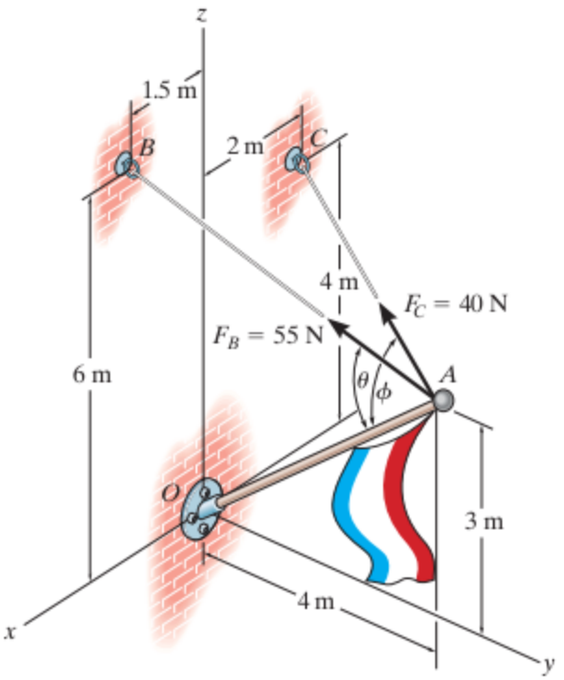
Probs. 2–83/84

2–85. Determine the projection of force $F = 80$ N along line *BC*. Express the result as a Cartesian vector.



Prob. 2–85

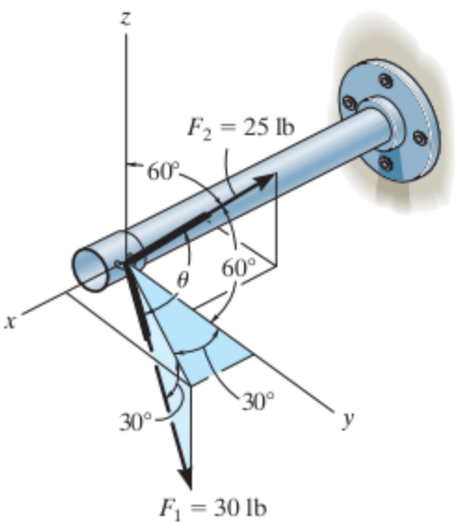
2–86. Determine the angles θ and ϕ made between the axes *OA* of the flag pole and *AB* and *AC*, respectively, of each cable.



Prob. 2–86

2–87. Two cables exert forces on the pipe. Determine the magnitude of the projected component of $\mathbf{F}_1$ along the line of action of $\mathbf{F}_2$.

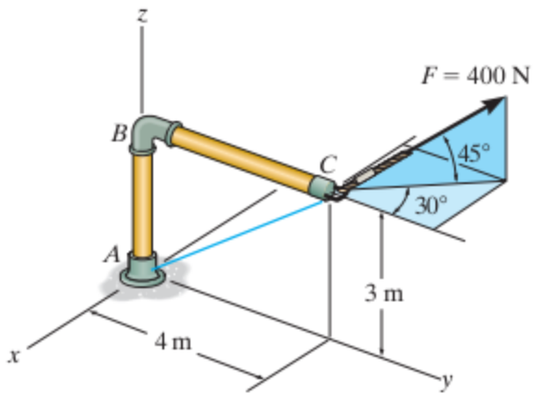
***2–88.** Determine the angle θ between the two cables attached to the pipe.



Probs. 2–87/88

2–89. Determine the projection of force $F = 400$ N acting along line AC of the pipe assembly. Express the result as a Cartesian vector.

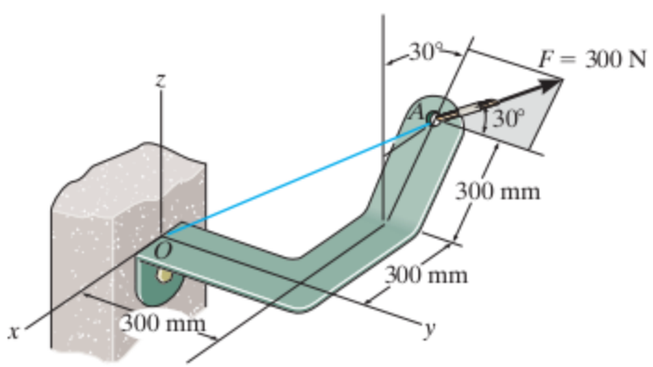
2–90. Determine the magnitudes of the components of force $F = 400$ N acting parallel and perpendicular to segment BC of the pipe assembly.



Probs. 2–89/90

2–91. Determine the magnitudes of the projected components of the force $F = 300$ N acting along the x and y axes.

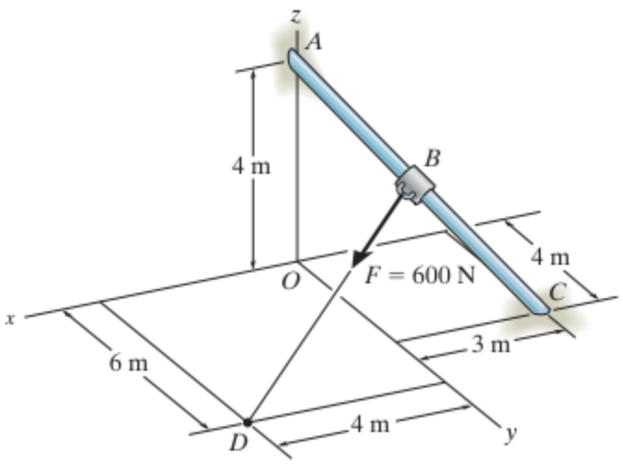
***2–92.** Determine the magnitude of the projected component of the force $F = 300$ N acting along line OA.



Probs. 2–91/92

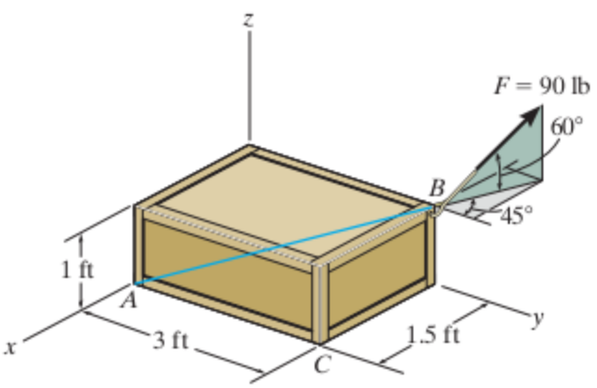
2–93. Determine the components of **F** that act along rod AC and perpendicular to it. Point B is located at the midpoint of the rod.

2–94. Determine the components of **F** that act along rod AC and perpendicular to it. Point B is located 3 m along the rod from end C.



Probs. 2–93/94

2–95. Determine the magnitudes of the components of force $F = 90$ lb acting parallel and perpendicular to diagonal AB of the crate.



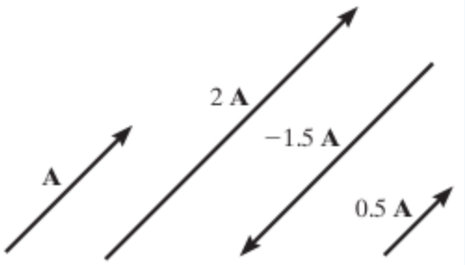
Prob. 2–95

CHAPTER REVIEW

A scalar is a positive or negative number; e.g., mass and temperature.

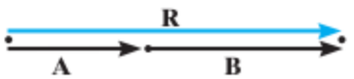
A vector has a magnitude and direction, where the arrowhead represents the sense of the vector.

Multiplication or division of a vector by a scalar will change only the magnitude of the vector. If the scalar is negative, the sense of the vector will change so that it acts in the opposite sense.



If vectors are collinear, the resultant is simply the algebraic or scalar addition.

$$R = A + B$$

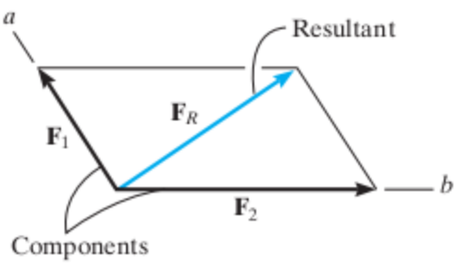


Parallelogram Law

Two forces add according to the parallelogram law. The *components* form the sides of the parallelogram and the *resultant* is the diagonal.

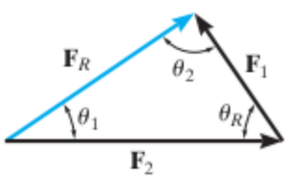
To find the components of a force along any two axes, extend lines from the head of the force, parallel to the axes, to form the components.

To obtain the components or the resultant, show how the forces add by tip-to-tail using the triangle rule, and then use the law of cosines and the law of sines to calculate their values.



$$F_R = \sqrt{F_1^2 + F_2^2 - 2F_1F_2\cos\theta_R}$$

$$\frac{F_1}{\sin\theta_1} = \frac{F_2}{\sin\theta_2} = \frac{F_R}{\sin\theta_R}$$



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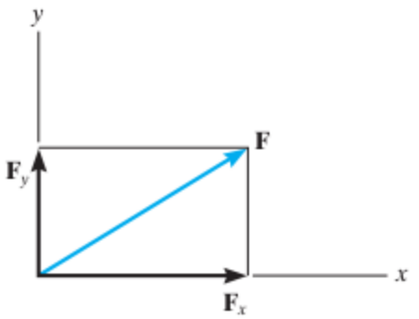
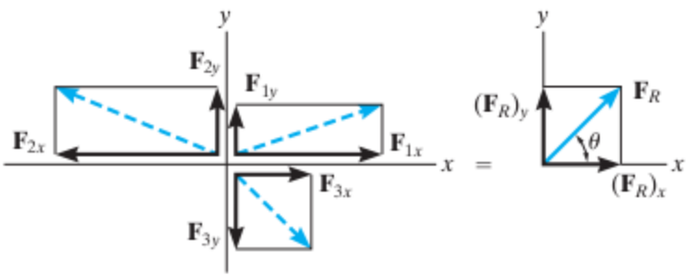
Rectangular Components: Two Dimensions

Vectors $\mathbf{F}_x$ and $\mathbf{F}_y$ are rectangular components of $\mathbf{F}$.



The resultant force is determined from the algebraic sum of its components.

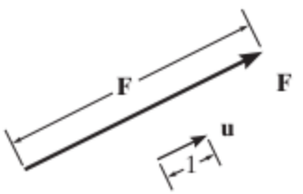
$$(F_R)_x = \Sigma F_x$$
$$(F_R)_y = \Sigma F_y$$
$$F_R = \sqrt{(F_{Rx})^2 + (F_{Ry})^2}$$
$$\theta = \tan^{-1}\left|\frac{(F_R)_y}{(F_R)_x}\right|$$



Cartesian Vectors

The unit vector $\mathbf{u}$ has a length of one, no units, and it points in the direction of the vector $\mathbf{F}$.

$$\mathbf{u} = \frac{\mathbf{F}}{F}$$



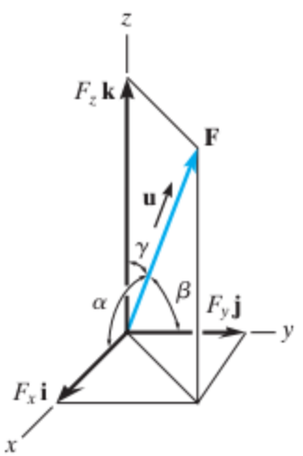
A force can be resolved into its Cartesian components along the x, y, z axes so that $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$.

The magnitude of $\mathbf{F}$ is determined from the positive square root of the sum of the squares of its components.

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

The coordinate direction angles α, β, γ are determined by formulating a unit vector in the direction of $\mathbf{F}$. The x, y, z components of $\mathbf{u}$ represent $\cos\alpha, \cos\beta, \cos\gamma$.

$$\mathbf{u} = \frac{\mathbf{F}}{F} = \frac{\mathbf{F}_x}{\mathbf{F}}\mathbf{i} + \frac{F_y}{F}\mathbf{j} + \frac{F_z}{F}\mathbf{k}$$
$$\mathbf{u} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$$



The coordinate direction angles are related so that only two of the three angles are independent of one another.

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

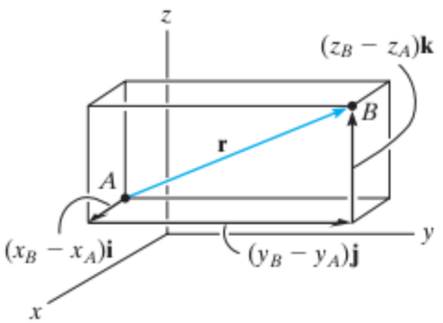
To find the resultant of a concurrent force system, express each force as a Cartesian vector and add the **i**, **j**, **k** components of all the forces in the system.

$$\mathbf{F}_R = \Sigma\mathbf{F} = \Sigma F_x\mathbf{i} + \Sigma F_y\mathbf{j} + \Sigma F_z\mathbf{k}$$

Position and Force Vectors

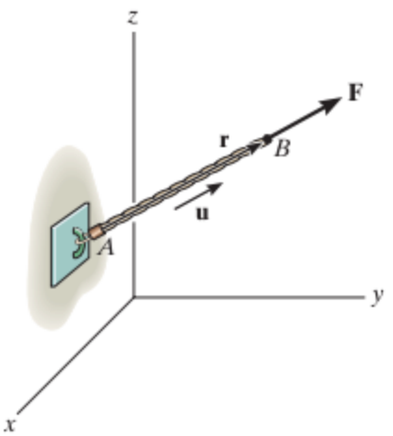
A position vector locates one point in space relative to another. The easiest way to formulate the components of a position vector is to determine the distance and direction that one must travel along the x, y, and z directions—going from the tail to the head of the vector.

$$\mathbf{r} = (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}$$



If the line of action of a force passes through points A and B, then the force acts in the same direction as the position vector **r**, which is defined by the unit vector **u**. The force can then be expressed as a Cartesian vector.

$$\mathbf{F} = F\mathbf{u} = F\left(\frac{\mathbf{r}}{r}\right)$$



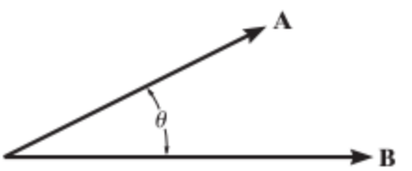
Dot Product

The dot product between two vectors **A** and **B** yields a scalar. If **A** and **B** are expressed in Cartesian vector form, then the dot product is the sum of the products of their x, y, and z components.

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta = A_xB_x + A_yB_y + A_zB_z$$

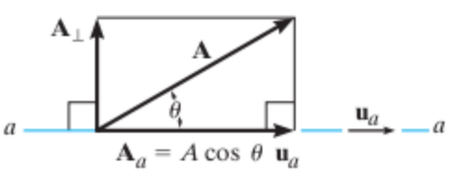
The dot product can be used to determine the angle between **A** and **B**.

$$\theta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{AB}\right)$$



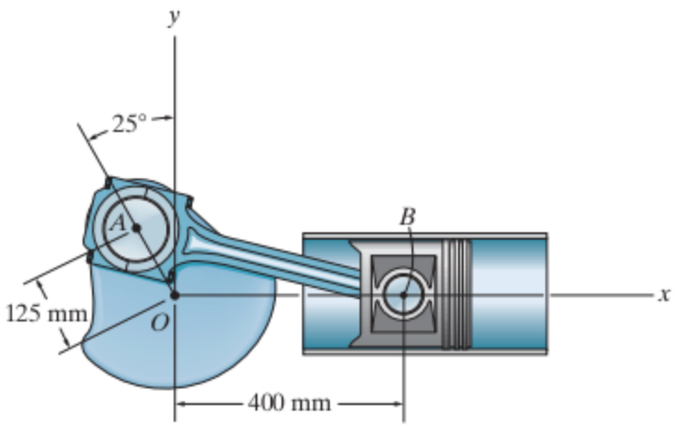
The dot product is also used to determine the projected component of a vector **A** onto an axis aa defined by its unit vector $\mathbf{u}_a$.

$$\mathbf{A}_a = A\cos\theta\,\mathbf{u}_a = (\mathbf{A}\cdot\mathbf{u}_a)\mathbf{u}_a$$



REVIEW PROBLEMS

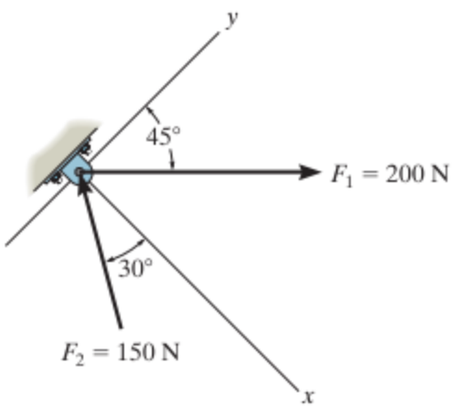
***2–96.** Determine the length of the connecting rod AB by first formulating a Cartesian position vector from A to B and then determining its magnitude.



Prob. 2–96

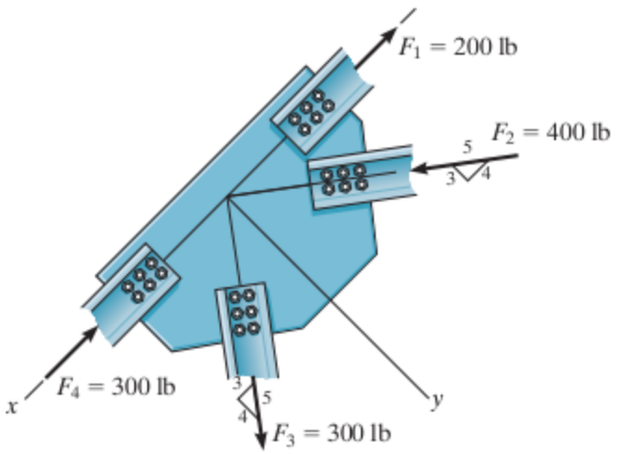
2–97. Determine the x and y components of $\mathbf{F}_1$ and $\mathbf{F}_2$.

2–98. Determine the magnitude of the resultant force and its direction, measured counterclockwise from the positive x axis.



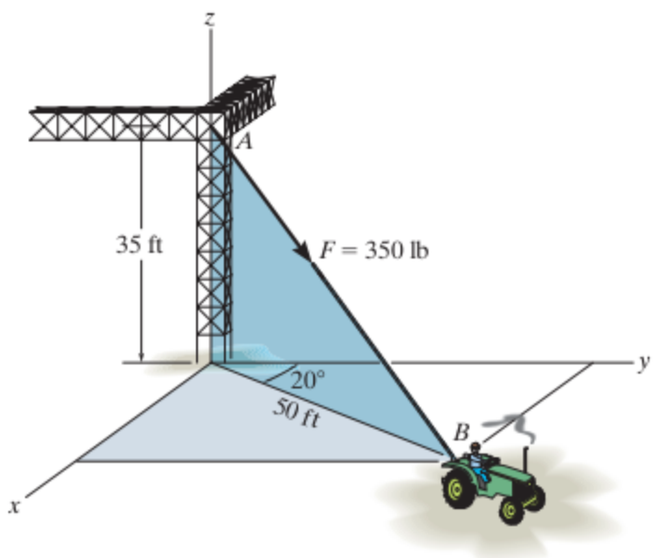
Probs. 2–97/98

2–99. Determine the x and y components of each force acting on the *gusset plate* of the bridge truss. Show that the resultant force is zero.



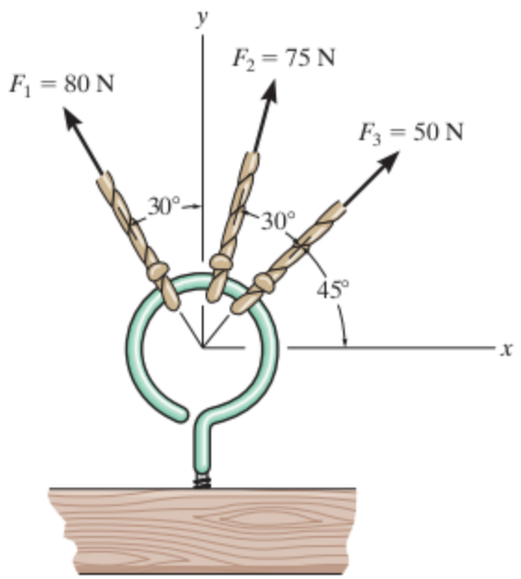
Prob. 2–99

***2–100.** The cable attached to the tractor at B exerts a force of 350 lb on the framework. Express this force as a Cartesian vector.



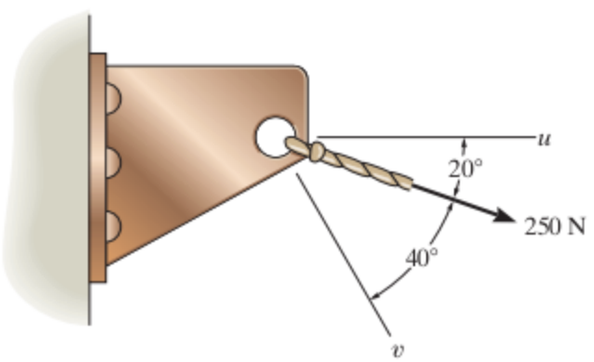
Prob. 2–100

2–101. Determine the magnitude and direction of the resultant $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$ of the three forces by first finding the resultant $\mathbf{F}' = \mathbf{F}_1 + \mathbf{F}_3$ and then forming $\mathbf{F}_R = \mathbf{F}' + \mathbf{F}_2$. Specify its direction measured counterclockwise from the positive x axis.



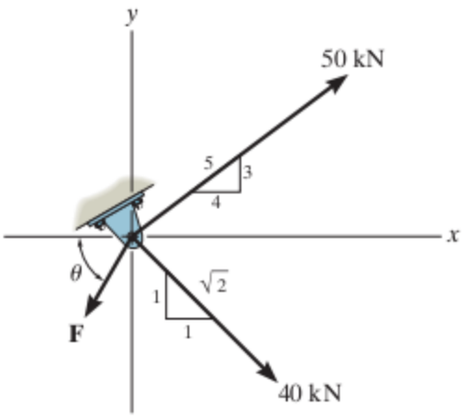
Prob. 2–101

2–102. Resolve the 250-N force into components acting along the u and v axes and determine the magnitudes of these components.



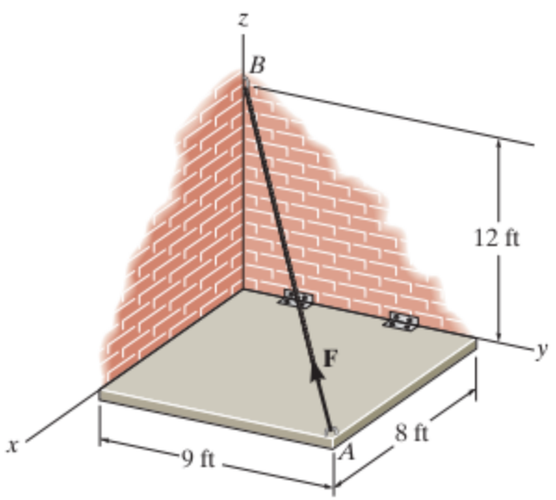
Prob. 2–102

2–103. If $\theta = 60°$ and $F = 20$ kN, determine the magnitude of the resultant force and its direction measured clockwise from the positive x axis.



Prob. 2–103

***2–104.** The hinged plate is supported by the cord AB. If the force in the cord is $F = 340$ lb, express this force, directed from A toward B, as a Cartesian vector. What is the length of the cord?



Prob. 2–104

Chapter 3

Forces applied to wrenches and wheels will produce rotation or a tendency for rotation. This effect is called a moment, and in this chapter we will study how to determine the moment of a system of forces and calculate their resultants.

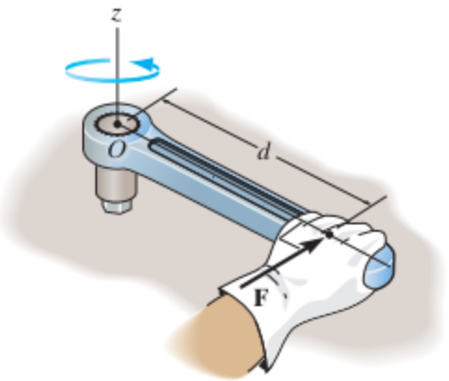
Force System Resultants

CHAPTER OBJECTIVES

- To discuss the concept of the moment of a force and show how to calculate it in two and three dimensions.
- To provide a method for finding the moment of a force about a specified axis.
- To define the moment of a couple.
- To present methods for determining the resultants of nonconcurrent force systems.
- To indicate how to reduce a simple distributed loading to a resultant force having a specified location.

3.1 Moment of a Force—Scalar Formulation

When a force is applied to a body it will produce a tendency for the body to rotate about a point that is not on the line of action of the force. This tendency to rotate is sometimes called a *torque*, but most often it is called the moment of a force or simply the *moment*. For example, consider a wrench used to unscrew the bolt in Fig. 3–1*a*. If a force is applied to the handle of the wrench it will tend to turn the bolt about point O (or the z axis). The magnitude of the moment is directly proportional to the magnitude of $\mathbf{F}$ and the perpendicular distance or *moment arm* d. The larger the force or the longer the moment arm, the greater the moment or turning effect. Note that if the force $\mathbf{F}$ is applied at an angle $\theta \neq 90°$, Fig. 3–1*b*, then it will be more difficult to turn the bolt since the moment arm $d' = d \sin\theta$ will be smaller than d. If $\mathbf{F}$ is applied along the wrench, Fig. 3–1*c*, its moment arm will be zero since the line of action of $\mathbf{F}$ will intersect point O (the z axis). As a result, the moment of $\mathbf{F}$ about O is also zero and no turning can occur.



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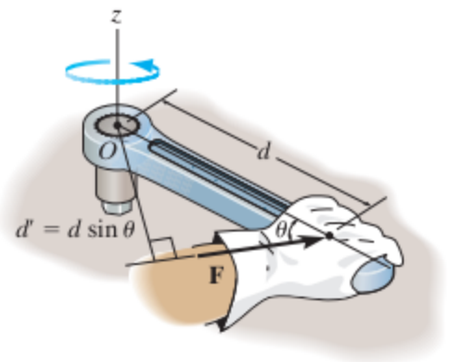

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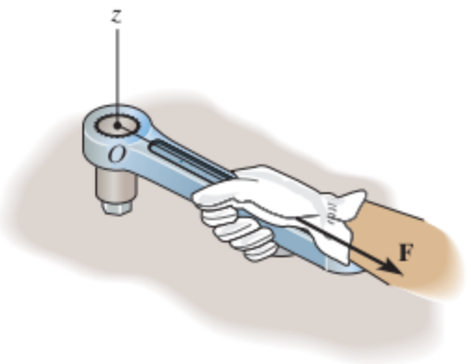


(c)

Fig. 3–1

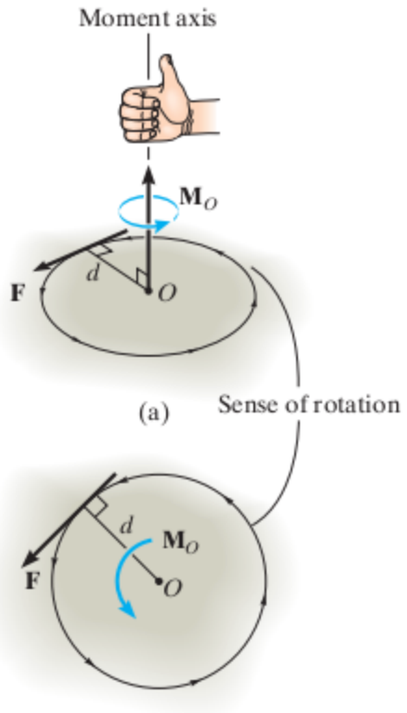


Fig. 3–2

We can generalize the above discussion and consider the force **F** and point O which lie in the shaded plane as shown in Fig. 3–2*a*. The moment $\mathbf{M}_O$ about point O, or about an axis passing through O and perpendicular to the plane, is a *vector quantity* since it has a specified magnitude and direction.

Magnitude. The magnitude of $\mathbf{M}_O$ is

$$M_O = Fd \tag{3–1}$$

where d is the *moment arm* or *perpendicular distance* from the axis at point O to the line of action of the force. Units of moment magnitude consist of force times distance, e.g., N · m or lb · ft.

Direction. The direction of $\mathbf{M}_O$ is defined by its *moment axis*, which is perpendicular to the plane that contains the force **F** and its moment arm d. The right-hand rule is used to establish the sense of direction of $\mathbf{M}_O$. According to this rule, the natural curl of the fingers of the right hand, as they are drawn towards the palm, represent the tendency for rotation caused by the moment. As this action is performed, the thumb of the right hand will give the directional sense of $\mathbf{M}_O$, Fig. 3–2*a*. Notice that the moment vector is represented three-dimensionally by a curl around an arrow. In two dimensions this vector is represented only by the curl as in Fig. 3–2*b*. Since in this case the moment will tend to cause a counterclockwise rotation, the moment vector is actually directed out of the page.

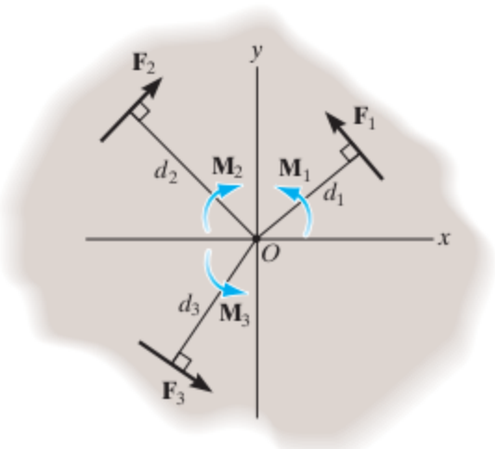


Fig. 3–3

Resultant Moment. For two-dimensional problems, where all the forces lie within the x–y plane, Fig. 3–3, the resultant moment $(\mathbf{M}_R)_o$ about point O (the z axis) can be determined by *finding the algebraic sum* of the moments caused by all the forces in the system. As a convention, we will generally consider *positive moments* as *counterclockwise* since they are directed along the positive z axis (out of the page). *Clockwise moments* will be *negative*. Doing this, the directional sense of each moment can be represented by a *plus or minus* sign. Using this sign convention, the resultant moment in Fig. 3–3 is therefore

$$\circlearrowleft + (M_R)_o = \Sigma Fd; \qquad (M_R)_o = F_1d_1 - F_2d_2 + F_3d_3$$

If the numerical result of this sum is a positive scalar, $(\mathbf{M}_R)_o$ will be a counterclockwise moment (out of the page); and if the result is negative, $(\mathbf{M}_R)_o$ will be a clockwise moment (into the page).

EXAMPLE 3.1

For each case illustrated in Fig. 3–4, determine the moment of the force about point O.

SOLUTION *(SCALAR ANALYSIS)*

The line of action of each force is extended as a dashed line in order to establish the moment arm d. Also illustrated is the tendency of rotation of the member as caused by the force. Furthermore, the orbit of the force about O is shown as a colored curl. Thus,

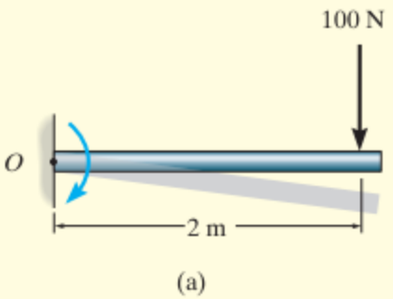
Fig. 3–4*a* $M_O = (100\ \text{N})(2\ \text{m}) = 200\ \text{N}\cdot\text{m}$ ↻ *Ans.*

Fig. 3–4*b* $M_O = (50\ \text{N})(0.75\ \text{m}) = 37.5\ \text{N}\cdot\text{m}$ ↻ *Ans.*

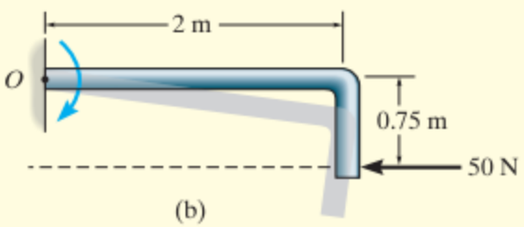
Fig. 3–4*c* $M_O = (40\ \text{lb})(4\ \text{ft} + 2\cos 30°\ \text{ft}) = 229\ \text{lb}\cdot\text{ft}$ ↻ *Ans.*

Fig. 3–4*d* $M_O = (60\ \text{lb})(1\sin 45°\ \text{ft}) = 42.4\ \text{lb}\cdot\text{ft}$ ↺ *Ans.*

Fig. 3–4*e* $M_O = (7\ \text{kN})(4\ \text{m} - 1\ \text{m}) = 21.0\ \text{kN}\cdot\text{m}$ ↺ *Ans.*



(a)



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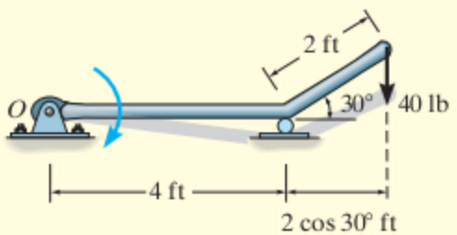

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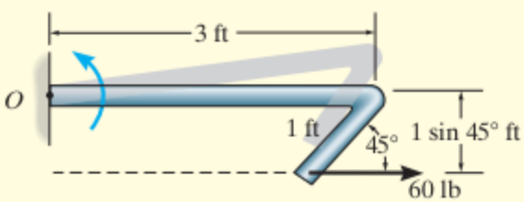

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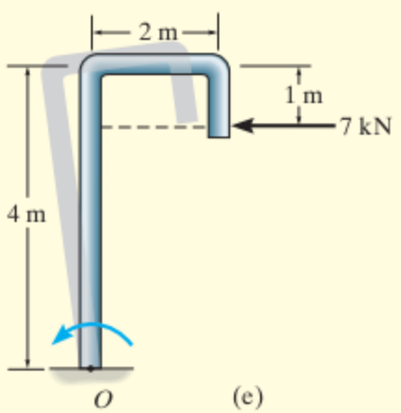


(e)

Fig. 3–4

EXAMPLE 3.2

Determine the resultant moment of the four forces acting on the rod shown in Fig. 3–5 about point O.

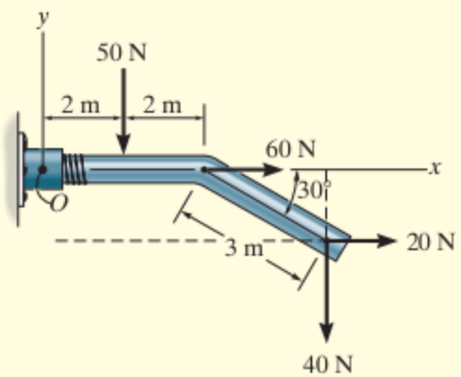


Fig. 3–5

SOLUTION

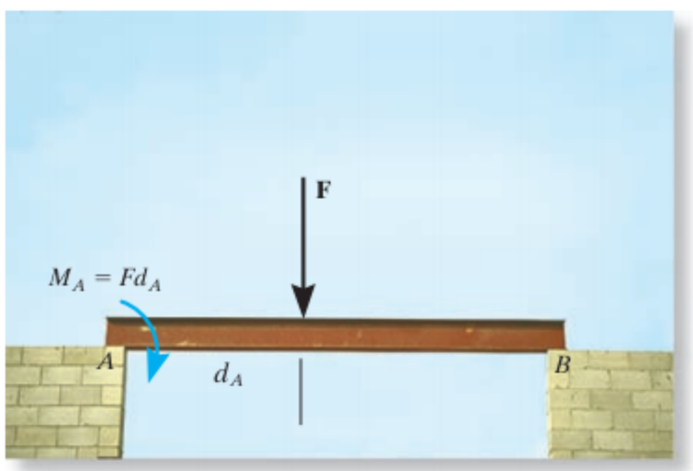
Assuming that positive moments act in the $+\mathbf{k}$ direction, i.e., counterclockwise, we have

$$\circlearrowleft + (M_R)_o = \Sigma Fd;$$

$$(M_R)_o = -50\,\text{N}(2\,\text{m}) + 60\,\text{N}(0) + 20\,\text{N}(3\sin 30^\circ\,\text{m})$$
$$-40\,\text{N}(4\,\text{m} + 3\cos 30^\circ\,\text{m})$$

$$(M_R)_o = -334\,\text{N}\cdot\text{m} = 334\,\text{N}\cdot\text{m}\ \circlearrowright \qquad \textit{Ans.}$$

For this calculation, note how the moment-arm distances for the 20-N and 40-N forces are established from the extended (dashed) lines of action of each of these forces.



As illustrated by the example problems, the moment of a force does not always cause a rotation. For example, the force **F** tends to rotate the beam clockwise about its support at A with a moment $M_A = Fd_A$. The actual rotation would occur if the support at B were removed.

The ability to remove the nail will require the moment of $\mathbf{F}_H$ about point O to be larger than the moment of the force $\mathbf{F}_N$ about O that is needed to pull the nail out.

3.2 Cross Product

The moment of a force will be formulated using Cartesian vectors in the next section. Before doing this, however, it is first necessary to expand our knowledge of vector algebra and introduce the cross-product method of vector multiplication.

The *cross product* of two vectors **A** and **B** yields the vector **C**, which is written

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \tag{3–2}$$

and is read "**C** equals **A** cross **B**."

Magnitude. The *magnitude* of **C** is defined as the product of the magnitudes of **A** and **B** and the sine of the angle θ between their tails ($0° \leq \theta \leq 180°$). Thus, $C = AB \sin\theta$.

Direction. Vector **C** has a *direction* that is perpendicular to the plane containing **A** and **B** such that **C** is specified by the right-hand rule; i.e., curling the fingers of the right hand from vector **A** (cross) to vector **B**, the thumb points in the direction of **C**, as shown in Fig. 3–6.

Knowing both the magnitude and direction of **C**, we can write

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = (AB \sin\theta)\mathbf{u}_C \tag{3–3}$$

where the scalar $AB \sin\theta$ defines the *magnitude* of **C** and the unit vector $\mathbf{u}_C$ defines the *direction* of **C**. The terms of Eq. 3–3 are illustrated graphically in Fig. 3–6.

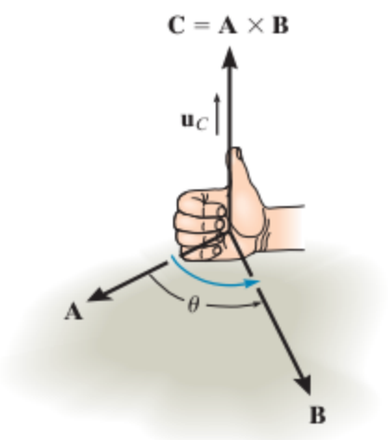


Fig. 3–6

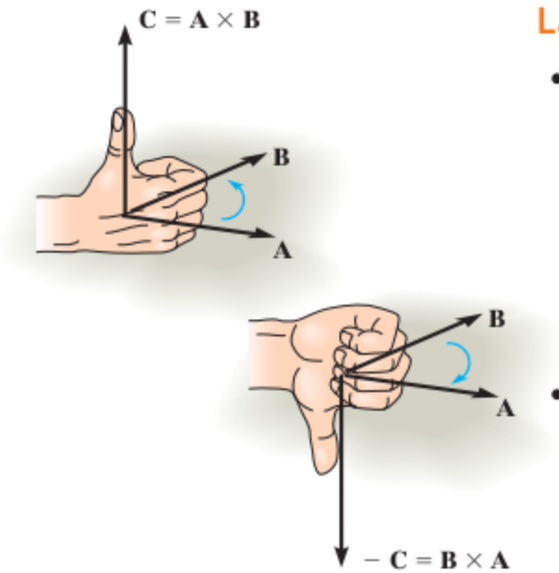


Fig. 3–7

Laws of Operation.

- The commutative law is *not* valid; i.e., $\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$. Rather,

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

This is shown in Fig. 3–7 by using the right-hand rule. The cross product $\mathbf{B} \times \mathbf{A}$ yields a vector that has the same magnitude but acts in the opposite direction to $\mathbf{C}$; i.e., $\mathbf{B} \times \mathbf{A} = -\mathbf{C}$.

- If the cross product is multiplied by a scalar a, it obeys the associative law;

$$a(\mathbf{A} \times \mathbf{B}) = (a\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (a\mathbf{B}) = (\mathbf{A} \times \mathbf{B})a$$

This property is easily shown since the magnitude of the resultant vector ($|a|AB \sin\theta$) and its direction are the same in each case.

- The vector cross product also obeys the distributive law of addition,

$$\mathbf{A} \times (\mathbf{B} + \mathbf{D}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{D})$$

- The proof of this identity is left as an exercise (see Prob. 3–1). It is important to note that *proper order* of the cross products must be maintained, since they are not commutative.

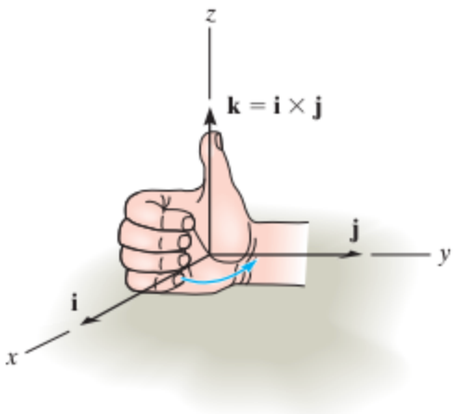


Fig. 3–8

Cartesian Vector Formulation. Equation 3–3 may be used to find the cross product of any pair of Cartesian unit vectors. For example, to find $\mathbf{i} \times \mathbf{j}$, the magnitude of the resultant vector is $(i)(j)(\sin 90°) = (1)(1)(1) = 1$, and its direction is determined using the right-hand rule. As shown in Fig. 3–8, the resultant vector points in the $+\mathbf{k}$ direction. Thus, $\mathbf{i} \times \mathbf{j} = (1)\mathbf{k}$. In a similar manner,

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= \mathbf{k} & \mathbf{i} \times \mathbf{k} &= -\mathbf{j} & \mathbf{i} \times \mathbf{i} &= \mathbf{0} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} & \mathbf{j} \times \mathbf{i} &= -\mathbf{k} & \mathbf{j} \times \mathbf{j} &= \mathbf{0} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j} & \mathbf{k} \times \mathbf{j} &= -\mathbf{i} & \mathbf{k} \times \mathbf{k} &= \mathbf{0} \end{aligned}$$

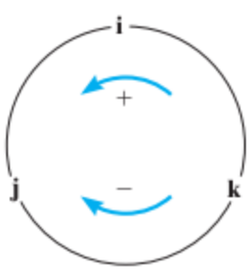


Fig. 3–9

These results should *not* be memorized; rather, it should be clearly understood how each is obtained by using the right-hand rule and the definition of the cross product. A simple scheme shown in Fig. 3–9 is helpful for obtaining the same results when the need arises. If the circle is constructed as shown, then "crossing" two unit vectors in a *counterclockwise* fashion around the circle yields the *positive* third unit vector; e.g., $\mathbf{k} \times \mathbf{i} = \mathbf{j}$. "Crossing" *clockwise*, a *negative* unit vector is obtained; e.g., $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

Let us now consider the cross product of two general vectors **A** and **B** which are expressed in Cartesian vector form. We have

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \times (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \\ &= A_xB_x(\mathbf{i} \times \mathbf{i}) + A_xB_y(\mathbf{i} \times \mathbf{j}) + A_xB_z(\mathbf{i} \times \mathbf{k}) \\ &\quad + A_y\,B_x(\mathbf{j} \times \mathbf{i}) + A_yB_y(\mathbf{j} \times \mathbf{j}) + A_yB_z(\mathbf{j} \times \mathbf{k}) \\ &\quad + A_zB_x(\mathbf{k} \times \mathbf{i}) + A_zB_y(\mathbf{k} \times \mathbf{j}) + A_zB_z(\mathbf{k} \times \mathbf{k})\end{aligned}$$

Carrying out the cross-product operations and combining terms yields

$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\mathbf{i} - (A_xB_z - A_zB_x)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k} \tag{3–4}$$

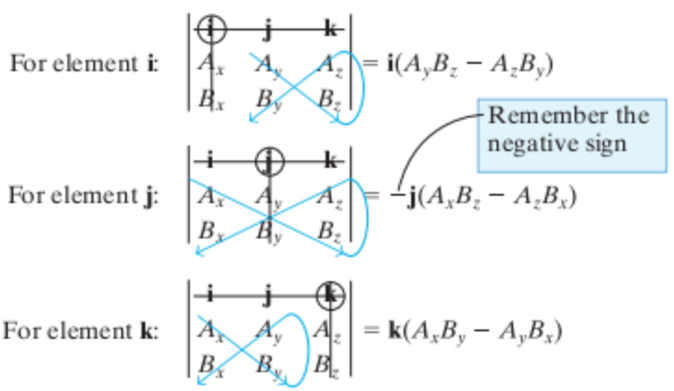
This equation may also be written in a more compact determinant form as

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{3–5}$$

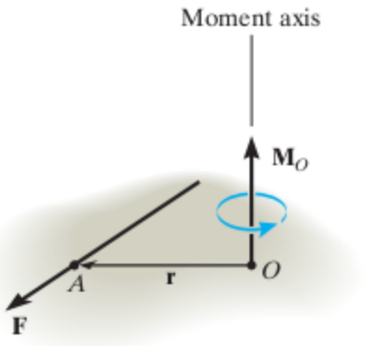
Thus, to find the cross product of any two Cartesian vectors **A** and **B**, it is necessary to expand a determinant whose first row of elements consists of the unit vectors **i**, **j**, and **k** and whose second and third rows represent the x, y, z components of the two vectors **A** and **B**, respectively.*

*A determinant having three rows and three columns can be expanded using three minors, each of which is multiplied by one of the three terms in the first row. There are four elements in each minor, for example,

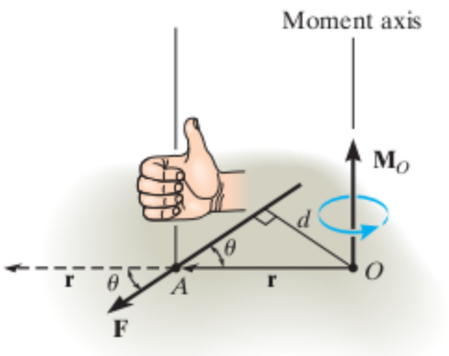
By *definition*, this determinant notation represents the terms $(A_{11}A_{22} - A_{12}A_{21})$, which is simply the product of the two elements intersected by the arrow slanting downward to the right $(A_{11}A_{22})$ *minus* the product of the two elements intersected by the arrow slanting downward to the left $(A_{12}A_{21})$. For a 3×3 determinant, such as Eq. 3–5, the three minors can be generated in accordance with the following scheme:



Adding the results and noting that the **j** element *must include the minus sign* yields the expanded form of $\mathbf{A} \times \mathbf{B}$ given by Eq. 3–4.



(a)



(b)

Fig. 3–10

3.3 Moment of a Force—Vector Formulation

The moment of a force **F** about point O, or actually about the moment axis passing through O and perpendicular to the plane containing O and **F**, Fig. 3–10*a*, can be expressed using the vector cross product, namely,

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} \tag{3–6}$$

Here **r** represents a position vector directed *from* O to *any point* on the line of action of **F**. We will now show that indeed the moment $\mathbf{M}_O$, when determined by this cross product, has the proper magnitude and direction.

Magnitude. The magnitude of the cross product is defined from Eq. 3–3 as $M_O = rF \sin\theta$, where the angle θ is measured between the *tails* of **r** and **F**. To establish this angle, **r** must be treated as a sliding vector so that θ can be constructed properly, Fig. 3–10*b*. Since the moment arm $d = r \sin\theta$, then

$$M_O = rF \sin\theta = F(r \sin\theta) = Fd$$

which agrees with Eq. 3–1.

Direction. The direction and sense of $\mathbf{M}_O$ in Eq. 3–6 are determined by the right-hand rule as it applies to the cross product. Thus, sliding **r** to the dashed position and curling the right-hand fingers from **r** toward **F**, "**r** cross **F**," the thumb is directed upward or perpendicular to the plane containing **r** and **F** and this is in the *same direction* as $\mathbf{M}_O$, the moment of the force about point O, Fig. 3–10*b*. Note that the "curl" of the fingers, like the curl around the moment vector, indicates the sense of rotation caused by the force. Since the cross product does not obey the commutative law, the order of $\mathbf{r} \times \mathbf{F}$ must be maintained to produce the correct sense of direction for $\mathbf{M}_O$.

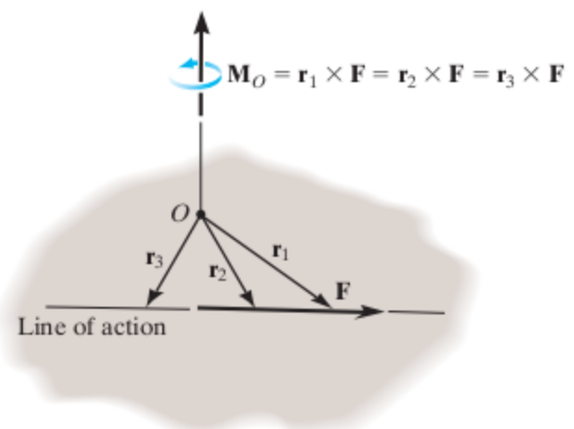


Fig. 3–11

Principle of Transmissibility. The cross product operation is often used in three dimensions since the perpendicular distance or moment arm from point O to the line of action of the force is not needed. In other words, we can use any position vector **r** measured from point O to any point on the line of action of the force **F**, Fig. 3–11. Thus,

$$\mathbf{M}_O = \mathbf{r}_1 \times \mathbf{F} = \mathbf{r}_2 \times \mathbf{F} = \mathbf{r}_3 \times \mathbf{F}$$

Since **F** can be applied at any point along its line of action and still create this *same moment* about point O, then **F** can be considered a *sliding vector*. This property is called the *principle of transmissibility* of a force.

Cartesian Vector Formulation. If we establish x, y, z coordinate axes, then the position vector $\mathbf{r}$ and force $\mathbf{F}$ can be expressed as Cartesian vectors, Fig. 3–12*a*. Applying Eq. 3–5 we have

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \tag{3–7}$$

where

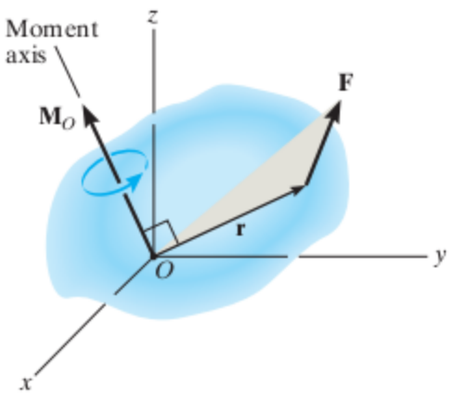
r_x, r_y, r_z represent the x, y, z components of the position vector drawn from point O to *any point* on the line of action of the force

F_x, F_y, F_z represent the x, y, z components of the force vector

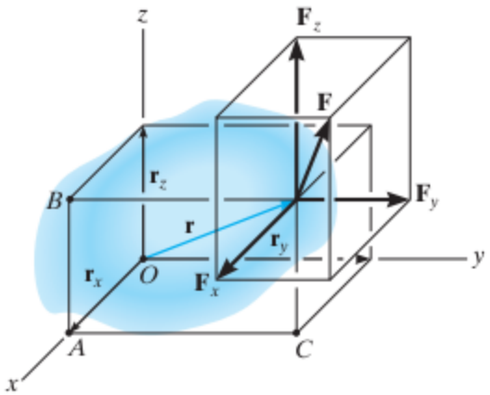
If the determinant is expanded, then like Eq. 3–4 we have

$$\mathbf{M}_O = (r_yF_z - r_zF_y)\mathbf{i} - (r_xF_z - r_zF_x)\mathbf{j} + (r_xF_y - r_yF_x)\mathbf{k} \tag{3–8}$$

The physical meaning of these three moment components becomes evident by studying Fig. 3–12*b*. For example, the **i** component of $\mathbf{M}_O$ can be determined from the moments of $\mathbf{F}_x$, $\mathbf{F}_y$, and $\mathbf{F}_z$ about the x axis. The component $\mathbf{F}_x$ does *not* create a moment or tendency to cause turning about the x axis since this force is *parallel* to the x axis. The line of action of $\mathbf{F}_y$ passes through point B, and so the magnitude of the moment of $\mathbf{F}_y$ about point A on the x axis is r_zF_y. By the right-hand rule this component acts in the *negative* **i** direction. Likewise, $\mathbf{F}_z$ passes through point C and so it contributes a moment component of $r_yF_z\,\mathbf{i}$ about the x axis. Thus, $(M_O)_x = (r_yF_z - r_zF_y)$ as shown in Eq. 3–8. As an exercise, establish the **j** and **k** components of $\mathbf{M}_O$ in this manner and show that indeed the expanded form of the determinant, Eq. 3–8, represents the moment of $\mathbf{F}$ about point O. Once $\mathbf{M}_O$ is determined, realize that it will always be *perpendicular* to the shaded plane containing vectors $\mathbf{r}$ and $\mathbf{F}$, Fig. 3–12*a*.



(a)



(b)

Fig. 3–12

Resultant Moment of a System of Forces. If a body is acted upon by a system of forces, Fig. 3–13, the resultant moment of the forces about point O can be determined by vector addition of the moment of each force. This resultant can be written symbolically as

$$(\mathbf{M}_R)_O = \Sigma(\mathbf{r} \times \mathbf{F}) \tag{3–9}$$

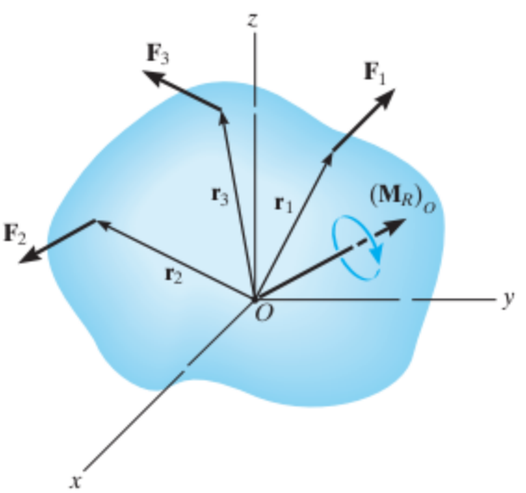
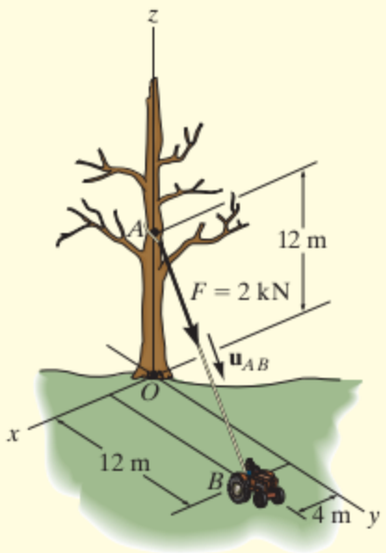


Fig. 3–13

EXAMPLE 3.3



(a)

Determine the moment produced by the force **F** in Fig. 3–14*a* about point *O*. Express the result as a Cartesian vector.

SOLUTION

As shown in Fig. 3–14*b*, either $\mathbf{r}_A$ or $\mathbf{r}_B$ can be used to determine the moment about point *O*. These position vectors are

$$\mathbf{r}_A = \{12\mathbf{k}\}\ \text{m} \quad \text{and} \quad \mathbf{r}_B = \{4\mathbf{i} + 12\mathbf{j}\}\ \text{m}$$

Force **F** expressed as a Cartesian vector is

$$\mathbf{F} = F\mathbf{u}_{AB} = 2\ \text{kN}\left[\frac{\{4\mathbf{i} + 12\mathbf{j} - 12\mathbf{k}\}\ \text{m}}{\sqrt{(4\ \text{m})^2 + (12\ \text{m})^2 + (-12\ \text{m})^2}}\right]$$

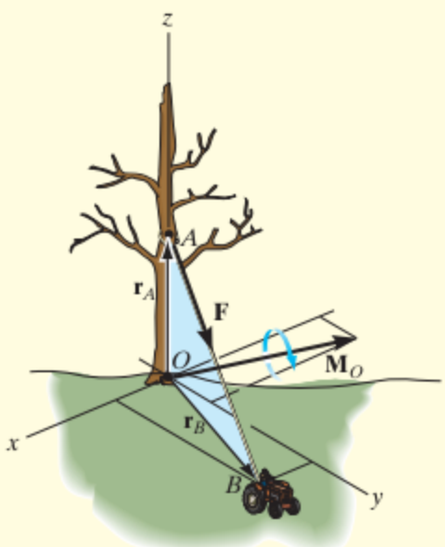
$$= \{0.4588\mathbf{i} + 1.376\mathbf{j} - 1.376\mathbf{k}\}\ \text{kN}$$

Thus

$$\mathbf{M}_O = \mathbf{r}_A \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 12 \\ 0.4588 & 1.376 & -1.376 \end{vmatrix}$$

$$= [0(-1.376) - 12(1.376)]\mathbf{i} - [0(-1.376) - 12(0.4588)]\mathbf{j} + [0(1.376) - 0(0.4588)]\mathbf{k}$$

$$= \{-16.5\mathbf{i} + 5.51\mathbf{j}\}\ \text{kN} \cdot \text{m} \qquad \textit{Ans.}$$



(b)

Fig. 3–14

or

$$\mathbf{M}_O = \mathbf{r}_B \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 12 & 0 \\ 0.4588 & 1.376 & -1.376 \end{vmatrix}$$

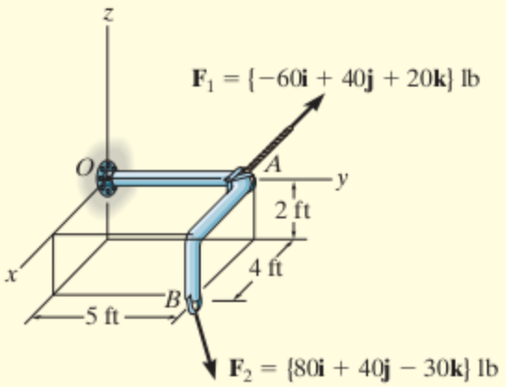
$$= [12(-1.376) - 0(1.376)]\mathbf{i} - [4(-1.376) - 0(0.4588)]\mathbf{j} + [4(1.376) - 12(0.4588)]\mathbf{k}$$

$$= \{-16.5\mathbf{i} + 5.51\mathbf{j}\}\ \text{kN} \cdot \text{m} \qquad \textit{Ans.}$$

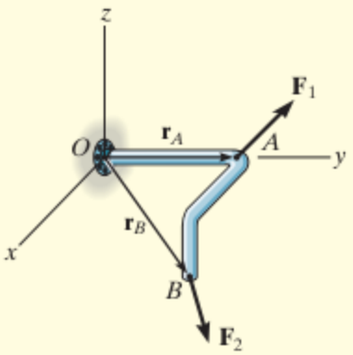
NOTE: As shown in Fig. 3–14*b*, $\mathbf{M}_O$ acts perpendicular to the plane that contains **F**, $\mathbf{r}_A$, and $\mathbf{r}_B$. Had this problem been worked using $M_O = Fd$, notice the difficulty that would arise in obtaining the moment arm *d*.

EXAMPLE 3.4

Two forces act on the rod shown in Fig. 3–15*a*. Determine the resultant moment they create about the flange at *O*. Express the result as a Cartesian vector.



(a)



(b)

SOLUTION

Position vectors are directed from point *O* to each force as shown in Fig. 3–15*b*. These vectors are

$$\mathbf{r}_A = \{5\mathbf{j}\}\ \text{ft}$$

$$\mathbf{r}_B = \{4\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}\}\ \text{ft}$$

The resultant moment about *O* is therefore

$$\mathbf{M}_{R_O} = \Sigma(\mathbf{r} \times \mathbf{F})$$

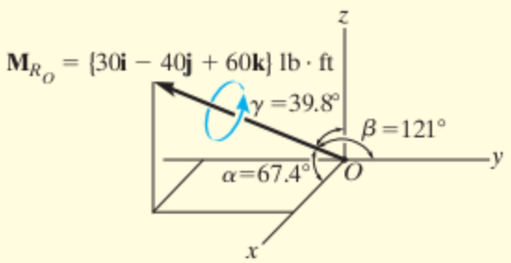
$$= \mathbf{r}_A \times \mathbf{F}_1 + \mathbf{r}_B \times \mathbf{F}_2$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 5 & 0 \\ -60 & 40 & 20 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 5 & -2 \\ 80 & 40 & -30 \end{vmatrix}$$

$$= [5(20) - 0(40)]\mathbf{i} - [0]\mathbf{j} + [0(40) - (5)(-60)]\mathbf{k}$$

$$+ [5(-30) - (-2)(40)]\mathbf{i} - [4(-30) - (-2)(80)]\mathbf{j} + [4(40) - 5(80)]\mathbf{k}$$

$$= \{30\mathbf{i} - 40\mathbf{j} + 60\mathbf{k}\}\ \text{lb}\cdot\text{ft} \qquad \textit{Ans.}$$



(c)

Fig. 3–15

NOTE: This result is shown in Fig. 3–15*c*. The coordinate direction angles were determined from the unit vector for $\mathbf{M}_{R_O}$. Realize that the two forces tend to cause the rod to rotate about the moment axis in the manner shown by the curl indicated on the moment vector.

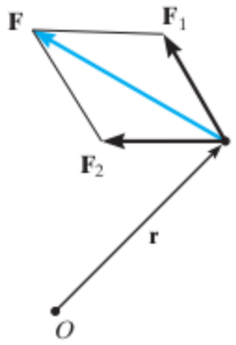


Fig. 3–16

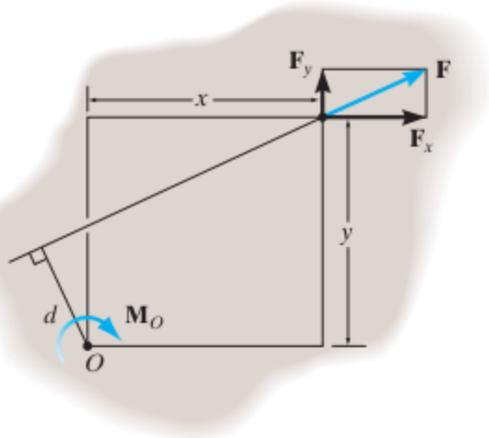


Fig. 3–17

The moment of the applied force **F** about point O is easy to determine if we use the principle of moments. It is simply $M_O = F_x d$.

3.4 Principle of Moments

A concept often used in mechanics is the *principle of moments*, which is sometimes referred to as *Varignon's theorem* since it was originally developed by the French mathematician Varignon (1654–1722). It states that *the moment of a force about a point is equal to the sum of the moments of the components of the force about the point*. This theorem can be proven easily using the vector cross product since the cross product obeys the *distributive law*. For example, consider the moments of the force **F** and two of its components about point O, Fig. 3–16. Since $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ we have

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2) = \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2$$

For two-dimensional problems, Fig. 3–17, we can use the principle of moments by resolving the force into its rectangular components and then determine the moment using a scalar analysis. Thus,

$$M_O = F_x y - F_y x$$

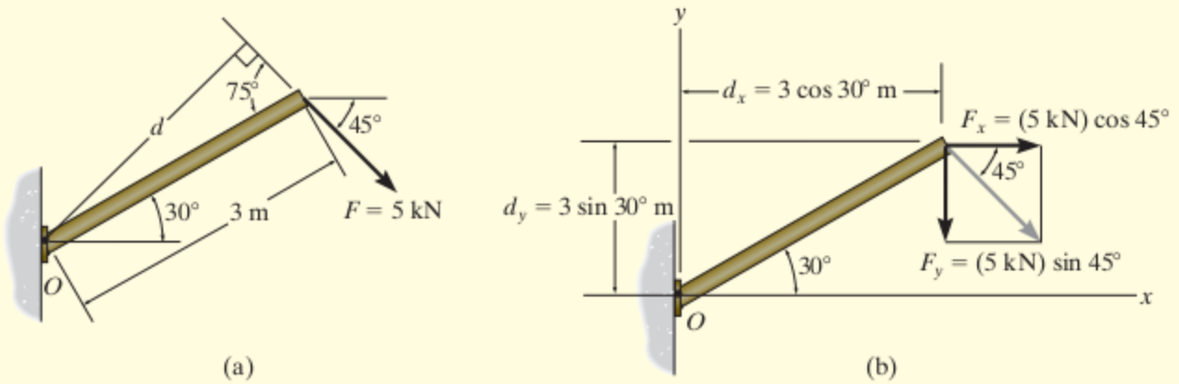
This method is generally easier than finding the same moment using $M_O = Fd$.

Important Points

- The moment of a force creates the tendency of a body to turn about an axis passing through a specific point O.
- Using the right-hand rule, the sense of rotation is indicated by the curl of the fingers, and the thumb is directed along the moment axis, or line of action of the moment.
- The magnitude of the moment is determined from $M_O = Fd$, where d is called the moment arm, which represents the perpendicular or shortest distance from point O to the line of action of the force.
- In three dimensions the vector cross product is used to determine the moment, i.e., $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$. Remember that **r** is directed *from* point O *to any point* on the line of action of **F**.
- The principle of moments states that the moment of a force about a point is equal to the sum of the moments of the force's components about the point. This is a very convenient method to use in two dimensions.

EXAMPLE 3.5

Determine the moment of the force in Fig. 3–18*a* about point *O*.



SOLUTION I

The moment arm d in Fig. 3–18*a* can be found from trigonometry.

$$d = (3 \text{ m}) \sin 75° = 2.898 \text{ m}$$

Thus,

$$M_O = Fd = (5 \text{ kN})(2.898 \text{ m}) = 14.5 \text{ kN} \cdot \text{m} \circlearrowright \qquad \textit{Ans.}$$

Since the force tends to rotate or orbit clockwise about point *O*, the moment is directed into the page.

SOLUTION II

The *x* and *y* components of the force are indicated in Fig. 3–18*b*. Considering counterclockwise moments as positive, and applying the principle of moments, we have

$$\begin{aligned} \circlearrowleft + M_O &= -F_x d_y - F_y d_x \\ &= -(5 \cos 45° \text{ kN})(3 \sin 30° \text{ m}) - (5 \sin 45° \text{ kN})(3 \cos 30° \text{ m}) \\ &= -14.5 \text{ kN} \cdot \text{m} = 14.5 \text{ kN} \cdot \text{m} \circlearrowright \end{aligned} \qquad \textit{Ans.}$$

SOLUTION III

The *x* and *y* axes can be set parallel and perpendicular to the rod's axis as shown in Fig. 3–18*c*. Here $\mathbf{F}_x$ produces no moment about point *O* since its line of action passes through this point. Therefore,

$$\begin{aligned} \circlearrowleft + M_O &= -F_y d_x \\ &= -(5 \sin 75° \text{ kN})(3 \text{ m}) \\ &= -14.5 \text{ kN} \cdot \text{m} = 14.5 \text{ kN} \cdot \text{m} \circlearrowright \end{aligned} \qquad \textit{Ans.}$$

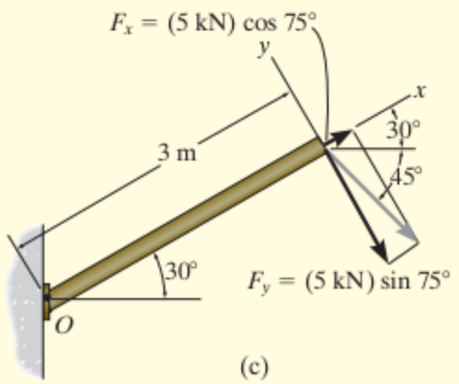
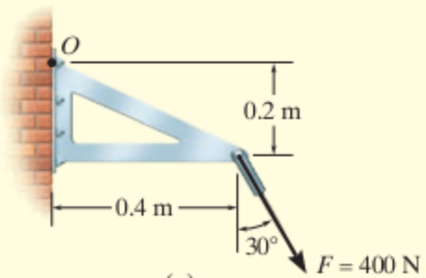


Fig. 3–18

EXAMPLE 3.6

Force **F** acts at the end of the angle bracket shown in Fig. 3–19*a*. Determine the moment of the force about point *O*.



(a)

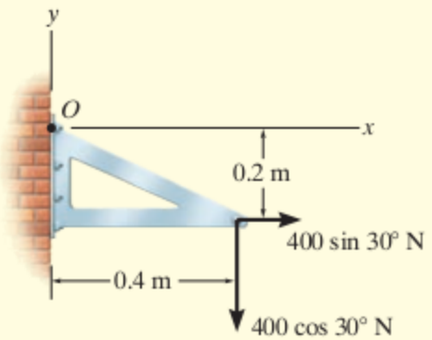
SOLUTION I *(SCALAR ANALYSIS)*

The force is resolved into its x and y components as shown in Fig. 3–19*b*, then

$$\circlearrowleft +M_O = 400 \sin 30^\circ \text{ N}(0.2 \text{ m}) - 400 \cos 30^\circ \text{ N}(0.4 \text{ m})$$
$$= -98.6 \text{ N} \cdot \text{m} = 98.6 \text{ N} \cdot \text{m} \circlearrowright$$

or

$$\mathbf{M}_O = \{-98.6\mathbf{k}\} \text{ N} \cdot \text{m} \qquad \textit{Ans.}$$

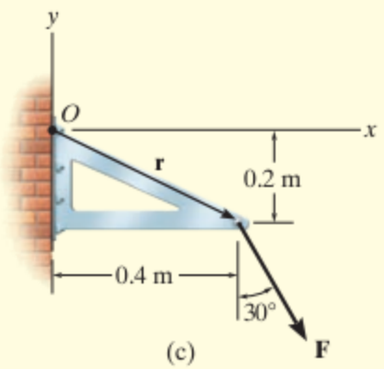


(b)

SOLUTION II *(VECTOR ANALYSIS)*

Using a Cartesian vector approach, the force and position vectors shown in Fig. 3–19*c* are

$$\mathbf{r} = \{0.4\mathbf{i} - 0.2\mathbf{j}\} \text{ m}$$
$$\mathbf{F} = \{400 \sin 30^\circ\mathbf{i} - 400 \cos 30^\circ\mathbf{j}\} \text{ N}$$
$$= \{200.0\mathbf{i} - 346.4\mathbf{j}\} \text{ N}$$



(c)

Fig. 3–19

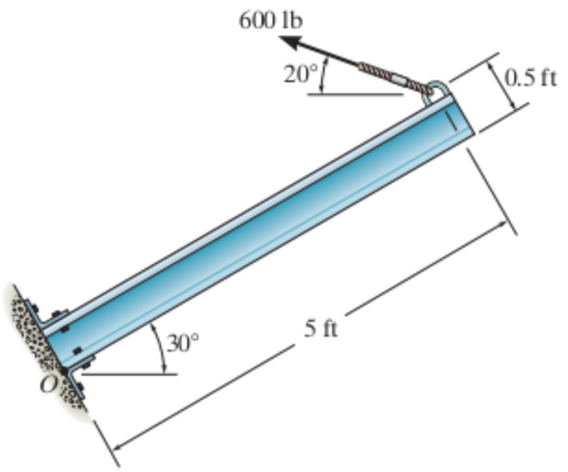
The moment is therefore

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.4 & -0.2 & 0 \\ 200.0 & -346.4 & 0 \end{vmatrix}$$
$$= 0\mathbf{i} - 0\mathbf{j} + [0.4(-346.4) - (-0.2)(200.0)]\mathbf{k}$$
$$= \{-98.6\mathbf{k}\} \text{ N} \cdot \text{m} \qquad \textit{Ans.}$$

NOTE: It is seen that the scalar analysis (Solution I) provides a more *convenient method* for analysis than Solution II since the direction of the moment and the moment arm for each component force are easy to establish. Hence, this method is generally recommended for solving problems displayed in two dimensions, whereas a Cartesian vector analysis is generally recommended only for solving three-dimensional problems.

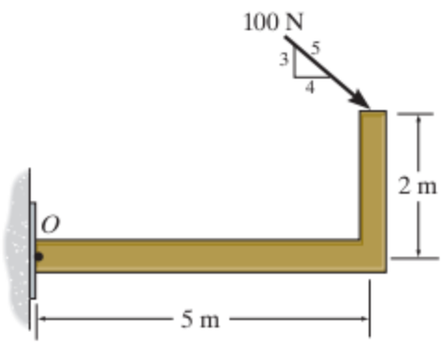
FUNDAMENTAL PROBLEMS

F3–1. Determine the moment of the force about point *O*.



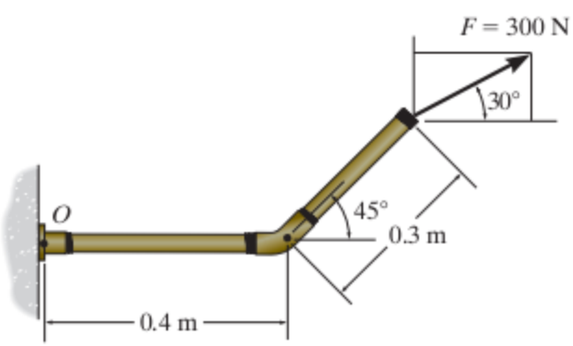
F3–1

F3–2. Determine the moment of the force about point *O*.



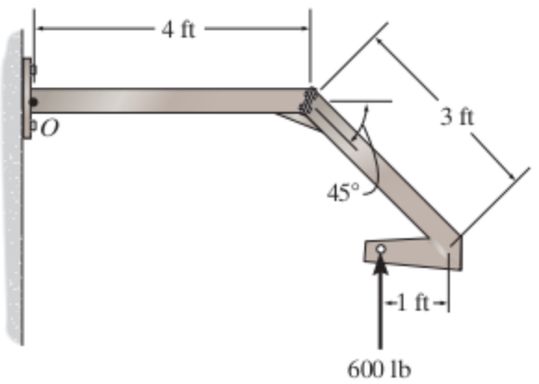
F3–2

F3–3. Determine the moment of the force about point *O*.



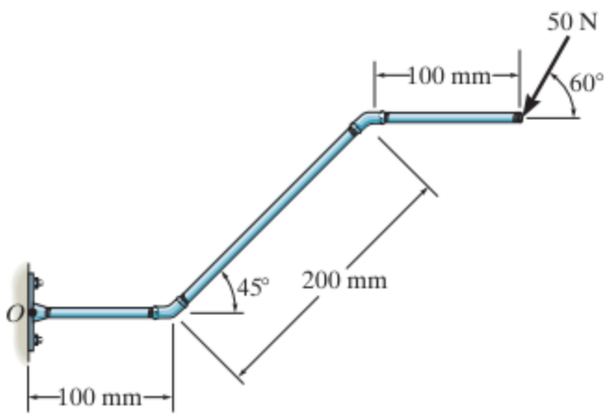
F3–3

F3–4. Determine the moment of the force about point *O*.



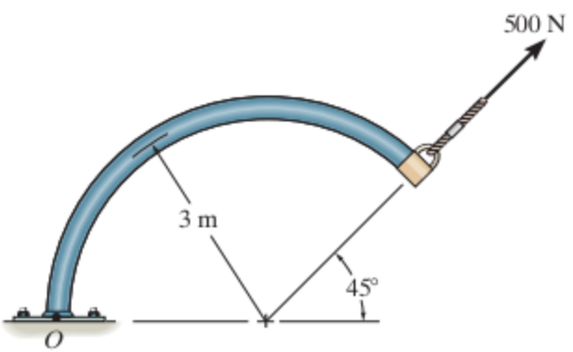
F3–4

F3–5. Determine the moment of the force about point *O*. Neglect the thickness of the member.



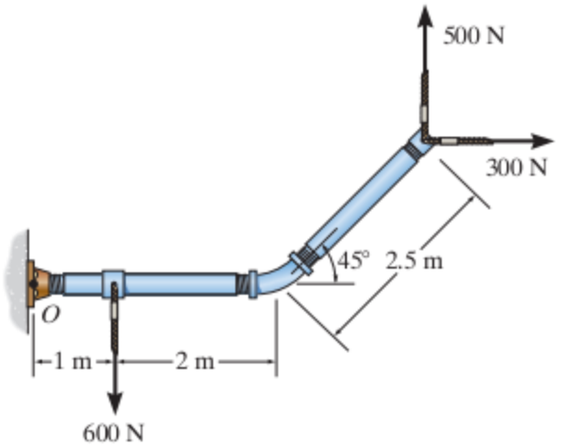
F3–5

F3–6. Determine the moment of the force about point *O*.



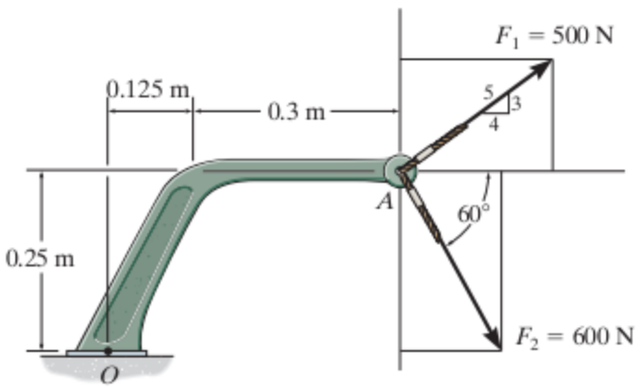
F3–6

F3–7. Determine the resultant moment produced by the forces about point *O*.



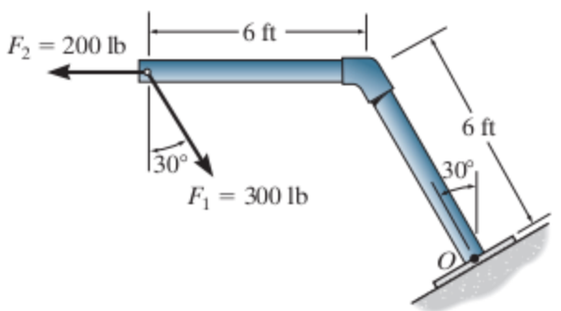
F3–7

F3–8. Determine the resultant moment produced by the forces about point *O*.



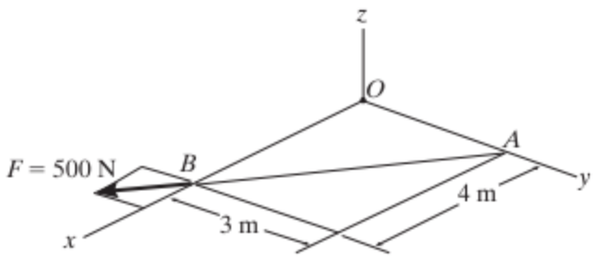
F3–8

F3–9. Determine the resultant moment produced by the forces about point *O*.



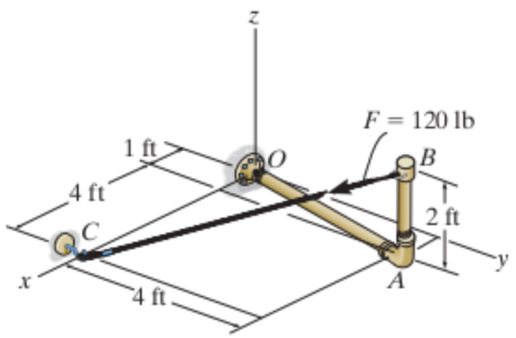
F3–9

F3–10. Determine the moment of force **F** about point *O*. Express the result as a Cartesian vector.



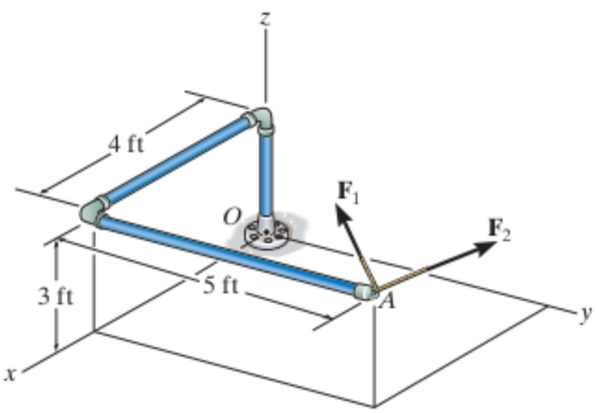
F3–10

F3–11. Determine the moment of force **F** about point *O*. Express the result as a Cartesian vector.



F3–11

F3–12. If $\mathbf{F}_1 = \{100\mathbf{i} - 120\mathbf{j} + 75\mathbf{k}\}$ lb and $\mathbf{F}_2 = \{-200\mathbf{i} + 250\mathbf{j} + 100\mathbf{k}\}$ lb, determine the resultant moment produced by these forces about point *O*. Express the result as a Cartesian vector.



F3–12

PROBLEMS

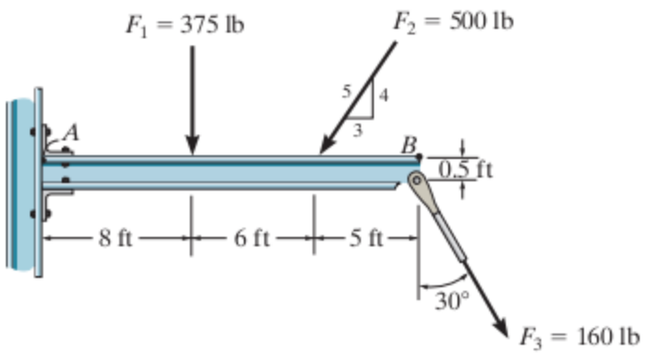
3–1. If **A**, **B**, and **D** are given vectors, prove the distributive law for the vector cross product, i.e., $\mathbf{A} \times (\mathbf{B} + \mathbf{D}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{D})$.

3–2. Prove the triple scalar product identity $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C}$.

3–3. Given the three nonzero vectors **A**, **B**, and **C**, show that if $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = 0$, the three vectors *must* lie in the same plane.

***3–4.** Determine the moment about point A of each of the three forces acting on the beam.

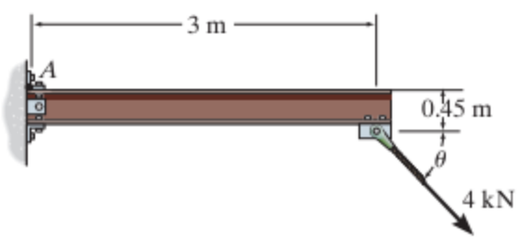
3–5. Determine the moment about point B of each of the three forces acting on the beam.



Probs. 3–4/5

3–6. If $\theta = 45°$, determine the moment produced by the 4-kN force about point A.

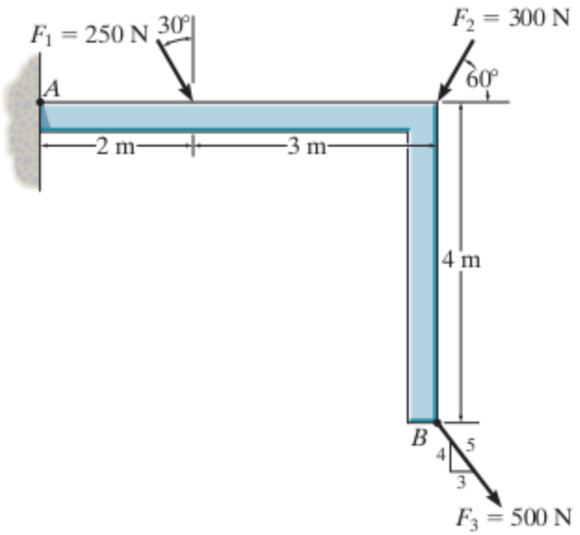
3–7. If the moment produced by the 4-kN force about point A is 10 kN · m clockwise, determine the angle θ, where $0° \leq \theta \leq 90°$.



Probs. 3–6/7

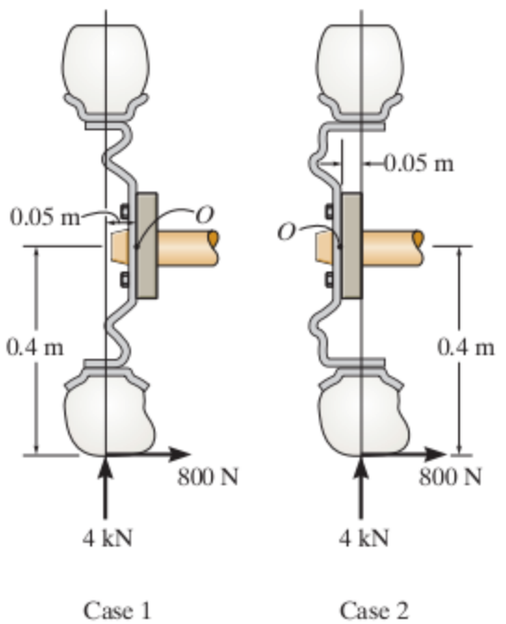
***3–8.** Determine the moment of each of the three forces about point A.

3–9. Determine the moment of each of the three forces about point B.



Probs. 3–8/9

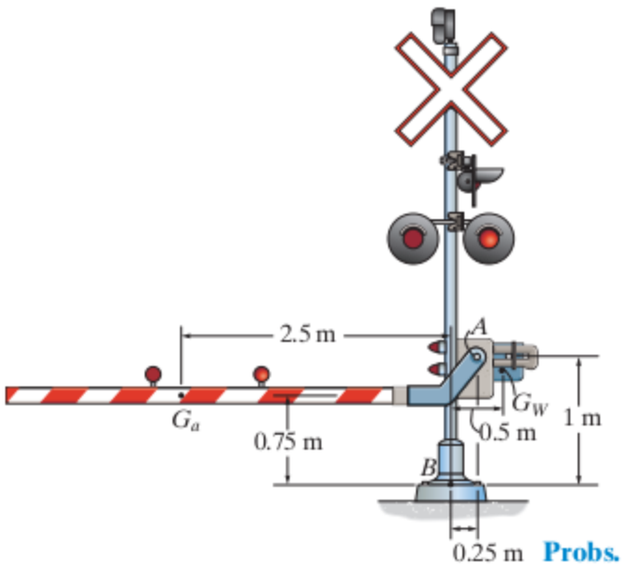
3–10. The hub of the wheel can be attached to the axle either with negative offset (left) or with positive offset (right). If the tire is subjected to both a normal and radial load as shown, determine the resultant moment of these loads about point O on the axle for both cases.



Prob. 3–10

3–11. The railway crossing gate consists of the 100-kg gate arm having a center of mass at G_a and the 250-kg counterweight having a center of mass at G_W. Determine the magnitude and directional sense of the resultant moment produced by the weights about point A.

***3–12.** The railway crossing gate consists of the 100-kg gate arm having a center of mass at G_a and the 250-kg counterweight having a center of mass at G_W. Determine the magnitude and directional sense of the resultant moment produced by the weights about point B.

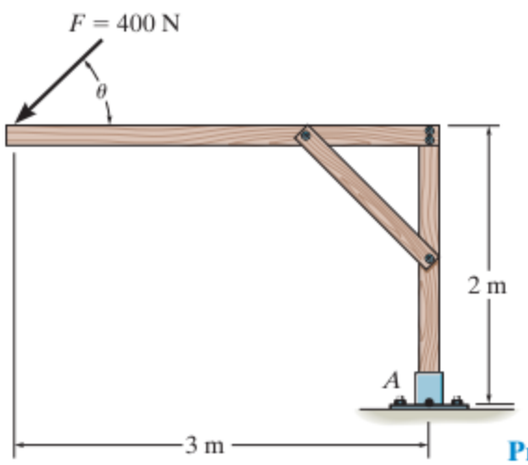


Probs. 3–11/12

3–13. Determine the direction θ for $0° \le \theta \le 180°$ of the force **F** so that it produces the maximum moment about point A. Calculate this moment.

3–14. Determine the moment of the force **F** about point A as a function of θ. Plot the results of M (ordinate) versus θ (abscissa) for $0° \le \theta \le 180°$.

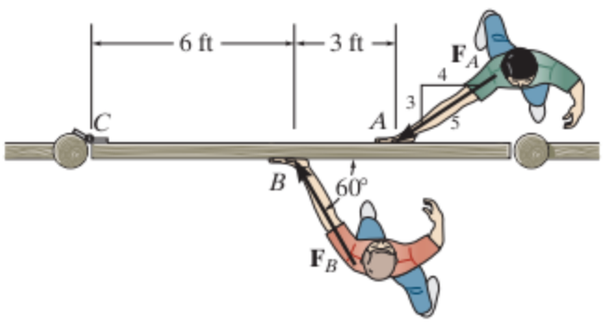
3–15. Determine the minimum moment produced by the force **F** about point A. Specify the angle θ $(0° \le \theta \le 180°)$.



Probs. 3–13/14/15

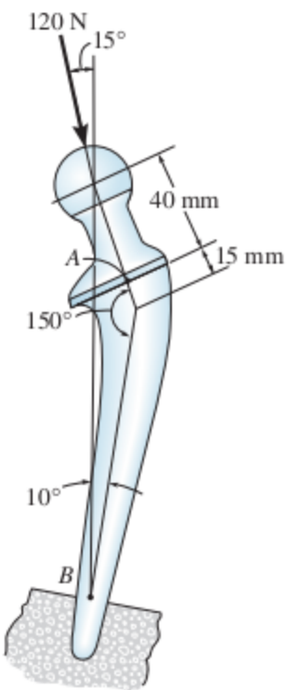
***3–16.** The two boys push on the gate with forces of $F_A = 30$ lb, and $F_B = 50$ lb, as shown. Determine the moment of each force about C. Which way will the gate rotate, clockwise or counterclockwise? Neglect the thickness of the gate.

3–17. Two boys push on the gate as shown. If the boy at B exerts a force of $F_B = 30$ lb, determine the magnitude of the force F_A the boy at A must exert in order to prevent the gate from turning. Neglect the thickness of the gate.



Probs. 3–16/17

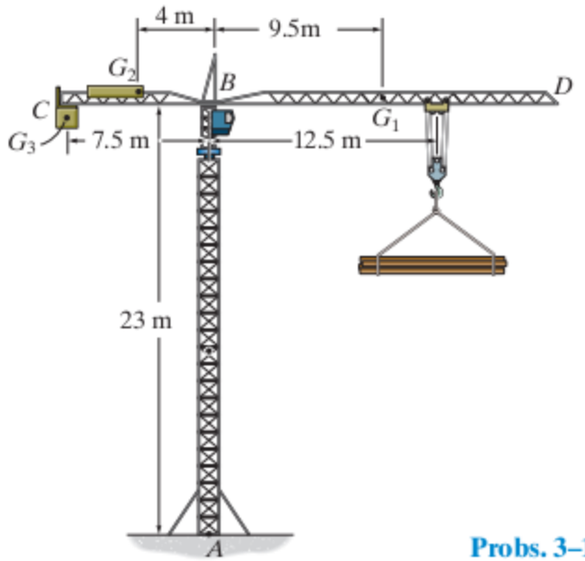
3–18. The total hip replacement is subjected to a force of $F = 120$ N. Determine the moment of this force about the neck at A and the stem at B.



Prob. 3–18

3–19. The tower crane is used to hoist the 2-Mg load upward at constant velocity. The 1.5-Mg jib BD, 0.5-Mg jib BC, and 6-Mg counterweight C have centers of mass at G_1, G_2, and G_3, respectively. Determine the resultant moment produced by the load and the weights of the tower crane jibs about point A and about point B.

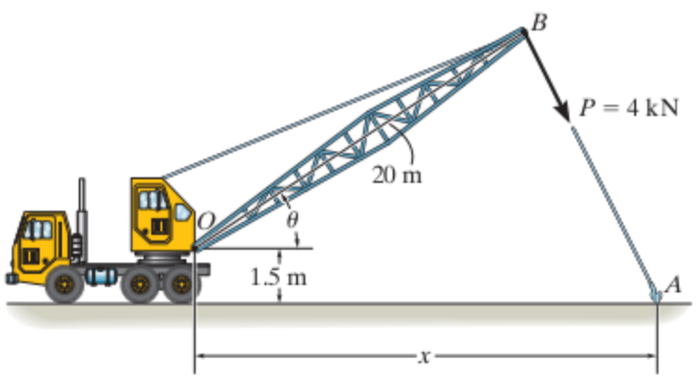
***3–20.** The tower crane is used to hoist a 2-Mg load upward at constant velocity. The 1.5-Mg jib BD and 0.5-Mg jib BC have centers of mass at G_1 and G_2, respectively. Determine the required mass of the counterweight C so that the resultant moment produced by the load and the weight of the tower crane jibs about point A is zero. The center of mass for the counterweight is located at G_3.



Probs. 3–19/20

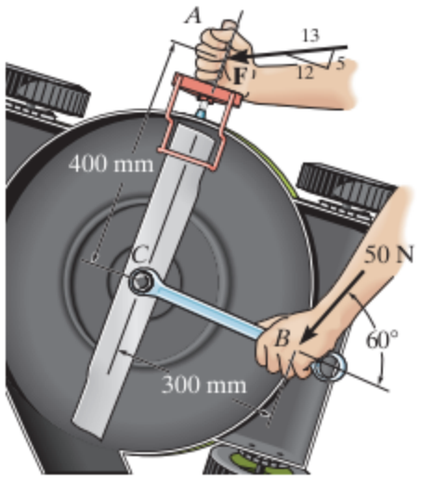
3–21. The towline exerts a force of $P = 4$ kN at the end of the 20-m-long crane boom. If $\theta = 30°$, determine the placement x of the hook at A so that this force creates a maximum moment about point O. What is this moment?

3–22. The towline exerts a force of $P = 4$ kN at the end of the 20-m-long crane boom. If $x = 25$ m, determine the position θ of the boom so that this force creates a maximum moment about point O. What is this moment?



Probs. 3–21/22

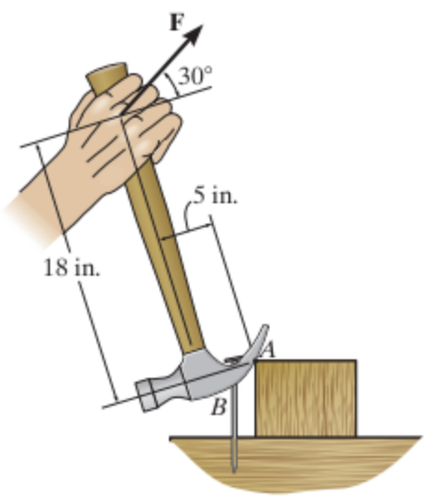
3–23. The tool at A is used to hold a power lawnmower blade stationary while the nut is being loosened with the wrench. If a force of 50 N is applied to the wrench at B in the direction shown, determine the moment it creates about the nut at C. What is the magnitude of force $\mathbf{F}$ at A so that it creates the opposite moment about C?



Prob. 3–23

***3–24.** The handle of the hammer is subjected to the force of $F = 20$ lb. Determine the moment of this force about the point A.

3–25. In order to pull out the nail at B, the force $\mathbf{F}$ exerted on the handle of the hammer must produce a clockwise moment of 500 lb · in. about point A. Determine the required magnitude of force $\mathbf{F}$.



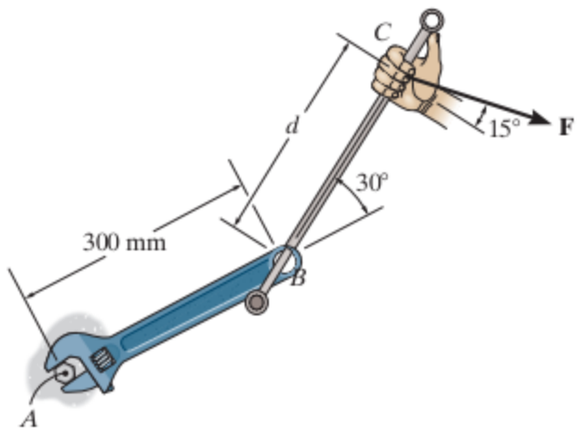
Probs. 3–24/25

3–26. The connected bar BC is used to increase the lever arm of the crescent wrench as shown. If the applied force is $F = 200\text{ N}$ and $d = 300\text{ mm}$, determine the moment produced by this force about the bolt at A.

3–27. The connected bar BC is used to increase the lever arm of the crescent wrench as shown. If a clockwise moment of $M_A = 120\text{ N}\cdot\text{m}$ is needed to tighten the bolt at A and the force $F = 200\text{ N}$, determine the required extension d in order to develop this moment.

3

***3–28.** The connected bar BC is used to increase the lever arm of the crescent wrench as shown. If a clockwise moment of $M_A = 120\text{ N}\cdot\text{m}$ is needed to tighten the nut at A and the extension $d = 300\text{ mm}$, determine the required force $\mathbf{F}$ in order to develop this moment.

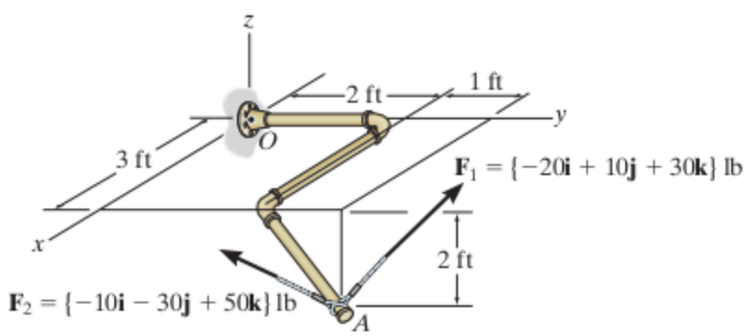


Probs. 3–26/27/28

3–29. Determine the moment produced by $\mathbf{F}_1$ about point O. Express the result as a Cartesian vector.

3–30. Determine the moment produced by $\mathbf{F}_2$ about point O. Express the result as a Cartesian vector.

3–31. Determine the resultant moment produced by the two forces about point O. Express the result as a Cartesian vector.

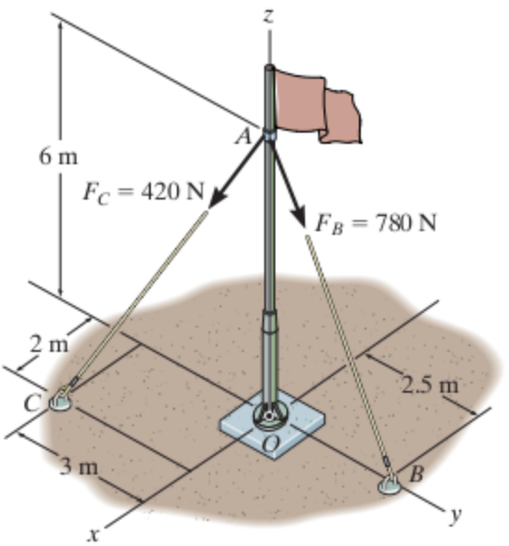


Probs. 3–29/30/31

***3–32.** Determine the moment produced by force $\mathbf{F}_B$ about point O. Express the result as a Cartesian vector.

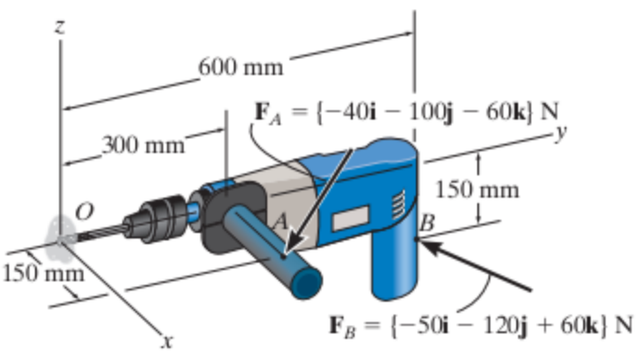
3–33. Determine the moment produced by force $\mathbf{F}_C$ about point O. Express the result as a Cartesian vector.

3–34. Determine the resultant moment produced by forces $\mathbf{F}_B$ and $\mathbf{F}_C$ about point O. Express the result as a Cartesian vector.



Probs. 3–32/33/34

3–35. Determine the moment produced by each force about point O located on the drill bit. Express the results as Cartesian vectors.

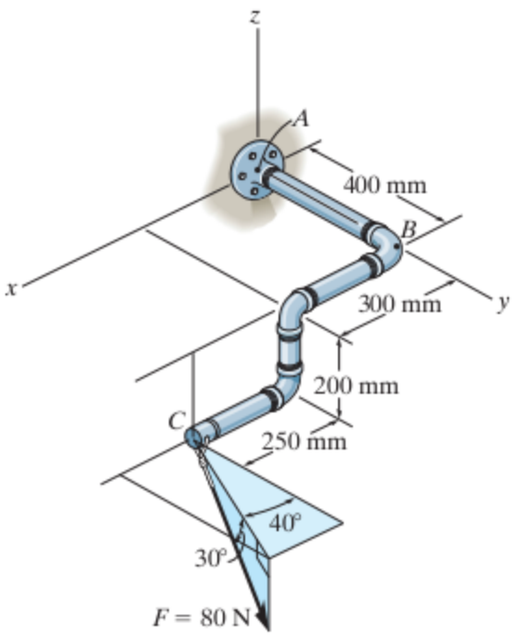


Prob. 3–35

*3–36. A force of $\mathbf{F} = \{6\mathbf{i} - 2\mathbf{j} + 1\mathbf{k}\}$ kN produces a moment of $\mathbf{M}_O = \{4\mathbf{i} + 5\mathbf{j} - 14\mathbf{k}\}$ kN·m about the origin of coordinates, point O. If the force acts at a point having an x coordinate of $x = 1$ m, determine the y and z coordinates.

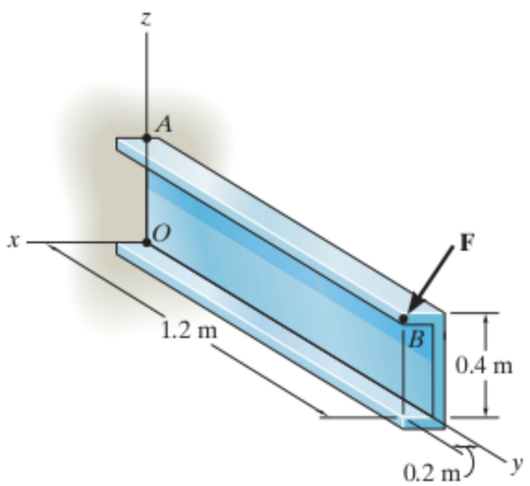
3–37. The pipe assembly is subjected to the 80-N force. Determine the moment of this force about point A.

3–38. The pipe assembly is subjected to the 80-N force. Determine the moment of this force about point B.



Probs. 3–37/38

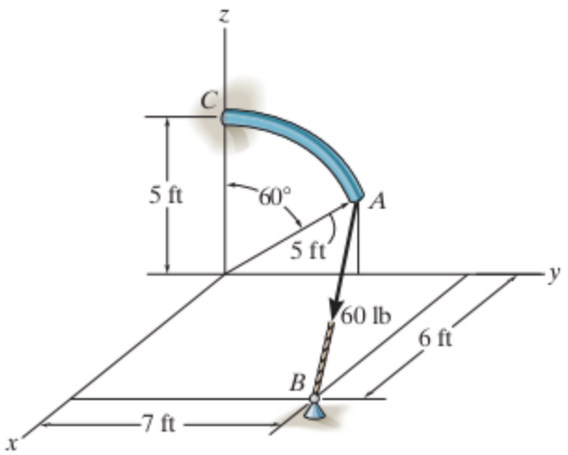
3–39. The force $\mathbf{F} = \{600\mathbf{i} + 300\mathbf{j} - 600\mathbf{k}\}$ N acts at the end of the beam. Determine the moment of the force about point A.



Prob. 3–39

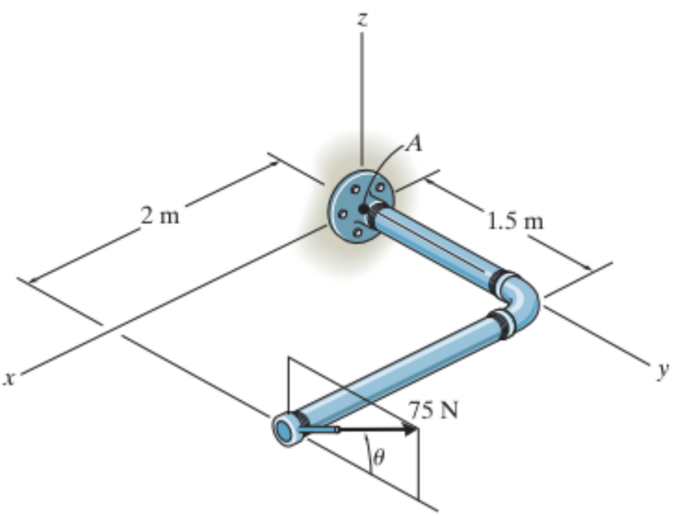
*3–40. The curved rod has a radius of 5 ft. If a force of 60 lb acts at its end as shown, determine the moment of this force about point C.

3–41. Determine the smallest force F that must be applied along the rope in order to cause the curved rod, which has a radius of 5 ft, to fail at the support C. This requires a moment of $M = 80$ lb·ft to be developed at C.



Probs. 3–40/41

3–42. Using a ring collar the 75-N force can act in the vertical plane at various angles θ. Determine the magnitude of the moment it produces about point A, plot the result of M (ordinate) versus θ (abscissa) for $0° \leq \theta \leq 180°$, and specify the angles that give the maximum and minimum moment.



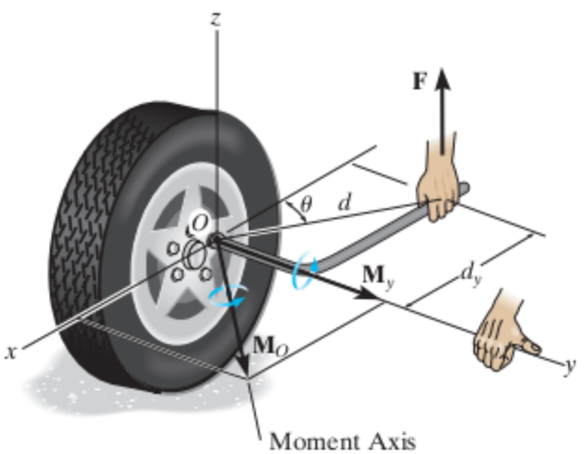
Prob. 3–42

3.5 Moment of a Force about a Specified Axis

Sometimes, the moment produced by a force about a *specified axis* must be determined. For example, suppose the lug nut at O on the car tire in Fig. 3–20*a* needs to be loosened. The force applied to the wrench will create a tendency for the wrench and the nut to rotate about the *moment axis* passing through O; however, the nut can only rotate about the y axis. Therefore, to determine the turning effect, only the y component of the moment is needed, and the total moment produced is not important. To determine this component, we can use either a scalar or vector analysis.

Scalar Analysis. To use a scalar analysis in the case of the lug nut in Fig. 3–20*a*, the moment arm perpendicular distance from the axis to the line of action of the force is $d_y = d\cos\theta$. Thus, the moment of $\mathbf{F}$ about the y axis is $M_y = F d_y = F(d\cos\theta)$. According to the right-hand rule, $\mathbf{M}_y$ is directed along the positive y axis as shown in the figure. In general, for any axis a, the moment is

$$M_a = Fd_a \tag{3–10}$$



(a)

Fig. 3–20

Vector Analysis. To find the moment of force **F** in Fig. 3–20*b* about the *y* axis using a vector analysis, we must first determine the moment of the force about *any point O* on the *y* axis by applying Eq. 3–7, $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$. The component $\mathbf{M}_y$ along the *y* axis is the *projection* of $\mathbf{M}_O$ onto the *y* axis. It can be found using the *dot product* discussed in Chapter 2, so that $M_y = \mathbf{j} \cdot \mathbf{M}_O = \mathbf{j} \cdot (\mathbf{r} \times \mathbf{F})$, where **j** is the unit vector for the *y* axis.

We can generalize this approach by letting $\mathbf{u}_a$ be the unit vector that specifies the direction of the *a* axis shown in Fig. 3–21. Then the moment of **F** about the axis is $M_a = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F})$. This combination is referred to as the *scalar triple product*. If the vectors are written in Cartesian form, we have

$$M_a = [u_{a_x}\mathbf{i} + u_{a_y}\mathbf{j} + u_{a_z}\mathbf{k}] \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= u_{a_x}(r_yF_z - r_zF_y) - u_{a_y}(r_xF_z - r_zF_x) + u_{a_z}(r_xF_y - r_yF_x)$$

This result can also be written in the form of a determinant, making it easier to memorize.*

$$M_a = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F}) = \begin{vmatrix} u_{a_x} & u_{a_y} & u_{a_z} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \tag{3–11}$$

where

$u_{a_x}, u_{a_y}, u_{a_z}$ represent the *x*, *y*, *z* components of the unit vector defining the direction of the *a* axis

r_x, r_y, r_z represent the *x*, *y*, *z* components of the position vector extended from *any point O* on the *a* axis to *any point A* on the line of action of the force

F_x, F_y, F_z represent the *x*, *y*, *z* components of the force vector.

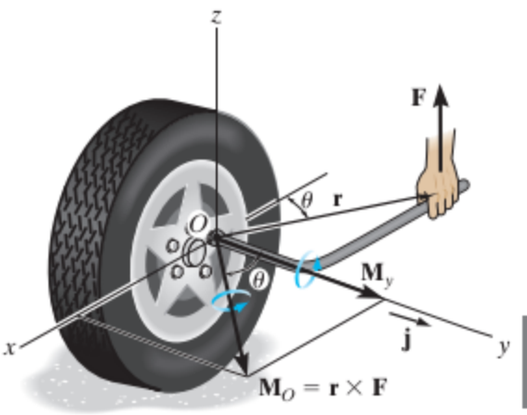
When M_a is evaluated from Eq. 3–11, it will yield a positive or negative scalar. The sign of this scalar indicates the sense of direction of $\mathbf{M}_a$ along the *a* axis. If it is positive, then $\mathbf{M}_a$ will have the same sense as $\mathbf{u}_a$, whereas if it is negative, then $\mathbf{M}_a$ will act opposite to $\mathbf{u}_a$.

Once M_a is determined, we can then express $\mathbf{M}_a$ as a Cartesian vector, namely,

$$\mathbf{M}_a = M_a\mathbf{u}_a \tag{3–12}$$

The examples which follow illustrate numerical applications of the above concepts.

*Take a moment to expand this determinant, to show that it will yield the above result.



(b)

Fig. 3–20 (cont.)

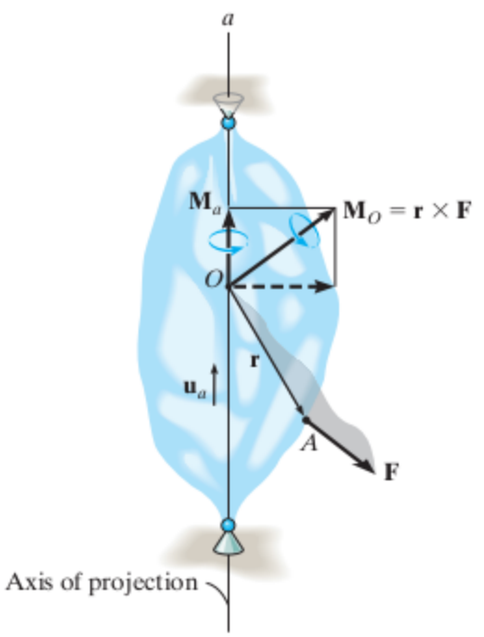


Fig. 3–21

Important Points

- The moment of a force about a specified axis can be determined provided the perpendicular distance d_a from the force line of action to the axis can be determined. $M_a = Fd_a$.
- If vector analysis is used, $M_a = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F})$, where $\mathbf{u}_a$ defines the direction of the axis and $\mathbf{r}$ is extended from *any point* on the axis to *any point* on the line of action of the force.
- If M_a is calculated as a negative scalar, then the sense of direction of $\mathbf{M}_a$ is opposite to $\mathbf{u}_a$.
- The moment $\mathbf{M}_a$ expressed as a Cartesian vector is determined from $\mathbf{M}_a = M_a\mathbf{u}_a$.

EXAMPLE 3.7

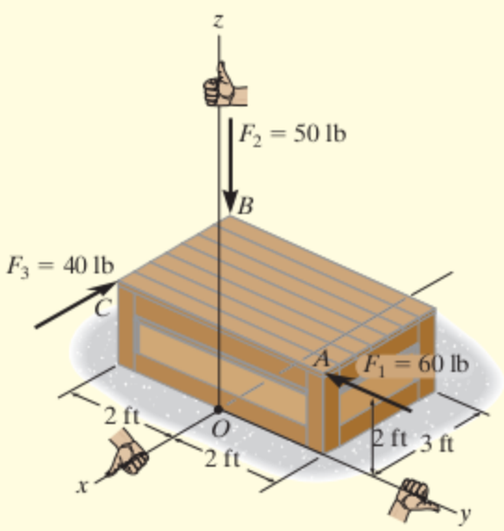


Fig. 3–22

Determine the resultant moment of the three forces in Fig. 3–22 about the x axis, the y axis, and the z axis.

SOLUTION

A force that is *parallel* to a coordinate axis or has a line of action that passes through the axis does *not* produce any moment or tendency for turning about that axis. Therefore, defining the positive direction of the moment of a force according to the right-hand rule, as shown in the figure, we have

$$M_x = (60\text{ lb})(2\text{ ft}) + (50\text{ lb})(2\text{ ft}) + 0 = 220\text{ lb}\cdot\text{ft} \qquad \textit{Ans.}$$

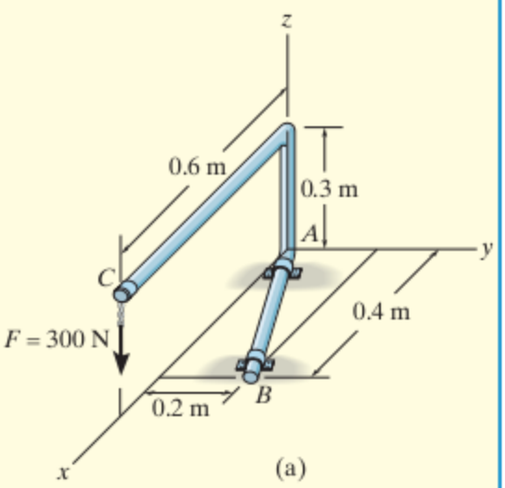
$$M_y = 0 - (50\text{ lb})(3\text{ ft}) - (40\text{ lb})(2\text{ ft}) = -230\text{ lb}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_z = 0 + 0 - (40\text{ lb})(2\text{ ft}) = -80\text{ lb}\cdot\text{ft} \qquad \textit{Ans.}$$

The negative signs indicate that $\mathbf{M}_y$ and $\mathbf{M}_z$ act in the $-y$ and $-z$ directions, respectively.

EXAMPLE 3.8

Determine the moment $\mathbf{M}_{AB}$ produced by the force $\mathbf{F}$ in Fig. 3–23*a*, which tends to rotate the rod about the AB axis.



(a)

SOLUTION

A vector t using $M_{AB} = \mathbf{u}_B \cdot (\mathbf{r} \times \mathbf{F})$ will be considered for the solution rather than trying to find the moment arm or perpendicular distance from the line of action of $\mathbf{F}$ to the AB axis. Each of the terms in the equation will now be identified.

Unit vector $\mathbf{u}_B$ defines the direction of the AB axis of the rod, Fig. 3–23*b*, where

$$\mathbf{u}_B = \frac{\mathbf{r}_B}{r_B} = \frac{\{0.4\mathbf{i} + 0.2\mathbf{j}\}\ \text{m}}{\sqrt{(0.4\ \text{m})^2 + (0.2\ \text{m})^2}} = 0.8944\mathbf{i} + 0.4472\mathbf{j}$$

Vector $\mathbf{r}$ is directed from *any point* on the AB axis to *any point* on the line of action of the force. For example, position vectors $\mathbf{r}_C$ and $\mathbf{r}_D$ are suitable, Fig. 3–23*b*. (Although not shown, $\mathbf{r}_{BC}$ or $\mathbf{r}_{BD}$ can also be used.) For simplicity, we choose $\mathbf{r}_D$, where

$$\mathbf{r}_D = \{0.6\mathbf{i}\}\ \text{m}$$

The force is

$$\mathbf{F} = \{-300\mathbf{k}\}\ \text{N}$$

Substituting these vectors into the determinant form and expanding, we have

$$M_{AB} = \mathbf{u}_B \cdot (\mathbf{r}_D \times \mathbf{F}) = \begin{vmatrix} 0.8944 & 0.4472 & 0 \\ 0.6 & 0 & 0 \\ 0 & 0 & -300 \end{vmatrix}$$

$$= 0.8944[0(-300) - 0(0)] - 0.4472[0.6(-300) - 0(0)] + 0[0.6(0) - 0(0)]$$

$$= 80.50\ \text{N} \cdot \text{m}$$

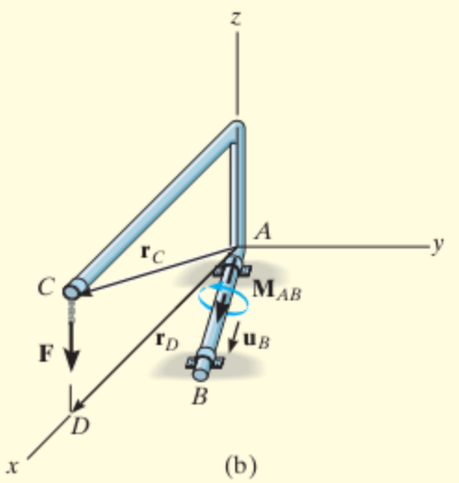
This positive result indicates that the sense of $\mathbf{M}_{AB}$ is in the same direction as $\mathbf{u}_B$.

Expressing $\mathbf{M}_{AB}$ in Fig. 3–23*b* as a Cartesian vector yields

$$\mathbf{M}_{AB} = M_{AB}\mathbf{u}_B = (80.50\ \text{N} \cdot \text{m})(0.8944\mathbf{i} + 0.4472\mathbf{j})$$

$$= \{72.0\mathbf{i} + 36.0\mathbf{j}\}\ \text{N} \cdot \text{m} \qquad \textit{Ans.}$$

NOTE: If axis AB is defined using a unit vector directed from B toward A, then in the above formulation $-\mathbf{u}_B$ would have to be used. This would lead to $M_{AB} = -80.50\ \text{N} \cdot \text{m}$. Consequently, $\mathbf{M}_{AB} = M_{AB}(-\mathbf{u}_B)$, and the same result would be obtained.



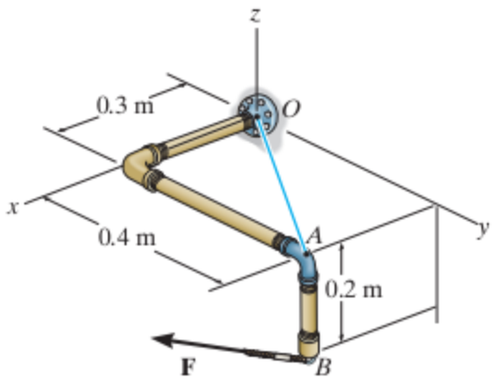
(b)

Fig. 3–23

FUNDAMENTAL PROBLEMS

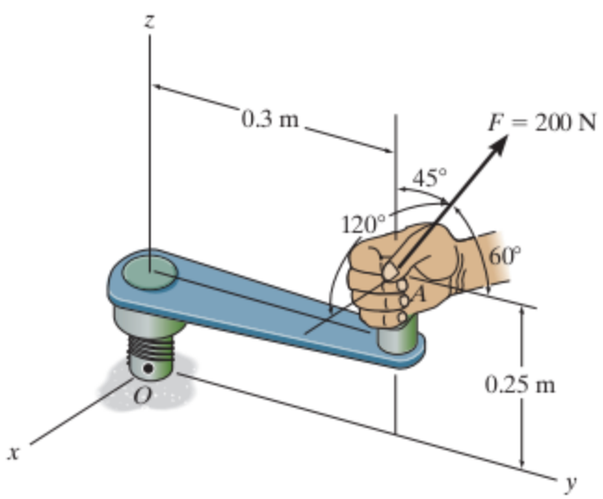
F3–13. Determine the magnitude of the moment of the force $\mathbf{F} = \{300\mathbf{i} - 200\mathbf{j} + 150\mathbf{k}\}$ N about the x axis.

F3–14. Determine the magnitude of the moment of the force $\mathbf{F} = \{300\mathbf{i} - 200\mathbf{j} + 150\mathbf{k}\}$ N about the OA axis.



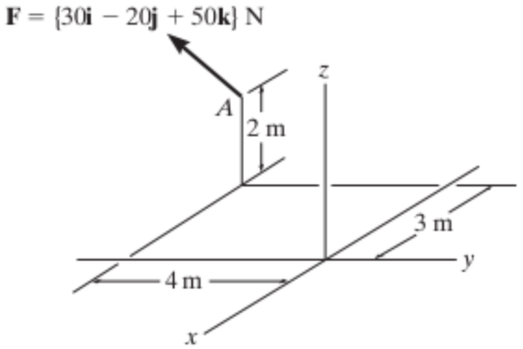
F3–13/14

F3–15. Determine the magnitude of the moment of the 200-N force about the x axis.



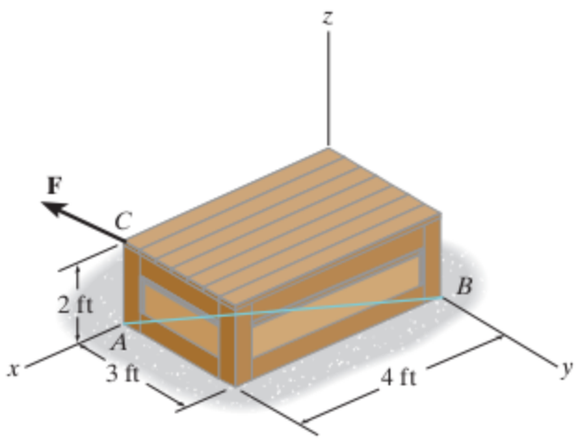
F3–15

F3–16. Determine the magnitude of the moment of the force about the y axis.



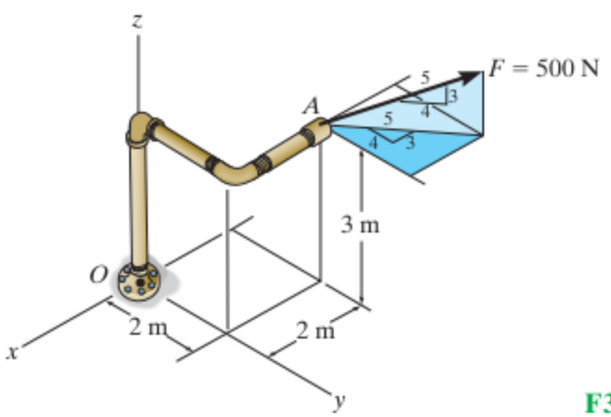
F3–16

F3–17. Determine the moment of the force $\mathbf{F} = \{50\mathbf{i} - 40\mathbf{j} + 20\mathbf{k}\}$ lb about the AB axis. Express the result as a Cartesian vector.



F3–17

F3–18. Determine the moment of force $\mathbf{F}$ about the x, the y, and the z axes. Use a scalar analysis.

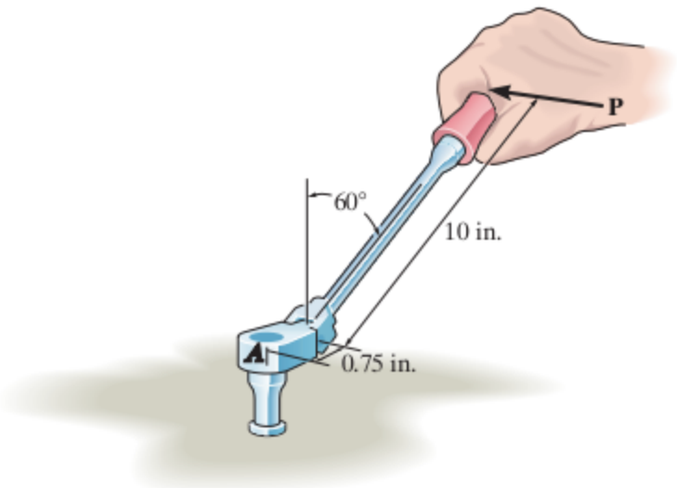


F3–18

PROBLEMS

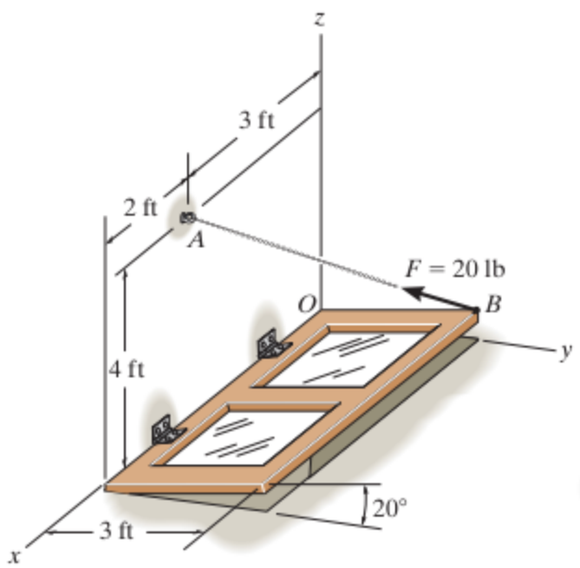
3–43. The flex-headed ratchet wrench is subjected to a force of $P = 16$ lb, applied perpendicular to the handle as shown. Determine the moment or torque this imparts along the vertical axis of the bolt at A.

***3–44.** If a torque or moment of 80 lb · in. is required to loosen the bolt at A, determine the force P that must be applied perpendicular to the handle of the flex-headed ratchet wrench.



Probs. 3–43/44

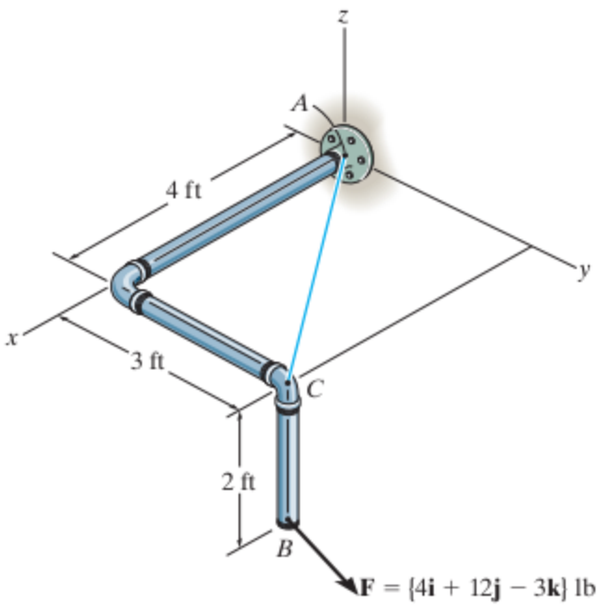
3–45. The chain AB exerts a force of 20 lb on the door at B. Determine the magnitude of the moment of this force along the hinged axis x of the door.



Prob. 3–45

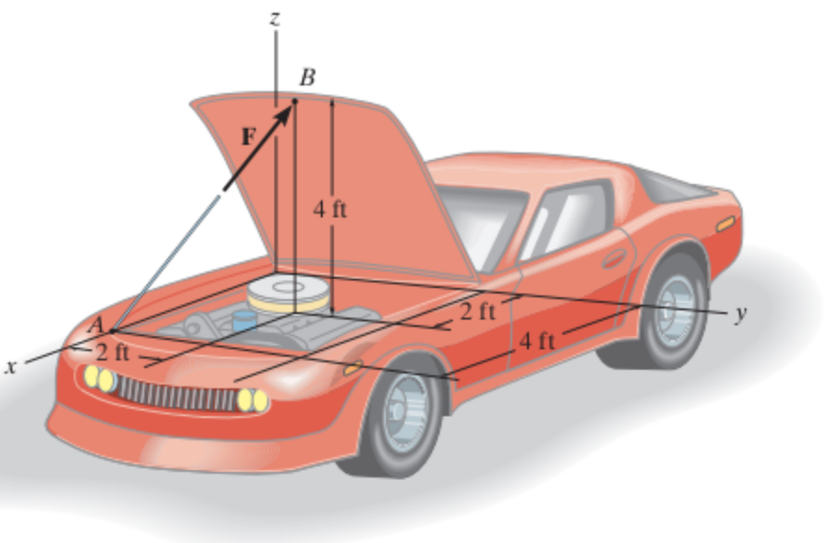
3–46. Determine the magnitude of the moments of the force $\mathbf{F}$ about the x, y, and z axes. Solve the problem (a) using a Cartesian vector approach and (b) using a scalar approach.

3–47. Determine the moment of the force $\mathbf{F}$ about an axis extending between A and C. Express the result as a Cartesian vector.



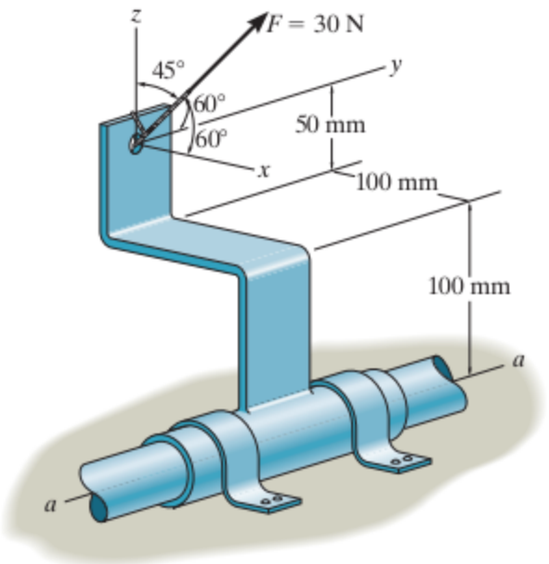
Probs. 3–46/47

***3–48.** The hood of the automobile is supported by the strut AB, which exerts a force of $F = 24$ lb on the hood. Determine the moment of this force about the hinged axis y.



Prob. 3–48

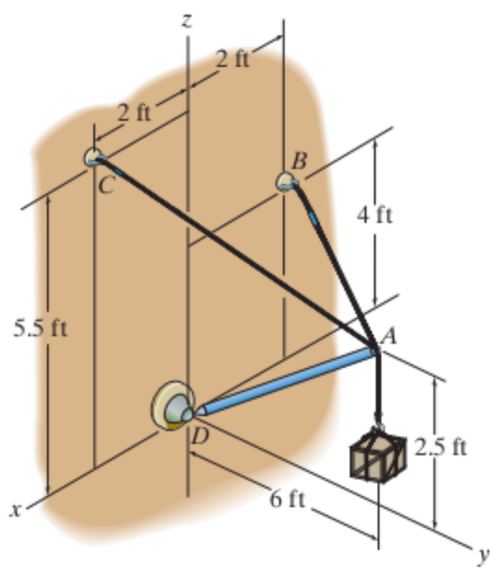
*3–49. The force of $F = 30\,\text{N}$ acts on the bracket as shown. Determine the moment of the force about the $a–a$ axis of the pipe. Also, determine the coordinate direction angles of F in order to produce the maximum moment about the $a–a$ axis. What is this moment?



Prob. 3–49

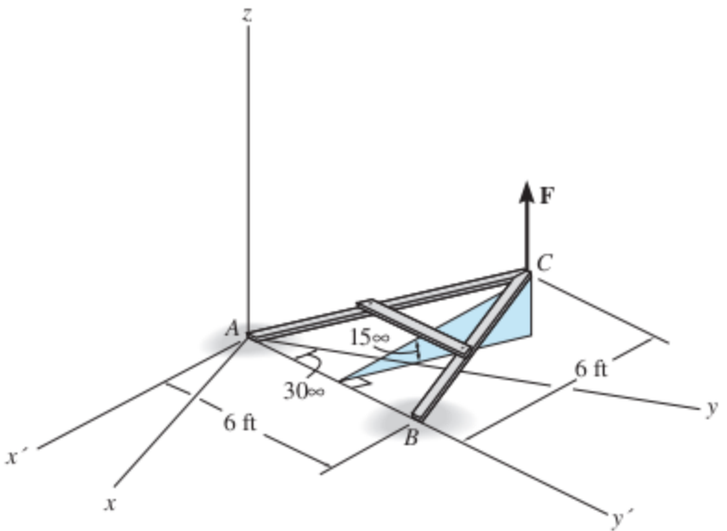
3–50. Determine the tension developed in cables AB and AC and the force developed along strut AD for equilibrium of the 400-lb crate.

3–51. If the tension developed in each of the cables cannot exceed 300 lb, determine the largest weight of the crate that can be supported. Also, what is the force developed along strut AD?



Probs. 3–50/51

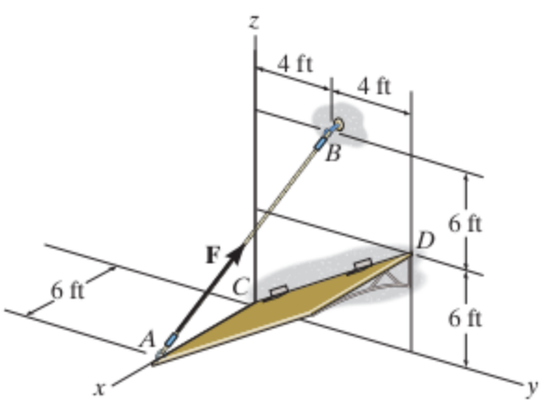
3–52. The A-frame is being hoisted into an upright position by the vertical force of $F = 80\,\text{lb}$. Determine the moment of this force about the y axis when the frame is in the position shown.



Prob. 3–52

3–53. If the tension in the cable is $F = 140\,\text{lb}$, determine the magnitude of the moment produced by this force about the hinged axis, CD, of the panel.

3–54. Determine the magnitude of force **F** in cable AB in order to produce a moment of $500\,\text{lb}\cdot\text{ft}$ about the hinged axis CD, which is needed to hold the panel in the position shown.



Probs. 3–53/54

3.6 Moment of a Couple

A *couple* is defined as two parallel forces that have the same magnitude, but opposite directions, and are separated by a perpendicular distance d, Fig. 3–24. Since the resultant force is zero, the only effect of a couple is to produce a rotation or tendency of rotation in a specified direction. For example, imagine that you are driving a car with both hands on the steering wheel and you are making a turn. One hand will push up on the wheel while the other hand pulls down, which causes the steering wheel to rotate.

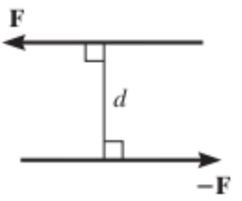


Fig. 3–24

The moment produced by a couple is called a *couple moment*. We can determine its value by finding the sum of the moments of both couple forces about *any* arbitrary point. For example, in Fig. 3–25, position vectors $\mathbf{r}_A$ and $\mathbf{r}_B$ are directed from point O to points A and B lying on the line of action of $-\mathbf{F}$ and $\mathbf{F}$. The couple moment determined about O is therefore

$$\mathbf{M} = \mathbf{r}_B \times \mathbf{F} + \mathbf{r}_A \times -\mathbf{F} = (\mathbf{r}_B - \mathbf{r}_A) \times \mathbf{F}$$

However $\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}$ or $\mathbf{r} = \mathbf{r}_B - \mathbf{r}_A$, so that

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} \tag{3–13}$$

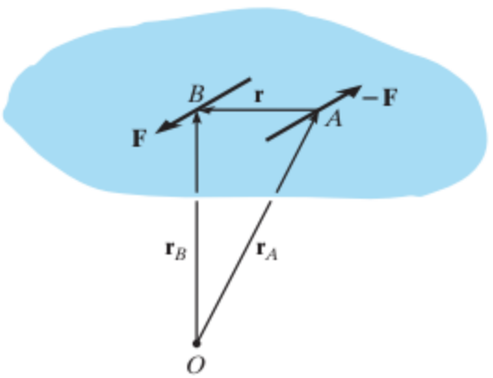


Fig. 3–25

This result indicates that a couple moment is a *free vector*, i.e., it can act at *any point* since $\mathbf{M}$ depends *only* upon the position vector $\mathbf{r}$ directed *between* the forces and *not* the position vectors $\mathbf{r}_A$ and $\mathbf{r}_B$, directed from the arbitrary point O to the forces. This concept is unlike the moment of a force, which requires a definite point (or axis) about which moments are determined.

Scalar Formulation. The moment of a couple, $\mathbf{M}$, Fig. 3–26, is defined as having a *magnitude* of

$$M = Fd \tag{3–14}$$

where F is the magnitude of one of the forces and d is the perpendicular distance or moment arm between the forces. The *direction* and sense of the couple moment are determined by the right-hand rule, where the thumb indicates this direction when the fingers are curled with the sense of rotation caused by the couple forces. In all cases, $\mathbf{M}$ will act perpendicular to the plane containing these forces.

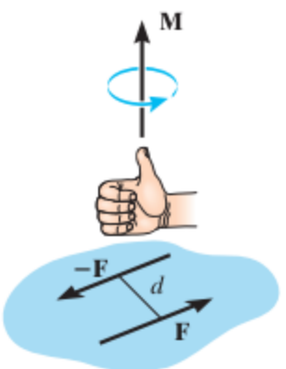


Fig. 3–26

Vector Formulation. The moment of a couple can also be expressed by the vector cross product using Eq. 3–13, i.e.,

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} \tag{3–15}$$

Application of theeis equation is easily remembered if one thinks of taking the moments of both forces about a point lying on the line of action of one of the forces. For example, if moments are taken about point A in Fig. 3–25, the moment of $-\mathbf{F}$ is *zero* about this point, and the moment of $\mathbf{F}$ is defined from Eq. 3–15. Therefore, in the formulation $\mathbf{r}$ is crossed with the force $\mathbf{F}$ to which it is directed.

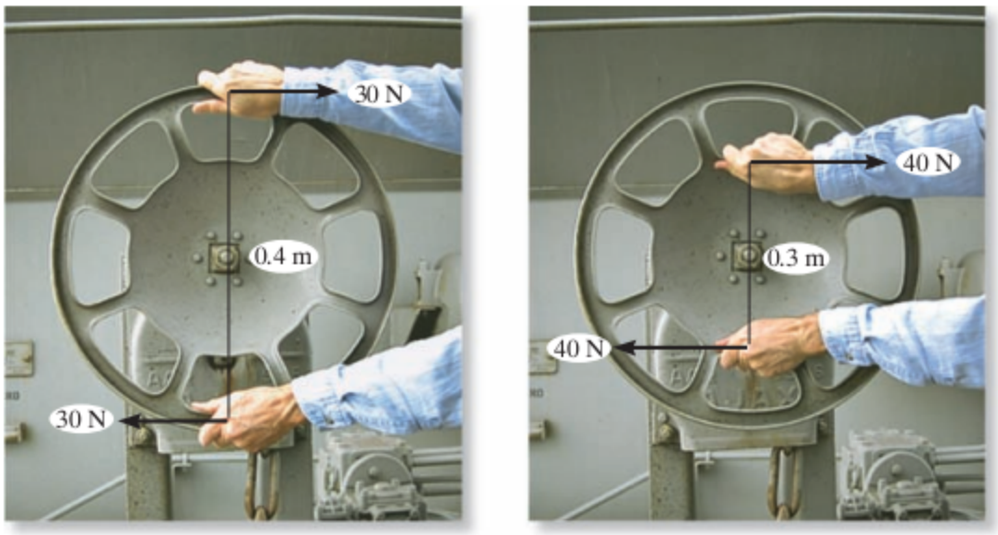


Fig. 3–27

Equivalent Couples. If two couples produce a moment with the *same magnitude and direction*, then these two couples are *equivalent*. For example, the two couples shown in Fig. 3–27 are *equivalent* because each couple moment has a magnitude of $M = 30\,\text{N}(0.4\,\text{m}) = 40\,\text{N}(0.3\,\text{m}) = 12\,\text{N}\cdot\text{m}$, and each is directed into the plane of the page. Notice that larger forces are required in the second case to create the same turning effect because the hands are placed closer together. Also, if the wheel was connected to the shaft at a point other than at its center, then the wheel would still turn when each couple is applied since the $12\,\text{N}\cdot\text{m}$ couple is a free vector.

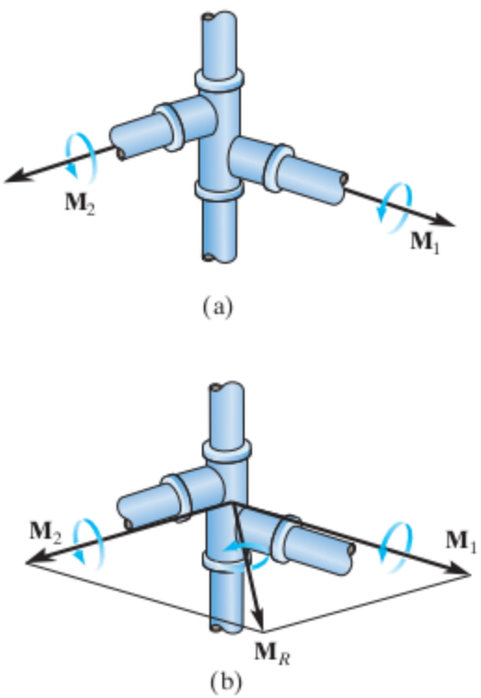


Fig. 3–28

Resultant Couple Moment. Since couple moments are vectors, their resultant can be determined by vector addition. For example, consider the couple moments $\mathbf{M}_1$ and $\mathbf{M}_2$ acting on the pipe in Fig. 3–28*a*. Since each couple moment is a free vector, we can join their tails at any arbitrary point and find the resultant couple moment, $\mathbf{M}_R = \mathbf{M}_1 + \mathbf{M}_2$ as shown in Fig. 3–28*b*.

If more than two couple moments act on the body, we may generalize this concept and write the vector resultant as

$$\mathbf{M}_R = \Sigma(\mathbf{r} \times \mathbf{F}) \tag{3–16}$$

These concepts are illustrated numerically in the examples that follow. In general, problems projected in two dimensions should be solved using a scalar analysis since the moment arms and force components are easy to determine.

Important Points

- A couple moment is produced by two noncollinear forces that are equal in magnitude but opposite in direction. Its effect is to produce pure rotation, or tendency for rotation in a specified direction.
- A couple moment is a free vector, and as a result it causes the same rotational effect on a body regardless of where the couple moment is applied to the body.
- The moment of the two couple forces can be determined about *any point*. For convenience, this point is often chosen on the line of action of one of the forces in order to eliminate the moment of this force about the point.
- In three dimensions the couple moment is often determined using the vector formulation, $\mathbf{M} = \mathbf{r} \times \mathbf{F}$, where $\mathbf{r}$ is directed from *any point* on the line of action of one of the forces to *any point* on the line of action of the other force $\mathbf{F}$.
- A resultant couple moment is simply the vector sum of all the couple moments of the system.

Steering wheels on vehicles have been made smaller than on older vehicles because power steering does not require the driver to apply a large couple moment to the rim of the wheel.

3

EXAMPLE 3.9

Determine the resultant couple moment of the three couples acting on the plate in Fig. 3–29.

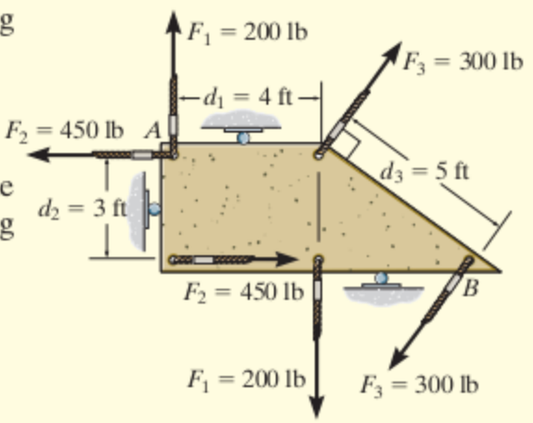


Fig. 3–29

SOLUTION

As shown the perpendicular distances between each pair of couple forces are $d_1 = 4$ ft, $d_2 = 3$ ft, and $d_3 = 5$ ft. Considering counterclockwise couple moments as positive, we have

$$\curvearrowleft + M_R = \Sigma M;\ M_R = -F_1 d_1 + F_2 d_2 - F_3 d_3$$

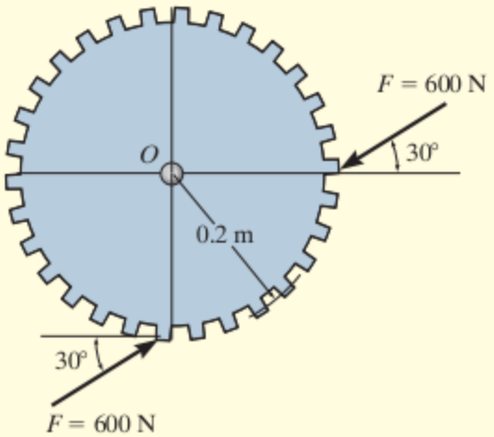
$$= -(200\text{ lb})(4\text{ ft}) + (450\text{ lb})(3\text{ ft}) - (300\text{ lb})(5\text{ ft})$$

$$= -950\text{ lb}\cdot\text{ft} = 950\text{ lb}\cdot\text{ft}\ \curvearrowright \qquad \textit{Ans.}$$

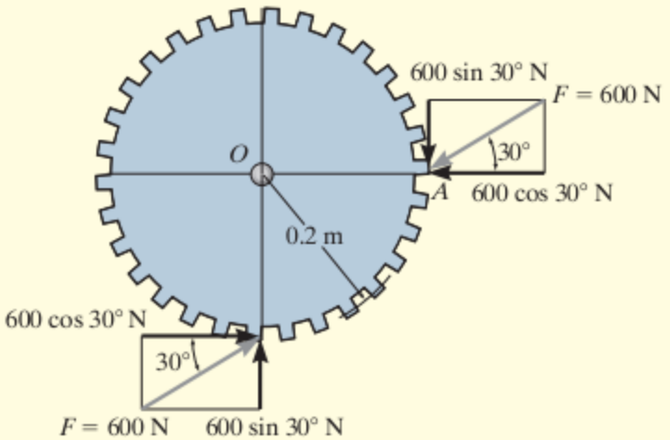
The negative sign indicates that $\mathbf{M}_R$ has a clockwise rotational sense.

EXAMPLE 3.10

Determine the magnitude and direction of the couple moment acting on the gear in Fig. 3–30*a*.



(a)



(b)

SOLUTION

The easiest solution requires resolving each force into its components as shown in Fig. 3–30*b*. The couple moment can be determined by summing the moments of these force components about any point, for example, the center O of the gear or point A. If we consider counterclockwise moments as positive, we have

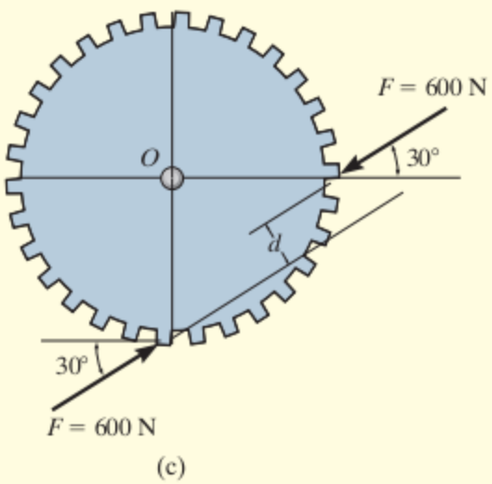
$$\circlearrowleft + M = \Sigma M_O;\ M = (600 \cos 30°\ \text{N})(0.2\ \text{m}) - (600 \sin 30°\ \text{N})(0.2\ \text{m})$$
$$= 43.9\ \text{N} \cdot \text{m} \circlearrowleft \qquad \textit{Ans.}$$

or

$$\circlearrowleft + M = \Sigma M_A;\ M = (600 \cos 30°\ \text{N})(0.2\ \text{m}) - (600 \sin 30°\ \text{N})(0.2\ \text{m})$$
$$= 43.9\ \text{N} \cdot \text{m} \circlearrowleft \qquad \textit{Ans.}$$

This positive result indicates that **M** has a counterclockwise rotational sense, so it is directed outward, perpendicular to the page.

NOTE: The same result can also be obtained using $M = Fd$, where d is the perpendicular distance between the lines of action of the couple forces, Fig. 3–30*c*. However, the computation for d is more involved. Realize that the couple moment is a free vector and can act at any point on the gear and produce the same turning effect about point O.

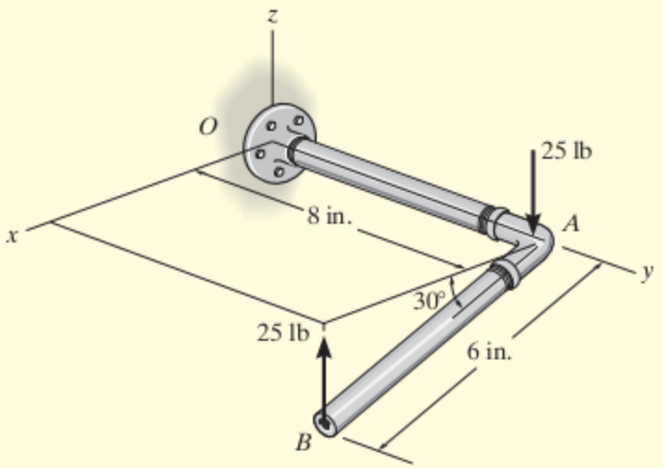


(c)

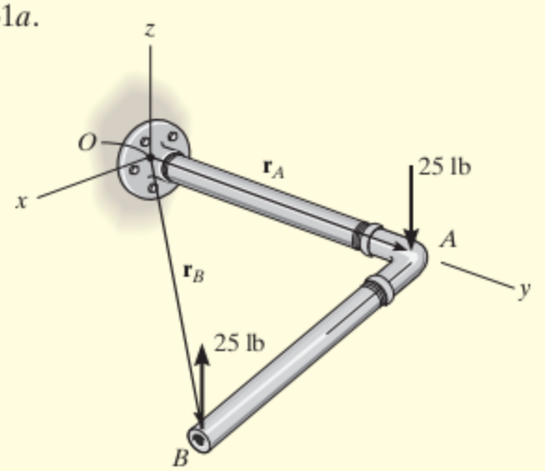
Fig. 3–30

EXAMPLE 3.11

Determine the couple moment acting on the pipe shown in Fig. 3–31*a*. Segment *AB* is directed 30° below the *x*–*y* plane.



(a)



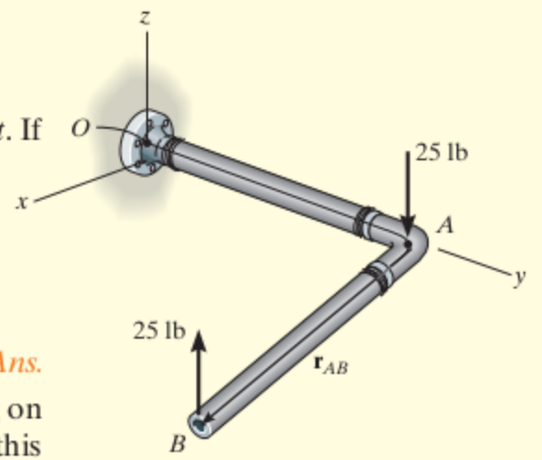
(b)

SOLUTION I *(VECTOR ANALYSIS)*

The moment of the two couple forces can be found about *any point*. If point *O* is considered, Fig. 3–31*b*, we have

$$\begin{aligned}\mathbf{M} &= \mathbf{r}_A \times (-25\mathbf{k}) + \mathbf{r}_B \times (25\mathbf{k}) \\ &= (8\mathbf{j}) \times (-25\mathbf{k}) + (6\cos 30°\mathbf{i} + 8\mathbf{j} - 6\sin 30°\mathbf{k}) \times (25\mathbf{k}) \\ &= -200\mathbf{i} - 129.9\mathbf{j} + 200\mathbf{i} \\ &= \{-130\mathbf{j}\}\ \text{lb}\cdot\text{in.}\end{aligned}$$

Ans.



(c)

It is *easier* to take moments of the couple forces about a point lying on the line of action of one of the forces, e.g., point *A*, Fig. 3–31*c*. In this case the moment of the force at *A* is zero, so that

$$\begin{aligned}\mathbf{M} &= \mathbf{r}_{AB} \times (25\mathbf{k}) \\ &= (6\cos 30°\mathbf{i} - 6\sin 30°\mathbf{k}) \times (25\mathbf{k}) \\ &= \{-130\mathbf{j}\}\ \text{lb}\cdot\text{in.}\end{aligned}$$

Ans.

SOLUTION II *(SCALAR ANALYSIS)*

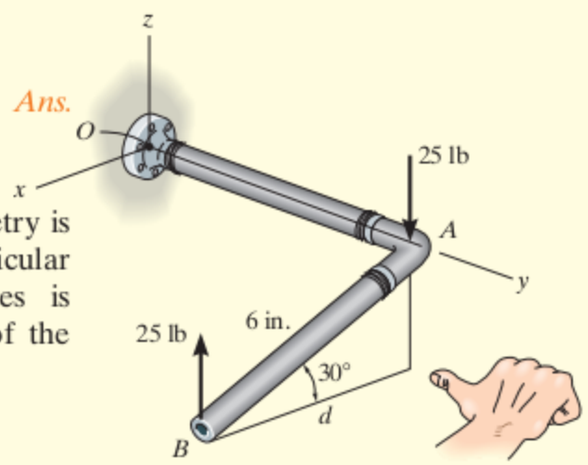
Although this problem is shown in three dimensions, the geometry is simple enough to use the scalar equation $M = Fd$. The perpendicular distance between the lines of action of the couple forces is $d = 6\cos 30° = 5.196$ in., Fig. 3–31*d*. Hence, taking moments of the forces about either point *A* or point *B* yields

$$M = Fd = 25\ \text{lb}\,(5.196\ \text{in.}) = 129.9\ \text{lb}\cdot\text{in.}$$

Applying the right-hand rule, **M** acts in the $-\mathbf{j}$ direction. Thus,

$$\mathbf{M} = \{-130\mathbf{j}\}\ \text{lb}\cdot\text{in.}$$

Ans.

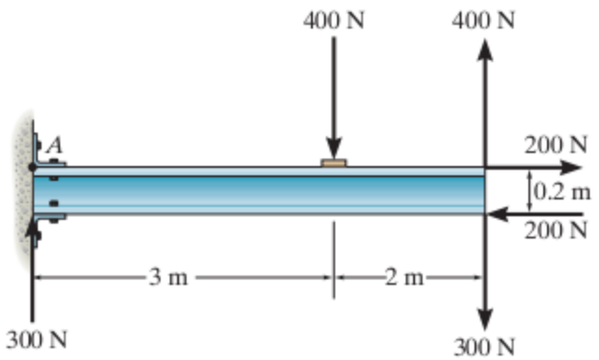


(d)

Fig. 3–31

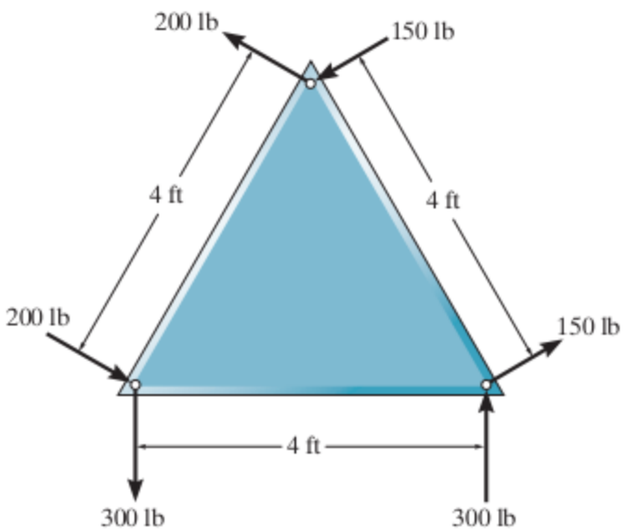
FUNDAMENTAL PROBLEMS

F3–19. Determine the resultant couple moment acting on the beam.



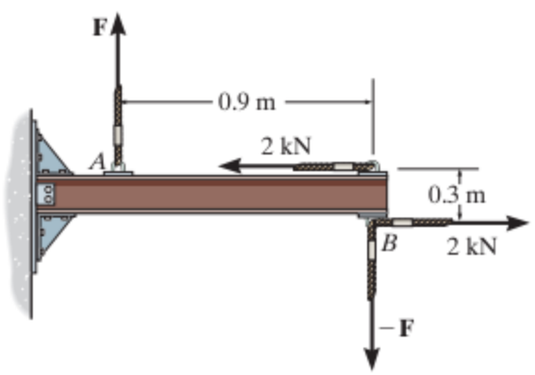
F3–19

F3–20. Determine the resultant couple moment acting on the triangular plate.



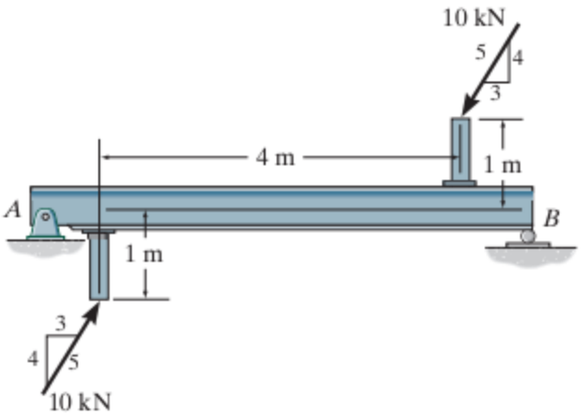
F3–20

F3–21. Determine the magnitude of **F** so that the resultant couple moment acting on the beam is 1.5 kN · m clockwise.



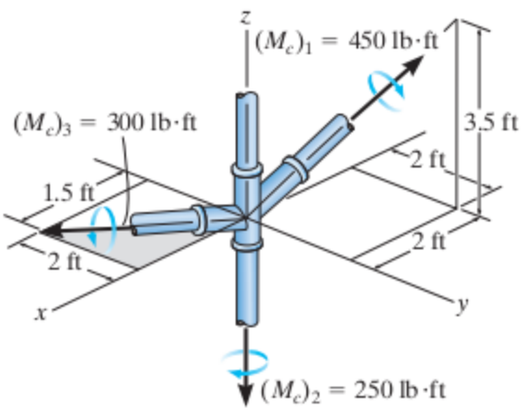
F3–21

F3–22. Determine the couple moment acting on the beam.



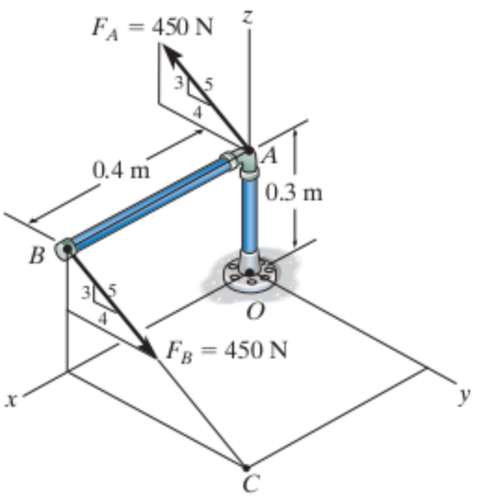
F3–22

F3–23. Determine the resultant couple moment acting on the pipe assembly.



F3–23

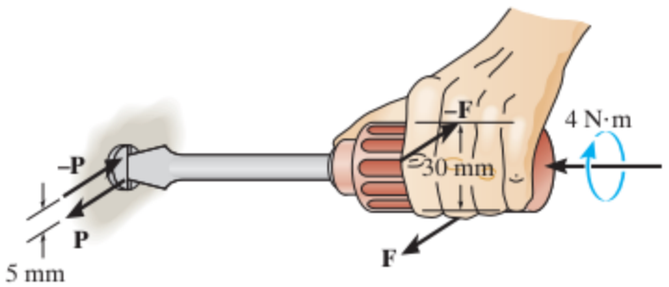
F3–24. Determine the couple moment acting on the pipe assembly and express the result as a Cartesian vector.



F3–24

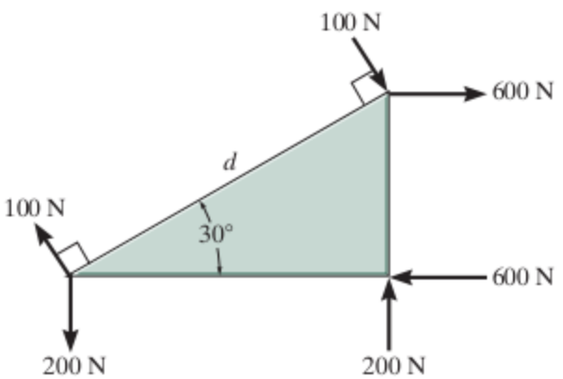
PROBLEMS

3–55. A twist of 4 N·m is applied to the handle of the screwdriver. Resolve this couple moment into a pair of couple forces **F** exerted on the handle and **P** exerted on the blade.



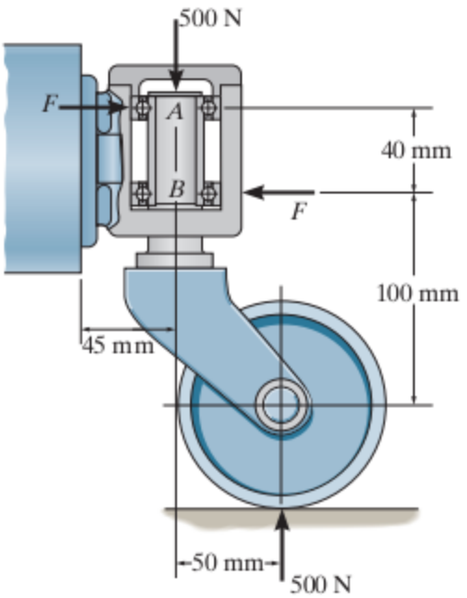
Prob. 3–55

*3–56. The ends of the triangular plate are subjected to three couples. Determine the plate dimension d so that the resultant couple is 350 N·m clockwise.



Prob. 3–56

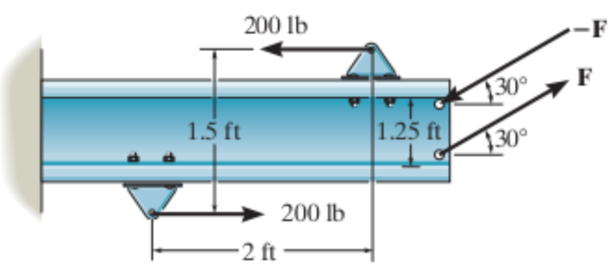
3–57. The caster wheel is subjected to the two couples. Determine the forces F that the bearings exert on the shaft so that the resultant couple moment on the caster is zero.



Prob. 3–57

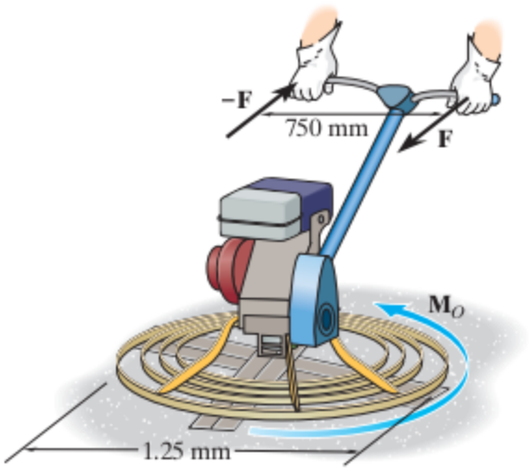
3–58. Two couples act on the beam. If $F = 125$ lb, determine the resultant couple moment.

3–59. Two couples act on the beam. Determine the magnitude of **F** so that the resultant couple moment is 450 lb·ft, counterclockwise. Where on the beam does the resultant couple moment act?



Probs. 3–58/59

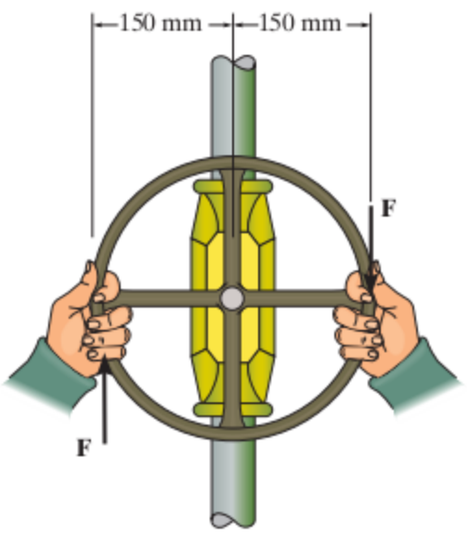
*3–60. Friction on the concrete surface creates a couple moment of $M_O = 100\,\text{N}\cdot\text{m}$ on the blades of the trowel. Determine the magnitude of the couple forces so that the resultant couple moment on the trowel is zero. The forces lie in the horizontal plane and act perpendicular to the handle of the trowel.



Prob. 3–60

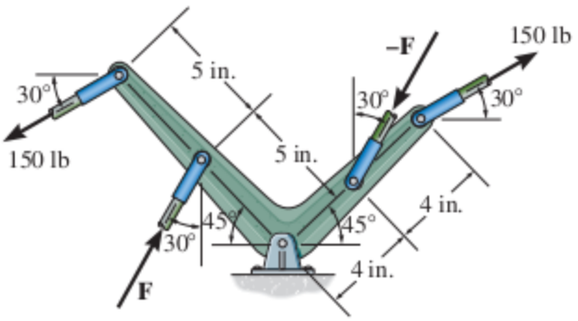
3–61. The man tries to open the valve by applying the couple forces of $F = 75\,\text{N}$ to the wheel. Determine the couple moment produced.

3–62. If the valve can be opened with a couple moment of $25\,\text{N}\cdot\text{m}$, determine the required magnitude of each couple force which must be applied to the wheel.



Probs. 3–61/62

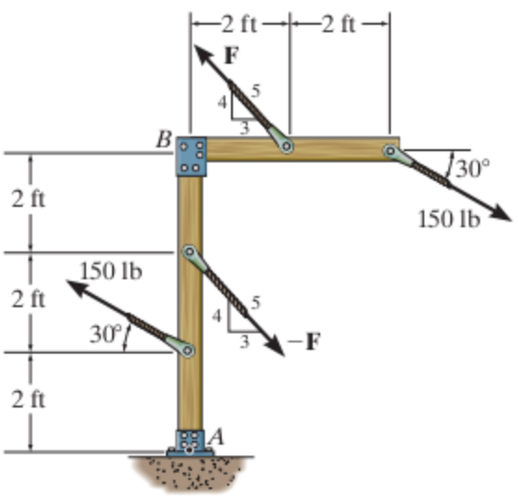
3–63. Determine the magnitude of the couple force **F** so that the resultant couple moment on the crank is zero.



Prob. 3–63

*3–64. If $F = 200$ lb, determine the resultant couple moment.

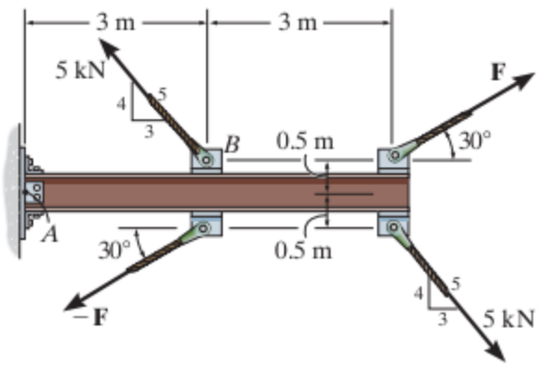
3–65. Determine the required magnitude of force **F** if the resultant couple moment on the frame is $200\,\text{lb}\cdot\text{ft}$, clockwise.



Probs. 3–64/65

3–66. Two couples act on the cantilever beam. If $F = 6$ kN, determine the resultant couple moment.

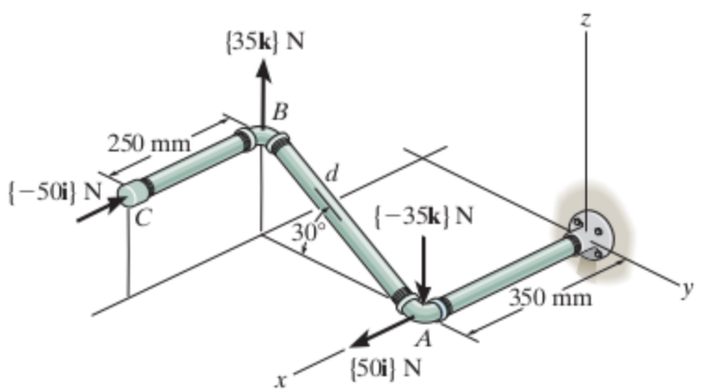
3–67. Determine the required magnitude of force **F**, if the resultant couple moment on the beam is to be zero.



Probs. 3–66/67

***3–68.** Determine the resultant couple moment of the two couples that act on the pipe assembly. The distance from A to B is $d = 400$ mm. Express the result as a Cartesian vector.

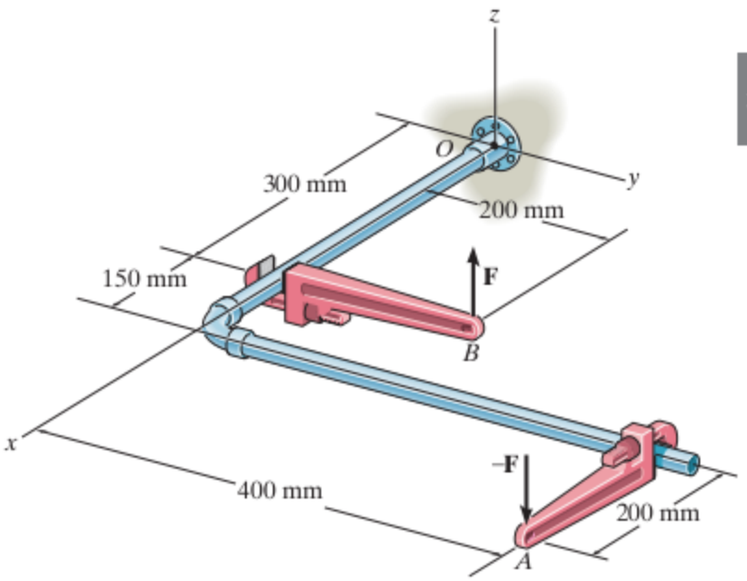
3–69. Determine the distance d between A and B so that the resultant couple moment has a magnitude of $M_R = 20$ N · m.



Probs. 3–68/69

3–70. Express the moment of the couple acting on the pipe assembly in Cartesian vector form. Solve the problem (a) using Eq. 3–13, and (b) summing the moment of each force about point O. Take $\mathbf{F} = \{25\mathbf{k}\}$ N.

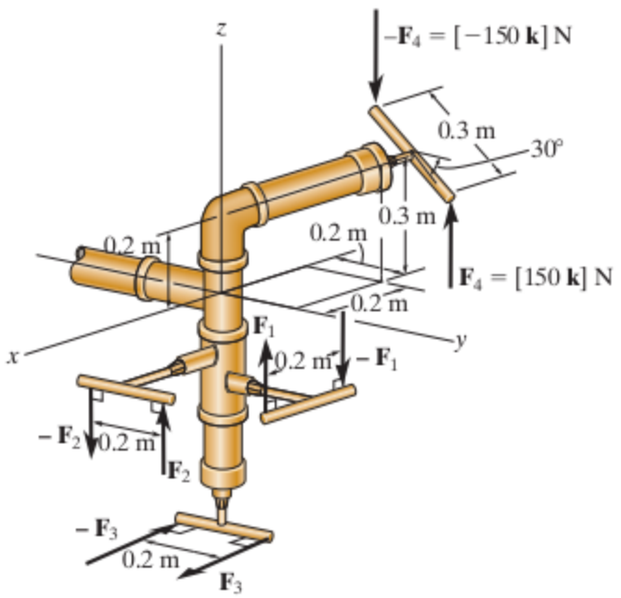
3–71. If the couple moment acting on the pipe has a magnitude of 400 N · m, determine the magnitude F of the vertical force applied to each wrench.



Probs. 3–70/71

***3–72** If $F_1 = 100$ N, $F_2 = 120$ N, and $F_3 = 80$ N, determine the magnitude and coordinate direction angles of the resultant couple moment.

3–73. Determine the required magnitude of $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ so that the resultant couple moment is $(\mathbf{M}_c)_R = [50\,\mathbf{i} - 45\,\mathbf{j} - 20\,\mathbf{k}]$ N · m.



Probs. 3–72/73

3.7 Simplification of a Force and Couple System

Sometimes it is convenient to reduce a system of forces and couple moments acting on a body to a simpler form by replacing it with an *equivalent system*, consisting of a single resultant force acting at a specific point and a resultant couple moment. A system is equivalent if the *external effects* it produces on a body are the same as those caused by the original force and couple moment system. In this context, the external effects of a system refer to the *translating and rotating motion* of the body if the body is free to move, or it refers to the *reactive forces* at the supports if the body is held fixed.

For example, consider holding the stick in Fig. 3–32*a*, which is subjected to the force **F** at point *A*. If we attach a pair of equal but opposite forces **F** and –**F** at point *B*, which is *on the line of action* of **F**, Fig. 3–32*b*, we observe that –**F** at *B* and **F** at *A* will cancel each other, leaving only **F** at *B*, Fig. 3–32*c*. Force **F** has now been moved from *A* to *B* without modifying its *external effects* on the stick; i.e., the reaction at the grip remains the same. This demonstrates the *principle of transmissibility*, which states that a force acting on a body (stick) is a *sliding vector* since it can be applied at any point along its line of action.

We can also use the above procedure to move a force to a point that is *not* on the line of action of the force. If **F** is applied perpendicular to the stick, as in Fig. 3–33*a*, then we can attach a pair of equal but opposite forces **F** and –**F** to *B*, Fig. 3–33*b*. Force **F** is now applied at *B*, and the other two forces, **F** at *A* and –**F** at *B*, form a couple that produces the couple moment $M = Fd$, Fig. 3–33*c*. Therefore, the force **F** can be moved from *A* to *B* provided a couple moment **M** is added to maintain an equivalent system. This couple moment is determined by taking the moment of **F** about *B*. Since **M** is actually a *free vector*, it can act at any point on the stick. In both cases the systems are equivalent which causes a downward force **F** and clockwise couple moment $M = Fd$ to be felt at the grip.

(a) (b) (c)

Fig. 3–32

(a) (b) (c)

Fig. 3–33

System of Forces and Couple Moments. Using the above method, a system of several forces and couple moments acting on a body can be reduced to an equivalent single resultant force acting at a point O and a resultant couple moment. For example, in Fig. 3–34*a*, O is not on the line of action of $\mathbf{F}_1$, and so this force can be moved to point O provided a couple moment $(\mathbf{M}_O)_1 = \mathbf{r}_1 \times \mathbf{F}$ is added to the body. Similarly, the couple moment $(\mathbf{M}_O)_2 = \mathbf{r}_2 \times \mathbf{F}_2$ should be added to the body when we move $\mathbf{F}_2$ to point O. Finally, since the couple moment $\mathbf{M}$ is a free vector, it can just be moved to point O. By doing this, we obtain the equivalent system shown in Fig. 3–34*b*, which produces the same external effects (support reactions) on the body as that of the force and couple system shown in Fig. 3–34*a*. If we sum the forces and couple moments, we obtain the resultant force $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$ and the resultant couple moment $(\mathbf{M}_R)_O = \mathbf{M} + (\mathbf{M}_O)_1 + (\mathbf{M}_O)_2$, Fig. 3–34*c*.

Notice that $\mathbf{F}_R$ is independent of the location of point O; however, $(\mathbf{M}_R)_O$ depends upon this location since the moments $\mathbf{M}_1$ and $\mathbf{M}_2$ are determined using the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. Also note that $(\mathbf{M}_R)_O$ is a free vector and can act at *any point* on the body, although point O is generally chosen as its point of application.

We can generalize the above method of reducing a force and couple system to an equivalent resultant force $\mathbf{F}_R$ acting at point O and a resultant couple moment $(\mathbf{M}_R)_O$ by using the following two equations.

$$\begin{aligned} \mathbf{F}_R &= \Sigma \mathbf{F} \\ (\mathbf{M}_R)_O &= \Sigma \mathbf{M}_O + \Sigma \mathbf{M} \end{aligned} \tag{3–17}$$

The first equation states that the resultant force of the system is equivalent to the sum of all the forces; and the second equation states that the resultant couple moment of the system is equivalent to the sum of all the couple moments $\Sigma \mathbf{M}$ plus the moments of all the forces $\Sigma \mathbf{M}_O$ about point O. If the force system lies in the x–y plane and any couple moments are perpendicular to this plane, then the above equations reduce to the following three scalar equations.

$$\begin{aligned} (F_R)_x &= \Sigma F_x \\ (F_R)_y &= \Sigma F_y \\ (M_R)_O &= \Sigma M_O + \Sigma M \end{aligned} \tag{3–18}$$

Here the resultant force is determined from the vector sum of its two components $(F_R)_x$ and $(F_R)_y$.

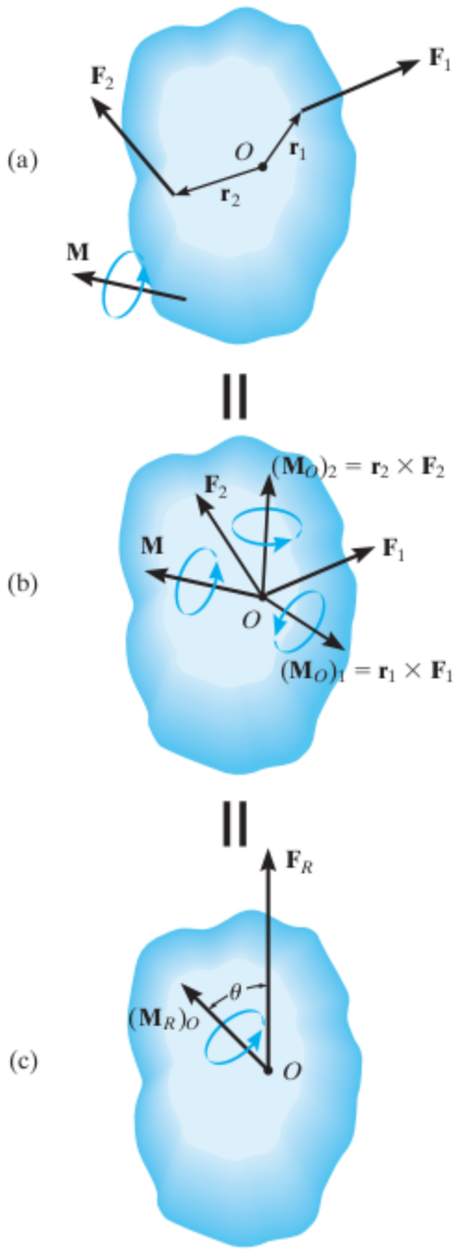
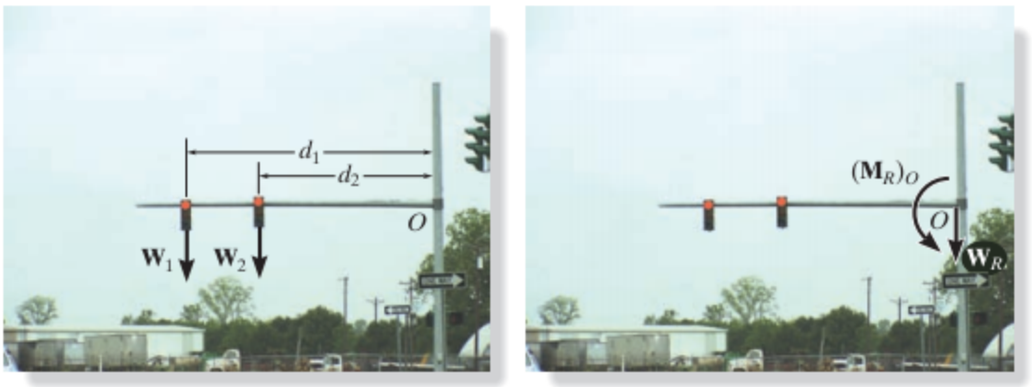


Fig. 3–34



The weights of these traffic lights can be replaced by their equivalent resultant force $W_R = W_1 + W_2$ and a couple moment $(M_R)_O = W_1 d_1 + W_2 d_2$ at the support, O. In both cases the support must provide the same resistance to translation and rotation in order to keep the member in the horizontal position.

Procedure for Analysis

The following points should be kept in mind when simplifying a force and couple moment system to an equivalent resultant force and couple system.

- Establish the coordinate axes with the origin located at point O and the axes having a selected orientation.

Force Summation.

- If the force system is *coplanar*, resolve each force into its x and y components. If a component is directed along the positive x or y axis, it represents a positive scalar; whereas if it is directed along the negative x or y axis, it is a negative scalar.

- In three dimensions, represent each force as a Cartesian vector before summing the forces.

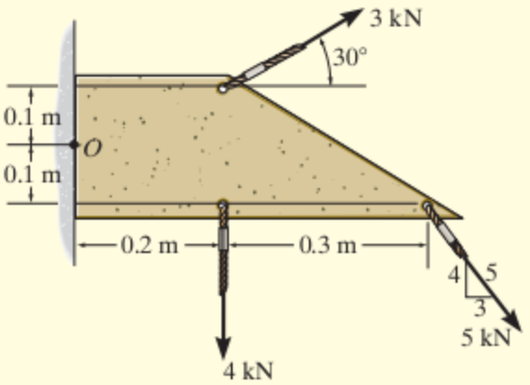
Moment Summation.

- When determining the moments of a *coplanar* force system about point O, it is generally advantageous to use the principle of moments, i.e., determine the moments of the components of each force, rather than the moment of the force itself.

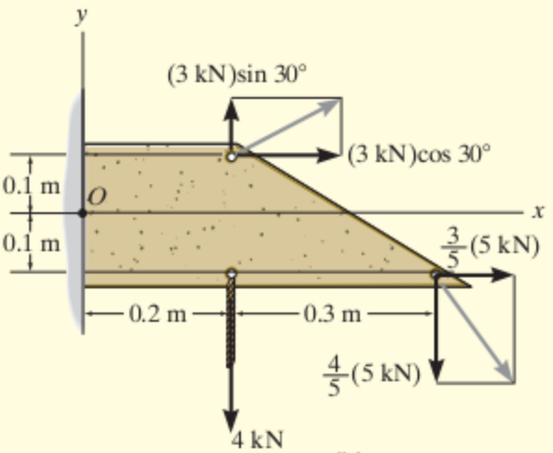
- In three dimensions use the vector cross product to determine the moment of each force about point O. Here the position vectors extend from O to any point on the line of action of each force.

EXAMPLE 3.12

Replace the force and couple system shown in Fig. 3–35*a* by an equivalent resultant force and couple moment acting at point *O*.



(a)



(b)

SOLUTION

Force Summation. The 3 kN and 5 kN forces are resolved into their *x* and *y* components as shown in Fig. 3–35*b*. We have

$$\overset{+}{\rightarrow}(F_R)_x = \Sigma F_x; \qquad (F_R)_x = (3\text{ kN})\cos 30° + \left(\tfrac{3}{5}\right)(5\text{ kN}) = 5.598\text{ kN} \rightarrow$$

$$+\uparrow(F_R)_y = \Sigma F_y; \qquad (F_R)_y = (3\text{ kN})\sin 30° - \left(\tfrac{4}{5}\right)(5\text{ kN}) - 4\text{ kN} = -6.50\text{ kN} = 6.50\text{ kN}\downarrow$$

Using the Pythagorean theorem, Fig. 3–35*c*, the magnitude of $\mathbf{F}_R$ is

$$F_R = \sqrt{(F_R)_x^2 + (F_R)_y^2} = \sqrt{(5.598\text{ kN})^2 + (6.50\text{ kN})^2} = 8.58\text{ kN} \qquad \textit{Ans.}$$

Its direction θ is

$$\theta = \tan^{-1}\left(\frac{(F_R)_y}{(F_R)_x}\right) = \tan^{-1}\left(\frac{6.50\text{ kN}}{5.598\text{ kN}}\right) = 49.3° \qquad \textit{Ans.}$$

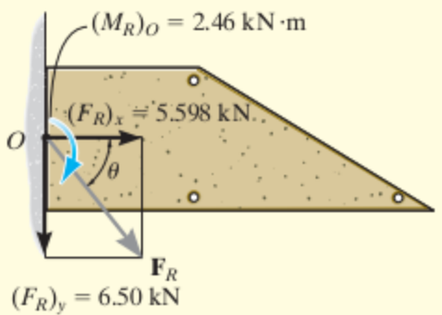
Moment Summation. The moments of 3 kN and 5 kN about point *O* will be determined using their *x* and *y* components. Referring to Fig. 3–35*b*, we have

$$\circlearrowleft + (M_R)_O = \Sigma M_O;$$

$$(M_R)_O = (3\text{ kN})\sin 30°(0.2\text{ m}) - (3\text{ kN})\cos 30°(0.1\text{ m}) + \left(\tfrac{3}{5}\right)(5\text{ kN})(0.1\text{ m}) - \left(\tfrac{4}{5}\right)(5\text{ kN})(0.5\text{ m}) - (4\text{ kN})(0.2\text{ m})$$

$$= -2.46\text{ kN}\cdot\text{m} = 2.46\text{ kN}\cdot\text{m}\,\circlearrowright \qquad \textit{Ans.}$$

This clockwise moment is shown in Fig. 3–35*c*.



(c)

Fig. 3–35

NOTE: Realize that the resultant force and couple moment in Fig. 3–35*c* will produce the same external effects or reactions at the supports as those produced by the force system, Fig. 3–35*a*.

EXAMPLE 3.13

Replace the force and couple system acting on the member in Fig. 3–36*a* by an equivalent resultant force and couple moment acting at point *O*.

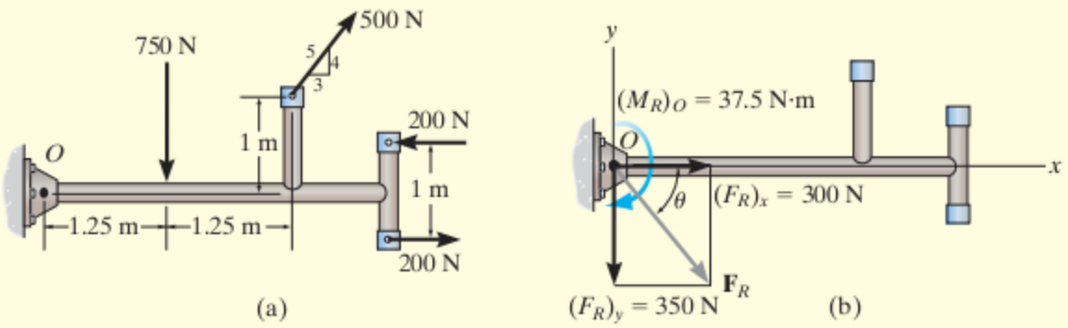


Fig. 3–36

SOLUTION

Force Summation. Since the couple forces of 200 N are equal but opposite, they produce a zero resultant force, and so it is not necessary to consider them in the force summation. The 500-N force is resolved into its *x* and *y* components, thus,

$$\overset{+}{\rightarrow}(F_R)_x = \Sigma F_x;\ (F_R)_x = \left(\tfrac{3}{5}\right)(500\ \text{N}) = 300\ \text{N} \rightarrow$$

$$+\uparrow(F_R)_y = \Sigma F_y;\ (F_R)_y = (500\ \text{N})\left(\tfrac{4}{5}\right) - 750\ \text{N} = -350\ \text{N} = 350\ \text{N}\downarrow$$

From Fig. 3–36*b*, the magnitude of $\mathbf{F}_R$ is

$$F_R = \sqrt{(F_R)_x^2 + (F_R)_y^2}$$
$$= \sqrt{(300\ \text{N})^2 + (350\ \text{N})^2} = 461\ \text{N} \qquad \textit{Ans.}$$

And the angle θ is

$$\theta = \tan^{-1}\left(\frac{(F_R)_y}{(F_R)_x}\right) = \tan^{-1}\left(\frac{350\ \text{N}}{300\ \text{N}}\right) = 49.4° \qquad \textit{Ans.}$$

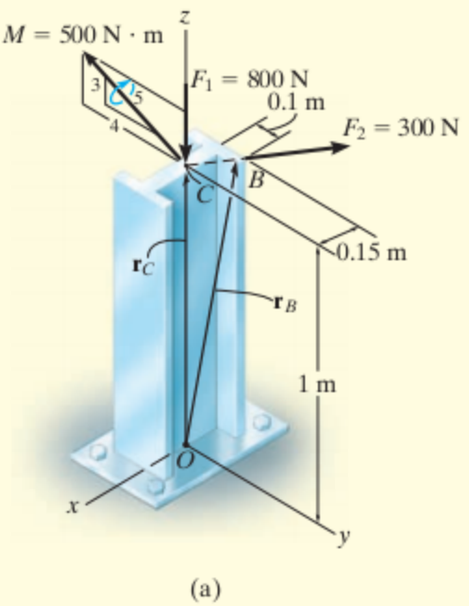
Moment Summation. Since the couple moment is a free vector, it can act at any point on the member. Referring to Fig. 3–36*a*, we have

$$\circlearrowleft + (M_R)_O = \Sigma M_O + \Sigma M$$
$$(M_R)_O = (500\ \text{N})\left(\tfrac{4}{5}\right)(2.5\ \text{m}) - (500\ \text{N})\left(\tfrac{3}{5}\right)(1\ \text{m})$$
$$- (750\ \text{N})(1.25\ \text{m}) + 200\ \text{N}\cdot\text{m}$$
$$= -37.5\ \text{N}\cdot\text{m} = 37.5\ \text{N}\cdot\text{m}\ \circlearrowright \qquad \textit{Ans.}$$

This clockwise moment is shown in Fig. 3–36*b*.

EXAMPLE 3.14

The structural member is subjected to a couple moment $\mathbf{M}$ and forces $\mathbf{F}_1$ and $\mathbf{F}_2$ in Fig. 3–37*a*. Replace this system by an equivalent resultant force and couple moment acting at its base, point O.



(a)

SOLUTION *(VECTOR ANALYSIS)*

The three-dimensional aspects of the problem can be simplified by using a Cartesian vector analysis. Expressing the forces and couple moment as Cartesian vectors, we have

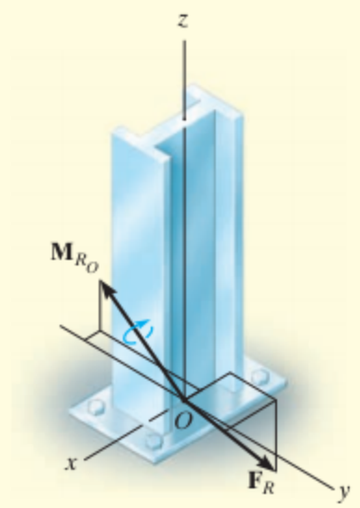
$$\mathbf{F}_1 = \{-800\mathbf{k}\}\ \text{N}$$

$$\mathbf{F}_2 = (300\ \text{N})\mathbf{u}_{CB}$$

$$= (300\ \text{N})\left(\frac{\mathbf{r}_{CB}}{r_{CB}}\right)$$

$$= 300\ \text{N}\left[\frac{\{-0.15\mathbf{i} + 0.1\mathbf{j}\}\ \text{m}}{\sqrt{(-0.15\ \text{m})^2 + (0.1\ \text{m})^2}}\right] = \{-249.6\mathbf{i} + 166.4\mathbf{j}\}\ \text{N}$$

$$\mathbf{M} = -500\left(\tfrac{4}{5}\right)\mathbf{j} + 500\left(\tfrac{3}{5}\right)\mathbf{k} = \{-400\mathbf{j} + 300\mathbf{k}\}\ \text{N}\cdot\text{m}$$



(b)

Fig. 3–37

Force Summation.

$$\mathbf{F}_R = \Sigma\mathbf{F}; \qquad \mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2 = -800\mathbf{k} - 249.6\mathbf{i} + 166.4\mathbf{j}$$

$$= \{-250\mathbf{i} + 166\mathbf{j} - 800\mathbf{k}\}\ \text{N} \qquad \textit{Ans.}$$

Moment Summation.

$$\mathbf{M}_{R_O} = \Sigma\mathbf{M} + \Sigma\mathbf{M}_O$$

$$\mathbf{M}_{R_O} = \mathbf{M} + \mathbf{r}_C \times \mathbf{F}_1 + \mathbf{r}_B \times \mathbf{F}_2$$

$$\mathbf{M}_{R_O} = (-400\mathbf{j} + 300\mathbf{k}) + (1\mathbf{k}) \times (-800\mathbf{k}) + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -0.15 & 0.1 & 1 \\ -249.6 & 166.4 & 0 \end{vmatrix}$$

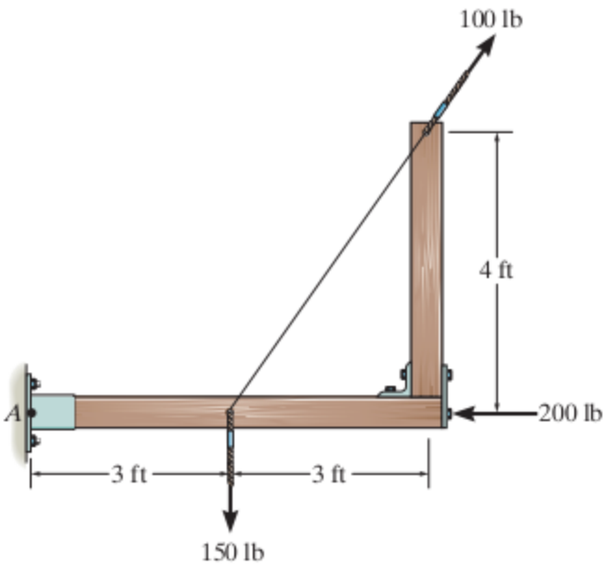
$$= (-400\mathbf{j} + 300\mathbf{k}) + (\mathbf{0}) + (-166.4\mathbf{i} - 249.6\mathbf{j})$$

$$= \{-166\mathbf{i} - 650\mathbf{j} + 300\mathbf{k}\}\ \text{N}\cdot\text{m} \qquad \textit{Ans.}$$

The results are shown in Fig. 3–37*b*.

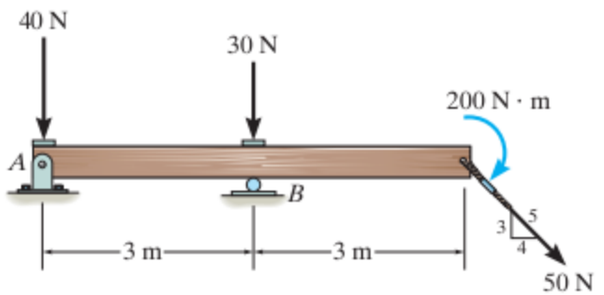
FUNDAMENTAL PROBLEMS

F3–25. Replace the loading system by an equivalent resultant force and couple moment acting at point A.



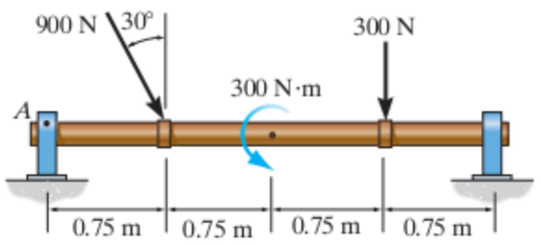
F3–25

F3–26. Replace the loading system by an equivalent resultant force and couple moment acting at point A.



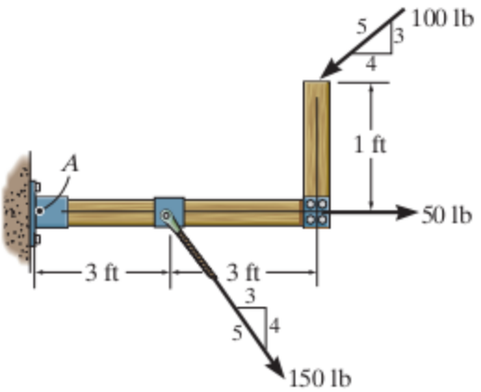
F3–26

F3–27. Replace the loading system by an equivalent resultant force and couple moment acting at point A.



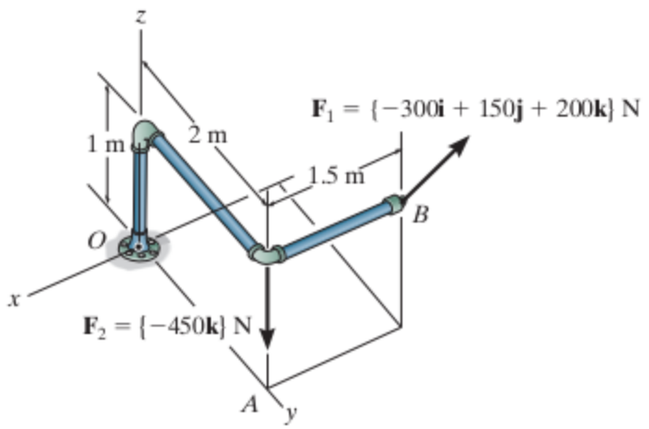
F3–27

F3–28. Replace the loading system by an equivalent resultant force and couple moment acting at point A.



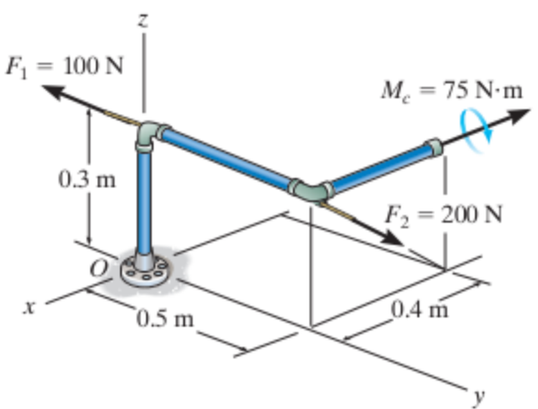
F3–28

F3–29. Replace the loading system by an equivalent resultant force and couple moment acting at point O.



F3–29

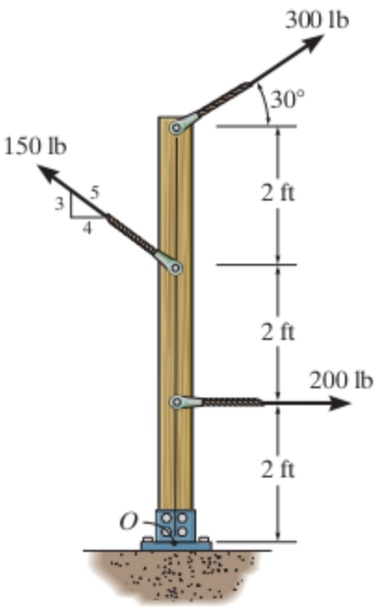
F3–30. Replace the loading system by an equivalent resultant force and couple moment acting at point O.



F3–30

PROBLEMS

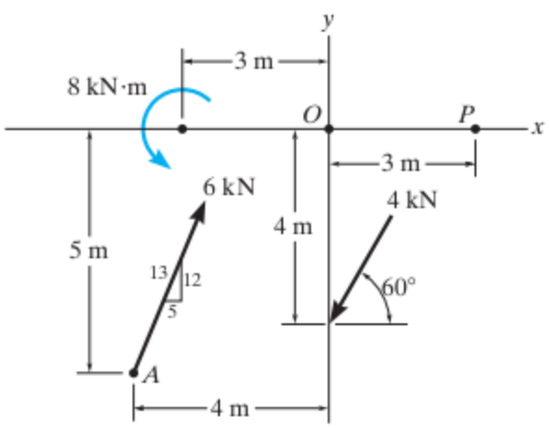
3–74. Replace the force system acting on the post by a resultant force and couple moment at point O.



Prob. 3–74

3–75. Replace the force and couple system by an equivalent force and couple moment at point O.

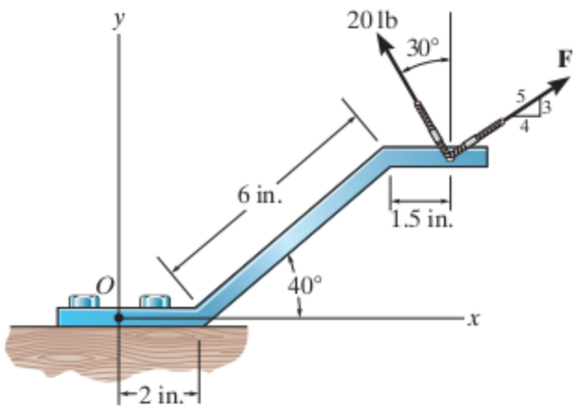
*3–76. Replace the force and couple system by an equivalent force and couple moment at point P.



Probs. 3–75/76

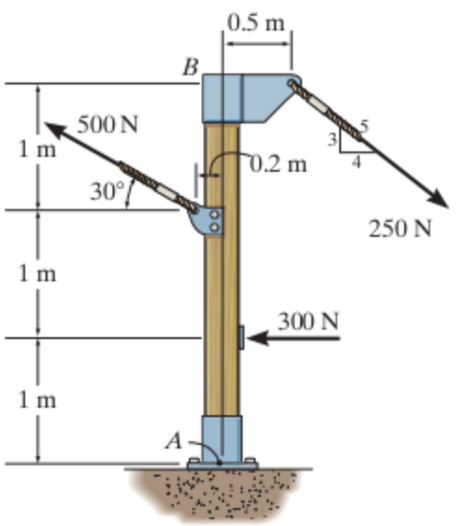
3–77. Replace the two forces by an equivalent resultant force and couple moment at point O. Set $F = 20$ lb.

3–78. Replace the two forces by an equivalent resultant force and couple moment at point O. Set $F = 15$ lb.



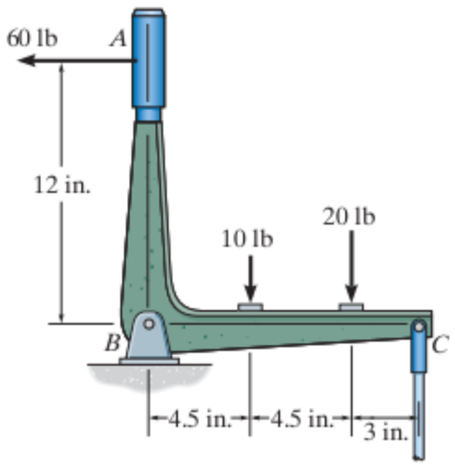
Probs. 3–77/78

3–79. Replace the force system acting on the post by a resultant force and couple moment at point A.



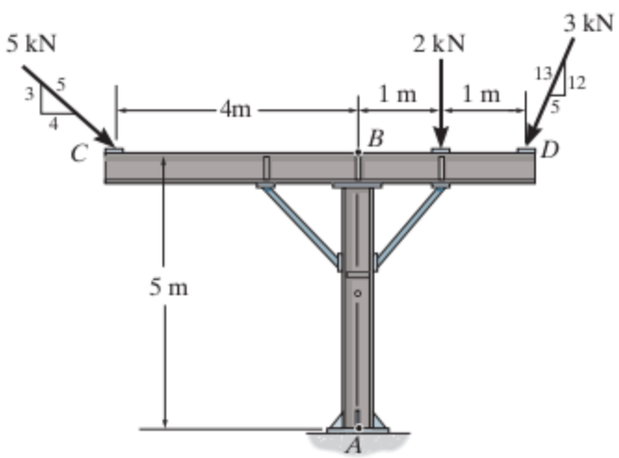
Prob. 3–79

*3–80. Replace the force system acting on the crank by a resultant force, and specify where its line of action intersects *BA* measured from the pin at *B*.



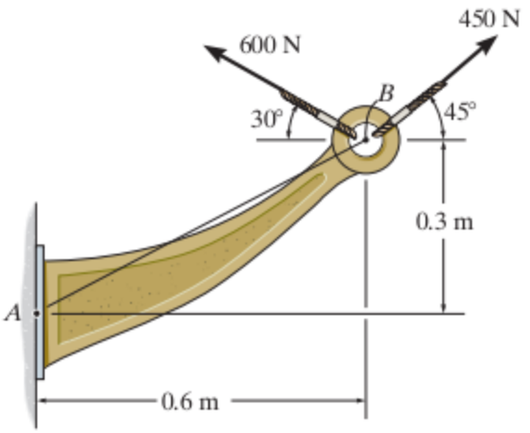
Prob. 3–80

3–81. Replace the force system acting on the frame by a resultant force and couple moment at point *A*.



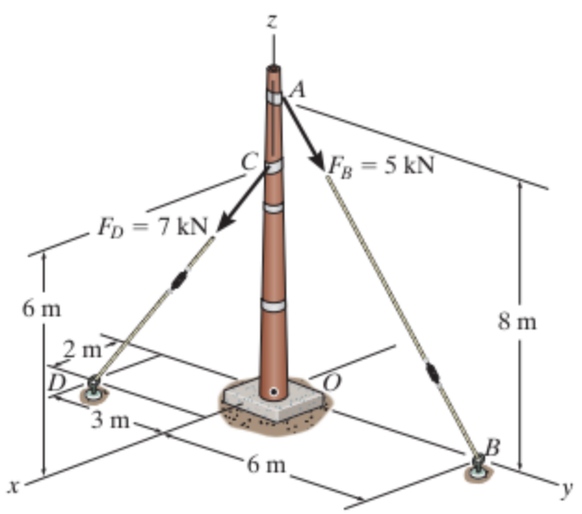
Prob. 3–81

3–82. Replace the force system acting on the bracket by a resultant force and couple moment at point *A*.



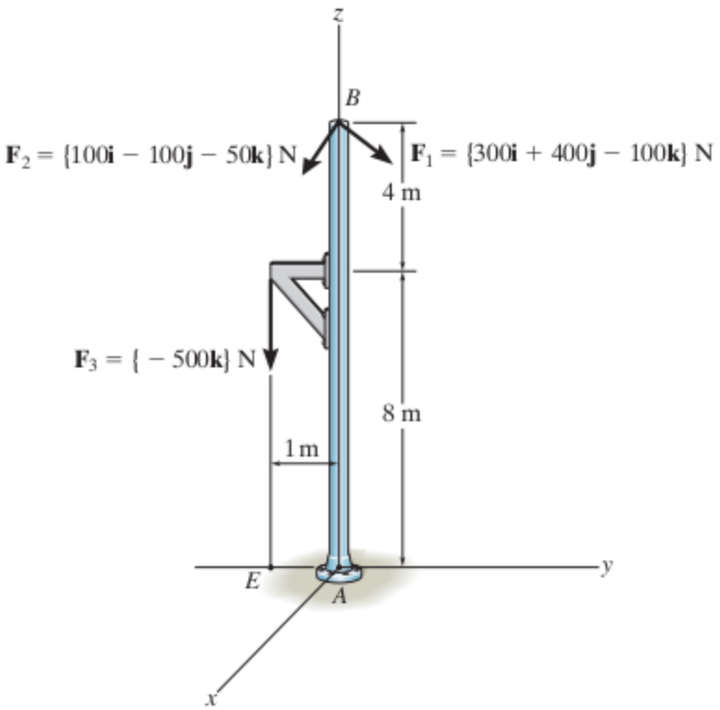
Prob. 3–82

3–83. Replace the two forces acting on the post by a resultant force and couple moment at point *O*. Express the results in Cartesian vector form.



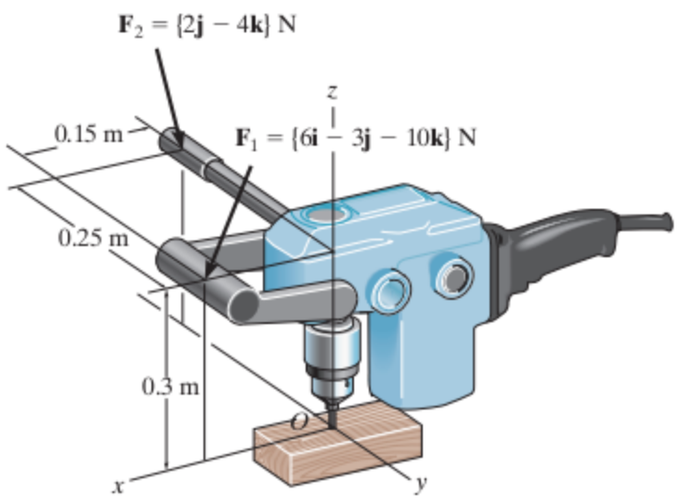
Prob. 3–83

***3–84.** Replace the force system by an equivalent force and couple moment at point A.



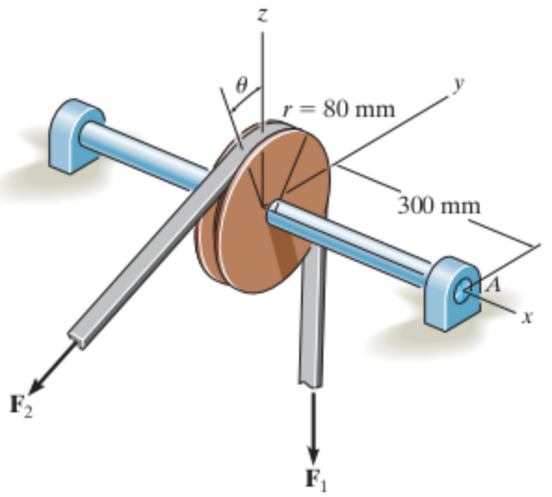
Prob. 3–84

3–85. Handle forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are applied to the electric drill. Replace this force system by an equivalent resultant force and couple moment acting at point O. Express the results in Cartesian vector form.



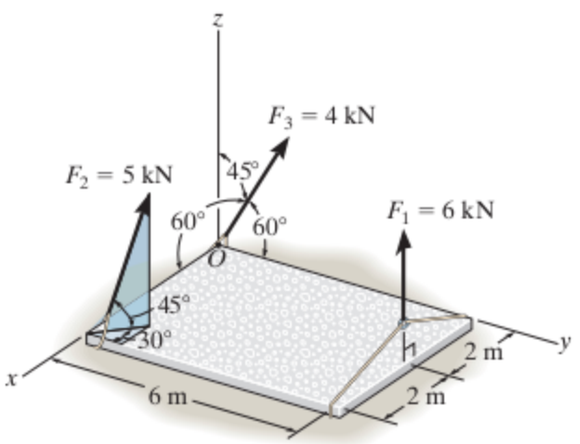
Prob. 3–85

3–86. The belt passing over the pulley is subjected to forces $\mathbf{F}_1$ and $\mathbf{F}_2$, each having a magnitude of 40 N. $\mathbf{F}_1$ acts in the $-\mathbf{k}$ direction. Replace these forces by an equivalent force and couple moment at point A. Express the result in Cartesian vector form. Set $\theta = 0°$ so that $\mathbf{F}_2$ acts in the $-\mathbf{j}$ direction.



Prob. 3–86

3–87. The slab is to be hoisted using the three slings shown. Replace the system of forces acting on slings by an equivalent force and couple moment at point O. The force $\mathbf{F}_1$ is vertical.



Prob. 3–87

3.8 Further Simplification of a Force and Couple System

In the preceding section, we developed a way to reduce a force and couple moment system acting on a rigid body into an equivalent resultant force $\mathbf{F}_R$ acting at a specific point O and a resultant couple moment $(\mathbf{M}_R)_O$. The force system can be further reduced to an equivalent single resultant force provided the lines of action of $\mathbf{F}_R$ and $(\mathbf{M}_R)_O$ are *perpendicular* to each other. Because of this condition, only concurrent, coplanar, and parallel force systems can be further simplified.

Concurrent Force System. Since a *concurrent force system* is one in which the lines of action of all the forces intersect at a common point O, Fig. 3–38*a*, then the force system produces no moment about this point. As a result, the equivalent system can be represented by a single resultant force $\mathbf{F}_R = \Sigma\mathbf{F}$ acting at O, Fig. 3–38*b*.

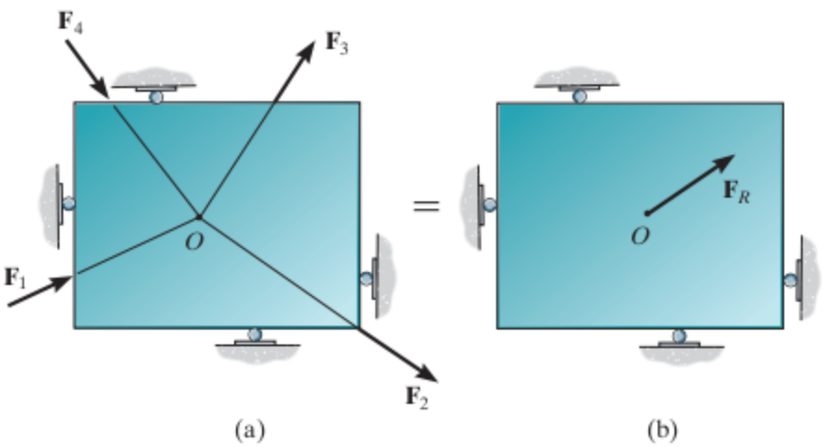


Fig. 3–38

Coplanar Force System. In the case of a *coplanar force system*, the lines of action of all the forces lie in the same plane, Fig. 3–39*a*, and so the resultant force $\mathbf{F}_R = \Sigma\mathbf{F}$ of this system also lies in this plane. Furthermore, the moment of each of the forces about any point O is directed perpendicular to this plane. Thus, the resultant moment $(\mathbf{M}_R)_O$ and resultant force $\mathbf{F}_R$ will be *mutually perpendicular*, Fig. 3–39*b*. The resultant moment can be replaced by moving the resultant force $\mathbf{F}_R$ a perpendicular or moment arm distance d away from point O such that $\mathbf{F}_R$ produces the *same moment* $(\mathbf{M}_R)_O$ about point O, Fig. 3–39*c*. This distance d can be determined from the scalar equation $(M_R)_O = F_R d = \Sigma M_O$ or $d = (M_R)_O / F_R$.

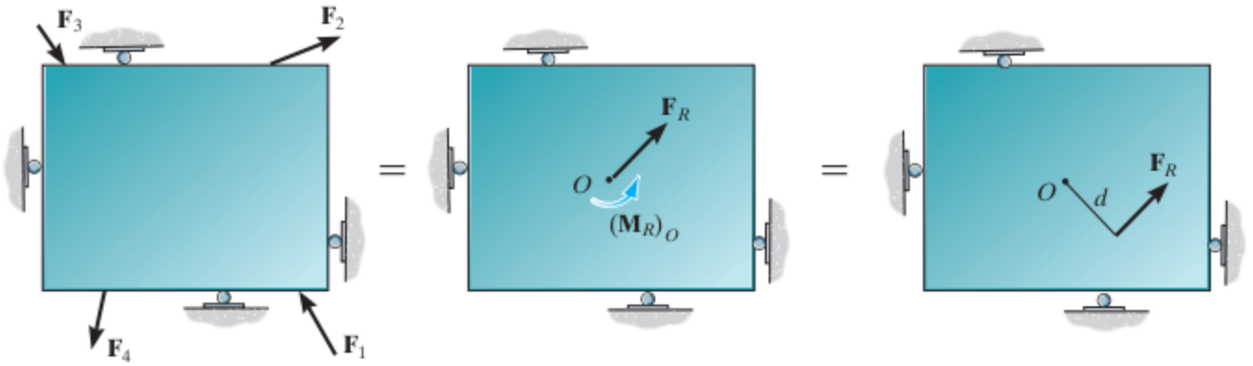
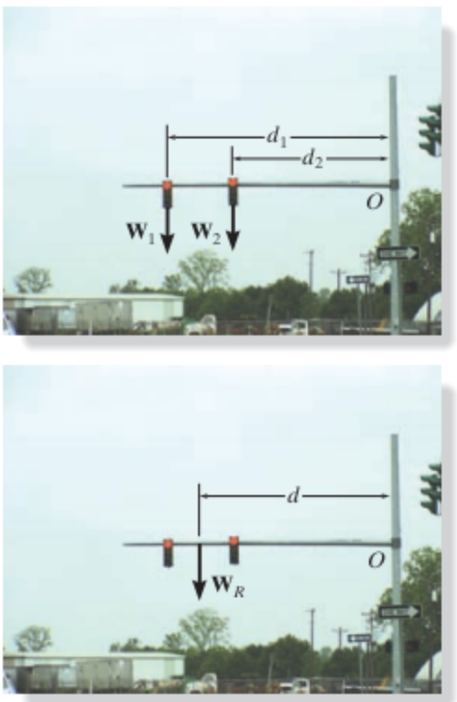


Fig. 3–39

Parallel Force System. The *parallel force system* shown in Fig. 3–40*a* consists of forces that are all parallel to the z axis. Thus, the resultant force $\mathbf{F}_R = \Sigma\mathbf{F}$ at point O must also be parallel to this axis, Fig. 3–40*b*. The moment produced by each force lies in the plane of the plate, and so the resultant couple moment, $(\mathbf{M}_R)_O$, will also lie in this plane, along the moment axis a since $\mathbf{F}_R$ and $(\mathbf{M}_R)_O$ are mutually perpendicular. As a result, the force system can be further reduced to an equivalent single resultant force $\mathbf{F}_R$, acting through point P located on the perpendicular b axis, Fig. 3–40*c*. The distance d along this axis from point O requires $(M_R)_O = F_R d = \Sigma M_O$ or $d = \Sigma M_O / F_R$.



Here the weights of the traffic lights are replaced by their resultant force $W_R = W_1 + W_2$ which acts at a distance $d = (W_1 d_1 + W_2 d_2)/W_R$ from O. Both systems are equivalent.

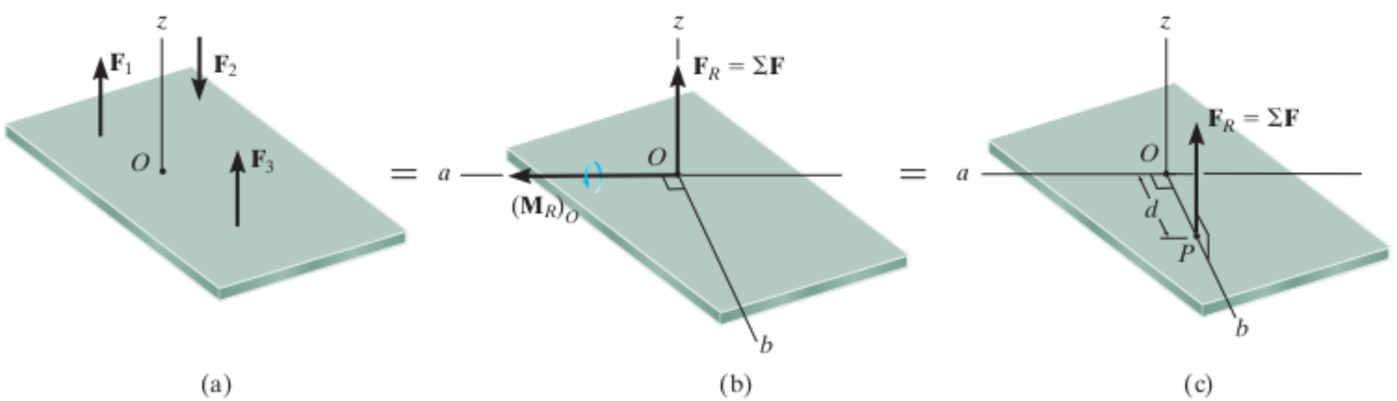


Fig. 3–40

The four cable forces are all concurrent at point O on this bridge tower. Consequently they produce no resultant moment there, only a resultant force $\mathbf{F}_R$. Note that the designers have positioned the cables so that $\mathbf{F}_R$ is directed *along* the bridge tower directly to the support, so that it does not cause any bending of the tower.

Procedure for Analysis

The technique used to reduce a coplanar or parallel force system to a single resultant force follows a similar procedure outlined in the previous section.

- Establish the x, y, z axes and locate the resultant force $\mathbf{F}_R$ an arbitrary distance away from the origin of the coordinates.

Force Summation.

- The resultant force is equal to the sum of all the forces in the system.
- For a coplanar force system, resolve each force into its x and y components. Positive components are directed along the positive x and y axes, and negative components are directed along the negative x and y axes.

Moment Summation.

- The moment of the resultant force about point O is equal to the sum of all the couple moments in the system plus the moments of all the forces in the system about O.
- This moment condition is used to find the location of the resultant force from point O.

EXAMPLE 3.15

Replace the force and couple moment system acting on the beam in Fig. 3–41*a* by an equivalent resultant force, and find where its line of action intersects the beam, measured from point *O*.

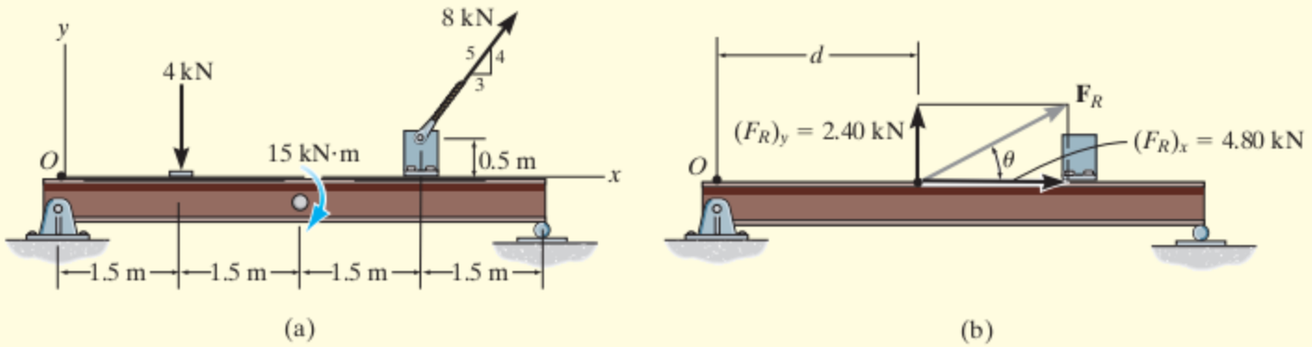


Fig. 3–41

SOLUTION

Force Summation. Summing the force components,

$$\overset{+}{\rightarrow}(F_R)_x = \Sigma F_x; \qquad (F_R)_x = 8\ \text{kN}\left(\tfrac{3}{5}\right) = 4.80\ \text{kN} \rightarrow$$

$$+\uparrow(F_R)_y = \Sigma F_y; \qquad (F_R)_y = -4\ \text{kN} + 8\ \text{kN}\left(\tfrac{4}{5}\right) = 2.40\ \text{kN}\uparrow$$

From Fig. 3–41*b*, the magnitude of $\mathbf{F}_R$ is

$$F_R = \sqrt{(4.80\ \text{kN})^2 + (2.40\ \text{kN})^2} = 5.37\ \text{kN} \qquad \textit{Ans.}$$

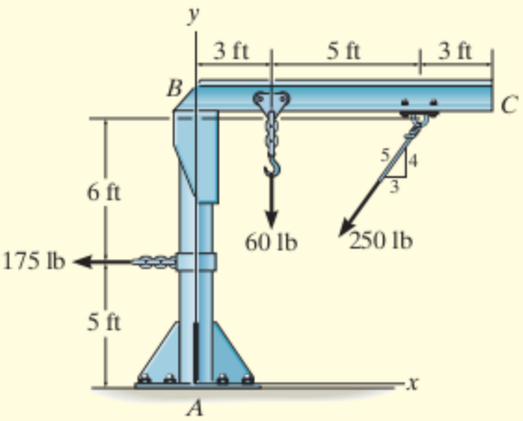
The angle θ is

$$\theta = \tan^{-1}\left(\frac{2.40\ \text{kN}}{4.80\ \text{kN}}\right) = 26.6° \qquad \textit{Ans.}$$

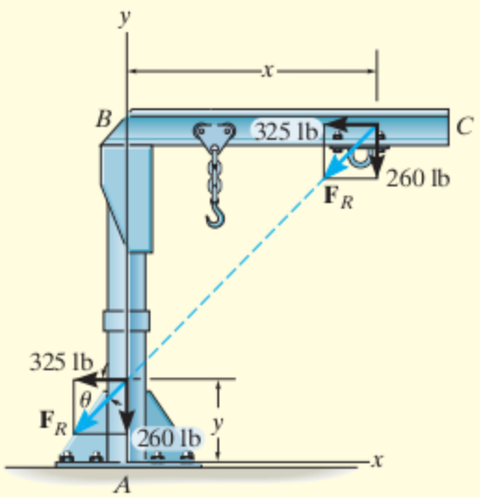
Moment Summation. We must equate the moment of $\mathbf{F}_R$ about point *O* in Fig. 3–41*b* to the sum of the moments of the force and couple moment system about point *O* in Fig. 3–41*a*. Since the line of action of $(\mathbf{F}_R)_x$ acts through point *O*, *only* $(\mathbf{F}_R)_y$ *produces a moment* about this point. Thus,

$$\circlearrowleft + (M_R)_O = \Sigma M_O; \qquad 2.40\ \text{kN}(d) = -(4\ \text{kN})(1.5\ \text{m}) - 15\ \text{kN}\cdot\text{m}$$
$$-\left[8\ \text{kN}\left(\tfrac{3}{5}\right)\right](0.5\ \text{m}) + \left[8\ \text{kN}\left(\tfrac{4}{5}\right)\right](4.5\ \text{m})$$
$$d = 2.25\ \text{m} \qquad \textit{Ans.}$$

EXAMPLE 3.16



(a)



(b)

Fig. 3–42

The jib crane shown in Fig. 3–42*a* is subjected to three coplanar forces. Replace this loading by an equivalent resultant force and specify where the resultant's line of action intersects the column *AB* and boom *BC*.

SOLUTION

Force Summation. Resolving the 250-lb force into x and y components and summing the force components yields

$$\overset{+}{\rightarrow}(F_R)_x = \Sigma F_x; \quad (F_R)_x = -250\text{ lb}\left(\tfrac{3}{5}\right) - 175\text{ lb} = -325\text{ lb} = 325\text{ lb}\leftarrow$$

$$+\uparrow(F_R)_y = \Sigma F_y; \quad (F_R)_y = -250\text{ lb}\left(\tfrac{4}{5}\right) - 60\text{ lb} = -260\text{ lb} = 260\text{ lb}\downarrow$$

As shown by the vector addition in Fig. 3–42*b*,

$$F_R = \sqrt{(325\text{ lb})^2 + (260\text{ lb})^2} = 416\text{ lb} \qquad \textit{Ans.}$$

$$\theta = \tan^{-1}\left(\frac{260\text{ lb}}{325\text{ lb}}\right) = 38.7°\theta\measuredangle \qquad \textit{Ans.}$$

Moment Summation. Moments will be summed about point A. Assuming the line of action of $\mathbf{F}_R$ *intersects* AB at a distance y from A, Fig. 3–42*b*, we have

$$\circlearrowleft +(M_R)_A = \Sigma M_A; \qquad 325\text{ lb}\,(y) + 260\text{ lb}\,(0)$$

$$= 175\text{ lb}\,(5\text{ ft}) - 60\text{ lb}\,(3\text{ ft}) + 250\text{ lb}\left(\tfrac{3}{5}\right)(11\text{ ft}) - 250\text{ lb}\left(\tfrac{4}{5}\right)(8\text{ ft})$$

$$y = 2.29\text{ ft} \qquad \textit{Ans.}$$

By the principle of transmissibility, $\mathbf{F}_R$ can be placed at a distance x where it intersects BC, Fig. 3–42*b*. In this case we have

$$\circlearrowleft +(M_R)_A = \Sigma M_A; \qquad 325\text{ lb}\,(11\text{ ft}) - 260\text{ lb}\,(x)$$

$$= 175\text{ lb}\,(5\text{ ft}) - 60\text{ lb}\,(3\text{ ft}) + 250\text{ lb}\left(\tfrac{3}{5}\right)(11\text{ ft}) - 250\text{ lb}\left(\tfrac{4}{5}\right)(8\text{ ft})$$

$$x = 10.9\text{ ft} \qquad \textit{Ans.}$$

EXAMPLE 3.17

The slab in Fig. 3–43*a* is subjected to four parallel forces. Determine the magnitude and direction of a resultant force equivalent to the given force system and locate its point of application on the slab.

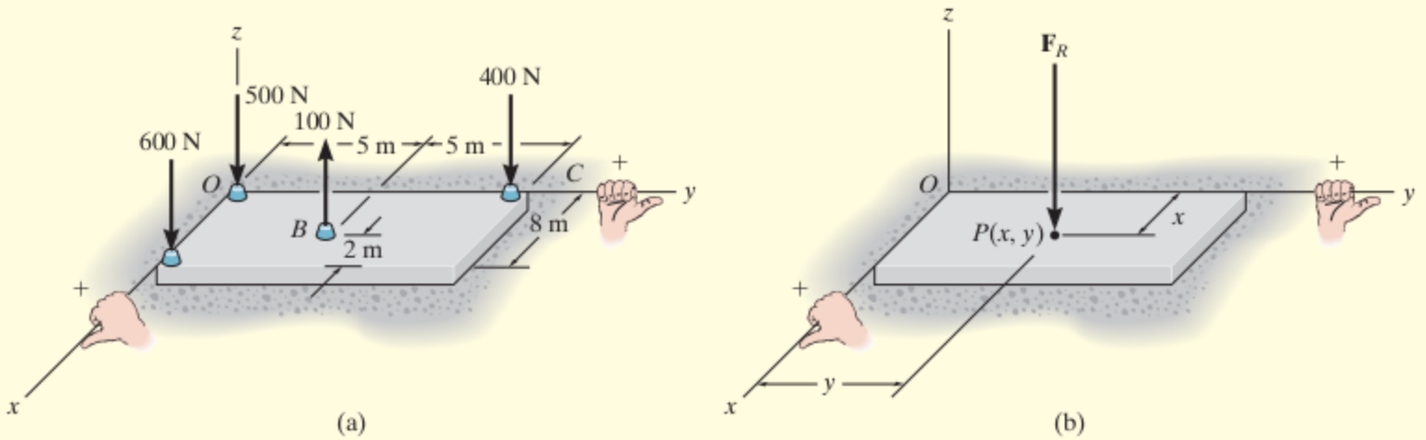


Fig. 3–43

SOLUTION *(SCALAR ANALYSIS)*

Force Summation. From Fig. 3–43*a*, the resultant force is

$$+\uparrow F_R = \Sigma F; \qquad F_R = -600\,\text{N} + 100\,\text{N} - 400\,\text{N} - 500\,\text{N}$$
$$= -1400\,\text{N} = 1400\,\text{N}\downarrow \qquad \textit{Ans.}$$

Moment Summation. We require the moment about the x axis of the resultant force, Fig. 3–43*b*, to be equal to the sum of the moments about the x axis of all the forces in the system, Fig. 3–43*a*. The moment arms are determined from the y coordinates since these coordinates represent the *perpendicular distances* from the x axis to the lines of action of the forces. Using the right-hand rule, we have

$$(M_R)_x = \Sigma M_x;$$

$$-(1400\,\text{N})y = 600\,\text{N}(0) + 100\,\text{N}(5\,\text{m}) - 400\,\text{N}(10\,\text{m}) + 500\,\text{N}(0)$$
$$-1400y = -3500 \quad y = 2.50\,\text{m} \qquad \textit{Ans.}$$

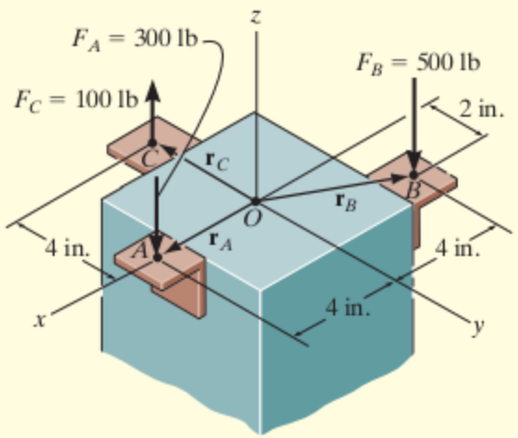
In a similar manner, a moment equation can be written about the y axis using moment arms defined by the x coordinates of each force.

$$(M_R)_y = \Sigma M_y;$$

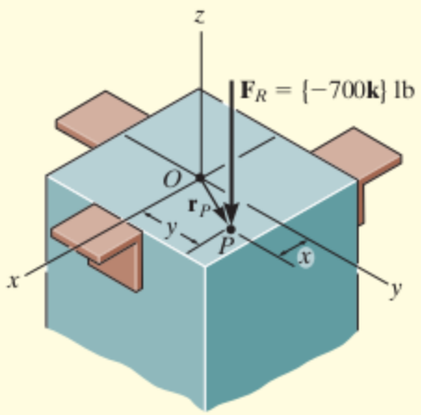
$$(1400\,\text{N})x = 600\,\text{N}(8\,\text{m}) - 100\,\text{N}(6\,\text{m}) + 400\,\text{N}(0) + 500\,\text{N}(0)$$
$$1400x = 4200$$
$$x = 3\,\text{m} \qquad \textit{Ans.}$$

NOTE: A force of $F_R = 1400$ N placed at point $P(3.00\,\text{m}, 2.50\,\text{m})$ on the slab, Fig. 3–43*b*, is therefore equivalent to the parallel force system acting on the slab in Fig. 3–43*a*.

EXAMPLE 3.18



(a)



(b)

Fig. 3–44

Replace the force system in Fig. 3–44*a* by an equivalent resultant force and specify its point of application on the pedestal.

SOLUTION

Force Summation. Here we will demonstrate a vector analysis. Summing forces,

$$\mathbf{F}_R = \Sigma\mathbf{F};\quad \mathbf{F}_R = \mathbf{F}_A + \mathbf{F}_B + \mathbf{F}_C$$
$$= \{-300\mathbf{k}\}\text{ lb} + \{-500\mathbf{k}\}\text{ lb} + \{100\mathbf{k}\}\text{ lb}$$
$$= \{-700\mathbf{k}\}\text{ lb} \qquad \textit{Ans.}$$

Location. Moments will be summed about point O. The resultant force $\mathbf{F}_R$ is assumed to act through point $P\,(x, y, 0)$, Fig. 3–44*b*. Thus

$$(\mathbf{M}_R)_O = \Sigma\mathbf{M}_O;$$
$$\mathbf{r}_P \times \mathbf{F}_R = (\mathbf{r}_A \times \mathbf{F}_A) + (\mathbf{r}_B \times \mathbf{F}_B) + (\mathbf{r}_C \times \mathbf{F}_C)$$
$$(x\mathbf{i} + y\mathbf{j}) \times (-700\mathbf{k}) = [(4\mathbf{i}) \times (-300\mathbf{k})]$$
$$+ [(-4\mathbf{i} + 2\mathbf{j}) \times (-500\mathbf{k})] + [(-4\mathbf{j}) \times (100\mathbf{k})]$$
$$-700x(\mathbf{i} \times \mathbf{k}) - 700y(\mathbf{j} \times \mathbf{k}) = -1200(\mathbf{i} \times \mathbf{k}) + 2000(\mathbf{i} \times \mathbf{k})$$
$$- 1000(\mathbf{j} \times \mathbf{k}) - 400(\mathbf{j} \times \mathbf{k})$$
$$700x\mathbf{j} - 700y\mathbf{i} = 1200\mathbf{j} - 2000\mathbf{j} - 1000\mathbf{i} - 400\mathbf{i}$$

Equating the **i** and **j** components,

$$-700y = -1400 \qquad (1)$$
$$y = 2\text{ in.} \qquad \textit{Ans.}$$
$$700x = -800 \qquad (2)$$
$$x = -1.14\text{ in.} \qquad \textit{Ans.}$$

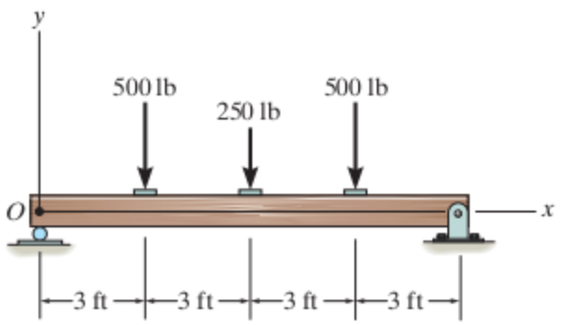
The negative sign indicates that the x coordinate of point P is negative.

NOTE: It is also possible to establish Eqs. (1) and (2) directly by summing moments about the x and y axes. Using the right-hand rule, we have

$$(M_R)_x = \Sigma M_x; \qquad -700y = -100\text{ lb}(4\text{ in.}) - 500\text{ lb}(2\text{ in.})$$
$$(M_R)_y = \Sigma M_y; \qquad 700x = 300\text{ lb}(4\text{ in.}) - 500\text{ lb}(4\text{ in.})$$

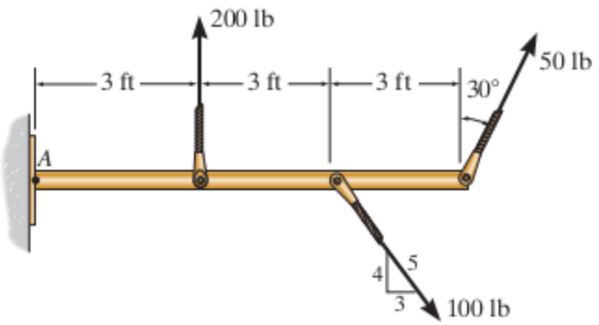
FUNDAMENTAL PROBLEMS

F3–31. Replace the loading system by an equivalent resultant force and specify where the resultant's line of action intersects the beam measured from O.



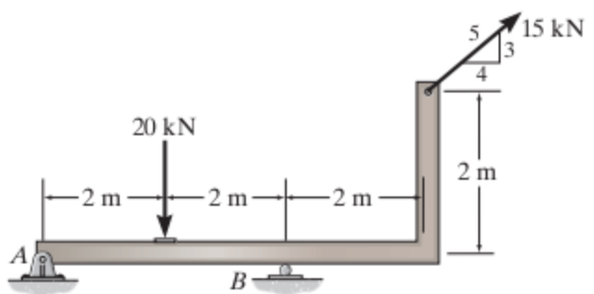
F3–31

F3–32. Replace the loading system by an equivalent resultant force and specify where the resultant's line of action intersects the member measured from A.



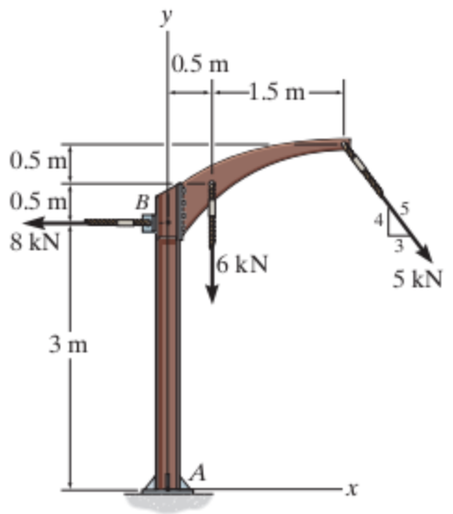
F3–32

F3–33. Replace the loading system by an equivalent resultant force and specify where the resultant's line of action intersects the member measured from A.



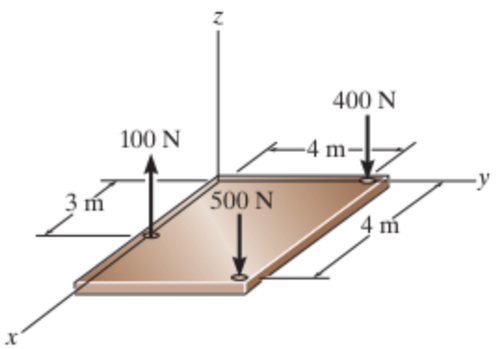
F3–33

F3–34. Replace the loading system by an equivalent resultant force and specify where the resultant's line of action intersects the member AB measured from A.



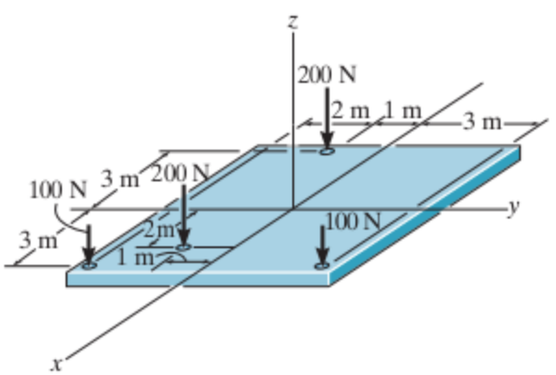
F3–34

F3–35. Replace the loading shown by an equivalent single resultant force and specify the x and y coordinates of its line of action.



F3–35

F3–36. Replace the loading shown by an equivalent single resultant force and specify the x and y coordinates of its line of action.

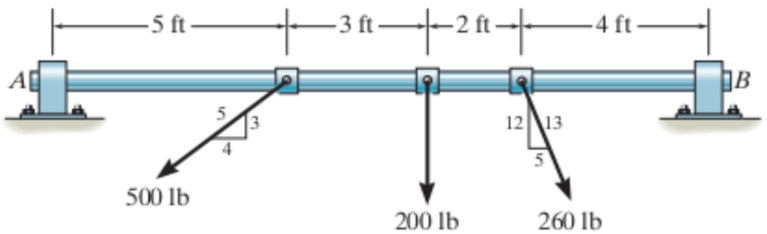


F3–36

PROBLEMS

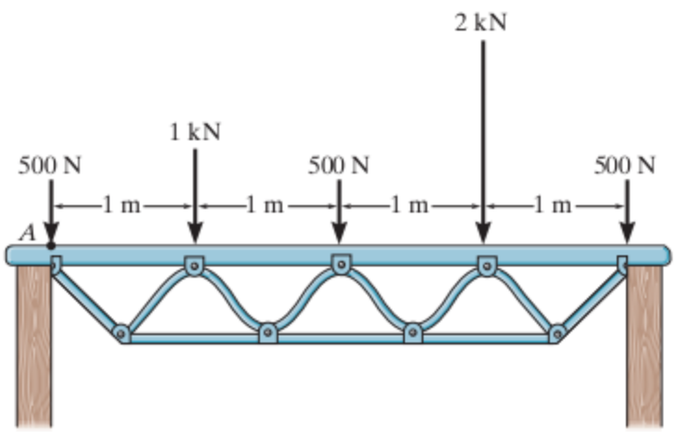
*3–88. Replace the three forces acting on the shaft by a single resultant force. Specify where the force acts, measured from end *A*.

3–89. Replace the three forces acting on the shaft by a single resultant force. Specify where the force acts, measured from end *B*.



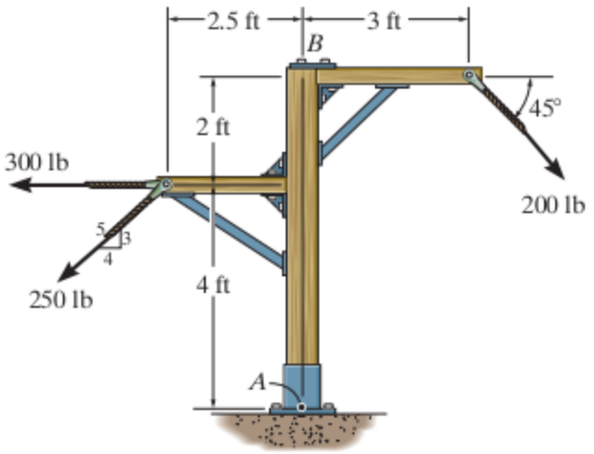
Probs. 3–88/89

3–90. The system of parallel forces acts on the top of the *Warren truss*. Determine the equivalent resultant force of the system and specify its location measured from point *A*.



Prob. 3–90

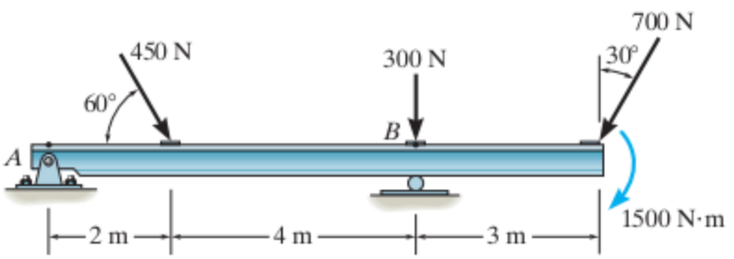
3–91. Replace the force system acting on the frame by a resultant force, and specify where its line of action intersects member *AB*, measured from point *A*.



Prob. 3–91

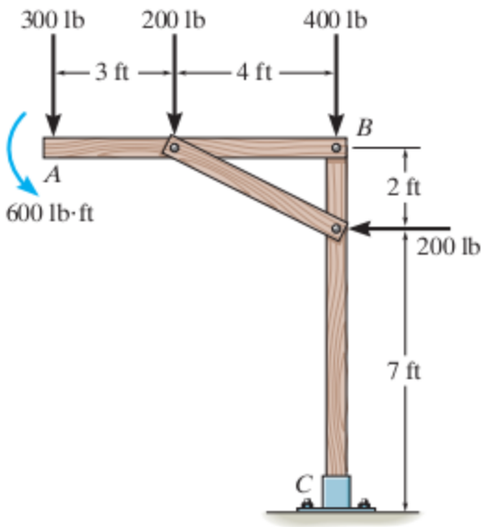
*3–92. Replace the loading acting on the beam by a single resultant force. Specify where the force acts, measured from end *A*.

3–93. Replace the loading acting on the beam by a single resultant force. Specify where the force acts, measured from *B*.



Probs. 3–92/93

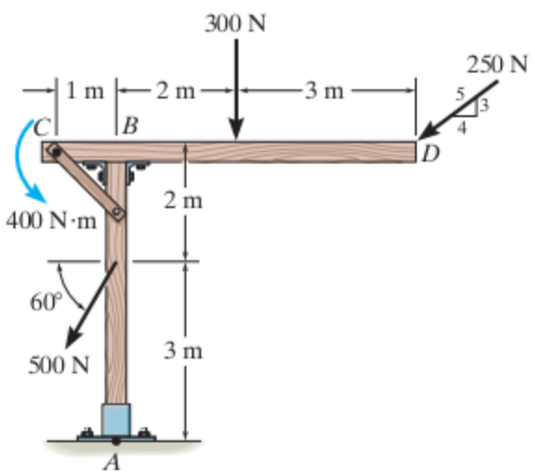
3–94. Replace the loading on the frame by a single resultant force. Specify where its line of action intersects member AB, measured from A.



Prob. 3–94

3–95. Replace the loading on the frame by a single resultant force. Specify where its line of action intersects member AB, measured from A.

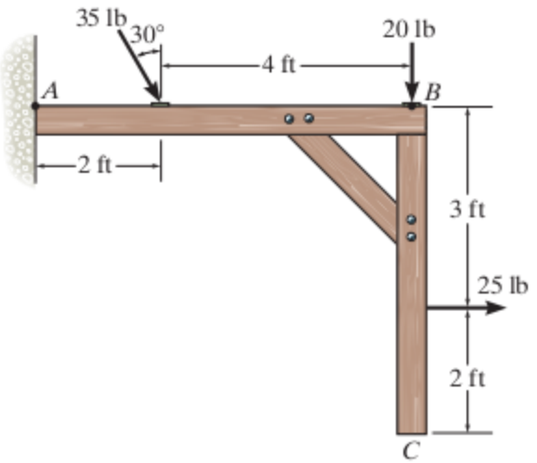
*3–96. Replace the loading on the frame by a single resultant force. Specify where its line of action intersects member CD, measured from end C.



Probs. 3–95/96

3–97. Replace the force system acting on the frame by an equivalent resultant force and specify where the resultant's line of action intersects member AB, measured from point A.

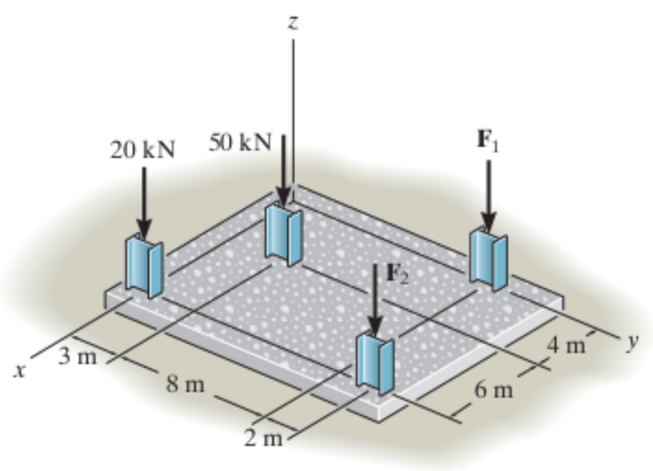
3–98. Replace the force system acting on the frame by an equivalent resultant force and specify where the resultant's line of action intersects member BC, measured from point B.



Probs. 3–97/98

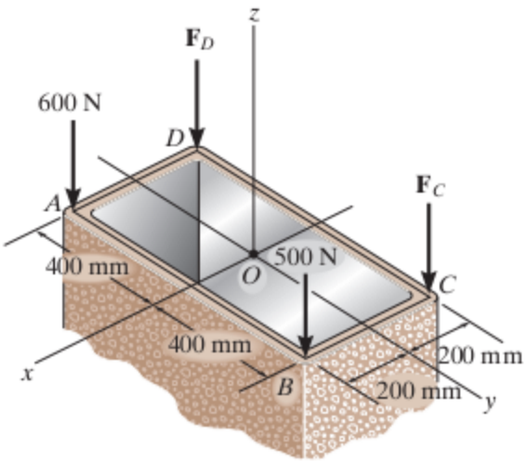
3–99. The building slab is subjected to four parallel column loadings. Determine the equivalent resultant force and specify its location (x, y) on the slab. Take $F_1 = 30$ kN, $F_2 = 40$ kN.

*3–100. The building slab is subjected to four parallel column loadings. Determine the equivalent resultant force and specify its location (x, y) on the slab. Take $F_1 = 20$ kN, $F_2 = 50$ kN.



Probs. 3–99/100

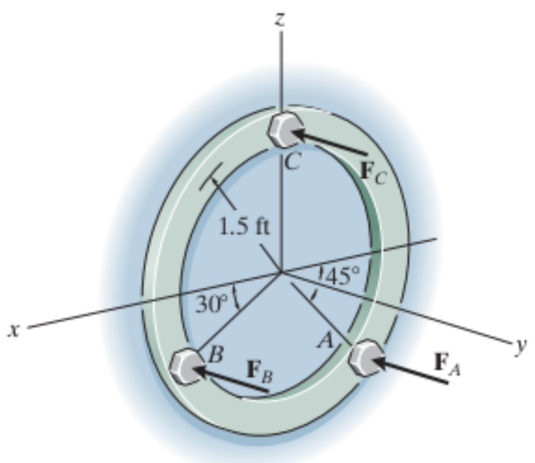
3–101. The tube supports the four parallel forces. Determine the magnitudes of forces $\mathbf{F}_C$ and $\mathbf{F}_D$ acting at C and D so that the equivalent resultant force of the force system acts through the midpoint O of the tube.



Prob. 3–101

3–102. Three parallel bolting forces act on the circular plate. Determine the resultant force, and specify its location (x, z) on the plate. $F_A = 200$ lb, $F_B = 100$ lb, and $F_C = 400$ lb.

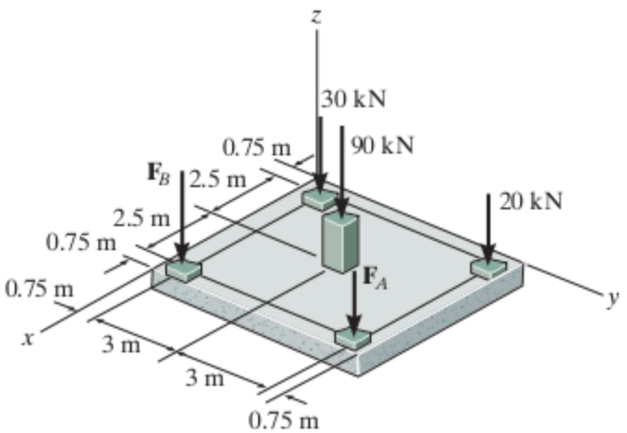
3–103. The three parallel bolting forces act on the circular plate. If the force at A has a magnitude of $F_A = 200$ lb, determine the magnitudes of $\mathbf{F}_B$ and $\mathbf{F}_C$ so that the resultant force $\mathbf{F}_R$ of the system has a line of action that coincides with the y axis. *Hint:* This requires $\Sigma M_x = 0$ and $\Sigma M_z = 0$.



Probs. 3–102/103

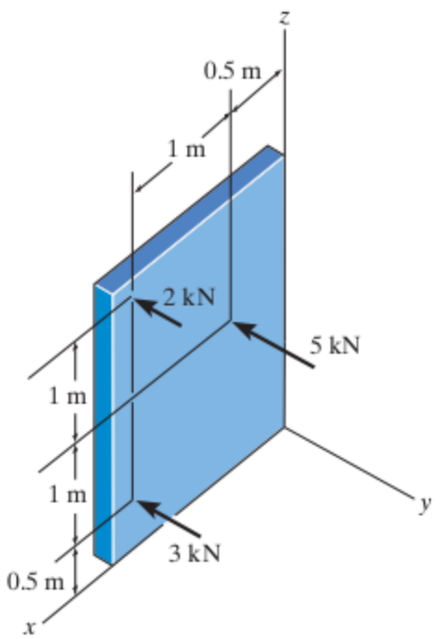
***3–104.** If $F_A = 40$ kN and $F_B = 35$ kN, determine the magnitude of the resultant force and specify the location of its point of application (x, y) on the slab.

3–105. If the resultant force is required to act at the center of the slab, determine the magnitude of the column loadings $\mathbf{F}_A$ and $\mathbf{F}_B$ and the magnitude of the resultant force.



Probs. 3–104/105

3–106. Replace the parallel force system acting on the plate by a resultant force and specify its location on the x–z plane.



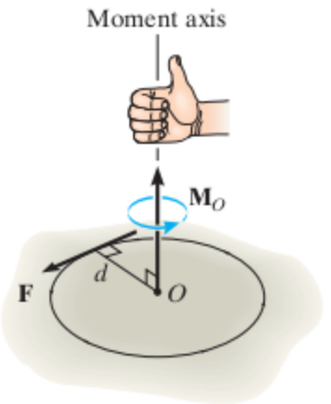
Prob. 3–106

CHAPTER REVIEW

Moment of Force—Scalar Definition

A force produces a turning effect or moment about a point O that does not lie on its line of action. In scalar form, the moment *magnitude* is the product of the force and the moment arm or perpendicular distance from point O to the line of action of the force.

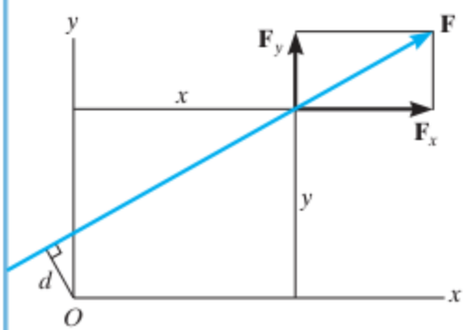
$$M_O = Fd$$



The *direction* of the moment is defined using the right-hatnd rule. $\mathbf{M}_O$ always acts along an axis perpendicular to the plane containing $\mathbf{F}$ and d, and passes through the point O.

Rather than finding d, it is normally easier to resolve the force into its x and y components, determine the moment of each component about the point, and then sum the results. This is called the principle of moments.

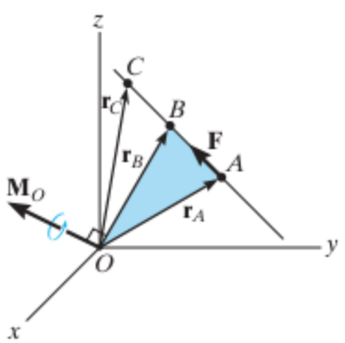
$$M_O = Fd = F_x y - F_y x$$



Moment of a Force—Vector Definition

Since three-dimensional geometry is generally more difficult to visualize, the vector cross product should be used to determine the moment. Here $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$, where $\mathbf{r}$ is a position vector that extends from point O to any point A, B, or C on the line of action of $\mathbf{F}$.

$$\mathbf{M}_O = \mathbf{r}_A \times \mathbf{F} = \mathbf{r}_B \times \mathbf{F} = \mathbf{r}_C \times \mathbf{F}$$



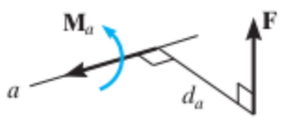
If the position vector $\mathbf{r}$ and force $\mathbf{F}$ are expressed as Cartesian vectors, then the cross product results from the expansion of a determinant.

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

Moment about an Axis

If the moment of a force **F** is to be determined about an arbitrary axis a, then for a scalar solution the moment arm, or shortest distance d_a from the line of action of the force to the axis must be used. This distance is perpendicular to both the axis and the force line of action.

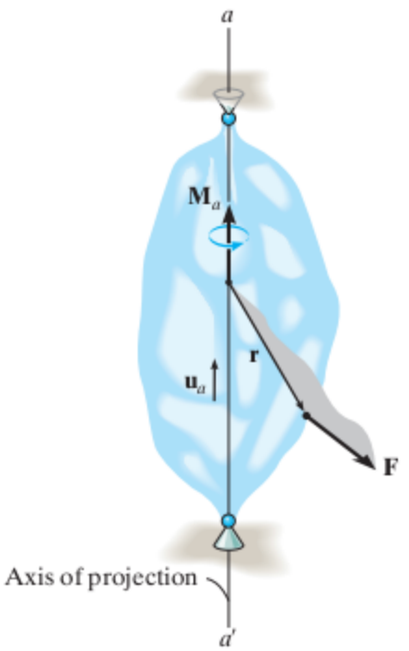
$$M_a = Fd_a$$



Note that when the line of action of **F** intersects the axis then the moment of **F** about the axis is zero. Also, when the line of action of **F** is parallel to the axis, the moment of **F** about the axis is zero.

In three dimensions, the scalar triple product should be used. Here $\mathbf{u}_a$ is the unit vector that specifies the direction of the axis, and **r** is a position vector that is directed from any point on the axis to any point on the line of action of the force. If M_a is calculated as a negative scalar, then the sense of direction of $\mathbf{M}_a$ is opposite to $\mathbf{u}_a$.

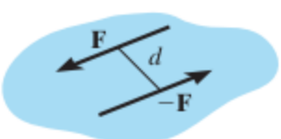
$$M_a = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F}) = \begin{vmatrix} u_{a_x} & u_{a_y} & u_{a_z} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$



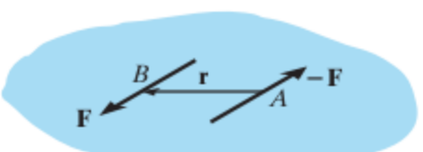
Couple Moment

A couple consists of two equal but opposite forces that act a perpendicular distance d apart. Couples tend to produce a rotation without translation.

$$M = Fd$$



The magnitude of the couple moment is $M = Fd$, and its direction is established using the right-hand rule.

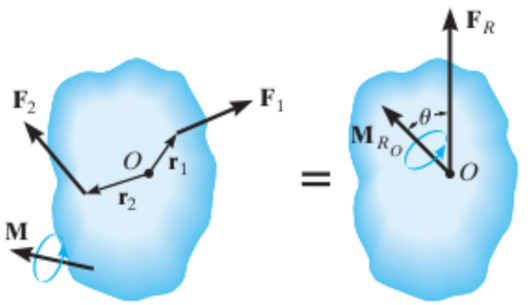


If the vector cross product is used to determine the moment of a couple, then **r** extends from any point on the line of action of one of the forces to any point on the line of action of the other force **F** that is used in the cross product.

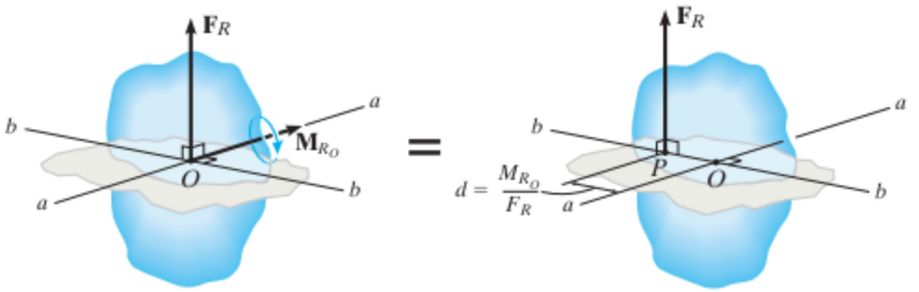
$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

Simplification of a Force and Couple System

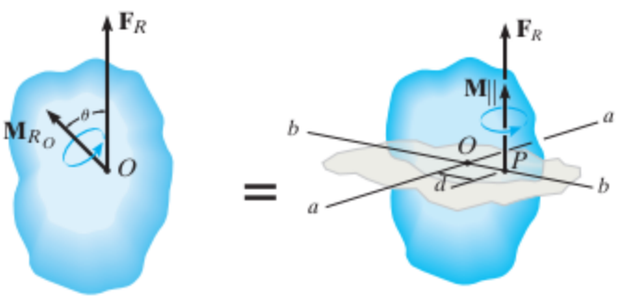
Any system of forces and couples can be reduced to a single resultant force and resultant couple moment acting at a point. The resultant force is the sum of all the forces in the system, $\mathbf{F}_R = \Sigma\mathbf{F}$, and the resultant couple moment is equal to the sum of all the moments of the forces about the point and couple moments. $\mathbf{M}_{R_O} = \Sigma\mathbf{M}_O + \Sigma\mathbf{M}$.



Further simplification to a single resultant force is possible, provided the force system is concurrent, coplanar, or parallel. To find the location of the resultant force from a point, it is necessary to equate the moment of the resultant force about the point to the moment of the forces and couples in the system about the same point.



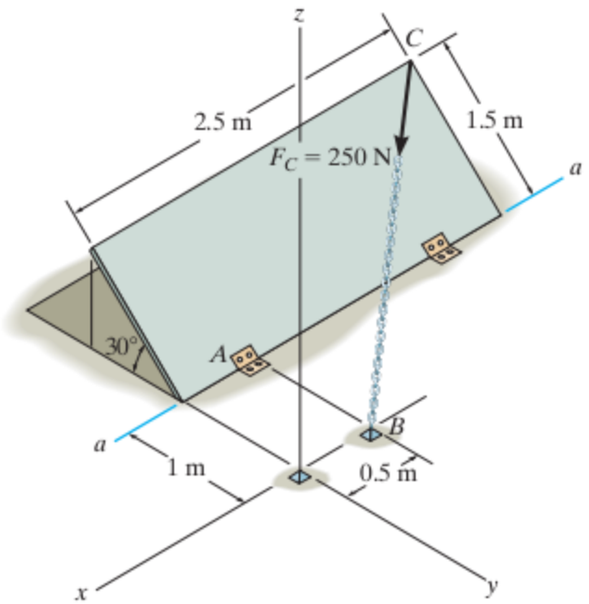
If the resultant force and couple moment at a point are not perpendicular to one another, then this system can be reduced to a wrench, which consists of the resultant force and collinear couple moment.



REVIEW PROBLEMS

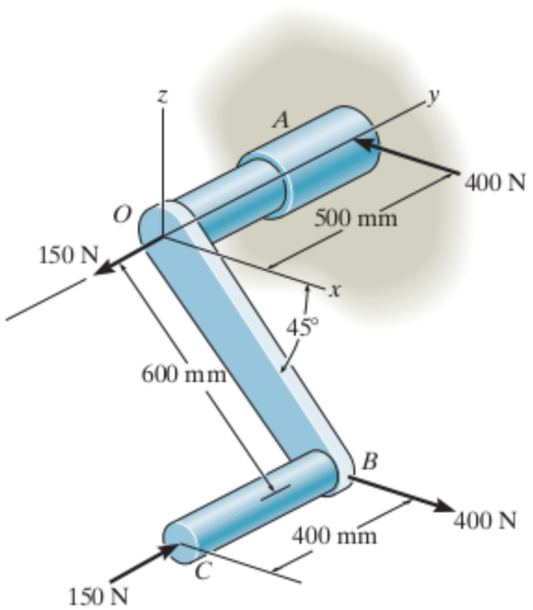
3–107. Determine the moment of the force $\mathbf{F}_C$ about the door hinge at A. Express the result as a Cartesian vector.

***3–108.** Determine the magnitude of the moment of the force $\mathbf{F}_C$ about the hinged axis a–a of the door.



Probs. 3–107/108

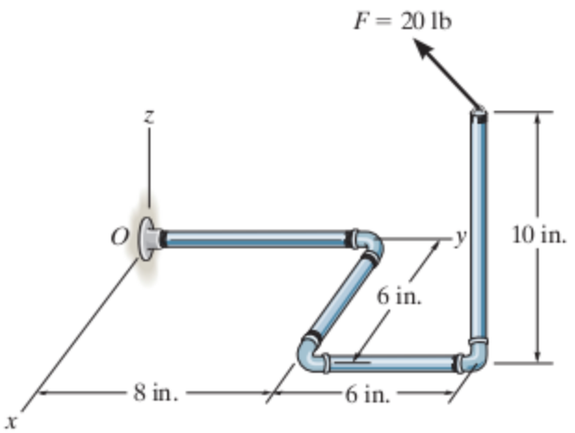
3–109. Determine the resultant couple moment of the two couples that act on the assembly. Member OB lies in the x–z plane.



Prob. 3–109

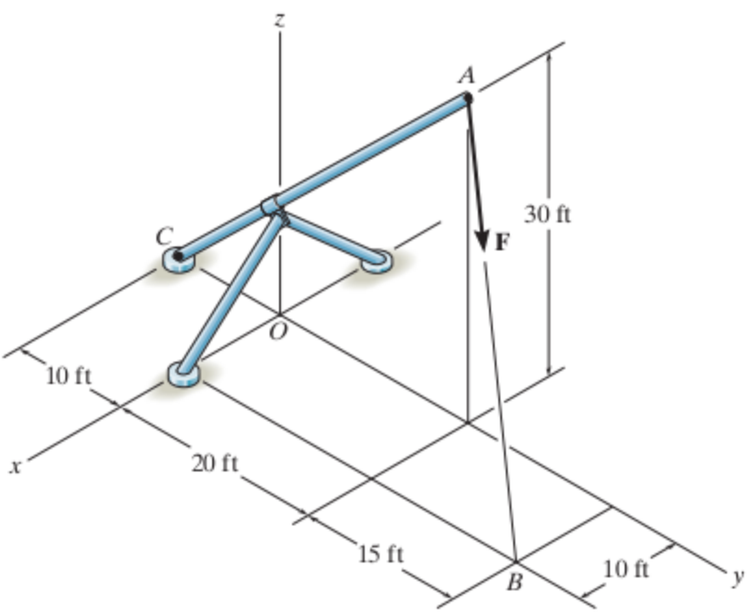
3–110. Determine the coordinate direction angles α, β, γ of $\mathbf{F}$, which is applied to the end of the pipe assembly, so that the moment of $\mathbf{F}$ about O is zero.

3–111. Determine the moment of the force $\mathbf{F}$ about point O. The force has coordinate direction angles of $\alpha = 60°$, $\beta = 120°, \gamma = 45°$. Express the result as a Cartesian vector.



Probs. 3–110/111

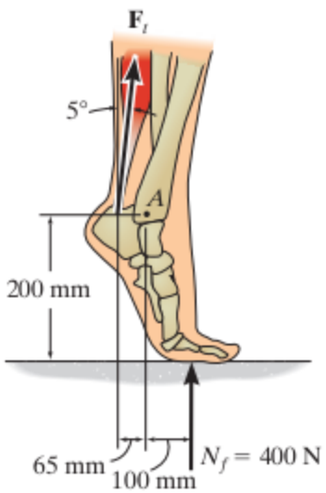
***3–112.** Replace the force $\mathbf{F}$ having a magnitude of $F = 50$ lb and acting at point A by an equivalent force and couple moment at point C.



Prob. 3–112

3–113. The Achilles tendon force of $F_t = 650\,\text{N}$ is mobilized when the man tries to stand on his toes. As this is done, each of his feet is subjected to a reactive force of $N_f = 400\,\text{N}$. Determine the resultant moment of $\mathbf{F}_t$ and $\mathbf{N}_f$ about the ankle joint A.

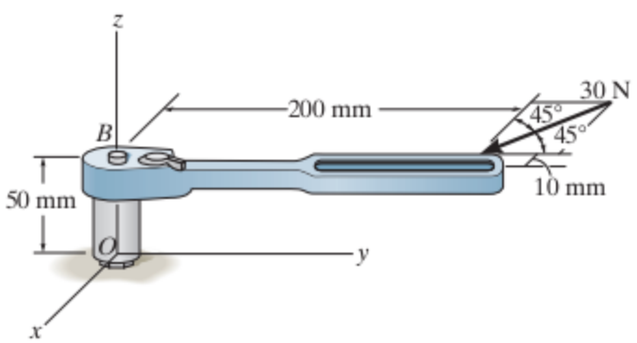
3–114. The Achilles tendon force $\mathbf{F}_t$ is mobilized when the man tries to stand on his toes. As this is done, each of his feet is subjected to a reactive force of $N_t = 400\,\text{N}$. If the resultant moment produced by forces $\mathbf{F}_t$ and $\mathbf{N}_t$ about the ankle joint A is required to be zero, determine the magnitude of $\mathbf{F}_t$.



Probs. 3–113/114

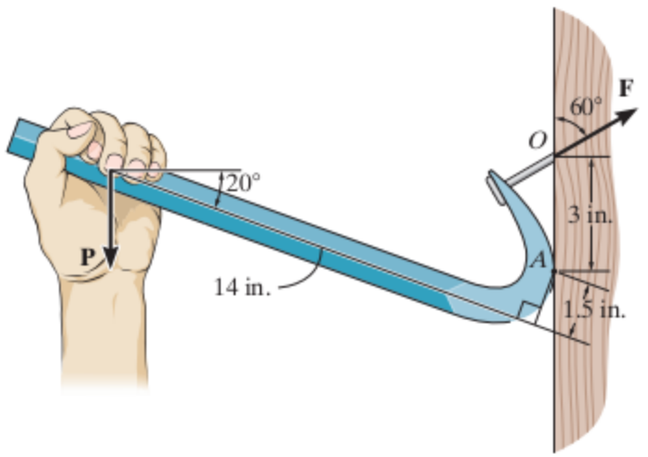
3–115. The horizontal 30-N force acts on the handle of the wrench. What is the magnitude of the moment of this force about the z axis?

***3–116.** The horizontal 30-N force acts on the handle of the wrench. Determine the moment of this force about point O. Specify the coordinate direction angles α, β, γ of the moment axis.



Probs. 3–115/116

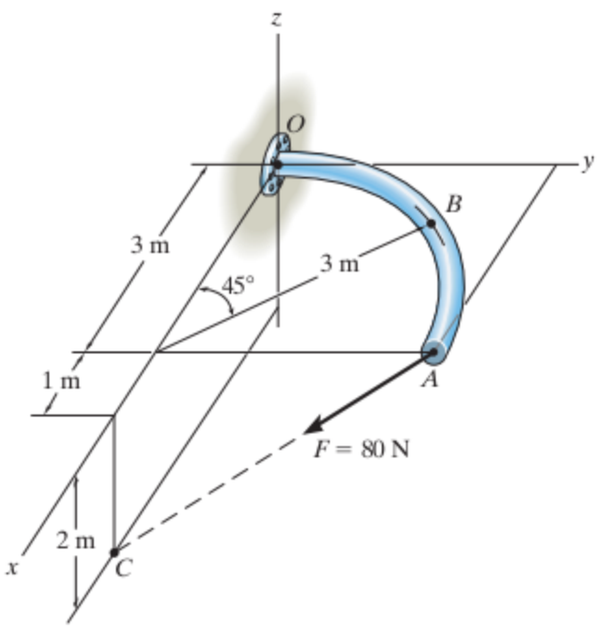
3–117. If it takes a force of $F = 125\,\text{lb}$ to pull the nail out, determine the smallest vertical force $\mathbf{P}$ that must be applied to the handle of the crowbar. *Hint:* This requires the moment of $\mathbf{F}$ about point A to be equal to the moment of $\mathbf{P}$ about A. Why?



Prob. 3–117

3–118. The curved rod lies in the x–y plane and has a radius of 3 m. If a force of $F = 80\,\text{N}$ acts at its end as shown, determine the moment of this force about point O.

3–119. The curved rod lies in the x–y plane and has a radius of 3 m. If a force of $F = 80\,\text{N}$ acts at its end as shown, determine the moment of this force about point B.



Probs. 3–118/119

Chapter 4

It is important to be able to determine the forces in the cables used to support this submarine to insure that they do not fail. In this chapter we will study how to apply equilibrium methods to determine the forces acting on the supports of a rigid body such as this.

Equilibrium of a Rigid Body

CHAPTER OBJECTIVES

- To develop the equations of equilibrium for a rigid body.
- To introduce the concept of the free-body diagram for a rigid body.
- To show how to solve rigid-body equilibrium problems using the equations of equilibrium.

4.1 Conditions for Rigid-Body Equilibrium

In this section, we will develop both the necessary and sufficient conditions for the equilibrium of the rigid body in Fig. 4–1*a*. As shown, this body is subjected to an external force and couple moment system that is the result of the effects of gravitational, electrical, magnetic, or contact forces caused by adjacent bodies. The internal forces caused by interactions between particles within the body are not shown in this figure because these forces occur in equal but opposite collinear pairs and hence will cancel out, a consequence of Newton's third law.

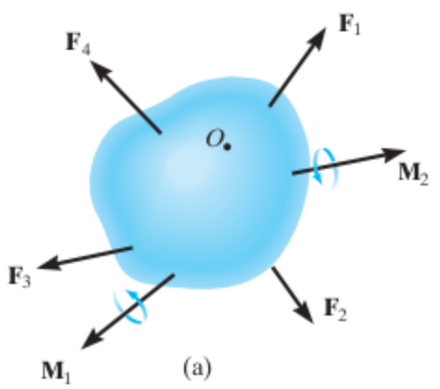
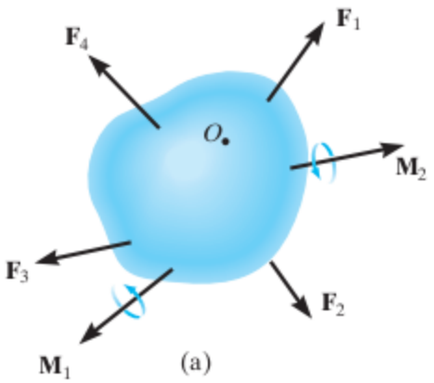
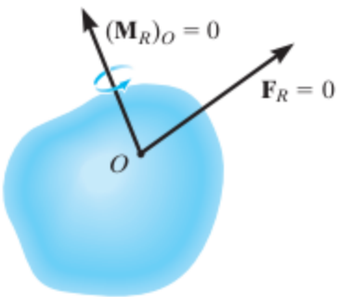


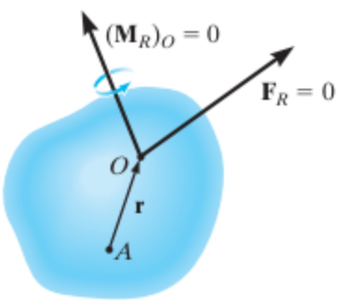
Fig. 4–1



(a)



(b)



(c)

Fig. 4–1 (cont.)

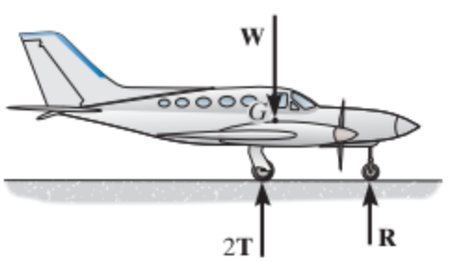


Fig. 4–2

Using the methods of the previous chapter, the force and couple moment system acting on a body can be reduced to an equivalent resultant force and resultant couple moment at any arbitrary point O on or off the body, Fig. 4–1*b*. If this resultant force and couple moment are both equal to zero, then the body is said to be in *equilibrium*. Mathematically, the equilibrium of a body is expressed as

$$\mathbf{F}_R = \Sigma \mathbf{F} = \mathbf{0}$$
$$(\mathbf{M}_R)_O = \Sigma \mathbf{M}_O = \mathbf{0} \qquad (4\text{–}1)$$

The first of these equations states that the sum of the forces acting on the body is equal to *zero*. The second equation states that the sum of the moments of all the forces in the system about point O, added to all the couple moments, is equal to *zero*. These two equations are not only necessary for equilibrium, they are also sufficient. To show this, consider summing moments about some other point, such as point A in Fig. 4–1*c*. We require

$$\Sigma \mathbf{M}_A = \mathbf{r} \times \mathbf{F}_R + (\mathbf{M}_R)_O = \mathbf{0}$$

Since $\mathbf{r} \neq \mathbf{0}$, this equation is satisfied only if Eqs. 4–1 are satisfied, namely $\mathbf{F}_R = \mathbf{0}$ and $(\mathbf{M}_R)_O = \mathbf{0}$.

When applying the equations of equilibrium, we will assume that the body remains rigid. In reality, however, all bodies deform when subjected to loads. Although this is the case, most engineering materials such as steel and concrete are very rigid and so their deformation is usually very small. Therefore, when applying the equations of equilibrium, we can generally assume that the body will remain *rigid* and *not deform* under the applied load without introducing any significant error. This way the direction of the applied forces and their moment arms with respect to a fixed reference remain unchanged before and after the body is loaded.

EQUILIBRIUM IN TWO DIMENSIONS

In the first part of the chapter, we will consider the case where the force system acting on a rigid body lies in or may be projected onto a *single* plane and, furthermore, any couple moments acting on the body are directed perpendicular to this plane. This type of force and couple system is often referred to as a two-dimensional or *coplanar* force system. For example, the airplane in Fig. 4–2 has a plane of symmetry through its center axis, and so the loads acting on the airplane are symmetrical with respect to this plane. Thus, each of the two wing tires will support the same load $\mathbf{T}$, which is represented on the side (two-dimensional) view of the plane as $2\mathbf{T}$.

4.2 Free-Body Diagrams

Successful application of the equations of equilibrium requires a complete specification of *all* the known and unknown external forces that act *on* the body. The best way to account for these forces is to draw a free-body diagram. This diagram is a sketch of the outlined shape of the body, which represents it as being *isolated* or "free" from its surroundings, i.e., a "free body." On this sketch it is necessary to show *all* the forces and couple moments that the surroundings exert *on the body* so that these effects can be accounted for when the equations of equilibrium are applied. *A thorough understanding of how to draw a free-body diagram is of primary importance for solving problems in mechanics.*

Support Reactions. Before presenting a formal procedure as to how to draw a free-body diagram, we will first consider the various types of reactions that occur at supports and points of contact between bodies subjected to coplanar force systems. As a general rule,

- If a support prevents the translation of a body in a given direction, then a force is developed on the body in that direction.
- If rotation is prevented, a couple moment is exerted on the body.

For example, let us consider three ways in which a horizontal member, such as a beam, is supported at its end. One method consists of a *roller* or cylinder, Fig. 4–3*a*. Since this support only prevents the beam from *translating* in the vertical direction, the roller will only exert a *force* on the beam in this direction, Fig. 4–3*b*.

The beam can be supported in a more restrictive manner by using a *pin*, Fig. 4–3*c*. The pin passes through a hole in the beam and two leaves which are fixed to the ground. Here the pin can prevent *translation* of the beam in *any direction* ϕ, Fig. 4–3*d*, and so the pin must exert a *force* $\mathbf{F}$ on the beam in this direction. For purposes of analysis, it is generally easier to represent this resultant force $\mathbf{F}$ by its two rectangular components $\mathbf{F}_x$ and $\mathbf{F}_y$, Fig. 4–3*e*. If F_x and F_y are known, then F and ϕ can be calculated.

The most restrictive way to support the beam would be to use a *fixed support* as shown in Fig. 4–3*f*. This support will prevent both *translation and rotation* of the beam. To do this a *force and couple moment* must be developed on the beam at its point of connection, Fig. 4–3*g*. As in the case of the pin, the force is usually represented by its rectangular components $\mathbf{F}_x$ and $\mathbf{F}_y$.

Table 4–1 lists other common types of supports for bodies subjected to coplanar force systems. (In all cases the angle θ is assumed to be known.) Carefully study each of the symbols used to represent these supports and the types of reactions they exert on their contacting members.

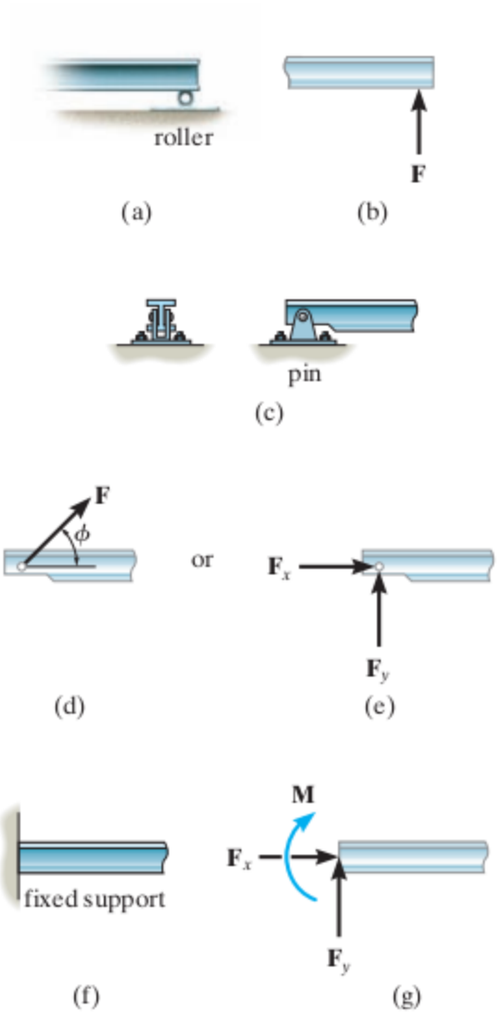
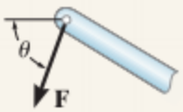
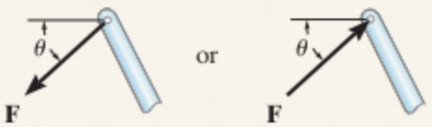
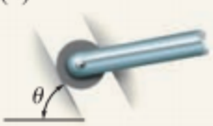
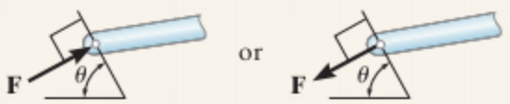
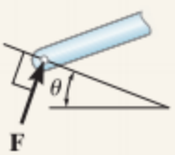
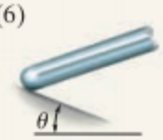
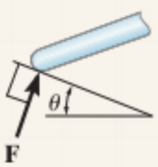
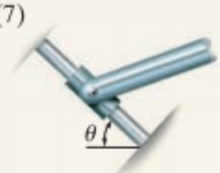
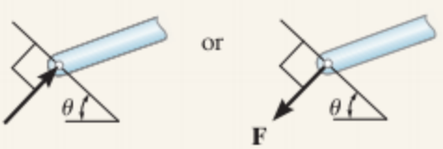


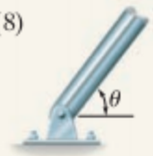
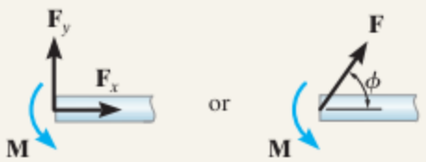
Fig. 4–3

TABLE 4–1 Supports for Rigid Bodies Subjected to Two-Dimensional Force Systems

Types of Connection	Reaction	Number of Unknowns
(1) cable		One unknown. The reaction is a tension force which acts away from the member in the direction of the cable.
(2) weightless link		One unknown. The reaction is a force which acts along the axis of the link.
(3) roller		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(4)  roller or pin in confined smooth slot		One unknown. The reaction is a force which acts perpendicular to the slot.
(5) rocker		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(6)  smooth contacting surface		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(7)  member pin connected to collar on smooth rod		One unknown. The reaction is a force which acts perpendicular to the rod.

continued

TABLE 4–1 Continued

Types of Connection	Reaction	Number of Unknowns
(8)  smooth pin or hinge		Two unknowns. The reactions are two components of force, or the magnitude and direction ϕ of the resultant force. Note that ϕ and θ are not necessarily equal [usually not, unless the rod shown is a link as in (2)].
(9) member fixed connected to collar on smooth rod		Two unknowns. The reactions are the couple moment and the force which acts perpendicular to the rod.
(10) fixed support		Three unknowns. The reactions are the couple moment and the two force components, or the couple moment and the magnitude and direction ϕ of the resultant force.

Typical examples of actual supports are shown in the following sequence of photos. The numbers refer to the connection types in Table 4–1.

The cable exerts a force on the bracket in the direction of the cable. (1)

The rocker support for this bridge girder allows horizontal movement so the bridge is free to expand and contract due to a change in temperature. (5)

This concrete girder rests on the ledge that is assumed to act as a smooth contacting surface. (6)

This utility building is pin supported at the top of the column. (8)

The floor beams of this building are welded together and thus form fixed connections. (10)

Springs. If a *linear elastic spring* is used to support a body, the length of the spring will change in direct proportion to the force acting on it. A characteristic that defines the "elasticity" of a spring is the *spring constant* or *stiffness* k. Specifically, the magnitude of force developed by a linear elastic spring which has a stiffness k, and is deformed (elongated or compressed) a distance s measured from its unloaded position, is

$$F = ks \tag{4–2}$$

Note that s is determined from the difference in the spring's deformed length l and its undeformed length l_0, i.e., $s = l - l_0$. Thus, if s is positive, **F** "pulls" on the spring; whereas if s is negative, **F** must "push" on it.

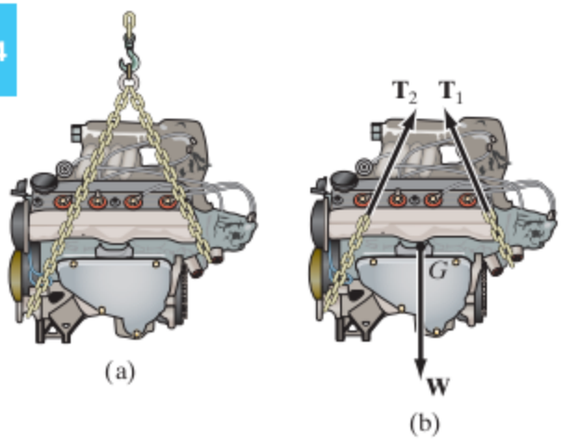


Fig. 4–4

Internal Forces. As stated in Sec. 4.1, the internal forces that act between adjacent particles in a body always occur in collinear pairs such that they have the same magnitude and act in opposite directions (Newton's third law). Since these forces cancel each other, they will not create an *external effect* on the body. It is for this reason that the internal forces should not be included on the free-body diagram if the entire body is to be considered. For example, the engine shown in Fig. 4–4*a* has a free-body diagram shown in Fig. 4–4*b*. The internal forces between all its connected parts such as the screws and bolts, will cancel out because they form equal and opposite collinear pairs. Only the external forces $\mathbf{T}_1$ and $\mathbf{T}_2$, exerted by the chains and the engine weight **W**, are shown on the free-body diagram.

Weight and the Center of Gravity. When a body is within a gravitational field, then each of its particles has a specified weight. It was shown in Sec. 3.8 that such a system of forces can be reduced to a single resultant force acting through a specified point. We refer to this force resultant as the *weight* **W** of the body and to the location of its point of application as the *center of gravity*. The methods used for its determination will be developed in Chapter 6.

In the examples and problems that follow, if the weight of the body is important for the analysis, this force will be reported in the problem statement. Also, when the body is *uniform* or made from the same material, the center of gravity will be located at the body's *geometric center* or *centroid*; however, if the body consists of a nonuniform distribution of material, or has an unusual shape, then the location of its center of gravity G will be given.

Idealized Models. When an engineer performs a force analysis of any object, he or she considers a corresponding analytical or idealized model that gives results that approximate as closely as possible the actual situation. To do this, careful choices have to be made so that selection of the type of supports, the material behavior, and the object's dimensions can be justified. This way one can feel confident that any design or analysis will yield results which can be trusted. In complex cases

this process may require developing several different models of the object that must be analyzed. In any case, this selection process requires both skill and experience.

The following two cases illustrate what is required to develop a proper model. In Fig. 4–5*a*, the steel beam is to be used to support the three roof joists of a building. For a force analysis it is reasonable to assume the material (steel) is rigid since only very small deflections will occur when the beam is loaded. A bolted connection at A will allow for any slight rotation that occurs here when the load is applied, and so a *pin* can be considered for this support. At B a *roller* can be considered since this support offers no resistance to horizontal movement. Building code is used to specify the roof loading A so that the joist loads $\mathbf{F}$ can be calculated. These forces will be larger than any actual loading on the beam since they account for extreme loading cases and for dynamic or vibrational effects. Finally, the weight of the beam is generally neglected when it is small compared to the load the beam supports. The idealized model of the beam is therefore shown with average dimensions a, b, c, and d in Fig. 4–5*b*.

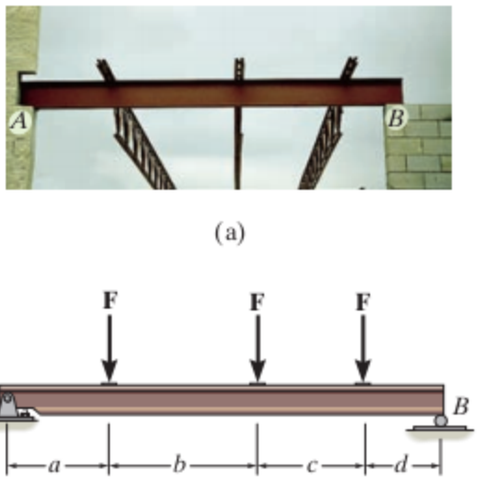


Fig. 4–5

As a second case, consider the lift boom in Fig. 4–6*a*. By inspection, it is supported by a pin at A and by the hydraulic cylinder BC, which can be approximated as a weightless link. The material can be assumed rigid, and with its density known, the weight of the boom and the location of its center of gravity G are determined. When a design loading $\mathbf{P}$ is specified, the idealized model shown in Fig. 4–6*b* can be used for a force analysis. Average dimensions (not shown) are used to specify the location of the loads and the supports.

Idealized models of specific objects will be given in some of the examples throughout the text. It should be realized, however, that each case represents the reduction of a practical situation using simplified assumptions like the ones illustrated here.

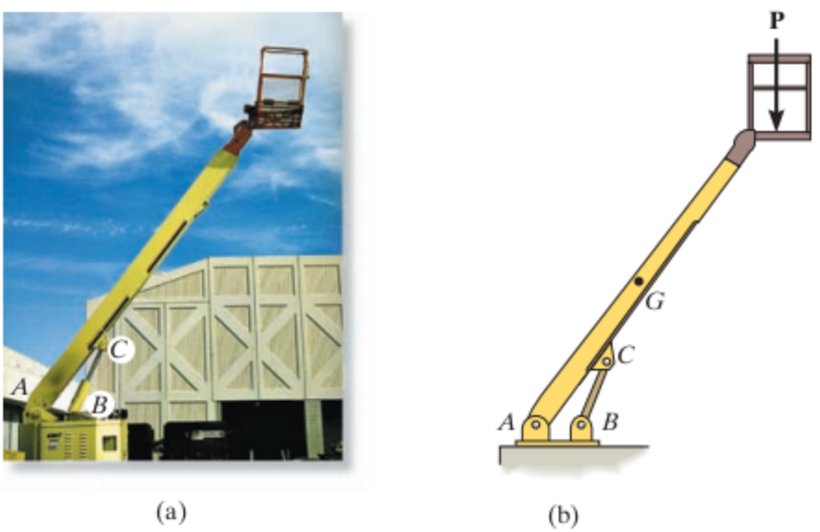


Fig. 4–6

Procedure for Analysis

To construct a free-body diagram for a rigid body or any group of bodies considered as a single system, the following steps should be performed:

Draw Outlined Shape.

Imagine the body to be *isolated* or cut "free" from its constraints and connections and draw (sketch) its outlined shape.

Show All Forces and Couple Moments.

Identify all the known and unknown *external forces* and couple moments that *act on the body*. Those generally encountered are due to (1) applied loadings, (2) reactions occurring at the supports or at points of contact with other bodies (see Table 4–1), and (3) the weight of the body. To account for all these effects, it may help to trace over the boundary, carefully noting each force or couple moment acting on it.

Identify Each Loading and Give Dimensions.

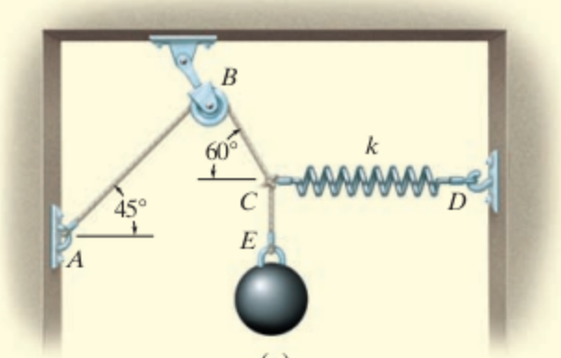
The forces and couple moments that are known should be labeled with their proper magnitudes and directions. Letters are used to represent the magnitudes and direction angles of forces and couple moments that are unknown. Establish an x, y coordinate system so that these unknowns, A_x, A_y, etc., can be identified. Finally, indicate the dimensions of the body necessary for calculating the moments of forces.

Important Points

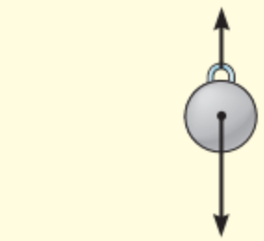
- No equilibrium problem should be solved without *first drawing the free-body diagram*, so as to account for all the forces and couple moments that act on the body.
- If a support *prevents translation* of a body in a particular direction, then the support exerts a *force* on the body in that direction.
- If *rotation is prevented*, then the support exerts a *couple moment* on the body.
- Study Table 4–1.
- Internal forces are never shown on the free-body diagram since they occur in equal but opposite collinear pairs and therefore cancel out.
- The weight of a body is an external force, and its effect is represented by a single resultant force acting through the body's center of gravity G.
- *Couple moments* can be placed anywhere on the free-body diagram since they are *free vectors*. *Forces* can act at any point along their lines of action since they are *sliding vectors*.

EXAMPLE 4.1

The sphere in Fig. 4–7*a* has a mass of 6 kg and is supported as shown. Draw a free-body diagram of the sphere, the cord *CE*, and the knot at *C*.



(a)

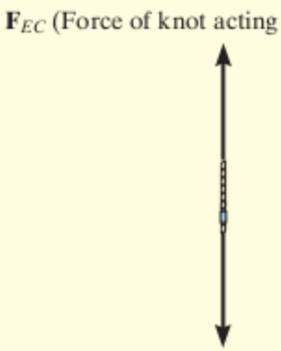


(b)

SOLUTION

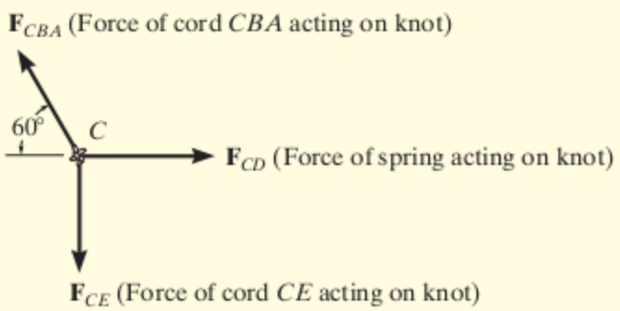
Sphere. By inspection, there are only two forces acting on the sphere, namely, its weight, 6 kg (9.81 m/s^2) = 58.9 N, and the force of cord *CE*. The free-body diagram is shown in Fig. 4–7*b*.

Cord *CE*. When the cord *CE* is isolated from its surroundings, its free-body diagram shows only two forces acting on it, namely, the force of the sphere and the force of the knot, Fig. 4–7*c*. Notice that $\mathbf{F}_{CE}$ shown here is equal but opposite to that shown in Fig. 4–7*b*, a consequence of Newton's third law of action–reaction. Also, $\mathbf{F}_{CE}$ and $\mathbf{F}_{EC}$ pull on the cord and keep it in tension so that it doesn't collapse. For equilibrium, $F_{CE} = F_{EC}$.



(c)

Knot. The knot at *C* is subjected to three forces, Fig. 4–7*d*. They are caused by the cords *CBA* and *CE* and the spring *CD*. As required, the free-body diagram shows all these forces labeled with their magnitudes and directions. It is important to recognize that the weight of the sphere does not directly act on the knot. Instead, the cord *CE* subjects the knot to this force.



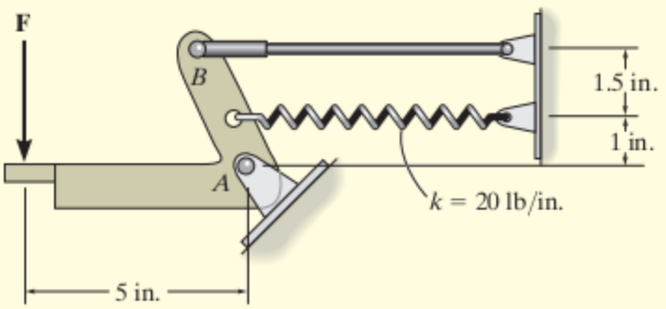
(d)

Fig. 4–7

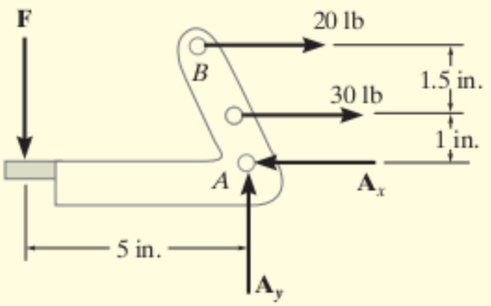
EXAMPLE 4.2

Draw the free-body diagram of the foot lever shown in Fig. 4–8*a*. The operator applies a vertical force to the pedal so that the spring is stretched 1.5 in. and the force in the short link at *B* is 20 lb.

(a)



(b)



(c)

Fig. 4–8

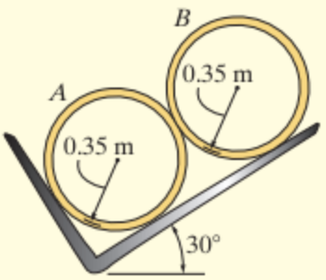
SOLUTION

By inspection of the photo, the lever is loosely bolted to the frame at *A*. The rod at *B* is pinned at its ends and acts as a "short link." After making the proper measurements, the idealized model of the lever is shown in Fig. 4–8*b*. From this, the free-body diagram is shown in Fig. 4–8*c*. The pin support at *A* exerts force components $\mathbf{A}_x$ and $\mathbf{A}_y$ on the lever. The link at *B* exerts a force of 20 lb, acting in the direction of the link. In addition the spring also exerts a horizontal force on the lever. If the stiffness is measured and found to be $k = 20\,\text{lb/in.}$, then since the stretch $s = 1.5\,\text{in.}$, using Eq. 4–2; $F_s = ks = 20\,\text{lb/in.}\,(1.5\,\text{in.}) = 30\,\text{lb}$. Finally, the operator's shoe applies a vertical force of **F** on the pedal. The dimensions of the lever are also shown on the free-body diagram, since this information will be useful when computing the moments of the forces. As usual, the senses of the unknown forces at *A* have been assumed. The correct senses will become apparent after solving the equilibrium equations.

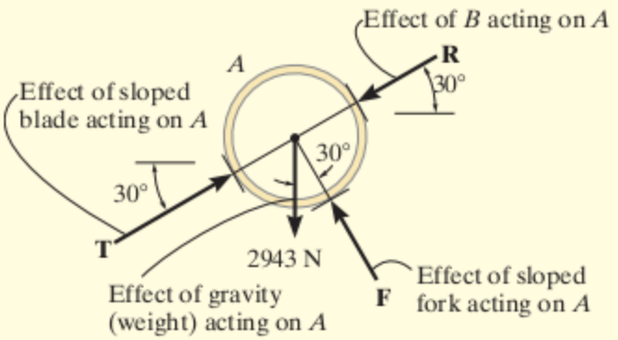
EXAMPLE 4.3

Two smooth pipes, each having a mass of 300 kg, are supported by the forked tines of the tractor in Fig. 4–9*a*. Draw the free-body diagrams for each pipe and both pipes together.

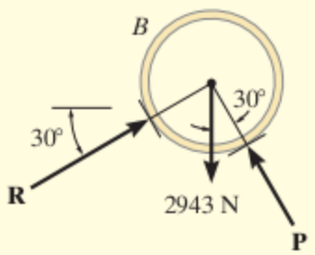
(a)



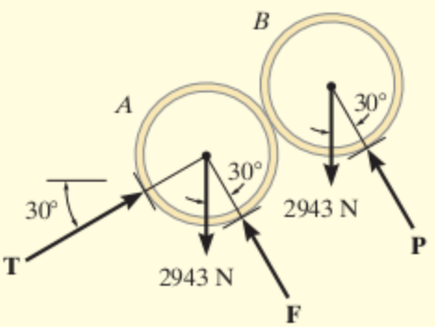
(b)



(c)



(d)



(e)

Fig. 4–9

SOLUTION

The idealized model from which we must draw the free-body diagrams is shown in Fig. 4–9*b*. Here the pipes are identified, the dimensions have been added, and the physical situation reduced to its simplest form.

The free-body diagram for pipe A is shown in Fig. 4–9*c*. Its weight is $W = 300(9.81)\ \text{N} = 2943\ \text{N}$. Assuming all contacting surfaces are *smooth*, the reactive forces **T**, **F**, **R** act in a direction *normal* to the tangent at their surfaces of contact.

The free-body diagram of pipe B is shown in Fig. 4–9*d*. Can you identify each of the three forces acting *on this pipe*? In particular, note that **R**, representing the force of A on B, Fig. 4–9*d*, is equal and opposite to **R** representing the force of B on A, Fig. 4–9*c*. This is a consequence of Newton's third law of motion.

The free-body diagram of both pipes combined ("system") is shown in Fig. 4–9*e*. Here the contact force **R**, which acts between A and B, is considered as an *internal* force and hence is not shown on the free-body diagram. That is, it represents a pair of equal but opposite collinear forces which cancel each other.

CONCEPTUAL PROBLEMS

P4–1. Draw the free-body diagram of the uniform trash bucket which has a significant weight. It is pinned at A and rests against the smooth horizontal member at B. Show your result in side view. Label any necessary dimensions.

P4–1

P4–2. Draw the free-body diagram of the outrigger ABC used to support a backhoe. The top pin B is connected to the hydraulic cylinder, which can be considered to be a short link (two-force member), the bearing shoe at A is smooth, and the outrigger is pinned to the frame at C.

P4–2

P4–3. Draw the free-body diagram of the wing on the passenger plane. The weights of the engine and wing are significant. The tires at B are smooth.

P4–3

P4–4. Draw the free-body diagram of the wheel and member ABC used as part of the landing gear on a jet plane. The hydraulic cylinder AD acts as a two-force member, and there is a pin connection at B.

P4–4

4.3 Equations of Equilibrium

In Sec. 4.1, we developed the two equations which are both necessary and sufficient for the equilibrium of a rigid body, namely, $\Sigma \mathbf{F} = \mathbf{0}$ and $\Sigma \mathbf{M}_O = \mathbf{0}$. When the body is subjected to a system of forces, which all lie in the x–y plane, then the forces can be resolved into their x and y components. Consequently, the conditions for equilibrium in two dimensions are

$$\begin{aligned} \Sigma F_x &= 0 \\ \Sigma F_y &= 0 \\ \Sigma M_O &= 0 \end{aligned} \tag{4–3}$$

Here ΣF_x and ΣF_y represent, respectively, the algebraic sums of the x and y components of all the forces acting on the body, and ΣM_O represents the algebraic sum of the couple moments and the moments of all the force components about the z axis, which is perpendicular to the x–y plane and passes through the arbitrary point O.

Alternative Sets of Equilibrium Equations. Although Eqs. 4–3 are *most often* used for solving coplanar equilibrium problems, two *alternative* sets of three independent equilibrium equations may also be used. One such set is

$$\begin{aligned} \Sigma F_x &= 0 \\ \Sigma M_A &= 0 \\ \Sigma M_B &= 0 \end{aligned} \tag{4–4}$$

When using these equations, it is required that a line passing through points A and B is *not parallel* to the y axis. To prove that Eqs. 4–4 provide the *conditions* for equilibrium, consider the free-body diagram of the plate shown in Fig. 4–10*a*. Using the methods of Sec. 3.7, all the forces on the free-body diagram may be replaced by an equivalent resultant force $\mathbf{F}_R = \Sigma \mathbf{F}$, acting at point A, and a resultant couple moment $(\mathbf{M}_R)_A = \Sigma \mathbf{M}_A$, Fig. 4–10*b*. If $\Sigma M_A = 0$ is satisfied, it is necessary that $(\mathbf{M}_R)_A = \mathbf{0}$. Furthermore, in order that $\mathbf{F}_R$ satisfy $\Sigma F_x = 0$, it must have *no component* along the x axis, and therefore $\mathbf{F}_R$ must be parallel to the y axis, Fig. 4–10*c*. Finally, if it is required that $\Sigma M_B = 0$, where B does not lie on the line of action of $\mathbf{F}_R$, then $\mathbf{F}_R = \mathbf{0}$. Since Eqs. 4–4 show that both of these resultants are zero, indeed the body in Fig. 4–10*a* must be in equilibrium.

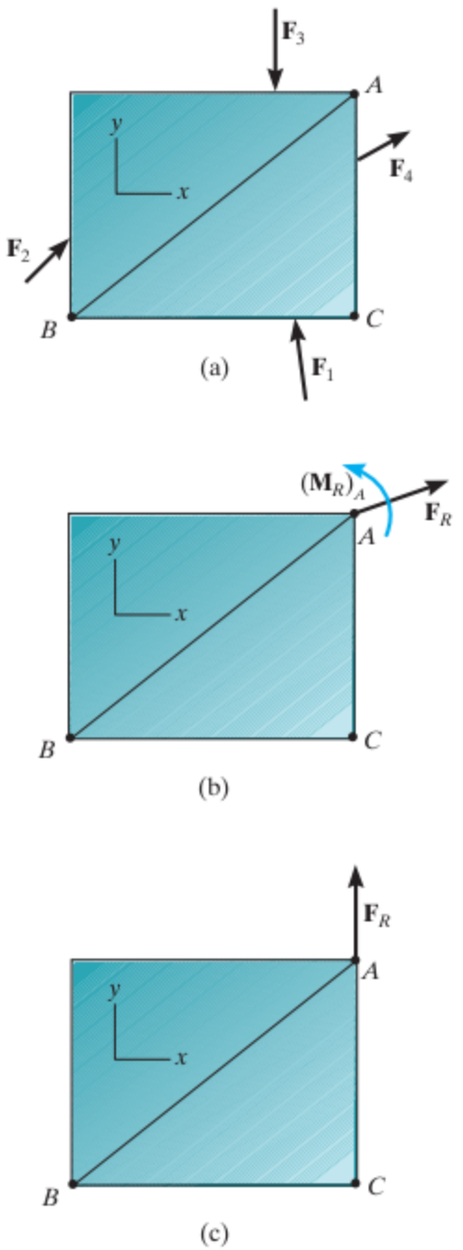


Fig. 4–10

A second alternative set of equilibrium equations is

$$\begin{aligned} \Sigma M_A &= 0 \\ \Sigma M_B &= 0 \\ \Sigma M_C &= 0 \end{aligned} \tag{4–5}$$

Here it is necessary that points A, B, and C do not lie on the same line. To prove that these equations, when satisfied, ensure equilibrium, consider again the free-body diagram in Fig. 4–10*b*. If $\Sigma M_A = 0$ is to be satisfied, then $(\mathbf{M}_R)_A = \mathbf{0}$. $\Sigma M_C = 0$ is satisfied if the line of action of $\mathbf{F}_R$ passes through point C as shown in Fig. 4–10*c*. Finally, if we require $\Sigma M_B = 0$, it is necessary that $\mathbf{F}_R = \mathbf{0}$, and so the plate in Fig. 4–10*a* must then be in equilibrium.

4

Procedure for Analysis

Coplanar force equilibrium problems for a rigid body can be solved using the following procedure.

Free-Body Diagram.

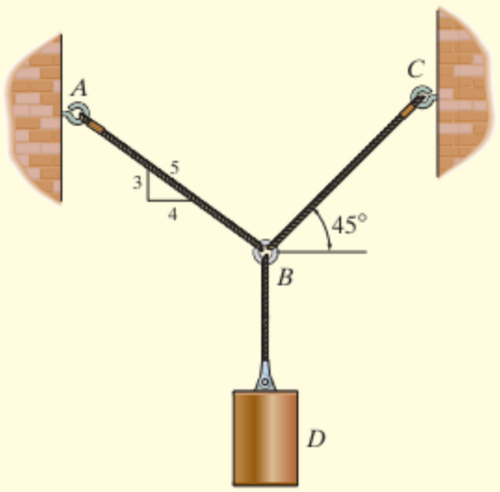
- Establish the x, y coordinate axes in any suitable orientation.
- Draw an outlined shape of the body.
- Show all the forces and couple moments acting on the body.
- Label all the loadings and specify their directions relative to the x or y axis. The sense of a force or couple moment having an *unknown* magnitude but known line of action can be *assumed*.
- Indicate the dimensions of the body necessary for computing the moments of forces.

Equations of Equilibrium.

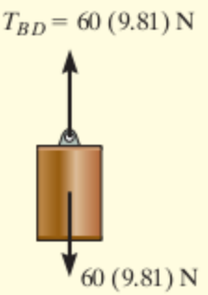
- Apply the moment equation of equilibrium, $\Sigma M_O = 0$, about a point (O) that lies at the intersection of the lines of action of two unknown forces. In this way, the moments of these unknowns are zero about O, and a *direct solution* for the third unknown can be determined.
- When applying the force equilibrium equations, $\Sigma F_x = 0$ and $\Sigma F_y = 0$, orient the x and y axes along lines that will provide the simplest resolution of the forces into their x and y components.
- If the solution of the equilibrium equations yields a negative scalar for a force or couple moment magnitude, this indicates that the sense is opposite to that which was assumed on the free-body diagram.

EXAMPLE 4.4

Determine the tension in cables BA and BC necessary to support the 60-kg cylinder in Fig. 4–11*a*.



(a)



(b)

SOLUTION

Free-Body Diagram. Due to equilibrium, the weight of the cylinder causes the tension in cable BD to be $T_{BD} = 60(9.81)$ N, Fig. 4–11*b*. The forces in cables BA and BC can be determined by investigating the equilibrium of ring B. Its free-body diagram is shown in Fig. 4–11*c*. The magnitudes of $\mathbf{T}_A$ and $\mathbf{T}_C$ are unknown, but their directions are known.

Equations of Equilibrium. Applying the equations of equilibrium along the x and y axes, we have

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad T_C \cos 45° - \left(\tfrac{4}{5}\right) T_A = 0 \tag{1}$$

$$+\uparrow \Sigma F_y = 0; \qquad T_C \sin 45° + \left(\tfrac{3}{5}\right) T_A - 60(9.81)\,\text{N} = 0 \tag{2}$$

Equation (1) can be written as $T_A = 0.8839 T_C$. Substituting this into Eq. (2) yields

$$T_C \sin 45° + \left(\tfrac{3}{5}\right)(0.8839 T_C) - 60(9.81)\,\text{N} = 0$$

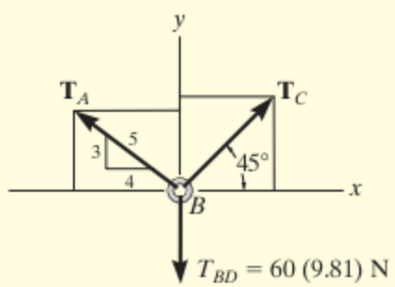
So that

$$T_C = 475.66\,\text{N} = 476\,\text{N} \qquad \textit{Ans.}$$

Substituting this result into either Eq. (1) or Eq. (2), we get

$$T_A = 420\,\text{N} \qquad \textit{Ans.}$$

NOTE: The accuracy of these results, of course, depends on the accuracy of the data, i.e., measurements of geometry and loads. For most engineering work involving a problem such as this, the data as measured to three significant figures would be sufficient.



(c)

Fig. 4–11

EXAMPLE 4.5

Determine the horizontal and vertical components of reaction on the beam caused by the pin at B and the rocker at A as shown in Fig. 4–12a. Neglect the weight of the beam.

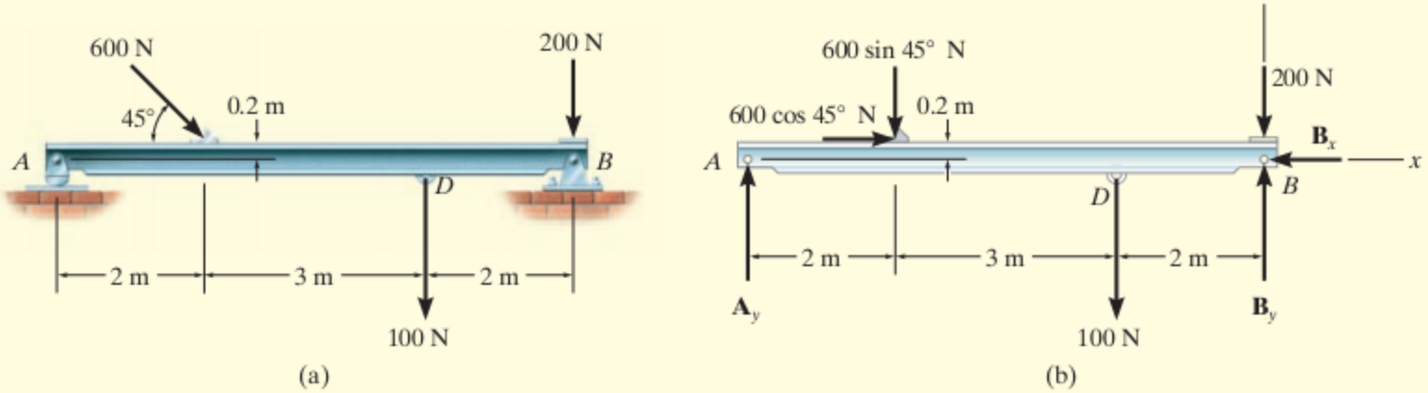
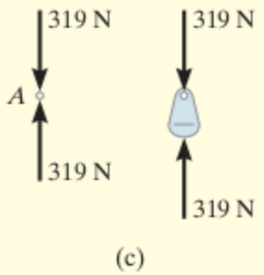


Fig. 4–12



SOLUTION

Free-Body Diagram. Identify each of the forces shown on the free-body diagram of the beam, Fig. 4–12b. For simplicity, the 600-N force is represented by its x and y components as shown in Fig. 4–12b.

Equations of Equilibrium. Summing forces in the x direction yields

$$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \qquad 600 \cos 45° \text{ N} - B_x = 0$$

$$B_x = 424 \text{ N} \qquad \textit{Ans.}$$

A direct solution for $\mathbf{A}_y$ can be obtained by applying the moment equation $\Sigma M_B = 0$ about point B.

$$\circlearrowleft + \Sigma M_B = 0; \qquad 100\,\text{N}(2 \text{ m}) + (600 \sin 45° \text{ N})(5 \text{ m}) - (600 \cos 45° \text{ N})(0.2 \text{ m}) - A_y(7 \text{ m}) = 0$$

$$A_y = 319 \text{ N} \qquad \textit{Ans.}$$

Summing forces in the y direction, using this result, gives

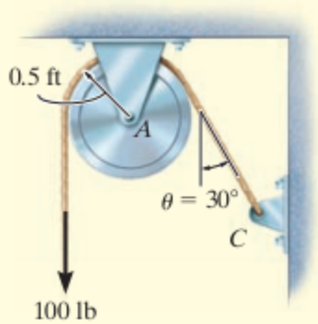
$$+\uparrow \Sigma F_y = 0; \qquad 319 \text{ N} - 600 \sin 45° \text{ N} - 100 \text{ N} - 200 \text{ N} + B_y = 0$$

$$B_y = 405 \text{ N} \qquad \textit{Ans.}$$

NOTE: Remember, the support forces in Fig. 4–12b are the result of pins that *act on the beam*. The opposite forces act on the pins. For example, Fig. 4–12c shows the equilibrium of the pin at A and the rocker.

EXAMPLE 4.6

The cord shown in Fig. 4–13*a* supports a force of 100 lb and wraps over the frictionless pulley. Determine the tension in the cord at C and the horizontal and vertical components of reaction at pin A.

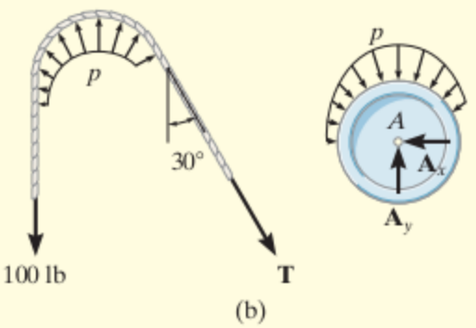


(a)

Fig. 4–13

SOLUTION

Free-Body Diagrams. The free-body diagrams of the cord and pulley are shown in Fig. 4–13*b*. Note that the principle of action, equal but opposite reaction must be carefully observed when drawing each of these diagrams: the cord exerts an unknown load distribution p on the pulley at the contact surface, whereas the pulley exerts an equal but opposite effect on the cord. For the solution, however, it is simpler to *combine* the free-body diagrams of the pulley and this portion of the cord, so that the distributed load becomes *internal* to this "system" and is therefore eliminated from the analysis, Fig. 4–13*c*.



(b)

Equations of Equilibrium. Summing moments about point A to eliminate $\mathbf{A}_x$ and $\mathbf{A}_y$, Fig. 4–13*c*, we have

$$\circlearrowleft + \Sigma M_A = 0; \qquad 100\text{ lb}\,(0.5\text{ ft}) - T(0.5\text{ ft}) = 0$$

$$T = 100\text{ lb} \qquad \textit{Ans.}$$

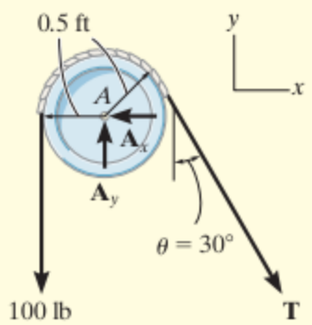
Using the result,

$$\pm \rightarrow \Sigma F_x = 0; \qquad -A_x + 100 \sin 30°\text{ lb} = 0$$

$$A_x = 50.0\text{ lb} \qquad \textit{Ans.}$$

$$+\uparrow \Sigma F_y = 0; \qquad A_y - 100\text{ lb} - 100 \cos 30°\text{ lb} = 0$$

$$A_y = 187\text{ lb} \qquad \textit{Ans.}$$

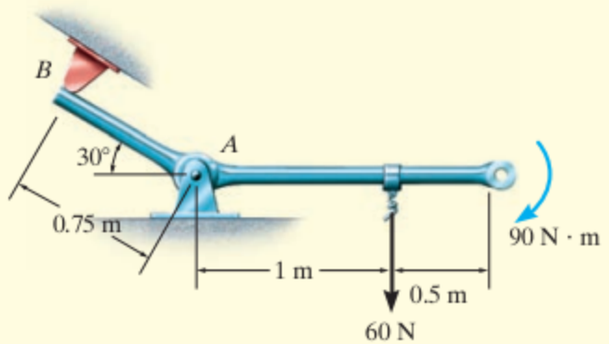


(c)

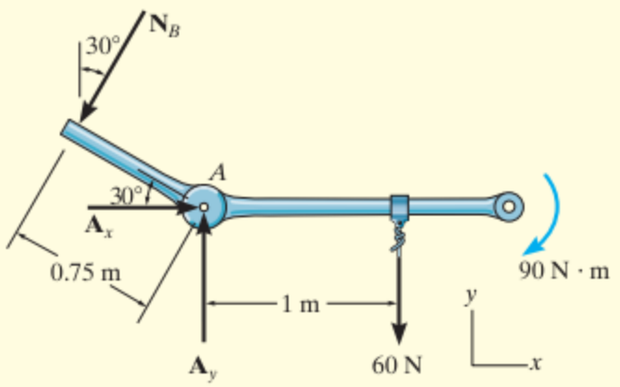
NOTE: It is seen that the tension remains *constant* as the cord passes over the pulley. (This of course is true for *any angle* θ at which the cord is directed and for *any radius* r of the pulley.)

EXAMPLE 4.7

The member shown in Fig. 4–14*a* is pin connected at A and rests against a smooth support at B. Determine the horizontal and vertical components of reaction at the pin A.



(a)



(b)

Fig. 4–14

SOLUTION

Free-Body Diagram. As shown in Fig. 4–14*b*, the reaction $\mathbf{N}_B$ is perpendicular to the member at B. Also, horizontal and vertical components of reaction are represented at A.

Equations of Equilibrium. Summing moments about A, we obtain a direct solution for N_B,

$$\circlearrowleft + \Sigma M_A = 0; \quad -90\ \text{N}\cdot\text{m} - 60\ \text{N}(1\ \text{m}) + N_B(0.75\ \text{m}) = 0$$

$$N_B = 200\ \text{N}$$

Using this result,

$$\pm \Sigma F_x = 0; \qquad A_x - 200 \sin 30°\ \text{N} = 0$$

$$A_x = 100\ \text{N} \qquad \textit{Ans.}$$

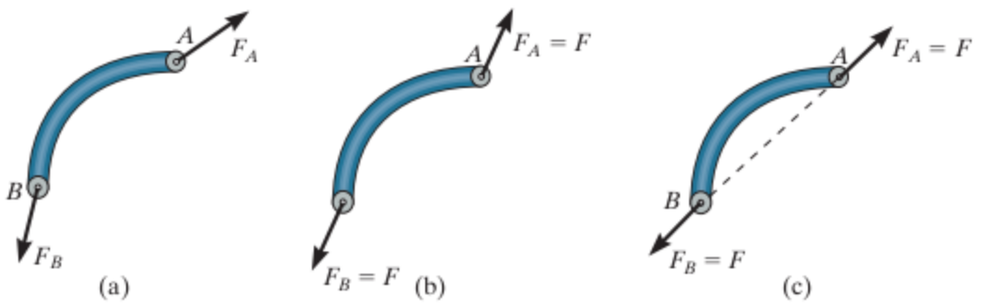
$$+\uparrow \Sigma F_y = 0; \qquad A_y - 200 \cos 30°\ \text{N} - 60\ \text{N} = 0$$

$$A_y = 233\ \text{N} \qquad \textit{Ans.}$$

4.4 Two- and Three-Force Members

The solutions to some equilibrium problems can be simplified by recognizing members that are subjected to only two or three forces.

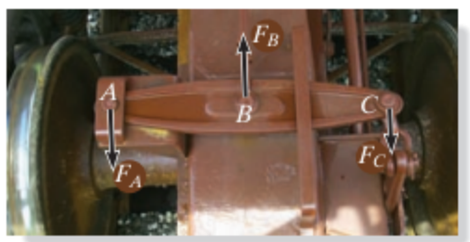
Two-Force Members. As the name implies, a *two-force member* has forces applied at only two points on the member. An example of a two-force member is shown in Fig. 4–15*a*. To satisfy force equilibrium, $\mathbf{F}_A$ and $\mathbf{F}_B$ must be equal in magnitude, $F_A = F_B = F$, but opposite in direction ($\Sigma \mathbf{F} = \mathbf{0}$), Fig. 4–15*b*. Furthermore, moment equilibrium requires that $\mathbf{F}_A$ and $\mathbf{F}_B$ share the same line of action, which can only happen if they are directed along the line joining points A and B ($\Sigma \mathbf{M}_A = \mathbf{0}$ or $\Sigma \mathbf{M}_B = \mathbf{0}$), Fig. 4–15*c*. Therefore, for any two-force member to be in equilibrium, the two forces acting on the member *must have the same magnitude, act in opposite directions, and have the same line of action, directed along the line joining the two points where these forces act.*



Two-force member

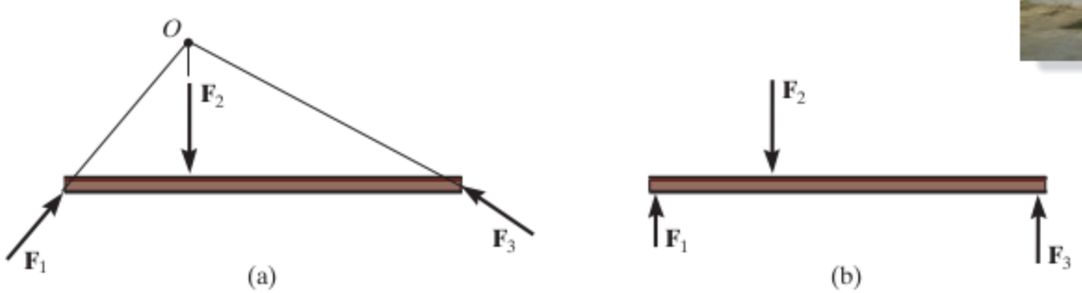
Fig. 4–15

The hydraulic cylinder AB is a typical example of a two-force member since it is pin connected at its ends and, provided its weight is neglected, only the pin forces act on this member.



The link used for this railroad car brake is a three-force member. Since the force $\mathbf{F}_B$ in the tie rod at B and $\mathbf{F}_C$ from the link at C are parallel, then for equilibrium the resultant force $\mathbf{F}_A$ at the pin A must also be parallel with these two forces.

Three-Force Members. If a member is subjected to only *three forces*, it is called a *three-force member*. Moment equilibrium can be satisfied only if the three forces form a *concurrent* or *parallel* force system. To illustrate, consider the member subjected to the three forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, shown in Fig. 4–16*a*. If the lines of action of $\mathbf{F}_1$ and $\mathbf{F}_2$ intersect at point O, then the line of action of $\mathbf{F}_3$ must *also* pass through point O so that the forces satisfy $\Sigma \mathbf{M}_O = \mathbf{0}$. As a special case, if the three forces are all parallel, Fig. 4–16*b*, the location of the point of intersection, O, will approach infinity.

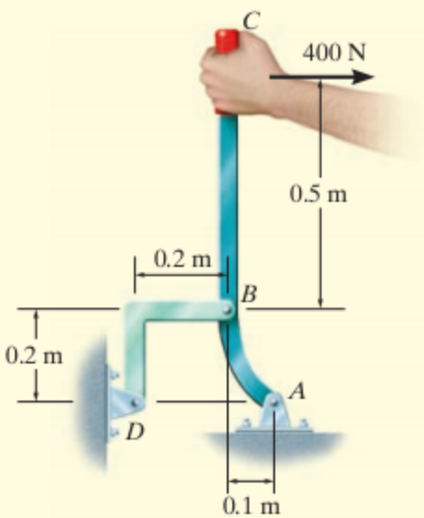


Three-force member

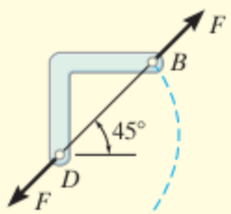
Fig. 4–16

The boom and bucket on this lift is a three-force member, provided its weight is neglected. Here the lines of action of the weight of the worker, $\mathbf{W}$, and the force of the two-force member (hydraulic cylinder) at B, $\mathbf{F}_B$, intersect at O. For moment equilibrium, the resultant force at the pin A, $\mathbf{F}_A$, must also be directed towards O.

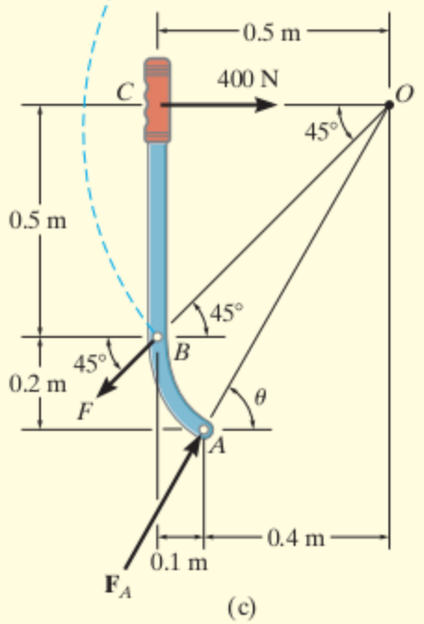
EXAMPLE 4.8



(a)



(b)



(c)

Fig. 4–17

The lever ABC is pin supported at A and connected to a short link BD as shown in Fig. 4–17*a*. If the weight of the members is negligible, determine the force of the pin on the lever at A.

SOLUTION

Free-Body Diagrams. As shown in Fig. 4–17*b*, the short link BD is a *two-force member*, so the *resultant forces* at pins D and B must be equal, opposite, and collinear. Although the magnitude of the force is unknown, the line of action is known since it passes through B and D.

Lever ABC is a *three-force member*, and therefore, in order to satisfy moment equilibrium, the three nonparallel forces acting on it must be concurrent at O, Fig. 4–17*c*. In particular, note that the force $\mathbf{F}$ on the lever at B is equal but opposite to the force $\mathbf{F}$ acting at B on the link. Why? The distance CO must be 0.5 m since the lines of action of $\mathbf{F}$ and the 400-N force are known.

Equations of Equilibrium. By requiring the force system to be concurrent at O, since $\Sigma M_O = 0$, the angle θ which defines the line of action of $\mathbf{F}_A$ can be determined from trigonometry,

$$\theta = \tan^{-1}\left(\frac{0.7}{0.4}\right) = 60.3^\circ$$

Using the x, y axes and applying the force equilibrium equations,

$$\stackrel{+}{\rightarrow}\Sigma F_x = 0; \qquad F_A \cos 60.3^\circ - F \cos 45^\circ + 400\ \text{N} = 0$$

$$+\uparrow \Sigma F_y = 0; \qquad F_A \sin 60.3^\circ - F \sin 45^\circ = 0$$

Solving, we get

$$F_A = 1.07\ \text{kN} \qquad \textit{Ans.}$$

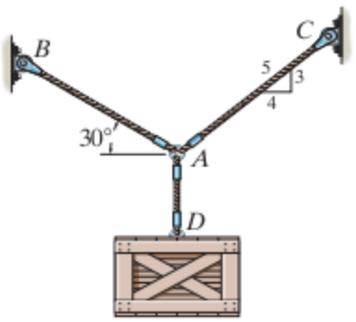
$$F = 1.32\ \text{kN}$$

NOTE: We can also solve this problem by representing the force at A by its two components $\mathbf{A}_x$ and $\mathbf{A}_y$ and applying $\Sigma M_A = 0$, $\Sigma F_x = 0$, $\Sigma F_y = 0$ to the lever. Once A_x and A_y are determined, we can get F_A and θ.

FUNDAMENTAL PROBLEMS

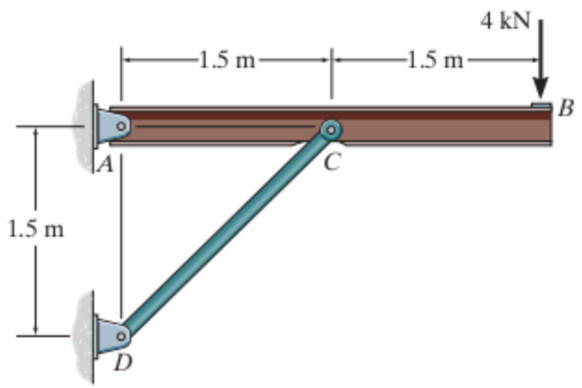
All problem solutions must include an FBD.

F4–1. The crate has a weight of 550 lb. Determine the force in each supporting cable.



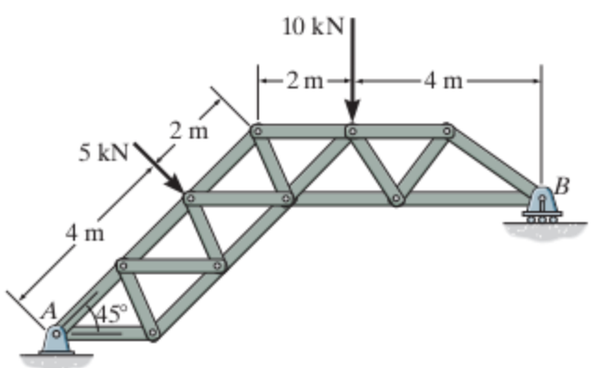
F4–1

F4–2. Determine the horizontal and vertical components of reaction at the pin *A* and the reaction on the beam at *C*.



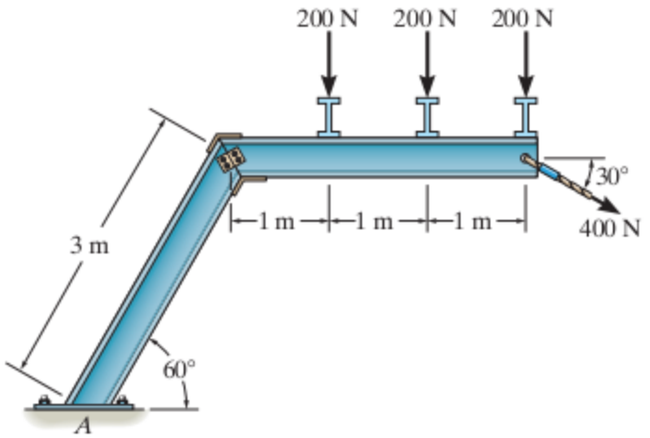
F4–2

F4–3. The truss is supported by a pin at *A* and a roller at *B*. Determine the support reactions.



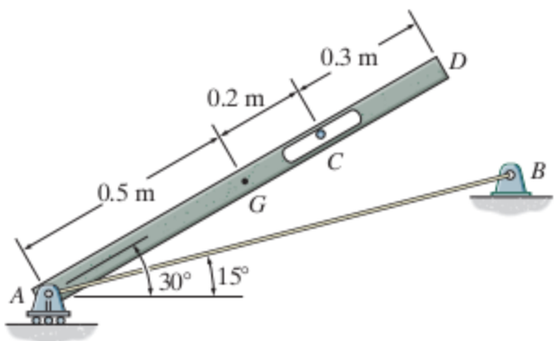
F4–3

F4–4. Determine the components of reaction at the fixed support *A*. Neglect the thickness of the beam.



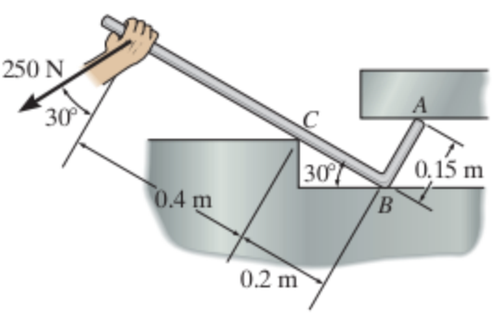
F4–4

F4–5. The 25-kg bar has a center of mass at *G*. If it is supported by a smooth peg at *C*, a roller at *A*, and cord *AB*, determine the reactions at these supports.



F4–5

F4–6. Determine the reactions at the smooth contact points *A*, *B*, and *C* on the bar.



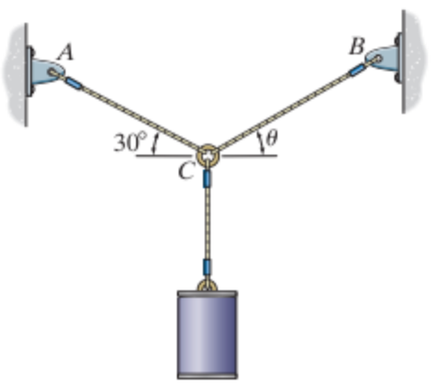
F4–6

PROBLEMS

All problem solutions must include an FBD.

4–1. Determine the tension developed in wires *CA* and *CB* required for equilibrium of the 10-kg cylinder. Take $\theta = 40°$.

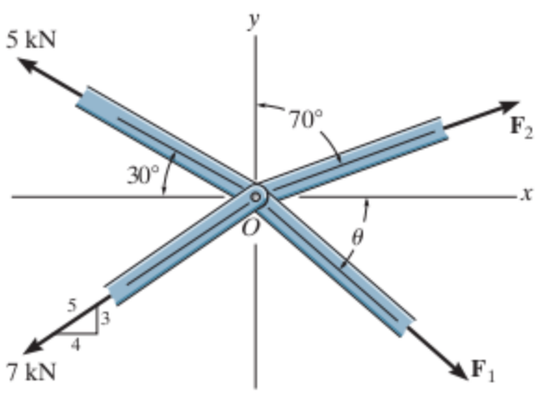
4–2. If cable *CB* is subjected to a tension that is twice that of cable *CA*, determine the angle θ for equilibrium of the 10-kg cylinder. Also, what are the tensions in wires *CA* and *CB*?



Probs. 4–1/2

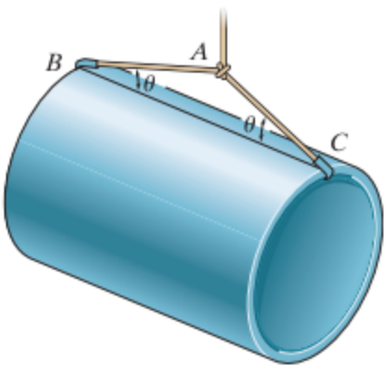
4–3. The members of a truss are pin connected at joint *O*. Determine the magnitudes of $\mathbf{F}_1$ and $\mathbf{F}_2$ for equilibrium. Set $\theta = 60°$.

***4–4.** The members of a truss are pin connected at joint *O*. Determine the magnitude of $\mathbf{F}_1$ and its angle θ for equilibrium. Set $F_2 = 6$ kN.



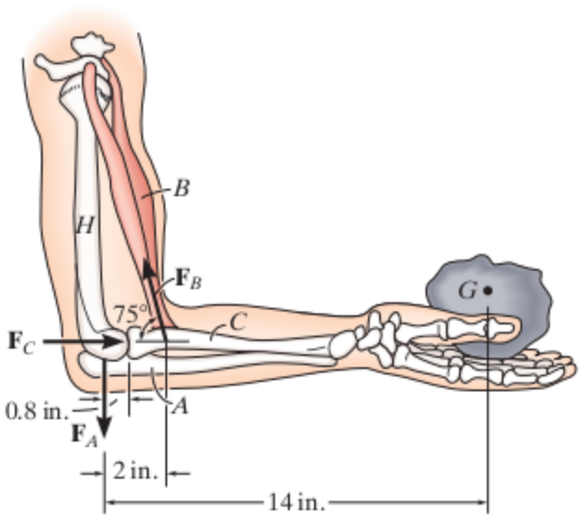
Probs. 4–3/4

4–5. Cords *AB* and *AC* can each sustain a maximum tension of 800 lb. If the drum has a weight of 900 lb, determine the smallest angle θ at which they can be attached to the drum.



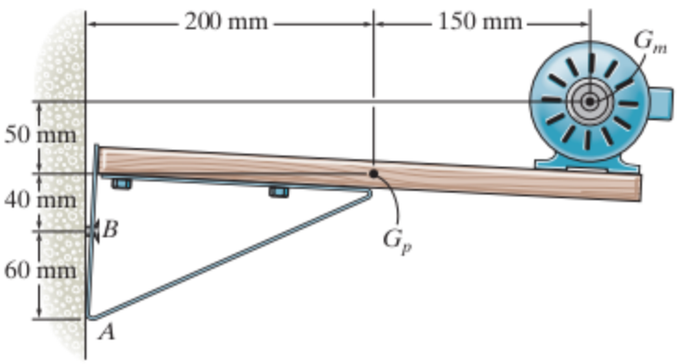
Prob. 4–5

4–6. When holding the 5-lb stone in equilibrium, the humerus *H*, assumed to be smooth, exerts normal forces $\mathbf{F}_C$ and $\mathbf{F}_A$ on the radius *C* and ulna *A* as shown. Determine these forces and the force $\mathbf{F}_B$ that the biceps *B* exerts on the radius for equilibrium. The stone has a center of mass at *G*. Neglect the weight of the arm.



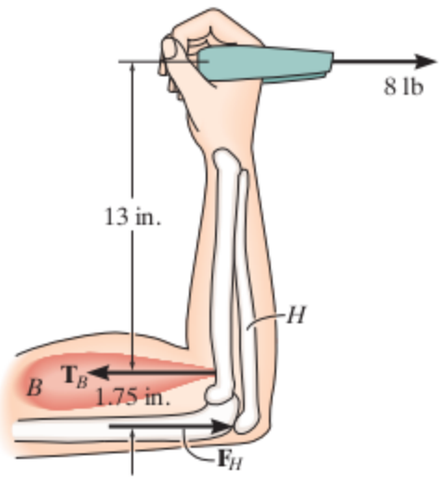
Prob. 4–6

4–7. The shelf supports the electric motor which has a mass of 15 kg and mass center at G_m. The platform upon which it rests has a mass of 4 kg and mass center at G_p. Assuming that a single bolt B holds the shelf up and the bracket bears against the smooth wall at A, determine this normal force at A and the horizontal and vertical components of reaction of the bolt on the bracket.



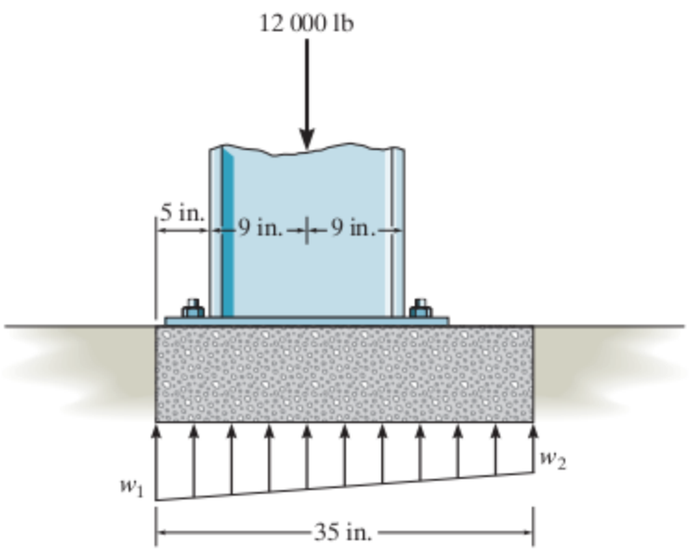
Prob. 4–7

4–9. The man is pulling a load of 8 lb with one arm held as shown. Determine the force $\mathbf{F}_H$ this exerts on the humerus bone H, and the tension developed in the biceps muscle B. Neglect the weight of the man's arm.



Prob. 4–9

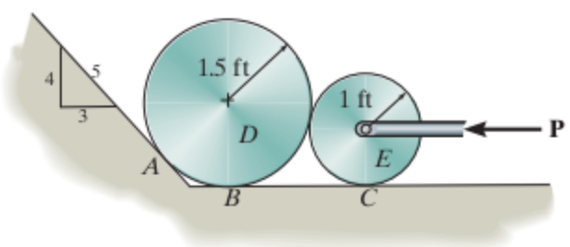
*__4–8.__ The pad footing is used to support the load of 12 000 lb. Determine the intensities w_1 and w_2 of the distributed loading acting on the base of the footing for the equilibrium.



Prob. 4–8

4–10. The smooth disks D and E have a weight of 200 lb and 100 lb, respectively. If a horizontal force of $P = 200$ lb is applied to the center of disk E, determine the normal reactions at the points of contact with the ground at A, B, and C.

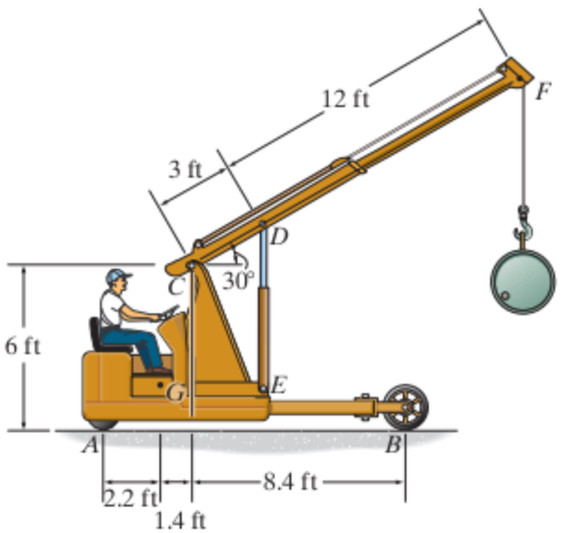
4–11. The smooth disks D and E have a weight of 200 lb and 100 lb, respectively. Determine the largest horizontal force P that can be applied to the center of disk E without causing the disk D to move up the incline.



Probs. 4–10/11

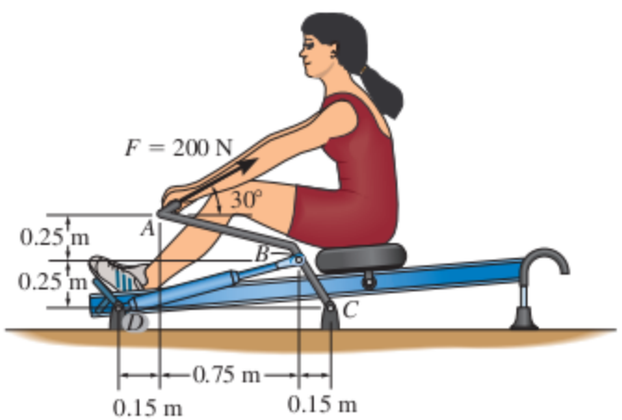
*4–12. The floor crane and the driver have a total weight of 2500 lb with a center of gravity at G. If the crane is required to lift the 500-lb drum, determine the normal reaction on *both* the wheels at A and *both* the wheels at B when the boom is in the position shown.

4–13. The floor crane and the driver have a total weight of 2500 lb with a center of gravity at G. Determine the largest weight of the drum that can be lifted without causing the crane to overturn when its boom is in the position shown.



Probs. 4–12/13

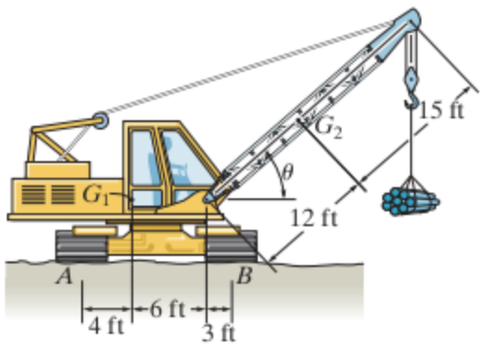
4–14. The woman exercises on the rowing machine. If she exerts a holding force of $F = 200$ N on handle ABC, determine the horizontal and vertical components of reaction at pin C and the force developed along the hydraulic cylinder BD on the handle.



Prob. 4–14

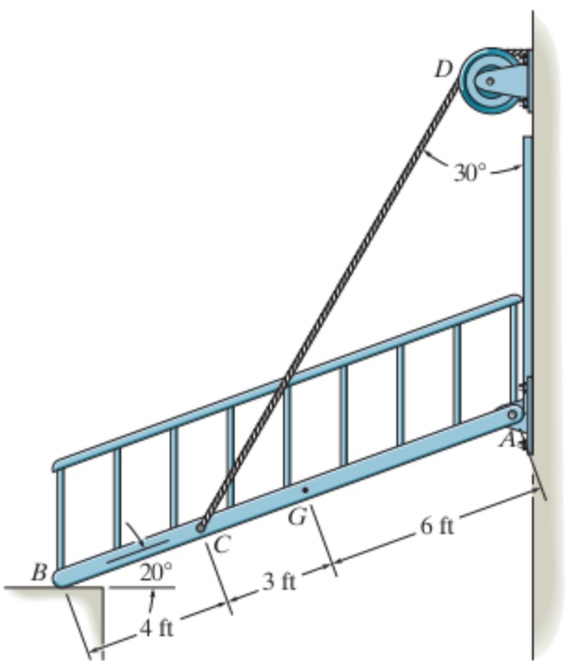
4–15. The mobile crane has a weight of 120 000 lb and center of gravity at G_1; the boom has a weight of 30 000 lb and center of gravity at G_2. Determine the smallest angle of tilt θ of the boom, without causing the crane to overturn if the suspended load is $W = 40\,000$ lb. Neglect the thickness of the tracks at A and B.

*4–16. The mobile crane has a weight of 120 000 lb and center of gravity at G_1; the boom has a weight of 30 000 lb and center of gravity at G_2. If the suspended load has a weight of $W = 16\,000$ lb, determine the normal reactions at the tracks A and B. For the calculation, neglect the thickness of the tracks and take $\theta = 30°$.



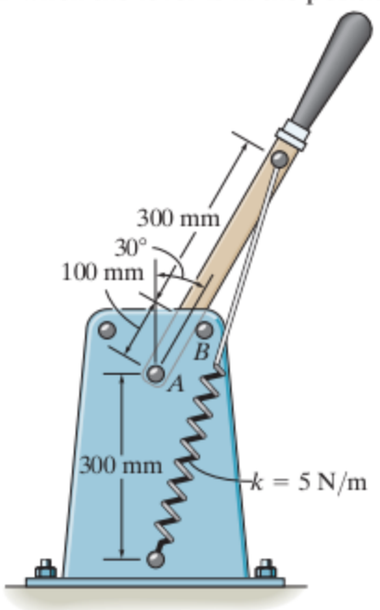
Probs. 4–15/16

4–17. The ramp of a ship has a weight of 200 lb and a center of gravity at G. Determine the cable force in CD needed to just start lifting the ramp, (i.e., so the reaction at B becomes zero). Also, determine the horizontal and vertical components of force at the hinge (pin) at A.



Prob. 4–17

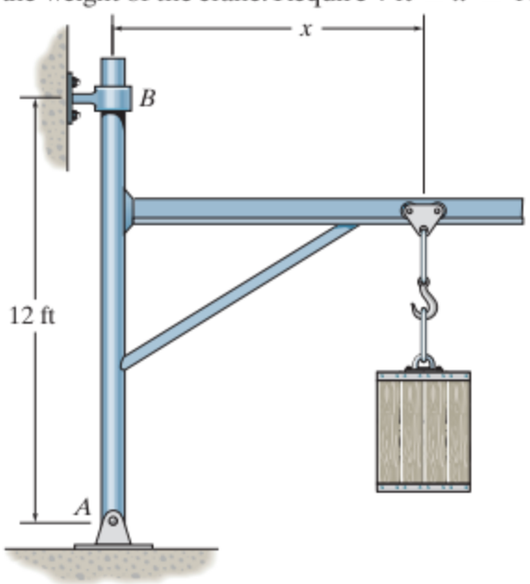
4–18. The toggle switch consists of a cocking lever that is pinned to a fixed frame at A and held in place by the spring which has an unstretched length of 200 mm. Determine the magnitude of the resultant force at A and the normal force on the peg at B when the lever is in the position shown.



Prob. 4–18

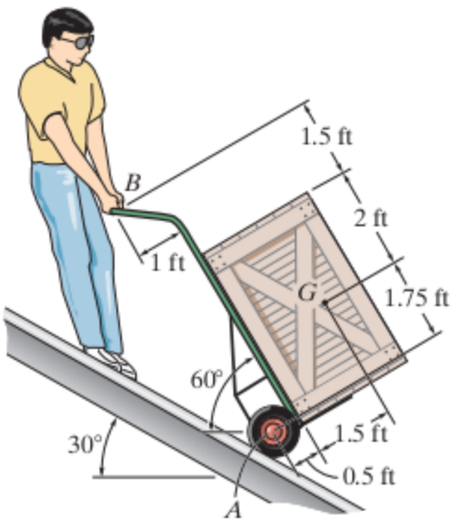
4–19. The jib crane is pin connected at A and supported by a smooth collar at B. If $x = 8$ ft, determine the reactions on the jib crane at the pin A and smooth collar B. The load has a weight of 5000 lb.

***4–20.** The jib crane is pin connected at A and supported by a smooth collar at B. Determine the roller placement x of the 5000-lb load so that it gives the maximum and minimum reactions at the supports. Calculate these reactions in each case. Neglect the weight of the crane. Require 4 ft $\leq x \leq$ 10 ft.



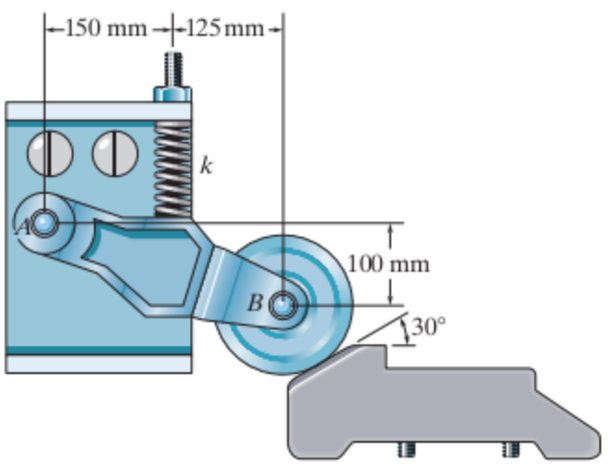
Probs. 4–19/20

4–21. The worker uses the hand truck to move material down the ramp. If the truck and its contents are held in the position shown and have a weight of 100 lb with center of gravity at G, determine the resultant normal force of both wheels on the ground A and the magnitude of the force required at the grip B.



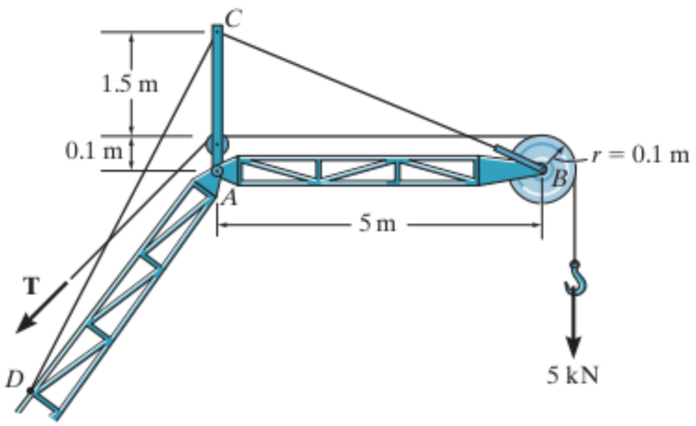
Prob. 4–21

4–22. The device is used to hold an elevator door open. If the spring has a stiffness of $k = 40$ N/m and it is compressed 0.2 m, determine the horizontal and vertical components of reaction at the pin A and the resultant force at the wheel bearing B.



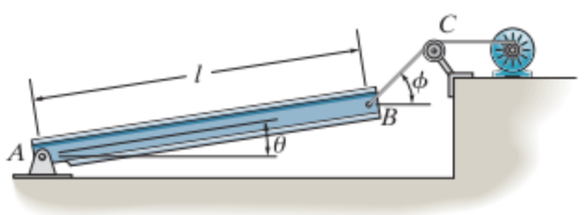
Prob. 4–22

4–23. The upper portion of the crane boom consists of the jib AB, which is supported by the pin at A, the guy line BC, and the backstay CD, each cable being separately attached to the mast at C. If the 5-kN load is supported by the hoist line, which passes over the pulley at B, determine the magnitude of the resultant force the pin exerts on the jib at A for equilibrium, the tension in the guy line BC, and the tension T in the hoist line. Neglect the weight of the jib. The pulley at B has a radius of 0.1 m.



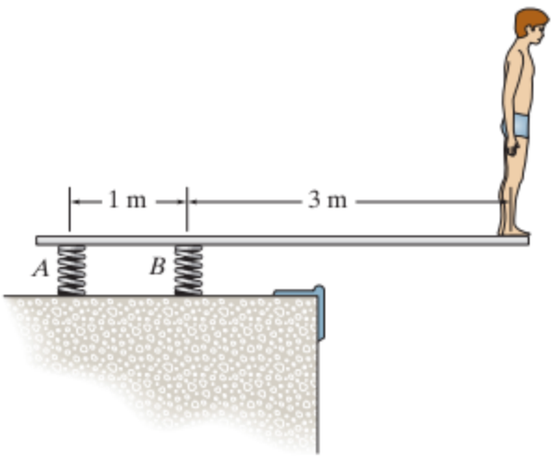
Prob. 4–23

*4–24. The uniform beam has a weight W and length l and is supported by a pin at A and a cable BC. Determine the horizontal and vertical components of reaction at A and the tension in the cable necessary to hold the beam in the position shown.



Prob. 4–24

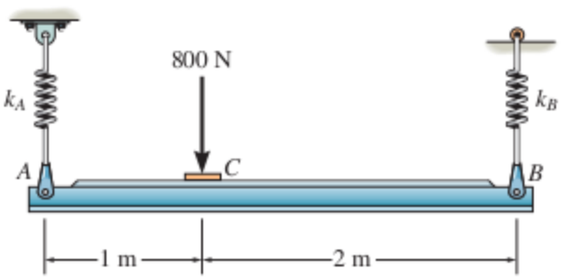
4–25. A boy stands out at the end of the diving board, which is supported by two springs A and B, each having a stiffness of $k = 15$ kN/m. In the position shown the board is horizontal. If the boy has a mass of 40 kg, determine the angle of tilt which the board makes with the horizontal after he jumps off. Neglect the weight of the board and assume it is rigid.



Prob. 4–25

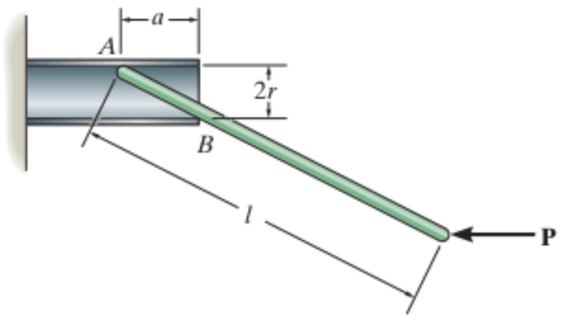
4–26. The horizontal beam is supported by springs at its ends. Each spring has a stiffness of $k = 5$ kN/m and is originally unstretched when the beam is in the horizontal position. Determine the angle of tilt of the beam if a load of 800 N is applied at point C as shown.

4–27. The horizontal beam is supported by springs at its ends. If the stiffness of the spring at A is $k_A = 5$ kN/m, determine the required stiffness of the spring at B so that if the beam is loaded with the 800-N force it remains in the horizontal position. The springs are originally constructed so that the beam is in the horizontal position when it is unloaded.



Probs. 4–26/27

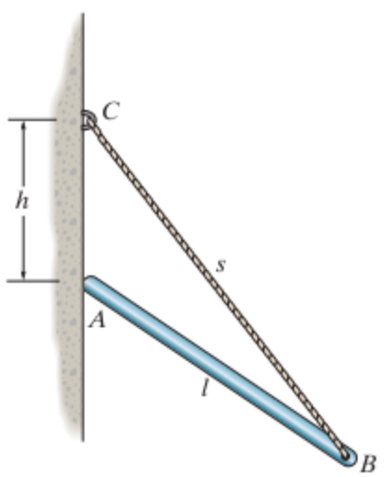
*4–28. The thin rod of length l is supported by the smooth tube. Determine the distance a needed for equilibrium if the applied load is **P**.



Prob. 4–28

4–29. The 30-N uniform rod has a length of $l = 1$ m. If $s = 1.5$ m, determine the distance h of placement at the end A along the smooth wall for equilibrium.

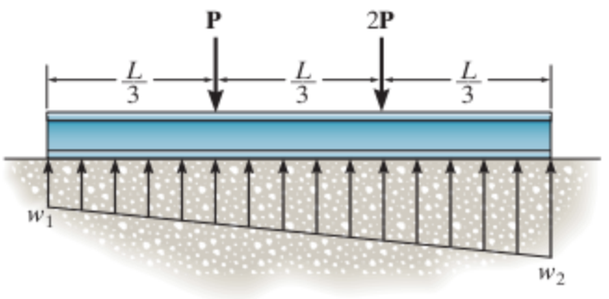
4–30. The uniform rod has a length l and weight W. It is supported at one end A by a smooth wall and the other end by a cord of length s which is attached to the wall as shown. Determine the placement h for equilibrium.



Probs. 4–29/30

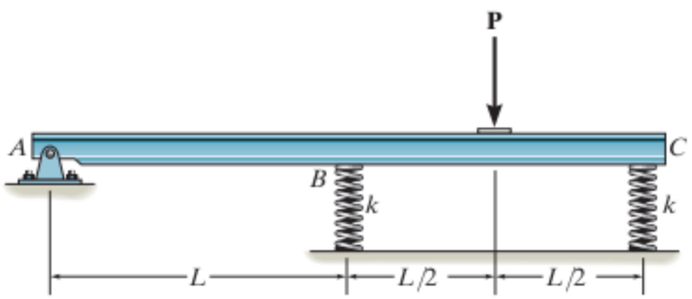
4–31. The beam is subjected to the two concentrated loads. Assuming that the foundation exerts a linearly varying load distribution on its bottom, determine the load intensities w_1 and w_2 for equilibrium if $P = 500$ lb and $L = 12$ ft.

*4–32. The beam is subjected to the two concentrated loads. Assuming that the foundation exerts a linearly varying load distribution on its bottom, determine the load intensities w_1 and w_2 for equilibrium in terms of the parameters shown.



Probs. 4–31/32

4–33. The rigid beam of negligible weight is supported horizontally by two springs and a pin. If the springs are uncompressed when the load is removed, determine the force in each spring when the load **P** is applied. Also, compute the vertical deflection of end C. Assume the spring stiffness k is large enough so that only small deflections occur. *Hint:* The beam rotates about A so the deflections in the springs can be related.



Prob. 4–33

CONCEPTUAL PROBLEMS

P4–5. The tie rod is used to support this overhang at the entrance of a building. If it is pin connected to the building wall at A and to the center of the overhang B, determine if the force in the rod will increase, decrease, or remain the same if (a) the support at A is moved to a lower position D, and (b) the support at B is moved to the outer position C. Explain your answer with an equilibrium analysis, using dimensions and loads. Assume the overhang is pin supported from the building wall.

P4–5

P4–6. The man attempts to pull the four wheeler up the incline and onto the trailer bed. From the position shown, is it more effective to keep the rope attached at A, or would it be better to attach it to the axle of the front wheels at B? Draw a free-body diagram and do an equilibrium analysis to explain your answer.

P4–6

P4–7. Like all aircraft, this jet plane rests on three wheels. Why not use an additional wheel at the tail for better support? (Can you think of any other reason for not including this wheel?) If there was a fourth tail wheel, draw a free-body diagram of the plane from a side (2 D) view, and show why one would not be able to determine all the wheel reactions using the equations of equilibrium.

P4–7

P4–8. Where is the best place to arrange most of the logs in the wheelbarrow so that it minimizes the amount of force on the backbone of the person transporting the load? Do an equilibrium analysis to explain your answer.

P4–8

EQUILIBRIUM IN THREE DIMENSIONS

4.5 Free-Body Diagrams

The first step in solving three-dimensional equilibrium problems, as in the case of two dimensions, is to draw a free-body diagram. Before we can do this, however, it is first necessary to discuss the types of reactions that can occur at the supports.

Support Reactions. The reactive forces and couple moments acting at various types of supports and connections, when the members are viewed in three dimensions, are listed in Table 4–2. It is important to recognize the symbols used to represent each of these supports and to understand clearly how the forces and couple moments are developed. As in the two-dimensional case:

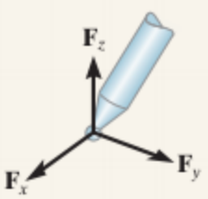
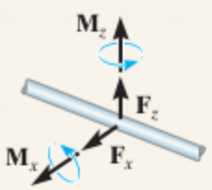
- A force is developed by a support that restricts the translation of its attached member.
- A couple moment is developed when rotation of the attached member is prevented.

For example, in Table 4–2, item (4), the ball-and-socket joint prevents any translation of the connecting member; therefore, a force must act on the member at the point of connection. This force has three components having unknown magnitudes, F_x, F_y, F_z. Provided these components are known, one can obtain the magnitude of force, $F = \sqrt{F_x^2 + F_y^2 + F_z^2}$, and the force's orientation defined by it's coordinate direction angles α, β, γ, Eqs. 2–5.* Since the connecting member is allowed to rotate freely about *any* axis, no couple moment is resisted by a ball-and-socket joint.

It should be noted that the *single* bearing supports in items (5) and (7), the *single* pin (8), and the *single* hinge (9) are shown to resist both force and couple-moment components. If, however, these supports are used in conjunction with *other* bearings, pins, or hinges to hold a rigid body in equilibrium and the supports are *properly aligned* when connected to the body, then the *force reactions* at these supports *alone* are adequate for supporting the body. In other words, the couple moments become redundant and are not shown on the free-body diagram. The reason for this should become clear after studying the examples which follow.

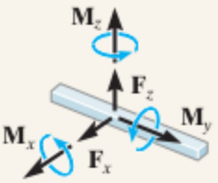
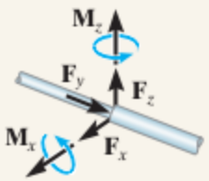
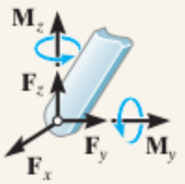
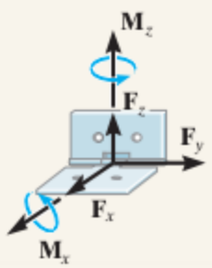
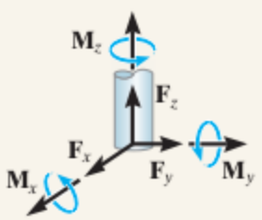
*The three unknowns may also be represented as an unknown force magnitude F and two unknown coordinate direction angles. The third direction angle is obtained using the identity $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, Eq. 2–8.

TABLE 4–2 Supports for Rigid Bodies Subjected to Three-Dimensional Force Systems

Types of Connection	Reaction	Number of Unknowns
(1) cable		One unknown. The reaction is a force which acts away from the member in the known direction of the cable.
(2) smooth surface support		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(3) roller		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(4) ball and socket		Three unknowns. The reactions are three rectangular force components.
(5) single journal bearing		Four unknowns. The reactions are two force and two couple-moment components which act perpendicular to the shaft. *Note*: The couple moments are generally not applied if the body is supported elsewhere. See the examples.

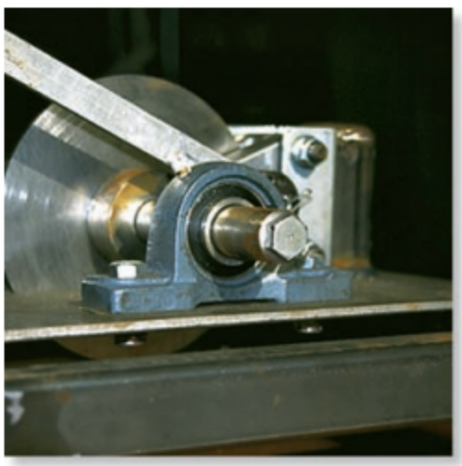
continued

TABLE 4–2 Continued

Types of Connection	Reaction	Number of Unknowns
(6) single journal bearing with square shaft		Five unknowns. The reactions are two force and three couple-moment components. *Note*: The couple moments are generally not applied if the body is supported elsewhere. See the examples.
(7) single thrust bearing		Five unknowns. The reactions are three force and two couple-moment components. *Note*: The couple moments are generally not applied if the body is supported elsewhere. See the examples.
(8) single smooth pin		Five unknowns. The reactions are three force and two couple-moment components. *Note*: The couple moments are generally not applied if the body is supported elsewhere. See the examples.
(9) single hinge		Five unknowns. The reactions are three force and two couple-moment components. *Note*: The couple moments are generally not applied if the body is supported elsewhere. See the examples.
(10) fixed support		Six unknowns. The reactions are three force and three couple-moment components.

Typical examples of actual supports that are referenced to Table 4–2 are shown in the following sequence of photos.

This ball-and-socket joint provides a connection for the housing of an earth grader to its frame. (4)



This journal bearing supports the end of the shaft. (5)

This thrust bearing is used to support the drive shaft on a machine. (7)

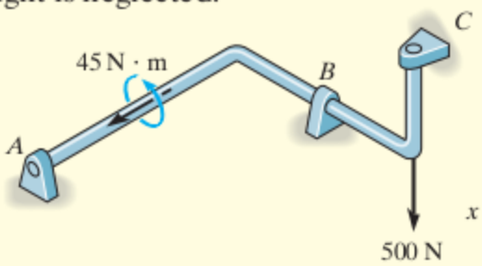
This pin is used to support the end of the strut used on a tractor. (8)

Free-Body Diagrams. The general procedure for establishing the free-body diagram of a rigid body has been outlined in Sec. 4.2. Essentially it requires first "isolating" the body by drawing its outlined shape. This is followed by a careful *labeling* of *all* the forces and couple moments with reference to an established x, y, z coordinate system. It is suggested to show the unknown components of reaction as acting on the free-body diagram in the *positive sense*. In this way, if any negative values are obtained, they will indicate that the components act in the negative coordinate directions.

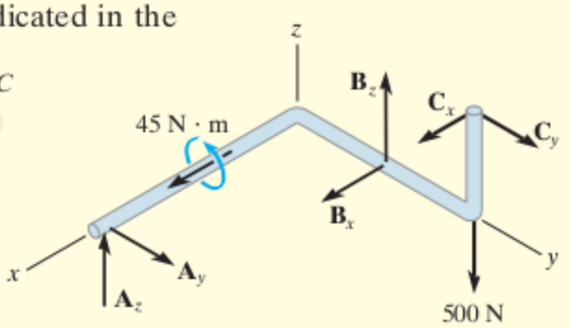
EXAMPLE 4.9

Consider the two rods and plate, along with their associated free-body diagrams shown in Fig. 4–18. The x, y, z axes are established on the diagram and the unknown reaction components are indicated in the *positive sense*. The weight is neglected.

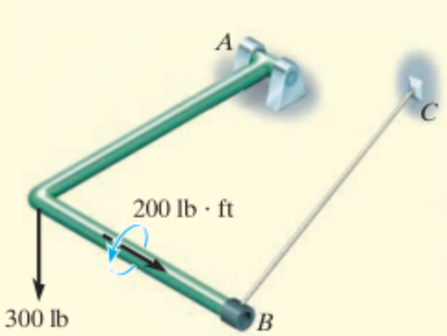
SOLUTION



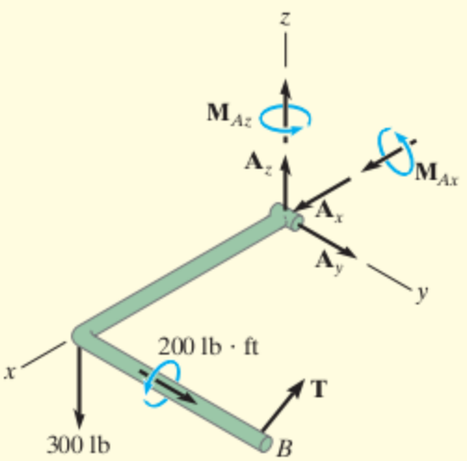
Properly aligned journal bearings at A, B, C.



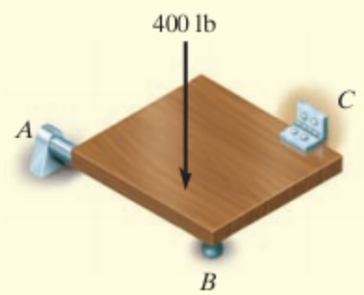
The force reactions developed by the bearings are sufficient for equilibrium since they prevent the shaft from rotating about each of the coordinate axes.



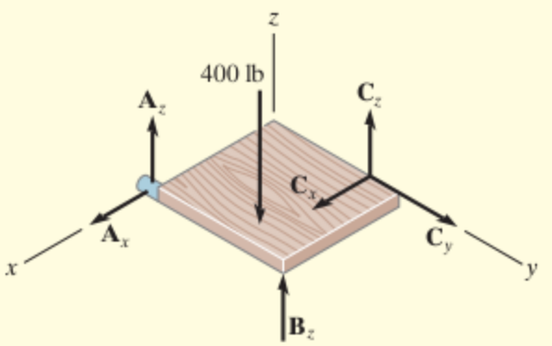
Pin at A and cable BC.



Moment components are developed by the pin on the rod to prevent rotation about the x and z axes.



Properly aligned journal bearing at A and hinge at C. Roller at B.



Only force reactions are developed by the bearing and hinge on the plate to prevent rotation about each coordinate axis. No moments at the hinge are developed.

Fig. 4–18

4.6 Equations of Equilibrium

As stated in Sec. 4.1, the conditions for equilibrium of a rigid body subjected to a three-dimensional force system require that both the *resultant* force and *resultant* couple moment acting on the body be equal to *zero*.

Vector Equations of Equilibrium. The two conditions for equilibrium of a rigid body may be expressed mathematically in vector form as

$$\begin{aligned} \Sigma \mathbf{F} &= \mathbf{0} \\ \Sigma \mathbf{M}_O &= \mathbf{0} \end{aligned} \tag{4–6}$$

where $\Sigma \mathbf{F}$ is the vector sum of all the external forces acting on the body and $\Sigma \mathbf{M}_O$ is the sum of the couple moments and the moments of all the forces about any point O located either on or off the body.

Scalar Equations of Equilibrium. If all the external forces and couple moments are expressed in Cartesian vector form and substituted into Eqs. 4–5, we have

$$\Sigma \mathbf{F} = \Sigma F_x \mathbf{i} + \Sigma F_y \mathbf{j} + \Sigma F_z \mathbf{k} = \mathbf{0}$$
$$\Sigma \mathbf{M}_O = \Sigma M_x \mathbf{i} + \Sigma M_y \mathbf{j} + \Sigma M_z \mathbf{k} = \mathbf{0}$$

Since the **i**, **j**, and **k** components are independent from one another, the above equations are satisfied provided

$$\begin{aligned} \Sigma F_x &= 0 \\ \Sigma F_y &= 0 \\ \Sigma F_z &= 0 \end{aligned} \tag{4–7a}$$

and

$$\begin{aligned} \Sigma M_x &= 0 \\ \Sigma M_y &= 0 \\ \Sigma M_z &= 0 \end{aligned} \tag{4–7b}$$

These *six scalar equilibrium equations* may be used to solve for at most six unknowns shown on the free-body diagram. Equations 4–7*a* require the sum of the external force components acting in the x, y, and z directions to be zero, and Eqs. 4–7*b* require the sum of the moment components about the x, y, and z axes to be zero.

Important Points

- Always draw the free-body diagram first when solving any equilibrium problem.
- If a support *prevents translation* of a body in a specific direction, then the support exerts a *force* on the body in that direction.
- If a support *prevents rotation about an axis*, then the support exerts a *couple moment* on the body about the axis.
- If a body is subjected to more unknown reactions than available equations of equilibrium, then the problem is *statically indeterminate*.
- A stable body requires that the lines of action of the reactive forces do not intersect a common axis and are not parallel to one another.

Procedure for Analysis

Three-dimensional equilibrium problems for a rigid body can be solved using the following procedure.

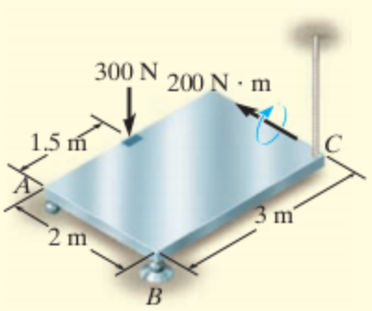
Free-Body Diagram.

- Draw an outlined shape of the body.
- Show all the forces and couple moments acting on the body.
- Establish the origin of the x, y, z axes at a convenient point and orient the axes so that they are parallel to as many of the external forces and moments as possible.
- Label all the loadings and specify their directions. In general, show all the *unknown* components having a *positive sense* along the x, y, z axes.
- Indicate the dimensions of the body necessary for computing the moments of forces.

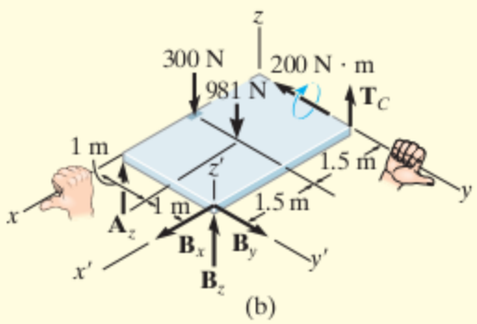
Equations of Equilibrium.

- If the x, y, z force and moment components seem easy to determine, then apply the six scalar equations of equilibrium; otherwise use the vector equations.
- It is not necessary that the set of axes chosen for force summation coincide with the set of axes chosen for moment summation. Actually, an axis in any arbitrary direction may be chosen for summing forces and moments.
- Choose the direction of an axis for moment summation such that it intersects the lines of action of as many unknown forces as possible. Realize that the moments of forces passing through points on this axis and the moments of forces which are parallel to the axis will then be zero.
- If the solution of the equilibrium equations yields a negative scalar for a force or couple moment magnitude, it indicates that the sense is opposite to that assumed on the free-body diagram.

EXAMPLE 4.10



(a)



(b)

Fig. 4–19

The homogeneous plate shown in Fig. 4–19*a* has a mass of 100 kg and is subjected to a force and couple moment along its edges. If it is supported in the horizontal plane by a roller at A, a ball-and-socket joint at B, and a cord at C, determine the components of reaction at these supports.

SOLUTION (*SCALAR ANALYSIS*)

Free-Body Diagram. There are five unknown reactions acting on the plate, as shown in Fig. 4–19*b*. Each of these reactions is assumed to act in a positive coordinate direction.

Equations of Equilibrium. Since the three-dimensional geometry is rather simple, a *scalar analysis* provides a *direct solution* to this problem. A force summation along each axis yields

$$\Sigma F_x = 0; \quad B_x = 0 \qquad \textit{Ans.}$$

$$\Sigma F_y = 0; \quad B_y = 0 \qquad \textit{Ans.}$$

$$\Sigma F_z = 0; \quad A_z + B_z + T_C - 300\ \text{N} - 981\ \text{N} = 0 \qquad (1)$$

Recall that the moment of a force about an axis is equal to the product of the force magnitude and the perpendicular distance (moment arm) from the line of action of the force to the axis. Also, forces that are parallel to an axis or pass through it create no moment about the axis. Hence, summing moments about the positive x and y axes, we have

$$\Sigma M_x = 0; \quad T_C(2\ \text{m}) - 981\ \text{N}(1\ \text{m}) + B_z(2\ \text{m}) = 0 \qquad (2)$$

$$\Sigma M_y = 0;$$

$$-300\ \text{N}(1.5\ \text{m}) + 981\ \text{N}(1.5\ \text{m}) - B_z(3\ \text{m}) - A_z(3\ \text{m}) - 200\ \text{N}\cdot\text{m} = 0 \qquad (3)$$

The components of the force at B can be eliminated if moments are summed about the x' and y' axes. We obtain

$$\Sigma M_{x'} = 0; \quad 981\ \text{N}(1\ \text{m}) + 300\ \text{N}(2\ \text{m}) - A_z(2\ \text{m}) = 0 \qquad (4)$$

$$\Sigma M_{y'} = 0;$$

$$-300\ \text{N}(1.5\ \text{m}) - 981\ \text{N}(1.5\ \text{m}) - 200\ \text{N}\cdot\text{m} + T_C(3\ \text{m}) = 0 \qquad (5)$$

Solving Eqs. (1) through (3) or the more convenient Eqs. (1), (4), and (5) yields

$$A_z = 790\ \text{N} \quad B_z = -217\ \text{N} \quad T_C = 707\ \text{N} \qquad \textit{Ans.}$$

The negative sign indicates that $\mathbf{B}_z$ acts downward.

NOTE: The solution of this problem does not require a summation of moments about the z axis. The plate is partially constrained since the supports cannot prevent it from turning about the z axis if a force is applied to it in the x–y plane.

EXAMPLE 4.11

Determine the components of reaction that the ball-and-socket joint at A, the smooth journal bearing at B, and the roller support at C exert on the rod assembly in Fig. 4–20*a*.

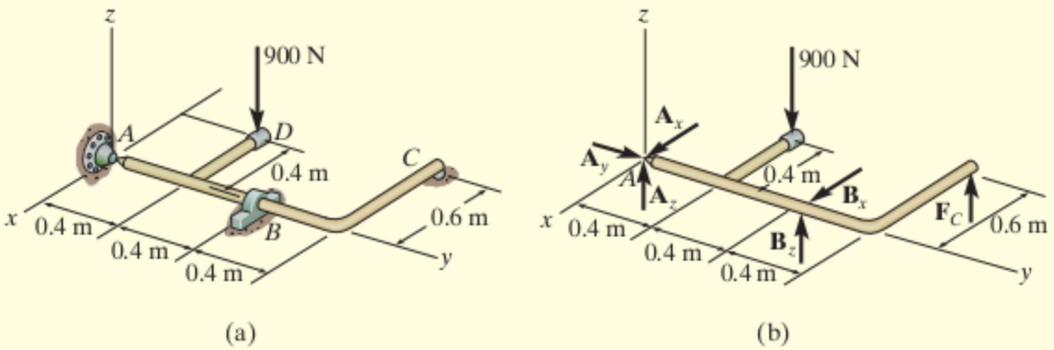


Fig. 4–20

SOLUTION

Free-Body Diagram. As shown on the free-body diagram, Fig. 4–20*b*, the reactive forces of the supports will prevent the assembly from rotating about each coordinate axis, and so the journal bearing at B only exerts reactive forces on the member.

Equations of Equilibrium. A direct solution for A_y can be obtained by summing forces along the y axis.

$$\Sigma F_y = 0; \qquad A_y = 0 \qquad \textit{Ans.}$$

The force F_C can be determined directly by summing moments about the y axis.

$$\Sigma M_y = 0; \qquad F_C(0.6\text{ m}) - 900\text{ N}(0.4\text{ m}) = 0$$
$$F_C = 600\text{ N} \qquad \textit{Ans.}$$

Using this result, B_z can be determined by summing moments about the x axis.

$$\Sigma M_x = 0; \qquad B_z(0.8\text{ m}) + 600\text{ N}(1.2\text{ m}) - 900\text{ N}(0.4\text{ m}) = 0$$
$$B_z = -450\text{ N} \qquad \textit{Ans.}$$

The negative sign indicates that $\mathbf{B}_z$ acts downward. The force $\mathbf{B}_x$ can be found by summing moments about the z axis.

$$\Sigma M_z = 0; \qquad -B_x(0.8\text{ m}) = 0 \quad B_x = 0 \qquad \textit{Ans.}$$

Thus,

$$\Sigma F_x = 0; \qquad A_x + 0 = 0 \quad A_x = 0 \qquad \textit{Ans.}$$

Finally, using the results of B_z and F_C.

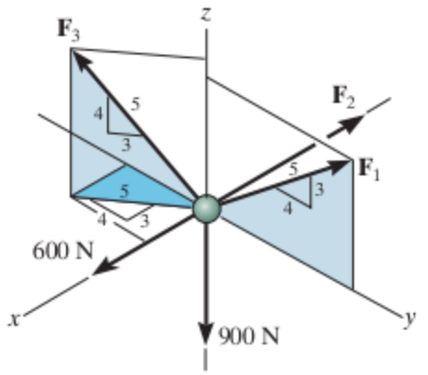
$$\Sigma F_z = 0; \qquad A_z + (-450\text{ N}) + 600\text{ N} - 900\text{ N} = 0$$
$$A_z = 750\text{ N} \qquad \textit{Ans.}$$

4

FUNDAMENTAL PROBLEMS

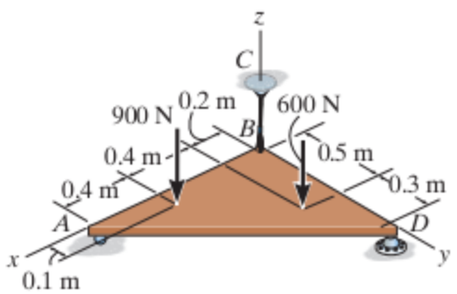
All problem solutions must include an FBD.

F4–7. Determine the magnitude of forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, so that the particle is held in equilibrium.



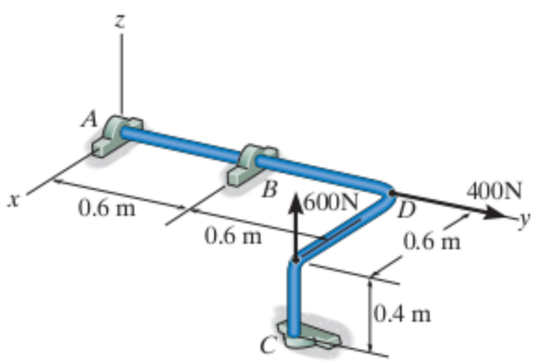
F4–7

F4–8. Determine the reactions at the roller support A, the ball-and-socket joint D, and the tension in cable BC for the plate.



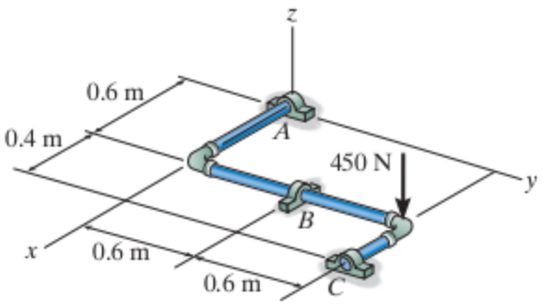
F4–8

F4–9. The rod is supported by smooth journal bearings at A, B and C and is subjected to the two forces. Determine the reactions at these supports.



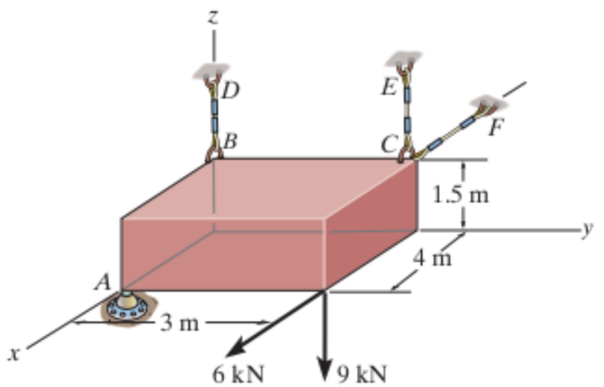
F4–9

F4–10. Determine the support reactions at the smooth journal bearings A, B, and C of the pipe assembly.



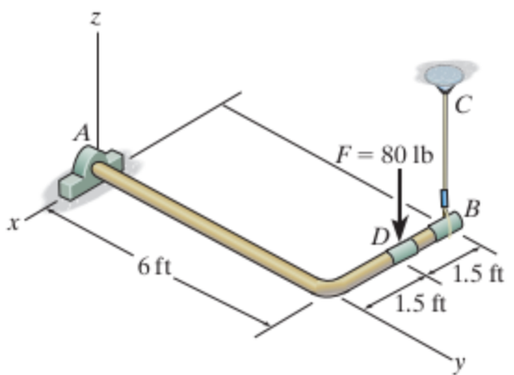
F4–10

F4–11. Determine the force developed in the link BD and the cords CE and CF, and the reactions of the ball-and-socket joint A on the block. Member DB is a short link.



F4–11

F4–12. Determine the components of reaction that the thrust bearing A and cable BC exert on the bar.

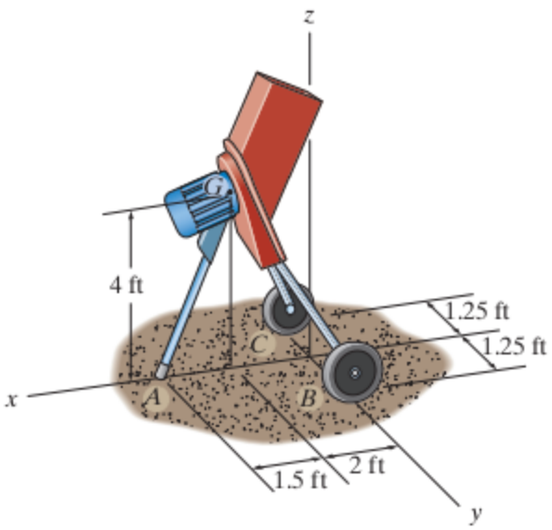


F4–12

PROBLEMS

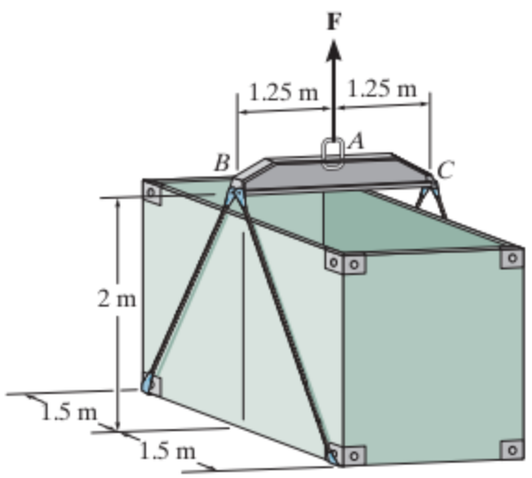
All problem solutions must include an FBD.

4–34. The 50-lb mulching machine has a center of gravity at G. Determine the vertical reactions at the wheels C and B and the smooth contact point A.



Prob. 4–34

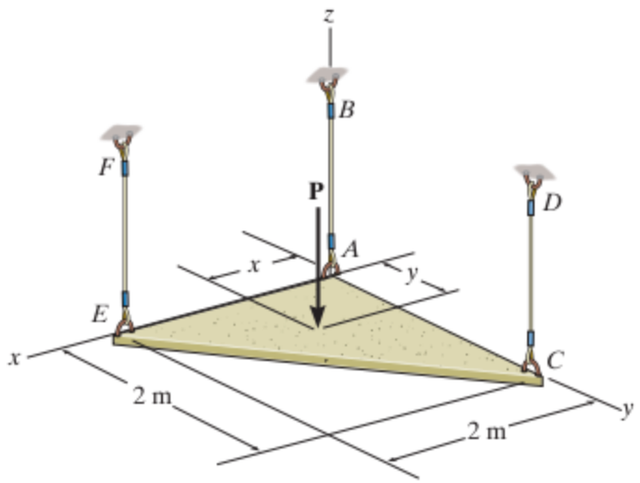
4–35. The uniform load has a mass of 600 kg and is lifted using a uniform 30-kg strongback beam BAC and four ropes as shown. Determine the tension in each rope and the force that must be applied at A.



Prob. 4–35

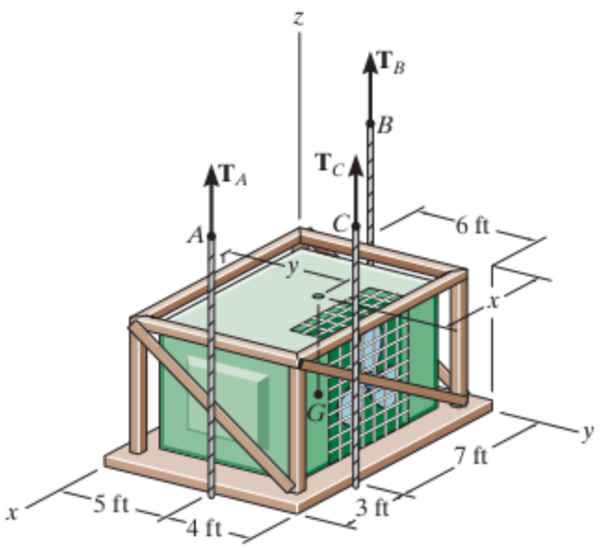
***4–36.** If $P = 6$ kN, $x = 0.75$ m and $y = 1$ m, determine the tension developed in cables AB, CD, and EF. Neglect the weight of the plate.

4–37. Determine the location x and y of the point of application of force **P** so that the tension developed in cables AB, CD, and EF is the same. Neglect the weight of the plate.



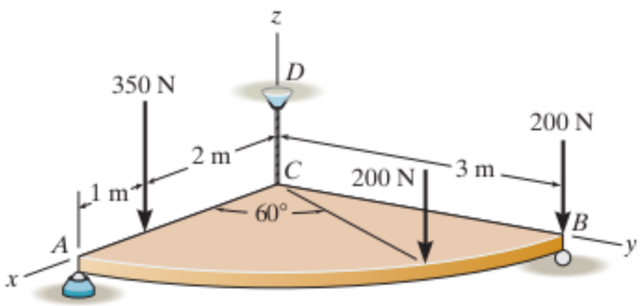
Probs. 4–36/37

4–38. The air-conditioning unit is hoisted to the roof of a building using the three cables. If the tensions in the cables are $T_A = 250$ lb, $T_B = 300$ lb, and $T_C = 200$ lb, determine the weight of the unit and the location (x, y) of its center of gravity G.



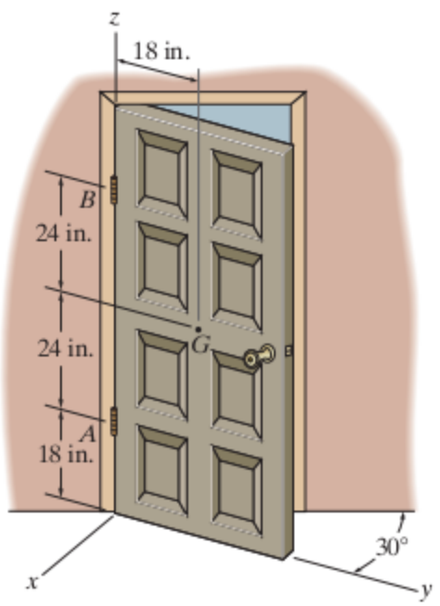
Prob. 4–38

4–39. Determine the force components acting on the ball-and-socket at A, the reaction at the roller B and the tension on the cord CD needed for equilibrium of the quarter circular plate.



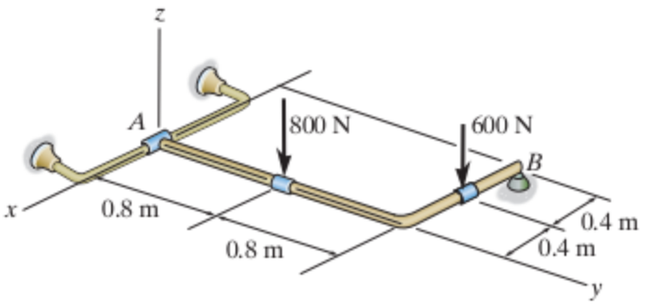
Prob. 4–39

***4–40.** The 100-lb door has its center of gravity at G. Determine the components of reaction at hinges A and B if hinge B resists only forces in the x and y directions and A resists forces in the x, y, z directions.



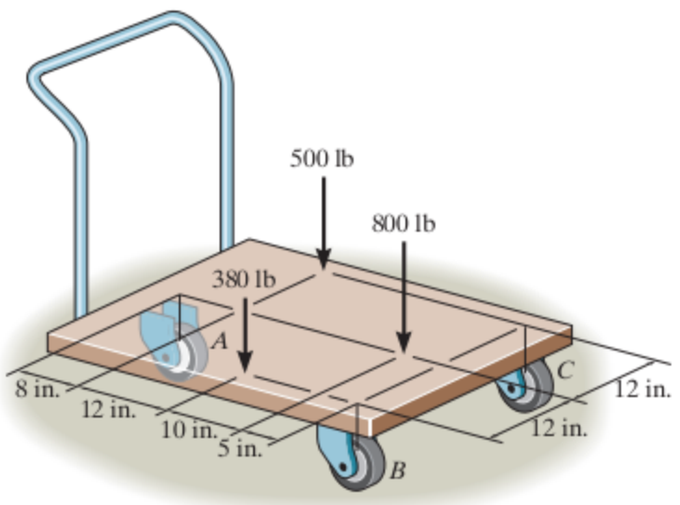
Prob. 4–40

4–41. Determine the support reactions at the smooth collar A and the normal reaction at the roller support B.



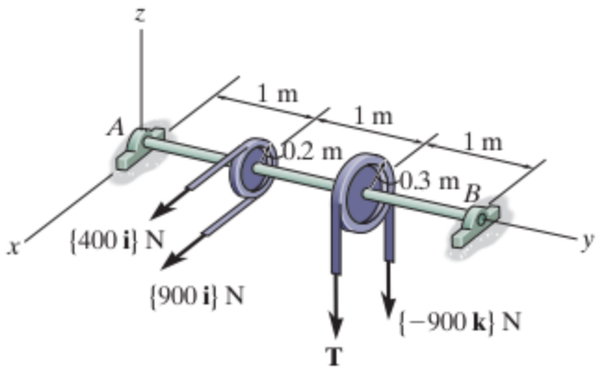
Prob. 4–41

4–42. The platform truck supports the three loadings shown. Determine the normal reactions on each of its three wheels.



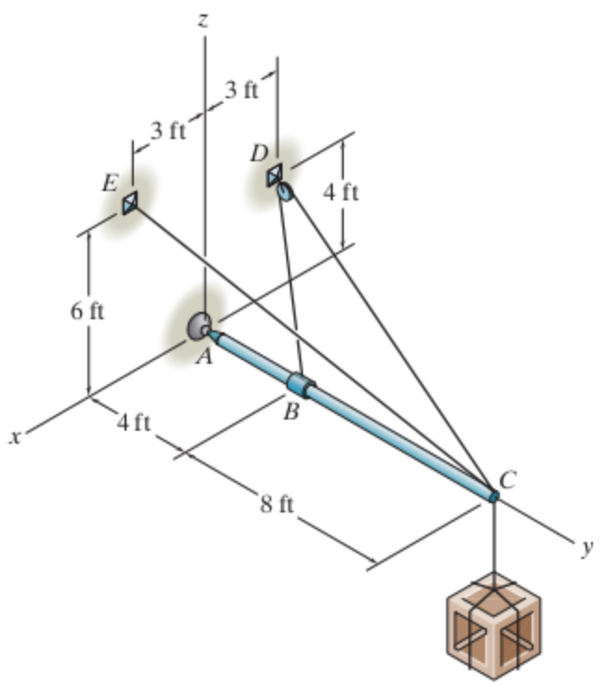
Prob. 4–42

4–43. If the pulleys are fixed to the shaft, determine the magnitude of tension **T** and the x, y, z components of reaction at the smooth thrust bearing A and smooth journal bearing B.



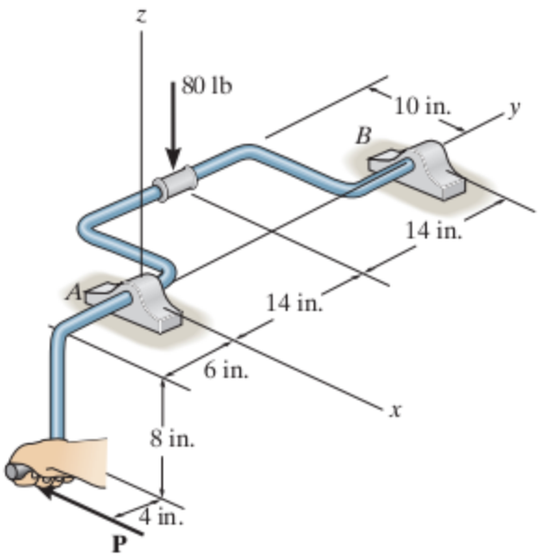
Prob. 4–43

***4–44.** The boom AC is supported at A by a ball-and-socket joint and by two cables BDC and CE. Cable BDC is continuous and passes over a pulley at D. Calculate the tension in the cables and the x, y, z components of reaction at A if a crate has a weight of 80 lb.



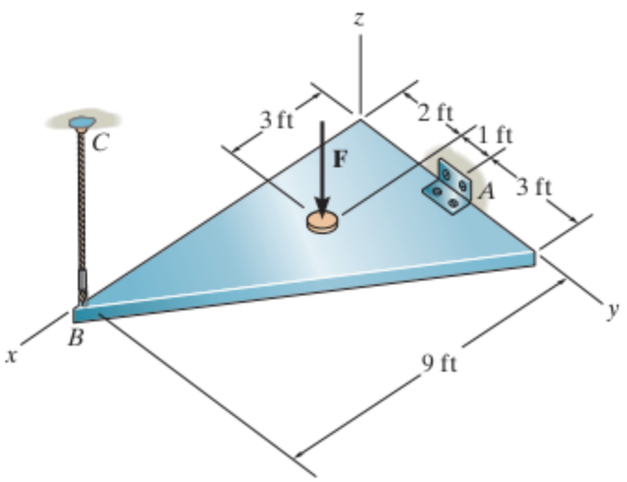
Prob. 4–44

4–45. A vertical force of 80 lb acts on the crankshaft. Determine the horizontal equilibrium force **P** that must be applied to the handle and the x, y, z components of reaction at the journal bearing A and thrust bearing B. The bearings are properly aligned and exert only force reactions on the shaft.

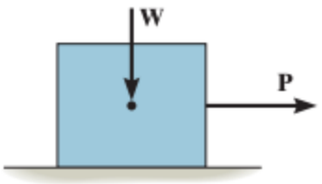


Prob. 4–45

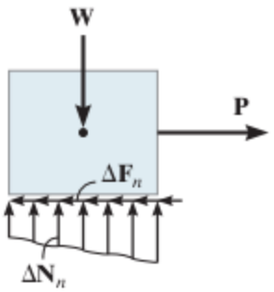
4–46. If the cable can be subjected to a maximum tension of 300 lb, determine the maximum force F which may be applied to the plate. Compute the x, y, z components of reaction at the hinge A for this loading.



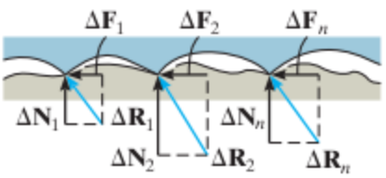
Prob. 4–46



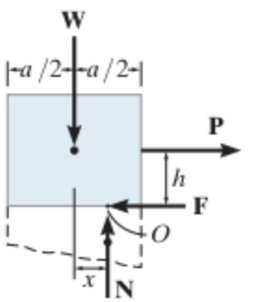
(a)



(b)



(c)



Resultant Normal and Frictional Forces

(d)

Fig. 4–21

4.7 Characteristics of Dry Friction

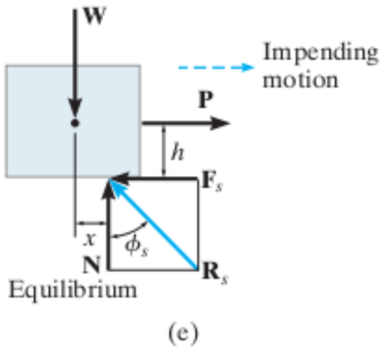
Friction is a force that resists the movement of two contacting surfaces that slide relative to one another. This force always acts *tangent* to the surface at the points of contact and is directed so as to oppose the possible or existing motion between the surfaces.

In this chapter, we will study the effects of *dry friction*, which is sometimes called *Coulomb friction* since its characteristics were studied extensively by C. A. Coulomb in 1781. Dry friction occurs between the contacting surfaces of bodies when there is no lubricating fluid.*

Theory of Dry Friction. The theory of dry friction can be explained by considering the effects caused by pulling horizontally on a block of uniform weight **W** which is resting on a rough horizontal surface that is *nonrigid or deformable,* Fig. 4–21*a*. The upper portion of the block, however, can be considered rigid. As shown on the free-body diagram of the block, Fig. 4–21*b*, the floor exerts an uneven *distribution* of both *normal force* $\Delta\mathbf{N}_n$ and *frictional force* $\Delta\mathbf{F}_n$ along the contacting surface. For equilibrium, the normal forces must act *upward* to balance the block's weight **W**, and the frictional forces act to the left to prevent the applied force **P** from moving the block to the right. Close examination of the contacting surfaces between the floor and block reveals how these frictional and normal forces develop, Fig. 4–21*c*. It can be seen that many microscopic irregularities exist between the two surfaces and, as a result, reactive forces $\Delta\mathbf{R}_n$ are developed at each point of contact.* As shown, each reactive force contributes both a frictional component $\Delta\mathbf{F}_n$ and a normal component $\Delta\mathbf{N}_n$.

Equilibrium. The effect of the *distributed* normal and frictional loadings is indicated by their *resultants* **N** and **F** on the free-body diagram, Fig. 4–21*d*. Notice that **N** acts a distance x to the right of the line of action of **W**, Fig. 4–21*d*. This location, which coincides with the centroid or geometric center of the normal force distribution in Fig. 4–21*b*, is necessary in order to balance the "tipping effect" caused by **P**. For example, if **P** is applied at a height h from the surface, Fig. 4–21*d*, then moment equilibrium about point O is satisfied if $Wx = Ph$ or $x = Ph/W$.

*Besides mechanical interactions as explained here, which is referred to as a classical approach, a detailed treatment of the nature of frictional forces must also include the effects of temperature, density, cleanliness, and atomic or molecular attraction between the contacting surfaces. See J. Krim, *Scientific American*, October, 1996.



(e)

Fig. 4–21 (cont.)

Regardless of the weight of the rake or shovel that is suspended, the device has been designed so that the small roller holds the handle in equilibrium due to frictional forces that develop at the points of contact, A, B, C.

Impending Motion. In cases where the surfaces of contact are rather "slippery," the frictional force **F** may *not* be great enough to balance **P**, and consequently the block will tend to slip. In other words, as P is slowly increased, F correspondingly increases until it attains a certain *maximum value* F_s, called the *limiting static frictional force*, Fig. 4–21*e*. When this value is reached, the block is in *unstable equilibrium* since any further increase in P will cause the block to move. Experimentally, it has been determined that this limiting static frictional force F_s is *directly proportional* to the resultant normal force N. Expressed mathematically,

$$F_s = \mu_s N \tag{4–8}$$

where the constant of proportionality, μ_s (mu "sub" s), is called the *coefficient of static friction*.

Thus, when the block is on the *verge of sliding*, the normal force **N** and frictional force $\mathbf{F}_s$ combine to create a resultant $\mathbf{R}_s$, Fig. 4–21*e*. The angle ϕ_s (phi "sub" s) that $\mathbf{R}_s$ makes with **N** is called the *angle of static friction*. From the figure,

$$\phi_s = \tan^{-1}\left(\frac{F_s}{N}\right) = \tan^{-1}\left(\frac{\mu_s N}{N}\right) = \tan^{-1}\mu_s$$

Typical values for μ_s are given in Table 4–3. Note that these values can vary since experimental testing was done under variable conditions of roughness and cleanliness of the contacting surfaces. For applications, therefore, it is important that both caution and judgment be exercised when selecting a coefficient of friction for a given set of conditions. When a more accurate calculation of F_s is required, the coefficient of friction should be determined directly by an experiment that involves the two materials to be used.

Table 4–3 Typical Values for μ_s

Contact Materials	Coefficient of Static Friction (μ_s)
Metal on ice	0.03–0.05
Wood on wood	0.30–0.70
Leather on wood	0.20–0.50
Leather on metal	0.30–0.60
Aluminum on aluminum	1.10–1.70

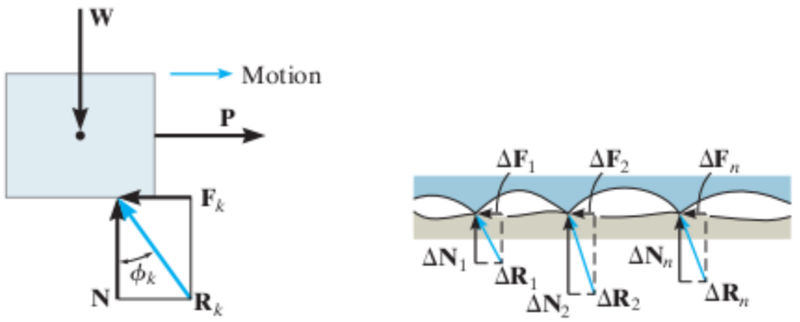


Fig. 4–22

4

Motion. If the magnitude of **P** acting on the block is increased so that it becomes slightly greater than F_s, the frictional force at the contacting surface will drop to a smaller value F_k, called the *kinetic frictional force*. The block will begin to slide with increasing speed, Fig. 4–22*a*. As this occurs, the block will "ride" on top of these peaks at the points of contact, as shown in Fig. 4–22*b*. The continued breakdown of the surface is the dominant mechanism creating kinetic friction.

Experiments with sliding blocks indicate that the magnitude of the kinetic friction force is directly proportional to the magnitude of the resultant normal force, expressed mathematically as

$$F_k = \mu_k N \tag{4–9}$$

Here the constant of proportionality, μ_k, is called the *coefficient of kinetic friction*. Typical values for μ_k are approximately 25 percent *smaller* than those listed in Table 4–3 for μ_s.

As shown in Fig. 4–22*a*, in this case, the resultant force at the surface of contact, $\mathbf{R}_k$, has a line of action defined by ϕ_k. This angle is referred to as the *angle of kinetic friction*, where

$$\phi_k = \tan^{-1}\left(\frac{F_k}{N}\right) = \tan^{-1}\left(\frac{\mu_k N}{N}\right) = \tan^{-1}\mu_k$$

By comparison, $\phi_s \geq \phi_k$.

The above effects regarding friction can be summarized by referring to the graph in Fig. 4–23, which shows the variation of the frictional force F versus the applied load P. Here the frictional force is categorized in three different ways:

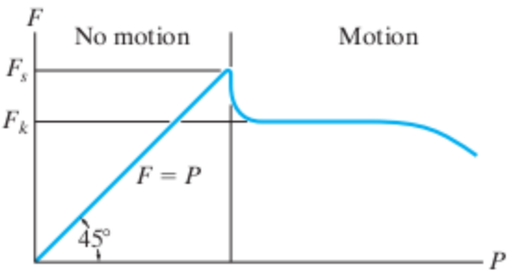


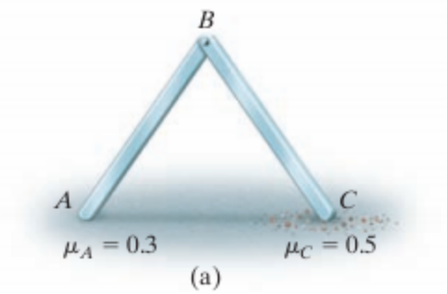
Fig. 4–23

- F is a *static frictional force* if equilibrium is maintained.
- F is a *limiting static frictional force* F_s when it reaches a maximum value needed to maintain equilibrium.
- F is termed a *kinetic frictional force* F_k when sliding occurs at the contacting surface.

Notice also from the graph that for very large values of P or for high speeds, aerodynamic effects will cause F_k and likewise μ_k to begin to decrease.

Characteristics of Dry Friction. As a result of *experiments* that pertain to the foregoing discussion, we can state the following rules that apply to bodies subjected to dry friction.

- The frictional force acts *tangent* to the contacting surfaces in a direction *opposed* to the *motion* or tendency for motion of one surface relative to another.
- The maximum static frictional force F_s that can be developed is independent of the area of contact, provided the normal pressure is not very low nor great enough to severely deform or crush the contacting surfaces of the bodies.
- The maximum static frictional force is generally greater than the kinetic frictional force for any two surfaces of contact. However, if one of the bodies is moving with a *very low velocity* over the surface of another, F_k becomes approximately equal to F_s, i.e., $\mu_s \approx \mu_k$.
- When *slipping* at the surface of contact is *about to occur*, the maximum static frictional force is proportional to the normal force, such that $F_s = \mu_s N$.
- When *slipping* at the surface of contact is *occurring*, the kinetic frictional force is proportional to the normal force, such that $F_k = \mu_k N$.



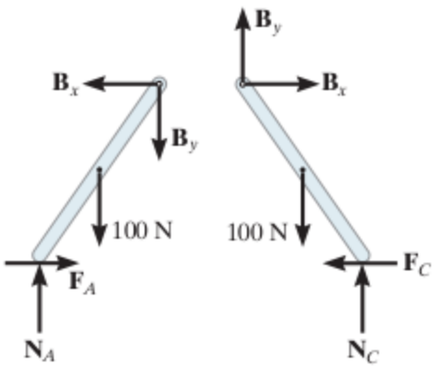
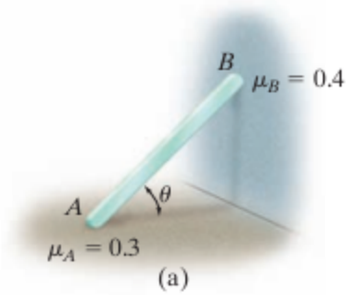


Fig. 4–24



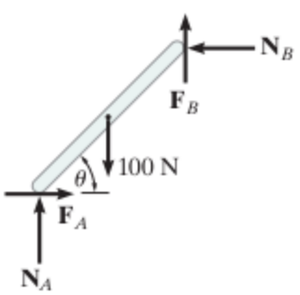


Fig. 4–25

4.8 Problems Involving Dry Friction

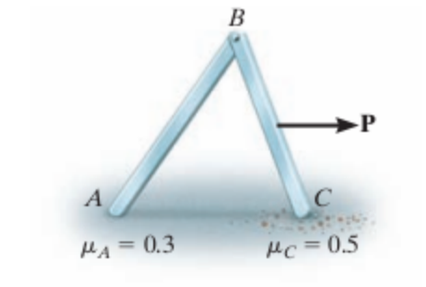
If a rigid body is in equilibrium when it is subjected to a system of forces that includes the effect of friction, the force system must satisfy not only the equations of equilibrium but *also* the laws that govern the frictional forces.

Types of Friction Problems. In general, there are three types of mechanics problems involving dry friction. They can easily be classified once free-body diagrams are drawn and the total number of unknowns are identified and compared with the total number of available equilibrium equations.

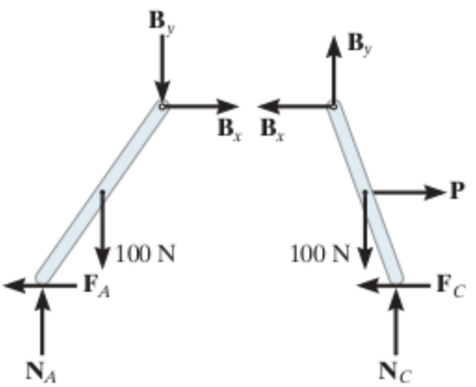
No Apparent Impending Motion. Problems in this category are strictly equilibrium problems, which require the number of unknowns to be *equal* to the number of available equilibrium equations. Once the frictional forces are determined from the solution, however, their numerical values must be checked to be sure they satisfy the inequality $F \leq \mu_s N$; otherwise, slipping will occur and the body will not remain in equilibrium. A problem of this type is shown in Fig. 4–24*a*. Here we must determine the frictional forces at A and C to check if the equilibrium position of the two-member frame can be maintained. If the bars are uniform and have known weights of 100 N each, then the free-body diagrams are as shown in Fig. 4–24*b*. There are six unknown force components which can be determined *strictly* from the six equilibrium equations (three for each member). Once F_A, N_A, F_C, and N_C are determined, then the bars will remain in equilibrium provided $F_A \leq 0.3N_A$ and $F_C \leq 0.5N_C$ are satisfied.

Impending Motion at All Points of Contact. In this case the total number of unknowns will *equal* the total number of available equilibrium equations *plus* the total number of available frictional equations, $F = \mu N$. When *motion is impending* at the points of contact, then $F_s = \mu_s N$; whereas if the body is *slipping*, then $F_k = \mu_k N$. For example, consider the problem of finding the smallest angle θ at which the 100-N bar in Fig. 4–25*a* can be placed against the wall without slipping. The free-body diagram is shown in Fig. 4–25*b*. Here the *five* unknowns are determined from the *three* equilibrium equations and *two* static frictional equations which apply at *both* points of contact, so that $F_A = 0.3N_A$ and $F_B = 0.4N_B$.

Impending Motion at Some Points of Contact. Here the number of unknowns will be *less* than the number of available equilibrium equations plus the number of available frictional equations or conditional equations for tipping. As a result, several possibilities for motion or impending motion will exist and the problem will involve a determination of the kind of motion which actually occurs. For example, consider the two-member frame in Fig. 4–26*a*. In this problem we wish to determine the horizontal force P needed to cause movement. If each member has a weight of 100 N, then the free-body diagrams are as shown in Fig. 4–26*b*. There are *seven* unknowns. For a unique solution we must satisfy the *six* equilibrium equations (three for each member) and only *one* of two possible static frictional equations. This means that as P increases it will either cause slipping at A and no slipping at C, so that $F_A = 0.3N_A$ and $F_C \leq 0.5N_C$; or slipping occurs at C and no slipping at A, in which case $F_C = 0.5N_C$ and $F_A \leq 0.3N_A$. The actual situation can be determined by calculating P for each case and then choosing the case for which P is *smaller*. If in both cases the *same value* for P is calculated, which in practice would be highly improbable, then slipping at both points occurs simultaneously; i.e., the *seven unknowns* would satisfy *eight equations*.

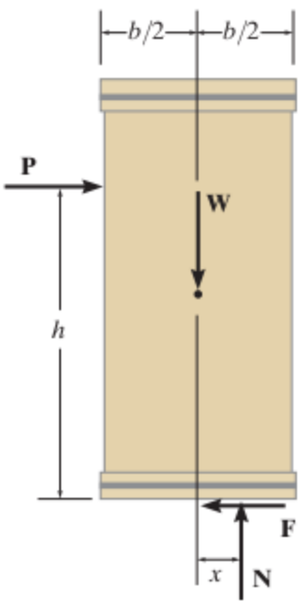


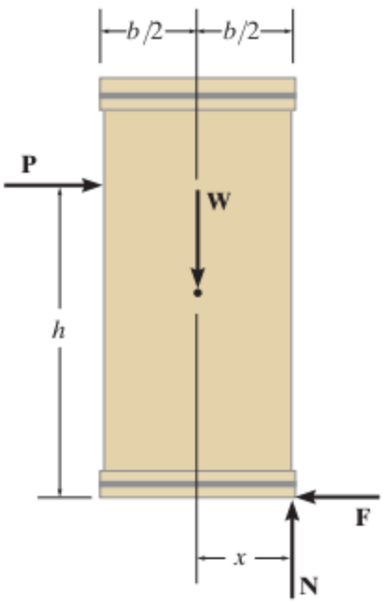
(a)



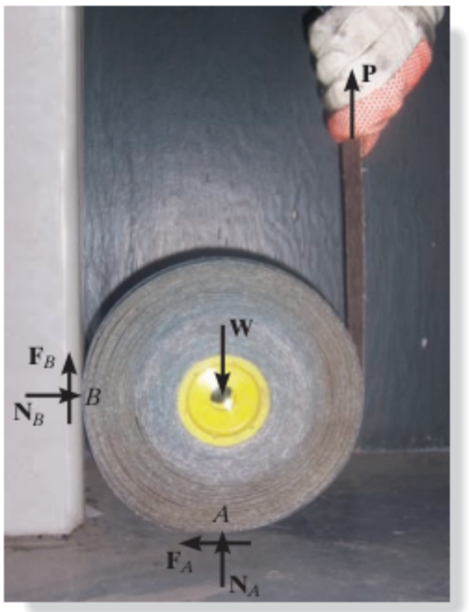
(b)

Fig. 4–26





Consider pushing on the uniform crate that has a weight W and sits on the rough surface. As shown on the first free-body diagram, if the magnitude of **P** is small, the crate will remain in equilibrium. As P increases the crate will either be on the verge of slipping on the surface ($F = \mu_s N$), or if the surface is very rough (large μ_s) then the resultant normal force will shift to the corner, $x = b/2$, as shown on the second free-body diagram. At this point the crate will begin to tip over. The crate also has a greater chance of tipping if **P** is applied at a greater height h above the surface, or if its width b is smaller.



The applied vertical force **P** on this roll must be large enough to overcome the resistance of friction at the contacting surfaces *A* and *B* in order to cause rotation.

Equilibrium versus Frictional Equations. Whenever we solve problems where the friction force F is to be an "equilibrium force" and satisfies the inequality $F < \mu_s N$, then we can assume the sense of direction of F on the free-body diagram. The correct sense is made known *after* solving the equations of equilibrium for F. If F is a negative scalar the sense of **F** is the reverse of that which was assumed. This convenience of *assuming* the sense of **F** is possible because the equilibrium equations equate to zero the *components of vectors* acting in the *same direction*. However, in cases where the frictional equation $F = \mu N$ is used in the solution of a problem, the convenience of *assuming* the sense of **F** is *lost*, since the frictional equation relates only the *magnitudes* of two *perpendicular* vectors. Consequently, **F** *must always* be shown acting with its *correct sense* on the free-body diagram, *whenever* the frictional equation is used for the solution of a problem.

Procedure for Analysis

Equilibrium problems involving dry friction can be solved using the following procedure.

Free-Body Diagrams.

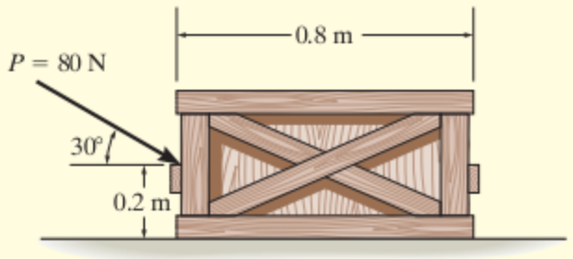
- Draw the necessary free-body diagrams, and unless it is stated in the problem that impending motion or slipping occurs, *always* show the frictional forces as unknowns (i.e., *do not assume* $F = \mu N$).
- Determine the number of unknowns and compare this with the number of available equilibrium equations.
- If there are more unknowns than equations of equilibrium, it will be necessary to apply the frictional equation at some, if not all, points of contact to obtain the extra equations needed for a complete solution.
- If the equation $F = \mu N$ is to be used, it will be necessary to show **F** acting in the correct sense of direction on the free-body diagram.

Equations of Equilibrium and Friction.

- Apply the equations of equilibrium and the necessary frictional equations (or conditional equations if tipping is possible) and solve for the unknowns.
- If the problem involves a three-dimensional force system such that it becomes difficult to obtain the force components or the necessary moment arms, apply the equations of equilibrium using Cartesian vectors.

EXAMPLE 4.12

The uniform crate shown in Fig. 4–27*a* has a mass of 20 kg. If a force $P = 80$ N is applied to the crate, determine if it remains in equilibrium. The coefficient of static friction is $\mu_s = 0.3$.

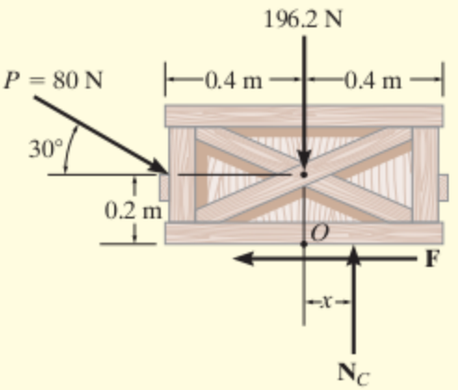


(a)

Fig. 4–27

SOLUTION

Free-Body Diagram. As shown in Fig. 4–27*b*, the *resultant* normal force $\mathbf{N}_C$ must act a distance x from the crate's center line in order to counteract the tipping effect caused by **P**. There are *three unknowns*, F, N_C, and x, which can be determined strictly from the *three* equations of equilibrium.



(b)

Equations of Equilibrium.

$$\pm\rightarrow \Sigma F_x = 0; \qquad 80\cos 30^\circ \text{ N} - F = 0$$

$$+\uparrow \Sigma F_y = 0; \qquad -80\sin 30^\circ \text{ N} + N_C - 196.2 \text{ N} = 0$$

$$\circlearrowleft + \Sigma M_O = 0; \qquad 80\sin 30^\circ \text{ N}(0.4 \text{ m}) - 80\cos 30^\circ \text{ N}(0.2 \text{ m}) + N_C(x) = 0$$

Solving,

$$F = 69.3 \text{ N}$$

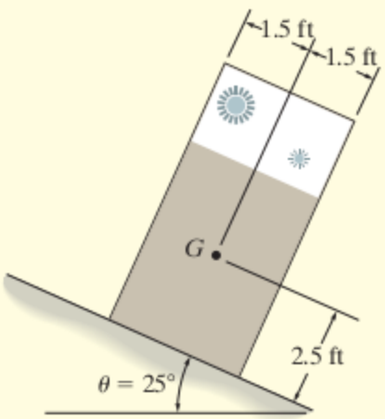
$$N_C = 236 \text{ N}$$

$$x = -0.00908 \text{ m} = -9.08 \text{ mm}$$

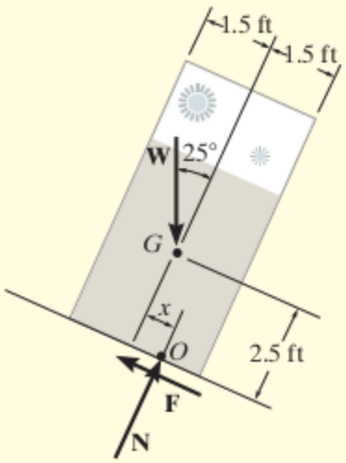
Since x is negative it indicates the *resultant* normal force acts (slightly) to the *left* of the crate's center line. No tipping will occur since $x < 0.4$ m. Also, the *maximum* frictional force which can be developed at the surface of contact is $F_{\max} = \mu_s N_C = 0.3(236 \text{ N}) = 70.8$ N. Since $F = 69.3 \text{ N} < 70.8$ N, the crate will *not slip*, although it is very close to doing so.

EXAMPLE 4.13

(a)



(b)



(c)

Fig. 4–28

It is observed that when the bed of the dump truck is raised to an angle of $\theta = 25°$ the vending machines will begin to slide off the bed, Fig. 4–28*a*. Determine the static coefficient of friction between a vending machine and the surface of the truckbed.

SOLUTION

An idealized model of a vending machine resting on the truckbed is shown in Fig. 4–28*b*. The dimensions have been measured and the center of gravity has been located. We will assume that the vending machine weighs W.

Free-Body Diagram. As shown in Fig. 4–28*c*, the dimension x is used to locate the position of the resultant normal force **N**. There are four unknowns, N, F, μ_s, and x.

Equations of Equilibrium.

$$+\searrow \Sigma F_x = 0; \qquad W \sin 25° - F = 0 \tag{1}$$

$$+\nearrow \Sigma F_y = 0; \qquad N - W \cos 25° = 0 \tag{2}$$

$$\circlearrowleft +\Sigma M_O = 0; \quad -W \sin 25°(2.5 \text{ ft}) + W \cos 25°(x) = 0 \tag{3}$$

Since slipping impends at $\theta = 25°$, using Eqs. (1) and (2), we have

$$F_s = \mu_s N; \qquad W \sin 25° = \mu_s (W \cos 25°)$$

$$\mu_s = \tan 25° = 0.466 \qquad \textit{Ans.}$$

The angle of $\theta = 25°$ is referred to as the *angle of repose*, and by comparison, it is equal to the angle of static friction, $\theta = \phi_s$. Notice from the calculation that θ is independent of the weight of the vending machine, and so knowing θ provides a convenient method for determining the coefficient of static friction.

NOTE: From Eq. (3), we find $x = 1.17$ ft. Since $1.17 \text{ ft} < 1.5 \text{ ft}$, indeed the vending machine will slip before it can tip as observed in Fig. 4–28*a*.

EXAMPLE 4.14

The uniform 10-kg ladder in Fig. 4–29*a* rests against the smooth wall at B, and the end A rests on the rough horizontal plane for which the coefficient of static friction is $\mu_s = 0.3$. Determine the angle of inclination θ of the ladder and the normal reaction at B if the ladder is on the verge of slipping.

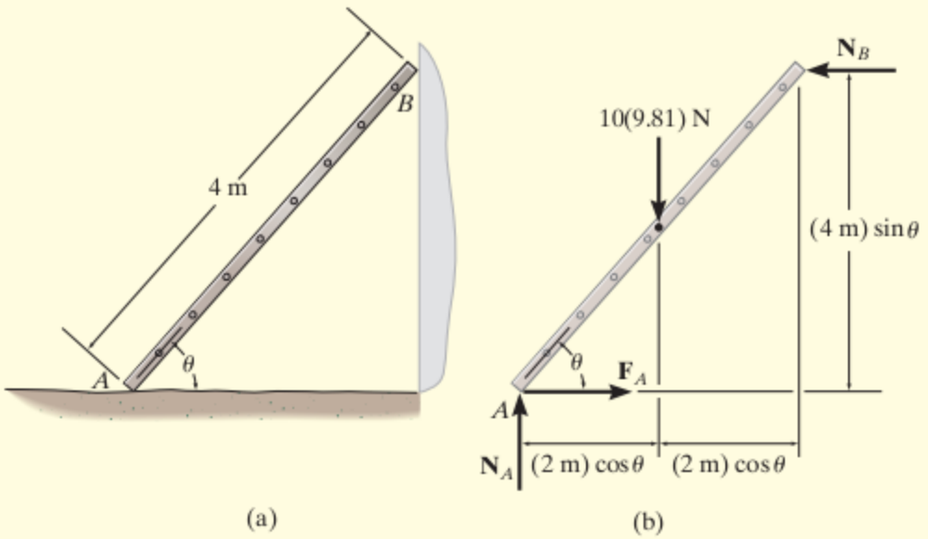


Fig. 4–29

SOLUTION

Free-Body Diagram. As shown on the free-body diagram, Fig. 4–29*b*, the frictional force $\mathbf{F}_A$ must act to the right since impending motion at A is to the left.

Equations of Equilibrium and Friction. Since the ladder is on the verge of slipping, then $F_A = \mu_s N_A = 0.3N_A$. By inspection, N_A can be obtained directly.

$$+\uparrow \Sigma F_y = 0; \qquad N_A - 10(9.81)\,\text{N} = 0 \qquad N_A = 98.1\,\text{N}$$

Using this result, $F_A = 0.3(98.1\,\text{N}) = 29.43\,\text{N}$. Now N_B can be found.

$$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \qquad 29.43\,\text{N} - N_B = 0$$

$$N_B = 29.43\,\text{N} = 29.4\,\text{N} \qquad \textit{Ans.}$$

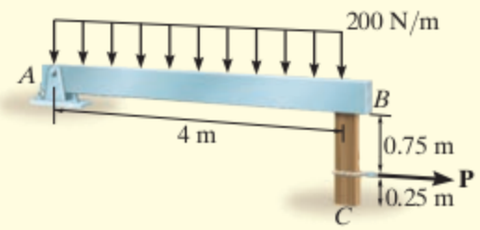
Finally, the angle θ can be determined by summing moments about point A.

$$\circlearrowleft + \Sigma M_A = 0; \qquad (29.43\,\text{N})(4\,\text{m})\sin\theta - [10(9.81)\,\text{N}](2\,\text{m})\cos\theta = 0$$

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = 1.6667$$

$$\theta = 59.04^\circ = 59.0^\circ \qquad \textit{Ans.}$$

EXAMPLE 4.15



(a)

Beam AB is subjected to a uniform load of 200 N/m and is supported at B by post BC, Fig. 4–30*a*. If the coefficients of static friction at B and C are $\mu_B = 0.2$ and $\mu_C = 0.5$, determine the force **P** needed to pull the post out from under the beam. Neglect the weight of the members and the thickness of the beam.

SOLUTION

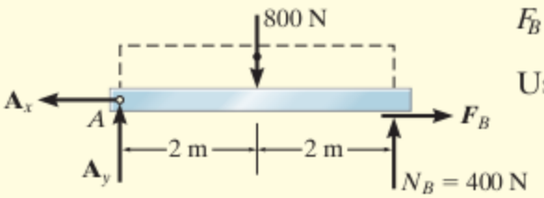
Free-Body Diagrams. The free-body diagram of the beam is shown in Fig. 4–30*b*. Applying $\Sigma M_A = 0$, we obtain $N_B = 400$ N. This result is shown on the free-body diagram of the post, Fig. 4–30*c*. Referring to this member, the *four* unknowns F_B, P, F_C, and N_C are determined from the *three* equations of equilibrium and *one* frictional equation applied either at B or C.

Equations of Equilibrium and Friction.

$$\pm\rightarrow \Sigma F_x = 0; \qquad P - F_B - F_C = 0 \tag{1}$$

$$+\uparrow \Sigma F_y = 0; \qquad N_C - 400\,\text{N} = 0 \tag{2}$$

$$\circlearrowleft + \Sigma M_C = 0; \qquad -P(0.25\,\text{m}) + F_B(1\,\text{m}) = 0 \tag{3}$$



(b)

(Post Slips at *B* and Rotates about *C*.) This requires $F_C \leq \mu_C N_C$ and

$$F_B = \mu_B N_B; \qquad F_B = 0.2(400\,\text{N}) = 80\,\text{N}$$

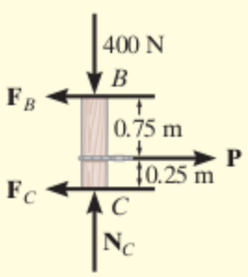
Using this result and solving Eqs. (1) through (3), we obtain

$$P = 320\,\text{N}$$
$$F_C = 240\,\text{N}$$
$$N_C = 400\,\text{N}$$

Since $F_C = 240\,\text{N} > \mu_C N_C = 0.5(400\,\text{N}) = 200\,\text{N}$, slipping at C occurs. Thus the other case of movement must be investigated.

(Post Slips at *C* and Rotates about *B*.) Here $F_B \leq \mu_B N_B$ and

$$F_C = \mu_C N_C; \qquad F_C = 0.5 N_C \tag{4}$$



(c)

Fig. 4–30

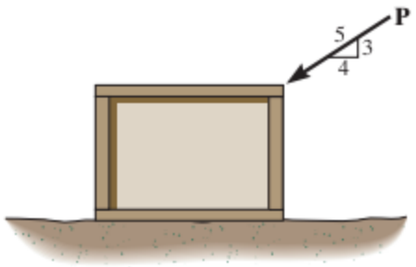
Solving Eqs. (1) through (4) yields

$$P = 267\,\text{N} \qquad \textit{Ans.}$$
$$N_C = 400\,\text{N}$$
$$F_C = 200\,\text{N}$$
$$F_B = 66.7\,\text{N}$$

Obviously, this case occurs first since it requires a *smaller* value for P.

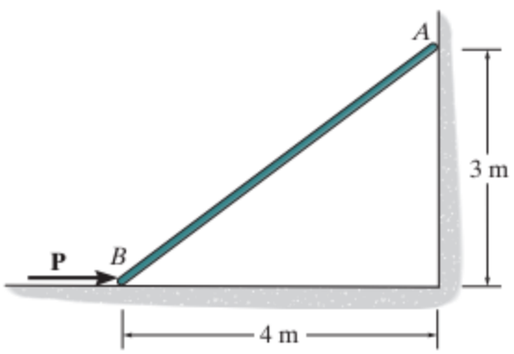
FUNDAMENTAL PROBLEMS

F4–13. If $P = 200$ N, determine the friction developed between the 50-kg crate and the ground. The coefficient of static friction between the crate and the ground is $\mu_s = 0.3$.



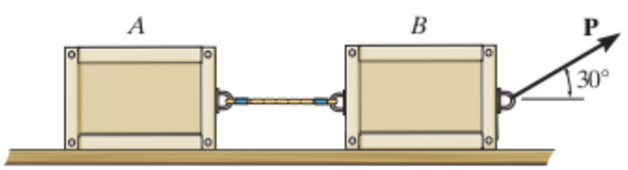
F4–13

F4–14. Determine the minimum force P to prevent the 80-kg rod AB from sliding. The contact surface at B is smooth, whereas the coefficient of static friction between the rod and the wall at A is $\mu_s = 0.2$.



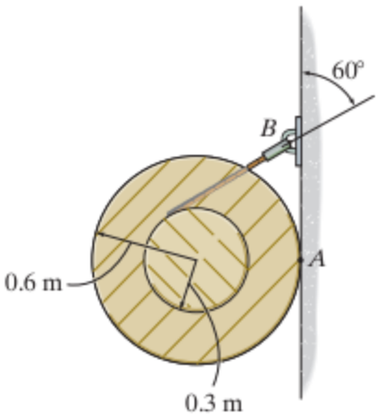
F4–14

F4–15. Determine the maximum force P that can be applied without causing the two 50-kg crates to move. The coefficient of static friction between each crate and the ground is $\mu_s = 0.25$.



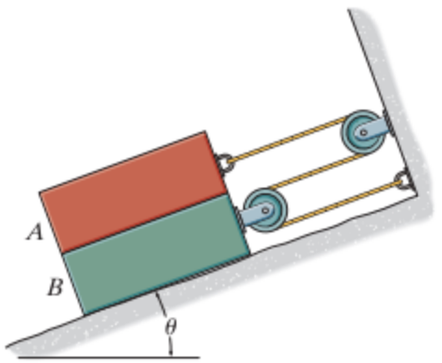
F4–15

F4–16. Determine the minimum coefficient of static friction between the uniform 50-kg spool and the wall so that the spool does not slip.



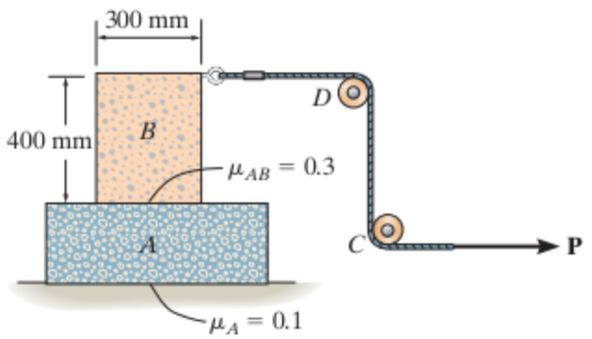
F4–16

F4–17. If the coefficient of static friction at all contacting surfaces is μ_s, determine the inclination θ at which the identical blocks, each of weight W, begin to slide.



F4–17

F4–18. Blocks A and B have a mass of 7 kg and 10 kg, respectively. Using the coefficients of static friction indicated, determine the largest force P which can be applied to the cord without causing motion. There are pulleys at C and D.



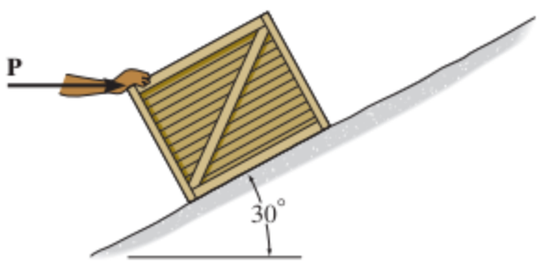
F4–18

PROBLEMS

4–47. Determine the minimum horizontal force P required to hold the crate from sliding down the plane. The crate has a mass of 50 kg and the coefficient of static friction between the crate and the plane is $\mu_s = 0.25$.

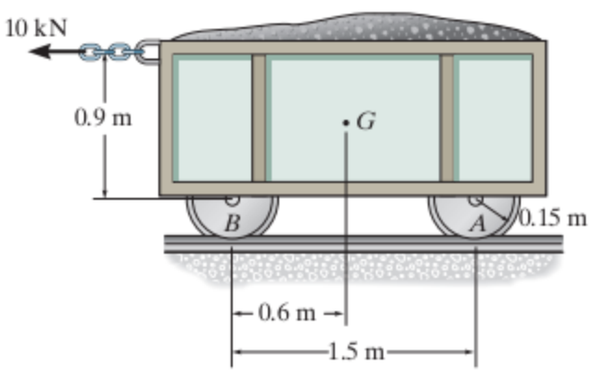
***4–48.** Determine the minimum force P required to push the crate up the plane. The crate has a mass of 50 kg and the coefficient of static friction between the crate and the plane is $\mu_s = 0.25$.

4–49. A horizontal force of $P = 100$ N is just sufficient to hold the crate from sliding down the plane, and a horizontal force of $P = 350$ N is required to just push the crate up the plane. Determine the coefficient of static friction between the plane and the crate, and find the mass of the crate.



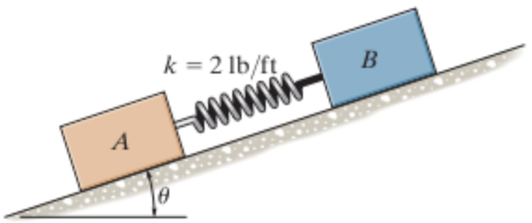
Probs. 4–47/48/49

4–50. The mine car and its contents have a total mass of 6 Mg and a center of gravity at G. If the coefficient of static friction between the wheels and the tracks is $\mu_s = 0.4$ when the wheels are locked, find the normal force acting on the front wheels at B and the rear wheels at A when the brakes at both A and B are locked. Does the car move?



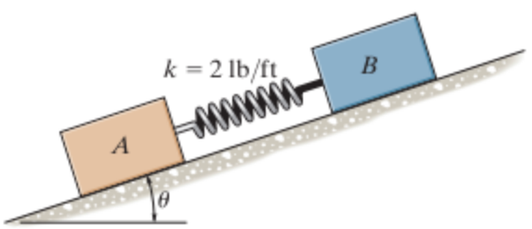
Prob. 4–50

4–51. Two blocks A and B have a weight of 10 lb and 6 lb, respectively. They are resting on the incline for which the coefficients of static friction are $\mu_A = 0.15$ and $\mu_B = 0.25$. Determine the incline angle θ for which both blocks begin to slide. Also find the required stretch or compression in the connecting spring for this to occur. The spring has a stiffness of $k = 2$ lb/ft.



Prob. 4–51

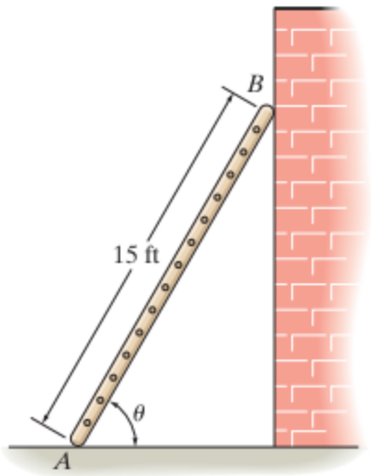
***4–52.** Two blocks A and B have a weight of 10 lb and 6 lb, respectively. They are resting on the incline for which the coefficients of static friction are $\mu_A = 0.15$ and $\mu_B = 0.25$. Determine the angle θ which will cause motion of one of the blocks. What is the friction force under each of the blocks when this occurs? The spring has a stiffness of $k = 2$ lb/ft and is originally unstretched.



Prob. 4–52

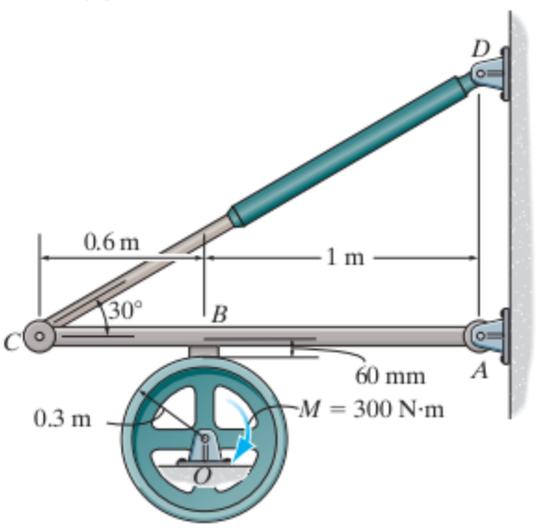
4–53. The ladder has a uniform weight of 80 lb and rests against the smooth wall at B. If the coefficient of static friction at A is $\mu_A = 0.4$, determine if the ladder will slip. Take $\theta = 60°$.

4–54. The ladder has a uniform weight of 80 lb and rests against the wall at B. If the coefficient of static friction at A and B is $\mu = 0.4$, determine the smallest angle θ at which the ladder will not slip.



Probs. 4–53/54

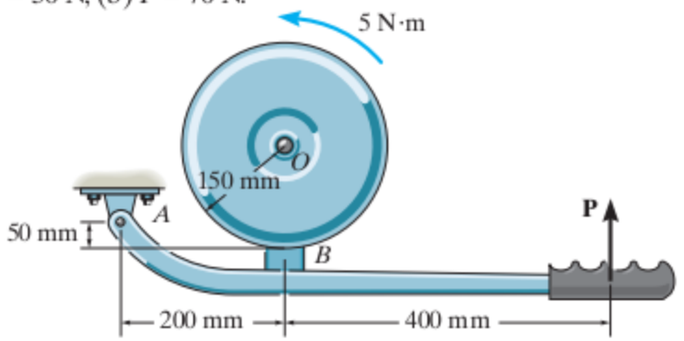
4–55. If a torque of $M = 300\ \text{N}\cdot\text{m}$ is applied to the flywheel, determine the force that must be developed in the hydraulic cylinder CD to prevent the flywheel from rotating. The coefficient of static friction between the friction pad at B and the flywheel is $\mu_s = 0.4$.



Prob. 4–55

***4–56.** The block brake consists of a pin-connected lever and friction block at B. The coefficient of static friction between the wheel and the lever is $\mu_s = 0.3$, and a torque of $5\ \text{N}\cdot\text{m}$ is applied to the wheel. Determine if the brake can hold the wheel stationary when the force applied to the lever is (a) $P = 30$ N, (b) $P = 70$ N.

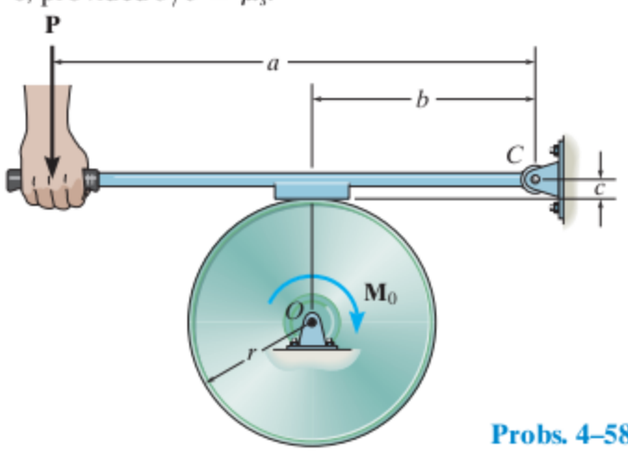
4–57. The block brake consists of a pin-connected lever and friction block at B. The coefficient of static friction between the wheel and the lever is $\mu_s = 0.3$, and a torque of $5\ \text{N}\cdot\text{m}$ is applied to the wheel. Determine if the brake can hold the wheel stationary when the force applied to the lever is (a) $P = 30$ N, (b) $P = 70$ N.



Probs. 4–56/57

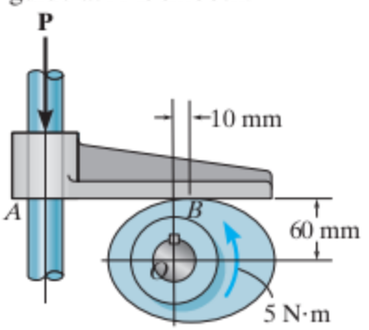
4–58. The block brake is used to stop the wheel from rotating when the wheel is subjected to a couple moment $\mathbf{M}_0$. If the coefficient of static friction between the wheel and the block is μ_s, determine the smallest force P that should be applied.

4–59. Show that the brake in Prob. 4–58 is self-locking, i.e., $P \le 0$, provided $b/c \le \mu_s$.



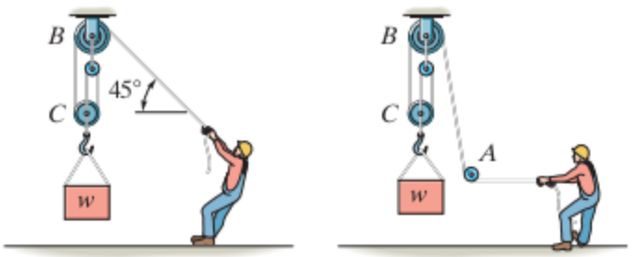
Probs. 4–58/59

***4–60.** The cam is subjected to a couple moment of $5\ \text{N}\cdot\text{m}$. Determine the minimum force P that should be applied to the follower in order to hold the cam in the position shown. The coefficient of static friction between the cam and the follower is $\mu = 0.4$. The guide at A is smooth.



Prob. 4–60

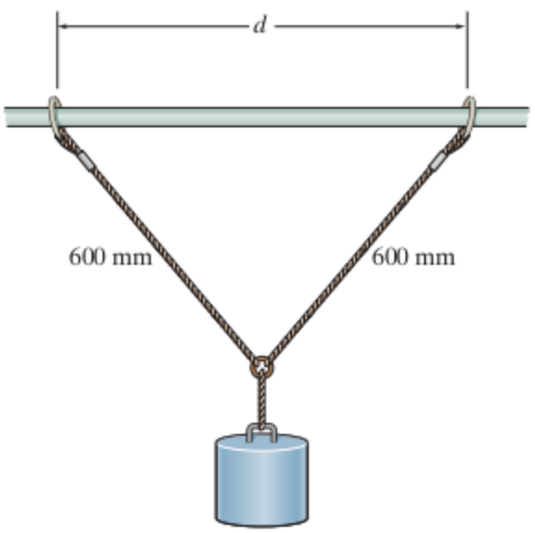
4–61. Determine the maximum weight W the man can lift with constant velocity using the pulley system, without and then with the "leading block" or pulley at A. The man has a weight of 200 lb and the coefficient of static friction between his feet and the ground is $\mu_s = 0.6$.



Prob. 4–61

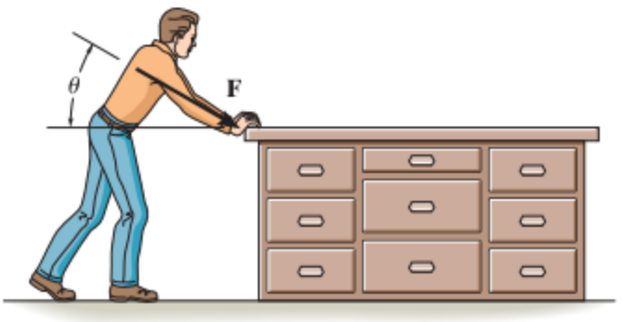
4–62. The 5-kg cylinder is suspended from two equal-length cords. The end of each cord is attached to a ring of negligible mass that passes along a horizontal shaft. If the rings can be separated by the greatest distance $d = 400$ mm and still support the cylinder, determine the coefficient of static friction between each ring and the shaft.

4–63. The 5-kg cylinder is suspended from two equal-length cords. The end of each cord is attached to a ring of negligible mass that passes along a horizontal shaft. If the coefficient of static friction between each ring and the shaft is $\mu_s = 0.5$, determine the greatest distance d by which the rings can be separated and still support the cylinder.



Probs. 4–62/63

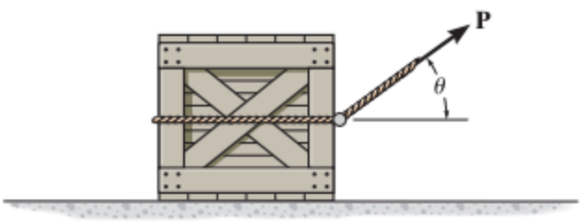
*4–64. The uniform dresser has a weight of 90 lb and rests on a tile floor for which $\mu_s = 0.25$. If the man pushes on it in the direction $\theta = 30°$, determine the smallest magnitude of force **F** needed to move the dresser. Also, if the man has a weight of 150 lb, determine the smallest coefficient of static friction between his shoes and the floor so that he does not slip.



Prob. 4–64

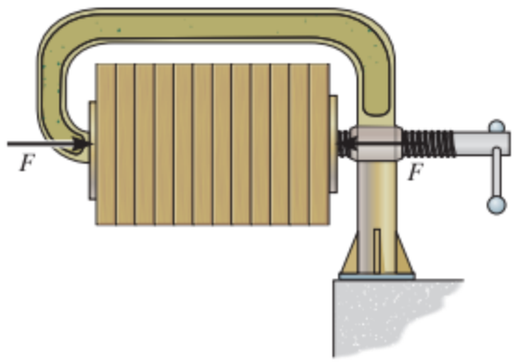
4–65. The crate has a weight of $W = 150$ lb, and the coefficients of static and kinetic friction are $\mu_s = 0.3$ and $\mu_k = 0.2$, respectively. Determine the friction force on the floor if $\theta = 30°$ and $P = 200$ lb.

4–66. The crate has a weight of $W = 350$ lb, and the coefficients of static and kinetic friction are $\mu_s = 0.3$ and $\mu_k = 0.2$, respectively. Determine the friction force on the floor if $\theta = 45°$ and $P = 100$ lb.



Probs. 4–65/66

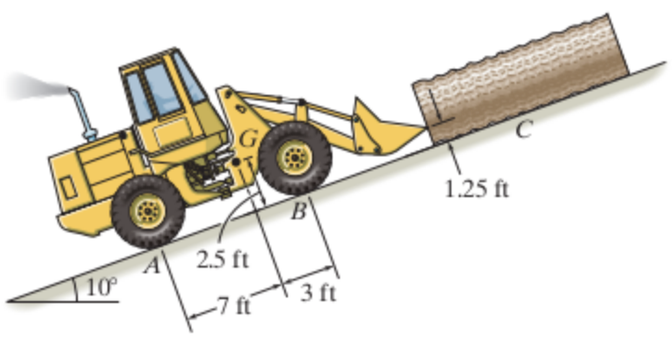
4–67. If the clamping force is $F = 200$ N and each board has a mass of 2 kg, determine the maximum number of boards the clamp can support. The coefficient of static friction between the boards is $\mu_s = 0.3$, and the coefficient of static friction between the boards and the clamp is $\mu_s' = 0.45$.



Prob. 4–67

*4–68. The tractor has a weight of 8000 lb with center of gravity at G. Determine if it can push the 550-lb log up the incline. The coefficient of static friction between the log and the ground is $\mu_s = 0.5$, and between the rear wheels of the tractor and the ground $\mu_s' = 0.8$. The front wheels are free to roll. Assume the engine can develop enough torque to cause the rear wheels to slip.

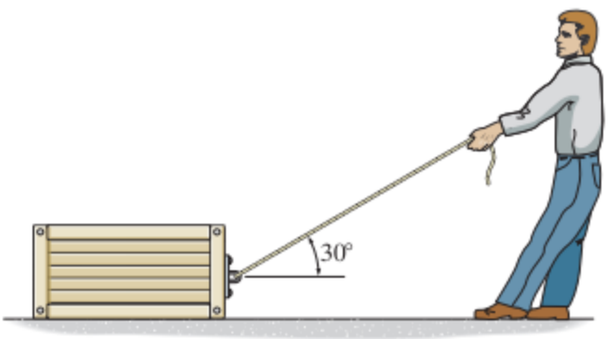
4–69. The tractor has a weight of 8000 lb with center of gravity at G. Determine the greatest weight of the log that can be pushed up the incline. The coefficient of static friction between the log and the ground is $\mu_s = 0.5$, and between the rear wheels of the tractor and the ground $\mu_s' = 0.7$. The front wheels are free to roll. Assume the engine can develop enough torque to cause the rear wheels to slip.



Probs. 4–68/69

4–70. The coefficient of static friction between the 150-kg crate and the ground is $\mu_s = 0.3$, while the coefficient of static friction between the 80-kg man's shoes and the ground is $\mu_s' = 0.4$. Determine if the man can move the crate.

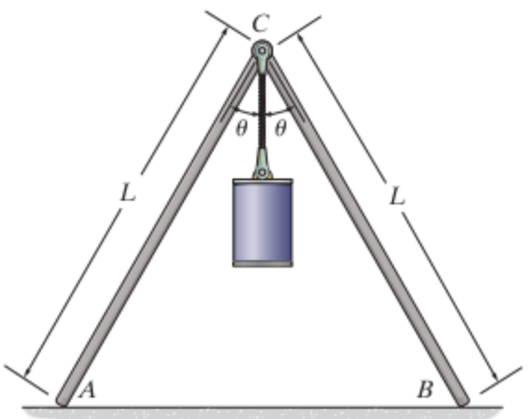
4–71. If the coefficient of static friction between the crate and the ground in Prob. 4–70 is $\mu_s = 0.3$, determine the minimum coefficient of static friction between the man's shoes and the ground so that the man can move the crate.



Probs. 4–70/71

*4–72. If $\theta = 30°$, determine the minimum coefficient of static friction at A and B so that equilibrium of the supporting frame is maintained regardless of the mass of the cylinder C. Neglect the mass of the rods.

4–73. If the coefficient of static friction at A and B is $\mu_s = 0.6$, determine the maximum angle θ so that the frame remains in equilibrium, regardless of the mass of the cylinder. Neglect the mass of the rods.



Probs. 4–72/73

CONCEPTUAL PROBLEMS

P4–9. Is it more effective to move the load forward at constant velocity with the boom fully extended as shown, or should the boom be fully retracted? Power is supplied to the rear wheels. The front wheels are free to roll. Do an equilibrium analysis to explain your answer.

P4–9

P4–10. The lug nut on the free-turning wheel is to be removed using the wrench. Which is the most effective way to apply force to the wrench? Also, why is it best to keep the car tire on the ground rather than first jacking it up? Explain your answers with an equilibrium analysis.

P4–10

P4–11. The rope is used to tow the refrigerator. Is it best to pull slightly up on the rope as shown, pull horizontally, or pull somewhat downwards? Also, is it best to attach the rope at a high position as shown, or at a lower position? Do an equilibrium analysis to explain your answer.

P4–12. The rope is used to tow the refrigerator. In order to prevent yourself from slipping while towing, is it best to pull up as shown, pull horizontally, or pull downwards on the rope? Do an equilibrium analysis to explain your answer.

P4–11/12

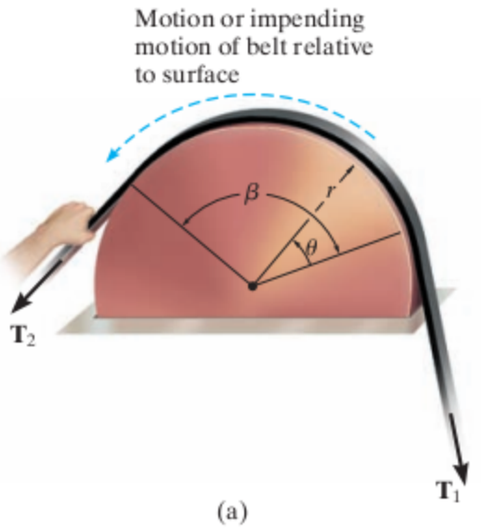
P4–13. Is it easier to tow the load by applying a force along the tow bar when it is in an almost horizontal position as shown, or is it better to pull on the bar when it has a steeper slope? Do an equilibrium analysis to explain your answer.

P4–13

4.9 Frictional Forces on Flat Belts

Whenever belt drives or band brakes are designed, it is necessary to determine the frictional forces developed between the belt and its contacting surface. In this section we will analyze the frictional forces acting on a flat belt, although the analysis of other types of belts, such as the V-belt, is based on similar principles.

Consider the flat belt shown in Fig. 4–31*a*, which passes over a fixed curved surface. The total angle of belt to surface contact in radians is β, and the coefficient of friction between the two surfaces is μ. We wish to determine the tension T_2 in the belt, which is needed to pull the belt counterclockwise over the surface, and thereby overcome both the frictional forces at the surface of contact and the tension T_1 in the other end of the belt. Obviously, $T_2 > T_1$.



(a)

Frictional Analysis. A free-body diagram of the belt segment in contact with the surface is shown in Fig. 4–31*b*. As shown, the normal and frictional forces, acting at different points along the belt, will vary both in magnitude and direction. Due to this *unknown* distribution, the analysis of the problem will first require a study of the forces acting on a differential element of the belt.

(b)

A free-body diagram of an element having a length ds is shown in Fig. 4–31*c*. Assuming either impending motion or motion of the belt, the magnitude of the frictional force $dF = \mu\, dN$. This force opposes the sliding motion of the belt, and so it will increase the magnitude of the tensile force acting in the belt by dT. Applying the two force equations of equilibrium, we have

$$\searrow + \Sigma F_x = 0; \qquad T\cos\left(\frac{d\theta}{2}\right) + \mu\, dN - (T + dT)\cos\left(\frac{d\theta}{2}\right) = 0$$

$$+\nearrow \Sigma F_y = 0; \qquad dN - (T + dT)\sin\left(\frac{d\theta}{2}\right) - T\sin\left(\frac{d\theta}{2}\right) = 0$$

Since $d\theta$ is of *infinitesimal size*, $\sin(d\theta/2) = d\theta/2$ and $\cos(d\theta/2) = 1$. Also, the *product* of the two infinitesimals dT and $d\theta/2$ may be neglected when compared to infinitesimals of the first order. As a result, these two equations become

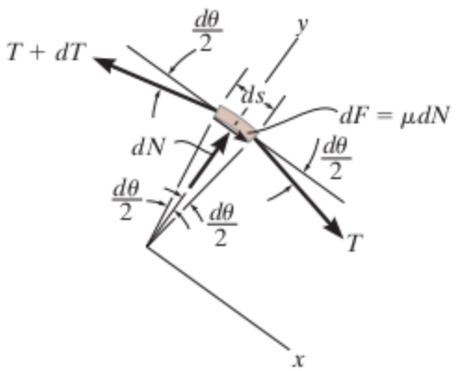
$$\mu\, dN = dT$$

and

$$dN = T\, d\theta$$

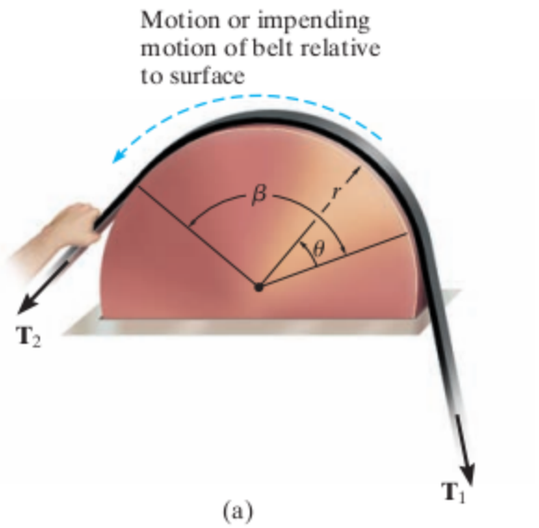
Eliminating dN yields

$$\frac{dT}{T} = \mu\, d\theta$$



(c)

Fig. 4–31



(a)

Fig. 4–31 (repeated)

Flat or V-belts are often used to transmit the torque developed by a motor to a wheel attached to a pump, fan or blower.

Integrating this equation between all the points of contact that the belt makes with the drum, and noting that $T = T_1$ at $\theta = 0$ and $T = T_2$ at $\theta = \beta$, yields

$$\int_{T_1}^{T_2} \frac{dT}{T} = \mu \int_0^{\beta} d\theta$$

$$\ln \frac{T_2}{T_1} = \mu\beta$$

Solving for T_2, we obtain

$$T_2 = T_1 e^{\mu\beta} \tag{4–10}$$

where

T_2, T_1 = belt tensions; T_1 opposes the direction of motion (or impending motion) of the belt measured relative to the surface, while T_2 acts in the direction of the relative belt motion (or impending motion); because of friction, $T_2 > T_1$

μ = coefficient of static or kinetic friction between the belt and the surface of contact

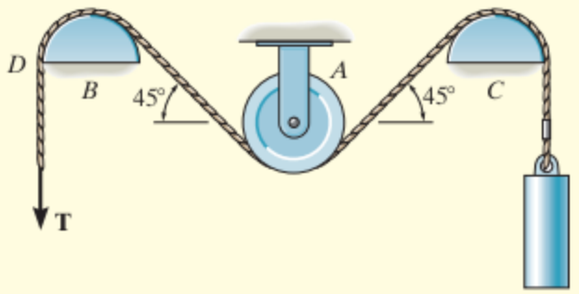
β = angle of belt to surface contact, measured in radians

e = 2.718 . . . , base of the natural logarithm

Note that T_2 is *independent* of the *radius* of the drum, and instead it is a function of the angle of belt to surface contact, β. As a result, this equation is valid for flat belts passing over any curved contacting surface.

EXAMPLE 4.16

The maximum tension that can be developed in the cord shown in Fig. 4–32*a* is 500 N. If the pulley at *A* is free to rotate and the coefficient of static friction at the fixed drums *B* and *C* is $\mu_s = 0.25$, determine the largest mass of the cylinder that can be lifted by the cord.



(a)

SOLUTION

Lifting the cylinder, which has a weight $W = mg$, causes the cord to move counterclockwise over the drums at *B* and *C*; hence, the maximum tension T_2 in the cord occurs at *D*. Thus, $F = T_2 = 500$ N. A section of the cord passing over the drum at *B* is shown in Fig. 4–32*b*. Since $180° = \pi$ rad the angle of contact between the drum and the cord is $\beta = (135°/180°)\pi = 3\pi/4$ rad. Using Eq. 4–10, we have

$$T_2 = T_1 e^{\mu_s \beta}; \qquad 500\ \text{N} = T_1\, e^{0.25[(3/4)\pi]}$$

Hence,

$$T_1 = \frac{500\ \text{N}}{e^{0.25[(3/4)\pi]}} = \frac{500\ \text{N}}{1.80} = 277.4\ \text{N}$$

Since the pulley at *A* is free to rotate, equilibrium requires that the tension in the cord remains the *same* on both sides of the pulley.

The section of the cord passing over the drum at *C* is shown in Fig. 4–32*c*. The weight $W < 277.4$ N. Why? Applying Eq. 4–10, we obtain

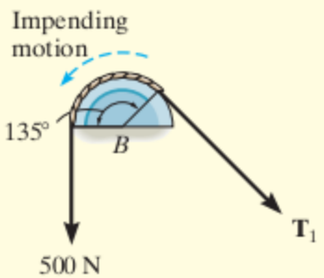
$$T_2 = T_1 e^{\mu_s \beta}; \qquad 277.4\ \text{N} = W e^{0.25[(3/4)\pi]}$$

$$W = 153.9\ \text{N}$$

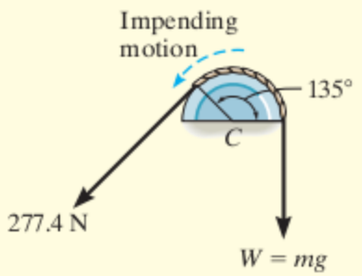
so that

$$m = \frac{W}{g} = \frac{153.9\ \text{N}}{9.81\ \text{m/s}^2}$$

$$= 15.7\ \text{kg} \qquad \textit{Ans.}$$



(b)



(c)

Fig. 4–32

Square-threaded screws find applications on valves, jacks, and vises, where particularly large forces must be developed along the axis of the screw.

4.10 Frictional Forces on Screws

In most cases screws are used as fasteners; however, in many types of machines they are incorporated to transmit power or motion from one part of the machine to another. A *square-threaded screw* is commonly used for the latter purpose, especially when large forces are applied along its axis. In this section we will analyze the forces acting on square-threaded screws. The analysis of other types of screws, such as the V-thread, is based on these same principles.

For analysis, a square-threaded screw, as in Fig. 4–33, can be considered a cylinder having an inclined square ridge or *thread* wrapped around it. If we unwind the thread by one revolution, as shown in Fig. 4–33*b*, the slope or the *lead angle* θ is determined from $\theta = \tan^{-1}(l/2\pi r)$. Here l and $2\pi r$ are the vertical and horizontal distances between A and B, where r is the mean radius of the thread. The distance l is called the *lead* of the screw and it is equivalent to the distance the screw advances when it turns one revolution.

Upward Impending Motion. Let us now consider the case of a square-threaded screw that is subjected to upward impending motion caused by the applied torsional moment **M**, Fig. 4–34.* A free-body diagram of the *entire unraveled thread* can be represented as a block as shown in Fig. 4–35*a*. The force **W** is the vertical force acting on the thread or the axial force applied to the shaft, and M/r is the resultant horizontal force produced by the couple moment M about the axis of the shaft. The reaction **R** of the groove on the thread, has both frictional and normal components, where $F = \mu_s N$. The angle of static friction is $\phi_s = \tan^{-1}(F/N) = \tan^{-1}\mu_s$. Applying the force equations of equilibrium along the horizontal and vertical axes, we have

$$\pm\rightarrow \Sigma F_x = 0; \qquad M/r - R\sin(\phi_s + \theta) = 0$$

$$+\uparrow \Sigma F_y = 0; \qquad R\cos(\phi_s + \theta) - W = 0$$

Eliminating R from these equations, we obtain

$$M = rW\tan(\phi_s + \theta) \tag{4–11}$$

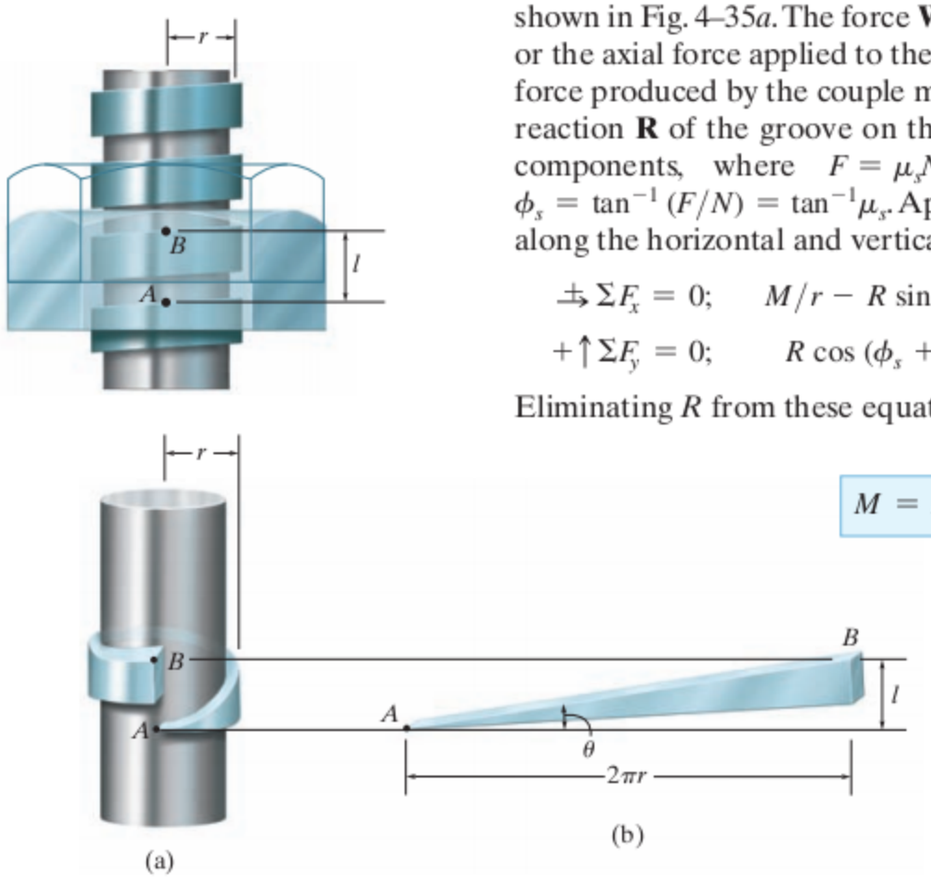


Fig. 4–33

*For applications, **M** is developed by applying a horizontal force **P** at a right angle to the end of a lever that would be fixed to the screw.

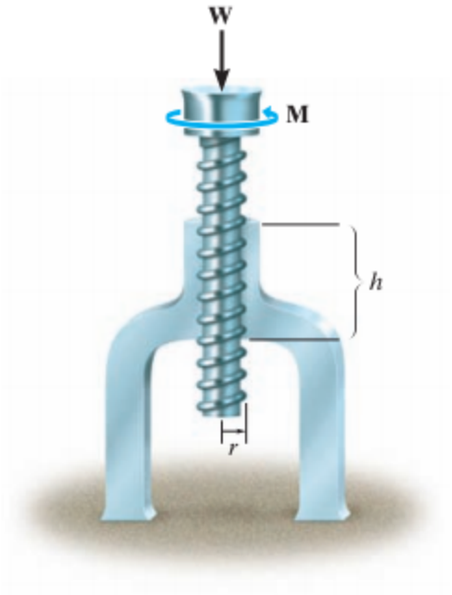


Fig. 4–34

Self-Locking Screw. A screw is said to be *self-locking* if it remains in place under any axial load **W** when the moment **M** is removed. For this to occur, the direction of the frictional force must be reversed so that **R** acts on the other side of **N**. Here the angle of static friction ϕ_s becomes greater than or equal to θ, Fig. 4–35*d*. If $\phi_s = \theta$, Fig. 4–35*b*, then **R** will act vertically to balance **W**, and the screw will be on the verge of winding downward.

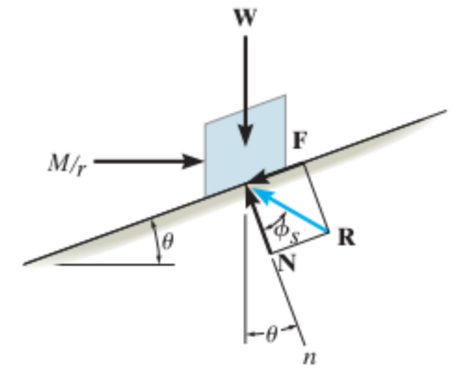
Downward Impending Motion ($\theta > \phi_s$). If the screw is not self-locking, it is necessary to apply a moment $\mathbf{M}'$ to prevent the screw from winding downward. Here, a horizontal force M'/r is required to push against the thread to prevent it from sliding down the plane, Fig. 4–35*c*. Using the same procedure as before, the magnitude of the moment $\mathbf{M}'$ required to prevent this unwinding is

$$M' = rW \tan(\theta - \phi_s) \tag{4–12}$$

Downward Impending Motion ($\phi_s > \theta$). If a screw is self-locking, a couple moment $\mathbf{M}''$ must be applied to the screw in the opposite direction to wind the screw downward ($\phi_s > \theta$). This causes a reverse horizontal force M''/r that pushes the thread down as indicated in Fig. 4–35*d*. In this case, we obtain

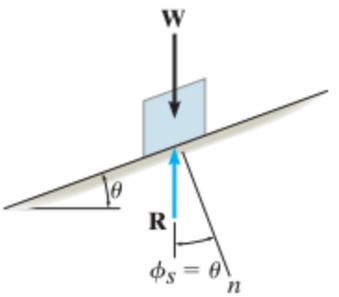
$$M'' = rW \tan(\phi_s - \theta) \tag{4–13}$$

If *motion of the screw* occurs, Eqs. 4–11, 4–12, and 4–13 can be applied by simply replacing ϕ_s with ϕ_k.



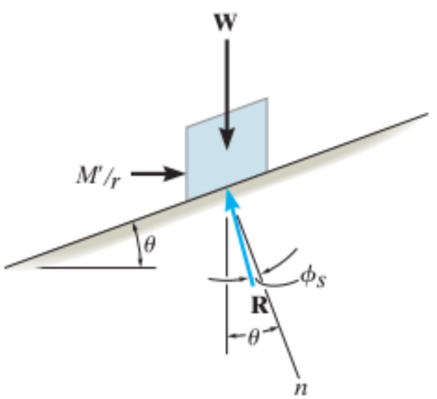
Upward screw motion

(a)



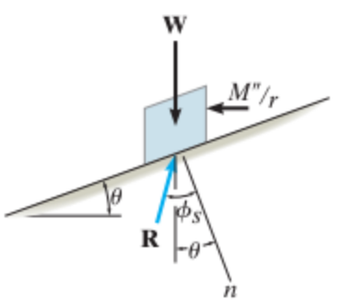
Self-locking screw ($\theta = \phi_s$)
(on the verge of rotating downward)

(b)



Downward screw motion ($\theta > \phi_s$)

(c)



Downward screw motion ($\theta < \phi_s$)

(d)

Fig. 4–35

EXAMPLE 4.17

The turnbuckle shown in Fig. 4–36 has a square thread with a mean radius of 5 mm and a lead of 2 mm. If the coefficient of static friction between the screw and the turnbuckle is $\mu_s = 0.25$, determine the moment **M** that must be applied to draw the end screws closer together.

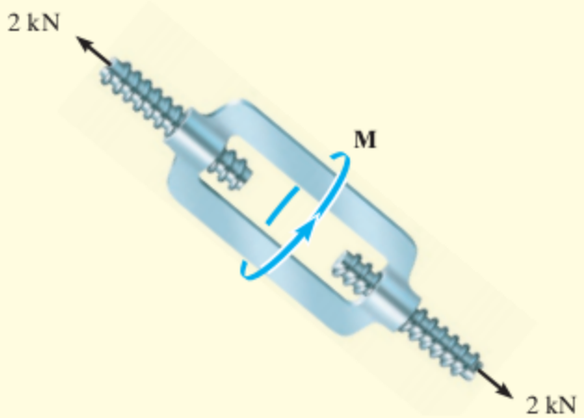


Fig. 4–36

SOLUTION

The moment can be obtained by applying Eq. 4–11. Since friction at *two screws* must be overcome, this requires

$$M = 2[Wr\tan(\theta + \phi)] \tag{1}$$

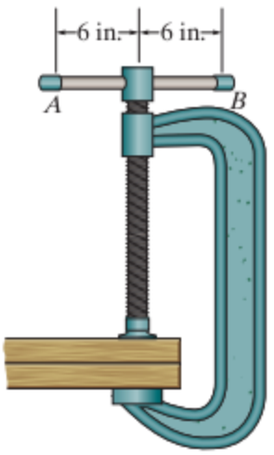
Here $W = 2000$ N, $r = 5$ mm, $\phi_s = \tan^{-1}\mu_s = \tan^{-1}(0.25) = 14.04°$, and $\theta = \tan^{-1}(l/2\pi r) = \tan^{-1}(2\text{ mm}/[2\pi(5\text{ mm})]) = 3.64°$. Substituting these values into Eq. (1) and solving gives

$$M = 2[(2000\text{ N})(5\text{ mm})\tan(14.04° + 3.64°)]$$
$$= 6374.7\text{ N}\cdot\text{mm} = 6.37\text{ N}\cdot\text{m} \quad \textit{Ans.}$$

NOTE: When the moment is *removed*, the turnbuckle will be self-locking; i.e., it will not unscrew since $\phi_s > \theta$.

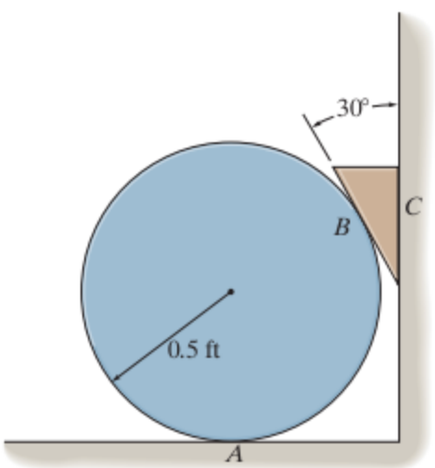
PROBLEMS

4–74. If the clamping force on the boards is 600 lb, determine the required magnitude of the couple forces that must be applied perpendicular to the lever AB of the clamp at A and B in order to loosen the screw. The single square-threaded screw has a mean diameter of 1 in. and a lead of 0.25 in. The coefficient of static friction is $\mu_s = 0.3$.



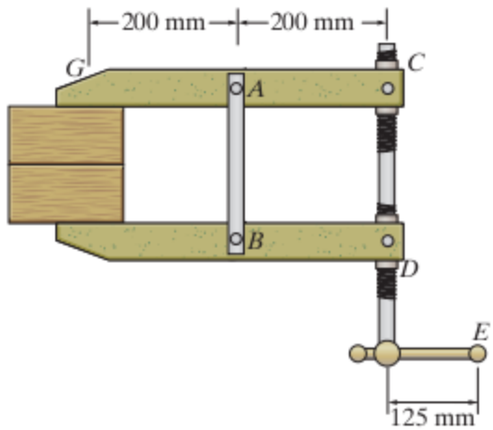
Prob. 4–74

4–75. Determine the largest weight of the wedge that can be placed between the 8-lb cylinder and the wall without upsetting equilibrium. The coefficient of static friction at A and C is $\mu_s = 0.5$ and at B, $\mu_s' = 0.6$.



Prob. 4–75

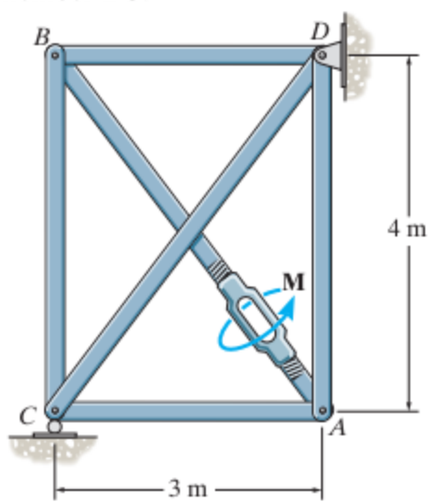
*4–76. If a horizontal force of $F = 50$ N is applied perpendicular to the handle of the lever at E, determine the clamping force developed at G. The mean diameter and lead of the single square-threaded screw at C and D are 25 mm and 5 mm, respectively. The coefficient of static friction is $\mu_s = 0.3$.



Prob. 4–76

4–77. A turnbuckle, similar to that shown in Fig. 4–36, is used to tension member AB of the truss. The coefficient of the static friction between the square threaded screws and the turnbuckle is $\mu_s = 0.5$. The screws have a mean radius of 6 mm and a lead of 3 mm. If a torque of $M = 10\ \text{N}\cdot\text{m}$ is applied to the turnbuckle to draw the screws closer together, determine the force in each member of the truss. No external forces act on the truss.

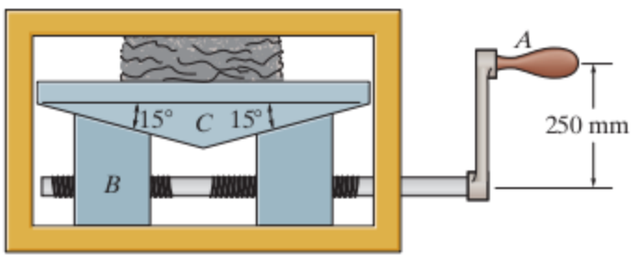
4–78. A turnbuckle, similar to that shown in Fig. 4–36, is used to tension member AB of the truss. The coefficient of the static friction between the square-threaded screws and the turnbuckle is $\mu_s = 0.5$. The screws have a mean radius of 6 mm and a lead of 3 mm. Determine the torque M which must be applied to the turnbuckle to draw the screws closer together, so that the compressive force of 500 N is developed in member BC.



Probs. 4–77/78

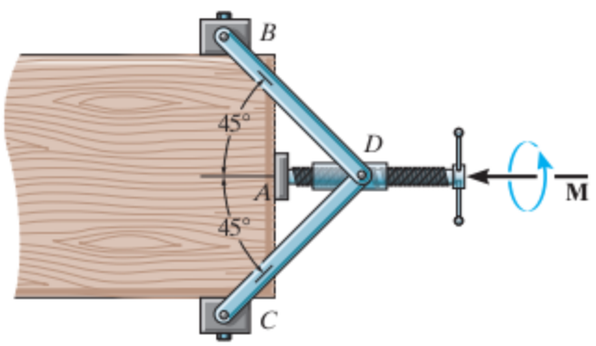
4–79. If a horizontal force of $P = 100$ N is applied perpendicular to the handle of the lever at A, determine the compressive force **F** exerted on the material. Each single square-threaded screw has a mean diameter of 25 mm and a lead of 7.5 mm. The coefficient of static friction at all contacting surfaces of the wedges is $\mu_s = 0.2$, and the coefficient of static friction at the screw is $\mu_s' = 0.15$.

***4–80.** Determine the horizontal force **P** that must be applied perpendicular to the handle of the lever at A in order to develop a compressive force of 12 kN on the material. Each single square-threaded screw has a mean diameter of 25 mm and a lead of 7.5 mm. The coefficient of static friction at all contacting surfaces of the wedges is $\mu_s = 0.2$, and the coefficient of static friction at the screw is $\mu_s' = 0.15$.



Probs. 4–79/80

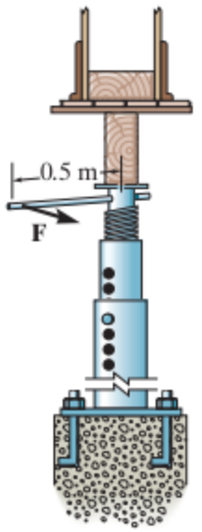
4–81. The clamp provides pressure from several directions on the edges of the board. If the square-threaded screw has a lead of 3 mm, mean radius of 10 mm, and the coefficient of static friction is $\mu_s = 0.4$, determine the horizontal force developed on the board at A and the vertical forces developed at B and C if a torque of $M = 1.5$ N · m is applied to the handle to tighten it further. The blocks at B and C are pin connected to the board.



Prob. 4–81

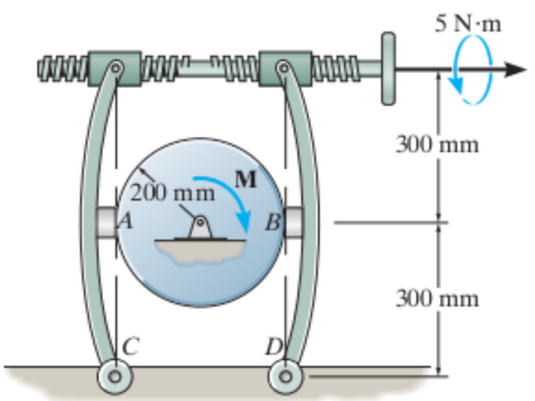
4–82. The column is used to support the upper floor. If a force $F = 80$ N is applied perpendicular to the handle to tighten the screw, determine the compressive force in the column. The square-threaded screw on the jack has a coefficient of static friction of $\mu_s = 0.4$, mean diameter of 25 mm, and a lead of 3 mm.

4–83. If the force **F** is removed from the handle of the jack in Prob. 4–82, determine if the screw is self-locking.



Probs. 4–82/83

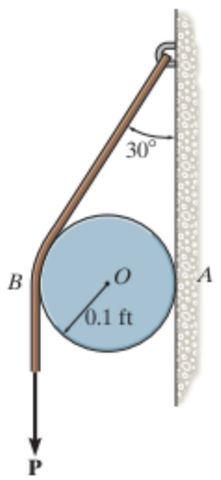
***4–84.** The braking mechanism consists of two pinned arms and a square-threaded screw with left- and right-hand threads. Thus when turned, the screw draws the two arms together. If the lead of the screw is 4 mm, the mean diameter 12 mm, and the coefficient of static friction is $\mu_s = 0.35$, determine the tension in the screw when a torque of 5 N · m is applied to tighten the screw. If the coefficient of static friction between the brake pads A and B and the circular shaft is $\mu_s' = 0.5$, determine the maximum torque M the brake can resist.



Prob. 4–84

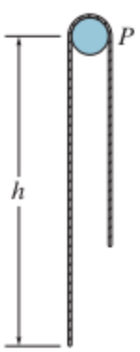
4–85. A minimum force of $P = 50$ lb is required to hold the cylinder from slipping against the belt and the wall. Determine the weight of the cylinder if the coefficient of friction between the belt and cylinder is $\mu_s = 0.3$ and slipping does not occur at the wall.

4–86. The cylinder weighs 10 lb and is held in equilibrium by the belt and wall. If slipping does not occur at the wall, determine the minimum vertical force P which must be applied to the belt for equilibrium. The coefficient of static friction between the belt and the cylinder is $\mu_s = 0.25$.



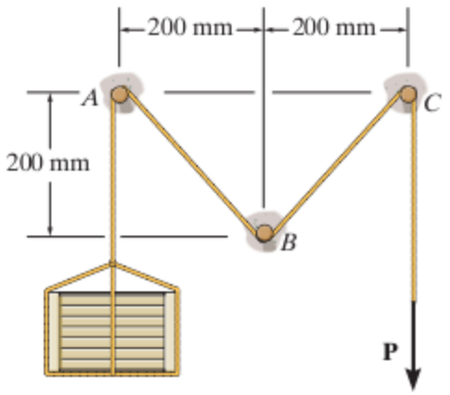
Probs. 4–85/86

4–87. A cord having a weight of 0.5 lb/ft and a total length of 10 ft is suspended over a peg P as shown. If the coefficient of static friction between the peg and cord is $\mu_s = 0.5$, determine the longest length h which one side of the suspended cord can have without causing motion. Neglect the size of the peg and the length of cord draped over it.



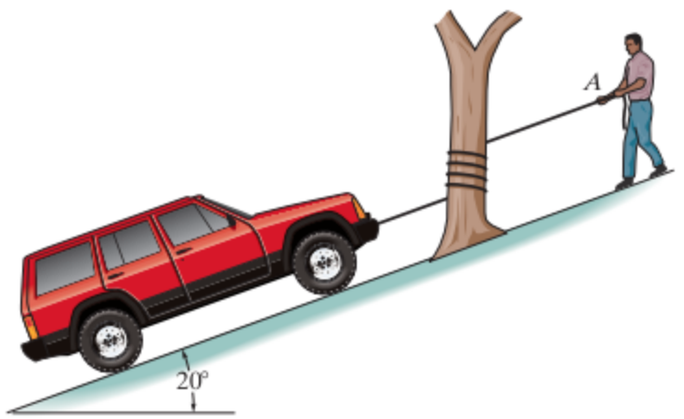
Prob. 4–87

***4–88.** Determine the smallest force **P** required to lift the 40-kg crate. The coefficient of static friction between the cable and each peg is $\mu_s = 0.1$.



Prob. 4–88

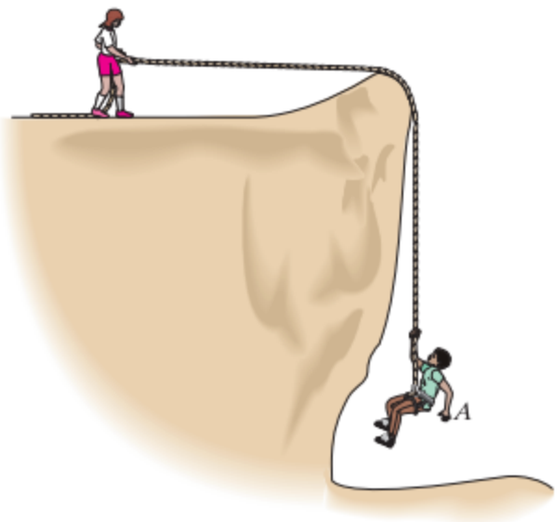
4–89. The truck, which has a mass of 3.4 Mg, is to be lowered down the slope by a rope that is wrapped around a tree. If the wheels are free to roll and the man at A can resist a pull of 300 N, determine the minimum number of turns the rope should be wrapped around the tree to lower the truck at a constant speed. The coefficient of kinetic friction between the tree and rope is $\mu_k = 0.3$.



Prob. 4–89

4–90. The 100-lb boy at A is suspended from the cable that passes over the quarter circular cliff rock. Determine if it is possible for the 185-lb woman to hoist him up; and if this is possible, what smallest force must she exert on the horizontal cable? The coefficient of static friction between the cable and the rock is $\mu_s = 0.2$, and between the shoes of the woman and the ground $\mu_s' = 0.8$.

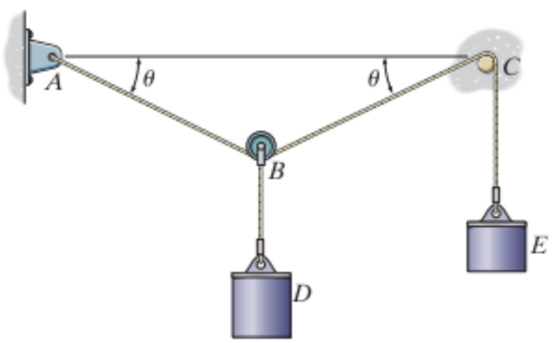
4–91. The 100-lb boy at A is suspended from the cable that passes over the quarter circular cliff rock. What horizontal force must the woman at A exert on the cable in order to let the boy descend at constant velocity? The coefficients of static and kinetic friction between the cable and the rock are $\mu_s = 0.4$ and $\mu_k = 0.35$, respectively.



Probs. 4–90/91

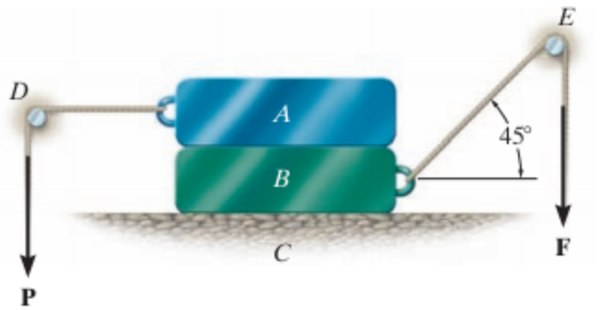
*4–92. A 10-kg cylinder D, which is attached to a small pulley B, is placed on the cord as shown. Determine the smallest angle θ so that the cord does not slip over the peg at C. The cylinder at E has a mass of 10 kg, and the coefficient of static friction between the cord and the peg is $\mu_s = 0.1$.

4–93. A 10-kg cylinder D, which is attached to a small pulley B, is placed on the cord as shown. Determine the largest angle θ so that the cord does not slip over the peg at C. The cylinder at E has a mass of 10 kg, and the coefficient of static friction between the cord and the peg is $\mu_s = 0.1$.



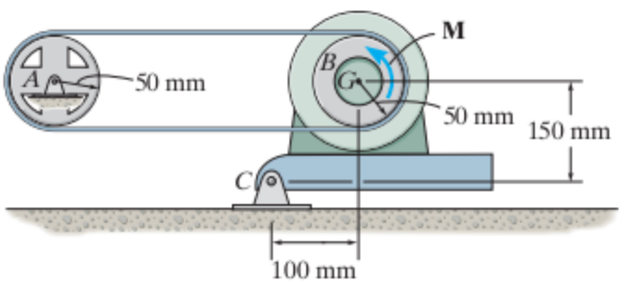
Probs. 4–92/93

4–94. Blocks A and B have a mass of 100 kg and 150 kg, respectively. If the coefficient of static friction between A and B and between B and C is $\mu_s = 0.25$ and between the ropes and the pegs D and E $\mu_s' = 0.5$ determine the smallest force F needed to cause motion of block B if $P = 30$ N.



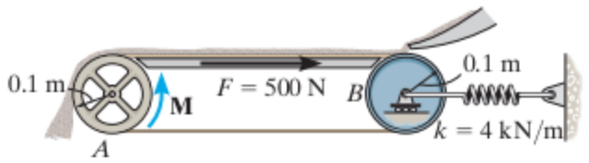
Prob. 4–94

4–95. The 20-kg motor has a center of gravity at G and is pin connected at C to maintain a tension in the drive belt. Determine the smallest counterclockwise twist or torque **M** that must be supplied by the motor to turn the disk B if wheel A locks and causes the belt to slip over the disk. No slipping occurs at A. The coefficient of static friction between the belt and the disk is $\mu_s = 0.3$.



Prob. 4–95

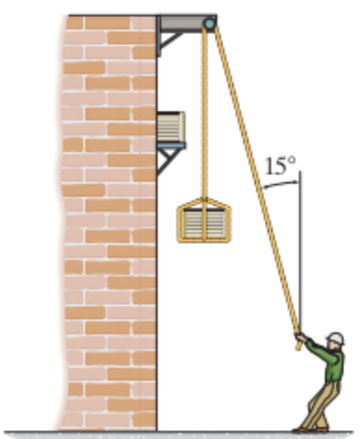
*4–96. A conveyer belt is used to transfer granular material and the frictional resistance on the top of the belt is $F = 500$ N. Determine the smallest stretch of the spring attached to the moveable axle of the idle pulley B so that the belt does not slip at the drive pulley A when the torque **M** is applied. What minimum torque **M** is required to keep the belt moving? The coefficient of static friction between the belt and the wheel at A is $\mu_s = 0.2$.



Prob. 4–96

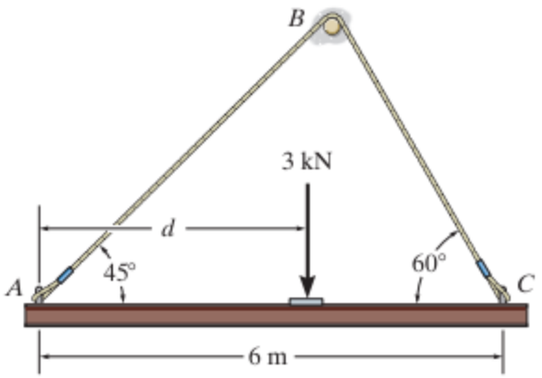
4–97. The 80-kg man tries to lower the 150-kg crate using a rope that passes over the rough peg. Determine the least number of full turns in addition to the basic wrap (165°) around the peg to do the job. The coefficients of static friction between the rope and the peg and between the man's shoes and the ground are $\mu_s = 0.1$ and $\mu'_s = 0.4$, respectively.

4–98. If the rope wraps three full turns plus the basic wrap (165°) around the peg, determine if the 80-kg man can keep the 300-kg crate from moving. The coefficients of static friction between the rope and the peg and between the man's shoes and the ground are $\mu_s = 0.1$ and $\mu'_s = 0.4$, respectively.



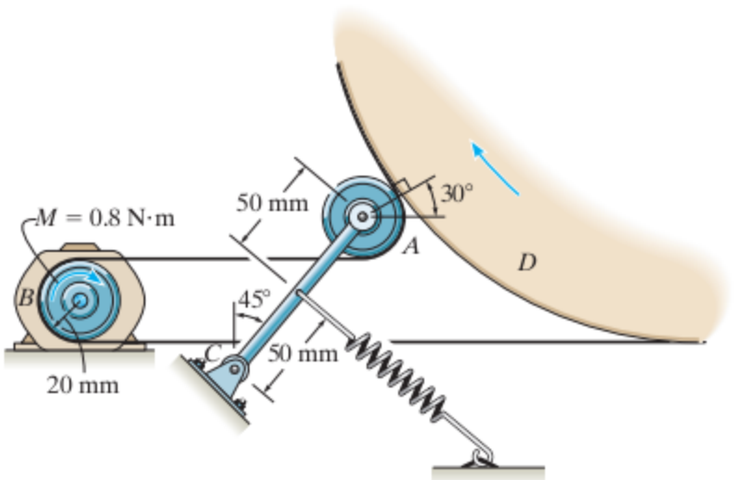
Probs. 4–97/98

4–99. Determine the minimum coefficient of static friction μ_s between the cable and the peg and the placement d of the 3-kN force for the uniform 100-kg beam to maintain equilibrium.



Prob. 4–99

*4–100. The belt on the portable dryer wraps around the drum D, idler pulley A, and motor pulley B. If the motor can develop a maximum torque of $M = 0.80$ N·m, determine the smallest spring tension required to hold the belt from slipping. The coefficient of static friction between the belt and the drum and motor pulley is $\mu_s = 0.3$.



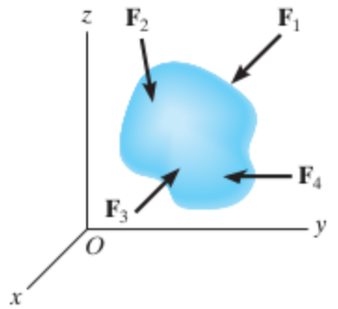
Prob. 4–100

CHAPTER REVIEW

Equilibrium

A body in equilibrium does not rotate but can translate with constant velocity, or it does not move at all.

$$\Sigma \mathbf{F} = \mathbf{0}$$
$$\Sigma \mathbf{M} = \mathbf{0}$$



Two Dimensions

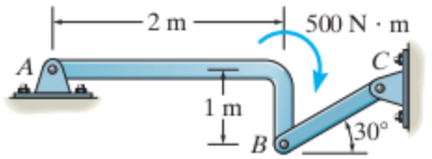
Before analyzing the equilibrium of a body, it is first necessary to draw its free-body diagram. This is an outlined shape of the body, which shows all the forces and couple moments that act on it.

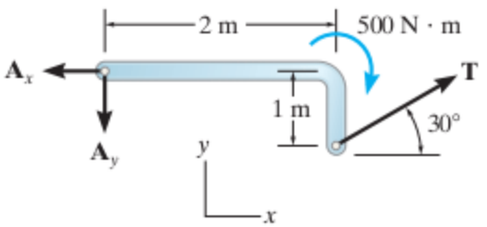
Couple moments can be placed anywhere on a free-body diagram since they are free vectors. Forces can act at any point along their line of action since they are sliding vectors.

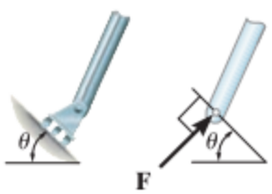
Angles used to resolve forces, and dimensions used to take moments of the forces, should also be shown on the free-body diagram.

Some common types of supports and their reactions are shown below in two dimensions.

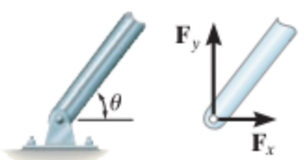
Remember that a support will exert a force on the body in a particular direction if it prevents translation of the body in that direction, and it will exert a couple moment on the body if it prevents rotation.



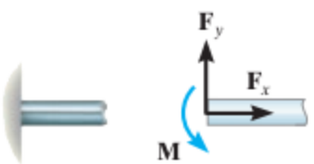




roller



smooth pin or hinge



fixed support

The three scalar equations of equilibrium can be applied when solving problems in two dimensions, since the geometry is easy to visualize.

$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$
$$\Sigma M_O = 0$$

For the most direct solution, try to sum forces along an axis that will eliminate as many unknown forces as possible. Sum moments about a point A that passes through the line of action of as many unknown forces as possible.

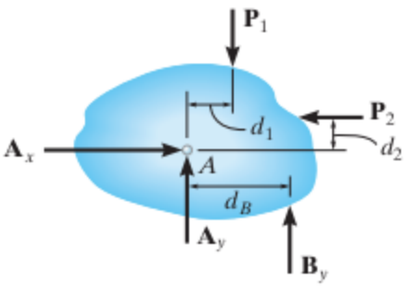
$$\Sigma F_x = 0;$$

$$A_x - P_2 = 0 \quad A_x = P_2$$

$$\Sigma M_A = 0;$$

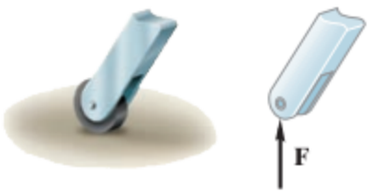
$$P_2 d_2 + B_y d_B - P_1 d_1 = 0$$

$$B_y = \frac{P_1 d_1 - P_2 d_2}{d_B}$$



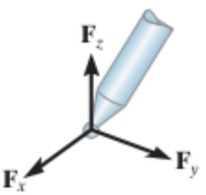
Three Dimensions

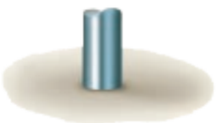
Some common types of supports and their reactions are shown here in three dimensions.



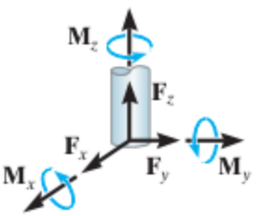
roller

ball and socket





fixed support



In three dimensions, it is often advantageous to use a Cartesian vector analysis when applying the equations of equilibrium. To do this, first express each known and unknown force and couple moment shown on the free-body diagram as a Cartesian vector. Then set the force summation equal to zero. Take moments about a point O that lies on the line of action of as many unknown force components as possible. From point O direct position vectors to each force, and then use the cross product to determine the moment of each force.

The six scalar equations of equilibrium are established by setting the respective **i**, **j**, and **k** components of these force and moment summations equal to zero.

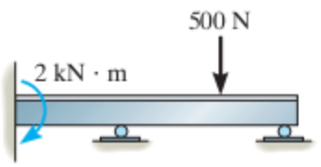
$$\Sigma \mathbf{F} = \mathbf{0}$$
$$\Sigma \mathbf{M}_O = \mathbf{0}$$

$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$
$$\Sigma F_z = 0$$
$$\Sigma M_x = 0$$
$$\Sigma M_y = 0$$
$$\Sigma M_z = 0$$

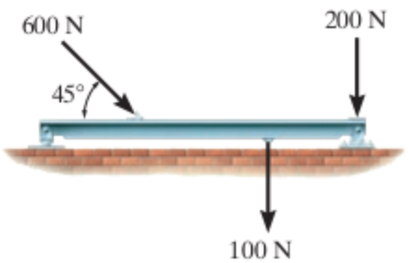
Determinacy and Stability

If a body is supported by a minimum number of constraints to ensure equilibrium, then it is statically determinate. If it has more constraints than required, then it is statically indeterminate.

To properly constrain the body, the reactions must not all be parallel to one another or concurrent.



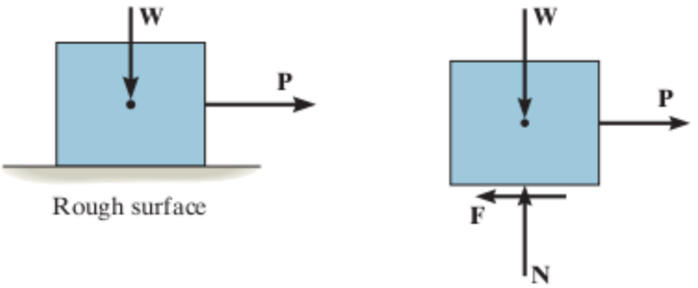
Statically indeterminate, five reactions, three equilibrium equations



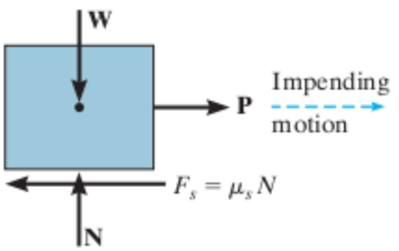
Proper constraint, statically determinate

Dry Friction

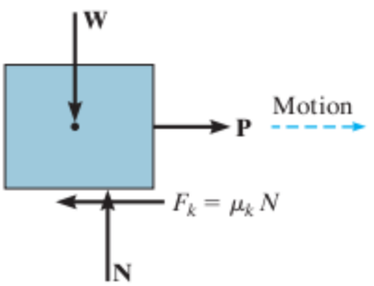
Frictional forces exist between two rough surfaces of contact. These forces act on a body so as to oppose its motion or tendency of motion.



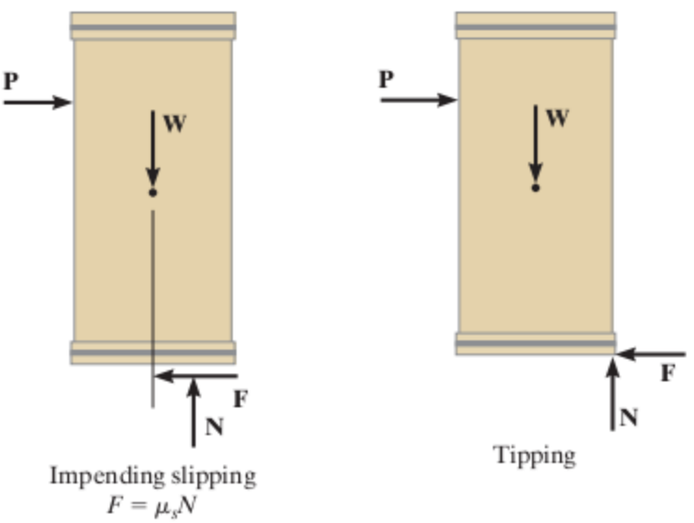
A static frictional force approaches a maximum value of $F_s = \mu_s N$, where μ_s is the *coefficient of static friction*. In this case, motion between the contacting surfaces is impending.



If slipping occurs, then the friction force remains essentially constant and equal to $F_k = \mu_k N$. Here μ_k is the *coefficient of kinetic friction*.



The solution of a problem involving friction requires first drawing the free-body diagram of the body. If the unknowns cannot be determined strictly from the equations of equilibrium, and the possibility of slipping occurs, then the friction equation should be applied at the appropriate points of contact in order to complete the solution.



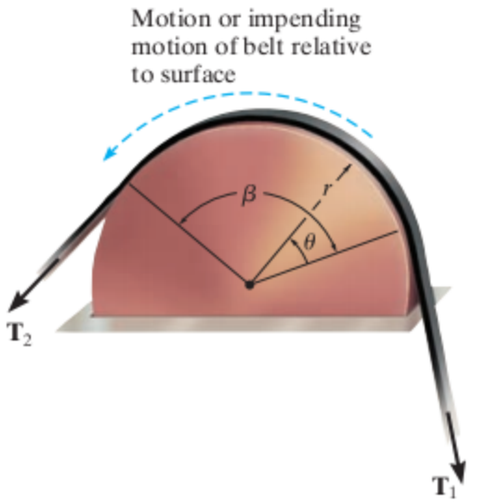
It may also be possible for slender objects, like crates, to tip over, and this situation should also be investigated.

Flat Belts

The force needed to move a flat belt over a rough curved surface depends only on the angle of belt contact, β, and the coefficient of friction.

$$T_2 = T_1 e^{\mu\beta}$$

$$T_2 > T_1$$



Screws

Square-threaded screws are used to move heavy loads. They represent an inclined plane, wrapped around a cylinder.

The moment needed to turn a screw depends upon the coefficient of friction and the screw's lead angle θ.

If the coefficient of friction between the surfaces is large enough, then the screw will support the load without tending to turn, i.e., it will be self-locking.

$$M = rW\tan(\phi_s + \theta)$$

Upward Impending Screw Motion

$$M' = rW\tan(\theta - \phi_s)$$

Downward Impending Screw Motion

$$\theta > \phi_s$$

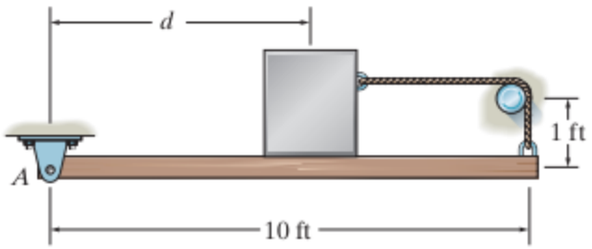
$$M'' = rW\tan(\phi_s - \theta)$$

Downward Impending Screw Motion

$$\phi_s > \theta$$

REVIEW PROBLEMS

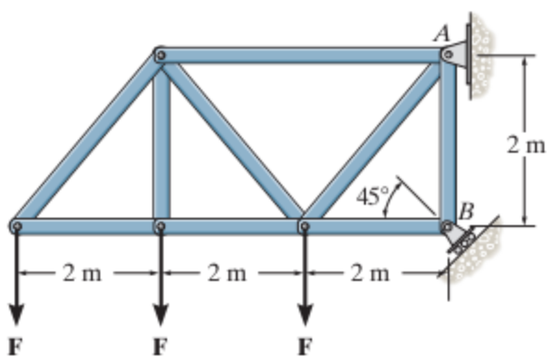
4–101. The uniform 50-lb beam is supported by the rope that is attached to the end of the beam, wraps over the rough peg, and is then connected to the 100-lb block. If the coefficient of static friction between the beam and the block, and between the rope and the peg, is $\mu_s = 0.4$, determine the maximum distance that the block can be placed from A and still remain in equilibrium. Assume the block will not tip.



Prob. 4–101

4–102. Determine the horizontal and vertical components of reaction at the pin A and the reaction at the roller B required to support the truss. Set $F = 600$ N.

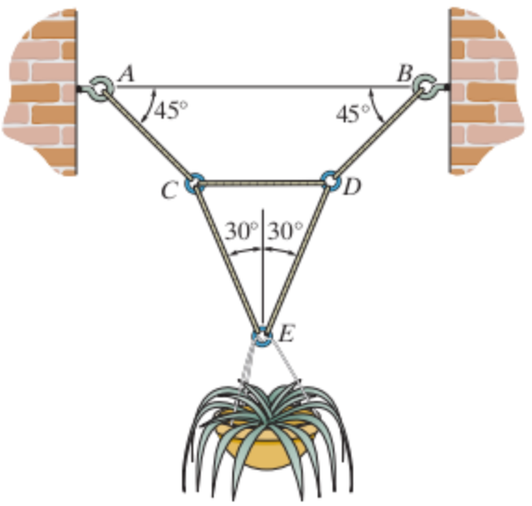
4–103. If the roller at B can sustain a maximum load of 3 kN, determine the largest magnitude of each of the three forces **F** that can be supported by the truss.



Probs. 4–102/103

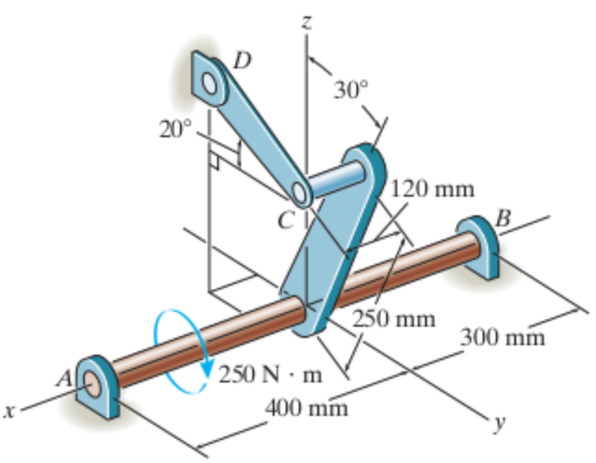
*4–104. Determine the tension developed in each wire which is needed to support the 50-lb flowerpot.

4–105. If the tension developed in each of the wires is not allowed to exceed 40 lb, determine the maximum weight of the flowerpot that can be safely supported.



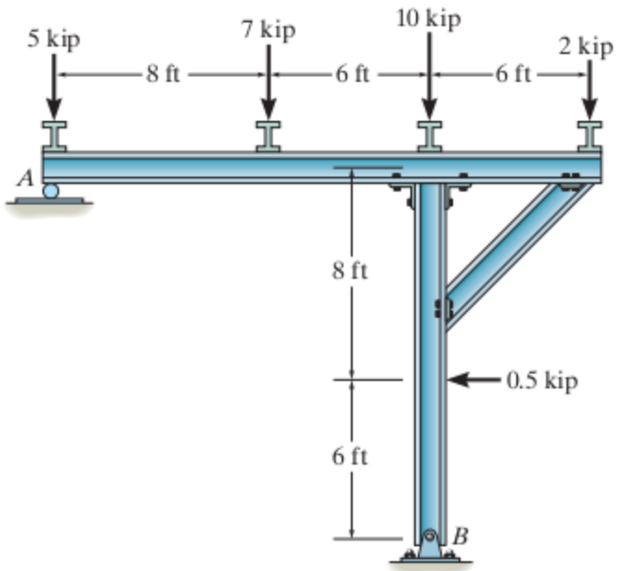
Probs. 4–104/105

4–106. The shaft assembly is supported by two smooth journal bearings A and B and a short link DC. If a couple moment is applied to the shaft as shown, determine the components of force reaction at the journal bearings and the force in the link. The link lies in a plane parallel to the y–z plane and the bearings are properly aligned on the shaft.



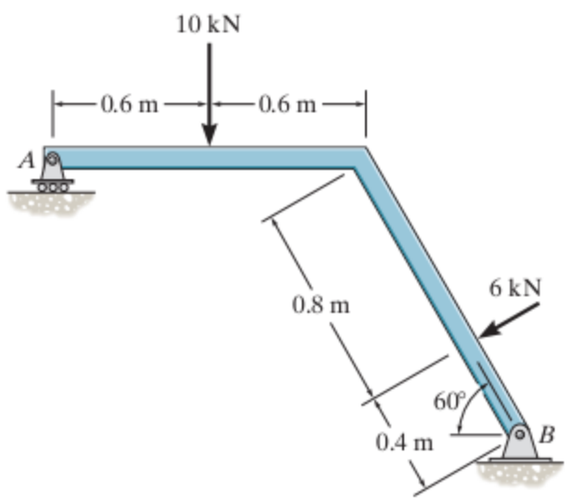
Prob. 4–106

4–107. Determine the reactions at the supports A and B of the frame.



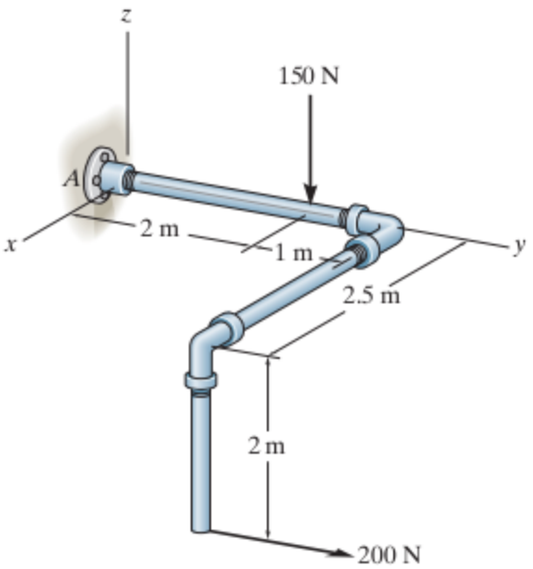
Prob. 4–107

***4–108.** Determine the normal reaction at the roller A and horizontal and vertical components at pin B for equilibrium of the member.



Prob. 4–108

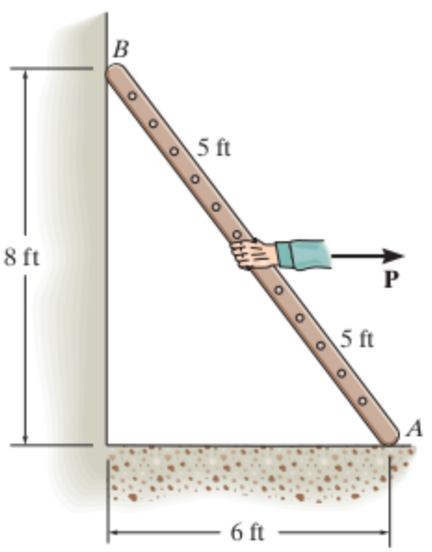
4–109. Determine the x, y, z components of reaction at the fixed wall A. The 150-N force is parallel to the z axis and the 200-N force is parallel to the y axis.



Prob. 4–109

4–110. The uniform 20-lb ladder rests on the rough floor for which the coefficient of static friction is $\mu_s = 0.8$ and against the smooth wall at B. Determine the horizontal force P the man must exert on the ladder in order to cause it to move.

4–111. The uniform 20-lb ladder rests on the rough floor for which the coefficient of static friction is $\mu_s = 0.4$ and against the smooth wall at B. Determine the horizontal force P the man must exert on the ladder in order to cause it to move.



Probs. 4–110/111

Chapter 5

In order to design the many parts of this boom assembly it is required that we know the forces that they must support. In this chapter we will show how to analyze such structures using the equations of equilibrium.

Structural Analysis

CHAPTER OBJECTIVES

- To show how to determine the forces in the members of a truss using the method of joints and the method of sections.
- To analyze the forces acting on the members of frames and machines composed of pin-connected members.

5.1 Simple Trusses

A *truss* is a structure composed of slender members joined together at their end points. The members commonly used in construction consist of wooden struts or metal bars. In particular, *planar* trusses lie in a single plane and are often used to support roofs and bridges. The truss shown in Fig. 5–1*a* is an example of a typical roof-supporting truss. In this figure, the roof load is transmitted to the truss *at the joints* by means of a series of *purlins*. Since this loading acts in the same plane as the truss, Fig. 5–1*b*, the analysis of the forces developed in the truss members will be two-dimensional.

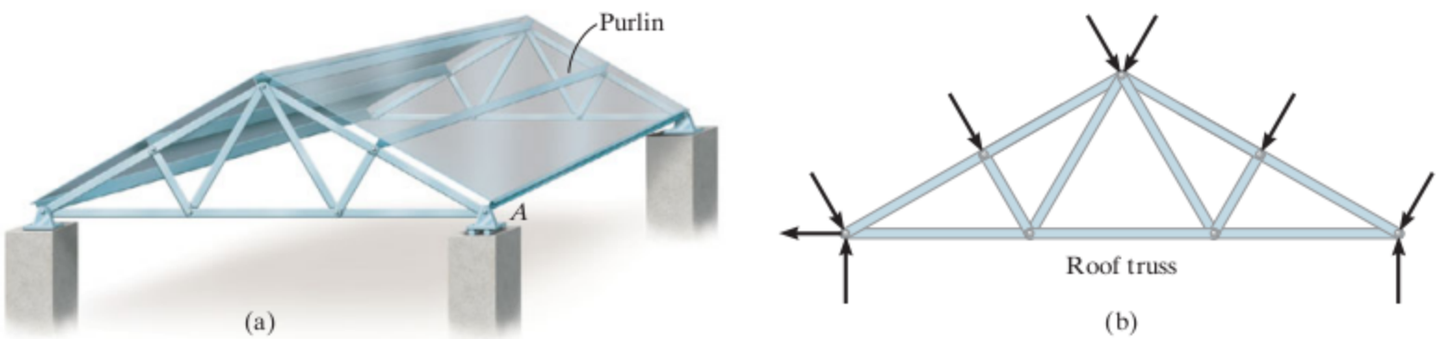


Fig. 5–1

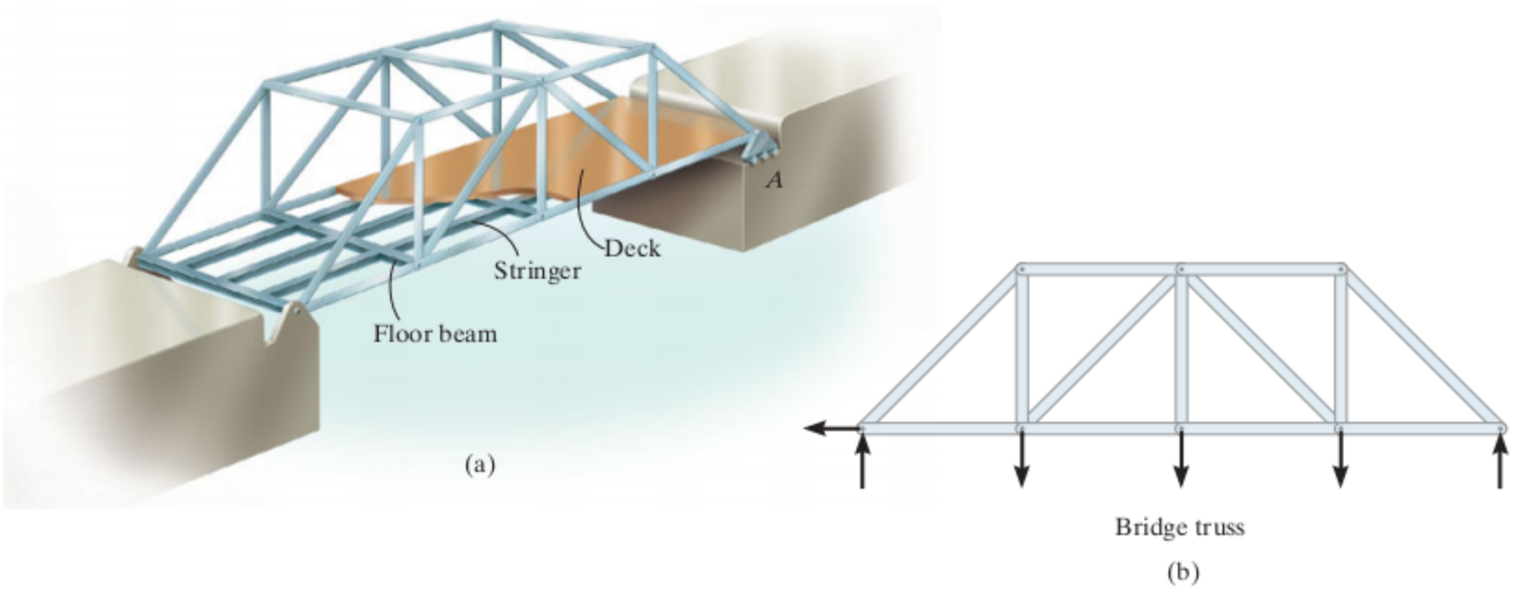
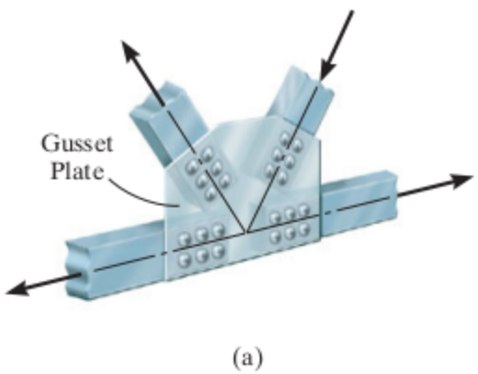


Fig. 5–2



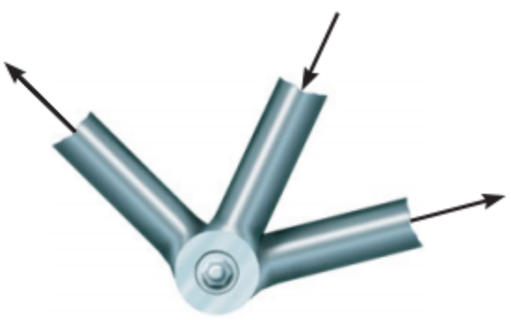


Fig. 5–3

In the case of a bridge, such as shown in Fig. 5–2*a*, the load on the deck is first transmitted to *stringers*, then to *floor beams*, and finally to the *joints* of the two supporting side trusses. Like the roof truss, the bridge truss loading is also coplanar, Fig. 5–2*b*.

When bridge or roof trusses extend over large distances, a rocker or roller is commonly used for supporting one end, for example, joint *A* in Figs. 5–1*a* and 5–2*a*. This type of support allows freedom for expansion or contraction of the members due to a change in temperature or application of loads.

Assumptions for Design. To design both the members and the connections of a truss, it is necessary first to determine the *force* developed in each member when the truss is subjected to a given loading. To do this we will make two important assumptions:

- ***All loadings are applied at the joints.*** In most situations, such as for bridge and roof trusses, this assumption is true. Frequently the weight of the members is neglected because the force supported by each member is usually much larger than its weight. However, if the weight is to be included in the analysis, it is generally satisfactory to apply it as a vertical force, with half of its magnitude applied at each end of the member.
- ***The members are joined together by smooth pins.*** The joint connections are usually formed by bolting or welding the ends of the members to a common plate, called a *gusset plate*, as shown in Fig. 5–3*a*, or by simply passing a large bolt or pin through each of the members, Fig. 5–3*b*. We can assume these connections act as pins provided the center lines of the joining members are *concurrent*, as in Fig. 5–3.

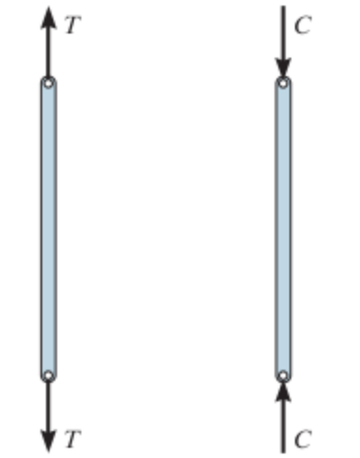


Fig. 5–4

Because of these two assumptions, *each truss member will act as a two-force member*, and therefore the force acting at each end of the member will be directed along the axis of the member. If the force tends to *elongate* the member, it is a *tensile force* (T), Fig. 5–4*a*; whereas if it tends to *shorten* the member, it is a *compressive force* (C), Fig. 5–4*b*. In the actual design of a truss it is important to state whether the nature of the force is tensile or compressive. Often, compression members must be made *thicker* than tension members because of the buckling or column effect that occurs when a member is in compression.

The use of metal gusset plates in the construction of these Warren trusses is clearly evident.

Simple Truss. If three members are pin connected at their ends, they form a *triangular truss* that will be *rigid*, Fig. 5–5. Attaching two more members and connecting these members to a new joint *D* forms a larger truss, Fig. 5–6. This procedure can be repeated as many times as desired to form an even larger truss. If a truss can be constructed by expanding the basic triangular truss in this way, it is called a *simple truss*.

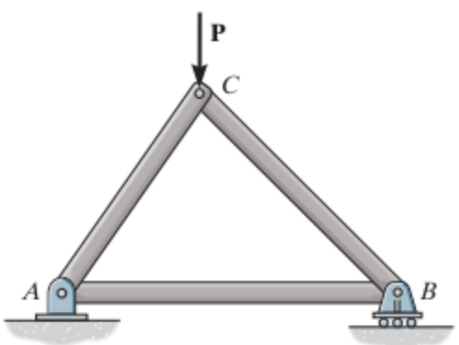


Fig. 5–5

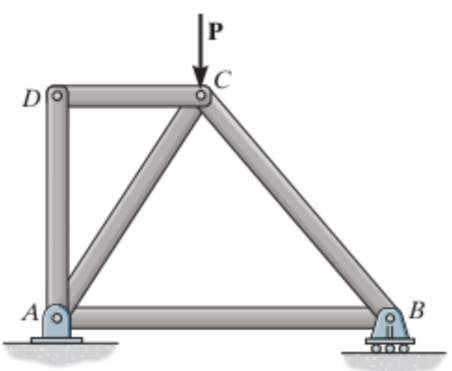


Fig. 5–6

5.2 The Method of Joints

In order to analyze or design a truss, it is necessary to determine the force in each of its members. One way to do this is to use the method of joints. This method is based on the fact that if the entire truss is in equilibrium, then each of its joints is also in equilibrium. Therefore, if the free-body diagram of each joint is drawn, the force equilibrium equations can then be used to obtain the member forces acting on each joint. Since the members of a *plane truss* are straight two-force members lying in a single plane, each joint is subjected to a force system that is *coplanar and concurrent*. As a result, only $\Sigma F_x = 0$ and $\Sigma F_y = 0$ need to be satisfied for equilibrium.

For example, consider the pin at joint B of the truss in Fig. 5–7*a*. Three forces act on the pin, namely, the 500-N force and the forces exerted by members BA and BC. The free-body diagram of the pin is shown in Fig. 5–7*b*. Here, $\mathbf{F}_{BA}$ is "pulling" on the pin, which means that member BA is in *tension;* whereas $\mathbf{F}_{BC}$ is "pushing" on the pin, and consequently member BC is in *compression*. These effects are clearly demonstrated by isolating the joint with small segments of the member connected to the pin, Fig. 5–7*c*. The pushing or pulling on these small segments indicates the effect of the member being either in compression or tension.

When using the method of joints, always start at a joint having at least one known force and at most two unknown forces, as in Fig. 5–7*b*. In this way, application of $\Sigma F_x = 0$ and $\Sigma F_y = 0$ yields two algebraic equations which can be solved for the two unknowns. When applying these equations, the correct sense of an unknown member force can be determined using one of two possible methods.

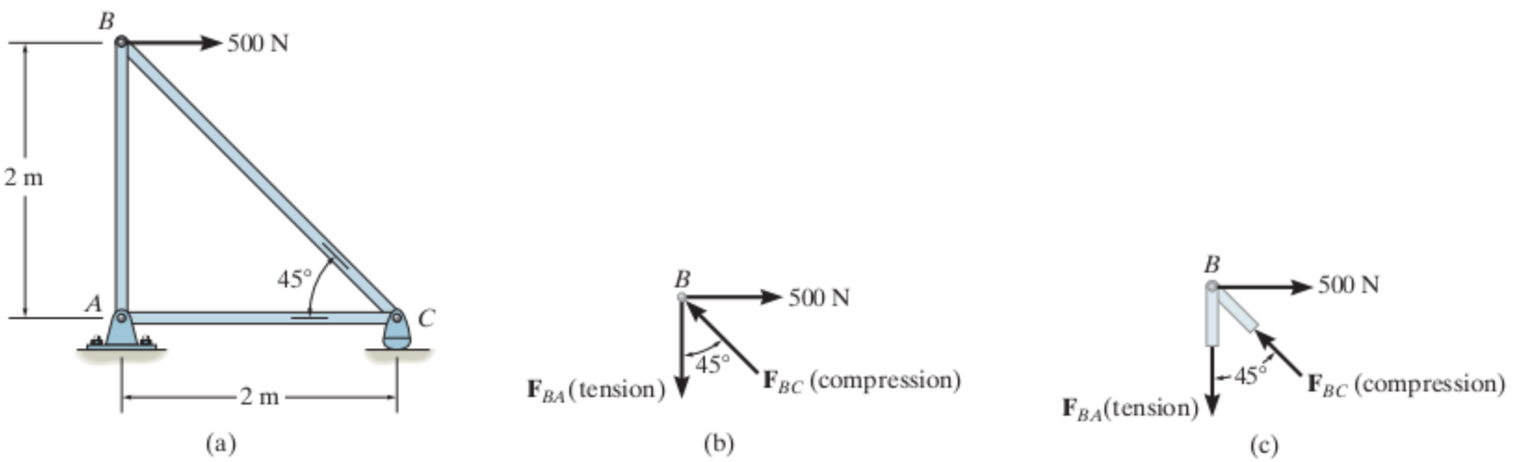


Fig. 5–7

- The *correct* sense of direction of an unknown member force can, in many cases, be determined "by inspection." For example, $\mathbf{F}_{BC}$ in Fig. 5–7*b* must push on the pin (compression) since its horizontal component, $F_{BC} \sin 45°$, must balance the 500-N force ($\Sigma F_x = 0$). Likewise, $\mathbf{F}_{BA}$ is a tensile force since it balances the vertical component, $F_{BC} \cos 45°$ ($\Sigma F_y = 0$). In more complicated cases, the sense of an unknown member force can be *assumed*; then, after applying the equilibrium equations, the assumed sense can be verified from the numerical results. A *positive* answer indicates that the sense is *correct*, whereas a *negative* answer indicates that the sense shown on the free-body diagram must be *reversed*.
- *Always assume* the *unknown member forces* acting on the joint's free-body diagram to be in *tension*; i.e., the forces "pull" on the pin. If this is done, then numerical solution of the equilibrium equations will yield *positive scalars for members in tension and negative scalars for members in compression*. Once an unknown member force is found, use its *correct* magnitude and sense (T or C) on subsequent joint free-body diagrams.

The forces in the members of this simple roof truss can be determined using the method of joints.

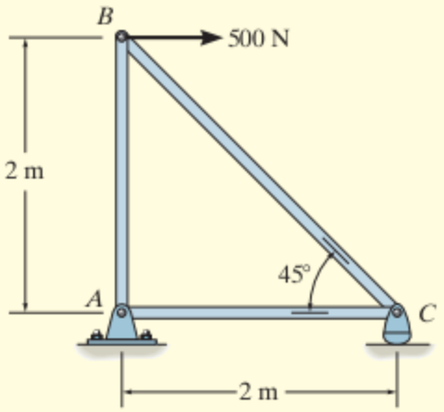
5

Procedure for Analysis

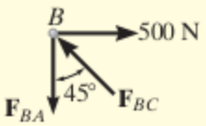
The following procedure provides a means for analyzing a truss using the method of joints.

- Draw the free-body diagram of a joint having at least one known force and at most two unknown forces. (If this joint is at one of the supports, then it may be necessary first to calculate the external reactions at the support.)
- Use one of the two methods described above for establishing the sense of an unknown force.
- Orient the x and y axes such that the forces on the free-body diagram can be easily resolved into their x and y components and then apply the two force equilibrium equations $\Sigma F_x = 0$ and $\Sigma F_y = 0$. Solve for the two unknown member forces and verify their correct sense.
- Using the calculated results, continue to analyze each of the other joints. Remember that a member in *compression* "pushes" on the joint and a member in *tension* "pulls" on the joint. Also, be sure to choose a joint having at most two unknowns and at least one known force.

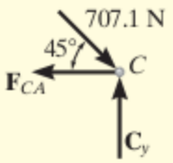
EXAMPLE 5.1



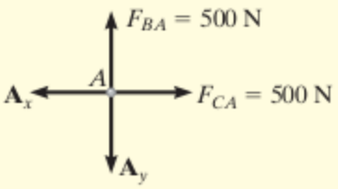
(a)



(b)



(c)



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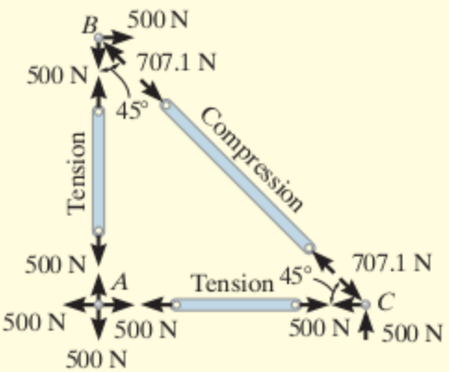


(e)

Fig. 5–8

Determine the force in each member of the truss shown in Fig. 5–8*a* and indicate whether the members are in tension or compression.

SOLUTION

Since we should have no more than two unknown forces at the joint and at least one known force acting there, we will begin our analysis at joint B.

Joint B. The free-body diagram of the joint at B is shown in Fig. 5–8*b*. Applying the equations of equilibrium, we have

$$\stackrel{+}{\rightarrow}\Sigma F_x = 0; \quad 500\ \text{N} - F_{BC}\sin 45° = 0 \quad F_{BC} = 707.1\ \text{N (C)} \quad \textit{Ans.}$$

$$+\uparrow\Sigma F_y = 0; \quad F_{BC}\cos 45° - F_{BA} = 0 \quad F_{BA} = 500\ \text{N (T)} \quad \textit{Ans.}$$

Since the force in member BC has been calculated, we can proceed to analyze joint C to determine the force in member CA and the support reaction at the rocker.

Joint C. From the free-body diagram of joint C, Fig. 5–8*c*, we have

$$\stackrel{+}{\rightarrow}\Sigma F_x = 0; \quad -F_{CA} + 707.1\cos 45°\ \text{N} = 0 \quad F_{CA} = 500\ \text{N (T)} \quad \textit{Ans.}$$

$$+\uparrow\Sigma F_y = 0; \quad C_y - 707.1\sin 45°\ \text{N} = 0 \quad C_y = 500\ \text{N} \quad \textit{Ans.}$$

Joint A. Although it is not necessary, we can determine the components of the support reactions at joint A using the results of F_{CA} and F_{BA}. From the free-body diagram, Fig. 5–8*d*, we have

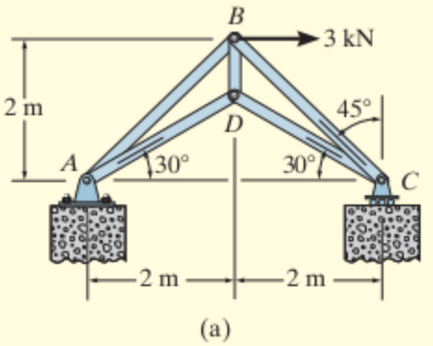
$$\stackrel{+}{\rightarrow}\Sigma F_x = 0; \quad 500\ \text{N} - A_x = 0 \quad A_x = 500\ \text{N}$$

$$+\uparrow\Sigma F_y = 0; \quad 500\ \text{N} - A_y = 0 \quad A_y = 500\ \text{N}$$

NOTE: The results of the analysis are summarized in Fig. 5–8*e*. Note that the free-body diagram of each joint (or pin) shows the effects of all the connected members and external forces applied to the joint, whereas the free-body diagram of each member shows only the effects of the end joints on the member.

EXAMPLE 5.2

Determine the forces acting in all the members of the truss shown in Fig. 5–9*a*.



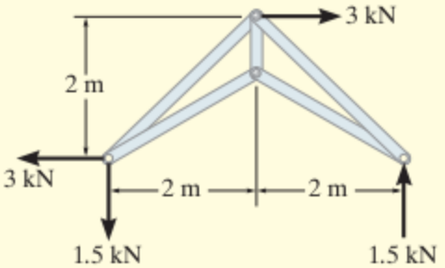
(a)

SOLUTION

By inspection, there are more than two unknowns at each joint. Consequently, the support reactions on the truss must first be determined. Show that they have been correctly calculated on the free-body diagram in Fig. 5–9*b*. We can now begin the analysis at joint *C*. Why?

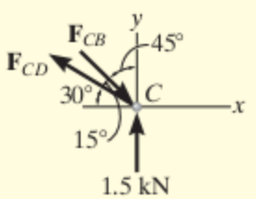
Joint C. From the free-body diagram, Fig. 5–9*c*,

$$\stackrel{+}{\rightarrow}\Sigma F_x = 0; \qquad -F_{CD}\cos 30° + F_{CB}\sin 45° = 0$$
$$+\uparrow \Sigma F_y = 0; \qquad 1.5\text{ kN} + F_{CD}\sin 30° - F_{CB}\cos 45° = 0$$



(b)

These two equations must be solved *simultaneously* for each of the two unknowns. Note, however, that a *direct solution* for one of the unknown forces may be obtained by applying a force summation along an axis that is *perpendicular* to the direction of the other unknown force. For example, summing forces along the y' axis, which is perpendicular to the direction of $\mathbf{F}_{CD}$, Fig. 5–9*d*, yields a *direct solution* for F_{CB}.

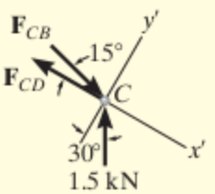


(c)

$$+\nearrow \Sigma F_{y'} = 0; \qquad 1.5\cos 30°\text{ kN} - F_{CB}\sin 15° = 0$$
$$F_{CB} = 5.019\text{ kN} = 5.02\text{ kN (C)} \qquad \textit{Ans.}$$

Then,

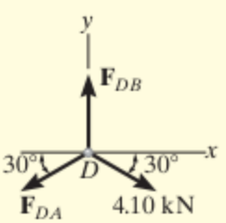
$$+\searrow \Sigma F_{x'} = 0;$$
$$-F_{CD} + 5.019\cos 15° - 1.5\sin 30° = 0;\ F_{CD} = 4.10\text{ kN (T)} \qquad \textit{Ans.}$$



(d)

Joint D. We can now proceed to analyze joint *D*. The free-body diagram is shown in Fig. 5–9*e*.

$$\stackrel{+}{\rightarrow}\Sigma F_x = 0; \qquad -F_{DA}\cos 30° + 4.10\cos 30°\text{ kN} = 0$$
$$F_{DA} = 4.10\text{ kN} \quad \text{(T)} \qquad \textit{Ans.}$$
$$+\uparrow \Sigma F_y = 0; \qquad F_{DB} - 2(4.10\sin 30°\text{ kN}) = 0$$
$$F_{DB} = 4.10\text{ kN} \quad \text{(T)} \qquad \textit{Ans.}$$



(e)

Fig. 5–9

NOTE: The force in the last member, *BA*, can be obtained from joint *B* or joint *A*. As an exercise, draw the free-body diagram of joint *B*, sum the forces in the horizontal direction, and show that $F_{BA} = 0.776$ kN (C).

EXAMPLE 5.3

Determine the force in each member of the truss shown in Fig. 5–10*a*. Indicate whether the members are in tension or compression.

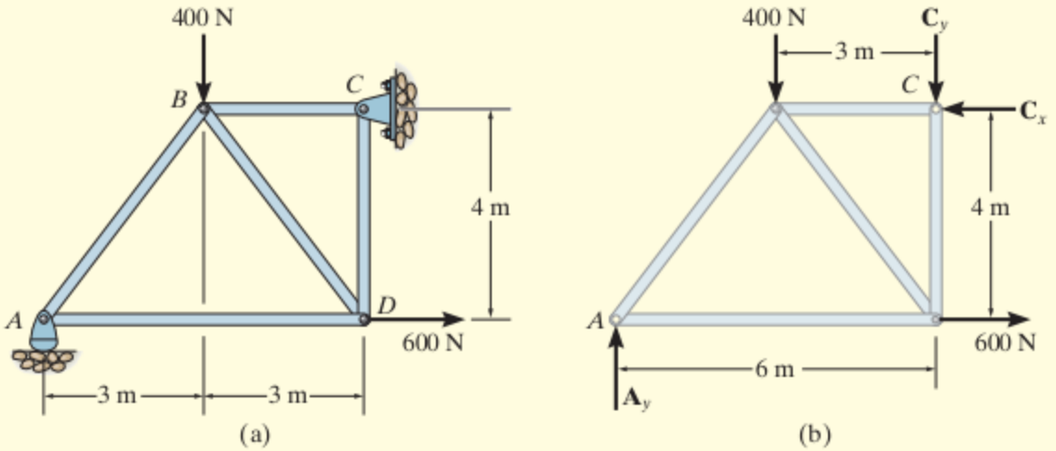


Fig. 5–10

SOLUTION

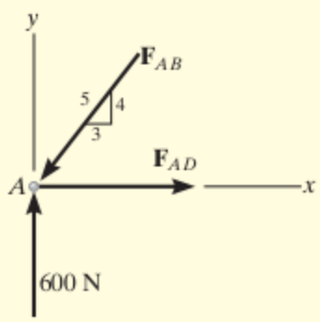
Support Reactions. No joint can be analyzed until the support reactions are determined, because each joint has at least three unknown forces acting on it. A free-body diagram of the entire truss is given in Fig. 5–10*b*. Applying the equations of equilibrium, we have

$$\stackrel{+}{\rightarrow}\Sigma F_x = 0; \qquad 600\ \text{N} - C_x = 0 \qquad C_x = 600\ \text{N}$$

$$\circlearrowleft +\Sigma M_C = 0; \qquad -A_y(6\ \text{m}) + 400\ \text{N}(3\ \text{m}) + 600\ \text{N}(4\ \text{m}) = 0$$

$$A_y = 600\ \text{N}$$

$$+\uparrow \Sigma F_y = 0; \qquad 600\ \text{N} - 400\ \text{N} - C_y = 0 \qquad C_y = 200\ \text{N}$$



(c)

The analysis can now start at either joint *A* or *C*. The choice is arbitrary since there are one known and two unknown member forces acting on the pin at each of these joints.

Joint A. (Fig. 5–10*c*). As shown on the free-body diagram, $\mathbf{F}_{AB}$ is assumed to be compressive and $\mathbf{F}_{AD}$ is tensile. Applying the equations of equilibrium, we have

$$+\uparrow \Sigma F_y = 0; \qquad 600\ \text{N} - \tfrac{4}{5}F_{AB} = 0 \qquad F_{AB} = 750\ \text{N}\ \text{(C)} \qquad \textit{Ans.}$$

$$\stackrel{+}{\rightarrow}\Sigma F_x = 0; \qquad F_{AD} - \tfrac{3}{5}(750\ \text{N}) = 0 \qquad F_{AD} = 450\ \text{N}\ \text{(T)} \qquad \textit{Ans.}$$

Joint *D*. (Fig. 5–10*d*). Using the result for F_{AD} and summing forces in the horizontal direction, Fig. 5–10*d*, we have

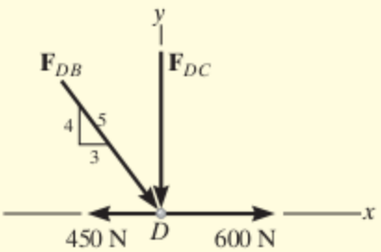
$$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \qquad -450\ \text{N} + \tfrac{3}{5}F_{DB} + 600\ \text{N} = 0 \qquad F_{DB} = -250\ \text{N}$$

The negative sign indicates that $\mathbf{F}_{DB}$ acts in the *opposite sense* to that shown in Fig. 5–10*d*.* Hence,

$$F_{DB} = 250\ \text{N (T)} \qquad \textit{Ans.}$$

To determine $\mathbf{F}_{DC}$, we can either correct the sense of $\mathbf{F}_{DB}$ on the free-body diagram, and then apply $\Sigma F_y = 0$, or apply this equation and retain the negative sign for F_{DB}, i.e.,

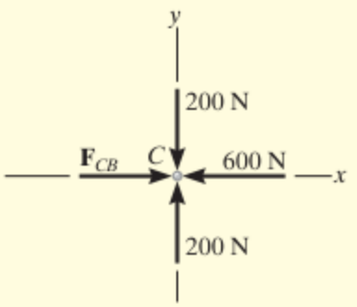
$$+\uparrow \Sigma F_y = 0; \qquad -F_{DC} - \tfrac{4}{5}(-250\ \text{N}) = 0 \qquad F_{DC} = 200\ \text{N (C)} \quad \textit{Ans.}$$



(d)

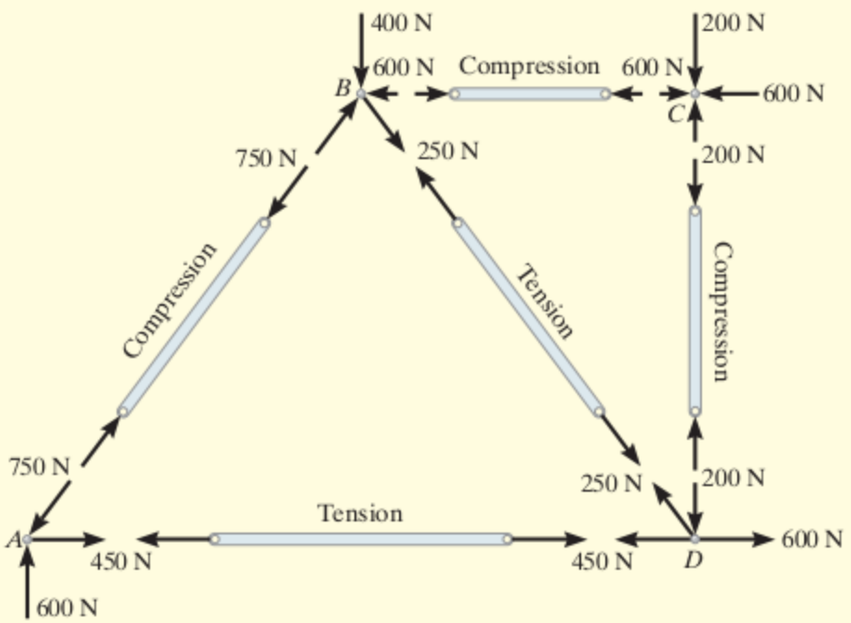
Joint *C*. (Fig. 5–10*e*).

$$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \qquad F_{CB} - 600\ \text{N} = 0 \qquad F_{CB} = 600\ \text{N (C)} \quad \textit{Ans.}$$
$$+\uparrow \Sigma F_y = 0; \qquad 200\ \text{N} - 200\ \text{N} \equiv 0 \quad \text{(check)}$$



(e)

NOTE: The analysis is summarized in Fig. 5–10*f*, which shows the free-body diagram for each joint and member.



(f)

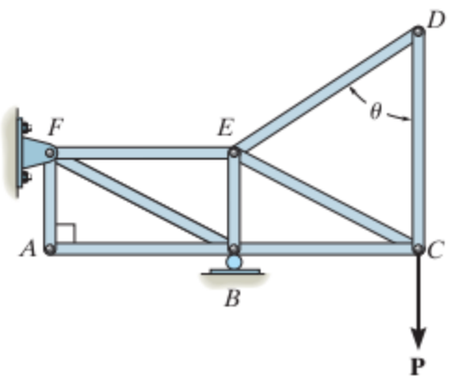
Fig. 5–10 (cont.)

*The proper sense could have been determined by inspection, prior to applying $\Sigma F_x = 0$.

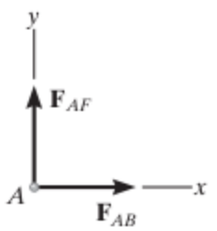
5.3 Zero-Force Members

Truss analysis using the method of joints is greatly simplified if we can first identify those members which support *no loading*. These *zero-force members* are used to increase the stability of the truss during construction and to provide added support if the loading is changed.

The zero-force members of a truss can generally be found *by inspection* of each of the joints. For example, consider the truss shown in Fig. 5–11*a*. If a free-body diagram of the pin at joint A is drawn, Fig. 5–11*b*, it is seen that members AB and AF are zero-force members. (We could not have come to this conclusion if we had considered the free-body diagrams of joints F or B simply because there are five unknowns at each of these joints.) In a similar manner, consider the free-body diagram of joint D, Fig. 5–11*c*. Here again it is seen that DC and DE are zero-force members. From these observations, we can conclude that *if only two non-collinear members form a truss joint and no external load or support reaction is applied to the joint, the two members must be zero-force members.* The load on the truss in Fig. 5–11*a* is therefore supported by only five members as shown in Fig. 5–11*d*.



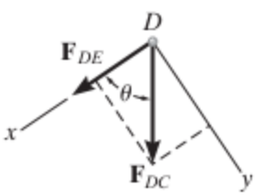
(a)



$\overset{+}{\rightarrow} \Sigma F_x = 0;\ F_{AB} = 0$

$+\uparrow \Sigma F_y = 0;\ F_{AF} = 0$

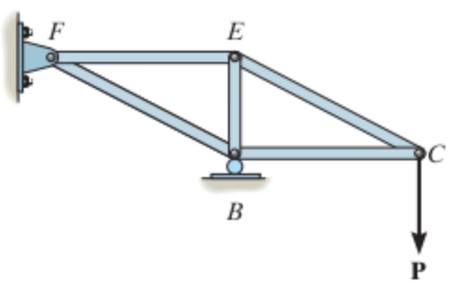
(b)



$+\searrow \Sigma F_y = 0;\ F_{DC} \sin\theta = 0;\quad F_{DC} = 0 \text{ since } \sin\theta \neq 0$

$+\swarrow \Sigma F_x = 0;\ F_{DE} + 0 = 0;\quad F_{DE} = 0$

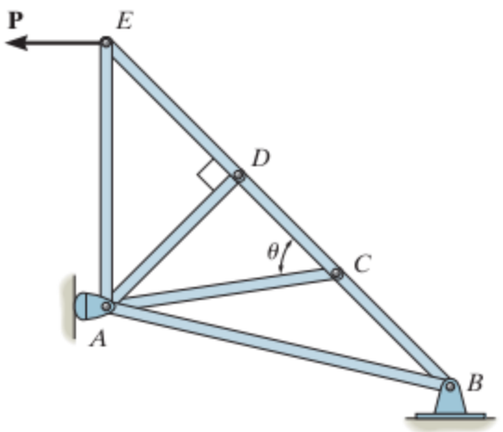
(c)



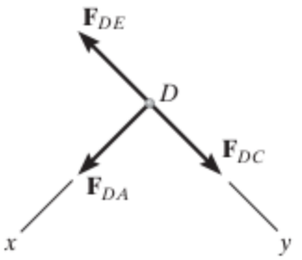
(d)

Fig. 5–11

Now consider the truss shown in Fig. 5–12*a*. The free-body diagram of the pin at joint D is shown in Fig. 5–12*b*. By orienting the y axis along members DC and DE and the x axis along member DA, it is seen that DA is a zero-force member. Note that this is also the case for member CA, Fig. 5–12*c*. In general then, *if three members form a truss joint for which two of the members are collinear, the third member is a zero-force member provided no external force or support reaction is applied to the joint.* The truss shown in Fig. 5–12*d* is therefore suitable for supporting the load **P**.



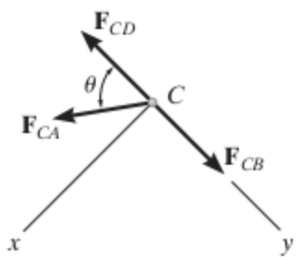
(a)



$+\swarrow \Sigma F_x = 0; \quad F_{DA} = 0$

$+\searrow \Sigma F_y = 0; \quad F_{DC} = F_{DE}$

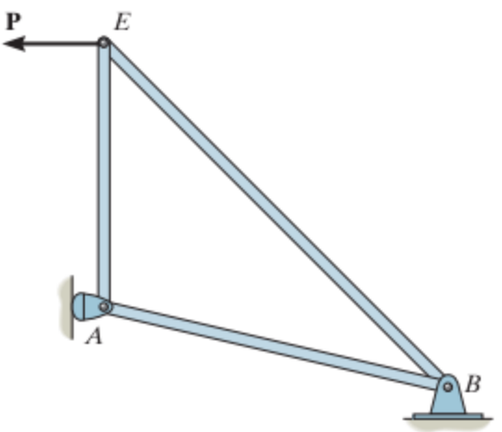
(b)



$+\swarrow \Sigma F_x = 0; \quad F_{CA} \sin\theta = 0; \quad F_{CA} = 0 \text{ since } \sin\theta \neq 0;$

$+\searrow \Sigma F_y = 0; \quad F_{CB} = F_{CD}$

(c)

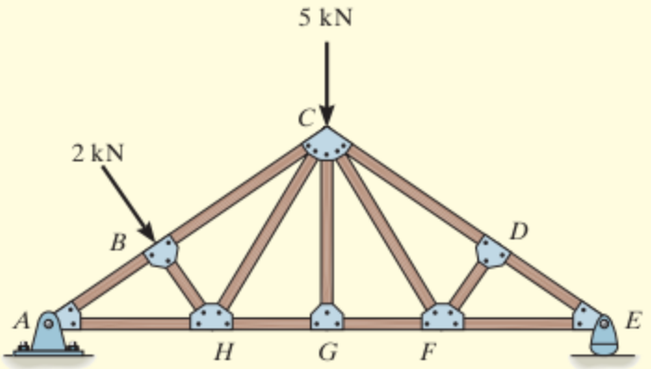


(d)

Fig. 5–12

EXAMPLE 5.4

Using the method of joints, determine all the zero-force members of the *Fink roof truss* shown in Fig. 5–13*a*. Assume all joints are pin connected.



(a)

Fig. 5–13

SOLUTION

Look for joint geometries that have three members for which two are collinear. We have

Joint G. (Fig. 5–13*b*).

$$+\uparrow \Sigma F_y = 0; \qquad F_{GC} = 0 \qquad \textit{Ans.}$$

Realize that we could not conclude that GC is a zero-force member by considering joint C, where there are five unknowns. The fact that GC is a zero-force member means that the 5-kN load at C must be supported by members CB, CH, CF, and CD.

Joint D. (Fig. 5–13*c*).

$$+\swarrow \Sigma F_x = 0; \qquad F_{DF} = 0 \qquad \textit{Ans.}$$

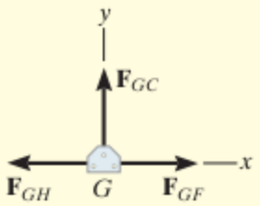
Joint F. (Fig. 5–13*d*).

$$+\uparrow \Sigma F_y = 0; \quad F_{FC}\cos\theta = 0 \quad \text{Since } \theta \neq 90°, \quad F_{FC} = 0 \qquad \textit{Ans.}$$

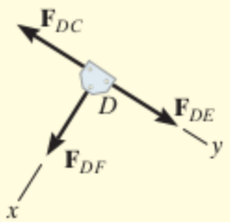
NOTE: If joint B is analyzed, Fig. 5–13*e*,

$$+\searrow \Sigma F_x = 0; \qquad 2\text{ kN} - F_{BH} = 0 \qquad F_{BH} = 2\text{ kN} \quad \text{(C)}$$

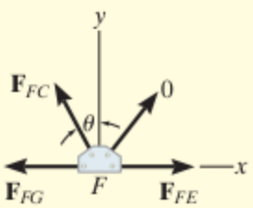
Also, F_{HC} must satisfy $\Sigma F_y = 0$, Fig. 5–13*f*, and therefore HC is *not* a zero-force member.



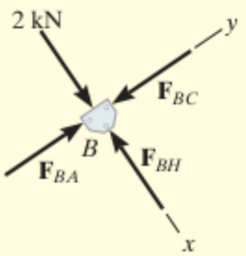
(b)



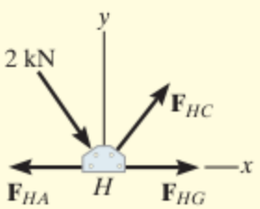
(c)



(d)



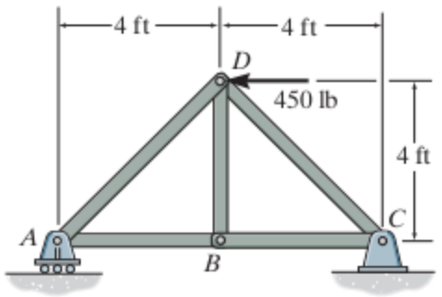
(e)



(f)

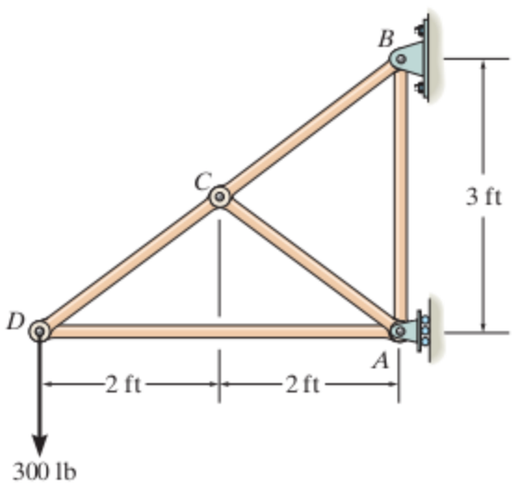
FUNDAMENTAL PROBLEMS

F5–1. Determine the force in each member of the truss. State if the members are in tension or compression.



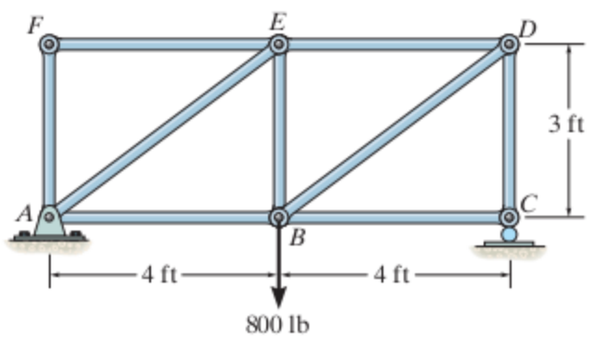
F5–1

F5–2. Determine the force in each member of the truss. State if the members are in tension or compression.



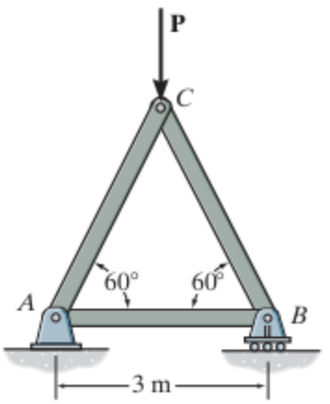
F5–2

F5–3. Determine the force in members AE and DC. State if the members are in tension or compression.



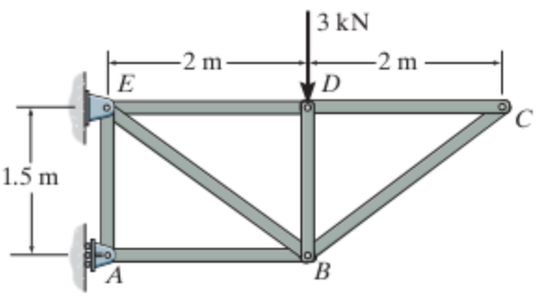
F5–3

F5–4. Determine the greatest load P that can be applied to the truss so that none of the members are subjected to a force exceeding either 2 kN in tension or 1.5 kN in compression.



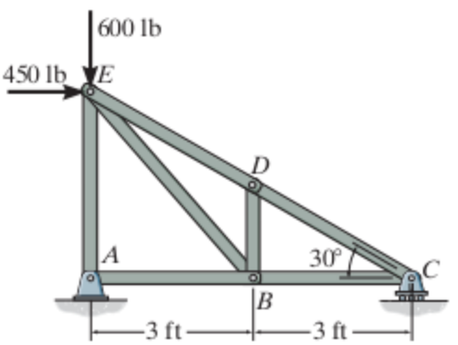
F5–4

F5–5. Identify the zero-force members in the truss.



F5–5

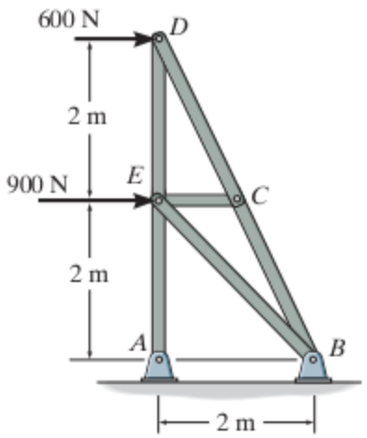
F5–6. Determine the force in each member of the truss. State if the members are in tension or compression.



F5–6

PROBLEMS

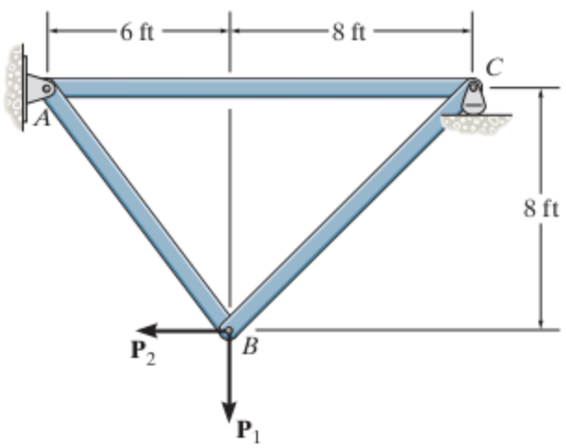
5–1. Determine the force in each member of the truss, and state if the members are in tension or compression.



Prob. 5–1

5–2. Determine the force in each member of the truss and state if the members are in tension or compression. Set $P_1 = 800$ lb and $P_2 = 400$ lb.

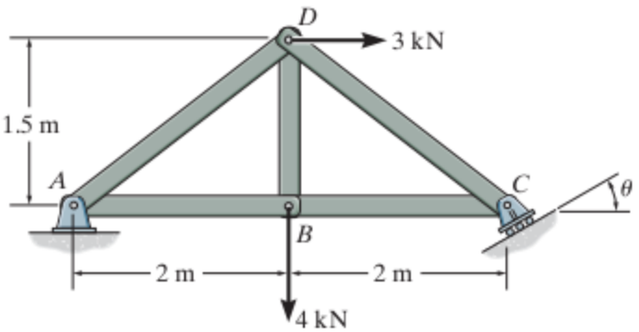
5–3. Determine the force on each member of the truss and state if the members are in tension or compression. Set $P_1 = 500$ lb and $P_2 = 100$ lb.



Probs. 5–2/3

*5–4. Determine the force in each member of the truss, and state if the members are in tension or compression. Set $\theta = 0°$.

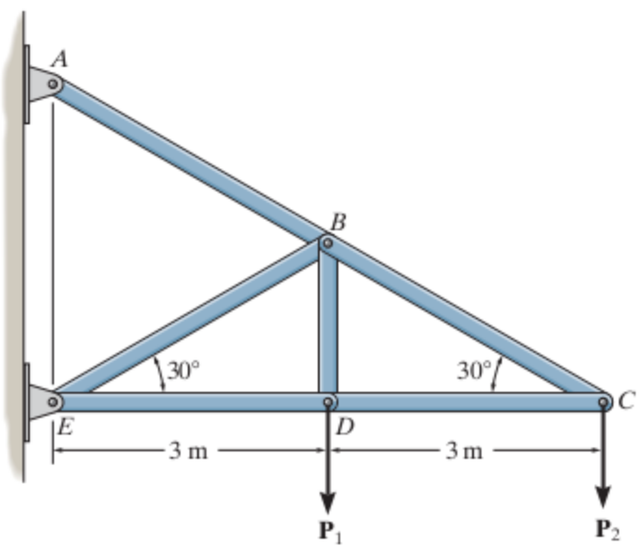
5–5. Determine the force in each member of the truss, and state if the members are in tension or compression. Set $\theta = 30°$.



Probs. 5–4/5

5–6. Determine the force in each member of the truss and state if the members are in tension or compression. Set $P_1 = 2$ kN and $P_2 = 1.5$ kN.

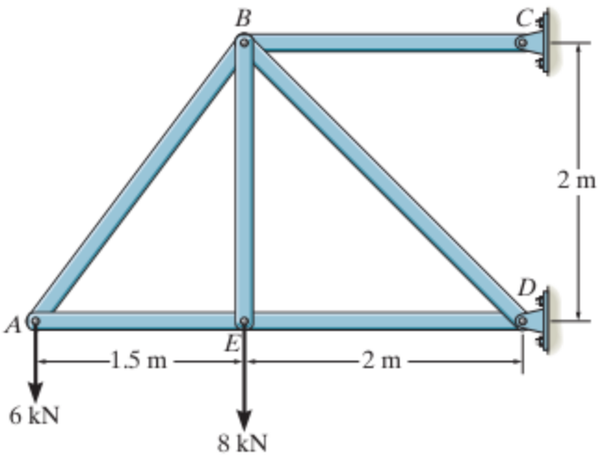
5–7. Determine the force in each member of the truss and state if the members are in tension or compression. Set $P_1 = P_2 = 4$ kN.



Probs. 5–6/7

*5–8. Determine the force in each member of the truss and state if the members are in tension or compression. *Hint:* The vertical component of force at C must equal zero. Why?

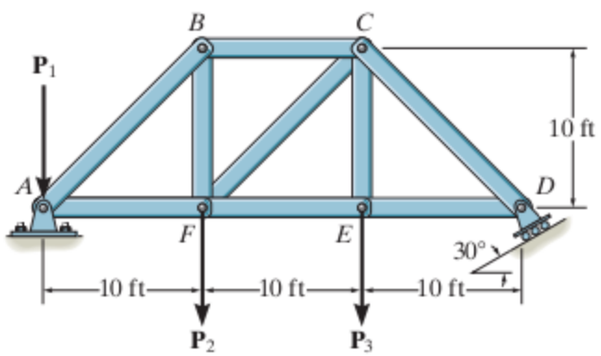
5–9. Each member of the truss is uniform and has a mass of 8 kg/m. Remove the external loads of 6 kN and 8 kN and determine the approximate force in each member due to the weight of the truss. State if the members are in tension or compression. Solve the problem by *assuming* the weight of each member can be represented as a vertical force, half of which is applied at each end of the member.



Probs. 5–8/9

5–10. Determine the force in each member of the truss and state if the members are in tension or compression. Set $P_1 = 100$ lb, $P_2 = 200$ lb, $P_3 = 300$ lb.

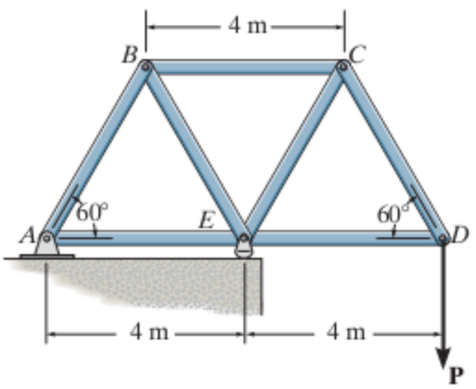
5–11. Determine the force in each member of the truss and state if the members are in tension or compression. Set $P_1 = 400$ lb, $P_2 = 400$ lb, $P_3 = 0$.



Probs. 5–10/11

*5–12. Determine the force in each member of the truss. State whether the members are in tension or compression. Set $P = 8$ kN.

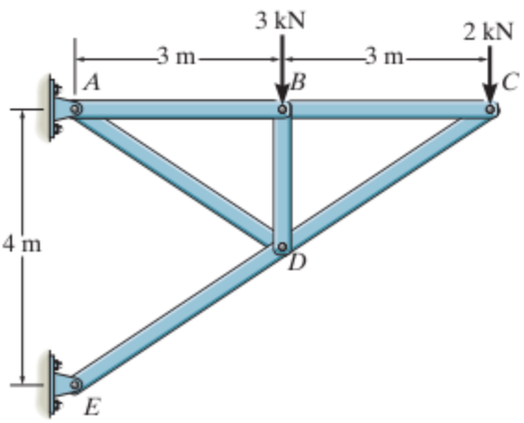
5–13. If the maximum force that any member can support is 8 kN in tension and 6 kN in compression, determine the maximum force P that can be supported at joint D.



Probs. 5–12/13

5–14. Determine the force in each member of the truss and state if the members are in tension or compression. *Hint:* The resultant force at the pin E acts along member ED. Why?

5–15. Each member of the truss is uniform and has a mass of 8 kg/m. Remove the external loads of 3 kN and 2 kN and determine the approximate force in each member due to the weight of the truss. State if the members are in tension or compression. Solve the problem by *assuming* the weight of each member can be represented as a vertical force, half of which is applied at each end of the member.

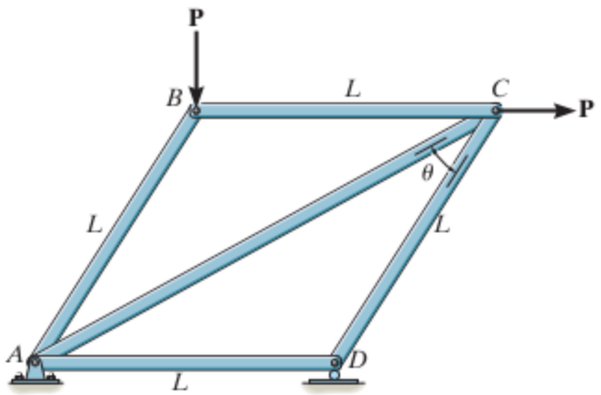


Probs. 5–14/15

5

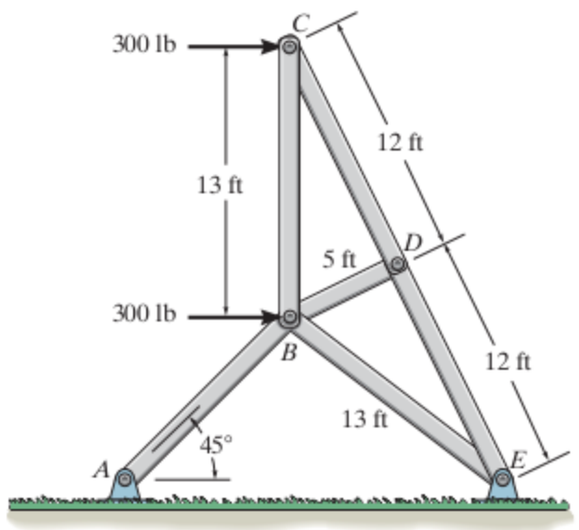
*5–16. Determine the force in each member of the truss in terms of the external loading and state if the members are in tension or compression.

5–17. The maximum allowable tensile force in the members of the truss is $(F_t)_{max} = 2$ kN, and the maximum allowable compressive force is $(F_c)_{max} = 1.2$ kN. Determine the maximum magnitude P of the two loads that can be applied to the truss. Take $L = 2$ m and $\theta = 30°$.



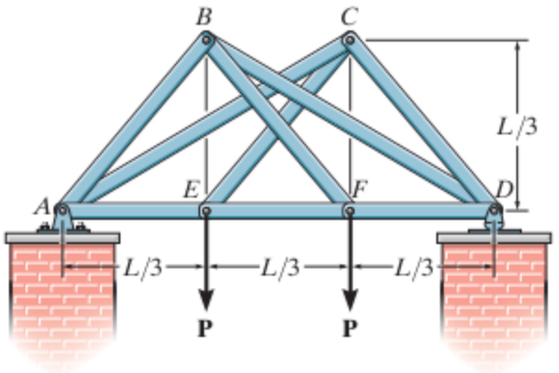
Probs. 5–16/17

5–18. A sign is subjected to a wind loading that exerts horizontal forces of 300 lb on joints B and C of one of the side supporting trusses. Determine the force in each member of the truss and state if the members are in tension or compression.



Prob. 5–18

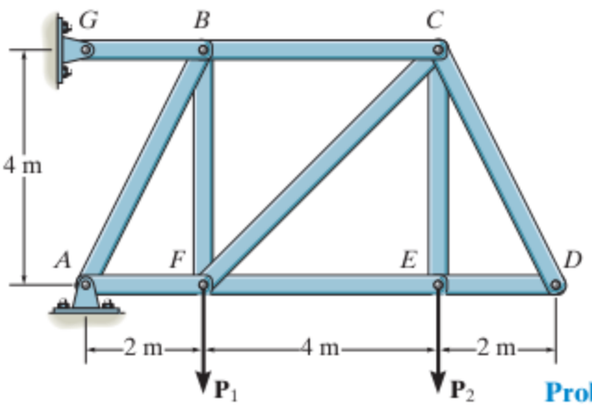
5–19. Determine the force in each member of the double scissors truss in terms of the load P and state if the members are in tension or compression.



Prob. 5–19

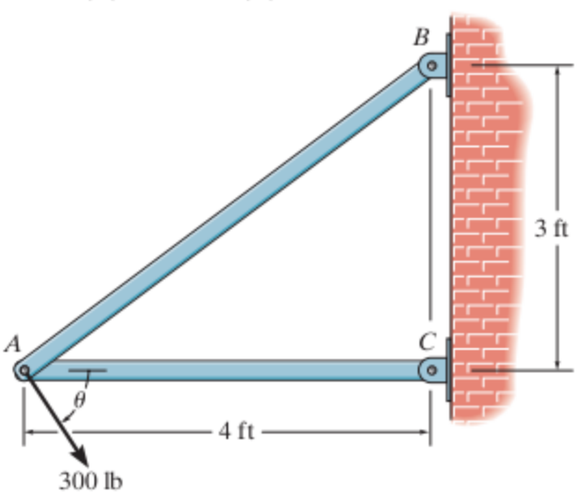
*5–20. Determine the force in each member of the truss and state if the members are in tension or compression. Set $P_1 = 10$ kN, $P_2 = 15$ kN.

5–21. Determine the force in each member of the truss and state if the members are in tension or compression. Set $P_1 = 0$, $P_2 = 20$ kN.



Probs. 5–20/21

5–22. The two-member truss is subjected to the force of 300 lb. Determine the range of θ for application of the load so that the force in either member does not exceed 400 lb (T) or 200 lb (C).



Prob. 5–22

5.4 The Method of Sections

When we need to find the force in only a few members of a truss, we can analyze the truss using the *method of sections*. It is based on the principle that if the truss is in equilibrium then any segment of the truss is also in equilibrium. For example, consider the two truss members shown on the left in Fig. 5–14. If the forces within the members are to be determined, then an imaginary section, indicated by the blue line, can be used to cut each member into two parts and thereby "expose" each internal force as "external" to the free-body diagrams shown on the right. Clearly, it can be seen that equilibrium requires that the member in tension (T) be subjected to a "pull," whereas the member in compression (C) is subjected to a "push."

The method of sections can also be used to "cut" or section the members of an entire truss. If the section passes through the truss and the free-body diagram of either of its two parts is drawn, we can then apply the equations of equilibrium to that part to determine the member forces at the "cut section." Since only *three* independent equilibrium equations ($\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma M_O = 0$) can be applied to the free-body diagram of any segment, then we should try to select a section that, in general, passes through not more than *three* members in which the forces are unknown. For example, consider the truss in Fig. 5–15*a*. If the forces in members *BC*, *GC*, and *GF* are to be determined, then section *aa* would be appropriate. The free-body diagrams of the two segments are shown in Figs. 5–15*b* and 5–15*c*. Note that the line of action of each member force is specified from the *geometry* of the truss, since the force in a member is along its axis. Also, the member forces acting on one part of the truss are equal but opposite to those acting on the other part—Newton's third law. Members *BC* and *GC* are assumed to be in *tension* since they are subjected to a "pull," whereas *GF* in *compression* since it is subjected to a "push."

The three unknown member forces $\mathbf{F}_{BC}$, $\mathbf{F}_{GC}$, and $\mathbf{F}_{GF}$ can be obtained by applying the three equilibrium equations to the free-body diagram in Fig. 5–15*b*. If, however, the free-body diagram in Fig. 5–15*c* is considered, the three support reactions $\mathbf{D}_x$, $\mathbf{D}_y$ and $\mathbf{E}_x$ will have to be known, because only three equations of equilibrium are available. (This, of course, is done in the usual manner by considering a free-body diagram of the *entire truss*.)

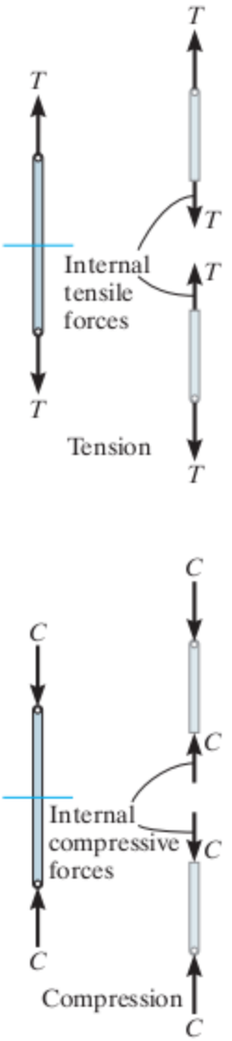


Fig. 5–14

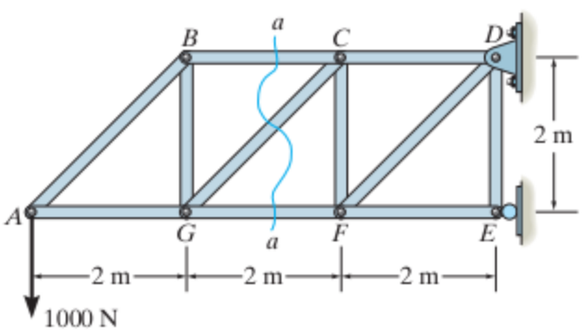


Fig. 5–15

5

The forces in selected members of this Pratt truss can readily be determined using the method of sections.

When applying the equilibrium equations, we should carefully consider ways of writing the equations so as to yield a *direct solution* for each of the unknowns, rather than having to solve simultaneous equations. For example, using the truss segment in Fig. 5–15*b* and summing moments about C would yield a direct solution for $\mathbf{F}_{GF}$ since $\mathbf{F}_{BC}$ and $\mathbf{F}_{GC}$ create zero moment about C. Likewise, $\mathbf{F}_{BC}$ can be directly obtained by summing moments about G. Finally, $\mathbf{F}_{GC}$ can be found directly from a force summation in the vertical direction since $\mathbf{F}_{GF}$ and $\mathbf{F}_{BC}$ have no vertical components. This ability to *determine directly* the force in a particular truss member is one of the main advantages of using the method of sections.*

As in the method of joints, there are two ways in which we can determine the correct sense of an unknown member force:

- The correct sense of an unknown member force can in many cases be determined "by inspection." For example, $\mathbf{F}_{BC}$ is a tensile force as represented in Fig. 5–15*b* since moment equilibrium about G requires that $\mathbf{F}_{BC}$ create a moment opposite to that of the 1000-N force. Also, $\mathbf{F}_{GC}$ is tensile since its vertical component must balance the 1000-N force which acts downward. In more complicated cases, the sense of an unknown member force may be *assumed.* If the solution yields a *negative* scalar, it indicates that the force's sense is *opposite* to that shown on the free-body diagram.
- *Always assume* that the unknown member forces at the cut section are *tensile* forces, i.e., "pulling" on the member. By doing this, the numerical solution of the equilibrium equations will yield *positive scalars for members in tension and negative scalars for members in compression.*

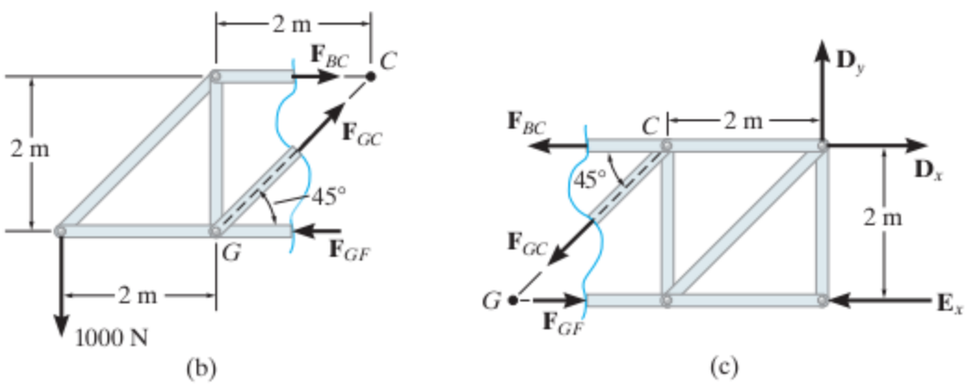


Fig. 5–15 (cont.)

*Notice that if the method of joints were used to determine, say, the force in member GC, it would be necessary to analyze joints A, B, and G in sequence.

Simple trusses are often used in the construction of large cranes in order to reduce the weight of the boom and tower.

Procedure for Analysis

The forces in the members of a truss may be determined by the method of sections using the following procedure.

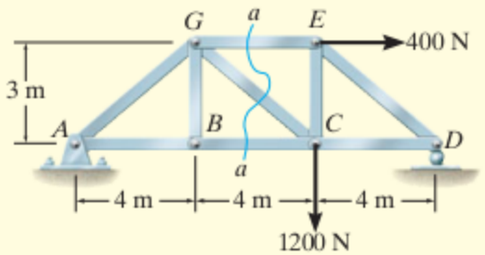
Free-Body Diagram.

- Make a decision on how to "cut" or section the truss through the members where forces are to be determined.
- Before isolating the appropriate section, it may first be necessary to determine the truss's support reactions. If this is done then the three equilibrium equations will be available to solve for member forces at the section.
- Draw the free-body diagram of that segment of the sectioned truss which has the least number of forces acting on it.
- Use one of the two methods described above for establishing the sense of the unknown member forces.

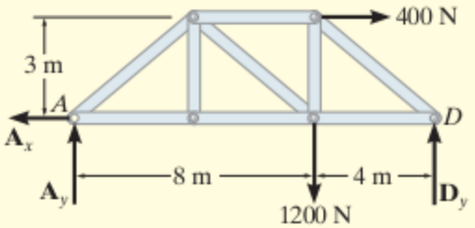
Equations of Equilibrium.

- Moments should be summed about a point that lies at the intersection of the lines of action of two unknown forces, so that the third unknown force can be determined directly from the moment equation.
- If two of the unknown forces are *parallel*, forces may be summed *perpendicular* to the direction of these unknowns to determine *directly* the third unknown force.

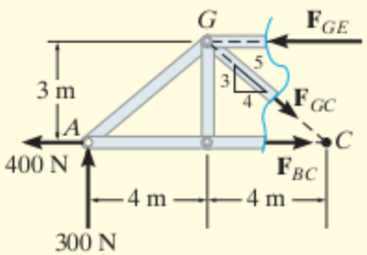
EXAMPLE 5.5



(a)



(b)



(c)

Fig. 5–16

Determine the force in members GE, GC, and BC of the truss shown in Fig. 5–16*a*. Indicate whether the members are in tension or compression.

SOLUTION

Section *aa* in Fig. 5–16*a* has been chosen since it cuts through the *three* members whose forces are to be determined. In order to use the method of sections, however, it is *first* necessary to determine the external reactions at A or D. Why? A free-body diagram of the entire truss is shown in Fig. 5–16*b*. Applying the equations of equilibrium, we have

$$\stackrel{+}{\rightarrow}\Sigma F_x = 0; \qquad 400\ \text{N} - A_x = 0 \qquad A_x = 400\ \text{N}$$

$$\circlearrowleft +\Sigma M_A = 0; \qquad -1200\ \text{N}(8\ \text{m}) - 400\ \text{N}(3\ \text{m}) + D_y(12\ \text{m}) = 0$$

$$D_y = 900\ \text{N}$$

$$+\uparrow\Sigma F_y = 0; \qquad A_y - 1200\ \text{N} + 900\ \text{N} = 0 \qquad A_y = 300\ \text{N}$$

Free-Body Diagram. For the analysis the free-body diagram of the left portion of the sectioned truss will be used, since it involves the least number of forces, Fig. 5–16*c*.

Equations of Equilibrium. Summing moments about point G eliminates $\mathbf{F}_{GE}$ and $\mathbf{F}_{GC}$ and yields a direct solution for F_{BC}.

$$\circlearrowleft +\Sigma M_G = 0; \quad -300\ \text{N}(4\ \text{m}) - 400\ \text{N}(3\ \text{m}) + F_{BC}(3\ \text{m}) = 0$$

$$F_{BC} = 800\ \text{N} \quad \text{(T)} \qquad \textit{Ans.}$$

In the same manner, by summing moments about point C we obtain a direct solution for F_{GE}.

$$\circlearrowleft +\Sigma M_C = 0; \quad -300\ \text{N}(8\ \text{m}) + F_{GE}(3\ \text{m}) = 0$$

$$F_{GE} = 800\ \text{N} \quad \text{(C)} \qquad \textit{Ans.}$$

Since $\mathbf{F}_{BC}$ and $\mathbf{F}_{GE}$ have no vertical components, summing forces in the y direction directly yields F_{GC}, i.e.,

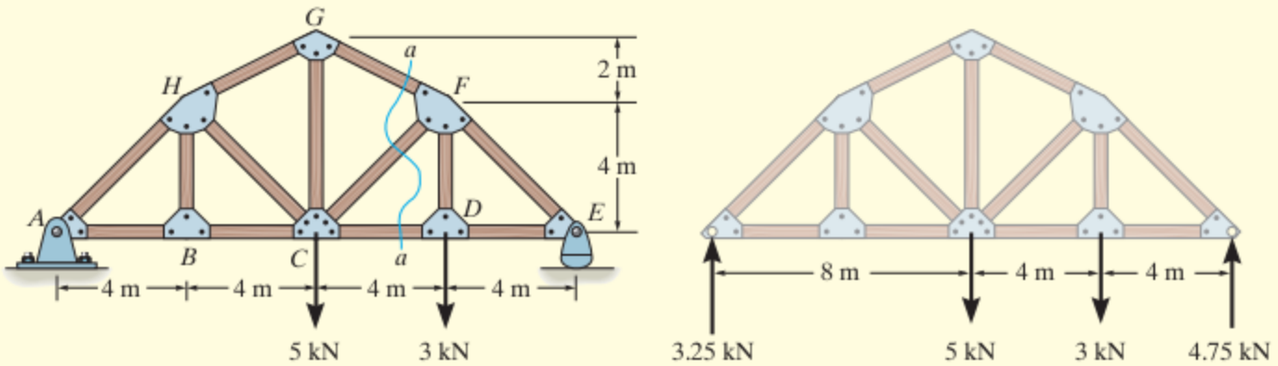
$$+\uparrow\Sigma F_y = 0; \qquad 300\ \text{N} - \tfrac{3}{5}F_{GC} = 0$$

$$F_{GC} = 500\ \text{N} \quad \text{(T)} \qquad \textit{Ans.}$$

NOTE: Here it is possible to tell, by inspection, the proper direction for each unknown member force. For example, $\Sigma M_C = 0$ requires $\mathbf{F}_{GE}$ to be *compressive* because it must balance the moment of the 300-N force about C.

EXAMPLE 5.6

Determine the force in member CF of the truss shown in Fig. 5–17*a*. Indicate whether the member is in tension or compression. Assume each member is pin connected.



SOLUTION

Free-Body Diagram. Section *aa* in Fig. 5–17*a* will be used since this section will "expose" the internal force in member CF as "external" on the free-body diagram of either the right or left portion of the truss. It is first necessary, however, to determine the support reactions on either the left or right side. Verify the results shown on the free-body diagram in Fig. 5–17*b*.

The free-body diagram of the right portion of the truss, which is the easiest to analyze, is shown in Fig. 5–17*c*. There are three unknowns, F_{FG}, F_{CF}, and F_{CD}.

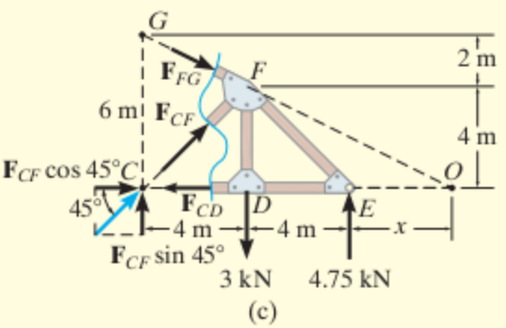


Fig. 5–17

Equations of Equilibrium. We will apply the moment equation about point O in order to eliminate the two unknowns F_{FG} and F_{CD}. The location of point O measured from E can be determined from proportional triangles, i.e., $4/(4 + x) = 6/(8 + x)$, $x = 4$ m. Or, stated in another manner, the slope of member GF has a drop of 2 m to a horizontal distance of 4 m. Since FD is 4 m, Fig. 5–17*c*, then from D to O the distance must be 8 m.

An easy way to determine the moment of $\mathbf{F}_{CF}$ about point O is to use the principle of transmissibility and slide $\mathbf{F}_{CF}$ to point C, and then resolve $\mathbf{F}_{CF}$ into its two rectangular components. We have

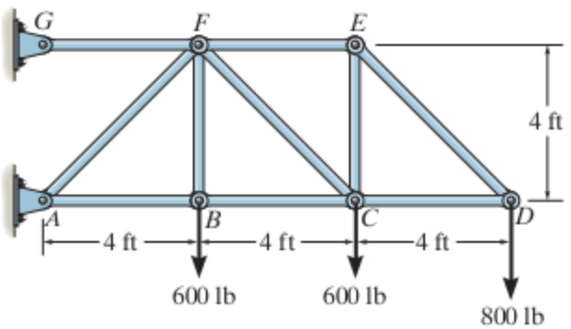
$$\circlearrowleft + \Sigma M_O = 0;$$

$$-F_{CF} \sin 45°(12\text{ m}) + (3\text{ kN})(8\text{ m}) - (4.75\text{ kN})(4\text{ m}) = 0$$

$$F_{CF} = 0.589\text{ kN} \quad \text{(C)} \qquad \textit{Ans.}$$

FUNDAMENTAL PROBLEMS

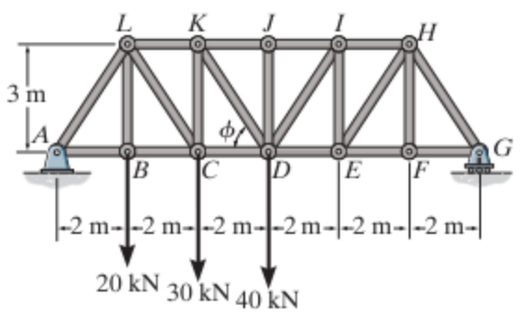
F5–7. Determine the force in members *BC*, *CF*, and *FE*. State if the members are in tension or compression.



F5–7

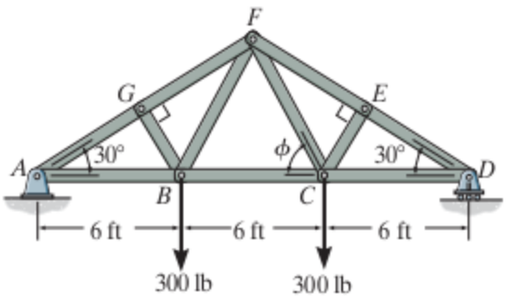
F5–8. Determine the force in members *LK*, *KC*, and *CD* of the Pratt truss. State if the members are in tension or compression.

F5–9. Determine the force in members *KJ*, *KD*, and *CD* of the Pratt truss. State if the members are in tension or compression.



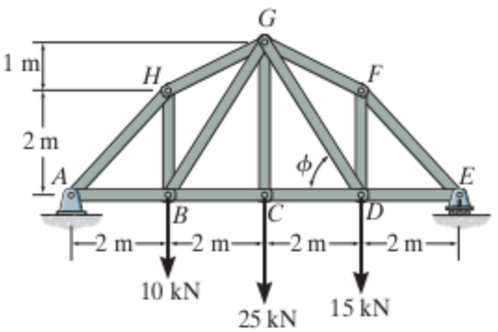
F5–8/9

F5–10. Determine the force in members *EF*, *CF*, and *BC* of the truss. State if the members are in tension or compression.



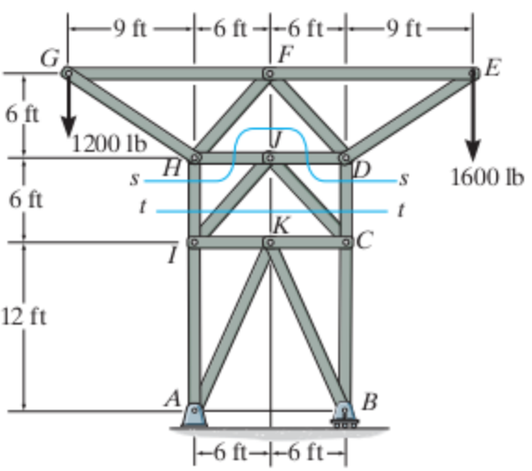
F5–10

F5–11. Determine the force in members *GF*, *GD*, and *CD* of the truss. State if the members are in tension or compression.



F5–11

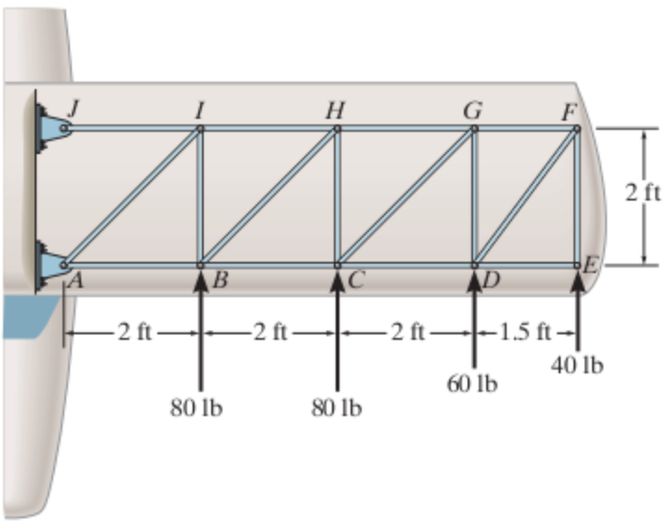
F5–12. Determine the force in members *DC*, *HI*, and *JI* of the truss. State if the members are in tension or compression.



F5–12

PROBLEMS

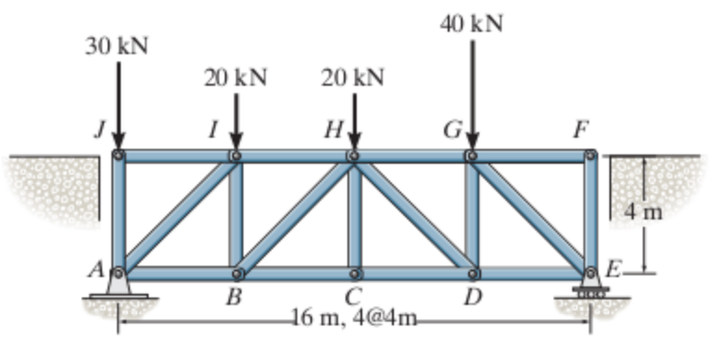
5–23. The internal drag truss for the wing of a light airplane is subjected to the forces shown. Determine the force in members *BC*, *BH*, and *HC*, and state if the members are in tension or compression.



Prob. 5–23

***5–24.** The Howe bridge *truss* is subjected to the loading shown. Determine the force in members *HD*, *CD*, and *GD*, and state if the members are in tension or compression.

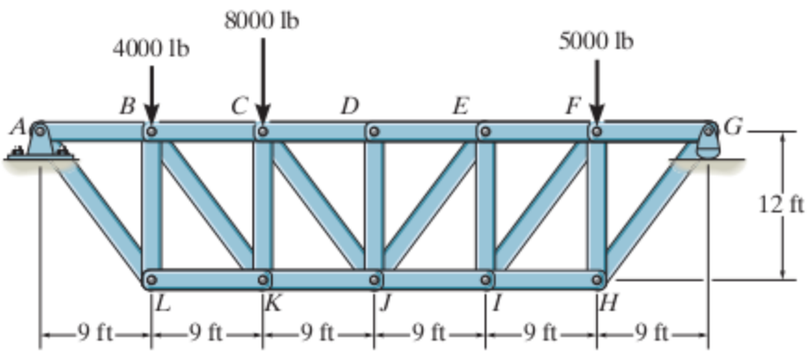
5–25. The Howe bridge *truss* is subjected to the loading shown. Determine the force in members *HI*, *HB*, and *BC*, and state if the members are in tension or compression.



Probs. 5–24/25

5–26. Determine the force in members *CD*, *CJ*, *KJ*, and *DJ* of the truss which serves to support the deck of a bridge. State if these members are in tension or compression.

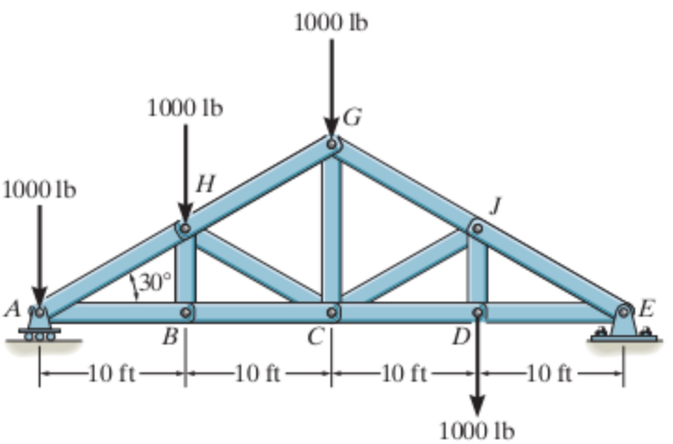
5–27. Determine the force in members *EI* and *JI* of the truss which serves to support the deck of a bridge. State if these members are in tension or compression.



Probs. 5–26/27

***5–28.** Determine the force in member *GJ* of the truss and state if this member is in tension or compression.

5–29. Determine the force in member *GC* of the truss and state if this member is in tension or compression.

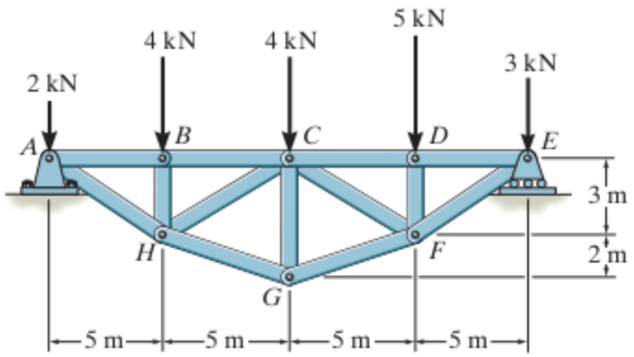


Probs. 5–28/29

5

5–30. Determine the force in members *BC*, *HC*, and *HG*. After the truss is sectioned use a single equation of equilibrium for the calculation of each force. State if these members are in tension or compression.

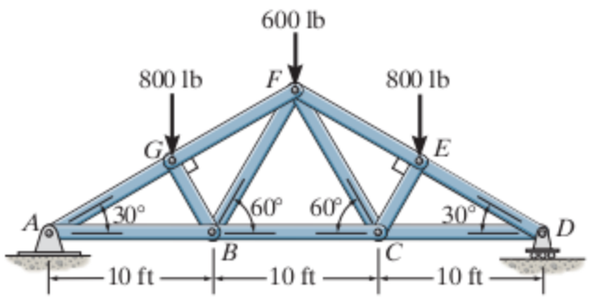
5–31. Determine the force in members *CD*, *CF*, and *CG* and state if these members are in tension or compression.



Probs. 5–30/31

5–34. Determine the force in members *GF*, *FB*, and *BC* of the *Fink truss* and state if the members are in tension or compression.

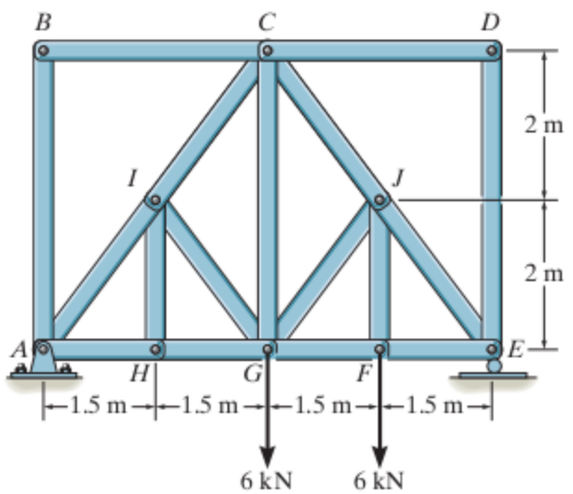
5–35. Determine the force in members *FE* and *EC* of the *Fink truss* and state if the members are in tension or compression.



Probs. 5–34/35

***5–32.** Determine the force in members *IC* and *CG* of the truss and state if these members are in tension or compression. Also, indicate all zero-force members.

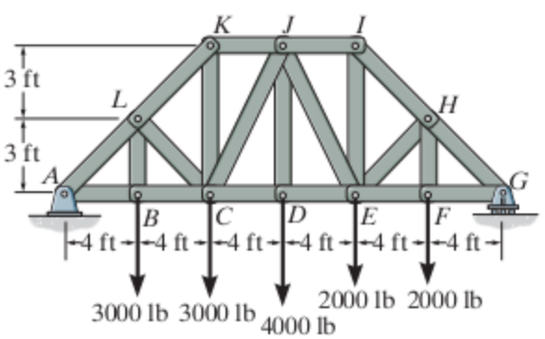
5–33. Determine the force in members *JE* and *GF* of the truss and state if these members are in tension or compression. Also, indicate all zero-force members.



Probs. 5–32/33

***5–36.** Determine the force in members *LK*, *LC*, and *BC* of the truss, and state if the members are in tension or compression.

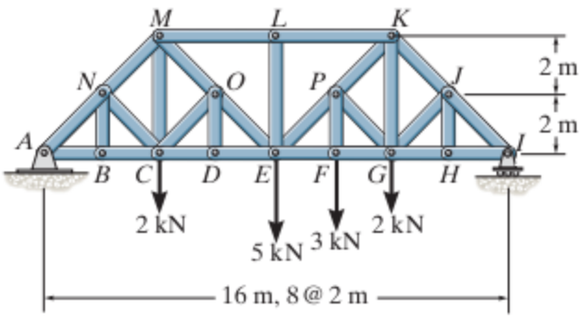
5–37. Determine the force in members *JI*, *JE*, and *DE* of the truss, and state if the members are in tension or compression.



Probs. 5–36/37

5–38. Determine the force in members *CD* and *CM* of the *Baltimore bridge truss* and state if the members are in tension or compression. Also, indicate all zero-force members.

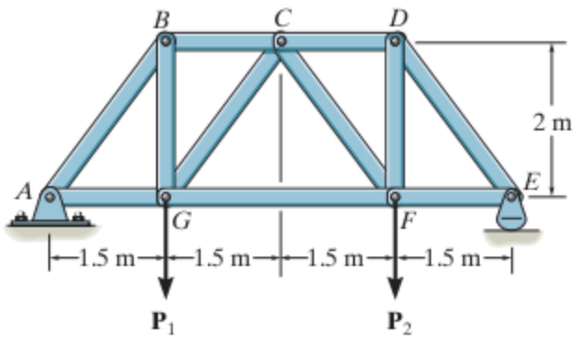
5–39. Determine the force in members *EF*, *EP*, and *LK* of the *Baltimore bridge truss* and state if the members are inxtension or compression. Also, indicate all zero-force members.



Probs. 5–38/39

5–42. Determine the force in each member of the truss and state if the members are in tension or compression. Set $P_1 = 20\,\text{kN}, P_2 = 10\,\text{kN}$.

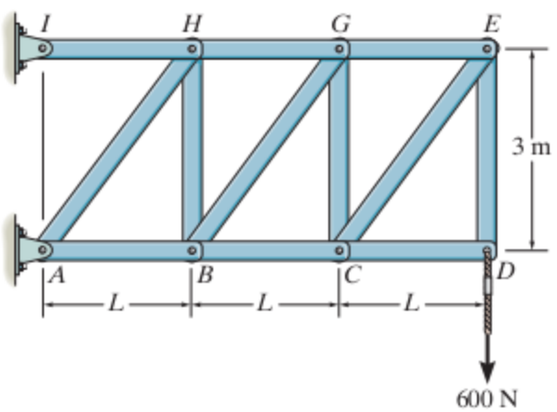
5–43. Determine the force in each member of the truss and state if the members are in tension or compression. Set $P_1 = 40\,\text{kN}, P_2 = 20\,\text{kN}$.



Probs. 5–42/43

*5–40. The truss supports the vertical load of 600 N. If $L = 2$ m, determine the force on members *HG* and *HB* of the truss and state if the members are in tension or compression.

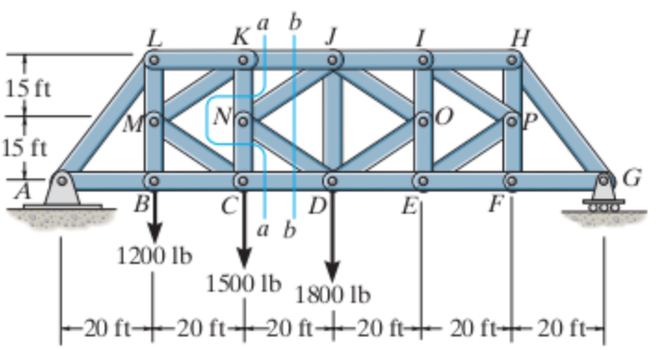
5–41. The truss supports the vertical load of 600 N. Determine the force in members *BC*, *BG*, and *HG* as the dimension *L* varies. Plot the results of *F* (ordinate with tension as positive) versus *L* (abscissa) for $0 \leq L \leq 3$ m.



Probs. 5–40/41

*5–44. Determine the force in members *KJ*, *NJ*, *ND*, and *CD* of the K truss. Indicate if the members are in tension or compression. *Hint:* Use sections *aa* and *bb*.

5–45. Determine the force in members *JI* and *DE* of the K truss. Indicate if the members are in tension or compression.



Probs. 5–44/45

5

5.5 Frames and Machines

This large crane is a typical example of a framework.

Common tools such as these pliers act as simple machines. Here the applied force on the handles creates a much larger force at the jaws.

Frames and machines are two types of structures which are often composed of pin-connected *multiforce members*, i.e., members that are subjected to more than two forces. *Frames* are used to support loads, whereas *machines* contain moving parts and are designed to transmit and alter the effect of forces. Provided a frame or machine contains no more supports or members than are necessary to prevent its collapse, the forces acting at the joints and supports can be determined by applying the equations of equilibrium to each of its members. Once these forces are obtained, it is then possible to *design* the size of the members, connections, and supports using the theory of mechanics of materials and an appropriate engineering design code.

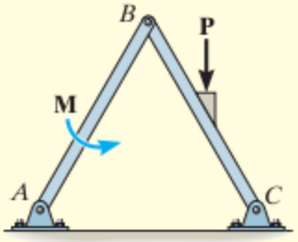
Free-Body Diagrams. In order to determine the forces acting at the joints and supports of a frame or machine, the structure must be disassembled and the free-body diagrams of its parts must be drawn. The following important points *must* be observed:

- Isolate each part by drawing its *outlined shape* Then show all the forces and/or couple moments that act on the part. Make sure to *label* or *identify* each known and unknown force and couple moment with reference to an established x, y coordinate system. Also, indicate any dimensions used for taking moments. Most often the equations of equilibrium are easier to apply if the forces are represented by their rectangular components. As usual, the sense of an unknown force or couple moment can be assumed.
- Identify all the two-force members in the structure and represent their free-body diagrams as having two equal but opposite collinear forces acting at their points of application. (See Sec. 4.4.) By recognizing the two-force members, we can avoid solving an unnecessary number of equilibrium equations.
- Forces common to any two *contacting* members act with equal magnitudes but opposite sense on the respective members. If the two members are treated as a "*system*" *of connected members*, then these forces are "*internal*" and are *not shown* on the *free-body diagram of the system*; however, if the free-body diagram of *each member* is drawn, the forces are "*external*" and *must* be shown on each of the free-body diagrams.

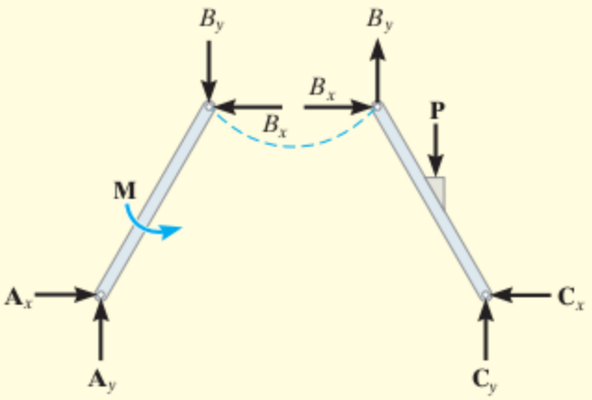
The following examples graphically illustrate how to draw the free-body diagrams of a dismembered frame or machine. In all cases, the weight of the members is neglected.

EXAMPLE 5.7

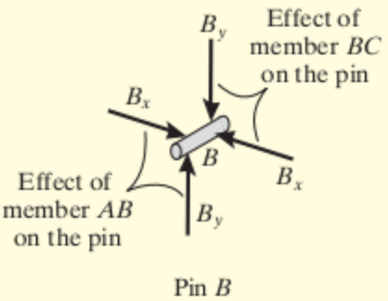
For the frame shown in Fig. 5–18*a*, draw the free-body diagram of (a) each member, (b) the pins at B and A, and (c) the two members connected together.



(a)



(b)



Pin B

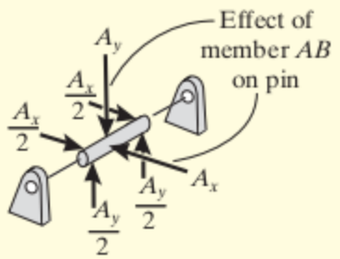
(c)

SOLUTION

Part (a). By inspection, members BA and BC are *not* two-force members. Instead, as shown on the free-body diagrams, Fig. 5–18*b*, BC is subjected to a force from each of the pins at B and C and the external force **P**. Likewise, AB is subjected to a force from each of the pins at A and B and the external couple moment **M**. The pin forces are represented by their x and y components.

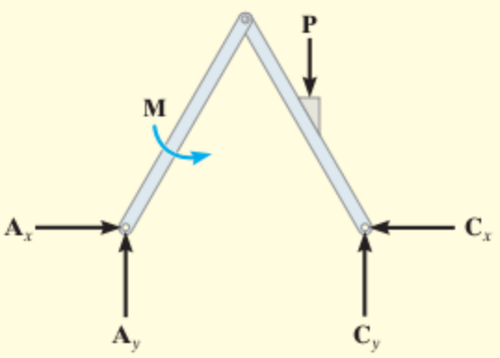
Part (b). The pin at B is subjected to only *two forces*, i.e., the force of member BC and the force of member AB. For *equilibrium* these forces (or their respective components) must be equal but opposite, Fig. 5–18*c*. Realize that Newton's third law is applied between the pin and its connected members, i.e., the effect of the pin on the two members, Fig. 5–18*b*, and the equal but opposite effect of the two members on the pin, Fig. 5–18*c*. In the same manner, there are three forces on pin A, Fig. 5–18*d*, caused by the force components of member AB and each of the two pin leafs.

Part (c). The free-body diagram of both members connected together, yet removed from the supporting pins at A and C, is shown in Fig. 5–18*e*. The force components $\mathbf{B}_x$ and $\mathbf{B}_y$ are *not shown* on this diagram since they are *internal* forces (Fig. 5–18*b*) and therefore cancel out. Also, to be consistent when later applying the equilibrium equations, the unknown force components at A and C must act in the *same sense* as those shown in Fig. 5–18*b*.



Pin A

(d)



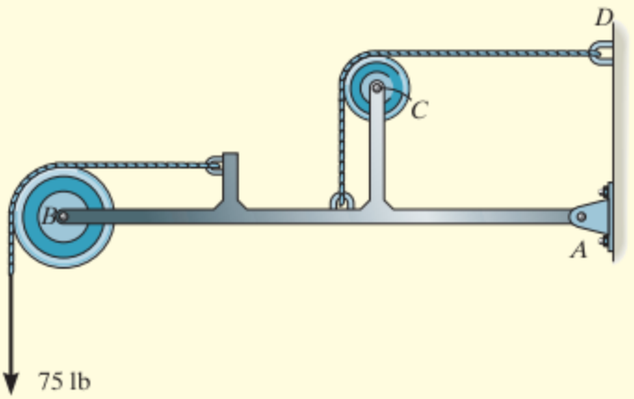
(e)

Fig. 5–18

5

EXAMPLE 5.8

For the frame shown in Fig. 5–19*a*, draw the free-body diagrams of (a) the entire frame including the pulleys and cords, (b) the frame without the pulleys and cords, and (c) each of the pulleys.



(a)

SOLUTION

Part (a). When the entire frame including the pulleys and cords is considered, the interactions at the points where the pulleys and cords are connected to the frame become pairs of *internal* forces which cancel each other and therefore are not shown on the free-body diagram, Fig. 5–19*b*.

Part (b). When the cords and pulleys are removed, their effect *on the frame* must be shown, Fig. 5–19*c*.

Part (c). The force components $\mathbf{B}_x$, $\mathbf{B}_y$, $\mathbf{C}_x$, $\mathbf{C}_y$ of the pins on the pulleys, Fig. 5–19*d*, are equal but opposite to the force components exerted by the pins on the frame, Fig. 5–19*c*. Why?

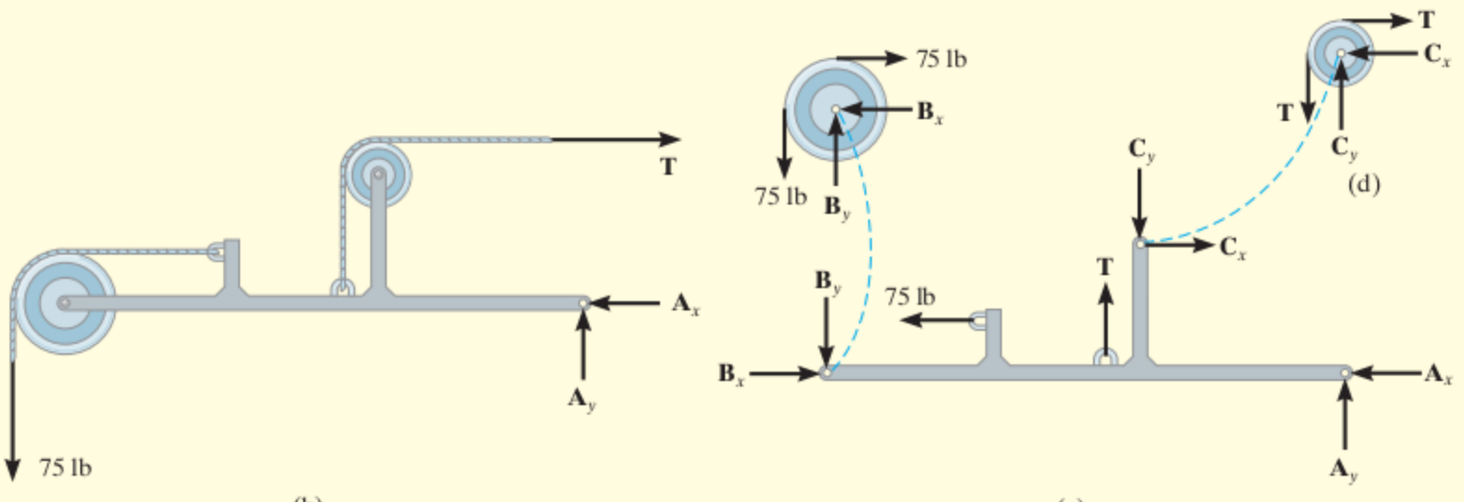


Fig. 5–19

Procedure for Analysis

The joint reactions on frames or machines (structures) composed of multiforce members can be determined using the following procedure.

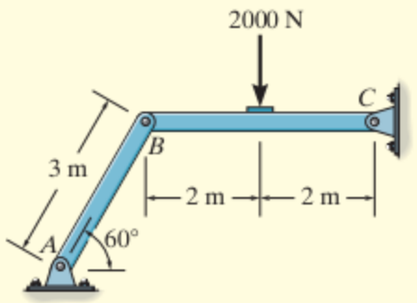
Free-Body Diagram.

- Draw the free-body diagram of the entire frame or machine, a portion of it, or each of its members. The choice should be made so that it leads to the most direct solution of the problem.
- When the free-body diagram of a group of members of a frame or machine is drawn, the forces between the connected parts of this group are internal forces and are not shown on the free-body diagram of the group.
- Forces common to two members which are in contact act with equal magnitude but opposite sense on the respective free-body diagrams of the members.
- Two-force members, regardless of their shape, have equal but opposite collinear forces acting at the ends of the member.
- In many cases it is possible to tell by inspection the proper sense of the unknown forces acting on a member; however, if this seems difficult, the sense can be assumed.
- Remember that a couple moment is a free vector and can act at any point on the free-body diagram. Also, a force is a sliding vector and can act at any point along its line of action.

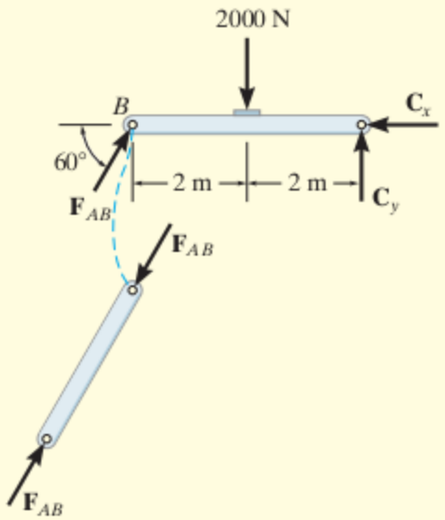
Equations of Equilibrium.

- Count the number of unknowns and compare it to the total number of equilibrium equations that are available. In two dimensions, there are three equilibrium equations that can be written for each member.
- Sum moments about a point that lies at the intersection of the lines of action of as many of the unknown forces as possible.
- If the solution of a force or couple moment magnitude is found to be negative, it means the sense of the force is the reverse of that shown on the free-body diagram.

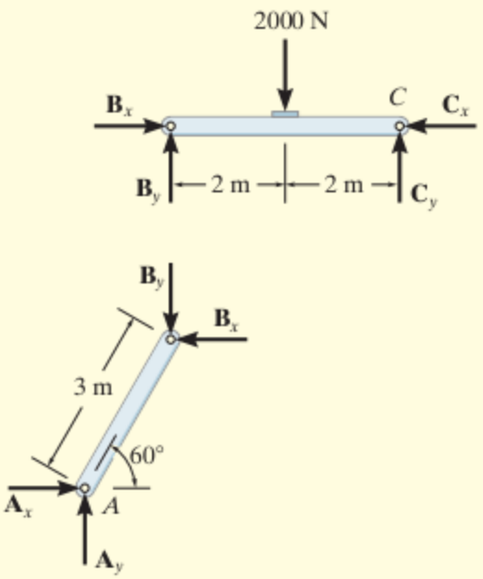
EXAMPLE 5.9



(a)



(b)



(c)

Fig. 5–20

Determine the horizontal and vertical components of force which the pin at C exerts on member BC of the frame in Fig. 5–20*a*.

SOLUTION I

Free-Body Diagrams. By inspection it can be seen that AB is a two-force member. The free-body diagrams are shown in Fig. 5–20*b*.

Equations of Equilibrium. The *three unknowns* can be determined by applying the three equations of equilibrium to member CB.

$$\circlearrowleft +\Sigma M_C = 0;\ 2000\ \text{N}(2\ \text{m}) - (F_{AB}\sin 60°)(4\ \text{m}) = 0\quad F_{AB} = 1154.7\ \text{N}$$

$$\pm\rightarrow \Sigma F_x = 0;\ 1154.7\cos 60°\ \text{N} - C_x = 0\quad C_x = 577\ \text{N}\qquad \textit{Ans.}$$

$$+\uparrow \Sigma F_y = 0;\ 1154.7\sin 60°\ \text{N} - 2000\ \text{N} + C_y = 0$$

$$C_y = 1000\ \text{N}\qquad \textit{Ans.}$$

SOLUTION II

Free-Body Diagrams. If one does not recognize that AB is a two-force member, then more work is involved in solving this problem. The free-body diagrams are shown in Fig. 5–20*c*.

Equations of Equilibrium. The *six unknowns* are determined by applying the three equations of equilibrium to each member.

Member AB

$$\circlearrowleft +\Sigma M_A = 0;\quad B_x(3\sin 60°\ \text{m}) - B_y(3\cos 60°\ \text{m}) = 0 \qquad (1)$$

$$\pm\rightarrow \Sigma F_x = 0;\quad A_x - B_x = 0 \qquad (2)$$

$$+\uparrow \Sigma F_y = 0;\quad A_y - B_y = 0 \qquad (3)$$

Member BC

$$\circlearrowleft +\Sigma M_C = 0;\quad 2000\ \text{N}(2\ \text{m}) - B_y(4\ \text{m}) = 0 \qquad (4)$$

$$\pm\rightarrow \Sigma F_x = 0;\quad B_x - C_x = 0 \qquad (5)$$

$$+\uparrow \Sigma F_y = 0;\quad B_y - 2000\ \text{N} + C_y = 0 \qquad (6)$$

The results for C_x and C_y can be determined by solving these equations in the following sequence: 4, 1, 5, then 6. The results are

$$B_y = 1000\ \text{N}$$

$$B_x = 577\ \text{N}$$

$$C_x = 577\ \text{N}\qquad \textit{Ans.}$$

$$C_y = 1000\ \text{N}\qquad \textit{Ans.}$$

By comparison, Solution I is simpler since the requirement that F_{AB} in Fig. 5–20*b* be equal, opposite, and collinear at the ends of member AB automatically satisfies Eqs. (1), (2), and (3) above and therefore eliminates the need to write these equations. *As a result, save yourself some time and effort by always identifying the two-force members before starting the analysis!*

EXAMPLE 5.10

The compound beam shown in Fig. 5–21*a* is pin connected at *B*. Determine the components of reaction at its supports. Neglect its weight and thickness.

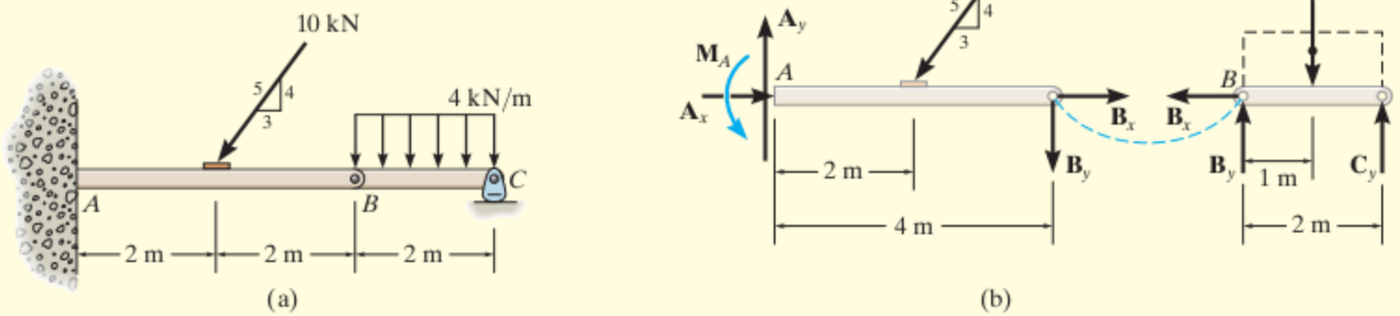


Fig. 5–21

SOLUTION

Free-Body Diagrams. By inspection, if we consider a free-body diagram of the *entire beam ABC*, there will be three unknown reactions at *A* and one at *C*. These four unknowns cannot all be obtained from the three available equations of equilibrium, and so for the solution it will become necessary to dismember the beam into its two segments, as shown in Fig. 5–21*b*.

Equations of Equilibrium. The six unknowns are determined as follows:

Segment BC

$$\pm\leftarrow \Sigma F_x = 0; \qquad B_x = 0$$

$$\circlearrowleft +\Sigma M_B = 0; \qquad -8\ \text{kN}(1\ \text{m}) + C_y(2\ \text{m}) = 0$$

$$+\uparrow \Sigma F_y = 0; \qquad B_y - 8\ \text{kN} + C_y = 0$$

Segment AB

$$\pm\rightarrow \Sigma F_x = 0; \qquad A_x - (10\ \text{kN})\left(\tfrac{3}{5}\right) + B_x = 0$$

$$\circlearrowleft +\Sigma M_A = 0; \qquad M_A - (10\ \text{kN})\left(\tfrac{4}{5}\right)(2\ \text{m}) - B_y(4\ \text{m}) = 0$$

$$+\uparrow \Sigma F_y = 0; \qquad A_y - (10\ \text{kN})\left(\tfrac{4}{5}\right) - B_y = 0$$

Solving each of these equations successively, using previously calculated results, we obtain

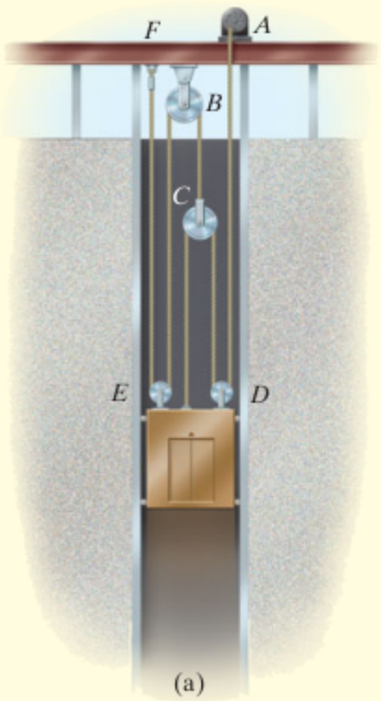
$$A_x = 6\ \text{kN} \qquad A_y = 12\ \text{kN} \qquad M_A = 32\ \text{kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$B_x = 0 \qquad B_y = 4\ \text{kN}$$

$$C_y = 4\ \text{kN} \qquad \textit{Ans.}$$

5

EXAMPLE 5.11



A 500-kg elevator car in Fig. 5–22*a* is being hoisted by motor *A* using the pulley system shown. If the car is traveling with a constant speed, determine the force developed in the two cables. Neglect the mass of the cable and pulleys.

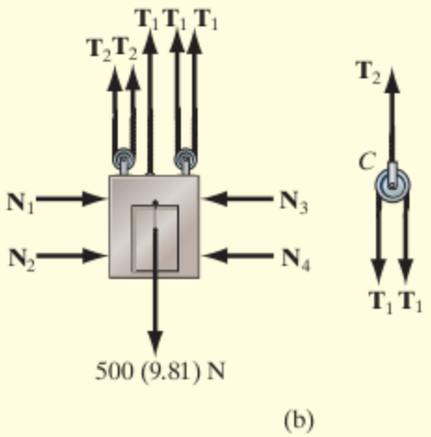


Fig. 5–22

SOLUTION

Free-Body Diagram. We can solve this problem using the free-body diagrams of the elevator car and pulley *C*, Fig. 5–22*b*. The tensile forces developed in the cables are denoted as T_1 and T_2.

Equations of Equilibrium. For pulley *C*,

$$+\uparrow \Sigma F_y = 0; \qquad T_2 - 2T_1 = 0 \qquad \text{or} \qquad T_2 = 2T_1 \tag{1}$$

For the elevator car,

$$+\uparrow \Sigma F_y = 0; \qquad 3T_1 + 2T_2 - 500(9.81)\,\text{N} = 0 \tag{2}$$

Substituting Eq. (1) into Eq. (2) yields

$$3T_1 + 2(2T_1) - 500(9.81)\,\text{N} = 0$$

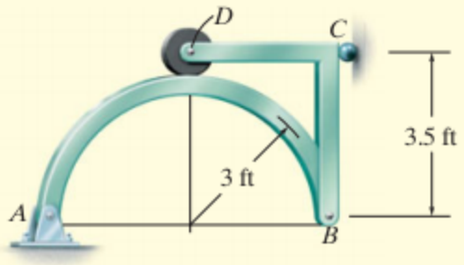
$$T_1 = 700.71\,\text{N} = 701\,\text{N} \qquad \textit{Ans.}$$

Substituting this result into Eq. (1),

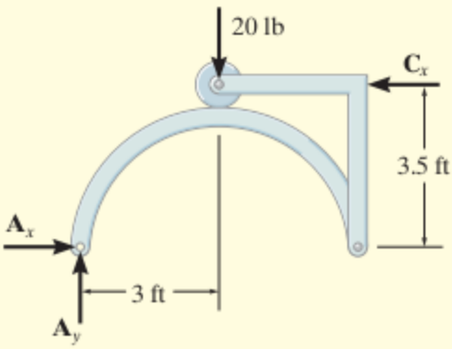
$$T_2 = 2(700.71)\,\text{N} = 1401\,\text{N} = 1.40\,\text{kN} \qquad \textit{Ans.}$$

EXAMPLE 5.12

The smooth disk shown in Fig. 5–23*a* is pinned at D and has a weight of 20 lb. Neglecting the weights of the other members, determine the horizontal and vertical components of reaction at pins B and D.



(a)



SOLUTION

Free-Body Diagrams. The free-body diagrams of the entire frame and each of its members are shown in Fig. 5–23*b*.

Equations of Equilibrium. The eight unknowns can of course be obtained by applying the eight equilibrium equations to each member—three to member AB, three to member BCD, and two to the disk. (Moment equilibrium is automatically satisfied for the disk.) If this is done, however, all the results can be obtained only from a simultaneous solution of some of the equations. (Try it and find out.) To avoid this situation, it is best first to determine the three support reactions on the *entire* frame; then, using these results, the remaining five equilibrium equations can be applied to two other parts in order to solve successively for the other unknowns.

Entire Frame

$$\circlearrowleft +\Sigma M_A = 0; \quad -20\text{ lb}(3\text{ ft}) + C_x(3.5\text{ ft}) = 0 \qquad C_x = 17.1\text{ lb}$$

$$\pm\rightarrow \Sigma F_x = 0; \quad A_x - 17.1\text{ lb} = 0 \qquad A_x = 17.1\text{ lb}$$

$$+\uparrow \Sigma F_y = 0; \quad A_y - 20\text{ lb} = 0 \qquad A_y = 20\text{ lb}$$

Member AB

$$\pm\rightarrow \Sigma F_x = 0; \quad 17.1\text{ lb} - B_x = 0 \qquad B_x = 17.1\text{ lb} \quad \textit{Ans.}$$

$$\circlearrowleft +\Sigma M_B = 0; \quad -20\text{ lb}(6\text{ ft}) + N_D(3\text{ ft}) = 0 \qquad N_D = 40\text{ lb}$$

$$+\uparrow \Sigma F_y = 0; \quad 20\text{ lb} - 40\text{ lb} + B_y = 0 \qquad B_y = 20\text{ lb} \quad \textit{Ans.}$$

Disk

$$\pm\rightarrow \Sigma F_x = 0; \quad D_x = 0 \quad \textit{Ans.}$$

$$+\uparrow \Sigma F_y = 0; \quad 40\text{ lb} - 20\text{ lb} - D_y = 0 \qquad D_y = 20\text{ lb} \quad \textit{Ans.}$$

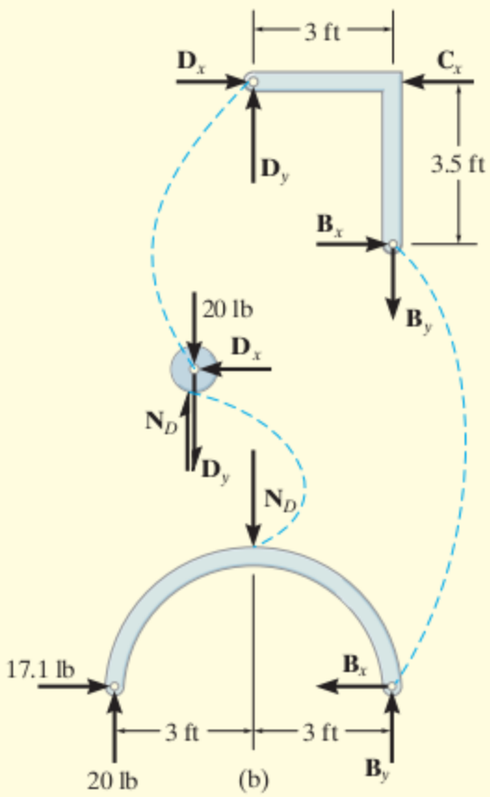
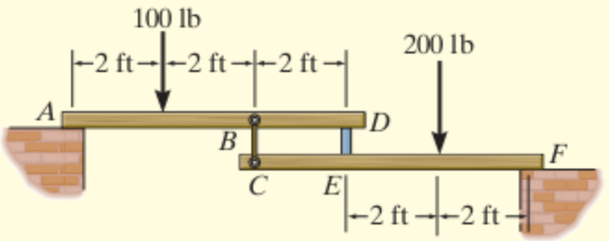


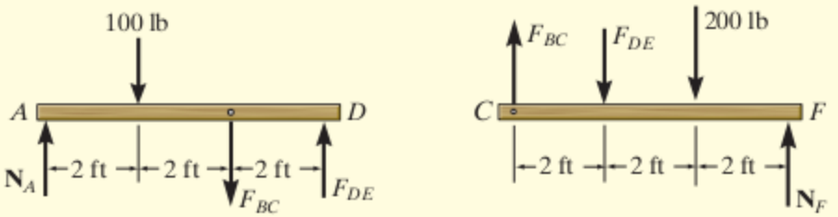
Fig. 5–23

EXAMPLE 5.13

The two planks in Fig. 5–24*a* are connected together by cable *BC* and a smooth spacer *DE*. Determine the reactions at the smooth supports *A* and *F*, and also find the force developed in the cable and spacer.



(a)



(b)

Fig. 5–24

SOLUTION

Free-Body Diagrams. The free-body diagram of each plank is shown in Fig. 5–24*b*. It is important to apply Newton's third law to the interaction forces as shown.

Equations of Equilibrium. For plank *AD*,

$$\circlearrowleft + \Sigma M_A = 0; \qquad F_{DE}(6\text{ ft}) - F_{BC}(4\text{ ft}) - 100\text{ lb}(2\text{ ft}) = 0$$

For plank *CF*,

$$\circlearrowleft + \Sigma M_F = 0; \qquad F_{DE}(4\text{ ft}) - F_{BC}(6\text{ ft}) + 200\text{ lb}(2\text{ ft}) = 0$$

Solving simultaneously,

$$F_{DE} = 140\text{ lb} \quad F_{BC} = 160\text{ lb} \qquad \textit{Ans.}$$

Using these results, for plank *AD*,

$$+\uparrow \Sigma F_y = 0; \qquad N_A + 140\text{ lb} - 160\text{ lb} - 100\text{ lb} = 0$$

$$N_A = 120\text{ lb} \qquad \textit{Ans.}$$

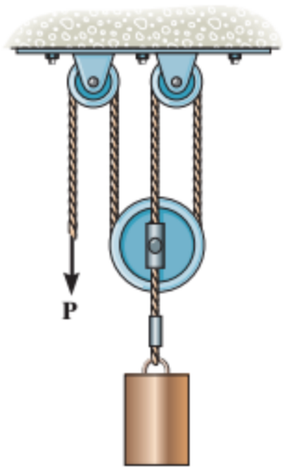
And for plank *CF*,

$$+\uparrow \Sigma F_y = 0; \qquad N_F + 160\text{ lb} - 140\text{ lb} - 200\text{ lb} = 0$$

$$N_F = 180\text{ lb} \qquad \textit{Ans.}$$

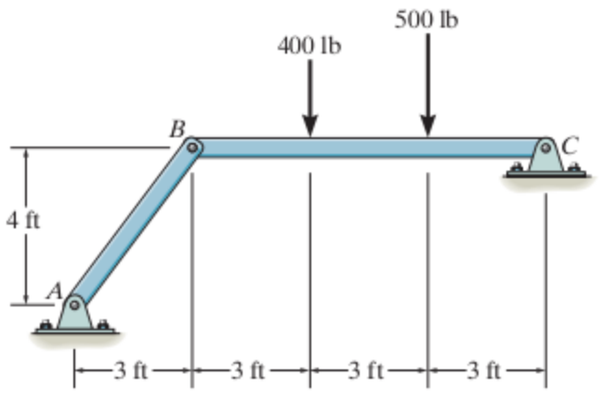
FUNDAMENTAL PROBLEMS

F5–13. Determine the force P needed to hold the 60-lb weight in equilibrium.



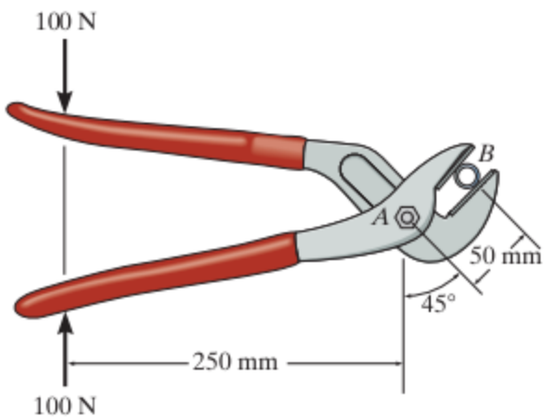
F5–13

F5–14. Determine the horizontal and vertical components of reaction at pin C.



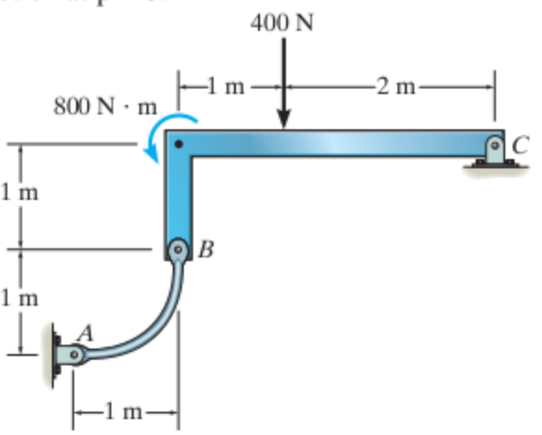
F5–14

F5–15. If a 100-N force is applied to the handles of the pliers, determine the clamping force exerted on the smooth pipe B and the magnitude of the resultant force that one of the members exerts on pin A.



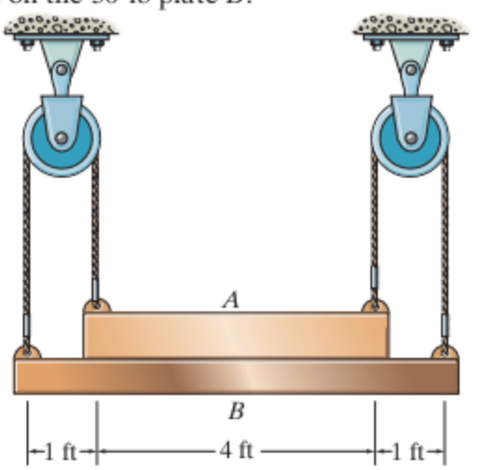
F5–15

F5–16. Determine the horizontal and vertical components of reaction at pin C.



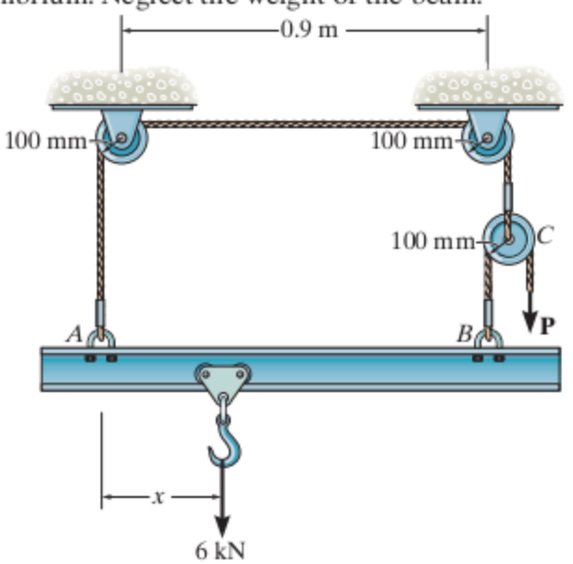
F5–16

F5–17. Determine the normal force that the 100-lb plate A exerts on the 30-lb plate B.



F5–17

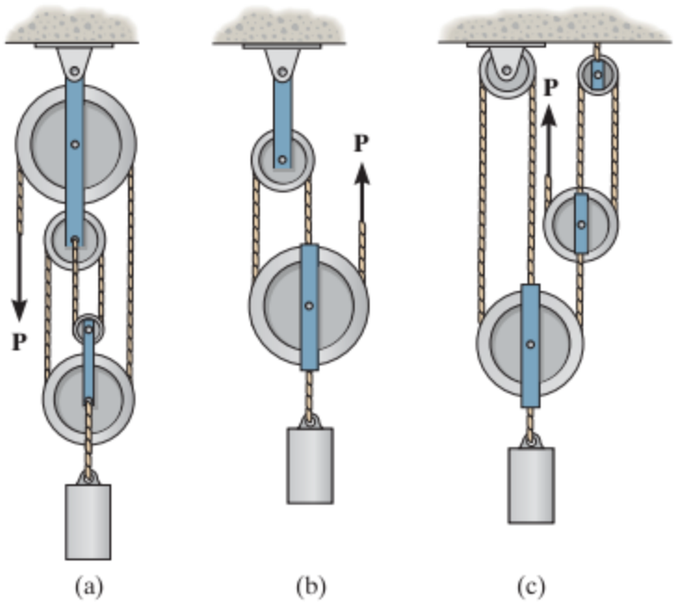
F5–18. Determine the force P needed to lift the load. Also, determine the proper placement x of the hook for equilibrium. Neglect the weight of the beam.



F5–18

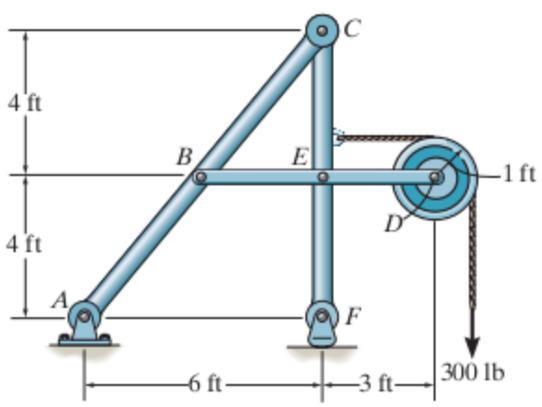
PROBLEMS

5–46. In each case, determine the force **P** required to maintain equilibrium. The block weighs 100 lb.



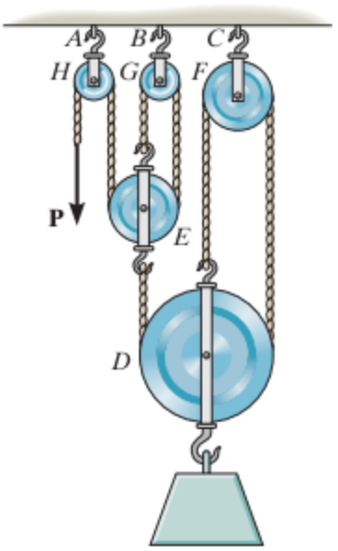
Prob. 5–46

5–47. Determine the horizontal and vertical components of force at C which member ABC exerts on member CEF.



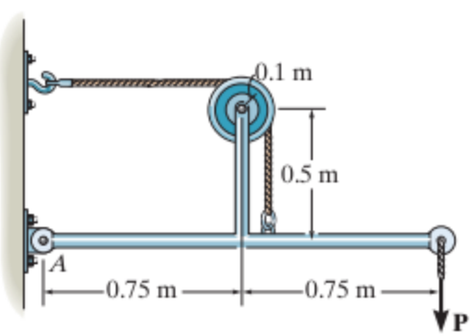
Prob. 5–47

***5–48.** Determine the force P needed to support the 20-kg mass using the *Spanish Burton rig*. Also, what are the reactions at the supporting hooks A, B, and C?



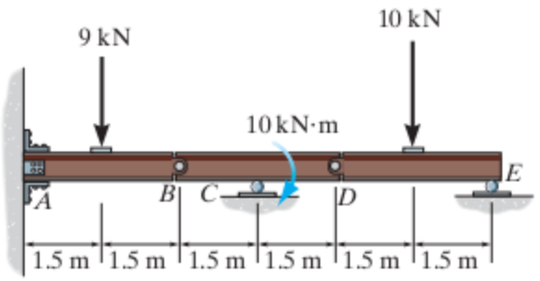
Prob. 5–48

5–49. Determine the greatest force P that can be applied to the frame if the largest force resultant acting at A can have a magnitude of 2 kN.



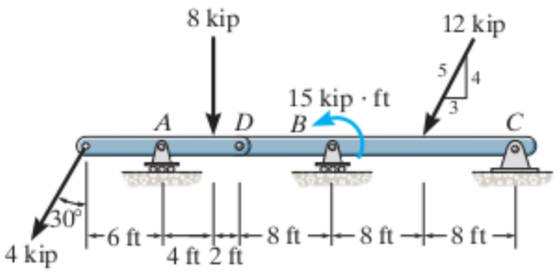
Prob. 5–49

5–50. Determine the support reactions at A, C, and E on the compound beam which is pin connected at B and D.



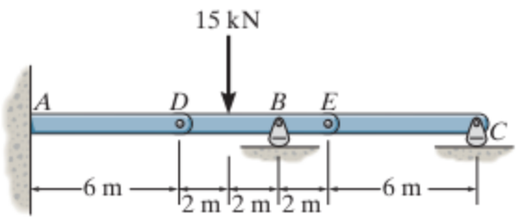
Prob. 5–50

5–51. The compound beam is pin-supported at C and supported by rollers at A and B. There is a hinge (pin) at D. Determine the components of reaction at the supports. Neglect the thickness of the beam.



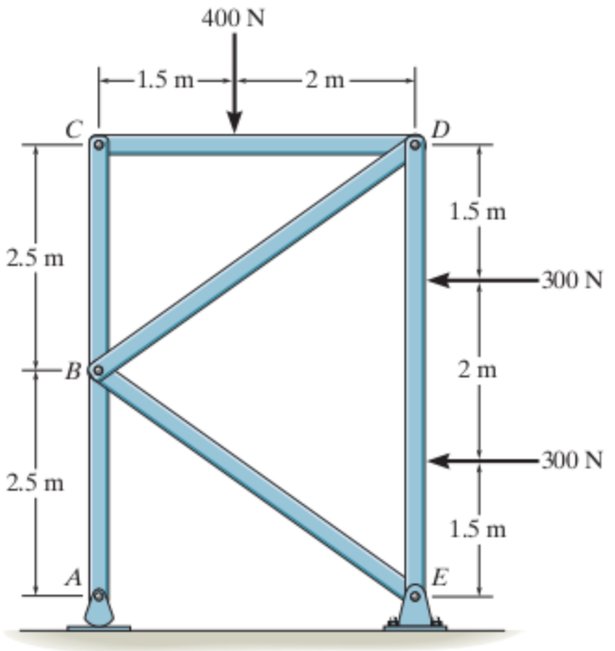
Prob. 5–51

***5–52.** The compound beam is fixed at A and supported by a rocker at B and C. There are hinges (pins) at D and E. Determine the reactions at the supports.



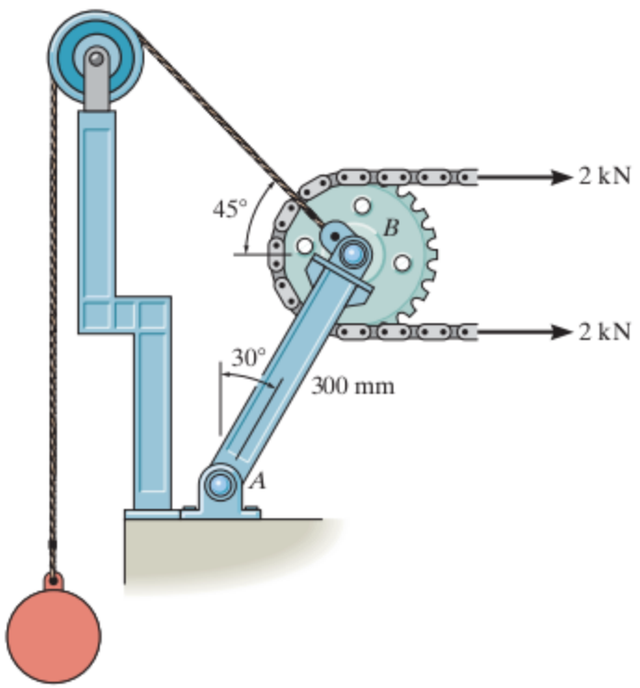
Prob. 5–52

5–53. Determine the horizontal and vertical components of force which the pins at A, B, and C exert on member ABC of the frame.



Prob. 5–53

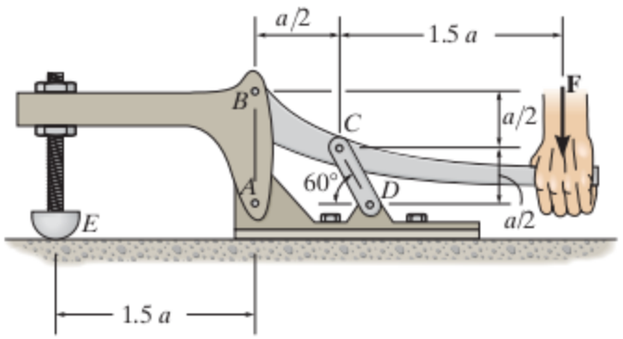
5–54. Determine the required mass of the suspended cylinder if the tension in the chain wrapped around the freely turning gear is to be 2 kN. Also, what is the magnitude of the resultant force on pin A?



Prob. 5–54

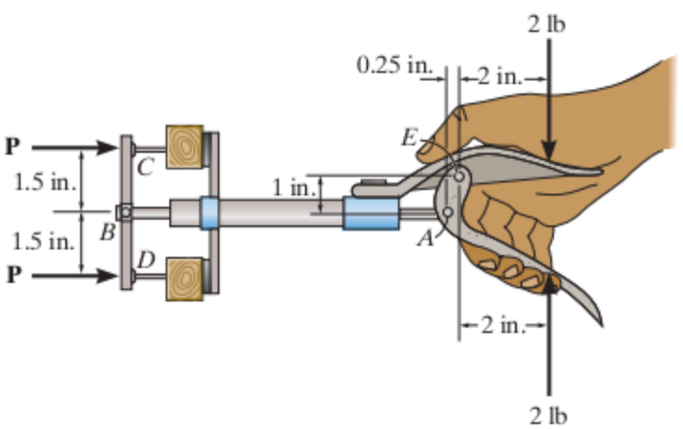
5

5–55. The toggle clamp is subjected to a force **F** at the handle. Determine the vertical clamping force acting at *E*.



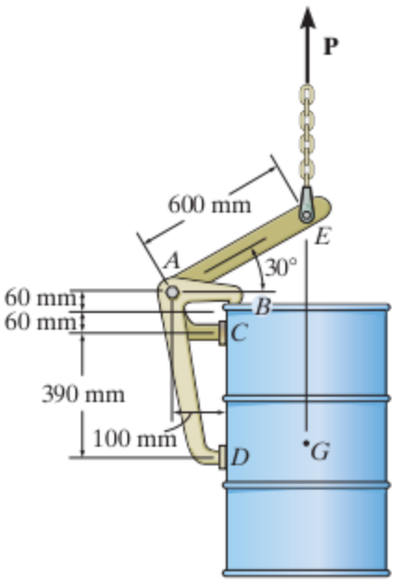
Prob. 5–55

*5–56. When a force of 2 lb is applied to the handles of the brad squeezer, it pulls in the smooth rod *AB*. Determine the force **P** exerted on each of the smooth brads at *C* and *D*.



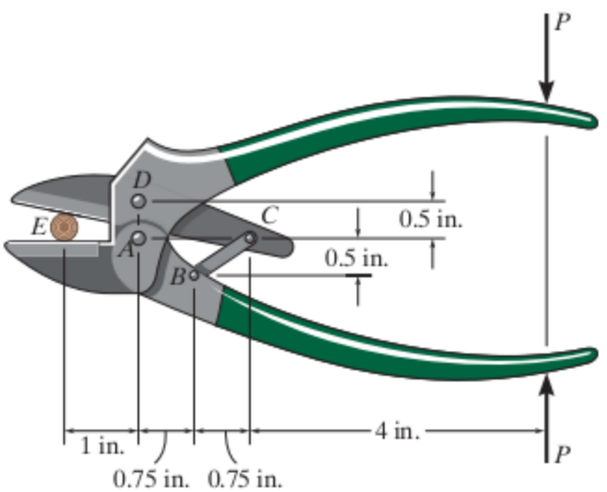
Prob. 5–56

5–57. If the 300-kg drum has a center of mass at point *G*, determine the horizontal and vertical components of force acting at pin *A* and the reactions on the smooth pads *C* and *D*. The grip at *B* on member *DAB* resists both horizontal and vertical components of force at the rim of the drum.



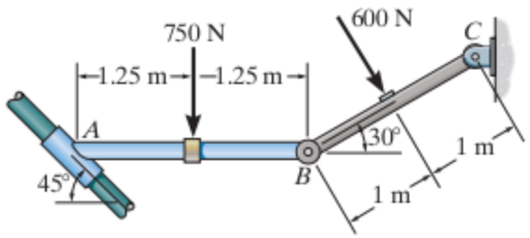
Prob. 5–57

5–58. Determine the required force *P* that must be applied at the blade of the pruning shears so that the blade exerts a normal force of 20 lb on the twig at *E*.



Prob. 5–58

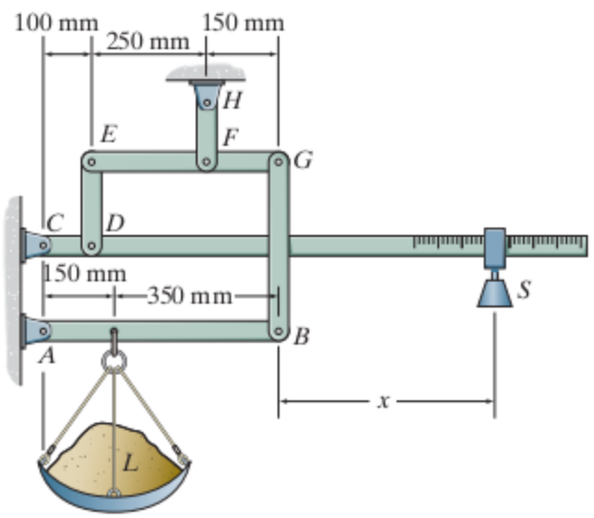
5–59. Determine the reactions on the collar at A and the pin at C. The collar fits over a smooth rod, and rod AB is fixed connected to the collar.



Prob. 5–59

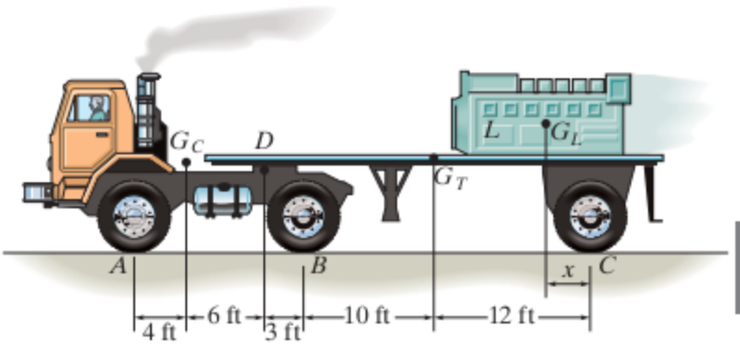
***5–60.** The platform scale consists of a combination of third and first class levers so that the load on one lever becomes the effort that moves the next lever. Through this arrangement, a small weight can balance a massive object. If $x = 450$ mm, determine the required mass of the counterweight S required to balance a 90-kg load, L.

5–61. The platform scale consists of a combination of third and first class levers so that the load on one lever becomes the effort that moves the next lever. Through this arrangement, a small weight can balance a massive object. If $x = 450$ mm and, the mass of the counterweight S is 2 kg, determine the mass of the load L required to maintain the balance.



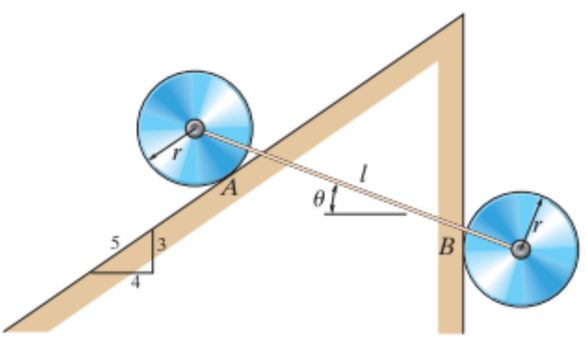
Probs. 5–60/61

5–62. The flat-bed trailer has a weight of 7000 lb and center of gravity at G_T. It is pin connected to the cab at D. The cab has a weight of 6000 lb and center of gravity at G_C. Determine the range of values x for the position of the 2000-lb load L so that no axle is subjected to more than 5500 lb. The load has a center of gravity at G_L.



Prob. 5–62

5–63. The two disks each have a mass of 20 kg and are attached at their centers by an elastic cord that has a stiffness of $k = 2$ kN/m. Determine the stretch of the cord when the system is in equilibrium and the angle θ of the cord.

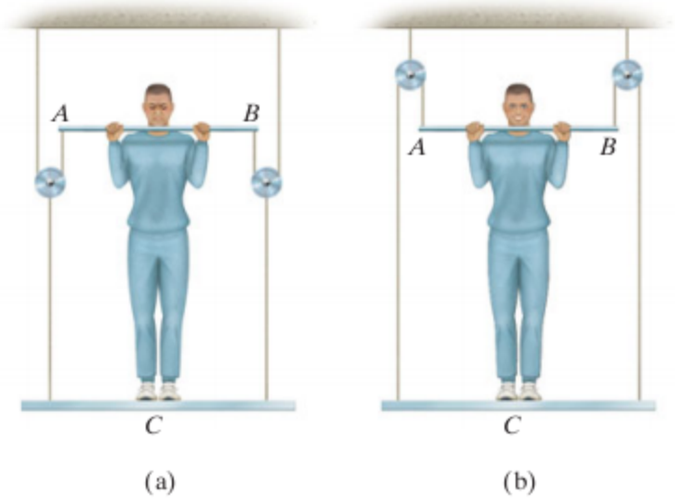


Prob. 5–63

5

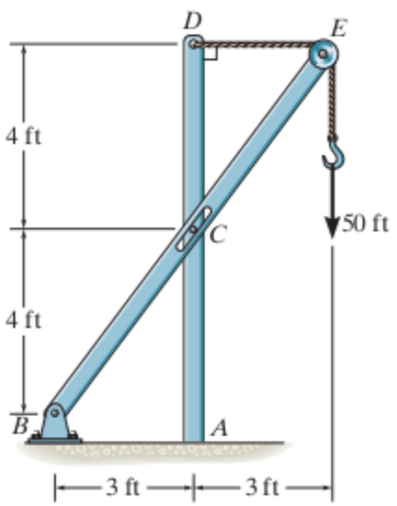
*5–64. A man having a weight of 175 lb attempts to hold himself using one of the two methods shown. Determine the total force he must exert on bar AB in each case and the normal reaction he exerts on the platform at C. Neglect the weight of the platform.

5–65. A man having a weight of 175 lb attempts to hold himself using one of the two methods shown. Determine the total force he must exert on bar AB in each case and the normal reaction he exerts on the platform at C. The platform has a weight of 30 lb.



Probs. 5–64/65

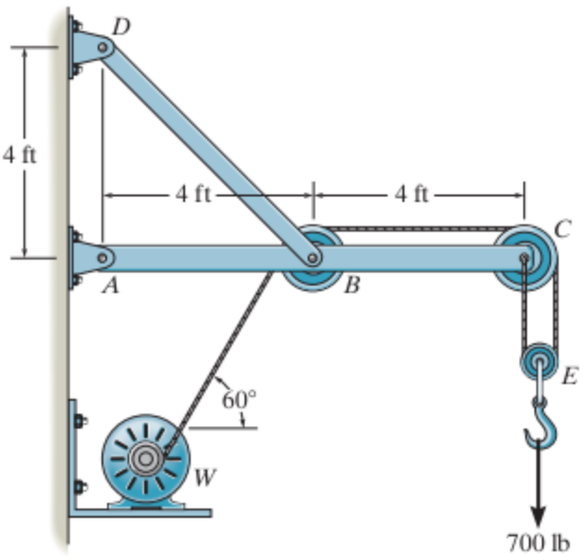
5–66. Determine the horizontal and vertical components of force at pin B and the normal force the pin at C exerts on the smooth slot. Also, determine the moment and horizontal and vertical reactions of force at A. There is a pulley at E.



Prob. 5–66

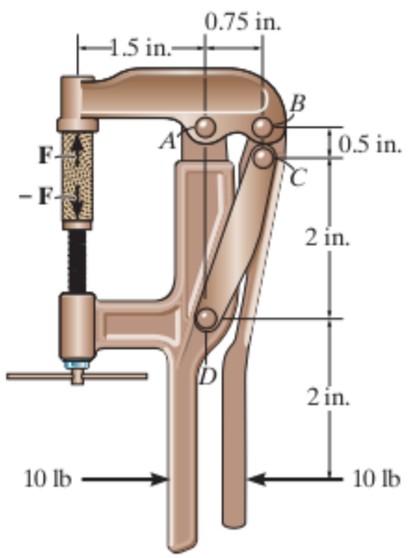
5–67. The wall crane supports a load of 700 lb. Determine the horizontal and vertical components of reaction at the pins A and D. Also, what is the force in the cable at the winch W?

*5–68. The wall crane supports a load of 700 lb. Determine the horizontal and vertical components of reaction at the pins A and D. Also, what is the force in the cable at the winch W? The jib ABC has a weight of 100 lb and member BD has a weight of 40 lb. Each member is uniform and has a center of gravity at its center.



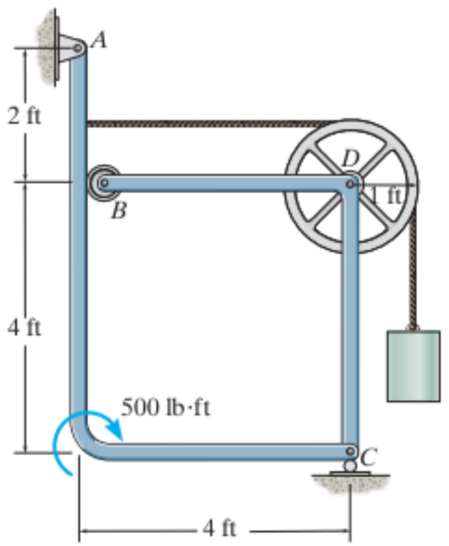
Probs. 5–67/68

5–69. If a force of 10 lb is applied to the grip of the clamp, determine the compressive force F that the wood block exerts on the clamp.



Prob. 5–69

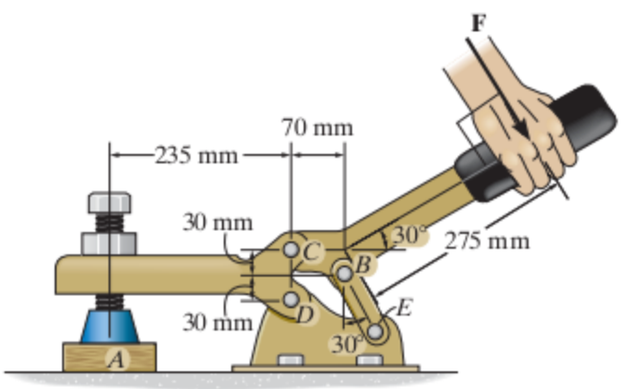
5–70. The two-member frame supports the 200-lb cylinder and 500-lb · ft couple moment. Determine the force of the roller at B on member AC, and the horizontal and vertical components of the forces that the pin at C exerts on member CB and the pin at A exerts on member AC. The roller C does not contact member CB.



Prob. 5–70

5–71. If a clamping force of 300 N is required at A, determine the amount of force **F** that must be applied to the handle of the toggle clamp.

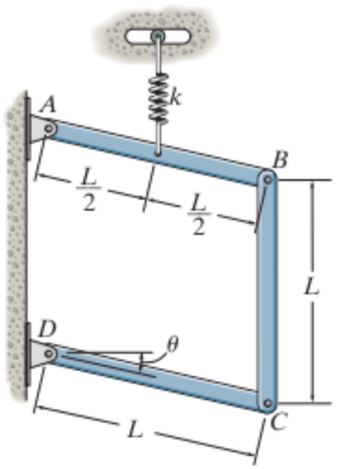
***5–72.** If a force of $F = 350$ N is applied to the handle of the toggle clamp, determine the resulting clamping force at A.



Probs. 5–71/72

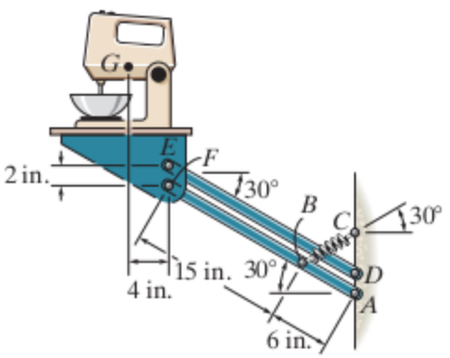
5–73. If each of the three uniform links of the mechanism has a length $L = 3$ ft and weight of $W = 10$ lb, determine the angle θ for equilibrium. The spring has a stiffness of $k = 20$ lb/in. It always remains vertical due to the roller guide and is unstretched when $\theta = 0$.

5–74. If each of the three uniform links of the mechanism has a length L and weight W, determine the angle θ for equilibrium. The spring, which always remains vertical, is unstretched when $\theta = 0°$.



Probs. 5–73/74

5–75. The mechanism is used to hide kitchen appliances under a cabinet by allowing the shelf to rotate downward. If the mixer weighs 10 lb, is centered on the shelf, and has a mass center at G, determine the stretch in the spring necessary to hold the shelf in the equilibrium position shown. There is a similar mechanism on each side of the shelf, so that each mechanism supports 5 lb of the load. The springs each have a stiffness of $k = 4$ lb/in.



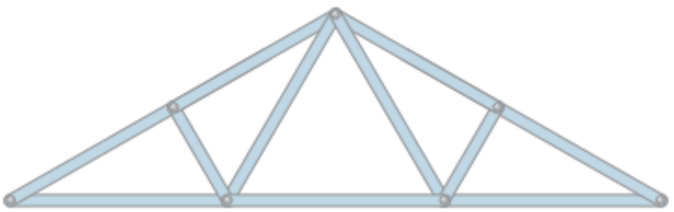
Prob. 5–75

5

CHAPTER REVIEW

Simple Truss

A simple truss consists of triangular elements connected together by pinned joints. The forces within its members can be determined by assuming the members are all two-force members, connected concurrently at each joint. The members are either in tension or compression, or carry no force.



Roof truss

Method of Joints

The method of joints states that if a truss is in equilibrium, then each of its joints is also in equilibrium. For a plane truss, the concurrent force system at each joint must satisfy force equilibrium.

To obtain a numerical solution for the forces in the members, select a joint that has a free-body diagram with at most two unknown forces and one known force. (This may require first finding the reactions at the supports.)

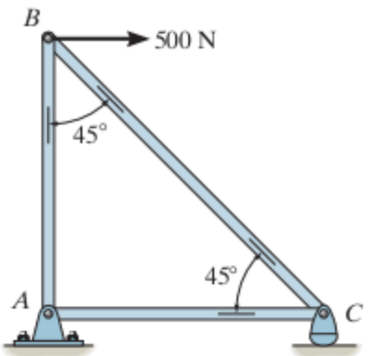
Once a member force is determined, use its value and apply it to an adjacent joint.

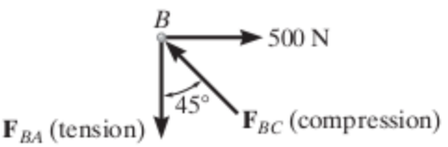
Remember that forces that are found to *pull* on the joint are *tensile forces*, and those that *push* on the joint are *compressive forces*.

To avoid a simultaneous solution of two equations, set one of the coordinate axes along the line of action of one of the unknown forces and sum forces perpendicular to this axis. This will allow a direct solution for the other unknown.

The analysis can also be simplified by first identifying all the zero-force members.

$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$





Method of Sections

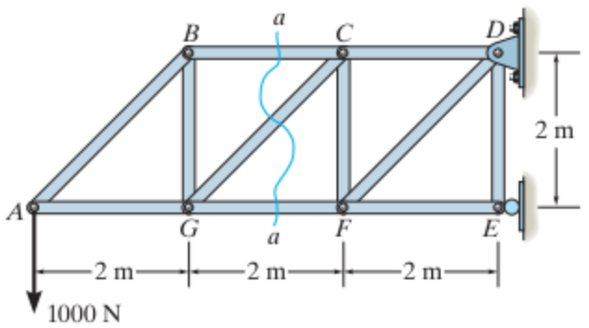
The method of sections states that if a truss is in equilibrium, then each segment of the truss is also in equilibrium. Pass a section through the truss and the member whose force is to be determined. Then draw the free-body diagram of the sectioned part having the least number of forces on it.

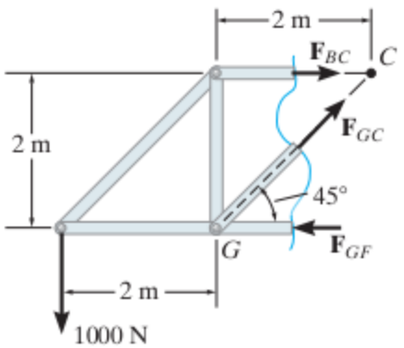
Sectioned members subjected to *pulling* are in *tension*, and those that are subjected to *pushing* are in *compression*.

Three equations of equilibrium are available to determine the unknowns.

If possible, sum forces in a direction that is perpendicular to two of the three unknown forces. This will yield a direct solution for the third force.

Sum moments about the point where the lines of action of two of the three unknown forces intersect, so that the third unknown force can be determined directly.





$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$
$$\Sigma M_O = 0$$

$$+\uparrow \Sigma F_y = 0$$
$$-1000\ \text{N} + F_{GC} \sin 45° = 0$$
$$F_{GC} = 1.41\ \text{kN (T)}$$

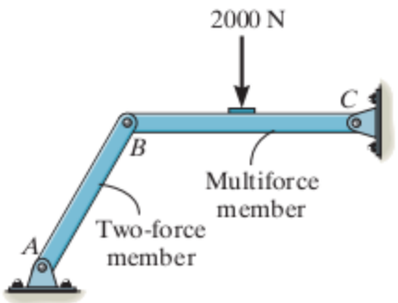
$$\circlearrowleft +\Sigma M_C = 0$$
$$1000\ \text{N}(4\ \text{m}) - F_{GF}(2\ \text{m}) = 0$$
$$F_{GF} = 2\ \text{kN (C)}$$

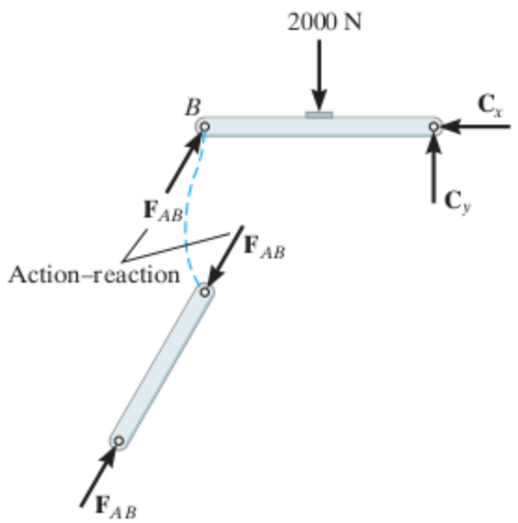
Frames and Machines

Frames and machines are structures that contain one or more multiforce members, that is, members with three or more forces or couples acting on them. Frames are designed to support loads, and machines transmit and alter the effect of forces.

The forces acting at the joints of a frame or machine can be determined by drawing the free-body diagrams of each of its members or parts. The principle of action–reaction should be carefully observed when indicating these forces on the free-body diagram of each adjacent member or pin. For a coplanar force system, there are three equilibrium equations available for each member.

To simplify the analysis, be sure to recognize all two-force members. They have equal but opposite collinear forces at their ends.

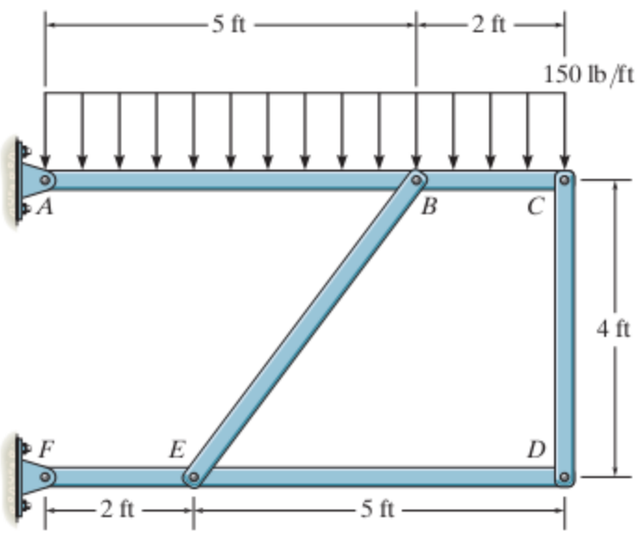




5

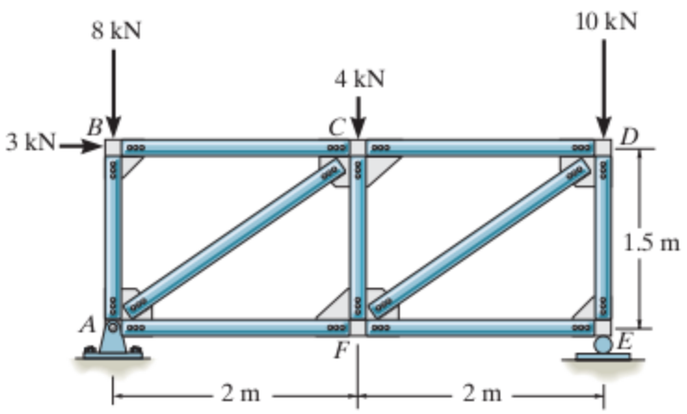
REVIEW PROBLEMS

***5–76.** Determine the resultant forces at pins B and C on member ABC of the four-member frame.



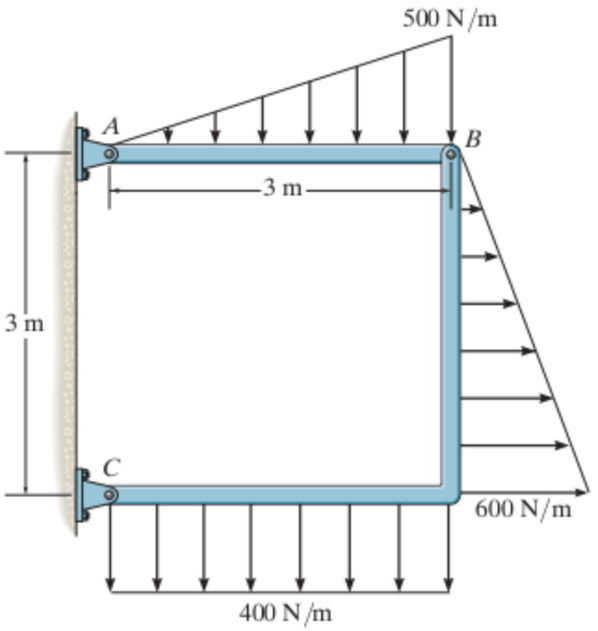
Prob. 5–76

5–77. Determine the force in each member of the truss and state if the members are in tension or compression.



Prob. 5–77

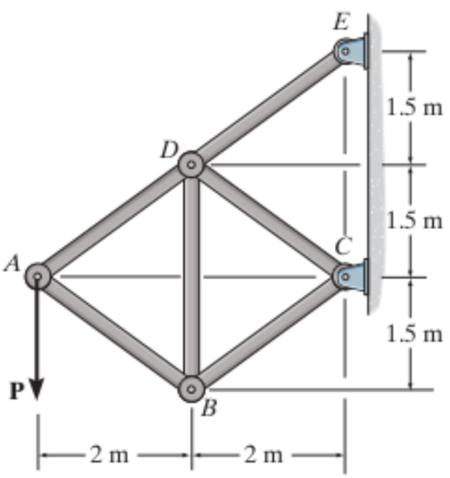
5–78. Determine the horizontal and vertical components of force at pins A and C of the two-member frame.



Prob. 5–78

5–79. Determine the greatest force P that can be applied to the truss so that none of the members are subjected to a force exceeding either 2.5 kN in tension or 2 kN in compression.

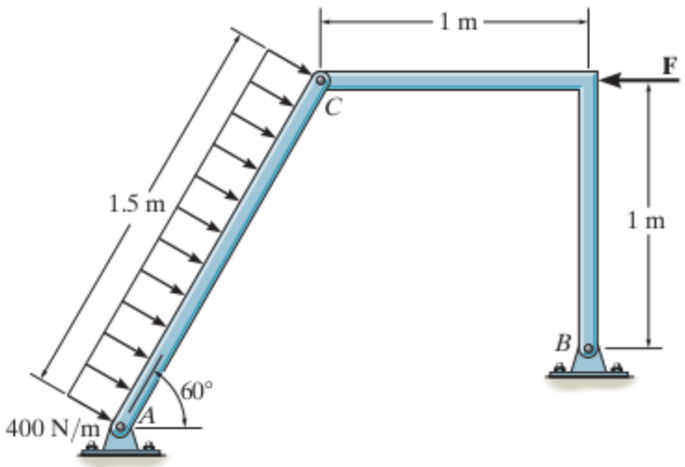
***5–80.** Determine the force in each member of the truss, and state if the members are in tension or compression. Set $P = 5$ kN.



Probs. 5–79/80

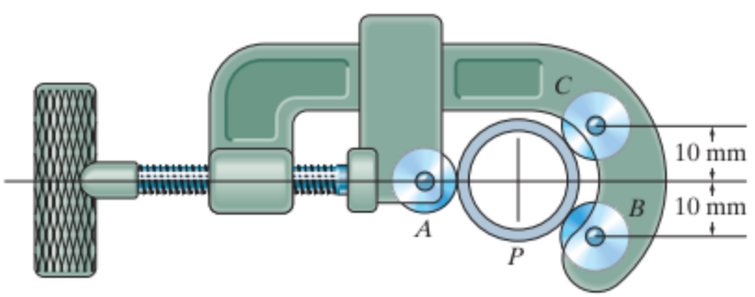
5–81. Determine the horizontal and vertical components of reaction that the pins A and B exert on the two-member frame. Set $F = 0$.

5–82. Determine the horizontal and vertical components of reaction that pins A and B exert on the two-member frame. Set $F = 500$ N.



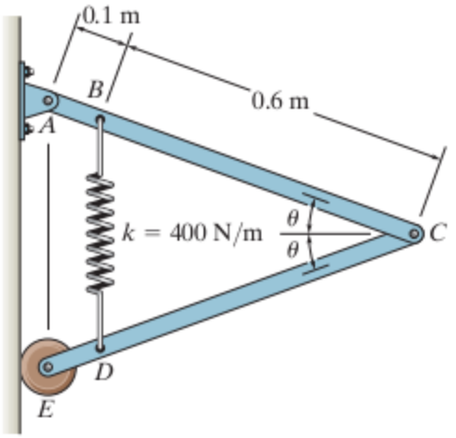
Probs. 5–81/82

5–83. The pipe cutter is clamped around the pipe P. If the wheel at A exerts a normal force of $F_A = 80$ N on the pipe, determine the normal forces of wheels B and C on the pipe. Also compute the pin reaction on the wheel at C. The three wheels each have a radius of 7 mm and the pipe has an outer radius of 10 mm.



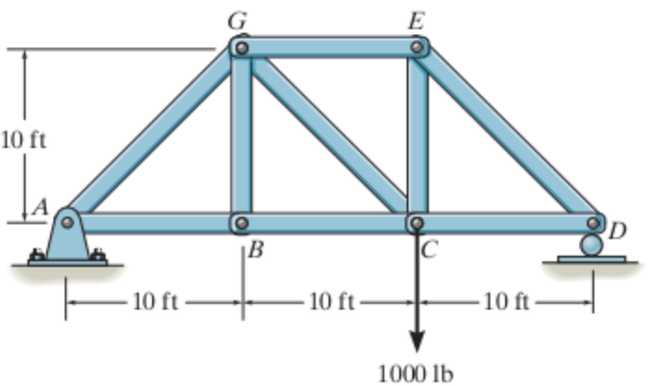
Prob. 5–83

***5–84.** The spring has an unstretched length of 0.3 m. Determine the mass m of each uniform link if the angle $\theta = 20°$ for equilibrium.



Prob. 5–84

5–85. Determine the force in each member of the truss and state if the members are in tension or compression.



Prob. 5–85

5

Chapter 6

The design of these structural members requires calculation of the centroid of the cross-sectional area and its moment of inertia. In this chapter, we will discuss how this is done.

Center of Gravity, Centroid, and Moment of Inertia

CHAPTER OBJECTIVES

- To discuss the concept of the center of gravity, center of mass, and the centroid.
- To show how to determine the location of the center of gravity and centroid for a body of arbitrary shape and a composite body.
- To present a method for finding the resultant of a general distributed loading.
- To show how to determine the moment of inertia of an area.

6.1 Center of Gravity, Center of Mass, and the Centroid of a Body

In this section we will first show how to locate the center of gravity for a body, and then we will show that the center of mass and the centroid of a body can be developed using this same method.

Center of Gravity. A body is composed of an infinite number of particles of differential size, and so if the body is located within a gravitational field, then each of these particles will have a weight dW, Fig. 6–1*a*. These weights will form an approximately parallel force system, and the resultant of this system is the total weight of the body, which passes through a single point called the *center of gravity*, G, Fig. 6–1*b*.*

*This is true as long as the gravity field is assumed to have the same magnitude and direction everywhere. That assumption is appropriate for most engineering applications, since gravity does not vary appreciably between, for instance, the bottom and the top of a building.

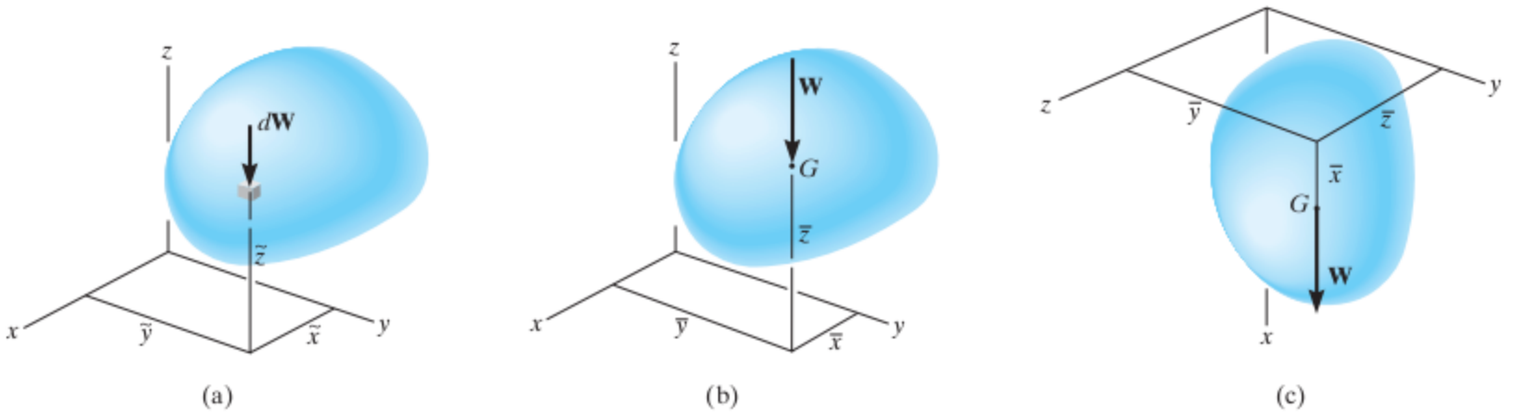


Fig. 6–1

Using the methods outlined in Sec. 3.8, the weight of the body is the sum of the weights of all of its particles, that is

$$+\downarrow F_R = \Sigma F_z; \qquad W = \int dW$$

The location of the center of gravity, measured from the y axis, is determined by equating the moment of W about the y axis, Fig. 6–1b, to the sum of the moments of the weights of the particles about this same axis. If dW is located at point $(\tilde{x}, \tilde{y}, \tilde{z})$, Fig. 6–1$a$, then

$$(M_R)_y = \Sigma M_y; \qquad \bar{x}W = \int \tilde{x}\,dW$$

Similarly, if moments are summed about the x axis,

$$(M_R)_x = \Sigma M_x; \qquad \bar{y}W = \int \tilde{y}\,dW$$

Finally, imagine that the body is fixed within the coordinate system and this system is rotated 90° about the y axis, Fig. 6–1c. Then the sum of the moments about the y axis gives

$$(M_R)_y = \Sigma M_y; \qquad \bar{z}W = \int \tilde{z}\,dW$$

Therefore, the location of the center of gravity G with respect to the x, y, z axes becomes

$$\bar{x} = \frac{\int \tilde{x}\,dW}{\int dW} \qquad \bar{y} = \frac{\int \tilde{y}\,dW}{\int dW} \qquad \bar{z} = \frac{\int \tilde{z}\,dW}{\int dW} \tag{6–1}$$

Here

$\bar{x}, \bar{y}, \bar{z}$ are the coordinates of the center of gravity G, Fig. 6–1b
$\tilde{x}, \tilde{y}, \tilde{z}$ are the coordinates of each particle in the body, Fig. 6–1a

Center of Mass of a Body. In order to study the *dynamic response* or accelerated motion of a body, it becomes important to locate the body's center of mass C_m, Fig. 6–2. This location can be determined by substituting $dW = g\,dm$ into Eqs. 6–1. Since g is constant, it cancels out, and so

$$\bar{x} = \frac{\int \tilde{x}\,dm}{\int dm} \qquad \bar{y} = \frac{\int \tilde{y}\,dm}{\int dm} \qquad \bar{z} = \frac{\int \tilde{z}\,dm}{\int dm} \tag{6–2}$$

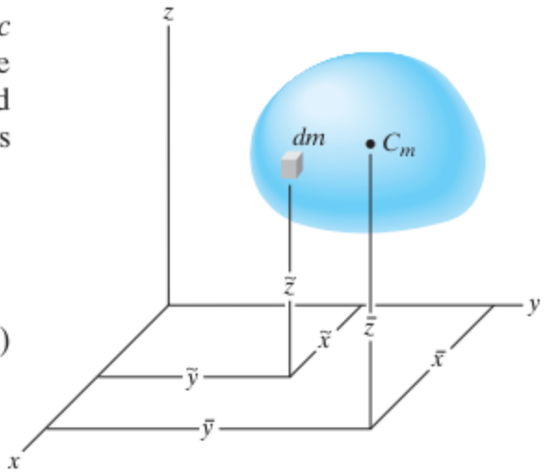


Fig. 6–2

Centroid of a Volume. If the body in Fig. 6–3 is made from a homogeneous material, then its density ρ (rho) will be constant. Therefore, a differential element of volume dV has a mass $dm = \rho\,dV$. Substituting this into Eqs. 6–2 and canceling out ρ, we obtain formulas that locate the *centroid* C or geometric center of the body; namely

$$\bar{x} = \frac{\int_V \tilde{x}\,dV}{\int_V dV} \qquad \bar{y} = \frac{\int_V \tilde{y}\,dV}{\int_V dV} \qquad \bar{z} = \frac{\int_V \tilde{z}\,dV}{\int_V dV} \tag{6–3}$$

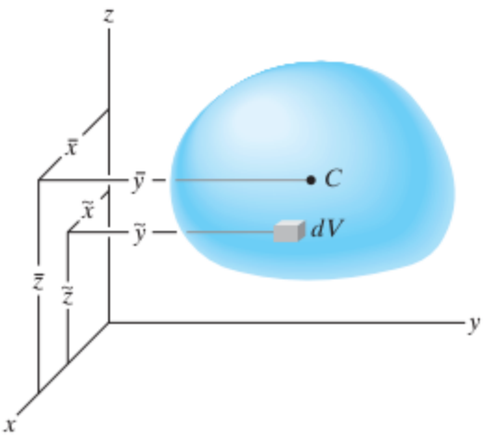


Fig. 6–3

These equations represent a balance of the moments of the volume of the body. Therefore, if the volume possesses two planes of symmetry, then its centroid must lie along the line of intersection of these two planes. For example, the cone in Fig. 6–4 has a centroid that lies on the y axis so that $\bar{x} = \bar{z} = 0$. The location $\bar{y}$ can be found using a single integration by choosing a differential element represented by a *thin disk* having a thickness dy and radius $r = z$. Its volume is $dV = \pi r^2\,dy = \pi z^2\,dy$ and its centroid is at $\tilde{x} = 0, \tilde{y} = y, \tilde{z} = 0$.

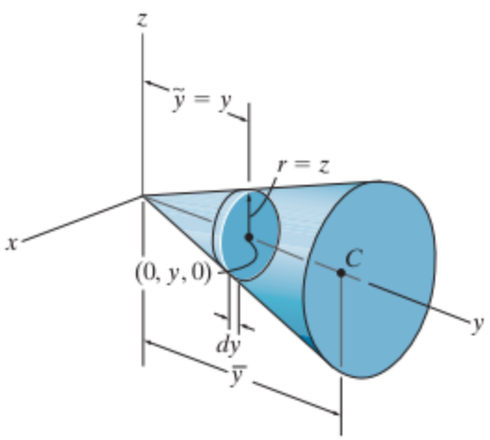
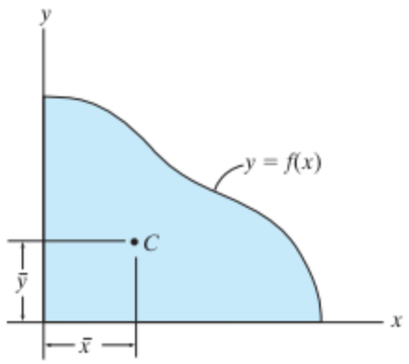
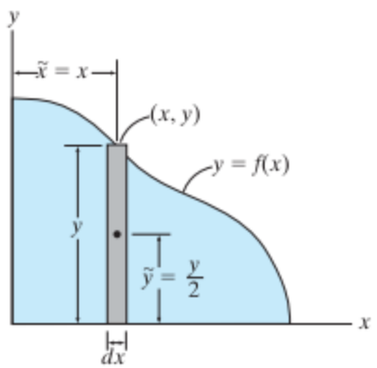


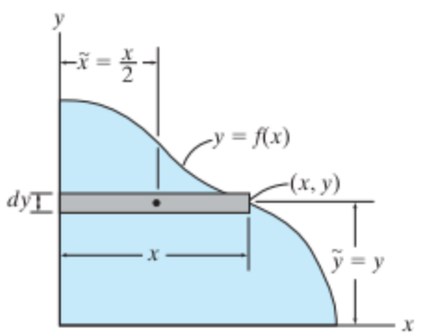
Fig. 6–4



(a)



(b)



(c)

Fig. 6–5

Centroid of an Area. If an area lies in the x–y plane and is bounded by the curve $y = f(x)$, as shown in Fig. 6–5*a*, then its centroid will be in this plane and can be determined from integrals similar to Eqs. 6–3, namely,

$$\bar{x} = \frac{\int_A \tilde{x}\, dA}{\int_A dA} \qquad \bar{y} = \frac{\int_A \tilde{y}\, dA}{\int_A dA} \tag{6–4}$$

These integrals can be evaluated by performing a *single integration* if we use a *rectangular strip* for the differential area element. For example, if a vertical strip is used, Fig. 6–5*b*, the area of the element is $dA = y\, dx$, and its centroid is located at $\tilde{x} = x$ and $\tilde{y} = y/2$. If we consider a horizontal strip, Fig. 6–5*c*, then $dA = x\, dy$, and its centroid is located at $\tilde{x} = x/2$ and $\tilde{y} = y$.

Important Points

- The centroid represents the geometric center of a body. This point coincides with the center of mass or the center of gravity only if the material composing the body is uniform or homogeneous.
- Formulas used to locate the center of gravity or the centroid simply represent a balance between the sum of moments of all the parts of the system and the moment of the "resultant" for the system.
- In some cases the centroid is located at a point that is not on the object, as in the case of a ring, where the centroid is at its center. Also, this point will lie on any axis of symmetry for the body.

Procedure for Analysis

The center of gravity or centroid of an object or shape can be determined by single integrations using the following procedure.

Differential Element.

- Select an appropriate coordinate system, specify the coordinate axes, and then choose a differential element for integration.
- For areas the element is generally a rectangle of area dA, having a finite length and differential width.
- For volumes the element can be a circular disk of volume dV, having a finite radius and differential thickness.
- Locate the element so that it touches the arbitrary point (x, y, z) on the curve that defines the boundary of the shape.

Size and Moment Arms.

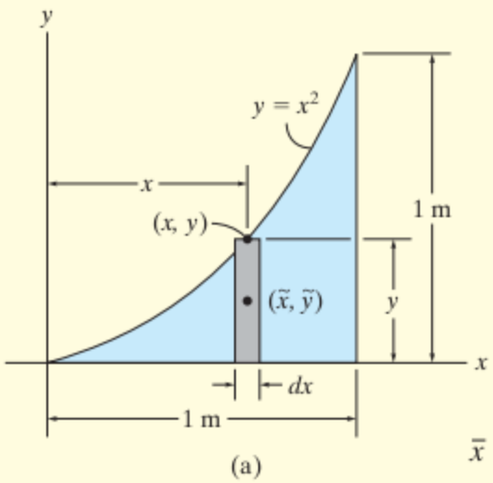
- Express the area dA, or volume dV of the element in terms of the coordinates describing the curve.
- Express the moment arms $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ for the centroid or center of gravity of the element in terms of the coordinates describing the curve.

Integrations.

- Substitute the formulations for $\tilde{x}$, $\tilde{y}$, $\tilde{z}$, and dA or dV into the appropriate equations (Eqs. 6–1 through 6–5).
- Express the function in the integrand in terms of the *same variable as the differential thickness of the element.*
- The limits of the integral are defined from the two extreme locations of the element's differential thickness, so that when the elements are "summed" or the integration performed, the entire region is covered.*

*Formulas for integration are given in Appendix A.

EXAMPLE 6.1



(a)

Locate the centroid of the area shown in Fig. 6–6*a*.

SOLUTION I

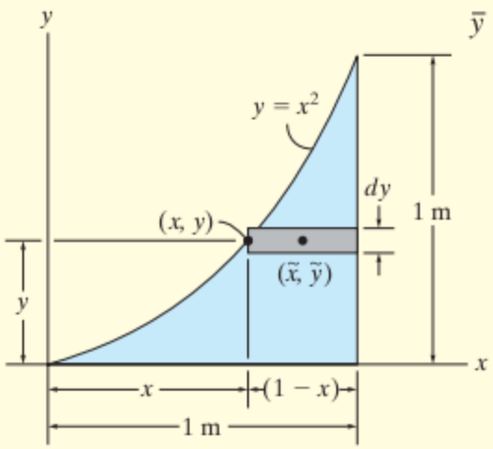
Differential Element. A differential element of thickness dx is shown in Fig. 6–6*a*. The element intersects the curve at the *arbitrary point* (x, y), and so it has a height y.

Area and Moment Arms. The area of the element is $dA = y\,dx$, and its centroid is located at $\tilde{x} = x$, $\tilde{y} = y/2$.

Integrations. Applying Eqs. 6–4 and integrating with respect to x yields

$$\bar{x} = \frac{\int_A \tilde{x}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} xy\,dx}{\int_0^{1\text{ m}} y\,dx} = \frac{\int_0^{1\text{ m}} x^3\,dx}{\int_0^{1\text{ m}} x^2\,dx} = \frac{0.250}{0.333} = 0.75\text{ m} \quad \textit{Ans.}$$

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} (y/2)y\,dx}{\int_0^{1\text{ m}} y\,dx} = \frac{\int_0^{1\text{ m}} (x^2/2)x^2\,dx}{\int_0^{1\text{ m}} x^2\,dx} = \frac{0.100}{0.333} = 0.3\text{ m} \quad \textit{Ans.}$$



(b)

Fig. 6–6

SOLUTION II

Differential Element. The differential element of thickness dy is shown in Fig. 6–6*b*. The element intersects the curve at the *arbitrary point* (x, y), and so it has a length $(1 - x)$.

Area and Moment Arms. The area of the element is $dA = (1 - x)\,dy$, and its centroid is located at

$$\tilde{x} = x + \left(\frac{1 - x}{2}\right) = \frac{1 + x}{2},\quad \tilde{y} = y$$

Integrations. Applying Eqs. 6–4 and integrating with respect to y, we obtain

$$\bar{x} = \frac{\int_A \tilde{x}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} [(1 + x)/2](1 - x)\,dy}{\int_0^{1\text{ m}} (1 - x)\,dy} = \frac{\frac{1}{2}\int_0^{1\text{ m}} (1 - y)\,dy}{\int_0^{1\text{ m}} (1 - \sqrt{y})\,dy} = \frac{0.250}{0.333} = 0.75\text{ m} \quad \textit{Ans.}$$

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} y(1 - x)\,dy}{\int_0^{1\text{ m}} (1 - x)\,dy} = \frac{\int_0^{1\text{ m}} (y - y^{3/2})\,dy}{\int_0^{1\text{ m}} (1 - \sqrt{y})\,dy} = \frac{0.100}{0.333} = 0.3\text{ m} \quad \textit{Ans.}$$

NOTE: Plot these results and notice that they seem reasonable. Also, for this problem, elements of thickness dx offer a simpler solution.

EXAMPLE 6.2

Locate the centroid of the semi-elliptical area shown in Fig. 6–7*a*.

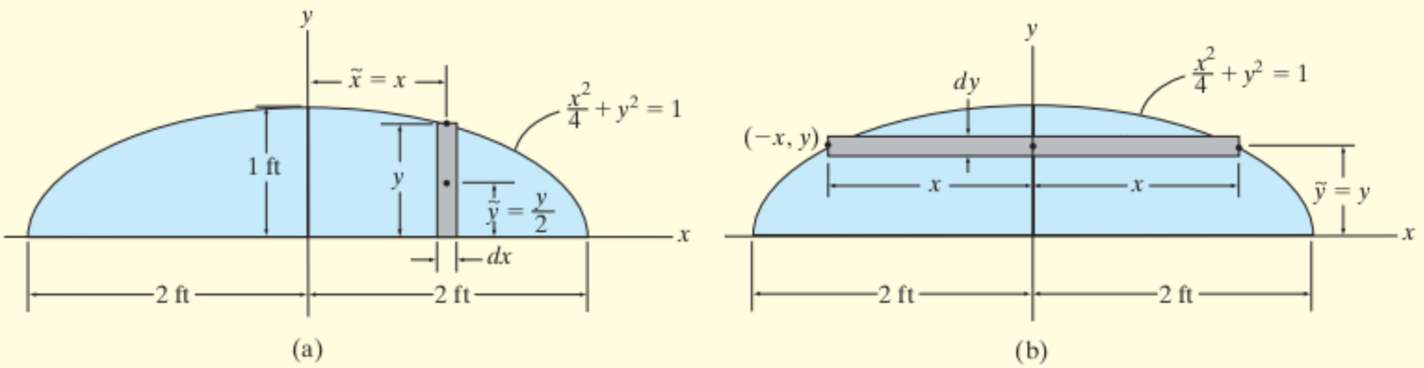


Fig. 6–7

SOLUTION I

Differential Element. The rectangular differential element parallel to the y axis shown shaded in Fig. 6–7*a* will be considered. This element has a thickness of dx and a height of y.

Area and Moment Arms. Thus, the area is $dA = y\,dx$, and its centroid is located at $\tilde{x} = x$ and $\tilde{y} = y/2$.

Integration. Since the area is symmetrical about the y axis,

$$\bar{x} = 0 \qquad \textit{Ans.}$$

Applying the second of Eqs. 6–4 with $y = \sqrt{1 - \frac{x^2}{4}}$, we have

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_{-2\,\text{ft}}^{2\,\text{ft}} \frac{y}{2}(y\,dx)}{\int_{-2\,\text{ft}}^{2\,\text{ft}} y\,dx} = \frac{\frac{1}{2}\int_{-2\,\text{ft}}^{2\,\text{ft}} \left(1 - \frac{x^2}{4}\right)dx}{\int_{-2\,\text{ft}}^{2\,\text{ft}} \sqrt{1 - \frac{x^2}{4}}\,dx} = \frac{4/3}{\pi} = 0.424\ \text{ft} \qquad \textit{Ans.}$$

SOLUTION II

Differential Element. The shaded rectangular differential element of thickness dy and width $2x$, parallel to the x axis, will be considered, Fig. 6–7*b*.

Area and Moment Arms. The area is $dA = 2x\,dy$, and its centroid is at $\tilde{x} = 0$ and $\tilde{y} = y$.

Integration. Applying the second of Eqs. 6–4, with $x = 2\sqrt{1 - y^2}$, we have

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^{1\,\text{ft}} y(2x\,dy)}{\int_0^{1\,\text{ft}} 2x\,dy} = \frac{\int_0^{1\,\text{ft}} 4y\sqrt{1 - y^2}\,dy}{\int_0^{1\,\text{ft}} 4\sqrt{1 - y^2}\,dy} = \frac{4/3}{\pi}\,\text{ft} = 0.424\ \text{ft} \qquad \textit{Ans.}$$

6

EXAMPLE 6.3

Locate the $\bar{y}$ centroid for the paraboloid of revolution, shown in Fig. 6–8.

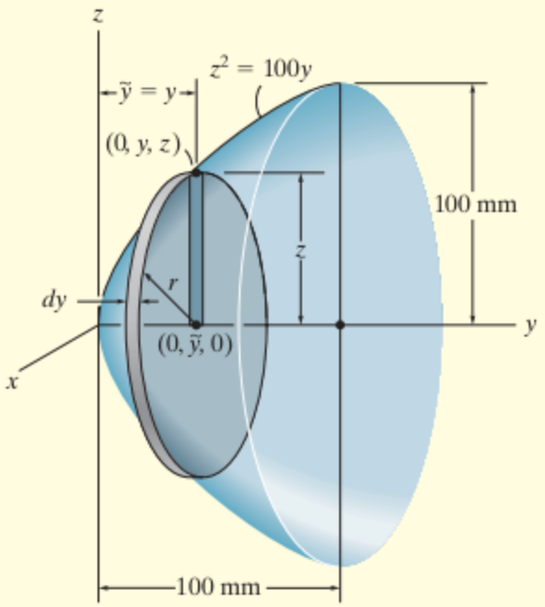


Fig. 6–8

SOLUTION

Differential Element. An element having the shape of a *thin disk* is chosen. This element has a thickness dy, it intersects the generating curve at the *arbitrary point* $(0, y, z)$, and so its radius is $r = z$.

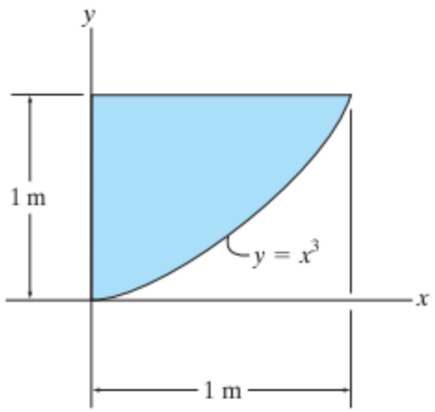
Volume and Moment Arm. The volume of the element is $dV = (\pi z^2)\, dy$, and its centroid is located at $\tilde{y} = y$.

Integration. Applying the second of Eqs. 6–3 and integrating with respect to y yields

$$\bar{y} = \frac{\int_V \tilde{y}\, dV}{\int_V dV} = \frac{\int_0^{100\text{ mm}} y(\pi z^2)\, dy}{\int_0^{100\text{ mm}} (\pi z^2)\, dy} = \frac{100\pi \int_0^{100\text{ mm}} y^2\, dy}{100\pi \int_0^{100\text{ mm}} y\, dy} = 66.7\text{ mm} \qquad \textit{Ans.}$$

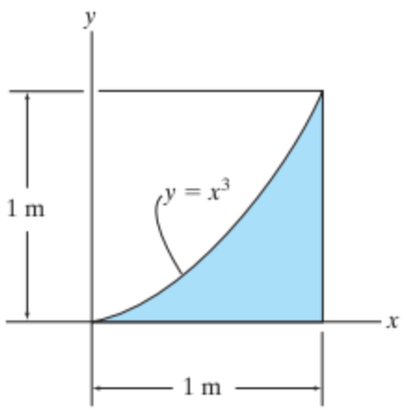
FUNDAMENTAL PROBLEMS

F6–1. Determine the centroid $(\bar{x}, \bar{y})$ of the shaded area.



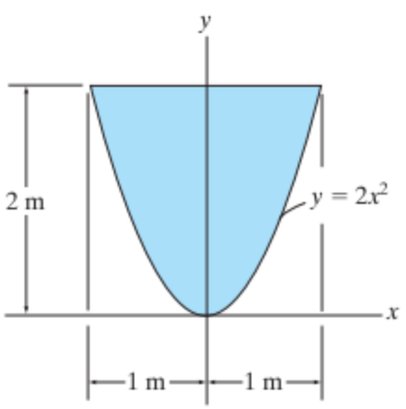
F6–1

F6–2. Determine the centroid $(\bar{x}, \bar{y})$ of the shaded area.



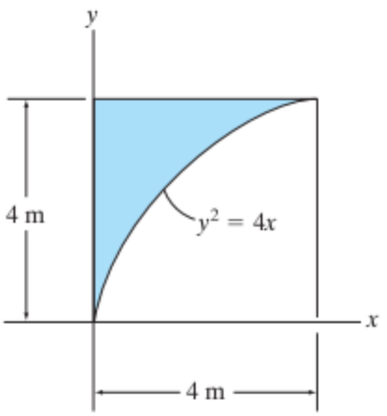
F6–2

F6–3. Determine the centroid $\bar{y}$ of the shaded area.



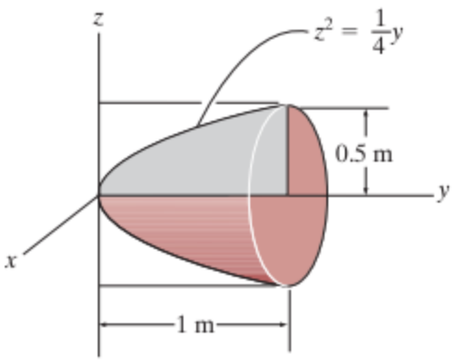
F6–3

F6–4. Determine the area and the centroid $(\bar{x}, \bar{y})$ of the area.



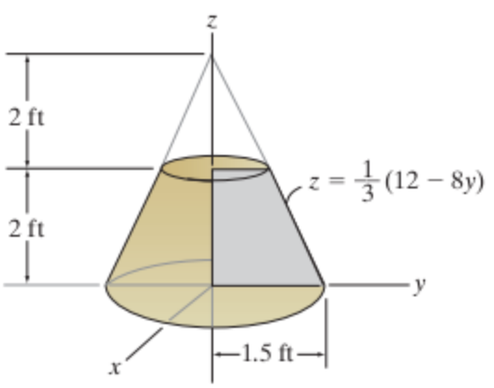
F6–4

F6–5. Locate the centroid $\bar{y}$ of the homogeneous solid formed by revolving the shaded area about the y axis.



F6–5

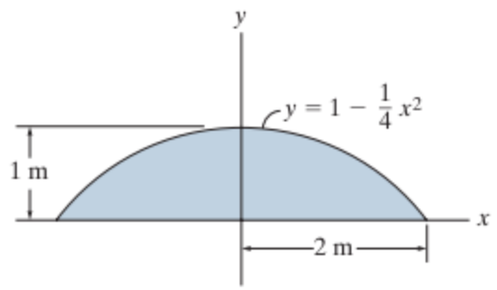
F6–6. Locate the centroid $\bar{z}$ of the homogeneous solid formed by revolving the shaded area about the z axis.



F6–6

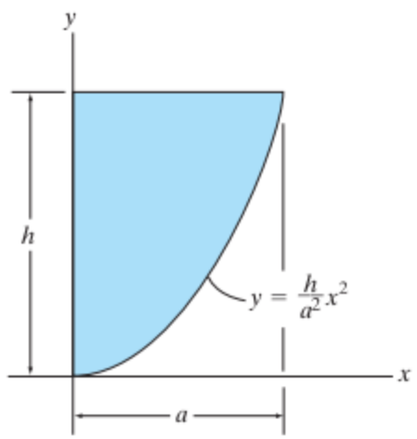
PROBLEMS

6–1. Locate the centroid $\bar{y}$ of the area.



Prob. 6–1

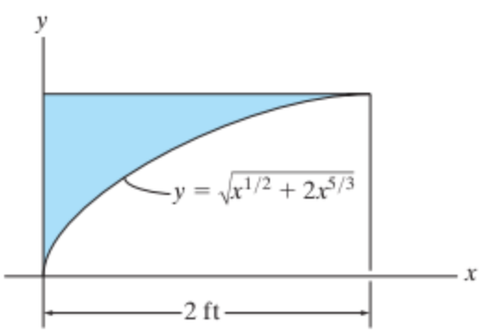
6–2. Locate the centroid $\bar{x}$ of the parabolic area.



Prob. 6–2

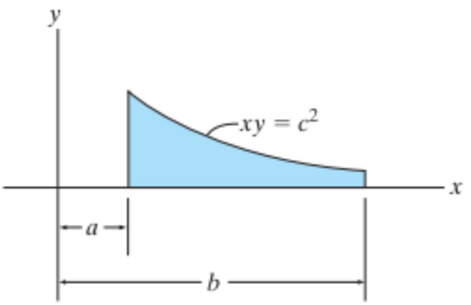
6–3. Locate the centroid $\bar{x}$ of the area.

***6–4.** Locate the centroid $\bar{y}$ of the area.



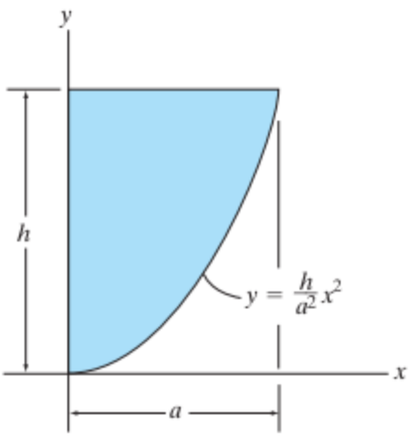
Probs. 6–3/4

6–5. Determine the area and the centroid $(\bar{x}, \bar{y})$ of the area.



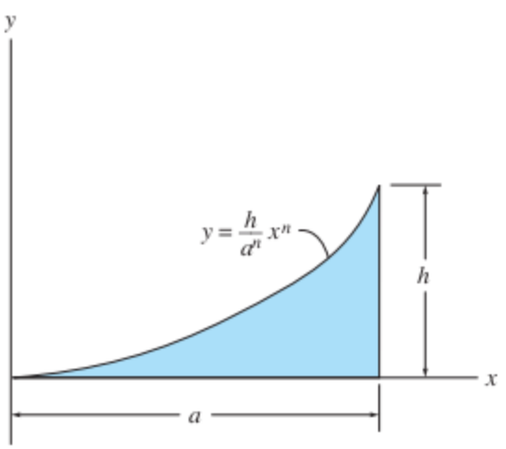
Prob. 6–5

6–6. Locate the centroid $\bar{y}$ of the parabolic area.



Prob. 6–6

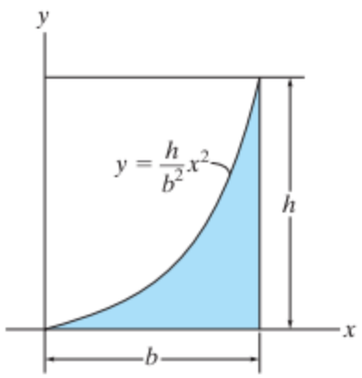
6–7. Locate the centroid $\bar{x}$ of the shaded area.



Prob. 6–7

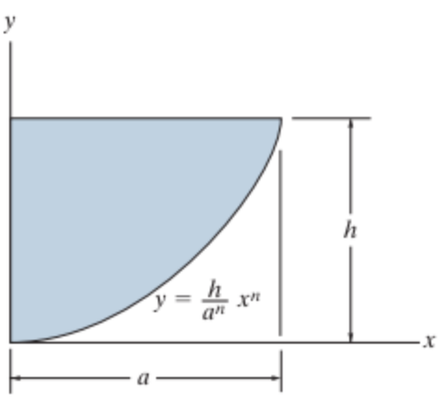
*6–8. Locate the centroid $\bar{x}$ of the area.

6–9. Locate the centroid $\bar{y}$ of the area.



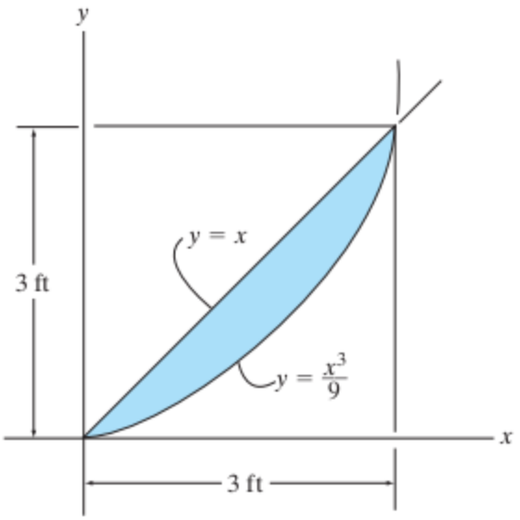
Probs. 6–8/9

6–10. Locate the centroid $\bar{x}$ of the shaded area.



Prob. 6–10

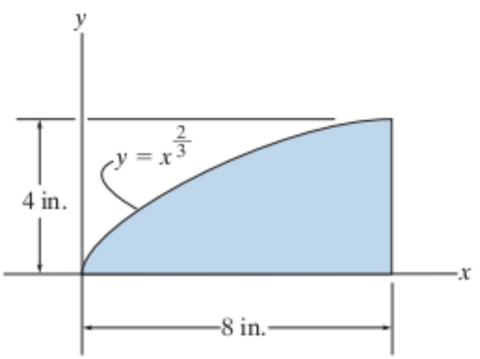
6–11. Determine the area and the centroid $(\bar{x}, \bar{y})$ of the area.



Prob. 6–11

*6–12. Locate the centroid $\bar{x}$ of the area.

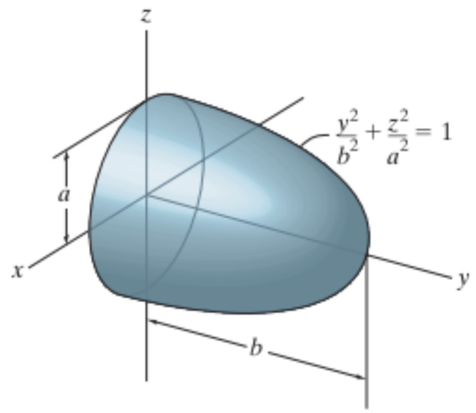
6–13. Locate the centroid $\bar{y}$ of the area.



Probs. 6–12/13

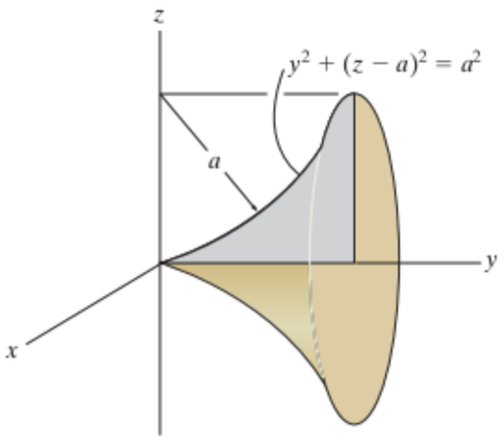
6

6–14. Locate the centroid of the ellipsoid of revolution.



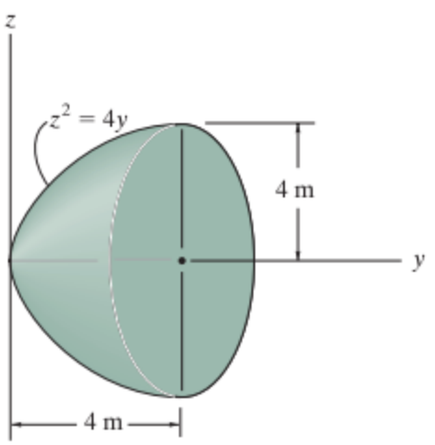
Prob. 6–14

6–15. Locate the centroid $\bar{y}$ of the homogeneous solid formed by revolving the shaded area about the y axis.



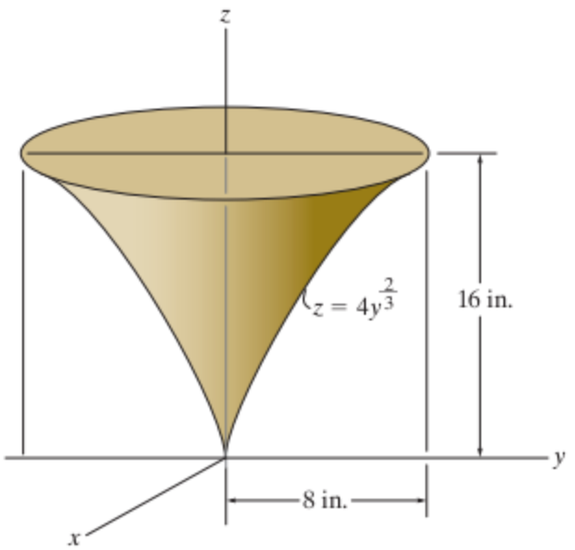
Prob. 6–15

*6–16. Locate the centroid $\bar{y}$ of the paraboloid.



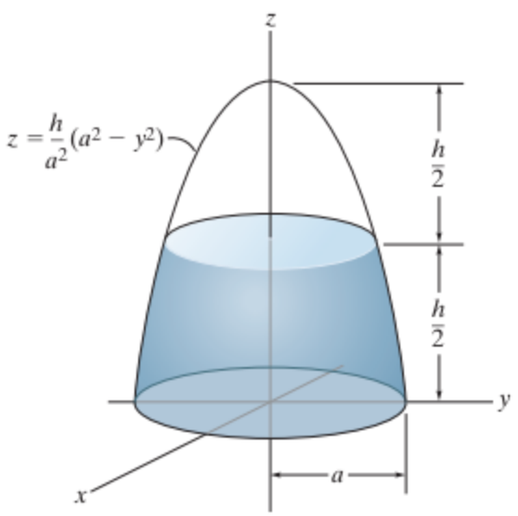
Prob. 6–16

6–17. Locate the center of gravity $\bar{z}$ of the solid.



Prob. 6–17

6–18. Locate the centroid $\bar{z}$ of the homogeneous solid frustum of the paraboloid formed by revolving the shaded area about the z axis.



Prob. 6–18

6.2 Composite Bodies

A *composite body* consists of a series of connected "simpler" shaped bodies, which may be rectangular, triangular, semicircular, etc. Such a body can often be sectioned or divided into its composite parts and, provided the *weight* and location of the center of gravity of each of these parts are known, we can then eliminate the need for integration to determine the center of gravity for the entire body. The method for doing this follows the same procedure outlined in Sec. 6.1. Formulas analogous to Eqs. 6–1 result; however, rather than account for an infinite number of differential weights, we have instead a finite number of weights. Therefore,

$$\bar{x} = \frac{\Sigma \tilde{x} W}{\Sigma W} \qquad \bar{y} = \frac{\Sigma \tilde{y} W}{\Sigma W} \qquad \bar{z} = \frac{\Sigma \tilde{z} W}{\Sigma W} \tag{6–5}$$

Where

$\bar{x}, \bar{y}, \bar{z}$ represent the coordinates of the center of gravity G of the composite body

$\tilde{x}, \tilde{y}, \tilde{z}$ represent the coordinates of the center of gravity of each composite part of the body

ΣW is the sum of the weights of all the composite parts of the body, or simply the total weight of the body

When the body has a *constant density or specific weight*, the center of gravity *coincides* with the centroid of the body. The centroid for composite lines, areas, and volumes can be found using relations analogous to Eqs. 6–5; however, the Ws are replaced by As, and Vs, respectively. Centroids for areas and volumes that often make up a composite body are given in Appendix B.

In order to determine the force required to tip over this concrete barrier it is first necessary to determine the location of its center of gravity G. Due to symmetry, G will lie on the vertical axis of symmetry.

Procedure for Analysis

The location of the center of gravity of a body or the centroid of a composite geometrical object represented by an area or volume can be determined using the following procedure.

Composite Parts.

- Using a sketch, divide the body or object into a finite number of composite parts that have simpler shapes.
- If a composite body has a *hole*, or a geometric region having no material, then consider the composite body without the hole and consider the hole as an *additional* composite part having *negative* weight or size.

Moment Arms.

- Establish the coordinate axes on the sketch and determine the coordinates $\tilde{x}, \tilde{y}, \tilde{z}$ of the center of gravity or centroid of each part.

Summations.

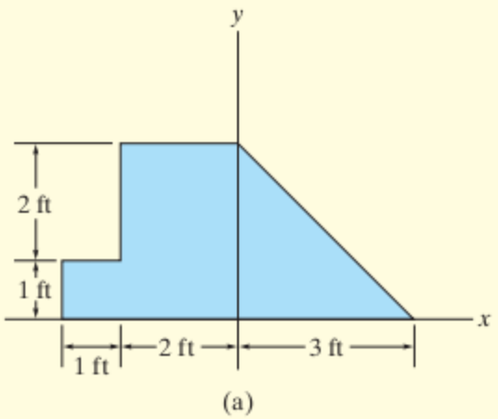
- Determine $\bar{x}, \bar{y}, \bar{z}$ by applying the center of gravity equations, Eqs. 6–5, or the analogous centroid equations.
- If an object is *symmetrical* about an axis, the centroid of the object lies on this axis.

If desired, the calculations can be arranged in tabular form, as indicated in the following two examples.

The center of gravity of this water tank can be determined by dividing it into composite parts and applying Eqs. 6–5.

EXAMPLE 6.4

Locate the centroid of the plate area shown in Fig. 6–9*a*.



(a)

Fig. 6–9

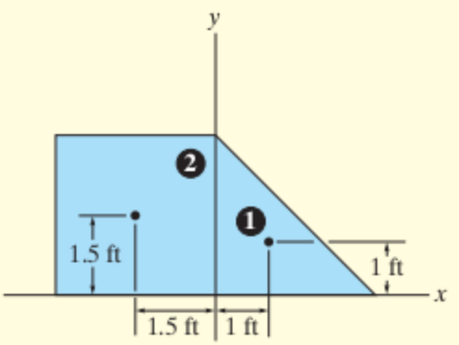
SOLUTION

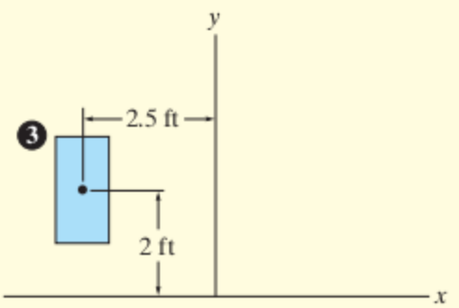
Composite Parts. The plate is divided into three segments as shown in Fig. 6–9*b*. Here the area of the small rectangle ③ is considered "negative" since it must be subtracted from the larger area ②.

Moment Arms. The centroid of each segment is located as indicated in the figure. Note that the $\tilde{x}$ coordinates of ② and ③ are *negative*.

Summations. Taking the data from Fig. 6–9*b*, the calculations are tabulated as follows:

Segment	A (ft^2)	$\tilde{x}$ (ft)	$\tilde{y}$ (ft)	$\tilde{x}\,A$ (ft^3)	$\tilde{y}\,A$ (ft^3)
1	$\frac{1}{2}(3)(3) = 4.5$	1	1	4.5	4.5
2	$(3)(3) = 9$	−1.5	1.5	−13.5	13.5
3	$-(2)(1) = -2$	−2.5	2	5	−4
	$\Sigma A = 11.5$			$\Sigma \tilde{x}A = -4$	$\Sigma \tilde{y}A = 14$





(b)

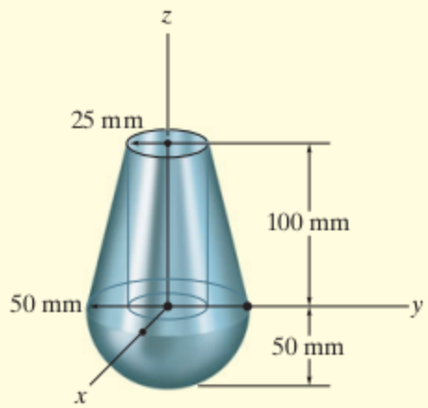
Thus,

$$\bar{x} = \frac{\Sigma \tilde{x}A}{\Sigma A} = \frac{-4}{11.5} = -0.348 \text{ ft} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\Sigma \tilde{y}A}{\Sigma A} = \frac{14}{11.5} = 1.22 \text{ ft} \qquad \textit{Ans.}$$

NOTE: If these results are plotted in Fig. 6–9*a*, the location of point C seems reasonable.

EXAMPLE 6.5



(a)

Fig. 6–10

Locate the center of mass of the assembly shown in Fig. 6–10*a*. The conical frustum has a density of $\rho_c = 8\ \text{Mg/m}^3$, and the hemisphere has a density of $\rho_h = 4\ \text{Mg/m}^3$. There is a 25-mm-radius cylindrical hole in the center of the frustum.

SOLUTION

Composite Parts. The assembly can be thought of as consisting of four segments as shown in Fig. 6–10*b*. For the calculations, ③ and ④ must be considered as "negative" segments in order that the four segments, when added together, yield the total composite shape shown in Fig. 6–10*a*.

Moment Arm. Using the table in Appendix B, the computations for the centroid $\tilde{z}$ of each piece are shown in the figure.

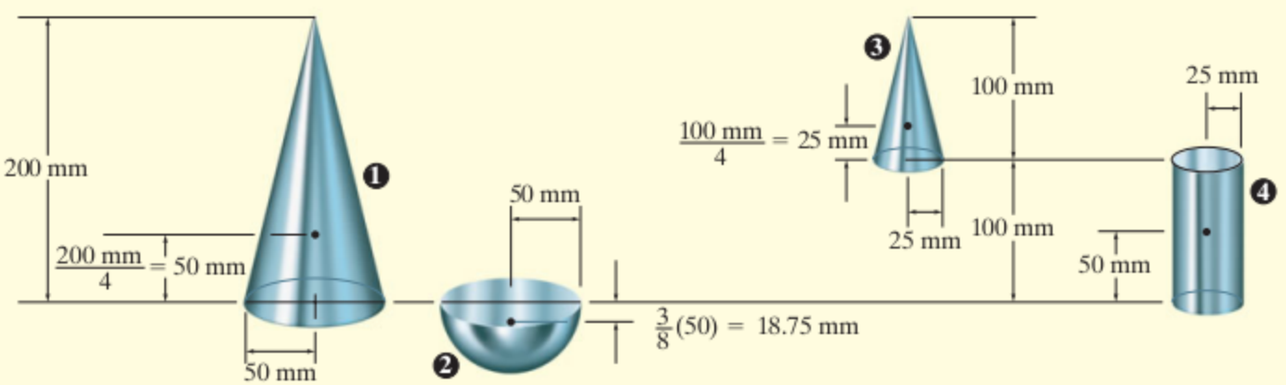
Summations. Because of *symmetry*, note that

$$\bar{x} = \bar{y} = 0 \qquad \textit{Ans.}$$

Since $W = mg$, and g is constant, the third of Eqs. 6–5 becomes $\bar{z} = \Sigma \tilde{z}m / \Sigma m$. The mass of each piece can be computed from $m = \rho V$ and used for the calculations. Also, $1\ \text{Mg/m}^3 = 10^{-6}\ \text{kg/mm}^3$, so that

Segment	m (kg)	$\tilde{z}$ (mm)	$\tilde{z}m$ (kg · mm)
1	$8(10^{-6})\left(\frac{1}{3}\right)\pi(50)^2(200) = 4.189$	50	209.440
2	$4(10^{-6})\left(\frac{2}{3}\right)\pi(50)^3 = 1.047$	−18.75	−19.635
3	$-8(10^{-6})\left(\frac{1}{3}\right)\pi(25)^2(100) = -0.524$	100 + 25 = 125	−65.450
4	$-8(10^{-6})\pi(25)^2(100) = -1.571$	50	−78.540
	$\Sigma m = 3.142$		$\Sigma \tilde{z}m = 45.815$

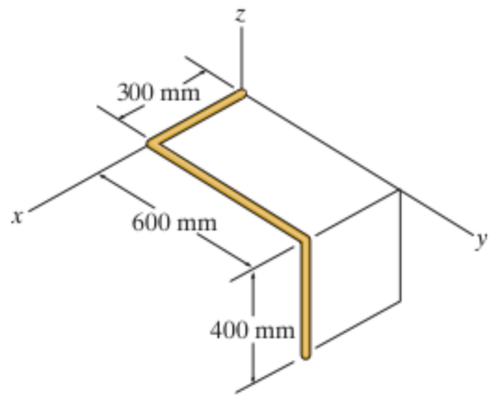
Thus, $$\tilde{z} = \frac{\Sigma \tilde{z}m}{\Sigma m} = \frac{45.815}{3.142} = 14.6\ \text{mm} \qquad \textit{Ans.}$$



(b)

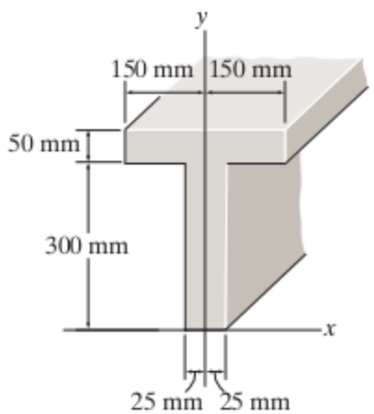
FUNDAMENTAL PROBLEMS

F6–7. Locate the centroid $(\bar{x}, \bar{y}, \bar{z})$ of the wire bent in the shape shown.



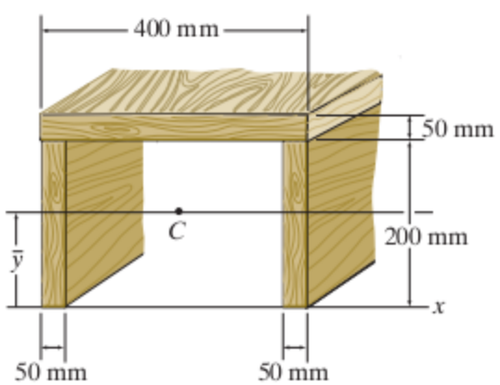
F6–7

F6–8. Locate the centroid $\bar{y}$ of the beam's cross-sectional area.



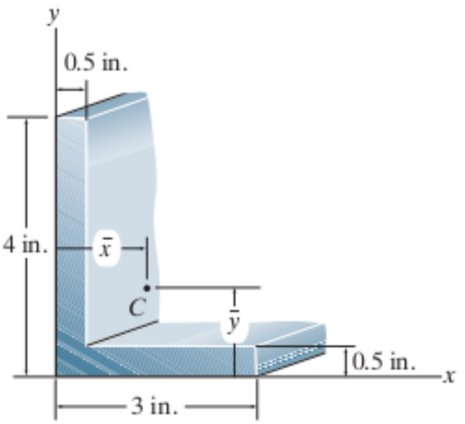
F6–8

F6–9. Locate the centroid $\bar{y}$ of the beam's cross-sectional area.



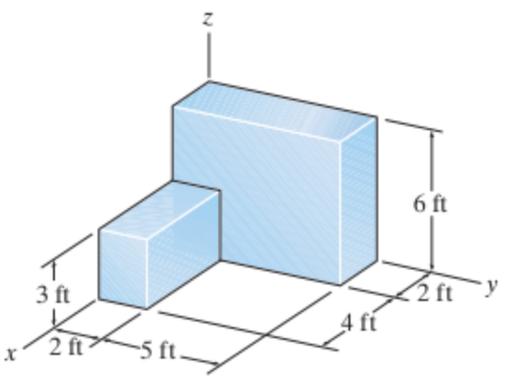
F6–9

F6–10. Locate the centroid $(\bar{x}, \bar{y})$ of the cross-sectional area.



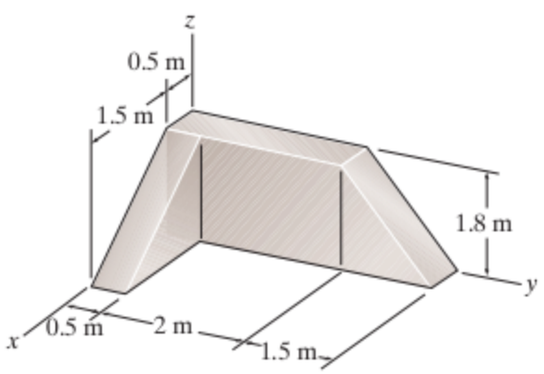
F6–10

F6–11. Locate the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of the homogeneous solid block.



F6–11

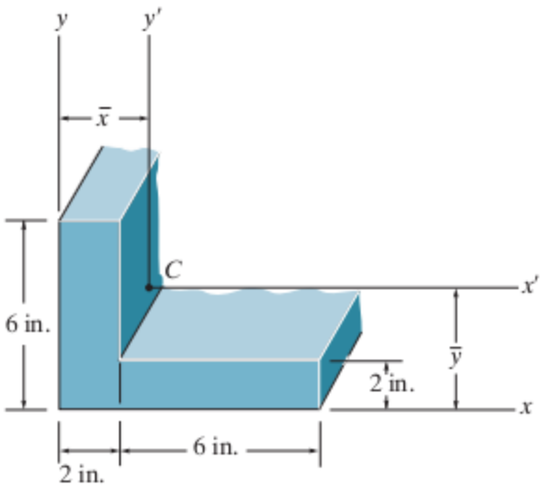
F6–12. Determine the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of the homogeneous solid block.



F6–12

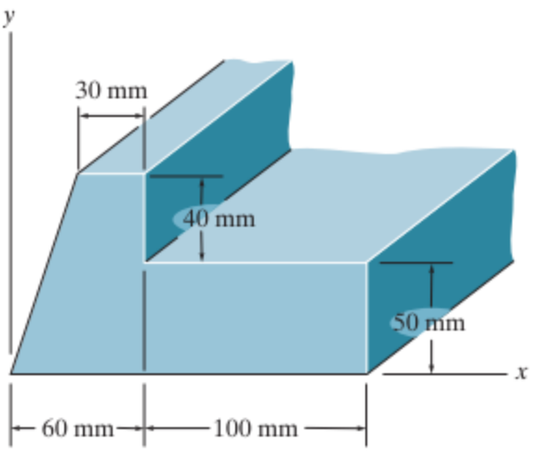
PROBLEMS

6–19. Locate the centroid $(\bar{x}, \bar{y})$ for the angle's cross-sectional area.



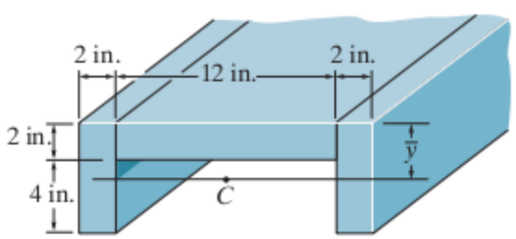
Prob. 6–19

6–21. Locate the centroid $(\bar{x}, \bar{y})$ of the member's cross-sectional area.



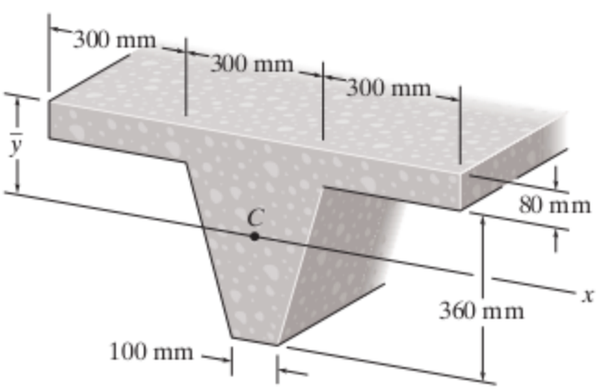
Prob. 6–21

***6–20.** Locate the centroid $\bar{y}$ of the channel's cross-sectional area.



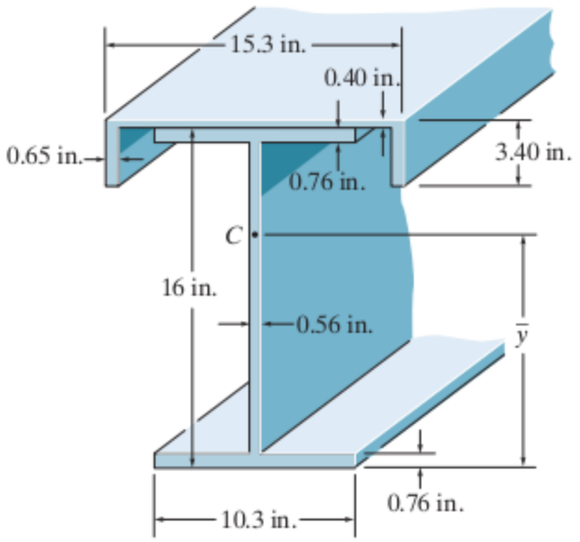
Prob. 6–20

6–22. Locate the centroid $\bar{y}$ of the concrete beam having the tapered cross section shown.



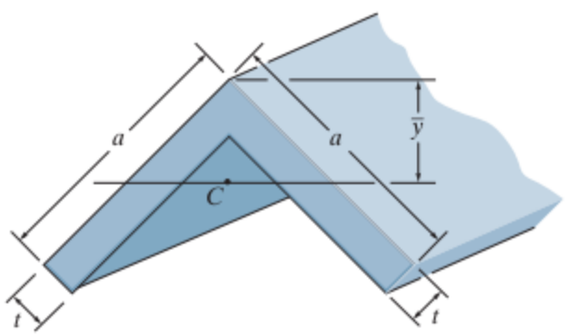
Prob. 6–22

6–23. Locate the centroid $\bar{y}$ of the beam's cross-section built up from a channel and a wide-flange beam.



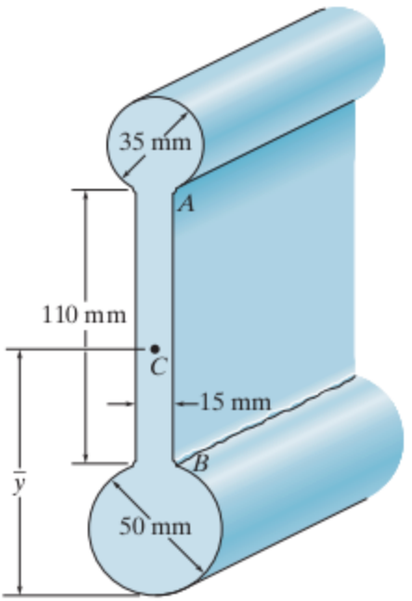
Prob. 6–23

*6–24. Locate the centroid $\bar{y}$ for the cross-sectional area of the angle.



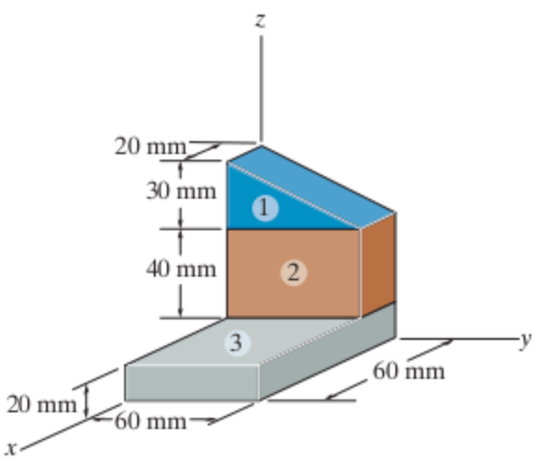
Prob. 6–24

6–25. Determine the location $\bar{y}$ of the centroid of the beam's cross-sectional area. Neglect the size of the corner welds at A and B for the calculation.



Prob. 6–25

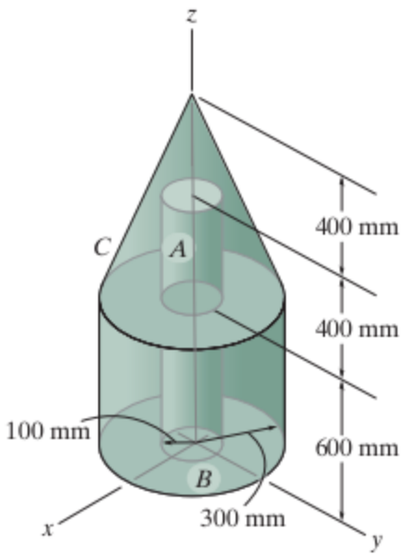
6–26. Locate the center of mass of the block. Materials 1, 2, and 3 have densities of 2.70 Mg/m^3, 5.70 Mg/m^3, and 7.80 Mg/m^3, respectively.



Prob. 6–26

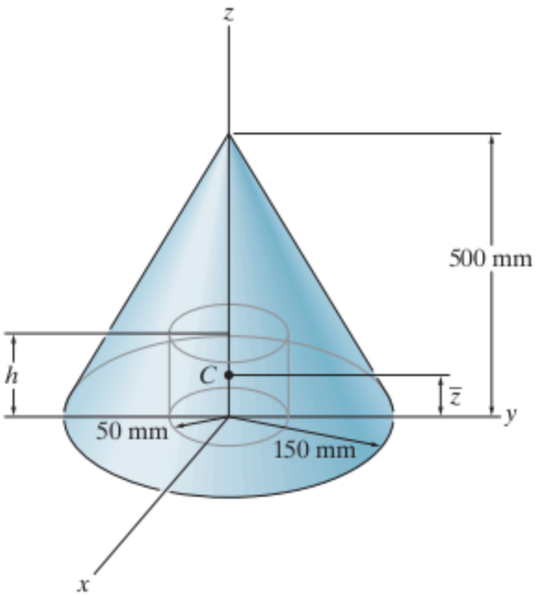
6

6–27. Locate the center of mass $\bar{z}$ of the assembly. The assembly consists of a cylindrical center core, A, having a density of 7.90 Mg/m^3, and a cylindrical outer part, B, and a cone cap, C, each having a density of 2.70 Mg/m^3.



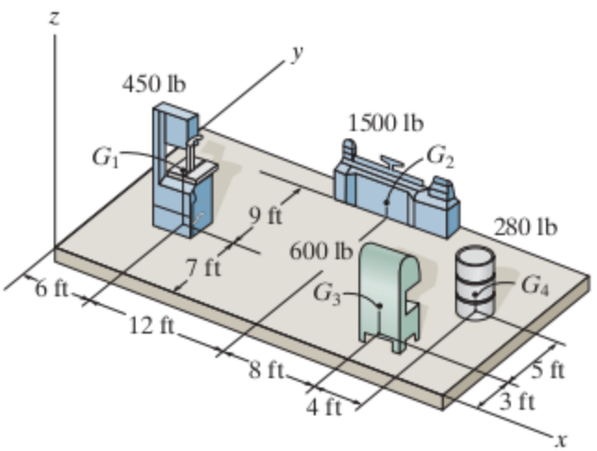
Prob. 6–27

6–29. Determine the distance h to which a 100-mm-diameter hole must be bored into the base of the cone so that the center of mass of the resulting shape is located at $\bar{z} = 115$ mm. The material has a density of 8 Mg/m^3.



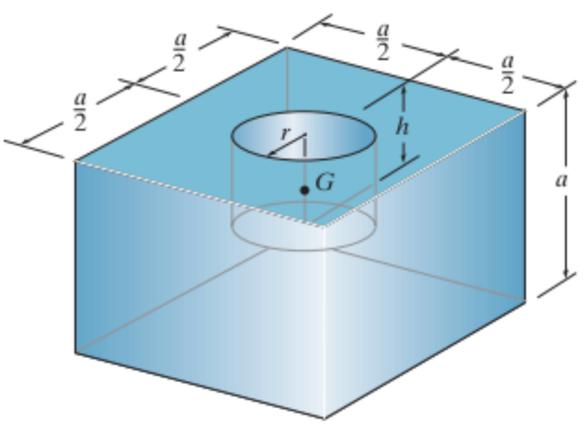
Prob. 6–29

*6–28. Major floor loadings in a shop are caused by the weights of the objects shown. Each force acts through its respective center of gravity G. Locate the center of gravity $(\bar{x}, \bar{y})$ of all these components.



Prob. 6–28

6–30. A hole having a radius r is to be drilled in the center of the homogeneous block. Determine the depth h of the hole so that the center of gravity G is as low as possible.



Prob. 6–30

6.3 Resultant of a Distributed Loading

Sometimes, a body may be subjected to a loading that is distributed over its surface. For example, the pressure of the wind on the face of a sign, the pressure of water within a tank, or the weight of sand on the floor of a storage container, are all *distributed loadings*. The pressure exerted at each point on the surface indicates the intensity of the loading. It is measured using pascals Pa (or N/m^2) in SI units or lb/ft^2 in the U.S. Customary system.

Consider, for example, the flat plate shown in Fig. 6–11*a*, which is subjected to the loading defined by $p = p(x, y)$ Pa, where 1 Pa (pascal) = 1 N/m^2. Knowing this function, we can determine the resultant force $\mathbf{F}_R$ acting on the plate and its location $(\bar{x}, \bar{y})$, Fig. 6–11*b*.

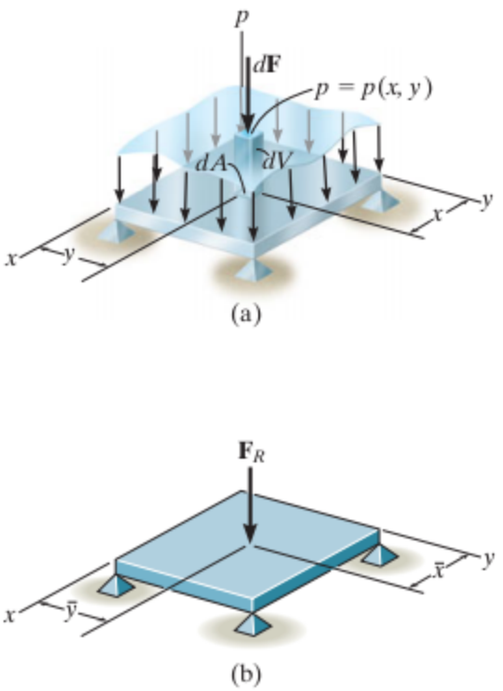


Fig. 6–11

Magnitude of Resultant Force. The force $d\mathbf{F}$ acting on the differential area dA m^2 of the plate, located at the arbitrary point (x, y), has a magnitude of $dF = [p(x, y)\ N/m^2](dA\ m^2) = [p(x, y)\ dA]$ N. Notice that $p(x, y)\ dA = dV$, the colored differential *volume element* shown in Fig. 6–11*a*. The *magnitude* of $\mathbf{F}_R$ is the sum of the differential forces acting over the plate's *entire surface area A*. Thus:

$$F_R = \Sigma F; \qquad \boxed{F_R = \int_A p(x, y)\, dA = \int_V dV = V} \tag{6–6}$$

This result indicates that the *magnitude of the resultant force is equal to the total volume under the distributed-loading diagram.*

Location of Resultant Force. The location $(\bar{x}, \bar{y})$ of $\mathbf{F}_R$ is determined by setting the moments of $\mathbf{F}_R$ equal to the moments of all the differential forces $d\mathbf{F}$ about the respective y and x axes: From Figs. 6–11*a* and 6–11*b*, using Eq. 6–6, this results in

$$\boxed{\bar{x} = \frac{\int_A xp(x, y)\, dA}{\int_A p(x, y)\, dA} = \frac{\int_V x\, dV}{\int_V dV} \qquad \bar{y} = \frac{\int_A yp(x, y)\, dA}{\int_A p(x, y)\, dA} = \frac{\int_V y\, dV}{\int_V dV}} \tag{6–7}$$

Hence, the *line of action of the resultant force passes through the geometric center or centroid of the volume under the distributed-loading diagram.*

The resultant of a wind loading that is distributed on the front or side walls of this building must be calculated using integration in order to design the framework that holds the building together.

Loading Along a Single Axis. The most common type of distributed loading encountered in engineering practice can be represented along a single axis. For example, consider the beam (or plate) in Fig. 6–12*a* that has a constant width and is subjected to a pressure loading that varies only along the x axis. This loading can be described by the function $p = p(x)$ N/m^2. It contains only one variable x, and for this reason, we can also represent it as a *coplanar distributed load*. To do so, we multiply the loading function by the width b m of the beam, so that $w(x) = p(x)b$ N/m, Fig. 6–12*b*. Using the methods of Sec. 6.8, we can replace this coplanar parallel force system with a single equivalent resultant force $\mathbf{F}_R$ acting at a specific location on the beam, Fig. 6–12*c*.

Magnitude of Resultant Force. From Eq. 3–17 ($F_R = \Sigma F$), the magnitude of $\mathbf{F}_R$ is equivalent to the sum of all the forces in the system. In this case integration must be used since there is an infinite number of parallel forces $d\mathbf{F}$ acting on the beam, Fig. 6–12*b*. Since $d\mathbf{F}$ is acting on an element of length dx, and $w(x)$ is a force per unit length, then $dF = w(x)\,dx = dA$. In other words, the magnitude of $d\mathbf{F}$ is determined from the colored differential *area* dA under the loading curve. For the entire length L,

6

$$+\downarrow F_R = \Sigma F; \qquad \boxed{F_R = \int_L w(x)\,dx = \int_A dA = A} \tag{6–8}$$

Therefore, the magnitude of the resultant force is equal to the total area A under the loading diagram, Fig. 6–12*c*.

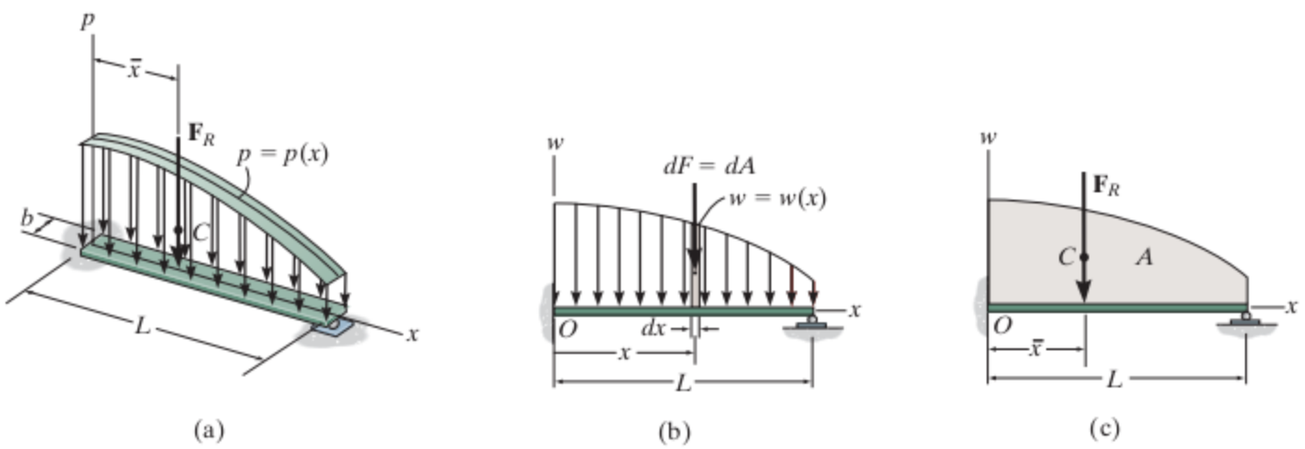


Fig. 6–12

Location of Resultant Force. Applying Eq. 3–18 ($(M_R)_O = \Sigma M_O$), the location $\bar{x}$ of the line of action of $\mathbf{F}_R$ can be determined by equating the moments of the force resultant and the parallel force distribution about point O (the y axis). Since $d\mathbf{F}$ produces a moment of $x\,dF = xw(x)\,dx$ about O, Fig. 6–12*b*, then for the entire length, Fig. 6–12*c*,

$$\curvearrowleft + (M_R)_O = \Sigma M_O; \qquad -\bar{x}F_R = -\int_L xw(x)\,dx$$

Solving for $\bar{x}$, using Eq. 6–8, we have

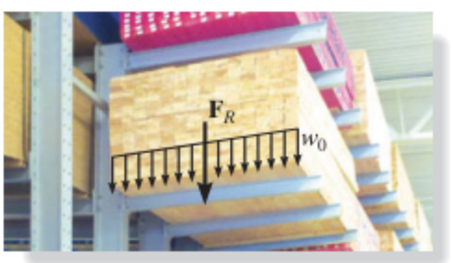
$$\bar{x} = \frac{\int_L xw(x)\,dx}{\int_L w(x)\,dx} = \frac{\int_A x\,dA}{\int_A dA} \tag{6–9}$$

This coordinate $\bar{x}$, locates the geometric center or *centroid* of the *area* under the distributed loading. *In other words, the resultant force has a line of action which passes through the centroid C (geometric center) of the area under the loading diagram*, Fig. 6–12*c*. In many cases, however, the distributed-loading diagram is in the shape of a rectangle, triangle, or some other simple geometric form. The centroid location for such common shapes does not have to be determined from the above equation. It can be obtained directly from the tabulation given in Appendix B.

Once $\bar{x}$ is determined, $\mathbf{F}_R$ by symmetry passes through point $(\bar{x}, 0)$ on the surface of the beam, Fig. 6–12*a*. *Therefore, in this case the resultant force has a magnitude equal to the volume under the loading curve $p = p(x)$ and a line of action which passes through the centroid (geometric center) of this volume.*

Important Points

- Coplanar distributed loadings are defined by using a loading function $w = w(x)$ that indicates the intensity of the loading along the length of a member. This intensity is measured in N/m or lb/ft.
- The external effects caused by a coplanar distributed load acting on a body can be represented by a single resultant force.
- This resultant force is equivalent to the *area* under the loading diagram, and has a line of action that passes through the *centroid* or geometric center of this area.



Each beam that supports this stack of lumber is subjected to a uniform loading of w_0. That resultant force is therefore equal to the area under the rectangular loading diagram. It acts through the centroid or geometric center of this area.

EXAMPLE 6.6

Determine the magnitude and location of the equivalent resultant force acting on the shaft in Fig. 6–13*a*.

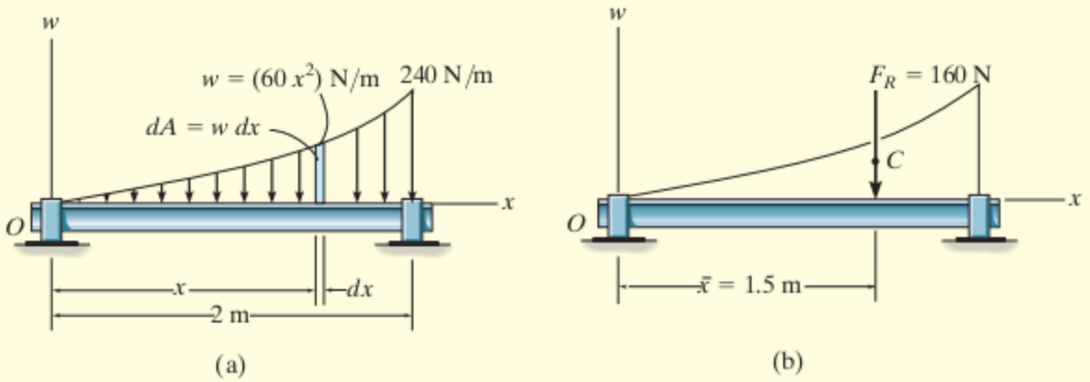


Fig. 6–13

SOLUTION

Since $w = w(x)$ is given, this problem will be solved by integration.

The differential element has an area $dA = w\,dx = 60x^2\,dx$. Applying Eq. 6–8,

$$+\downarrow F_R = \Sigma F;$$

$$F_R = \int_A dA = \int_0^{2\text{ m}} 60x^2\,dx = 60\left(\frac{x^3}{3}\right)\Big|_0^{2\text{ m}} = 60\left(\frac{2^3}{3} - \frac{0^3}{3}\right)$$

$$= 160\text{ N} \qquad \textit{Ans.}$$

The location $\bar{x}$ of $\mathbf{F}_R$ measured from O, Fig. 6–13*b*, is determined from Eq. 6–9.

$$\bar{x} = \frac{\int_A x\,dA}{\int_A dA} = \frac{\int_0^{2\text{ m}} x(60x^2)\,dx}{160\text{ N}} = \frac{60\left(\frac{x^4}{4}\right)\Big|_0^{2\text{ m}}}{160\text{ N}} = \frac{60\left(\frac{2^4}{4} - \frac{0^4}{4}\right)}{160\text{ N}}$$

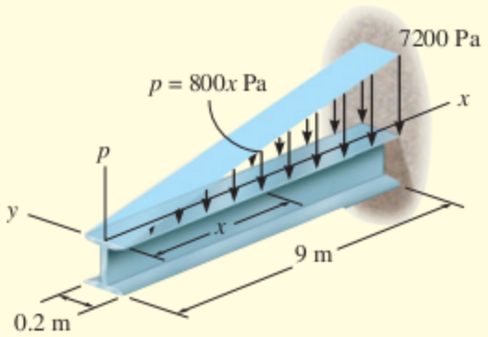
$$= 1.5\text{ m} \qquad \textit{Ans.}$$

NOTE: These results can be checked by using the table in Appendix B, where it is shown that for an exparabolic area of length a, height b, and shape shown in Fig. 6–13*a*, we have

$$A = \frac{ab}{3} = \frac{2\text{ m}(240\text{ N/m})}{3} = 160\text{ N and } \bar{x} = \frac{3}{4}a = \frac{3}{4}(2\text{ m}) = 1.5\text{ m}$$

EXAMPLE 6.7

A distributed loading of $p = (800x)$ Pa acts over the top surface of the beam shown in Fig. 6–14*a*. Determine the magnitude and location of the equivalent resultant force.



(a)

SOLUTION

Since the loading intensity is uniform along the width of the beam (the y axis), the loading can be viewed in two dimensions as shown in Fig. 6–14*b*. Here

$$w = (800x\ \text{N/m}^2)(0.2\ \text{m})$$
$$= (160x)\ \text{N/m}$$

At $x = 9$ m, note that $w = 1440$ N/m. We may again apply Eqs. 6–8 and 6–9 as in the previous example.

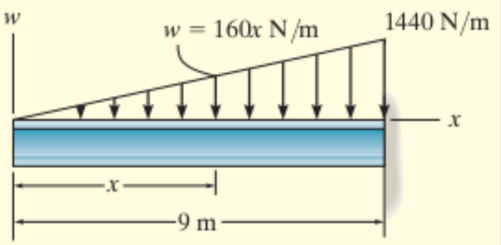
The magnitude of the resultant force is equivalent to the area of the triangle.

$$F_R = \tfrac{1}{2}(9\ \text{m})(1440\ \text{N/m}) = 6480\ \text{N} = 6.48\ \text{kN} \qquad \textit{Ans.}$$

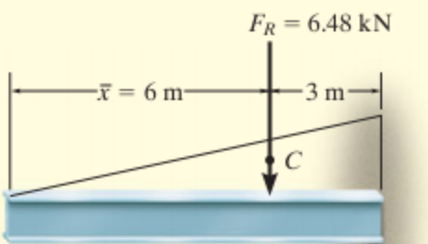
The line of action of $\mathbf{F}_R$ passes through the *centroid* C of this triangle. Hence,

$$\bar{x} = 9\ \text{m} - \tfrac{1}{3}(9\ \text{m}) = 6\ \text{m} \qquad \textit{Ans.}$$

The results are shown in Fig. 6–14*c*.



(b)



(c)

Fig. 6–14

NOTE: We may also view the resultant $\mathbf{F}_R$ as *acting* through the *centroid* of the *volume* of the loading diagram $p = p(x)$ in Fig. 6–14*a*. Hence $\mathbf{F}_R$ intersects the x–y plane at the point (6 m, 0). Furthermore, the magnitude of $\mathbf{F}_R$ is equal to the volume under the loading diagram; i.e.,

$$F_R = V = \tfrac{1}{2}(7200\ \text{N/m}^2)(9\ \text{m})(0.2\ \text{m}) = 6.48\ \text{kN} \qquad \textit{Ans.}$$

EXAMPLE 6.8

The granular material exerts the distributed loading on the beam as shown in Fig. 6–15*a*. Determine the magnitude and location of the equivalent resultant of this load.

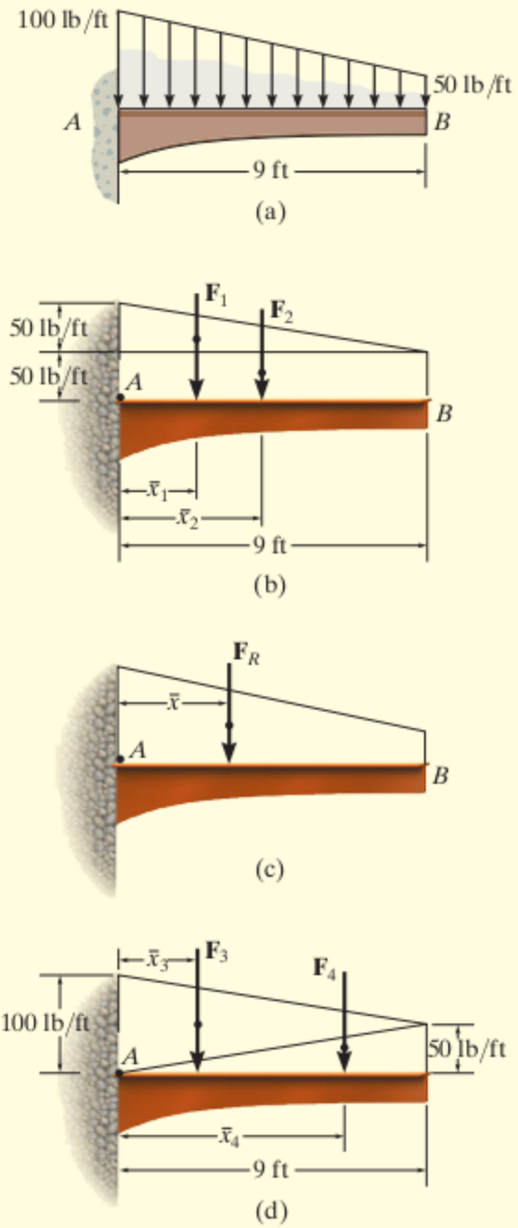


Fig. 6–15

SOLUTION

The area of the loading diagram is a *trapezoid*, and therefore the solution can be obtained directly from the area and centroid formulas for a trapezoid listed in Appendix B. Since these formulas are not easily remembered, instead we will solve this problem by using "composite" areas. Here we will divide the trapezoidal loading into a rectangular and triangular loading as shown in Fig. 6–15*b*. The magnitude of the force represented by each of these loadings is equal to its associated *area*,

$$F_1 = \tfrac{1}{2}(9\text{ ft})(50\text{ lb/ft}) = 225\text{ lb}$$

$$F_2 = (9\text{ ft})(50\text{ lb/ft}) = 450\text{ lb}$$

The lines of action of these parallel forces act through the respective *centroids* of their associated areas and therefore intersect the beam at

$$\bar{x}_1 = \tfrac{1}{3}(9\text{ ft}) = 3\text{ ft}$$

$$\bar{x}_2 = \tfrac{1}{2}(9\text{ ft}) = 4.5\text{ ft}$$

The two parallel forces $\mathbf{F}_1$ and $\mathbf{F}_2$ can be reduced to a single resultant $\mathbf{F}_R$. The magnitude of $\mathbf{F}_R$ is

$$+\downarrow F_R = \Sigma F; \qquad F_R = 225 + 450 = 675\text{ lb} \qquad \textit{Ans.}$$

We can find the location of $\mathbf{F}_R$ with reference to point A, Fig. 6–15*b* and 6–15*c*. We require

$$\circlearrowleft + (M_R)_A = \Sigma M_A; \quad \bar{x}(675) = 3(225) + 4.5(450)$$

$$\bar{x} = 4\text{ ft} \qquad \textit{Ans.}$$

NOTE: The trapezoidal area in Fig. 6–15*a* can also be divided into two triangular areas as shown in Fig. 6–15*d*. In this case

$$F_3 = \tfrac{1}{2}(9\text{ ft})(100\text{ lb/ft}) = 450\text{ lb}$$

$$F_4 = \tfrac{1}{2}(9\text{ ft})(50\text{ lb/ft}) = 225\text{ lb}$$

and

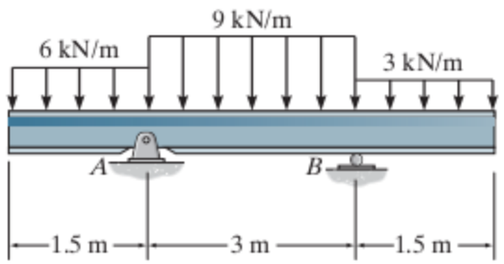
$$\bar{x}_3 = \tfrac{1}{3}(9\text{ ft}) = 3\text{ ft}$$

$$\bar{x}_4 = 9\text{ ft} - \tfrac{1}{3}(9\text{ ft}) = 6\text{ ft}$$

Using these results, show that again $F_R = 675$ lb and $\bar{x} = 4$ ft.

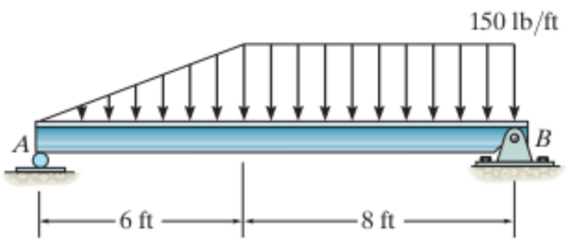
FUNDAMENTAL PROBLEMS

F6–13. Determine the resultant force and specify where it acts on the beam measured from *A*.



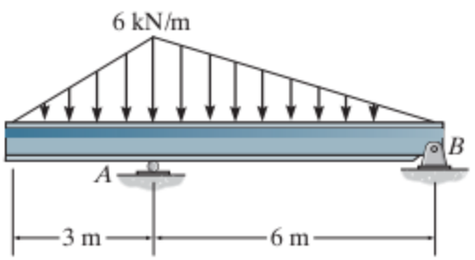
F6–13

F6–14. Determine the resultant force and specify where it acts on the beam measured from *A*.



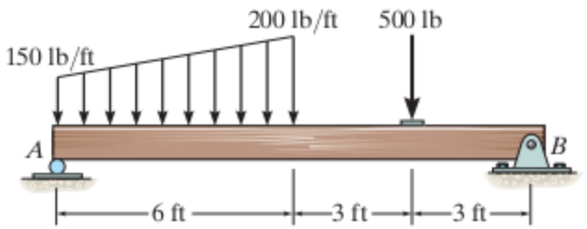
F6–14

F6–15. Determine the resultant force and specify where it acts on the beam measured from *A*.



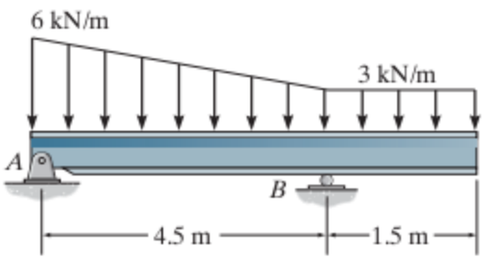
F6–15

F6–16. Determine the resultant force and specify where it acts on the beam measured from *A*.



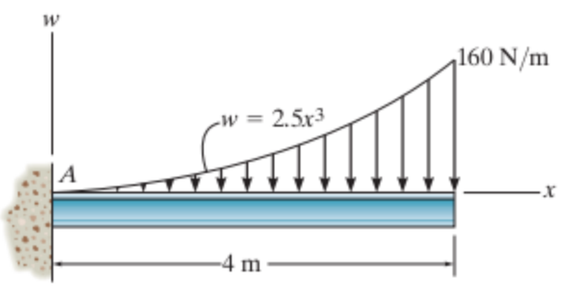
F6–16

F6–17. Determine the resultant force and specify where it acts on the beam measured from *A*.



F6–17

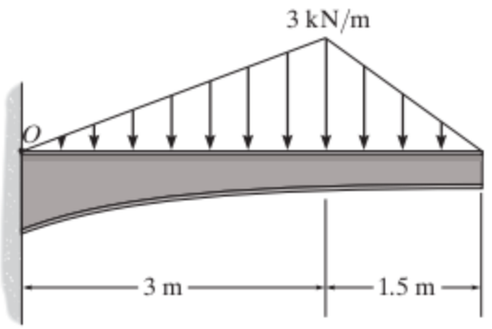
F6–18. Determine the resultant force and specify where it acts on the beam measured from *A*.



F6–18

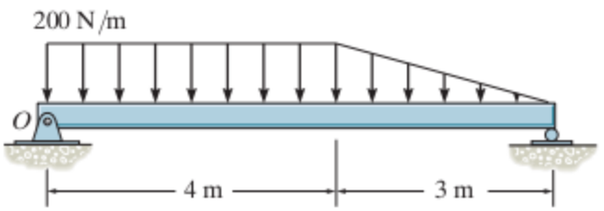
PROBLEMS

6–31. Replace the distributed loading with an equivalent resultant force, and specify its location on the beam measured from point O.



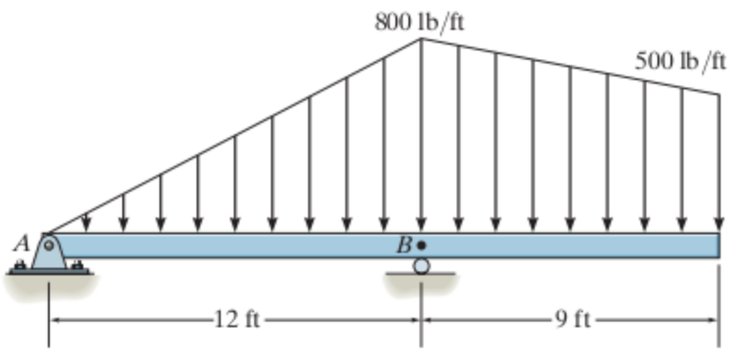
Prob. 6–31

*6–32. Replace the loading by an equivalent force and couple moment acting at point O.



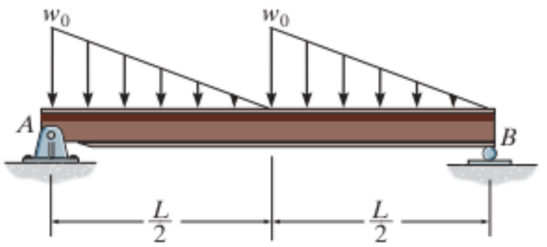
Prob. 6–32

6–33. Replace the loading by an equivalent resultant force and specify its location on the beam, measured from point B.



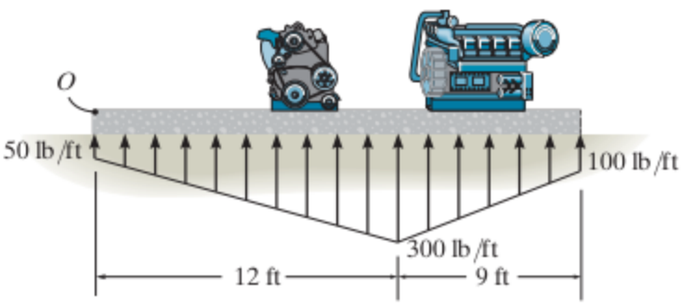
Prob. 6–33

6–34. Replace the distributed loading with an equivalent resultant force, and specify its location on the beam measured from point A.



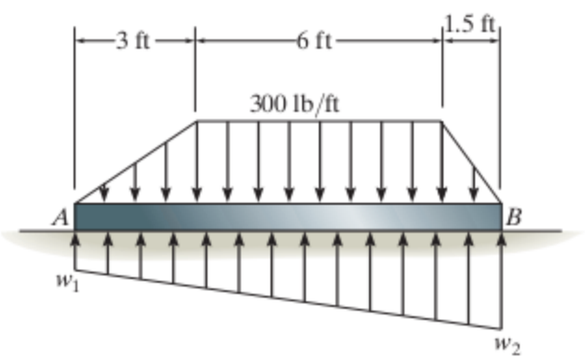
Prob. 6–34

6–35. The distribution of soil loading on the bottom of a building slab is shown. Replace this loading by an equivalent resultant force and specify its location, measured from point O.



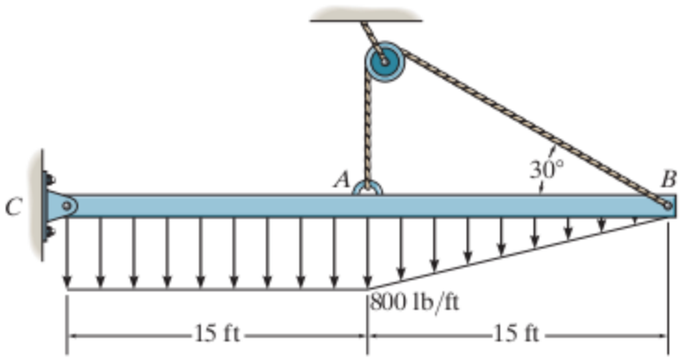
Prob. 6–35

*6–36. Determine the intensities w_1 and w_2 of the distributed loading acting on the bottom of the slab so that this loading has an equivalent resultant force that is equal but opposite to the resultant of the distributed loading acting on the top of the plate.



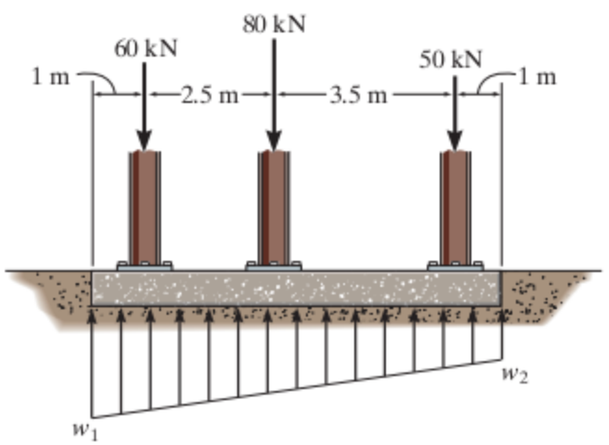
Prob. 6–36

6–37. Replace the distributed loading by an equivalent resultant force, and specify its location on the beam, measured from the pin at *C*.



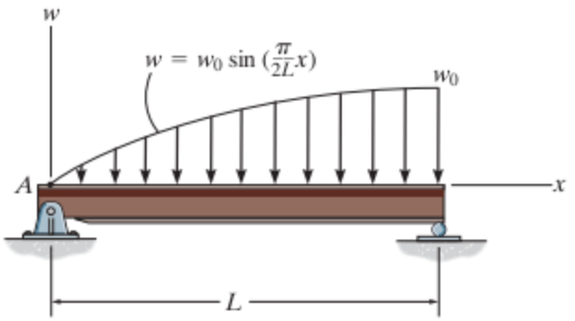
Prob. 6–37

6–38. If the soil exerts a trapezoidal distribution of load on the bottom of the footing, determine the intensities w_1 and w_2 of this distribution needed to support the column loadings.



Prob. 6–38

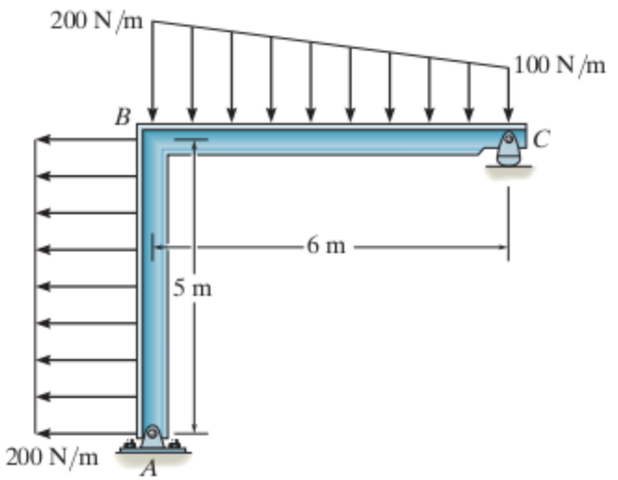
6–39. Replace the distributed loading with an equivalent resultant force, and specify its location on the beam measured from point *A*.



Prob. 6–39

***6–40.** Replace the distributed loading by an equivalent resultant force and specify where its line of action intersects member *AB*, measured from *A*.

6–41. Replace the distributed loading by an equivalent resultant force and specify where its line of action intersects member *BC*, measured from *C*.



Probs. 6–40/41

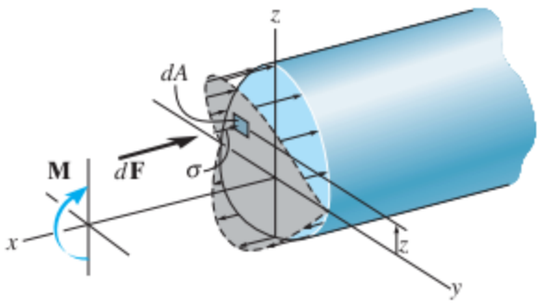


Fig. 6–16

6.4 Moments of Inertia for Areas

In the first few sections of this chapter, we determined the centroid for an area by considering the first moment of the area about an axis; that is, for the computation we had to evaluate an integral of the form $\int x\,dA$. An integral of the second moment of an area, such as $\int x^2\,dA$, is referred to as the *moment of inertia* for the area. The terminology "moment of inertia" as used here is actually a misnomer; however, it has been adopted because of the similarity with integrals of the same form related to mass.

The moment of inertia of an area orginates whenever one relates the normal stress σ (sigma), or force per unit area, acting on the transverse cross section of an elastic beam, to the applied external moment **M**, which causes bending of the beam. From the theory of mechanics of materials, it can be shown that the stress within the beam varies linearly with its distance from an axis passing through the centroid C of the beam's cross-sectional area; i.e., $\sigma = kz$, Fig. 6–16. The magnitude of force acting on the area element dA, shown in the figure, is therefore $dF = \sigma\,dA = kz\,dA$. Since this force is located a distance z from the y axis, the moment of $d\mathbf{F}$ about the y axis is $dM = dF\,z = kz^2\,dA$. The resulting moment of the entire stress distribution is equal to the applied moment **M**; hence, $M = k\int z^2\,dA$. Here the integral represents the moment of inertia of the area about the y axis. Since integrals of this form often arise in formulas used in mechanics of materials, structural mechanics, fluid mechanics, and machine design, the engineer should become familiar with the methods used for their computation.

Moment of Inertia. Consider the area A, shown in Fig. 6–17, which lies in the x–y plane. By definition, the moments of inertia of the differential planar area dA about the x and y axes are $dI_x = y^2\,dA$ and $dI_y = x^2\,dA$, respectively. For the entire area the *moments of inertia* are determined by integration; i.e.,

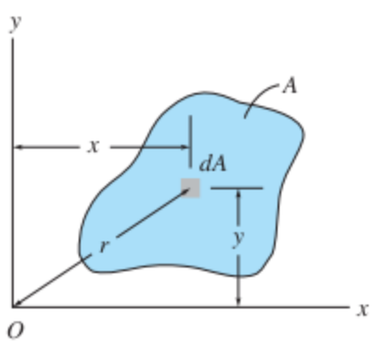


Fig. 6–17

$$I_x = \int_A y^2\,dA \qquad I_y = \int_A x^2\,dA \tag{6–10}$$

We can also formulate this quantity for dA about the "pole" O or z axis, Fig. 6–18. This is referred to as the *polar moment of inertia*. It is defined as $dJ_O = r^2 dA$, where r is the perpendicular distance from the pole (z axis) to the element dA. For the entire area the *polar moment of inertia* is

$$J_O = \int_A r^2 dA = I_x + I_y \tag{6–11}$$

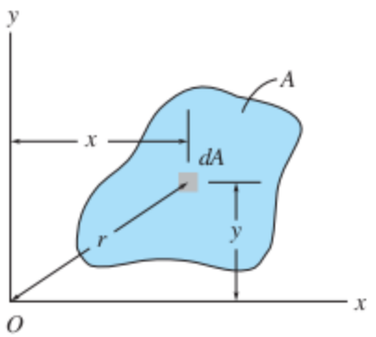


Fig. 6–18

This relation between J_O and I_x, I_y is possible since $r^2 = x^2 + y^2$, Fig. 6–18.

From the above formulations it is seen that I_x, I_y, and J_O will always be positive since they involve the product of distance squared and area. Furthermore, the units for moment of inertia involve length raised to the fourth power, e.g., m^4, mm^4 or ft^4, $in.^4$.

6.5 Parallel-Axis Theorem for an Area

The *parallel-axis theorem* can be used to find the moment of inertia of an area about *any axis* that is parallel to an axis passing through the centroid and about which the moment of inertia is known. To develop this theorem, we will consider finding the moment of inertia of the shaded area shown in Fig. 6–19 about the x axis. To start, we choose a differential element dA located at an arbitrary distance y' from the *centroidal* x' axis. If the distance between the parallel x and x' axes is d_y, then the moment of inertia of dA about the x axis is $dI_x = (y' + d_y)^2\, dA$. For the entire area,

$$I_x = \int_A (y' + d_y)^2\, dA$$

$$= \int_A y'^2\, dA + 2d_y \int_A y'\, dA + d_y^2 \int_A dA$$

The first integral represents the moment of inertia of the area about the centroidal axis, $\bar{I}_{x'}$. The second integral is zero since the x' axis passes through the area's centroid C; i.e., $\int y'\, dA = \bar{y}' \int dA = 0$ since $\bar{y}' = 0$. Since the third integral represents the total area A, the final result is therefore

$$I_x = \bar{I}_{x'} + A d_y^2 \tag{6–12}$$

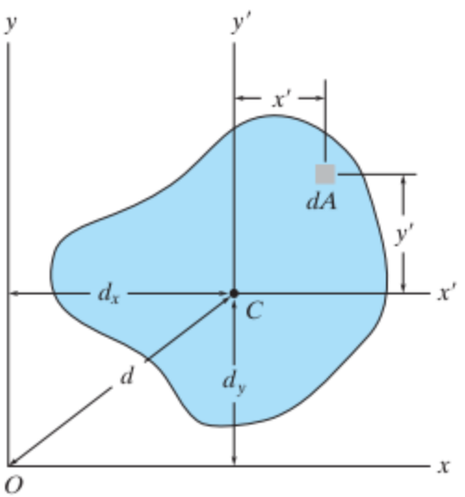


Fig. 6–19

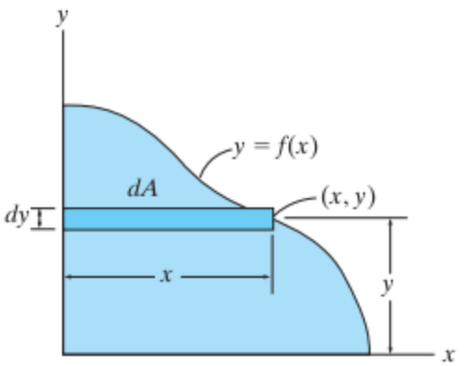
A similar expression can be written for I_y; i.e.,

$$I_y = \bar{I}_{y'} + Ad_x^2 \tag{6–13}$$

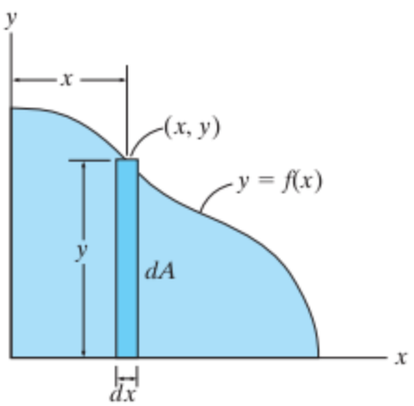
And finally, for the polar moment of inertia, since $\bar{J}_C = \bar{I}_{x'} + \bar{I}_{y'}$ and $d^2 = d_x^2 + d_y^2$, we have

$$J_O = \bar{J}_C + Ad^2 \tag{6–14}$$

The form of each of these three equations states that *the moment of inertia for an area about an axis is equal to its moment of inertia about a parallel axis passing through the area's centroid, plus the product of the area and the square of the perpendicular distance between the axes.*



(a)



(b)

Fig. 6–20

Procedure for Analysis

In most cases the moment of inertia can be determined using a single integration. The following procedure shows two ways in which this can be done.

- If the curve defining the boundary of the area is expressed as $y = f(x)$, then select a rectangular differential element such that it has a finite length and differential width.
- The element should be located so that it intersects the curve at the *arbitrary point* (x, y).

Case 1:

- Orient the element so that its length is *parallel* to the axis about which the moment of inertia is computed. This situation occurs when the rectangular element shown in Fig. 6–20*a* is used to determine I_x for the area. Here the entire element is at a distance y from the x axis since it has a thickness dy. Thus $I_x = \int y^2 dA$. To find I_y, the element is oriented as shown in Fig. 6–20*b*. This element lies at the *same* distance x from the y axis so that $I_y = \int x^2 dA$.

Case 2:

- The length of the element can be oriented *perpendicular* to the axis about which the moment of inertia is computed; however, Eq. 6–10 *does not apply* since all points on the element will *not* lie at the same moment-arm distance from the axis. For example, if the rectangular element in Fig. 6–20*a* is used to determine I_y, it will first be necessary to calculate the moment of inertia of the *element* about an axis parallel to the y axis that passes through the element's centroid, and then determine the moment of inertia of the *element* about the y axis using the parallel-axis theorem. Integration of this result will yield I_y. See Example 6–10.

EXAMPLE 6.9

Determine the moment of inertia for the rectangular area shown in Fig. 6–21 with respect to (a) the centroidal x' axis, (b) the axis x_b passing through the base of the rectangle, and (c) the pole or z' axis perpendicular to the x'–y' plane and passing through the centroid C.

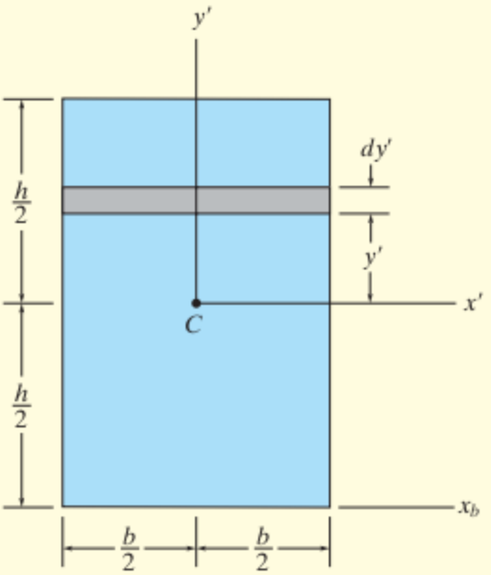


Fig. 6–21

SOLUTION (CASE 1)

Part (a). The differential element shown in Fig. 6–21 is chosen for integration. Because of its location and orientation, the *entire element* is at a distance y' from the x' axis. Here it is necessary to integrate from $y' = -h/2$ to $y' = h/2$. Since $dA = b\,dy'$, then

$$\bar{I}_{x'} = \int_A y'^2\,dA = \int_{-h/2}^{h/2} y'^2(b\,dy') = b\int_{-h/2}^{h/2} y'^2\,dy'$$

$$\bar{I}_{x'} = \frac{1}{12}bh^3 \qquad \textit{Ans.}$$

Part (b). The moment of inertia about an axis passing through the base of the rectangle can be obtained by using the above result of part (a) and applying the parallel-axis theorem, Eq. 6–12.

$$I_{x_b} = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3}bh^3 \qquad \textit{Ans.}$$

Part (c). To obtain the polar moment of inertia about point C, we must first obtain $\bar{I}_{y'}$, which may be found by interchanging the dimensions b and h in the result of part (a), i.e.,

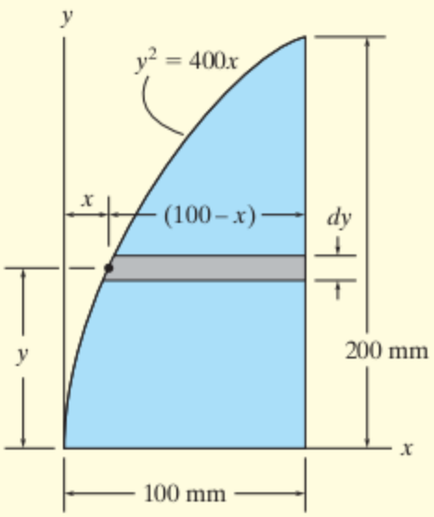
$$\bar{I}_{y'} = \frac{1}{12}hb^3$$

Using Eq. 6–11, the polar moment of inertia about C is therefore

$$\bar{J}_C = \bar{I}_{x'} + \bar{I}_{y'} = \frac{1}{12}bh(h^2 + b^2) \qquad \textit{Ans.}$$

6

EXAMPLE 6.10



(a)

Determine the moment of inertia for the shaded area shown in Fig. 6–22*a* about the *x* axis.

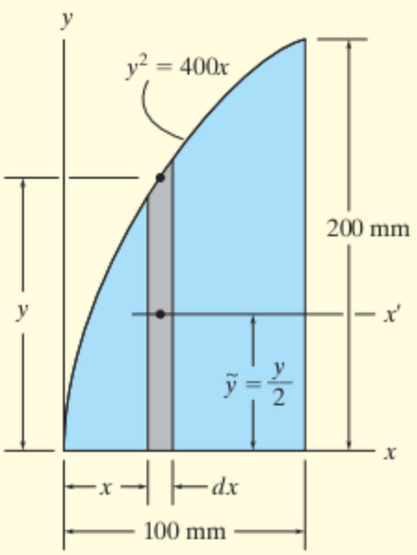
SOLUTION I (CASE 1)

A differential element of area that is *parallel* to the *x* axis, as shown in Fig. 6–22*a*, is chosen for integration. Since this element has a thickness dy and intersects the curve at the *arbitrary point* (x, y), its area is $dA = (100 - x)\,dy$. Furthermore, the element lies at the same distance y from the *x* axis. Hence, integrating with respect to y, from $y = 0$ to $y = 200$ mm, yields

$$I_x = \int_A y^2\,dA = \int_0^{200\text{ mm}} y^2(100 - x)\,dy$$

$$= \int_0^{200\text{ mm}} y^2\left(100 - \frac{y^2}{400}\right)dy = \int_0^{200\text{ mm}} \left(100y^2 - \frac{y^4}{400}\right)dy$$

$$= 107(10^6)\text{ mm}^4 \qquad \textit{Ans.}$$



(b)

Fig. 6–22

SOLUTION II (CASE 2)

A differential element *parallel* to the *y* axis, as shown in Fig. 6–22*b*, is chosen for integration. It intersects the curve at the *arbitrary point* (x, y). In this case, all points of the element do *not* lie at the same distance from the *x* axis, and therefore the parallel-axis theorem must be used to determine the *moment of inertia of the element* with respect to this axis. For a rectangle having a base b and height h, the moment of inertia about its centroidal axis has been determined in part (a) of Example 6–9. There it was found that $\bar{I}_{x'} = \frac{1}{12}bh^3$. For the differential element shown in Fig. 6–22*b*, $b = dx$ and $h = y$, and thus $d\bar{I}_{x'} = \frac{1}{12}dx\,y^3$. Since the centroid of the element is $\tilde{y} = y/2$ from the *x* axis, the moment of inertia of the element about this axis is

$$dI_x = d\bar{I}_{x'} + dA\,\tilde{y}^2 = \frac{1}{12}dx\,y^3 + y\,dx\left(\frac{y}{2}\right)^2 = \frac{1}{3}y^3\,dx$$

(This result can also be concluded from part (b) of Example 6–9.) Integrating with respect to *x*, from $x = 0$ to $x = 100$ mm, yields

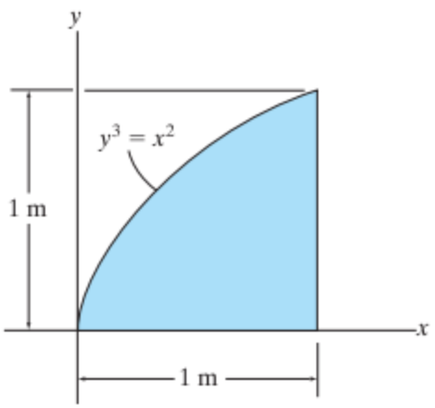
$$I_x = \int dI_x = \int_0^{100\text{ mm}} \frac{1}{3}y^3\,dx = \int_0^{100\text{ mm}} \frac{1}{3}(400x)^{3/2}\,dx$$

$$= 107(10^6)\text{ mm}^4 \qquad \textit{Ans.}$$

6

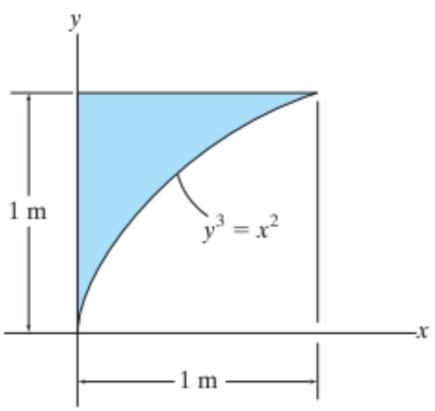
FUNDAMENTAL PROBLEMS

F6–19. Determine the moment of inertia of the shaded area about the x axis.



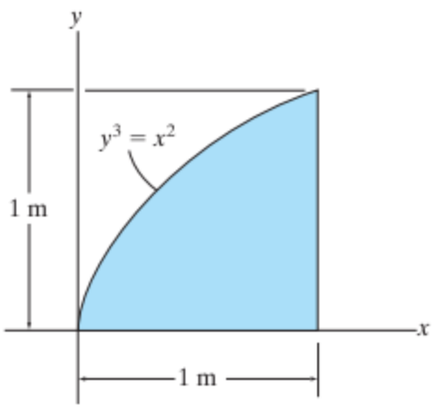
F6–19

F6–20. Determine the moment of inertia of the shaded area about the x axis.



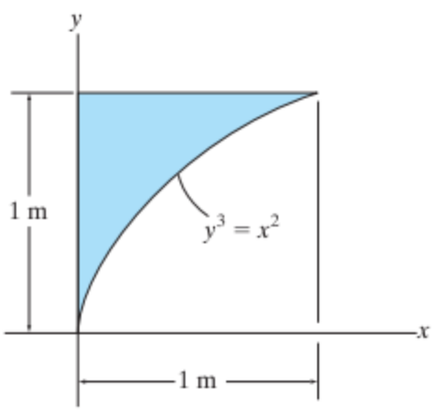
F6–20

F6–21. Determine the moment of inertia of the shaded area about the y axis.



F6–21

F6–22. Determine the moment of inertia of the shaded area about the y axis.

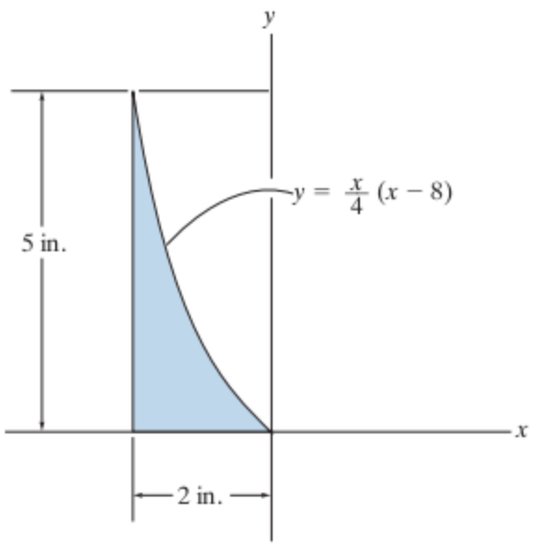


F6–22

PROBLEMS

6–42. Determine the moment of inertia of the shaded area about the x axis.

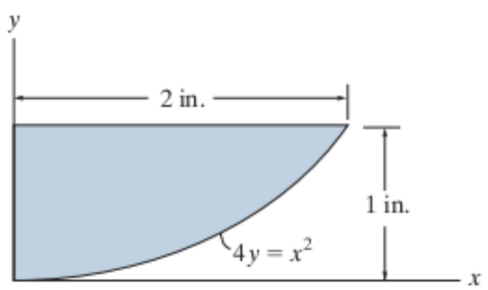
6–43. Determine the moment of inertia of the shaded area about the y axis.



Probs. 6–42/43

*__6–44.__ Determine the moment of inertia of the area about the x axis.

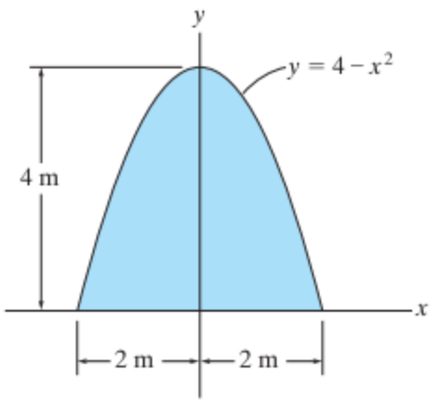
6–45. Determine the moment of inertia of the area about the y axis.



Probs. 6–44/45

6–46. Determine the moment of inertia of the area about the x axis.

6–47. Determine the moment of inertia of the area about the y axis.

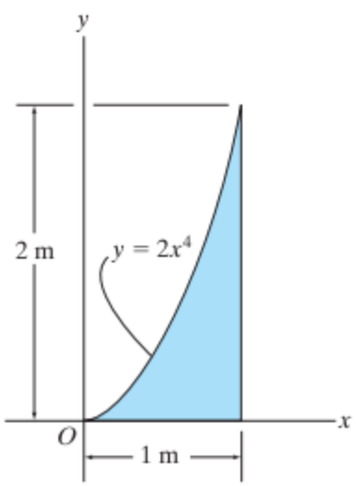


Probs. 6–46/47

*__6–48.__ Determine the moment of inertia of the area about the x axis.

6–49. Determine the moment of inertia of the area about the y axis.

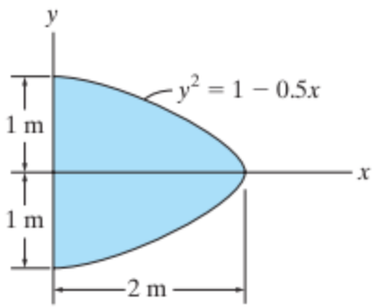
6–50. Determine the polar moment of inertia of the area about the z axis passing through point O.



Probs. 6–48/49/50

6–51. Determine the moment of inertia of the area about the x axis.

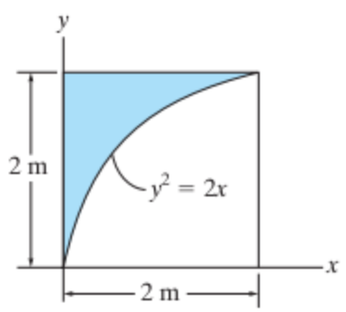
***6–52.** Determine the moment of inertia of the area about the y axis.



Probs. 6–51/52

6–53. Determine the moment of inertia of the area about the x axis.

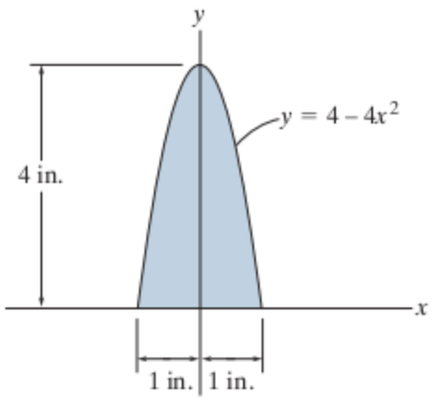
6–54. Determine the moment of inertia of the area about the y axis.



Probs. 6–53/54

6–55. Determine the moment of inertia of the area about the x axis. Solve the problem in two ways, using rectangular differential elements: (a) having a thickness of dx, and (b) having a thickness of dy.

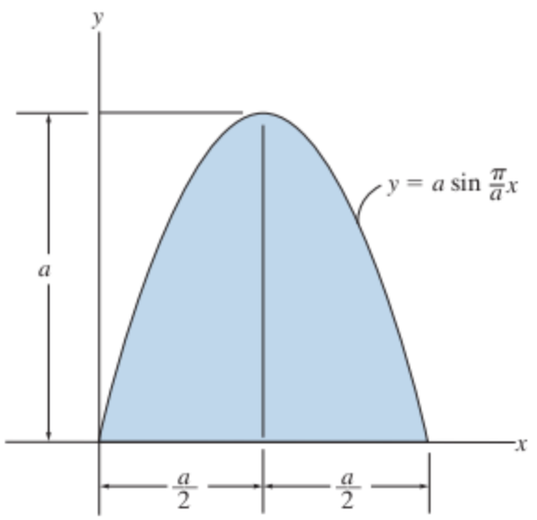
***6–56.** Determine the moment of inertia of the area about the y axis. Solve the problem in two ways, using rectangular differential elements: (a) having a thickness of dx, and (b) having a thickness of dy.



Probs. 6–55/56

6–57. Determine the moment of inertia of the shaded area about the x axis.

6–58. Determine the moment of inertia of the shaded area about the y axis.



Probs. 6–57/58

6.6 Moments of Inertia for Composite Areas

A composite area consists of a series of connected "simpler" parts or shapes, such as rectangles, triangles, and circles. Provided the moment of inertia of each of these parts is known or can be determined about a common axis, then the moment of inertia for the composite area about this axis equals the *algebraic sum* of the moments of inertia of all its parts.

Procedure for Analysis

The moment of inertia for a composite area about a reference axis can be determined using the following procedure.

Composite Parts.

- Using a sketch, divide the area into its composite parts and indicate the perpendicular distance from the centroid of each part to the reference axis.

Parallel-Axis Theorem.

- If the centroidal axis for each part does not coincide with the reference axis, the parallel-axis theorem, $I = \bar{I} + Ad^2$, should be used to determine the moment of inertia of the part about the reference axis. For the calculation of $\bar{I}$, use Appendix B.

Summation.

- The moment of inertia of the entire area about the reference axis is determined by summing the results of its composite parts about this axis.

- If a composite part has a "hole," its moment of inertia is found by "subtracting" the moment of inertia of the hole from the moment of inertia of the entire part including the hole.

For design or analysis of this T-beam, engineers must be able to locate the centroid of its cross-sectional area, and then find the moment of inertia of this area about the centroidal axis.

EXAMPLE 6.11

Determine the moment of inertia of the area shown in Fig. 6–23*a* about the *x* axis.

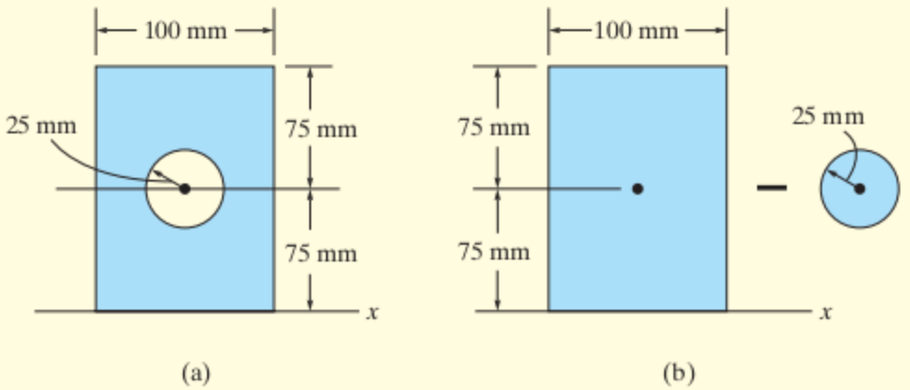


Fig. 6–23

SOLUTION

Composite Parts. The area can be obtained by *subtracting* the circle from the rectangle shown in Fig. 6–23*b*. The centroid of each area is located in the figure.

Parallel-Axis Theorem. The moments of inertia about the *x* axis are determined using the parallel-axis theorem and the geometric properties formulae for circular and rectangular areas $I_x = \frac{1}{4}\pi r^4$; $I_x = \frac{1}{12}bh^3$, found on the inside back cover.

Circle

$$I_x = \bar{I}_{x'} + Ad_y^2$$
$$= \frac{1}{4}\pi(25)^4 + \pi(25)^2(75)^2 = 11.4(10^6)\,\text{mm}^4$$

Rectangle

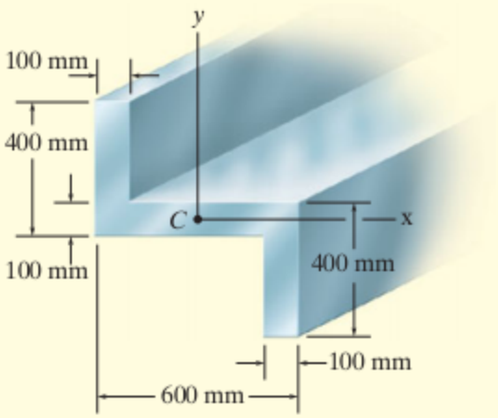
$$I_x = \bar{I}_{x'} + Ad_y^2$$
$$= \frac{1}{12}(100)(150)^3 + (100)(150)(75)^2 = 112.5(10^6)\,\text{mm}^4$$

Summation. The moment of inertia for the area is therefore

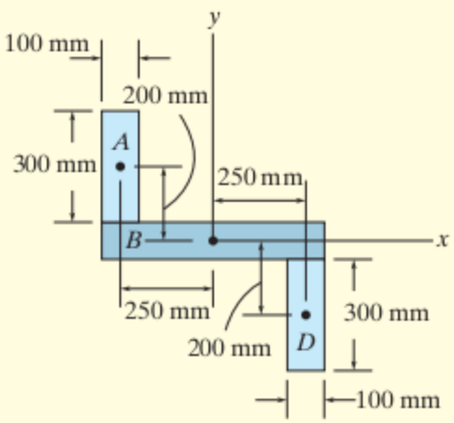
$$I_x = -11.4(10^6) + 112.5(10^6)$$
$$= 101(10^6)\,\text{mm}^4 \qquad \textit{Ans.}$$

6

EXAMPLE 6.12



(a)



(b)

Fig. 6–24

Determine the moments of inertia for the cross-sectional area of the member shown in Fig. 6–24*a* about the x and y centroidal axes.

SOLUTION

Composite Parts. The cross section can be subdivided into the three rectangular areas A, B, and D shown in Fig. 6–24*b*. For the calculation, the centroid of each of these rectangles is located in the figure.

Parallel-Axis Theorem. From the table in Appendix B, in Example 6–9, the moment of inertia of a rectangle about its centroidal axis is $\bar{I} = \frac{1}{12}bh^3$. Hence, using the parallel-axis theorem for rectangles A and D, the calculations are as follows:

Rectangles A and D

$$I_x = \bar{I}_{x'} + Ad_y^2 = \frac{1}{12}(100)(300)^3 + (100)(300)(200)^2$$
$$= 1.425(10^9)\ \text{mm}^4$$

$$I_y = \bar{I}_{y'} + Ad_x^2 = \frac{1}{12}(300)(100)^3 + (100)(300)(250)^2$$
$$= 1.90(10^9)\ \text{mm}^4$$

Rectangle B

$$I_x = \frac{1}{12}(600)(100)^3 = 0.05(10^9)\ \text{mm}^4$$
$$I_y = \frac{1}{12}(100)(600)^3 = 1.80(10^9)\ \text{mm}^4$$

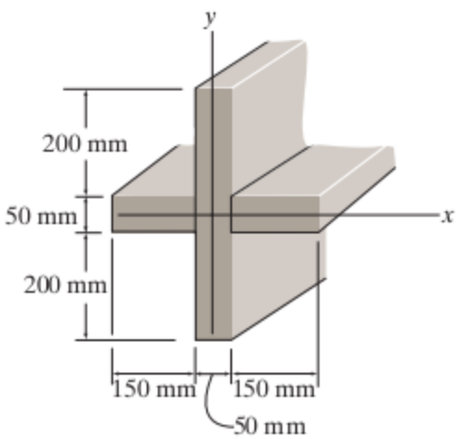
Summation. The moments of inertia for the entire cross section are thus

$$I_x = 2[1.425(10^9)] + 0.05(10^9)$$
$$= 2.90(10^9)\ \text{mm}^4 \qquad \textit{Ans.}$$

$$I_y = 2[1.90(10^9)] + 1.80(10^9)$$
$$= 5.60(10^9)\ \text{mm}^4 \qquad \textit{Ans.}$$

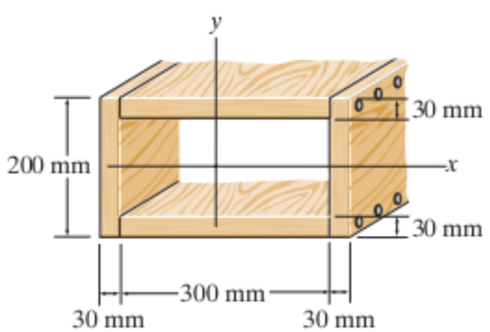
FUNDAMENTAL PROBLEMS

F6–23. Determine the moment of inertia of the beam's cross-sectional area about the centroidal x and y axes.



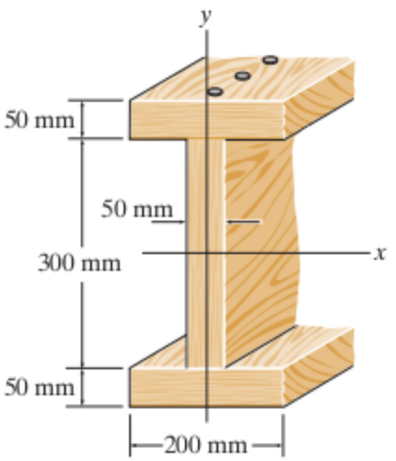
F6–23

F6–24. Determine the moment of inertia of the beam's cross-sectional area about the centroidal x and y axes.



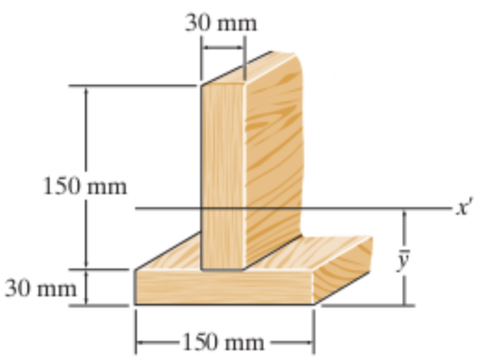
F6–24

F6–25. Determine the moment of inertia of the cross-sectional area of the channel with respect to the y axis.



F6–25

F6–26. Determine the moment of inertia of the cross-sectional area of the T-beam with respect to the x' axis passing through the centroid of the cross section.



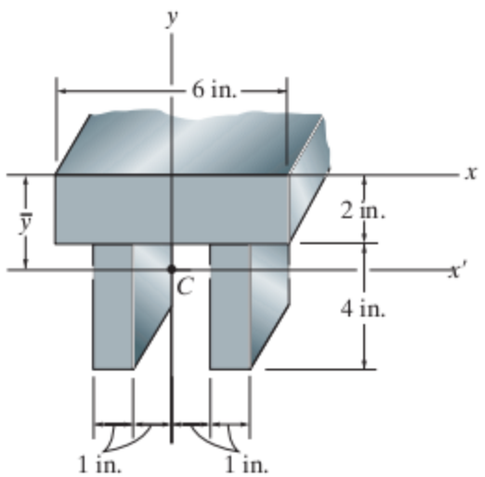
F6–26

PROBLEMS

6–59. Determine the distance $\bar{y}$ to the centroid of the beam's cross-sectional area; then find the moment of inertia about the x' axis.

***6–60.** Determine the moment of inertia of the beam's cross-sectional area about the x axis.

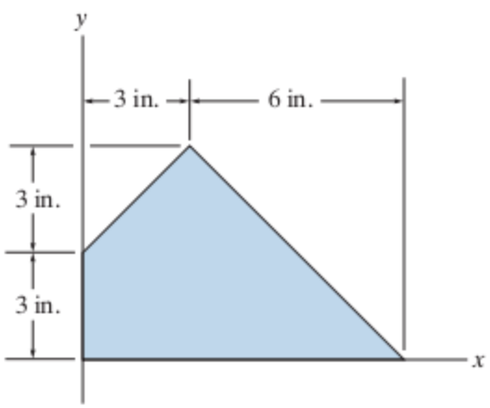
6–61. Determine the moment of inertia of the beam's cross-sectional area about the y axis.



Probs. 6–59/60/61

6–62. Determine the moment of inertia of the composite area about the x axis.

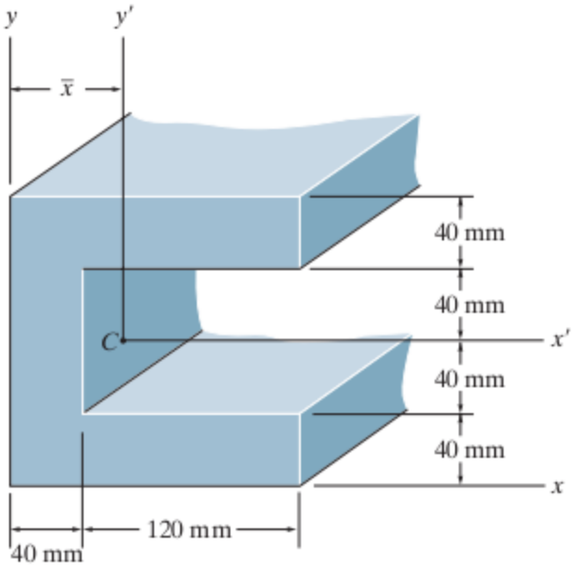
6–63. Determine the moment of inertia of the composite area about the y axis.



Probs. 6–62/63

***6–64.** Determine the distance $\bar{x}$ to the centroid of the beam's cross-sectional area, then find the moment of inertia about the y' axis.

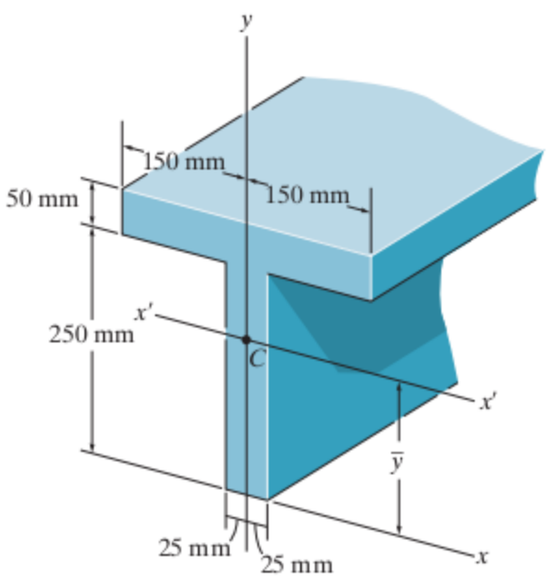
6–65. Determine the moment of inertia of the beam's cross-sectional area about the x' axis.



Probs. 6–64/65

6–66. Determine the moment of inertia of the beam's cross-sectional area about the y axis.

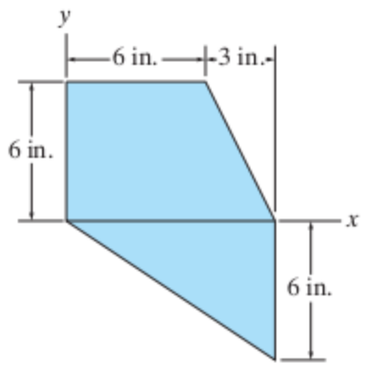
6–67. Determine $\bar{y}$, which locates the centroidal axis x' for the cross-sectional area of the T-beam, and then find the moment of inertia about the x' axis.



Probs. 6–66/67

*6–68. Determine the moment of inertia I_x of the shaded area about the x axis.

6–69. Determine the moment of inertia I_y of the shaded area about the y axis.

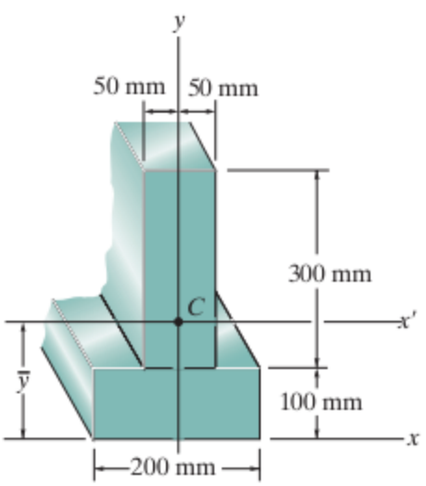


Probs. 6–68/69

6–70. Determine the distance $\bar{y}$ to the centroid of the beam's cross-sectional area; then find the moment of inertia about the x' axis.

6–71. Determine the moment of inertia of the beam's cross-sectional area about the x axis.

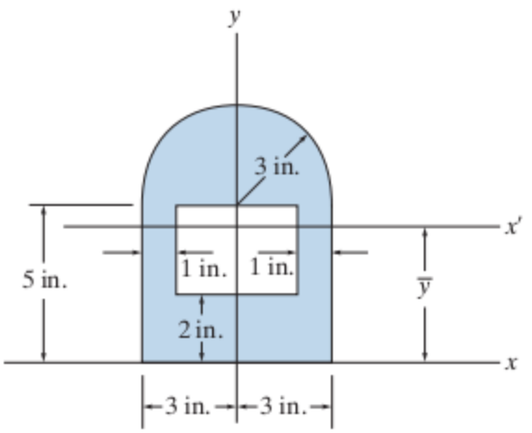
*6–72. Determine the moment of inertia of the beam's cross-sectional area about the y axis.



Probs. 6–70/71/72

6–73. Locate the centroid $\bar{y}$ of the composite area, then determine the moment of intertia of this area about the centroidal x' axis.

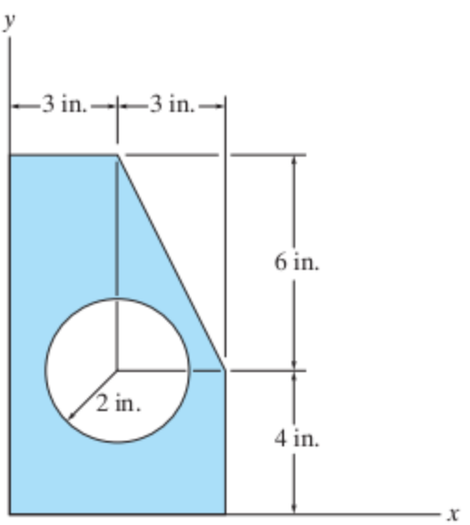
6–74. Determine the moment of inertia of the composite area about the centroidal y axis.



Probs. 6–73/74

6–75. Determine the moment of inertia of the area about the x axis.

*6–76. Determine the moment of inertia of the area about the y axis.



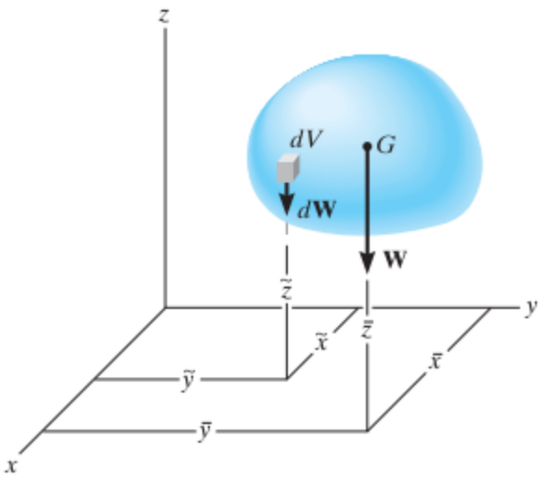
Probs. 6–75/76

CHAPTER REVIEW

Center of Gravity and Centroid

The *center of gravity* G represents a point where the weight of the body can be considered concentrated. The distance from an axis to this point can be determined from a balance of moments, which requires that the moment of the weight of all the particles of the body about this axis must equal the moment of the entire weight of the body about the axis.

$$\bar{x} = \frac{\int \tilde{x}\, dW}{\int dW} \qquad \bar{y} = \frac{\int \tilde{y}\, dW}{\int dW} \qquad \bar{z} = \frac{\int \tilde{z}\, dW}{\int dW}$$



The center of mass will coincide with the center of gravity provided the acceleration of gravity is constant.

The *centroid* is the location of the geometric center for the body. It is determined in a similar manner, using a moment balance of geometric elements such as area, or volume segments. For bodies having a continuous shape, moments are summed (integrated) using differential elements.

$$\bar{x} = \frac{\int_A \tilde{x}\, dA}{\int_A dA} \qquad \bar{y} = \frac{\int_A \tilde{y}\, dA}{\int_A dA} \qquad \bar{z} = \frac{\int_A \tilde{z}\, dA}{\int_A dA}$$

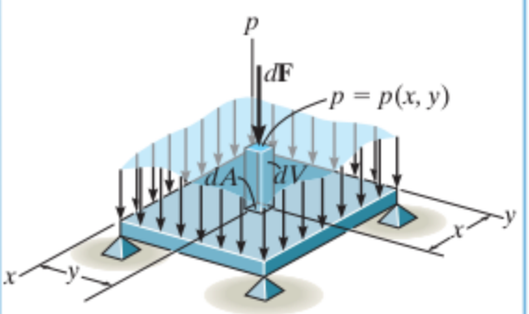
$$\bar{x} = \frac{\int_V \tilde{x}\, dV}{\int_V dV} \qquad \bar{y} = \frac{\int_V \tilde{y}\, dV}{\int_V dV} \qquad \bar{z} = \frac{\int_V \tilde{z}\, dV}{\int_V dV}$$

General Distributed Loading

The magnitude of the resultant force is equal to the total volume under the distributed-loading diagram. The line of action of the resultant force passes through the geometric center or centroid of this volume.

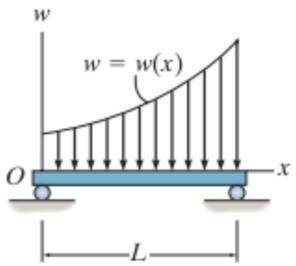
$$F_R = \int_A p(x, y)\, dA = \int_V dV$$

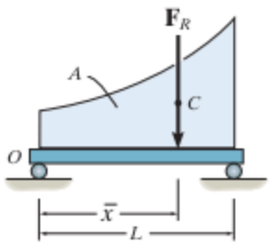
$$\bar{x} = \frac{\int_V x\, dV}{\int_V dV} \qquad \bar{y} = \frac{\int_V y\, dV}{\int_V dV}$$



Coplanar Distributed Loading

A simple distributed loading can be represented by its resultant force, which is equivalent to the *area* under the loading curve. This resultant has a line of action that passes through the *centroid* or geometric center of the area or volume under the loading diagram.





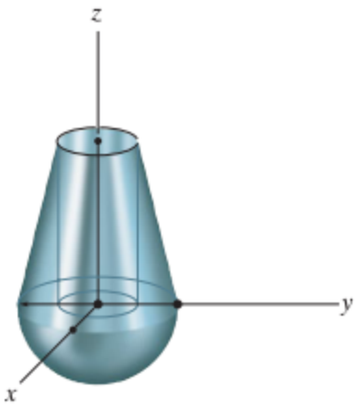
Composite Body

If the body is a composite of several shapes, each having a known location for its center of gravity or centroid, then the location of the center of gravity or centroid of the body can be determined from a discrete summation using its composite parts.

$$\bar{x} = \frac{\Sigma \tilde{x} W}{\Sigma W}$$

$$\bar{y} = \frac{\Sigma \tilde{y} W}{\Sigma W}$$

$$\bar{z} = \frac{\Sigma \tilde{z} W}{\Sigma W}$$



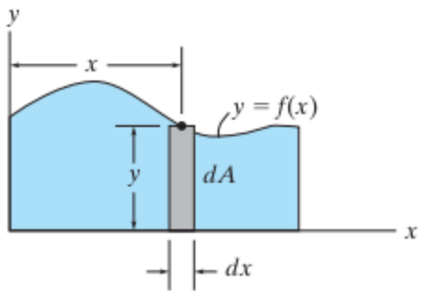
Area Moment of Inertia

The *area moment of inertia* represents the second moment of the area about an axis. It is frequently used in formulas related to the strength and stability of structural members or mechanical elements.

If the area shape is irregular but can be described mathematically, then a differential element must be selected and integration over the entire area must be performed to determine the moment of inertia.

$$I_x = \int_A y^2 \, dA$$

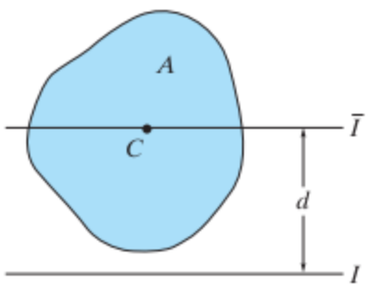
$$I_y = \int_A x^2 \, dA$$



Parallel-Axis Theorem

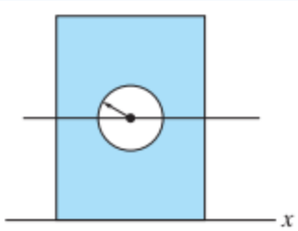
If the moment of inertia for an area is known about a centroidal axis, then its moment of inertia about a parallel axis can be determined using the parallel-axis theorem.

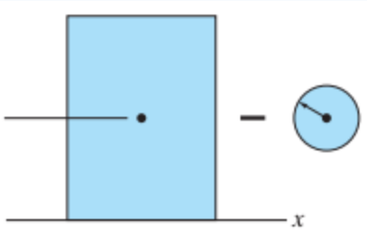
$$I = \bar{I} + Ad^2$$



Composite Area

If an area is a composite of common shapes, then its moment of inertia is equal to the algebraic sum of the moments of inertia of each of its parts.

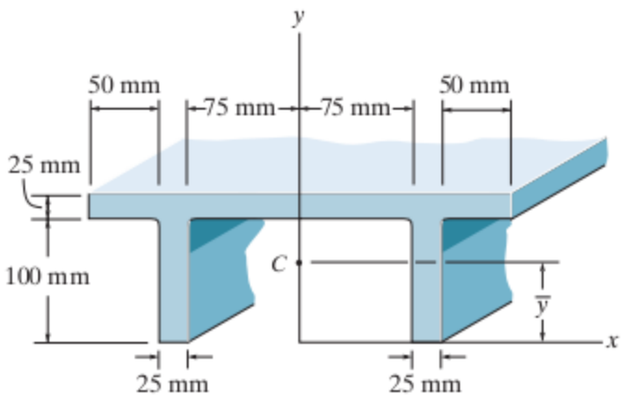




6

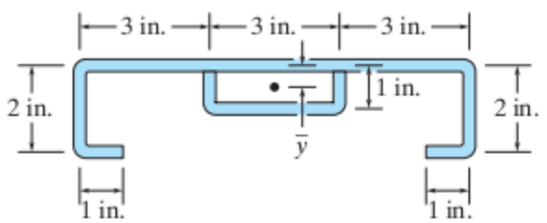
REVIEW PROBLEMS

6–77. Locate the centroid $\bar{y}$ of the beam's cross-sectional area.



Prob. 6–77

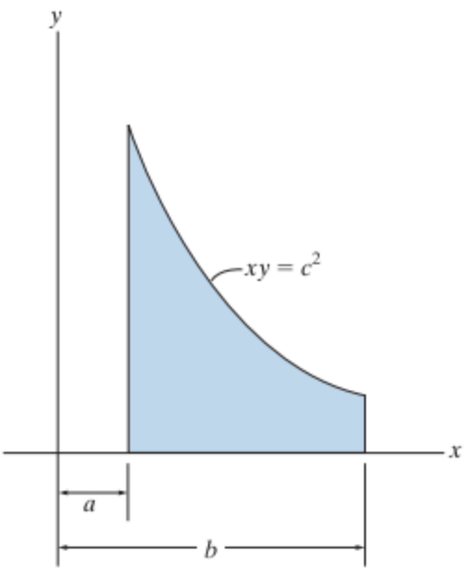
6–78. The thin-walled channel and stiffener have the cross section shown. If the material has a constant thickness, determine the location ($\bar{y}$) of its centroid. The dimensions are indicated to the center of each segment.



Prob. 6–78

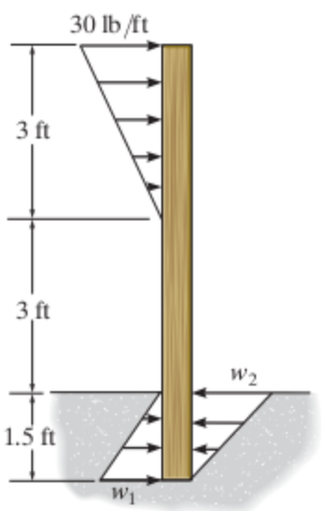
6–79. Locate the centroid $\bar{x}$ of the area.

***6–80.** Locate the centroid $\bar{y}$ of the area.



Probs. 6–79/80

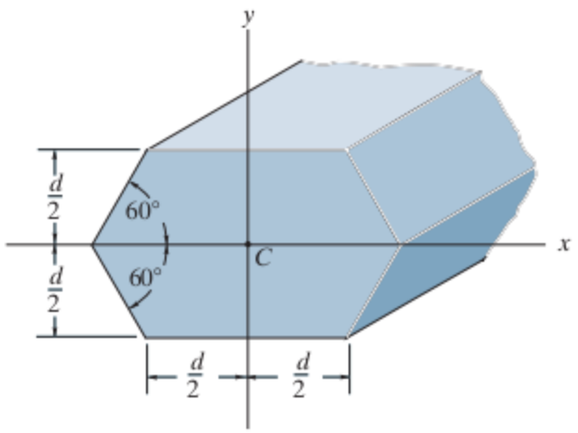
6–81. The post is embedded into a concrete footing so that it is fixed supported. If the reaction of the concrete on the post can be approximated by the distributed loading shown, determine the intensity of w_1 and w_2 so that the resultant force and couple moment on the post due to the loadings are both zero.



Prob. 6–81

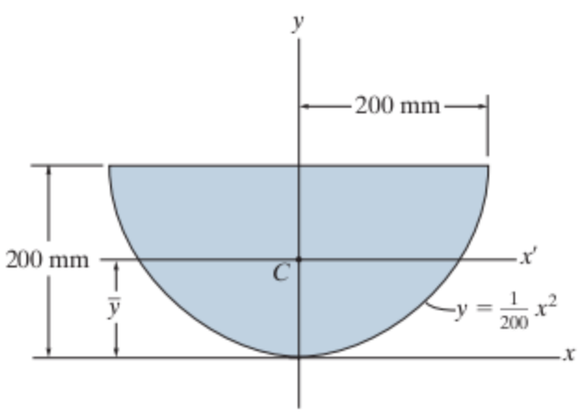
6–82. Determine the moment of inertia of the beam's cross-sectional area about the x axis which passes through the centroid C.

6–83. Determine the moment of inertia of the beam's cross-sectional area about the y axis which passes through the centroid C.



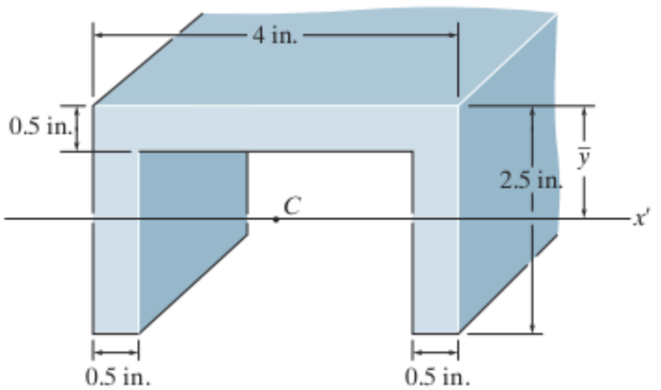
Probs. 6–82/83

*6–84. Determine the area moment of inertia of the area about the x axis. Then, using the parallel-axis theorem, find the area moment of inertia about the x' axis that passes through the centroid C of the area. $\bar{y} = 120$ mm.



Prob. 6–84

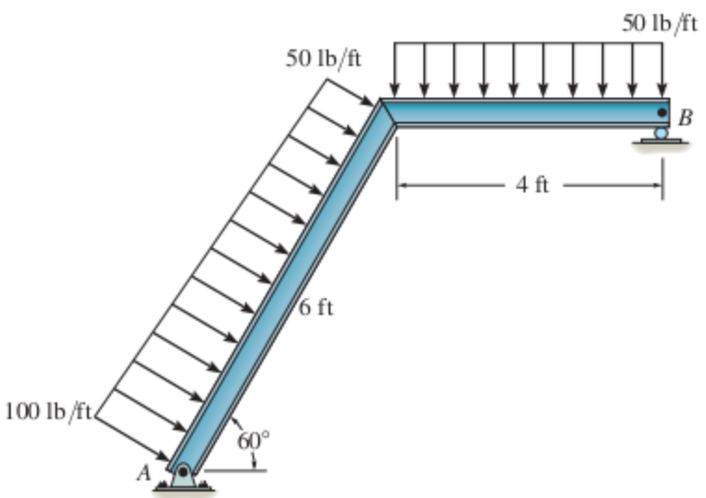
6–85. Determine the moment of inertia of the beam's cross-sectional area with respect to the x' axis passing through the centroid C.



Prob. 6–85

6–86. Replace the loading by an equivalent resultant force and couple moment at point A.

6–87. Replace the loading by an equivalent resultant force and couple moment acting at point B.



Probs. 6–86/87

Chapter 7

The bolts used for the connections of this steel framework are subjected to stress. In this chapter, we will discuss how engineers design these connections and their fasteners.

Stress and Strain

CHAPTER OBJECTIVES

- To show how to use the method of sections for determining the internal loadings in a member.
- To introduce the concepts of normal and shear stress, and to use them in the analysis and design of members subject to axial load and direct shear.
- To define normal and shear strain, and show how they can be determined for various types of problems.

7.1 Introduction

It was stated in Chapter 1 that ***Mechanics of materials*** is a branch of mechanics that studies the relationships between the *external* loads applied to a deformable body and the intensity of *internal* forces acting within the body. This subject also involves computing the *deformations* of the body, and it provides a study of the body's *stability* when the body is subjected to external forces.

In the design of any structure or machine, it is *first* necessary to use the principles of statics to determine the forces acting both on and within its various members. The size of the members, their deflection, and their stability depend not only on the internal loadings, but also on the type of material from which the members are made. Consequently, an accurate determination and fundamental understanding of *material behavior* will be of vital importance for developing the necessary equations used in mechanics of materials. Realize that many formulas and rules for design, as defined in engineering codes and used in practice, are based on the fundamentals of mechanics of materials, and for this reason an understanding of the principles of this subject is very important.

7.2 Internal Resultant Loadings

In order to design the horizontal members of this building frame, it is first necessary to find the internal loadings at various points along their length.

In mechanics of materials, statics is primarily used to determine the resultant loadings that act within a body. For example, consider the body shown in Fig. 7–1*a*, which is held in equilibrium by the four external forces.* In order to obtain the internal loadings acting on a specific region within the body, it is necessary to pass an imaginary section or "cut" through the region where the internal loadings are to be determined. The two parts of the body are then separated, and a free-body diagram of one of the parts is drawn, Fig. 7–1*b*. Notice that there is actually a distribution of internal force acting on the "exposed" area of the section. These forces represent the effects of the material of the top part of the body acting on the adjacent material of the bottom part.

Although the exact distribution of this internal loading may be *unknown*, we can use the equations of equilibrium to relate the external forces on the bottom part of the body to the distribution's *resultant force and moment*, $\mathbf{F}_R$ and $\mathbf{M}_{R_O}$, *at any specific point* O on the sectioned area, Fig. 7–1*c*. It will be shown in later portions of the text that point O is most often chosen at the *centroid* of the sectioned area, and so we will always choose this location for O, unless otherwise stated. Also, if a member is long and slender, as in the case of a rod or beam, the section to be considered is generally taken *perpendicular* to the longitudinal axis of the member. This section is referred to as the ***cross section***.

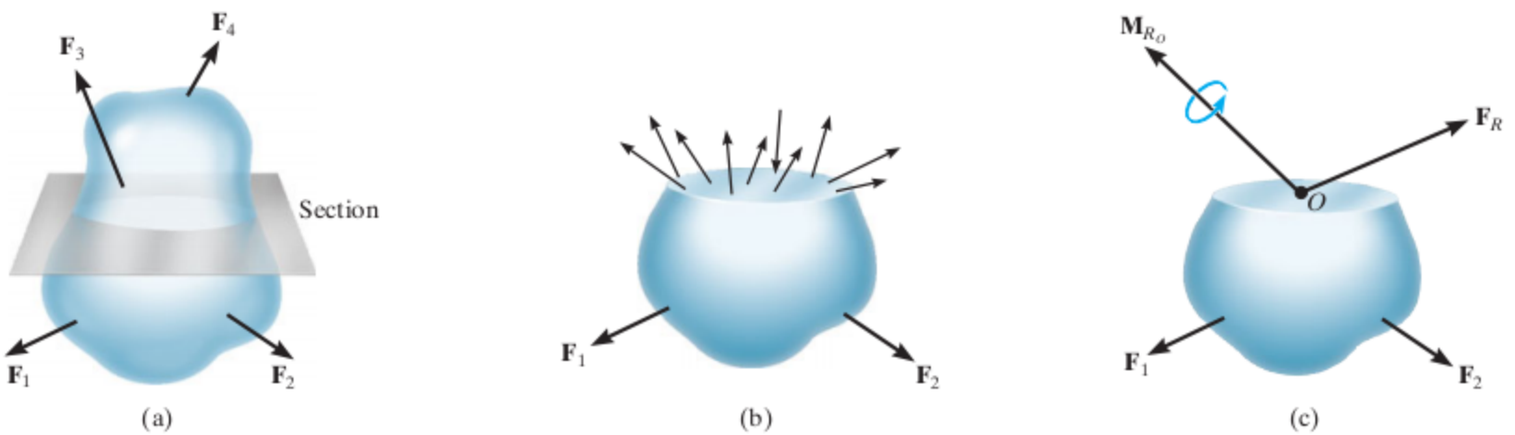


Fig. 7–1

*The body's weight is not shown, since it is assumed to be quite small, and therefore negligible compared with the other loads.

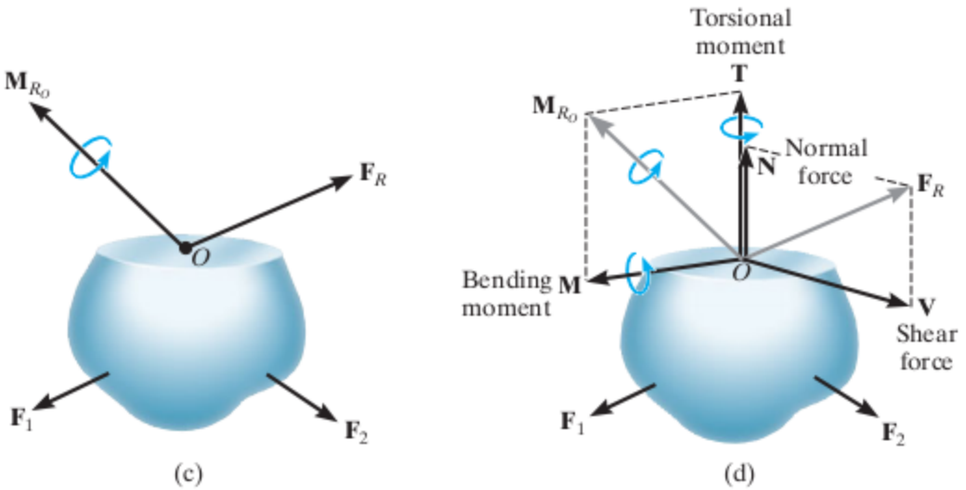


Fig. 7–1 (cont.)

Three Dimensions. Later in this text we will show how to relate the resultant loadings, $\mathbf{F}_R$ and $\mathbf{M}_{R_O}$, to the *distribution of force* on the sectioned area, and thereby develop equations that can be used for analysis and design. To do this, however, the components of $\mathbf{F}_R$ and $\mathbf{M}_{R_O}$ acting both normal and perpendicular to the sectioned area must be considered, Fig. 7–1*d*. Four different types of resultant loadings can then be defined as follows:

Normal force, N. This force acts perpendicular to the area. It is developed whenever the external loads tend to push or pull on the two segments of the body.

Shear force, V. The shear force lies in the plane of the area and it is developed when the external loads tend to cause the two segments of the body to slide over one another.

Torsional moment or torque, T. This effect is developed when the external loads tend to twist one segment of the body with respect to the other about an axis perpendicular to the area.

Bending moment, M. The bending moment is caused by the external loads that tend to bend the body about an axis lying within the plane of the area.

In this text, note that graphical representation of a moment or torque is shown in three dimensions as a vector with an associated curl. By the *right-hand rule*, the thumb gives the arrowhead sense of this vector and the fingers or curl indicate the tendency for rotation (twisting or bending).

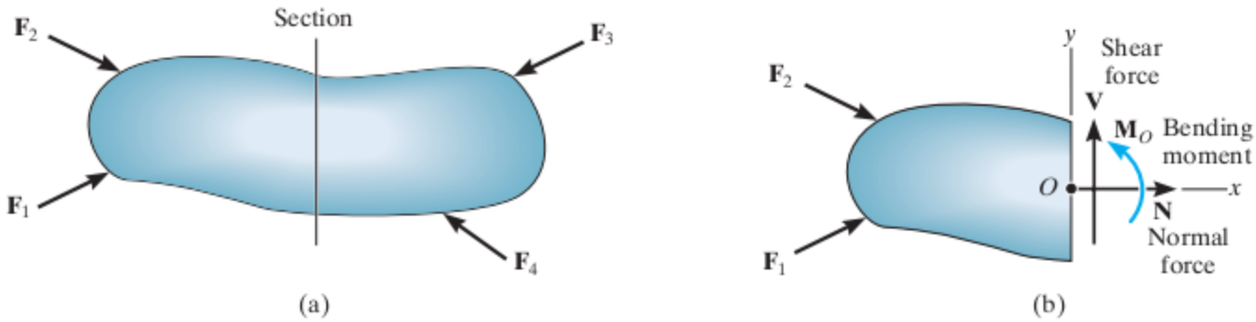


Fig. 7–2

Coplanar Loadings. If the body is subjected to a *coplanar system of forces*, Fig. 7–2*a*, then only normal-force, shear-force, and bending-moment components will exist at the section, Fig. 7–2*b*. If we use the x, y, z coordinate axes, as shown on the left segment, then $\mathbf{N}$ can be obtained by applying $\Sigma F_x = 0$, and $\mathbf{V}$ can be obtained from $\Sigma F_y = 0$. Finally, the bending moment $\mathbf{M}_O$ can be determined by summing moments about point O (the z axis), $\Sigma M_O = 0$, in order to eliminate the moments caused by the unknowns $\mathbf{N}$ and $\mathbf{V}$.

Important Points

- *Mechanics of materials* is a study of the relationship between the external loads applied to a body and the stress and strain caused by the internal loads within the body.
- External forces can be applied to a body as *distributed* or *concentrated surface loadings*, or as *body forces* that act throughout the volume of the body.
- Linear distributed loadings produce a *resultant force* having a *magnitude* equal to the *area* under the load diagram, and having a *location* that passes through the *centroid* of this area.
- A support produces a *force* in a particular direction on its attached member if it *prevents translation* of the member in that direction, and it produces a *couple moment* on the member if it *prevents rotation*.
- The equations of equilibrium $\Sigma \mathbf{F} = \mathbf{0}$ and $\Sigma \mathbf{M} = \mathbf{0}$ must be satisfied in order to prevent a body from translating with accelerated motion and from rotating.
- When applying the equations of equilibrium, it is important to first draw the free-body diagram for the body in order to account for all the terms in the equations.
- The method of sections is used to determine the internal resultant loadings acting on the surface of the sectioned body. In general, these resultants consist of a normal force, shear force, torsional moment, and bending moment.

Procedure for Analysis

The resultant *internal* loadings at a point located on the section of a body can be obtained using the method of sections. This requires the following steps.

Support Reactions.

- First decide which segment of the body is to be considered. If the segment has a support or connection to another body, then *before* the body is sectioned, it will be necessary to determine the reactions acting on the chosen segment. To do this draw the free-body diagram of the *entire body* and then apply the necessary equations of equilibrium to obtain these reactions.

Free-Body Diagram.

- Keep all external distributed loadings, couple moments, torques, and forces in their *exact locations*, before passing an imaginary section through the body at the point where the resultant internal loadings are to be determined.
- Draw a free-body diagram of one of the "cut" segments and indicate the unknown resultants **N**, **V**, **M**, and **T** at the section. These resultants are normally placed at the point representing the geometric center or *centroid* of the sectioned area.
- If the member is subjected to a *coplanar* system of forces, only **N**, **V**, and **M** act at the centroid.
- Establish the x, y, z coordinate axes with origin at the centroid and show the resultant internal loadings acting along the axes.

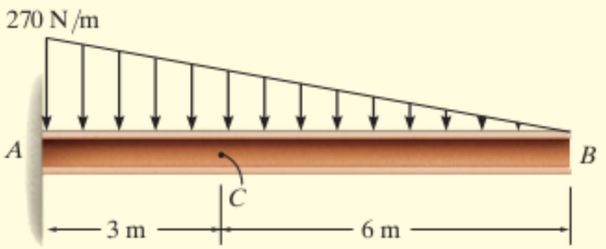
Equations of Equilibrium.

- Moments should be summed at the section, about each of the coordinate axes where the resultants act. Doing this eliminates the unknown forces **N** and **V** and allows a direct solution for **M** (and **T**).
- If the solution of the equilibrium equations yields a negative value for a resultant, the assumed *directional sense* of the resultant is *opposite* to that shown on the free-body diagram.

The following examples illustrate this procedure numerically and also provide a review of some of the important principles of statics.

EXAMPLE 7.1

Determine the resultant internal loadings acting on the cross section at C of the cantilevered beam shown in Fig. 7–3*a*.

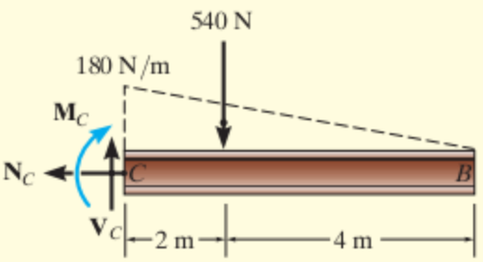


(a)

Fig. 7–3

SOLUTION

Support Reactions. The support reactions at A do not have to be determined if segment CB is considered.



(b)

Free-Body Diagram. The free-body diagram of segment CB is shown in Fig. 7–3*b*. It is important to keep the distributed loading on the segment until *after* the section is made. Only then should this loading be replaced by a single resultant force. Notice that the intensity of the distributed loading at C is found by proportion, i.e., from Fig. 7–3*a*, $w/6\text{ m} = (270\text{ N/m})/9\text{ m}$, $w = 180\text{ N/m}$. The magnitude of the resultant of the distributed load is equal to the area under the loading curve (triangle) and acts through the centroid of this area. Thus, $F = \frac{1}{2}(180\text{ N/m})(6\text{ m}) = 540\text{ N}$, which acts $\frac{1}{3}(6\text{ m}) = 2\text{ m}$ from C as shown in Fig. 7–3*b*.

Equations of Equilibrium. Applying the equations of equilibrium we have

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad -N_C = 0$$

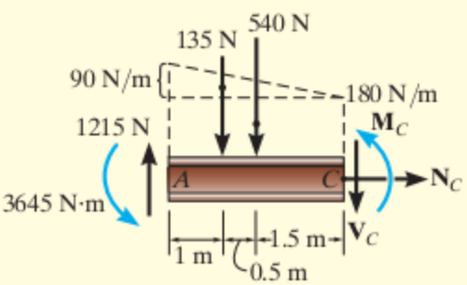
$$N_C = 0 \qquad \textit{Ans.}$$

$$+\uparrow\Sigma F_y = 0; \qquad V_C - 540\text{ N} = 0$$

$$V_C = 540\text{ N} \qquad \textit{Ans.}$$

$$\circlearrowleft +\Sigma M_C = 0; \qquad -M_C - 540\text{ N}(2\text{ m}) = 0$$

$$M_C = -1080\text{ N}\cdot\text{m} \qquad \textit{Ans.}$$



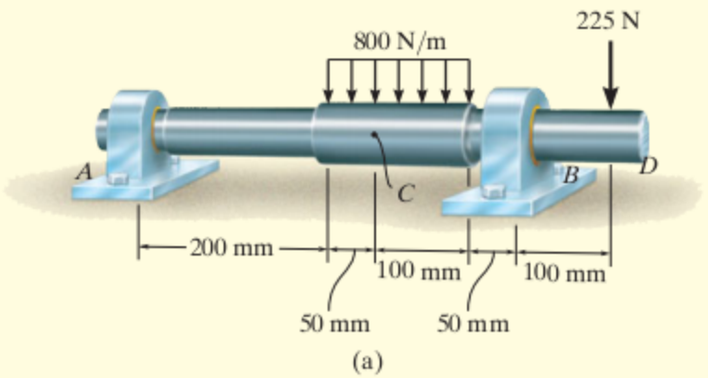
(c)

NOTE: The negative sign indicates that $\mathbf{M}_C$ acts in the opposite direction to that shown on the free-body diagram. Try solving this problem using segment AC, by first obtaining the support reactions at A, which are given in Fig. 7–3*c*.

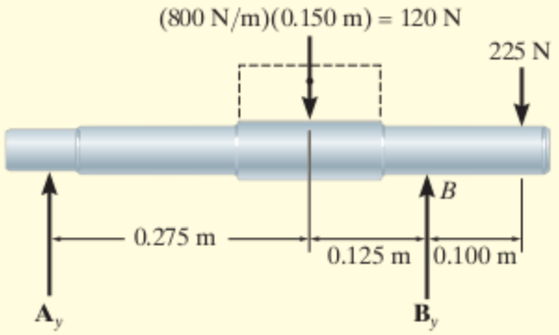
7

EXAMPLE 7.2

Determine the resultant internal loadings acting on the cross section at C of the machine shaft shown in Fig. 7–4*a*. The shaft is supported by journal bearings at A and B, which only exert vertical forces on the shaft.



(a)



(b)

Fig. 7–4

SOLUTION

We will solve this problem using segment AC of the shaft.

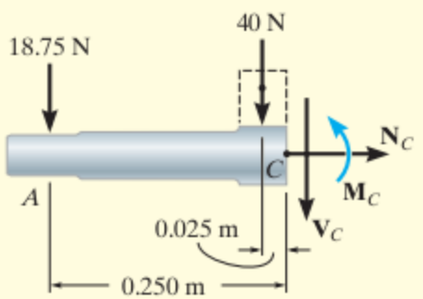
Support Reactions. The free-body diagram of the entire shaft is shown in Fig. 7–4*b*. Since segment AC is to be considered, only the reaction at A has to be determined. Why?

$$\circlearrowleft + \Sigma M_B = 0; \quad -A_y(0.400\text{ m}) + 120\text{ N}(0.125\text{ m}) - 225\text{ N}(0.100\text{ m}) = 0$$

$$A_y = -18.75\text{ N}$$

The negative sign indicates that $\mathbf{A}_y$ acts in the *opposite sense* to that shown on the free-body diagram.

Free-Body Diagram. The free-body diagram of segment AC is shown in Fig. 7–4*c*.



(c)

Equations of Equilibrium.

$$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \qquad N_C = 0 \qquad \textit{Ans.}$$

$$+\uparrow \Sigma F_y = 0; \qquad -18.75\text{ N} - 40\text{ N} - V_C = 0$$

$$V_C = -58.8\text{ N} \qquad \textit{Ans.}$$

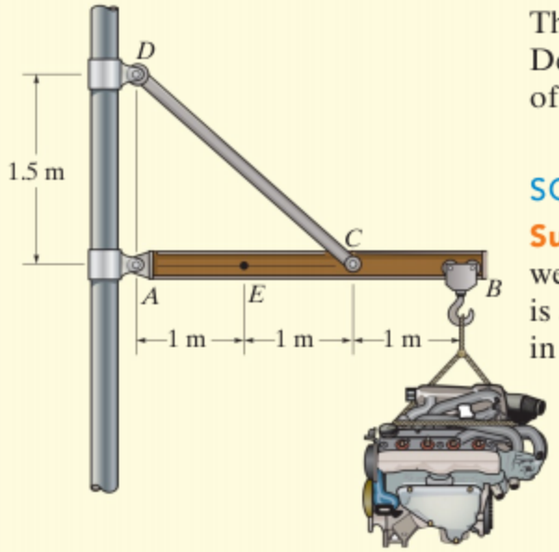
$$\circlearrowleft + \Sigma M_C = 0; \quad M_C + 40\text{ N}(0.025\text{ m}) + 18.75\text{ N}(0.250\text{ m}) = 0$$

$$M_C = -5.69\text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

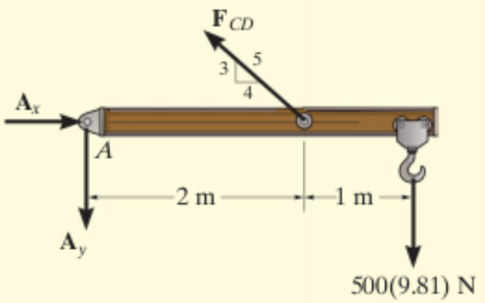
NOTE: The negative signs for V_C and M_C indicate they act in the opposite directions on the free-body diagram. As an exercise, calculate the reaction at B and try to obtain the same results using segment CBD of the shaft.

7

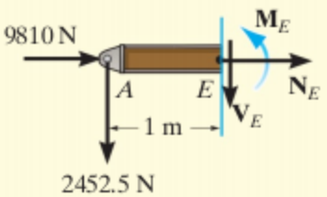
EXAMPLE 7.3



(a)



(b)



(c)

Fig. 7–5

The 500-kg engine is suspended from the crane boom in Fig. 7–5*a*. Determine the resultant internal loadings acting on the cross section of the boom at point E.

SOLUTION

Support Reactions. We will consider segment AE of the boom so we must first determine the pin reactions at A. Notice that member CD is a two-force member. The free-body diagram of the boom is shown in Fig. 7–5*b*. Applying the equations of equilibrium,

$$\circlearrowleft + \Sigma M_A = 0; \quad F_{CD}\left(\tfrac{3}{5}\right)(2\text{ m}) - [500(9.81)\text{ N}](3\text{ m}) = 0$$

$$F_{CD} = 12\,262.5\text{ N}$$

$$\pm \rightarrow \Sigma F_x = 0; \quad A_x - (12\,262.5\text{ N})\left(\tfrac{4}{5}\right) = 0$$

$$A_x = 9810\text{ N}$$

$$+ \uparrow \Sigma F_y = 0; \quad -A_y + (12\,262.5\text{ N})\left(\tfrac{3}{5}\right) - 500(9.81)\text{ N} = 0$$

$$A_y = 2452.5\text{ N}$$

Free-Body Diagram. The free-body diagram of segment AE is shown in Fig. 7–5*c*.

Equations of Equilibrium.

$$\pm \rightarrow \Sigma F_x = 0; \quad N_E + 9810\text{ N} = 0$$

$$N_E = -9810\text{ N} = -9.81\text{ kN} \qquad \textit{Ans.}$$

$$+ \uparrow \Sigma F_y = 0; \quad -V_E - 2452.5\text{ N} = 0$$

$$V_E = -2452.5\text{ N} = -2.45\text{ kN} \qquad \textit{Ans.}$$

$$\circlearrowleft + \Sigma M_E = 0; \quad M_E + (2452.5\text{ N})(1\text{ m}) = 0$$

$$M_E = -2452.5\text{ N}\cdot\text{m} = -2.45\text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

7

EXAMPLE 7.4

Determine the resultant internal loadings acting on the cross section at B of the pipe shown in Fig. 7–6*a*. The pipe has a mass of 2 kg/m and is subjected to both a vertical force of 50 N and a couple moment of 70 N · m at its end A. It is fixed to the wall at C.

SOLUTION

The problem can be solved by considering segment AB, so we do not need to calculate the support reactions at C.

Free-Body Diagram. The x, y, z axes are established at B and the free-body diagram of segment AB is shown in Fig. 7–6*b*. The resultant force and moment components at the section are assumed to act in the positive coordinate directions and to pass through the *centroid* of the cross-sectional area at B. The weight of each segment of pipe is calculated as follows:

$$W_{BD} = (2\text{ kg/m})(0.5\text{ m})(9.81\text{ N/kg}) = 9.81\text{ N}$$
$$W_{AD} = (2\text{ kg/m})(1.25\text{ m})(9.81\text{ N/kg}) = 24.525\text{ N}$$

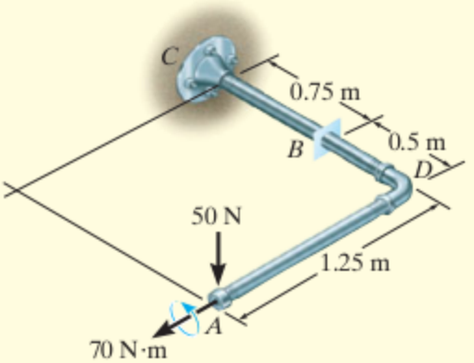
These forces act through the center of gravity of each segment.

Equations of Equilibrium. Applying the six scalar equations of equilibrium, we have*

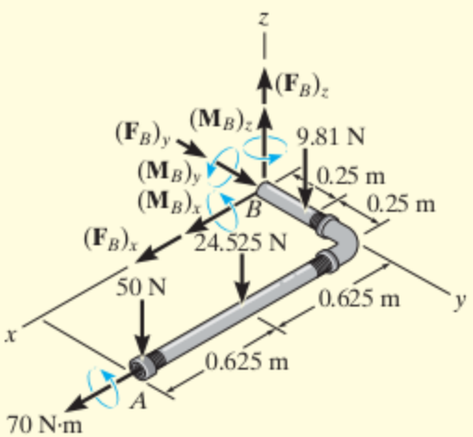
$$\Sigma F_x = 0; \qquad (F_B)_x = 0 \qquad \textit{Ans.}$$
$$\Sigma F_y = 0; \qquad (F_B)_y = 0 \qquad \textit{Ans.}$$
$$\Sigma F_z = 0; \qquad (F_B)_z - 9.81\text{ N} - 24.525\text{ N} - 50\text{ N} = 0$$
$$(F_B)_z = 84.3\text{ N} \qquad \textit{Ans.}$$
$$\Sigma (M_B)_x = 0; \qquad (M_B)_x + 70\text{ N} \cdot \text{m} - 50\text{ N}\,(0.5\text{ m})$$
$$- 24.525\text{ N}\,(0.5\text{ m}) - 9.81\text{ N}\,(0.25\text{ m}) = 0$$
$$(M_B)_x = -30.3\text{ N} \cdot \text{m} \qquad \textit{Ans.}$$
$$\Sigma (M_B)_y = 0; \qquad (M_B)_y + 24.525\text{ N}\,(0.625\text{ m}) + 50\text{ N}\,(1.25\text{ m}) = 0$$
$$(M_B)_y = -77.8\text{ N} \cdot \text{m} \qquad \textit{Ans.}$$
$$\Sigma (M_B)_z = 0; \qquad (M_B)_z = 0 \qquad \textit{Ans.}$$

NOTE: What do the negative signs for $(M_B)_x$ and $(M_B)_y$ indicate? Note that the normal force $N_B = (F_B)_y = 0$, whereas the shear force is $V_B = \sqrt{(0)^2 + (84.3)^2} = 84.3$ N. Also, the torsional moment is $T_B = (M_B)_y = 77.8\text{ N} \cdot \text{m}$ and the bending moment is $M_B = \sqrt{(30.3)^2 + (0)^2} = 30.3\text{ N} \cdot \text{m}$.

*The *magnitude* of each moment about an axis is equal to the magnitude of each force times the perpendicular distance from the axis to the line of action of the force. The *direction* of each moment is determined using the right-hand rule, with positive moments (thumb) directed along the positive coordinate axes.



(a)



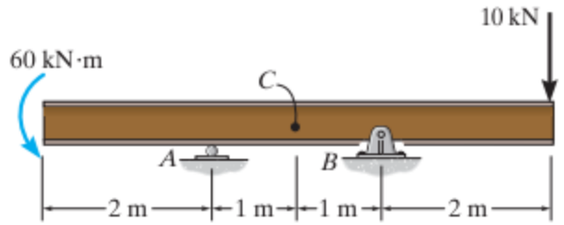
(b)

Fig. 7–6

7

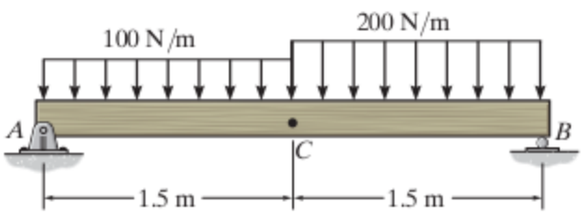
FUNDAMENTAL PROBLEMS

F7–1. Determine the internal normal force, shear force, and bending moment at point *C* in the beam.



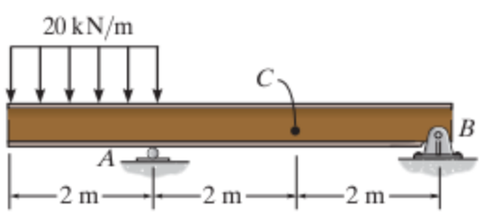
F7–1

F7–2. Determine the internal normal force, shear force, and bending moment at point *C* in the beam.



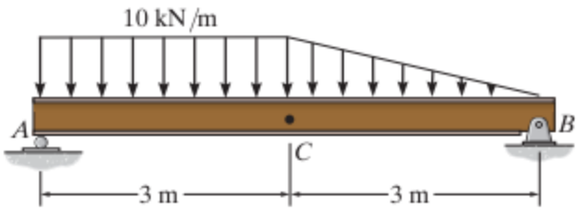
F7–2

F7–3. Determine the internal normal force, shear force, and bending moment at point *C* in the beam.



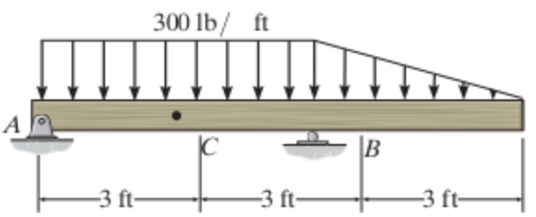
F7–3

F7–4. Determine the internal normal force, shear force, and bending moment at point *C* in the beam.



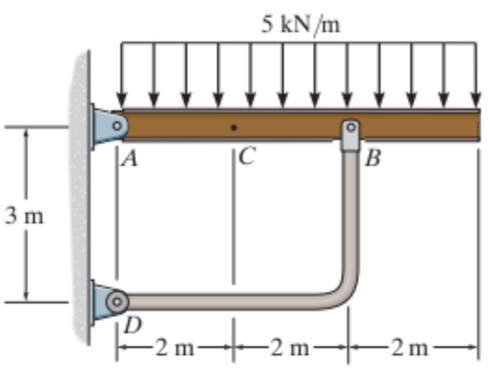
F7–4

F7–5. Determine the internal normal force, shear force, and bending moment at point *C* in the beam.



F7–5

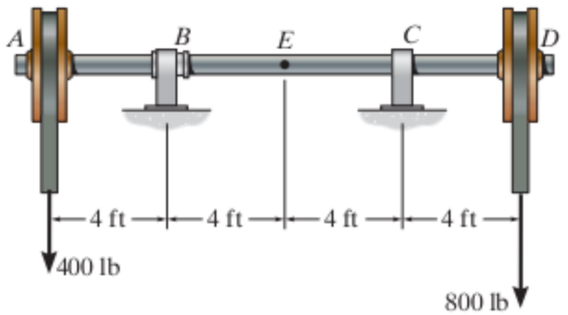
F7–6. Determine the internal normal force, shear force, and bending moment at point *C* in the beam.



F7–6

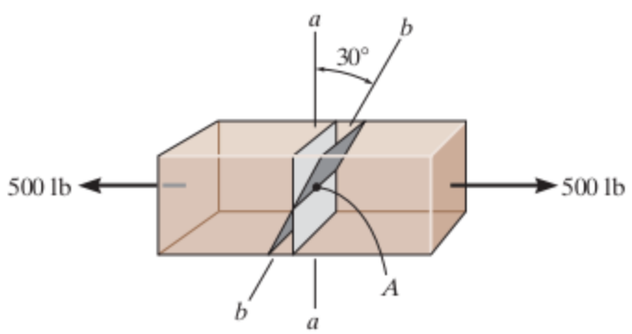
PROBLEMS

7–1. The shaft is supported by a smooth thrust bearing at B and a journal bearing at C. Determine the resultant internal loadings acting on the cross section at E.



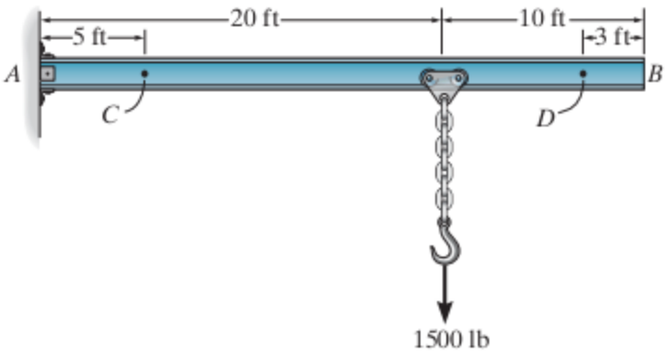
Prob. 7–1

7–2. Determine the resultant internal normal and shear force in the member at (a) section a–a and (b) section b–b, each of which passes through point A. The 500-lb load is applied along the centroidal axis of the member.



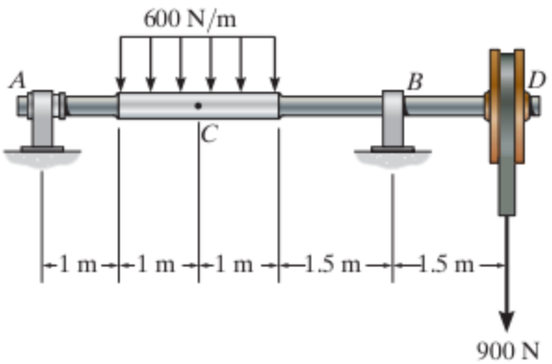
Prob. 7–2

7–3. The beam AB is fixed to the wall and has a uniform weight of 80 lb/ft. If the trolley supports a load of 1500 lb, determine the resultant internal loadings acting on the cross sections through points C and D.



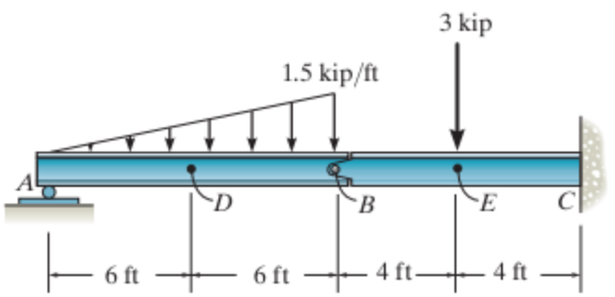
Prob. 7–3

*7–4. The shaft is supported by a smooth thrust bearing at A and a smooth journal bearing at B. Determine the resultant internal loadings acting on the cross section at C.



Prob. 7–4

7–5. Determine the resultant internal loadings in the beam at cross sections through points D and E. Point E is just to the right of the 3-kip load.

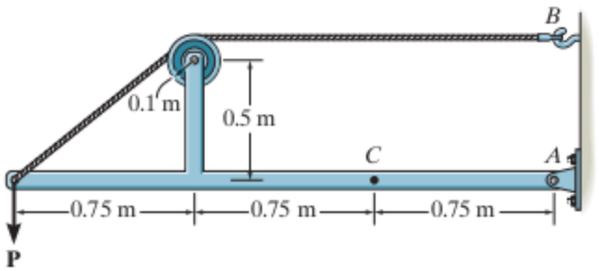


Prob. 7–5

7

7–6. Determine the normal force, shear force, and moment at a section through point C. Take $P = 8$ kN.

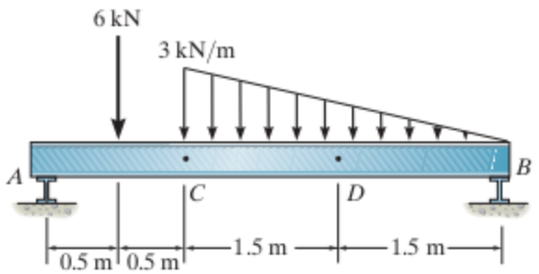
7–7. The cable will fail when subjected to a tension of 2 kN. Determine the largest vertical load P the frame will support and calculate the internal normal force, shear force, and moment at the cross section through point C for this loading.



Probs. 7–6/7

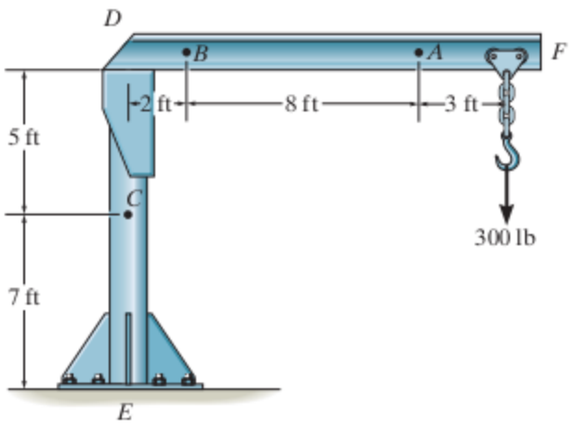
***7–8.** Determine the resultant internal loadings on the cross section through point C. Assume the reactions at the supports A and B are vertical.

7–9. Determine the resultant internal loadings on the cross section through point D. Assume the reactions at the supports A and B are vertical.



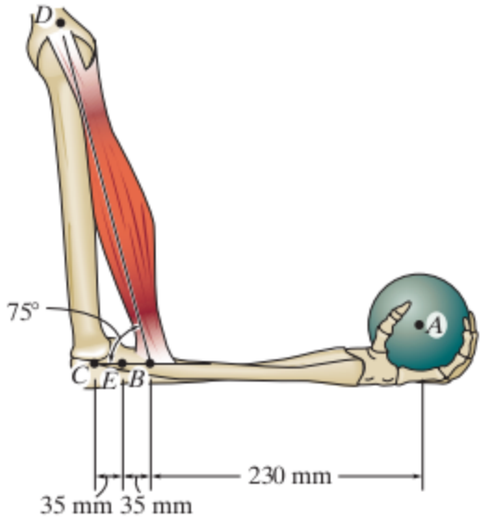
Probs. 7–8/9

7–10. The boom DF of the jib crane and the column DE have a uniform weight of 50 lb/ft. If the hoist and load weigh 300 lb, determine the resultant internal loadings in the crane on cross sections through points A, B, and C.



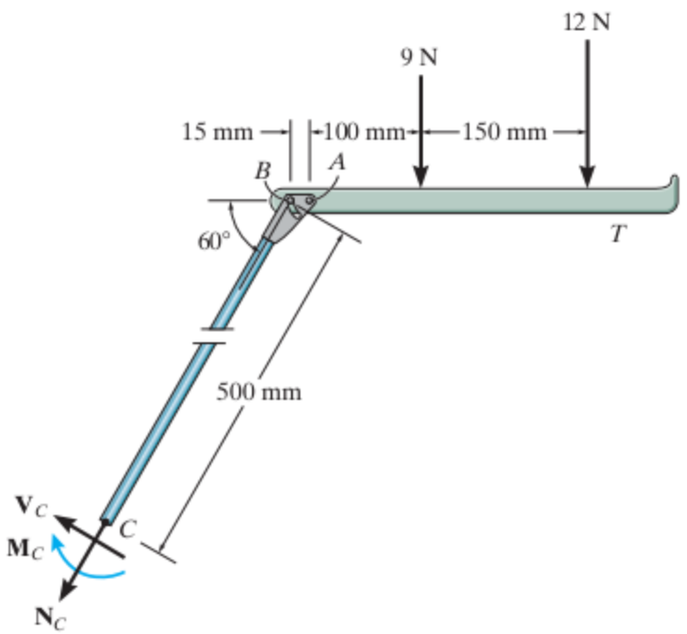
Prob. 7–10

7–11. The forearm and biceps support the 2-kg load at A. If C can be assumed as a pin support, determine the resultant internal loadings acting on the cross section of the bone of the forearm at E. The biceps pulls on the bone along BD.



Prob. 7–11

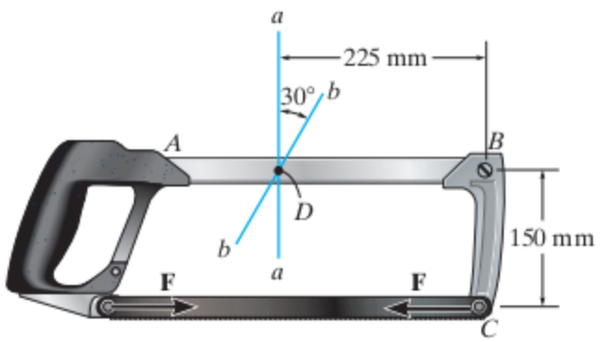
***7–12.** The serving tray T used on an airplane is supported on *each side* by an arm. The tray is pin connected to the arm at A, and at B there is a smooth pin. (The pin can move within the slot in the arms to permit folding the tray against the front passenger seat when not in use.) Determine the resultant internal loadings acting on the cross section of the arm through point C when the tray arm supports the loads shown.



Prob. 7–12

7–13. The blade of the hacksaw is subjected to a pretension force of $F = 100$ N. Determine the resultant internal loadings acting on section a–a that passes through point D.

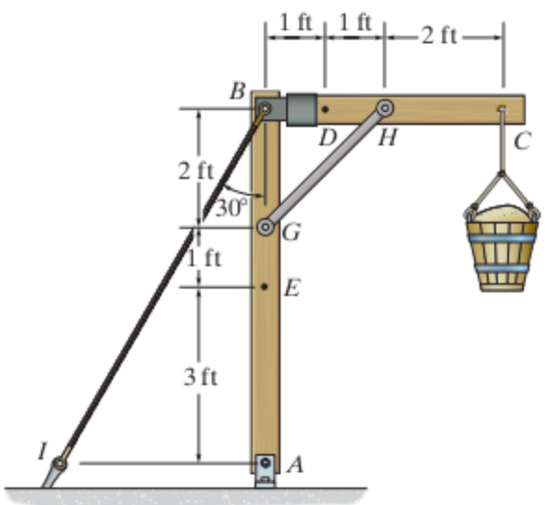
7–14. The blade of the hacksaw is subjected to a pretension force of $F = 100$ N. Determine the resultant internal loadings acting on section b–b that passes through point D.



Probs. 7–13/14

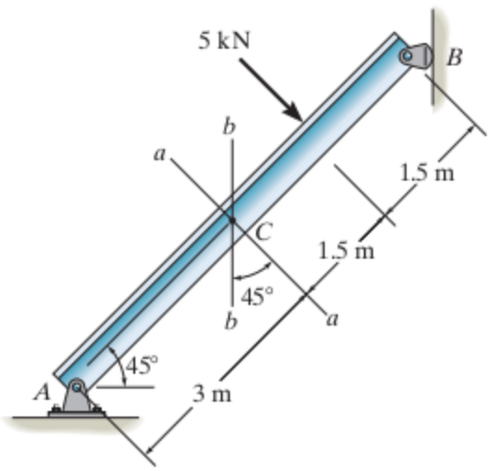
7–15. A 150-lb bucket is suspended from a cable on the wooden frame. Determine the resultant internal loadings on the cross section at D.

***7–16.** A 150-lb bucket is suspended from a cable on the wooden frame. Determine the resultant internal loadings acting on the cross section at E.



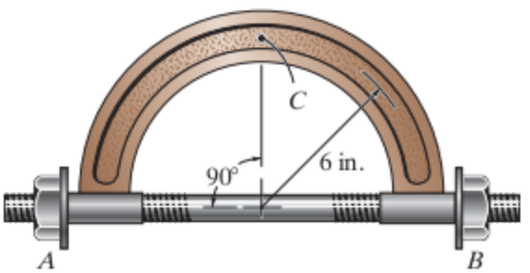
Probs. 7–15/16

7–17. Determine resultant internal loadings acting on section a–a and section b–b. Each section passes through the centerline at point C.



Prob. 7–17

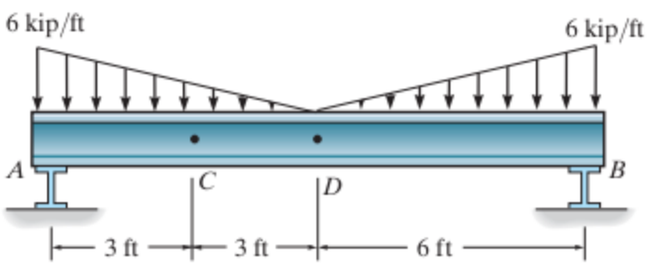
7–18. The bolt shank is subjected to a tension of 80 lb. Determine the resultant internal loadings acting on the cross section at point C.



Prob. 7–18

7–19. Determine the resultant internal loadings acting on the cross section through point C. Assume the reactions at the supports A and B are vertical.

***7–20.** Determine the resultant internal loadings acting on the cross section through point D. Assume the reactions at the supports A and B are vertical.



Probs. 7–19/20

7

7.3 Stress

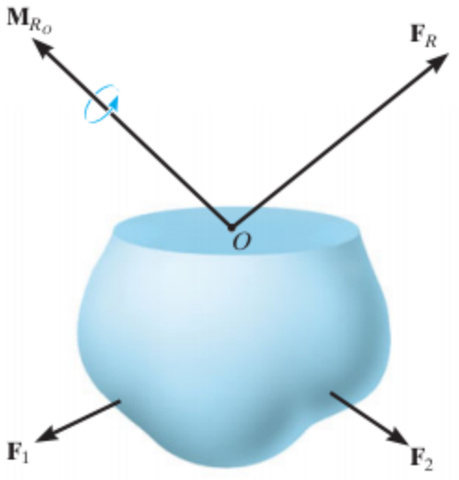


Fig. 7–7

It was stated in Section 7.2 that the force and moment acting at a specified point O on the sectioned area of the body, Fig. 7–7, represents the resultant effects of the actual *distribution of loading* acting over the sectioned area, Fig. 7–8*a*. Obtaining this *distribution* is of primary importance in mechanics of materials. To solve this problem it is necessary to establish the concept of stress.

We begin by considering the sectioned area to be subdivided into small areas, such as ΔA shown in Fig. 7–8*a*. As we reduce ΔA to a smaller and smaller size, we must make two assumptions regarding the properties of the material. We will consider the material to be ***continuous***, that is, to consist of a *continuum* or uniform distribution of matter having no voids. Also, the material must be ***cohesive***, meaning that all portions of it are connected together, without having breaks, cracks, or separations. A typical finite yet very small force $\Delta\mathbf{F}$, acting on ΔA, is shown in Fig. 7–8*a*. This force, like all the others, will have a unique direction, but for further discussion we will replace it by its *three components*, namely, $\Delta\mathbf{F}_x$, $\Delta\mathbf{F}_y$, and $\Delta\mathbf{F}_z$, which are taken tangent, tangent, and normal to the area, respectively. As ΔA approaches zero, so do $\Delta\mathbf{F}$ and its components; however, the quotient of the force and area will, in general, approach a finite limit. This quotient is called *stress*, and as noted, it describes the *intensity of the internal force* acting on a *specific plane* (area) passing through a point.

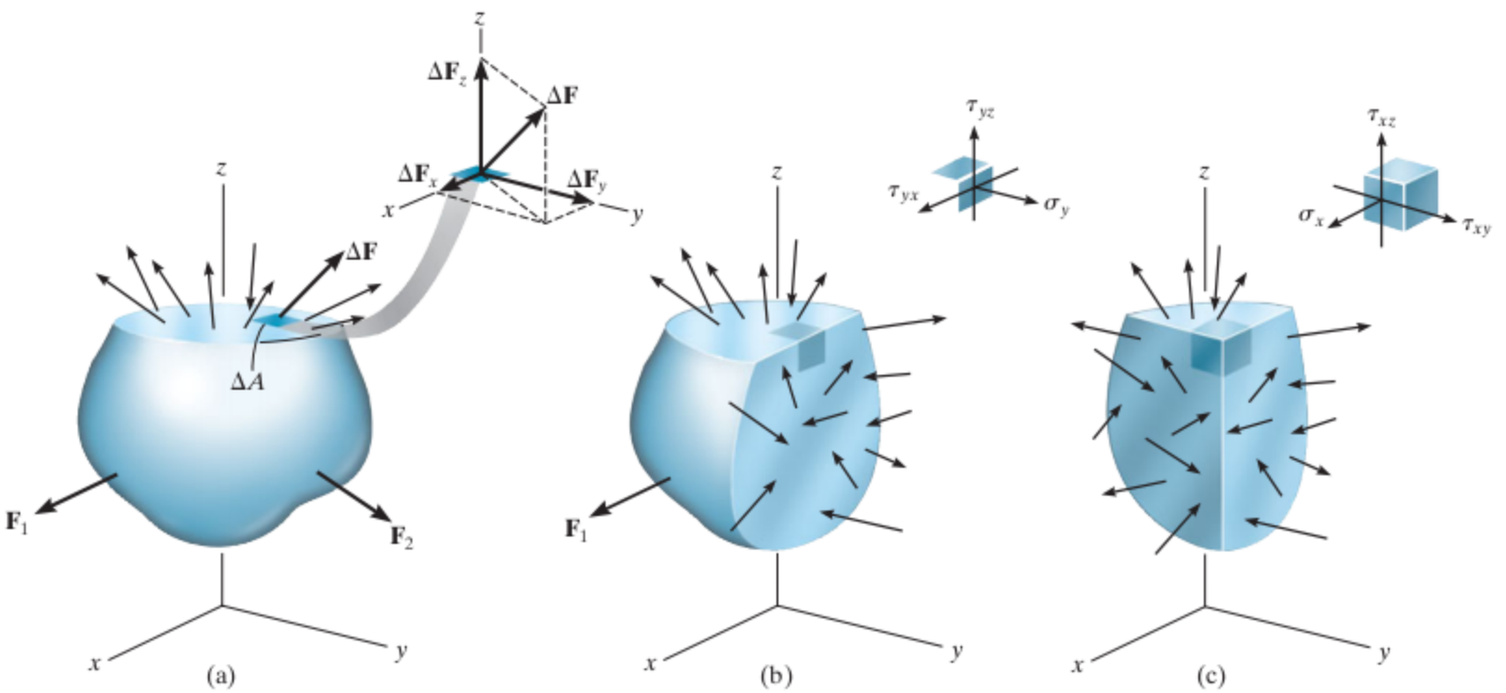


Fig. 7–8

Normal Stress. The *intensity* of the force acting normal to ΔA is defined as the ***normal stress***, σ (sigma). Since $\Delta \mathbf{F}_z$ is normal to the area then

$$\sigma_z = \lim_{\Delta A \to 0} \frac{\Delta F_z}{\Delta A} \tag{7–1}$$

If the normal force or stress "pulls" on ΔA as shown in Fig. 7–8*a*, it is referred to as *tensile stress*, whereas if it "pushes" on ΔA it is called *compressive stress*.

Shear Stress. The intensity of force acting tangent to ΔA is called the ***shear stress***, τ (tau). Here we have shear stress components,

$$\tau_{zx} = \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A}$$
$$\tau_{zy} = \lim_{\Delta A \to 0} \frac{\Delta F_y}{\Delta A} \tag{7–2}$$

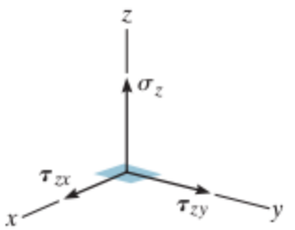


Fig. 7–9

Note that in this subscript notation z specifies the orientation of the area ΔA, Fig. 7–9, and x and y indicate the axes along which each shear stress acts.

General State of Stress. If the body is further sectioned by planes parallel to the x–z plane, Fig. 7–8*b*, and the y–z plane, Fig. 7–8*c*, we can then "cut out" a cubic volume element of material that represents the ***state of stress*** acting around the chosen point in the body. This state of stress is then characterized by three components acting on each face of the element, Fig. 7–10.

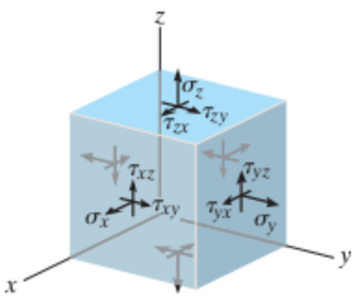
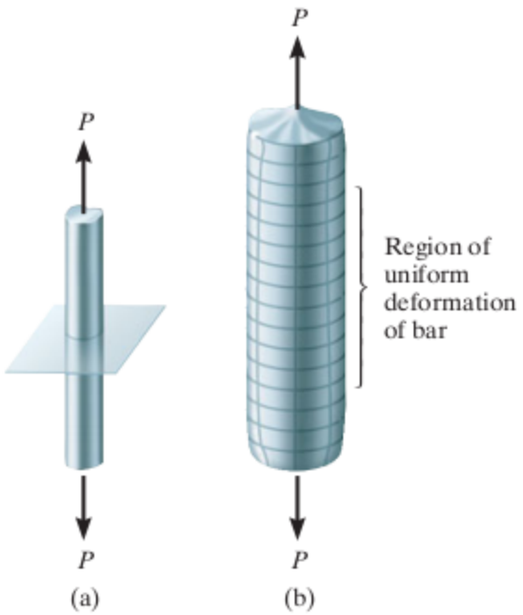


Fig. 7–10

Units. Since stress represents a force per unit area, in the International Standard or SI system, the magnitudes of both normal and shear stress are specified in the basic units of newtons per square meter (N/m^2). This unit, called a pascal ($1\ Pa = 1\ N/m^2$) is rather small, and in engineering work prefixes such as kilo- (10^3), symbolized by k, mega- (10^6), symbolized by M, or giga- (10^9), symbolized by G, are used to represent larger, more realistic values of stress.* Likewise, in the Foot-Pound-Second system of units, engineers usually express stress in pounds per square inch (psi) or kilopounds per square inch (ksi), where 1 kilopound (kip) = 1000 lb.

*Sometimes stress is expressed in units of N/mm^2, where $1\ mm = 10^{-3}\ m$. However, in the SI system. Prefixes are not allowed in the denominator of a fraction and therefore it is better to use the equivalent $1\ N/mm^2 = 1\ MN/m^2 = 1\ MPa$.

7.4 Average Normal Stress in an Axially Loaded Bar



In this section we will determine the average stress distribution acting on the cross-sectional area of an axially loaded bar such as the one shown in Fig. 7–11*a*. This bar is **prismatic** since all cross sections are the same throughout its length. When the load P is applied to the bar through the centroid of its cross-sectional area, then the bar will deform uniformly throughout the central region of its length, as shown in Fig. 7–11*b*, provided the material of the bar is both homogeneous and isotropic.

Homogeneous material has the same physical and mechanical properties throughout its volume, and ***isotropic material*** has these same properties in all directions. Many engineering materials may be approximated as being both homogeneous and isotropic as assumed here. Steel, for example, contains thousands of randomly oriented crystals in each cubic millimeter of its volume, and since most problems involving this material have a physical size that is very much larger than a single crystal, the above assumption regarding its material composition is quite realistic.

Note that anisotropic materials such as wood have different properties in different directions, and although this is the case, like wood if the anisotropy is oriented along the bar's axis, then the bar will also deform uniformly when subjected to the axial load P.

Average Normal Stress Distribution. If we pass a section through the bar, and separate it into two parts, then equilibrium requires the resultant normal force at the section to be P, Fig. 7–11*c*. Due to the *uniform* deformation of the material, it is necessary that the cross section be subjected to a *constant normal stress distribution*, Fig. 7–11*d*.

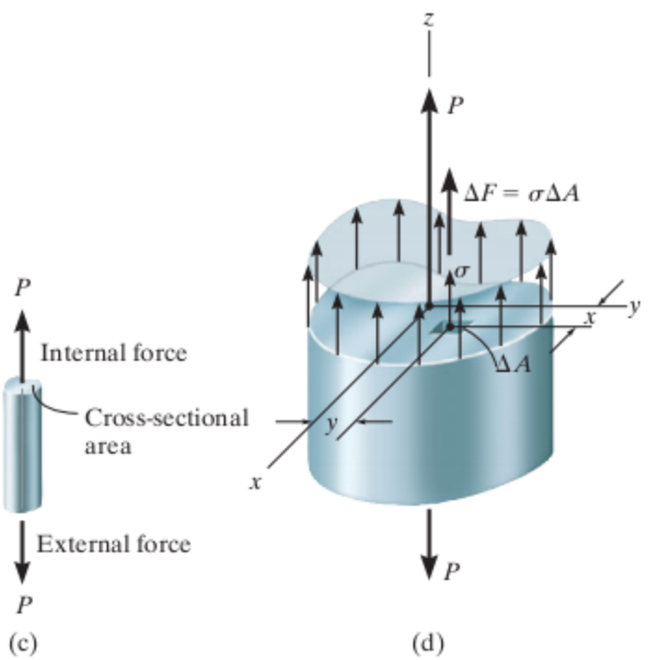


Fig. 7–11

As a result, each small area ΔA on the cross section is subjected to a force $\Delta F = \sigma\,\Delta A$, and the *sum* of these forces acting over the entire cross-sectional area must be equivalent to the internal resultant force **P** at the section. If we let $\Delta A \rightarrow dA$ and therefore $\Delta F \rightarrow dF$, then, recognizing σ is *constant*, we have

$$+\uparrow F_{Rz} = \Sigma F_z; \qquad \int dF = \int_A \sigma\, dA$$

$$P = \sigma A$$

$$\boxed{\sigma = \frac{P}{A}} \tag{7–3}$$

Here

σ = average normal stress at any point on the cross-sectional area

P = *internal resultant normal force*, which acts through the *centroid* of the cross-sectional area. P is determined using the method of sections and the equations of equilibrium

A = cross-sectional area of the bar where σ is determined

Since the internal load P passes through the centroid of the cross section, the uniform stress distribution will produce zero moments about the x and y axes passing through this point, Fig. 7–11*d*. To show this, we require the moment of P about each axis to be equal to the moment of the stress distribution about the axes, namely,

$$(M_R)_x = \Sigma M_x; \qquad 0 = \int_A y\, dF = \int_A y\sigma\, dA = \sigma \int_A y\, dA$$

$$(M_R)_y = \Sigma M_y; \qquad 0 = -\int_A x\, dF = -\int_A x\sigma\, dA = -\sigma \int_A x\, dA$$

These equations are indeed satisfied, since by definition of the centroid, $\int y\, dA = 0$ and $\int x\, dA = 0$.

Equilibrium. It should be apparent that only a normal stress exists on any small volume element of material located at each point on the cross section of an axially loaded bar. If we consider vertical equilibrium of the element, Fig. 7–12, then apply the equation of force equilibrium,

$$\Sigma F_z = 0; \qquad \sigma(\Delta A) - \sigma'(\Delta A) = 0$$

$$\sigma = \sigma'$$

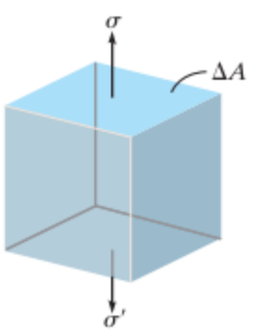


Fig. 7–12

7

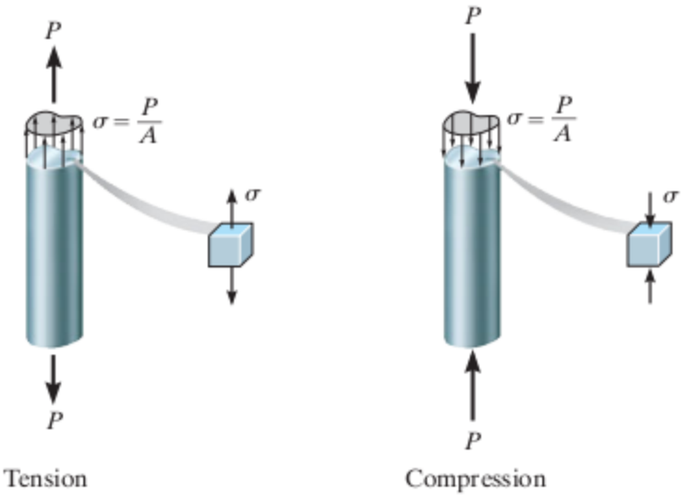


Fig. 7–13

In other words, the two normal stress components on the element must be equal in magnitude but opposite in direction. This is referred to as *uniaxial stress*.

The previous analysis applies to members subjected to either tension or compression, as shown in Fig. 7–13. As a graphical interpretation, the ***magnitude*** of the internal resultant force **P** is ***equivalent*** to the ***volume*** under the stress diagram; that is, $P = \sigma A$ (volume = height × base). Furthermore, as a consequence of the balance of moments, ***this resultant passes through the centroid of this volume***.

Although we have developed this analysis for *prismatic* bars, this assumption can be relaxed somewhat to include bars that have a *slight taper*. For example, it can be shown, using the more exact analysis of the theory of elasticity, that for a tapered bar of rectangular cross section, for which the angle between two adjacent sides is 15°, the average normal stress, as calculated by $\sigma = P/A$, is only 2.2% *less* than its value found from the theory of elasticity.

Maximum Average Normal Stress. In our analysis both the internal force P and the cross-sectional area A were *constant* along the longitudinal axis of the bar, and as a result the normal stress $\sigma = P/A$ is also *constant* throughout the bar's length. Occasionally, however, the bar may be subjected to *several* external loads along its axis, or a change in its cross-sectional area may occur. As a result, the normal stress within the bar could be different from one section to the next, and, if the *maximum* average normal stress is to be determined, then it becomes important to find the location where the ratio P/A is a *maximum*. To do this it is necessary to determine the internal force P at various sections along the bar. Here it may be helpful to show this variation by drawing an ***axial*** or ***normal force diagram***. Specifically, this diagram is a plot of the normal force P versus its position x along the bar's length. As a sign convention, P will be positive if it causes tension in the member, and negative if it causes compression. Once the internal loading throughout the bar is known, the maximum ratio of P/A can then be identified.

This steel tie rod is used as a hanger to suspend a portion of a staircase, and as a result it is subjected to tensile stress.

Important Points

- When a body subjected to external loads is sectioned, there is a distribution of force acting over the sectioned area which holds each segment of the body in equilibrium. The intensity of this internal force at a point in the body is referred to as *stress*.
- Stress is the limiting value of force per unit area, as the area approaches zero. For this definition, the material is considered to be continuous and cohesive.
- The magnitude of the stress components at a point depends upon the type of loading acting on the body, and the orientation of the element at the point.
- When a prismatic bar is made from homogeneous and isotropic material, and is subjected to an axial force acting through the centroid of the cross-sectional area, then the center region of the bar will deform uniformly. As a result, the material will be subjected *only to normal stress*. This stress is uniform or *averaged* over the cross-sectional area.

Procedure for Analysis

The equation $\sigma = P/A$ gives the *average* normal stress on the cross-sectional area of a member when the section is subjected to an internal resultant normal force **P**. For axially loaded members, application of this equation requires the following steps.

Internal Loading.

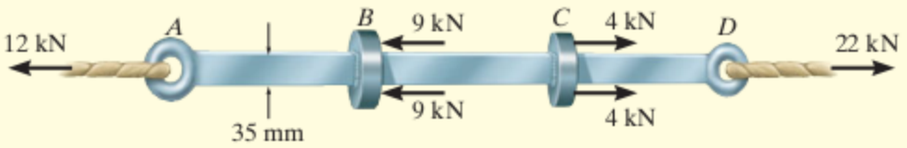
- Section the member *perpendicular* to its longitudinal axis at the point where the normal stress is to be determined and use the necessary free-body diagram and force equation of equilibrium to obtain the *internal axial force* **P** at the section.

Average Normal Stress.

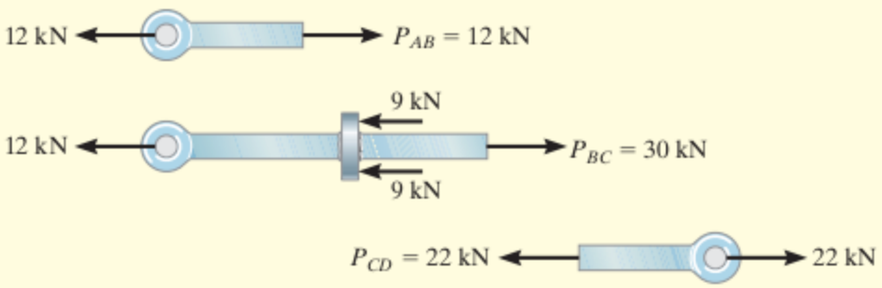
- Determine the member's cross-sectional area at the section and calculate the average normal stress $\sigma = P/A$.
- It is suggested that σ be shown acting on a small volume element of the material located at a point on the section where stress is calculated. To do this, first draw σ on the face of the element coincident with the sectioned area A. Here σ acts in the *same direction* as the internal force **P** since all the normal stresses on the cross section develop this resultant. The normal stress σ on the other face of the element acts in the opposite direction.

EXAMPLE 7.5

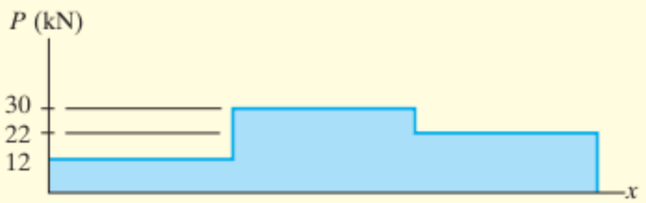
The bar in Fig. 7–14*a* has a constant width of 35 mm and a thickness of 10 mm. Determine the maximum average normal stress in the bar when it is subjected to the loading shown.



(a)



(b)



(c)

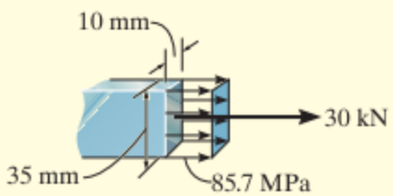
SOLUTION

Internal Loading. By inspection, the internal axial forces in regions *AB*, *BC*, and *CD* are all constant yet have different magnitudes. Using the method of sections, these loadings are determined in Fig. 7–14*b*; and the normal force diagram which represents these results graphically is shown in Fig. 7–14*c*. The largest loading is in region *BC*, where $P_{BC} = 30$ kN. Since the cross-sectional area of the bar is *constant*, the largest average normal stress also occurs within this region of the bar.

Average Normal Stress. Applying Eq. 7–3, we have

$$\sigma_{BC} = \frac{P_{BC}}{A} = \frac{30(10^3)\text{ N}}{(0.035\text{ m})(0.010\text{ m})} = 85.7\text{ MPa} \qquad \textit{Ans.}$$

NOTE: The stress distribution acting on an arbitrary cross section of the bar within region *BC* is shown in Fig. 7–14*d*. Graphically the *volume* (or "block") represented by this distribution of stress is equivalent to the load of 30 kN; that is, 30 kN = (85.7 MPa)(35 mm)(10 mm).

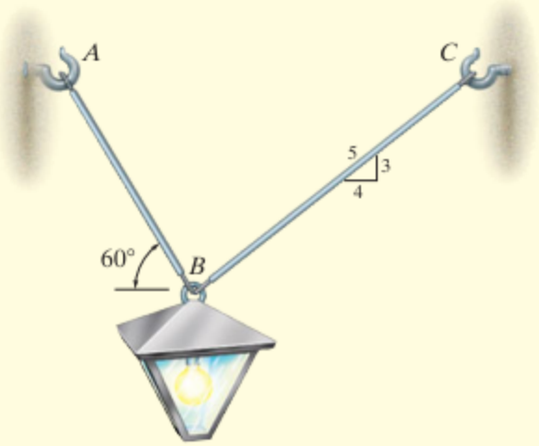


(d)

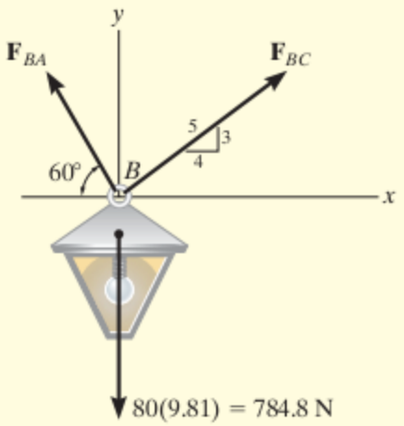
Fig. 7–14

EXAMPLE 7.6

The 80-kg lamp is supported by two rods AB and BC as shown in Fig. 7–15*a*. If AB has a diameter of 10 mm and BC has a diameter of 8 mm, determine the average normal stress in each rod.



(a)



(b)

Fig. 7–15

SOLUTION

Internal Loading. We must first determine the axial force in each rod. A free-body diagram of the lamp is shown in Fig. 7–15*b*. Applying the equations of force equilibrium,

$$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \qquad F_{BC}\left(\tfrac{4}{5}\right) - F_{BA}\cos 60° = 0$$

$$+\uparrow \Sigma F_y = 0; \qquad F_{BC}\left(\tfrac{3}{5}\right) + F_{BA}\sin 60° - 784.8\ \text{N} = 0$$

$$F_{BC} = 395.2\ \text{N}, \qquad F_{BA} = 632.4\ \text{N}$$

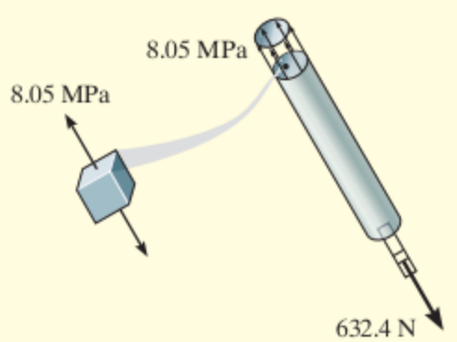
By Newton's third law of action, equal but opposite reaction, these forces subject the rods to tension throughout their length.

Average Normal Stress. Applying Eq. 7–3,

$$\sigma_{BC} = \frac{F_{BC}}{A_{BC}} = \frac{395.2\ \text{N}}{\pi(0.004\ \text{m})^2} = 7.86\ \text{MPa} \qquad \textit{Ans.}$$

$$\sigma_{BA} = \frac{F_{BA}}{A_{BA}} = \frac{632.4\ \text{N}}{\pi(0.005\ \text{m})^2} = 8.05\ \text{MPa} \qquad \textit{Ans.}$$

NOTE: The average normal stress distribution acting over a cross section of rod AB is shown in Fig. 7–15*c*, and at a point on this cross section, an element of material is stressed as shown in Fig. 7–15*d*.



(d) (c)

7

EXAMPLE 7.7

Member AC shown in Fig. 7–16*a* is subjected to a vertical force of 3 kN. Determine the position x of this force so that the average compressive stress at the smooth support C is equal to the average tensile stress in the tie rod AB. The rod has a cross-sectional area of 400 mm^2 and the contact area at C is 650 mm^2.

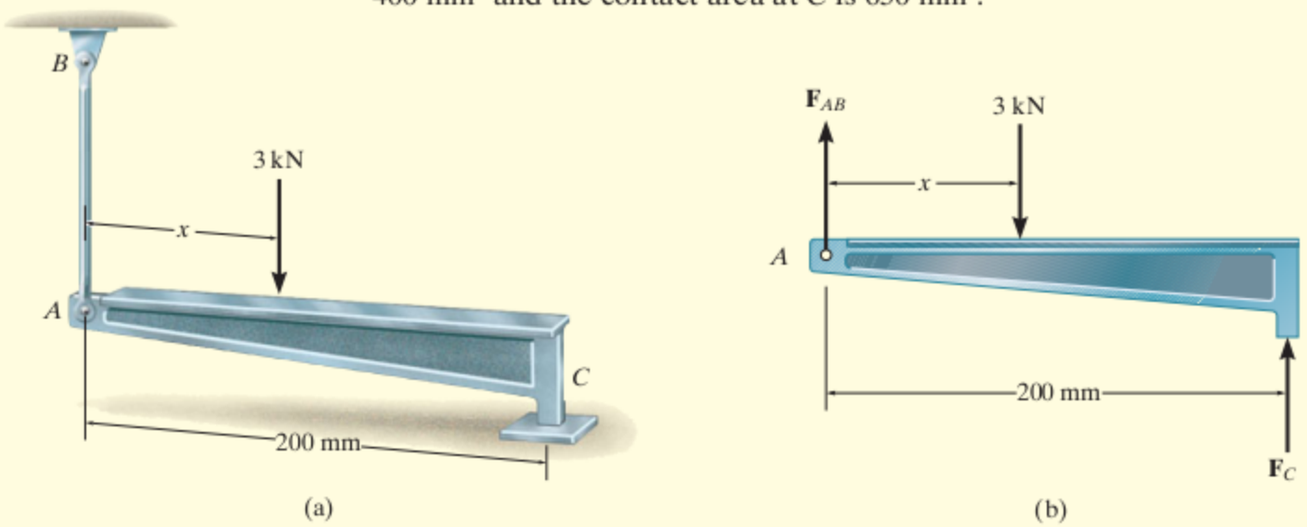


Fig. 7–16

SOLUTION

Internal Loading. The forces at A and C can be related by considering the free-body diagram for member AC, Fig. 7–16*b*. There are three unknowns, namely, F_{AB}, F_C, and x. To solve this problem we will work in units of newtons and millimeters.

$$+\uparrow \Sigma F_y = 0; \qquad F_{AB} + F_C - 3000\ \text{N} = 0 \tag{1}$$

$$\circlearrowleft +\Sigma M_A = 0; \qquad -3000\ \text{N}(x) + F_C(200\ \text{mm}) = 0 \tag{2}$$

Average Normal Stress. A necessary third equation can be written that requires the tensile stress in the bar AB and the compressive stress at C to be equivalent, i.e.,

$$\sigma = \frac{F_{AB}}{400\ \text{mm}^2} = \frac{F_C}{650\ \text{mm}^2}$$

$$F_C = 1.625F_{AB}$$

Substituting this into Eq. (1), solving for F_{AB}, then solving for F_C, we obtain

$$F_{AB} = 1143\ \text{N}$$

$$F_C = 1857\ \text{N}$$

The position of the applied load is determined from Eq. (2),

$$x = 124\ \text{mm} \qquad \textit{Ans.}$$

NOTE: $0 < x < 200$ mm, as required.

7.5 Average Shear Stress

Shear stress has been defined in Section 7.3 as the stress component that acts *in the plane* of the sectioned area. To show how this stress can develop, consider the effect of applying a force **F** to the bar in Fig. 7–17*a*. If the supports are considered rigid, and **F** is large enough, it will cause the material of the bar to deform and fail along the planes identified by *AB* and *CD*. A free-body diagram of the unsupported center segment of the bar, Fig. 7–17*b*, indicates that the shear force $V = F/2$ must be applied at each section to hold the segment in equilibrium. The ***average shear stress*** distributed over each sectioned area that develops this shear force is defined by

$$\tau_{avg} = \frac{V}{A} \tag{7–4}$$

where

τ_{avg} = average shear stress at the section, which is assumed to be the *same* at each point located on the section
V = internal resultant shear force on the section determined from the equations of equilibrium
A = area at the section

The distribution of average shear stress acting over the sections is shown in Fig. 7–17*c*. Notice that τ_{avg} is in the *same direction* as **V**, since the shear stress must create associated forces, all of which contribute to the internal resultant force **V** at the section.

The loading case discussed here is an example of ***simple*** or ***direct shear***, since the shear is caused by the *direct action* of the applied load **F**. This type of shear often occurs in various types of simple connections that use bolts, pins, welding material, etc. In all these cases, however, application of Eq. 7–4 is *only approximate*. A more precise investigation of the shear-stress distribution over the section often reveals that much larger shear stresses occur in the material than those predicted by this equation. Although this may be the case, application of Eq. 7–4 is generally acceptable for many problems in engineering design and analysis. For example, engineering codes allow its use when considering design sizes for fasteners such as bolts and for obtaining the bonding strength of glued joints subjected to shear loadings.

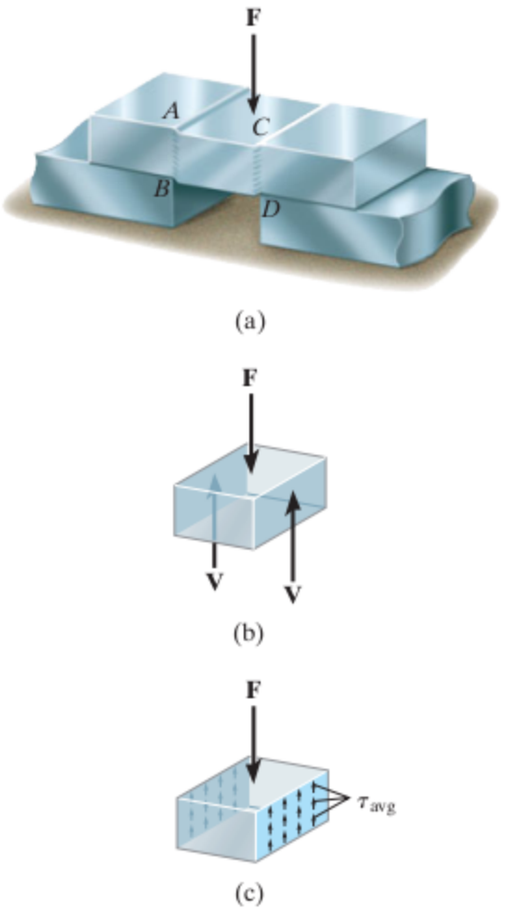


Fig. 7–17

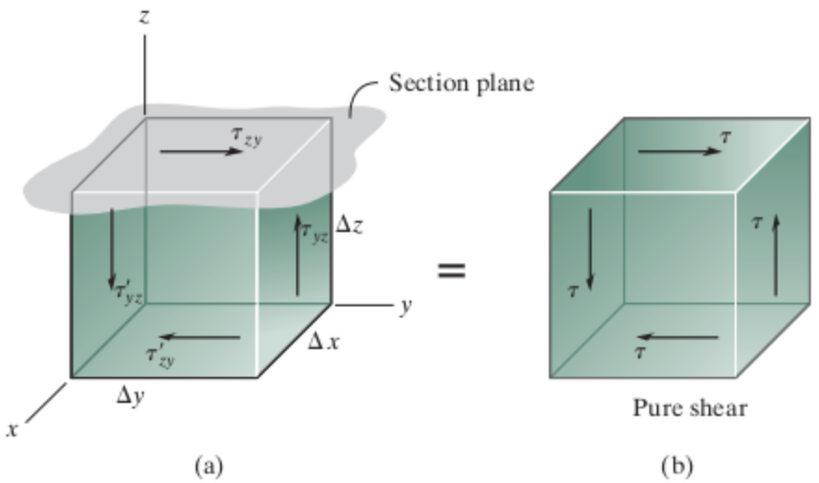


Fig. 7–18

Shear Stress Equilibrium. Figure 7–18*a* shows a volume element of material taken at a point located on the surface of a sectioned area which is subjected to a shear stress τ_{zy}. Force and moment equilibrium requires the shear stress acting on this face of the element to be accompanied by shear stress acting on three other faces. To show this we will first consider force equilibrium in the y direction. Then

$$\Sigma F_y = 0; \qquad \underbrace{\tau_{zy}}_{\text{force}}\underbrace{(\Delta x\ \Delta y)}_{\text{stress area}} - \underbrace{\tau'_{zy}}_{\text{force}}\underbrace{\Delta x\ \Delta y}_{\text{stress area}} = 0$$

$$\tau_{zy} = \tau'_{zy}$$

In a similar manner, force equilibrium in the z direction yields $\tau_{yz} = \tau'_{yz}$. Finally, taking moments about the x axis,

$$\Sigma M_x = 0; \qquad \underbrace{-\tau_{zy}}_{\text{moment}}\underbrace{(\Delta x\ \Delta y)}_{\text{force}}\underbrace{\Delta z}_{\text{arm}} + \underbrace{\tau_{yz}}_{\text{moment}}\underbrace{(\Delta x\ \Delta z)}_{\text{force}}\underbrace{\Delta y}_{\text{arm}} = 0$$

$$\tau_{zy} = \tau_{yz}$$

so that

$$\tau_{zy} = \tau'_{zy} = \tau_{yz} = \tau'_{yz} = \tau$$

In other words, ***all four shear stresses must have equal magnitude and be directed either toward or away from each other at opposite edges of the element***, Fig. 7–18*b*. This is referred to as the *complementary property of shear*, and under the conditions shown in Fig. 7–18, the material is subjected to *pure shear*.

Important Points

- If two parts are *thin or small* when joined together, the applied loads may cause shearing of the material with negligible bending. If this is the case, it is generally assumed that an *average shear stress* acts over the cross-sectional area.
- When shear stress τ acts on a plane, then equilibrium of a volume element of material at a point on the plane requires associated shear stress of the same magnitude act on three adjacent sides of the element.

Procedure for Analysis

The equation $\tau_{avg} = V/A$ is used to determine the *average shear stress* in the material. Application requires the following steps.

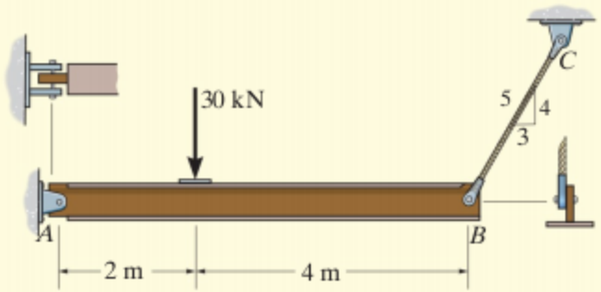
Internal Shear.

- Section the member at the point where the average shear stress is to be determined.
- Draw the necessary free-body diagram, and calculate the internal shear force **V** acting at the section that is necessary to hold the part in equilibrium.

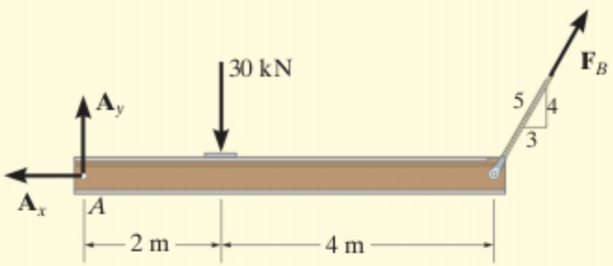
Average Shear Stress.

- Determine the sectioned area A, and determine the average shear stress $\tau_{avg} = V/A$.
- It is suggested that τ_{avg} be shown on a small volume element of material located at a point on the section where it is determined. To do this, first draw τ_{avg} on the face of the element, coincident with the sectioned area A. This stress acts in the *same direction* as **V**. The shear stresses acting on the three adjacent planes can then be drawn in their appropriate directions following the scheme shown in Fig. 7–18.

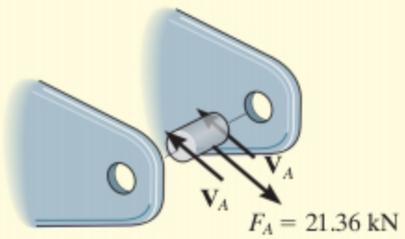
EXAMPLE 7.8



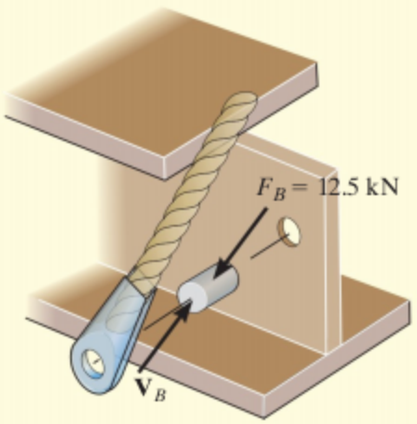
(a)



(b)



(c)



(d)

Fig. 7–19

Determine the average shear stress in the 20-mm-diameter pin at A and the 30-mm-diameter pin at B that support the beam in Fig. 7–19*a*.

SOLUTION

Internal Loadings. The forces on the pins can be obtained by considering the equilibrium of the beam, Fig. 7–19*b*.

$$\circlearrowleft + \Sigma M_A = 0; \quad F_B\left(\frac{4}{5}\right)(6\text{ m}) - 30\text{ kN}(2\text{ m}) = 0 \qquad F_B = 12.5\text{ kN}$$

$$\pm \Sigma F_x = 0; \quad (12.5\text{ kN})\left(\frac{3}{5}\right) - A_x = 0 \qquad A_x = 7.50\text{ kN}$$

$$+\uparrow \Sigma F_y = 0; \quad A_y + (12.5\text{ kN})\left(\frac{4}{5}\right) - 30\text{ kN} = 0$$

$$A_y = 20\text{ kN}$$

Thus, the resultant force acting on pin A is

$$F_A = \sqrt{A_x^2 + A_y^2} = \sqrt{(7.50\text{ kN})^2 + (20\text{ kN})^2} = 21.36\text{ kN}$$

The pin at A is supported by two fixed "leaves" and so the free-body diagram of the center segment of the pin shown in Fig. 7–19*c* has two shearing surfaces between the beam and each leaf. The force of the beam (21.36 kN) acting on the pin is therefore supported by shear force on each of these surfaces. This case is called *double shear*. Thus,

$$V_A = \frac{F_A}{2} = \frac{21.36\text{ kN}}{2} = 10.68\text{ kN}$$

In Fig. 7–19*a*, note that pin B is subjected to *single shear*, which occurs on the section between the cable and beam, Fig. 7–19*d*. For this pin segment,

$$V_B = F_B = 12.5\text{ kN}$$

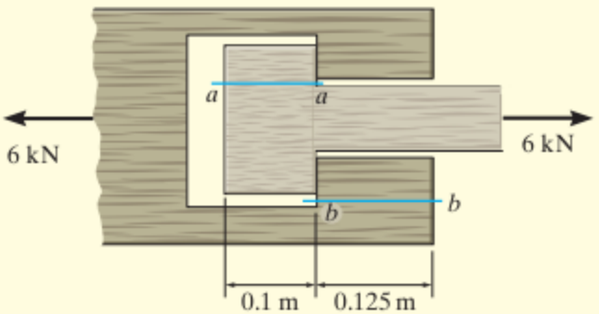
Average Shear Stress.

$$(\tau_A)_{\text{avg}} = \frac{V_A}{A_A} = \frac{10.68(10^3)\text{ N}}{\frac{\pi}{4}(0.02\text{ m})^2} = 34.0\text{ MPa} \qquad \textit{Ans.}$$

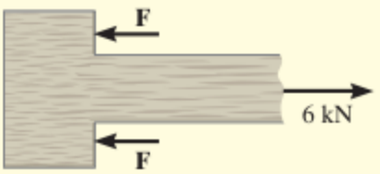
$$(\tau_B)_{\text{avg}} = \frac{V_B}{A_B} = \frac{12.5(10^3)\text{ N}}{\frac{\pi}{4}(0.03\text{ m})^2} = 17.7\text{ MPa} \qquad \textit{Ans.}$$

EXAMPLE 7.9

If the wood joint in Fig. 7–20*a* has a width of 150 mm, determine the average shear stress developed along shear planes *a*–*a* and *b*–*b*. For each plane, represent the state of stress on an element of the material.



(a)



(b)

Fig. 7–20

SOLUTION

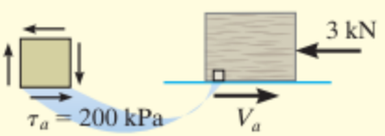
Internal Loadings. Referring to the free-body diagram of the member, Fig. 7–20*b*,

$$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \quad 6\ \text{kN} - F - F = 0 \quad F = 3\ \text{kN}$$

Now consider the equilibrium of segments cut across shear planes *a*–*a* and *b*–*b*, shown in Figs. 7–20*c* and 7–20*d*.

$$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \quad V_a - 3\ \text{kN} = 0 \quad V_a = 3\ \text{kN}$$

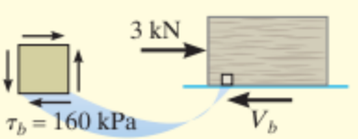
$$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \quad 3\ \text{kN} - V_b = 0 \quad V_b = 3\ \text{kN}$$



(c)

Average Shear Stress.

$$(\tau_a)_{\text{avg}} = \frac{V_a}{A_a} = \frac{3(10^3)\ \text{N}}{(0.1\ \text{m})(0.15\ \text{m})} = 200\ \text{kPa} \qquad \textit{Ans.}$$



(d)

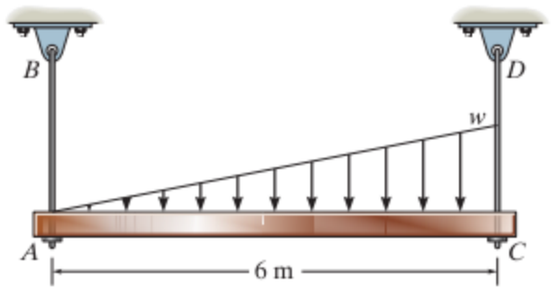
$$(\tau_b)_{\text{avg}} = \frac{V_b}{A_b} = \frac{3(10^3)\ \text{N}}{(0.125\ \text{m})(0.15\ \text{m})} = 160\ \text{kPa} \qquad \textit{Ans.}$$

The state of stress on elements located on sections *a*–*a* and *b*–*b* is shown in Figs. 7–20*c* and 7–20*d*, respectively.

7

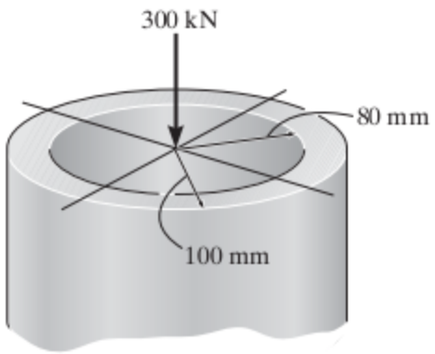
FUNDAMENTAL PROBLEMS

F7–7. The uniform beam is supported by two rods AB and CD that have cross-sectional areas of 10 mm^2 and 15 mm^2, respectively. Determine the intensity w of the distributed load so that the average normal stress in each rod does not exceed 300 kPa.



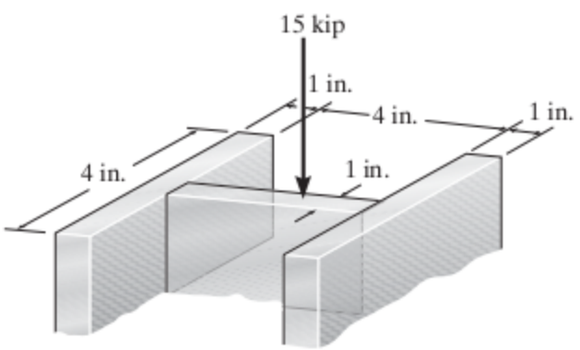
F7–7

F7–8. Determine the average normal stress developed on the cross section. Sketch the normal stress distribution over the cross section.



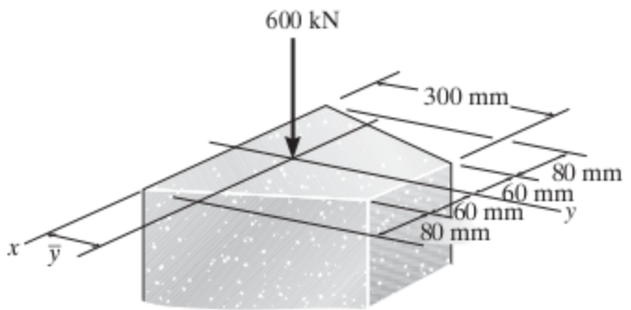
F7–8

F7–9. Determine the average normal stress developed on the cross section. Sketch the normal stress distribution over the cross section.



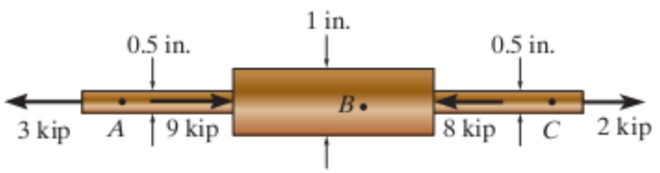
F7–9

F7–10. If the 600-kN force acts through the centroid of the cross section, determine the location $\bar{y}$ of the centroid and the average normal stress developed on the cross section. Also, sketch the normal stress distribution over the cross section.



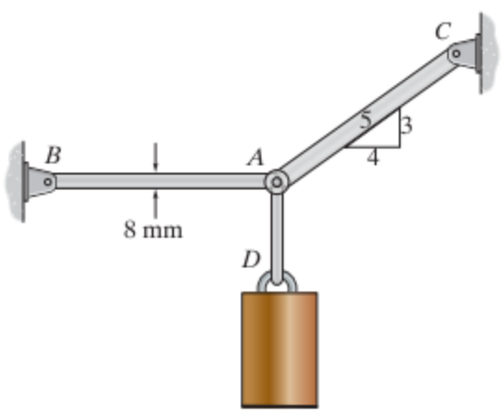
F7–10

F7–11. Determine the average normal stress developed at points A, B, and C. The diameter of each segment is indicated in the figure.



F7–11

F7–12. Determine the average normal stress developed in rod AB if the load has a mass of 50 kg. The diameter of rod AB is 8 mm.



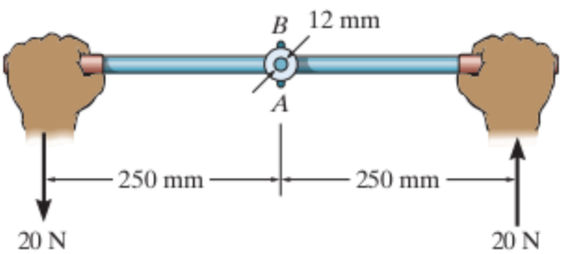
F7–12

PROBLEMS

7–21. The supporting wheel on a scaffold is held in place on the leg using a 4-mm-diameter pin as shown. If the wheel is subjected to a normal force of 3 kN, determine the average shear stress developed in the pin. Neglect friction between the inner scaffold puller leg and the tube used on the wheel.

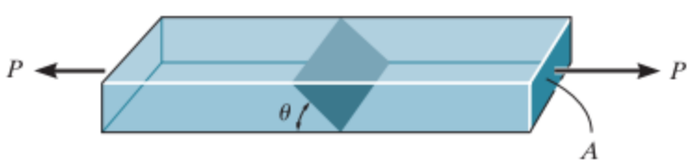
Prob. 7–21

7–22. The lever is held to the fixed shaft using a tapered pin AB, which has a mean diameter of 6 mm. If a couple is applied to the lever, determine the average shear stress in the pin between the pin and lever.



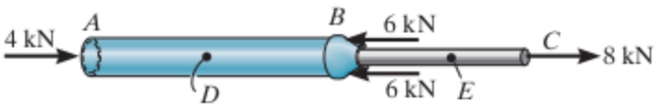
Prob. 7–22

7–23. The bar has a cross-sectional area A and is subjected to the axial load P. Determine the average normal and average shear stresses acting over the shaded section, which is oriented at θ from the horizontal. Plot the variation of these stresses as a function of θ ($0 \leq \theta \leq 90°$).



Prob. 7–23

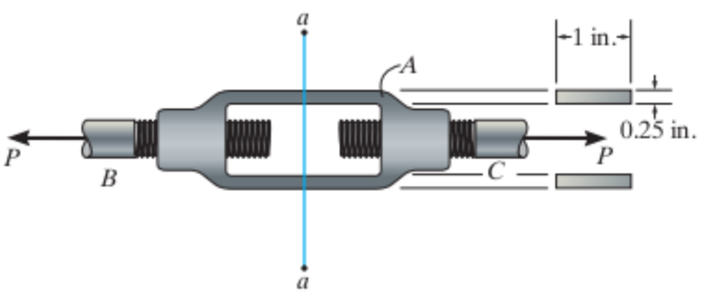
***7–24.** The built-up shaft consists of a pipe AB and solid rod BC. The pipe has an inner diameter of 20 mm and outer diameter of 28 mm. The rod has a diameter of 12 mm. Determine the average normal stress at points D and E and represent the stress on a volume element located at each of these points.



Prob. 7–24

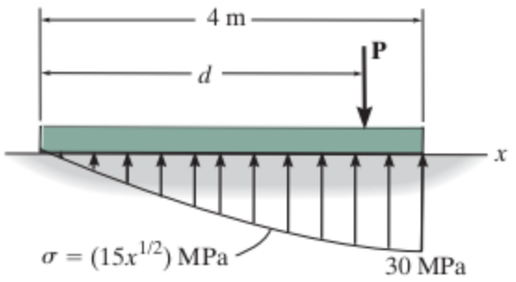
7–25. If the turnbuckle is subjected to an axial force of $P = 900$ lb, determine the average normal stress developed in section a–a and in each of the bolt shanks at B and C. Each bolt shank has a diameter of 0.5 in.

7–26. The average normal stresses developed in section a–a of the turnbuckle, and the bolts shanks at B and C, are not allowed to exceed 15 ksi and 45 ksi, respectively. Determine the maximum axial force **P** that can be applied to the turnbuckle. Each bolt shank has a diameter of 0.5 in.



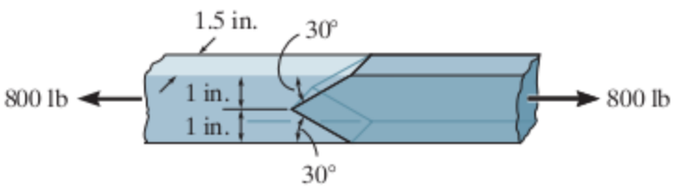
Probs. 7–25/26

7–27. The plate has a width of 0.5 m. If the stress distribution at the support varies as shown, determine the force **P** applied to the plate and the distance d to where it is applied.



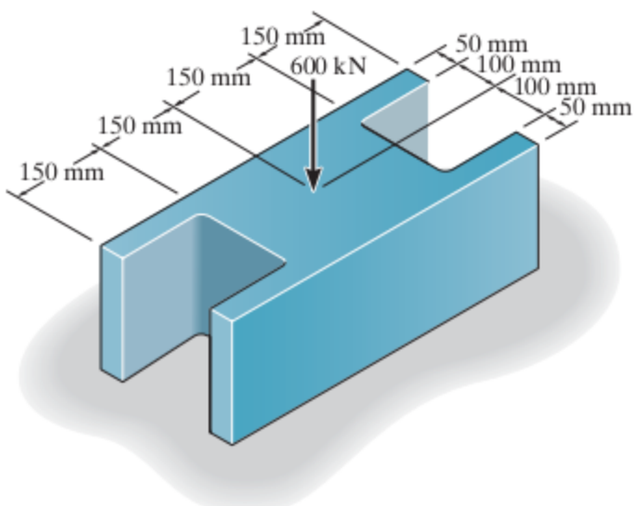
Prob. 7–27

*7–28. The two members used in the construction of an aircraft fuselage are joined together using a 30° fish-mouth weld. Determine the average normal and average shear stress on the plane of each weld. Assume each inclined plane supports a horizontal force of 400 lb.



Prob. 7–28

7–29. If the block is subjected to the centrally applied force of 600 kN, determine the average normal stress in the material. Show the stress acting on a differential volume element of the material.

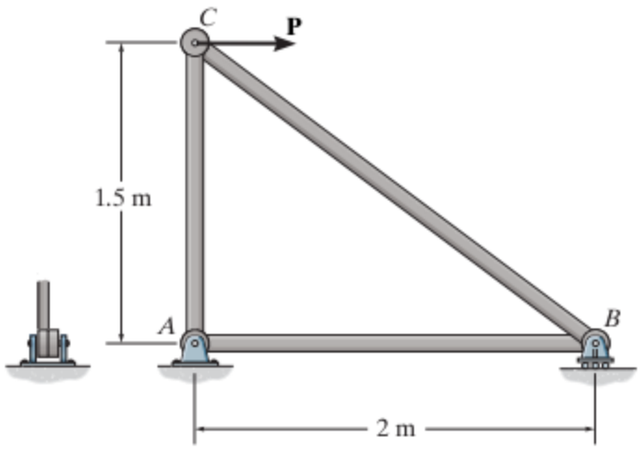


Prob. 7–29

7–30. Determine the average normal stress in each of the 20-mm diameter bars of the truss. Set $P = 40$ kN.

7–31. If the average normal stress in each of the 20-mm diameter bars is not allowed to exceed 150 MPa, determine the maximum force **P** that can be applied to joint C.

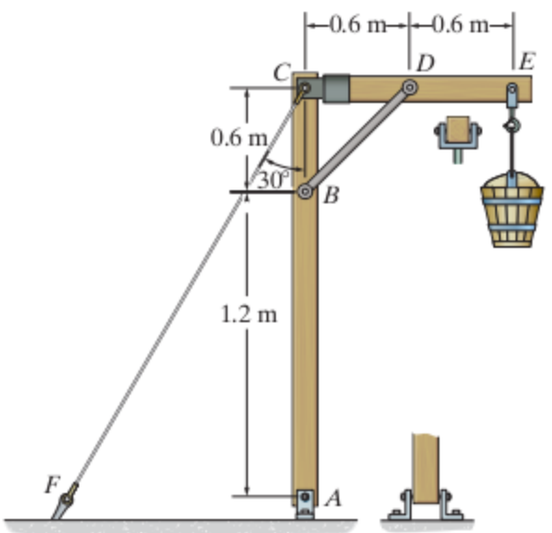
*7–32. Determine the average shear stress developed in pin A of the truss. A horizontal force of $P = 40$ kN is applied to joint C. Each pin has a diameter of 25 mm and is subjected to double shear.



Probs. 7–30/31/32

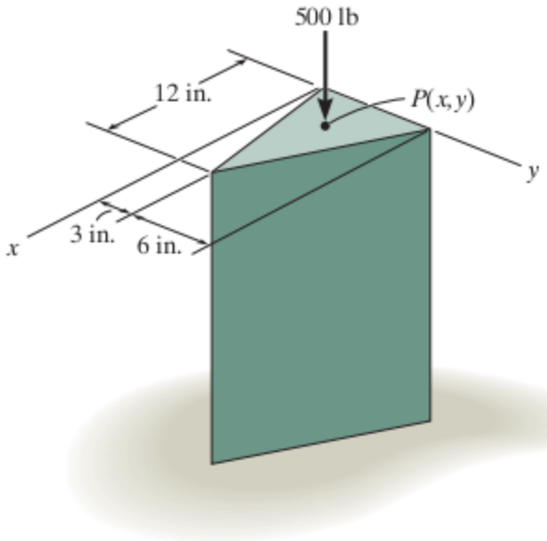
7–33. The 150-kg bucket is suspended from end E of the frame. Determine the average normal stress in the 6-mm diameter wire CF and the 15-mm diameter short strut BD.

7–34. The 150-kg bucket is suspended from end E of the frame. If the diameters of the pins at A and D are 6 mm and 10 mm, respectively, determine the average shear stress developed in these pins. Each pin is subjected to double shear.



Probs. 7–33/34

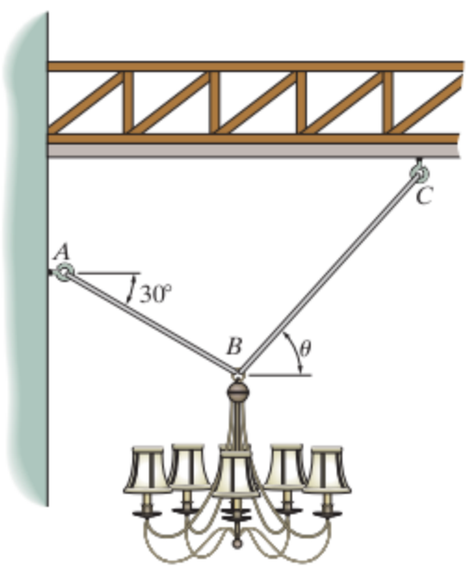
7–35. The pedestal has a triangular cross section as shown. If it is subjected to a compressive force of 500 lb, specify the x and y coordinates for the location of point $P(x, y)$, where the load must be applied on the cross section, so that the average normal stress is uniform. Compute the stress and sketch its distribution acting on the cross section at a location removed from the point of load application.



Prob. 7–35

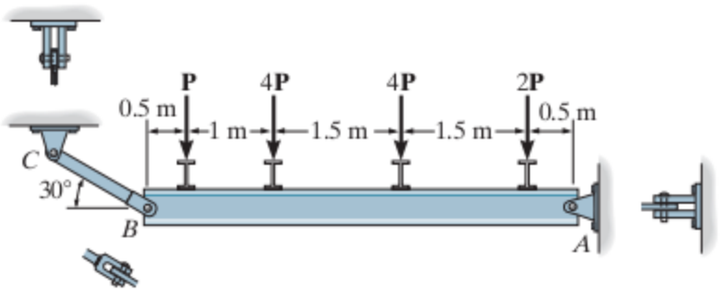
***7–36.** The 20-kg chandelier is suspended from the wall and ceiling using rods AB and BC, which have diameters of 3 mm and 4 mm, respectively. Determine the angle θ so that the average normal stress in both rods is the same.

7–37. The chandelier is suspended from the wall and ceiling using rods AB and BC, which have diameters of 3 mm and 4 mm, respectively. If the average normal stress in both rods is not allowed to exceed 150 MPa, determine the largest mass of the chandelier that can be supported if $\theta = 45°$.



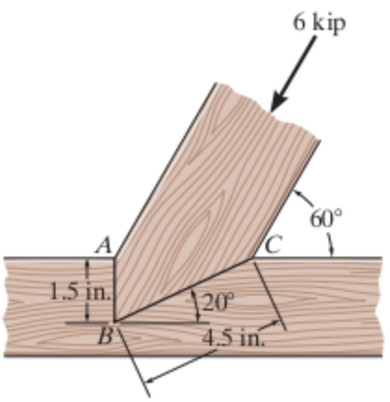
Probs. 7–36/37

7–38. The beam is supported by a pin at A and a short link BC. If $P = 15$ kN, determine the average shear stress developed in the pins at A, B, and C. All pins are in double shear as shown, and each has a diameter of 18 mm.



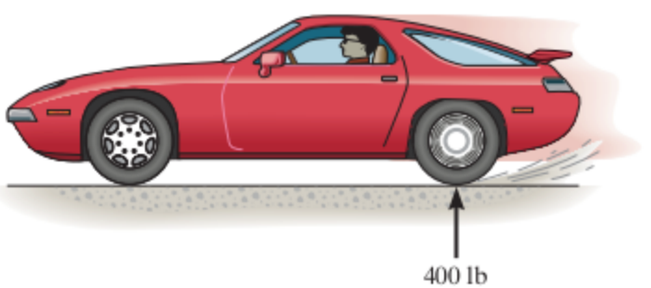
Prob. 7–38

7–39. The joint is subjected to the axial member force of 6 kip. Determine the average normal stress acting on sections AB and BC. Assume the member is smooth and is 1.5-in. thick.



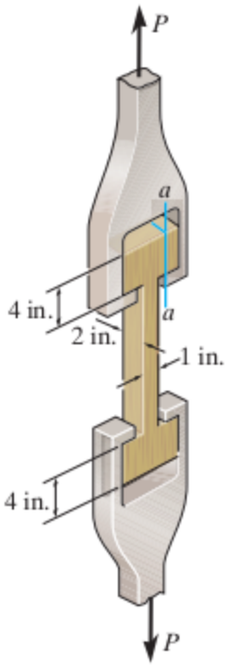
Prob. 7–39

***7–40.** The driver of the sports car applies his rear brakes and causes the tires to slip. If the normal force on each rear tire is 400 lb and the coefficient of kinetic friction between the tires and the pavement is $\mu_k = 0.5$, determine the average shear stress developed by the friction force on the tires. Assume the rubber of the tires is flexible and each tire is filled with an air pressure of 32 psi.



Prob. 7–40

7–41. During the tension test, the wooden specimen is subjected to an average normal stress of 2 ksi. Determine the axial force **P** applied to the specimen. Also, find the average shear stress developed along section *a–a* of the specimen.

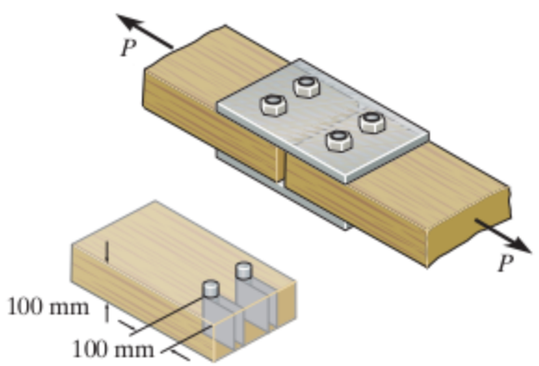


Prob. 7–41

7

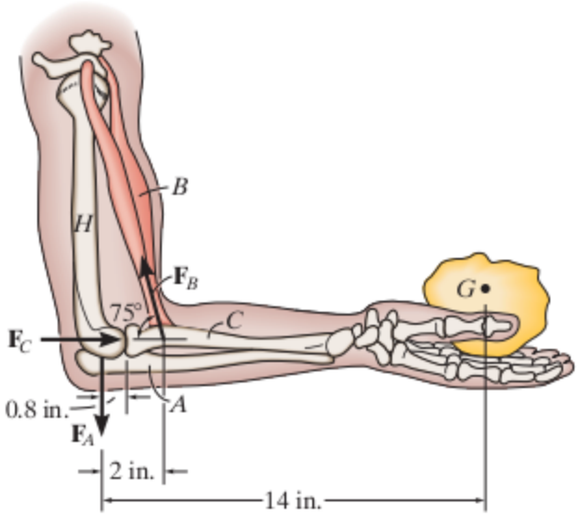
7–42. If the joint is subjected to an axial force of $P = 9$ kN, determine the average shear stress developed in each of the 6-mm diameter bolts between the plates and the members and along each of the four shaded shear planes.

7–43. The average shear stress in each of the 6-mm diameter bolts and along each of the four shaded shear planes is not allowed to exceed 80 MPa and 500 kPa, respectively. Determine the maximum axial force **P** that can be applied to the joint.



Probs. 7–42/43

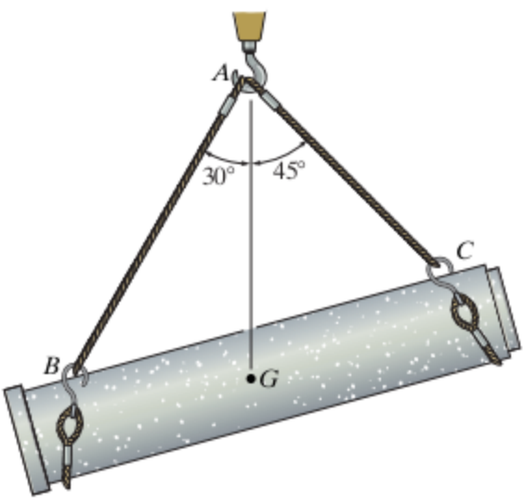
*7–44. When the hand is holding the 5-lb stone, the humerus *H*, assumed to be smooth, exerts normal forces F_C and F_A on the radius *C* and ulna *A*, respectively, as shown. If the smallest cross-sectional area of the ligament at *B* is 0.30 in^2, determine the greatest average tensile stress to which it is subjected.



Prob. 7–44

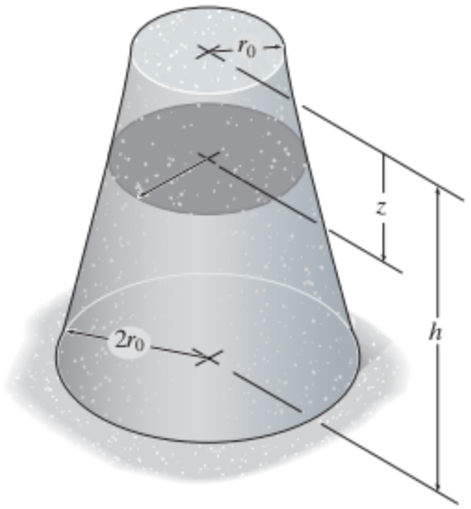
7–45. The 2-Mg concrete pipe has a center of mass at point *G*. If it is suspended from cables *AB* and *AC*, determine the average normal stress developed in the cables. The diameters of *AB* and *AC* are 12 mm and 10 mm, respectively.

7–46. The 2-Mg concrete pipe has a center of mass at point *G*. If it is suspended from cables *AB* and *AC*, determine the diameter of cable *AB* so that the average normal stress developed in this cable is the same as in the 10-mm diameter cable *AC*.



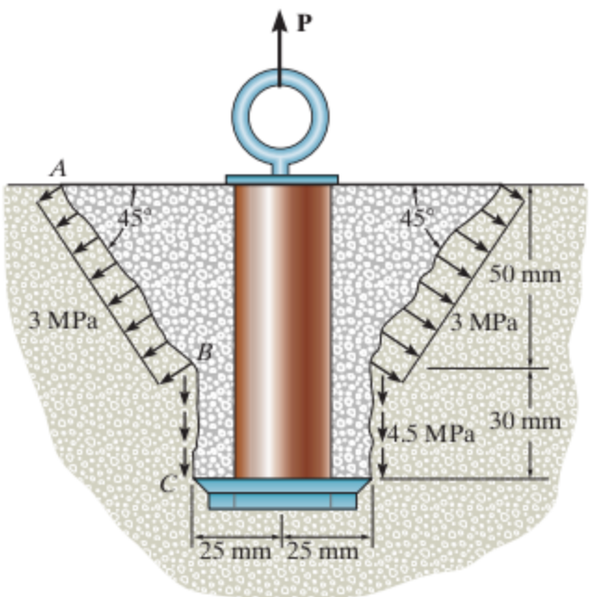
Probs. 7–45/46

7–47. If the concrete pedestal has a specific weight of γ, determine the average normal stress developed in the pedestal as a function of z.



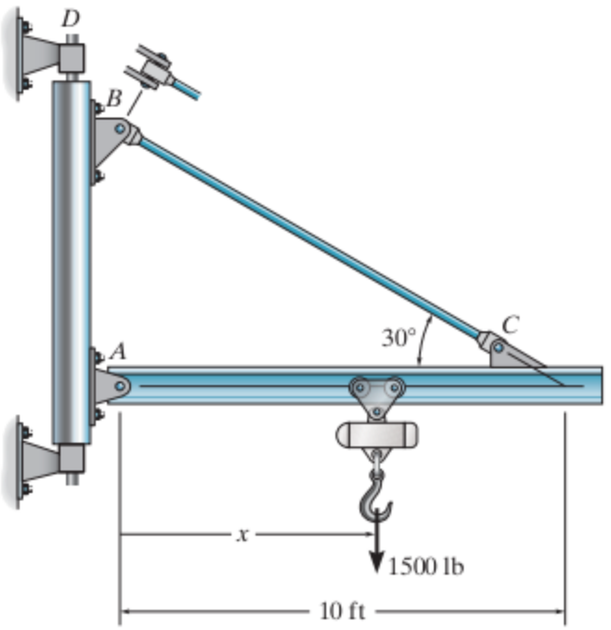
Prob. 7–47

***7–48.** The anchor bolt was pulled out of the concrete wall and the failure surface formed part of a frustum and cylinder. This indicates a shear failure occurred along the cylinder BC and tension failure along the frustum AB. If the shear and normal stresses along these surfaces have the magnitudes shown, determine the force **P** that must have been applied to the bolt.



Prob. 7–48

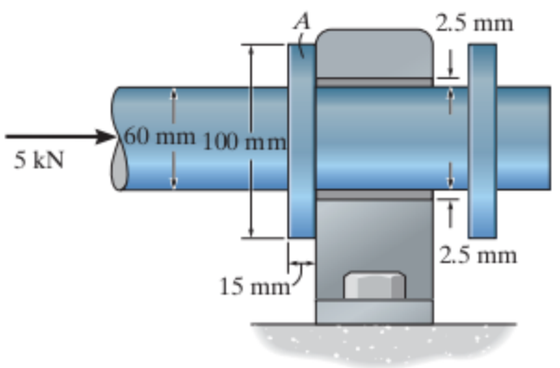
7–49. The jib crane is pinned at A and supports a chain hoist that can travel along the bottom flange of the beam, $1\text{ ft} \le x \le 12\text{ ft}$. If the hoist is rated to support a maximum of 1500 lb, determine the maximum average normal stress in the $\frac{3}{4}$-in.-diameter tie rod BC and the maximum average shear stress in the $\frac{5}{8}$-in.-diameter pin at B.



Prob. 7–49

7–50. If the shaft is subjected to an axial force of 5 kN, determine the bearing stress acting on the collar A.

7–51. If the 60-mm diameter shaft is subjected to an axial force of 5 kN, determine the average shear stress developed in the shear plane where the collar A and shaft are connected.



Probs. 7–50/51

7.6 Allowable Stress

To properly *design* a structural member or mechanical element, it is necessary to restrict the stress in the material to a level that will be safe. To ensure this safety, it is therefore necessary to choose an allowable stress that restricts the applied load to one that is *less* than the load the member can fully support. There are many reasons for doing this. For example, the load for which the member is designed may be different from actual loadings placed on it. The intended measurements of a structure or machine may not be exact due to errors in fabrication or in the assembly of its component parts. Unknown vibrations, impact, or accidental loadings can occur that may not be accounted for in the design. Atmospheric corrosion, decay, or weathering tend to cause materials to deteriorate during service. And lastly, some materials, such as wood, concrete, or fiber-reinforced composites, can show high variability in mechanical properties.

One method of specifying the allowable load for a member is to use a number called the factor of safety. The ***factor of safety*** (F.S.) is a ratio of the failure load F_{fail} to the allowable load F_{allow}. Here F_{fail} is found from experimental testing of the material, and the factor of safety is selected based on experience so that the above mentioned uncertainties are accounted for when the member is used under similar conditions of loading and geometry. Stated mathematically,

$$\text{F.S.} = \frac{F_{\text{fail}}}{F_{\text{allow}}} \tag{7–5}$$

If the load applied to the member is *linearly related* to the stress developed within the member, as in the case of using $\sigma = P/A$ and $\tau_{\text{avg}} = V/A$, then we can also express the factor of safety as a ratio of the failure stress σ_{fail} (or τ_{fail}) to the allowable stress σ_{allow} (or τ_{allow});* that is,

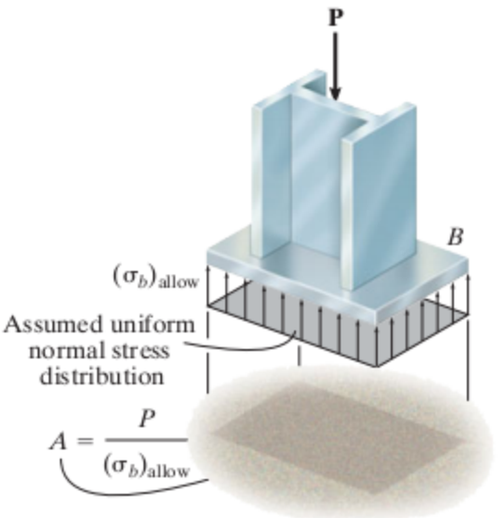
$$\text{F.S.} = \frac{\sigma_{\text{fail}}}{\sigma_{\text{allow}}} \tag{7–6}$$

or

$$\text{F.S.} = \frac{\tau_{\text{fail}}}{\tau_{\text{allow}}} \tag{7–7}$$

*In some cases, such as columns, the applied load is not linearly related to stress and therefore only Eq. 7–5 can be used to determine the factor of safety. See Chapter 17.

In any of these equations, the factor of safety must be *greater* than 1 in order to avoid the potential for failure. Specific values depend on the types of materials to be used and the intended purpose of the structure or machine. For example, the F.S. used in the design of aircraft or space-vehicle components may be close to 1 in order to reduce the weight of the vehicle. Or, in the case of a nuclear power plant, the factor of safety for some of its components may be as high as 3 due to uncertainties in loading or material behavior. In many cases, the factor of safety for a specific case can be found in design codes and engineering handbooks. These values are intended to form a balance of ensuring public and environmental safety and providing a reasonable economic solution to design.



The area of the column base plate B is determined from the allowable bearing stress for the concrete.

7.7 Design of Simple Connections

By making simplifying assumptions regarding the behavior of the material, the equations $\sigma = P/A$ and $\tau_{avg} = V/A$ can often be used to analyze or design a simple connection or mechanical element. In particular, if a member is subjected to *normal force* at a section, its required area at the section is determined from

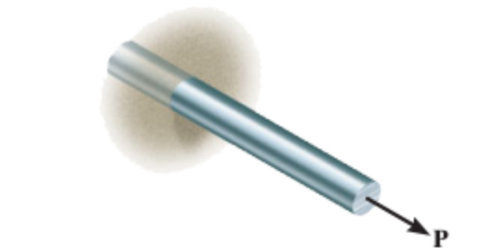
$$A = \frac{P}{\sigma_{allow}} \tag{7–8}$$

Then the other hand, if the section is subjected to an average *shear force*, then the required area at the section is

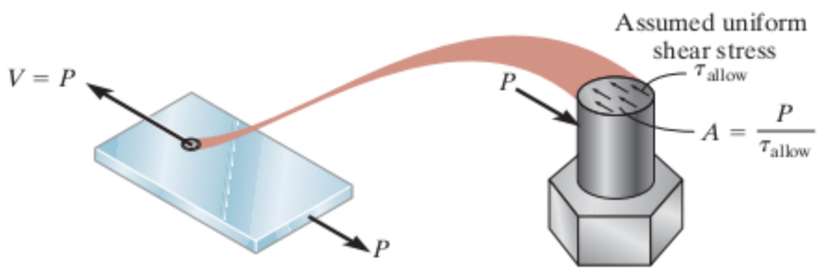
$$A = \frac{V}{\tau_{allow}} \tag{7–9}$$

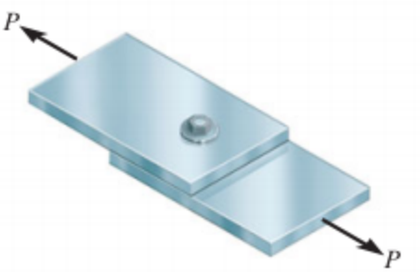
As discussed in Sec. 7.6, the allowable stress used in each of these equations is determined either by applying a factor of safety to the material's normal or shear failure stress or by finding these stresses directly from an appropriate design code.

Three examples of where the above equations apply are shown in Fig. 7–21.



The embedded length l of this rod in concrete can be determined using the allowable shear stress of the bonding glue.





The area of the bolt for this lap joint is determined from the shear stress which is largest between the plates.

Fig. 7–21

Important Point

- Design of a member for strength is based on selecting an allowable stress that will enable it to safely support its intended load. Since there are many unknown factors that can influence the actual stress in a member, then depending upon the intended use of the member, a *factor of safety* is applied to obtain the allowable load the member can support.

Procedure for Analysis

When solving problems using the average normal and shear stress equations, a careful consideration should first be made as to choose the section over which the critical stress is acting. Once this section is determined, the member must then be designed to have a sufficient area at the section to resist the stress that acts on it. This area is determined using the following steps.

Internal Loading.

- Section the member through the area and draw a free-body diagram of a segment of the member. The internal resultant force at the section is then determined using the equations of equilibrium.

Required Area.

- Provided the allowable stress is known or can be determined, the required area needed to sustain the load at the section is then determined from $A = P/\sigma_{\text{allow}}$ or $A = V/\tau_{\text{allow}}$.

Appropriate factors of safety must be considered when designing cranes and cables used to transfer heavy loads.

EXAMPLE 7.10

The control arm is subjected to the loading shown in Fig. 7–22*a*. Determine to the nearest $\frac{1}{4}$ in. the required diameter of the steel pin at *C* if the allowable shear stress for the steel is $\tau_{allow} = 8$ ksi.

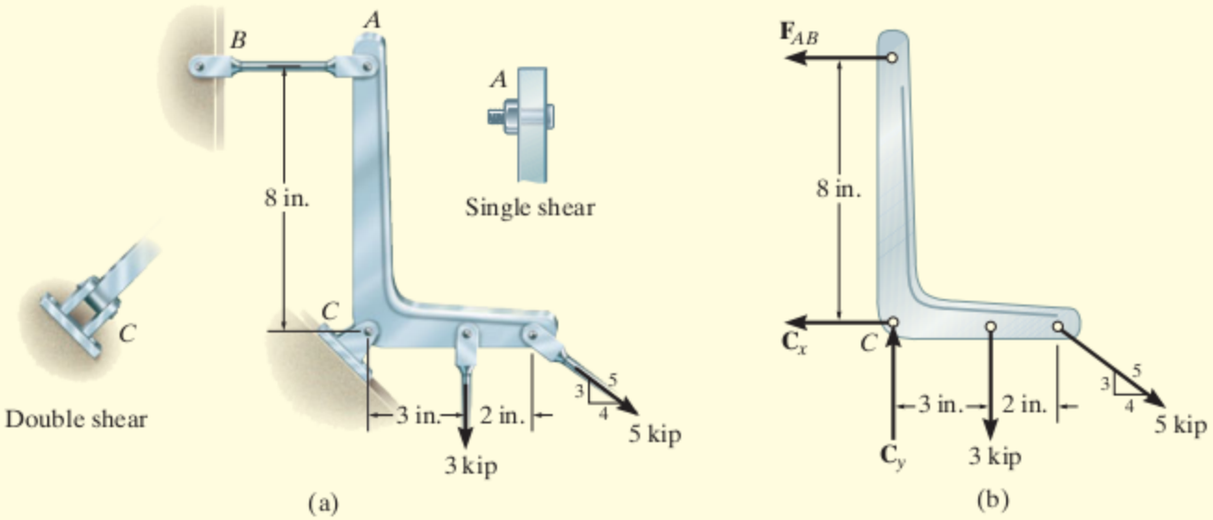


Fig. 7–22

SOLUTION

Internal Shear Force. A free-body diagram of the arm is shown in Fig. 7–22*b*. For equilibrium we have

$$\circlearrowleft +\Sigma M_C = 0; \quad F_{AB}(8\text{ in.}) - 3\text{ kip}\,(3\text{ in.}) - 5\text{ kip}\left(\tfrac{3}{5}\right)(5\text{ in.}) = 0$$

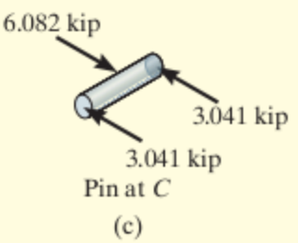
$$F_{AB} = 3\text{ kip}$$

$$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \quad -3\text{ kip} - C_x + 5\text{ kip}\left(\tfrac{4}{5}\right) = 0 \qquad C_x = 1\text{ kip}$$

$$+\uparrow \Sigma F_y = 0; \quad C_y - 3\text{ kip} - 5\text{ kip}\left(\tfrac{3}{5}\right) = 0 \qquad C_y = 6\text{ kip}$$

The pin at *C* resists the resultant force at C, which is

$$F_C = \sqrt{(1\text{ kip})^2 + (6\text{ kip})^2} = 6.082\text{ kip}$$



Since the pin is subjected to double shear, a shear force of 3.041 kip acts over its cross-sectional area *between* the arm and each supporting leaf for the pin, Fig. 7–22*c*.

Required Area. We have

$$A = \frac{V}{\tau_{allow}} = \frac{3.041\text{ kip}}{8\text{ kip/in}^2} = 0.3802\text{ in}^2$$

$$\pi\left(\frac{d}{2}\right)^2 = 0.3802\text{ in}^2$$

$$d = 0.696\text{ in.}$$

Use a pin having a diameter of

$$d = \tfrac{3}{4}\text{ in.} = 0.750\text{ in.} \qquad \textit{Ans.}$$

7

EXAMPLE 7.11

The suspender rod is supported at its end by a fixed-connected circular disk as shown in Fig. 7–23*a*. If the rod passes through a 40-mm-diameter hole, determine the minimum required diameter of the rod and the minimum thickness of the disk needed to support the 20-kN load. The allowable normal stress for the rod is $\sigma_{allow} = 60$ MPa, and the allowable shear stress for the disk is $\tau_{allow} = 35$ MPa.

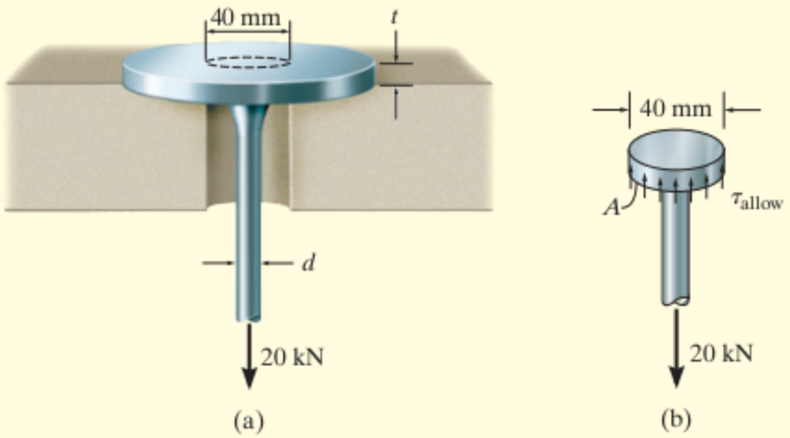


Fig. 7–23

SOLUTION

Diameter of Rod. By inspection, the axial force in the rod is 20 kN. Thus the required cross-sectional area of the rod is

$$A = \frac{P}{\sigma_{allow}}; \qquad \frac{\pi}{4}d^2 = \frac{20(10^3)\ \text{N}}{60(10^6)\ \text{N/m}^2}$$

so that

$$d = 0.0206\ \text{m} = 20.6\ \text{mm} \qquad \textit{Ans.}$$

Thickness of Disk. As shown on the free-body diagram in Fig. 7–23*b*, the material at the sectioned area of the disk must resist *shear stress* to prevent movement of the disk through the hole. If this shear stress is *assumed* to be uniformly distributed over the sectioned area, then, since $V = 20$ kN, we have

$$A = \frac{V}{\tau_{allow}}; \qquad 2\pi(0.02\ \text{m})(t) = \frac{20(10^3)\ \text{N}}{35(10^6)\ \text{N/m}^2}$$

$$t = 4.55(10^{-3})\ \text{m} = 4.55\ \text{mm} \qquad \textit{Ans.}$$

EXAMPLE 7.12

The shaft shown in Fig. 7–24*a* is supported by the collar at *C*, which is attached to the shaft and located on the right side of the bearing at *B*. Determine the largest value of *P* for the axial forces at *E* and *F* so that the bearing stress on the collar does not exceed an allowable stress of $(\sigma_b)_{\text{allow}} = 75$ MPa and the average normal stress in the shaft does not exceed an allowable stress of $(\sigma_t)_{\text{allow}} = 55$ MPa.

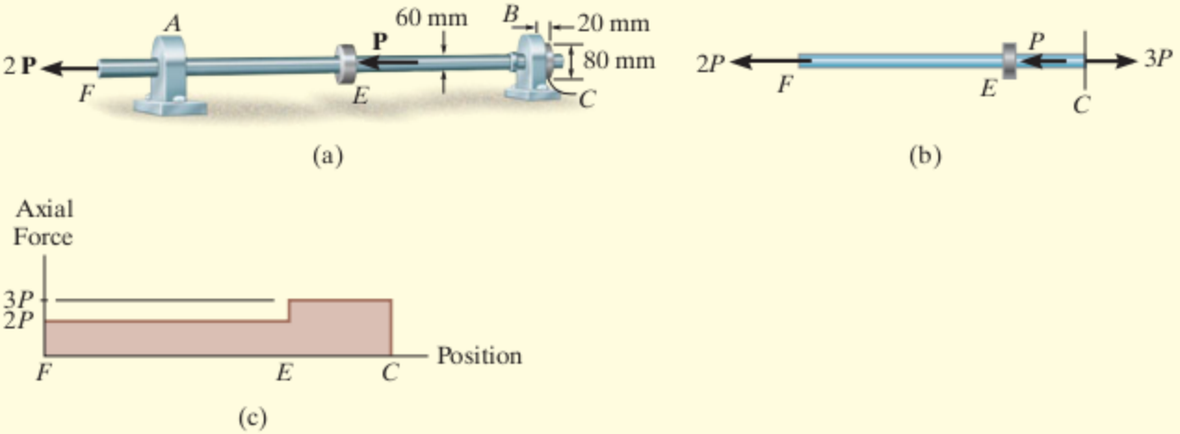


Fig. 7–24

SOLUTION

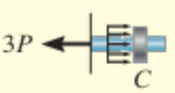
To solve the problem we will determine *P* for each possible failure condition. Then we will choose the *smallest* value. Why?

Normal Stress. Using the method of sections, the axial load within region *FE* of the shaft is $2P$, whereas the *largest* axial force, $3P$, occurs within region *EC*, Fig. 7–24*b*. The variation of the internal loading is clearly shown on the normal-force diagram, Fig. 7–24*c*. Since the cross-sectional area of the entire shaft is constant, region *EC* is subjected to the maximum average normal stress. Applying Eq. 7–8, we have

$$A = \frac{P}{\sigma_{\text{allow}}}; \qquad \pi(0.03\ \text{m})^2 = \frac{3P}{55(10^6)\ \text{N/m}^2}$$

$$P = 51.8\ \text{kN} \qquad \textit{Ans.}$$

Bearing Stress. As shown on the free-body diagram in Fig. 7–24*d*, the collar at *C* must resist the load of $3P$, which acts over a bearing area of $A_b = [\pi(0.04\ \text{m})^2 - \pi(0.03\ \text{m})^2] = 2.199(10^{-3})\ \text{m}^2$. Thus,



(d)

$$A = \frac{P}{\sigma_{\text{allow}}}; \qquad 2.199(10^{-3})\ \text{m}^2 = \frac{3P}{75(10^6)\ \text{N/m}^2}$$

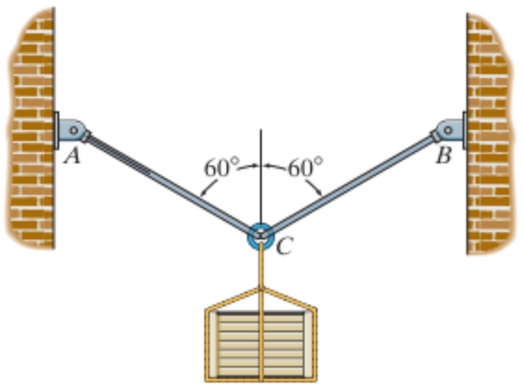
$$P = 55.0\ \text{kN}$$

By comparison, the largest load that can be applied to the shaft is $P = 51.8$ kN, since any load larger than this will cause the allowable normal stress in the shaft to be exceeded.

NOTE: Here we have not considered a possible shear failure of the collar as in Example 7.11.

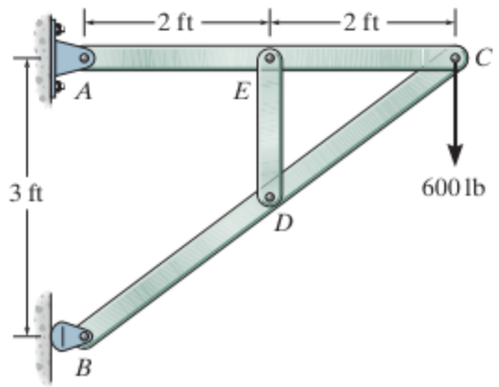
FUNDAMENTAL PROBLEMS

F7–13. Rods *AC* and *BC* are used to suspend the 200-kg mass. If each rod is made of a material for which the average normal stress can not exceed 150 MPa, determine the minimum required diameter of each rod to the nearest mm.



F7–13

F7–14. The frame supports the loading shown. The pin at *A* has a diameter of 0.25 in. If it is subjected to double shear, determine the average shear stress in the pin.

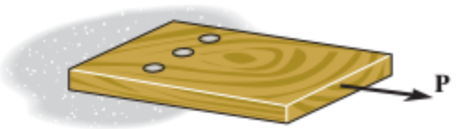


F7–14

F7–15. Determine the maximum average shear stress developed in each $\frac{3}{4}$-in.-diameter bolt.

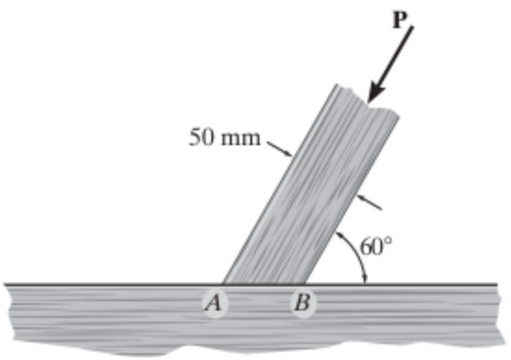
F7–15

F7–16. If each of the three nails has a diameter of 4 mm and can withstand an average shear stress of 60 MPa, determine the maximum allowable force **P** that can be applied to the board.



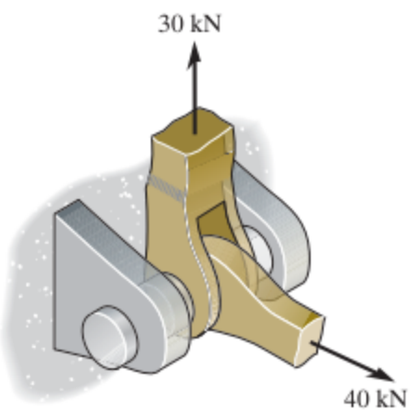
F7–16

F7–17. The strut is glued to the horizontal member at surface *AB*. If the strut has a thickness of 25 mm and the glue can withstand an average shear stress of 600 kPa, determine the maximum force **P** that can be applied to the strut.



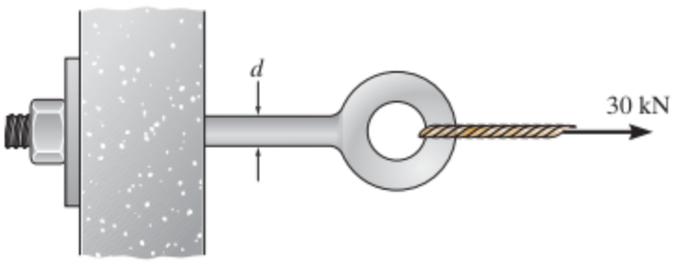
F7–17

F7–18. Determine the maximum average shear stress developed in the 30-mm-diameter pin.



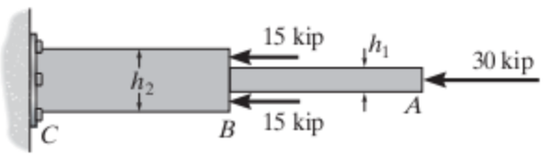
F7–18

F7–19. If the eyebolt is made of a material having a yield stress of $\sigma_Y = 250$ MPa, determine the minimum required diameter d of its shank. Apply a factor of safety F.S. = 1.5 against yielding.



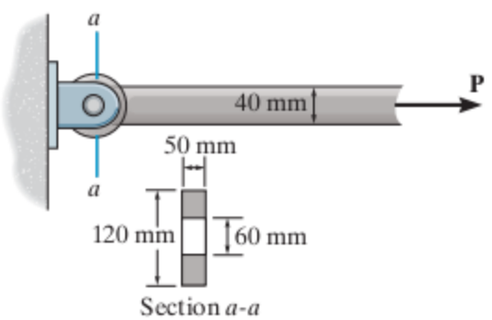
F7–19

F7–20. If the bar assembly is made of a material having a yield stress of $\sigma_Y = 50$ ksi, determine the minimum required dimensions h_1 and h_2 to the nearest $\frac{1}{8}$ in. Apply a factor of safety F.S. = 1.5 against yielding. Each bar has a thickness of 0.5 in.



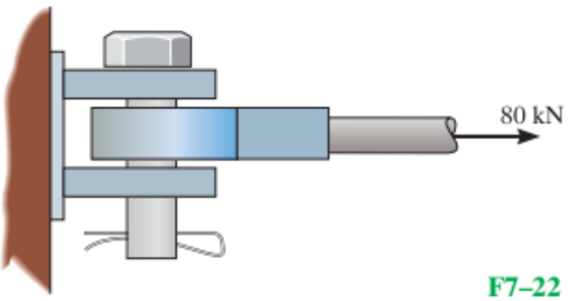
F7–20

F7–21. Determine the maximum force **P** that can be applied to the rod if it is made of material having a yield stress of $\sigma_Y = 250$ MPa. Consider the possibility that failure occurs in the rod and at section a–a. Apply a factor of safety F.S. = 2 against yielding.



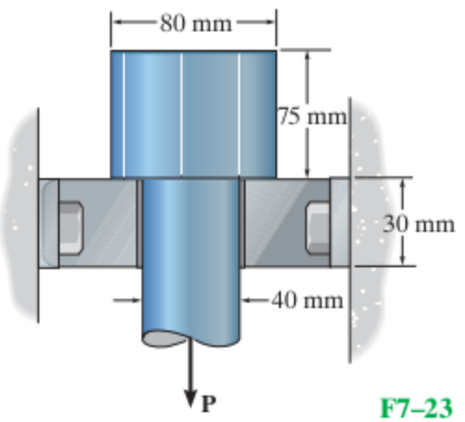
F7–21

F7–22. The pin is made of a material having a failure shear stress of $\tau_{fail} = 100$ MPa. Determine the minimum required diameter of the pin to the nearest mm. Apply a factor of safety F.S. = 2.5 against shear failure.



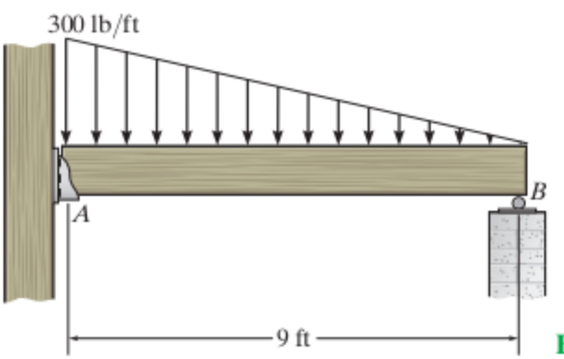
F7–22

F7–23. If the bolt head and the supporting bracket are made of the same material having a failure shear stress of $\tau_{fail} = 120$ MPa, determine the maximum allowable force **P** that can be applied to the bolt so that it does not pull through the plate. Apply a factor of safety F.S. = 2.5 against shear failure.



F7–23

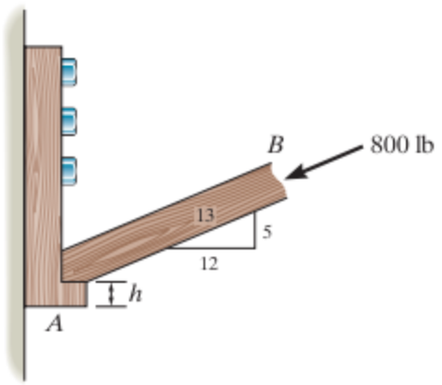
F7–24. Six nails are used to hold the hanger at A against the column. Determine the minimum required diameter of each nail to the nearest $\frac{1}{16}$ in. if it is made of material having $\tau_{fail} = 16$ ksi. Apply a factor of safety F.S. = 2 against shear failure.



F7–24

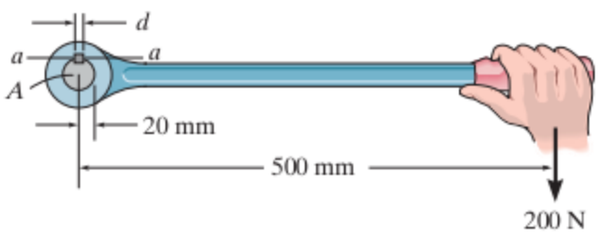
PROBLEMS

***7–52.** Member B is subjected to a compressive force of 800 lb. If A and B are both made of wood and are $\frac{3}{8}$ in. thick, determine to the nearest $\frac{1}{4}$ in. the smallest dimension h of the horizontal segment so that it does not fail in shear. The average shear stress for the segment is $\tau_{allow} = 300$ psi.



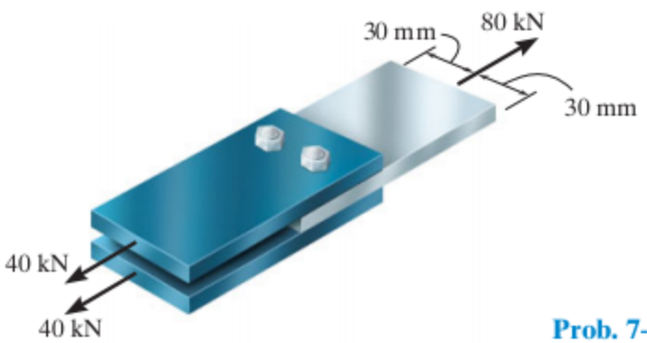
Prob. 7–52

7–53. The lever is attached to the shaft A using a key that has a width d and length of 25 mm. If the shaft is fixed and a vertical force of 200 N is applied perpendicular to the handle, determine the dimension d if the allowable shear stress for the key is $\tau_{allow} = 35$ MPa.



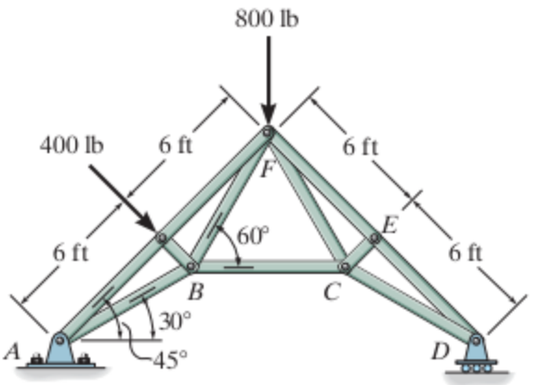
Prob. 7–53

7–54. The joint is fastened together using two bolts. Determine the required diameter of the bolts if the failure shear stress for the bolts is $\tau_{fail} = 350$ MPa. Use a factor of safety for shear of F.S. $= 2.5$.



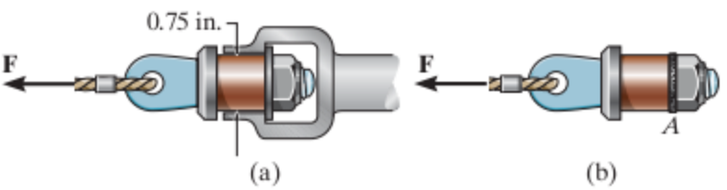
Prob. 7–54

7–55. The truss is used to support the loading shown. Determine the required cross-sectional area of member BC if the allowable normal stress is $\sigma_{allow} = 24$ ksi.



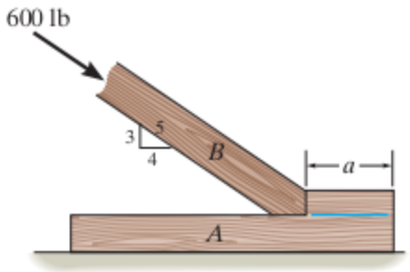
Prob. 7–55

***7–56.** The steel swivel bushing in the elevator control of an airplane is held in place using a nut and washer as shown in Fig. (a). Failure of the washer A can cause the push rod to separate as shown in Fig. (b). If the maximum average normal stress for the washer is $\sigma_{max} = 60$ ksi and the maximum average shear stress is $\tau_{max} = 21$ ksi, determine the force **F** that must be applied to the bushing that will cause this to happen. The washer is $\frac{1}{16}$ in. thick.



Prob. 7–56

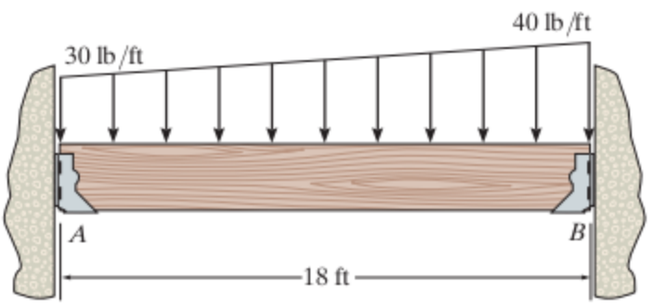
7–57. Member B is subjected to a compressive force of 600 lb. If A and B are both made of wood and are 1.5 in. thick, determine to the nearest $\frac{1}{8}$ in. the smallest dimension a of the support so that the average shear stress along the blue line does not exceed $\tau_{allow} = 50$ psi. Neglect friction.



Prob. 7–57

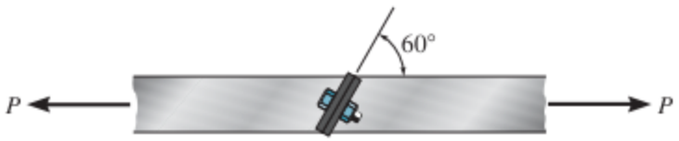
7–58. The hangers support the joist uniformly, so that it is assumed the four nails on each hanger carry an equal portion of the load. If the joist is subjected to the loading shown, determine the average shear stress in each nail of the hanger at ends A and B. Each nail has a diameter of 0.25 in. The hangers only support vertical loads.

7–59. The hangers support the joists uniformly, so that it is assumed the four nails on each hanger carry an equal portion of the load. Determine the smallest diameter of the nails at A and at B if the allowable stress for the nails is $\tau_{allow} = 4$ ksi. The hangers only support vertical loads.



Probs. 7–58/59

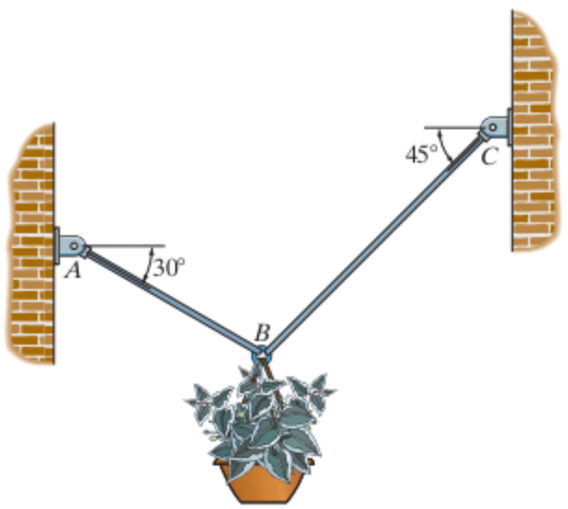
***7–60.** The tension member is fastened together using *two* bolts, one on each side of the member as shown. Each bolt has a diameter of 0.3 in. Determine the maximum load P that can be applied to the member if the allowable shear stress for the bolts is $\tau_{allow} = 12$ ksi and the allowable average normal stress is $\sigma_{allow} = 20$ ksi.



Prob. 7–60

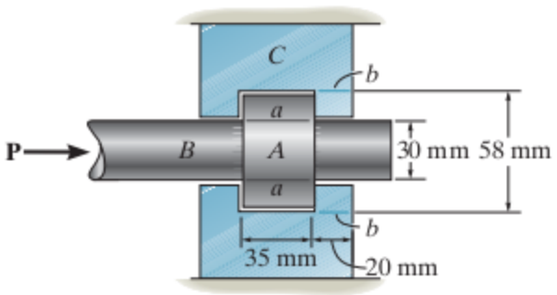
7–61. The 50-kg flowerpot is suspended from wires AB and BC. If the wires have a normal failure stress of $\sigma_{fail} = 350$ MPa, determine the minimum diameter of each wire. Use a factor of safety of 2.5.

7–62. The 50-kg flowerpot is suspended from wires AB and BC which have diameters of 1.5 mm and 2 mm, respectively. If the wires have a normal failure stress of $\sigma_{fail} = 350$ MPa, determine the factor of safety of each wire.



Probs. 7–61/62

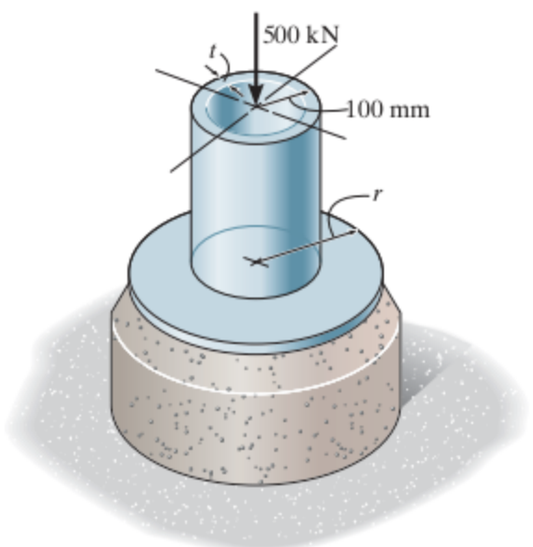
7–63. The thrust bearing consists of a circular collar A fixed to the shaft B. Determine the maximum axial force P that can be applied to the shaft so that it does not cause the shear stress along a cylindrical surface a or b to exceed an allowable shear stress of $\tau_{allow} = 170$ MPa.



Prob. 7–63

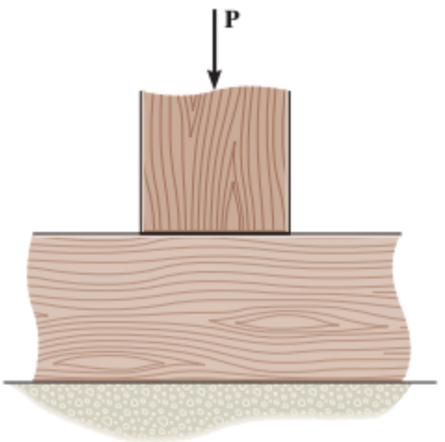
***7–64.** The steel pipe is supported on the circular base plate and concrete pedestal. If the normal failure stress for the steel is $(\sigma_{fail})_{st} = 350$ MPa, determine the minimum thickness t of the pipe if it supports the force of 500 kN. Use a factor of safety against failure of 1.5. Also, find the minimum radius r of the base plate so that the minimum factor of safety against failure of the concrete due to bearing is 2.5. The failure bearing stress for concrete is $(\sigma_{fail})_{con} = 25$ MPa.

7–65. The steel pipe is supported on the circular base plate and concrete pedestal. If the thickness of the pipe is $t = 5$ mm and the base plate has a radius of 150 mm, determine the factors of safety against failure of the steel and concrete. The applied force is 500 kN, and the normal failure stresses for steel and concrete are $(\sigma_{fail})_{st} = 350$ MPa and $(\sigma_{fail})_{con} = 25$ MPa, respectively.



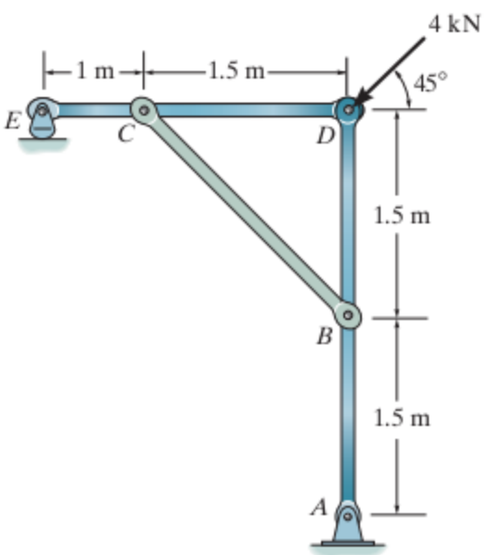
Probs. 7–64/65

7–66. The 60 mm × 60 mm oak post is supported on the pine block. If the allowable bearing stresses for these materials are $\sigma_{oak} = 43$ MPa and $\sigma_{pine} = 25$ MPa, determine the greatest load P that can be supported. If a rigid bearing plate is used between these materials, determine its required area so that the maximum load P can be supported. What is this load?



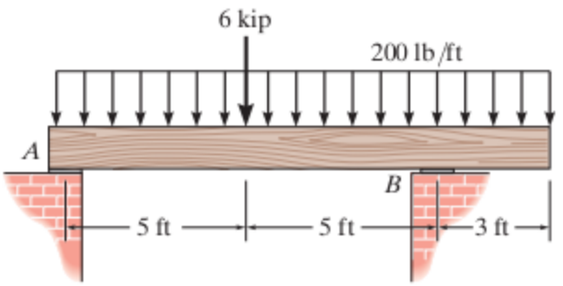
Prob. 7–66

7–67. The frame is subjected to the load of 4 kN which acts on member ABD at D. Determine the required diameter of the pins at D and C if the allowable shear stress for the material is $\tau_{allow} = 40$ MPa. Pin C is subjected to double shear, whereas pin D is subjected to single shear.



Prob. 7–67

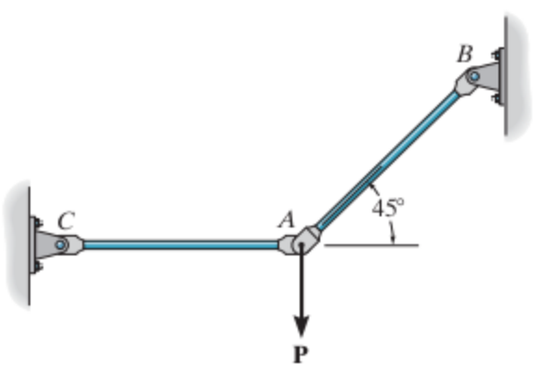
*7–68. The beam is made from southern pine and is supported by base plates resting on brick work. If the allowable bearing stresses for the materials are $(\sigma_{pine})_{allow} = 2.81$ ksi and $(\sigma_{brick})_{allow} = 6.70$ ksi, determine the required length of the base plates at A and B to the nearest $\frac{1}{4}$ inch in order to support the load shown. The plates are 3 in. wide.



Prob. 7–68

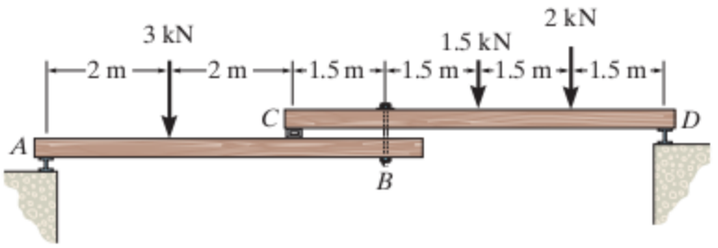
7–69. The two aluminum rods support the vertical force of $P = 20$ kN. Determine their required diameters if the allowable tensile stress for the aluminum is $\sigma_{allow} = 150$ MPa.

7–70. The two aluminum rods AB and AC have diameters of 10 mm and 8 mm, respectively. Determine the largest vertical force **P** that can be supported. The allowable tensile stress for the aluminum is $\sigma_{allow} = 150$ MPa.



Probs. 7–69/70

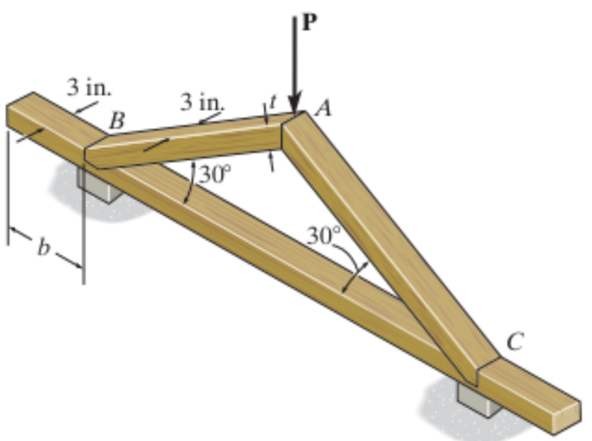
7–71. The compound wooden beam is connected together by a bolt at B. Assuming that the connections at A, B, C, and D exert only vertical forces on the beam, determine the required diameter of the bolt at B and the required outer diameter of its washers if the allowable tensile stress for the bolt is $(\sigma_t)_{allow} = 150$ MPa and the allowable bearing stress for the wood is $(\sigma_b)_{allow} = 28$ MPa. Assume that the hole in the washers has the same diameter as the bolt.



Prob. 7–71

*7–72. Determine the required minimum thickness t of member AB and edge distance b of the frame if $P = 9$ kip and the factor of safety against failure is 2. The wood has a normal failure stress of $\sigma_{fail} = 6$ ksi, and shear failure stress of $\tau_{fail} = 1.5$ ksi.

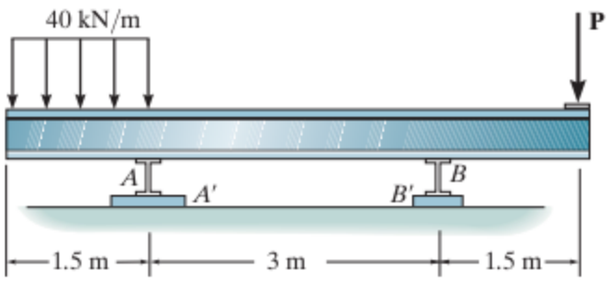
7–73. Determine the maximum allowable load **P** that can be safely supported by the frame if $t = 1.25$ in. and $b = 3.5$ in. The wood has a normal failure stress of $\sigma_{fail} = 6$ ksi, and shear failure stress of $\tau_{fail} = 1.5$ ksi. Use a factor of safety against failure of 2.



Probs. 7–72/73

7–74. If the allowable bearing stress for the material under the supports at A and B is $(\sigma_b)_{allow} = 1.5$ MPa, determine the size of *square* bearing plates A' and B' required to support the load. Dimension the plates to the nearest mm. The reactions at the supports are vertical. Take $P = 100$ kN.

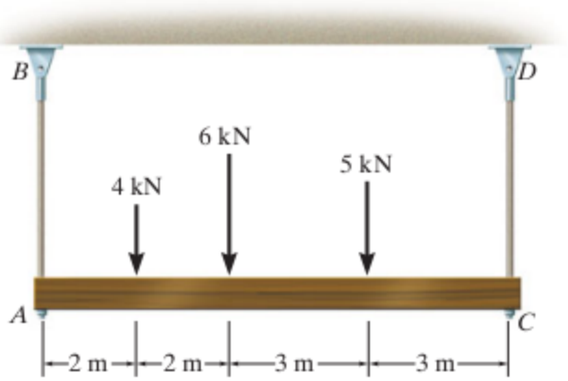
7–75. If the allowable bearing stress for the material under the supports at A and B is $(\sigma_b)_{allow} = 1.5$ MPa, determine the maximum load P that can be applied to the beam. The bearing plates A' and B' have square cross sections of 150 mm × 150 mm and 250 mm × 250 mm, respectively.



Probs. 7–74/75

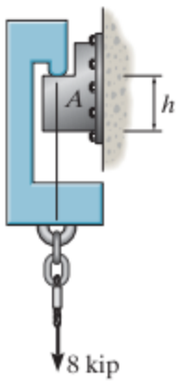
7

***7–76.** The rods AB and CD are made of steel having a failure tensile stress of $\sigma_{fail} = 510$ MPa. Using a factor of safety of F.S. = 1.75 for tension, determine their smallest diameter so that they can support the load shown. The beam is assumed to be pin connected at A and C.



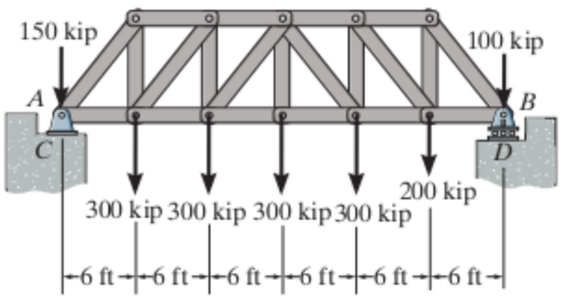
Prob. 7–76

7–77. The aluminum bracket A is used to support the centrally applied load of 8 kip. If it has a constant thickness of 0.5 in., determine the smallest height h in order to prevent a shear failure. The failure shear stress is $\tau_{fail} = 23$ ksi. Use a factor of safety for shear of F.S. = 2.5.



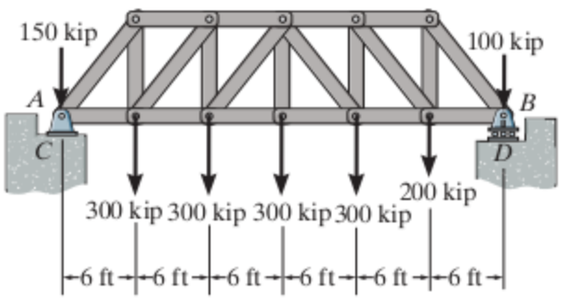
Prob. 7–77

7–78. The pin support A and roller support B of the bridge truss are supported on concrete abutments. If the bearing failure stress of the concrete is $(\sigma_{fail})_b = 4$ ksi, determine the required minimum dimension of the square bearing plates at C and D to the nearest $\frac{1}{16}$ in. Apply a factor of safety of 2 against failure.



Prob. 7–78

7–79. The pin support A and roller support B of the bridge truss are supported on the concrete abutments. If the square bearing plates at C and D are 21 in. × 21 in., and the bearing failure stress for concrete is $(\sigma_{fail})_b = 4$ ksi, determine the factor of safety against bearing failure for the concrete under each plate.

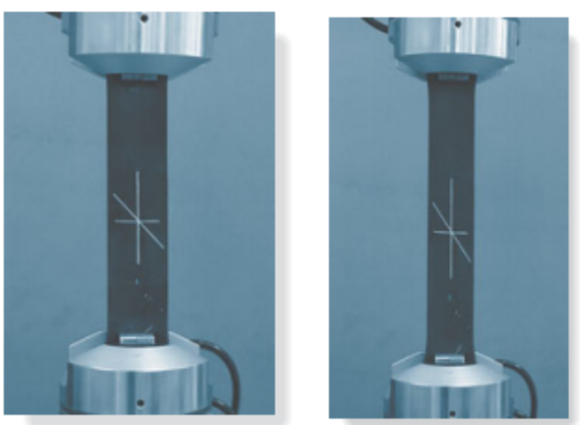


Prob. 7–79

7.8 Deformation

Whenever a force is applied to a body, it will tend to change the body's shape and size. These changes are referred to as ***deformation***, and they may be either highly visible or practically unnoticeable. For example, a rubber band will undergo a very large deformation when stretched, whereas only slight deformations of structural members occur when a building is occupied by people walking about. Deformation of a body can also occur when the temperature of the body is changed. A typical example is the thermal expansion or contraction of a roof caused by the weather.

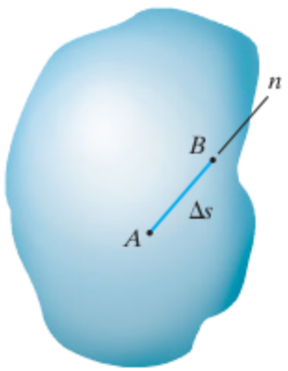
In a general sense, the deformation of a body will not be uniform throughout its volume, and so the change in geometry of any line segment within the body may vary substantially along its length. Hence, to study deformational changes in a more uniform manner, we will consider line segments that are very short and located in the neighborhood of a point. Realize, however, that these changes will also depend on the orientation of the line segment at the point. For example, a line segment may elongate if it is oriented in one direction, whereas it may contract if it is oriented in another direction.



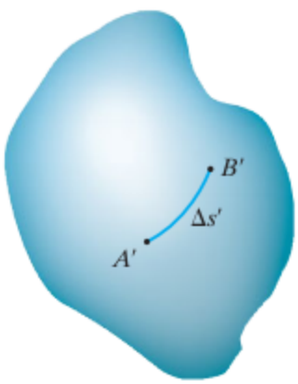
Note the before and after positions of three different line segments on this rubber membrane which is subjected to tension. The vertical line is lengthened, the horizontal line is shortened, and the inclined line changes its length and rotates.

7.9 Strain

In order to describe the deformation of a body by changes in length of line segments and the changes in the angles between them, we will develop the concept of strain. Strain is actually measured by experiments, and once the strain is obtained, it will be shown in the next chapter how it can be related to the stress acting within the body.



Undeformed body
(a)



Deformed body
(b)

Fig. 7–25

Normal Strain. If we define the normal strain as the change in length of a line per unit length, then we will not have to specify the *actual length* of any particular line segment. Consider, for example, the line AB, which is contained within the undeformed body shown in Fig. 7–25*a*. This line lies along the n axis and has an original length of Δs. After deformation, points A and B are displaced to A' and B', and the line becomes a curve having a length of $\Delta s'$, Fig. 7–25*b*. The change in length of the line is therefore $\Delta s' - \Delta s$. If we define the *average normal strain* using the symbol ϵ_{avg} (epsilon), then

$$\epsilon_{avg} = \frac{\Delta s' - \Delta s}{\Delta s} \tag{7–10}$$

As point B is chosen closer and closer to point A, the length of the line will become shorter and shorter, such that $\Delta s \to 0$. Also, this causes B' to approach A', such that $\Delta s' \to 0$. Consequently, in the limit the normal strain at point A and in the direction of n is

$$\epsilon = \lim_{B \to A \text{ along } n} \frac{\Delta s' - \Delta s}{\Delta s} \tag{7–11}$$

Hence, when ϵ (or ϵ_{avg}) is positive the initial line will elongate, whereas if ϵ is negative the line contracts.

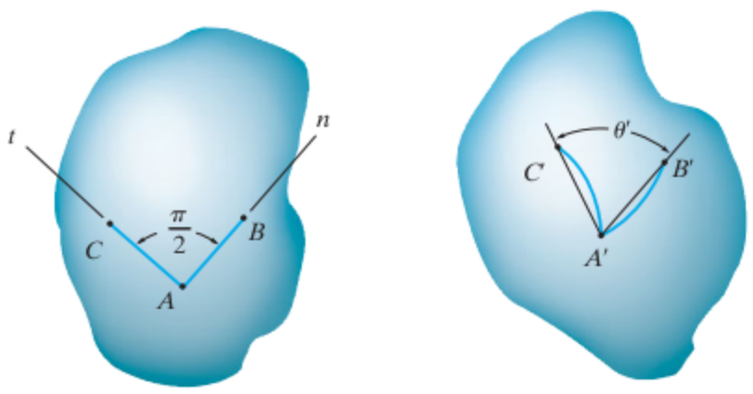
Note that normal strain is a *dimensionless quantity*, since it is a ratio of two lengths. Although this is the case, it is sometimes stated in terms of a ratio of length units. If the SI system is used, then the basic unit for length is the meter (m). Ordinarily, for most engineering applications ϵ will be very small, so measurements of strain are in micrometers per meter (μm/m), where $1\ \mu\text{m} = 10^{-6}\ \text{m}$. In the Foot-Pound-Second system, strain is often stated in units of inches per inch (in./in.). Sometimes

for experimental work, strain is expressed as a percent, e.g., 0.001 m/m = 0.1%. As an example, a normal strain of $480(10^{-6})$ can be reported as $480(10^{-6})$ in./in., 480 μm/m, or 0.0480%. Also, one can state this answer as simply 480 μ (480 "micros").

Shear Strain. Deformations not only cause line segments to elongate or contract, but they also cause them to change direction. If we select two line segments that are originally perpendicular to one another, then the change in angle that occurs between these two line segments is referred to as ***shear strain***. This angle is denoted by γ (gamma) and is always measured in radians (rad), which are dimensionless. For example, consider the line segments AB and AC originating from the same point A in a body, and directed along the perpendicular n and t axes, Fig. 7–26*a*. After deformation, the ends of both lines are displaced, and the lines themselves become curves, such that the angle between them at A is θ', Fig. 7–26*b*. Hence the shear strain at point A associated with the n and t axes becomes

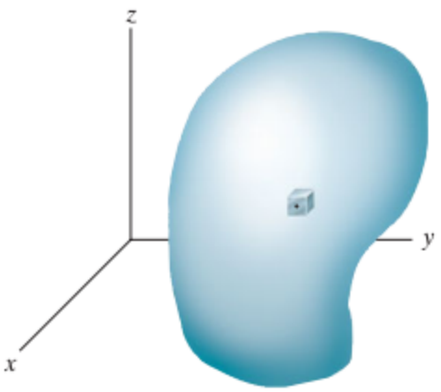
$$\gamma_{nt} = \frac{\pi}{2} - \lim_{\substack{B \to A \text{ along } n \\ C \to A \text{ along } t}} \theta' \tag{7–12}$$

Notice that if θ' is smaller than $\pi/2$ the shear strain is positive, whereas if θ' is larger than $\pi/2$ the shear strain is negative.



Undeformed body (a) Deformed body (b)

Fig. 7–26



(a)

Cartesian Strain Components. Using the definitions of normal and shear strain, we will now show how they can be used to describe the deformation of the body in Fig. 7–27*a*. To do so, imagine the body is subdivided into small elements such as the one shown in Fig. 7–27*b*. This element is rectangular, has undeformed dimensions Δx, Δy, and Δz, and is located in the neighborhood of a point in the body, Fig. 7–27*a*. If the element's dimensions are very small, then its deformed shape will be a parallelepiped, Fig. 7–27*c*, since very small line segments will remain approximately straight after the body is deformed. In order to achieve this deformed shape, we will first consider how the normal strain changes the lengths of the sides of the rectangular element, and then how the shear strain changes the angles of each side. For example, Δx elongates $\epsilon_x \Delta x$, so its new length is $\Delta x + \epsilon_x \Delta x$. Therefore, the approximate lengths of the three sides of the parallelepiped are

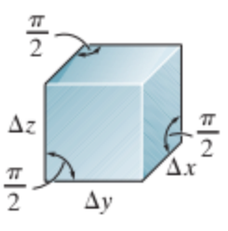
$$(1 + \epsilon_x)\,\Delta x \qquad (1 + \epsilon_y)\,\Delta y \qquad (1 + \epsilon_z)\,\Delta z$$

And the approximate angles between these sides are

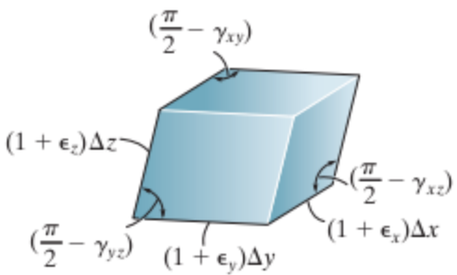
$$\frac{\pi}{2} - \gamma_{xy} \qquad \frac{\pi}{2} - \gamma_{yz} \qquad \frac{\pi}{2} - \gamma_{xz}$$

Notice that the ***normal strains cause a change in volume*** of the element, whereas the ***shear strains cause a change in its shape***. Of course, both of these effects occur simultaneously during the deformation.

In summary, then, the *state of strain* at a point in a body requires specifying three normal strains, ϵ_x, ϵ_y, ϵ_z, and three shear strains, γ_{xy}, γ_{yz}, γ_{xz}. These strains completely describe the deformation of a rectangular volume element of material located at the point and oriented so that its sides are originally parallel to the x, y, z axes. Provided these strains are defined at all points in the body, then the deformed shape of the body can be determined.



Undeformed element
(b)



Deformed element
(c)

Fig. 7–27

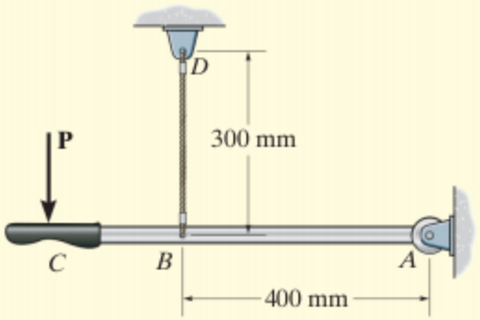
Small Strain Analysis. Most engineering design involves applications for which only *small deformations* are allowed. In this text, therefore, we will assume that the deformations that take place within a body are almost infinitesimal. In particular, the *normal strains* occurring within the material are *very small* compared to 1, so that $\epsilon \ll 1$. This assumption has wide practical application in engineering, and it is often referred to as a *small strain analysis*. It can be used, for example, to approximate $\sin\theta = \theta$, $\cos\theta = 1$, and $\tan\theta = \theta$, provided θ is very small.

The rubber bearing support under this concrete bridge girder is subjected to both normal strain caused by the weight and bridge loads on the girder, and shear strain caused by the horizontal movement of the girder due to temperature changes.

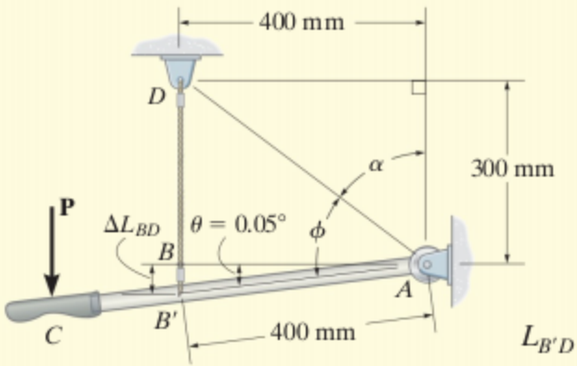
Important Points

- Loads will cause all material bodies to deform and, as a result, points in a body will undergo *displacements or changes in position.*
- *Normal strain* is a measure per unit length of the elongation or contraction of a small line segment in the body, whereas *shear strain* is a measure of the change in angle that occurs between two small line segments that are originally perpendicular to one another.
- The state of strain at a point is characterized by six strain components: three normal strains ϵ_x, ϵ_y, ϵ_z and three shear strains γ_{xy}, γ_{yz}, γ_{xz}. These components depend upon the original orientation of the line segments and their location in the body.
- Strain is the geometrical quantity that is measured using experimental techniques. Once obtained, the stress in the body can then be determined from material property relations, as discussed in the next chapter.
- Most engineering materials undergo very small deformations, and so the normal strain $\epsilon \ll 1$. This assumption of "small strain analysis" allows the calculations for normal strain to be simplified, since first-order approximations can be made about their size.

EXAMPLE 7.13



(a)



(b)

Fig. 7–28

When force **P** is applied to the rigid lever arm ABC in Fig. 7–28*a*, the arm rotates counterclockwise about pin A through an angle of 0.05°. Determine the normal strain developed in wire BD.

SOLUTION I

Geometry. The orientation of the lever arm after it rotates about point A is shown in Fig. 7–28*b*. From the geometry of this figure,

$$\alpha = \tan^{-1}\left(\frac{400\ \text{mm}}{300\ \text{mm}}\right) = 53.1301°$$

Then

$$\phi = 90° - \alpha + 0.05° = 90° - 53.1301° + 0.05° = 36.92°$$

For triangle ABD the Pythagorean theorem gives

$$L_{AD} = \sqrt{(300\ \text{mm})^2 + (400\ \text{mm})^2} = 500\ \text{mm}$$

Using this result and applying the law of cosines to triangle $AB'D$,

$$L_{B'D} = \sqrt{L_{AD}^2 + L_{AB'}^2 - 2(L_{AD})\,(L_{AB'})\cos\phi}$$

$$= \sqrt{(500\ \text{mm})^2 + (400\ \text{mm})^2 - 2(500\ \text{mm})\,(400\ \text{mm})\cos 36.92°}$$

$$= 300.3491\ \text{mm}$$

Normal Strain.

$$\epsilon_{BD} = \frac{L_{B'D} - L_{BD}}{L_{BD}} = \frac{300.3491\ \text{mm} - 300\ \text{mm}}{300\ \text{mm}} = 0.00116\ \text{mm/mm} \qquad \textit{Ans.}$$

SOLUTION II

Since the strain is small, this same result can be obtained by approximating the elongation of wire BD as ΔL_{BD}, shown in Fig. 7–28*b*. Here,

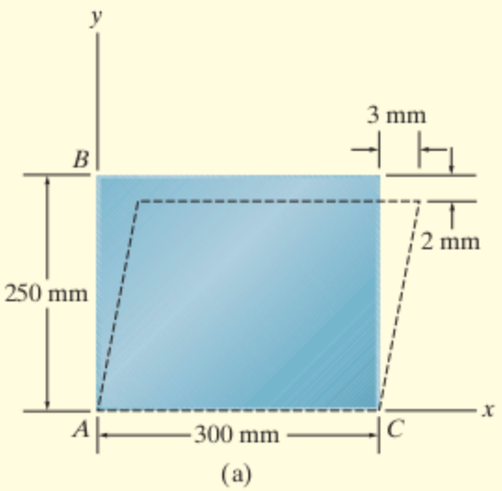
$$\Delta L_{BD} = \theta L_{AB} = \left[\left(\frac{0.05°}{180°}\right)(\pi\ \text{rad})\right](400\ \text{mm}) = 0.3491\ \text{mm}$$

Therefore,

$$\epsilon_{BD} = \frac{\Delta L_{BD}}{L_{BD}} = \frac{0.3491\ \text{mm}}{300\ \text{mm}} = 0.00116\ \text{mm/mm} \qquad \textit{Ans.}$$

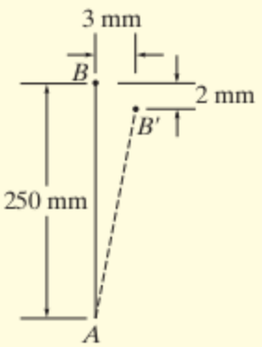
EXAMPLE 7.14

Due to a loading, the plate is deformed into the dashed shape shown in Fig. 7–29*a*. Determine (a) the average normal strain along the side AB, and (b) the average shear strain in the plate at A relative to the x and y axes.



(a)

Fig. 7–29



(b)

SOLUTION

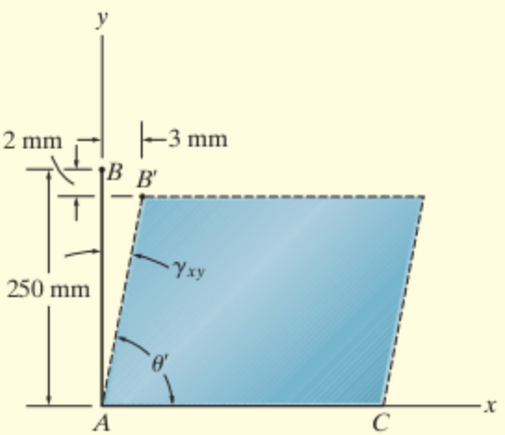
Part (a). Line AB, coincident with the y axis, becomes line AB' after deformation, as shown in Fig. 7–29*b*. The length of AB' is

$$AB' = \sqrt{(250\text{ mm} - 2\text{ mm})^2 + (3\text{ mm})^2} = 248.018\text{ mm}$$

The average normal strain for AB is therefore

$$(\epsilon_{AB})_{\text{avg}} = \frac{AB' - AB}{AB} = \frac{248.018\text{ mm} - 250\text{ mm}}{250\text{ mm}}$$
$$= -7.93(10^{-3})\text{ mm/mm} \qquad \textit{Ans.}$$

The negative sign indicates the strain causes a contraction of AB.



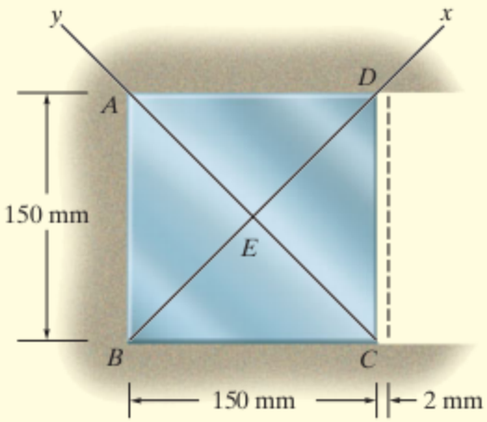
(c)

Part (b). As noted in Fig. 7–29*c*, the once 90° angle BAC between the sides of the plate at A changes to θ' due to the displacement of B to B'. Since $\gamma_{xy} = \pi/2 - \theta'$, then γ_{xy} is the angle shown in the figure. Thus,

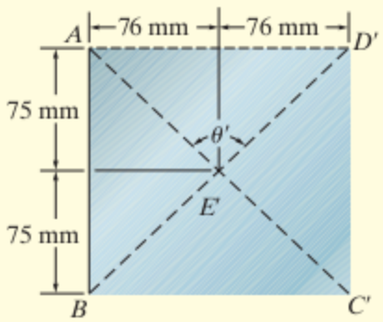
$$\gamma_{xy} = \tan^{-1}\left(\frac{3\text{ mm}}{250\text{ mm} - 2\text{ mm}}\right) = 0.0121\text{ rad} \qquad \textit{Ans.}$$

7

EXAMPLE 7.15



(a)



(b)

Fig. 7–30

The plate shown in Fig. 7–30*a* is fixed connected along AB and held in the horizontal guides at its top and bottom, AD and BC. If its right side CD is given a uniform horizontal displacement of 2 mm, determine (a) the average normal strain along the diagonal AC, and (b) the shear strain at E relative to the x, y axes.

SOLUTION

Part (a). When the plate is deformed, the diagonal AC becomes AC', Fig. 7–30*b*. The length of diagonals AC and AC' can be found from the Pythagorean theorem. We have

$$AC = \sqrt{(0.150\text{ m})^2 + (0.150\text{ m})^2} = 0.21213\text{ m}$$

$$AC' = \sqrt{(0.150\text{ m})^2 + (0.152\text{ m})^2} = 0.21355\text{ m}$$

Therefore the average normal strain along the diagonal is

$$(\epsilon_{AC})_{avg} = \frac{AC' - AC}{AC} = \frac{0.21355\text{ m} - 0.21213\text{ m}}{0.21213\text{ m}}$$

$$= 0.00669\text{ mm/mm} \qquad \textit{Ans.}$$

Part (b). To find the shear strain at E relative to the x and y axes, it is first necessary to find the angle θ' after deformation, Fig. 7–30*b*. We have

$$\tan\left(\frac{\theta'}{2}\right) = \frac{76\text{ mm}}{75\text{ mm}}$$

$$\theta' = 90.759^\circ = \left(\frac{\pi}{180^\circ}\right)(90.759^\circ) = 1.58404\text{ rad}$$

Applying Eq. 7–12, the shear strain at E is therefore

$$\gamma_{xy} = \frac{\pi}{2} - 1.58404\text{ rad} = -0.0132\text{ rad} \qquad \textit{Ans.}$$

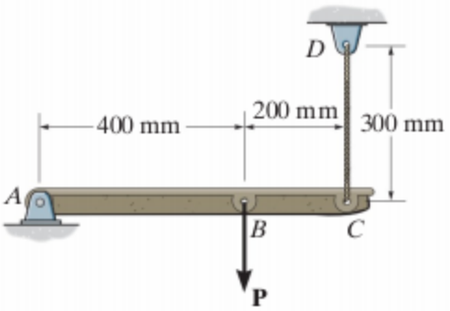
The *negative sign* indicates that the angle θ' is *greater than* 90°.

NOTE: If the x and y axes were horizontal and vertical at point E, then the 90° angle between these axes would not change due to the deformation, and so $\gamma_{xy} = 0$ at point E.

7

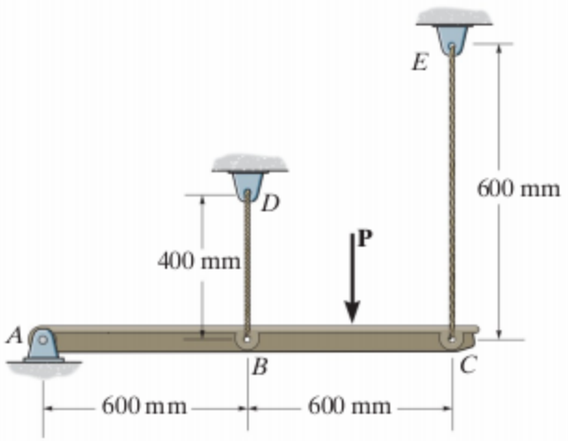
FUNDAMENTAL PROBLEMS

F7–25. When force **P** is applied to the rigid arm ABC, point B displaces vertically downward through a distance of 0.2 mm. Determine the normal strain developed in wire CD.



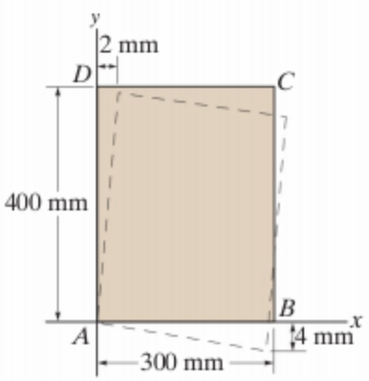
F7–25

F7–26. If the applied force **P** causes the rigid arm ABC to rotate clockwise about pin A through an angle of 0.02°, determine the normal strain developed in wires BD and CE.



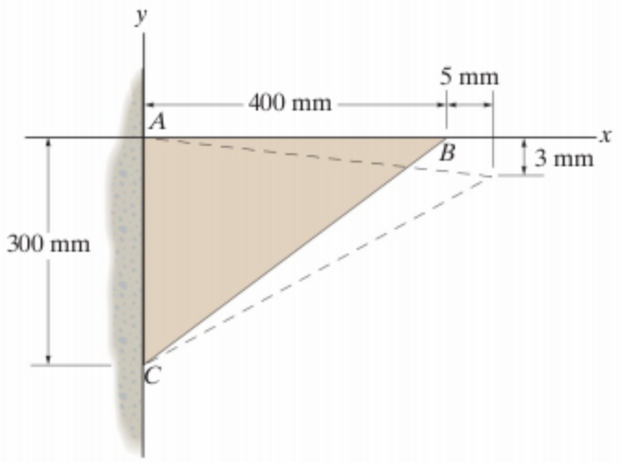
F7–26

F7–27. The rectangular plate is deformed into the shape of a rhombus shown by the dashed line. Determine the average shear strain at corner A with respect to the x and y axes.



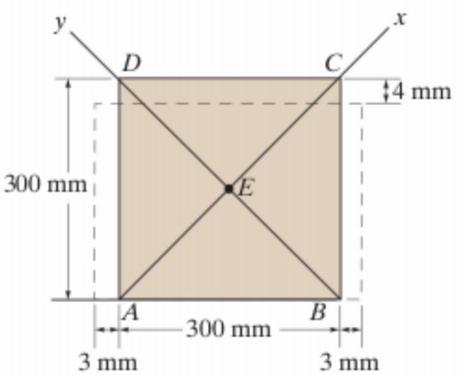
F7–27

F7–28. The triangular plate is deformed into the shape shown by the dashed line. Determine the normal strain developed along edge BC and the average shear strain at corner A with respect to the x and y axes.



F7–28

F7–29. The square plate is deformed into the shape shown by the dashed line. Determine the average normal strain along diagonal AC and the shear strain of point E with respect to the x and y axes.



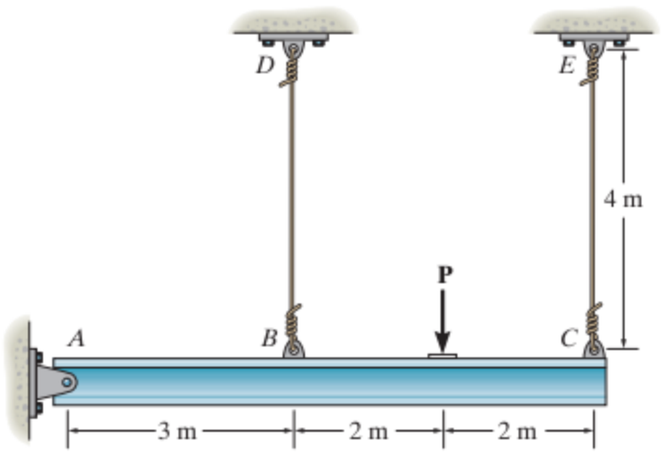
F7–29

PROBLEMS

***7–80.** An air-filled rubber ball has a diameter of 6 in. If the air pressure within it is increased until the ball's diameter becomes 7 in., determine the average normal strain in the rubber.

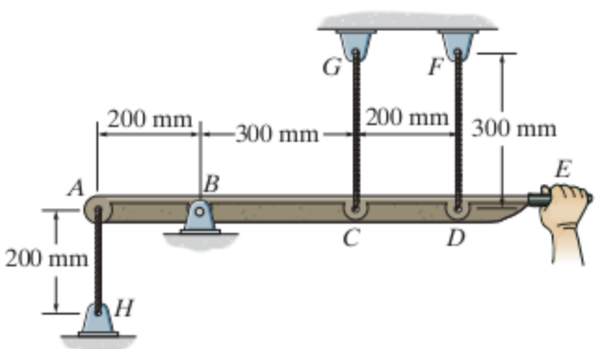
7–81. A thin strip of rubber has an unstretched length of 15 in. If it is stretched around a pipe having an outer diameter of 5 in., determine the average normal strain in the strip.

7–82. The rigid beam is supported by a pin at A and wires BD and CE. If the load **P** on the beam causes the end C to be displaced 10 mm downward, determine the normal strain developed in wires CE and BD.



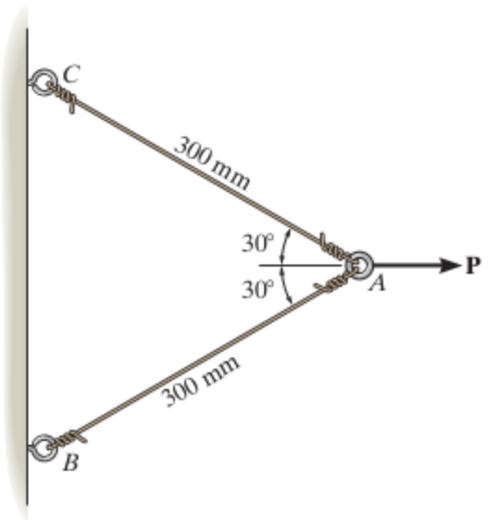
Prob. 7–82

7–83. The force applied at the handle of the rigid lever causes the lever to rotate clockwise about the pin B through an angle of 2°. Determine the average normal strain developed in each wire. The wires are unstretched when the lever is in the horizontal position.



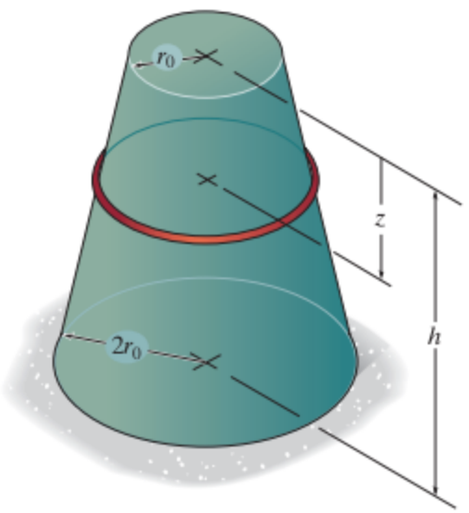
Prob. 7–83

***7–84.** The two wires are connected together at A. If the force **P** causes point A to be displaced horizontally 2 mm, determine the normal strain developed in each wire.



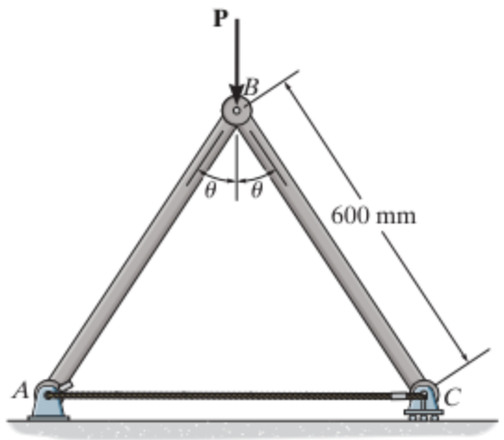
Prob. 7–84

7–85. The rubber band of unstretched length $2r_0$ is forced down the frustum of the cone. Determine the average normal strain in the band as a function of z.



Prob. 7–85

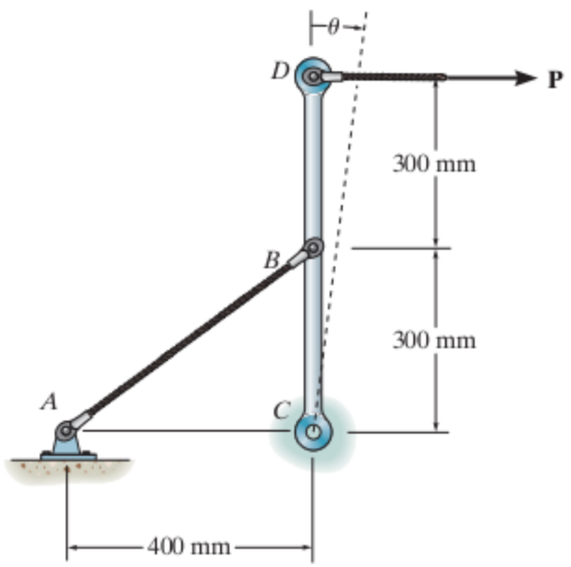
7–86. The pin-connected rigid rods AB and BC are inclined at $\theta = 30°$ when they are unloaded. When the force **P** is applied θ becomes 30.2°. Determine the average normal strain developed in wire AC.



Prob. 7–86

7–87. Part of a control linkage for an airplane consists of a rigid member CBD and a flexible cable AB. If a force is applied to the end D of the member and causes it to rotate by $\theta = 0.3°$, determine the normal strain in the cable. Originally the cable is unstretched.

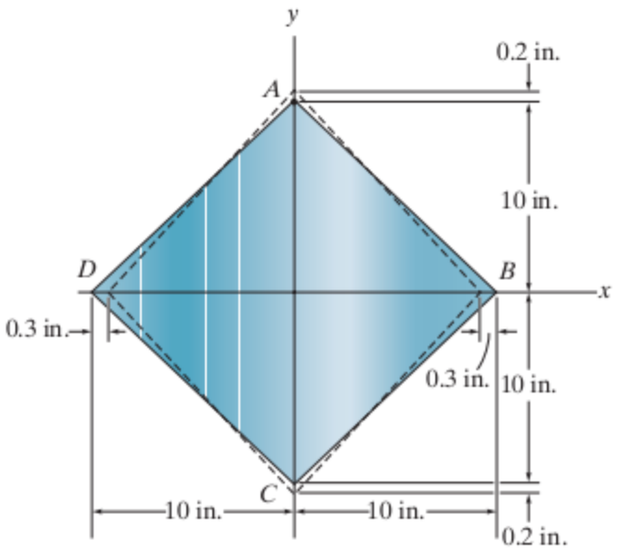
*__7–88.__ Part of a control linkage for an airplane consists of a rigid member CBD and a flexible cable AB. If a force is applied to the end D of the member and causes a normal strain in the cable of 0.0035 mm/mm, determine the displacement of point D. Originally the cable is unstretched.



Probs. 7–87/88

7–89. The corners of the square plate are given the displacements indicated. Determine the shear strain along the edges of the plate at A and B.

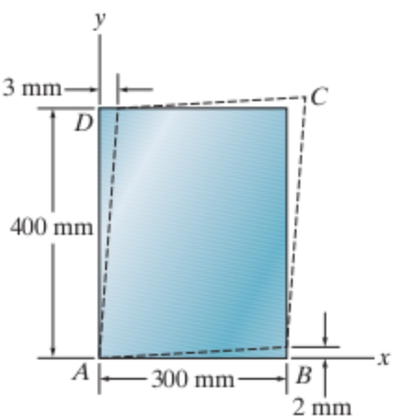
7–90. The corners of the square plate are given the displacements indicated. Determine the average normal strains along side AB and diagonals AC and DB.



Probs. 7–89/90

7–91. The piece of rubber is originally rectangular. Determine the average shear strain γ_{xy} at A if the corners B and D are subjected to the displacements that cause the rubber to distort as shown by the dashed lines.

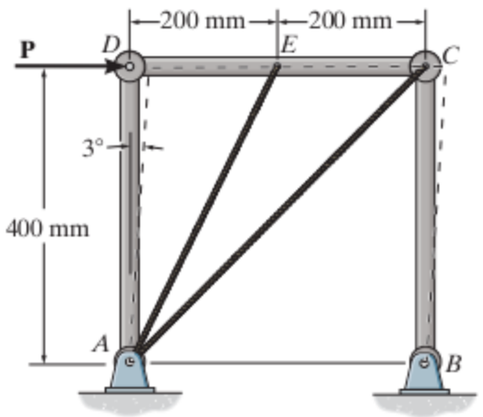
*__7–92.__ The piece of rubber is originally rectangular and subjected to the deformation shown by the dashed lines. Determine the average normal strain along the diagonal DB and side AD.



Probs. 7–91/92

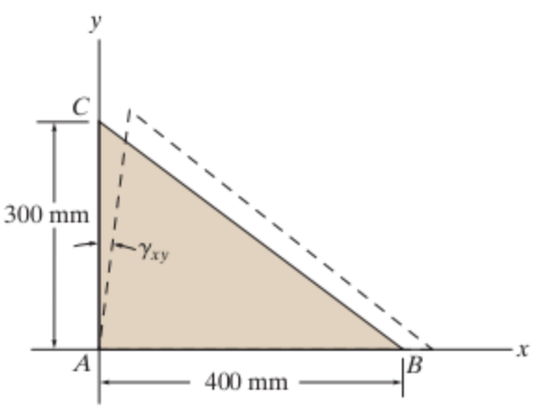
7–93. The force **P** applied at joint D of the square frame causes the frame to sway and form the dashed rhombus. Determine the average normal strain developed in wire AC. Assume the three rods are rigid.

7–94. The force **P** applied at joint D of the square frame causes the frame to sway and form the dashed rhombus. Determine the average normal strain developed in wire AE. Assume the three rods are regular.



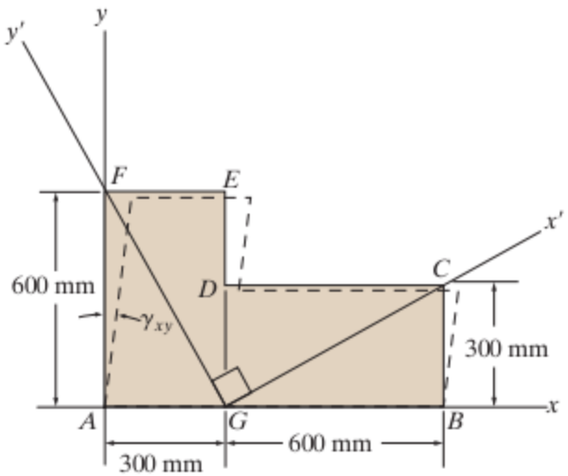
Probs. 7–93/94

7–95. The triangular plate ABC is deformed into the shape shown by the dashed lines. If at A, $\epsilon_{AB} = 0.0075$, $\epsilon_{AC} = 0.01$ and $\gamma_{xy} = 0.005$ rad, determine the average normal strain along edge BC.



Prob. 7–95

*7–96. The plate is deformed uniformly into the shape shown by the dashed lines. If at A, $\gamma_{xy} = 0.0075$ rad., while $\epsilon_{AB} = \epsilon_{AF} = 0$, determine the average shear strain at point G with respect to the x' and y' axes.

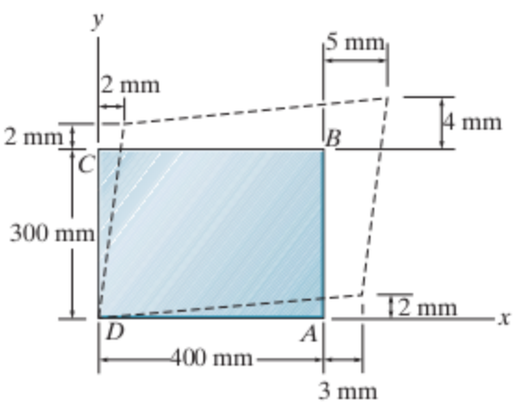


Prob. 7–96

7–97. The piece of plastic is originally rectangular. Determine the shear strain γ_{xy} at corners A and B if the plastic distorts as shown by the dashed lines.

7–98. The piece of plastic is originally rectangular. Determine the shear strain γ_{xy} at corners D and C if the plastic distorts as shown by the dashed lines.

7–99. The piece of plastic is originally rectangular. Determine the average normal strain that occurs along the diagonals AC and DB.



Probs. 7–97/98/99

CONCEPTUAL PROBLEMS

P7–1. Here hurricane winds caused the failure of this highway sign. Assuming the wind creates a uniform pressure on the sign of 2 kPa, use reasonable dimensions for the sign and determine the resultant shear and moment at the two connections where the failure occurred.

P7–1

P7–2. The two structural tubes are connected by the pin which passes through them. If the vertical load being supported is 100 kN, draw a free-body diagram of the pin and then use the method of sections to find the maximum average shear force in the pin. If the pin has a diameter of 50 mm, what is the maximum average shear stress in the pin?

P7–2

P7–3. The hydraulic cylinder H applies a horizontal force F on the pin at A. Draw the free-body diagram of the pin and show the forces acting on it. Using the method of sections, explain why the average shear stress in the pin is largest at sections through the gaps D and E and not at some intermediate section.

P7–3

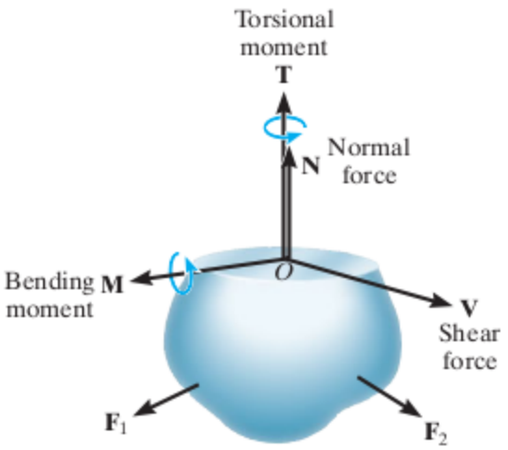
P7–4. The vertical load on the hook is 1000 lb. Draw the appropriate free-body diagrams and determine the maximum average shear force on the pins at A, B, and C. Note that due to symmetry four wheels are used to support the loading on the railing.

P7–4

CHAPTER REVIEW

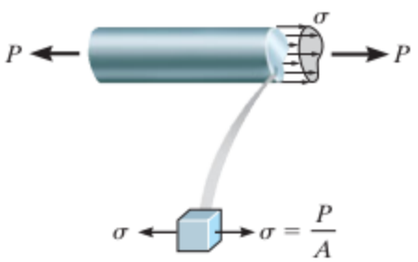
The internal loadings in a body consist of a normal force, shear force, bending moment, and torsional moment. They represent the resultants of both a normal and shear stress distribution that acts over the cross section. To obtain these resultants, use the method of sections and the equations of equilibrium.

$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$
$$\Sigma F_z = 0$$
$$\Sigma M_x = 0$$
$$\Sigma M_y = 0$$
$$\Sigma M_z = 0$$



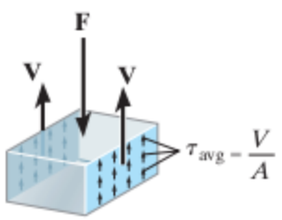
If a bar is made from homogeneous isotropic material and it is subjected to a series of external axial loads that pass through the centroid of the cross section, then a uniform normal stress distribution will act over the cross section. This average normal stress can be determined from $\sigma = P/A$, where P is the internal axial load at the section.

$$\sigma = \frac{P}{A}$$



The average shear stress can be determined using $\tau_{avg} = V/A$, where V is the shear force acting on the cross-sectional area A. This formula is often used to find the average shear stress in fasteners or in parts used for connections.

$$\tau_{avg} = \frac{V}{A}$$

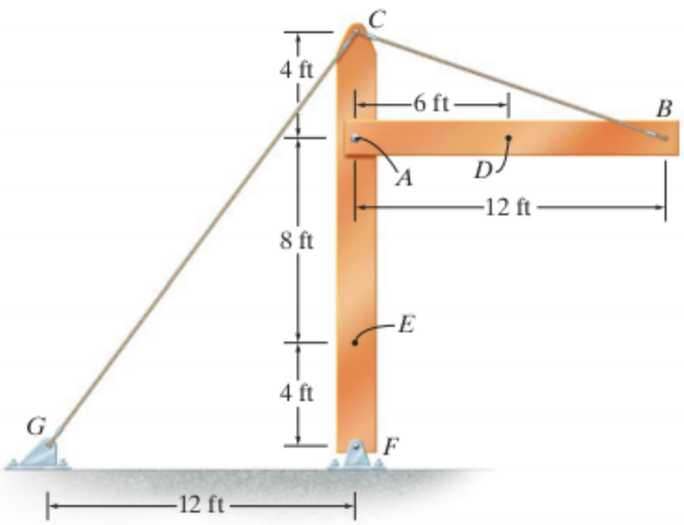


7

The design of any simple connection requires that the average stress along any cross section not exceed an allowable stress of σ_{allow} or τ_{allow}. These values are reported in codes and are considered safe on the basis of experiments or through experience. Sometimes a factor of safety is reported provided the ultimate stress is known. Deformation is defined as the change in the shape and size of a body. It causes line segments to change length and orientation.	$$\text{F.S.} = \frac{\sigma_{fail}}{\sigma_{allow}} = \frac{\tau_{fail}}{\tau_{allow}}$$	
Normal strain ϵ is the change in length per unit length of a line segment. If ϵ is positive, the line segment elongates. If it is negative, the line segment contracts. Shear strain γ is a measure of the change in angle made between two line segments that are originally perpendicular to one another. Strain is dimensionless, however, ϵ is sometimes reported in in-/in., mm/mm, and γ is in radians.	$$\epsilon_{avg} = \frac{\Delta s' - \Delta s}{\Delta s}$$ $$\gamma = \frac{\pi}{2} - \theta$$	

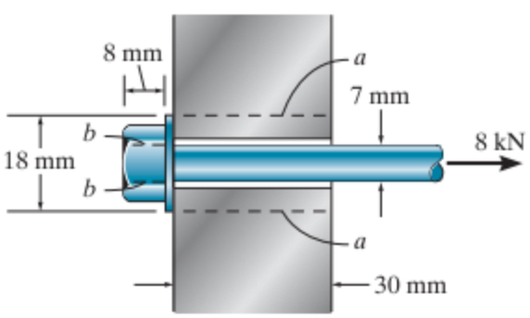
REVIEW PROBLEMS

***7–100.** The beam AB is pin supported at A and supported by a cable BC. A *separate* cable CG is used to hold up the frame. If AB weighs 120 lb/ft and the column FC has a weight of 180 lb/ft, determine the resultant internal loadings acting on cross sections located at points D and E. Neglect the thickness of both the beam and column in the calculation.



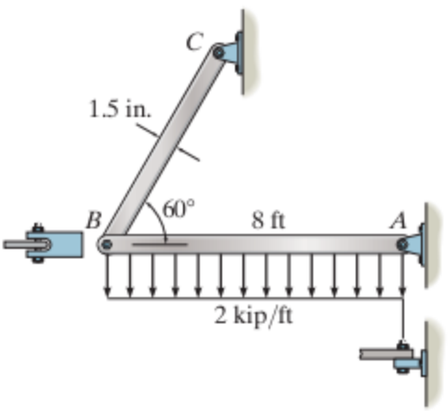
Prob. 7–100

7–101. The long bolt passes through the 30-mm-thick plate. If the force in the bolt shank is 8 kN, determine the average normal stress in the shank, the average shear stress along the cylindrical area of the plate defined by the section lines a–a, and the average shear stress in the bolt head along the cylindrical area defined by the section lines b–b.



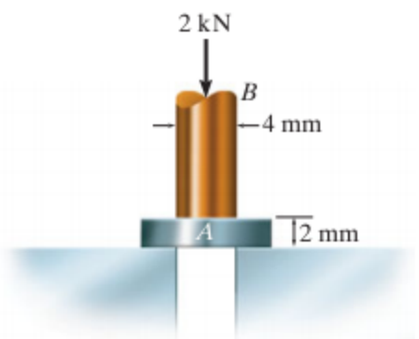
Prob. 7–101

7–102. Determine the required thickness of member BC and the diameter of the pins at A and B if the allowable normal stress for member BC is $\sigma_{allow} = 29$ ksi, and the allowable shear stress for the pins is $\tau_{allow} = 10$ ksi.



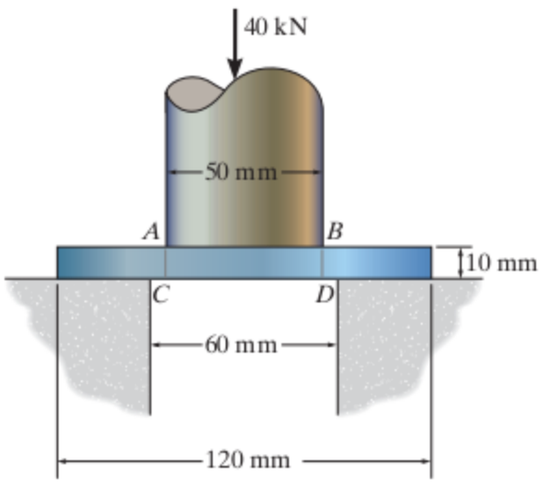
Prob. 7–102

7–103. The circular punch B exerts a force of 2 kN on the top of the plate A. Determine the average shear stress in the plate due to this loading.



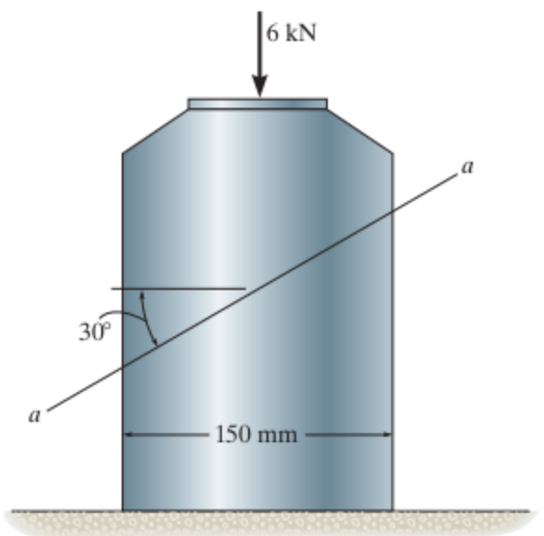
Prob. 7–103

*7–104. Determine the average punching shear stress the circular shaft creates in the metal plate through section AC and BD. Also, what is the bearing stress developed on the surface of the plate under the shaft?



Prob. 7–104

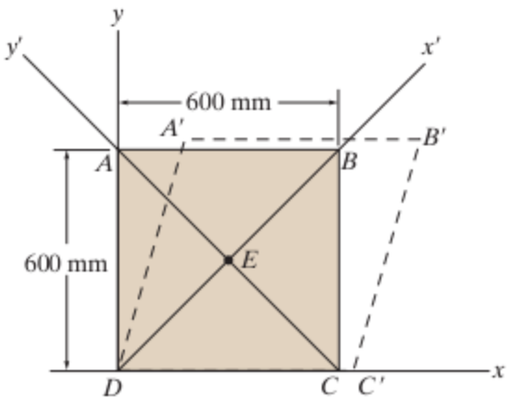
7–105. The bearing pad consists of a 150 mm by 150 mm block of aluminum that supports a compressive load of 6 kN. Determine the average normal and shear stress acting on the plane through section a–a. Show the results on a differential volume element located on the plane.



Prob. 7–105

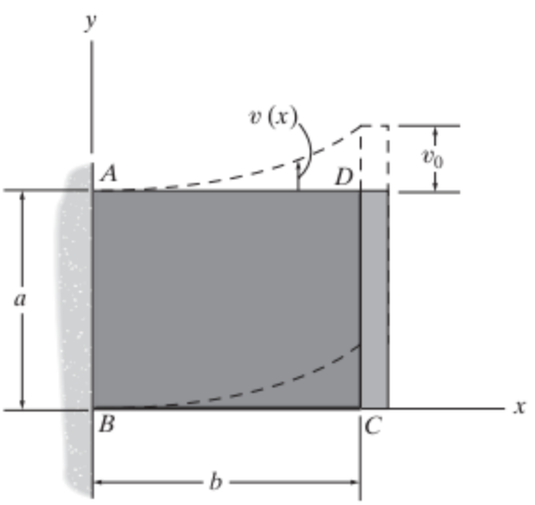
7–106. The square plate is deformed into the shape shown by the dashed lines. If DC has a normal strain $\epsilon_x = 0.004$, DA has a normal strain $\epsilon_y = 0.005$ and at D, $\gamma_{xy} = 0.02$ rad, determine the average normal strain along diagonal CA.

7–107. The square plate is deformed into the shape shown by the dashed lines. If DC has a normal strain $\epsilon_x = 0.004$, DA has a normal strain $\epsilon_y = 0.005$ and at D, $\gamma_{xy} = 0.02$ rad, determine the shear strain at point E with respect to the x' and y' axes.



Probs. 7–106/107

*7–108. The rubber block is fixed along edge AB, and edge CD is moved so that the vertical displacement of any point in the block is given by $v(x) = (v_0/b^3)\,x^3$. Determine the shear strain γ_{xy} at points $(b/2, a/2)$ and (b, a).



Prob. 7–108

Chapter 8

Horizontal ground displacements caused by an earthquake produced excessive strains in this bridge pier until it fractured. The material properties of the concrete and steel reinforcement must be known so that engineers can properly design this structure and thereby avoid such failures.

Mechanical Properties of Materials

CHAPTER OBJECTIVES

- To show how stress can be related to strain by using experimental methods to determine the stress–strain diagram for a particular material.
- To discuss the properties of the stress–strain diagram for materials commonly used in engineering.
- To discuss other mechanical properties and tests related to the development of the mechanics of materials.

8.1 The Tension and Compression Test

The strength of a material depends on its ability to sustain a load without undue deformation or failure. This property is inherent in the material itself and must be determined by *experiment*. One of the most important tests to perform in this regard is the ***tension*** or ***compression test***. Although several important mechanical properties of a material can be determined from this test, it is used primarily to determine the relationship between the average normal stress and average normal strain in many engineering materials such as metals, ceramics, polymers, and composites.

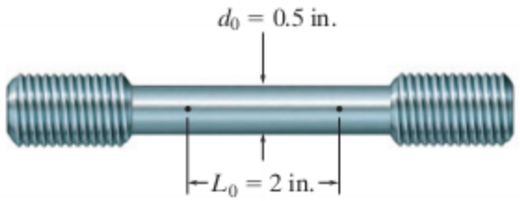
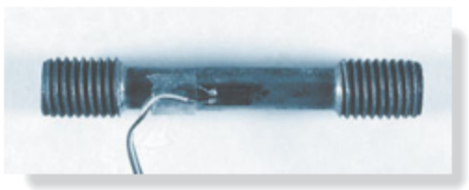


Fig. 8–1



Typical steel specimen with attached strain gauge.

To perform a tension or compression test a specimen of the material is made into a "standard" shape and size. It has a constant circular cross section with enlarged ends, so that failure will not occur at the grips. Before testing, two small punch marks are placed along the specimen's uniform length. Measurements are taken of both the specimen's initial cross-sectional area, A_0, and the ***gauge-length*** distance L_0 between the punch marks. For example, when a metal specimen is used in a tension test it generally has an initial diameter of $d_0 = 0.5$ in. (13 mm) and a gauge length of $L_0 = 2$ in. (50 mm), Fig. 8–1. In order to apply an axial load with no bending of the specimen, the ends are usually seated into ball-and-socket joints. A testing machine like the one shown in Fig. 8–2 is then used to stretch the specimen at a very slow, constant rate until it fails. The machine is designed to read the load required to maintain this uniform stretching.

At frequent intervals during the test, data is recorded of the applied load P, as read on the dial of the machine or taken from a digital readout. Also, the elongation $\delta = L - L_0$ between the punch marks on the specimen may be measured using either a caliper or a mechanical or optical device called an ***extensometer***. This value of δ (delta) is then used to calculate the average normal strain in the specimen. Sometimes, however, this measurement is not taken, since it is also possible to read the strain *directly* by using an ***electrical-resistance strain gauge***, which looks like the one shown in Fig. 8–3. The operation of this gauge is based on the change in electrical resistance of a very thin wire or piece of metal foil under strain. Essentially the gauge is cemented to the specimen along its length. If the cement is very strong in comparison to the gauge, then the gauge is in effect an integral part of the specimen, so that when the specimen is strained in the direction of the gauge, the wire and specimen will experience the same strain. By measuring the electrical resistance of the wire, the gauge may be calibrated to read values of normal strain directly.

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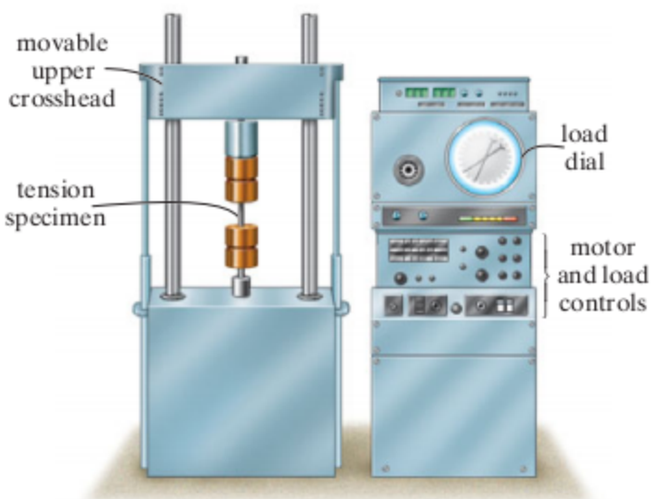
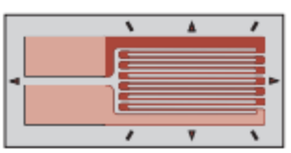


Fig. 8–2



Electrical–resistance strain gauge

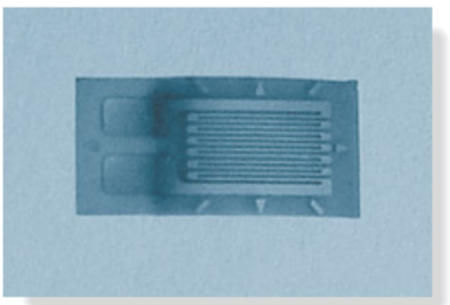


Fig. 8–3

8.2 The Stress–Strain Diagram

It is not feasible to prepare a test specimen to match the size, A_0 and L_0, of each structural member. Rather, the test results must be reported so they apply to a member of *any size*. To achieve this, the load and corresponding deformation data are used to calculate various values of the stress and corresponding strain in the specimen. A plot of the results produces a curve called the ***stress–strain diagram***. There are two ways in which it is normally described.

Conventional Stress–Strain Diagram. We can determine the ***nominal*** or ***engineering stress*** by dividing the applied load P by the specimen's *original* cross-sectional area A_0. This calculation assumes that the stress is *constant* over the cross section and throughout the gauge length. We have

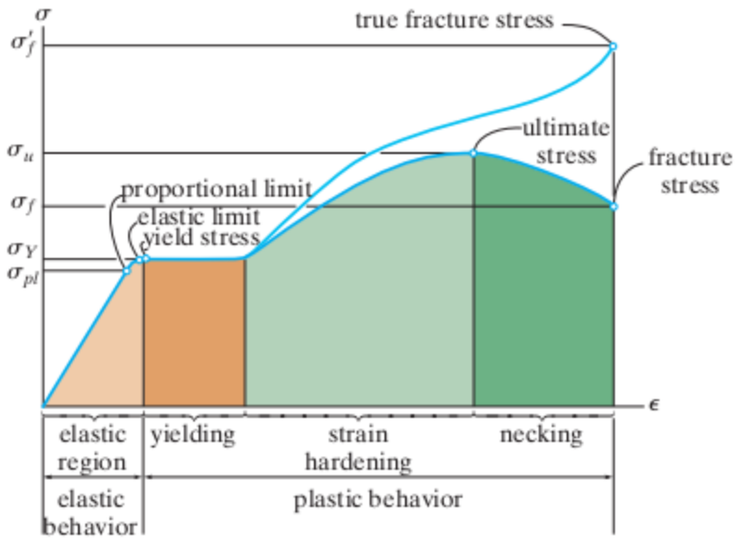
$$\sigma = \frac{P}{A_0} \tag{8–1}$$

Likewise, the ***nominal*** or ***engineering strain*** is found directly from the strain gauge reading, or by dividing the change in the specimen's gauge length, δ, by the specimen's original gauge length L_0. Here the strain is assumed to be constant throughout the region between the gauge points. Thus,

$$\epsilon = \frac{\delta}{L_0} \tag{8–2}$$

If the corresponding values of σ and ϵ are plotted so that the vertical axis is the stress and the horizontal axis is the strain, the resulting curve is called a ***conventional stress–strain diagram***. Realize, however, that two stress–strain diagrams for a particular material will be quite similar, but will never be exactly the same. This is because the results actually depend on variables such as the material's composition, microscopic imperfections, the way it is manufactured, the rate of loading, and the temperature during the time of the test.

We will now discuss the characteristics of the conventional stress–strain curve as it pertains to *steel*, a commonly used material for fabricating both structural members and mechanical elements. Using the method described above, the characteristic stress–strain diagram for a steel specimen is shown in Fig. 8–4. From this curve we can identify four different ways in which the material behaves, depending on the amount of strain induced in the material.



Conventional and true stress–strain diagrams for ductile material (steel) (not to scale)

Fig. 8–4

Elastic Behavior. Elastic behavior of the material occurs when the strains in the specimen are within the light orange region shown in Fig. 8–4. Here the curve is actually a *straight line* throughout most of this region, so that the stress is *proportional* to the strain. The material in this region is said to be *linear elastic*. The upper stress limit to this linear relationship is called the ***proportional limit***, σ_{pl}. If the stress slightly exceeds the proportional limit, the curve tends to bend and flatten out as shown. This continues until the stress reaches the ***elastic limit***. Upon reaching this point, if the load is removed the specimen will still return back to its original shape. Normally for steel, however, the elastic limit is seldom determined, since it is very close to the proportional limit and therefore rather difficult to detect.

Yielding. A slight increase in stress above the elastic limit will result in a breakdown of the material and cause it to *deform permanently*. This behavior is called ***yielding***, and it is indicated by the rectangular dark orange region of the curve. The stress that causes yielding is called the ***yield stress*** or ***yield point***, σ_Y, and the deformation that occurs is called ***plastic deformation***. Although not shown in Fig. 8–4, for low-carbon steels or those that are hot rolled, the yield point is often distinguished by two values. The ***upper yield point*** occurs first, followed by a sudden decrease in load-carrying capacity to a ***lower yield point***. Notice that once the yield point is reached, then as shown in Fig. 8–4, the specimen will continue to elongate (strain) *without any increase in load*. When the material is in this state, it is often referred to as being ***perfectly plastic***.

Strain Hardening. When yielding has ended, an increase in load can be supported by the specimen, resulting in a curve that rises continuously but becomes flatter until it reaches a maximum stress referred to as the ***ultimate stress***, σ_u. The rise in the curve in this manner is called ***strain hardening***, and it is identified in Fig. 8–4 as the region in light green.

Necking. Up to the ultimate stress, as the specimen elongates, its cross-sectional area will decrease. This decrease is fairly *uniform* over the specimen's entire gauge length; however, just after, at the ultimate stress, the cross-sectional area will begin to decrease in a *localized* region of the specimen. As a result, a constriction or "neck" tends to form in this region as the specimen elongates further, Fig. 8–5*a*. This region of the curve due to necking is indicated in dark green in Fig. 8–4. Here the stress–strain diagram tends to curve downward until the specimen breaks at the ***fracture stress***, σ_f, Fig. 8–5*b*.

True Stress–Strain Diagram. Instead of always using the *original* cross-sectional area and specimen length to calculate the (engineering) stress and strain, we could have used the *actual* cross-sectional area and specimen length at the *instant* the load is measured. The values of stress and strain found from these measurements are called *true stress* and *true strain*, and a plot of their values is called the ***true stress–strain diagram***. When this diagram is plotted, it has a form shown by the light-blue curve in Fig. 8–4. Note that the conventional and true σ–ϵ diagrams are practically coincident when the strain is small. The differences between the diagrams begin to appear in the strain-hardening range, where the magnitude of strain becomes more significant. In particular, there is a large divergence within the necking region. Here it can be seen from the conventional σ–ϵ diagram that the specimen *actually* supports a *decreasing load*, since A_0 is constant when calculating engineering stress, $\sigma = P/A_0$. However, from the true σ–ϵ diagram, the actual area A within the necking region is always decreasing until fracture, σ_f', and so the material actually sustains *increasing stress*, hence $\sigma = P/A$.

Necking

Failure of a ductile material

Fig. 8–5

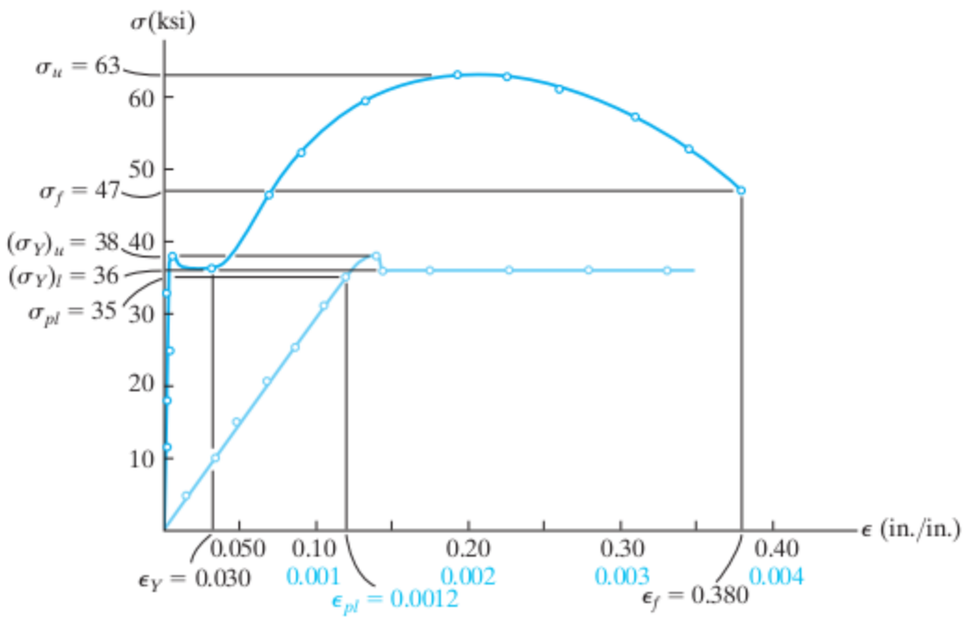
Typical necking pattern which has occurred on this steel specimen just before fracture.

This steel specimen clearly shows the necking that occurred just before the specimen failed. This resulted in the formation of a "cup-cone" shape at the fracture location, which is characteristic of ductile materials.

8

Although the true and conventional stress–strain diagrams are different, most engineering design is done so that the material supports a stress within the elastic range. This is because the deformation of the material is generally not severe and the material will restore itself when the load is removed. The true strain up to the elastic limit will remain small enough so that the error in using the engineering values of σ and ϵ is very small (about 0.1%) compared with their true values. This is one of the primary reasons for using conventional stress–strain diagrams.

The above concepts can be summarized with reference to Fig. 8–6, which shows an actual conventional stress–strain diagram for a mild steel specimen. In order to enhance the details, the elastic region of the curve has been shown in light blue color using an exaggerated strain scale, also shown in light blue. Tracing the behavior, the proportional limit is reached at $\sigma_{pl} = 35$ ksi (241 MPa), where $\epsilon_{pl} = 0.0012$ in./in. This is followed by an upper yield point of $(\sigma_Y)_u = 38$ ksi (262 MPa), then suddenly a lower yield point of $(\sigma_Y)_l = 36$ ksi (248 MPa). The end of yielding occurs at a strain of $\epsilon_Y = 0.030$ in./in., which is 25 times greater than the strain at the proportional limit! Continuing, the specimen undergoes strain hardening until it reaches the ultimate stress of $\sigma_u = 63$ ksi (434 MPa), then it begins to neck down until a fracture occurs, $\sigma_f = 47$ ksi (324 MPa). By comparison, the strain at failure, $\epsilon_f = 0.380$ in./in., is 317 times greater than ϵ_{pl}!



Stress-strain diagram for mild steel

Fig. 8–6

8.3 Stress–Strain Behavior of Ductile and Brittle Materials

Materials can be classified as either being ductile or brittle, depending on their stress–strain characteristics.

Ductile Materials. Any material that can be subjected to large strains before it fractures is called a ***ductile material***. Mild steel, as discussed previously, is a typical example. Engineers often choose ductile materials for design because these materials are capable of absorbing shock or energy, and if they become overloaded, they will usually exhibit large deformation before failing.

One way to specify the ductility of a material is to report its percent elongation or percent reduction in area at the time of fracture. The ***percent elongation*** is the specimen's fracture strain expressed as a percent. Thus, if the specimen's original gauge length is L_0 and its length at fracture is L_f, then

$$\text{Percent elongation} = \frac{L_f - L_0}{L_0}(100\%) \tag{8–3}$$

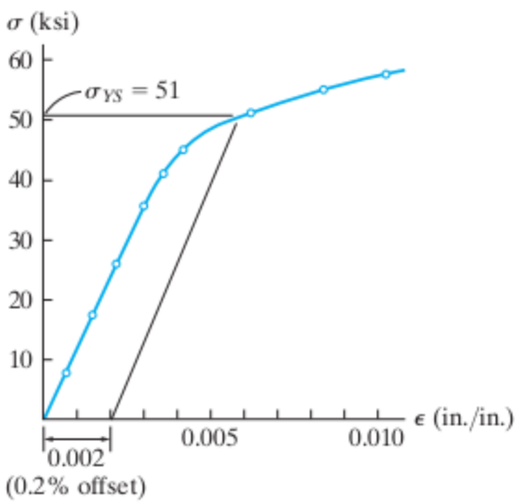
As seen in Fig. 8–6, since $\epsilon_f = 0.380$, this value would be 38% for a mild steel specimen.

The ***percent reduction in area*** is another way to specify ductility. It is defined within the region of necking as follows:

$$\text{Percent reduction of area} = \frac{A_0 - A_f}{A_0}(100\%) \tag{8–4}$$

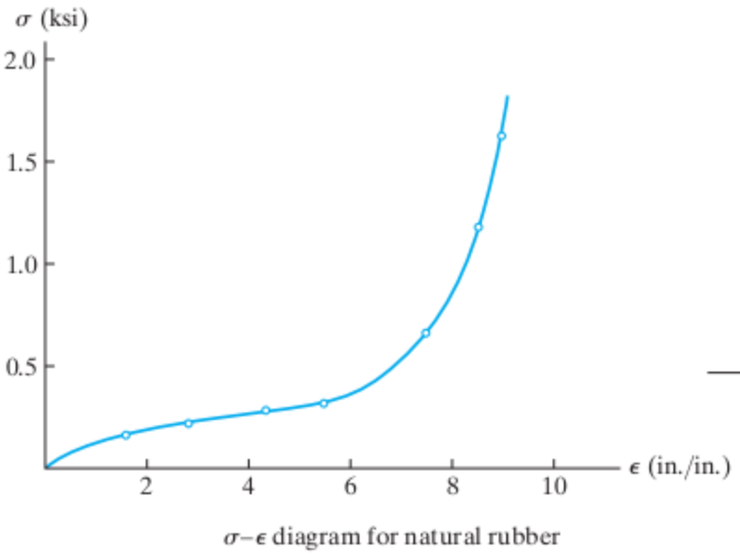
Where A_0 is the specimen's original cross-sectional area and A_f is the area of the neck at fracture. Mild steel has a typical value of 60%.

Besides steel, other metals such as brass, molybdenum, and zinc may also exhibit ductile stress–strain characteristics similar to steel, whereby they undergo elastic stress–strain behavior, yielding at constant stress, strain hardening, and finally necking until fracture. In most metals, however, constant yielding will *not occur* beyond the elastic range. One metal for which this is the case is aluminum. Actually, this metal often does not have a well-defined *yield point*, and consequently it is standard practice to define a ***yield strength*** using a graphical procedure called the ***offset method***. Normally, for structural design a 0.2% strain (0.002 in./in.) is chosen, and from this point on the ϵ axis, a line parallel to the initial straight-line portion of the stress–strain diagram is drawn. The point where this line intersects the curve defines the yield strength. An example of the construction for determining the yield strength for an aluminum alloy is shown in Fig. 8–7. From the graph, the yield strength is $\sigma_{YS} = 51$ ksi (352 MPa). Apart from metals, 0.2% strain is used as an offset to determine the yield strength of many plastics.



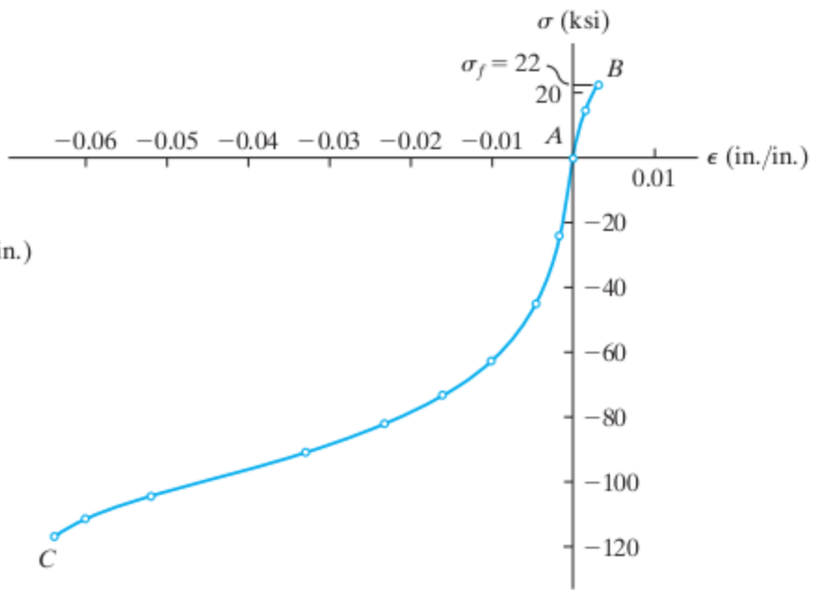
Yield strength for an aluminum alloy

Fig. 8–7



σ–ϵ diagram for natural rubber

Fig. 8–8



σ–ϵ diagram for gray cast iron

Fig. 8–9

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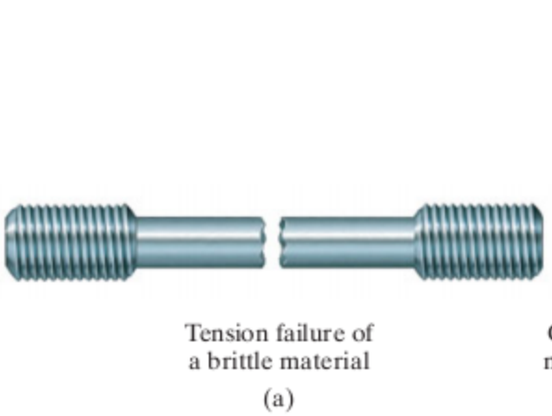
Concrete used for structural purposes must be routinely tested in compression to be sure it provides the necessary design strength for this bridge deck. The concrete cylinders shown are compression tested for ultimate stress after curing for 30 days.

Realize that the yield strength is not a physical property of the material, since it is a stress that causes a *specified* permanent strain in the material. In this text, however, we will assume that the yield strength, yield point, elastic limit, and proportional limit all *coincide* unless otherwise stated. An exception would be natural rubber, which in fact does not even have a proportional limit, since stress and strain are *not* linearly related. Instead, as shown in Fig. 8–8, this material, which is known as a polymer, exhibits *nonlinear elastic behavior*.

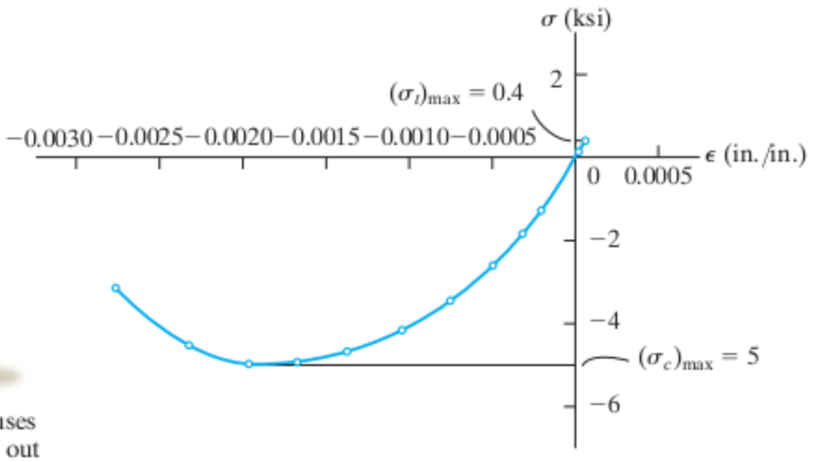
Wood is a material that is often moderately ductile, and as a result it is usually designed to respond only to elastic loadings. The strength characteristics of wood vary greatly from one species to another, and for each species they depend on the moisture content, age, and the size and arrangement of knots in the wood. Since wood is a fibrous material, its tensile or compressive characteristics will differ greatly when it is loaded either parallel or perpendicular to its grain. Specifically, wood splits easily when it is loaded in tension perpendicular to its grain, and consequently, tensile loads are almost always intended to be applied parallel to the grain of wood members.



Tension failure of a brittle material
(a)

Compression causes material to bulge out
(b)

Fig. 8–10



σ − ϵ diagram for typical concrete mix

Fig. 8–11

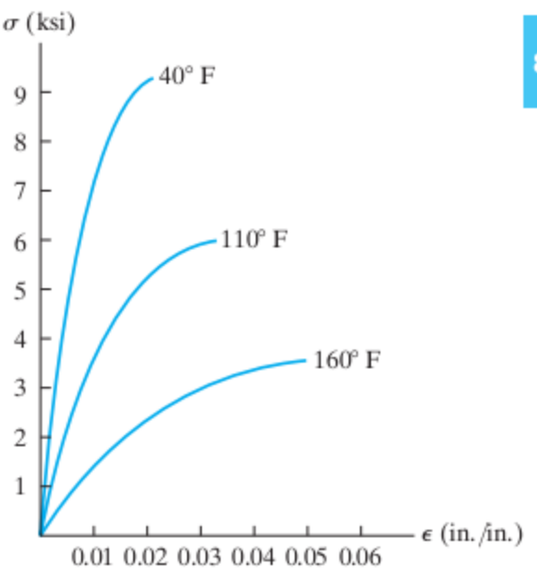
Brittle Materials. Materials that exhibit little or no yielding before failure are referred to as ***brittle materials***. Gray cast iron is an example, having a stress–strain diagram in tension as shown by portion AB of the curve in Fig. 8–9. Here fracture at $\sigma_f = 22$ ksi (152 MPa) took place initially at an imperfection or microscopic crack and then spread rapidly across the specimen, causing complete fracture. Since the appearance of initial cracks in a specimen is quite random, brittle materials do not have a well-defined tensile fracture stress. Instead the *average* fracture stress from a set of observed tests is generally reported. A typical failed specimen is shown in Fig. 8–10*a*.

Compared with their behavior in tension, brittle materials, such as gray cast iron, exhibit a much higher resistance to axial compression, as evidenced by portion AC of the curve in Fig. 8–9. For this case any cracks or imperfections in the specimen tend to close up, and as the load increases the material will generally bulge or become barrel shaped as the strains become larger, Fig. 8–10*b*.

Like gray cast iron, concrete is classified as a brittle material, and it also has a low strength capacity in tension. The characteristics of its stress–strain diagram depend primarily on the mix of concrete (water, sand, gravel, and cement) and the time and temperature of curing. A typical example of a "complete" stress–strain diagram for concrete is given in Fig. 8–11. By inspection, its maximum compressive strength is almost 12.5 times greater than its tensile strength, $(\sigma_c)_{max} = 5$ ksi (34.5 MPa) versus $(\sigma_t)_{max} = 0.40$ ksi (2.76 MPa). For this reason, concrete is almost always reinforced with steel bars or rods whenever it is designed to support tensile loads.

It can generally be stated that most materials exhibit both ductile and brittle behavior. For example, steel has brittle behavior when it contains a high carbon content, and it is ductile when the carbon content is reduced. Also, at low temperatures materials become harder and more brittle, whereas when the temperature rises they become softer and more ductile. This effect is shown in Fig. 8–12 for a methacrylate plastic.

Steel rapidly loses its strength when heated. For this reason engineers often require main structural members to be insulated in case of fire.



σ − ϵ diagrams for a methacrylate plastic

Fig. 8–12

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8.4 Hooke's Law

As noted in the previous section, the stress–strain diagrams for most engineering materials exhibit a *linear relationship* between stress and strain within the elastic region. Consequently, an increase in stress causes a proportionate increase in strain. This fact was discovered by Robert Hooke in 1676 using springs and is known as *Hooke's law*. It may be expressed mathematically as

$$\sigma = E\epsilon \tag{8–5}$$

Here E represents the constant of proportionality, which is called the ***modulus of elasticity*** or ***Young's modulus***, named after Thomas Young, who published an account of it in 1807.

Equation 8–5 actually represents the equation of the *initial straight-lined portion* of the stress–strain diagram up to the proportional limit. Furthermore, the modulus of elasticity represents the *slope* of this line. Since strain is dimensionless, from Eq. 8–5, E will have the same units as stress, such as psi, ksi, or pascals. As an example of its calculation, consider the stress–strain diagram for steel shown in Fig. 8–6. Here $\sigma_{pl} = 35$ ksi and $\epsilon_{pl} = 0.0012$ in./in. so that

$$E = \frac{\sigma_{pl}}{\epsilon_{pl}} = \frac{35 \text{ ksi}}{0.0012 \text{ in./in.}} = 29(10^3) \text{ ksi}$$

As shown in Fig. 8–13, the proportional limit for a particular type of steel alloy depends on its carbon content; however, most grades of steel, from the softest rolled steel to the hardest tool steel, have about the

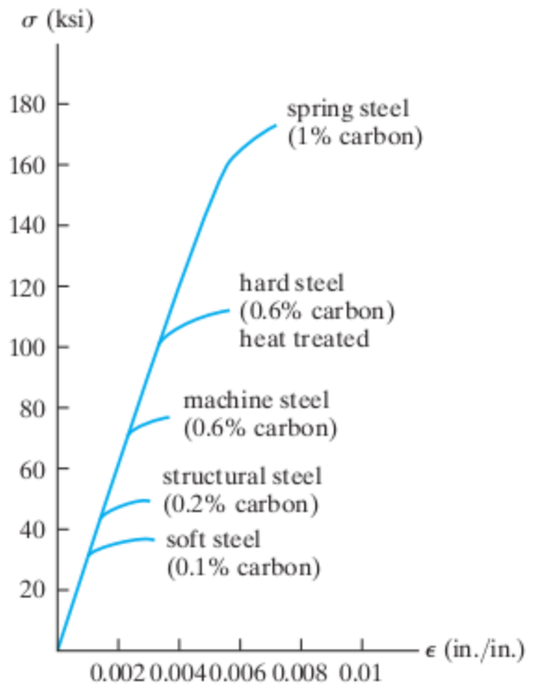


Fig. 8–13

same modulus of elasticity, generally accepted to be $E_{st} = 29(10^3)$ ksi or 200 GPa. Values of E for other commonly used engineering materials are often tabulated in engineering codes and reference books. Representative values are also listed on the inside back cover of this book. It should be noted that the modulus of elasticity is a mechanical property that indicates the *stiffness* of a material. Materials that are very stiff, such as steel, have large values of E [$E_{st} = 29(10^3)$ ksi or 200 GPa], whereas spongy materials such as vulcanized rubber may have low values [$E_r = 0.10$ ksi or 0.70 MPa].

The modulus of elasticity is one of the most important mechanical properties used in the development of equations presented in this text. It must always be remembered, though, that E can be used only if a material has *linear elastic behavior*. Also, if the stress in the material is *greater* than the proportional limit, the stress–strain diagram ceases to be a straight line and so Eq. 8–5 is no longer valid.

Strain Hardening. If a specimen of ductile material, such as steel, is loaded into the *plastic region* and then unloaded, *elastic strain is recovered* as the material returns to its equilibrium state. The *plastic strain remains*, however, and as a result the material is subjected to a ***permanent set***. For example, a wire when bent (plastically) will spring back a little (elastically) when the load is removed; however, it will not fully return to its original position. This behavior can be illustrated on the stress–strain diagram shown in Fig. 8–14*a*. Here the specimen is first loaded beyond its yield point A to point A'. Since interatomic forces have to be overcome to elongate the specimen *elastically*, then these same forces pull the atoms back together when the load is removed, Fig. 8–14*a*. Consequently, the modulus of elasticity, E, is the same, and therefore the slope of line $O'A'$ is the same as line OA.

If the load is reapplied, the atoms in the material will again be displaced until yielding occurs at or near the stress A', and the stress–strain diagram continues along the same path as before, Fig. 8–14*b*. It should be noted, however, that this new stress–strain diagram, defined by $O'A'B$, now has a *higher* yield point (A'), a consequence of strain-hardening. In other words, the material now has a *greater elastic region*; however, it has *less ductility*, a smaller plastic region, than when it was in its original state.

This pin was made from a hardened steel alloy, that is, one having a high carbon content. It failed due to brittle fracture.

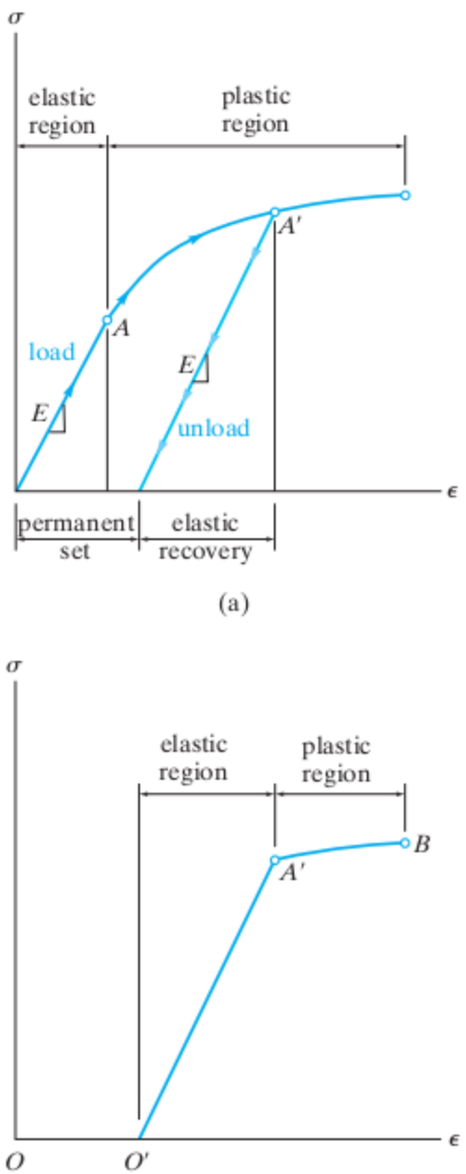


Fig. 8–14

8.5 Strain Energy

As a material is deformed by an external load, the load will do external work, which in turn will be stored in the material as internal energy. This energy is related to the strains in the material, and so it is referred to as ***strain energy***. To obtain this strain energy let us consider a volume element of material from a tension test specimen, Fig. 8–15. It is subjected to the uniaxial stress σ. This stress develops a force $\Delta F = \sigma\,\Delta A = \sigma(\Delta x\,\Delta y)$ on the top and bottom faces of the element *after* the element of length Δz undergoes a vertical displacement $\epsilon\,\Delta z$. By definition, the *work* of ΔF is determined by the product of the force and the displacement in the direction of the force. Since the force is increased uniformly from zero to its final magnitude ΔF when the displacement $\epsilon\,\Delta z$ occurs, the work done on the element by the force is then equal to the *average* force magnitude $(\Delta F/2)$ times the displacement $\epsilon\,\Delta z$. The conservation of energy requires this "external work" on the element to be equivalent to the "internal work" or strain energy stored in the element—assuming that no energy is lost in the form of heat. Consequently, the strain energy ΔU is $\Delta U = (\frac{1}{2}\Delta F)\,\epsilon\,\Delta z = (\frac{1}{2}\,\sigma\,\Delta x\,\Delta y)\,\epsilon\,\Delta z$. Since the volume of the element is $\Delta V = \Delta x\,\Delta y\,\Delta z$, then $\Delta U = \frac{1}{2}\sigma\epsilon\,\Delta V$.

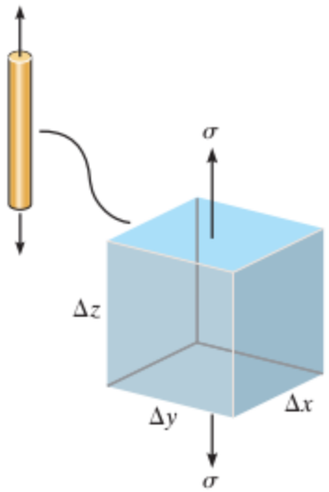


Fig. 8–15

For applications, it is often convenient to specify the strain energy per unit volume of material. This is called the ***strain-energy density***, and it can be expressed as

$$u = \frac{\Delta U}{\Delta V} = \frac{1}{2}\sigma\epsilon \tag{8–6}$$

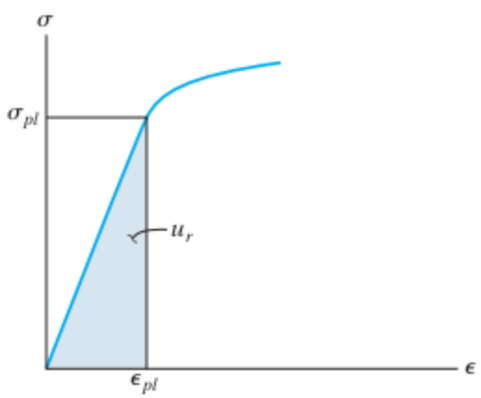
Finally, if the material behavior is *linear elastic*, then Hooke's law applies, $\sigma = E\epsilon$, and therefore we can express the *elastic strain-energy density* in terms of the uniaxial stress σ as

$$u = \frac{1}{2}\frac{\sigma^2}{E} \tag{8–7}$$

Modulus of Resilience. In particular, when the stress σ reaches the proportional limit, the strain-energy density, as calculated by Eq. 8–6 or 8–7, is referred to as the ***modulus of resilience***, i.e.,

$$u_r = \frac{1}{2}\sigma_{pl}\epsilon_{pl} = \frac{1}{2}\frac{\sigma_{pl}^2}{E} \tag{8–8}$$

From the elastic region of the stress–strain diagram, Fig. 8–16*a*, notice that u_r is equivalent to the shaded *triangular area* under the diagram. Physically, the modulus of resilience represents the amount of internal strain energy per unit volume the material can absorb without causing any permanent damage to the material. Certainly, this becomes important when designing bumpers or shock absorbers.

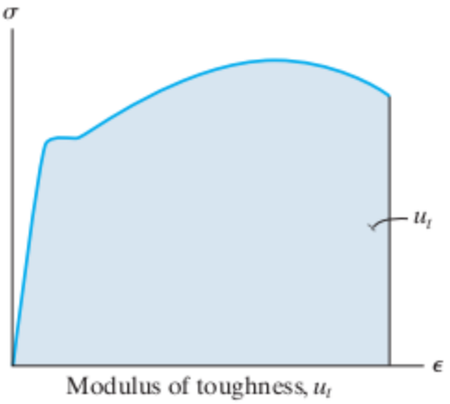


Modulus of resilience, u_r

(a)

Fig. 8–16

Modulus of Toughness. Another important property of a material is the ***modulus of toughness***, u_t. This quantity represents the *entire area* under the stress–strain diagram, Fig. 8–16*b*, and therefore it indicates the maximum amount of strain-energy the material can absorb just before it fractures. This property becomes important when designing members that may be accidentally overloaded. Note that alloying metals can also change their resilience and toughness. For example, by changing the percentage of carbon in steel, the resulting stress–strain diagrams in Fig. 8–17 show how the degrees of resilience (Fig. 8–16*a*) and toughness (Fig. 8–16*b*) can be changed.



Modulus of toughness, u_t

(b)

Fig. 8–16 (cont.)

Important Points

- A *conventional stress–strain diagram* is important in engineering since it provides a means for obtaining data about a material's tensile or compressive strength without regard for the material's physical size or shape.
- *Engineering stress and strain* are calculated using the *original* cross-sectional area and gauge length of the specimen.
- A *ductile material*, such as mild steel, has four distinct behaviors as it is loaded. They are *elastic behavior, yielding, strain hardening,* and *necking*.
- A material is *linear elastic* if the stress is proportional to the strain within the elastic region. This behavior is described by *Hooke's law*, $\sigma = E\epsilon$, where the *modulus of elasticity* E is the slope of the line.
- Important points on the stress–strain diagram are the *proportional limit, elastic limit, yield stress, ultimate stress*, and *fracture stress*.
- The *ductility* of a material can be specified by the specimen's *percent elongation* or the *percent reduction in area*.
- If a material does not have a distinct yield point, a *yield strength* can be specified using a graphical procedure such as the *offset method*.
- *Brittle materials*, such as gray cast iron, have very little or no yielding and so they can fracture suddenly.
- *Strain hardening* is used to establish a higher yield point for a material. This is done by straining the material beyond the elastic limit, then releasing the load. The modulus of elasticity remains the same; however, the material's ductility *decreases*.
- *Strain energy* is energy stored in a material due to its deformation. This energy per unit volume is called *strain-energy density*. If it is measured up to the proportional limit it is referred to as the *modulus of resilience*, and if it is measured up to the point of fracture it is called the *modulus of toughness*. It can be determined from the area under the $\sigma-\epsilon$ diagram.

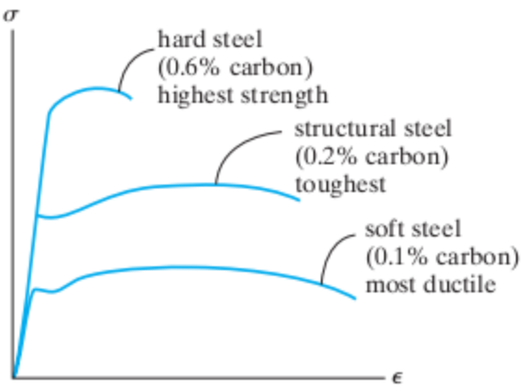


Fig. 8–17

This nylon specimen exhibits a high degree of toughness as noted by the large amount of necking that has occurred just before fracture.

8

EXAMPLE 8.1

A tension test for a steel alloy results in the stress–strain diagram shown in Fig. 8–18. Calculate the modulus of elasticity and the yield strength based on a 0.2% offset. Identify on the graph the ultimate stress and the fracture stress.

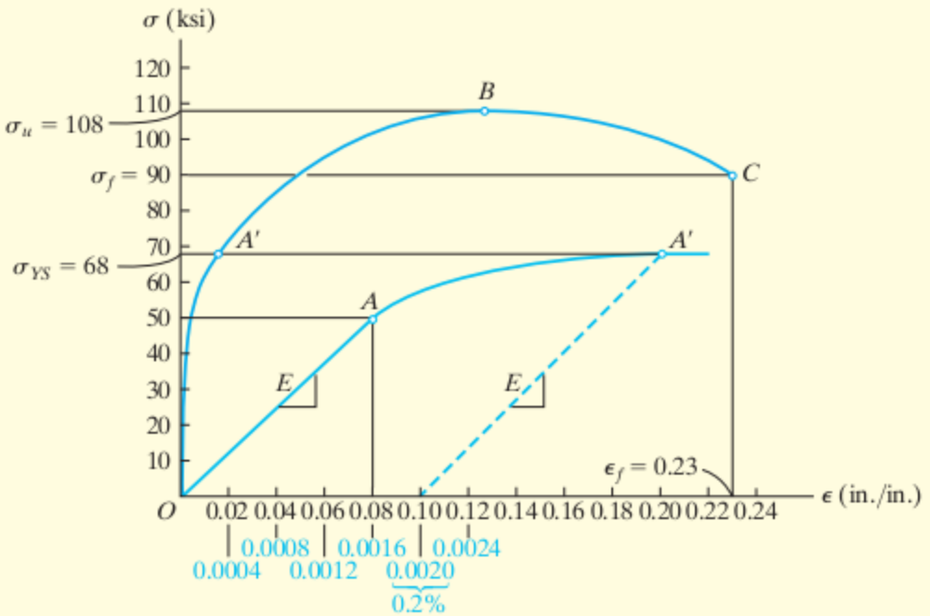


Fig. 8–18

SOLUTION

Modulus of Elasticity. We must calculate the *slope* of the initial straight-line portion of the graph. Using the magnified curve and scale shown in blue, this line extends from point O to an estimated point A, which has coordinates of approximately (0.0016 in./in., 50 ksi). Therefore,

$$E = \frac{50 \text{ ksi}}{0.0016 \text{ in./in.}} = 31.2(10^3) \text{ ksi} \qquad \textit{Ans.}$$

Note that the equation of line OA is thus $\sigma = 31.2(10^3)\epsilon$.

Yield Strength. For a 0.2% offset, we begin at a strain of 0.2% or 0.0020 in./in. and graphically extend a (dashed) line parallel to OA until it intersects the σ–ϵ curve at A'. The yield strength is approximately

$$\sigma_{YS} = 68 \text{ ksi} \qquad \textit{Ans.}$$

Ultimate Stress. This is defined by the peak of the σ–ϵ graph, point B in Fig. 8–18.

$$\sigma_u = 108 \text{ ksi} \qquad \textit{Ans.}$$

Fracture Stress. When the specimen is strained to its maximum of $\epsilon_f = 0.23$ in./in. it fractures at point C. Thus,

$$\sigma_f = 90 \text{ ksi} \qquad \textit{Ans.}$$

EXAMPLE 8.2

The stress–strain diagram for an aluminum alloy that is used for making aircraft parts is shown in Fig. 8–19. If a specimen of this material is stressed to 600 MPa, determine the permanent strain that remains in the specimen when the load is released. Also, find the modulus of resilience both before and after the load application.

SOLUTION

Permanent Strain. When the specimen is subjected to the load, it strain-hardens until point B is reached on the σ–ϵ diagram. The strain at this point is approximately 0.023 mm/mm. When the load is released, the material behaves by following the straight line BC, which is parallel to line OA. Since both lines have the same slope, the strain at point C can be determined analytically. The slope of line OA is the modulus of elasticity, i.e.,

$$E = \frac{450\text{ MPa}}{0.006\text{ mm/mm}} = 75.0\text{ GPa}$$

From triangle CBD, we require

$$E = \frac{BD}{CD}; \qquad 75.0(10^9)\text{ Pa} = \frac{600(10^6)\text{ Pa}}{CD}$$

$$CD = 0.008\text{ mm/mm}$$

This strain represents the amount of *recovered elastic strain*. The permanent strain, ϵ_{OC}, is thus

$$\epsilon_{OC} = 0.023\text{ mm/mm} - 0.008\text{ mm/mm}$$
$$= 0.0150\text{ mm/mm} \qquad \textit{Ans.}$$

Note: If gauge marks on the specimen were originally 50 mm apart, then after the load is *released* these marks will be 50 mm + (0.0150)(50 mm) = 50.75 mm apart.

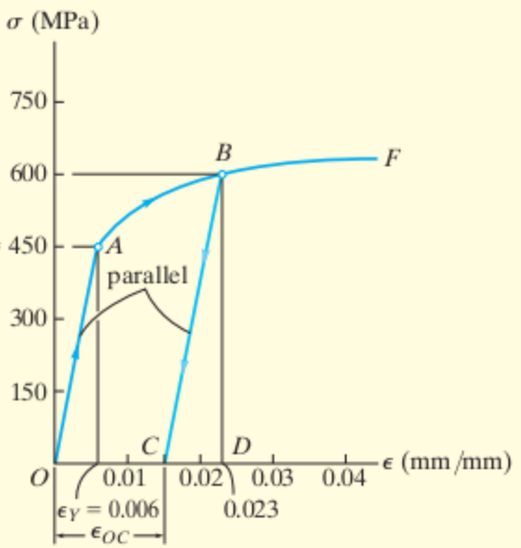


Fig. 8–19

Modulus of Resilience. Applying Eq. 8–8, we have*

$$(u_r)_{\text{initial}} = \frac{1}{2}\sigma_{pl}\epsilon_{pl} = \frac{1}{2}(450\text{ MPa})(0.006\text{ mm/mm})$$

$$= 1.35\text{ MJ/m}^3 \qquad \textit{Ans.}$$

$$(u_r)_{\text{final}} = \frac{1}{2}\sigma_{pl}\epsilon_{pl} = \frac{1}{2}(600\text{ MPa})(0.008\text{ mm/mm})$$

$$= 2.40\text{ MJ/m}^3 \qquad \textit{Ans.}$$

NOTE: By comparison, the effect of strain-hardening the material has caused an increase in the modulus of resilience; however, note that the modulus of toughness for the material has decreased since the area under the original curve, $OABF$, is larger than the area under curve CBF.

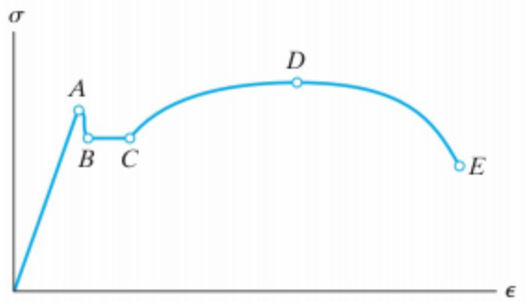
*Work in the SI system of units is measured in joules, where 1 J = 1 N · m.

8

FUNDAMENTAL PROBLEMS

F8–1. Define homogeneous material.

F8–2. Indicate the points on the stress–strain diagram which represent the proportional limit and the ultimate stress.



F8–2

F8–3. Define the modulus of elasticity E.

F8–4. At room temperature, mild steel is a ductile material. True or false?

F8–5. Engineering stress and strain are calculated using the *actual* cross-sectional area and length of the specimen. True or false?

F8–6. As the temperature increases the modulus of elasticity will increase. True or false?

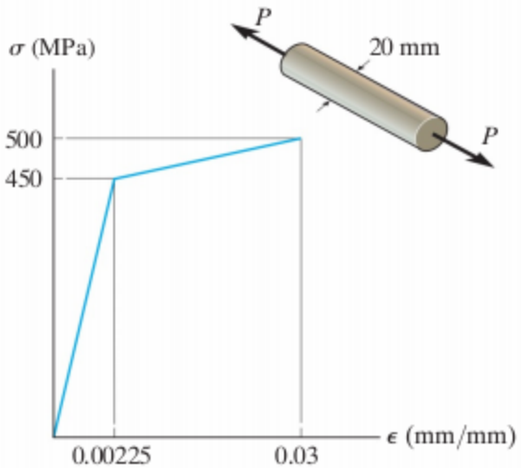
F8–7. A 100-mm-long rod has a diameter of 15 mm. If an axial tensile load of 100 kN is applied, determine its change is length. $E = 200$ GPa.

F8–8. A bar has a length of 8 in. and cross-sectional area of 12 in^2. Determine the modulus of elasticity of the material if it is subjected to an axial tensile load of 10 kip and stretches 0.003 in. The material has linear-elastic behavior.

F8–9. A 10-mm-diameter brass rod has a modulus of elasticity of $E = 100$ GPa. If it is 4 m long and subjected to an axial tensile load of 6 kN, determine its elongation.

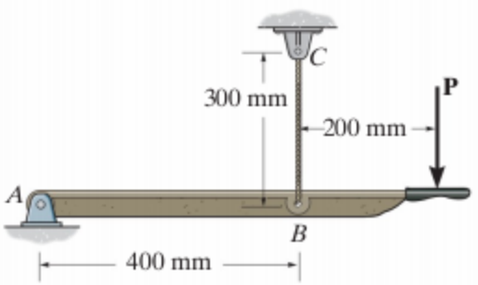
F8–10. The material for the 50-mm-long specimen has the stress–strain diagram shown. If $P = 100$ kN, determine the elongation of the specimen.

F8–11. The material for the 50-mm-long specimen has the stress–strain diagram shown. If $P = 150$ kN is applied and then released, determine the permanent elongation of the specimen.



F8–10/11

F8–12. If the elongation of wire BC is 0.2 mm after the force **P** is applied, determine the magnitude of **P**. The wire is A-36 steel and has a diameter of 3 mm.



F8–12

PROBLEMS

8–1. A tension test was performed on a steel specimen having an original diameter of 0.503 in. and gauge length of 2.00 in. The data is listed in the table. Plot the stress–strain diagram and determine approximately the modulus of elasticity, the yield stress, the ultimate stress, and the rupture stress. Use a scale of 1 in. = 20 ksi and 1 in. = 0.05 in./in. Redraw the elastic region, using the same stress scale but a strain scale of 1 in. = 0.001 in./in.

Load (kip)	Elongation (in.)
0	0
1.50	0.0005
4.60	0.0015
8.00	0.0025
11.00	0.0035
11.80	0.0050
11.80	0.0080
12.00	0.0200
16.60	0.0400
20.00	0.1000
21.50	0.2800
19.50	0.4000
18.50	0.4600

Prob. 8–1

8–2. Data taken from a stress–strain test for a ceramic are given in the table. The curve is linear between the origin and the first point. Plot the diagram, and determine the modulus of elasticity and the modulus of resilience.

8–3. Data taken from a stress–strain test for a ceramic are given in the table. The curve is linear between the origin and the first point. Plot the diagram, and determine approximately the modulus of toughness. The rupture stress is $\sigma_r = 53.4$ ksi.

σ (ksi)	ϵ (in./in.)
0	0
33.2	0.0006
45.5	0.0010
49.4	0.0014
51.5	0.0018
53.4	0.0022

Probs. 8–2/3

*__8–4.__ A tension test was performed on a steel specimen having an original diameter of 0.503 in. and a gauge length of 2.00 in. The data is listed in the table. Plot the stress–strain diagram and determine approximately the modulus of elasticity, the ultimate stress, and the rupture stress. Use a scale of 1 in. = 15 ksi and 1 in. = 0.05 in./in. Redraw the linear-elastic region, using the same stress scale but a strain scale of 1 in. = 0.001 in.

8–5. A tension test was performed on a steel specimen having an original diameter of 0.503 in. and gauge length of 2.00 in. Using the data listed in the table, plot the stress–strain diagram and determine approximately the modulus of toughness.

Load (kip)	Elongation (in.)
0	0
2.50	0.0009
6.50	0.0025
8.50	0.0040
9.20	0.0065
9.80	0.0098
12.0	0.0400
14.0	0.1200
14.5	0.2500
14.0	0.3500
13.2	0.4700

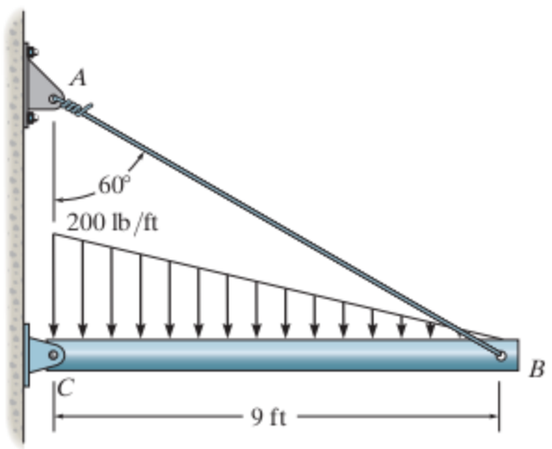
Probs. 8–4/5

8–6. A specimen is originally 1 ft long, has a diameter of 0.5 in., and is subjected to a force of 500 lb. When the force is increased from 500 lb to 1800 lb, the specimen elongates 0.009 in. Determine the modulus of elasticity for the material if it remains linear elastic.

8–7. A structural member in a nuclear reactor is made of a zirconium alloy. If an axial load of 4 kip is to be supported by the member, determine its required cross-sectional area. Use a factor of safety of 3 relative to yielding. What is the load on the member if it is 3 ft long and its elongation is 0.02 in.? $E_{zr} = 14(10^3)$ ksi, $\sigma_Y = 57.5$ ksi. The material has elastic behavior.

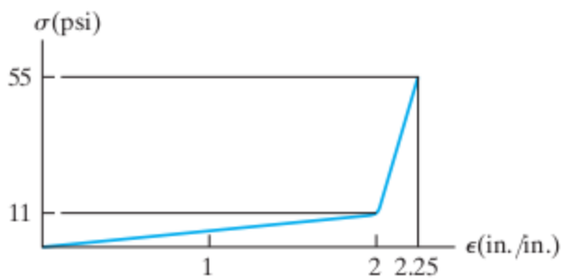
8

*8–8. The strut is supported by a pin at C and an A-36 steel guy wire AB. If the wire has a diameter of 0.2 in., determine how much it stretches when the distributed load acts on the strut.



Prob. 8–8

8–9. The σ–ϵ diagram for elastic fibers that make up human skin and muscle is shown. Determine the modulus of elasticity of the fibers and estimate their modulus of toughness and modulus of resilience.

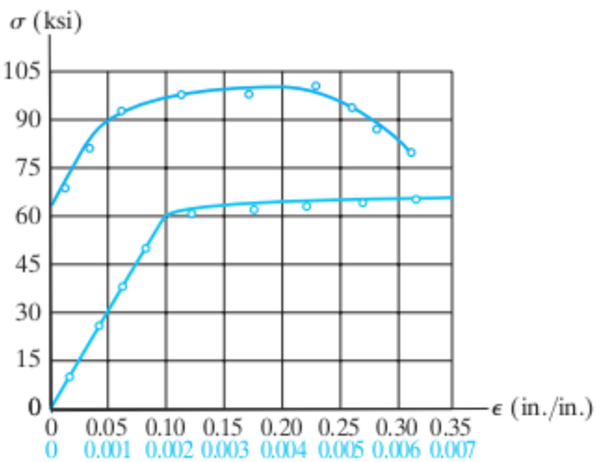


Prob. 8–9

8–10. The stress–strain diagram for a metal alloy having an original diameter of 0.5 in. and a gauge length of 2 in. is given in the figure. Determine approximately the modulus of elasticity for the material, the load on the specimen that causes yielding, and the ultimate load the specimen will support.

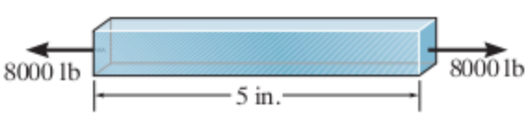
8–11. The stress–strain diagram for a steel alloy having an original diameter of 0.5 in. and a gauge length of 2 in. is given in the figure. If the specimen is loaded until it is stressed to 90 ksi, determine the approximate amount of elastic recovery and the increase in the gauge length after it is unloaded.

*8–12. The stress–strain diagram for a steel alloy having an original diameter of 0.5 in. and a gauge length of 2 in. is given in the figure. Determine approximately the modulus of resilience and the modulus of toughness for the material.



Probs. 8–10/11/12

8–13. A bar having a length of 5 in. and cross-sectional area of 0.7 in^2 is subjected to an axial force of 8000 lb. If the bar stretches 0.002 in., determine the modulus of elasticity of the material. The material has linear-elastic behavior.

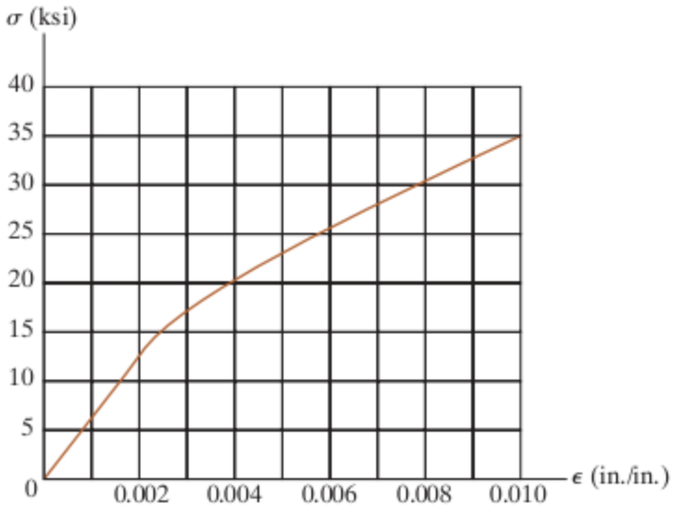


Prob. 8–13

8

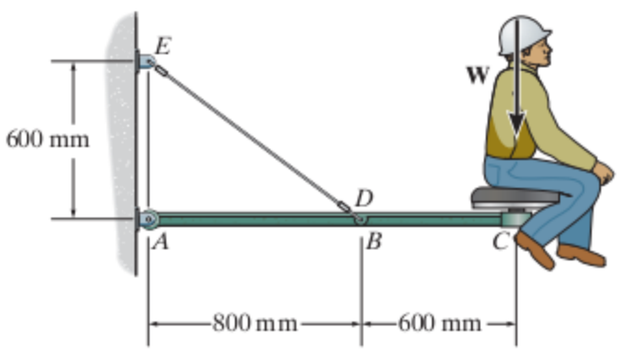
8–14. A tension test was performed on a magnesium alloy specimen having a diameter 0.5 in. and gauge length 2 in. The resulting stress–strain diagram is shown in the figure. Determine the approximate modulus of elasticity and the yield strength of the alloy using the 0.2% strain offset method.

8–15. A tension test was performed on a magnesium alloy specimen having a diameter 0.5 in. and gauge length of 2 in. The resulting stress–strain diagram is shown in the figure. If the specimen is stressed to 30 ksi and unloaded, determine the permanent elongation of the specimen.



Probs. 8–14/15

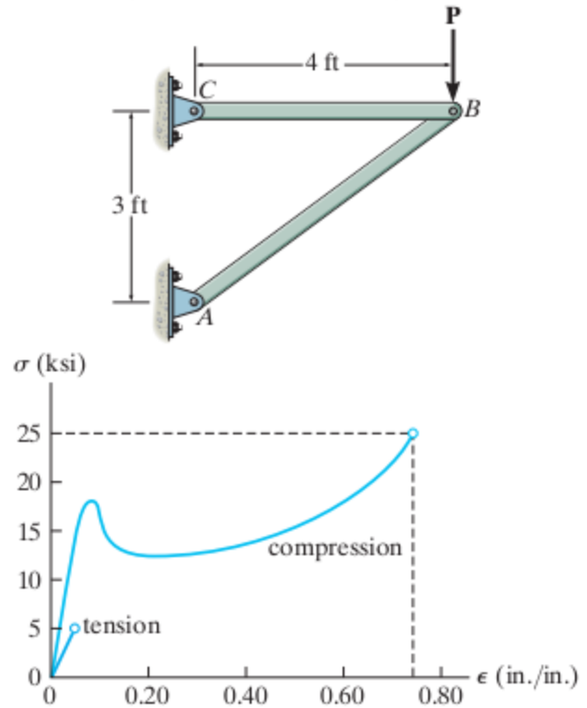
***8–16.** The wire has a diameter of 5 mm and is made from A-36 steel. If a 80-kg man is sitting on seat *C*, determine the elongation of wire *DE*.



Prob. 8–16

8–17. The two bars are made of polystyrene, which has the stress–strain diagram shown. If the cross-sectional area of bar *AB* is 1.5 in^2 and *BC* is 4 in^2, determine the largest force *P* that can be supported before any member ruptures. Assume that buckling does not occur.

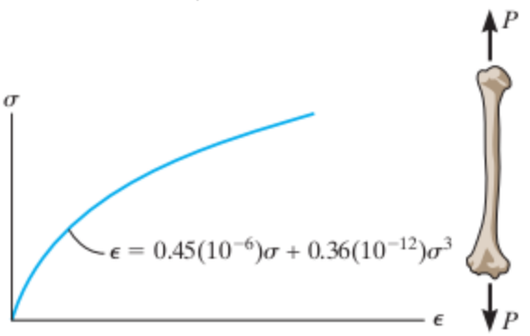
8–18. The two bars are made of polystyrene, which has the stress–strain diagram shown. Determine the cross-sectional area of each bar so that the bars rupture simultaneously when the load $P = 3$ kip. Assume that buckling does not occur.



Probs. 8–17/18

8–19. The stress–strain diagram for a bone is shown, and can be described by the equation $\epsilon = 0.45(10^{-6})\sigma + 0.36(10^{-12})\sigma^3$, where σ is in kPa. Determine the yield strength assuming a 0.3% offset.

***8–20.** The stress–strain diagram for a bone is shown and can be described by the equation $\epsilon = 0.45(10^{-6})\sigma + 0.36(10^{-12})\sigma^3$, where σ is in kPa. Determine the modulus of toughness and the amount of elongation of a 200-mm-long region just before it fractures if failure occurs at $\epsilon = 0.12$ mm/mm.



Probs. 8–19/20

8.6 Poisson's Ratio

When a deformable body is subjected to an axial tensile force, not only does it elongate but it also contracts laterally. For example, if a rubber band is stretched, it can be noted that both the thickness and width of the band are decreased. Likewise, a compressive force acting on a body causes it to contract in the direction of the force and yet its sides expand laterally.

Consider a bar having an original radius r and length L and subjected to the tensile force P in Fig. 8–20. This force elongates the bar by an amount δ, and its radius contracts by an amount δ'. Strains in the longitudinal or axial direction and in the lateral or radial direction are, respectively,

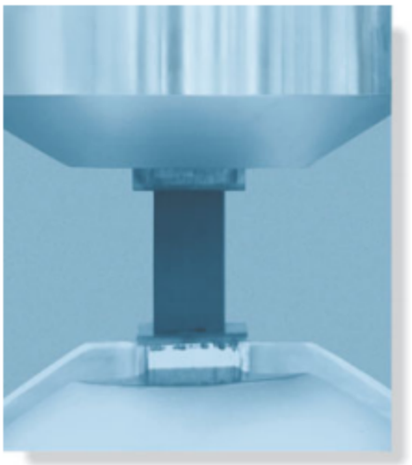
$$\epsilon_{\text{long}} = \frac{\delta}{L} \quad \text{and} \quad \epsilon_{\text{lat}} = \frac{\delta'}{r}$$

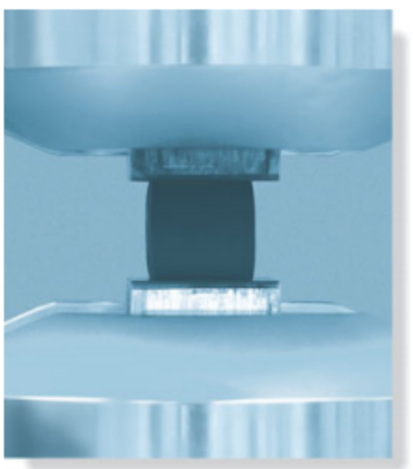
In the early 1800s, the French scientist S. D. Poisson realized that within the *elastic range* the *ratio* of these strains is a *constant*, since the deformations δ and δ' are proportional. This constant is referred to as ***Poisson's ratio***, ν (nu), and it has a numerical value that is unique for a particular material that is both *homogeneous and isotropic*. Stated mathematically it is

$$\nu = -\frac{\epsilon_{\text{lat}}}{\epsilon_{\text{long}}} \tag{8–9}$$

The negative sign is included here since *longitudinal elongation* (positive strain) causes *lateral contraction* (negative strain), and vice versa. Notice that these strains are caused only by the axial or longitudinal force P; i.e., no force or stress acts in a lateral direction in order to strain the material in this direction.

Poisson's ratio is a *dimensionless* quantity, and for most nonporous solids it has a value that is generally between $\frac{1}{4}$ and $\frac{1}{3}$. Typical values of ν for common engineering materials are listed on the inside back cover. For an "ideal material" having no lateral deformation when it is stretched or compressed Poisson's ratio will be 0. Furthermore, it will be shown in Sec. 14.10 that the *maximum* possible value for Poisson's ratio is 0.5. Therefore, $0 \leq \nu \leq 0.5$.





When the rubber block is compressed (negative strain) its sides will expand (positive strain). The ratio of these strains remains constant.

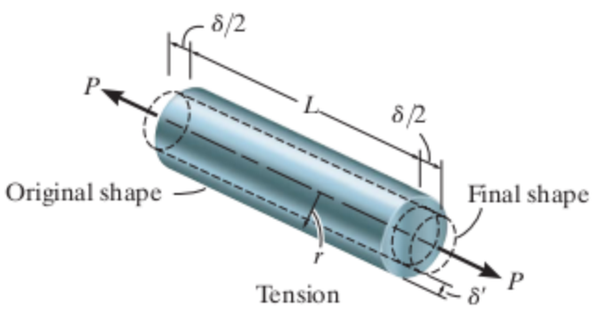


Fig. 8–20

EXAMPLE 8.3

A bar made of A-36 steel has the dimensions shown in Fig. 8–21. If an axial force of $P = 80$ kN is applied to the bar, determine the change in its length and the change in the dimensions of its cross section after applying the load. The material behaves elastically.

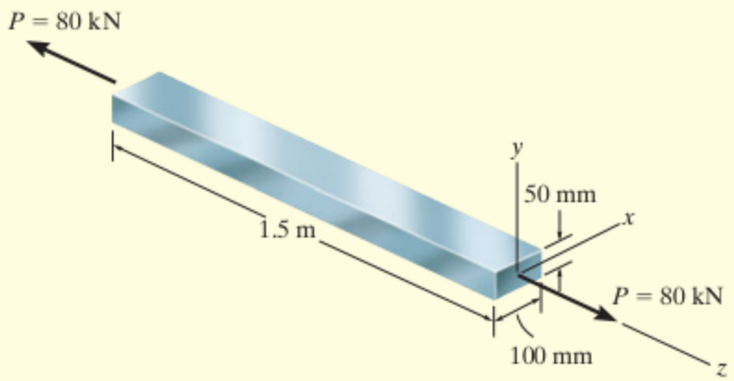


Fig. 8–21

SOLUTION

The normal stress in the bar is

$$\sigma_z = \frac{P}{A} = \frac{80(10^3)\text{ N}}{(0.1\text{ m})(0.05\text{ m})} = 16.0(10^6)\text{ Pa}$$

From the table on the inside back cover for A-36 steel $E_{st} = 200$ GPa, and so the strain in the z direction is

$$\epsilon_z = \frac{\sigma_z}{E_{st}} = \frac{16.0(10^6)\text{ Pa}}{200(10^9)\text{ Pa}} = 80(10^{-6})\text{ mm/mm}$$

The axial elongation of the bar is therefore

$$\delta_z = \epsilon_z L_z = [80(10^{-6})](1.5\text{ m}) = 120\,\mu\text{m} \qquad \textit{Ans.}$$

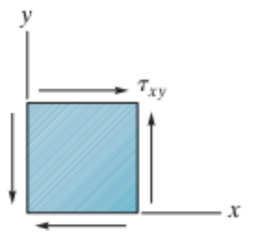
Using Eq. 8–9, where $\nu_{st} = 0.32$ as found from the inside back cover, the lateral contraction strains in *both* the x and y directions are

$$\epsilon_x = \epsilon_y = -\nu_{st}\,\epsilon_z = -0.32[80(10^{-6})] = -25.6\,\mu\text{m/m}$$

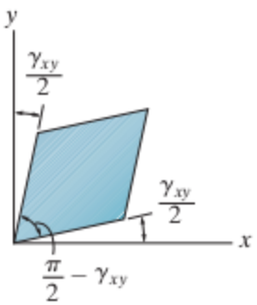
Thus the changes in the dimensions of the cross section are

$$\delta_x = \epsilon_x L_x = -[25.6(10^{-6})](0.1\text{ m}) = -2.56\,\mu\text{m} \qquad \textit{Ans.}$$

$$\delta_y = \epsilon_y L_y = -[25.6(10^{-6})](0.05\text{ m}) = -1.28\,\mu\text{m} \qquad \textit{Ans.}$$



(a)



(b)

Fig. 8–22

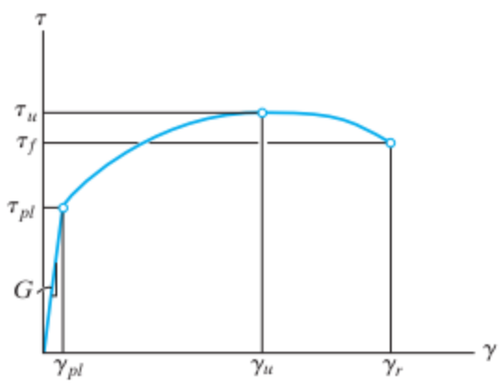


Fig. 8–23

8.7 The Shear Stress–Strain Diagram

In Sec. 7.5 it was shown that when a small element of material is subjected to *pure shear*, equilibrium requires that equal shear stresses must be developed on four faces of the element. These stresses τ_{xy} must be directed toward or away from diagonally opposite corners of the element, as shown in Fig. 8–22*a*. Furthermore, if the material is *homogeneous* and *isotropic*, then this shear stress will distort the element uniformly, Fig. 8–22*b*. As mentioned in Sec. 7.9, the shear strain γ_{xy} measures the angular distortion of the element relative to the sides originally along the x and y axes.

The behavior of a material subjected to pure shear can be studied in a laboratory using specimens in the shape of thin tubes and subjecting them to a torsional loading. If measurements are made of the applied torque and the resulting angle of twist, then by the methods to be explained in Chapter 10, the data can be used to determine the shear stress and shear strain, and a shear stress–strain diagram plotted. An example of such a diagram for a ductile material is shown in Fig. 8–23. Like the tension test, this material when subjected to shear will exhibit linear-elastic behavior and it will have a defined *proportional limit* τ_{pl}. Also, strain hardening will occur until an *ultimate shear stress* τ_u is reached. And finally, the material will begin to lose its shear strength until it reaches a point where it fractures, τ_f.

For most engineering materials, like the one just described, the elastic behavior is *linear*, and so Hooke's law for shear can be written as

$$\tau = G\gamma \tag{8–10}$$

Here G is called the ***shear modulus of elasticity*** or the ***modulus of rigidity***. Its value represents the slope of the line on the τ–γ diagram, that is, $G = \tau_{pl}/\gamma_{pl}$. Typical values for common engineering materials are listed on the inside back cover. Notice that the units of measurement for G will be the *same* as those for τ (Pa or psi), since γ is measured in radians, a dimensionless quantity.

It will be shown in Sec. 14.10 that the three material constants, E, ν, and G are actually *related* by the equation

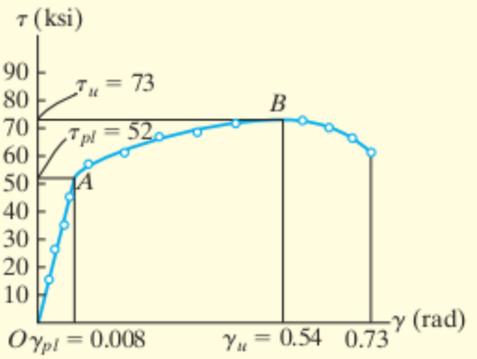
$$G = \frac{E}{2(1 + \nu)} \tag{8–11}$$

Provided E and G are known, the value of ν can then be determined from this equation rather than through experimental measurement. For example, in the case of A-36 steel, $E_{st} = 29(10^3)$ ksi and $G_{st} = 11.0(10^3)$ ksi, so that, from Eq. 8–11, $\nu_{st} = 0.32$.

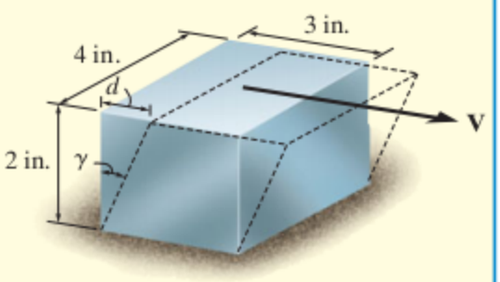
8

EXAMPLE 8.4

A specimen of titanium alloy is tested in torsion and the shear stress–strain diagram is shown in Fig. 8–24*a*. Determine the shear modulus G, the proportional limit, and the ultimate shear stress. Also, determine the maximum distance d that the top of a block of this material, shown in Fig. 8–24*b*, could be displaced horizontally if the material behaves elastically when acted upon by a shear force **V**. What is the magnitude of **V** necessary to cause this displacement?



(a)



(b)

Fig. 8–24

SOLUTION

Shear Modulus. This value represents the slope of the straight-line portion OA of the τ–γ diagram. The coordinates of point A are (0.008 rad, 52 ksi). Thus,

$$G = \frac{52 \text{ ksi}}{0.008 \text{ rad}} = 6500 \text{ ksi} \qquad \textit{Ans.}$$

The equation of line OA is therefore $\tau = G\gamma = 6500\gamma$, which is Hooke's law for shear.

Proportional Limit. By inspection, the graph ceases to be linear at point A. Thus,

$$\tau_{pl} = 52 \text{ ksi} \qquad \textit{Ans.}$$

Ultimate Stress. This value represents the maximum shear stress, point B. From the graph,

$$\tau_u = 73 \text{ ksi} \qquad \textit{Ans.}$$

Maximum Elastic Displacement and Shear Force. Since the maximum elastic shear strain is 0.008 rad, a very small angle, the top of the block in Fig. 8–24*b*, will be displaced horizontally:

$$\tan(0.008 \text{ rad}) \approx 0.008 \text{ rad} = \frac{d}{2 \text{ in.}}$$

$$d = 0.016 \text{ in.} \qquad \textit{Ans.}$$

The corresponding *average* shear stress in the block is $\tau_{pl} = 52$ ksi. Thus, the shear force V needed to cause the displacement is

$$\tau_{\text{avg}} = \frac{V}{A}; \qquad 52 \text{ ksi} = \frac{V}{(3 \text{ in.})(4 \text{ in.})}$$

$$V = 624 \text{ kip} \qquad \textit{Ans.}$$

8

EXAMPLE 8.5

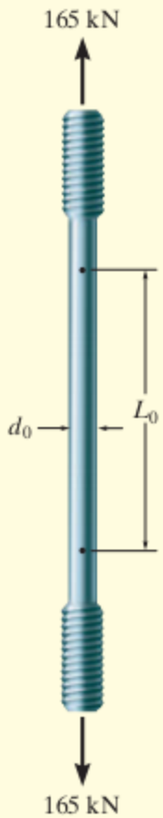


Fig. 8–25

An aluminum specimen shown in Fig. 8–25 has a diameter of $d_0 = 25$ mm and a gauge length of $L_0 = 250$ mm. If a force of 165 kN elongates the gauge length 1.20 mm, determine the modulus of elasticity. Also, determine by how much the force causes the diameter of the specimen to contract. Take $G_{al} = 26$ GPa and $\sigma_Y = 440$ MPa.

SOLUTION

Modulus of Elasticity. The average normal stress in the specimen is

$$\sigma = \frac{P}{A} = \frac{165\,(10^3)\text{ N}}{(\pi/4)\,(0.025\text{ m})^2} = 336.1\text{ MPa}$$

and the average normal strain is

$$\epsilon = \frac{\delta}{L} = \frac{1.20\text{ mm}}{250\text{ mm}} = 0.00480\text{ mm/mm}$$

Since $\sigma < \sigma_Y = 440$ MPa, the material behaves elastically. The modulus of elasticity is therefore

$$E_{al} = \frac{\sigma}{\epsilon} = \frac{336.1\,(10^6)\text{ Pa}}{0.00480} = 70.0\text{ GPa} \qquad \textit{Ans.}$$

Contraction of Diameter. First we will determine Poisson's ratio for the material using Eq. 8–11.

$$G = \frac{E}{2(1+\nu)}$$

$$26\text{ GPa} = \frac{70.0\text{ GPa}}{2(1+\nu)}$$

$$\nu = 0.347$$

Since $\epsilon_{long} = 0.00480$ mm/mm, then by Eq. 8–9,

$$\nu = -\frac{\epsilon_{lat}}{\epsilon_{long}}$$

$$0.347 = -\frac{\epsilon_{lat}}{0.00480\text{ mm/mm}}$$

$$\epsilon_{lat} = -0.00166\text{ mm/mm}$$

The contraction of the diameter is therefore

$$\delta' = (0.00166)\,(25\text{ mm})$$

$$= 0.0416\text{ mm} \qquad \textit{Ans.}$$

8

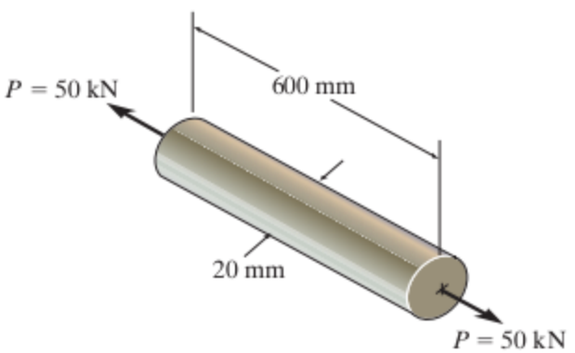
Important Points

- *Poisson's ratio*, ν, is a ratio of the lateral strain of a homogeneous and isotropic material to its longitudinal strain. Generally these strains are of opposite signs, that is, if one is an elongation, the other will be a contraction.
- The *shear stress–strain diagram* is a plot of the shear stress versus the shear strain. If the material is homogeneous and isotropic, and is also linear elastic, the slope of the straight line within the elastic region is called the modulus of rigidity or the shear modulus, G.
- There is a mathematical relationship between G, E, and ν.

FUNDAMENTAL PROBLEMS

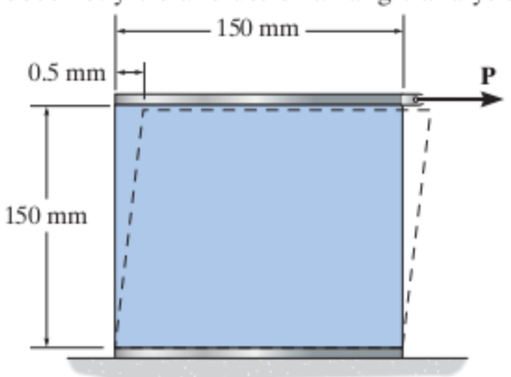
F8–13. A 100-mm long rod has a diameter of 15 mm. If an axial tensile load of 10 kN is applied to it, determine the change in its diameter. $E = 70$ GPa, $\nu = 0.35$.

F8–14. A solid circular rod that is 600 mm long and 20 mm in diameter is subjected to an axial force of $P = 50$ kN. The elongation of the rod is $\delta = 1.40$ mm, and its diameter becomes $d' = 19.9837$ mm. Determine the modulus of elasticity and the modulus of rigidity of the material. Assume that the material does not yield.



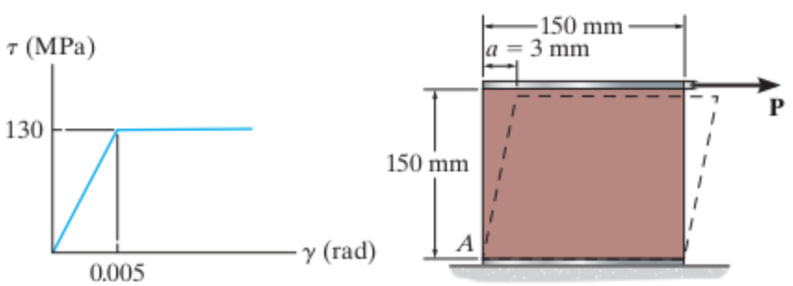
F8–14

F8–15. A 20-mm-wide block is firmly bonded to rigid plates at its top and bottom. When the force **P** is applied the block deforms into the shape shown by the dashed line. Determine the magnitude of **P**. The block's material has a modulus of rigidity of $G = 26$ GPa. Assume that the material does not yield and use small angle analysis.



F8–15

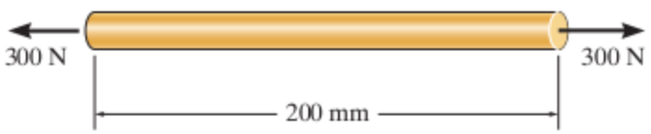
F8–16. A 20-mm-wide block is bonded to rigid plates at its top and bottom. When the force **P** is applied the block deforms into the shape shown by the dashed line. If $a = 3$ mm and **P** is released, determine the permanent shear strain in the block.



F8–16

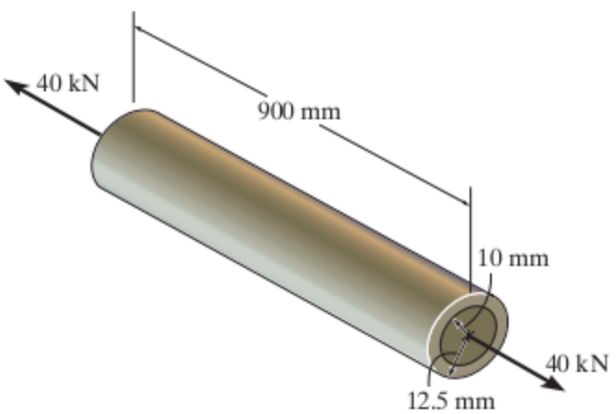
PROBLEMS

8–21. The acrylic plastic rod is 200 mm long and 15 mm in diameter. If an axial load of 300 N is applied to it, determine the change in its length and the change in its diameter. $E_p = 2.70$ GPa, $\nu_p = 0.4$.



Prob. 8–21

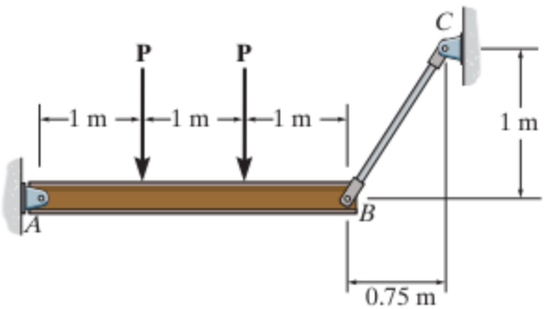
8–22. The thin-walled tube is subjected to an axial force of 40 kN. If the tube elongates 3 mm and its circumference decreases 0.09 mm, determine the modulus of elasticity, Poisson's ratio, and the shear modulus of the tube's material. The material behaves elastically.



Prob. 8–22

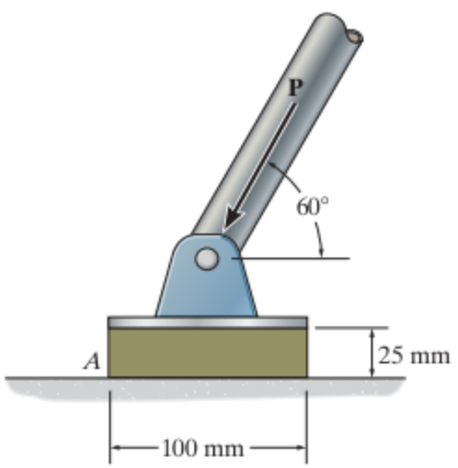
8–23. When the two forces are placed on the beam, the diameter of the A-36 steel rod BC decreases from 40 mm to 39.99 mm. Determine the magnitude of each force **P**.

***8–24.** If $P = 150$ kN, determine the elastic elongation of rod BC and the decrease in its diameter. Rod BC is made of A-36 streel and has a diameter of 40 mm.



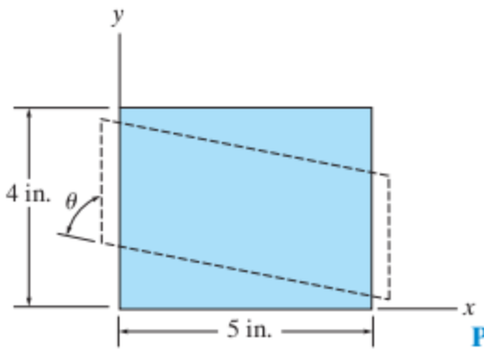
Probs. 8–23/24

8–25. The friction pad A is used to support the member, which is subjected to an axial force of $P = 2$ kN. The pad is made from a material having a modulus of elasticity of $E = 4$ MPa and Poisson's ratio $\nu = 0.4$. If slipping does not occur, determine the normal and shear strains in the pad. The width is 50 mm. Assume that the material is linearly elastic. Also, neglect the effect of the moment acting on the pad.



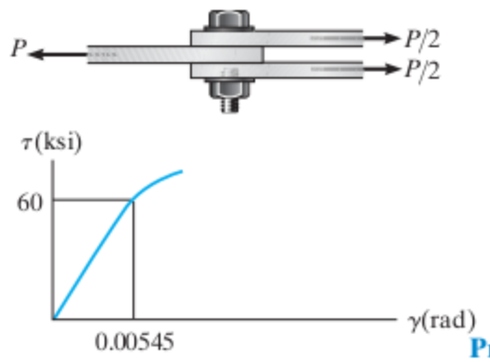
Prob. 8–25

8–26. The block is made of titanium Ti-6A1-4V and is subjected to a compression of 0.06 in. along the y axis, and its shape is given a tilt of $\theta = 89.7°$. Determine ϵ_x, ϵ_y, and γ_{xy}.



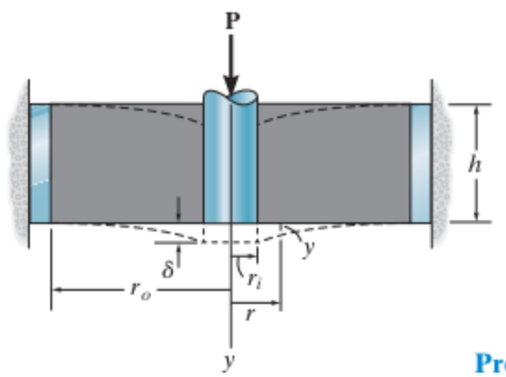
Prob. 8–26

8–27. The shear stress–strain diagram for a steel alloy is shown in the figure. If a bolt having a diameter of 0.75 in. is made of this material and used in the double lap joint, determine the modulus of elasticity E and the force P required to cause the material to yield. Take $\nu = 0.3$.



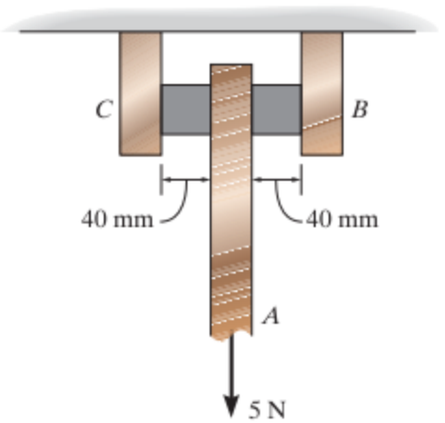
Prob. 8–27

*8–28. A shear spring is made by bonding the rubber annulus to a rigid fixed ring and a plug. When an axial load **P** is placed on the plug, show that the slope at point y in the rubber is $dy/dr = -\tan\gamma = -\tan(P/(2\pi hGr))$. For small angles we can write $dy/dr = -P/(2\pi hGr)$. Integrate this expression and evaluate the constant of integration using the condition that $y = 0$ at $r = r_o$. From the result compute the deflection $y = \delta$ of the plug.



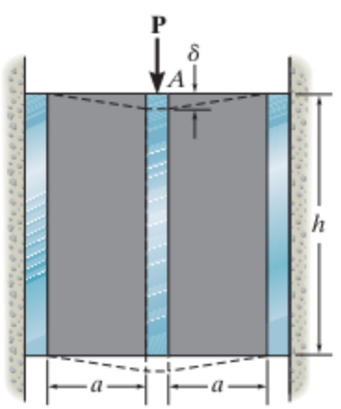
Prob. 8–28

8–29. The support consists of three rigid plates, which are connected together using two symmetrically placed rubber pads. If a vertical force of 5 N is applied to plate A, determine the approximate vertical displacement of this plate due to shear strains in the rubber. Each pad has cross-sectional dimensions of 30 mm and 20 mm. $G_r = 0.20$ MPa.



Prob. 8–29

8–30. A shear spring is made from two blocks of rubber, each having a height h, width b, and thickness a. The blocks are bonded to three plates as shown. If the plates are rigid and the shear modulus of the rubber is G, determine the displacement of plate A if a vertical load **P** is applied to this plate. Assume that the displacement is small so that $\delta = a\tan\gamma \approx a\gamma$.



Prob. 8–30

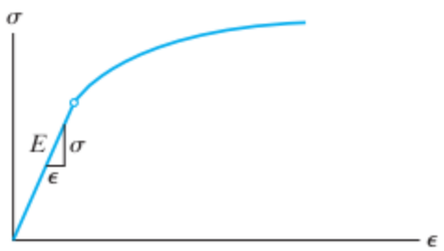
8

CHAPTER REVIEW

One of the most important tests for material strength is the tension test. The results, found from stretching a specimen of known size, are plotted as normal stress on the vertical axis and normal strain on the horizontal axis.

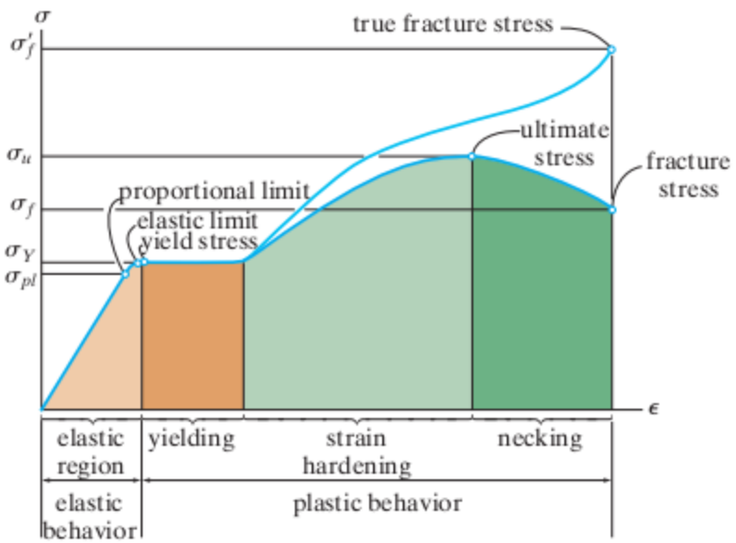
Many engineering materials exhibit initial linear elastic behavior, whereby stress is proportional to strain, defined by Hooke's law, $\sigma = E\epsilon$. Here E, called the modulus of elasticity, is the slope of this straight line on the stress–strain diagram.

$$\sigma = E\epsilon$$



Ductile material

When the material is stressed beyond the yield point, permanent deformation will occur. In particular, steel has a region of yielding, whereby the material will exhibit an increase in strain with no increase in stress. The region of strain hardening causes further yielding of the material with a corresponding increase in stress. Finally, at the ultimate stress, a localized region on the specimen will begin to constrict, forming a neck. It is after this that the fracture occurs.



Conventional and true stress–strain diagrams for ductile material (steel) (not to scale)

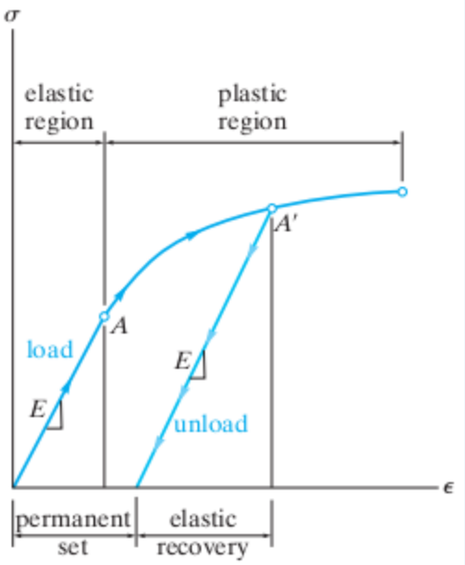
Ductile materials, such as most metals, exhibit both elastic and plastic behavior. Wood is moderately ductile. Ductility is usually specified by the percent elongation to failure or by the percent reduction in the cross-sectional area.

$$\text{Percent elongation} = \frac{L_f - L_0}{L_0}(100\%)$$

$$\text{Percent reduction of area} = \frac{A_0 - A_f}{A_0}(100\%)$$

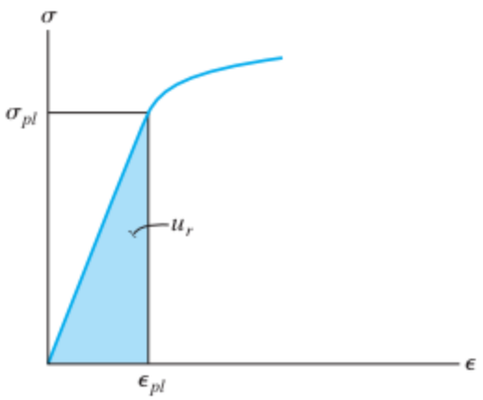
Brittle materials exhibit little or no yielding before failure. Cast iron, concrete, and glass are typical examples.

The yield point of a material at A can be increased by strain hardening. This is accomplished by applying a load that causes the stress to be greater than the yield stress, then releasing the load. The larger stress A' becomes the new yield point for the material.

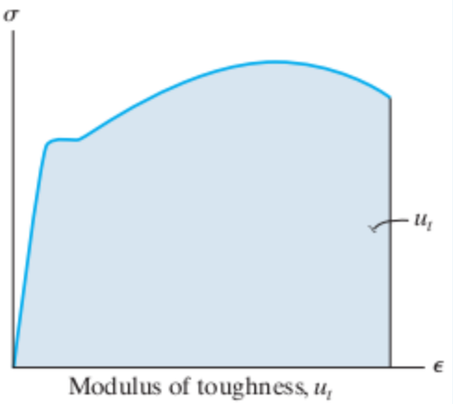


When a load is applied to a member, the deformations cause strain energy to be stored in the material. The strain energy per unit volume or strain energy density is equivalent to the area under the stress–strain curve. This area up to the yield point is called the modulus of resilience. The entire area under the stress–strain diagram is called the modulus of toughness.

Poisson's ratio ν is a dimensionless material property that relates the lateral strain to the longitudinal strain. Its range of values is $0 \leq \nu \leq 0.5$.



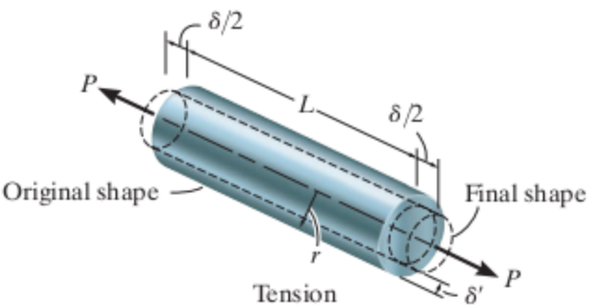
Modulus of resilience



Modulus of toughness, u_t

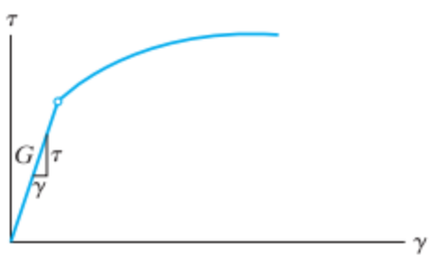
(b)

$$\nu = -\frac{\epsilon_{lat}}{\epsilon_{long}}$$



Shear stress versus shear strain diagrams can also be established for a material. Within the elastic region, $\tau = G\gamma$, where G is the shear modulus found from the slope of the line. The value of ν can be obtained from the relationship that exists between G, E, and ν.

$$G = \frac{E}{2(1 + \nu)}$$

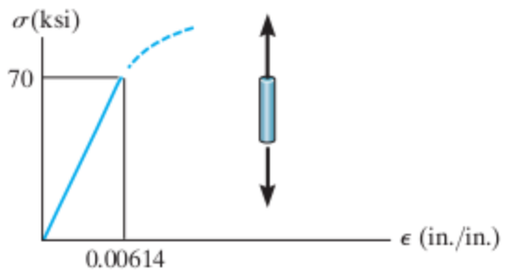


8

REVIEW PROBLEMS

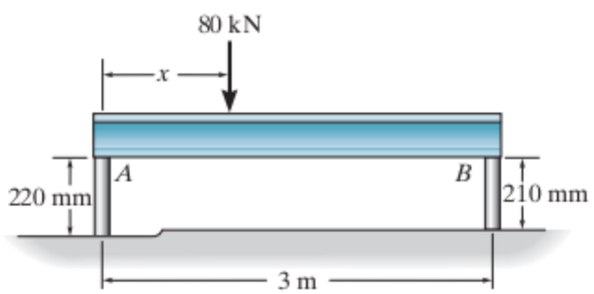
8–31. The elastic portion of the tension stress–strain diagram for an aluminum alloy is shown in the figure. The specimen used for the test has a gauge length of 2 in. and a diameter of 0.5 in. When the applied load is 9 kip, the new diameter of the specimen is 0.49935 in. Compute the shear modulus G_{al} for the aluminum.

***8–32.** The elastic portion of the tension stress–strain diagram for an aluminum alloy is shown in the figure. The specimen used for the test has a gauge length of 2 in. and a diameter of 0.5 in. If the applied load is 10 kip, determine the new diameter of the specimen. The shear modulus is $G_{al} = 3.8(10^3)$ ksi.



Probs. 8–31/32

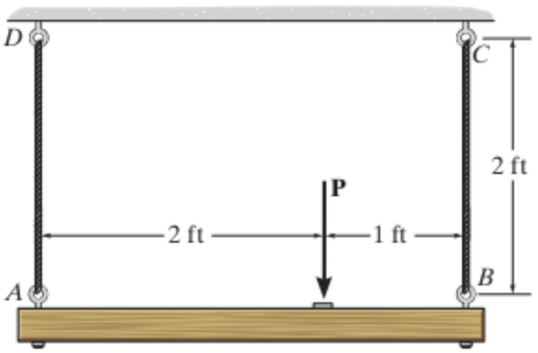
8–33. The rigid beam rests in the horizontal position on two 2014-T6 aluminum cylinders having the *unloaded* lengths shown. If each cylinder has a diameter of 30 mm, determine the placement x of the applied 80-kN load so that the beam remains horizontal. What is the new diameter of cylinder A after the load is applied? $\nu_{al} = 0.35$.



Prob. 8–33

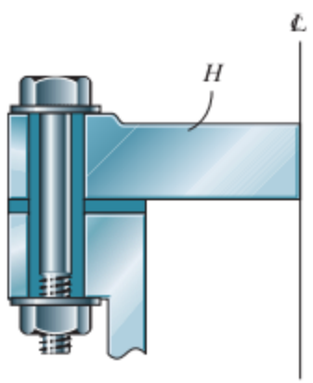
8–34. The wires each have a diameter of $\frac{1}{2}$ in., length of 2 ft, and are made from 304 stainless steel. If $P = 6$ kip, determine the angle of tilt of the rigid beam AB.

8–35. The wires each have a diameter of $\frac{1}{2}$ in., length of 2 ft, and are made from 304 stainless steel. Determine the magnitude of force **P** so that the rigid beam tilts 0.015°.



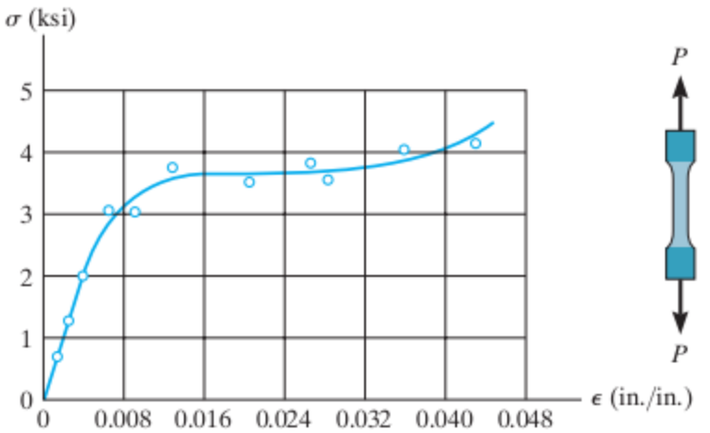
Probs. 8–34/35

***8–36.** The head H is connected to the cylinder of a compressor using six steel bolts. If the clamping force in each bolt is 800 lb, determine the normal strain in the bolts. Each bolt has a diameter of $\frac{3}{16}$ in. If $\sigma_Y = 40$ ksi and $E_{st} = 29(10^3)$ ksi, what is the strain in each bolt when the nut is unscrewed so that the clamping force is released?



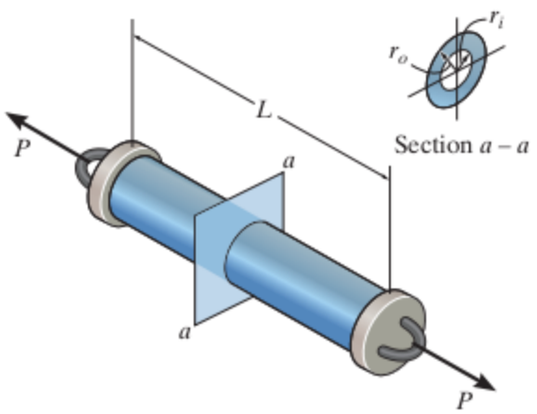
Prob. 8–36

8–37. The stress–strain diagram for polyethylene, which is used to sheath coaxial cables, is determined from testing a specimen that has a gauge length of 10 in. If a load P on the specimen develops a strain of $\epsilon = 0.024$ in./in., determine the approximate length of the specimen, measured between the gauge points, when the load is removed. Assume the specimen recovers elastically.



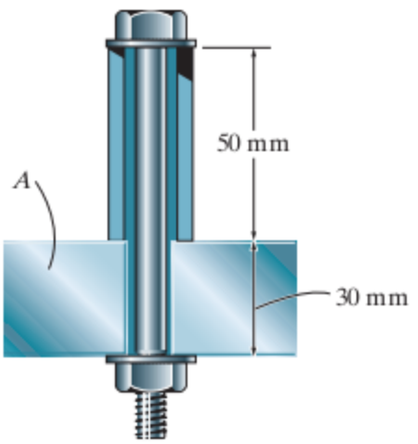
Prob. 8–37

8–38. The pipe with two rigid caps attached to its ends is subjected to an axial force P. If the pipe is made from a material having a modulus of elasticity E and Poisson's ratio ν, determine the change in volume of the material.



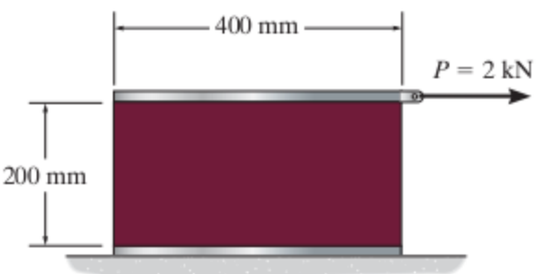
Prob. 8–38

8–39. The 8-mm-diameter bolt is made of an aluminum alloy. It fits through a magnesium sleeve that has an inner diameter of 12 mm and an outer diameter of 20 mm. If the original lengths of the bolt and sleeve are 80 mm and 50 mm, respectively, determine the strains in the sleeve and the bolt if the nut on the bolt is tightened so that the tension in the bolt is 8 kN. Assume the material at A is rigid. $E_{al} = 70$ GPa, $E_{mg} = 45$ GPa.



Prob. 8–39

*8–40. An acetal polymer block is fixed to the rigid plates at its top and bottom surfaces. If the top plate displaces 2 mm horizontally when it is subjected to a horizontal force $P = 2$ kN, determine the shear modulus of the polymer. The width of the block is 100 mm. Assume that the polymer is linearly elastic and use small angle analysis.



Prob. 8–40

8

Chapter 9

The string of drill pipe suspended from this traveling block on an oil rig is subjected to extremely large loadings and axial deformations.

Axial Load

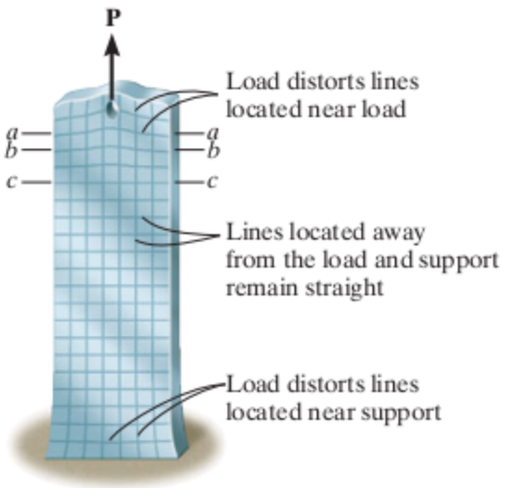
CHAPTER OBJECTIVES

In Chapter 8 we developed the method for finding the normal stress in axially loaded members. In this chapter, our objectives are:

- To determine the deformation of axially loaded members.
- To determine the support reactions when these reactions cannot be determined solely from the equations of equilibrium.
- To analyze the effects of thermal stresses.

9.1 Saint-Venant's Principle

In the previous chapters, we have developed the concept of *stress* as a means of measuring the force distribution within a body and *strain* as a means of measuring a body's deformation. We have also shown that the mathematical relationship between stress and strain depends on the type of material from which the body is made. In particular, if the material behaves in a linear elastic manner, then Hooke's law applies, and there is a proportional relationship between stress and strain.



(a)

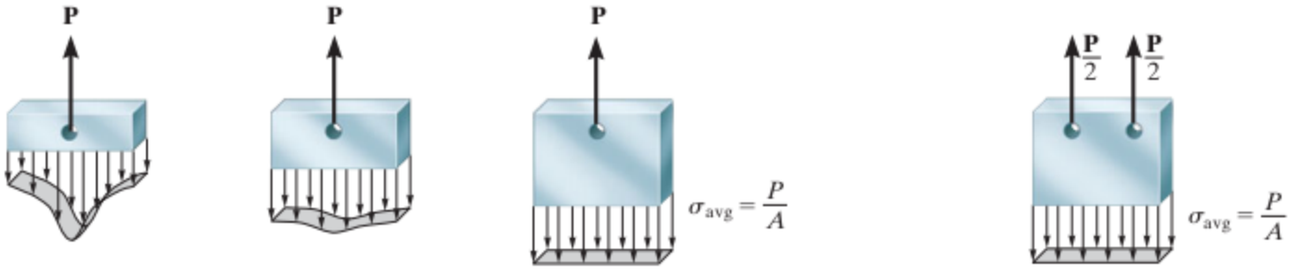
Fig. 9–1

Using this idea, consider the manner in which a rectangular bar will deform elastically when the bar is subjected to a force **P** applied along its centroidal axis, Fig. 9–1*a*. Here the bar is fixed connected at one end, with the force applied through a hole at its other end. Due to the loading, the bar deforms as indicated by the once horizontal and vertical grid lines drawn on the bar. Notice how the *localized deformation* that occurs at each end tends to even out and become uniform throughout the midsection of the bar.

If the material remains elastic then the strains caused by this deformation are directly related to the stress in the bar. As a result, the stress will be distributed more uniformly throughout the cross-sectional area when the section is taken farther and farther from the point where any external load is applied. For example, consider a profile of the variation of the stress distribution acting at sections *a–a*, *b–b*, and *c–c*, each of which is shown in Fig. 9–1*b*. By comparison, the stress tends to reach a uniform value at section *c–c*, which is sufficiently removed from the end since the localized deformation caused by **P** *vanishes*. The minimum distance from the bar's end where this occurs can be determined using a mathematical analysis based on the theory of elasticity.

It has been found that this distance should at least be equal to the *largest dimension* of the loaded cross section. Hence, section *c–c* should be located at a distance at least equal to the width (not the thickness) of the bar.*

*When section *c–c* is so located, the theory of elasticity predicts the maximum stress to be $\sigma_{max} = 1.02\sigma_{avg}$.



Section *a–a* Section *b–b* Section *c–c*

(b)

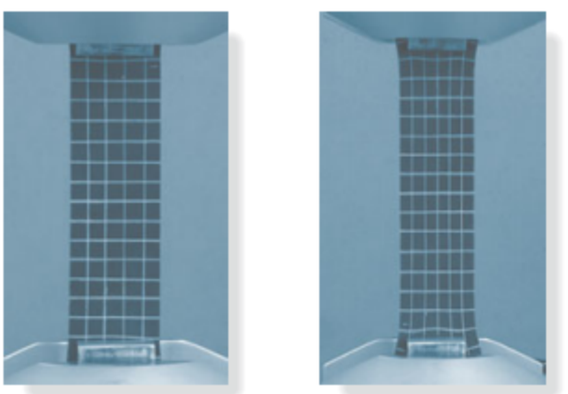
Section *c–c*

(c)

Fig. 9–1 (cont.)

In the same way, the stress distribution at the support will also even out and become uniform over the cross section located the same distance away from the support.

The fact that stress and deformation behave in this manner is referred to as *Saint-Venant's principle*, since it was first noticed by the French scientist Barré de Saint-Venant in 1855. Essentially, it states that the *stress and strain produced at points in a body sufficiently removed from the region of load application will be the same as the stress and strain produced by any applied loadings that have the same statically equivalent resultant, and are applied to the body within the same region.* For example, if two symmetrically applied forces $P/2$ act on the bar, Fig. 9–1*c*, the stress distribution at section *c–c* will be uniform and therefore equivalent to $\sigma_{avg} = P/A$ as in Fig. 9–1*b*.



Notice how the lines on this rubber membrane distort after it is stretched. The localized distortions at the grips smooth out as stated by Saint-Venant's principle.

9.2 Elastic Deformation of an Axially Loaded Member

Using Hooke's law and the definitions of stress and strain, we will now develop an equation that can be used to determine the *elastic* displacement of a member subjected to axial loads. To generalize the development, consider the bar shown in Fig. 9–2*a*, which has a cross-sectional area that *gradually* varies along its length L. The bar is subjected to concentrated loads at its ends and a variable external load distributed along its length. This distributed load could, for example, represent the weight of the bar if it is in the vertical position, or friction forces acting on the bar's surface. Here we wish to find the ***relative displacement*** δ (delta) of one end of the bar with respect to the other end as caused by this loading. We will *neglect* the localized deformations that occur at points of concentrated loading and where the cross section suddenly changes. From Saint-Venant's principle, these effects occur within small regions of the bar's length and will therefore have only a slight effect on the final result. For the most part, the bar will deform uniformly, so the normal stress will be uniformly distributed over the cross section.

Using the method of sections, a differential element (or wafer) of length dx and cross-sectional area $A(x)$ is isolated from the bar at the arbitrary position x. The free-body diagram of this element is shown in Fig. 9–2*b*. The resultant internal axial force will be a function of x since the external distributed loading will cause it to vary along the length of the bar. This load, $P(x)$, will deform the element into the shape indicated by the dashed outline, and therefore the displacement of one end of the element with respect to the other end is $d\delta$. The stress and strain in the element are

$$\sigma = \frac{P(x)}{A(x)} \quad \text{and} \quad \epsilon = \frac{d\delta}{dx}$$

Provided the stress does not exceed the proportional limit, we can apply Hooke's law; i.e.,

$$\sigma = E(x)\epsilon$$

$$\frac{P(x)}{A(x)} = E(x)\left(\frac{d\delta}{dx}\right)$$

$$d\delta = \frac{P(x)\,dx}{A(x)E(x)}$$

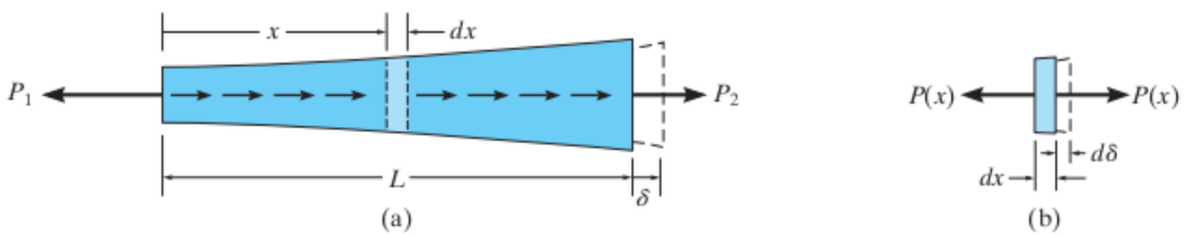


Fig. 9–2

For the entire length L of the bar, we must integrate this expression to find δ. This yields

$$\delta = \int_0^L \frac{P(x)\,dx}{A(x)\,E(x)} \tag{9–1}$$

where

δ = displacement of one point on the bar relative to the other point
L = original length of bar
$P(x)$ = internal axial force at the section, located a distance x from one end
$A(x)$ = cross-sectional area of the bar expressed as a function of x
$E(x)$ = modulus of elasticity for the material expressed as a function of x.

Constant Load and Cross-Sectional Area. In many cases the bar will have a constant cross-sectional area A; and the material will be homogeneous, so E is constant. Furthermore, if a constant external force is applied at each end, Fig. 9–3, then the internal force P throughout the length of the bar is also constant. As a result, Eq. 9–1 can be integrated to yield

$$\delta = \frac{PL}{AE} \tag{9–2}$$

If the bar is subjected to several different axial forces along its length, or the cross-sectional area or modulus of elasticity changes abruptly from one region of the bar to the next, the above equation can be applied to each *segment* of the bar where these quantities remain *constant*. The displacement of one end of the bar with respect to the other is then found from the *algebraic addition* of the relative displacements of the ends of each segment. For this general case,

$$\delta = \sum \frac{PL}{AE} \tag{9–3}$$

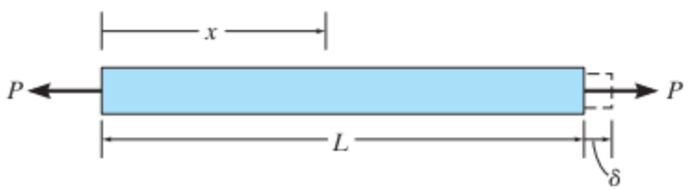
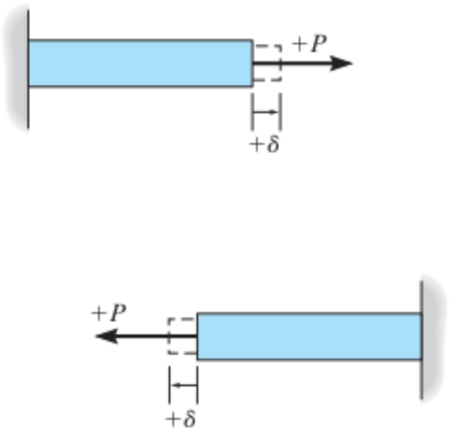


Fig. 9–3

The vertical displacement at the top of these building columns depends upon the loading applied on the roof and to the floor attached to their midpoint.



Positive sign convention for P and δ

Fig. 9–4

Sign Convention. In order to apply Eq. 9–3, we must develop a sign convention for the internal axial force and the displacement of one end of the bar with respect to the other end. To do so, we will consider both the force and displacement to be *positive* if they cause *tension and elongation*, respectively, Fig. 9–4; whereas a *negative* force and displacement will cause *compression* and *contraction*, respectively.

For example, consider the bar shown in Fig. 9–5*a*. The *internal axial forces* "P" are determined by the method of sections for each segment, Fig. 9–5*b*. They are $P_{AB} = +5\,\text{kN}$, $P_{BC} = -3\,\text{kN}$, $P_{CD} = -7\,\text{kN}$. This variation in axial load is shown on the axial or *normal force diagram* for the bar, Fig. 9–5*c*. Since we now know how the *internal* force varies throughout the bar's length, the displacement of end A relative to end D is determined from

$$\delta_{A/D} = \sum \frac{PL}{AE} = \frac{(5\,\text{kN})L_{AB}}{AE} + \frac{(-3\,\text{kN})L_{BC}}{AE} + \frac{(-7\,\text{kN})L_{CD}}{AE}$$

If the other data are substituted and a positive answer is calculated, it means that end A will move away from end D (the bar elongates), whereas a negative result would indicate that end A moves toward end D (the bar shortens). The double subscript notation is used to indicate this relative displacement ($\delta_{A/D}$); however, if the displacement is to be determined relative to a *fixed point*, then only a single subscript will be used. For example, if D is located at a *fixed* support, then the displacement will be denoted as simply δ_A.

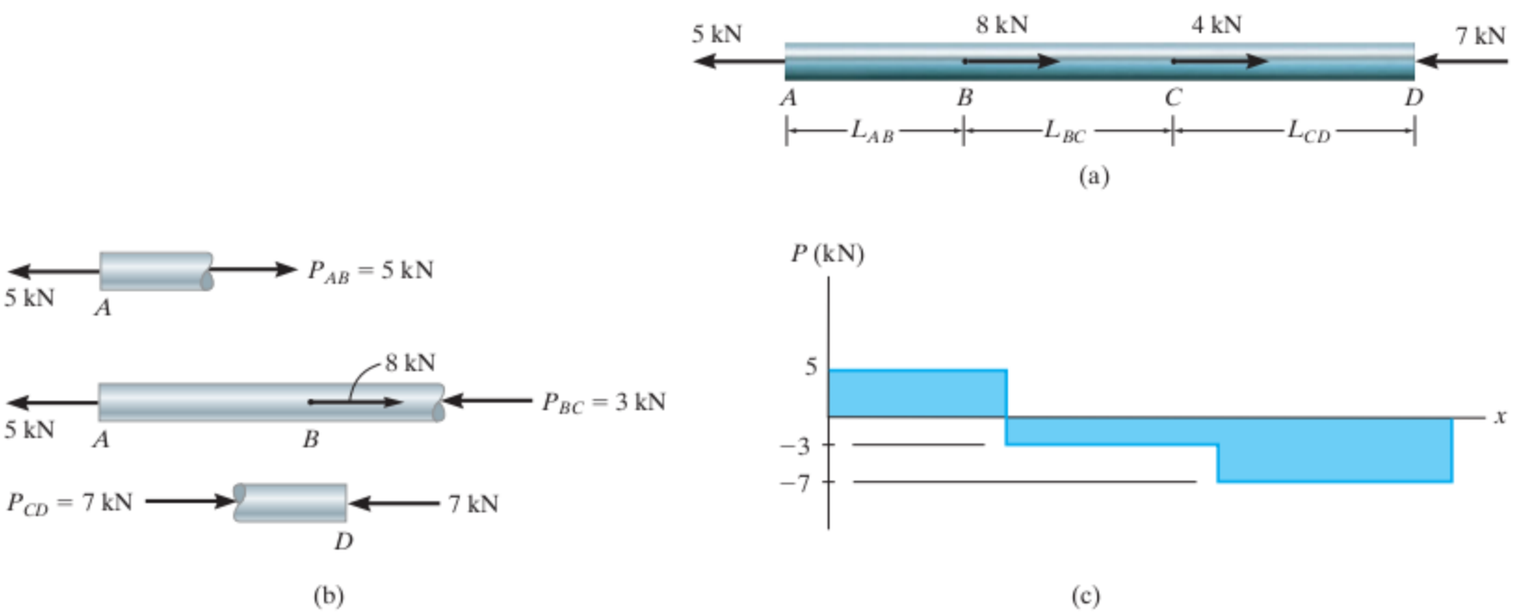


Fig. 9–5

Important Points

- *Saint-Venant's principle* states that both the localized deformation and stress which occur within the regions of load application or at the supports tend to "even out" at a distance sufficiently removed from these regions.
- The displacement of one end of an axially loaded member relative to the other end is determined by relating the applied *internal* load to the stress using $\sigma = P/A$ and relating the displacement to the strain using $\epsilon = d\delta/dx$. Finally, these two equations are combined using Hooke's law, $\sigma = E\epsilon$, which yields Eq. 9–1.
- Since Hooke's law has been used in the development of the displacement equation, it is important that no internal load causes yielding of the material, and that the material is homogeneous and behaves in a linear elastic manner.

Procedure for Analysis

The relative displacement between any two points A and B on an axially loaded member can be determined by applying Eq. 9–1 (or Eq. 9–2). Application requires the following steps.

Internal Force.

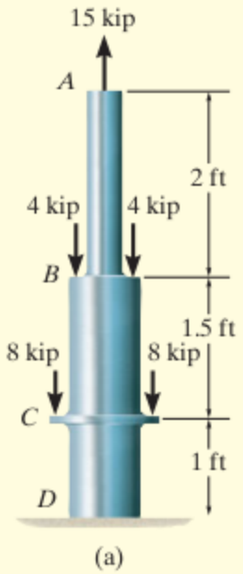
- Use the method of sections to determine the internal axial force P within the member.
- If this force varies along the member's length due to an *external distributed loading*, a section should be made at the arbitrary location x from one end of the member and the force represented as a function of x, i.e., $P(x)$.
- If several *constant external forces* act on the member, the internal force in each *segment* of the member, between any two external forces, must be determined.
- For any segment, an internal *tensile force* is *positive* and an internal *compressive force* is *negative*. For convenience, the results of the internal loading can be shown graphically by constructing the normal-force diagram.

Displacement.

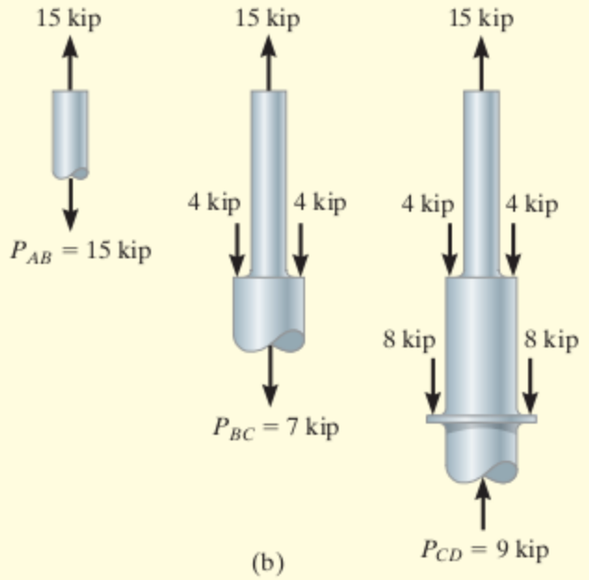
- When the member's cross-sectional area *varies* along its length, the area must be expressed as a function of its position x, i.e., $A(x)$.
- If the cross-sectional area, the modulus of elasticity, or the internal loading *suddenly changes*, then Eq. 9–2 should be applied to each segment for which these quantities are constant.
- When substituting the data into Eqs. 9–1 through 9–3, be sure to account for the proper sign for the internal force **P**. Tensile loadings are positive and compressive loadings are negative. Also, use a consistent set of units. For any segment, if the result is a *positive* numerical quantity, it indicates *elongation*; if it is *negative*, it indicates a *contraction*.

EXAMPLE 9.1

The A-36 steel bar shown in Fig. 9–6*a* is made from two segments having cross-sectional areas of $A_{AB} = 1\ \text{in}^2$ and $A_{BD} = 2\ \text{in}^2$. Determine the vertical displacement of end A and the displacement of B relative to C.



(a)



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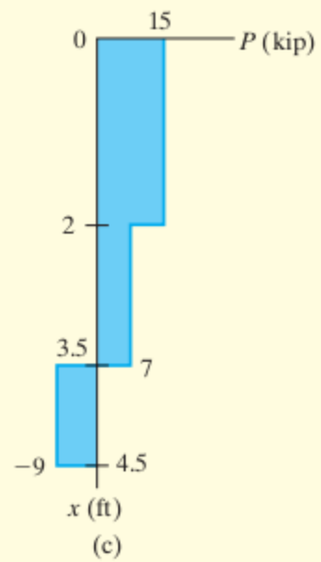


(c)

Fig. 9–6

SOLUTION

Internal Force. Due to the application of the external loadings, the internal axial forces in regions AB, BC, and CD will all be different. These forces are obtained by applying the method of sections and the equation of vertical force equilibrium as shown in Fig. 9–6*b*. This variation is plotted in Fig. 9–6*c*.

Displacement. From the inside back cover, $E_{st} = 29(10^3)\ \text{ksi}$. Using the sign convention, i.e., internal tensile forces are positive and compressive forces are negative, the vertical displacement of A relative to the *fixed* support D is

$$\delta_A = \sum \frac{PL}{AE} = \frac{[+15\ \text{kip}](2\ \text{ft})(12\ \text{in./ft})}{(1\ \text{in}^2)[29(10^3)\ \text{kip/in}^2]} + \frac{[+7\ \text{kip}](1.5\ \text{ft})(12\ \text{in./ft})}{(2\ \text{in}^2)[29(10^3)\ \text{kip/in}^2]} + \frac{[-9\ \text{kip}](1\ \text{ft})(12\ \text{in./ft})}{(2\ \text{in}^2)[29(10^3)\ \text{kip/in}^2]}$$

$$= +0.0127\ \text{in.} \qquad \textit{Ans.}$$

Since the result is *positive*, the bar *elongates* and so the displacement at A is upward.

Applying Eq. 9–2 between points B and C, we obtain,

$$\delta_{B/C} = \frac{P_{BC}L_{BC}}{A_{BC}E} = \frac{[+7\ \text{kip}](1.5\ \text{ft})(12\ \text{in./ft})}{(2\ \text{in}^2)[29(10^3)\ \text{kip/in}^2]} = +0.00217\ \text{in.} \quad \textit{Ans.}$$

Here B moves away from C, since the segment elongates.

EXAMPLE 9.2

The assembly shown in Fig. 9–7*a* consists of an aluminum tube *AB* having a cross-sectional area of 400 mm^2. A steel rod having a diameter of 10 mm is attached to a rigid collar and passes through the tube. If a tensile load of 80 kN is applied to the rod, determine the displacement of the end *C* of the rod. Take $E_{st} = 200$ GPa, $E_{al} = 70$ GPa.

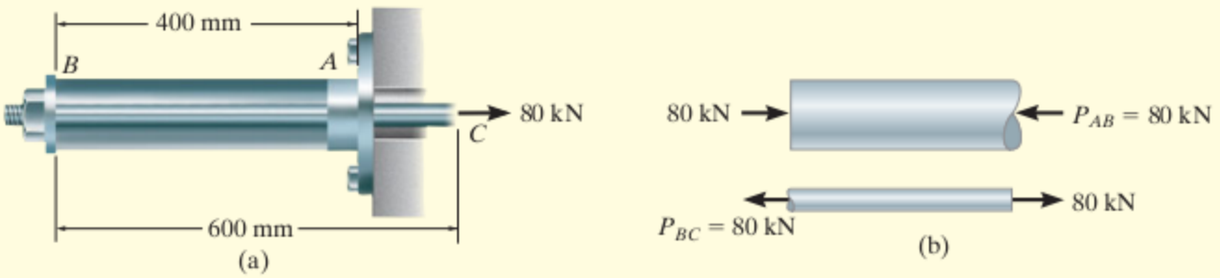


Fig. 9–7

SOLUTION

Internal Force. The free-body diagram of the tube and rod segments in Fig. 9–7*b* shows that the rod is subjected to a tension of 80 kN and the tube is subjected to a compression of 80 kN.

Displacement. We will first determine the displacement of end *C* with respect to end *B*. Working in units of newtons and meters, we have

$$\delta_{C/B} = \frac{PL}{AE} = \frac{[+80(10^3)\ \text{N}](0.6\ \text{m})}{\pi(0.005\ \text{m})^2[200(10^9)\ \text{N/m}^2]} = +0.003056\ \text{m} \rightarrow$$

The positive sign indicates that end *C* moves *to the right* relative to end *B*, since the bar elongates.

The displacement of end *B* with respect to the *fixed* end *A* is

$$\delta_B = \frac{PL}{AE} = \frac{[-80(10^3)\ \text{N}](0.4\ \text{m})}{[400\ \text{mm}^2(10^{-6})\ \text{m}^2/\text{mm}^2][70(10)^9\ \text{N/m}^2]}$$

$$= -0.001143\ \text{m} = 0.001143\ \text{m} \rightarrow$$

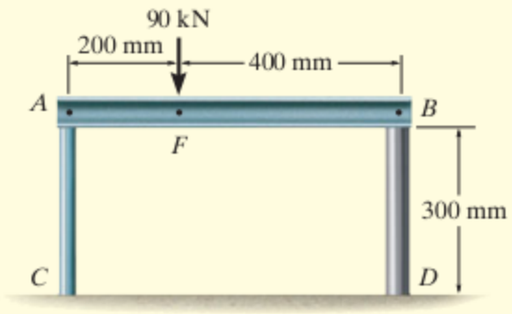
Here the negative sign indicates that the tube shortens, and so *B* moves to the *right* relative to *A*.

Since both displacements are to the right, the displacement of *C* relative to the fixed end *A* is therefore

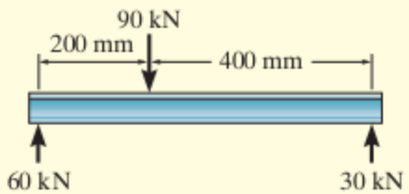
$$(\pm\rightarrow)^{+} \qquad \delta_C = \delta_B + \delta_{C/B} = 0.001143\ \text{m} + 0.003056\ \text{m}$$

$$= 0.00420\ \text{m} = 4.20\ \text{mm} \rightarrow \qquad \textit{Ans.}$$

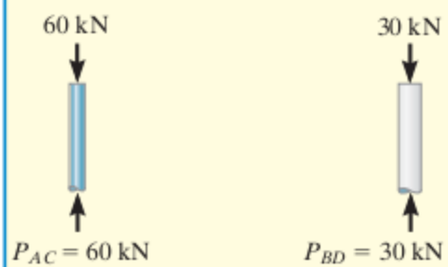
EXAMPLE 9.3



(a)



(b)



(c)

Rigid beam AB rests on the two short posts shown in Fig. 9–8*a*. AC is made of steel and has a diameter of 20 mm, and BD is made of aluminum and has a diameter of 40 mm. Determine the displacement of point F on AB if a vertical load of 90 kN is applied over this point. Take $E_{st} = 200$ GPa, $E_{al} = 70$ GPa.

SOLUTION

Internal Force. The compressive forces acting at the top of each post are determined from the equilibrium of member AB, Fig. 9–8*b*. These forces are equal to the internal forces in each post, Fig. 9–8*c*.

Displacement. The displacement of the top of each post is

Post *AC*:

$$\delta_A = \frac{P_{AC}L_{AC}}{A_{AC}E_{st}} = \frac{[-60(10^3)\text{ N}](0.300\text{ m})}{\pi(0.010\text{ m})^2[200(10^9)\text{ N/m}^2]} = -286(10^{-6})\text{ m}$$

$$= 0.286\text{ mm}\downarrow$$

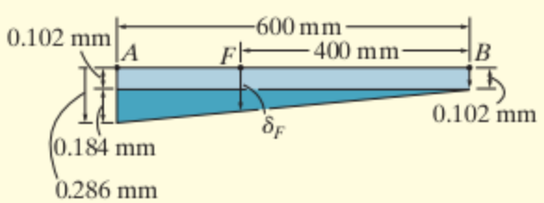
Post *BD*:

$$\delta_B = \frac{P_{BD}L_{BD}}{A_{BD}E_{al}} = \frac{[-30(10^3)\text{ N}](0.300\text{ m})}{\pi(0.020\text{ m})^2[70(10^9)\text{ N/m}^2]} = -102(10^{-6})\text{ m}$$

$$= 0.102\text{ mm}\downarrow$$

A diagram showing the centerline displacements at A, B, and F on the beam is shown in Fig. 9–8*d*. By proportion of the blue shaded triangle, the displacement of point F is therefore

$$\delta_F = 0.102\text{ mm} + (0.184\text{ mm})\left(\frac{400\text{ mm}}{600\text{ mm}}\right) = 0.225\text{ mm}\downarrow \qquad \textit{Ans.}$$

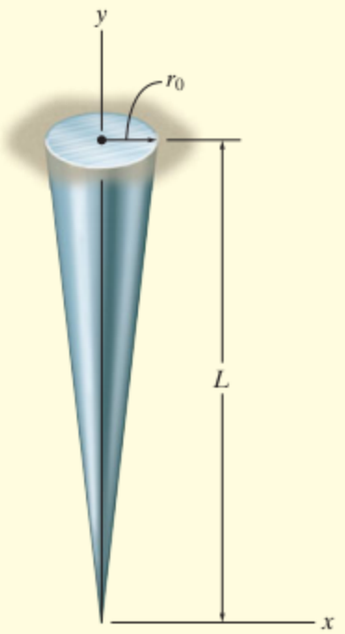


(d)

Fig. 9–8

EXAMPLE 9.4

A member is made from a material that has a specific weight γ and modulus of elasticity E. If it is in the form of a *cone* having the dimensions shown in Fig. 9–9*a*, determine how far its end is displaced due to gravity when it is suspended in the vertical position.



(a)

SOLUTION

Internal Force. The internal axial force varies along the member since it is dependent on the weight $W(y)$ of a segment of the member below any section, Fig. 9–9*b*. Hence, to calculate the displacement, we must use Eq. 9–1. At the section located a distance y from its free end, the radius x of the cone as a function of y is determined by proportion; i.e.,

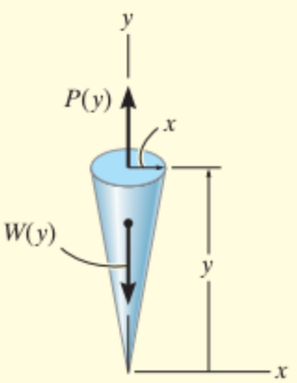
$$\frac{x}{y} = \frac{r_0}{L}; \qquad x = \frac{r_0}{L}y$$

The volume of a cone having a base of radius x and height y is

$$V = \frac{1}{3}\pi y x^2 = \frac{\pi r_0^2}{3L^2}y^3$$

Since $W = \gamma V$, the internal force at the section becomes

$$+\uparrow \Sigma F_y = 0; \qquad P(y) = \frac{\gamma \pi r_0^2}{3L^2}y^3$$



(b)

Fig. 9–9

Displacement. The area of the cross section is also a function of position y, Fig. 9–9*b*. We have

$$A(y) = \pi x^2 = \frac{\pi r_0^2}{L^2}y^2$$

Applying Eq. 9–1 between the limits of $y = 0$ and $y = L$ yields

$$\delta = \int_0^L \frac{P(y)\,dy}{A(y)E} = \int_0^L \frac{\left[(\gamma \pi r_0^2/3L^2)y^3\right]dy}{\left[(\pi r_0^2/L^2)y^2\right]E}$$

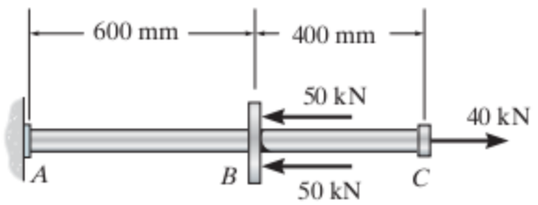
$$= \frac{\gamma}{3E}\int_0^L y\,dy$$

$$= \frac{\gamma L^2}{6E} \qquad \textit{Ans.}$$

NOTE: As a partial check of this result, notice how the units of the terms, when canceled, give the displacement in units of length as expected.

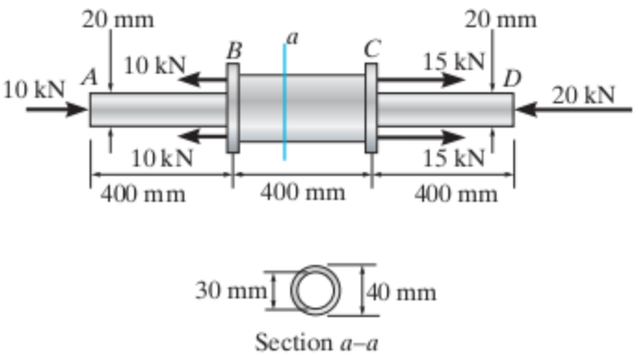
FUNDAMENTAL PROBLEMS

F9–1. The 20-mm-diameter A-36 steel rod is subjected to the axial forces shown. Determine the displacement of end C with respect to the fixed support at A.



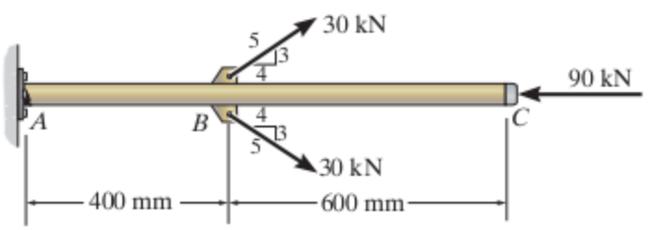
F9–1

F9–2. Segments AB and CD of the assembly are solid circular rods, and segment BC is a tube. If the assembly is made of 6061-T6 aluminum, determine the displacement of end D with respect to end A.



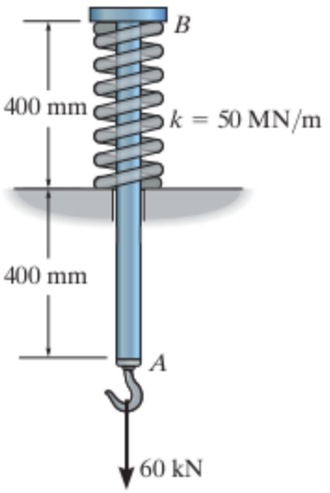
F9–2

F9–3. The 30-mm-diameter A-36 steel rod is subjected to the loading shown. Determine the displacement of end A with respect to end C.



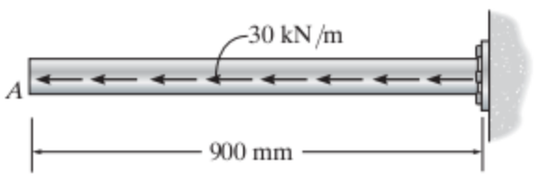
F9–3

F9–4. If the 20-mm-diameter rod is made of A-36 steel and the stiffness of the spring is $k = 50\ \text{MN/m}$, determine the displacement of end A when the 60-kN force is applied.



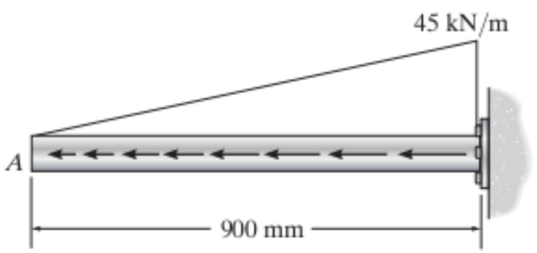
F9–4

F9–5. The 20-mm-diameter 2014-T6 aluminum rod is subjected to the uniform distributed axial load. Determine the displacement of end A.



F9–5

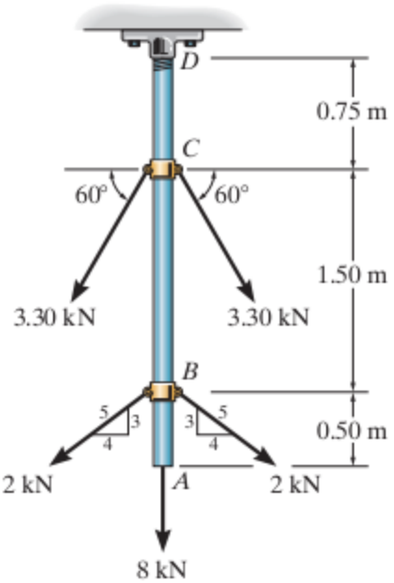
F9–6. The 20-mm-diameter 2014-T6 aluminum rod is subjected to the triangular distributed axial load. Determine the displacement of end A.



F9–6

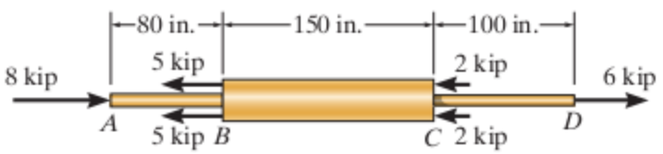
PROBLEMS

9–1. The A992 steel rod is subjected to the loading shown. If the cross-sectional area of the rod is 60 mm^2, determine the displacement of B and A, Neglect the size of the couplings at B, C, and D.



Prob. 9–1

9–2. The copper shaft is subjected to the axial loads shown. Determine the displacement of end A with respect to end D if the diameters of each segment are $d_{AB} = 0.75$ in., $d_{BC} = 1$ in., and $d_{CD} = 0.5$ in. Take $E_{cu} = 18(10^3)$ ksi.

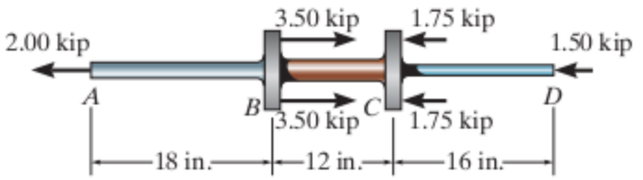


Prob. 9–2

9–3. The composite shaft, consisting of aluminum, copper, and steel sections, is subjected to the loading shown. Determine the displacement of end A with respect to end D and the normal stress in each section. The cross-sectional area and modulus of elasticity for each section are shown in the figure. Neglect the size of the collars at B and C.

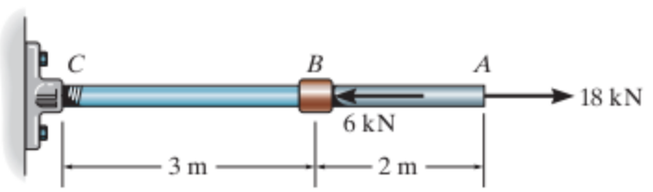
*__9–4.__ Determine the displacement of B with respect to C of the composite shaft in Prob. 9–3.

Aluminum	Copper	Steel
$E_{al} = 10(10^3)$ ksi	$E_{cu} = 18(10^3)$ ksi	$E_{st} = 29(10^3)$ ksi
$A_{AB} = 0.09$ in^2	$A_{BC} = 0.12$ in^2	$A_{CD} = 0.06$ in^2



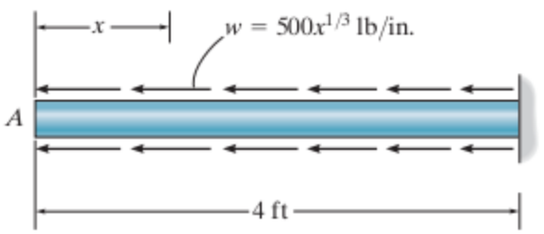
Probs. 9–3/4

9–5. The assembly consists of a steel rod CB and an aluminum rod BA, each having a diameter of 12 mm. If the rod is subjected to the axial loadings at A and at the coupling B, determine the displacement of the coupling B and the end A. The unstretched length of each segment is shown in the figure. Neglect the size of the connections at B and C, and assume that they are rigid. $E_{st} = 200$ GPa, $E_{al} = 70$ GPa.



Prob. 9–5

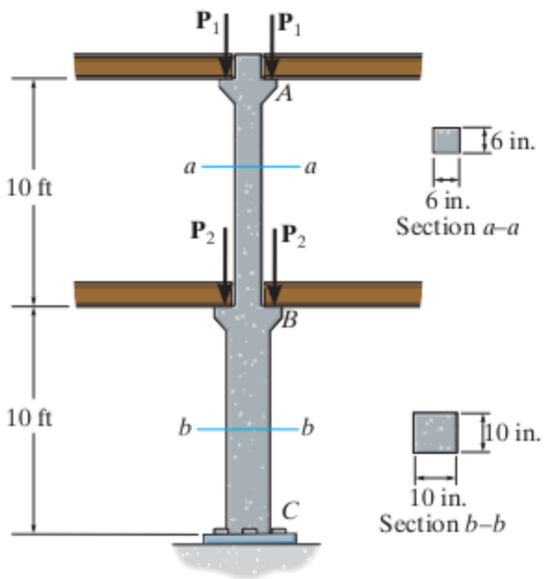
9–6. The bar has a cross-sectional area of 3 in^2, and $E = 35\ (10^3)$ ksi. Determine the displacement of its end A when it is subjected to the distributed loading.



Prob. 9–6

9–7. If $P_1 = 50$ kip and $P_2 = 150$ kip, determine the vertical displacement of end A of the high strength precast concrete column.

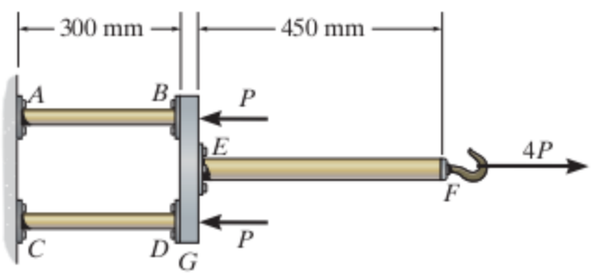
***9–8.** If the vertical displacements of end A of the high strength precast concrete column relative to B and B relative to C are 0.08 in. and 0.1 in., respectively, determine the magnitudes of $\mathbf{P}_1$ and $\mathbf{P}_2$.



Probs. 9–7/8

9–9. The assembly consists of two 10-mm diameter red brass C83400 copper rods AB and CD, a 15-mm diameter 304 stainless steel rod EF, and a rigid bar G. If $P = 5$ kN, determine the horizontal displacement of end F of rod EF.

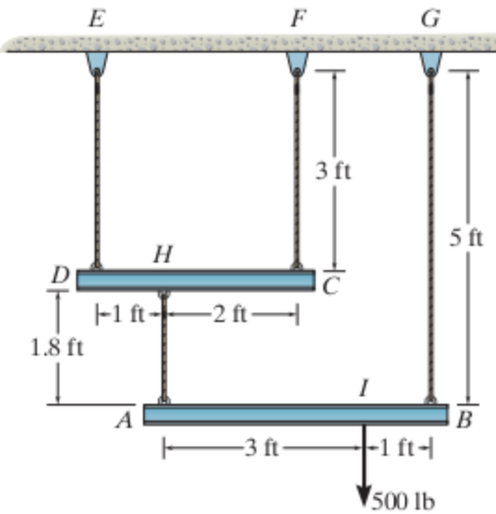
9–10. The assembly consists of two 10-mm diameter red brass C83400 copper rods AB and CD, a 15-mm diameter 304 stainless steel rod EF, and a rigid bar G. If the horizontal displacement of end F of rod EF is 0.45 mm, determine the magnitude of P.



Probs. 9–9/10

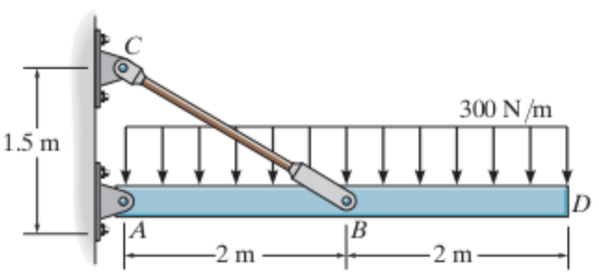
9–11. The load is supported by the four 304 stainless steel wires that are connected to the rigid members AB and DC. Determine the vertical displacement of the 500-lb load if the members were originally horizontal when the load was applied. Each wire has a cross-sectional area of 0.025 in^2.

***9–12.** The load is supported by the four 304 stainless steel wires that are connected to the rigid members AB and DC. Determine the angle of tilt of each member after the 500-lb load is applied. The members were originally horizontal, and each wire has a cross-sectional area of 0.025 in^2.



Probs. 9–11/12

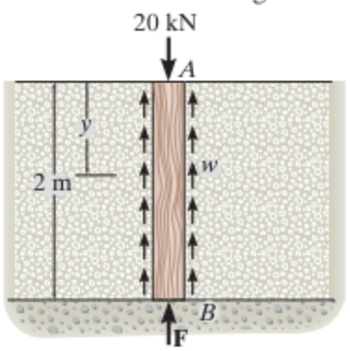
9–13. The rigid bar is supported by the pin-connected rod CB that has a cross-sectional area of 14 mm^2 and is made from 6061-T6 aluminum. Determine the vertical deflection of the bar at D when the distributed load is applied.



Prob. 9–13

9–14. The post is made of Douglas fir and has a diameter of 60 mm. If it is subjected to the load of 20 kN and the soil provides a frictional resistance that is uniformly distributed along its sides of $w = 4$ kN/m, determine the force **F** at its bottom needed for equilibrium. Also, what is the displacement of the top of the post A with respect to its bottom B? Neglect the weight of the post.

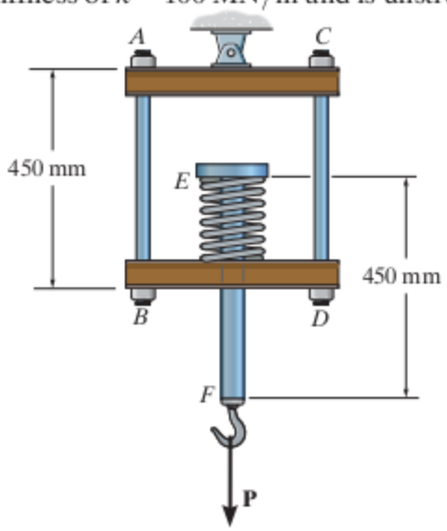
9–15. The post is made of Douglas fir and has a diameter of 60 mm. If it is subjected to the load of 20 kN and the soil provides a frictional resistance that is distributed along its length and varies linearly from $w = 0$ at $y = 0$ to $w = 3$ kN/m at $y = 2$ m, determine the force **F** at its bottom needed for equilibrium. Also, what is the displacement of the top of the post A with respect to its bottom B? Neglect the weight of the post.



Probs. 9–14/15

***9–16.** The hanger consists of three 2014-T6 aluminum alloy rods, rigid beams AC and BD, and a spring. If the hook supports a load of $P = 60$ kN, determine the vertical displacement of F. Rods AB and CD each have a diameter of 10 mm, and rod EF has a diameter of 15 mm. The spring has a stiffness of $k = 100$ MN/m and is unstretched when $P = 0$.

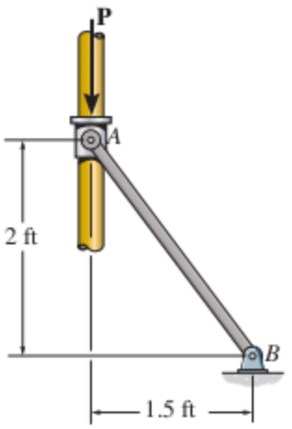
9–17. The hanger consists of three 2014-T6 aluminum alloy rods, rigid beams AC and BD, and a spring. If the vertical displacement of end F is 5 mm, determine the magnitude of the load **P**. Rods AB and CD each have a diameter of 10 mm, and rod EF has a diameter of 15 mm. The spring has a stiffness of $k = 100$ MN/m and is unstretched when $P = 0$.



Probs. 9–16/17

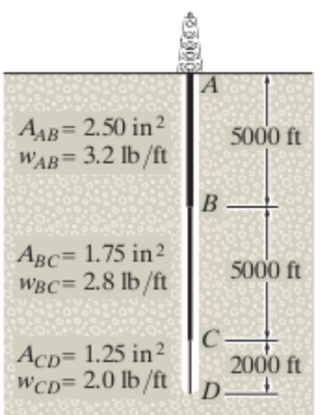
9–18. Collar A can slide freely along the smooth vertical guide. If the supporting rod AB is made of 304 stainless steel and has a diameter of 0.75 in., determine the vertical displacement of the collar when $P = 10$ kip.

9–19. Collar A can slide freely along the smooth vertical guide. If the vertical displacement of the collar is 0.035 in. and the supporting 0.75 in. diameter rod AB is made of 304 stainless steel, determine the magnitude of **P**.



Probs. 9–18/19

***9–20.** The A992 steel drill shaft of an oil well extends 12000 ft into the ground. Assuming that the pipe used to drill the well is suspended freely from the derrick at A, determine the maximum average normal stress in each pipe segment and the elongation of its end D with respect to the fixed end at A. The shaft consists of three different sizes of pipe, AB, BC, and CD, each having the length, weight per unit length, and cross-sectional area indicated.

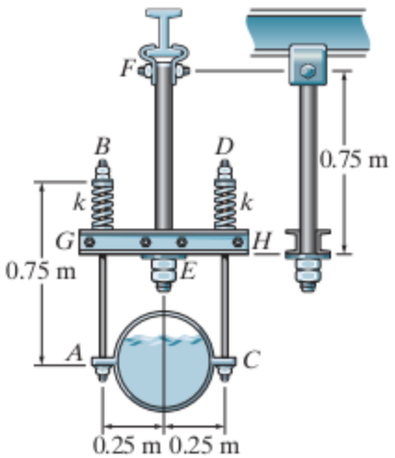


Prob. 9–20

9

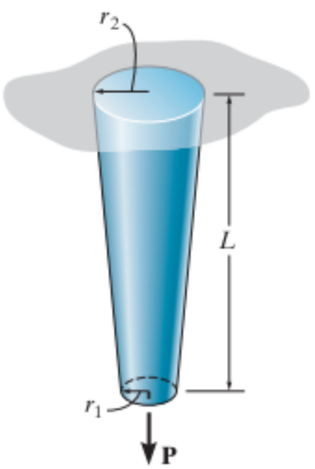
9–21. A spring-supported pipe hanger consists of two springs which are originally unstretched and have a stiffness of $k = 60\ \text{kN/m}$, three 304 stainless steel rods, AB and CD, which have a diameter of 5 mm, and EF, which has a diameter of 12 mm, and a rigid beam GH. If the pipe and the fluid it carries have a total weight of 4 kN, determine the displacement of the pipe when it is attached to the support.

9–22. A spring-supported pipe hanger consists of two springs, which are originally unstretched and have a stiffness of $k = 60\ \text{kN/m}$, three 304 stainless steel rods, AB and CD, which have a diameter of 5 mm, and EF, which has a diameter of 12 mm, and a rigid beam GH. If the pipe is displaced 82 mm when it is filled with fluid, determine the combined weight of the pipe and fluid.



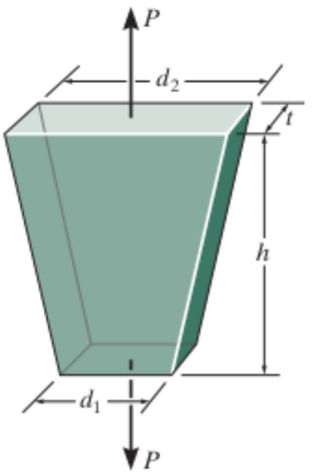
Probs. 9–21/22

9–23. The rod has a slight taper and length L. It is suspended from the ceiling and supports a load **P** at its end. Show that the displacement of its end due to this load is $\delta = PL/(\pi E r_2 r_1)$. Neglect the weight of the material. The modulus of elasticity is E.



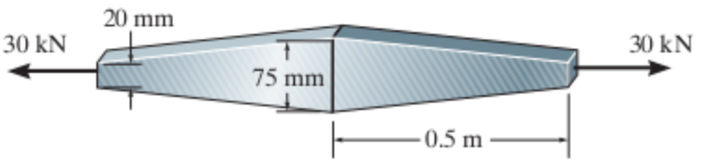
Prob. 9–23

***9–24.** Determine the relative displacement of one end of the tapered plate with respect to the other end when it is subjected to an axial load P.



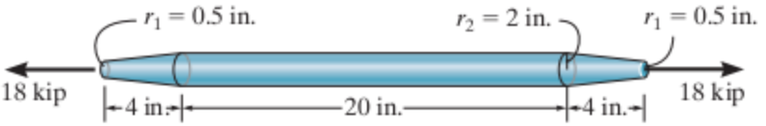
Prob. 9–24

9–25. Determine the elongation of the A-36 steel member when it is subjected to an axial force of 30 kN. The member is 10 mm thick. Use the result of Prob. 9–24.



Prob. 9–25

9–26. Determine the elongation of the tapered A992 steel shaft when it is subjected to an axial force of 18 kip. *Hint:* Use the result of Prob. 9–23.



Prob. 9–26

9.3 Principle of Superposition

The principle of superposition is often used to determine the stress or displacement at a point in a member when the member is subjected to a complicated loading. By subdividing the loading into components, the ***principle of superposition*** states that the resultant stress or displacement at the point can be determined by algebraically summing the stress or displacement caused by each load component applied separately to the member.

The following two conditions must be satisfied if the principle of superposition is to be applied.

- ***The loading must be linearly related to the stress or displacement that is to be determined.*** For example, the equations $\sigma = P/A$ and $\delta = PL/AE$ involve a linear relationship between P and σ or δ.

- ***The loading must not significantly change the original geometry or configuration of the member.*** If significant changes do occur, the direction and location of the applied forces and their moment arms will change. For example, consider the slender rod shown in Fig. 9–10*a*, which is subjected to the load **P**. In Fig. 9–10*b*, **P** is replaced by two of its components, $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$. If **P** causes the rod to deflect a large amount, as shown, the moment of the load about its support, Pd, will not equal the sum of the moments of its component loads, $Pd \neq P_1d_1 + P_2d_2$, because $d_1 \neq d_2 \neq d$.

This principle will be used throughout this text whenever we assume Hooke's law applies and also, the bodies that are considered will be such that the loading will produce deformations that are so small that the change in position and direction of the loading will be insignificant and can be neglected.

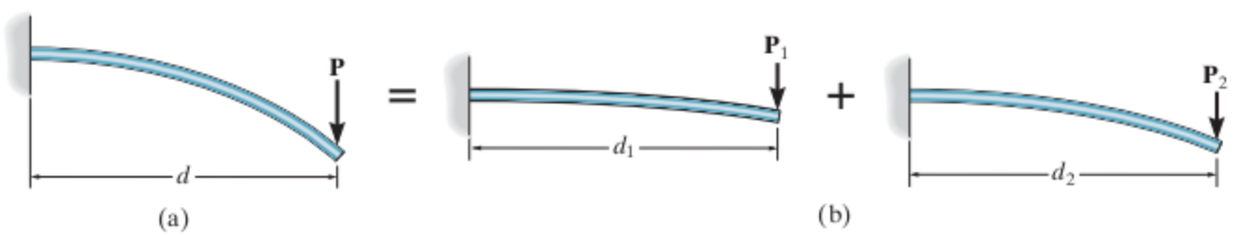
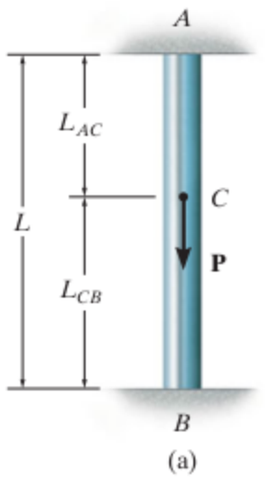


Fig. 9–10

9.4 Statically Indeterminate Axially Loaded Member



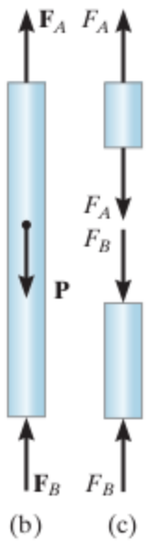


Fig. 9–11

Consider the bar shown in Fig. 9–11*a* which is fixed supported at both of its ends. From the free-body diagram, Fig. 9–11*b*, equilibrium requires

$$+\uparrow \Sigma F = 0; \qquad F_B + F_A - P = 0$$

This type of problem is called ***statically indeterminate***, since the equilibrium equation(s) are not sufficient to determine the two reactions on the bar.

In order to establish an additional equation needed for solution, it is necessary to consider how points on the bar displace. Specifically, an equation that specifies the conditions for displacement is referred to as a ***compatibility*** or ***kinematic condition***. In this case, a suitable compatibility condition would require the displacement of one end of the bar with respect to the other end to be equal to zero, since the end supports are fixed. Hence, the compatibility condition becomes

$$\delta_{A/B} = 0$$

This equation can be expressed in terms of the applied loads by using a *load–displacement relationship*, which depends on the material behavior. For example, if linear-elastic behavior occurs, $\delta = PL/AE$ can be used. Realizing that the internal force in segment AC is $+F_A$, and in segment CB the internal force is $-F_B$, Fig. 9–11*c*, the above equation can be written as

$$\frac{F_A L_{AC}}{AE} - \frac{F_B L_{CB}}{AE} = 0$$

Since AE is constant, then $F_A = F_B(L_{CB}/L_{AC})$, so that using the equilibrium equation, the equations for the reactions become

$$F_A = P\left(\frac{L_{CB}}{L}\right) \quad \text{and} \quad F_B = P\left(\frac{L_{AC}}{L}\right)$$

Since both of these results are positive, the direction of the reactions is shown correctly on the free-body diagram.

Important Points

- The *principle of superposition* is sometimes used to simplify stress and displacement problems having complicated loadings. This is done by subdividing the loading into components, then algebraically adding the results.
- Superposition requires that the loading be linearly related to the stress or displacement, and the loading does not significantly change the original geometry of the member.
- A problem is *statically indeterminate* if the equations of equilibrium are not sufficient to determine all the reactions on a member.
- *Compatibility conditions* specify the displacement constraints that occur at the supports or other points on a member.

Procedure for Analysis

The support reactions for statically indeterminate problems are determined by satisfying equilibrium, compatibility, and force-displacement requirements for the member.

Equilibrium.

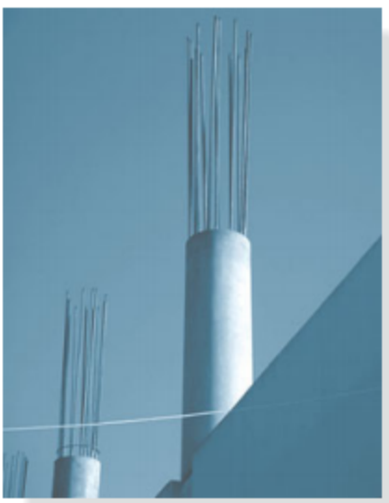
- Draw a free-body diagram of the member in order to identify all the forces that act on it.
- The problem can be classified as statically indeterminate if the number of unknown reactions on the free-body diagram is greater than the number of available equations of equilibrium.
- Write the equations of equilibrium for the member.

Compatibility.

- Consider drawing a displacement diagram in order to investigate the way the member will elongate or contract when subjected to the external loads.
- Express the compatibility conditions in terms of the displacements caused by the loading.

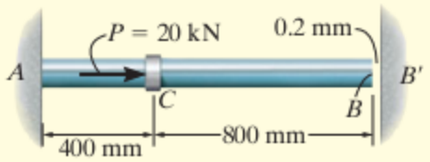
Load–Displacement.

- Use a load–displacement relation, such as $\delta = PL/AE$, to relate the unknown displacements to the reactions.
- Solve the equilibrium and compatibility equations for the reactions. If any of the results has a negative numerical value, it indicates that this force acts in the opposite sense of direction to that indicated on the free-body diagram.



Most concrete columns are reinforced with steel rods; and since these two materials work together in supporting the applied load, the forces in each material become statically indeterminate.

EXAMPLE 9.5



(a)

The steel rod shown in Fig. 9–12*a* has a diameter of 10 mm. It is fixed to the wall at A, and before it is loaded there is a gap of 0.2 mm between the wall at B' and the rod. Determine the reactions at A and B' if the rod is subjected to an axial force of $P = 20\,\text{kN}$ as shown. Neglect the size of the collar at C. Take $E_{st} = 200\,\text{GPa}$.

SOLUTION

Equilibrium. As shown on the free-body diagram, Fig. 9–12*b*, we will *assume* that force P is large enough to cause the rod's end B to contact the wall at B'. The problem is statically indeterminate since there are two unknowns and only one equation of equilibrium.

$$+\uparrow \Sigma F = 0; \qquad -F_A - F_B + 20(10^3)\,\text{N} = 0 \tag{1}$$

Compatibility. The force P causes point B to move to B', with no further displacement. Therefore the compatibility condition for the rod is

$$\delta_{B/A} = 0.0002\,\text{m}$$

(b)

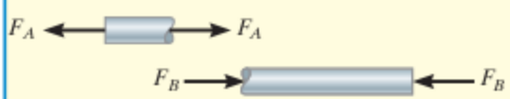
Load–Displacement. This displacement can be expressed in terms of the unknown reactions using the load–displacement relationship, Eq. 9–2, applied to segments AC and CB, Fig. 9–12*c*. Working in units of newtons and meters, we have

$$\delta_{B/A} = 0.0002\,\text{m} = \frac{F_A L_{AC}}{AE} - \frac{F_B L_{CB}}{AE}$$

$$0.0002\,\text{m} = \frac{F_A(0.4\,\text{m})}{\pi(0.005\,\text{m})^2[200(10^9)\text{N/m}^2]} - \frac{F_B(0.8\,\text{m})}{\pi(0.005\,\text{m})^2[200(10^9)\text{N/m}^2]}$$

or

$$F_A(0.4\,\text{m}) - F_B(0.8\,\text{m}) = 3141.59\,\text{N}\cdot\text{m} \tag{2}$$



(c)

Fig. 9–12

Solving Eqs. (1) and (2) yields

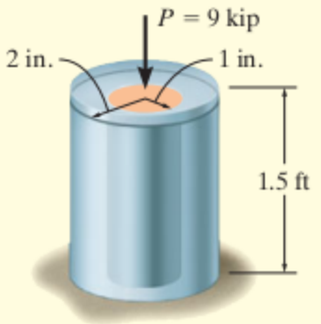
$$F_A = 16.0\,\text{kN} \qquad F_B = 4.05\,\text{kN} \qquad \textit{Ans.}$$

Since the answer for F_B is *positive*, indeed end B contacts the wall at B' as originally assumed.

NOTE: If F_B were a negative quantity, the problem would be statically determinate, so that $F_B = 0$ and $F_A = 20\,\text{kN}$.

EXAMPLE 9.6

The aluminum post shown in Fig. 9–13*a* is reinforced with a brass core. If this assembly supports an axial compressive load of $P = 9$ kip applied to the rigid cap, determine the average normal stress in the aluminum and the brass. Take $E_{al} = 10(10^3)$ ksi and $E_{br} = 15(10^3)$ ksi.



(a)

SOLUTION

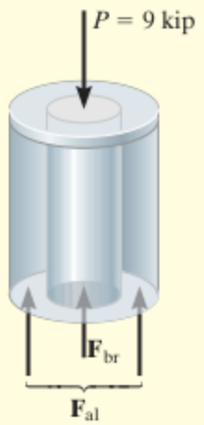
Equilibrium. The free-body diagram of the post is shown in Fig. 9–13*b*. Here the resultant axial force at the base is represented by the unknown components carried by the aluminum, $\mathbf{F}_{al}$, and brass, $\mathbf{F}_{br}$. The problem is statically indeterminate. Why?

Vertical force equilibrium requires

$$+\uparrow \Sigma F_y = 0; \qquad -9 \text{ kip} + F_{al} + F_{br} = 0 \tag{1}$$

Compatibility. The rigid cap at the top of the post causes both the aluminum and brass to displace the same amount. Therefore,

$$\delta_{al} = \delta_{br}$$



(b)

Load–Displacement. Using the load–displacement relationships,

$$\frac{F_{al}L}{A_{al}E_{al}} = \frac{F_{br}L}{A_{br}E_{br}}$$

$$F_{al} = F_{br}\left(\frac{A_{al}}{A_{br}}\right)\left(\frac{E_{al}}{E_{br}}\right)$$

$$F_{al} = F_{br}\left[\frac{\pi[(2 \text{ in.})^2 - (1 \text{ in.})^2]}{\pi(1 \text{ in.})^2}\right]\left[\frac{10(10^3) \text{ ksi}}{15(10^3) \text{ ksi}}\right]$$

$$F_{al} = 2F_{br} \tag{2}$$

Solving Eqs. (1) and (2) simultaneously yields

$$F_{al} = 6 \text{ kip} \qquad F_{br} = 3 \text{ kip}$$

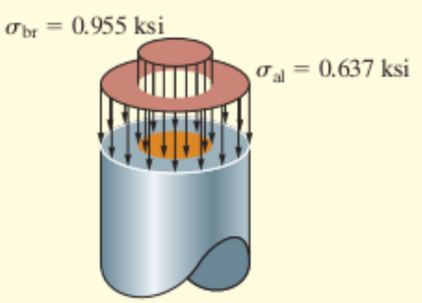
Since the results are positive, indeed the stress will be compressive.

The average normal stress in the aluminum and brass is therefore

$$\sigma_{al} = \frac{6 \text{ kip}}{\pi[(2 \text{ in.})^2 - (1 \text{ in.})^2]} = 0.637 \text{ ksi} \qquad \textit{Ans.}$$

$$\sigma_{br} = \frac{3 \text{ kip}}{\pi(1 \text{ in.})^2} = 0.955 \text{ ksi} \qquad \textit{Ans.}$$

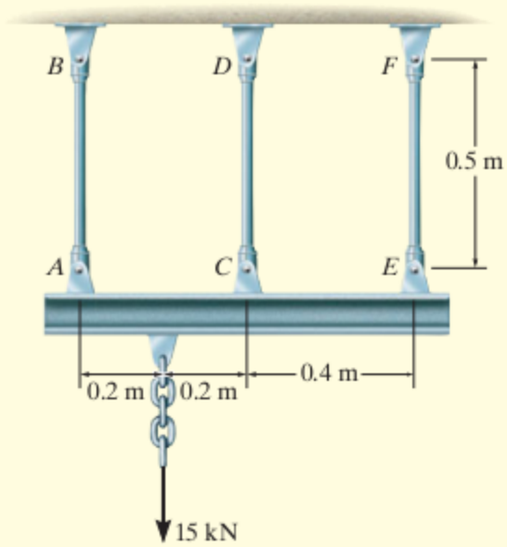
NOTE: Using these results, the stress distributions are shown in Fig. 9–13*c*.



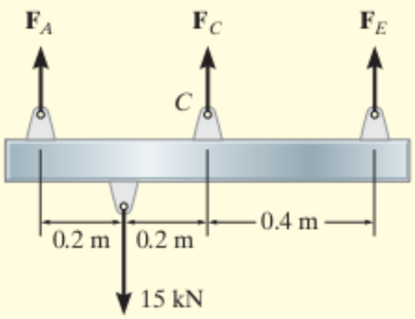
(c)

Fig. 9–13

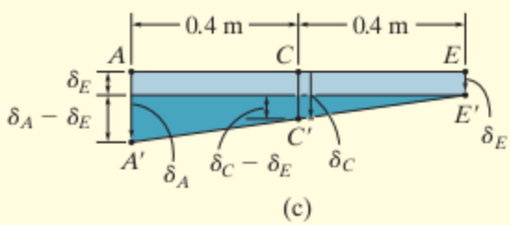
EXAMPLE 9.7



(a)



(b)



(c)

Fig. 9–14

The three A-36 steel bars shown in Fig. 9–14*a* are pin connected to a *rigid* member. If the applied load on the member is 15 kN, determine the force developed in each bar. Bars AB and EF each have a cross-sectional area of 50 mm^2, and bar CD has a cross-sectional area of 30 mm^2.

SOLUTION

Equilibrium. The free-body diagram of the rigid member is shown in Fig. 9–14*b*. This problem is statically indeterminate since there are three unknowns and only two available equilibrium equations.

$$+\uparrow \Sigma F_y = 0; \qquad F_A + F_C + F_E - 15\ \text{kN} = 0 \tag{1}$$

$$\circlearrowleft +\Sigma M_C = 0; \quad -F_A(0.4\ \text{m}) + 15\ \text{kN}(0.2\ \text{m}) + F_E(0.4\ \text{m}) = 0 \tag{2}$$

Compatibility. The applied load will cause the horizontal line ACE shown in Fig. 9–14*c* to move to the inclined line $A'C'E'$. The displacements of points A, C, and E can be related by similar triangles. Thus the compatibility equation that relates these displacements is

$$\frac{\delta_A - \delta_E}{0.8\ \text{m}} = \frac{\delta_C - \delta_E}{0.4\ \text{m}}$$

$$\delta_C = \frac{1}{2}\delta_A + \frac{1}{2}\delta_E$$

Load–Displacement. Using the load–displacement relationship, Eq. 9–2, we have

$$\frac{F_C L}{(30\ \text{mm}^2)E_{\text{st}}} = \frac{1}{2}\left[\frac{F_A L}{(50\ \text{mm}^2)E_{\text{st}}}\right] + \frac{1}{2}\left[\frac{F_E L}{(50\ \text{mm}^2)E_{\text{st}}}\right]$$

$$F_C = 0.3F_A + 0.3F_E \tag{3}$$

Solving Eqs. (1) through (3) simultaneously yields

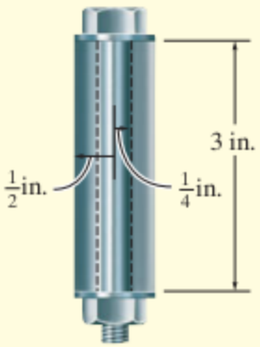
$$F_A = 9.52\ \text{kN} \quad \textit{Ans.}$$

$$F_C = 3.46\ \text{kN} \quad \textit{Ans.}$$

$$F_E = 2.02\ \text{kN} \quad \textit{Ans.}$$

EXAMPLE 9.8

The bolt shown in Fig. 9–15*a* is made of 2014-T6 aluminum alloy and is tightened so it compresses a cylindrical tube made of Am 1004-T61 magnesium alloy. The tube has an outer radius of $\frac{1}{2}$ in., and it is assumed that both the inner radius of the tube and the radius of the bolt are $\frac{1}{4}$ in. The washers at the top and bottom of the tube are considered to be rigid and have a negligible thickness. Initially the nut is hand tightened snugly; then, using a wrench, the nut is further tightened one-half turn. If the bolt has 20 threads per inch, determine the stress in the bolt.



(a)

SOLUTION

Equilibrium. The free-body diagram of a section of the bolt and the tube, Fig. 9–15*b*, is considered in order to relate the force in the bolt F_b to that in the tube, F_t. Equilibrium requires

$$+\uparrow \Sigma F_y = 0; \qquad F_b - F_t = 0 \tag{1}$$

(b)

Compatibility. When the nut is tightened on the bolt, the tube will shorten δ_t, and the bolt will *elongate* δ_b, Fig. 9–15*c*. Since the nut undergoes one-half turn, it advances a distance of $(\frac{1}{2})(\frac{1}{20}\text{ in.}) = 0.025$ in. along the bolt. Thus, the compatibility of these displacements requires

$$(+\uparrow) \qquad \delta_t = 0.025 \text{ in.} - \delta_b$$

Load–Displacement. Taking the moduli of elasticity from the table on the inside back cover, and applying Eq. 9–2, yields

$$\frac{F_t(3 \text{ in.})}{\pi[(0.5 \text{ in.})^2 - (0.25 \text{ in.})^2]\,[6.48\,(10^3)\text{ ksi}]} =$$

$$0.025 \text{ in.} - \frac{F_b(3 \text{ in.})}{\pi(0.25 \text{ in.})^2[10.6(10^3)\text{ ksi}]}$$

$$0.78595F_t = 25 - 1.4414F_b \tag{2}$$

Solving Eqs. (1) and (2) simultaneously, we get

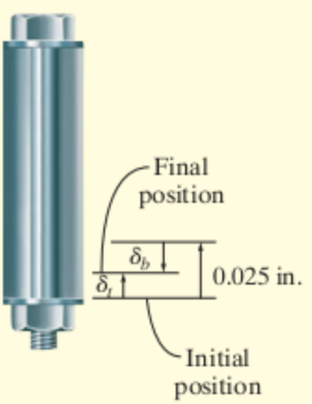
$$F_b = F_t = 11.22 \text{ kip}$$

The stresses in the bolt and tube are therefore

$$\sigma_b = \frac{F_b}{A_b} = \frac{11.22 \text{ kip}}{\pi(0.25 \text{ in.})^2} = 57.2 \text{ ksi} \qquad \textit{Ans.}$$

$$\sigma_t = \frac{F_t}{A_t} = \frac{11.22 \text{ kip}}{\pi[(0.5 \text{ in.})^2 - (0.25 \text{ in.})^2]} = 19.1 \text{ ksi}$$

These stresses are less than the reported yield stress for each material, $(\sigma_Y)_{al} = 60$ ksi and $(\sigma_Y)_{mg} = 22$ ksi (see the inside back cover), and therefore this "elastic" analysis is valid.



(c)

Fig. 9–15

9.5 The Force Method of Analysis for Axially Loaded Members

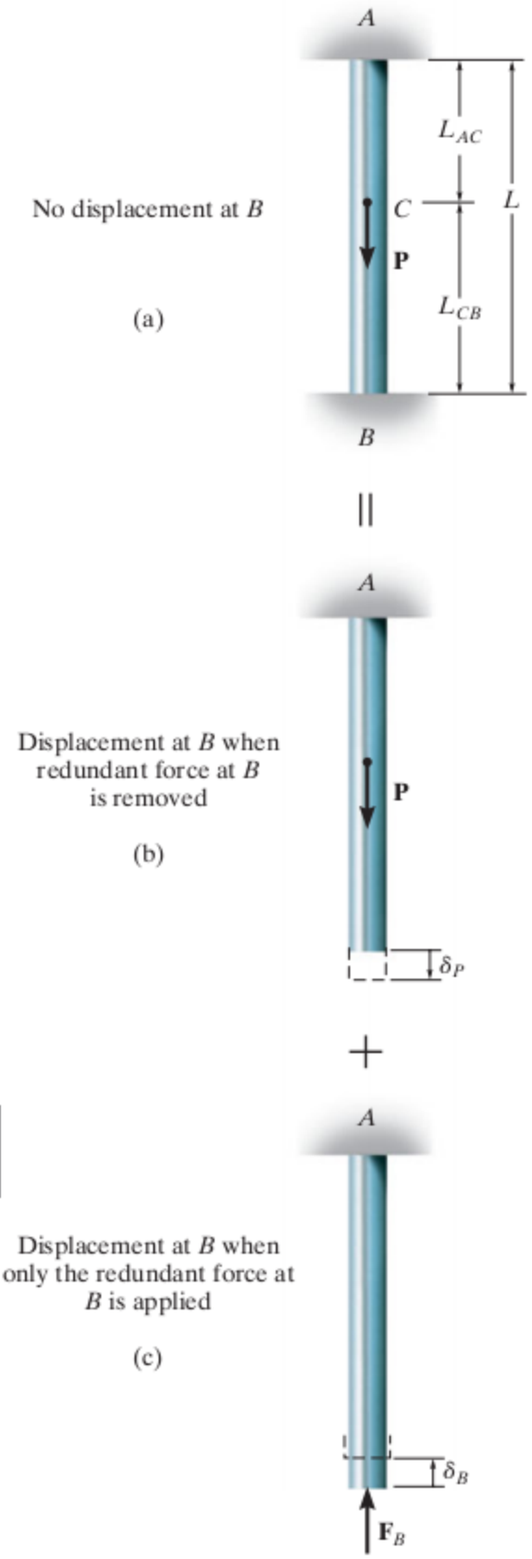


Fig. 9–16

It is also possible to solve statically indeterminate problems by writing the compatibility equation using the principle of superposition. This method of solution is often referred to as the ***flexibility*** or ***force method of analysis***. To show how it is applied, consider again the bar in Fig. 9–16*a*. If we choose the support at B as "redundant" and *temporarily* remove its effect on the bar, then the bar will become statically determinate as in Fig. 9–16*b*. By using the principle of superposition, we must add back the unknown redundant load $\mathbf{F}_B$, as shown in Fig. 9–16*c*.

If load $\mathbf{P}$ causes B to be displaced *downward* by an amount δ_P, the reaction $\mathbf{F}_B$ must displace end B of the bar *upward* by an amount δ_B, such that no displacement occurs at B when the two loadings are superimposed. Thus,

$$(+\downarrow) \qquad 0 = \delta_P - \delta_B$$

This equation represents the compatibility equation for displacements at point B, for which we have assumed that displacements are positive downward.

Applying the load–displacement relationship to each case, we have $\delta_P = PL_{AC}/AE$ and $\delta_B = F_BL/AE$. Consequently,

$$0 = \frac{PL_{AC}}{AE} - \frac{F_BL}{AE}$$

$$F_B = P\left(\frac{L_{AC}}{L}\right)$$

From the free-body diagram of the bar, Fig. 9–11*b*, the reaction at A can now be determined from the equation of equilibrium,

$$+\uparrow \Sigma F_y = 0; \qquad P\left(\frac{L_{AC}}{L}\right) + F_A - P = 0$$

Since $L_{CB} = L - L_{AC}$, then

$$F_A = P\left(\frac{L_{CB}}{L}\right)$$

These results are the same as those obtained in Sec. 9.4, except that here we have applied the condition of compatibility to obtain one reaction *and then* the equilibrium condition to obtain the other.

Procedure for Analysis

The force method of analysis requires the following steps.

Compatibility.

- Choose one of the supports as redundant and write the equation of compatibility. To do this, the known displacement at the redundant support, which is usually zero, is equated to the displacement at the support caused *only* by the external loads acting on the member, *plus* (vectorially) the displacement at this support caused *only* by the redundant reaction acting on the member.
- Express the external load and redundant displacements in terms of the loadings by using a load–displacement relationship, such as $\delta = PL/AE$.
- Once established, the compatibility equation can then be solved for the magnitude of the redundant force.

Equilibrium.

- Draw a free-body diagram and write the appropriate equations of equilibrium for the member using the calculated result for the redundant. Solve these equations for any other reactions.

EXAMPLE 9.9

The A-36 steel rod shown in Fig. 9–17*a* has a diameter of 10 mm. It is fixed to the wall at A, and before it is loaded there is a gap of 0.2 mm between the wall at B' and the rod. Determine the reactions at A and B'. Neglect the size of the collar at C. Take $E_{st} = 200$ GPa.

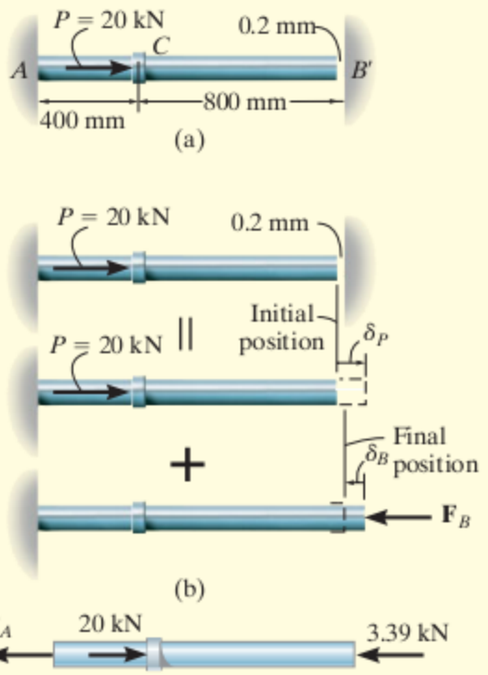


Fig. 9–17

SOLUTION

Compatibility. Here we will consider the support at B' as redundant. Using the principle of superposition, Fig. 9–17*b*, we have

$$(\stackrel{+}{\rightarrow}) \qquad 0.0002\ \text{m} = \delta_P - \delta_B \qquad (1)$$

The deflections δ_P and δ_B are determined from Eq. 9–2.

$$\delta_P = \frac{PL_{AC}}{AE} = \frac{[20(10^3)\ \text{N}](0.4\ \text{m})}{\pi(0.005\ \text{m})^2[200(10^9)\ \text{N/m}^2]} = 0.5093(10^{-3})\ \text{m}$$

$$\delta_B = \frac{F_B L_{AB}}{AE} = \frac{F_B(1.20\ \text{m})}{\pi(0.005\ \text{m})^2[200(10^9)\ \text{N/m}^2]} = 76.3944(10^{-9})F_B$$

Substituting into Eq. (1), we get

$$0.0002\ \text{m} = 0.5093(10^{-3})\ \text{m} - 76.3944(10^{-9})F_B$$

$$F_B = 4.05(10^3)\ \text{N} = 4.05\ \text{kN} \qquad \textit{Ans.}$$

Equilibrium. From the free-body diagram, Fig. 9–17*c*,

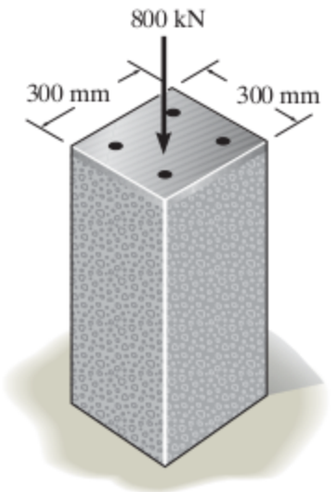
$$\stackrel{+}{\rightarrow} \Sigma F_y = 0; \quad -F_A + 20\ \text{kN} - 4.05\ \text{kN} = 0 \quad F_A = 16.0\ \text{kN} \qquad \textit{Ans.}$$

9

PROBLEMS

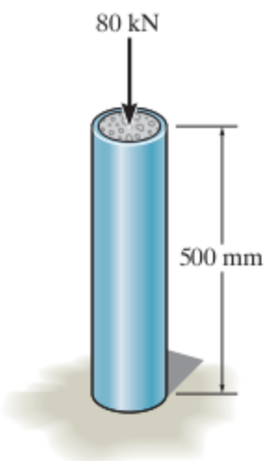
9–27. The concrete column is reinforced using four steel rods, each having a diameter of 18 mm. Determine the stress in the concrete and the steel if the column is subjected to an axial load of 800 kN. $E_{st} = 200$ GPa, $E_c = 25$ GPa.

***9–28.** The column is constructed from high-strength concrete and four A-36 steel reinforcing rods. If it is subjected to an axial force of 800 kN, determine the required diameter of each rod so that one-fourth of the load is carried by the steel and three-fourths by the concrete. $E_{st} = 200$ GPa, $E_c = 25$ GPa.



Probs. 9–27/28

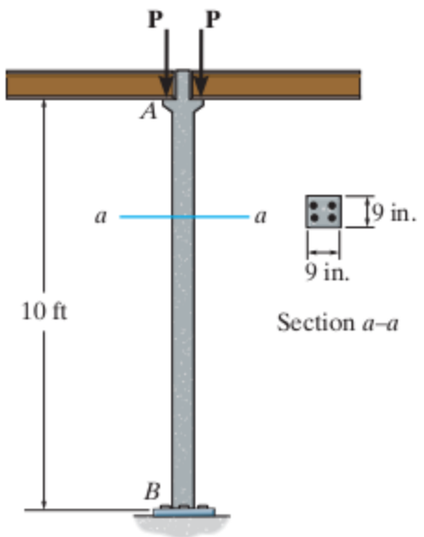
9–29. The steel pipe is filled with concrete and subjected to a compressive force of 80 kN. Determine the average normal stress in the concrete and the steel due to this loading. The pipe has an outer diameter of 80 mm and an inner diameter of 70 mm. $E_{st} = 200$ GPa, $E_c = 24$ GPa.



Prob. 9–29

9–30. If column AB is made from high strength pre-cast concrete and reinforced with four $\frac{3}{4}$ in. diameter A-36 steel rods, determine the average normal stress developed in the concrete and in each rod. Set $P = 75$ kip.

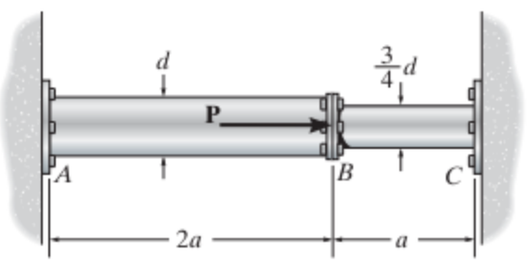
9–31. If column AB is made from high strength pre-cast concrete and reinforced with four $\frac{3}{4}$ in. diameter A-36 steel rods, determine the maximum allowable floor loadings **P**. The allowable normal stress for the high strength concrete and the steel are $(\sigma_{allow})_{con} = 2.5$ ksi and $(\sigma_{allow})_{st} = 24$ ksi, respectively.



Probs. 9–30/31

***9–32.** Determine the support reactions at the rigid supports A and C. The material has a modulus of elasticity of E.

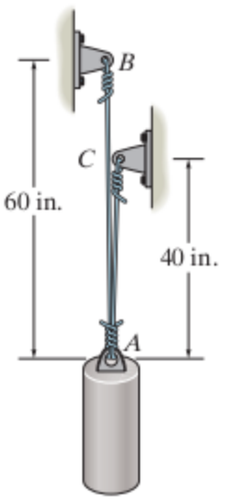
9–33. If the supports at A and C are flexible and have a stiffness k, determine the support reactions at A and C. The material has a modulus of elasticity of E.



Probs. 9–32/33

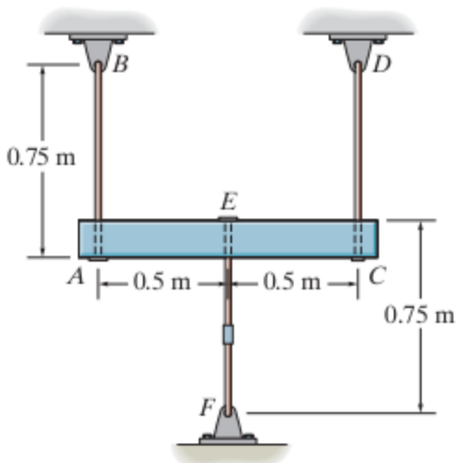
9–34. The load of 2800 lb is to be supported by the two essentially vertical A-36 steel wires. If originally wire AB is 60 in. long and wire AC is 40 in. long, determine the force developed in each wire after the load is suspended. Each wire has a cross-sectional area of 0.02 in^2.

9–35. The load of 2800 lb is to be supported by the two essentially vertical A-36 steel wires. If originally wire AB is 60 in. long and wire AC is 40 in. long, determine the cross-sectional area of AB if the load is to be shared equally between both wires. Wire AC has a cross-sectional area of 0.02 in^2.



Probs. 9–34/35

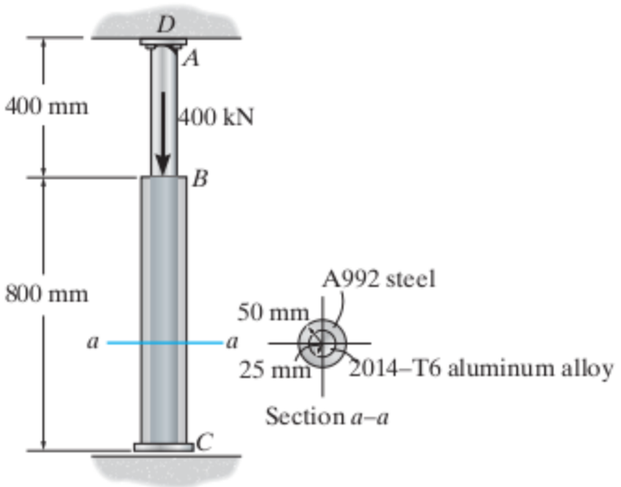
***9–36.** The rigid member is held in the position shown by three A-36 steel tie rods. Each rod has an unstretched length of 0.75 m and a cross-sectional area of 125 mm^2. Determine the forces in the rods if a turnbuckle on rod EF undergoes one full turn. The lead of the screw is 1.5 mm. Neglect the size of the turnbuckle and assume that it is rigid. *Note:* The lead would cause the rod, when *unloaded*, to shorten 1.5 mm when the turnbuckle is rotated one revolution.



Prob. 9–36

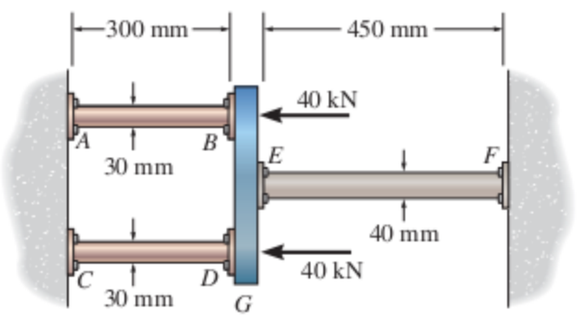
9–37. The 2014-T6 aluminum rod AC is reinforced with the firmly bonded A992 steel tube BC. If the assembly fits snugly between the rigid supports so that there is no gap at C, determine the support reactions when the axial force of 400 kN is applied. The assembly is attached at D.

9–38. The 2014-T6 aluminum rod AC is reinforced with the firmly bonded A992 steel tube BC. When no load is applied to the assembly, the gap between end C and the rigid support is 0.5 mm. Determine the support reactions when the axial force of 400 kN is applied.



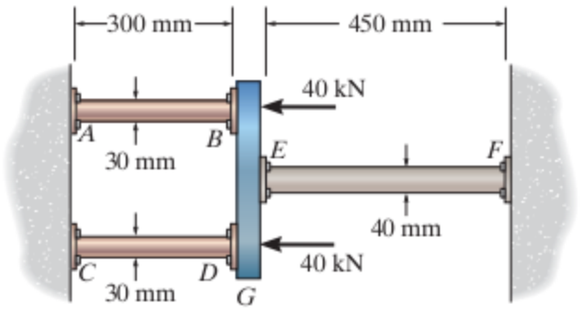
Probs. 9–37/38

9–39. The assembly consists of two red brass C83400 copper alloy rods AB and CD of diameter 30 mm, a stainless 304 steel alloy rod EF of diameter 40 mm, and a rigid cap G. If the supports at A, C, and F are rigid, determine the average normal stress developed in rods AB, CD, and EF.



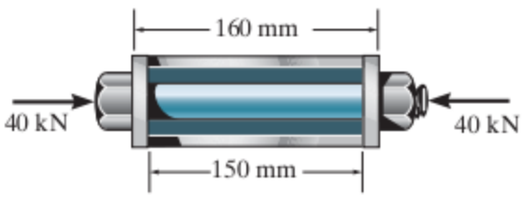
Prob. 9–39

*9–40. The assembly consists of two red brass C83400 copper rods AB and CD having a diameter of 30 mm, a 304 stainless steel rod EF having a diameter of 40 mm, and a rigid member G. If the supports at A, C, and F each have a stiffness of $k = 200$ MN/m, determine the average normal stress developed in the rods when the load is applied.



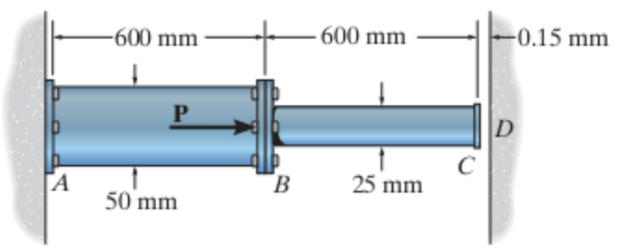
Prob. 9–40

9–41. The bolt has a diameter of 20 mm and passes through a tube that has an inner diameter of 50 mm and an outer diameter of 60 mm. If the bolt and tube are made of A-36 steel, determine the normal stress in the tube and bolt when a force of 40 kN is applied to the bolt. Assume the end caps are rigid.



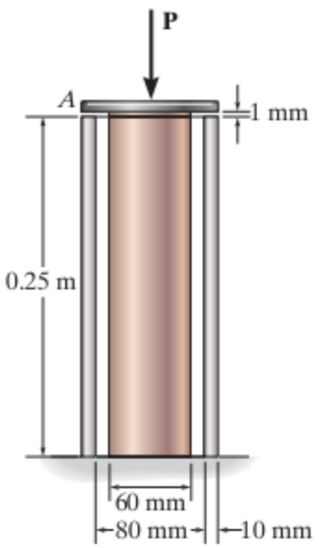
Prob. 9–41

9–42. If the gap between C and the rigid wall at D is initially 0.15 mm, determine the support reactions at A and D when the force $\mathbf{P} = 200$ kN is applied. The assembly is made of A36 steel.



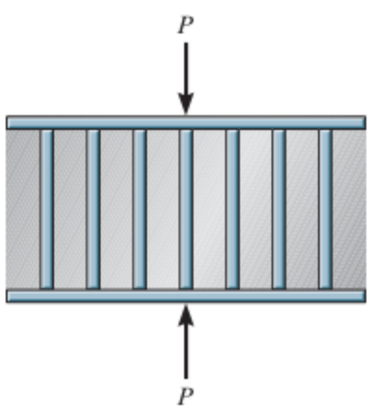
Prob. 9–42

9–43. The support consists of a solid red brass C83400 post surrounded by a 304 stainless steel tube. Before the load is applied the gap between these two parts is 1 mm. Given the dimensions shown, determine the greatest axial load that can be applied to the rigid cap A without causing yielding of any one of the materials.



Prob. 9–43

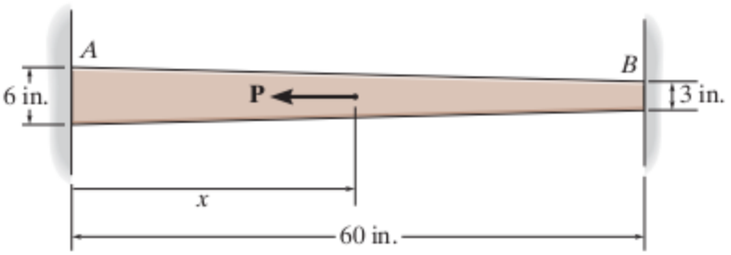
*9–44. The specimen represents a filament-reinforced matrix system made from plastic (matrix) and glass (fiber). If there are n fibers, each having a cross-sectional area of A_f and modulus of E_f, embedded in a matrix having a cross-sectional area of A_m and modulus of E_m, determine the stress in the matrix and each fiber when the force P is imposed on the specimen.



Prob. 9–44

9–45. The tapered member is fixed connected at its ends A and B and is subjected to a load $P = 7$ kip at $x = 30$ in. Determine the reactions at the supports. The material is 2 in. thick and is made from 2014-T6 aluminum.

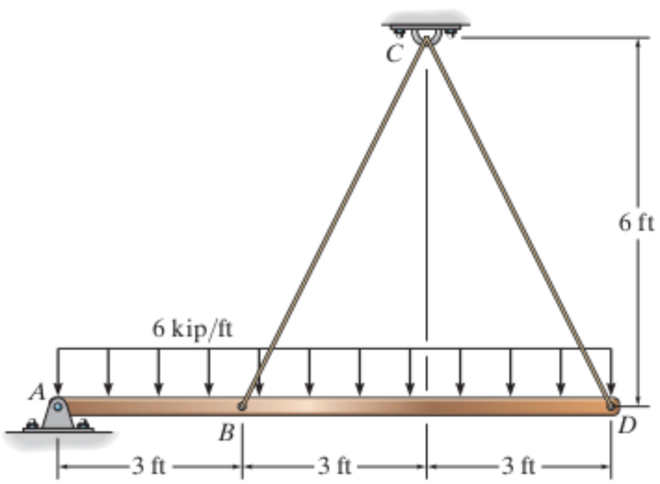
9–46. The tapered member is fixed connected at its ends A and B and is subjected to a load **P**. Determine the location x of the load and its greatest magnitude so that the average normal stress in the bar does not exceed $\sigma_{allow} = 4$ ksi. The member is 2 in. thick.



Probs. 9–45/46

9–47. The rigid bar supports the uniform distributed load of 6 kip/ft. Determine the force in each cable if each cable has a cross-sectional area of 0.05 in^2, and $E = 31(10^3)$ ksi.

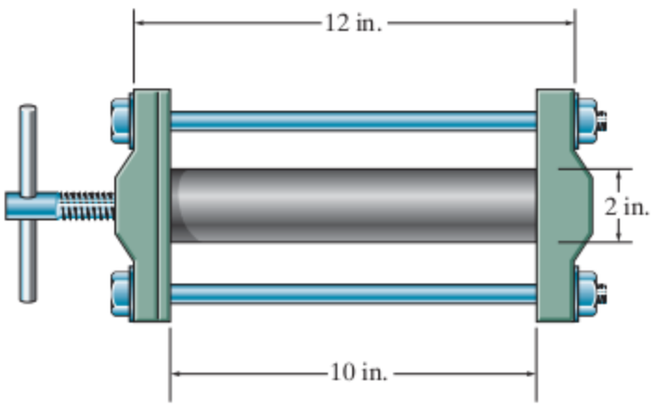
***9–48.** The rigid bar is originally horizontal and is supported by two cables each having a cross-sectional area of 0.05 in^2, and $E = 31(10^3)$ ksi. Determine the slight rotation of the bar when the uniform load is applied.



Probs. 9–47/48

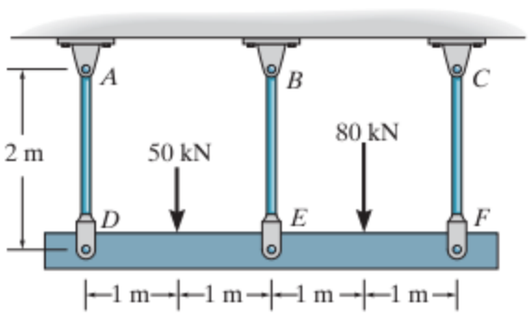
9–49. The press consists of two rigid heads that are held together by the two A-36 steel $\frac{1}{2}$-in.-diameter rods. A 6061-T6 solid aluminum cylinder is placed in the press and the screw is adjusted so that it just presses up against the cylinder. If it is then tightened one-half turn, determine the average normal stress in the rods and in the cylinder. The single-threaded screw on the bolt has a lead of 0.01 in. *Note:* The lead represents the distance the screw advances along its axis for one complete turn of the screw.

9–50. The press consists of two rigid heads that are held together by the two A-36 steel $\frac{1}{2}$-in.-diameter rods. A 6061-T6 solid aluminum cylinder is placed in the press and the screw is adjusted so that it just presses up against the cylinder. Determine the angle through which the screw can be turned before the rods or the specimen begin to yield. The single-threaded screw on the bolt has a lead of 0.01 in. *Note:* The lead represents the distance the screw advances along its axis for one complete turn of the screw.



Probs. 9–49/50

9–51. The three suspender bars are made of A-36 steel and have equal cross-sectional areas of 450 mm^2. Determine the average normal stress in each bar if the rigid beam is subjected to the loading shown.



Prob. 9–51

9

9.6 Thermal Stress

Most traffic bridges are designed with expansion joints to accommodate the thermal movement of the deck and thus avoid any thermal stress.

A change in temperature can cause a body to change its dimensions. Generally, if the temperature increases, the body will expand, whereas if the temperature decreases, it will contract.* Ordinarily this expansion or contraction is *linearly* related to the temperature increase or decrease that occurs. If this is the case, and the material is homogeneous and isotropic, it has been found from experiment that the displacement of a member having a length L can be calculated using the formula

$$\delta_T = \alpha \Delta T L \tag{9–4}$$

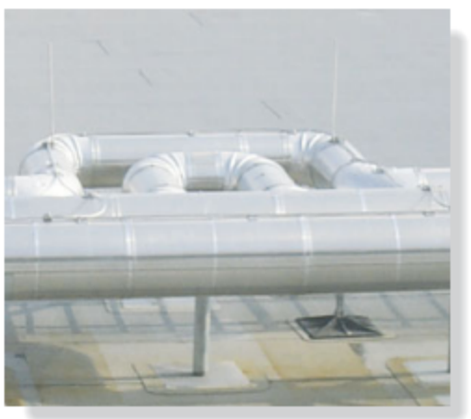
where

α = a property of the material, referred to as the ***linear coefficient of thermal expansion***. The units measure strain per degree of temperature. They are 1/°F (Fahrenheit) in the FPS system, and 1/°C (Celsius) or 1/K (Kelvin) in the SI system. Typical values are given on the inside back cover

ΔT = the algebraic change in temperature of the member

L = the original length of the member

δ_T = the algebraic change in the length of the member



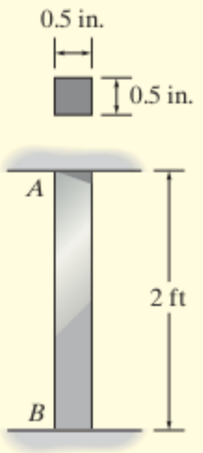
Long extensions of ducts and pipes that carry fluids are subjected to variations in climate that will cause them to expand and contract. Expansion joints, such as the one shown, are used to mitigate thermal stress in the material.

The change in length of a *statically determinate* member can easily be calculated using Eq. 9–4, since the member is free to expand or contract when it undergoes a temperature change. However, in a *statically indeterminate* member, these thermal displacements will be constrained by the supports, thereby producing ***thermal stresses*** that must be considered in design. Determining these thermal stresses is possible using the methods outlined in the previous sections. The following examples illustrate some applications.

*There are some materials that behave in the opposite way, but we will not consider this here. They include Invar®, which is an iron-nickel alloy, and scandium trifluoride.

EXAMPLE 9.10

The A-36 steel bar shown in Fig. 9–18*a* is constrained to just fit between two fixed supports when $T_1 = 60°F$. If the temperature is raised to $T_2 = 120°F$, determine the average normal thermal stress developed in the bar.



(a)

SOLUTION

Equilibrium. The free-body diagram of the bar is shown in Fig. 9–18*b*. Since there is no external load, the force at A is equal but opposite to the force at B; that is,

$$+\uparrow \Sigma F_y = 0; \qquad F_A = F_B = F$$

The problem is statically indeterminate since this force cannot be determined from equilibrium.

Compatibility. Since $\delta_{A/B} = 0$, the thermal displacement δ_T at A that occurs, Fig. 9–18*c*, is counteracted by the force **F** that is required to push the bar δ_F back to its original position. The compatibility condition at A becomes

$$(+\uparrow) \qquad \delta_{A/B} = 0 = \delta_T - \delta_F$$

Load–Displacement. Applying the thermal and load–displacement relationships, we have

$$0 = \alpha \Delta T L - \frac{FL}{AE}$$

Thus, from the data on the inside back cover,

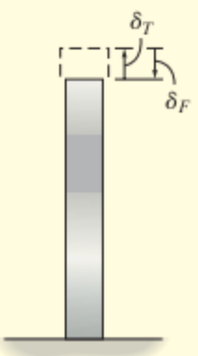
$$\begin{aligned} F &= \alpha \Delta T A E \\ &= [6.60(10^{-6})/°F](120°F - 60°F)(0.5 \text{ in.})^2[29(10^3) \text{ kip/in}^2] \\ &= 2.871 \text{ kip} \end{aligned}$$

Since **F** also represents the internal axial force within the bar, the average normal compressive stress is thus

$$\sigma = \frac{F}{A} = \frac{2.871 \text{ kip}}{(0.5 \text{ in.})^2} = 11.5 \text{ ksi} \qquad \textit{Ans.}$$

NOTE: From the magnitude of **F**, it should be apparent that changes in temperature can cause large reaction forces in statically indeterminate members.

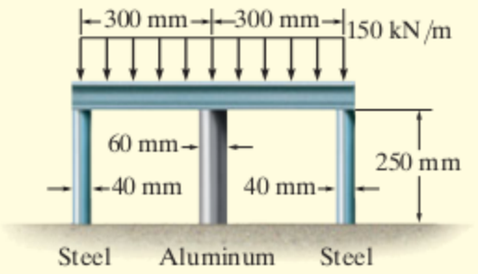
(b)



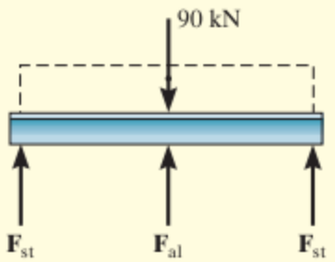
(c)

Fig. 9–18

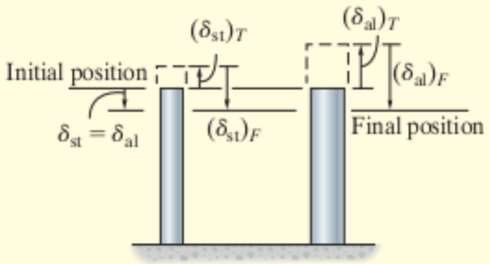
EXAMPLE 9.11



(a)



(b)



(c)

Fig. 9–19

The rigid beam shown in Fig. 9–19*a* is fixed to the top of the three posts made of A-36 steel and 2014-T6 aluminum. The posts each have a length of 250 mm when no load is applied to the beam, and the temperature is $T_1 = 20°C$. Determine the force supported by each post if the bar is subjected to a uniform distributed load of 150 kN/m and the temperature is raised to $T_2 = 80°C$.

SOLUTION

Equilibrium. The free-body diagram of the beam is shown in Fig. 9–19*b*. Moment equilibrium about the beam's center requires the forces in the steel posts to be equal. Summing forces on the free-body diagram, we have

$$+\uparrow \Sigma F_y = 0; \qquad 2F_{st} + F_{al} - 90(10^3)\,\text{N} = 0 \tag{1}$$

Compatibility. Due to load, geometry, and material symmetry, the top of each post is displaced by an equal amount. Hence,

$$(+\downarrow) \qquad \delta_{st} = \delta_{al} \tag{2}$$

The final position of the top of each post is equal to its displacement caused by the temperature increase, plus its displacement caused by the internal axial compressive force, Fig. 9–19*c*. Thus, for the steel and aluminum post, we have

$$(+\downarrow) \qquad \delta_{st} = -(\delta_{st})_T + (\delta_{st})_F$$
$$(+\downarrow) \qquad \delta_{al} = -(\delta_{al})_T + (\delta_{al})_F$$

Applying Eq. (2) gives

$$-(\delta_{st})_T + (\delta_{st})_F = -(\delta_{al})_T + (\delta_{al})_F$$

Load–Displacement. Using Eqs. 9–2 and 9–4 and the material properties on the inside back cover, we get

$$-[12(10^{-6})/°C](80°C - 20°C)(0.250\,\text{m}) + \frac{F_{st}(0.250\,\text{m})}{\pi(0.020\,\text{m})^2[200(10^9)\,\text{N/m}^2]}$$
$$= -[23(10^{-6})/°C](80°C - 20°C)(0.250\,\text{m}) + \frac{F_{al}(0.250\,\text{m})}{\pi(0.030\,\text{m})^2[73.1(10^9)\,\text{N/m}^2]}$$

$$F_{st} = 1.216F_{al} - 165.9(10^3) \tag{3}$$

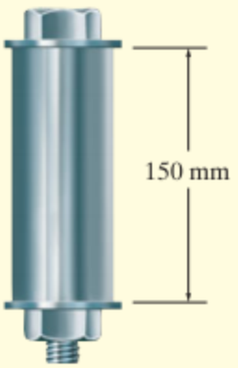
To be *consistent*, all numerical data has been expressed in terms of newtons, meters, and degrees Celsius. Solving Eqs. (1) and (3) simultaneously yields

$$F_{st} = -16.4\,\text{kN} \quad F_{al} = 123\,\text{kN} \qquad \textit{Ans.}$$

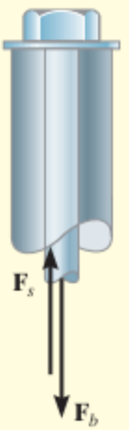
The negative value for F_{st} indicates that this force acts opposite to that shown in Fig. 9–19*b*. In other words, the steel posts are in tension and the aluminum post is in compression.

EXAMPLE 9.12

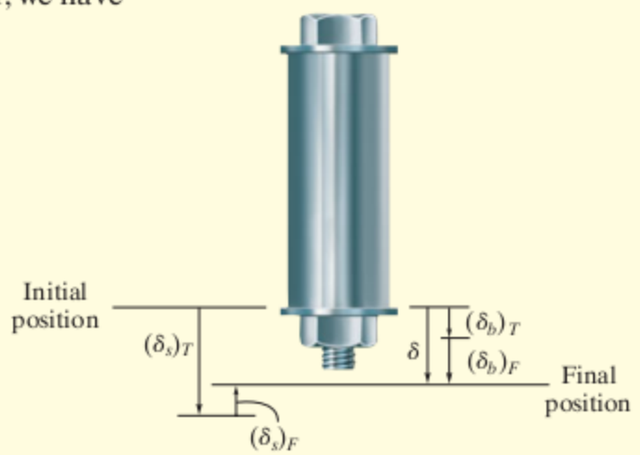
A 2014-T6 aluminum tube having a cross-sectional area of 600 mm^2 is used as a sleeve for an A-36 steel bolt having a cross-sectional area of 400 mm^2, Fig. 9–20*a*. When the temperature is $T_1 = 15°C$, the nut holds the assembly in a snug position such that the axial force in the bolt is negligible. If the temperature increases to $T_2 = 80°C$, determine the force in the bolt and sleeve.



(a)



(b)



(c)

Fig. 9–20

SOLUTION

Equilibrium. The free-body diagram of a top segment of the assembly is shown in Fig. 9–20*b*. The forces F_b and F_s are produced since the sleeve has a higher coefficient of thermal expansion than the bolt, and therefore the sleeve will expand more when the temperature is increased. It is required that

$$+\uparrow \Sigma F_y = 0; \qquad F_s = F_b \tag{1}$$

Compatibility. The temperature increase causes the sleeve and bolt to expand $(\delta_s)_T$ and $(\delta_b)_T$, Fig. 9–20*c*. However, the redundant forces F_b and F_s elongate the bolt and shorten the sleeve. Consequently, the end of the assembly reaches a final position, which is not the same as its initial position. Hence, the compatibility condition becomes

$$(+\downarrow) \qquad \delta = (\delta_b)_T + (\delta_b)_F = (\delta_s)_T - (\delta_s)_F$$

Load–Displacement. Applying Eqs. 9–2 and 9–4, and using the mechanical properties from the table on the inside back cover, we have

$$[12(10^{-6})/°C](80°C - 15°C)(0.150\ m) + \frac{F_b(0.150\ m)}{(400\ mm^2)(10^{-6}\ m^2/mm^2)[200(10^9)\ N/m^2]} = [23(10^{-6})/°C](80°C - 15°C)(0.150\ m) - \frac{F_s(0.150\ m)}{(600\ mm^2)(10^{-6}\ m^2/mm^2)[73.1(10^9)\ N/m^2]}$$

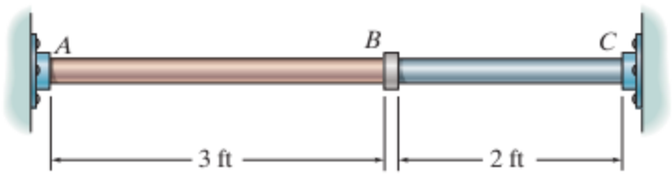
Using Eq. (1) and solving gives

$$F_s = F_b = 20.3\ kN \qquad \textit{Ans.}$$

NOTE: Since linear elastic material behavior was assumed in this analysis, the average normal stresses should be checked to make sure that they do not exceed the proportional limits for the material.

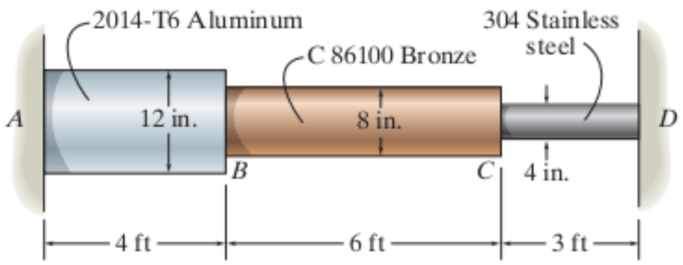
PROBLEMS

***9–52.** The C83400 red brass rod AB and 2014-T6 aluminum rod BC are joined at the collar B and fixed connected at their ends. If there is no load in the members when $T_1 = 50°F$, determine the average normal stress in each member when $T_2 = 120°F$. Also, how far will the collar be displaced? The cross-sectional area of each member is 1.75 in^2.



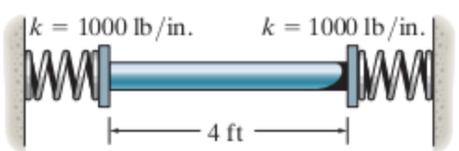
Prob. 9–52

9–53. The assembly has the diameters and material make-up indicated. If it fits securely between its fixed supports when the temperature is $T_1 = 70°F$, determine the average normal stress in each material when the temperature reaches $T_2 = 110°F$.



Prob. 9–53

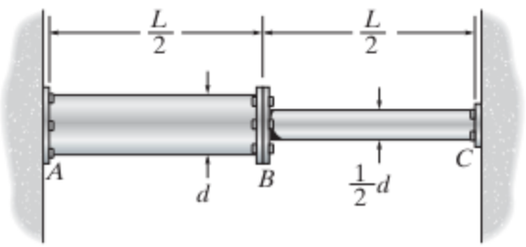
9–54. The rod is made of A-36 steel and has a diameter of 0.25 in. If the rod is 4 ft long when the springs are compressed 0.5 in. and the temperature of the rod is $T = 40°F$, determine the force in the rod when its temperature is $T = 160°F$.



Prob. 9–54

9–55. If the assembly fits snugly between two rigid supports A and C when the temperature is at T_1, determine the normal stress developed in both rod segments when the temperature rises to T_2. Both segments are made of the same material, having a modulus of elasticity of E and coefficient of thermal expansion of α.

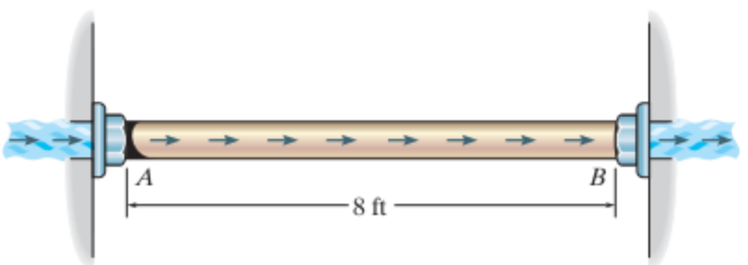
***9–56.** If the assembly fits snugly between the two supports A and C when the temperature is at T_1, determine the normal stress developed in both segments when the temperature rises to T_2. Both segments are made of the same material having a modulus of elasticity of E and coefficient of the thermal expansion of α. The flexible supports at A and C each have a stiffness k.



Probs. 9–55/56

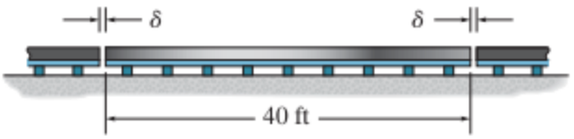
9–57. The pipe is made of A-36 steel and is connected to the collars at A and B. When the temperature is 60°F, there is no axial load in the pipe. If hot gas traveling through the pipe causes its temperature to rise by $\Delta T = (40 + 15x)°F$, where x is in feet, determine the average normal stress in the pipe. The inner diameter is 2 in., the wall thickness is 0.15 in.

9–58. The bronze C86100 pipe has an inner radius of 0.5 in. and a wall thickness of 0.2 in. If the gas flowing through it changes the temperature of the pipe uniformly from $T_A = 200°F$ at A to $T_B = 60°F$ at B, determine the axial force it exerts on the walls. The pipe was fitted between the walls when $T = 60°F$.



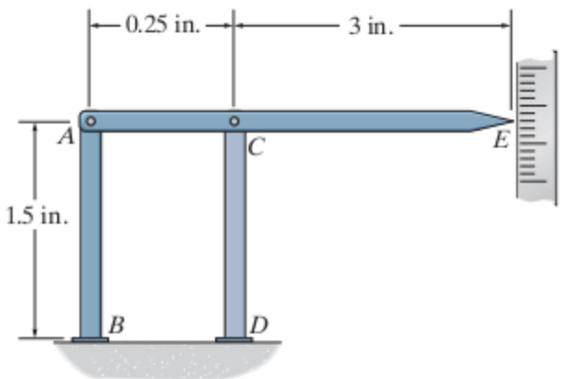
Probs. 9–57/58

9–59. The 40-ft-long A-36 steel rails on a train track are laid with a small gap between them to allow for thermal expansion. Determine the required gap δ so that the rails just touch one another when the temperature is increased from $T_1 = -20°F$ to $T_2 = 90°F$. Using this gap, what would be the axial force in the rails if the temperature were to rise to $T_3 = 110°F$? The cross-sectional area of each rail is 5.10 in^2.



Prob. 9–59

***9–60.** The device is used to measure a change in temperature. Bars AB and CD are made of A-36 steel and 2014-T6 aluminum alloy, respectively. When the temperature is at 75°F, ACE is in the horizontal position. Determine the vertical displacement of the pointer at E when the temperature rises to 150°F.

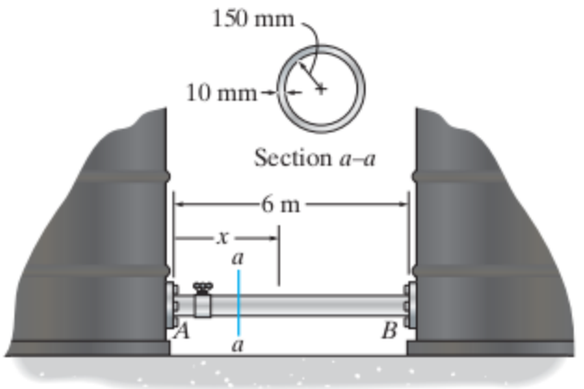


Prob. 9–60

9–61. The bar has a cross-sectional area A, length L, modulus of elasticity E, and coefficient of thermal expansion α. The temperature of the bar changes uniformly along its length from T_A at A to T_B at B so that at any point x along the bar $T = T_A + x(T_B - T_A)/L$. Determine the force the bar exerts on the rigid walls. Initially no axial force is in the bar and the bar has a temperature of T_A.

Prob. 9–61

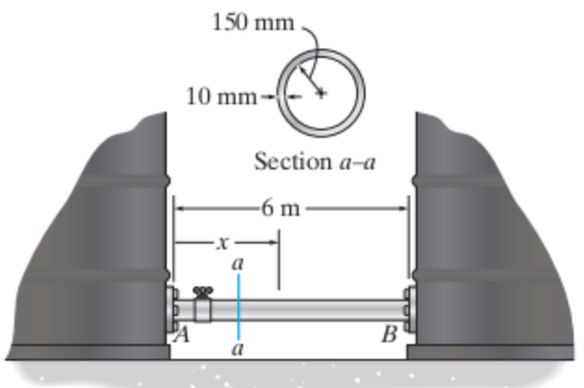
9–62. When the temperature is at 30°C, the A-36 steel pipe fits snugly between the two fuel tanks. When fuel flows through the pipe, the temperatures at ends A and B rise to 130°C and 80°C, respectively. If the temperature drop along the pipe is linear, determine the average normal stress developed in the pipe. Assume each tank provides a rigid support at A and B.



Prob. 9–62

9–63. When the temperature is at 30°C, the A-36 steel pipe fits snugly between the two fuel tanks. When fuel flows through the pipe, the temperatures at ends A and B rise to 130°C and 80°C, respectively. If the temperature drop along the pipe is linear, determine the average normal stress developed in the pipe. Assume the walls of each tank act as a spring, each having a stiffness of k= 900 MN/m.

***9–64.** When the temperature is at 30°C, the A-36 steel pipe fits snugly between the two fuel tanks. When fuel flows through the pipe, it causes the temperature to vary along the pipe as $T = (\frac{5}{3}x^2 - 20x + 120)°C$, where x is in meters. Determine the normal stress developed in the pipe. Assume each tank provides a rigid support at A and B.



Probs. 9–63/64

9.7 Stress Concentrations

This saw blade has grooves cut into it in order to relieve both the dynamic that develops within it as it rotates and the thermal stress that develops as it heats up. Note the small circles at the end of each groove. These serve to reduce the stress concentrations that develop at the end of each groove.

In Sec. 9.1 it was pointed out that when an axial force is applied to a member, it creates a complex stress distribution within the localized region of the point of load application. Not only do complex stress distributions arise just under a concentrated loading, they can also arise at sections where the member's cross-sectional area changes. For example, consider the bar in Fig. 9–21*a*, which is subjected to an axial force P. Here the once horizontal and vertical grid lines deflect into an irregular pattern around the hole centered in the bar. The maximum normal stress in the bar occurs on section *a*–*a*, which is taken through the bar's *smallest* cross-sectional area. Provided the material behaves in a linear-elastic manner, the stress distribution acting on this section can be determined either from a mathematical analysis, using the theory of elasticity, or experimentally by measuring the strain normal to section *a*–*a* and then calculating the stress using Hooke's law, $\sigma = E\epsilon$. Regardless of the method used, the general shape of the stress distribution will be like that shown in Fig. 9–21*b*. In a similar manner, if the bar has a reduction in its cross section, achieved using shoulder fillets as in Fig. 9–22*a*, then again the maximum normal stress in the bar will occur at the *smallest* cross-sectional area, section *a*–*a*, and the stress distribution will look like that shown in Fig. 9–22*b*.

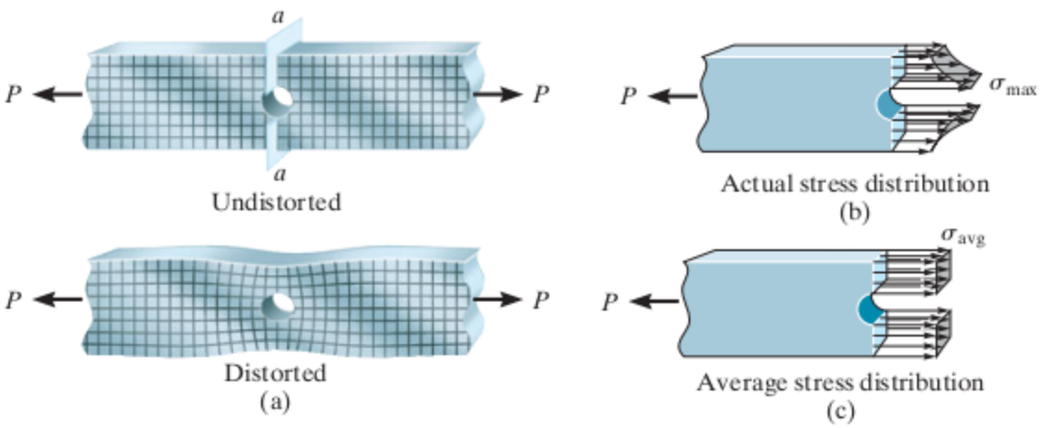


Fig. 9–21

In both of these cases, *force equilibrium* requires the magnitude of the *resultant force* developed by the stress distribution to be equal to P. In other words,

$$P = \int_A \sigma \, dA \tag{9–5}$$

This integral *graphically* represents the total *volume* under each of the stress-distribution diagrams shown in Fig. 9–21*b* or Fig. 9–22*b*. The resultant **P** must act through the *centroid* of each *volume*.

In engineering practice, the actual stress distributions in Fig. 9–21*b* and Fig. 9–22*b* do *not* have to be determined. Instead, only the *maximum stress* at these sections must be known, and the member is then designed to resist this stress when the axial load **P** is applied. Specific values of this maximum normal stress can be determined by experimental methods or by advanced mathematical techniques using the theory of elasticity. The results of these investigations are usually reported in graphical form using a ***stress-concentration factor*** K. We define K as a ratio of the maximum stress to the average normal stress acting at the cross section; i.e.,

$$K = \frac{\sigma_{\max}}{\sigma_{\text{avg}}} \tag{9–6}$$

Provided K is known, and the average normal stress has been calculated from $\sigma_{\text{avg}} = P/A$, where A is the *smallest* cross-sectional area, Figs. 9–21*c* and 9–22*c*, then the maximum normal stress at the cross section is $\sigma_{\max} = K(P/A)$.

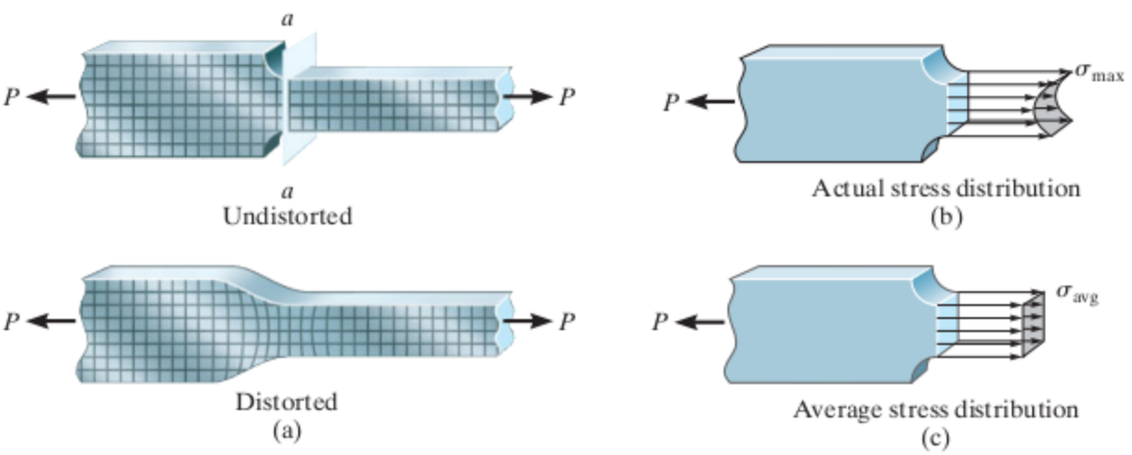


Fig. 9–22

Stress concentrations often arise at sharp corners on heavy machinery. Engineers can mitigate this effect by using stiffeners welded to the corners.

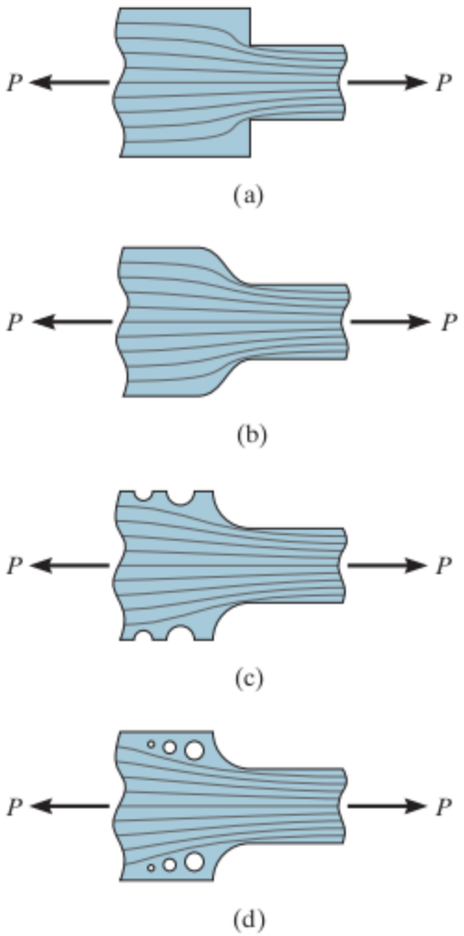


Fig. 9–23

Specific values of K are generally reported in handbooks related to stress analysis. Examples are given in Figs. 9–24 and 9–25. Note that K is independent of the bar's material properties; rather, it *depends only* on the bar's *geometry* and the type of discontinuity. As the size r of the discontinuity is *decreased*, the stress concentration is increased. For example, if a bar requires a change in cross section, it has been determined that a sharp corner, Fig. 9–23*a*, produces a stress-concentration factor greater than 3. In other words, the maximum normal stress will be three times greater than the average normal stress on the smallest cross section. However, this can be reduced to, say, 1.5 by introducing a fillet, Fig. 9–23*b*. A further reduction can be made by means of small grooves or holes placed at the transition, Fig. 9–23*c* and Fig. 9–23*d*. In all of these cases these designs help to reduce the rigidity of the material surrounding the corners, so that both the strain and the stress are more evenly spread throughout the bar.

The stress-concentration factors given in Figs. 9–24 and 9–25 were determined on the basis of a static loading, with the assumption that the stress in the material does not exceed the proportional limit. If the material is *very brittle*, the proportional limit may be at the fracture stress, and so for this material, failure will begin *at* the point of stress concentration. Essentially a crack begins to form at this point, and a higher stress concentration will develop at the *tip* of this crack. This, in turn, causes the crack to propagate over the cross section, resulting in sudden fracture. For this reason, it is very important to use stress-concentration factors in design when using brittle materials. On the other hand, if the material is ductile and subjected to a static load, it is often not necessary to use stress-concentration factors since any stress that exceeds the proportional limit will not result in a crack. Instead, the material will have reserve strength due to yielding and strain-hardening. In the next section we will discuss the effects caused by this phenomenon.

Stress concentrations are also responsible for many failures of structural members or mechanical elements subjected to *fatigue loadings*. For these cases, a stress concentration will cause the material to crack if the stress exceeds the material's endurance limit, whether or not the material is ductile or brittle. Here, the material *localized* at the tip of the crack remains in a *brittle state*, and so the crack continues to grow, leading to a progressive fracture. As a result, one must seek ways to limit the amount of damage that can be caused by fatigue.

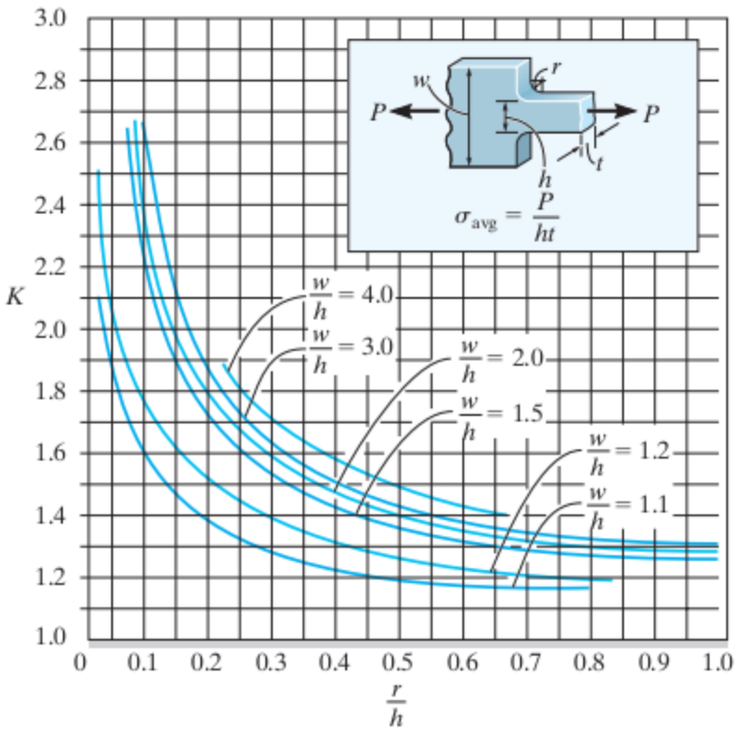


Fig. 9–24

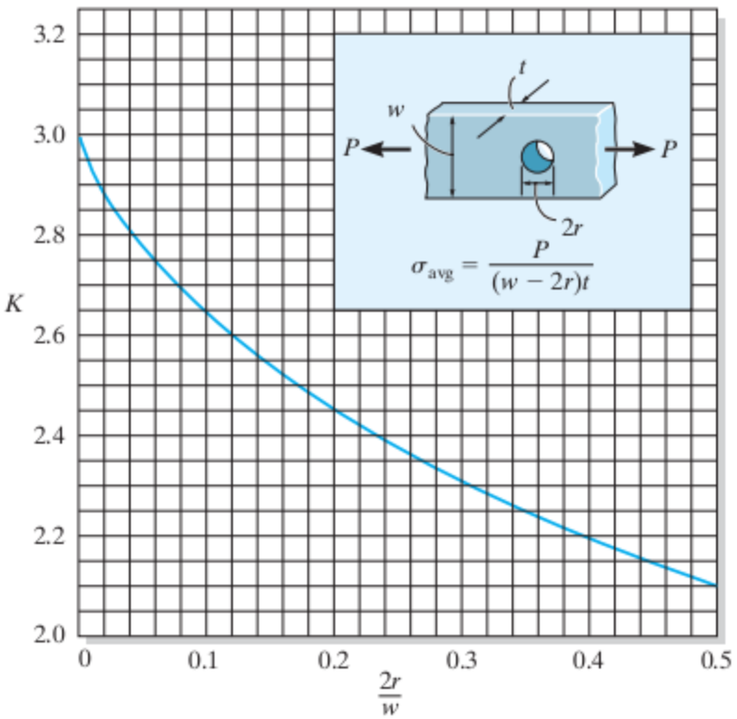
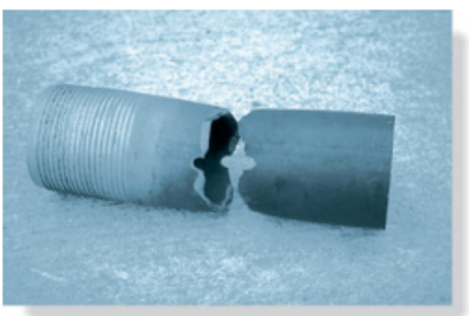


Fig. 9–25

Important Points

- *Stress concentrations* occur at sections where the cross-sectional area suddenly changes. The more severe the change, the larger the stress concentration.
- For design or analysis, it is only necessary to determine the maximum stress acting on the smallest cross-sectional area. This is done using a *stress-concentration factor, K*, that has been determined through experiment and is only a function of the geometry of the specimen.
- Normally the stress concentration in a ductile specimen that is subjected to a static loading will *not* have to be considered in design; however, if the material is *brittle*, or subjected to *fatigue* loadings, then stress concentrations become important.



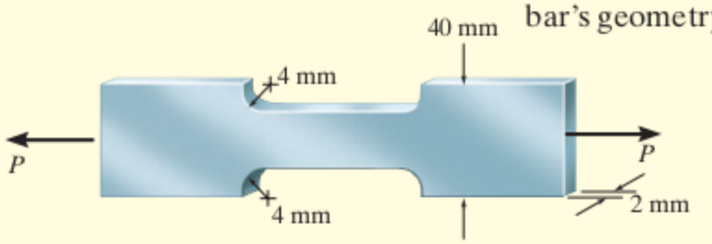
Failure of this steel pipe in tension occurred at its smallest cross-sectional area, which is through the hole. Notice how the material yielded around the fractured surface.

EXAMPLE 9.13

The bar in Fig. 9–26*a* is made of steel that is assumed to be elastic perfectly plastic, with $\sigma_Y = 250\,\text{MPa}$. Determine (a) the maximum value of the applied load P that can be applied without causing the steel to yield and (b) the maximum value of P that the bar can support. Sketch the stress distribution at the critical section for each case.

SOLUTION

Part (a). When the material behaves elastically, we must use a stress-concentration factor determined from Fig. 9–24 that is unique for the bar's geometry. Here



(a)

$$\frac{r}{h} = \frac{4\,\text{mm}}{(40\,\text{mm} - 8\,\text{mm})} = 0.125$$

$$\frac{w}{h} = \frac{40\,\text{mm}}{(40\,\text{mm} - 8\,\text{mm})} = 1.25$$

From the figure $K \approx 1.75$. The maximum load, without causing yielding, occurs when $\sigma_{\max} = \sigma_Y$. The average normal stress is $\sigma_{\text{avg}} = P/A$. Using Eq. 9–6, we have

$$\sigma_{\max} = K\sigma_{\text{avg}}; \qquad \sigma_Y = K\left(\frac{P_Y}{A}\right)$$

$$250(10^6)\,\text{Pa} = 1.75\left[\frac{P_Y}{(0.002\,\text{m})(0.032\,\text{m})}\right]$$

$$P_Y = 9.14\,\text{kN} \qquad \textit{Ans.}$$

(b)

This load has been calculated using the *smallest* cross section. The resulting stress distribution is shown in Fig. 9–26*b*. For equilibrium, the "volume" contained within this distribution must equal 9.14 kN.

(c)

Fig. 9–26

Part (b). The maximum load sustained by the bar will cause *all the material* at the smallest cross section to yield. Therefore, as P is increased to the *plastic load* P_p, it gradually changes the stress distribution from the elastic state shown in Fig. 9–26*b* to the plastic state shown in Fig. 9–26*c*. We require

$$\sigma_Y = \frac{P_p}{A}$$

$$250(10^6)\,\text{Pa} = \frac{P_p}{(0.002\,\text{m})(0.032\,\text{m})}$$

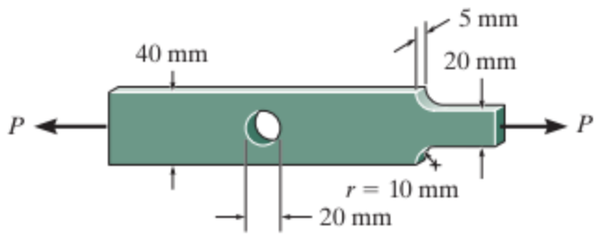
$$P_p = 16.0\,\text{kN} \qquad \textit{Ans.}$$

Here P_p equals the "volume" contained within the stress distribution, which in this case is $P_p = \sigma_Y A$.

PROBLEMS

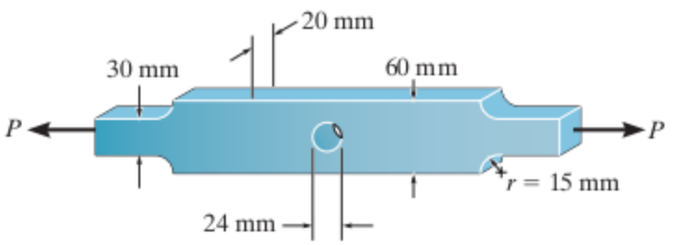
9–65. Determine the maximum normal stress developed in the bar when it is subjected to a tension of $P = 8$ kN.

9–66. If the allowable normal stress for the bar is $\sigma_{allow} = 120$ MPa, determine the maximum axial force P that can be applied to the bar.



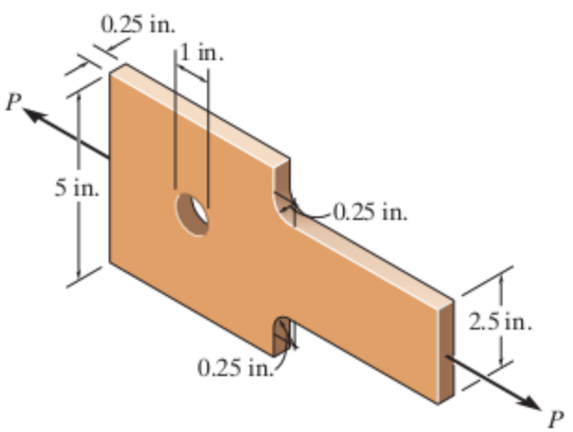
Probs. 9–65/66

9–67. The steel bar has the dimensions shown. Determine the maximum axial force P that can be applied so as not to exceed an allowable tensile stress of $\sigma_{allow} = 150$ MPa.



Prob. 9–67

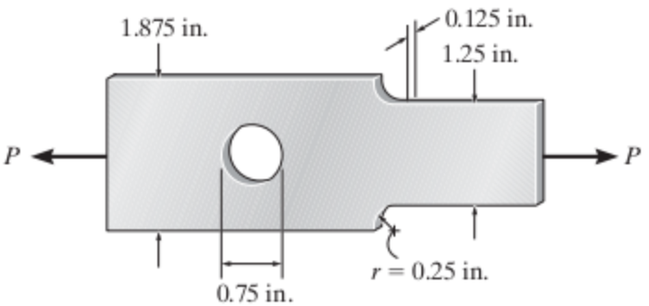
***9–68.** Determine the maximum axial force P that can be applied to the steel plate. The allowable stress is $\sigma_{allow} = 21$ ksi.



Prob. 9–68

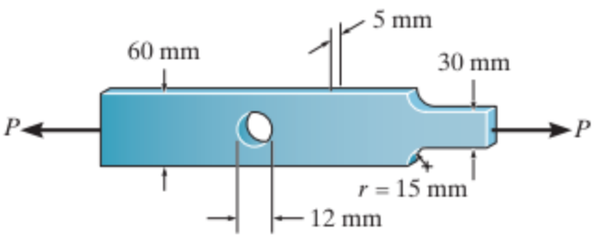
9–69. Determine the maximum axial force P that can be applied to the bar. The bar is made from steel and has an allowable stress of $\sigma_{allow} = 21$ ksi.

9–70. Determine the maximum normal stress developed in the bar when it is subjected to a tension of $P = 2$ kip.



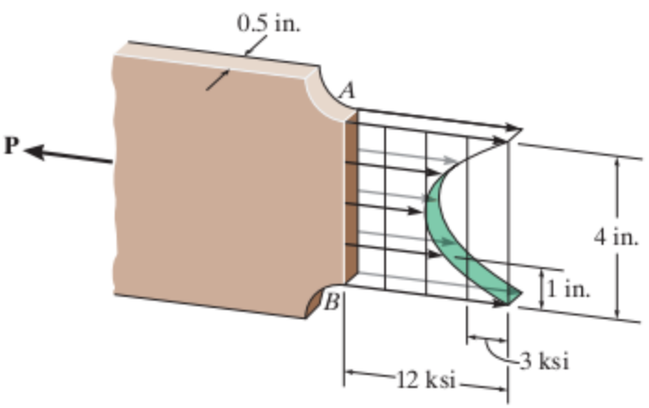
Probs. 9–69/70

9–71. Determine the maximum normal stress developed in the bar when it is subjected to a tension of $P = 8$ kN.



Prob. 9–71

***9–72.** The resulting stress distribution along section AB for the bar is shown. From this distribution, determine the approximate resultant axial force P applied to the bar. Also, what is the stress-concentration factor for this geometry?



Prob. 9–72

9

CONCEPTUAL PROBLEMS

P9–1. The concrete footing A was poured when this column was put in place. Later the rest of the foundation slab was poured. Can you explain why the 45° cracks occurred at each corner? Can you think of a better design that would avoid such cracks?

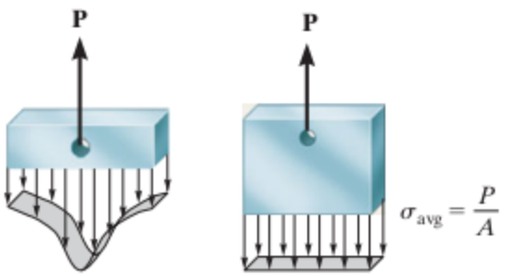
P9–1

P9–2. The row of bricks, along with mortar and an internal steel reinforcing rod, was intended to serve as a lintel beam to support the bricks above this ventilation opening on an exterior wall of a building. Explain what may have caused the bricks to fail in the manner shown.

P9–2

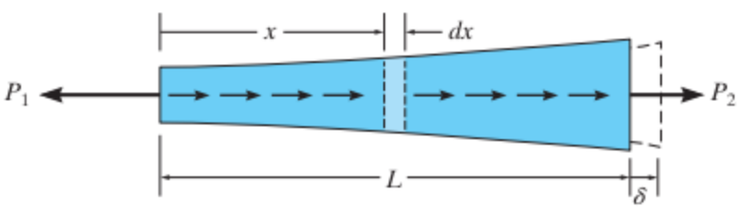
CHAPTER REVIEW

When a loading is applied at a point on a body, it tends to create a stress distribution within the body that becomes more uniformly distributed at regions removed from the point of application of the load. This is called Saint-Venant's principle.



The relative displacement at the end of an axially loaded member relative to the other end is determined from

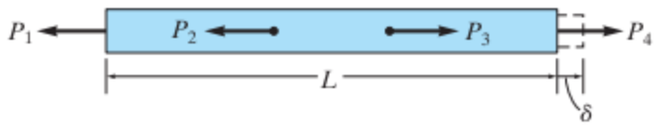
$$\delta = \int_0^L \frac{P(x)\,dx}{AE}$$



If a series of concentrated external axial forces are applied to a member and AE is also piecewise constant, then

$$\delta = \Sigma \frac{PL}{AE}$$

For application, it is necessary to use a sign convention for the internal load P and displacement δ. We considered tension and elongation as positive values. Also, the material must not yield, but rather it must remain linear elastic.



Superposition of load and displacement is possible provided the material remains linear elastic and no significant changes in the geometry of the member occur after loading.

The reactions on a statically indeterminate bar can be determined using the equilibrium equations and compatibility conditions that specify the displacement at the supports. These displacements are related to the loads using a load–displacement relationship such as $\delta = PL/AE$.

A change in temperature can cause a member made of homogeneous isotropic material to change its length by

$$\delta = \alpha \Delta T L$$

If the member is confined, this change will produce thermal stress in the member.

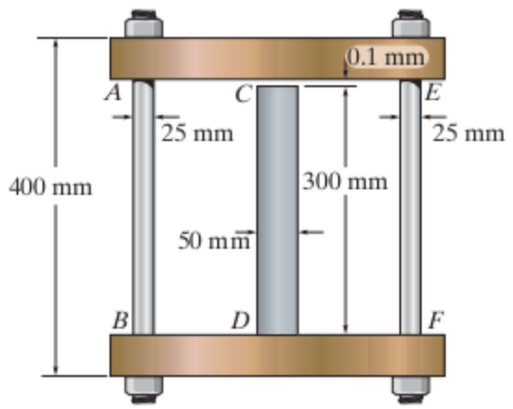
Holes and sharp transitions at a cross section will create stress concentrations. For the design of a member made of brittle material one obtains the stress concentration factor K from a graph, which has been determined from experiment. This value is then multiplied by the average stress to obtain the maximum stress at the cross section.

$$\sigma_{max} = K\sigma_{avg}$$

REVIEW PROBLEMS

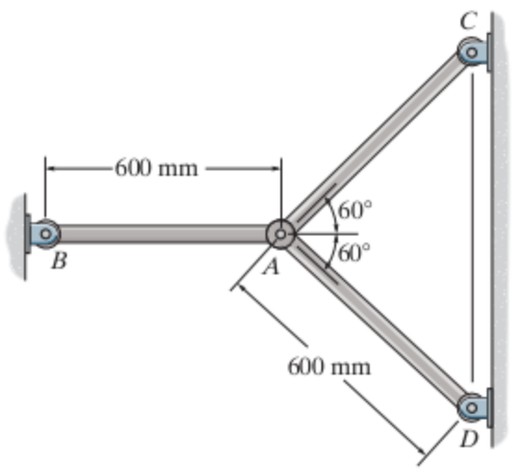
9–73. The assembly consists of two A992 steel bolts *AB* and *EF* and an 6061-T6 aluminum rod *CD*. When the temperature is at 30° C, the gap between the rod and rigid member *AE* is 0.1 mm. Determine the normal stress developed in the bolts and the rod if the temperature rises to 130° C. Assume *BF* is also rigid.

9–74. The assembly shown consists of two A992 steel bolts *AB* and *EF* and an 6061-T6 aluminum rod *CD*. When the temperature is at 30° C, the gap between the rod and rigid member *AE* is 0.1 mm. Determine the highest temperature to which the assembly can be raised without causing yielding either in the rod or the bolts. Assume *BF* is also rigid.



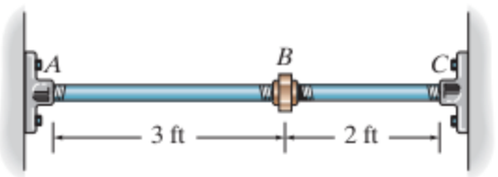
Probs. 9–73/74

9–75. The rods each have the same 25-mm diameter and 600-mm length. If they are made of A-36 steel, determine the forces developed in each rod when the temperature increases to 50°C.



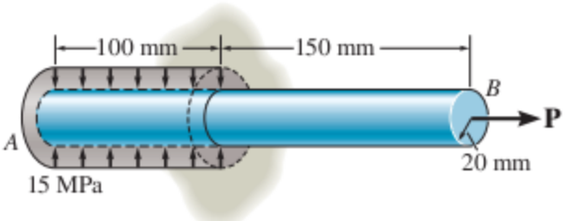
Prob. 9–75

***9–76.** Two A-36 steel pipes, each having a cross-sectional area of 0.32 in^2, are screwed together using a union at *B* as shown. Originally the assembly is adjusted so that no load is on the pipe. If the union is then tightened so that its screw, having a lead of 0.15 in., undergoes two full turns, determine the average normal stress developed in the pipe. Assume that the union at *B* and couplings at *A* and *C* are rigid. Neglect the size of the union. *Note:* The lead would cause the pipe, when *unloaded*, to shorten 0.15 in. when the union is rotated one revolution.



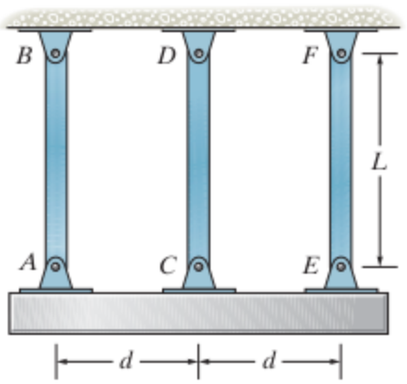
Prob. 9–76

9–77. The brass plug is force-fitted into the rigid casting. The uniform normal bearing pressure on the plug is estimated to be 15 MPa. If the coefficient of static friction between the plug and casting is $\mu_s = 0.3$, determine the axial force *P* needed to pull the plug out. Also, calculate the displacement of end *B* relative to end *A* just before the plug starts to slip out. $E_{br} = 98$ GPa.



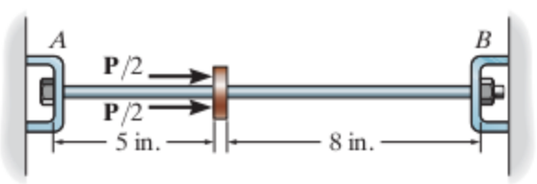
Prob. 9–77

9–78. The assembly consists of two bars AB and CD of the same material having a modulus of elasticity E_1 and coefficient of thermal expansion α_1, and a bar EF having a modulus of elasticity E_2 and coefficient of thermal expansion α_2. All the bars have the same length L and cross-sectional area A. If the rigid beam is originally horizontal at temperature T_1, determine the angle it makes with the horizontal when the temperature is increased to T_2.



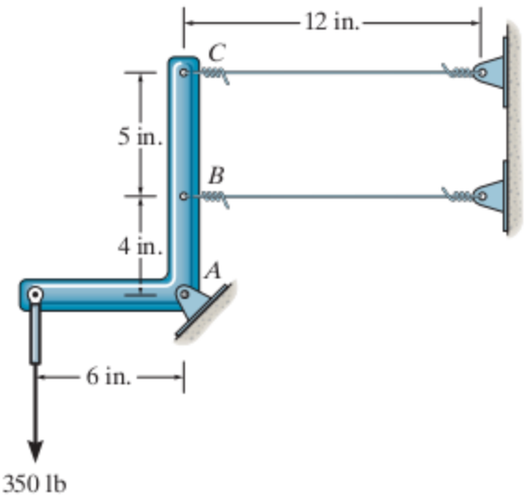
Prob. 9–78

9–79. The 2014-T6 aluminum rod has a diameter of 0.5 in. and is lightly attached to the rigid supports at A and B when $T_1 = 70°F$. Determine the force P that must be applied to the collar so that, when $T = 0°F$, the reaction at B is zero.



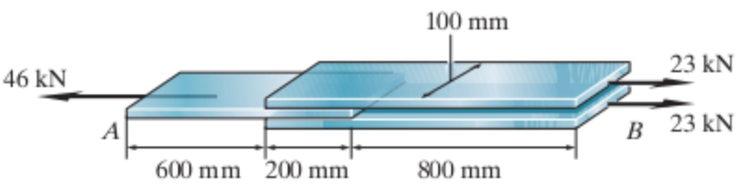
Prob. 9–79

***9–80.** The rigid link is supported by a pin at A and two A-36 steel wires, each having an unstretched length of 12 in. and cross-sectional area of 0.0125 in^2. Determine the force developed in the wires when the link supports the vertical load of 350 lb.



Prob. 9–80

9–81. The joint is made from three A992 steel plates that are bonded together at their seams. Determine the displacement of end A with respect to end B when the joint is subjected to the axial loads shown. Each plate has a thickness of 5 mm.



Prob. 9–81

Chapter 10

The torsional stress and angle of twist of this soil auger depend upon the output of the machine turning the bit as well as the resistance of the soil in contact with the shaft.

Torsion

CHAPTER OBJECTIVES

- To determine the torsional deformation of a perfectly elastic circular shaft.
- To determine the support reactions when these reactions cannot be determined solely from the moment equilibrium equation.
- To determine the maximum power that can be transmitted by a shaft.

10.1 Torsional Deformation of a Circular Shaft

Torque is a moment that tends to twist a member about its longitudinal axis. Its effect is of primary concern in the design of drive shafts used in vehicles and machinery. We can illustrate physically what happens when a torque is applied to a circular shaft by considering the shaft to be made of a highly deformable material such as rubber, Fig. 10–1*a*. When the torque is applied, the circles and longitudinal grid lines originally marked on the shaft tend to distort into the pattern shown in Fig. 10–1*b*. Note that twisting causes the circles to *remain circles*, and each longitudinal grid line deforms into a helix that intersects the circles at equal angles. Also, the cross sections at the ends of the shaft will remain *flat*—that is, they do not warp or bulge in or out—and radial lines *remain straight* during the deformation, Fig. 10–1*b*. From these observations we can assume that if the angle of twist is *small*, the *length of the shaft* and *its radius* will *remain unchanged*.

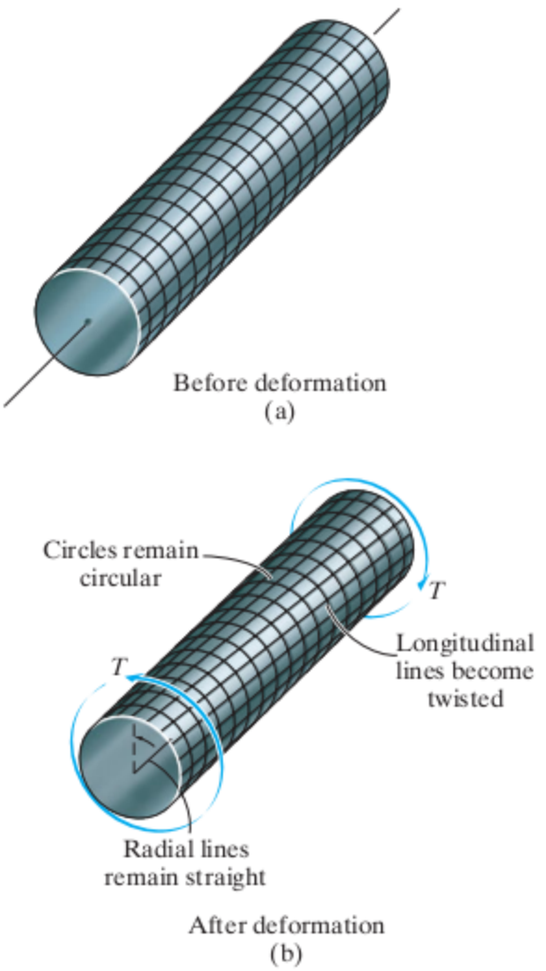


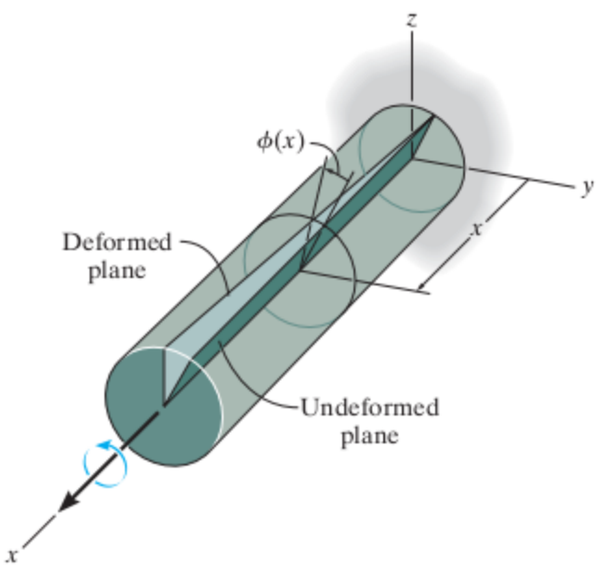
Fig. 10–1

If the shaft is fixed at one end and a torque is applied to its other end, the dark green shaded plane in Fig. 10–2 will distort into a skewed form as shown. Here a radial line located on the cross section at a distance x from the fixed end of the shaft will rotate through an angle $\phi(x)$. The angle $\phi(x)$, so defined, is called the *angle of twist*. It depends on the position x and will vary along the shaft as shown.

In order to understand how this distortion strains the material, we will now isolate a small element located at a radial distance ρ (rho) from the axis of the shaft, Fig. 10–3. Due to the deformation as noted in Fig. 10–2, the front and rear faces of the element will undergo a rotation—the back face by $\phi(x)$, and the front face by $\phi(x) + \Delta\phi$. As a result, the *difference* in these rotations, $\Delta\phi$, causes the element to be subjected to a *shear strain*. To calculate this strain, note that before deformation the angle between the edges AB and AC is 90°; after deformation, however, the edges of the element are AD and AC and the angle between them is θ'. From the definition of shear strain, Eq. 7–12, we have

$$\gamma = \frac{\pi}{2} - \theta'$$

Notice the deformation of the rectangular element when this rubber bar is subjected to a torque.



The angle of twist $\phi(x)$ increases as x increases.

Fig. 10–2

10

This angle, γ, is indicated on the element. It can be related to the length Δx of the element and the angle $\Delta\phi$ between the shaded planes by considering the length of arc BD, that is

$$BD = \rho\Delta\phi = \Delta x\gamma$$

Therefore, if we let $\Delta x \to dx$ and $\Delta\phi \to d\phi$,

$$\gamma = \rho\frac{d\phi}{dx} \tag{10–1}$$

Since dx and $d\phi$ are the *same* for *all elements* located at points on the cross section at x, then $d\phi/dx$ is constant over the cross section, and Eq. 10–1 states that the magnitude of the shear strain for any of these elements varies only with its radial distance ρ from the axis of the shaft. In other words, the shear strain within the shaft varies linearly along any radial line, from zero at the axis of the shaft to a maximum γ_{max} at its outer boundary, Fig. 10–4. Since $d\phi/dx = \gamma/\rho = \gamma_{max}/c$, then

$$\gamma = \left(\frac{\rho}{c}\right)\gamma_{max} \tag{10–2}$$

The results obtained here are also valid for circular tubes. They depend only on the assumptions regarding the deformations mentioned above.

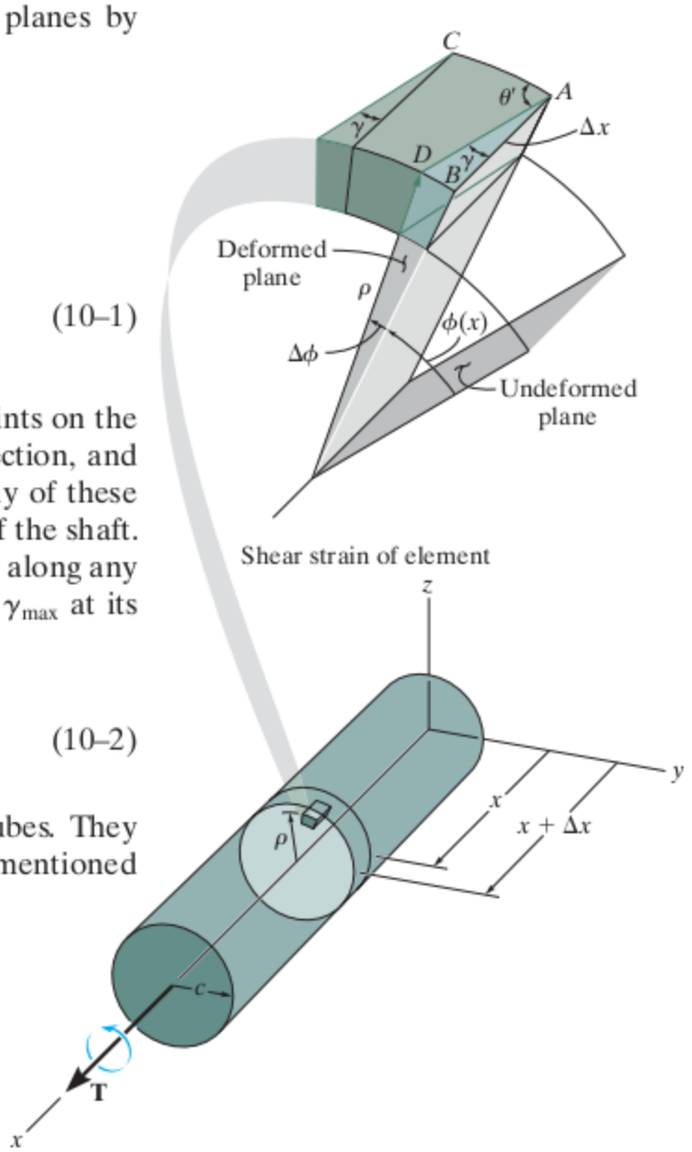
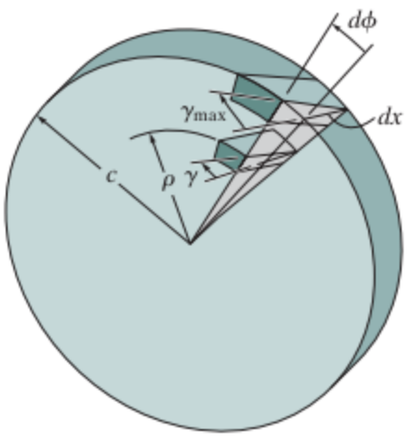


Fig. 10–3



The shear strain at points on the cross section increases linearly with ρ, i.e., $\gamma = (\rho/c)\gamma_{max}$.

Fig. 10–4

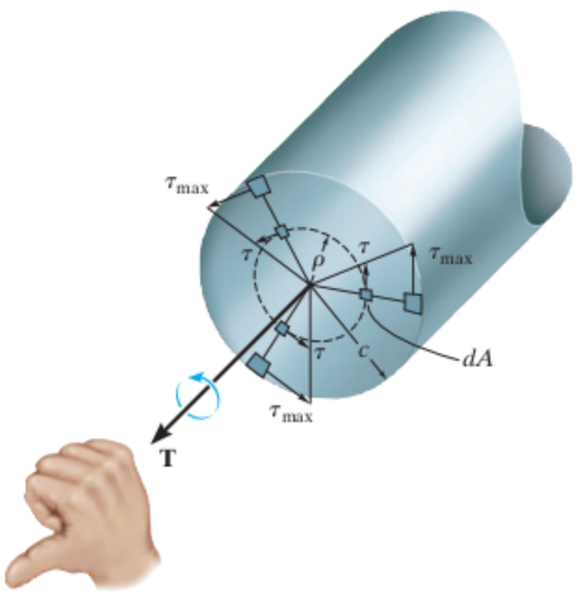
10.2 The Torsion Formula

When an external torque is applied to a shaft it creates a corresponding internal torque within the shaft. In this section, we will develop an equation that relates this internal torque to the shear stress distribution on the cross section of a circular shaft or tube.

If the material is linear-elastic, then Hooke's law applies, $\tau = G\gamma$, and consequently a ***linear variation in shear strain***, as noted in the previous section, leads to a corresponding ***linear variation in shear stress*** along any radial line on the cross section. Hence, τ will vary from zero at the shaft's longitudinal axis to a maximum value, τ_{max}, at its outer surface. This variation is shown in Fig. 10–5 on the front faces of a selected number of elements, located at an intermediate radial position ρ and at the outer radius c. Due to the proportionality of triangles, we can write

$$\tau = \left(\frac{\rho}{c}\right)\tau_{max} \tag{10–3}$$

This equation expresses the shear-stress distribution over the cross section in terms of the radial position ρ of the element. Using it, we can now apply the condition that requires the torque produced by the stress distribution over the entire cross section to be equivalent to the resultant internal torque T at the section, which holds the shaft in equilibrium, Fig. 10–5.



Shear stress varies linearly along each radial line of the cross section.

Fig. 10–5

Specifically, each element of area dA, located at ρ, is subjected to a force of $dF = \tau\, dA$. The torque produced by this force is $dT = \rho(\tau\, dA)$. We therefore have for the entire cross section

$$T = \int_A \rho(\tau\, dA) = \int_A \rho\left(\frac{\rho}{c}\right)\tau_{\max}\, dA \tag{10–4}$$

Since $\tau_{\max}/c$ is constant,

$$T = \frac{\tau_{\max}}{c}\int_A \rho^2\, dA \tag{10–5}$$

The integral depends only on the geometry of the shaft. It represents the ***polar moment of inertia*** of the shaft's cross-sectional area about the shaft's longitudinal axis. We will symbolize its value as J, and therefore the above equation can be rearranged and written in a more compact form, namely,

$$\boxed{\tau_{\max} = \frac{Tc}{J}} \tag{10–6}$$

Here

$\tau_{\max}$ = the maximum shear stress in the shaft, which occurs at the outer surface

T = the resultant *internal torque* acting at the cross section. Its value is determined from the method of sections and the equation of moment equilibrium applied about the shaft's longitudinal axis

J = the polar moment of inertia of the cross-sectional area

c = the outer radius of the shaft

Combining Eqs. 10–3 and 10–6, the shear stress at the intermediate distance ρ can be determined from

$$\boxed{\tau = \frac{T\rho}{J}} \tag{10–7}$$

Either of the above two equations is often referred to as the ***torsion formula***. Recall that it is used only if the shaft is circular and the material is homogeneous and behaves in a linear elastic manner, since the derivation is based on Hooke's law.

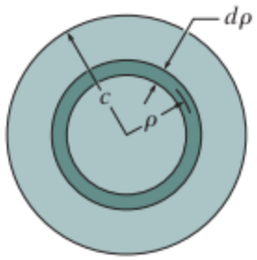


Fig. 10–6

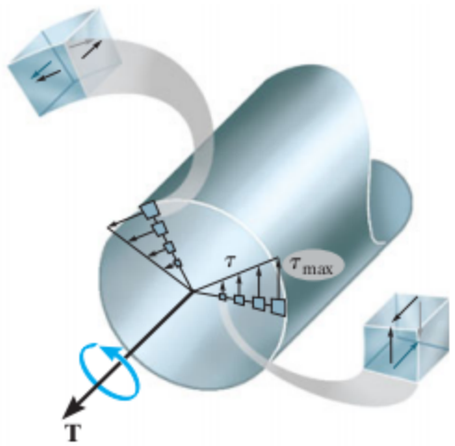
Solid Shaft. If the shaft has a solid circular cross section, the polar moment of inertia J can be determined using an area element in the form of a *differential ring* or annulus having a thickness $d\rho$ and circumference $2\pi\rho$, Fig. 10–6. For this ring, $dA = 2\pi\rho\, d\rho$, and so

$$J = \int_A \rho^2\, dA = \int_0^c \rho^2 (2\pi\rho\, d\rho) = 2\pi \int_0^c \rho^3\, d\rho = 2\pi\left(\frac{1}{4}\right)\rho^4\Big|_0^c$$

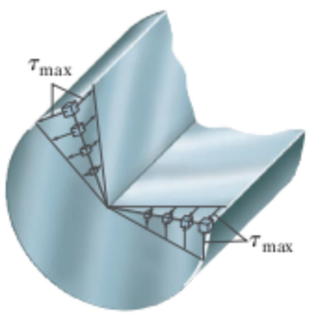
$$\boxed{J = \frac{\pi}{2}c^4} \tag{10–8}$$

Note that J is a *geometric property* of the circular area and is always positive. Common units used for its measurement are mm^4 or in^4.

The shear stress has been shown to vary linearly along each radial line of the cross section of the shaft. However, if an element of material on the cross section is isolated, then due to the complementary property of shear, equal shear stresses must also act on four of its adjacent faces as shown in Fig. 10–7*a*. Hence, ***not only does the internal torque T develop a linear distribution of shear stress along each radial line in the plane of the cross-sectional area, but also an associated shear-stress distribution is developed along an axial plane***, Fig. 10–7*b*. It is interesting to note that because of this axial distribution of shear stress, shafts made from wood tend to *split* along the axial plane when subjected to excessive torque, Fig. 10–8. This is because wood is an anisotropic material. Its shear resistance parallel to its grains or fibers, directed along the axis of the shaft, is much less than its resistance perpendicular to the fibers, directed in the plane of the cross section.



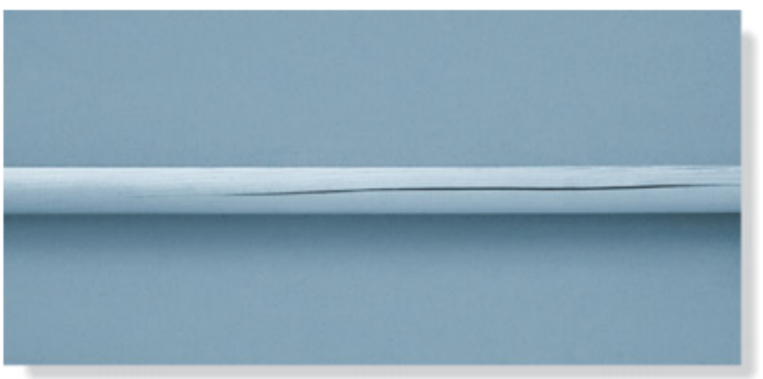
(a)



Shear stress varies linearly along each radial line of the cross section.

(b)

Fig. 10–7



Failure of a wooden shaft due to torsion.

Fig. 10–8

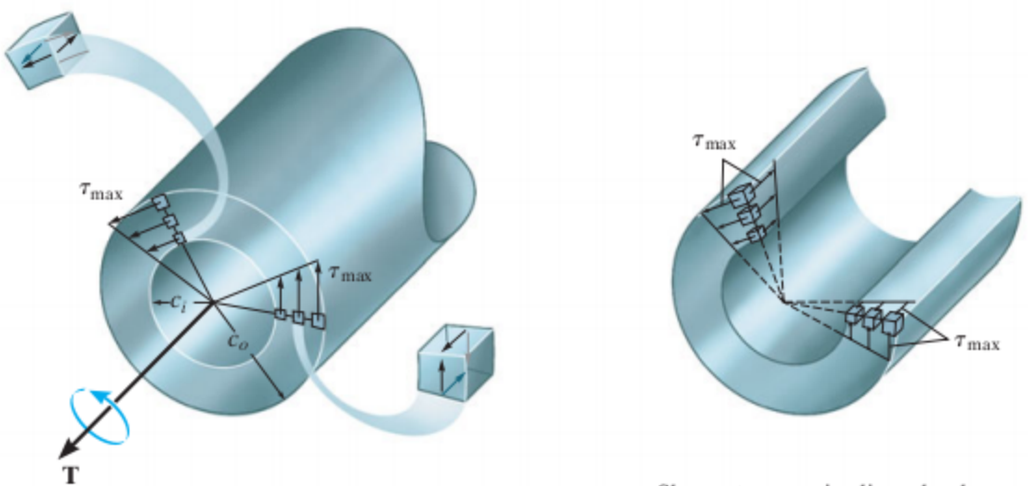
Tubular Shaft. If a shaft has a tubular cross section, with inner radius c_i and outer radius c_o, then from Eq. 10–8 we can determine its polar moment of inertia by subtracting J for a shaft of radius c_i from that determined for a shaft of radius c_o. The result is

$$J = \frac{\pi}{2}(c_o^4 - c_i^4) \tag{10–9}$$

Like the solid shaft, the shear stress distributed over the tube's cross-sectional area varies linearly along any radial line, Fig. 10–9*a*. Furthermore, the shear stress varies along an axial plane in this same manner, Fig. 10–9*b*.

Absolute Maximum Torsional Stress. If the absolute maximum torsional stress is to be determined, then it becomes important to find the location where the ratio Tc/J is a maximum. In this regard, it may be helpful to show the variation of the internal torque T at each section along the axis of the shaft by drawing a ***torque diagram***, which is a plot of the internal torque T versus its position x along the shaft's length. Once the internal torque throughout the shaft is determined, the maximum ratio of Tc/J can then be identified.

This tubular drive shaft for a truck was subjected to an excessive torque, resulting in failure caused by yielding of the material.



(a) Shear stress varies linearly along each radial line of the cross section. (b)

Fig. 10–9

Important Points

- When a shaft having a *circular cross section* is subjected to a torque, the cross section *remains plane* while radial lines rotate. This causes a *shear strain* within the material that *varies linearly* along any radial line, from zero at the axis of the shaft to a maximum at its outer boundary.
- For linear elastic homogeneous material the *shear stress* along any radial line of the shaft also *varies linearly*, from zero at its axis to a maximum at its outer boundary. This maximum shear stress *must not* exceed the proportional limit.
- Due to the complementary property of shear, the linear shear stress distribution within the plane of the cross section is also distributed along an adjacent axial plane of the shaft.
- The torsion formula is based on the requirement that the resultant torque on the cross section is equal to the torque produced by the shear stress distribution about the longitudinal axis of the shaft. It is required that the shaft or tube have a *circular* cross section and that it is made of *homogeneous* material which has *linear-elastic* behavior.

Procedure for Analysis

The torsion formula can be applied using the following procedure.

Internal Loading.

- Section the shaft perpendicular to its axis at the point where the shear stress is to be determined, and use the necessary free-body diagram and equations of equilibrium to obtain the internal torque at the section.

Section Property.

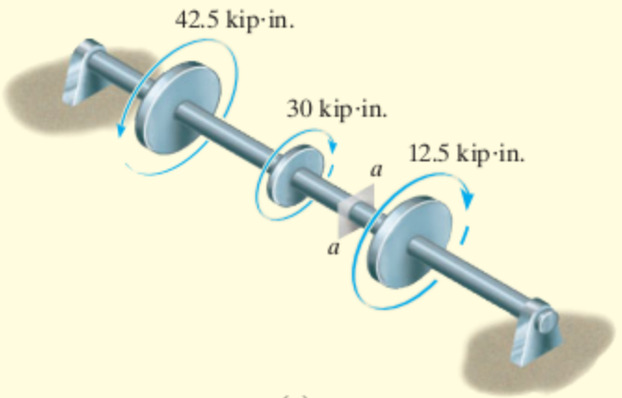
- Calculate the polar moment of inertia of the cross-sectional area. For a solid section of radius c, $J = \pi c^4/2$, and for a tube of outer radius c_o and inner radius c_i, $J = \pi(c_o^4 - c_i^4)/2$.

Shear Stress.

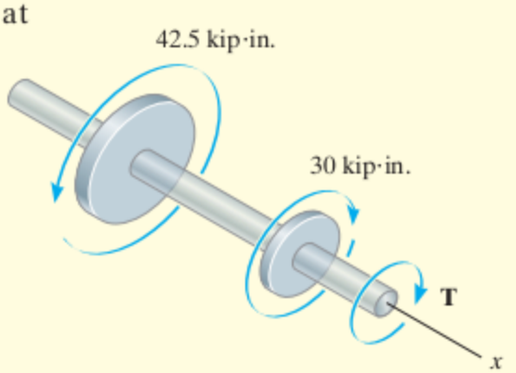
- Specify the radial distance ρ, measured from the center of the cross section to the point where the shear stress is to be found. Then apply the torsion formula $\tau = T\rho/J$, or if the maximum shear stress is to be determined use $\tau_{\max} = Tc/J$. When substituting the data, make sure to use a consistent set of units.
- The shear stress acts on the cross section in a direction that is always perpendicular to ρ. The force it creates must contribute a torque about the axis of the shaft that is in the *same direction* as the internal resultant torque $\mathbf{T}$ acting on the section. Once this direction is established, a volume element located at the point where τ is determined can be isolated, and the direction of τ acting on the remaining three adjacent faces of the element can be shown.

EXAMPLE 10.1

The shaft shown in Fig. 10–10*a* is supported by two bearings and is subjected to three torques. Determine the shear stress developed at points *A* and *B*, located at section *a–a* of the shaft, Fig. 10–10*c*.



(a)



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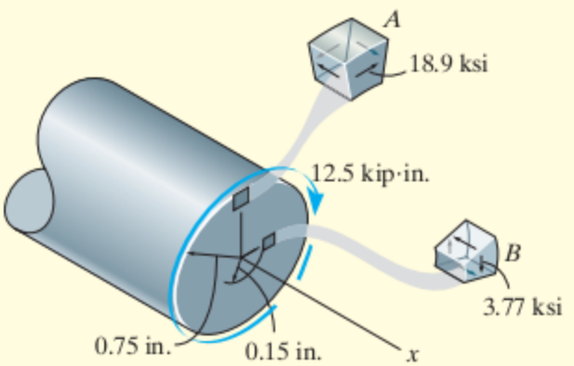


(c)

Fig. 10–10

SOLUTION

Internal Torque. Since the bearing reactions do not offer resistance to shaft rotation, the applied torques satisfy moment equilibrium about the shaft's axis.

The internal torque at section *a–a* will be determined from the free-body diagram of the left segment, Fig. 10–10*b*. We have

$$\Sigma M_x = 0; \quad 42.5 \text{ kip} \cdot \text{in.} - 30 \text{ kip} \cdot \text{in.} - T = 0 \quad T = 12.5 \text{ kip} \cdot \text{in.}$$

Section Property. The polar moment of inertia for the shaft is

$$J = \frac{\pi}{2}(0.75 \text{ in.})^4 = 0.497 \text{ in}^4$$

Shear Stress. Since point *A* is at $\rho = c = 0.75$ in.,

$$\tau_A = \frac{Tc}{J} = \frac{(12.5 \text{ kip} \cdot \text{in.})(0.75 \text{ in.})}{(0.497 \text{ in}^4)} = 18.9 \text{ ksi} \qquad \textit{Ans.}$$

Likewise for point *B*, at $\rho = 0.15$ in., we have

$$\tau_B = \frac{T\rho}{J} = \frac{(12.5 \text{ kip} \cdot \text{in.})(0.15 \text{ in.})}{(0.497 \text{ in}^4)} = 3.77 \text{ ksi} \qquad \textit{Ans.}$$

NOTE: The directions of these stresses on each element at *A* and *B*, Fig. 10–10*c*, are established from the direction of the resultant internal torque **T**, shown in Fig. 10–10*b*. Note carefully how the shear stress acts on the planes of each of these elements.

EXAMPLE 10.2

The pipe shown in Fig. 10–11*a* has an inner diameter of 80 mm and an outer diameter of 100 mm. If its end is tightened against the support at *A* using a torque wrench at *B*, determine the shear stress developed in the material at the inner and outer walls along the central portion of the pipe when the 80-N forces are applied to the wrench.

SOLUTION

Internal Torque. A section is taken at an intermediate location *C* along the pipe's axis, Fig. 10–11*b*. The only unknown at the section is the internal torque **T**. We require

$$\Sigma M_y = 0; \quad 80\ \text{N}\,(0.3\ \text{m}) + 80\ \text{N}\,(0.2\ \text{m}) - T = 0$$

$$T = 40\ \text{N}\cdot\text{m}$$

Section Property. The polar moment of inertia for the pipe's cross-sectional area is

$$J = \frac{\pi}{2}\left[(0.05\ \text{m})^4 - (0.04\ \text{m})^4\right] = 5.796(10^{-6})\ \text{m}^4$$

Shear Stress. For any point lying on the outside surface of the pipe, $\rho = c_o = 0.05$ m, we have

$$\tau_o = \frac{Tc_o}{J} = \frac{40\ \text{N}\cdot\text{m}\ (0.05\ \text{m})}{5.796(10^{-6})\ \text{m}^4} = 0.345\ \text{MPa} \qquad \textit{Ans.}$$

And for any point located on the inside surface, $\rho = c_i = 0.04$ m, so that

$$\tau_i = \frac{Tc_i}{J} = \frac{40\ \text{N}\cdot\text{m}\ (0.04\ \text{m})}{5.796(10^{-6})\text{m}^4} = 0.276\ \text{MPa} \qquad \textit{Ans.}$$

NOTE: To show how these stresses act at representative points *D* and *E* on the cross section, we will first view the cross section from the front of segment *CA* of the pipe, Fig. 10–11*a*. On this section, Fig. 10–11*c*, the resultant internal torque is equal but opposite to that shown in Fig. 10–11*b*. The shear stresses at *D* and *E* contribute to this torque and therefore act on the shaded faces of the elements in the directions shown. As a consequence, notice how the shear-stress components act on the other three faces. Furthermore, since the top face of *D* and the inner face of *E* are in stress-free regions taken from the pipe's outer and inner walls, no shear stress can exist on these faces or on the other corresponding faces of the elements.

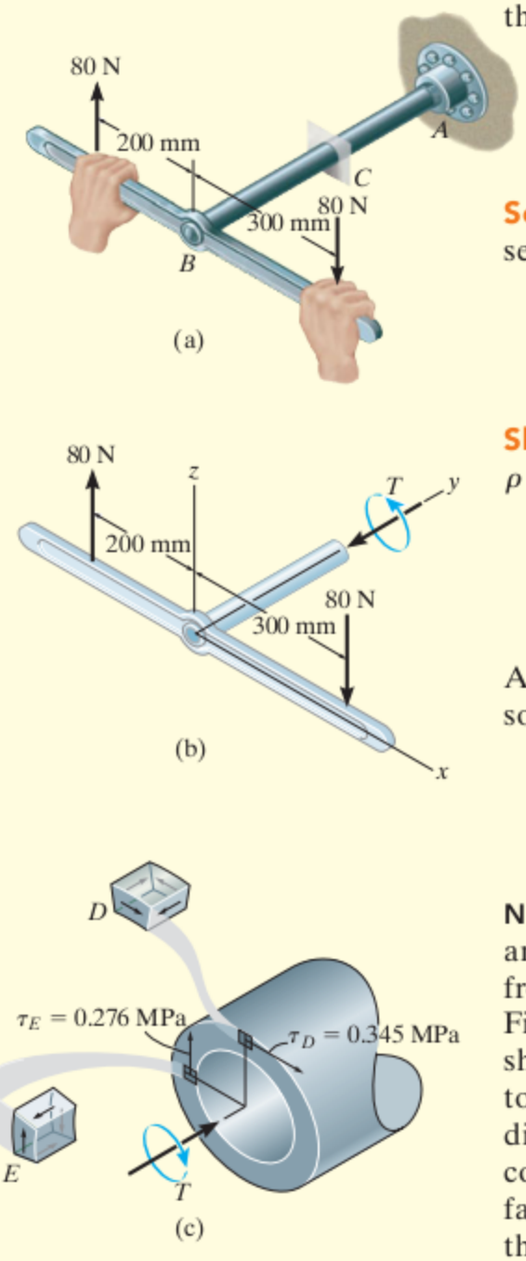


Fig. 10–11

10.3 Power Transmission

Shafts and tubes having circular cross sections are often used to transmit power developed by a machine. When used for this purpose, they are subjected to a torque that depends on the power generated by the machine and the angular speed of the shaft. ***Power*** is defined as the work performed per unit of time. Also, the work transmitted by a rotating shaft equals the torque applied times the angle of rotation. Therefore, if during an instant of time dt an applied torque $\mathbf{T}$ causes the shaft to rotate $d\theta$, then the instantaneous power is

$$P = \frac{T\,d\theta}{dt}$$

Since the shaft's angular velocity is $\omega = d\theta/dt$, we can express the power as

$$\boxed{P = T\omega} \tag{10–10}$$

In the SI system, power is expressed in *watts* when torque is measured in newton-meters (N · m) and ω is in radians per second (rad/s) (1 W = 1 N · m/s). In the FPS system, the basic units of power are foot-pounds per second (ft · lb/s); however, horsepower (hp) is often used in engineering practice, where

$$1\text{ hp} = 550\text{ ft} \cdot \text{lb/s}$$

For machinery, the *frequency* of a shaft's rotation, f, is often reported. This is a measure of the number of revolutions or cycles the shaft makes per second and is expressed in hertz (1 Hz = 1 cycle/s). Since 1 cycle = 2π rad, then $\omega = 2\pi f$, and so the above equation for power becomes

$$\boxed{P = 2\pi f T} \tag{10–11}$$

Shaft Design. When the power transmitted by a shaft and its frequency of rotation are known, the torque developed in the shaft can be determined from Eq. 10–11, that is, $T = P/2\pi f$. Knowing T and the allowable shear stress for the material, τ_{allow}, we can determine the size of the shaft's cross section using the torsion formula, provided the material behavior is linear elastic. Specifically, the design or geometric parameter J/c becomes

$$\frac{J}{c} = \frac{T}{\tau_{\text{allow}}} \tag{10–12}$$

For a *solid shaft*, $J = (\pi/2)c^4$, and thus, upon substitution, a *unique value* for the shaft's radius c can be determined. If the shaft is *tubular*, so that $J = (\pi/2)(c_o^4 - c_i^4)$, design permits a wide range of possibilities for the solution. This is because an *arbitrary choice* can be made for either c_o or c_i and the other radius can then be determined from Eq. 10–12.

The chain drive transmits the torque developed by the electric motor to the shaft. The stress developed in the shaft depends upon the power transmitted by the motor and the rate of rotation of the connecting shaft. $P = T\omega$.

EXAMPLE 10.3

A solid steel shaft AB shown in Fig. 10–12 is to be used to transmit 5 hp from the motor M to which it is attached. If the shaft rotates at $\omega = 175$ rpm and the steel has an allowable shear stress of $\tau_{allow} = 14.5$ ksi, determine the required diameter of the shaft to the nearest $\frac{1}{8}$ in.

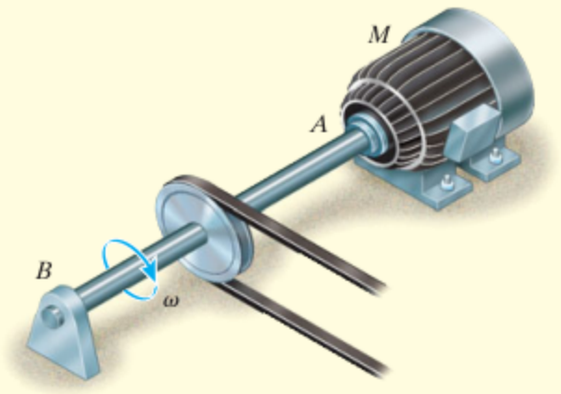


Fig. 10–12

SOLUTION

The torque on the shaft is determined from Eq. 10–10, that is, $P = T\omega$. Expressing P in foot-pounds per second and ω in radians/second, we have

$$P = 5\text{ hp}\left(\frac{550\text{ ft}\cdot\text{lb/s}}{1\text{ hp}}\right) = 2750\text{ ft}\cdot\text{lb/s}$$

$$\omega = \frac{175\text{ rev}}{\text{min}}\left(\frac{2\pi\text{ rad}}{1\text{ rev}}\right)\left(\frac{1\text{ min}}{60\text{ s}}\right) = 18.33\text{ rad/s}$$

Thus,

$$P = T\omega; \qquad 2750\text{ ft}\cdot\text{lb/s} = T(18.33\text{ rad/s})$$

$$T = 150.1\text{ ft}\cdot\text{lb}$$

Applying Eq. 10–12 yields

$$\frac{J}{c} = \frac{\pi c^4}{2\,c} = \frac{T}{\tau_{allow}}$$

$$c = \left(\frac{2T}{\pi\tau_{allow}}\right)^{1/3} = \left(\frac{2(150.1\text{ ft}\cdot\text{lb})(12\text{ in./ft})}{\pi(14\,500\text{ lb/in}^2)}\right)^{1/3}$$

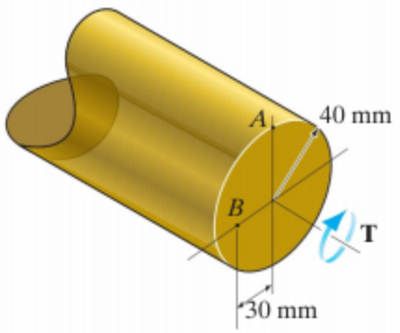
$$c = 0.429\text{ in.}$$

Since $2c = 0.858$ in., select a shaft having a diameter of

$$d = \frac{7}{8}\text{ in.} = 0.875\text{ in.} \qquad \textit{Ans.}$$

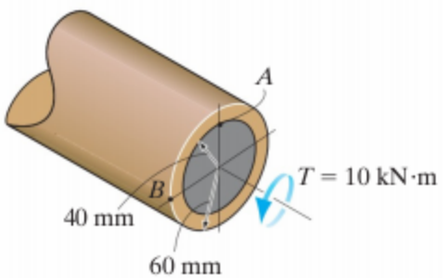
FUNDAMENTAL PROBLEMS

F10–1. The solid circular shaft is subjected to an internal torque of $T = 5$ kN·m. Determine the shear stress developed at points A and B. Represent each state of stress on a volume element.



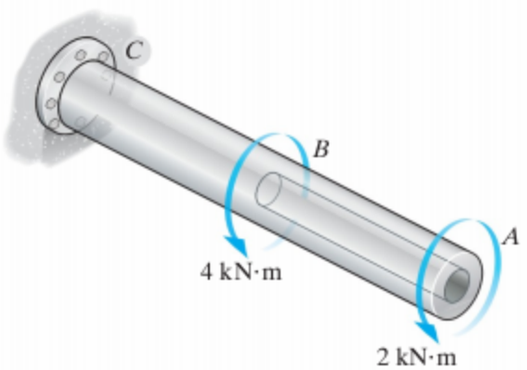
F10–1

F10–2. The hollow circular shaft is subjected to an internal torque of $T = 10$ kN·m. Determine the shear stress developed at points A and B. Represent each state of stress on a volume element.



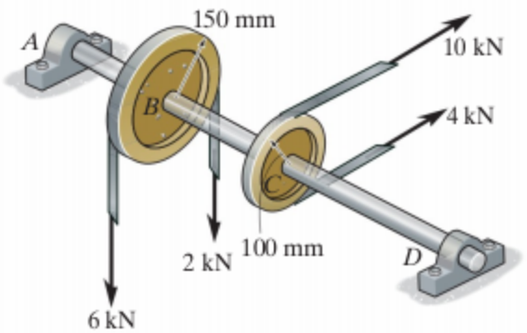
F10–2

F10–3. The shaft is hollow from A to B and solid from B to C. Determine the maximum shear stress developed in the shaft. The shaft has an outer diameter of 80 mm, and the thickness of the wall of the hollow segment is 10 mm.



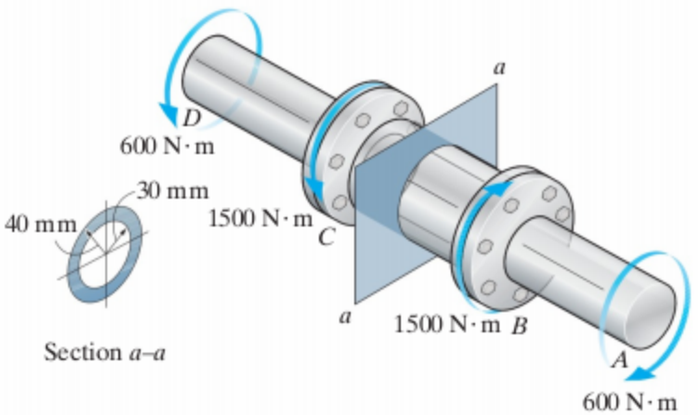
F10–3

F10–4. Determine the maximum shear stress developed in the 40-mm diameter shaft.



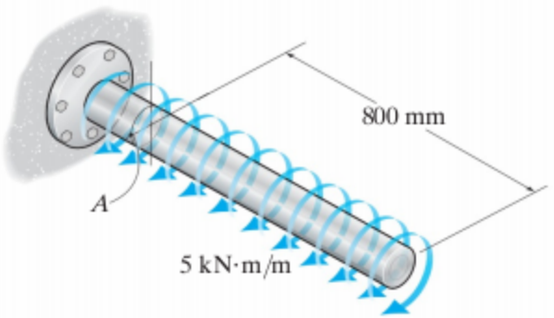
F10–4

F10–5. Determine the maximum shear stress developed in the shaft at section $a-a$.



F10–5

F10–6. Determine the shear stress developed at point A on the surface of the shaft. Represent the state of stress on a volume element at this point. The shaft has a radius of 40 mm.

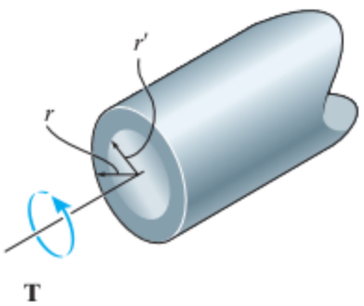


F10–6

PROBLEMS

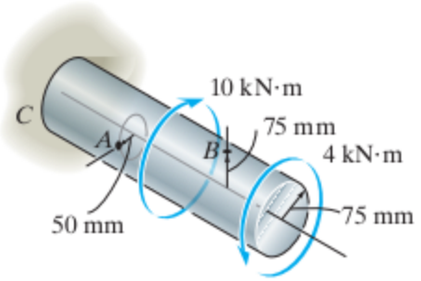
10–1. The solid shaft of radius r is subjected to a torque **T**. Determine the radius r' of the inner core of the shaft that resists one-half of the applied torque ($T/2$). Solve the problem two ways: (a) by using the torsion formula, (b) by finding the resultant of the shear-stress distribution.

10–2. The solid shaft of radius r is subjected to a torque **T**. Determine the radius r' of the inner core of the shaft that resists one-quarter of the applied torque ($T/4$). Solve the problem two ways: (a) by using the torsion formula, (b) by finding the resultant of the shear-stress distribution.



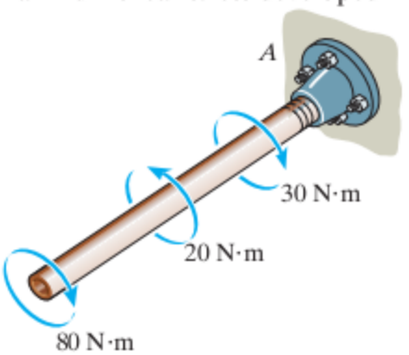
Probs. 10–1/2

10–3. The solid shaft is fixed to the support at C and subjected to the torsional loadings shown. Determine the shear stress at points A and B and sketch the shear stress on volume elements located at these points.



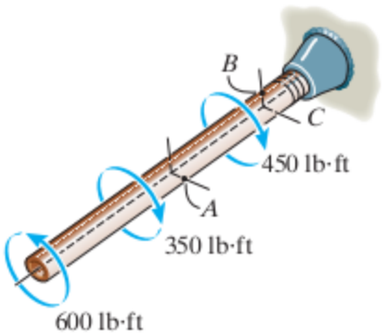
Prob. 10–3

***10–4.** The copper pipe has an outer diameter of 40 mm and an inner diameter of 37 mm. If it is tightly secured to the wall at A and three torques are applied to it as shown, determine the absolute maximum shear stress developed in the pipe.



Prob. 10–4

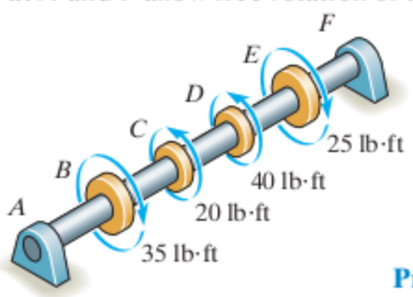
10–5. The copper pipe has an outer diameter of 2.50 in. and an inner diameter of 2.30 in. If it is tightly secured to the wall at C and three torques are applied to it as shown, determine the shear stress developed at points A and B. These points lie on the pipe's outer surface. Sketch the shear stress on volume elements located at A and B.



Prob. 10–5

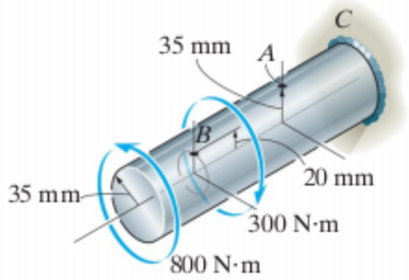
10–6. The solid shaft has a diameter of 0.75 in. If it is subjected to the torques shown, determine the maximum shear stress developed in regions BC and DE of the shaft. The bearings at A and F allow free rotation of the shaft.

10–7. The solid shaft has a diameter of 0.75 in. If it is subjected to the torques shown, determine the maximum shear stress developed in regions CD and EF of the shaft. The bearings at A and F allow free rotation of the shaft.



Probs. 10–6/7

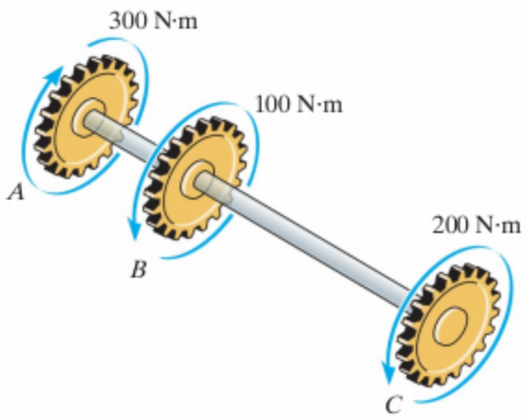
*10–8. The solid shaft is fixed to the support at C and subjected to the torsional loadings shown. Determine the shear stress at points A and B on the surface, and sketch the shear stress on volume elements located at these points.



Prob. 10–8

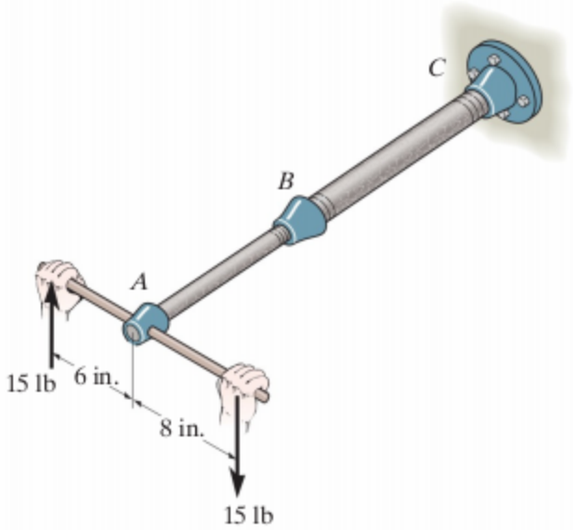
10–9. The A-36 steel shaft is supported on smooth bearings that allow it to rotate freely. If the gears are subjected to the torques shown, determine the maximum shear stress developed in the segments AB and BC. The shaft has a diameter of 40 mm.

10–10. The A-36 steel shaft is supported on smooth bearings that allow it to rotate freely. If the gears are subjected to the torques shown, determine the required diameter of the shaft to the nearest mm if $\tau_{allover} = 60$ MPa.



Probs. 10–9/10

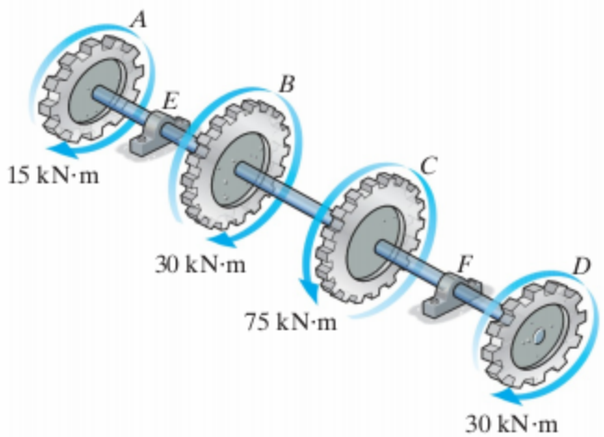
10–11. The assembly consists of two sections of galvanized steel pipe connected together using a reducing coupling at B. The smaller pipe has an outer diameter of 0.75 in. and an inner diameter of 0.68 in., whereas the larger pipe has an outer diameter of 1 in. and an inner diameter of 0.86 in. If the pipe is tightly secured to the wall at C, determine the maximum shear stress developed in each section of the pipe when the couple shown is applied to the handles of the wrench.



Prob. 10–11

*10–12. The 150-mm-diameter shaft is supported by a smooth journal bearing at E and a smooth thrust bearing at F. Determine the maximum shear stress developed in each segment of the shaft.

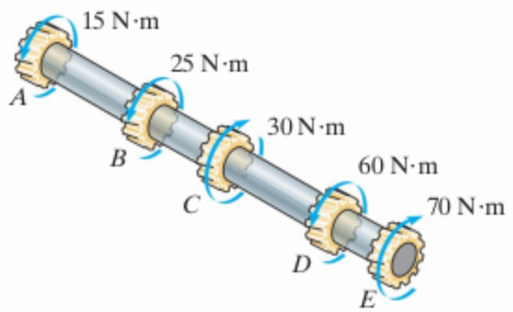
10–13. If the tubular shaft is made from material having an allowable shear stress of $\tau_{allow} = 85$ MPa, determine the required minimum wall thickness of the shaft to the nearest millimeter. The shaft has an outer diameter of 150 mm.



Probs. 10–12/13

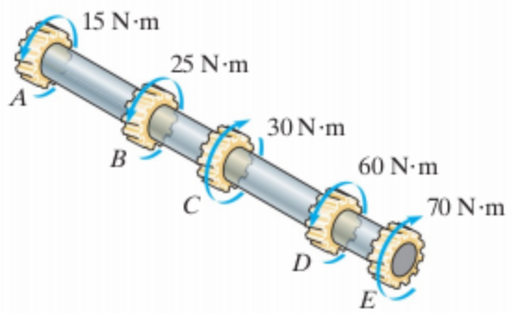
10–14. A steel tube having an outer diameter of 2.5 in. is used to transmit 9 hp when turning at 27 rev/min. Determine the inner diameter d of the tube to the nearest $\frac{1}{8}$ in. if the allowable shear stress is $\tau_{allow} = 10$ ksi.

10–15. The solid shaft is made of material that has an allowable shear stress of $\tau_{allow} = 10$ MPa. Determine the required diameter of the shaft to the nearest mm.



Prob. 10–15

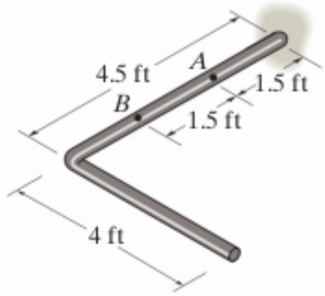
*10–16. The solid shaft has a diameter of 40 mm. Determine the absolute maximum shear stress in the shaft and sketch the shear-stress distribution along a radial line of the shaft where the shear stress is maximum.



Prob. 10–16

10–17. The rod has a diameter of 1 in. and a weight of 10 lb/ft. Determine the maximum torsional stress in the rod at a section located at A due to the rod's weight.

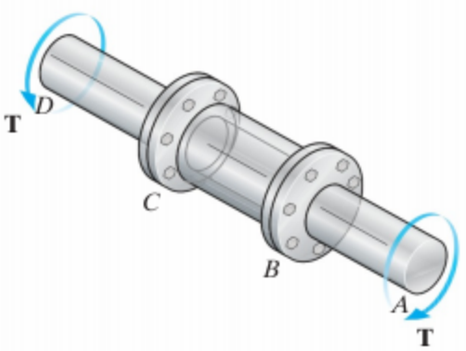
10–18. The rod has a diameter of 1 in. and a weight of 15 lb/ft. Determine the maximum torsional stress in the rod at a section located at B due to the rod's weight.



Probs. 10–17/18

10–19. The shaft consists of solid 80-mm-diameter rod segments AB and CD, and the tubular segment BC which has an outer diameter of 100 mm and inner diameter of 80 mm. If the material has an allowable shear stress of $\tau_{allow} = 75$ MPa, determine the maximum allowable torque **T** that can be applied to the shaft.

*10–20. The shaft consists of rod segments AB and CD, and the tubular segment BC. If the torque $T = 10$ kN · m is applied to the shaft, determine the required minimum diameter of the rod and the maximum inner diameter of the tube. The outer diameter of the tube is 120 mm, and the material has an allowable shear stress of $\tau_{allow} = 75$ MPa.

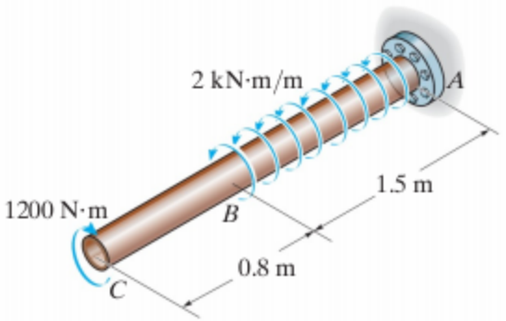


Probs. 10–19/20

10

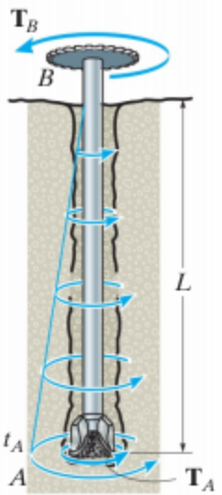
10–21. The 60-mm-diameter solid shaft is subjected to the distributed and concentrated torsional loadings shown. Determine the absolute maximum and minimum shear stresses on the outer surface of the shaft and specify their locations, measured from the fixed end A.

10–22. The solid shaft is subjected to the distributed and concentrated torsional loadings shown. Determine the required diameter d of the shaft to the nearest mm if the allowable shear stress for the material is $\tau_{allow} = 50$ MPa.



Probs. 10–21/22

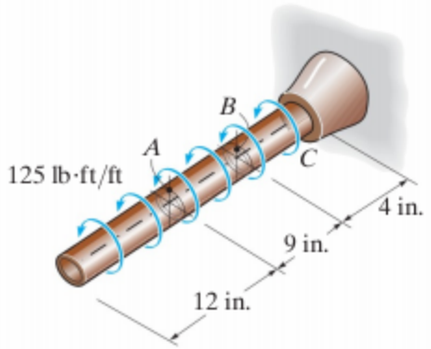
10–23. When drilling a well at constant angular velocity, the bottom end of the drill pipe encounters a torsional resistance T_A. Also, soil along the sides of the pipe creates a distributed frictional torque along its length, varying uniformly from zero at the surface B to t_A at A. Determine the minimum torque T_B that must be supplied by the drive unit to overcome the resisting torques, and compute the maximum shear stress in the pipe. The pipe has an outer radius r_o and an inner radius r_i.



Prob. 10–23

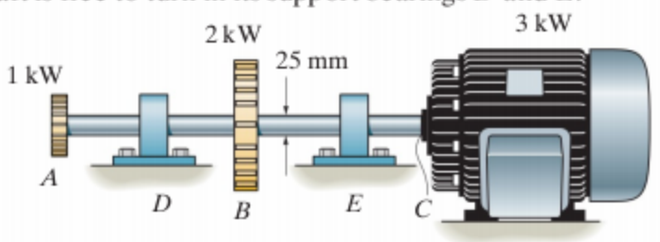
***10–24.** The copper pipe has an outer diameter of 2.50 in. and an inner diameter of 2.30 in. If it is tightly secured to the wall at C and a uniformly distributed torque is applied to it as shown, determine the shear stress developed at points A and B. These points lie on the pipe's outer surface. Sketch the shear stress on volume elements located at A and B.

10–25. The copper pipe has an outer diameter of 2.50 in. and an inner diameter of 2.30 in. If it is tightly secured to the wall at C and it is subjected to the uniformly distributed torque along its entire length, determine the absolute maximum shear stress in the pipe. Discuss the validity of this result.



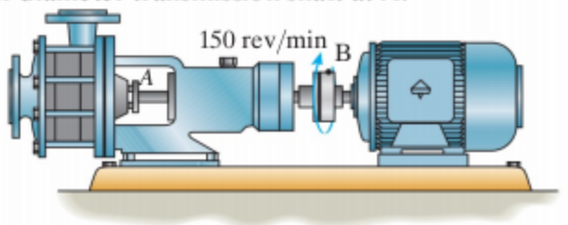
Probs. 10–24/25

10–26. The solid steel shaft AC has a diameter of 25 mm and is supported by smooth bearings at D and E. It is coupled to a motor at C, which delivers 3 kW of power to the shaft while it is turning at 50 rev/s. If gears A and B remove 1 kW and 2 kW, respectively, determine the maximum shear stress developed in the shaft within regions AB and BC. The shaft is free to turn in its support bearings D and E.



Prob. 10–26

10–27. The pump operates using the motor that has a power of 85 W. If the impeller at B is turning at 150 rev/min, determine the maximum shear stress developed in the 20-mm-diameter transmission shaft at A.



Prob. 10–27

10.4 Angle of Twist

Oil wells are commonly drilled to depths exceeding one thousand meters. As a result, the total angle of twist of a string of drill pipe can be substantial and must be determined.

Occasionally the design of a shaft depends on restricting the amount of rotation or twist that may occur when the shaft is subjected to a torque. Furthermore, being able to compute the angle of twist for a shaft is important when analyzing the reactions on statically indeterminate shafts.

In this section we will develop a formula for determining the ***angle of twist*** ϕ (phi) of one end of a shaft with respect to its other end. The shaft is assumed to have a circular cross section that can gradually vary along its length, Fig. 10–13*a*. Also, the material is assumed to be homogeneous and to behave in a linear-elastic manner when the torque is applied. Like the case of an axially loaded bar, we will neglect the localized deformations that occur at points of application of the torques and where the cross section changes abruptly. By Saint-Venant's principle, these effects occur within small regions of the shaft's length and generally they will have only a slight effect on the final result.

Using the method of sections, a differential disk of thickness dx, located at position x, is isolated from the shaft, Fig. 10–13*b*. The internal resultant torque is $T(x)$, since the external loading may cause it to vary along the axis of the shaft. Due to $T(x)$, the disk will twist, such that the *relative rotation* of one of its faces with respect to the other face is $d\phi$, Fig. 10–13*b*. As a result, an element of material located at an arbitrary radius ρ within the disk will undergo a shear strain γ. The values of γ and $d\phi$ are related by Eq. 10–1, namely,

$$d\phi = \gamma \frac{dx}{\rho} \tag{10–13}$$

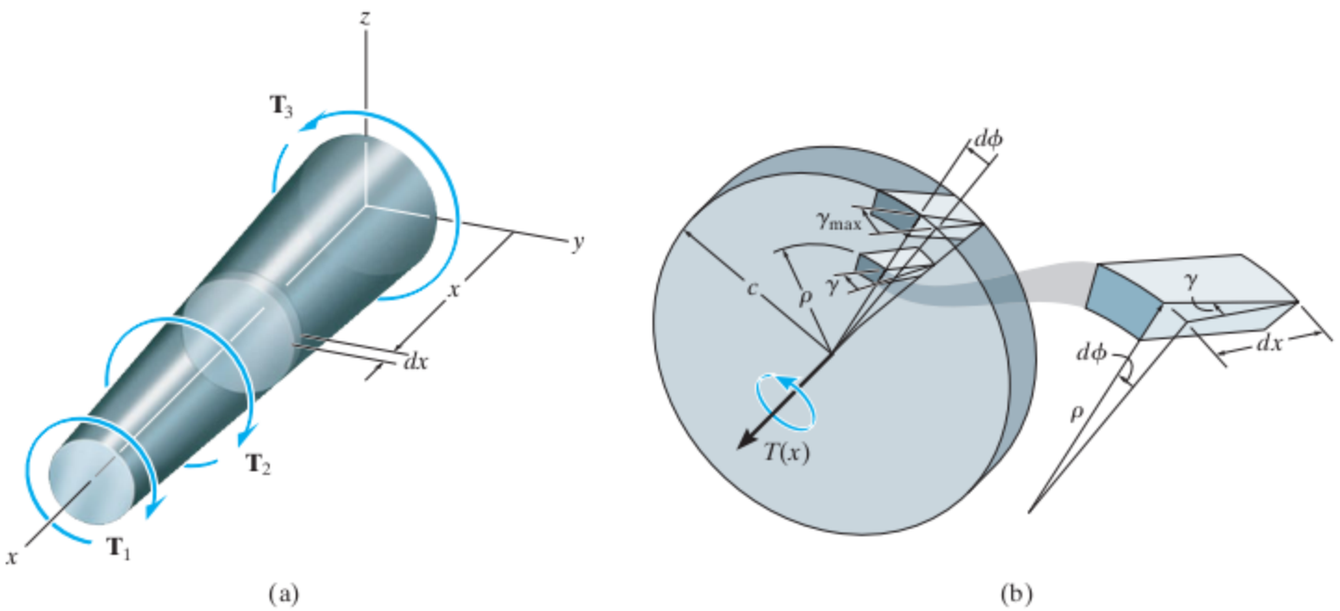


Fig. 10–13

Since Hooke's law, $\gamma = \tau/G$, applies and the shear stress can be expressed in terms of the applied torque using the torsion formula $\tau = T(x)\rho/J(x)$, then $\gamma = T(x)\rho/J(x)G(x)$. Substituting this into Eq. 10–13, the angle of twist for the disk is

$$d\phi = \frac{T(x)}{J(x)G(x)}\,dx$$

Integrating over the entire length L of the shaft, we obtain the angle of twist for the entire shaft, namely,

$$\phi = \int_0^L \frac{T(x)\,dx}{J(x)G(x)} \tag{10–14}$$

Here

ϕ = the angle of twist of one end of the shaft with respect to the other end, measured in radians
$T(x)$ = the *internal torque* at the arbitrary position x, found from the method of sections and the equation of moment equilibrium applied about the shaft's axis
$J(x)$ = the shaft's polar moment of inertia expressed as a function of x
$G(x)$ = the shear modulus of elasticity for the material expressed as a function of x

Constant Torque and Cross-Sectional Area. Usually in engineering practice the material is homogeneous so that G is constant. Also, the shaft's cross-sectional area and the external torque are constant along the length of the shaft, Fig. 10–14. If this is the case, the internal torque $T(x) = T$, the polar moment of inertia $J(x) = J$, and Eq. 10–13 can be integrated, which gives

$$\phi = \frac{TL}{JG} \tag{10–15}$$

The similarities between the above two equations and those for an axially loaded bar ($\delta = \int P(x)\,dx/A(x)E$ and $\delta = PL/AE$) should be noted.

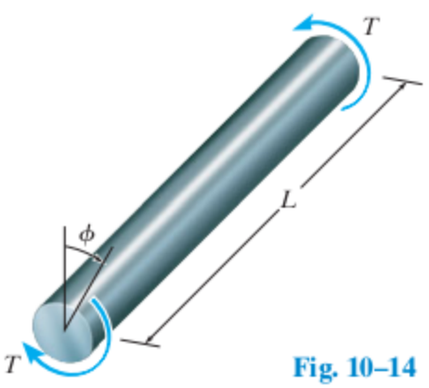


Fig. 10–14

When computing both the stress and the angle of twist of this soil auger, it is necessary to consider the variable torsional loading which acts along its length.

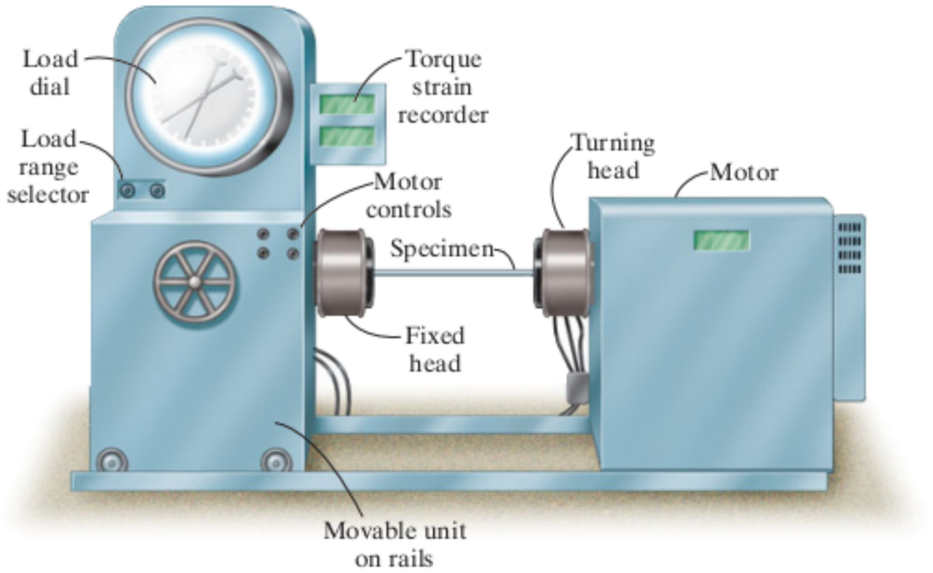


Fig. 10–15

Equation 10–15 is often used to determine the shear modulus of elasticity G of a material. To do so, a specimen of known length and diameter is placed in a torsion testing machine like the one shown in Fig. 10–15. The applied torque T and angle of twist ϕ are then measured along the length L. Using Eq. 10–15, $G = TL/J\phi$. Usually, to obtain a more reliable value of G, several of these tests are performed and the average value is used.

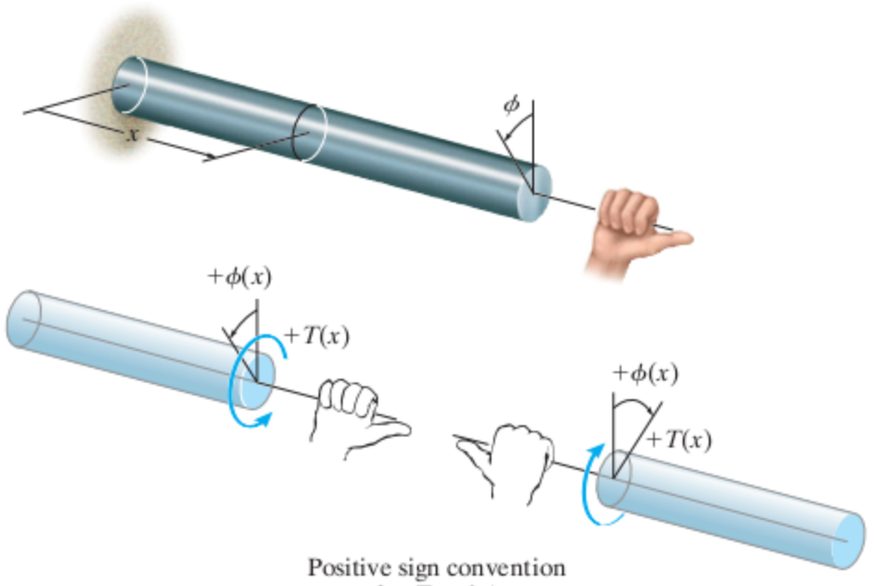
Multiple Torques. If the shaft is subjected to several different torques, or the cross-sectional area or shear modulus changes abruptly from one region of the shaft to the next, Eq. 10–15 can be applied to each segment of the shaft where these quantities are all constant. The angle of twist of one end of the shaft with respect to the other is then found from the vector addition of the angles of twist of each segment. For this case,

$$\phi = \sum \frac{TL}{JG} \tag{10–16}$$

10

Sign Convention. In order to apply this equation, we must develop a sign convention for both the internal torque and the angle of twist of one end of the shaft with respect to the other end. To do this, we will use the right-hand rule, whereby both the torque and angle will be *positive*, provided the *thumb* is directed *outward* from the shaft when the fingers curl to give the tendency for rotation, Fig. 10–16.

To illustrate the use of this sign convention, consider the shaft shown in Fig. 10–17*a*. The angle of twist of end A with respect to end D is to be determined. Three segments of the shaft must be considered, since the



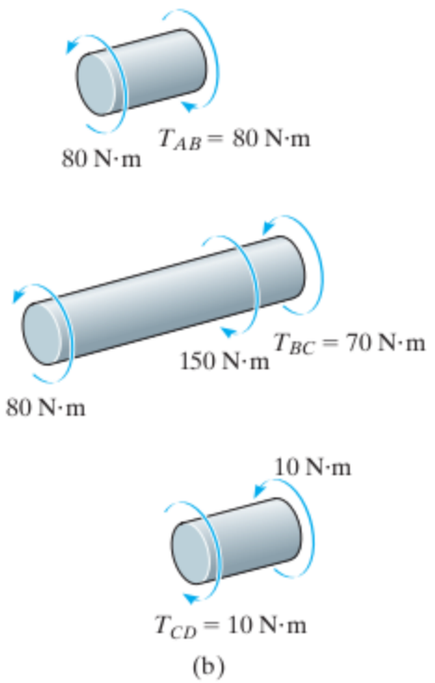
Positive sign convention for T and ϕ.

Fig. 10–16

internal torque will change at B and at C. Using the method of sections, the *internal torques* are found for each segment, Fig. 10–17*b*. By the right-hand rule, with positive torques directed away from the *sectioned end* of the shaft, we have $T_{AB} = +80\,\text{N}\cdot\text{m}$, $T_{BC} = -70\,\text{N}\cdot\text{m}$, and $T_{CD} = -10\,\text{N}\cdot\text{m}$. These results are also shown on the *torque diagram* for the shaft, Fig. 10–17*c*. Applying Eq. 10–16, we have

$$\phi_{A/D} = \frac{(+80\,\text{N}\cdot\text{m})\,L_{AB}}{JG} + \frac{(-70\,\text{N}\cdot\text{m})\,L_{BC}}{JG} + \frac{(-10\,\text{N}\cdot\text{m})\,L_{CD}}{JG}$$

If the other data is substituted and the answer is found as a *positive* quantity, it means that end A will rotate as indicated by the curl of the right-hand fingers when the thumb is directed *away* from the shaft, Fig. 10–17*a*. The double subscript notation is used to indicate this relative angle of twist ($\phi_{A/D}$); however, if the angle of twist is to be determined relative to a *fixed support*, then only a single subscript will be used. For example, if D is a fixed support, then the angle of twist will be denoted as ϕ_A.



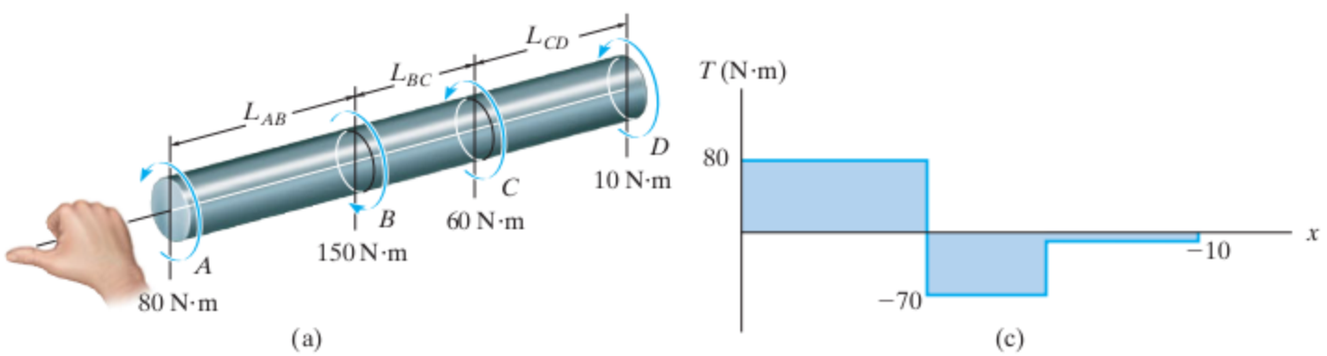


Fig. 10–17

Important Point

- When applying Eq. 10–14 to determine the angle of twist, it is important that the applied torques do not cause yielding of the material and that the material is homogeneous and behaves in a linear elastic manner.

Procedure for Analysis

The angle of twist of one end of a shaft or tube with respect to the other end can be determined using the following procedure.

Internal Torque.

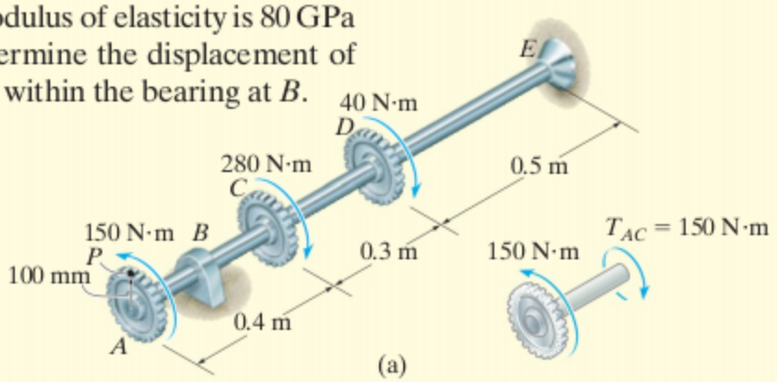
- The *internal torque* is found at a point on the axis of the shaft by using the method of sections and the equation of moment equilibrium, applied along the shaft's axis.
- If the torque varies along the shaft's length, a section should be made at the arbitrary position x along the shaft and the *internal torque* represented as a function of x, i.e., $T(x)$.
- If several constant external torques act on the shaft between its ends, the internal torque in each *segment* of the shaft, between any two external torques, must be determined. The results can be represented graphically as a torque diagram.

Angle of Twist.

- When the circular cross-sectional area of the shaft varies along the shaft's axis, the polar moment of inertia must be expressed as a function of its position x along the axis, $J(x)$.
- If the polar moment of inertia or the internal torque *suddenly changes* between the ends of the shaft, then $\phi = \int (T(x)/J(x)G)\,dx$ or $\phi = TL/JG$ must be applied to *each segment* for which J, G, and T are continuous or constant.
- When the internal torque in each segment is determined, be sure to use a consistent sign convention for the shaft, such as the one discussed in Fig. 10–16. Also make sure that a consistent set of units is used when substituting numerical data into the equations.

EXAMPLE 10.4

The gears attached to the fixed-end steel shaft are subjected to the torques shown in Fig. 10–18*a*. If the shear modulus of elasticity is 80 GPa and the shaft has a diameter of 14 mm, determine the displacement of the tooth P on gear A. The shaft turns freely within the bearing at B.



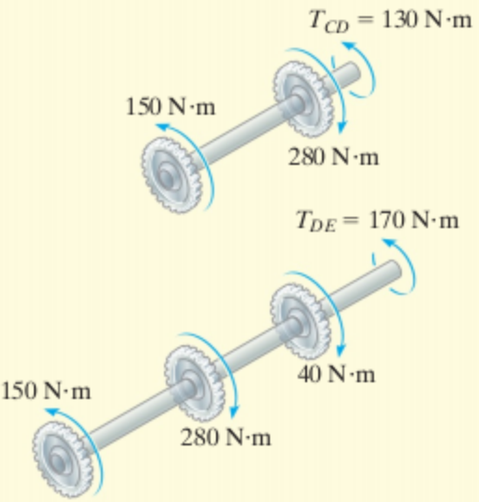
(a)

SOLUTION

Internal Torque. By inspection, the torques in segments AC, CD, and DE are different yet *constant* throughout each segment. Free-body diagrams of appropriate segments of the shaft along with the calculated internal torques are shown in Fig. 10–18*b*. Using the right-hand rule and the established sign convention that positive torque is directed away from the sectioned end of the shaft, we have

$$T_{AC} = +150\ \text{N}\cdot\text{m} \qquad T_{CD} = -130\ \text{N}\cdot\text{m} \qquad T_{DE} = -170\ \text{N}\cdot\text{m}$$

These results are also shown on the torque diagram, Fig. 10–18*c*.



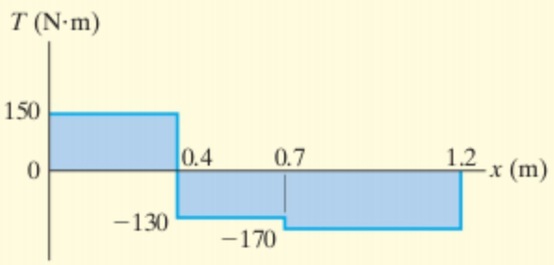
(b)

Angle of Twist. The polar moment of inertia for the shaft is

$$J = \frac{\pi}{2}(0.007\ \text{m})^4 = 3.771(10^{-9})\ \text{m}^4$$

Applying Eq. 10–16 to each segment and adding the results algebraically, we have

$$\phi_A = \sum \frac{TL}{JG} = \frac{(+150\ \text{N}\cdot\text{m})(0.4\ \text{m})}{3.771(10^{-9})\ \text{m}^4\left[80(10^9)\ \text{N/m}^2\right]} + \frac{(-130\ \text{N}\cdot\text{m})(0.3\ \text{m})}{3.771(10^{-9})\ \text{m}^4\left[80(10^9)\ \text{N/m}^2\right]} + \frac{(-170\ \text{N}\cdot\text{m})(0.5\ \text{m})}{3.771(10^{-9})\ \text{m}^4\left[80(10^9)\ \text{N/m}^2\right]} = -0.2121\ \text{rad}$$

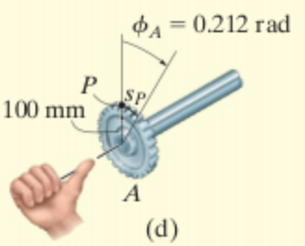


(c)

Since the answer is negative, by the right-hand rule the thumb is directed *toward* the end E of the shaft, and therefore gear A will rotate as shown in Fig. 10–18*d*.

The displacement of tooth P on gear A is

$$s_P = \phi_A r = (0.2121\ \text{rad})(100\ \text{mm}) = 21.2\ \text{mm} \qquad \textit{Ans.}$$



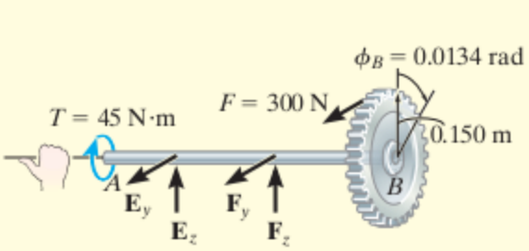
(d)

Fig. 10–18

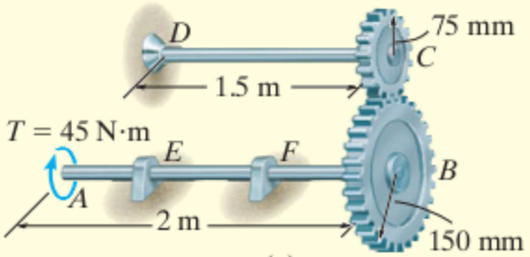
NOTE: Remember that this analysis is valid only if the shear stress does not exceed the proportional limit of the material.

EXAMPLE 10.5

The two solid steel shafts shown in Fig. 10–19*a* are coupled together using the meshed gears. Determine the angle of twist of end A of shaft AB when the torque $T = 45\ \text{N}\cdot\text{m}$ is applied. Take $G = 80\ \text{GPa}$. Shaft AB is free to rotate within bearings E and F, whereas shaft DC is fixed at D. Each shaft has a diameter of 20 mm.

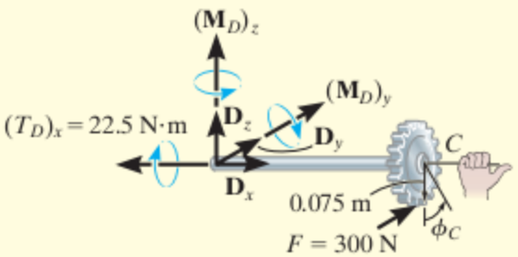


(b)



(a)

Fig. 10–19



(c)

SOLUTION

Internal Torque. Free-body diagrams for each shaft are shown in Figs. 10–19*b* and 10–19*c*. Summing moments along the x axis of shaft AB yields the tangential reaction between the gears of $F = 45\ \text{N}\cdot\text{m}/0.15\ \text{m} = 300\ \text{N}$. Summing moments about the x axis of shaft DC, this force then creates a torque of $(T_D)_x = 300\ \text{N}\ (0.075\ \text{m}) = 22.5\ \text{N}\cdot\text{m}$ on shaft DC.

Angle of Twist. To solve the problem, we will first calculate the rotation of gear C due to the torque of $22.5\ \text{N}\cdot\text{m}$ in shaft DC, Fig. 10–19*c*. This angle of twist is

$$\phi_C = \frac{TL_{DC}}{JG} = \frac{(+22.5\ \text{N}\cdot\text{m})(1.5\ \text{m})}{(\pi/2)(0.010\ \text{m})^4\left[80(10^9)\ \text{N/m}^2\right]} = +0.0269\ \text{rad}$$

Since the gears at the end of the shaft are in mesh, the rotation ϕ_C of gear C causes gear B to rotate ϕ_B, Fig. 10–19*b*, where

$$\phi_B\,(0.15\ \text{m}) = (0.0269\ \text{rad})(0.075\ \text{m})$$
$$\phi_B = 0.0134\ \text{rad}$$

We will now determine the angle of twist of end A with respect to end B of shaft AB caused by the $45\ \text{N}\cdot\text{m}$ torque, Fig. 10–19*b*. We have

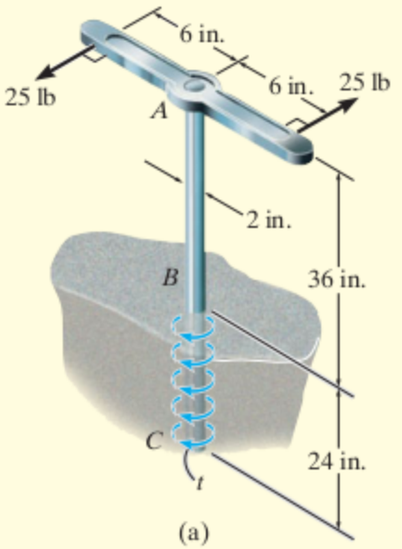
$$\phi_{A/B} = \frac{T_{AB}L_{AB}}{JG} = \frac{(+45\ \text{N}\cdot\text{m})(2\ \text{m})}{(\pi/2)(0.010\ \text{m})^4\left[80(10^9)\ \text{N/m}^2\right]} = +0.0716\ \text{rad}$$

The rotation of end A is therefore determined by adding ϕ_B and $\phi_{A/B}$, since both angles are in the *same direction*, Fig. 10–19*b*. We have

$$\phi_A = \phi_B + \phi_{A/B} = 0.0134\ \text{rad} + 0.0716\ \text{rad} = +0.0850\ \text{rad}\quad \textit{Ans.}$$

EXAMPLE 10.6

The 2-in.-diameter solid cast-iron post shown in Fig. 10–20*a* is buried 24 in. in soil. If a torque is applied to its top using a rigid wrench, determine the maximum shear stress in the post and the angle of twist at its top. Assume that the torque is about to turn the post, and the soil exerts a uniform torsional resistance of t lb · in./in. along its 24-in. buried length. $G = 5.5(10^3)$ ksi.



(a)

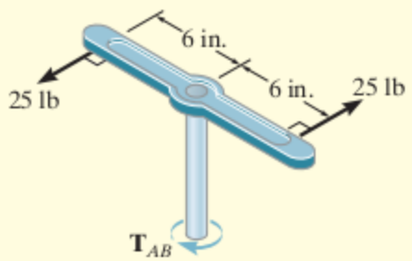
SOLUTION

Internal Torque. The internal torque in segment AB of the post is constant. From the free-body diagram, Fig. 10–20*b*, we have

$$\Sigma M_z = 0; \qquad T_{AB} = 25 \text{ lb}\,(12 \text{ in.}) = 300 \text{ lb} \cdot \text{in.}$$

The magnitude of the uniform distribution of torque along the buried segment BC can be determined from equilibrium of the entire post, Fig. 10–20*c*. Here

$$\Sigma M_z = 0 \qquad 25 \text{ lb}\,(12 \text{ in.}) - t(24 \text{ in.}) = 0$$
$$t = 12.5 \text{ lb} \cdot \text{in./in.}$$



(b)

Hence, from a free-body diagram of a section of the post located at the position x, Fig. 10–20*d*, we have

$$\Sigma M_z = 0; \qquad T_{BC} - 12.5x = 0$$
$$T_{BC} = 12.5x$$

Maximum Shear Stress. The largest shear stress occurs in region AB, since the torque is largest there and J is constant for the post. Applying the torsion formula, we have

$$\tau_{\max} = \frac{T_{AB}c}{J} = \frac{(300 \text{ lb} \cdot \text{in.})(1 \text{ in.})}{(\pi/2)(1 \text{ in.})^4} = 191 \text{ psi} \qquad \textit{Ans.}$$

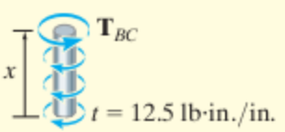
Angle of Twist. The angle of twist at the top can be determined relative to the bottom of the post, since it is fixed and yet is about to turn. Both segments AB and BC twist, and so in this case we have

$$\phi_A = \frac{T_{AB}\,L_{AB}}{JG} + \int_0^{L_{BC}} \frac{T_{BC}\,dx}{JG}$$

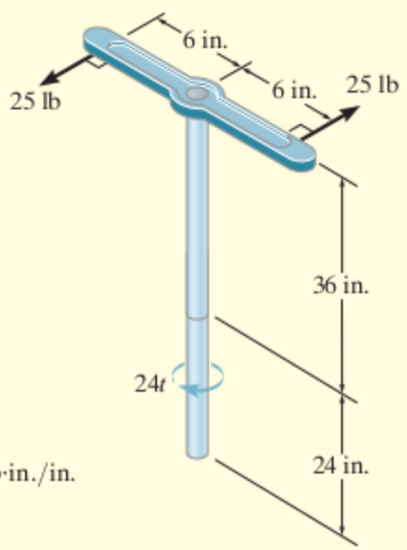
$$= \frac{(300 \text{ lb} \cdot \text{in.})\,36 \text{ in.}}{JG} + \int_0^{24 \text{ in.}} \frac{12.5x\,dx}{JG}$$

$$= \frac{10\,800 \text{ lb} \cdot \text{in}^2}{JG} + \frac{12.5[(24)^2/2] \text{ lb} \cdot \text{in}^2}{JG}$$

$$= \frac{14\,400 \text{ lb} \cdot \text{in}^2}{(\pi/2)(1 \text{ in.})^4\,5500(10^3) \text{ lb/in}^2} = 0.00167 \text{ rad} \qquad \textit{Ans.}$$



(d)

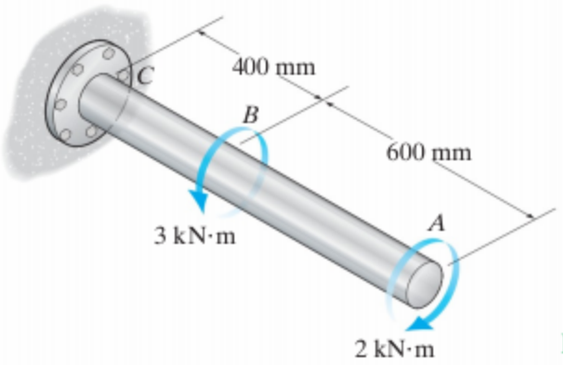


(c)

Fig. 10–20

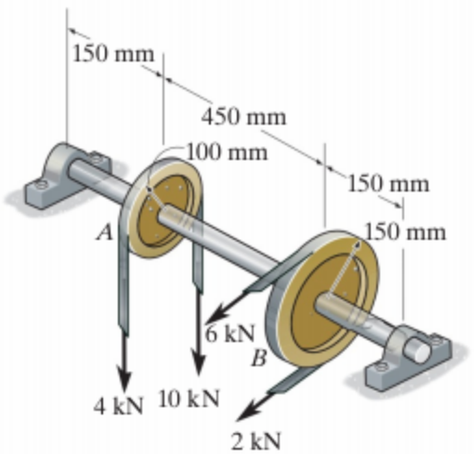
FUNDAMENTAL PROBLEMS

F10–7. The 60-mm-diameter steel shaft is subjected to the torques shown. Determine the angle of twist of end A with respect to C. Take $G = 75$ GPa.



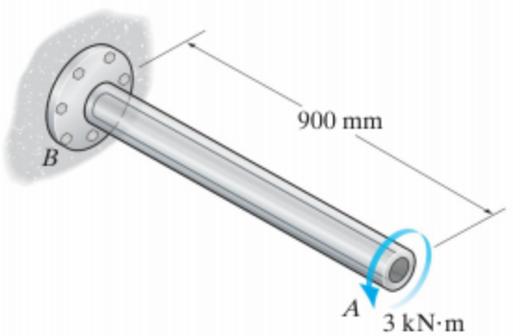
F10–7

F10–8. Determine the angle of twist of wheel B with respect to wheel A. The shaft has a diameter of 40 mm and is made of steel for which $G = 75$ GPa.



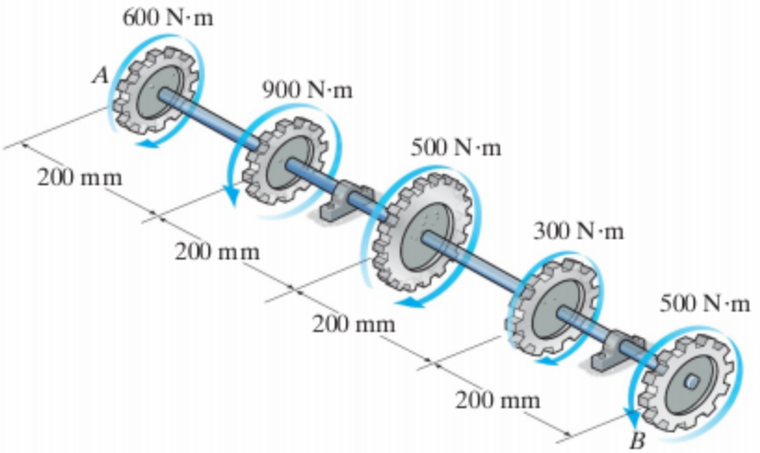
F10–8

F10–9. The hollow 6061-T6 aluminum shaft has an outer and inner radius of $c_o = 40$ mm and $c_i = 30$ mm, respectively. Determine the angle of twist of end A. The flexible support at B has a torsional stiffness of $k = 90\ \text{kN} \cdot \text{m/rad}$.



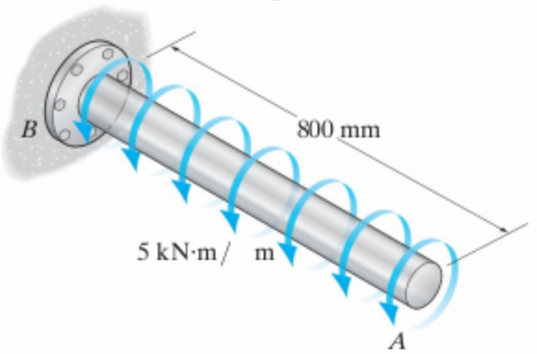
F10–9

F10–10. A series of gears are mounted on the 40-mm-diameter steel shaft. Determine the angle of twist of gear B relative to gear A. Take $G = 75$ GPa.



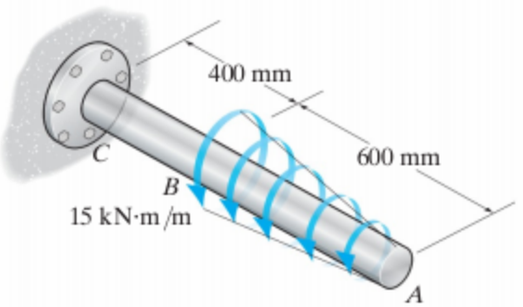
F10–10

F10–11. The 80-mm-diameter shaft is made of steel. If it is subjected to the uniform distributed torque, determine the angle of twist of end A with respect to B. Take $G = 75$ GPa.



F10–11

F10–12. The 80-mm-diameter shaft is made of steel. If it is subjected to the triangular distributed load, determine the angle of twist of end A with respect to C. Take $G = 75$ GPa.

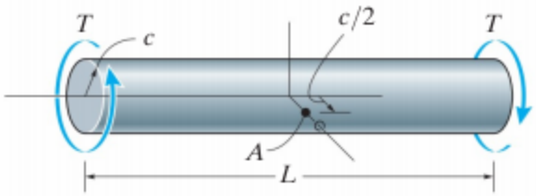


F10–12

PROBLEMS

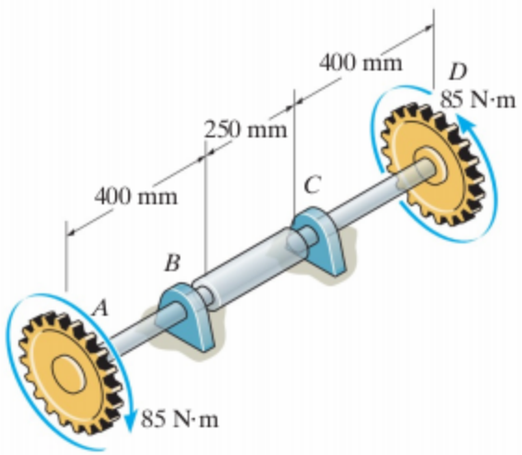
*10–28. The propellers of a ship are connected to a A-36 steel shaft that is 60 m long and has an outer diameter of 340 mm and inner diameter of 260 mm. If the power output is 4.5 MW when the shaft rotates at 20 rad/s, determine the maximum torsional stress in the shaft and its angle of twist.

10–29. The solid shaft of radius c is subjected to a torque **T** at its ends. Show that the maximum shear strain developed in the shaft is $\gamma_{max} = Tc/JG$. What is the shear strain on an element located at point A, $c/2$ from the center of the shaft? Sketch the strain distortion of this element.



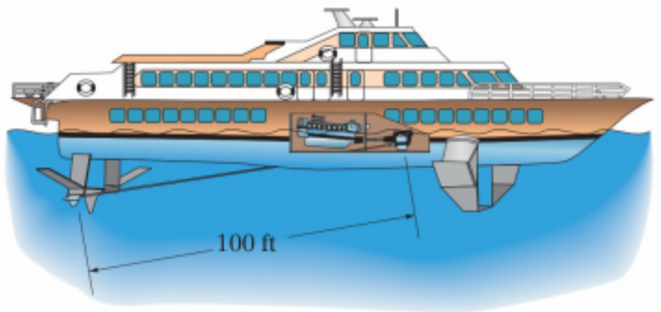
Prob. 10–29

10–30. The A-36 steel axle is made from tubes AB and CD and a solid section BC. It is supported on smooth bearings that allow it to rotate freely. If the gears, fixed to its ends, are subjected to 85-N · m torques, determine the angle of twist of gear A relative to gear D. The tubes have an outer diameter of 30 mm and an inner diameter of 20 mm. The solid section has a diameter of 40 mm.



Prob. 10–30

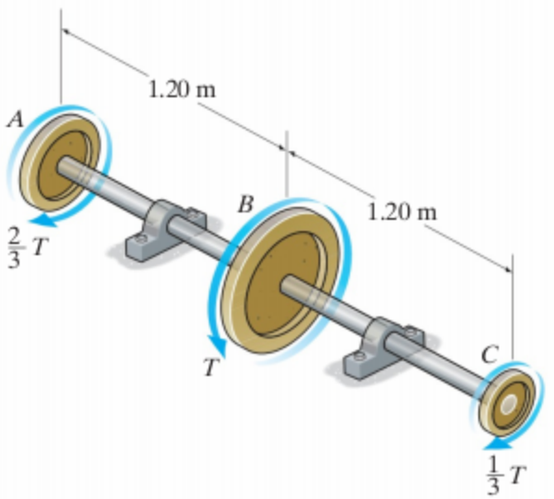
10–31. The hydrofoil boat has an A-36 steel propeller shaft that is 100 ft long. It is connected to an in-line diesel engine that delivers a maximum power of 2500 hp and causes the shaft to rotate at 1700 rpm. If the outer diameter of the shaft is 8 in. and the wall thickness is $\frac{3}{8}$ in., determine the maximum shear stress developed in the shaft. Also, what is the "wind up," or angle of twist in the shaft at full power?



Prob. 10–31

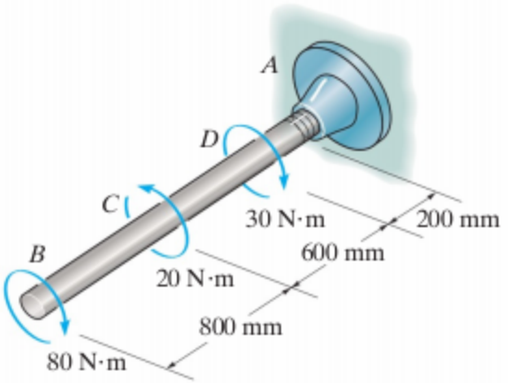
*10–32. The 60-mm-diameter shaft is made of 6061-T6 aluminum having an allowable shear stress of $\tau_{allow} = 80$ MPa. Determine the maximum allowable torque **T**. Also, find the corresponding angle of twist of disk A relative to disk C.

10–33. The 60-mm-diameter shaft is made of 6061-T6 aluminum. If the allowable shear stress is $\tau_{allow} = 80$ MPa, and the angle of twist of disk A relative to disk C is limited so that it does not exceed 0.06 rad, determine the maximum allowable torque **T**.



Probs. 10–32/33

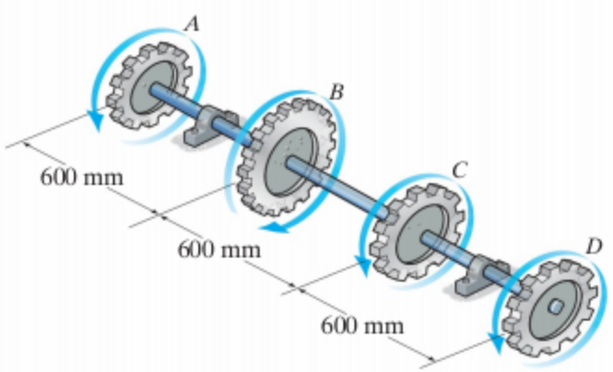
10–34. The 20-mm-diameter A-36 steel shaft is subjected to the torques shown. Determine the angle of twist of the end B.



Prob. 10–34

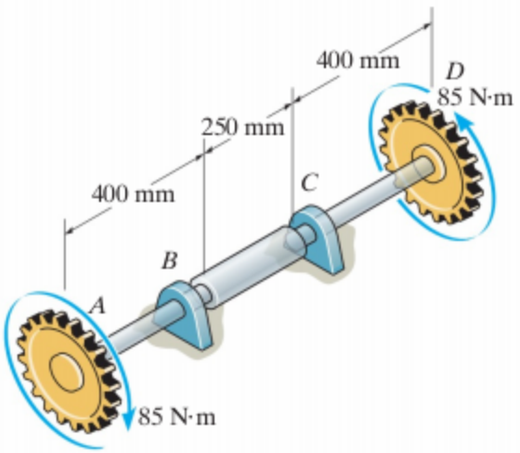
10–35. The shaft is made of A992 steel with the allowable shear stress of $\tau_{allow} = 75$ MPa. If gear B supplies 15 kW of power, while gears A, C, and D withdraw 6 kW, 4 kW, and 5 kW, respectively, determine the required minimum diameter d of the shaft to the nearest millimeter. Also, find the corresponding angle of twist of gear A relative to gear D. The shaft is rotating at 600 rpm.

***10–36.** Gear B supplies 15 kW of power, while gears A, C, and D withdraw 6 kW, 4 kW, and 5 kW, respectively. If the shaft is made of steel with the allowable shear stress of $\tau_{allow} = 75$ MPa, and the relative angle of twist between any two gears cannot exceed 0.05 rad, determine the required minimum diameter d of the shaft to the nearest millimeter. The shaft is rotating at 600 rpm.



Probs. 10–35/36

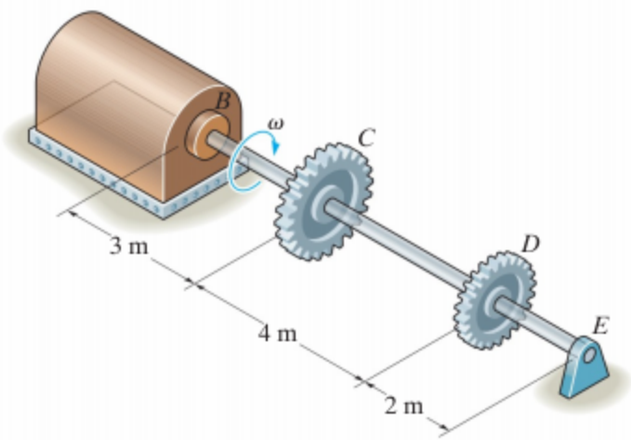
10–37. The A-36 steel axle is made from tubes AB and CD and a solid section BC. It is supported on smooth bearings that allow it to rotate freely. If the gears, fixed to its ends, are subjected to 85-N · m torques, determine the angle of twist of the end B of the solid section relative to end C. The tubes have an outer diameter of 30 mm and an inner diameter of 20 mm. The solid section has a diameter of 40 mm.



Prob. 10–37

10–38. The turbine develops 150 kW of power, which is transmitted to the gears such that C receives 70% and D receives 30%. If the rotation of the 100-mm-diameter A-36 steel shaft is $\omega = 800$ rev/min., determine the absolute maximum shear stress in the shaft and the angle of twist of end E of the shaft relative to B. The journal bearing at E allows the shaft to turn freely about its axis.

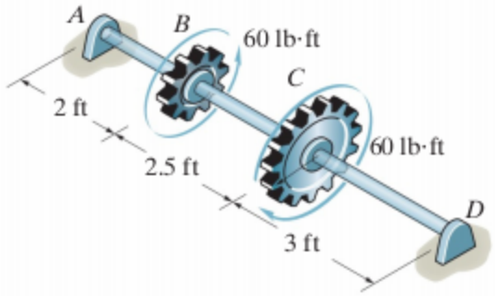
10–39. The turbine develops 150 kW of power, which is transmitted to the gears such that both C and D receive an equal amount. If the rotation of the 100-mm-diameter A-36 steel shaft is $\omega = 500$ rev/min., determine the absolute maximum shear stress in the shaft and the rotation of end B of the shaft relative to E. The journal bearing at E allows the shaft to turn freely about its axis.



Probs. 10–38/39

*10–40. The shaft is made of A-36 steel. It has a diameter of 1 in. and is supported by bearings at A and D, which allow free rotation. Determine the angle of twist of B with respect to D.

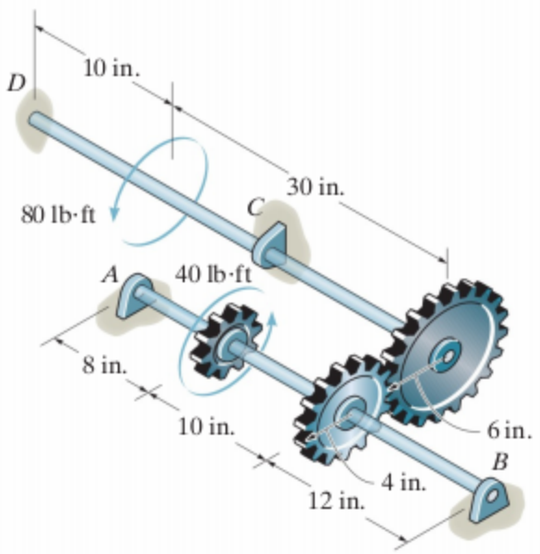
10–41. The shaft is made of A-36 steel. It has a diameter of 1 in. and is supported by bearings at A and D, which allow free rotation. Determine the angle of twist of gear C with respect to B.



Probs. 10–40/41

10–42. The two shafts are made of A-36 steel. Each has a diameter of 1 in., and they are supported by bearings at A, B, and C, which allow free rotation. If the support at D is fixed, determine the angle of twist of end B when the torques are applied to the assembly as shown.

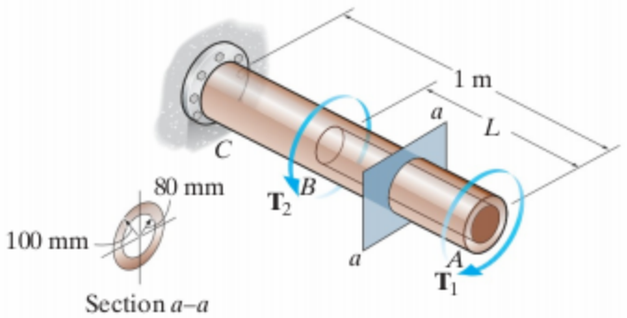
10–43. The two shafts are made of A-36 steel. Each has a diameter of 1 in., and they are supported by bearings at A, B, and C, which allow free rotation. If the support at D is fixed, determine the angle of twist of end A when the torques are applied to the assembly as shown.



Probs. 10–42/43

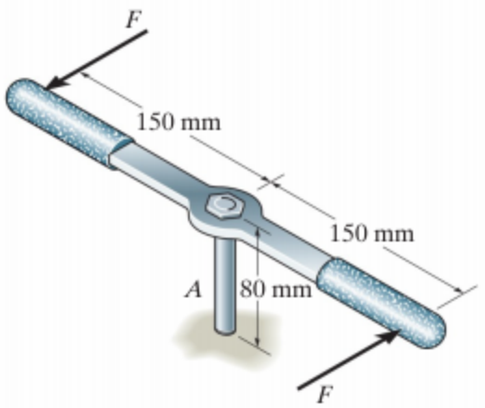
*10–44. If the shaft is made of red brass C83400 copper with an allowable shear stress of $\tau_{allow} = 20$ MPa, determine the maximum allowable torques T_1 and T_2 that can be applied at A and B. Also, find the corresponding angle of twist of end A. Set $L = 0.75$ m.

10–45. If the shaft is made of red brass C83400 copper and is subjected to torques $T_1 = 20$ kN · m and $T_2 = 50$ kN · m, determine the distance L so that the angle of twist at end A is zero.



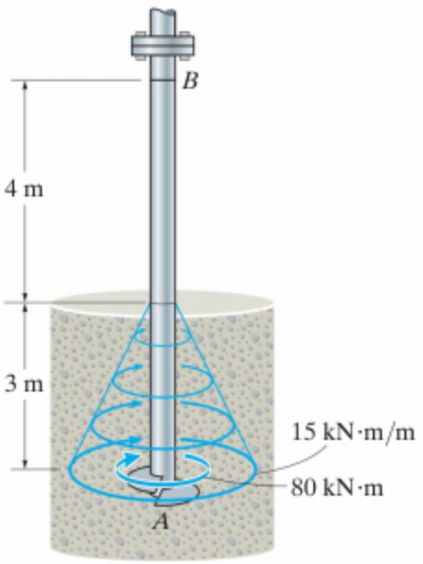
Probs. 10–44/45

10–46. The 8-mm-diameter A-36 steel bolt is screwed tightly into a block at A. Determine the couple forces F that should be applied to the wrench so that the maximum shear stress in the bolt becomes 18 MPa. Also, compute the corresponding displacement of each force F needed to cause this stress. Assume that the wrench is rigid.



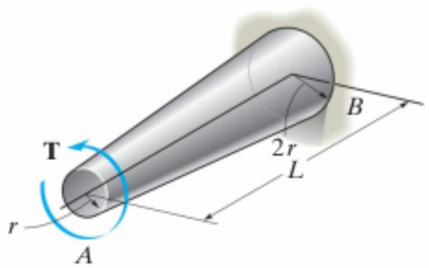
Prob. 10–46

10–47. The A992 steel posts are "drilled" at constant angular speed into the soil using the rotary installer. If the post has an inner diameter of 200 mm and an outer diameter of 225 mm, determine the relative angle of twist of end A of the post with respect to end B when the post reaches the depth indicated. Due to soil friction, assume the torque along the post varies linearly as shown, and a concentrated torque of 80 kN · m acts at the bit.



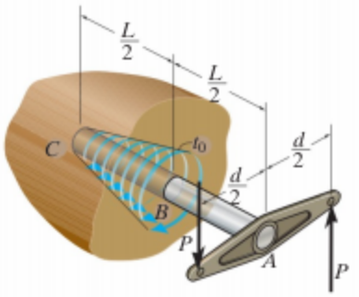
Prob. 10–47

*10–48. The tapered shaft has a length L and a radius r at end A and $2r$ at end B. If it is fixed at end B and is subjected to a torque T, determine the angle of twist of end A. The shear modulus is G.



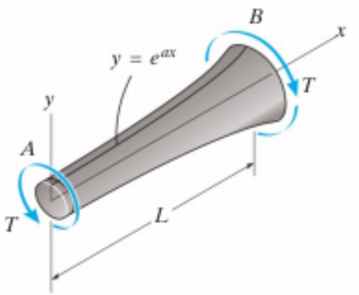
Prob. 10–48

10–49. The rod ABC of radius c is embedded into a medium where the distributed torque reaction varies linearly from zero at C to t_0 at B. If couple forces P are applied to the lever arm, determine the value of t_0 for equilibrium. Also, find the angle of twist of end A. The rod is made from material having a shear modulus of G.



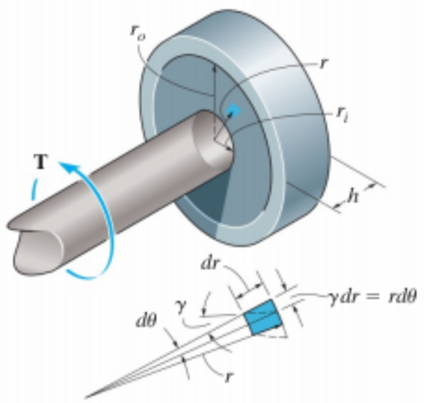
Prob. 10–49

10–50. The contour of the surface of the shaft is defined by the equation $y = e^{ax}$, where a is a constant. If the shaft is subjected to a torque T at its ends, determine the angle of twist of end A with respect to end B. The shear modulus is G.



Prob. 10–50

10–51. A cylindrical spring consists of a rubber annulus bonded to a rigid ring and shaft. If the ring is held fixed and a torque T is applied to the rigid shaft, determine the angle of twist of the shaft. The shear modulus of the rubber is G. *Hint:* As shown in the figure, the deformation of the element at radius r can be determined from $rd\theta = dr\gamma$. Use this expression along with $\tau = T/(2\pi r^2 h)$ to obtain the result.

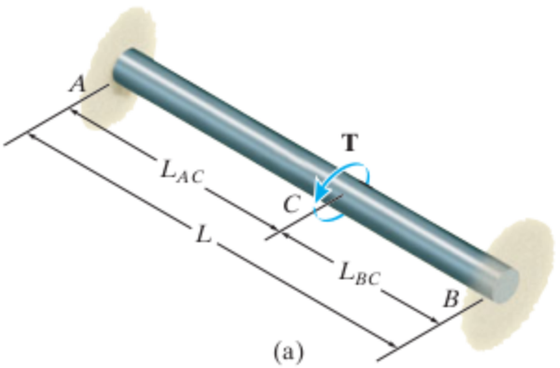


Prob. 10–51

10.5 Statically Indeterminate Torque-Loaded Members

A torsionally loaded shaft may be classified as statically indeterminate if the moment equation of equilibrium, applied about the axis of the shaft, is not adequate to determine the unknown torques acting on the shaft. An example of this situation is shown in Fig. 10–21*a*. As shown on the free-body diagram, Fig. 10–21*b*, the reactive torques at the supports A and B are unknown. We require that

$$\Sigma M_x = 0; \qquad T - T_A - T_B = 0$$



(a)

In order to obtain a solution, we will use the method of analysis discussed in Sec. 9.4. The necessary condition of compatibility, or the kinematic condition, requires the angle of twist of one end of the shaft with respect to the other end to be equal to zero, since the end supports are fixed. Therefore,

$$\phi_{A/B} = 0$$

Provided the material is linear-elastic, we can apply the load–displacement relation $\phi = TL/JG$ to express the compatibility condition in terms of the unknown torques. Realizing that the internal torque in segment AC is $+T_A$ and in segment CB it is $-T_B$, Fig. 10–21*c*, we have

$$\frac{T_A L_{AC}}{JG} - \frac{T_B L_{BC}}{JG} = 0$$

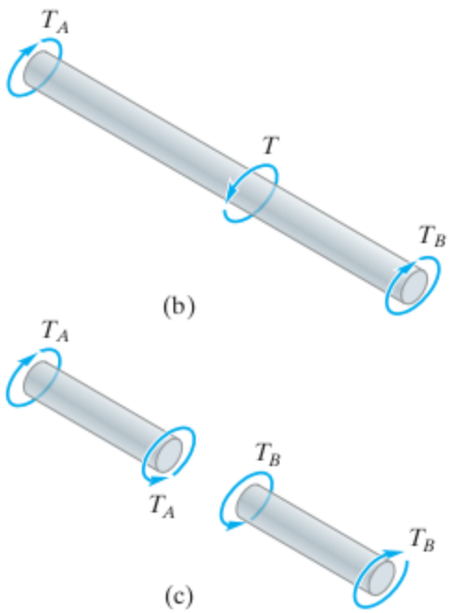


Fig. 10–21

Solving the above two equations for the reactions, realizing that $L = L_{AC} + L_{BC}$, we get

$$T_A = T\left(\frac{L_{BC}}{L}\right) \quad \text{and} \quad T_B = T\left(\frac{L_{AC}}{L}\right)$$

Procedure for Analysis

The unknown torques in statically indeterminate shafts are determined by satisfying equilibrium, compatibility, and torque-displacement requirements for the shaft.

Equilibrium.

- Draw a free-body diagram of the shaft in order to identify all the external torques that act on it. Then write the equation of moment equilibrium about the axis of the shaft.

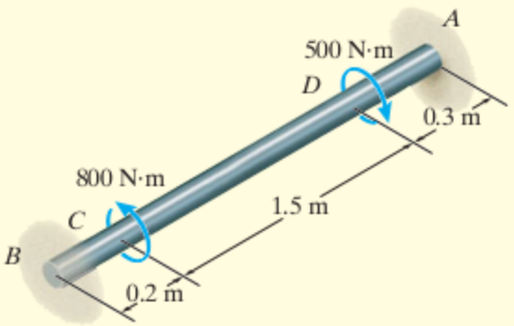
Compatibility.

- Write the compatibility equation between two points along the shaft. Give consideration as to how the supports constrain the shaft when it is twisted.
- Express the angles of twist in the compatibility condition in terms of the torques, using a torque-displacement relation, such as $\phi = TL/JG$.
- Solve the equilibrium and compatibility equations for the unknown reactive torques. If any of the magnitudes have a negative numerical value, it indicates that this torque acts in the opposite sense of direction to that shown on the free-body diagram.

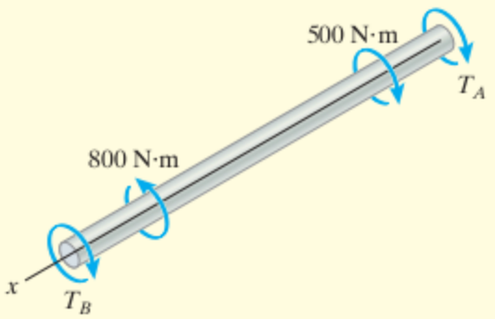
The shaft of this cutting machine is fixed at its ends and subjected to a torque at its center, allowing it to act as a torsional spring.

EXAMPLE 10.7

The solid steel shaft shown in Fig. 10–22*a* has a diameter of 20 mm. If it is subjected to the two torques, determine the reactions at the fixed supports *A* and *B*.



(a)



(b)

SOLUTION

Equilibrium. By inspection of the free-body diagram, Fig. 10–22*b*, it is seen that the problem is statically indeterminate since there is only *one* available equation of equilibrium and there are two unknowns. We require

$$\Sigma M_x = 0; \qquad -T_B + 800\ \text{N}\cdot\text{m} - 500\ \text{N}\cdot\text{m} - T_A = 0 \qquad (1)$$

Compatibility. Since the ends of the shaft are fixed, the angle of twist of one end of the shaft with respect to the other must be zero. Hence, the compatibility equation becomes

$$\phi_{A/B} = 0$$

This condition can be expressed in terms of the unknown torques by using the load–displacement relationship, $\phi = TL/JG$. Here there are three regions of the shaft where the internal torque is constant. On the free-body diagrams in Fig. 10–22*c* we have shown the internal torques acting on the left segments of the shaft which are sectioned in each of these regions. This way the internal torque is only a function of T_B. Using the sign convention established in Sec. 10.4, we have

$$\frac{-T_B\,(0.2\ \text{m})}{JG} + \frac{(800 - T_B)(1.5\ \text{m})}{JG} + \frac{(300 - T_B)(0.3\ \text{m})}{JG} = 0$$

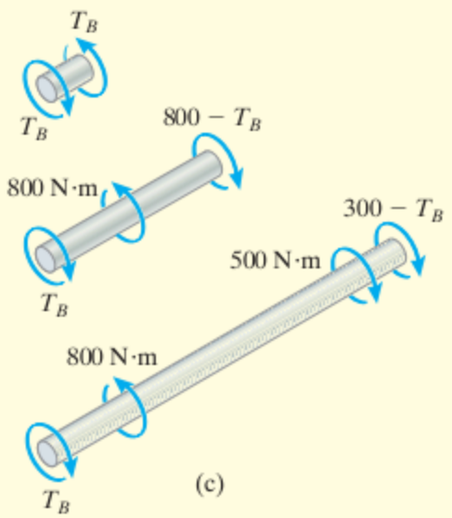
so that

$$T_B = 645\ \text{N}\cdot\text{m} \qquad \textit{Ans.}$$

Using Eq. (1),

$$T_A = -345\ \text{N}\cdot\text{m} \qquad \textit{Ans.}$$

The negative sign indicates that $\mathbf{T}_A$ acts in the opposite direction of that shown in Fig. 10–22*b*.

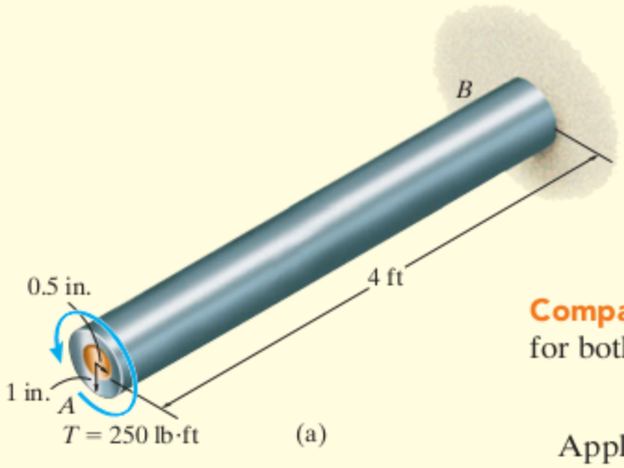


(c)

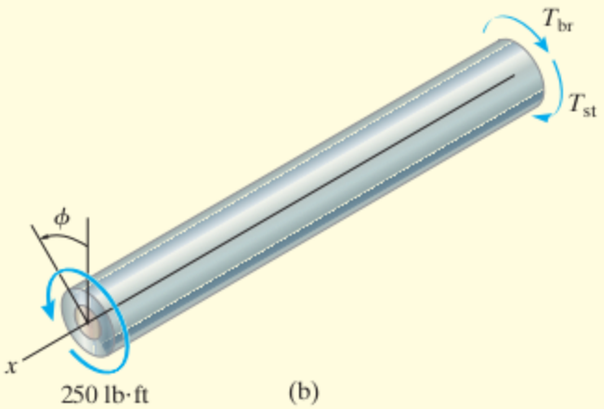
Fig. 10–22

EXAMPLE 10.8

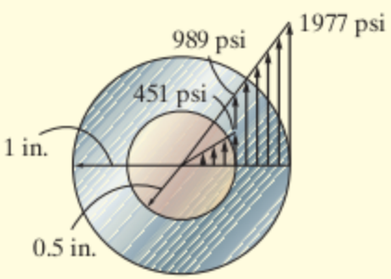
The shaft shown in Fig. 10–23*a* is made from a steel tube, which is bonded to a brass core. If a torque of $T = 250\ \text{lb}\cdot\text{ft}$ is applied at its end, plot the shear-stress distribution along a radial line of its cross-sectional area. Take $G_{st} = 11.4(10^3)$ ksi, $G_{br} = 5.20(10^3)$ ksi.



(a)



(b)



Shear–stress distribution

(c)

Fig. 10–23

SOLUTION

Equilibrium. A free-body diagram of the shaft is shown in Fig. 10–23*b*. The reaction at the wall has been represented by the unknown amount of torque resisted by the steel, T_{st}, and by the brass, T_{br}. Working in units of pounds and inches, equilibrium requires

$$-T_{st} - T_{br} + (250\ \text{lb}\cdot\text{ft})(12\ \text{in./ft}) = 0 \qquad (1)$$

Compatibility. We require the angle of twist of end A to be the same for both the steel and brass since they are bonded together. Thus,

$$\phi = \phi_{st} = \phi_{br}$$

Applying the load–displacement relationship, $\phi = TL/JG$,

$$\frac{T_{st}L}{(\pi/2)[(1\ \text{in.})^4 - (0.5\ \text{in.})^4]11.4(10^3)\ \text{kip/in}^2} = \frac{T_{br}L}{(\pi/2)(0.5\ \text{in.})^4\,5.20(10^3)\ \text{kip/in}^2}$$

$$T_{st} = 32.88\,T_{br} \qquad (2)$$

Solving Eqs. (1) and (2), we get

$$T_{st} = 2911.5\ \text{lb}\cdot\text{in.} = 242.6\ \text{lb}\cdot\text{ft}$$
$$T_{br} = 88.5\ \text{lb}\cdot\text{in.} = 7.38\ \text{lb}\cdot\text{ft}$$

The shear stress in the brass core varies from zero at its center to a maximum at the interface where it contacts the steel tube. Using the torsion formula,

$$(\tau_{br})_{max} = \frac{(88.5\ \text{lb}\cdot\text{in.})(0.5\ \text{in.})}{(\pi/2)(0.5\ \text{in.})^4} = 451\ \text{psi}$$

For the steel, the minimum and maximum shear stresses are

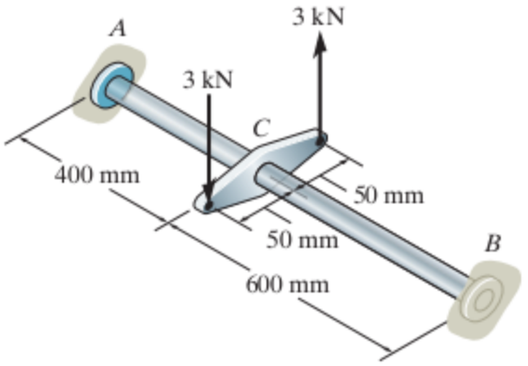
$$(\tau_{st})_{min} = \frac{(2911.5\ \text{lb}\cdot\text{in.})(0.5\ \text{in.})}{(\pi/2)[(1\ \text{in.})^4 - (0.5\ \text{in.})^4]} = 989\ \text{psi}$$

$$(\tau_{st})_{max} = \frac{(2911.5\ \text{lb}\cdot\text{in.})(1\ \text{in.})}{(\pi/2)[(1\ \text{in.})^4 - (0.5\ \text{in.})^4]} = 1977\ \text{psi}$$

The results are plotted in Fig. 10–23*c*. Note the discontinuity of *shear stress* at the brass and steel interface. This is to be expected, since the materials have different moduli of rigidity; i.e., steel is stiffer than brass ($G_{st} > G_{br}$) and thus it carries more shear stress at the interface. Although the shear stress is discontinuous here, the *shear strain* is not. Rather, the shear strain is the *same* for both the brass and the steel.

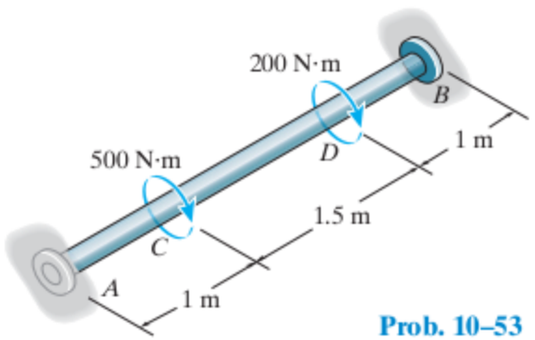
PROBLEMS

*10–52. The steel shaft has a diameter of 40 mm and is fixed at its ends A and B. If it is subjected to the couple determine the maximum shear stress in regions AC and CB of the shaft. $G_{st} = 75$ GPa.



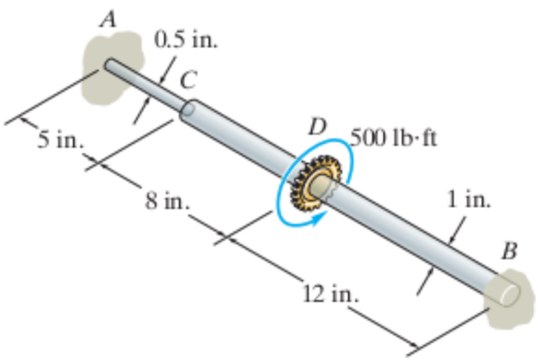
Prob. 10–52

10–53. The A992 steel shaft has a diameter of 60 mm and is fixed at its ends A and B. If it is subjected to the torques shown, determine the absolute maximum shear stress in the shaft.



Prob. 10–53

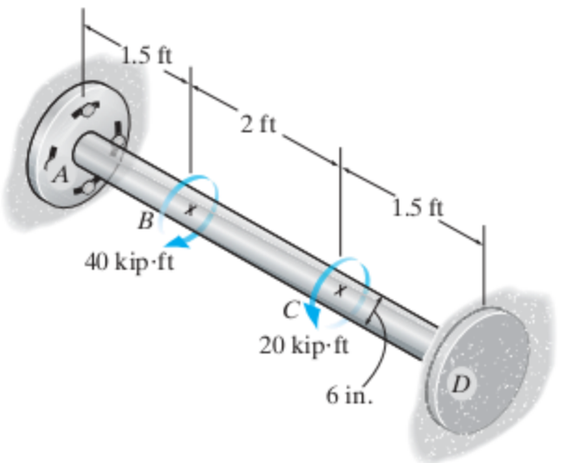
10–54. The steel shaft is made from two segments: AC has a diameter of 0.5 in, and CB has a diameter of 1 in. If the shaft is fixed at its ends A and B and subjected to a torque of 500 lb · ft, determine the maximum shear stress in the shaft. $G_{st} = 10.8(10^3)$ ksi.



Prob. 10–54

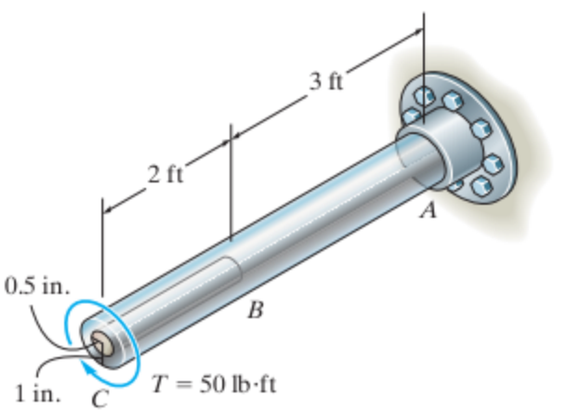
10–55. The shaft is made of A-36 steel and is fixed at its ends A and D. Determine the torsional reactions at these supports.

*10–56. The shaft is made of A-36 steel and is fixed at end D, while end A is allowed to rotate 0.005 rad when the torque is applied. Determine the torsional reactions at these supports.



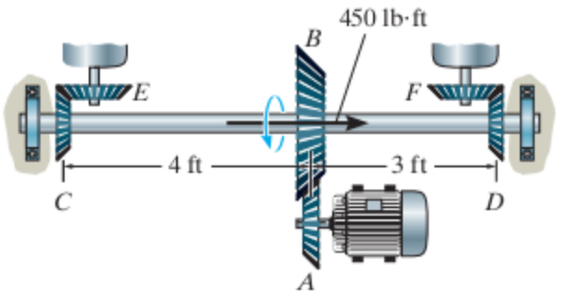
Probs. 10–55/56

10–57. The shaft is made from a solid steel section AB and a tubular portion made of steel and having a brass core. If it is fixed to a rigid support at A, and a torque of $T = 50$ lb · ft is applied to it at C, determine the angle of twist that occurs at C and compute the maximum shear stress and maximum shear strain in the brass and steel. Take $G_{st} = 11.5(10^3)$ ksi, $G_{br} = 5.6(10^3)$ ksi.



Prob. 10–57

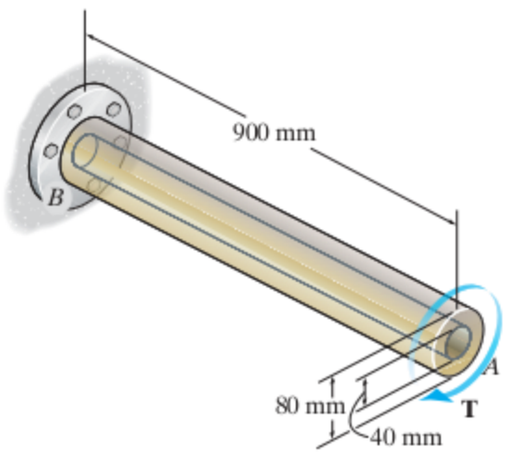
10–58. The motor A develops a torque at gear B of 450 lb · ft, which is applied along the axis of the 2-in.-diameter steel shaft CD. This torque is to be transmitted to the pinion gears at E and F. If these gears are temporarily fixed, determine the maximum shear stress in segments CB and BD of the shaft. Also, what is the angle of twist of each of these segments? The bearings at C and D only exert force reactions on the shaft and do not resist torque. $G_{st} = 12(10^3)$ ksi.



Prob. 10–58

10–59. The Am1004-T61 magnesium tube is bonded to the A-36 steel rod. If the allowable shear stress for the magnesium and steel are $(\tau_{allow})_{mg} = 45$ MPa and $(\tau_{allow})_{st} = 75$ MPa, respectively, determine the maximum allowable torque that can be applied at A. Also, find the corresponding angle of twist of end A.

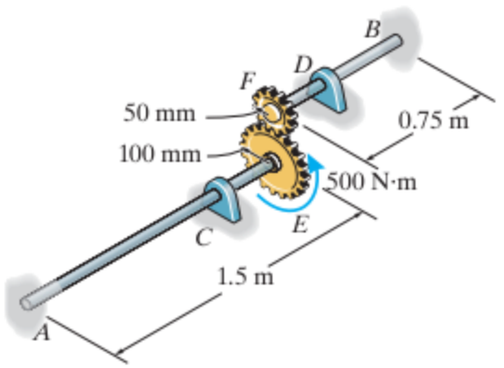
***10–60.** The Am1004-T61 magnesium tube is bonded to the A-36 steel rod. If a torque of $T = 5$ kN · m is applied to end A, determine the maximum shear stress in each material. Sketch the shear stress distribution.



Probs. 10–59/60

10–61. The two shafts are made of A-36 steel. Each has a diameter of 25 mm and they are connected using the gears fixed to their ends. Their other ends are attached to fixed supports at A and B. They are also supported by journal bearings at C and D, which allow free rotation of the shafts along their axes. If a torque of 500 N · m is applied to the gear at E as shown, determine the reactions at A and B.

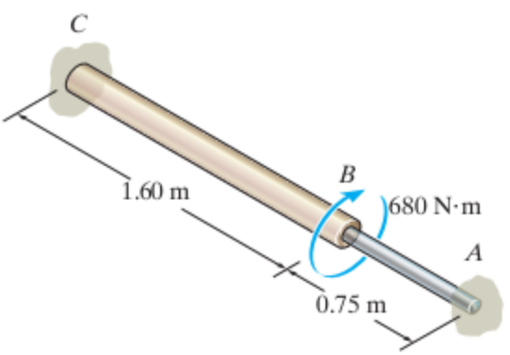
10–62. Determine the rotation of the gear at E in Prob. 10–61.



Probs. 10–61/62

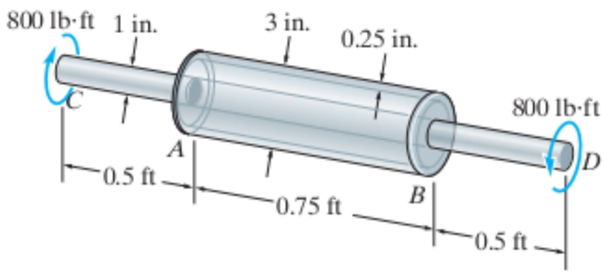
10–63. A rod is made from two segments: AB is steel and BC is brass. It is fixed at its ends and subjected to a torque of $T = 680$ N · m. If the steel portion has a diameter of 30 mm, determine the required diameter of the brass portion so the reactions at the walls will be the same. $G_{st} = 75$ GPa, $G_{br} = 39$ GPa.

***10–64.** Determine the absolute maximum shear stress in the shaft of Prob. 10–59.



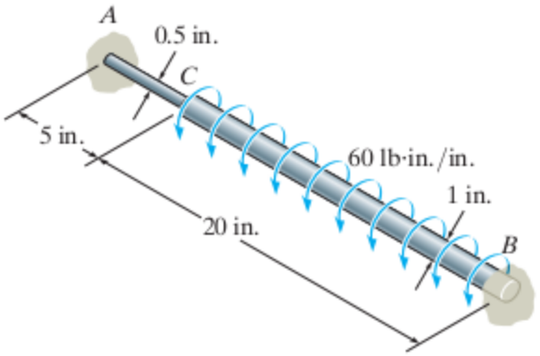
Probs. 10–63/64

10–65. The composite shaft consists of a mid-section that includes the 1-in.-diameter solid shaft and a tube that is welded to the rigid flanges at A and B. Neglect the thickness of the flanges and determine the angle of twist of end C of the shaft relative to end D. The shaft is subjected to a torque of 800 lb · ft. The material is A-36 steel.



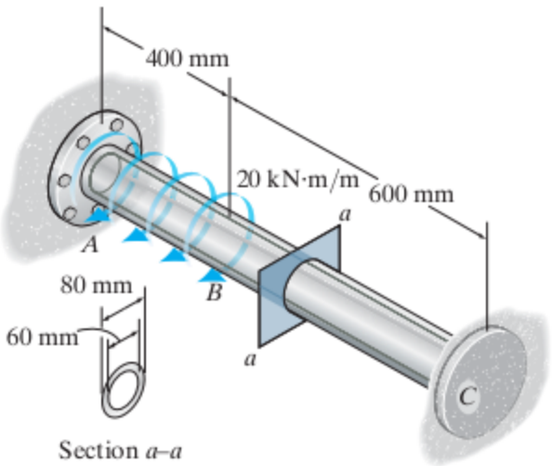
Prob. 10–65

10–66. The A992 steel shaft is made from two segments. AC has a diameter of 0.5 in. and CB has a diameter of 1 in. If the shaft is fixed at its ends A and B and subjected to a uniform distributed torque of 60 lb · in./in. along segment CB, determine the absolute maximum shear stress in the shaft.



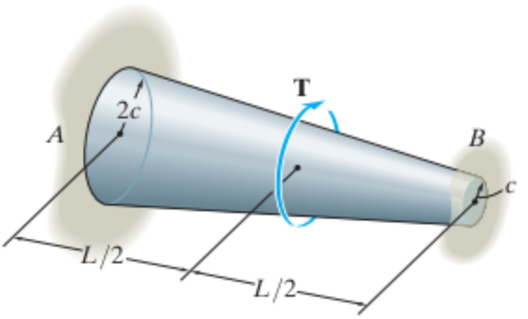
Prob. 10–66

10–67. If the shaft is subjected to a uniform distributed torque of $t = 20$ kN · m/m, determine the maximum shear stress developed in the shaft. The shaft is made of 2014-T6 aluminum alloy and is fixed at A and C.



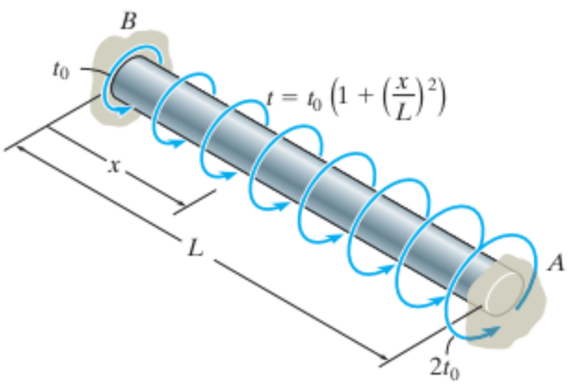
Prob. 10–67

*10–68. The tapered shaft is confined by the fixed supports at A and B. If a torque $\mathbf{T}$ is applied at its mid-point, determine the reactions at the supports.



Prob. 10–68

10–69. The shaft of radius C is subjected to a distributed torque t, measured as torque/length of shaft. Determine the reactions at the fixed supports A and B.

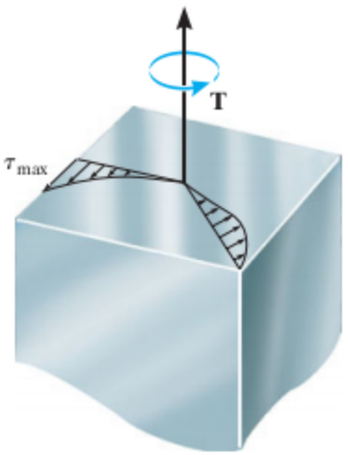


Prob. 10–69

*10.6 Solid Noncircular Shafts

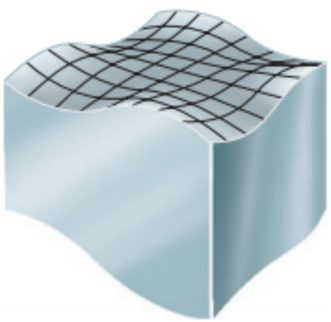
It was demonstrated in Sec. 10.1 that when a torque is applied to a shaft having a circular cross section—that is, one that is axisymmetric—the shear strains vary linearly from zero at its center to a maximum at its outer surface. Furthermore, due to the uniformity of the shear strain at all points on the same radius, the cross sections do not deform, but rather remain plane after the shaft has twisted. Shafts that have a noncircular cross section, however, are *not* axisymmetric, and so their cross sections will ***bulge*** or ***warp*** when the shaft is twisted. Evidence of this can be seen from the way grid lines deform on a shaft having a square cross section when the shaft is twisted, Fig. 10–24. As a consequence of this deformation the torsional analysis of *noncircular* shafts becomes considerably more complicated and will not be considered in this text.

Using a mathematical analysis based on the theory of elasticity, however, it is possible to determine the shear-stress distribution within a shaft of square cross section. Examples of how this shear stress varies along two radial lines of the shaft are shown in Fig. 10–25*a*. Because these shear-stress distributions vary in a complex manner, the shear strains they create will *warp* the cross section as shown in Fig. 10–25*b*. In particular notice that the corner points of the shaft must be subjected to zero shear stress and therefore zero shear strain. The reason for this can be shown by considering an element of material located at one of these points, Fig. 10–25*c*. One would expect the top face of this element to be subjected to a shear stress in order to aid in resisting the applied torque **T**. This, however, cannot occur since the complementary shear stresses τ and τ', acting on the *outer surface* of the shaft, must be zero.



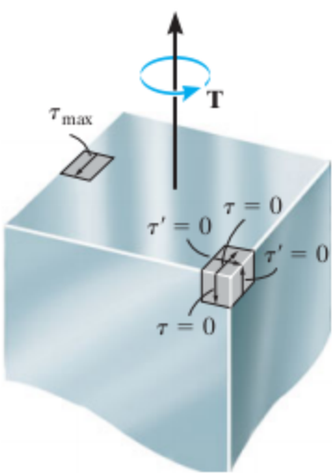
Shear stress distribution along two radial lines.

(a)



Warping of cross-sectional area.

(b)



(c)

Fig. 10–25

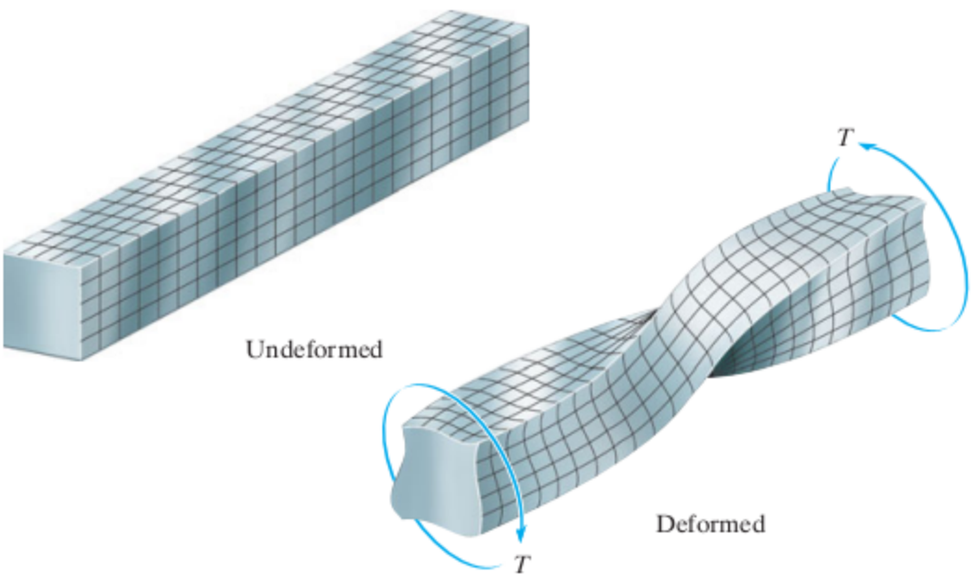
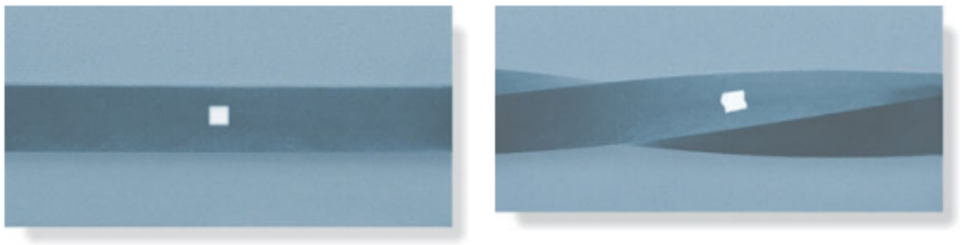
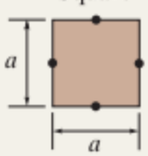
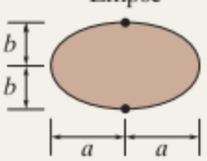


Fig. 10–24



Notice the deformation of the square element when this rubber bar is subjected to a torque.

The results of the analysis for square cross sections, along with other results from the theory of elasticity, for shafts having triangular and elliptical cross sections, are reported in Table 10–1. In all cases the *maximum shear stress* occurs at a point on the edge of the cross section that is *closest to* the center axis of the shaft. In Table Table 10–1 these points are indicated as "dots" on the cross sections. Also given are formulas for the angle of twist of each shaft. By extending these results to a shaft having an *arbitrary* cross section, it can also be shown that the *most efficient shaft* has a *circular* cross section, since it is subjected to both a *smaller* maximum shear stress and a *smaller* angle of twist than one having the same cross-sectional area, but not circular, and subjected to the same torque.

Shape of cross section	τ_{max}	ϕ
Square 	$\frac{4.81\,T}{a^3}$	$\frac{7.10\,TL}{a^4G}$
Equilateral triangle	$\frac{20\,T}{a^3}$	$\frac{46\,TL}{a^4G}$
Ellipse 	$\frac{2\,T}{\pi ab^2}$	$\frac{(a^2+b^2)TL}{\pi a^3b^3G}$

The drill shaft is connected to the soil auger using a shaft having a square cross section.

EXAMPLE 10.9

The 6061-T6 aluminum shaft shown in Fig. 10–26 has a cross-sectional area in the shape of an equilateral triangle. Determine the largest torque **T** that can be applied to the end of the shaft if the allowable shear stress is $\tau_{allow} = 8$ ksi and the angle of twist at its end is restricted to $\phi_{allow} = 0.02$ rad. How much torque can be applied to a shaft of circular cross section made from the same amount of material?

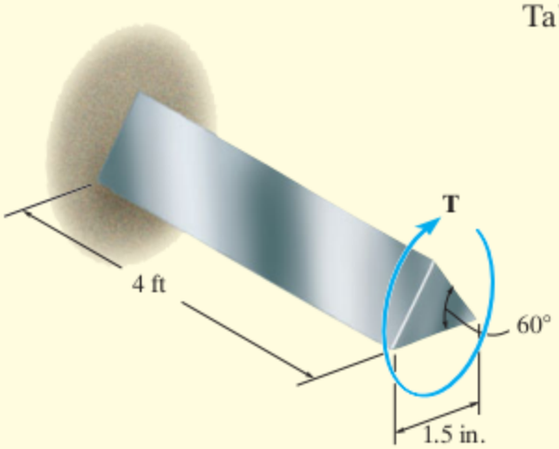


Fig. 10–26

SOLUTION

By inspection, the resultant internal torque at any cross section along the shaft's axis is also **T**. Using the formulas for τ_{max} and ϕ in Table 10–1, we require

$$\tau_{allow} = \frac{20T}{a^3}; \qquad 8(10^3)\,\text{lb/in}^2 = \frac{20T}{(1.5\,\text{in.})^3}$$

$$T = 1350\,\text{lb}\cdot\text{in.}$$

Also,

$$\phi_{allow} = \frac{46TL}{a^4 G_{al}}; \qquad 0.02\,\text{rad} = \frac{46T(4\,\text{ft})(12\,\text{in./ft})}{(1.5\,\text{in.})^4[3.7(10^6)\,\text{lb/in}^2]}$$

$$T = 170\,\text{lb}\cdot\text{in.} \qquad \textit{Ans.}$$

By comparison, the torque is limited due to the angle of twist.

Circular Cross Section. If the same amount of aluminum is to be used in making the same length of shaft having a circular cross section, then the radius of the cross section can be calculated. We have

$$A_{circle} = A_{triangle}; \qquad \pi c^2 = \frac{1}{2}(1.5\,\text{in.})(1.5\sin 60°)$$

$$c = 0.557\,\text{in.}$$

The limitations of stress and angle of twist then require

$$\tau_{allow} = \frac{Tc}{J}; \qquad 8\,(10^3)\,\text{lb/in}^2 = \frac{T(0.557\,\text{in.})}{(\pi/2)\,(0.557\,\text{in.})^4}$$

$$T = 2170\,\text{lb}\cdot\text{in.}$$

$$\phi_{allow} = \frac{TL}{JG_{al}}; \qquad 0.02\,\text{rad} = \frac{T(4\,\text{ft})(12\,\text{in./ft})}{(\pi/2)(0.557\,\text{in.})^4[3.7(10^6)\,\text{lb/in}^2]}$$

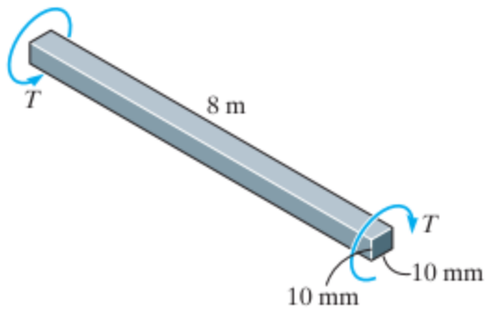
$$T = 233\,\text{lb}\cdot\text{in.} \qquad \textit{Ans.}$$

Again, the angle of twist limits the applied torque.

NOTE: Comparing this result (233 lb·in.) with that given above (170 lb·in.), it is seen that a shaft of circular cross section can support 37% more torque than the one having a triangular cross section.

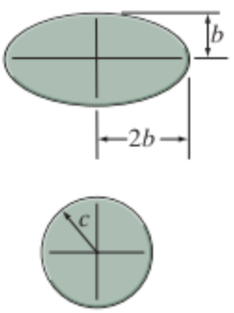
PROBLEMS

10–70. The aluminum rod has a square cross section of 10 mm by 10 mm. If it is 8 m long, determine the torque T that is required to rotate one end relative to the other end by 90°. $G_{al} = 28$ GPa, $(\tau_Y)_{al} = 240$ MPa.



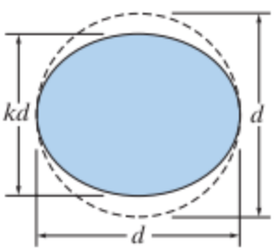
Prob. 10–70

10–71. The shafts have elliptical and circular cross sections and are to be made from the same amount of a similar material. Determine the percent of increase in the maximum shear stress and the angle of twist for the elliptical shaft compared to the circular shaft when both shafts are subjected to the same torque and have the same length.



Prob. 10–71

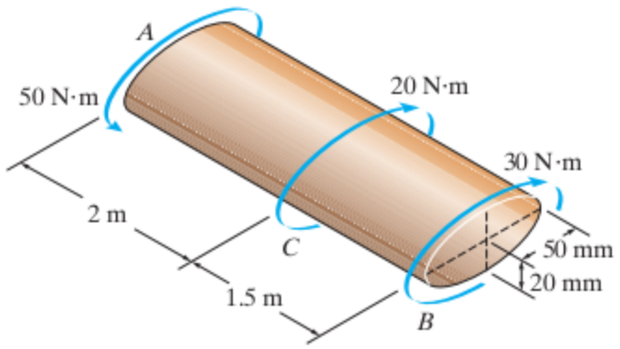
*10–72. It is intended to manufacture a circular bar to resist torque; however, the bar is made elliptical in the process of manufacturing, with one dimension smaller than the other by a factor k as shown. Determine the factor by which the maximum shear stress is increased.



Prob. 10–72

10–73. The shaft is made of red brass C83400 and has an elliptical cross section. If it is subjected to the torsional loading shown, determine the maximum shear stress within regions AC and BC, and the angle of twist ϕ of end B relative to end A.

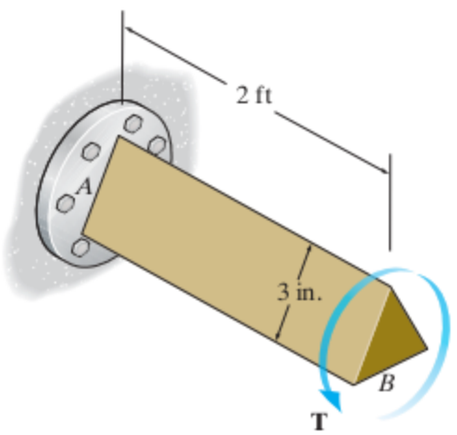
10–74. Solve Prob. 10–73 for the maximum shear stress within regions AC and BC, and the angle of twist ϕ of end B relative to C.



Probs. 10–73/74

10–75. If end B of the shaft, which has an equilateral triangle cross section, is subjected to a torque of $T = 900$ lb·ft, determine the maximum shear stress developed in the shaft. Also, find the angle of twist of end B. The shaft is made from 6061-T1 aluminum.

*10–76. If the shaft has an equilateral triangle cross section and is made from an alloy that has an allowable shear stress of $\tau_{allow} = 12$ ksi, determine the maximum allowable torque **T** that can be applied to end B. Also, find the corresponding angle of twist of end B.



Probs. 10–75/76

10.7 Stress Concentration

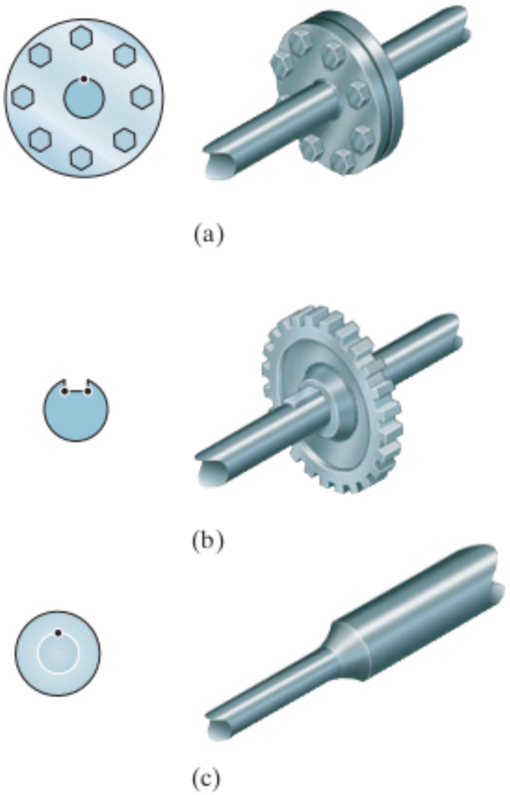


Fig. 10–27

The torsion formula, $\tau_{max} = Tc/J$, cannot be applied to regions of a shaft having a sudden change in the cross section. Here the shear-stress and shear-strain distributions in the shaft become complex and can be obtained only by using experimental methods or possibly by a mathematical analysis based on the theory of elasticity. Three common discontinuities of the cross section that occur in practice are shown in Fig. 10–27. They are at *couplings*, which are used to connect two collinear shafts together, Fig. 10–27*a*, at *keyways*, used to connect gears or pulleys to a shaft, Fig. 10–27*b*, and in *step shafts*, which are fabricated or machined from a single shaft, Fig. 10–27*c*. In each case the maximum shear stress will occur at the point (dot) indicated on the cross section.

The necessity to perform a complex stress analysis at a shaft discontinuity to obtain the maximum shear stress can be eliminated by using a ***torsional stress-concentration factor***, K. As in the case of axially loaded members, Sec. 9.7, K is usually taken from a graph based on experimental data. An example, for the shoulder-fillet shaft, is shown in Fig. 10–28. To use this graph, one first finds the geometric ratio D/d to define the appropriate curve, and then once the abscissa r/d is calculated, the value of K is found along the ordinate.

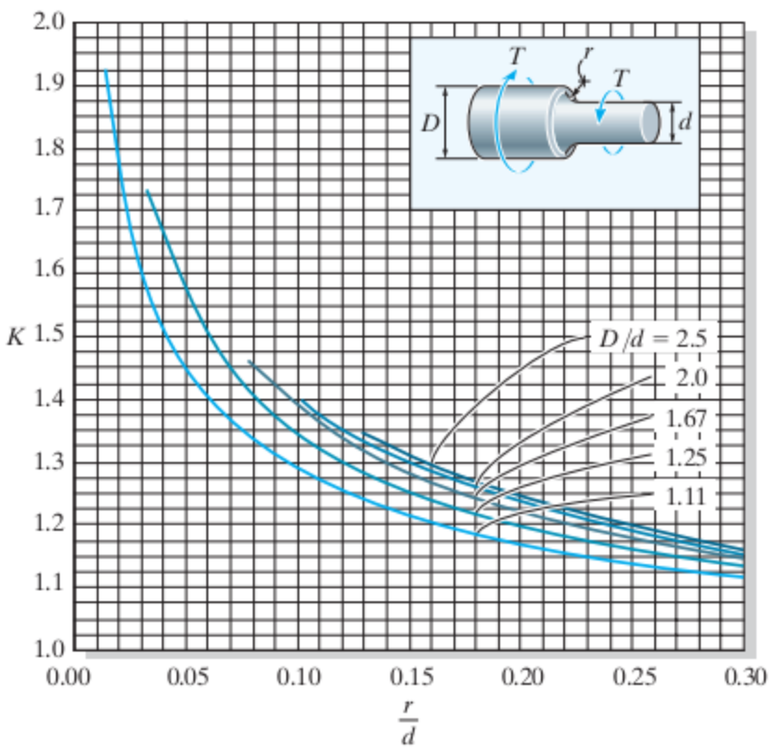


Fig. 10–28

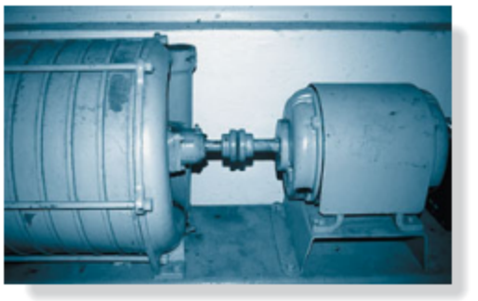
The maximum shear stress is then determined from

$$\tau_{\max} = K\frac{Tc}{J} \tag{10–17}$$

Here the torsion formula is applied to the *smaller* of the two connected shafts, since $\tau_{\max}$ occurs at the base of the fillet, Fig. 10–27*c*.

Note from the graph that an *increase* in radius r causes a *decrease* in K. Hence the maximum shear stress in the shaft can be *reduced* by *increasing* the radius. Also, if the diameter of the larger section is reduced, the Dy$/d$ ratio will be lower and so the value of K and therefore $\tau_{\max}$ will be lower.

Like the case of axially loaded members, torsional stress concentration factors should *always* be used when designing shafts made from *brittle materials*, or when designing shafts that will be subjected to *fatigue* or *cyclic torsional loadings*. These conditions give rise to the formation of cracks at the stress concentration, and this can often lead to a sudden fracture. On the other hand, if large *static* torsional loadings are applied to a shaft made from *ductile material*, then *inelastic strains* will develop within the shaft. Yielding of the material will cause the stress distribution to become more *evenly distributed* throughout the shaft, so that the maximum stress will not be limited to the region of stress concentration. This phenomenon will be discussed further in the next section.



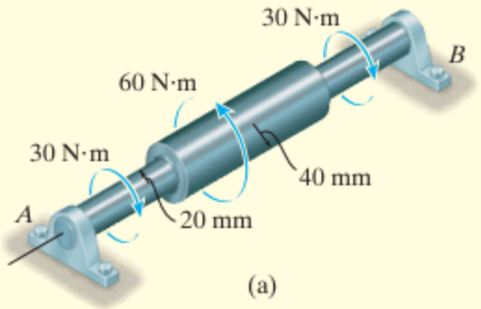
Stress concentrations can arise at the coupling of these shaft and this must be taken into account when the shaft is designed.

Important Points

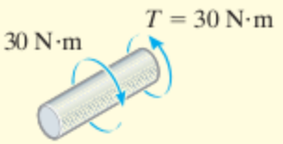
- *Stress concentrations* in shafts occur at points of sudden cross-sectional change, such as couplings, keyways, and in step shafts. The more severe the change in geometry, the larger the stress concentration.
- For design or analysis, it is not necessary to know the exact shear-stress distribution on the cross section. Instead, it is possible to obtain the maximum shear stress using a stress concentration factor, K, that has been determined through experiment, and is only a function of the geometry of the shaft.
- Normally a stress concentration in a ductile shaft subjected to a static torque will *not* have to be considered in design; however, if the material is *brittle*, or subjected to *fatigue* loadings, then stress concentrations become important.

EXAMPLE 10.10

The stepped shaft shown in Fig. 10–29*a* is supported by bearings at *A* and *B*. Determine the maximum stress in the shaft due to the applied torques. The shoulder fillet at the junction of each shaft has a radius of $r = 6$ mm.



(a)



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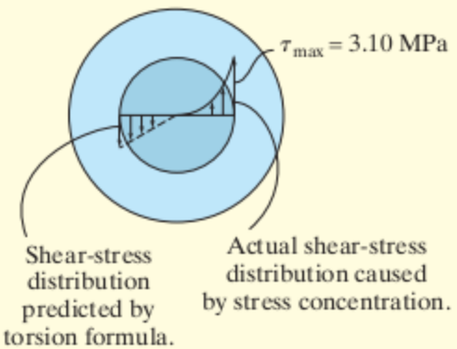


(c)

Fig. 10–29

SOLUTION

Internal Torque. By inspection, moment equilibrium about the axis of the shaft is satisfied. Since the maximum shear stress occurs at the rooted ends of the *smaller* diameter shafts, the internal torque (30 N · m) can be found there by applying the method of sections, Fig. 10–29*b*.

Maximum Shear Stress. The stress-concentration factor can be determined by using Fig. 10–28. From the shaft geometry we have

$$\frac{D}{d} = \frac{2(40\text{ mm})}{2(20\text{ mm})} = 2$$

$$\frac{r}{d} = \frac{6\text{ mm}}{2(20\text{ mm})} = 0.15$$

Thus, the value of $K = 1.3$ is obtained.

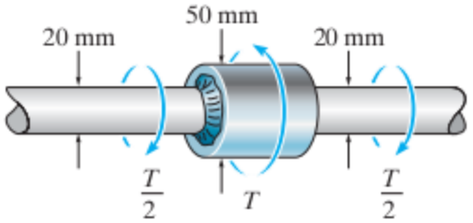
Applying Eq. 10–17, we have

$$\tau_{\max} = K\frac{Tc}{J};\quad \tau_{\max} = 1.3\left[\frac{30\text{ N}\cdot\text{m}\,(0.020\text{ m})}{(\pi/2)(0.020\text{ m})^4}\right] = 3.10\text{ MPa} \qquad \textit{Ans.}$$

NOTE: From experimental evidence, the actual stress distribution along a radial line of the cross section at the critical section looks similar to that shown in Fig. 10–29*c*. Notice how this compares with the linear stress distribution found from the torsion formula.

PROBLEMS

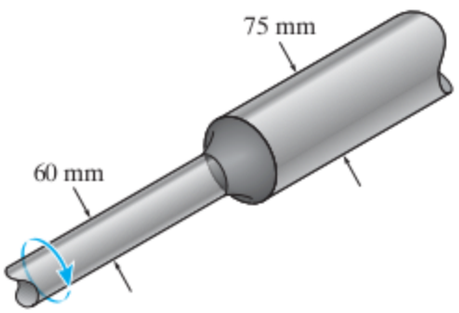
10–77. The steel step shaft has an allowable shear stress of $\tau_{allow} = 8$ MPa. If the transition between the cross sections has a radius $r = 4$ mm, determine the maximum torque T that can be applied.



Prob. 10–77

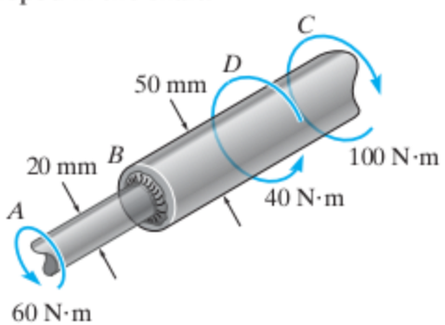
10–78. The step shaft is to be designed to rotate at 720 rpm while transmitting 30 kW of power. Is this possible? The allowable shear stress is $\tau_{allow} = 12$ MPa and the radius at the transition on the shaft is 7.5 mm.

10–79. The built-up shaft is designed to rotate at 540 rpm. If the radius at the transition on the shaft is $r = 7.2$ mm, and the allowable shear stress for the material is $\tau_{allow} = 55$ MPa, determine the maximum power the shaft can transmit.



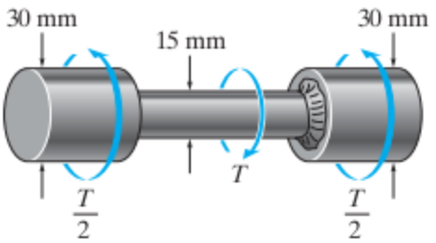
Probs. 10–78/79

*10–80. The transition at the cross sections of the step shaft has a radius of 2.8 mm. Determine the maximum shear stress developed in the shaft.



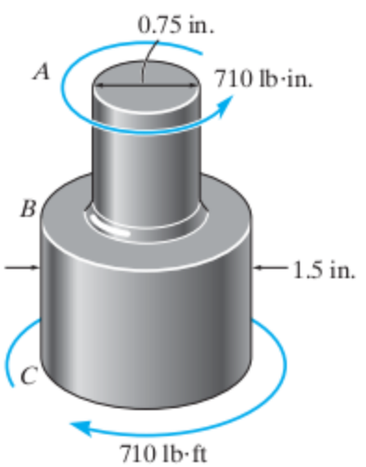
Prob. 10–80

10–81. The steel used for the step shaft has an allowable shear stress of $\tau_{allow} = 8$ MPa. If the radius at the transition between the cross sections is $r = 2.25$ mm, determine the maximum torque T that can be applied.



Prob. 10–81

10–82. The step shaft is subjected to a torque of 710 lb · in. If the allowable shear stress for the material is $\tau_{allow} = 12$ ksi, determine the smallest radius at the junction between the cross sections that can be used to transmit the torque.



Prob. 10–82

CHAPTER REVIEW

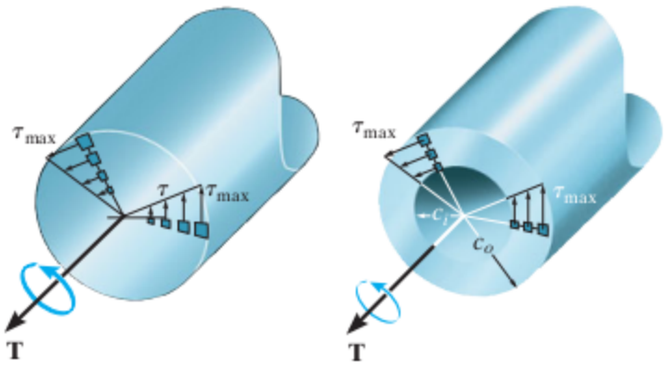
Torque causes a shaft having a circular cross section to twist, such that the shear strain in the shaft is proportional to its radial distance from the center of the shaft. Provided the material is homogeneous and linear elastic, then the shear stress is determined from the torsion formula,

$$\tau = \frac{T\rho}{J}$$

The design of a shaft requires finding the geometric parameter,

$$\frac{J}{c} = \frac{T}{\tau_{\text{allow}}}$$

Often the power P supplied to a shaft rotating at ω is reported, in which case the torque is determined from $P = T\omega$.



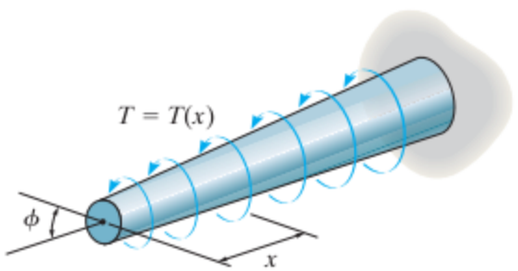
The angle of twist of a circular shaft is determined from

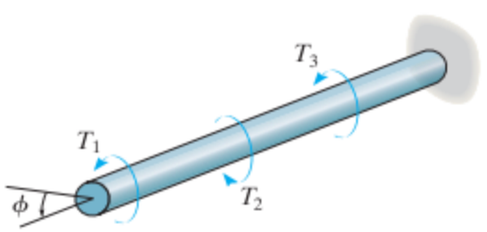
$$\phi = \int_0^L \frac{T(x)\,dx}{JG}$$

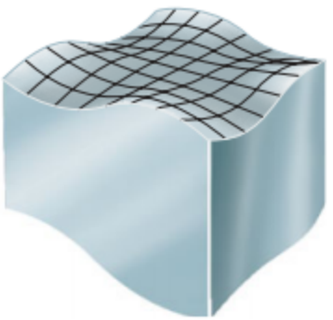
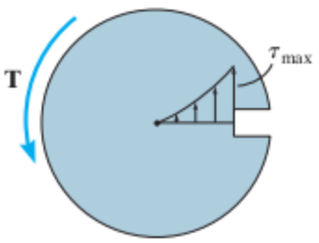
If the internal torque and JG are constant within each segment of the shaft then

$$\phi = \sum \frac{TL}{JG}$$

For application, it is necessary to use a sign convention for the internal torque and to be sure the material remains linear elastic.



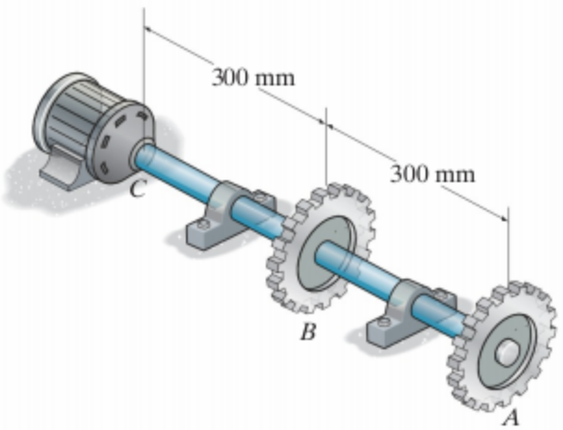


If the shaft is statically indeterminate, then the reactive torques are determined from equilibrium, compatibility of twist, and a torque-twist relationship, such as $\phi = TL/JG$.	
Solid noncircular shafts tend to warp out of plane when subjected to a torque. Formulas are available to determine the maximum elastic shear stress and the twist for these cases.	
Stress concentrations occur in shafts when the cross section suddenly changes. The maximum shear stress is determined using a stress concentration factor K, which is determined from experiment and represented in graphical form. Once obtained, $$\tau_{\max} = K\left(\frac{Tc}{J}\right)$$	

REVIEW PROBLEMS

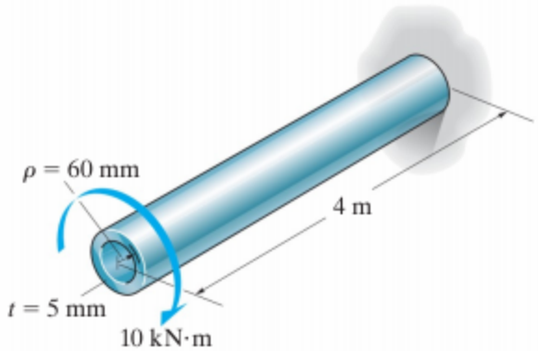
10–83. The shaft is made of A992 steel and has an allowable shear stress of $\tau_{allow} = 75$ MPa. When the shaft is rotating at 300 rpm, the motor supplies 8 kW of power, while gears A and B withdraw 5 kW and 3 kW, respectively. Determine the required minimum diameter of the shaft to the nearest millimeter. Also, find the rotation of gear A relative to C.

***10–84.** The shaft is made of A992 steel and has an allowable shear stress of $\tau_{allow} = 75$ MPa. when the shaft is rotating at 300 rpm, the motor supplies 8 kW of power, while gears A and B withdraw 5 kW and 3 kW, respectively. If the angle of twist of gear A relative to C is not allowed to exceed 0.03 rad, determine the required minimum diameter of the shaft to the nearest millimeter.



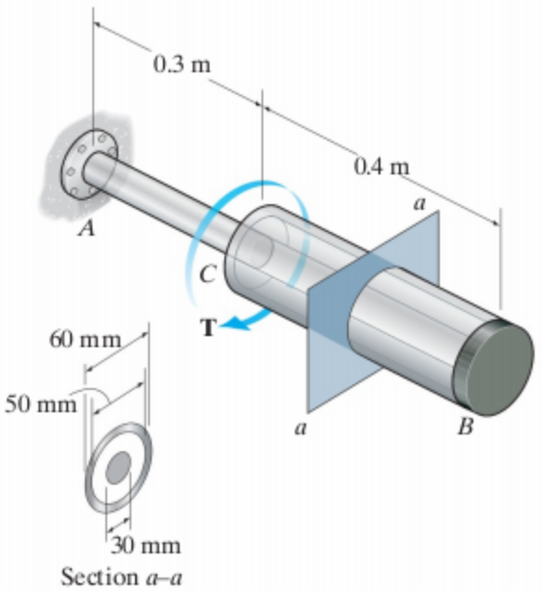
Probs. 10–83/84

10–85. The A-36 steel circular tube is subjected to a torque of 10 kN · m. Determine the shear stress at the mean radius $\rho = 60$ mm and compute the angle of twist of the tube if it is 4 m long and fixed at its far end. Solve the problem using Eqs. 10–7 and 10–15.



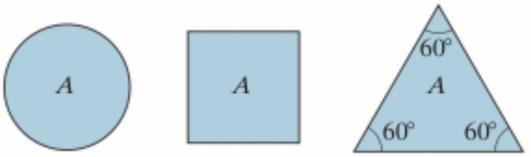
Prob. 10–85

10–86. Rod AB is made of A-36 steel with an allowable shear stress of $(\tau_{allow})_{st} = 75$ MPa, and tube BC is made of AM1004-T61 magnesium alloy with an allowable shear stress of $(\tau_{allow})_{mg} = 45$ MPa. The angle of twist of end C is not allowed to exceed 0.05 rad. Determine the maximum allowable torque **T** that can be applied to the assembly.



Prob. 10–86

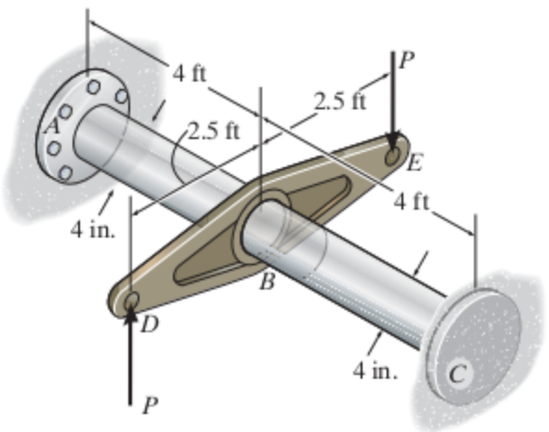
10–87. The material of which each of three shafts is made has a yield stress of τ_Y and a shear modulus of G. Determine which shaft geometry will resist the largest torque without yielding. What percentage of this torque can be carried by the other two shafts? Assume that each shaft is made of the same amount of material and that it has the same cross-sectional area A.



Prob. 10–87

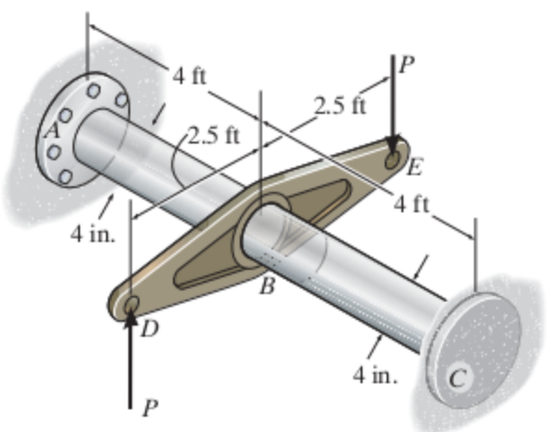
10

*10–88. Segments AB and BC of the assembly are made from 6061-T6 aluminum and A992 steel, respectively. If couple forces $P = 3$ kip are applied to the lever arm, determine the maximum shear stress developed in each segment. The assembly is fixed at A and C.



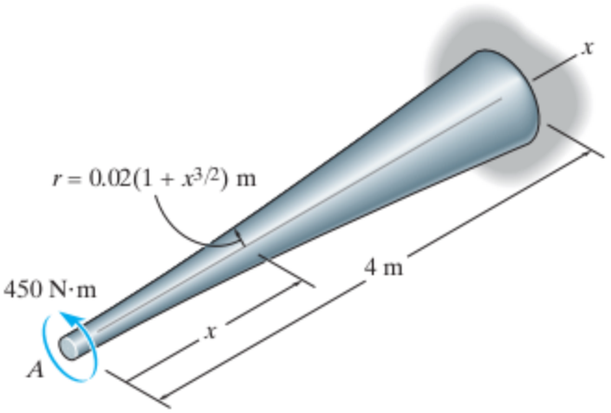
Prob. 10–88

10–89. Segments AB and BC of the assembly are made from 6061-T6 aluminum and A992 steel, respectively. If the allowable shear stress for the aluminum is $(\tau_{allow})_{al} = 12$ ksi and for the steel $(\tau_{allow})_{st} = 10$ ksi, determine the maximum allowable couple forces P that can be applied to the lever arm. The assembly is fixed at A and C.



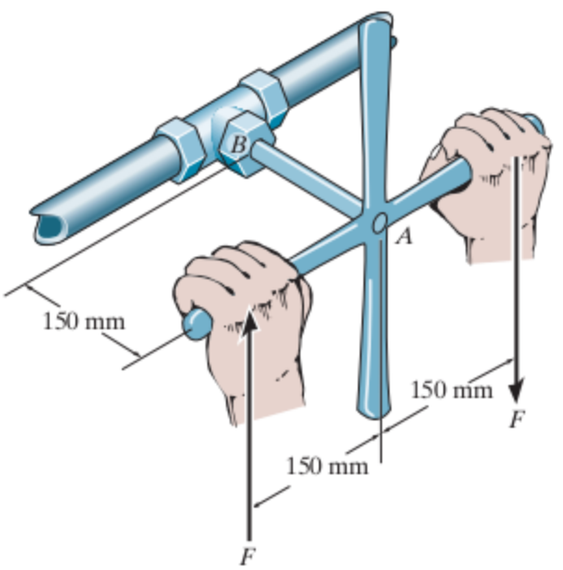
Prob. 10–89

10–90. The tapered shaft is made from 2014-T6 aluminum alloy, and has a radius which can be described by the function $r = 0.02(1 + x^{3/2})$ m, where x is in meters. Determine the angle of twist of its end A if it is subjected to a torque of 450 N · m.



Prob. 10–90

10–91. If the solid shaft AB to which the valve handle is attached is made of C83400 red brass and has a diameter of 10 mm, determine the maximum couple forces F that can be applied to the handle just before the material starts to fail. Take $\tau_{allow} = 40$ MPa. What is the angle of twist of the handle? The shaft is fixed at A.



Prob. 10–91

Chapter 11

Beams are important structural members used in building construction. Their design is often based upon their ability to resist bending stress, which forms the subject matter of this chapter.

Bending

CHAPTER OBJECTIVES

- To generalize the procedure by formulating equations that can be plotted so that they describe the internal shear and moment throughout a member.
- To use the relations between distributed load, shear, and moment to draw shear and moment diagrams.
- To determine the stress in elastic symmetric members subject to bending.
- To develop methods to determine the stress in unsymmetric beams subject to bending.

11.1 Shear and Moment Diagrams

Members that are slender and support loadings that are applied perpendicular to their longitudinal axis are called ***beams***. In general, beams are long, straight bars having a constant cross-sectional area. Often they are classified as to how they are supported. For example, a *simply supported beam* is pinned at one end and roller supported at the other, Fig. 11–1, a *cantilevered beam* is fixed at one end and free at the other, and an *overhanging beam* has one or both of its ends freely extended over the supports. Beams are considered among the most important of all structural elements. They are used to support the floor of a building, the deck of a bridge, or the wing of an aircraft. Also, the axle of an automobile, the boom of a crane, even many of the bones of the body act as beams.

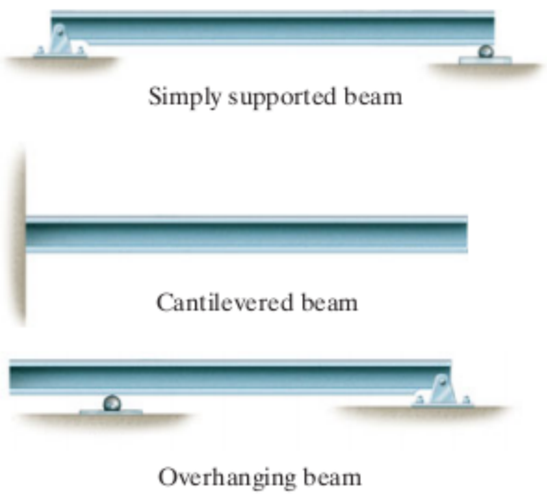


Fig. 11–1

Because of the applied loadings, beams develop an internal shear force and bending moment that, in general, vary from point to point along the axis of the beam. In order to properly design a beam it therefore becomes necessary to determine the *maximum* shear and moment in the beam. One way to do this is to express V and M as functions of their arbitrary position x along the beam's axis. These *shear and moment functions* can then be plotted and represented by graphs called ***shear and moment diagrams***. The maximum values of V and M can then be obtained from these graphs. Also, since the shear and moment diagrams provide detailed information about the *variation* of the shear and moment along the beam's axis, they are often used by engineers to decide where to place reinforcement materials within the beam or how to proportion the size of the beam at various points along its length.

In order to formulate V and M in terms of x we must choose the origin and the positive direction for x. Although the choice is arbitrary, most often the origin is located at the left end of the beam and the positive direction is to the right.

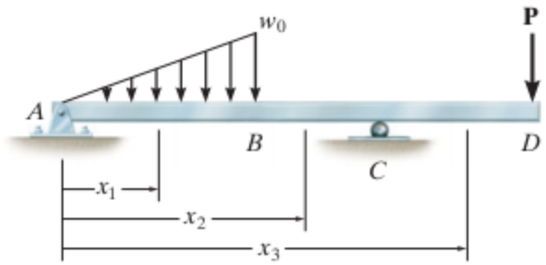


Fig. 11–2

In general, the internal shear and moment functions of x will be *discontinuous*, or their slope will be discontinuous, at points where a distributed load changes or where concentrated forces or couple moments are applied. Because of this, the shear and moment functions must be determined for *each region* of the beam *between* any two discontinuities of loading. For example, coordinates x_1, x_2, and x_3 will have to be used to describe the variation of V and M throughout the length of the beam in Fig. 11–2. These coordinates will be valid *only* within the regions from A to B for x_1, from B to C for x_2, and from C to D for x_3.

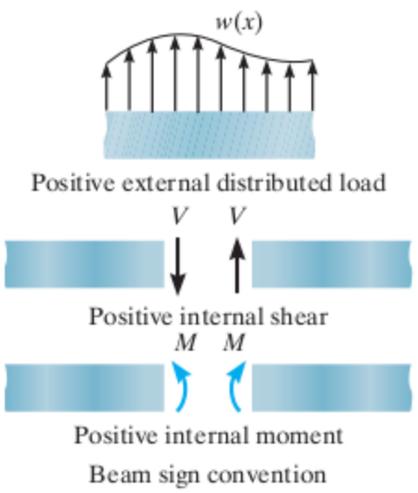


Fig. 11–3

Beam Sign Convention. Before presenting a method for determining the shear and moment as functions of x and later plotting these functions (shear and moment diagrams), it is first necessary to establish a *sign convention* so as to define "positive" and "negative" values for V and M. Although the choice of a sign convention is arbitrary, here we will use the one often used in engineering practice and shown in Fig. 11–3. The *positive directions* are as follows: the *distributed load* acts *upward* on the beam; the internal *shear force* causes a *clockwise* rotation of the beam segment on which it acts; and the internal *moment* causes *compression* in the *top fibers* of the segment such that it bends the segment so that it holds water. Loadings that are opposite to these are considered negative.

Important Points

- *Beams* are long straight members that are subjected to loads perpendicular to their longitudinal axis. They are classified according to the way they are supported, e.g., simply supported, cantilevered, or overhanging.
- In order to properly design a beam, it is important to know the *variation* of the internal shear and moment along its axis in order to find the points where these values are a maximum.
- Using an established sign convention for positive shear and moment, the shear and moment in the beam can be determined as a function of its position x on the beam, and then these functions can be plotted to form the shear and moment diagrams.

Procedure for Analysis

The shear and moment diagrams for a beam can be constructed using the following procedure.

Support Reactions.

- Determine all the reactive forces and couple moments acting on the beam, and resolve all the forces into components acting perpendicular and parallel to the beam's axis.

Shear and Moment Functions.

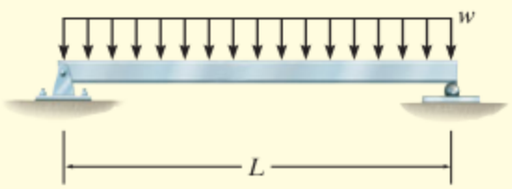
- Specify separate coordinates x having an origin at the beam's *left end* and extending to regions of the beam between concentrated forces and/or couple moments, or where there is no discontinuity of distributed loading.
- Section the beam at each distance x, and draw the free-body diagram of one of the segments. Be sure $\mathbf{V}$ and $\mathbf{M}$ are shown acting in their positive sense, in accordance with the sign convention given in Fig. 11–3.
- The shear is obtained by summing forces perpendicular to the beam's axis.
- To eliminate V, the moment is obtained directly by summing moments about the sectioned end of the segment.

Shear and Moment Diagrams.

- Plot the shear diagram (V versus x) and the moment diagram (M versus x). If numerical values of the functions describing V and M are *positive*, the values are plotted above the x axis, whereas negative values are plotted below the axis.
- Generally it is convenient to show the shear and moment diagrams below the free-body diagram of the beam.

EXAMPLE 11.1

Draw the shear and moment diagrams for the beam shown in Fig. 11–4*a*.

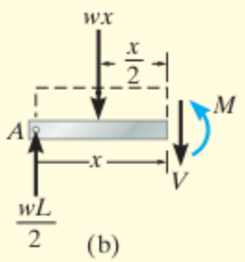


(a)

SOLUTION

Support Reactions. The support reactions are shown in Fig. 11–4*c*.

Shear and Moment Functions. A free-body diagram of the left segment of the beam is shown in Fig. 11–4*b*. The distributed loading on this segment, wx, is represented by its resultant force only *after* the segment is isolated as a free-body diagram. This force acts through the centroid of the area comprising the distributed loading, a distance of $x/2$ from the right end. Applying the two equations of equilibrium yields



(b)

$$+\uparrow \Sigma F_y = 0; \qquad \frac{wL}{2} - wx - V = 0$$

$$V = w\left(\frac{L}{2} - x\right) \tag{1}$$

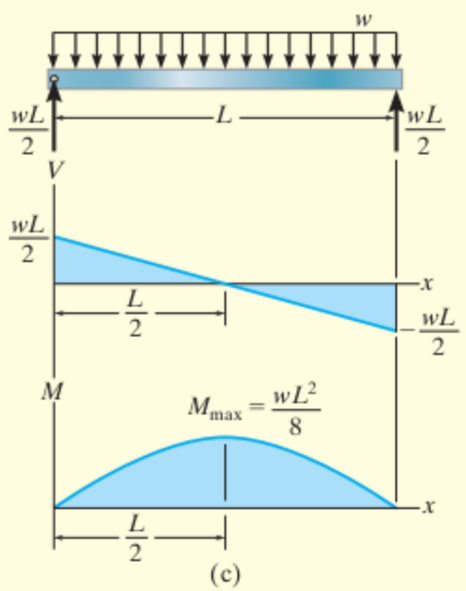
$$\circlearrowleft +\Sigma M = 0; \qquad -\left(\frac{wL}{2}\right)x + (wx)\left(\frac{x}{2}\right) + M = 0$$

$$M = \frac{w}{2}(Lx - x^2) \tag{2}$$

Shear and Moment Diagrams. The shear and moment diagrams shown in Fig. 11–4*c* are obtained by plotting Eqs. (1) and (2). The point of *zero shear* can be found from Eq. (1):

$$V = w\left(\frac{L}{2} - x\right) = 0$$

$$x = \frac{L}{2}$$



(c)

Fig. 11–4

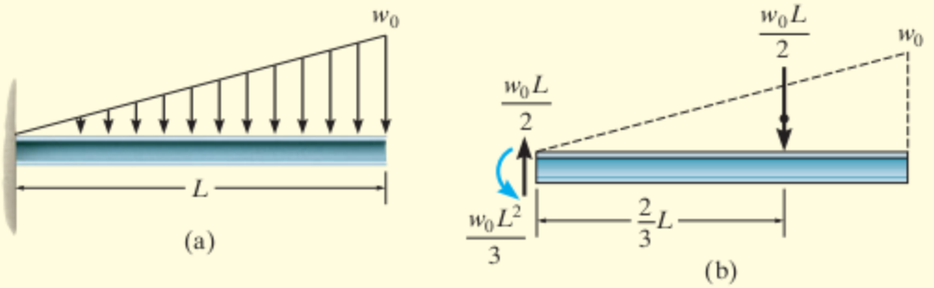
NOTE: From the moment diagram, this value of x represents the point on the beam where the *maximum moment* occurs, since by Eq. 11–2 (see Sec. 11.2) the *slope* $V = dM/dx = 0$. From Eq. (2), we have

$$M_{\max} = \frac{w}{2}\left[L\left(\frac{L}{2}\right) - \left(\frac{L}{2}\right)^2\right]$$

$$= \frac{wL^2}{8}$$

EXAMPLE 11.2

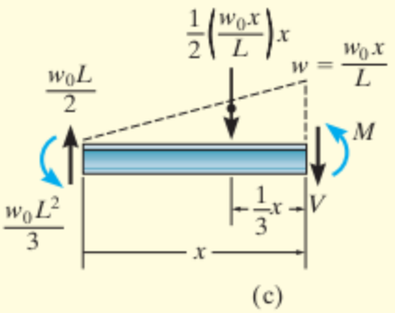
Draw the shear and moment diagrams for the beam shown in Fig. 11–5*a*.



SOLUTION

Support Reactions. The distributed load is replaced by its resultant force and the reactions have been determined as shown in Fig. 11–5*b*.

Shear and Moment Functions. A free-body diagram of a beam segment of length x is shown in Fig. 11–5*c*. Note that the intensity of the triangular load at the section is found by proportion, that is, $w/x = w_0/L$ or $w = w_0 x/L$. With the load intensity known, the resultant of the distributed loading is determined from the area under the diagram. Thus,



$$+\uparrow \Sigma F_y = 0; \qquad \frac{w_0 L}{2} - \frac{1}{2}\left(\frac{w_0 x}{L}\right)x - V = 0$$

$$V = \frac{w_0}{2L}(L^2 - x^2) \tag{1}$$

$$\circlearrowleft +\Sigma M = 0; \qquad \frac{w_0 L^2}{3} - \frac{w_0 L}{2}(x) + \frac{1}{2}\left(\frac{w_0 x}{L}\right)x\left(\frac{1}{3}x\right) + M = 0$$

$$M = \frac{w_0}{6L}(-2L^3 + 3L^2 x - x^3) \tag{2}$$

These results can be checked by applying Eqs. 11–1 and 11–2 of Sec. 11.2, that is,

$$w = \frac{dV}{dx} = \frac{w_0}{2L}(0 - 2x) = -\frac{w_0 x}{L} \qquad \text{OK}$$

$$V = \frac{dM}{dx} = \frac{w_0}{6L}(0 + 3L^2 - 3x^2) = \frac{w_0}{2L}(L^2 - x^2) \qquad \text{OK}$$

Shear and Moment Diagrams. The graphs of Eqs. (1) and (2) are shown in Fig. 11–5*d*.

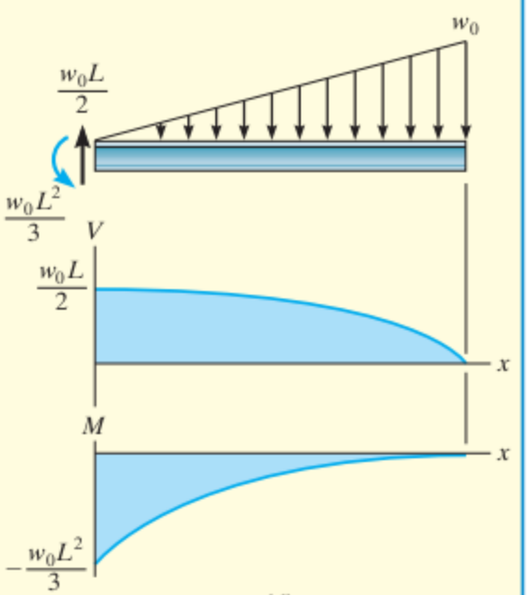
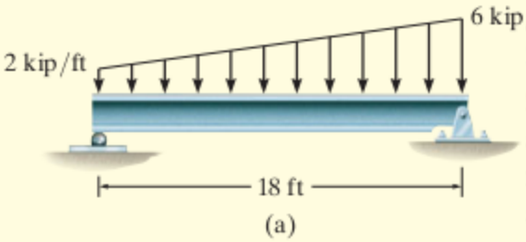


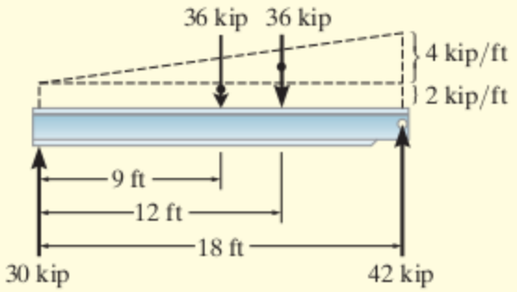
Fig. 11–5

EXAMPLE 11.3

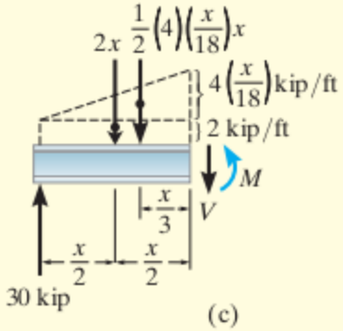
Draw the shear and moment diagrams for the beam shown in Fig. 11–6*a*.



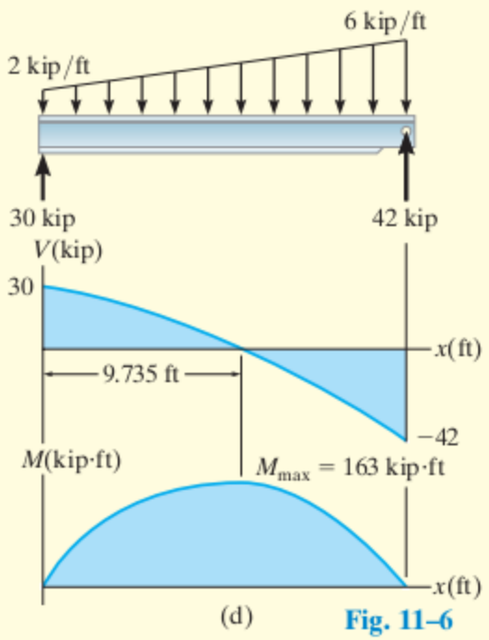
(a)



(b)



(c)



(d)

Fig. 11–6

SOLUTION

Support Reactions. The distributed load is divided into triangular and rectangular component loadings and these loadings are then replaced by their resultant forces. The reactions have been determined as shown on the beam's free-body diagram, Fig. 11–6*b*.

Shear and Moment Functions. A free-body diagram of the left segment is shown in Fig. 11–6*c*. As above, the trapezoidal loading is replaced by rectangular and triangular distributions. Note that the intensity of the triangular load at the section is found by proportion. The resultant force and the location of each distributed loading are also shown. Applying the equilibrium equations, we have

$$+\uparrow \Sigma F_y = 0; \qquad 30\ \text{kip} - (2\ \text{kip/ft})x - \frac{1}{2}(4\ \text{kip/ft})\left(\frac{x}{18\ \text{ft}}\right)x - V = 0$$

$$V = \left(30 - 2x - \frac{x^2}{9}\right)\text{kip} \tag{1}$$

$$\circlearrowleft + \Sigma M = 0;$$

$$-30\ \text{kip}(x) + (2\ \text{kip/ft})x\left(\frac{x}{2}\right) + \frac{1}{2}(4\ \text{kip/ft})\left(\frac{x}{18\ \text{ft}}\right)x\left(\frac{x}{3}\right) + M = 0$$

$$M = \left(30x - x^2 - \frac{x^3}{27}\right)\text{kip}\cdot\text{ft} \tag{2}$$

Equation (2) may be checked by noting that $dM/dx = V$, that is, Eq. (1). Also, $w = dV/dx = -2 - \frac{2}{9}x$. This equation checks, since when $x = 0$, $w = -2$ kip/ft, and when $x = 18$ ft, $w = -6$ kip/ft, Fig. 11–6*a*.

Shear and Moment Diagrams. Equations (1) and (2) are plotted in Fig. 11–6*d*. Since the point of maximum moment occurs when $dM/dx = V = 0$ (Eq. 11–2), then, from Eq. (1),

$$V = 0 = 30 - 2x - \frac{x^2}{9}$$

Choosing the positive root,

$$x = 9.735\ \text{ft}$$

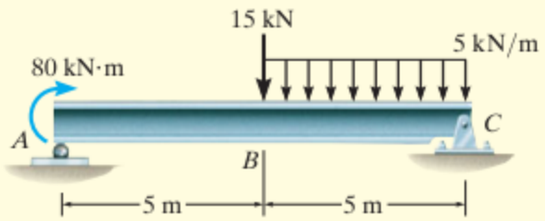
Thus, from Eq. (2),

$$M_{max} = 30(9.735) - (9.735)^2 - \frac{(9.735)^3}{27}$$

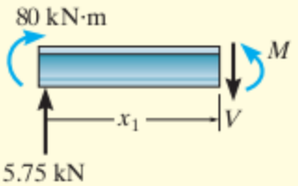
$$= 163\ \text{kip}\cdot\text{ft}$$

EXAMPLE 11.4

Draw the shear and moment diagrams for the beam shown in Fig. 11–7*a*.



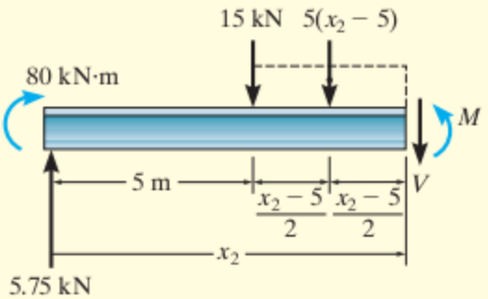
(a)



(b)

SOLUTION

Support Reactions. The reactions at the supports have been determined and are shown on the free-body diagram of the beam, Fig. 11–7*d*.



(c)

Shear and Moment Functions. Since there is a discontinuity of distributed load and also a concentrated load at the beam's center, two regions of x must be considered in order to describe the shear and moment functions for the entire beam.

$0 \le x_1 < 5$ m, Fig. 11–7*b*:

$$+\uparrow \Sigma F_y = 0; \qquad 5.75\ \text{kN} - V = 0$$

$$V = 5.75\ \text{kN}$$

$$\circlearrowleft +\Sigma M = 0; \qquad -80\ \text{kN}\cdot\text{m} - 5.75\ \text{kN}\, x_1 + M = 0$$

$$M = (5.75x_1 + 80)\ \text{kN}\cdot\text{m} \qquad (1)$$

$5\ \text{m} < x_2 \le 10$ m, Fig. 11–7*c*:

$$+\uparrow \Sigma F_y = 0; \quad 5.75\ \text{kN} - 15\ \text{kN} - 5\ \text{kN/m}(x_2 - 5\ \text{m}) - V = 0$$

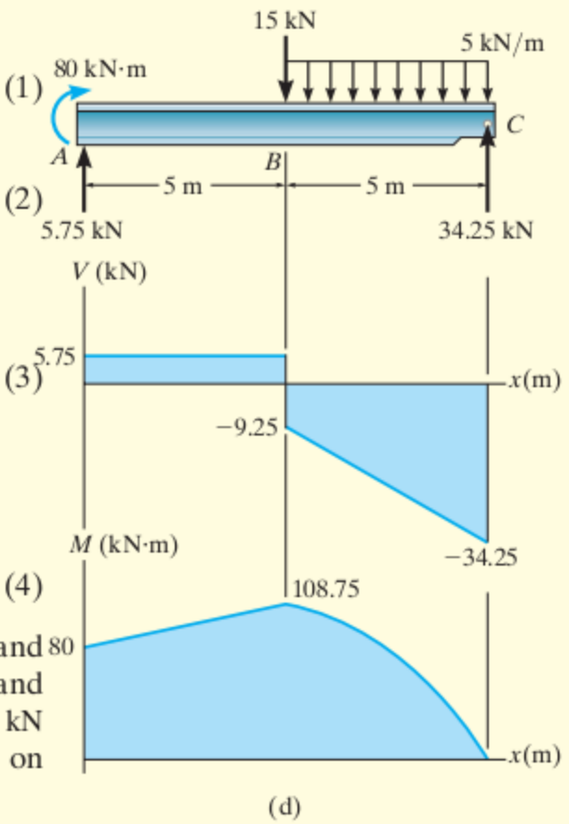
$$V = (15.75 - 5x_2)\ \text{kN}$$

$$\circlearrowleft +\Sigma M = 0; \quad -80\ \text{kN}\cdot\text{m} - 5.75\ \text{kN}\, x_2 + 15\ \text{kN}(x_2 - 5\ \text{m})$$

$$+5\ \text{kN/m}(x_2 - 5\ \text{m})\left(\frac{x_2 - 5\ \text{m}}{2}\right) + M = 0$$

$$M = (-2.5x_2^2 + 15.75x_2 + 92.5)\ \text{kN}\cdot\text{m} \qquad (4)$$

These results can be checked in part by noting that $w = dV/dx$ and $V = dM/dx$. Also, when $x_1 = 0$, Eqs. (1) and (2) give $V = 5.75$ kN and $M = 80$ kN·m; when $x_2 = 10$ m, Eqs. (3) and (4) give $V = -34.25$ kN and $M = 0$. These values check with the support reactions shown on the free-body diagram, Fig. 11–7*d*.



(d)

Fig. 11–7

Shear and Moment Diagrams. Equations (2) through (4) are plotted in Fig. 11–7*d*.

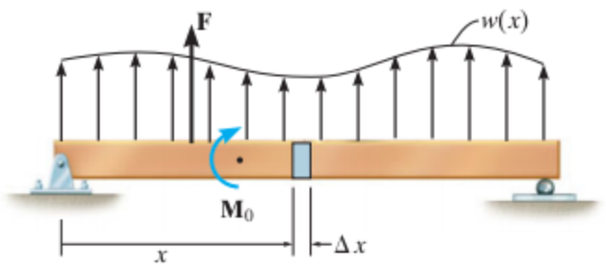
Failure of this table occurred at the brace support on its right side. If drawn, the bending moment diagram for the table loading would indicate this to be the point of maximum internal moment.

11.2 Graphical Method for Constructing Shear and Moment Diagrams

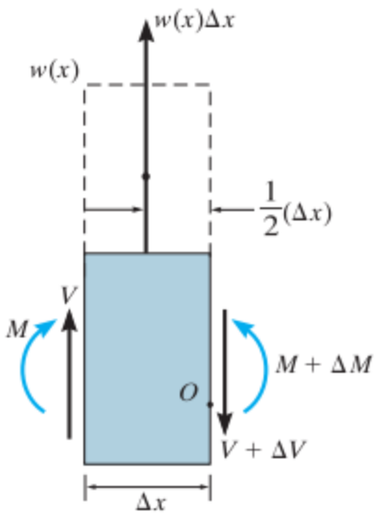
In cases where a beam is subjected to *several* different loadings, determining V and M as functions of x and then plotting these equations can become quite tedious. In this section a simpler method for constructing the shear and moment diagrams is discussed—a method based on two differential relations, one that exists between distributed load and shear, and the other between shear and moment.

Regions of Distributed Load. For purposes of generality, consider the beam shown in Fig. 11–8*a*, which is subjected to an arbitrary loading. A free-body diagram for a very small segment Δx of the beam is shown in Fig. 11–8*b*. Since this segment has been chosen at a position x where there is no concentrated force or couple moment, the results to be obtained will *not* apply at these points of concentrated loading.

Notice that all the loadings shown on the segment act in their positive directions according to the established sign convention, Fig. 11–3. Also, both the internal resultant shear and moment, acting on the right face of the segment, must be changed by a small amount in order to keep the segment in equilibrium. The distributed load has been replaced by an approximate resultant force $w(x)\,\Delta x$ that acts at $\frac{1}{2}(\Delta x)$. Applying the equations of equilibrium to the segment, we have



(a)



Free-body diagram of segment Δx

(b)

Fig. 11–8

$$+\uparrow \Sigma F_y = 0; \qquad V + w(x)\,\Delta x - (V + \Delta V) = 0$$
$$\Delta V = w(x)\,\Delta x$$

$$\zeta + \Sigma M_O = 0; \qquad -V\,\Delta x - M - w(x)\,\Delta x[\tfrac{1}{2}(\Delta x)] + (M + \Delta M) = 0$$
$$\Delta M = V\,\Delta x + w(x)\,\tfrac{1}{2}(\Delta x)^2$$

Dividing by Δx and taking the limit as $\Delta x \to 0$, the above two equations become

$$\frac{dV}{dx} = w(x) \tag{11–1}$$

slope of shear diagram at each point = distributed load intensity at each point

$$\frac{dM}{dx} = V \tag{11–2}$$

slope of moment diagram at each point = shear at each point

These two equations provide a convenient means for quickly obtaining the shear and moment diagrams for a beam. Equation 11–1 states that at a point the *slope* of the shear diagram equals the intensity of the distributed loading. For example, consider the beam in Fig. 11–9*a*. The distributed loading is negative and increases from zero to w_B. Therefore, the shear diagram will be a curve that has a *negative slope*, increasing from zero to $-w_B$. Specific slopes $w_A = 0$, $-w_C$, $-w_D$, and $-w_B$ are shown in Fig. 11–9*b*.

In a similar manner, Eq. 11–2 states that at a point the *slope* of the moment diagram is equal to the shear. Notice that the shear diagram in Fig. 11–9*b* starts at $+V_A$, decreases to zero, and then becomes negative and decreases to $-V_B$. The moment diagram will then have an initial slope of $+V_A$ which decreases to zero, then the slope becomes negative and decreases to $-V_B$. Specific slopes V_A, V_C, V_D, 0, and $-V_B$ are shown in Fig. 11–9*c*.

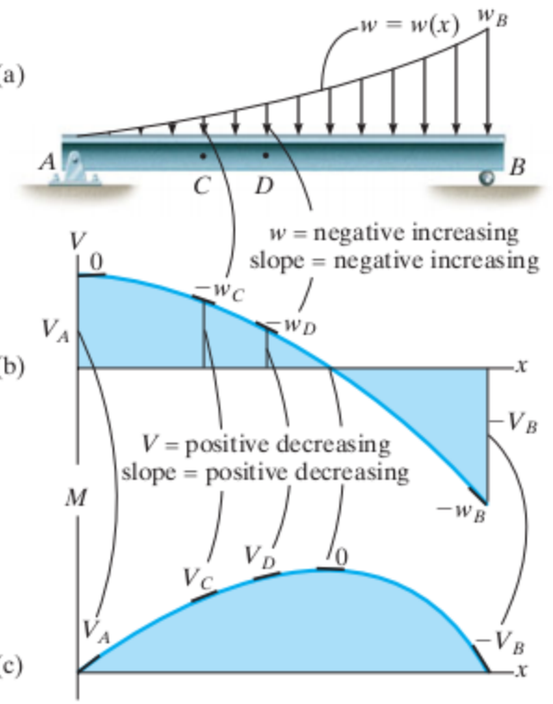


Fig. 11–9

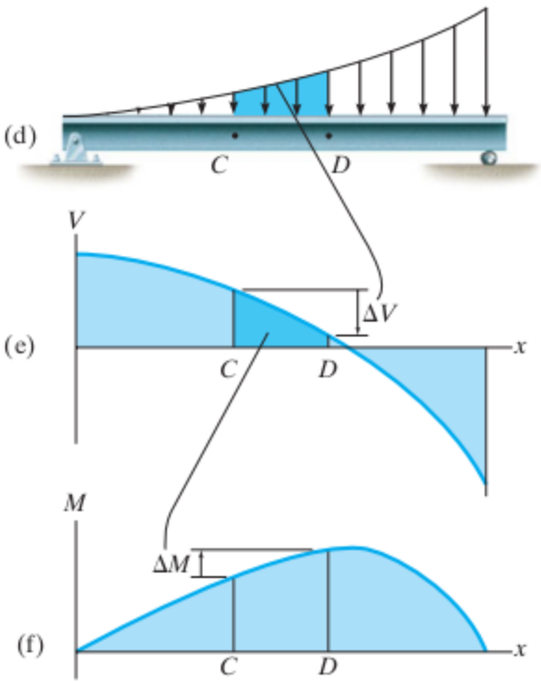


Fig. 11–9 (cont.)

Equations 11–1 and 11–2 may also be rewritten in the form $dV = w(x)\,dx$ and $dM = V\,dx$. Noting that $w(x)\,dx$ and $V\,dx$ represent differential areas under the distributed loading and shear diagram, respectively, we can integrate these areas between any two points C and D on the beam, Fig. 11–9*d*, and write

$$\Delta V = \int w(x)\,dx$$

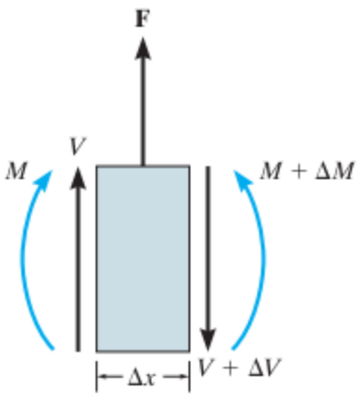
$$\text{change in shear} = \text{area under distributed loading} \tag{11–3}$$

$$\Delta M = \int V(x)\,dx$$

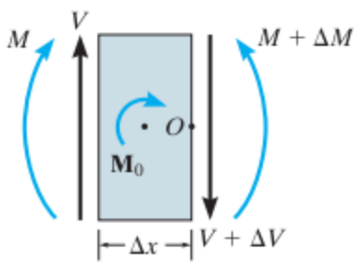
$$\text{change in moment} = \text{area under shear diagram} \tag{11–4}$$

Equation 11–3 states that the *change in shear* between C and D is equal to the *area* under the distributed-loading curve between these two points, Fig. 11–9*d*. In this case the change is negative since the distributed load acts downward. Similarly, from Eq. 11–4, the change in moment between C and D, Fig. 11–9*f*, is equal to the area under the shear diagram within the region from C to D. Here the change is positive.

Since the above equations do not apply at points where a concentrated force or couple moment acts, we will now consider each of these cases.



(a)



(b)

Fig. 11–10

Regions of Concentrated Force and Moment. A free-body diagram of a small segment of the beam in Fig. 11–10*a* taken from under the force is shown in Fig. 11–10*a*. Here it can be seen that force equilibrium requires

$$+\uparrow \Sigma F_y = 0; \qquad V + F - (V + \Delta V) = 0$$

$$\Delta V = F \tag{11–5}$$

Thus, when **F** acts *upward* on the beam, ΔV is *positive* so the shear will "jump" *upward*. Likewise, if **F** acts *downward*, the jump (ΔV) will be *downward*.

When the beam segment includes the couple moment M_0, Fig. 11–10*b*, then moment equilibrium requires the change in moment to be

$$\circlearrowleft + \Sigma M_O = 0; \qquad M + \Delta M - M_0 - V\,\Delta x - M = 0$$

Letting $\Delta x \rightarrow 0$, we get

$$\Delta M = M_0 \tag{11–6}$$

In this case, if $\mathbf{M}_0$ is applied *clockwise*, ΔM is *positive* so the moment diagram will "jump" *upward*. Likewise, when $\mathbf{M}_0$ acts *counterclockwise*, the jump (ΔM) will be *downward*.

Procedure for Analysis

The following procedure provides a method for constructing the shear and moment diagrams for a beam based on the relations among distributed load, shear, and moment.

Support Reactions.

- Determine the support reactions and resolve the forces acting on the beam into components that are perpendicular and parallel to the beam's axis.

Shear Diagram.

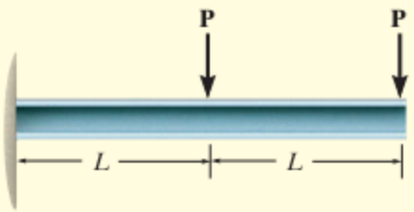
- Establish the V and x axes and plot the known values of the shear at the two *ends* of the beam.
- Notice how the values of the distributed load vary along the beam, and realize that each of these values indicates the way the shear diagram will slope ($dV/dx = w$). Here w is positive when it acts upward.
- If a numerical value of the shear is to be determined at a point, one can find this value either by using the method of sections and the equation of force equilibrium, or by using $\Delta V = \int w(x)\,dx$, which states that the *change in the shear* between any two points is equal to the *area under the load diagram* between the two points.

Moment Diagram.

- Establish the M and x axes and plot the known values of the moment at the *ends* of the beam.
- Notice how the values of the shear diagram vary along the beam, and realize that each of these values indicates the way the moment diagram will slope ($dM/dx = V$).
- At the point where the shear is zero, $dM/dx = 0$, and therefore this would be a point of maximum or minimum moment.
- If a numerical value of the moment is to be determined at the point, one can find this value either by using the method of sections and the equation of moment equilibrium, or by using $\Delta M = \int V(x)\,dx$, which states that the *change in moment* between any two points is equal to the *area under the shear diagram* between the two points.
- Since $w(x)$ must be *integrated* to obtain ΔV, and $V(x)$ is integrated to obtain $M(x)$, then if $w(x)$ is a curve of degree n, $V(x)$ will be a curve of degree $n + 1$ and $M(x)$ will be a curve of degree $n + 2$. For example, if $w(x)$ is uniform, $V(x)$ will be linear and $M(x)$ will be parabolic.

EXAMPLE 11.5

Draw the shear and moment diagrams for the beam shown in Fig. 11–11*a*.



(a)

SOLUTION

Support Reactions. The reaction at the fixed support is shown on the free-body diagram, Fig. 11–11*b*.

Shear Diagram. The shear at each end of the beam is plotted first, Fig. 11–11*c*. Since there is no distributed loading on the beam, the slope of the shear diagram is zero as indicated. Note how the force P at the center of the beam causes the shear diagram to jump downward an amount P, since this force acts downward.

Moment Diagram. The moments at the ends of the beam are plotted, Fig. 11–11*d*. Here the moment diagram consists of two sloping lines, one with a slope of $+2P$ and the other with a slope of $+P$.

The value of the moment in the center of the beam can be determined by the method of sections, or from the area under the shear diagram. If we choose the left half of the shear diagram,

$$M|_{x=L} = M|_{x=0} + \Delta M$$

$$M|_{x=L} = -3PL + (2P)(L) = -PL$$

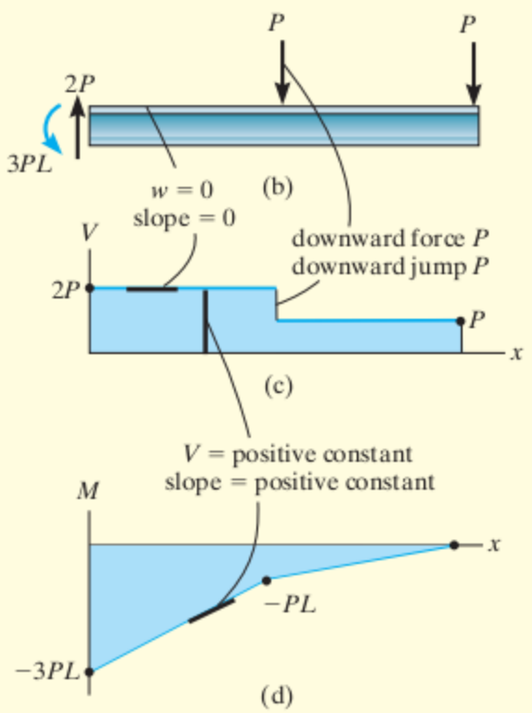
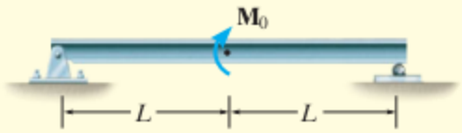


Fig. 11–11

EXAMPLE 11.6

Draw the shear and moment diagrams for the beam shown in Fig. 11–12*a*.



(a)

SOLUTION

Support Reactions. The reactions are shown on the free-body diagram in Fig. 11–12*b*.

Shear Diagram. The shear at each end is plotted first, Fig. 11–12*c*. Since there is no distributed load on the beam, the shear diagram has zero slope and is therefore a horizontal line.

Moment Diagram. The moment is zero at each end, Fig. 11–12*d*. The moment diagram has a constant negative slope of $-M_0/2L$ since this is the shear in the beam at each point. Note that the couple moment M_0 causes a jump in the moment diagram at the beam's center, but it does not affect the shear diagram at this point.

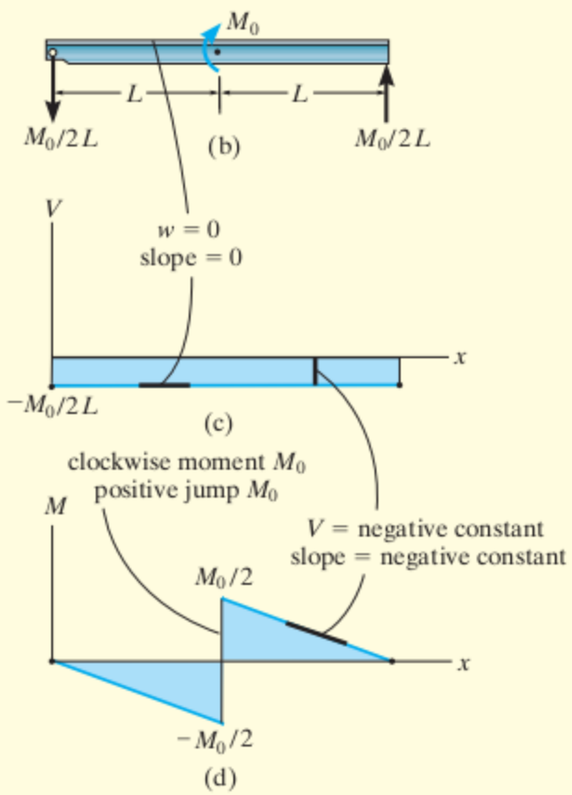


Fig. 11–12

EXAMPLE 11.7

Draw the shear and moment diagrams for each of the beams shown in Figs. 11–13*a* and 11–14*a*.

SOLUTION

Support Reactions. The reactions at the fixed support are shown on each free-body diagram, Figs. 11–13*b* and 11–14*b*.

Shear Diagram. The shear at each end point is plotted first, Figs. 11–13*c* and 11–14*c*. The distributed loading on each beam indicates the slope of the shear diagram and thus produces the shapes shown.

Moment Diagram. The moment at each end point is plotted first, Figs. 11–13*d* and 11–14*d*. Various values of the shear at each point on the beam indicate the slope of the moment diagram at the point. Notice how this variation produces the curves shown.

NOTE: Observe how the degree of the curves from w to V to M increases by one due to the integration of $dV = w dx$ and $dM = V dx$. For example, in Fig. 11–14, the linear distributed load produces a parabolic shear diagram and cubic moment diagram.

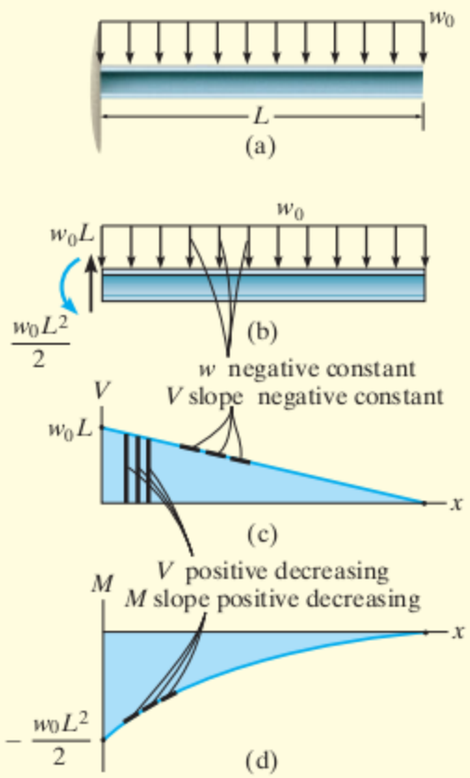


Fig. 11–13

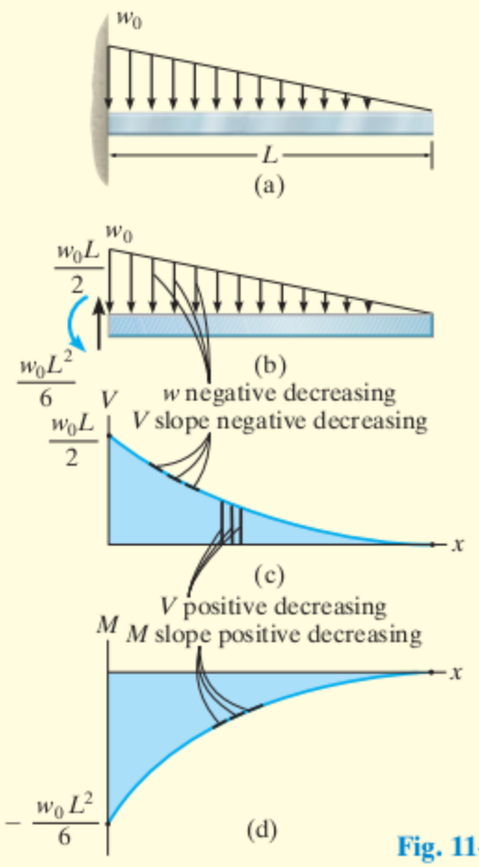
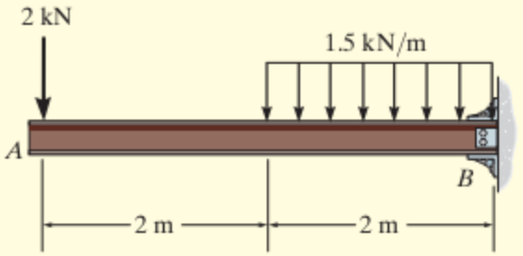


Fig. 11–14

EXAMPLE 11.8

Draw the shear and moment diagrams for the cantilever beam in Fig. 11–15*a*.



(a)

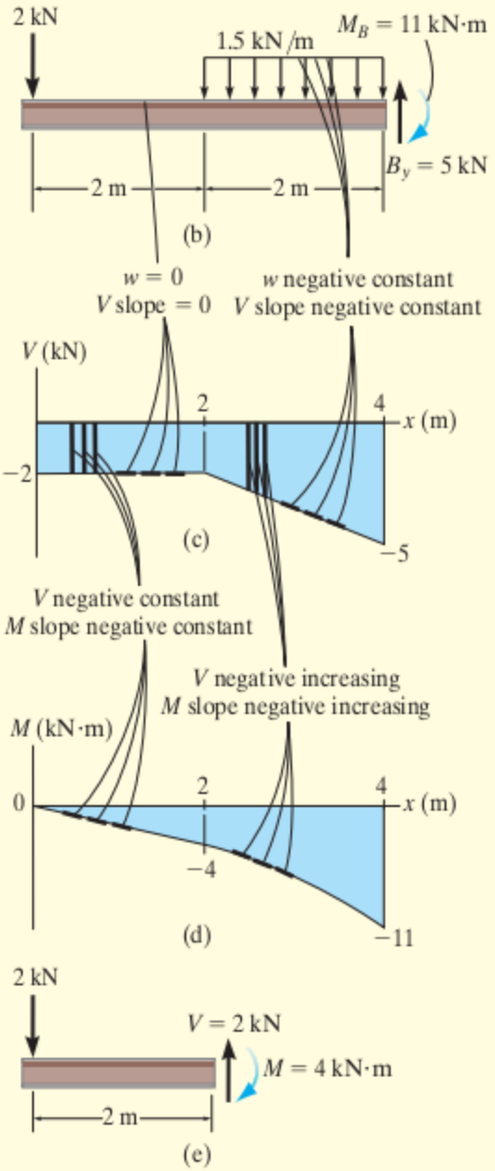


Fig. 11–15

SOLUTION

Support Reactions. The support reactions at the fixed support B are shown in Fig. 11–15*b*.

Shear Diagram. The shear at end A is –2 kN. This value is plotted at $x = 0$, Fig. 11–15*c*. Notice how the shear diagram is constructed by following the slopes defined by the loading w. The shear at $x = 4$ m is –5 kN, the reaction on the beam. This value can be verified by finding the area under the distributed loading, Eq. 11–3.

$$V|_{x=4\text{ m}} = V|_{x=2\text{ m}} + \Delta V = -2\text{ kN} - (1.5\text{ kN/m})(2\text{ m}) = -5\text{ kN}$$

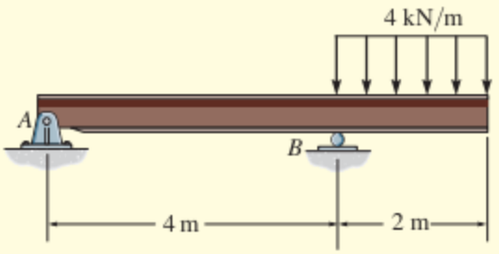
Moment Diagram. The moment of zero at $x = 0$ is plotted in Fig. 11–15*d*. Notice how the moment diagram is constructed based on knowing its slope, which is equal to the shear at each point. The change of moment from $x = 0$ to $x = 2$ m is determined from the area under the shear diagram. Hence, the moment at $x = 2$ m is

$$M|_{x=2\text{ m}} = M|_{x=0} + \Delta M = 0 + [-2\text{ kN}(2\text{ m})] = -4\text{ kN}\cdot\text{m}$$

This same value can be determined from the method of sections, Fig. 11–15*e*.

EXAMPLE 11.9

Draw the shear and moment diagrams for the overhang beam in Fig. 11–16*a*.



(a)

SOLUTION

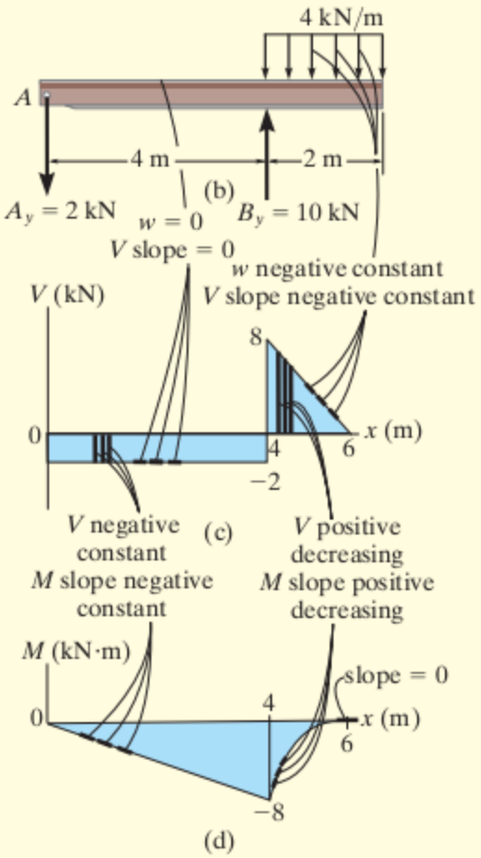
Support Reactions. The support reactions are shown in Fig. 11–16*b*.

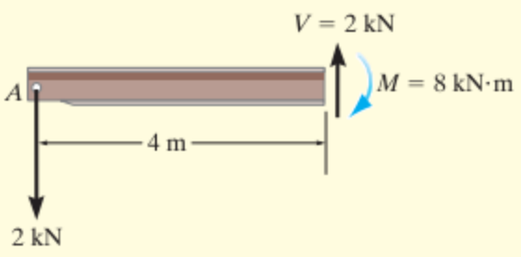
Shear Diagram. The shear of −2 kN at end A of the beam is plotted at $x = 0$, Fig. 11–16*c*. The slopes are determined from the loading and from this the shear diagram is constructed, as indicated in the figure. In particular, notice the positive jump of 10 kN at $x = 4$ m due to the force B_y, as indicated in the figure.

Moment Diagram. The moment of zero at $x = 0$ is plotted, Fig. 11–16*d*. Then following the behavior of the slope found from the shear diagram, the moment diagram is constructed. The moment at $x = 4$ m is found from the area under the shear diagram.

$$M|_{x=4\,\text{m}} = M|_{x=0} + \Delta M = 0 + [-2\,\text{kN}(4\,\text{m})] = -8\,\text{kN}\cdot\text{m}$$

We can also obtain this value by using the method of sections, as shown in Fig. 11–16*e*.



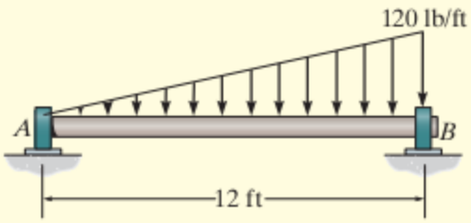


(e)

Fig. 11–16

EXAMPLE 11.10

The shaft in Fig. 11–17*a* is supported by a thrust bearing at *A* and a journal bearing at *B*. Draw the shear and moment diagrams.



(a)

SOLUTION

Support Reactions. The support reactions are shown in Fig. 11–17*b*.

Shear Diagram. As shown in Fig. 11–17*c*, the shear at $x = 0$ is +240 lb. Following the slope defined by the loading, the shear diagram is constructed, where at *B* its value is −480 lb. Since the shear changes sign, the point where $V = 0$ must be located. To do this we will use the method of sections. The free-body diagram of the left segment of the shaft, sectioned at an arbitrary position *x*, is shown in Fig. 11–17*e*. Notice that the intensity of the distributed load at *x* is $w = 10x$, which has been found by proportional triangles, i.e., $120/12 = w/x$.

Thus, for $V = 0$,

$$+\uparrow \Sigma F_y = 0; \qquad 240\text{ lb} - \tfrac{1}{2}(10x)x = 0$$

$$x = 6.93\text{ ft}$$

Moment Diagram. The moment diagram starts at 0 since there is no moment at *A*; then it is constructed based on the slope as determined from the shear diagram. The maximum moment occurs at $x = 6.93$ ft, where the shear is equal to zero, since $dM/dx = V = 0$, Fig. 11–17*d*,

$$\circlearrowleft +\Sigma M = 0;$$

$$M_{\max} + \tfrac{1}{2}[(10)(6.93)]\,6.93\left(\tfrac{1}{3}(6.93)\right) - 240(6.93) = 0$$

$$M_{\max} = 1109\text{ lb}\cdot\text{ft}$$

Finally, notice how integration, first of the loading *w* which is linear, produces a shear diagram which is parabolic, and then a moment diagram which is cubic.

NOTE: Having studied these examples, test yourself by covering over the shear and moment diagrams in Examples 11–1 through 11–4 and see if you can construct them using the concepts discussed here.

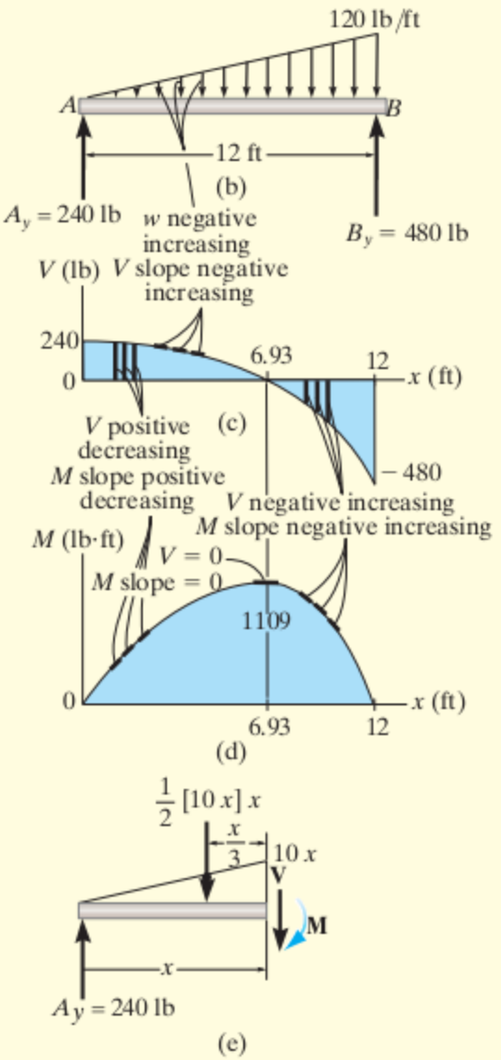
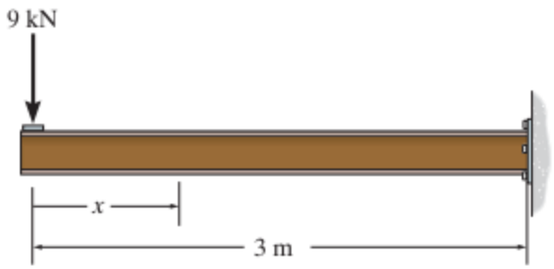


Fig. 11–17

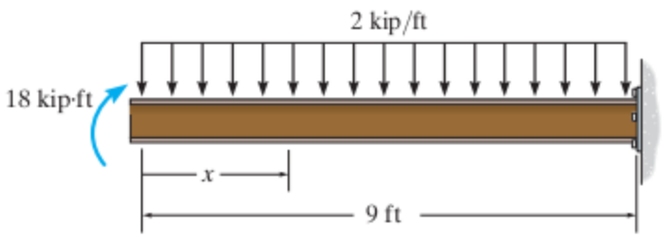
FUNDAMENTAL PROBLEMS

F11–1. Express the shear and moment functions in terms of x, and then draw the shear and moment diagrams for the cantilever beam.



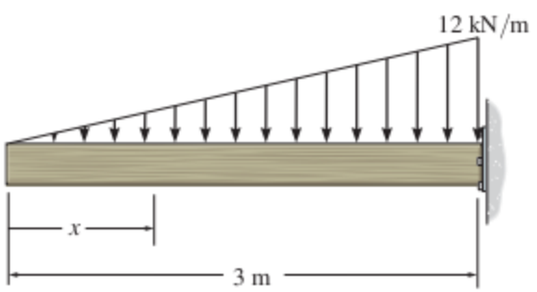
F11–1

F11–2. Express the shear and moment functions in terms of x, and then draw the shear and moment diagrams for the cantilever beam.



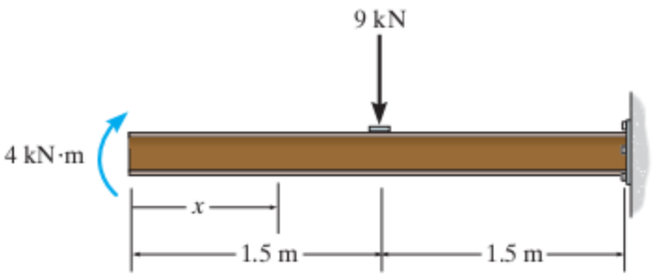
F11–2

F11–3. Express the shear and moment functions in terms of x, and then draw the shear and moment diagrams for the cantilever beam.



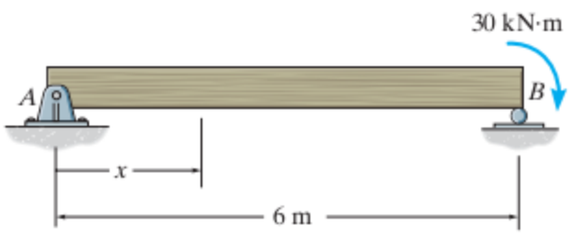
F11–3

F11–4. Express the shear and moment functions in terms of x, where $0 < x < 1.5$ m and 1.5 m $< x < 3$ m, and then draw the shear and moment diagrams for the cantilever beam.



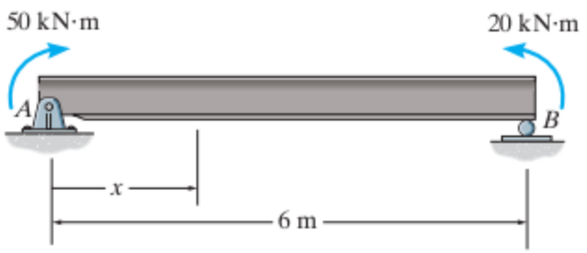
F11–4

F11–5. Express the shear and moment functions in terms of x, and then draw the shear and moment diagrams for the simply supported beam.



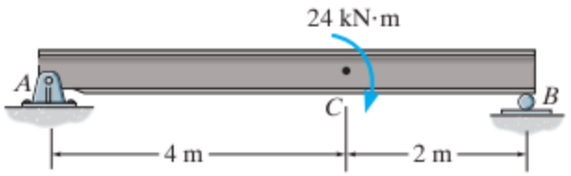
F11–5

F11–6. Express the shear and moment functions in terms of x, and then draw the shear and moment diagrams for the simply supported beam.



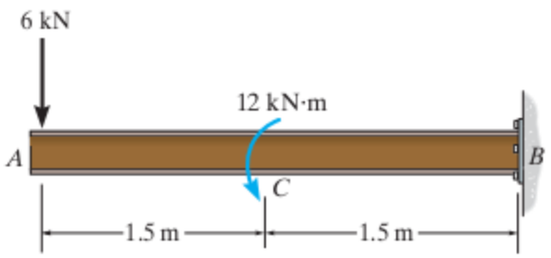
F11–6

F11–7. Draw the shear and moment diagrams for the simply supported beam.



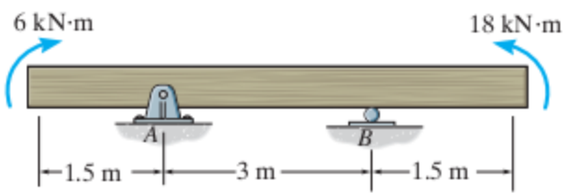
F11–7

F11–8. Draw the shear and moment diagrams for the cantilever beam.



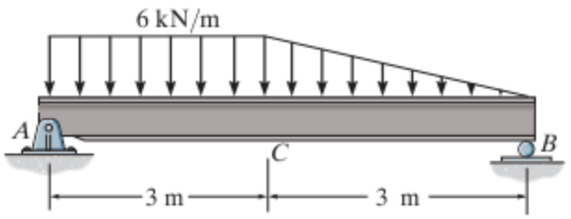
F11–8

F11–9. Draw the shear and moment diagrams for the double-overhang beam.



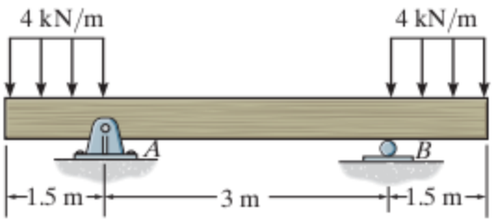
F11–9

F11–10. Draw the shear and moment diagrams for the simply supported beam.



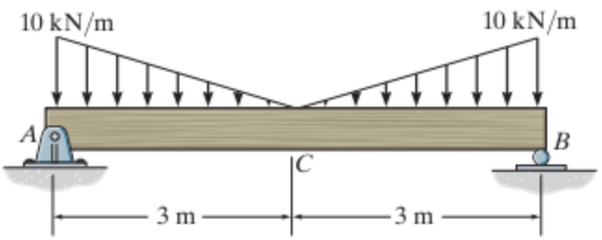
F11–10

F11–11. Draw the shear and moment diagrams for the double-overhang beam.



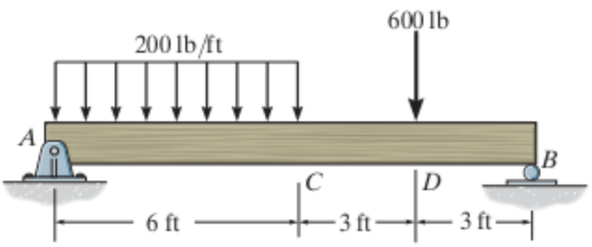
F11–11

F11–12. Draw the shear and moment diagrams for the simply supported beam.



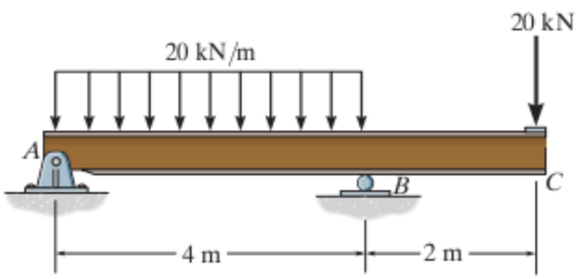
F11–12

F11–13. Draw the shear and moment diagrams for the simply supported beam.



F11–13

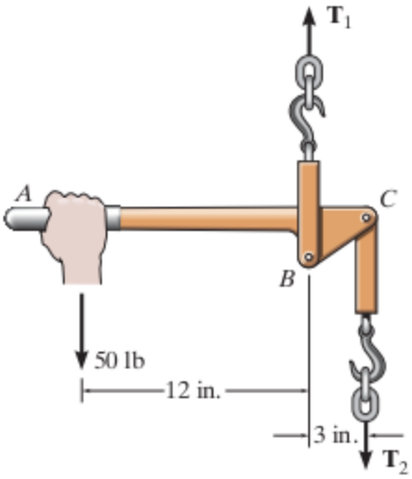
F11–14. Draw the shear and moment diagrams for the overhang beam.



F11–14

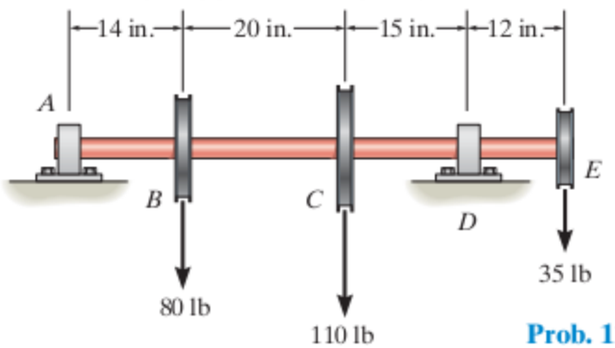
PROBLEMS

11–1. The load binder is used to support a load. If the force applied to the handle is 50 lb, determine the tensions T_1 and T_2 in each end of the chain and then draw the shear and moment diagrams for the arm ABC.



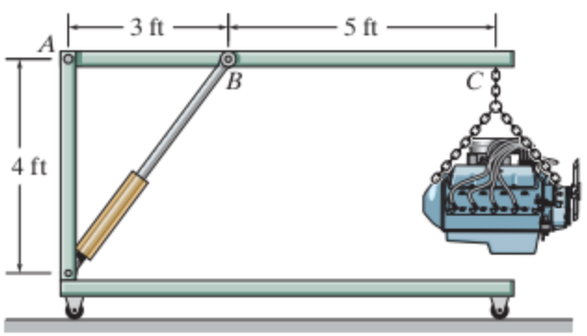
Prob. 11–1

11–2. Draw the shear and moment diagrams for the shaft The bearings at A and D exert only vertical reaction on the shaft. The loading is applied to the pulleys at B and C and E.



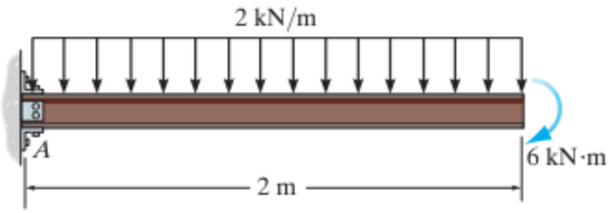
Prob. 11–2

11–3. The engine crane is used to support the engine, which has a weight of 1200 lb. Draw the shear and moment diagrams of the boom ABC when it is in the horizontal positions shown.



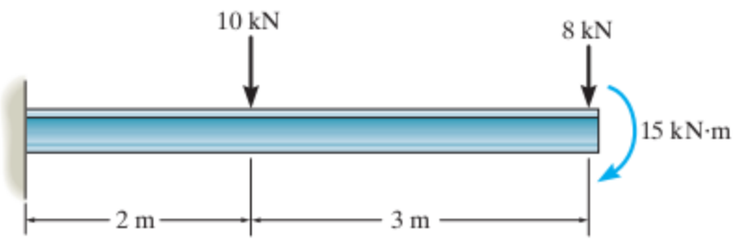
Prob. 11–3

***11–4.** Draw the shear and moment diagrams for the cantilever beam.



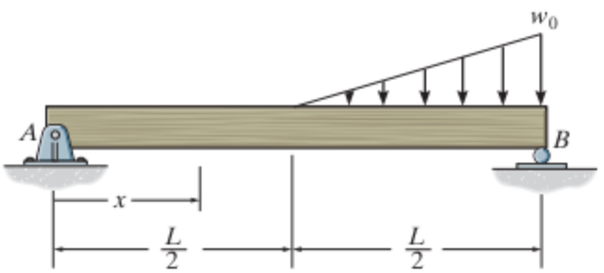
Prob. 11–4

11–5. Draw the shear and moment diagrams for the beam.



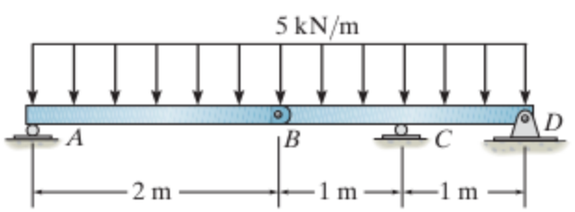
Prob. 11–5

11–6. Express the internal shear and moment in terms of x and then draw the shear and moment diagrams.



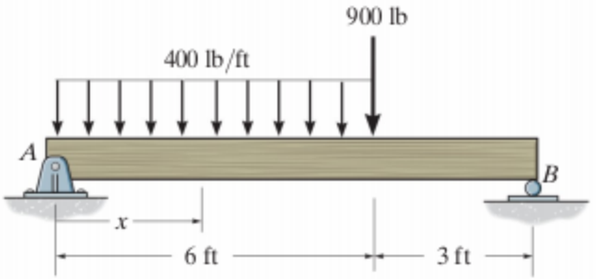
Prob. 11–6

11–7. Draw the shear and moment diagrams for the compound beam.



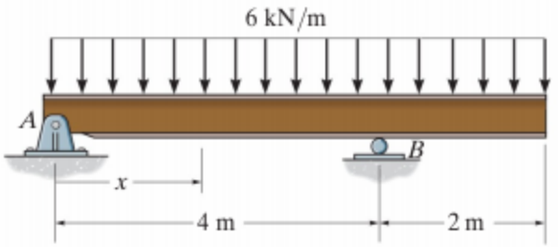
Prob. 11–7

***11–8.** Express the internal shear and moment in terms of x and then draw the shear and moment diagrams for the beam.



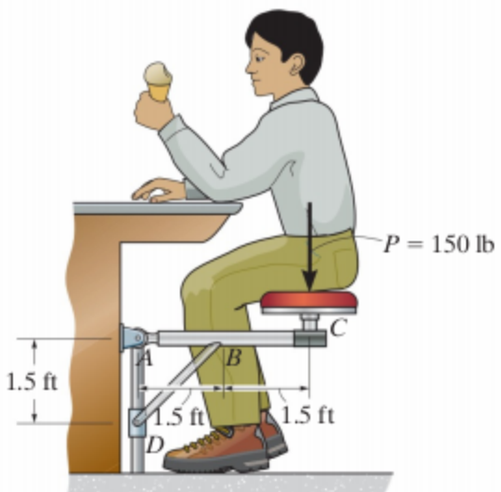
Prob. 11–8

11–9. Express the internal shear and moment in terms of x and then draw the shear and moment diagrams for the overhanging beam.



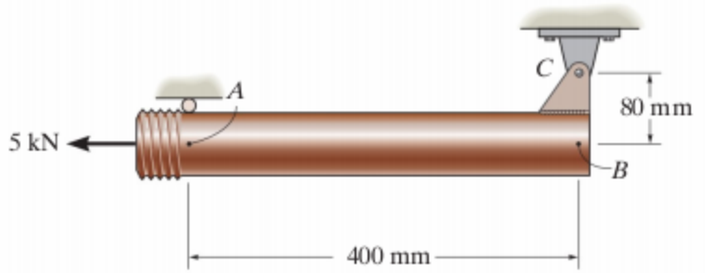
Prob. 11–9

11–10. Members ABC and BD of the counter chair are rigidly connected at B and the smooth collar at D is allowed to move freely along the vertical slot. Draw the shear and moment diagrams for member ABC.



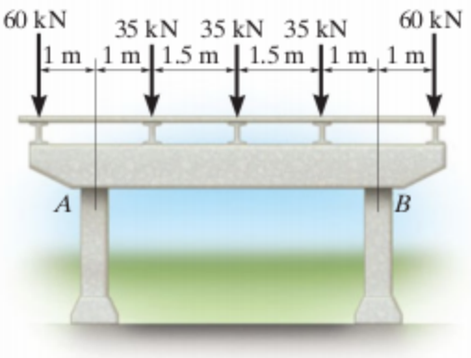
Prob. 11–10

11–11. Draw the shear and moment diagrams for the pipe. The end screw is subjected to a horizontal force of 5 kN. *Hint:* The reactions at the pin C must be replaced by an equivalent loading at point B on the axis of the pipe.



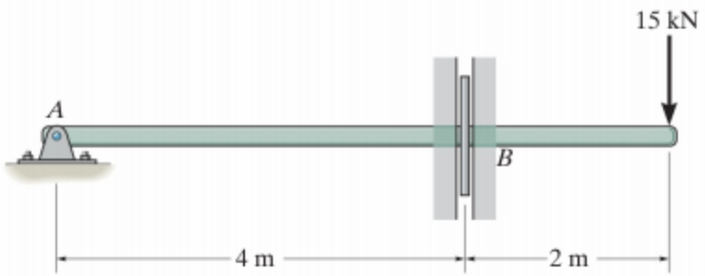
Prob. 11–11

***11–12.** A reinforced concrete pier is used to support the stringers for a bridge deck. Draw the shear and moment diagrams for the pier when it is subjected to the stringer loads shown. Assume the columns at A and B exert only vertical reactions on the pier.



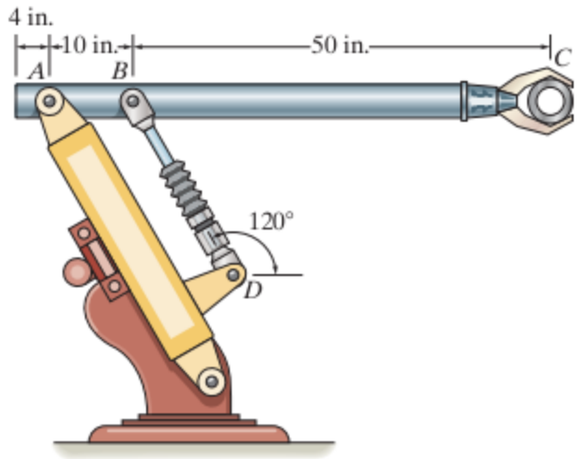
Prob. 11–12

11–13. Draw the shear and moment diagrams for the rod. It is supported by a pin at A and a smooth plate at B. The plate slides within the groove and so it cannot support a vertical force, although it can support a moment.



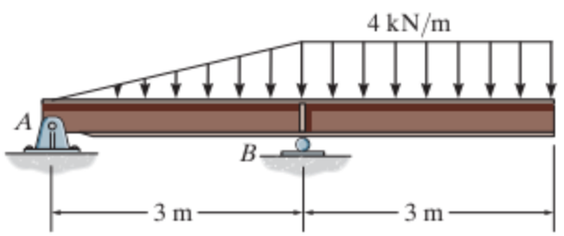
Prob. 11–13

11–14. The industrial robot is held in the stationary position shown. Draw the shear and moment diagrams of the arm ABC if it is pin connected at A and connected to a hydraulic cylinder (two-force member) BD. Assume the arm and grip have a uniform weight of 1.5 lb/in. and support the load of 40 lb at C.



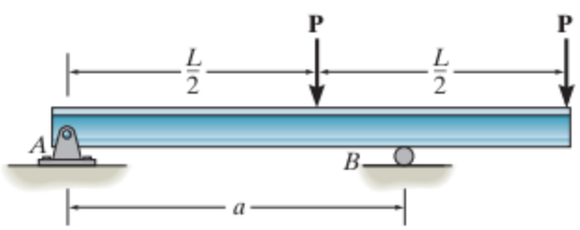
Prob. 11–14

11–15. Draw the shear and moment diagrams for the overhang beam.



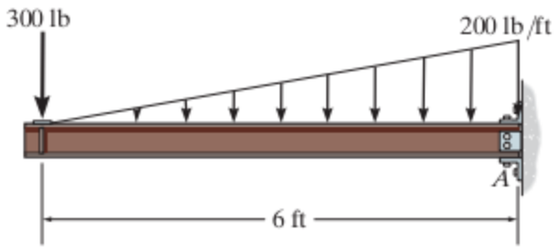
Prob. 11–15

***11–16.** Determine the placement distance a of the roller support so that the largest absolute value of the moment is a minimum. Draw the shear and moment diagrams for this condition.



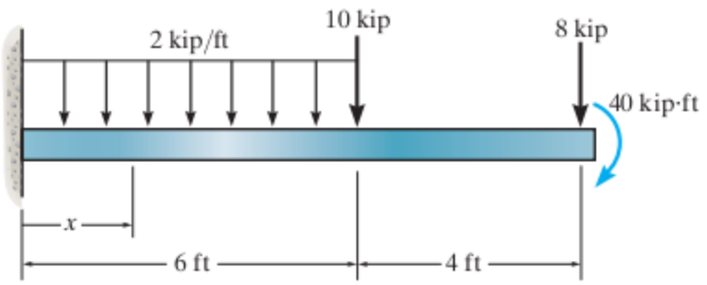
Prob. 11–16

11–17. Draw the shear and moment diagrams for the cantilevered beam.



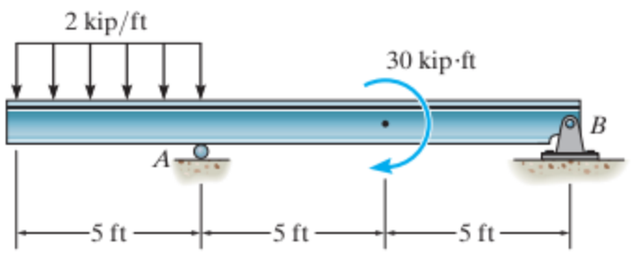
Prob. 11–17

11–18. Draw the shear and moment diagrams for the beam, and determine the shear and moment throughout the beam as functions of x.



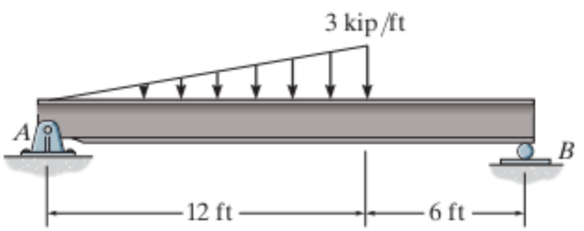
Prob. 11–18

11–19. Draw the shear and moment diagrams for the beam.



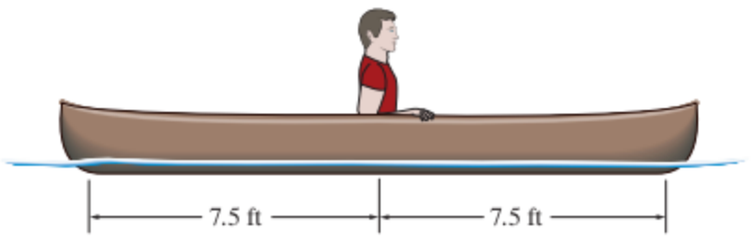
Prob. 11–19

***11–20.** Draw the shear and moment diagrams for the overhanging beam.



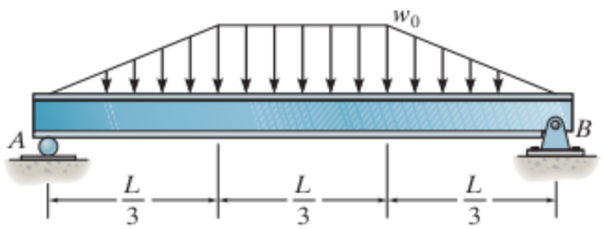
Prob. 11–20

11–21. The 150-lb man sits in the center of the boat, which has a uniform width and a weight per linear foot of 3 lb/ft. Determine the maximum bending moment exerted on the boat. Assume that the water exerts a uniform distributed load upward on the bottom of the boat.



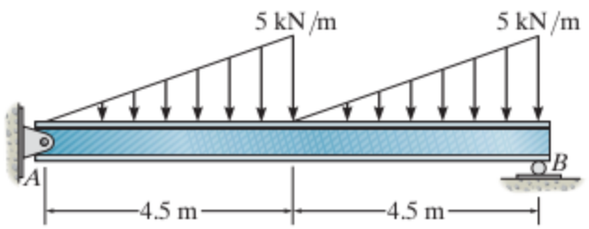
Prob. 11–21

11–22. Draw the shear and moment diagrams for the beam.



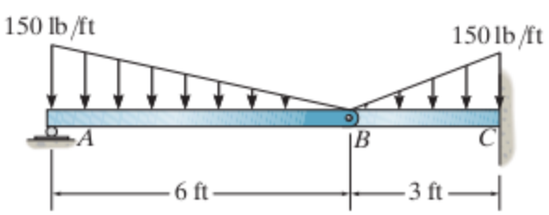
Prob. 11–22

11–23. Draw the shear and moment diagrams for the beam.



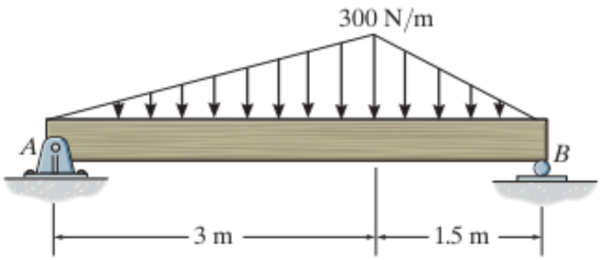
Prob. 11–23

***11–24.** Draw the shear and moment diagrams for the compound beam.



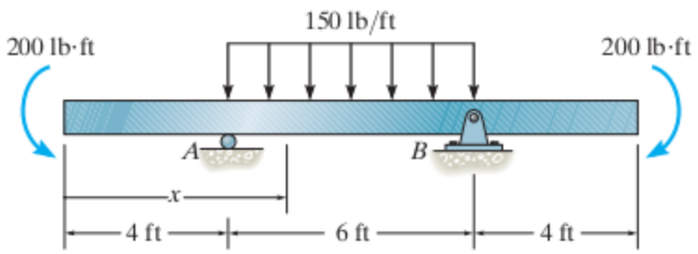
Prob. 11–24

11–25. Express the shear and moment in terms of x and then draw the shear and moment diagrams for the simply supported beam.



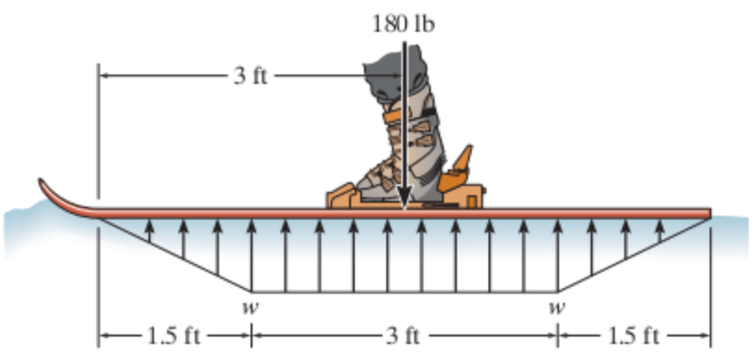
Prob. 11–25

11–26. Draw the shear and moment diagrams for the beam and determine the shear and moment in the beam as functions of x, where 4 ft < x < 10 ft.



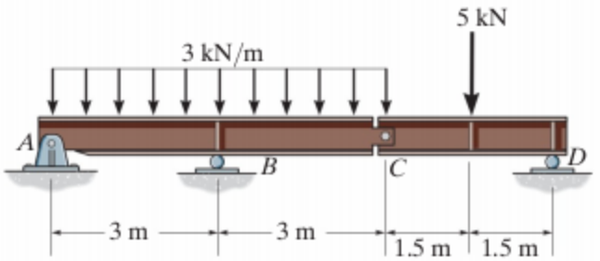
Prob. 11–26

11–27. The ski supports the 180-lb weight of the man. If the snow loading on its bottom surface is trapezoidal as shown, determine the intensity w, and then draw the shear and moment diagrams for the ski.



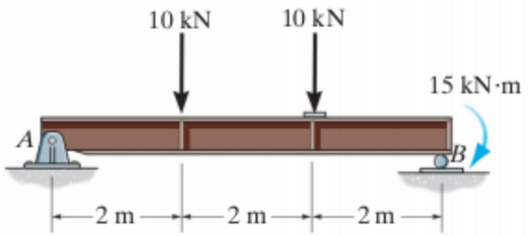
Prob. 11–27

*11–28. Draw the shear and moment diagrams for the compound beam.



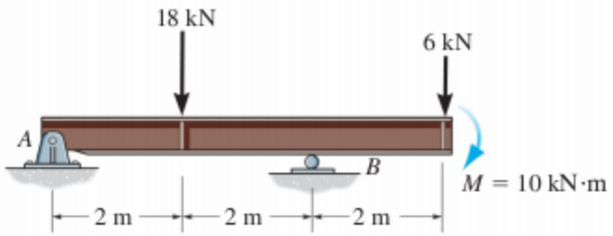
Prob. 11–28

11–29. Draw the shear and moment diagrams for the simply supported beam.



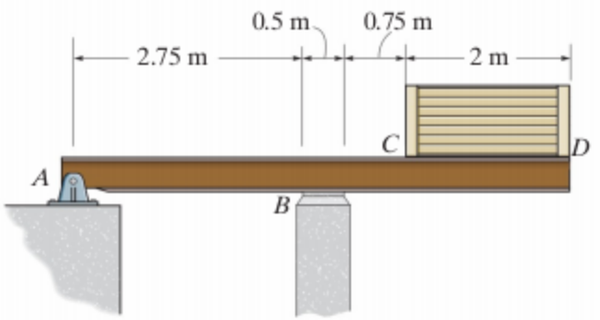
Prob. 11–29

11–30. Draw the shear and moment diagrams for the overhang beam.



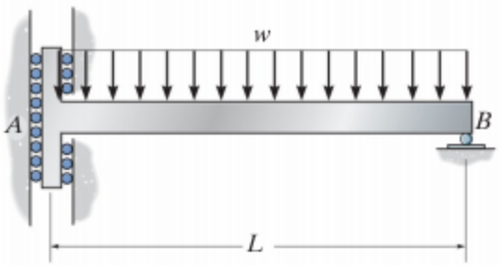
Prob. 11–30

11–31. The beam is used to support a uniform load along *CD* due to the 6-kN weight of the crate. If the reaction at bearing support *B* can be assumed uniformly distributed along its width, draw the shear and moment diagrams for the beam.



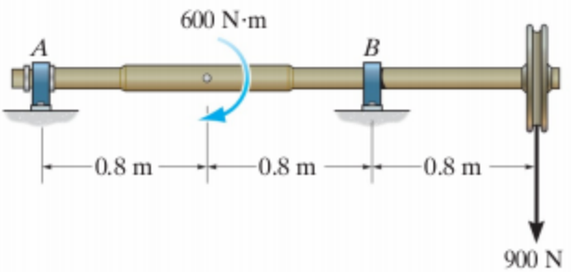
Prob. 11–31

*11–32. The support at *A* allows the beam to slide freely along the vertical guide so that it cannot support a vertical force. Draw the shear and moment diagrams for the beam.



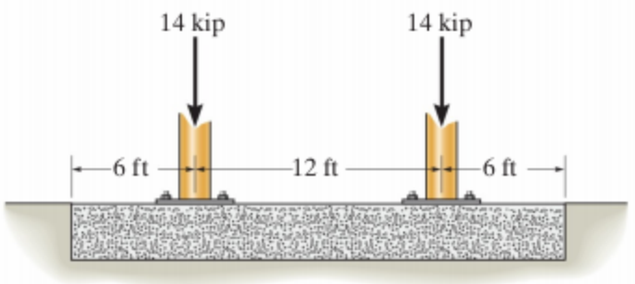
Prob. 11–32

11–33. The shaft is supported by a smooth thrust bearing at *A* and a smooth journal bearing at *B*. Draw the shear and moment diagrams for the shaft.



Prob. 11–33

11–34. The footing supports the load transmitted by the two columns. Draw the shear and moment diagrams for the footing if the reaction of soil pressure on the footing is assumed to be uniform.



Prob. 11–34

11.3 Bending Deformation of a Straight Member

In this section, we will discuss the deformations that occur when a straight prismatic beam, made of homogeneous material, is subjected to bending. The discussion will be limited to beams having a cross-sectional area that is symmetrical with respect to an axis, and the bending moment is applied about an axis perpendicular to this axis of symmetry as shown in Fig. 11–18. The behavior of members that have unsymmetrical cross sections, or are made from several different materials, is based on similar observations and will be discussed separately in later sections of this chapter.

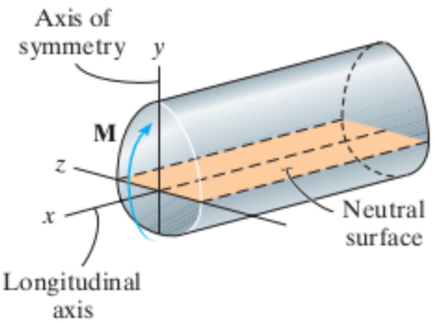


Fig. 11–18

By using a highly deformable material such as rubber, we can illustrate what happens when a straight prismatic member is subjected to a bending moment. Consider, for example, the undeformed bar in Fig. 11–19*a*, which has a square cross section and is marked with longitudinal and transverse grid lines. When a bending moment is applied, it tends to distort these lines into the pattern shown in Fig. 11–19*b*. Notice that the longitudinal lines become *curved* and the vertical transverse lines *remain straight* and yet undergo a *rotation*.

The bending moment causes the material within the *bottom* portion of the bar to *stretch* and the material within the *top* portion to *compress*. Consequently, between these two regions there must be a surface, called the *neutral surface*, in which longitudinal fibers of the material will not undergo a change in length, Fig. 11–18.

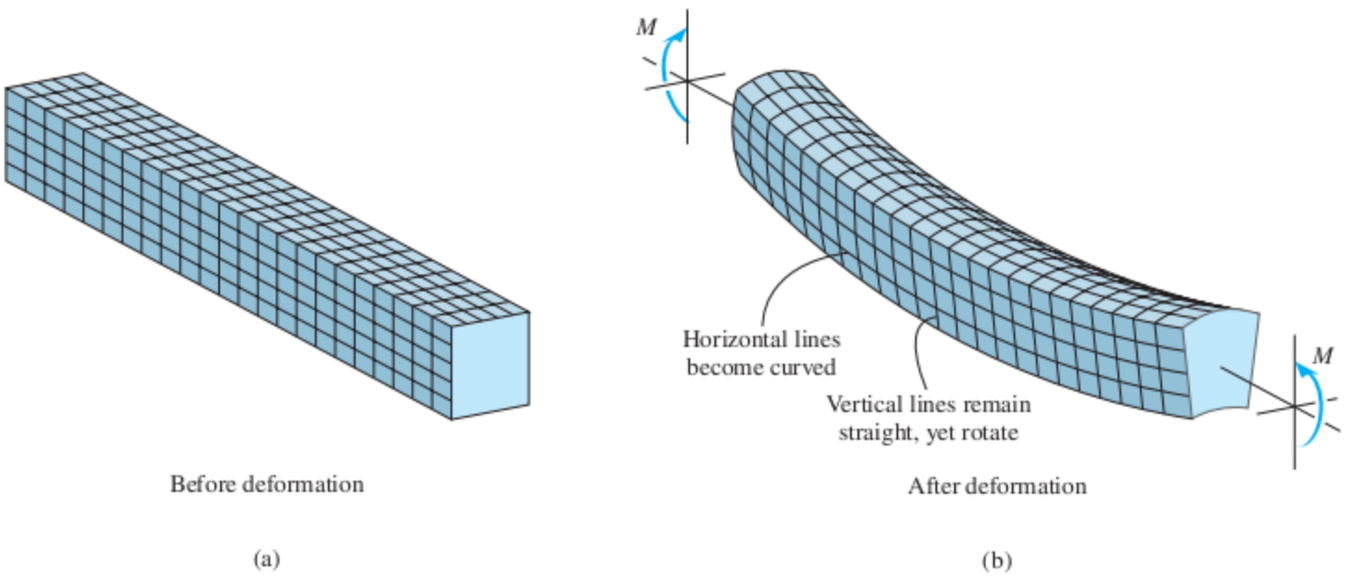
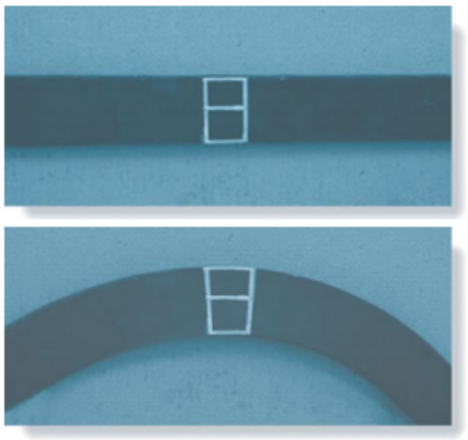


Fig. 11–19



Note the distortion of the lines due to bending of this rubber bar. The top line stretches, the bottom line compresses, and the center line remains the same length. Furthermore the vertical lines rotate and yet remain straight.

From these observations we will make the following three assumptions regarding the way the stress deforms the material. First, the *longitudinal axis* x, which lies within the neutral surface, Fig. 11–20*a*, does *not* experience any *change in length*. Rather the moment will tend to deform the beam so that this line *becomes a curve* that lies in the x–y plane of symmetry, Fig. 11–20*b*. Second, all ***cross sections*** of the beam ***remain plane*** and perpendicular to the longitudinal axis during the deformation. And third, *any deformation* of the *cross section* within its own plane, as noticed in Fig. 11–19*b*, will be *neglected*. In particular, the z axis, lying in the plane of the cross section and about which the cross section rotates, is called the *neutral axis*, Fig. 11–20*b*.

In order to show how this distortion will strain the material, we will isolate a small segment of the beam located a distance x along the beam's length and having an undeformed thickness Δx, Fig. 11–20*a*. This element, taken from the beam, is shown in profile view in the undeformed and

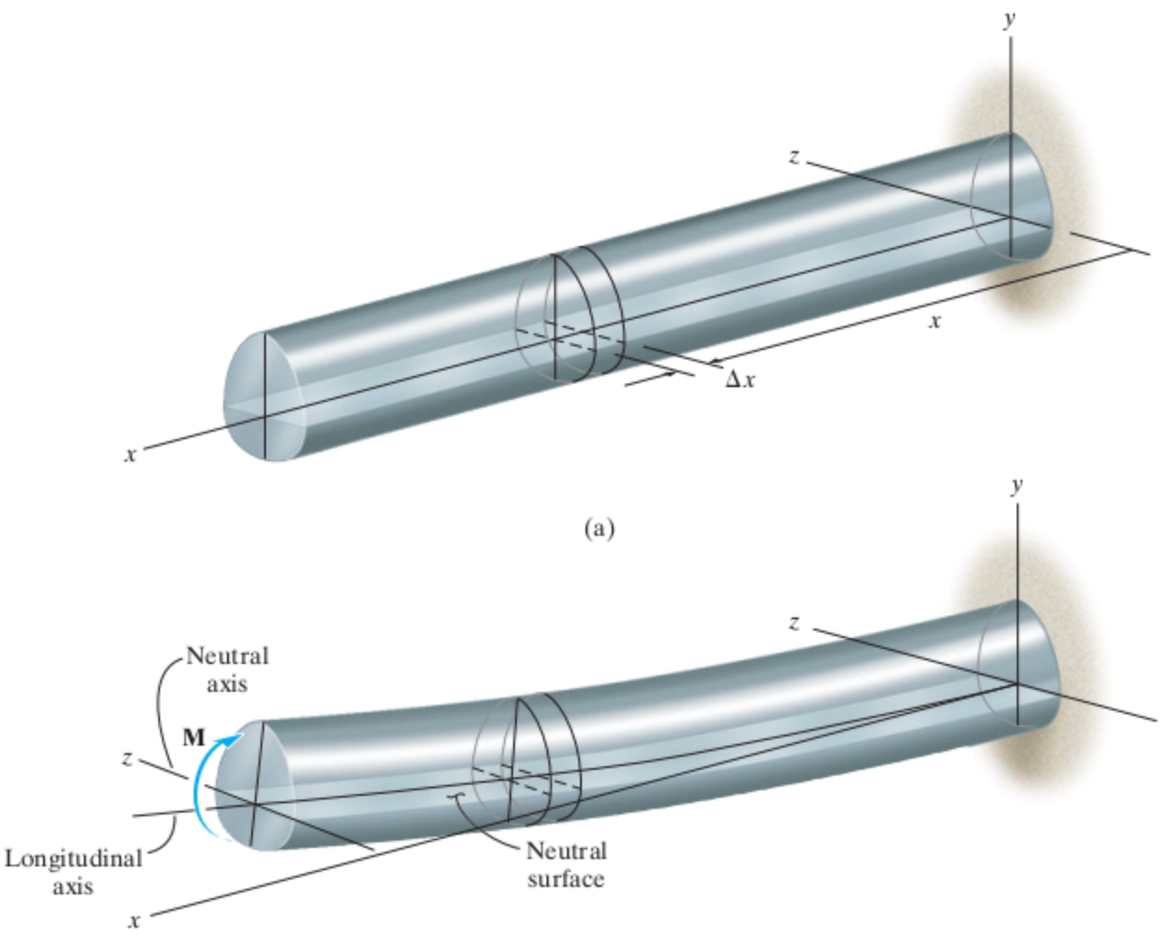


Fig. 11–20

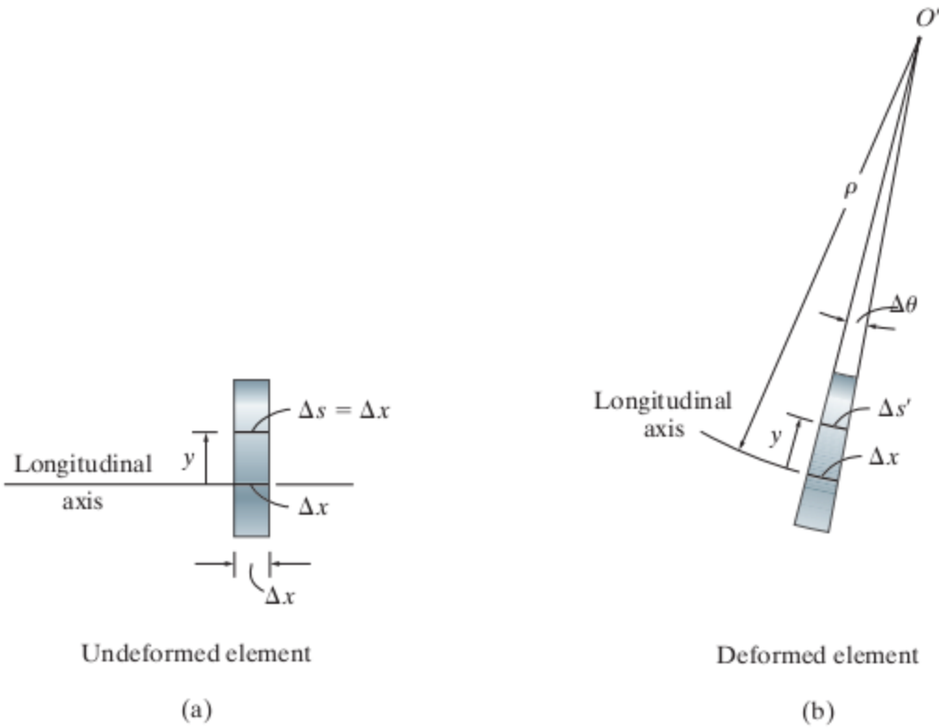


Fig. 11–21

deformed positions in Fig. 11–21. Notice that any line segment Δx, located on the neutral surface, does not change its length, whereas any line segment Δs, located at the arbitrary distance y above the neutral surface, will contract and become $\Delta s'$ after deformation. By definition, the normal strain along Δs is determined from Eq. 7–11, namely,

$$\epsilon = \lim_{\Delta s \to 0} \frac{\Delta s' - \Delta s}{\Delta s}$$

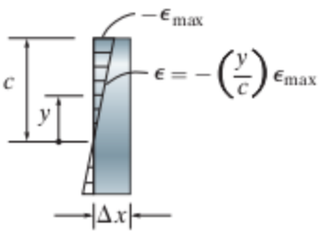
We will now represent this strain in terms of the location y of the segment and the radius of curvature ρ of the longitudinal axis of the element. Before deformation, $\Delta s = \Delta x$, Fig. 11–21*a*. After deformation Δx has a radius of curvature ρ, with center of curvature at point O', Fig. 11–21*b*. Since $\Delta\theta$ defines the angle between the sides of the element, $\Delta x = \Delta s = \rho\Delta\theta$. In the same manner, the deformed length of Δs becomes $\Delta s' = (\rho - y)\Delta\theta$. Substituting into the above equation, we get

$$\epsilon = \lim_{\Delta\theta \to 0} \frac{(\rho - y)\Delta\theta - \rho\Delta\theta}{\rho\Delta\theta}$$

or

$$\epsilon = -\frac{y}{\rho} \tag{11–7}$$

This important result indicates that the longitudinal normal strain of any element within the beam depends on its location y on the cross



Normal strain distribution

Fig. 11–22

section and the radius of curvature of the beam's longitudinal axis at the point. In other words, for any specific cross section, the ***longitudinal normal strain will vary linearly*** with y from the neutral axis. A contraction $(-\epsilon)$ will occur in fibers located above the neutral axis $(+y)$, whereas elongation $(+\epsilon)$ will occur in fibers located below the axis $(-y)$. This variation in strain over the cross section is shown in Fig. 11–22. Here the maximum strain occurs at the outermost fiber, located a distance of $y = c$ from the neutral axis. Using Eq. 11–7, since $\epsilon_{max} = c/\rho$, then by division,

$$\frac{\epsilon}{\epsilon_{max}} = -\left(\frac{y/\rho}{c/\rho}\right)$$

So that

$$\epsilon = -\left(\frac{y}{c}\right)\epsilon_{max} \tag{11–8}$$

This normal strain depends only on the assumptions made with regards to the *deformation*. When a moment is applied to the beam, therefore, it will only cause a *normal stress* in the longitudinal or x direction. All the other components of normal and shear stress will be zero. It is this uniaxial state of stress that causes the material to have the longitudinal normal strain component ϵ_x, defined by Eq. 11–8. Furthermore, by Poisson's ratio, there must *also* be associated strain components $\epsilon_y = -\nu\epsilon_x$ and $\epsilon_z = -\nu\epsilon_x$, which deform the plane of the cross-sectional area, although here we have neglected these deformations. Such deformations will, however, cause the *cross-sectional dimensions* to become smaller below the neutral axis and larger above the neutral axis. For example, if the beam has a square cross section, it will actually deform as shown in Fig. 11–23.

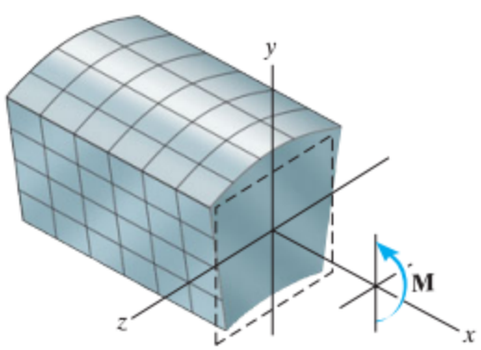


Fig. 11–23

11.4 The Flexure Formula

In this section, we will develop an equation that relates the stress distribution in a beam to the internal resultant bending moment acting on the beam's cross section. To do this we will assume that the material behaves in a linear-elastic manner and therefore a ***linear variation of normal strain***, Fig. 11–24*a*, must then be the result of a ***linear variation in normal stress***, Fig. 11–24*b*. Hence, like the normal strain variation, σ will vary from zero at the member's neutral axis to a maximum value, σ_{max}, a distance c farthest from the neutral axis. Because of the proportionality of triangles, Fig. 11–23*b*, or by using Hooke's law, $\sigma = E\epsilon$, and Eq. 11–8, we can write

$$\sigma = -\left(\frac{y}{c}\right)\sigma_{max} \tag{11–9}$$

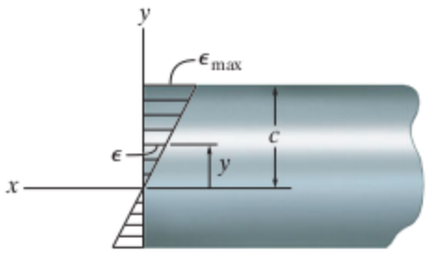
This equation describes the stress distribution over the cross-sectional area. The sign convention established here is significant. For positive **M**, which acts in the $+z$ direction, positive values of y give negative values for σ, that is, a compressive stress since it acts in the negative x direction. Similarly, negative y values will give positive or tensile values for σ. If a volume element of material is selected at a specific point on the cross section, only these tensile or compressive normal stresses will act on it. For example, the element located at $+y$ is shown in Fig. 11–24*c*.

We can locate the position of the neutral axis on the cross section by satisfying the condition that the *resultant force* produced by the stress distribution over the cross-sectional area must be equal to *zero*. Noting that the force $dF = \sigma\, dA$ acts on the arbitrary element dA in Fig. 11–24*c*, we require

$$F_R = \Sigma F_x; \qquad 0 = \int_A dF = \int_A \sigma\, dA$$

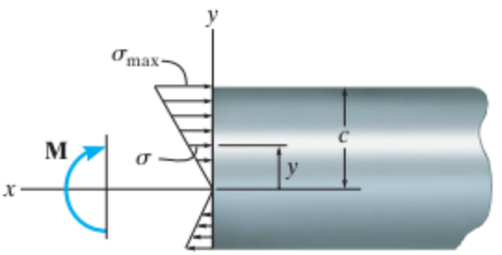
$$= \int_A -\left(\frac{y}{c}\right)\sigma_{max}\, dA$$

$$= \frac{-\sigma_{max}}{c}\int_A y\, dA$$



Normal strain variation (profile view)

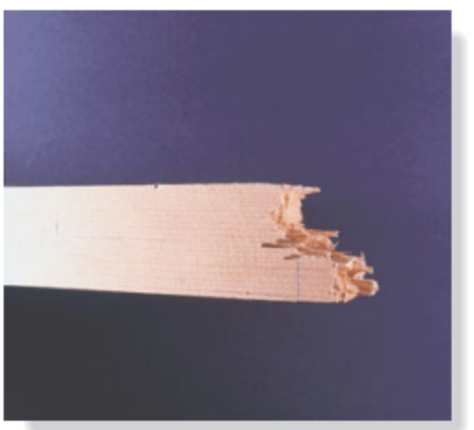
(a)



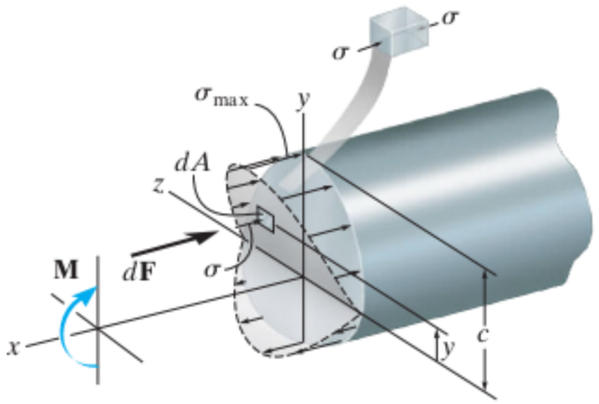
Bending stress variation (profile view)

(b)

Fig. 11–24



This wood specimen failed in bending due to its fibers being crushed at its top and torn apart at its bottom.



Bending stress variation

(c)

Fig. 11–24 (cont.)

Since σ_{max}/c is not equal to zero, then

$$\int_A y\, dA = 0 \tag{11–10}$$

In other words, the first moment of the member's cross-sectional area about the neutral axis must be zero. This condition can only be satisfied if the *neutral axis* is also the horizontal *centroidal axis* for the cross section.* Consequently, once the centroid for the member's cross-sectional area is determined, the location of the neutral axis is known

We can determine the stress in the beam from the requirement that the resultant internal moment M must be equal to the moment produced by the stress distribution about the neutral axis. The moment of $d\mathbf{F}$ in Fig. 11–24*c* about the neutral axis is $dM = y\, dF$. Since $dF = \sigma\, dA$, using Eq. 11–9, we have for the entire cross section,

$$(M_R)_z = \Sigma M_z; \qquad M = \int_A y\, dF = \int_A y(\sigma\, dA) = \int_A y\left(\frac{y}{c}\sigma_{max}\right) dA$$

or

$$M = \frac{\sigma_{max}}{c}\int_A y^2\, dA \tag{11–11}$$

*Recall that the location $\bar{y}$ for the centroid of the cross-sectional area is defined from the equation $\bar{y} = \int y\, dA / \int dA$. If $\int y\, dA = 0$, then $\bar{y} = 0$, and so the centroid lies on the reference (neutral) axis. See Eq. 6–4.

The integral represents the *moment of inertia* of the cross-sectional area about the neutral axis. We will symbolize its value as I. Hence, Eq. 11–11 can be solved for σ_{max} and written as

$$\sigma_{max} = \frac{Mc}{I} \tag{11–12}$$

Here

σ_{max} = maximum normal stress in the member, which occurs at a point on the cross-sectional area *farthest away* from the neutral axis

M = resultant internal moment, determined from the method of sections and the equations of equilibrium, and calculated about the neutral axis of the cross section

c = perpendicular distance from the neutral axis to a point farthest away from the neutral axis. This is where σ_{max} acts

I = moment of inertia of the cross-sectional area about the neutral axis

Since $\sigma_{max}/c = -\sigma/y$, Eq. 11–9, the normal stress at the location distance y can be determined from an equation similar to Eq. 11–12. We have

$$\sigma = -\frac{My}{I} \tag{11–13}$$

Note that the negative sign is necessary since it agrees with the established x, y, z axes. By the right-hand rule, M is positive along the $+z$ axis, y is positive upward, and σ therefore must be negative (compressive) since it acts in the negative x direction, Fig. 11–24*c*.

Either of the above two equations is often referred to as the ***flexure formula***. It is used to determine the normal stress in a straight member, having a cross section that is symmetrical with respect to an axis, and the moment is applied perpendicular to this axis. Although we have assumed that the member is prismatic, we can in most cases of engineering design also use the flexure formula to determine the normal stress in members that have a *slight taper*. For example, using a mathematical analysis based on the theory of elasticity, a member having a rectangular cross section and a length that is tapered 15° will have an actual maximum normal stress that is about 5.4% *less* than that calculated using the flexure formula.

Important Points

- The cross section of a straight beam *remains plane* when the beam deforms due to bending. This causes tensile stress on one portion of the cross section and compressive stress on the other portion. In between these portions, there exists the *neutral axis* which is subjected to *zero stress*.
- Due to the deformation, the *longitudinal strain* varies *linearly* from zero at the neutral axis to a maximum at the outer fibers of the beam. Provided the material is homogeneous and linear elastic, then the *stress* also varies in a *linear* fashion over the cross section.
- The neutral axis passes through the *centroid* of the cross-sectional area. This result is based on the fact that the resultant normal force acting on the cross section must be zero.
- The flexure formula is based on the requirement that the resultant internal moment on the cross section is equal to the moment produced by the normal stress distribution about the neutral axis.

Procedure for Analysis

In order to apply the flexure formula, the following procedure is suggested.

Internal Moment.

- Section the member at the point where the bending or normal stress is to be determined, and obtain the internal moment M at the section. The centroidal or neutral axis for the cross section must be known, since M *must* be calculated about this axis.
- If the absolute maximum bending stress is to be determined, then draw the moment diagram in order to determine the maximum moment in the member.

Section Property.

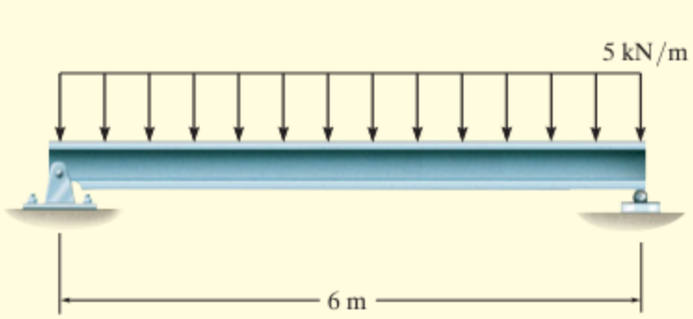
- Determine the moment of inertia of the cross-sectional area about the neutral axis. Methods used for its calculation as well as a table listing values of I for several common shapes can be found in Appendix B.

Normal Stress.

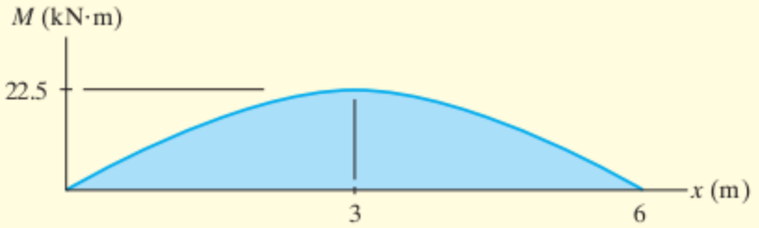
- Specify the location y, measured perpendicular to the neutral axis to the point where the normal stress is to be determined. Then apply the equation $\sigma = -My/I$, or if the maximum bending stress is to be calculated, use $\sigma_{\max} = Mc/I$. When substituting the data, make sure the units are consistent.
- The stress acts in a direction such that the force it creates at the point contributes a moment about the neutral axis that is in the same direction as the internal moment **M**, Fig. 11–24*c*. In this manner the stress distribution acting over the entire cross section can be sketched, or a volume element of the material can be isolated and used to graphically represent the normal stress acting at the point.

EXAMPLE 11.11

The simply supported beam in Fig. 11–25*a* has the cross-sectional area shown in Fig. 11–25*b*. Determine the absolute maximum bending stress in the beam and draw the stress distribution over the cross section at this location.



(a)



(c)

SOLUTION

Maximum Internal Moment. The maximum internal moment in the beam, $M = 22.5\ \text{kN} \cdot \text{m}$, occurs at the center.

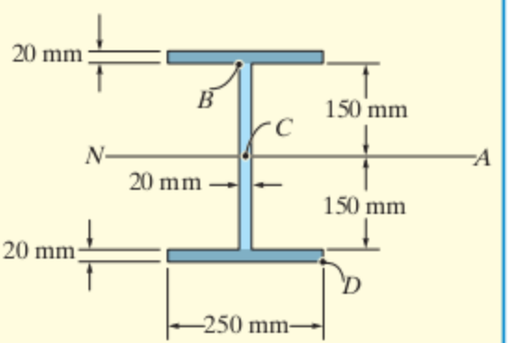
Section Property. By reasons of symmetry, the neutral axis passes through the centroid C at the midheight of the beam, Fig. 11–25*b*. The area is subdivided into the three parts shown, and the moment of inertia of each part is calculated about the neutral axis using the parallel-axis theorem. (See Eq. 6–12.) Choosing to work in meters, we have

$$\begin{aligned} I &= \Sigma(\bar{I} + Ad^2) \\ &= 2\left[\frac{1}{12}(0.25\ \text{m})(0.020\ \text{m})^3 + (0.25\ \text{m})(0.020\ \text{m})(0.160\ \text{m})^2\right] \\ &\quad + \left[\frac{1}{12}(0.020\ \text{m})(0.300\ \text{m})^3\right] \\ &= 301.3(10^{-6})\ \text{m}^4 \end{aligned}$$

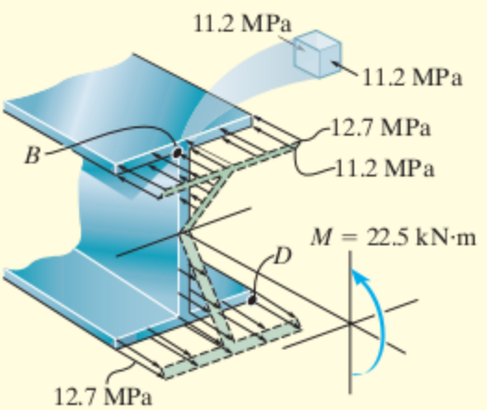
$$\sigma_{\max} = \frac{Mc}{I};\quad \sigma_{\max} = \frac{22.5(10^3)\ \text{N} \cdot \text{m}(0.170\ \text{m})}{301.3(10^{-6})\ \text{m}^4} = 12.7\ \text{MPa} \quad \textit{Ans.}$$

A three-dimensional view of the stress distribution is shown in Fig. 11–25*d*. Notice how the stress at points B and D on the cross section develops a force that contributes a moment about the neutral axis that has the same direction as **M**. Specifically, at point B, $y_B = 150\ \text{mm}$, and so as shown in Fig. 6–26*d*,

$$\sigma_B = -\frac{My_B}{I};\qquad \sigma_B = -\frac{22.5(10^3)\ \text{N} \cdot \text{m}(0.150\ \text{m})}{301.3(10^{-6})\ \text{m}^4} = -11.2\ \text{MPa}$$



(b)



(d)

Fig. 11–25

EXAMPLE 11.12

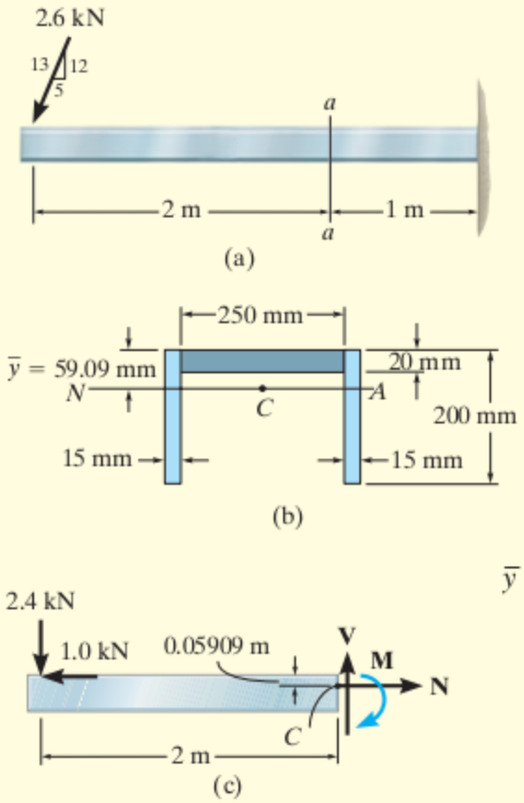


Fig. 11–26

The beam shown in Fig. 11–26*a* has a cross-sectional area in the shape of a channel, Fig. 11–26*b*. Determine the maximum bending stress that occurs in the beam at section *a–a*.

SOLUTION

Internal Moment. Here the beam's support reactions do not have to be determined. Instead, by the method of sections, the segment to the left of section *a–a* can be used, Fig. 11–26*c*. In particular, note that the resultant internal axial force **N** passes through the centroid of the cross section. Also, realize that *the resultant internal moment must be calculated about the beam's neutral axis* at section *a–a*.

To find the location of the neutral axis, the cross-sectional area is subdivided into three composite parts as shown in Fig. 11–26*b*. Using Eq. 6–5 for areas, we have

$$\bar{y} = \frac{\Sigma \tilde{y} A}{\Sigma A} = \frac{2[0.100\text{ m}](0.200\text{ m})(0.015\text{ m}) + [0.010\text{ m}](0.02\text{ m})(0.250\text{ m})}{2(0.200\text{ m})(0.015\text{ m}) + 0.020\text{ m}(0.250\text{ m})}$$
$$= 0.05909\text{ m} = 59.09\text{ mm}$$

This dimension is shown in Fig. 11–26*c*.

Applying the moment equation of equilibrium about the neutral axis, we have

$$\zeta + \Sigma M_{NA} = 0; \quad 2.4\text{ kN}(2\text{ m}) + 1.0\text{ kN}(0.05909\text{ m}) - M = 0$$
$$M = 4.859\text{ kN}\cdot\text{m}$$

Section Property. The moment of inertia about the neutral axis is determined using $I = \Sigma\,(\bar{I} + Ad^2)$ applied to each of the three composite parts of the cross-sectional area. Working in meters, we have

$$I = \left[\frac{1}{12}(0.250\text{ m})(0.020\text{ m})^3 + (0.250\text{ m})(0.020\text{ m})(0.05909\text{ m} - 0.010\text{ m})^2\right]$$
$$+ 2\left[\frac{1}{12}(0.015\text{ m})(0.200\text{ m})^3 + (0.015\text{ m})(0.200\text{ m})(0.100\text{ m} - 0.05909\text{ m})^2\right]$$
$$= 42.26(10^{-6})\text{ m}^4$$

Maximum Bending Stress. The maximum bending stress occurs at points farthest away from the neutral axis. This is at the bottom of the beam, $c = 0.200\text{ m} - 0.05909\text{ m} = 0.1409\text{ m}$. Thus,

$$\sigma_{\max} = \frac{Mc}{I} = \frac{4.859(10^3)\text{ N}\cdot\text{m}(0.1409\text{ m})}{42.26(10^{-6})\text{ m}^4} = 16.2\text{ MPa} \qquad \textit{Ans.}$$

Show that at the top of the beam the bending stress is $\sigma' = 6.79$ MPa.

NOTE: The normal force of $N = 1$ kN and shear force $V = 2.4$ kN will also contribute additional stress on the cross section. The superposition of all these effects will be discussed in Chapter 13.

EXAMPLE 11.13

The member having a rectangular cross section, Fig. 11–27*a*, is designed to resist a moment of 40 N · m. In order to increase its strength and rigidity, it is proposed that two small ribs be added at its bottom, Fig. 11–27*b*. Determine the maximum normal stress in the member for both cases.

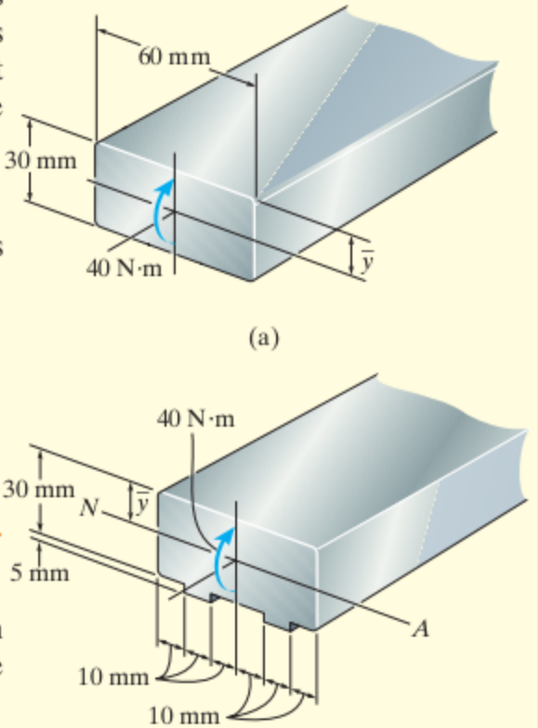


Fig. 11–27

SOLUTION

Without Ribs. Clearly the neutral axis is at the center of the cross section, Fig. 11–27*a*, so $\bar{y} = c = 15 \text{ mm} = 0.015 \text{ m}$. Thus,

$$I = \frac{1}{12}bh^3 = \frac{1}{12}(0.060 \text{ m})(0.030 \text{ m})^3 = 0.135(10^{-6}) \text{ m}^4$$

Therefore the maximum normal stress is

$$\sigma_{\max} = \frac{Mc}{I} = \frac{(40 \text{ N} \cdot \text{m})(0.015 \text{ m})}{0.135(10^{-6}) \text{ m}^4} = 4.44 \text{ MPa} \qquad \textit{Ans.}$$

With Ribs. From Fig. 11–27*b*, segmenting the area into the large main rectangle and the bottom two rectangles (ribs), the location $\bar{y}$ of the centroid and the neutral axis is determined as follows:

$$\bar{y} = \frac{\Sigma \bar{y} A}{\Sigma A}$$

$$= \frac{[0.015 \text{ m}](0.030 \text{ m})(0.060 \text{ m}) + 2[0.0325 \text{ m}](0.005 \text{ m})(0.010 \text{ m})}{(0.03 \text{ m})(0.060 \text{ m}) + 2(0.005 \text{ m})(0.010 \text{ m})}$$

$$= 0.01592 \text{ m}$$

This value does not represent *c*. Instead

$$c = 0.035 \text{ m} - 0.01592 \text{ m} = 0.01908 \text{ m}$$

Using the parallel-axis theorem, the moment of inertia about the neutral axis is

$$I = \left[\frac{1}{12}(0.060 \text{ m})(0.030 \text{ m})^3 + (0.060 \text{ m})(0.030 \text{ m})(0.01592 \text{ m} - 0.015 \text{ m})^2\right]$$
$$+ 2\left[\frac{1}{12}(0.010 \text{ m})(0.005 \text{ m})^3 + (0.010 \text{ m})(0.005 \text{ m})(0.0325 \text{ m} - 0.01592 \text{ m})^2\right]$$
$$= 0.1642(10^{-6}) \text{ m}^4$$

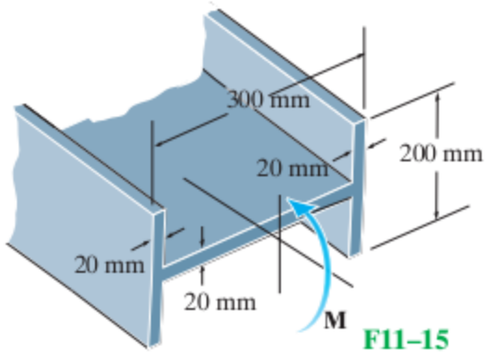
Therefore, the maximum normal stress is

$$\sigma_{\max} = \frac{Mc}{I} = \frac{40 \text{ N} \cdot \text{m}(0.01908 \text{ m})}{0.1642(10^{-6}) \text{ m}^4} = 4.65 \text{ MPa} \qquad \textit{Ans.}$$

NOTE: This surprising result indicates that the addition of the ribs to the cross section will *increase* the maximum normal stress rather than decrease it, and for this reason they should be omitted.

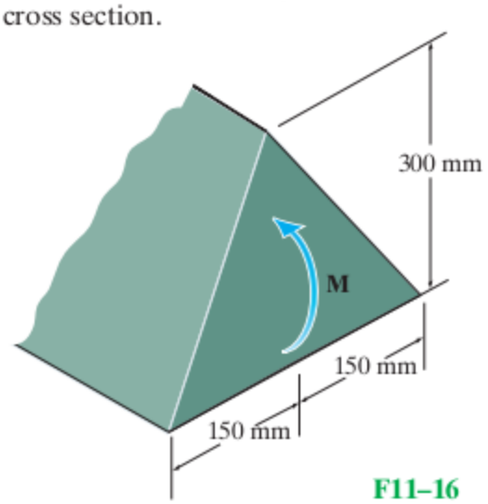
FUNDAMENTAL PROBLEMS

F11–15. If the beam is subjected to a bending moment of $M = 20\ \text{kN} \cdot \text{m}$, determine the maximum bending stress in the beam.



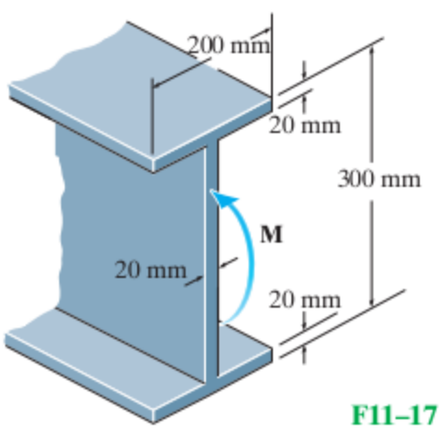
F11–15

F11–16. If the beam is subjected to a bending moment of $M = 50\ \text{kN} \cdot \text{m}$, sketch the bending stress distribution over the beam's cross section.



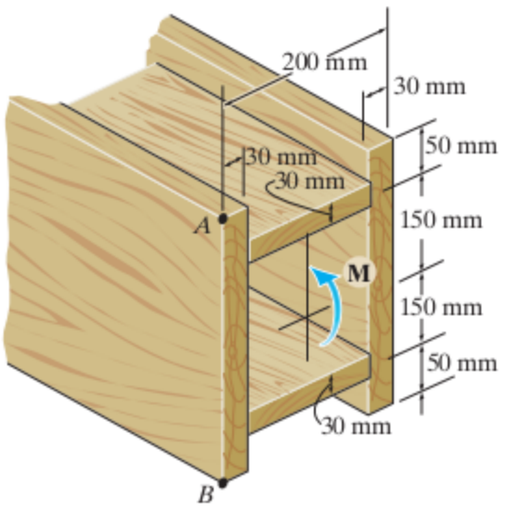
F11–16

F11–17. If the beam is subjected to a bending moment of $M = 50\ \text{kN} \cdot \text{m}$, determine the maximum bending stress in the beam.



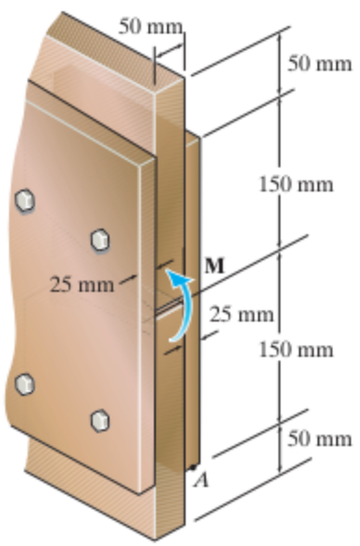
F11–17

F11–18. If the beam is subjected to a bending moment of $M = 10\ \text{kN} \cdot \text{m}$, determine the bending stress in the beam at points A and B, and sketch the results on a differential element at each of these points.



F11–18

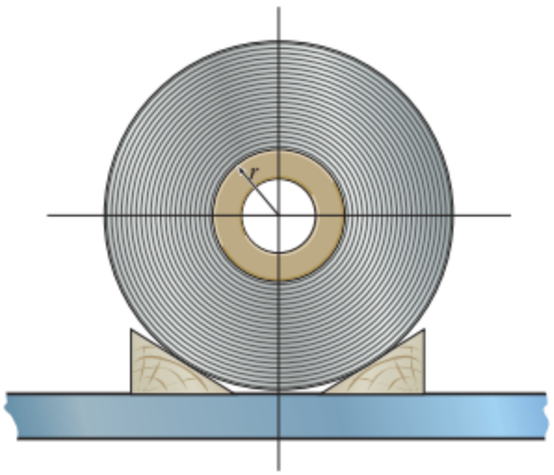
F11–19. If the beam is subjected to a bending moment of $M = 5\ \text{kN} \cdot \text{m}$, determine the bending stress developed at point A and sketch the result on a differential element at the point.



F11–19

PROBLEMS

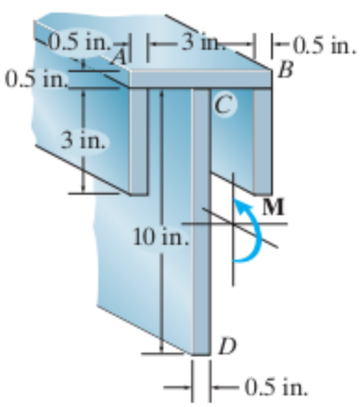
11–35. If the A-36 steel sheet roll is supported as shown and the allowable bending stress is 165 MPa, determine the smallest radius r of the spool if the steel sheet has a width of 1 m and a thickness of 1.5 mm. Also, find the corresponding maximum internal moment developed in the sheet.



Prob. 11–35

*11–36. Determine the moment M that will produce a maximum stress of 10 ksi on the cross section.

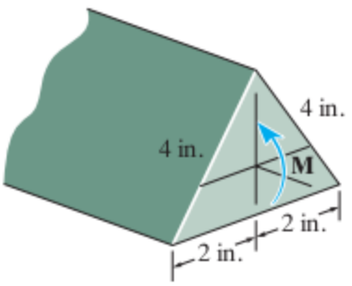
11–37. Determine the maximum tensile and compressive bending stress in the beam if it is subjected to a moment of $M = 4$ kip · ft.



Probs. 11–36/37

11–38. A member has the triangular cross section shown. Determine the largest internal moment M that can be applied to the cross section without exceeding allowable tensile and compressive stresses of $(\sigma_{allow})_t = 22$ ksi and $(\sigma_{allow})_c = 15$ ksi, respectively.

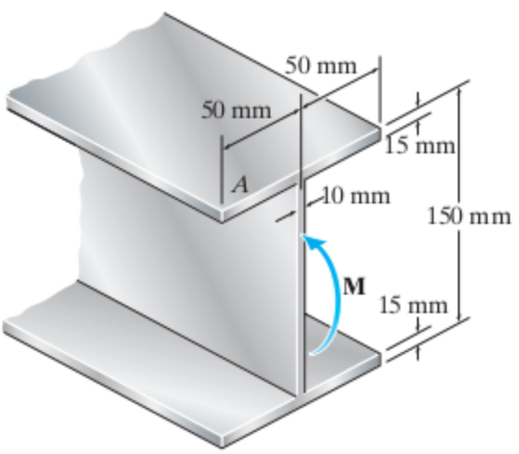
11–39. A member has the triangular cross section shown. If a moment of $M = 800$ lb · ft is applied to the cross section, determine the maximum tensile and compressive bending stresses in the member. Also, sketch a three-dimensional view of the stress distribution action over the cross section.



Probs. 11–38/39

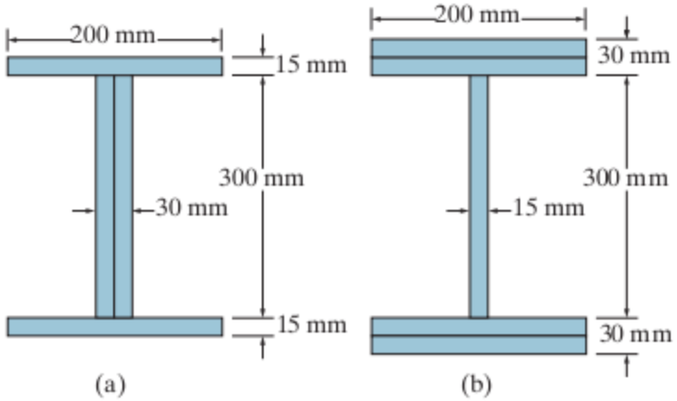
*11–40. If the beam is subjected to an internal moment of $M = 30$ kN · m, determine the maximum bending stress in the beam. The beam is made from A992 steel. Sketch the bending stress distribution on the cross section.

11–41. If the beam is subjected to an internal moment of $M = 30$ kN · m, determine the resultant force caused by the bending stress distribution acting on the top flange A.



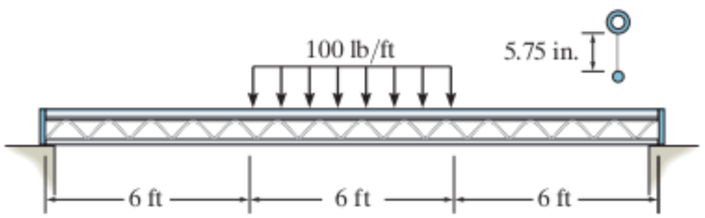
Probs. 11–40/41

11–42. Two designs for a beam are to be considered. Determine which one will support a moment of $M = 150\ \text{kN}\cdot\text{m}$ with the least amount of bending stress. What is that stress?



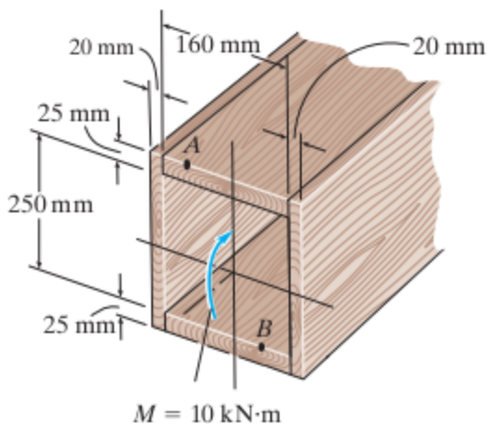
Prob. 11–42

11–43. The simply supported truss is subjected to the central distributed load. Neglect the effect of the diagonal lacing and determine the absolute maximum bending stress in the truss. The top member is a pipe having an outer diameter of 1 in. and thickness of $\frac{3}{16}$ in., and the bottom member is a solid rod having a diameter of $\frac{1}{2}$ in.



Prob. 11–43

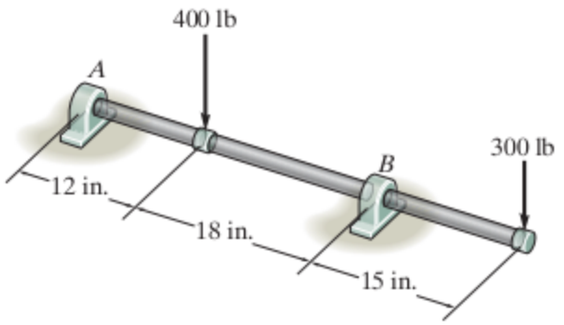
***11–44.** A box beam is constructed from four pieces of wood, glued together as shown. If the moment acting on the cross section is $10\ \text{kN}\cdot\text{m}$, determine the stress at points A and B and show the results acting on volume elements located at these points.



Prob. 11–44

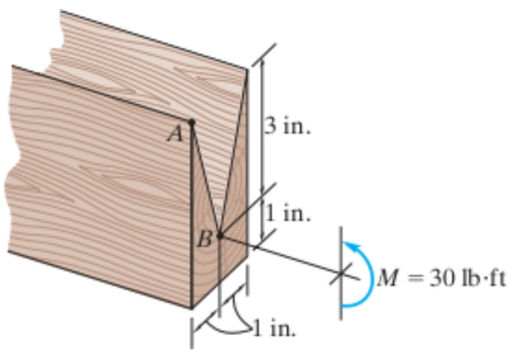
11–45. Determine the absolute maximum bending stress in the 1.5-in.-diameter shaft which is subjected to the concentrated forces. The sleeve bearings at A and B support only vertical forces.

11–46. Determine the smallest allowable diameter of the shaft which is subjected to the concentrated forces. The sleeve bearings at A and B support only vertical forces, and the allowable bending stress is $\sigma_{\text{allow}} = 22$ ksi.



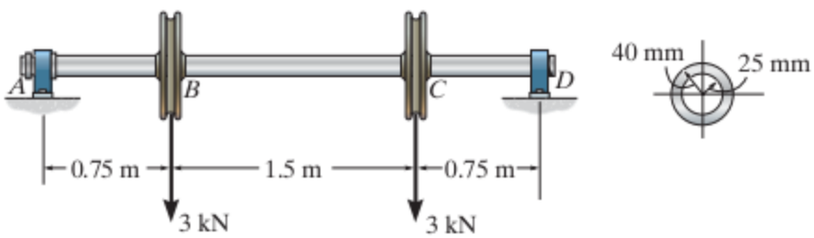
Probs. 11–45/46

11–47. The beam is subjected to a moment of $M = 30\ \text{lb}\cdot\text{ft}$. Determine the bending stress acting at point A and B. Also, stetch a three-dimensional view of the stress distribution acting over the entire cross-sectional area.



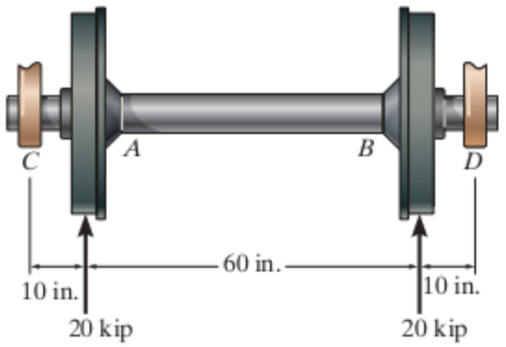
Prob. 11–47

***11–48.** The shaft is supported by a smooth thrust bearing at A and smooth journal bearing at D. If the shaft has the cross section shown, determine the absolute maximum bending stress in the shaft.



Prob. 11–48

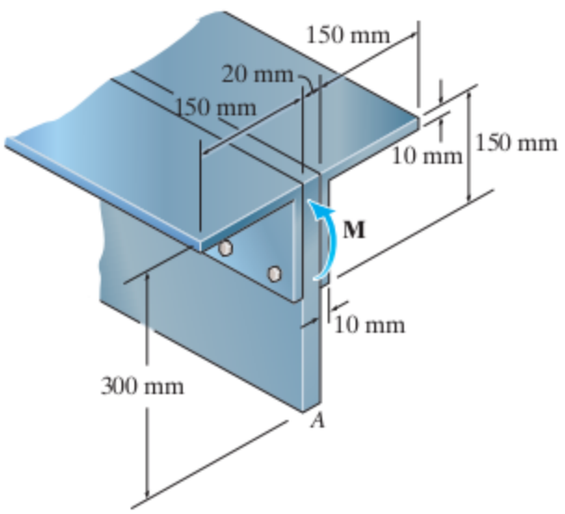
11–49. The axle of the freight car is subjected to wheel loadings of 20 kip. If it is supported by two journal bearings at C and D, determine the maximum bending stress developed at the center of the axle, where the diameter is 5.5 in.



Prob. 11–49

11–50. If the built-up beam is subjected to an internal moment of $M = 75\ \text{kN}\cdot\text{m}$, determine the maximum tensile and compressive stress acting in the beam.

11–51. If the built-up beam is subjected to an internal moment of $M = 75\ \text{kN}\cdot\text{m}$, determine the amount of this internal moment resisted by plate A.



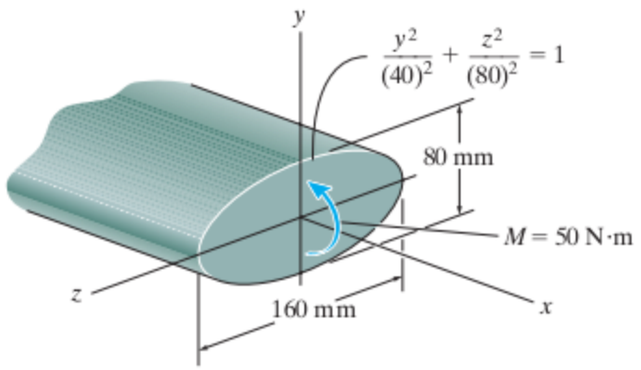
Probs. 11–50/51

***11–52.** If the compound beam in Prob. 11–7 has a square cross section of side length a, determine the minimum value of a if the allowable bending stress is $\sigma_{\text{allow}} = 150\ \text{MPa}$.

11–53. If the crane boom ABC in Prob. 11–3 has a rectangular cross section with a base of 2.5 in., determine its required height h to the nearest $\frac{1}{4}$ in. if the allowable bending stress is $\sigma_{\text{allow}} = 24\ \text{ksi}$.

11–54. A shaft is made of a polymer having an elliptical cross section. If it resists an internal moment of $M = 50\ \text{N}\cdot\text{m}$, determine the maximum bending stress developed in the material (a) using the flexure formula, where $I_z = \frac{1}{4}\pi\,(0.08\ \text{m})(0.0\ \text{m})^3$, (b) using integration. Sketch a three-dimensional view of the stress distribution acting over the cross-sectional area.

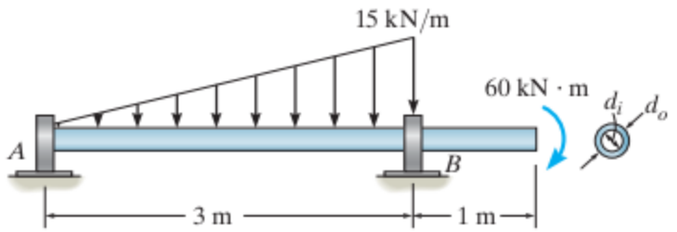
11–55. Solve Prob. 11–54 if the moment $M = 50\ \text{N}\cdot\text{m}$ is applied about the y axis instead of the x axis. Here $I_y = \frac{1}{4}\pi\,(0.04\ \text{m})(0.08\ \text{m})^3$.



Probs. 11–54/55

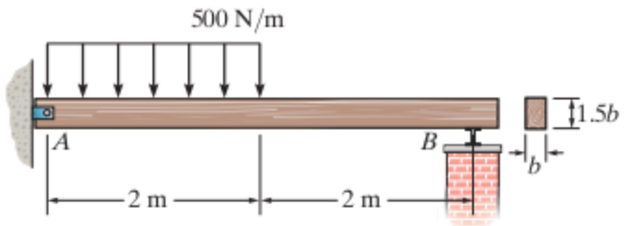
***11–56.** Determine the absolute maximum bending stress in the tubular shaft if $d_i = 160\ \text{mm}$ and $d_o = 200\ \text{mm}$.

11–57. The tubular shaft is to have a cross section such that its inner diameter and outer diameter are related by $d_i = 0.8d_o$. Determine these required dimensions if the allowable bending stress is $\sigma_{\text{allow}} = 155\ \text{MPa}$.



Probs. 11–56/57

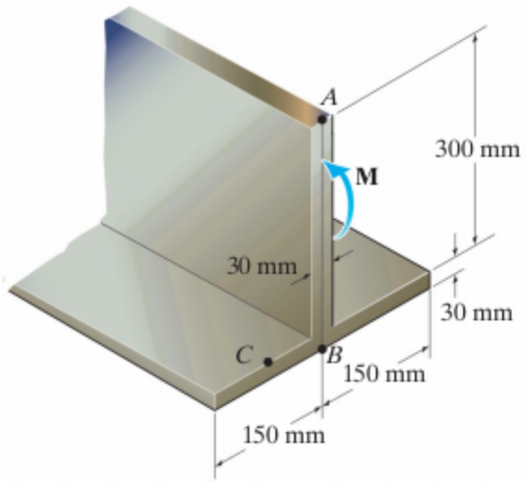
11–58. The wood beam has a rectangular cross section in the proportion shown. Determine its required dimension b if the allowable bending stress is $\sigma_{\text{allow}} = 10\ \text{MPa}$.



Prob. 11–58

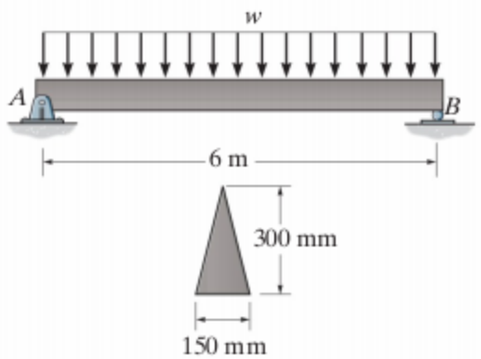
11–59. If the beam is subjected to an internal moment of $M = 100\,\text{kN}\cdot\text{m}$, determine the bending stress developed at points A, B, and C. Sketch the bending stress distribution on the cross section.

***11–60.** If the beam is made of material having an allowable tensile and compressive stress of $(\sigma_{\text{allow}})_t = 125$ MPa and $(\sigma_{\text{allow}})_c = 150$ MPa, respectively, determine the maximum allowable internal moment **M** that can be applied to the beam.



Probs. 11–59/60

11–61. If the material of the beam has an allowable bending stress of $\sigma_{\text{allow}} = 150$ MPa, determine the maximum allowable intensity w_0 of the uniform distributed load.



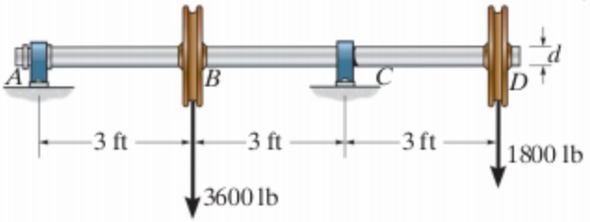
Prob. 11–61

11–62. If the compound beam in Prob. 11–24 has a square cross section, determine its dimension a if the allowable bending stress is $\sigma_{\text{allow}} = 150$ MPa.

11–63. If the beam in Prob. 11–22 has a rectangular cross section with a width b and a height h, determine the absolute maximum bending stress in the beam.

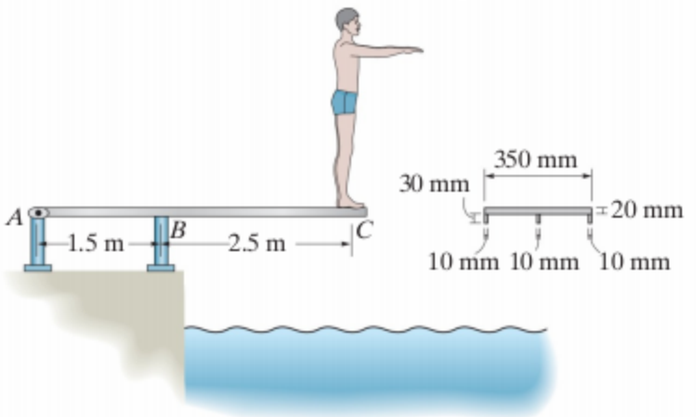
***11–64.** The shaft is supported by a smooth thrust bearing at A and smooth journal bearing at C. If $d = 3$ in., determine the absolute maximum bending stress in the shaft.

11–65. The shaft is supported by a smooth thrust bearing at A and smooth journal bearing at C. If the material has an allowable bending stress of $\sigma_{\text{allow}} = 24$ ksi, determine the required minimum diameter d of the shaft to the nearest $\frac{1}{16}$ in.



Probs. 11–64/65

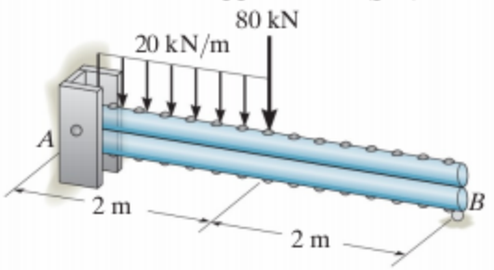
11–66. The man has a mass of 78 kg and stands motionless at the end of the diving board. If the board has the cross section shown, determine the maximum normal strain developed in the board. The modulus of elasticity for the material is $E = 125$ GPa. Assume A is a pin and B is a roller.



Prob. 11–66

11–67. The two solid steel rods are bolted together along their length and support the loading shown. Assume the support at A is a pin and B is a roller. Determine the required diameter d of each of the rods if the allowable bending stress is $\sigma_{\text{allow}} = 130$ MPa.

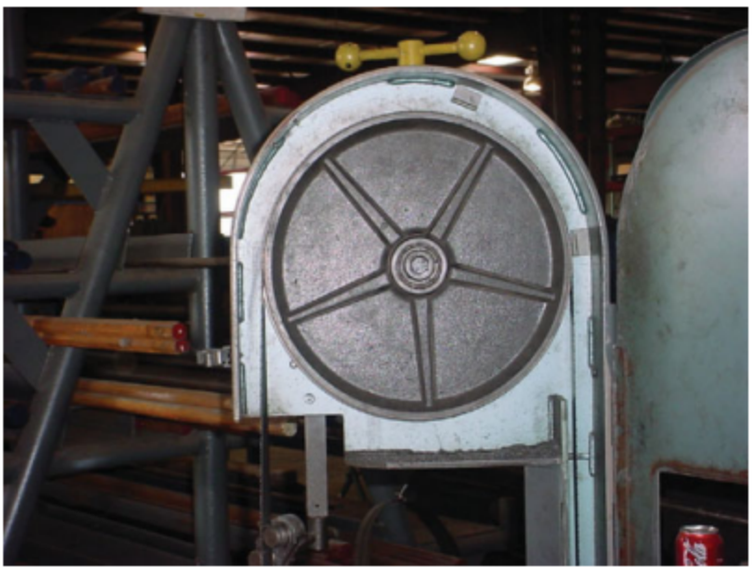
***11–68.** Solve Prob. 11–67 if the rods are rotated 90° so that both rods rest on the supports at A (pin) and B (roller).



Probs. 11–67/68

CONCEPTUAL PROBLEMS

P11–1. The steel saw blade passes over the drive wheel of the band saw. Using appropriate measurements and data, explain how to determine the bending stress in the blade.



P11–1

P11–2. This crane boom on a ship has a moment of inertia that varies along its length. Draw the moment diagram for the boom to explain why the boom tapers as shown.

P11–2

P11–3. Hurricane winds caused failure of this highway sign by bending the supporting pipes at their connections with the column. Assuming the pipes are made of A-36 steel, use reasonable dimensions for the sign and pipes, and try and estimate the smallest uniform wind pressure acting on the face of the sign that caused the pipes to yield.

P11–3

(a)

(b)

P11–4

P11–4. These garden shears were manufactured using an inferior material. Using a loading of 50 lb applied normal to the blades, and appropriate dimensions for the shears, determine the absolute maximum bending stress in the material and show why the failure occurred at the critical location on the handle.

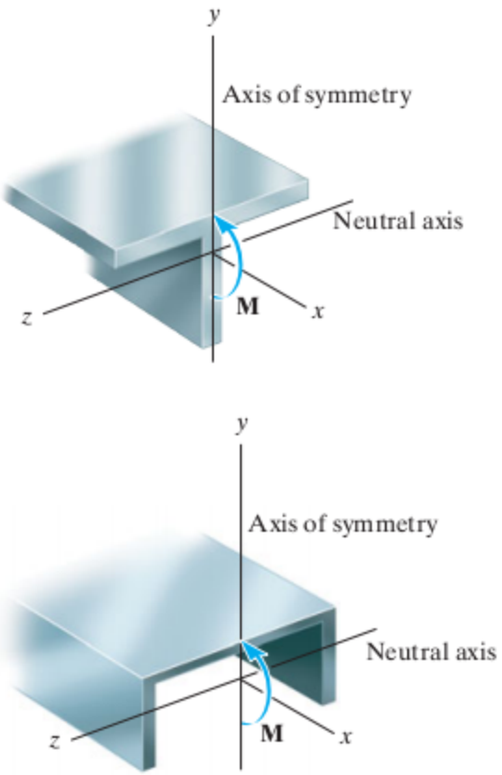


Fig. 11–28

11.5 Unsymmetric Bending

When developing the flexure formula we imposed a condition that the cross-sectional area be *symmetric* about an axis perpendicular to the neutral axis; furthermore, the resultant internal moment **M** acts along the neutral axis. Such is the case for the "T" or channel sections shown in Fig. 11–28. These conditions, however, are unnecessary, and in this section we will show that the flexure formula can also be applied either to a beam having a cross-sectional area of any shape or to a beam having a resultant internal moment that acts in any direction.

Moment Applied About Principal Axis. Consider the beam's cross section to have the unsymmetrical shape shown in Fig. 11–29*a*. As in Sec. 11.4, the right-handed x, y, z coordinate system is established such that the origin is located at the centroid C on the cross section, and the resultant internal moment **M** acts along the $+z$ axis. We require the stress distribution acting over the entire cross-sectional area to have a zero force resultant, the resultant internal moment about the y axis to be zero, and the resultant internal moment about the z axis to equal **M**.* These three conditions can be expressed mathematically by considering the force acting on the differential element dA located at $(0, y, z)$, Fig. 11–29*a*. This force is $dF = \sigma\, dA$, and therefore we have

$$F_R = \Sigma F_x; \qquad 0 = -\int_A \sigma\, dA \tag{11–14}$$

$$(M_R)_y = \Sigma M_y; \qquad 0 = -\int_A z\sigma\, dA \tag{11–15}$$

$$(M_R)_z = \Sigma M_z; \qquad M = \int_A y\sigma\, dA \tag{11–16}$$

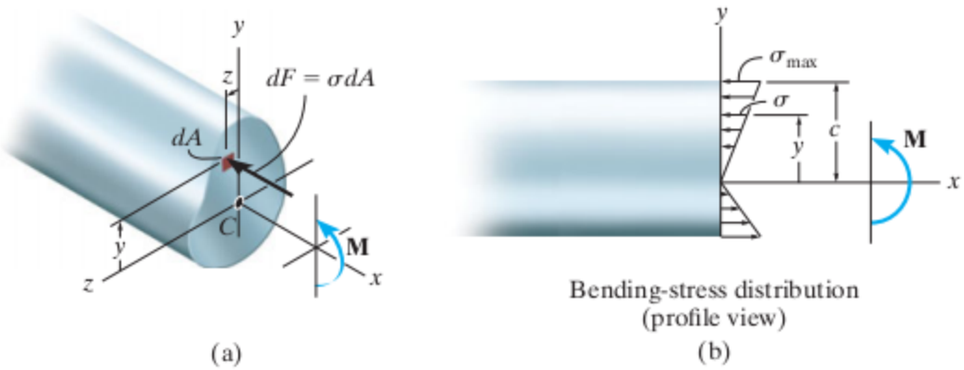


Fig. 11–29

*The condition that moments about the y axis be equal to zero was not considered in Sec. 11.4, since the bending-stress distribution was *symmetric* with respect to the y axis, and such a distribution of stress automatically produces zero moment about the y axis See Fig. 11–24*c*.

As shown in Sec. 11.4, Eq. 11–14 is satisfied since the z axis passes through the *centroid* of the area. Also, since the z axis represents the *neutral axis* for the cross section, the normal stress will vary linearly from zero at the neutral axis, to a maximum at $|y| = c$, Fig. 11–29*b*. Hence the stress distribution is defined by $\sigma = -(y/c)\sigma_{max}$. When this equation is substituted into Eq. 11–16 and integrated, it leads to the flexure formula $\sigma_{\max} = Mc/I$. When it is substituted into Eq. 11–15, we get

$$0 = \frac{-\sigma_{\max}}{c}\int_A yz\, dA$$

which requires

$$\int_A yz\, dA = 0$$

This integral is called the ***product of inertia*** for the area. As indicated in Appendix A, it will indeed be zero provided the y and z axes are chosen as ***principal axes of inertia*** for the area. For an arbitrarily shaped area, the orientation of the principal axes can always be determined, using either the inertia transformation equations or Mohr's circle of inertia. If the area has an axis of symmetry, however, the ***principal axes*** can easily be established ***since they will always be oriented along the axis of symmetry and perpendicular to it.***

For example, consider the members shown in Fig. 11–30. In each of these cases, y and z must define the principal axes of inertia for the cross section in order to satisfy Eqs. 11–14 through 11–16. In Fig. 11–30*a* the principal axes are located by symmetry, and in Figs. 11–30*b* and 11–30*c* their orientation is determined using the methods of Appendix A. Since **M** is applied about one of the principal axes (z axis), the stress distribution is determined from the flexure formula, $\sigma = -My/I_z$, and is shown for each case.

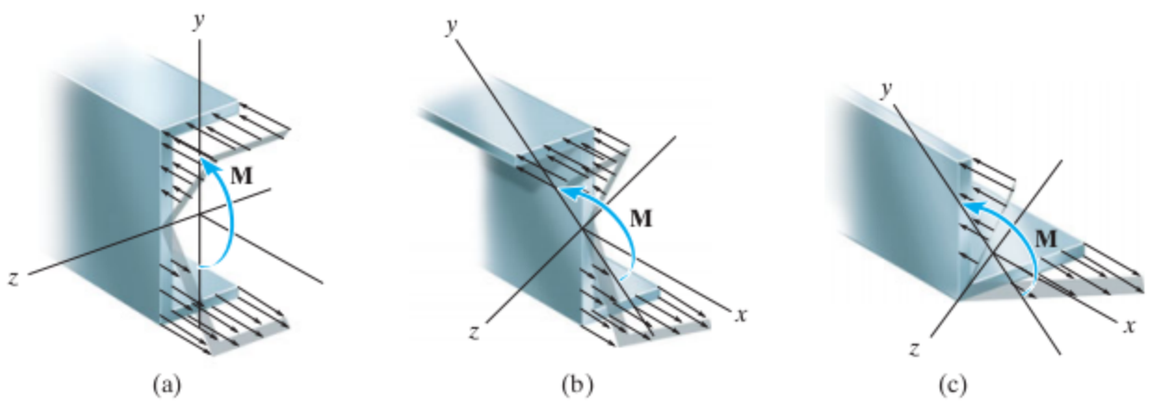


Fig. 11–30

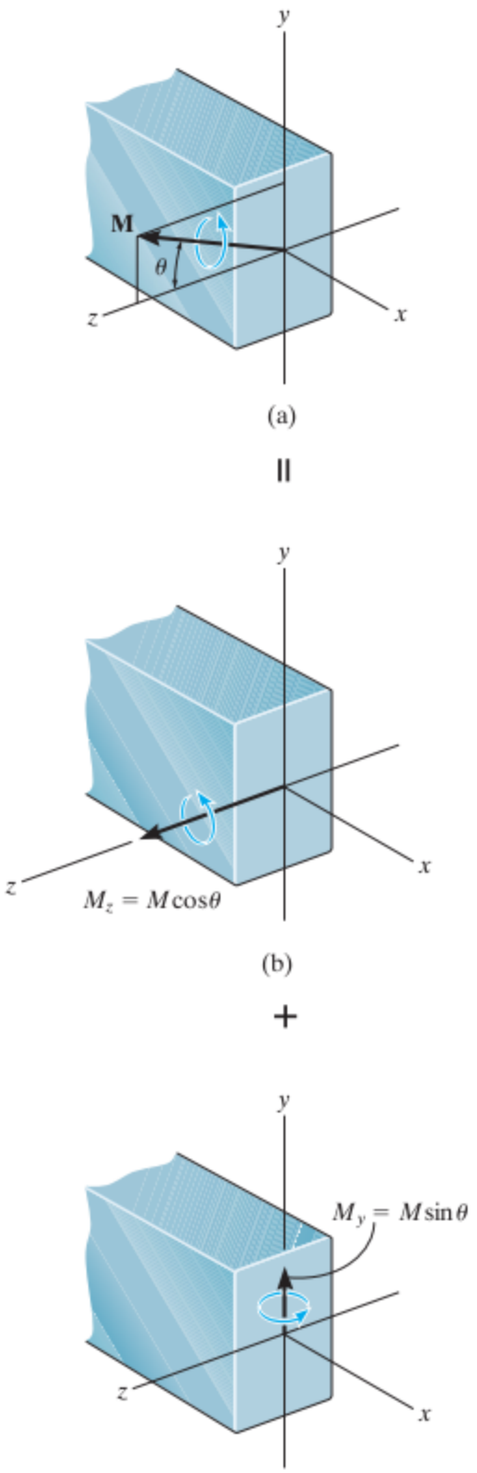


Fig. 11–31

Moment Arbitrarily Applied. Sometimes a member may be loaded such that **M** does not act about one of the principal axes of the cross section. When this occurs, the moment should first be resolved into components directed along the principal axes, then the flexure formula can be used to determine the normal stress caused by *each* moment component. Finally, using the principle of superposition, the resultant normal stress at the point can be determined.

To show this, consider the beam to have a rectangular cross section and to be subjected to the moment **M**, Fig. 11–31*a*. Here **M** makes an angle θ with the maximum *principal* z axis, that is, the axis of maximum moment of inertia for the cross section. We will assume θ is positive when it is directed from the $+z$ axis toward the $+y$ axis, the minimum principal axis, as shown. Resolving **M** into components along the z and y axes, we have $M_z = M\cos\theta$ and $M_y = M\sin\theta$, as shown in Figs. 11–31*b* and 11–31*c*. The normal stress distributions that produce **M** and its components $\mathbf{M}_z$ and $\mathbf{M}_y$ are shown in Figs. 11–31*d*, 11–31*e*, and 11–31*f*, where it is assumed that $(\sigma_x)_{\max} > (\sigma'_x)_{\max}$. By inspection, the maximum tensile and compressive stresses $[(\sigma_x)_{\max} + (\sigma'_x)_{\max}]$ occur at two opposite corners of the cross section, Fig. 11–31*d*.

Applying the flexure formula to each moment component in Figs. 11–31*b* and 11–31*c*, and adding the results algebraically, the resultant normal stress at any point on the cross section, Fig. 11–31*d*, is then

$$\sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} \tag{11–17}$$

Here,

σ = the normal stress at the point

y, z = the coordinates of the point measured from x, y, z axes having their origin at the centroid of the cross-sectional area and forming a right-handed coordinate system. The x axis is directed outward from the cross section and the y and z axes represent, respectively, the principal axes of minimum and maximum moment of inertia for the area

M_y, M_z = the resultant internal moment components directed along the maximum z and minimum y principal axes. They are positive if directed along the $+y$ and $+z$ axes, otherwise they are negative. Or, stated another way, $M_y = M\sin\theta$ and $M_z = M\cos\theta$, where θ is measured positive from the $+z$ axis toward the $+y$ axis

I_y, I_z = the maximum and minimum *principal moments of inertia* calculated about the z and y axes, respectively. See Appendix B

The x, y, z axes form a right-handed system, and the proper algebraic signs must be assigned to the moment components and the coordinates when applying this equation. When this is the case, the resulting stress will be *tensile* if it is *positive* and *compressive* if it is *negative*.

Orientation of the Neutral Axis. The angle α of the neutral axis in Fig. 11–31*d* can be determined by applying Eq. 11–17 with $\sigma = 0$, since by definition no normal stress acts on the neutral axis. We have

$$y = \frac{M_y I_z}{M_z I_y} z$$

Since $M_z = M \cos\theta$ and $M_y = M \sin\theta$, then

$$y = \left(\frac{I_z}{I_y} \tan\theta\right) z \tag{11–18}$$

This equation defines the neutral axis for the cross section. Since the slope of this line is $\tan\alpha = y/z$, then

$$\tan\alpha = \frac{I_z}{I_y} \tan\theta \tag{11–19}$$

Here it can be seen that unless $I_z = I_y$ the angle θ, defining the direction of the moment $\mathbf{M}$, Fig. 11–31*a*, will *not equal* α, the angle defining the inclination of the neutral axis, Fig. 11–31*d*.

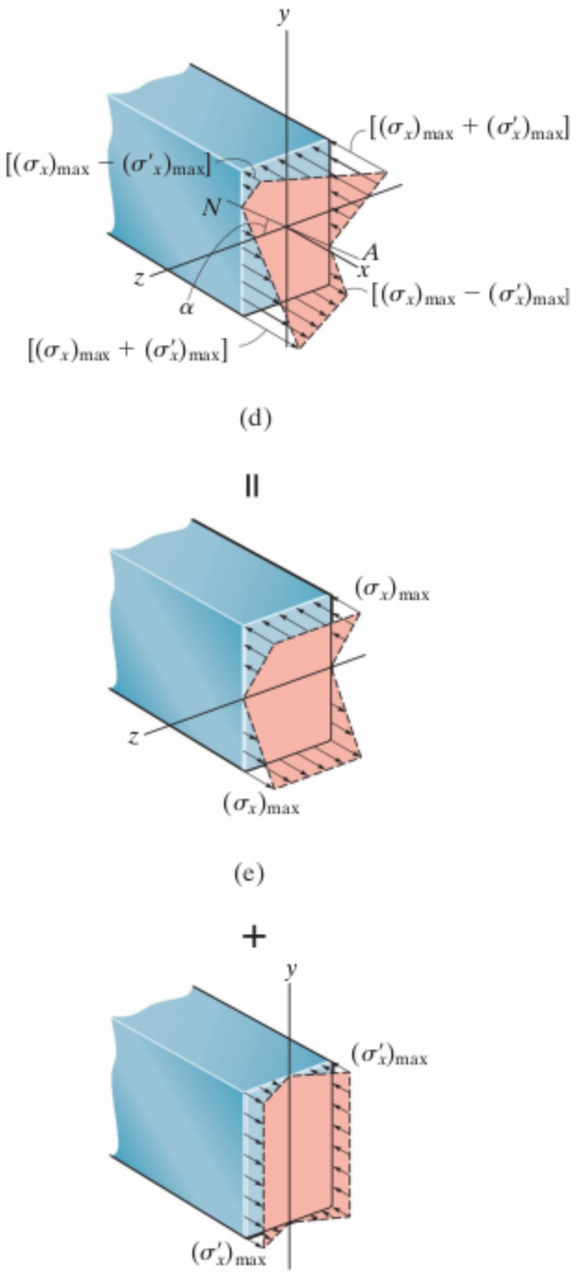


Fig. 11–31 (cont.)

Important Points

- The flexure formula can be applied only when bending occurs about axes that represent the *principal axes of inertia* for the cross section. These axes have their origin at the centroid and are oriented along an axis of symmetry, if there is one, and perpendicular to it.
- If the moment is applied about some arbitrary axis, then the moment must be resolved into components along each of the principal axes, and the stress at a point is determined by superposition of the stress caused by each of the moment components.

EXAMPLE 11.14

The rectangular cross section shown in Fig. 11–32*a* is subjected to a bending moment of $M = 12\ \text{kN}\cdot\text{m}$. Determine the normal stress developed at each corner of the section, and specify the orientation of the neutral axis.

SOLUTION

Internal Moment Components. By inspection it is seen that the y and z axes represent the principal axes of inertia since they are axes of symmetry for the cross section. As required we have established the z *axis* as the principal axis for *maximum* moment of inertia. The moment is resolved into its y and z components, where

$$M_y = -\frac{4}{5}(12\ \text{kN}\cdot\text{m}) = -9.60\ \text{kN}\cdot\text{m}$$

$$M_z = \frac{3}{5}(12\ \text{kN}\cdot\text{m}) = 7.20\ \text{kN}\cdot\text{m}$$

Section Properties. The moments of inertia about the y and z axes are

$$I_y = \frac{1}{12}(0.4\ \text{m})(0.2\ \text{m})^3 = 0.2667(10^{-3})\ \text{m}^4$$

$$I_z = \frac{1}{12}(0.2\ \text{m})(0.4\ \text{m})^3 = 1.067(10^{-3})\ \text{m}^4$$

Bending Stress. Thus,

$$\sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

$$\sigma_B = -\frac{7.20(10^3)\ \text{N}\cdot\text{m}(0.2\ \text{m})}{1.067(10^{-3})\ \text{m}^4} + \frac{-9.60(10^3)\ \text{N}\cdot\text{m}(-0.1\ \text{m})}{0.2667(10^{-3})\ \text{m}^4} = 2.25\ \text{MPa} \quad \textit{Ans.}$$

$$\sigma_C = -\frac{7.20(10^3)\ \text{N}\cdot\text{m}(0.2\ \text{m})}{1.067(10^{-3})\ \text{m}^4} + \frac{-9.60(10^3)\ \text{N}\cdot\text{m}(0.1\ \text{m})}{0.2667(10^{-3})\ \text{m}^4} = -4.95\ \text{MPa} \quad \textit{Ans.}$$

$$\sigma_D = -\frac{7.20(10^3)\ \text{N}\cdot\text{m}(-0.2\ \text{m})}{1.067(10^{-3})\ \text{m}^4} + \frac{-9.60(10^3)\ \text{N}\cdot\text{m}(0.1\ \text{m})}{0.2667(10^{-3})\ \text{m}^4} = -2.25\ \text{MPa} \quad \textit{Ans.}$$

$$\sigma_E = -\frac{7.20(10^3)\ \text{N}\cdot\text{m}(-0.2\ \text{m})}{1.067(10^{-3})\ \text{m}^4} + \frac{-9.60(10^3)\ \text{N}\cdot\text{m}(-0.1\ \text{m})}{0.2667(10^{-3})\ \text{m}^4} = 4.95\ \text{MPa} \quad \textit{Ans.}$$

The resultant normal-stress distribution has been sketched using these values, Fig. 11–32*b*. Since superposition applies, the distribution is linear as shown.

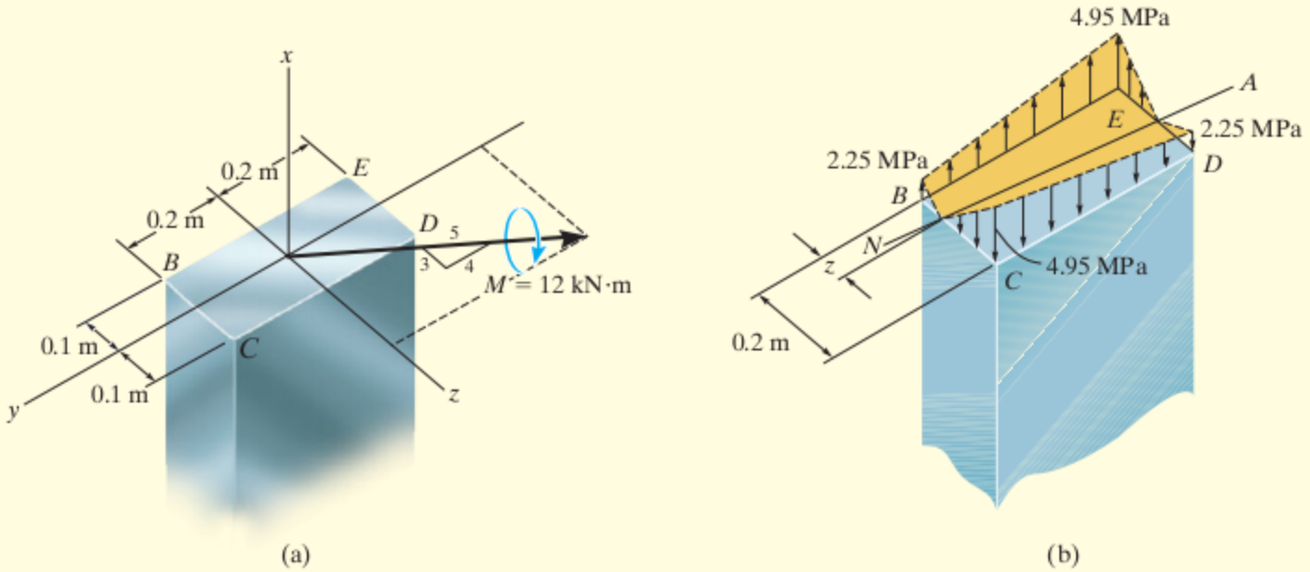


Fig. 11–32

Orientation of Neutral Axis. The location z of the neutral axis (NA), Fig. 11–32*b*, can be established by proportion. Along the edge BC, we require

$$\frac{2.25\text{ MPa}}{z} = \frac{4.95\text{ MPa}}{(0.2\text{ m} - z)}$$

$$0.450 - 2.25z = 4.95z$$

$$z = 0.0625\text{ m}$$

In the same manner this is also the distance from D to the neutral axis in Fig. 11–32*b*.

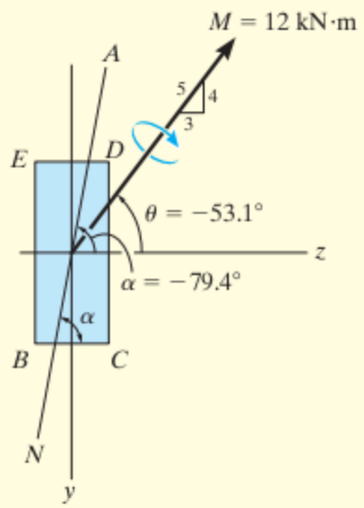
We can also establish the orientation of the NA using Eq. 11–19, which is used to specify the angle α that the axis makes with the z or *maximum* principal axis. According to our sign convention, θ must be measured from the $+z$ axis toward the $+y$ axis. By comparison, in Fig. 11–32*c*, $\theta = -\tan^{-1}\frac{4}{3} = -53.1°$ (or $\theta = +306.9°$). Thus,

$$\tan\alpha = \frac{I_z}{I_y}\tan\theta$$

$$\tan\alpha = \frac{1.067(10^{-3})\text{ m}^4}{0.2667(10^{-3})\text{ m}^4}\tan(-53.1°)$$

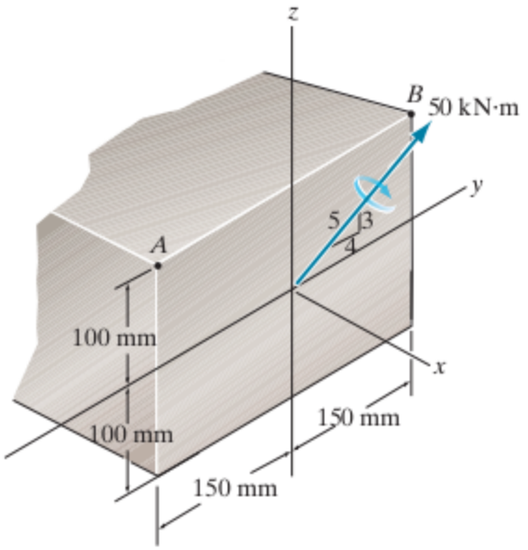
$$\alpha = -79.4° \qquad \textit{Ans.}$$

This result is shown in Fig. 11–32*c*. Using the value of z calculated above, verify, using the geometry of the cross section, that one obtains the same answer.



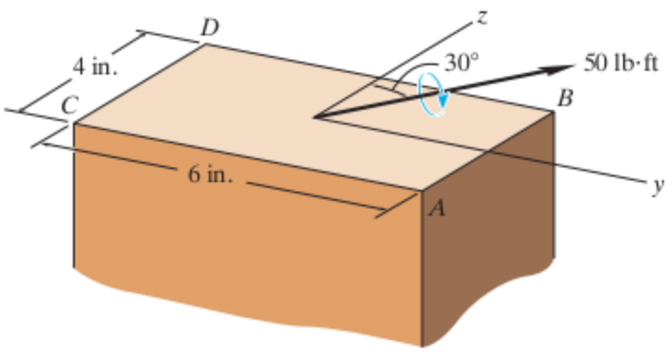
FUNDAMENTAL PROBLEMS

F11–20. Determine the bending stress developed at corners A and B. What is the orientation of the neutral axis?



F11–20

F11–21. Determine the maximum stress in the beam's cross section.

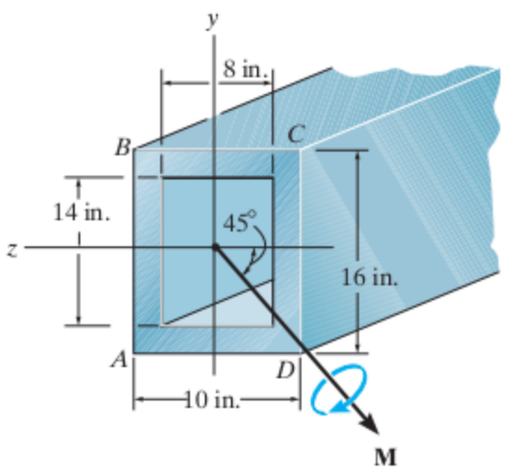


F11–21

PROBLEMS

11–69. The beam is subjected to a bending moment of $M = 20\,\text{kip}\cdot\text{ft}$ directed as shown. Determine the maximum bending stress in the beam and the orientation of the neutral axis.

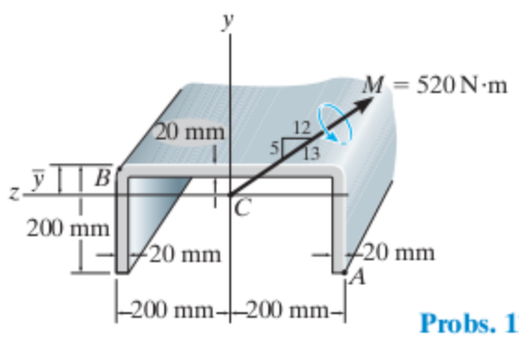
11–70. Determine the maximum magnitude of the bending moment **M** that can be applied to the beam so that the bending stress in the member does not exceed 12 ksi.



Probs. 11–69/70

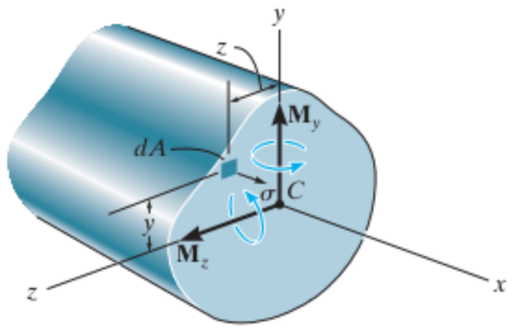
11–71. If the resultant internal moment acting on the cross section of the aluminum strut has a magnitude of $M = 520\,\text{N}\cdot\text{m}$ and is directed as shown, determine the bending stress at points A and B. The location $\bar{y}$ of the centroid C of the strut's cross-sectional area must be determined. Also, specify the orientation of the neutral axis.

***11–72.** The resultant internal moment acting on the cross section of the aluminum strut has a magnitude of $M = 520\,\text{N}\cdot\text{m}$ and is directed as shown. Determine maximum bending stress in the strut. The location $\bar{y}$ of the centroid C of the strut's cross-sectional area must be determined. Also, specify the orientation of the neutral axis.



Probs. 11–71/72

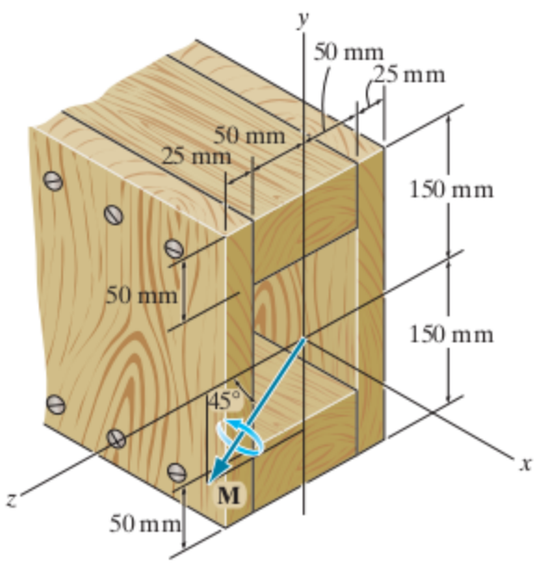
11–73. Consider the general case of a prismatic beam subjected to bending-moment components $\mathbf{M}_y$ and $\mathbf{M}_z$, as shown, when the x, y, z axes pass through the centroid of the cross section. If the material is linear-elastic, the normal stress in the beam is a linear function of position such that $\sigma = a + by + cz$. Using the equilibrium conditions $0 = \int_A \sigma\, dA$, $M_y = \int_A z\sigma\, dA$, $M_z = \int_A -y\sigma\, dA$, determine the constants a, b, and c, and show that the normal stress can be determined from the equation $\sigma = [-(M_z I_y + M_y I_{yz})y + (M_y I_z + M_z I_{yz})z]/(I_y I_z - I_{yz}^2)$, where the moments and products of inertia are defined in Appendix B.



Prob. 11–73

11–74. The box beam is subjected to the internal moment of $M = 4\ \text{kN}\cdot\text{m}$, which is directed as shown. Determine the maximum bending stress developed in the beam and the orientation of the neutral axis.

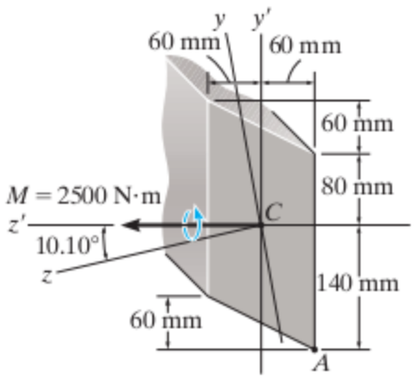
11–75. If the wood used for the box beam has an allowable bending stress of $(\sigma_{\text{allow}}) = 6$ MPa, determine the maximum allowable internal moment $\mathbf{M}$ that can be applied to the beam.



Probs. 11–74/75

***11–76.** For the section, $I_{y'} = 31.7(10^{-6})\ \text{m}^4$, $I_{z'} = 114(10^{-6})\ \text{m}^4$, $I_{y'z'} = 15.1(10^{-6})\ \text{m}^4$. Using the techniques outlined in Appendix A, the member's cross-sectional area has principal moments of inertia of $I_y = 29.0(10^{-6})\ \text{m}^4$ and $I_z = 117(10^{-6})\ \text{m}^4$, computed about the principal axes of inertia y and z, respectively. If the section is subjected to a moment of $M = 2500\ \text{N}\cdot\text{m}$ directed as shown, determine the stress produced at point A, using Eq. 11–17.

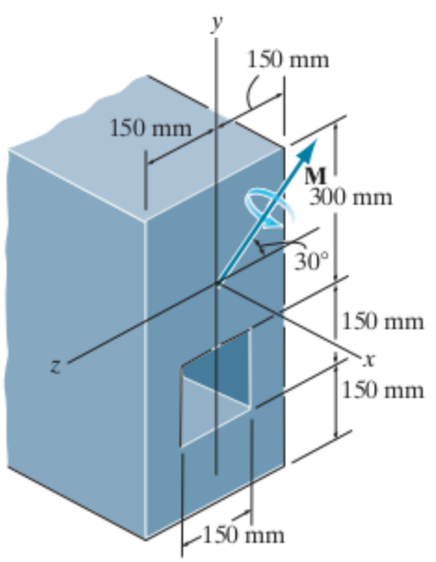
11–77. Solve Prob. 11–76 using the equation developed in Prob. 11–73.



Probs. 11–76/77

11–78. If the beam is subjected to the internal moment of, $M = 1200\ \text{kN}\cdot\text{m}$, determine the maximum bending stress acting on the beam and the orientation of the neutral axis.

11–79. If the beam is made from a material having an allowable tensile and compressive stress of $(\sigma_{\text{allow}})_t = 125$ MPa and $(\sigma_{\text{allow}})_C = 150$ MPa, respectively, determine the maximum allowable internal moment $\mathbf{M}$ that can be applied to the beam.



Probs. 11–78/79

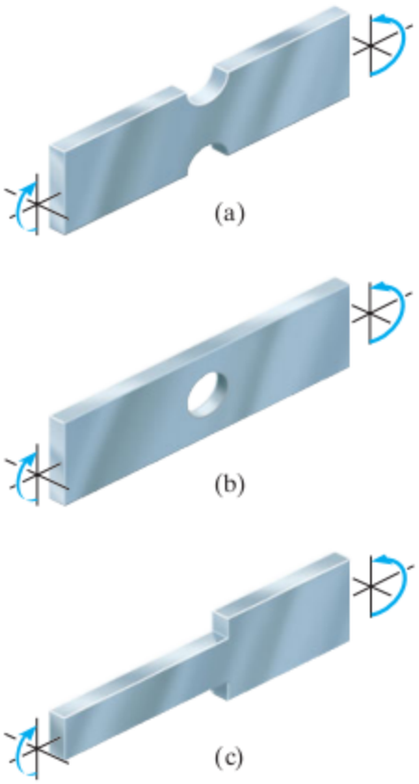


Fig. 11–33

11.6 Stress Concentrations

The flexure formula cannot be used to determine the stress distribution within regions of a member where the cross-sectional area suddenly changes, since the normal-stress and strain distributions at the section become *nonlinear*. The results can only be obtained through experiment or, in some cases, by using the theory of elasticity. Common discontinuities include members having notches on their surfaces, Fig. 11–33*a*, holes for passage of fasteners or other items, Fig. 11–33*b*, or abrupt changes in the outer dimensions of the member's cross section, Fig. 11–33*c*. The *maximum* normal stress at each of these discontinuities occurs at the section taken through the *smallest* cross-sectional area.

For design, it is generally important to only know the maximum normal stress developed at these sections, not the actual stress distribution. As in the previous cases of axially loaded bars and torsionally loaded shafts, we can obtain the maximum normal stress due to bending using a stress-concentration factor K. For example, Fig. 11–34 gives values of K for a flat bar that has a change in cross section using shoulder fillets. To use this graph simply find the geometric ratios w/h and r/h and then find the corresponding value of K for a particular

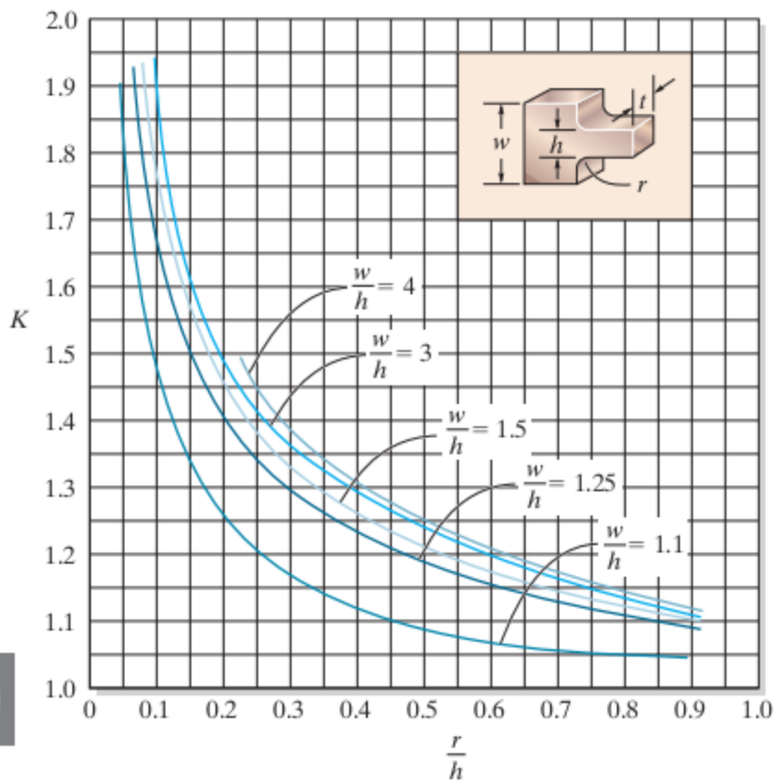


Fig. 11–34

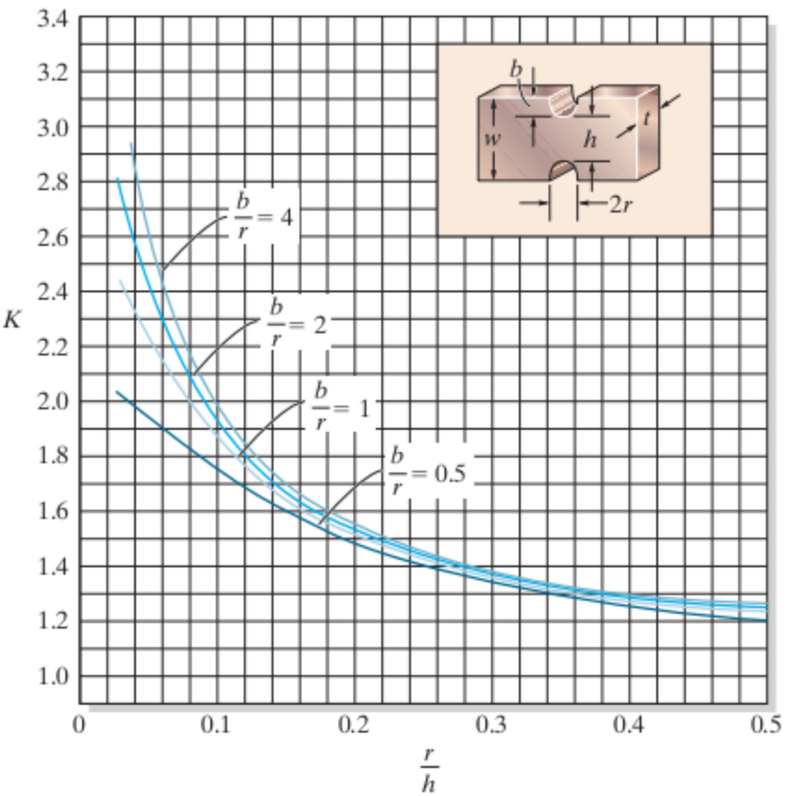


Fig. 11–35

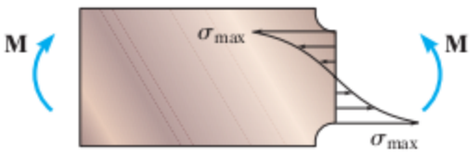


Fig. 11–36

geometry. Once K is obtained, the maximum bending stress shown in Fig. 11–36 is determined using

$$\sigma_{max} = K\frac{Mc}{I} \tag{11–20}$$

In the same manner, Fig. 11–35 can be used if the discontinuity consists of circular grooves or notches.

Like axial load and torsion, stress concentration for bending should always be considered when designing members made of brittle materials or those that are subjected to fatigue or cyclic loadings. Also, realize that stress-concentration factors apply *only* when the material is subjected to *elastic behavior*. If the applied moment causes yielding of the material, as is the case with ductile materials, the stress becomes redistributed throughout the member, and the maximum stress that results will be *lower* than that determined using stress-concentration factors. This phenomenon is discussed further in the next section.

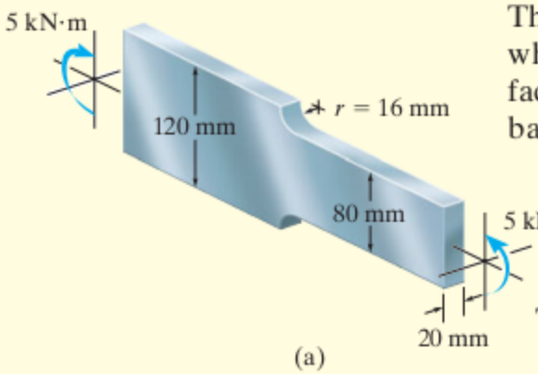
Stress concentrations caused by bending occur at the sharp corners of this window lintel and are responsible for the crack at the corner.

Important Points

- Stress concentrations occur at points of sudden cross-sectional change, caused by notches and holes, because here the stress and strain become nonlinear. The more severe the change, the larger the stress concentration.
- For design or analysis, the maximum normal stress occurs on the *smallest* cross-sectional area. This stress can be obtained by using a stress concentration factor, K, that has been determined through experiment and is only a function of the geometry of the member.
- Normally, the stress concentration in a ductile material subjected to a static moment will not have to be considered in design; however, if the material is *brittle*, or subjected to *fatigue* loading, then stress concentrations become important.

EXAMPLE 11.15

The transition in the cross-sectional area of the steel bar is achieved using shoulder fillets as shown in Fig. 11–37*a*. If the bar is subjected to a bending moment of 5 kN · m, determine the maximum normal stress developed in the steel. The yield stress is $\sigma_Y = 500$ MPa.



(a)

SOLUTION

The moment creates the largest stress in the bar at the base of the fillet, where the cross-sectional area is smallest. The stress-concentration factor can be determined by using Fig. 11–34. From the geometry of the bar, we have $r = 16$ mm, $h = 80$ mm, $w = 120$ mm. Thus,

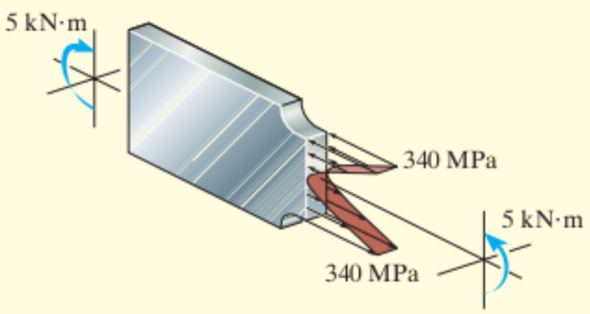
$$\frac{r}{h} = \frac{16\text{ mm}}{80\text{ mm}} = 0.2 \qquad \frac{w}{h} = \frac{120\text{ mm}}{80\text{ mm}} = 1.5$$

These values give $K = 1.45$. Applying Eq. 11–20, we have

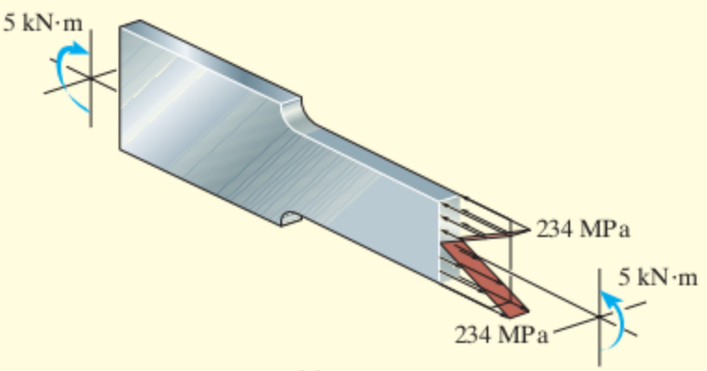
$$\sigma_{max} = K\frac{Mc}{I} = (1.45)\frac{(5(10^3)\text{ N} \cdot \text{m})(0.04\text{ m})}{\left[\frac{1}{12}(0.020\text{ m})(0.08\text{ m})^3\right]} = 340\text{ MPa} \qquad \textit{Ans.}$$

This result indicates that the steel remains elastic since the stress is below the yield stress (500 MPa).

NOTE: The normal-stress distribution is nonlinear and is shown in Fig. 11–37*b*. Realize, however, that by Saint-Venant's principle, Sec. 9.1, these localized stresses smooth out and become linear when one moves (approximately) a distance of 80 mm or more to the right of the transition. In this case, the flexure formula gives $\sigma_{max} = 234$ MPa, Fig. 11–37*c*. Also note that the choice of a larger-radius fillet will significantly reduce σ_{max}, since as r increases in Fig. 11–34, K will decrease.



(b)

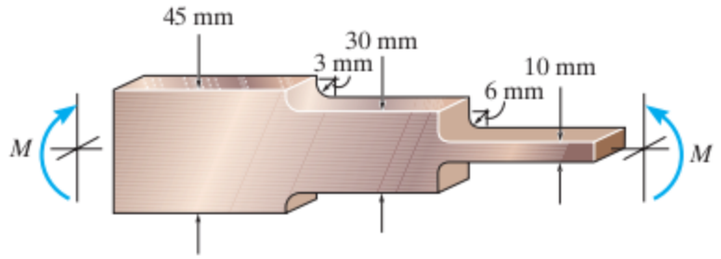


(c)

Fig. 11–37

PROBLEMS

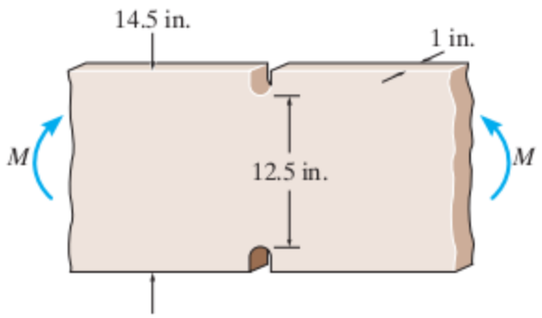
***11–80.** The stepped bar has a thickness of 15 mm. Determine the maximum moment that can be applied to its ends if it is made of a material having an allowable bending stress of $\sigma_{allow} = 200$ MPa.



Prob. 11–80

11–81. If the radius of each notch on the plate is $r = 0.5$ in., determine the largest moment that can be applied. The allowable bending stress for the material is $\sigma_{allow} = 18$ ksi.

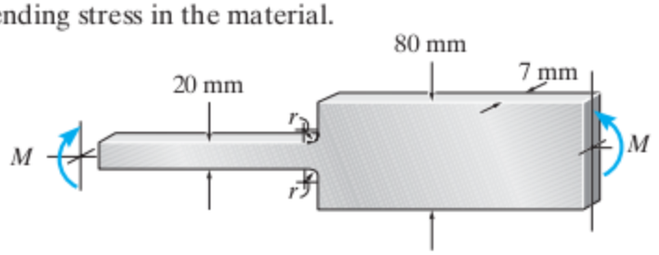
11–82. The symmetric notched plate is subjected to bending. If the radius of each notch is $r = 0.5$ in. and the applied moment is $M = 10$ kip · ft, determine the maximum bending stress in the plate.



Probs. 11–81/82

11–83. The bar is subjected to a moment of $M = 40$ N · m. Determine the smallest radius r of the fillets so that an allowable bending stress of $\sigma_{allow} = 124$ MPa is not exceeded.

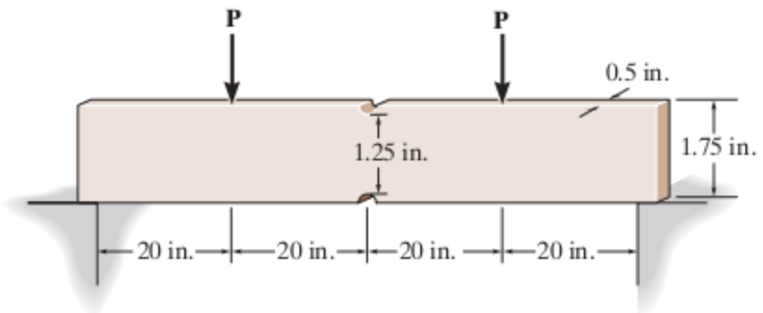
***11–84.** The bar is subjected to a moment of $M = 17.5$ N · m If $r = 5$ mm, determine the maximum bending stress in the material.



Probs. 11–83/84

11–85. The simply supported notched bar is subjected to two forces **P**. Determine the largest magnitude of **P** that can be applied without causing the material to yield. The material is A-36 steel. Each notch has a radius of $r = 0.125$ in.

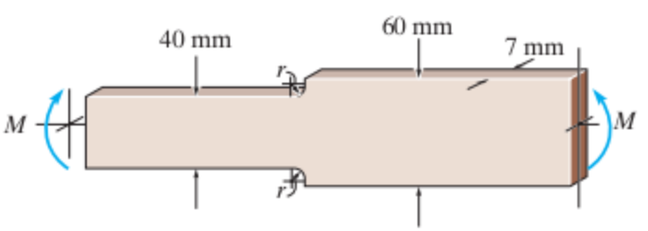
11–86. The simply supported notched bar is subjected to the two loads, each having a magnitude of $P = 100$ lb. Determine the maximum bending stress developed in the bar, and sketch the bending-stress distribution acting over the cross section at the center of the bar. Each notch has a radius of $r = 0.125$ in.



Probs. 11–85/86

11–87. The bar is subjected to a moment of $M = 153$ N · m. Determine the smallest radius r of the fillets so that an allowable bending stress of $\sigma_{allow} = 120$ MPa is not exceeded.

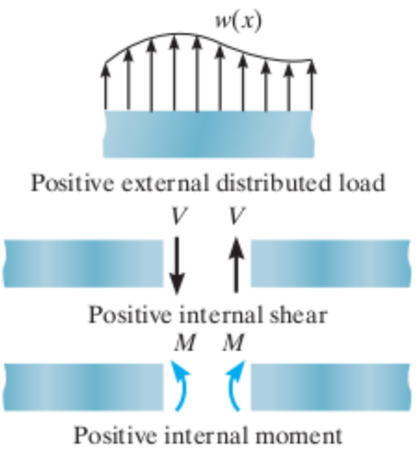
***11–88.** The bar is subjected to a moment of $M = 17.5$ N · m. If $r = 6$ mm determine the maximum bending stress in the material.



Probs. 11–87/88

CHAPTER REVIEW

Shear and moment diagrams are graphical representations of the internal shear and moment within a beam. They can be constructed by sectioning the beam an arbitrary distance x from the left end, using the equilibrium equations to find V and M as functions of x, and then plotting the results. A sign convention for positive distributed load, shear, and moment must be followed.



It is also possible to plot the shear and moment diagrams by realizing that at each point the slope of the shear diagram is equal to the intensity of the distributed loading at the point.

$$w = \frac{dV}{dx}$$

Likewise, the slope of the moment diagram is equal to the shear at the point.

$$V = \frac{dM}{dx}$$

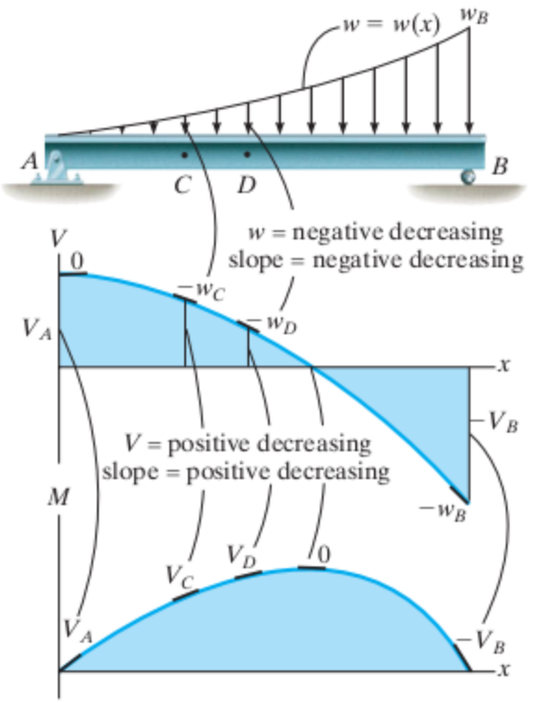
The area under the distributed loading diagram between the points represents the change in shear.

$$\Delta V = \int w\, dx$$

The area under the shear diagram represents the change in moment.

$$\Delta M = \int V\, dx$$

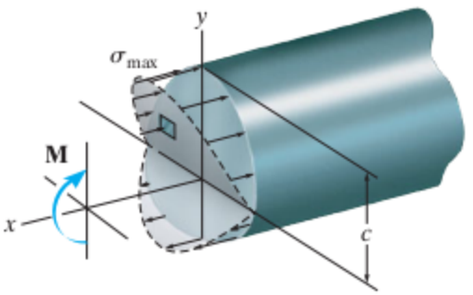
The shear and moment at any point can be obtained using the method of sections. The maximum (or minimum) moment occurs where the shear is zero.



A bending moment tends to produce a linear variation of normal strain within a straight beam. Provided the material is homogeneous, and linear elastic, then equilibrium can be used to relate the internal moment in the beam to the stress distribution. The result is the flexure formula,

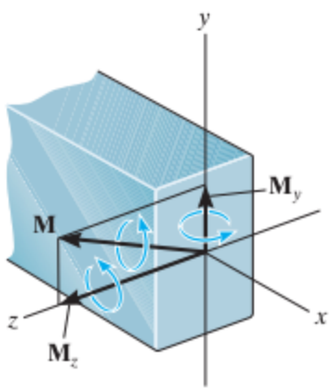
$$\sigma_{max} = \frac{Mc}{I}$$

where I and c are determined from the neutral axis that passes through the centroid of the cross section.



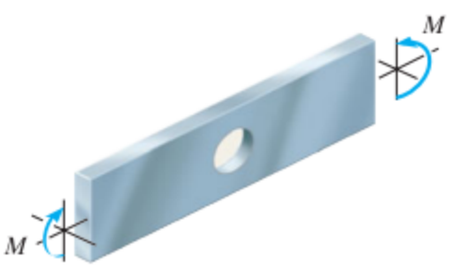
If the cross-sectional area of the beam is not symmetric about an axis that is perpendicular to the neutral axis, then unsymmetrical bending will occur. The maximum stress can be determined from formulas, or the problem can be solved by considering the superposition of bending caused by the moment components $\mathbf{M}_y$ and $\mathbf{M}_z$ about the principal axes of inertia for the area.

$$\sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$



Stress concentrations occur in members having a sudden change in their cross section, caused, for example, by holes and notches. The maximum bending stress at these locations is determined using a stress concentration factor K that is found from graphs determined from experiment.

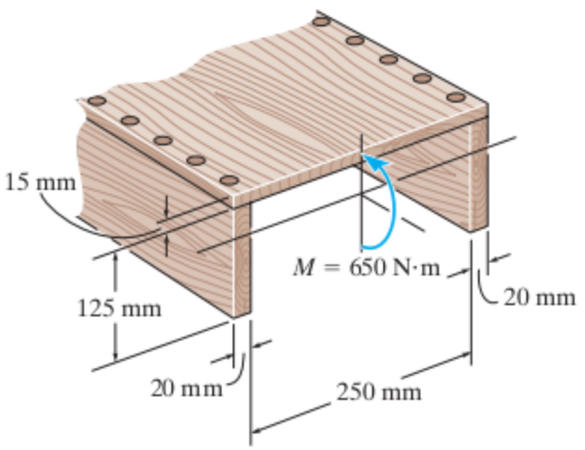
$$\sigma_{max} = k\frac{Mc}{I}$$



REVIEW PROBLEMS

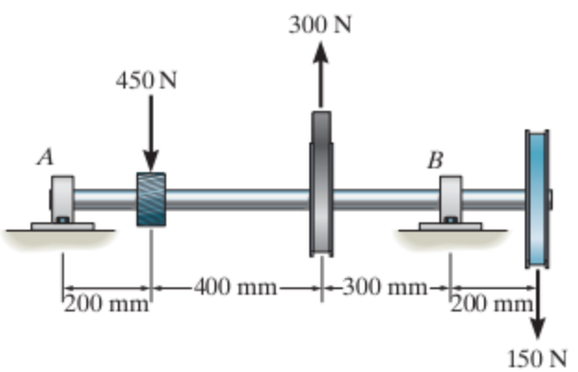
11–89. The beam is made from three boards nailed together as shown. If the moment acting on the cross section is $M = 650\ \text{N}\cdot\text{m}$, determine the resultant force the bending stress produces on the top board.

11–90. The beam is made from three boards nailed together as shown. Determine the maximum tensile and compressive stresses in the beam.



Probs. 11–89/90

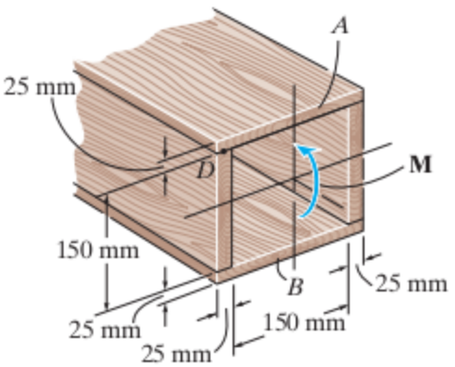
11–91. Draw the shear and moment diagrams for the shaft if it is subjected to the vertical loadings of the belt, gear, and flywheel. The bearings at A and B exert only vertical reactions on the shaft.



Prob. 11–91

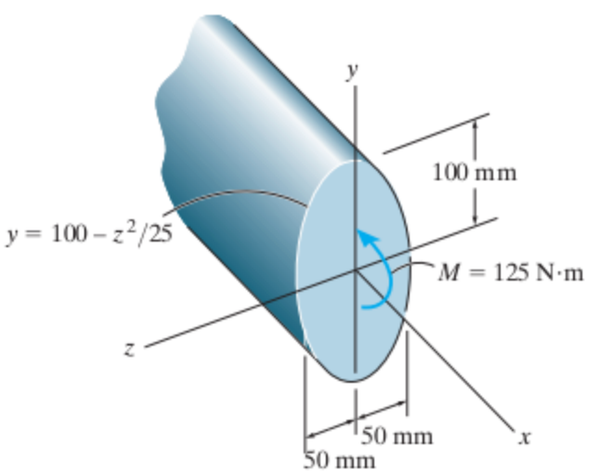
***11–92.** The beam is subjected to a moment **M**. Determine the percentage of this moment that is resisted by the stresses acting on both the top and bottom boards, A and B, of the beam.

11–93. Determine the moment M that should be applied to the beam in order to create a compressive stress at point D of $\sigma_D = 30$ MPa. Also sketch the stress distribution acting over the cross section and compute the maximum stress developed in the beam.



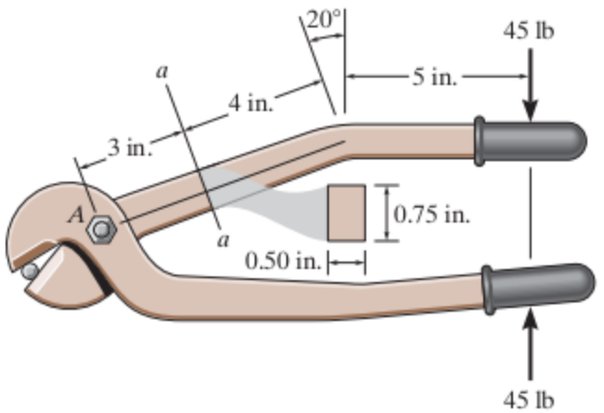
Probs. 11–92/93

11–94. A shaft is made of a polymer having a parabolic cross section. If it resists an internal moment of $M = 125\ \text{N}\cdot\text{m}$, determine the maximum bending stress developed in the material (a) using the flexure formula and (b) using integration. Sketch a three-dimensional view of the stress distribution acting over the cross-sectional area.



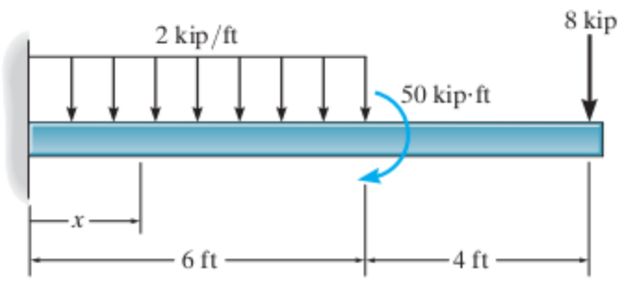
Prob. 11–94

11–95. Determine the maximum bending stress in the handle of the cable cutter at section *a–a*. A force of 45 lb is applied to the handles. The cross-sectional area is shown in the figure.



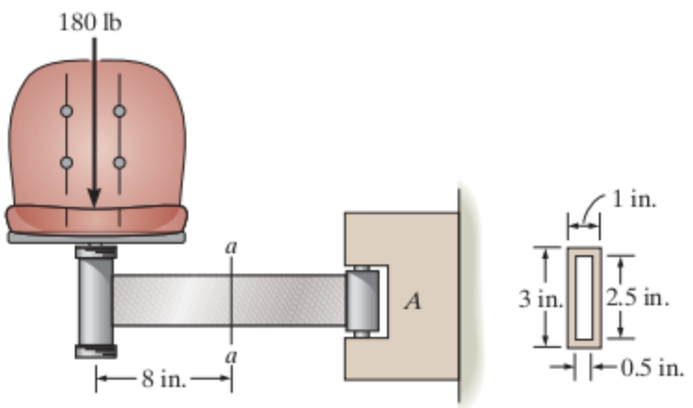
Prob. 11–95

11–97. Draw the shear and moment diagrams for the beam and determine the shear and moment in the beam as functions of x, where $0 \leq x < 6$ ft.



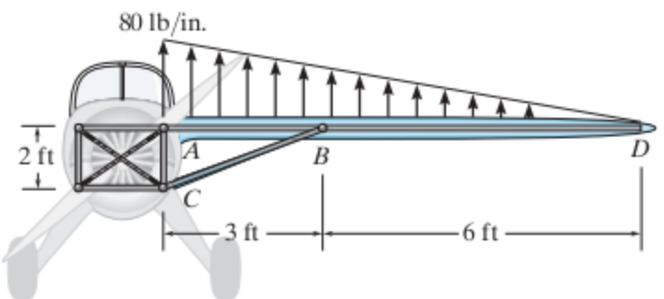
Prob. 11–97

***11–96.** The chair is supported by an arm that is hinged so it rotates about the vertical axis at *A*. If the load on the chair is 180 lb and the arm is a hollow tube section having the dimensions shown, determine the maximum bending stress at section *a–a*.



Prob. 11–96

11–98. The wing spar *ABD* of a light plane is made from 2014–T6 aluminum and has a cross-sectional area of 1.27 in.^{2}, a depth of 3 in., and a moment of inertia about its neutral axis of 2.68 in^4. Determine the absolute maximum bending stress in the spar if the anticipated loading is to be as shown. Assume *A*, *B*, and *C* are pins. Connection is made along the central longitudinal axis of the spar.



Prob. 11–98

Chapter 12

Railroad ties act as beams that support very large transverse shear loadings. As a result, if they are made of wood they will tend to split at their ends, where the shear loads are the largest.

Transverse Shear

CHAPTER OBJECTIVES

- To determine the shear stress in straight beams subject to transverse loading.
- To calculate the shear flow in beams composed of several members.

12.1 Shear in Straight Members

In general, a beam will support both shear and moment. The shear **V** is the result of a transverse shear-stress distribution that acts over the beam's cross section. Due to the complementary property of shear, however, this stress will create corresponding longitudinal shear stresses which will act along longitudinal planes of the beam as shown in Fig. 12–1.

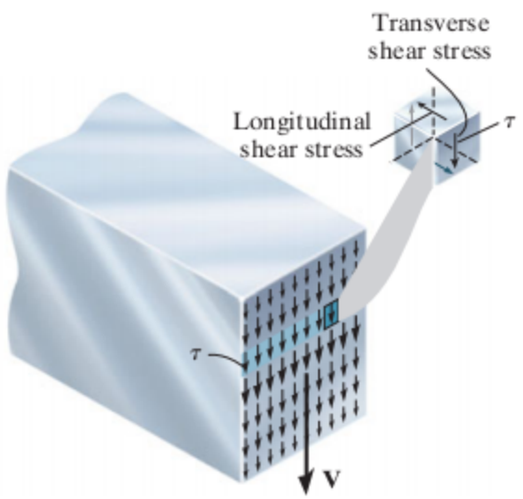


Fig. 12–1

12

Boards not bonded together
(a)

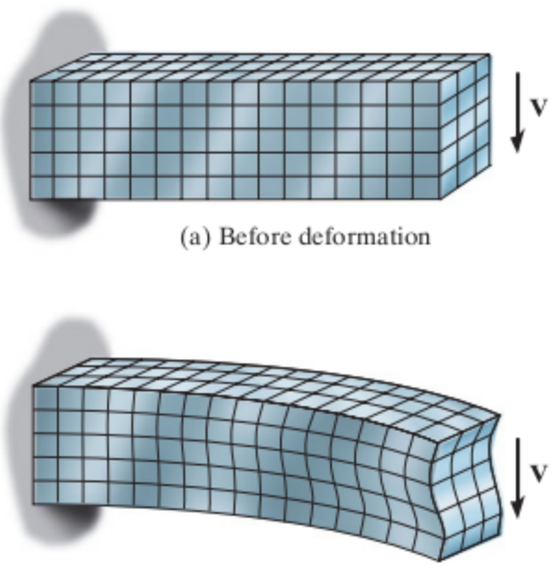
Boards bonded together
(b)

Fig. 12–2

Shear connectors are "tack welded" to this corrugated metal floor liner so that when the concrete floor is poured, the connectors will prevent the concrete slab from slipping on the liner surface. The two materials will thus act as a composite slab.

To illustrate this effect, consider the beam to be made from three boards, Fig. 12–2*a*. If the top and bottom surfaces of each board are smooth, and the boards are *not* bonded together, then application of the load **P** will cause the boards to *slide* relative to one another when the beam deflects. However, if the boards are bonded together, then the longitudinal shear stresses acting between the boards will prevent their relative sliding, and consequently the beam will act as a single unit, Fig. 12–2*b*.

As a result of the shear stress, shear strains will be developed and these will tend to distort the cross section in a rather complex manner. For example, consider the short bar in Fig. 12–3*a* made of a highly deformable material and marked with horizontal and vertical grid lines. When a shear **V** is applied, it tends to deform these lines into the pattern shown in Fig. 12–3*b*. This nonuniform shear-strain distribution will cause the cross section to *warp*.



(a) Before deformation

(b) After deformation

Fig. 12–3

As a result, when a beam is subjected to *both* bending and shear, the cross section will not remain plane as assumed in the development of the flexure formula. Although this is the case, we can generally assume the cross-sectional warping due to shear is small enough so that it can be *neglected*. This assumption is particularly true for the most common case of a *slender beam*; that is, one that has a small depth compared with its length.

12.2 The Shear Formula

Because the strain distribution for shear is not easily defined, as in the case of axial load, torsion, and bending, we will develop the shear formula in an indirect manner. To do this we will consider the *horizontal force equilibrium* of a portion of the element taken from the beam in Fig. 12–4*a*. A free-body diagram of this *element* is shown in Fig. 12–4*b*. This distribution is caused by the bending moments M and $M + dM$. We have excluded the effects of V, $V + dV$, and $w(x)$ on the free-body diagram since these loadings are vertical and will therefore not be involved in a horizontal force summation. The element in Fig. 12–4*b* will indeed satisfy $\Sigma F_x = 0$ since the stress distribution on each side of the element forms only a couple moment and therefore a zero force resultant.

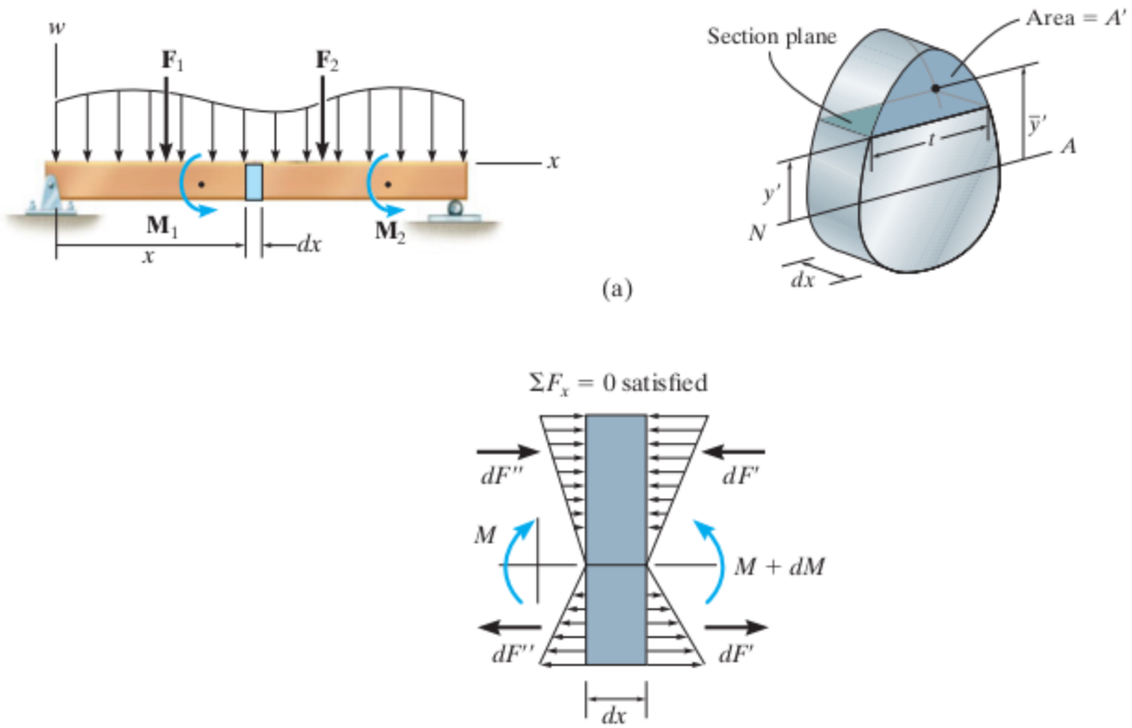


Fig. 12–4

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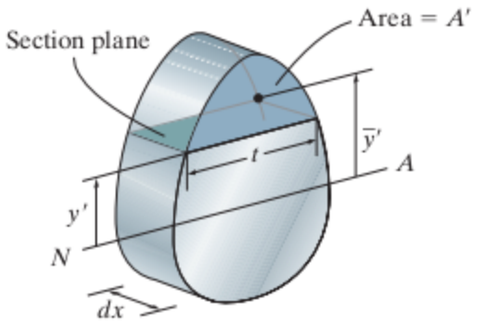


(a)

Now consider the shaded top *portion* of the element that has been sectioned at y' from the neutral axis, Fig. 12–4*a*. This segment has a width t at the section, and the two cross-sectional sides each have an area A'. Because the resultant moments on each side of the element differ by dM, it can be seen in Fig. 12–4*c* that $\Sigma F_x = 0$ will not be satisfied *unless* a longitudinal shear stress τ acts over the bottom face of the segment. We will assume this shear stress is *constant* across the width t of the bottom face. It acts on the area $t\,dx$. Applying the equation of horizontal force equilibrium, and using the flexure formula, Eq. 11–13, we have

$$\overset{+}{\leftarrow} \Sigma F_x = 0; \qquad \int_{A'} \sigma'\,dA' - \int_{A'} \sigma\,dA' - \tau(t\,dx) = 0$$

$$\int_{A'} \left(\frac{M + dM}{I}\right) y\,dA' - \int_{A'} \left(\frac{M}{I}\right) y\,dA' - \tau(t\,dx) = 0$$

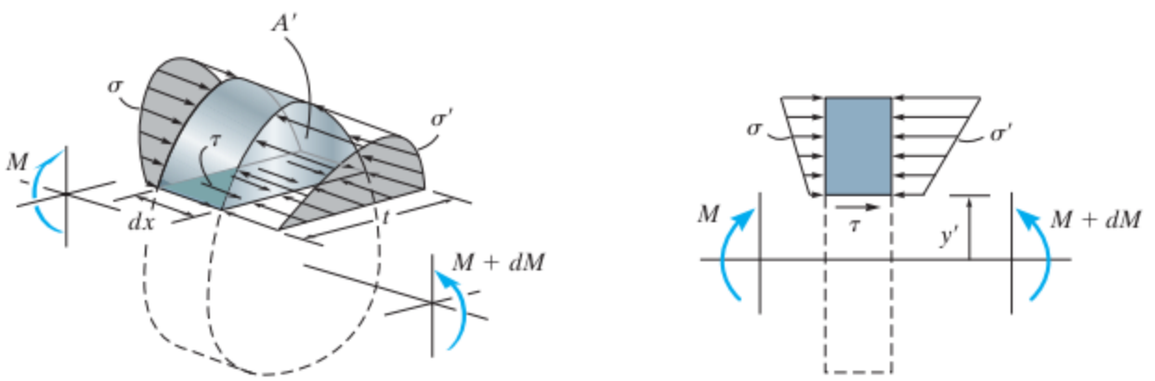
$$\left(\frac{dM}{I}\right) \int_{A'} y\,dA' = \tau(t\,dx) \tag{12–1}$$

Solving for τ, we get

$$\tau = \frac{1}{It}\left(\frac{dM}{dx}\right) \int_{A'} y\,dA'$$

This equation can be simplified by noting that $V = dM/dx$ (Eq. 11–2). Also, the integral represents the moment of the area A' about the neutral axis. We will denote this by the symbol Q. Since the location of the centroid of the area A' is determined from $\bar{y}' = \int_{A'} y\,dA'/A'$, we can also write

$$Q = \int_{A'} y\,dA' = \bar{y}'A' \tag{12–2}$$



Three-dimensional view (c) Profile view

Fig. 12–4 (cont.)

The final result is therefore

$$\tau = \frac{VQ}{It} \tag{12–3}$$

Here, as shown in Fig. 12–5,

τ = the shear stress in the member at the point located a distance y' from the neutral axis. This stress is assumed to be constant and therefore *averaged* across the width t of the member

V = the internal resultant shear force, determined from the method of sections and the equations of equilibrium

I = the moment of inertia of the *entire* cross-sectional area calculated about the neutral axis

t = the width of the member's cross-sectional area, measured at the point where τ is to be determined

$Q = \bar{y}'A'$, where A' is the area of the top (or bottom) portion of the member's cross-sectional area, above (or below) the section plane where t is measured, and $\bar{y}'$ is the distance from the neutral axis to the centroid of A'

The above equation is referred to as the *shear formula*. Although in the derivation we considered only the shear stresses acting on the beam's longitudinal plane, the formula applies as well for finding the transverse shear stress on the beam's cross section. Recall that these stresses are complementary and numerically equal.

Also, because the flexure formula was used in the derivation, it is necessary that the material behave in a linear elastic manner and have a modulus of elasticity that is the *same* in tension as it is in compression.

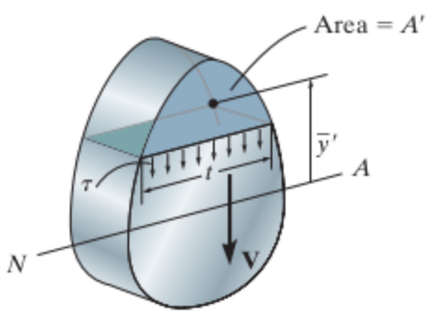


Fig. 12–5

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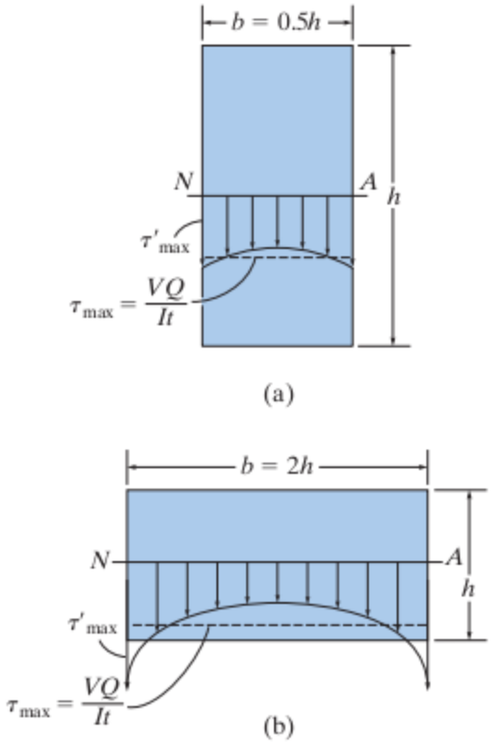


Fig. 12–6

Limitations on the Use of the Shear Formula. One of the major assumptions used in the development of the shear formula is that the shear stress is *uniformly* distributed over the *width* t at the section. In other words, the *average shear stress* is calculated across the width. We can test the accuracy of this assumption by comparing it with a more exact mathematical analysis based on the theory of elasticity. For example, if the beam's cross section is rectangular, the shear-stress distribution across the neutral axis as calculated from the theory of elasticity varies as shown in Fig. 12–6. The maximum value, τ'_{max}, occurs at the *sides* of the cross section, and its magnitude depends on the ratio b/h (width/depth). For sections having a $b/h = 0.5$, τ'_{max} is only about 3% greater than the shear stress calculated from the shear formula, Fig. 12–6*a*. However, for *flat sections*, say $b/h = 2$, τ'_{max} is about 40% greater than τ_{max}, Fig. 12–6*b*. The error becomes even greater as the section becomes flatter, or as the b/h ratio increases. Errors of this magnitude are certainly intolerable if one uses the shear formula to determine the shear stress in the *flange* of the wide-flange beam shown in Fig. 12–7.

It should also be pointed out that the shear formula will not give accurate results when used to determine the shear stress at the flange–web junction of a wide-flange beam, since this is a point of sudden cross-sectional change and therefore a *stress concentration* occurs here. Fortunately, these limitations for applying the shear formula to the flanges of a wide-flange beam are not important in engineering practice. Most often engineers must only calculate the *average maximum shear stress* in the beam, which occurs at the neutral axis, where the b/h (width/depth) ratio for the web is *very small*, and therefore the calculated result is very close to the *actual* maximum shear stress as explained above.

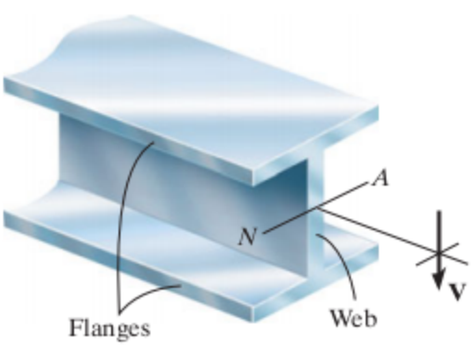


Fig. 12–7

Another important limitation on the use of the shear formula can be illustrated with reference to Fig. 12–8*a*, which shows a member having a cross section with an irregular or nonrectangular boundary. If we apply the shear formula to determine the (average) shear stress τ along the line AB, it will be directed downward as shown in Fig. 12–8*b*. However, consider an element of material taken from the boundary point B, Fig. 12–8*c*. Here τ on the front face of the element is resolved into components, τ' and τ'' acting perpendicular and parallel to the boundary. By inspection, τ' must be equal to zero since its corresponding longitudinal component τ', on the stress-free boundary surface, must be zero. To satisfy this boundary condition, therefore, the shear stress acting on this element must *actually be directed tangent to the boundary*. As a result, the shear-stress distribution across line AB is directed as shown in Fig. 12–8*d*. Here specific values for the shear stress must be obtained using the theory of elasticity. Note, however, that we can apply the shear formula to obtain the shear stress acting across each of the colored lines in Fig. 12–8*a*. These lines intersect the tangents to the boundary at *right angles*, and as shown in Fig. 12–8*e*, the transverse shear stress is vertical and constant along each line.

To summarize the above points, the shear formula does not give accurate results when applied to members having cross sections that are *short or flat*, or at points where the cross section suddenly changes. Nor should it be applied across a section that intersects the boundary of the member at an angle other than 90°. Instead, for these cases the shear stress should be determined using more advanced methods based on the theory of elasticity.

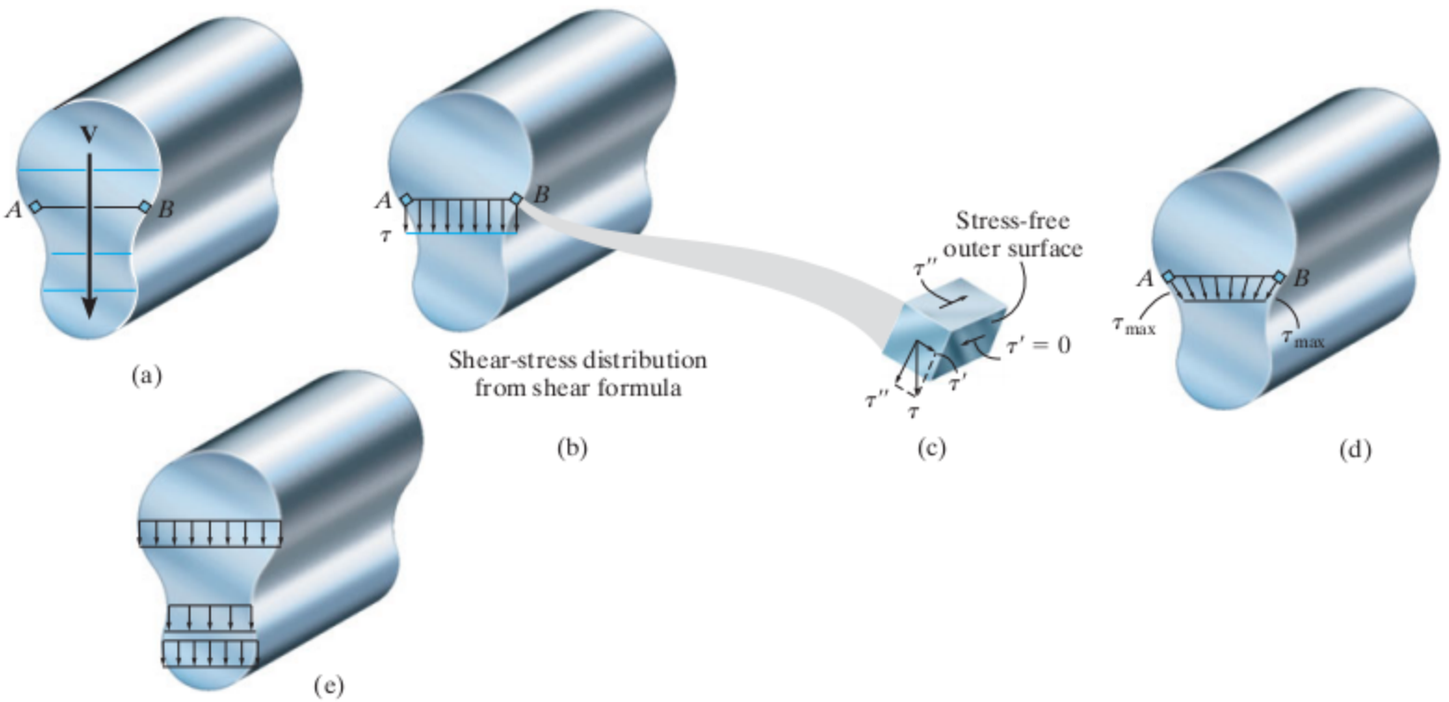


Fig. 12–8

12

Important Points

- Shear forces in beams cause *nonlinear shear-strain* distributions over the cross section, causing it to *warp*.
- Due to the complementary property of shear stress, the shear stress developed in a beam acts over the cross section of the beam and along its longitudinal planes.
- The *shear formula* was derived by considering horizontal force equilibrium of the longitudinal shear-stress and bending-stress distributions acting on a portion of a differential segment of the beam.
- The shear formula is to be used on straight prismatic members made of homogeneous material that has linear elastic behavior. Also, the internal resultant shear force must be directed along an axis of symmetry for the cross-sectional area.
- The shear formula should not be used to determine the shear stress on cross sections that are short or flat, at points of sudden cross-sectional changes, or at a point on an inclined boundary.

Procedure for Analysis

In order to apply the shear formula, the following procedure is suggested.

Internal Shear.

- Section the member perpendicular to its axis at the point where the shear stress is to be determined, and obtain the internal shear **V** at the section.

Section Properties.

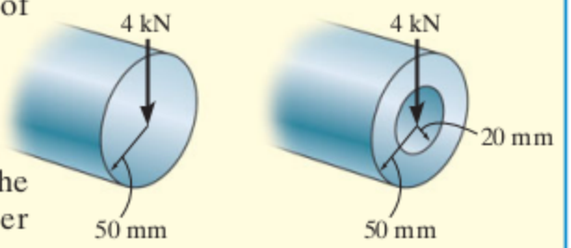
- Determine the location of the neutral axis, and determine the moment of inertia I of the *entire cross-sectional area* about the neutral axis.
- Pass an imaginary horizontal section through the point where the shear stress is to be determined. Measure the width t of the cross-sectional area at this section.
- The portion of the area lying either above or below this width is A'. Determine Q by using $Q = \bar{y}'A'$. Here $\bar{y}'$ is the distance to the centroid of A', measured from the neutral axis. It may be helpful to realize that A' is the portion of the member's cross-sectional area that is being "held onto the member" by the longitudinal shear stresses. See Fig. 12–4*c*.

Shear Stress.

- Using a consistent set of units, substitute the data into the shear formula and calculate the shear stress τ.
- It is suggested that the direction of the transverse shear stress τ be established on a volume element of material located at the point where it is calculated. This can be done by realizing that τ acts on the cross section in the same direction as **V**. From this, the corresponding shear stresses acting on the other three planes of the element can then be established.

EXAMPLE 12.1

The solid shaft and tube shown in Fig. 12–9*a* are subjected to the shear force of 4 kN. Determine the shear stress acting over the diameter of each cross section.



(a)

SOLUTION

Section Properties. Using the table on the inside front cover, the moment of inertia of each section, calculated about its diameter (or neutral axis), is

$$I_{\text{solid}} = \frac{1}{4}\pi c^4 = \frac{1}{4}\pi(0.05\ \text{m})^4 = 4.909(10^{-6})\ \text{m}^4$$

$$I_{\text{tube}} = \frac{1}{4}\pi(c_o^4 - c_i^4) = \frac{1}{4}\pi\left[(0.05\ \text{m})^4 - (0.02\ \text{m})^4\right] = 4.783(10^{-6})\ \text{m}^4$$

The semicircular area shown shaded in Fig. 12–9*b*, above (or below) each diameter, represents Q, because this area is "held onto the member" by the longitudinal shear stress along the diameter.

$$Q_{\text{solid}} = \bar{y}'A' = \frac{4c}{3\pi}\left(\frac{\pi c^2}{2}\right) = \frac{4(0.05\ \text{m})}{3\pi}\left(\frac{\pi(0.05\ \text{m})^2}{2}\right) = 83.33\,(10^{-6})\ \text{m}^3$$

$$Q_{\text{tube}} = \Sigma\bar{y}'A' = \frac{4c_o}{3\pi}\left(\frac{\pi c_o^2}{2}\right) - \frac{4c_i}{3\pi}\left(\frac{\pi c_i^2}{2}\right)$$

$$= \frac{4(0.05\ \text{m})}{3\pi}\left(\frac{\pi(0.05\ \text{m})^2}{2}\right) - \frac{4(0.02\ \text{m})}{3\pi}\left(\frac{\pi(0.02\ \text{m})^2}{2}\right)$$

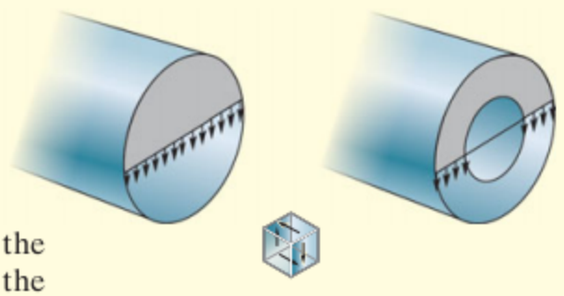
$$= 78.0(10^{-6})\ \text{m}^3$$

Shear Stress. Applying the shear formula where $t = 0.1$ m for the solid section, and $t = 2(0.03\ \text{m}) = 0.06$ m for the tube, we have

$$\tau_{\text{solid}} = \frac{VQ}{It} = \frac{4(10^3)\ \text{N}(83.33(10^{-6})\ \text{m}^3)}{4.909(10^{-6})\ \text{m}^4\,(0.1\ \text{m})} = 679\ \text{kPa} \qquad \textit{Ans.}$$

$$\tau_{\text{tube}} = \frac{VQ}{It} = \frac{4(10^3)\ \text{N}(78.0(10^{-6})\ \text{m}^3)}{4.783(10^{-6})\ \text{m}^4\,(0.06\ \text{m})} = 1.09\ \text{MPa} \qquad \textit{Ans.}$$

NOTE: As discussed in the limitations for the shear formula, the calculations performed here are valid since the shear stress along the diameter is vertical and therefore tangent to the boundary of the cross section. An element of material on the diameter is subjected to "pure shear" as shown in Fig. 12–9*b*.



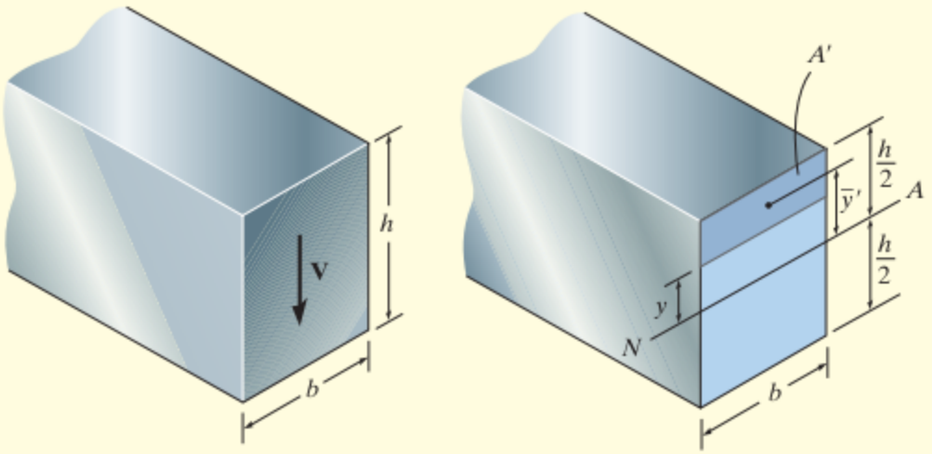
(b)

Fig. 12–9

12

EXAMPLE 12.2

Determine the distribution of the shear stress over the cross section of the beam shown in Fig. 12–10*a*.

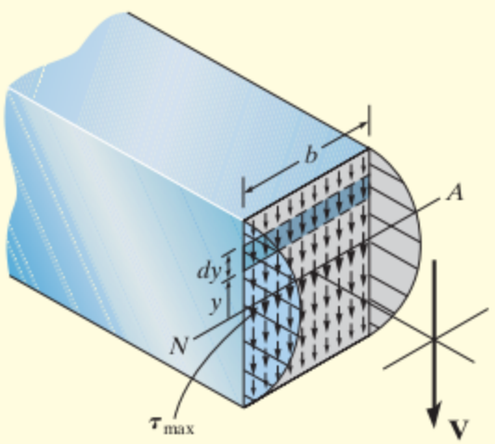


(a) (b)

SOLUTION

The distribution can be determined by finding the shear stress at an *arbitrary height y* from the neutral axis, Fig. 12–10*b*, and then plotting this function. Here, the dark colored area A' will be used for Q.* Hence

$$Q = \bar{y}'A' = \left[y + \frac{1}{2}\left(\frac{h}{2} - y\right)\right]\left(\frac{h}{2} - y\right)b = \frac{1}{2}\left(\frac{h^2}{4} - y^2\right)b$$



Shear–stress distribution

(c)

Fig. 12–10

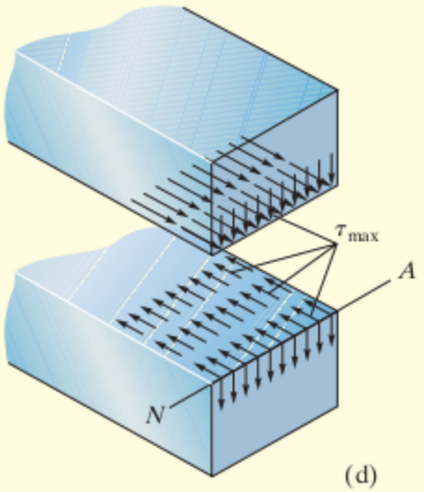
Applying the shear formula, we have

$$\tau = \frac{VQ}{It} = \frac{V\left(\frac{1}{2}\right)\left[(h^2/4) - y^2\right]b}{\left(\frac{1}{12}bh^3\right)b} = \frac{6V}{bh^3}\left(\frac{h^2}{4} - y^2\right) \qquad (1)$$

This result indicates that the shear-stress distribution over the cross section is ***parabolic***. As shown in Fig. 12–10*c*, the intensity varies from zero at the top and bottom, $y = \pm h/2$, to a maximum value at the neutral axis, $y = 0$. Specifically, since the area of the cross section is $A = bh$, then at $y = 0$ we have

$$\tau_{\max} = 1.5\frac{V}{A} \qquad (2)$$

*The area below y can also be used $[A' = b(h/2 + y)]$, but doing so involves a bit more algebraic manipulation.



(d)

Fig. 12–10 (cont.)

Typical shear failure of this wooden beam occurred at the support and through the approximate center of its cross section.

This same value for τ_{max} can be obtained directly from the shear formula, $\tau = VQ/It$, by realizing that τ_{max} occurs where Q is *largest*, since V, I, and t are *constant*. By inspection, Q will be a maximum when the entire area above (or below) the neutral axis is considered; that is, $A' = bh/2$ and $\bar{y}' = h/4$. Thus,

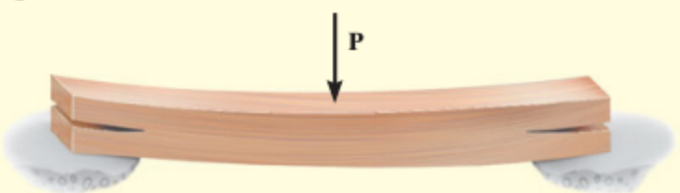
$$\tau_{max} = \frac{VQ}{It} = \frac{V(h/4)(bh/2)}{\left[\frac{1}{12}bh^3\right]b} = 1.5\,\frac{V}{A}$$

By comparison, τ_{max} is 50% greater than the *average* shear stress determined from Eq. 7–4; that is, $\tau_{avg} = V/A$.

It is important to realize that τ_{max} also acts in the longitudinal direction of the beam, Fig. 12–10*d*. It is this stress that can cause a timber beam to fail as shown Fig. 12–10*e*. Here horizontal splitting of the wood starts to occur through the neutral axis at the beam's ends, since there the vertical reactions subject the beam to large shear stress and wood has a low resistance to shear along its grains, which are oriented in the longitudinal direction.

It is instructive to show that when the shear-stress distribution, Eq. (1), is integrated over the cross section it yields the resultant shear V. To do this, a differential strip of area $dA = b\,dy$ is chosen, Fig. 12–10*c*, and since τ acts uniformly over this strip, we have

$$\begin{aligned}\int_A \tau\,dA &= \int_{-h/2}^{h/2} \frac{6V}{bh^3}\left(\frac{h^2}{4} - y^2\right)b\,dy \\ &= \frac{6V}{h^3}\left[\frac{h^2}{4}y - \frac{1}{3}y^3\right]_{-h/2}^{h/2} \\ &= \frac{6V}{h^3}\left[\frac{h^2}{4}\left(\frac{h}{2} + \frac{h}{2}\right) - \frac{1}{3}\left(\frac{h^3}{8} + \frac{h^3}{8}\right)\right] = V\end{aligned}$$

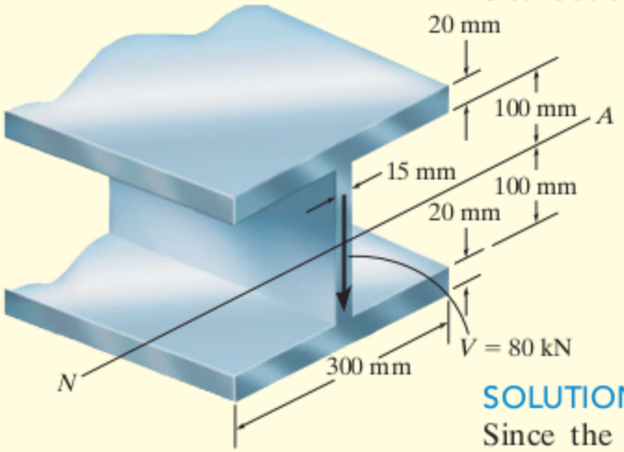


(e)

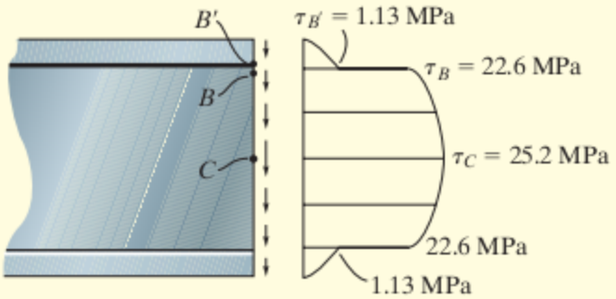
12

EXAMPLE 12.3

A steel wide-flange beam has the dimensions shown in Fig. 12–11*a*. If it is subjected to a shear of $V = 80\,\text{kN}$, plot the shear-stress distribution acting over the beam's cross-sectional area.



(a)



(b)

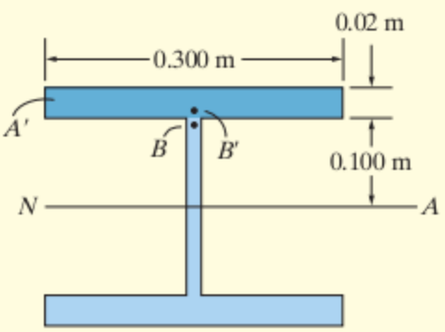
SOLUTION

Since the flange and web are rectangular elements, then like the previous example, the shear-stress distribution will be parabolic and in this case it will vary in the manner shown in Fig. 12–11*b*. Due to symmetry, only the shear stresses at points B', B, and C have to be determined. To show how these values are obtained, we must first determine the moment of inertia of the cross-sectional area about the neutral axis. Working in meters, we have

$$\begin{aligned} I &= \left[\frac{1}{12}(0.015\text{ m})(0.200\text{ m})^3\right] \\ &\quad + 2\left[\frac{1}{12}(0.300\text{ m})(0.02\text{ m})^3 + (0.300\text{ m})(0.02\text{ m})(0.110\text{ m})^2\right] \\ &= 155.6(10^{-6})\text{ m}^4 \end{aligned}$$

For point B', $t_{B'} = 0.300\text{ m}$, and A' is the dark shaded area shown in Fig. 12–11*c*. Thus,

$$Q_{B'} = \bar{y}'A' = [0.110\text{ m}](0.300\text{ m})(0.02\text{ m}) = 0.660(10^{-3})\text{ m}^3$$



(c)

Fig. 12–11

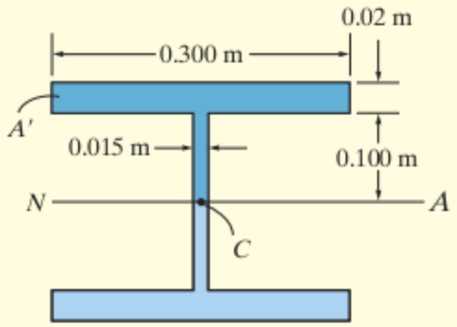
so that

$$\tau_{B'} = \frac{VQ_{B'}}{It_{B'}} = \frac{80(10^3)\text{ N}(0.660(10^{-3})\text{ m}^3)}{155.6(10^{-6})\text{ m}^4\,(0.300\text{ m})} = 1.13\text{ MPa}$$

For point B, $t_B = 0.015\text{ m}$ and $Q_B = Q_{B'}$, Fig. 12–11*c*. Hence

$$\tau_B = \frac{VQ_B}{It_B} = \frac{80(10^3)\text{ N}(0.660(10^{-3})\text{ m}^3)}{155.6(10^{-6})\text{ m}^4(0.015\text{ m})} = 22.6\text{ MPa}$$

Note from the discussion of "Limitations on the Use of the Shear Formula" that the calculated value for both $\tau_{B'}$ and τ_B will actually be very misleading. Why?



(d)

Fig. 12–11 (cont.)

For point C, $t_C = 0.015$ m and A' is the dark shaded area shown in Fig. 12–11*d*. Considering this area to be composed of two rectangles, we have

$$\begin{aligned} Q_C &= \Sigma \bar{y}' A' = [0.110\text{ m}](0.300\text{ m})(0.02\text{ m}) \\ &\quad + [0.05\text{ m}](0.015\text{ m})(0.100\text{ m}) \\ &= 0.735(10^{-3})\text{ m}^3 \end{aligned}$$

Thus,

$$\tau_C = \tau_{\max} = \frac{VQ_C}{It_C} = \frac{80(10^3)\text{ N}[0.735(10^{-3})\text{ m}^3]}{155.6(10^{-6})\text{ m}^4\,(0.015\text{ m})} = 25.2\text{ MPa}$$

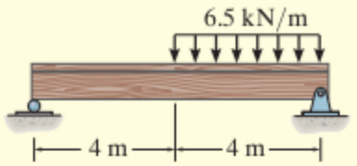
NOTE: From Fig. 12–11*b*, note that most of the shear stress occurs in the web and is almost uniform throughout its depth, varying from 22.6 MPa to 25.2 MPa. It is for this reason that for design, some codes permit the use of calculating the *average* shear stress on the cross section of the web, rather than using the shear formula. That is,

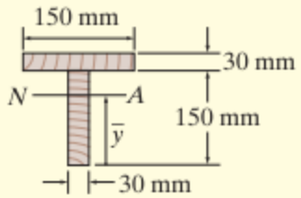
$$\tau_{\text{avg}} = \frac{V}{A_w} = \frac{80(10^3)\text{ N}}{(0.015\text{ m})(0.2\text{ m})} = 26.7\text{ MPa}$$

This will be discussed further in Chapter 15.

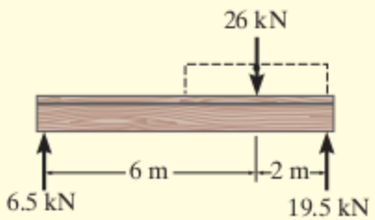
12

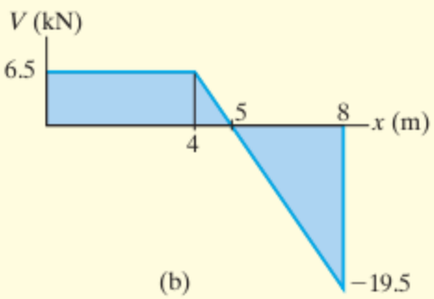
EXAMPLE 12.4



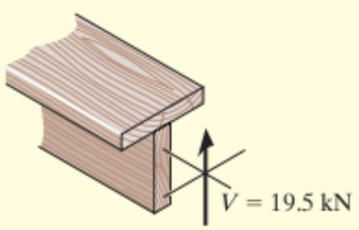


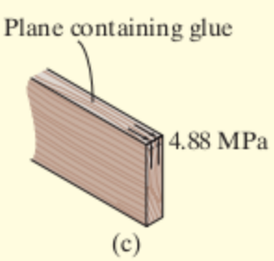
(a)





(b)





(c)

Fig. 12–12

The beam shown in Fig. 12–12*a* is made from two boards. Determine the maximum shear stress in the glue necessary to hold the boards together along the seam where they are joined.

SOLUTION

Internal Shear. The support reactions and the shear diagram for the beam are shown in Fig. 12–12*b*. It is seen that the maximum shear in the beam is 19.5 kN.

Section Properties. The centroid and therefore the neutral axis will be determined from the reference axis placed at the bottom of the cross-sectional area, Fig. 12–12*a*. Working in units of meters, we have

$$\bar{y} = \frac{\Sigma \tilde{y} A}{\Sigma A}$$

$$= \frac{[0.075\text{ m}](0.150\text{ m})(0.030\text{ m}) + [0.165\text{ m}](0.030\text{ m})(0.150\text{ m})}{(0.150\text{ m})(0.030\text{ m}) + (0.030\text{ m})(0.150\text{ m})} = 0.120\text{ m}$$

The moment of inertia, about the neutral axis, Fig. 12–12*a*, is therefore

$$I = \left[\frac{1}{12}(0.030\text{ m})(0.150\text{ m})^3 + (0.150\text{ m})(0.030\text{ m})(0.120\text{ m} - 0.075\text{ m})^2\right]$$

$$+ \left[\frac{1}{12}(0.150\text{ m})(0.030\text{ m})^3 + (0.030\text{ m})(0.150\text{ m})(0.165\text{ m} - 0.120\text{ m})^2\right]$$

$$= 27.0(10^{-6})\text{ m}^4$$

The top board (flange) is being held onto the bottom board (web) by the glue, which is applied over the thickness $t = 0.03$ m. Consequently A' is defined as the area of the top board, Fig. 12–12*a*. We have

$$Q = \bar{y}'A' = [0.180\text{ m} - 0.015\text{ m} - 0.120\text{ m}](0.03\text{ m})(0.150\text{ m})$$

$$= 0.2025(10^{-3})\text{ m}^3$$

Shear Stress. Using the above data and applying the shear formula yields

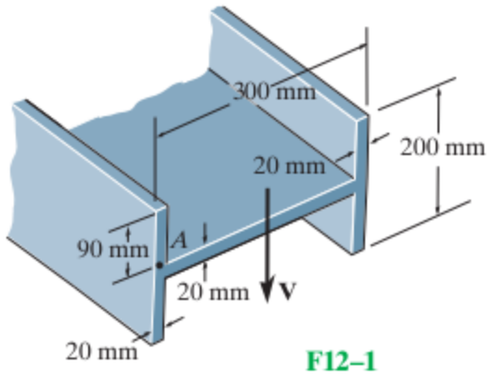
$$\tau_{max} = \frac{VQ}{It} = \frac{19.5(10^3)\text{ N}(0.2025(10^{-3})\text{ m}^3)}{27.0(10^{-6})\text{ m}^4\,(0.030\text{ m})} = 4.88\text{ MPa} \qquad \textit{Ans.}$$

The shear stress acting at the top of the bottom board is shown in Fig. 12–12*c*.

NOTE: It is the glue's resistance to this longitudinal *shear* stress that holds the boards from slipping at the right-hand support.

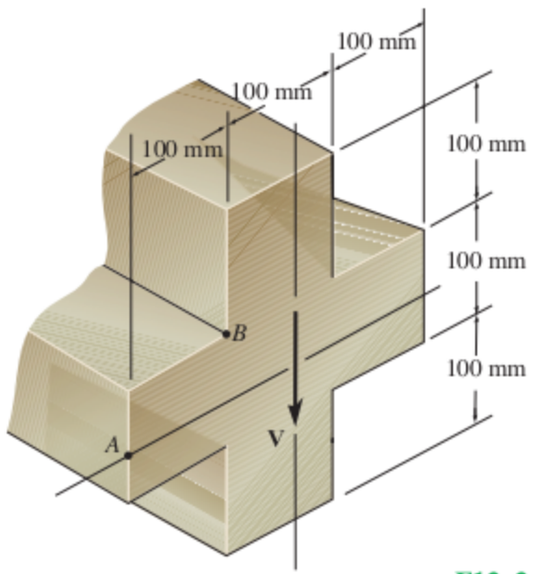
FUNDAMENTAL PROBLEMS

F12–1. If the beam is subjected to a shear force of $V = 100$ kN, determine the shear stress developed at point A. Represent the state of stress on a volume element at this point.



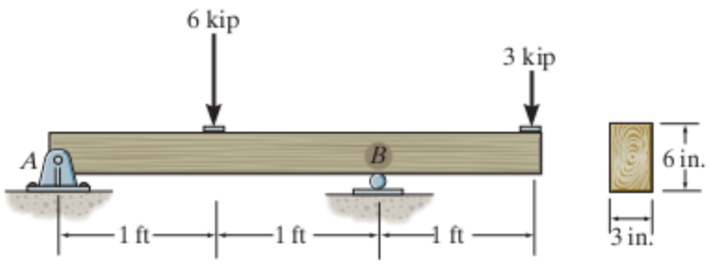
F12–1

F12–2. Determine the shear stress at points A and B on the beam if it is subjected to a shear force of $V = 600$ kN. Represent the state of stress on a volume element at each of these points.



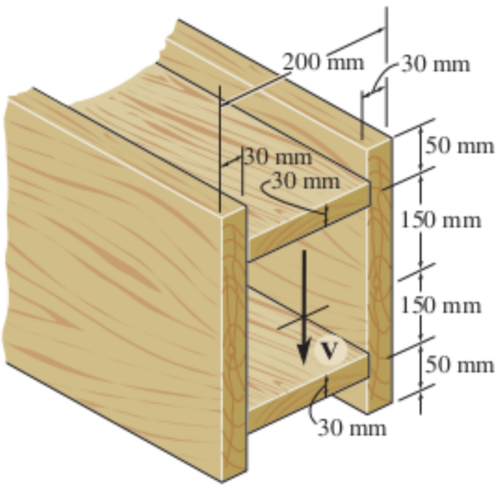
F12–2

F12–3. Determine the absolute maximum shear stress developed in the beam.



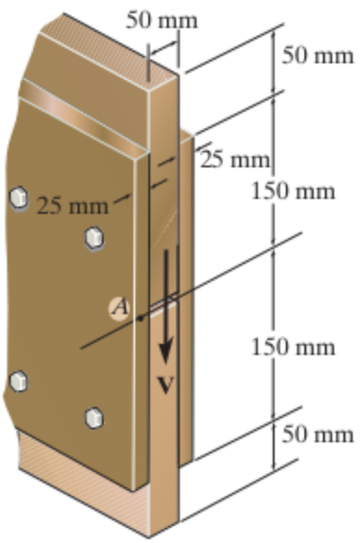
F12–3

F12–4. If the beam is subjected to a shear force of $V = 20$ kN, determine the maximum shear stress developed in the beam.



F12–4

F12–5. If the beam is made from four plates and subjected to a shear force of $V = 20$ kN, determine the maximum shear stress developed in the beam at point A. Represent the state of stress on a volume element at this point.



F12–5

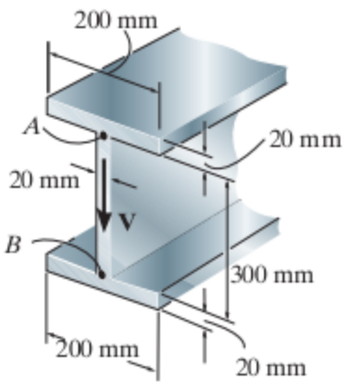
12

PROBLEMS

12–1. If the wide-flange beam is subjected to a shear of $V = 20$ kN, determine the shear stress on the web at A. Indicate the shear-stress components on a volume element located at this point.

12–2. If the wide-flange beam is subjected to a shear of $V = 20$ kN, determine the maximum shear stress in the beam.

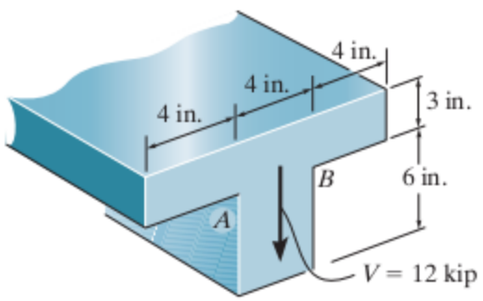
12–3. If the wide-flange beam is subjected to a shear of $V = 20$ kN, determine the shear force resisted by the web of the beam.



Probs. 12–1/2/3

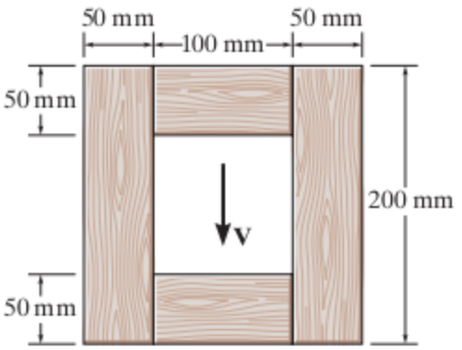
*12–4. If the T-beam is subjected to a vertical shear of $V = 12$ kip, determine the maximum shear stress in the beam. Also, compute the shear-stress jump at the flange-web junction AB. Sketch the variation of the shear-stress intensity over the entire cross section.

12–5. If the T-beam is subjected to a vertical shear of $V = 12$ kip, determine the vertical shear force resisted by the flange.



Probs. 12–4/5

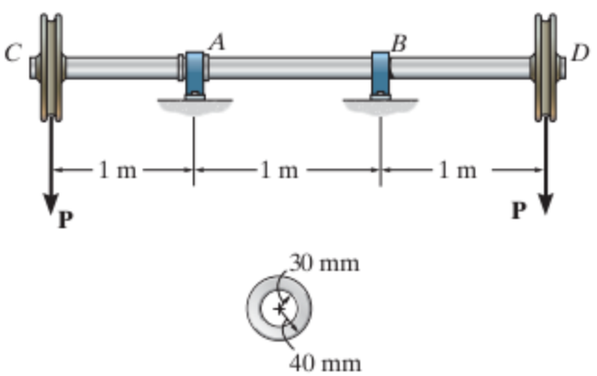
12–6. The wood beam has an allowable shear stress of $\tau_{allow} = 7$ MPa. Determine the maximum shear force V that can be applied to the cross section.



Prob. 12–6

12–7. The shaft is supported by a smooth thrust bearing at A and a smooth journal bearing at B. If $P = 20$ kN, determine the absolute maximum shear stress in the shaft.

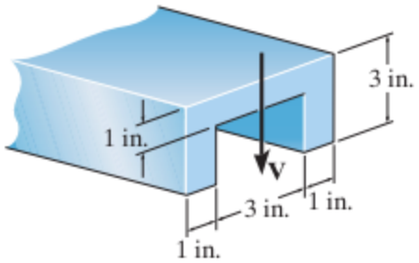
*12–8. The shaft is supported by a smooth thrust bearing at A and a smooth journal bearing at B. If the shaft is made from a material having an allowable shear stress of $\tau_{allow} = 75$ MPa, determine the maximum value for P.



Probs. 12–7/8

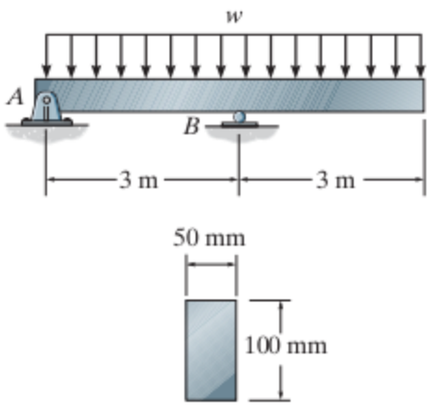
12–9. Determine the largest shear force V that the member can sustain if the allowable shear stress is $\tau_{allow} = 8$ ksi.

12–10. If the applied shear force $V = 18$ kip, determine the maximum shear stress in the member.



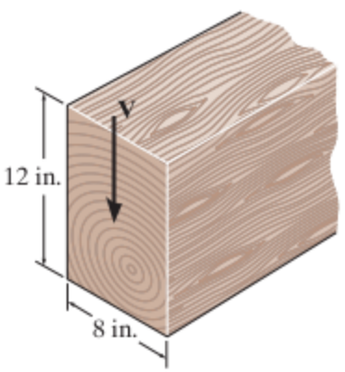
Probs. 12–9/10

12–11. The overhang beam is subjected to the uniform distributed load having an intensity of $w = 50$ kN/m. Determine the maximum shear stress developed in the beam.



Prob. 12–11

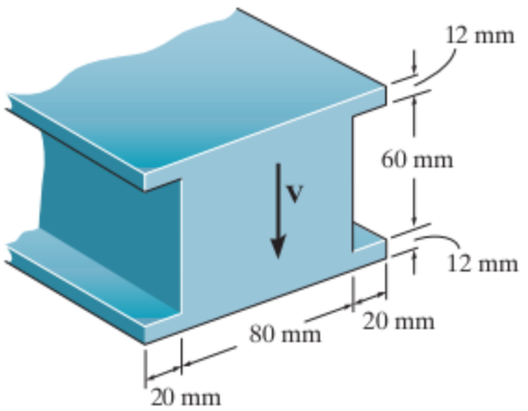
***12–12.** The beam has a rectangular cross section and is made of wood having an allowable shear stress of $\tau_{allow} = 200$ psi. Determine the maximum shear force V that can be developed in the cross section of the beam. Also, plot the shear-stress variation over the cross section.



Prob. 12–12

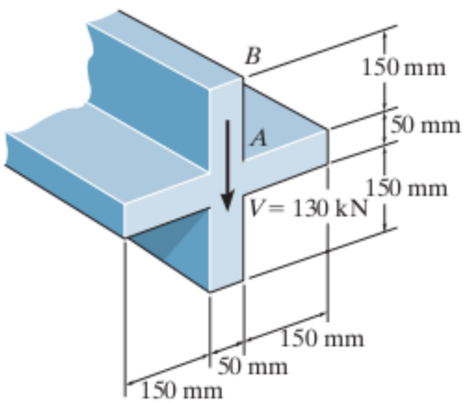
12–13. Determine the maximum shear stress in the strut if it is subjected to a shear force of $V = 20$ kN.

12–14. Determine the maximum shear force V that the strut can support if the allowable shear stress for the material is $\tau_{allow} = 40$ MPa.



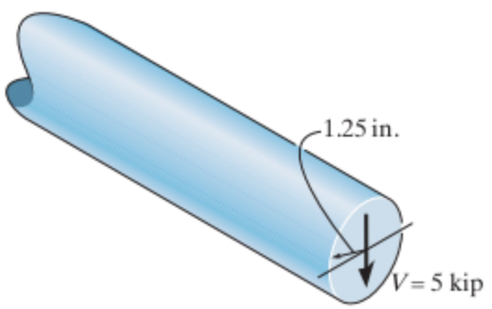
Probs. 12–13/14

12–15. The strut is subjected to a vertical shear of $V = 130$ kN. Plot the intensity of the shear-stress distribution acting over the cross-sectional area, and compute the resultant shear force developed in the vertical segment AB.



Prob. 12–15

***12–16.** The steel rod has a radius of 1.25 in. If it is subjected to a shear of $V = 5$ kip, determine the maximum shear stress.



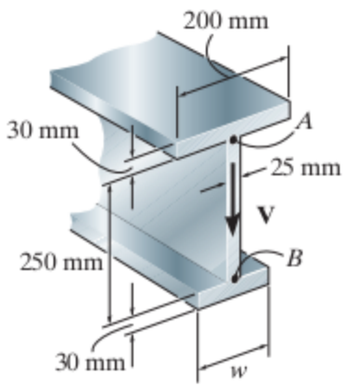
Prob. 12–16

12

12–17. If the beam is subjected to a shear of $V = 15$ kN, determine the web's shear stress at A and B. Indicate the shear-stress components on a volume element located at these points. Set $w = 125$ mm. Show that the neutral axis is located at $\bar{y} = 0.1747$ m from the bottom and $I_{NA} = 0.2182(10^{-3})$ m^4.

12–18. If the wide-flange beam is subjected to a shear of $V = 30$ kN, determine the maximum shear stress in the beam. Set $w = 200$ mm.

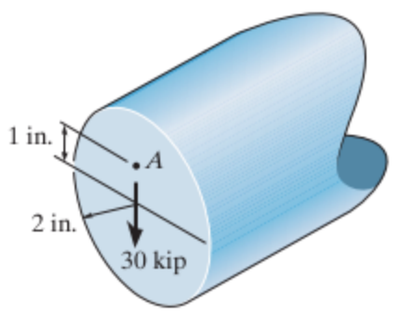
12–19. If the wide-flange beam is subjected to a shear of $V = 30$ kN, determine the shear force resisted by the web of the beam. Set $w = 200$ mm.



Probs. 12–17/18/19

*12–20. The steel rod is subjected to a shear of 30 kip. Determine the maximum shear stress in the rod.

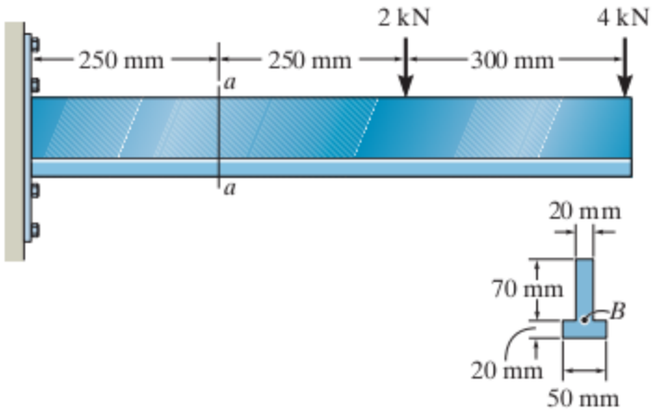
12–21. The steel rod is subjected to a shear of 30 kip. Determine the shear stress at point A. Show the result on a volume element at this point.



Probs. 12–20/21

12–22. Determine the shear stress at point B on the web of the cantilevered strut at section a–a.

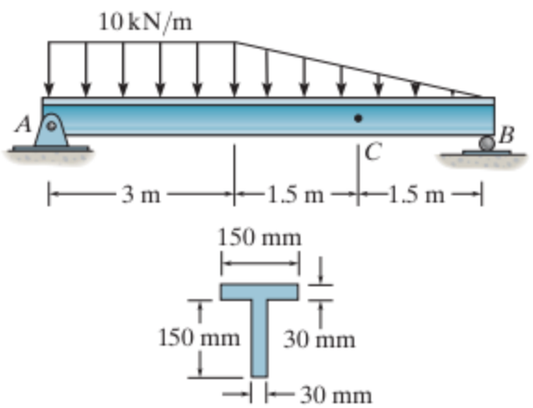
12–23. Determine the maximum shear stress acting at section a–a of the cantilevered strut.



Probs. 12–22/23

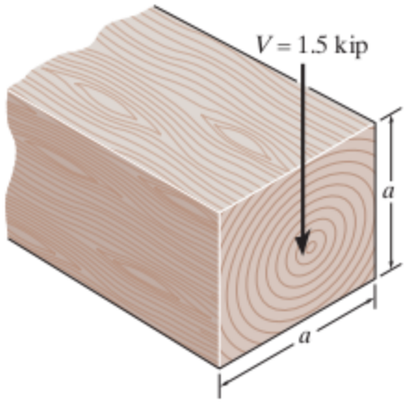
*12–24. Determine the maximum shear stress in the T-beam at the critical section where the internal shear force is maximum.

12–25. Determine the maximum shear stress in the T-beam at section C. Show the result on a volume element at this point.



Probs. 12–24/25

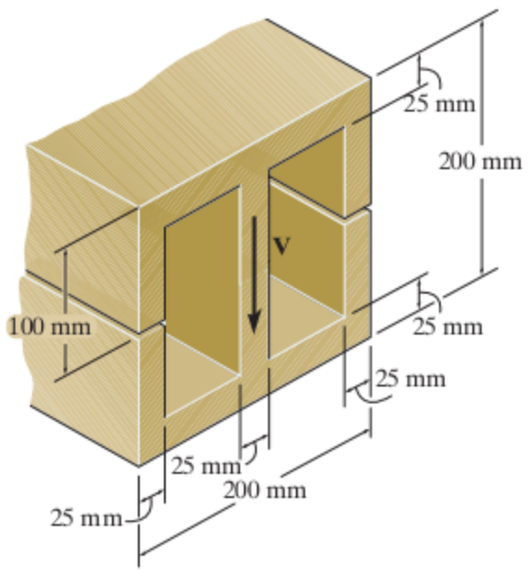
12–26. The beam has a square cross section and is made of wood having an allowable shear stress of $\tau_{allow} = 1.4$ ksi. If it is subjected to a shear of $V = 1.5$ kip, determine the smallest dimension a of its sides.



Prob. 12–26

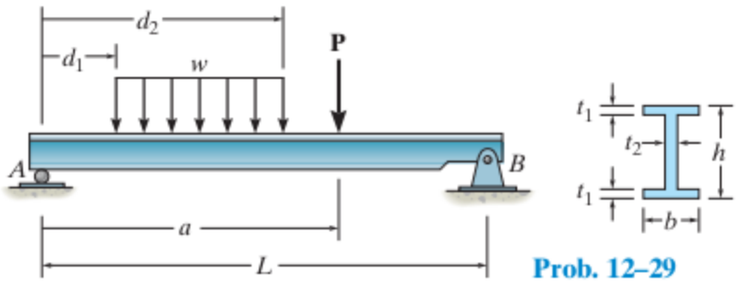
12–27. The beam is slit longitudinally along both sides as shown. If it is subjected to an internal shear of $V = 250$ kN, compare the maximum shear stress developed in the beam before and after the cuts were made.

***12–28.** The beam is to be cut longitudinally along both sides as shown. If it is made from a material having an allowable shear stress of $\tau_{allow} = 75$ MPa, determine the maximum allowable internal shear force **V** that can be applied before and after the cut is made.



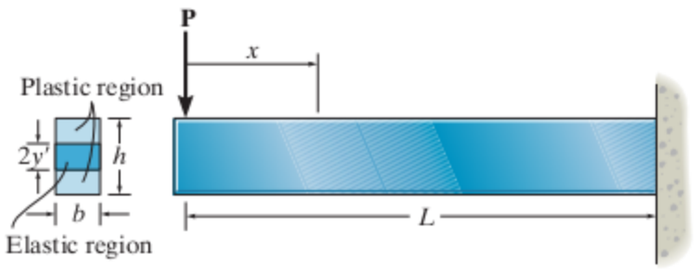
Probs. 12–27/28

12–29. Write a computer program that can be used to determine the maximum shear stress in the beam that has the cross section shown, and is subjected to a specified constant distributed load w and concentrated force P. Show an application of the program using the values $L = 4$ m, $a = 2$ m, $P = 1.5$ kN, $d_1 = 0$, $d_2 = 2$ m, $w = 400$ N/m, $t_1 = 15$ mm, $t_2 = 20$ mm, $b = 50$ mm, and $h = 150$ mm.



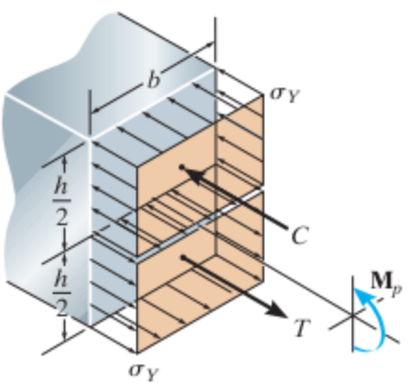
Prob. 12–29

12–30. The beam has a rectangular cross section and is subjected to a load P that is just large enough to develop a fully plastic moment $M_p = PL$ at the fixed support. If the material is elastic-plastic, then at a distance $x < L$ the moment $M = Px$ creates a region of plastic yielding with an associated elastic core having a height $2y'$. Prove that the maximum shear stress developed in the beam is given by $\tau_{max} = \frac{3}{2}(P/A')$, where $A' = 2y'b$, the cross-sectional area of the elastic core.



Prob. 12–30

12–31. The beam in the figure below is subjected to a fully plastic moment $\mathbf{M}_p$. Prove that the longitudinal and transverse shear stresses in the beam are zero. *Hint:* Consider an element of the beam as shown in Fig. 12–4*c*.



Plastic moment

Prob. 12–31

12

Fig. 12–13

12.3 Shear Flow in Built-Up Members

Occasionally in engineering practice, members are "built up" from several composite parts in order to achieve a greater resistance to loads. Examples are shown in Fig. 12–13. If the loads cause the members to bend, fasteners such as nails, bolts, welding material, or glue may be needed to keep the component parts from sliding relative to one another, Fig. 12–2. In order to design these fasteners or determine their spacing, it is necessary to know the shear force that must be resisted by the fastener. This loading, when measured as a force per unit length of beam, is referred to as ***shear flow q***.

The magnitude of the shear flow can be obtained using a development similar to that for finding the shear stress in the beam. To show this, we will consider finding the shear flow along the juncture where the segment in Fig. 12–14*a* is connected to the flange of the beam. As shown in Fig. 12–14*b*, three horizontal forces must act on this segment. Two of these forces, F and $F + dF$, are developed by normal stresses caused by the moments M and $M + dM$, respectively. The third force, which for equilibrium equals dF, acts at the juncture and it is to be supported by the fastener. Realizing that dF is the result of dM, then, like Eq. 12–1, we have

$$dF = \frac{dM}{I}\int_{A'} y\, dA'$$

The integral represents Q, that is, the moment of the segment's area A' in Fig. 12–14*b* about the neutral axis for the entire cross section. Since the segment has a length dx, the shear flow, or force per unit length along the beam, is $q = dF/dx$. Hence dividing both sides by dx and noting that $V = dM/dx$, Eq. 11–2, we can write

$$q = \frac{VQ}{I} \tag{12–4}$$

Here

q = the shear flow, measured as a force per unit length along the beam

V = the internal resultant shear force, determined from the method of sections and the equations of equilibrium

I = the moment of inertia of the *entire* cross-sectional area computed about the neutral axis

$Q = \bar{y}'A'$, where A' is the cross-sectional area of the segment that is connected to the beam at the juncture where the shear flow is to be calculated, and $\bar{y}'$ is the distance from the neutral axis to the centroid of A'

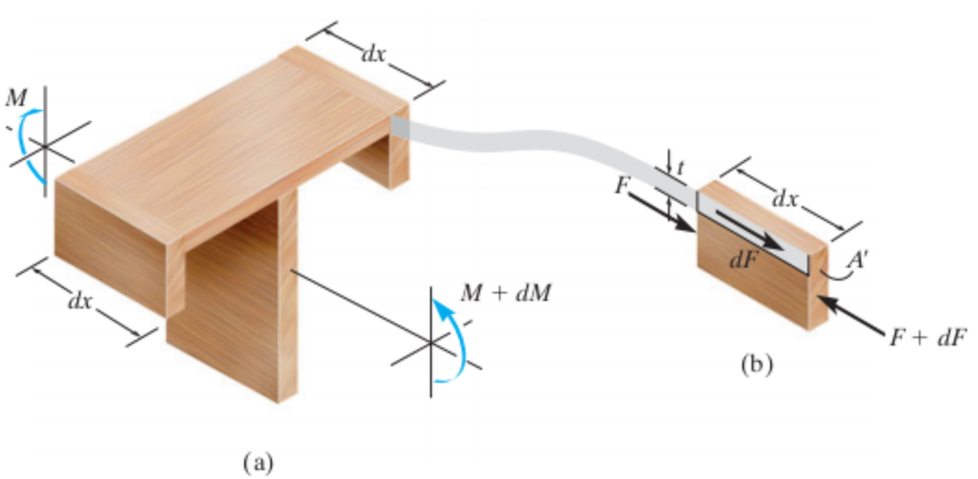


Fig. 12–14

Application of this equation follows the same "procedure for analysis" as outlined in Sec. 12.2 for the shear formula. It is very important to identify Q correctly when determining the shear flow at a particular junction on the cross section. A few examples should serve to illustrate how this is done. Consider the beam cross sections shown in Fig. 12–15. The shaded segments are connected to the beam by fasteners and at the planes of connection, identified by the thick black lines, the shear flow q is determined by using a value of Q calculated from A' and $\bar{y}'$ indicated in each figure. This value of q will be resisted by a *single* fastener in Fig. 12–15*a*, by *two* fasteners in Fig. 12–15*b*, and by *three* fasteners in Fig. 12–15*c*. In other words, the fastener in Fig. 12–15*a* supports the calculated value of q, and in Figs. 12–15*b* and 12–15*c* each fastener supports $q/2$ and $q/3$, respectively.

Important Point

- *Shear flow* is a measure of the force per unit length along the axis of a beam. This value is found from the shear formula and is used to determine the shear force developed in fasteners and glue that holds the various segments of a composite beam together.

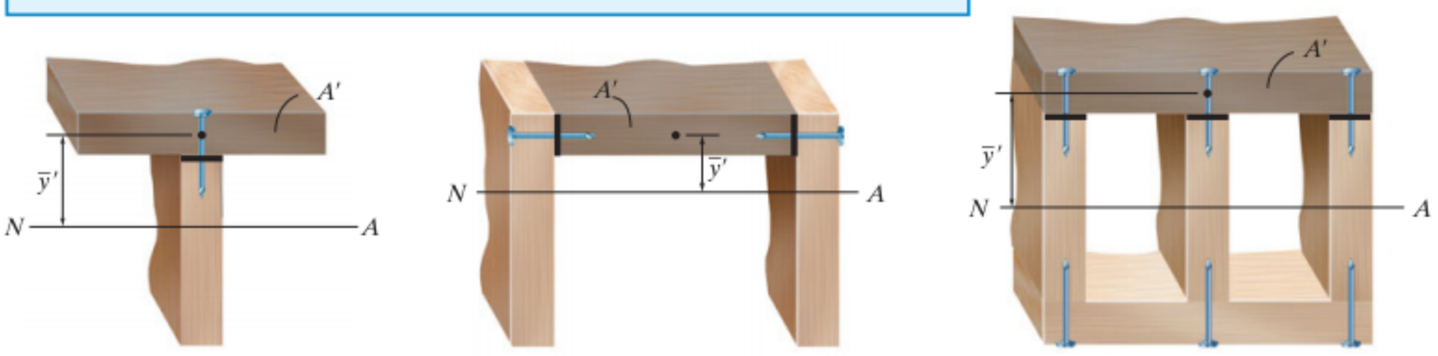
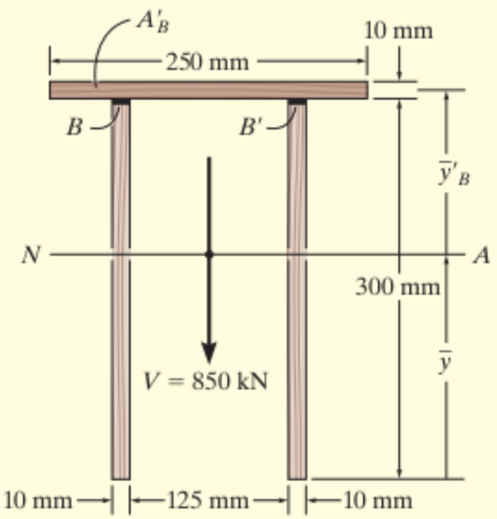


Fig. 12–15

12

EXAMPLE 12.5

The beam is constructed from three boards glued together as shown in Fig. 12–16*a*. If it is subjected to a shear of $V = 850$ kN, determine the shear flow at B and B' that must be resisted by the glue.



(a)

SOLUTION

Section Properties. The neutral axis (centroid) will be located from the bottom of the beam, Fig. 12–16*a*. Working in units of meters, we have

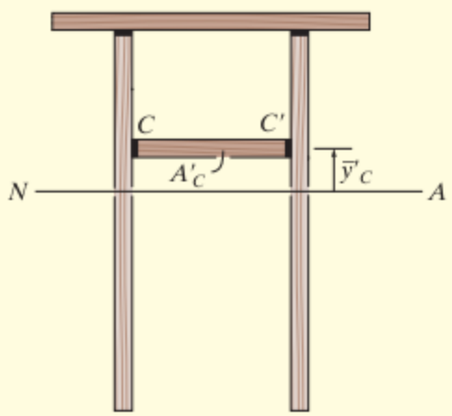
$$\bar{y} = \frac{\Sigma \tilde{y}A}{\Sigma A} = \frac{2[0.15\text{ m}](0.3\text{ m})(0.01\text{ m}) + [0.305\text{ m}](0.250\text{ m})(0.01\text{ m})}{2(0.3\text{ m})(0.01\text{ m}) + 0.250\text{ m}(0.01\text{ m})}$$
$$= 0.1956\text{ m}$$

The moment of inertia about the neutral axis is thus

$$I = 2\left[\frac{1}{12}(0.01\text{ m})(0.3\text{ m})^3 + (0.01\text{ m})(0.3\text{ m})(0.1956\text{ m} - 0.150\text{ m})^2\right]$$
$$+ \left[\frac{1}{12}(0.250\text{ m})(0.01\text{ m})^3 + (0.250\text{ m})(0.01\text{ m})(0.305\text{ m} - 0.1956\text{ m})^2\right]$$
$$= 87.42(10^{-6})\text{ m}^4$$

Since the glue at B and B' in Fig. 12–16*a* "holds" the top board to the beam, we have

$$Q_B = \bar{y}'_B A'_B = [0.305\text{ m} - 0.1956\text{ m}](0.250\text{ m})(0.01\text{ m})$$
$$= 0.2735(10^{-3})\text{ m}^3$$



(b)

Fig. 12–16

Shear Flow. For B and B' we have

$$q'_B = \frac{VQ_B}{I} = \frac{850(10^3)\text{ N}(0.2735(10^{-3})\text{ m}^3)}{87.42(10^{-6})\text{ m}^4} = 2.66\text{ MN/m}$$

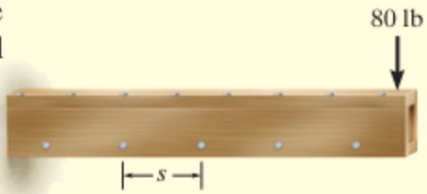
Since *two seams* are used to secure the board, the glue per meter length of beam at the seam must be strong enough to resist *one-half* of the calculated value of q'_B. Thus,

$$q_B = q_{B'} = 1.33\text{ MN/m} \qquad \textit{Ans.}$$

If the board CC' is added to the beam, Fig. 12–16*b*, then $\bar{y}$ and I would have to be recalculated, and the shear flow at C and C' determined from $q'_C = V\,y'_C A'_C / I$. Finally, this value is divided by one-half to obtain q_C and q'_C.

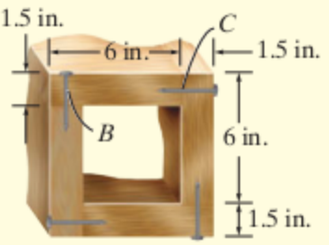
EXAMPLE 12.6

A box beam is constructed from four boards nailed together as shown in Fig. 12–17*a*. If each nail can support a shear force of 30 lb, determine the maximum spacing s of the nails at B and at C so that the beam will support the force of 80 lb.



SOLUTION

Internal Shear. If the beam is sectioned at an *arbitrary point* along its length, the internal shear required for equilibrium is always $V = 80$ lb, and so the shear diagram is shown in Fig. 12–17*b*.



(a)

Section Properties. The moment of inertia of the cross-sectional area about the neutral axis can be determined by considering a 7.5-in. × 7.5-in. square minus a 4.5-in. × 4.5-in. square.

$$I = \frac{1}{12}(7.5\text{ in.})(7.5\text{ in.})^3 - \frac{1}{12}(4.5\text{ in.})(4.5\text{ in.})^3 = 229.5\text{ in}^4$$

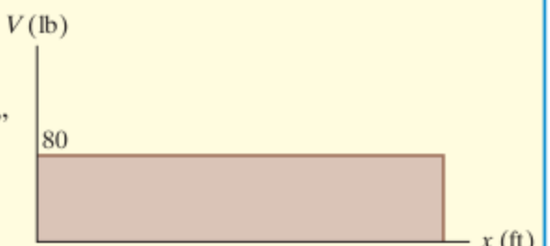
The shear flow at B is determined using Q_B found from the darker shaded area shown in Fig. 12–17*c*. It is this "symmetric" portion of the beam that is to be "held" onto the rest of the beam by nails on the left side and by the fibers of the board on the right side.

Thus,

$$Q_B = \bar{y}'A' = [3\text{ in.}](7.5\text{ in.})(1.5\text{ in.}) = 33.75\text{ in}^3$$

Likewise, the shear flow at C can be determined using the "symmetric" shaded area shown in Fig. 12–17*d*. We have

$$Q_C = \bar{y}'A' = [3\text{ in.}](4.5\text{ in.})(1.5\text{ in.}) = 20.25\text{ in}^3$$

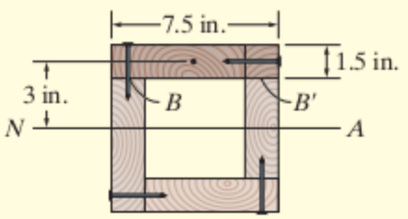


(b)

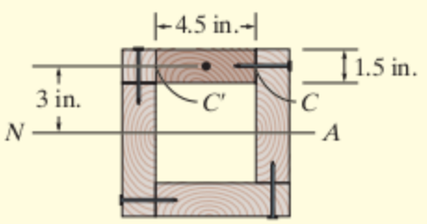
Shear Flow.

$$q_B = \frac{VQ_B}{I} = \frac{80\text{ lb}(33.75\text{ in}^3)}{229.5\text{ in}^4} = 11.76\text{ lb/in.}$$

$$q_C = \frac{VQ_C}{I} = \frac{80\text{ lb}(20.25\text{ in}^3)}{229.5\text{ in}^4} = 7.059\text{ lb/in.}$$



(c)



(d)

Fig. 12–17

These values represent the shear force per unit length of the beam that must be resisted by the nails at B and the fibers at B', Fig. 12–17*c*, and the nails at C and the fibers at C', Fig. 12–17*d*, respectively. Since in each case the shear flow is resisted at *two* surfaces and each nail can resist 30 lb, for B the spacing is

$$s_B = \frac{30\text{ lb}}{(11.76/2)\text{ lb/in.}} = 5.10\text{ in.} \qquad \text{Use } s_B = 5\text{ in.} \qquad \textit{Ans.}$$

And for C,

$$s_C = \frac{30\text{ lb}}{(7.059/2)\text{ lb/in.}} = 8.50\text{ in.} \qquad \text{Use } s_C = 8.5\text{ in.} \qquad \textit{Ans.}$$

12

EXAMPLE 12.7

Nails having a total shear strength of 40 lb are used in a beam that can be constructed either as in Case I or as in Case II, Fig. 12–18. If the nails are spaced at 9 in., determine the largest vertical shear that can be supported in each case so that the fasteners will not fail.

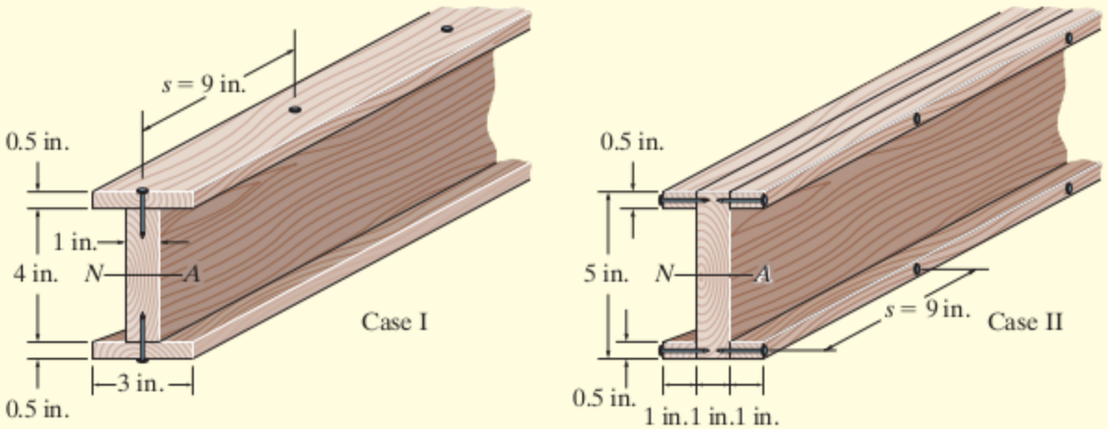


Fig. 12–18

SOLUTION

Since the cross section is the same in both cases, the moment of inertia about the neutral axis is

$$I = \frac{1}{12}(3\text{ in.})(5\text{ in.})^3 - 2\left[\frac{1}{12}(1\text{ in.})(4\text{ in.})^3\right] = 20.58\text{ in}^4$$

Case I. For this design a single row of nails holds the top or bottom flange onto the web. For one of these flanges,

$$Q = \bar{y}'A' = [2.25\text{ in.}](3\text{ in.}(0.5\text{ in.})) = 3.375\text{ in}^3$$

so that

$$q = \frac{VQ}{I}$$

$$\frac{40\text{ lb}}{9\text{ in.}} = \frac{V(3.375\text{ in}^3)}{20.58\text{ in}^4}$$

$$V = 27.1\text{ lb} \qquad \textit{Ans.}$$

Case II. Here a single row of nails holds one of the side boards onto the web. Thus,

$$Q = \bar{y}'A' = [2.25\text{ in.}](1\text{ in.}(0.5\text{ in.})) = 1.125\text{ in}^3$$

$$q = \frac{VQ}{I}$$

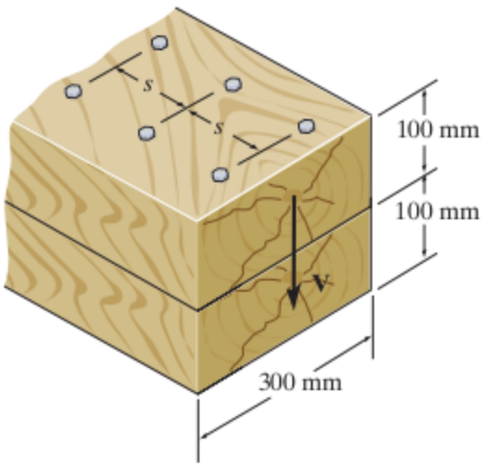
$$\frac{40\text{ lb}}{9\text{ in.}} = \frac{V(1.125\text{ in}^3)}{20.58\text{ in}^4}$$

$$V = 81.3\text{ lb} \qquad \textit{Ans.}$$

FUNDAMENTAL PROBLEMS

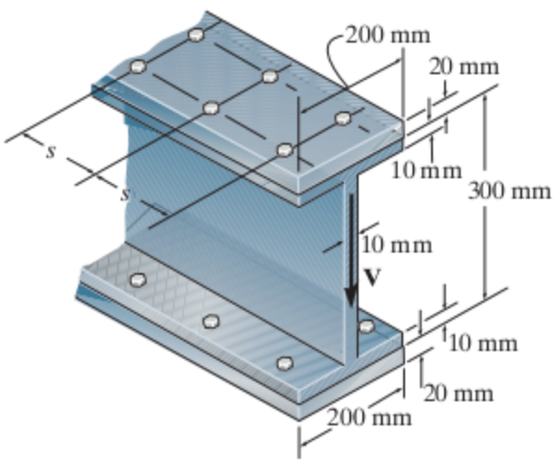
F12–6. The two identical boards are bolted together to form the beam. Determine the maximum allowable spacing s of the bolts to the nearest mm if each bolt has a shear strength of 15 kN. The beam is subjected to a shear force of $V = 50$ kN.

F12–7. The two identical boards are bolted together to form the beam. If the spacing of the bolts is $s = 100$ mm and each bolt has a shear strength of 15 kN, determine the maximum shear force **V** the beam can resist.



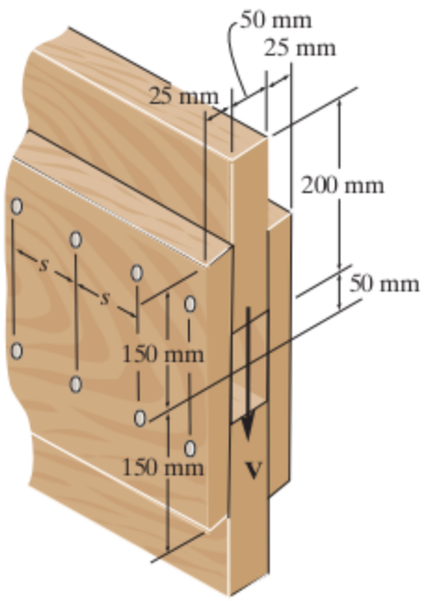
F12–6/7

F12–8. Two identical 20-mm thick plates are bolted to the top and bottom flange to form the built-up beam. If the beam is subjected to a shear force of $V = 300$ kN, determine the allowable maximum spacing s of the bolts to the nearest mm. Each bolt has a shear strength of 30 kN.



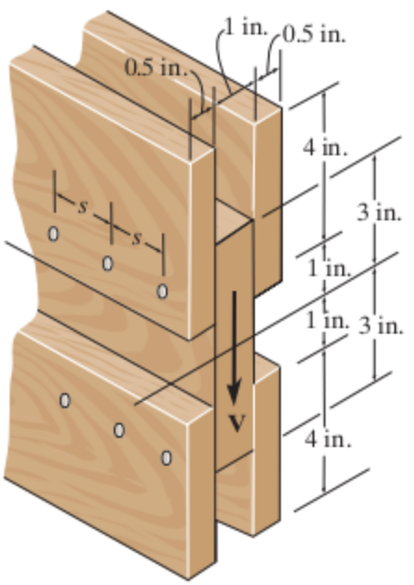
F12–8

F12–9. The boards are bolted together to form the built-up beam. If the beam is subjected to a shear force of $V = 20$ kN, determine the allowable maximum spacing of the bolts to the nearest mm. Each bolt has a shear strength of 8 kN.



F12–9

F12–10. The boards are bolted together to form the built-up beam. If the beam is subjected to a shear force of $V = 15$ kip, determine the allowable maximum spacing of the bolts to the nearest $\frac{1}{8}$ in. Each bolt has a shear strength of 6 kip.



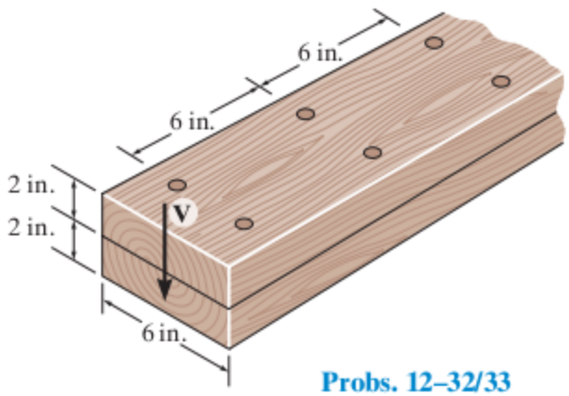
F12–10

12

PROBLEMS

*12–32. The beam is constructed from two boards fastened together at the top and bottom with two rows of nails spaced every 6 in. If each nail can support a 500-lb shear force, determine the maximum shear force V that can be applied to beam.

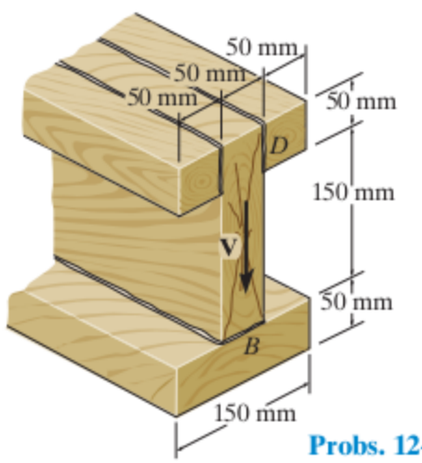
12–33. The beam is construced from two boards fastened together at the top and bottom with two rows of nails spaced every 6 in. If an internal shear force of $V = 600$ lb is applied to the boards, determine the shear force resisted by each nail.



Probs. 12–32/33

12–34. The boards are glued together to form the built-up beam. If the wood has an allowable shear stress of $\tau_{allow} = 3$ MPa, and the glue seam at B can withstand a maximum shear stress of 1.5 MPa, determine the maximum allowable internal shear **V** that can be developed in the beam.

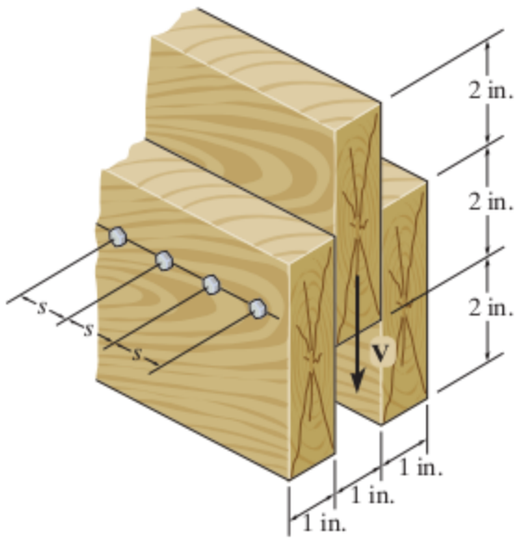
12–35. The boards are glued together to form the built-up beam. If the wood has an allowable shear stress of $\tau_{allow} = 3$ MPa, and the glue seam at D can withstand a maximum shear stress of 1.5 MPa, determine the maximum allowable shear **V** that can be developed in the beam.



Probs. 12–34/35

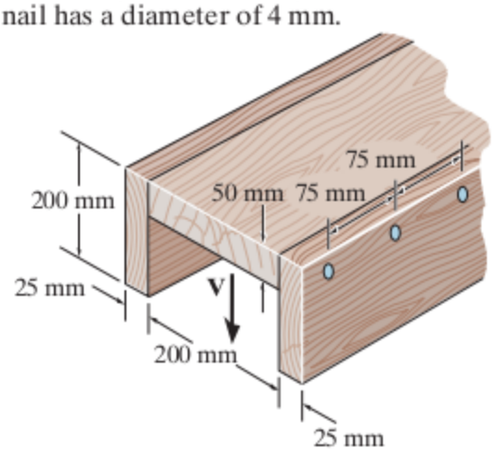
*12–36. Three identical boards are bolted together to form the built-up beam. Each bolt has a shear strength of 1.5 kip and the bolts are spaced at a distance of $s = 6$ in. If the wood has an allowable shear stress of $\tau_{allow} = 450$ psi, determine the maximum allowable internal shear **V** that can act on the beam.

12–37. Three identical boards are bolted together to form the built-up beam. If the wood has an allowable shear stress of $\tau_{allow} = 450$ psi, determine the maximum allowable internal shear **V** that can act on the beam. Also, find the corresponding average shear stress in the $\frac{3}{8}$ in. diameter bolts which are spaced equally at $s = 6$ in.



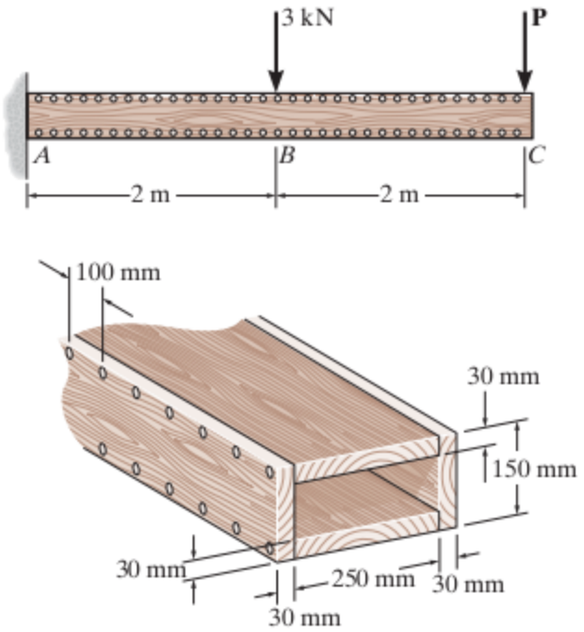
Probs. 12–36/37

12–38. The beam is subjected to a shear of $V = 2$ kN. Determine the average shear stress developed in each nail if the nails are spaced 75 mm apart on each side of the beam. Each nail has a diameter of 4 mm.



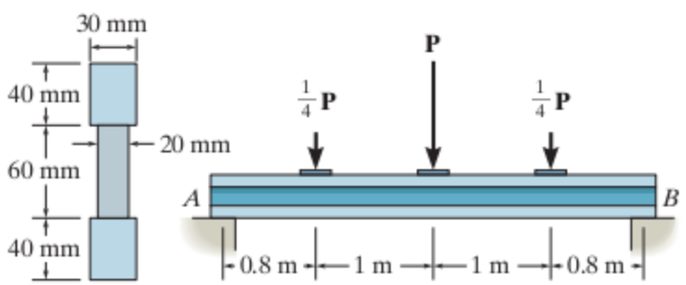
Prob. 12–38

12–39. The beam is constructed from four boards which are nailed together. If the nails are on both sides of the beam and each can resist a shear of 3 kN, determine the maximum load P that can be applied to the end of the beam.



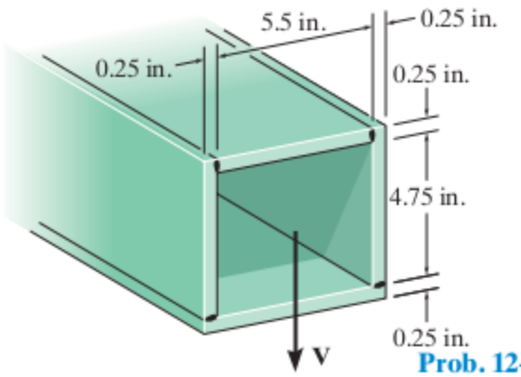
Prob. 12–39

***12–40.** The beam is made from three polystyrene strips that are glued together as shown. If the glue has a shear strength of 80 kPa, determine the maximum load P that can be applied without causing the glue to lose its bond.



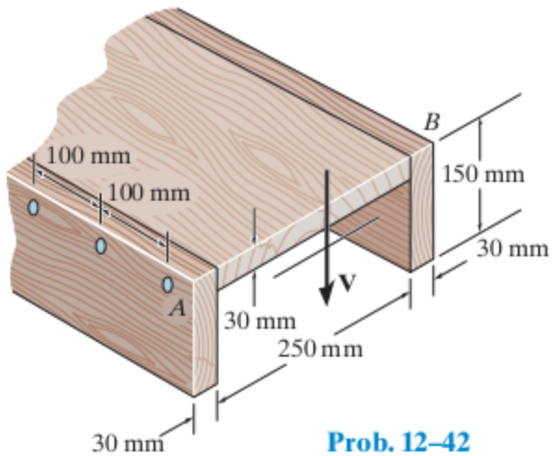
Prob. 12–40

12–41. The box beam is made from four pieces of plastic that are glued together as shown. If $V = 2$ kip, determine the shear stress resisted by the seam at each of the glued joints.



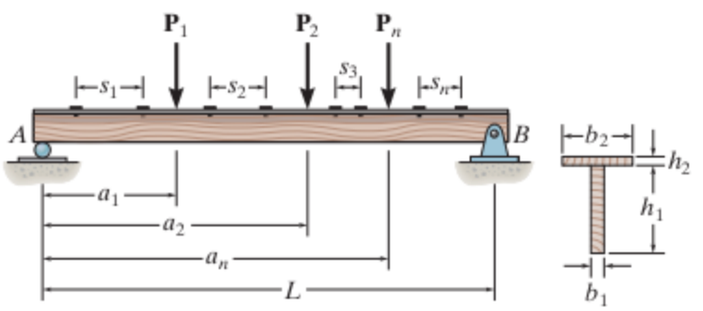
Prob. 12–41

12–42. The beam is subjected to a shear of $V = 800$ N. Determine the average shear stress developed in the nails along the sides A and B if the nails are spaced $s = 100$ mm apart. Each nail has a diameter of 2 mm.



Prob. 12–42

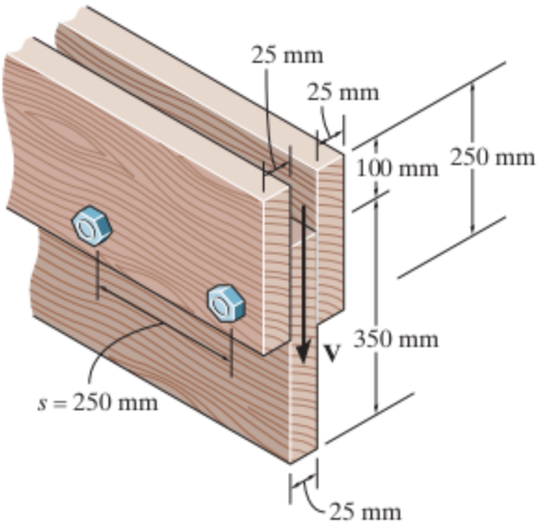
12–43. The timber T-beam is subjected to a load consisting of n concentrated forces, P_n. If the allowable shear V_{nail} for each of the nails is known, write a computer program that will specify the nail spacing between each load. Show an application of the program using the values $L = 15$ ft, $a_1 = 4$ ft, $P_1 = 600$ lb, $a_2 = 8$ ft, $P_2 = 1500$ lb, $b_1 = 1.5$ in., $h_1 = 10$ in., $b_2 = 8$ in., $h_2 = 1$ in., and $V_{nail} = 200$ lb.



Prob. 12–43

12

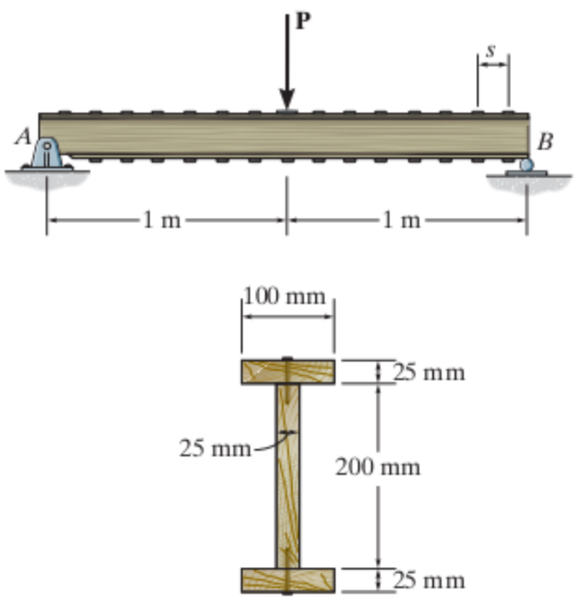
***12–44.** A beam is constructed from three boards bolted together as shown. Determine the shear force developed in each bolt if the bolts are spaced $s = 250$ mm apart and the applied shear is $V = 35$ kN.



Prob. 12–44

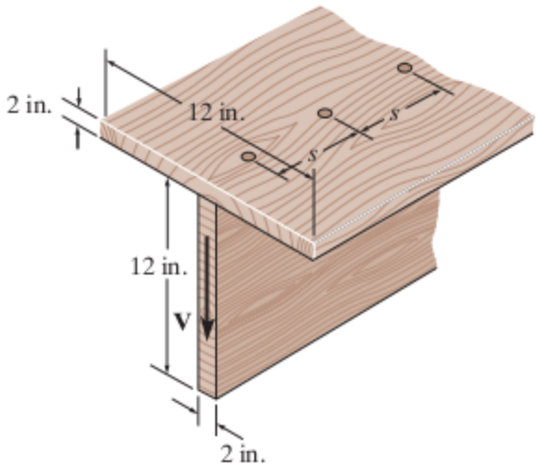
12–45. The simply-supported beam is built-up from three boards by nailing them together as shown. The wood has an allowable shear stress of $\tau_{allow} = 1.5$ MPa, and an allowable bending stress of $\sigma_{allow} = 9$ MPa. The nails are spaced at $s = 75$ mm, and each has a shear strength of 1.5 kN. Determine the maximum allowable force **P** that can be applied to the beam.

12–46. The simply-supported beam is built-up from three boards by nailing them together as shown. If $P = 12$ kN, determine the maximum allowable spacing s of the nails to support that load, if each nail can resist a shear force of 1.5 kN.



Probs. 12–45/46

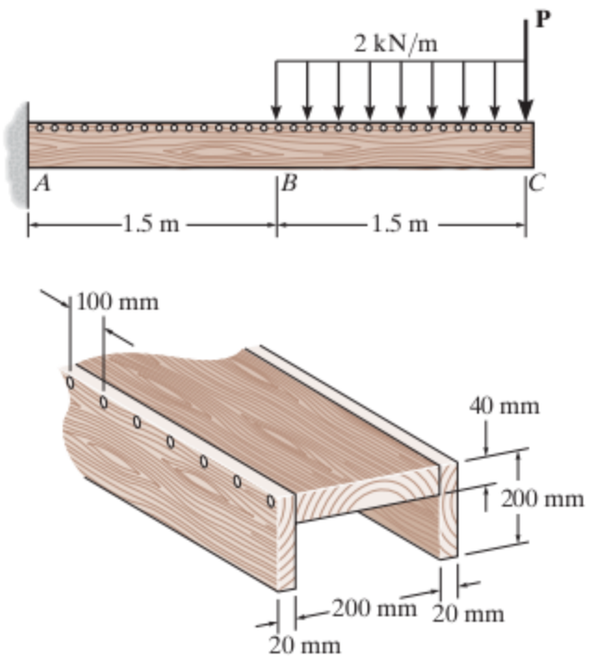
12–47. The T-beam is nailed together as shown. If the nails can each support a shear force of 950 lb, determine the maximum shear force V that the beam can support and the corresponding maximum nail spacing s to the nearest $\frac{1}{8}$ in. The allowable shear stress for the wood is $\tau_{allow} = 450$ psi.



Prob. 12–47

***12–48.** Determine the average shear stress developed in the nails within region AB of the beam. The nails are located on each side of the beam and are spaced 100 mm apart. Each nail has a diameter of 4 mm. Take $P = 2$ kN.

12–49. The nails are on both sides of the beam and each can resist a shear of 2 kN. In addition to the distributed loading, determine the maximum load P that can be applied to the end of the beam. The nails are spaced 100 mm apart and the allowable shear stress for the wood is $\tau_{allow} = 3$ MPa.



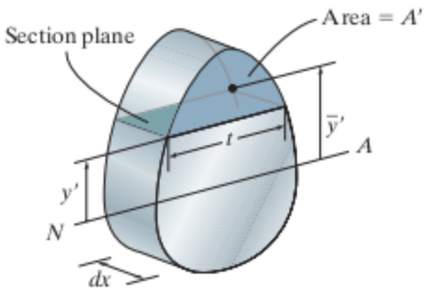
Probs. 12–48/49

CHAPTER REVIEW

Transverse shear stress in beams is determined indirectly by using the flexure formula and the relationship between moment and shear ($V = dM/dx$). The result is the shear formula

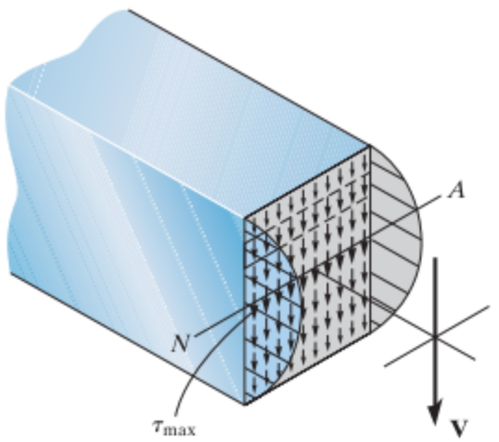
$$\tau = \frac{VQ}{It}$$

In particular, the value for Q is the moment of the area A' about the neutral axis, $Q = \bar{y}'A'$. This area is the portion of the cross-sectional area that is "held on" to the beam above (or below) the thickness t where τ is to be determined.



If the beam has a *rectangular* cross section, then the shear-stress distribution will be parabolic, having a maximum value at the neutral axis. The maximum shear stress can be determined using

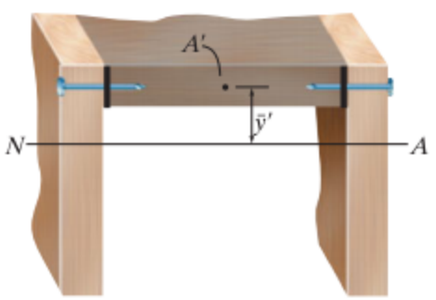
$$\tau = 1.5\,\frac{V}{A}$$



Shear–stress distribution

Fasteners, such as nails, bolts, glue, or welds, are used to connect the composite parts of a "built-up" section. The shear force resisted by these fasteners is determined from the shear flow, q, or force per unit length, that must be carried by the beam. The shear flow is

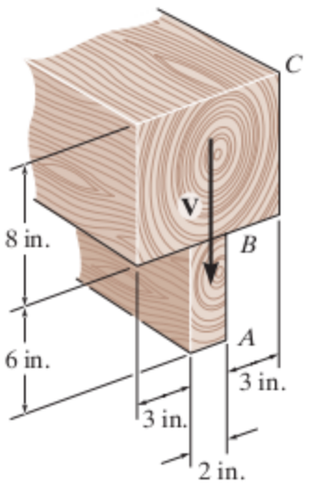
$$q = \frac{VQ}{I}$$



12

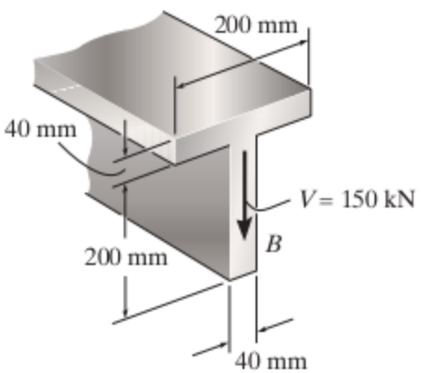
REVIEW PROBLEMS

12–50. Sketch the intensity of the shear-stress distribution acting over the beam's cross-sectional area, and determine the resultant shear force acting on the segment AB. The shear acting at the section is $V = 35$ kip. Show that $I_{NA} = 872.49\ \text{in}^4$.



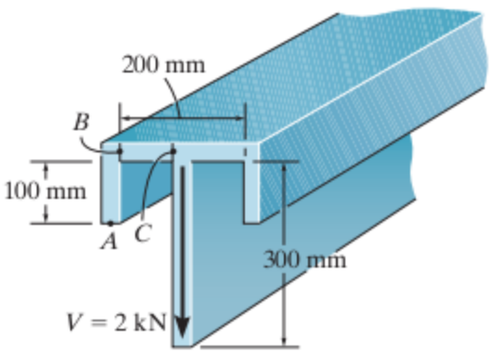
Prob. 12–50

12–51. The T-beam is subjected to a shear of $V = 150$ kN. Determine the amount of this force that is supported by the web B.



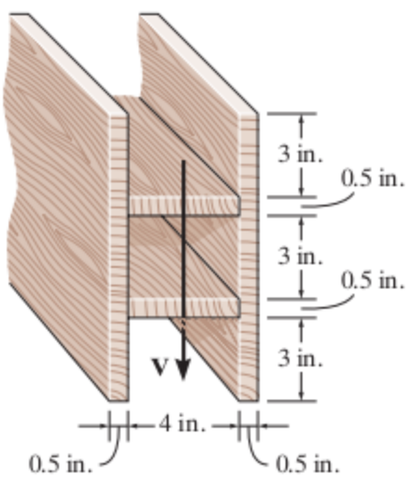
Prob. 12–51

***12–52.** The member is subjected to a shear force of $V = 2$ kN. Determine the shear flow at points A, B, and C. The thickness of each thin-walled segment is 15 mm.



Prob. 12–52

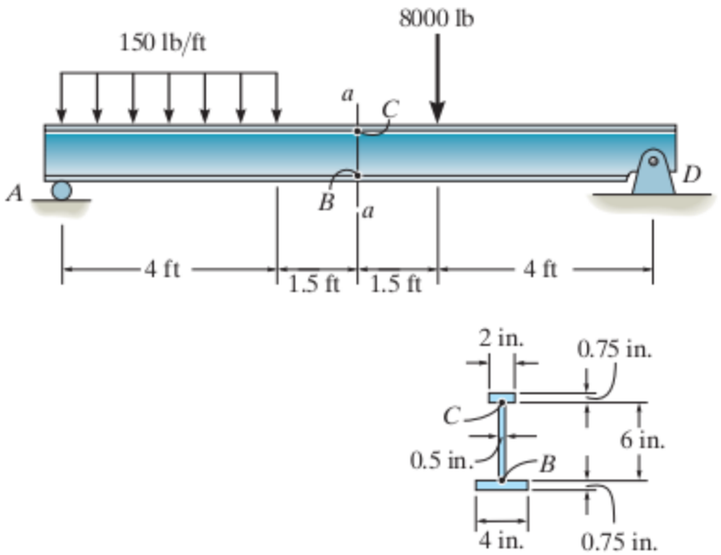
12–53. The beam is constructed from four boards glued together at their seams. If the glue can withstand 75 lb/in., what is the maximum vertical shear V that the beam can support? What is the maximum vertical shear V that the beam can support if it is rotated 90° from the position shown?



Prob. 12–53

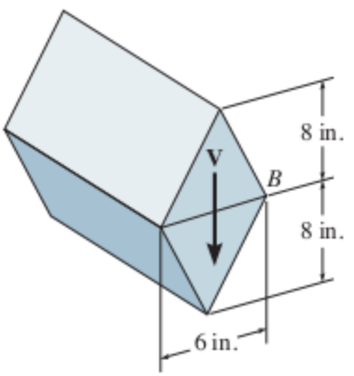
12–54. Determine the shear stress at points B and C on the web of the beam located at section a–a.

12–55. Determine the maximum shear stress acting at section a–a in the beam.



Probs. 12–54/55

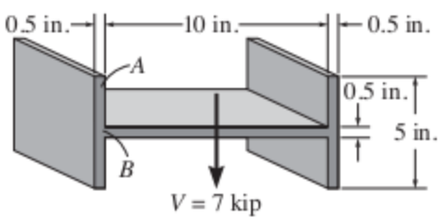
12–57. The member consists of two triangular plastic strips bonded together along AB. If the glue can support an allowable shear stress of $\tau_{allow} = 600$ psi, determine the maximum vertical shear V that can be applied to the member based on the strength of the glue.



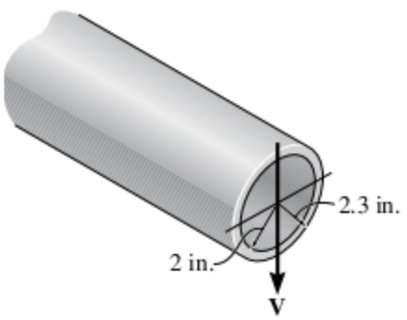
Prob. 12–57

12–58. If the pipe is subjected to a shear of $V = 15$ kip, determine the maximum shear stress in the pipe.

***12–56.** The beam supports a vertical shear of $V = 7$ kip. Determine the resultant force this develops in segment AB of the beam.



Prob. 12–56



Prob. 12–58

Chapter 13

The offset hanger supporting this ski gondola is subjected to the combined loadings of axial force and bending moment.

Combined Loadings

CHAPTER OBJECTIVES

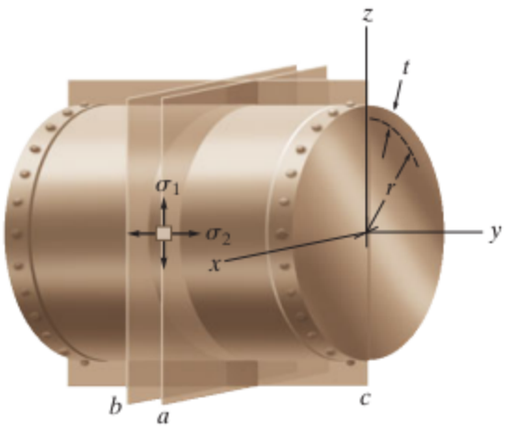
- To analyze the stresses in thin-walled pressure vessels.
- To develop methods to analyze the stresses in members subject to combined loadings (e.g., tension or compression, shear, torsion, bending moments).

13.1 Thin-Walled Pressure Vessels

Cylindrical or spherical vessels are commonly used in industry to serve as boilers or tanks. When under pressure, the material of which they are made is subjected to a loading from all directions. Although this is the case, the vessel can be analyzed in a simple manner provided it has a thin wall. In general, "***thin wall***" refers to a vessel having an inner-radius-to-wall-thickness ratio of 10 or more ($r/t \geq 10$). Specifically, when $r/t = 10$ the results of a thin-wall analysis will predict a stress that is approximately 4% *less* than the actual maximum stress in the vessel. For larger r/t ratios this error will be even smaller.

Provided the vessel wall is "thin," the stress distribution throughout its thickness will not vary significantly, and so we will assume that it is *uniform* or *constant*. Using this assumption, we will now analyze the state of stress in thin-walled cylindrical and spherical pressure vessels. In both cases, the pressure in the vessel is understood to be the *gauge pressure*, that is, it measures the pressure *above* atmospheric pressure, since atmospheric pressure is assumed to exist both inside and outside the vessel's wall before the vessel is pressurized.

Cylindrical pressure vessels, such as this gas tank, have semi-spherical end caps rather than flat ones in order to reduce the stress in the tank.

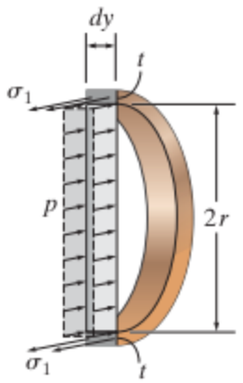


(a)

Cylindrical Vessels. Consider the cylindrical vessel in Fig. 13–1*a*, having a wall thickness t, inner radius r, and subjected to a gauge pressure p that developed within the vessel by a contained gas. Due to this loading, a small element of the vessel that is sufficiently removed from the ends and oriented as shown in Fig. 13–1*a*, is subjected to normal stresses σ_1 in the ***circumferential*** or ***hoop direction*** and σ_2 in the ***longitudinal*** or ***axial direction***.

The hoop stress can be determined by considering the vessel to be sectioned by planes a, b, and c. A free-body diagram of the back segment along with the contained gas is shown in Fig. 13–1*b*. Here only the loadings in the x direction are shown. These loadings are developed by the uniform hoop stress σ_1, acting on the vessel's wall, and the pressure acting on the vertical face of the gas. For equilibrium in the x direction, we require

$$\Sigma F_x = 0; \qquad 2[\sigma_1(t\,dy)] - p(2r\,dy) = 0$$

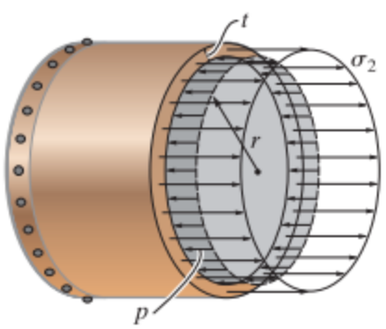


(b)

$$\sigma_1 = \frac{pr}{t} \tag{13–1}$$

The longitudinal stress can be determined by considering the left portion of section b of the cylinder, Fig. 13–1*a*. As shown in Fig. 13–1*c*, σ_2 acts uniformly throughout the wall, and p acts on the section of the contained gas. Since the mean radius is approximately equal to the vessel's inner radius, equilibrium in the y direction requires

$$\Sigma F_y = 0; \qquad \sigma_2(2\pi rt) - p(\pi r^2) = 0$$



(c)

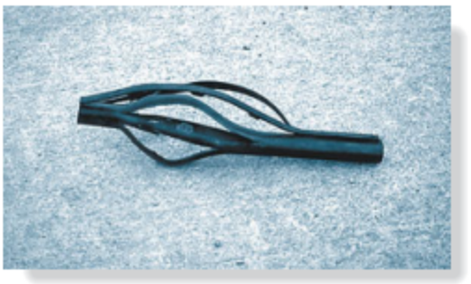
$$\sigma_2 = \frac{pr}{2t} \tag{13–2}$$

In the above equations,

σ_1, σ_2 = the normal stress in the hoop and longitudinal directions, respectively. Each is assumed to be *constant* throughout the wall of the cylinder, and each subjects the material to tension
p = the internal gauge pressure developed by the contained gas
r = the inner radius of the cylinder
t = the thickness of the wall ($r/t \geq 10$)

Fig. 13–1

By comparison, note that the hoop or circumferential stress is twice as large as the longitudinal or axial stress. Consequently, when fabricating cylindrical pressure vessels from rolled-formed plates, the longitudinal joints must be designed to carry twice as much stress as the circumferential joints.



Shown is the barrel of a shotgun which was clogged with debris just before firing. Gas pressure from the charge increased the circumferential stress within the barrel enough to cause the rupture.

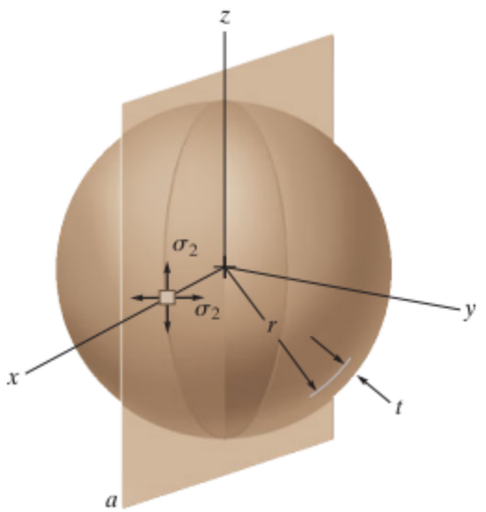
Spherical Vessels. We can analyze a spherical pressure vessel in a similar manner. To do this, consider the vessel to have a wall thickness t, inner radius r, and subjected to an internal gauge pressure p, Fig. 13–2*a*. If the vessel is sectioned in half, the resulting free-body diagram is shown in Fig. 13–2*b*. Like the cylinder, equilibrium in the y direction requires

$$\Sigma F_y = 0; \qquad \sigma_2 (2\pi rt) - p(\pi r^2) = 0$$

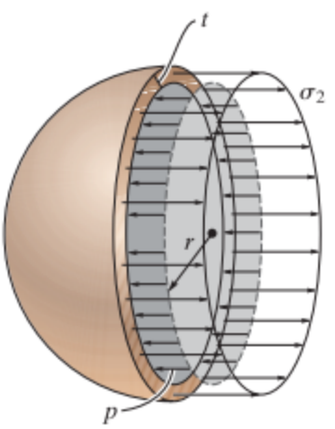
$$\sigma_2 = \frac{pr}{2t} \tag{13–3}$$

This is the *same result* as that obtained for the longitudinal stress in the cylindrical pressure vessel. Furthermore, from the analysis, this stress will be the same *regardless* of the orientation of the hemispheric free-body diagram. Consequently, a small element of the material is subjected to the state of stress shown in Fig. 13–2*a*.

The above analysis indicates that an element of material taken from either a cylindrical or a spherical pressure vessel is subjected to ***biaxial stress***, i.e., normal stress existing in only two directions. Actually, the pressure also subjects the material to a ***radial stress***, σ_3, which acts along a radial line. This stress has a maximum value equal to the pressure p at the interior wall and it decreases through the wall to zero at the exterior surface of the vessel, since the gauge pressure there is zero. For thin-walled vessels, however, we will *ignore* this radial-stress component, since our limiting assumption of $r/t = 10$ results in σ_2 and σ_1 being, respectively, 5 and 10 times *higher* than the maximum radial stress, $(\sigma_3)_{\max} = p$. Finally, if the vessel is subjected to an *external pressure*, the compressive stress developed within the thin wall may cause the vessel to become unstable, and collapse may occur by buckling rather than causing the material to fracture.



(a)



(b)

Fig. 13–2

EXAMPLE 13.1

A cylindrical pressure vessel has an inner diameter of 4 ft and a thickness of $\frac{1}{2}$ in. Determine the maximum internal pressure it can sustain so that neither its circumferential nor its longitudinal stress component exceeds 20 ksi. Under the same conditions, what is the maximum internal pressure that a similar-size spherical vessel can sustain?

SOLUTION

Cylindrical Pressure Vessel. The maximum stress occurs in the circumferential direction. From Eq. 13–1 we have

$$\sigma_1 = \frac{pr}{t}; \qquad 20 \text{ kip/in}^2 = \frac{p(24 \text{ in.})}{\frac{1}{2} \text{ in.}}$$

$$p = 417 \text{ psi} \qquad \textit{Ans.}$$

Note that when this pressure is reached, from Eq. 13–2, the stress in the longitudinal direction will be $\sigma_2 = \frac{1}{2}(20 \text{ ksi}) = 10 \text{ ksi}$. Furthermore, the *maximum stress* in the *radial direction* occurs on the material at the inner wall of the vessel and is $(\sigma_3)_{\max} = p = 417 \text{ psi}$. This value is 48 times smaller than the circumferential stress (20 ksi), and as stated earlier, its effects will be neglected.

Spherical Vessel. Here the maximum stress occurs in any two perpendicular directions on an element of the vessel, Fig. 13–2*a*. From Eq. 13–3, we have

$$\sigma_2 = \frac{pr}{2t}; \qquad 20 \text{ kip/in}^2 = \frac{p(24 \text{ in.})}{2\left(\frac{1}{2} \text{ in.}\right)}$$

$$p = 833 \text{ psi} \qquad \textit{Ans.}$$

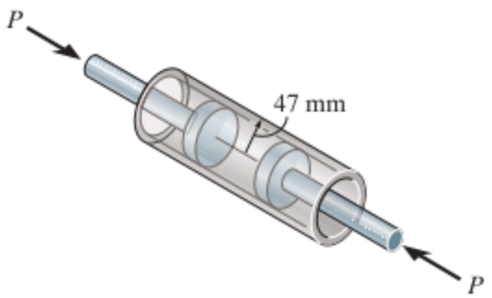
NOTE: Although it is more difficult to fabricate, the spherical pressure vessel will carry twice as much internal pressure as a cylindrical vessel.

PROBLEMS

13–1. A spherical gas tank has an inner radius of $r = 1.5$ m. If it is subjected to an internal pressure of $p = 300$ kPa, determine its required thickness if the maximum normal stress is not to exceed 12 MPa.

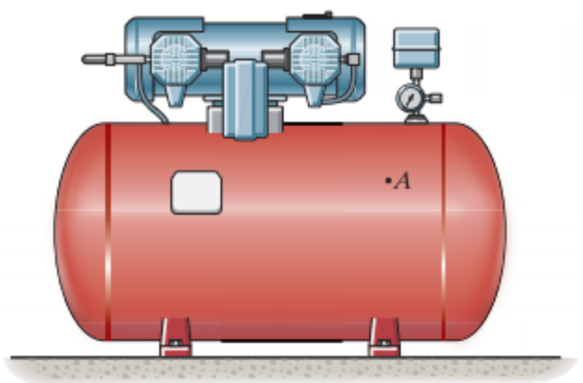
13–2. A pressurized spherical tank is to be made of 0.5-in.-thick steel. If it is subjected to an internal pressure of $p = 200$ psi, determine its outer radius if the maximum normal stress is not to exceed 15 ksi.

13–3. The thin-walled cylinder can be supported in one of two ways as shown. Determine the state of stress in the wall of the cylinder for both cases if the piston P causes the internal pressure to be 65 psi. The wall has a thickness of 0.25 in. and the inner diameter of the cylinder is 8 in.



Prob. 13–3

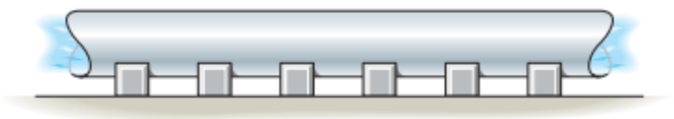
*13–4. The tank of the air compressor is subjected to an internal pressure of 90 psi. If the internal diameter of the tank is 22 in., and the wall thickness is 0.25 in., determine the stress components acting at point A. Draw a volume element of the material at this point, and show the results on the element.



Prob. 13–4

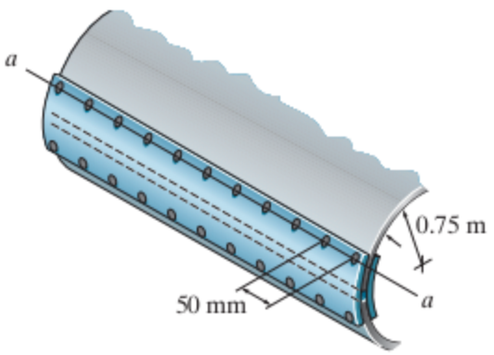
13–5. The open-ended polyvinyl chloride pipe has an inner diameter of 4 in. and thickness of 0.2 in. If it carries flowing water at 60 psi pressure, determine the state of stress in the walls of the pipe.

13–6. If the flow of water within the pipe in Prob. 13–5 is stopped due to the closing of a valve, determine the state of stress in the walls of the pipe. Neglect the weight of the water. Assume the supports only exert vertical forces on the pipe.



Probs. 13–5/6

13–7. A boiler is constructed of 8-mm thick steel plates that are fastened together at their ends using a butt joint consisting of two 8-mm cover plates and rivets having a diameter of 10 mm and spaced 50 mm apart as shown. If the steam pressure in the boiler is 1.35 MPa, determine (a) the circumferential stress in the boiler's plate apart from the seam, (b) the circumferential stress in the outer cover plate along the rivet line a–a, and (c) the shear stress in the rivets.

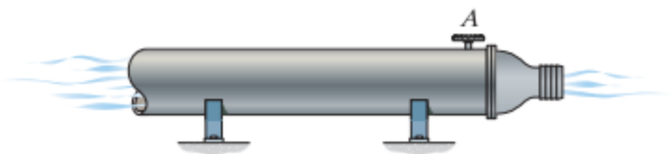


Prob. 13–7

13

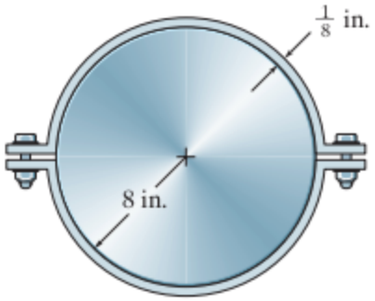
*13–8. The steel water pipe has an inner diameter of 12 in. and wall thickness 0.25 in. If the valve A is opened and the flowing water is under a gauge pressure of 250 psi, determine the longitudinal and hoop stress developed in the wall of the pipe.

13–9. The steel water pipe has an inner diameter of 12 in. and wall thickness 0.25 in. If the valve A is closed and the water pressure is 300 psi, determine the longitudinal and hoop stress developed in the wall of the pipe. Draw the state of stress on a volume element located on the wall.



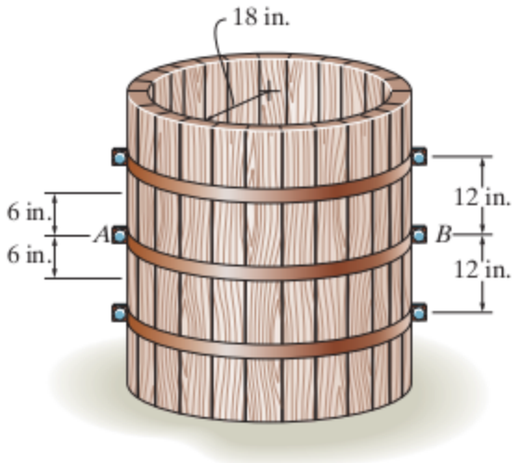
Probs. 13–8/9

13–10. The A-36 steel band is 2 in. wide and is secured around the smooth rigid cylinder. If the bolts are tightened so that the tension in them is 400 lb, determine the normal stress in the band, the pressure exerted on the cylinder, and the distance half the band stretches.



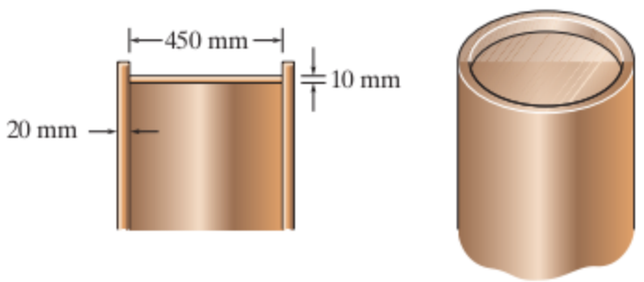
Prob. 13–10

13–11. The staves or vertical members of the wooden tank are held together using semicircular hoops having a thickness of 0.5 in. and a width of 2 in. Determine the normal stress in hoop AB if the tank is subjected to an internal gauge pressure of 2 psi and this loading is transmitted directly to the hoops. Also, if 0.25-in.-diameter bolts are used to connect each hoop together, determine the tensile stress in each bolt at A and B. Assume hoop AB supports the pressure loading within a 12-in. length of the tank as shown.



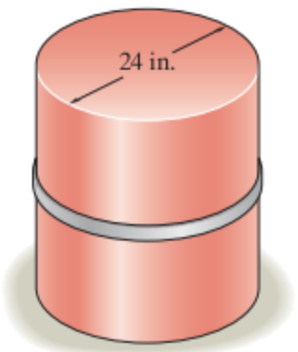
Prob. 13–11

*13–12. A pressure-vessel head is fabricated by gluing the circular plate to the end of the vessel as shown. If the vessel sustains an internal pressure of 450 kPa, determine the average shear stress in the glue and the state of stress in the wall of the vessel.



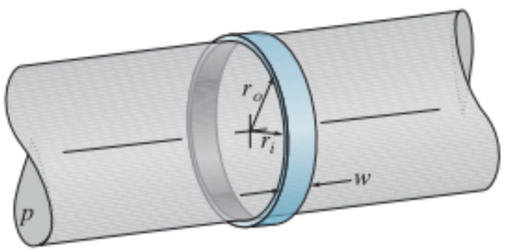
Prob. 13–12

13–13. An A-36 steel hoop has an inner diameter of 23.99 in., thickness of 0.25 in., and width of 1 in. If it and the 24-in.-diameter rigid cylinder have a temperature of 65° F, determine the temperature to which the hoop should be heated in order for it to just slip over the cylinder. What is the pressure the hoop exerts on the cylinder, and the tensile stress in the ring when it cools back down to 65° F?



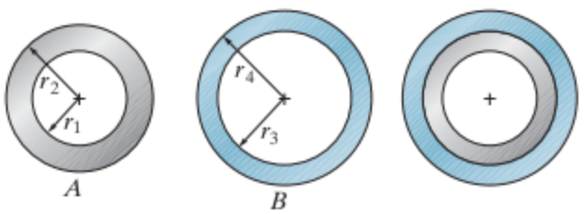
Prob. 13–13

13–14. The ring, having the dimensions shown, is placed over a flexible membrane which is pumped up with a pressure p. Determine the change in the internal radius of the ring after this pressure is applied. The modulus of elasticity for the ring is E.



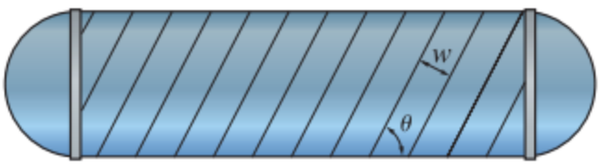
Prob. 13–14

13–15. The inner ring A has an inner radius r_1 and outer radius r_2. Before heating, the outer ring B has an inner radius r_3 and an outer radius r_4, and $r_2 > r_3$. If the outer ring is heated and then fitted over the inner ring, determine the pressure between the two rings when ring B reaches the temperature of the inner ring. The material has a modulus of elasticity of E and a coefficient of thermal expansion of α.



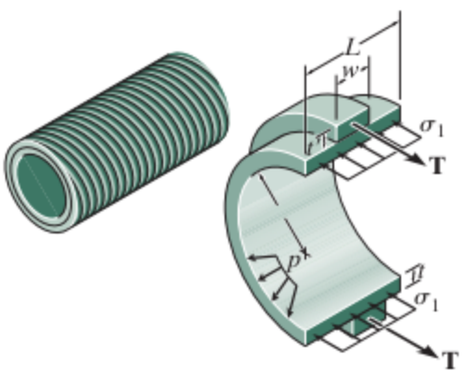
Prob. 13–15

***13–16.** The cylindrical tank is fabricated by welding a strip of thin plate helically, making an angle θ with the longitudinal axis of the tank. If the strip has a width w and thickness t, and the gas within the tank of diameter d is pressured to p, show that the normal stress developed a long the strip is given by $\sigma_\theta = (pd/8t)(3 - \cos 2\theta)$.



Prob. 13–16

13–17. In order to increase the strength of the pressure vessel, filament winding of the same material is wrapped around the circumference of the vessel as shown. If the pretension in the filament is T and the vessel is subjected to an internal pressure p, determine the hoop stresses in the filament and in the wall of the vessel. Use the free-body diagram shown, and assume the filament winding has a thickness t' and width w for a corresponding length L of the vessel.



Prob. 13–17

This chimney is subjected to the combined loading of wind and weight. It is important to investigate the tensile stress in the chimney since masonry is weak in tension.

13.2 State of Stress Caused by Combined Loadings

In previous chapters we developed methods for determining the stress distributions in a member subjected to either an internal axial force, a shear force, a bending moment, or a torsional moment. Most often, however, the cross section of a member is subjected to *several* of these loadings *simultaneously*. When this occurs, the method of superposition can be used to determine the *resultant* stress distribution. Recall from Sec. 9.3 that the principle of superposition can be used for this purpose provided a *linear relationship* exists between the *stress* and the *loads*. Also, the geometry of the member should *not* undergo *significant change* when the loads are applied. These conditions are necessary in order to ensure that the stress produced by one load is not related to the stress produced by any other load.

Procedure for Analysis

The following procedure provides a general means for establishing the normal and shear stress components at a point in a member when the member is subjected to several different types of loadings simultaneously. It is assumed that the material is homogeneous and behaves in a linear elastic manner. Also, Saint-Venant's principle requires that the point where the stress is to be determined is far removed from any discontinuities in the cross section or points of applied load.

Internal Loading.

- Section the member perpendicular to its axis at the point where the stress is to be determined and obtain the resultant internal normal and shear force components and the bending and torsional moment components.
- The force components should act through the *centroid* of the cross section, and the moment components should be computed about *centroidal axes*, which represent the principal axes of inertia for the cross section.

Stress Components.

- Determine the stress component associated with *each* internal loading. For each case, represent the effect either as a distribution of stress acting over the entire cross-sectional area, or show the stress on an element of the material located at a specified point on the cross section.

Normal Force.

- The internal normal force is developed by a uniform normal-stress distribution determined from $\sigma = P/A$.

Shear Force.

- The internal shear force in a member is developed by a shear-stress distribution determined from the shear formula, $\tau = VQ/It$. Special care, however, must be exercised when applying this equation, as noted in Sec. 12.2.

Bending Moment.

- For *straight members* the internal bending moment is developed by a normal-stress distribution that varies linearly from zero at the neutral axis to a maximum at the outer boundary of the member. This stress distribution is determined from the flexure formula, $\sigma = -My/I$. If the member is *curved*, the stress distribution is nonlinear and is determined from $\sigma = My/[Ae(R - y)]$.

Torsional Moment.

- For circular shafts and tubes the internal torsional moment is developed by a shear-stress distribution that varies linearly from zero at the central axis of the shaft to a maximum at the shaft's outer boundary. This stress distribution is determined from the torsional formula, $\tau = T\rho/J$.

Thin-Walled Pressure Vessels.

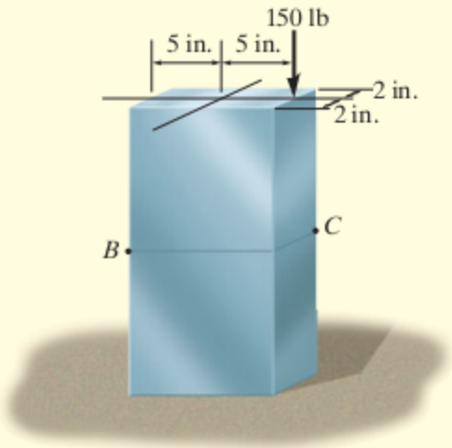
- If the vessel is a thin-walled cylinder, the internal pressure p will cause a biaxial state of stress in the material such that the hoop or circumferential stress component is $\sigma_1 = pr/t$ and the longitudinal stress component is $\sigma_2 = pr/2t$. If the vessel is a thin-walled sphere, then the biaxial state of stress is represented by two equivalent components, each having a magnitude of $\sigma_2 = pr/2t$.

Superposition.

- Once the normal and shear stress components for each loading have been calculated, use the principle of superposition and determine the resultant normal and shear stress components.
- Represent the results on an element of material located at the point, or show the results as a distribution of stress acting over the member's cross-sectional area.

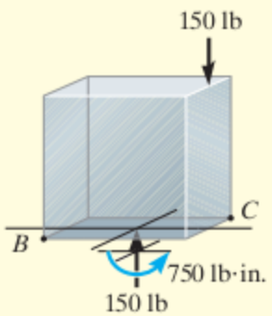
Problems in this section, which involve combined loadings, serve as a basic *review* of the application of the stress equations mentioned above. A thorough understanding of how these equations are applied, as indicated in the previous chapters, is necessary if one is to successfully solve the problems at the end of this section. The following examples should be carefully studied before proceeding to solve the problems.

EXAMPLE 13.2



(a)

Fig. 13–3



(b)

A force of 150 lb is applied to the edge of the member shown in Fig. 13–3*a*. Neglect the weight of the member and determine the state of stress at points *B* and *C*.

SOLUTION

Internal Loadings. The member is sectioned through *B* and *C*. For equilibrium at the section there must be an axial force of 150 lb acting through the *centroid* and a bending moment of 750 lb · in. about the centroidal principal axis, Fig. 13–3*b*.

Stress Components.

Normal Force. The uniform normal-stress distribution due to the normal force is shown in Fig. 13–3*c*. Here

$$\sigma = \frac{P}{A} = \frac{150\text{ lb}}{(10\text{ in.})(4\text{ in.})} = 3.75\text{ psi}$$

Bending Moment. The normal-stress distribution due to the bending moment is shown in Fig. 13–3*d*. The maximum stress is

$$\sigma_{\max} = \frac{Mc}{I} = \frac{750\text{ lb} \cdot \text{in.}(5\text{ in.})}{\frac{1}{12}(4\text{ in.})(10\text{ in.})^3} = 11.25\text{ psi}$$

Superposition. If the above normal-stress distributions are added algebraically, the resultant stress distribution is shown in Fig. 13–3*e*.

Elements of material at *B* and *C* are subjected only to normal or *uniaxial stress* as shown in Figs. 13–3*f* and 13–3*g*. Hence,

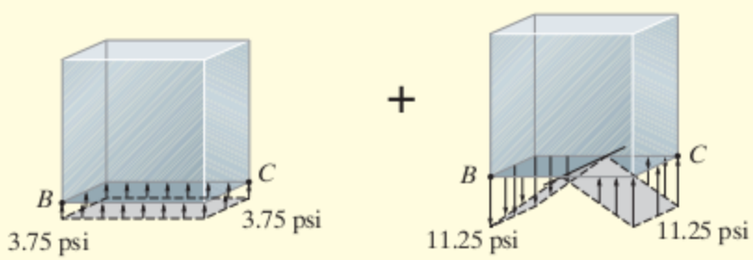
$$\sigma_B = -\frac{P}{A} + \frac{Mc}{I} = -3.75\text{ psi} + 11.25\text{ psi} = 7.5\text{ psi (tension)} \quad \textit{Ans.}$$

$$\sigma_C = -\frac{P}{A} - \frac{Mc}{I} = -3.75\text{ psi} - 11.25\text{ psi} = 15\text{ psi (compression)} \quad \textit{Ans.}$$

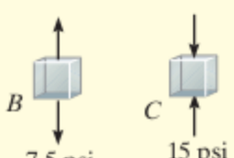
Although it is not needed here, the location of the line of zero stress can be determined by proportional triangles; i.e.,

$$\frac{7.5\text{ psi}}{x} = \frac{15\text{ psi}}{(10\text{ in.} - x)}$$

$$x = 3.33\text{ in.}$$



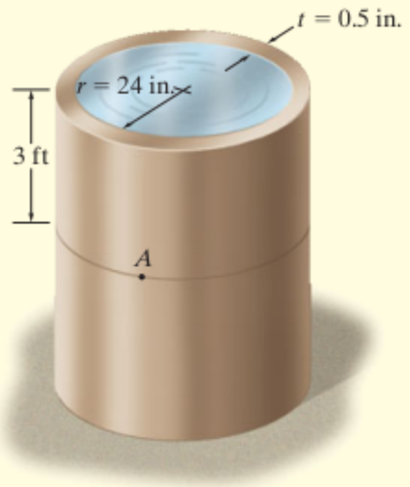
(c) Normal force (d) Bending moment (e) Combined loading



(f) (g)

EXAMPLE 13.3

The tank in Fig. 13–4*a* has an inner radius of 24 in. and a thickness of 0.5 in. It is filled to the top with water having a specific weight of $\gamma_w = 62.4\ \text{lb/ft}^3$. If it is made of steel having a specific weight of $\gamma_{st} = 490\ \text{lb/ft}^3$, determine the state of stress at point A. The tank is open at the top.



(a)

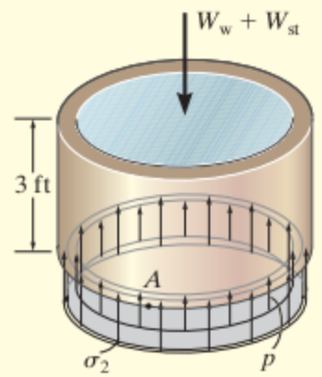
SOLUTION

Internal Loadings. The free-body diagram of the section of both the tank and the water above point A is shown in Fig. 13–4*b*. Notice that the weight of the water is supported by the water surface just *below* the section, *not* by the walls of the tank. In the vertical direction, the walls simply hold up the weight of the tank. This weight is

$$W_{st} = \gamma_{st} V_{st} = (490\ \text{lb/ft}^3)\left[\pi\left(\frac{24.5}{12}\ \text{ft}\right)^2 - \pi\left(\frac{24}{12}\ \text{ft}\right)^2\right](3\ \text{ft})$$
$$= 777.7\ \text{lb}$$

The stress in the circumferential direction is developed by the water pressure at level A. To obtain this (gage) pressure we must use $p = \gamma_w z$, which gives the pressure at a point located a depth z in the water. Consequently, the pressure on the tank at level A is

$$p = \gamma_w z = (62.4\ \text{lb/ft}^3)(3\ \text{ft}) = 187.2\ \text{lb/ft}^2 = 1.30\ \text{psi}$$



(b)

Stress Components.

Circumferential Stress. Since $r/t = 24\ \text{in.}/0.5\ \text{in.} = 48 > 10$, the tank is a thin-walled vessel. Applying Eq. 13–1, using the inner radius $r = 24$ in., we have

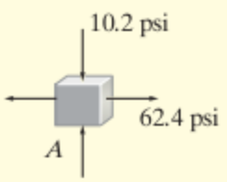
$$\sigma_1 = \frac{pr}{t} = \frac{1.30\ \text{lb/in}^2\ (24\ \text{in.})}{0.5\ \text{in.}} = 62.4\ \text{psi} \qquad \textit{Ans.}$$

Longitudinal Stress. Since the weight of the tank is supported uniformly by the walls, we have

$$\sigma_2 = \frac{W_{st}}{A_{st}} = \frac{777.7\ \text{lb}}{\pi[(24.5\ \text{in.})^2 - (24\ \text{in.})^2]} = 10.2\ \text{psi} \qquad \textit{Ans.}$$

NOTE: Equation 13–2, $\sigma_2 = pr/2t$, does *not apply* here, since the tank is open at the top and therefore, as stated previously, the water cannot develop a loading on the walls in the longitudinal direction.

Point A is therefore subjected to the biaxial stress shown in Fig. 13–4*c*.



(c)

Fig. 13–4

13

EXAMPLE 13.4

The member shown in Fig. 13–5*a* has a rectangular cross section. Determine the state of stress that the loading produces at point *C*.

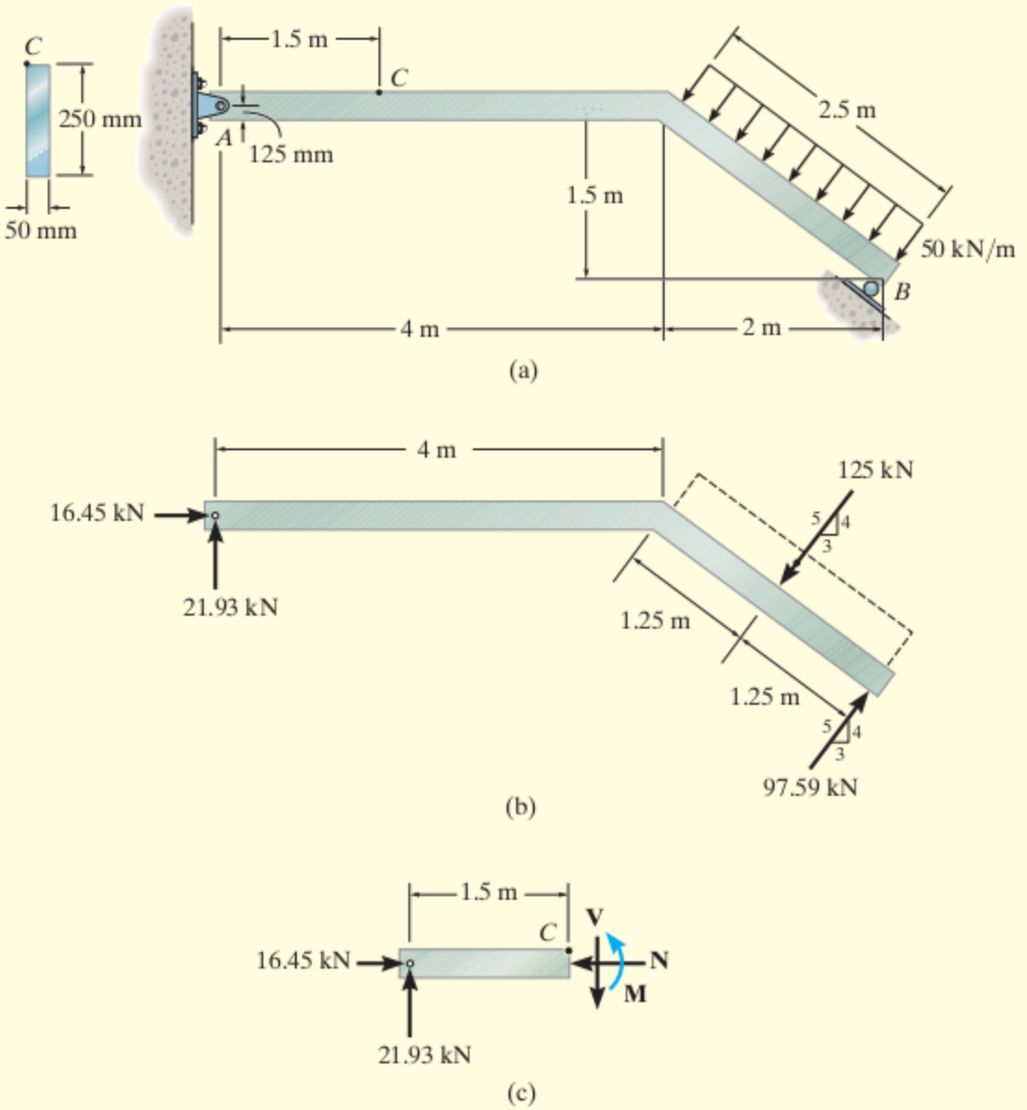
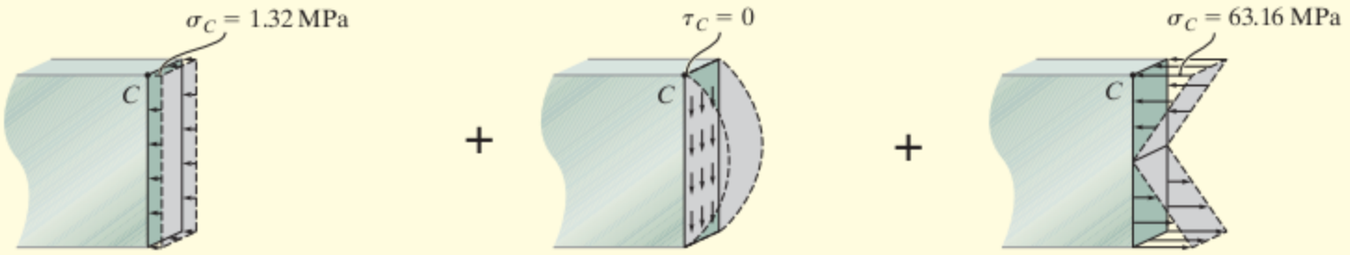


Fig. 13–5

SOLUTION

Internal Loadings. The support reactions on the member have been determined and are shown in Fig. 13–5*b*. If the left segment *AC* of the member is considered, Fig. 13–5*c*, the resultant internal loadings at the section consist of a normal force, a shear force, and a bending moment. Solving,

$$N = 16.45\ \text{kN} \quad V = 21.93\ \text{kN} \quad M = 32.89\ \text{kN} \cdot \text{m}$$



Normal force (d) + Shear force (e) + Bending moment (f)

Fig. 13–5 (cont.)

Stress Components.

Normal Force. The uniform normal-stress distribution acting over the cross section is produced by the normal force, Fig. 13–5*d*. At point *C*,

$$\sigma_C = \frac{P}{A} = \frac{16.45\,(10^3)\,\text{N}}{(0.050\,\text{m})(0.250\,\text{m})} = 1.32\,\text{MPa}$$

Shear Force. Here the area $A' = 0$, since point *C* is located at the top of the member. Thus $Q = \bar{y}'A' = 0$ and for *C*, Fig. 13–5*e*, the shear stress

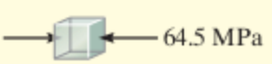
$$\tau_C = 0$$

Bending Moment. Point *C* is located at $y = c = 0.125$ m from the neutral axis, so the normal stress at *C*, Fig. 13–5*f*, is

$$\sigma_C = \frac{Mc}{I} = \frac{(32.89\,(10^3)\,\text{N}\cdot\text{m})(0.125\,\text{m})}{\left[\frac{1}{12}(0.050\,\text{m})(0.250\,\text{m})^3\right]} = 63.16\,\text{MPa}$$

Superposition. The shear stress is zero. Adding the normal stresses determined above gives a compressive stress at *C* having a value of

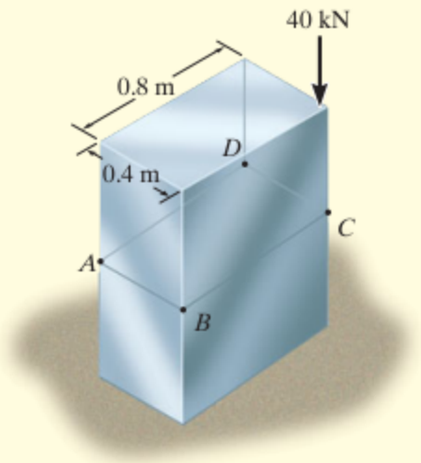
$$\sigma_C = 1.32\,\text{MPa} + 63.16\,\text{MPa} = 64.5\,\text{MPa} \qquad \textit{Ans.}$$



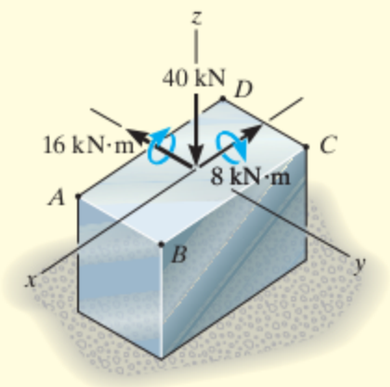
(g)

This result, acting on an element at *C*, is shown in Fig. 13–5*g*.

EXAMPLE 13.5



(a)



(b)

Fig. 13–6

The rectangular block of negligible weight in Fig. 13–6*a* is subjected to a vertical force of 40 kN, which is applied to its corner. Determine the largest normal stress acting on a section through *ABCD*.

SOLUTION

Internal Loadings. If we consider the equilibrium of the bottom segment of the block, then the 40-kN force must act through the centroid of the cross section and *two* bending-moment components of 16 kN · m and 8 kN · m must also act about the centroidal or principal axes of inertia for the section. Verify these results.

Stress Components.

Normal Force. The uniform normal-stress distribution is shown in Fig. 13–6*c*. We have

$$\sigma = \frac{P}{A} = \frac{40\,(10^3)\text{ N}}{(0.8\text{ m})(0.4\text{ m})} = 125\text{ kPa}$$

Bending Moments. The normal-stress distribution for the 8-kN · m moment is shown in Fig. 13–6*d*. The maximum stress is

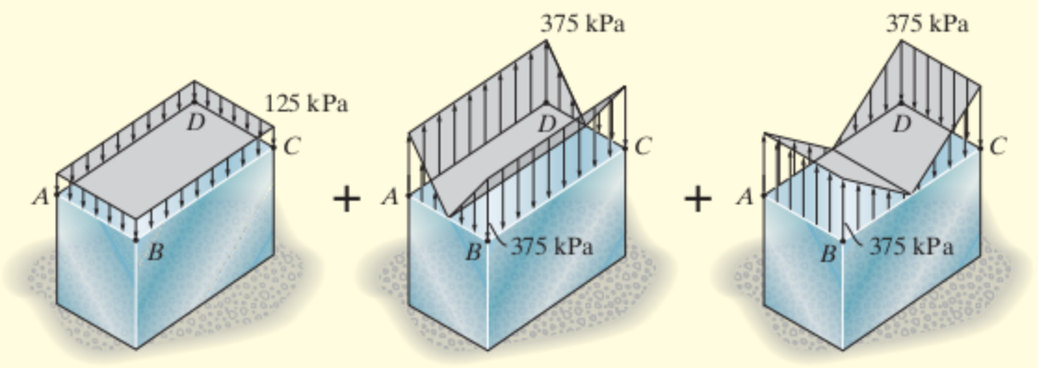
$$\sigma_{\max} = \frac{M_x c_y}{I_x} = \frac{8\,(10^3)\text{ N}\cdot\text{m}(0.2\text{ m})}{\left[\frac{1}{12}(0.8\text{ m})(0.4\text{ m})^3\right]} = 375\text{ kPa}$$

Likewise, for the 16-kN · m moment, Fig. 13–6*e*, the maximum normal stress is

$$\sigma_{\max} = \frac{M_y c_x}{I_y} = \frac{16\,(10^3)\text{ N}\cdot\text{m}(0.4\text{ m})}{\left[\frac{1}{12}(0.4\text{ m})(0.8\text{ m})^3\right]} = 375\text{ kPa}$$

Superposition. By inspection the normal stress at point *C* is the largest since each loading creates a compressive stress there. Therefore

$$\sigma_C = -125\text{ kPa} - 375\text{ kPa} - 375\text{ kPa} = -875\text{ kPa} \quad \textit{Ans.}$$



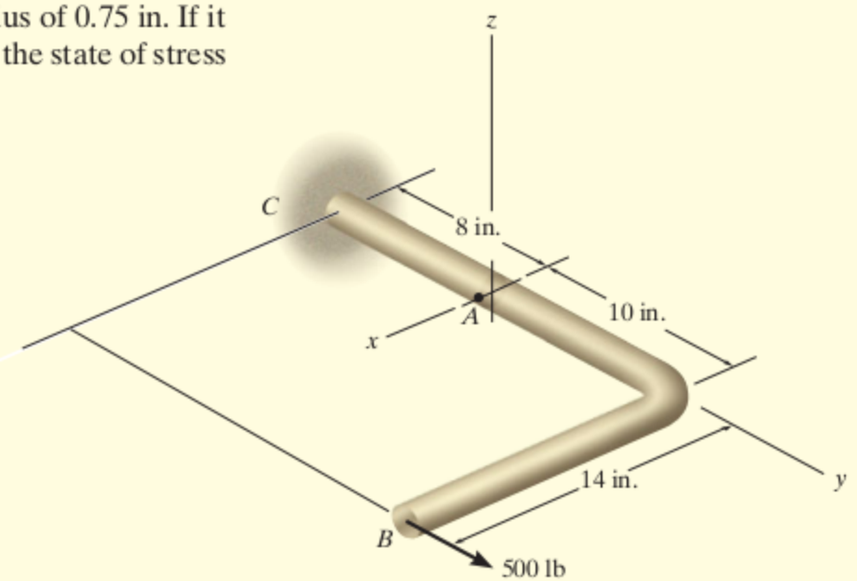
Normal force (40 kN) (c)

Bending moment (8 kN·m) (d)

Bending moment (16 kN·m) (e)

EXAMPLE 13.6

The solid rod shown in Fig. 13–7*a* has a radius of 0.75 in. If it is subjected to the force of 500 lb, determine the state of stress at point *A*.



(a)

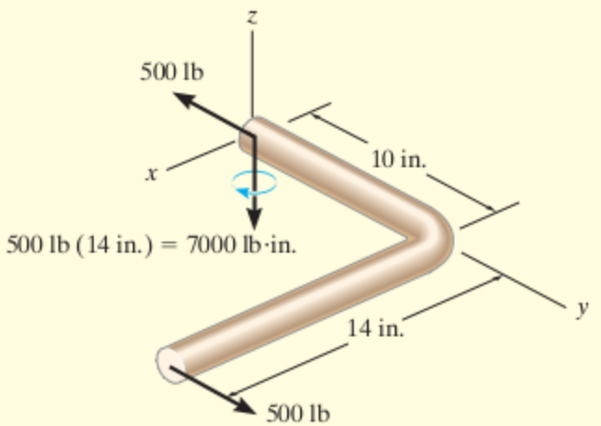
SOLUTION

Internal Loadings. The rod is sectioned through point *A*. Using the free-body diagram of segment *AB*, Fig. 13–7*b*, the resultant internal loadings are determined from the equations of equilibrium. Verify these results. In order to better "visualize" the stress distributions due to these loadings, we can consider the *equal but opposite resultants* acting on segment *AC*, Fig. 13–7*c*.

Stress Components.

Normal Force. The normal-stress distribution is shown in Fig. 13–7*d*. For point *A*, we have

$$(\sigma_A)_y = \frac{P}{A} = \frac{500\text{ lb}}{\pi(0.75\text{ in.})^2} = 283\text{ psi} = 0.283\text{ ksi}$$



(b)

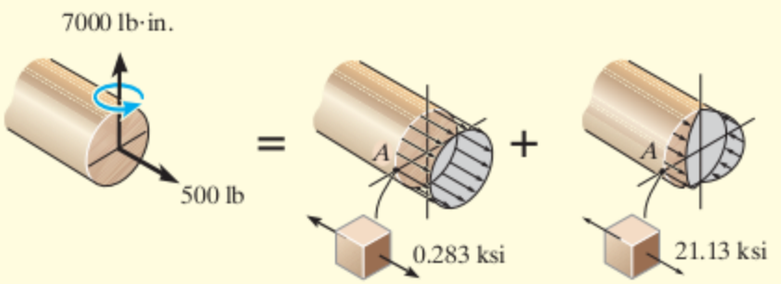
Bending Moment. For the moment, $c = 0.75$ in., so the normal stress at point *A*, Fig. 13–7*e*, is

$$(\sigma_A)_y = \frac{Mc}{I} = \frac{7000\text{ lb}\cdot\text{in.}(0.75\text{ in.})}{[\frac{1}{4}\pi(0.75\text{ in.})^4]}$$

$$= 21{,}126\text{ psi} = 21.13\text{ ksi}$$

Superposition. When the above results are superimposed, it is seen that an element of material at *A* is subjected to the normal stress

$$(\sigma_A)_y = 0.283\text{ ksi} + 21.13\text{ ksi} = 21.4\text{ ksi}\quad \textit{Ans.}$$

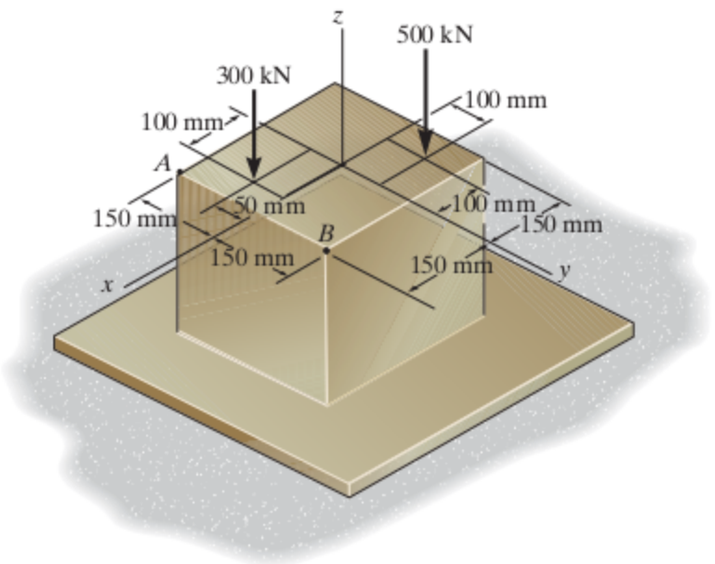


(c) (d) (e)

Fig. 13–7

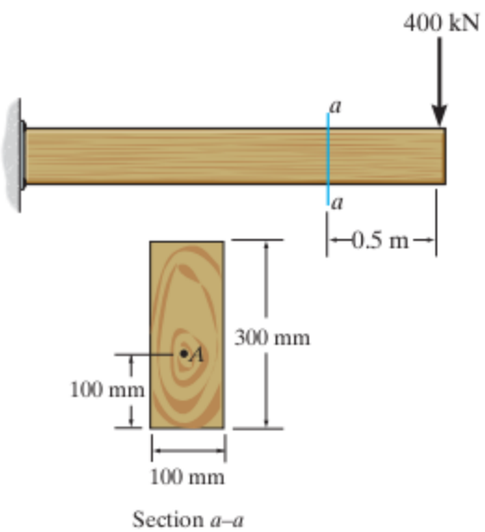
FUNDAMENTAL PROBLEMS

F13–1. Determine the normal stress developed at corners A and B of the column.



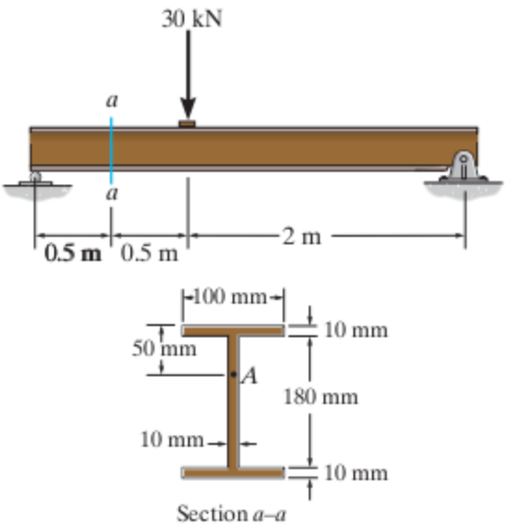
F13–1

F13–2. Determine the state of stress at point A on the cross section at section a–a of the cantilever beam. Show the results on a differential volume element at the point.



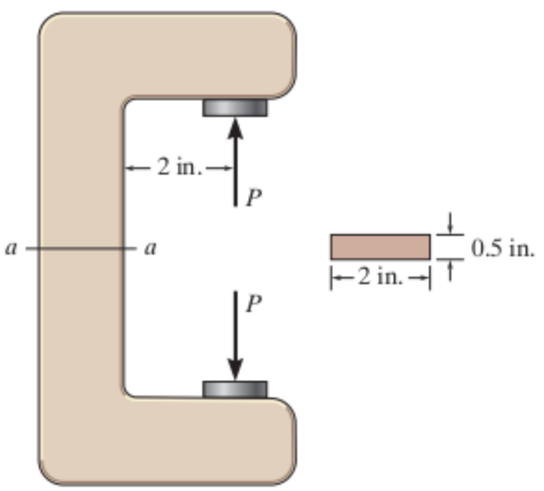
F13–2

F13–3. Determine the state of stress at point A on the cross section of the beam at section a–a. Show the results on a differential volume element at the point.



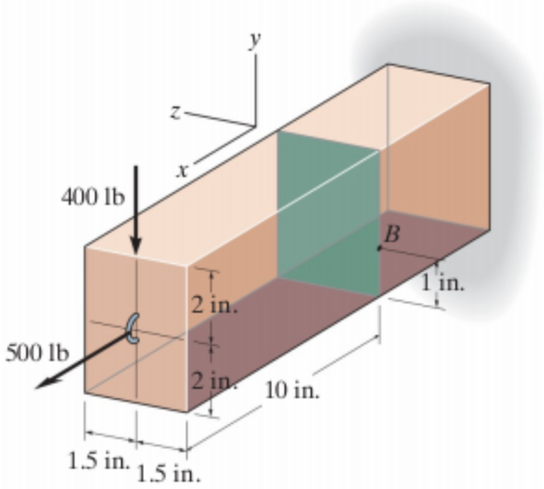
F13–3

F13–4. Determine the magnitude of the load P that will cause a maximum normal stress of $\sigma_{max} = 30$ ksi in the link along section a–a. Show the results on a differential volume element at the point.



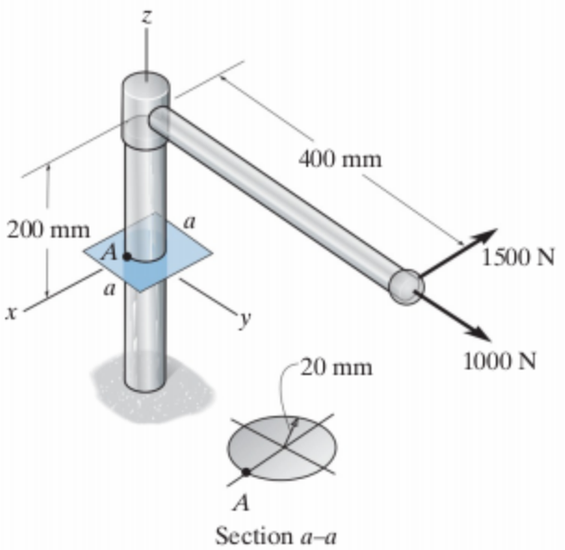
F13–4

F13–5. The beam has a rectangular cross section and is subjected to the loading shown. Determine the state of stress at point B. Show the results on a differential volume element at the point.



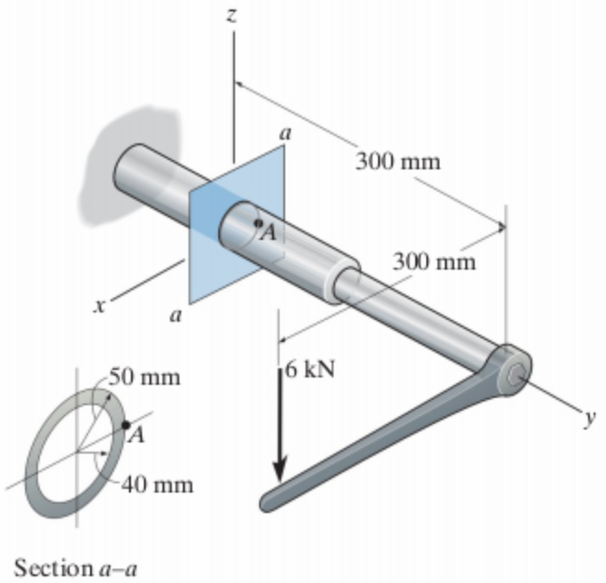
F13–5

F13–6. Determine the state of stress at point A on the cross section of the pipe assembly at section a–a. Show the results on a differential volume element at the point.



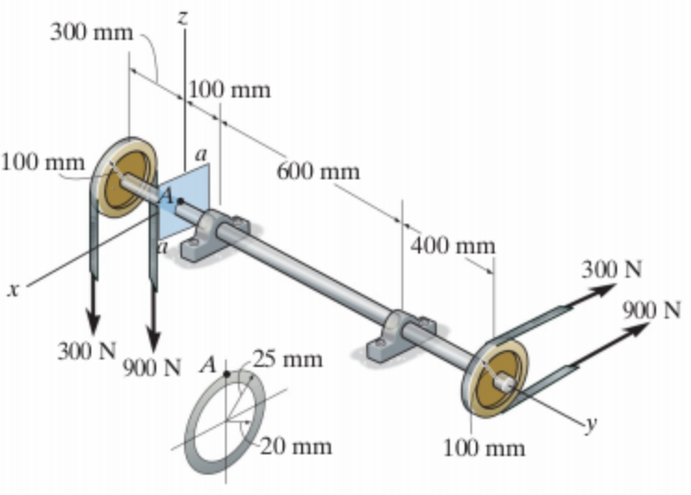
F13–6

F13–7. Determine the state of stress at point A on the cross section of the pipe at section a–a. Show the results on a differential volume element at the point.



F13–7

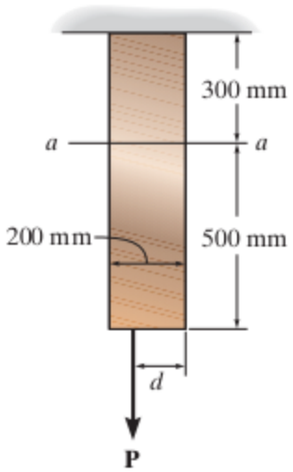
F13–8. Determine the state of stress at point A on the cross section of the shaft at section a–a. Show the results on a differential volume element at the point.



F13–8

PROBLEMS

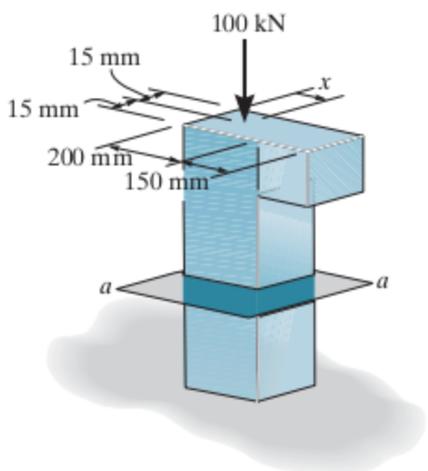
13–18. The vertical force **P** acts on the bottom of the plate having a negligible weight. Determine the shortest distance d to the edge of the plate at which it can be applied so that it produces no compressive stresses on the plate at section a–a. The plate has a thickness of 10 mm and **P** acts along the center line of this thickness.



Prob. 13–18

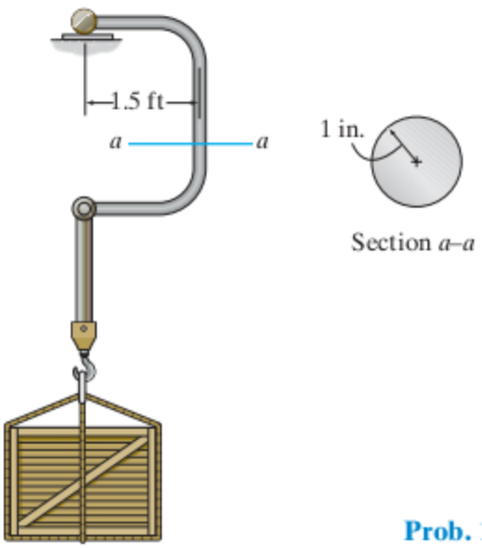
13–19. Determine the maximum and minimum normal stress in the bracket at section a–a when the load is applied at $x = 0$.

***13–20.** Determine the maximum and minimum normal stress in the bracket at section a–a when the load is applied at $x = 300$ mm.



Probs. 13–19/20

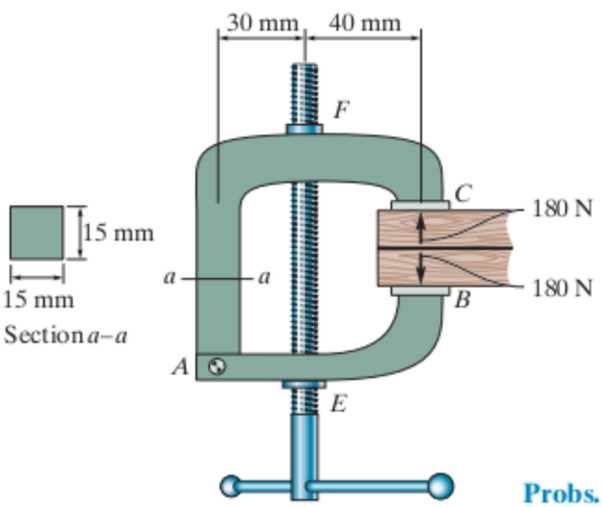
13–21. If the load has a weight of 600 lb, determine the maximum normal stress developed on the cross section of the supporting member at section a–a. Also, plot the normal stress distribution over the cross section.



Prob. 13–21

13–22. The clamp is made from members AB and AC, which are pin connected at A. If it exerts a compressive force at C and B of 180 N, determine the maximum compressive stress in the clamp at section a–a. The screw EF is subjected only to a tensile force along its axis.

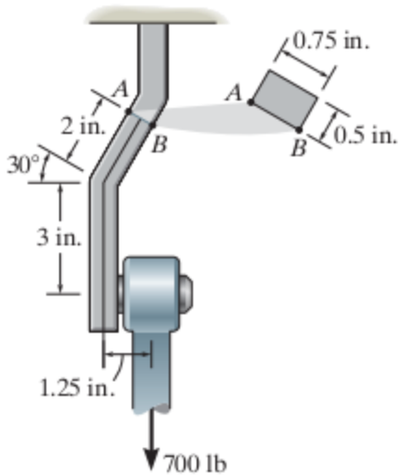
13–23. The clamp is made from members AB and AC, which are pin connected at A. If it exerts a compressive force at C and B of 180 N, sketch the stress distribution acting over section a–a. The screw EF is subjected only to a tensile force along its axis.



Probs. 13–22/23

*13–24. The bearing pin supports the load of 700 lb. Determine the stress components in the support member at point A. The support is 0.5 in. thick.

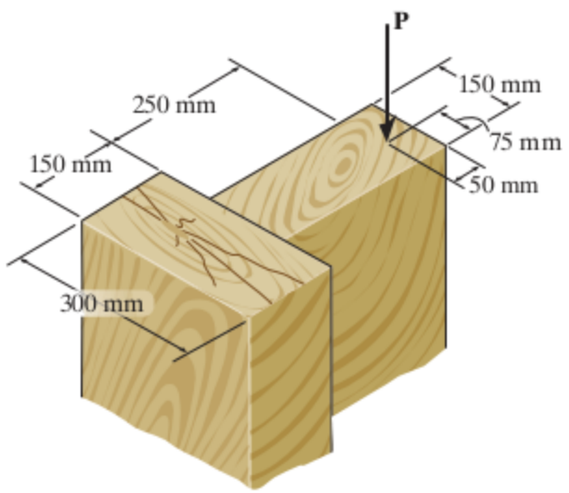
13–25. The bearing pin supports the load of 700 lb. Determine the stress components in the support member at point B. The support is 0.5 in. thick.



Probs. 13–24/25

13–26. The column is built up by gluing the two identical boards together. Determine the maximum normal stress developed on the cross section when the eccentric force of $P = 50$ kN is applied.

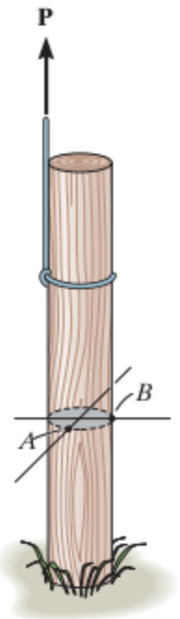
13–27. The column is built up by gluing the two identical boards together. If the wood has an allowable normal stress of $\sigma_{allow} = 6$ MPa, determine the maximum allowable eccentric force **P** that can be applied to the column.



Probs. 13–26/27

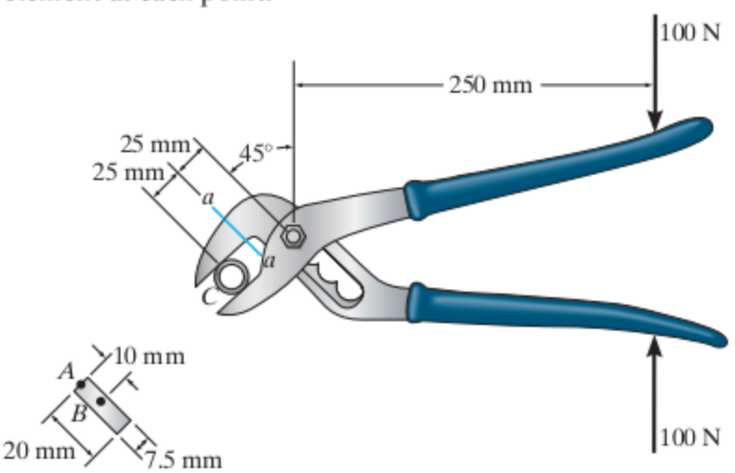
*13–28. The cylindrical post, having a diameter of 40 mm, is being pulled from the ground using a sling of negligible thickness. If the rope is subjected to a vertical force of $P = 500$ N, determine the normal stress at points A and B. Show the results on a volume element located at each of these points.

13–29. Determine the maximum load P that can be applied to the sling having a negligible thickness so that the normal stress in the post does not exceed $\sigma_{allow} = 30$ MPa. The post has a diameter of 50 mm.



Probs. 13–28/29

13–30. The rib-joint pliers are used to grip the smooth pipe C. If the force of 100 N is applied to the handles, determine the state of stress at points A and B on the cross section of the jaw at section a–a. Indicate the results on an element at each point.



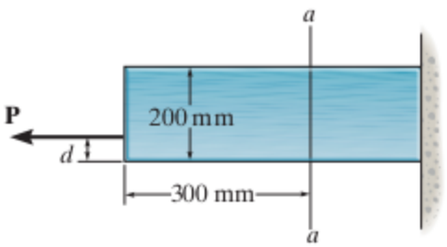
Section a–a

Prob. 13–30

13

13–31. Determine the smallest distance d to the edge of the plate at which the force **P** can be applied so that it produces no compressive stresses in the plate at section a–a. The plate has a thickness of 20 mm and **P** acts along the centerline of this thickness.

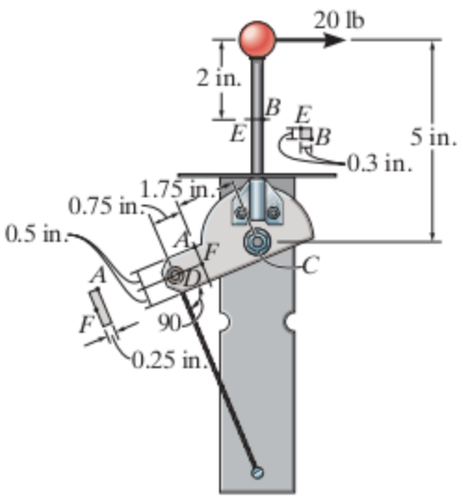
***13–32.** The horizontal force of $P = 80$ kN acts at the end of the plate. The plate has a thickness of 10 mm and **P** acts along the centerline of this thickness such that $d = 50$ mm. Plot the distribution of normal stress acting along section a–a.



Probs. 13–31/32

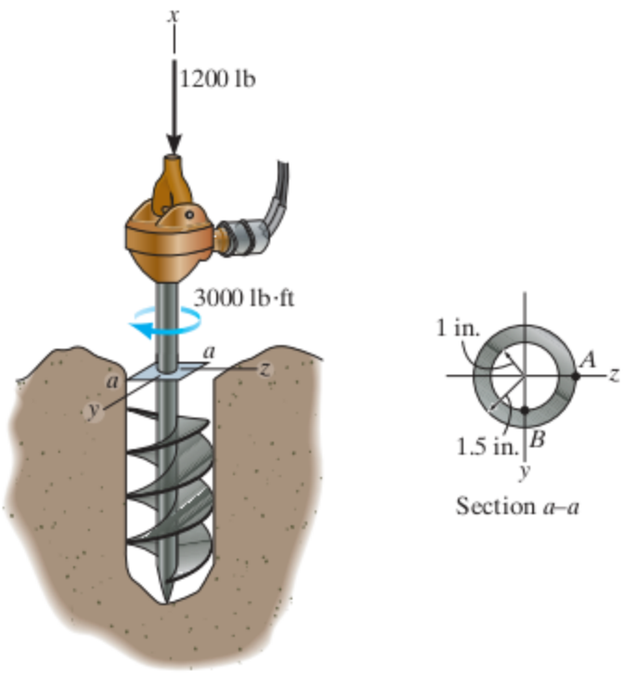
13–33. The control lever is subjected to a horizontal force of 20 lb on the handle. Determine the state of stress at points A and B. Sketch the results on differential elements located at each of these points. The assembly is pin-connected at C and attached to a cable at D.

13–34. The control lever is subjected to a horizontal force of 20 lb on the handle. Determine the state of stress at points E and F. Sketch the results on differential elements located at each of these points. The assembly is pin connected at C and attached to a cable at D.



Probs. 13–33/34

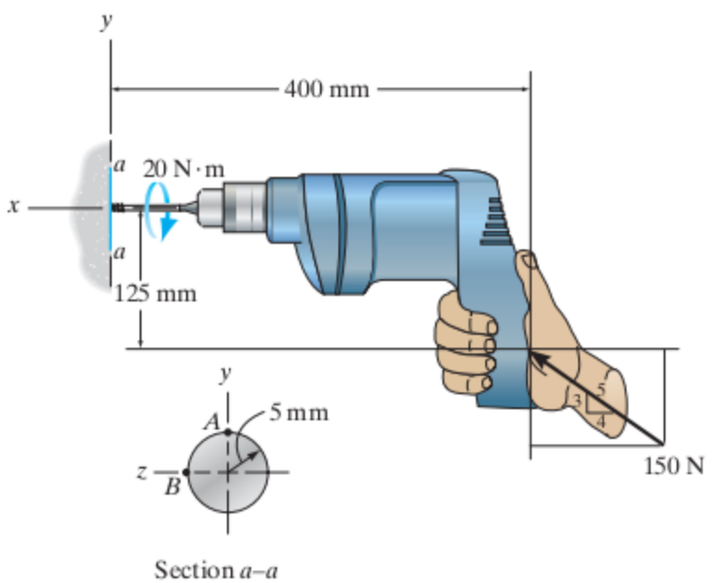
13–35. The tubular shaft of the soil auger is subjected to the axial force and torque shown. If the auger is rotating at a constant rate, determine the state of stress at points A and B on the cross section of the shaft at section a–a.



Prob. 13–35

***13–36.** The drill is jammed in the wall and is subjected to the torque and force shown. Determine the state of stress at point A on the cross section of drill bit at section a–a.

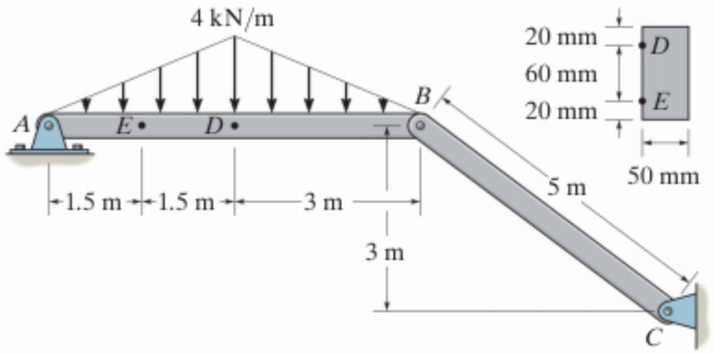
13–37. The drill is jammed in the wall and is subjected to the torque and force shown. Determine the state of stress at point B on the cross section of drill bit, in back, at section a–a.



Probs. 13–36/37

13–38. The frame supports the distributed load shown. Determine the state of stress acting at point D. Show the results on a differential element at this point.

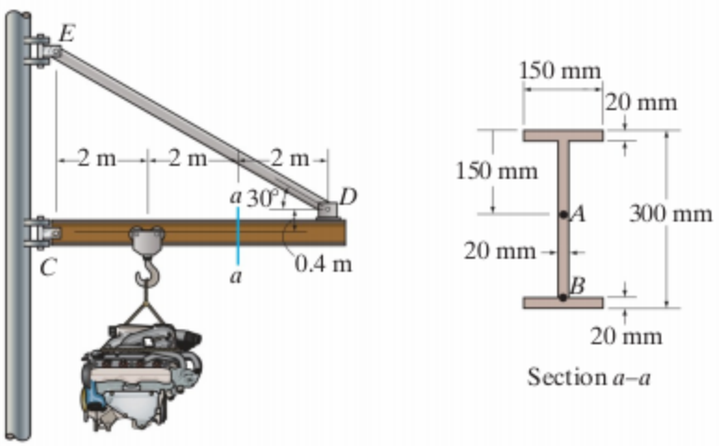
13–39. The frame supports the distributed load shown. Determine the state of stress acting at point E. Show the results on a differential element at this point.



Probs. 13–38/39

***13–40.** The 500-kg engine is suspended from the jib crane at the position shown. Determine the state of stress at point A on the cross section of the boom at section a–a.

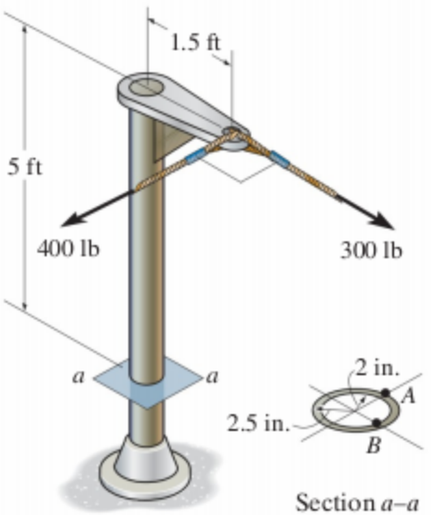
13–41. The 500-kg engine is suspended from the jib crane at the position shown. Determine the state of stress at point B on the cross section of the boom at section a–a. Point B is just above the bottom flange.



Probs. 13–40/41

13–42. Determine the state of stress at point A on the cross section of the post at section a–a. Indicate the results on a differential element at the point.

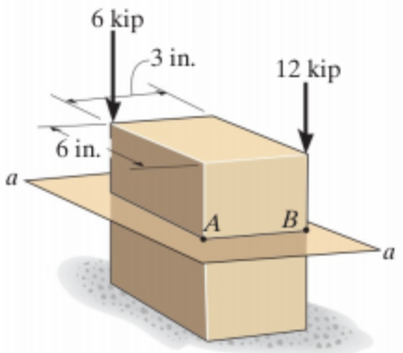
13–43. Determine the state of stress at point B on the cross section of the post at section a–a. Indicate the results on a differential element at the point.



Probs. 13–42/43

***13–44.** Determine the normal stress developed at points A and B. Neglect the weight of the block.

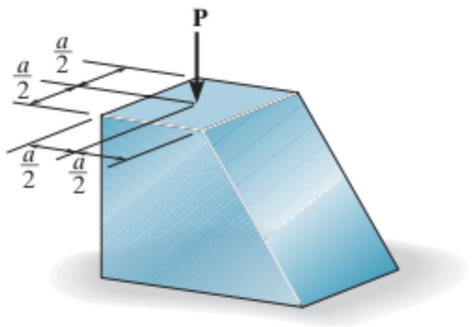
13–45. Sketch the normal stress distribution acting over the cross section at section a–a. Neglect the weight of the block.



Probs. 13–44/45

13

13–46. The support is subjected to the compressive load **P**. Determine the absolute maximum and minimum normal stress acting in the material.

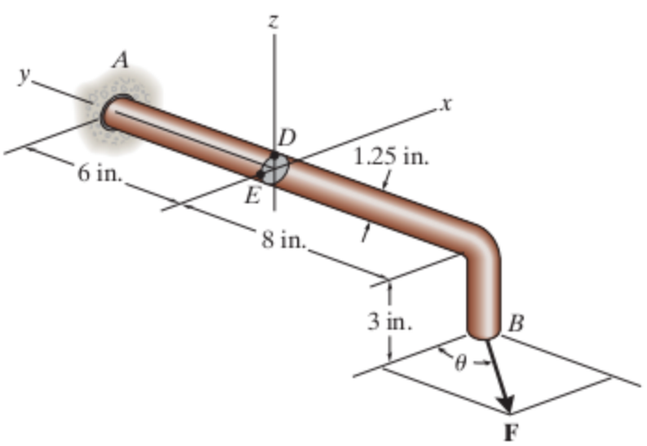


Prob. 13–46

13–47. The bent shaft is fixed in the wall at A. If a force **F** is applied at B, determine the stress components at points D and E. Show the results on a differential element located at each of these points. Take $F = 12$ lb and $\theta = 0°$.

***13–48.** The bent shaft is fixed in the wall at A. If a force **F** is applied at B, determine the stress components at points D and E. Show the results on a differential element located at each of these points. Take $F = 12$ lb and $\theta = 90°$.

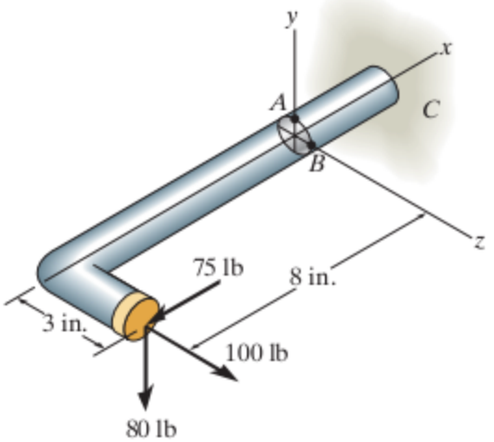
13–49. The bent shaft is fixed in the wall at A. If a force **F** is applied at B, determine the stress components at points D and E. Show the results on a volume element located at each of these points. Take $F = 12$ lb and $\theta = 45°$.



Probs. 13–47/48/49

13–50. The 1-in.-diameter rod is subjected to the loads shown. Determine the state of stress at point A, and show the results on a differential element located at this point.

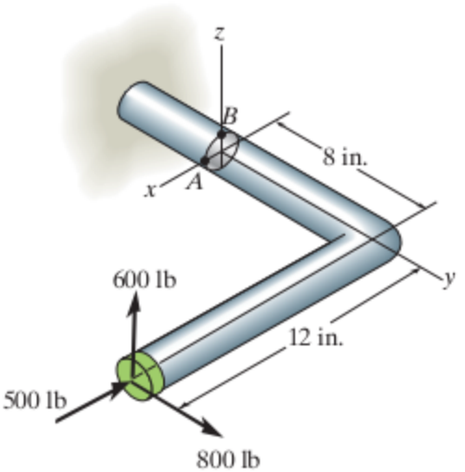
13–51. The 1-in.-diameter rod is subjected to the loads shown. Determine the state of stress at point B, and show the results on a differential element located at this point.



Probs. 13–50/51

***13–52.** The 2-in.-diameter rod is subjected to the loads shown. Determine the state of stress at point A, and show the results on a differential element located at this point.

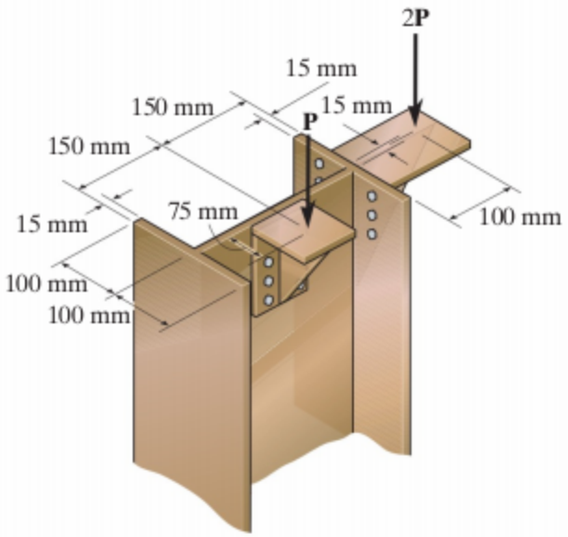
13–53. The 2-in.-diameter rod is subjected to the loads shown. Determine the state of stress at point B, and show the results on a differential element located at this point.



Probs. 13–52/53

13–54. If $P = 60$ kN, determine the maximum normal stress developed on the cross section of the column.

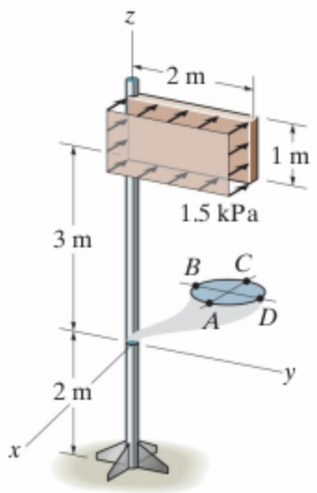
13–55. Determine the maximum allowable force **P**, if the column is made from material having an allowable normal stress of $\sigma_{allow} = 100$ MPa.



Probs. 13–54/55

*13–56. The sign is subjected to the uniform wind loading. Determine the stress components at points A and B on the 100-mm-diameter supporting post. Show the results on a volume element located at each of these points.

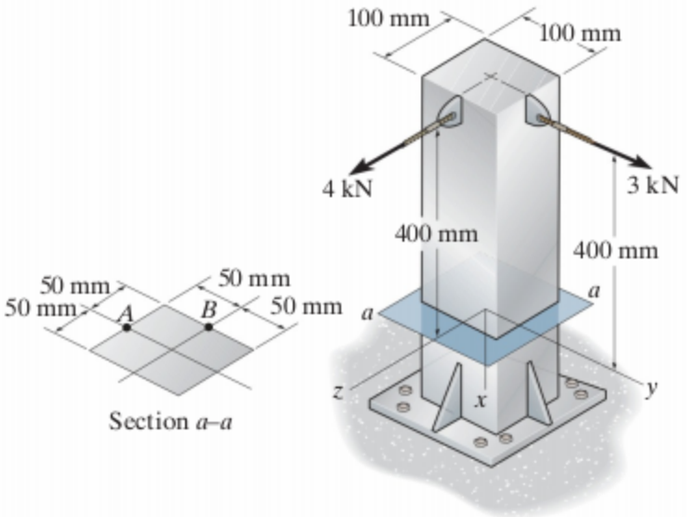
13–57. The sign is subjected to the uniform wind loading. Determine the stress components at points C and D on the 100-mm-diameter supporting post. Show the results on a volume element located at each of these points.



Probs. 13–56/57

13–58. Determine the state of stress at point A on the cross section of the post at section a–a. Indicate the results on a differential element at the point.

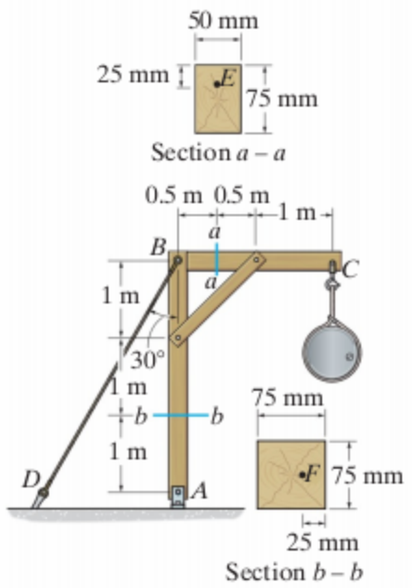
13–59. Determine the state of stress at point B on the cross section of the post at section a–a. Indicate the results on a differential element at the point.



Probs. 13–58/59

*13–60. The 20-kg drum is suspended from the hook mounted on the wooden frame. Determine the state of stress at point E on the cross section of the frame at section a–a. Indicate the results on an element.

13–61. The 20-kg drum is suspended from the hook mounted on the wooden frame. Determine the state of stress at point F on the cross section of the frame at section b–b. Indicate the results on an element.



Probs. 13–60/61

CONCEPTUAL PROBLEMS

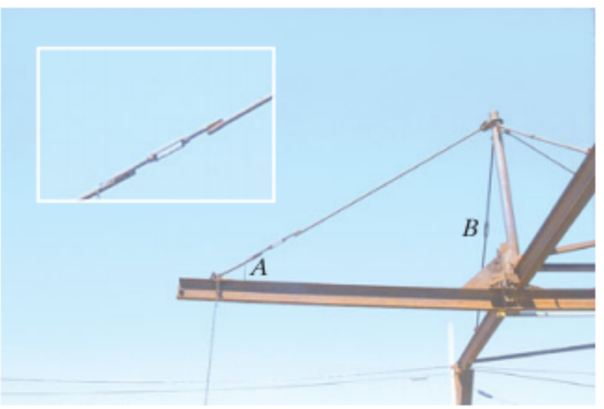
P13–1. Explain why failure of this garden hose occurred near its end and why the tear occurred along its length. Use numerical values to explain your result. Assume the water pressure is 30 psi.

P13–1

P13–2. This open-ended silo contains granular material. It is constructed from wood slats and held together with steel bands. Explain, using numerical values, why the bands are not spaced evenly along the height of the cylinder. Also, how would you find this spacing if each band is to be subjected to the same stress?

P13–2

P13–3. Unlike the turnbuckle at B, which is connected along the axis of the rod, the one at A has been welded to the edges of the rod, and so it will be subjected to additional stress. Use the same numerical values for the tensile load in each rod and the rod's diameter, and compare the stress in each rod.



P13–3

P13–4. A constant wind blowing against the side of this chimney has caused creeping strains in the mortar joints, such that the chimney has a noticeable deformation. Explain how to obtain the stress distribution over a section at the base of the chimney, and sketch this distribution over the section.

P13–4

CHAPTER REVIEW

A pressure vessel is considered to have a thin wall provided $r/t \geq 10$. For a thin-walled cylindrical vessel, the circumferential or hoop stress is

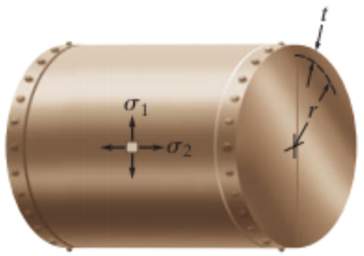
$$\sigma_1 = \frac{pr}{t}$$

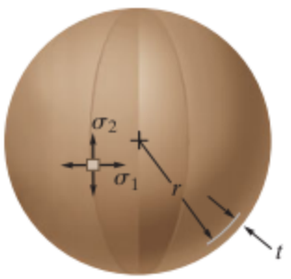
This stress is twice as great as the longitudinal stress,

$$\sigma_2 = \frac{pr}{2t}$$

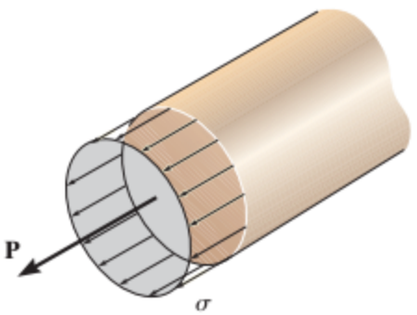
Thin-walled spherical vessels have the same stress within their walls in all directions. It is

$$\sigma_1 = \sigma_2 = \frac{pr}{2t}$$

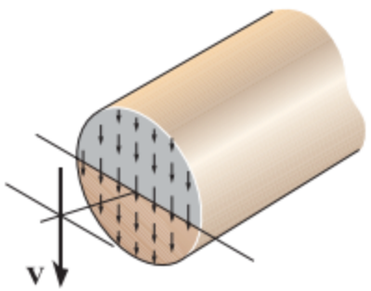




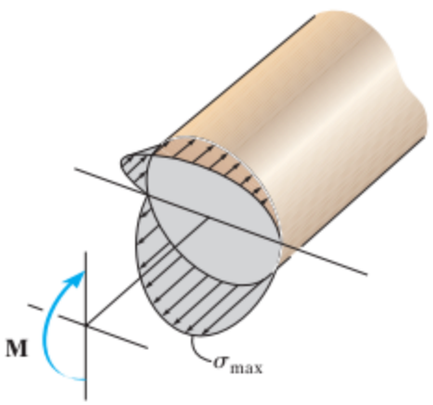
Superposition of stress components can be used to determine the normal and shear stress at a point in a member subjected to a combined loading. To do this it is first necessary to determine the resultant axial and shear forces and the internal resultant torsional and bending moments at the section where the point is located. Then the normal and shear stress resultant components at the point are determined by algebraically adding the normal and shear stress components of each loading.



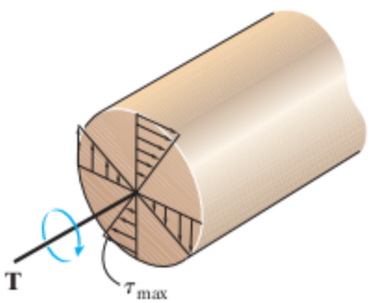
$$\sigma = \frac{P}{A}$$



$$\tau = \frac{VQ}{It}$$



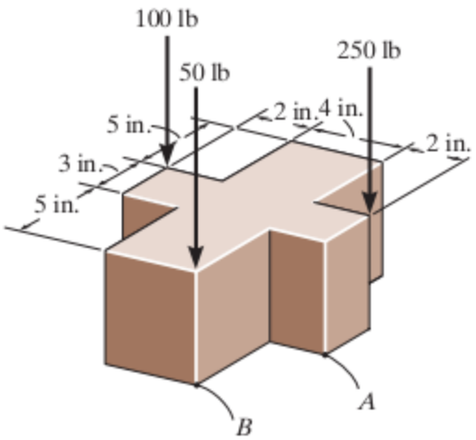
$$\sigma = -\frac{My}{I}$$



$$\tau = \frac{T\rho}{J}$$

REVIEW PROBLEMS

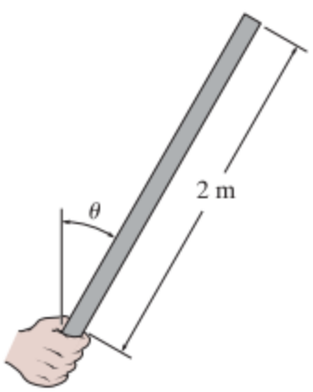
13–62. The block is subjected to the three axial loads shown. Determine the normal stress developed at points A and B. Neglect the weight of the block.



Prob. 13–62

13–63. A bar having a square cross section of 30 mm by 30 mm is 2 m long and is held upward. If it has a mass of 5 kg/m, determine the largest angle θ, measured from the vertical, at which it can be supported before it is subjected to a tensile stress along its axis near the grip.

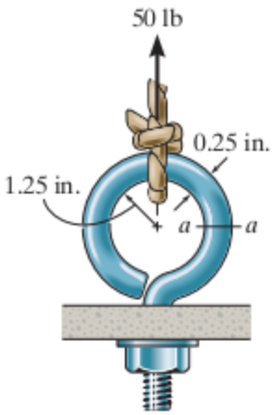
***13–64.** Solve Prob. 13–63 if the bar has a circular cross section of 30-mm diameter.



Probs. 13–63/64

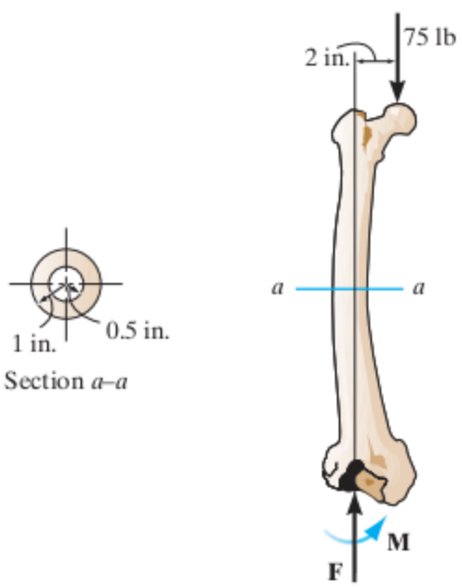
13–65. The eye is subjected to the force of 50 lb. Determine the maximum tensile and compressive stresses at section $a-a$. The cross section is circular and has a diameter of 0.25 in. Use the curved-beam formula to compute the bending stress.

13–66. Solve Prob. 13–65 if the cross section is square, having dimensions of 0.25 in. by 0.25 in.



Probs. 13–65/66

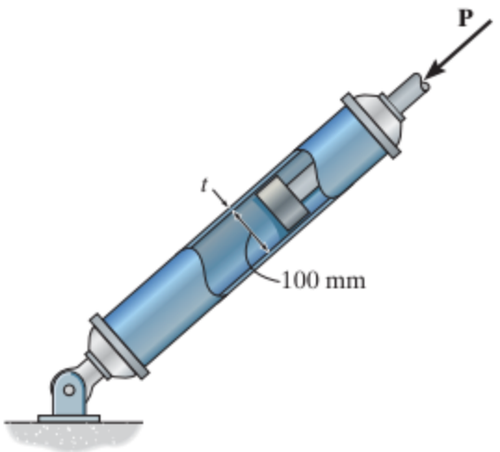
13–67. If the cross section of the femur at section a–a can be approximated as a circular tube as shown, determine the maximum normal stress developed on the cross section at section a–a due to the load of 75 lb.



Prob. 13–67

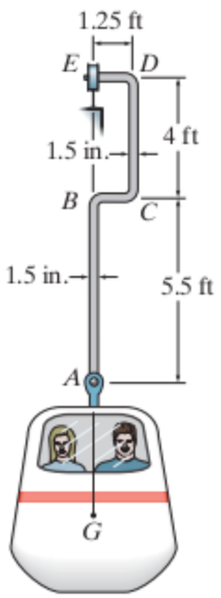
*13–68. The hydraulic cylinder is required to support a force of $P = 100$ kN. If the cylinder has an inner diameter of 100 mm and is made from a material having an allowable normal stress of $\sigma_{allow} = 150$ MPa, determine the required minimum thickness t of the wall of the cylinder.

13–69. The hydraulic cylinder has an inner diameter of 100 mm and wall thickness of $t = 4$ mm. If it is made from a material having an allowable normal stress of $\sigma_{allow} = 150$ MPa, determine the maximum allowable force **P**.



Probs. 13–68/69

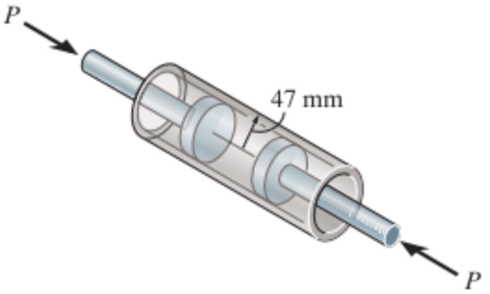
13–70. The gondola and passengers have a weight of 1500 lb and center of gravity at G. The suspender arm AE has a square cross-sectional area of 1.5 in. by 1.5 in., and is pin connected at its ends A and E. Determine the largest tensile stress developed in regions AB and DC of the arm.



Prob. 13–70

13–71. Air pressure in the cylinder is increased by exerting forces $P = 2$ kN on the two pistons, each having a radius of 45 mm. If the cylinder has a wall thickness of 2 mm, determine the state of stress in the wall of the cylinder.

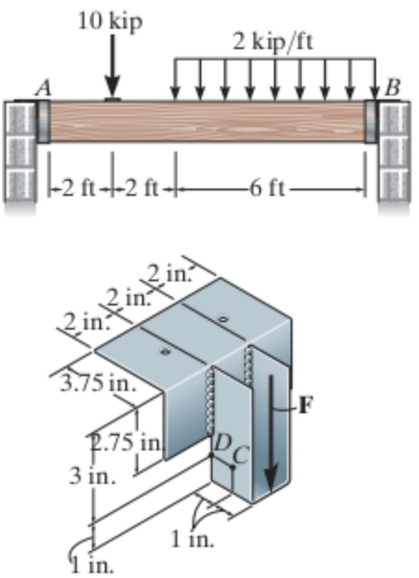
*13–72. Determine the maximum force P that can be exerted on each of the two pistons so that the circumferential stress component in the cylinder does not exceed 3 MPa. Each piston has a radius of 45 mm and the cylinder has a wall thickness of 2 mm.



Probs. 13–71/72

13–73. The wall hanger has a thickness of 0.25 in. and is used to support the vertical reactions of the beam that is loaded as shown. If the load is transferred uniformly to each strap of the hanger, determine the state of stress at points C and D on the strap at A. Assume the vertical reaction **F** at this end acts in the center and on the edge of the bracket as shown.

13–74. The wall hanger has a thickness of 0.25 in. and is used to support the vertical reactions of the beam that is loaded as shown. If the load is transferred uniformly to each strap of the hanger, determine the state of stress at points C and D on the strap at B. Assume the vertical reaction F at this end acts in the center and on the edge of the bracket as shown.



Probs. 13–73/74

Chapter 14

These turbine blades are subjected to a complex pattern of stress. For design it is necessary to determine where and in what direction the maximum stress occurs.

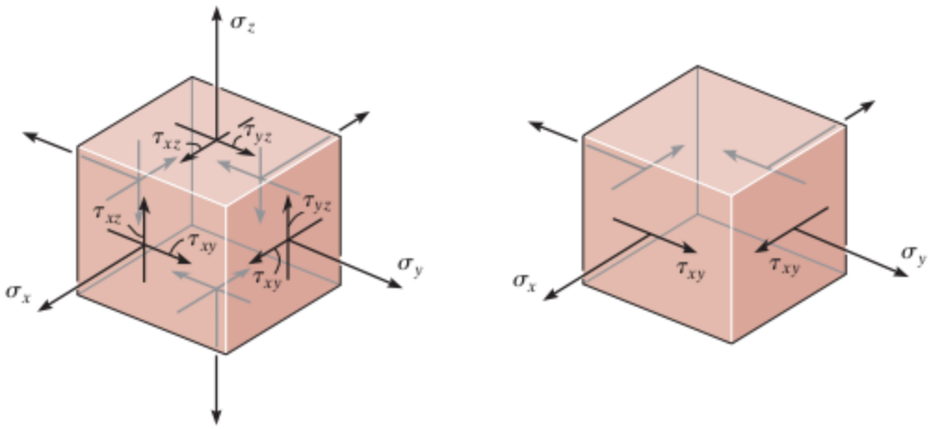
Stress and Strain Transformation

CHAPTER OBJECTIVES

- To develop the transformation of stress components from one orientation of the coordinate system to another orientation.
- To determine the principal stress and maximum in-plane shear stress at a point.
- To develop the transformation of plane strain components from one orientation of the coordinate system to another orientation.
- To determine the principal strain and maximum in-plane shear strain at a point.
- To develop Mohr's circle for analyzing stress and strain components.
- To discuss strain rosettes for strain components.
- To present the relationships between material properties, such as the elastic modulus, shear modulus, and Poisson's ratio.

14.1 Plane-Stress Transformation

It was shown in Sec. 7.3 that the general state of stress at a point is characterized by *six* independent normal and shear stress components, which act on the faces of an element of material located at the point, Fig. 14–1*a*. This state of stress, however, is not often encountered in engineering practice. Instead, engineers frequently make approximations or simplifications of the loadings on a body in order that the stress produced in a structural member or mechanical element can be analyzed in a *single plane*. When this is the case, the material is said to be subjected to *plane stress*, Fig. 14–1*b*. For example, if there is no load on the surface of a body, then the normal and shear stress components will be zero on the face of an element that lies on this surface. Consequently, the corresponding stress components on the opposite face will also be zero, and so the material at the point will be subjected to plane stress. This case occurred throughout the previous chapter.



General state of stress (a)

Plane stress (b)

Fig. 14–1

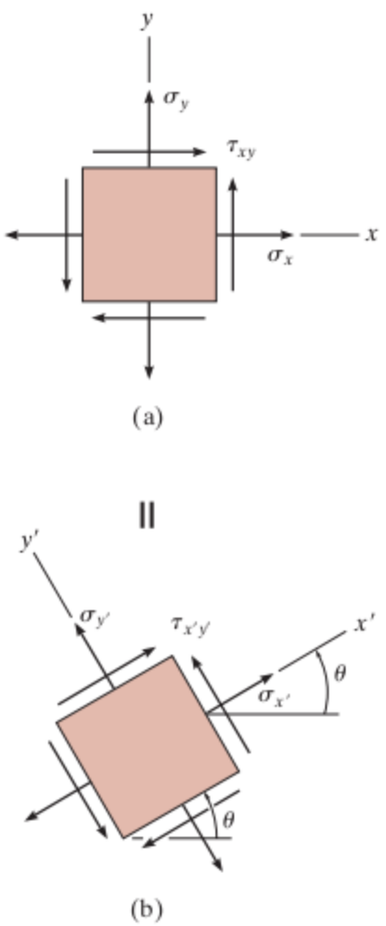


Fig. 14–2

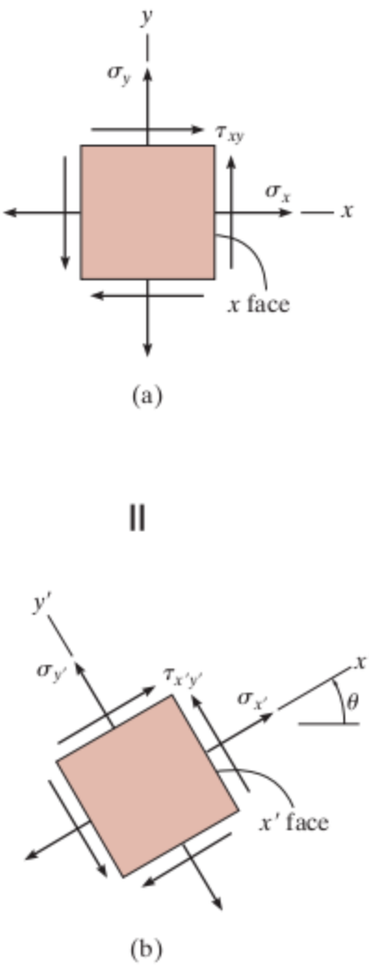
The general state of ***plane stress*** at a point is therefore represented by a combination of two normal-stress components, σ_x, σ_y, and one shear stress component, τ_{xy}, which act on four faces of the element. For convenience, in this text we will view this state of stress in the x–y plane, with the x, y axes as shown in Fig. 14–2*a*. If this state of stress is defined on an element having a *different orientation* θ as in Fig. 14–2*b*, then it will be subjected to three *different* stress components defined as $\sigma_{x'}$, $\sigma_{y'}$, $\tau_{x'y'}$ relative to the x', y' axes. In other words, ***the state of plane stress at the point is uniquely represented by two normal stress components and one shear stress component acting on an element. These three components will be different for each specific orientation θ of the element at the point.***

In this section, we will show how to *transform* the stress components from the orientation of an element in Fig. 14–2*a* to the orientation of the element in Fig. 14–2*b*. This is like knowing two force components, say, $\mathbf{F}_x$ and $\mathbf{F}_y$, directed along the x, y axes, that produce a resultant force $\mathbf{F}_R$, and then trying to find the force components $\mathbf{F}_{x'}$ and $\mathbf{F}_{y'}$, directed along the x', y' axes, so they produce the *same* resultant. The transformation for force must only account for the force component's magnitude and direction. The transformation of stress components, however, is more difficult since it must account for the magnitude and direction of each stress component *and* the orientation of the area upon which each component acts.

Procedure for Analysis

If the state of stress at a point is known for a given orientation of an element of material, Fig. 14–3*a*, then the state of stress on an element having some other orientation, θ, Fig. 14–3*b*, can be determined using the following procedure.

- To determine the normal and shear stress components $\sigma_{x'}, \tau_{x'y'}$ acting on the $+x'$ face of the element, Fig. 14–3*b*, section the element in Fig. 14–3*a* as shown in Fig. 14–3*c*. If the sectioned area is ΔA, then the adjacent areas of the segment will be $\Delta A \sin\theta$ and $\Delta A \cos\theta$.
- Draw the free-body diagram of the segment, which requires showing the *forces* that act on the segment, Fig. 14–3*d*. This is done by multiplying the stress components on each face by the area upon which they act.
- When $\Sigma F_{x'} = 0$ is applied to the free-body diagram, the area ΔA will cancel out of each term and a *direct* solution for $\sigma_{x'}$ will be possible. Likewise, $\Sigma F_{y'} = 0$ will yield $\tau_{x'y'}$.
- If $\sigma_{y'}$, acting on the $+y'$ face of the element in Fig. 14–3*b*, is to be determined, then it is necessary to consider a segment of the element as shown in Fig. 14–3*e* and follow the same procedure just described. Here, however, the shear stress $\tau_{x'y'}$ does not have to be determined if it was previously calculated, since it is complementary, that is, it must have the same magnitude on each of the four faces of the element, Fig. 14–3*b*.



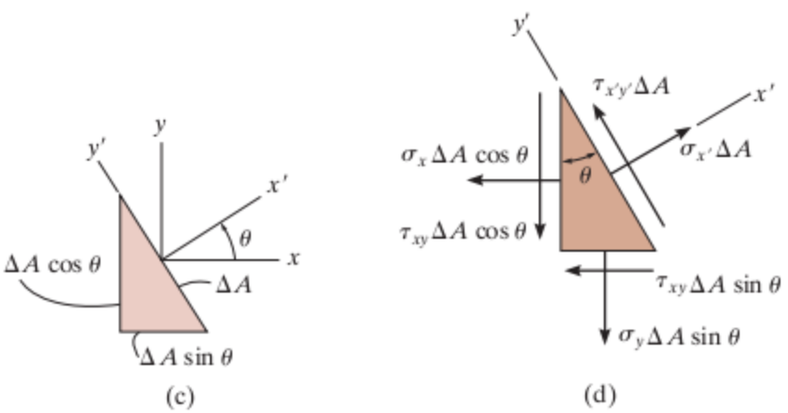
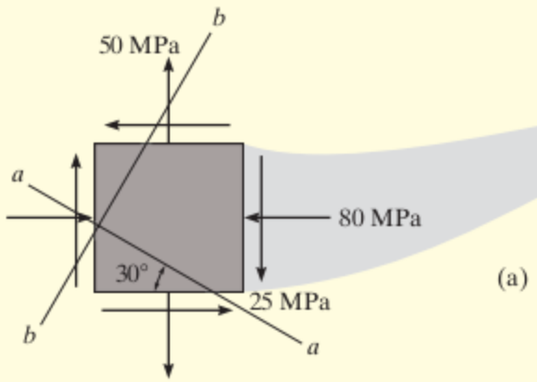


Fig. 14–3

EXAMPLE 14.1

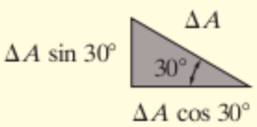
The state of plane stress at a point on the surface of the airplane fuselage is represented on the element oriented as shown in Fig. 14–4*a*. Represent the state of stress at the point on an element that is oriented 30° clockwise from the position shown.



(a)

SOLUTION

The rotated element is shown in Fig. 14–4*d*. To obtain the stress components on this element we will first section the element in Fig. 14–4*a* by the line *a*–*a*. The bottom segment is removed, and assuming the sectioned (inclined) plane has an area ΔA, the horizontal and vertical planes have the areas shown in Fig. 14–4*b*. The free-body diagram of this segment is shown in Fig. 14–4*c*. Notice that the sectioned x' face is defined by the *outward normal* x' axes, and the y' axis is along the face.



(b)

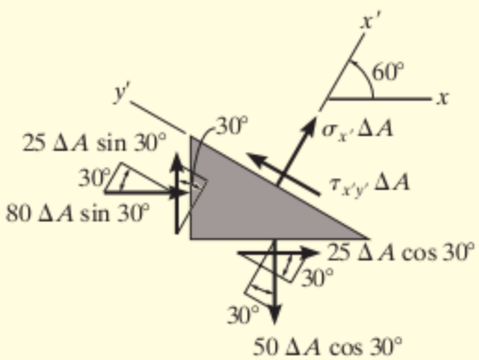
Equilibrium. Applying the equations of force equilibrium in the x' and y' directions to avoid a simultaneous solution for the two unknowns $\sigma_{x'}$ and $\tau_{x'y'}$, we have

$$+\nearrow \Sigma F_{x'} = 0; \quad \sigma_{x'}\Delta A - (50\Delta A \cos 30°)\cos 30° + (25\Delta A \cos 30°)\sin 30° + (80\Delta A \sin 30°)\sin 30° + (25\,\Delta A \sin 30°)\cos 30° = 0$$

$$\sigma_{x'} = -4.15 \text{ MPa} \qquad \textit{Ans.}$$

$$+\nwarrow \Sigma F_{y'} = 0; \quad \tau_{x'y'}\Delta A - (50\,\Delta A \cos 30°)\sin 30° - (25\,\Delta A \cos 30°)\cos 30° - (80\Delta A \sin 30°)\cos 30° + (25\,\Delta A \sin 30°)\sin 30° = 0$$

$$\tau_{x'y'} = 68.8 \text{ MPa} \qquad \textit{Ans.}$$

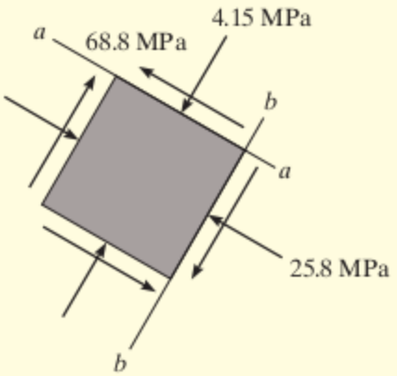


(c)

Fig. 14–4

Since $\sigma_{x'}$ is negative, it acts in the opposite direction of that shown in Fig. 14–4*c*. The results are shown on the *top* of the element in Fig. 14–4*d*, since this surface is the one considered in Fig. 14–4*c*.

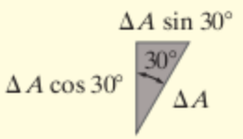
We must now repeat the procedure to obtain the stress on the *perpendicular* plane b–b. Sectioning the element in Fig. 14–4*a* along b–b results in a segment having sides with areas shown in Fig. 14–4*e*. Orienting the $+x'$ axis outward, perpendicular to the sectioned face, the associated free-body diagram is shown in Fig. 14–4*f*. Thus,



(d)

$$+\searrow \Sigma F_{x'} = 0; \quad \sigma_{x'}\Delta A - (25\ \Delta A \cos 30°) \sin 30°$$
$$+ (80\ \Delta A \cos 30°) \cos 30° - (25\ \Delta A \sin 30°) \cos 30°$$
$$- (50\ \Delta A \sin 30°) \sin 30° = 0$$
$$\sigma_{x'} = -25.8 \text{ MPa} \qquad \textit{Ans.}$$

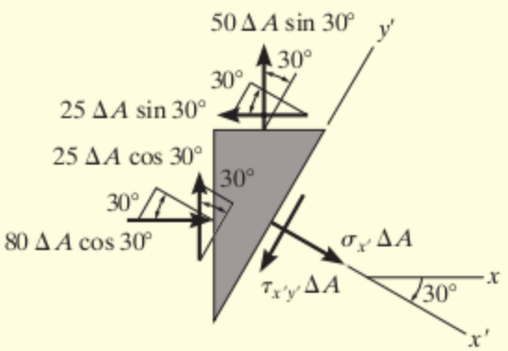
$$+\nearrow \Sigma F_{y'} = 0; \quad -\tau_{x'y'}\Delta A + (25\ \Delta A \cos 30°) \cos 30°$$
$$+ (80\ \Delta A \cos 30°) \sin 30° - (25\ \Delta A \sin 30°) \sin 30°$$
$$+ (50\ \Delta A \sin 30°) \cos 30° = 0$$
$$\tau_{x'y'} = 68.8 \text{ MPa} \qquad \textit{Ans.}$$



(e)

Since $\sigma_{x'}$ is a negative quantity, it acts opposite to its direction shown in Fig. 14–4*f*. The stress components are shown acting on the *right side* of the element in Fig. 14–4*d*.

From this analysis we may therefore conclude that the state of stress at the point can be represented by choosing an element oriented as shown in Fig. 14–4*a*, or by choosing one oriented as shown in Fig. 14–4*d*. In other words, these states of stress are equivalent.



(f)

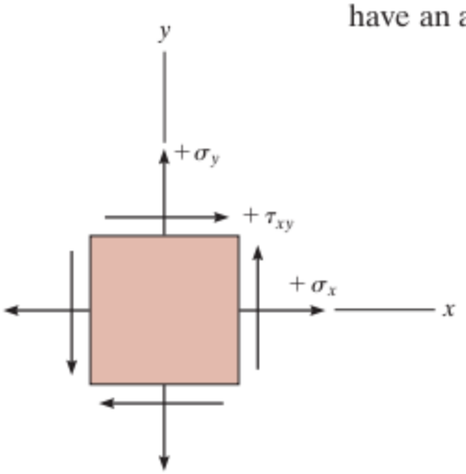
14.2 General Equations of Plane-Stress Transformation

The method of transforming the normal and shear stress components from the x, y to the x', y' coordinate axes, as discussed in the previous section, can be developed in a general manner and expressed as a set of stress-transformation equations.

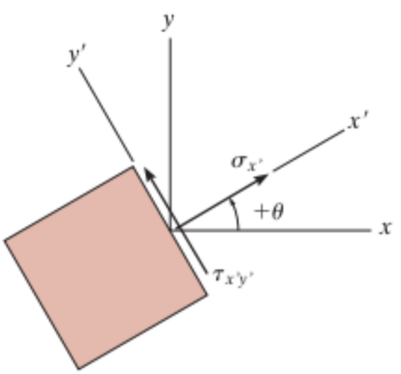
Sign Convention. First we must establish a sign convention for the stress components. To do this the $+x$ and $+x'$ axes are used to define the outward normal from a side of the element. Then σ_x and $\sigma_{x'}$ are positive when they act in the positive x and x' directions, and τ_{xy} and $\tau_{x'y'}$ are positive when they act in the positive y and y' directions, Fig. 14–5.

The orientation of the plane on which the normal and shear stress components are to be determined will be defined by the angle θ, which is measured from the $+x$ axis to the $+x'$ axis, Fig. 14–5*b*. Notice that the unprimed and primed sets of axes in this figure both form right-handed coordinate systems; that is, the positive z (or z') axis is established by the right-hand rule. Curling the fingers from x (or x') toward y (or y') gives the direction for the positive z (or z') axis that points outward, along the thumb. The *angle* θ will be *positive* provided it follows the curl of the right-hand fingers, i.e., counterclockwise as shown in Fig. 14–5*b*.

Normal and Shear Stress Components. Using the established sign convention, the element in Fig. 14–6*a* is sectioned along the inclined plane and the segment shown in Fig. 14–6*b* is isolated. Assuming the sectioned area is ΔA, then the horizontal and vertical faces of the segment have an area of $\Delta A \sin\theta$ and $\Delta A \cos\theta$, respectively.



(a)



(b)

Positive sign convention

Fig. 14–5

The resulting *free-body diagram* of the segment is shown in Fig. 14–6*c*. Applying the equations of equilibrium to determine the unknown normal and shear stress components $\sigma_{x'}$ and $\tau_{x'y'}$, we have

$$+\nearrow \Sigma F_{x'} = 0; \quad \sigma_{x'}\Delta A - (\tau_{xy}\Delta A \sin\theta)\cos\theta - (\sigma_y \Delta A \sin\theta)\sin\theta - (\tau_{xy}\Delta A \cos\theta)\sin\theta - (\sigma_x \Delta A \cos\theta)\cos\theta = 0$$

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}(2\sin\theta\cos\theta)$$

$$+\Sigma F_{y'} = 0; \quad \tau_{x'y'}\Delta A + (\tau_{xy}\Delta A \sin\theta)\sin\theta - (\sigma_y \Delta A \sin\theta)\cos\theta - (\tau_{xy}\Delta A \cos\theta)\cos\theta + (\sigma_x \Delta A \cos\theta)\sin\theta = 0$$

$$\tau_{x'y'} = (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

These two equations may be simplified by using the trigonometric identities $\sin 2\theta = 2\sin\theta\cos\theta$, $\sin^2\theta = (1 - \cos 2\theta)/2$, and $\cos^2\theta = (1 + \cos 2\theta)/2$, in which case,

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \tag{14–1}$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \tag{14–2}$$

If the normal stress acting in the y' direction is needed, it can be obtained by simply substituting $(\theta = \theta + 90°)$ for θ into Eq. 14–1, Fig. 14–6*d*. This yields

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \tag{14–3}$$

If $\sigma_{y'}$ is calculated as a positive quantity, it indicates that it acts in the positive y' direction as shown in Fig. 14–6*d*.

Procedure for Analysis

To apply the stress transformation Eqs. 14–1 and 14–2, it is simply necessary to substitute in the known data for σ_x, σ_y, τ_{xy}, and θ in accordance with the established sign convention, Fig. 14–5. Remember that the x' axis is *always* directed *positive outward* from the plane upon which the normal stress is to be determined. The angle θ is *positive counterclockwise*, from the x to the x' axis. If $\sigma_{x'}$ and $\tau_{x'y'}$ are calculated as positive quantities, then these stresses act in the positive direction of the x' and y' axes.

For convenience, these equations can easily be programmed on a pocket calculator.

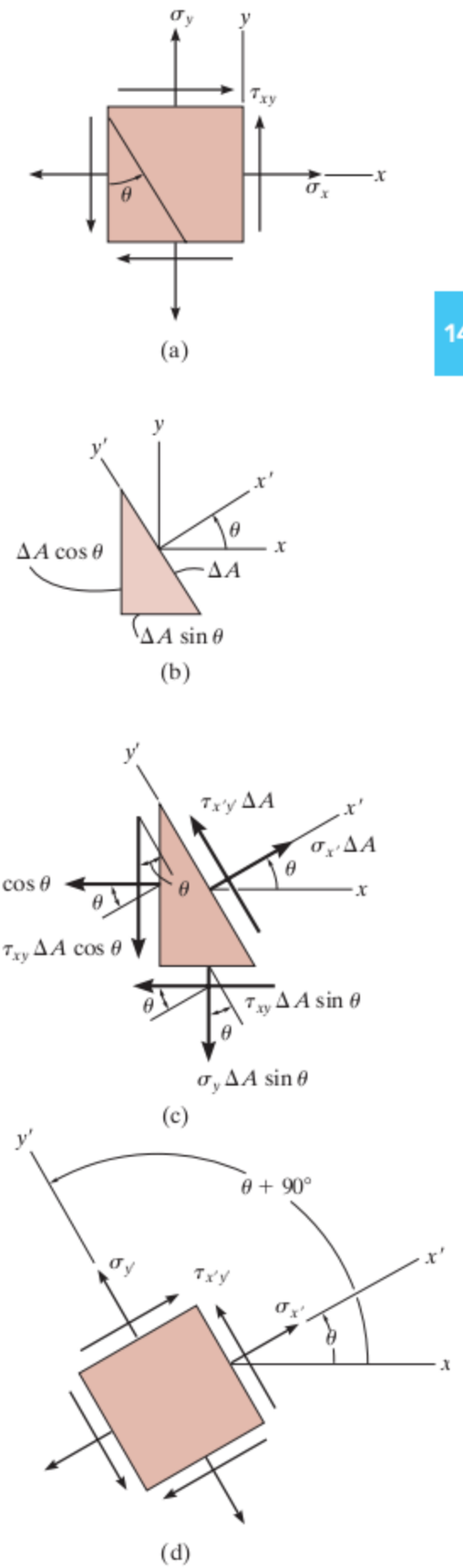
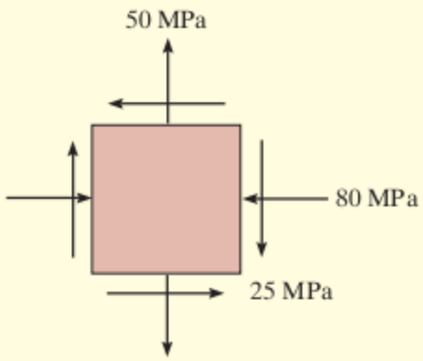


Fig. 14–6

EXAMPLE 14.2



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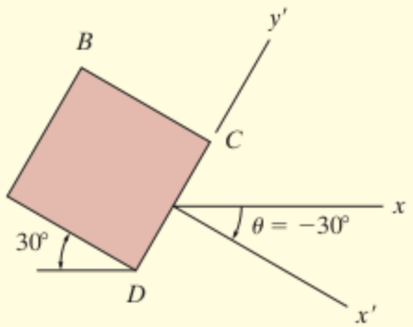

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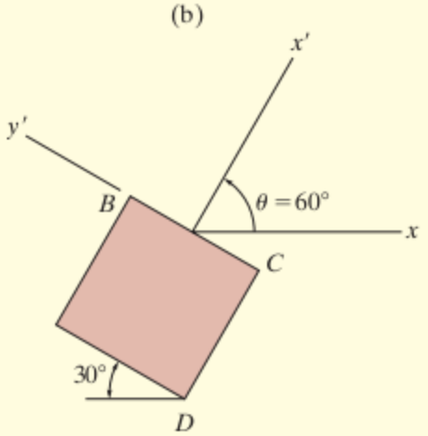

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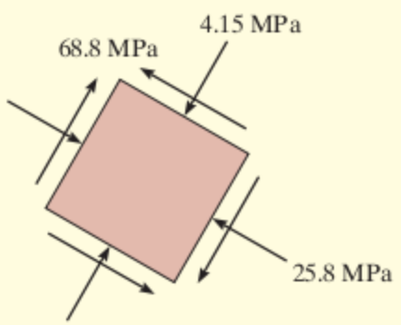


(d)

Fig. 14–7

The state of plane stress at a point is represented by the element shown in Fig. 14–7*a*. Determine the state of stress at the point on another element oriented 30° clockwise from the position shown.

SOLUTION

This problem was solved in Example 14.1 using basic principles. Here we will apply Eqs. 14–1 and 14–2. From the established sign convention, Fig. 14–5, it is seen that

$$\sigma_x = -80\ \text{MPa} \qquad \sigma_y = 50\ \text{MPa} \qquad \tau_{xy} = -25\ \text{MPa}$$

Plane *CD*. To obtain the stress components on plane *CD*, Fig. 14–7*b*, the positive x' axis must be directed outward, perpendicular to *CD*, and the associated y' axis is directed along *CD*. The angle measured from the x to the x' axis is $\theta = -30°$ (clockwise). Applying Eqs. 14–1 and 14–2 yields

$$\begin{aligned}\sigma_{x'} &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ &= \frac{-80 + 50}{2} + \frac{-80 - 50}{2}\cos 2(-30°) + (-25)\sin 2(-30°) \\ &= -25.8\ \text{MPa} \qquad \textit{Ans.}\end{aligned}$$

$$\begin{aligned}\tau_{x'y'} &= -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \\ &= -\frac{-80 - 50}{2}\sin 2(-30°) + (-25)\cos 2(-30°) \\ &= -68.8\ \text{MPa} \qquad \textit{Ans.}\end{aligned}$$

The negative signs indicate that $\sigma_{x'}$ and $\tau_{x'y'}$ act in the negative x' and y' directions, respectively. The results are shown acting on the element in Fig. 14–7*d*.

Plane *BC*. Establishing the x' axis outward from plane *BC*, Fig. 14–7*c*, then between the x and x' axes, $\theta = 60°$ (counterclockwise). Applying Eqs. 14–1 and 14–2,* we get

$$\begin{aligned}\sigma_{x'} &= \frac{-80 + 50}{2} + \frac{-80 - 50}{2}\cos 2(60°) + (-25)\sin 2(60°) \\ &= -4.15\ \text{MPa} \qquad \textit{Ans.}\end{aligned}$$

$$\begin{aligned}\tau_{x'y'} &= -\frac{-80 - 50}{2}\sin 2(60°) + (-25)\cos 2(60°) \\ &= 68.8\ \text{MPa} \qquad \textit{Ans.}\end{aligned}$$

Here $\tau_{x'y'}$ has been calculated twice in order to provide a check. The negative sign for $\sigma_{x'}$ indicates that this stress acts in the negative x' direction, Fig. 14–7*c*. The results are shown on the element in Fig. 14–7*d*.

*Alternatively, we could apply Eq. 14–3 with $\theta = -30°$ rather than Eq. 14–1.

14.3 Principal Stresses and Maximum In-Plane Shear Stress

From Eqs. 14–1 and 14–2, it can be seen that the magnitudes of $\sigma_{x'}$ and $\tau_{x'y'}$ depend on the angle of inclination θ of the planes on which these stresses act. In engineering practice it is often important to determine the orientation of the element that causes the normal stress to be a maximum and a minimum and the orientation that causes the shear stress to be a maximum. In this section each of these problems will be considered.

In-Plane Principal Stresses. To determine the maximum and minimum *normal stress*, we must differentiate Eq. 14–1 with respect to θ and set the result equal to zero. This gives

$$\frac{d\sigma_{x'}}{d\theta} = -\frac{\sigma_x - \sigma_y}{2}(2\sin 2\theta) + 2\tau_{xy}\cos 2\theta = 0$$

Solving this equation we obtain the orientation $\theta = \theta_p$ of the planes of maximum and minimum normal stress.

$$\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2} \tag{14–4}$$

The solution has two roots, θ_{p_1}, and θ_{p_2}. Specifically, the values of $2\theta_{p_1}$ and $2\theta_{p_2}$ are 180° apart, so θ_{p_1} and θ_{p_2} will be 90° apart.

The values of θ_{p_1} and θ_{p_2} must be substituted into Eq. 14–1 if we are to obtain the required normal stresses. To do this we can obtain the necessary sine and cosine of $2\theta_{p_1}$ and $2\theta_{p_2}$ from the shaded triangles shown in Fig. 14–8. The construction of these triangles is based on Eq. 14–4, assuming that τ_{xy} and $(\sigma_x - \sigma_y)$ are both positive or both negative quantities.

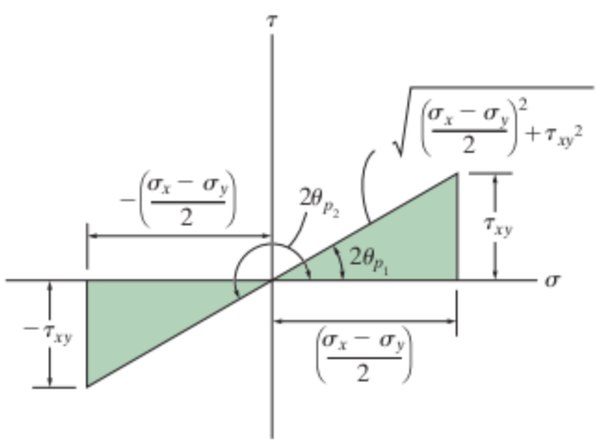


Fig. 14–8

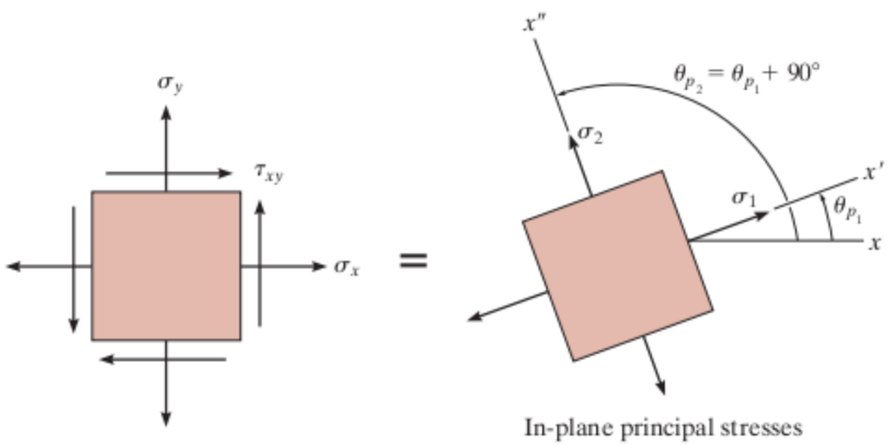
The cracks in this concrete beam were caused by tension stress, even though the beam was subjected to both an internal moment and shear. The stress-transformation equations can be used to predict the direction of the cracks, and the principal normal stresses that caused them.

14

Substituting these trigonometric values into Eq. 14–1 and simplifying, we obtain

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Depending upon the sign chosen, this result gives the maximum or minimum in-plane normal stress acting at a point, where $\sigma_1 \geq \sigma_2$. This particular set of values are called the in-plane ***principal stresses***, and the corresponding planes on which they act are called the ***principal planes*** of stress, Fig. 14–9. Furthermore, if the trigonometric relations for θ_{p_1} or θ_{p_2} are substituted into Eq. 14–2, it can be seen that $\tau_{x'y'} = 0$; in other words, ***no shear stress acts on the principal planes.***



In-plane principal stresses

Fig. 14–9

Maximum In-Plane Shear Stress. The orientation of an element that is subjected to maximum shear stress on its sides can be determined by taking the derivative of Eq. 14–2 with respect to θ and setting the result equal to zero. This gives

$$\tan 2\theta_s = \frac{-(\sigma_x - \sigma_y)/2}{\tau_{xy}} \tag{14–6}$$

The two roots of this equation, θ_{s_1} and θ_{s_2}, can be determined from the shaded triangles shown in Fig. 14–10. By comparison with Eq. 14–4, $\tan 2\theta_s$ is the negative reciprocal of $\tan 2\theta_p$ and so each root $2\theta_s$ is 90° from $2\theta_p$, and the roots θ_s and θ_p are 45° apart. Therefore, an element subjected to ***maximum shear stress will be 45° from the position of an element that is subjected to the principal stress***.

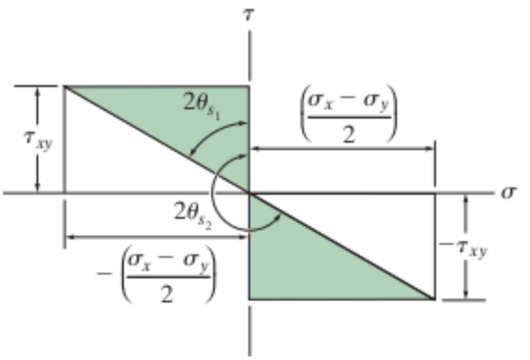


Fig. 14–10

Using either one of the roots θ_{s_1} or θ_{s_2}, the maximum shear stress can be found by taking the trigonometric values of $\sin 2\theta_s$ and $\cos 2\theta_s$ from Fig. 14–10 and substituting them into Eq. 14–2. The result is

$$\tau_{\substack{\max \\ \text{in-plane}}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{14–7}$$

The value of $\tau_{\substack{\max \\ \text{in-plane}}}$ as calculated from this equation is referred to as the ***maximum in-plane shear stress*** because it acts on the element in the x–y plane.

Substituting the values for $\sin 2\theta_s$ and $\cos 2\theta_s$ into Eq. 14–1, we see that there is *also* an average normal stress on the planes of maximum in-plane shear stress. We get

$$\sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2} \tag{14–8}$$

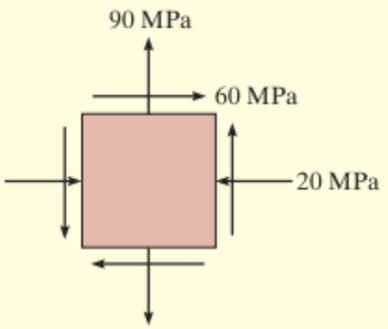
Like the stress-transformation equations, it may be convenient to program Eqs. 14–4 through Eq. 14–8 for use on a pocket calculator.

Important Points

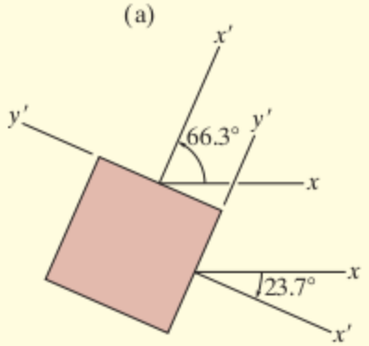
- The *principal stresses* represent the maximum and minimum normal stress at the point.
- When the state of stress is represented by the principal stresses, *no shear stress* will act on the element.
- The state of stress at the point can also be represented in terms of the *maximum in-plane shear stress*. In this case an *average normal stress* will also act on the element.
- The element representing the maximum in-plane shear stress with the associated average normal stresses is *oriented* 45° from the element representing the principal stresses.

EXAMPLE 14.3

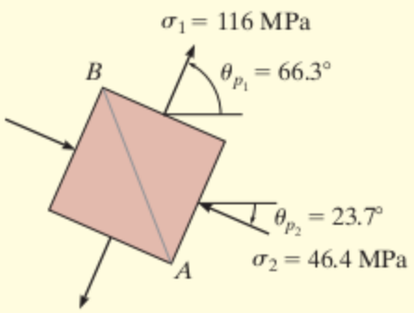
Notice how the failure plane is at an angle (23.7°) due to tearing of the material, Fig. 14–11*c*.



(a)



(b)



(c)

Fig. 14–11

The state of plane stress at a failure point on the shaft is shown on the element in Fig. 14–11*a*. Represent this stress state in terms of the principal stresses.

SOLUTION

From the established sign convention, we have

$$\sigma_x = -20 \text{ MPa} \qquad \sigma_y = 90 \text{ MPa} \qquad \tau_{xy} = 60 \text{ MPa}$$

Orientation of Element. Applying Eq. 14–4,

$$\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2} = \frac{60}{(-20 - 90)/2}$$

Solving, and referring to this root as θ_{p_2}, as will be shown below, yields

$$2\theta_p = -47.49° \qquad \theta_{p_2} = -23.7°$$

Since the difference between $2\theta_{p_1}$ and $2\theta_{p_2}$ is 180°, we have

$$2\theta_{p_1} = 180° + 2\theta_{p_2} = 132.51° \qquad \theta_{p_1} = 66.3°$$

Recall that θ must be measured positive *counterclockwise* from the x axis to the outward normal (x' axis) on the face of the element, and so the results are shown in Fig. 14–11*b*.

Principal Stress. We have

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + {\tau_{xy}}^2}$$

$$= \frac{-20 + 90}{2} \pm \sqrt{\left(\frac{-20 - 90}{2}\right)^2 + (60)^2}$$

$$= 35.0 \pm 81.4$$

$$\sigma_1 = 116 \text{ MPa} \qquad \textit{Ans.}$$

$$\sigma_2 = -46.4 \text{ MPa} \qquad \textit{Ans.}$$

The principal plane on which each normal stress acts can be determined by applying Eq. 14–1 with, say, $\theta = \theta_{p_2} = -23.7°$. We have

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

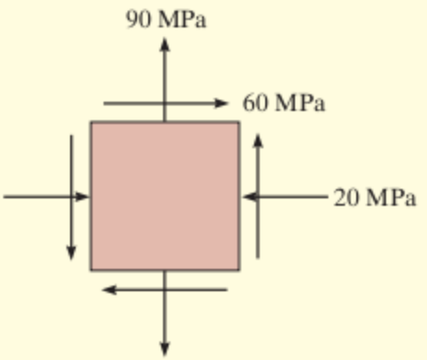
$$= \frac{-20 + 90}{2} + \frac{-20 - 90}{2}\cos 2(-23.7°) + 60 \sin 2(-23.7°)$$

$$= -46.4 \text{ MPa}$$

Hence, $\sigma_2 = -46.4$ MPa acts on the plane defined by $\theta_{p_2} = -23.7°$, whereas $\sigma_1 = 116$ MPa acts on the plane defined by $\theta_{p_1} = 66.3°$. The results are shown on the element in Fig. 14–11*c*. Recall that no shear stress acts on this element.

EXAMPLE 14.4

The state of plane stress at a point on a body is represented on the element shown in Fig. 14–12*a*. Represent this stress state in terms of the maximum in-plane shear stress and associated average normal stress.



(a)

SOLUTION

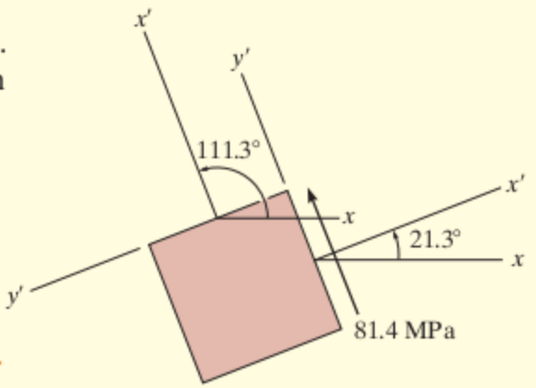
Orientation of Element. Since $\sigma_x = -20$ MPa, $\sigma_y = 90$ MPa, and $\tau_{xy} = 60$ MPa, applying Eq. 14–6, we have

$$\tan 2\theta_s = \frac{-(\sigma_x - \sigma_y)/2}{\tau_{xy}} = \frac{-(-20 - 90)/2}{60}$$

$$2\theta_{s_2} = 42.5^\circ \qquad \theta_{s_2} = 21.3^\circ$$

$$2\theta_{s_1} = 180^\circ + 2\theta_{s_2} \qquad \theta_{s_1} = 111.3^\circ$$

Note how these angles are formed between the x and x' axes, Fig. 14–12*b*. They happen to be 45° away from the principal planes of stress, which were determined in Example 14.3.



(b)

Maximum In-Plane Shear Stress. Applying Eq. 14–7,

$$\tau_{\substack{\max \\ \text{in-plane}}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{-20 - 90}{2}\right)^2 + (60)^2}$$

$$= \pm 81.4 \text{ MPa} \qquad \textit{Ans.}$$

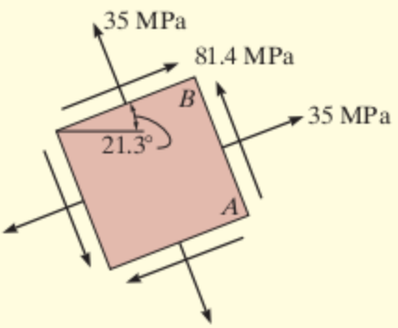
The proper direction of $\tau_{\substack{\max \\ \text{in-plane}}}$ on the element can be determined by substituting $\theta = \theta_{s_2} = 21.3^\circ$ into Eq. 14–2. We have

$$\tau_{x'y'} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$= -\left(\frac{-20 - 90}{2}\right)\sin 2(21.3^\circ) + 60\cos 2(21.3^\circ)$$

$$= 81.4 \text{ MPa}$$

This positive result indicates that $\tau_{\substack{\max \\ \text{in-plane}}} = \tau_{x'y'}$ acts in the *positive* y' direction on this face ($\theta = 21.3^\circ$) Fig. 14–12*b*. The shear stresses on the other three faces are directed as shown in Fig. 14–12*c*.



(c)

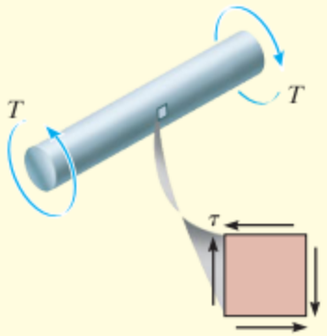
Fig. 14–12

Average Normal Stress. Besides the maximum shear stress, as calculated above, the element is also subjected to an average normal stress determined from Eq. 14–8; that is,

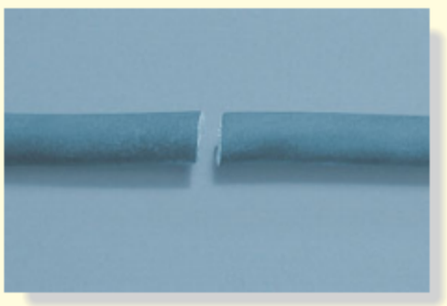
$$\sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2} = \frac{-20 + 90}{2} = 35 \text{ MPa} \qquad \textit{Ans.}$$

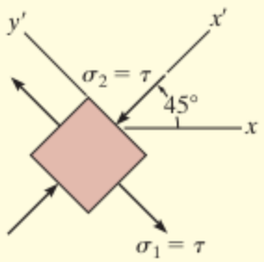
This is a tensile stress. The results are shown in Fig. 14–12*c*.

EXAMPLE 14.5



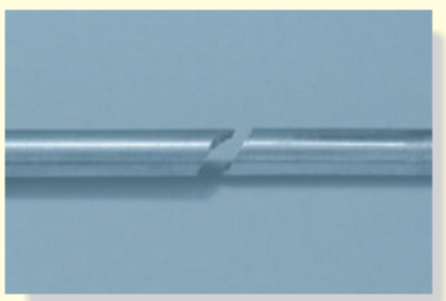
(a)





(b)

Fig. 14–13



When the torsional loading T is applied to the bar in Fig. 14–13*a*, it produces a state of pure shear stress in the material. Determine (a) the maximum in-plane shear stress and the associated average normal stress, and (b) the principal stress.

SOLUTION

From the established sign convention,

$$\sigma_x = 0 \qquad \sigma_y = 0 \qquad \tau_{xy} = -\tau$$

Maximum In-Plane Shear Stress. Applying Eqs. 14–7 and 14–8, we have

$$\tau_{\substack{\max \\ \text{in-plane}}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{(0)^2 + (-\tau)^2} = \pm\tau \qquad \textit{Ans.}$$

$$\sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2} = \frac{0 + 0}{2} = 0 \qquad \textit{Ans.}$$

Thus, as expected, the maximum in-plane shear stress is represented by the element in Fig. 14–13*a*.

NOTE: Through experiment it has been found that materials that are *ductile* will *fail* due to *shear stress*. As a result, if the bar in Fig. 14–13*a* is made of mild steel, the maximum in-plane shear stress will cause it to fail as shown in the adjacent photo.

Principal Stress. Applying Eqs. 14–4 and 14–5 yields

$$\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2} = \frac{-\tau}{(0 - 0)/2}, \quad \theta_{p_2} = 45°, \theta_{p_1} = -45°$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 0 \pm \sqrt{(0)^2 + \tau^2} = \pm\tau \qquad \textit{Ans.}$$

If we now apply Eq. 14–1 with $\theta_{p_2} = 45°$, then

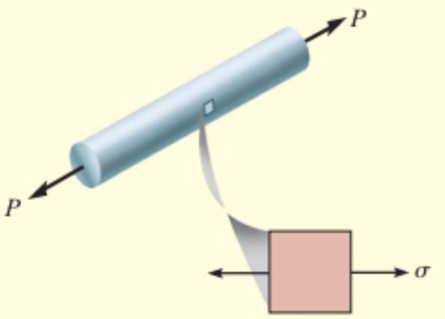
$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$= 0 + 0 + (-\tau)\sin 90° = -\tau$$

Thus, $\sigma_2 = -\tau$ acts at $\theta_{p_2} = 45°$ as shown in Fig. 14–13*b*, and $\sigma_1 = \tau$ acts on the other face, $\theta_{p_1} = -45°$.

NOTE: Materials that are *brittle* fail due to *normal stress*. Therefore, if the bar in Fig. 14–13*a* is made of cast-iron it will fail in tension at a 45° inclination as seen in the adjacent photo.

EXAMPLE 14.6

When the axial loading P is applied to the bar in Fig. 14–14*a*, it produces a tensile stress in the material. Determine (a) the principal stress and (b) the maximum in-plane shear stress and associated average normal stress.



(a)

SOLUTION

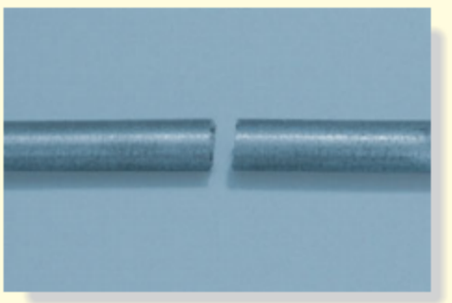
From the established sign convention,

$$\sigma_x = \sigma \qquad \sigma_y = 0 \qquad \tau_{xy} = 0$$

Principal Stress. By observation, the element oriented as shown in Fig. 14–14*a* illustrates a condition of principal stress since no shear stress acts on this element. This can also be shown by direct substitution of the above values into Eqs. 14–4 and 14–5. Thus,

$$\sigma_1 = \sigma \qquad \sigma_2 = 0 \qquad \textit{Ans.}$$

NOTE: Experiments have shown that *brittle materials* will fail due to normal stress. Thus if the bar in Fig. 14–14*a* is made of cast-iron, it will cause failure as shown in the adjacent photo.

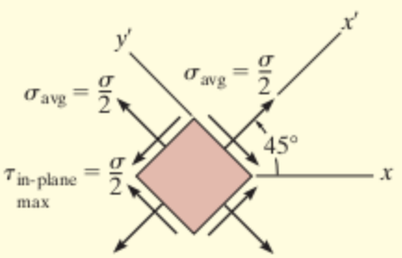


Maximum In-Plane Shear Stress. Applying Eqs. 14–6, 14–7, and 14–8, we have

$$\tan 2\theta_s = \frac{-(\sigma_x - \sigma_y)/2}{\tau_{xy}} = \frac{-(\sigma - 0)/2}{0}; \quad \theta_{s_1} = 45°, \ \theta_{s_2} = -45°$$

$$\tau_{\substack{\max \\ \text{in-plane}}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + {\tau_{xy}}^2} = \sqrt{\left(\frac{\sigma - 0}{2}\right)^2 + (0)^2} = \pm\frac{\sigma}{2} \qquad \textit{Ans.}$$

$$\sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma + 0}{2} = \frac{\sigma}{2} \qquad \textit{Ans.}$$



(b)

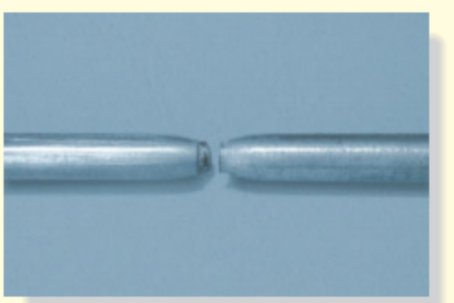
Fig. 14–14

To determine the proper orientation of the element, apply Eq. 14–2.

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta = -\frac{\sigma - 0}{2}\sin 90° + 0 = -\frac{\sigma}{2}$$

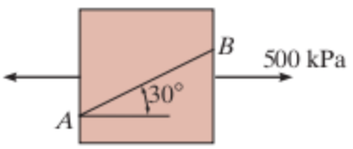
This negative shear stress acts on the x' face, in the negative y' direction as shown in Fig. 14–14*b*.

NOTE: If the bar in Fig. 14–14*a* is made of a *ductile material* such as mild steel then *shear stress* will cause it to fail. This can be noted in the adjacent photo, where within the region of necking, shear stress has caused "slipping" along the steel's crystalline boundaries, resulting in a plane of failure that has formed a *cone* around the bar oriented at approximately 45° as calculated above.



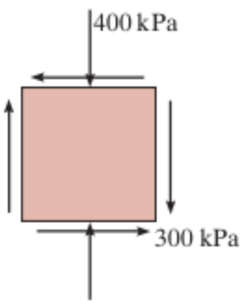
FUNDAMENTAL PROBLEMS

F14–1. Determine the normal stress and shear stress acting on the inclined plane AB. Sketch the result on the sectioned element.



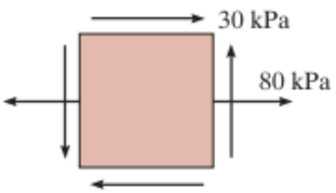
F14–1

F14–2. Determine the equivalent state of stress on an element at the same point oriented 45° clockwise with respect to the element shown.



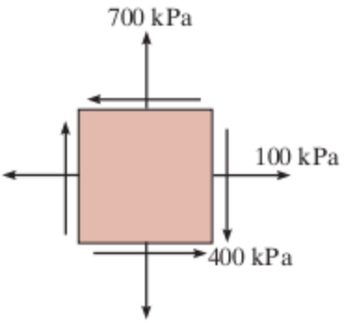
F14–2

F14–3. Determine the equivalent state of stress on an element at the same point that represents the principal stresses at the point. Also, find the corresponding orientation of the element with respect to the element shown.



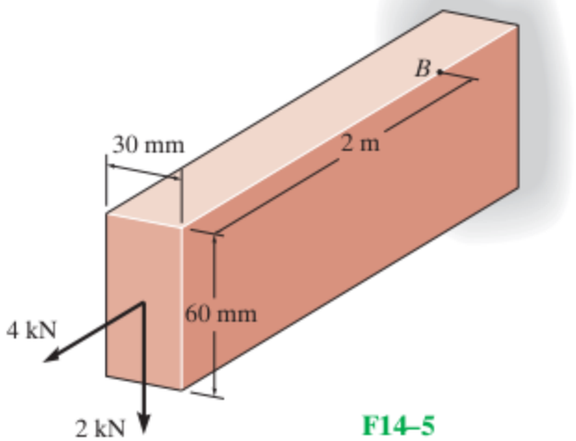
F14–3

F14–4. Determine the equivalent state of stress on an element at the same point that represents the maximum in-plane shear stress at the point.



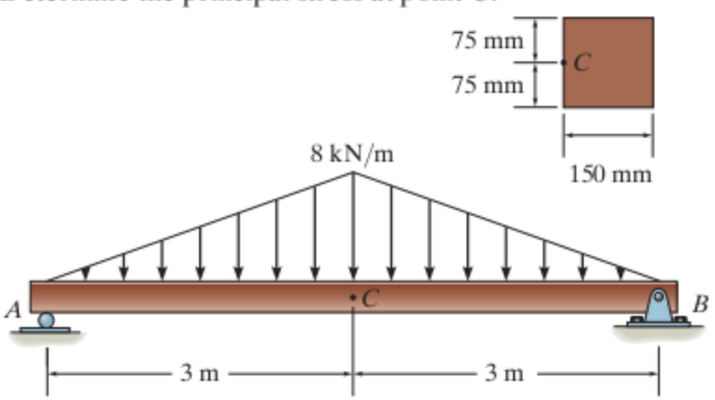
F14–4

F14–5. The beam is subjected to the load at its end. Determine the maximum principal stress at point B.



F14–5

F14–6. The beam is subjected to the loading shown. Determine the principal stress at point C.

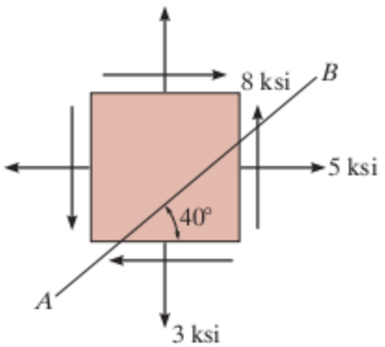


F14–6

PROBLEMS

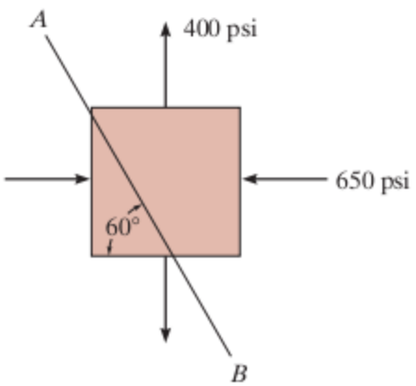
14–1. Prove that the sum of the normal stresses $\sigma_x + \sigma_y = \sigma_{x'} + \sigma_{y'}$ is constant. See Figs. 14–2*a* and 14–2*b*.

14–2. The state of stress at a point in a member is shown on the element. Determine the stress components acting on the inclined plane *AB*. Solve the problem using the method of equilibrium described in Sec. 14.1.



Prob. 14–2

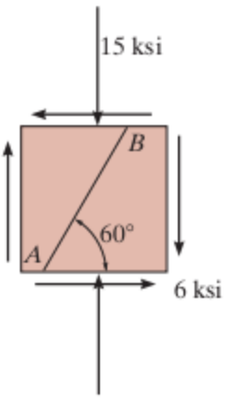
14–3. The state of stress at a point in a member is shown on the element. Determine the stress components acting on the inclined plane *AB*. Solve the problem using the method of equilibrium described in Sec. 14.1.



Prob. 14–3

***14–4.** Determine the normal stress and shear stress acting on the inclined plane *AB*. Solve the problem using the method of equilibrium described in Sec. 14.1.

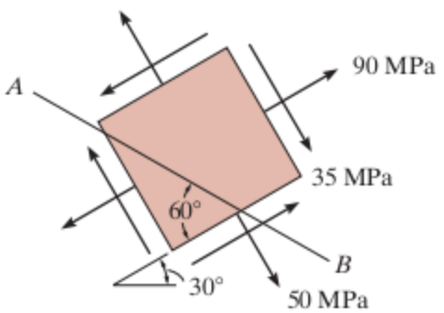
14–5. Determine the normal stress and shear stress acting on the inclined plane *AB*. Solve the problem using the stress transformation equations. Show the results on the sectional element.



Probs. 14–4/5

14–6. The state of stress at a point in a member is shown on the element. Determine the stress components acting on the inclined plane *AB*. Solve the problem using the method of equilibrium described in Sec. 14.1.

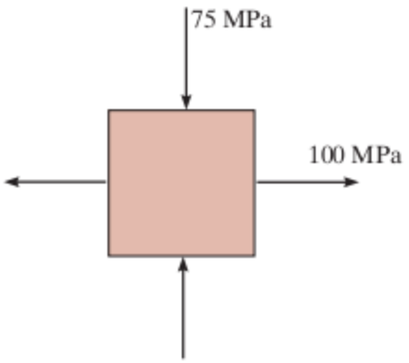
14–7. Solve Prob. 14–6 using the stress-transformation equations developed in Sec. 14.2. Show the result on a sketch.



Probs. 14–6/7

*14–8. Determine the equivalent state of stress on an element at the same point oriented 30° clockwise with respect to the element shown. Sketch the results on the element.

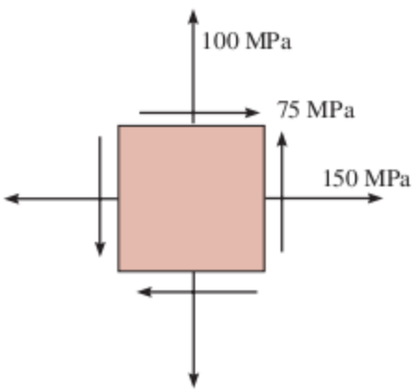
14–9. Determine the equivalent state of stress on an element at the same point oriented 30° counterclockwise with respect to the element shown. Sketch the results on the element.



Probs. 14–8/9

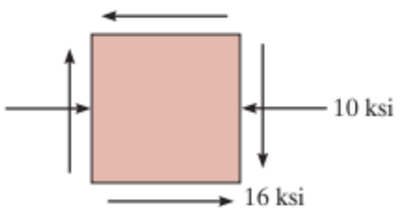
14–10. Determine the equivalent state of stress on an element at the same point oriented 60° clockwise with respect to the element shown. Sketch the results on the element.

14–11. Determine the equivalent state of stress on an element at the same point oriented 60° counterclockwise with respect to the element shown. Sketch the results on the element.



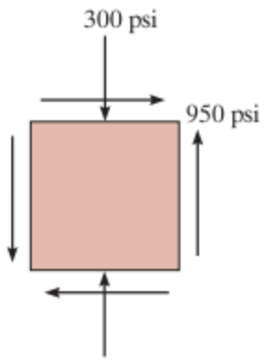
Probs. 14–10/11

*14–12. Determine the equivalent state of stress on an element if it is oriented 50° counterclockwise from the element shown. Use the stress-transformation equations.



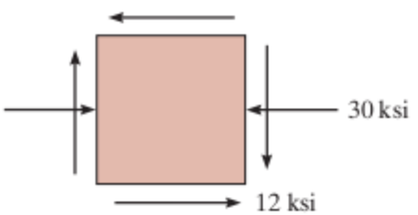
Prob. 14–12

14–13. Determine the equivalent state of stress on an the element if it is oriented 30° clockwise from the element shown. Use the stress-transformation equations.



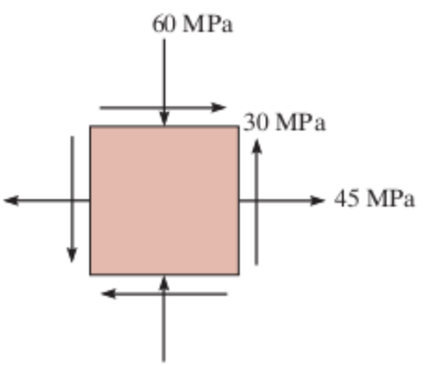
Prob. 14–13

14–14. The state of stress at a point is shown on the element. Determine (a) the principal stress and (b) the maximum in-plane shear stress and average normal stress at the point. Specify the orientation of the element in each case. Show the results on each element.



Prob. 14–14

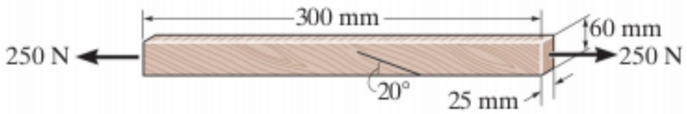
14–15. The state of stress at a point is shown on the element. Determine (a) the principal stress and (b) the maximum in-plane shear stress and average normal stress at the point. Specify the orientation of the element in each case.



Prob. 14–15

The following problems involve material covered in Chapter 13.

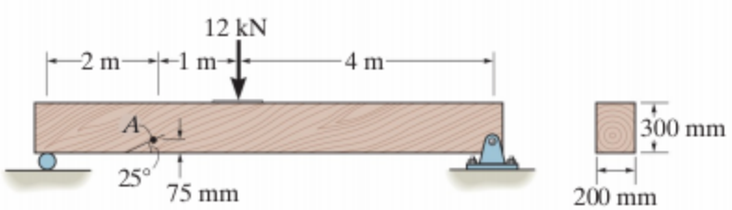
*14–16. The grains of wood in the board make an angle of 20° with the horizontal as shown. Determine the normal and shear stress that act perpendicular and parallel to the grains if the board is subjected to an axial load of 250 N.



Prob. 14–16

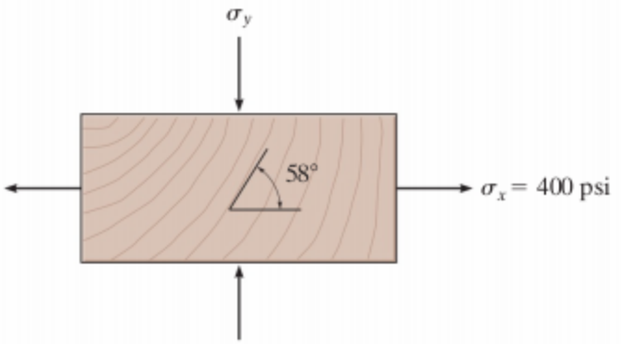
14–17. The wood beam is subjected to a load of 12 kN. If a grain of wood in the beam at point A makes an angle of 25° with the horizontal as shown, determine the normal and shear stress that act perpendicular and parallel to the grain due to the loading.

14–18. The wood beam is subjected to a load of 12 kN. Determine the principal stress at point A and specify the orientation of the element.



Probs. 14–17/18

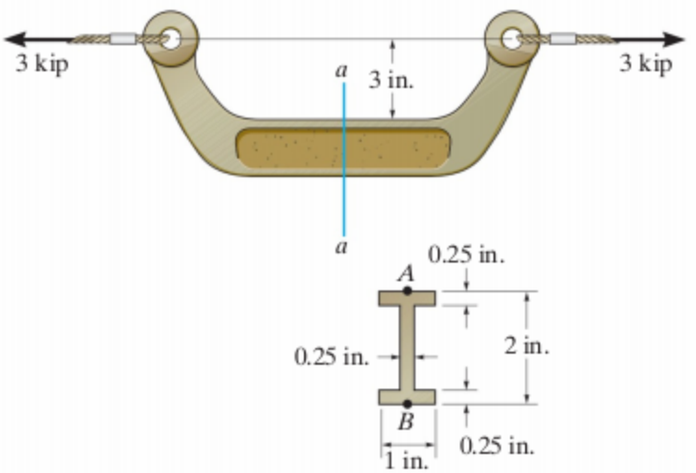
14–19. The wooden block will fail if the shear stress acting along the grain is 550 psi. If the normal stress $\sigma_x = 400$ psi, determine the necessary compressive stress σ_y that will cause failure.



Prob. 14–19

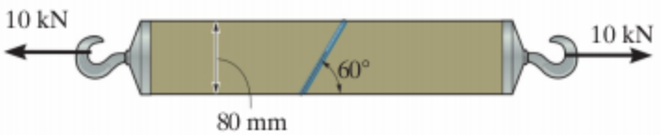
*14–20. The bracket is subjected to the force of 3 kip. Determine the principal stress and maximum in-plane shear stress at point A on the cross section at section a–a. Specify the orientation of this state of stress and show the results on elements.

14–21. The bracket is subjected to the force of 3 kip. Determine the principal stress and maximum in-plane shear stress at point B on the cross section at section a–a. Specify the orientation of this state of stress and show the results on elements.



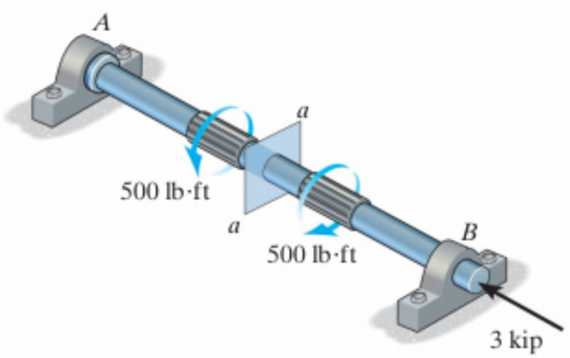
Probs. 14–20/21

14–22. The 25-mm thick rectangular bar is subjected to the axial load of 10 kN. If the bar is joined by the weld, which makes an angle of 60° with the horizontal, determine the shear stress parallel to the weld and the normal stress perpendicular to the weld.



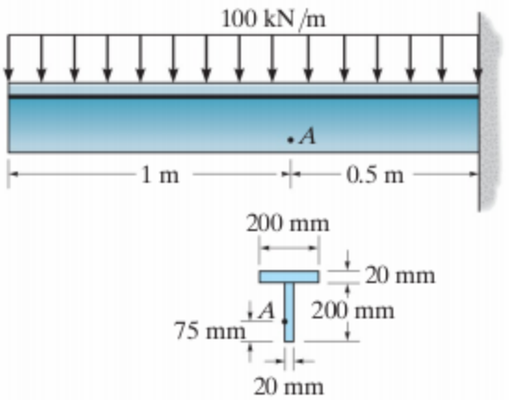
Prob. 14–22

14–23. The 3-in.-diameter shaft is supported by a smooth thrust bearing at *A* and a smooth journal bearing at *B*. Determine the principal stresses and maximum in-plane shear stress at a point on the outer surface of the shaft at section *a–a*.



Prob. 14–23

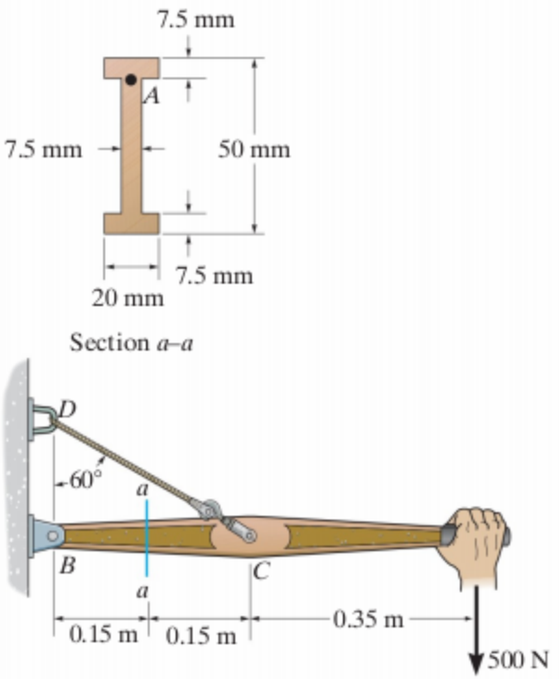
***14–24.** The T-beam is subjected to the distributed loading that is applied along its centerline. Determine the principal stress at point *A* and show the results on an element located at this point.



Prob. 14–24

14–25. Determine the principal stress at point *A* on the cross section of the arm at section *a–a*. Specify the orientation of this state of stress and indicate the results on an element at the point.

14–26. Determine the maximum in-plane shear stress developed at point *A* on the cross section of the arm at section *a–a*. Specify the orientation of this state of stress and indicate the results on an element at the point.



Probs. 14–25/26

14.4 Mohr's Circle—Plane Stress

In this section, we will show how to apply the equations for plane stress transformation using a graphical solution that is often convenient to use and easy to remember. Furthermore, this approach will allow us to "visualize" how the normal and shear stress components $\sigma_{x'}$ and $\tau_{x'y'}$ vary as the plane on which they act is oriented in different directions, Fig. 14–15*a*.

If we write Eqs. 14–1 and 14–2 in the form

$$\sigma_{x'} - \left(\frac{\sigma_x + \sigma_y}{2}\right) = \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta \tag{14–9}$$

$$\tau_{x'y'} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta \tag{14–10}$$

then the parameter θ can be *eliminated* by squaring each equation and adding the equations together. The result is

$$\left[\sigma_{x'} - \left(\frac{\sigma_x + \sigma_y}{2}\right)\right]^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

For a specific problem, σ_x, σ_y, τ_{xy} are *known constants*. Thus the above equation can be written in a more compact form as

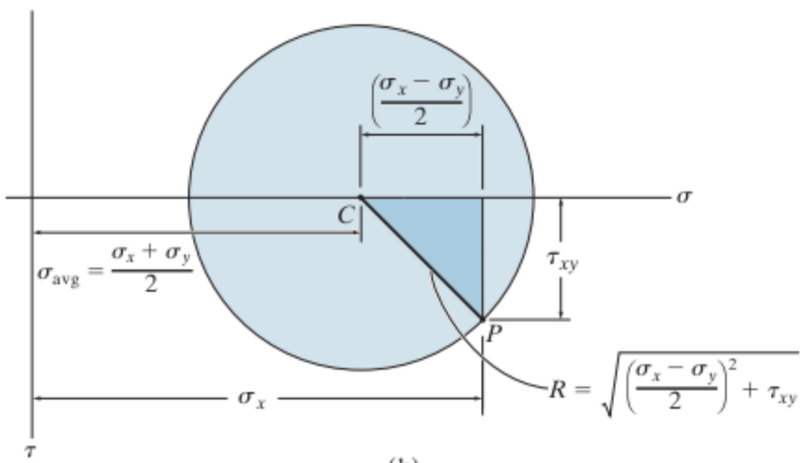
$$(\sigma_{x'} - \sigma_{\text{avg}})^2 + \tau_{x'y'}^2 = R^2 \tag{14–11}$$

where

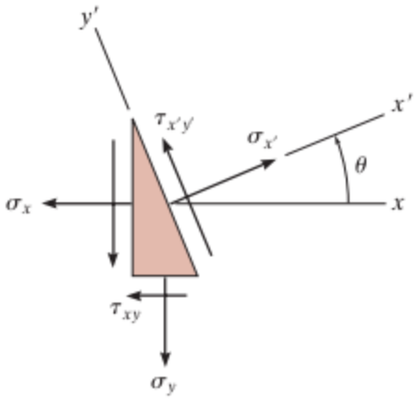
$$\sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{14–12}$$

If we establish coordinate axes, σ *positive to the right* and τ *positive downward*, and then plot Eq. 14–11, it will be seen that this equation represents a *circle* having a radius R and center on the σ axis at point $C(\sigma_{\text{avg}}, 0)$, Fig. 14–15*b*. This circle is called *Mohr's circle*, because it was developed by the German engineer Otto Mohr.



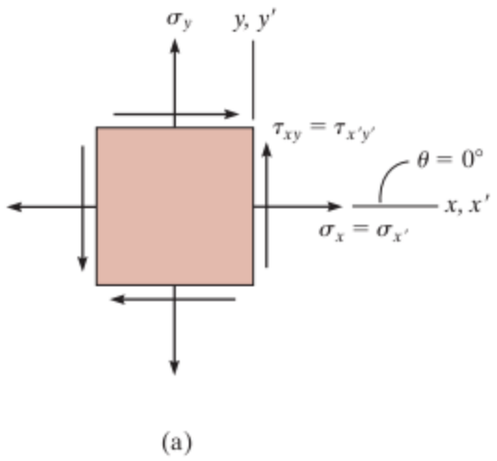
(b)



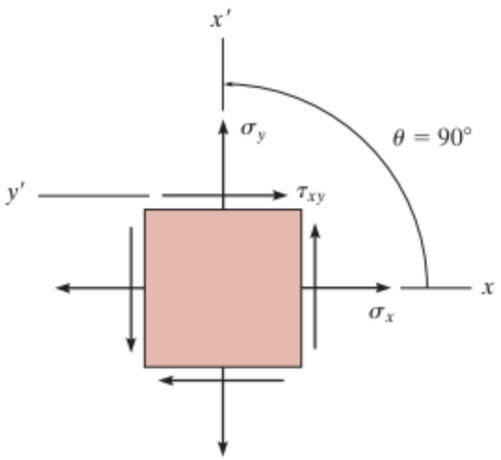
(a)

Fig. 14–15

14



(a)

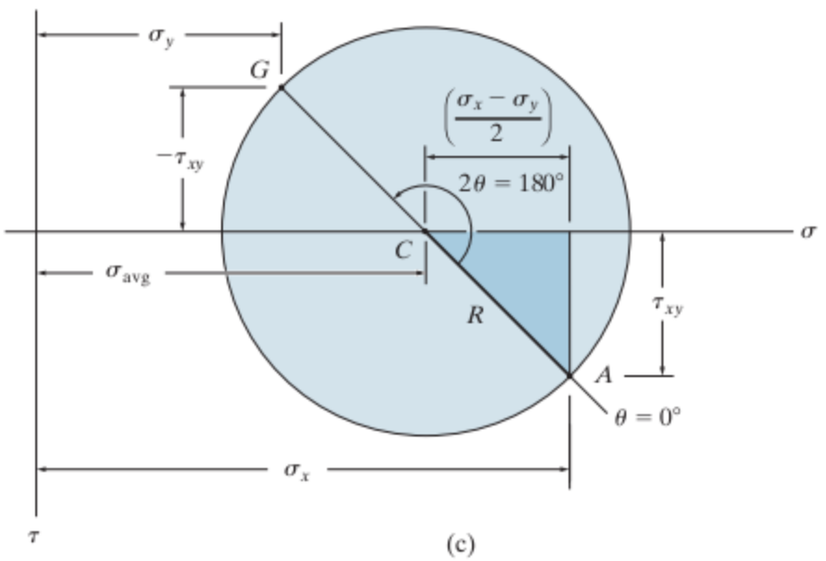


(b)

Each point on Mohr's circle represents the two stress components $\sigma_{x'}$ and $\tau_{x'y'}$ acting on the side of the element defined by the x' axis, when the axis is in a specific direction θ. For example, when x' is coincident with the x axis as shown in Fig. 14–16*a*, then $\theta = 0°$ and $\sigma_{x'} = \sigma_x$, $\tau_{x'y'} = \tau_{xy}$. We will refer to this as the "reference point" A and plot its coordinates $A(\sigma_x, \tau_{xy})$, Fig. 14–16*c*.

Now consider rotating the x' axis 90° counterclockwise, Fig. 14–16*b*. Then $\sigma_{x'} = \sigma_y$, $\tau_{x'y'} = -\tau_{xy}$. These values are the coordinates of point $G(\sigma_y, -\tau_{xy})$ on the circle, Fig. 14–16*c*. Hence, the radial line CG is 180° counterclockwise from the "reference line" CA. In other words, a rotation θ of the x' axis on the element will correspond to a rotation 2θ on the circle in the *same direction*.*

Once constructed, Mohr's circle can be used to determine the principal stresses, the maximum in-plane shear stress and associated average normal stress, or the stress on any arbitrary plane.



(c)

Fig. 14–16

*If instead the τ axis were established *positive upwards*, then the angle 2θ on the circle would be measured in the *opposite direction* to the orientation θ of the plane.

Procedure for Analysis

The following steps are required to draw and use Mohr's circle.

Construction of the Circle.

- Establish a coordinate system such that the horizontal axis represents the normal stress σ, with *positive to the right*, and the vertical axis represents the shear stress τ, with *positive downwards*, Fig. 14–17*a*.*
- Using the positive sign convention for σ_x, σ_y, τ_{xy}, as shown in Fig. 14–17*b*, plot the center of the circle C, which is located on the σ axis at a distance $\sigma_{avg} = (\sigma_x + \sigma_y)/2$ from the origin, Fig. 14–17*a*.
- Plot the "reference point" A having coordinates $A(\sigma_x, \tau_{xy})$. This point represents the normal and shear stress components on the element's right-hand vertical face, and since the x' axis coincides with the x axis, this represents $\theta = 0°$, Fig. 14–17*a*.
- Connect point A with the center C of the circle and determine CA by trigonometry. This distance represents the radius R of the circle, Fig. 14–17*a*.
- Once R has been determined, sketch the circle.

Principal Stress.

- The principal stresses σ_1 and σ_2 ($\sigma_1 \geq \sigma_2$) are the coordinates of points B and D where the circle intersects the σ axis, i.e., where $\tau = 0$, Fig. 14–17*a*.
- These stresses act on planes defined by angles θ_{p_1} and θ_{p_2}, Fig. 14–17*c*. They are represented on the circle by angles $2\theta_{p_1}$ (shown) and $2\theta_{p_2}$ (not shown) and are measured *from* the radial reference line CA *to* lines CB and CD, respectively.
- Using trigonometry, only one of these angles needs to be calculated from the circle, since θ_{p_1} and θ_{p_2} are 90° apart. Remember that the direction of rotation $2\theta_p$ on the circle (here it happens to be counterclockwise) represents the *same* direction of rotation θ_p from the reference axis ($+x$) to the principal plane ($+x'$), Fig. 14–17*c*.*

Maximum In-Plane Shear Stress.

- The average normal stress and maximum in-plane shear stress components are determined from the circle as the coordinates of either point E or F, Fig. 14–17*a*.
- In this case the angles θ_{s_1} and θ_{s_2} give the orientation of the planes that contain these components, Fig. 14–17*d*. The angle $2\theta_{s_1}$ is shown in Fig. 14–17*a* and can be determined using trigonometry. Here the rotation happens to be clockwise, from CA to CE, and so θ_{s_1} must be clockwise on the element, Fig. 14–17*d*.*

14

Procedure for Analysis (continued)

Stresses on Arbitrary Plane.

- The normal and shear stress components $\sigma_{x'}$ and $\tau_{x'y'}$ acting on a specified plane or x' axis, defined by the angle θ, Fig. 14–17*e*, can be obtained from the circle using trigonometry to determine the coordinates of point P, Fig. 14–17*a*.
- To locate P, the known angle θ (in this case counterclockwise), Fig. 14–17*e*, must be measured on the circle in the *same direction* 2θ (counterclockwise), *from* the radial reference line CA *to* the radial line CP, Fig. 14–17*a*.*

*If the τ axis were constructed *positive upwards*, then the angle 2θ on the circle would be measured in the *opposite direction* to the orientation θ of the x' axis.

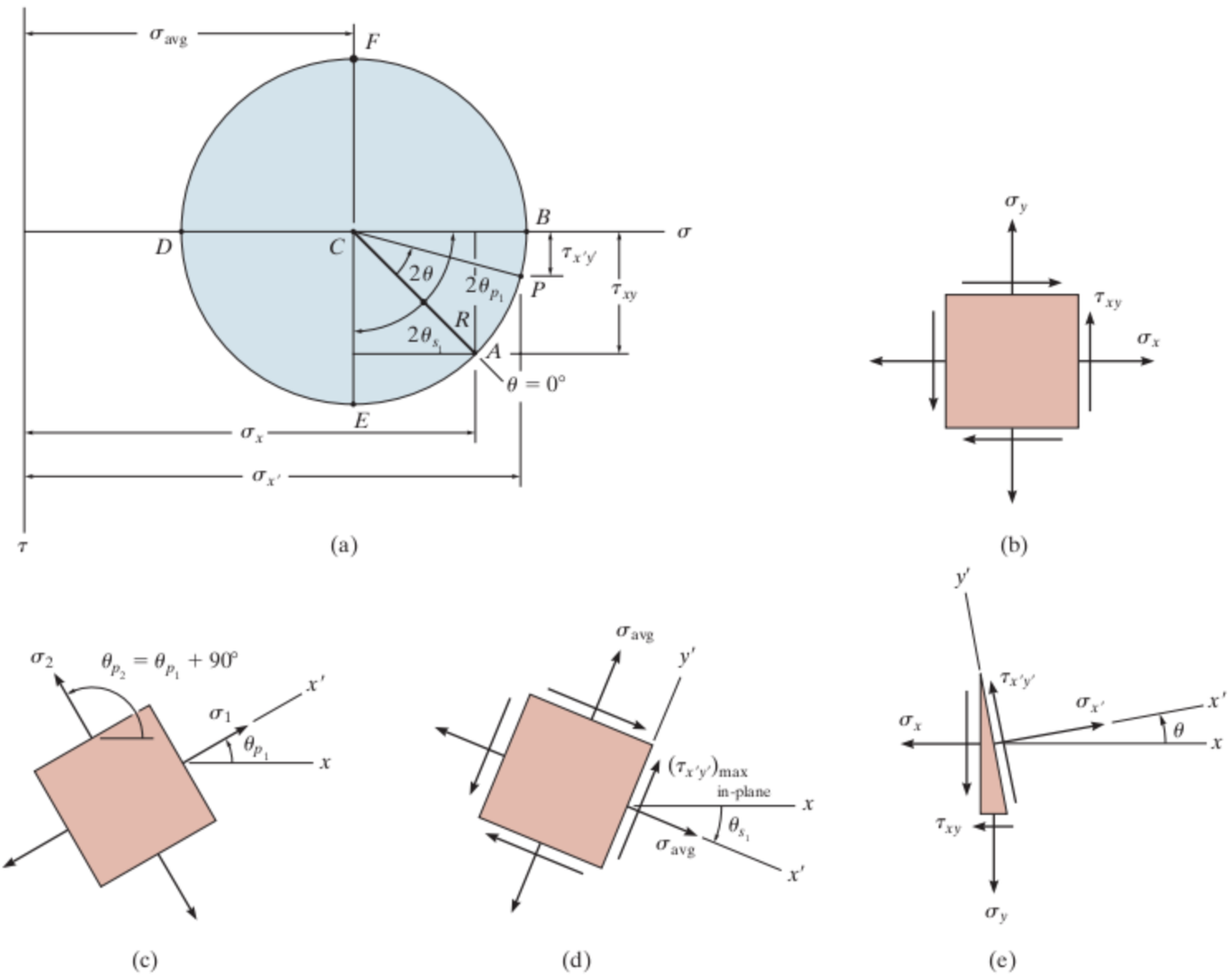
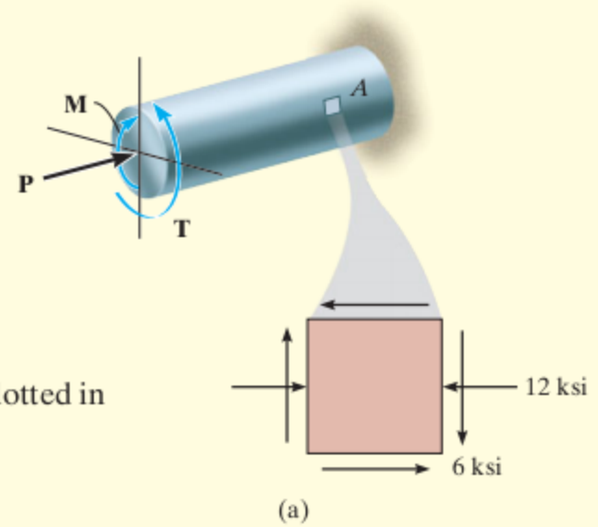


Fig. 14–17

EXAMPLE 14.7

Due to the applied loading, the element at point A on the solid shaft in Fig. 14–18*a* is subjected to the state of stress shown. Determine the principal stresses acting at this point.



(a)

SOLUTION

Construction of the Circle. From Fig. 14–18*a*,

$$\sigma_x = -12 \text{ ksi} \qquad \sigma_y = 0 \qquad \tau_{xy} = -6 \text{ ksi}$$

The center of the circle is at

$$\sigma_{avg} = \frac{-12 + 0}{2} = -6 \text{ ksi}$$

The reference point $A(-12, -6)$ and the center $C(-6, 0)$ are plotted in Fig. 14–18*b*. The circle is constructed having a radius of

$$R = \sqrt{(12 - 6)^2 + (6)^2} = 8.49 \text{ ksi}$$

Principal Stresses. The principal stresses are indicated by the coordinates of points B and D. We have, for $\sigma_1 > \sigma_2$,

$$\sigma_1 = 8.49 - 6 = 2.49 \text{ ksi} \qquad \textit{Ans.}$$

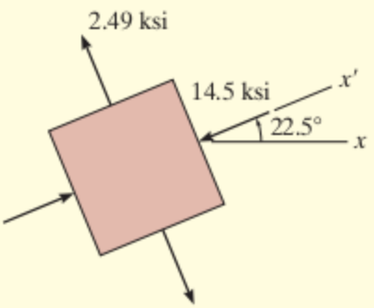
$$\sigma_2 = -6 - 8.49 = -14.5 \text{ ksi} \qquad \textit{Ans.}$$

The orientation of the element can be determined by calculating the angle $2\theta_{p_2}$ in Fig. 14–18*b*, which here is measured *counterclockwise* from CA to CD. It defines the direction θ_{p_2} of σ_2 and its associated principal plane. We have

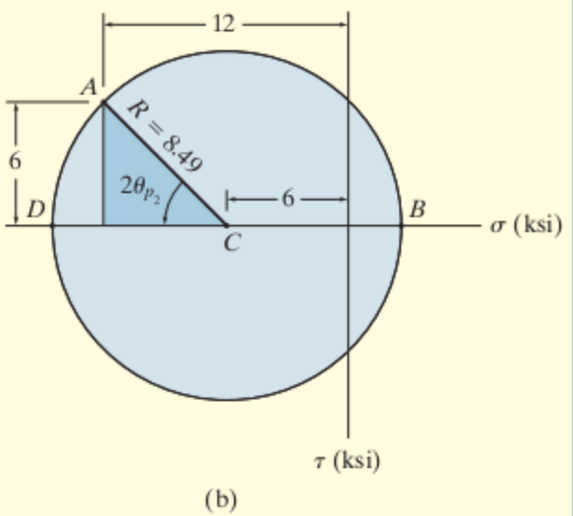
$$2\theta_{p_2} = \tan^{-1}\frac{6}{12 - 6} = 45.0°$$

$$\theta_{p_2} = 22.5°$$

The element is oriented such that the x' axis or σ_2 is directed 22.5° *counterclockwise* from the horizontal (x axis) as shown in Fig. 14–18*c*.



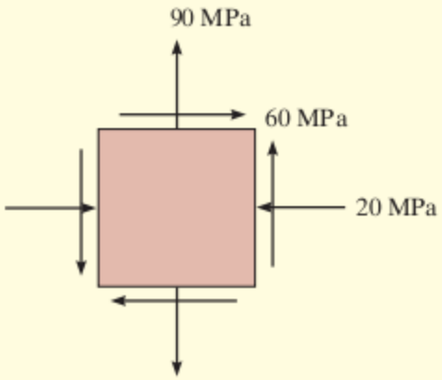
(c)



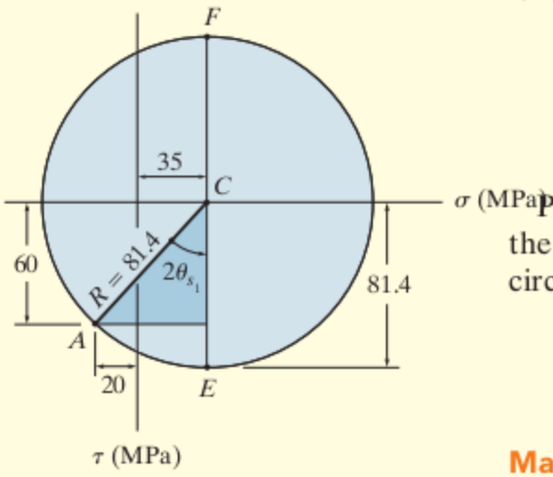
(b)

Fig. 14–18

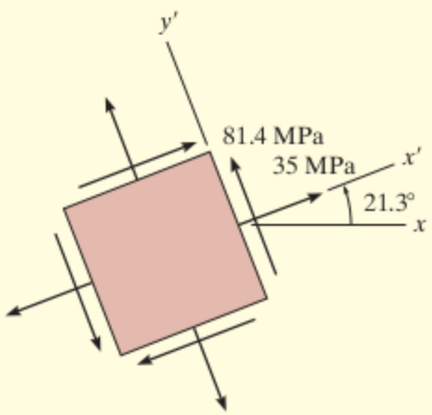
EXAMPLE 14.8



(a)



(b)



(c)

Fig. 14–19

The state of plane stress at a point is shown on the element in Fig. 14–19*a*. Determine the maximum in-plane shear stress at this point.

SOLUTION

Construction of the Circle. From the problem data,

$$\sigma_x = -20 \text{ MPa} \qquad \sigma_y = 90 \text{ MPa} \qquad \tau_{xy} = 60 \text{ MPa}$$

The σ, τ axes are established in Fig.14–19*b*. The center of the circle C is located on the σ axis, at the point

$$\sigma_{\text{avg}} = \frac{-20 + 90}{2} = 35 \text{ MPa}$$

Point C and the reference point $A\,(-20, 60)$ are plotted. Applying the Pythagorean theorem to the shaded triangle to determine the circle's radius CA, we have

$$R = \sqrt{(60)^2 + (55)^2} = 81.4 \text{ MPa}$$

Maximum In-Plane Shear Stress. The maximum in-plane shear stress and the average normal stress are identified by point E (or F) on the circle. The coordinates of point $E(35, 81.4)$ give

$$\tau_{\substack{\max \\ \text{in-plane}}} = 81.4 \text{ MPa} \qquad \textit{Ans.}$$

$$\sigma_{\text{avg}} = 35 \text{ MPa} \qquad \textit{Ans.}$$

The angle θ_{s_1}, measured *counterclockwise* from CA to CE, can be found from the circle, identified as $2\theta_{s_1}$. We have

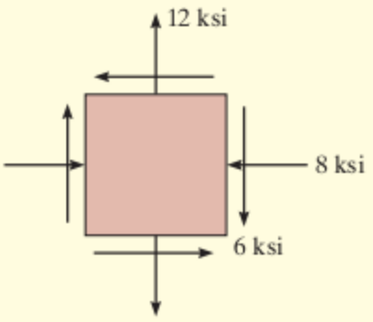
$$2\theta_{s_1} = \tan^{-1}\left(\frac{20 + 35}{60}\right) = 42.5°$$

$$\theta_{s_1} = 21.3° \qquad \textit{Ans.}$$

This *counterclockwise* angle defines the direction of the x' axis, Fig. 14–19*c*. Since point E has *positive* coordinates, then the average normal stress and the maximum in-plane shear stress both act in the *positive* x' and y' directions as shown.

EXAMPLE 14.9

The state of plane stress at a point is shown on the element in Fig. 14–20*a*. Represent this state of stress on an element oriented 30° counterclockwise from the position shown.



(a)

SOLUTION

Construction of the Circle. From the problem data,

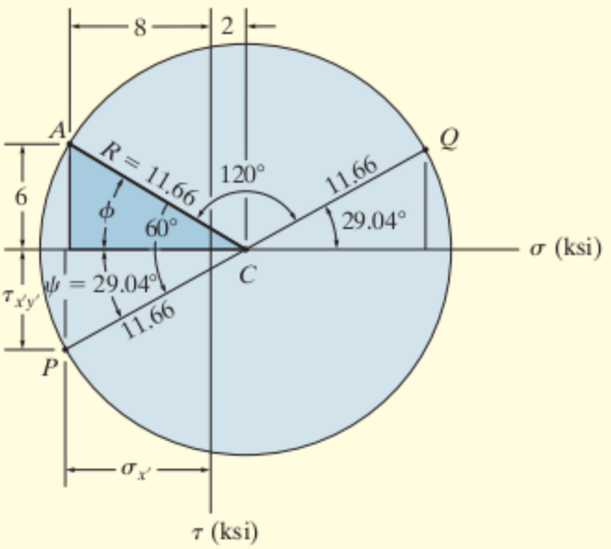
$$\sigma_x = -8 \text{ ksi} \qquad \sigma_y = 12 \text{ ksi} \qquad \tau_{xy} = -6 \text{ ksi}$$

The σ and τ axes are established in Fig. 14–20*b*. The center of the circle C is on the σ axis at

$$\sigma_{avg} = \frac{-8 + 12}{2} = 2 \text{ ksi}$$

The reference point for $\theta = 0°$ has coordinates $A(-8, -6)$. Hence from the shaded triangle the radius CA is

$$R = \sqrt{(10)^2 + (6)^2} = 11.66$$



(b)

Stresses on 30° Element. Since the element is to be rotated 30° *counterclockwise*, we must construct a radial line CP, $2(30°) = 60°$ *counterclockwise*, measured from CA ($\theta = 0°$), Fig. 14–20*b*. The coordinates of point $P(\sigma_{x'}, \tau_{x'y'})$ must now be obtained. From the geometry of the circle,

$$\phi = \tan^{-1}\frac{6}{10} = 30.96° \qquad \psi = 60° - 30.96° = 29.04°$$

$$\sigma_{x'} = 2 - 11.66 \cos 29.04° = -8.20 \text{ ksi} \qquad \textit{Ans.}$$

$$\tau_{x'y'} = 11.66 \sin 29.04° = 5.66 \text{ ksi} \qquad \textit{Ans.}$$

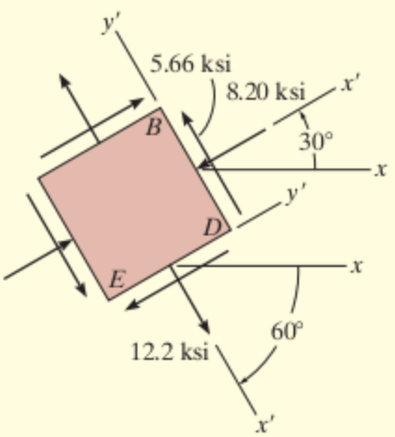
These two stress components act on face BD of the element shown in Fig. 14–20*c* since the x' axis for this face is oriented 30° *counterclockwise* from the x axis.

The stress components acting on the adjacent face DE of the element, which is 60° *clockwise* from the positive x axis, Fig. 14–20*c*, are represented by the coordinates of point Q on the circle. This point lies on the radial line CQ, which is 180° from CP. The coordinates of point Q are

$$\sigma_{x'} = 2 + 11.66 \cos 29.04° = 12.2 \text{ ksi} \qquad \textit{Ans.}$$

$$\tau_{x'y'} = -(11.66 \sin 29.04) = -5.66 \text{ ksi} \quad \text{(check)} \qquad \textit{Ans.}$$

NOTE: Here $\tau_{x'y'}$ acts in the $-y'$ direction.

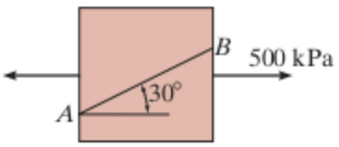


(c)

Fig. 14–20

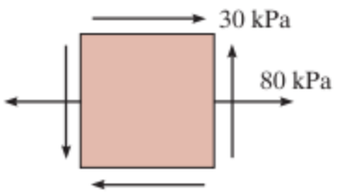
FUNDAMENTAL PROBLEMS

F14–7. Determine the normal stress and shear stress acting on the inclined plane AB. Sketch the result on the sectioned element.



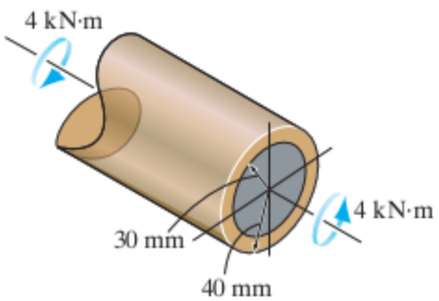
F14–7

F14–8. Determine the equivalent state of stress on an element at the same point that represents the principal stresses at the point. Also, find the corresponding orientation of the element with respect to the element shown. Sketch the results on the element.



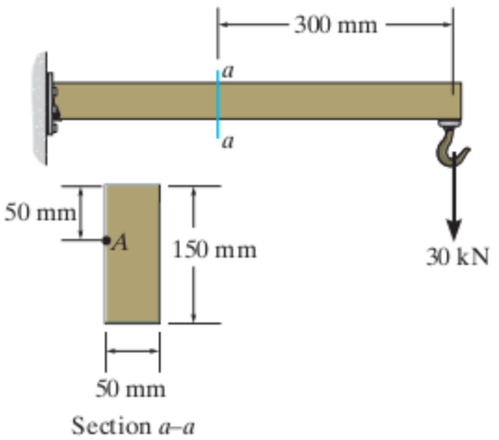
F14–8

F14–9. The hollow circular shaft is subjected to the torque of 4 kN · m. Determine the principal stress developed at a point on the surface of the shaft.



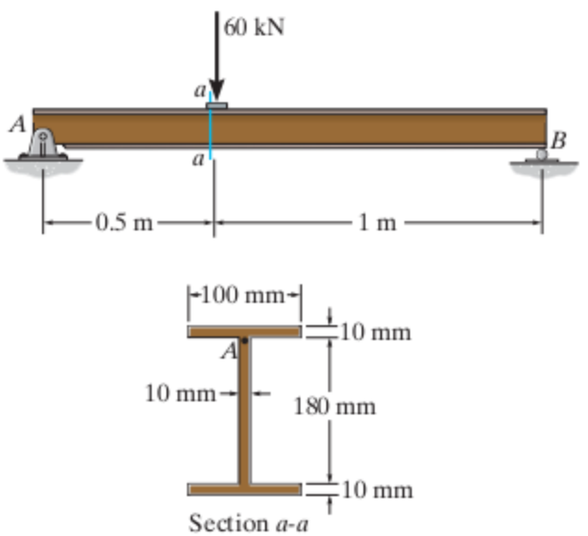
F14–9

F14–10. Determine the principal stress developed at point A on the cross section of the beam at section a–a.



F14–10

F14–11. Determine the maximum in-plane shear stress developed at point A on the cross section of the beam at section a–a, which is located just to the left of the 60-kN force. Point A is just below the flange.



F14–11

PROBLEMS

14–27. Solve Prob. 14–3 using Mohr's circle.

***14–28.** Solve Prob. 14–6 using Mohr's circle.

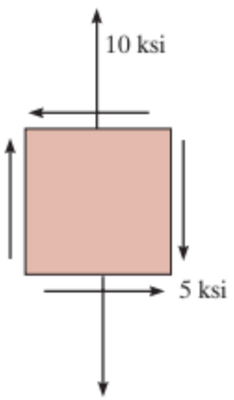
14–29. Solve Prob. 14–14 using Mohr's circle.

14–30. Solve Prob. 14–12 using Mohr's circle.

14–31. Solve Prob. 14–10 using Mohr's circle.

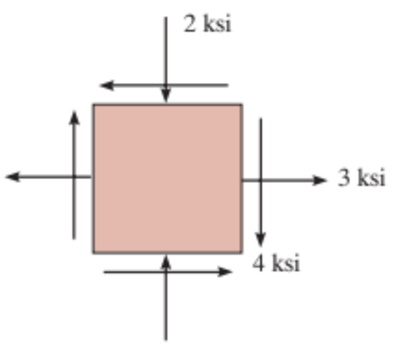
***14–32.** Mohr's circle for the state of stress in Fig. 14–15*a* is shown in Fig. 14–15*b*. Show that finding the coordinates of point $P(\sigma_{x'}, \tau_{x'y'})$ on the circle gives the same value as the stress-transformation Eqs. 14–1 and 14–2.

14–33. Determine the equivalent state of stress if an element is oriented 45° clockwise from the element shown.



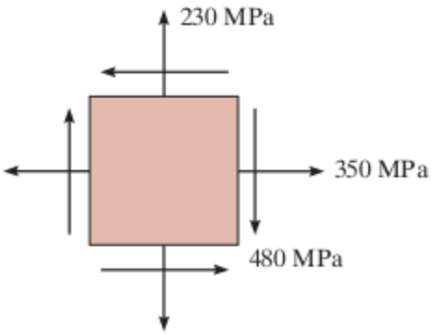
Prob. 14–33

14–34. Determine the equivalent state of stress if an element is oriented 20° clockwise from the element shown.



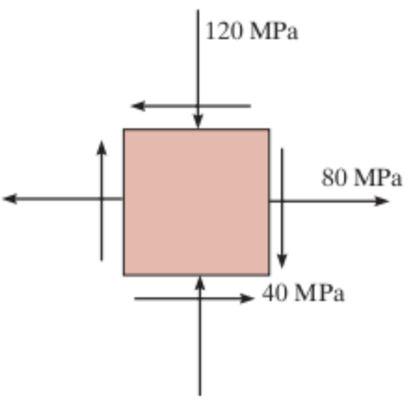
Prob. 14–34

14–35. Determine the equivalent state of stress if an element is oriented 30° clockwise from the element shown.



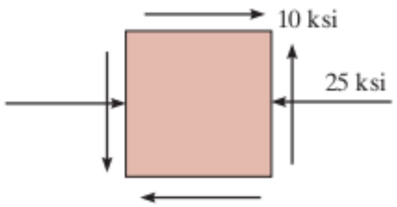
Prob. 14–35

***14–36.** Determine the equivalent state of stress which represents (a) the principal stress and (b) the maximum in-plane shear stress and the associated average normal stress. For each case, determine the corresponding orientation of the element with respect to the element shown.



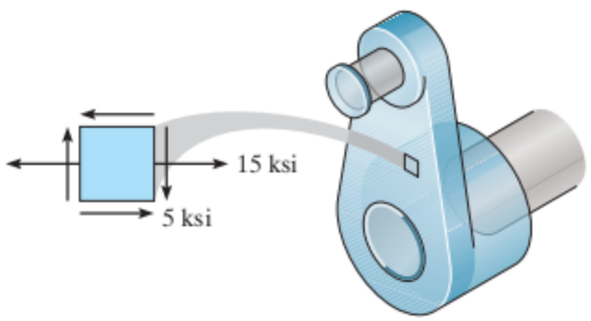
Prob. 14–36

14–37. Determine the equivalent state of stress which represents (a) the principal stress and (b) the maximum in-plane shear stress and the associated average normal stress. For each case, determine the corresponding orientation of the element with respect to the element shown.



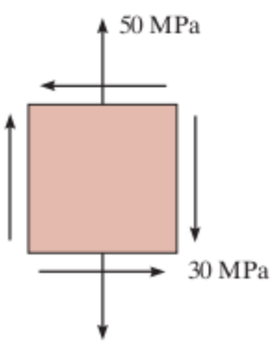
Prob. 14–37

14–38. Determine the principal stress, the maximum in-plane shear stress, and average normal stress. Specify the orientation of the element in each case.



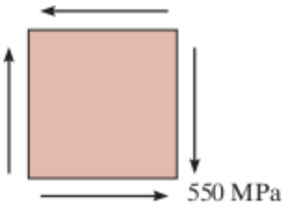
Prob. 14–38

14–39. Determine (a) the principal stresses and (b) the maximum in-plane shear stress and average normal stress. Specify the orientation of the element in each case.



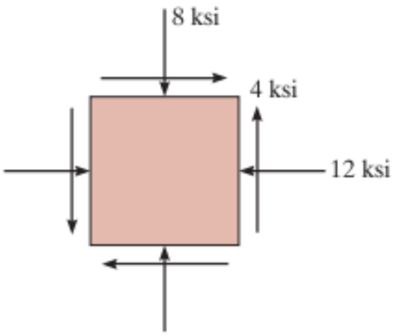
Prob. 14–39

***14–40.** Determine the equivalent state of stress if an element is oriented 25° counterclockwise from the element shown.



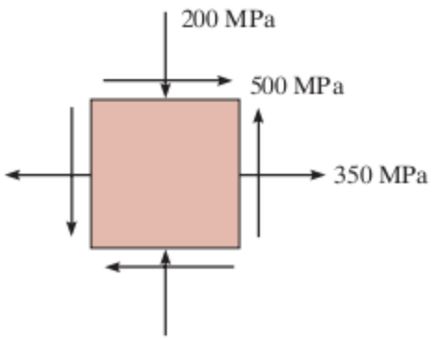
Prob. 14–40

14–41. Determine (a) the principal stresses and (b) the maximum in-plane shear stress and average normal stress. Specify the orientation of the element in each case.



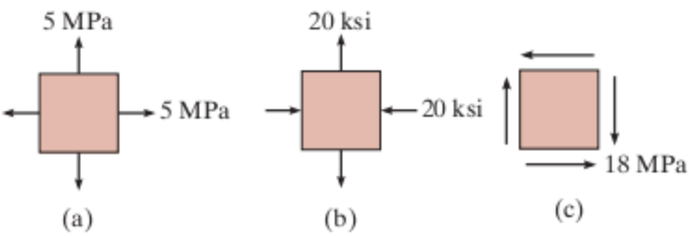
Prob. 14–41

14–42. Determine the principal stress, the maximum in-plane shear stress, and average normal stress. Specify the orientation of the element in each case.



Prob. 14–42

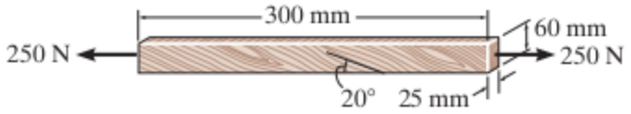
14–43. Draw Mohr's circle that describes each of the following states of stress.



Prob. 14–43

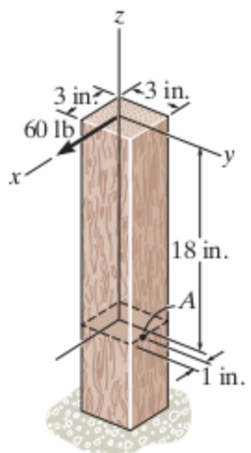
The following problems involve material covered in Chapter 13.

*14–44. The grains of wood in the board make an angle of 20° with the horizontal as shown. Using Mohr's circle, determine the normal and shear stresses that act perpendicular and parallel to the grains if the board is subjected to an axial load of 250 N.



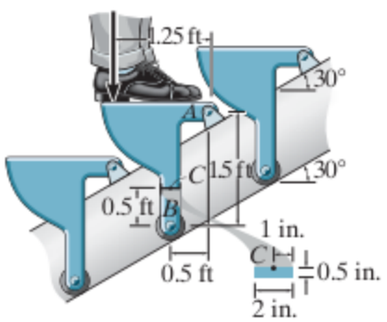
Prob. 14–44

14–45. The post has a square cross-sectional area. If it is fixed supported at its base and a horizontal force is applied at its end as shown, determine (a) the maximum in-plane shear stress developed at A and (b) the principal stresses at A.



Prob. 14–45

14–46. The stair tread of the escalator is supported on two of its sides by the moving pin at A and the roller at B. If a man having a weight of 300 lb stands in the center of the tread, determine the principal stresses developed in the supporting truck on the cross section at point C. The stairs move at constant velocity.



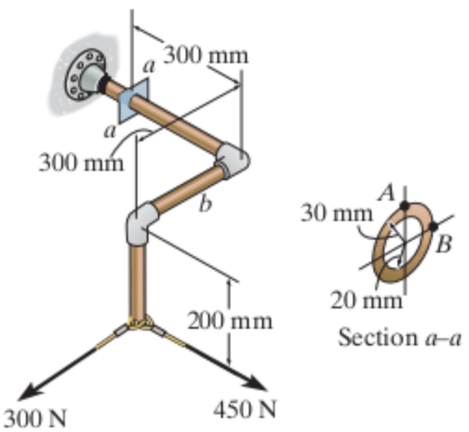
Prob. 14–46

14–47. The thin-walled pipe has an inner diameter of 0.5 in. and a thickness of 0.025 in. If it is subjected to an internal pressure of 500 psi and the axial tension and torsional loadings shown, determine the principal stress at a point on the surface of the pipe.

Prob. 14–47

*14–48. Determine the principal stress and maximum in-plane shear stress at point A on the cross section of the pipe at section a–a.

14–49. Determine the principal stress and maximum in-plane shear stress at point B on the cross section of the pipe at section a–a.



Probs. 14–48/49

14–50. The rotor shaft of the helicopter is subjected to the tensile force and torque shown when the rotor blades provide the lifting force to suspend the helicopter at midair. If the shaft has a diameter of 6 in., determine the principal stress and maximum in-plane shear stress at a point located on the surface of the shaft.

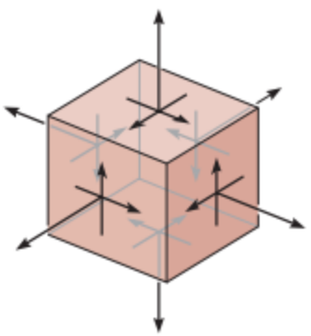
Prob. 14–50

14

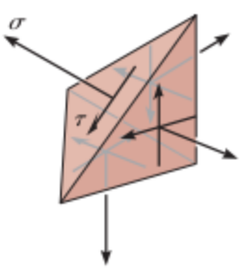
14.5 Absolute Maximum Shear Stress

When a point in a body is subjected to a general three-dimensional state of stress, an element of material has a normal-stress and two shear-stress components acting on each of its faces, Fig. 14–21*a*. Like the case of plane stress, it is possible to develop stress-transformation equations that can be used to determine the normal and shear stress components σ and τ acting on *any* skewed plane of the element, Fig. 14–21*b*. Furthermore, at the point it is also possible to determine the unique orientation of an element having only principal stresses acting on its faces. As shown in Fig. 14–21*c*, in general these principal stresses will have magnitudes of maximum, intermediate, and minimum intensity, i.e., $\sigma_{max} \geq \sigma_{int} \geq \sigma_{min}$. This is a condition known as ***triaxial stress***.

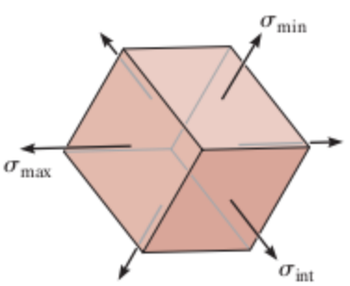
A discussion of the transformation of stress in three dimensions is beyond the scope of this text; however, it is discussed in books related to the theory of elasticity. For our purposes, we will confine our attention only to the case of plane stress. For example, consider the



(a)



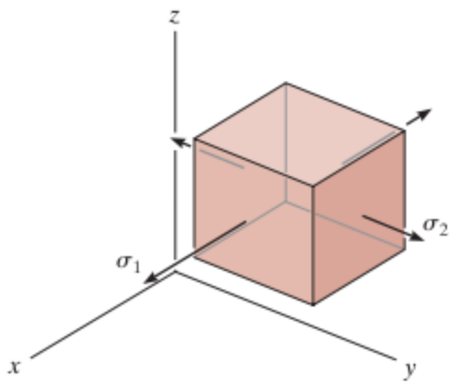
(b)



Triaxial stress

(c)

Fig. 14–21



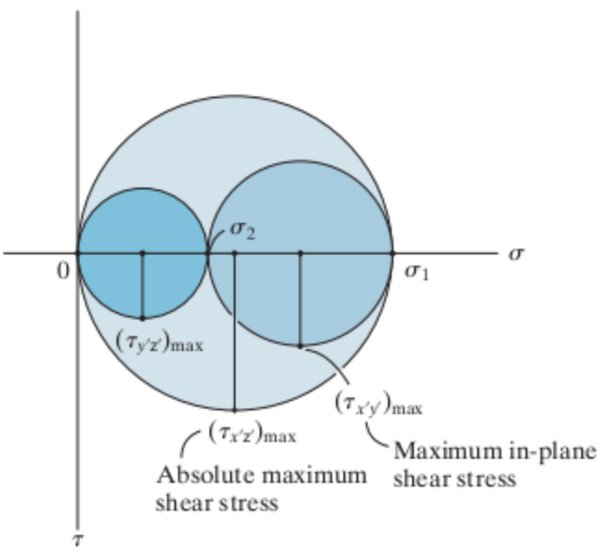
x–y plane stress

(a)

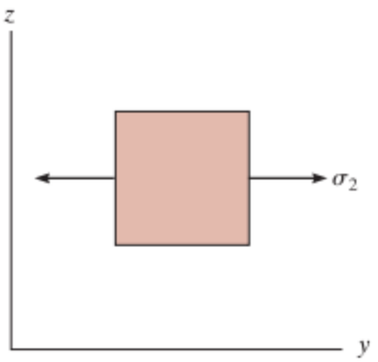
material to be subjected to the in-plane principal stresses σ_1 and σ_2 shown in Fig. 14–22*a*, where *both* of these stresses are tensile. If we view the element in two dimensions, that is, in the y–z, x–z, and x–y planes, Fig. 14–22*b*, 14–22*c*, and 14–22*d*, then we can use Mohr's circle to determine the maximum in-plane shear for each case and from this, determine the *absolute maximum shear stress* in the material. For example, the diameter of Mohr's circle extends between *0* and σ_2 for the case shown in Fig. 14–22*b*. From this circle, Fig. 14–22*e*, the maximum in-plane shear stress is $\tau_{y'z'} = \sigma_2/2$. For all three circles, it is seen that although the maximum in-plane shear stress is $\tau_{x'y'} = (\sigma_1 - \sigma_2)/2$, this value is *not* the absolute maximum shear stress. Instead, from Fig. 14–22*e*,

$$\tau_{\substack{\text{abs}\\\text{max}}} = \frac{\sigma_1}{2} \qquad (14\text{–}13)$$

σ_1 and σ_2 have the same sign



(e)



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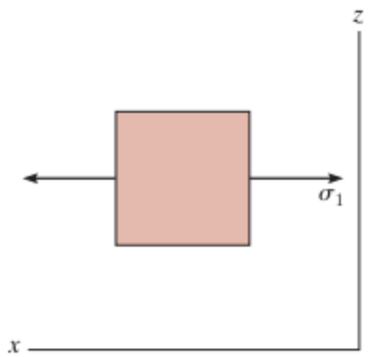
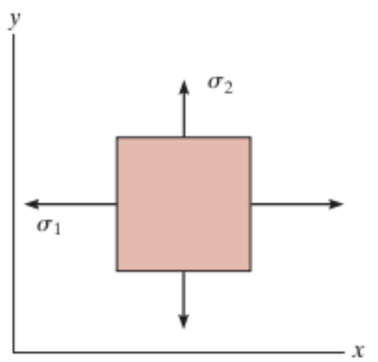


(c)



(d)

Fig. 14–22

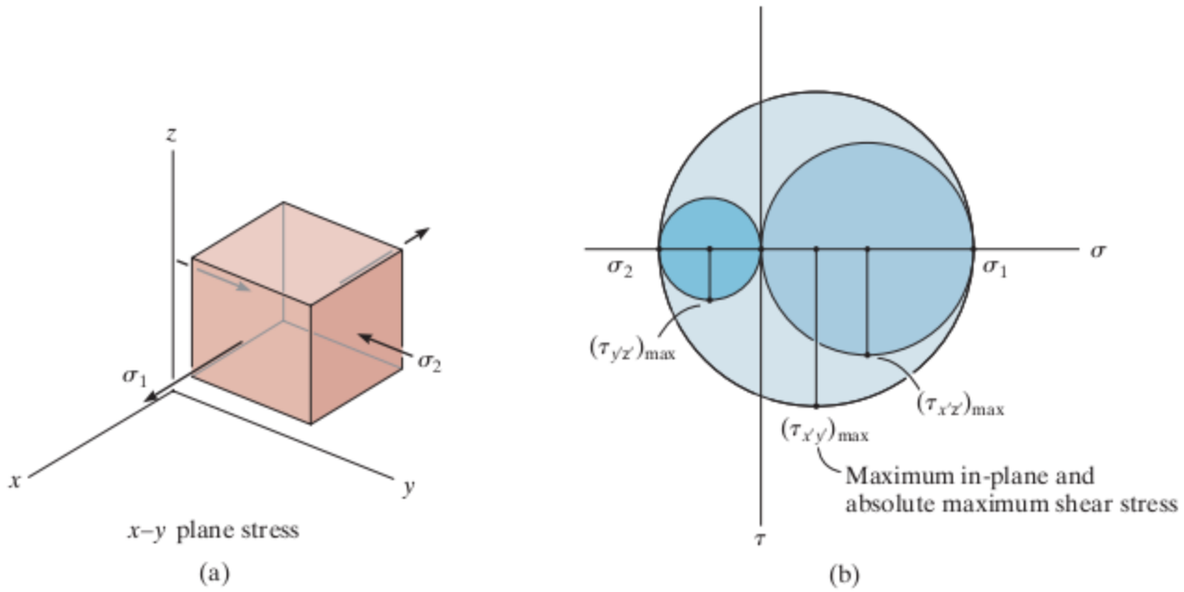


Fig. 14–23

If one of the in-plane principal stresses has the *opposite sign* of that of the other, Fig. 14–23*a*, then the three Mohr's circles that describe the state of stress for element orientations about each coordinate axis are shown in Fig. 14–23*b*. Clearly, in this case

$$\tau_{\text{max}}^{\text{abs}} = \frac{\sigma_1 - \sigma_2}{2} \qquad \sigma_1 \text{ and } \sigma_2 \text{ have opposite signs} \tag{14–14}$$

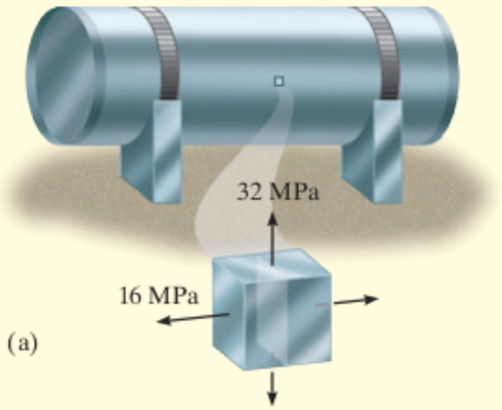
Calculation of the absolute maximum shear stress as indicated here is important when designing members made of a ductile material, since the strength of the material depends on its ability to resist shear stress.

Important Points

- The general three-dimensional state of stress at a point can be represented by an element oriented so that only three principal stresses σ_{max}, σ_{int}, σ_{min} act on it.
- In the case of plane stress, if the in-plane principal stresses both have the *same sign*, the *absolute maximum shear stress* will occur *out of the plane* and has a value of $\tau_{\text{max}}^{\text{abs}} = \sigma_{\text{max}}/2$. This value is greater than the in-plane shear stress.
- If the in-plane principal stresses are of *opposite signs*, then the *absolute maximum shear stress will equal the maximum in-plane shear stress*; that is, $\tau_{\text{max}}^{\text{abs}} = (\sigma_{\text{max}} - \sigma_{\text{min}})/2$.

EXAMPLE 14.10

The point on the surface of the cylindrical pressure vessel in Fig. 14–24*a* is subjected to the state of plane stress. Determine the absolute maximum shear stress at this point.



(a)

SOLUTION

The principal stresses are $\sigma_1 = 32$ MPa, $\sigma_2 = 16$ MPa. If these stresses are plotted along the σ axis, the three Mohr's circles can be constructed that describe the stress state viewed in each of the three perpendicular planes, Fig. 14–24*b*. The largest circle has a radius of 16 MPa and describes the state of stress in the plane only containing $\sigma_1 = 32$ MPa, shown shaded in Fig. 14–24*a*. An orientation of an element 45° within this plane yields the state of absolute maximum shear stress and the associated average normal stress, namely,

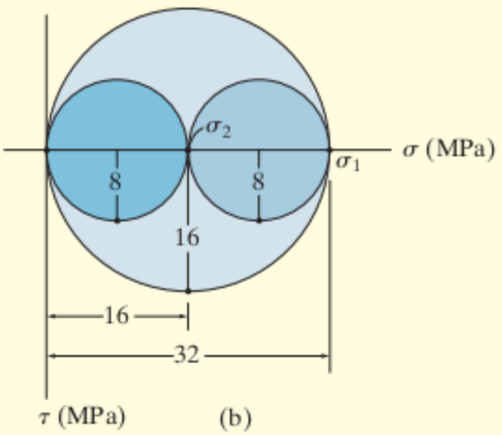
$$\tau_{\substack{\text{abs}\\\text{max}}} = 16 \text{ MPa} \qquad \textit{Ans.}$$
$$\sigma_{\text{avg}} = 16 \text{ MPa}$$

This same result for $\tau_{\substack{\text{abs}\\\text{max}}}$ can be obtained from direct application of Eq. 14–13.

$$\tau_{\substack{\text{abs}\\\text{max}}} = \frac{\sigma_1}{2} = \frac{32}{2} = 16 \text{ MPa} \qquad \textit{Ans.}$$
$$\sigma_{\text{avg}} = \frac{32 + 0}{2} = 16 \text{ MPa}$$

By comparison, the maximum in-plane shear stress can be determined from the Mohr's circle drawn between $\sigma_1 = 32$ MPa and $\sigma_2 = 16$ MPa, Fig. 14–24*b*. This gives a value of

$$\tau_{\substack{\text{max}\\\text{in-plane}}} = \frac{32 - 16}{2} = 8 \text{ MPa}$$
$$\sigma_{\text{avg}} = \frac{32 + 16}{2} = 24 \text{ MPa}$$



(b)

Fig. 14–24

EXAMPLE 14.11

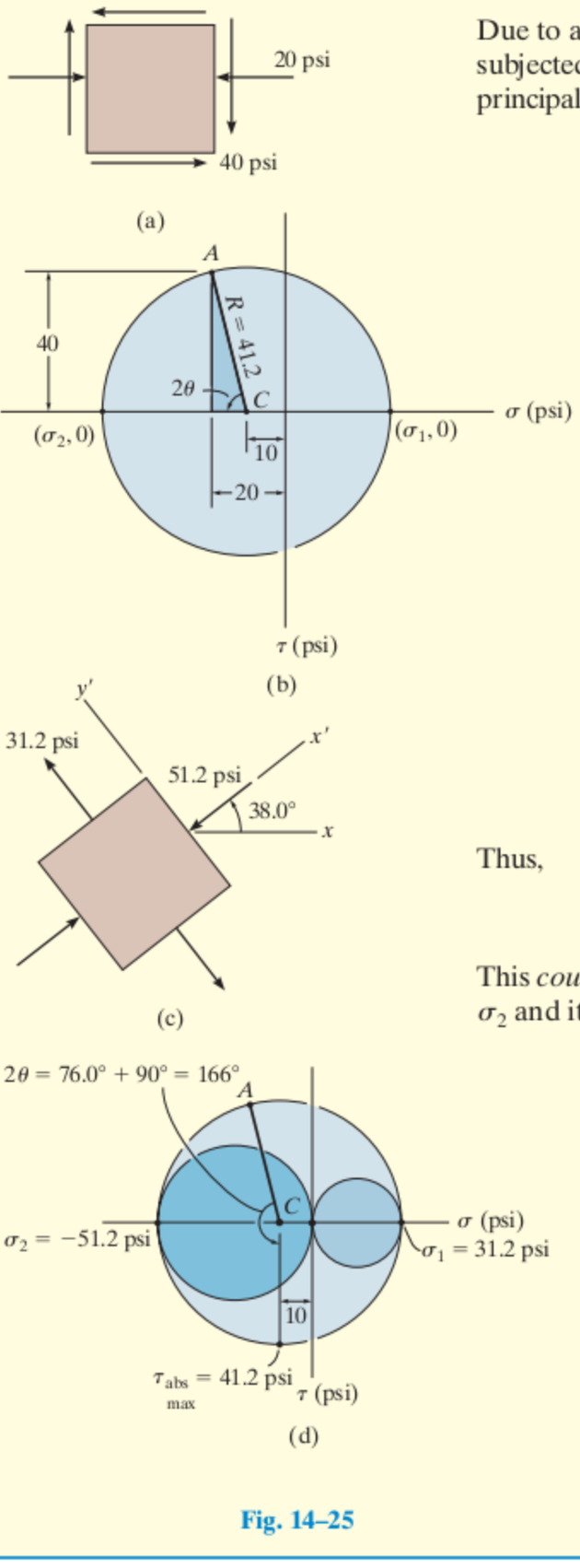


Fig. 14–25

Due to an applied loading, an element at the point on a machine shaft is subjected to the state of plane stress shown in Fig. 14–25*a*. Determine the principal stresses and the absolute maximum shear stress at the point.

SOLUTION

Principal Stresses. The in-plane principal stresses can be determined from Mohr's circle. The center of the circle is on the σ axis at $\sigma_{avg} = (-20 + 0)/2 = -10$ psi. Plotting the reference point $A(-20, -40)$, the radius CA is established and the circle is drawn as shown in Fig. 14–25*b*. The radius is

$$R = \sqrt{(20 - 10)^2 + (40)^2} = 41.2 \text{ psi}$$

The principal stresses are at the points where the circle intersects the σ axis; i.e.,

$$\sigma_1 = -10 + 41.2 = 31.2 \text{ psi}$$
$$\sigma_2 = -10 - 41.2 = -51.2 \text{ psi}$$

From the circle, the *counterclockwise* angle 2θ, measured from CA to the $-\sigma$ axis, is

$$2\theta = \tan^{-1}\left(\frac{40}{20 - 10}\right) = 76.0°$$

Thus,

$$\theta = 38.0°$$

This *counterclockwise* rotation defines the direction of the x' axis and σ_2 and its associated principal plane, Fig. 14–25*c*. We have

$$\sigma_1 = 31.2 \text{ psi} \quad \sigma_2 = -51.2 \text{ psi} \qquad \textit{Ans.}$$

Absolute Maximum Shear Stress. Since these stresses have opposite signs, applying Eq. 14–14 we have

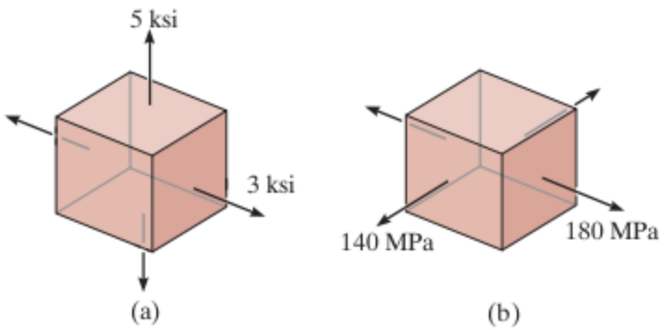
$$\tau_{\substack{abs \\ max}} = \frac{\sigma_1 - \sigma_2}{2} = \frac{31.2 - (-51.2)}{2} = 41.2 \text{ psi}$$

$$\sigma_{avg} = \frac{31.2 - 51.2}{2} = -10 \text{ psi} \qquad \textit{Ans.}$$

NOTE: These same results can also be obtained by drawing Mohr's circle for each orientation of an element about the x, y, and z axes, Fig. 14–25*d*. Since σ_1 and σ_2 are of *opposite signs*, then the absolute maximum shear stress equals the maximum in-plane shear stress.

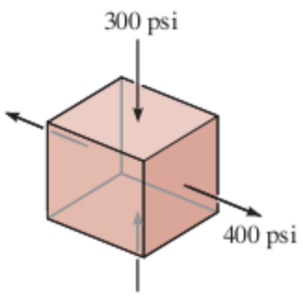
PROBLEMS

14–51. Draw the three Mohr's circles that describe each of the following states of stress.



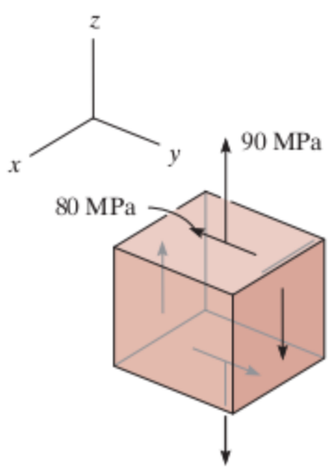
Prob. 14–51

*__14–52.__ Draw the three Mohr's circles that describe the following state of stress.



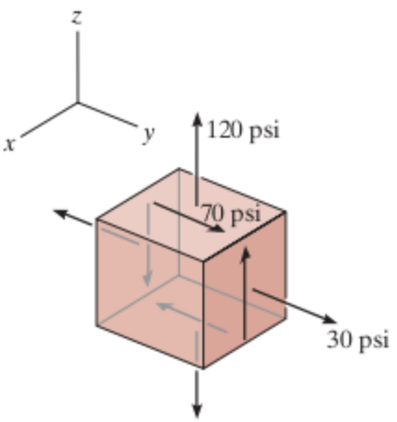
Prob. 14–52

14–53. The stress at a point is shown on the element. Determine the principal stress and the absolute maximum shear stress.



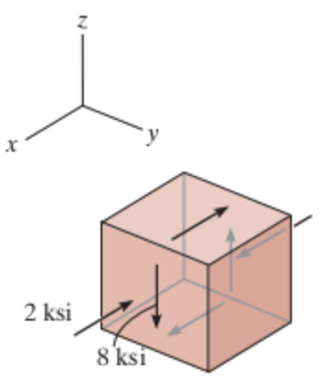
Prob. 14–53

14–54. The stress at a point is shown on the element. Determine the principal stress and the absolute maximum shear stress.



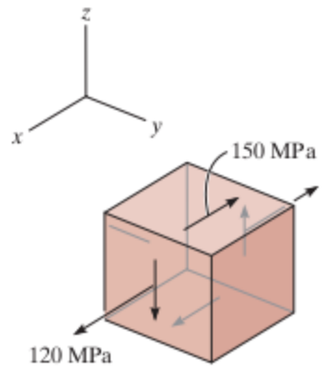
Prob. 14–54

14–55. The stress at a point is shown on the element. Determine the principal stress and the absolute maximum shear stress.



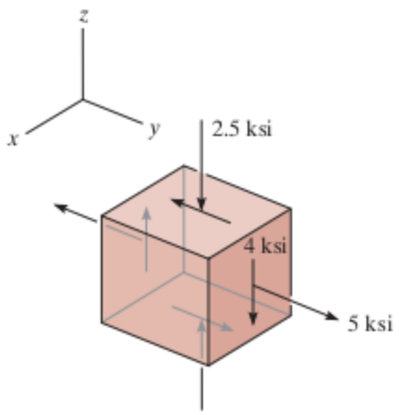
Prob. 14–55

*14–56. The stress at a point is shown on the element. Determine the principal stress and the absolute maximum shear stress.



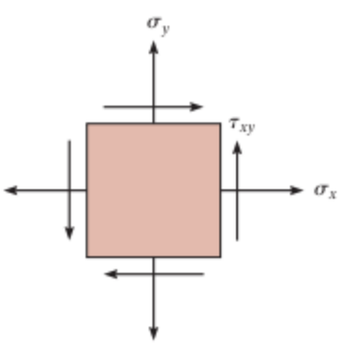
Prob. 14–56

14–57. The state of stress at a point is shown on the element. Determine the principal stress and the absolute maximum shear stress.



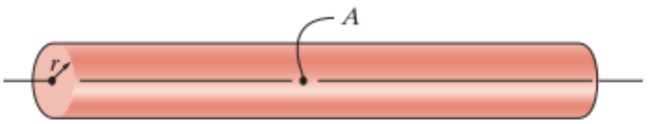
Prob. 14–57

14–58. Consider the general case of plane stress as shown. Write a computer program that will show a plot of the three Mohr's circles for the element, and will also calculate the maximum in-plane shear stress and the absolute maximum shear stress.



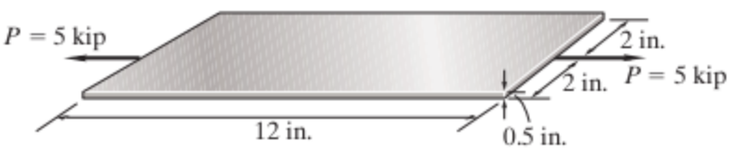
Prob. 14–58

14–59. The solid cylinder having a radius r is placed in a sealed container and subjected to a pressure p. Determine the stress components acting at point A located on the center line of the cylinder. Draw Mohr's circles for the element at this point.



Prob. 14–59

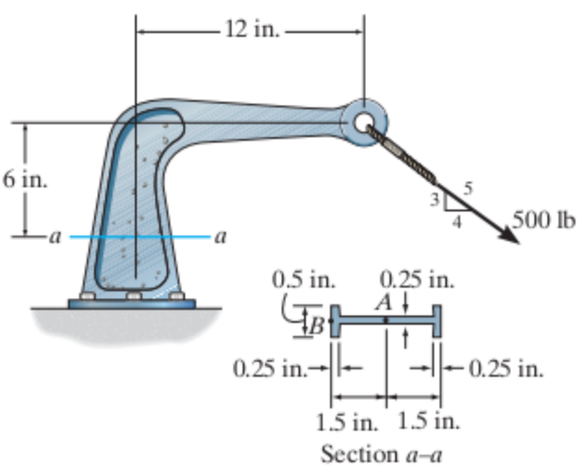
*14–60. The plate is subjected to a tensile force $P = 5$ kip. If it has the dimensions shown, determine the principal stresses and the absolute maximum shear stress. If the material is ductile it will fail in shear. Make a sketch of the plate showing how this failure would appear. If the material is brittle the plate will fail due to the principal stresses. Show how this failure occurs.



Prob. 14–60

14–61. Determine the principal stress and absolute maximum shear stress developed at point A on the cross section of the bracket at section a–a.

14–62. Determine the principal stress and absolute maximum shear stress developed at point B on the cross section of the bracket at section a–a.

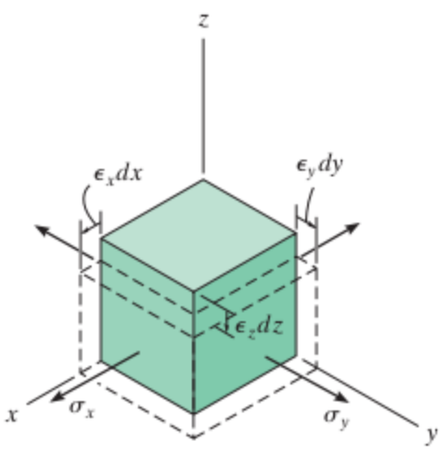


Probs. 14–61/62

14.6 Plane Strain

As outlined in Sec. 14.1, the general state of strain at a point in a body is represented by a combination of three components of normal strain, ϵ_x, ϵ_y, ϵ_z, and three components of shear strain γ_{xy}, γ_{xz}, γ_{yz}. These six components tend to deform each face of an element of the material, and like stress, the normal and shear strain *components* at the point will vary according to the orientation of the element. The *strains* at a point are often determined by using strain gauges, which measure normal strain in *specified directions*. For both analysis and design, however, engineers must sometimes transform this data in order to obtain the strain in other directions.

To understand how this is done, we will first confine our attention to a study of ***plane strain***. Specifically, we will not consider the effects of the components ϵ_z, γ_{xz}, and γ_{yz}. In general, then, a plane-strained element is subjected to two components of normal strain, ϵ_x, ϵ_y, and one component of shear strain, γ_{xy}. Although plane strain and plane stress each have three components lying in the same plane, realize that plane stress *does not* necessarily cause plane strain or vice versa. The reason for this has to do with the Poisson effect discussed in Sec. 8.6. For example, if the element in Fig. 14–26 is subjected to plane stress σ_x and σ_y, not only are normal strains ϵ_x and ϵ_y produced, but there is *also* an associated normal strain, ϵ_z. This is obviously *not* a case of plane strain. In general, then, unless $\nu = 0$, the Poisson effect will *prevent* the simultaneous occurrence of plane strain and plane stress.



Plane stress, σ_x, σ_y, does not cause plane strain in the x–y plane since $\epsilon_z \neq 0$.

Fig. 14–26

14.7 General Equations of Plane-Strain Transformation

It is important in plane-strain analysis to establish transformation equations that can be used to determine the x', y' components of normal and shear strain at a point, provided the x, y components of strain are known. Essentially this problem is one of geometry and requires relating the deformations and rotations of line segments, which represent the sides of differential elements that are parallel to each set of axes.

Sign Convention. Before the strain-transformation equations can be developed, we must first establish a sign convention for the strains. With reference to the differential element shown in Fig. 14–27*a*, *normal strains* ϵ_x and ϵ_y are *positive* if they cause *elongation* along the x and y axes, respectively, and the *shear strain* γ_{xy} is *positive* if the interior angle AOB *becomes smaller* than 90°. This sign convention also follows the corresponding one used for plane stress, Fig. 14–5*a*, that is, positive σ_x, σ_y, τ_{xy} will cause the element to *deform* in the positive ϵ_x, ϵ_y, γ_{xy} directions, respectively.

The problem here will be to determine at a point the normal and shear strains $\epsilon_{x'}$, $\epsilon_{y'}$, $\gamma_{x'y'}$, measured relative to the x', y' axes, if we know ϵ_x, ϵ_y, γ_{xy}, measured relative to the x, y axes. If the angle between the x and x' axes is θ, then, like the case of plane stress, θ will be *positive* provided it follows the curl of the right-hand fingers, i.e., counterclockwise, as shown in Fig. 14–27*b*.

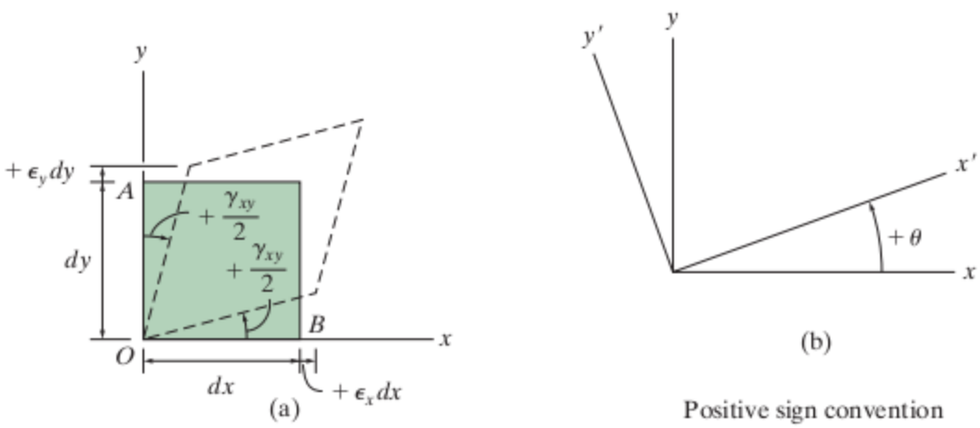


Fig. 14–27

Normal and Shear Strains. In order to develop the strain-transformation equation for $\epsilon_{x'}$, we must determine the elongation of a line segment dx' that lies along the x' axis and is subjected to strain components ϵ_x, ϵ_y, γ_{xy}. As shown in Fig. 14–28*a*, the components of the line dx' along the x and y axes are

$$dx = dx' \cos\theta \tag{14–15}$$

$$dy = dx' \sin\theta$$

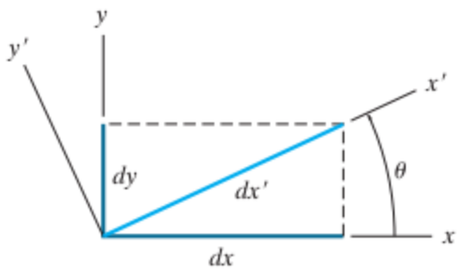
When the positive normal strain ϵ_x occurs, the line dx is elongated $\epsilon_x dx$, Fig. 14–28*b*, which causes line dx' to elongate $\epsilon_x dx \cos\theta$. Likewise, when ϵ_y occurs, line dy elongates $\epsilon_y dy$, Fig. 14–28*c*, which causes line dx' to elongate $\epsilon_y dy \sin\theta$. Finally, assuming that dx remains fixed in position, the shear strain γ_{xy}, which is the change in angle between dx and dy, causes the top of line dy to be displaced $\gamma_{xy} dy$ to the right, as shown in Fig. 14–28*d*. This causes dx' to elongate $\gamma_{xy} dy \cos\theta$. If all three of these elongations are added together, the resultant elongation of dx' is then

$$\delta x' = \epsilon_x dx \cos\theta + \epsilon_y dy \sin\theta + \gamma_{xy} dy \cos\theta$$

From Eq. 7–11, the normal strain along the line dx' is $\epsilon_{x'} = \delta x'/dx'$. Using Eqs. 14–15, we therefore have

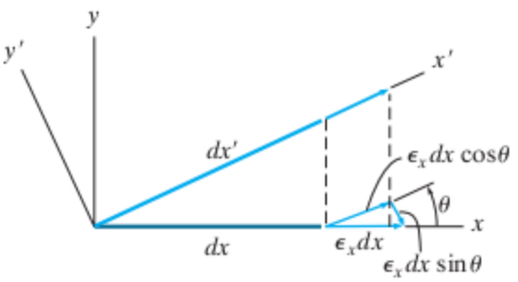
$$\epsilon_{x'} = \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy} \sin\theta \cos\theta \tag{14–16}$$

The rubber specimen is constrained between the two fixed supports, and so it will undergo plane strain when loads are applied to it in the horizontal plane.



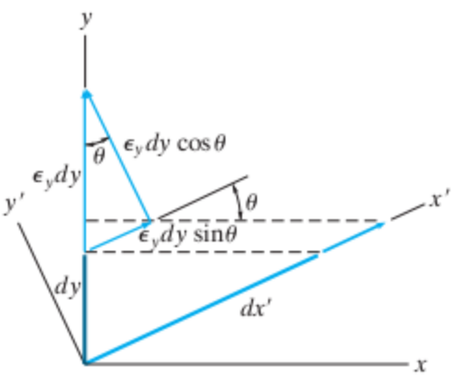
Before deformation

(a)



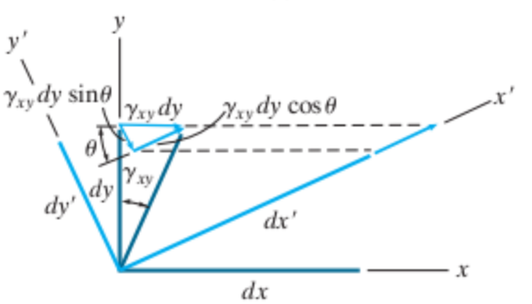
Normal strain ϵ_x

(b)



Normal strain ϵ_y

(c)



Shear strain γ_{xy}

(d)

Fig. 14–28

14

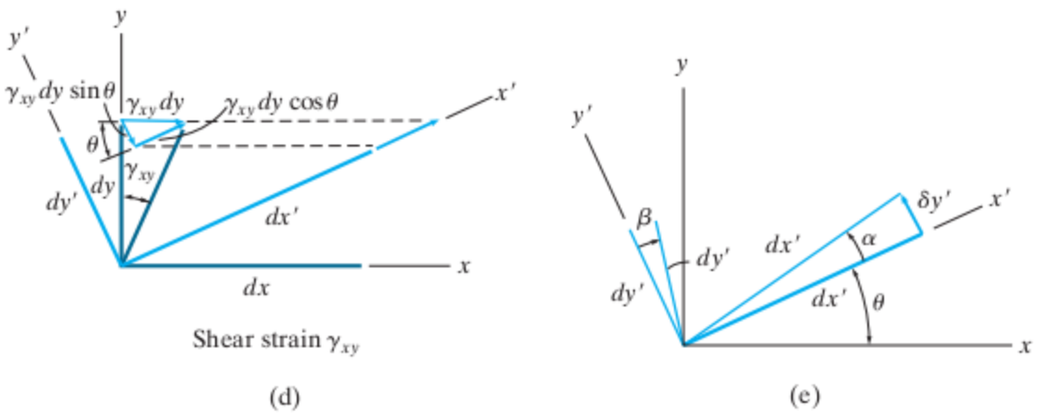


Fig. 14–28 (cont.)

The strain-transformation equation for $\gamma_{x'y'}$ can be developed by considering the amount of rotation each of the line segments dx' and dy' undergo when subjected to the strain components ϵ_x, ϵ_y, γ_{xy}. First we will consider the rotation of dx', which is defined by the counterclockwise angle α shown in Fig. 14–28*e*. It can be determined by the displacement caused by $\delta y'$ using $\alpha = \delta y'/dx'$. To obtain $\delta y'$, consider the following three displacement components acting in the y' direction: one from ϵ_x, giving $-\epsilon_x\, dx \sin\theta$, Fig. 14–28*b*; another from ϵ_y, giving $\epsilon_y\, dy \cos\theta$, Fig. 14–28*c*; and the last from γ_{xy}, giving $-\gamma_{xy}\, dy \sin\theta$, Fig. 14–28*d*. Thus, $\delta y'$, as caused by all three strain components, is

$$\delta y' = -\epsilon_x\, dx \sin\theta + \epsilon_y\, dy \cos\theta - \gamma_{xy}\, dy \sin\theta$$

Dividing each term by dx' and using Eq. 14–15, with $\alpha = \delta y'/dx'$, we have

$$\alpha = (-\epsilon_x + \epsilon_y)\sin\theta\cos\theta - \gamma_{xy}\sin^2\theta \qquad (14\text{–}17)$$

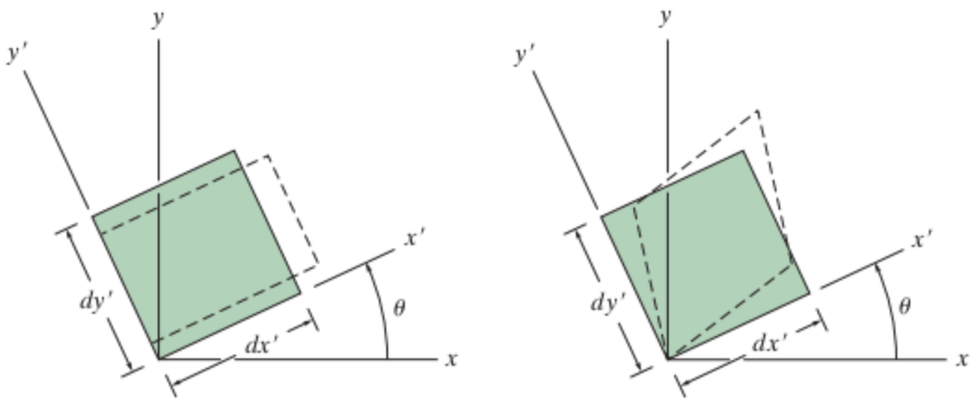
As shown in Fig. 14–28*e*, the line dy' rotates by an amount β. We can determine this angle by a similar analysis, or by simply substituting $\theta + 90°$ for θ into Eq. 14–17. Using the identities $\sin(\theta + 90°) = \cos\theta$, $\cos(\theta + 90°) = -\sin\theta$, we have

$$\beta = (-\epsilon_x + \epsilon_y)\sin(\theta + 90°)\cos(\theta + 90°) - \gamma_{xy}\sin^2(\theta + 90°)$$

$$= -(-\epsilon_x + \epsilon_y)\cos\theta\sin\theta - \gamma_{xy}\cos^2\theta$$

Since α and β represent the rotation of the sides dx' and dy' of a differential element whose sides were originally oriented along the x' and y' axes, Fig. 14–28*e*, the element is then subjected to a shear strain of

$$\gamma_{x'y'} = \alpha - \beta = -2(\epsilon_x - \epsilon_y)\sin\theta\cos\theta + \gamma_{xy}(\cos^2\theta - \sin^2\theta) \qquad (14\text{–}18)$$



Positive normal strain, $\epsilon_{x'}$ (a)

Positive shear strain, $\gamma_{x'y'}$ (b)

Fig. 14–29

Using the trigonometric identities $\sin 2\theta = 2 \sin\theta \cos\theta$, $\cos^2\theta = (1 + \cos 2\theta)/2$, and $\sin^2\theta + \cos^2\theta = 1$, we can rewrite Eqs. 14–16 and 14–18 in the final form

$$\epsilon_{x'} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta \tag{14–19}$$

$$\frac{\gamma_{x'y'}}{2} = -\left(\frac{\epsilon_x - \epsilon_y}{2}\right)\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta \tag{14–20}$$

These strain-transformation equations give the normal strain $\epsilon_{x'}$ in the x' direction and the shear strain $\gamma_{x'y'}$ of an element oriented at an angle θ, as shown in Fig. 14–29. According to the established sign convention, if $\epsilon_{x'}$ is *positive*, the element *elongates* in the positive x' direction, Fig. 14–29*a*, and if $\gamma_{x'y'}$ is positive, the element deforms as shown in Fig. 14–29*b*.

If the normal strain in the y' direction is required, it can be obtained from Eq. 14–19 by simply substituting $(\theta + 90°)$ for θ. The result is

$$\epsilon_{y'} = \frac{\epsilon_x + \epsilon_y}{2} - \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta \tag{14–21}$$

The similarity between the above three equations and those for plane-stress transformation, Eqs. 14–1, 14–2, and 14–3, should be noted. By comparison, σ_x, σ_y, $\sigma_{x'}$, $\sigma_{y'}$ correspond to ϵ_x, ϵ_y, $\epsilon_{x'}$, $\epsilon_{y'}$; and τ_{xy}, $\tau_{x'y'}$ correspond to $\gamma_{xy}/2$, $\gamma_{x'y'}/2$.

14

Complex stresses are often developed at the joints where the cylindrical and hemispherical vessels are joined together. The stresses are determined by making measurements of strain.

Principal Strains. Like stress, an element can be oriented at a point so that the element's deformation is caused only by normal strains, with *no* shear strain. When this occurs the normal strains are referred to as *principal strains*, and if the material is isotropic, the axes along which these strains occur will coincide with the axes that define the planes of principal stress.

From Eqs. 14–4 and 14–5, and the correspondence between stress and strain mentioned above, the direction of the x' axis and the two values of the principal strains ϵ_1 and ϵ_2 are determined from

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y} \tag{14–22}$$

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \tag{14–23}$$

Maximum In-Plane Shear Strain. Using Eqs. 14–6, 14–7, and 14–8, the direction of the x' axis, and the maximum in-plane shear strain and associated average normal strain are determined from the following equations:

$$\tan 2\theta_s = -\left(\frac{\epsilon_x - \epsilon_y}{\gamma_{xy}}\right) \tag{14–24}$$

$$\frac{\gamma_{\text{in-plane}}^{\max}}{2} = \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \tag{14–25}$$

$$\epsilon_{\text{avg}} = \frac{\epsilon_x + \epsilon_y}{2} \tag{14–26}$$

Important Points

- In the case of plane stress, plane-strain analysis may be used within the plane of the stresses to analyze the data from strain gauges. Remember, though, there will be a normal strain that is perpendicular to the gauges due to the Poisson effect.
- When the state of strain is represented by the principal strains, no shear strain will act on the element.
- The state of strain at a point can also be represented in terms of the maximum in-plane shear strain. In this case an average normal strain will also act on the element.
- The element representing the maximum in-plane shear strain and its associated average normal strains is 45° from the orientation of an element representing the principal strains.

EXAMPLE 14.12

A differential element of material at a point is subjected to a state of plane strain $\epsilon_x = 500(10^{-6})$, $\epsilon_y = -300(10^{-6})$, $\gamma_{xy} = 200(10^{-6})$, which tends to distort the element as shown in Fig. 14–30*a*. Determine the equivalent strains acting on an element of the material oriented at the point, *clockwise* 30° from the original position.

SOLUTION

The strain-transformation Eqs. 14–19 and 14–20 will be used to solve the problem. Since θ is *positive counterclockwise*, then for this problem $\theta = -30°$. Thus,

$$\epsilon_{x'} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$= \left[\frac{500 + (-300)}{2}\right](10^{-6}) + \left[\frac{500 - (-300)}{2}\right](10^{-6})\cos(2(-30°))$$

$$+ \left[\frac{200(10^{-6})}{2}\right]\sin(2(-30°))$$

$$\epsilon_{x'} = 213(10^{-6}) \qquad \textit{Ans.}$$

$$\frac{\gamma_{x'y'}}{2} = -\left(\frac{\epsilon_x - \epsilon_y}{2}\right)\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$$= -\left[\frac{500 - (-300)}{2}\right](10^{-6})\sin(2(-30°)) + \frac{200(10^{-6})}{2}\cos(2(-30°))$$

$$\gamma_{x'y'} = 793(10^{-6}) \qquad \textit{Ans.}$$

The strain in the y' direction can be obtained from Eq. 14–21 with $\theta = -30°$. However, we can also obtain $\epsilon_{y'}$ using Eq. 14–19 with $\theta = 60°(\theta = -30° + 90°)$, Fig. 14–30*b*. We have with $\epsilon_{y'}$ replacing $\epsilon_{x'}$,

$$\epsilon_{y'} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$= \left[\frac{500 + (-300)}{2}\right](10^{-6}) + \left[\frac{500 - (-300)}{2}\right](10^{-6})\cos(2(60°))$$

$$+ \frac{200(10^{-6})}{2}\sin(2(60°))$$

$$\epsilon_{y'} = -13.4(10^{-6}) \qquad \textit{Ans.}$$

These results tend to distort the element as shown in Fig. 14–30*c*.

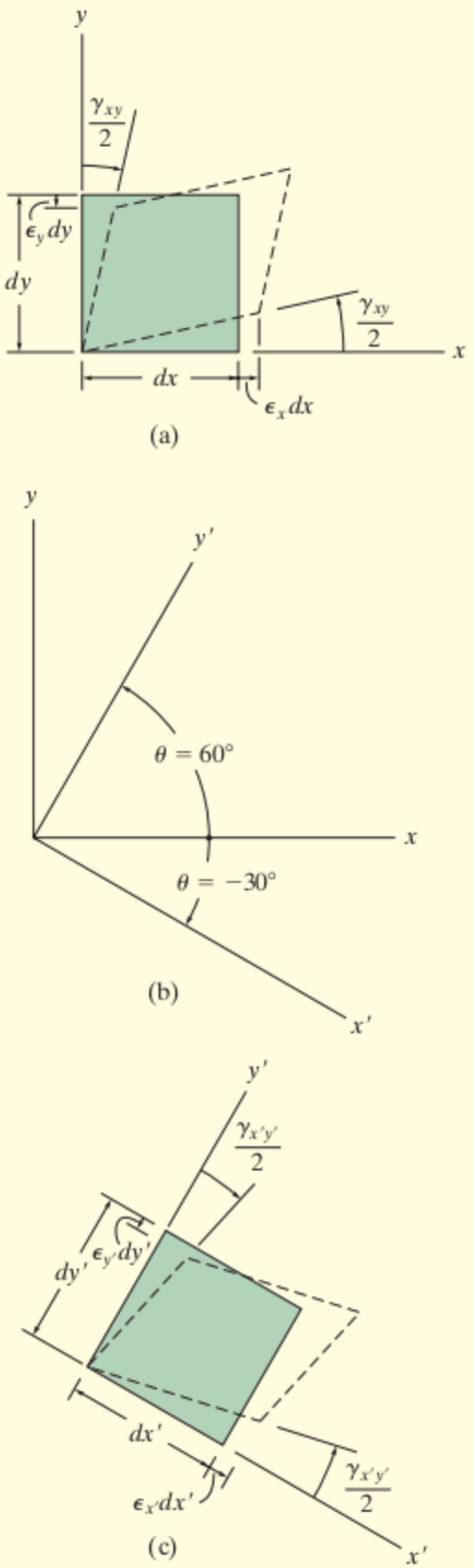
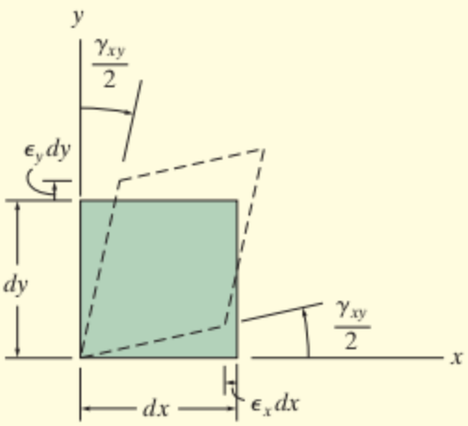
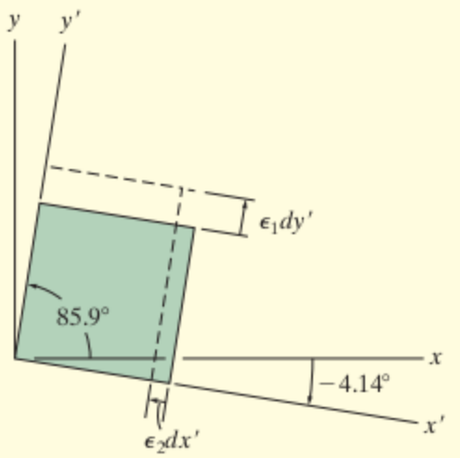


Fig. 14–30

EXAMPLE 14.13



(a)



(b)

Fig. 14–31

A differential element of material at a point is subjected to a state of plane strain defined by $\epsilon_x = -350(10^{-6})$, $\epsilon_y = 200(10^{-6})$, $\gamma_{xy} = 80(10^{-6})$, which tends to distort the element as shown in Fig. 14–31*a*. Determine the principal strains at the point and the associated orientation of the element.

SOLUTION

Orientation of the Element. From Eq. 14–22 we have

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y}$$

$$= \frac{80(10^{-6})}{(-350 - 200)(10^{-6})}$$

Thus, $2\theta_p = -8.28°$ and $-8.28° + 180° = 171.72°$, so that

$$\theta_p = -4.14° \text{ and } 85.9° \qquad \textit{Ans.}$$

Each of these angles is measured *positive counterclockwise*, from the x axis to the outward normals on each face of the element, Fig. 14–31*b*.

Principal Strains. The principal strains are determined from Eq. 14–23. We have

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= \frac{(-350 + 200)(10^{-6})}{2} \pm \left[\sqrt{\left(\frac{-350 - 200}{2}\right)^2 + \left(\frac{80}{2}\right)^2}\right](10^{-6})$$

$$= -75.0(10^{-6}) \pm 277.9(10^{-6})$$

$$\epsilon_1 = 203(10^{-6}) \qquad \epsilon_2 = -353(10^{-6}) \qquad \textit{Ans.}$$

We can determine which of these two strains deforms the element in the x' direction by applying Eq. 14–19 with $\theta = -4.14°$. Thus,

$$\epsilon_{x'} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$= \left(\frac{-350 + 200}{2}\right)(10^{-6}) + \left(\frac{-350 - 200}{2}\right)(10^{-6})\cos 2(-4.14°)$$

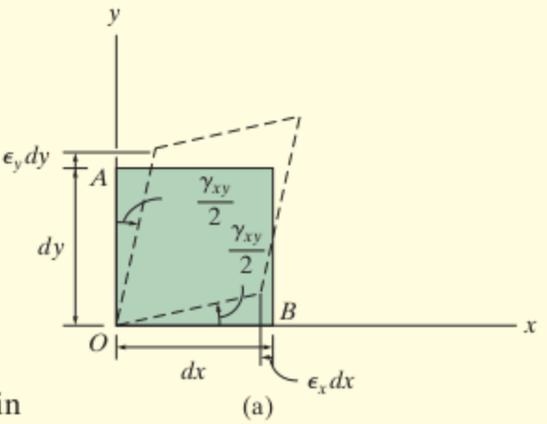
$$+ \frac{80(10^{-6})}{2}\sin 2(-4.14°)$$

$$\epsilon_{x'} = -353(10^{-6})$$

Hence $\epsilon_{x'} = \epsilon_2$. When subjected to the principal strains, the element is distorted as shown in Fig. 14–31*b*.

EXAMPLE 14.14

A differential element of material at a point is subjected to a state of plane strain defined by $\epsilon_x = -350(10^{-6})$, $\epsilon_y = 200(10^{-6})$, $\gamma_{xy} = 80(10^{-6})$, which tends to distort the element as shown in Fig. 14–32*a*. Determine the maximum in-plane shear strain at the point and the associated orientation of the element.



(a)

SOLUTION

Orientation of the Element. From Eq. 14–24 we have

$$\tan 2\theta_s = -\left(\frac{\epsilon_x - \epsilon_y}{\gamma_{xy}}\right) = -\frac{(-350 - 200)(10^{-6})}{80(10^{-6})}$$

Thus, $2\theta_s = 81.72°$ and $81.72° + 180° = 261.72°$, so that

$$\theta_s = 40.9° \text{ and } 131°$$

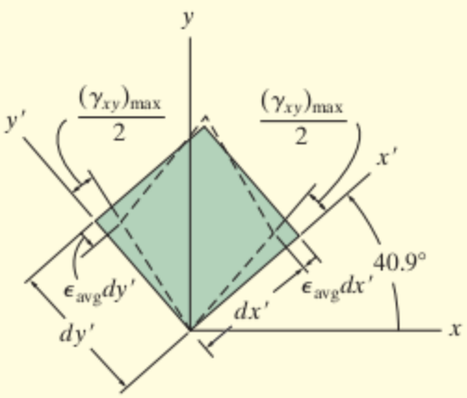
Note that this orientation is 45° from that shown in Fig. 14–31*b* in Example 14.13 as expected.

Maximum In-Plane Shear Strain. Applying Eq. 14–25 gives

$$\frac{\gamma_{\text{in-plane}}^{\max}}{2} = \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= \left[\sqrt{\left(\frac{-350 - 200}{2}\right)^2 + \left(\frac{80}{2}\right)^2}\right](10^{-6})$$

$$\gamma_{\text{in-plane}}^{\max} = 556(10^{-6}) \qquad \textit{Ans.}$$



(b)

Fig. 14–32

Due to the square root, the proper sign of $\gamma_{\text{in-plane}}^{\max}$ can be obtained by applying Eq. 14–20 with $\theta_s = 40.9°$. We have

$$\frac{\gamma_{x'y'}}{2} = -\frac{\epsilon_x - \epsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$$= -\left(\frac{-350 - 200}{2}\right)(10^{-6})\sin 2(40.9°) + \frac{80(10^{-6})}{2}\cos 2(40.9°)$$

$$\gamma_{x'y'} = 556(10^{-6})$$

This result is positive and so tends to distort the element so that the right angle between dx' and dy' is *decreased* (positive sign convention), Fig. 14–32*b*.

Also, there are associated average normal strains imposed on the element that are determined from Eq. 14–26:

$$\epsilon_{\text{avg}} = \frac{\epsilon_x + \epsilon_y}{2} = \frac{-350 + 200}{2}(10^{-6}) = -75(10^{-6})$$

These strains tend to cause the element to contract, Fig. 14–32*b*.

14

*14.8 Mohr's Circle—Plane Strain

Since the equations of plane-strain transformation are mathematically similar to the equations of plane-stress transformation, we can also solve problems involving the transformation of strain using Mohr's circle.

Like the case for stress, the parameter θ in Eqs. 14–19 and 14–20 can be eliminated and the result rewritten in the form

$$(\epsilon_{x'} - \epsilon_{\text{avg}})^2 + \left(\frac{\gamma_{x'y'}}{2}\right)^2 = R^2 \tag{14–27}$$

where

$$\epsilon_{\text{avg}} = \frac{\epsilon_x + \epsilon_y}{2}$$

$$R = \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

Equation 14–27 represents the equation of Mohr's circle for strain. It has a center on the ϵ axis at point $C(\epsilon_{\text{avg}}, 0)$ and a radius R.

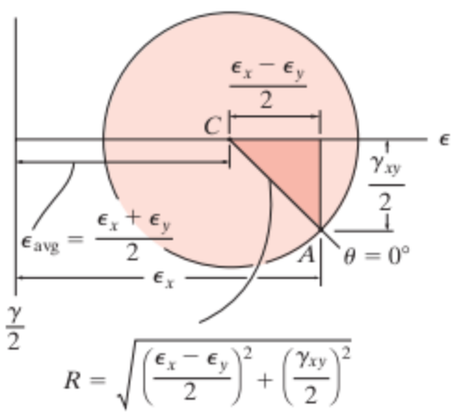


Fig. 14–33

Procedure for Analysis

The procedure for drawing Mohr's circle for strain follows the same one established for stress.

Construction of the Circle.

- Establish a coordinate system such that the abscissa represents the normal strain ϵ, with *positive to the right*, and the ordinate represents *half* the value of the shear strain, $\gamma/2$, with *positive downward*, Fig. 14–33.
- Using the positive sign convention for ϵ_x, ϵ_y, γ_{xy}, as shown in Fig. 14–27, determine the center of the circle C, which is located on the ϵ axis at a distance $\epsilon_{\text{avg}} = (\epsilon_x + \epsilon_y)/2$ from the origin, Fig. 14–33.
- Plot the reference point A having coordinates $A(\epsilon_x, \gamma_{xy}/2)$. This point represents the case for which the x' axis coincides with the x axis. Hence $\theta = 0°$, Fig. 14–33.
- Connect point A with the center C of the circle and from the shaded triangle determine the radius R of the circle, Fig. 14–33.
- Once R has been determined, sketch the circle.

Principal Strains.

- The principal strains ϵ_1 and ϵ_2 are determined from the circle as the coordinates of points B and D, that is where $\gamma/2 = 0$, Fig. 14–34*a*.
- The orientation of the plane on which ϵ_1 acts can be determined from the circle by calculating $2\theta_{p_1}$ using trigonometry. Here this angle happens to be counterclockwise *from* the radial reference line CA *to* line CB, Fig. 14–34*a*. Remember that the *rotation* of θ_{p_1} must be in this *same direction*, from the element's reference axis x to the x' axis, Fig. 14–34*b*.*
- When ϵ_1 and ϵ_2 are indicated as being positive as in Fig. 14–34*a*, the element in Fig. 14–34*b* will elongate in the x' and y' directions as shown by the dashed outline.

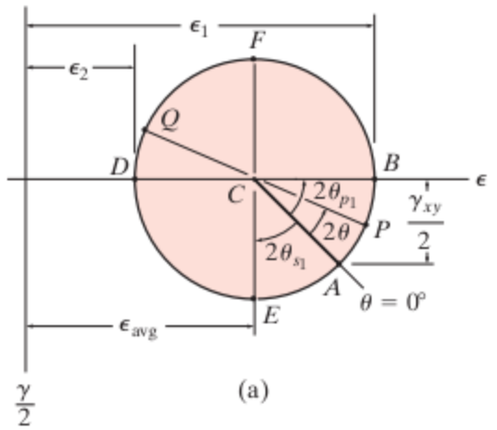
Maximum In-Plane Shear Strain.

- The average normal strain and half the maximum in-plane shear strain are determined from the circle as the coordinates of point E or F, Fig. 14–34*a*.
- The orientation of the plane on which $\gamma_{\text{in-plane}}^{\max}$ and ϵ_{avg} act can be determined from the circle by calculating $2\theta_{s_1}$ using trigonometry. Here this angle happens to be clockwise *from* the radial reference line CA *to* line CE, Fig. 14–34*a*. Remember that the *rotation* of θ_{s_1} must be in this *same direction*, from the element's reference axis x to the x' axis, Fig. 14–34*c*.*

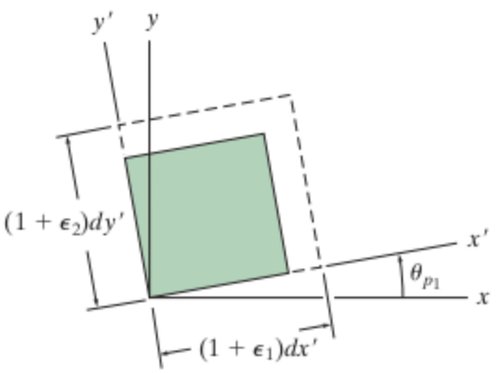
Strains on Arbitrary Plane.

- The normal and shear strain components $\epsilon_{x'}$ and $\gamma_{x'y'}$ for a plane oriented at an angle θ, Fig. 14–34*d*, can be obtained from the circle using trigonometry to determine the coordinates of point P, Fig. 14–34*a*.
- To locate P, the known angle θ of the x' axis is measured on the circle as 2θ. This measurement is made *from* the radial reference line CA *to* the radial line CP. Remember that measurements for 2θ on the circle must be in the same direction as θ for the x' axis.*
- If the value of $\epsilon_{y'}$ is required, it can be determined by calculating the ϵ coordinate of point Q in Fig. 14–34*a*. The line CQ lies 180° away from CP and thus represents a 90° rotation of the x' axis.

*If the $\gamma/2$ axis were constructed *positive upwards*, then the angle 2θ on the circle would be measured in the *opposite direction* to the orientation θ of the plane.



(a)



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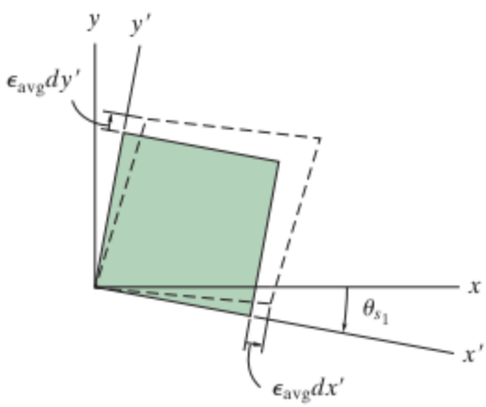
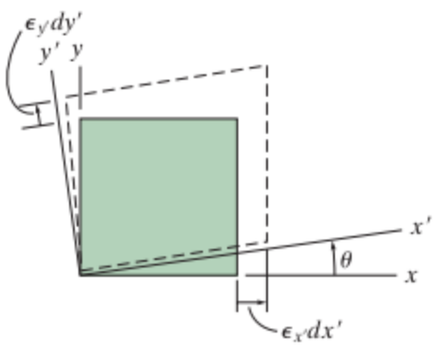


(c)



(d)

Fig. 14–34

14

EXAMPLE 14.15

The state of plane strain at a point is represented by the components $\epsilon_x = 250(10^{-6})$, $\epsilon_y = -150(10^{-6})$, and $\gamma_{xy} = 120(10^{-6})$. Determine the principal strains and the orientation of the element.

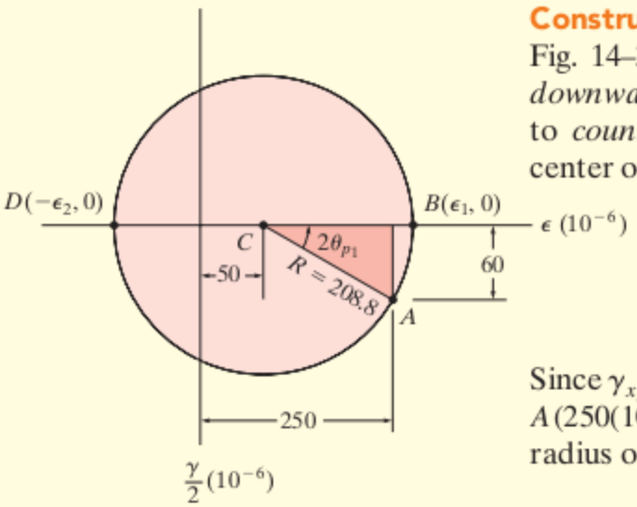
SOLUTION

Construction of the Circle. The ϵ and $\gamma/2$ axes are established in Fig. 14–35*a*. Remember that the *positive* $\gamma/2$ axis must be directed *downward* so that *counterclockwise* rotations of the element correspond to *counterclockwise* rotation around the circle, and vice versa. The center of the circle C is located on the ϵ axis at

$$\epsilon_{avg} = \frac{250 + (-150)}{2}(10^{-6}) = 50(10^{-6})$$

Since $\gamma_{xy}/2 = 60(10^{-6})$, the reference point A ($\theta = 0°$) has coordinates $A(250(10^{-6}), 60(10^{-6}))$. From the shaded triangle in Fig. 14–35*a*, the radius of the circle is CA; that is,

$$R = \left[\sqrt{(250 - 50)^2 + (60)^2}\right](10^{-6}) = 208.8(10^{-6})$$



(a)

Principal Strains. The ϵ coordinates of points B and D represent the principal strains. They are

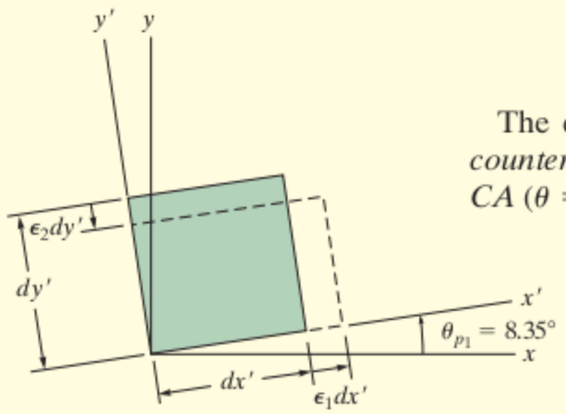
$$\epsilon_1 = (50 + 208.8)(10^{-6}) = 259(10^{-6}) \quad \textit{Ans.}$$

$$\epsilon_2 = (50 - 208.8)(10^{-6}) = -159(10^{-6}) \quad \textit{Ans.}$$

The direction of the positive principal strain ϵ_1 is defined by the *counterclockwise* angle $2\theta_{p_1}$, measured from the radial reference line CA ($\theta = 0°$) to the line CB. We have

$$\tan 2\theta_{p_1} = \frac{60}{(250 - 50)}$$

$$\theta_{p_1} = 8.35° \quad \textit{Ans.}$$



(b)

Fig. 14–35

Hence, the side dx' of the element is oriented *counterclockwise* 8.35° as shown in Fig. 14–35*b*. This also defines the direction of ϵ_1. The deformation of the element is also shown in the figure.

EXAMPLE 14.16

The state of plane strain at a point is represented by the components $\epsilon_x = 250(10^{-6})$, $\epsilon_y = -150(10^{-6})$, and $\gamma_{xy} = 120(10^{-6})$. Determine the maximum in-plane shear strains and the orientation of an element.

SOLUTION

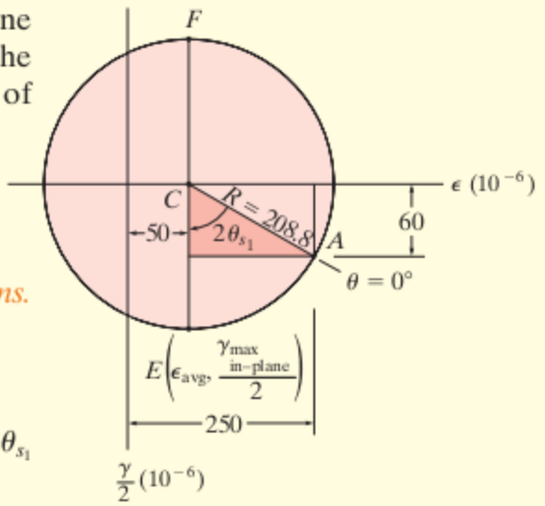
The circle has been established in the previous example and is shown in Fig. 14–36*a*.

Maximum In-Plane Shear Strain. Half the maximum in-plane shear strain and average normal strain are represented by the coordinates of point E or F on the circle. From the coordinates of point E,

$$\frac{(\gamma_{x'y'})_{\text{in-plane}}^{\max}}{2} = 208.8(10^{-6})$$

$$(\gamma_{x'y'})_{\text{in-plane}}^{\max} = 418(10^{-6}) \qquad \textit{Ans.}$$

$$\epsilon_{\text{avg}} = 50(10^{-6})$$



(a)

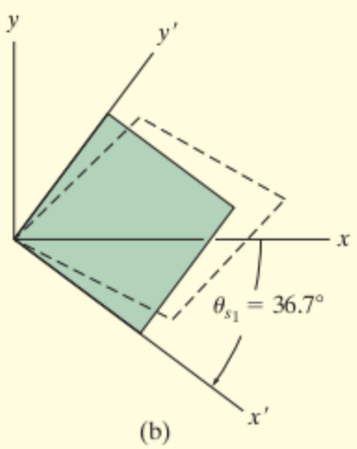
Fig. 14–36

To orient the element, we can determine the clockwise angle $2\theta_{s_1}$ measured from CA ($\theta = 0°$) to CE.

$$2\theta_{s_1} = 90° - 2(8.35°)$$

$$\theta_{s_1} = 36.7° \qquad \textit{Ans.}$$

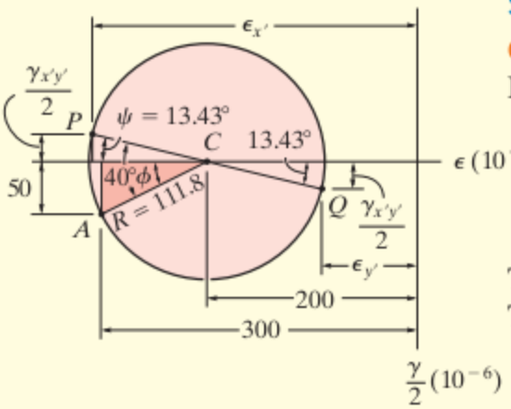
This angle is shown in Fig. 14–36*b*. Since the shear strain defined from point E on the circle has a positive value and the average normal strain is also positive, these strains deform the element into the dashed shape shown in the figure.



(b)

EXAMPLE 14.17

The state of plane strain at a point is represented on an element having components $\epsilon_x = -300(10^{-6})$, $\epsilon_y = -100(10^{-6})$, and $\gamma_{xy} = 100(10^{-6})$. Determine the state of strain on an element oriented 20° clockwise from this reported position.



(a)

SOLUTION

Construction of the Circle. The ϵ and $\gamma/2$ axes are established in Fig. 14–37*a*. The center of the circle is on the ϵ axis at

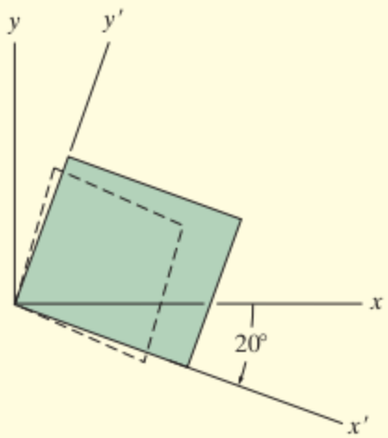
$$\epsilon_{avg} = \left(\frac{-300 - 100}{2}\right)(10^{-6}) = -200(10^{-6})$$

The reference point A has coordinates $A(-300(10^{-6}), 50(10^{-6}))$. The radius CA determined from the shaded triangle is therefore

$$R = \left[\sqrt{(300 - 200)^2 + (50)^2}\right](10^{-6}) = 111.8(10^{-6})$$

Strains on Inclined Element. Since the element is to be oriented 20° *clockwise*, we must establish a radial line CP, $2(20°) = 40°$ *clockwise*, measured from CA ($\theta = 0°$), Fig. 14–37*a*. The coordinates of point P $(\epsilon_{x'}, \gamma_{x'y'}/2)$ are obtained from the geometry of the circle. Note that

$$\phi = \tan^{-1}\left(\frac{50}{(300 - 200)}\right) = 26.57°, \quad \psi = 40° - 26.57° = 13.43°$$



(b)

Fig. 14–37

Thus,

$$\epsilon_{x'} = -(200 + 111.8\cos 13.43°)(10^{-6})$$
$$= -309(10^{-6}) \quad \textit{Ans.}$$

$$\frac{\gamma_{x'y'}}{2} = -(111.8\sin 13.43°)(10^{-6})$$
$$\gamma_{x'y'} = -52.0(10^{-6}) \quad \textit{Ans.}$$

The normal strain $\epsilon_{y'}$ can be determined from the ϵ coordinate of point Q on the circle, Fig. 14–37*a*. Why?

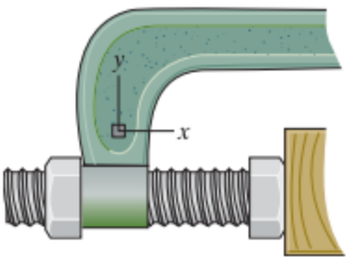
$$\epsilon_{y'} = -(200 - 111.8\cos 13.43°)(10^{-6}) = -91.3(10^{-6}) \quad \textit{Ans.}$$

As a result of these strains, the element deforms relative to the x', y' axes as shown in Fig. 14–37*b*.

PROBLEMS

14–63. Prove that the sum of the normal strains in perpendicular directions is constant.

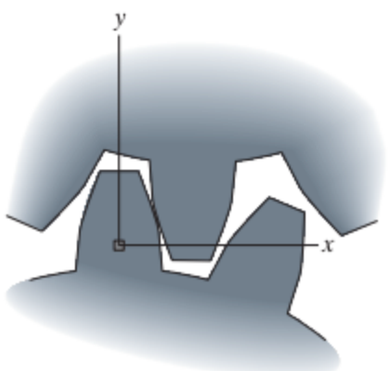
***14–64.** The state of strain at the point has components of $\epsilon_x = 200(10^{-6})$, $\epsilon_y = -300(10^{-6})$, and $\gamma_{xy} = 400(10^{-6})$. Use the strain-transformation equations to determine the equivalent in-plane strains on an element oriented at an angle of 30° counterclockwise from the original position. Sketch the deformed element due to these strains within the x–y plane.



Prob. 14–64

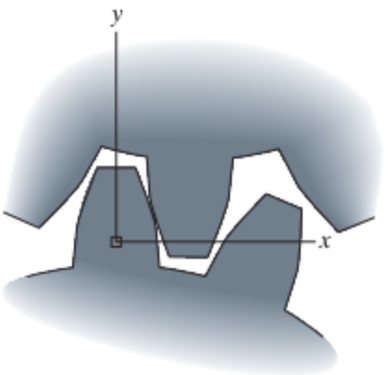
14–65. The state of strain at a point on a wrench has components $\epsilon_x = 120(10^{-6})$, $\epsilon_y = -180(10^{-6})$, $\gamma_{xy} = 150(10^{-6})$. Use the strain-transformation equations to determine (a) the in-plane principal strains and (b) the maximum in-plane shear strain and average normal strain. In each case specify the orientation of the element and show how the strains deform the element within x–y plane.

14–66. The state of strain at the point on the gear tooth has components $\epsilon_x = 850(10^{-6})$, $\epsilon_y = 480(10^{-6})$, $\gamma_{xy} = 650(10^{-6})$. Use the strain-transformation equations to determine (a) the in-plane principal strains and (b) the maximum in-plane shear strain and average normal strain. In each case specify the orientation of the element and show how the strains deform the element within the x–y plane.



Probs. 14–65/66

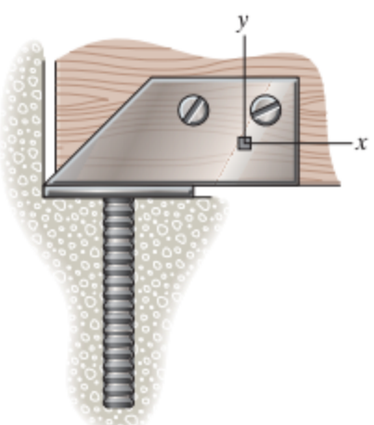
14–67. The state of strain at the point on the gear tooth has the components $\epsilon_x = 520(10^{-6})$, $\epsilon_y = -760(10^{-6})$, $\gamma_{xy} = -750(10^{-6})$. Use the strain-transformation equations to determine (a) the in-plane principal strains and (b) the maximum in-plane shear strain and average normal strain. In each case specify the orientation of the element and show how the strains deform the element within the x–y plane.



Prob. 14–67

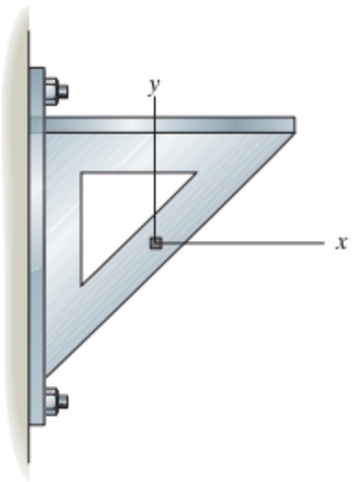
***14–68.** A differential element on the bracket is subjected to plane strain that has the following components: $\epsilon_x = 150(10^{-6})$, $\epsilon_y = 200(10^{-6})$, $\gamma_{xy} = -700(10^{-6})$. Use the strain-transformation equations and determine the equivalent in plane strains on an element oriented at an angle of $\theta = 60°$ counterclockwise from the original position. Sketch the deformed element within the x–y plane due to these strains.

14–69. Solve Prob. 14–68 for an element oriented $\theta = 30°$ clockwise.



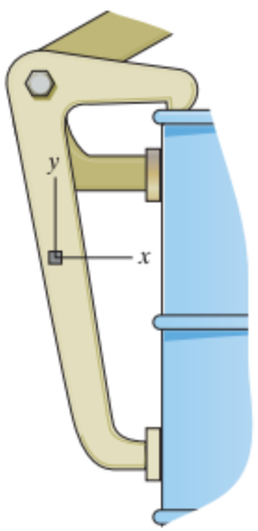
Probs. 14–68/69

14–70. The state of strain at the point on the bracket has components $\epsilon_x = -200(10^{-6})$, $\epsilon_y = -650(10^{-6})$, $\gamma_{xy} = -175(10^{-6})$. Use the strain-transformation equations to determine the equivalent in-plane strains on an element oriented at an angle of $\theta = 20°$ counterclockwise from the original position. Sketch the deformed element due to these strains within the x–y plane.



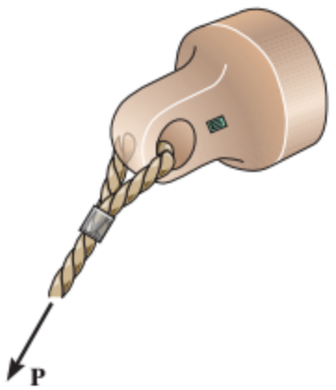
Prob. 14–70

14–71. The state of strain at the point has components of $\epsilon_x = 180(10^{-6})$, $\epsilon_y = -120(10^{-6})$, and $\gamma_{xy} = -100(10^{-6})$. Use the strain-transformation equations to determine (a) the in-plane principal strains and (b) the maximum in-plane shear strain and average normal strain. In each case specify the orientation of the element and show how the strains deform the element within the x–y plane.



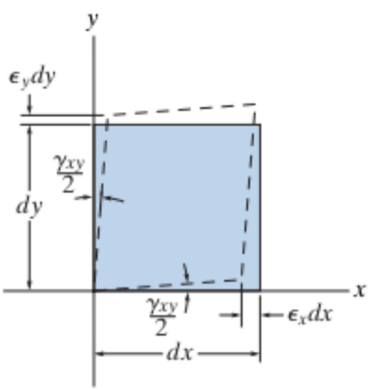
Prob. 14–71

***14–72.** The state of strain at the point on the support has components of $\epsilon_x = 350(10^{-6})$, $\epsilon_y = 400(10^{-6})$, $\gamma_{xy} = -675(10^{-6})$. Use the strain-transformation equations to determine (a) the in-plane principal strains and (b) the maximum in-plane shear strain and average normal strain. In each case specify the orientation of the element and show how the strains deform the element within the x–y plane.



Prob. 14–72

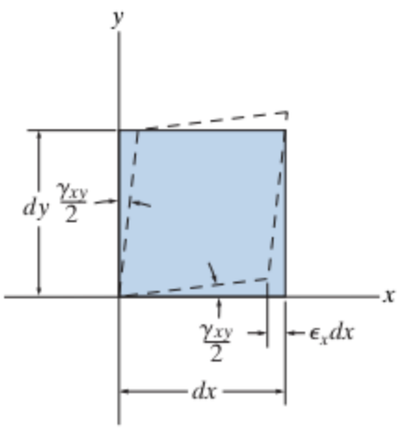
14–73. The state of strain on an element has components $\epsilon_x = -150(10^{-6})$, $\epsilon_y = 450(10^{-6})$, $\gamma_{xy} = 200(10^{-6})$. Determine the equivalent state of strain on an element at the same point oriented 30° counterclockwise with respect to the original element. Sketch the results on this element.



Prob. 14–73

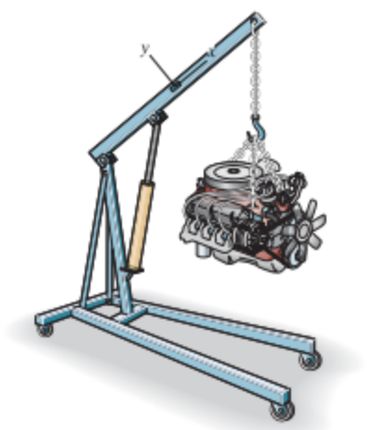
14–74. The state of strain on an element has components $\epsilon_x = -400(10^{-6})$, $\epsilon_y = 0$, $\gamma_{xy} = 150(10^{-6})$. Determine the equivalent state of strain on an element at the same point oriented 30° clockwise with respect to the original element. Sketch the results on this element.

14–75. The state of plane strain on an element is $\epsilon_x = -300(10^{-6})$, $\epsilon_y = 0$, and $\gamma_{xy} = 150(10^{-6})$. Determine the equivalent state of strain which represents (a) the principal strains, and (b) the maximum in-plane shear strain and the associated average normal strain. Specify the orientation of the corresponding elements for these states of strain with respect to the original element.



Probs. 14–74/75

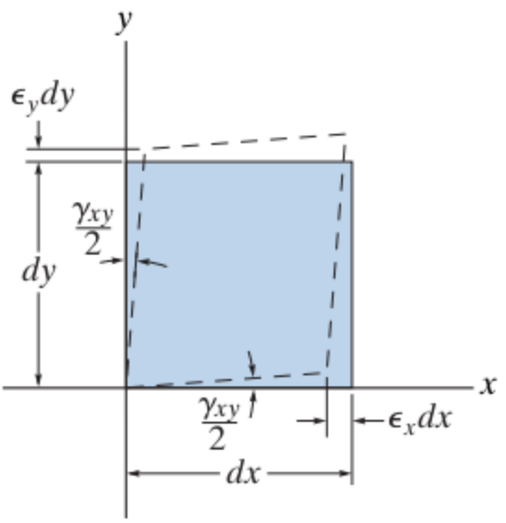
*14–76. The state of strain at the point on a boom of an hydraulic engine crane has components of $\epsilon_x = 250(10^{-6})$, $\epsilon_y = 300(10^{-6})$, and $\gamma_{xy} = -180(10^{-6})$. Use the strain-transformation equations to determine (a) the in-plane principal strains and (b) the maximum in-plane shear strain and average normal strain. In each case, specify the orientation of the element and show how the strains deform the element within the x–y plane.



Prob. 14–76

14–77. Consider the general case of plane strain where ϵ_x, ϵ_y, and γ_{xy} are known. Write a computer program that can be used to determine the normal and shear strain, $\epsilon_{x'}$ and $\gamma_{x'y'}$, on the plane of an element oriented θ from the horizontal. Also, include the principal strains and the element's orientation, and the maximum in-plane shear strain, and the average normal strain and the element's orientation.

14–78. The state of strain on an element has components $\epsilon_x = -300(10^{-6})$, $\epsilon_y = 100(10^{-6})$, $\gamma_{xy} = 150(10^{-6})$. Determine the equivalent state of strain, which represents (a) the principal strains, and (b) the maximum in-plane shear strain and the associated average normal strain. Specify the orientation of the corresponding elements for these states of strain with respect to the original element.



Prob. 14–78

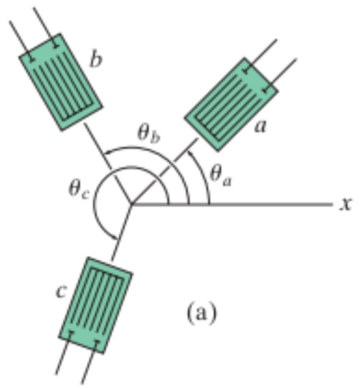
14–79. Solve Prob. 14–65 using Mohr's circle.

***14–80.** Solve Prob. 14–66 using Mohr's circle.

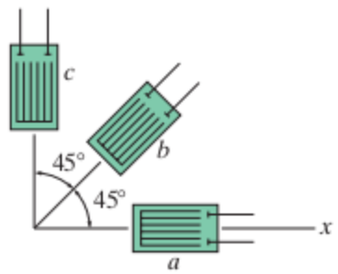
14–81. Solve Prob. 14–67 using Mohr's circle.

14–82. Solve Prob. 14–70 using Mohr's circle.

14–83. Solve Prob. 14–76 using Mohr's circle.

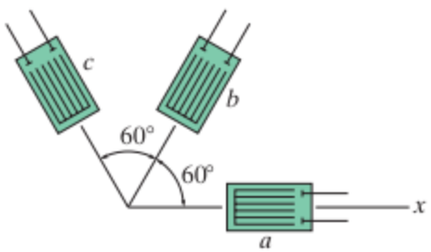


(a)



45° strain rosette

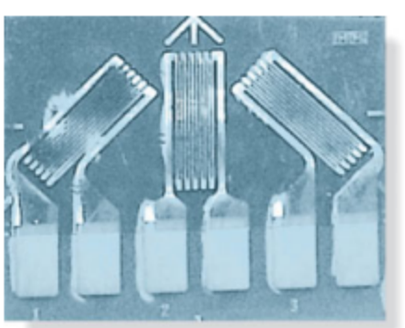
(b)



60° strain rosette

(c)

Fig. 14–38



Typical electrical resistance 45° strain rosette.

14.9 Strain Rosettes

When performing a tension test on a specimen as discussed in Sec. 8.1, the normal strain in the material is measured using an ***electrical-resistance strain gauge***, which consists of a wire grid or piece of metal foil bonded to the specimen. For a general loading on a body, however, the strains at a point on its free surface are determined using a cluster of three electrical-resistance strain gauges, arranged in a specified pattern. This pattern is referred to as a ***strain rosette***, and once the normal strains on the three gauges are measured, the data can then be transformed to specify the state of strain at the point. Since these strains are measured *only* in the plane of the gauges, and since the body is stress-free on its surface, the gauges may be subjected to *plane stress* but *not* plane strain. Although the strain normal to the surface is not measured, realize that the out-of-plane displacement caused by this strain will *not* affect the in-plane measurements of the gauges.

In the general case, the axes of the three gauges are arranged at the angles $\theta_a, \theta_b, \theta_c$ shown in Fig. 14–38*a*. If the readings $\epsilon_a, \epsilon_b, \epsilon_c$ are taken, we can determine the strain components $\epsilon_x, \epsilon_y, \gamma_{xy}$ at the point by applying the strain-transformation equation, Eq. 14–16, for each gauge. We have

$$\begin{aligned}
\epsilon_a &= \epsilon_x \cos^2\theta_a + \epsilon_y \sin^2\theta_a + \gamma_{xy} \sin\theta_a \cos\theta_a \\
\epsilon_b &= \epsilon_x \cos^2\theta_b + \epsilon_y \sin^2\theta_b + \gamma_{xy} \sin\theta_b \cos\theta_b \qquad (14\text{–}28) \\
\epsilon_c &= \epsilon_x \cos^2\theta_c + \epsilon_y \sin^2\theta_c + \gamma_{xy} \sin\theta_c \cos\theta_c
\end{aligned}$$

The values of $\epsilon_x, \epsilon_y, \gamma_{xy}$ are determined by solving these three equations simultaneously.

Strain rosettes are often arranged in 45° or 60° patterns. In the case of the 45° or "rectangular" strain rosette shown in Fig. 14–38*b*, $\theta_a = 0°$, $\theta_b = 45°, \theta_c = 90°$, so that Eq. 14–28 gives

$$\begin{aligned}
\epsilon_x &= \epsilon_a \\
\epsilon_y &= \epsilon_c \\
\gamma_{xy} &= 2\epsilon_b - (\epsilon_a + \epsilon_c)
\end{aligned}$$

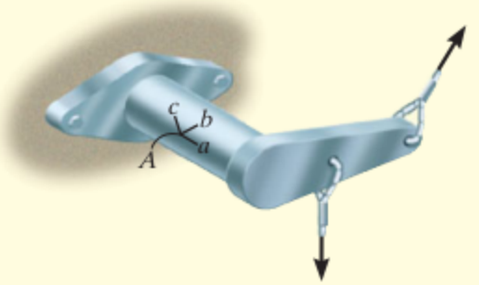
And for the 60° strain rosette in Fig. 14–38*c*, $\theta_a = 0°$, $\theta_b = 60°$, $\theta_c = 120°$. Here Eq. 14–28 gives

$$\begin{aligned}
\epsilon_x &= \epsilon_a \\
\epsilon_y &= \frac{1}{3}(2\epsilon_b + 2\epsilon_c - \epsilon_a) \qquad (14\text{–}29) \\
\gamma_{xy} &= \frac{2}{\sqrt{3}}(\epsilon_b - \epsilon_c)
\end{aligned}$$

Once $\epsilon_x, \epsilon_y, \gamma_{xy}$ are determined, the transformation equations of Sec. 14.6 or Mohr's circle can then be used to determine the principal in-plane strains and the maximum in-plane shear strain at the point.

EXAMPLE 14.18

The state of strain at point A on the bracket in Fig. 14–39*a* is measured using the strain rosette shown in Fig. 14–39*b*. Due to the loadings, the readings from the gauges give $\epsilon_a = 60(10^{-6})$, $\epsilon_b = 135(10^{-6})$, and $\epsilon_c = 264(10^{-6})$. Determine the in-plane principal strains at the point and the directions in which they act.

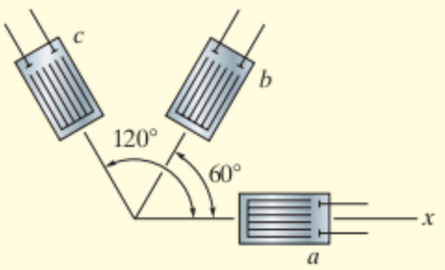


(a)

SOLUTION

We will use Eqs. 14–28 for the solution. Establishing an x axis as shown in Fig. 14–39*b* and measuring the angles counterclockwise from the $+x$ axis to the centerlines of each gauge, we have $\theta_a = 0°, \theta_b = 60°$, and $\theta_c = 120°$. Substituting these results, along with the problem data, into the equations gives

$$\begin{aligned} 60(10^{-6}) &= \epsilon_x \cos^2 0° + \epsilon_y \sin^2 0° + \gamma_{xy} \sin 0° \cos 0° \\ &= \epsilon_x \qquad (1) \\ 135(10^{-6}) &= \epsilon_x \cos^2 60° + \epsilon_y \sin^2 60° + \gamma_{xy} \sin 60° \cos 60° \\ &= 0.25\epsilon_x + 0.75\epsilon_y + 0.433\gamma_{xy} \qquad (2) \\ 264(10^{-6}) &= \epsilon_x \cos^2 120° + \epsilon_y \sin^2 120° + \gamma_{xy} \sin 120° \cos 120° \\ &= 0.25\epsilon_x + 0.75\epsilon_y - 0.433\gamma_{xy} \qquad (3) \end{aligned}$$



(b)

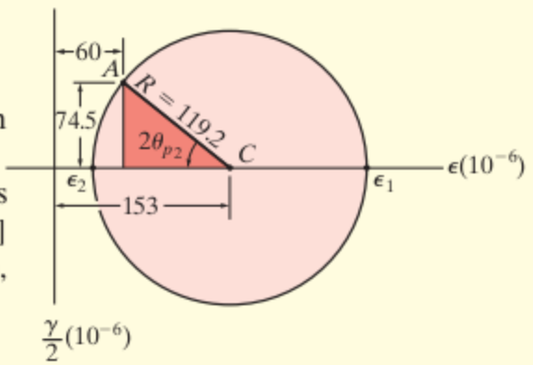
Using Eq. (1) and solving Eq. (2) and (3) simultaneously, we get

$$\epsilon_x = 60(10^{-6}) \quad \epsilon_y = 246(10^{-6}) \quad \gamma_{xy} = -149(10^{-6})$$

These same results can also be obtained in a more direct manner from Eq. 14–29.

The in-plane principal strains can be determined using Mohr's circle. The reference point on the circle is at A $[60(10^{-6}), -74.5(10^{-6})]$ and the center of the circle, C, is on the ϵ axis at $\epsilon_{avg} = 153(10^{-6})$, Fig. 14–39*c*. From the shaded triangle, the radius is

$$R = \left[\sqrt{(153 - 60)^2 + (74.5)^2}\right](10^{-6}) = 119.1(10^{-6})$$



(c)

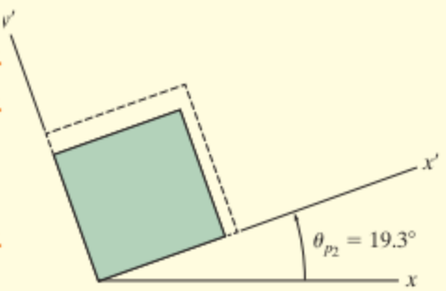
The in-plane principal strains are thus

$$\epsilon_1 = 153(10^{-6}) + 119.1(10^{-6}) = 272(10^{-6}) \quad \textit{Ans.}$$

$$\epsilon_2 = 153(10^{-6}) - 119.1(10^{-6}) = 33.9(10^{-6}) \quad \textit{Ans.}$$

$$2\theta_{p_2} = \tan^{-1}\frac{74.5}{(153 - 60)} = 38.7°$$

$$\theta_{p_2} = 19.3° \quad \textit{Ans.}$$



(d)

Fig. 14–39

NOTE: The deformed element is shown in the dashed position in Fig. 14–39*d*. Realize that, due to the Poisson effect, the element is *also* subjected to an out-of-plane strain, i.e., in the z direction, although this value will not influence the calculated results.

14.10 Material-Property Relationships

In this section we will present some important relationships involving a material's properties that are used when the material is subjected to multiaxial stress and strain. To do so we will assume that the material is homogeneous and isotropic and behaves in a linear elastic manner.

Generalized Hooke's Law. If the material at a point is subjected to a state of triaxial stress, σ_x, σ_y, σ_z, Fig. 14–40*a*, associated normal strains ϵ_x, ϵ_y, ϵ_z will be developed in the material. The stresses can be related to these strains by using the principle of superposition, Poisson's ratio, $\epsilon_{\text{lat}} = -\nu\epsilon_{\text{long}}$, and Hooke's law, as it applies in the uniaxial direction, $\epsilon = \sigma/E$. For example, consider the normal strain of the element in the x direction, caused by separate application of each normal stress. When σ_x is applied, Fig. 14–40*b*, the element elongates in the x direction and the strain ϵ'_x is

$$\epsilon'_x = \frac{\sigma_x}{E}$$

Application of σ_y causes the element to contract with a strain ϵ''_x, Fig. 14–40*c*. Here

$$\epsilon''_x = -\nu\frac{\sigma_y}{E}$$

Likewise, application of σ_z, Fig. 14–40*d*, causes a contraction such that

$$\epsilon'''_x = -\nu\frac{\sigma_z}{E}$$

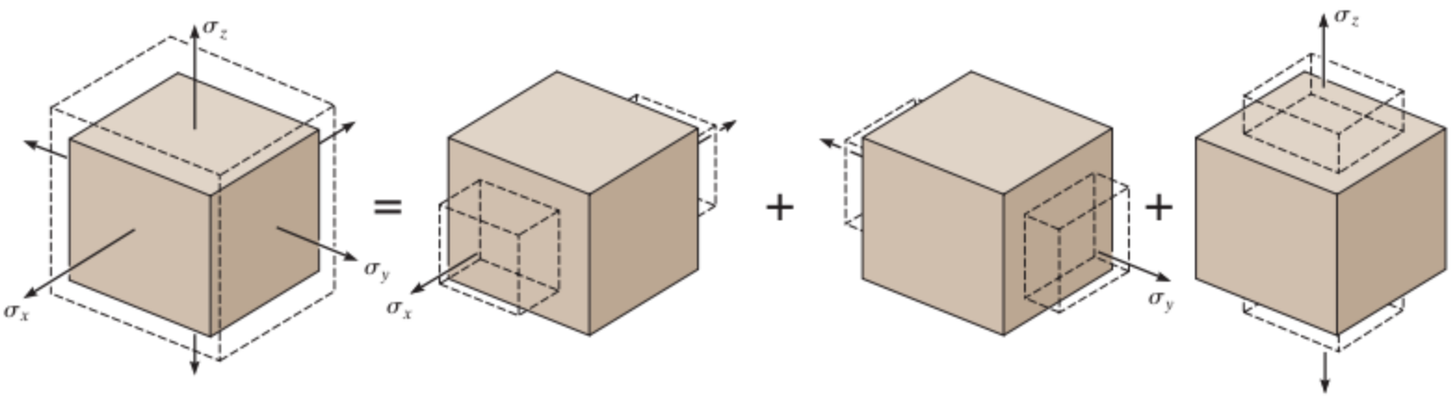


Fig. 14–40

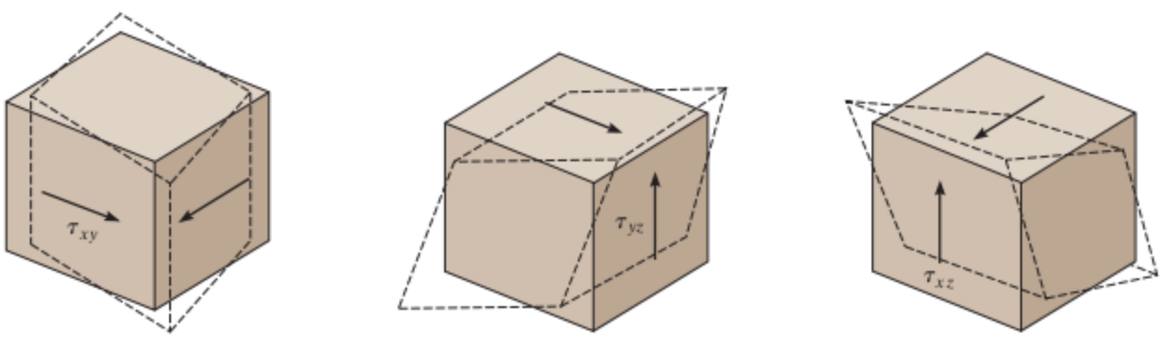
When these three normal strains are superimposed, the normal strain ϵ_x is determined for the state of stress in Fig. 14–40*a*. Similar equations can be developed for the normal strains in the y and z directions. The final results can be written as

$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$
$$\epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \qquad (14\text{–}30)$$
$$\epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

These three equations express Hooke's law in a general form for a triaxial state of stress. For application tensile stresses are considered positive quantities, and compressive stresses are negative. If a resulting normal strain is *positive*, it indicates that the material *elongates*, whereas a *negative* normal strain indicates the material *contracts*.

If we now apply a shear stress τ_{xy} to the element, Fig. 14–40*e*, experimental observations indicate that the material will deform *only* due to a shear strain γ_{xy}; that is, τ_{xy} will not cause other strains in the material. Likewise, τ_{yz} and τ_{xz} will only cause shear strains γ_{yz} and γ_{xz}, Figs. 14–40*f* and 14–40*g*, and so Hooke's law for shear stress and shear strain can be written as

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} \quad \gamma_{yz} = \frac{1}{G}\tau_{yz} \quad \gamma_{xz} = \frac{1}{G}\tau_{xz} \qquad (14\text{–}31)$$



(e) (f) (g)

Fig. 14–40 (cont.)

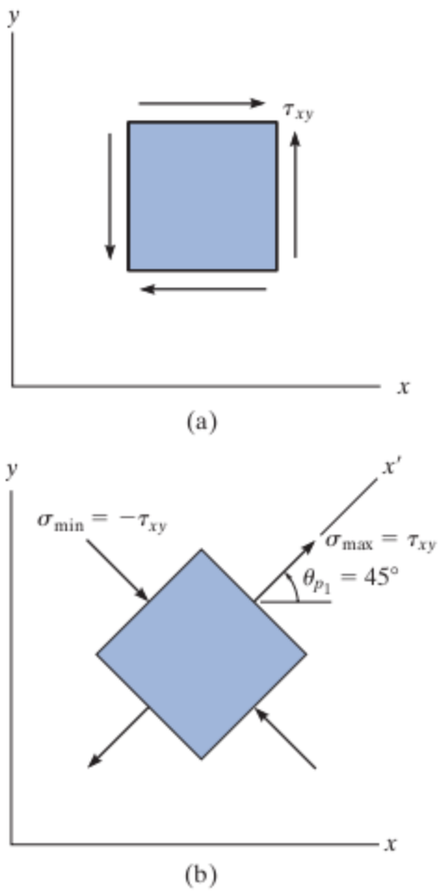


Fig. 14–41

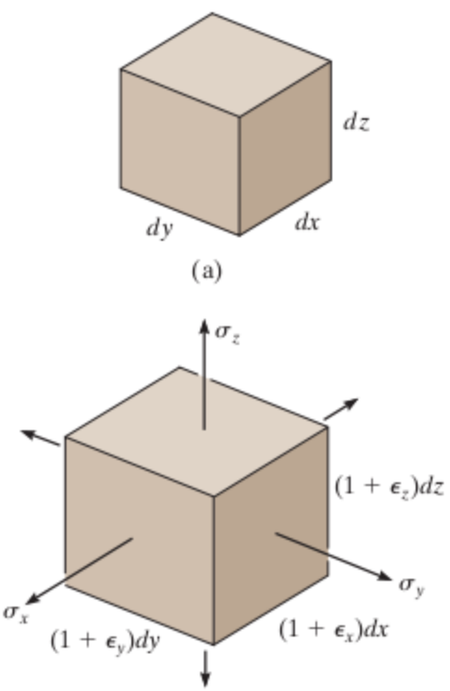


Fig. 14–42

Relationship Involving *E*, *ν*, and *G*. In Sec. 8.7 it was stated that the modulus of elasticity E is related to the shear modulus G by Eq. 8–11, namely,

$$G = \frac{E}{2(1 + \nu)} \tag{14–32}$$

One way to derive this relationship is to consider an element of the material to be subjected to pure shear ($\sigma_x = \sigma_y = \sigma_z = 0$), Fig. 14–41*a*. Applying Eq. 14–5 to obtain the principal stresses yields $\sigma_{max} = \tau_{xy}$ and $\sigma_{min} = -\tau_{xy}$. This element must be oriented $\theta_{p_1} = 45°$ counterclockwise from the x axis as shown in Fig. 14–41*b*. If the three principal stresses $\sigma_{max} = \tau_{xy}$, $\sigma_{int} = 0$, and $\sigma_{min} = -\tau_{xy}$ are substituted into the first of Eqs. 14–30, the principal strain ϵ_{max} can be related to the shear stress τ_{xy}. The result is

$$\epsilon_{max} = \frac{\tau_{xy}}{E}(1 + \nu) \tag{14–33}$$

This strain, which deforms the element along the x' axis, can also be related to the shear strain γ_{xy}. To do this, first note that since $\sigma_x = \sigma_y = \sigma_z = 0$, then from the first and second Eqs. 14–30, $\epsilon_x = \epsilon_y = 0$. Substituting these results into the strain transformation Eq. 14–23, we get

$$\epsilon_1 = \epsilon_{max} = \frac{\gamma_{xy}}{2}$$

By Hooke's law, $\gamma_{xy} = \tau_{xy}/G$, so that $\epsilon_{max} = \tau_{xy}/2G$. Substituting into Eq. 14–33 and rearranging terms gives the final result, namely, Eq. 14–32.

Dilatation and Bulk Modulus. When an elastic material is subjected to normal stress, its volume will change. For example, consider a volume element which is subjected to the principal stresses $\sigma_x, \sigma_y, \sigma_z$. If the sides of the element are originally dx, dy, dz, Fig. 14–42*a*, then after application of the stress they become $(1 + \epsilon_x)\,dx$, $(1 + \epsilon_y)\,dy$, $(1 + \epsilon_z)\,dz$, Fig. 14–42*b*. The change in volume of the element is therefore

$$\delta V = (1 + \epsilon_x)(1 + \epsilon_y)(1 + \epsilon_z)\,dx\,dy\,dz - dx\,dy\,dz$$

Neglecting the products of the strains since the strains are very small, we have

$$\delta V = (\epsilon_x + \epsilon_y + \epsilon_z)\,dx\,dy\,dz$$

The change in volume per unit volume is called the "volumetric strain" or the ***dilatation*** e. It can be written as

$$e = \frac{\delta V}{dV} = \epsilon_x + \epsilon_y + \epsilon_z \tag{14–34}$$

By comparison, the shear strains will *not* change the volume of the element, rather they will only change its rectangular shape.

Also, if we use Hooke's law, as defined by Eqs. 14–30, we can write the dilatation in terms of the applied stress. We have

$$e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) \tag{14–35}$$

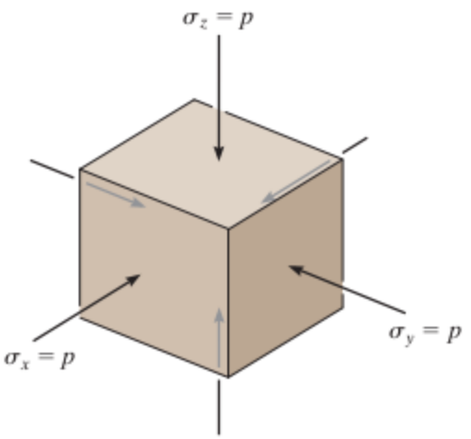
When a volume element of material is subjected to the uniform pressure p of a liquid, the pressure on the body is the same in all directions and is always normal to any surface on which it acts. Shear stresses are *not present*, since the shear resistance of a liquid is zero. This state of "hydrostatic" loading requires the normal stresses to be equal in any and all directions, and therefore an element of the body is subjected to principal stresses $\sigma_x = \sigma_y = \sigma_z = -p$, Fig. 14–43. Substituting into Eq. 14–35 and rearranging terms yields

$$\frac{p}{e} = -\frac{E}{3(1 - 2\nu)} \tag{14–36}$$

Since this ratio is *similar* to the ratio of linear elastic stress to strain, which defines E, i.e., $\sigma/\epsilon = E$, the term on the right is called the *volume modulus of elasticity* or the ***bulk modulus***. It has the same units as stress and will be symbolized by the letter k; that is,

$$k = \frac{E}{3(1 - 2\nu)} \tag{14–37}$$

Note that for most metals $\nu \approx \frac{1}{3}$ so $k \approx E$. If a material existed that did not change its volume then $\delta V = e = 0$, and k would have to be infinite. From Eq. 14–37 the theoretical *maximum* value for Poisson's ratio is therefore $\nu = 0.5$. During yielding, no actual volume change of the material is observed, and so $\nu = 0.5$ is used when plastic yielding occurs.



Hydrostatic stress

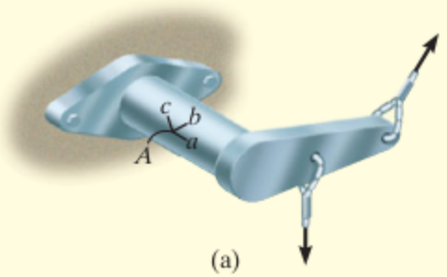
Fig. 14–43

Important Points

- When a homogeneous isotropic material is subjected to a state of triaxial stress, the strain in each direction is influenced by the strains produced by *all* the stresses. This is the result of the Poisson effect, and results in the form of a generalized Hooke's law.
- Unlike normal stress, a shear stress applied to homogeneous isotropic material will only produce shear strain in the same plane.
- The material constants E, G, and ν are related mathematically.
- *Dilatation*, or *volumetric strain*, is caused only by normal strain, not shear strain.
- The *bulk modulus* is a measure of the stiffness of a volume of material. This material property provides an upper limit to Poisson's ratio of $\nu = 0.5$, which remains at this value while plastic yielding occurs.

EXAMPLE 14.19

The bracket in Example 14.18, Fig. 14–44*a*, is made of steel for which $E_{st} = 200\,\text{GPa}$, $\nu_{st} = 0.3$. Determine the principal stresses at point A.



(a)

Fig. 14–44

SOLUTION I

From Example 14.18 the principal strains have been determined as

$$\epsilon_1 = 272(10^{-6})$$

$$\epsilon_2 = 33.9(10^{-6})$$

Since point A is on the *surface* of the bracket for which there is no loading, the stress on the surface is zero, and so point A is subjected to plane stress. Applying Hooke's law with $\sigma_3 = 0$, we have

$$\epsilon_1 = \frac{\sigma_1}{E} - \frac{\nu}{E}\sigma_2; \qquad 272(10^{-6}) = \frac{\sigma_1}{200(10^9)} - \frac{0.3}{200(10^9)}\sigma_2$$

$$54.4(10^6) = \sigma_1 - 0.3\sigma_2 \tag{1}$$

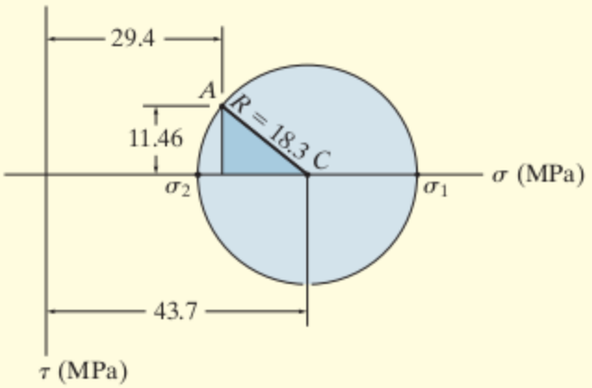
$$\epsilon_2 = \frac{\sigma_2}{E} - \frac{\nu}{E}\sigma_1; \qquad 33.9(10^{-6}) = \frac{\sigma_2}{200(10^9)} - \frac{0.3}{200(10^9)}\sigma_1$$

$$6.78(10^6) = \sigma_2 - 0.3\sigma_1 \tag{2}$$

Solving Eqs. (1) and (2) simultaneously yields

$$\sigma_1 = 62.0\,\text{MPa} \qquad \textit{Ans.}$$

$$\sigma_2 = 25.4\,\text{MPa} \qquad \textit{Ans.}$$



(b)

Fig. 14–44 (cont.)

SOLUTION II

It is also possible to solve the problem using the given state of strain,

$$\epsilon_x = 60(10^{-6}) \quad \epsilon_y = 246(10^{-6}) \quad \gamma_{xy} = -149(10^{-6})$$

as specified in Example 14.18. Applying Hooke's law in the x–y plane, we have

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}\sigma_y; \qquad 60(10^{-6}) = \frac{\sigma_x}{200(10^9)\ \text{Pa}} - \frac{0.3\sigma_y}{200(10^9)\ \text{Pa}}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E}\sigma_x; \qquad 246(10^{-6}) = \frac{\sigma_y}{200(10^9)\ \text{Pa}} - \frac{0.3\sigma_x}{200(10^9)\ \text{Pa}}$$

$$\sigma_x = 29.4\ \text{MPa} \quad \sigma_y = 58.0\ \text{MPa}$$

The shear stress is determined using Hooke's law for shear. First, however, we must calculate G.

$$G = \frac{E}{2(1+\nu)} = \frac{200\ \text{GPa}}{2(1+0.3)} = 76.9\ \text{GPa}$$

Thus,

$$\tau_{xy} = G\gamma_{xy}; \qquad \tau_{xy} = 76.9(10^9)[-149(10^{-6})] = -11.46\ \text{MPa}$$

The Mohr's circle for this state of plane stress has a reference point A(29.4 MPa, −11.46 MPa) and center at $\sigma_{avg} = 43.7$ MPa, Fig. 14–44*b*. The radius is determined from the shaded triangle.

$$R = \sqrt{(43.7 - 29.4)^2 + (11.46)^2} = 18.3\ \text{MPa}$$

Therefore,

$$\sigma_1 = 43.7\ \text{MPa} + 18.3\ \text{MPa} = 62.0\ \text{MPa} \qquad \textit{Ans.}$$

$$\sigma_2 = 43.7\ \text{MPa} - 18.3\ \text{MPa} = 25.4\ \text{MPa} \qquad \textit{Ans.}$$

NOTE: Each of these solutions is valid provided the material is both linear elastic and isotropic, since then the principal planes of stress and strain coincide.

EXAMPLE 14.20

The copper bar in Fig. 14–45 is subjected to a uniform loading along its edges as shown. If it has a length $a = 300$ mm, width $b = 50$ mm, and thickness $t = 20$ mm before the load is applied, determine its new length, width, and thickness after application of the load. Take $E_{cu} = 120$ GPa, $\nu_{cu} = 0.34$.

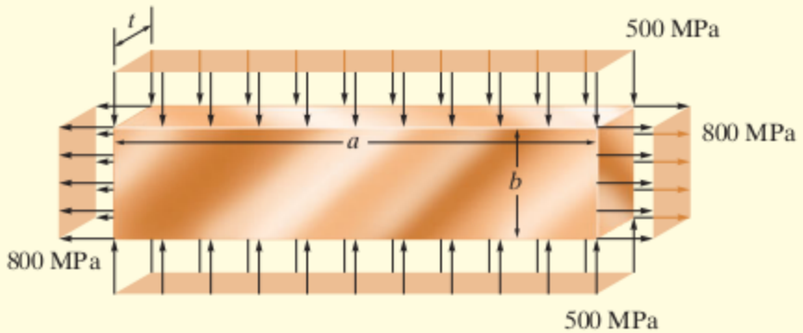


Fig. 14–45

SOLUTION

By inspection, the bar is subjected to a state of plane stress. From the loading we have

$$\sigma_x = 800\text{ MPa} \quad \sigma_y = -500\text{ MPa} \quad \tau_{xy} = 0 \quad \sigma_z = 0$$

The associated normal strains are determined from the generalized Hooke's law, Eq. 14–30; that is,

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z)$$

$$= \frac{800\text{ MPa}}{120(10^3)\text{ MPa}} - \frac{0.34}{120(10^3)\text{ MPa}}(-500\text{ MPa} + 0) = 0.00808$$

$$\epsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_x + \sigma_z)$$

$$= \frac{-500\text{ MPa}}{120(10^3)\text{ MPa}} - \frac{0.34}{120(10^3)\text{ MPa}}(800\text{ MPa} + 0) = -0.00643$$

$$\epsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y)$$

$$= 0 - \frac{0.34}{120(10^3)\text{ MPa}}(800\text{ MPa} - 500\text{ MPa}) = -0.000850$$

The new bar length, width, and thickness are therefore

$$a' = 300\text{ mm} + 0.00808(300\text{ mm}) = 302.4\text{ mm} \quad \textit{Ans.}$$

$$b' = 50\text{ mm} + (-0.00643)(50\text{ mm}) = 49.68\text{ mm} \quad \textit{Ans.}$$

$$t' = 20\text{ mm} + (-0.000850)(20\text{ mm}) = 19.98\text{ mm} \quad \textit{Ans.}$$

EXAMPLE 14.21

If the rectangular block shown in Fig. 14–46 is subjected to a uniform pressure of $p = 20$ psi, determine the dilatation and the change in length of each side. Take $E = 600$ psi, $\nu = 0.45$.

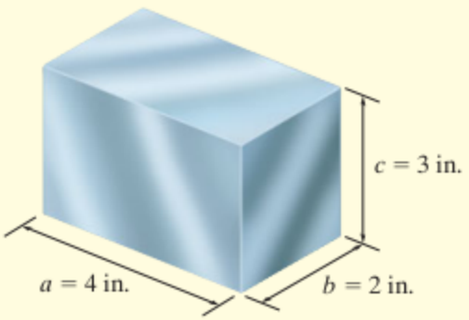


Fig. 14–46

SOLUTION

Dilatation. The dilatation can be determined using Eq. 14–35 with $\sigma_x = \sigma_y = \sigma_z = -20$ psi. We have

$$e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z)$$

$$= \frac{1 - 2(0.45)}{600\ \text{psi}}[3(-20\ \text{psi})]$$

$$= -0.01\ \text{in}^3/\text{in}^3 \qquad \textit{Ans.}$$

Change in Length. The normal strain on each side can be determined from Hooke's law, Eq. 14–30; that is,

$$\epsilon = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$= \frac{1}{600\ \text{psi}}[-20\ \text{psi} - (0.45)(-20\ \text{psi} - 20\ \text{psi})] = -0.00333\ \text{in./in.}$$

Thus, the change in length of each side is

$$\delta a = -0.00333(4\ \text{in.}) = -0.0133\ \text{in.} \qquad \textit{Ans.}$$

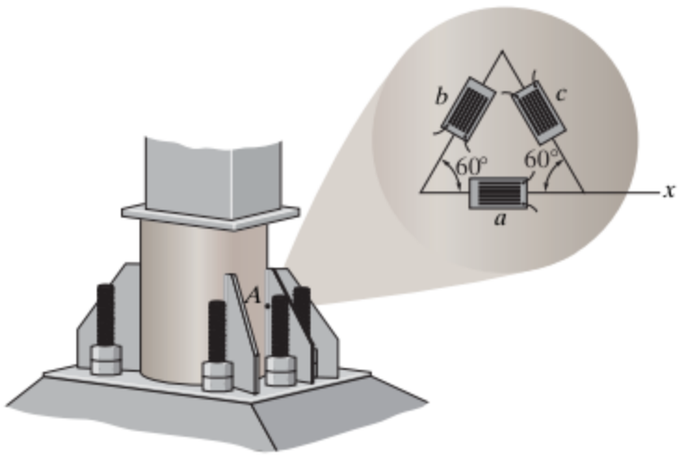
$$\delta b = -0.00333(2\ \text{in.}) = -0.00667\ \text{in.} \qquad \textit{Ans.}$$

$$\delta c = -0.00333(3\ \text{in.}) = -0.0100\ \text{in.} \qquad \textit{Ans.}$$

The negative signs indicate that each dimension is decreased.

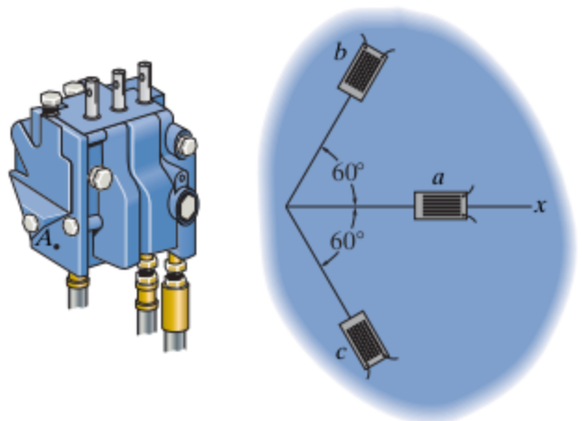
PROBLEMS

***14–84.** The 60° strain rosette is attached to point A on the surface of the support. Due to the loading the strain gauges give a reading of $\epsilon_a = 300(10^{-6})$, $\epsilon_b = -150(10^{-6})$, and $\epsilon_c = -450(10^{-6})$. Use Mohr's circle and determine (a) the in-plane principal strains and (b) the maximum in-plane shear strain and the associated average normal strain. Specify the orientation of each element that has these states of strain with respect to the x axis.



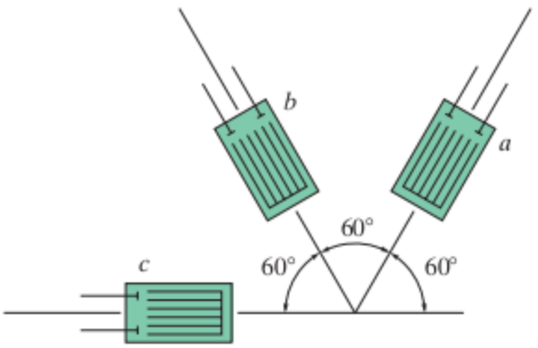
Prob. 14–84

14–85. The strain rosette is attached at the point on the surface of the pump. Due to the loading, the strain gauges give a reading of $\epsilon_a = -250(10^{-6})$, $\epsilon_b = 300(10^{-6})$, and $\epsilon_c = -200(10^{-6})$. Determine (a) the in-plane principal strains, and (b) the maximum in-plane shear strain. Specify the orientation of each element that has these states of strain with respect to the x axis.



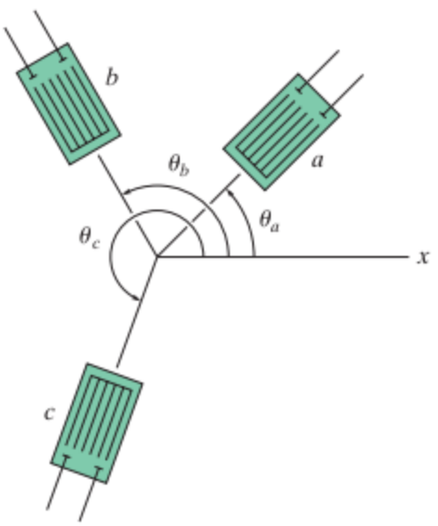
Prob. 14–85

14–86. The 60° strain rosette is mounted on a beam. The following readings are obtained from each gauge: $\epsilon_a = 250(10^{-6})$, $\epsilon_b = -400(10^{-6})$, $\epsilon_c = 280(10^{-6})$. Determine (a) the in-plane principal strains and their orientation, and (b) the maximum in-plane shear strain and average normal strain. In each case show the deformed element due to these strains.



Prob. 14–86

14–87. Consider the general orientation of three strain gauges at a point as shown. Write a computer program that can be used to determine the principal in-plane strains and the maximum in-plane shear strain at the point. Show an application of the program using the values $\theta_a = 40°$, $\epsilon_a = 160(10^{-6})$, $\theta_b = 125°$, $\epsilon_b = 100(10^{-6})$, $\theta_c = 220°$, $\epsilon_c = 80(10^{-6})$.



Prob. 14–87

*14–88. For the case of plane stress, show that Hooke's law can be written as

$$\sigma_x = \frac{E}{(1-\nu^2)}(\epsilon_x + \nu\epsilon_y), \quad \sigma_y = \frac{E}{(1-\nu^2)}(\epsilon_y + \nu\epsilon_x)$$

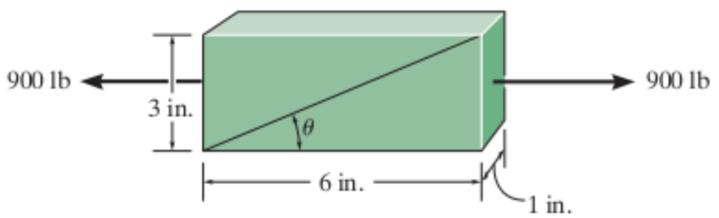
14–89. Use Hooke's law, Eq. 14–30, to develop the strain-transformation equations, Eqs. 14–19 and 14–20, from the stress-transformation equations, Eqs. 14–1 and 14–2.

14–90. The principal plane stresses and associated strains in a plane at a point are $\sigma_1 = 36$ ksi, $\sigma_2 = 16$ ksi, $\epsilon_1 = 1.02(10^{-3})$, $\epsilon_2 = 0.180(10^{-3})$. Determine the modulus of elasticity and Poisson's ratio.

14–91. A rod has a radius of 10 mm. If it is subjected to an axial load of 15 N such that the axial strain in the rod is $\epsilon_x = 2.75(10^{-6})$, determine the modulus of elasticity E and the change in its diameter. $\nu = 0.23$.

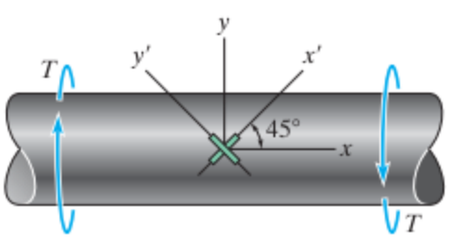
*14–92. The polyvinyl chloride bar is subjected to an axial force of 900 lb. If it has the original dimensions shown determine the *change* in the angle θ after the load is applies. $E_{pvc} = 800(10^3)$ psi, $\nu_{pvc} = 0.20$.

14–93. The polyvinyl chloride bar is subjected to an axial force of 900 lb. If it has the original dimensions shown determine the value of Poisson's ratio if the angle θ decreases by $\Delta\theta = 0.01°$ after the load is applied. $E_{pvc} = 800(10^3)$ psi.



Probs. 14–92/93

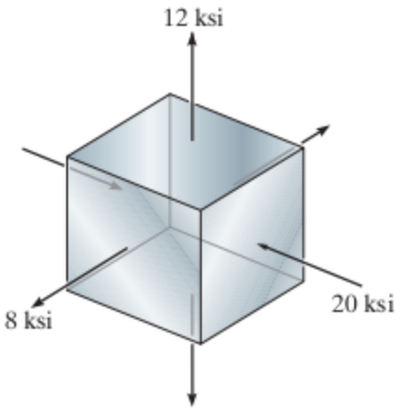
14–94. The steel shaft has a radius of 15 mm. Determine the torque T in the shaft if the two strain gauges, attached to the surface of the shaft, report strains of $\epsilon_{x'} = -80(10^{-6})$ and $\epsilon_{y'} = 80(10^{-6})$. Also, compute the strains acting in the x and y directions. $E_{st} = 200$ GPa, $\nu_{st} = 0.3$.



Prob. 14–94

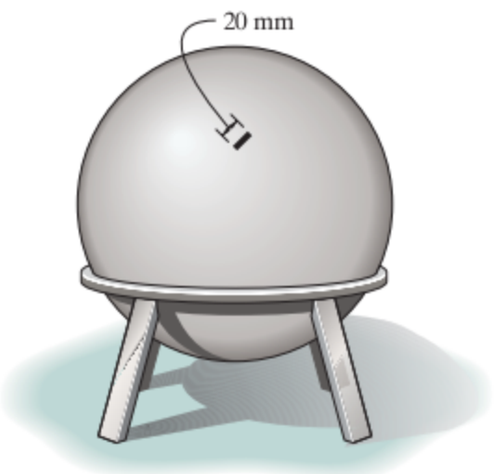
14–95. Determine the bulk modulus for each of the following materials: (a) rubber, $E_r = 0.4$ ksi, $\nu_r = 0.48$, and (b) glass, $E_g = 8(10^3)$ ksi, $\nu_g = 0.24$.

*14–96. The principal stresses at a point are shown in the figure. If the material is A-36 steel, determine the principal strains.



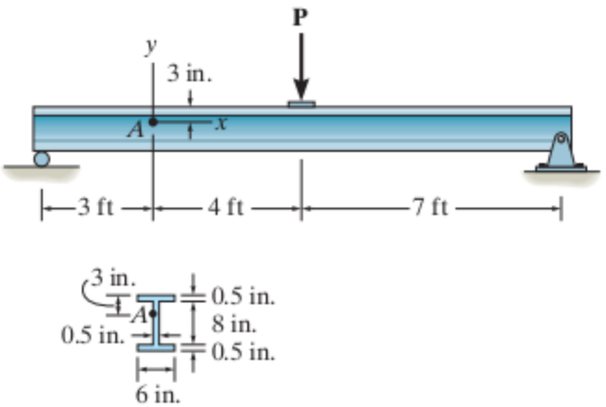
Prob. 14–96

14–97. The spherical pressure vessel has an inner diameter of 2 m and a thickness of 10 mm. A strain gauge having a length of 20 mm is attached to it, and it is observed to increase in length by 0.012 mm when the vessel is pressurized. Determine the pressure causing this deformation, and find the maximum in-plane shear stress, and the absolute maximum shear stress at a point on the outer surface of the vessel. The material is steel, for which $E_{st} = 200$ GPa and $\nu_{st} = 0.3$.



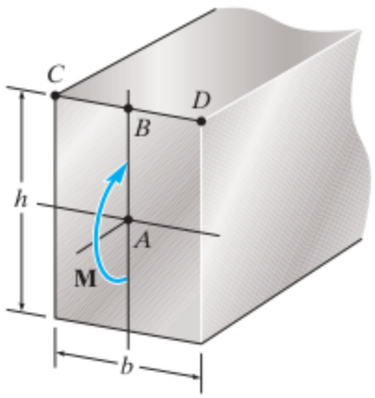
Prob. 14–97

14–98. The strain in the x direction at point A on the steel beam is measured and found to be $\epsilon_x = -100(10^{-6})$. Determine the applied load P. What is the shear strain γ_{xy} at point A? $E_{st} = 29(10^3)$ ksi, $\nu_{st} = 0.3$.



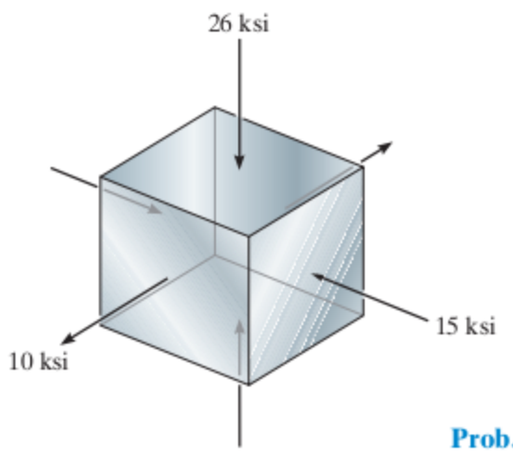
Prob. 14–98

14–99. The cross section of the rectangular beam is subjected to the bending moment **M**. Determine an expression for the increase in length of lines AB and CD. The material has a modulus of elasticity E and Poisson's ratio is ν.



Prob. 14–99

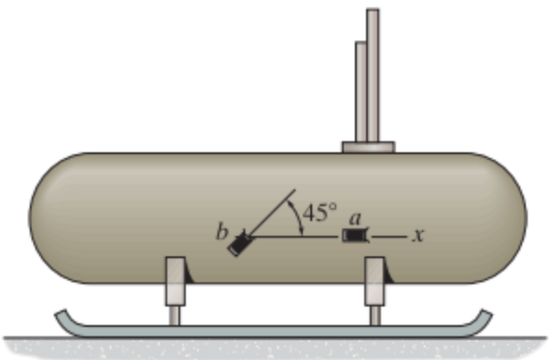
***14–100.** The principal stresses at a point are shown in the figure. If the material is aluminum for which $E_{al} = 10(10^3)$ ksi and $\nu_{al} = 0.33$, determine the principal strains.



Prob. 14–100

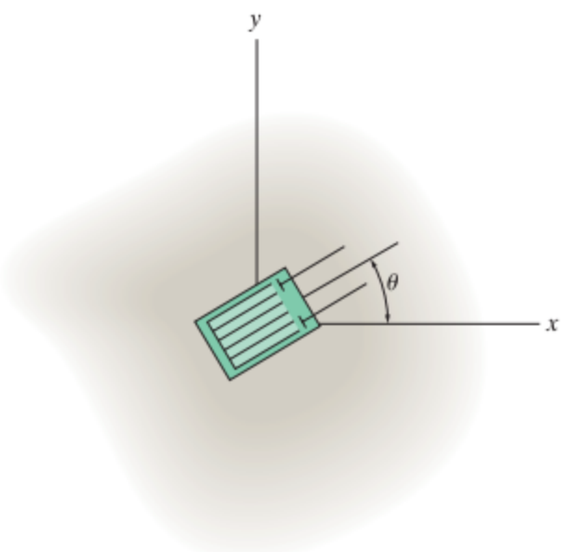
14–101. A strain gauge a is attached in the longitudinal direction (x axis) on the surface of the gas tank. When the tank is pressurized, the strain gauge gives a reading of $\epsilon_a = 100(10^{-6})$. Determine the pressure p in the tank. The tank has an inner diameter of 1.5 m and wall thickness of 25 mm. It is made of steel having a modulus of elasticity $E = 200$ GPa and Poisson's ratio $\nu = \frac{1}{3}$.

14–102. Strain gauge b is attached to the surface of the gas tank at an angle of 45° with x axis as shown. When the tank is pressurized, the strain gauge gives a reading of $\epsilon_b = 250(10^{-6})$. Determine the pressure in the tank. The tank has an inner diameter of 1.5 m and wall thickness of 25 mm. It is made of steel having a modulus of elasticity $E = 200$ GPa and Poisson's ratio $\nu = \frac{1}{3}$.



Probs. 14–101/102

14–103. A material is subjected to principal stresses σ_x and σ_y. Determine the orientation θ of a strain gauge placed at the point so that its reading of normal strain responds only to σ_y and not σ_x. The material constants are E and ν.



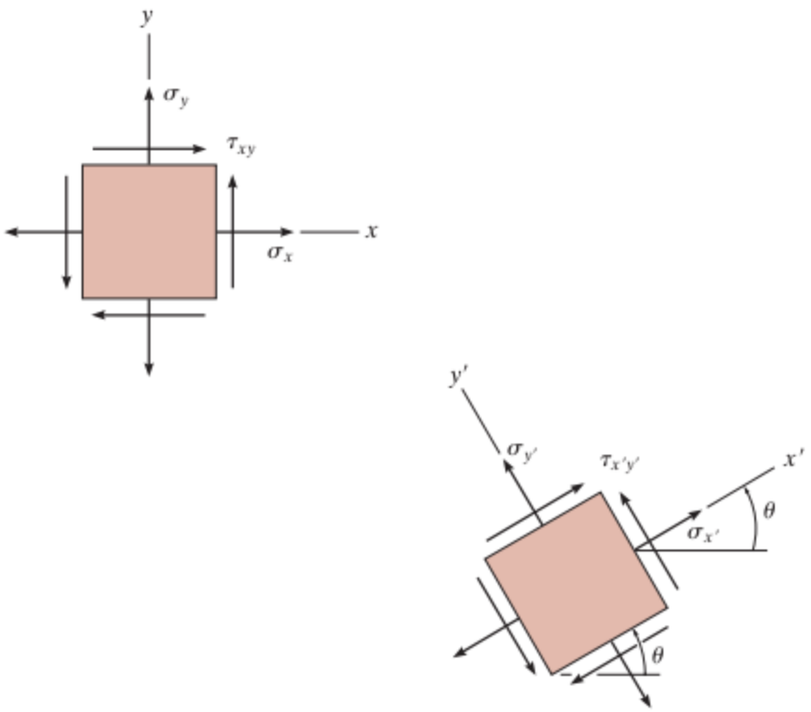
Prob. 14–103

CHAPTER REVIEW

Plane stress occurs when the material at a point is subjected to two normal stress components σ_x and σ_y and a shear stress τ_{xy}. Provided these components are known, then the stress components acting on an element having a different orientation θ can be determined using the two force equations of equilibrium or the equations of stress transformation.

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$



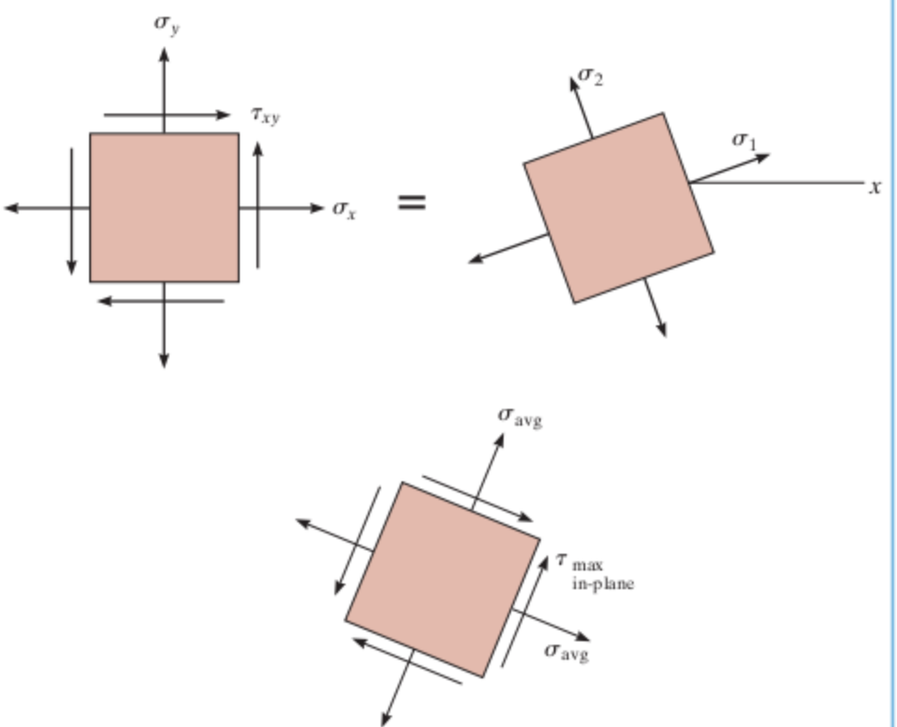
For design, it is important to determine the orientation of the element that produces the maximum principal normal stresses and the maximum in-plane shear stress. Using the stress transformation equations, it is found that no shear stress acts on the planes of principal stress. The principal stresses are

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

The planes of maximum in-plane shear stress are oriented 45° from this orientation, and on these shear planes there is an associated average normal stress.

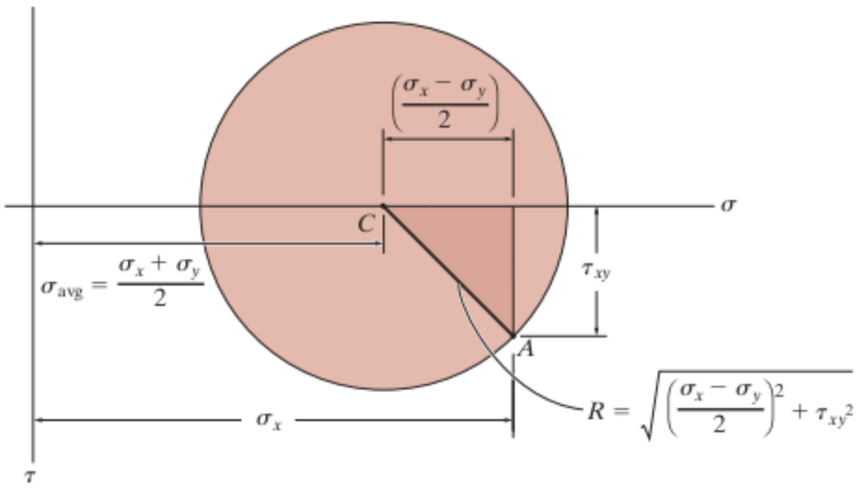
$$\tau_{\text{max in-plane}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2}$$



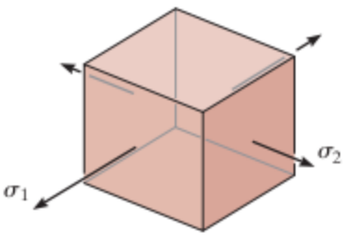
14

Mohr's circle provides a semi-graphical method for finding the stress on any plane, the principal normal stresses, and the maximum in-plane shear stress. To draw the circle, the σ and τ axes are established, the center of the circle $C[(\sigma_x + \sigma_y)/2, 0]$ and the reference point $A(\sigma_x, \tau_{xy})$ are plotted. The radius R of the circle extends between these two points and is determined from trigonometry.



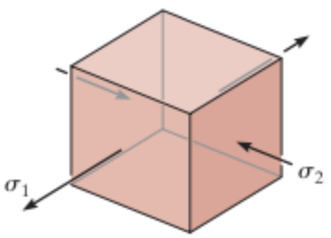
If σ_1 and σ_2 are of the same sign, then the absolute maximum shear stress will lie out of plane.

In the case of plane stress, the absolute maximum shear stress will be equal to the maximum in-plane shear stress provided the principal stresses σ_1 and σ_2 have the opposite sign.



x–y plane stress

$$\tau_{\text{max}}^{\text{abs}} = \frac{\sigma_1}{2}$$



x–y plane stress

$$\tau_{\text{max}}^{\text{abs}} = \frac{\sigma_1 - \sigma_2}{2}$$

When an element of material is subjected to deformations that only occur in a single plane, then it undergoes plane strain. If the strain components ϵ_x, ϵ_y, and γ_{xy} are known for a specified orientation of the element, then the strains acting for some other orientation of the element can be determined using the plane-strain transformation equations. Likewise, the principal normal strains and maximum in-plane shear strain can be determined using transformation equations.

$$\epsilon_{x'} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

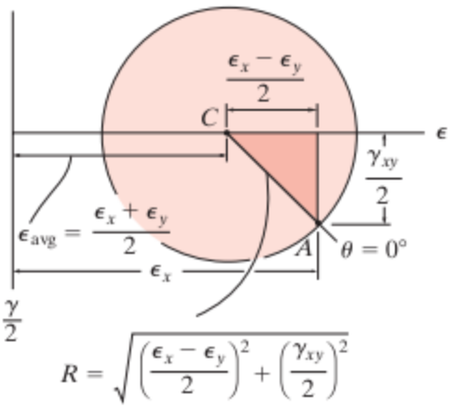
$$\epsilon_{y'} = \frac{\epsilon_x + \epsilon_y}{2} - \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\frac{\gamma_{x'y'}}{2} = -\left(\frac{\epsilon_x - \epsilon_y}{2}\right)\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\frac{(\gamma)_{\text{in-plane}}^{\text{max}}}{2} = \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

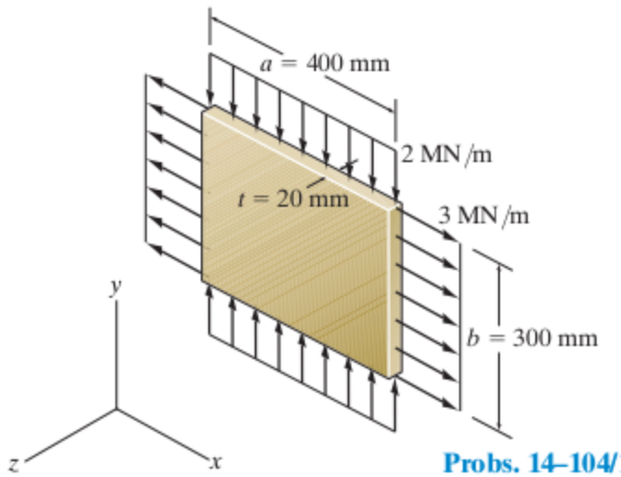
$$\epsilon_{\text{avg}} = \frac{\epsilon_x + \epsilon_y}{2}$$

Strain transformation problems can also be solved in a semi-graphical manner using Mohr's circle. To draw the circle, the ϵ and $\gamma/2$ axes are established and the center of the circle $C[(\epsilon_x + \epsilon_y)/2, 0]$ and the "reference point" A $(\epsilon_x, \gamma_{xy}/2)$ are plotted. The radius of the circle extends between these two points and is determined from trigonometry.		
If the material is subjected to triaxial stress, then the strain in each direction is influenced by the strain produced by all three stresses. Hooke's law then involves the material properties E and ν.	$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$ $\epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)]$ $\epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$	
If E and ν are known, then G can be determined.	$G = \frac{E}{2(1 + \nu)}$	
The dilatation is a measure of volumetric strain. The bulk modulus is used to measure the stiffness of a volume of material.	$e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z)$ $k = \frac{E}{3(1 - 2\nu)}$	

REVIEW PROBLEMS

***14–104.** In the case of plane stress, where the in-plane principal strains are given by ϵ_1 and ϵ_2, show that the third principal strain can be obtained from $\epsilon_3 = -[\nu/(1-\nu)](\epsilon_1 + \epsilon_2)$, where ν is Poisson's ratio for the material.

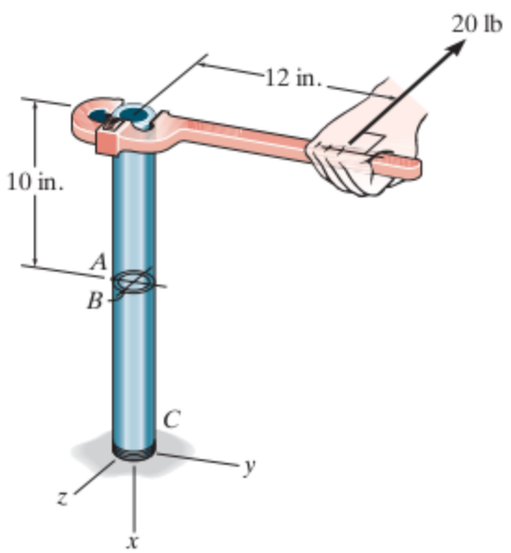
14–105. The plate is made of material having a modulus of elasticity $E = 200$ GPa and Poisson's ratio $\nu = \frac{1}{3}$. Determine the change in width a, height b, and thickness t when it is subjected to the uniform distributed loading shown.



Probs. 14–104/105

14–106. The steel pipe has an inner diameter of 2.75 in. and an outer diameter of 3 in. If it is fixed at C and subjected to the horizontal 20-lb force acting on the handle of the pipe wrench at its end, determine the principal stresses in the pipe at point A, which is located on the surface of the pipe.

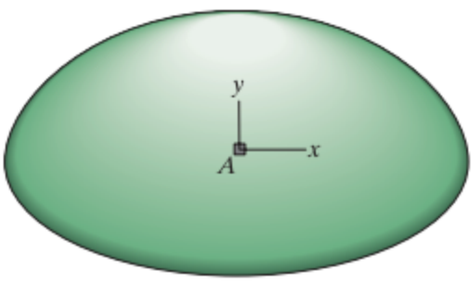
14–107. Solve Prob. 14–106 for point B, which is located on the surface of the pipe.



Probs. 14–106/107

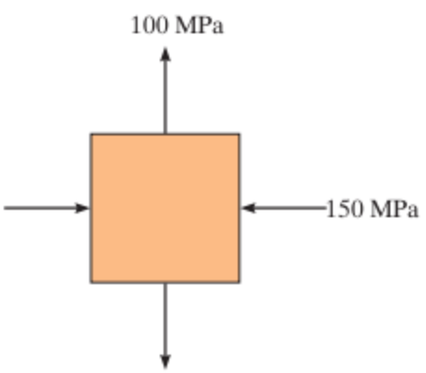
***14–108.** A thin-walled spherical pressure vessel has an inner radius r, thickness t, and is subjected to an internal pressure p. If the material constants are E and ν, determine the strain in the circumferential direction in terms of the stated parameters.

14–109. The strain at point A on the shell has components $\epsilon_x = 250(10^{-6})$, $\epsilon_y = 400(10^{-6})$, $\gamma_{xy} = 275(10^{-6})$, $\epsilon_z = 0$. Determine (a) the principal strains at A, (b) the maximum shear strain in the x–y plane, and (c) the absolute maximum shear strain.



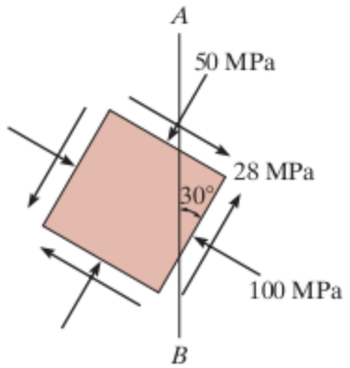
Prob. 14–109

14–110. The princpal plane stresses acting a point are shown in the figure If the material is machine steel having a yield stress of $\sigma_Y = 500$ MPa, determine the factor of safety with respect to yielding if the maximum-shear-stress theory is considered.



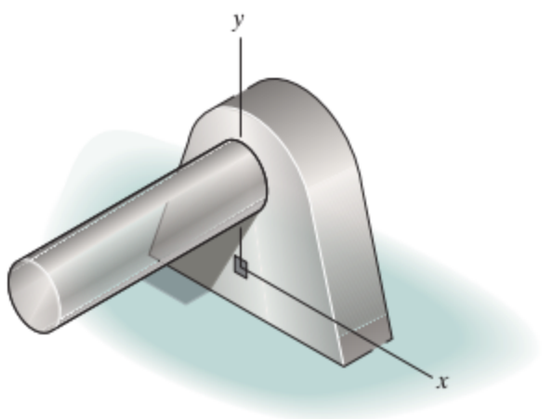
Prob. 14–110

14–111. The state of stress at a point in a member is shown on the element. Determine the stress components acting on the plane AB.



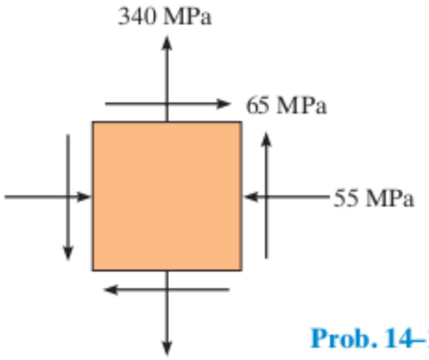
Prob. 14–111

*14–112. The state of strain at the point on the bracket has components $\epsilon_x = 350(10^{-6})$, $\epsilon_y = -860(10^{-6})$, $\gamma_{xy} = 250(10^{-6})$. Use the strain-transformation equations to determine the equivalent in-plane strains on an element oriented at an angle of $\theta = 45°$ clockwise from the original position. Sketch the deformed element within the x–y plane due to these strains.



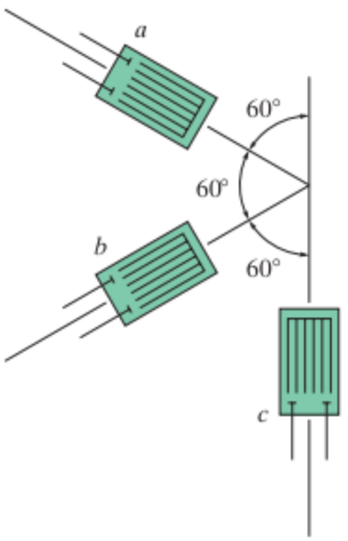
Prob. 14–112

14–113. The components of plane stress at a critical point on a thin steel shell are shown. Determine if failure (yielding) has occurred on the basis of the maximum-distortion-energy theory. The yield stress for the steel is $\sigma_Y = 650$ MPa.



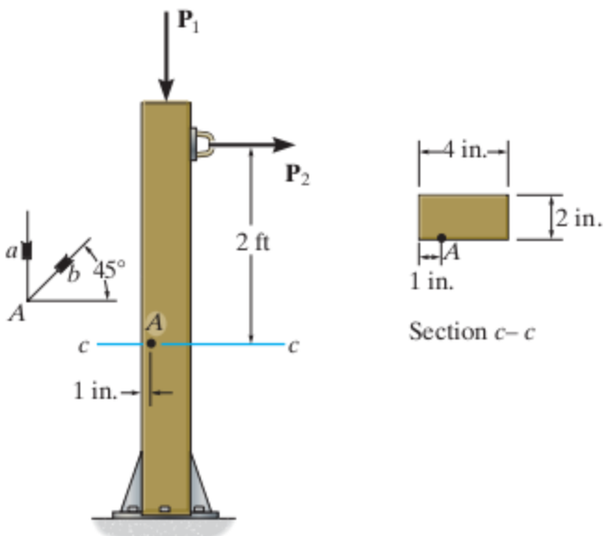
Prob. 14–113

14–114. The 60° strain rosette is mounted on a beam. The following reading are obtained for each gauge; $\epsilon_a = 600(10^{-6})$, $\epsilon_b = -700(10^{-6})$, and $\epsilon_c = 350(10^{-6})$, Determine (a) the in-plane principal strains and (b) the maximum in plane shear strain and average normal strain. In each case show the deformed element due to these stains.



Prob. 14–114

14–115. The A-36 steel post is subjected to the forces shown. If the strain gauges a and b at point A give readings of $\epsilon_a = 300(10^{-6})$ and $\epsilon_b = 175(10^{-6})$, determine the magnitudes of $\mathbf{P}_1$ and $\mathbf{P}_2$.



Prob. 14–115

Chapter 15

Beams are important structural members that are used to support roof and floor loadings.

Design of Beams and Shafts

CHAPTER OBJECTIVES

- To develop methods for designing beams to resist both bending and shear loads.
- To develop methods for determining the deflection of beams subject to bending and shear loads.
- To use beam deflection methods to solve statically indeterminate equilibrium problems.

15.1 Basis for Beam Design

Beams are said to be designed on the basis of strength so that they can resist the internal shear and moment developed along their length. To design a beam in this way requires application of the shear and flexure formulas provided the material is homogeneous and has linear elastic behavior. Although some beams may also be subjected to an axial force, the effects of this force are often neglected in design since the axial stress is generally much smaller than the stress developed by shear and bending.

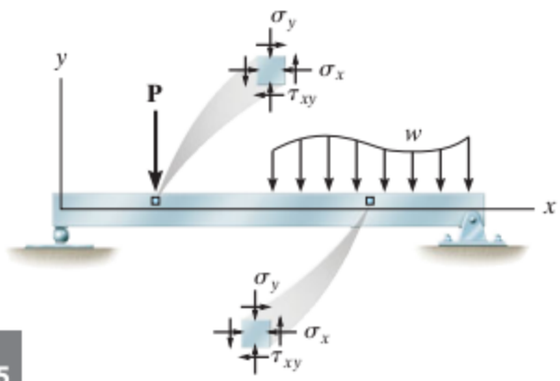


Fig. 15–1

As shown in Fig. 15–1, the external loadings on a beam will create additional stresses in the beam *directly under the load*. Notably, a compressive stress σ_y will be developed, in addition to the bending stress σ_x and shear stress τ_{xy} discussed previously. Using advanced methods of analysis, as treated in the theory of elasticity, it can be shown that σ_y diminishes rapidly throughout the beam's depth, and for *most* beam span-to-depth ratios used in engineering practice, the maximum value of σ_y generally represents only a small percentage compared to the bending stress σ_x, that is, $\sigma_x \gg \sigma_y$. Furthermore, the direct application of concentrated loads is generally avoided in beam design. Instead, *bearing plates* are used to spread these loads more evenly onto the surface of the beam.

Although beams are designed mainly for strength, they must also be braced properly along their sides so that they do not buckle or suddenly become unstable. Furthermore, in some cases beams must be designed to resist a limited amount of *deflection*, as when they support ceilings made of brittle materials such as plaster. Methods for finding beam deflections will be discussed in Chapter 16, and limitations placed on beam buckling are often discussed in codes related to structural or mechanical design.

Since the shear and flexure formulas are used for beam design, we will discuss the general results obtained when these equations are applied to various points in a cantilevered beam that has a rectangular cross section and supports a load **P** at its end, Fig. 15–2*a*.

In general, at an arbitrary section *a–a* along the beam's axis, Fig. 15–2*b*, the internal shear **V** and moment **M** are developed from a *parabolic* shear-stress distribution, and a *linear* normal-stress distribution, Fig. 15–2*c*. As a result, the stresses acting on elements located at points 1 through 5 along the section will be as shown in Fig. 15–2*d*. Note that elements 1 and 5 are subjected only to the maximum normal stress, whereas element 3, which is on the neutral axis, is subjected only to the maximum shear stress. The intermediate elements 2 and 4 resist *both* normal and shear stress.

In each case the state of stress can be transformed into *principal stresses*, using either the stress-transformation equations or Mohr's circle. The results are shown in Fig. 15–2*e*. Here each successive element, 1 through 5, undergoes a counterclockwise orientation. Specifically, relative to element 1, considered to be at the 0° position, element 3 is oriented at 45° and element 5 is oriented at 90°.

Whenever large shear loads occur on a beam it is important to use stiffeners such as at *A*, in order to prevent any localized failure such as crimping of the beam flanges.

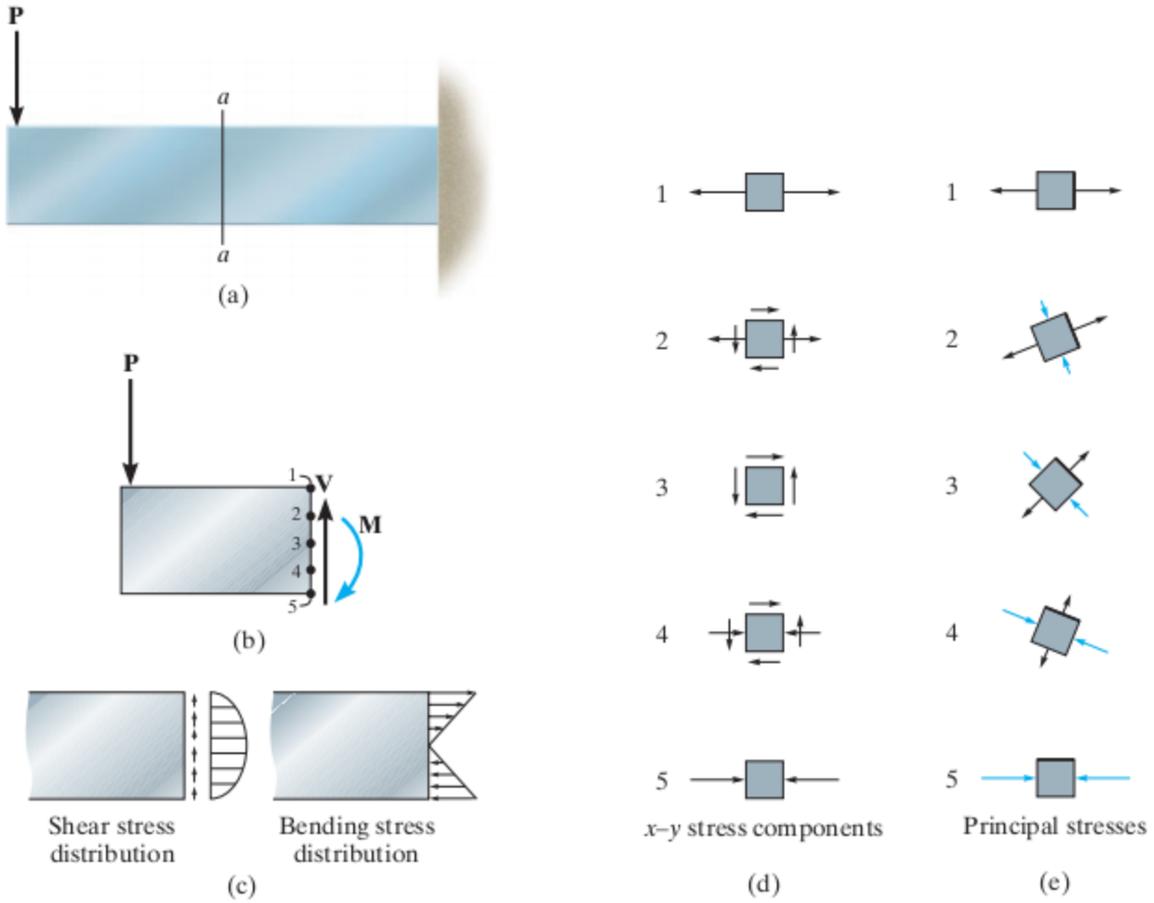
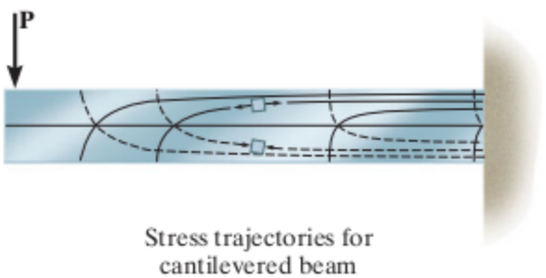


Fig. 15–2



Stress trajectories for cantilevered beam

Fig. 15–3

If this analysis is extended to many vertical sections along the beam other than *a–a*, a profile of the results can be represented by curves called *stress trajectories*. Each of these curves indicates the *direction* of a principal stress having a constant magnitude. Some of these trajectories are shown for the cantilevered beam in Fig. 15–3. Here the solid lines represent the direction of the tensile principal stresses and the dashed lines represent the direction of the compressive principal stresses. As expected, the lines intersect the neutral axis at 45° angles (like element 3) and the solid and dashed lines will intersect at 90° because the principal stresses are always 90° apart. Knowing the direction of these lines can help engineers decide where to reinforce a beam if it is made of brittle material so that it does not crack or become unstable.

15.2 Prismatic Beam Design

Most beams are made of ductile materials and when this is the case it is generally not necessary to plot the stress trajectories for the beam. Instead, it is simply necessary to be sure the actual bending stress and shear stress in the beam do not exceed allowable bending and shear stress for the material as defined by structural or mechanical codes. In the majority of cases the suspended span of the beam will be relatively long, so that the internal moments become large. When this occurs the engineer will first consider a design based upon bending and then check the shear strength. A bending design requires a determination of the beam's ***section modulus***, a geometric property which is the ratio of I and c, that is, $S = I/c$. Using the flexure formula, $\sigma = Mc/I$, we have

$$S_{\text{req'd}} = \frac{M_{\max}}{\sigma_{\text{allow}}} \tag{15–1}$$

Here M is determined from the beam's moment diagram, and the allowable bending stress, σ_{allow}, is specified in a design code. In many cases the beam's as yet unknown weight will be small and can be neglected in comparison with the loads the beam must carry. However, if the additional moment caused by the weight is to be included in the design, a selection for S is made so that it slightly *exceeds* $S_{\text{req'd}}$.

Once $S_{\text{req'd}}$ is known, if the beam has a simple cross-sectional shape, such as a square, a circle, or a rectangle of known width-to-height proportions, its *dimensions* can be determined directly from $S_{\text{req'd}}$, since $S_{\text{req'd}} = I/c$. However, if the cross section is made from several elements, such as a wide-flange section, then an infinite number of web and flange dimensions can be determined that satisfy the value of $S_{\text{req'd}}$. In practice, however, engineers choose a particular beam meeting the requirement that $S > S_{\text{req'd}}$ from a handbook that lists the standard shapes available from manufacturers. Often several beams that have the same section modulus can be selected from these tables. If deflections are not restricted, usually the beam having the smallest cross-sectional area is chosen, since it is made of less material and is therefore both lighter and more economical than the others.

The two floor beams are connected to the beam AB, which transmits the load to the columns of this building frame. For design, all the connections can be considered to act as pins.

Once the beam has been selected, the shear formula $\tau_{allow} \geq VQ/It$ can then be used to check that the allowable shear stress is not exceeded. Often this requirement will not present a problem. However, if the beam is "short" and supports large concentrated loads, the shear-stress limitation may dictate the size of the beam. This limitation is particularly important in the design of wood beams, because wood tends to split along its grain due to shear (see Fig. 12–10*e*).

Fabricated Beams. Since beams are often made of steel or wood, we will now discuss some of the tabulated properties of beams made from these materials.

Steel Sections. Most manufactured steel beams are produced by rolling a hot ingot of steel until the desired shape is formed. These so-called *rolled shapes* have properties that are tabulated in the American Institute of Steel Construction (AISC) manual. A representative listing for wide-flange beams taken from this manual is given in Appendix C. As noted in this appendix, the wide-flange shapes are designated by their depth and weight per unit length; for example, W18 × 46 indicates a wide-flange cross section (W) having a depth of 18 in. and a weight of 46 lb/ft, Fig. 15–4. For any given section, the weight per length, dimensions, cross-sectional area, moment of inertia, and section modulus are reported. Also included is the radius of gyration r, which is a geometric property related to the section's buckling strength. This will be discussed in Chapter 17. Appendix C and the *AISC Manual* also list data for other members such as channels and angles.

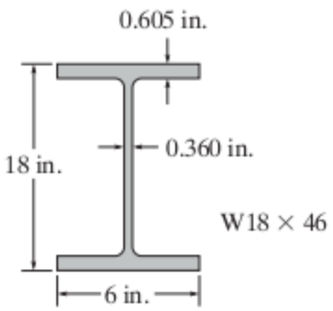
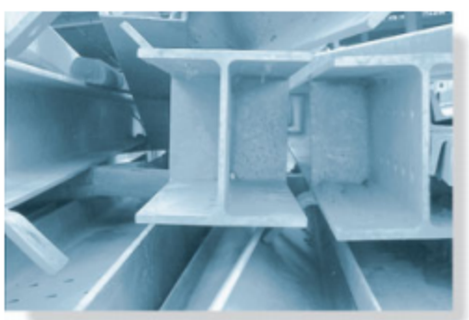
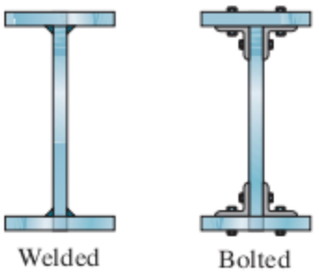


Fig. 15–4



Typical profile view of a steel wide-flange beam.

Wood Sections. Most beams made of wood have rectangular cross sections because such beams are easy to manufacture and handle. Manuals, such as that of the National Forest Products Association, list the dimensions of lumber often used in the design of wood beams. Often, both the nominal and actual dimensions are reported. Lumber is identified by its *nominal* dimensions, such as 2 × 4 (2 in. by 4 in.); however, its actual or "dressed" dimensions are smaller, being 1.5 in. by 3.5 in. The reduction in the dimensions occurs in order to obtain a smooth surface from lumber that is rough sawn. Obviously, the *actual dimensions* must be used whenever stress calculations are performed on wood beams.



Steel plate girders

Fig. 15–5

Built-up Sections. A *built-up section* is constructed from two or more parts joined together to form a single unit. Since $S_{req'd} = M/\sigma_{allow}$, the capacity of the beam to resist a moment will vary directly with its section modulus, and since $S_{req'd} = I/c$, then $S_{req'd}$ is *increased* if I is *increased*. In order to increase I, *most of the material* should be placed as far *away* from the neutral axis as practical. This, of course, is what makes a deep wide-flange beam so efficient in resisting a moment. For a very large load, however, an available rolled-steel section may not have a section modulus great enough to support the load. Rather than using several available beams, instead engineers will usually "build up" a beam made from plates and angles. A deep I-shaped section having this form is called a *plate girder*. For example, the steel plate girder in Fig. 15–5 has two flange plates that are either welded or, using angles, bolted to the web plate.

Wood beams are also "built up," usually in the form of a box beam section, Fig. 15–6*a*. They may be made having plywood webs and larger boards for the flanges. For very large spans, *glulam beams* are used. These members are made from several boards glue-laminated together to form a single unit, Fig. 15–6*b*.

Just as in the case of rolled sections or beams made from a single piece, the design of built-up sections requires that the bending and shear stresses be checked. In addition, the shear stress in the fasteners, such as weld, glue, nails, etc., must be checked to be certain the beam acts as a single unit. The principles for doing this were outlined in Sec. 12.4.

Wooden box beam

(a)

Glulam beam

(b)

Fig. 15–6

Important Points

- Beams support loadings that are applied perpendicular to their axes. If they are designed on the basis of strength, they must resist allowable shear and bending stresses.
- The maximum bending stress in the beam is assumed to be much greater than the localized stresses caused by the application of loadings on the surface of the beam.

Procedure for Analysis

Based on the previous discussion, the following procedure provides a rational method for the design of a beam on the basis of strength.

Shear and Moment Diagrams.

- Determine the maximum shear and moment in the beam. Often this is done by constructing the beam's shear and moment diagrams.
- For built-up beams, shear and moment diagrams are useful for identifying *regions* where the shear and moment are excessively large and may require additional structural reinforcement or fasteners.

Bending Stress.

- If the beam is relatively long, it is designed by finding its section modulus using the flexure formula, $S_{req'd} = M_{max}/\sigma_{allow}$.
- Once $S_{req'd}$ is determined, the cross-sectional dimensions for simple shapes can then be computed, since $S_{req'd} = I/c$.
- If rolled-steel sections are to be used, several possible values of S may be selected from the tables in Appendix C. Of these, choose the one having the smallest cross-sectional area, since this beam has the least weight and is therefore the most economical.
- Make sure that the selected section modulus, S, is *slightly greater* than $S_{req'd}$, so that the additional moment created by the beam's weight is considered.

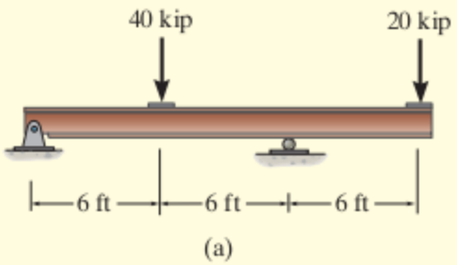
Shear Stress.

- Normally beams that are short and carry large loads, especially those made of wood, are first designed to resist shear and then later checked against the allowable-bending-stress requirements.
- Using the shear formula, check to see that the allowable shear stress is not exceeded; that is, use $\tau_{allow} \geq V_{max}Q/It$.
- If the beam has a solid *rectangular* cross section, the shear formula becomes $\tau_{allow} \geq 1.5(V_{max}/A)$ (See Eq. 2 of Example 12.2.), and if the cross section is a *wide flange*, it is generally appropriate to assume that the shear stress is *constant* over the cross-sectional area of the beam's web so that $\tau_{allow} \geq V_{max}/A_{web}$, where A_{web} is determined from the product of the beam's depth and the web's thickness. (See the note at the end of Example 12.3.)

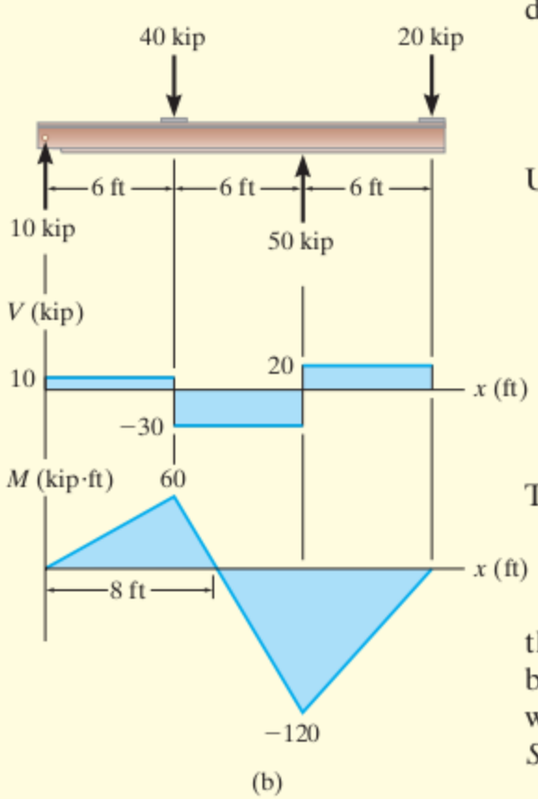
Adequacy of Fasteners.

- The adequacy of fasteners used on built-up beams depends upon the shear stress the fasteners can resist. Specifically, the required spacing of nails or bolts of a particular size is determined from the allowable shear flow, $q_{allow} = VQ/I$, calculated at points on the cross section where the fasteners are located. (See Sec. 15.3.)

EXAMPLE 15.1



(a)



(b)

Fig. 15–7

A beam is to be made of steel that has an allowable bending stress of $\sigma_{allow} = 24$ ksi and an allowable shear stress of $\tau_{allow} = 14.5$ ksi. Select an appropriate W shape that will carry the loading shown in Fig. 15–7*a*.

SOLUTION

Shear and Moment Diagrams. The support reactions have been calculated, and the shear and moment diagrams are shown in Fig. 15–7*b*. From these diagrams, $V_{max} = 30$ kip and $M_{max} = 120$ kip · ft.

Bending Stress. The required section modulus for the beam is determined from the flexure formula,

$$S_{req'd} = \frac{M_{max}}{\sigma_{allow}} = \frac{120 \text{ kip} \cdot \text{ft}\,(12 \text{ in./ft})}{24 \text{ kip/in}^2} = 60 \text{ in}^3$$

Using the table in Appendix C, the following beams are adequate:

W18 × 40	$S = 68.4 \text{ in}^3$
W16 × 45	$S = 72.7 \text{ in}^3$
W14 × 43	$S = 62.7 \text{ in}^3$
W12 × 50	$S = 64.7 \text{ in}^3$
W10 × 54	$S = 60.0 \text{ in}^3$
W8 × 67	$S = 60.4 \text{ in}^3$

The beam having the least weight per foot is chosen, i.e.,

$$\text{W18} \times 40$$

The *actual* maximum moment M_{max}, which includes the weight of the beam, can be calculated and the adequacy of the selected beam can be checked. In comparison with the applied loads, however, the beam's weight, $(0.040 \text{ kip/ft})(18 \text{ ft}) = 0.720$ kip, will only *slightly increase* $S_{req'd}$. In spite of this,

$$S_{req'd} = 60 \text{ in}^3 < 68.4 \text{ in}^3 \qquad \text{OK}$$

Shear Stress. Since the beam is a *wide-flange section*, the *average shear stress* within the web will be considered. (See Example 12.3.) Here the web is assumed to extend from the very top to the very bottom of the beam. From Appendix C, for a W18 × 40, $d = 17.90$ in., $t_w = 0.315$ in. Thus,

$$\tau_{avg} = \frac{V_{max}}{A_w} = \frac{30 \text{ kip}}{(17.90 \text{ in.})(0.315 \text{ in.})} = 5.32 \text{ ksi} < 14.5 \text{ ksi} \qquad \text{OK}$$

Use a W18 × 40. *Ans.*

EXAMPLE 15.2

The laminated wooden beam shown in Fig. 15–8*a* supports a uniform distributed loading of 12 kN/m. If the beam is to have a height-to-width ratio of 1.5, determine its smallest width. The allowable bending stress is $\sigma_{allow} = 9\,\text{MPa}$ and the allowable shear stress is $\tau_{allow} = 0.6\,\text{MPa}$. Neglect the weight of the beam.

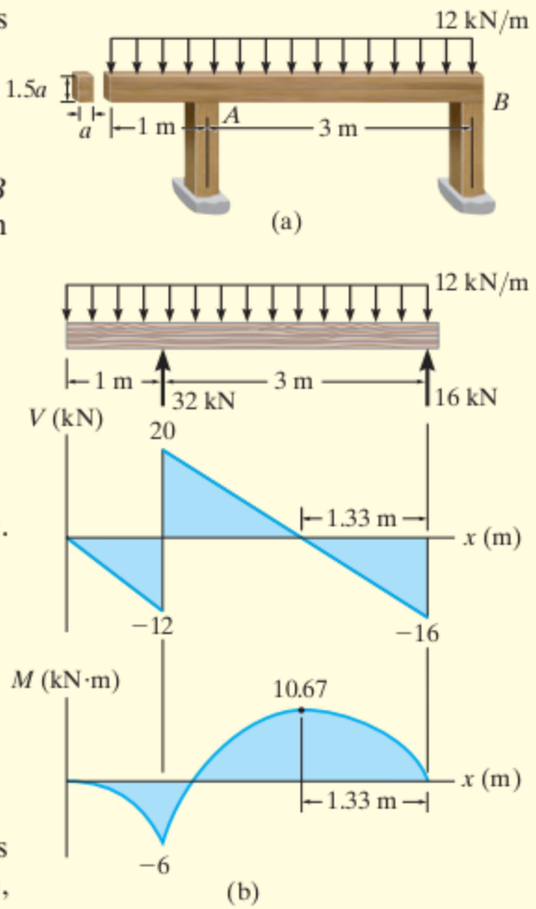


Fig. 15–8

SOLUTION

Shear and Moment Diagrams. The support reactions at A and B have been calculated and the shear and moment diagrams are shown in Fig. 15–8*b*. Here $V_{max} = 20\,\text{kN}$, $M_{max} = 10.67\,\text{kN}\cdot\text{m}$.

Bending Stress. Applying the flexure formula,

$$S_{req'd} = \frac{M_{max}}{\sigma_{allow}} = \frac{10.67(10^3)\,\text{N}\cdot\text{m}}{9(10^6)\,\text{N/m}^2} = 0.00119\,\text{m}^3$$

Assuming that the width is a, then the height is $1.5a$, Fig. 15–8*a*. Thus,

$$S_{req'd} = \frac{I}{c} = 0.00119\,\text{m}^3 = \frac{\frac{1}{12}(a)(1.5a)^3}{(0.75a)}$$

$$a^3 = 0.003160\,\text{m}^3$$
$$a = 0.147\,\text{m}$$

Shear Stress. Applying the shear formula for rectangular sections (which is a special case of $\tau_{max} = VQ/It$, as shown in Example 12.2), we have

$$\tau_{max} = 1.5\frac{V_{max}}{A} = (1.5)\frac{20(10^3)\,\text{N}}{(0.147\,\text{m})(1.5)(0.147\,\text{m})}$$
$$= 0.929\,\text{MPa} > 0.6\,\text{MPa}$$

EQUATION

Since the design fails the shear criterion, the beam must be redesigned on the basis of shear.

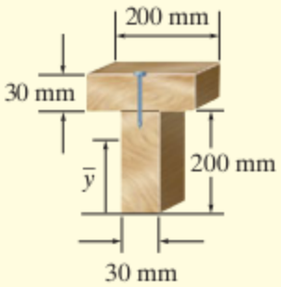
$$\tau_{allow} = 1.5\frac{V_{max}}{A}$$
$$600\,\text{kN/m}^2 = 1.5\frac{20(10^3)\,\text{N}}{(a)(1.5a)}$$
$$a = 0.183\,\text{m} = 183\,\text{mm} \qquad \textit{Ans.}$$

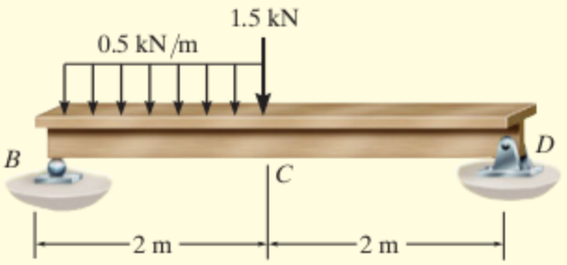
This larger section will also adequately resist the normal stress.

15

EXAMPLE 15.3

The wooden T-beam shown in Fig. 15–9*a* is made from two 200 mm × 30 mm boards. If the allowable bending stress is $\sigma_{allow} = 12$ MPa and the allowable shear stress is $\tau_{allow} = 0.8$ MPa, determine if the beam can safely support the loading shown. Also, specify the maximum spacing of nails needed to hold the two boards together if each nail can safely resist 1.50 kN in shear.

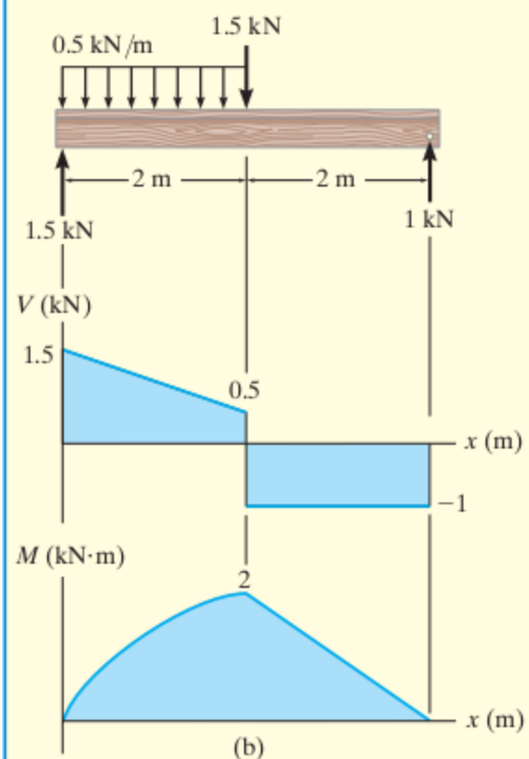




(a)

SOLUTION

Shear and Moment Diagrams. The reactions on the beam are shown, and the shear and moment diagrams are drawn in Fig. 15–9*b*. Here $V_{max} = 1.5$ kN, $M_{max} = 2$ kN · m.



(b)

Fig. 15–9

Bending Stress. The neutral axis (centroid) will be located from the bottom of the beam. Working in units of meters, we have

$$\bar{y} = \frac{\Sigma \bar{y} A}{\sigma A} = \frac{(0.1\text{ m})(0.03\text{ m})(0.2\text{ m}) + 0.215\text{ m}(0.03\text{ m})(0.2\text{ m})}{0.03\text{ m}(0.2\text{ m}) + 0.03\text{ m}(0.2\text{ m})} = 0.1575\text{ m}$$

Thus,

$$I = \left[\frac{1}{12}(0.03\text{ m})(0.2\text{ m})^3 + (0.03\text{ m})(0.2\text{ m})(0.1575\text{ m} - 0.1\text{ m})^2\right] + \left[\frac{1}{12}(0.2\text{ m})(0.03\text{ m})^3 + (0.03\text{ m})(0.2\text{ m})(0.215\text{ m} - 0.1575\text{ m})^2\right] = 60.125(10^{-6})\text{ m}^4$$

Since $c = 0.1575$ m (not $0.230\text{ m} - 0.1575\text{ m} = 0.0725$ m), we require

$$\sigma_{allow} \geq \frac{M_{max} c}{I}$$

$$12(10^6)\text{ Pa} \geq \frac{2(10^3)\text{ N} \cdot \text{m}(0.1575\text{ m})}{60.125(10^{-6})\text{ m}^4} = 5.24(10^6)\text{ Pa} \qquad \text{OK}$$

Shear Stress. Maximum shear stress in the beam depends upon the magnitude of Q and t. It occurs at the neutral axis, since Q is a maximum there and the neutral axis is in the web, where the thickness $t = 0.03$ m is smallest for the cross section. For simplicity, we will use the rectangular area below the neutral axis to calculate Q, rather than a two-part composite area above this axis, Fig. 15–9*c*. We have

$$Q = \bar{y}'A' = \left(\frac{0.1575\text{ m}}{2}\right)[(0.1575\text{ m})(0.03\text{ m})] = 0.372(10^{-3})\text{ m}^3$$

so that

$$\tau_{\text{allow}} \geq \frac{V_{\max}Q}{It}$$

$$800(10^3)\text{ Pa} \geq \frac{1.5(10^3)\text{ N}[0.372(10^{-3})]\text{ m}^3}{60.125(10^{-6})\text{ m}^4(0.03\text{ m})} = 309(10^3)\text{ Pa} \qquad \text{OK}$$

Nail Spacing. From the shear diagram it is seen that the shear varies over the entire span. Since the nail spacing depends on the magnitude of shear in the beam, for simplicity (and to be conservative), we will design the spacing on the basis of $V = 1.5$ kN for region BC and $V = 1$ kN for region CD. Since the nails join the flange to the web, Fig. 15–9*d*, we have

$$Q = \bar{y}'A' = (0.0725\text{ m} - 0.015\text{ m})[(0.2\text{ m})(0.03\text{ m})] = 0.345(10^{-3})\text{ m}^3$$

The shear flow for each region is therefore

$$q_{BC} = \frac{V_{BC}Q}{I} = \frac{1.5(10^3)\text{ N}[0.345(10^{-3})\text{ m}^3]}{60.125(10^{-6})\text{ m}^4} = 8.61\text{ kN/m}$$

$$q_{CD} = \frac{V_{CD}Q}{I} = \frac{1(10^3)\text{ N}[0.345(10^{-3})\text{ m}^3]}{60.125(10^{-6})\text{ m}^4} = 5.74\text{ kN/m}$$

One nail can resist 1.50 kN in shear, so the maximum spacing becomes

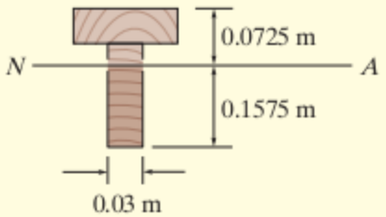
$$s_{BC} = \frac{1.50\text{ kN}}{8.61\text{ kN/m}} = 0.174\text{ m}$$

$$s_{CD} = \frac{1.50\text{ kN}}{5.74\text{ kN/m}} = 0.261\text{ m}$$

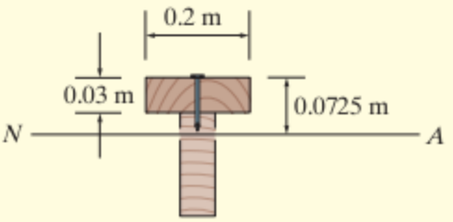
For ease of measuring, use

$$s_{BC} = 150\text{ mm} \qquad \textit{Ans.}$$

$$s_{CD} = 250\text{ mm} \qquad \textit{Ans.}$$



(c)



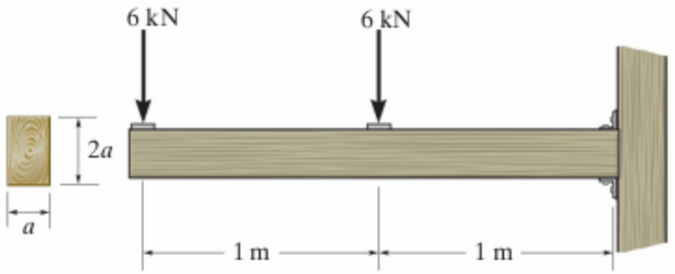
(d)

Fig. 15–9 (cont.)

15

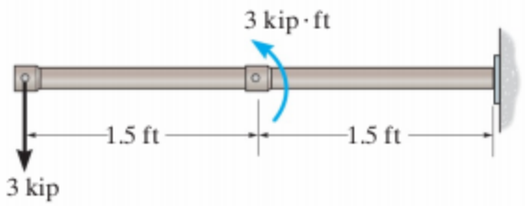
FUNDAMENTAL PROBLEMS

F15–1. Determine the minimum dimension a to the nearest mm of the beam's cross section to safely support the load. The wood has an allowable normal stress of $\sigma_{allow} = 10$ MPa and an allowable shear stress of $\tau_{allow} = 1$ MPa.



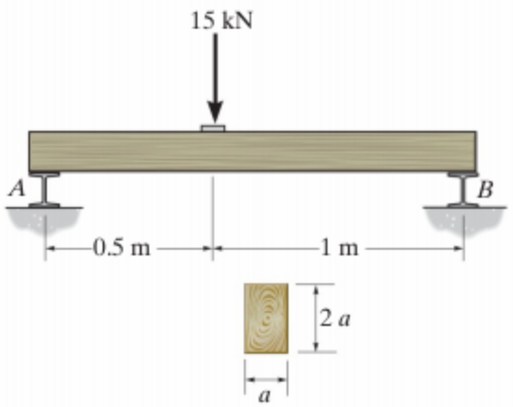
F15–1

F15–2. Determine the minimum diameter d to the nearest $\frac{1}{8}$ in. of the rod to safely support the load. The rod is made of a material having an allowable normal stress of $\sigma_{allow} = 20$ ksi and an allowable shear stress of $\tau_{allow} = 10$ ksi.



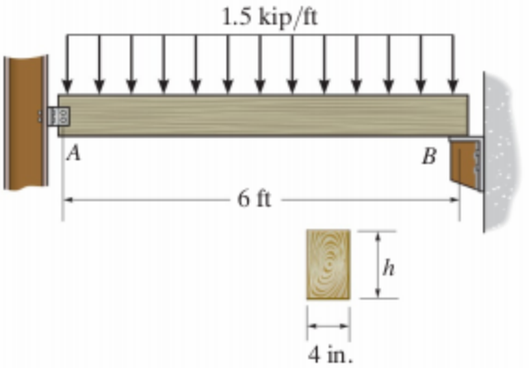
F15–2

F15–3. Determine the minimum dimension a to the nearest mm of the beam's cross section to safely support the load. The wood has an allowable normal stress of $\sigma_{allow} = 12$ MPa and an allowable shear stress of $\tau_{allow} = 1.5$ MPa.



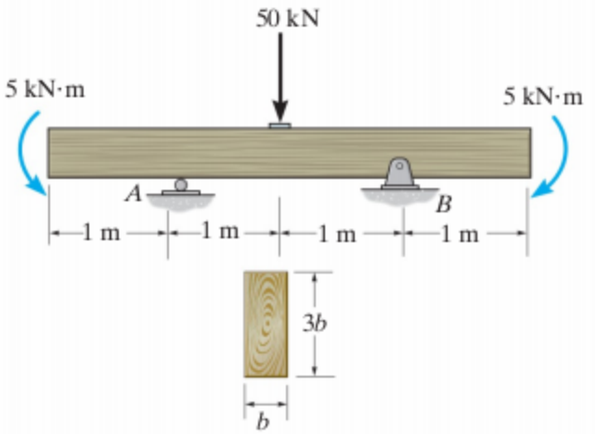
F15–3

F15–4. Determine the minimum dimension h to the nearest $\frac{1}{8}$ in. of the beam's cross section to safely support the load. The wood has an allowable normal stress of $\sigma_{allow} = 2$ ksi and an allowable shear stress of $\tau_{allow} = 200$ psi.



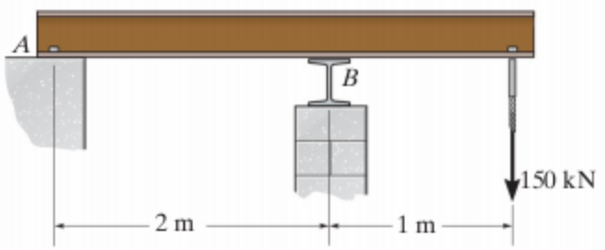
F15–4

F15–5. Determine the minimum dimension b to the nearest mm of the beam's cross section to safely support the load. The wood has an allowable normal stress of $\sigma_{allow} = 12$ MPa and an allowable shear stress of $\tau_{allow} = 1.5$ MPa.



F15–5

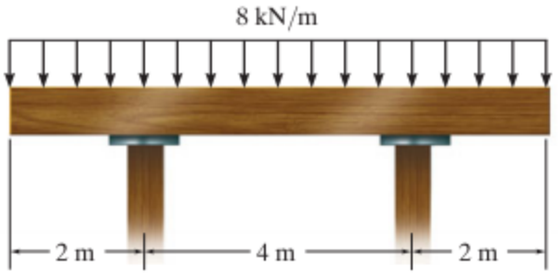
F15–6. Select the lightest W410-shaped section that can safely support the load. The beam is made of steel having an allowable normal stress of $\sigma_{allow} = 150$ MPa and an allowable shear stress of $\tau_{allow} = 75$ MPa.



F15–6

PROBLEMS

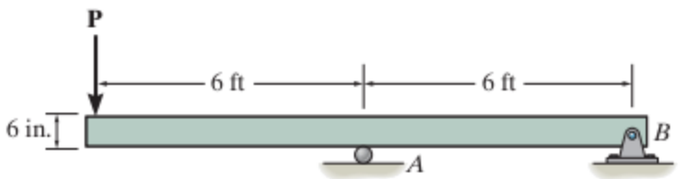
15–1. The simply supported beam is made of timber that has an allowable bending stress of $\sigma_{allow} = 6.5$ MPa and an allowable shear stress of $\tau_{allow} = 500$ kPa. Determine its dimensions if it is to be rectangular and have a height-to-width ratio of 1.25.



Prob. 15–1

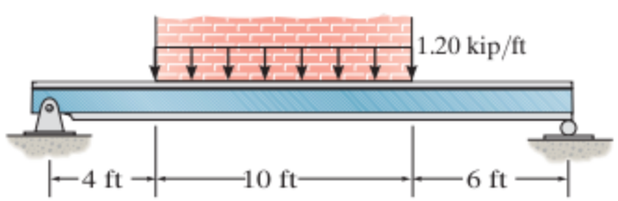
15–2. Determine the minimum width of the beam to the nearest $\frac{1}{4}$ in. that will safely support the loading of $P = 8$ kip. The allowable bending stress is $\sigma_{allow} = 24$ ksi and the allowable shear stress is $\tau_{allow} = 15$ ksi.

15–3. Solve Prob. 15–2 if $P = 10$ kip.



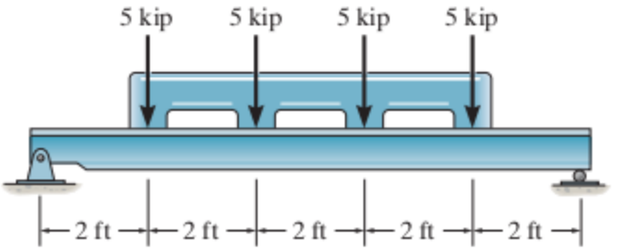
Probs. 15–2/3

*15–4. The brick wall exerts a uniform distributed load of 1.20 kip/ft on the beam. If the allowable bending stress is $\sigma_{allow} = 22$ ksi and the allowable shear stress is $\tau_{allow} = 12$ ksi, select the lightest wide-flange section with the shortest depth from Appendix B that will safely support the load. If there are several choices of equal weight, choose the one with the shortest height.



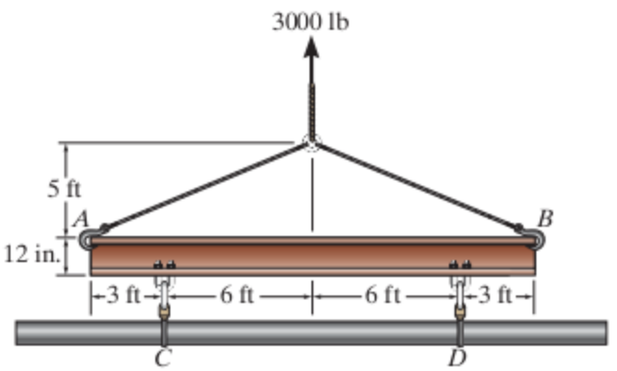
Prob. 15–4

15–5. Select the lightest-weight steel wide-flange beam from Appendix C that will safely support the machine loading shown. The allowable bending stress is $\sigma_{allow} = 24$ ksi and the allowable shear stress is $\tau_{allow} = 14$ ksi.



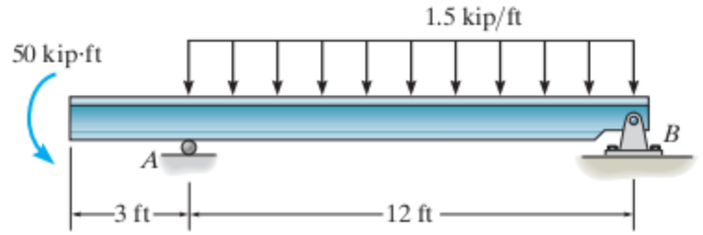
Prob. 15–5

15–6. The spreader beam AB is used to slowly lift the 3000-lb pipe that is centrally located on the straps at C and D. If the beam is a W12 × 45, determine if it can safely support the load. The allowable bending stress is $\sigma_{allow} = 22$ ksi and the allowable shear stress is $\tau_{allow} = 12$ ksi.



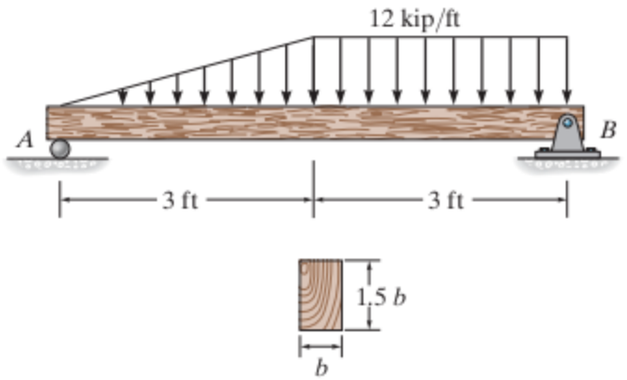
Prob. 15–6

15–7. Draw the shear and moment diagrams for the W12 × 14 beam and check if the beam will safely support the loading. Take $\sigma_{allow} = 22$ ksi and $\tau_{allow} = 12$ ksi.



Prob. 15–7

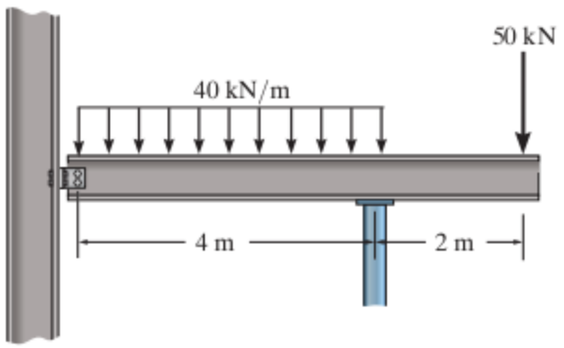
***15–8.** The simply supported beam is made of timber that has an allowable bending stress of $\sigma_{allow} = 1.20$ ksi and an allowable shear stress of $\tau_{allow} = 100$ psi. Determine its smallest dimensions to the nearest $\frac{1}{8}$ in. if it is rectangular and has a height-to-width ratio of 1.5.



Prob. 15–8

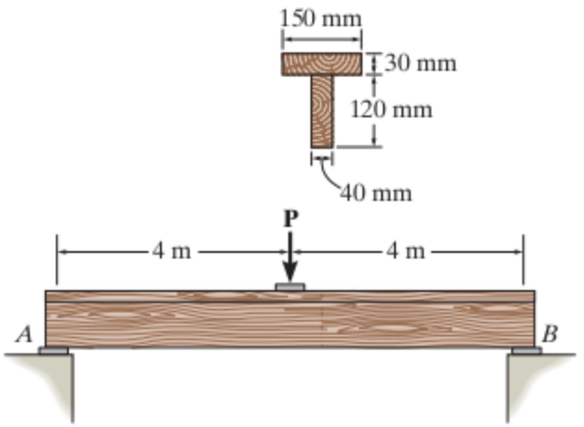
15–9. Select the lightest W 360 shape section from Appendix B that can safely support the loading acting on the overhanging beam. The beam is made from steel having an allowable normal stress of $\sigma_{allow} = 150$ MPa and an allowable shear stress of $\tau_{allow} = 80$ MPa.

15–10. Investigate if a W 250 × 58 can safely support the loading acting on the overhanging beam. The beam is made from steel having an allowable normal stress of $\sigma_{allow} = 150$ MPa and an allowable shear stress of $\tau_{allow} = 80$ MPa.



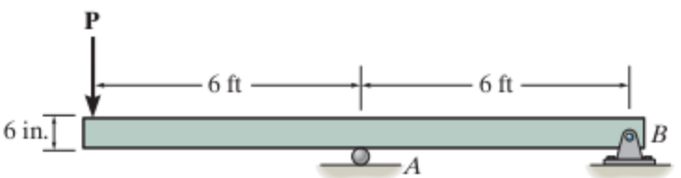
Probs. 15–9/10

15–11. The timber beam is to be loaded as shown. If the ends support only vertical forces, determine the greatest magnitude of **P** that can be applied. $\sigma_{allow} = 25$ MPa, $\tau_{allow} = 700$ kPa.



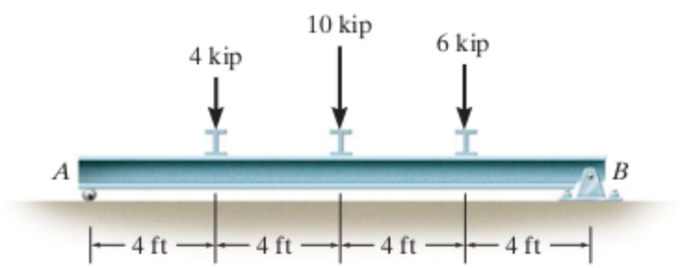
Prob. 15–11

***15–12.** Determine the minimum width of the beam to the nearest $\frac{1}{4}$ in. that will safely support the loading of $P = 8$ kip. The allowable bending stress is $\sigma_{allow} = 24$ ksi and the allowable shear stress is $\tau_{allow} = 15$ ksi.



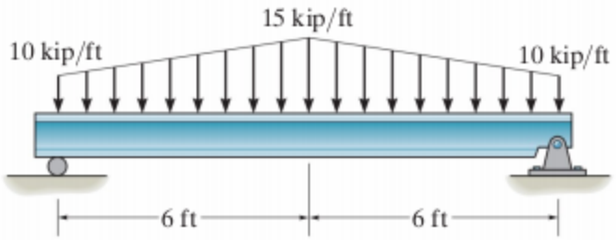
Prob. 15–12

15–13. Select the shortest and lightest-weight steel wide-flange beam from Appendix C that will safely support the loading shown. The allowable bending stress is $\sigma_{allow} = 22$ ksi and the allowable shear stress is $\tau_{allow} = 12$ ksi.



Prob. 15–13

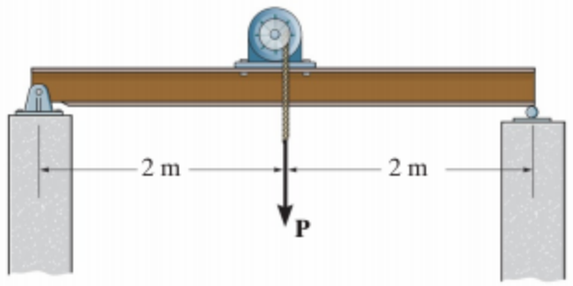
15–14. Select the lightest-weight steel wide-flange beam from Appendix B that will safely support the loading shown. The allowable bending stress $\sigma_{allow} = 24$ ksi and the allowable shear stress of $\tau_{allow} = 14$ ksi.



Prob. 15–14

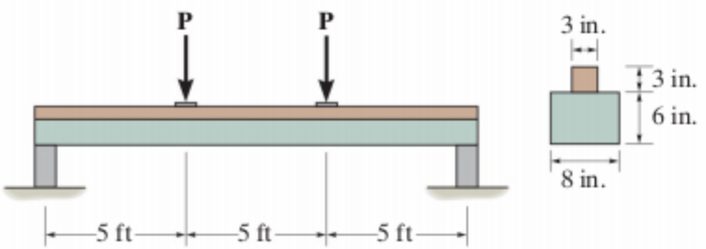
***15–16.** If the cable is subjected to a maximum force of $P = 50$ kN, select the lightest W 310 shape that can safely support the load. The beam is made from steel having an allowable normal stress of $\sigma_{allow} = 150$ MPa and an allowable shear stress of $\tau_{allow} = 85$ MPa.

15–17. If the W 360 × 45 beam is made from steel having an allowable normal stress of $\sigma_{allow} = 150$ MPa and an allowable shear stress of $\tau_{allow} = 85$ MPa, determine the maximum cable force P that can safely be supported by the beam.



Probs. 15–16/17

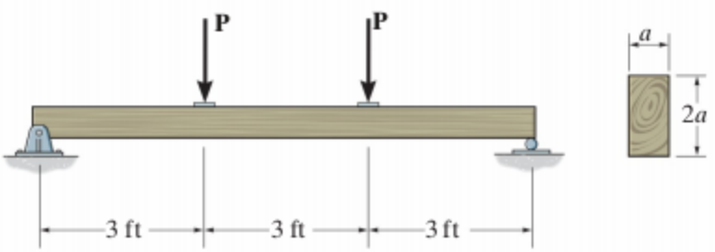
15–15. Two acetyl plastic members are to be glued together and used to support the loading shown. If the allowable bending stress for the plastic is $\sigma_{allow} = 13$ ksi and the allowable shear stress is $\tau_{allow} = 4$ ksi, determine the greatest load P that can be supported and specify the required shear stress capacity of the glue.



Prob. 15–15

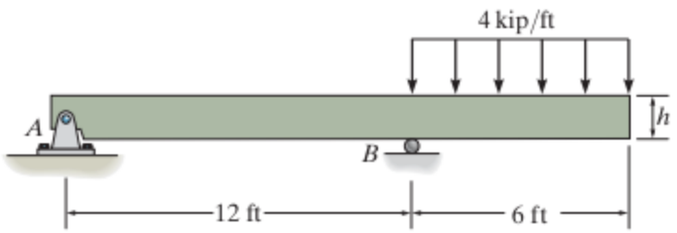
15–18. If $P = 800$ lb, determine the minimum dimension a of the beam's cross section to the nearest $\frac{1}{8}$ in. to safely support the load. The wood species has an allowable normal stress of $\sigma_{allow} = 1.5$ ksi and an allowable shear stress of $\tau_{allow} = 150$ psi.

15–19. If $a = 3$ in. and the wood has an allowable normal stress of $\sigma_{allow} = 1.5$ ksi, and an allowable shear stress of $\tau_{allow} = 150$ psi, determine the maximum allowable value of P acting on the beam.



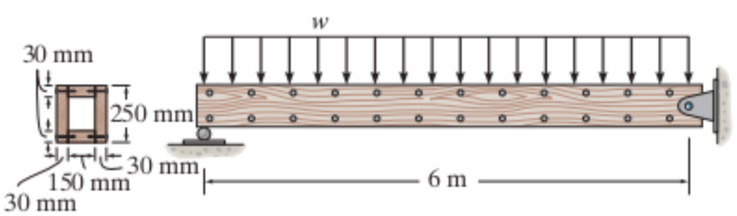
Probs. 15–18/19

*15–20. Determine the minimum depth h of the beam to the nearest $\frac{1}{8}$ in. that will safely support the loading shown. The allowable bending stress is $\sigma_{allow} = 21$ ksi and the allowable shear stress is $\tau_{allow} = 10$ ksi. The beam has a uniform thickness of 3 in.



Prob. 15–20

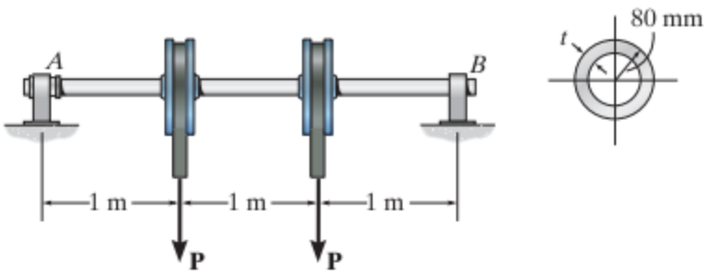
15–21. The box beam has an allowable bending stress of $\sigma_{allow} = 10$ MPa and an allowable shear stress of $\tau_{allow} = 775$ kPa. Determine the maximum intensity w of the distributed loading that it can safely support. Also, determine the maximum safe nail spacing for each third of the length of the beam. Each nail can resist a shear force of 200 N.



Prob. 15–21

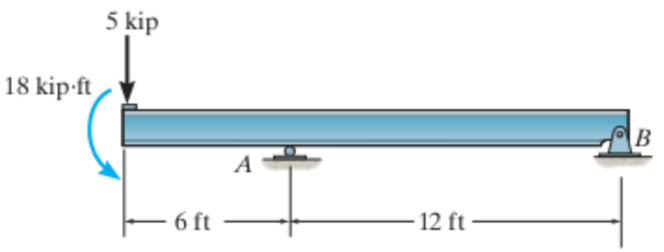
15–22. The shaft is supported by a smooth thrust bearing at A and smooth journal bearing at B. If $P = 10$ kN and the shaft is made from steel having an allowable normal stress of $\sigma_{allow} = 150$ MPa and an allowable shear stress of $\tau_{allow} = 85$ MPa, determine the required minimum wall thickness t of the shaft to the nearest millimeter to safely support the load.

15–23. The circular hollow shaft is supported by a smooth thrust bearing at A and smooth journal bearing at B. If the shaft is made from steel having an allowable normal stress of $\sigma_{allow} = 150$ MPa and allowable shear stress of $\tau_{allow} = 85$ MPa, determine the maximum allowable magnitude of the two forces **P** that can be applied to the shaft. The thickness of the shaft's wall is $t = 5$ mm.



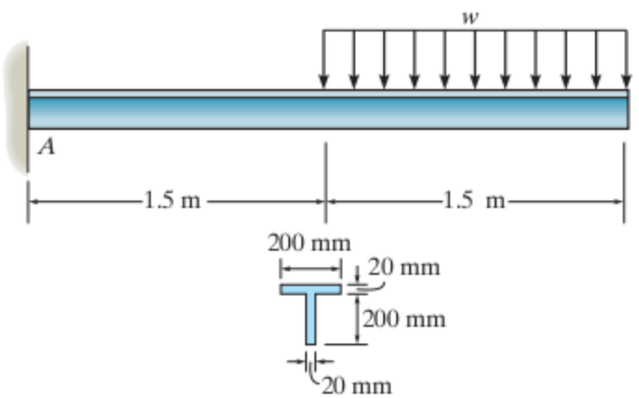
Probs. 15–22/23

*15–24. Select the lightest-weight steel wide-flange beam from Appendix C that will safely support the loading shown. The allowable bending stress is $\sigma_{allow} = 22$ ksi and the allowable shear stress is $\tau_{allow} = 12$ ksi.



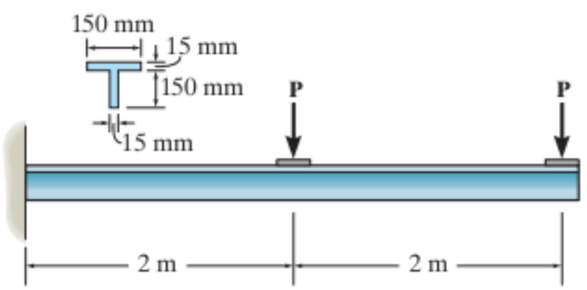
Prob. 15–24

15–25. The T-beam is made from two plates welded together as shown. Determine the maximum uniform distributed load w that can be safely supported on the beam if the allowable bending stress is $\sigma_{allow} = 150$ MPa and the allowable shear stress is $\tau_{allow} = 70$ MPa.



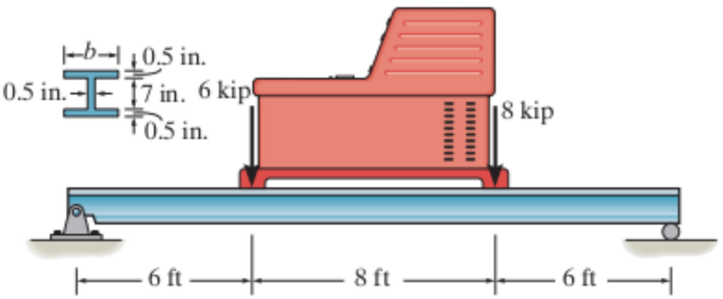
Prob. 15–25

15–26. The steel cantilevered T-beam is made from two plates welded together as shown. Determine the maximum loads P that can be safely supported on the beam if the allowable bending stress is $\sigma_{allow} = 170$ MPa and the allowable shear stress is $\tau_{allow} = 95$ MPa.



Prob. 15–26

15–27. The beam is to be used to support the machine, which exerts the forces of 6 kip and 8 kip as shown. If the maximum bending stress is not to exceed $\sigma_{allow} = 22$ ksi, determine the required width b of the flanges.

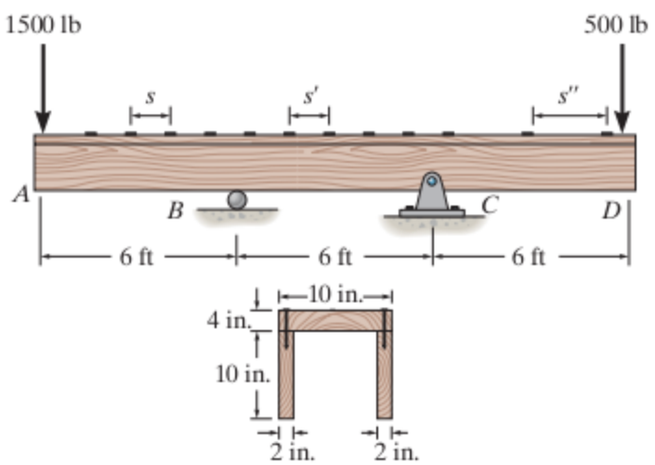


Prob. 15–27

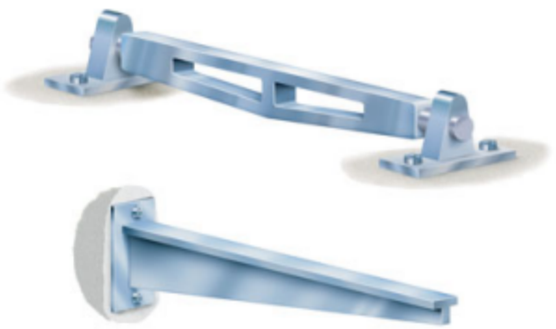
*15

15

***15–28.** The beam is constructed from three boards as shown. If each nail can support a shear force of 300 lb, determine the maximum allowable spacing of the nails, s, s', s'', for regions AB, BC, and CD respectively. Also, if the allowable bending stress is $\sigma_{allow} = 1.5$ ksi and the allowable shear stress is $\tau_{allow} = 150$ psi, determine if it can safely support the load.



Prob. 15–28



(a)

Haunched concrete beam
(b)

Wide-flange beam with cover plates
(c)

Fig. 15–10

The beam for this bridge pier has a variable moment of inertia. This design will reduce material weight and save cost.

*15.3 Fully Stressed Beams

Since the moment in a beam generally *varies* along its length, the choice of a prismatic beam is usually inefficient since it is never fully stressed at points where the internal moment is less than the maximum moment in the beam. In order to reduce the weight of the beam, engineers sometimes choose a beam having a *variable* cross-sectional area, such that at each cross section along the beam, the bending stress reaches its maximum allowable value. Beams having a variable cross-sectional area are called *nonprismatic beams*. They are often used in machines since they can be readily formed by casting. Examples are shown in Fig. 15–10*a*. In structures such beams may be "haunched" at their ends as shown in Fig. 15–10*b*. Also, beams may be "built up" or fabricated in a shop using plates. An example is a girder made from a rolled-shaped wide-flange beam and having cover plates welded to it in the region where the moment is a maximum, Fig. 15–10*c*.

The stress analysis of a nonprismatic beam is generally very difficult to perform and is beyond the scope of this text. Most often these shapes are analyzed by using a computer or the theory of elasticity. The results obtained from such an analysis, however, do indicate that the assumptions used in the derivation of the flexure formula are approximately correct for predicting the bending stresses in nonprismatic sections, provided the taper or slope of the upper or lower boundary of the beam is not too severe. On the other hand, the shear formula cannot be used for nonprismatic beam design, since the results obtained from it are very misleading.

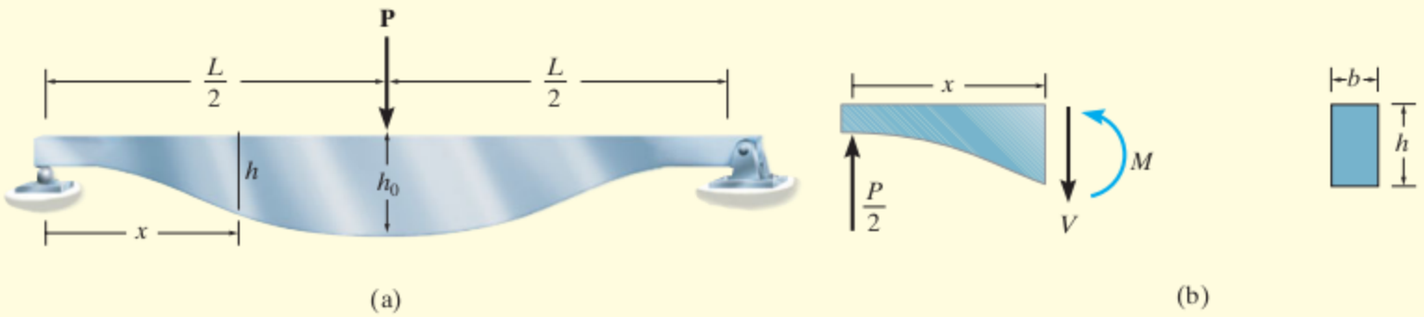
Although caution is advised when applying the flexure formula to nonprismatic beam design, we will show here, in principle, how this formula can be used as an approximate means for obtaining the beam's general shape. In this regard, the *size* of the cross section of a nonprismatic beam that supports a given loading can be determined using the flexure formula written as

$$S = \frac{M}{\sigma_{\text{allow}}}$$

If we express the internal moment M in terms of its position x along the beam, then since σ_{allow} is a known constant, the section modulus S or the beam's dimensions become a function of x. A beam designed in this manner is called a ***fully stressed beam***. Although *only* bending stresses have been considered in approximating its final shape, attention must also be given to ensure that the beam will resist shear, especially at points where concentrated loads are applied.

EXAMPLE 15.4

Determine the shape of a fully stressed, simply supported beam that supports a concentrated force at its center, Fig. 15–11*a*. The beam has a rectangular cross section of constant width b, and the allowable stress is σ_{allow}.



(a) (b)

Fig. 15–11

SOLUTION

The internal moment in the beam, Fig. 15–11*b*, expressed as a function of position, $0 \le x < L/2$, is

$$M = \frac{P}{2}x$$

Hence the required section modulus is

$$S = \frac{M}{\sigma_{\text{allow}}} = \frac{P}{2\sigma_{\text{allow}}}x$$

Since $S = I/c$, then for a cross-sectional area h by b we have

$$\frac{I}{c} = \frac{\frac{1}{12}bh^3}{h/2} = \frac{P}{2\sigma_{\text{allow}}}x$$

$$h^2 = \frac{3P}{\sigma_{\text{allow}}b}x$$

If $h = h_0$ at $x = L/2$, then

$${h_0}^2 = \frac{3PL}{2\sigma_{\text{allow}}b}$$

so that

$$h^2 = \left(\frac{2{h_0}^2}{L}\right)x \qquad \textit{Ans.}$$

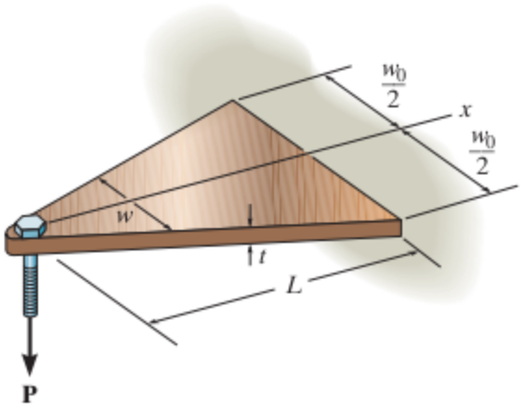
By inspection, the depth h must therefore vary in a *parabolic* manner with the distance x.

NOTE: In practice this *shape* is the basis for the design of leaf springs used to support the rear-end axles of most heavy trucks or train cars as shown in the adjacent photo. Note that although this result indicates that $h = 0$ at $x = 0$, it is necessary that the beam resist shear stress at the supports, and so practically speaking, it must be required that $h > 0$ at the supports, Fig. 15–11*a*.

15

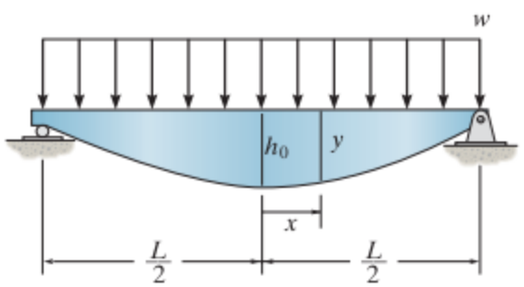
PROBLEMS

15–29. Determine the variation in the depth w as a function of x for the cantilevered beam that supports a concentrated force **P** at its end so that it has a maximum bending stress σ_{allow} throughout its length. The beam has a constant thickness t.



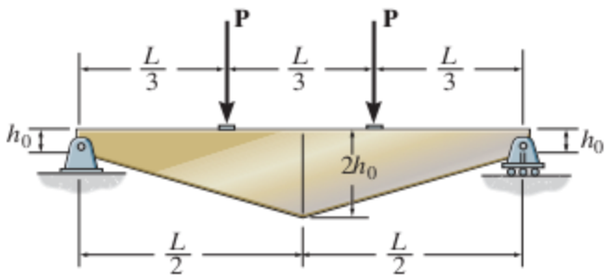
Prob. 15–29

15–30. The beam is made from a plate that has a constant thickness b. If it is simply supported and carries a uniform load w, determine the variation of its depth as a function of x so that it maintains a constant maximum bending stress σ_{allow} throughout its length.



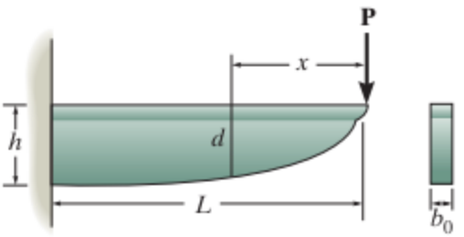
Prob. 15–30

15–31. The simply supported tapered rectangular beam with constant width b supports the concentrated forces **P**. Determine the absolute maximum normal stress developed in the beam and specify its location.



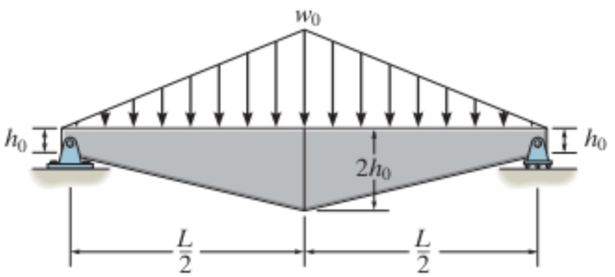
Prob. 15–31

***15–32.** Determine the variation in the depth d of a cantilevered beam that supports a concentrated force **P** at its end so that it has a constant maximum bending stress σ_{allow} throughout its length. The beam has a constant width b_0.



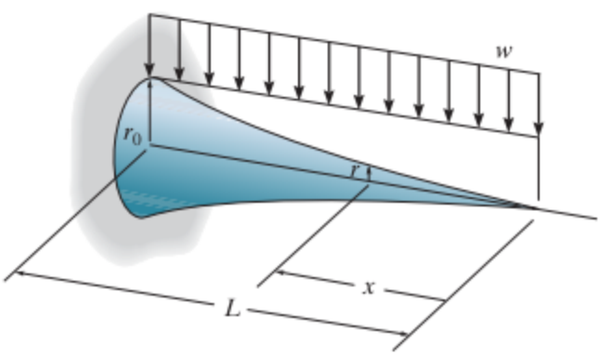
Prob. 15–32

15–33. The beam is made from a plate that has a constant thickness b. If it is simply supported and carries the distributed loading shown, determine the maximum bending stress in the beam.



Prob. 15–33

15–34. Determine the variation of the radius r of the cantilevered beam that supports the uniform distributed load so that it has a constant maximum bending stress σ_{max} throughout its length.



Prob. 15–34

CHAPTER REVIEW

Failure of a beam occurs when the internal shear or moment in the beam is a maximum. To resist these loadings, it is therefore important that the associated maximum shear and bending stress not exceed allowable values as stated in codes. Normally, the cross section of a beam is first designed to resist the allowable bending stress,

$$\sigma_{\text{allow}} = \frac{M_{\max} c}{I}$$

Then the allowable shear stress is checked. For rectangular sections, $\tau_{\text{allow}} \geq 1.5(V_{\max}/A)$ and for wide-flange sections it is appropriate to use $\tau_{\text{allow}} \geq V_{\max}/A_{\text{web}}$. In general, use

$$\tau_{\text{allow}} = \frac{VQ}{It}$$

For built-up beams, the spacing of fasteners or the strength of glue or weld is determined using an allowable shear flow

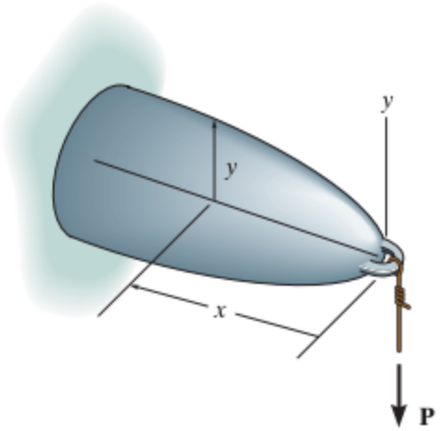
$$q_{\text{allow}} = \frac{VQ}{I}$$

Fully stressed beams are nonprismatic and designed such that each cross section along the beam will resist the allowable bending stress. This will define the shape of the beam.

15

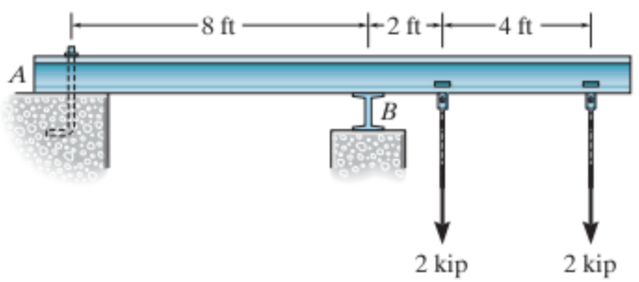
REVIEW PROBLEMS

15–35. The cantilevered beam has a circular cross section. If it supports a force **P** at its end, determine its radius y as a function of x so that it is subjected to a constant maximum bending stress σ_{allow} throughout its length.



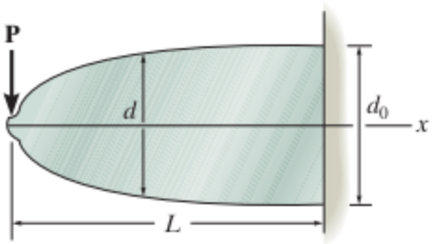
Prob. 15–35

***15–36.** Select the lightest-weight steel wide-flange overhanging beam from Appendix B that will safely support the loading. Assume the support at A is a pin and the support at B is a roller. The allowable bending stress is $\sigma_{allow} = 24$ ksi and the allowable shear stress is $\sigma_{allow} = 14$ ksi.



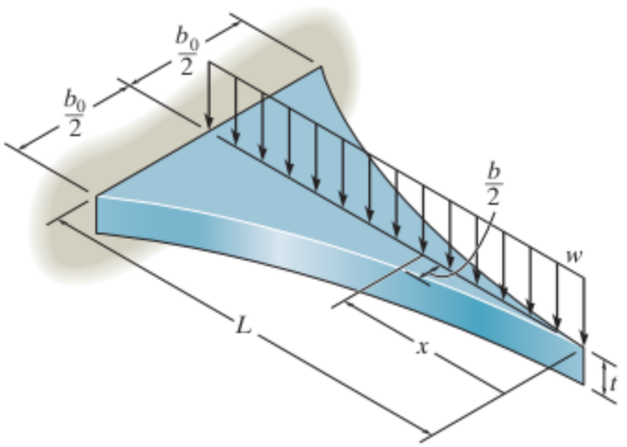
Prob. 15–36

15–37. Determine the variation in the depth d of a cantilevered beam that supports a concentrated force **P** at its end so that it has a constant maximum bending stress σ_{allow} throughout its length. The beam has a constant width b_0.



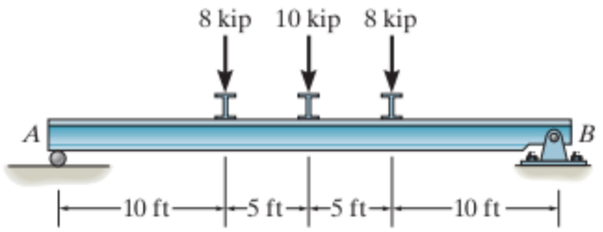
Prob. 15–37

15–38. Determine the variation in the width b as a function of x for the cantilevered beam that supports a uniform distributed load along its centerline so that it has the same maximum bending stress σ_{allow} throughout its length. The beam has a constant depth t.



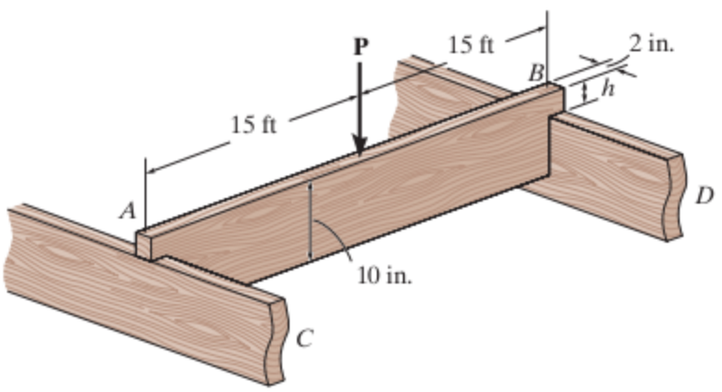
Prob. 15–38

15–39. Select the lightest-weight steel wide-flange beam from Appendix B that will safely support the loading shown. The allowable bending stress is $\sigma_{allow} = 12$ ksi and the allowable shear stress is $\tau_{allow} = 12$ ksi.



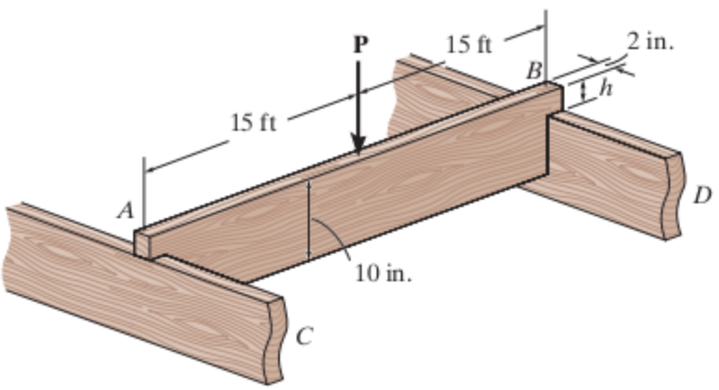
Prob. 15–39

*__15–40.__ The simply-supported joist is used in the construction of a floor for a building. In order to keep the floor low with respect to the sill beams C and D, the ends of the joists are notched as shown. If the allowable shear stress for the wood is $\tau_{allow} = 350$ psi and the allowable bending stress is $\sigma_{allow} = 1500$ psi determine the height h that will cause the beam to reach both allowable stresses at the same time. Also, what load P causes this to happen? Neglect the stress concentration at the notch.



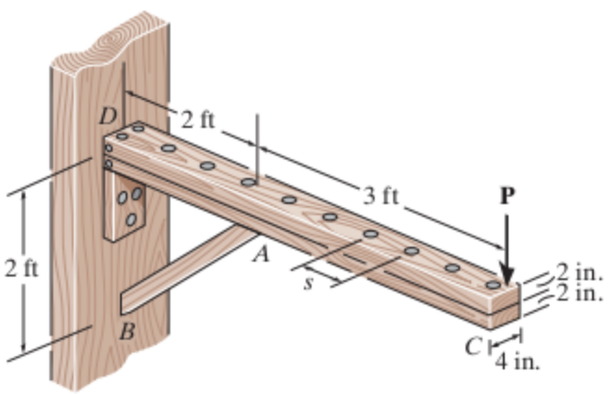
Prob. 15–40

15–41. The simply-supported joist is used in the construction of a floor for a building. In order to keep the floor low with respect to the sill beams C and D, the ends of the joists are notched as shown. If the allowable shear stress for the wood is $\tau_{allow} = 350$ psi and the allowable bending stress is $\sigma_{allow} = 1700$ psi, determine the smallest height h so that the beam will support a load of $P = 600$ lb. Also, will the entire joist safely support the load? Neglect the stress concentration at the notch.



Prob. 15–41

15–42. The overhang beam is constructed using two 2-in. by 4-in. pieces of wood braced as shown. If the allowable bending stress is $\sigma_{allow} = 600$ psi, determine the largest load P that can be applied. Also, determine the associated maximum spacing of nails, s, along the beam section AC if each nail can resist a shear force of 800 lb. Assume the beam is pin-connected at A, B, and D. Neglect the axial force developed in the beam along DA.



Prob. 15–42

Chapter 16

If the curvature of this pole is measured, it is then possible to determine the bending stress developed within it.

Deflection of Beams and Shafts

CHAPTER OBJECTIVES

- To determine the deflection and slope at specific points on beams and shafts using the integration method, discontinuity functions, and the method of superposition.
- To use the method of superposition to solve for the support reactions on a beam or shaft that is statically indeterminate.

16.1 The Elastic Curve

The deflection of a beam or shaft must often be limited in order to provide integrity and stability of a structure or machine, and prevent the cracking of any attached brittle materials such as concrete or glass. Furthermore, code restrictions often require these members not vibrate or deflect severely in order to safely support their intended loading. Most important, though, deflections at specific points on a beam or shaft must be determined if one is to analyze those that are statically indeterminate.

Before the slope or the displacement at a point on a beam (or shaft) is determined, it is often helpful to sketch the deflected shape of the beam when it is loaded, in order to "visualize" any computed results and thereby partially check these results. The deflection curve of the longitudinal axis that passes through the centroid of each cross-sectional area of a beam is called the *elastic curve*. For most beams the elastic curve can be sketched without much difficulty. When doing so, however, it is necessary to know how the slope or displacement is restricted at various types of supports. In general, supports that resist a *force*, such as a pin, restrict *displacement*, and those that resist a *moment*, such as a fixed wall, restrict *rotation* or *slope* as well as displacement. With this in mind, two typical examples of the elastic curves for loaded beams (or shafts), sketched to an exaggerated scale, are shown in Fig. 16–1.

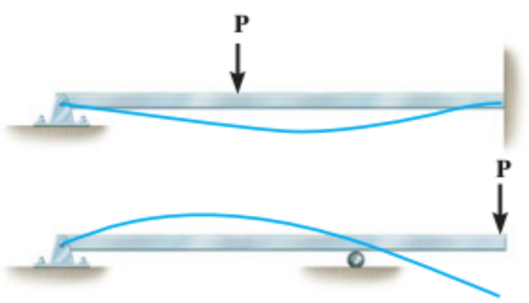


Fig. 16–1

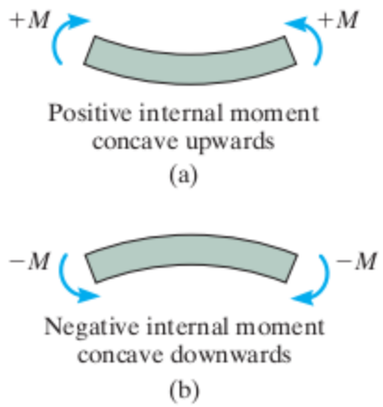


Fig. 16–2

If the elastic curve for a beam seems difficult to establish, it is suggested that the moment diagram for the beam be drawn first. Using the beam sign convention established in Sec. 11.1, a positive internal moment tends to bend the beam concave upward, Fig. 16–2*a*. Likewise, a negative moment tends to bend the beam concave downward, Fig. 16–2*b*. Therefore, if the moment diagram is *known*, it will be easy to construct the elastic curve. For example, consider the beam in Fig. 16–3*a* with its associated moment diagram shown in Fig. 16–3*b*. Due to the roller and pin supports, the displacement at B and D must be zero. Within the region of negative moment, AC, Fig. 16–3*b*, the elastic curve must be concave downward, and within the region of positive moment, CD, the elastic curve must be concave upward. Hence, there must be an *inflection point* at point C, where the curve changes from concave up to concave down, since this is a point of zero moment. Using these facts, the beam's elastic curve is sketched in Fig. 16–3*c*. It should also be noted that the displacements Δ_A and Δ_E are especially critical. At point E the *slope* of the elastic curve is *zero*, and there the beam's *deflection* may be a *maximum*. Whether Δ_E is actually greater than Δ_A depends on the relative magnitudes of $\mathbf{P}_1$ and $\mathbf{P}_2$ and the location of the roller at B.

Following these same principles, note how the elastic curve in Fig. 16–4 was constructed. Here the beam is cantilevered from a fixed support at A and therefore the elastic curve must have both zero displacement and zero slope at this point. Also, the largest displacement will occur either at D, where the slope is zero, or at C.

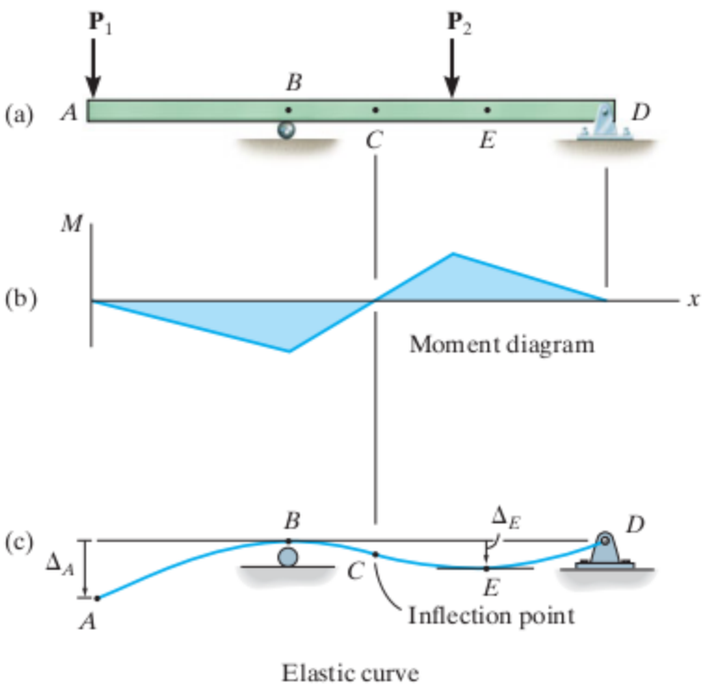


Fig. 16–3

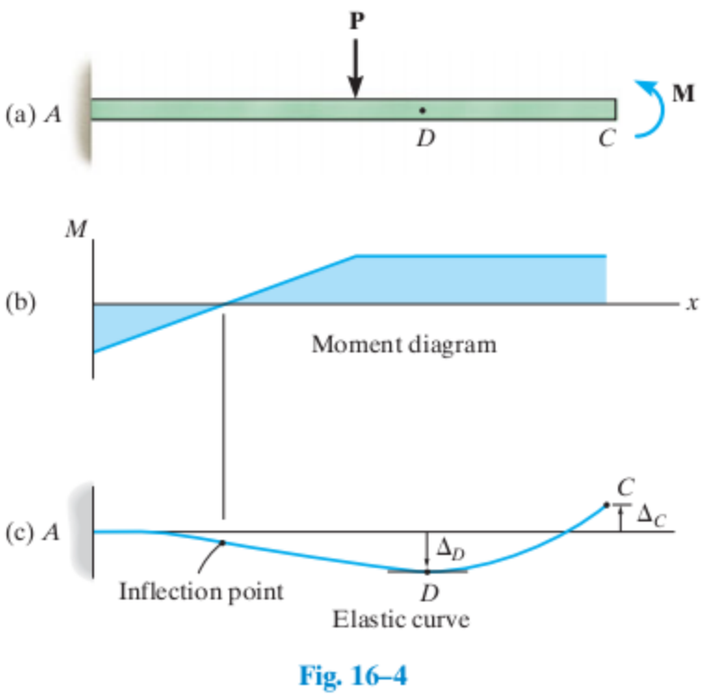


Fig. 16–4

Moment-Curvature Relationship. We will now develop an important relationship between the internal moment and the radius of curvature ρ (rho) of the elastic curve at a point. The resulting equation will be used for establishing each of the methods presented in the chapter for finding the slope and displacement at points on the elastic curve.

The following analysis, here and in the next section, will require the use of three coordinates. As shown in Fig. 16–5*a*, the x axis extends positive to the right, along the initially straight longitudinal axis of the beam. It is used to locate the differential element, having an undeformed width dx. The v axis extends *positive upward* from the x axis. It measures the *displacement* of the elastic curve. Lastly, a "localized" y coordinate is used to specify the position of a fiber in the beam element. It is measured *positive upward* from the neutral axis (or elastic curve) as shown in Fig. 16–5*b*. Recall that this same sign convention for x and y was used in the derivation of the flexure formula.

To derive the relationship between the internal moment and ρ, we will limit the analysis to the most common case of an initially straight beam that is elastically deformed by loads applied perpendicular to the beam's x axis and lying in the x–v plane of symmetry for the beam's cross-sectional area. Due to the loading, the deformation of the beam is caused by both the internal shear force and bending moment. If the beam has a length that is much greater than its depth, the greatest deformation will be caused by bending, and therefore we will direct our attention to its effects.

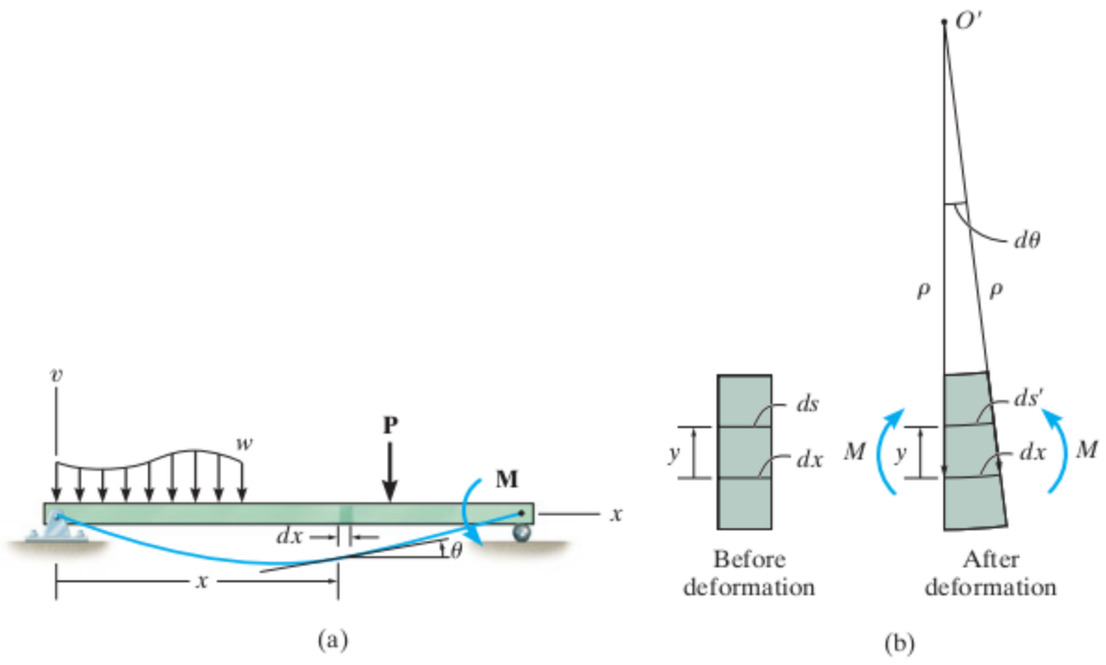
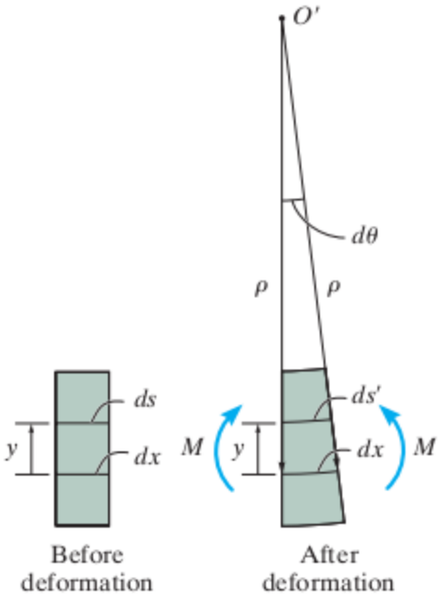


Fig. 16–5



(b)

Fig 16-5 (cont.)

When the internal moment M deforms the element of the beam, the angle between the cross sections becomes $d\theta$, Fig. 16–5*b*. The arc dx represents a portion of the elastic curve that intersects the neutral axis for each cross section. The *radius of curvature* for this arc is defined as the distance ρ, which is measured from the *center of curvature* O' to dx. Any arc on the element other than dx is subjected to a normal strain. For example, the strain in arc ds, located at a position y from the neutral axis, is $\epsilon = (ds' - ds)/ds$. However, $ds = dx = \rho\, d\theta$ and $ds' = (\rho - y)\, d\theta$, and so $\epsilon = [(\rho - y)\, d\theta - \rho\, d\theta]/\rho\, d\theta$ or

$$\frac{1}{\rho} = -\frac{\epsilon}{y} \tag{16–1}$$

If the material is homogeneous and behaves in a linear-elastic manner, then Hooke's law applies, $\epsilon = \sigma/E$. Also, since the flexure formula applies, $\sigma = -My/I$. Combining these two equations and substituting into the above equation, we have

$$\boxed{\frac{1}{\rho} = \frac{M}{EI}} \tag{16–2}$$

where

ρ = the radius of curvature at the point on the elastic curve ($1/\rho$ is referred to as the *curvature*)
M = the internal moment in the beam at the point
E = the material's modulus of elasticity
I = the beam's moment of inertia about the neutral axis

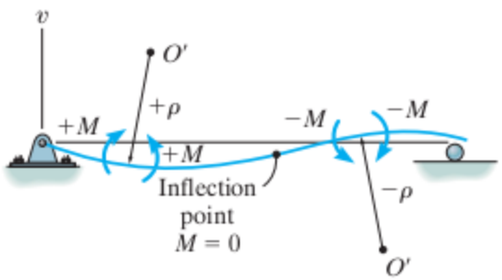


Fig. 16–6

The product EI in this equation is referred to as the *flexural rigidity*, and it is always a positive quantity. The sign for ρ therefore depends on the direction of the moment. As shown in Fig. 16–6, when M is *positive*, ρ extends *above* the beam, i.e., in the positive v direction; when M is *negative*, ρ extends *below* the beam, or in the negative v direction.

Using the flexure formula, $\sigma = -My/I$, we can also express the curvature in terms of the stress in the beam, namely,

$$\frac{1}{\rho} = -\frac{\sigma}{Ey} \tag{16–3}$$

Both Eqs. 16–2 and 16–3 are valid for either small or large radii of curvature. However, the value of ρ is almost always calculated as a *very large quantity*. For example, consider an A-36 steel beam made from a W14 × 53 (Appendix C), where $E_{st} = 29(10^3)$ ksi and $\sigma_Y = 36$ ksi. When the material at the outer fibers, $y = \pm 7$ in., is about to *yield*, then, from Eq. 16–3, $\rho = \pm 5639$ in. Values of ρ calculated at other points along the beam's elastic curve may be even *larger*, since σ cannot exceed σ_Y at the outer fibers.

16.2 Slope and Displacement by Integration

The equation of the elastic curve for a beam can be expressed mathematically as $v = f(x)$. To obtain this equation, we must first represent the curvature $(1/\rho)$ in terms of v and x. In most calculus books it is shown that this relationship is

$$\frac{1}{\rho} = \frac{d^2v/dx^2}{[1 + (dv/dx)^2]^{3/2}}$$

Substituting into Eq. 16–2, we have

$$\frac{d^2v/dx^2}{[1 + (dv/dx)^2]^{3/2}} = \frac{M}{EI} \tag{16–4}$$

This equation represents a nonlinear second-order differential equation. Its solution, which is called the *elastica*, gives the exact shape of the elastic curve, assuming, of course, that beam deflections occur only due to bending. Through the use of higher mathematics, elastica solutions have been obtained only for simple cases of beam geometry and loading.

In order to facilitate the solution of a greater number of deflection problems, Eq. 16–4 can be modified. Most engineering design codes specify *limitations* on deflections for tolerance or esthetic purposes, and as a result the elastic deflections for the majority of beams and shafts form a shallow curve. Consequently, the *slope* of the elastic curve, which is determined from dv/dx, will be *very small*, and its square will be negligible compared with unity.* Therefore the curvature, as defined above, can be *approximated* by $1/\rho = d^2v/dx^2$. Using this simplification, Eq. 16–4 can now be written as

$$\frac{d^2v}{dx^2} = \frac{M}{EI} \tag{16–5}$$

It is also possible to write this equation in two alternative forms. If we differentiate each side with respect to x and substitute $V = dM/dx$ (Eq. 11–2), we get

$$\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right) = V(x) \tag{16–6}$$

Differentiating again, using (Eq. 11–1), yields

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = w(x) \tag{16–7}$$

*See Example 16.1.

For most problems the flexural rigidity (EI) will be constant along the length of the beam. Assuming this to be the case, the above results may be reordered into the following set of three equations:

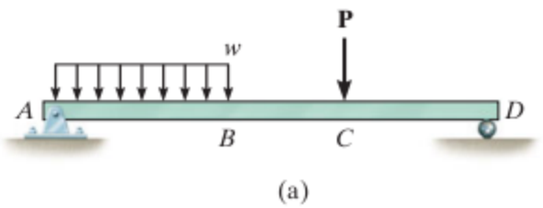
$$EI\frac{d^4v}{dx^4} = w(x) \tag{16–8}$$

$$EI\frac{d^3v}{dx^3} = V(x) \tag{16–9}$$

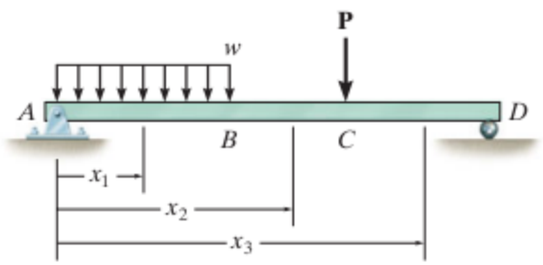
$$EI\frac{d^2v}{dx^2} = M(x) \tag{16–10}$$

Solution of any of these equations requires successive integrations to obtain the deflection v of the elastic curve. For each integration it is necessary to introduce a "constant of integration" and then solve for all the constants to obtain a unique solution for a particular problem. For example, if the distributed load w is expressed as a function of x and Eq. 16–8 is used, then four constants of integration must be evaluated; however, if the internal moment M is determined and Eq. 16–10 is used, only two constants of integration must be found. The choice of which equation to start with depends on the problem. Generally, however, it is easier to determine the internal moment M as a function of x, integrate twice, and evaluate only two integration constants.

Recall from Sec. 11.1 that if the loading on a beam is discontinuous, that is, consists of a series of several distributed and concentrated loads, then several functions must be written for the internal moment, each valid within the region between the discontinuities. Also, for convenience in writing each moment expression, *the origin* for each x coordinate can be *selected arbitrarily*. For example, consider the beam shown in Fig. 16–7*a*. The internal moment in regions AB, BC, and CD can be written in terms of the x_1, x_2, and x_3 coordinates selected, as shown in either Fig. 16–7*b* or Fig. 16–7*c*, or in fact in any manner that will yield $M = f(x)$ in as simple a form as possible. Once these functions are integrated twice through the use of Eq. 16–10 and the constants of integration determined, the functions will give the slope and deflection (elastic curve) for each region of the beam for which they are valid.



(a)



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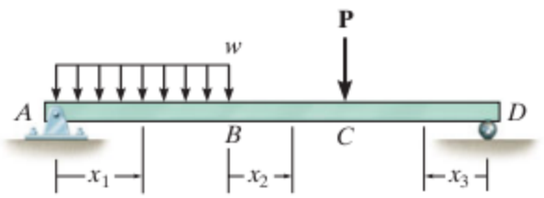


(c)

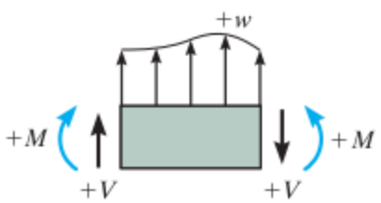
Fig. 16–7

Sign Convention and Coordinates. When applying Eqs. 16–8 through 16–10, it is important to use the proper signs for M, V, or w, as established by the sign convention that was used in the derivation of these equations. For review, these terms are shown in their *positive directions* in Fig. 16–8*a*. Furthermore, recall that *positive deflection*, v, is *upward*, and as a result, the *positive slope angle* θ will be measured *counterclockwise* from the x axis when x is *positive to the right*. The reason for this is shown in Fig. 16–8*b*. Here positive increases dx and dv in x and v create an increased θ that is counterclockwise. If, however, *positive* x is directed to the *left*, then θ will be *positive clockwise*, Fig. 16–8*c*.

Realize that by assuming dv/dx to be very small, the original horizontal length of the beam's axis and the arc of its elastic curve will be about the same. In other words, ds in Fig. 16–8*b* and 16–8*c* is approximately equal to dx, since $ds = \sqrt{(dx)^2 + (dv)^2} = \sqrt{1 + (dv/dx)^2}\,dx \approx dx$. As a result, points on the elastic curve are assumed to be *displaced vertically*, and not horizontally. Also, since the *slope angle* θ will be *very small*, its value in radians can be determined *directly* from $\theta \approx \tan\theta = dv/dx$.

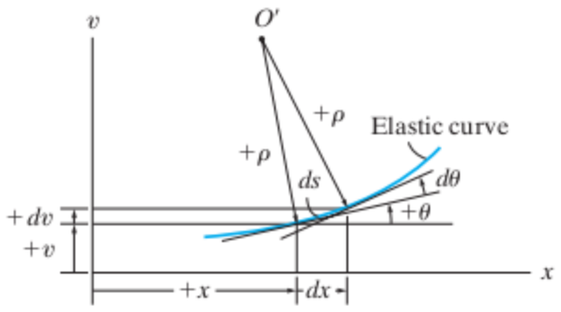
The design of a roof system requires a careful consideration of deflection. For example, rain can accumulate on areas of the roof, which then causes ponding, leading to further deflection, then further ponding, and finally possible failure of the roof.

16



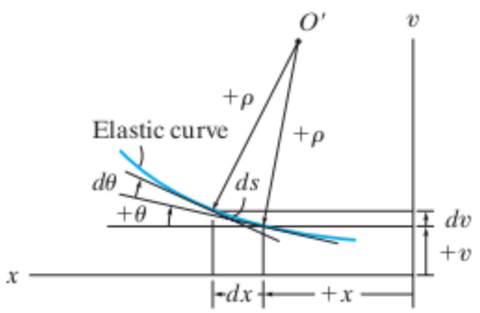
Positive sign convention

(a)



Positive sign convention

(b)

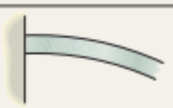


Positive sign convention

(c)

Fig. 16–8

TABLE 16–1

1		$\Delta = 0$ $M = 0$ Roller
2		$\Delta = 0$ $M = 0$ Pin
3		$\Delta = 0$ Roller
4		$\Delta = 0$ Pin
5		$\theta = 0$ $\Delta = 0$ Fixed end
6		$V = 0$ $M = 0$ Free end
7		$M = 0$ Internal pin or hinge

16

Boundary and Continuity Conditions. When solving Eqs. 16–8, 16–9, or 16–10, the constants of integration are determined by evaluating the functions for shear, moment, slope, or displacement at a particular point on the beam where the value of the function is known. These values are called ***boundary conditions***. Several possible boundary conditions that are often used to solve beam (or shaft) deflection problems are listed in Table 16–1. For example, if the beam is supported by a roller or pin (1, 2, 3, 4), then it is required that the displacement be *zero* at these points. Furthermore, if these supports are located at the *ends of the beam* (1, 2), the internal moment in the beam must also be zero. At the fixed support (5), the slope and displacement are both zero, whereas the free-ended beam (6) has both zero moment and zero shear. Lastly, if two segments of a beam are connected by an "internal" pin or hinge (7), the moment must be zero at this connection.

If the elastic curve cannot be expressed using a single coordinate, then ***continuity conditions*** must be used to evaluate some of the integration constants. For example, consider the beam in Fig. 16–9*a*. Here two x coordinates are chosen with origins at A. Each is valid only within the regions $0 \le x_1 \le a$ and $a \le x_2 \le (a + b)$ Once the functions for the slope and deflection are obtained, they must give the *same values* for the slope and deflection at point B so the elastic curve is physically *continuous*. Expressed mathematically, this requires that $\theta_1(a) = \theta_2(a)$ and $v_1(a) = v_2(a)$. These conditions can be used to evaluate two constants of integration. If instead the elastic curve is expressed in terms of the coordinates $0 \le x_1 \le a$ and $0 \le x_2 \le b$, shown in Fig. 16–9*b*, then the continuity of slope and deflection at B requires $\theta_1(a) = -\theta_2(b)$ and $v_1(a) = v_2(b)$. In this particular case, a *negative* sign is necessary to match the slopes at B since x_1 extends positive to the right, whereas x_2 extends positive to the left. Consequently, θ_1 is positive counterclockwise, and θ_2 is positive clockwise. See Figs. 16–8*b* and 16–8*c*.

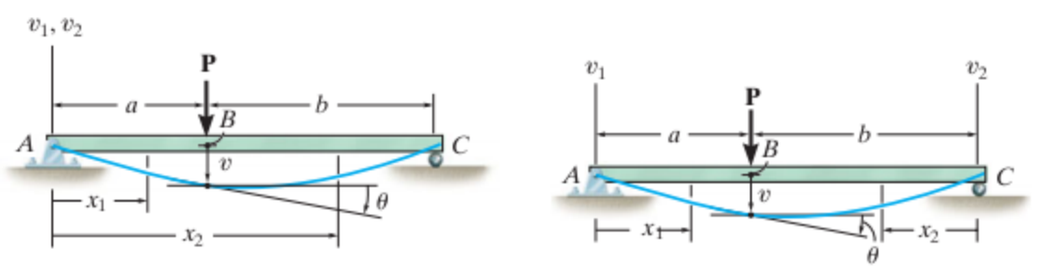


Fig. 16–9

Procedure for Analysis

The following procedure provides a method for determining the slope and deflection of a beam (or shaft) using the method of integration.

Elastic Curve.

- Draw an exaggerated view of the beam's elastic curve. Recall that zero slope and zero displacement occur at all fixed supports, and zero displacement occurs at all pin and roller supports.
- Establish the x and v coordinate axes. The x axis must be parallel to the undeflected beam and can have an origin at any point along the beam, with a positive direction either to the right or to the left. In all cases, the associated positive v axis should be directed upward.
- If several discontinuous loads are present, establish x coordinates that are valid for each region of the beam between the discontinuities. Choose these coordinates so that they will simplify subsequent algebraic work.

Load or Moment Function.

- For each region in which there is an x coordinate, express the loading w or the internal moment M as a function of x. In particular, *always* assume that M acts in the *positive direction* when applying the equation of moment equilibrium to determine $M = f(x)$.

Slope and Elastic Curve.

- Provided EI is constant, apply either the load equation $EI\, d^4v/dx^4 = w(x)$, which requires four integrations to get $v = v(x)$, or the moment equation $EI\, d^2v/dx^2 = M(x)$, which requires only two integrations. For each integration it is important to include a constant of integration.
- The constants are evaluated using the boundary conditions for the supports (Table 16–1) and the continuity conditions that apply to slope and displacement at points where two functions meet. Once the constants are evaluated and substituted back into the slope and deflection equations, the slope and displacement at *specific points* on the elastic curve can then be determined.
- The numerical values obtained can be checked graphically by comparing them with the sketch of the elastic curve. Realize that *positive* values for *slope* are *counterclockwise* if the x axis extends *positive* to the *right*, and *clockwise* if the x axis extends *positive* to the *left*. In either of these cases, *positive displacement* is *upward*.

EXAMPLE 16.1

The cantilevered beam shown in Fig. 16–10*a* is subjected to a vertical load **P** at its end. Determine the equation of the elastic curve. *EI* is constant.

SOLUTION I

Elastic Curve. The load tends to deflect the beam as shown in Fig. 16–10*a*. By inspection, the internal moment can be represented throughout the beam using a single x coordinate.

Moment Function. From the free-body diagram, with **M** acting in the *positive direction*, Fig. 16–10*b*, we have

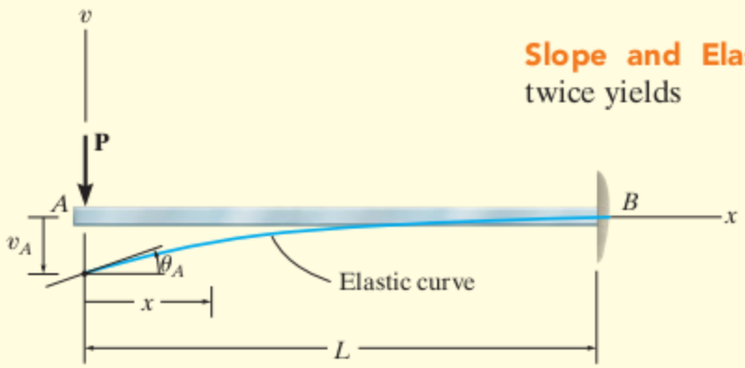
$$M = -Px$$

Slope and Elastic Curve. Applying Eq. 16–10 and integrating twice yields

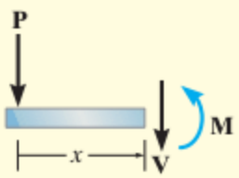
$$EI\frac{d^2v}{dx^2} = -Px \tag{1}$$

$$EI\frac{dv}{dx} = -\frac{Px^2}{2} + C_1 \tag{2}$$

$$EIv = -\frac{Px^3}{6} + C_1x + C_2 \tag{3}$$



(a)



(b)

Fig. 16–10

Using the boundary conditions $dv/dx = 0$ at $x = L$ and $v = 0$ at $x = L$, Eqs. (2) and (3) become

$$0 = -\frac{PL^2}{2} + C_1$$

$$0 = -\frac{PL^3}{6} + C_1L + C_2$$

Thus, $C_1 = PL^2/2$ and $C_2 = -PL^3/3$. Substituting these results into Eqs. (2) and (3) with $\theta = dv/dx$, we get

$$\theta = \frac{P}{2EI}(L^2 - x^2)$$

$$v = \frac{P}{6EI}(-x^3 + 3L^2x - 2L^3) \qquad \textit{Ans.}$$

Maximum slope and displacement occur at $A\,(x = 0)$, for which

$$\theta_A = \frac{PL^2}{2EI} \tag{4}$$

$$v_A = -\frac{PL^3}{3EI} \tag{5}$$

The *positive* result for θ_A indicates *counterclockwise* rotation and the *negative* result for v_A indicates that v_A is *downward*. This agrees with the results sketched in Fig. 16–10*a*.

In order to obtain some idea as to the actual *magnitude* of the slope and displacement at the end A, consider the beam in Fig. 16–10*a* to have a length of 15 ft, support a load of $P = 6$ kip, and be made of A-36 steel having $E_{st} = 29(10^3)$ ksi. Using the methods of Sec. 15.2, if this beam was designed without a factor of safety by assuming the allowable normal stress is equal to the yield stress $\sigma_{allow} = 36$ ksi; then a W12 × 26 would be found to be adequate ($I = 204\ \text{in}^4$). From Eqs. (4) and (5) we get

$$\theta_A = \frac{6\ \text{kip}(15\ \text{ft})^2(12\ \text{in.}/\text{ft})^2}{2[29(10^3)\ \text{kip}/\text{in}^2](204\ \text{in}^4)} = 0.0164\ \text{rad}$$

$$v_A = -\frac{6\ \text{kip}(15\ \text{ft})^3(12\ \text{in.}/\text{ft})^3}{3[29(10^3)\ \text{kip}/\text{in}^2](204\ \text{in}^4)} = -1.97\ \text{in.}$$

Since $\theta_A^2 = (dv/dx)^2 = 0.000270\ \text{rad}^2 \ll 1$, this justifies the use of Eq. 16–10, rather than applying the more exact Eq. 16–4, for computing the deflection of beams. Also, since this numerical application is for a *cantilevered beam*, we have obtained *larger values* for θ and v than would have been obtained if the beam were supported using pins, rollers, or other fixed supports.

SOLUTION II

This problem can also be solved using Eq. 16–8, $EI\, d^4v/dx^4 = w(x)$. Here $w(x) = 0$ for $0 \le x \le L$, Fig. 16–10*a*, so that upon integrating once we get the form of Eq. 16–9, i.e.,

$$EI\frac{d^4v}{dx^4} = 0$$

$$EI\frac{d^3v}{dx^3} = C_1' = V$$

The shear constant C_1' can be evaluated at $x = 0$, since $V_A = -P$ (negative according to the beam sign convention, Fig. 16–8*a*). Thus, $C_1' = -P$. Integrating again yields the form of Eq. 16–10, i.e.,

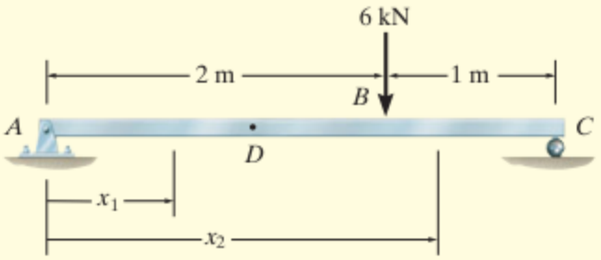
$$EI\frac{d^3v}{dx^3} = -P$$

$$EI\frac{d^2v}{dx^2} = -Px + C_2' = M$$

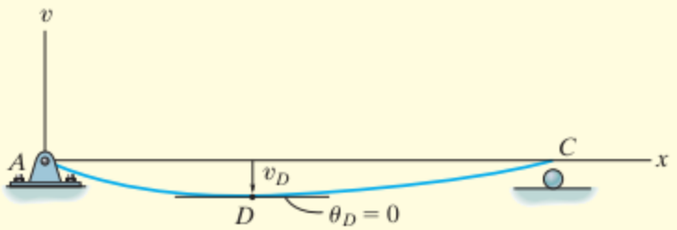
Here $M = 0$ at $x = 0$, so $C_2' = 0$, and as a result one obtains Eq. (1) and the solution proceeds as before.

EXAMPLE 16.2

The simply supported beam shown in Fig. 16–11*a* is subjected to the concentrated force. Determine the maximum deflection of the beam. *EI* is constant.



(a)

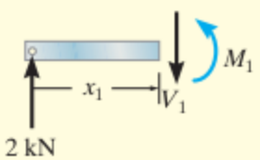


(b)

SOLUTION

Elastic Curve. The beam deflects as shown in Fig. 16–11*b*. Two coordinates must be used, since the moment function will change at *B*. Here we will take x_1 and x_2, having the *same origin* at *A*.

Moment Function. From the free-body diagrams shown in Fig. 16–11*c*,



$$M_1 = 2x_1$$

$$M_2 = 2x_2 - 6(x_2 - 2) = 4(3 - x_2)$$

Slope and Elastic Curve. Applying Eq. 16–10 for M_1, for $0 \le x_1 < 2$ m, and integrating twice yields

$$EI\frac{d^2v_1}{dx_1^2} = 2x_1$$

$$EI\frac{dv_1}{dx_1} = x_1^2 + C_1 \qquad (1)$$

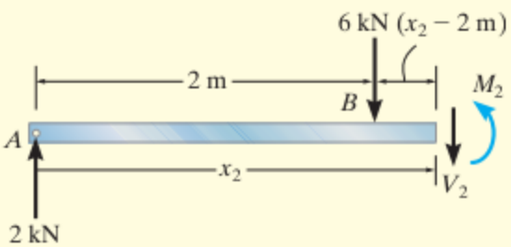
$$EIv_1 = \frac{1}{3}x_1^3 + C_1x_1 + C_2 \qquad (2)$$

Likewise for M_2, for $2\text{ m} < x_2 \le 3$ m,

$$EI\frac{d^2v_2}{dx_2^2} = 4(3 - x_2)$$

$$EI\frac{dv_2}{dx_2} = 4\left(3x_2 - \frac{x_2^2}{2}\right) + C_3 \qquad (3)$$

$$EIv_2 = 4\left(\frac{3}{2}x_2^2 - \frac{x_2^3}{6}\right) + C_3x_2 + C_4 \qquad (4)$$



(c)

Fig. 16–11

The four constants are evaluated using *two* boundary conditions, namely, $x_1 = 0$, $v_1 = 0$ and $x_2 = 3\,\text{m}$, $v_2 = 0$. Also, *two* continuity conditions must be applied at B, that is, $dv_1/dx_1 = dv_2/dx_2$ at $x_1 = x_2 = 2\,\text{m}$ and $v_1 = v_2$ at $x_1 = x_2 = 2\,\text{m}$. Substitution as specified results in the following four equations:

$$v_1 = 0 \text{ at } x_1 = 0; \qquad 0 = 0 + 0 + C_2$$

$$v_2 = 0 \text{ at } x_2 = 3\,\text{m}; \qquad 0 = 4\left(\frac{3}{2}(3)^2 - \frac{(3)^3}{6}\right) + C_3(3) + C_4$$

$$\left.\frac{dv_1}{dx_1}\right|_{x=2\,\text{m}} = \left.\frac{dv_2}{dx_2}\right|_{x=2\,\text{m}}; \quad (2)^2 + C_1 = 4\left(3(2) - \frac{(2)^2}{2}\right) + C_3$$

$$v_1(2\,\text{m}) = v_2(2\,\text{m}); \qquad \frac{1}{3}(2)^3 + C_1(2) + C_2 = 4\left(\frac{3}{2}(2)^2 - \frac{(2)^3}{6}\right) + C_3(2) + C_4$$

Solving, we get

$$C_1 = -\frac{8}{3} \qquad C_2 = 0$$

$$C_3 = -\frac{44}{3} \qquad C_4 = 8$$

Thus Eqs. (1) through (4) become

$$EI\frac{dv_1}{dx_1} = x_1^2 - \frac{8}{3} \tag{5}$$

$$EIv_1 = \frac{1}{3}x_1^3 - \frac{8}{3}x_1 \tag{6}$$

$$EI\frac{dv_2}{dx_2} = 12x_2 - 2x_2^2 - \frac{44}{3} \tag{7}$$

$$EIv_2 = 6x_2^2 - \frac{2}{3}x_2^3 - \frac{44}{3}x_2 + 8 \tag{8}$$

By inspection of the elastic curve, Fig. 16–11*b*, the maximum deflection occurs at D, somewhere within region AB. Here the slope must be zero. From Eq. (5),

$$x_1^2 - \frac{8}{3} = 0$$

$$x_1 = 1.633$$

Substituting into Eq. (6),

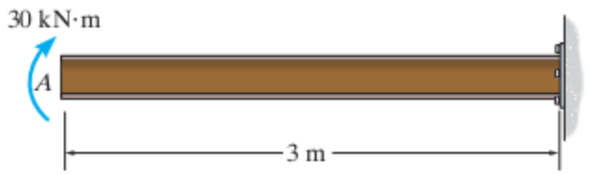
$$v_{\max} = -\frac{2.90\,\text{kN}\cdot\text{m}^3}{EI} \qquad \textit{Ans.}$$

The negative sign indicates that the deflection is downward.

16

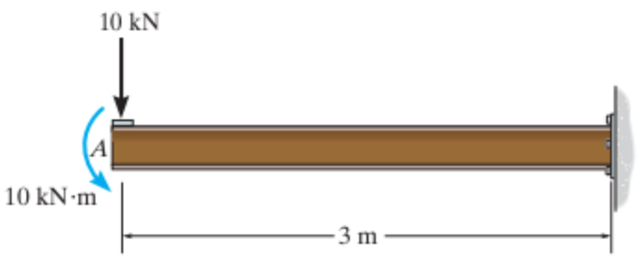
FUNDAMENTAL PROBLEMS

F16–1. Determine the slope and deflection of end A of the cantilevered beam. $E = 200$ GPa and $I = 65.0(10^6)$ mm^4.



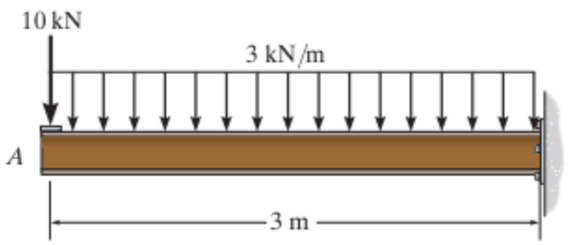
F16–1

F16–2. Determine the slope and deflection of end A of the cantilevered beam. $E = 200$ GPa and $I = 65.0(10^6)$ mm^4.



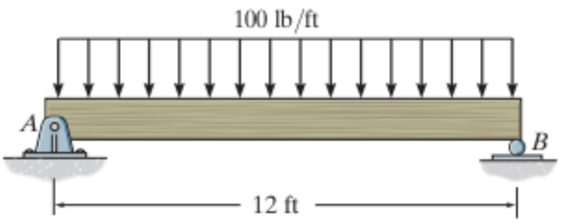
F16–2

F16–3. Determine the slope of end A of the cantilevered beam. $E = 200$ GPa and $I = 65.0(10^6)$ mm^4.



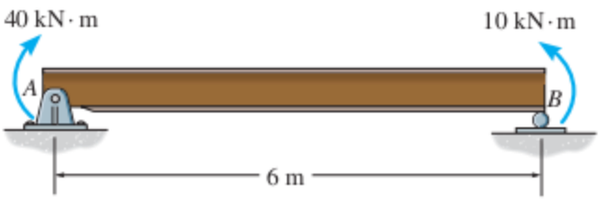
F16–3

F16–4. Determine the maximum deflection of the simply supported beam. The beam is made of wood having a modulus of elasticity of $E_w = 1.5(10^3)$ ksi and a rectangular cross section of $b = 3$ in. and $h = 6$ in.



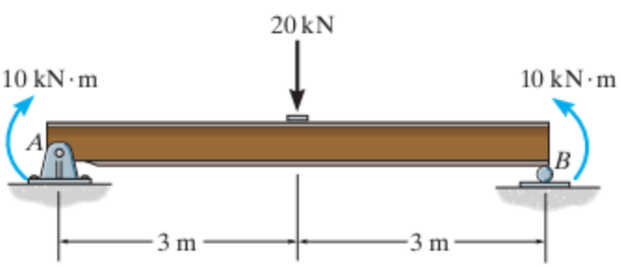
F16–4

F16–5. Determine the maximum deflection of the simply supported beam. $E = 200$ GPa and $I = 39.9(10^{-6})$ m^4.



F16–5

F16–6. Determine the slope of the simply supported beam at A. $E = 200$ GPa and $I = 39.9(10^{-6})$ m^4.

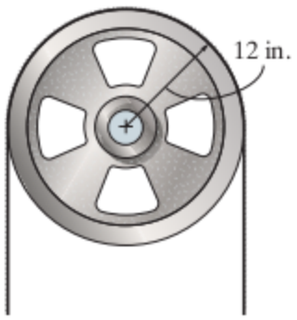


F16–6

PROBLEMS

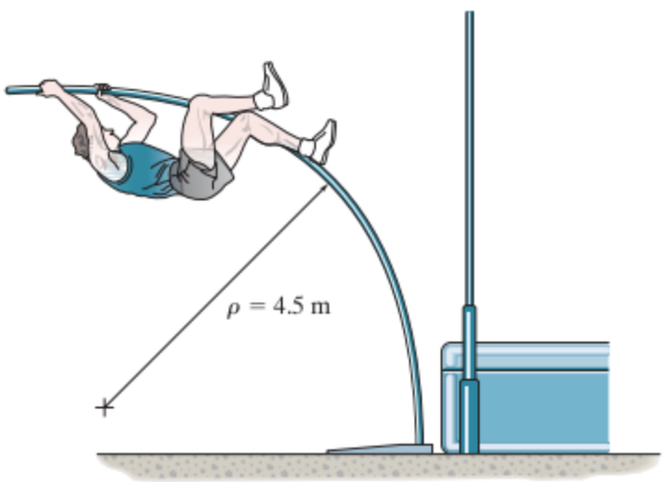
16–1. An L2 steel strap having a thickness of 0.125 in. and a width of 2 in. is bent into a circular arc of radius 600 in. Determine the maximum bending stress in the strap.

16–2. The L2 steel blade of the band saw wraps around the pulley having a radius of 12 in. Determine the maximum normal stress in the blade. The blade has a width of 0.75 in. and a thickness of 0.0625 in.



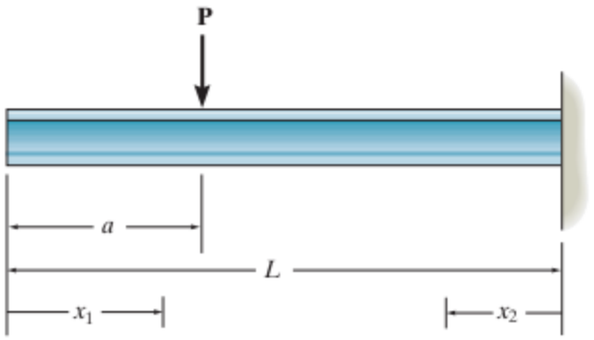
Prob. 16–2

16–3. A picture is taken of a man performing a pole vault, and the minimum radius of curvature of the pole is estimated by measurement to be 4.5 m. If the pole is 40 mm in diameter and it is made of a glass-reinforced plastic for which $E_g = 131$ GPa, determine the maximum bending stress in the pole.



Prob. 16–3

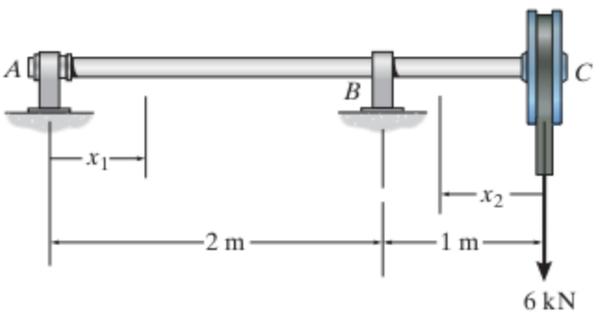
***16–4.** Determine the equations of the elastic curve using the x_1 and x_2 coordinates. EI is constant.



Prob. 16–4

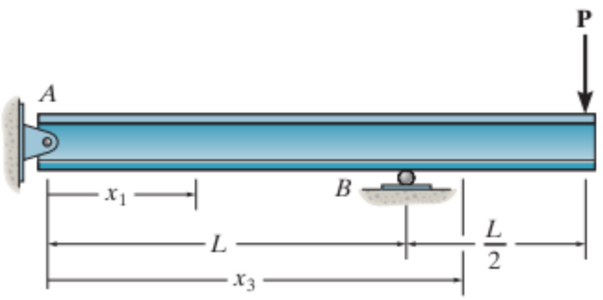
16

16–5. Determine the deflection of end C of the 100-mm-diameter solid circular shaft. The shaft is made of steel having a modulus elasticity of $E = 200$ GPa.



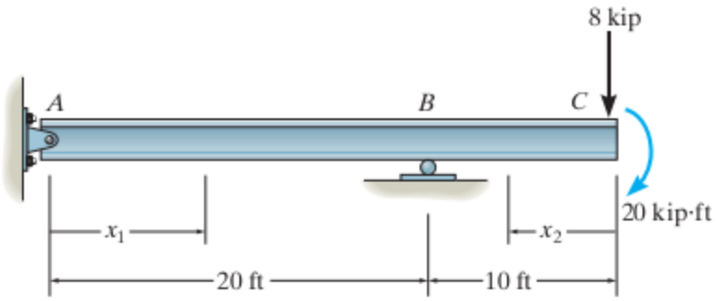
Prob. 16–5

16–6. Determine the equations of the elastic curve for the beam using the x_1 and x_3 coordinates. Specify the beam's maximum deflection. EI is constant.



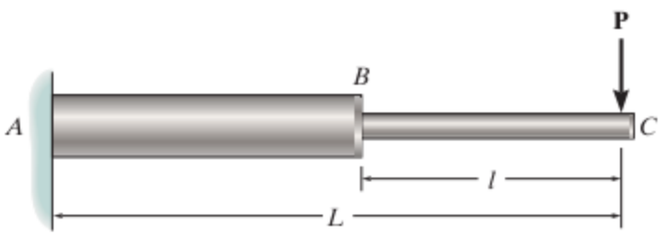
Prob. 16–6

16–7. Determine the equations of the elastic curve using the x_1 and x_2 coordinates, and specify the slope at A and the deflection at C. EI is constant.



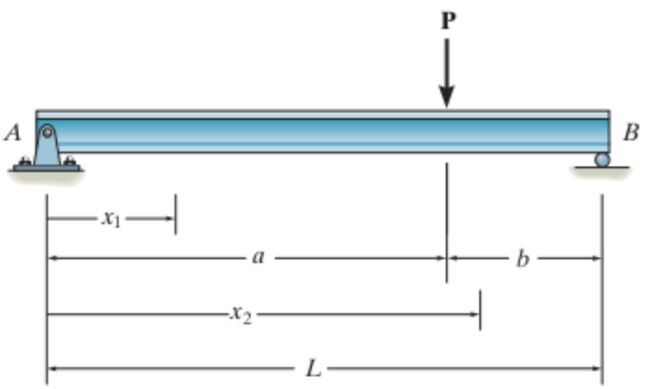
Prob. 16–7

***16–8.** The beam is made of two rods and is subjected to the concentrated load **P**. Determine the maximum deflection of the beam if the moments of inertia of the rods are I_{AB} and I_{BC}, and the modulus of elasticity is E.



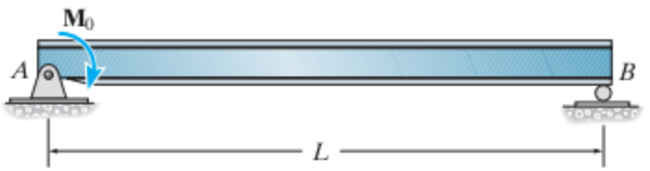
Prob. 16–8

16–9. Determine the equations of the elastic curve using the x_1 and x_2 coordinates. EI is constant.



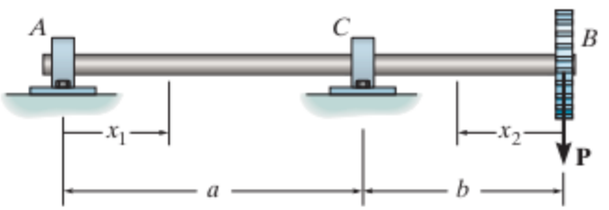
Prob. 16–9

16–10. Determine the maximum slope and maximum deflection of the simply supported beam which is subjected to the couple moment $\mathbf{M}_0$. EI is constant.



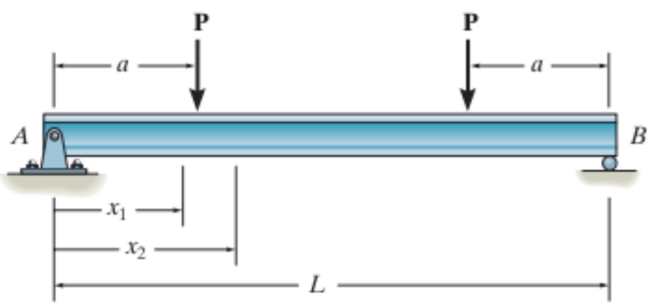
Prob. 16–10

16–11. The shaft is supported at A by a journal bearing that exerts only vertical reactions on the shaft, and at C by a thrust bearing that exerts horizontal and vertical reactions on the shaft. Determine the equations of the elastic curve using the coordinates x_1 and x_2. EI is constant.



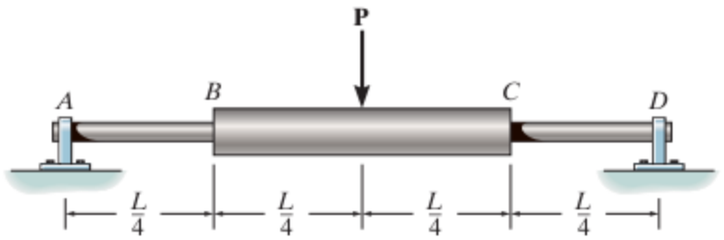
Prob. 16–11

***16–12.** Determine the equations of the elastic curve for the beam using the x_1 and x_2 coordinates. Specify the slope at A and the maximum displacement of the shaft. EI is constant.



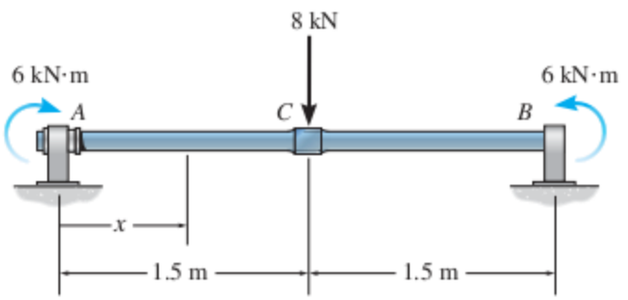
Prob. 16–12

16–13. The simply supported shaft has a moment of inertia of $2I$ for region BC and a moment of inertia I for regions AB and CD. Determine the maximum deflection of the shaft due to the load **P**.



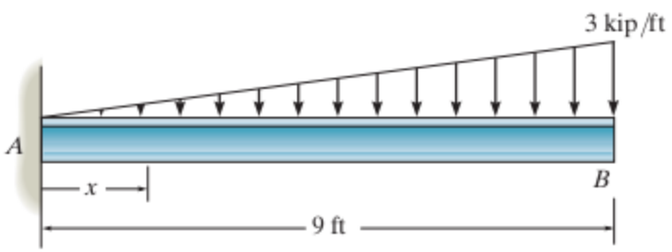
Prob. 16–13

16–14. Determine the maximum deflection of the solid circular shaft. The shaft is made of steel having $E = 200$ GPa. It has a diameter of 100 mm.



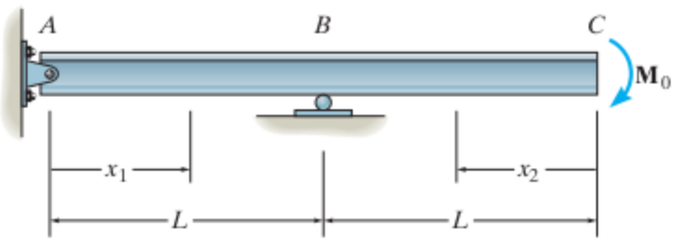
Prob. 16–14

16–15. Determine the elastic curve for the cantilevered W14 × 30 beam using the x coordinate. Specify the maximum slope and maximum deflection. $E = 29(10^3)$ ksi.



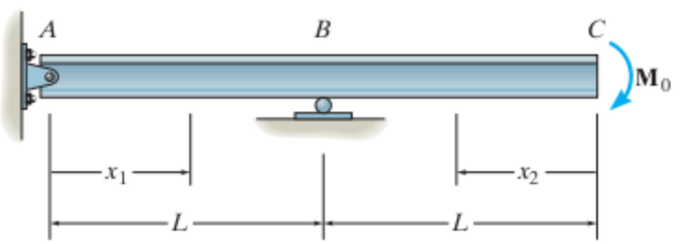
Prob. 16–15

***16–16.** Determine the equations of the elastic curve using the coordinates x_1 and x_2, and specify the deflection and slope at C. EI is constant.



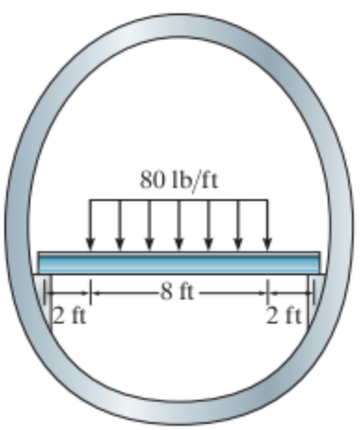
Prob. 16–16

16–17. Determine the equations of the elastic curve using the coordinates x_1 and x_2, and specify the slope at A. EI is constant.



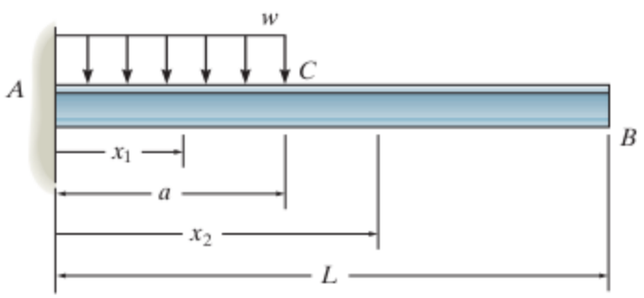
Prob. 16–17

16–18. The floor beam of the airplane is subjected to the loading shown. Assuming that the fuselage exerts only vertical reactions on the ends of the beam, determine the maximum deflection of the beam. EI is constant.



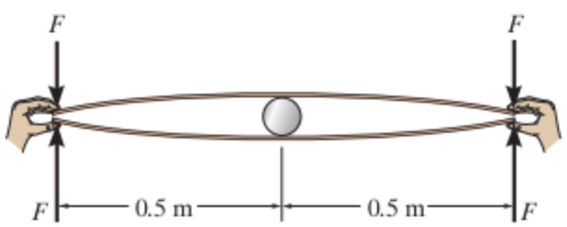
Prob. 16–18

16–19. Determine the equations of the elastic curve using the coordinates x_1 and x_2, and specify the slope and deflection at B. EI is constant.



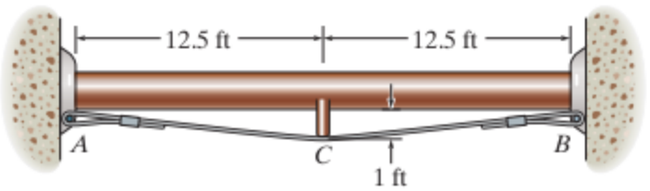
Prob. 16–19

*16–20. The two wooden meter sticks are separated at their centers by a smooth rigid cylinder having a diameter of 50 mm. Determine the force F that must be applied at each end in order to just make their ends touch. Each stick has a width of 20 mm and a thickness of 5 mm. $E_w = 11$ GPa.



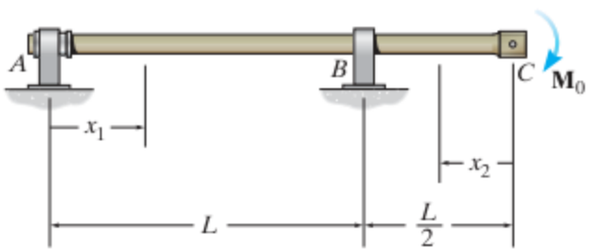
Prob. 16–20

16–21. The pipe can be assumed roller supported at its ends and by a rigid saddle C at its center. The saddle rests on a cable that is connected to the supports. Determine the force that should be developed in the cable if the saddle keeps the pipe from sagging or deflecting at its center. The pipe and fluid within it have a combined weight of 125 lb/ft. EI is constant.



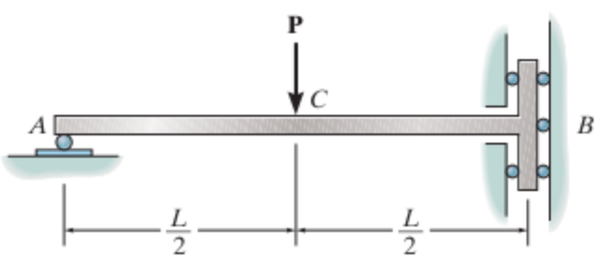
Prob. 16–21

16–22. Determine the elastic curve in terms of the x_1 and x_2 coordinates. What is the deflection of end C of the shaft? EI is constant.



Prob. 16–22

16–23. The bar is supported by a roller constraint at B, which allows vertical displacement but resists axial load and moment. If the bar is subjected to the loading shown, determine the slope at A and the deflection at C. EI is constant.



Prob. 16–23

*16.3 Discontinuity Functions

The method of integration, used to find the equation of the elastic curve for a beam or shaft, is convenient if the load or internal moment can be expressed as a continuous function throughout the beam's entire length. If several different loadings act on the beam, however, the method becomes more tedious to apply, because separate loading or moment functions must be written for each region of the beam. Furthermore, integration of these functions requires the evaluation of integration constants using both boundary and continuity conditions. For example, the beam shown in Fig. 16–12 requires four moment functions to be written. They describe the moment in regions AB, BC, CD, and DE. When applying the moment-curvature relationship, $EI\,d^2v/dx^2 = M$, and integrating each moment equation twice, we must evaluate eight constants of integration. These involve two boundary conditions that require zero displacement at points A and E, and six continuity conditions for both slope and displacement at points B, C, and D.

In this section, we will discuss a method for finding the equation of the elastic curve for a *multiply loaded beam* using a *single expression*, either formulated from the loading on the beam, $w = w(x)$, or from the beam's internal moment, $M = M(x)$. If the expression for w is substituted into $EI\,d^4v/dx^4 = w(x)$ and integrated four times, or if the expression for M is substituted into $EI\,d^2v/dx^2 = M(x)$ and integrated twice, the constants of integration will be determined only from the boundary conditions. Since the continuity equations will not be involved, the analysis will be greatly simplified.

For safety purposes theses cantilevered beams that support sheets of plywood must be designed for both strength and a restricted amount of deflection.

16

Discontinuity Functions. In order to express the load on the beam or the internal moment within it using a single expression, we will use two types of mathematical operators known as *discontinuity functions*.

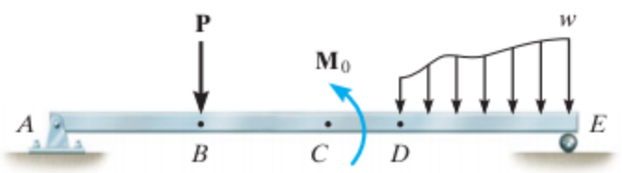
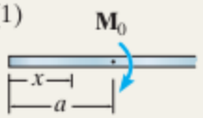
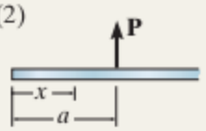
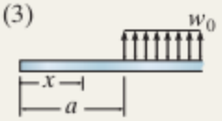
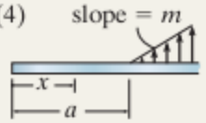


Fig. 16–12

TABLE 16–2

Loading	Loading Function $w = w(x)$	Shear $V = \int w(x)dx$	Moment $M = \int V dx$
(1) 	$w = M_0\langle x-a\rangle^{-2}$	$V = M_0\langle x-a\rangle^{-1}$	$M = M_0\langle x-a\rangle^{0}$
(2) 	$w = P\langle x-a\rangle^{-1}$	$V = P\langle x-a\rangle^{0}$	$M = P\langle x-a\rangle^{1}$
(3) 	$w = w_0\langle x-a\rangle^{0}$	$V = w_0\langle x-a\rangle^{1}$	$M = \frac{w_0}{2}\langle x-a\rangle^{2}$
(4) 	$w = m\langle x-a\rangle^{1}$	$V = \frac{m}{2}\langle x-a\rangle^{2}$	$M = \frac{m}{6}\langle x-a\rangle^{3}$

16

Macaulay Functions. For purposes of beam or shaft deflection, Macaulay functions, named after the mathematician W. H. Macaulay, can be used to describe ***distributed loadings***. These functions can be written in general form as

$$\langle x-a\rangle^n = \begin{cases} 0 & \text{for } x < a \\ (x-a)^n & \text{for } x \geq a \end{cases} \qquad n \geq 0 \tag{16–11}$$

Here x represents the coordinate position of a point along the beam, and a is the location on the beam where a "discontinuity" occurs, namely the point where a distributed loading *begins*. Note that the Macaulay function $\langle x-a\rangle^n$ is written with angle (or Macaulay) brackets to distinguish it from the ordinary function $(x-a)^n$, written with parentheses. As stated by the equation, only when $x \geq a$ is $\langle x-a\rangle^n = (x-a)^n$, otherwise it is zero. Furthermore, these functions are valid only for exponential values $n \geq 0$. Integration of Macaulay functions follows the same rules as for ordinary functions, i.e.,

$$\int \langle x-a\rangle^n dx = \frac{\langle x-a\rangle^{n+1}}{n+1} + C \tag{16–12}$$

Note how the Macaulay functions describe both the *uniform load* w_0 $(n = 0)$ and *triangular load* $(n = 1)$, shown in Table 16–2, items 3 and 4. This type of description can, of course, be extended to distributed loadings having other forms. Also, it is possible to use superposition with

the uniform and triangular loadings to create the Macaulay function for a trapezoidal loading. Using integration, the Macaulay functions for shear, $V = \int w(x)\,dx$, and moment, $M = \int V\,dx$, are also shown in the table.

Singularity Functions. These functions are only used to describe the point location of concentrated forces or couple moments acting on a beam or shaft. Specifically, a concentrated force **P** can be considered as a special case of infinitely large distributed loading, where the intensity of the loading is $w = P/\epsilon$ such that its length is ϵ, where $\epsilon \to 0$, Fig. 16–13. The area under this loading diagram is equivalent to P, *positive upwards*, and has this value only when $x = a$. We will use a symbolic representation w to express this result,

$$w = P\langle x - a\rangle^{-1} = \begin{cases} 0 & \text{for } x \neq a \\ P & \text{for } x = a \end{cases} \tag{16–13}$$

This expression is referred to as a ***singularity function***, since it takes on the value P only at the point $x = a$ where the load acts, otherwise it is zero.*

In a similar manner, a couple moment $\mathbf{M}_0$, considered *positive clockwise*, is a limit as $\epsilon \to 0$ of two distributed loadings as shown in Fig. 16–14. Here the following function describes its value.

$$w = M_0\langle x - a\rangle^{-2} = \begin{cases} 0 & \text{for } x \neq a \\ M_0 & \text{for } x = a \end{cases} \tag{16–14}$$

The exponent $n = -2$, in order to ensure that the units of w, force per length, are maintained.

Integration of the above two singularity functions follow the rules of operational calculus and yields results that are *different* from those of Macaulay functions. Specifically,

$$\int \langle x - a\rangle^n dx = \langle x - a\rangle^{n+1}, n = -1, -2 \tag{16–15}$$

Using this formula, notice how M_0 and P, described in Table 16–2, items 1 and 2, are integrated once, then twice, to obtain the internal shear and moment in the beam.

Application of Eqs. 16–11 through Eq. 16–15 provides a rather direct means for expressing the loading or the internal moment in a beam as a function of x. When doing so, close attention must be paid to the signs of the external loadings. As stated above, and as shown in Table 16–2, *concentrated forces and distributed loads are positive upward, and couple moments are positive clockwise*. If this sign convention is followed, then the internal shear and moment are in accordance with the beam sign convention established in Sec. 11.1.

*This point load function $\langle x - a\rangle^{-1}$ is sometimes referred to as a unit impulse function or the Dirac delta.

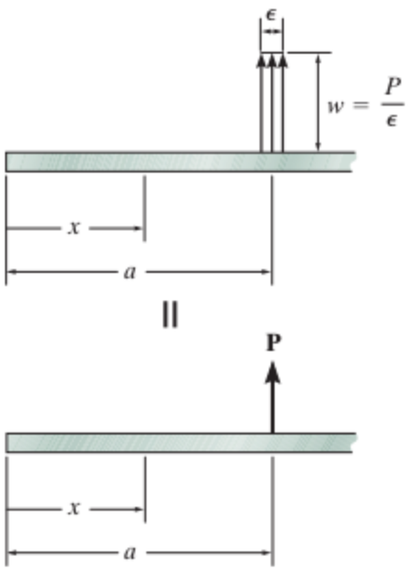


Fig. 16–13

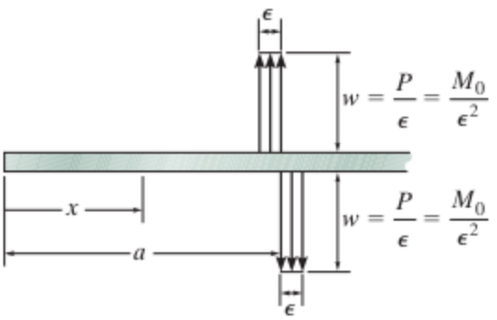


Fig. 16–14

As an example of how to apply discontinuity functions to describe the loading or internal moment consider the beam loaded as shown in Fig. 16–15*a*. Here the reactive 2.75-kN force created by the roller, Fig. 16–15*b*, is positive since it acts upward, and the 1.5-kN · m couple moment is also positive since it acts clockwise. Finally, the trapezoidal loading is negative and has been separated into triangular and uniform loadings. From Table 16–2, the loading at any point x on the beam is therefore

$$w = 2.75\,\text{kN}\langle x - 0\rangle^{-1} + 1.5\,\text{kN}\cdot\text{m}\langle x - 3\,\text{m}\rangle^{-2} - 3\,\text{kN/m}\langle x - 3\,\text{m}\rangle^{0} - 1\,\text{kN/m}^2\langle x - 3\,\text{m}\rangle^{1}$$

The reactive force at B is not included here since x is never greater than 6 m, and furthermore, this value is of no consequence in calculating the slope or deflection. We can determine the moment expression directly from Table 16–2, rather than integrating this expression twice. In either case,

16

$$\begin{aligned} M &= 2.75\,\text{kN}\langle x - 0\rangle^{1} + 1.5\,\text{kN}\cdot\text{m}\langle x - 3\,\text{m}\rangle^{0} - \frac{3\,\text{kN/m}}{2}\langle x - 3\,\text{m}\rangle^{2} - \frac{1\,\text{kN/m}^2}{6}\langle x - 3\,\text{m}\rangle^{3} \\ &= 2.75x + 1.5\langle x - 3\rangle^{0} - 1.5\langle x - 3\rangle^{2} - \frac{1}{6}\langle x - 3\rangle^{3} \end{aligned}$$

The deflection of the beam can now be determined after this equation is integrated two successive times and the constants of integration are evaluated using the boundary conditions of zero displacement at A and B.

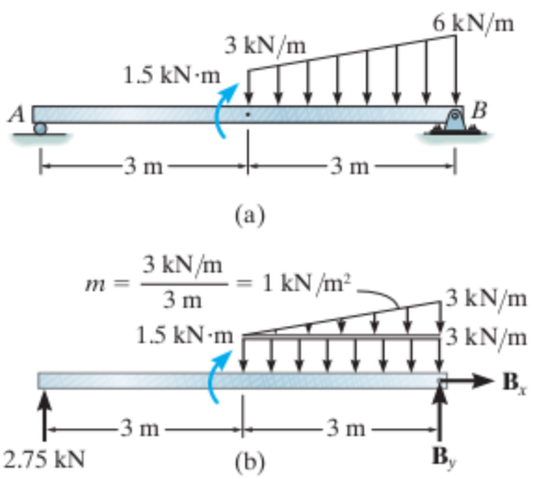


Fig. 16–15

Procedure for Analysis

The following procedure provides a method for using discontinuity functions to determine a beam's elastic curve. This method is particularly advantageous for solving problems involving beams or shafts subjected to *several loadings*, since the constants of integration can be evaluated by using *only* the boundary conditions, while the compatibility conditions are automatically satisfied.

Elastic Curve.

- Sketch the beam's elastic curve and identify the boundary conditions at the supports.
- Zero displacement occurs at all pin and roller supports, and zero slope and zero displacement occur at fixed supports.
- Establish the x axis so that it extends to the right and has its origin at the beam's left end.

Load or Moment Function.

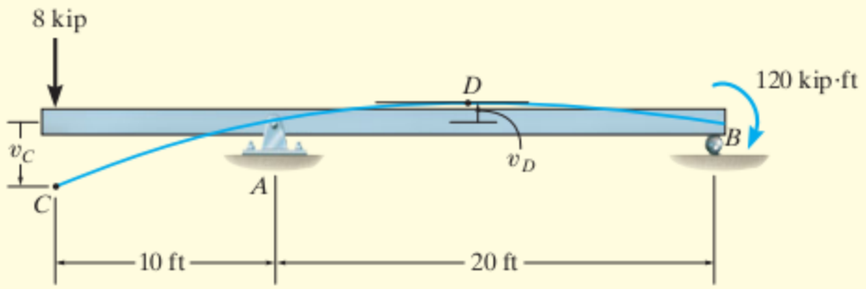
- Calculate the support reactions at $x = 0$ and then use the discontinuity functions in Table 16–2 to express either the loading w or the internal moment M as a function of x. Make sure to follow the sign convention for each loading as it applies for this equation.
- Note that the distributed loadings must extend all the way to the beam's right end to be valid. If this does not occur, use the method of superposition, which is illustrated in Example 16.4.

Slope and Elastic Curve.

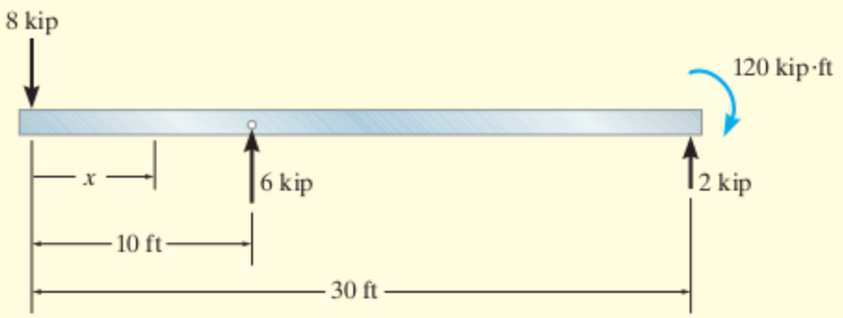
- Substitute w into $EI\, d^4v/dx^4 = w(x)$, or M into the moment curvature relation $EI\, d^2v/dx^2 = M$, and integrate to obtain the equations for the beam's slope and deflection.
- Evaluate the constants of integration using the boundary conditions, and substitute these constants into the slope and deflection equations to obtain the final results.
- When the slope and deflection equations are evaluated at any point on the beam, a *positive slope* is *counterclockwise*, and a *positive displacement* is *upward*.

EXAMPLE 16.3

Determine the maximum deflection of the beam shown in Fig. 16–16*a*. *EI* is constant.



(a)



(b)

Fig. 16–16

SOLUTION

Elastic Curve. The beam deflects as shown in Fig. 16–16*a*. The boundary conditions require zero displacement at *A* and *B*.

Loading Function. The reactions have been calculated and are shown on the free-body diagram in Fig. 16–16*b*. The loading function for the beam can be written as

$$w = -8\,\text{kip}\,\langle x - 0\rangle^{-1} + 6\,\text{kip}\,\langle x - 10\,\text{ft}\rangle^{-1}$$

The couple moment and force at *B* are not included here, since they are located at the right end of the beam, and *x* cannot be greater than 30 ft. Integrating $dV/dx = w(x)$, we get

$$V = -8\langle x - 0\rangle^{0} + 6\langle x - 10\rangle^{0}$$

In a similar manner, $dM/dx = V$ yields

$$\begin{aligned} M &= -8\langle x - 0\rangle^{1} + 6\langle x - 10\rangle^{1} \\ &= (-8x + 6\langle x - 10\rangle^{1})\,\text{kip}\cdot\text{ft} \end{aligned}$$

Notice how this equation can also be established *directly* using the results of Table 16–2 for moment.

Slope and Elastic Curve. Integrating twice yields

$$EI\frac{d^2v}{dx^2} = -8x + 6\langle x - 10\rangle^1$$

$$EI\frac{dv}{dx} = -4x^2 + 3\langle x - 10\rangle^2 + C_1$$

$$EIv = -\frac{4}{3}x^3 + \langle x - 10\rangle^3 + C_1x + C_2 \qquad (1)$$

From Eq. (1), the boundary condition $v = 0$ at $x = 10$ ft and $v = 0$ at $x = 30$ ft gives

$$0 = -1333 + (10 - 10)^3 + C_1(10) + C_2$$
$$0 = -36\,000 + (30 - 10)^3 + C_1(30) + C_2$$

Solving these equations simultaneously for C_1 and C_2, we get $C_1 = 1333$ and $C_2 = -12\,000$. Thus,

$$EI\frac{dv}{dx} = -4x^2 + 3\langle x - 10\rangle^2 + 1333 \qquad (2)$$

$$EIv = -\frac{4}{3}x^3 + \langle x - 10\rangle^3 + 1333x - 12\,000 \qquad (3)$$

From Fig. 16–16*a*, maximum displacement may occur either at C, or at D, where the slope $dv/dx = 0$. To obtain the displacement of C, set $x = 0$ in Eq. (3). We get

$$v_C = -\frac{12\,000\ \text{kip}\cdot\text{ft}^3}{EI} \qquad \textit{Ans.}$$

The *negative* sign indicates that the displacement is *downward* as shown in Fig. 16–16*a*. To locate point D, use Eq. (2) with $x > 10$ ft and $dv/dx = 0$. This gives

$$0 = -4x_D^2 + 3(x_D - 10)^2 + 1333$$
$$x_D^2 + 60x_D - 1633 = 0$$

Solving for the positive root,

$$x_D = 20.3\ \text{ft}$$

Hence, from Eq. (3),

$$EIv_D = -\frac{4}{3}(20.3)^3 + (20.3 - 10)^3 + 1333(20.3) - 12\,000$$
$$v_D = \frac{5006\ \text{kip}\cdot\text{ft}^3}{EI}$$

Comparing this value with v_C, we see that $v_{\max} = v_C$.

16

EXAMPLE 16.4

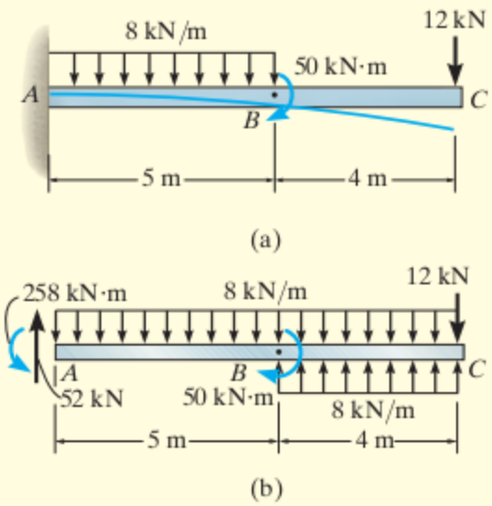


Fig. 16–17

Determine the equation of the elastic curve for the cantilevered beam shown in Fig. 16–17*a*. *EI* is constant.

SOLUTION

Elastic Curve. The loads cause the beam to deflect as shown in Fig. 16–17*a*. The boundary conditions require zero slope and displacement at *A*.

Loading Function. The support reactions at *A* have been calculated and are shown on the free-body diagram in Fig. 16–17*b*. Since the distributed loading in Fig. 16–17*a* does not extend to *C* as required, we can use the superposition of loadings shown in Fig. 16–17*b* to represent the same effect. By our sign convention, the beam's loading is therefore

$$w = 52\,\text{kN}\langle x-0\rangle^{-1} - 258\,\text{kN}\cdot\text{m}\langle x-0\rangle^{-2} - 8\,\text{kN/m}\langle x-0\rangle^{0} + 50\,\text{kN}\cdot\text{m}\langle x-5\,\text{m}\rangle^{-2} + 8\,\text{kN/m}\langle x-5\,\text{m}\rangle^{0}$$

The 12-kN load is *not included* here, since x cannot be greater than 9 m. Because $dV/dx = w(x)$, then by integrating, neglecting the constant of integration since the reactions are included in the load function, we have

$$V = 52\langle x-0\rangle^{0} - 258\langle x-0\rangle^{-1} - 8\langle x-0\rangle^{1} + 50\langle x-5\rangle^{-1} + 8\langle x-5\rangle^{1}$$

Furthermore, $dM/dx = V$, so that integrating again yields

$$M = -258\langle x-0\rangle^{0} + 52\langle x-0\rangle^{1} - \frac{1}{2}(8)\langle x-0\rangle^{2} + 50\langle x-5\rangle^{0} + \frac{1}{2}(8)\langle x-5\rangle^{2}$$
$$= (-258 + 52x - 4x^2 + 50\langle x-5\rangle^{0} + 4\langle x-5\rangle^{2})\,\text{kN}\cdot\text{m}$$

This same result can be obtained *directly* from Table 16–2.

Slope and Elastic Curve. Applying Eq. 16–10 and integrating twice, we have

$$EI\frac{d^2v}{dx^2} = -258 + 52x - 4x^2 + 50\langle x-5\rangle^{0} + 4\langle x-5\rangle^{2}$$

$$EI\frac{dv}{dx} = -258x + 26x^2 - \frac{4}{3}x^3 + 50\langle x-5\rangle^{1} + \frac{4}{3}\langle x-5\rangle^{3} + C_1$$

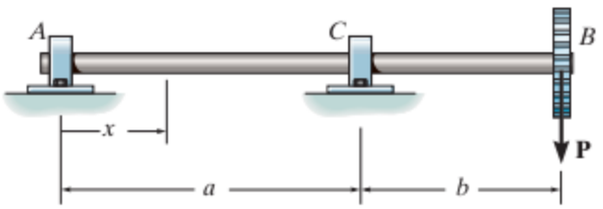
$$EIv = -129x^2 + \frac{26}{3}x^3 - \frac{1}{3}x^4 + 25\langle x-5\rangle^{2} + \frac{1}{3}\langle x-5\rangle^{4} + C_1x + C_2$$

Since $dv/dx = 0$ at $x = 0$, $C_1 = 0$; and $v = 0$ at $x = 0$, so $C_2 = 0$. Thus,

$$v = \frac{1}{EI}\left(-129x^2 + \frac{26}{3}x^3 - \frac{1}{3}x^4 + 25\langle x-5\rangle^{2} + \frac{1}{3}\langle x-5\rangle^{4}\right)\text{m} \quad \textit{Ans.}$$

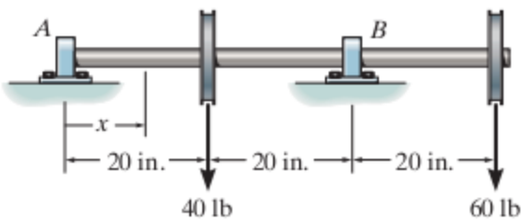
PROBLEMS

***16–24.** The shaft is supported at A by a journal bearing that exerts only vertical reactions on the shaft, and at C by a thrust bearing that exerts horizontal and vertical reactions on the shaft. Determine the equation of the elastic curve. EI is constant.



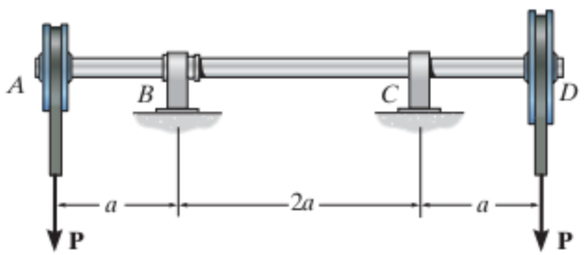
Prob. 16–24

16–25. The shaft supports the two pulley loads shown. Determine the equation of the elastic curve. The bearings at A and B exert only vertical reactions on the shaft. EI is constant.



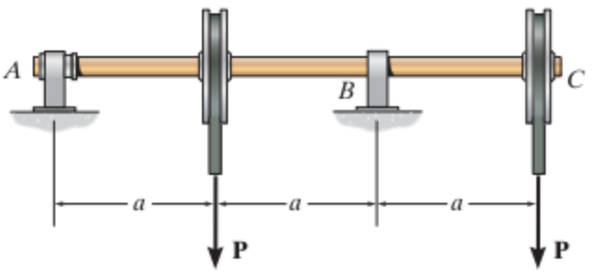
Prob. 16–25

16–26. Determine the equation of the elastic curve, the maximum deflection in region BC, and the deflection of end A of the shaft. EI is constant.



Prob. 16–26

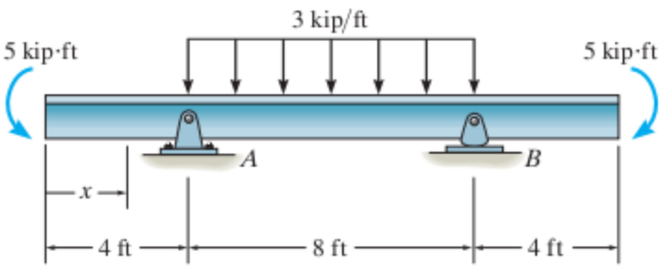
16–27. Determine the equation of the elastic curve, the maximum deflection in region AB, and the deflection of end C of the shaft. EI is constant.



Prob. 16–27

16

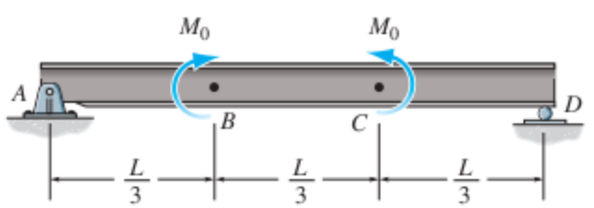
***16–28.** The beam is subjected to the load shown. Determine the equation of the elastic curve. EI is constant.



Prob. 16–28

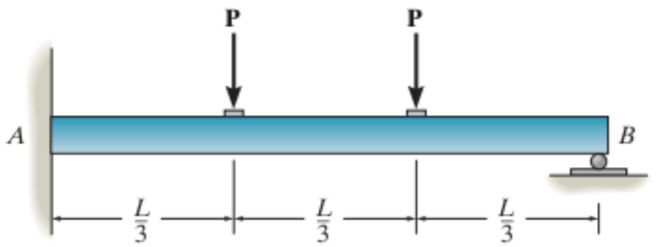
16–29. Determine the equation of the elastic curve, the slope at A, and the deflection at B of the simply supported beam. EI is constant.

16–30. Determine the equation of the elastic curve and the maximum deflection of the simply supported beam. EI is constant.



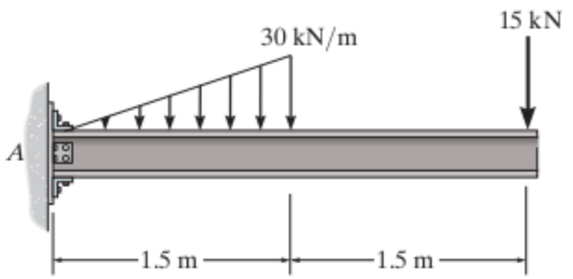
Probs. 16–29/30

16–31. Determine the equation of the elastic curve, the slope at A, and the maximum deflection of the simply supported beam. EI is constant.



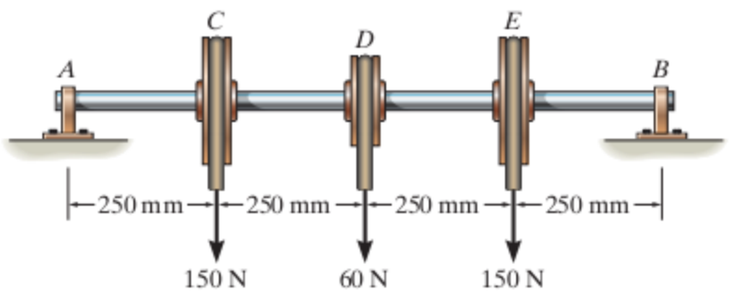
Prob. 16–31

*16–32. Determine the maximum deflection of the cantilevered beam. The beam is made of material having an $E = 200$ GPa and $I = 65.0(10^6)$ mm^6.



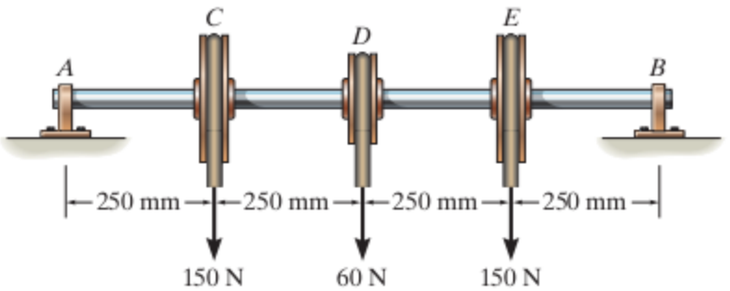
Prob. 16–32

16–33. Determine the deflection at each of the pulleys C, D, and E. The shaft is made of steel and has a diameter of 30 mm. The bearings at A and B exert only vertical reactions on the shaft. $E_{st} = 200$ GPa.



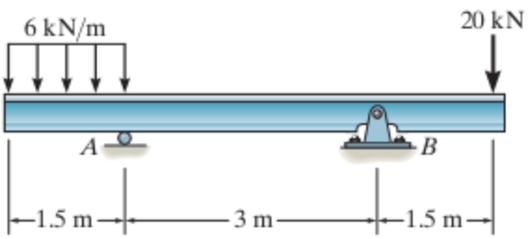
Prob. 16–33

16–34. Determine the slope of the shaft at the bearings at A and B. The shaft is made of steel and has a diameter of 30 mm. The bearings at A and B exert only vertical reactions on the shaft. $E_{st} = 200$ GPa.



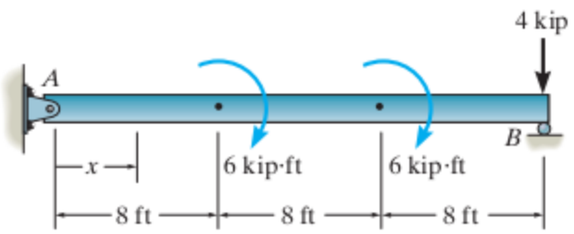
Prob. 16–34

16–35. The beam is subjected to the load shown. Determine the equation of the elastic curve. EI is constant.



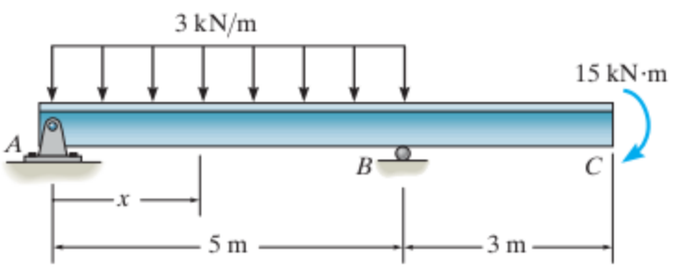
Prob. 16–35

*16–36. Determine the equation of the elastic curve. EI is constant.



Prob. 16–36

16–37. The beam is subjected to the load shown. Determine the equations of the slope and elastic curve. EI is constant.



Prob. 16–37

16.4 Method of Superposition

The differential equation $EI\, d^4v/dx^4 = w(x)$ satisfies the two necessary requirements for applying the principle of superposition; i.e., the load $w(x)$ is linearly related to the deflection $v(x)$, and the load is assumed not to change significantly the original geometry of the beam or shaft. As a result, the deflections for a series of separate loadings acting on a beam may be superimposed. For example, if v_1 is the deflection for one load and v_2 is the deflection for another load, the total deflection for both loads acting together is the algebraic sum $v_1 + v_2$. Using tabulated results for various beam loadings, such as the ones listed in Appendix D, or those found in various engineering handbooks, it is therefore possible to find the slope and displacement at a point on a beam subjected to several different loadings by algebraically adding the effects of its various component parts.

The following examples illustrate how to use the method of superposition to solve deflection problems, where the deflection is caused not only by beam deformations, but also by rigid-body displacements, such as those that occur when the beam is supported by springs.

The resultant deflection at any point on this beam can be determined from the superposition of the deflections caused by each of the separate loadings acting on the beam.

EXAMPLE 16.5

Determine the displacement at point C and the slope at the support A of the beam shown in Fig. 16–18*a*. EI is constant.

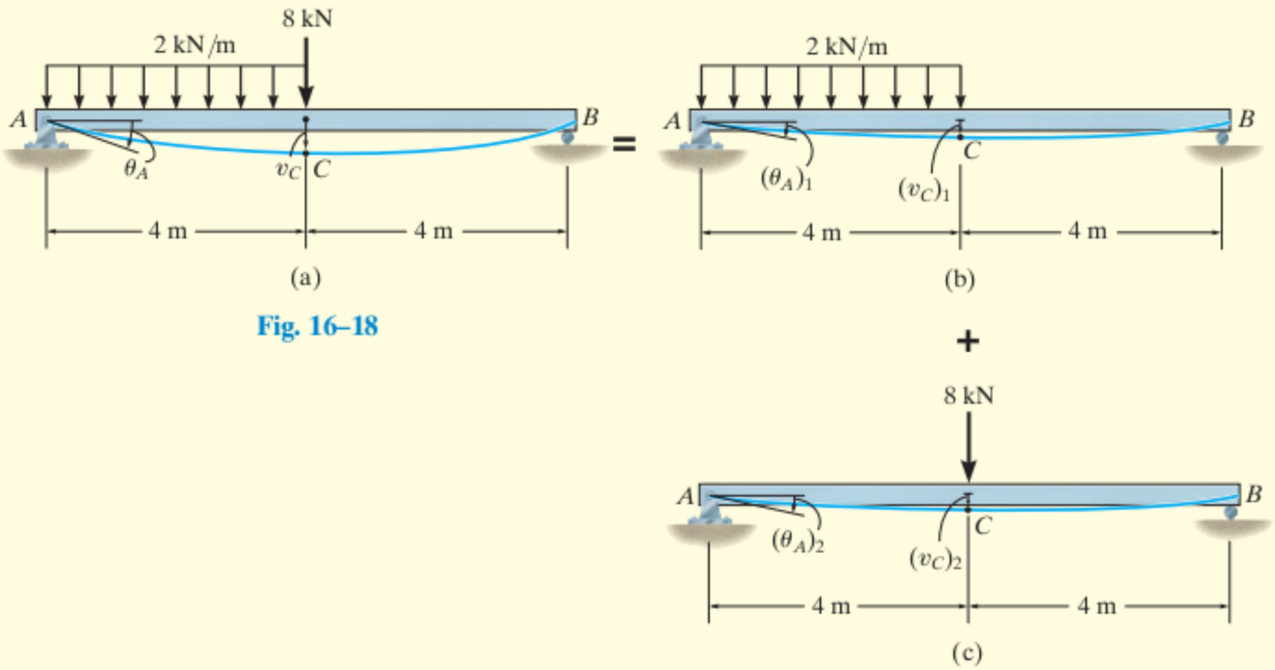


Fig. 16–18

SOLUTION

The loading can be separated into two component parts as shown in Figs. 16–18*b* and 16–18*c*. The displacement at C and slope at A are found using the table in Appendix D for each part.

For the distributed loading,

$$(\theta_A)_1 = \frac{3wL^3}{128EI} = \frac{3(2\text{ kN/m})(8\text{ m})^3}{128EI} = \frac{24\text{ kN}\cdot\text{m}^2}{EI}\curvearrowright$$

$$(v_C)_1 = \frac{5wL^4}{768EI} = \frac{5(2\text{ kN/m})(8\text{ m})^4}{768EI} = \frac{53.33\text{ kN}\cdot\text{m}^3}{EI}\downarrow$$

For the 8-kN concentrated force,

$$(\theta_A)_2 = \frac{PL^2}{16EI} = \frac{8\text{ kN}(8\text{ m})^2}{16EI} = \frac{32\text{ kN}\cdot\text{m}^2}{EI}\curvearrowright$$

$$(v_C)_2 = \frac{PL^3}{48EI} = \frac{8\text{ kN}(8\text{ m})^3}{48EI} = \frac{85.33\text{ kN}\cdot\text{m}^3}{EI}\downarrow$$

The displacement at C and the slope at A are the algebraic sums of these components. Hence,

$$(+\curvearrowright) \qquad \theta_A = (\theta_A)_1 + (\theta_A)_2 = \frac{56\text{ kN}\cdot\text{m}^2}{EI}\curvearrowright \qquad \textit{Ans.}$$

$$(+\downarrow) \qquad v_C = (v_C)_1 + (v_C)_2 = \frac{139\text{ kN}\cdot\text{m}^3}{EI}\downarrow \qquad \textit{Ans.}$$

EXAMPLE 16.6

Determine the displacement at the end C of the overhanging beam shown in Fig. 16–19*a*. EI is constant.

SOLUTION

Since the table in Appendix D *does not* include beams with overhangs, the beam will be separated into a simply supported and a cantilevered portion. First we will calculate the slope at B, as caused by the distributed load acting on the simply supported span, Fig. 16–19*b*.

$$(\theta_B)_1 = \frac{wL^3}{24EI} = \frac{5\ \text{kN/m}(4\ \text{m})^3}{24EI} = \frac{13.33\ \text{kN}\cdot\text{m}^2}{EI}\circlearrowleft$$

Since this angle is *small*, $(\theta_B)_1 \approx \tan(\theta_B)_1$, and the vertical displacement at point C is

$$(v_C)_1 = (2\ \text{m})\left(\frac{13.33\ \text{kN}\cdot\text{m}^2}{EI}\right) = \frac{26.67\ \text{kN}\cdot\text{m}^3}{EI}\uparrow$$

Next, the 10-kN load on the overhang causes a statically equivalent force of 10 kN and couple moment of 20 kN · m at the support B of the simply supported span, Fig. 16–19*c*. The 10-kN force does not cause a displacement or slope at B; however, the 20-kN · m couple moment does cause a slope. The slope at B due to this moment is

$$(\theta_B)_2 = \frac{M_0 L}{3EI} = \frac{20\ \text{kN}\cdot\text{m}(4\ \text{m})}{3EI} = \frac{26.67\ \text{kN}\cdot\text{m}^2}{EI}\circlearrowright$$

so that the extended point C is displaced

$$(v_C)_2 = (2\ \text{m})\left(\frac{26.7\ \text{kN}\cdot\text{m}^2}{EI}\right) = \frac{53.33\ \text{kN}\cdot\text{m}^3}{EI}\downarrow$$

Finally, the cantilevered portion BC is displaced by the 10-kN force, Fig. 16–19*d*. We have

$$(v_C)_3 = \frac{PL^3}{3EI} = \frac{10\ \text{kN}(2\ \text{m})^3}{3EI} = \frac{26.67\ \text{kN}\cdot\text{m}^3}{EI}\downarrow$$

Summing these results algebraically, we obtain the displacement of point C,

$$(+\downarrow) \qquad v_C = -\frac{26.7}{EI} + \frac{53.3}{EI} + \frac{26.7}{EI} = \frac{53.3\ \text{kN}\cdot\text{m}^3}{EI}\downarrow \qquad \textit{Ans.}$$

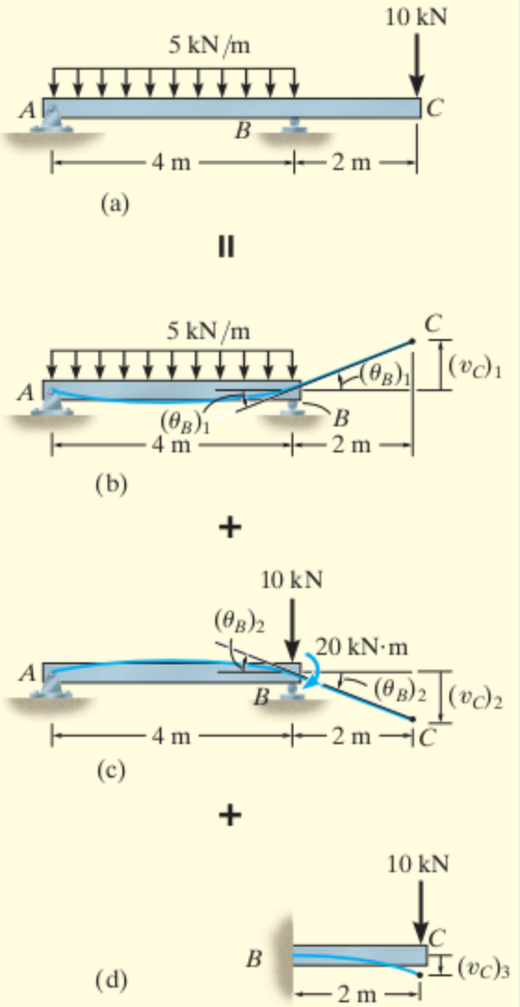


Fig. 16–19

16

EXAMPLE 16.7

Determine the displacement at the end C of the cantilever beam shown in Fig. 16–20. EI is constant.

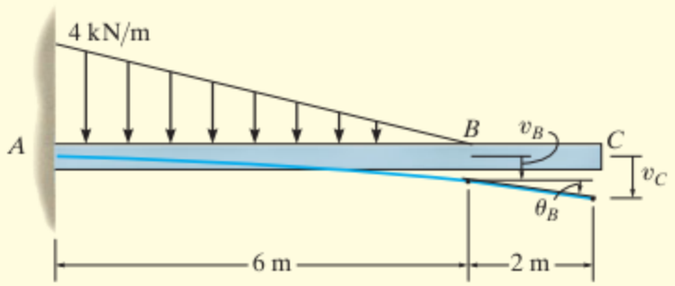


Fig. 16–20

SOLUTION

Using the table in Appendix D for the triangular loading, the slope and displacement at point B are

$$\theta_B = \frac{w_0 L^3}{24EI} = \frac{4\ \text{kN/m}(6\ \text{m})^3}{24EI} = \frac{36\ \text{kN}\cdot\text{m}^2}{EI}$$

$$v_B = \frac{w_0 L^4}{30EI} = \frac{4\ \text{kN/m}(6\ \text{m})^4}{30EI} = \frac{172.8\ \text{kN}\cdot\text{m}^3}{EI}$$

The unloaded region BC of the beam remains straight, as shown in Fig. 16–20. Since θ_B is small, the displacement at C becomes

$$(+\downarrow) \qquad v_C = v_B + \theta_B(L_{BC})$$

$$= \frac{172.8\ \text{kN}\cdot\text{m}^3}{EI} + \frac{36\ \text{kN}\cdot\text{m}^2}{EI}(2\ \text{m})$$

$$= \frac{244.8\ \text{kN}\cdot\text{m}^3}{EI}\downarrow \qquad \textit{Ans.}$$

EXAMPLE 16.8

The steel bar shown in Fig. 16–21*a* is supported by two springs at its ends A and B. Each spring has a stiffness of $k = 15$ kip/ft and is originally unstretched. If the bar is loaded with a force of 3 kip at point C, determine the vertical displacement of the force. Neglect the weight of the bar and take $E_{st} = 29(10^3)$ ksi, $I = 12$ in^4.

SOLUTION

The end reactions at A and B are calculated and shown in Fig. 16–21*b*. Each spring deflects by an amount

$$(v_A)_1 = \frac{2 \text{ kip}}{15 \text{ kip/ft}} = 0.1333 \text{ ft}$$

$$(v_B)_1 = \frac{1 \text{ kip}}{15 \text{ kip/ft}} = 0.0667 \text{ ft}$$

If the bar is considered to be *rigid*, these displacements cause it to move into the position shown in Fig. 16–21*b*. For this case, the vertical displacement at C is

$$(v_C)_1 = (v_B)_1 + \frac{6 \text{ ft}}{9 \text{ ft}}[(v_A)_1 - (v_B)_1]$$

$$= 0.0667 \text{ ft} + \frac{2}{3}[0.1333 \text{ ft} - 0.0667 \text{ ft}] = 0.1111 \text{ ft} \downarrow$$

We can find the displacement at C caused by the *deformation* of the bar, Fig. 16–21*c*, by using the table in Appendix D. We have

$$(v_C)_2 = \frac{Pab}{6EIL}(L^2 - b^2 - a^2)$$

$$= \frac{3 \text{ kip}(3 \text{ ft})(6 \text{ ft})[(9 \text{ ft})^2 - (6 \text{ ft})^2 - (3 \text{ ft})^2]}{6[29(10^3)\text{kip/in}^2](144 \text{ in}^2/1 \text{ ft}^2)(12 \text{ in}^4)(1 \text{ ft}^4/20\,736 \text{ in}^4)(9 \text{ ft})}$$

$$= 0.0149 \text{ ft} \downarrow$$

Adding the two displacement components, we get

$$(+\downarrow) \quad v_C = 0.1111 \text{ ft} + 0.0149 \text{ ft} = 0.126 \text{ ft} = 1.51 \text{ in.} \downarrow \qquad \textit{Ans.}$$

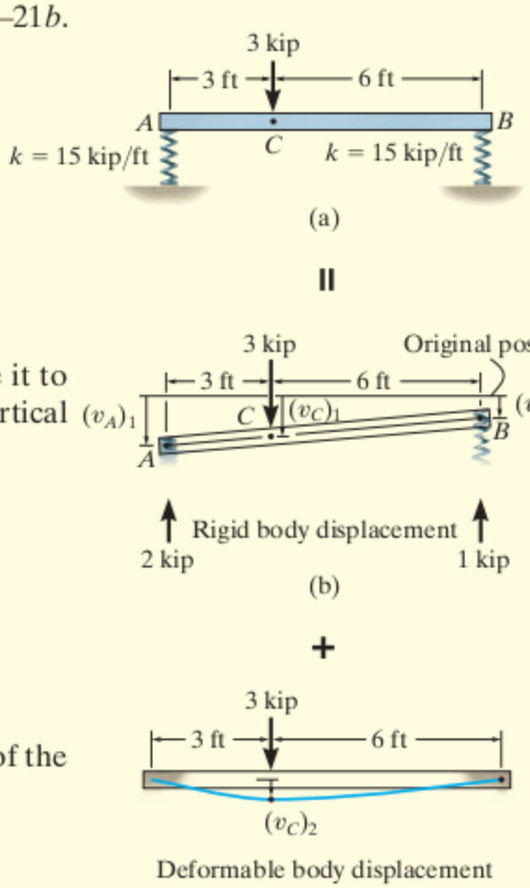
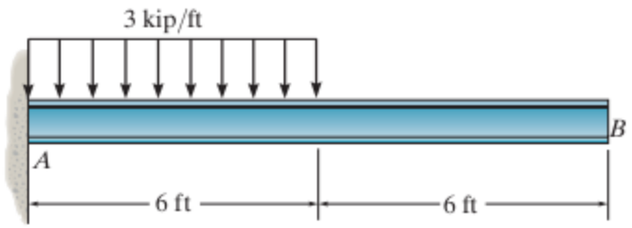


Fig. 16–21

16

PROBLEMS

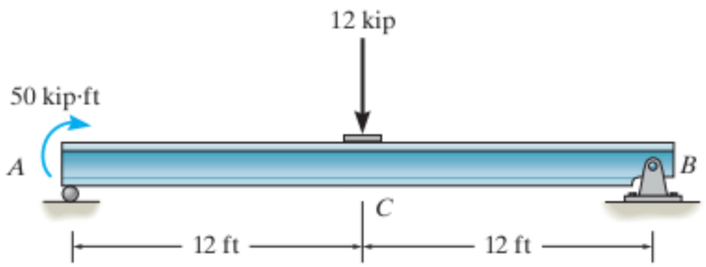
16–38. The W10 × 15 cantilevered beam is made of A-36 steel and is subjected to the loading shown. Determine the slope and displacement at its end *B*.



Prob. 16–38

16

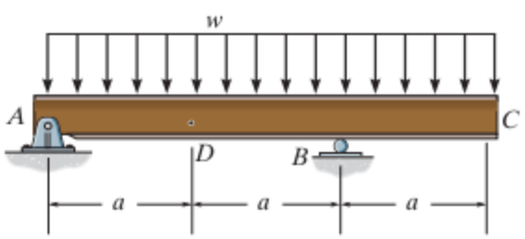
16–39. The W12 × 45 simply supported beam is made of A-36 steel and is subjected to the loading shown. Determine the deflection at its center *C*.



Prob. 16–39

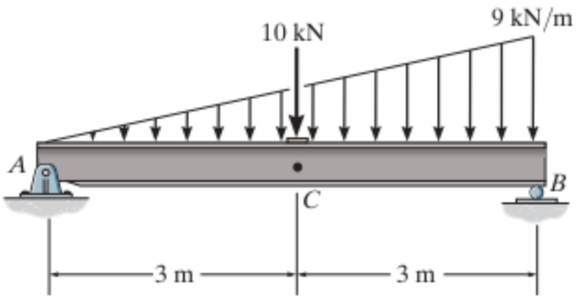
***16–40.** Determine the slope and deflection at end *C* of the overhang beam. *EI* is constant.

16–41. Determine the slope at *A* and the deflection at point *D* of the overhang beam. *EI* is constant.



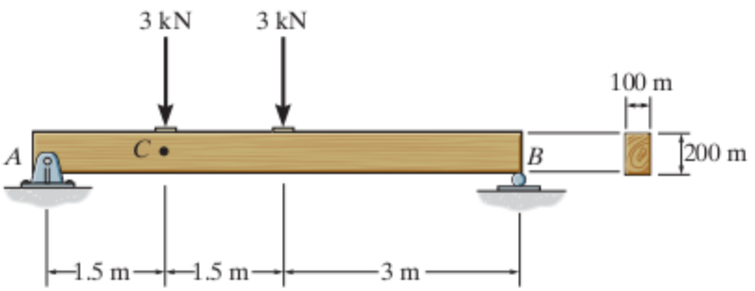
Probs. 16–40/41

16–42. Determine the slope at *B* and the deflection at point *C* of the simply supported beam. $E = 200$ GPa and $I = 45.5(10^6)\text{mm}^4$.



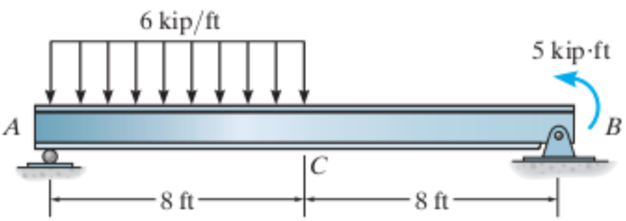
Prob. 16–42

16–43. Determine the slope at *A* and the deflection at point *C* of the simply supported beam. The modulus of elasticity of the wood is $E = 10$ GPa.



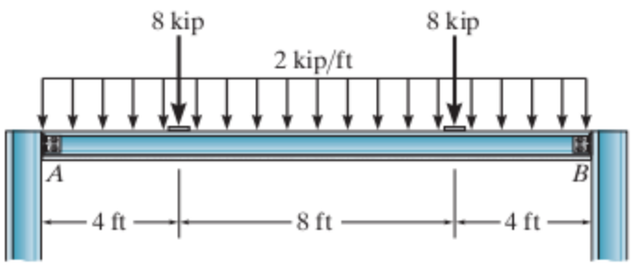
Prob. 16–43

***16–44.** The W8 × 24 simply supported beam is made of A-36 steel and is subjected to the loading shown. Determine the deflection at its center *C*.



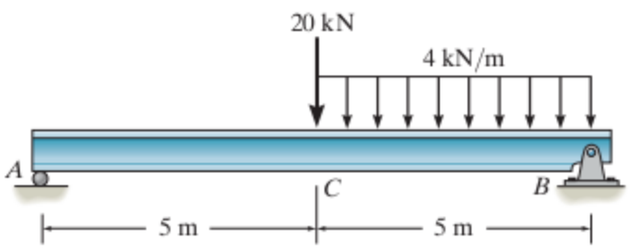
Prob. 16–44

16–45. The simply supported beam carries a uniform load of 2 kip/ft. Code restrictions, due to a plaster ceiling, require the maximum deflection not to exceed 1/360 of the span length. Select the lightest-weight A992 steel wide-flange beam from Appendix B that will satisfy this requirement and safely support the load. The allowable bending stress is $\sigma_{allow} = 24$ ksi and the allowable shear stress is $\tau_{allow} = 14$ ksi. Assume A is a pin and B a roller support.



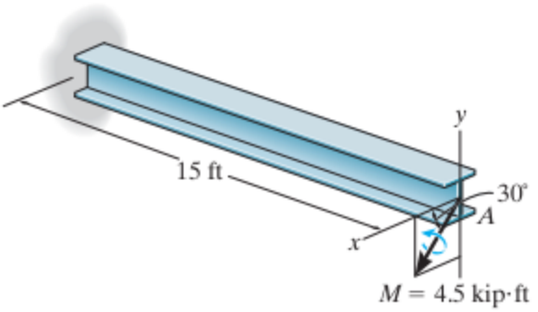
Prob. 16–45

16–46. The simply supported beam is made of A-36 steel and is subjected to the loading shown. Determine the deflection at its center C. $I = 0.1457(10^{-3})$ m^4.



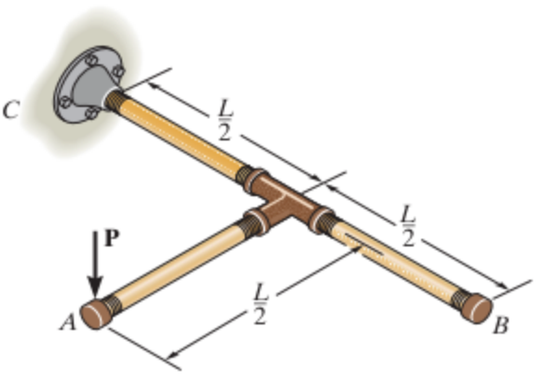
Prob. 16–46

16–47. The W10 × 30 cantilevered beam is made of A-36 steel and is subjected to unsymmetrical bending caused by the applied moment. Determine the deflection of the centroid at its end A due to the loading. *Hint:* Resolve the moment into components and use superposition.



Prob. 16–47

***16–48.** The pipe assembly consists of three equal-sized pipes with flexibility stiffness EI and torsional stiffness GJ. Determine the vertical deflection at point A.



Prob. 16–48

16

16.5 Statically Indeterminate Beams and Shafts–Method of Superposition

The analysis of statically indeterminate axially loaded bars and torsionally loaded shafts has been discussed in Secs. 9.4 and 10.5, respectively. In this section we will illustrate a general method for determining the reactions on statically indeterminate beams and shafts. Specifically, a member of any type is classified as *statically indeterminate* if the number of unknown reactions *exceeds* the available number of equilibrium equations.

The additional support reactions on the beam or shaft that are *not needed* to keep it in stable equilibrium are called *redundants*. The number of these redundants is referred to as the *degree of indeterminacy*. For example, consider the beam shown in Fig. 16–22*a*. If the free-body diagram is drawn, Fig. 16–22*b*, there will be four unknown support reactions, and since three equilibrium equations are available for solution, the beam is classified as being indeterminate to the first degree. Either $\mathbf{A}_y$, $\mathbf{B}_y$, or $\mathbf{M}_A$ can be classified as the redundant, for if any one of these reactions is removed, the beam remains stable and in equilibrium ($\mathbf{A}_x$ cannot be classified as the redundant, for if it were removed, $\Sigma F_x = 0$ would not be satisfied.) In a similar manner, the *continuous beam* in Fig. 16–23*a* is indeterminate to the second degree, since there are five unknown reactions and only three available equilibrium equations, Fig. 16–23*b*. Here the two redundant support reactions can be chosen among $\mathbf{A}_y$, $\mathbf{B}_y$, $\mathbf{C}_y$, and $\mathbf{D}_y$.

To determine the reactions on a beam (or shaft) that is statically indeterminate, it is first necessary to specify the redundant reactions. We can determine these redundants from conditions of geometry known as *compatibility conditions*. Once determined, the redundants are then applied to the beam, and the remaining reactions are determined from the equations of equilibrium.

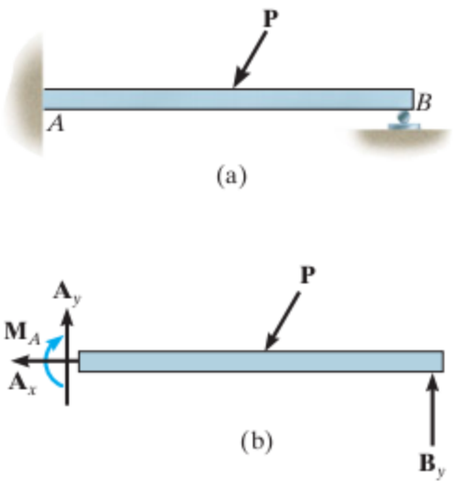


Fig. 16–22

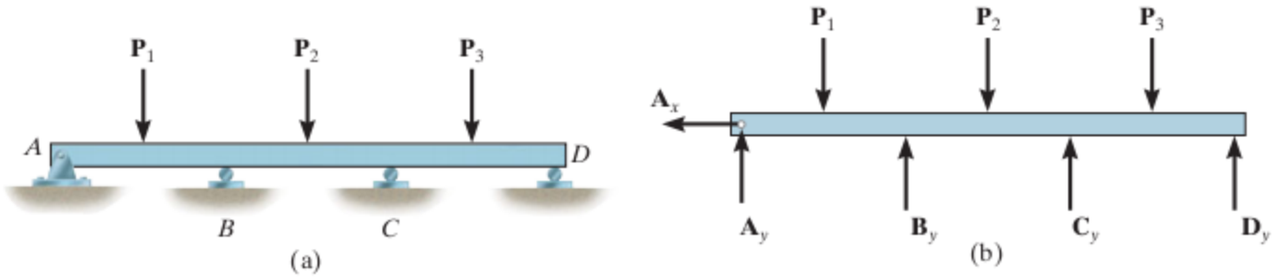


Fig. 16–23

The method of superposition has been used previously to solve for the redundant loading on axially loaded bars and torsionally loaded shafts. In order to apply this method to the solution of statically indeterminate beams (or shafts), it is first necessary to identify the redundant support reactions. By *removing* them from the beam we obtain the so-called *primary beam*, which is statically determinate and stable, and is subjected *only* to the external load. If we add to this beam a succession of similarly supported beams, each loaded with a *separate* redundant, then by the principle of superposition, we obtain the actual loaded beam. Finally, in order to solve for the redundants, we must write the ***conditions of compatibility*** that exist at the supports where each of the redundants acts. Since the redundant forces are determined directly in this manner, this method of analysis is sometimes called the ***force method***. Once the redundants are obtained, the other reactions on the beam can then be determined from the three equations of equilibrium.

To clarify these concepts, consider the beam shown in Fig. 16–24*a*. If we choose the reaction $\mathbf{B}_y$ at the roller as the redundant, then the primary beam is shown in Fig. 16–24*b*, and the beam with the redundant $\mathbf{B}_y$ acting on it is shown in Fig. 16–24*c*. The displacement at the roller is to be zero, and since the displacement of point B on the primary beam is v_B, and $\mathbf{B}_y$ causes point B to be displaced upward v'_B, we can write the compatibility equation at B as

$$(+\uparrow) \qquad 0 = -v_B + v'_B$$

The displacements v_B and v'_B can be obtained using the method of superposition. From the table in Appendix D we have

$$v_B = \frac{5PL^3}{48EI} \quad \text{and} \quad v'_B = \frac{B_y L^3}{3EI}$$

Substituting into the compatibility equation, we get

$$0 = -\frac{5PL^3}{48EI} + \frac{B_y L^3}{3EI}$$

$$B_y = \frac{5}{16}P$$

Now that $\mathbf{B}_y$ is known, the reactions at the wall are determined from the three equations of equilibrium applied to the free-body diagram of the beam, Fig. 16–24*d*. The results are

$$A_x = 0 \qquad A_y = \frac{11}{16}P$$

$$M_A = \frac{3}{16}PL$$

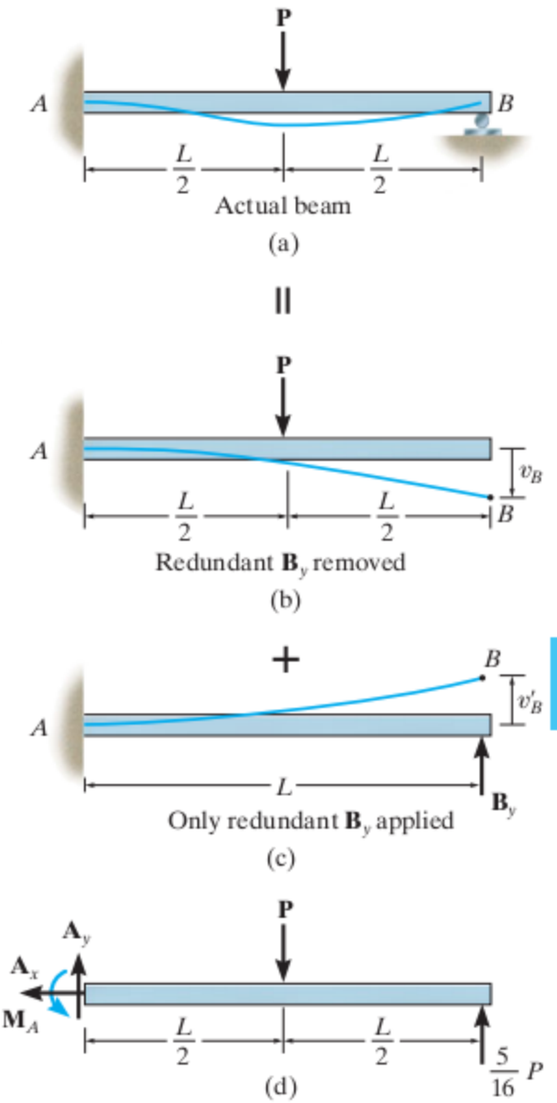


Fig. 16–24

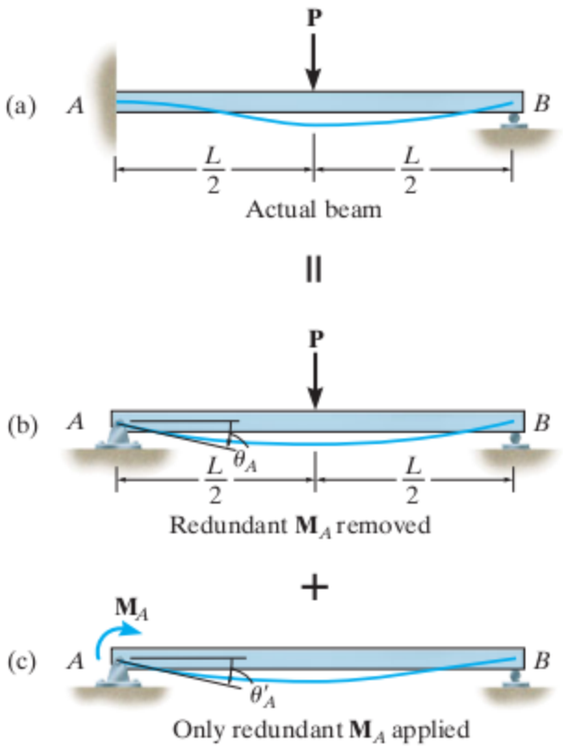


Fig. 16–25

The choice of the redundant is *arbitrary*, provided the primary beam remains stable. For example, the moment at A for the beam in Fig. 16–25*a* can also be chosen as the redundant. In this case the capacity of the beam to resist $\mathbf{M}_A$ is removed, and so the primary beam is then pin supported at A, Fig. 16–25*b*. To it we add the beam with the redundant at A acting on it, Fig. 16–25*c*. Referring to the slope at A caused by the load $\mathbf{P}$ as θ_A, and the slope at A caused by the redundant $\mathbf{M}_A$ as θ'_A, the compatibility equation for the slope at A requires

$$(\curvearrowleft +) \qquad 0 = \theta_A + \theta'_A$$

Again using the table in Appendix D, we have

$$\theta_A = \frac{PL^2}{16EI} \quad \text{and} \quad \theta'_A = \frac{M_A L}{3EI}$$

Thus,

$$0 = \frac{PL^2}{16EI} + \frac{M_A L}{3EI}$$

$$M_A = -\frac{3}{16} PL$$

This is the same result determined previously. Here the negative sign for M_A simply means that $\mathbf{M}_A$ acts in the opposite sense of direction of that shown in Fig. 16–25*c*.

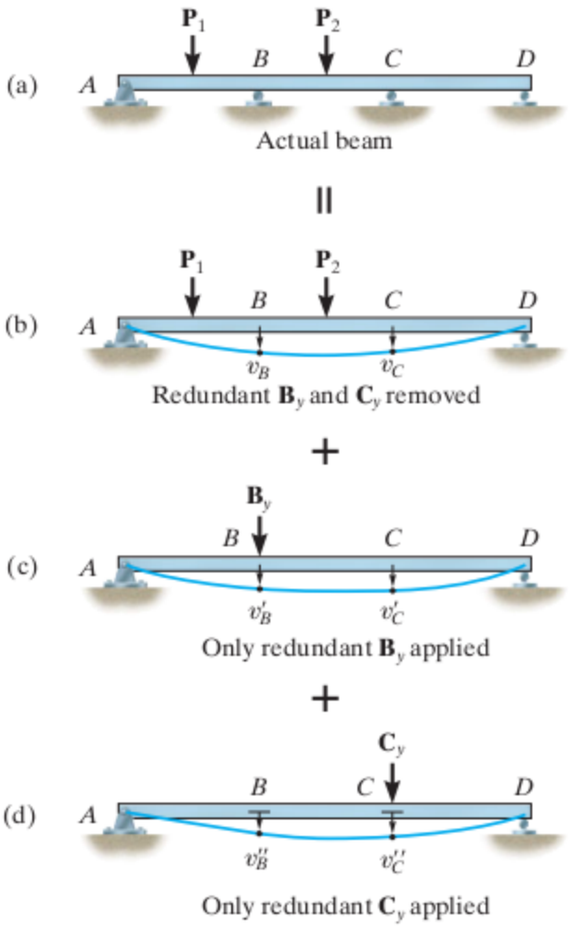


Fig. 16–26

Another example that illustrates this method is given in Fig. 16–26*a*. In this case the beam is indeterminate to the second degree and therefore *two* compatibility equations will be necessary for the solution. We will choose the forces at the roller supports B and C as redundants. The primary (statically determinate) beam deforms as shown in Fig. 16–26*b* when the redundants are removed. Each redundant force deforms this beam as shown in Figs. 16–26*c* and 16–26*d*, respectively. By superposition, the compatibility equations for the displacements at B and C are

$$(+\downarrow) \qquad 0 = v_B + v'_B + v''_B$$

$$(+\downarrow) \qquad 0 = v_C + v'_C + v''_C \tag{16–16}$$

Here the displacement components v'_B and v'_C will be expressed in terms of the unknown $\mathbf{B}_y$, and the components v''_B and v''_C will be expressed in terms of the unknown $\mathbf{C}_y$. When these displacements have been determined and substituted into Eq. 16–16, these equations may then be solved simultaneously for the two unknowns $\mathbf{B}_y$ and $\mathbf{C}_y$.

Procedure for Analysis

The following procedure provides a means for applying the method of superposition (or the force method) to determine the reactions on statically indeterminate beams or shafts.

Elastic Curve.

- Specify the unknown redundant forces or moments that must be removed from the beam in order to make it statically determinate and stable.
- Using the principle of superposition, draw the statically indeterminate beam and show it equal to a sequence of corresponding *statically determinate beams*.
- The first of these beams, the primary beam, supports the same external loads as the statically indeterminate beam, and each of the other beams "added" to the primary beam shows the beam loaded with a separate redundant force or moment.
- Sketch the deflection curve for each beam and indicate symbolically the displacement (slope) at the point of each redundant force (moment).

Compatibility Equations.

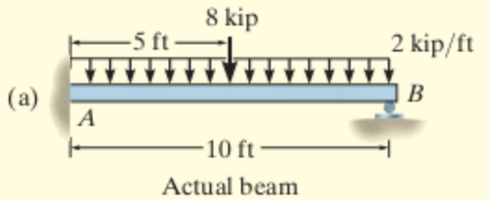
- Write a compatibility equation for the displacement (slope) at each point where there is a redundant force (moment).
- Determine all the displacements or slopes using the table in Appendix D.
- Substitute the results into the compatibility equations and solve for the unknown redundants.
- If a numerical value for a redundant is *positive*, it has the *same sense of direction* as originally assumed. Similarly, a *negative* numerical value indicates the redundant acts *opposite* to its assumed *sense of direction*.

Equilibrium Equations.

- Once the redundant forces and/or moments have been determined, the remaining unknown reactions can be found from the equations of equilibrium applied to the loadings shown on the beam's free-body diagram.

EXAMPLE 16.9

Determine the reactions at the roller support B of the beam shown in Fig. 16–27*a*, then draw the shear and moment diagrams. EI is constant.



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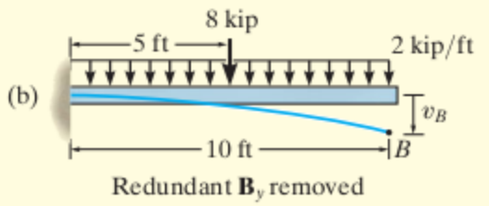

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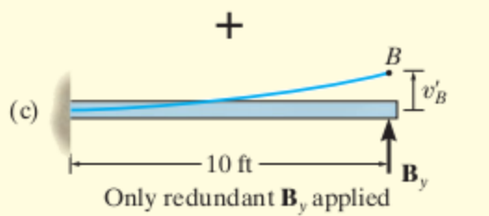


SOLUTION

Principle of Superposition. By inspection, the beam is statically indeterminate to the first degree. The roller support at B will be chosen as the redundant so that $\mathbf{B}_y$ will be determined *directly*. Figures 16–27*b* and 16–27*c* show application of the principle of superposition. Here we have assumed that $\mathbf{B}_y$ acts upward on the beam.

Compatibility Equation. Taking positive displacement as down ward, the compatibility equation at B is

$$(+\downarrow) \qquad 0 = v_B - v'_B \qquad (1)$$

These displacements can be obtained directly from the table in Appendix D.

$$v_B = \frac{wL^4}{8EI} + \frac{5PL^3}{48EI}$$

$$= \frac{2\ \text{kip/ft}(10\ \text{ft})^4}{8EI} + \frac{5(8\ \text{kip})(10\ \text{ft})^3}{48EI} = \frac{3333\ \text{kip}\cdot\text{ft}^3}{EI}\downarrow$$

$$v'_B = \frac{PL^3}{3EI} = \frac{B_y(10\ \text{ft})^3}{3EI} = \frac{333.3\ \text{ft}^3\, B_y}{EI}\uparrow$$

Substituting into Eq. (1) and solving yields

$$0 = \frac{3333}{EI} - \frac{333.3B_y}{EI}$$

$$B_y = 10\ \text{kip} \qquad \textit{Ans.}$$

Equilibrium Equations. Using this result and applying the three equations of equilibrium, we obtain the results shown on the beam's free-body diagram in Fig. 16–27*d*. The shear and moment diagrams are shown in Fig. 16–27*e*.

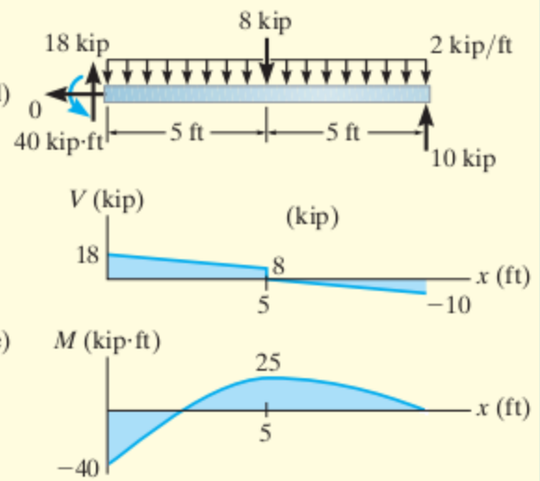


Fig. 16–27

16

EXAMPLE 16.10

The beam in Fig. 16–28*a* is fixed supported to the wall at A and pin-connected to a $\frac{1}{2}$-in.-diameter rod BC. If $E = 29(10^3)$ ksi for both members, determine the force developed in the rod due to the loading. The moment of inertia of the beam about its neutral axis is $I = 475\ \text{in}^4$.

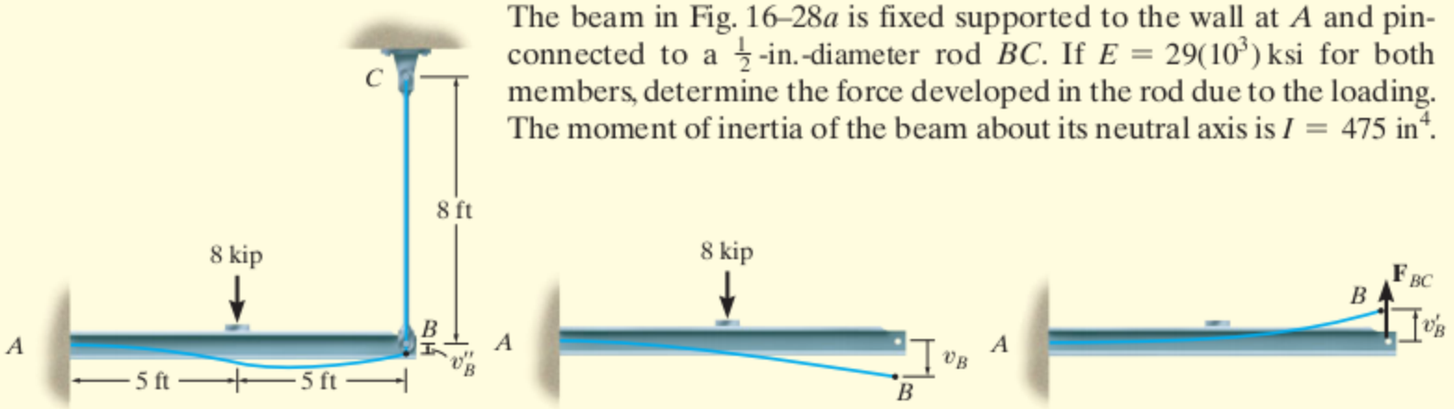


Fig. 16–28

16

SOLUTION I

Principle of Superposition. By inspection, this problem is indeterminate to the first degree. Here B will undergo an unknown displacement v''_B, since the rod will stretch. The rod will be treated as the redundant and hence the force of the rod is removed from the beam at B, Fig. 16–28*b*, and then reapplied, Fig. 16–28*c*.

Compatibility Equation. At point B we require

$$(+\downarrow) \qquad v''_B = v_B - v'_B \tag{1}$$

The displacements v_B and v'_B are determined from the table in Appendix D. v''_B is calculated from Eq. 9–2. Working in kilopounds and inches, we have

$$v''_B = \frac{PL}{AE} = \frac{F_{BC}(8\ \text{ft})(12\ \text{in./ft})}{(\pi/4)\left(\frac{1}{2}\ \text{in.}\right)^2[29(10^3)\ \text{kip/in}^2]} = 0.01686F_{BC}\ \downarrow$$

$$v_B = \frac{5PL^3}{48EI} = \frac{5(8\ \text{kip})(10\ \text{ft})^3(12\ \text{in./ft})^3}{48[29(10^3)\ \text{kip/in}^2](475\ \text{in}^4)} = 0.1045\ \text{in.}\ \downarrow$$

$$v'_B = \frac{PL^3}{3EI} = \frac{F_{BC}(10\ \text{ft})^3(12\ \text{in./ft})^3}{3[29(10)^3\ \text{kip/in}^2](475\ \text{in}^4)} = 0.04181F_{BC}\ \uparrow$$

Thus, Eq. (1) becomes

$$(+\downarrow) \qquad 0.01686F_{BC} = 0.1045 - 0.04181F_{BC}$$

$$F_{BC} = 1.78\ \text{kip} \qquad \textit{Ans.}$$

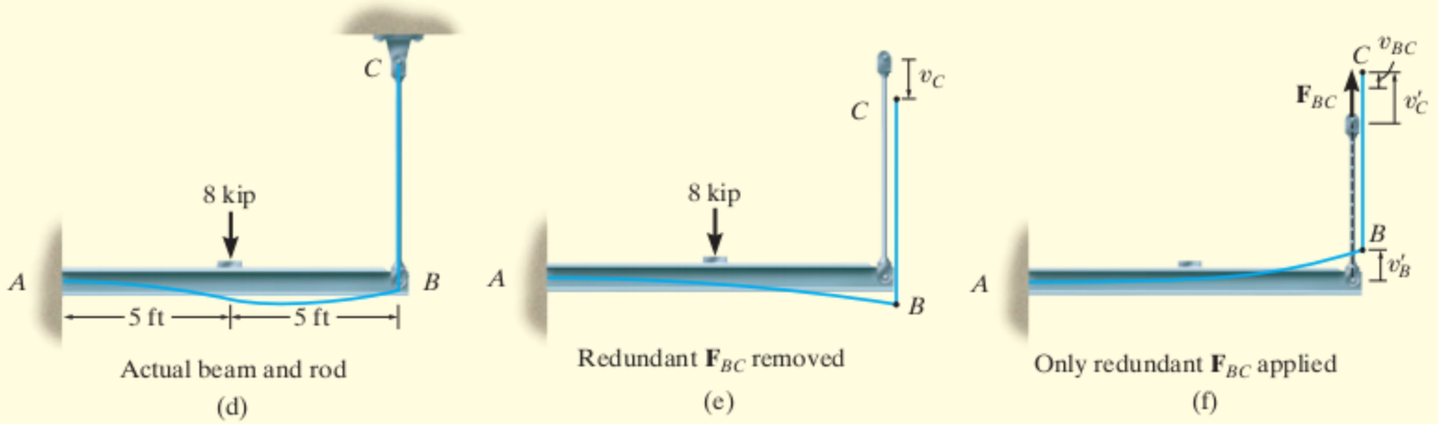


Fig. 16–28 (cont.)

16

SOLUTION II

Principle of Superposition. We can also solve this problem by removing the pin support at C and keeping the rod attached to the beam. In this case the 8-kip load will cause points B and C to be displaced downward the *same amount* v_C, Fig. 16–28*e*, since no force exists in rod BC. When the redundant force $\mathbf{F}_{BC}$ is applied at point C, it causes the end C of the rod to be displaced upward v'_C and the end B of the beam to be displaced upward v'_B, Fig. 16–28*f*. The difference in these two displacements, v_{BC}, represents the stretch of the rod due to $\mathbf{F}_{BC}$, so that $v'_C = v_{BC} + v'_B$. Hence, from Figs. 16–28*d*, 16–28*e*, and 16–28*f*, the compatibility of displacement at point C is

$$(+\downarrow) \qquad 0 = v_C - (v_{BC} + v'_B) \qquad (2)$$

From Solution I, we have

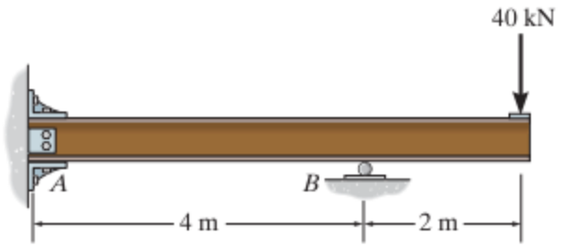
$$v_C = v_B = 0.1045 \text{ in.} \downarrow$$
$$v_{BC} = v''_B = 0.01686F_{BC} \uparrow$$
$$v'_B = 0.04181F_{BC} \uparrow$$

Therefore, Eq. (2) becomes

$$(+\downarrow) \qquad 0 = 0.1045 - (0.01686F_{BC} + 0.04181F_{BC})$$
$$F_{BC} = 1.78 \text{ kip} \qquad \textit{Ans.}$$

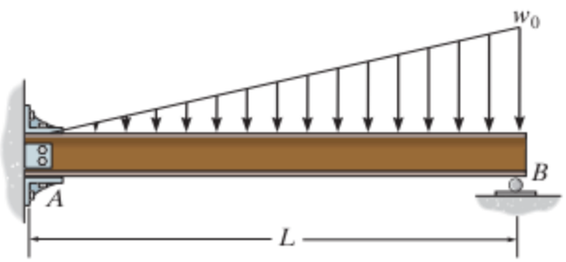
FUNDAMENTAL PROBLEMS

F16–7. Determine the reactions at the fixed support A and the roller B. EI is constant.



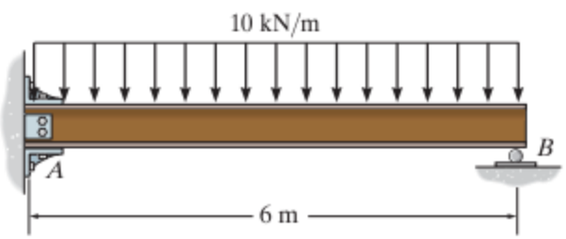
F16–7

F16–8. Determine the reactions at the fixed support A and the roller B. EI is constant.



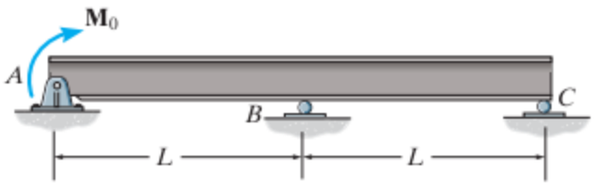
F16–8

F16–9. Determine the reactions at the fixed support A and the roller B. Support B settles 2 mm. $E = 200$ GPa, $I = 65.0(10^{-6})$ m^4.



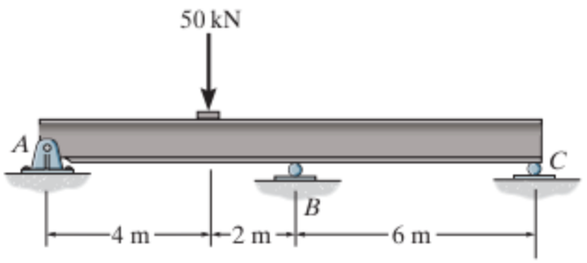
F16–9

F16–10. Determine the reaction at the roller B. EI is constant.



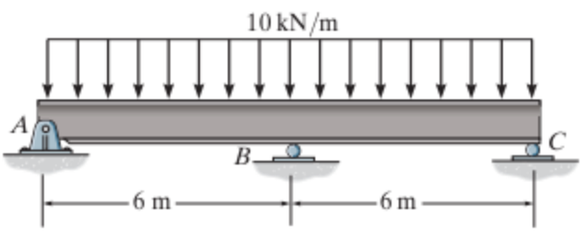
F16–10

F16–11. Determine the reaction at the roller B. EI is constant.



F16–11

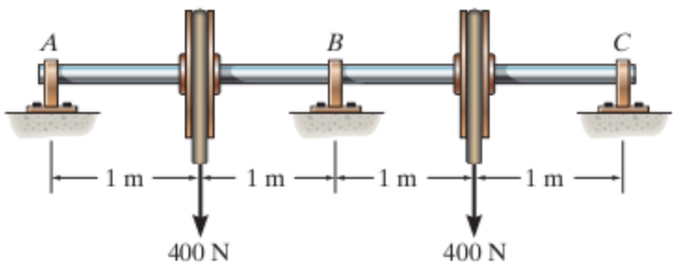
F16–12. Determine the reaction at the roller support B if it settles 5 mm. $E = 200$ GPa and $I = 65.0(10^{-6})$ m^4.



F16–12

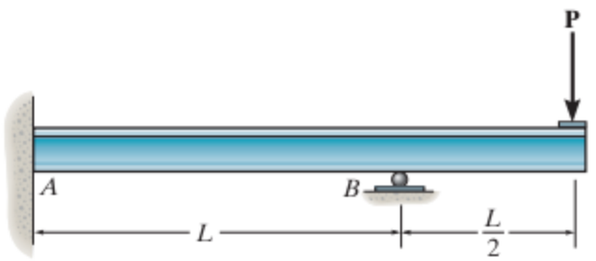
PROBLEMS

16–49. Determine the reactions at the bearing supports A, B, and C of the shaft, then draw the shear and moment diagrams. EI is constant. Each bearing exerts only vertical reactions on the shaft.



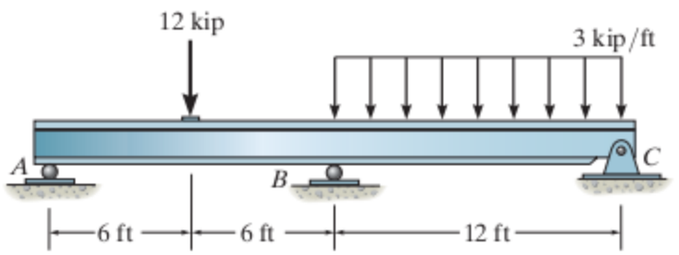
Prob. 16–49

16–50. Determine the reactions at the supports A and B. EI is constant.



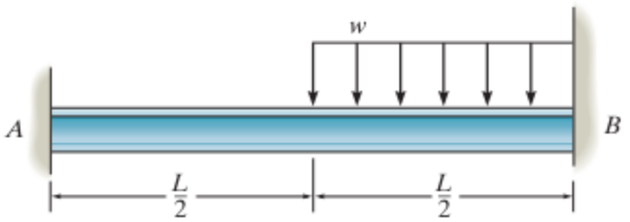
Prob. 16–50

16–51. Determine the reactions at the supports A, B, and C, then draw the shear and moment diagrams. EI is constant.



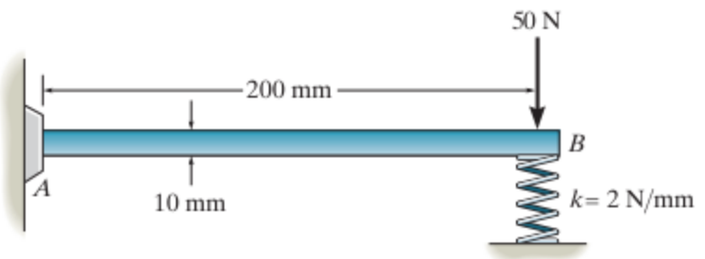
Prob. 16–51

***16–52.** Determine the moment reactions at the supports A and B. EI is constant.



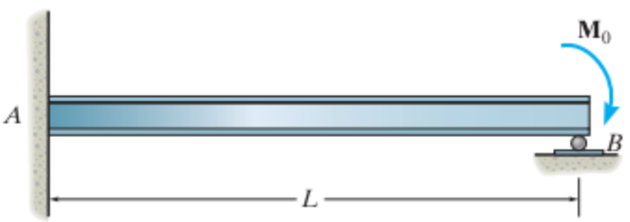
Prob. 16–52

16–53. Determine the deflection at the end B of the clamped A-36 steel strip. The spring has a stiffness of $k = 2\ \text{N/mm}$. The strip is 5 mm wide and 10 mm high. Also, draw the shear and moment diagrams for the strip.



Prob. 16–53

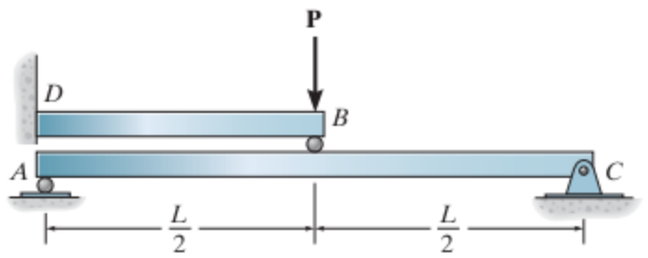
16–54. Determine the reactions at the supports A and B. EI is constant.



Prob. 16–54

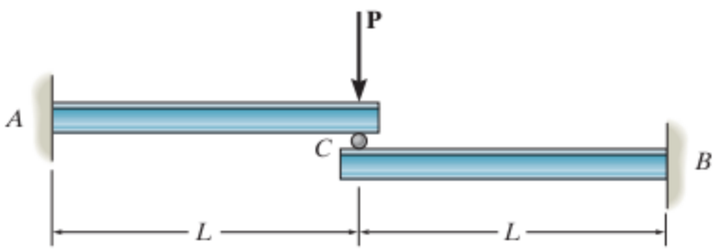
16

16–55. Determine the reactions at support C. EI is constant for both beams.



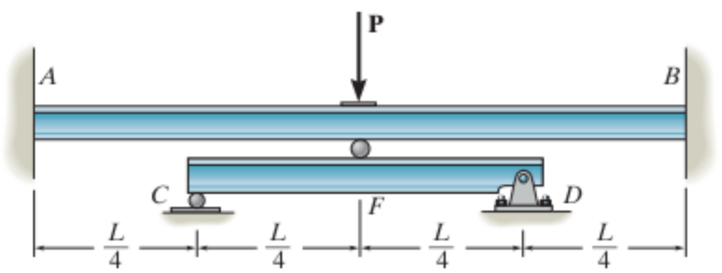
Prob. 16–55

*16–56. The compound beam segments meet in the center using a smooth contact (roller). Determine the reactions at the fixed supports A and B when the load **P** is applied. EI is constant.



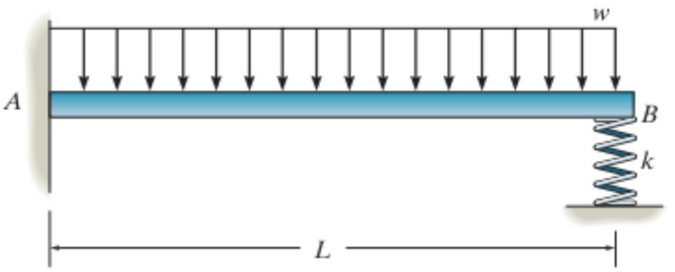
Prob. 16–56

16–57. The fixed supported beam AB is strengthened using the simply supported beam CD and the roller at F which is set in place just before application of the load **P**. Determine the reactions at the supports if EI is constant.



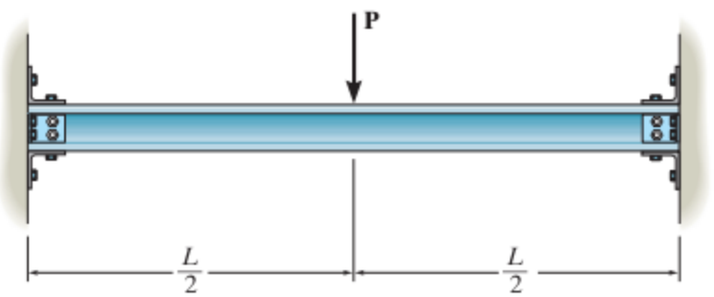
Prob. 16–57

16–58. Determine the force in the spring. EI is constant.



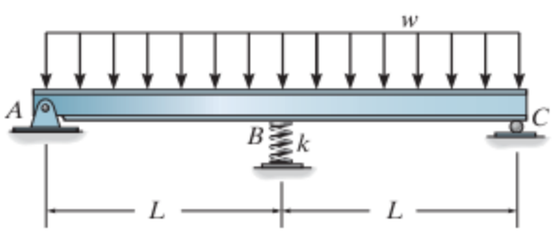
Prob. 16–58

16–59. The beam is supported by the bolted supports at its ends. When loaded these supports do not provide an actual fixed connection, but instead allow a slight rotation α before becoming fixed. Determine the moment at the connections and the maximum deflection of the beam.



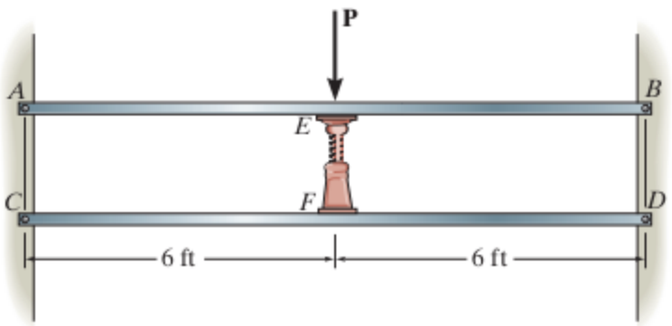
Prob. 16–59

*16–60. The beam is supported by a pin at A, a spring having a stiffness k at B, and a roller at C. Determine the force the spring exerts on the beam. EI is constant.



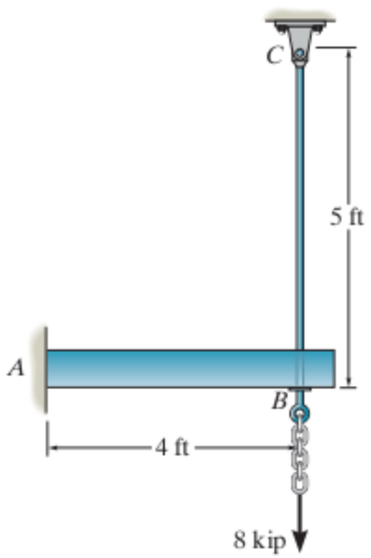
Prob. 16–60

16–61. Each of the two members is made from 6061-T6 aluminum and has a square cross section 1 in. × 1 in. They are pin-connected at their ends and a jack is placed between them and opened until the force it exerts on each member is 50 lb. Determine the greatest force P that can be applied to the center of the top member without causing either of the two members to yield. For the analysis neglect the axial force in each member. Assume the jack is rigid.



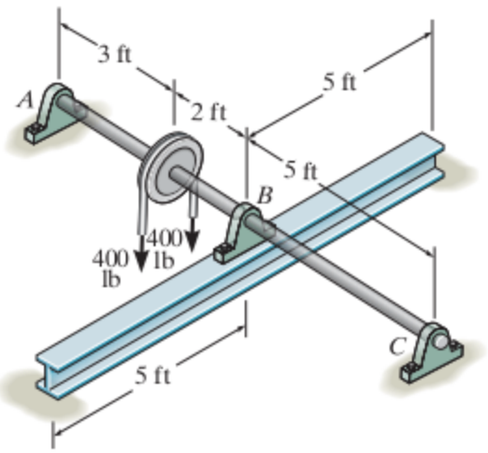
Prob. 16–61

16–62. The A992 steel beam and rod are used to support the load of 8 kip. If it is required that the allowable normal stress for the steel is $\sigma_{allow} = 18$ ksi, and the maximum deflection not exceed 0.05 in., determine the smallest diameter rod that should be used. The beam is rectangular, having a height of 5 in. and a thickness of 3 in.



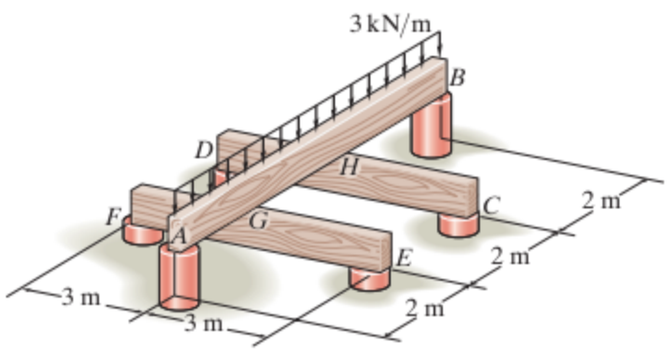
Prob. 16–62

16–63. The 1-in.-diameter A-36 steel shaft is supported by unyielding bearings at A and C. The bearing at B rests on a simply supported steel wide-flange beam having a moment of inertia of $I = 500\ \text{in}^4$. If the belt loads on the pulley are 400 lb each, determine the vertical reactions at A, B, and C.



Prob. 16–63

*16–64. The assembly consists of three simply supported beams for which the bottom of the top beam rests on the top of the bottom two. If a uniform load of 3 kN/m is applied to the beam, determine the vertical reactions at each of the supports. EI is constant.



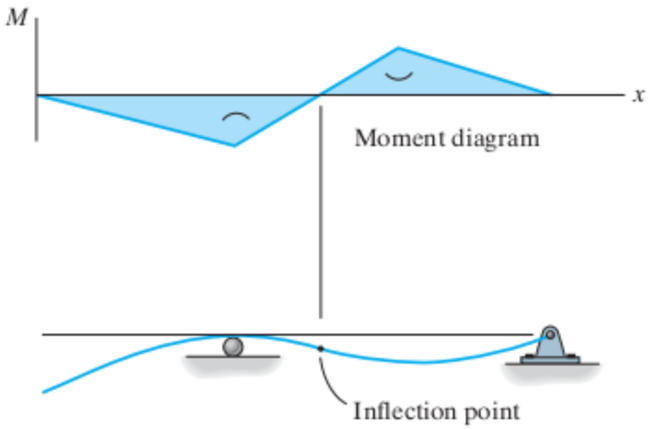
Prob. 16–64

16

CHAPTER REVIEW

The elastic curve represents the centerline deflection of a beam or shaft. Its shape can be determined using the moment diagram. Positive moments cause the elastic curve to be concave upwards and negative moments cause it to be concave downwards. The radius of curvature at any point is determined from

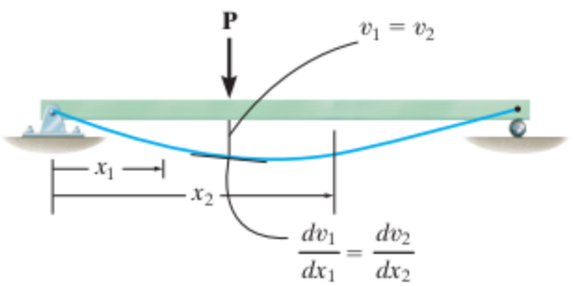
$$\frac{1}{\rho} = \frac{M}{EI}$$



Elastic curve

The equation of the elastic curve and its slope can be obtained by first finding the internal moment in the member as a function of x. If several loadings act on the member, then separate moment functions must be determined between each of the loadings. Integrating these functions once using $EI(d^2v/dx^2) = M(x)$ gives the equation for the slope of the elastic curve, and integrating again gives the equation for the deflection. The constants of integration are determined from the boundary conditions at the supports, or in cases where several moment functions are involved, continuity of slope and deflection at points where these functions join must be satisfied.

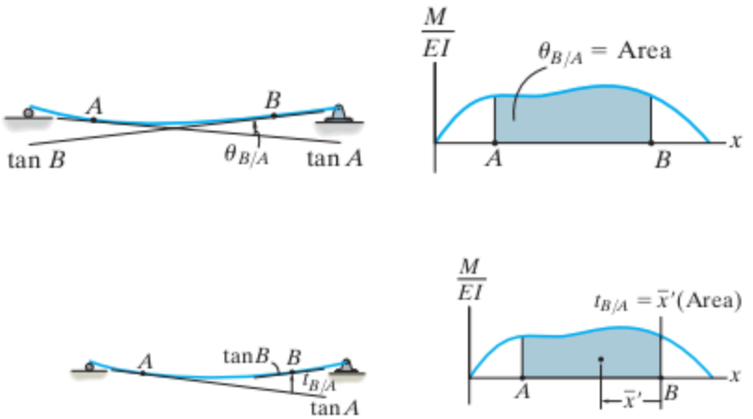
Boundary conditions



Continuity conditions

Discontinuity functions allow one to express the equation of the elastic curve as a continuous function, regardless of the number of loadings on the member. This method eliminates the need to use continuity conditions, since the two constants of integration can be determined solely from the two boundary conditions.

The moment-area method is a semi-graphical technique for finding the slope of tangents or the vertical distance between tangents at specific points on the elastic curve. It requires finding area segments under the M/EI diagram, or the moment of these segments about points on the elastic curve. The method works well for M/EI diagrams composed of simple shapes, such as those produced by concentrated forces and couple moments.



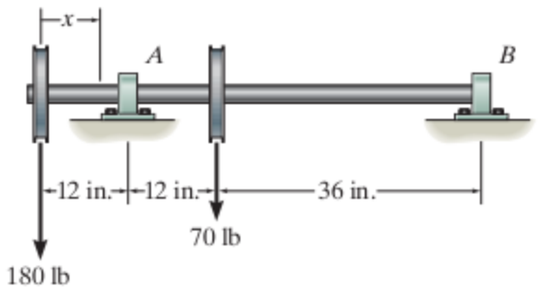
The deflection or slope at a point on a member subjected to combinations of loadings can be determined using the method of superposition. The table in Appendix D is available for this purpose.

Statically indeterminate beams and shafts have more unknown support reactions than available equations of equilibrium. To solve, one first identifies the redundant reactions. The method of integration or the moment-area theorems can then be used to solve for the unknown redundants. It is also possible to determine the redundants by using the method of superposition, where one considers the conditions of continuity at the redundant. Here the displacement due to the external loading is determined with the redundant removed, and again with the redundant applied and the external loading removed. The tables in Appendix D can be used to determine these necessary displacements.

16

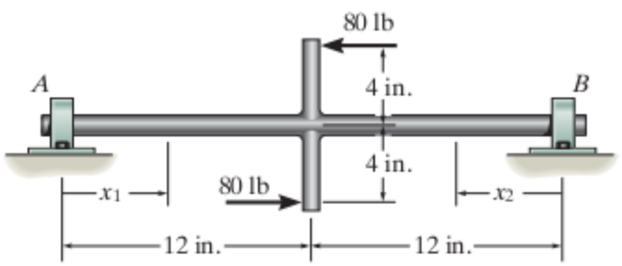
REVIEW PROBLEMS

16–65. The shaft supports the two pulley loads shown. Using discontinuity functions, determine the equation of the elastic curve. The bearings at A and B exert only vertical reactions on the shaft. EI is constant.



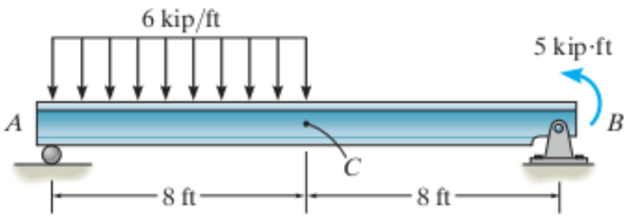
Prob. 16–65

16–66. The shaft is supported by a journal bearing at A, which exerts only vertical reactions on the shaft, and by a thrust bearing at B, which exerts both horizontal and vertical reactions on the shaft. Draw the bending-moment diagram for the shaft and then, from this diagram, sketch the deflection or elastic curve for the shaft's centerline. Determine the equations of the elastic curve using the coordinates x_1 and x_2. EI is constant.



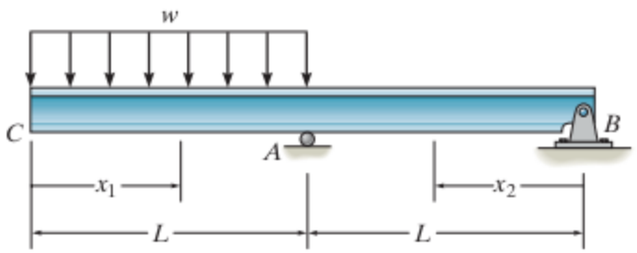
Prob. 16–66

16–67. The W8 × 24 simply supported beam is subjected to the loading shown. Using the method of superposition, determine the deflection at its center C. The beam is made of A-36 steel.



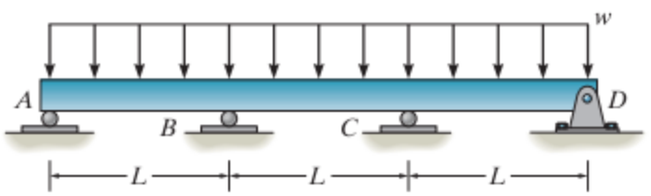
Prob. 16–67

***16–68.** Determine the equations of the elastic curve for the beam using the x_1 and x_2 coordinates. Specify the slope at A and the maximum deflection. EI is constant. Use the method of integration.



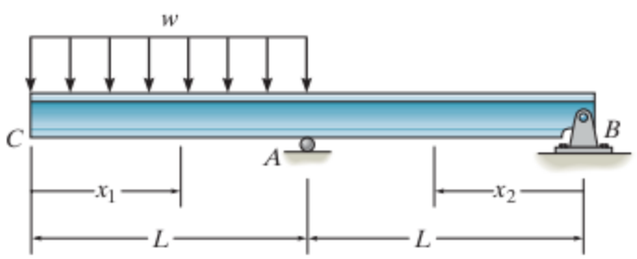
Prob. 16–68

16–69. Determine the reactions at the supports. EI is constant. Use the method of superposition.



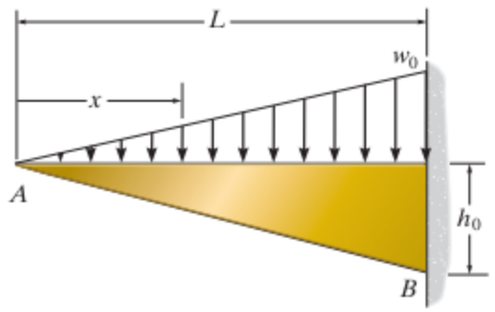
Prob. 16–69

16–70. Determine the maximum deflection between the supports A and B. EI is constant. Use the method of integration.



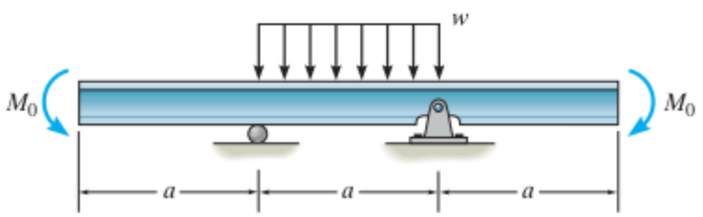
Prob. 16–70

16–71. If the cantilever beam has a constant thickness t, determine the deflection at end A. The beam is made of material having a modulus of elasticity E.



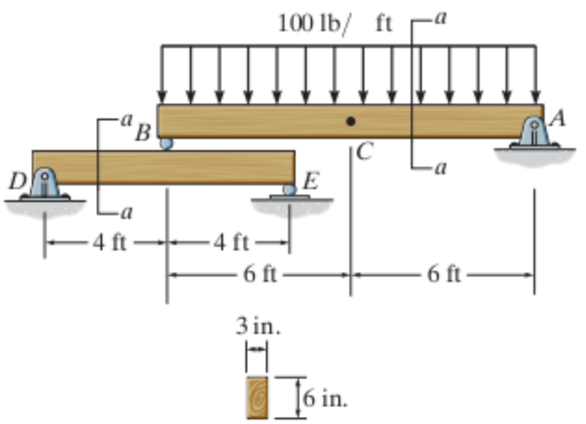
Prob. 16–71

*16–72. Using the method of superposition, determine the magnitude of $\mathbf{M}_0$ in terms of the distributed load w and dimension a so that the deflection at the center of the beam is zero. EI is constant.



Prob. 16–72

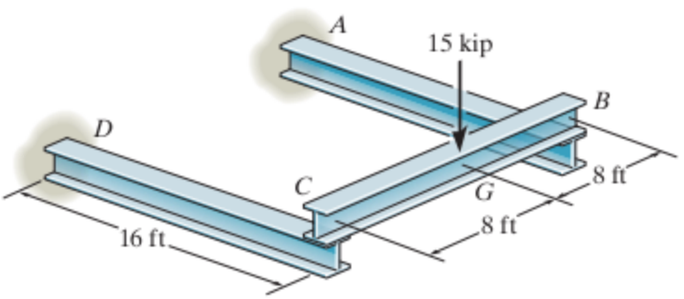
16–73. Using the method of superposition, determine the deflection at C of beam AB. The beams are made of wood having a modulus of elasticity of $E = 1.5(10^3)$ ksi.



Section a–a

Prob. 16–73

16–74. The framework consists of two A-36 steel cantilevered beams CD and BA and a simply supported beam CB. If each beam is made of steel and has a moment of inertia about its principal axis of $I_x = 118\ \text{in}^4$, determine the deflection at the center G of beam CB. ***Use the method of superposition.***



Prob. 16–74

16

Chapter 17

The columns used to support this water tank are braced at their mid-height in order to reduce their chance of buckling.

Buckling of Columns

CHAPTER OBJECTIVES

- To develop methods to calculate the critical loads causing columns to buckle in the elastic region of the stress-strain curve.
- To consider the effect of different supports on the critical load for buckling.
- To introduce approximate methods for critical buckling loads in the inelastic region.

17.1 Critical Load

Whenever a member is designed, it is necessary that it satisfy specific strength, deflection, and stability requirements. In the preceding chapters we have discussed some of the methods used to determine a member's strength and deflection, while assuming that the member was always in stable equilibrium. Some members, however, may be subjected to compressive loadings, and if these members are long and slender the loading may be large enough to cause the member to deflect laterally or sidesway. To be specific, long slender members subjected to an axial compressive force are called ***columns***, and the lateral deflection that occurs is called ***buckling***. Quite often the buckling of a column can lead to a sudden and dramatic failure of a structure or mechanism, and as a result, special attention must be given to the design of columns so that they can safely support their intended loadings without buckling.

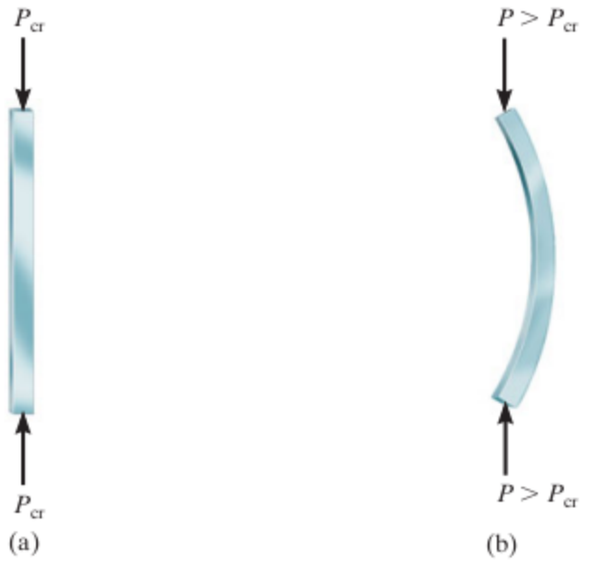


Fig. 17–1

The maximum axial load that a column can support when it is on the *verge* of buckling is called the ***critical load***, P_{cr}, Fig. 17–1*a*. Any additional loading will cause the column to buckle and therefore deflect laterally as shown in Fig. 17–1*b*. In order to better understand the nature of this instability, consider a two-bar mechanism consisting of weightless bars that are rigid and pin-connected as shown in Fig. 17–2*a*. When the bars are in the vertical position, the spring, having a stiffness k, is unstretched, and a *small* vertical force **P** is applied at the top of one of the bars. We can upset this equilibrium position by displacing the pin at A by a small amount Δ, Fig. 17–2*b*. As shown on the free-body diagram of the pin when the bars are displaced, Fig. 17–2*c*, the spring will produce a restoring force $F = k\Delta$, while the applied load **P** develops two horizontal components, $P_x = P \tan\theta$, which tend to push the pin (and the bars) further out of equilibrium. Since θ is small, $\Delta \approx \theta(L/2)$ and $\tan\theta \approx \theta$. Thus the *restoring* spring force becomes $F = k\theta L/2$, and the *disturbing* force is $2P_x = 2P\theta$.

If the restoring force is greater than the disturbing force, that is, $k\theta L/2 > 2P\theta$, then, noticing that θ cancels out, we can solve for P, which gives

$$P < \frac{kL}{4} \quad \text{stable equilibrium}$$

This is a condition for *stable equilibrium* since the force developed by the spring would be adequate to restore the bars back to their vertical position. However, if $kL\theta/2 < 2P\theta$, or

$$P > \frac{kL}{4} \quad \text{unstable equilibrium}$$

then the mechanism would be in *unstable equilibrium*. In other words, if this load **P** is applied, and a slight displacement occurs at A, the mechanism will tend to move out of equilibrium and not be restored to its original position.

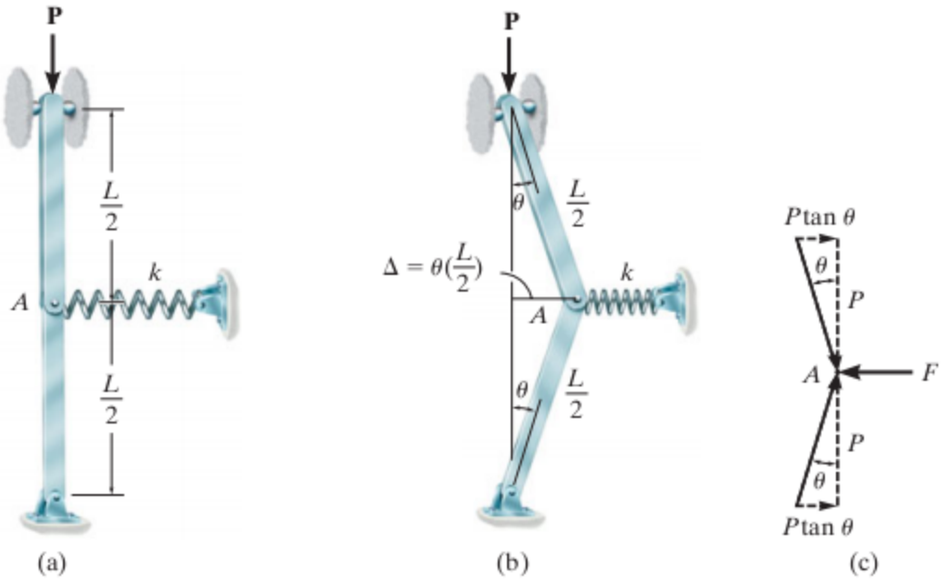


Fig. 17–2

The intermediate value of P, which requires $kL\theta/2 = 2P\theta$, is the *critical load*. Here

$$P_{cr} = \frac{kL}{4} \quad \text{neutral equilibrium}$$

This loading represents a case of the mechanism being in *neutral equilibrium*. Since P_{cr} is *independent* of the (small) displacement θ of the bars, any slight disturbance given to the mechanism will not cause it to move further out of equilibrium, nor will it be restored to its original position. Instead, the bars will *remain* in the deflected position.

These three different states of equilibrium are represented graphically in Fig. 17–3. The transition point where the load is equal to the critical value $P = P_{cr}$ is called the *bifurcation point*. At this point the mechanism will be in equilibrium for any *small value* of θ, measured either to the right or to the left of the vertical. Physically, P_{cr} represents the load for which the mechanism is on the verge of buckling. It is quite reasonable to determine this value by assuming *small displacements* as done here; however, it should be understood that P_{cr} may *not* be the largest value of P that the mechanism can support. Indeed, if a larger load is placed on the bars, then the mechanism may have to undergo a further deflection before the spring is compressed or elongated enough to hold the mechanism in equilibrium.

Like the two-bar mechanism just discussed, the critical buckling loads on columns supported in various ways can be obtained, and the method used to do this will be explained in the next section. Although in engineering design the critical load may be considered to be the largest load the column can support, realize that, like the two-bar mechanism in the deflected or buckled position, a column may actually support an

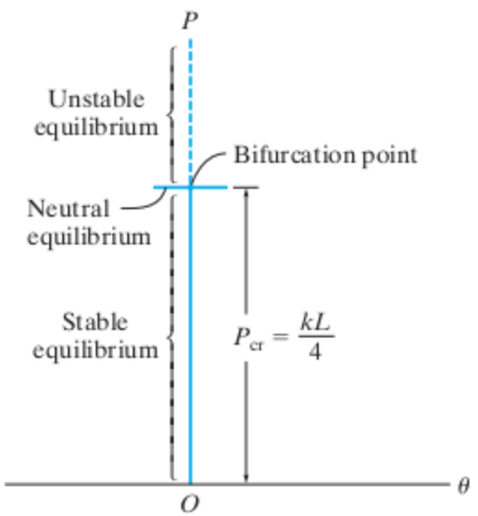


Fig. 17–3

even greater load than P_{cr}. Unfortunately, however, this loading may require the column to undergo a *large* deflection, which is generally not tolerated in engineering structures or machines. For example, it may take only a few newtons of force to buckle a meterstick, but the additional load it may support can be applied only after the stick undergoes a relatively large lateral deflection.

17.2 Ideal Column with Pin Supports

Some slender pin-connected members used in moving machinery, such as this short link, are subjected to compressive loads and thus act as columns.

In this section we will determine the critical buckling load for a column that is pin-supported as shown in Fig. 17–4*a*. The column to be considered is an ***ideal column***, meaning one that is perfectly straight before loading, is made of homogeneous material, and upon which the load is applied through the centroid of the cross section. It is further assumed that the material behaves in a linear-elastic manner and that the column buckles or bends in a single plane. In reality, the conditions of column straightness and load application are never accomplished; however, the analysis to be performed on an "ideal column" is similar to that used to analyze initially crooked columns or those having an eccentric load application. These more realistic cases will be discussed later in this chapter.

Since an ideal column is straight, theoretically the axial load P could be increased until failure occurs by either fracture or yielding of the material. However, when the critical load P_{cr} is reached, the column will be on the verge of becoming unstable, so that a small lateral force F, Fig. 17–4*b*, will cause the column to remain in the deflected position when F is removed, Fig. 17–4*c*. Any slight reduction in the axial load P from P_{cr} will allow the column to straighten out, and any slight increase in P, beyond P_{cr}, will cause further increases in lateral deflection.

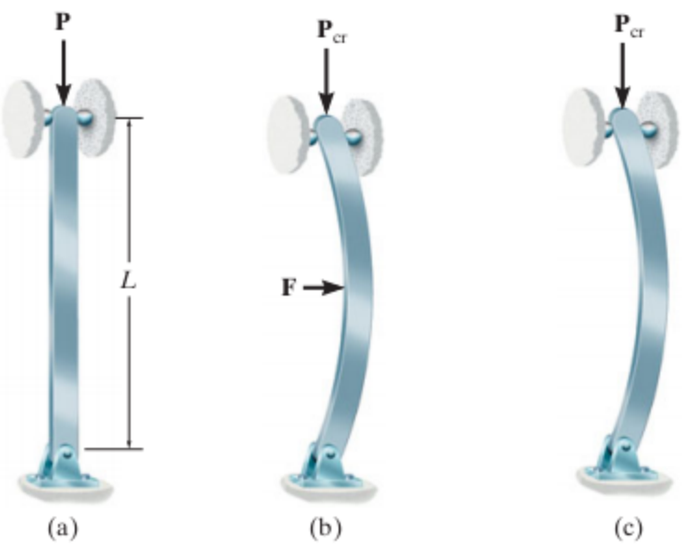


Fig. 17–4

Whether or not a column will remain stable or become unstable when subjected to an axial load will depend on its ability to restore itself, which is based on its resistance to bending. Hence, in order to determine the critical load and the buckled shape of the column, we will apply Eq. 16–5, which relates the internal moment in the column to its deflected shape, i.e.,

$$EI\frac{d^2v}{dx^2} = M \tag{17–1}$$

Recall that this equation assumes that the slope of the elastic curve is small and that deflections occur only by bending. When the column is in its deflected position, Fig. 17–5*a*, the internal bending moment can be determined by using the method of sections. The free-body diagram of a segment in the deflected position is shown in Fig. 17–5*b*. Here both the deflection v and the internal moment M are shown in the *positive direction* according to the sign convention used to establish Eq. 17–1. Moment equilibrium requires, $M = -Pv$. Thus Eq. 17–1 becomes

$$EI\frac{d^2v}{dx^2} = -Pv$$

$$\frac{d^2v}{dx^2} + \left(\frac{P}{EI}\right)v = 0 \tag{17–2}$$

This is a homogeneous, second-order, linear differential equation with constant coefficients. It can be shown by using the methods of differential equations, or by direct substitution into Eq. 17–2, that the general solution is

$$v = C_1 \sin\left(\sqrt{\frac{P}{EI}}x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}x\right) \tag{17–3}$$

The two constants of integration are determined from the boundary conditions at the ends of the column. Since $v = 0$ at $x = 0$, then $C_2 = 0$. And since $v = 0$ at $x = L$, then

$$C_1 \sin\left(\sqrt{\frac{P}{EI}}L\right) = 0$$

This equation is satisfied if $C_1 = 0$; however, then $v = 0$, which is a *trivial solution* that requires the column to always remain straight, even though the load may cause the column to become unstable. The other possibility is for

$$\sin\left(\sqrt{\frac{P}{EI}}L\right) = 0$$

which is satisfied if

$$\sqrt{\frac{P}{EI}}L = n\pi$$

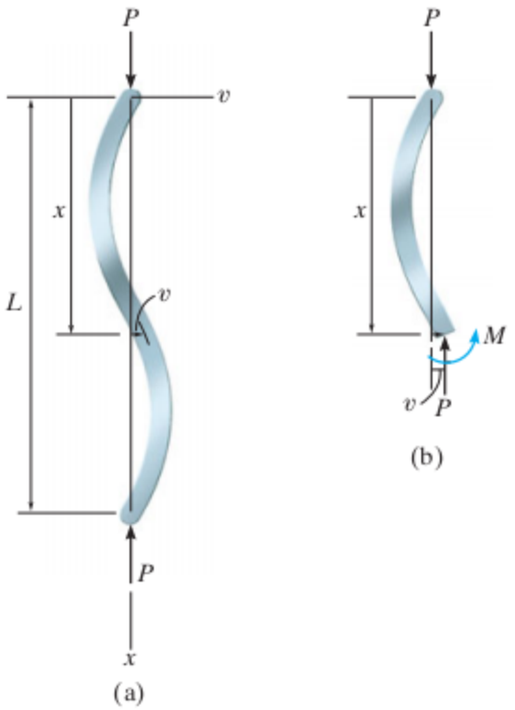
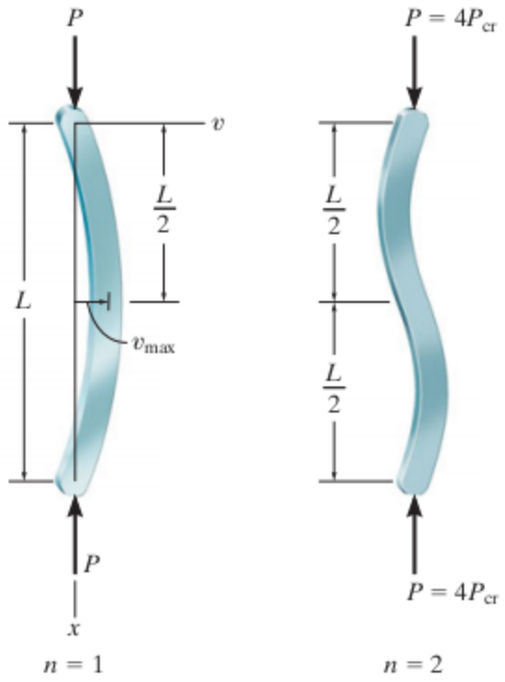


Fig. 17–5



(c)

Fig. 17–5 (cont.)

17

or

$$P = \frac{n^2\pi^2 EI}{L^2} \quad n = 1, 2, 3, \ldots \tag{17–4}$$

The *smallest value* of P is obtained when $n = 1$, so the *critical load* for the column is therefore*

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

This load is sometimes referred to as the *Euler load*, named after the Swiss mathematician Leonhard Euler, who originally solved this problem in 1757. The corresponding buckled shape is defined by the equation

$$v = C_1 \sin\frac{\pi x}{L}$$

Here the constant C_1 represents the maximum deflection, v_{max}, which occurs at the midpoint of the column, Fig. 17–5*c*. Specific values for C_1 cannot be obtained, since the exact deflected form for the column is unknown once it has buckled. It has been assumed, however, that this deflection is small.

Note that the critical load is independent of the strength of the material; rather it only depends on the column's dimensions (I and L) and the material's stiffness or modulus of elasticity E. For this reason, as far as elastic buckling is concerned, columns made, for example, of high-strength steel offer no advantage over those made of lower-strength steel, since the modulus of elasticity for both is approximately the same. Also note that the load-carrying capacity of a column will increase as the moment of inertia of the cross section increases. Thus, efficient columns are designed so that most of the column's cross-sectional area is located as far away as possible from the principal centroidal axes for the section. This is why hollow sections such as tubes are more economical than solid sections. Furthermore, wide-flange sections, and columns that are "built up" from channels, angles, plates, etc., are better than sections that are solid and rectangular.

*n represents the number of curves in the deflected shape of the column. For example, if $n = 2$, then *two* curves will appear, Fig. 17–5*c*. Here the critical load is 4 P_{cr} just prior to buckling, which practically speaking will not exist.

It is also important to realize that a column will buckle about the principal axis of the cross section having the ***least moment of inertia*** (the weakest axis) provided it is supported the same way about each axis. For example, a column having a rectangular cross section, like a meter stick, as shown in Fig. 17–6, will buckle about the a–a axis, not the b–b axis. As a result, engineers usually try to achieve a balance, keeping the moments of inertia the same in all directions. Geometrically, then, circular tubes would make excellent columns. Also, square tubes or those shapes having $I_x \approx I_y$ are often selected for columns.

Summarizing the above discussion, the buckling equation for a pin-supported long slender column can be rewritten, and the terms defined as follows:

$$P_{cr} = \frac{\pi^2 EI}{L^2} \tag{17–5}$$

where

P_{cr} = critical or maximum axial load on the column just before it begins to buckle. This load must *not* cause the stress in the column to exceed the proportional limit

E = modulus of elasticity for the material

I = *least* moment of inertia for the column's cross-sectional area

L = unsupported length of the column, whose ends are pinned

Fig. 17–6

For purposes of design, the above equation can also be written in a more useful form by expressing $I = Ar^2$, where A is the cross-sectional area and r is the ***radius of gyration*** of the cross-sectional area. Thus,

$$P_{cr} = \frac{\pi^2 E(Ar^2)}{L^2}$$

$$\left(\frac{P}{A}\right)_{cr} = \frac{\pi^2 E}{(L/r)^2}$$

or

$$\sigma_{cr} = \frac{\pi^2 E}{(L/r)^2} \tag{17–6}$$

Here

σ_{cr} = critical stress, which is an average normal stress in the column just before the column buckles. This stress is an *elastic stress* and therefore $\sigma_{cr} \leq \sigma_Y$

E = modulus of elasticity for the material

L = unsupported length of the column, whose ends are pinned

r = *smallest* radius of gyration of the column, determined from $r = \sqrt{I/A}$, where I is the *least* moment of inertia of the column's cross-sectional area A

The geometric ratio L/r in Eq. 17–6 is known as the ***slenderness ratio***. It is a measure of the column's flexibility, and as will be discussed later, it serves to classify columns as long, intermediate, or short.

Typical interior steel pipe columns used to support the roof of a single story building.

It is possible to graph Eq. 17–6 using axes that represent the critical stress versus the slenderness ratio. Examples of this graph for columns made of a typical structural steel and aluminum alloy are shown in Fig. 17–7. Note that the curves are hyperbolic and are valid only for critical stresses below the material's yield point (proportional limit), since the material must behave elastically. For the steel the yield stress is $(\sigma_Y)_{st} = 36$ ksi $[E_{st} = 29(10^3)$ ksi], and for the aluminum it is $(\sigma_Y)_{al} = 27$ ksi $[E_{al} = 10(10^3)$ ksi]. Substituting $\sigma_{cr} = \sigma_Y$ into Eq. 17–6, the *smallest* allowable slenderness ratios for the steel and aluminum columns are therefore $(L/r)_{st} = 89$ and $(L/r)_{al} = 60.5$, respectively. Thus, for a steel column, if $(L/r)_{st} \geq 89$, Euler's formula can be used to determine the critical load since the stress in the column remains elastic. On the other hand, if $(L/r)_{st} < 89$, the column's stress will exceed the yield point before buckling can occur, and therefore the Euler formula is not valid in this case.

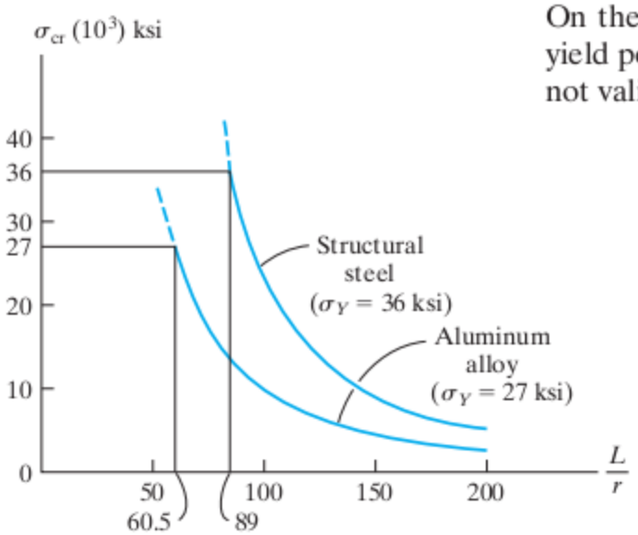


Fig. 17–7

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Important Points

- *Columns* are long slender members that are subjected to axial compressive loads.
- The *critical load* is the maximum axial load that a column can support when it is on the verge of buckling. This loading represents a case of *neutral equilibrium*.
- An *ideal column* is initially perfectly straight, made of homogeneous material, and the load is applied through the centroid of the cross section.
- A pin-connected column will buckle about the principal axis of the cross section having the *least* moment of inertia.
- The *slenderness ratio* is L/r, where r is the smallest radius of gyration of the cross section. Buckling will occur about the axis where this ratio gives the greatest value.

EXAMPLE 17.1

The A-36 steel W8 × 31 member shown in Fig. 17–8 is to be used as a pin-connected column. Determine the largest axial load it can support before it either begins to buckle or the steel yields.

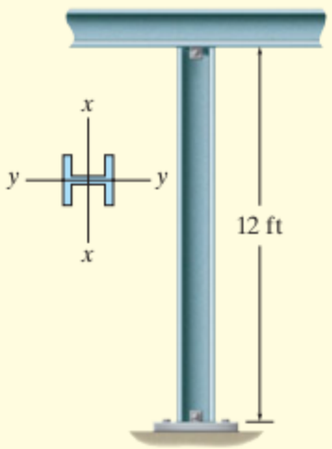


Fig. 17–8

SOLUTION

From the table in Appendix C, the column's cross-sectional area and moments of inertia are $A = 9.13\,\text{in}^2$, $I_x = 110\,\text{in}^4$, and $I_y = 37.1\,\text{in}^4$. By inspection, buckling will occur about the y–y axis. Why? Applying Eq. 17–5, we have

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2[29(10^3)\,\text{kip/in}^2](37.1\,\text{in}^4)}{[12\,\text{ft}(12\,\text{in./ft})]^2} = 512\,\text{kip}$$

When fully loaded, the average compressive stress in the column is

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{512\,\text{kip}}{9.13\,\text{in}^2} = 56.1\,\text{ksi}$$

Since this stress exceeds the yield stress (36 ksi), the load P is determined from simple compression:

$$36\,\text{ksi} = \frac{P}{9.13\,\text{in}^2}; \qquad P = 329\,\text{kip} \qquad \textit{Ans.}$$

In actual practice, a factor of safety would be placed on this loading.

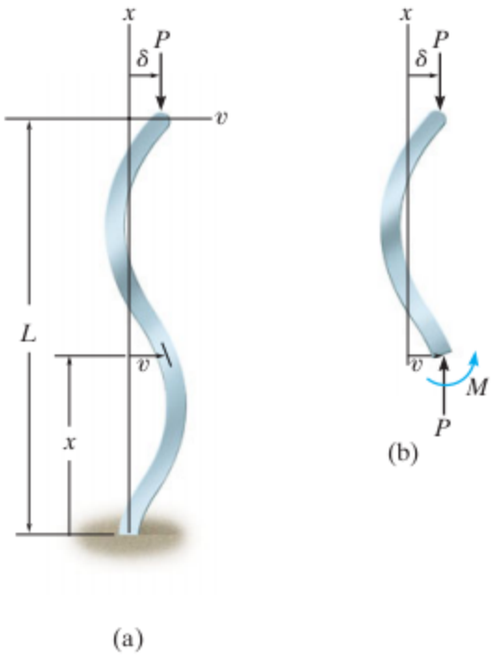


Fig. 17–9

17.3 Columns Having Various Types of Supports

The Euler load was derived for a column that is pin-connected or free to rotate at its ends. Oftentimes, however, columns may be supported in some other way. For example, consider the case of a column fixed at its base and free at the top, Fig. 17–9*a*. As the column buckles the load displaces δ and at x the displacement is v. From the free-body diagram in Fig. 17–9*b*, the internal moment at the arbitrary section is $M = P(\delta - v)$. Consequently, the differential equation for the deflection curve is

$$EI\frac{d^2v}{dx^2} = P(\delta - v)$$

$$\frac{d^2v}{dx^2} + \frac{P}{EI}v = \frac{P}{EI}\delta \tag{17–7}$$

Unlike Eq. 17–2, this equation is nonhomogeneous because of the nonzero term on the right side. The solution consists of both a complementary and a particular solution, namely,

$$v = C_1 \sin\left(\sqrt{\frac{P}{EI}}x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}x\right) + \delta$$

The constants are determined from the boundary conditions. At $x = 0$, $v = 0$, so that $C_2 = -\delta$. Also,

$$\frac{dv}{dx} = C_1\sqrt{\frac{P}{EI}}\cos\left(\sqrt{\frac{P}{EI}}x\right) - C_2\sqrt{\frac{P}{EI}}\sin\left(\sqrt{\frac{P}{EI}}x\right)$$

At $x = 0$, $dv/dx = 0$, so that $C_1 = 0$. The deflection curve is therefore

$$v = \delta\left[1 - \cos\left(\sqrt{\frac{P}{EI}}x\right)\right] \tag{17–8}$$

Since the deflection at the top of the column is δ, that is, at $x = L$, $v = \delta$, we require

$$\delta\cos\left(\sqrt{\frac{P}{EI}}L\right) = 0$$

The trivial solution $\delta = 0$ indicates that no buckling occurs, regardless of the load P. Instead,

$$\cos\left(\sqrt{\frac{P}{EI}}L\right) = 0 \quad \text{or} \quad \sqrt{\frac{P}{EI}}L = \frac{n\pi}{2}, n = 1, 3, 5 \ldots$$

The smallest critical load occurs when $n = 1$, so that

$$P_{cr} = \frac{\pi^2 EI}{4L^2} \tag{17–9}$$

By comparison with Eq. 17–5, it is seen that a column fixed-supported at its base and free at its top will support only one-fourth the critical load that can be applied to a column pin-supported at both ends.

17

Other types of supported columns are analyzed in much the same way and will not be covered in detail here. Instead, we will tabulate the results for the most common types of column support and show how to apply these results by writing Euler's formula in a general form.

Effective Length. As stated previously, the Euler formula, Eq. 17–5, was developed for the case of a column having ends that are pinned or free to rotate. In other words, L in the equation represents the unsupported distance between the points of zero moment. This formula can be used to determine the critical load on columns having other types of support provided "L" represents the distance between the zero-moment points. This distance is called the column's ***effective length***, L_e. Obviously, for a pin-ended column $L_e = L$, Fig. 17–10*a*. For the fixed- and free-ended column, the deflection curve, Eq. 17–8, was found to be onehalf that of a column that is pin-connected and has a length of $2L$, Fig. 17–10*b*. Thus the effective length between the points of zero moment is $L_e = 2L$. Examples for two other columns with different end supports are also shown in Fig. 17–10. The column fixed at its ends, Fig. 17–10*c*, has inflection points or points of zero moment $L/4$ from each support. The effective length is therefore represented by the middle half of its length, that is, $L_e = 0.5L$. Lastly, the pin- and fixed-ended column, Fig. 17–10*d*, has an inflection point at approximately $0.7L$ from its pinned end, so that $L_e = 0.7L$.

Rather than specifying the column's effective length, many design codes provide column formulas that employ a dimensionless coefficient K called the ***effective-length factor***. This factor is defined from

$$L_e = KL \tag{17–10}$$

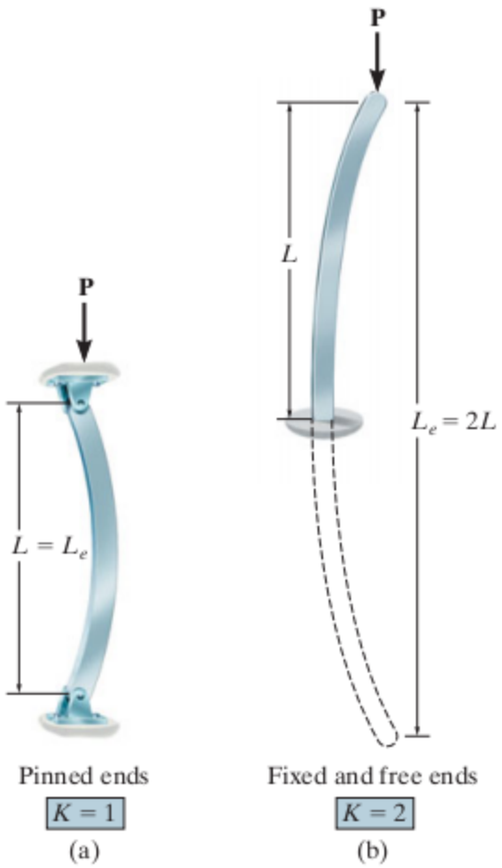


Fig. 17–10

Specific values of K are also given in Fig. 17–10. Based on this generality, we can therefore write Euler's formula as

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2} \tag{17–11}$$

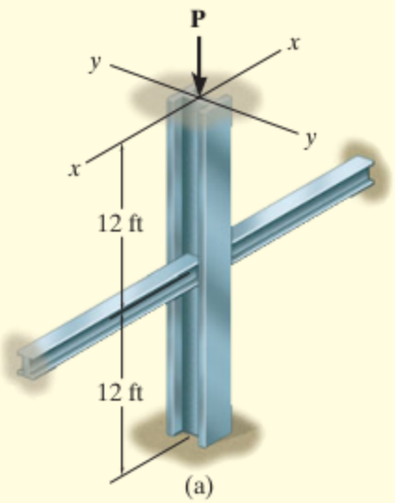
or

$$\sigma_{cr} = \frac{\pi^2 E}{(KL/r)^2} \tag{17–12}$$

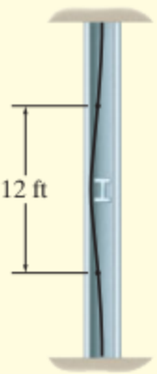
Here (KL/r) is the column's ***effective-slenderness ratio***. For example, if the column is fixed at its base and free at its end, we have $K = 2$, and therefore Eq. 17–11 gives the same result as Eq. 17–9.

17

EXAMPLE 17.2

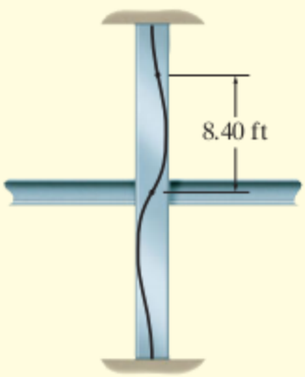


(a)



x–x axis buckling

(b)



y–y axis buckling

(c)

Fig. 17–11

A W6 × 15 steel column is 24 ft long and is fixed at its ends as shown in Fig. 17–11*a*. Its load-carrying capacity is increased by bracing it about the *y–y* (weak) axis using struts that are assumed to be pin-connected to its midheight. Determine the load it can support so that the column does not buckle nor the material exceed the yield stress. Take $E_{st} = 29(10^3)$ ksi and $\sigma_Y = 60$ ksi.

SOLUTION

The buckling behavior of the column will be *different* about the *x–x* and *y–y* axes due to the bracing. The buckled shape for each of these cases is shown in Figs. 17–11*b* and 17–11*c*. From Fig. 17–11*b*, the effective length for buckling about the *x–x* axis is $(KL)_x = 0.5(24\text{ ft}) = 12\text{ ft} = 144\text{ in.}$, and from Fig. 17–11*c*, for buckling about the *y–y* axis, $(KL)_y = 0.7(24\text{ ft}/2) = 8.40\text{ ft} = 100.8\text{ in.}$ The moments of inertia for a W6 × 15 are found from the table in Appendix C. We have $I_x = 29.1\text{ in}^4$, $I_y = 9.32\text{ in}^4$.

Applying Eq. 17–11,

$$(P_{cr})_x = \frac{\pi^2 E I_x}{(KL)_x^2} = \frac{\pi^2[29(10^3)\text{ ksi}]29.1\text{ in}^4}{(144\text{ in.})^2} = 401.7\text{ kip} \qquad (1)$$

$$(P_{cr})_y = \frac{\pi^2 E I_y}{(KL)_y^2} = \frac{\pi^2[29(10^3)\text{ ksi}]9.32\text{ in}^4}{(100.8\text{ in.})^2} = 262.5\text{ kip} \qquad (2)$$

By comparison, buckling will occur about the *y–y* axis.

The area of the cross section is 4.43 in^2, so the average compressive stress in the column is

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{262.5\text{ kip}}{4.43\text{ in}^2} = 59.3\text{ ksi}$$

Since this stress is less than the yield stress, buckling will occur before the material yields. Thus,

$$P_{cr} = 263\text{ kip} \qquad \textit{Ans.}$$

NOTE: From Eq. 17–12 it can be seen that buckling will always occur about the column axis having the *largest* slenderness ratio, since a large slenderness ratio will give a small critical stress. Thus, using the data for the radius of gyration from the table in Appendix C, we have

$$\left(\frac{KL}{r}\right)_x = \frac{144\text{ in.}}{2.56\text{ in.}} = 56.2$$

$$\left(\frac{KL}{r}\right)_y = \frac{100.8\text{ in.}}{1.46\text{ in.}} = 69.0$$

Hence, *y–y* axis buckling will occur, which is the same conclusion reached by comparing Eqs. (1) and (2).

17

EXAMPLE 17.3

The aluminum column is fixed at its bottom and is braced at its top by cables so as to prevent movement at the top along the x axis, Fig. 17–12*a*. If it is assumed to be fixed at its base, determine the largest allowable load P that can be applied. Use a factor of safety for buckling of F.S. = 3.0. Take $E_{al} = 70$ GPa, $\sigma_Y = 215$ MPa, $A = 7.5(10^{-3})\ \text{m}^2$, $I_x = 61.3(10^{-6})\ \text{m}^4$, $I_y = 23.2(10^{-6})\ \text{m}^4$.

SOLUTION

Buckling about the x and y axes is shown in Fig. 17–12*b* and 17–12*c*, respectively. Using Fig. 17–10*a*, for x–x axis buckling, $K = 2$, so $(KL)_x = 2(5\ \text{m}) = 10\ \text{m}$. Also, for y–y axis buckling, $K = 0.7$, so $(KL)_y = 0.7(5\ \text{m}) = 3.5\ \text{m}$.

Applying Eq. 17–11, the critical loads for each case are

$$(P_{cr})_x = \frac{\pi^2 E I_x}{(KL)_x^2} = \frac{\pi^2[70(10^9)\ \text{N/m}^2](61.3(10^{-6})\ \text{m}^4)}{(10\ \text{m})^2}$$

$$= 424\ \text{kN}$$

$$(P_{cr})_y = \frac{\pi^2 E I_y}{(KL)_y^2} = \frac{\pi^2[70(10^9)\ \text{N/m}^2](23.2(10^{-6})\ \text{m}^4)}{(3.5\ \text{m})^2}$$

$$= 1.31\ \text{MN}$$

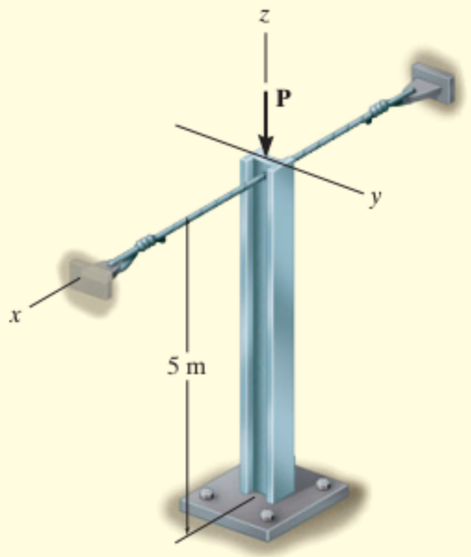
By comparison, as P is increased the column will buckle about the x–x axis. The allowable load is therefore

$$P_{allow} = \frac{P_{cr}}{\text{F.S.}} = \frac{424\ \text{kN}}{3.0} = 141\ \text{kN} \qquad \textit{Ans.}$$

Since

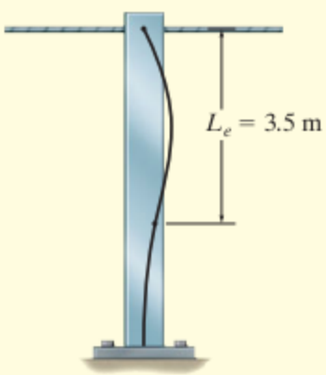
$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{424\ \text{kN}}{7.5(10^{-3})\ \text{m}^2} = 56.5\ \text{MPa} < 215\ \text{MPa}$$

Euler's equation can be applied.



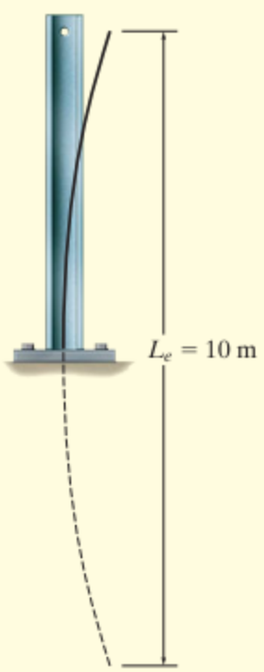
(a)

Fig. 17–12



y–y axis buckling

(c)



x–x axis buckling

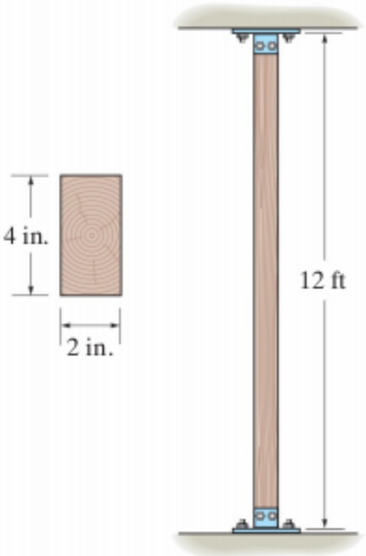
(b)

17

FUNDAMENTAL PROBLEMS

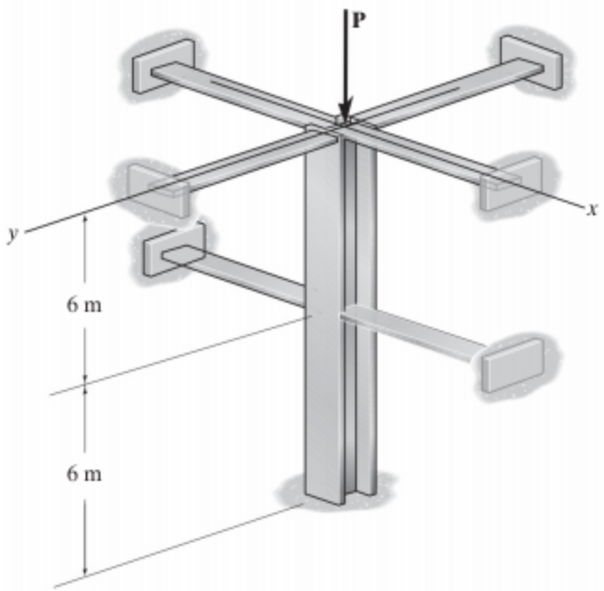
F17–1. A 50-in.-long rod is made from a 1-in.-diameter steel rod. Determine the critical buckling load if the ends are fixed supported. $E = 29(10^3)$ ksi, $\sigma_Y = 36$ ksi.

F17–2. A 12-ft wooden rectangular column has the dimensions shown. Determine the critical load if the ends are assumed to be pin-connected. $E = 1.6(10^3)$ ksi. Yielding does not occur.



F17–2

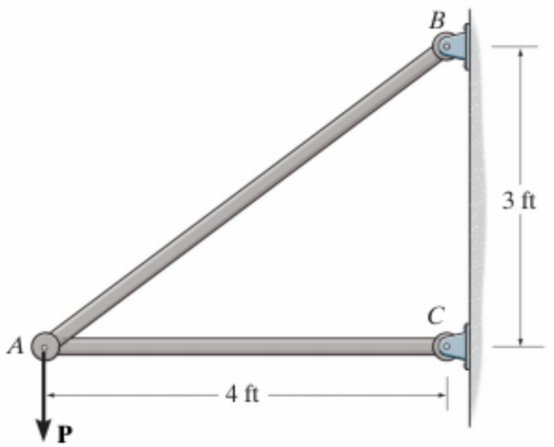
F17–3. The A-36 steel column can be considered pinned at its top and bottom and braced against its weak axis at the mid-height. Determine the maximum allowable force **P** that the column can support without buckling. Apply a F.S. = 2 against buckling. Take $A = 7.4(10^{-3})$ m^2, $I_x = 87.3(10^{-6})$ m^4, and $I_y = 18.8(10^{-6})$ m^4.



F17–3

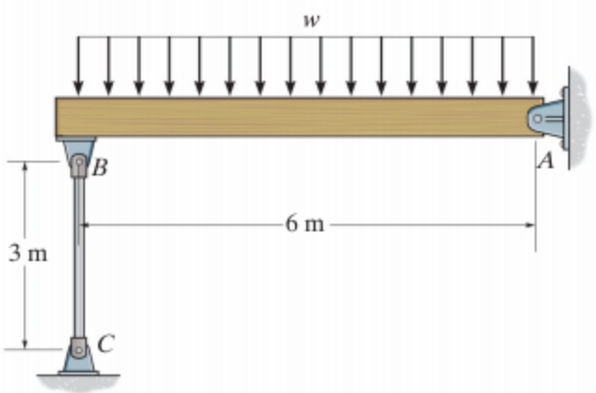
F17–4. A steel pipe is fixed supported at its ends. If it is 5 m long and has an outer diameter of 50 mm and a thickness of 10 mm, determine the maximum axial load P that it can carry without buckling. $E_{st} = 200$ GPa, $\sigma_Y = 250$ MPa.

F17–5. Determine the maximum force **P** that can be supported by the assembly without causing member AC to buckle. The member is made of A-36 steel and has a diameter of 2 in. Take F.S. = 2 against buckling.



F17–5

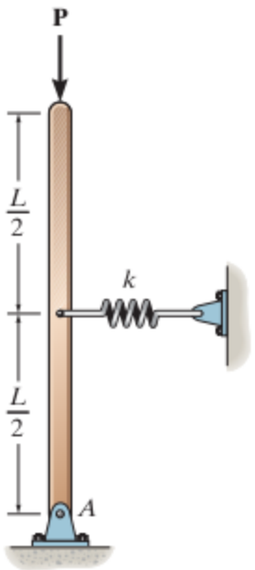
F17–6. The A-36 steel rod BC has a diameter of 50 mm and is used as a strut to support the beam. Determine the maximum intensity w of the uniform distributed load that can be applied to the beam without causing the strut to buckle. Take F.S. = 2 against buckling.



F17–6

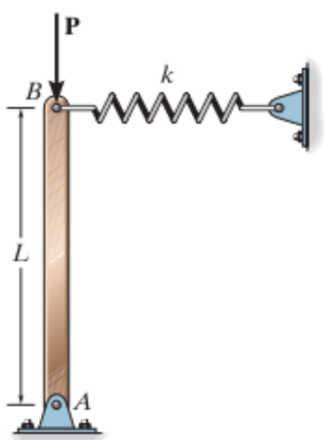
PROBLEMS

17–1. Determine the critical buckling load for the column. The material can be assumed rigid.



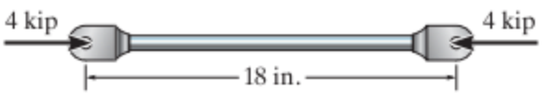
Prob. 17–1

17–2. The column consists of a rigid member that is pinned at its bottom and attached to a spring at its top. If the spring is unstretched when the column is in the vertical position, determine the critical load that can be placed on the column.



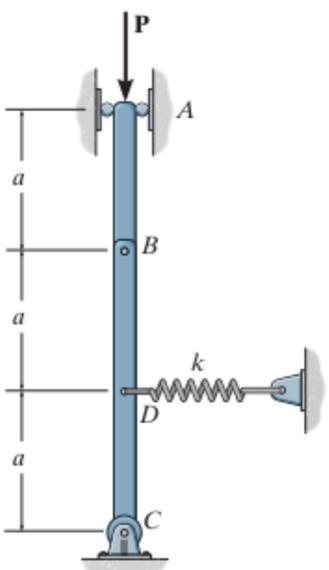
Prob. 17–2

17–3. The aircraft link is made from an A992 steel rod. Determine the smallest diameter of the rod, to the nearest $\frac{1}{16}$ in., that will support the load of 4 kip without buckling. The ends are pin-connected.



Prob. 17–3

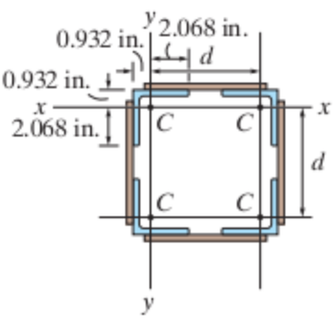
*17–4. Rigid bars AB and BC are pin-connected at B. If the spring at D has a stiffness k, determine the critical load P_{cr} for the system.



Prob. 17–4

17–5. A column is constructed using four A992 steel angles that are laced together as shown. The length of the column is to be 25 ft and the ends are assumed to be pin-connected. Each angle shown below has an area of $A = 2.75\ \text{in}^2$ and moments of inertia of $I_x = I_y = 2.22\ \text{in}^4$. Determine the distance d between the centroids C of the angles so that the column can support an axial load of $P = 350$ kip without buckling. Neglect the effect of the lacing.

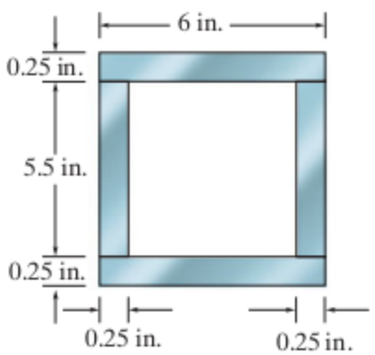
17–6. A column is constructed using four A992 steel angles that are laced together as shown. The length of the column is to be 40 ft and the ends are assumed to be fixed connected. Each angle shown below has an area of $A = 2.75\ \text{in}^2$ and moments of inertia of $I_x = I_y = 2.22\ \text{in}^4$. Determine the distance d between the centroids C of the angles so that the column can support an axial load of $P = 350$ kip without buckling. Neglect the effect of the lacing.



Probs. 17–5/6

17–7. A column is made of A-36 steel, has a length of 20 ft, and is pinned at both ends. If the cross-sectional area has the dimensions shown, determine the critical load.

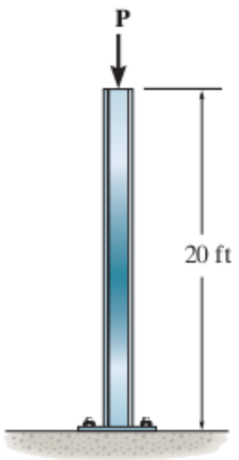
***17–8.** A column is made of 2014-T6 aluminum, has a length of 30 ft, and is fixed at its bottom and pinned at its top. If the cross-sectional area has the dimensions shown, determine the critical load.



Probs. 17–7/8

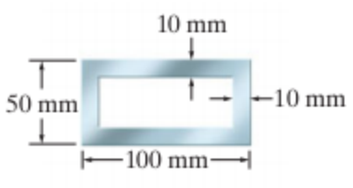
17–9. The W14 × 38 column is made of A-36 steel and is fixed supported at its base. If it is subjected to an axial load of $P = 15$ kip, determine the factor of safety with respect to buckling.

17–10. The W14 × 38 column is made of A-36 steel. Determine the critical load if its bottom end is fixed supported and its top is free to move about the strong axis and is pinned about the weak axis.



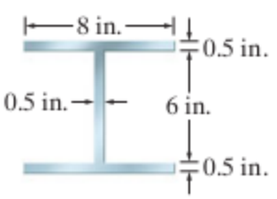
Probs. 17–9/10

17–11. An A992 steel column has a length of 5 m and is fixed at both ends. If the cross-sectional area has the dimensions shown, determine the critical load.



Prob. 17–11

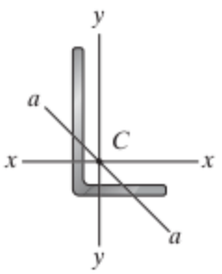
***17–12.** An A-36 steel column has a length of 15 ft and is pinned at both ends. If the cross-sectional area has the dimensions shown, determine the critical load.



Prob. 17–12

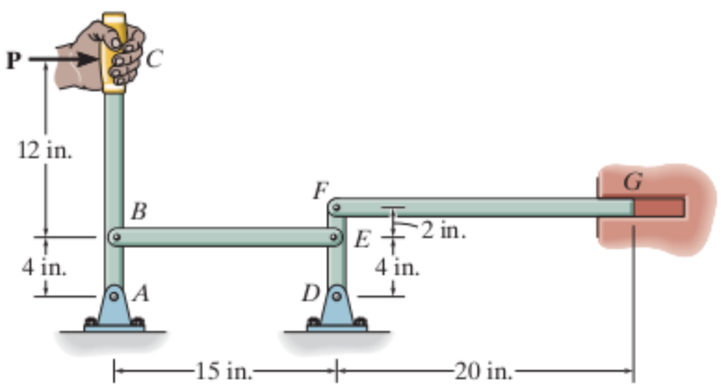
17

17–13. The A992 steel angle has a cross-sectional area of $A = 2.48\,\text{in}^2$ and a radius of gyration about the x axis of $r_x = 1.26$ in. and about the y axis of $r_y = 0.879$ in. The smallest radius of gyration occurs about the a–a axis and is $r_a = 0.644$ in. If the angle is to be used as a pin-connected 10-ft-long column, determine the largest axial load that can be applied through its centroid C without causing it to buckle.



Prob. 17–13

17–14. The control linkage for a machine consists of two L2 steel rods BE and FG, each with a diameter of 1 in. If a device at G causes the end G to freeze up and become pin-connected, determine the maximum horizontal force P that can be applied to the handle without causing either of the two rods to buckle. The members are pin-connected at $A, B, D, E,$ and F.



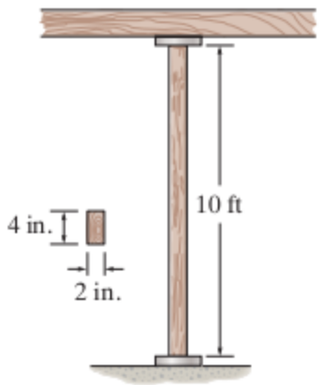
Prob. 17–14

17–15. An A-36-steel W8 × 24 column is fixed at one end and free at its other end. If it is subjected to an axial load of 20 kip, determine the maximum allowable length of the column if F.S. = 2 against buckling is desired.

***17–16.** An A-36-steel W8 × 24 column is fixed at one end and pinned at the other end. If it is subjected to an axial load of 60 kip, determine the maximum allowable length of the column if F.S. = 2 against buckling is desired.

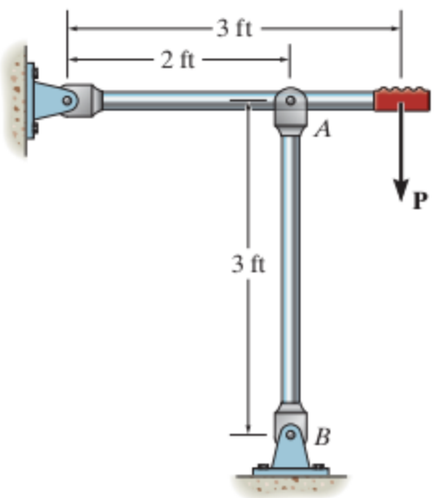
17–17. The 10-ft wooden rectangular column has the dimensions shown. Determine the critical load if the ends are assumed to be pin-connected. $E_w = 1.6(10^3)$ ksi, $\sigma_Y = 5$ ksi.

17–18. The 10-ft wooden column has the dimensions shown. Determine the critical load if the bottom is fixed and the top is pinned. $E_w = 1.6(10^3)$ ksi, $\sigma_Y = 5$ ksi.



Probs. 17–17/18

17–19. Determine the maximum force P that can be applied to the handle so that the A992 steel control rod AB does not buckle. The rod has a diameter of 1.25 in. It is pin-connected at its ends.

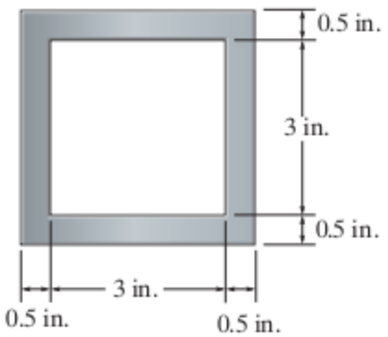


Prob. 17–19

17

*17–20. The A992 steel tube has the cross-sectional area shown. If it has a length of 15 ft and is pinned at both ends, determine the maximum axial load that the tube can support without causing it to buckle.

17–21. The A992 steel tube has the cross-sectional area shown. If it has a length of 15 ft and is fixed at one end and free at the other end, determine the maximum axial load that the tube can support without causing it to buckle.

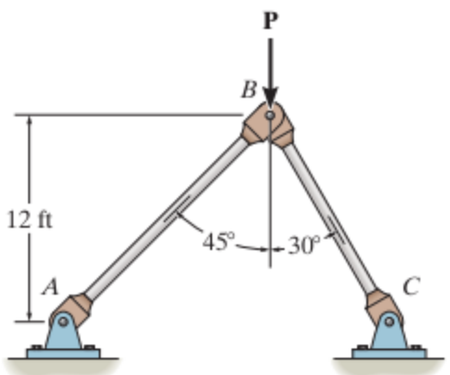


Probs. 17–20/21

17

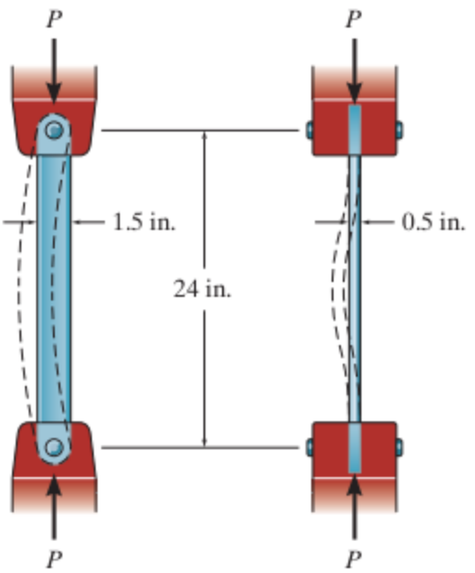
17–22. The linkage is made using two A992 steel rods, each having a circular cross section. Determine the diameter of each rod to the nearest $\frac{1}{8}$ in. that will support a load of $P = 6$ kip. Assume that the rods are pin-connected at their ends. Use a factor of safety with respect to buckling of 1.8.

17–23. The linkage is made using two A992 steel rods, each having a circular cross section. If each rod has a diameter of $\frac{3}{4}$ in., determine the largest load it can support without causing any rod to buckle. Assume that the rods are pin-connected at their ends.



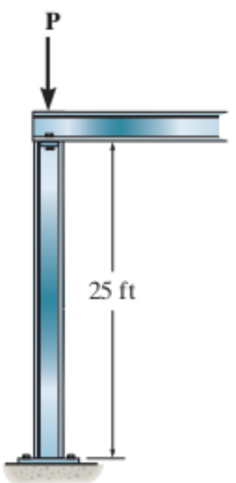
Probs. 17–22/23

*17–24. An L-2 tool steel link in a forging machine is pin-connected to the forks at its ends as shown. Determine the maximum load P it can carry without buckling. Use a factor of safety with respect to buckling of F.S. = 1.75. Note from the figure on the left that the ends are pinned for buckling, whereas from the figure on the right the ends are fixed.



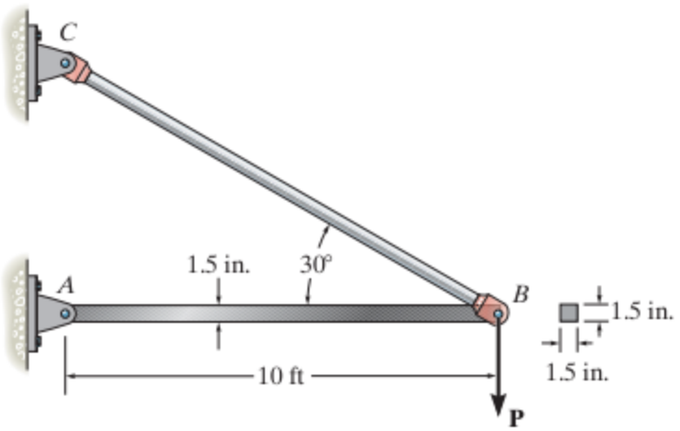
Prob. 17–24

17–25. The W14 × 30 is used as a structural A-36 steel column that can be assumed pinned at both of its ends. Determine the largest axial force P that can be applied without causing it to buckle.



Prob. 17–25

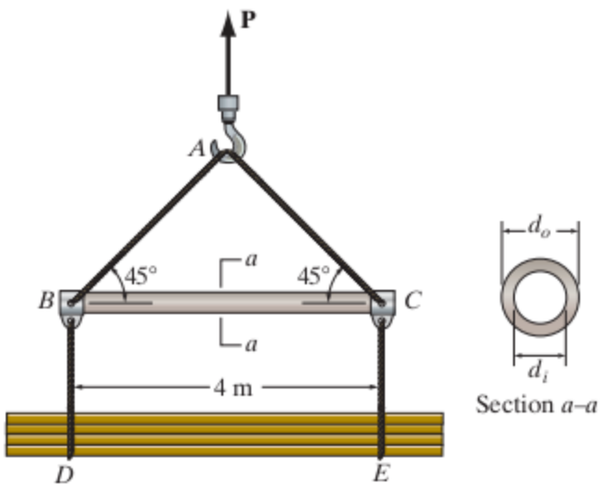
17–26. The A-36 steel bar AB has a square cross section. If it is pin-connected at its ends, determine the maximum allowable load P that can be applied to the frame. Use a factor of safety with respect to buckling of 2.



Prob. 17–26

17–27. The strongback BC is made of an A992 steel hollow circular section with $d_o = 60$ mm and $d_i = 40$ mm. Determine the allowable maximum lifting force **P** without causing the strongback to buckle. F.S. = 2 against buckling is desired.

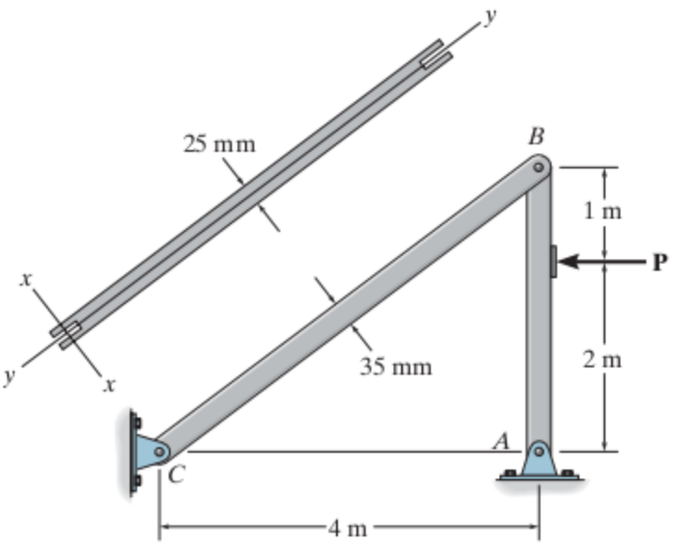
***17–28.** The strongback is made of an A992 steel hollow circular section with the outer diameter of $d_o = 60$ mm. If it is designed to withstand the lifting force of $P = 60$ kN, determine the minimum required wall thickness of the strongback so that it will not buckle. Use F.S. = 2 against buckling.



Probs. 17–27/28

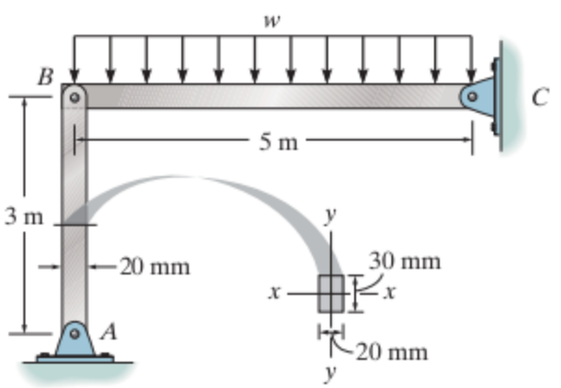
17–29. The frame supports the load of $P = 4$ kN. As a result, the A992 steel member BC is subjected to a compressive load. Due to the forked ends on this member, consider the supports at B and C to act as pins for x–x axis buckling and as fixed supports for y–y axis buckling. Determine the factor of safety with respect to buckling about each of these axes.

17–30. Determine the greatest load P the frame will support without causing the A992 steel member BC to buckle. Due to the forked ends on the member, consider the supports at B and C to act as pins for x–x axis buckling and as fixed supports for y–y axis buckling.



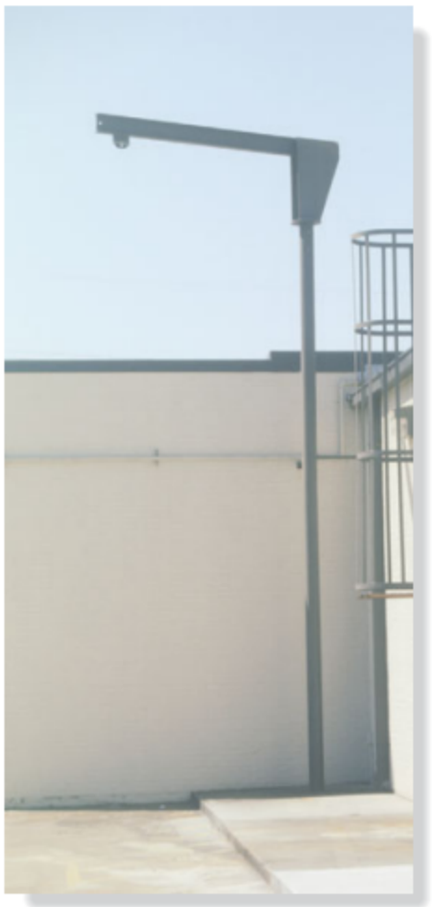
Probs. 17–29/30

17–31. The steel bar AB has a rectangular cross section. If it is pin-connected at its ends, determine the maximum allowable intensity w of the distributed load that can be applied to BC without causing bar AB to buckle. Use a factor of safety with respect to buckling of 1.5. $E_{st} = 200$ GPa, $\sigma Y = 360$ MPa.



Prob. 17–31

17



The column supporting this crane is unusually long. It will be subjected not only to uniaxial load, but also a bending moment. To ensure it will not buckle, it should be braced at the roof as a pin connection.

*17.4 The Secant Formula

The Euler formula was derived assuming the load P is always applied through the centroid of the column's cross-sectional area and that the column is perfectly straight. This is actually quite unrealistic, since manufactured columns are never perfectly straight, nor is the application of the load known with great accuracy. In reality, then, columns never suddenly buckle; instead they begin to bend, although ever so slightly, immediately upon application of the load. As a result, the actual criterion for load application should be limited either to a specified deflection of the column or by not allowing the maximum stress in the column to exceed an allowable stress.

To study this effect, we will apply the load P to the column at a short *eccentric distance* e from its centroid, Fig. 17–13*a*. This loading on the column is statically equivalent to the axial load P and bending moment $M' = Pe$ shown in Fig. 17–13*b*. As shown, in both cases, the ends A and B are supported so that they are free to rotate (pin-supported). As before, we will only consider small slopes and deflections and linear-elastic material behavior. Furthermore, the x–v plane is a plane of symmetry for the cross-sectional area.

From the free-body diagram of the arbitrary section, Fig. 17–13*c*, the internal moment in the column is

$$M = -P(e + v) \tag{17–13}$$

The differential equation for the deflection curve is therefore

$$EI\frac{d^2v}{dx^2} = -P(e + v)$$

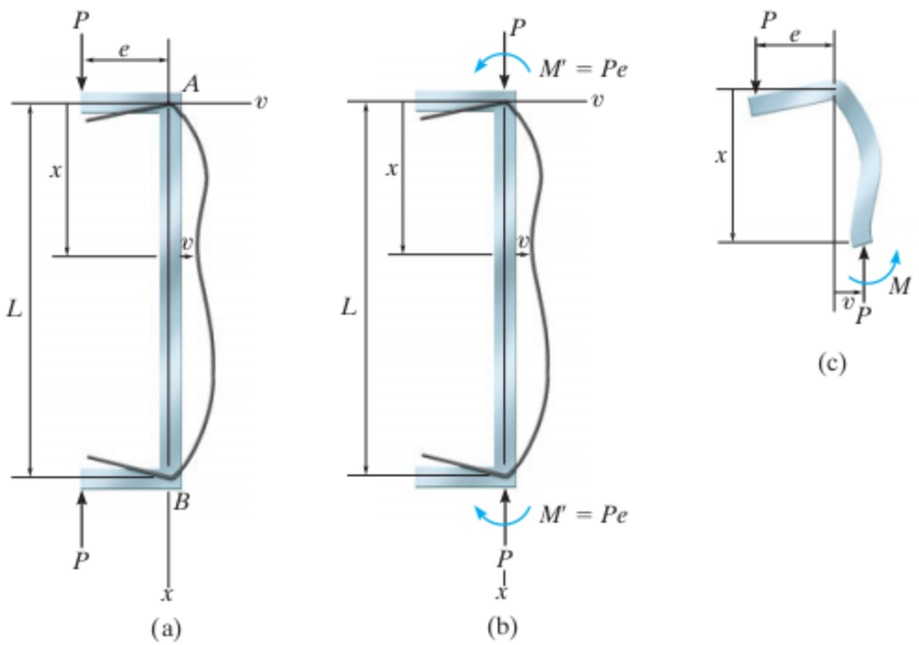


Fig. 17–13

or

$$\frac{d^2v}{dx^2} + \frac{P}{EI}v = -\frac{P}{EI}e$$

This equation is similar to Eq. 17–7 and has a general solution consisting of the complementary and particular solutions, namely,

$$v = C_1 \sin\sqrt{\frac{P}{EI}}x + C_2 \cos\sqrt{\frac{P}{EI}}x - e \tag{17–14}$$

To evaluate the constants we must apply the boundary conditions. At $x = 0, v = 0$, so $C_2 = e$. And at $x = L, v = 0$, which gives

$$C_1 = \frac{e[1 - \cos(\sqrt{P/EI}L)]}{\sin(\sqrt{P/EI}L)}$$

Since $1 - \cos(\sqrt{P/EI}L) = 2\sin^2(\sqrt{P/EI}L/2)$ and $\sin(\sqrt{P/EI}L) = 2\sin(\sqrt{P/EI}L/2)\cos(\sqrt{P/EI}L/2)$, we have

$$C_1 = e\tan\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)$$

Hence, the deflection curve, Eq. 17–14, can be written as

$$v = e\left[\tan\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)\sin\left(\sqrt{\frac{P}{EI}}x\right) + \cos\left(\sqrt{\frac{P}{EI}}x\right) - 1\right] \tag{17–15}$$

Maximum Deflection. Due to symmetry of loading, both the maximum deflection and maximum stress occur at the column's midpoint. Therefore, when $x = L/2, v = v_{max}$, so

$$v_{max} = e\left[\sec\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right) - 1\right] \tag{17–16}$$

Notice that if e approaches zero, then v_{max} approaches zero. However, if the terms in the brackets approach infinity as e approaches zero, then v_{max} will have a nonzero value. Mathematically, this would represent the behavior of an axially loaded column at failure when subjected to the critical load P_{cr}. Therefore, to find P_{cr} we require

$$\sec\left(\sqrt{\frac{P_{cr}}{EI}}\frac{L}{2}\right) = \infty$$

$$\sqrt{\frac{P_{cr}}{EI}}\frac{L}{2} = \frac{\pi}{2}$$

$$P_{cr} = \frac{\pi^2 EI}{L^2} \tag{17–17}$$

which is the same result found from the Euler formula, Eq. 17–5.

If Eq. 17–16 is plotted as load P versus deflection v_{max} for various values of eccentricity e, the family of colored curves shown in Fig. 17–14 results. Here the critical load becomes an asymptote to the curves, and of

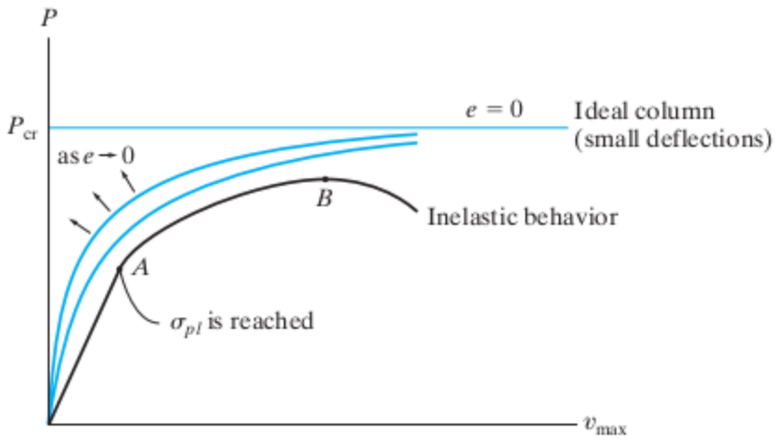


Fig. 17–14

course represents the unrealistic case of an ideal column ($e = 0$). As stated earlier, e is *never zero* due to imperfections in initial column straightness and load application; however, as $e \to 0$, the curves tend to approach the ideal case. Furthermore, these curves are appropriate only for *small deflections*, since the curvature was approximated by d^2v/dx^2 when Eq. 17–16 was developed. Had a more exact analysis been performed, all these curves would tend to turn upward, intersecting and then rising above the line $P = P_{cr}$. This, of course, indicates that a larger load P is needed to create larger column deflections. We have not considered this analysis here, however, since most often engineering design restricts the deflection of columns to small values.

It should also be noted that the colored curves in Fig. 17–14 apply only for linear-elastic material behavior. Such is the case if the column is long and slender. However, if a short or intermediate-length stocky column is considered, then the applied load, as it is increased, may eventually cause the material to yield, and the column will begin to behave in an *inelastic manner*. This occurs at point A for the black curve in Fig. 17–14. As the load is further increased, the curve never reaches the critical load, and instead the load reaches a maximum value at B. Afterwards, a sudden decrease in load-carrying capacity occurs as the column continues to yield and deflect by larger amounts.

Lastly, the colored curves in Fig. 17–14 also illustrate that a *nonlinear* relationship occurs between the load P and the deflection v. As a result, the principle of superposition *cannot be used* to determine the total deflection of a column caused by applying *successive loads* to the column. Instead, the loads must first be added, and then the corresponding deflection due to their resultant can be determined. Physically, the reason that successive loads and deflections cannot be superimposed is that the column's internal moment *depends* on *both* the *load* P and the *deflection* v, that is, $M = -P(e + v)$, Eq. 17–13.

The Secant Formula. The maximum stress in the column can be determined by realizing that it is caused by both the axial load and the moment, Fig. 17–15*a*. Maximum moment occurs at the column's midpoint, and using Eqs. 17–13 and 17–16, it has a magnitude of

$$M = |P(e + v_{\max})| \quad M = Pe \sec\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right) \tag{17–18}$$

As shown in Fig. 17–15*b*, the maximum stress in the column is compressive, and it has a value of

$$\sigma_{\max} = \frac{P}{A} + \frac{Mc}{I}; \qquad \sigma_{\max} = \frac{P}{A} + \frac{Pec}{I}\sec\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)$$

Since the radius of gyration is defined as $r^2 = I/A$, the above equation can be written in a form called the *secant formula*:

$$\sigma_{\max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right] \tag{17–19}$$

Here

- $\sigma_{\max}$ = maximum *elastic stress* in the column, which occurs at the inner concave side at the column's midpoint. This stress is compressive
- P = vertical load applied to the column. $P < P_{cr}$ unless $e = 0$; then $P = P_{cr}$ (Eq. 17–5)
- e = eccentricity of the load P, measured from the centroidal axis of the column's cross-sectional area to the line of action of P
- c = distance from the centroidal axis to the outer fiber of the column where the maximum compressive stress $\sigma_{\max}$ occurs
- A = cross-sectional area of the column
- L = unsupported length of the column *in the plane of bending*. For supports other than pins, the effective length $L_e = KL$ should be used. See Fig. 17–10
- E = modulus of elasticity for the material
- r = radius of gyration, $r = \sqrt{I/A}$, where I is calculated about the centroidal or bending axis

Like Eq. 17–16, Eq. 17–19 indicates that there is a nonlinear relationship between the load and the stress. Hence, the principle of superposition does not apply, and therefore the loads have to be added *before* the stress is determined. Furthermore, due to this nonlinear relationship, any factor of safety used for design purposes applies to the load and not to the stress.

For a given value of $\sigma_{\max}$, graphs of Eq. 17–19 can be plotted as the slenderness ratio KL/r versus the average stress P/A for various values of the *eccentricity ratio* ec/r^2. A specific set of graphs for a structural-grade A-36 steel having a yield point of $\sigma_{\max} = \sigma_Y = 36$ ksi

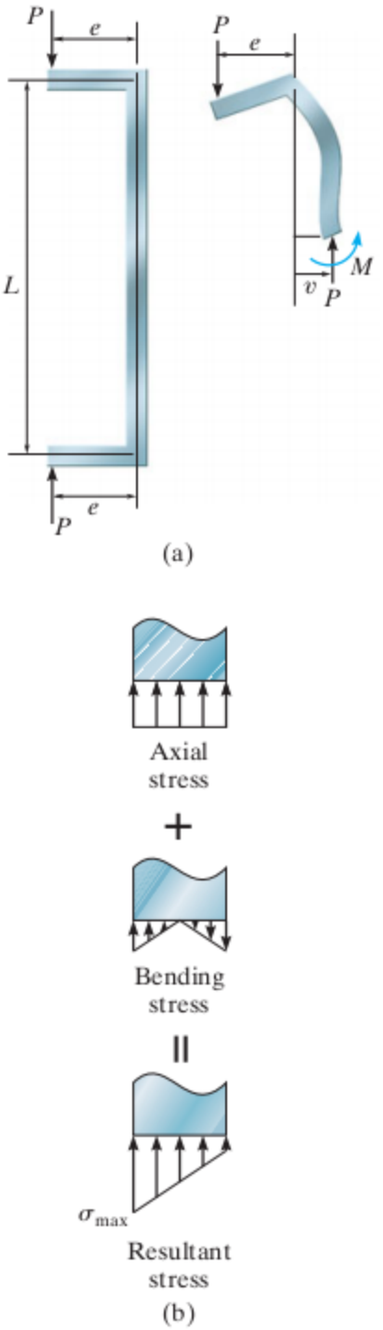


Fig. 17–15

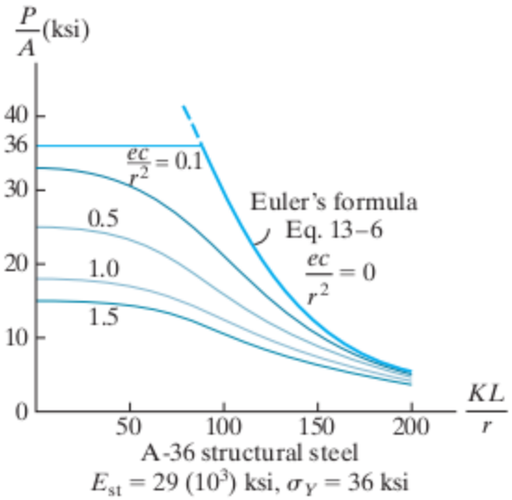


Fig. 17–16

and a modulus of elasticity of $E_{st} = 29(10^3)$ ksi is shown in Fig. 17–16. Note that when $e \to 0$, or when $ec/r^2 \to 0$, Eq. 17–19 gives $\sigma_{max} = P/A$, where P is the critical load on the column, defined by Euler's formula. This results in Eq. 17–6, which has been plotted in Fig. 17–7 and repeated in Fig. 17–16. Since both Eqs. 17–6 and 17–19 are valid only for elastic loadings, the stresses shown in Fig. 17–16 cannot exceed $\sigma_Y = 36$ ksi, represented here by the horizontal line.

The curves in Fig. 17–16 indicate that differences in the eccentricity ratio have a marked effect on the load-carrying capacity of columns that have small slenderness ratios. However, columns that have large slenderness ratios tend to fail at or near the Euler critical load regardless of the eccentricity ratio. When using Eq. 17–19 for design purposes, it is therefore important to have a somewhat accurate value for the eccentricity ratio for shorter-length columns.

Design. Once the eccentricity ratio has been determined, the column data can be substituted into Eq. 17–19. If a value of $\sigma_{max} = \sigma_Y$ is chosen, then the corresponding load P_Y can be determined from a trial-and-error procedure, since the equation is transcendental and cannot be solved explicitly for P_Y. As a design aid, computer software, or graphs such as those in Fig. 17–16, can also be used to determine P_Y directly.

Realize that P_Y is the load that will cause the column to develop a maximum compressive stress of σ_Y at its inner concave fibers. Due to the eccentric application of P_Y, this load will always be smaller than the critical load P_{cr}, which is determined from the Euler formula that assumes (unrealistically) that the column is axially loaded. Once P_Y is obtained, an appropriate factor of safety can then be applied in order to specify the column's safe load.

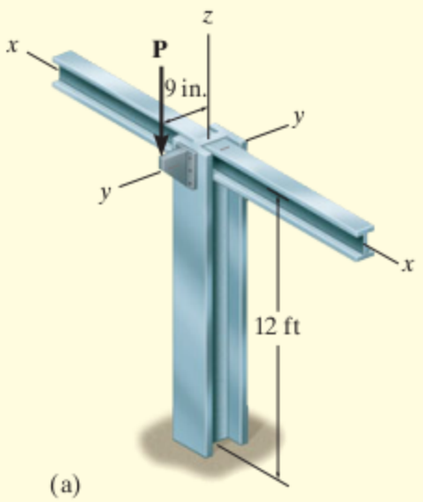
17

Important Points

- Due to imperfections in manufacturing or specific application of the load, a column will never suddenly buckle; instead, it begins to bend.
- The load applied to a column is related to its deflection in a nonlinear manner, and so the principle of superposition does not apply.
- As the slenderness ratio increases, eccentrically loaded columns tend to fail at or near the Euler buckling load.

EXAMPLE 17.4

The W8 × 40 A-36 steel column shown in Fig. 17–17*a* is fixed at its base and braced at the top so that it is fixed from displacement, yet free to rotate about the *y*–*y* axis. Also, it can sway to the side in the *y*–*z* plane. Determine the maximum eccentric load the column can support before it either begins to buckle or the steel yields.



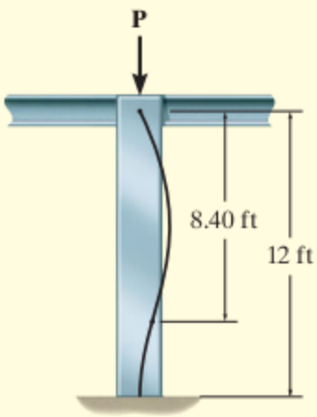
(a)

SOLUTION

From the support conditions it is seen that about the *y*–*y* axis the column behaves as if it were pinned at its top and fixed at the bottom and subjected to an axial load *P*, Fig. 17–17*b*. About the *x*–*x* axis the column is free at the top and fixed at the bottom, and it is subjected to both an axial load *P* and moment $M = P(9\text{ in.})$, Fig. 17–17*c*.

y–y Axis Buckling. From Fig. 17–10*d* the effective length factor is $K_y = 0.7$, so $(KL)_y = 0.7(12)\text{ ft} = 8.40\text{ ft} = 100.8\text{ in.}$ Using the table in Appendix C to determine I_y for the W8 × 40 section and applying Eq. 17–11, we have

$$(P_{cr})_y = \frac{\pi^2 EI_y}{(KL)_y^2} = \frac{\pi^2[29(10^3)\text{ ksi}](49.1\text{ in}^4)}{(100.8\text{ in.})^2} = 1383\text{ kip}$$



(b) *y*–*y* axis buckling

x–x Axis Yielding. From Fig. 17–10*b*, $K_x = 2$, so $(KL)_x = 2(12)\text{ ft} = 24\text{ ft} = 288\text{ in.}$ Again using the table in Appendix C to determine $A = 11.7\text{ in}^2$, $c = 8.25\text{ in.}/2 = 4.125\text{ in.}$, and $r_x = 3.53\text{ in.}$, and applying the secant formula, we have

$$\sigma_Y = \frac{P_x}{A}\left[1 + \frac{ec}{r_x^2}\sec\left(\frac{(KL)_x}{2r_x}\sqrt{\frac{P_x}{EA}}\right)\right]$$

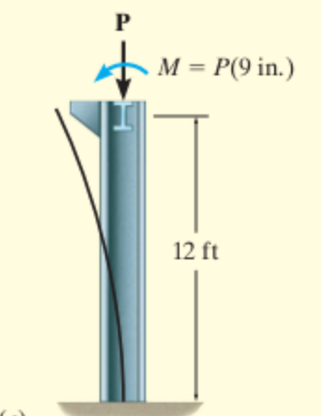
Substituting the data and simplifying yields

$$421.2 = P_x[1 + 2.979\sec(0.0700\sqrt{P_x})]$$

Solving for P_x by trial and error, noting that the argument for the secant is in radians, we get

$$P_x = 88.4\text{ kip} \qquad \textit{Ans.}$$

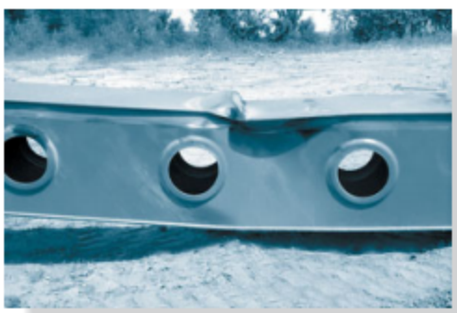
Since this value is less than $(P_{cr})_y = 1383\text{ kip}$, failure will occur about the *x*–*x* axis.



(c) *x*–*x* axis yielding

Fig. 17–17

*17.5 Inelastic Buckling

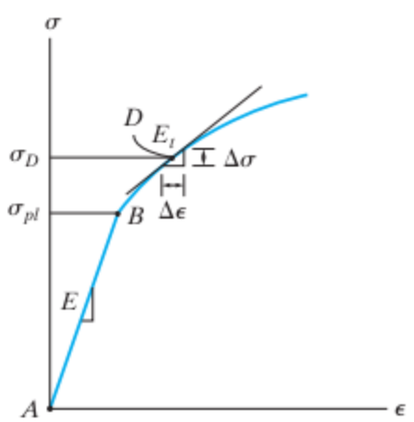


This crane boom failed by buckling caused by an overload. Note the region of localized collapse.

In engineering practice, columns are generally classified according to the type of stresses developed within the column at the time of failure. *Long slender columns* will become unstable when the compressive stress remains elastic. The failure that occurs is referred to as *elastic instability*. *Intermediate columns* fail due to *inelastic instability*, meaning that the compressive stress at failure is greater than the material's proportional limit. And *short columns*, sometimes called *posts*, do not become unstable; rather the material simply yields or fractures.

Application of the Euler equation requires that the stress in the column remain *below* the material's yield point (actually the proportional limit) when the column buckles, and so this equation applies only to long columns. In practice, however, most columns are selected to have intermediate lengths. The behavior of these columns can be studied by modifying the Euler equation so that it applies for inelastic buckling. To show how this can be done, consider the material to have a stress–strain diagram as shown in Fig. 17–18*a*. Here the proportional limit is σ_{pl}, and the modulus of elasticity, or slope of the line AB, is E.

If the column has a slenderness ratio that is *less* than $(KL/r)_{pl}$, then the critical stress in the column must be greater than σ_{pl}. For example, suppose a column has a slenderness ratio of $(KL/r)_1 < (KL/r)_{pl}$, with corresponding critical stress $\sigma_D > \sigma_{pl}$ needed to cause instability. When the column is *about to buckle*, the change in stress and strain that occurs in the column is within a *small range* $\Delta\sigma$ and $\Delta\epsilon$, so that the modulus of elasticity or stiffness for the material can be taken as the ***tangent modulus*** $E_t = \Delta\sigma/\Delta\epsilon$ defined as the slope of the σ–ϵ diagram at point D, Fig. 17–18*a*. In other words, at the time of failure, the column behaves as if it were made from a material that has a *lower stiffness* than when it behaves elastically, $E_t < E$.



(a)

Fig. 17–18

In general, therefore, as the slenderness ratio (KL/r) decreases, the *critical stress* for a column continues to rise; and from the $\sigma-\epsilon$ diagram, the *tangent modulus* for the material *decreases*. Using this idea, we can modify Euler's equation to include these cases of inelastic buckling by substituting the material's tangent modulus E_t for E, so that

$$\sigma_{cr} = \frac{\pi^2 E_t}{(KL/r)^2} \tag{17–20}$$

This is the so-called *tangent modulus* or *Engesser equation*, proposed by F. Engesser in 1889. A plot of this equation for intermediate and short-length columns of a material defined by the $\sigma-\epsilon$ diagram in Fig. 17–18*a* is shown in Fig. 17–18*b*.

No *actual column* can be considered to be either perfectly straight or loaded along its centroidal axis, as assumed here, and therefore it is indeed very difficult to develop an expression that will provide a complete analysis of this phenomenon. As a result, other methods of describing the inelastic buckling of columns have been considered. One of these methods was developed by the aeronautical engineer F. R. Shanley and is called the *Shanley theory* of inelastic buckling. Although it provides a better description of the phenomenon than the tangent modulus theory, as explained here, experimental testing of a large number of columns, each of which approximates the ideal column, has shown that Eq. 17–20 is *reasonably accurate* in predicting the column's critical stress. Furthermore, the tangent modulus approach to modeling inelastic column behavior is relatively easy to apply.

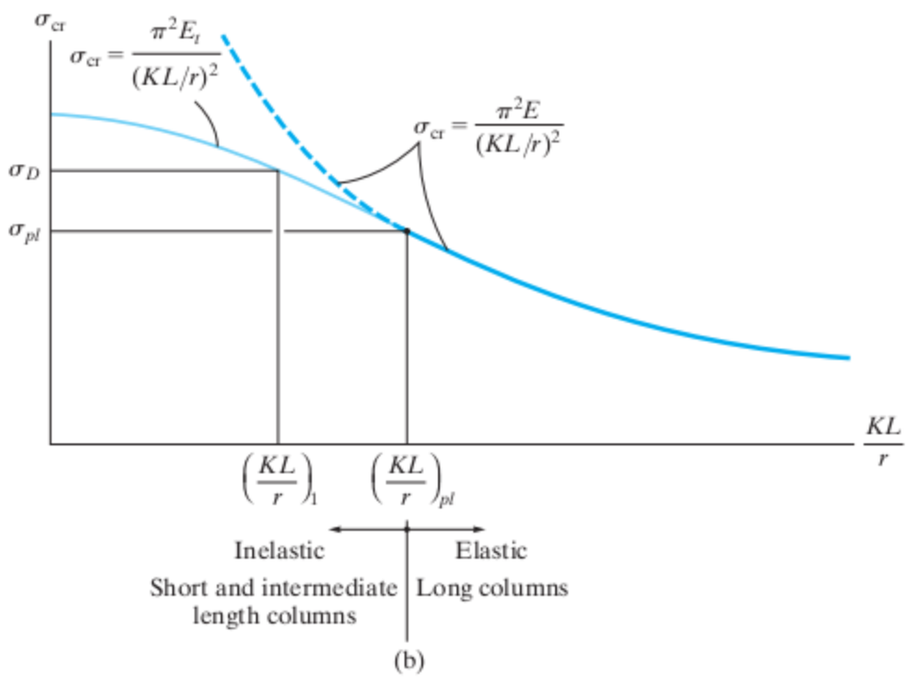


Fig. 17–18 (cont.)

EXAMPLE 17.5

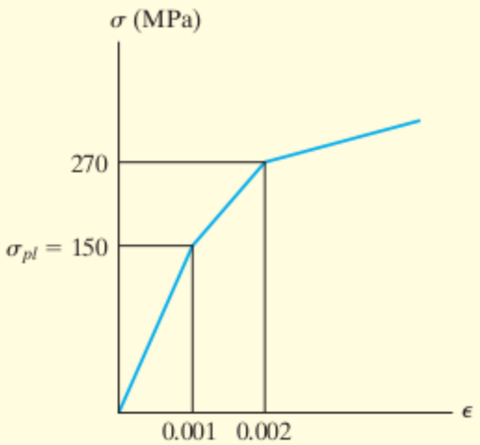


Fig. 17–19

A solid rod has a diameter of 30 mm and is 600 mm long. It is made of a material that can be modeled by the stress–strain diagram shown in Fig. 17–19. If it is used as a pin-supported column, determine the critical load.

SOLUTION

The radius of gyration is

$$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{(\pi/4)(15\text{ mm})^4}{\pi(15\text{ mm})^2}} = 7.5\text{ mm}$$

and therefore the slenderness ratio is

$$\frac{KL}{r} = \frac{1(600\text{ mm})}{7.5\text{ mm}} = 80$$

Applying Eq. 17–20 we have,

$$\sigma_{\text{cr}} = \frac{\pi^2 E_t}{(KL/r)^2} = \frac{\pi^2 E_t}{(80)^2} = 1.542(10^{-3})E_t \qquad (1)$$

First we will assume that the critical stress is elastic. From Fig. 17–19,

$$E = \frac{150\text{ MPa}}{0.001} = 150\text{ GPa}$$

Thus, Eq. (1) becomes

$$\sigma_{\text{cr}} = 1.542(10^{-3})[150(10^3)]\text{ MPa} = 231.3\text{ MPa}$$

Since $\sigma_{\text{cr}} > \sigma_{pl} = 150$ MPa, inelastic buckling occurs.

From the second line segment of the σ–ϵ diagram, Fig. 17–19, we have

$$E_t = \frac{\Delta\sigma}{\Delta\epsilon} = \frac{270\text{ MPa} - 150\text{ MPa}}{0.002 - 0.001} = 120\text{ GPa}$$

Applying Eq. (1) yields

$$\sigma_{\text{cr}} = 1.542(10^{-3})[120(10^3)]\text{ MPa} = 185.1\text{ MPa}$$

Since this value falls within the limits of 150 MPa and 270 MPa, it is indeed the critical stress.

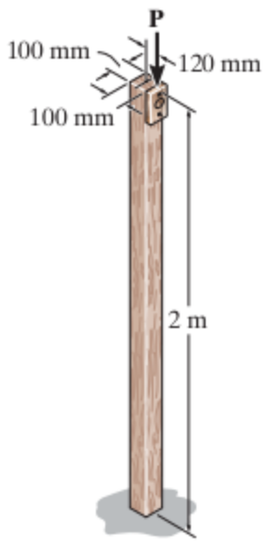
The critical load on the rod is therefore

$$P_{\text{cr}} = \sigma_{\text{cr}}A = 185.1(10^6)\text{ Pa}[\pi(0.015\text{ m})^2 = 131\text{ kN} \quad \textit{Ans.}$$

17

PROBLEMS

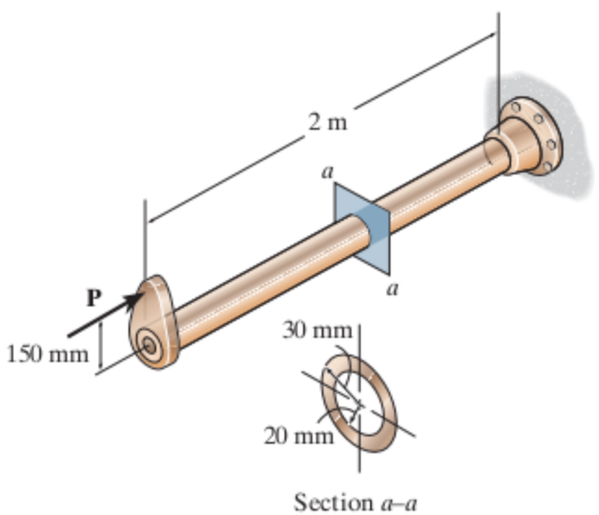
*17–32. The wood column has a square cross section with dimensions 100 mm by 100 mm. It is fixed at its base and free at its top. Determine the load P that can be applied to the edge of the column without causing the column to fail either by buckling or by yielding. $E_w = 12$ GPa, $\sigma_Y = 55$ MPa.



Prob. 17–32

17–33. The hollow red brass C83400 copper alloy shaft is fixed at one end but free at the other end. Determine the maximum eccentric force P the shaft can support without causing it to buckle or yield. Also, find the corresponding maximum deflection of the shaft.

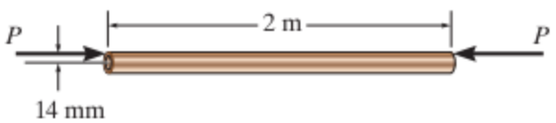
17–34. The hollow red brass C83400 copper alloy shaft is fixed at one end but free at the other end. If the eccentric force $P = 5$ kN is applied to the shaft as shown, determine the maximum normal stress and the maximum deflection.



Probs. 17–33/34

17–35. The tube is made of copper and has an outer diameter of 35 mm and a wall thickness of 7 mm. Using a factor of safety with respect to buckling and yielding of F.S. = 2.5, determine the allowable eccentric load P. The tube is pin-supported at its ends. $E_{cu} = 120$ GPa, $\sigma_Y = 750$ MPa.

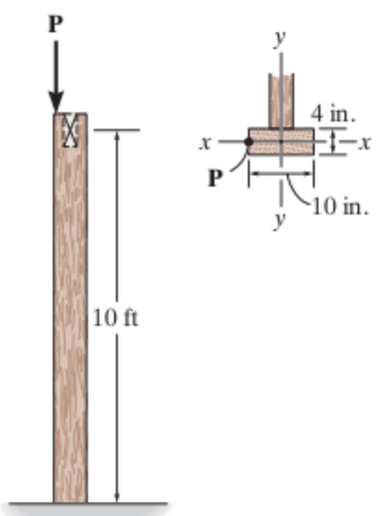
*17–36. The tube is made of copper and has an outer diameter of 35 mm and a wall thickness of 7 mm. Using a factor of safety with respect to buckling and yielding of F.S. = 2.5, determine the allowable eccentric load P that it can support without failure. The tube is fixed-supported at its ends. $E_{cu} = 120$ GPa, $\sigma_Y = 750$ MPa.



Probs. 17–35/36

17–37. Assume that the wood column is pin-connected at its base and top. Determine the maximum eccentric load P that can be applied without causing the column to buckle or yield. $E_w = 1.8\,(10^3)$ ksi, $\sigma_Y = 8$ ksi.

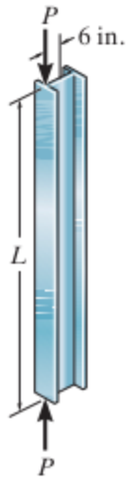
17–38. Assume that the wood column is pinned top and bottom for movement about the x–x axis, and fixed at the bottom and free at the top for movement about the y–y axis. Determine the maximum eccentric load P that can be applied without causing the column to buckle or yield. $E_w = 1.8\,(10^3)$ ksi, $\sigma_Y = 8$ ksi.



Probs. 17–37/38

17

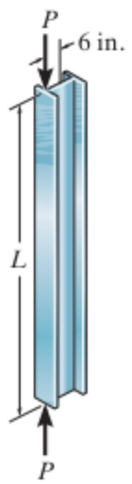
17–39. A W12 × 26 structural A992 steel column is pin-connected at its ends and has a length $L = 11.5$ ft. Determine the maximum eccentric load P that can be applied so the column does not buckle or yield. Compare this value with an axial critical load P' applied through the centroid of the column.



Prob. 17–39

17

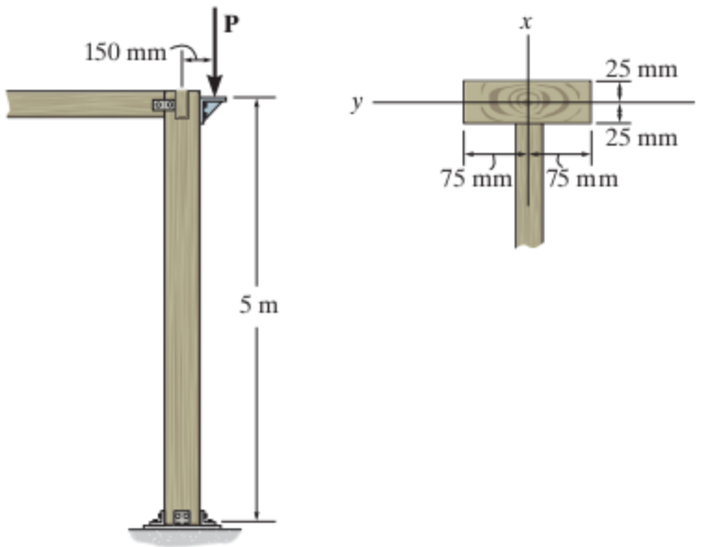
***17–40.** A W14 × 30 structural A-36 steel column is pin-connected at its ends and has a length $L = 10$ ft. Determine the maximum eccentric load P that can be applied so the column does not buckle or yield. Compare this value with an axial critical load P' applied through the centroid of the column.



Prob. 17–40

17–41. The wood column is fixed at its base, and its top can be considered pinned. If the eccentric force $P = 10$ kN is applied to the column, investigate whether the column is adequate to support this loading without buckling or yielding. Take $E = 10$ GPa and $\sigma_Y = 15$ MPa.

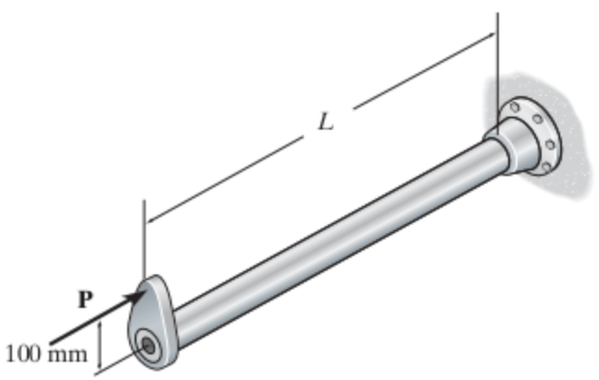
17–42. The wood column is fixed at its base, and its top can be considered pinned. Determine the maximum eccentric force P the column can support without causing it to either buckle or yield. Take $E = 10$ GPa and $\sigma_Y = 15$ MPa.



Probs. 17–41/42

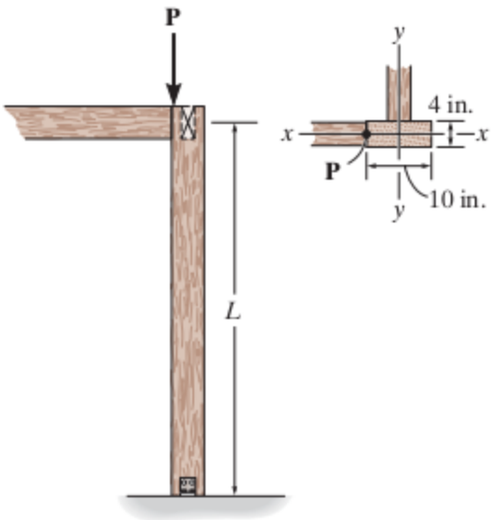
17–43. The 6061-T6 aluminum alloy solid shaft is fixed at one end but free at the other end. If the shaft has a diameter of 100 mm, determine its maximum allowable length L if it is subjected to the eccentric force $P = 80$ kN.

***17–44.** The 6061-T6 aluminum alloy solid shaft is fixed at one end but free at the other end. If the length is $L = 3$ m, determine its minimum required diameter if it is subjected to the eccentric force $P = 60$ kN.



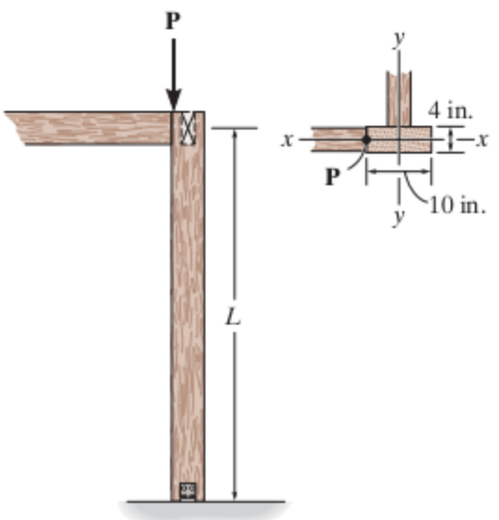
Probs. 17–43/44

17–45. The wood column is pinned at its base and top. If $L = 7$ ft, determine the maximum eccentric load P that can be applied without causing the column to buckle or yield. $E_w = 1.8(10^3)$ ksi, $\sigma_Y = 8$ ksi.



Prob. 17–45

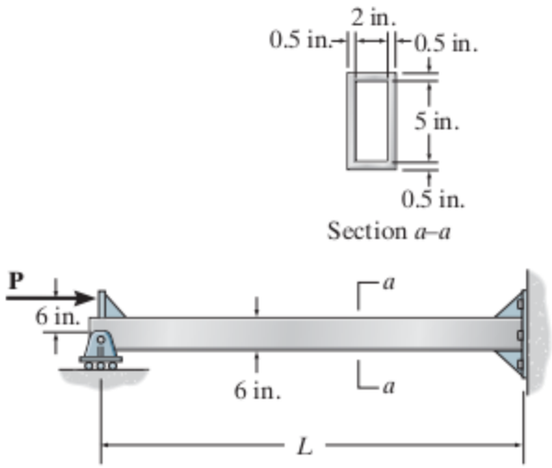
17–46. The wood column is pinned at its base and top. If $L = 5$ ft, determine the maximum eccentric load P that can be applied without causing the column to buckle or yield. $E_w = 1.8(10^3)$ ksi, $\sigma_Y = 8$ ksi.



Prob. 17–46

17–47. The A992 steel rectangular hollow section column is pinned at both ends. If it has a length of $L = 14$ ft, determine the maximum allowable eccentric force **P** it can support without causing it to either buckle or yield.

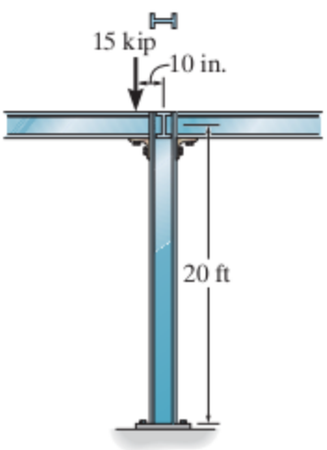
***17–48.** The A992 steel rectangular hollow section column is pinned at both ends. If it is subjected to the eccentric force $P = 45$ kip, determine its maximum allowable length L without causing it to either buckle or yield.



Probs. 17–47/48

17–49. The W14 × 26 structural A-36-steel member is used as a 20-ft-long column that is assumed to be fixed at its top and fixed at its bottom. If the 15-kip load is applied at an eccentric distance of 10 in., determine the maximum stress in the column.

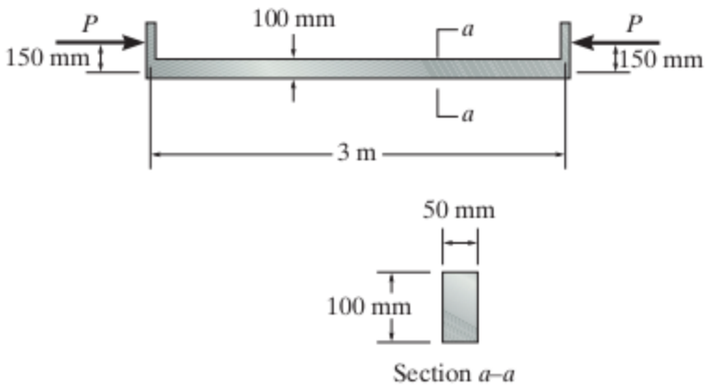
17–50. The W14 × 26 structural A-36-steel member is used as a column that is assumed to be fixed at its top and pinned at its bottom. If the 15-kip load is applied at an eccentric distance of 10 in., determine the maximum stress in the column.



Probs. 17–49/50

17

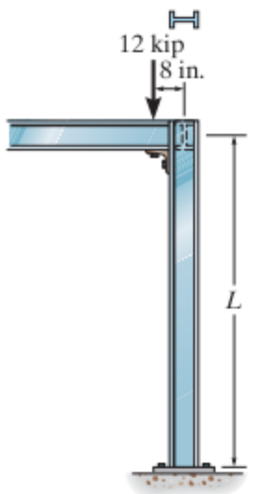
17–51. Determine the maximum eccentric load P the 2014-T6-aluminum-alloy strut can support without causing it either to buckle or yield. The ends of the strut are pin-connected.



Prob. 17–51

***17–52.** The W10 × 45 structural A992 steel column is assumed to be pinned at its top and bottom. If the 12-kip load is applied at an eccentric distance of 8 in., determine the maximum stress in the column. Take $L = 12.6$ ft.

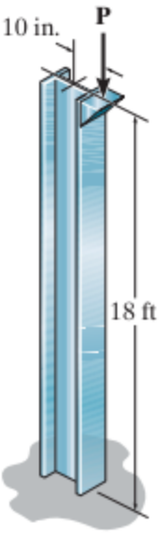
17–53. The W10 × 45 structural A992 steel column is assumed to be pinned at its top and bottom. If the 12-kip load is applied at an eccentric distance of 8 in., determine the maximum stress in the column. Take $L = 9$ ft.



Probs. 17–52/53

17–54. The W14 × 53 structural A992 steel column is fixed at its base and free at its top. If $P = 75$ kip, determine the sidesway deflection at its top and the maximum stress in the column.

17–55. The W14 × 53 column is fixed at its base and free at its top. Determine the maximum eccentric load P that it can support without causing it to buckle or yield. $E_{st} = 29(10^3)$ ksi, $\sigma_Y = 50$ ksi.

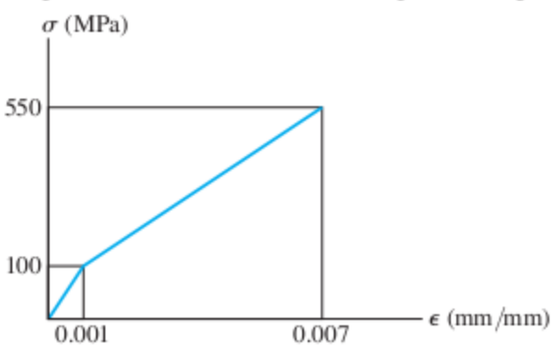


Probs. 17–54/55

***17–56.** The stress–strain diagram of the material can be approximated by the two line segments shown. If a bar having a diameter of 80 mm and a length of 1.5 m is made from this material, determine the critical load provided the ends are pinned. Assume that the load acts through the axis of the bar. Use Engesser's equation.

17–57. The stress–strain diagram of the material can be approximated by the two line segments shown. If a bar having a diameter of 80 mm and a length of 1.5 m is made from this material, determine the critical load provided the ends are fixed. Assume that the load acts through the axis of the bar. Use Engesser's equation.

17–58. The stress–strain diagram of the material can be approximated by the two line segments shown. If a bar having a diameter of 80 mm and a length of 1.5 m is made from this material, determine the critical load provided one end is pinned and the other is fixed. Assume that the load acts through the axis of the bar. Use Engesser's equation.



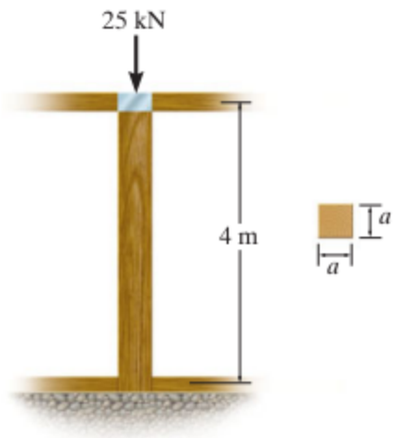
Probs. 17–56/57/58

CHAPTER REVIEW

Buckling is the sudden instability that occurs in columns or members that support an axial compressive load. The maximum axial load that a member can support just before buckling is called the critical load P_{cr}.		
The critical load for an ideal column is determined from Euler's formula, where $K = 1$ for pin supports, $K = 0.5$ for fixed supports, $K = 0.7$ for a pin and a fixed support, and $K = 2$ for a fixed support and a free end.	$P_{cr} = \dfrac{\pi^2 EI}{(KL)^2}$	
If the axial loading is applied eccentrically to the column, then the secant formula can be used to determine the maximum stress in the column.	$\sigma_{max} = \dfrac{P}{A}\left[1 + \dfrac{ec}{r^2}\sec\left(\dfrac{L}{2r}\sqrt{\dfrac{P}{EA}}\right)\right]$	
When the axial load causes yielding of the material, then the tangent modulus should be used with Euler's formula to determine the critical load for the column. This is referred to as Engesser's equation.	$\sigma_{cr} = \dfrac{\pi^2 E_t}{(KL/r)^2}$	
Empirical formulas based on experimental data have been developed for use in the design of steel, aluminum, and timber columns.		

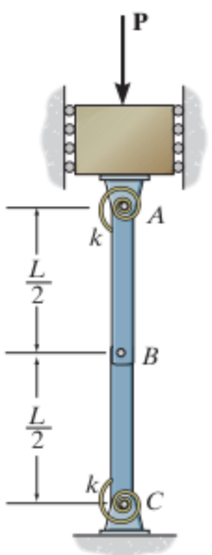
REVIEW PROBLEMS

17–59. The wood column is 4 m long and is required to support the axial load of 25 kN. If the cross section is square, determine the dimension a of each of its sides using a factor of safety against buckling of F.S. = 2.5. The column is assumed to be pinned at its top and bottom. Use the Euler equation. $E_w = 11$ GPa, and $\sigma_Y = 10$ MPa.



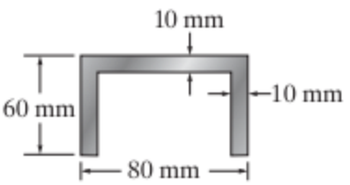
Prob. 17–59

*17–60. If the torsional springs attached to ends A and C of the rigid members AB and BC have a stiffness k, determine the critical load P_{cr}.



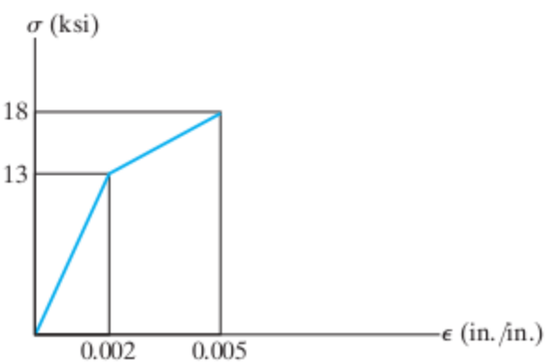
Prob. 17–60

17–61. A steel column has a length of 5 m and is free at one end and fixed at the other end. If the cross-sectional area has the dimensions shown, determine the critical load. $E_{st} = 200$ GPa, $\sigma_Y = 360$ MPa.



Prob. 17–61

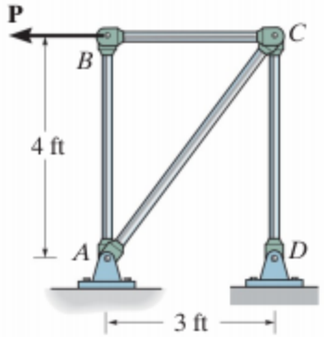
17–62. Construct the buckling curve, P/A versus L/r, for a column that has a bilinear stress–strain curve in compression as shown.



Prob. 17–62

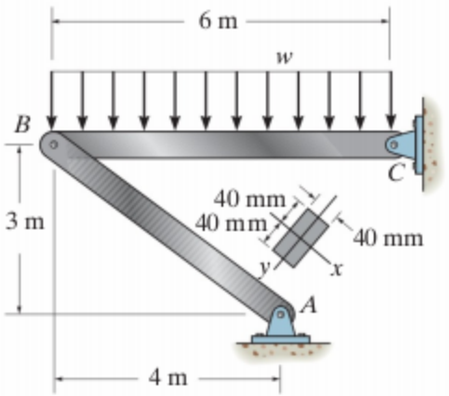
17

17–63. The members of the truss are assumed to be pin-connected. If member AC is an A-36 steel rod of 2 in. diameter, determine the maximum load P that can be supported by the truss without causing the member to buckle.



Prob. 17–63

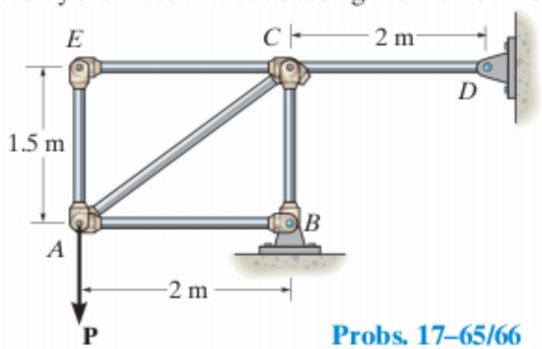
***17–64.** The steel bar AB of the frame is assumed to be pin-connected at its ends for y–y axis buckling. If $w = 3$ kN/m, determine the factor of safety with respect to buckling about the y–y axis due to the applied loading. $E_{st} = 200$ GPa, $\sigma_Y = 360$ MPa.



Prob. 17–64

17–65. The members of the truss are assumed to be pin-connected. If member AB is an A-36 steel rod of 40 mm diameter, determine the maximum force P that can be supported by the truss without causing the member to buckle.

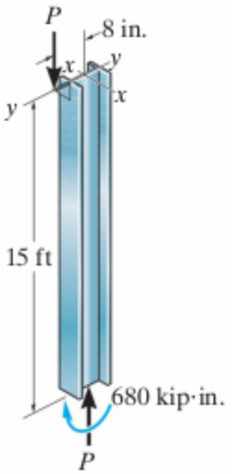
17–66. The members of the truss are assumed to be pin-connected. If member CB is an A-36 steel rod of 40 mm diameter, determine the maximum load P that can be supported by the truss without causing the member to buckle.



Probs. 17–65/66

17–67. The W10 × 30 structural A992 steel columm is pinned at its top and bottom. Determine the maximum load P it can support.

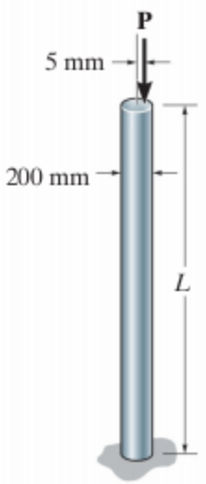
***17–68.** The W10 × 30 structural A992 steel columm is fixed at its bottom and free at its top. If it is subjected to the eccentric load of $P = 85$ kip, determine if the column fails by yielding. The column is braced so that it does not buckle about the y–y axis.



Probs. 17–67/68

17–69. The aluminum rod is fixed at its base free at its top. If the eccentric load $P = 200$ kN is applied, determine the greatest allowable length L of the rod so that it does not buckle or yield. $E_{al} = 72$ GPa, $\sigma_Y = 410$ MPa.

17–70. The aluminum rod is fixed at its base free at its top. If the length of the rod is $L = 2$ m, determine the greatest allowable load P that can be applied so that the rod does not buckle or yield. Also, determine the largest sidesway deflection of the rod due to the loading. $E_{al} = 72$ GPa, $\sigma_Y = 410$ MPa.



Probs. 17–69/70

APPENDIX

A

Mathematical Review and Expressions

Geometry and Trigonometry Review

The angles θ in Fig. A–1 are equal between the transverse and two parallel lines.

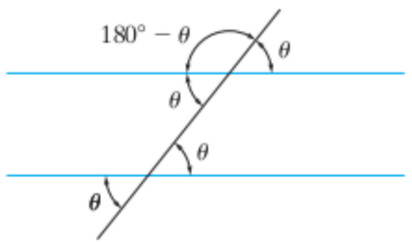


Fig. A–1

A

For a line and its normal, the angles θ in Fig. A–2 are equal.

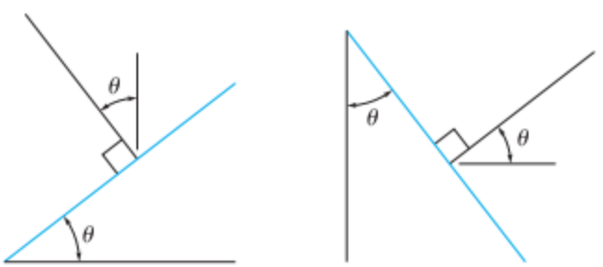


Fig. A–2

For the circle in Fig. A–3 $s = \theta r$, so that when $\theta = 360° = 2\pi$ rad then the circumference is $s = 2\pi r$. Also, since $180° = \pi$ rad, then θ (rad) $= (\pi/180°)\theta°$. The area of the circle is $A = \pi r^2$.

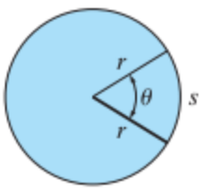


Fig. A–3

The sides of a similar triangle can be obtained by proportion as in Fig. A–4, where $\frac{a}{A} = \frac{b}{B} = \frac{c}{C}$.

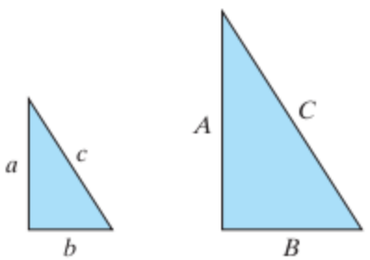


Fig. A–4

For the right triangle in Fig. A–5, the Pythagorean theorem is

$$h = \sqrt{(o)^2 + (a)^2}$$

The trigonometric functions are

$$\sin\theta = \frac{o}{h}$$

$$\cos\theta = \frac{a}{h}$$

$$\tan\theta = \frac{o}{a}$$

This is easily remembered as "soh, cah, toa," i.e., the sine is the opposite over the hypotenuse, etc. The other trigonometric functions follow from this.

$$\csc\theta = \frac{1}{\sin\theta} = \frac{h}{o}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{h}{a}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{a}{o}$$

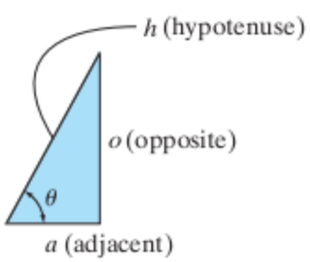


Fig. A–5

A

Trigonometric Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos\theta = \pm\sqrt{\frac{1+\cos 2\theta}{2}}, \sin\theta = \pm\sqrt{\frac{1-\cos 2\theta}{2}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

A

$$1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

Quadratic Formula

$$\text{If } ax^2 + bx + c = 0, \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2},$$

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

Power-Series Expansions

$$\sin x = x - \frac{x^3}{3!} + \cdots, \cos x = 1 - \frac{x^2}{2!} + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \cdots, \cosh x = 1 + \frac{x^2}{2!} + \cdots$$

Derivatives

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx} \qquad \frac{d}{dx}(\sin u) = \cos u\frac{du}{dx}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \frac{d}{dx}(\cos u) = -\sin u\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \qquad \frac{d}{dx}(\tan u) = \sec^2 u\frac{du}{dx}$$

$$\frac{d}{dx}(\cot u) = -\csc^2 u\frac{du}{dx} \qquad \frac{d}{dx}(\sinh u) = \cosh u\frac{du}{dx}$$

$$\frac{d}{dx}(\sec u) = \tan u\sec u\frac{du}{dx} \qquad \frac{d}{dx}(\cosh u) = \sinh u\frac{du}{dx}$$

$$\frac{d}{dx}(\csc u) = -\csc u\cot u\frac{du}{dx}$$

Integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$$

$$\int \frac{dx}{a+bx} = \frac{1}{b}\ln(a+bx) + C$$

$$\int \frac{dx}{a+bx^2} = \frac{1}{2\sqrt{-ab}}\ln\left[\frac{a+x\sqrt{-ab}}{a-x\sqrt{-ab}}\right] + C,\ ab < 0$$

$$\int \frac{x\,dx}{a+bx^2} = \frac{1}{2b}\ln(bx^2+a) + C$$

$$\int \frac{x^2\,dx}{a+bx^2} = \frac{x}{b} - \frac{a}{b\sqrt{ab}}\tan^{-1}\frac{x\sqrt{ab}}{a} + C,\ ab > 0$$

$$\int \sqrt{a+bx}\,dx = \frac{2}{3b}\sqrt{(a+bx)^3} + C$$

$$\int x\sqrt{a+bx}\,dx = \frac{-2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2} + C$$

$$\int x^2\sqrt{a+bx}\,dx = \frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3} + C$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right] + C,\ a > 0$$

$$\int x\sqrt{a^2-x^2}\,dx = -\frac{1}{3}\sqrt{(a^2-x^2)^3} + C$$

$$\int x^2\sqrt{a^2-x^2}\,dx = -\frac{x}{4}\sqrt{(a^2-x^2)^3} + \frac{a^2}{8}\left(x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right) + C,\ a > 0$$

$$\int \sqrt{x^2 \pm a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2 \pm a^2} \pm a^2\ln\left(x + \sqrt{x^2 \pm a^2}\right)\right] + C$$

$$\int x\sqrt{x^2 \pm a^2}\,dx = \frac{1}{3}\sqrt{(x^2 \pm a^2)^3} + C$$

$$\int x^2\sqrt{x^2 \pm a^2}\,dx = \frac{x}{4}\sqrt{(x^2 \pm a^2)^3} \mp \frac{a^2}{8}x\sqrt{x^2 \pm a^2} - \frac{a^4}{8}\ln\left(x + \sqrt{x^2 \pm a^2}\right) + C$$

$$\int \frac{dx}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b} + C$$

$$\int \frac{x\,dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2} + C$$

$$\int \frac{dx}{\sqrt{a+bx+cx^2}} = \frac{1}{\sqrt{c}}\ln\left[\sqrt{a+bx+cx^2} + x\sqrt{c} + \frac{b}{2\sqrt{c}}\right] + C,\ c > 0$$

$$= \frac{1}{\sqrt{-c}}\sin^{-1}\left(\frac{-2cx-b}{\sqrt{b^2-4ac}}\right) + C,\ c < 0$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax) + C$$

$$\int x^2\cos(ax)\,dx = \frac{2x}{a^2}\cos(ax) + \frac{a^2x^2-2}{a^3}\sin(ax) + C$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1) + C$$

$$\int \sinh x\,dx = \cosh x + C$$

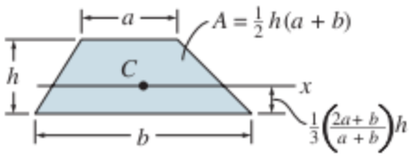
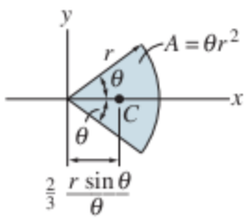
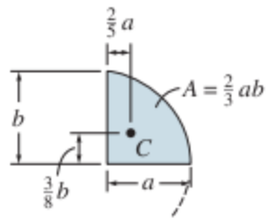
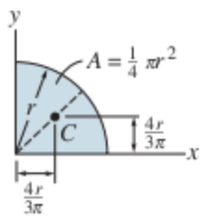
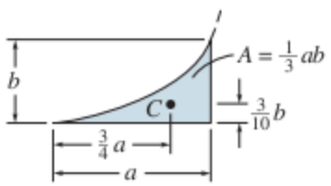
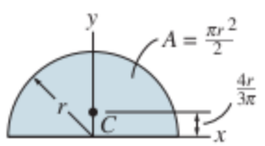
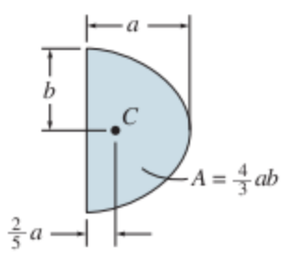
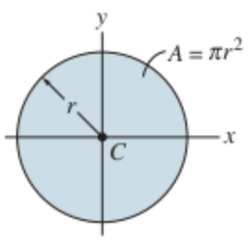
$$\int \cosh x\,dx = \sinh x + C$$

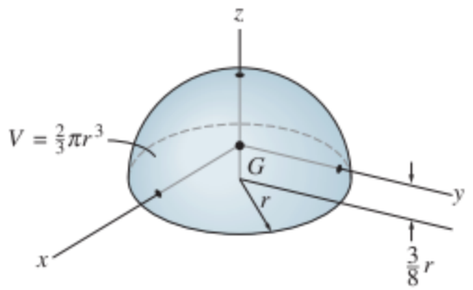
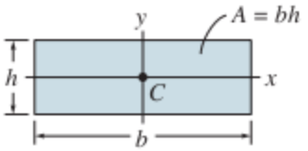
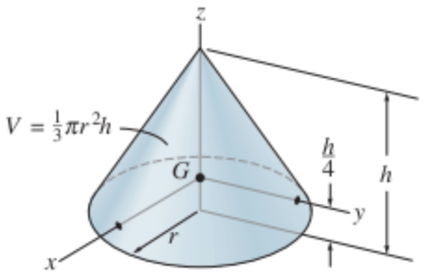
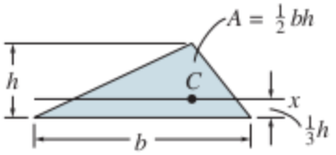
A

APPENDIX

B

Geometric Properties of An Area and Volume

Centroid Location	Centroid Location	Area Moment of Inertia
 Trapezoidal area	 Circular sector area	$I_x = \frac{1}{4}r^4(\theta - \frac{1}{2}\sin 2\theta)$ $I_y = \frac{1}{4}r^4(\theta + \frac{1}{2}\sin 2\theta)$
 Semiparabolic area	 Quarter circular area	$I_x = \frac{1}{16}\pi r^4$ $I_y = \frac{1}{16}\pi r^4$
 Exparabolic area	 Semicircular area	$I_x = \frac{1}{8}\pi r^4$ $I_y = \frac{1}{8}\pi r^4$
 Parabolic area	 Circular area	$I_x = \frac{1}{4}\pi r^4$ $I_y = \frac{1}{4}\pi r^4$

Centroid Location	Centroid Location	Area Moment of Inertia
 Hemisphere	 Rectangular area	$I_x = \frac{1}{12}bh^3$ $I_y = \frac{1}{12}hb^3$
 Cone	 Triangular area	$I_x = \frac{1}{36}bh^3$

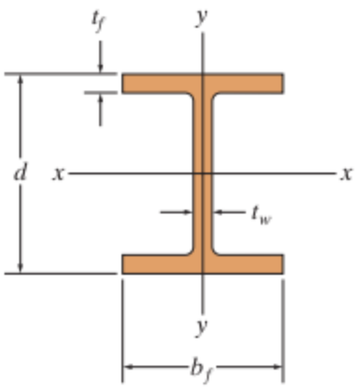
B

APPENDIX

C Geometric Properties of Wide-Flange Sections

Wide-Flange Sections or W Shapes FPS Units

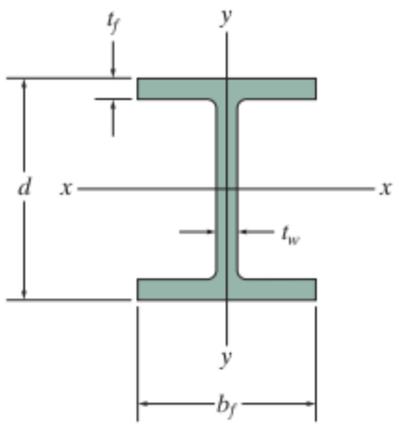
Designation	Area A	Depth d	Web thickness t_w	Flange width b_f	Flange thickness t_f	x–x axis I	x–x axis S	x–x axis r	y–y axis I	y–y axis S	y–y axis r
in. × lb/ft	in^2	in.	in.	in.	in.	in^4	in^3	in.	in^4	in^3	in.
W24 × 104	30.6	24.06	0.500	12.750	0.750	3100	258	10.1	259	40.7	2.91
W24 × 94	27.7	24.31	0.515	9.065	0.875	2700	222	9.87	109	24.0	1.98
W24 × 84	24.7	24.10	0.470	9.020	0.770	2370	196	9.79	94.4	20.9	1.95
W24 × 76	22.4	23.92	0.440	8.990	0.680	2100	176	9.69	82.5	18.4	1.92
W24 × 68	20.1	23.73	0.415	8.965	0.585	1830	154	9.55	70.4	15.7	1.87
W24 × 62	18.2	23.74	0.430	7.040	0.590	1550	131	9.23	34.5	9.80	1.38
W24 × 55	16.2	23.57	0.395	7.005	0.505	1350	114	9.11	29.1	8.30	1.34
W18 × 65	19.1	18.35	0.450	7.590	0.750	1070	117	7.49	54.8	14.4	1.69
W18 × 60	17.6	18.24	0.415	7.555	0.695	984	108	7.47	50.1	13.3	1.69
W18 × 55	16.2	18.11	0.390	7.530	0.630	890	98.3	7.41	44.9	11.9	1.67
W18 × 50	14.7	17.99	0.355	7.495	0.570	800	88.9	7.38	40.1	10.7	1.65
W18 × 46	13.5	18.06	0.360	6.060	0.605	712	78.8	7.25	22.5	7.43	1.29
W18 × 40	11.8	17.90	0.315	6.015	0.525	612	68.4	7.21	19.1	6.35	1.27
W18 × 35	10.3	17.70	0.300	6.000	0.425	510	57.6	7.04	15.3	5.12	1.22
W16 × 57	16.8	16.43	0.430	7.120	0.715	758	92.2	6.72	43.1	12.1	1.60
W16 × 50	14.7	16.26	0.380	7.070	0.630	659	81.0	6.68	37.2	10.5	1.59
W16 × 45	13.3	16.13	0.345	7.035	0.565	586	72.7	6.65	32.8	9.34	1.57
W16 × 36	10.6	15.86	0.295	6.985	0.430	448	56.5	6.51	24.5	7.00	1.52
W16 × 31	9.12	15.88	0.275	5.525	0.440	375	47.2	6.41	12.4	4.49	1.17
W16 × 26	7.68	15.69	0.250	5.500	0.345	301	38.4	6.26	9.59	3.49	1.12
W14 × 53	15.6	13.92	0.370	8.060	0.660	541	77.8	5.89	57.7	14.3	1.92
W14 × 43	12.6	13.66	0.305	7.995	0.530	428	62.7	5.82	45.2	11.3	1.89
W14 × 38	11.2	14.10	0.310	6.770	0.515	385	54.6	5.87	26.7	7.88	1.55
W14 × 34	10.0	13.98	0.285	6.745	0.455	340	48.6	5.83	23.3	6.91	1.53
W14 × 30	8.85	13.84	0.270	6.730	0.385	291	42.0	5.73	19.6	5.82	1.49
W14 × 26	7.69	13.91	0.255	5.025	0.420	245	35.3	5.65	8.91	3.54	1.08
W14 × 22	6.49	13.74	0.230	5.000	0.335	199	29.0	5.54	7.00	2.80	1.04



Wide-Flange Sections or W Shapes FPS Units

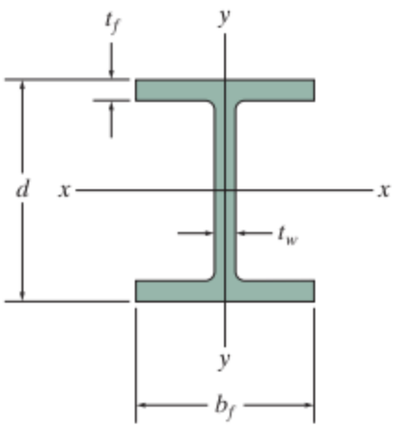
Designation	Area A	Depth d	Web thickness t_w	Flange width b_f	Flange thickness t_f	x–x axis I	x–x axis S	x–x axis r	y–y axis I	y–y axis S	y–y axis r
in. × lb/ft	in^2	in.	in.	in.	in.	in^4	in^3	in.	in^4	in^3	in.
W12 × 87	25.6	12.53	0.515	12.125	0.810	740	118	5.38	241	39.7	3.07
W12 × 50	14.7	12.19	0.370	8.080	0.640	394	64.7	5.18	56.3	13.9	1.96
W12 × 45	13.2	12.06	0.335	8.045	0.575	350	58.1	5.15	50.0	12.4	1.94
W12 × 26	7.65	12.22	0.230	6.490	0.380	204	33.4	5.17	17.3	5.34	1.51
W12 × 22	6.48	12.31	0.260	4.030	0.425	156	25.4	4.91	4.66	2.31	0.847
W12 × 16	4.71	11.99	0.220	3.990	0.265	103	17.1	4.67	2.82	1.41	0.773
W12 × 14	4.16	11.91	0.200	3.970	0.225	88.6	14.9	4.62	2.36	1.19	0.753
W10 × 100	29.4	11.10	0.680	10.340	1.120	623	112	4.60	207	40.0	2.65
W10 × 54	15.8	10.09	0.370	10.030	0.615	303	60.0	4.37	103	20.6	2.56
W10 × 45	13.3	10.10	0.350	8.020	0.620	248	49.1	4.32	53.4	13.3	2.01
W10 × 39	11.5	9.92	0.315	7.985	0.530	209	42.1	4.27	45.0	11.3	1.98
W10 × 30	8.84	10.47	0.300	5.810	0.510	170	32.4	4.38	16.7	5.75	1.37
W10 × 19	5.62	10.24	0.250	4.020	0.395	96.3	18.8	4.14	4.29	2.14	0.874
W10 × 15	4.41	9.99	0.230	4.000	0.270	68.9	13.8	3.95	2.89	1.45	0.810
W10 × 12	3.54	9.87	0.190	3.960	0.210	53.8	10.9	3.90	2.18	1.10	0.785
W8 × 67	19.7	9.00	0.570	8.280	0.935	272	60.4	3.72	88.6	21.4	2.12
W8 × 58	17.1	8.75	0.510	8.220	0.810	228	52.0	3.65	75.1	18.3	2.10
W8 × 48	14.1	8.50	0.400	8.110	0.685	184	43.3	3.61	60.9	15.0	2.08
W8 × 40	11.7	8.25	0.360	8.070	0.560	146	35.5	3.53	49.1	12.2	2.04
W8 × 31	9.13	8.00	0.285	7.995	0.435	110	27.5	3.47	37.1	9.27	2.02
W8 × 24	7.08	7.93	0.245	6.495	0.400	82.8	20.9	3.42	18.3	5.63	1.61
W8 × 15	4.44	8.11	0.245	4.015	0.315	48.0	11.8	3.29	3.41	1.70	0.876
W6 × 25	7.34	6.38	0.320	6.080	0.455	53.4	16.7	2.70	17.1	5.61	1.52
W6 × 20	5.87	6.20	0.260	6.020	0.365	41.4	13.4	2.66	13.3	4.41	1.50
W6 × 16	4.74	6.28	0.260	4.030	0.405	32.1	10.2	2.60	4.43	2.20	0.966
W6 × 15	4.43	5.99	0.230	5.990	0.260	29.1	9.72	2.56	9.32	3.11	1.46
W6 × 12	3.55	6.03	0.230	4.000	0.280	22.1	7.31	2.49	2.99	1.50	0.918
W6 × 9	2.68	5.90	0.170	3.940	0.215	16.4	5.56	2.47	2.19	1.11	0.905

C



Wide-Flange Sections or W Shapes SI Units

Designation	Area A	Depth d	Web thickness t_w	Flange width b_f	Flange thickness t_f	x–x axis I	x–x axis S	x–x axis r	y–y axis I	y–y axis S	y–y axis r
mm × kg/m	mm^2	mm	mm	mm	mm	$10^6\ mm^4$	$10^3\ mm^3$	mm	$10^6\ mm^4$	$10^3\ mm^3$	mm
W610 × 155	19 800	611	12.70	324.0	19.0	1 290	4 220	255	108	667	73.9
W610 × 140	17 900	617	13.10	230.0	22.2	1 120	3 630	250	45.1	392	50.2
W610 × 125	15 900	612	11.90	229.0	19.6	985	3 220	249	39.3	343	49.7
W610 × 113	14 400	608	11.20	228.0	17.3	875	2 880	247	34.3	301	48.8
W610 × 101	12 900	603	10.50	228.0	14.9	764	2 530	243	29.5	259	47.8
W610 × 92	11 800	603	10.90	179.0	15.0	646	2 140	234	14.4	161	34.9
W610 × 82	10 500	599	10.00	178.0	12.8	560	1 870	231	12.1	136	33.9
W460 × 97	12 300	466	11.40	193.0	19.0	445	1 910	190	22.8	236	43.1
W460 × 89	11 400	463	10.50	192.0	17.7	410	1 770	190	20.9	218	42.8
W460 × 82	10 400	460	9.91	191.0	16.0	370	1 610	189	18.6	195	42.3
W460 × 74	9 460	457	9.02	190.0	14.5	333	1 460	188	16.6	175	41.9
W460 × 68	8 730	459	9.14	154.0	15.4	297	1 290	184	9.41	122	32.8
W460 × 60	7 590	455	8.00	153.0	13.3	255	1 120	183	7.96	104	32.4
W460 × 52	6 640	450	7.62	152.0	10.8	212	942	179	6.34	83.4	30.9
W410 × 85	10 800	417	10.90	181.0	18.2	315	1 510	171	18.0	199	40.8
W410 × 74	9 510	413	9.65	180.0	16.0	275	1 330	170	15.6	173	40.5
W410 × 67	8 560	410	8.76	179.0	14.4	245	1 200	169	13.8	154	40.2
W410 × 53	6 820	403	7.49	177.0	10.9	186	923	165	10.1	114	38.5
W410 × 46	5 890	403	6.99	140.0	11.2	156	774	163	5.14	73.4	29.5
W410 × 39	4 960	399	6.35	140.0	8.8	126	632	159	4.02	57.4	28.5
W360 × 79	10 100	354	9.40	205.0	16.8	227	1 280	150	24.2	236	48.9
W360 × 64	8 150	347	7.75	203.0	13.5	179	1 030	148	18.8	185	48.0
W360 × 57	7 200	358	7.87	172.0	13.1	160	894	149	11.1	129	39.3
W360 × 51	6 450	355	7.24	171.0	11.6	141	794	148	9.68	113	38.7
W360 × 45	5 710	352	6.86	171.0	9.8	121	688	146	8.16	95.4	37.8
W360 × 39	4 960	353	6.48	128.0	10.7	102	578	143	3.75	58.6	27.5
W360 × 33	4 190	349	5.84	127.0	8.5	82.9	475	141	2.91	45.8	26.4



Wide-Flange Sections or W Shapes SI Units

Designation	Area A	Depth d	Web thickness t_w	Flange width b_f	Flange thickness t_f	x–x axis I	x–x axis S	x–x axis r	y–y axis I	y–y axis S	y–y axis r
mm × kg/m	mm^2	mm	mm	mm	mm	$10^6\ mm^4$	$10^3\ mm^3$	mm	$10^6\ mm^4$	$10^3\ mm^3$	mm
W310 × 129	16 500	318	13.10	308.0	20.6	308	1940	137	100	649	77.8
W310 × 74	9 480	310	9.40	205.0	16.3	165	1060	132	23.4	228	49.7
W310 × 67	8 530	306	8.51	204.0	14.6	145	948	130	20.7	203	49.3
W310 × 39	4 930	310	5.84	165.0	9.7	84.8	547	131	7.23	87.6	38.3
W310 × 33	4 180	313	6.60	102.0	10.8	65.0	415	125	1.92	37.6	21.4
W310 × 24	3 040	305	5.59	101.0	6.7	42.8	281	119	1.16	23.0	19.5
W310 × 21	2 680	303	5.08	101.0	5.7	37.0	244	117	0.986	19.5	19.2
W250 × 149	19 000	282	17.30	263.0	28.4	259	1840	117	86.2	656	67.4
W250 × 80	10 200	256	9.40	255.0	15.6	126	984	111	43.1	338	65.0
W250 × 67	8 560	257	8.89	204.0	15.7	104	809	110	22.2	218	50.9
W250 × 58	7 400	252	8.00	203.0	13.5	87.3	693	109	18.8	185	50.4
W250 × 45	5 700	266	7.62	148.0	13.0	71.1	535	112	7.03	95	35.1
W250 × 28	3 620	260	6.35	102.0	10.0	39.9	307	105	1.78	34.9	22.2
W250 × 22	2 850	254	5.84	102.0	6.9	28.8	227	101	1.22	23.9	20.7
W250 × 18	2 280	251	4.83	101.0	5.3	22.5	179	99.3	0.919	18.2	20.1
W200 × 100	12 700	229	14.50	210.0	23.7	113	987	94.3	36.6	349	53.7
W200 × 86	11 000	222	13.00	209.0	20.6	94.7	853	92.8	31.4	300	53.4
W200 × 71	9 100	216	10.20	206.0	17.4	76.6	709	91.7	25.4	247	52.8
W200 × 59	7 580	210	9.14	205.0	14.2	61.2	583	89.9	20.4	199	51.9
W200 × 46	5 890	203	7.24	203.0	11.0	45.5	448	87.9	15.3	151	51.0
W200 × 36	4 570	201	6.22	165.0	10.2	34.4	342	86.8	7.64	92.6	40.9
W200 × 22	2 860	206	6.22	102.0	8.0	20.0	194	83.6	1.42	27.8	22.3
W150 × 37	4 730	162	8.13	154.0	11.6	22.2	274	68.5	7.07	91.8	38.7
W150 × 30	3 790	157	6.60	153.0	9.3	17.1	218	67.2	5.54	72.4	38.2
W150 × 22	2 860	152	5.84	152.0	6.6	12.1	159	65.0	3.87	50.9	36.8
W150 × 24	3 060	160	6.60	102.0	10.3	13.4	168	66.2	1.83	35.9	24.5
W150 × 18	2 290	153	5.84	102.0	7.1	9.19	120	63.3	1.26	24.7	23.5
W150 × 14	1 730	150	4.32	100.0	5.5	6.84	91.2	62.9	0.912	18.2	23.0

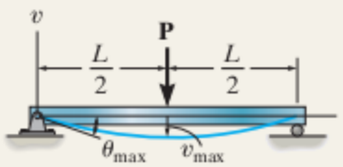
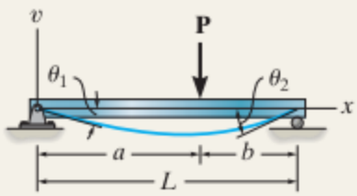
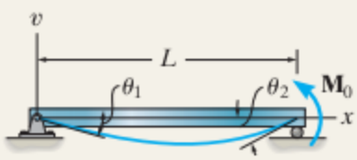
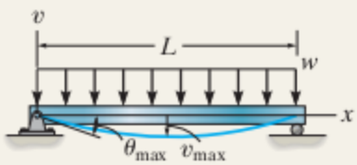
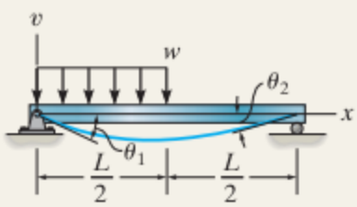
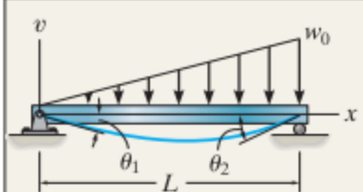
C

APPENDIX

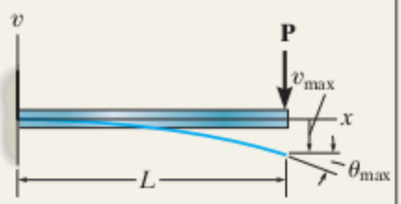
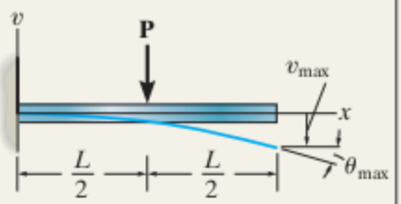
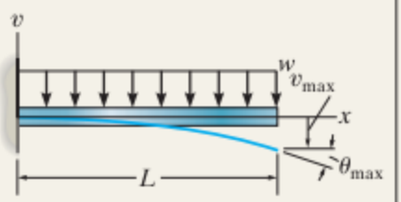
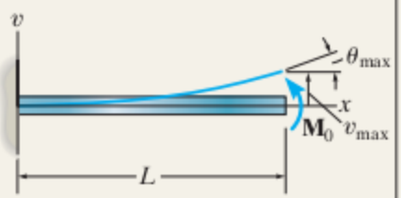
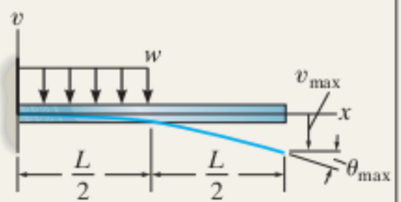
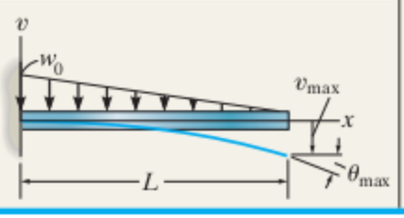
D

Slopes and Deflections of Beams

Simply Supported Beam Slopes and Deflections

Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \frac{-PL^2}{16EI}$	$v_{max} = \frac{-PL^3}{48EI}$	$v = \frac{-Px}{48EI}(3L^2 - 4x^2)$ $0 \le x \le L/2$
	$\theta_1 = \frac{-Pab(L + b)}{6EIL}$ $\theta_2 = \frac{Pab(L + a)}{6EIL}$	$v\Big\vert_{x=a} = \frac{-Pba}{6EIL}(L^2 - b^2 - a^2)$	$v = \frac{-Pbx}{6EIL}(L^2 - b^2 - x^2)$ $0 \le x \le a$
	$\theta_1 = \frac{-M_0L}{6EI}$ $\theta_2 = \frac{M_0L}{3EI}$	$v_{max} = \frac{-M_0L^2}{9\sqrt{3}EI}$ at $x = 0.5774L$	$v = \frac{-M_0x}{6EIL}(L^2 - x^2)$
	$\theta_{max} = \frac{-wL^3}{24EI}$	$v_{max} = \frac{-5wL^4}{384EI}$	$v = \frac{-wx}{24EI}(x^3 - 2Lx^2 + L^3)$
	$\theta_1 = \frac{-3wL^3}{128EI}$ $\theta_2 = \frac{7wL^3}{384EI}$	$v\Big\vert_{x=L/2} = \frac{-5wL^4}{768EI}$ $v_{max} = -0.006563\frac{wL^4}{EI}$ at $x = 0.4598L$	$v = \frac{-wx}{384EI}(16x^3 - 24Lx^2 + 9L^3)$ $0 \le x \le L/2$ $v = \frac{-wL}{384EI}(8x^3 - 24Lx^2 + 17L^2x - L^3)$ $L/2 \le x < L$
	$\theta_1 = \frac{-7w_0L^3}{360EI}$ $\theta_2 = \frac{w_0L^3}{45EI}$	$v_{max} = -0.00652\frac{w_0L^4}{EI}$ at $x = 0.5193L$	$v = \frac{-w_0x}{360EIL}(3x^4 - 10L^2x^2 + 7L^4)$

Cantilevered Beam Slopes and Deflections

Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \frac{-PL^2}{2EI}$	$v_{max} = \frac{-PL^3}{3EI}$	$v = \frac{-Px^2}{6EI}(3L - x)$
	$\theta_{max} = \frac{-PL^2}{8EI}$	$v_{max} = \frac{-5PL^3}{48EI}$	$v = \frac{-Px^2}{12EI}(3L - 2x) \quad 0 \le x \le L/2$ $v = \frac{-PL^2}{48EI}(6x - L) \quad L/2 \le x \le L$
	$\theta_{max} = \frac{-wL^3}{6EI}$	$v_{max} = \frac{-wL^4}{8EI}$	$v = \frac{-wx^2}{24EI}(x^2 - 4Lx + 6L^2)$
	$\theta_{max} = \frac{M_0 L}{EI}$	$v_{max} = \frac{M_0 L^2}{2EI}$	$v = \frac{M_0 x^2}{2EI}$
	$\theta_{max} = \frac{-wL^3}{48EI}$	$v_{max} = \frac{-7wL^4}{384EI}$	$v = \frac{-wx^2}{24EI}\left(x^2 - 2Lx + \frac{3}{2}L^2\right) \quad 0 \le x \le L/2$ $v = \frac{-wL^3}{384EI}(8x - L) \quad L/2 \le x \le L$
	$\theta_{max} = \frac{-w_0 L^3}{24EI}$	$v_{max} = \frac{-w_0 L^4}{30EI}$	$v = \frac{-w_0 x^2}{120EIL}(10L^3 - 10L^2 x + 5Lx^2 - x^3)$

D

Fundamental Problems Partial Solutions and Answers

Chapter 2

F2–1

$$F_R = \sqrt{(2\text{ kN})^2 + (6\text{ kN})^2 - 2(2\text{ kN})(6\text{ kN})\cos 105^\circ}$$
$$= 6.798\text{ kN} = 6.80\text{ kN} \quad \textit{Ans.}$$
$$\frac{\sin\phi}{6\text{ kN}} = \frac{\sin 105^\circ}{6.798\text{ kN}}, \quad \phi = 58.49^\circ$$
$$\theta = 45^\circ + \phi = 45^\circ + 58.49^\circ = 103^\circ \quad \textit{Ans.}$$

F2–2

$$F_R = \sqrt{200^2 + 500^2 - 2(200)(500)\cos 140^\circ}$$
$$= 666\text{ N} \quad \textit{Ans.}$$

F2–3

$$F_R = \sqrt{600^2 + 800^2 - 2(600)(800)\cos 60^\circ}$$
$$= 721.11\text{ N} = 721\text{ N} \quad \textit{Ans.}$$
$$\frac{\sin\alpha}{800} = \frac{\sin 60^\circ}{721.11}; \quad \alpha = 73.90^\circ$$
$$\phi = \alpha - 30^\circ = 73.90^\circ - 30^\circ = 43.9^\circ \quad \textit{Ans.}$$

F2–4

$$\frac{F_u}{\sin 45^\circ} = \frac{30}{\sin 105^\circ}; \quad F_u = 22.0\text{ lb} \quad \textit{Ans.}$$
$$\frac{F_v}{\sin 30^\circ} = \frac{30}{\sin 105^\circ}; \quad F_v = 15.5\text{ lb} \quad \textit{Ans.}$$

F2–5

$$\frac{F_{AB}}{\sin 105^\circ} = \frac{450}{\sin 30^\circ}$$
$$F_{AB} = 869\text{ lb} \quad \textit{Ans.}$$
$$\frac{F_{AC}}{\sin 45^\circ} = \frac{450}{\sin 30^\circ}$$
$$F_{AC} = 636\text{ lb} \quad \textit{Ans.}$$

F2–6

$$\frac{F}{\sin 30^\circ} = \frac{6}{\sin 105^\circ} \quad F = 3.11\text{ kN} \quad \textit{Ans.}$$
$$\frac{F_v}{\sin 45^\circ} = \frac{6}{\sin 105^\circ} \quad F_v = 4.39\text{ kN} \quad \textit{Ans.}$$

F2–7

$$(F_1)_x = 0 \quad (F_1)_y = 300\text{ N} \quad \textit{Ans.}$$
$$(F_2)_x = -(450\text{ N})\cos 45^\circ = -318\text{ N} \quad \textit{Ans.}$$
$$(F_2)_y = (450\text{ N})\sin 45^\circ = 318\text{ N} \quad \textit{Ans.}$$
$$(F_3)_x = \left(\tfrac{3}{5}\right)600\text{ N} = 360\text{ N} \quad \textit{Ans.}$$
$$(F_3)_y = \left(\tfrac{4}{5}\right)600\text{ N} = 480\text{ N} \quad \textit{Ans.}$$

F2–8

$$F_{Rx} = 300 + 400\cos 30^\circ - 250\left(\tfrac{4}{5}\right) = 446.4\text{ N}$$
$$F_{Ry} = 400\sin 30^\circ + 250\left(\tfrac{3}{5}\right) = 350\text{ N}$$
$$F_R = \sqrt{(446.4)^2 + 350^2} = 567\text{ N} \quad \textit{Ans.}$$
$$\theta = \tan^{-1}\tfrac{350}{446.4} = 38.1^\circ\measuredangle \quad \textit{Ans.}$$

F2–9

$$\overset{+}{\rightarrow}(F_R)_x = \Sigma F_x;$$
$$(F_R)_x = -(700\text{ lb})\cos 30^\circ + 0 + \left(\tfrac{3}{5}\right)(600\text{ lb})$$
$$= -246.22\text{ lb}$$
$$+\uparrow(F_R)_y = \Sigma F_y;$$
$$(F_R)_y = -(700\text{ lb})\sin 30^\circ - 400\text{ lb} - \left(\tfrac{4}{5}\right)(600\text{ lb})$$
$$= -1230\text{ lb}$$
$$F_R = \sqrt{(246.22\text{ lb})^2 + (1230\text{ lb})^2} = 1254\text{ lb} \quad \textit{Ans.}$$
$$\phi = \tan^{-1}\left(\tfrac{1230\text{ lb}}{246.22\text{ lb}}\right) = 78.68^\circ$$
$$\theta = 180^\circ + \phi = 180^\circ + 78.68^\circ = 259^\circ \quad \textit{Ans.}$$

F2–10

$$\overset{+}{\rightarrow}(F_R)_x = \Sigma F_x;$$
$$750\text{ N} = F\cos\theta + \left(\tfrac{5}{13}\right)(325\text{ N}) + (600\text{ N})\cos 45^\circ$$
$$+\uparrow(F_R)_y = \Sigma F_y;$$
$$0 = F\sin\theta + \left(\tfrac{12}{13}\right)(325\text{ N}) - (600\text{ N})\sin 45^\circ$$
$$\tan\theta = 0.6190 \quad \theta = 31.76^\circ = 31.8^\circ\measuredangle \quad \textit{Ans.}$$
$$F = 236\text{ N} \quad \textit{Ans.}$$

F2–11

$$\overset{+}{\rightarrow}(F_R)_x = \Sigma F_x;$$
$$(80\text{ lb})\cos 45^\circ = F\cos\theta + 50\text{ lb} - \left(\tfrac{3}{5}\right)90\text{ lb}$$
$$+\uparrow(F_R)_y = \Sigma F_y;$$
$$-(80\text{ lb})\sin 45^\circ = F\sin\theta - \left(\tfrac{4}{5}\right)(90\text{ lb})$$
$$\tan\theta = 0.2547 \quad \theta = 14.29^\circ = 14.3^\circ\measuredangle \quad \textit{Ans.}$$
$$F = 62.5\text{ lb} \quad \textit{Ans.}$$

F2–12

$$(F_R)_x = 15\left(\tfrac{4}{5}\right) + 0 + 15\left(\tfrac{4}{5}\right) = 24\text{ kN} \rightarrow$$
$$(F_R)_y = 15\left(\tfrac{3}{5}\right) + 20 - 15\left(\tfrac{3}{5}\right) = 20\text{ kN} \uparrow$$
$$F_R = 31.2\text{ kN} \quad \textit{Ans.}$$
$$\theta = 39.8^\circ \quad \textit{Ans.}$$

F2–13

$$F_x = 75\cos 30^\circ \sin 45^\circ = 45.93\text{ lb}$$
$$F_y = 75\cos 30^\circ \cos 45^\circ = 45.93\text{ lb}$$
$$F_z = -75\sin 30^\circ = -37.5\text{ lb}$$
$$\alpha = \cos^{-1}\left(\tfrac{45.93}{75}\right) = 52.2^\circ \quad \textit{Ans.}$$
$$\beta = \cos^{-1}\left(\tfrac{45.93}{75}\right) = 52.2^\circ \quad \textit{Ans.}$$
$$\gamma = \cos^{-1}\left(\tfrac{-37.5}{75}\right) = 120^\circ \quad \textit{Ans.}$$

F2–14

$$\cos\beta = \sqrt{1 - \cos^2 120^\circ - \cos^2 60^\circ} = \pm 0.7071$$
Require $\beta = 135^\circ$.
$$\mathbf{F} = F\mathbf{u}_F = (500\text{ N})(-0.5\mathbf{i} - 0.7071\mathbf{j} + 0.5\mathbf{k})$$
$$= \{-250\mathbf{i} - 354\mathbf{j} + 250\mathbf{k}\}\text{ N} \quad \textit{Ans.}$$

F2–15 $\cos^2\alpha + \cos^2 135° + \cos^2 120° = 1$

$\alpha = 60°$

$\mathbf{F} = F\mathbf{u}_F = (500\text{ N})(0.5\mathbf{i} - 0.7071\mathbf{j} - 0.5\mathbf{k})$

$= \{250\mathbf{i} - 354\mathbf{j} - 250\mathbf{k}\}\text{ N}$ *Ans.*

F2–16 $F_z = (50\text{ lb})\sin 45° = 35.36\text{ lb}$

$F' = (50\text{ lb})\cos 45° = 35.36\text{ lb}$

$F_x = \left(\frac{3}{5}\right)(35.36\text{ lb}) = 21.21\text{ lb}$

$F_y = \left(\frac{4}{5}\right)(35.36\text{ lb}) = 28.28\text{ lb}$

$\mathbf{F} = \{-21.2\mathbf{i} + 28.3\mathbf{j} + 35.4\mathbf{k}\}\text{ lb}$ *Ans.*

F2–17 $F_z = (750\text{ N})\sin 45° = 530.33\text{ N}$

$F' = (750\text{ N})\cos 45° = 530.33\text{ N}$

$F_x = (530.33\text{ N})\cos 60° = 265.1\text{ N}$

$F_y = (530.33\text{ N})\sin 60° = 459.3\text{ N}$

$\mathbf{F}_2 = \{265\mathbf{i} - 459\mathbf{j} + 530\mathbf{k}\}\text{ N}$ *Ans.*

F2–18 $\mathbf{F}_1 = \left(\frac{4}{5}\right)(500\text{ lb})\,\mathbf{j} + \left(\frac{3}{5}\right)(500\text{ lb})\mathbf{k}$

$= \{400\mathbf{j} + 300\mathbf{k}\}\text{ lb}$

$\mathbf{F}_2 = [(800\text{ lb})\cos 45°]\cos 30°\,\mathbf{i}$

$+ [(800\text{ lb})\cos 45°]\sin 30°\mathbf{j}$

$+ (800\text{ lb})\sin 45°(-\mathbf{k})$

$= \{489.90\mathbf{i} + 282.84\mathbf{j} - 565.69\mathbf{k}\}\text{ lb}$

$\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2 = \{490\mathbf{i} + 683\mathbf{j} - 266\mathbf{k}\}\text{ lb}$ *Ans.*

F2–19 $\mathbf{r}_{AB} = \{-6\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}\}\text{ m}$ *Ans.*

$r_{AB} = \sqrt{(-6\text{ m})^2 + (6\text{ m})^2 + (3\text{ m})^2} = 9\text{ m}$ *Ans.*

$\alpha = 132°,\quad \beta = 48.2°,\quad \gamma = 70.5°$ *Ans.*

F2–20 $\mathbf{r}_{AB} = \{-4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}\}\text{ ft}$ *Ans.*

$r_{AB} = \sqrt{(-4\text{ ft})^2 + (2\text{ ft})^2 + (4\text{ ft})^2} = 6\text{ ft}$ *Ans.*

$\alpha = \cos^{-1}\left(\frac{-4\text{ ft}}{6\text{ ft}}\right) = 131.8°$

$\theta = 180° - 131.8° = 48.2°$ *Ans.*

F2–21 $\mathbf{r}_B = \{2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}\}\text{ m}$

$\mathbf{F}_B = F_B\mathbf{u}_B$

$= (630\text{ N})\left(\frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} - \frac{6}{7}\mathbf{k}\right)$

$= \{180\mathbf{i} + 270\mathbf{j} - 540\mathbf{k}\}\text{ N}$ *Ans.*

F2–22 $\mathbf{F} = F\mathbf{u}_{AB} = 900\text{ N}\left(-\frac{4}{9}\mathbf{i} + \frac{7}{9}\mathbf{j} - \frac{4}{9}\mathbf{k}\right)$

$= \{-400\mathbf{i} + 700\mathbf{j} - 400\mathbf{k}\}\text{ N}$

F2–23 $\mathbf{F}_B = F_B\mathbf{u}_B$

$= (840\text{ N})\left(\frac{3}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} - \frac{6}{7}\mathbf{k}\right)$

$= \{360\mathbf{i} - 240\mathbf{j} - 720\mathbf{k}\}\text{ N}$

$\mathbf{F}_C = F_C\mathbf{u}_C$

$= (420\text{ N})\left(\frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} - \frac{6}{7}\mathbf{k}\right)$

$= \{120\mathbf{i} + 180\mathbf{j} - 360\mathbf{k}\}\text{ N}$

$F_R = \sqrt{(480\text{ N})^2 + (-60\text{ N})^2 + (-1080\text{ N})^2}$

$= 1.18\text{ kN}$ *Ans.*

F2–24 $\mathbf{F}_B = F_B\mathbf{u}_B$

$= (600\text{ lb})\left(-\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right)$

$= \{-200\mathbf{i} + 400\mathbf{j} - 400\mathbf{k}\}\text{ lb}$

$\mathbf{F}_C = F_C\mathbf{u}_C$

$= (490\text{ lb})\left(-\frac{6}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} - \frac{2}{7}\mathbf{k}\right)$

$= \{-420\mathbf{i} + 210\mathbf{j} - 140\mathbf{k}\}\text{ lb}$

$\mathbf{F}_R = \mathbf{F}_B + \mathbf{F}_C = \{-620\mathbf{i} + 610\mathbf{j} - 540\mathbf{k}\}\text{ lb}$ *Ans.*

F2–25 $\mathbf{u}_{AO} = -\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$

$\mathbf{u}_F = -0.5345\mathbf{i} + 0.8018\mathbf{j} + 0.2673\mathbf{k}$

$\theta = \cos^{-1}(\mathbf{u}_{AO}\cdot\mathbf{u}_F) = 57.7°$ *Ans.*

F2–26 $\mathbf{u}_{AB} = -\frac{3}{5}\mathbf{j} + \frac{4}{5}\mathbf{k}$

$\mathbf{u}_F = \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$

$\theta = \cos^{-1}(\mathbf{u}_{AB}\cdot\mathbf{u}_F) = 68.9°$ *Ans.*

F2–27 $\mathbf{u}_{OA} = \frac{12}{13}\mathbf{i} + \frac{5}{13}\mathbf{j}$

$\mathbf{u}_{OA}\cdot\mathbf{j} = u_{OA}(1)\cos\theta$

$\cos\theta = \frac{5}{13};\quad \theta = 67.4°$ *Ans.*

F2–28 $\mathbf{u}_{OA} = \frac{12}{13}\mathbf{i} + \frac{5}{13}\mathbf{j}$

$\mathbf{F} = F\mathbf{u}_F = [650\mathbf{j}]\text{ N}$

$F_{OA} = \mathbf{F}\cdot\mathbf{u}_{OA} = 250\text{ N}$

$\mathbf{F}_{OA} = F_{OA}\,\mathbf{u}_{OA} = \{231\mathbf{i} + 96.2\mathbf{j}\}\text{ N}$ *Ans.*

F2–29 $\mathbf{F} = (400\text{ N})\dfrac{\{4\,\mathbf{i} + 1\,\mathbf{j} - 6\,\mathbf{k}\}\text{m}}{\sqrt{(4\text{ m})^2 + (1\text{ m})^2 + (-6\text{ m})^2}}$

$= \{219.78\mathbf{i} + 54.94\mathbf{j} - 329.67\mathbf{k}\}\text{ N}$

$\mathbf{u}_{AO} = \dfrac{\{-4\,\mathbf{j} - 6\,\mathbf{k}\}\text{m}}{\sqrt{(-4\text{ m})^2 + (-6\text{ m})^2}}$

$= -0.5547\mathbf{j} - 0.8321\mathbf{k}$

$(F_{AO})_{\text{proj}} = \mathbf{F}\cdot\mathbf{u}_{AO} = 244\text{ N}$ *Ans.*

F2–30
$$\mathbf{F} = [(-600\text{ lb})\cos 60°]\sin 30°\mathbf{i} + [(600\text{ lb})\cos 60°]\cos 30°\,\mathbf{j} + [(600\text{ lb})\sin 60°]\,\mathbf{k}$$
$$= \{-150\mathbf{i} + 259.81\mathbf{j} + 519.62\mathbf{k}\}\text{ lb}$$
$$\mathbf{u}_A = -\tfrac{2}{3}\mathbf{i} + \tfrac{2}{3}\mathbf{j} + \tfrac{1}{3}\mathbf{k}$$
$$(F_A)_{\text{proj}} = \mathbf{F}\cdot\mathbf{u}_A = 446.41\text{ lb} = 446\text{ lb} \quad \textit{Ans.}$$
$$(F_A)_{\text{per}} = \sqrt{(600\text{ lb})^2 - (446.41\text{ lb})^2} = 401\text{ lb} \quad \textit{Ans.}$$

Chapter 3

F3–1
$$\circlearrowleft +M_O = 600\sin 50°\,(5) + 600\cos 50°\,(0.5) = 2.49\text{ kip}\cdot\text{ft} \quad \textit{Ans.}$$

F3–2
$$\circlearrowleft +M_O = -\left(\tfrac{4}{5}\right)(100\text{ N})(2\text{ m}) - \left(\tfrac{3}{5}\right)(100\text{ N})(5\text{ m}) = -460\text{ N}\cdot\text{m} = 460\text{ N}\cdot\text{m}\,\circlearrowright \quad \textit{Ans.}$$

F3–3
$$\circlearrowleft +M_O = [(300\text{ N})\sin 30°][0.4\text{m} + (0.3\text{ m})\cos 45°] - [(300\text{ N})\cos 30°][(0.3\text{ m})\sin 45°] = 36.7\text{ N}\cdot\text{m} \quad \textit{Ans.}$$

F3–4
$$\circlearrowleft +M_O = (600\text{ lb})(4\text{ ft} + (3\text{ ft})\cos 45° - 1\text{ ft}) = 3.07\text{ kip}\cdot\text{ft} \quad \textit{Ans.}$$

F3–5
$$\circlearrowleft +M_O = 50\sin 60°\,(0.1 + 0.2\cos 45° + 0.1) - 50\cos 60°(0.2\sin 45°) = 11.2\text{ N}\cdot\text{m} \quad \textit{Ans.}$$

F3–6
$$\circlearrowleft +M_O = 500\sin 45°\,(3 + 3\cos 45°) - 500\cos 45°\,(3\sin 45°) = 1.06\text{ kN}\cdot\text{m} \quad \textit{Ans.}$$

F3–7
$$\circlearrowleft +(M_R)_O = \Sigma Fd;$$
$$(M_R)_O = -(600\text{ N})(1\text{ m}) + (500\text{ N})[3\text{ m} + (2.5\text{ m})\cos 45°] - (300\text{ N})[(2.5\text{ m})\sin 45°]$$
$$= 1254\text{ N}\cdot\text{m} = 1.25\text{ kN}\cdot\text{m} \quad \textit{Ans.}$$

F3–8
$$\circlearrowleft +(M_R)_O = \Sigma Fd;$$
$$(M_R)_O = \left[\left(\tfrac{3}{5}\right)500\text{ N}\right](0.425\text{ m}) - \left[\left(\tfrac{4}{5}\right)500\text{ N}\right](0.25\text{ m}) - [(600\text{ N})\cos 60°](0.25\text{ m}) - [(600\text{ N})\sin 60°](0.425\text{ m})$$
$$= -268\text{ N}\cdot\text{m} = 268\text{ N}\cdot\text{m}\,\circlearrowright$$

F3–9
$$\circlearrowleft +(M_R)_O = \Sigma Fd;$$
$$(M_R)_O = (300\cos 30°\text{ lb})(6\text{ ft} + 6\sin 30°\text{ ft}) - (300\sin 30°\text{ lb})(6\cos 30°\text{ ft}) + (200\text{ lb})(6\cos 30°\text{ ft})$$
$$= 2.60\text{ kip}\cdot\text{ft} \quad \textit{Ans.}$$

F3–10
$$\mathbf{F} = F\mathbf{u}_{AB} = 500\text{ N}\left(\tfrac{4}{5}\mathbf{i} - \tfrac{3}{5}\mathbf{j}\right) = \{400\mathbf{i} - 300\mathbf{j}\}\text{N}$$
$$\mathbf{M}_O = \mathbf{r}_{OA}\times\mathbf{F} = \{3\mathbf{j}\}\text{ m}\times\{400\mathbf{i} - 300\mathbf{j}\}\text{ N} = \{-1200\mathbf{k}\}\text{ N}\cdot\text{m} \quad \textit{Ans.}$$
or
$$\mathbf{M}_O = \mathbf{r}_{OB}\times\mathbf{F} = \{4\mathbf{i}\}\text{ m}\times\{400\mathbf{i} - 300\mathbf{j}\}\text{ N} = \{-1200\mathbf{k}\}\text{ N}\cdot\text{m} \quad \textit{Ans.}$$

F3–11
$$\mathbf{F} = F\mathbf{u}_{BC} = 120\text{ lb}\left[\frac{\{4\,\mathbf{i} - 4\,\mathbf{j} - 2\,\mathbf{k}\}\text{ ft}}{\sqrt{(4\text{ ft})^2 + (-4\text{ ft})^2 + (-2\text{ ft})^2}}\right] = \{80\mathbf{i} - 80\mathbf{j} - 40\mathbf{k}\}\text{ lb}$$
$$\mathbf{M}_O = \mathbf{r}_C\times\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 5 & 0 & 0\\ 80 & -80 & -40\end{vmatrix} = \{200\mathbf{j} - 400\mathbf{k}\}\text{ lb}\cdot\text{ft} \quad \textit{Ans.}$$
or
$$\mathbf{M}_O = \mathbf{r}_B\times\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 4 & 2\\ 80 & -80 & -40\end{vmatrix} = \{200\mathbf{j} - 400\mathbf{k}\}\text{ lb}\cdot\text{ft} \quad \textit{Ans.}$$

F3–12
$$\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2 = \{(100 - 200)\mathbf{i} + (-120 + 250)\mathbf{j} + (75 + 100)\mathbf{k}\}\text{ lb} = \{-100\mathbf{i} + 130\mathbf{j} + 175\mathbf{k}\}\text{lb}$$
$$(\mathbf{M}_R)_O = \mathbf{r}_A\times\mathbf{F}_R = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & 5 & 3\\ -100 & 130 & 175\end{vmatrix} = \{485\mathbf{i} - 1000\mathbf{j} + 1020\mathbf{k}\}\text{ lb}\cdot\text{ft} \quad \textit{Ans.}$$

F3–13
$$M_x = \mathbf{i}\cdot(\mathbf{r}_{OB}\times\mathbf{F}) = \begin{vmatrix}1 & 0 & 0\\ 0.3 & 0.4 & -0.2\\ 300 & -200 & 150\end{vmatrix} = 20\text{ N}\cdot\text{m} \quad \textit{Ans.}$$

F3–14
$$\mathbf{u}_{OA'} = \frac{\mathbf{r}_A}{r_A} = \frac{\{0.3\mathbf{i} + 0.4\mathbf{j}\}\text{ m}}{\sqrt{(0.3\text{ m})^2 + (0.4\text{ m})^2}}$$
$$M_{OA} = \mathbf{u}_{OA}\cdot(\mathbf{r}_{AB}\times\mathbf{F}) = \begin{vmatrix}0.6 & 0.8 & 0\\ 0 & 0 & -0.2\\ 300 & -200 & 150\end{vmatrix} = -72\text{ N}\cdot\text{m} \quad \textit{Ans.}$$

F3–15

$$\mathbf{F} = (200\text{ N})\cos 120°\,\mathbf{i} + (200\text{ N})\cos 60°\,\mathbf{j} + (200\text{ N})\cos 45°\,\mathbf{k}$$
$$= \{-100\mathbf{i} + 100\mathbf{j} + 141.42\mathbf{k}\}\text{ N}$$
$$\mathbf{M}_O = \mathbf{i}\cdot(\mathbf{r}_A \times \mathbf{F}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0.3 & 0.25 \\ -100 & 100 & 141.42 \end{vmatrix}$$
$$= 17.4\text{ N}\cdot\text{m}$$

F3–16

$$\mathbf{M}_p = \mathbf{j}\cdot(\mathbf{r}_A \times \mathbf{F}) = \begin{vmatrix} 0 & 1 & 0 \\ -3 & -4 & 2 \\ 30 & -20 & 50 \end{vmatrix}$$
$$= 210\text{ N}\cdot\text{m} \quad \textit{Ans.}$$

F3–17

$$\mathbf{u}_{AB} = \frac{\mathbf{r}_{AB}}{r_{AB}} = \frac{\{-4\mathbf{i} + 3\mathbf{j}\}\text{ ft}}{\sqrt{(-4\text{ ft})^2 + (3\text{ ft})^2}} = -0.8\mathbf{i} + 0.6\mathbf{j}$$
$$M_{AB} = \mathbf{u}_{AB}\cdot(\mathbf{r}_{AC} \times \mathbf{F})$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -0.8 & 0.6 & 0 \\ 0 & 0 & 2 \\ 50 & -40 & 20 \end{vmatrix} = -4\text{ lb}\cdot\text{ft}$$
$$\mathbf{M}_{AB} = M_{AB}\mathbf{u}_{AB} = \{3.2\mathbf{i} - 2.4\mathbf{j}\}\text{ lb}\cdot\text{ft} \quad \textit{Ans.}$$

F3–18

$$F_x = \left[\left(\tfrac{4}{5}\right)500\text{ N}\right]\left(\tfrac{3}{5}\right) = 240\text{ N}$$
$$F_y = \left[\left(\tfrac{4}{5}\right)500\text{ N}\right]\left(\tfrac{4}{5}\right) = 320\text{ N}$$
$$F_z = (500\text{ N})\left(\tfrac{3}{5}\right) = 300\text{ N}$$
$$M_x = 300\text{ N}(2\text{ m}) - 320\text{ N}(3\text{ m})$$
$$= -360\text{ N}\cdot\text{m} \quad \textit{Ans.}$$
$$M_y = 300\text{ N}(2\text{ m}) - 240\text{ N}(3\text{ m})$$
$$= -120\text{ N}\cdot\text{m} \quad \textit{Ans.}$$
$$M_z = 240\text{ N}(2\text{ m}) - 320\text{ N}(2\text{ m})$$
$$= -160\text{ N}\cdot\text{m} \quad \textit{Ans.}$$

F3–19

$$\circlearrowleft + M_{C_R} = \Sigma M_A = 400(3) - 400(5) + 300(5) + 200(0.2) = 740\text{ N}\cdot\text{m} \quad \textit{Ans.}$$

Also,

$$\circlearrowleft + M_{C_R} = 300(5) - 400(2) + 200(0.2)$$
$$= 740\text{ N}\cdot\text{m} \quad \textit{Ans.}$$

F3–20

$$\curvearrowright + M_{C_R} = 300(4) + 200(4) + 150(4)$$
$$= 2600\text{ lb}\cdot\text{ft} \quad \textit{Ans.}$$

F3–21

$$\curvearrowright + (M_B)_R = \Sigma M_B$$
$$-1.5\text{ kN}\cdot\text{m} = (2\text{ kN})(0.3\text{ m}) - F(0.9\text{ m})$$
$$F = 2.33\text{ kN} \quad \textit{Ans.}$$

F3–22

$$\curvearrowright + M_C = 10\left(\tfrac{3}{5}\right)(2) - 10\left(\tfrac{4}{5}\right)(4) = -20\text{ kN}\cdot\text{m}$$
$$= 20\text{ kN}\cdot\text{m}\,\circlearrowright$$

F3–23

$$\mathbf{u}_1 = \frac{\mathbf{r}_1}{r_1} = \frac{[-2\mathbf{i} + 2\mathbf{j} + 3.5\mathbf{k}]\text{ ft}}{\sqrt{(-2\text{ ft})^2 + (2\text{ ft})^2 + (3.5\text{ ft})^2}}$$
$$= -\tfrac{2}{4.5}\mathbf{i} + \tfrac{2}{4.5}\mathbf{j} + \tfrac{3.5}{4.5}\mathbf{k}$$
$$\mathbf{u}_2 = -\mathbf{k}$$
$$\mathbf{u}_3 = \tfrac{1.5}{2.5}\mathbf{i} - \tfrac{2}{2.5}\mathbf{j}$$
$$(\mathbf{M}_c)_1 = (M_c)_1\mathbf{u}_1$$
$$= (450\text{ lb}\cdot\text{ft})\left(-\tfrac{2}{4.5}\mathbf{i} + \tfrac{2}{4.5}\mathbf{j} + \tfrac{3.5}{4.5}\mathbf{k}\right)$$
$$= \{-200\mathbf{i} + 200\mathbf{j} + 350\mathbf{k}\}\text{ lb}\cdot\text{ft}$$
$$(\mathbf{M}_c)_2 = (M_c)_2\mathbf{u}_2 = (250\text{ lb}\cdot\text{ft})(-\mathbf{k})$$
$$= \{-250\mathbf{k}\}\text{ lb}\cdot\text{ft}$$
$$(\mathbf{M}_c)_3 = (M_c)_3\mathbf{u}_3 = (300\text{ lb}\cdot\text{ft})\left(\tfrac{1.5}{2.5}\mathbf{i} - \tfrac{2}{2.5}\mathbf{j}\right)$$
$$= \{180\mathbf{i} - 240\mathbf{j}\}\text{ lb}\cdot\text{ft}$$
$$(\mathbf{M}_c)_R = \Sigma M_c;$$
$$(\mathbf{M}_c)_R = \{-20\mathbf{i} - 40\mathbf{j} + 100\mathbf{k}\}\text{ lb}\cdot\text{ft} \quad \textit{Ans.}$$

F3–24

$$\mathbf{F}_B = \left(\tfrac{4}{5}\right)(450\text{ N})\mathbf{j} - \left(\tfrac{3}{5}\right)(450\text{ N})\,\mathbf{k}$$
$$= \{360\mathbf{j} - 270\mathbf{k}\}\text{ N}$$
$$\mathbf{M}_c = \mathbf{r}_{AB} \times \mathbf{F}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.4 & 0 & 0 \\ 0 & 360 & -270 \end{vmatrix}$$
$$= \{108\mathbf{j} + 144\mathbf{k}\}\text{ N}\cdot\text{m} \quad \textit{Ans.}$$

Also,

$$\mathbf{M}_c = (\mathbf{r}_A \times \mathbf{F}_A) + (\mathbf{r}_B \times \mathbf{F}_B)$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0.3 \\ 0 & -360 & 270 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.4 & 0 & 0.3 \\ 0 & 360 & -270 \end{vmatrix}$$
$$= \{108\mathbf{j} + 144\mathbf{k}\}\text{ N}\cdot\text{m} \quad \textit{Ans.}$$

F3–25

$$\overset{+}{\leftarrow} F_{R_x} = \Sigma F_x;\ F_{R_x} = 200 - \tfrac{3}{5}(100) = 140\text{ lb}$$
$$+\downarrow F_{R_y} = \Sigma F_y;\ F_{R_y} = 150 - \tfrac{4}{5}(100) = 70\text{ lb}$$
$$F_R = \sqrt{140^2 + 70^2} = 157\text{ lb} \quad \textit{Ans.}$$
$$\theta = \tan^{-1}\left(\tfrac{70}{140}\right) = 26.6°\ \swarrow \quad \textit{Ans.}$$
$$\circlearrowleft + M_{A_R} = \Sigma M_A;$$
$$M_{A_R} = \tfrac{3}{5}(100)(4) - \tfrac{4}{5}(100)(6) + 150(3)$$
$$M_{R_A} = 210\text{ lb}\cdot\text{ft} \quad \textit{Ans.}$$

F3–26

$$\overset{+}{\rightarrow} F_{Rx} = \Sigma F_x;\quad F_{Rx} = \tfrac{4}{5}(50) = 40\ \text{N}$$
$$+\downarrow F_{Ry} = \Sigma F_y;\quad F_{Ry} = 40 + 30 + \tfrac{3}{5}(50)$$
$$= 100\ \text{N}$$
$$F_R = \sqrt{(40)^2 + (100)^2} = 108\ \text{N} \quad \textit{Ans.}$$
$$\theta = \tan^{-1}\left(\tfrac{100}{40}\right) = 68.2^\circ \measuredangle \quad \textit{Ans.}$$
$$\circlearrowleft + M_{A_R} = \Sigma M_A;$$
$$M_{A_R} = 30(3) + \tfrac{3}{5}(50)(6) + 200$$
$$= 470\ \text{N}\cdot\text{m} \quad \textit{Ans.}$$

F3–27

$$\overset{+}{\rightarrow} (F_R)_x = \Sigma F_x;$$
$$(F_R)_x = 900 \sin 30^\circ = 450\ \text{N} \rightarrow$$
$$+\uparrow (F_R)_y = \Sigma F_y;$$
$$(F_R)_y = -900 \cos 30^\circ - 300$$
$$= -1079.42\ \text{N} = 1079.42\ \text{N} \downarrow$$
$$F_R = \sqrt{450^2 + 1079.42^2}$$
$$= 1169.47\ \text{N} = 1.17\ \text{kN} \quad \textit{Ans.}$$
$$\theta = \tan^{-1}\left(\tfrac{1079.42}{450}\right) = 67.4^\circ \measuredangle \quad \textit{Ans.}$$
$$\circlearrowleft + (M_R)_A = \Sigma M_A;$$
$$(M_R)_A = 300 - 900 \cos 30^\circ\,(0.75) - 300(2.25)$$
$$= -959.57\ \text{N}\cdot\text{m}$$
$$= 960\ \text{N}\cdot\text{m}\circlearrowright \quad \textit{Ans.}$$

F3–28

$$\overset{+}{\rightarrow} (F_R)_x = \Sigma F_x;$$
$$(F_R)_x = 150\left(\tfrac{3}{5}\right) + 50 - 100\left(\tfrac{4}{5}\right) = 60\ \text{lb} \rightarrow$$
$$+\uparrow (F_R)_y = \Sigma F_y;$$
$$(F_R)_y = -150\left(\tfrac{4}{5}\right) - 100\left(\tfrac{3}{5}\right)$$
$$= -180\ \text{lb} = 180\ \text{lb} \downarrow$$
$$F_R = \sqrt{60^2 + 180^2} = 189.74\ \text{lb} = 190\ \text{lb} \quad \textit{Ans.}$$
$$\theta = \tan^{-1}\left(\tfrac{180}{60}\right) = 71.6^\circ \measuredangle \quad \textit{Ans.}$$
$$\circlearrowleft + (M_R)_A = \Sigma M_A;$$
$$(M_R)_A = 100\left(\tfrac{4}{5}\right)(1) - 100\left(\tfrac{3}{5}\right)(6) - 150\left(\tfrac{4}{5}\right)(3)$$
$$= -640 = 640\ \text{lb}\cdot\text{ft}\circlearrowright \quad \textit{Ans.}$$

F3–29

$$\mathbf{F}_R = \Sigma \mathbf{F};$$
$$F_R = \mathbf{F}_1 + \mathbf{F}_2$$
$$= (-300\mathbf{i} + 150\mathbf{j} + 200\mathbf{k}) + (-450\mathbf{k})$$
$$= \{-300\mathbf{i} + 150\mathbf{j} - 250\mathbf{k}\}\ \text{N} \quad \textit{Ans.}$$
$$\mathbf{r}_{OA} = (2 - 0)\mathbf{j} = \{2\mathbf{j}\}\ \text{m}$$
$$\mathbf{r}_{OB} = (-1.5 - 0)\mathbf{i} + (2 - 0)\mathbf{j} + (1 - 0)\mathbf{k}$$
$$= \{-1.5\mathbf{i} + 2\mathbf{j} + 1\mathbf{k}\}\ \text{m}$$
$$(\mathbf{M}_R)_O = \Sigma \mathbf{M};$$
$$(\mathbf{M}_R)_O = \mathbf{r}_{OB} \times \mathbf{F}_1 + \mathbf{r}_{OA} \times \mathbf{F}_2$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1.5 & 2 & 1 \\ -300 & 150 & 200 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ 0 & 0 & -450 \end{vmatrix}$$
$$= \{-650\mathbf{i} + 375\mathbf{k}\}\ \text{N}\cdot\text{m} \quad \textit{Ans.}$$

F3–30

$$\mathbf{F}_1 = \{-100\mathbf{j}\}\ \text{N}$$
$$\mathbf{F}_2 = (200\ \text{N})\left[\frac{\{-0.4\mathbf{i} - 0.3\mathbf{k}\}\ \text{m}}{\sqrt{(-0.4\ \text{m})^2 + (-0.3\ \text{m})^2}}\right]$$
$$= \{-160\mathbf{i} - 120\mathbf{k}\}\ \text{N}$$
$$\mathbf{M}_c = \{-75\mathbf{i}\}\ \text{N}\cdot\text{m}$$
$$\mathbf{F}_R = \{-160\mathbf{i} - 100\mathbf{j} - 120\mathbf{k}\}\ \text{N} \quad \textit{Ans.}$$
$$(\mathbf{M}_R)_O = (0.3\mathbf{k}) \times (-100\mathbf{j})$$
$$+ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0.5 & 0.3 \\ -160 & 0 & -120 \end{vmatrix} + (-75\mathbf{i})$$
$$= \{-105\mathbf{i} - 48\mathbf{j} + 80\mathbf{k}\}\ \text{N}\cdot\text{m} \quad \textit{Ans.}$$

F3–31

$$+\downarrow F_R = \Sigma F_y;\quad F_R = 500 + 250 + 500$$
$$= 1250\ \text{lb} \quad \textit{Ans.}$$
$$\circlearrowleft + F_R x = \Sigma M_O;$$
$$1250(x) = 500(3) + 250(6) + 500(9)$$
$$x = 6\ \text{ft} \quad \textit{Ans.}$$

F3–32

$$\overset{+}{\rightarrow} (F_R)_x = \Sigma F_x;$$
$$(F_R)_x = 100\left(\tfrac{3}{5}\right) + 50 \sin 30^\circ = 85\ \text{lb} \rightarrow$$
$$+\uparrow (F_R)_y = \Sigma F_y;$$
$$(F_R)_y = 200 + 50 \cos 30^\circ - 100\left(\tfrac{4}{5}\right)$$
$$= 163.30\ \text{lb}\uparrow$$
$$F_R = \sqrt{85^2 + 163.30^2} = 184\ \text{lb}$$
$$\theta = \tan^{-1}\left(\tfrac{163.30}{85}\right) = 62.5^\circ \measuredangle \quad \textit{Ans.}$$
$$\circlearrowleft + (M_R)_A = \Sigma M_A;$$
$$163.30(d) = 200(3) - 100\left(\tfrac{4}{5}\right)(6) + 50 \cos 30^\circ(9)$$
$$d = 3.12\ \text{ft} \quad \textit{Ans.}$$

F3–33

$$\overset{+}{\rightarrow} (F_R)_x = \Sigma F_x;$$
$$(F_R)_x = 15\left(\tfrac{4}{5}\right) = 12\ \text{kN} \rightarrow$$
$$+\uparrow (F_R)_y = \Sigma F_y;$$
$$(F_R)_y = -20 + 15\left(\tfrac{3}{5}\right) = -11\ \text{kN} = 11\ \text{kN} \downarrow$$
$$F_R = \sqrt{12^2 + 11^2} = 16.3\ \text{kN} \quad \textit{Ans.}$$
$$\theta = \tan^{-1}\left(\tfrac{11}{12}\right) = 42.5^\circ \measuredangle \quad \textit{Ans.}$$

$$\circlearrowleft + (M_R)_A = \Sigma M_A;$$
$$-11(d) = -20(2) - 15\left(\tfrac{4}{5}\right)(2) + 15\left(\tfrac{3}{5}\right)(6)$$
$$d = 0.909 \text{ m} \quad \textit{Ans.}$$

F3–34
$$\overset{+}{\rightarrow}(F_R)_x = \Sigma F_x;$$
$$(F_R)_x = \left(\tfrac{3}{5}\right) 5 \text{ kN} - 8 \text{ kN}$$
$$= -5 \text{ kN} = 5 \text{ kN} \leftarrow$$
$$+\uparrow (F_R)_y = \Sigma F_y;$$
$$(F_R)_y = -6 \text{ kN} - \left(\tfrac{4}{5}\right) 5 \text{ kN}$$
$$= -10 \text{ kN} = 10 \text{ kN} \downarrow$$
$$F_R = \sqrt{5^2 + 10^2} = 11.2 \text{ kN} \quad \textit{Ans.}$$
$$\theta = \tan^{-1}\left(\tfrac{10 \text{ kN}}{5 \text{ kN}}\right) = 63.4° \measuredangle \quad \textit{Ans.}$$
$$\circlearrowleft + (M_R)_A = \Sigma M_A;$$
$$5 \text{ kN}(d) = 8 \text{ kN}(3 \text{ m}) - 6 \text{ kN}(0.5 \text{ m})$$
$$- \left[\left(\tfrac{4}{5}\right)5 \text{ kN}\right](2 \text{ m})$$
$$- \left[\left(\tfrac{3}{5}\right)5 \text{ kN}\right](4 \text{ m})$$
$$d = 0.2 \text{ m} \quad \textit{Ans.}$$

F3–35
$$+\downarrow F_R = \Sigma F_z; \quad F_R = 400 + 500 - 100$$
$$= 800 \text{ N} \quad \textit{Ans.}$$
$$M_{Rx} = \Sigma M_x; \quad -800y = -400(4) - 500(4)$$
$$y = 4.50 \text{ m} \quad \textit{Ans.}$$
$$M_{Ry} = \Sigma M_y; \quad 800x = 500(4) - 100(3)$$
$$x = 2.125 \text{ m} \quad \textit{Ans.}$$

F3–36
$$+\downarrow F_R = \Sigma F_z;$$
$$F_R = 200 + 200 + 100 + 100$$
$$= 600 \text{ N} \quad \textit{Ans.}$$
$$M_{Rx} = \Sigma M_x;$$
$$-600y = 200(1) + 200(1) + 100(3) - 100(3)$$
$$y = -0.667 \text{ m} \quad \textit{Ans.}$$
$$M_{Ry} = \Sigma M_y;$$
$$600x = 100(3) + 100(3) + 200(2) - 200(3)$$
$$x = 0.667 \text{ m} \quad \textit{Ans.}$$

Chapter 4

F4–1
$$\overset{+}{\rightarrow}\Sigma F_x = 0; \tfrac{4}{5}F_{AC} - F_{AB}\cos 30° = 0$$
$$+\uparrow \Sigma F_y = 0; \tfrac{3}{5}F_{AC} + F_{AB}\sin 30° - 550 = 0$$
$$F_{AB} = 478 \text{ lb} \quad \textit{Ans.}$$
$$F_{AC} = 518 \text{ lb} \quad \textit{Ans.}$$

F4–2
$$\circlearrowleft + \Sigma M_A = 0;$$
$$F_{CD}\sin 45°(1.5 \text{ m}) - 4 \text{ kN}(3 \text{ m}) = 0$$
$$F_{CD} = 11.31 \text{ kN} = 11.3 \text{ kN} \quad \textit{Ans.}$$
$$\overset{+}{\rightarrow}\Sigma F_x = 0; A_x + (11.31 \text{ kN})\cos 45° = 0$$
$$A_x = -8 \text{ kN} = 8 \text{ kN} \leftarrow \quad \textit{Ans.}$$
$$+\uparrow \Sigma F_y = 0;$$
$$A_y + (11.31 \text{ kN})\sin 45° - 4 \text{ kN} = 0$$
$$A_y = -4 \text{ kN} = 4 \text{ kN} \downarrow \quad \textit{Ans.}$$

F4–3
$$\circlearrowleft + \Sigma M_A = 0;$$
$$N_B[6 \text{ m} + (6 \text{ m})\cos 45°]$$
$$- 10 \text{ kN}[2 \text{ m} + (6 \text{ m})\cos 45°]$$
$$- 5 \text{ kN}(4 \text{ m}) = 0$$
$$N_B = 8.047 \text{ kN} = 8.05 \text{ kN} \quad \textit{Ans.}$$
$$\overset{+}{\rightarrow}\Sigma F_x = 0;$$
$$(5 \text{ kN})\cos 45° - A_x = 0$$
$$A_x = 3.54 \text{ kN} \quad \textit{Ans.}$$
$$+\uparrow \Sigma F_y = 0;$$
$$A_y + 8.047 \text{ kN} - (5 \text{ kN})\sin 45° - 10 \text{ kN} = 0$$
$$A_y = 5.49 \text{ kN} \quad \textit{Ans.}$$

F4–4
$$\overset{+}{\rightarrow}\Sigma F_x = 0; \quad -A_x + 400\cos 30° = 0$$
$$A_x = 346 \text{ N} \quad \textit{Ans.}$$
$$+\uparrow \Sigma F_y = 0;$$
$$A_y - 200 - 200 - 200 - 400\sin 30° = 0$$
$$A_y = 800 \text{ N} \quad \textit{Ans.}$$
$$\circlearrowleft + \Sigma M_A = 0;$$
$$M_A - 200(2.5) - 200(3.5) - 200(4.5)$$
$$- 400\sin 30°(4.5) - 400\cos 30°(3\sin 60°) = 0$$
$$M_A = 3.90 \text{ kN} \cdot \text{m} \quad \textit{Ans.}$$

F4–5
$$\circlearrowleft + \Sigma M_A = 0;$$
$$N_C(0.7 \text{ m}) - [25(9.81) \text{ N}](0.5 \text{ m})\cos 30° = 0$$
$$N_C = 151.71 \text{ N} = 152 \text{ N} \quad \textit{Ans.}$$
$$\overset{+}{\rightarrow}\Sigma F_x = 0;$$
$$T_{AB}\cos 15° - (151.71 \text{ N})\cos 60° = 0$$
$$T_{AB} = 78.53 \text{ N} = 78.5 \text{ N} \quad \textit{Ans.}$$
$$+\uparrow \Sigma F_y = 0;$$
$$F_A + (78.53 \text{ N})\sin 15°$$
$$+ (151.71 \text{ N})\sin 60° - 25(9.81) \text{ N} = 0$$
$$F_A = 93.5 \text{ N} \quad \textit{Ans.}$$

F4–6
$$\overset{+}{\rightarrow}\Sigma F_x = 0;$$
$$N_C\sin 30° - (250 \text{ N})\sin 60° = 0$$
$$N_C = 433.0 \text{ N} = 433 \text{ N} \quad \textit{Ans.}$$
$$\circlearrowleft + \Sigma M_B = 0;$$
$$-N_A\sin 30°(0.15 \text{ m}) - 433.0 \text{ N}(0.2 \text{ m})$$

$+ [(250\text{ N})\cos 30°](0.6\text{ m}) = 0$

$N_A = 577.4\text{ N} = 577\text{ N}$ *Ans.*

$+\uparrow \Sigma F_y = 0;$

$N_B - 577.4\text{ N} + (433.0\text{ N})\cos 30°$

$- (250\text{ N})\cos 60° = 0$

$N_B = 327\text{ N}$ *Ans.*

F4–7 $\Sigma F_x = 0; \quad \left[\left(\frac{3}{5}\right)F_3\right]\left(\frac{3}{5}\right) + 600\text{ N} - F_2 = 0 \quad (1)$

$\Sigma F_y = 0; \quad \left(\frac{4}{5}\right)F_1 - \left[\left(\frac{3}{5}\right)F_3\right]\left(\frac{4}{5}\right) = 0 \quad (2)$

$\Sigma F_z = 0; \quad \left(\frac{4}{5}\right)F_3 + \left(\frac{3}{5}\right)F_1 - 900\text{ N} = 0 \quad (3)$

$F_3 = 776\text{ N}$ *Ans.*

$F_1 = 466\text{ N}$ *Ans.*

$F_2 = 879\text{ N}$ *Ans.*

F4–8 $\Sigma M_y = 0;$

$600\text{ N}(0.2\text{ m}) + 900\text{ N}(0.6\text{ m}) - F_A(1\text{ m}) = 0$

$F_A = 660\text{ N}$ *Ans.*

$\Sigma M_x = 0;$

$D_z(0.8\text{ m}) - 600\text{ N}(0.5\text{ m}) - 900\text{ N}(0.1\text{ m}) = 0$

$D_z = 487.5\text{ N}$ *Ans.*

$\Sigma F_x = 0; \quad D_x = 0$ *Ans.*

$\Sigma F_y = 0; \quad D_y = 0$ *Ans.*

$\Sigma F_z = 0;$

$T_{BC} + 660\text{ N} + 487.5\text{ N} - 900\text{ N} - 600\text{ N} = 0$

$T_{BC} = 352.5\text{ N}$ *Ans.*

F4–9 $\Sigma F_y = 0; \quad 400\text{ N} + C_y = 0;$

$C_y = -400\text{ N}$ *Ans.*

$\Sigma M_y = 0; \quad -C_x(0.4\text{ m}) - 600\text{ N}(0.6\text{ m}) = 0$

$C_x = -900\text{ N}$ *Ans.*

$\Sigma M_x = 0; \quad B_z(0.6\text{ m}) + 600\text{ N}(1.2\text{ m})$

$+ (-400\text{ N})(0.4\text{ m}) = 0$

$B_z = -933.3\text{ N}$ *Ans.*

$\Sigma M_z = 0;$

$-B_x(0.6\text{ m}) + (-900\text{ N})(1.2\text{ m})$

$+ (-400\text{ N})(0.6\text{ m}) = 0$

$B_x = 1400\text{ N}$ *Ans.*

$\Sigma F_x = 0; \quad 1400\text{ N} + (-900\text{ N}) + A_x = 0$

$A_x = -500\text{ N}$ *Ans.*

$\Sigma F_z = 0; \quad A_z - 933.3\text{N} + 600\text{ N} = 0$

$A_z = 333.3\text{ N}$ *Ans.*

F4–10 $\Sigma F_x = 0; \quad B_x = 0$ *Ans.*

$\Sigma M_z = 0;$

$C_y(0.4\text{ m} + 0.6\text{ m}) = 0 \quad C_y = 0$ *Ans.*

$\Sigma F_y = 0; \quad A_y + 0 = 0 \quad A_y = 0$ *Ans.*

$\Sigma M_x = 0; C_z(0.6\text{ m} + 0.6\text{ m}) + B_z(0.6\text{ m})$

$- 450\text{ N}(0.6\text{ m} + 0.6\text{ m}) = 0$

$1.2C_z + 0.6B_z - 540 = 0$

$\Sigma M_y = 0; \quad -C_z(0.6\text{ m} + 0.4\text{ m})$

$- B_z(0.6\text{ m}) + 450\text{ N}(0.6\text{ m}) = 0$

$-C_z = 0.6B_z + 270 = 0$

$C_z = 1350\text{ N} \quad B_z = -1800\text{ N}$ *Ans.*

$\Sigma F_z = 0;$

$A_z + 1350\text{ N} + (-1800\text{ N}) - 450\text{ N} = 0$

$A_z = 900\text{ N}$ *Ans.*

F4–11 $\Sigma F_y = 0; \quad A_y = 0$ *Ans.*

$\Sigma M_x = 0; \quad -9(3) + F_{CE}(3) = 0$

$F_{CE} = 9\text{ kN}$ *Ans.*

$\Sigma M_z = 0; \quad F_{CF}(3) - 6(3) = 0$

$F_{CF} = 6\text{ kN}$ *Ans.*

$\Sigma M_y = 0; \quad 9(4) - A_z(4) - 6(1.5) = 0$

$A_z = 6.75\text{ kN}$ *Ans.*

$\Sigma F_x = 0; \quad A_x + 6 - 6 = 0 \quad A_x = 0$ *Ans.*

$\Sigma F_z = 0; \quad F_{DB} + 9 - 9 + 6.75 = 0$

$F_{DB} = -6.75\text{ kN}$ *Ans.*

F4–12 $\Sigma F_x = 0; \quad A_x = 0$ *Ans.*

$\Sigma F_y = 0; \quad A_y = 0$ *Ans.*

$\Sigma F_z = 0; \quad A_z + F_{BC} - 80 = 0$

$\Sigma M_x = 0; (M_A)_x + 6F_{BC} - 80(6) = 0$

$\Sigma M_y = 0; 3F_{BC} - 80(1.5) = 0 \quad F_{BC} = 40\text{ lb}$ *Ans.*

$\Sigma M_z = 0; (M_A)_z = 0$ *Ans.*

$A_z = 40\text{ lb} \quad (M_A)_x = 240\text{ lb}\cdot\text{ft}$ *Ans.*

F4–13 $+\uparrow \Sigma F_y = 0; \quad N - 50(9.81) - 200\left(\frac{3}{5}\right) = 0$

$N = 610.5\text{ N}$

$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \quad F - 200\left(\frac{4}{5}\right) = 0$

$F = 160\text{ N}$

$F < F_{\max} = \mu_s N = 0.3(610.5) = 183.15\text{ N},$

therefore $F = 160\text{ N}$ *Ans.*

F4–14 $\circlearrowleft + \Sigma M_B = 0;$

$N_A(3) + 0.2N_A(4) - 30(9.81)(2) = 0$

$N_A = 154.89\text{ N}$

$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \quad P - 154.89 = 0$

$P = 154.89\text{ N} = 155\text{ N}$ *Ans.*

F4–15 Crate A

$+\uparrow\Sigma F_y = 0;\quad N_A - 50(9.81) = 0$

$N_A = 490.5\text{ N}$

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad T - 0.25(490.5) = 0$

$T = 122.62\text{ N}$

Crate B

$+\uparrow\Sigma F_y = 0;\quad N_B + P\sin 30° - 50(9.81) = 0$

$N_B = 490.5 - 0.5P$

$\overset{+}{\rightarrow}\Sigma F_x = 0;$

$P\cos 30° - 0.25(490.5 - 0.5\,P) - 122.62 = 0$

$P = 247\text{ N}$ *Ans.*

F4–16

$\circlearrowleft +\Sigma M_A = 0;\quad 490.5(0.6) - T\cos 60°(0.3\cos 60° + 0.6)$

$- T\sin 60°(0.3\sin 60°) = 0$

$T = 490.5\text{ N}$

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad 490.5\sin 60° - N_A = 0;\quad N_A = 424.8\text{ N}$

$+\uparrow\Sigma F_y = 0;\quad \mu_s(424.8) + 490.5\cos 60° - 490.5 = 0$

$\mu_s = 0.577$ *Ans.*

F4–17 A is about to move down the plane and B moves upward.

Block A

$+\nwarrow\Sigma F_y = 0;\quad N = W\cos\theta$

$+\nearrow\Sigma F_x = 0;\quad T + \mu_s(W\cos\theta) - W\sin\theta = 0$

$T = W\sin\theta - \mu_s W\cos\theta \qquad (1)$

Block B

$+\nwarrow\Sigma F_y = 0;\quad N' = 2W\cos\theta$

$+\nearrow\Sigma F_x = 0;\quad 2T - \mu_s W\cos\theta - \mu_s(2W\cos\theta)$

$- W\sin\theta = 0$

Using Eq.(1);

$\theta = \tan^{-1} 5\mu_s$ *Ans.*

F4–18 Assume B is about to slip on A, $F_B = 0.3\,N_B$.

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad P - 0.3(10)(9.81) = 0$

$P = 29.4\text{ N}$

Assume B is about to slip on A, $x = 0$.

$\circlearrowleft +\Sigma M_O = 0;\quad 10(9.81)(0.15) - P(0.4) = 0$

$P = 36.8\text{ N}$

Assume A is about to slip, $F_A = 0.1\,N_A$.

$\overset{+}{\rightarrow}\Sigma F_x = 0\quad P - 0.1[7(9.81) + 10(9.81)] = 0$

$P = 16.7\text{ N}$

Choose the smallest result. $P = 16.7\text{ N}$ *Ans.*

Chapter 5

F5–1 *Joint A.*

$+\uparrow\Sigma F_y = 0;\quad 225\text{ lb} - F_{AD}\sin 45° = 0$

$F_{AD} = 318.20\text{ lb} = 318\text{ lb (C)}$ *Ans.*

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad F_{AB} - (318.20\text{ lb})\cos 45° = 0$

$F_{AB} = 225\text{ lb (T)}$ *Ans.*

Joint B.

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad F_{BC} - 225\text{ lb} = 0$

$F_{BC} = 225\text{ lb (T)}$ *Ans.*

$+\uparrow\Sigma F_y = 0;\quad F_{BD} = 0$ *Ans.*

Joint D.

$\overset{+}{\rightarrow}\Sigma F_x = 0;$

$F_{CD}\cos 45° + (318.20\text{ lb})\cos 45° - 450\text{ lb} = 0$

$F_{CD} = 318.20\text{ lb} = 318\text{ lb (T)}$ *Ans.*

F5–2 *Joint D:*

$+\uparrow\Sigma F_y = 0;\ \frac{3}{5}F_{CD} - 300 = 0;$

$F_{CD} = 500\text{ lb (T)}$ *Ans.*

$\overset{+}{\rightarrow}\Sigma F_x = 0;\ -F_{AD} + \frac{4}{5}(500) = 0$

$F_{AD} = 400\text{ lb (C)}$ *Ans.*

$F_{BC} = 500\text{ lb (T)},\ F_{AC} = F_{AB} = 0$ *Ans.*

F5–3 $A_x = 0,\ A_y = C_y = 400\text{ lb}$

Joint A:

$+\uparrow\Sigma F_y = 0;\ -\frac{3}{5}F_{AE} + 400 = 0$

$F_{AE} = 667\text{ lb (C)}$ *Ans.*

Joint C:

$+\uparrow\Sigma F_y = 0;\ -F_{DC} + 400 = 0;$

$F_{DC} = 400\text{ lb (C)}$ *Ans.*

F5–4 *Joint C.*

$+\uparrow\Sigma F_y = 0;\quad 2F\cos 30° - P = 0$

$F_{AC} = F_{BC} = F = \frac{P}{2\cos 30°} = 0.5774P\text{ (C)}$

Joint B.

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad 0.5774P\cos 60° - F_{AB} = 0$

$F_{AB} = 0.2887P\text{ (T)}$

$F_{AB} = 0.2887P = 2\text{ kN}$

$P = 6.928\text{ kN}$

$F_{AC} = F_{BC} = 0.5774P = 1.5\text{ kN}$

$P = 2.598\text{ kN}$

The *smaller value* of P is chosen,

$P = 2.598\text{ kN} = 2.60\text{ kN}$ *Ans.*

F5–5 $F_{CB} = 0$ *Ans.*

$F_{CD} = 0$ *Ans.*

$F_{AE} = 0$ *Ans.*

$F_{DE} = 0$ *Ans.*

F5–6 *Joint C.*

$+\uparrow \Sigma F_y = 0;\quad 259.81\text{ lb} - F_{CD}\sin 30° = 0$

$F_{CD} = 519.62\text{ lb} = 520\text{ lb (C)}$ *Ans.*

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad (519.62\text{ lb})\cos 30° - F_{BC} = 0$

$F_{BC} = 450\text{ lb (T)}$ *Ans.*

Joint D.

$+\nearrow \Sigma F_{y'} = 0;\quad F_{BD}\cos 30° = 0\quad F_{BD} = 0$ *Ans.*

$+\searrow \Sigma F_{x'} = 0;\quad F_{DE} - 519.62\text{ lb} = 0$

$F_{DE} = 519.62\text{ lb} = 520\text{ lb (C)}$ *Ans.*

Joint B.

$\uparrow \Sigma F_y = 0;\quad F_{BE}\sin\phi = 0\quad F_{BE} = 0$ *Ans.*

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad 450\text{ lb} - F_{AB} = 0$

$F_{AB} = 450\text{ lb (T)}$ *Ans.*

Joint A.

$+\uparrow \Sigma F_y = 0;\quad 340.19\text{ lb} - F_{AE} = 0$

$F_{AE} = 340\text{ lb (C)}$ *Ans.*

F5–7 $+\uparrow \Sigma F_y = 0;\ F_{CF}\sin 45° - 600 - 800 = 0$

$F_{CF} = 1980\text{ lb (T)}$ *Ans.*

$\circlearrowleft +\Sigma M_C = 0;\ F_{FE}(4) - 800(4) = 0$

$F_{FE} = 800\text{ lb (T)}$ *Ans.*

$\circlearrowleft +\Sigma M_F = 0;\ F_{BC}(4) - 600(4) - 800(8) = 0$

$F_{BC} = 2200\text{ lb (C)}$ *Ans.*

F5–8 $+\uparrow \Sigma F_y = 0;\quad F_{KC} + 33.33\text{ kN} - 40\text{ kN} = 0$

$F_{KC} = 6.67\text{ kN (C)}$ *Ans.*

$\circlearrowleft +\Sigma M_K = 0;$

$33.33\text{ kN}(8\text{ m}) - 40\text{ kN}(2\text{ m}) - F_{CD}(3\text{ m}) = 0$

$F_{CD} = 62.22\text{ kN} = 62.2\text{ kN (T)}$ *Ans.*

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad F_{LK} - 62.22\text{ kN} = 0$

$F_{LK} = 62.2\text{ kN (C)}$ *Ans.*

F5–9 $\circlearrowleft +\Sigma M_A = 0;\quad G_y(12\text{ m}) - 20\text{ kN}(2\text{ m})$

$- 30\text{ kN}(4\text{ m}) - 40\text{ kN}(6\text{ m}) = 0$

$G_y = 33.33\text{ kN}$

From the geometry of the truss,

$\phi = \tan^{-1}(3\text{ m}/2\text{ m}) = 56.31°.$

$\circlearrowleft +\Sigma M_K = 0;$

$33.33\text{ kN}(8\text{ m}) - 40\text{ kN}(2\text{ m}) - F_{CD}(3\text{ m}) = 0$

$F_{CD} = 62.2\text{ kN (T)}$ *Ans.*

$\circlearrowleft +\Sigma M_D = 0;\quad 33.33\text{ kN}(6\text{ m}) - F_{KJ}(3\text{ m}) = 0$

$F_{KJ} = 66.7\text{ kN (C)}$ *Ans.*

$+\uparrow \Sigma F_y = 0;$

$33.33\text{ kN} - 40\text{ kN} + F_{KD}\sin 56.31° = 0$

$F_{KD} = 8.01\text{ kN (T)}$ *Ans.*

F5–10 From the geometry of the truss,

$\tan\phi = \frac{(9\text{ ft})\tan 30°}{3\text{ ft}} = 1.732\quad \phi = 60°$

$\circlearrowleft +\Sigma M_C = 0;$

$F_{EF}\sin 30°(6\text{ ft}) + 300\text{ lb}(6\text{ ft}) = 0$

$F_{EF} = -600\text{ lb} = 600\text{ lb (C)}$ *Ans.*

$\circlearrowleft +\Sigma M_D = 0;$

$300\text{ lb}(6\text{ ft}) - F_{CF}\sin 60°(6\text{ ft}) = 0$

$F_{CF} = 346.41\text{ lb} = 346\text{ lb (T)}$ *Ans.*

$\circlearrowleft +\Sigma M_F = 0;$

$300\text{ lb}(9\text{ ft}) - 300\text{ lb}(3\text{ ft}) - F_{BC}(9\text{ ft})\tan 30° = 0$

$F_{BC} = 346.41\text{ lb} = 346\text{ lb (T)}$ *Ans.*

F5–11 From the geometry of the truss,

$\theta = \tan^{-1}(1\text{ m}/2\text{ m}) = 26.57°$

$\phi = \tan^{-1}(3\text{ m}/2\text{ m}) = 56.31°.$

The location of G can be found using similar triangles.

$$\frac{1\text{ m}}{2\text{ m}} = \frac{2\text{ m}}{2\text{ m} + x}$$

$$4\text{ m} = 2\text{ m} + x$$

$$x = 2\text{ m}$$

$\circlearrowleft +\Sigma M_G = 0;$

$26.25\text{ kN}(4\text{ m}) - 15\text{ kN}(2\text{ m}) - F_{CD}(3\text{ m}) = 0$

$F_{CD} = 25\text{ kN (T)}$ *Ans.*

$\circlearrowleft +\Sigma M_D = 0;$

$26.25\text{ kN}(2\text{ m}) - F_{GF}\cos 26.57°(2\text{ m}) = 0$

$F_{GF} = 29.3\text{ kN (C)}$ *Ans.*

$\circlearrowleft +\Sigma M_O = 0;\quad 15\text{ kN}(4\text{ m}) - 26.25\text{ kN}(2\text{ m})$

$- F_{GD}\sin 56.31°(4\text{ m}) = 0$

$F_{GD} = 2.253\text{ kN} = 2.25\text{ kN (T)}$ *Ans.*

F5–12 $\circlearrowleft +\Sigma M_H = 0;$

$F_{DC}(12\text{ ft}) + 1200\text{ lb}(9\text{ ft}) - 1600\text{ lb}(21\text{ ft}) = 0$

$F_{DC} = 1900\text{ lb (C)}$ *Ans.*

$\circlearrowleft +\Sigma M_D = 0;$

$1200\text{ lb}(21\text{ ft}) - 1600\text{ lb}(9\text{ ft}) - F_{HI}(12\text{ ft}) = 0$

$F_{HI} = 900\text{ lb (C)}$ *Ans.*

$\circlearrowleft +\Sigma M_C = 0;\ F_{JI}\cos 45°(12\text{ ft}) + 1200\text{ lb}(21\text{ ft})$

$- 900\text{ lb}(12\text{ ft}) - 1600\text{ lb}(9\text{ ft}) = 0$

$F_{JI} = 0$ *Ans.*

F5–13 $+\uparrow \Sigma F_y = 0;\quad 3P - 60 = 0$

$P = 20\text{ lb}$ *Ans.*

F5–14 $\circlearrowleft +\Sigma M_C = 0;$

$-\left(\frac{4}{5}\right)(F_{AB})(9) + 400(6) + 500(3) = 0$

$$F_{AB} = 541.67 \text{ lb}$$

$$\stackrel{+}{\rightarrow}\Sigma F_x = 0; -C_x + \tfrac{3}{5}(541.67) = 0$$

$$C_x = 325 \text{ lb} \quad \textit{Ans.}$$

$$+\uparrow\Sigma F_y = 0; C_y + \tfrac{4}{5}(541.67) - 400 - 500 = 0$$

$$C_y = 467 \text{ lb} \quad \textit{Ans.}$$

F5–15 $\circlearrowleft +\Sigma M_A = 0; 100 \text{ N}(250 \text{ mm}) - N_B(50 \text{ mm}) = 0$

$$N_B = 500 \text{ N} \quad \textit{Ans.}$$

$$\stackrel{+}{\rightarrow}\Sigma F_x = 0; \quad (500 \text{ N})\sin 45° - A_x = 0$$

$$A_x = 353.55 \text{ N}$$

$$+\uparrow\Sigma F_y = 0; A_y - 100 \text{ N} - (500 \text{ N})\cos 45° = 0$$

$$A_y = 453.55 \text{ N}$$

$$F_A = \sqrt{(353.55 \text{ N})^2 + (453.55 \text{ N})^2}$$

$$= 575 \text{ N} \quad \textit{Ans.}$$

F5–16 $\circlearrowleft +\Sigma M_C = 0;$

$$F_{AB}\cos 45°(1) - F_{AB}\sin 45°(3) + 800 + 400(2) = 0$$

$$F_{AB} = 1131.37 \text{ N}$$

$$\stackrel{+}{\rightarrow}\Sigma F_x = 0; \; -C_x + 1131.37\cos 45° = 0$$

$$C_x = 800 \text{ N} \quad \textit{Ans.}$$

$$+\uparrow\Sigma F_y = 0; \quad -C_y + 1131.37\sin 45° - 400 = 0$$

$$C_y = 400 \text{ N} \quad \textit{Ans.}$$

F5–17 Plate A:

$$+\uparrow\Sigma F_y = 0; \; 2T + N_{AB} - 100 = 0$$

Plate B:

$$+\uparrow\Sigma F_y = 0; \; 2T - N_{AB} - 30 = 0$$

$$T = 32.5 \text{ lb}, N_{AB} = 35 \text{ lb} \quad \textit{Ans.}$$

F5–18 Pulley C:

$$+\uparrow\Sigma F_y = 0; \; T - 2P = 0; T = 2P$$

Beam:

$$+\uparrow\Sigma F_y = 0; \quad 2P + P - 6 = 0$$

$$P = 2 \text{ kN} \quad \textit{Ans.}$$

$$\circlearrowleft +\Sigma M_A = 0; 2(1) - 6(x) = 0$$

$$x = 0.333 \text{ m} \quad \textit{Ans.}$$

Chapter 6

F6–1

$$\bar{x} = \frac{\int_A \bar{x}\,dA}{\int_A dA} = \frac{\frac{1}{2}\int_0^{1\text{ m}} y^{2/3}\,dy}{\int_0^{1\text{ m}} y^{1/3}dy} = 0.4 \text{ m} \quad \textit{Ans.}$$

$$\bar{y} = \frac{\int_A \bar{y}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} y^{4/3}\,dy}{\int_0^{1\text{ m}} y^{1/3}dy} = 0.571 \text{ m} \quad \textit{Ans.}$$

F6–2

$$\bar{x} = \frac{\int_A \bar{x}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} x(x^3\,dx)}{\int_0^{1\text{ m}} x^3\,dx}$$

$$= 0.8 \text{ m} \quad \textit{Ans.}$$

$$\bar{y} = \frac{\int_A \bar{y}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} \frac{1}{2}x^3\left(x^3\,dx\right)}{\int_0^{1\text{ m}} x^3\,dx}$$

$$= 0.286 \text{ m} \quad \textit{Ans.}$$

F6–3

$$\bar{y} = \frac{\int_A \bar{y}\,dA}{\int_A dA} = \frac{\int_0^{2\text{ m}} y\left(2\left(\frac{y^{1/2}}{\sqrt{2}}\right)\right)dy}{\int_0^{2\text{ m}} 2\left(\frac{y^{1/2}}{\sqrt{2}}\right)dy}$$

$$= 1.2 \text{ m} \quad \textit{Ans.}$$

F6–4

$$A = \int_0^{4\text{ m}} \frac{y^2}{4}\,dy = 5.33 \text{ m}^2 \quad \textit{Ans.}$$

$$\bar{x} = \frac{\int_0^{4\text{ m}} \frac{y^2}{8}\left(\frac{y^2}{4}\,dy\right)}{5.333} = 1.2 \text{ m} \quad \textit{Ans.}$$

$$\bar{y} = \frac{\int_0^{4\text{ m}} y\left(\frac{y^2}{4}\,dy\right)}{5.333} = 3 \text{ m} \quad \textit{Ans.}$$

F6–5

$$\bar{y} = \frac{\int_V \bar{y}\,dV}{\int_V dV} = \frac{\int_0^{1\text{ m}} y\left(\frac{\pi}{4}y\,dy\right)}{\int_0^{1\text{ m}} \frac{\pi}{4}y\,dy}$$

$$= 0.667 \text{ m} \quad \textit{Ans.}$$

F6–6

$$\bar{z} = \frac{\int_V \bar{z}\,dV}{\int_V dV} = \frac{\int_0^{2\text{ ft}} z\left[\frac{9\pi}{64}(4-z)^2\,dz\right]}{\int_0^{2\text{ ft}} \frac{9\pi}{64}(4-z)^2\,dz}$$

$$= 0.786 \text{ ft} \quad \textit{Ans.}$$

F6–7

$$\bar{x} = \frac{\Sigma \bar{x}\,L}{\Sigma L}$$

$$= \frac{150(300) + 300(600) + 300(400)}{300 + 600 + 400}$$

$$= 265 \text{ mm} \quad \textit{Ans.}$$

$$\bar{y} = \frac{\Sigma \bar{y}\,L}{\Sigma L}$$

$$= \frac{0(300) + 300(600) + 600(400)}{300 + 600 + 400}$$

$$= 323 \text{ mm} \quad \textit{Ans.}$$

$$\bar{z} = \frac{\Sigma \bar{z}\, L}{\Sigma L}$$
$$= \frac{0(300) + 0(600) + (-200)(400)}{300 + 600 + 400}$$
$$= -61.5 \text{ mm} \quad \textit{Ans.}$$

F6–8
$$\bar{y} = \frac{\Sigma \bar{y}\, A}{\Sigma A} = \frac{150[300(50)] + 325[50(300)]}{300(50) + 50(300)}$$
$$= 237.5 \text{ mm} \quad \textit{Ans.}$$

F6–9
$$\bar{y} = \frac{\Sigma \bar{y}\, A}{\Sigma A} = \frac{100[2(200)(50)] + 225[50(400)]}{2(200)(50) + 50(400)}$$
$$= 162.5 \text{ mm} \quad \textit{Ans.}$$

F6–10
$$\bar{x} = \frac{\Sigma \bar{x}\, A}{\Sigma A} = \frac{0.25[4(0.5)] + 1.75[0.5(2.5)]}{4(0.5) + 0.5(2.5)}$$
$$= 0.827 \text{ in.} \quad \textit{Ans.}$$
$$\bar{y} = \frac{\Sigma \bar{y}\, A}{\Sigma A} = \frac{2[4(0.5)] + 0.25[(0.5)(2.5)]}{4(0.5) + (0.5)(2.5)}$$
$$= 1.33 \text{ in.} \quad \textit{Ans.}$$

F6–11
$$\bar{x} = \frac{\Sigma \bar{x}\, V}{\Sigma V} = \frac{1[2(7)(6)] + 4[4(2)(3)]}{2(7)(6) + 4(2)(3)}$$
$$= 1.67 \text{ ft} \quad \textit{Ans.}$$
$$\bar{y} = \frac{\Sigma \bar{y}\, V}{\Sigma V} = \frac{3.5[2(7)(6)] + 1[4(2)(3)]}{2(7)(6) + 4(2)(3)}$$
$$= 2.94 \text{ ft} \quad \textit{Ans.}$$
$$\bar{z} = \frac{\Sigma \bar{z}\, V}{\Sigma V} = \frac{3[2(7)(6)] + 1.5[4(2)(3)]}{2(7)(6) + 4(2)(3)}$$
$$= 2.67 \text{ ft} \quad \textit{Ans.}$$

F6–12
$$\bar{x} = \frac{\Sigma \bar{x}\, V}{\Sigma V}$$
$$= \frac{0.25[0.5(2.5)(1.8)] + 0.25\left[\frac{1}{2}(1.5)(1.8)(0.5)\right] + \left[\frac{1}{2}(1.5)(1.8)(0.5)\right]}{0.5(2.5)(1.8) + \frac{1}{2}(1.5)(1.8)(0.5) + \frac{1}{2}(1.5)(1.8)(0.5)}$$
$$= 0.391 \text{ m} \quad \textit{Ans.}$$
$$\bar{y} = \frac{\Sigma \bar{y}\, V}{\Sigma V} = \frac{5.00625}{3.6} = 1.39 \text{ m} \quad \textit{Ans.}$$
$$\bar{z} = \frac{\Sigma \bar{z}\, V}{\Sigma V} = \frac{2.835}{3.6} = 0.7875 \text{ m} \quad \textit{Ans.}$$

F6–13
$$+\uparrow F_R = \Sigma F_y;$$
$$-F_R = -6(1.5) - 9(3) - 3(1.5)$$
$$F_R = 40.5 \text{ kN} \downarrow \quad \textit{Ans.}$$
$$\circlearrowleft + (M_R)_A = \Sigma M_A;$$
$$-40.5(d) = 6(1.5)(0.75) - 9(3)(1.5) - 3(1.5)(3.75)$$
$$d = 1.25 \text{ m} \quad \textit{Ans.}$$

F6–14
$$F_R = \tfrac{1}{2}(6)(150) + 8(150) = 1650 \text{ lb} \quad \textit{Ans.}$$
$$\circlearrowleft + M_{A_R} = \Sigma M_A;$$
$$1650d = \left[\tfrac{1}{2}(6)(150)\right](4) + [8(150)](10)$$
$$d = 8.36 \text{ ft} \quad \textit{Ans.}$$

F6–15
$$+\uparrow F_R = \Sigma F_y;$$
$$-F_R = -\tfrac{1}{2}(6)(3) - \tfrac{1}{2}(6)(6)$$
$$F_R = 27 \text{ kN} \downarrow \quad \textit{Ans.}$$
$$\circlearrowleft + (M_R)_A = \Sigma M_A;$$
$$-27(d) = \tfrac{1}{2}(6)(3)(1) - \tfrac{1}{2}(6)(6)(2)$$
$$d = 1 \text{ m} \quad \textit{Ans.}$$

F6–16
$$+\downarrow F_R = \Sigma F_y;$$
$$F_R = \tfrac{1}{2}(50)(6) + 150(6) + 500$$
$$= 1550 \text{ lb} \quad \textit{Ans.}$$
$$\circlearrowleft + M_{A_R} = \Sigma M_A;$$
$$1550d = \left[\tfrac{1}{2}(50)(6)\right](4) + [150(6)](3) + 500(9)$$
$$d = 5.03 \text{ ft} \quad \textit{Ans.}$$

F6–17
$$+\uparrow F_R = \Sigma F_y;$$
$$-F_R = -\tfrac{1}{2}(3)(4.5) - 3(6)$$
$$F_R = 24.75 \text{ kN} \downarrow \quad \textit{Ans.}$$
$$\circlearrowleft + (M_R)_A = \Sigma M_A;$$
$$-24.75(d) = -\tfrac{1}{2}(3)(4.5)(1.5) - 3(6)(3)$$
$$d = 2.59 \text{ m} \quad \textit{Ans.}$$

F6–18
$$F_R = \int w(x)\, dx = \int_0^4 2.5x^3\, dx = 160 \text{ N}$$
$$\circlearrowleft + M_{A_R} = \Sigma M_A;$$
$$x = \frac{\int xw(x)\, dx}{\int w(x)\, dx} = \frac{\int_0^4 2.5x^4\, dx}{160} = 3.20 \text{ m}$$

F6–19
$$I_x = \int_A y^2\, dA = \int_0^{1\text{ m}} y^2\left[\left(1 - y^{3/2}\right)dy\right] = 0.111 \text{ m}^4 \quad \textit{Ans.}$$

F6–20
$$I_x = \int_A y^2\, dA = \int_0^{1\text{ m}} y^2\left(y^{3/2}dy\right) = 0.222 \text{ m}^4 \quad \textit{Ans.}$$

F6–21
$$I_y = \int_A x^2\, dA = \int_0^{1\text{ m}} x^2\left(x^{2/3}\right)dx = 0.273 \text{ m}^4 \quad \textit{Ans.}$$

F6–22

$$I_y = \int_A x^2\,dA = \int_0^{1\text{ m}} x^2\left[(1 - x^{2/3})\,dx\right] = 0.0606\text{ m}^4 \quad \textit{Ans.}$$

F6–23

$$I_x = \left[\tfrac{1}{12}(50)(450^3) + 0\right] + \left[\tfrac{1}{12}(300)(50^3) + 0\right]$$
$$= 383(10^6)\text{ mm}^4 \quad \textit{Ans.}$$
$$I_y = \left[\tfrac{1}{12}(450)(50^3) + 0\right] + 2\left[\tfrac{1}{12}(50)(150^3) + (150)(50)(100)^2\right]$$
$$= 183(10^6)\text{ mm}^4 \quad \textit{Ans.}$$

F6–24

$$I_x = \tfrac{1}{12}(360)(200^3) - \tfrac{1}{12}(300)(140^3)$$
$$= 171(10^6)\text{ mm}^4 \quad \textit{Ans.}$$
$$I_y = \tfrac{1}{12}(200)(360^3) - \tfrac{1}{12}(140)(300^3)$$
$$= 463(10^6)\text{ mm}^4 \quad \textit{Ans.}$$

F6–25

$$I_y = 2\left[\tfrac{1}{12}(50)(200^3) + 0\right] + \left[\tfrac{1}{12}(300)(50^3) + 0\right]$$
$$= 69.8\,(10^6)\text{ mm}^4 \quad \textit{Ans.}$$

F6–26.

$$\bar{y} = \frac{\Sigma \bar{y} A}{\Sigma A} = \frac{15(150)(30) + 105(30)(150)}{150(30) + 30(150)} = 60\text{ mm}$$
$$\bar{I}_{x'} = \Sigma(\bar{I} + A d^2)$$
$$= \left[\tfrac{1}{12}(150)(30)^3 + (150)(30)(60 - 15)^2\right] + \left[\tfrac{1}{12}(30)(150)^3 + 30(150)(105 - 60)^2\right]$$
$$= 27.0\,(10^6)\text{ mm}^4 \quad \textit{Ans.}$$

Chapter 7

F7–1

Entire beam:

$\circlearrowleft + \Sigma M_B = 0;$ $\quad 60 - 10(2) - A_y(2) = 0$ $\quad A_y = 20\text{ kN}$

Left segment:

$\pm\rightarrow \Sigma F_x = 0;$ $\quad N_C = 0$ $\quad \textit{Ans.}$

$+\uparrow \Sigma F_y = 0;$ $\quad 20 - V_C = 0$ $\quad V_C = 20\text{ kN}$ $\quad \textit{Ans.}$

$\circlearrowleft + \Sigma M_C = 0;$ $\quad M_C + 60 - 20(1) = 0$ $\quad M_C = -40\text{ kN}\cdot\text{m}$ $\quad \textit{Ans.}$

F7–2

Entire beam:

$\circlearrowleft + \Sigma M_A = 0;$ $\quad B_y(3) - 100(1.5)(0.75) - 200(1.5)(2.25) = 0$

$B_y = 262.5\text{ N}$

Right segment:

$\pm\rightarrow \Sigma F_x = 0;$ $\quad N_C = 0$ $\quad \textit{Ans.}$

$+\uparrow \Sigma F_y = 0;$ $\quad V_C + 262.5 - 200(1.5) = 0$ $\quad V_C = 37.5\text{ N}$ $\quad \textit{Ans.}$

$\circlearrowleft + \Sigma M_C = 0;$ $\quad 262.5(1.5) - 200(1.5)(0.75) - M_C = 0$

$M_C = 169\text{ N}\cdot\text{m}$ $\quad \textit{Ans.}$

F7–3

Entire beam:

$\pm\rightarrow \Sigma F_x = 0;$ $\quad B_x = 0$

$\circlearrowleft + \Sigma M_A = 0;$ $\quad 20(2)(1) - B_y(4) = 0$ $\quad B_y = 10\text{ kN}$

Right segment:

$\pm\rightarrow \Sigma F_x = 0;$ $\quad N_C = 0$ $\quad \textit{Ans.}$

$+\uparrow \Sigma F_y = 0;$ $\quad V_C - 10 = 0$ $\quad V_C = 10\text{ kN}$ $\quad \textit{Ans.}$

$\circlearrowleft + \Sigma M_C = 0;$ $\quad -M_C - 10(2) = 0$ $\quad M_C = -20\text{ kN}\cdot\text{m}$ $\quad \textit{Ans.}$

F7–4

Entire beam:

$\circlearrowleft + \Sigma M_B = 0;$ $\quad \frac{1}{2}(10)(3)(2) + 10(3)(4.5) - A_y(6) = 0$ $\quad A_y = 27.5\text{ kN}$

Left segment:

$\pm\rightarrow \Sigma F_x = 0;$ $N_C = 0$ *Ans.*

$+\uparrow \Sigma F_y = 0;$ $27.5 - 10(3) - V_C = 0$ $V_C = -2.5\ \text{kN}$ *Ans.*

$\circlearrowleft +\Sigma M_C = 0;$ $M_C + 10(3)(1.5) - 27.5(3) = 0$ $M_C = 37.5\ \text{kN}\cdot\text{m}$ *Ans.*

F7–5 Entire beam:

$\pm\rightarrow \Sigma F_x = 0;$ $A_x = 0$

$\circlearrowleft +\Sigma M_B = 0;$ $300(6)(3) - \frac{1}{2}(300)(3)(1) - A_y(6) = 0$ $A_y = 825\ \text{lb}$

Left segment:

$\pm\rightarrow \Sigma F_x = 0;$ $N_C = 0$ *Ans.*

$+\uparrow \Sigma F_y = 0;$ $825 - 300(3) - V_C = 0$ $V_C = -75\ \text{lb}$ *Ans.*

$\circlearrowleft +\Sigma M_C = 0;$ $M_C + 300(3)(1.5) - 825(3) = 0$ $M_C = 1125\ \text{lb}\cdot\text{ft}$ *Ans.*

F7–6 Entire beam:

$\circlearrowleft +\Sigma M_A = 0;$ $F_{BD}\left(\frac{3}{5}\right)(4) - 5(6)(3) = 0$ $F_{BD} = 37.5\ \text{kN}$

$\pm\rightarrow \Sigma F_x = 0;$ $37.5\left(\frac{4}{5}\right) - A_x = 0$ $A_x = 30\ \text{kN}$

$+\uparrow \Sigma F_y = 0;$ $A_y + 37.5\left(\frac{3}{5}\right) - 5(6) = 0$ $A_y = 7.5\ \text{kN}$

Left segment:

$\pm\rightarrow \Sigma F_x = 0;$ $N_C - 30 = 0$ $N_C = 30\ \text{kN}$ *Ans.*

$+\uparrow \Sigma F_y = 0;$ $7.5 - 5(2) - V_C = 0$ $V_C = -2.5\ \text{kN}$ *Ans.*

$\circlearrowleft +\Sigma M_C = 0;$ $M_C + 5(2)(1) - 7.5(2) = 0$ $M_C = 5\ \text{kN}\cdot\text{m}$ *Ans.*

F7–7 Beam:

$\Sigma M_A = 0;\ T_{CD} = 2w$

$\Sigma F_y = 0;\ T_{CD} = w$

Rod AB:

$\sigma = \frac{P}{A};\ 300(10^3) = \frac{w}{10};$

$w = 3\ \text{N/m}$

Rod CD:

$\sigma = \frac{P}{A};\ 300(10^3) = \frac{2w}{15};$

$w = 2.25\ \text{N/m}$ *Ans.*

F7–8 $A = \pi(0.1^2 - 0.08^2) = 3.6(10^{-3})\pi\ \text{m}^2$

$\sigma_{\text{avg}} = \frac{P}{A} = \frac{300(10^3)}{3.6(10^{-3})\pi} = 26.5\ \text{MPa}$ *Ans.*

F7–9 $A = 3[4(1)] = 12\ \text{in}^2$

$\sigma_{\text{avg}} = \frac{P}{A} = \frac{15}{12} = 1.25\ \text{ksi}$ *Ans.*

F7–10 Consider the cross section to be a rectangle and two triangles.

$$\bar{y} = \frac{\Sigma \tilde{y}A}{\Sigma A} = \frac{0.15[(0.3)(0.12)] + (0.1)\left[\frac{1}{2}(0.16)(0.3)\right]}{0.3(0.12) + \frac{1}{2}(0.16)(0.3)}$$

$= 0.13\ \text{m} = 130\ \text{mm}$ *Ans.*

$\sigma_{\text{avg}} = \frac{P}{A} = \frac{600(10^3)}{0.06} = 10\ \text{MPa}$ *Ans.*

F7–11

$A_A = A_C = \frac{\pi}{4}(0.5^2) = 0.0625\pi\ \text{in}^2,\ A_B = \frac{\pi}{4}(1^2) = 0.25\pi\ \text{in}^2$

$\sigma_A = \frac{N_A}{A_A} = \frac{3}{0.0625\pi} = 15.3\ \text{ksi (T)}$ *Ans.*

$$\sigma_B = \frac{N_B}{A_B} = \frac{-6}{0.25\pi} = -7.64 \text{ ksi} = 7.64 \text{ ksi (C)} \quad \textit{Ans.}$$

$$\sigma_C = \frac{N_C}{A_C} = \frac{2}{0.0625\pi} = 10.2 \text{ ksi (T)} \quad \textit{Ans.}$$

F7–12

$F_{AD} = 50(9.81) \text{ N} = 490.5 \text{ N}$

$+\uparrow \Sigma F_y = 0; \quad F_{AC}\left(\frac{3}{5}\right) - 490.5 = 0 \quad F_{AC} = 817.5 \text{ N}$

$\pm\rightarrow \Sigma F_x = 0; \quad 817.5\left(\frac{4}{5}\right) - F_{AB} = 0 \quad F_{AB} = 654 \text{ N}$

$A_{AB} = \frac{\pi}{4}(0.008^2) = 16(10^{-6})\pi \text{ m}^2$

$(\sigma_{AB})_{\text{avg}} = \frac{F_{AB}}{A_{AB}} = \frac{654}{16(10^{-6})\pi} = 13.0 \text{ MPa} \quad \textit{Ans.}$

F7–13 Ring C:

$+\uparrow \Sigma F_y = 0; \quad 2F\cos 60^\circ - 200(9.81) = 0 \quad F = 1962 \text{ N}$

$(\sigma_{\text{allow}})_{\text{avg}} = \frac{F}{A}; \quad 150(10^6) = \frac{1962}{\frac{\pi}{4}d^2}$

$d = 0.00408 \text{ m} = 4.08 \text{ mm}$

Use $d = 5$ mm *Ans.*

F7–14 Entire frame:

$\Sigma F_y = 0; \; A_y = 600 \text{ lb}$

$\Sigma M_B = 0; \; A_x = 800 \text{ lb}$

$F_A = \sqrt{(600)^2 + (800)^2} = 1000 \text{ lb}$

$(\tau_A)_{\text{avg}} = \frac{F_A/2}{A} = \frac{1000/2}{\frac{\pi}{4}(0.25)^2} = 10.2 \text{ ksi} \quad \textit{Ans.}$

F7–15 Double shear:

$\Sigma F_x = 0; \quad 4V - 10 = 0 \quad V = 2.5 \text{ kip}$

$A = \frac{\pi}{4}\left(\frac{3}{4}\right)^2 = 0.140625\pi \text{ in}^2$

$\tau_{\text{avg}} = \frac{V}{A} = \frac{2.5}{0.140625\pi} = 5.66 \text{ ksi} \quad \textit{Ans.}$

F7–16 Single shear:

$\Sigma F_x = 0; \quad P - 3V = 0 \quad V = \frac{P}{3}$

$A = \frac{\pi}{4}(0.004^2) = 4(10^{-6})\pi \text{ m}^2$

$(\tau_{\text{avg}})_{\text{allow}} = \frac{V}{A}; \quad 60(10^6) = \frac{\frac{P}{3}}{4(10^{-6})\pi}$

$P = 2.262(10^3) \text{ N} = 2.26 \text{ kN} \quad \textit{Ans.}$

F7–17 $\pm\rightarrow \Sigma F_x = 0; \quad V - P\cos 60^\circ = 0 \quad V = 0.5P$

$A = \left(\frac{0.05}{\sin 60^\circ}\right)(0.025) = 1.4434(10^{-3}) \text{ m}^2$

$(\tau_{\text{avg}})_{\text{allow}} = \frac{V}{A}, \quad 600(10^3) = \frac{0.5P}{1.4434(10^{-3})}$

$P = 1.732(10^3) \text{ N} = 1.73 \text{ kN} \quad \textit{Ans.}$

F7–18 The resultant force on the pin is

$F = \sqrt{30^2 + 40^2} = 50 \text{ kN}.$

Here we have double shear:

$V = \frac{F}{2} = \frac{50}{2} = 25 \text{ kN}$

$A = \frac{\pi}{4}(0.03^2) = 0.225(10^{-3})\pi \text{ m}^2$

$\tau_{\text{avg}} = \frac{V}{A} = \frac{25(10^3)}{0.225(10^{-3})\pi} = 35.4 \text{ MPa} \quad \textit{Ans.}$

F7–19 $\pm\rightarrow \Sigma F_x = 0; \quad 30 - N = 0 \quad N = 30 \text{ kN}$

$\sigma_{\text{allow}} = \frac{\sigma_Y}{\text{F.S.}} = \frac{250}{1.5} = 166.67 \text{ MPa}$

$\sigma_{\text{allow}} = \frac{N}{A}; \quad 166.67(10^6) = \frac{30(10^3)}{\frac{\pi}{4}d^2}$

$d = 15.14 \text{ mm}$

Use $d = 16$ mm *Ans.*

F7–20.

$\pm\rightarrow \Sigma F_x = 0; \quad N_{AB} - 30 = 0 \quad N_{AB} = 30 \text{ kip}$

$\pm\rightarrow \Sigma F_x = 0; \quad N_{BC} - 15 - 15 - 30 = 0 \quad N_{BC} = 60 \text{ kip}$

$\sigma_{\text{allow}} = \frac{\sigma_Y}{\text{F.S.}} = \frac{50}{1.5} = 33.33 \text{ ksi}$

Segment AB:

$\sigma_{\text{allow}} = \frac{N_{AB}}{A_{AB}}; \quad 33.33 = \frac{30}{h_1(0.5)}$

$h_1 = 1.8 \text{ in.}$

Segment BC:

$\sigma_{\text{allow}} = \frac{N_{BC}}{A_{BC}}; \quad 33.33 = \frac{60}{h_2(0.5)}$

$h_2 = 3.6 \text{ in.}$

Use $h_1 = 1\frac{7}{8}$ in. and $h_2 = 3\frac{5}{8}$ in. *Ans.*

F7–21 $N = P$

$$\sigma_{\text{allow}} = \frac{\sigma_Y}{\text{F.S.}} = \frac{250}{2} = 125 \text{ MPa}$$

$$A_r = \frac{\pi}{4}(0.04^2) = 1.2566(10^{-3}) \text{ m}^2$$

$$A_{a-a} = 2(0.06 - 0.03)(0.05) = 3(10^{-3}) \text{ m}^2$$

The rod will fail first.

$$\sigma_{\text{allow}} = \frac{N}{A_r}; \quad 125(10^6) = \frac{P}{1.2566(10^{-3})}$$

$$P = 157.08(10^3) \text{ N} = 157 \text{ kN} \qquad \textit{Ans.}$$

F7–22 $\stackrel{+}{\rightarrow} \Sigma F_x = 0; \quad 80 - 2V = 0 \quad V = 40 \text{ kN}$

$$\tau_{\text{allow}} = \frac{\tau_{\text{fail}}}{\text{F.S.}} = \frac{100}{2.5} = 40 \text{ MPa}$$

$$\tau_{\text{allow}} = \frac{V}{A}; \quad 40(10^6) = \frac{40(10^3)}{\frac{\pi}{4}d^2}$$

$$d = 0.03568 \text{ m} = 35.68 \text{ mm}$$

Use $d = 36$ mm *Ans.*

F7–23 $V = P$

$$\tau_{\text{allow}} = \frac{\tau_{\text{fail}}}{\text{F.S.}} = \frac{120}{2.5} = 48 \text{ MPa}$$

Area of shear plane for bolt head and plate:

$$A_b = \pi dt = \pi(0.04)(0.075) = 0.003\pi \text{ m}^2$$

$$A_p = \pi dt = \pi(0.08)(0.03) = 0.0024\pi \text{ m}^2$$

Since the area of shear plane for the plate is smaller,

$$\tau_{\text{allow}} = \frac{V}{A_p}; \quad 48(10^6) = \frac{P}{0.0024\pi}$$

$$P = 361.91(10^3) \text{ N} = 362 \text{ kN} \qquad \textit{Ans.}$$

F7–24

$$\circlearrowleft + \Sigma M_B = 0; \quad \frac{1}{2}(300)(9)(6) - 6V(9) = 0 \quad V = 150 \text{ lb}$$

$$\tau_{\text{allow}} = \frac{\tau_{\text{fail}}}{\text{F.S.}} = \frac{16}{2} = 8 \text{ ksi}$$

$$\tau_{\text{allow}} = \frac{V}{A}; \quad 8(10^3) = \frac{150}{\frac{\pi}{4}d^2}$$

$$d = 0.1545 \text{ in.}$$

Use $d = \frac{3}{16}$ in. *Ans.*

F7–25 $\frac{\delta_C}{600} = \frac{0.2}{400}; \quad \delta_C = 0.3 \text{ mm}$

$$\epsilon_{CD} = \frac{\delta_C}{L_{CD}} = \frac{0.3}{300} = 0.001 \text{ mm/mm} \qquad \textit{Ans.}$$

F7–26 $\theta = \left(\frac{0.02^\circ}{180^\circ}\right)\pi \text{ rad} = 0.3491(10^{-3}) \text{ rad}$

$$\delta_B = \theta L_{AB} = 0.3491(10^{-3})(600) = 0.2094 \text{ mm}$$

$$\delta_C = \theta L_{AC} = 0.3491(10^{-3})(1200) = 0.4189 \text{ mm}$$

$$\epsilon_{BD} = \frac{\delta_B}{L_{BD}} = \frac{0.2094}{400} = 0.524(10^{-3}) \text{ mm/mm} \qquad \textit{Ans.}$$

$$\epsilon_{CE} = \frac{\delta_C}{L_{CE}} = \frac{0.4189}{600} = 0.698(10^{-3}) \text{ mm/mm} \qquad \textit{Ans.}$$

F7–27

$$\alpha = \frac{2}{400} = 0.005 \text{ rad} \qquad \beta = \frac{4}{300} = 0.01333 \text{ rad}$$

$$(\gamma_A)_{xy} = \frac{\pi}{2} - \theta$$

$$= \frac{\pi}{2} - \left(\frac{\pi}{2} - \alpha + \beta\right)$$

$$= \alpha - \beta$$

$$= 0.005 - 0.01333$$

$$= -0.00833 \text{ rad} \qquad \textit{Ans.}$$

F7–28

$$L_{BC} = \sqrt{300^2 + 400^2} = 500 \text{ mm}$$

$$L_{B'C} = \sqrt{(300 - 3)^2 + (400 + 5)^2} = 502.2290 \text{ mm}$$

$$\alpha = \frac{3}{405} = 0.007407 \text{ rad}$$

$$(\epsilon_{BC})_{\text{avg}} = \frac{L_{B'C} - L_{BC}}{L_{BC}} = \frac{502.2290 - 500}{500}$$

$$= 0.00446 \text{ mm/mm} \qquad \textit{Ans.}$$

$$(\gamma_A)_{xy} = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \left(\frac{\pi}{2} + \alpha\right) = -\alpha = -0.00741 \text{ rad} \qquad \textit{Ans.}$$

F7–29

$$L_{AC} = \sqrt{L_{CD}{}^2 + L_{AD}{}^2} = \sqrt{300^2 + 300^2} = 424.2641 \text{ mm}$$

$$L_{A'C'} = \sqrt{L_{C'D'}{}^2 + L_{A'D'}{}^2} = \sqrt{306^2 + 296^2} = 425.7370 \text{ mm}$$

$$\frac{\theta}{2} = \tan^{-1}\left(\frac{L_{C'D'}}{L_{A'D'}}\right); \quad \theta = 2\tan^{-1}\left(\frac{306}{296}\right) = 1.6040 \text{ rad}$$

$$(\epsilon_{AC})_{\text{avg}} = \frac{L_{A'C'} - L_{AC}}{L_{AC}} = \frac{425.7370 - 424.2641}{424.2641}$$

$= 0.00347$ mm/mm *Ans.*

$(\gamma_E)_{xy} = \frac{\pi}{2} - \theta = \frac{\pi}{2} - 1.6040 = -0.0332$ rad *Ans.*

Chapter 8

F8–1 Material has uniform properties throughout. *Ans.*

F8–2 Proportional limit is A. *Ans.*

Ultimate stress is D. *Ans.*

F8–3 The initial slope of the diagram. *Ans.*

F8–4 True. *Ans.*

F8–5 False. Use the *original* cross-sectional area and length. *Ans.*

F8–6 False. It will normally decrease. *Ans.*

F8–7
$$\epsilon = \frac{\sigma}{E} = \frac{P}{AE}$$
$$\delta = \epsilon L = \frac{PL}{AE} = \frac{100(10^3)(0.100)}{\frac{\pi}{4}(0.015)^2\,200(10^9)}$$
$= 0.283$ mm *Ans.*

F8–8
$$\epsilon = \frac{\sigma}{E} = \frac{P}{AE}$$
$$\delta = \epsilon L = \frac{PL}{AE};$$
$$0.003 = \frac{(10\,000)(8)}{12E}$$
$E = 2.22(10^6)$ psi *Ans.*

F8–9
$$\epsilon = \frac{\sigma}{E} = \frac{P}{AE}$$
$$\delta = \epsilon L = \frac{PL}{AE} = \frac{6(10^3)4}{\frac{\pi}{4}(0.01)^2 100(10^9)}$$
$= 3.06$ mm *Ans.*

F8–10
$$\sigma = \frac{P}{A} = \frac{100(10^3)}{\frac{\pi}{4}(0.02^2)} = 318.31 \text{ MPa}$$
Since $\sigma < \sigma_Y = 450$ MPa, Hooke's Law is applicable.
$$E = \frac{\sigma_Y}{\epsilon_Y} = \frac{450(10^6)}{0.00225} = 200 \text{ GPa}$$
$$\epsilon = \frac{\sigma}{E} = \frac{318.31(10^6)}{200(10^9)} = 0.001592 \text{ mm/mm}$$
$$\delta = \epsilon L = 0.001592(50) = 0.0796 \text{ mm} \quad \textit{Ans.}$$

F8–11
$$\sigma = \frac{P}{A} = \frac{150(10^3)}{\frac{\pi}{4}(0.02^2)} = 477.46 \text{ MPa}$$
Since $\sigma > \sigma_Y = 450$ MPa, Hooke's Law is not applicable. From the geometry of the stress-strain diagram,
$$\frac{\epsilon - 0.00225}{0.03 - 0.00225} = \frac{477.46 - 450}{500 - 450}$$
$$\epsilon = 0.017493$$
When the load is removed, the strain recovers along a line parallel to the original elastic line.
$$\text{Here } E = \frac{\sigma_Y}{\epsilon_Y} = \frac{450(10^6)}{0.00225} = 200 \text{ GPa}.$$
The elastic recovery is
$$\epsilon_r = \frac{\sigma}{E} = \frac{477.46(10^6)}{200(10^9)} = 0.002387 \text{ mm/mm}$$
$$\epsilon_p = \epsilon - \epsilon_r = 0.017493 - 0.002387$$
$$= 0.01511 \text{ mm/mm}$$
$$\delta_p = \epsilon_p L = 0.01511(50) = 0.755 \text{ mm} \quad \textit{Ans.}$$

F8–12
$$\epsilon_{BC} = \frac{\delta_{BC}}{L_{BC}} = \frac{0.2}{300} = 0.6667(10^{-3}) \text{ mm/mm}$$
$$\sigma_{BC} = E\epsilon_{BC} = 200(10^9)[0.6667(10^{-3})]$$
$$= 133.33 \text{ MPa}$$
Since $\sigma_{BC} < \sigma_Y = 250$ MPa, Hooke's Law is valid.
$$\sigma_{BC} = \frac{F_{BC}}{A_{BC}}, \qquad 133.33(10^6) = \frac{F_{BC}}{\frac{\pi}{4}(0.003^2)}$$
$$F_{BC} = 942.48 \text{ N}$$
$$\circlearrowleft + \Sigma M_A = 0; \qquad 942.48(0.4) - P(0.6) = 0$$
$$P = 628.31 \text{ N} = 628 \text{ N} \quad \textit{Ans.}$$

F8–13
$$\sigma = \frac{P}{A} = \frac{10(10^3)}{\frac{\pi}{4}(0.015)^2} = 56.59 \text{ MPa}$$
$$\epsilon_{\text{long}} = \frac{\sigma}{E} = \frac{56.59(10^6)}{70(10^9)} = 0.808(10^{-3})$$
$$\epsilon_{\text{lat}} = -\nu\epsilon_{\text{long}} = -0.35(0.808(10^{-3}))$$
$$= -0.283(10^{-3})$$
$$\delta d = (-0.283(10^{-3}))(15 \text{ mm}) = -4.24(10^{-3}) \text{ mm}$$
Ans.

F8–14
$$\sigma = \frac{P}{A} = \frac{50(10^3)}{\frac{\pi}{4}(0.02^2)} = 159.15 \text{ MPa}$$
$$\epsilon_a = \frac{\delta}{L} = \frac{1.40}{600} = 0.002333 \text{ mm/mm}$$
$$E = \frac{\sigma}{\epsilon_a} = \frac{159.15(10^6)}{0.002333} = 68.2 \text{ GPa} \quad \textit{Ans.}$$
$$\epsilon_e = \frac{d' - d}{d} = \frac{19.9837 - 20}{20} = -0.815(10^{-3}) \text{ mm/mm}$$

$$\nu = -\frac{\epsilon_e}{\epsilon_a} = -\frac{-0.815(10^{-3})}{0.002333} = 0.3493 = 0.349$$

$$G = \frac{E}{2(1+\nu)} = \frac{68.21}{2(1+0.3493)} = 25.3\ \text{GPa} \quad \textit{Ans.}$$

F8–15 $\alpha = \dfrac{0.5}{150} = 0.003333\ \text{rad}$

$$\gamma = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \left(\frac{\pi}{2} - \alpha\right)$$
$$= \alpha = 0.003333\ \text{rad}$$
$$\tau = G\gamma = [26(10^9)](0.003333) = 86.67\ \text{MPa}$$
$$\tau = \frac{V}{A}; \qquad 86.67(10^6) = \frac{P}{0.15(0.02)}$$
$$P = 260\ \text{kN} \quad \textit{Ans.}$$

F8–16 $\alpha = \dfrac{3}{150} = 0.02\ \text{rad}$

$$\gamma = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \left(\frac{\pi}{2} - \alpha\right) = \alpha = 0.02\ \text{rad}$$

When P is removed, the shear strain recovers along a line parallel to the original elastic line.

$$\gamma_r = \gamma_Y = 0.005\ \text{rad}$$
$$\gamma_p = \gamma - \gamma_r = 0.02 - 0.005 = 0.015\ \text{rad} \quad \textit{Ans.}$$

Chapter 9

F9–1 $A = \dfrac{\pi}{4}(0.02^2) = 0.1(10^{-3})\pi\ \text{m}^2$

$$\delta_C = \frac{1}{AE}\{40(10^3)(400) + [-60(10^3)(600)]\}$$
$$= \frac{-20(10^6)\ \text{N}\cdot\text{mm}}{AE}$$
$$= -0.318\ \text{mm} \quad \textit{Ans.}$$

F9–2 $A_{AB} = A_{CD} = \dfrac{\pi}{4}(0.02^2) = 0.1(10^{-3})\pi\ \text{m}^2$

$$A_{BC} = \frac{\pi}{4}(0.04^2 - 0.03^2) = 0.175(10^{-3})\pi\ \text{m}^2$$
$$\delta_{D/A} = \frac{[-10(10^3)](400)}{[0.1(10^{-3})\pi][68.9(10^9)]} + \frac{[10(10^3)](400)}{[0.175(10^{-3})\pi][68.9(10^9)]} + \frac{[-20(10^3)](400)}{[0.1(10^{-3})\pi][68.9(10^9)]}$$
$$= -0.449\ \text{mm} \quad \textit{Ans.}$$

F9–3 $A = \dfrac{\pi}{4}(0.03^2) = 0.225(10^{-3})\pi\ \text{m}^2$

$$\delta_C = \frac{1}{0.225(10^{-3})\pi[200(10^9)]}\left\{\left[-90(10^3) - 2\left(\frac{4}{5}\right)30(10^3)\right](0.4) + [-90(10^3)(0.6)]\right\}$$
$$= -0.772(10^{-3})\ \text{m} = -0.772\ \text{mm} \quad \textit{Ans.}$$

F9–4 $\delta_{A/B} = \dfrac{PL}{AE} = \dfrac{[60(10^3)](0.8)}{[0.1(10^{-3})\pi][200(10^9)]}$

$$= 0.7639(10^{-3})\ \text{m} \downarrow$$
$$\delta_B = \frac{F_{sp}}{k} = \frac{60(10^3)}{50(10^6)} = 1.2(10^{-3})\ \text{m} \downarrow$$
$$+\downarrow \delta_A = \delta_B + \delta_{A/B}$$
$$\delta_A = 1.2(10^{-3}) + 0.7639(10^{-3})$$
$$= 1.9639(10^{-3})\ \text{m} = 1.96\ \text{mm} \downarrow \quad \textit{Ans.}$$

F9–5 $A = \dfrac{\pi}{4}(0.02^2) = 0.1(10^{-3})\pi\ \text{m}^2$

Internal load $P(x) = 30(10^3)x$

$$\delta_A = \int \frac{P(x)dx}{AE}$$
$$= \frac{1}{[0.1(10^{-3})\pi][73.1(10^9)]}\int_0^{0.9\ \text{m}} 30(10^3)x\,dx$$
$$= 0.529(10^{-3})\ \text{m} = 0.529\ \text{mm} \quad \textit{Ans.}$$

F9–6 Distributed load $P(x) = \dfrac{45(10^3)}{0.9}x = 50(10^3)x\ \text{N/m}$

Internal load $P(x) = \frac{1}{2}(50(10^3))x(x) = 25(10^3)x^2$

$$\delta_A = \int_0^L \frac{P(x)dx}{AE}$$
$$= \frac{1}{[0.1(10^{-3})\pi][73.1(10^9)]}\int_0^{0.9\ \text{m}} [25(10^3)x^2]dx$$
$$= 0.265\ \text{mm} \quad \textit{Ans.}$$

Chapter 10

F10–1 $J = \dfrac{\pi}{2}(0.04^4) = 1.28(10^{-6})\pi\ \text{m}^4$

$$\tau_A = \tau_{\max} = \frac{T_C}{J} = \frac{5(10^3)(0.04)}{1.28(10^{-6})\pi} = 49.7\ \text{MPa} \quad \textit{Ans.}$$
$$\tau_B = \frac{T\rho_B}{J} = \frac{5(10^3)(0.03)}{1.28(10^{-6})\pi} = 37.3\ \text{MPa} \quad \textit{Ans.}$$

F10–2 $J = \dfrac{\pi}{2}(0.06^4 - 0.04^4) = 5.2(10^{-6})\pi\ \text{m}^4$

$$\tau_A = \tau_{\max} = \frac{Tc}{J} = \frac{10(10^3)(0.06)}{5.2(10^{-6})\pi} = 36.7\ \text{MPa} \quad \textit{Ans.}$$
$$\tau_A = \frac{T\rho_A}{J} = \frac{10(10^3)(0.04)}{5.2(10^{-6})\pi} = 24.5\ \text{MPa} \quad \textit{Ans.}$$

F10–3 $J_{AB} = \frac{\pi}{2}(0.04^4 - 0.03^4) = 0.875(10^{-6})\pi \text{ m}^4$

$J_{BC} = \frac{\pi}{2}(0.04^4) = 1.28(10^{-6})\pi \text{ m}^4$

$(\tau_{AB})_{\max} = \frac{T_{AB}c_{AB}}{J_{AB}} = \frac{[2(10^3)](0.04)}{0.875(10^{-6})\pi} = 29.1 \text{ MPa}$

$(\tau_{BC})_{\max} = \frac{T_{BC}c_{BC}}{J_{BC}} = \frac{[6(10^3)](0.04)}{1.28(10^{-6})\pi}$

$= 59.7 \text{ MPa}$ *Ans.*

F10–4 $T_{AB} = 0, T_{BC} = 600 \text{ N} \cdot \text{m}, T_{CD} = 0$

$J = \frac{\pi}{2}(0.02^4) = 80(10^{-9})\pi \text{ m}^4$

$\tau_{\max} = \frac{Tc}{J} = \frac{600(0.02)}{80(10^{-9})\pi} = 47.7 \text{ MPa}$ *Ans.*

F10–5 $J_{BC} = \frac{\pi}{2}(0.04^4 - 0.03^4) = 0.875(10^{-6})\pi \text{ m}^4$

$(\tau_{BC})_{\max} = \frac{T_{BC}c_{BC}}{J_{BC}} = \frac{2100(0.04)}{0.875(10^{-6})\pi}$

$= 30.6 \text{ MPa}$ *Ans.*

F10–6 $t = 5(10^3) \text{ N} \cdot \text{m/m}$

Internal torque is $T = 5(10^3)(0.8) = 4000 \text{ N} \cdot \text{m}$

$J = \frac{\pi}{2}(0.04^4) = 1.28(10^{-6})\pi \text{ m}^4$

$\tau_A = \frac{T_A c}{J} = \frac{4000(0.04)}{1.28(10^{-6})\pi} = 39.8 \text{ MPa}$ *Ans.*

F10–7 $J = \frac{\pi}{2}(0.03^4) = 0.405(10^{-6})\pi \text{ m}^4$

$\phi_{A/C} = \frac{1}{[0.405(10^{-6})\pi][75(10^9)]}\{[-2(10^3)](0.6)$

$+ 1(10^3)(0.4)\}$

$= -0.00838 \text{ rad} = -0.480°$ *Ans.*

F10–8 $J = \frac{\pi}{2}(0.02^4) = 80(10^{-9})\pi \text{ m}^4$

$\phi_{B/A} = \frac{600(0.45)}{[80(10^{-9})\pi][75(10^9)]}$

$= 0.01432 \text{ rad} = 0.821°$ *Ans.*

F10–9 $J = \frac{\pi}{2}(0.04^4 - 0.03^4) = 0.875(10^{-6})\pi \text{ m}^4$

$\phi_{A/B} = \frac{T_{AB}L_{AB}}{JG} = \frac{3(10^3)(0.9)}{[0.875(10^{-6})\pi][26(10^9)]}$

$= 0.03778 \text{ rad}$

$\phi_B = \frac{T_B}{k} = \frac{3(10^3)}{90(10^3)} = 0.03333 \text{ rad}$

$\phi_A = \phi_B + \phi_{A/B}$

$= 0.03333 + 0.03778$

$= 0.07111 \text{ rad} = 4.07°$ *Ans.*

F10–10 $J = \frac{\pi}{2}(0.02^4) = 80(10^{-9})\pi \text{ m}^4$

$\phi_{B/A} = \frac{0.2}{[80(10^{-9})\pi][75(10^9)]}[600 + (-300)$

$+ 200 + 500]$

$= 0.01061 \text{ rad} = 0.608°$ *Ans.*

F10–11 $J = \frac{\pi}{2}(0.04^4) = 1.28(10^{-6})\pi \text{ m}^4$

$t = 5(10^3) \text{ N} \cdot \text{m/m}$

Internal torque is $5(10^3)x \text{ N} \cdot \text{m}$

$\phi_{A/B} = \int_0^L \frac{T(x)dx}{JG}$

$= \frac{1}{[1.28(10^{-6})\pi][75(10^9)]}\int_0^{0.8\text{ m}} 5(10^3)x dx$

$= 0.00531 \text{ rad} = 0.304°$ *Ans.*

F10–12 $J = \frac{\pi}{2}(0.04^4) = 1.28(10^{-6})\pi \text{ m}^4$

Distributed torque is $t = \frac{15(10^3)}{0.6}(x)$

$= 25(10^3)x \text{ N} \cdot \text{m/m}$

Internal torque is $T(x) = \frac{1}{2}(25x)(x)$

$= 12.5(10^3)x^2 \text{ N} \cdot \text{m}$

$\phi_{A/C} = \int_0^L \frac{T(x)dx}{JG} + \frac{T_{BC}L_{BC}}{JG}$

$= \frac{1}{[1.28(10^{-6})\pi][75(10^9)]}\left[\int_0^{0.6\text{ m}} 12.5(10^3)x^2 dx + 4500(0.4)\right]$

$= 0.008952 \text{ rad} = 0.513°$ *Ans.*

Chapter 11

F11–1 $+\uparrow \Sigma F_y = 0; \quad -V - 9 = 0 \quad V = -9 \text{ kN}$ *Ans.*

$\zeta + \Sigma M_o = 0; \quad M + 9x = 0 \quad M = \{-9x\} \text{ kN} \cdot \text{m}$ *Ans.*

F11–2 $+\uparrow \Sigma F_y = 0; \; -V - 2x = 0 \; V = \{-2x\} \text{ kip}$ *Ans.*

$$\zeta+\Sigma M_O = 0;\ M + 2x\left(\frac{x}{2}\right) - 18 = 0$$
$$M = \{18 - x^2\}\ \text{kip}\cdot\text{ft} \qquad \textit{Ans.}$$

F11–3
$$+\uparrow\Sigma F_y = 0;\quad -V - \frac{1}{2}(4x)(x) = 0$$
$$V = \{-2x^2\}\ \text{kN} \qquad \textit{Ans.}$$
$$\Sigma M_O = 0;\ M + \left[\frac{1}{2}(4x)(x)\right]\left(\frac{x}{3}\right) = 0$$
$$M = \left\{-\frac{2}{3}x^3\right\}\ \text{kN}\cdot\text{m} \qquad \textit{Ans.}$$

F11–4
$0 \le x < 1.5$ m
$+\uparrow\Sigma F_y = 0;\quad V = 0$ *Ans.*
$\zeta+\Sigma M_O = 0;\quad M - 4 = 0 \quad M = 4\ \text{kN}\cdot\text{m}$ *Ans.*
$1.5\ \text{m} < x \le 3$ m
$+\uparrow\Sigma F_y = 0;\quad -V - 9 = 0 \quad V = -9\ \text{kN}$ *Ans.*
$\zeta+\Sigma M_O = 0;\quad M + 9(x - 1.5) - 4 = 0$
$M = \{17.5 - 9x\}\ \text{kN}\cdot\text{m}$ *Ans.*

F11–5
$\zeta+\Sigma M_B = 0;\quad A_y(6) - 30 = 0 \quad A_y = 5\ \text{kN}$
$+\uparrow\Sigma F_y = 0;\quad -V - 5 = 0 \quad V = -5\ \text{kN}$
Ans.
$\zeta+\Sigma M_O = 0;\quad M + 5x = 0 \quad M = \{-5x\}\ \text{kN}\cdot\text{m}$
Ans.

F11–6
$\zeta+\Sigma M_B = 0;\quad A_y(6) + 20 - 50 = 0 \quad A_y = 5\ \text{kN}$
$+\uparrow\Sigma F_y = 0;\quad -V - 5 = 0 \quad V = -5\ \text{kN}$
Ans.
$\zeta+\Sigma M_O = 0;\quad M + 5x - 50 = 0$
$M = \{50 - 5x\}\ \text{kN}\cdot\text{m}$ *Ans.*

F11–7 *Shear diagram.* $V = -4, x = 0$. Zero slope to $x = 6$.
Moment diagram. $M = 0, x = 0$. Constant negative slope to $M = -16, x = 4^-, M = 8, x = 4^+$.
Constant negative slope to $M = 0, x = 6$.

F11–8 *Shear diagram.* $V = -6, x = 0$. Zero slope to $x = 3$.
Moment diagram. $M = 0, x = 0$. Constant negative slope to $M = -9, x = 1.5^-, M = -21, x = 1.5^+$.
Constant negative slope to $M = -30, x = 3$.

F11–9 *Shear diagram.* $V = 0, x = 0$. Zero slope to $x = 1.5^-$. $V = 4, x = 1.5^+$. Zero slope to $x = 4.5^-$, $V = 0, x = 4.5^+$. Zero slope to $V = 0, x = 6$.
Moment diagram. $M = 6, x = 0$. Zero slope to $x = 1.5$. $M = 6, x = 1.5$. Constant positive slope to $x = 4.5$. $M = 18, x = 4.5$. Zero slope to $M = 18, x = 6$.

F11–10 *Shear diagram.* $V = 16.5, x = 0$. Constant negative slope to $x = 3$. $V = 0, x = 2.75, V = -1.5, x = 3$. Negative decreasing slope, $V = -10.5, x = 6$.
Moment diagram. $M = 0, x = 0$. Positive decreasing slope. $M = 22.7, x = 2.75$. Negative decreasing slope to $M = 0, x = 6$.

F11–11 *Shear diagram.* $V = 0, x = 0$. Negative constant slope, $V = -6, x = 1.5^-, V = 0, x = 1.5^+$. Zero slope to $x = 4.5$. $V = 0, x = 4.5^-$. $V = 6$, $x = 4.5^+$. Constant negative slope, $V = 0, x = 6$.
Moment diagram. $M = 0, x = 0$. Negative increasing slope, $M = -4.5, x = 1.5$. Zero slope, $M = -4.5, x = 4.5$. Positive decreasing slope, $M = 0, x = 6$.

F11–12 *Shear diagram.* $V = 15, x = 0$. Negative decreasing slope to zero slope at $x = 3$. $V = 0, x = 0$. Negative increasing slope to $V = -15, x = 6$.
Moment diagram. $M = 0, x = 0$. Positive decreasing slope to zero slope at $x = 3$. $M = 15$, $x = 3$. Negative increasing slope. $M = 0, x = 6$.

F11–13 *Shear diagram.* $V = 1050, x = 0$. Constant negative slope. $V = 0, x = 5.25, V = -150, x = 6$. Zero slope. $V = -150, x = 9^-, V = -750, x = 9^+$. Zero slope. $V = -750, x = 12$.

Moment diagram. $M = 0, x = 0$. Positive decreasing slope to $x = 5.25$. $M = 2756, x = 5.25$. Negative increasing slope to $M = 2700, x = 6$. Constant negative slope. $M = 2250, x = 9$. Constant negative slope. $M = 0, x = 12$.

F11–14 *Shear diagram.* $V = 30, x = 0$. Constant negative slope, $V = 0, x = 1.5, V = -50, x = 4^-$. $V = 20$, $x = 4^+$. Zero slope, $V = 20, x = 6$.
Moment diagram. $M = 0, x = 0$. Positive decreasing slope, to zero slope at $x = 1.5$. $M = 22.5, x = 1.5$. Negative increasing slope, $M = -40, x = 4$. Positive constant slope, $M = 0, x = 6$.

F11–15
$$I = 2\left[\frac{1}{12}(0.02)(0.2^3)\right] + \frac{1}{12}(0.26)(0.02^3)$$
$$= 26.84(10^{-6})\ \text{m}^4$$
$$\sigma_{\max} = \frac{Mc}{I} = \frac{20(10^3)(0.1)}{26.84(10^{-6})} = 74.5\ \text{MPa} \qquad \textit{Ans.}$$

F11–16
$$\bar{y} = \frac{0.3}{3} = 0.1\ \text{m}$$
$$I = \frac{1}{36}(0.3)(0.3^3) = 0.225(10^{-3})\ \text{m}^4$$

$$(\sigma_{\max})_c = \frac{Mc}{I} = \frac{50(10^3)(0.3 - 0.1)}{0.225(10^{-3})}$$
$$= 44.4 \text{ MPa (C)} \quad \textit{Ans.}$$
$$(\sigma_{\max})_t = \frac{My}{I} = \frac{50(10^3)(0.1)}{0.225(10^{-3})} = 22.2 \text{ MPa (T)} \quad \textit{Ans.}$$

F11–17 $I = \frac{1}{12}(0.2)(0.3^3) - \frac{1}{12}(0.18)(0.26^3)$
$= 0.18636(10^{-3}) \text{ m}^4$
$$\sigma_{\max} = \frac{Mc}{I} = \frac{50(10^3)(0.15)}{0.18636(10^{-3})} = 40.2 \text{ MPa} \quad \textit{Ans.}$$

F11–18
$$I = 2\left[\frac{1}{12}(0.03)(0.4^3)\right] + 2\left[\frac{1}{12}(0.14)(0.03^3) + 0.14(0.03)(0.15^2)\right]$$
$$= 0.50963(10^{-3}) \text{ m}^4$$
$$\sigma_{\max} = \frac{Mc}{I} = \frac{10(10^3)(0.2)}{0.50963(10^{-3})} = 3.92 \text{ MPa}$$
$\sigma_A = 3.92$ MPa (C) *Ans.*
$\sigma_B = 3.92$ MPa (T) *Ans.*

F11–19
$$I = \frac{1}{12}(0.05)(0.4)^3 + 2\left[\frac{1}{12}(0.025)(0.3)^3\right]$$
$$= 0.37917(10^{-3}) \text{ m}^4$$
$$\sigma_A = \frac{My_A}{I} = -\frac{5(10^3)(-0.15)}{0.37917(10^{-3})} = 1.98 \text{ MPa (T)} \quad \textit{Ans.}$$

F11–20 $M_y = 50\left(\frac{4}{5}\right) = 40 \text{ kN} \cdot \text{m}$
$M_z = 50\left(\frac{3}{5}\right) = 30 \text{ kN} \cdot \text{m}$
$I_y = \frac{1}{12}(0.3)(0.2^3) = 0.2(10^{-3}) \text{ m}^4$
$I_z = \frac{1}{12}(0.2)(0.3^3) = 0.45(10^{-3}) \text{ m}^4$
$$\sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$
$$\sigma_A = -\frac{[30(10^3)](-0.15)}{0.45(10^{-3})} + \frac{[40(10^3)](0.1)}{0.2(10^{-3})}$$
$= 30$ MPa (T) *Ans.*
$$\sigma_B = -\frac{[30(10^3)](0.15)}{0.45(10^{-3})} + \frac{[40(10^3)](0.1)}{0.2(10^{-3})}$$
$= 10$ MPa (T) *Ans.*
$$\tan\alpha = \frac{I_z}{I_y}\tan\theta$$
$$\tan\alpha = \left[\frac{0.45(10^{-3})}{0.2(10^{-3})}\right]\left(\frac{4}{3}\right)$$
$\alpha = 71.6°$ *Ans.*

F11–21 Maximum stress occurs at D or A.
$$(\sigma_{\max})_D = \frac{(50\cos 30°)12(3)}{\frac{1}{12}(4)(6)^3} + \frac{(50\sin 30°)12(2)}{\frac{1}{12}(6)(4)^3}$$
$= 40.4$ psi *Ans.*

Chapter 12

F12–1 $I = 2\left[\frac{1}{12}(0.02)(0.2^3)\right] + \frac{1}{12}(0.26)(0.02^3)$
$= 26.84(10^{-6}) \text{ m}^4$
$Q_A = 0.055(0.09)(0.02) = 99(10^{-6}) \text{ m}^3$
$$\tau_A = \frac{VQ_A}{It} = \frac{100(10^3)[99(10^{-6})]}{[26.84(10^{-6})](0.02)}$$
$= 18.4$ MPa *Ans.*

F12–2 $I = \frac{1}{12}(0.1)(0.3^3) + \frac{1}{12}(0.2)(0.1^3) = 0.24167(10^{-3}) \text{ m}^4$
$Q_A = y'_1A'_1 + y'_2A'_2$
$= \left[\frac{1}{2}(0.05)\right](0.05)(0.3) + 0.1(0.1)(0.1)$
$= 1.375(10^{-3}) \text{ m}^3$
$Q_B = y'_3A'_3 = 0.1(0.1)(0.1) = 1(10^{-3}) \text{ m}^3$
$$\tau_A = \frac{VQ}{It} = \frac{600(10^3)[1.375(10^{-3})]}{[0.24167(10^{-3})](0.3)} = 11.4 \text{ MPa} \quad \textit{Ans.}$$
$$\tau_B = \frac{VQ}{It} = \frac{600(10^3)[1(10^{-3})]}{[0.24167(10^{-3})](0.1)} = 24.8 \text{ MPa} \quad \textit{Ans.}$$

F12–3 $V_{\max} = 4.5$ kip
$I = \frac{1}{12}(3)(6^3) = 54 \text{ in}^4$
$Q_{\max} = y'A' = 1.5(3)(3) = 13.5 \text{ in}^3$
$$(\tau_{\max})_{\text{abs}} = \frac{V_{\max}Q_{\max}}{It} = \frac{4.5(10^3)(13.5)}{54(3)} = 375 \text{ psi}$$
Ans.

F12–4 $I = 2\left[\frac{1}{12}(0.03)(0.4)^3\right] + 2\left[\frac{1}{12}(0.14)(0.03)^3\right.$
$\left. + 0.14(0.03)(0.15^2)\right] = 0.50963(10^{-3}) \text{ m}^4$

$$Q_{max} = 2y'_1A'_1 + y'_2A'_2 = 2(0.1)(0.2)(0.03) + (0.15)(0.14)(0.03) = 1.83(10^{-3})\ m^3$$

$$\tau_{max} = \frac{VQ_{max}}{It} = \frac{20(10^3)[1.83(10^{-3})]}{0.50963(10^{-3})[2(0.3)]} = 1.20\ MPa \quad \textit{Ans.}$$

F12–5 $I = \frac{1}{12}(0.05)(0.4)^3 + 2\left[\frac{1}{12}(0.025)(0.3)^3\right]$

$= 0.37917(10^3)\ m^4$

$$Q_{max} = 2y'_1A'_1 + y'_2A'_2 = 2(0.075)(0.025)(0.15) + (0.1)(0.05)(0.2) = 1.5625(10^{-3})\ m^3$$

$$\tau_{max} = \frac{VQ_{max}}{It} = \frac{20(10^3)[1.5625(10^{-3})]}{[0.37917(10^{-3})][2(0.025)]} = 1.65\ MPa \quad \textit{Ans.}$$

F12–6 $I = \frac{1}{12}(0.3)(0.2^3) = 0.2(10^{-3})\ m^4$

$$Q = y'A' = 0.05(0.1)(0.3) = 1.5(10^{-3})\ m^3$$

$$q_{allow} = 2\left(\frac{F}{s}\right) = \frac{2[15(10^3)]}{s} = \frac{30(10^3)}{s}$$

$$q_{allow} = \frac{VQ}{I}; \qquad \frac{30(10^3)}{s} = \frac{50(10^3)[1.5(10^{-3})]}{0.2(10^{-3})}$$

$$s = 0.08\ m = 80\ mm \quad \textit{Ans.}$$

F12–7 $I = \frac{1}{12}(0.3)(0.2^3) = 0.2(10^{-3})\ m^4$

$$Q = y'A' = 0.05(0.1)(0.3) = 1.5(10^{-3})\ m^3$$

$$q_{allow} = 2\left(\frac{F}{s}\right) = \frac{2[15(10^3)]}{0.1} = 300(10^3)\ N/m$$

$$q_{allow} = \frac{VQ}{I}; \quad 300(10^3) = \frac{V[1.5(10^{-3})]}{0.2(10^{-3})}$$

$$V = 40(10^3)\ N = 40\ kN \quad \textit{Ans.}$$

F12–8 $I = \frac{1}{12}(0.2)(0.34^3) - \frac{1}{12}(0.19)(0.28^3)$

$= 0.3075(10^{-3})\ m^4$

$$Q = y'A' = 0.16(0.02)(0.2) = 0.64(10^{-3})\ m^3$$

$$q_{allow} = 2\left(\frac{F}{s}\right) = \frac{2[30(10^3)]}{s} = \frac{60(10^3)}{s}$$

$$q_{allow} = \frac{VQ}{I}; \qquad \frac{60(10^3)}{s} = \frac{300(10^3)[0.64(10^{-3})]}{0.3075(10^{-3})}$$

$s = 0.09609\ m = 96.1\ mm$

Use $s = 96\ mm$ *Ans.*

F12–9

$I = 2\left[\frac{1}{12}(0.025)(0.3^3)\right] + 2\left[\frac{1}{12}(0.05)(0.2^3) + 0.05(0.2)(0.15^2)\right]$

$= 0.62917(10^{-3})\ m^4$

$$Q = y'A' = 0.15(0.2)(0.05) = 1.5(10^{-3})\ m^3$$

$$q_{allow} = 2\left(\frac{F}{s}\right) = \frac{2[8(10^3)]}{s} = \frac{16(10^3)}{s}$$

$$q_{allow} = \frac{VQ}{I}; \qquad \frac{16(10^3)}{s} = \frac{20(10^3)[1.5(10^{-3})]}{0.62917(10^{-3})}$$

$$s = 0.3356\ m = 335.56\ mm$$

Use $s = 335\ mm$ *Ans.*

F12–10 $I = \frac{1}{12}(1)(6^3) + 4\left[\frac{1}{12}(0.5)(4^3) + 0.5(4)(3^2)\right]$

$= 100.67\ in^4$

$$Q = y'A' = 3(4)(0.5) = 6\ in^3$$

$$q_{allow} = \frac{F}{s} = \frac{6}{s}$$

$$q_{allow} = \frac{VQ}{I}; \qquad \frac{6}{s} = \frac{15(6)}{100.67}$$

$$s = 6.711\ in.$$

Use $s = 6\frac{5}{8}\ in.$ *Ans.*

Chapter 13

F13–1 $+\uparrow \Sigma F_z = (F_R)_z; \quad -500 - 300 = P$

$P = -800\ kN$

$\Sigma M_x = 0; \qquad 300(0.05) - 500(0.1) = M_x$

$M_x = -35\ kN \cdot m$

$\Sigma M_y = 0; \qquad 300(0.1) - 500(0.1) = M_y$

$M_y = -20\ kN \cdot m$

$A = 0.3(0.3) = 0.09\ m^2$

$I_x = I_y = \frac{1}{12}(0.3)(0.3^3) = 0.675(10^{-3})\ m^4$

$$\sigma_A = \frac{-800(10^3)}{0.09} + \frac{[20(10^3)](0.15)}{0.675(10^{-3})} + \frac{[35(10^3)](0.15)}{0.675(10^{-3})}$$

$= 3.3333\ MPa = 3.33\ MPa\ (T)$ *Ans.*

$$\sigma_B = \frac{-800(10^3)}{0.09} + \frac{[20(10^3)](0.15)}{0.675(10^{-3})} - \frac{[35(10^3)](0.15)}{0.675(10^{-3})}$$

$= -12.22\ MPa = 12.2\ MPa\ (C)$ *Ans.*

F13–2 $+\uparrow \Sigma F_y = 0; \qquad V - 400 = 0 \qquad V = 400\ kN$

$\circlearrowleft + \Sigma M_A = 0; \ -M - 400(0.5) = 0 \ M = -200\ kN \cdot m$

$$I = \frac{1}{12}(0.1)(0.3^3) = 0.225(10^{-3})\ \text{m}^4$$

$$Q_A = y'A' = 0.1(0.1)(0.1) = 1(10^{-3})\ \text{m}^3$$

$$\sigma_A = \frac{My}{I} = \frac{[200(10^3)](-0.05)}{0.225(10^{-3})}$$

$$= -44.44\ \text{MPa} = 44.4\ \text{MPa (C)} \qquad \textit{Ans.}$$

$$\tau_A = \frac{VQ}{It} = \frac{400(10^3)[1(10^{-3})]}{0.225(10^{-3})(0.1)} = 17.8\ \text{MPa} \qquad \textit{Ans.}$$

F13–3 Left reaction is 20 kN.

Left segment:

$$+\uparrow \Sigma F_y = 0; \quad 20 - V = 0 \qquad V = 20\ \text{kN}$$

$$\circlearrowleft + \Sigma M_s = 0; \quad M - 20(0.5) = 0 \qquad M = 10\ \text{kN}\cdot\text{m}$$

$$I = \frac{1}{12}(0.1)(0.2^3) - \frac{1}{12}(0.09)(0.18^3)$$

$$= 22.9267(10^{-6})\ \text{m}^4$$

$$Q_A = y'_1A'_1 + y'_2A'_2 = 0.07(0.04)(0.01)$$

$$+ 0.095(0.1)(0.01) = 0.123(10^{-3})\ \text{m}^3$$

$$\sigma_A = -\frac{My_A}{I} = -\frac{[10(10^3)](0.05)}{22.9267(10^{-6})}$$

$$= -21.81\text{MPa} = 21.8\text{MPa (C)} \qquad \textit{Ans.}$$

$$\tau_A = \frac{VQ_A}{It} = \frac{20(10^3)[0.123(10^{-3})]}{[22.9267(10^{-6})](0.01)}$$

$$= 10.7\ \text{MPa} \qquad \textit{Ans.}$$

F13–4 At the section through centroidal axis:

$$N = P$$

$$V = 0$$

$$M = (2 + 1)P = 3P$$

$$\sigma = \frac{P}{A} + \frac{Mc}{I}$$

$$30 = \frac{P}{2(0.5)} + \frac{(3P)(1)}{\frac{1}{12}(0.5)(2)^3}$$

$$P = 3\ \text{kip} \qquad \textit{Ans.}$$

F13–5 At section through B:

$$N = 500\ \text{lb},\ V = 400\ \text{lb}$$

$$M = 400(10) = 4000\ \text{lb}\cdot\text{in.}$$

Axial load:

$$\sigma_x = \frac{P}{A} = \frac{500}{4(3)} = 41.667\ \text{psi (T)}$$

Shear load:

$$\tau_{xy} = \frac{VQ}{It} = \frac{400[(1.5)(3)(1)]}{[\frac{1}{12}(3)(4)^3]3} = 37.5\ \text{psi}$$

Bending moment:

$$\sigma_x = \frac{My}{I} = \frac{4000(1)}{\frac{1}{12}(3)(4)^3} = 250\ \text{psi (C)}$$

Thus

$$\sigma_x = 41.667 - 250 = 208\ \text{psi (C)} \qquad \textit{Ans.}$$

$$\sigma_y = 0 \qquad \textit{Ans.}$$

$$\tau_{xy} = 37.5\ \text{psi} \qquad \textit{Ans.}$$

F13–6 Top segment:

$$\Sigma F_y = 0; \quad V_y + 1000 = 0 \qquad V_y = -1000\ \text{N}$$

$$\Sigma F_x = 0; \quad V_x - 1500 = 0 \qquad V_x = 1500\ \text{N}$$

$$\Sigma M_z = 0; \quad T_z - 1500(0.4) = 0 \qquad T_z = 600\ \text{N}\cdot\text{m}$$

$$\Sigma M_y = 0; \quad M_y - 1500(0.2) = 0 \qquad M_y = 300\ \text{N}\cdot\text{m}$$

$$\Sigma M_x = 0; \quad M_x - 1000(0.2) = 0 \qquad M_x = 200\ \text{N}\cdot\text{m}$$

$$I_y = I_x = \frac{\pi}{4}(0.02^4) = 40(10^{-9})\pi\ \text{m}^4$$

$$J = \frac{\pi}{2}(0.02^4) = 80(10^{-9})\pi\ \text{m}^4$$

$$(Q_y)_A = \frac{4(0.02)}{3\pi}\left[\frac{\pi}{2}(0.02^2)\right] = 5.3333(10^{-6})\ \text{m}^3$$

$$\sigma_A = \frac{M_x y}{I_x} + \frac{M_y z}{I_y} = \frac{200(0)}{40(10^{-9})\pi} + \frac{300(0.02)}{40(10^{-9})\pi}$$

$$= 47.7\ \text{MPa (T)} \qquad \textit{Ans.}$$

$$[(\tau_{zy})_T]_A = \frac{T_z c}{J} = \frac{600(0.02)}{80(10^{-9})\pi} = 47.746\ \text{MPa}$$

$$[(\tau_{zy})_V]_A = \frac{V_y(Q_y)_A}{I_x t} = \frac{1000[5.3333(10^{-6})]}{[40(10^{-9})\pi](0.04)}$$

$$= 1.061\ \text{MPa}$$

Combining these two shear stress components,

$$(\tau_{zy})_A = 47.746 + 1.061 = 48.8\ \text{MPa} \qquad \textit{Ans.}$$

F13–7 Right Segment:

$$\Sigma F_z = 0; \quad V_z - 6 = 0 \qquad V_z = 6\ \text{kN}$$

$$\Sigma M_y = 0; \quad T_y - 6(0.3) = 0 \qquad T_y = 1.8\ \text{kN}\cdot\text{m}$$

$$\Sigma M_x = 0; \quad M_x - 6(0.3) = 0 \qquad M_x = 1.8\ \text{kN}\cdot\text{m}$$

$$I_x = \frac{\pi}{4}(0.05^4 - 0.04^4) = 0.9225(10^{-6})\pi\ \text{m}^4$$

$$J = \frac{\pi}{2}(0.05^4 - 0.04^4) = 1.845(10^{-6})\pi\ \text{m}^4$$

$$(Q_z)_A = y'_2A'_2 - y'_1A'_1$$

$$= \frac{4(0.05)}{3\pi}\left[\frac{\pi}{2}(0.05^2)\right] - \frac{4(0.04)}{3\pi}\left[\frac{\pi}{2}(0.04^2)\right]$$

$$= 40.6667(10^{-6})\ \text{m}^3$$

$$\sigma_A = \frac{M_x z}{I_x} = \frac{1.8(10^3)}{0.9225(10^{-6})\pi} = 0 \qquad \textit{Ans.}$$

$$[(\tau_{yz})_T]_A = \frac{T_y c}{J} = \frac{[1.8(10^3)](0.05)}{1.845(10^{-6})\pi} = 15.53 \text{ MPa}$$

$$[(\tau_{yz})_V]_A = \frac{V_z(Q_z)_A}{I_x t} = \frac{6(10^3)[40.6667(10^{-6})]}{[0.9225(10^{-6})\pi](0.02)} = 4.210 \text{ MPa}$$

Combining these two shear stress components,

$(\tau_{yz})_A = 15.53 - 4.210 = 11.3$ MPa *Ans.*

F13–8 Left Segment:

$\Sigma F_z = 0;\ V_z - 900 - 300 = 0 \qquad V_z = 1200$ N

$\Sigma M_y = 0;\ T_y + 300(0.1) - 900(0.1) = 0 \quad T_y = 60$ N·m

$\Sigma M_x = 0;\ M_x + (900 + 300)0.3 = 0 \qquad M_x = -360$ N·m

$$I_x = \frac{\pi}{4}(0.025^4 - 0.02^4) = 57.65625(10^{-9})\pi \text{ m}^4$$

$$J = \frac{\pi}{2}(0.025^4 - 0.02^4) = 0.1153125(10^{-6})\pi \text{ m}^4$$

$(Q_y)_A = 0$

$$\sigma_A = \frac{M_x y}{I_x} = \frac{(360)(0.025)}{57.65625(10^{-9})\pi} = 49.7 \text{ MPa} \quad \textit{Ans.}$$

$$[(\tau_{xy})_T]_A = \frac{T_y \rho_A}{J} = \frac{60(0.025)}{0.1153125(10^{-6})\pi} = 4.14 \text{ MPa} \quad \textit{Ans.}$$

$$[(\tau_{yz})_V]_A = \frac{V_z(Q_z)_A}{I_x t} = 0 \quad \textit{Ans.}$$

Chapter 14

F14–1 $\theta = 120° \quad \sigma_x = 500$ kPa $\quad \tau_y = 0 \quad \tau_{xy} = 0$

Apply Eqs. 9–1, 9–2.

$\sigma_{x'} = 125$ kPa *Ans.*

$\tau_{x'y'} = 217$ kPa *Ans.*

F14–2 $\theta = 45° \quad \sigma_x = 0 \quad \sigma_y = -400$ kPa

$\tau_{xy} = -300$ kPa

Apply Eqs. 9–1, 9–3, 9–2.

$\sigma_{x'} = 100$ kPa *Ans.*

$\sigma_{y'} = -500$ kPa *Ans.*

$\tau_{x'y'} = 200$ kPa *Ans.*

F14–3 $\sigma_x = 80$ MPa $\quad \sigma_y = 0 \quad \tau_{xy} = 30$ MPa

Apply Eqs. 9–5, 9–4.

$\sigma_1 = 90$ MPa $\quad \sigma_2 = -10$ MPa *Ans.*

$\theta_p = 18.43°$ and $108.43°$

From Eq. 9–1,

$$\sigma_{x'} = \frac{80 + 0}{2} + \frac{80 - 0}{2}\cos 2(18.43°) + 30 \sin 2(18.43°) = 90 \text{MPa} = \sigma_1$$

Thus,

$(\theta_p)_1 = 18.4°$ and $(\theta_p)_2 = 108°$ *Ans.*

F14–4 $\sigma_x = 100$ kPa $\quad \sigma_y = 700$ kPa

$\tau_{xy} = -400$ kPa

Apply Eqs. 9–7, 9–8.

$\tau_{\text{max}}_{\text{in-plane}} = 500$ kPa *Ans.*

$\sigma_{\text{avg}} = 400$ kPa *Ans.*

F14–5 At the cross section through B:

$N = 4$ kN $\qquad V = 2$ kN

$M = 2(2) = 4$ kN·m

$$\sigma_B = \frac{P}{A} + \frac{Mc}{I} = \frac{4(10^3)}{0.03(0.06)} + \frac{4(10^3)(0.03)}{\frac{1}{12}(0.03)(0.06)^3} = 224\text{MPa (T)}$$

Note $\tau_B = 0$ since $Q = 0$.

Thus

$\sigma_1 = 224$ MPa *Ans.*

$\sigma_2 = 0$

F14–6 $A_y = B_y = 12$ kN

Segment AC:

$V_C = 0 \qquad M_C = 24$ kN·m

$\tau_C = 0$ (since $V_C = 0$)

$\sigma_C = 0$ (since C is on neutral axis)

$\sigma_1 = \sigma_2 = 0$ *Ans.*

F14–7 $$\sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2} = \frac{500 + 0}{2} = 250 \text{ kPa}$$

The coordinates of the center C of the circle and the reference point A are

$A(500, 0) \qquad C(250, 0)$

$R = CA = 500 - 250 = 250$ kPa

$\theta = 120°$ (counterclockwise). Rotate the radial line CA counterclockwise $2\theta = 240°$ to the coordinates of point $P(\sigma_{x'}, \tau_{x'y'})$.

$\alpha = 240° - 180° = 60°$

$\sigma_{x'} = 250 - 250 \cos 60° = 125$ kPa *Ans.*

$\tau_{x'y'} = 250 \sin 60° = 217$ kPa *Ans.*

F14–8 $$\sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2} = \frac{80 + 0}{2} = 40 \text{ kPa}$$

The coordinates of the center C of the circle and the reference point A are

$A(80, 30) \qquad C(40, 0)$

$R = CA = \sqrt{(80 - 40)^2 + 30^2} = 50$MPa

$\sigma_1 = 40 + 50 = 90\text{MPa}$ *Ans.*

$\sigma_2 = 40 - 50 = -10\text{MPa}$ *Ans.*

$\tan 2(\theta_p)_1 = \dfrac{30}{80 - 40} = 0.75$

$(\theta_p)_1 = 18.4°$ (counterclockwise) *Ans.*

F14–9 $J = \dfrac{\pi}{2}(0.04^4 - 0.03^4) = 0.875(10^{-6})\pi\ \text{m}^4$

$\tau = \dfrac{Tc}{J} = \dfrac{4(10^3)(0.04)}{0.875(10^{-6})\pi} = 58.21\ \text{MPa}$

$\sigma_x = \sigma_y = 0$ and $\tau_{xy} = -58.21\ \text{MPa}$

$\sigma_{\text{avg}} = \dfrac{\sigma_x + \sigma_y}{2} = 0$

The coordinates of the reference point A and the center C of the circle are

$A(0, -58.21)$ $C(0, 0)$

$R = CA = 58.21\ \text{MPa}$

$\sigma_1 = 0 + 58.21 = 58.2\ \text{MPa}$ *Ans.*

$\sigma_2 = 0 - 58.21 = -58.2\ \text{MPa}$ *Ans.*

F14–10

$+\uparrow \Sigma F_y = 0;$ $V - 30 = 0$ $V = 30\ \text{kN}$

$\circlearrowleft + \Sigma M_O = 0;$ $-M - 30(0.3) = 0$ $M = -9\ \text{kN}\cdot\text{m}$

$I = \dfrac{1}{12}(0.05)(0.15^3) = 14.0625(10^{-6})\ \text{m}^4$

$Q_A = y'A' = 0.05(0.05)(0.05) = 0.125(10^{-3})\ \text{m}^3$

$\sigma_A = -\dfrac{My_A}{I} = \dfrac{[-9(10^3)](0.025)}{14.0625(10^{-6})} = 16\ \text{MPa (T)}$

$\tau_A = \dfrac{VQ_A}{It} = \dfrac{30(10^3)[0.125(10^{-3})]}{14.0625(10^{-6})(0.05)} = 5.333\ \text{MPa}$

$\sigma_x = 16\ \text{MPa}, \sigma_y = 0$, and $\tau_{xy} = -5.333\ \text{MPa}$

$\sigma_{\text{avg}} = \dfrac{\sigma_x + \sigma_y}{2} = \dfrac{16 + 0}{2} = 8\ \text{MPa}$

The coordinates of the reference point A and the center C of the circle are

$A\ (16, -5.333)$ $C(8, 0)$

$R = CA = \sqrt{(16 - 8)^2 + (-5.333)^2} = 9.615\ \text{MPa}$

$\sigma_1 = 8 + 9.615 = 17.6\ \text{MPa}$ *Ans.*

$\sigma_2 = 8 - 9.615 = -1.61\ \text{MPa}$ *Ans.*

F14–11

$\circlearrowleft + \Sigma M_B = 0;$ $60(1) - A_y(1.5) = 0$ $A_y = 40\ \text{kN}$

$+\uparrow \Sigma F_y = 0;$ $40 - V = 0$ $V = 40\ \text{kN}$

$\circlearrowleft + \Sigma M_O = 0;$ $M - 40(0.5) = 0$ $M = 20\ \text{kN}\cdot\text{m}$

$I = \dfrac{1}{12}(0.1)(0.2^3) - \dfrac{1}{12}(0.09)(0.18^3) = 22.9267(10^{-6})\ \text{m}^4$

$Q_A = y'A' = 0.095(0.01)(0.1) = 95(10^{-6})\ \text{m}^3$

$\sigma_A = -\dfrac{My_A}{I} = -\dfrac{[20(10^3)](0.09)}{22.9267(10^{-6})} = -78.51\ \text{MPa}$

$= 78.51\ \text{MPa (C)}$

$\tau_A = \dfrac{VQ_A}{It} = \dfrac{40(10^3)[95(10^{-6})]}{[22.9267(10^{-6})](0.01)} = 16.57\ \text{MPa}$

$\sigma_x = -78.51\ \text{MPa}, \sigma_y = 0$, and $\tau_{xy} = -16.57\ \text{MPa}$

$\sigma_{\text{avg}} = \dfrac{\sigma_x + \sigma_y}{2} = \dfrac{-78.51 + 0}{2} = -39.26\ \text{MPa}$

The coordinates of the reference point A and the center C of the circle are

$A(-78.51, -16.57)$ $C(-39.26, 0)$

$R = CA = \sqrt{[-78.51 - (-39.26)]^2 + (-16.57)^2}$

$= 42.61\ \text{MPa}$

$\tau_{\substack{\text{max}\\ \text{in-plane}}} = |R| = 42.6\ \text{MPa}$ *Ans.*

Chapter 15

F15–1

$V_{\text{max}} = 12\ \text{kN}$ $M_{\text{max}} = 18\ \text{kN}\cdot\text{m}$

$\sigma_{\text{allow}} = \dfrac{M_{\text{max}}c}{I};$ $10(10^6) = \dfrac{18(10^3)(a)}{\frac{2}{3}a^4}$

$a = 0.1392\ \text{m} = 139.2\ \text{mm}$

Use $a = 140\ \text{mm}$ *Ans.*

$I = \dfrac{2}{3}(0.14^4) = 0.2561(10^{-3})\ \text{m}^4$

$Q_{\text{max}} = \dfrac{0.14}{2}(0.14)(0.14) = 1.372(10^{-3})\ \text{m}^3$

$\tau_{\text{max}} = \dfrac{V_{\text{max}}Q_{\text{max}}}{It} = \dfrac{12(10^3)[1.372(10^{-3})]}{[0.2561(10^{-3})](0.14)}$

$= 0.459\ \text{MPa} < \tau_{\text{allow}} = 1\ \text{MPa (OK)}$

F15–2

$V_{\text{max}} = 3\ \text{kip}$ $M_{\text{max}} = 12\ \text{kip}\cdot\text{ft}$

$I = \dfrac{\pi}{4}\left(\dfrac{d}{2}\right)^4 = \dfrac{\pi d^4}{64}$

$$\sigma_{allow} = \frac{M_{max}c}{I}; \qquad 20 = \frac{12(12)\left(\frac{d}{2}\right)}{\frac{\pi d^4}{64}}$$

$$d = 4.19 \text{ in.}$$

Use $d = 4\frac{1}{4}$ in. *Ans.*

$$I = \frac{\pi}{64}(4.25^4) = 16.015 \text{ in}^4$$

$$Q_{max} = \frac{4(4.25/2)}{3\pi}\left[\frac{1}{2}\left(\frac{\pi}{4}\right)(4.25^2)\right] = 6.397 \text{ in}^3$$

$$\tau_{max} = \frac{V_{max}Q_{max}}{It} = \frac{3(6.397)}{16.015(4.25)}$$

$$= 0.282 \text{ ksi} < \tau_{allow} = 10 \text{ ksi (OK)}$$

F15–3

$V_{max} = 10$ kN $\qquad M_{max} = 5$ kN · m

$$I = \frac{1}{12}(a)(2a)^3 = \frac{2}{3}a^4$$

$$\sigma_{allow} = \frac{M_{max}c}{I}; \qquad 12(10^6) = \frac{5(10^3)(a)}{\frac{2}{3}a^4}$$

$$a = 0.0855 \text{ m} = 85.5 \text{ mm}$$

Use $a = 86$ mm *Ans.*

$$I = \frac{2}{3}(0.086^4) = 36.4672(10^{-6}) \text{ m}^4$$

$$Q_{max} = \frac{0.086}{2}(0.086)(0.086)$$

$$= 0.318028(10^{-3}) \text{ m}^3$$

$$\tau_{max} = \frac{V_{max}Q_{max}}{It} = \frac{10(10^3)[0.318028(10^{-3})]}{[36.4672(10^{-6})](0.086)}$$

$$= 1.01 \text{ MPa} < \tau_{allow} = 1.5 \text{ MPa (OK)}$$

F15–4

$V_{max} = 4.5$ kip $\qquad M_{max} = 6.75$ kip · ft

$$I = \frac{1}{12}(4)(h^3) = \frac{h^3}{3}$$

$$\sigma_{allow} = \frac{M_{max}c}{I}; \qquad 2 = \frac{6.75(12)\left(\frac{h}{2}\right)}{\frac{h^3}{3}}$$

$$h = 7.794 \text{ in.}$$

$$Q_{max} = y'A' = \frac{h}{4}\left(\frac{h}{2}\right)(4) = \frac{h^2}{2}$$

$$\tau_{max} = \frac{V_{max}Q_{max}}{It}; \; 0.2 = \frac{4.5\left(\frac{h^2}{2}\right)}{\frac{h^3}{3}(4)}$$

$$h = 8.4375 \text{ in. (controls)}$$

Use $h = 8\frac{1}{2}$ in. *Ans.*

F15–5

$V_{max} = 25$ kN $\qquad M_{max} = 20$ kN · m

$$I = \frac{1}{12}(b)(3b)^3 = 2.25b^4$$

$$\sigma_{allow} = \frac{M_{max}c}{I}; \qquad 12(10^6) = \frac{20(10^3)(1.5b)}{2.25b^4}$$

$$b = 0.1036\ m = 103.6 \text{ mm}$$

Use $b = 104$ mm *Ans.*

$$I = 2.25(0.104^4) = 0.2632(10^{-3}) \text{ m}^4$$

$$Q_{max} = 0.75(0.104)[1.5(0.104)(0.104)] = 1.2655(10^{-3}) \text{ m}^3$$

$$\tau_{max} = \frac{V_{max}Q_{max}}{It} = \frac{25(10^3)[1.2655(10^{-3})]}{[0.2632(10^{-3})](0.104)}$$

$$= 1.156 \text{ MPa} < \tau_{allow} = 1.5 \text{ MPa (OK).}$$

F15–6

$V_{max} = 150$ kN $\qquad M_{max} = 150$ kN · m

$$S_{req'd} = \frac{M_{max}}{\sigma_{allow}} = \frac{150(10^3)}{150(10^6)} = 0.001 \text{ m}^3 = 1000(10^3) \text{ mm}^3$$

Select W410 × 67 [$S_x = 1200(10^3)$ mm^3, $d = 410$ mm, and $t_w = 8.76$ mm]. *Ans.*

$$\tau_{max} = \frac{V}{t_w d} = \frac{150(10^3)}{0.00876(0.41)}$$

$$= 41.76 \text{ MPa} < \tau_{allow} = 75 \text{ MPa (OK)}$$

Chapter 16

F16–1

$\zeta + \Sigma M_O = 0;$ $\qquad M(x) = 30$ kN · m

$$EI\frac{d^2v}{dx^2} = 30$$

$$EI\frac{dv}{dx} = 30x + C_1$$

$$EIv = 15x^2 + C_1x + C_2$$

At $x = 3$ m, $\frac{dv}{dx} = 0.$

$C_1 = -90$ kN · m^2

At $x = 3$ m, $v = 0.$

$C_2 = 135$ kN · m^3

$$\frac{dv}{dx} = \frac{1}{EI}(30x - 90)$$

$$v = \frac{1}{EI}(15x^2 - 90x + 135)$$

For end A, $x = 0$

$$\theta_A = \left.\frac{dv}{dx}\right|_{x=0} = -\frac{90(10^3)}{200(10^9)[65.0(10^{-6})]} = -0.00692 \text{ rad}$$

Ans.

$$v_A = v|_{x=0} = -\frac{135(10^3)}{200(10^9)[65.0(10^{-6})]} = 0.01038 \text{ m} = 10.4 \text{ mm}$$

Ans.

F16–2

$$\zeta + \Sigma M_O = 0; \qquad M(x) = (-10x - 10) \text{ kN} \cdot \text{m}$$

$$EI\frac{d^2x}{dx^2} = -10x - 10$$

$$EI\frac{dv}{dx} = 5x^2 - 10x + C_1$$

$$EIv = -\frac{5}{3}x^3 - 5x^2 + C_1x + C_2$$

At $x = 3$ m, $\frac{dv}{dx} = 0$.

$$EI(0) = -5(3^2) - 10(3) + C_1 \qquad C_1 = 75 \text{ kN} \cdot \text{m}^2$$

At $x = 3$ m, $v = 0$.

$$EI(0) = -\frac{5}{3}(3^3) - 5(3^2) + 75(3) + C_2 \qquad C_2 = -135 \text{ kN} \cdot \text{m}^3$$

$$\frac{dv}{dx} = \frac{1}{EI}(-5x^2 - 10x + 75)$$

$$v = \frac{1}{EI}\left(-\frac{5}{3}x^3 - 5x^2 + 75x - 135\right)$$

For end A, $x = 0$

$$\theta_A = \left.\frac{dv}{dx}\right|_{x=0} = \frac{1}{EI}[-5(0) - 10(0) + 75]$$

$$= \frac{75(10^3)}{200(10^9)[65.0(10^{-6})]} = 0.00577 \text{ rad}$$

Ans.

$$v_A = v|_{x=0} = \frac{1}{EI}\left[-\frac{5}{3}(0^3) - 5(0^2) + 75(0) - 135\right]$$

$$= -\frac{135(10^3)}{200(10^9)[65.0(10^{-6})]} = -0.01038 \text{ m} = -10.4 \text{ mm}$$

Ans.

F16–3

$$\zeta + \Sigma M_O = 0; \qquad M(x) = \left(-\frac{3}{2}x^2 - 10x\right) \text{ kN} \cdot \text{m}$$

$$EI\frac{d^2x}{dx^2} = -\frac{3}{2}x^2 - 10x$$

$$EI\frac{dv}{dx} = -\frac{1}{2}x^3 - 5x^2 + C_1$$

At $x = 3$ m, $\frac{dv}{dx} = 0$.

$$EI(0) = -\frac{1}{2}(3^3) - 5(3^2) + C_1 \qquad C_1 = 58.5 \text{ kN} \cdot \text{m}^2$$

$$\frac{dv}{dx} = \frac{1}{EI}\left(-\frac{1}{2}x^3 - 5x^2 + 58.5\right)$$

For end A, $x = 0$

$$\theta A = \frac{dv}{dx}|_{x=0} = \frac{58.5(10^3)}{200(10^9)[65.0(10^{-6})]} = 0.0045 \text{ rad}$$

Ans.

F16–4

$$A_y = 600 \text{ lb}$$

$$\zeta + \Sigma M_O = 0; \qquad M(x) = (600x - 50x^2) \text{ lb} \cdot \text{ft}$$

$$EI\frac{d^2x}{dx^2} = 600x - 50x^2$$

$$EI\frac{dv}{dx} = 300x^2 - 16.667x^3 + C_1$$

$$EIv = 100x^3 - 4.1667x^4 + C_1x + C_2$$

At $x = 0$, $v = 0$.

$$EI(0) = 100(0^3) - 4.1667(0^4) + C_1(0) + C_2 \qquad C_2 = 0$$

At $x = 12$ ft, $v = 0$.

$$EI(0) = 100(12^3) - 4.1667(12^4) + C_1(12)$$

$$C_1 = -7200 \text{ lb} \cdot \text{ft}^2$$

$$\frac{dv}{dx} = \frac{1}{EI}(300x^2 - 16.667x^3 - 7200)$$

$$v = \frac{1}{EI}(100x^3 - 4.1667x^4 - 7200x)$$

$v_{\max}$ occurs where $\frac{dv}{dx} = 0$.

$$300x^2 - 16.667x^3 - 7200 = 0$$

$$x = 6 \text{ ft}$$

Ans.

$$v = \frac{1}{EI}[100(6^3) - 4.1667(6^4) - 7200(6)]$$

$$= \frac{-27\,000(12 \text{ in./ft})^3}{1.5(10^6)\left[\frac{1}{12}(3)(6^3)\right]}$$

$$= -0.576 \text{ in.}$$

Ans.

F16–5

$$\zeta + \Sigma M_O = 0; \qquad M(x) = (40 - 5x) \text{ kN} \cdot \text{m}$$

$$EI\frac{d^2x}{dx^2} = 40 - 5x$$

$$EI\frac{dv}{dx} = 40x - 2.5x^2 + C_1$$

$$EIv = 20x^2 - 0.8333x^3 + C_1x + C_2$$

At $x = 0$, $v = 0$.

$$EI(0) = 20(0^2) - 0.8333(0^3) + C_1(0) + C_2 \qquad C_2 = 0$$

At $x = 6$ m, $v = 0$.

$$EI(0) = 20(6^2) - 0.8333(6^3) + C_1(6) + 0$$

$$C_1 = -90 \text{ kN} \cdot \text{m}^2$$

$$\frac{dv}{dx} = \frac{1}{EI}(40x - 2.5x^2 - 90)$$

$$v = \frac{1}{EI}(20x^2 - 0.8333x^3 - 90x)$$

v_{max} occurs where $\frac{dv}{dx} = 0$.

$$40x - 2.5x^2 - 90 = 0$$

$$x = 2.7085 \text{ m}$$

$$v = \frac{1}{EI}[20(2.7085^2) - 0.83333(2.7085^3) - 90(2.7085)]$$

$$= -\frac{113.60(10^3)}{200(10^9)[39.9(10^{-6})]} = -0.01424 \text{ m} = -14.2 \text{ mm} \quad \textit{Ans.}$$

F16–6

$$\circlearrowleft + \Sigma M_O = 0; \qquad M(x) = (10x + 10) \text{ kN} \cdot \text{m}$$

$$EI\frac{d^2x}{dx^2} = 10x + 10$$

$$EI\frac{dv}{dx} = 5x^2 + 10x + C_1$$

Due to symmetry, $\frac{dv}{dx} = 0$ at $x = 3$ m.

$$EI(0) = 5(3^2) + 10(3) + C_1 \qquad C_1 = -75 \text{ kN} \cdot \text{m}^2$$

$$\frac{dv}{dx} = \frac{1}{EI}[5x^2 + 10x - 75]$$

At $x = 0$,

$$\frac{dv}{dx} = \frac{-75(10^3)}{200(10^9)(39.9(10^{-6}))} = -9.40(10^{-3}) \text{ rad} \quad \textit{Ans.}$$

F16–7

$$(v_B)_1 = \frac{Px^2}{6EI}(3L - x) = \frac{40(4^2)}{6EI}[3(6) - 4] = \frac{1493.33}{EI}\downarrow$$

$$(v_B)_2 = \frac{PL^3}{3EI} = \frac{B_y(4^3)}{3EI} = \frac{21.33B_y}{EI}\uparrow$$

$$(+\uparrow) v_B = 0 = (v_B)_1 + (v_B)_2$$

$$0 = -\frac{1493.33}{EI} + \frac{21.33B_y}{EI}$$

$$By = 70 \text{ kN} \quad \textit{Ans.}$$

$$\pm\rightarrow \Sigma F_x = 0; \qquad A_x = 0 \quad \textit{Ans.}$$

$$+\uparrow \Sigma F_y = 0; \qquad 70 - 40 - A_y = 0 \quad A_y = 30 \text{ kN} \quad \textit{Ans.}$$

$$\circlearrowleft + \Sigma M_A = 0; \qquad 70(4) - 40(6) - M_A = 0$$

$$M_A = 40 \text{ kN} \cdot \text{m} \quad \textit{Ans.}$$

F16–8

To use the deflection tables, consider loading as a superposition of uniform distributed load minus a triangular load.

$$(v_B)_1 = \frac{w_0L^4}{8EI}\downarrow \qquad (v_B)_2 = \frac{w_0L^4}{30EI}\uparrow \qquad (v_B)_3 = \frac{B_yL^3}{3EI}\uparrow$$

$$(+\uparrow) \qquad v_B = 0 = (v_B)_1 + (v_B)_2 + (v_B)_3$$

$$0 = -\frac{w_0L^4}{8EI} + \frac{w_0L^4}{30EI} + \frac{B_yL^3}{3EI}$$

$$B_y = \frac{11w_0L}{40} \quad \textit{Ans.}$$

$$\pm\rightarrow \Sigma F_x = 0; \qquad A_x = 0 \quad \textit{Ans.}$$

$$+\uparrow \Sigma F_y = 0; \qquad A_y + \frac{11w_0L}{40} - \frac{1}{2}w_0L = 0$$

$$A_y = \frac{9w_0L}{40} \quad \textit{Ans.}$$

$$\circlearrowleft + \Sigma M_A = 0; \qquad M_A + \frac{11w_0L}{40}(L) - \frac{1}{2}w_0L\left(\frac{2}{3}L\right) = 0$$

$$M_A = \frac{7w_0L^2}{120} \quad \textit{Ans.}$$

F16–9

$$(v_B)_1 = \frac{wL^4}{8EI} = \frac{[10(10^3)](6^4)}{8[200(10^9)][65.0(10^{-6})]} = 0.12461 \text{ m} \downarrow$$

$$(v_B)_2 = \frac{B_yL^3}{3EI} = \frac{B_y(6^3)}{3[200(10^9)][65.0(10^{-6})]} = 5.5385(10^{-6})B_y\uparrow$$

$$(+\downarrow) \qquad v_B = (v_B)_1 + (v_B)_2$$

$$0.002 = 0.12461 - 5.5385(10^{-6})B_y$$

$$B_y = 22.14(10^3) \text{ N} = 22.1 \text{ kN} \quad \textit{Ans.}$$

$$\pm\rightarrow \Sigma F_x = 0; \qquad A_x = 0 \quad \textit{Ans.}$$

$$+\uparrow \Sigma F_y = 0; \qquad A_y + 22.14 - 10(6) = 0 \qquad A_y = 37.9 \text{ kN}$$

Ans.

$$\circlearrowleft + \Sigma M_A = 0; \qquad M_A + 22.14(6) - 10(6)(3) = 0$$

$$M_A = 47.2 \text{ kN} \cdot \text{m} \quad \textit{Ans.}$$

F16–10

$$(v_B)_1 = \frac{M_O L}{6EI(2L)}[(2L)^2 - L^2] = \frac{M_O L^2}{4EI}\downarrow$$

$$(v_B)_2 = \frac{B_y(2L)^3}{48EI} = \frac{B_y L^3}{6EI}\uparrow$$

$$(+\uparrow) \quad v_B = 0 = (v_B)_1 + (v_B)_2$$

$$0 = -\frac{M_O L^2}{4EI} + \frac{B_y L^3}{6EI}$$

$$B_y = \frac{3M_O}{2L} \qquad \textit{Ans.}$$

F16–11

$$(v_B)_1 = \frac{Pbx}{6EIL}(L^2 - b^2 - x^2) = \frac{50(4)(6)}{6EI(12)}(12^2 - 4^2 - 6^2)$$

$$= \frac{1533.3\,\text{kN}\cdot\text{m}^3}{EI}\downarrow$$

$$(v_B)_2 = \frac{B_y L^3}{48EI} = \frac{B_y(12^3)}{48EI} = \frac{36B_y}{EI}\uparrow$$

$$(+\uparrow) \quad v_B = 0 = (v_B)_1 + (v_B)_2$$

$$0 = -\frac{1533.3\,\text{kN}\cdot\text{m}^3}{EI} + \frac{36B_y}{EI}$$

$$B_y = 42.6\ \text{kN} \qquad \textit{Ans.}$$

F16–12

$$(v_B)_1 = \frac{5wL^4}{384EI} = \frac{5[10(10^3)](12^4)}{384[200(10^9)][65.0(10^{-6})]} = 0.20769\ \downarrow$$

$$(v_B)_2 = \frac{B_y L^3}{48EI} = \frac{B_y(12^3)}{48[200(10^9)][65.0(10^{-6})]} = 2.7692(10^{-6})B_y\ \uparrow$$

$$(+\uparrow) \quad v_B = (v_B)_1 + (v_B)_2$$

$$0.005 = 0.20769 - 2.7692(10^{-6})B_y$$

$$B_y = 73.19(10^3)\ \text{N} = 73.2\ \text{kN} \qquad \textit{Ans.}$$

Chapter 17

F17–1

$$P = \frac{\pi^2 EI}{(KL)^2} = \frac{\pi^2(29(10^3))(\frac{\pi}{4}(0.5)^4)}{[0.5(50)]^2} = 22.5\ \text{kip} \qquad \textit{Ans.}$$

$$\sigma = \frac{P}{A} = \frac{22.5}{\pi(0.5)^2} = 28.6\ \text{ksi} < \sigma_Y \qquad \text{OK}$$

F17–2

$$P = \frac{\pi^2 EI}{(KL)^2} = \frac{\pi^2(1.6)(10^3)[\frac{1}{12}(4)(2)^3]}{[1(12)(12)]^2}$$

$$= 2.03\ \text{kip} \qquad \textit{Ans.}$$

F17–3

For buckling about the x axis, $K_x = 1$ and $L_x = 12$ m.

$$P_{\text{cr}} = \frac{\pi^2 EI_x}{(K_x L_x)^2} = \frac{\pi^2[200(10^9)][87.3(10^{-6})]}{[1(12)]^2} = 1.197(10^6)\ \text{N}$$

For buckling about the y axis, $L = 6$ m and $K_y = 1$

$$P_{\text{cr}} = \frac{\pi^2 EI_y}{(K_y L_y)^2} = \frac{\pi^2[200(10^9)][18.8(10^{-6})]}{[1(6)]^2}$$

$$= 1.031(10^6)\ \text{N (controls)}$$

$$P_{\text{allow}} = \frac{P_{\text{cr}}}{\text{F.S.}} = \frac{1.031(10^6)}{2} = 515\ \text{kN} \qquad \textit{Ans.}$$

$$\sigma_{\text{cr}} = \frac{P_{\text{cr}}}{A} = \frac{1.031(10^6)}{7.4(10^{-3})} = 139.30\ \text{MPa} < \sigma_Y = 250\ \text{MPa}$$

(OK)

F17–4

$$A = \pi((0.025)^2 - (0.015)^2) = 1.257\ (10^{-3})\ \text{m}^2$$

$$I = \frac{1}{4}\pi((0.025)^4 - (0.015)^4) = 267.04(10^{-9})\ \text{m}^4$$

$$P = \frac{\pi^2 EI}{(KL)^2} = \frac{\pi^2(200(10^9))(267.04)(10^{-9})}{[0.5(5)]^2} = 84.3\ \text{kN} \qquad \textit{Ans.}$$

$$\sigma = \frac{P}{A} = \frac{84.3(10^3)}{1.257(10^{-3})} = 67.1\ \text{MPa} < 250\ \text{MPa} \qquad \text{(OK)}$$

F17–5

$$+\uparrow \Sigma F_y = 0; \quad F_{AB}\left(\frac{3}{5}\right) - P = 0 \quad F_{AB} = 1.6667P\,\text{(T)}$$

$$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \quad 1.6667P\left(\frac{4}{5}\right) - F_{AC} = 0$$

$$F_{AC} = 1.3333P\ \text{(C)}$$

$$A = \frac{\pi}{4}(2^2) = \pi\ \text{in}^2 \qquad I = \frac{\pi}{4}(1^4) = \frac{\pi}{4}\ \text{in}^4$$

$$P_{\text{cr}} = F_{AC}\,(\text{F.S.}) = 1.3333P(2) = 2.6667P$$

$$P_{\text{cr}} = \frac{\pi^2 EI}{(KL)^2}$$

$$2.6667P = \frac{\pi^2[29(10^3)]\left[\frac{\pi}{4}\right]}{[1(4)(12)]^2}$$

$P = 36.59 \text{ kip} = 36.6 \text{ kip}$ *Ans.*

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{2.6667(36.59)}{\pi} = 31.06 \text{ ksi} < \sigma_Y = 36 \text{ ksi} \quad \text{(OK)}$$

F17–6

$\circlearrowleft + \Sigma M_A = 0; \qquad w(6)(3) - F_{BC}(6) = 0 \qquad F_{BC} = 3w$

$$A = \frac{\pi}{4}(0.05^2) = 0.625(10^{-3})\pi \text{ m}^2 \qquad I = \frac{\pi}{4}(0.025^4) = 97.65625(10^{-9})\pi \text{ m}^4$$

$P_{cr} = F_{BC}(\text{F.S.}) = 3w(2) = 6w$

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2}$$

$$6w = \frac{\pi^2[200(10^9)][97.65625(10^{-9})\pi]}{[1(3)]^2}$$

$w = 11.215(10^3) \text{ N/m} = 11.2 \text{ kN/m}$ *Ans.*

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{6[11.215(10^3)]}{0.625(10^{-3})\pi} = 34.27 \text{ MPa} < \sigma_Y = 250 \text{ MPa} \quad \text{(OK)}$$

Answers to Selected Problems

Chapter 1

1–1. **a.** 58.3 km
b. 68.5 s
c. 2.55 kN
d. 7.56 Mg

1–2. 2.42 Mg/m^3

1–3. **a.** GN/s
b. Gg/N
c. GN/(kg · s)

1–5. **a.** 0.431 g
b. 35.3 kN
c. 5.32 m

1–6. 88.5 km/h
24.6 m/s

1–7. 1 Pa = $20.9(10^{-3})$ lb/ft^2
1 ATM = 101 kPa

1–9. **a.** 3.65 Gg
b. 35.8 MN
c. 5.89 MN
d. 3.65 Gg

1–10. **a.** 8.53 km/kg^2
b. 135 $m^2 \cdot kg^3$

1–11. **a.** 0.447 kg · m/N
b. 0.911 kg · s
c. 18.8 GN/m

1–13. **a.** 27.1 N · m
b. 70.7 kN/m^3
c. 1.27 mm/s

1–14. **a.** 0.185 Mg^2
b. 4 μg^2
c. 0.0122 km^3

1–15. **a.** 2.04 g
b. 15.3 Mg
c. 6.12 Gg

1–17. 584 kg

1–18. 7.41 μN

1–19. 1.00 Mg/m^3

1–21. **a.** 4.81 slug
b. 70.2 kg
c. 689 N
d. 25.5 lb
e. 70.2 kg

Chapter 2

2–1. $F_R = 605$ N
$\phi = 85.4°$

2–2. $F_{1u} = 205$ N
$F_{1v} = 160$ N

2–3. $F_{2u} = 376$ N
$F_{2v} = 482$ N

2–5. $\frac{F_u}{\sin 105°} = \frac{200}{\sin 30°}$ $F_u = 386$ lb
$F_v = 283$ lb

2–6. $F_u = 150$ lb
$F_v = 260$ lb

2–7. $F_R = 497$ N
$\phi = 155°$

2–9. $F_{AB} = 448$ N
$F_{AC} = 366$ N

2–10. $F_{AB} = 314$ lb
$F_{AC} = 256$ lb

2–11. $F_R = 400$ N
$\theta = 60°$

2–13. $F_{2v} = 77.6$ N
$F_{2u} = 150$ N

2–14. $F_a = 30.6$ lb
$F_b = 26.9$ lb

2–15. $F = 917$ lb
$\theta = 31.8°$

2–17. $\theta = 54.9°$
$F_R = 10.4$ kN

2–18. $\theta = 75.5°$

2–19. $\phi = \frac{\theta}{2}$
$F_R = 2F\cos(\frac{\theta}{2})$

2–21. $F_R = 4.01$ kN
$\phi = 16.2°$

2–22. $\theta = 90°$
$F_B = 1$ kN
$F_R = 1.73$ kN

2–23. $\theta = 36.9°$
$F_R = 920$ N

2–25. $F_{R1} = 264.6$ lb, $\theta = 10.9°$
$F_{min} = 235$ lb

2–26. $F_x = 514$ lb,
$F_y = -613$ lb

2–27. $\mathbf{F}_1 = \{900\mathbf{i}\}$ N
$\mathbf{F}_2 = \{530\mathbf{i} + 530\mathbf{j}\}$ N
$\mathbf{F}_3 = \{520\mathbf{i} - 390\mathbf{j}\}$ N

2–29. $\mathbf{F}_1 = \{90\mathbf{i}, -120\mathbf{j}\}$ lb
$\mathbf{F}_2 = \{-275\mathbf{j}\}$ lb
$\mathbf{F}_3 = \{-37.5\mathbf{i}, -65.0\mathbf{j}\}$ lb
$\mathbf{F}_R = 463$ lb

2–30. $F_R = 0$
$F_1 = -417\ \text{N}$
2–31. $F_1 = 143\ \text{N}$
$F_R = 91.9\ \text{N}$
2–33. $F_R = 702\ \text{N}$
$\theta = 44.6°$
2–34. $\phi = 42.4°\quad F_1 = 731\ \text{N}$
2–35. $\theta = 103°$
$F_2 = 88.1\ \text{lb}$
2–37. $F_R = 463\ \text{lb}$
$\theta = 39.6°$
2–38. $\phi = 58.2°$
$F_1 = 494\ \text{lb}$
2–39. $0 = F_1 \sin\phi - 180 - 240$
$F_R = F_1 \cos\phi + 240 - 100$
$F_1 = 420\ \text{lb}$
$F_R = 140\ \text{lb}$
2–41. $F_R = 114\ \text{lb}$
$\alpha = 62.1°$
$\beta = 113°$
$\gamma = 142°$
2–42. $\mathbf{F}_1 = \{53.1\mathbf{i} - 44.5\mathbf{j} + 40\mathbf{k}\}\ \text{lb}$
$\alpha_1 = 48.4°$
$\beta_1 = 124°$
$\gamma_1 = 60°$
$\mathbf{F}_2 = \{-130\mathbf{k}\}\ \text{lb}$
$\alpha_2 = 90°$
$\beta_2 = 90°$
$\gamma_2 = 180°$
2–43. $\alpha = 46.1°$
$\beta = 114°$
$\gamma = 53.1°$
2–45. $F_x = -200\ \text{N}$
$F_y = 200\ \text{N}$
$F_z = 283\ \text{N}$
2–46. $F = 775\ \text{N}$
$F_y = 387\ \text{N}$
$\alpha = 113°$
$\gamma = 39.2°$
2–47. $\mathbf{F}_1 = [480\mathbf{i} + 360\mathbf{k}]\ \text{lb}$
$\mathbf{F}_2 = [200\mathbf{i} + 283\mathbf{j} - 200\mathbf{k}]\ \text{lb}$
2–49. $F_1 = 87.7\ \text{N}$
$\alpha_1 = 46.9°$
$\beta_1 = 125°$
$\gamma_1 = 62.9°$
$F_2 = 98.6\ \text{N}$
$\alpha_2 = 114°$
$\beta_2 = 150°$
$\gamma_2 = 72.3°$
2–50. $\mathbf{F} = \{217\mathbf{i} + 85.5\mathbf{j} - 91.2\mathbf{k}\}\ \text{lb}$

2–51. $F_3 = 166\ \text{N}$
$\alpha = 97.5°$
$\beta = 63.7°$
$\gamma = 27.5°$
2–53. $\beta = 45°\ F_R = 718\ \text{lb}$
$\alpha_R = 86.8°$
$\beta_R = 13.3°$
$\gamma_R = 103°$
2–54. $F = 775\ \text{N}$
$\alpha = 125°\quad \beta = 60.3°\quad \gamma = 48.9°$
2–55. $\beta = 120°$
$F = \{30\mathbf{i} - 30\mathbf{j} + 42.4\mathbf{k}\}\ \text{N}$
2–57. $F_R = 733\ \text{N}$
$\alpha = 53.5°$
$\beta = 65.3°$
$\gamma = 133°$
2–58. $F_2 = 363\ \text{N}$
$\alpha_2 = 15.8°$
$\beta_2 = 104°$
$\gamma_2 = 82.6°$
2–59. $F_2 = 180\ \text{N}$
$\alpha_2 = 147°$
$\beta_2 = 119°$
$\gamma_2 = 75.0°$
2–61. $\alpha = 121°$
$\gamma = 53.1°$
$F_R = 754\ \text{N}$
$\beta = 52.5°$
2–62. $\mathbf{r}_{AB} = \{-3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}\}\ \text{m}$
$r_{AB} = 7\ \text{m}$
2–63. $z = 5.35\ \text{m}$
2–65. $r_{AD} = 1.50\ \text{m}$
$r_{BD} = 1.50\ \text{m}$
$r_{CD} = 1.73\ \text{m}$
2–66. $x = -5.06\ \text{m}$
$y = 3.61\ \text{m}$
$z = 6.51\ \text{m}$
2–67. $d = 6.71\ \text{km}$
2–69. $\mathbf{F}_B = \{-400\mathbf{i} - 200\mathbf{j} + 400\mathbf{k}\}\ \text{N}$
$\mathbf{F}_C = \{-200\mathbf{i} + 200\mathbf{j} + 350\mathbf{k}\}\ \text{N}$
2–70. $F_R = 960\ \text{N}$
$\alpha = 129°$
$\beta = 90°$
$\gamma = 38.7°$
2–71. $F_R = 1.50\ \text{kN}$
$\alpha = 77.6°$
$\beta = 90.6°$
$\gamma = 168°$
2–73. $F_R = 110\ \text{lb}$
$\alpha = 35.4°$
$\beta = 68.8°$
$\gamma = 117°$

2–74. $\mathbf{F} = \{13.4\mathbf{i} + 23.2\mathbf{j} + 53.7\mathbf{k}\}$ lb

2–75. $F_R = 316$ N
$\alpha = 60.1°$
$\beta = 74.6°$
$\gamma = 146°$

2–77. $\dfrac{2400x}{\sqrt{x^2 + z^2 + 36}} = 46\theta$, $z = 2.20$ m
$x = 1.25$ m
$F_R = 3.59$ kN

2–79. $\theta = 82.0°$

2–81. $r_{BC} = 5.39$ m

2–82. $\theta = 70.5°$

2–83. $(F_{BC})_{pa} = 28.33$ lb $= 28.3$ lb
$(F_{BC})_{pr} = 68.0$ lb

2–85. $F_{BC} = 45.2$ N

2–86. $\theta = 74.4°$
$\phi = 55.4°$

2–87. $|(\mathbf{F}_1)_{F_2}| = 5.44$ lb

2–89. $F_{AC} = 366$ lb
$\mathbf{F}_{AC} = \{293\mathbf{j} + 219\mathbf{k}\}$ lb

2–90. $(F_{BC})_{||} = 245$ N
$(F_{BC})_{\perp} = 316$ N

2–91. $\boldsymbol{F} = -300 \sin 30° \sin 30° \mathbf{i} +$
$300 \cos 30° \mathbf{j} + 300 \sin 30° \cos 30° \mathbf{k}$
$F_x = 75$ N
$F_y = 260$ N

2–93. $F_{||} = 99.1$ N
$F_{\perp} = 592$ N

2–94. $F_{||} = 82.4$ N
$F_{\perp} = 594$ N

2–95. $[(F)_{AB}]_{||} = 63.2$ lb
$[(F)_{AB}]_{\perp} = 64.1$ lb

2–97. $F_{1x} = 141$ N
$F_{1y} = 141$ N
$F_{2x} = -130$ N
$F_{2y} = 75$ N

2–98. $F_R = 217$ N
$\theta = 87.0°$

2–99. $F_{1x} = -200$ lb, $F_{1y} = 0$
$F_{2x} = 320$ lb, $F_{2y} = -240$ lb
$F_{3x} = 180$ lb, $F_{3y} = 240$ lb
$F_{4x} = -300$ lb, $F_{4y} = 0$

2–101. $F_R = 178$ N
$\theta = 85.2°$

2–102. $\dfrac{250}{\sin 120°} = \dfrac{F_u}{\sin 40°}$ $F_u = 186$ N
$F_v = 98.7$ N

2–103. $F_R = 60.3$ kN
$\phi = 15.0°$

Chapter 3

3–1. $\mathbf{A} \times (\mathbf{B} + \mathbf{D}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{D}$
$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$
$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$
$\mathbf{D} = D_x\mathbf{i} + D_y\mathbf{j} + D_z\mathbf{k}$
$= (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{D})$

3–2. Volume $= |\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}|$
Since $|\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}|$ represents this same volume then
$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C}$ **(QED)**
Thus, *LHS* = *RHS*
$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C}$ **(QED)**

3–3. If $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = 0$, then the volume equals zero, so that A, B, and C are coplanar.

3–5. $(M_{F_1})_B = 4.125$ kip · ft (*Counterclockwise*)
$(M_{F_2})_B = 2.00$ kip · ft (*Counterclockwise*)
$(M_{F_3})_B = 40.0$ lb · ft (*Counterclockwise*)

3–6. $M_A = 7.21$ kN · m ⟲

3–7. $\theta = 64.0°$

3–9. $\zeta + M_B = 150$ N · m ↻
$\zeta + M_B = 600$ N · m ↻
$M_B = 0$

3–10. $M_O = 120$ N · m ↻
$M_O = 520$ N · m ↻

3–11. $(M_R)_A = 2.08$ kN · m (*Counterclockwise*)

3–13. $M_A = 400\sqrt{(3)^2 + (2)^2}$
$M_A = 1.44$ kN · m ↻
$\theta = 56.3°$

3–14. $\zeta + M_A = 1200 \sin\theta + 800 \cos\theta$
$M_{max} = 1.44$ kN · m ↻
$\theta_{max} = 56.3°$

3–15. $M_{min} = 0$
$\theta_{min} = 146°$

3–17. $F_A = 28.9$ lb

3–18. $\zeta + M_A = 0.418$ N · m (*Counterclockwise*)
$\zeta + M_B = 4.92$ N · m (*Clockwise*)

3–19. $(M_R)_A = 76.0$ kN · m (*Counterclockwise*)

3–21. $\zeta + (M_O)_{max} = 80$ kN · m ⟲
$x = 24.0$ m

3–22. $\circlearrowleft + (M_O)_{max} = 80.0$ kN · m
$\theta = 33.6°$
$\theta = 33.6°$

3–23. $M_A = 13.0$ N · m
$F = 35.2$ N

3–25. $F = 27.6$ lb

3–26. $M_A = 100$ N · m (*Clockwise*)

3–27. $\zeta + (M_R)_A = 402$ mm

3–29. $\mathbf{M}_O = \mathbf{r}_{OA} \times \mathbf{F}_1 = \{110\mathbf{i} - 50\mathbf{j} + 90\mathbf{k}\}$ lb · ft

3–30. $\mathbf{M}_O = \{90\mathbf{i} - 130\mathbf{j} - 60\mathbf{k}\}$ lb · ft

3–31. $(\mathbf{M}_R)_O = \{200\mathbf{i} - 180\mathbf{j} + 30\mathbf{k}\}$ lb · ft

3–33. $\mathbf{M}_O = \mathbf{r}_{OA} \times \mathbf{F}_C = \{1080\mathbf{i} + 720\mathbf{j}\}\ \text{N} \cdot \text{m}$
3–34. $\mathbf{M}_O = \{-720\mathbf{i} + 720\mathbf{j}\}\ \text{N} \cdot \text{m}$
3–35. $(\mathbf{M}_A)_O = \{-18\mathbf{i} + 9\mathbf{j} - 3\mathbf{k}\}\ \text{N} \cdot \text{m}$
$(\mathbf{M}_B)_O = \{18\mathbf{i} + 7.5\mathbf{j} + 30\,\mathbf{k}\}\ \text{N} \cdot \text{m}$
3–37. $\mathbf{M}_A = \mathbf{r}_{AC} \times \mathbf{F}$
$= \{-5.39\mathbf{i} + 13.1\mathbf{j} + 11.4\mathbf{k}\}\ \text{N} \cdot \text{m}$
3–38. $\mathbf{M}_B = \{10.6\mathbf{i} + 13.1\mathbf{j} + 29.2\mathbf{k}\}\ \text{N} \cdot \text{m}$
3–39. $F = 27.6\ \text{lb}$
3–41. $F = 18.6\ \text{lb}.$
3–42. $\theta_{max} = 90°$
$\theta_{min} = 0,\ 180°$
3–43. $M = 151\ \text{lb} \cdot \text{in.}$
3–45. $M_x = 44.4\ \text{lb} \cdot \text{ft}$
3–46. $M_x = 15.0\ \text{lb} \cdot \text{ft}$
$M_y = 4.00\ \text{lb} \cdot \text{ft}$
$M_z = 36.0\ \text{lb} \cdot \text{ft}$
3–47. $\mathbf{M}_{AC} = \{11.5\mathbf{i} + 8.64\mathbf{j}\}\ \text{lb} \cdot \text{ft}$
3–49. $M_a = 4.37\ \text{N} \cdot \text{m}$
$\alpha = 33.7°$
$\beta = 90°$
$\gamma = 56.3°$
$M = 5.41\ \text{N} \cdot \text{m}$
3–50. $F_{AB} = 274\ \text{lb},\ F_{AC} = 295\ \text{lb}$
$F_{AD} = 547\ \text{lb}$
3–51. $F_{AD} = 557\ \text{lb}$
$W = 407\ \text{lb}$
3–53. $M_{CD} = \mathbf{u}_{CD} \cdot \mathbf{r}_{CA} \times \mathbf{F}$
$= u_{CD} \cdot \mathbf{r}_{DB} \times \mathbf{F} = -432\ \text{lb} \cdot \text{ft}$
3–54. $F = 162\ \text{lb}$
3–55. $F = 133\ \text{N}$
$P = 800\ \text{N}$
3–57. $F = 625\ \text{N}$
3–58. $\circlearrowleft + (M_R)_C = 435\ \text{lb} \cdot \text{ft}\ \circlearrowright$
3–59. $F = 139\ \text{lb}$
3–61. $M_C = 22.5\ \text{N} \cdot \text{m}\ \circlearrowright$
3–62. $F = 83.3\ \text{N}$
3–63. $F = 194\ \text{lb}$
3–65. $F = 221\ \text{lb}$
3–66. $(M_c)_R = 5.20\ \text{kN} \cdot \text{m}\ (\textit{Clockwise})$
3–67. $F = 14.2\ \text{kN} \cdot \text{m}$
3–69. $d = 342\ \text{mm}$
3–70. $M_C = \{-5\mathbf{i} + 8.75\mathbf{j}\}\ \text{N} \cdot \text{m}$
3–71. $F = \dfrac{400}{\sqrt{(-0.2)^2 + (0.35)^2}} = 992\ \text{N}$
3–73. $F_2 = 112\ \text{N}$
$F_1 = 87.2\ \text{N}$
$F_3 = 100\ \text{N}$
3–74. $F_R = 416\ \text{lb}$
$\theta = 35.2°\ \measuredangle$
$(M_R)_A = 1.48\ \text{kip} \cdot \text{ft}\ (\textit{Clockwise})$
3–75. $F_R = 2.10\ \text{kN}$
$\theta = 81.6°\ \measuredangle$
$M_O = 10.6\ \text{kN} \cdot \text{m}\ \circlearrowright$
3–77. $F_R = 29.9\ \text{lb}$
$\theta = 78.4°\ \measuredangle$
$M_{R_O} = 214\ \text{lb} \cdot \text{in.}\ \circlearrowleft$
3–78. $F_R = 26.4\ \text{lb}$
$\theta = 85.7°\ \measuredangle$
$M_{R_O} = 205\ \text{lb} \cdot \text{in.}\ \circlearrowleft$
3–79. $F_R = \sqrt{533.01^2 + 100^2} = 542\ \text{N}$
$\theta = 10.6°\ \measuredangle$
$(M_R)_A = 441\ \text{N} \cdot \text{m}\ \circlearrowleft$
3–81. $F_R = 8.27\ \text{kN}$
$\theta = 69.9°\ \measuredangle$
$(M_R)_A = 9.77\ \text{kN} \cdot \text{m}\ (\textit{Clockwise})$
3–82. $F_R = 650\ \text{N}$
$\theta = 72.0°\ \measuredangle$
3–83. $\mathbf{F}_R = [2\mathbf{i} - 10\mathbf{k}]\ \text{kN}$
$(\mathbf{M}_R)_O = [-6\mathbf{i} + 12\mathbf{j}]\ \text{kN} \cdot \text{m}$
3–85. $\mathbf{F}_R = \{6\mathbf{i} - 1\mathbf{j} - 14\mathbf{k}\}\ \text{N}$
$\mathbf{M}_{R_O} = \{1.30\mathbf{i} + 3.30\mathbf{j} - 0.450\mathbf{k}\}\ \text{N} \cdot \text{m}$
3–86. $\mathbf{F}_R = \{-40\mathbf{j} - 40\mathbf{k}\}\ \text{N}$
$\mathbf{M}_{RA} = \{-12\mathbf{j} + 12\mathbf{k}\}\ \text{N} \cdot \text{m}$
3–87. $\mathbf{F}_2 = \{-1.768\mathbf{i} + 3.062\mathbf{j} + 3.536\mathbf{k}\}\ \text{kN}$
$\mathbf{F}_R = \{0.232\mathbf{i} + 5.06\mathbf{j} + 12.4\mathbf{k}\}\ \text{kN}$
$\mathbf{M}_{R_O} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2$
$= \{36.0\mathbf{i} - 26.1\mathbf{j} + 12.2\mathbf{k}\}\ \text{kN} \cdot \text{m}$
3–89. $F = 798\ \text{lb}$
$\theta = 67.9°\ \measuredangle$
$x = 6.57\ \text{ft}$
3–90. $F_R = 4.50\ \text{kN}$
$d = 2.22\ \text{m}$
3–91. $F_R = 462\ \text{lb}$
$\theta = 39.1°\ \measuredangle$
$d = 3.07\ \text{ft}$
3–93. $F = 1302\ \text{N}$
$\theta = 84.5°\ \measuredangle$
$x = 1.36\ \text{m}\ (\text{to the right})$
3–94. $F = 922\ \text{lb}$
$\theta = 77.5°\ \measuredangle$
$x = 3.56\ \text{ft}$
3–95. $F_R = 991\ \text{N}$
$y = 1.78\ \text{m}$
3–97. $F_R = 65.9\ \text{lb}$
$\theta = 49.8°\ \measuredangle$
$d = 2.10\ \text{ft}$
3–98. $F_R = 65.9\ \text{lb}$
$\theta = 49.8°\ \measuredangle$
$d = 4.62\ \text{ft}$

3–99. $F_R = 140\ \text{kN}$
$y = 7.14\ \text{m}$
$x = 5.71\ \text{m}$
3–101. $F_C = 600\ \text{N} \quad F_D = 500\ \text{N}$
3–102. $F_R = 700\ \text{lb}$
$z = 0.447\ \text{ft}$
$x = -0.117\ \text{ft}$
3–103. $0 = 200(1.5\cos 45°) - F_B(1.5\cos 30°)$
$F_B = 163\ \text{lb}$
$F_C = 223\ \text{lb}$
3–105. $F_A = 30\ \text{kN} \quad F_B = 20\ \text{kN} \quad F_R = 190\ \text{kN}$
3–106. $F_R = -10\ \text{kN}$
$z = 1.40\ \text{m}$
$x = 1.00\ \text{m}$
3–107. $\mathbf{M}_A = \{-59.7\mathbf{i} - 159\mathbf{k}\}\ \text{N}\cdot\text{m}$
3–109. $\mathbf{M}_{CR} = \{63.6\mathbf{i} - 170\mathbf{j} + 264\mathbf{k}\}\ \text{N}\cdot\text{m}$
3–110. $\alpha = 109°$ or $70.8°$
$\beta = 140°$ or $39.8°$
$\gamma = 123°$ or $56.7°$
3–111. $\mathbf{M}_O = \mathbf{r}_{OA} \times \mathbf{F} = \{298\mathbf{i} + 15.1\mathbf{j} - 200\mathbf{k}\}\ \text{lb}\cdot\text{in}$
3–113. $(M_R)_A = 2.09\ \text{N}\cdot\text{m}$ (*Clockwise*)
3–114. $F = 618\ \text{N}$
3–115. $M_z = -4.03\ \text{N}\cdot\text{m}$
$M_z = -4.03\ \text{N}\cdot\text{m}$
3–117. $P = 23.8\ \text{lb}$
3–118. $\mathbf{M}_O = \{-128\mathbf{i} + 128\mathbf{j} - 257\mathbf{k}\}\ \text{N}\cdot\text{m}$
3–119. $\mathbf{M}_B = \{-37.6\mathbf{i} + 90.7\mathbf{j} - 155\mathbf{k}\}\ \text{N}\cdot\text{m}$

Chapter 4

4–1. $F_{CA} = 80.0\ \text{N}, \quad F_{CB} = 90.4\ \text{N}$
4–2. $\theta = 64.3°$
$F_{CB} = 85.2\ \text{N}$
$F_{CA} = 42.6\ \text{N}$
4–3. $F_2 = 9.60\ \text{kN}$
$F_1 = 1.83\ \text{kN}$
4–5. $\theta = 34.2°$
4–6. $F_A = 30\ \text{lb}, F_B = 36.2\ \text{lb}, F_C = 9.38\ \text{lb}$
4–7. $B_x = 989\ \text{N}, A_x = 989\ \text{N}, B_y = 186\ \text{N}$
4–9. $F_H = 59.43 = 59.4\ \text{lb}$
$T_B = 67.4\ \text{lb}$
4–10. $N_A = 250\ \text{lb}, N_B = 9.18\ \text{lb}, N_C = 141\ \text{lb}$
4–11. $P_{max} = 210\ \text{lb}$
4–13. $W = 5.34\ \text{kip}$
4–14. $F_{BD} = 628\ \text{N}$
$C_y = 68.2\ \text{N}, C_x = 432\ \text{N}$
4–15. $\theta = 26.4°$
4–17. $F_{CD} = 195\ \text{lb}, A_x = 97.5\ \text{lb}, A_y = 31.2\ \text{lb}$
4–18. $l = 0.67664\ \text{m}, \quad \dfrac{\sin\theta}{0.3} = \dfrac{\sin 150°}{0.67664}, F_s = 2.383\ \text{N},$
$N_B = 2.11\ \text{N}, F_A = 2.81\ \text{N}$
4–19. $N_B = 3.33\ \text{kip}$
$A_y = 5.00\ \text{kip}$
$A_x = 3.33\ \text{kip}$
4–21. $F_B = 50.2\ \text{lb}$
4–22. $F_B = 6.38\ \text{N}, A_x = 3.19\ \text{N}, A_y = 2.48\ \text{N}$
4–23. $T_{BC} = 16.4\ \text{kN}$
$F_A = 20.6\ \text{kN}$
4–25. $\alpha = \tan^{-1}\left(\dfrac{0.10464 + 0.07848}{1}\right) = 10.4°$
4–26. $\Delta_A = \dfrac{533.33}{5(10^3)}, \Delta_B = \dfrac{266.7}{5(10^3)}, \alpha = 1.02°$
4–27. $k_B = 2.50\ \text{kN/m}$
4–29. $h = 0.645\ \text{m}$
4–30. $h = \sqrt{\dfrac{s^2 - l^2}{3}}$
4–31. $w_1 = 83.3\ \text{lb/ft}, w_2 = 167\ \text{lb/ft}$
4–33. $\dfrac{L}{\Delta_B} = \dfrac{2L}{\Delta_C}, \dfrac{F_C}{k} = \dfrac{2F_B}{k}, F_B = 0.3P$
$F_C = 0.6P$
$x_C = 0.6P/k$
4–34. $N_A = 28.6\ \text{lb}$
$N_B = 10.7\ \text{lb}, N_C = 10.7\ \text{lb}$
4–35. $T = 1.84\ \text{kN}, F = 6.18\ \text{kN}$
4–37. $y = 0.667\ \text{m}, x = 0.667\ \text{m}$
4–38. $w = 750\ \text{lb}, x = 5.20\ \text{ft}, y = 5.27\ \text{ft}$
4–39. $N_B = 373\ \text{N}$
$A_z = 333\ \text{N}$
$T_{CD} = 43.5\ \text{N}$
$A_x = 0$
$A_y = 0$
4–41. $N_B = 1000\ \text{N}$
$(M_A)_z = 0$
$A_y = 0$
$(M_A)_y = -560\ \text{N}\cdot\text{m}$
$A_z = 400\ \text{N}$
4–42. $F_A = 663\ \text{lb}, F_C = 569\ \text{lb}, F_B = 449\ \text{lb}$
4–43. $A_y = 0$
$T = 1.23\ \text{kN}$
$B_x = 433\ \text{N}$
$B_z = 1.42\ \text{kN}$
$A_x = 867\ \text{N}$
$A_z = 711\ \text{N}$
4–45. $P = 100\ \text{lb}, B_z = 40\ \text{lb}, B_x = -35.7\ \text{lb},$
$A_x = 136\ \text{lb}, A_z = 40\ \text{lb}, B_y = 0$
4–46. $F = 900\ \text{lb}$
$A_x = 0$
$A_y = 0$
$A_z = 600\ \text{lb}$
$M_{Ax} = 0$
$M_{Az} = 0$

4–47. $P\cos 30° + 0.25N - 50(9.81)\sin 30° = 0$
$P = 140$ N
$N = 494.94$ N
4–49. $\mu_s = 0.256$
4–50. $N_A = 16.5$ kN,
$N_B = 42.3$ kN, the mine car does not move.
4–51. $\theta = 10.6°$, $x = 0.184$ ft of stretch
4–53. The ladder will not slip.
4–54. $\theta = 46.4°$
4–55. $F_{CD} = 3.05$ kN
4–57. **a.** $P = 30$ N < 34.06 N No
b. $P = 70$ N > 34.26 N Yes
4–58. $P = \dfrac{M_0}{\mu_s ra}(b - \mu_s c)$
4–59. $\mu_s \geq \dfrac{b}{c}$
4–61. **a.** $W = 318$ lb,
b. $W = 360$ lb
4–62. $\mu = 0.354$
4–63. $d = 537$ mm
4–65. $F_f = 10$ lb
4–66. $F_f = 70.7$ lb
4–67. Slip at clamp, $n = 9.17$, slip between end boards;
$n = 8.11$, use $n = 8$.
4–69. $W = 836$ lb
4–70. Thus, **he can move the crate.**
4–71. $\mu_s' = 0.376$
4–73. $\theta = 31.0°$
4–74. $F = 5.38$ lb
4–75. $W = 66.64$ lb $= 66.6$ lb
4–77. $F_{AB} = 1.38$ kN (T), $F_{BD} = 828$ N (C)
$F_{BC} = 1.10$ kN (C), $F_{AC} = 828$ N (C)
$F_{AD} = 1.10$ kN (C), $F_{CD} = 1.38$ kN (T)
4–78. $M = 4.53$ N · m
4–79. $T = 4.02$ kN
4–81. $A_x = 328.6$ N
$B_y = C_y = 164$ N
4–82. $W = 7.19$ kN
4–83. Since $\phi_s > \theta_p$, the screw is self-locking.
4–85. $W = 9.17$ lb
4–86. $P = 78.7$ lb
4–87. $h = 8.28$ ft
4–89. Approx. 2 turns (695°)
4–90. $N = 185$ lb
$F = 136.9$ lb
Yes, just barely.
4–91. $T_1 = 57.7$ lb
4–93. $\theta = 38.8°$
4–94. $F = 2.49$ kN
4–95. $M = 3.37$ N · m
4–97. $T = 486.55$ N $N = 314.82$ N
$\beta = (2n + 0.9167)\pi$ rad
Thus, the required number of full turns is
$n = 2$.
4–98. The man can hold the crate in equilibrium.
4–99. $d = 4.07$ m
$\mu_s = 0.189$
4–101. $d = 4.60$ ft
4–102. $600(6) + 600(4) + 600(2) - N_B\cos 45°(2) = 0$
$N_B = 5.09$ kN
$A_x = 3.60$ kN
$A_y = 1.80$ kN
4–103. $F = 354$ N
4–105. $W = 56.6$ lb
4–106. $F_{CD} = 1.02$ kN
$A_z = -208$ N
$B_z = -139$ N
$A_y = 573$ N
$B_y = 382$ N
4–107. $5(14) + 7(6) + 0.5(6) - 2(6) - A_y(14) = 0$
$A_y = 7.36$ kip
$B_x = 0.5$ kip
$B_y = 16.6$ kip
4–109. $A_x = 0$,
$A_y = -200$ N, $A_z = 150$ N, $(M_A)_x = 100$ N · m
$(M_A)_y = 0$, $(M_A)_z = 500$ N · m
4–110. $P = 15$ lb
4–111. $P = 1$ lb

Chapter 5

5–1. Joint D: $600 - F_{DC}\sin 26.57° = 0$
$F_{DC} = 1.34$ kN (C)
$F_{DE} = 1.20$ kN (T)
Joint C: $-F_{CE}\cos 26.57° = 0$
$F_{CE} = 0$
$F_{CB} = 1.34$ kN (C)
Joint E: $900 - F_{EB}\sin 45° = 0$
$F_{EB} = 1.27$ kN (C)
$F_{EA} = 2.10$ kN (T)
5–2. $F_{AB} = 286$ lb (T)
$F_{AC} = 571$ lb (C)
$F_{BC} = 808$ lb (T)
5–3. $F_{AB} = 286$ lb (T)
$F_{AC} = 271$ lb (C)
$F_{BC} = 384$ lb (T)
5–5. $F_{CD} = 5.21$ kN (C)
$F_{CB} = 2.36$ kN (T)
$F_{AD} = 1.46$ kN (C)
$F_{AB} = 2.36$ kN(T)
$F_{BD} = 4$ kN (T)

5–6. $F_{CB} = 3.00$ kN (T)
$F_{CD} = 2.60$ kN (C)
$F_{DE} = 2.60$ kN (C)
$F_{DB} = 2.00$ kN (T)
$F_{BE} = 2.00$ kN (C)
$F_{BA} = 5.00$ kN (T)

5–7. $F_{CB} = 8.00$ kN (T)
$F_{CD} = 6.93$ kN (C)
$F_{DE} = 6.93$ kN (C)
$F_{DB} = 4.00$ kN (T)
$F_{BE} = 4.00$ kN (C)
$F_{BA} = 12.0$ kN (T)

5–9. $F_{AB} = 196$ N (T), $F_{AE} = 118$ N (C)
$F_{ED} = 118$ N (C), $F_{EB} = 216$ N (T)
$F_{BD} = 1.04$ kN (C), $F_{BC} = 857$ N (T)

5–10. $F_{AB} = 330$ lb (C), $F_{AF} = 79.4$ lb (T)
$F_{BF} = 233$ lb (T), $F_{BC} = 233$ lb (C)
$F_{FC} = 47.1$ lb (C), $F_{FE} = 113$ lb (T)
$F_{EC} = 300$ lb (T), $F_{ED} = 113$ lb (T)
$F_{CD} = 377$ lb (C)

5–11. $F_{AB} = 377$ lb (C), $F_{AF} = 190$ lb (T)
$F_{BF} = 267$ lb (T), $F_{BC} = 267$ lb (C)
$F_{FC} = 189$ lb (T), $F_{FE} = 56.4$ lb (T)
$F_{ED} = 56.4$ lb (T), $F_{EC} = 0$, $F_{CD} = 189$ lb (C)

5–13. Maximum force:
$F_{DC} = F_{CB} = F_{CE} = F_{BE} = F_{BA} = 1.1547P$
$P = 5.20$ kN

5–14. $F_{CD} = 3.61$ kN (C), $F_{CB} = 3$ kN (T)
$F_{BA} = 3$ kN (T), $F_{BD} = 3$ kN (C)
$F_{DA} = 2.70$ kN (T), $F_{DE} = 6.31$ kN (C)

5–15. $F_{CD} = 467$ N (C), $F_{CB} = 389$ N (T)
$F_{BA} = 389$ N (T), $F_{BD} = 314$ N (C)
$F_{DE} = 1.20$ kN (C), $F_{DA} = 736$ N (T)

5–17. Maximum force: $F_{CA} = 2.732P$ (T),
$F_{CD} = 1.577P$ (C), $P_{max} = 732$ N

5–18. $F_{CD} = 780$ lb (C)
$F_{CB} = 720$ lb (T)
$F_{DB} = 0$
$F_{DE} = 780$ lb (C)
$F_{BE} = 297$ lb (T)
$F_{BA} = 722$ lb (T)

5–19. $F_{FE} = 0.667P$ (T)
$F_{FD} = 1.67P$ (T)
$F_{AB} = 0.471P$ (C)
$F_{AE} = 1.67P$ (T)
$F_{AC} = 1.49P$ (C)
$F_{BF} = 1.41P$ (T)
$F_{BD} = 1.49P$ (C)
$F_{EC} = 1.41P$ (T)
$F_{CD} = 0.471P$ (C)

5–21. $F_{GB} = 30$ kN (T),
$F_{AF} = 20$ kN (C), $F_{AB} = 22.4$ kN (C)
$F_{BF} = 20$ kN (T), $F_{BC} = 20$ kN (T)
$F_{FC} = 28.3$ kN (C), $F_{FE} = 0$
$F_{ED} = 0$, $F_{EC} = 20.0$ kN (T)
$F_{DC} = 0$

5–22. $127° \leq \theta \leq 196°$
$336° \leq \theta \leq 347°$

5–23. $F_{BH} = 255$ lb (T)
$F_{BC} = 130$ lb (T)
$F_{HC} = 180$ lb (C)

5–25. $A_y = 65.0$ kN
$A_x = 0$
$F_{BC}(4) + 20(4) + 30(8) - 65.0(8) = 0$
$F_{BC} = 50.0$ kN (T)
$F_{HI} = 35.0$ kN (C)
$F_{HB} = 21.2$ kN (C)

5–26. $F_{KJ} = 11.2$ kip (T), $F_{CD} = 9.38$ kip (C),
$F_{CJ} = 3.12$ kip (C), $F_{DJ} = 0$

5–27. $F_{JI} = 7.50$ kip (T)
$F_{EI} = 2.50$ kip (C)

5–29. $F_{GC} = 1.00$ kip (T)

5–30. $F_{BC} = 10.4$ kN (C), $F_{HG} = 9.15$ kN (T),
$F_{HC} = 2.24$ kN (T)

5–31. $F_{CD} = 11.2$ kN (C)
$F_{CF} = 3.21$ kN (T)
$F_{CG} = 6.80$ kN (C)

5–33. *AB, BC, CD, DE, HI,* and *GI* are zero-force members.
$F_{JE} = 9.38$ kN (C), $F_{GF} = 5.62$ kN (T)

5–34. $F_{GF} = 1.80$ kip (C), $F_{FB} = 693$ lb (T)
$F_{BC} = 1.21$ kip (T)

5–35. $F_{FE} = 1.80$ kip (C)
$F_{EC} = 693$ lb (C)

5–37. $F_{JI} = 7333$ lb (C), $F_{DE} = 9000$ lb (T)
$F_{JE} = 3005$ lb (C)

5–38. ***Method of Joints***: By inspection, members *BN, NC, DO, OC, HJ, LE* and *JG* are zero-force members.
$F_{CD} = 5.625$ kN (T), $F_{CM} = 2.00$ kN (T)

5–39. ***Method of Joints***: By inspection, members *BN, NC, DO, OC, HJ, LE* and *JG* are zero-force members.
$F_{EF} = 7.88$ kN (T), $F_{LK} = 9.25$ kN (C)
$F_{ED} = 1.94$ kN (T)

5–41. $F_{BG} = \{-600 \csc \theta\}$ N
$F_{BC} = -200L$ N
$F_{HG} = 400L$ N

5–42. $F_{AB} = 21.9$ kN (C), $F_{AG} = 13.1$ kN (T),
$F_{BC} = 13.1$ kN (C), $F_{BG} = 17.5$ kN (T),
$F_{CG} = 3.12$ kN (T), $F_{FG} = 11.2$ kN (T),
$F_{CF} = 3.12$ kN (C), $F_{CD} = 9.38$ kN (C),
$F_{DE} = 15.6$ kN (C), $F_{DF} = 12.5$ kN (T),
$F_{EF} = 9.38$ kN (T)

Chapter 6

6–1. $\bar{y} = \frac{2}{5}\,\text{m}$

6–2. $\bar{x} = \frac{3}{8}a$

6–3. $\bar{x} = 0.649\ \text{in.}$

6–5. $A = c^2 \ln \frac{b}{a}$

$\bar{x} = \dfrac{b - a}{\ln \frac{b}{a}}$

$\bar{y} = \dfrac{c^2(b - a)}{2ab \ln \frac{b}{a}}$

6–6. $A = \frac{2}{3}ah$

$\bar{y} = \frac{3}{5}h$

6–7. $\bar{x} = 1.61\ \text{in.}$

6–9. $\bar{y} = \frac{3}{10}h$

6–10. $\bar{x} = \dfrac{n + 1}{2(n + 2)}a$

6–11. $dA = \left(x - \frac{x^3}{9}\right) dx$

$\tilde{x} = x$

$\tilde{y} = \frac{1}{2}\left(x + \frac{x^3}{9}\right)$

$A = 2.25\ \text{ft}^2$

$\bar{x} = 1.6\ \text{ft}$

$\bar{y} = 1.14\ \text{ft}$

6–13. $\bar{y} = 1.43\ \text{in.}$

6–14. $\bar{y} = \frac{3}{8}b$

$\bar{x} = 0$

6–15. $\bar{y} = \dfrac{a}{2(10 - 3\pi)}$

6–17. $\bar{z} = 12.8\ \text{in.}$

6–18. $\bar{z} = \frac{2}{9}h$

6–19. $\bar{x} = 3\ \text{in.}, \bar{y} = 2\ \text{in.}$

6–21. $\bar{x} = 77.2\ \text{mm}, \bar{y} = 31.7\ \text{mm}$

6–22. $\bar{y} = 135\ \text{mm}$

6–23. $\bar{y} = 10.2\ \text{in.}$

6–25. $\bar{y} = 85.9\ \text{mm}$

6–26. $\bar{x} = 22.7\ \text{mm}$

$\bar{y} = 29.5\ \text{mm}$

$\bar{z} = 22.6\ \text{mm}$

6–27. $\bar{z} = 463\,\text{mm}$

6–29. $h = 323\ \text{mm}$

6–30. $h = \dfrac{a^3 - a^2\sqrt{a^2 - \pi r^2}}{\pi r^2}$

6–31. $F_R = 6.75\ \text{kN} \downarrow$

$\bar{x} = 2.5\ \text{m}$

6–33. $F_R = 10.6\ \text{kip} \downarrow$

$x = 0.479\ \text{ft}$

6–34. $F_R = \frac{1}{2}w_0 L \downarrow$

$\bar{x} = \frac{5}{12}L$

6–35. $F_R = 3.90\ \text{kip} \uparrow$

$d = 11.3\ \text{ft}$

6–37. $F_R = 18.0\ \text{kip} \downarrow$

$x = 11.7\ \text{ft}$

6–38. $w_2 = 17.2\ \text{kN/m}$

$w_1 = 30.3\ \text{kN/m}$

6–39. $F_R = \dfrac{2w_0 L}{\pi} \downarrow$

$\bar{x} = \dfrac{2L}{\pi}$

6–41. $F_R = 1.35\ \text{kN}$

$x = 0.556\ \text{m}$

6–42. $I_x = 18.5\ \text{in}^4$

6–43. $I_y = 9.6\ \text{in}^4$

6–45. $I_y = 1.07\ \text{in}^4$

6–46. $I_x = 39.0\ \text{m}^4$

6–47. $I_y = 8.53\ \text{m}^4$

6–49. $I_y = 0.286\ \text{m}^4$

6–50. $dA = \left[1 - \left(\frac{y}{2}\right)^{1/4}\right] dy$

$I_x = 0.2051\ \text{m}^4$

$dA = 2x^4\,dx$

$I_y = 0.2857\ \text{m}^4$

$J_O = 0.491\ \text{m}^4$

6–51. $I_x = 0.533\ \text{m}^4$

6–53. $I_x = 3.20\ \text{m}^4$

6–54. $I_y = 0.762\ \text{m}^4$

6–55. $I_x = 19.5\ \text{in}^4$

6–57. $I_x = \dfrac{4a^4}{9\pi}$

6–58. $I_y = \left(\dfrac{\pi^2 - 4}{\pi^3}\right)a^4$

6–59. $\bar{y} = 2.20\ \text{in.}$

$I_{x'} = 57.9\ \text{in}^4$

6–61. $I_y = \frac{1}{12}(2)(6)^3 + 2\left[\frac{1}{12}(4)(1)^3 + 1(4)(1.5)^2\right]$

$= 54.7\ \text{in}^4$

6–62. $I_x = 209\ \text{in}^4$

6–63. $I_y = 533\ \text{in}^4$

6–65. $\bar{I}_{x'} = 49.5(10^6)\ \text{mm}^4$

6–66. $I_y = 115(10^6)\ \text{mm}^4$

6–67. $\bar{y} = 207\,\text{mm}$

$\bar{I}_{x'} = 222(10^6)\,\text{mm}^4$

6–69. $I_y = 1971\,\text{in}^4$

6–70. $\bar{y} = 170\ \text{mm}$

$I_{x'} = 722(10)^6\,\text{mm}^4$

6–71. $I_x = 2.17(10^{-3})\,\text{m}^4$

Chapter 6

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$\bar{x} = 2.5\ \text{m}$

6–33. $F_R = 10.6\ \text{kip} \downarrow$

$x = 0.479\ \text{ft}$

6–34. $F_R = \frac{1}{2}w_0 L \downarrow$

$\bar{x} = \frac{5}{12}L$

6–35. $F_R = 3.90\ \text{kip} \uparrow$

$d = 11.3\ \text{ft}$

6–37. $F_R = 18.0\ \text{kip} \downarrow$

$x = 11.7\ \text{ft}$

6–38. $w_2 = 17.2\ \text{kN/m}$

$w_1 = 30.3\ \text{kN/m}$

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6–55. $I_x = 19.5\ \text{in}^4$

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6–70. $\bar{y} = 170\ \text{mm}$

$I_{x'} = 722(10)^6\,\text{mm}^4$

6–71. $I_x = 2.17(10^{-3})\,\text{m}^4$

6–73. $\bar{y} = 3.79$ in.
6–74. $\bar{y} = 3.79$ in. $I_{x'} = 198\ \text{in}^4$
6–75. $I_x = 1.19(10^3)\ \text{in}^4$
6–77. $\bar{y} = 87.5$ mm
6–78. $\bar{y} = 0.600$ in.
6–79. $\bar{x} = \dfrac{b-a}{\ln\dfrac{b}{a}}$
6–81. $w_1 = 660$ lb/ft
$w_2 = 720$ lb/ft
6–82. $I_x = 0.0954\, d^4$
6–83. Consider four triangles and a rectangle.
$I_y = 0.187\, d^4$
6–85. $\bar{y} = 0.875$ in., $I_{x'} = 2.27\ \text{in}^4$
6–86. $F_R = 577$ lb
$\theta = 47.5°$ ⦤
$M_{RA} = 2.20$ kip · ft ↻
6–87. $F_R = 577$ lb
$\theta = 47.5°$ ⦤
$M_{RB} = 2.80$ kip · ft ↺

Chapter 7

7–1. $N_E = 0, V_E = -200$ lb, $M_E = -2.40$ kip · ft
7–2. $N_a = 500$ lb, $V_a = 0$,
$N_b = 4331$ lb, $V_b = 250$ lb
7–3. $N_C = 0, V_C = 3.50$ kip, $M_C = -47.5$ kip · ft,
$N_D = 0, V_D = 0.240$ kip, $M_D = -0.360$ kip · ft
7–5. $9.00(4) - A_y(12) = 0, A_y = 3.00$ kip,
$B_y = 6.00$ kip, $N_D = 0, V_D = 0.750$ kip,
$M_D = 13.5$ kip · ft, $N_E = 0, V_E = -9.00$ kip,
$M_E = -24.0$ kip · ft
7–6. $N_C = -30.0$ kN, $V_C = -8.00$ kN,
$M_C = 6.00$ kN · m
7–7. $P = 0.533$ kN, $N_C = -2.00$ kN,
$V_C = -0.533$ kN, $M_C = 0.400$ kN · m
7–9. $N_D = 0, V_D = -1.875$ kN,
$M_D = 3.94$ kN · m
7–10. $N_A = 0, V_A = 450$ lb, $M_A = -1.125$ kip · ft,
$N_B = 0, V_B = 850$ lb, $M_B = -6.325$ kip · ft,
$V_C = 0, N_C = -1.20$ kip, $M_C = -8.125$ kip · ft
7–11. $N_E = -22.5$ N, $V_E = -64.5$ N,
$M_E = -2.26$ N · m
7–13. $N_{a-a} = -100$ N, $V_{a-a} = 0, M_{a-a} = -15$ N · m
7–14. $N_{b-b} = -86.6$ N, $V_{b-b} = 50$ N,
$M_{b-b} = -15$ N · m
7–15. $N_D = 300$ lb, $V_D = -150$ lb, $M_D = -150$ lb · ft
7–17. $N_B = 5.303$ kN, $N_{a-a} = -3.75$ kN,
$V_{a-a} = 1.25$ kN, $M_{a-a} = 3.75$ kN · m,
$N_{b-b} = -1.77$ kN, $V_{b-b} = 3.54$ kN,
$M_{b-b} = 3.75$ kN · m
7–18. $N_C = -80$ lb, $V_C = 0, M_C = -480$ lb · in.
7–19. $N_C = 0, V_C = 4.50$ kip, $M_C = 31.5$ kip · ft
7–21. $\tau_{\text{avg}} = 119$ MPa
7–22. $\tau_{\text{avg}} = 29.5$ MPa
7–23. $V = P\cos\theta, N = P\sin\theta,$
$\sigma = \dfrac{P}{A}\sin^2\theta, \tau_{\text{avg}} = \dfrac{P}{2A}\sin 2\theta$
7–25. $(\sigma_{a-a})_{\text{avg}} = 1.80$ ksi, $\sigma_b = 4.58$ ksi
7–26. $15(10^3) = 7.50$ kip (controls)
7–27. $dF = 7.5(10^6)\, x^{1/2}\, dx,$
$P = 40$ MN, $d = 2.40$ m
7–29. $\sigma_{\text{avg}} = 5$ MPa
7–30. $(\sigma_{\text{avg}})_{BC} = 159$ MPa
$(\sigma_{\text{avg}})_{AC} = 95.5$ MPa
$(\sigma_{\text{avg}})_{AB} = 127$ MPa
7–31. $P = 37.7$ kN
7–33. $(\sigma_{\text{avg}})_{BD} = 23.6$ MPa, $(\sigma_{\text{avg}})_{CF} = 69.4$ MPa
7–34. $(\tau_{\text{avg}})_A = 58.7$ MPa
$(\tau_{\text{avg}})_D = 26.5$ MPa
7–35. $x = 4$ in., $y = 4$ in., $\sigma = 9.26$ psi
7–37. $m = 148$ kg
7–38. $\tau_B = \tau_C = 324$ MPa
$\tau_A = 324$ MPa
7–39. $\sigma_{AB} = 2.17$ ksi, $\sigma_{BC} = 0.819$ ksi
7–41. $P = 4$ kip, $(\tau_{a-a})_{\text{avg}} = 250$ psi
7–42. $(\tau_{\text{avg}})_b = 79.6$ MPa
$(\tau_{\text{avg}})_p = 225$ kPa
7–43. $P = 9.05$ kN
7–45. $\sigma_{AB} = 127$ MPa, $\sigma_{AC} = 129$ MPa
7–46. $d_{AB} = 11.9$ mm
7–47. $\sigma_{\text{avg}} = \dfrac{\gamma}{3}\left[\dfrac{(z+h)^3 - h^3}{(z+h)^2}\right]$
7–49. $\sigma = 8.15$ ksi, $\tau = 5.87$ ksi
7–50. $\sigma_b = 1.10$ MPa
7–51. $\tau_{\text{avg}} = 1.77$ MPa
7–53. $d = 5.71$ mm
7–54. $d = 13.5$ mm
7–55. $A = 0.0356\ \text{in}^2$
7–57. $a = 6\dfrac{1}{2}$ in.
7–58. For nails at A: $\tau_{\text{avg}} = 1.53$ ksi
For nails at B: $\tau_{\text{avg}} = 1.68$ ksi
7–59. $d_A = 0.155$ in.
$d_B = 0.162$ in.
7–61. $d_{AB} = 1.81$ mm, $d_{BC} = 2.00$ mm
7–62. $(\text{F.S.})_{AB} = 1.72, (\text{F.S.})_{BC} = 2.50$
7–63. $P = 561$ kN (controls)
7–65. $(\text{F.S.})_{st} = 2.14, (\text{F.S.})_{con} = 3.53$
7–66. $P = 90$ kN, $A = 6.19(10^{-3})\ \text{m}^2,$
$P_{\max} = 155$ kN
7–67. Use $d_C = 12$ mm
Use $d_D = 14$ mm
7–69. $d_{AB} = 15.5$ mm, $d_{AC} = 13.0$ mm

7–70. $P = 7.54\text{ kN}$
7–71. $d_B = 6.11\text{ mm}$
$d_w = 15.4\text{ mm}$
7–73. $P = 9.09\text{ kip}$
7–74. $a_{A'} = 130\text{ mm}, a_{B'} = 300\text{ mm}$
7–75. $P = 72.5\text{ kN}$
7–77. $h = 1.74\text{ in.}$
7–78. Use $a_D = 19\frac{5}{8}\text{ in.}, a_C = 21\frac{1}{16}\text{ in.}$
7–79. $(\text{F.S.})_C = 2.00$
$(\text{F.S.})_D = 2.30$
7–81. $\epsilon = 0.0472\text{ in./in.}$
7–82. $\epsilon_{CE} = 0.00250\text{ mm/mm}, \epsilon_{BD} = 0.00107\text{ mm/mm}$
7–83. $(\epsilon_{\text{avg}})_{AH} = 0.0349\text{ mm/mm}$
$(\epsilon_{\text{avg}})_{CG} = 0.0349\text{ mm/mm}$
$(\epsilon_{\text{avg}})_{DF} = 0.0582\text{ mm/mm}$
7–85. $\epsilon_{\text{avg}} = \frac{\pi}{h}(z + h) - 1$
7–86. $(\epsilon_{\text{avg}})_{AC} = 6.04\,(10^{-3})\text{ mm/mm}$
7–87. $AB = 0.00251\text{ mm/mm}$
7–89. $(\gamma_A)_{nt} = 0.0502\text{ rad}, (\gamma_B)_{nt} = -0.0502\text{ rad}$
7–90. $\epsilon_{AB} = -0.00469\text{ in./in.}, \epsilon_{AC} = 0.0200\text{ in./in.}$,
$\epsilon_{DB} = -0.0300\text{ in./in.}$
7–91. $\gamma_{xy} = 0.0142\text{ rad}$
7–93. $(\epsilon_{\text{avg}})_{AC} = 0.0258\text{ mm/mm}$
7–94. $(\epsilon_{\text{avg}})_{AE} = 0.0207\text{ mm/mm}$
7–95. $\epsilon_{BC} = 5.98(10^{-3})\text{ mm/mm}$
7–97. $(\gamma_B)_{xy} = 11.6(10^{-3})\text{ rad}$,
$(\gamma_A)_{xy} = 11.6(10^{-3})\text{ rad}$
7–98. $(\gamma_C)_{xy} = 11.6(10^{-3})\text{ rad}$,
$(\gamma_D)_{xy} = 11.6(10^{-3})\text{ rad}$
7–99. $\epsilon_{AC} = 1.60(10^{-3})\text{ mm/mm}$
$\epsilon_{DB} = 12.8(10^{-3})\text{ mm/mm}$
7–101. $\sigma_s = 208\text{ MPa}, (\tau_{\text{avg}})_a = 4.72\text{ MPa}$,
$(\tau_{\text{avg}})_b = 45.5\text{ MPa}$
7–102. $t = \frac{1}{4}\text{ in.}, d_A = 1\frac{1}{8}\text{ in.}, d_B = \frac{13}{16}\text{ in.}$
7–103. $\tau_{\text{avg}} = 79.6\text{ MPa}$
7–105. $\sigma_{a-a} = 200\text{ kPa}, \tau_{a-a} = 115\text{ kPa}$
7–106. $(\epsilon_{\text{avg}})_{CA} = -5.59\,(10^{-3})\text{ mm/mm}$
7–107. $(\gamma_E)_{x'y'} = 0.996\,(10^{-3})\text{ rad}$

Chapter 8

8–1. $(\sigma_{\text{ult}})_{\text{approx}} = 110\text{ ksi}, (\sigma_R)_{\text{approx}} = 93.1\text{ ksi}$,
$(\sigma_Y)_{\text{approx}} = 55\text{ ksi}, E_{\text{approx}} = 32.0\,(10^3)\text{ ksi}$
8–2. $E = 55.3\,(10^3)\text{ ksi}, u_r = 9.96\,\frac{\text{in}\cdot\text{lb}}{\text{in}^3}$
8–3. $(u_t)_{\text{approx}} = 85.0\,\frac{\text{in}\cdot\text{lb}}{\text{in}^3}$
8–5. $u_r = 16.3\,\frac{\text{in.}\cdot\text{kip}}{\text{in}^3}$
8–6. $E = 8.83\,(10^3)\text{ ksi}$
8–7. $A = 0.209\text{ in}^2, P = 1.62\text{ kip}$
8–9. $E = 5.5\text{ psi}, u_t = 19.25\text{ psi}, u_r = 11\text{ psi}$
8–10. $E = 30.0\,(10^3)\text{ ksi}, P_Y = 11.8\text{ kip}, P_{\text{ult}} = 19.6\text{ kip}$
8–11. Elastic Recovery $= 0.003\text{ in./in.}$,
$\Delta L = 0.094\text{ in.}$
8–13. $\sigma = 11.43\text{ ksi}, \epsilon = 0.000400\text{ in./in.}$,
$E = 28.6\,(10^3)\text{ ksi}$
8–14. $E_{\text{approx}} = 6.50\,(10^3)\text{ ksi}, \sigma_{YS} = 25.9\text{ ksi}$
8–15. $\delta_P = 0.00637\text{ in.}$
8–17. $P = 15.0\text{ kip}$
8–18. $A_{BC} = 0.8\text{ in}^2, A_{BA} = 0.2\text{ in}^2$
8–19. $\sigma = 2.22\text{ MPa}$
8–21. $\sigma = 1.697\text{ MPa}$,
$\delta = 0.126\text{ mm}, \Delta d = -0.00377\text{ mm}$
8–22. $E = 67.9\text{ GPa}, \nu = 0.344, G = 25.3\text{ GPa}$
8–23. $P = 157\text{ kN}$
8–25. $\epsilon = 0.08660\text{ mm/mm}, \gamma = 0.140\text{ rad}$
8–26. $\epsilon_y = -0.0150\text{ in./in.}, \epsilon_x = 0.00540\text{ in./in.}$,
$\gamma_{xy} = -0.00524\text{ rad}$
8–27. $P = 53.0\text{ kip}, E = 28.6\,(10^3)\text{ksi}$
8–29. $\tau_{\text{avg}} = 4166.67\text{ Pa}, \gamma = 0.02083\text{ rad}$,
$\delta = 0.833\text{ mm}$
8–30. $\delta = \frac{Pa}{2bhG}$
8–31. $G_{\text{al}} = 4.31\,(10^3)\text{ksi}$
8–33. $x = 1.53\text{ m}, d'_A = 30.008\text{ mm}$
8–34. $\theta = 0.0139°$
8–35. $P = 6.48\text{ kip}$
8–37. $L = 10.17\text{ in.}$
8–38. $\delta V = \frac{PL}{E}(1 - 2\nu)$
8–39. $\epsilon_b = 0.00227\text{ mm/mm}$,
$\epsilon_r = 0.000884\text{ mm/mm}$

Chapter 9

9–1. $\delta_B = 2.31\text{ mm}, \delta_A = 2.64\text{ mm}$
9–2. $\delta_{A/D} = 0.111\text{ in. away from end } D.$
9–3. $\sigma_{AB} = 22.2\text{ ksi (T)}, \sigma_{BC} = 41.7\text{ ksi (C)}$,
$\sigma_{CD} = 25.0\text{ ksi (C)}, \delta_{A/D} = 0.00157\text{ in.}$
towards end D
9–5. $\delta_A = 0.0128\text{ in.}$
9–6. $\delta_A = 0.0128\text{ in.}$
9–7. $\delta_A = -0.194\text{ in.}$
9–9. $\delta_F = 0.453\text{ mm}$
9–10. $P = 4.97\text{ kN}$
9–11. $\delta_l = 0.0260\text{ in.}$
9–13. $\delta_D = 17.3\text{ mm}$
9–14. $\delta_{A/B} = -0.864\text{ mm}$

9–15. $\delta_{A/B} = -1.03$ mm
9–17. $P = 59.5$ kN
9–18. $(\delta_A)_v = 0.0379$ in.
9–19. $P = 9.24$ kip
9–21. $\delta_D = 0.1374$ mm, $\delta_{A/B} = 0.3958$ mm, $\delta_C = 0.5332$ mm, $\delta_{\text{tot}} = 33.9$ mm
9–22. $W = 9.69$ kN
9–25. $\delta = 0.360$ mm
9–26. $\delta = 0.00257$ in.
9–27. $\sigma_{st} = 65.9$ MPa, $\sigma_{con} = 8.24$ MPa
9–29. $P_{\text{st}} = 57.47$ kN, $P_{\text{con}} = 22.53$ kN, $\sigma_{\text{st}} = 48.8$ MPa, $\sigma_{\text{con}} = 5.85$ MPa
9–30. $\sigma_{con} = 1.64$ ksi, $\sigma_{st} = 11.3$ ksi
9–31. $P = 114$ kip
9–33. $F_C = \left[\dfrac{9(8ka + \pi d^2E)}{136ka + 18\pi d^2E}\right]P$, $F_A = \left(\dfrac{64ka + 9\pi d^2E}{136ka + 18\pi d^2E}\right)P$
9–34. $T_{AB} = 1.12$ kip, $T_{AC} = 1.68$ kip
9–35. $A_{AB} = 0.03$ in^2
9–37. $F_D = 71.4$ kN, $F_C = 329$ kN
9–38. $F_D = 219$ kN, $F_C = 181$ kN
9–39. $\sigma_{AB} = 26.5$ MPa, $\sigma_{EF} = 33.8$ MPa
9–41. $F_b = 10.17\,(10^3)$ N, $F_t = 29.83\,(10^3)$ N, $\sigma_b = 32.4$ MPa, $\sigma_t = 34.5$ MPa
9–42. $F_D = 20.4$ kN, $F_A = 180$ kN
9–43. $P = 198$ kN
9–45. $y = 3 - 0.025x$, $F_A = 4.09$ kip, $F_B = 2.91$ kip
9–46. $x = 28.9$ in., $P = 60.4$ kip
9–47. $T_{CD} = 27.2$ kip, $T_{CD} = 9.06$ kip
9–49. $F_{\text{st}} = 1.822$ kip, $F_{\text{al}} = 3.644$ kip, $\sigma_{\text{rod}} = 9.28$ ksi, $\sigma_{\text{cyl}} = 1.16$ ksi
9–50. $\theta = 698°$
9–51. $\sigma_{BE} = 96.3$ MPa, $\sigma_{AD} = 79.6$ MPa, $\sigma_{CF} = 113$ MPa
9–53. $\sigma_{al} = 2.46$ ksi, $\sigma_{br} = 5.52$ ksi, $\sigma_{st} = 22.1$ ksi
9–54. $F = 0.509$ kip
9–55. $\sigma_{AB} = \dfrac{2}{5}\alpha(T_2 - T_1)E$, $\sigma_{BC} = \dfrac{8}{5}\alpha(T_2 - T_1)E$
9–57. $0 = \delta_T - \delta_F$, $F = 19.14A$, $\sigma = 19.1$ ksi
9–58. $F = 7.60$ kip
9–59. $\delta = 0.348$ in., $F = 19.5$ kip
9–61. $0 = \Delta_T - \delta_F$, $F = \dfrac{\alpha A E}{2}(T_B - T_A)$
9–62. $\sigma = 180$ MPa
9–63. $\sigma = 105$ MPa
9–65. $\sigma_{\max} = 168$ MPa
9–66. $P = 5.71$ kN
9–67. $P = 49.1$ kN
9–69. $P = 1.34$ kip
9–70. $\sigma_{\max} = 31.3$ ksi
9–71. $\sigma_{\max} = 81.7$ MPa
9–73. $\sigma_b = 33.5$ MPa, $\sigma_r = 16.8$ MPa
9–74. $T = 507°$ C
9–75. $F_{AB} = F_{AC} = F_{AD} = 58.9$ kN
9–77. $P = 56.5$ kN, $\delta_{B/A} = 0.0918$ mm
9–78. $\theta = \dfrac{3E_2L(T_2 - T_1)(\alpha_2 - \alpha_1)}{d(5E_2 + E_1)}$
9–79. $P = 4.85$ kip
9–81. $\delta_{A/B} = 0.491$ mm

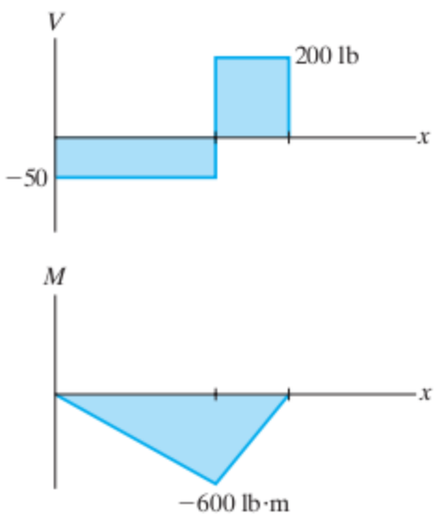
Chapter 10

10–1. $r' = 0.841r$
10–2. $r' = 0.707r$
10–3. $\tau_B = 6.04$ MPa, $\tau_A = 6.04$ MPa
10–5. $\tau_A = 3.45$ ksi, $\tau_B = 2.76$ ksi
10–6. $(\tau_{BC})_{\max} = 5.07$ ksi, $(\tau_{DE})_{\max} = 3.62$ ksi
10–7. $(\tau_{EF})_{\max} = 0$, $(\tau_{CD})_{\max} = 2.17$ ksi
10–9. $(\tau_{AB})_{\max} = 23.9$ MPa, $(\tau_{BC})_{\max} = 15.9$ MPa
10–10. $d = 30$ mm
10–11. $\tau_{AB} = 7.82$ ksi, $\tau_{BC} = 2.36$ ksi
10–13. Use $t = 25$ mm
10–14. Use $d = 1\dfrac{3}{4}$ in.
10–15. $d = 33$ mm
10–17. $\tau_{\max} = 4.89$ ksi
10–18. $\tau_{\max} = 7.33$ ksi
10–19. $T = 7.54$ kN · m
10–21. $\tau_{AB} = (2000x - 1200)$ N · m, $d = 0.9$ m, $\tau_{\min} = 0$, $d = 0$, $\tau_{\max} = 42.4$ MPa
10–22. $d = 57$ mm
10–23. $T_A + \dfrac{1}{2}t_AL - T_B = 0$, $T_B = \dfrac{2T_A + t_AL}{2}$, $\tau_{\max} = \dfrac{(2T_A + t_AL)r_o}{\pi(r_o^4 - r_i^4)}$
10–25. $T_{\max} = 260.42$ lb · ft, $\tau_{\max}^{\text{abs}} = 3.59$ ksi
10–26. $(\tau_{AB})_{\max} = 1.04$ MPa, $(\tau_{BC})_{\max} = 3.11$ MPa
10–27. $\tau_{\max} = 3.44$ MPa
10–29. $\gamma = \dfrac{Tc}{2JG}$
10–30. $T_{AB} = -85$ N · m, $T_{BC} = -85$ N · m, $\phi_{A/D} = 0.879°$
10–31. $\tau_{\max} = 2.83$ ksi, $\phi = 4.43°$
10–33. $T = 4.96$ kN · m (controls)

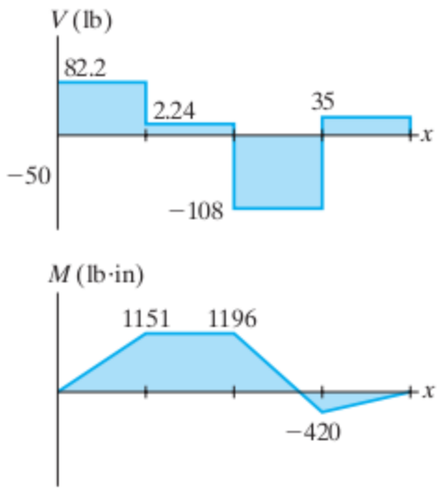
10–34. $T_{BC} = -80\ \text{N}\cdot\text{m}$,
$T_{CD} = -60\ \text{N}\cdot\text{m}$,
$T_{DA} = -90\ \text{N}\cdot\text{m}$,
$\phi_B = |5.74°|$
10–35. Use $d = 22$ mm, $\phi_{A/D} = 2.54°$
10–37. $\phi_{B/C} = 0.0646°$
10–38. $\tau_{\max} = 9.12$ MPa, $\phi_{E/B} = 0.585°$
10–39. $\tau_{\max} = 14.6$ MPa, $\phi_{B/E} = 1.11°$
10–41. $\phi_{C/B} = 1.15°$
10–42. $\phi_E = 0.01778$ rad,
$\phi_F = 0.02667$ rad,
$\phi_B = 1.53°$
10–43. $\phi_A = 1.78°$
10–45. $L = 470$ mm
10–46. $F = 6.03$ N, $s = 0.720$ mm
10–47. $\phi = \dfrac{T}{2a\pi G}(1 - e^{-4aL})$
10–49. $t_o = \dfrac{4\,pd}{L}$,
$\phi = \dfrac{4PLd}{3\pi r^4 G}$
10–50. $\phi_{A/B} = 5.62°$
10–51. $\theta = \dfrac{T}{4\pi\, hG}\left[\dfrac{1}{r_i^2} - \dfrac{1}{r_o^2}\right]$
10–53. $\tau_{AC} = 9.77$ MPa
10–54. $\tau_{\max} = 29.3$ ksi
10–55. $T_A = 22\ \text{kip}\cdot\text{ft}$
$T_D = 2\ \text{kip}\cdot\text{ft}$
10–57. $\phi_C = 0.116°$,
$(\tau_s)_{\max} = 395$ psi,
$(\gamma_{st})_{\max} = 34.3(10^{-6})$ rad,
$(\tau_{br})_{\max} = 96.1$ psi,
$(\gamma_{bt})_{\max} = 17.2(10^{-6})$ rad
10–58. $(\tau_{BC})_{\max} = 1.47$ ksi,
$(\tau_{BD})_{\max} = 1.96$ ksi,
$\phi = 0.338°$
10–59. $d = 2c = 0.04269\ \text{m} = 42.7\ \text{mm}$
10–61. $T_B = 222\ \text{N}\cdot\text{m}$,
$T_A = 55.6\ \text{N}\cdot\text{m}$
10–62. $\phi_E = 1.66°$
10–63. $T = 4.34\ \text{kN}\cdot\text{m}$ (control!)
$\phi_A = 2.58°$
10–65. $\phi_{C/D} = 6.22°$
10–66. $\tau_{\substack{\text{abs}\\\max}} = 5.50$ ksi
10–67. $(\tau_{\max})_{\text{abs}} = 93.1$ MPa
10–69. $T_B = \dfrac{7t_0 L}{12}$,
$T_A = \dfrac{3t_0 L}{4}$
10–70. $T = 7.74\ \text{N}\cdot\text{m}$
10–71. % of increase in shear stress $= 41.4\%$
% of increase in angle of twist $= 25\%$
10–73. $(\tau_{BC})_{\max} = 0.955$ MPa,
$(\tau_{AC})_{\max} = 1.59$ MPa,
$\phi_{B/A} = 0.207°$
10–74. $(\tau_{BC})_{\max} = 0.955$ MPa,
$(\tau_{AC})_{\max} = 1.59$ MPa,
$\phi_{B/C} = |0.0643°|$
10–75. $\tau_{\max} = 8$ ksi
$\phi = 2.28°$
10–77. $T = 20.1\ \text{N}\cdot\text{m}$
10–78. No, it is not possible.
10–79. $P = 101$ kW
10–81. $T = 8.16\ \text{N}\cdot\text{m}$
10–82. $r = 0.075$ in.
10–83. Use $d = 26$ mm, $\phi_{A/C} = 2.11°$
10–85. $r_o = 0.0625$ m, $r_i = 0.0575$ m,
Eq. 5–7: $\tau_{\rho=0.06\,\text{m}} = 88.27$ MPa,
Eq. 5–18: $\tau_{\text{avg}} = 88.42$ MPa,
Eq. 5–15: $\phi = 4.495°$,
Eq. 5–20: $\phi = 4.503°$
10–86. $T = 331\ \text{N}\cdot\text{m}$
10–87. The circular shaft will resist the largest torque.
For the square shaft: 73.7%,
For the triangular shaft: 62.2%
10–89. $P = 2.80$ kip
10–90. $\phi_A = 1.59°$
10–91. $F = 26.2$ N, $\phi = 1.86°$

Chapter 11

11–1. $T_1 = 250$ lb, $T_2 = 200$ lb



11–2.



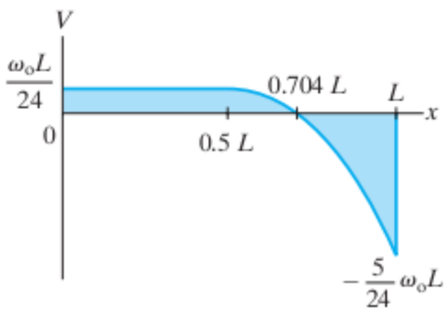
11–3. $x = 3^-,\ V = -2000,\ M = -6000$

11–5. $x = 2^+, V = 8, M = -39$

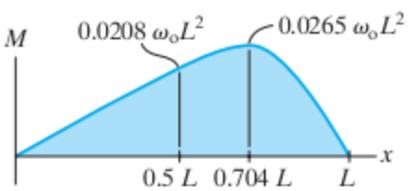
11–6. For $0 \le x < \frac{L}{2}: V = \frac{w_0 L}{24}, M = \frac{w_0 L}{24}x,$

For $\frac{L}{2} < x \le L: V = \frac{w_0}{24L}[L^2 - 6(2x - L)^2],$

$M = \frac{w_0}{24L}[L^2 x - (2x - L)^3]$

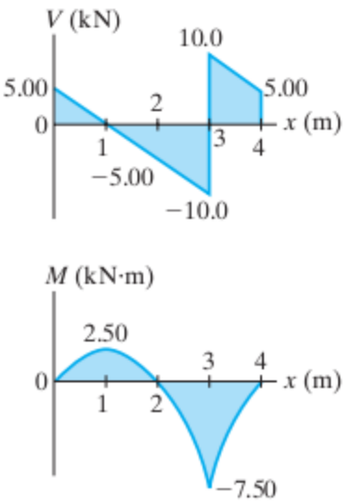


(d)

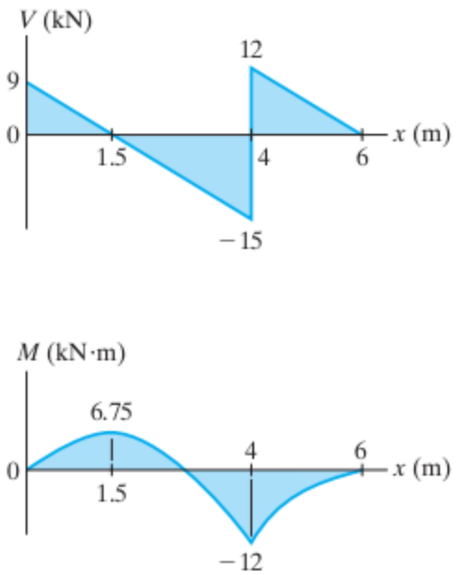


(e)

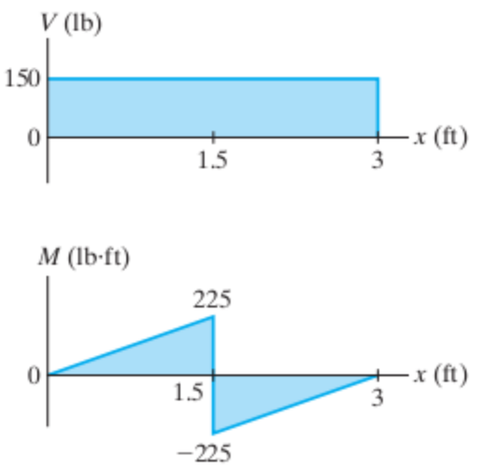
11–7.



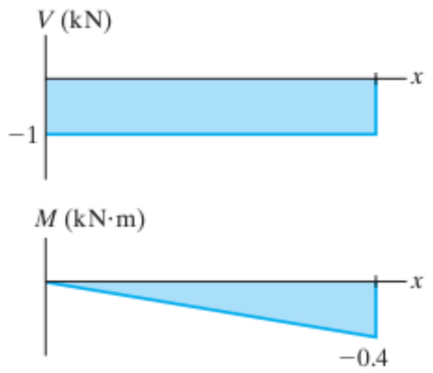
11–9. For $0 \le x < 4$ m: $V = \{9 - 6x\}$ kN,
$M = \{9x - 3x^2\}$ kN · m,
For 4 m $< x \le 6$ m: $V = \{6(6 - x)\}$ kN · m,
$M = -\{3(6 - x)^2\}$ kN · m



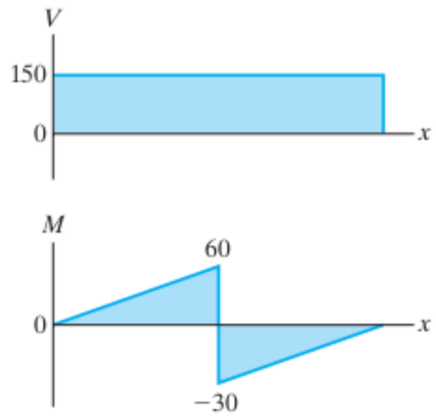
11–10.



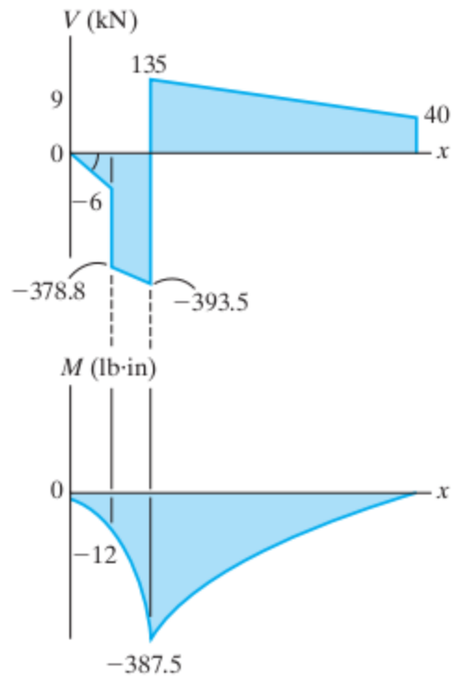
11–11.



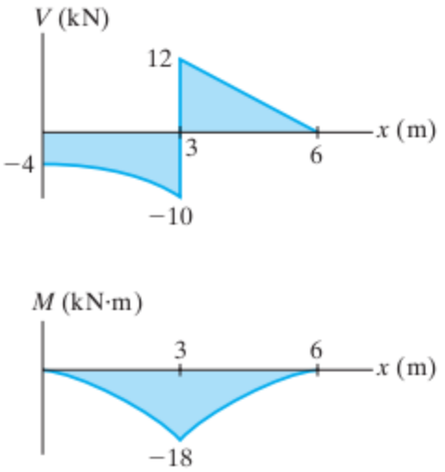
11–13.



11–14.



11–15.

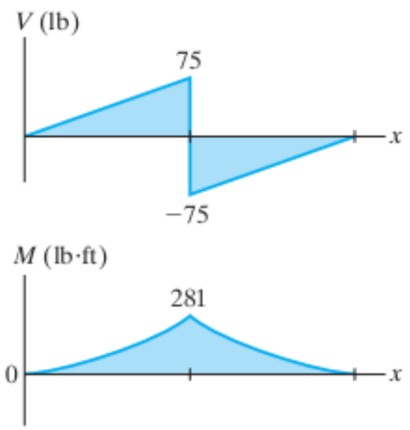


11–17. $x = 6, V = -900, M = -3000,$
$V = \{-300 - 16.67x^2\}$ lb,
$M = \{-300x - 5.556x^3\}$ lb · ft

11–18. $V = \{30.0 - 2x\}$ kip,
$M = \{-x^2 + 30.0x - 216\}$ kip · ft,
$V = 8.00$ kip
$M = \{8.00x - 120\}$ kip · ft

11–19. $x = 5^-, V = -10, M = -25$

11–21. $M_{max} = 281$ lb · ft



11–22. $x = L/3, V = w_0L/6, M = 5w_0L^2/54$

11–23. $x = 4.11, V = 0, M = 25.7,$
At $x = 4.108$ m, $M = 25.67$ kN · m,
At $x = 4.5$ m, $M = 25.31$ kN · m

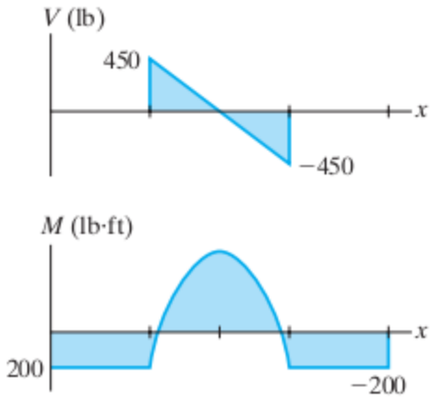
11–25. $V = \{300 - 50x^2\}$ N

$$M = \left\{300x - \frac{50}{3}x^3\right\} \text{N} \cdot \text{m}$$

$V = \{100(4.5 - x)^2 - 375\}$ N

$$M = \left\{375(4.5 - x) - \frac{100}{3}(4.5 - x)^3\right\} \text{N} \cdot \text{m}$$

11–26. $V = 1050 - 150x$
$M = -75x^2 + 1050x - 3200$

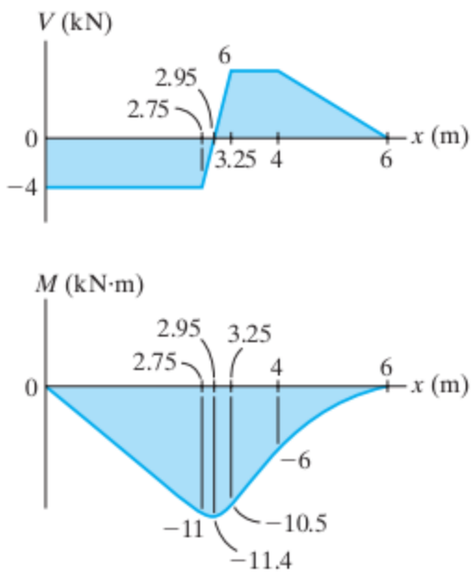


11–27. $w = 40.0$ lb/ft,
$V = 30.0$ lb,
$M = 15.0$ lb · ft

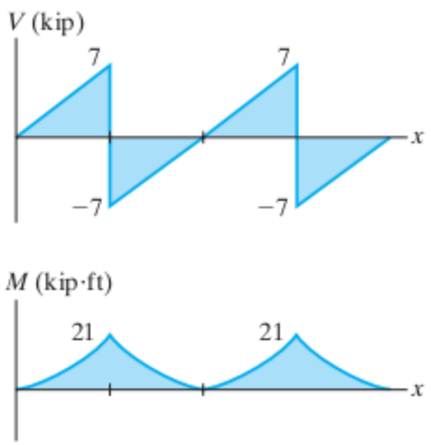
11–29. $x = 2^+, V = -2.5, M = 15$

11–30. $x = 2^+, V = -14.5, M = 7$

11–31.



11–34.



11–35. $(\sigma_t)_{max} = 3.72$ ksi, $(\sigma_c)_{max} = 1.78$ ksi

11–37. $\bar{y} = 3.40$ in.,
$I_{NA} = 91.73$ in^4,
$(\sigma_t)_{max} = 3.72$ ksi, $(\sigma_c)_{max} = 1.78$ ksi

11–38. $M = 2.50$ kip · ft

11–39. $(\sigma_{max})_t = 2.40$ ksi, $(\sigma_{max})_c = 4.80$ ksi

11–41. $F_R = 200$ kN

11–42. $I_a = 0.21645(10^{-3})$ m^4,
$I_b = 0.36135(10^{-3})$ m^4,
$\sigma_{max} = 74.7$ MPa

11–43. $\sigma_{max} = 22.1$ ksi

11–45. $\sigma = 13.6$ ksi

11–46. $d = 1.28$ in.

11–47. $\sigma_A = 214$ psi (C), $\sigma_B = 33.0$ psi (T),
$\sigma_C = 115$ psi (T)

11–49. $\sigma_{max} = 12.2$ ksi

11–50. $(\sigma_{max})_c = 78.1$ MPa,
$(\sigma_{max})_t = 165$ MPa

11–51. $M = 50.3$ kN · m

11–53. Use $h = 275$ in.

11–54. (a) $\sigma_{max} = 497$ kPa,
(b) $\sigma_{max} = 497$ kPa

11–55. (a) $\sigma_{max} = 249$ kPa,
(b) $\sigma_{max} = 249$ kPa

11–57. $d_o = 188$ mm
$d_i = 151$ mm

11–58. $b = 53.1$ mm

11–59. $M = 123$ kN · m

11–61. $w = 18.75$ kN/m

11–62. $M_{max} = 7.50$ kN · m, $a = 66.9$ mm

11–63. $\sigma_{max} = \dfrac{23w_0L^2}{36bh^2}$

11–65. $(\sigma_{max})_c = 120$ MPa (C)
$(\sigma_{max})_c = 60$ MPa (T)

11–66. $\bar{y} = 0.012848$ m, $I = 0.79925(10^{-6})$ m^4,
$\epsilon_{max} = 0.711(10^{-3})$ mm/mm

11–67. $d = 116$ mm

11–69. $M_y = -14.14$ kip · ft, $M_z = -14.14$ kip · ft,
$I_y = 736$ in^4, $I_z = 1584$ in^4,
$\sigma_{max} = 2.01$ ksi (T), $\sigma_{max} = 2.01$ ksi (C),
$\alpha = 65.1°$

11–70. $M = 119$ kip · ft

11–71. $\bar{y} = 57.4$ mm, $\sigma_A = 1.30$ MPa (C), $\sigma_B = 0.587$ MPa (T), $\sigma = -3.74°$
11–73. $z'_A = 1.155$ in., $y'_A = -2.828$ in., $\sigma_A = 21.0$ ksi (C)
11–74. $\sigma_{max} = 4.70$ MPa, $\alpha = -76.0°$
11–75. $M = 5.11$ kN · m
11–77. $\sigma_A = 2.60$ MPa
11–78. $\sigma_B = 131$ MPa (C), $\alpha = -66.5°$
11–79. $M = 1186$ kN · m
11–81. $K = 2.60$, $M = 15.0$ kip · ft
11–82. $\sigma_{max} = 12.0$ ksi
11–83. $r = 5.00$ mm
11–85. $K = 1.92$, $P = 122$ lb
11–86. $\sigma_{max} = 29.5$ ksi
11–87. $r = 8.0$ mm
11–89. $F_R = 5.88$ kN
11–90. $(\sigma_{max})_t = 3.43$ MPa (T), $(\sigma_{max})_c = 1.62$ MPa (C)
11–91. $x = 0.6^-$, $V = -233$ N, $M = -50$ N · m
11–93. $I = 91.14583(10^{-6})$ m^4, $M = 36.5$ kN · m, $\sigma_{max} = 40.0$ MPa
11–94. **a.** $\sigma_{max} = 0.410$ MPa, **b.** $\sigma_{max} = 0.410$ MPa
11–95. $\sigma_{max} = 8.41$ ksi
11–97. $V = 20 - 2x$, $M = -x^2 + 20x - 166$
11–98. $\sigma_{max} = 25.8$ ksi

Chapter 12

12–1. $I = 0.2501(10^{-3})$ m^4, $Q_A = 0.64(10^{-3})$ m^3, $\tau_A = 2.56$ MPa
12–2. $\tau_{max} = 3.46$ MPa
12–3. $V_w = 19.0$ kN
12–5. $\bar{y} = 3.30$ in., $I_{NA} = 390.60$ in^4, $Q = 65.34 - 6y^2$, $V_f = 3.82$ kip
12–6. $V_{max} = 100$ kN
12–7. $\tau_{max} = 17.9$ MPa
12–9. $\bar{y} = 1.1667$ in., $I = 6.75$ in^4, $V = 32.1$ kip
12–10. $\tau_{max} = 4.48$ ksi
12–11. $\tau_{max} = 45.0$ MPa
12–13. $\bar{y} = 0.080196$ m, $I = 4.8646(10^{-6})$ m^4, $\tau_{max} = 4.22$ MPa
12–14. $V = 190$ kN
12–15. $V_{AB} = 50.3$ kN
12–17. $\tau_A = 1.99$ MPa, $\tau_B = 1.65$ MPa
12–18. $\tau_{max} = 4.62$ MPa
12–19. $V_w = 27.1$ kN
12–21. $I = 4\pi$ in^4, $Q = \frac{2}{3}(4 - y^2)^{3/2}$, $\tau_A = 2.39$ ksi
12–22. $\tau_B = 4.41$ MPa
12–23. $\tau_{max} = 4.85$ MPa
12–25. $V_C = -13.75$ kN, $I = 27.0(10^{-6})$ m^4, $Q_{max} = 0.216(10^{-3})$ m^3, $\tau_{max} = 3.67$ MPa
12–26. $a = 1.27$ in.
12–27. $\tau_{max} = 22.0$ MPa, $(\tau_{max})_s = 66.0$ MPa
12–33. $F = 675$ lb
12–34. $V = 16.7$ kN
12–35. $V = 28.2$ kN
12–37. $V = 2571$ lb, $\tau_b = 14.0$ ksi
12–38. $\tau_n = 35.2$ MPa
12–39. $I_{NA} = 72.0(10^{-6})$ m^4, $Q = 0.450(10^{-3})$ m^3, $P = 6.60$ kN
12–41. $\tau_B = 646$ psi, $\tau_A = 592$ psi
12–42. $\tau_{avg} = 97.2$ MPa
12–45. $P = 11.4$ kN (controls)
12–46. $s = 71.3$ mm
12–47. $V = 8.82$ kip, $s = 1\frac{1}{8}$ in.
12–49. $P = 3.67$ kN
12–50. $V_{AB} = 9.96$ kip
12–51. $V = 131$ kN
12–53. $V = 4.10$ kip
12–54. $\tau_B = 795$ psi, $\tau_C = 596$ psi
12–55. $\tau_{max} = 928$ psi
12–57. $V = 28.8$ kip
12–58. $\tau_{max} = 7.38$ ksi

Chapter 13

13–1. $t = 18.8$ mm
13–2. $r_o = 75.5$ in.
13–3. $P = 848$ N
13–5. $\sigma_1 = 600$ ksi, $\sigma_2 = 0$
13–6. $\sigma_1 = 600$ ksi, $\sigma_2 = 300$ psi
13–7. **a.** $\sigma_1 = 127$ MPa, **b.** $\sigma'_1 = 79.1$ MPa, **c.** $(\tau_{avg})_r = 322$ MPa
13–9. $\sigma_{hoop} = 7.20$ ksi, $\sigma_{long} = 3.60$ ksi
13–10. $\sigma_1 = 1.60$ ksi, $p = 25$ psi, $\delta = 0.00140$ in.
13–11. $\sigma_h = 432$ psi, $\sigma_b = 8.80$ ksi
13–13. $T_1 = 128°$, $\sigma_1 = 12.1$ ksi, $p = 252$ psi
13–14. $$p = \frac{E(r_2 - r_3)}{\dfrac{r_2^2}{r_2 - r_1} + \dfrac{r_3^2}{r_4 - r_3}}$$
13–15. $$\delta_{r_i} = \frac{pr_i^2}{E(r_o - r_i)}$$

13–17. $\sigma_{fil} = \dfrac{pr}{t + t'} + \dfrac{T}{wt'}$,
$\sigma_w = \dfrac{pr}{t + t'} - \dfrac{T}{wt}$
13–18. $d = 66.7$ mm
13–19. $\sigma_L = 66.7$ MPa (C), $\sigma_R = 33.3$ MPa (T)
13–21. $\sigma_{\max} = \sigma_L = 13.9$ ksi (T), $\sigma_R = 13.6$ ksi (C)
13–22. $\sigma_{\max} = 1.07$ MPa
13–23. $\sigma_{\text{const}} = 1.07$ MPa
13–25. $N = 606.218$ lb, $V = 350$ lb,
$M = 175$ lb · in.,
$\sigma_B = 5.35$ ksi, $\tau_B = 0$
13–26. $\sigma_{\max} = 2.34$ MPa (C)
13–27. $P = 128$ kN
13–29. $P = 11.8$ kN
13–30. $\sigma_A = 25$ MPa (C), $\sigma_B = 0$,
$\tau_A = 0$, $\tau_B = 5$ MPa
13–31. $d = 66.7$ mm
13–33. $\sigma_B = 8.89$ ksi (C), $\tau_B = 0$,
$\sigma_A = 720$ psi (T), $\tau_A = 0$
13–34. $\sigma_E = 8.89$ ksi (T), $\tau_E = 0, \sigma_F = 0, \tau_F = 240$ psi
13–35. $\sigma_A = \sigma_B = 306$ psi (C),
$\tau_A = 8.46$ ksi, $\tau_B = 5.64$ ksi
13–37. $\sigma_B = 1.53$ MPa (C), $\tau_B = 100$ MPa
13–38. $\sigma_D = -88.0$ MPa, $\tau_D = 0$
13–39. $\sigma_E = 57.8$ MPa, $\tau_E = 864$ kPa
13–41. $\sigma_B = 3.26$ MPa (T), $\tau_B = 0.209$ MPa
13–42. $\sigma_A = 3.31$ ksi (T), $\tau_A = 0.581$ ksi
13–43. $\sigma_B = 1.99$ ksi (C), $\tau_B = 0.510$ ksi
13–45. $A = 18.0$ in^2, $I_y = 13.5$ in^4, $I_z = 54.0$ in^4,
$\sigma_A = 1.00$ ksi (C), $\sigma_B = 3.00$ ksi (C)
13–46. $\sigma_{\max} = \dfrac{1.33P}{a^2}$ (C), $\sigma_{\min} = \dfrac{P}{3a^2}$ (T)
13–47. $\sigma_D = 0, \tau_D = 80.8$ psi,
$\sigma_E = -501$ psi, $\tau_E = 93.9$ psi
13–49. $\sigma_D = -126$ psi, $\tau_D = 57.2$ psi,
$\sigma_E = -347$ psi, $\tau_E = 66.4$ psi
13–50. $\sigma_A = 6.61$ ksi (T), $\tau_A = 1.39$ ksi
13–51. $\sigma_B = 5.76$ ksi (C), $\tau_B = 1.36$ ksi
13–53. $\sigma = 5.86$ ksi (C), $\tau = 4.80$ ksi
13–54. $\sigma_{\max} = 71.0$ MPa (C)
13–55. $P = 84.5$ kN
13–57. $\sigma_C = 107$ MPa (C), $\tau_C = 15.3$ MPa,
$\sigma_D = 0, \tau_D = 15.8$ MPa
13–58. $\sigma_A = 7.20$ MPa (T), $\tau_A = 0.6$ MPa
13–59. $\sigma_B = 9.60$ MPa (T)
$\left[(\tau_{xy})_V\right]_B = 0$
$\left[(\tau_{xy})_V\right]_B = 0.45$ MPa
13–61. $\sigma_F = 695$ kPa (C), $\tau_F = 31.0$ kPa
13–62. $\sigma_A = -21.3$ psi, $\sigma_B = -12.2$ psi
13–63. $\theta = 0.286°$
13–65. $\displaystyle\int_A \frac{dA}{r} = 0.035774$ in., $A = 0.049087$ in^2,
$(\sigma_t)_{\max} = 49.0$ ksi (T), $(\sigma_c)_{\max} = 40.8$ ksi (C)
13–66. $(\sigma_t)_{\max} = 28.8$ ksi (T), $(\sigma_c)_{\max} = 24.0$ ksi (C)
13–67. $\sigma_{\max} = 236$ psi (C)
13–69. $P = 94.2$ kN
13–70. $(\sigma_{\max})_{AB} = 667$ psi, $(\sigma_{\max})_{CD} = 40.7$ ksi
13–71. $\sigma_1 = 7.07$ MPa, $\sigma_2 = 0$
13–73. $\sigma_C = 11.6$ ksi, $\tau_C = 0$,
$\sigma_D = -23.2$ ksi, $\tau_D = 0$
13–74. $\sigma_C = 10.4$ ksi, $\tau_C = 0$,
$\sigma_D = -20.8$ ksi, $\tau_D = 0$

Chapter 14

14–2. $\sigma_{x'} = -4.05$ ksi, $\tau_{x'y'} = -0.404$ ksi
14–3. $\sigma_{x'} = -388$ psi, $\tau_{x'y'} = 455$ psi
14–5. $\sigma_{x'} = 1.45$ ksi, $\tau_{x'y'} = 3.50$ ksi
14–6. $\sigma_{x'} = 49.7$ MPa, $\tau_{x'y'} = -34.8$ MPa
14–7. $\sigma_{x'} = 49.7$ MPa, $\tau_{x'y'} = -34.8$ MPa
14–9. $\sigma_{x'} = 56.25$ MPa, $\sigma_{y'} = -31.25$ MPa,
$\tau_{x'y'} = -75.8$ MPa
14–10. $\sigma_{x'} = 47.5$ MPa, $\sigma_{y'} = 202$ MPa,
$\tau_{x'y'} = -15.8$ MPa
14–11. $\sigma_{x'} = 177$ MPa, $\sigma_{y'} = 72.5$ MPa,
$\tau_{x'y'} = -59.2$ MPa
14–13. $\sigma_{x'} = -898$ psi, $\tau_{x'y'} = 605$ psi, $\sigma_{y'} = 598$ psi
14–14. **a.** $\sigma_1 = 4.21$ ksi, $\sigma_2 = -34.2$ ksi,
$\theta_{p_2} = 19.3°, \theta_{p_1} = -70.7°$
b. $\tau_{\substack{\max\\ \text{in-plane}}} = 19.2$ ksi, $\sigma_{\text{avg}} = -15$ ksi,
$\theta_s = -25.7°, 64.3°$
14–15. $\sigma_1 = 53.0$ MPa, $\sigma_2 = -68.0$ MPa,
$\theta_{p1} = 14.9°$ and $\theta_{p2} = -75.1°$,
$\sigma_{\text{avg}} = -7.50$ Mpa, $\tau_{\substack{\max\\ \text{in-plane}}} = 60.5$ Mpa,
$\theta_s = -30.1°$ and $59.9°$
14–17. $I = 0.45(10^{-3})$ m^4, $Q_A = 1.6875(10^{-3})$ m^3,
$\sigma_{x'} = 0.507$ MPa, $\tau_{x'y'} = 0.958$ MPa
14–18. $\sigma_1 = 2.29$ MPa
$\sigma_2 = -7.20$ kPa
14–19. $\sigma_y = -824$ psi
14–21. $\sigma_1 = 0, \sigma_2 = -22.9$ ksi,
$\tau_{\substack{\max\\ \text{in-plane}}} = 11.5$ ksi, $\theta_s = 45°, 135°$,
$\sigma_{\text{avg}} = -11.5$ ksi
14–22. $\sigma_{x'} = 3.75$ MPa
$\tau_{x'y'} = 2.17$ MPa

14–23. $\sigma_1 = 0.939$ ksi, $\sigma_2 = -1.36$ ksi, $\tau_{\text{in-plane}}^{\max} = 1.15$ ksi

14–25. $\sigma_1 = 6.38$ MPa, $\sigma_2 = -0.360$ MPa, $(\theta_p)_1 = 13.4°$, $(\theta_p)_2 = 103°$

14–26. $\tau_{\text{in-plane}}^{\max} = 3.37$ Mpa, $\theta_s = -31.6°, 58.4°$, $\sigma_{\text{avg}} = 3.01$ MPa

14–27. $\sigma_{x'} = -388$ psi, $\tau_{x'y'} = 455$ psi

14–29. $R = 19.21$ ksi, $\sigma_1 = 4.21$ ksi, $\sigma_2 = -34.2$ ksi, $\theta_{p2} = 19.3°$, $\tau_{\text{in-plane}}^{\max} = 19.2$ ksi, $\sigma_{\text{avg}} = -15$ ksi, $\theta_{s2} = 64.3°$

14–30. $\sigma_{x'} = -19.9$ ksi, $\tau_{x'y'} = 7.70$ ksi, $\sigma_{y'} = 9.89$ ksi

14–31. $\sigma_{x'} = 47.5$ MPa, $\sigma_{y'} = 202$ MPa, $\tau_{x'y'} = -15.8$ MPa

14–33. $\sigma_{x'} = 10$ ksi, $\tau_{x'y'} = -5$ ksi, $\sigma_{y'} = 0$

14–34. $\sigma_{x'} = 4.99$ ksi, $\tau_{x'y'} = -1.46$ ksi, $\sigma_{y'} = -3.99$ ksi

14–35. $\sigma_{x'} = 736$ MPa, $\sigma_{y'} = -156$ MPa, $\tau_{x'y'} = -188$ MPa

14–37. $\sigma_1 = 3.51$ ksi, $\sigma_2 = -28.5$ ksi, $(\theta_p)_1 = 19.3°$(clockwise), $\tau_{\text{in-plane}}^{\max} = 16.0$ ksi, $\sigma_{\text{avg}} = -12.5$ ksi, $\theta_s = 25.7°$(counterclockwise)

14–38. $\sigma_{\text{avg}} = 7.50$ ksi

a. $\sigma_1 = 16.5$ ksi, $\sigma_2 = -1.51$ ksi, $\theta_{p_1} = 16.8°$ ↻

b. $\tau_{\text{in-plane}}^{\max} = -9.01$ ksi, $\theta_s = 28.2°$ ↺

14–39. $\sigma_1 = 64.1$ MPa, $\sigma_2 = -14.1$ MPa, $\theta_{p2} = 25.1°$ $\sigma_{\text{avg}} = 25.0$ MPa, $\tau_{\text{in-plane}}^{\max} = 39.1$ MPa, $\theta_s = -19.9°$

14–41. **a.** $\sigma_1 = -5.53$ ksi, $\sigma_2 = -14.5$ ksi, $\theta_p = -31.7°$

b. $\tau_{\text{in-plane}}^{\max} = 4.47$ ksi, $\sigma_{\text{avg}} = -10$ ksi, $\theta_s = 13.3°$

14–42. **a.** $\sigma_1 = 646$ MPa, $\sigma_2 = -496$ MPa, $\theta_{p_1} = 30.6°$ ↺

b. $\tau_{\text{in-plane}}^{\max} = 571$ MPa, $\theta_s = 14.4°$ ↻

14–45. $\tau_{\text{in-plane}}^{\max} = 41.0$ psi, $\sigma_1 = 0.976$ psi, $\sigma_2 = -81.0$ psi

14–46. $\sigma_1 = 68.6$ psi, $\sigma_2 = -206$ psi

14–47. $\sigma_1 = 29.4$ ksi, $\sigma_2 = -17.0$ ksi

14–49. $\sigma_1 = 0.942$ MPa, $\sigma_2 = -3.30$ MPa, $\tau_{\text{in-plane}}^{\max} = 2.12$ MPa

14–50. $\sigma_1 = 3.85$ ksi, $\sigma_2 = -2.08$ ksi, $\tau_{\text{in-plane}}^{\max} = 2.96$ ksi

14–53. $\sigma_1 = 0$, $\sigma_2 = 137$ MPa, $\sigma_3 = -46.8$ MPa, $\tau_{\text{max}}^{\text{abs}} = 91.8$ MPa

14–54. $\sigma_{\max} = 158$ psi, $\sigma_{\min} = -8.22$ psi, $\sigma_{\text{int}} = 0$ psi, $\tau_{\text{max}}^{\text{abs}} = 83.2$ psi

14–55. $\sigma_{\text{int}} = 0$ ksi, $\sigma_{\max} = 7.06$ ksi, $\sigma_{\min} = -9.06$ ksi, $\tau_{\text{max}}^{\text{abs}} = 8.06$ ksi

14–57. $\sigma_1 = 6.73$ ksi, $\sigma_2 = 0$, $\sigma_3 = -4.23$ ksi, $\tau_{\text{max}}^{\text{abs}} = 5.48$ ksi

14–59. The stress in every direction is $\sigma_1 = \sigma_2 = \sigma_3 = -p$

14–61. $\sigma_{\max} = 582$ psi, $\sigma_{\text{int}} = 0$, $\sigma_{\min} = -926$ psi, $\tau_{\text{max}}^{\text{abs}} = 755$ psi

14–62. $\sigma_{\max} = 10.9$ ksi, $\sigma_{\text{int}} = \sigma_{\min} = 0$, $\tau_{\text{max}}^{\text{abs}} = 5.46$ ksi

14–65. $\epsilon_1 = 138(10^{-6})$, $\epsilon_2 = -198(10^{-6})$, $\theta_{p1} = 13.3°$, $\theta_{p2} = -76.7°$, $\gamma_{\text{in-plane}}^{\max} = 335(10^{-6})$, $\epsilon_{\text{avg}} = -30.0(10^{-6})$, $\theta_s = -31.7°$ and $58.3°$

14–66. $\epsilon_1 = 1039(10^{-6})$, $\epsilon_2 = 291(10^{-6})$ Therefore, $\theta_{p1} = 30.2°$, $\theta_{p2} = 120°$ $\gamma_{\text{in-plane}}^{\max} = 748(10^{-6})$ $\epsilon_{\text{avg}} = 665(10^{-6})$ $\theta_t = -14.8°$ and $75.2°$

14–67. $\epsilon_1 = 622(10^{-6})$, $\epsilon_2 = -862(10^{-6})$, $\theta_{p1} = -15.2°$ and $\theta_{p2} = 74.8°$, $\gamma_{\text{in-plane}}^{\max} = -1484(10^{-6})$, $\epsilon_{\text{avg}} = -120(10^{-6})$, $\theta_s = 29.8°$ and $-60.2°$

14–69. $\epsilon_{x'} = 466(10^{-6})$, $\epsilon_{y'} = -116(10^{-6})$, $\gamma_{x'y'} = -393(10^{-6})$

14–70. $\epsilon_{x'} = -309(10^{-6})$ $\epsilon_{y'} = -541(10^{-6})$ $\gamma_{x'y'} = -423(10^{-6})$

14–71. $\epsilon_1 = 188(10^{-6})$, $\epsilon_2 = -128(10^{-6})$,
$(\theta_p)_1 = -9.22°$, $(\theta_p)_2 = 80.8°$,
$\gamma_{\underset{\text{in-plane}}{\max}} = 316(10^{-6})$,
$\theta_s = 35.8°, -54.2°$, $\epsilon_{\text{avg}} = 30(10^{-6})$

14–73. $\epsilon_{x'} = 86.6(10^{-6})$, $\gamma_{x'y'} = 620(10^{-6})$,
$\epsilon_{y'} = 213(10^{-6})$

14–74. $\epsilon_{x'} = -365(10^{-6})$
$\frac{\gamma_{x'y'}}{2} = -271(10^{-6})$
$\epsilon_{y'} = -35.0(10^{-6})$

14–75. $\epsilon_1 = 17.7(10^{-6})$, $\epsilon_2 = -318(10^{-6})$,
$(\theta_p)_1 = 76.7°$, $(\theta_p)_2 = -13.3°$,
$\gamma_{\underset{\text{in-plane}}{\max}} = 335(10^{-6})$, $\theta_s = 31.7°, 122°$,
$\epsilon_{\text{avg}} = -150(10^{-6})$

14–78. $\epsilon_1 = 114(10^{-6})$, $\epsilon_2 = -314(10^{-6})$
$(\theta_p)_1 = -10.3°$
$\gamma_{\underset{\text{in-plane}}{\max}} = 427(10^{-6})$
$\theta_s = 34.7°$ and $125°$
$\epsilon_{\text{avg}} = -100(10^{-6})$

14–79. $R = 167.71(10^{-6})$, $\epsilon_1 = 138(10^{-6})$,
$\epsilon_2 = -198(10^{-6})$, $\theta_p = 13.3°$

14–81. **a.** $\varepsilon_1 = 622(10^{-6})$
$\varepsilon_2 = -862(10^{-6})$
$\theta_{p_1} = 15.2°$
b. $\gamma_{\underset{\text{in-plane}}{\max}} = -1484(10^{-6})$
$\varepsilon_{\text{avg}} = -120(10^{-6})$
$\theta_s = 29.8°$

14–82. $\epsilon_{x'} = 103(10^{-6})$
$\epsilon_{y'} = 46.7(10^{-6})$
$\gamma_{x'y'} = 718(10^{-6})$

14–83. **a.** $R = 93.408$, $\epsilon_1 = 368(10^{-6})$,
$\epsilon_2 = 182(10^{-6})$, $\theta_{p1} = -52.8°$, $\theta_{p2} = 37.2°$
b. $\gamma_{\underset{\text{in-plane}}{\max}} = 187(10^{-6})$,
$\theta_s = -7.76°, 82.2°$, $\epsilon_{\text{avg}} = 275(10^{-6})$

14–85. $\epsilon_1 = 301(10^{-6})$, $\epsilon_2 = -401(10^{-6})$,
$\theta_{p2} = 27.6°$(clockwise),
$\gamma_{\underset{\text{in-plane}}{\max}} = 702\,(10^{-6})$, $\epsilon_{\text{avg}} = -50(10^{-6})$,
$\theta_s = 17.4°$ (counterclockwise)

14–86. **a.** $\epsilon_1 = 487(10^{-6})$
$\epsilon_2 = -400(10^{-6})$
b. $\gamma_{\underset{\text{in-plane}}{\max}} = 887(10^{-6})$
$\epsilon_{\text{avg}} = 43.3(10^{-6})$

14–90. $E = 30.7(10^3)$ ksi
$\nu = 0.291$

14–91. $E = 17.4$ GPa, $\Delta d = -12.6(10^{-6})$ mm

14–93. $\nu_{pvc} = 0.164$

14–94. $\epsilon_x = \epsilon_y = 0$
$\gamma_{xy} = -160(10^{-6})$
$T = 65.2\ \text{N} \cdot \text{m}$

14–95. **a.** $K_r = 3.33$ ksi,
b. $K_g = 5.13(10^3)$ ksi

14–97. $\rho = 3.43$ MPa,
$\tau_{\underset{\text{in-plane}}{\max}} = 0$,
$\tau_{\underset{\text{max}}{\text{abs}}} = 85.7$ MPa

14–98. $P = 13.9$ kip
$\gamma_{xy} = 0.156(10^{-3})$ rad

14–99. $\sigma_z = -\dfrac{12\,My}{bh^3}$, $\epsilon_y = \dfrac{12\,\nu My}{Ebh^3}$,
$\Delta L_{AB} = \dfrac{3\,\nu M}{2Ebh}$,
$\Delta L_{CD} = \dfrac{6\,\nu M}{Eh^2}$

14–101. $p = 4$ MPa

14–102. $p = 3.33$ MPa

14–103. $\theta = \tan^{-1}\left(\dfrac{1}{\sqrt{\nu}}\right)$

14–105. $\delta_a = 0.367$ mm, $\delta_b = -0.255$ mm,
$\delta_t = -0.00167$ mm

14–106. $\sigma_1 = 119$ psi, $\sigma_2 = -119$ psi

14–107. $\sigma_1 = 329$ psi, $\sigma_2 = -72.1$ psi

14–109. **a.** $\epsilon_1 = 482(10^{-6})$, $\epsilon_2 = 168(10^{-6})$
b. $\gamma_{\underset{\text{in-plane}}{\max}} = 313(10^{-6})$
c. $\gamma_{\underset{\text{max}}{\text{abs}}} = 482(10^{-6})$

14–110. F.S. $= 2$

14–111. $\sigma_1 = 3.03$ ksi, $\sigma_2 = -33.0$ ksi,
$\theta_{p1} = -16.8°$ and $\theta_{p2} = 73.2°$,
$\tau_{\underset{\text{in-plane}}{\max}} = 18.0$ ksi, $\sigma_{\text{avg}} = -15$ ksi, $\theta_s = 28.2°$

14–113. No.

14–114. $\epsilon_{\text{avg}} = 83.3(10^{-6})$, $\epsilon_1 = 880\,(10^{-6})$,
$\epsilon_2 = -713(10^{-6})$, $\theta_p = 54.8°$ (clockwise),
$\gamma_{\underset{\text{in-plane}}{\max}} = -1593(10^{-6})$,
$\theta_s = 9.78°$(clockwise)

14–115. $P_2 = 11.4$ kip
$P_1 = 136$ kip

Chapter 15

15–1. $b = 211$ mm, $h = 264$ mm

15–2. Use $b = 4$ in.

15–3. Use $b = 5$ in.

15–5. $S_{req'd} = 15.0\ \text{in}^3$, Use W12 × 16

15–6. Yes.

15–7. No.

15–9. Use W360 × 45

15–10. Yes.

15–11. $P = 2.49$ kN

15–13. $S_{req'd} = 32.73\ \text{in}^3$, Use W12 × 26

15–14. Use W12 × 26

15–15. Use W24 × 62

15–17. $P = 12.5$ kip, $\tau_{req'd} = 466$ psi
$P = 103$ kN

15–18. Use $a = 3\frac{1}{8}$ in.

15–19. $P = 750$ lb

15–21. $w = 3.02$ kN/m, $s_{ends} = 16.7$ mm, $s_{mid} = 50.2$ mm

15–22. Use $t = 4$ mm

15–23. $P = 13.7$ kN

15–25. $w = 10.8$ kN/m

15–26. $P = 2.90$ kN

15–27. $b = 5.86$ in.

15–29. $w = \frac{w_0}{L}x$

15–30. $\frac{y^2}{h_0^2} + \frac{4x^2}{L^2} = 1$

15–31. $x = \frac{L}{3}, \frac{2L}{3}, \sigma_{\substack{abs\\max}} = \frac{18PL}{25bh_0^2}$

15–33. $\sigma_{\substack{abs\\max}} = \frac{0.155w_0L^2}{bh_0^2}$

15–34. $r^3 = \frac{r_0^3}{L^2}x^2$

15–35. $y = \left[\frac{4P}{\pi\sigma_{allow}}x\right]^{\frac{1}{2}}$

15–37. $\sigma_{allow} = \frac{Px}{b_0d^2/6}, d = d_0\sqrt{\frac{x}{L}}$

15–38. $b = \frac{b_0}{L^2}x^2$

15–39. Use W18 × 50

15–41. $h = 0.643$ in. Yes, the joist will support the load.

15–42. $P = 178$ lb, $s = 12.0$ in.

Chapter 16

16–1. $\sigma = 3.02$ ksi

16–2. $\sigma = 75.5$ ksi

16–3. $\sigma = 582$ MPa

16–5. $v_C = 6.11\ \text{mm}\downarrow$

16–6. $v_1 = \frac{Px_1}{12EI}(-x_1^3 + L^2)$,

$v_3 = \frac{P}{12EI}(2x_3^3 - 9Lx_3^3 + 10L^2x_3 - 3L^3)$,

$v_{max} = \frac{PL^3}{8EI}$

16–7. $\theta_A = \frac{333\ \text{kip}\cdot\text{ft}^2}{EI}$

$v_1 = \frac{1}{EI}\left(-\frac{5}{6}x_1^3 + 333\,x_1\right)\text{kip}\cdot\text{ft}^3$

$v_2 = \frac{1}{EI}\left(-\frac{4}{3}x_2^3 - 10x_2^2 + 1267\,x_2 - 10333\right)\text{kip}\cdot\text{ft}^3$

$v_C = \frac{10\,333\ \text{kip}\cdot\text{ft}^3}{\text{EI}}\downarrow$

16–9. $M_1 = \frac{Pb}{L}x_1,\ M_2 = Pa\left(1 - \frac{x_2}{L}\right)$,

$v_1 = \frac{Pb}{6EIL}(x_1^3 - (L^2 - b^2)x_1)$,

$v_2 = \frac{Pa}{6EIL}(3x_2^3L - x_2^3 - (2L^2 + a^2)x_2 + a^2L)$

16–10. $\theta_{max} = \frac{M_0L}{3EI}$,

$v_{max} = -\frac{\sqrt{3}M_0L^2}{27EI}$

16–11. $v_1 = \frac{-Pb}{6aEI}[x_1^3 - a^2x_1]$

$v_2 = \frac{P}{6EI}(-x_2^3 + b(2a + 3b)x_2 - 2b^2(a + b))$

16–13. $v_{max} = \frac{3PL^3}{256EI}\downarrow$

16–14. $v_{max} = 11.5\ \text{mm}\downarrow$

16–15. $\theta_{max} = 0.00466$ rad, $v_{max} = 0.369$ in.$\downarrow$

16–17. $v_1 = \frac{M_0}{6EIL}[-x_1^3 + L^2x_1]$

$v_2 = \frac{M_0}{6EIL}[-3Lx_2^2 + 8L^2x_2 - 5L^3]$

$\theta_A = \frac{M_0L}{6EI}$

16–18. $v_{max} = \frac{-18.8\ \text{kip}\cdot\text{ft}^3}{EI}$

16–19. $\theta_B = -\dfrac{wa^3}{6EI}$,

$v_1 = \dfrac{w}{24EI}\left(-x_1^4 + 4ax_1^3 - 6a^2x_1^2\right)$,

$v_2 = \dfrac{wa^3}{24EI}(-4x_2 + a)$,

$v_B = \dfrac{wa^3}{24EI}(-4L + a)$

16–21. $R = 12.2$ kip

16–22. $v_1 = \dfrac{M_0}{6EIL}(-x_1^3 + L^2x_1)$,

$v_2 = \dfrac{M_0}{24EI}(-12x_2^2 + 20Lx_2 - 7L^2)$,

$v_C = \dfrac{7M_0L^2}{24EI}\downarrow$

16–23. $\theta_A = -\dfrac{3PL^2}{8EI}$, $v_C = \dfrac{-PL^3}{6EI}$

16–25. $v = \dfrac{1}{EI}[-1.67x^3 - 6.67\langle x - 20\rangle^3$

$+ 18.3\langle x - 40\rangle^3 + 4000x]$ lb · in^3

16–26. $v = \dfrac{P}{6EI}\left[-x^3 + \langle x - a\rangle^3 + \langle x - 3a\rangle^3\right.$

$\left. + 9a^2x - 8a^3\right]$,

$(v_{\max})_{BC} = \dfrac{Pa^3}{2EI}\uparrow$, $v_A = \dfrac{4Pa^3}{3EI}\downarrow$

16–27. $v = \dfrac{P}{12EI}[-2\langle x - a\rangle^3 + 4\langle x - 2a\rangle^3 + a^2x]$,

$(v_{\max})_{AB} = \dfrac{0.106Pa^3}{EI}\uparrow$, $v_C = \dfrac{3Pa^3}{4EI}\downarrow$

16–29. $\theta_A = \dfrac{M_0L}{6\,EI}$

$v = \dfrac{M_0}{6EI}\left[3\left\langle x - \dfrac{L}{3}\right\rangle^2 - 3\left\langle x - \dfrac{2}{3}L\right\rangle^2 - Lx\right]$

$v_B = \dfrac{M_0L^2}{18EI}\downarrow$

16–30. $M = M_0\left\langle x - \dfrac{L}{3}\right\rangle^0 - M_0\left\langle x - \dfrac{2}{3}L\right\rangle$,

$v = \dfrac{M_0}{6EI}\left[3\left\langle x - \dfrac{L}{3}\right\rangle^2 - 3\left\langle x - \dfrac{2}{3}L\right\rangle^2 - Lx\right]$,

$v_{\max} = \dfrac{5M_0L^2}{72EI}\downarrow$

16–31. $\theta_A = \dfrac{PL^2}{9EI}$,

$v = \dfrac{P}{18EI}\left[3x^3 - 3\left\langle x - \dfrac{L}{3}\right\rangle^3\right.$

$\left. - 3\left\langle x - \dfrac{2}{3}L\right\rangle^3 - 2L^2x\right]$,

$v_{\max} = \dfrac{23PL^3}{648EI}\downarrow$

16–33. $v = \dfrac{1}{EI}\left[-\dfrac{10}{3}x^3 + \dfrac{10}{3}\langle x - 1.5\rangle^3 + \dfrac{10}{3}\langle x - 4.5\rangle^3\right.$

$\left. + 67.5x - 90\right]$ kN · m^3

16–34. $v_C = -0.501$ mm, $v_D = -0.698$ mm,
$v_E = -0.501$ mm

16–35. $v = \dfrac{1}{EI}[-0.25x^4 + 0.208\langle x - 1.5\rangle^3$

$+ 0.25\langle x - 1.5\rangle^4 + 4.625\langle x - 4.5\rangle^3$
$+ 25.1x - 36.4]$ kN · m^3

16–37. $\dfrac{dv}{dx} = \dfrac{1}{EI}[2.25x^2 - 0.5x^3 + 5.25\langle x - 5\rangle^2$

$+ 0.5\langle x - 5\rangle^3 - 3.125]$ kN · m^2,

$v = \dfrac{1}{EI}[0.75x^3 - 0.125x^4 + 1.75\langle x - 5\rangle^3$

$+ 0.125\langle x - 5\rangle^4 - 3.125x]$ kN · m^3

16–38. $\theta_B = -0.00778$ rad, $v_B = 0.981$ in.$\downarrow$

16–39. $\Delta_C = 0.895$ in.$\downarrow$

16–41. $\theta_A = \dfrac{wa^3}{6EI}$, $\Delta_D = \dfrac{wa^4}{12EI}\downarrow$

16–42. $\theta_B = 0.00722$ rad, $\Delta_C = 13.3$ mm $\downarrow$

16–43. $\theta_A = 0.0190$ rad
$\Delta_C = 25.3$ mm

16–45. Use W14 × 34

16–46. $\Delta_C = 23.2$ m $\downarrow$

16–47. $\Delta_A = 0.916$ in.

16–49. $v_B' = \dfrac{366.67\text{ N} \cdot \text{m}^3}{EI}\downarrow$,

$v_B'' = \dfrac{1.3333B_y\text{ m}^3}{EI}\uparrow$,

$B_y = 550$ N, $A_y = 125$ N, $C_y = 125$ N

16–50. $B_y = \dfrac{7P}{4}$, $A_y = \dfrac{3P}{4}$, $M_A = \dfrac{PL}{4}$

16–51. $C_x = 0$, $B_y = 30.75$ kip,
$A_y = 2.625$ kip, $C_y = 14.625$ kip

16–53. $\Delta_B = 1.50$ mm$\downarrow$

16–54. $A_x = 0$, $B_y = \dfrac{3M_0}{2L}$, $A_y = \dfrac{3M_0}{2L}$, $M_A = \dfrac{M_0}{2}$

16–55. $C_x = 0,\ C_y = \frac{P}{3}$

16–57. $M_A = M_B = \frac{1}{24}PL, A_y = B_y = \frac{1}{6}P,$

$C_y = D_y = \frac{1}{3}P, D_x = 0$

16–58. $F_{sp} = \frac{3kwL^4}{24EI + 8kL^3}$

16–59. $M = \left(\frac{PL}{8} - \frac{2EI}{L}\alpha\right), \Delta_{\max} = \frac{PL^3}{192EI} + \frac{\alpha L}{4}$

16–61. $P = 243$ lb

16–62. $d = 0.708$ in.

16–63. $B_y = 634$ lb, $A_y = 243$ lb, $C_y = 76.8$ lb

16–65. $v = \frac{1}{EI}(-30x^3 + 46.25\langle x - 12\rangle^3$

$- 11.7\langle x - 24\rangle^3 + 38{,}700x - 412{,}560)$ lb · in^3

16–66. $v_1 = \frac{1}{EI}(4.44x_1^3 - 640x_1)$ lb · in^3,

$v_2 = \frac{1}{EI}(-4.44x_2^3 + 640x_2)$ lb · in^3

16–67. $\Delta_C = 1.90$ in.↓

16–69. $\Delta_B = \Delta_C = \frac{11wL^4}{12EI}, \Delta_{BB} = \Delta_{CC} = \frac{4B_yL^3}{9EI},$

$\Delta_{BC} = \Delta_{CB} = \frac{7B_yL^3}{18EI}, B_y = C_y = \frac{11wL}{10},$

$A_y = \frac{2wL}{5}, D_y = \frac{2wL}{5}, D_x = 0$

16–70. $(v_2)_{\max} = \frac{wL^4}{18\sqrt{3}EI}$

16–71. $v_A = \frac{w_0L^4}{Eth_0^3}\downarrow$

16–73. $\Delta_D = \frac{6400\text{ lb} \cdot \text{ft}^3}{EI}\downarrow, (\Delta_C)_1 = \frac{3200\text{ lb} \cdot \text{ft}^3}{EI}\downarrow,$

$(\Delta_C)_2 = \frac{27000\text{ lb} \cdot \text{ft}^3}{EI}\downarrow, \Delta_C = 0.644$ in.↓

16–74. $\Delta_G = 5.82$ in.↓

Chapter 17

17–1. $F = 2P\theta, F_s = \frac{kL\theta}{2}, P_{cr} = \frac{kL}{4}$

17–2. $P_{cr} = kL$

17–3. Use $d = \frac{9}{16}$ in.

17–5. $d = 6.07$ in.

17–6. $d = 4.73$ in.

17–7. $P_{cr} = 158$ kip

17–9. $P_{cr} = 33.17$ kip, F.S. $= 2.21$

17–10. $P_{cr} = 271$ kip

17–11. $P_{cr} = 272$ kN

17–13. $P_{cr} = 20.4$ kip

17–14. $P = 13.2$ kip (controls)

17–15. $L = 15.1$ ft

17–17. $A = 8.00$ in^2, $I_x = 10.667$ in^4,
$I_y = 2.6667$ in^4, $P_{cr} = 2.92$ kip

17–18. $P_{cr} = 5.97$ kip

17–19. $P = 17.6$ kip

17–21. $P_{cr} = 32.2$ kip

17–22. Use $d_{AB} = 2\frac{1}{8}$ in., $d_{BC} = 2$ in.

17–23. $P = 207$ lb

17–25. $A = 8.85$ in^2, $I_y = 19.6$ in^4, $P = 62.3$ kip

17–26. $P = 2.42$ kip

17–27. $P = 63.0$ kN

17–29. F.S. $= 2.12$
F.S. $= 4.32$

17–30. $P = 8.46$ kN

17–31. $w = 1.17$ kN/m

17–33. $P = 5.87$ kN, $v_{\max} = 42.1$ mm

17–34. $\sigma_{\max} = 57.4$ MPa
$v_{\max} = 34.7$ mm

17–35. $A = 0.61575(10^{-3})$ m^2,
$I = 64.1152(10^{-9})$ m^4, $P_{\max} = P_{cr} = 18.98$ kN,
$P = 6.75$ kN

17–37. $\sigma_{\max} = 65.8$ kip

17–38. $P = 45.7$ kip (controls)

17–39. $P = 156$ kip

17–41. The column is adequate.

17–42. $P_{cr} = 12.6$ kN

17–43. $L = 2.53$ m

17–45. $P = 73.5$ kip

17–46. $P = 76.6$ kip (controls)

17–47. $P_{\max} = 61.2$ kip

17–49. $\sigma_{\max} = 6.22$ ksi

17–50. $\sigma_{\max} = 6.24$ ksi

17–51. $(KL)_x = (KL)_y = 3$ m, $P_{cr} = 83.5$ kN

17–53. $\sigma_{\max} = 2.86$ ksi

17–54. $v_{\max} = 1.23$ in.
$\sigma_{\max} = 15.6$ ksi $< \sigma_Y$

17–55. $P = 88.5$ kip

17–57. $P_{cr} = 2645.9$ kN

17–58. $P_{cr} = 1350$ kN

17–59. $a = 103$ mm

17–61. $P_{cr} = 12.1$ kN

17–62. $\sigma_1 = 50.0$ MPa, $\sigma_2 = 25.0$ MPa, $F_b = 133$ kN

17–63. $P = 37.5$ kip

17–65. $P = 46.5$ kN

17–66. $P = 110$ kN

17–67. $P = 129$ kip

17–69. $L = 8.34$ m

17–70. $P = 3.20$ MN (controls)
$v_{\max} = 70.5$ mm

Index

Allowable stress, 342–343
Angle of twist (ϕ), 451–453, 468–475, 496
  circular shafts, 451–453, 468–475
  constant torque and, 469–470
  cross-sectional area and, 453, 469–470
  external torque and, 451–453
  Hooke's formula and, 468–469
  internal torque and, 468–475, 496
  multiple torques and, 470
  procedure for analysis of, 472
  right-hand rule for, 470–471
  shear strain (γ) and, 452–453
  sign convention for, 470–471
Angles (θ), 17, 39–40, 42, 61, 70–71, 181–182, 200–201. *See also* Angle of twist
  coordinate direction (α, β, γ), 39–40, 42, 70–71
  dot products and, 61
  dry friction (ϕ), 181–182
  kinetic friction (ϕ_k), 182
  lead (slope), 200–201
  line of action (direction) and, 17
  static friction (ϕ_s), 181
Area (A), 264, 290–300, 304–305, 379, 808–809
  axis cross-sectional, 290–294, 305
  centroid (C) of, 264, 290–300, 304–305, 808–809
  composite bodies, 298–300, 305
  geometric properties of 808–809
  moment of inertia (I) for, 290–300, 305, 808–809
  parallel-axis theorem for, 291–292, 305
  percent reduction of, 379
  polar moment of inertia, 291
Axes, 76, 96–99, 134, 282–283, 290–294, 305, 526–527, 530, 542–547, 555
  area (A) cross section, 290–294, 305
  bending applied to, 526–527, 542–543, 555
  centroid (C) location, 282–283, 290–294
  direction and, 76
  distributed loading along single, 282–283
  line of action, 98, 134
  longitudinal, 526–527, 717
  moment of force about, 96–99, 134
  moment of inertia of, 290–294, 305, 544
  neutral, 526, 530, 545
  parallel-axis theorem, 291–292
  principal, 542–543, 555
  projection, 97
  resultant force on, 96–99, 134, 282–283
  scalar analysis of, 96
  unsymmetric bending and, 542–547, 555
  vector analysis of, 97
Axial (longitudinal) direction, 592
Axial loads, 324–330, 368, 404–449. *See also* Critical load
  constant external force and, 409, 447
  cross-sectional area of, 406–407, 409, 440–444, 447
  deformation and displacement from, 404–449
  displacement (δ), 440–449
  elastic deformation of members, 406–415, 446
  equations of equilibrium, 325–326
  force diagrams for, 326
  force method of analysis for, 428–429
  normal stress (σ) distribution, 324–330, 368
  principle of superposition for, 421, 447
  prismatic bars, 324–330
  procedures for analysis of, 327, 411, 423, 429
  relative displacement (δ), 408–415, 446
  Saint-Venant's principle, 405–407, 446
  sign convention for, 410
  statically indeterminate members, 422–427, 434, 447
  stress concentrations and, 440–444, 447
  thermal stress and, 434–437, 447

Ball and socket supports, 168, 170
Beams, 500–557, 559–572, 587, 692–715, 716–767, 810–815. *See also* Structural design
  bending, 500–557
  bonded to prevent sliding, 560
  built-up members, 578–582, 587, 698, 713
  cantilevered, 501, 815
  cross-sectional area of, 525–535, 542–547, 555, 560–572, 587, 696–697, 713
  deflection of, 694, 716–767, 814–815
  deformation of, 525–528
  design of, 692–715
  fabricated, 697–698
  flexure formula, 529–535, 555
  fully stressed, 710–711, 713
  geometric properties of, 810–813
  glulam, 698
  irregular (nonrectangular), 565
  nonprismatic, 710
  overhanging, 501
  prismatic, 696–703
  procedures for analysis of, 503, 511, 532, 566, 699
  shear and moment diagrams, 501–517, 554
  shear formula for, 561–572, 587
  sign conventions for, 502, 510, 545
  simply supported, 501, 814
  slender, 561
  stress concentrations, 550–552, 555
  stress distribution, 529–535, 555
  stress trajectories, 695
  transverse shear in, 559–572, 587
  unsymmetric bending, 542–547, 555
  warping, 560–561
  wide flange, 564, 697, 713, 810–813
Bearing plates, 694
Bearing supports, 167–170
Belts (flat), frictional forces on, 197–199, 211
Bending, 311, 368, 500–557, 694–695, 713
  beams, 500–557
  cross-sectional area and, 525–530, 542–547, 555
  flexure formula, 529–535, 555
  internal moment (M), 542–547, 555
  longitudinal strain (elongation) from, 527–529
  moment of inertia (I) for, 531–532, 544
  moments (M), 311, 368, 542–547, 555
  procedures for analysis of, 503, 511, 532
  shear and moment diagrams, 501–517, 554
  straight members, deformation of by, 525–528
  stress concentrations, 550–552, 555
  stress distribution, 529–535, 555, 694–695, 713
  structural design and, 694–695, 713
  unsymmetric, 542–547, 555
Biaxial stress, 593
Bifurcation point, 771
Bonded beams to prevent sliding, 560
Boundary conditions, 724, 764
Bridges, trusses for, 216
Brittle materials, 381, 400, 442–443, 493
Buckling, 768–803
  critical loads, 769–781, 801
  deflection and, 772–777, 788–793
  effective-length factor (K) for, 779
  Engesser equation, 795, 801
  equilibrium and, 770–772
  Euler load, 774–775, 801
  fixed-supported columns, 778–779
  ideal columns, 772–777
  inelastic, 794–796, 801
  least moment of inertia and, 775
  pin-supported columns, 772–777
  radius of gyration for, 775
  Secant formula, 788–793, 801
  Shanley theory, 795
  slenderness ratio and, 775–776, 779, 794–795
  tangent modulus (E_t) for, 794–795
Built-up members, 578–582, 587, 698, 713
Bulging cross sections, 488
Bulk modulus (k), 679, 689

Cable supports, 142, 168
Cantilevered beams, 501, 815
Cartesian strain components, 358
Cartesian vectors, 31, 38–45, 50–52, 60, 70–71, 80–81, 83–85, 133
  addition of, 41–45
  coplanar forces, notation of, 31
  cross product, formulation of, 80–81, 133
  direction of, 39–42, 70–71
  dot product and, 60, 71
  magnitude of, 39, 70
  position vectors for, 50–52
  rectangular components, 38
  resultant forces, formulation of, 40–41, 71
  resultant moments, formulation of, 83–85, 133
  right-hand rule for, 38, 50

three-dimensional components, 38–39
unit vectors, 38, 70
Center of gravity (G), 144, 260–307
center of mass (C_m) and, 263, 304
centroid (C) for, 144, 261–307
composite bodies, 273–276, 305
constant density and, 273
distributed loading resultants and, 281–286, 304–305
free-body diagrams and, 144
integration for, 262–265
location of, 144, 261–262, 304
procedures for analysis of, 265, 274
rigid-body equilibrium and, 144
specific weight and, 273
weight (W) and, 144, 261–262, 273, 304–305
Center of mass, 263, 304
Centroid (C), 144, 260–307, 808–809. *See also* Center of gravity
area, of a, 264, 808–809
distributed loads and location of, 281–283, 304–305
moments of inertia and, 290–300, 305
procedure for analysis of, 265
volume, of a, 263, 808–809
Circular shafts, 452–461, 468–475, 481–484, 496–497
Circumferential (hoop) direction, 592
Cohesive material, 322
Collinear vectors, 19, 69
Columns, 768–803
buckling, 768–803
critical loads, 769–781, 801
eccentric distance (e) of, 788–789
effective length (L_e) of, 779
effective-slenderness ratio, 779
fixed-supported, 778–781
flexural rigidity (EI), 773–774
ideal, 772–777
inelastic buckling, 794–796
pin-supported, 772–777
radius of gyration of, 775
Secant formula for, 788–793, 801
short (posts), 794
slender, 794
slenderness ratio of, 775–776, 779, 794–795
Combined loadings, 590–617
procedure for analysis of, 598–599
state of stress caused by, 598–605, 615
superposition of stress components, 598–605, 615
thin-walled pressure vessels, 591–594, 615
stresses from, 590–617
Compatibility (kinematic) conditions, 422, 752–759
Components of force, 18, 20–21, 30–35, 38–45, 50–55, 61, 69–70
Cartesian vectors for 38–45
coplanar forces, 30–37
directed along a line, 53–55, 61, 71
dot products and, 61
parallelogram law for, 18, 69
position vectors for, 50–55
rectangular, 30–35, 38, 70
scalar notation of, 30
three-dimensional, 38–45, 50–55
vector addition for, 18, 20–21, 30–35
Composite bodies, 273–276, 298–300, 305
area (A), 298–300, 305
center of gravity (G) of, 273–276, 305
moment of inertia for, 298–300, 305
procedures for analysis of, 274, 298
specific weight and, 273
Compression test, 373–374
Compressive forces, truss members, 217–219, 231, 256
Compressive stress, 323
Concentrated forces, 5, 510
Concurrent force systems, 122, 135, 157, 218
Constant density, center of gravity and, 273
Constant loads, elastic deformation and, 409
Continuity conditions, 724, 764
Continuous material, 322
Contraction, 379, 392, 410–411
Conventional stress–strain diagram, 375–377
Conversion of units, 9
Coordinate direction (α, β, γ), 39–40, 42, 70–71
Coplanar forces, 30–34, 70, 114, 122–128, 135, 140–166, 208–209, 282–283, 305, 312. *See also* Two-dimensional equilibrium
Cartesian notation for, 31
centroid (C) of, 282–283, 305
couple systems and, 114, 122–128
distributed load resultants, 282–283, 305
equilibrium of, 140–166, 208
equations of equilibrium for, 151–156, 208–209
free-body diagrams of, 141–149
internal resultant loadings, 312
magnitude of, 30–31
procedures for analysis of, 146, 152
rectangular vector components, 30–34, 70
resultants, 31–32, 70
resultant moments, simplification of, 114, 122–128
scalar notation for, 30
support reactions, 141–143
systems, 114, 122–123, 135, 140–166, 208
three-force members, 157–158
two-force members, 157–158
Coulomb friction, 180. *See also* Dry friction
Couple moments, 103–107, 112–117, 122–128, 134–135, 510, 737–738
collinear, 135
concurrent force systems and, 122, 135
coplanar force systems and, 114, 122–128, 135
direction (right-hand rule), 103
equivalent, 104
force systems and, 103–107, 112–117, 122–128, 135
magnitude of, 103, 134
parallel force systems and, 123–128, 135
procedures for analysis of, 114, 124
resultant, 104
scalar formulation, 103
shear and moment diagrams, 510
simplification of systems, 112–117, 135
singularity functions for, 737–738
vector formulation, 103–104, 134
Couplings, 492
Critical loads, 769–781, 801
bifurcation point, 771
column buckling and, 769–781, 801
disturbing force and, 770
effective-length factor (L_e) for, 779
equilibrium and, 770–772
Euler formula for, 774–775, 779–781
fixed-supported columns, 778–779
pin-supported columns, 772–777
restoring force and, 770
Critical stress, 775–776
Cross product, 79–81, 133
Cross sections, 290–294, 310–311, 324, 368, 406–407, 409, 440–444, 453–460, 446–447, 469–470, 488–490, 492–494, 525–535, 542–547, 555, 560–572, 587, 696–697, 713, 775
angle of twist (ϕ) and, 453, 469–470
axially loaded members, 324, 406–407, 409, 440–444, 446–447
axis, 290–294
beams, 525–535, 542–547, 555, 560–572, 587, 696–697, 713
bending deformation and, 525–528, 555
bulging, 488
centroid (C) of, 290–300
changes in, 492–494, 497
circular shafts, 453–460, 469–470, 492–494, 496–497
columns, 775
constant external force and, 409, 447
flexural formula and, 529–535, 555
internal resultant loadings, 310–311, 368
irregular (nonrectangular) beams, 565
localized deformation and, 406–407
moments of inertia (I), 290–294, 531, 775
non-circular shafts, 488–490
prismatic bars, 324
Saint Venant's principle, 406–407, 446
shear formula and, 561–572, 587
stress concentrations and, 440–444, 447, 492–494, 497
torque and, 469–470
torsional formula for, 454–460
transverse shear stress and, 560–572, 567
unsymmetric bending and, 542–547, 555
warping, 488, 560–561
wide flange, 564, 697, 713

Curvature, deflection and, 719–720, 764
Cylindrical pressure vessels, 592–593

Deflection, 694, 716–767, 772–777, 788–793, 814–815
  beams, 694, 716–767, 814–815
  boundary conditions, 724, 764
  cantilevered beams, 815
  column buckling, 772–777, 788–793
  continuity conditions, 724, 764
  discontinuity functions for, 735–742, 764
  displacement, 717, 721–729, 764
  elastic curve, 717–729, 764
  flexural rigidity (EI), 720, 722, 773–774
  integration of slope and displacement for, 721–729, 465
  Macaulay functions for, 736–737
  maximum, 774, 789–791
  moment-curve relationship, 719–720, 765
  procedures for analysis of, 725, 739, 756
  Secant formula for, 789–791, 801
  shafts, 694, 716–767
  sign conventions for, 723
  simply supported beams, 814
  singularity functions for, 737–738
  slope and, 717–718, 721–729, 764, 814–815
  statically indeterminate beams and shafts, 752–759, 765
  structural design and, 694, 716–767
  superposition, method of for, 745–749, 765
Deformation, 140, 309, 355–362, 369, 375–380, 383–387, 392–397, 400–401, 404–449, 451–453, 496, 525–528
  axially loaded members, 404–449
  bending, 525–528
  displacement (δ), 404–449
  elastic behavior and, 406–415, 446
  elastic limit for, 376
  equilibrium and, 140
  failure of materials from, 377–380
  load–displacement relationship, 422, 447
  localized, 406–407
  mechanics of materials and, 309, 369
  modulus of rigidity (G) and, 394–397
  necking, 377
  plastic, 376
  Poisson's ratio (ν) and, 392–394
  principle of superposition for, 421, 447
  proportional limit, 376
  relative displacement (δ), 408–415, 446
  Saint Venant's principle and, 406–407, 446
  small strain analysis, 359
  statically indeterminate members, 422–427, 447
  strain and, 309, 355–362, 369, 382–387
  strain energy and, 384–387, 401
  strain hardening, 377, 383
  stress and, 309, 355, 369, 376–378
  stress concentrations and, 440–444, 447
  stress–strain diagrams for, 375–380, 394–397, 400–401
  thermal stress and, 434–437, 447
  torsional, 451–453, 496
  true stress and true strain, 377–378
  twisting, 451–453, 496
  yielding, 376, 400
Degree of indeterminacy, 752
Derivatives, 806
Determinacy, 173, 209, 752. *See also* Equilibrium; Statically indeterminate systems
Differential ring, 456
Dilatation (e), 678–679, 689
Dimensional homogeneity, 10–11
Dimensionless quantities, 356
Direct (simple) shear stress (τ), 331
Direction, 17, 38–42, 50, 53–55, 69–71, 76, 79–80, 82, 103, 133–134, 219, 224, 232, 592–593, 615, 695
  angle (θ) for, 17
  axial (longitudinal), 592
  by inspection, 219, 224, 232
  Cartesian vectors, 39–41, 70–71
  circumferential (hoop), 592
  coordinate angles (α, β, γ), 39–40, 42, 70–71
  combined loadings, 592–593, 615
  cosines, 39–41
  couple moments, 103, 134
  cross product, 79–80, 133
  force and, 17, 38–42, 50, 53–55, 69–71, 76, 79–80, 82, 103, 133–134
  line of action, 17
  moment of a force, 82, 133
  pressure vessels, stresses in, 592–593, 615
  resultant moments, 76, 79–80, 82, 133
  right-hand rule for, 38, 50, 76, 79–80, 82, 103, 133
  scalars and, 76, 133
  sense of, 17, 69
  stress trajectories, 695
  truss member forces, 219, 224, 232
  vectors and, 17, 39–42, 53–55, 69–70, 79–80, 82, 103, 133–134
  unit vectors, 53–55, 71
Discontinuity functions, 735–742, 764
  distributed loadings and, 736–737
  Macaulay functions, 736–737
  moments (M) and, 735–739
  procedure for analysis using, 739
  singularity functions, 737–738
Displacement (δ), 404–449, 717, 721–729, 764. *See also* Deformation
  axial loads, 404–449
  boundary conditions, 724, 764
  compatibility (kinematic) conditions, 422
  continuity conditions, 724, 764
  deflection and, 717, 721–729, 764
  elastic deformation and, 406–415
  force method of analysis, 428–429
  integration of slope and, 721–729, 764
  principle of superposition, 421, 447
  procedure for analysis of, 411, 423, 429
  relative, 408–415, 446
  sign convention, 410
  statically indeterminate members, 422–427, 434, 447
  stress concentrations and, 440–444, 447
  thermal stress and, 434–437, 447
Distributed loading, 281–286, 304–305, 309–317, 322–335, 368, 508–510, 554, 736–737
  axially loaded members, 324–330
  axis (single), along, 282–283, 305
  bending moment (M), 311, 368
  center of gravity (G) and, 281–286, 304–305
  centroid (C) location, 281–283, 301–305
  coplanar, 282–283, 305
  deflection and, 736–737
  flat plates, 281, 304
  internal resultant, 309–317, 368
  Macaulay functions for, 736–737
  magnitude of resultant force, 281–282
  normal force (N), 311, 368
  pressure (p) and, 281
  resultant forces of, 281–286, 304–305
  shear and moment diagram regions, 508–510, 554
  shear force (V), 311, 368
  stress (σ) and, 322–335, 368
  torsional moment (T), 311, 368
  volume (V) element for, 281
Disturbing force, 770
Dot product, 60–64, 71, 97
Double shear, 334
Dry friction, 180–190, 210
  angles (ϕ) of, 181–182
  coefficients of (μ), 181–183, 210
  equilibrium and, 180–190, 210
  friction equations and, 184–186
  frictional forces and, 180–183
  impending motion and, 181, 184–190
  kinetic frictional forces, 182–183, 210
  motion and, 182–183
  normal forces and, 180–181
  problems involving, 184–190
  procedure for analysis of, 186
  rigid-body equilibrium and, 180–190, 210
  rolling and, 186
  slipping and, 181–184, 210
  static frictional forces, 181, 183, 210
  tipping and, 180, 185, 210
Ductile materials, 377, 379–380, 400

Eccentric distance (e), 788–789
Effective length (L_e), 779
Effective-length factor (K), 779
Effective-slenderness ratio, 779
Elastic curve, 717–729, 764
  boundary conditions, 724, 764
  continuity conditions, 724, 764
  deflection and, 717–729, 764
  displacement of, 719, 721–729

flexural rigidity (EI), 720, 722
inflection point, 718, 764
integration of slope and displacement, 721–729, 764
internal moments (M) and, 718–720
moment-curvature relationship, 719–720
radius of curvature (ρ), 719–720, 764
sign convention for, 723
slope and, 717–718, 721–729, 764
Elastic limit, 376, 400
Elastic material behavior, 376, 379–380, 382–384, 394, 400–401, 406–415, 446, 678–679, 689. *See also* Buckling
axial loads, 406–415, 446
bulk modulus (k), 679, 689
contraction, 379, 392, 410–411
deformation of members, 406–415, 446
ductile materials, 379–380
elongation, 379, 392, 400, 410–411
Hooke's law and, 382–383
linear, 383–384
modulus of rigidity (G), 394, 401
nonlinear, 380
procedure for analysis of, 411
proportional limit, 376, 394
relative displacement (δ), 408–415, 446
Saint Venant's principle and, 406–407, 446
shear stress (τ) and strain (γ) applications, 394
strain energy and, 384
strain hardening and, 383
stress–strain diagrams for, 376, 394, 400–401
stress and strain transformations and, 678–679, 689
Young's modulus (E), 382–383
Elastic strain, 383
Electrical-resistance strain gauge, 374, 674
Elongation, 379, 392, 400, 410–411
Engesser equation, 795, 801
Engineering, 2–15
dimensional homogeneity and, 10–11
fundamental quantities and concepts for, 4–7
mechanics of materials, 3–4
mechanics, 3–4
Newton's laws of gravitational attraction, 7
Newton's laws of motion, 6
numerical calculations, 10–11
procedure for analysis, 12
rounding off numbers, 11
significant figures and, 11
statics, 3–4
units of measurement, 7–10
Engineering (nominal) stress and strain, 375
Equilibrium, 138–213, 325–326, 332, 770–772
axial loaded members, 325–326
bifurcation point, 771
column buckling and, 770–772
concurrent forces and, 157
conditions for, 139–140
critical loads and, 770–772
deformation and, 140
determinacy and, 173, 209
dry friction and, 180–190, 210
equations of, 151–156, 172–175, 208–209, 325–326, 332
free-body diagrams for, 141–149, 167–171, 208–209
friction equations for, 184–186
frictional forces and, 197–207, 211
neutral, 771–772
normal stress (σ) and, 325–326
parallel forces and, 157
procedures for analysis of, 146, 152, 173, 186
rigid bodies, 138–213
scalar equations of, 172
shear stress (τ) and, 332
stability and, 173, 209, 770
support reactions and, 141–143, 168–170, 208–209
three-force members, 157–158
two- and three-force members, 157–158
two-dimensional (coplanar), 140–166, 208–209
unstable, 770
vector equations of, 172
Equivalent couple moments, 104
Equivalent force systems, 112–117, 135
Euler load, 774–775, 801
Extensometer, 374

Fabricated beams, 697–698
Factor of safety (F.S.), 342–343, 369
Failure of materials, 342–343, 377–381, 442–443, 493
brittle materials, 381, 442–443, 493
deformation and, 377–380
ductile materials, 377, 379–380
fatigue loadings, 442–443, 493
fracture stress, 377
inelastic strain, 493
necking, 377
percent elongation, 379
percent reduction of area, 379
stress and, 342–343, 377–378
stress concentrations and, 442–443, 493
stress–strain diagrams for, 379–381
true stress–strain diagrams for, 377–378
torsional loading, 493
yield strength, 379–380
Fatigue loadings, failure from, 442–443, 493
Feet (ft), unit of, 8
Fixed supports, 141, 143, 169, 471
Flat plates, distributed loadings of, 281, 304
Flexural rigidity (EI), 720, 722, 773–774
Flexure formula, 529–535, 555
Floor beams, 216
Foot-pound per second, unit of, 461
Foot-pound-second (psi), unit of, 323, 356–357

Force (F), 4–5, 8, 16–73, 96, 112, 139, 140, 144, 157–158, 180–190, 197–202, 218–129, 311–312, 770. *See also* Components of force; Dry friction; Frictional forces; Weight
addition of, 20–25
concentrated, 5
concurrent, 157
coplanar, 30–34, 312
directed along a line, 52–55, 61, 71
disturbing, 770
frictional, 180–190, 197–202
fundamental quantity of, 4
internal resultant loadings, 311–312, 368
internal, 139, 144
kinetic, 182–183
normal (N), 180–181, 311–312, 343
parallel, 157
procedure for analysis of, 22
reactive, 112
restoring, 770
resultants of, 18–20, 31–32, 41–42, 69–71
rigid-body equilibrium and, 139, 144
scalar determination of, 17–18, 30, 96
shear (V), 311, 343
two-and three-force members, equilibrium of, 157–158
units of, 8
unknown, 218–219
vector determination of, 16–73
zero, 140
Force (axial) diagrams, 326
Force method of analysis, 428–429, 753
Force systems, 74–137, 138–213, 218–219, 231
axis, moment of about, 96–99, 134
Cartesian vector formulation, 80–81, 83–85, 133
compressive, 217–218, 231
concurrent, 122, 135, 157, 218
coplanar, 114, 122–128, 135, 140–166, 208–209, 218
couple moments, 103–107, 112–117, 122–128, 134–135
cross product, 79–81, 133
direction "by inspection", 219
dry friction and, 180–190, 210
equilibrium of, 138–213
equivalent, 112–117, 135
external effects on, 112
frictional forces on, 180–190, 197–202, 210–211
moments (M_o), 75–78, 82–88, 96–99, 103–107
parallel, 123–128, 135, 157
principle of moments, 86–88, 133
principle of transmissibility, 82, 112
procedures for analysis of, 114, 124
resultants, 74–137
rigid-bodies, 138–213
rotational motion and, 112–114, 141, 146, 167, 173

scalar formulation of, 75–78, 96, 103, 133
simplification of, 112–117, 122–128, 135
statically indeterminate, 173, 209
support reactions and, 141–143, 167–170, 208–209
tensile, 217–218, 231, 257
three-dimensional, equilibrium of, 167–179, 209
translational motion and, 112–114, 141, 146, 167, 173
truss members, 218–219, 231
two-dimensional, equilibrium of, 140–166, 208
vector formulation of, 82–85, 97, 103–104, 133
Fracture stress, 377, 400
Frames, structural analysis of, 240–248, 257
Free-body diagrams, 141–149, 152, 167–171, 173, 208–209, 218–219, 224–225, 231–232, 240–242
center of gravity (G), 144
equilibrium and, 141–149, 152, 208–209
frames, 240–242
idealized models of, 144–145
internal forces and, 144
machine parts, 240–242
procedures for analysis using, 146, 152, 173
rigid bodies, 141–149
springs, 144
structural analysis using, 218–219, 224–225, 231–232, 240–242
support reactions, 141–143, 167–170, 208–209
three-dimensional equilibrium, 167–171, 173, 209
truss members, 218–219, 224–225, 231–232
two-dimensional (coplanar forces), 141–149, 152, 208–209
weight (W), 144
Free vector, 112–113
Frequency (f) of rotation, 461
Frictional forces, 180–190, 197–202, 210–211. *See also* Dry friction
angles (ϕ) of, 181–182
coefficients of (μ), 181–183, 210
dry friction and, 180–190
equilibrium and, 180–190, 197–202, 211
flat belts, 197–199, 211
kinetic, 182–183, 210
normal, 180–181
procedure for analysis of, 186
screws, 200–202, 211
static, 181, 183, 210
Fully stressed beams, 710–711, 713

Gauge-length distance, 374
Gauge pressure, 591
Geometric center, 281, 283, 305. *See also* Centroid
Glulam beams, 698
Gravitational attraction, 7
Gravity, *see* Center of gravity; Weight
Gusset plates, 216

Hertz (Hz), units of, 461
Hinge supports, 141–143, 167, 169
Homogeneous material, 324
Hooke's law, 382–383, 394, 400, 408–409, 676–677, 689
axially loaded members, 408–409
elastic deformation using, 408–409
material properties and, 382–383, 394
shear stress (τ) and strain (γ) applications, 394
strain hardening and, 383
stress–strain diagrams and, 382–383
triaxial stress transformations, 676–677, 689
Young's modulus (E), 382–383
Hoop (circumferential) direction, 592
Horsepower (hp), units of, 461
Hyperbolic functions, 806

Ideal columns, 772–777
Idealizations, 5
Idealized models, 144–145. *See also* Free-body diagrams
Impending motion, 181, 184–190, 200–201
angle of static friction for, 181
coefficient of static friction for, 181
downwards, 201
equilibrium and, 181, 184–190
frictional forces and, 181, 184–190, 200–201
screws, 200–201
slipping (all points of contact), 181, 184
tipping (some points of contact), 185
upward, 200
Inelastic buckling, 794–796
Inelastic strains, 493
Inertia, *see* Moment of inertia
Inflection point, 718, 764
Integrals, 807
Integration of slope and displacement, 721–729, 764
Internal forces, 139, 144, 309–317, 368, 542–547, 555, 735–739. *See also* Strain; Stress
bending moment (M), 311, 368, 542–547, 555, 735–739
coplanar systems, 312
cross sections for, 310–311, 368
deflection, moments for, 718–720, 735–739
free-body diagrams of, 139, 144
mechanics of materials and, 309–317, 368
normal force (N), 311–312, 368
procedure for analysis of, 313
resultant loadings, 309–317, 368
right-hand rule for, 311
shear force (V), 311, 368
torsional moment (T), 311, 368
Internal moments (M), 718–720, 735–736
International System (SI) of units, 8–10, 323
Isotropic material, 324

Joints, 215–226, 256
bridges and, 216
gusset plates for, 216
loadings on, 215–216
method of, 218–226, 256
pin connections, 216–217
roof loads and, 215
tensile and compressive forces and, 217–219, 256
trusses, 215–226
two-force members, as, 217
zero-force members and, 224–226
Journal bearing supports, 167–169

Keyways, 492
Kilogram (kg), unit of, 8
Kinetic friction (ϕ_k), angle of, 182
Kinetic friction (μ_k), coefficient of, 182–183, 210

Lateral contraction (negative strain), 392, 410–411
Lead angle (θ), 181–182
Length, fundamental quantity of, 4
Length, units of, 8
Line of action, 17, 98, 112, 134, 281–283
Linear coefficient of thermal expansion, 434
Load-displacement relationship, 422–427
Loadings, *see* Axial loads; Combined loadings; Critical load; Distributed loadings
Localized deformation, 406–407
Longitudinal (axial) direction, 592
Longitudinal elongation (positive strain), 392, 400, 410–411
Lower yield point, 376

Macaulay functions, 736–737
Machines, structural analysis of, 240–248, 257
Magnitude, 17, 22, 30–31, 39, 69–70, 76, 79–80, 82, 103, 133–134, 281–282
Cartesian vectors, 39, 70
centroid (C) location and, 281–282
coplanar force systems, 30–31
couple moments, 103, 134
cross product and, 79–80
distributed loadings and, 281–282
law of cosines for, 22
moments of a force and, 76, 79–80, 82, 133–134
resultant forces, 20, 281–282
scalars and, 17, 69, 76
vectors and, 17, 22, 30–31, 39, 69–70, 82
Mass, quantity of, 4
Mass, units of, 8
Material properties, 372–403, 676–683, 689
brittle materials, 381, 400
bulk modulus (k), 679, 689
compression test for, 373–374

deformation and, 375–381, 400–401
dilatation (e), 678–679, 689
ductile materials, 377, 379–380, 400
elastic behavior, 376, 400
engineering (nominal) stress and strain, 375
failure of materials and, 377, 379–381, 400
Hooke's law for, 382–383, 400, 676–677, 689
mechanical, 373–403
modulus of elasticity (E), 382–383, 394
modulus of resilience (u_r), 384
modulus of rigidity (G), 394
modulus of toughness (u_t), 385
offset method for, 379–380
plane transformations and, 676–683, 689
plastic behavior, 376, 383, 400
Poisson's ratio (ν), 392–394, 401
relationships involving E, ν, and G, 678–679, 689
shear modulus (G), 394, 678, 689
shear stress–strain diagrams for, 394–397, 401
stiffness, 383
strain energy, 384–387, 401
strain hardening, 377, 383
stress–strain diagrams for, 375–381, 394–397, 400–401
tension test for, 373–374, 400
triaxial stress transformations, 676–677, 689
true stress–strain diagrams, 377–378
yielding, 376, 400
Young's modulus (E), 382–383
Mechanics of materials, 3–4, 309–317. *See also* Material properties
engineering study of, 3–4
idealizations for, 5
internal resultant loadings and, 309–317
Newton's laws for, 6–7
quantities in, 4–5
Meter (m), unit of, 8, 356, 461
Method of joints, 218–226, 256
Method of sections, 231–235, 257
Modulus of elasticity (E), 382–383, 400
Modulus of resilience (u_r), 384, 401
Modulus of rigidity (G), 394, 401
Modulus of toughness (u_t), 385, 401
Mohr's circle, 639–645, 688
plane strain, 666–670, 689
plane stress, 639–645, 688
procedures for analysis using, 641–642, 666–667
Moment of inertia (I), 290–294, 298–300, 305, 455, 531–532, 543–544, 775, 808–809
area (A), 290–294, 298–300, 305, 808–809
axis cross-sectional area, 290–294
beam cross-sectional area, 531
centroid (C) location and, 290–294, 298–300, 305
column cross-sectional area, 775
composite bodies, 298–300, 305
flexure formula and, 531–532
least, 775
parallel-axis theorem for, 291–292, 305
polar, 291, 455
procedures for analysis of, 292, 298
unsymmetric bending and, 543–544
Moments, 74–137, 311, 525, 542–547, 555, 718–720, 735–742. *See also* Moments of inertia
arbitrarily applied, 544–545
arm (distance), 75
axis (direction), 76
bending (M), 311, 525, 542–547, 555
Cartesian vector formulation, 83–85, 133
couple, 103–107, 112–117, 134–135, 737–738
curvature relationship, 719–720
deflection and, 718–720, 735–742
direction of, 76, 79–80, 82, 133
discontinuity functions and, 735–742
dot product for, 97
elastic curve and, 718–720
force about an axis, 96–99, 134
force and couple systems, simplification of, 112–117, 122–128, 135
internal (M0), 311, 718–720, 735–736
magnitude of, 76, 79–80, 82, 133
principal axis, applied to, 542–543, 555
principle of, 86–88
principle of transmissibility, 82, 112
procedures for analysis of, 114, 124
resultant (M_o), 74–137
right-hand rule for, 76, 79–80, 133
scalar formulation of, 75–78, 96, 103, 133
sign convention for, 76, 544–545
torsional (T), 311
unsymmetric bending and, 542–547, 555
vector formulation of, 82–85, 97, 103, 133
Motion, 6, 112–114, 141, 146, 167, 173, 181–190, 200–201
equilibrium and, 181–184
force systems and, 112–114
frictional forces and, 181–183, 200–201
impending, 181, 183–190, 200–201
kinetic frictional force and, 182–183
Newton's laws of, 6
rolling, 186
rotational, 112–114, 141, 146, 167, 173
slipping (sliding), 181–184
static frictional force and, 181, 183
supports for prevention of, 141, 146, 167, 173
tipping, 185
translational, 112–114, 141, 146, 167, 173
Multiforce members, 240–242

Necking, 377
Neutral axis, 526, 530, 545
Neutral equilibrium, 771–772
Newton (N), unit of, 8, 323, 461
Newton-meters, units of, 461
Newton's law of gravitational attraction, 7
Newton's laws of motion, 6
Noncircular shafts, 488–490, 497
Nonlinear elastic behavior, 380
Nonprismatic beams, 710
Normal forces, 180–181, 311–312, 343, 368
Normal strain (ϵ), 356–357, 369, 659–662
average, 356–357, 369
plane strain, components of, 659–662
Normal stress (σ), 323–330, 368, 624–625, 627–628, 687
average, 324–330, 368
axially loaded bars, 324–330
compressive, 323
equilibrium equations for, 325–326
maximum, 326
normal distribution and, 324–325
plane stress, components of, 624–625, 627–628, 687
procedure for analysis of, 327
tensile, 323
Numerical calculations, engineering use of, 10–11

Offset method, 379–380
Overhanging beams, 501

Parallel-axis theorem, 291–292, 305
Parallel force systems, 123–128, 135, 157
Parallelogram law, 18, 20, 22, 69
Particles, concept of, 5
Pascal (Pa), unit of, 281, 323
Percent elongation, 379, 400
Percent reduction of area, 379, 400
Perfectly plastic materials, 376
Permanent set, 383, 401
Pin connections, 216–217
Pin supports, 141–143, 167, 169, 772–777
Planar truss, 215
Plane strain, 657–683, 688–689
bulk modulus (k) and, 679, 689
components, 657
dilatation (e) and, 678–679, 689
general equations of transformation, 658–665, 688
in-plane, 662
material-property relationships, 676–683, 689
Mohr's circle for, 666–670, 689
normal, 659–662
principle strains, 662
procedure for analysis of, 666–667
relationships involving E, ν, and G, 678–679, 689
shear, 658–662
sign convention for, 658
strain rosettes, 674–675
transformation, 657–683, 688–689
Plane stress, 619–656, 676–683, 687–689
absolute maximum shear, 650–654, 688
bulk modulus (k) and, 679, 689

components, 624–625
dilatation (e) and, 678–679, 689
general equations of transformation, 624–626, 687
Hooke's law and, 676–677, 689
in-plane, 627–633, 687
material-property relationships, 676–683, 689
Mohr's circle for, 639–645, 688
normal, 624–625, 627–628, 687
principle stresses, 627–633, 687
procedures for analysis of, 621, 625, 641–642
relationships involving E, ν, and G, 678–679, 689
shear, 624–625, 629, 650–654, 687–688
sign convention for, 624
state of, 619–623
transformation, 619–656, 676–683, 687–689
triaxial, 650–654, 676–677, 689
Plastic material behavior, 376, 383, 400
Plate girders, 698
Poisson's ratio (ν), 392–394, 401
Polar moment of inertia, 291, 455
Position vectors, 50–55, 71
Pound (lb), unit of, 8
Power-series expansion, 806
Power transmission by shafts, 461–462
Pressure (p), units of, 281
Pressure vessels, 591–594, 615
Principal axis, bending moment applied to, 542–545, 555
Principle of moments, 86–88, 133
Principle of superposition, 421, 447
Principle of transmissibility, 82, 112
Prismatic beams, 696–703, 713
Prismatic members (bars), 324–330
Procedure for engineering analysis, 12
Product of inertia, 543
Proportional limit, 376
Pure shear, 332
Purlins, 215

Quadratic formula, 806

Radial stress, 593
Radians per second, units of, 461
Radius of curvature (ρ), 719–720, 764
Radius of gyration (r), 775
Redundants, 752–755
Relative displacement (δ), 408–415, 446
Restoring force, 770
Resultants, 18–20, 31–32, 69–71, 74–137, 281–286, 304–305, 309–317
axis, moment of force about, 96–99, 134, 282–283, 305
Cartesian vector formulation, 41–42, 71, 80–81, 83–85, 133
centroid (C) location and, 281–283, 304–305
collinear vectors, 19, 69
concurrent force systems, 122
coplanar force systems, 122–128
coplanar force, 31–32, 70
couple moments, 103–107, 112–117, 122–128, 134–135
cross product, 79–81, 133
direction of, 76, 79–80, 82, 133
directions (right-hand rule) for, 76, 79–80, 82, 103
distributed loadings, 281–286, 304–305
equivalent force systems, 112–117
force and couple moments, simplification of, 112–117, 122–128, 135
force systems, 74–137
internal loadings, 309–317
location of force, 281, 283, 304–305
magnitude of, 20, 76, 79, 82, 133, 281–282
moments (M_o), 75–78, 82–88, 96–99, 103–107, 112–117
parallel force systems, 123–128
parallelogram law for, 18, 20, 69
perpendicular force systems, 122–128
principle of moments, 86–88
principle of transmissibility, 82, 112
procedures for analysis of, 114, 124
rectangular components, 31–32, 70
scalar formulation of moment, 75–78, 96, 103, 133
triangle rule for, 18–19
Varignon's theorem, 86–88
vector addition for, 18–20
vector formulation of moment, 82–85, 97, 103–104, 133
vector subtraction for, 19
Right-hand rule, 38, 50, 76, 79–80, 82, 103, 133, 311, 454, 470–471
Rigid bodies, 5, 138–213
center of gravity (G), 144
deformation of, 140
dry friction and, 180–190, 210
equilibrium, 138–213
flat belts, 197–199, 211
free-body diagrams for, 141–149, 167–171, 208–209
frictional forces on, 197–207, 211
idealized models of, 144–145
impending motion of, 181, 184–190, 200–201
internal forces and, 139, 144
procedures for analysis of, 146, 152, 173, 186
rotational motion of, 112–114, 141, 146, 167, 173
screws, 200–202, 211
springs, 144
statically indeterminate, 173, 209
support reactions, 141–143, 167–170, 208–209
three-dimensional equilibrium, 167–179, 209
three-force members, 157–158
translational motion of, 112–114, 141, 146, 167, 173
two-dimensional (coplanar) equilibrium, 140–166, 208–209
two-force members, 157–158
weight (W), 144
Rocker supports, 142
Roller supports, 141–142, 145
Rolling, frictional forces and, 186
Roof loads, 215
Rotational motion, 112–114, 141, 146, 167, 173
force systems and, 112–114
supports for prevention of, 141, 146, 167, 173
three-dimensional rigid bodies, 167, 173
two-dimensional rigid bodies, 141, 146
Rounding off numbers, 11

Saint-Venant's principle, 405–407, 446
Scalars, 17–18, 30, 69, 75–78, 96, 103, 133, 172
coplanar forces, notation for, 30
couple moments, 103
division of a vector by, 18
equations of equilibrium, 172
moment of force about an axis, 96
multiplication of a vector by, 18
quantity, 17, 69
resultant moment, formulation of, 75–78, 96, 103, 133
sign convention for, 76, 133
Screws, frictional forces on, 200–202, 211
Secant formula, 788–793, 801
Second (s), unit of, 8
Section modulus (S), 696
Shafts, 451–462, 468–475, 481–484, 488–490, 496–497, 692–715, 716–767. *See also* Structural design
angle of twist (ϕ) and, 451–453, 468–475, 496
circular, 452–461, 468–475, 481–484, 496–497
cross sections of, 453–460, 488–490, 492–494, 496–497
deflection of, 694, 716–767
design of, 461, 692–715
frequency (f) of rotation, 461
internal torque (T) of, 454–460
noncircular, 488–490, 497
polar moment of inertia of, 455
power transmission by, 461–462
procedures for analysis of, 458, 472, 482
shear strain (γ) in, 452–453
shear stress (τ) in, 454–455
solid, 456, 461, 488–491, 497
statically indeterminate, 481–484
torque (T) applied to, 451–462, 481–484, 488–490, 496–497
torque diagram for, 457, 471
torsional deformation of, 451–453
torsional formula for, 454–460, 496

tubular, 457, 461
twisting, 451–453, 496
Shanley theory, 795
Shear, *see* Shear stress; Shear strain
Shear and moment diagrams, 501–517, 554
beams, 501–517, 554
concentrated force and moment regions, 510
couple moments in, 510
discontinuous functions of, 502–503
distributed load regions, 508–510, 554
graphical method for construction of, 508–517, 554
procedures for analysis of, 503, 511
sign convention for beams, 502, 510
slopes of, 509–511, 554
Shear flow (q), 578–582, 587
Shear force (V), 311, 343, 368
Shear formula, 561–572, 587
Shear strain (γ), 357, 369, 394, 452–453, 658–665, 688–689
angle of twist (ϕ) and, 452–453
deformation of materials by, 357, 369
maximum in-plane, 662, 665
modulus of rigidity (G) and, 394
plane strain transformation, 658–665, 688–689
torsional deformation by, 452–453
Shear stress (τ), 323, 331–335, 368, 394, 454–455, 496, 558–589, 624–625, 629, 650–654, 687–688, 694–695, 713
absolute maximum, 650–654, 688
average, 331–335, 368
beam design and, 694–695, 713
complementary property of, 332
direct (simple), 331
distribution, 323, 331–335, 368, 694–695, 713
double, 334
equilibrium equations for, 332
Hooke's law for, 394
in-plane, 629, 687
linear variation in shafts, 454–455
modulus of rigidity (G) and, 394
plane stress transformation, 624–625, 629, 650–654, 687–688
procedure for analysis of, 333
pure, 332, 394
single, 334
torsion formula and, 454–455, 496
transverse shear, 558–589
ultimate, 394
Shear stress–strain diagrams, 394–397, 401
Significant figures, 11
Simple connections, stress and design of, 343–347, 369
Simply supported beams, 501, 814
Single shear, 334
Singularity functions, 737–738
Slenderness ratio, 775–776, 779, 794–795
Sliding vector, 82, 112
Slipping, frictional forces and, 181–184, 210
Slopes, 509–511, 554, 717–718, 721–729, 764, 814–815
cantilevered beams, 815
deflections and, 717, 721–729, 814–815
displacement and, 721–729, 764
elastic curve and, 717–718, 721–729, 764, 814–815
integration of, 721–729, 764
negative, 509
procedure for analysis of, 511, 725
shear and moment diagrams, 509–511, 554
sign convention for, 510, 723
simply supported beams, 814
Slug, unit of, 8
Small strain analysis, 359
Specific weight, 273
Spherical pressure vessels, 593
Springs, free-body diagrams of, 144
Stability, equilibrium and, 173, 209, 770
Static friction (ϕ_s), angle of, 181
Static friction (μ_s), coefficient of, 181–182, 210
Statically indeterminate systems, 173, 209, 422–427, 434, 447, 481–484, 497, 752–759, 765
axially loaded members, 422–427, 434, 447
compatibility (kinematic) conditions for, 422, 752–759
deflection of, 752–759, 765
degree of indeterminacy, 752
equilibrium and, 173, 209
force method for, 753
load–displacement relationship, 422, 447
procedures for analysis of, 423, 482, 756
redundants, 752–755
shafts, 481–484, 497
superposition, method of for, 752–759, 765
thermal stresses in, 434
torque-loaded members, 481–484, 497
Statics, engineering study of, 3–4
Steel sections, 697
Stiffness of materials, 383
Strain (ϵ), 308–367, 375, 377–378, 383, 392–393, 493, 527–529, 657–683, 688–689. *See also* Shear strain
average, 356–357
bending deformation and, 527–529
Cartesian components, 358
deformation and, 309, 355–362, 369, 377–378
elastic, 383
engineering (nominal), 375
inelastic, 493
internal resultant loadings and, 309–317, 368
lateral contraction (negative), 392
longitudinal elongation (positive), 392, 527–529
mechanics of materials and, 309–317
normal (ϵ), 356–357, 369, 659–662
plane transformation, 657–683, 688–689
plastic, 383
Poisson's ratio (ν) and, 392–394
small strain analysis, 359
state of, 358
true, 377–378
units of, 356–357, 369
Strain energy, 384–387, 401
Strain hardening, 377, 383, 400
Strain rosettes, 674–675
Strain transformation, *see* Plane strain
Stress (σ), 308–367, 375–378, 434–437, 440–444, 447, 529–535, 555, 590–617, 619–656, 676–683, 687–689, 694–695, 713, 775–776. *See also* Shear stress
allowable, 342–343
average, 324–335, 368
axially loaded members, 324–330, 368, 434–437, 440–444, 447
beam design and, 694–695, 713
bending deformation and, 529–535, 555
biaxial, 593
combined loadings and, 590–617
compressive, 323
critical, 775–776
deformation and, 309, 355, 369, 376–378
direction of for pressure vessels, 592–593
distribution of, 322–335, 368, 529–535, 694–695, 713
engineering (nominal), 375
equilibrium equations for, 325–326, 332
factor of safety (F.S.), 342–343, 369
failure of, 342–343
flexure formula, 529–535, 555
force diagrams for, 326
fracture, 377
internal resultant loadings and, 309–317, 368
load distribution and, 322–334
material-property relationships, 676–683, 689
maximum, 326
mechanics of materials and, 309–317
normal (σ), 323–330, 368
plane transformation, 619–656, 676–683, 687–689
pressure vessels, 591–594
procedures for analysis of, 327, 333, 344, 598–599
radial, 593
simple connections, design of and, 343–347, 369
state of, 323, 598–605, 619–623
structural design and, 343–347, 694–695, 713
superposition of components, 598–605, 615
tensile, 323
thermal, 434–437, 447
trajectories, 695
triaxial, 650–654, 676–677, 689
true, 377–378
ultimate, 377

uniaxial, 326
units of, 323
yield point, 376
Stress concentrations, 440–444, 447, 492–494, 497, 550–552, 555
beams, 550–552, 555
bending and, 550–552, 555
brittle material failure from, 442, 493
cross-sectional area of, 440–444, 447, 492–494, 497
factor (K), 441–442, 447, 492–493, 497, 550–551, 555
fatigue loading failure from, 442, 493
inelastic strains and, 493
torsional loads, 492–494, 497
Stress–strain diagrams, 375–383, 394–397, 400–401
brittle materials, 381, 400
conventional, 375–377
ductile materials, 379–380, 400
elastic behavior, 376, 380, 382–383, 400
elastic limit, 376
engineering (nominal) stress and strain, 375
failure of materials and, 377–381, 400
fracture stress, 377
Hooke's law, 382–383
necking, 377
offset method, 379–380
plastic deformation, 376, 400
proportional limit, 376
shear, 394–397, 401
strain hardening, 377, 383, 400
true, 377–378
ultimate stress, 377
yield point (stress), 376
yield strength, 379–380
yielding, 376, 400
Stress transformation, *see* Plane stress
Stringers, 216
Structural analysis, 214–259
frames, 240–248, 257
free-body diagrams, 218–219, 224–225, 231–232, 240–242
machines, 240–248, 257
method of joints, 218–226, 256
method of sections, 231–235, 257
procedures for analysis, 219, 233, 243
trusses, 215–239, 256–257
zero-force members, 224–226
Structural design, 215–216, 343–347, 692–715, 716–767, 768–803
allowable stress, 343–347
beams, 692–715, 716–767
bearing plates, 694
bending and, 694–695, 713
built-up sections, 698, 713
columns, 768–803
cross-sectional areas and, 696–699, 713
deflection and, 694, 716–767
fabricated, 697–698
factor of safety (F.S.), 343–347
floor beams, 216
fully stressed, 710–711, 713
glulam, 698
joints, 215–226
plate girders, 698
prismatic, 696–703, 413
procedures for analysis of, 344, 699
section modulus (S) for, 696
shafts, 692–715, 716–767
stress distribution, 694–695, 713
simple connections, 343–347
steel sections, 697
stress trajectories for, 695
trusses, 215–216
wood sections, 697
Subscript designation, 51
Superposition, 421, 447, 544–545, 555, 598–605, 615, 745–749, 752–759, 765
combined loadings, stress components in, 598–605, 617
deflection and, 745–749, 752–759, 765
principle of, 421, 447
procedure for analysis using, 598–599, 756
statically indeterminate beams and shafts, 752–759, 765
unsymmetric bending and, 544–545, 555
Support reactions, 141–143, 167–170, 208–209
ball and sockets, 168, 170
bearings, 167–170
cables, 142, 168
fixed, 141, 143, 169
free-body diagrams of, 141–143, 167–170, 208–209
hinges, 141–143, 167, 169
pins, 141–143, 167, 169
rigid-body equilibrium and, 141–143, 167–170, 208–209
rockers, 142
rollers, 141–142, 168
rotational motion and, 141, 146, 167, 173
three-dimensional systems, 167–170, 209
translation and, 141, 146, 167, 173
two-dimensional (coplanar) systems, 141–143, 208

Tangent modulus (E_t), 794–795
Tensile forces, truss members, 217–219, 231, 256
Tensile stress, 323
Tension test, 373–374, 400
Thermal stress, 434–437, 447
Thin-walled pressure vessels, 591–594, 615
Threads, impending motion and, 200–210
Three-dimensional equilibrium, 167–179, 209
equations of equilibrium, 172–175, 209
free-body diagrams for, 167–171, 209
procedure for analysis of, 173
scalar equations of, 172
support reactions and, 167–170, 209
vector equations of, 172
Three-force members, equilibrium of, 157–158
Time, fundamental quantity of, 4
Time, units of, 8
Tipping frictional forces and, 180, 185, 210
Torque (T), 75, 451–462, 468–475, 481–484, 488–490, 496–497. *See also* Moments
angle of twist (ϕ) and, 451–453, 468–475, 496
circular shafts, 451–462, 468–475, 496–497
constant, 469–470
external, 451–453
internal, 454–460, 468–475
moment of a force, as, 75
multiple, 470
noncircular shafts, 488–490, 497
procedures for analysis of, 458, 472, 482
sign convention for, 470–471, 496
statically indeterminate members and, 481–484, 497
twisting and, 451–453, 496
Torque diagram, 457, 471
Torsion, 450–499. *See also* Torque
angle of twist (ϕ), 451–453, 468–475
deformation, 451–453
power transmission, 461–462
procedures for analysis of, 458, 472, 482
shafts, 451–462, 468–475, 481–484, 488–490, 492–494, 496–497
shear strain (γ) and, 452–453
shear stress (τ) and, 454–455
statically indeterminate torque-loaded members, 481–484, 497
stress concentration and, 492–494, 497
torsional formula, 454–460, 496
Torsional formula, 454–460, 496
polar moment of inertia for, 455
procedure for analysis using, 458
shear stress (τ), linear variation in, 454–455, 496
solid circular shafts, 456
torque diagram for, 457
tubular circular shafts, 457
Torsional moment (T), 311, 368
Transformation equations, 624–626, 658–665, 687–688
plane strain, 658–665, 688
plane stress, 624–626, 687
sign conventions for, 624, 658
Translational motion, 112–114, 141, 146, 167, 173
force systems and, 112–114
supports for prevention of, 141, 146, 167, 173
three-dimensional rigid bodies, 167, 173
two-dimensional rigid bodies, 141, 146
Transverse shear, 558–589
beams, 559–572, 587
built-up members, 578–582, 587
procedure for analysis of, 566
shear flow (q), 578–582, 587
shear formula for, 561–572, 587
straight members, 559–561

Triangle rule, 18–19, 69
Triaxial stress, 650–654, 676–677, 689
Trigonometric identities, 806
True stress–strain diagrams, 377–378
Trusses, 215–239, 256–257
  bridges, 216
  design assumptions, 216–217
  force direction "by inspection," 219, 224, 232
  free-body diagrams, 218–219, 224–225, 231–232
  joints, 215–226, 256
  member forces of, 217–219, 231, 256–257
  method of joints, 218–226, 256
  method of sections, 231–235, 257
  planar, 215
  procedures for analysis of, 219, 233
  purlins, 215
  roof loads, 215
  simple, 217–219, 256
  structural analysis of, 215–239, 256–257
  tensile and compressive forces of members, 217–219, 231, 256
  zero-force members, 224–226
Twisting, 451–453, 496. *See also* Angle of twist; Torque
Two-dimensional equilibrium, 140–166, 208
  center of gravity (G) and, 144
  coplanar force systems, 140–166, 208
  equations of equilibrium, 151–156, 208–209
  free-body diagrams for, 141–149, 208
  internal forces and, 144
  procedure for analysis of, 146
  springs, 144
  three-force members, 157–158
  two-force members, 157–158
  weight (W) and, 144
Two-force members, equilibrium of, 157–158, 217

U.S. Customary units, 8–9
Ultimate stress, 377, 394, 400
Uniaxial stress, 326
Unit vectors, 53–55, 71
Units of measurement, 7–10, 281, 323, 356–357, 369, 461
  base units, 7, 10
  conversion of, 9
  derived units, 7–8, 10
  International System (SI), 8–10
  force, 8
  length, 8
  mass, 8
  power, 461
  pressure, 281
  SI prefixes, 9, 323
  SI symbols, 10
  stress, 323
  strain, 356–357, 369
  time, 8
  U.S. Customary, 8–9
Unsymmetric bending, 542–547, 555
  internal bending moment (M), 542–547, 555
  moment arbitrarily applied, 544–545
  moment of inertia (I) for, 544
  neutral axis orientation, 545
  principal axis, moment applied to, 542–543, 555
  superposition for, 544–545, 555
Upper yield point, 376

Varignon's theorem, 86–88
Vectors, 16–73, 79–85, 97, 103–107, 112–117, 133–135, 172
  addition, 18–19, 20–25, 30–32, 41–45, 50–52, 69
  angles, 17, 39–41, 61, 70–71
  Cartesian, 31, 38–45, 50–52, 60, 70, 80–81, 83–85, 133
  collinear, 19, 69
  components of force, 18, 20–21, 30–35
  coplanar forces, 30–34
  couple moments, 103–107, 112–117, 134–135
  cross product, 79–81, 133
  directed along a line, 53–55, 61, 71
  direction of, 17, 39–42, 53–55, 79–80, 82
  division of by a scalar, 18, 69
  dot product, 60–64, 71
  equations of equilibrium, 172
  force and, 16–73, 112–117, 134–135
  free, 112–113
  line of action, 17, 98, 112, 134
  magnitude of, 17, 22, 30–31, 39, 69–70
  moment of force about an axis, 97
  multiplication by a scalar, 18, 69
  parallelogram law for, 18, 20, 22, 69
  position, 50–52, 71
  procedures of analysis for, 22
  resultant moments and, 82–85, 97, 103–104, 133
  resultants of a force, 18–20, 31–32, 41–42, 69–71, 79–85, 133
  right-hand rule for, 38, 50
  scalars and, 17–18, 30, 69
  several forces, 21
  sliding, 82, 112
  subtraction, 19
  three-dimensional components, 38–39, 50–55
  triangle rule for, 18–19, 69
  unit, 53–55, 71
  x, y, z coordinates, 38, 50, 70–71
Volume (V), 263, 281, 304, 808–809

Warping cross sections, 488, 560–561
Watts (W), units of, 461
Weight (W), 7, 144, 261–262, 273, 304–305
  center of gravity (G) and, 144, 261–262, 273, 304–305
  composite bodies and, 273, 305
  free-body diagrams and, 144
  rigid-body equilibrium and, 144
  specific, 273
Wide-flange beams, 564, 697, 713, 810–813
Wood sections, 697

x, y, z coordinates, 38, 50, 70–71

Yield strength, 379–380
Yield stress (point), 376, 400–401
Yielding, 376, 400
Young's modulus (E), 382–383

Zero force, 140
Zero-force members, 224–226